ESTIMATING SPEED AND DAMPING IN THE STOCHASTIC WAVE EQUATION

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Abstract. A parameter estimation problem is considered for a one-dimensional stochastic wave equation driven by additive space-time Gaussian white noise. The estimator is of spectral type and utilizes a finite number of the spatial Fourier coefficients of the solution. The asymptotic properties of the estimator are studied as the number of the Fourier coefficients increases, while the observation time and the noise intensity are fixed.

1. Introduction

Consider the stochastic wave equation

\[ \frac{\partial^2 u}{\partial t^2} = \theta_1 \frac{\partial^2 u}{\partial x^2} + \theta_2 \frac{\partial u}{\partial t} + \dot{W}(t), \quad 0 < t < T, \ 0 < x < \pi, \]

with zero initial boundary conditions, driven by space-time white noise \( \dot{W} \). The solution of this equation can be written as a Fourier series

\[ u(t, x) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} u_k(t) \sin(kx). \]

The objective is to construct and investigate the maximum likelihood estimators of the unknown numbers \( \theta_1 > 0 \) and \( \theta_2 \in \mathbb{R} \), given \( \{u_1(t), \ldots, u_N(t)\}, t \in [0, T] \), the first \( N \) Fourier coefficients of the solution.

A similar problem for stochastic parabolic equations is relatively well studied, with the first result announced in the paper by Huebner, Khasminskii, and Rozovskii [3]. While most of the existing papers concentrate on estimating either a single parameter or a function of time, estimation of several parameters in parabolic equations has also been studied [2, 8]. The objective of the current paper is to extend some of the results from [2] to stochastic hyperbolic equations. In both parabolic and hyperbolic setting, the underlying assumption is that the solution \( u = u(t, x) \) of the equation can be measured at every point in time and space. Then the Fourier coefficients \( u_k \) of the solution can be computed and used to construct the estimator.

The main result of the paper is as follows.

2000 Mathematics Subject Classification. Primary 62F12; Secondary 60G15, 60H15, 60G30, 62M05.

Key words and phrases. Cylindrical Brownian motion, Second-Order Stochastic Equations, Stochastic Hyperbolic Equations.

The work was partially supported by the NSF Grant DMS-0803378.
Theorem 1.1. The (joint) maximum likelihood estimator of the parameters $\theta_1, \theta_2$ is strongly consistent and asymptotically normal as $N \to \infty$. The normalizing matrix is diagonal, with the diagonal elements $N^{3/2}$ and $N^{1/2}$; these elements specify the rate of convergence of the estimator to $\theta_1$ and $\theta_2$, respectively.

This theorem is proved in Section 3. In Section 2, we establish existence, uniqueness, and regularity of the solution of (1.1).

Throughout the presentation below, we fix a stochastic basis

$$\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$$

with the usual assumptions (completeness of $\mathcal{F}_0$ and right-continuity of $\mathcal{F}_t$). We also assume that $\mathbb{F}$ is large enough to support countably many independent standard Brownian motions. For a random variable $\xi$, $\mathbb{E}\xi$ denotes the expectation. $\mathbb{R}^n$ is an $n$-dimensional Euclidean space; $\mathcal{C}(A; B)$ is the space of continuous functions from $A$ to $B$; $\mathcal{N}(m, \sigma^2)$ is a Gaussian random variable with mean $m$ and variance $\sigma^2$.

