ON PROPER $\mathbb{R}$-ACTIONS ON HYPERBOLIC STEIN SURFACES

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Abstract. In this paper we investigate proper $\mathbb{R}$-actions on hyperbolic Stein surfaces and prove in particular the following result: Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy which admits a proper $\mathbb{R}$-action by holomorphic transformations. The quotient $D/\mathbb{Z}$ with respect to the induced proper $\mathbb{Z}$-action is a Stein manifold. A normal form for the domain $D$ is deduced.

1. Introduction

Let $X$ be a Stein manifold endowed with a real Lie transformation group $G$ of holomorphic automorphisms. In this situation it is natural to ask whether there exists a $G$-invariant holomorphic map $\pi: X \to X//G$ onto a complex space $X//G$ such that $\mathcal{O}_{X//G} = (\pi_* \mathcal{O}_X)^G$ and, if yes, whether this quotient $X//G$ is again Stein. If the group $G$ is compact, both questions have a positive answer as is shown in [Hei91].

For non-compact $G$ even the existence of a complex quotient in the above sense of $X$ by $G$ cannot be guaranteed. In this paper we concentrate on the most basic and already non-trivial case $G = \mathbb{R}$. We suppose that $G$ acts properly on $X$. Let $\Gamma = \mathbb{Z}$. Then $X/\Gamma$ is a complex manifold and if, moreover, it is Stein, we can define $X//G := (X/\Gamma)//(G/\Gamma)$. The following was conjectured by Alan Huckleberry.

Let $X$ be a contractible bounded domain of holomorphy in $\mathbb{C}^n$ with a proper action of $G = \mathbb{R}$. Then the complex manifold $X/\Gamma$ is Stein.

In [P101] this conjecture is proven for the unit ball and in [Mie08] for arbitrary bounded homogeneous domains in $\mathbb{C}^n$. In this paper we make a first step towards a proof in the general case by showing

Theorem. Let $D$ be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on $D$. Then the complex manifold $D/\mathbb{Z}$ is Stein. Moreover, $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.

As an application of this theorem we deduce a normal form for domains of holomorphy whose identity component of the automorphism group is non-compact as well as for proper $\mathbb{R}$-actions on them. Notice that we make no assumption on smoothness of their boundaries.

We first discuss the following more general situation. Let $X$ be a hyperbolic Stein manifold with a proper $\mathbb{R}$-action. Then there is an induced local holomorphic $\mathbb{C}$-action on $X$ which can be globalized in the sense of [HI97]. The following result is central for the proof of the above theorem.

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Theorem. Let $X$ be a hyperbolic Stein surface with a proper $\mathbb{R}$–action. Suppose that either $X$ is taut or that it admits the Bergman metric and $H^1(X, \mathbb{R}) = 0$. Then the universal globalization $X^*$ of the induced local $\mathbb{C}$–action is Hausdorff and $\mathbb{C}$ acts properly on $X^*$. Furthermore, for simply-connected $X$ one has that $X^* \to X^*/\mathbb{C}$ is a holomorphically trivial $\mathbb{C}$–principal bundle over a simply-connected Riemann surface.

Finally, we discuss several examples of hyperbolic Stein manifolds $X$ with proper $\mathbb{R}$–actions such that $X/\mathbb{Z}$ is not Stein. If one does not require the existence of an $\mathbb{R}$–action, there are bounded Reinhardt domains in $\mathbb{C}^2$ with proper $\mathbb{Z}$–actions for which the quotients are not Stein.

2. Hyperbolic Stein $\mathbb{R}$–manifolds

In this section we present the general set-up.

2.1. The induced local $\mathbb{C}$–action and its globalization. Let $X$ be a hyperbolic Stein manifold. It is known that the group $\text{Aut}(X)$ of holomorphic automorphisms of $X$ is a real Lie group with respect to the compact-open topology which acts properly on $X$ (see [Kol98]). Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a closed one parameter subgroup of $\text{Aut}(D)$. Consequently, the action $\mathbb{R} \times X \to X$, $t \cdot x := \varphi_t(x)$, is proper. By restriction, we obtain also a proper $\mathbb{Z}$–action on $X$. Since every such action must be free, the quotient $X/\mathbb{Z}$ is a complex manifold. This complex manifold $X/\mathbb{Z}$ carries an action of $S^1 \cong \mathbb{R}/\mathbb{Z}$ which is induced by the $\mathbb{R}$–action on $X$.

Integrating the holomorphic vector field on $X$ which corresponds to this $\mathbb{R}$–action we obtain a local $\mathbb{C}$–action on $X$ in the following sense. There are an open neighborhood $\Omega \subset \mathbb{C} \times X$ of $\{0\} \times X$ and a holomorphic map $\Phi: \Omega \to X$, $\Phi(t, x) =: t \cdot x$, such that the following holds:

1. For every $x \in X$ the set $\Omega(x) := \{t \in \mathbb{C}; (t, x) \in \Omega\} \subset \mathbb{C}$ is connected;
2. for all $x \in X$ we have $0 \cdot x = x$;
3. we have $(t + t') \cdot x = t \cdot (t' \cdot x)$ whenever both sides are defined.

Following [Pal57] (compare [HI97] for the holomorphic setting) we say that a globalization of the local $\mathbb{C}$–action on $X$ is an open $\mathbb{R}$–equivariant holomorphic embedding $\iota: X \hookrightarrow X^*$ into a (not necessarily Hausdorff) complex manifold $X^*$ endowed with a holomorphic $\mathbb{C}$–action such that $\mathbb{C} \cdot \iota(X) = X^*$. A globalization $\iota: X \hookrightarrow X^*$ is called universal if for every $\mathbb{R}$–equivariant holomorphic map $f: X \to Y$ into a holomorphic $\mathbb{C}$–manifold $Y$ there exists a holomorphic $\mathbb{C}$–equivariant map $F: X^* \to Y$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\iota} & X^* \\
\downarrow{f} & & \downarrow{F} \\
Y & & 
\end{array}
$$

commutes. It follows that a universal globalization is unique up to isomorphism if it exists.

Since $X$ is Stein, the universal globalization $X^*$ of the induced local $\mathbb{C}$–action exists as is proven in [HI97]. We will always identify $X$ with its image $\iota(X) \subset X^*$. Then the local $\mathbb{C}$–action on $X$ coincides with the restriction of the global $\mathbb{C}$–action on $X^*$ to $X$. 
Recall that \(X\) is said to be orbit-connected in \(X^*\) if for every \(x \in X^*\) the set
\[
\Sigma(x) := \{ t \in \mathbb{C}; \ t \cdot x \in X \}
\]
is connected. The following criterion for a globalization to be universal is proven in [CTIT00].

**Lemma 2.1.** Let \(X^*\) be any globalization of the induced local \(\mathbb{C}\)–action on \(X\). Then \(X^*\) is universal if and only if \(X\) is orbit-connected in \(X^*\).

