Single Particle Hopping Between Luttinger Liquids: A Spectral Function Approach

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Abstract

We present a pedagogical account of our approach to the problem of Luttinger liquids coupled by interliquid single particle hopping. It is shown that the key issue is that of coherence/incoherence of interliquid hopping, and not of relevance/irrelevance in a renormalization group sense. A clear signal of coherence, present in the case of coupled Fermi liquids, is absent for Luttinger liquids, and we argue for the existence of an incoherent regime when the interliquid hopping rate is sufficiently small. The problem is relevant to any sufficiently anisotropic, strongly correlated metal, and in particular to understanding the anomalous c-axis conductivity in the cuprate superconductors, and the physics of the quasi-1D organic conductors.

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I. INTRODUCTION

The development of a complete theoretical understanding of the high-temperature superconducting cuprates (HTSC’s) represents one of the most challenging tasks presently facing the field of condensed matter physics. Apart from the extraordinarily high superconducting transition temperatures, $T_c$, these materials exhibit a bewildering array of anomalous normal state properties. It is clear that the normal state is not a (Landau) Fermi liquid: on this point there is practically universal agreement. On the other hand, the mechanism giving rise to the non-Fermi liquid normal state is still controversial. Nevertheless, it would seem perverse to reject the notion that there is an intimate connection between high $T_c$ and the fact that the normal state is an anomalous metal.

For a long time, Anderson has emphasized another aspect of the experimental data for the normal state. This is the qualitative difference between the in-plane (ab-plane) and inter-plane (c-axis) physics. Indeed, the very use of the term ‘in-plane’ presupposes such a qualitative difference: the c-axis transport is not simply the same as the ab-plane transport up to a scaling factor to account for anisotropy. This is an important point, for recent studies [1] on Sr$_2$RuO$_4$, a structural analogue of La$_2$CuO$_4$, show that the low-temperature dc-resistivity has the same temperature dependence in both the ab-plane and c-axis directions, with an anisotropy of up to $\approx 500$. It is no coincidence that this temperature dependence is Fermi-liquid like.

An extensive account of c-axis experiments in the cuprates has been given by Cooper and Gray [2]. As far as the c-axis conductivity, $\sigma_c(\omega)$, is concerned, the key observations for superconducting samples are a dc-conductivity of order of or well below the Mott limit of minimum metallic conductivity, and a frequency dependence which is completely incoherent in the sense of the absence of a Drude term [2–4]. It has been argued that YBa$_2$Cu$_3$O$_7$ has a Drude-like component [4,5], but the width is anomalously large.

The underlying idea of Anderson’s “confinement” hypothesis [6] toward understanding the strange behavior of the c-axis conductivity is to associate the observed incoherent con-
ductivity with the non-Fermi liquid ab-plane properties. From this point of view, it is significant that in overdoped samples of La$_{2-x}$Sr$_x$CuO$_4$ the appearance of a Drude-like term in $\sigma_c(\omega)$ coincides with a crossover to Fermi liquid-like ab-plane properties [3].

In Bi$_2$Sr$_2$CaCu$_2$O$_8$, this picture of incoherent c-axis transport is confirmed by the photoemission experiments of Ding et al. [7]. These authors observe a single Fermi surface, up to a resolution of $\sim 10 meV$. This is in strong disagreement with the band theory prediction of two Fermi surfaces split by an energy of $\sim 400 meV$, this energy splitting being the direct result of coherent single particle hopping between adjacent CuO$_2$ planes.

The issue we address in this paper is relevant to any sufficiently anisotropic, strongly correlated metal, and therefore in particular to the quasi-1D metals. We have already proposed one application of our ideas to partially understanding the anomalous magnetoresistance in the quasi-1D organic conductor (TMTSF)$_2$PF$_6$ [8]. An extensive discussion of this and of more recent experiments supporting our point of view [9] will appear elsewhere [10].

Given the acceptance of a non-Fermi liquid (NFL) state in the planes, the experimental observations in the cuprates force us to carefully examine the problem of 2D NFL’s coupled by single particle hopping terms which transfer real electrons from one liquid to a physically adjacent one. The Hamiltonian we consider is simply

$$H = \sum_i H_{NFL}^{(i)} + t_\perp \sum_{i,x} \{c_{i,\sigma}^\dagger(x)c_{i+1,\sigma}(x) + h.c.\}$$

i.e., $t_\perp$ is local in real space, and thus $k$-diagonal in momentum space. Of great import is the realization that real electrons are not eigenstates of the NFL: there are no electron-like quasiparticles, i.e., $Z = 0$. As such, removing an electron from, or adding an electron to, one of the liquids will necessarily “disturb” the liquid in some sense. The effect is similar to the way in which the flipping of a Kondo spin causes an orthogonal rearrangement of the Fermi sea with which it is in contact.

It is also important to realize that the central issue of concern here is not one of relevance or irrelevance of $t_\perp$ in the renormalization group (RG) sense. In fact, the existence of a sharp Fermi surface is sufficient evidence of itself to point to the relevance of $t_\perp$. Within a Luttinger
liquid framework, the singularity of \( n(k) \) near \( k_F \) behaves as

\[
\frac{\partial n(k)}{\partial k} \sim -|k - k_F|^{2\alpha - 1}
\]  

At the bare level of the RG \( t_\perp \) is irrelevant iff \( 2\alpha > 1 \), in which case there would not be a sharp Fermi surface at all! A sharp Fermi surface therefore implies the relevance of \( t_\perp \). Nevertheless, the relevance of \( t_\perp \) in the RG sense does not guarantee that single particle interliquid hopping will be coherent. An example is afforded by the problem of a two-level system (TLS) coupled to a dissipative bath. Here it is known that there are situations where the tunneling operator between the two (usually degenerate) states is relevant, yet nevertheless tunneling is completely incoherent.