Finally, for the convenience of the reader, we recall that a cylindrical Brownian motion $W = W(t), t \geq 1$, over (or on) a Hilbert space $H$ is a linear mapping

$$W : f \mapsto W_f(\cdot)$$

from $H$ to the space of zero-mean Gaussian processes such that, for every $f, g \in H$ and $t, s > 0$,

$$\mathbb{E}(W_f(t)W_g(s)) = \min(t, s)(f, g)_H. \tag{1.2}$$

If $\{h_k, k \geq 1\}$ is an orthonormal basis in $H$ and $w_k, k \geq 1$, are independent standard Brownian motions, then

$$f \mapsto \sum_{k \geq 1} (f, h_k)_H w_k(t) \tag{1.3}$$

is a cylindrical Brownian motion. Thus, a cylindrical Brownian motion $W$ is often represented by a generalized Fourier series

$$W(t) = \sum_{k \geq 1} w_k(t)h_k, \tag{1.4}$$

where $w_k = W_{h_k}$. The corresponding space-time white noise is then

$$\dot{W}(t) = \sum_{k \geq 1} \dot{w}_k(t)h_k.$$

2. Stochastic Wave Equation

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - 2b \frac{\partial u}{\partial t} + \dot{W}(t), \quad 0 < t < T, \quad 0 < x < \pi, \tag{2.1}$$
where $W$ is a cylindrical Brownian motion over $L_2((0, \pi))$. For simplicity, we assume

\begin{equation}
(2.2) \quad a \geq 1, \quad 2|b| \leq 1;
\end{equation}

\begin{equation}
(2.3) \quad u|_{t=0} = \frac{\partial u}{\partial t} \bigg|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0;
\end{equation}

see Remark 2.3 below about relaxing these assumptions. In physical models, $a > 0$ represents the speed of the wave and $b$ characterizes damping (amplification, if $b < 0$).

For $\gamma \in \mathbb{R}$, define the Hilbert space $H^\gamma$ as the closure of the set of smooth compactly supported functions on $(0, \pi)$ with respect to the norm

\begin{equation}
(2.4) \quad \|f\|_\gamma = \left( \sum_{k \geq 1} k^{2\gamma} f_k^2 \right)^{1/2}, \text{ where } f_k = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin(kx) dx.
\end{equation}

Note that each of the functions $\sin(kx)$ belongs to every $H^\gamma$, and if $f$ is twice continuously-differentiable on $(0, \pi)$ with $f(0) = f(\pi) = 0$, then, after two integrations by parts, $|f_k| \leq k^{-2} \sup_{x \in (0, \pi)} |f''(x)|$, so that, in particular, $f \in H^1$. More generally, every $f \in H^\gamma$ can be identified with a sequence $\{f_k, \ k \geq 1\}$ of real numbers such that $\sum_{k \geq 1} k^{2\gamma} f_k^2 < \infty$. Even though $f$ is a generalized function when $\gamma < 0$, we will still occasionally write $f = f(x)$, keeping in mind a generalized Fourier series representation $f(x) = \sqrt{2/\pi} \sum_{k \geq 1} f_k \sin(kx)$.

Given $\gamma > 0$, $f \in H^{-\gamma}$ and $g \in H^\gamma$, we define

$$(f, g) = \sum_{k \geq 1} f_k g_k;$$

if $f, g \in L_2((0, \pi))$, then

$$(f, g) = \int_0^\pi f(x) g(x) dx.$$

In other words, $(\cdot, \cdot)$ is the duality between $H^\gamma$ and $H^{-\gamma}$ relative to the inner product in $H^0 = L_2((0, \pi))$; see [5, Section IV.1.10].

Equation (2.1) is interpreted as a system of two first-order Itô equations

\begin{equation}
(2.5) \quad du = v dt, \quad dv = (a^2 u_{xx} - 2b) dt + dW(t).
\end{equation}

More precisely, we have the following definition.

**Definition 2.1.** An adapted process $u \in L_2(\Omega \times (0, T) \times (0, \pi))$ is called a solution of (2.1) if there exists an adapted process $v$ such that

1. $v \in L_2(\Omega; L_2((0, T); H^{-1}))$;
2. For every twice continuously-differentiable on $(0, \pi)$ function $f = f(x)$ with $f(0) = f(\pi) = 0$, the equalities

\begin{equation}
(2.6) \quad (u(t, \cdot, f)) = \int_0^t (v(t', \cdot, f)) ds,
\end{equation}

\begin{equation}
(2.6) \quad (v(t, \cdot, f)) = \int_0^t (a^2(u(t', \cdot, f'') - 2b(v(t', \cdot, f))) ds + W_f(t)
\end{equation}
hold for all $t \in [0, T]$ on the same set of probability one.