**Remark.** The results about (universal) globalizations hold for a bigger class of groups ([CTIT00]). However, we will need it only for the groups \(\mathbb{C}\) and \(\mathbb{C}^*\) and thus will not give the most general formulation.

For later use we also note the following

**Lemma 2.2.** The \(\mathbb{C}\)–action on \(X^*\) is free.

**Proof.** Suppose that there exists a point \(x \in X^*\) such that \(\mathbb{C}_x\) is non-trivial. Because of \(\mathbb{C} \cdot X = X^*\) we can assume that \(x \in X\) holds. Since \(\mathbb{C}_x\) is a non-trivial closed subgroup of \(\mathbb{C}\), it is either a lattice of rank 1 or 2, or \(\mathbb{C}\). The last possibility means that \(x\) is a fixed point under \(\mathbb{C}\) which is not possible since \(\mathbb{R}\) acts freely on \(X\).

We observe that the lattice \(\mathbb{C}_x\) is contained in the connected \(\mathbb{R}\)–invariant set \(\Sigma(x) = \{ t \in \mathbb{C}; \ t \cdot x \in X \}\). By \(\mathbb{R}\)–invariance \(\Sigma(x)\) is a strip. Since \(X\) is hyperbolic, this strip cannot coincide with \(\mathbb{C}\). The only lattice in \(\mathbb{C}\) which can possibly be contained in such a strip is of the form \(\mathbb{Z}r\) for some \(r \in \mathbb{R}\). Since this contradicts the fact that \(\mathbb{R}\) acts freely on \(X\), the lemma is proven. \(\square\)

Note that we do not know whether \(X^*\) is Hausdorff. In order to guarantee the Hausdorff property of \(X^*\), we make further assumptions on \(X\). The following result is proven in [Ian03] and [IST04].

**Theorem 2.3.** Let \(X\) be a hyperbolic Stein manifold with a proper \(\mathbb{R}\)–action. Suppose in addition that \(X\) is taut or admits the Bergman metric. Then \(X^*\) is Hausdorff. If \(X\) is simply-connected, then the same is true for \(X^*\).

We refer the reader to Chapter 4.10 and Chapter 5 in [Kob98] for the definitions and examples of tautness and the Bergman metric.

**Remark.** Every bounded domain in \(\mathbb{C}^n\) admits the Bergman metric.

### 2.2. The quotient \(X/\mathbb{Z}\).

We assume from now on that \(X\) fulfills the hypothesis of Theorem 2.3. Since \(X^*\) is covered by the translates \(t \cdot X\) for \(t \in \mathbb{C}\) and since the action of \(\mathbb{Z}\) on each domain \(t \cdot X\) is proper, we conclude that the quotient \(X^*/\mathbb{Z}\) fulfills all axioms of a complex manifold except for possibly not being Hausdorff.

We have the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & X^* \\
\downarrow & & \downarrow \\
X/\mathbb{Z} & \longrightarrow & X^*/\mathbb{Z}.
\end{array}
\]

Note that the group \(\mathbb{C}^* = (S^1)^\mathbb{C} \cong \mathbb{C}/\mathbb{Z}\) acts on \(X^*/\mathbb{Z}\). Concretely, if we identify \(\mathbb{C}/\mathbb{Z}\) with \(\mathbb{C}^*\) via \(\mathbb{C} \rightarrow \mathbb{C}^*, \ t \mapsto e^{2\pi it}\), the quotient map \(p: X^* \rightarrow X^*/\mathbb{Z}\) fulfills \(p(t \cdot x) = e^{2\pi it} \cdot p(x)\).
Lemma 2.4. The induced map $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$ is the universal globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.

Proof. The open embedding $X \hookrightarrow X^*$ induces an open embedding $X/\mathbb{Z} \hookrightarrow X^*/\mathbb{Z}$. This embedding is $S^1$–equivariant and we have $\mathbb{C}^* \cdot X/\mathbb{Z} = X^*/\mathbb{Z}$. This implies that $X^*/\mathbb{Z}$ is a globalization of the local $\mathbb{C}^*$–action on $X/\mathbb{Z}$.

In order to prove that this globalization is universal, by the globalization theorem in [CTIT00] it is enough to show that $X/\mathbb{Z}$ is orbit-connected in $X^*/\mathbb{Z}$. Hence, we must show that for every $[x] \in X/\mathbb{Z}$ the set $\Sigma([x]) := \{ t \in \mathbb{C}^*; t \cdot [x] \in X/\mathbb{Z} \}$ is connected in $\mathbb{C}^*$. For this we consider the set $\Sigma(x) = \{ t \in \mathbb{C}^*; t \cdot x \in X \}$. Since the map $X \to X/\mathbb{Z}$ intertwines the local $\mathbb{C}^*$– and $\mathbb{C}^*$–actions, we conclude that $t \in \Sigma(x)$ holds if and only if $e^{2\pi i t} \in \Sigma([x])$ holds. Since $X^*$ is universal, $\Sigma(x)$ is connected which implies that $\Sigma([x])$ is likewise connected. Thus $X^*/\mathbb{Z}$ is universal. \qed

Remark. The globalization $X^*/\mathbb{Z}$ is Hausdorff if and only if $\mathbb{Z}$ or, equivalently, $\mathbb{R}$ act properly on $X^*$. As we shall see in Lemma 3.3, this is the case if $X$ is taut.

2.3. A sufficient condition for $X/\mathbb{Z}$ to be Stein. If $\dim X = 2$, we have the following sufficient condition for $X/\mathbb{Z}$ to be a Stein surface.

Proposition 2.5. If the $\mathbb{C}$–action on $X^*$ is proper and if the Riemann surface $X^*/\mathbb{C}$ is not compact, then $X/\mathbb{Z}$ is Stein.

Proof. Under the above hypothesis we have the $\mathbb{C}$–principal bundle $X^* \to X^*/\mathbb{C}$. If the base $X^*/\mathbb{C}$ is not compact, then this bundle is holomorphically trivial, i.e. $X^*$ is biholomorphic to $\mathbb{C} \times \mathbb{R}$ where $\mathbb{R}$ is a non-compact Riemann surface. Since $\mathbb{R}$ is Stein, the same is true for $X^*$ and for $X^*/\mathbb{Z} \cong \mathbb{C}^* \times \mathbb{R}$. Since $X/\mathbb{Z}$ is locally Stein, see [Mie08], in the Stein manifold $X^*/\mathbb{Z}$, the claim follows from [DG60]. \qed

Therefore, the crucial step in the proof of our main result consists in showing that $\mathbb{C}$ acts properly on $X^*$ under the assumption $\dim X = 2$.

3. Local properness

Let $X$ be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that $X$ is taut or that it admits the Bergman metric and $H^1(X, \mathbb{R}) = \{0\}$. We show that then $\mathbb{C}$ acts locally properly on $X^*$.