Before briefly discussing the TLS problem, it is helpful to begin with some very simple, yet instructive, considerations.

**II. FERMI’S “GOLDEN RULE”**

Consider the textbook derivation of the so-called “Golden Rule” of Fermi: a particle is in state \( |i\rangle \), an eigenstate for all times \( t < 0 \). At time \( t = 0 \) a perturbation \( V \) is turned on. To \( O(V^2) \) the probability for the particle to be observed in state \( |f\rangle \) at time \( t > 0 \) is

\[
\left| \frac{-i}{\hbar} \int_0^t dt' \langle f | V | i \rangle e^{i\hbar(E_f - E_i)t'} \right|^2
\]

\[
= |V_{if}|^2 \left| \frac{e^{i\Delta E t/\hbar} - 1}{\Delta E} \right|^2
\]

\[
= 4 \frac{|V_{if}|^2}{(\Delta E)^2} \sin^2 \left( \frac{\Delta E t}{2\hbar} \right)
\]

Assuming \( V_{if} \) depends only on \( \Delta E \), the probability to find the particle in its original state \( |i\rangle \) at time \( t > 0 \) is then just

\[
P(t) = 1 - 4 \sum_f \frac{|V_{if}|^2}{(E_f - E_i)^2} \sin^2 \left( \frac{(E_f - E_i)t}{2\hbar} \right) + O(V^4)
\]

\[
\to 1 - 4 \int d\epsilon |V(\epsilon)|^2 \rho(\epsilon) \frac{\sin^2(\epsilon t/2\hbar)}{\epsilon^2}
\]

(3)
where \( \rho(\epsilon) \) is the density of final states of energy \( \epsilon \) relative to \( E_i \). If it is now assumed that 
\[
|V(\epsilon)|^2 \rho(\epsilon) \approx V^2 \rho_0,
\]
a constant, over some interval \([-\Lambda, \Lambda]\) of \( \epsilon \), then
\[
1 - P(t) \approx 4V^2 \rho_0 \int_{-\Lambda}^{\Lambda} d\epsilon \frac{\sin^2(\epsilon t/2\hbar)}{\epsilon^2}
\]
\[
\xleftarrow{\Lambda t/2\hbar \gg 1} 4V^2 \rho_0 \int_{-\infty}^{\infty} d\epsilon \frac{\sin^2(\epsilon t/2\hbar)}{\epsilon^2}
\]
\[
= \frac{2\pi}{\hbar} (V^2 \rho_0) t
\]
(4)

Thus, to lowest order in the perturbation, the rate at which the particle leaves \( |i\rangle \) is given by
\[
\Gamma = \frac{2\pi}{\hbar} (V^2 \rho_0)
\]
which is Fermi’s Golden Rule.

In many applications of the Golden Rule it is assumed, usually without proof, that this lowest order result can be extended to all \( t \) via exponentiation to give \( P(t) = e^{-\Gamma t} \).

There are two crucial steps in the derivation of the Golden rule. The first is the replacement of 
\[
|V(\epsilon)|^2 \rho(\epsilon)
\]
by a constant \( V^2 \rho_0 \). This is a reasonable approximation as long as 
\[
|V(\epsilon)|^2 \rho(\epsilon)
\]
is nonsingular in the interval \([-\Lambda, \Lambda]\). The second is the assumption that 
\( \Lambda t/2\hbar \gg 1 \). This requires the spectral weight to be spread out over an energy scale larger than \( \hbar/t \).

A trivial example of a situation where the Golden Rule is invalid is the case where there is just one final state, \( |f\rangle \). Then
\[
P(t) = 1 - \frac{|V_{if}|^2}{(E_f - E_i)^2} \sin^2 \left( \frac{(E_f - E_i)t}{2\hbar} \right)
\]
\[
\xrightarrow{E_f \rightarrow E_i} 1 - \frac{|V_{if}|^2 t^2}{\hbar^2}
\]
(5)

Such a result precludes any interpretation in terms of a decay rate. Indeed, this trivial problem is solved exactly by diagonalization of the Hamiltonian
\[
H = \begin{pmatrix}
E_i & V^* \\
V & E_f
\end{pmatrix}
\]
The oscillation frequency of \( P(t) \) is
\[
\hbar \omega_{osc} = [(\Delta E)^2 + 4|V|^2]^{1/2}
\]
Note that the \( O(V^2) \) perturbative calculation does not directly give this oscillation frequency. Care must be taken in interpretation even in the case \( 4|V|^2/(E_f - E_i)^2 \ll 1 \). Here, the perturbation expansion appears controlled, \( P(t) \) oscillating between 1 and \( 1 - 4|V|^2/(E_f - E_i)^2 \) with frequency \((E_f - E_i)/\hbar\), yet the exact solution shows that the true oscillation frequency is \( \omega_{osc} \). The \( O(V^2) \) calculation fails to pick up the frequency shift. The oscillation of \( P(t) \) is most striking in the degenerate case \( E_f = E_i \). In this case the \( O(V^2) \) calculation picks up the first term in an expansion of the exact result
\[
P(t) = \cos^2(|V|t/\hbar) = \frac{1}{2} \left\{ \cos \left( \frac{2|V|t}{\hbar} \right) + 1 \right\}
\]
(6)
The particle oscillates between \(|i\rangle\) and \(|f\rangle\) with frequency \( 2|V|/\hbar \). Such spectacular oscillation effects occur in the famous \((K_L, K_S)\) system in particle physics, and were at one time proposed as a possible resolution of the solar neutrino problem. The effect is the hallmark of a truly “quantum” system and is at the heart of measurement theory and the kinds of \textit{gedanken} problems which began with Schrödinger’s worries about cats.

