Isogrowth Cosmology (and How to Map the Universe)

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While general relativity ties together the cosmic expansion history and growth history of large scale structure, beyond the standard model these can have independent behaviors. We derive expressions for cosmologies with identical growth histories but different expansion histories, or other deviations. This provides a relation for isogrowth cosmologies, but also highlights in general the need for observations to measure each of the growth, expansion, gravity, and dark matter property histories.

I. INTRODUCTION

Mapping the expansion history of the universe, e.g. through supernova distances [1, 2], led to the discovery of cosmic acceleration, replacing the then standard model of a matter dominated universe. Within general relativity, the growth of linear density perturbations into large scale structure – the cosmic growth history – is determined by the expansion history (so long as the matter density behaves in the standard way).

However, shortly after cosmic acceleration became part of the new standard model, the recognition that it could arise not only from an additional energy density component with strongly negative pressure but also from a modification of the gravitational sector severed the lockstep between expansion and growth histories [3–6]. Cosmologies with identical expansion histories could have different growth histories, due to the differing gravity. In this sense the standard model has moved from having two types of cosmic history, with one determining the other, to possessing three cosmic histories, with any two determining the third.

Here we turn the situation around and study the inverse case of cosmologies with identical growth histories, and derive the necessary relations for the expansion and gravity histories. Cosmic growth history can be mapped through observations such as redshift space distortions in galaxy clustering, and as a blend with expansion history (and gravity) in probes such as weak gravitational lensing. As the upcoming generation of experiments prepares to deliver such data, and planning is underway for the science cases of the next generation of experiments, it behooves us to study the relation – and freedom – between the different types of cosmic history. That is, what should be the vision of how to map the universe so as to truly understand our cosmology (cf. [7, 8])?

Another element is the division between the matter whose clustering the observations measure, and other energy density components that do not cluster effectively. In one sense, this occurs at the background expansion level, where distances measure the Hubble expansion rate $H(a)$, but the division into matter and dark energy is done separately. This is sometimes referred to as the “dark degeneracy” [9, 10]. Here we are more interested in the perturbative level: what enters as the matter density perturbation $\delta \rho$ in the Poisson equation, and hence the growth evolution equation – i.e. what clusters. Having only a fraction of the nonrelativistic matter cluster would imply physics beyond the standard cosmological model, whether arising from interactions with another component or internal properties. Such “dark matter property history” can have interesting implications, and be constrained in turn by cosmic probes (see, e.g., [11]).

In Section II we derive the relations for isogrowth cosmology, and in Section III investigate three subcases where certain aspects of the physics have freedom while fixing others (i.e. the matter, gravity, and expansion behaviors respectively). We discuss the use of isogrowth cosmology as a clear demonstration of freedoms and connections, and conclude with the general vision of mapping all four histories, in Section IV.

II. COSMIC GROWTH HISTORY

The growth of matter perturbations $\delta \equiv \delta \rho_m/\rho_m$ into large scale structure is given by

$$\dddot{g} + \left[ 4 + \frac{1}{2} \left( \ln H^2 \right)^3 \right] \ddot{g} + \left[ 3 + \frac{1}{2} \left( \ln H^2 \right)^2 - \frac{3}{2} G_N G(a) \Omega_m^l(a) \right] g = 0 ,$$  (1)

in the linear density regime on subhorizon scales, with prime denoting $d/d \ln a$ for $a$ the cosmic scale factor (see, e.g., [6]). Here $g(a) = [\delta(a)/a]/[\delta(a_i)/a_i]$ is the normalized growth factor (equal to one during standard matter domination), $H(a)$ is the Hubble parameter describing the expansion history, $G_N$ is Newton’s constant, $G(a)$ a
dimensionless modification of the gravitational coupling to matter perturbations, and \( \Omega_{m}^{1}(a) \) is the fraction of the critical density contributed by clustered matter. In the standard model there is no distinction between \( \Omega_{m}(a) \) and \( \Omega_{m}^{1}(a) \) since there all matter (i.e. nonrelativistic, zero pressure energy density) clusters. However here we want to allow the possibility that only part of the matter clusters; we will abbreviate \( \Omega(a) \equiv \Omega_{m}^{1}(a) \) since we will be adding subscripts labeling different cosmologies.

