A NONPARAMETRIC PROCESS FOR MULTI-RESOLUTION ADAPTIVE SHRINKAGE

By Li Ma

Duke University

We introduce a nonparametric prior for probability densities that achieves multi-resolution adaptive shrinkage in posterior inference. The prior applies a varying amount of shrinkage to data structures of different scales and/or at different locations. It is adaptive in that the appropriate amount of shrinkage is determined through the behavior of the data at different scale-location combinations. The prior employs two important additional features: a Markov feature that introduces dependence into the shrinkage across different scale-location combinations, and a randomized partitioning feature that allows adaptive construction of multi-scale grids to most effectively characterize the underlying distribution. We study the theoretical properties of the process, showing that it possesses large support, posterior conjugacy, and posterior consistency. We then provide analytic recipes for marginalization and for computing the posterior through recursion. We illustrate through several numerical examples that the multi-resolution adaptiveness so achieved can substantially improve inference in nonparametric problems.

1. Introduction. In his seminal works that jump-started modern Bayesian nonparametric inference, Ferguson [8, 9] formalized the notion of a Dirichlet process (DP) and introduced a tail-free process that contains the DP as a special case. This tail-free process was later named the Pólya tree (PT) due to its relationship to the Pólya urn model [16], and was popularized in the early 1990’s by a sequence of works [13, 16, 14] that carefully investigated its various theoretical properties. The PT process generates probability measures through a partition-and-probability-assignment procedure. The sample space is recursively bisected into smaller and smaller sets, and at each stage of division, the probability mass that has been assigned to a set is randomly split among its children through the drawing of a Beta random variable. In essence, the PT transforms the inference of a nonparametric probability measure into that of a collection of binomial experiments organized on a multi-resolution grid.

This multi-resolution design of the PT process resembles those of other multi-scale methods such as wavelet decomposition. A nonparametric quantity of interest—an unknown probability distribution for the PT or an unknown function for the wavelet decomposition—is transformed into a collection of coefficients on a multi-resolution grid. In the case of wavelets, these are the familiar wavelet coefficients, whereas for PTs they are the probability assignment coefficients (PACs) specifying how much probability mass goes into the children. Bayesian inference proceeds by placing priors on these coefficients—adopting a PT prior boils down to placing a Beta prior on each of the PACs. The mean of the Beta priors collectively determine the mean of the PT process, whereas the variances of these Beta priors determine how tightly around its mean the process is.

\[\text{Supported in part by NSF grant DMS-1309057.}\
\text{AMS 2000 subject classifications: Primary 62F15, 62G99; secondary 62G07.}\
\text{Keywords and phrases: Pólya tree, multi-scale modeling, multi-resolution inference, Bayesian nonparametrics, adaptive inference, recursive partition, density estimation.} \]
The prior variability of the PACs controls the amount of shrinkage the prior applies to the data. More specifically, the Beta variance can be specified distinctly at different scales and locations, thereby allowing different amounts of shrinkage to be applied to empirical features of different scales and/or at different locations in the sample space. In practical applications of the PT, the statistician usually chooses the Beta pseudo-counts to be of the form $c \cdot m(k)$, where $c$ is a constant and $m(k)$ a (typically non-decreasing) function of the scale index $k$. Intuitively, $c$ corresponds to the overall level of shrinkage, while $m(k)$ the amount of scale-specific shrinkage applied to coefficients at scale $k$. The most popular choice is $m(k) = k^2$.

We note that no matter what $m(k)$ the statistician chooses, it corresponds to a particular rate of shrinkage that may be appropriate for some scale-location combinations but not for others. We illustrate this point through an example.

**Example 1.** We simulate 750 i.i.d data from the following mixture distribution on $[0, 1]$

$$0.1 \ U(0, 1) + 0.3 \ U(0.25, 0.5) + 0.4 \ \text{Beta}_{(0.25, 0.5)}(2, 2) + 0.2 \ \text{Beta}(5000, 2000)$$

where $\text{Beta}_{(0.25, 0.5)}(2, 2)$ represents a Beta$(2, 2)$ recentered and rescaled to be supported on the interval $(0.25, 0.5)$—that is, the distribution with density $8(4x - 1)(1 - 2x)$ on $(0.25, 0.5)$. The red dashed curves in Figure 1 indicates this density function. The “hump” on the interval $(0.25, 0.5)$ constitute a distributional structure of a relatively large scale or low resolution, while the spike corresponding to Beta$(5000, 2000)$ constitutes a small-scale or high-resolution feature.

![Fig 1. PPD of the PT prior and the true density. The middle and right plots give the zoom-in views of the low-resolution and high-resolution features respectively.](image)

Let us place a PT prior on the underlying distribution with the pseudo-counts for each Beta prior at scale $k$ set to $c \cdot m(k)$ with $c = 1$ and $m(k) = k^2$, the most common choice. The dark gray solid curves in Figure 1 shows the posterior predictive density (PPD). The middle and right plots give the zoom-in views of the large-scale and small-scale features respectively. We see that overall the $k^2$ pseudo-counts provide a decent amount of shrinkage for capturing the low-resolution feature, as the general shape of that feature is adequately recovered in the PPD. For the high-resolution feature, however, they result in too much shrinkage. Interestingly, if we zoom into higher resolutions within the low-resolution feature, the PPD shows jumpy patterns of overfitting, indicating that more shrinkage is needed at higher resolutions in those locations to ensure proper smoothness.

The above example represents a typical situation—the appropriate amount of shrinkage depends on the scale and location of the data feature. Only an oracle can a priori choose the pseudo-counts
that give the right amount of shrinkage at all resolutions and locations. Popular choices such as \( c \cdot m(k) \) for the pseudo-counts cannot always provide the right amount of shrinkage on the entire scale-location spectrum.

Fortunately, as we learn from other multi-scale inference problems such as wavelet denoising, the appropriate amounts of shrinkage for different scale-location combinations can be inferred from the data—that is, through adaptive shrinkage. A Bayesian strategy to achieving adaptive shrinkage is to place a hyper-prior on the parameters that control the amount of shrinkage, here the Beta pseudo-counts. In an earlier work, Hanson \([10]\) proposed mixing over the global parameter \( c \), which achieves adaptive shrinkage at the overall level. However, the function \( m(k) \) must still be determined \( a \text{ priori} \), and so this mixture of PT model does not attain the desired scale-location specific shrinkage.

To achieve scale-location specific shrinkage, we need to allow the prior variability of each PAC to be individually modeled. Moreover, our (prior) model should allow the PACs to be correlated according to their scale-location “distance”. This is because interesting distributional features often cluster in scale and/or in location. For instance, in Example 1, each of the distributional features is represented by a cluster of PACs with values far from their prior mean 0.5. The high-resolution feature corresponds to an especially tight cluster. Similar phenomena have been noted in other multi-resolution inference contexts such as wavelet-based image processing \([5]\).