The situation where there is just a single final state corresponds to the spectral density \( \rho(\omega) = \delta(\omega) \). One can consider other types of singular densities of states, perhaps the simplest being those of “edge” type,
\[
\rho(\epsilon) = (1 - \gamma)\Lambda^{\gamma-1}\theta_+ (\epsilon)\epsilon^{-\gamma}
\]
\( (\gamma < 1) \). To \( O(V^2) \) (assuming \( V \approx \text{constant} \))
\[
1 - P(t) = 4V^2 \int_0^\Lambda d\epsilon \epsilon^{-(\gamma+2)} \sin^2(\epsilon t/2\hbar)
\]
\[\xrightarrow{\Lambda \gg \hbar} \Lambda^{-1}4V^2 \left( \frac{t}{2\hbar} \right)^{1+\gamma} \int_0^\infty dx \frac{\sin^2 x}{x^{2+\gamma}}\]
(7)
For \( \gamma < -1 \), \( P(t) \to 1 \) for all \( t \) in the limit \( V \to 0 \): the particle is localized in the state \(|i\rangle\), and \( V \) is an irrelevant perturbation. \( \gamma \to 0 \) corresponds to a uniform density of states, in
which case the Golden Rule is applicable, while \( \gamma \to 1 \) corresponds to the extreme coherence limit in which the Golden Rule fails. A natural question to ask is: what is the true nature of \( P(t) \) in the intermediate cases \(-1 < \gamma < 1\)? Such is the question studied in the context of a more particular problem, that of a two-level system (TLS) in contact with a dissipative bath.

III. A SUMMARY OF THE TWO-LEVEL SYSTEM PROBLEM

Following Ref. [11], we define the two level system model by the Hamiltonian:

\[
H_{\text{TLS}} = \frac{1}{2} \Delta \sigma_x + \frac{1}{2} \epsilon \sigma_z + \sum_i \left\{ \frac{1}{2} m_i \omega_i x_i^2 + \frac{1}{2} p_i^2 / m_i \right\} + \frac{1}{2} \sigma_z \sum_i C_i x_i
\]

(8)

Here \( C_i \) is the coupling to the \( i \)th oscillator, and \( m_i, \omega_i, x_i \) and \( p_i \) are the mass, frequency, position and momentum of the \( i \)th oscillator, respectively.

The model describes a single quantum mechanical degree of freedom which can be in either of two states and which is coupled to a bath of harmonic oscillators. The \( \sigma_i \) are Pauli matrices, so we will refer to the two-state degree of freedom as a ‘spin’ for convenience. We are primarily interested in the \( \epsilon = 0 \) case and subsequent discussion refers to this case unless otherwise stated. In this case, one may think of the model as one in which a particle tunnels at a rate \( \Delta / 2 \) between two degenerate states (labelled by \( \sigma_z = \pm 1 \)). The environment, represented by the bath of oscillators, influences the tunneling because the bath is sensitive to which of the states the spin is in.

The TLS model provides the prototypical example of a quantum to classical crossover. For \( C_i = 0 \) the model represents the quantum mechanics of an isolated two state system, whereas for sufficiently strong coupling to the environment the dynamics of the spin, if followed without reference to the oscillator bath, are dissipative and no quantum coherence effects are observable [11]. In fact, this is how one generally expects classical behavior to emerge for macroscopic systems: the macroscopic degrees of freedom exchange energy
with an enormous number of unobserved microscopic degrees of freedom and are therefore unable to maintain a definite phase long enough for quantum interference effects to manifest themselves.

Consider first the case where the coupling of the spin to the environment vanishes, i.e. $\alpha = 0$, and where the spin is prepared in an eigenstate of $\sigma_z$. The exact eigenstates of the spin are the $\sigma_x$ eigenstates which are split by an energy $\Delta$. The initial state of the system is therefore a superposition of these two states with a definite phase between them. Since the two states have different energy, this phase is not time independent. For vanishing coupling to the environment the phase remains well defined indefinitely. The time dependence of the phase therefore results in observable oscillations in the expectation value of $\sigma_z$, in fact (in units where $\hbar = 1$) $\langle \sigma_z(t) \rangle = \cos \Delta t$. Such oscillations are a quantum interference effect.

In general, we would expect such oscillations to also occur when the spin is coupled to the environment provided the spin is capable of flipping without exchanging an amount of energy with its environment sufficient for the randomization of the phases of the various states in the superposition. Indeed [11], in the TLS model the oscillations persist for a range of couplings to the environment, albeit with coupling dependent damping of the oscillations. Such damping results from the exchange of energy between the spin and the environment during the course of a typical flip.

In order to study the quantum oscillations, or lack thereof, in a TLS, it is natural, in the light of the discussion above, to invoke the following prescription. First, the system is prepared by clamping the spin into the $\sigma_z = 1$ state for all $t < 0$, allowing the oscillator bath to relax to equilibrium. The spin is then released at $t = 0$, and one attempts to determine $\langle \sigma_z(t) \rangle$ for positive times, $t$. This is, quite generally, an appropriate quantity to study for questions about macroscopic quantum coherence, for if the spin represents a generic macroscopic quantum degree of freedom which the experimenter can observe and control, whereas the oscillators represent microscopic degrees of freedom which are beyond both control and observational capacities of the experimenter, then the above “clamping” prescription is exactly the sort of preparation which is possible experimentally.
of $\langle \sigma_z(t) \rangle$ is equivalent to determining $P(t)$, the probability of observing the spin in the $\sigma_z = 1$ state at time $t > 0$ \[12\]. The two are simply connected by $\langle \sigma_z(t) \rangle = 2P(t) - 1$. The signature of quantum coherence in $\langle \sigma_z(t) \rangle$ will be the presence of oscillations (damped or otherwise) in contrast to the incoherent relaxation ($\langle \sigma_z(t) \rangle \sim e^{-\Gamma t}$) which must result if the spin exchanges sufficient energy with the environment to randomize its phase in tunneling between the two $\sigma_z$ eigenstates. The transition to purely incoherent relaxation is indeed found to occur in the TLS problem even at short times \[11\] when $\alpha > \frac{1}{2}$.