3.1. Locally proper actions. Recall that the action of a Lie group $G$ on a manifold $M$ is called locally proper if every point in $M$ admits a $G$–invariant open neighborhood on which the $G$ acts properly.

Lemma 3.1. Let $G \times M \to M$ be locally proper.

1. For every $x \in M$ the isotropy group $G_x$ is compact.
2. Every $G$–orbit admits a geometric slice.
3. The orbit space $M/G$ is a smooth manifold which is in general not Hausdorff.
4. All $G$–orbits are closed in $M$.
5. The $G$–action on $M$ is proper if and only if $M/G$ is Hausdorff.
Proof. The first claim is elementary to check. The second claim is proven in [DK00]. The third one is a consequence of (2) since the slices yield charts on \( M/G \) which are smoothly compatible because the transitions are given by the smooth action of \( G \) on \( M \). Assertion (4) follows from (3) because in locally Euclidean topological spaces points are closed. The last claim is proven in [Pal61]. \( \square \)

Remark. Since \( \mathbb{R} \) acts properly on \( X \), the \( \mathbb{R} \)-action on \( X^* \) is locally proper.

3.2. Local properness of the \( \mathbb{C} \)-action on \( X^* \). Recall that we assume that

\[
(3.1) \quad X \text{ is taut}
\]

or that

\[
(3.2) \quad X \text{ admits the Bergman metric and } H^1(X, \mathbb{R}) = \{0\}.
\]

We first show that assumption (3.1) implies that \( \mathbb{C} \) acts locally properly on \( X^* \).

Since \( X^* \) is the universal globalization of the induced local \( \mathbb{C} \)-action on \( X \), we know that \( X \) is orbit-connected in \( X^* \). This means that for every \( x \in X^* \) the set \( \Sigma(x) = \{ t \in \mathbb{C}; t \cdot x \in X \} \) is a strip in \( \mathbb{C} \). In the following we will exploit the properties of the thickness of this strip.

Since \( \Sigma(x) \) is \( \mathbb{R} \)-invariant, there are “numbers” \( u(x) \in \mathbb{R} \cup \{-\infty\} \) and \( o(x) \in \mathbb{R} \cup \{\infty\} \) for every \( x \in X^* \) such that

\[
\Sigma(x) = \{ t \in \mathbb{C}; u(x) < \text{Im}(t) < o(x) \}.
\]

The functions \( u: X^* \to \mathbb{R} \cup \{-\infty\} \) and \( o: X^* \to \mathbb{R} \cup \{\infty\} \) so obtained are upper and lower semicontinuous, respectively. Moreover, \( u \) and \( o \) are \( \mathbb{R} \)-equivariant and \( i\mathbb{R} \)-equivariant:

\[
u(it \cdot x) = u(x) - t \quad \text{and} \quad o(it \cdot x) = o(x) - t.
\]

Proposition 3.2. The functions \( u, -o: X^* \to \mathbb{R} \cup \{-\infty\} \) are plurisubharmonic. Moreover, \( u \) and \( o \) are continuous on \( X^* \setminus \{u = -\infty\} \) and \( X^* \setminus \{o = \infty\} \), respectively.

Proof. It is proven in [For93] that \( u \) and \( -o \) are plurisubharmonic on \( X \). By equivariance, we obtain this result for \( X^* \).

Now we prove that the function \( u: X \setminus \{u = -\infty\} \to \mathbb{R} \) is continuous which was remarked without complete proof in [Lam03]. For this let \( (x_n) \) be a sequence in \( X \) which converges to \( x_0 \in X \setminus \{u = -\infty\} \). Since \( u \) is upper semi-continuous, we have \( \limsup_{n \to \infty} u(x_n) \leq u(x_0) \). Suppose that \( u \) is not continuous in \( x_0 \). Then, after replacing \( (x_n) \) by a subsequence, we find \( \varepsilon > 0 \) such that \( u(x_n) \leq u(x_0) - \varepsilon < u(x_0) \) holds for all \( n \in \mathbb{N} \). Consequently, we have \( \Sigma(x_0) = \{ t \in \mathbb{C}; u(x_0) < \text{Im}(t) < o(x_0) \} \subset \Sigma := \{ t \in \mathbb{C}; u(x_0) - \varepsilon < \text{Im}(t) < o(x_0) \} \subset \Sigma(x_n) \) for all \( n \in \mathbb{N} \) and hence obtain the sequence of holomorphic functions \( f_n: \Sigma \to X, f_n(t) := t \cdot x_n \). Since \( X \) is taut and \( f_n(0) = x_n \to x_0 \), the sequence \( (f_n) \) has a subsequence which compactly converges to a holomorphic function \( f_0: \Sigma \to X \). Because of \( f_0(iu(x_0)) = \lim_{n \to \infty} f_n(iu(x_0)) = \lim_{n \to \infty} iu(x_0) \cdot x_n = iu(x_0) \cdot x_0 \notin X \) we arrive at a contradiction. Thus the function \( u: X \setminus \{u = -\infty\} \to \mathbb{R} \) is continuous. By \( (i\mathbb{R}) \)-equivariance, \( u \) is also continuous on \( X^* \setminus \{u = -\infty\} \). A similar argument shows continuity of \( -o: X^* \setminus \{o = \infty\} \to \mathbb{R}. \) \( \square \)

Let us consider the sets

\[
\mathcal{N}(o) := \{ x \in X^*; o(x) = 0 \} \quad \text{and} \quad \mathcal{P}(o) := \{ x \in X^*; o(x) = \infty \}.
\]
The sets $\mathcal{N}(u)$ and $\mathcal{P}(u)$ are similarly defined. Since $X = \{x \in X^*; u(x) < 0 < o(x)\}$, we can recover $X$ from $X^*$ with the help of $u$ and $o$.

**Lemma 3.3.** The action of $\mathbb{R}$ on $X^*$ is proper.

**Proof.** Let $\partial^*X$ denote the boundary of $X$ in $X^*$. Since the functions $u$ and $-o$ are continuous on $X^* \setminus \mathcal{P}(u)$ and $X^* \setminus \mathcal{P}(o)$ one verifies directly that $\partial^*X = \mathcal{N}(u) \cup \mathcal{N}(o)$ holds. As a consequence, we note that if $x \in \partial^*X$, then for every $\varepsilon > 0$ the element $(i \varepsilon) \cdot x$ is not contained in $\partial^*X$.