Furthermore, in multi-dimensional settings, it is worthwhile to allow the construction of the multi-resolution grid—that is, the recursive partition of the sample space in the case of PTs—to be data-adaptive as well \([20]\). In standard PTs, the partition sequence of the sample space is \( a \text{ priori} \) fixed. But in multivariate cases, there are often a variety of ways to partition the sample space, each resulting in a different multi-resolution decomposition of the underlying distribution. Some of the partitions—namely those that divide more finely in the dimensions that the density changes more abruptly—are more effective in capturing the distributional features than others. Allowing the multi-resolution grid to be part of the inference can substantially improve efficiency.

In this work we introduce a new process that satisfy all the above considerations—achieving scale-location specific adaptive shrinkage on a data-adaptive partitioning of the sample space. Moreover, it is designed in a way that maintains analytic and computational tractability, which allows us to investigate its theoretical properties and construct efficient inferential recipes. The process again takes the form of a generative partition-and-probability-assignment procedure but now the partitioning is randomized and the process can fall into one of several hidden states at each scale-location combination. The randomized partitioning allows inference on an effective multi-resolution grid. The hidden states, on the other hand, correspond to different prior variances of the PACs and thus amounts of shrinkage. The process transitions from one state into another in a Markovian fashion as it progresses toward finer resolutions, allowing prior dependencies in shrinkage across different scale-location combinations.

In Section 2, we present the construction of our new process, called the adaptive Pólya tree, and provide default suggestions for prior specification. We then carry out a relatively complete study of its theoretical properties. In particular, we show that the process achieves large prior support, posterior consistency, and posterior conjugacy—the corresponding posterior is of the same form with updated parameter values that can be computed analytically through recursion. In Section 3 we illustrate the multi-resolution adaptive shrinkage feature of the process through three numerical examples and compare its performance to those of two existing methods in density estimation. We then conclude with some remarks on the computational efficiency of inference using the process.

We close this introduction by relating this paper to previous works. Multi-resolution adaptive shrinkage has been extensively studied in the context of wavelet denoising from both classical and
Bayesian perspectives. The literature is enormous and interested readers may look at [19] for an excellent overview. Some notable Bayesian examples include [2, 18, 4, 3]. The two main techniques we use to construct the adaptive Pólya tree process—namely randomized recursive partitioning and a hidden Markov model (HMM) on multi-resolution grids—have respectively been adopted in earlier works. See for example [6, 1, 20] and [5]. Through bringing these techniques together, our new process achieves the desirable features of both—adaptive shrinkage on adaptive multiscale decomposition. Previous works using HMMs in multi-resolution inference such as [5] have mostly focused on the modeling and algorithmic aspects. In this work, we provide a theoretical investigation in addition to those perspectives. Finally, while the process we introduce is a general-purpose nonparametric prior, it is worth mentioning that for the particular application of density estimation, there is a large literature on wavelet-based methods. See for example [7, 17, 12, 11].

2. Method.

2.1. Partition rules and elementary regions. We start by introducing some basic concepts related to recursive partitioning that will be used for formulating our multi-resolution framework. These notions have been introduced in earlier works [20, 15] and so this subsection is a brief review.

Let $\Omega$ denote the sample space of the data, which can be finite or a (possibly unbounded) Euclidean rectangle. Let $\mu$ denote the natural measure associated with $\Omega$, which is the counting measure if $\Omega$ is finite and the Lebesgue measure if $\Omega$ is Euclidean.

A partition on $A$ is a collection of subsets $\{A_1, A_2, \ldots, A_K\}$ such that $\mu(A_i \cap A_j) = 0$ for $i \neq j$ while $A = \bigcup_{i=1}^{K} A_i$. A partition rule $R$ is a function that maps any subset $A$ of $\Omega$ to a collection of partitions on $A$. In general, we shall use $\{A_1^i, A_2^i, \ldots, A_K^i(A)\}$ to denote the $j$th way of dividing $A$ under $R$, where $A_i^j$ denotes the $i$th child under the $j$th partition, and $K^j(A)$ the total number of children under that partition.

Example 2 (The dimensionwise dyadic split rule). Let $\Omega$ be a possible unbounded rectangle $[l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_d, u_d]$, and let $G_1, G_2, \ldots, G_d$ be the cdfs of $d$ univariate distributions with each $G_i$ supported on $[l_i, u_i]$. Then one can define a partition rule as follows. For each rectangular set $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, $R(A) = \{\{A_1^1, A_2^1\}, \{A_1^2, A_2^2\}, \ldots, \{A_1^d, A_2^d\}\}$ with

$$A_1^j = \{(y_1, y_2, \ldots, y_d) \in A : G_j(y_j) \leq (G_j(a_j) + G_j(b_j))/2\}$$
$$A_2^j = \{(y_1, y_2, \ldots, y_d) \in A : G_j(y_j) \geq (G_j(a_j) + G_j(b_j))/2\},$$

and $R(A) = \emptyset$ for non-rectangular $A$’s. This rule is called a dimensionwise dyadic split (DDS) rule with respect to $(G_1, G_2, \ldots, G_d)$. In particular, if $\Omega$ is a bounded rectangle—that is when all $l_i$’s and $u_i$’s are finite—and $G_i$ is the uniform distribution on $[a_i, b_i]$, then we call the partition rule the uniform dimensionwise dyadic split (UDDS) rule on $\Omega$, and denote it by $R_U$.

Given a partition rule $R$, we can now define the notion of elementary regions in an inductive manner. First, we say that $\Omega$ is the level-0 (or scale-0) elementary region. From now on, we shall use $A_k$ for $k = 0, 1, 2, \ldots$ to denote the collection of level-$k$ elementary regions. We have just defined $A^0 = \{\Omega\}$. Now, suppose we have defined $A^0, A^1, \ldots, A^k$, then we define $A^{k+1}$, the level-$(k+1)$ elementary regions, as follows. A set $A$ is called a level-$(k+1)$ elementary region if there exists a set $A^p$ in $A^k$ such that $A$ is a child of $A^p$ in at least one of the partitions allowed under $R$. In words, the level-$k$ elementary regions are the subsets of $\Omega$ that can arise as children after $k$ steps
of nested partitions on \( \Omega \) allowed under \( \mathcal{R} \). Thus each level-\( k \) elementary region correspond to a particular location (or subset) of scale \( k \) in the sample space. Also, let
\[
\mathcal{A}^{(l)} = \bigcup_{k=0}^{l} \mathcal{A}^k \quad \text{and} \quad \mathcal{A}^{(\infty)} = \bigcup_{k=0}^{\infty} \mathcal{A}^k.
\]
That is, \( \mathcal{A}^{(l)} \) is the collection of all elementary regions up to level \( l \) and \( \mathcal{A}^{(\infty)} \) is the collection of all elementary regions. For the DDS rule introduced in Example 2, all elementary regions are rectangular sets.