Here is the main result about existence and uniqueness of solution of (2.1).

**Theorem 2.2.** Under assumptions (2.2) and (2.3), equation (2.1) has a unique solution and, for every $\gamma < 1/2$,

$$
(2.7) \quad u \in L_2(\Omega; L_2((0, T); H^\gamma)); \quad v \in L_2(\Omega; L_2((0, T); H^{\gamma-1})).
$$

**Proof.** While the result can be derived from the general theory of stochastic hyperbolic equations (see, for example, Chow [1, Theorem 6.8.4]), we present a different, and a more direct, proof. This proof will also help in the construction and analysis of the estimators.

Take in (2.6) $f(x) = \sqrt{2/\pi} \sin(kx)$ and write $u_k(t) = (u(t, \cdot), f)$, $v_k(t) = (v(t, \cdot), f)$, $w_k = W_f$. Then

$$
(2.8) \quad u_k(t) = \int_0^t v_k(s)ds, \quad v_k(t) = -a^2 k^2 \int_0^t u_k(s)ds - 2b \int_0^t v_k(s)ds + w_k(t),
$$
or

$$
(2.9) \quad \ddot{u}_k(t) + 2b\dot{u}_k(t) + a^2 k^2 u_k(t) = \dot{w}_k(t), \quad u_k(0) = \dot{u}_k(0) = 0.
$$

By assumption (2.2),

$$
(2.10) \quad a^2 k^2 > b^2
$$
for all $k \geq 1$. Define

$$
(2.11) \quad \ell_k = \sqrt{a^2 k^2 - b^2}.
$$

Using the variation of parameters formula for the linear second-order equation with constant coefficients, we conclude that the solution of (2.8) is

$$
(2.12) \quad u_k(t) = \frac{1}{\ell_k} \int_0^t e^{-b(t-s)} \sin \left( \ell_k(t-s) \right) dw_k(s),
$$

$$
(2.13) \quad v_k(t) = \frac{1}{\ell_k} \int_0^t e^{-b(t-s)} \left( \ell_k \cos \left( \ell_k(t-s) \right) - b \sin \left( \ell_k(t-s) \right) \right) dw_k(s).
$$

By direct computation, there exists a number $C = C(T, a, b)$ such that, for all $t, s \in [0, T]$,

$$
(2.14) \quad \mathbb{E} u_k^2(t) \leq \ell_k^{-2} C(T) = \frac{C(T)}{a^2 k^2 - b^2}, \quad \mathbb{E} v_k^2(t) \leq C(T).
$$

Then the Gaussian processes

$$
(2.15) \quad u(t, x) = \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} u_k(t) \sin(kx), \quad v(t, x) = \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} v_k(t) \sin(kx)
$$
satisfy (2.6) and (2.7). Uniqueness of the solution follows from the completeness of the system $\{\sqrt{2/\pi} \sin(kx), \ k \geq 1\}$ in $L_2((0, \pi))$. \qed
Remark 2.3. We can now comment on the significance of assumptions (2.2) and (2.3). Assumption (2.2) can be relaxed to $a > 0$, because we will still have $a^2 k^2 > b^2$ for all sufficiently large $k$, and so representation formulas (2.12) for the solution of equation (2.9) will continue to hold for all sufficiently large $k$. In other words, if $a > 0$, then the free motion (any solution of the homogeneous version of (2.9)) is oscillatory for all sufficiently large $k \geq 1$; the oscillations are damped if $b > 0$, harmonic if $b = 0$, and amplified if $b > 0$. This is also the reason to call $b$ the damping coefficient, with an understanding that negative damping means amplification. Thus, (2.2) is only needed to simplify the computations by ensuring that equalities (2.12) hold for all $k \geq 1$.