Let $(t_n)$ and $(x_n)$ be sequences in $\mathbb{R}$ and $X^*$ such that $(t_n \cdot x_n, x_n)$ converges to $(y_0, x_0)$ in $X^* \times X^*$. We may assume without loss of generality that $x_0$ and hence $x_n$ are contained in $X$ for all $n$. Consequently, we have $y_0 \in X \cup \partial^*X$. If $y_0 \in \partial^*X$ holds, we may choose an $\varepsilon > 0$ such that $(i \varepsilon) \cdot y_0$ and $(i \varepsilon) \cdot x_0$ lie in $X$. Since the $\mathbb{R}$–action on $X$ is proper, we find a convergent subsequence of $(t_n)$ which was to be shown. □

**Lemma 3.4.** We have:

1. $\mathcal{N}(u)$ and $\mathcal{N}(o)$ are $\mathbb{R}$–invariant.
2. We have $\mathcal{N}(u) \cap \mathcal{N}(o) = \emptyset$.
3. The sets $\mathcal{P}(u)$ and $\mathcal{P}(o)$ are closed, $\mathbb{C}$–invariant and pluripolar in $X^*$.
4. $\mathcal{P}(u) \cap \mathcal{P}(o) = \emptyset$.

**Proof.** The first claim follows from the $\mathbb{R}$–invariance of $u$ and $o$.

The second claim follows from $u(x) < o(x)$.

The third one is a consequence of the $\mathbb{R}$–invariance and $i\mathbb{R}$–equivariance of $u$ and $o$.

If there was a point $x \in \mathcal{P}(u) \cap \mathcal{P}(o)$, then $\mathbb{C} \cdot x$ would be a subset of $X$ which is impossible since $X$ is hyperbolic. □

**Lemma 3.5.** If $o$ is not identically $\infty$, then the map

$$\varphi: i\mathbb{R} \times \mathcal{N}(o) \to X^* \setminus \mathcal{P}(o), \quad \varphi(it, z) = it \cdot z,$$

is an $i\mathbb{R}$–equivariant homeomorphism. Since $\mathbb{R}$ acts properly on $\mathcal{N}(o)$, it follows that $\mathbb{C}$ acts properly on $X^* \setminus \mathcal{P}(o)$. The same holds when $o$ is replaced by $u$.

**Proof.** The inverse map $\varphi^{-1}$ is given by $x \mapsto (-io(x), io(x) \cdot x)$. □

**Corollary 3.6.** The $\mathbb{C}$–action on $X^*$ is locally proper. If $\mathcal{P}(o) = \emptyset$ or $\mathcal{P}(u) = \emptyset$ hold, then $\mathbb{C}$ acts properly on $X^*$.

From now on we suppose that $X$ fulfills the assumption (3.2). Recall that the Bergman form $\omega$ is a Kähler form on $X$ invariant under the action of $\text{Aut}(X)$. Let $\xi$ denote the complete holomorphic vector field on $X$ which corresponds to the $\mathbb{R}$–action, i.e. we have $\xi(x) = \frac{\partial}{\partial n} \varphi_n(x)$. Hence, $t \xi \omega = \omega(\cdot, \xi)$ is a 1–form on $X$ and since $H^1(X, \mathbb{R}) = \{0\}$ there exists a function $\mu^\xi \in \mathcal{C}^\infty(X)$ with $d\mu^\xi = t \xi \omega$.

**Remark.** This means that $\mu^\xi$ is a momentum map for the $\mathbb{R}$–action on $X$.

**Lemma 3.7.** The map $\mu^\xi: X \to \mathbb{R}$ is an $\mathbb{R}$–invariant submersion.

**Proof.** The claim follows from $d\mu^\xi(x) J\xi_x = \omega_x(J\xi_x, \xi_x) > 0$. □

**Proposition 3.8.** The $\mathbb{C}$–action on $X^*$ is locally proper.
therefore there exists a function $f$ may assume that $f$ be two different orbits in $X$. Then
\[
\frac{d}{dt}\bigg|_0 \mu^c(it \cdot x) = \omega_x(J\xi_x, \xi_x) > 0
\]
implies that every $i\mathbb{R}$–orbit intersects $(\mu^c)^{-1}(c)$ transversally. Since $X$ is orbit-connected in $X^*$, the map $i\mathbb{R} \times (\mu^c)^{-1}(c) \to X^*$ is injective and therefore a diffeomorphism onto its open image. Together with the fact that $(\mu^c)^{-1}(c)$ is $\mathbb{R}$–invariant this yields the existence of differentiable local slices for the $\mathbb{C}$–action. \hfill \Box

3.3. A necessary condition for $X/\mathbb{Z}$ to be Stein. We have the following necessary condition for $X/\mathbb{Z}$ to be a Stein manifold.

Proposition 3.9. If the quotient manifold $X/\mathbb{Z}$ is Stein, then $X^*$ is Stein and the $\mathbb{C}$–action on $X^*$ is proper.

Proof. Suppose that $X/\mathbb{Z}$ is a Stein manifold. By [CTIT00] this implies that $X^*$ is Stein as well.

Next we will show that the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ is proper. For this we will use as above a moment map for the $S^1$–action on $X^*/\mathbb{Z}$.

By compactness of $S^1$ we may apply the complexification theorem from [Hei91] which shows that $X^*/\mathbb{Z}$ is also a Stein manifold and in particular Hausdorff. Hence, there exists a smooth strictly plurisubharmonic exhaustion function $\rho: X^*/\mathbb{Z} \to \mathbb{R}^{>0}$ invariant under $S^1$. Consequently, $\omega := \frac{i}{2}\partial \bar{\partial} \rho \in A^{1,1}(X^*)$ is an $S^1$–invariant Kähler form. Associated to $\omega$ we have the $S^1$–invariant moment map
\[
\mu: X^*/\mathbb{Z} \to \mathbb{R}, \quad \mu^c(x) := \frac{d}{dt}\bigg|_0 \rho(\exp(it\xi) \cdot x),
\]
where $\xi$ is the complete holomorphic vector field on $X^*/\mathbb{Z}$ which corresponds to the $S^1$–action. Now we can apply the same argument as above in order to deduce that $\mathbb{C}^*$ acts locally properly on $X^*/\mathbb{Z}$.

We still must show that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. To see this, let $\mathbb{C}^* \cdot x_j, \ j = 0, 1$, be two different orbits in $X^*/\mathbb{Z}$. Since $\mathbb{C}^*$ acts locally properly, these are closed and therefore there exists a function $f \in O(X^*/\mathbb{Z})$ with $f|_{\mathbb{C}^* \cdot x_j} = j$ for $j = 0, 1$. Again we may assume that $f$ is $S^1$– and consequently $\mathbb{C}^*$–invariant. Hence, there is a continuous function on $(X^*/\mathbb{Z})/\mathbb{C}^*$ which separates the two orbits, which implies that $(X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. This proves that $\mathbb{C}^*$ acts properly on $X^*/\mathbb{Z}$.