Thus we see that outside the standard model there is freedom to loosen the connection between expansion \( H(a) \) and growth \( g(a) \). For example, at the background expansion level one can define an effective energy density corresponding to gravitational modifications that will exactly match the contribution of a physical energy density to the Hubble parameter, giving identical \( H(a) \); however the gravitational modifications will enter in \( G(a) \) and change the growth. Here we consider identical growth histories \( g(a) \), i.e. \( \delta(a) \), and derive the different expansion histories.

Two pedagogical points: The \( G(a) \) here corresponds to \( G_{\text{matter}}(a) \), also sometimes called \( \mu(a) \), the gravitational coupling strength to matter that appears in the modified Poisson equation for matter. There is further gravitational freedom beyond this, which does not enter here. Second, one could ask for identical growth history in terms of \( \delta(t) \) rather than \( \delta(a) \). This is less interesting in that we measure matter clustering at different redshifts, not times, and if we allow differing expansion histories then the \( a(t) \) relation is not fixed.

Considering two different cosmologies, 1 and 2, with identical growth histories \( g(a) \), and hence \( g' \) and \( g'' \). Eq. (1) yields the matching condition

\[
G_{2}(a) = G_{1}(a) \frac{\Omega_{1}(a)}{\Omega_{2}(a)} - \frac{f(a)}{3\Omega_{2}(a)} \left( \ln \frac{H_{2}^{2}}{H_{2}^{2}} \right)^{\prime},
\]

where we have used that the growth rate \( f(a) \equiv \delta' / \delta = 1 + g' / g \). Note that the growth rate enters observationally in redshift space distortions. Equation (2) shows the condition needed on the gravitational coupling modification for the growth histories to match between the two cosmologies. One could adopt cosmology 1 as being a general relativity standard cosmology, say, in which case \( G_{1}(a) = 1, \Omega_{1}(a) = \Omega_{m}(a) = \Omega_{m,0}a^{-3}/[H_{1}(a)/H_{1}(a = 1)]^{2} \).

We can write the matching condition needed on the expansion history instead, obtaining

\[
H_{2}(a) = H_{1}(a) \frac{H_{2}(a = 1)}{H_{1}(a = 1)} e^{(3/2) \int_{0}^{a} d \ln x [G_{2}(x)\Omega_{2}(x) - G_{1}(x)\Omega_{1}(x)]/f(x)}.
\]

Either Eq. (2) or Eq. (3) demonstrate that isogrowth cosmologies impose a relation among expansion history, gravity history, and matter clustering, while allowing freedom to trade between them. This is unlike general relativity standard cosmology where isogrowth determines exactly the expansion history (and vice versa).

## III. THREE CASES

Given the general relation, we can describe three special cases where we have both identical growth histories and one other type of history.

### A. Identical Growth and Clustering Matter

In standard cosmology all the nonrelativistic matter clusters, \( \Omega(a) = \Omega_{m}(a) \). If we preserve this, then we still retain the freedom to trade off a modified expansion vs a modified gravity. That is, identical growth will result when either of the two following equivalent expressions holds

\[
G_{2}(a) = G_{1}(a) - \frac{f(a)}{3\Omega_{m}(a)} \left( \ln \frac{H_{1}^{2}}{H_{1}^{2}} \right)^{\prime},
\]

\[
H_{2}(a) = H_{1}(a) \frac{H_{2}(a = 1)}{H_{1}(a = 1)} e^{(3/2) \int_{0}^{a} d \ln x [G_{2}(x)\Omega_{m}(x) - G_{1}(x)\Omega_{m}(x)]/f(x)}.
\]

It is interesting to write the expansion derivative in terms of the total background equation of state \( w(a) \) for the respective cosmology, giving

\[
G_{2}(a) = G_{1}(a) - \frac{f(a)}{\Omega_{m}(a)} [w_{2}(a) - w_{1}(a)].
\]