Non-zero initial conditions, if sufficiently regular, will not affect existence and regularity of the solution. Similarly, the analysis will not change much for zero Neumann or other homogeneous boundary conditions.

3. Estimating the Coefficients

In this section, we assume that the solution $u = u(t, x), v = v(t, x)$ of equation (2.1) is observed for all $0 \leq t \leq T$ and $x \in (0, \pi)$, and study the question of estimating the numbers $a^2, b$ from these observations. It will be convenient to introduce the notations

(3.1) $\theta_1 = a^2, \theta_2 = -2b,$

so that (2.1) becomes

(3.2) $\frac{\partial^2 u}{\partial t^2} = \theta_1 \frac{\partial^2 u}{\partial x^2} + \theta_2 \frac{\partial u}{\partial t} + \dot{W}(t), \ t < 0 < T, \ 0 < x < \pi.$

To simplify the presentation, we keep the assumptions (2.2) and (2.3). By Theorem 2.2, the solution of (3.2) has a Fourier series expansion (2.12). We will construct the maximum likelihood estimators of $\theta_1$ and $\theta_2$ using the observations of the $2N$-dimensional process \{$u_k(t), v_k(t), k = 1, \ldots, N, t \in [0, T]$\} and study the asymptotic properties of the estimators in the limit $N \to \infty$. Note that both the amplitude of noise and the observation time are fixed.

By (2.8),

(3.3) $u_k(t) = \int_0^t v_k(s)ds, \ v_k(t) = -\theta_1 k^2 \int_0^t u_k(s)ds + \theta_2 \int_0^t v_k(s)ds + w_k(t).$

For each $k \geq 1$, the processes $u_k, v_k$, and $w_k$ generate measures $P^u_k, P^v_k, P^w_k$ in the space $C((0, T); \mathbb{R})$ of continuous, real-valued functions on $[0, T]$. Since $u_k$ is a continuously-differentiable function, the measures $P^u_k$ and $P^w_k$ are mutually singular. On the other hand, we can write

(3.4) $dv_k(t) = F_k(v)dt + dw_k,$

where $F_k(v) = -\theta_1 k^2 \int_0^t v_k(s)ds + \theta_2 v_k(t)$ is a non-anticipating functional of $v$. Thus, the process $v$ is a process of diffusion type in the sense of Liptser and Shiryaev [7].
Definition 4.2.7. Further analysis shows that the measure $\mathbf{P}_k^v$ is absolutely continuous with respect to the measure $\mathbf{P}_k^w$, and

$$
\frac{d\mathbf{P}_k^v}{d\mathbf{P}_k^w}(v_k) = \exp \left( \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right) \, dv_k(t) \right)
$$

(3.5)

and

$$
- \frac{1}{2} \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right)^2 \, dt;
$$

see [1, Theorem 7.6]. Since the processes $w_k$ are independent for different $k$, so are the processes $v_k$. Therefore, the measure $\mathbf{P}^w_N$ generated in $C((0, T); \mathbb{R}^N)$ by the vector process $\{v_k, k = 1, \ldots, N\}$ is absolutely continuous with respect to the measure $\mathbf{P}^w_N$ generated in $C((0, T); \mathbb{R}^N)$ by the vector process $\{w_k, k = 1, \ldots, N\}$, and the density is

$$
\frac{d\mathbf{P}^w_N}{d\mathbf{P}^w_N}(v_k) = \exp \left( \sum_{k=1}^N \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right) \, dv_k(t) \right)
$$

(3.6)

$$
- \frac{1}{2} \sum_{k=1}^N \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right)^2 \, dt;
$$

the corresponding log-likelihood ratio is

$$
Z_N(\theta_1, \theta_2) = \sum_{k=1}^N \left( \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right) \, dv_k(t) \right)
$$

(3.7)

$$
- \frac{1}{2} \int_0^T \left( - \theta_1 k^2 u_k(t) + \theta_2 v_k(t) \right)^2 \, dt.
$$