Since we know already that the $\mathbb{C}$–action on $X^*$ is locally proper, it is enough to show that $X^*/\mathbb{C}$ is Hausdorff. But this follows from the properness of the $\mathbb{C}^*$–action on $X^*/\mathbb{Z}$ since $X^*/\mathbb{C} \cong (X^*/\mathbb{Z})/\mathbb{C}^*$ is Hausdorff. \hfill \Box

4. Properness of the $\mathbb{C}$–action

Let $X$ be a hyperbolic Stein $\mathbb{R}$–manifold. Suppose that $X$ fulfills (3.1) or (3.2). We have seen that $\mathbb{C}$ acts locally properly on $X^*$. In this section we prove that under the additional assumption $\dim X = 2$ the orbit space $X^*/\mathbb{C}$ is Hausdorff. This implies that $\mathbb{C}$ acts properly on $X^*$ if $\dim X = 2$. 


4.1. **Stein surfaces with $\mathbb{C}$–actions.** For every function $f \in \mathcal{O}(\Delta)$ which vanishes only at the origin, we define

$$X_f := \{(x, y, z) \in \Delta \times \mathbb{C}^2; f(x)y - z^2 = 1\}.$$ 

Since the differential of the defining equation of $X_f$ is given by $(f'(x)y f(x) - 2z)$, we see that 1 is a regular value of $(x, y, z) \mapsto f(x)y - z^2$. Hence, $X_f$ is a smooth Stein surface in $\Delta \times \mathbb{C}^2$.

There is a holomorphic $\mathbb{C}$–action on $X_f$ defined by

$$t \cdot (x, y, z) := (x, y + 2tz + t^2 f(x), z + tf(x)).$$

One can directly check that this defines an action.

**Lemma 4.1.** The $\mathbb{C}$–action on $X_f$ is free, and all orbits are closed.

**Proof.** Let $t \in \mathbb{C}$ such that $(x, y + 2tz + t^2 f(x), z + tf(x)) = (x, y, z)$ for some $(x, y, z) \in X_f$. If $f(x) \neq 0$, then $z + tf(x) = z$ implies $t = 0$. If $f(x) = 0$, then $z \neq 0$ and $y + 2tz = y$ gives $t = 0$.

The map $\pi: X_f \to \Delta$, $(x, y, z) \mapsto x$, is $\mathbb{C}$–invariant. If $a \in \Delta^*$, then $f(a) \neq 0$ and we have

$$\frac{z}{f(a)} \cdot (a, f(a)^{-1}, 0) = (a, y, z) \in X_f,$$

which implies $\pi^{-1}(a) = \mathbb{C} \cdot (a, f(a)^{-1}, 0)$. A similar calculation gives $\pi^{-1}(0) = \mathbb{C} \cdot p_1 \cup \mathbb{C} \cdot p_2$ with $p_1 = (0, 0, i)$ and $p_2 = (0, 0, -i)$. Consequently, every $\mathbb{C}$–orbit is closed. \(\square\)

**Remark.** The orbit space $X_f/\mathbb{C}$ is the unit disc with a doubled origin and in particular not Hausdorff.

We calculate slices at the point $p_j$, $j = 1, 2$, as follows. Let $\varphi_j: \Delta \times \mathbb{C} \to X_f$ be given by $\varphi_1(z, t) := t \cdot (z, 0, i)$ and $\varphi_2(w, s) = s \cdot (w, 0, -i)$. Solving the equation $s \cdot (w, 0, -i) = t \cdot (z, 0, i)$ for $(w, s)$ yields the transition function $\varphi_{12} = \varphi_2^{-1} \circ \varphi_1: \Delta^* \times \mathbb{C} \to \Delta^* \times \mathbb{C}$,

$$(z, t) \mapsto \left(z, t + \frac{2i}{f(z)}\right).$$

The function $\frac{1}{f}$ is a meromorphic function on $\Delta$ without zeros and with the unique pole 0.

**Lemma 4.2.** Let $\mathbb{R}$ act on $X_f$ via $\mathbb{R} \looparrowright \mathbb{C}$, $t \mapsto ta$, for some $a \in \mathbb{C}^*$. Then there is no $\mathbb{R}$–invariant domain $D \subset X_f$ with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$ on which $\mathbb{R}$ acts properly.

**Proof.** Suppose that $D \subset X_f$ is an $\mathbb{R}$–invariant domain with $D \cap \mathbb{C} \cdot p_j \neq \emptyset$ for $j = 1, 2$. Without loss of generality we may assume that $p_1 \in D$ and $\zeta \cdot p_2 = (0, -2\zeta i, -i) \in D$ for some $\zeta \in \mathbb{C}$. We will show that the orbits $\mathbb{R} \cdot p_1$ and $\mathbb{R} \cdot (\zeta \cdot p_2)$ cannot be separated by $\mathbb{R}$–invariant open neighborhoods.

Let $U_1 \subset D$ be an $\mathbb{R}$–invariant open neighborhood of $p_1$. Then there are $r, r' > 0$ such that $\Delta_r^* \times \Delta_{r'} \times \{i\} \subset U_1$ holds. Here, $\Delta_r = \{z \in \mathbb{C}; |z| < r\}$. For $(\varepsilon_1, \varepsilon_2) \in \Delta_r^* \times \Delta_{r'}$ and $t \in \mathbb{R}$ we have

$$t \cdot (\varepsilon_1, \varepsilon_2, i) = (\varepsilon_1, \varepsilon_2 + 2tai + (ta)^2 f(\varepsilon_1), i + (ta)f(\varepsilon_1)) \in U_1.$$
We have to show that for all \( r_2, r_3 > 0 \) there exist \((\tilde{\varepsilon}_2, \tilde{\varepsilon}_3) \in \Delta_{r_2} \times \Delta_{r_3}, (\varepsilon_1, \varepsilon_2) \in \Delta_+ \times \Delta_-\) and \( t \in \mathbb{R} \) such that

\[
(\varepsilon_1, \varepsilon_2 + (ta)^2 f(\varepsilon_1), i + (ta^2 f(\varepsilon_1))) = (\varepsilon_1, -2\zeta_1 + \tilde{\varepsilon}_2, -i + \tilde{\varepsilon}_3)
\]

holds.

Let \( r_2, r_3 > 0 \) be given. From (4.1) we obtain \( \tilde{\varepsilon}_3 = taf(\varepsilon_1) + 2i \) or, equivalently, \( ta = \frac{\tilde{\varepsilon}_2}{af(\varepsilon_1)} \). Setting \( \varepsilon_2 = \varepsilon_2 \) we obtain from \( (ta)^2 f(\varepsilon_1) = -2\zeta_1 \) the equivalent expression

\[
(4.2) \quad f(\varepsilon_1) = -2\zeta_1 + \frac{ta}{(ta)^2}.
\]

for \( t \neq 0 \). Choosing a real number \( t \gg 1 \), we find an \( \varepsilon_1 \in \Delta_+ \) such that (4.2) is fulfilled. After possibly enlarging \( t \) we have \( \tilde{\varepsilon}_3 := taf(\varepsilon_1) + 2i = -2\zeta_1 \tilde{\varepsilon}_2 \in \Delta_{r_3} \). Together with \( \varepsilon_2 = \tilde{\varepsilon}_2 \) equation (4.1) is fulfilled and the proof is finished. \( \square \)

Thus, the Stein surface \( X_f \) cannot be obtained as globalization of the local \( \mathbb{C} \)-action on any \( \mathbb{R} \)-invariant domain \( D \subset X_f \) on which \( \mathbb{R} \) acts properly.