2.2. Adaptive Pólya trees. We are now ready to introduce our nonparametric prior constructively through a generative procedure. The procedure generates a probability measure \( Q \) through randomly and recursively dividing the sample space into smaller and smaller elementary regions according to \( \mathcal{R} \) and assigning probability among the children regions at each step of partition. The procedure falls into one of several hidden states on each elementary region, and the hidden state governs the mechanism for further partitioning and probability assignment. Later we will associate these different states with differential amounts of shrinkage, but we defer such connection to keep the current presentation general.

The procedure can be described inductively. Suppose after \( k \) steps of the procedure, we end up with two collections of disjoint elementary regions that together form a partition of \( \Omega \), and a probability measure \( Q^{(k)} \) on \( \Omega \). The two collection of elementary regions are
\[
\mathcal{T}_0^k = \{A_1, A_2, \ldots, A_I\} \quad \text{and} \quad \mathcal{T}_1^k = \{A'_1, A'_2, \ldots, A'_{I'}\}
\]
where each \( A_i \in \mathcal{A}^{(k-1)} \) and each \( A'_i \in \mathcal{A}^k \). As will be explained below, \( \mathcal{T}_0^k \) represents the elementary regions on which the process has “stopped” during the first \( k \) steps of the procedure, and \( \mathcal{T}_1^k \) represents the elementary regions that “arise as children” during the \( k \)th step. Also, \( Q^{(k)} \) is a probability measure such that
\[
Q^{(k)}(\cdot \mid A) = Q_0(\cdot \mid A) \quad \text{for all} \quad A \in \mathcal{T}_0^k \cup \mathcal{T}_1^k
\]
where \( Q_0 \) is a base measure on \( \Omega \), and \( Q_0(\cdot \mid A) \) is the corresponding conditional base distribution on \( A \). Throughout this work, we shall always assume that \( Q_0 \ll \mu \). To initiate the induction, we have \( k = 0, \mathcal{T}_0^0 = \emptyset, \mathcal{T}_1^0 = \{\Omega\} \), and \( Q^{(0)} = Q_0 \).

At the beginning of the \((k+1)\)th step of the procedure, we let \( \mathcal{T}_0^{k+1} = \mathcal{T}_0^k \) and \( \mathcal{T}_1^{k+1} = \emptyset \). During the \((k+1)\)th step, we apply a sequence of three operations on each \( A \in \mathcal{T}_1^k \)—state transition, random partition, and probability assignment. We update \( \mathcal{T}_0^{k+1} \) and \( \mathcal{T}_1^{k+1} \) during the state transition and random partition operations, and define a new measure \( Q^{(k+1)} \) through the probability assignment operation. We describe each of the three operations in turn.

I. State transition: On each \( A \in \mathcal{T}_1^k \), we allow the procedure to be in one of \( T + 1 \) different states. We let \( \mathcal{S} = \{1, 2, \ldots, T, \infty\} \) denote the set of states. We use \( \infty \), rather than \( T + 1 \) for the last state to emphasize its difference from the first \( T \) states. In particular, we use “\( \infty \)” to represent the “stopping” state, which in the most generic sense, is a state that once the procedure falls into, it cannot move into another state. Let \( A^p \) denote the parent of \( A \), that is, the elementary region in \( \mathcal{T}_1^{k-1} \) whose partitioning during the \( k \)th step gave rise to \( A \) as a child.

We generate the state of the procedure on \( A \) by drawing a random variable \( S(A) \), which we call the state variable on \( A \), from a multinomial trial distribution that depends on the state of the procedure on \( A \)’s parent:
\[
S(A) \mid S(A^p) = s \sim \text{Multinomial} (\{1, 2, \ldots, T, \infty\}; (\rho_{s,1}(A), \rho_{s,2}(A), \ldots, \rho_{s,T}(A), \rho_{s,\infty}(A)))
\]
where \( \rho_{s,t}(A) \in [0,1] \), for \( s, t \in S \) and \( \sum_{t \in S} \rho_{s,t}(A) = 1 \) for all \( s \in S \setminus \{\infty\} \). (We will see in
the random partition operation that by construction \( S(A^p) \) can only be a non-stopping state for otherwise \( A^p \) will not have children.)

If \( S(A) = \infty \), we say that the procedure “stops on” \( A \) and we add \( A \) into \( T_0^{k+1} \). If instead
\( S(A) = t \in S \setminus \{\infty\} \), then we say that the procedure does not stop on \( A \) and is in state \( t \) on \( A \). (In
the following we will use language such as “\( A \) is in state \( t \)” or “the state of \( A \)” to mean things like
“the procedure is in state \( t \) on \( A \)” or “the state of the procedure on \( A \)”.) We emphasize that the
state is a feature of the procedure, not of a set, but the former language is adopted for simplicity without causing confusion.)

We call the \( \rho_{s,t}(A) \) terms \((state)\ transition probabilities\), because \( \rho_{s,t}(A) \) represents the probability
for the procedure to transition into state \( t \) on \( A \) from its parent’s state \( s \). Accordingly, we can
organize them into a \((T + 1) \times (T + 1)\) \((state)\ transition matrix:

\[
\rho(A) = \begin{pmatrix}
\rho_{1,1}(A) & \rho_{1,2}(A) & \cdots & \rho_{1,T}(A) & \rho_{1,\infty}(A) \\
\rho_{2,1}(A) & \rho_{2,2}(A) & \cdots & \rho_{2,T}(A) & \rho_{2,\infty}(A) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_{T,1}(A) & \rho_{T,2}(A) & \cdots & \rho_{T,T}(A) & \rho_{T,\infty}(A) \\
\rho_{\infty,1}(A) & \rho_{\infty,2}(A) & \cdots & \rho_{\infty,T}(A) & \rho_{\infty,\infty}(A)
\end{pmatrix}.
\]

Because by definition the procedure cannot transition from the stopping state into the \( T \) non-stopping states, we have \( \rho_{\infty,t}(A) = 0 \) for \( t = 1, 2, \ldots, T \) while \( \rho_{\infty,\infty}(A) = 1 \).