Within this formulation of the TLS problem it is convenient to make the canonical transformation

$$H'_{TLS} = \hat{U} H_{TLS} \hat{U}^{-1}$$

(9)

where

$$\hat{U} = \exp \left\{ -\frac{1}{2} \sum_i C_i \frac{\sigma_z}{m_i \omega_i^2} \hat{p}_i \right\}$$

(10)

$\hat{p}_i$ is the momentum operator of the $i$th oscillator. The new Hamiltonian takes the form

$$H'_{TLS} = \frac{1}{2}\Delta(\sigma^+ e^{-i\Omega} + h.c.) + H_{oscillators}$$

(11)

where $\Omega = \sum_i \frac{C_i}{m_i \omega_i} \hat{p}_i$. This transformation is very helpful in guiding one’s intuition for it has removed all coupling of the spin to the oscillator bath. One can now introduce oscillator creation and annihilation operators, $\hat{a}_i^\dagger$, $\hat{a}_i$, and in so doing one observes that the price paid for removing all coupling of the spin to the bath is that the tunneling operator between the two states has been replaced by an operator which creates and destroys excitations of the bath as well as changing the state of the spin. It is then clear that quantum oscillations will be in danger of being wiped out should the low-frequency oscillator density of states and/or the low-frequency couplings $C_i$ be sufficiently large.

For our purposes, we shall only consider the so-called ohmic regime \[11\] where the density of states of the bath, and its couplings, $C_i$, to the spin, are such that the “spectral density”, $\tilde{J}(\omega)$, of the bath is given by
\[ J(\omega) \equiv \frac{\pi}{2} \sum_i \frac{C_i}{m_i \omega_i} \delta(\omega - \omega_i) \]
\[ = 2\pi \alpha \omega \exp(-\omega/\omega_c) \]

\( \alpha \) is a positive constant measuring the strength of the coupling to the bath and \( \omega_c \) is an ultraviolet cutoff.

In the ohmic regime the two point correlation function of \( \sigma^+ e^{-i\Omega} \) is
\[
\langle \sigma^+ e^{-i\Omega(t)} \sigma^- e^{i\Omega(0)} \rangle = \exp\left\{- \int_0^\infty \frac{1 - e^{-i\omega t}}{\omega^2} J(\omega) \right\}
\]
\[ = \exp\left\{-2\alpha \int_0^\infty \frac{1 - e^{-i\omega t}}{\omega} e^{-\omega/\omega_c} \right\} \]
\[ \sim e^{i\pi \alpha (\omega_c t)^{-2\alpha}} \]

From the correlation function we can immediately construct the spectral function of the operator \( e^{i\Omega} \) in the low energy, universal regime:
\[
\rho_\Omega(\omega) = \sum_m |\langle m | e^{i\Omega} | GS \rangle|^2 \delta(\omega - E_m)
\]
\[ = \Gamma^{-1}(2\alpha) \theta_+(\omega) \omega^{-1+2\alpha} \omega^{-2\alpha} e^{-\omega/\omega_c} \]

where \( \{m\} \) is a complete set of oscillator eigenstates with energies \( E_m \) and \( |GS\rangle \) is the oscillator ground state. The spectral function is of “edge” type, and the physics is largely determined by whether or not the edge singularity is divergent as \( \omega \to 0 \).

We may now construct the short time approximation to \( P(t) \) by using the spectral function above and ordinary time dependent perturbation theory. We find
\[
P(t) = 1 - \frac{\Delta^2}{2} \int d\omega \rho_\Omega(\omega) \frac{\sin^2(\omega t)}{\omega^2} + \cdots
\]
Notice that when \( \alpha > 1 \), \( \rho_\Omega(\omega) \sim \omega^{-1+2\alpha} \) results in an infrared convergent \( P(t) \). In the limit \( \Delta \to 0 \), \( P(t) \to 1 \) for all \( t \). This corresponds to the irrelevance of \( \Delta \) and to localization of the spin.

For \( \alpha \to 0 \), \( \rho_\Omega(\omega) \to \delta(\omega) \) and \( P(t) \to 1 - \frac{\Delta^2}{2} t^2 + ... \), in agreement with the expansion of the exact \( \alpha = 0 \) result \( P(t) = \cos^2(\Delta t/2) \) or, equivalently, of \( \langle \sigma_z(t) \rangle = \cos(\Delta(t)) \). For \( 0 < \alpha < 1 \) we are in a more complicated region. Clearly the difference between \( P(t) \)
and $1$ grows to order unity for any arbitrarily small $\Delta$ throughout this region (this simply reflects the renormalization group relevance of $\Delta$) and one might at first sight be tempted to conclude that $P(t)$ would undergo damped oscillations with a period approximately given by the time at which $\Delta^2 \int d\omega \rho_0(\omega) \frac{\sin^2(\omega t)}{\omega^2} \sim 1$. However, for $\alpha = \frac{1}{2}$, the spectral function is flat and featureless out to the cutoff scale. A flat spectral function is exactly the condition under which the Golden Rule approximation should be valid, implying incoherent decay without any recurrence effects or oscillations.