4.2. The quotient \( X^*/\mathbb{C} \) is Hausdorff. Suppose that \( X^*/\mathbb{C} \) is not Hausdorff and let \( x_1, x_2 \in X \) be such that the corresponding \( \mathbb{C} \)-orbits cannot be separated in \( X^*/\mathbb{C} \). Since we already know that \( \mathbb{C} \) acts locally proper on \( X^* \) we find local holomorphic slices \( \varphi_j : \Delta \times \mathbb{C} \to U_j \subset X \), \( \varphi_j(z, t) = t \cdot s_j(z) \) at each \( \Delta \cdot x_j \) where \( s_j : \Delta \to X \) is holomorphic with \( s_j(0) = x_j \). Consequently, we obtain the transition function \( \varphi_{12} : (\Delta \setminus A) \times \mathbb{C} \to (\Delta \setminus A) \times \mathbb{C} \) for some closed subset \( A \subset \Delta \) which must be of the form \( (z, t) \mapsto (z, t + f(z)) \) for some \( f \in \mathcal{O}(\Delta \setminus A) \). The following lemma applies to show that \( A \) is discrete and that \( f \) is meromorphic on \( \Delta \). Hence, we are in one of the model cases discussed in the previous subsection.

**Lemma 4.3.** Let \( \Delta_1 \) and \( \Delta_2 \) denote two copies of the unit disk \( \{ z \in \mathbb{C} ; |z| < 1 \} \). Let \( U \subset \Delta_j, j = 1, 2 \), be a connected open subset and \( f : U \subset \Delta_1 \to \mathbb{C} \) a non-constant holomorphic function on \( U \). Define the complex manifold

\[
M := (\Delta_1 \times \mathbb{C}) \cup (\Delta_2 \times \mathbb{C})/\sim,
\]

where \( \sim \) is the relation \( (z_1, t_1) \sim (z_2, t_2) : \iff z_1 = z_2 = z \in U \) and \( t_2 = t_1 + f(z) \).

Suppose that \( M \) is Hausdorff. Then the complement \( A \) of \( U \) is discrete and \( f \) extends to a meromorphic function on \( \Delta_1 \).

**Proof.** We first prove that for every sequence \( (x_n), x_n \in U \), with \( \lim_{n \to \infty} x_n = p \in \partial U \), one has \( \lim_{n \to \infty} f(x_n) = \infty \in \mathbb{P}(\mathbb{C}) \). Assume the contrary, i.e. there is a sequence \( (x_n) \), \( x_n \in U \), with \( \lim_{n \to \infty} x_n = p \in \partial U \) such that \( \lim_{n \to \infty} f(x_n) = a \in \mathbb{C} \). Choose now \( t_1 \in \mathbb{C} \), consider the two points \((p, t_1) \in \Delta_1 \times \mathbb{C} \) and \((p, t_1 + a) \in \Delta_2 \times \mathbb{C} \) and note their corresponding points in \( M \) as \( q_1 \) and \( q_2 \). Then \( q_1 \neq q_2 \). The sequences \((x_n, t_1) \in \Delta_1 \times \mathbb{C} \) and \((x_n, t_1 + f(x_n)) \in \Delta_2 \times \mathbb{C} \) define the same sequence in \( M \) having \( q_1 \) and \( q_2 \) as accumulation points. So \( M \) is not Hausdorff, a contradiction.

In particular we have proved that the zeros of \( f \) do not accumulate to \( \partial U \) in \( \Delta_1 \). So there is an open neighborhood \( V \) of \( \partial U \) in \( \Delta_1 \) such that the restriction of \( f \) to \( W := U \cap V \) does not vanish. Let \( g := 1/f \) on \( W \). Then \( g \) extends to a continuous function on \( V \) taking the value zero outside of \( U \). The theorem of Rado implies that
this function is holomorphic on $V$. It follows that the boundary $\partial U$ is discrete in $\Delta_1$ and that $f$ has a pole in each of the points of this set, so $f$ is a meromorphic function on $\Delta_1$.

\[\text{Theorem 4.4. The orbit space } X^*/\mathbb{C} \text{ is Hausdorff. Consequently, } \mathbb{C} \text{ acts properly on } X^*.\]

\[\text{Proof.} \text{ By virtue of the above lemma, in a neighborhood of two non-separable } \mathbb{C}\text{–orbits } X \text{ is isomorphic to a domain in one of the model Stein surfaces discussed in the previous subsection. Since we have seen there that these surfaces are never globalizationes, we arrive at a contradiction. Hence, all } \mathbb{C}\text{–orbits are separable.} \Box\]

5. Examples

In this section we discuss several examples which illustrate our results.

5.1. Hyperbolic Stein surfaces with proper $\mathbb{R}\text{–actions.}$ Let $R$ be a compact Riemann surface of genus $g \geq 2$. It follows that the universal covering of $R$ is given by the unit disc $\Delta \subset \mathbb{C}$ and hence that $R$ is hyperbolic. The fundamental group $\pi_1(R)$ of $R$ contains a normal subgroup $N$ such that $\pi_1(R)/N \cong \mathbb{Z}$. Let $\tilde{R} \to R$ denote the corresponding normal covering. Then $\tilde{R}$ is a hyperbolic Riemann surface with a holomorphic $\mathbb{Z}$–action such that $\tilde{R}/\mathbb{Z} = R$. Note that $\mathbb{Z}$ is not contained in a one parameter group of automorphisms of $\tilde{R}$.

We have two mappings

$$X := \mathbb{H} \times_{\mathbb{Z}} \tilde{R} \xrightarrow{q} \tilde{R}/\mathbb{Z} = R$$

$$\mathbb{H}/\mathbb{Z} \cong \Delta \setminus \{0\}.$$

The map $p: X \to \Delta \setminus \{0\}$ is a holomorphic fiber bundle with fiber $\tilde{R}$. Since the Serre problem has a positive answer if the fiber is a non-compact Riemann surface ($[\text{Mok82}]$), the suspension $X = \mathbb{H} \times_{\mathbb{Z}} \tilde{R}$ is a hyperbolic Stein surface. The group $\mathbb{R}$ acts on $\mathbb{H} \times \tilde{R}$ by $t \cdot (z, x) = (z + t, x)$ and this action commutes with the diagonal action of $\mathbb{Z}$. Consequently, we obtain an action of $\mathbb{R}$ on $X$.