Introduce the following notations:

$$
J_{1,N} = \sum_{k=1}^N k^4 \int_0^T u_k^2(t) \, dt, \quad J_{2,N} = \sum_{k=1}^N \int_0^T v_k^2(t) \, dt,
$$

$$
J_{12,N} = \sum_{k=1}^N k^2 \int_0^T u_k(t) v_k(t) \, dt;
$$

(3.8)

$$
B_{1,N} = - \sum_{k=1}^N k^2 \int_0^T u_k(t) \, dv_k(t), \quad \xi_{1,N} = \sum_{k=1}^N k^2 \int_0^T u_k(t) \, dw_k(t);
$$

$$
B_{2,N} = \sum_{k=1}^N \int_0^T v_k(t) \, dv_k(t), \quad \xi_{2,N} = \sum_{k=1}^N \int_0^T v_k(t) \, dw_k(t).
$$

Note that the numbers $J$ and $B$ are computable from the observations of $u_k$ and $v_k$, $k = 1, \ldots, N$, and also

(3.9) \hspace{1cm}
B_{1,N} = \theta_1 J_{1,N} - \theta_2 J_{12,N} - \xi_{1,N}, \quad B_{2,N} = -\theta_1 J_{12,N} + \theta_2 J_{2,N} + \xi_{2,N},

(3.10) \hspace{1cm} J_{12,N} = \frac{1}{2} \sum_{k=1}^N k^2 u_k^2(T).
We consider the problem of estimating simultaneously both $\theta_1$ and $\theta_2$ from the observations
$$\{u_k(t), v_k(t), \ k = 1, \ldots, N, \ t \in [0, T]\}.$$

The maximum likelihood estimators $\hat{\theta}_{1,N}$, $\hat{\theta}_{2,N}$ satisfy
$$\left.\frac{\partial Z_N(\theta_1, \theta_2)}{\partial \theta_1}\right|_{\theta_1=\hat{\theta}_{1,N}, \theta_2=\hat{\theta}_{2,N}} = 0 \ \text{and} \ \left.\frac{\partial Z_N(\theta_1, \theta_2)}{\partial \theta_2}\right|_{\theta_1=\hat{\theta}_{1,N}, \theta_2=\hat{\theta}_{2,N}} = 0,$$

or, after solving the system of equations,
$$\hat{\theta}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \ \hat{\theta}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}. \quad (3.11)$$

For $T > 0$ and $\theta_2 \in \mathbb{R}$, define
$$C(\theta_2, T) = \begin{cases} \frac{e^{\theta_2 T} - \theta_2 T - 1}{2\theta_2^2}, & \text{if } \theta_2 \neq 0; \\ \frac{T^2}{4}, & \text{if } \theta_2 = 0. \end{cases} \quad (3.12)$$

Note that $C(\theta_2, T) > 0$ for all $T > 0$ and $\theta_2 \in \mathbb{R}$.

The following theorem describes the asymptotic behavior of the estimators $(3.11)$.

**Theorem 3.1.** Under assumptions $(2.2)$ and $(2.3)$,
$$\lim_{N \to \infty} \hat{\theta}_{1,N}^{(3)} = \theta_1, \ \lim_{N \to \infty} \hat{\theta}_{2,N}^{(3)} = \theta_2$$
with probability one and
$$\lim_{N \to \infty} N^{3/2}(\hat{\theta}_{1,N} - \theta_1) = \mathcal{N} \left(0, \frac{3\theta_1}{C(\theta_2, T)}\right),$$
$$\lim_{N \to \infty} N^{1/2}(\hat{\theta}_{2,N} - \theta_2) = \mathcal{N} \left(0, \frac{1}{C(\theta_2, T)}\right)$$
in distribution.