We may scale out the time dependence in (15) to obtain to $O(\Delta^2)$ and for $\omega_c t \gg 1$

$$P(t) \approx 1 - \alpha \Delta^2 \omega_c^{-2\alpha} t^{2-2\alpha} \int_0^{\infty} dx \frac{\sin^2 x}{x^{3-2\alpha}}$$

(16)

(for simplicity, we have replaced the cutoff $e^{-\omega/\omega_c}$ by a hard cutoff at $\omega_c$). Thus, for $\alpha > \frac{1}{2}$, where the spectral function for the tunneling operator is vanishing at low frequencies, we see that the $O(\Delta^2)$ term in $P(t)$ grows even slower than $t$, suggesting an even “more incoherent” decay of $P(t)$. If we define $\Gamma(t) = -dP(t)/dt$, the rate at which the spin flips, then in this regime $\Gamma(t)$ is bounded for all $t$. For the special value $\alpha = 1/2$, $\Gamma(t) = \Gamma$, a constant, and a naive re-exponentiation of the Golden Rule is $P(t) = (1 + e^{-\Gamma t})/2$, corresponding to $\langle \sigma_z(t) \rangle = e^{-\Gamma t}$. For $\alpha > 1/2$ it would appear reasonable to expect exponential relaxation, too. A self-consistent approximation to determining the relaxation rate $\Gamma$ for small $\Delta/\omega_c$ involves cutting off the $\omega$-integral at $\omega \sim \Gamma$ to give $\Gamma \sim \Delta^2 \omega_c^{-2\alpha} \Gamma^{2\alpha-1}$ yielding

$$\Gamma \approx \Delta \left( \frac{\Delta}{\omega_c} \right)^{\alpha/(1-\alpha)}$$

The right hand side is in fact nothing but $\Delta_{\text{ren}}$, the renormalized tunneling rate which emerges from an RG analysis.

The true behavior of $P(t)$ in the region $1/2 < \alpha < 1$ is actually not rigorously known, but there are reasons for believing that the self-consistent argument given above is not too far from the truth. The true decay of $P(t)$ is probably not simply exponential relaxation, but the key point is that there are not any oscillations. Thus, despite the fact that the RG approach yields the same scale $\Delta_{\text{ren}}$ as the self-consistent approach, it fails to distinguish between an essentially coherent $\Delta_{\text{ren}}$ and a completely incoherent one.
The important physical effect of finite $\alpha$ is that there is a substantial contribution to $P(t)$ from transitions to states with energies that are larger than the putative renormalized oscillation frequency. When the amount of weight in these transitions is larger than the amount of weight in transitions to low energy states it no longer makes sense to consider the effects of $\Delta$ to be coherent. Effectively, each change of state is accompanied by the creation or annihilation of sufficient numbers of bosons in the environmental bath that the phase of the spin is randomized. Intuitively, one crosses over from degenerate or nearly degenerate perturbation theory to non-degenerate perturbation theory (as opposed to the transition to irrelevant $\Delta$ where the long time perturbation theory becomes convergent).

For $0 < \alpha < 1/2$, $\Gamma(t)$ is unbounded and any attempt at characterizing $P(t)$ by exponential relaxation fails, as indeed it must as $\alpha \to 0$. It is believed [11] that the true behavior of $P(t)$ in this region of $\alpha$ is a damped oscillation with oscillation frequency $\omega_{\text{osc}} = \cos(\pi\alpha/(2-2\alpha))$ and damping $\Gamma = \sin(\pi\alpha/(2-2\alpha))$, plus an incoherent background.

In conclusion, the key point we wish to make here is that in the TLS model the qualitative behavior of $P(t)$, in the sense of whether or not it exhibits oscillations, \textit{i.e.} quantum coherence, can actually be determined from lowest order perturbation theory. The special point $\alpha = 1/2$, at which the Golden Rule is believed applicable, separates the region of completely incoherent behavior, $1/2 \leq \alpha < 1$, from that of damped oscillations, $0 < \alpha < 1/2$. In the former region, an “extended Golden Rule” (\textit{i.e.} the self-consistency argument above) works and $P(t)$ exhibits incoherent behavior. In the latter, the extended Golden Rule fails and, in fact, $P(t)$ exhibits (damped) oscillations.

IV. COUPLED LUTTINGER LIQUIDS

With this preparation, we now turn to the problem of interest, that of NFL’s coupled by interliquid electron hopping operators. There are several issues at stake here, of varying complexity, but the central one is whether or not interliquid single particle hopping is coherent in the limit of small $t_\perp$. If it were, one would have to then address some other
questions; for example, would the resulting ground state be a NFL in one higher dimension with a warped Fermi surface, or would it be a Fermi liquid?

On the other hand, if the interliquid hopping is incoherent, there cannot be any coherent interliquid velocity nor any warping of the Fermi surface. There will be dramatic implications for interliquid properties. In particular, the interliquid conductivity will not exhibit a Drude term.

In order to be able to make precise calculations, we restrict ourselves to the case of coupled 1D liquids, which are Luttinger liquids (LL), a specific type of NFL. To the extent that the Luttinger liquid concept can be extended to two dimensions, we expect our results to be generalizable.

Ideally, we would like to tackle the problem of \( N \) coupled liquids, for \( N \to \infty \). However, the problem of two coupled liquids ought to be sufficient to settle the coherence issue. In any case, for the purposes of the calculation it makes no difference if we restrict ourselves to just two liquids since we calculate only to \( O(t_\perp^2) \).