The notions of transition probability and transition matrix do not apply when \( A = \Omega \) however,
because the sample space does not have a parent. For \( A = \Omega \), instead of specifying a set of transition
probabilities, we specify a set of \( initial \ state \ probabilities \)—the probabilities that the procedure starts
off on \( \Omega \) in each of the \( T + 1 \) states. We denote these probabilities by

\[
(\rho_{0,1}, \rho_{0,2}, \ldots, \rho_{0,T}, \rho_{0,\infty})
\]

where \( \rho_{0,t} \in [0,1] \) for \( t \in S \), is the probability for the procedure to start in state \( t \) on \( \Omega \). In order
to let the notions of transition probability and transition matrix apply to \( A = \Omega \) as well, we can simply \( define \) the transition probabilities for \( \Omega \) as follows

\[
\rho_{1,t}(\Omega) = \rho_{2,t}(\Omega) = \cdots = \rho_{T,t}(\Omega) = \rho_{0,t}, \quad \text{for all } t \in S.
\]

This completes the description of the state transition operation on \( A \), and the procedure moves
onto the random partition on \( A \).

II. Random partition: If the procedure stops on \( A \), that is \( S(A) = \infty \), then we do not partition \( A \)
and the random partition on \( A \) is completed. If \( S(A) = t \in S \setminus \{\infty\} \), then we continue on to divide
\( A \) into smaller sets as follows.

Suppose under \( R \) there are \( N(A) \) ways to divide \( A \). We randomly select one by drawing a
multinomial trial variable, \( J(A) \), which we call the \((partition) selection variable\), that can take
values from \( 1, 2, \ldots, N(A) \). The probability that \( J(A) \) takes each value is

\[
P(J(A) = j) = \lambda_j(A, t)
\]

with \( \sum_{j=1}^{N(A)} \lambda_j(A, t) = 1 \) for each \( t \in S \setminus \{\infty\} \). The \( \lambda_j(A, t) \)'s are called the \((partition) selection probabilities\) of \( A \) in state \( t \). If \( J(A) = j \), then we partition \( A \) in the \( j \)th way under \( R \), and add the
\( K_j(A) \) children, \( A_1^j, A_2^j, \ldots, A_{K_j(A)}^j \), into \( T_1^{k+1} \). This completes the random partition on \( A \) and we
move onto probability assignment.
III. Probability assignment: If the procedure stops on $A$, or $S(A) = \infty$, then $A$ has not been partitioned, and we assign probability to $A$ in the same way as we did in the $k$th step. That is,

$$Q^{(k+1)}(A) = Q^{(k)}(A) \quad \text{and} \quad Q^{(k+1)}(\cdot | A) = Q_0(\cdot | A).$$

If $S(A) = t \in S \setminus \{\infty\}$, and $J(A) = j$, then $A$ has been divided in the $j$th way. To assign probabilities to the $K^j(A)$ children, we draw a Dirichlet random vector

$$Y^j(A) = \left( Y^j_1(A), Y^j_2(A), \ldots, Y^j_{K^j(A)}(A) \right) \sim \text{Dirichlet} \left( \alpha^j(A, t) \right)$$

where

$$\alpha^j(A, t) = \left( \alpha^j_1(A, t), \alpha^j_2(A, t), \ldots, \alpha^j_{K^j(A)}(A, t) \right)$$

are the pseudo-count parameters for $A$ in state $t$. We then assign $Y^j_i(A)$ proportion of the probability we gave to $A$ during the $k$th step to the $i$th child, $A^j_i$. That is, we let

$$Q^{(k+1)}(A^j_i) = Q^{(k)}(A) \cdot Y^j_i(A) \quad \text{and} \quad Q^{(k+1)}(\cdot | A_i) = Q_0(\cdot | A^j_i)$$

for $i = 1, 2, \ldots, K^j(A)$. This completes the probability assignment on $A$.

After applying these three operations to each set in $T^k$, we have our new collections of stopped sets $T^k_0$ and “new-born” children $T^k_1$, and a fully specified probability measure $Q^{(k+1)}$. This completes our inductive description of the $(k+1)$th step of the constructive procedure. Note that the specification of this procedure requires three sets of parameters: $\rho(A)$, $\lambda_j(A, t)$ and $\alpha^j(A, t)$ for all $A \in A^{(\infty)}$, $t \in S \setminus \{\infty\}$, and $j = 1, 2, \ldots, N(A)$. From now on we shall use $\rho$, $\alpha$ and $\lambda$ to respectively represent the totality of these three sets of parameters.

The next theorem shows that if the probability to stop on each elementary region is not too small, then the procedure will eventually produce a well-defined probability measure on $\Omega$ that is absolutely continuous with respect to the base distribution.

**Theorem 1.** If there exists $\delta > 0$ and some $k_0 \in \{0, 1, 2, \ldots\}$ such that for all $s \in S$ and $A \in \bigcup_{k \geq k_0} A^k$, we have $\rho_{s,\infty}(A) > \delta$, then with probability 1, there is a probability measure $Q$ such that $Q \ll Q_0$ and $Q^{(k)}$ converges to $Q$ in total variation as $k \to \infty$.

**Proof of Theorem 1.** The sequence of measures $Q^{(k)}$ can be equivalently generated in two stages. In the first stage, consider a “non-stopping” version of the above constructive procedure whose collection of states is $S \setminus \{\infty\}$, with the state transition matrix given by

$$\rho^*_{s,t}(A) = \rho_{s,t}(A) / (1 - \rho_{s,\infty}(A))$$

for each $A \in A^{(\infty)}$ and $s, t \in S \setminus \{\infty\}$.

Let $S^{*(k)}$ and $J^{*(k)}$ be the collection of the state and partition selection variables in the first $k$ steps of the non-stopping version of the constructive procedure. For the non-stopping version, $T^*_0 = \emptyset$ and $T^*_1 = A^k(J^{*(k)})$, where $A^k(J^{*(k)})$ represents the collection of elementary regions in $A^k$ that arise as children in the $(k-1)$th step, which depends just on the partition selection variables. Now let $Q^{*(k)}$ be the sequence of measures corresponding to the non-stopping version of the procedure.

In the second stage, starting from $k = 0$, for each $A \in A^k(J^{*(k)})$, we draw a stopping indicator variable $S^{**}(A)$ which given the state of $A$’s parent in $A^{k-1}(J^{*(k-1)})$, denoted by $s$, has probability...
of step functions \( f \) as \( \gamma \rightarrow 0 \) there is a partition \( \Omega = \bigcup_{k=0}^{\infty} A^{(k)} \), we modify \( Q^{(k)} \) to get \( Q \) by setting \( Q^{(k)}(\cdot \mid A) = Q_0(\cdot \mid A) \). The rest of the proof follows from the argument in Theorem 1 in [20] with \( \mu \) replaced by \( Q_0 \).

Now we can formally define the constructive procedure as a distribution on the space of probability measures on \( \Omega \).

**Definition 1 (Adaptive Pólya tree).** The random probability measure \( Q \) is said to have an adaptive Pólya tree (APT) distribution with parameters \( \rho, \alpha, \lambda \), partition rule \( \mathcal{R} \), and a base distribution \( Q_0 \). We write \( Q \sim \text{APT}(\mathcal{R}, \rho, \alpha, \lambda, Q_0) \).