**Proof.** Define
$$D_N = \frac{J_{12,N}^2}{J_{1,N}J_{2,N}}. \quad (3.13)$$

It follows from $(3.9)$ and $(3.11)$ that
$$\hat{\theta}_{1,N}^{(3)} = \theta_1 + \frac{1}{1 - D_N} \left(\frac{\xi_{1,N}}{J_{1,N}} + \xi_{2,N} \frac{J_{12,N}}{J_{1,N}J_{2,N}}\right),$$
$$\hat{\theta}_{2,N}^{(3)} = \theta_2 + \frac{1}{1 - D_N} \left(\frac{\xi_{2,N}}{J_{2,N}} + \xi_{1,N} \frac{J_{12,N}}{J_{1,N}J_{2,N}}\right).$$

By direct computations using $(2.12)$ (and keeping in mind $(3.11)$),
$$\lim_{k \to \infty} k^2 \mathbb{E} \int_0^T u_k^2(t)dt = \frac{C(\theta_2, T)}{\theta_1}, \quad (3.14)$$
and
\begin{equation}
\lim_{N \to \infty} N^{-3} \mathbb{E} J_{1,N} = \frac{C(\theta_2, T)}{3\theta_1}.
\end{equation}

Since each \(u_k\) is a Gaussian process,
\[
\sup_k k^4 \mathbb{E} \int_0^T u_k^4(t) dt < \infty,
\]
and then the strong law of large numbers implies
\[
\lim_{N \to \infty} \frac{J_{1,N}}{\mathbb{E} J_{1,N}} = 1, \quad \lim_{N \to \infty} \frac{\xi_{1,N}}{\mathbb{E} J_{1,N}} = 0,
\]
both with probability one [apply the first theorem in Appendix, taking \(\xi_k = k^4 \int_0^T u_k^2 dt\) and then, \(\xi_k = k^2 \int_0^T u_k(t) dw_k(t)\)]. The central limit theorem implies
\[
\lim_{N \to \infty} \frac{\xi_{1,N}}{\sqrt{\mathbb{E} J_{1,N}}} = \mathcal{N}(0, 1)
\]
in distribution [apply the second theorem in Appendix, taking \(f_k(t) = k^2 u_k(t)\)]. Similarly,
\begin{equation}
\lim_{k \to \infty} \mathbb{E} \int_0^T v_k^2(t) dt = C(\theta_2, T),
\end{equation}
and
\begin{equation}
\lim_{N \to \infty} N^{-1} \mathbb{E} J_{2,N} = C(\theta_2, T).
\end{equation}

Since each \(v_k\) is a Gaussian process,
\[
\sup_k \mathbb{E} \int_0^T v_k^4(t) dt < \infty,
\]
and then the strong law of large numbers implies
\[
\lim_{N \to \infty} \frac{J_{2,N}}{\mathbb{E} J_{2,N}} = 1, \quad \lim_{N \to \infty} \frac{\xi_{2,N}}{\mathbb{E} J_{2,N}} = 0,
\]
both with probability one. The central limit theorem implies
\[
\lim_{N \to \infty} \frac{\xi_{2,N}}{\sqrt{\mathbb{E} J_{2,N}}} = \mathcal{N}(0, 1)
\]
in distribution. Finally, define
\[
\tilde{C}(\theta_2, T) = \begin{cases} 
\frac{e^{\theta_2 T} - 1}{2\theta_2}, & \text{if } \theta_2 \neq 0; \\
\frac{T}{2}, & \text{if } \theta_2 = 0.
\end{cases}
\]

Then (3.10) and (2.12) imply
\[
\lim_{N \to \infty} N^{-1} \mathbb{E} J_{12,N} = \frac{\tilde{C}(\theta_2, T)}{2\theta_1},
\]
and, by the strong law of large numbers,

$$\lim_{N \to \infty} \frac{J_{12,N}}{E J_{12,N}} = 1$$

with probability one. Then (3.15) and (3.17) imply

$$\lim_{N \to \infty} D_N = 0, \quad \lim_{N \to \infty} J_{12,N} J_{2,N} = \tilde{C}(\theta_2, T) \frac{\tilde{C}(\theta_2, T)}{2 \theta_1 C(\theta_2, T)},$$

both with probability one. The conclusions of the theorem now follow. \(\square\)

4. ACKNOWLEDGEMENT

The work of SVL was partially supported by the NSF Grant DMS-0803378.