It is a non-trivial matter to determine how one should go about settling the coherence/incoherence issue. Our approach utilizes the similarity to the TLS problem with ohmic dissipation. We have

\[
H = H_{\text{LL}}^{(1)} + H_{\text{LL}}^{(2)} + t_\perp \sum_x \{c_{\sigma}^{(1)\dagger}(x)c_{\sigma}^{(2)}(x) + \text{h.c.}\} \tag{17}
\]

The connection to the TLS-type physics is made by first bosonizing the Luttinger liquids, which then play the role of two baths of spin and charge bosons. Under the bosonization the interliquid hopping operators become exponentials of spin and charge boson creation and annihilation operators. They resemble the operators \( e^{\pm i\Omega} \) of the TLS Hamiltonian, \( H_{\text{TLS}}' \). The \( t_\perp \) operator acts to raise the particle number of one chain by 1, and lower the other by 1, analogous to the action of the spin flip operators in the TLS. Moreover, the \( t_\perp \) operator has a power law two-point function. Thus, it is very similar to the tunneling operator \( (\sigma^+ e^{-i\Omega} + \sigma^- e^{i\Omega}) \) in the ohmic regime of the TLS, and \( H_{\text{LL}}^{(1)} + H_{\text{LL}}^{(2)} \) plays a role similar to the oscillator bath in the TLS.
Despite the striking similarity, however, there is no precise mapping of $H$ to $H_{\text{TLS}}$ for the simple reason that in the TLS problem there is just a \textit{single} tunneling particle, and this particle is \textit{distinct} from the oscillator bath, while in the coupled LL problem there are $N$ particles which can hop from liquid to liquid and, moreover, these particles are themselves the \textit{source} of the dissipative bath. The most natural variable analogous to $\sigma_z$ is $\Delta N = N_2 - N_1$, the particle number difference between the two liquids, with the obvious difference that $\Delta N$ is not simply two-valued.

However, despite these problems with the analogy, the function $P(t)$ in the TLS is readily generalized to the coupled LL problem. For the LL problem we define

$$P(t) \equiv |\langle O_1 O_2 | e^{iH_0 t} e^{-iHt} | O_1 O_2 \rangle|^2$$  \hspace{1cm} (18)$$

Here $|O_1 O_2\rangle$ denotes the product of the ground states $|O_1\rangle$, $|O_2\rangle$ of each Luttinger liquid in the absence of $t_\perp$. For $t < 0$, $H = H^{(1)}_{\text{LL}} + H^{(2)}_{\text{LL}}$, and at $t = 0$ the interliquid hopping is turned on. The particle number difference $\Delta N$ entails a Fermi momentum difference $\Delta k$ and a chemical potential difference $\Delta \mu$.

Suppose that, instead of being Luttinger liquids, the 1D liquids were free Fermi gasses. The Hamiltonian becomes a direct product $H = \bigotimes_k H_k$, where

$$H = \begin{pmatrix} E_k & t_\perp \\ t_\perp & E_k \end{pmatrix}$$

so that

$$P(t) = \cos^2 \Delta N (t_\perp t)$$

Perturbation theory picks up the $O(t_\perp^2)$ term correctly,

$$P(t) \sim 1 - \Delta N t_\perp^2 t^2$$

This is precisely the type of behavior for which Golden Rule or extended Golden Rule, \textit{i.e.} incoherent, type methods fail and for a very clear reason: quantum coherence is established separately for each $k$ in a very trivial way.
When interactions between electrons within a given liquid are included, $H$ can no longer be written in this direct product form. We might suspect, however, that for true (Landau) Fermi liquids, where the Landau quasiparticle concept is valid, an approximate decomposition into a direct product of quasiparticle Hamiltonians would be possible. On the other hand, the situation for coupled Luttinger liquids, where the quasiparticle concept completely breaks down, is not at all obvious.

In what follows, we outline an approach to calculating $P(t)$ using spectral functions (complete details will be given elsewhere [10]). This method is perhaps more illuminating than the space-time Green’s function method used by us previously [13]. The “shape” of the spectral function which determines $P(t)$ can be examined to determine the nature of the interliquid hopping processes.

To $O(t^2)$ we have

$$1 - P(t) = 2t_1^2 L \Re \int_0^t dt_1 \int_0^{t_1} dt_2 \int dx \left\{ \langle c^{(1)}(x, t_1) c^{(1)\dagger}(0, t_2) \rangle \langle c^{(2)\dagger}(x, t_1) c^{(2)}(0, t_2) \rangle + (1 \leftrightarrow 2) \right\}$$

(19)

where the superscripts on the electron operators label the chain in which the operator acts. For convenience, we again define $\Gamma(t) \equiv -dP(t)/dt$ which can be written in a spectral function form as

$$\Gamma(t) = 2t_1^2 L \int \frac{d\omega \sin \omega t}{2\pi \omega} \{ A_{12}(\omega) + A_{21}(\omega) \}$$

(20)

where

$$A_{ij}(\omega) = \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} J_{1}^{(i)}(k, \omega') J_{2}^{(j)}(k, \omega' - \omega)$$

(21)

and $J_{1,2}(k, \omega)$ are the Fourier transforms of

$$J_1(k, t) \equiv \langle c(k, t)c^\dagger(k, 0) \rangle$$

$$J_2(k, t) \equiv \langle c^\dagger(k, 0)c(k, t) \rangle$$

In the $T = 0$ limit,
\[ J_{1,2}(k, \omega') = \theta_{\pm}(\omega' - \mu) \rho(k, \omega' - \mu) \]  

(22)

where \( \rho(k, \omega) \) is the electron spectral function as conventionally defined.

Physically, \( A_{12}(\omega) \) is the effective spectral function governing hops in which an electron hops to liquid 1, from liquid 2, and \( A_{21}(\omega) \) the opposite.

A. Free Fermi Gasses, and Fermi Liquids

For the sake of comparison, it is worthwhile considering first the (hypothetical in 1D) situation of coupled free Fermi gasses or Fermi liquids. For free Fermi gasses, \( A_{12}(\omega) \propto \Delta \mu \delta(\omega), \ A_{21}(\omega) = 0 \). Thus \( \Gamma(t) \propto \Delta \mu t \), a clear signal of coherent hopping and hence of a fundamental rearrangement of the ground state.