\[\text{Lemma 5.1. The universal globalization of the local } \mathbb{C}\text{–action on } X \text{ is given by } X^* = \mathbb{C} \times_{\mathbb{Z}} \tilde{R}. \text{ Moreover, } \mathbb{C} \text{ acts properly on } X^*.\]

\[\text{Proof. One checks directly that } t \cdot [z, x] := [z + t, x] \text{ defines a holomorphic } \mathbb{C}\text{–action on } X^* = \mathbb{C} \times_{\mathbb{Z}} \tilde{R} \text{ which extends the } \mathbb{R}\text{–action on } X. \text{ We will show that } X \text{ is orbit-connected in } X^*: \text{ Since } [z + t, x] \text{ lies in } X \text{ if and only if there exist elements } (z', x') \in \mathbb{H} \times \tilde{R} \text{ and } m \in \mathbb{Z} \text{ such that } (z + t, x) = (z' + m, m \cdot x'), \text{ we conclude } \mathbb{C}[z, x] = \{t \in \mathbb{C}; \text{ Im}(t) > -\text{ Im}(z)\} \text{ which is connected.} \]

\[\text{In order to show that } \mathbb{C} \text{ acts properly on } X^* \text{ it is sufficient to show that } \mathbb{C} \times \mathbb{Z} \text{ acts properly on } \mathbb{C} \times \tilde{R}. \text{ Hence, we choose sequences } \{t_n\} \text{ in } \mathbb{C}, \{m_n\} \text{ in } \mathbb{Z} \text{ and } \{(z_n, x_n)\} \text{ in } \mathbb{C} \times \tilde{R} \text{ such that} \]

$$((t_n, m_n) \cdot (z_n, x_n), (z_n, x_n)) = ((z_n + t_n + m_n, m_n \cdot x_n), (z_n, x_n)) \to ((z_1, x_1), (z_0, x_0))$$
holds. Since $\mathbb{Z}$ acts properly on $\tilde{R}$, it follows that $\{m_n\}$ has a convergent subsequence, which in turn implies that $\{t_n\}$ has a convergent subsequence. Hence, the lemma is proven.

**Proposition 5.2.** The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein. The quotient $X^*/\mathbb{C}$ is holomorphically equivalent to $\tilde{R}/\mathbb{Z} = R$.

**Proof.** It is sufficient to note that the map $\Phi : \mathbb{H} \times \mathbb{R} \to \Delta^* \times R$, $\Phi(z,x) := (e^{2\pi i z}, x)$, induces a biholomorphic map $X/\mathbb{Z} \to \Delta^* \times R$. □

**Proposition 5.3.** The quotient $X/\mathbb{Z} \cong \Delta^* \times R$ is not holomorphically separable and in particular not Stein.

Thus we have found an example for a hyperbolic Stein surface $X$ endowed with a proper $\mathbb{R}$–action such that the associated $\mathbb{Z}$–quotient is not holomorphically separable. Moreover, the $\mathbb{R}$–action on $X$ extends to a proper $\mathbb{C}$–action on a Stein manifold $X^*$ containing $X$ as an orbit-connected domain such that $X^*/\mathbb{C}$ is any given compact Riemann surface of genus $g \geq 2$.

### 5.2. Counterexamples with domains in $\mathbb{C}^n$.

There is a bounded Reinhardt domain $D$ in $\mathbb{C}^2$ endowed with a holomorphic action of $\mathbb{Z}$ such that $D/\mathbb{Z}$ is not Stein. However, this $\mathbb{Z}$–action does not extend to an $\mathbb{R}$–action. We give quickly the construction.

Let $\lambda := \frac{1}{2}(3 + \sqrt{5})$ and

$$D := \{ (x,y) \in \mathbb{C}^2 \mid |x| > |y|^\lambda, |y| > |x|^\lambda \}.$$ 

It is obvious that $D$ is a bounded Reinhardt domain in $\mathbb{C}^2$ avoiding the coordinate hyperplanes. The holomorphic automorphism group of $D$ is a semidirect product $\Gamma \rtimes (S^1)^2$, where the group $\Gamma \simeq \mathbb{Z}$ is generated by the automorphism $(x,y) \mapsto (x^3 y^{-1}, x)$ and $(S^1)^2$ is the rotation group. Therefore the group $\Gamma$ is not contained in a one-parameter group. Furthermore the quotient $D/\Gamma$ is the (non-Stein) complement of the singular point in a 2-dimensional normal complex Stein space, a so-called "cusp singularity". These singularities are intensively studied in connection with Hilbert modular surfaces and Inoue-Hirzebruch surfaces, see e.g. [vdG88] and [Zaf01].

In the rest of this subsection we give an example of a hyperbolic domain of holomorphy in a 3–dimensional Stein solvmanifold endowed with a proper $\mathbb{R}$–action such that the $\mathbb{Z}$–quotient is not Stein. While this domain is not simply-connected, its fundamental group is much simpler than the fundamental groups of our two-dimensional examples.

Let $G := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; \ a, b, c \in \mathbb{C} \right\}$ be the complex Heisenberg group and let us consider its discrete subgroup

$$\Gamma := \left\{ \begin{pmatrix} 1 & m & \frac{m^2}{2} + 2\pi ik \\ 0 & 1 & m + 2\pi il \\ 0 & 0 & 1 \end{pmatrix} ; \ m, k, l \in \mathbb{Z} \right\}.$$ 

Note that $\Gamma$ is isomorphic to $\mathbb{Z}_m \ltimes \mathbb{Z}_{(k,l)}^2$. We let $\Gamma$ act on $\mathbb{C}^2$ by

$$(z, w) \mapsto \left( z + mw - \frac{m^2}{2} - 2\pi ik, w - m - 2\pi il \right).$$
Proposition 5.4. The group \( \Gamma \) acts properly and freely on \( \mathbb{C}^2 \), and the quotient manifold \( \mathbb{C}^2/\Gamma \) is holomorphically separable but not Stein.

Proof. Since \( \Gamma' \cong \mathbb{Z}^2 \) is a normal subgroup of \( \Gamma \), we obtain \( \mathbb{C}^2/\Gamma \cong (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma') \). The map \( \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^* \), \( (z, w) \mapsto (\exp(z), \exp(w)) \), identifies \( \mathbb{C}^2/\Gamma' \) with \( \mathbb{C}^* \times \mathbb{C}^* \). The induced action of \( \Gamma/\Gamma' \cong \mathbb{Z} \) on \( \mathbb{C}^* \times \mathbb{C}^* \) is given by

\[
(z, w) \mapsto \left( e^{-m^2/2}zw^m, e^{-m}w \right)
\]

which shows that \( \Gamma \) acts properly and freely on \( \mathbb{C}^2 \). Moreover, we obtain the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & Y := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z} \\
\downarrow_{(z,w)\mapsto w} & & \downarrow \\
\mathbb{C}^* & \longrightarrow & T := \mathbb{C}^*/\mathbb{Z}.
\end{array}
\]

The group \( \mathbb{C}^* \) acts by multiplication in the first factor on \( \mathbb{C}^* \times \mathbb{C}^* \) and this action commutes with the \( \mathbb{Z} \)-action. One checks directly that the joint \( (\mathbb{C}^* \times \mathbb{Z}) \)-action on \( \mathbb{C}^* \times \mathbb{C}^* \) is proper which implies that the map \( Y \to T \) is a \( \mathbb{C}^* \)-principal bundle. Consequently, \( Y \) is not Stein.