This ties together key cosmological quantities of gravity, matter density, and equation of state histories.
If we preserve general relativity, so that $G(a) = 1$, then we can still obtain identical growth with different expansion histories if we adapt the matter clustering. This could be done through a certain fraction of the dark matter, say, interacting with another energy density component, or a self interaction. If the full matter density does not cluster, this can affect the initial conditions for the growth equation, introduce an extra integrated Sachs-Wolfe effect in the cosmic microwave background (CMB), and generically affect redshift space distortion observations (except here we are fixing the growth rate). See [11] for discussion concerning the interplay of effects, and some experimental constraints.

The matching conditions are

$$
\Omega_2(a) = \Omega_1(a) - \frac{f(a)}{3} \left( \ln \frac{H_2^2}{H_1^2} \right),
$$

$$
H_2(a) = H_1(a) \left( \frac{H_2(a = 1)}{H_1(a = 1)} \right)^{(3/2) \int_0^a d\ln a \left( \Omega_2(x) - \Omega_1(x) \right)/f(x)},
$$

equivalently. Note that if we take matter to cluster in the standard way, so $\Omega_2(a) = \Omega_1(a)$, then since we are in general relativity this means that isogrowth cosmologies would imply isoeexpansion cosmologies. (In fact, this holds as well with $G_2(a) = G_1(a) \neq 1$, i.e. the same gravitational cosmologies even if the gravity is not general relativity, so long as the form of the density perturbation growth equation is not altered.)

### C. Identical Growth and Expansion

The last case is a curious one, where the two main cosmic histories remain identical, in particular the expansion history $H(a)$ in addition to growth, yet there is freedom distinct from the standard model. The matching condition is

$$
G_2(a) \Omega_2(a) = G_1(a) \Omega_1(a).
$$

That is, the effective gravitating clustering matter densities $G\Omega$ must be identical, but we can trade off the gravitational coupling strength against the clustering matter density. For example, enhanced gravitational strength can give the same growth even with a smaller clustering matter density.

Of course this freedom is not available within general relativity, where $G_2 = G_1 = 1$. Nor may we do a simple change in the present matter density assuming all the matter clusters, $\Omega_{m,2,0} \neq \Omega_{m,1,0}$ since in the matter dominated epoch the expansion goes as $H^2 = H_0^2 \Omega_{m,i,0} a^{-3}$ for $i = 1, 2$ so for the same expansion one must have $\Omega_{m,2,0} = \Omega_{m,1,0}$.

### IV. CONCLUSIONS

Just as observations determining the expansion history alone are not sufficient to characterize the cosmology, neither is growth history data alone. We presented explicit expressions for isogrowth cosmologies, where variations in some set of expansion, gravity, and matter clustering histories could still yield identical growth history.

However, isogrowth imposes relations among the three other histories and if we further adopt one of those to remain identical when varying cosmology, we obtain three subcases. The first, with all of the matter clustering, keeps the matter sector as standard, and trades off the gravity and expansion histories. The second, with gravity unchanged (e.g. standard general relativity) shows that variations in the dark matter sector can compensate for a differing expansion history. The last, where both growth and expansion are preserved, highlights the important role of the effective gravitating clustering matter density.

Thus, to truly understand our cosmology we must measure at least three of the growth, expansion, gravity, and dark matter property histories, while mapping the fourth serves as an important crosscheck. Note that there are further aspects of gravity not entering in the discussion here: gravity affects not only matter but light, and the quantity $G_{\text{light}}(a)$ (sometimes called $\Sigma$) entering the modified Poisson equation for light affects gravitational lensing and other light propagation. Gravitational waves also probe gravity in a somewhat distinct manner, e.g. through the running of the effective Planck mass. Furthermore there are new aspects that enter beyond the linear density regime, including scale dependence.

The mapping of all these histories through cosmological observations should be a central part of the vision for understanding our universe in the next two decades.
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