The next definition and the following theorem show that under general choices of the parameters, the APT has large \( L_1 \) support and so is a nonparametric prior.

**Definition 2 (Fine partition criterion).** An APT(\( \mathcal{R}, \rho, \alpha, \lambda, Q_0 \)) process is said to satisfy the **fine partition criterion** if for any \( \gamma > 0 \), there exists a compact set \( E \subseteq \Omega \) with \( Q_0(E^c) < \gamma \) such that for any \( \epsilon > 0 \), there exists some \( k \) and a partition allowed under \( \mathcal{R}, \Omega = \bigcup_{i} A_i \), with \( A_i \in \mathcal{A}^{(k)} \) such that the diameter of \( A_i \cap E \) is less than \( \epsilon \) for all \( A_i \).

**Theorem 2 (Large \( L_1 \) support).** Suppose \( Q \) has an \( \text{APT}(\mathcal{R}, \rho, \alpha, \lambda, Q_0) \) distribution that satisfies (i) \( \exists \delta > 0 \) such that for all \( s \in \mathcal{S} \) and \( A \in \mathcal{A}^{(\infty)} \), \( \rho_{s, \infty}(A) > \delta \); (ii) the fine partition criterion, and (iii) the selection probabilities \( \lambda_i(A, s) \) and \( \alpha^*_i(A, s) = \sum_{i'} \alpha^{*_i}_{i'}(A, s) \) are uniformly bounded away from 0 and 1 for all \( A \in \mathcal{A}^{(\infty)} \) and \( s \in \mathcal{S} \setminus \{\infty\} \). Then for any distribution \( F \) absolutely continuous w.r.t \( Q_0 \) and any \( \tau > 0 \), we have

\[
P \left( \int |q - f|d\mu < \tau \right) > 0
\]

where \( q = dQ/d\mu \) and \( f = dF/d\mu \) are the corresponding densities with respect to \( \mu \).

**Proof.** Let \( \tilde{q} = q/q_0 \) and \( \tilde{f} = f/f_0 \) where \( q_0 = dQ_0/d\mu \). Our goal is to prove that for any \( \tau > 0 \),

\[
P \left( \int |\tilde{q} - \tilde{f}|dQ_0 < \tau \right) > 0.
\]

First we assume that \( \tilde{f} \) is continuous and bounded, and let \( M \) be a finite upperbound of \( \tilde{f} \). By the fine partition criterion, for any \( \gamma > 0 \), there exists a compact set \( E \) such that for any \( \epsilon > 0 \) there is a partition \( \Omega = \bigcup_{i} A_i \) such that the diameter of each \( A_i \cap E \) is less than \( \epsilon \). By the absolute continuity of \( F \) w.r.t \( Q_0 \), there exists \( \beta(\gamma) > 0 \) such that \( F(E^c) < \beta(\gamma) \) if \( Q_0(E^c) \gamma \downarrow 0 \) as \( \gamma \downarrow 0 \). We define the modulus of continuity of \( \tilde{f} \) on \( E \) as

\[
\delta_E(\epsilon) = \sup_{x, y \in E: |x - y| < \epsilon} |\tilde{f}(x) - \tilde{f}(y)|.
\]

Note that by the continuity of \( \tilde{f} \) and the compactness of \( E \), \( \delta_E(\epsilon) \downarrow 0 \) as \( \epsilon \downarrow 0 \). Now we approximate \( \tilde{f} \) by a step function \( \tilde{f}^*(x) = \sum_{i} \tilde{f}_i^* I_{A_i} \) where \( \tilde{f}_i^* = \int_{A_i \cap E} \tilde{f}dQ_0/Q_0(A_i \cap E) \). Let \( D_\epsilon(\tilde{f}) \) be the set of step functions \( g(\cdot) = \sum_{i} g_i I_{A_i}(\cdot) \) such that \( \sup_{x} |g(x) - \tilde{f}(x)| < \delta_E(\epsilon) + M\gamma \).
Suppose \( g \in D_\epsilon(\tilde{f}) \). For any \( B \in \mathcal{B} \), the Borel sets, we have \( B_i = B \cap A_i \). Then

\[
\left| \int_B (g - \tilde{f})dQ_0 \right| \leq \sum_i |g_i - \tilde{f}_i|Q_0(B_i) + \sum_i |\tilde{f}_i^*|Q_0(B_i) - \int_{B_i} \tilde{f}dQ_0
\]

\[
\leq (\delta_E(\epsilon) + M\gamma)Q_0(B) + \sum_i |\tilde{f}_i^*|Q_0(B_i \cap E) - \int_{B_i \cap E} \tilde{f}dQ_0 \leq \sum_i |\tilde{f}_i^*|Q_0(B_i \cap E^c) - \int_{B_i \cap E^c} \tilde{f}dQ_0
\]

\[
< (\delta_E(\epsilon) + M\gamma)Q_0(B) + \sum_i r_i + 2M \cdot Q_0(E^c)
\]

where

\[
r_i = Q_0(B_i \cap E) \left| \int_{A_i \cap E} \tilde{f}dQ_0 \right| \frac{\int_{B_i \cap E} \tilde{f}dQ_0}{Q_0(A_i \cap E)} - Q_0(B_i \cap E) \left| \int_{A_i \cap E} \left( \tilde{f}(x) - \tilde{f}(x_i) \right) q_0(x) dx \right| \frac{\int_{B_i \cap E} \left( \tilde{f}(x) - \tilde{f}(x_i) \right) q_0(x) dx}{Q_0(B_i \cap E)}
\]

for some \( x_i \in B_i \). Thus

\[
|r_i| < 2\delta_E(\epsilon)Q_0(B_i)
\]

and so

\[
\left| \int_B (g - \tilde{f})dQ_0 \right| < 3\delta_E(\epsilon)Q_0(B) + 3M\gamma \quad \text{for all } B \in \mathcal{B}.
\]

Therefore by taking \( B = \{ x : g > \tilde{f} \} \) and \( B = \{ x : g \leq \tilde{f} \} \), we get

\[
\int |g - \tilde{f}|dQ_0 < 3\delta_E(\epsilon) + 6M\gamma.
\]

By condition (iii) in the theorem, we have for \( q^{(k)} = q^{(k)}/q_0 \) where \( q^{(k)} = dQ^{(k)}/d\mu \),

\[
P \left( \tilde{q}^{(k)} \in D_\epsilon(\tilde{f}) \text{ for all large } k \right) > 0.
\]

Thus

\[
P \left( \int |\tilde{q}^{(k)} - \tilde{f}|dQ_0 < 3\delta_E(\epsilon) + 6M\gamma \text{ for all large } k \right) > 0.
\]

By Theorem 1, we have

\[
P \left( \int |\tilde{q}^{(k)} - \tilde{q}|dQ_0 \rightarrow 0 \right) = 1.
\]

Therefore, combining these we get

\[
P \left( \int |\tilde{q} - \tilde{f}|dQ_0 < 4\delta_E(\epsilon) + 6M\gamma \right) > 0.
\]

The result follows by letting \( \epsilon \downarrow 0 \) and \( \gamma \downarrow 0 \).