In order to show that \( Y \) is holomorphically separable, note that by \([\text{Oel92}]\) this \( \mathbb{C}^* \)-principal bundle \( Y \to T \) extends to a line bundle \( p: L \to T \) with first Chern class \( c_1(L) = -1 \). Therefore the zero section of \( p: L \to T \) can be blown down and we obtain a singular normal Stein space \( \overline{Y} = Y \cup \{y_0\} \) where \( y_0 = \text{Sing}(\overline{Y}) \) is the blown down zero section. Thus \( Y \) is holomorphically separable. \( \square \)

Let us now choose a neighborhood of the singularity \( y_0 \in \overline{Y} \) biholomorphic to the unit ball and let \( U \) be its inverse image in \( \mathbb{C}^2 \). It follows that \( U \) is a hyperbolic domain with smooth strictly Levi-convex boundary in \( \mathbb{C}^2 \) and in particular Stein. In order to obtain a proper action of \( \mathbb{R} \) we form the suspension \( D = \mathbb{H} \times_{\Gamma} U \) where \( \Gamma \) acts on \( \mathbb{H} \times U \) by \( (t, z, w) \mapsto (t + m, z + mw - \frac{m^2}{2} - 2\pi ik, w - m - 2\pi il) \).

Proposition 5.5. The suspension \( D = \mathbb{H} \times_{\Gamma} U \) is isomorphic to a Stein domain in the Stein manifold \( G/\Gamma \).

Proof. We identify \( \mathbb{H} \times U \) with the \( \mathbb{R} \times \Gamma \)-invariant domain

\[
\Omega := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; \; \text{Im}(a) > 0, (c, b) \in U \right\}
\]

in \( G \).

Since \( \mathbb{H} \times U \) is Stein, it follows that \( \mathbb{H} \times_{\Gamma} U \) is locally Stein in \( G/\Gamma \). Hence, by virtue of \([\text{DG60}]\) we only have to show that \( G/\Gamma \) is Stein.

For this we note first that \( G \) is a closed subgroup of \( \text{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \) which implies that \( G/\Gamma \) is a closed complex submanifold of \( X := (\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2)/\Gamma \). By \([\text{Oel92}]\) the manifold \( X \) is holomorphically separable, hence \( G/\Gamma \) is holomorphically separable. Since \( G \) is solvable, a result of Huckleberry and Oeljeklaus \([\text{HOS86}]\) yields the Steinness of \( G/\Gamma \).
One checks directly that the action of $\mathbb{R} \times \Gamma$ on $\mathbb{H} \times U$ is proper which implies that $\mathbb{R}$ acts properly on $\mathbb{H} \times \Gamma U$.

Because of $(\mathbb{H} \times \Gamma U)/\mathbb{Z} \cong \Delta^* \times (U/\Gamma)$ this quotient manifold is not Stein but holomorphically separable.

6. Bounded domains with proper $\mathbb{R}$–actions

In this section we give the proof of our main result.

6.1. Proper $\mathbb{R}$–actions on $D$. Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\text{Aut}(D)^0$ be the connected component of the identity in $\text{Aut}(D)$.

Lemma 6.1. A proper $\mathbb{R}$–action by holomorphic transformations on $D$ exists if and only if the group $\text{Aut}(D)^0$ is non-compact.

The proof follows from the existence of a diffeomorphism $K \times V \to \text{Aut}(D)^0$ where $K$ is a maximal compact subgroup of $\text{Aut}(D)^0$ and $V$ is a linear subspace of the Lie algebra of $\text{Aut}(D)^0$.

6.2. Steinness of $D/\mathbb{Z}$. Now we give the proof of our main result.

Theorem 6.2. Let $D$ be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$. Suppose that the group $\mathbb{R}$ acts properly by holomorphic transformations on $D$. Then the complex manifold $D/\mathbb{Z}$ is biholomorphically equivalent to a domain of holomorphy in $\mathbb{C}^2$.

Proof. Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy. Since the Serre problem is solvable if the fiber is $D$, see [Sin76], the universal globalization $D^*$ is a simply-connected Stein surface, [CTIT00]. Moreover, we have shown in Theorem 4.4 that $\mathbb{C}$ acts properly on $D^*$. Since the Riemann surface $D^*/\mathbb{C}$ is also simply-connected, it must be $\Delta$, $\mathbb{C}$ or $\mathbb{P}_1(\mathbb{C})$. In all three cases the bundle $D^* \to D^*/\mathbb{C}$ is holomorphically trivial. So we can exclude the case that $D^*/\mathbb{C}$ is compact and it follows that $D/\mathbb{Z} \cong \mathbb{C}^* \times (D^*/\mathbb{C})$ is a Stein domain in $\mathbb{C}^2$. 

6.3. A normal form for domains with non-compact $\text{Aut}(D)^0$. Let $D \subset \mathbb{C}^2$ be a simply-connected bounded domain of holomorphy such that its automorphism group is non-compact. As we have seen, this yields a proper $\mathbb{R}$–action on $D$ by holomorphic transformations and the universal globalization of the induced local $\mathbb{C}$–action on $D$ is isomorphic to $\mathbb{C} \times S$ where $S$ is either $\Delta$ or $\mathbb{C}$ and where $\mathbb{C}$ acts by translation in the first factor.

Moreover, there are plurisubharmonic functions $u,-o: \mathbb{C} \times S \to \mathbb{R} \cup \{-\infty\}$ which fulfill

$$u(t \cdot (z_1, z_2)) = u(z_1, z_2) - \text{Im}(t) \quad \text{and} \quad o(t \cdot (z_1, z_2)) = o(z_1, z_2) - \text{Im}(t)$$

such that $D = \{(z_1, z_2) \in \mathbb{C} \times S; u(z_1, z_2) < 0 < o(z_1, z_2)\}$. From this we conclude $u(z_1, z_2) = u(0, z_2) - \text{Im}(z_1)$, $o(z_1, z_2) = o(0, z_2) - \text{Im}(z_1)$ and define $u'(z_2) := u(0, z_2)$, $o'(z_2) := o(0, z_2)$.

We summarize our remarks in the following
Theorem 6.3. Let $D$ be a simply-connected bounded domain of holomorphy in $\mathbb{C}^2$ admitting a non-compact connected identity component of its automorphism group. Then $D$ is biholomorphic to a domain of the form

$$\tilde{D} = \{(z_1, z_2) \in \mathbb{C} \times S; u'(z_2) < \text{Im}(z_1) < o'(z_2)\},$$

where the functions $u', -o'$ are subharmonic in $S$.

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