For a true Fermi liquid one finds, using Fermi liquid spectral functions,

\[ A_{12}(\omega) \sim \frac{1}{v_F} \theta_{\pm}(\omega + \Delta \mu) \{ Z^2 \Delta \mu \delta(\omega) + \gamma Z (1 - Z) \omega \} \]  

(23)

where \( 0 < Z < 1 \) is the quasiparticle renormalization factor, and \( \gamma \) characterizes the strength of the electron-electron interactions. \( \Gamma(t) \) is therefore a sum of a term \( \propto Z^2 \Delta \mu t^2 \) representing fundamentally coherent processes, and a term \( \propto \gamma Z (1 - Z) t^{-1} \) which is on the border of incoherent and irrelevant. By choosing a sufficiently small \( t_\perp \) one can find a time \( t \) such that \( (1 - P(t))/N \ll 1 \) (i.e. we are not outside of the reasonable range of our \( O(t_\perp^2) \) expansion), yet the ratio of the coherent contribution to the incoherent contribution is arbitrarily large.

This is true regardless of how small \( Z \) is. Thus, a perturbative calculation in \( t_\perp \) does not reveal any likelihood of a loss of coherence of interliquid tunneling, and there is therefore no impediment to the formation of an interliquid band of width \( \sim Z t_\perp \).

B. Luttinger Liquids

To apply the above method here, we need the electron spectral function for a Luttinger liquid. In [13] we used the space-time Luttinger liquid Green’s functions, \( G(x, t) \), to calculate
$P(t)$, since these are more directly calculated within the bosonization framework than are the corresponding $G(k, \omega)$. The electron spectral function $\rho(k, \omega)$ can be calculated by direct Fourier transform of $G(x, t)$ \[14\], but it can actually be determined in a much simpler way \[10\] by writing the electron space-time Green’s function as a product of “fracton” Green’s functions, where the fracton operators are exponentials of spin and charge boson operators. The fracton spectral functions are sharp $\delta$-functions. By “glueing” the fracton spectral functions together via convolution, one obtains simple integral expressions for the electron spectral function. For example, for the case of a spin-1/2 Luttinger liquid with spin-independent interactions (this is the case we are referring to when we use the term ‘Luttinger liquid’, unless explicitly stated otherwise) we find

\[
J_1(k, \omega) \propto \int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega - \mu - \sum_i \omega_i) \delta \left( k - k_F - \frac{(\omega_1 - \omega_3)}{v_c} - \frac{\omega_2}{v_s} \right) (\omega_1/v_c)^{\alpha - 1/2} (\omega_2/v_s)^{-1/2} (\omega_3/v_c)^{\alpha - 1/2}
\]

(24)

The various singularities near $\omega = \pm v_c k, v_s k$ can be readily determined from this expression. $2\alpha$ is the Luttinger liquid exponent which characterizes the singularity in $n(k)$ near $k_F$ (see \[4\]). For the 1D Hubbard model, $0 < 2\alpha < 1/8$, regardless of the magnitude of the (on-site) repulsion.

For reasons of pedagogy, it is convenient to first consider the case of Luttinger liquid models with forward scattering only (FSO), i.e. where there is no coupling between left- and right-moving electrons. Then we shall consider the generic Luttinger liquid case.

1. Forward-Scattering-Only Luttinger liquid

This model exhibits spin-charge separation, but no anomalous exponent \[5\]. The eigenexcitations are spin and charge bosons with velocities $v_s$ and $v_c$, respectively, and $v_c - v_s \equiv \Delta v > 0$. We find the rather simple expression

\[
A_{12}(\omega) \propto \frac{1}{\Delta v} \theta_+ (v_c \Delta k - \Delta \mu - \omega) \theta_+ (\omega + \Delta \mu - v_s \Delta k)
\]

(25)
and $A_{21}(\omega) = 0$.

The spectral function for $\Gamma(t)$ is therefore flat $\propto 1/\Delta v$ and non-vanishing only in the region $v_s \Delta k - \Delta \mu < \omega < v_c \Delta k - \Delta \mu$. It is a simple matter to explicitly determine $\Gamma(t)$ from $A_{12}(\omega)$, but the essential physics can be obtained by inspection. In the limit of $\Delta v \to 0$, $A_{12}(\omega) \to \delta(\omega)$. For $\Delta v \neq 0$, $A_{12}(\omega)$ is peaked around $\omega = 0$, but has a width $\tau_{\Delta k}^{-1} \sim \Delta k \Delta v$. Thus $\Gamma(t) \propto (\Delta k)t$ for all times $t \lesssim \tau_{\Delta k}$, which can be made arbitrarily long by choice of sufficiently small $\Delta k$, while remaining in the perturbative regime, $N^{-1} \int_0^t \Gamma(t')dt' \ll 1$.

We conclude, therefore, that in this case, too, it appears that there is interliquid coherence for arbitrarily small $t_{\perp}$.

2. Generic Luttinger liquid

We now turn to the case where there is spin-charge separation, $v_c - v_s = \Delta v > 0$, and an anomalous exponent, $\alpha$. This case is relevant, for example, to the 1D Hubbard model, and to most physical models which are not “chiral”. The expressions for $A_{12}(\omega)$, $A_{21}(\omega)$ can be reduced to a one-dimensional integral form \[10\]. The key features for $A_{12}(\omega)$ are

(i) a “spikey” low-frequency part much like the $A_{12}(\omega)$ in the FSOLL, above;

(ii) a broad incoherent part, $\propto \omega^{4\alpha}$. More precisely, for $\omega \gtrsim \Delta \mu$

$$A_{12}(\omega) \sim \omega^{4\alpha} + \lambda \Delta \mu \omega^{4\alpha - 1} + O(\Delta \mu)^2$$

so that for $\alpha > 1/4$ the $O(\Delta \mu)$ contribution is also incoherent.