Finally, if \( \tilde{f} \) is not continuous and bounded, then since \( Q_0 \) is a probability measure, \( \tilde{f} \) can be approximately arbitrarily well in \( L_1 \) w.r.t \( Q_0 \) by a continuous bounded density. \( \square \)
The next result shows that an APT process can be centered at its base distribution, and therefore can be used as a nonparametric extension to parametric distributions.

**Theorem 3 (Centering of an APT).** Suppose an APT($R, \rho, \lambda, \alpha, Q_0$) satisfies

(i) There exists $\delta > 0$ such that for all $s \in S$ and $A \in A^{(\infty)}$, $\rho_{s,\infty}(A) > \delta$;

(ii) For all $A \in A^{(\infty)}$ such that $Q_0(A) > 0$ and all $s \in S\setminus\{\infty\}$, we have

$$\frac{\alpha_i^j(A, s)}{\sum_{i=1}^{K_j(A)} \alpha_i^j(A, s)} = \frac{Q_0(A^j_i)}{Q_0(A)},$$

then the process has mean $Q_0$ in the following sense. If a random measure $Q$ has this APT distribution, we have

$$EQ(B) = Q_0(B) \text{ for any Borel subset } B \text{ of } \Omega.$$

Accordingly, the predictive density with respect to $\mu$ of the APT process is

$$E\mu(x) = q_0(x)$$

where $q = dQ/d\mu$ and $q_0 = dQ_0/d\mu$.

Remark I: We shall from now on refer to Condition (ii) as the centering condition. Under the conditions of this theorem, we say that the APT process is “centered” at its base distribution $Q_0$.

Remark II: To satisfy the centering condition, when $Q_0(A^j_i)/Q_0(A) = 0$, we let $\alpha_i^j(A, s) = 0$ for all $s \in S\setminus\{\infty\}$. When $Q_0(A^j_i)/Q_0(A) = 1$, we let $\alpha_i^j(A, s) = \infty$ for all $s \in S\setminus\{\infty\}$.

**Proof of Theorem 3.** The proof for the first claim follows from the same argument as the proof for Theorem 1 in [15]. The second claim then follows from taking the limit of the first claim.

The APT process we have introduced so far is over-parametrized when $T > 1$ in that a relabeling of the nonstopping states and a corresponding permutation of the $\alpha$ and $\lambda$ parameters with respect to the state labels will result in essentially the same process. Without loss of generality, one may impose the further (identifiability) condition that

$$\alpha_i^j(A, s) \leq \alpha_i^j(A, t) \text{ if } s < t$$

for all $A \in A^{(\infty)}$, $j \in \{1, 2, \ldots, N(A)\}$, $i = 1, 2, \ldots, K_j(A)$, and $s, t \in \{1, 2, \ldots, T\}$, and for every distinct pair of $s$ and $t$, there exists $A \in A^{(\infty)}$, $i$, and $j$ such that strict inequality holds.

The above condition renders the different states an intrinsic ordering and an interesting new meaning. If we think of the variability around the base distribution as “speed”, then the $T + 1$ states correspond to $T + 1$ different speeds with a higher state corresponding to lower speed, or less prior variance. The stopping state ($\infty$) intuitively corresponds to zero speed. The subclass of APTs satisfying this constraint will be of most interest and therefore deserves a separate definition.

**Definition 3 (Canonical APT).** An APT($R, \rho, \alpha, \lambda, Q_0$) with condition (*) on $\alpha$ is called a canonical adaptive Pólya tree.

From now on, unless explicitly stated otherwise, all APTs we consider are canonical. So when there is no confusion, we shall refer to a canonical APT simply as an APT.
2.3. Prior specifications and multi-resolution adaptive shrinkage. The APT involves a lot of parameters and default guidelines for choosing them a priori is needed. Ideally, the default choices should be robust and yet parsimonious—involving only a small number of tuning parameters. This is the objective of this subsection. Later we will show how one can use empirical Bayes strategies to specify the tuning parameters. We begin with the default choice for the selection probabilities $\lambda$, as it is the most simple. Then we will move onto the choice of prior $\alpha$ and $\rho$. The choice of $\alpha$ is the key to achieving adaptive shrinkage, while that of $\rho$ allows us to incorporate dependence in shrinkage across the scale-location spectrum.

Prior choices of $\lambda$. Although we allow $\lambda_j(A,t)$ to depend on the state $t$, there is typically no a priori reasons to believe that the preference for different divisions should depend on the shrinkage state, so we shall typically let $\lambda_j(A,t) = \lambda_j(A)$ a priori. In most situations the statistician has little or no prior knowledge about which dimensions should more likely be partitions than others. Thus a default choice is to evenly spread the partition probability over all possible divisions. That is, $\lambda_j(A,t) = 1/N(A)$ for all $A^{(\infty)}$, $j = 1, 2, \ldots, N(A)$, and $t \in \mathcal{S} \setminus \{\infty\}$.

Prior choices of $\alpha$. A useful way for choosing $\alpha$ is through a three-part specification

$$\alpha_i^j(A,t) = c \cdot l(A,t) \cdot \frac{Q_0(A_t^j)}{Q_0(A)}$$

for all $A \in A^{(\infty)}$ and $t \in \mathcal{S} \setminus \{\infty\}$. The three parts involved are:

- A positive constant $c$ shared by all $A$'s, called the global shrinkage parameter, which controls the overall prior variance around the base distribution.
- The mapping $l(\cdot, \cdot) : A^{(\infty)} \times \mathcal{S} \setminus \{\infty\} \to [0,\infty]$, called the local shrinkage function, which specifies the elementary region specific pseudo-count and thus the local prior variability around the base measure under each of the $T$ non-stopping states.
- The ratio $Q_0(A_t^j)/Q_0(A)$, which ensures the centering condition. When $Q_0(A) = 0$, such as when $A$ falls outside of the support of $Q_0$, this ratio can be arbitrary, so we can just define $Q_0(A_t^j)/Q_0(A) = 1$.

A useful choice of $l(\cdot, \cdot)$ is to make it depend on the level or scale of $A$. That is, for all $A \in A^k$ and $t = 1, 2, \ldots, T$, we let

$$l(A,t) = h(k,t)$$

where $h : \mathbb{N} \times \mathcal{S} \setminus \{\infty\} \to [0,\infty]$, which we call the “local shrinkage rate function” (LSRF). The LSRF characterizes the scale specific prior variability around the base distribution.