REFERENCES

[1] P.-L. Chow. *Stochastic partial differential equations*. Chapman & Hall/CRC, Boca Raton, FL, 2007.
[2] M. Huebner. A characterization of asymptotic behaviour of maximum likelihood estimators for stochastic PDE’s. *Math. Methods Statist.*, 6(4):395–415, 1997.
[3] M. Huebner, R. Z. Khas’minskii, and B. L. Rozovskii. Two examples of parameter estimation. In S. Cambanis, J. K. Ghosh, R. L. Karandikar, and P. K. Sen, editors, *Stochastic Processes: A volume in honor of G. Kallianpur*, pages 149–160. Springer, New York, 1992.
[4] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes, 2nd Ed.*, volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Springer, New York, 2003.
[5] S. G. Krein, Yu. I. Petunin, and E. M. Semenov. *Interpolation of linear operators*, volume 54 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1982.
[6] R. Sh. Liptser and A. N. Shiryaev. *Theory of martingales*, volume 49 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers, Dordrecht, 1989.
[7] R. Sh. Liptser and A. N. Shiryaev. *Statistics of random processes, I: General Theory, 2nd Ed.*, volume 5 of *Applications of Mathematics*. Springer, 2001.
[8] S. V. Lototsky. Parameter estimation for stochastic parabolic equations: asymptotic properties of a two-dimensional projection-based estimator. *Stat. Inference Stoch. Process.*, 6(1):65–87, 2003.
[9] A. N. Shiryaev. *Probability, 2nd Ed.*, volume 95 of *Graduate Texts in Mathematics*. Springer, 1996.

APPENDIX

Below, we formulate the strong law of large numbers and the central limit theorem used in the proof of Theorem 3.1.

**Theorem 4.1 (Strong Law of Large Numbers).** Let \( \xi_k, \ k \geq 1 \), be independent random variables with the following properties:

- \( \mathbb{E} \xi_k = 0, \mathbb{E} \xi_k^2 > 0 \),
- There exist real numbers \( c > 0 \) and \( \alpha \geq -1 \) such that \( \lim_{k \to \infty} k^{-\alpha} \mathbb{E} \xi_k^2 = c \).
Then, with probability one,
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \xi_k}{\sum_{k=1}^{N} \mathbb{E} \xi_k^2} = 0.
\]
If, in addition, \( \mathbb{E} \xi_k^4 \leq c_1 \left( \mathbb{E} \xi_k^2 \right)^2 \) for all \( k \geq 1 \), with \( c_1 > 0 \) independent of \( k \), then, also with probability one,
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \xi_k^2}{\sum_{k=1}^{N} \mathbb{E} \xi_k^2} = 1.
\]

Proof. This is a particular case of Kolmogorov’s strong law of large numbers; see, for example, Shiryaev [9, Theorem IV.3.2].

Theorem 4.2 (Central Limit Theorem). Let \( w_k = w_k(t) \) be independent standard Brownian motions and let \( f_k = f_k(t) \) be adapted, continuous, square-integrable processes such that
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \int_0^T f_k(t) \, dt}{\sum_{k=1}^{N} \mathbb{E} \int_0^T f_k^2(t) \, dt} = 1
\]
in probability. Then
\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \int_0^T f_k(t) \, dw_k(t)}{\left( \sum_{k=1}^{N} \mathbb{E} \int_0^T f_k^2(t) \, dt \right)^{1/2}} = \mathcal{N}(0, 1)
\]
in distribution.

Proof. This is a particular case of a martingale limit theorem; see, for example Jacod and Shiryaev [4, Theorem VIII.4.17] or Liptser and Shiryaev [5, Theorem 5.5.4(I)].