The resulting $\Gamma(t)$ is as given in \[13\], but with this spectral representation the picture is somewhat clearer. The potentially coherent contribution to $\Gamma(t)$ from the low-frequency part of $A_{12}(\omega)$ is $\propto (\Delta \mu)t^{1-4\alpha}$ provided $t \lesssim \tau_{\Delta k}$; for $t \gtrsim \tau_{\Delta k}$ this term crosses over to incoherent behavior. The broad background contributes a piece to $\Gamma(t)$ proportional to $t^{-4\alpha}$.

Qualitatively, we have the following: there is a (short time) coherent part with weight $\propto \Delta k$ (or, equivalently, $\Delta \mu$) coming from the low-frequency part of $A_{12}(\omega)$, and an incoherent part coming from all but the very lowest frequencies. Coherence would be favoured by a
large coherent contribution, which suggests making $\Delta k$ large. Increasing $\Delta k$, however, has the effect of increasing the width, i.e. reducing the “lifetime” of the coherent part. The result is that for arbitrarily small $t_\perp$ one cannot find a time $t$ such that the ratio of the coherent and incoherent contributions to $P(t)$ is arbitrarily large. This puts the likelihood of coherence in great doubt.

To clarify the physics, let us consider two simple morphologies of $A_{12}(\omega)$ and their consequences. First, suppose $A_{12}(\omega)$ was the sum of a term $z\delta(\omega)$ and a constant part, $\gamma \theta_+(\omega)\theta_+(\Lambda - \omega)$. Then no matter how small $z$ is, there would always exist a sufficiently small $t_\perp$ such that there was a time $t$, not outside the reasonable range of validity of the $O(t_\perp^2)$ perturbation expansion, for which the coherent contribution to $P(t)$ is arbitrarily larger than the incoherent contribution. This is true for the simple reason that the coherent contribution grows as $t^2$, faster than the incoherent contribution which only grows as $t$.

Now suppose the $\delta(\omega)$ piece is broadened by a lifetime, $\tau$. Then the contribution to $P(t)$ is now coherent only for $t < \sim \tau$. It is then clear that the argument given above for the $z\delta(\omega)$ type contribution will run into trouble if $z$ is too small, for then

$$1 - P(t) = \delta P_{coh}(t) + \delta P_{incoh}(t)$$

with

$$\frac{1}{N}\delta P_{coh}(t) \sim z t_\perp^2 t^2 \quad t < \sim \tau$$

$$zt_\perp^2 \tau t \quad t \sim \tau$$

$$\frac{1}{N}\delta P_{incoh}(t) \sim t_\perp^2 \gamma t$$

Clearly, coherence is in doubt if $\gamma \sim z\tau$.

We can now see how the Fermi liquid and FSO Luttinger liquid compare qualitatively to this simple example. In the former, one effectively has $\tau = \infty$, so the above problem does not arise. In the latter, while $\tau \sim (\Delta k \Delta v)^{-1}$ there is no incoherent contribution, and is therefore analogous to the simple model above with $\gamma = 0$.

Finally, we point out that in the spinless case $A_{12}(\omega)$ goes over to
\[ A_{12}(\omega) \propto \theta_{\pm} (\omega - (v_c \Delta k - \Delta \mu))(\omega - (v_c \Delta k - \Delta \mu))^{2\alpha-1}(\omega + (v_c \Delta k + \Delta \mu))^{2\alpha+1} \]

The presence of a divergent edge singularity for \(2\alpha < 1\) as \(\omega \to (v_c \Delta k - \Delta \mu)\) implies that the above argument for a breakdown of coherence in the spin-1/2 case is not easily extended to the spinless case.

V. CONCLUSION

We conclude that, within this perturbative approach to the problem of 1D Luttinger liquids coupled by interliquid hopping, there is no unambiguous signal of interliquid coherence. This is in contrast to the case of coupled Fermi liquids and of coupled FSO Luttinger liquids where a dominant coherent signal emerges at arbitrarily small interliquid hopping rate, \(t_\perp\).

Moreover, the \(O(t_\perp^2)\) result will be modified at higher order by the presence of interhop “interactions” (\textit{i.e.} correlations). Such interactions can be expected to favour incoherence, interrupting the smooth growth of the coherent-like amplitude. An estimate of this effect was given in [13], leading to an approximate expression for the critical interliquid hopping rate, \(t_{\perp}^c\), below which coherent interliquid hopping will not exist. While the estimate was crude, the \textit{existence} of such a \(t_{\perp}^c\) would appear beyond doubt. An extensive discussion of this point will be given in [10]. The emphasis in this paper is on the absence of an unambiguous signal of coherence within a simple perturbative “Golden Rule” calculation.

We emphasize that, unlike other approaches to this problem [16], we directly address the coherence/incoherence issue, and our method is \textit{exact}, albeit perturbative. Some other approaches which claim to go beyond lowest order perturbation theory do so only at the expense of introducing an uncontrolled approximation. For example, Boise \textit{et al.} [17] use a Wick-theorem resummation to calculate the single-particle Green’s function for coupled liquids. Such a resummation is invalid, for the \(2n\)-point correlation functions in a Luttinger liquid ground state \textit{do not} satisfy a Wick theorem at all. The approximation appears to miss the all-important effects of the incoherent background.
The application of these ideas to interplanar conduction in the cuprates, and to understanding the anomalous magnetoresistance in the quasi-1D organic conductor (TMTSF)$_2$PF$_6$ have been given elsewhere [18,8].

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