To ensure that the APT is canonical, we further require that $h(k,t)$ be a non-decreasing function in $t$ for each $k$. Under this specification, the higher the state, the less the prior variability (the “speed”) and therefore the more shrinkage toward the base. For example, we can let

$$h(k,t) = 10^{t-1}, \quad h(k,t) = k^{t-1} \quad \text{or} \quad h(k,t) = e^{k(t-1)} \quad \text{for } t = 1, 2, \ldots, T.$$

Prior choices of $\rho$. The Markov dependence is incorporated through the transition matrix $\rho$. To allow clustering of the shrinkage levels, we want the shrinkage state to be “sticky”—the procedure is more likely to remain in a state on $A$ that is similar to its state on $A$’s parent. A parsimonious way (involving only one hyperparameter) to achieving this stickiness in shrinkage is to let

$$\rho_{s,t}(A) = e^{-\gamma |t-s|} / \sum_{i \in \mathcal{S}} e^{-\gamma |i-s|}$$

...
for \( s \in S \setminus \{\infty\} \) and \( t \in S \) with the convention that \( t - s = T + 1 - s \) for \( t = \infty \). The hyperparameter \( \gamma \) controls the “stickiness” of the shrinkage state. The larger \( \gamma \) is, the more likely that the process will remain in similar shrinkage states as it transitions from a set to its children. For this reason, we shall refer to \( \gamma \) as the stickiness coefficient.

In many situations, it is appropriate to impose stronger shrinkage on deeper levels of the partitions (or higher resolutions) to ensure smoothness. This can be easily incorporated by imposing a further constraint that the process can only transition into higher shrinkage states, that is, lower prior variability (i.e., the speed). We introduce a corresponding notion of deceleration.

**Definition 4.** A (canonical) APT process is said to be decelerating if for all \( A \in \mathcal{A}(\infty) \), the transition matrix \( \rho(A) \) is upper triangular. We call this condition the deceleration condition.

A decelerating version of the above specification for \( \rho \) is such that for \( s \in S \setminus \{\infty\} \) and \( t \in S \),

\[
\rho_{s,t}(A) = \begin{cases} 
  e^{-\gamma (t-s)} \sum_{i \geq s} e^{-\gamma (i-s)} & \text{for } t \geq s \\
  0 & \text{otherwise}
\end{cases}
\]

again with the convention that \( t - s = T + 1 - s \) for \( t = \infty \).

**Choosing the hyperparameters.** Our default choices of \( \lambda, \alpha, \) and \( \rho \) recommended above are highly parsimonious in that they reduce the number of free parameters down to three. The three hyperparameters are the global shrinkage parameter \( c \), the number of non-stopping shrinkage states \( T \), and the stickiness coefficient \( \gamma \). They can be specified through empirical Bayes by maximizing the marginal likelihood. A prerequisite of this strategy is an efficient way for computing the marginal likelihood under the APT, which is given later in Theorem 5.

**2.4. Bayesian inference under the APT prior.** To carry out inference using the APT prior, one must address two basic questions. What is the corresponding posterior distribution of an APT prior? Can we effectively summarize and sample from it? In this subsection we show that the answers are both positive. Our main result (Theorem 4) establishes the conjugacy of the APT distribution—that is, the posterior of an APT is still an APT—and provides explicit analytic expressions of the parameters in a posterior APT. To pave the way for this result, we investigate the likelihood function and the marginal likelihood under the process, which, as in other Bayesian inference problems, are essential components for finding the posterior through Bayes’ theorem.

Suppose \( Q \sim \text{APT}(\mathcal{R}, \rho, \lambda, \alpha, Q_0) \). We let \( q_0 = dQ_0/d\mu \) and \( q = dQ/d\mu \) be the corresponding densities with respect to the natural measure \( \mu \) for \( Q_0 \) and \( Q \). Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) be \( n \) i.i.d. data with distribution \( Q \). In the following, for any \( A \subset \Omega \), we let \( \mathbf{x}(A) \) be the subset of \( \mathbf{x} \) that fall in \( A \), and \( n(A) \) the number of data points in \( A \). The likelihood, as a function of \( q \), is

\[
q(\mathbf{x}) = \prod_{i=1}^{n} q(x_i).
\]

Now if \( S(\Omega) = \infty \), then \( q(\cdot) = q_0(\cdot) \), and so \( q(\mathbf{x}) = \prod_{i=1}^{n} q_0(x_i) \). If \( S(\Omega) = t \in S \setminus \{\infty\} \), then

\[
q(x_i) = \prod_{i=1}^{K^{J(\Omega)}(\Omega)} \left( Y_i^{J(\Omega)}(\Omega) \cdot q(x_i \mid \Omega_i^{J(\Omega)}) \right)^{1(x_i \in \Omega_i^{J(\Omega)})}
\]
and thus
\[ q(x) = \prod_{i=1}^{n} q(x_i) = \prod_{i=1}^{K^{J(A)}(\Omega)} \left( Y_i^{J(A)}(\Omega) \right)^{n_i^{J(A)}} \cdot q \left( x \left| \Omega_i^{J(A)} \right| \Omega_i^{J(A)} \right). \]

More generally, for any elementary region \( A \) that arises during the random partitioning, the conditional likelihood
\[ (2.3) \]
\[ q(x(A) | A) := \prod_{x_i \in A} q(x_i | A) = \begin{cases} \prod_{i=1}^{K^{J(A)}(A)} \left( Y_i^{J(A)}(A) \right)^{n_i^{J(A)}} \cdot q \left( x \left| A \right| A \right) & \text{if } S(A) \in S \backslash \{\infty\} \\ \prod_{x_j \in A} q_0(x_j | A) & \text{if } S(A) = \infty. \end{cases} \]

Given \( x \), we define a mapping \( \Phi(\cdot, \cdot, x) : \mathcal{A}(\infty) \times S \to \mathbb{R} \) as follows. For \( A \in \mathcal{A}(\infty) \) and \( s \in S \backslash \{\infty\} \),
\[ (2.4) \]
\[ \Phi(A, s, x) = \int q(x(A) | A) \pi \left( dq \left| E_1(A), E_2, s(A) \right. \right), \]
where the two events \( E_1(A) \) and \( E_{2, s}(A) \) are
- \( E_1(A) = \{ A \text{ arises in the random partitioning} \} \),
- \( E_{2, s}(A) = \{ \text{The process is in state } s \text{ on } A\text{'s parent} \} \).

Intuitively, \( \Phi(A, s, x) \) represents the marginal likelihood under the APT process restricted on \( A \) given \( E_1(A) \) and \( E_{2, s}(A) \). Due to the self-similarity of the APT process, given \( E_1(A) \) and \( E_{2, s}(A) \), the APT restricted on \( A \) is still an APT. On the other hand, for \( A \in \mathcal{A}(\infty) \) and \( s = \infty \), we define
\[ \Phi(A, \infty, x) = \prod_{x_i \in A} q_0(x_i | A). \]

As we will see later, the corresponding posterior of an APT can be written in terms of the prior parameters and the \( \Phi(A, s, x) \) terms. Intuitively this is not surprising as Bayes’ theorem combines the prior with the likelihood. The following three lemmas address how we can compute the \( \Phi(A, s, x) \) terms given the data. In particular, they provide a recursive recipe for computing \( \Phi(A, s, x) \) for all \( A \in \mathcal{A}(\infty) \) and \( s \in S \).

**Lemma 1.** For each \( A \in \mathcal{A}(\infty) \) and \( s \in S \), \( \Phi(A, s, x) \) can be written as follows
\[ \Phi(A, s, x) = \sum_{t \in S} \rho_{s,t}(A) \cdot Z(A, t, x), \quad \text{for } s \in S, \]
with
\[ Z(A, t, x) = \begin{cases} \sum_{j=1}^{N(A)} Z_j(A, t, x) & \text{for } t = 1, 2, \ldots, T \\ \prod_{x_i \in A} q_0(x_i | A) & \text{for } t = \infty, \end{cases} \]
and
\[ Z_j(A, t, x) = \lambda_j(A, t) \cdot \frac{D \left( n_j(A) + \alpha_j(A, t) \right)}{D \left( \alpha_j(A, t) \right)} \cdot \prod_{i=1}^{K(A)} \Phi(A_i^j, t, x) \]
for \( t \in S \backslash \{\infty\} \) and \( j = 1, 2, \ldots, N(A) \), where
\[ n_j(A) = \left( n(A_1^j), n(A_2^j), \ldots, n(A_{K(A)}^j) \right) \]
and
\[ D(w_1, w_2, \ldots, w_k) = \frac{\Gamma(w_1)\Gamma(w_2) \cdots \Gamma(w_k)}{\Gamma(w_1 + w_2 + \cdots + w_k)}. \]
Remark: This lemma shows that the mapping $\Phi(\cdot, \cdot, x) : A^{(\infty)} \times S \to \mathbb{R}$ is recursive in its first argument, in the sense that one can compute $\Phi(A, \cdot, x)$ based on $\Phi(A_{i}^{j}, \cdot, x)$ for all $j = 1, 2, \ldots, N(A)$ and $i = 1, 2, \ldots, K^{j}(A)$.

**Proof of Lemma 1.** Let us carry out the integration in (2.4). Let $A$ be an elementary region that arises in the random partitioning as a child of $A^{p}$. Then given $S(A^{p}) = s$, the chance of $S(A) = t$ for $t \in S$ is $\rho_{s,t}(A)$ by definition. We define $Z(A, t, x)$ as the marginal likelihood under the APT process restricted on $A$, given $E_{1}(A)$ and that $S(A) = t$ for $t \in S$. Then

$$\Phi(A, s, x) = \sum_{i \in S} \rho_{s,t}(A) \cdot Z(A, t, x).$$

Now, if $S(A) = \infty$, then $q(\cdot|A) = q_{0}(\cdot|A)$, and so $Z(A, \infty, x) = \prod_{x_{i} \in A} q_{0}(x_{i} | A)$. For $S(A) = t \in S \setminus \{\infty\}$, by integrating over (2.3), we have

$$Z(A, t, x) = \sum_{j=1}^{N(A)} \lambda_{j}(A, t) \cdot E \left( \prod_{i=1}^{K^{j}(A)} \left( \theta_{i}^{j}(A) \right)^{n(A_{i}^{j})} \right) \cdot \prod_{i=1}^{K^{j}(A)} \left( \int q(x(A_{i}^{j})|A) \cdot \pi \left( dq | E_{1}(A_{i}^{j}), E_{2,t}(A_{i}^{j}) \right) \right)$$

$$= \sum_{j=1}^{N(A)} \lambda_{j}(A, t) \cdot \frac{D(n_{j}(A) + \alpha_{j}(A, t))}{D(\alpha_{j}(A, t))} \cdot \prod_{i=1}^{K^{j}(A)} \Phi(A_{i}^{j}, t, x)$$

$$= \sum_{j=1}^{N(A)} Z_{j}(A, t, x).$$

**Lemma 2.** For three types of elementary regions, the $\Phi(A, s, x)$ terms are known analytically:

- **Type 1.** For an elementary region $A$ that cannot be partitioned under $\mathcal{R}$,

  $$\Phi(A, s, x) = \prod_{x_{i} \in A} q_{0}(x_{i} | A) \quad \text{for all } s = 1, 2, \ldots, T.$$

- **Type 2.** For an elementary region $A$ that contains no data points,

  $$\Phi(A, s, x) = 1 \quad \text{for all } s = 1, 2, \ldots, T.$$

- **Type 3.** For an elementary region $A$ containing exactly one data point, then under the centering condition,

  $$\Phi(A, s, x) = q_{0}(x | A) \quad \text{for all } s = 1, 2, \ldots, T.$$

Remark: From now on we shall refer to these three types of elementary regions as Type 1, Type 2, Type 3 (terminal) regions, respectively.

**Proof of Lemma 2.** The first claim holds because if $A$ cannot be further partitioned, then $\rho_{s,\infty}(A) = 1$, and so $\Phi(A, s, x) = Z(A, \infty, x) = \prod_{x_{i} \in A} q_{0}(x_{i} | A)$. The second claim holds because $\Phi(A, s, x)$ is the conditional predictive likelihood, which is 1 if there are no data points in $A$. For the last claim, note that the APT process is self-similar. More specifically, given that an elementary region $A$ arises in the random generative partitioning, the process is again a APT process on the
new sample space $A$ with the parameters and partition rule restricted to $A$ and the elementary regions that are descendants of $A$, with $Q_0(\cdot|A)$ being the base measure. The centering condition then ensures that the predictive density of this restricted APT is $q_0(\cdot|A)$, the base conditional density. The result now follows because when there is but one data point $x$ in $A$, $\Phi(A, s, x)$ is just the conditional predictive density at $x$. \hfill \Box

Lemma 2 implies that the recursion for computing $\Phi(\cdot, \cdot, x)$ does not need to proceed further on the above three types of elementary regions, as the mapping is known analytically on all descendants of those regions. To see this, note that the descendants of terminal regions are still terminal regions because (i) Type 1 regions have no children by definition, (ii) the children of Type 2 regions are still Type 2 regions, and (iii) the children of Type 3 regions are either Type 2 or Type 3 regions.

Will the recursion \textit{always} end up in these three types of terminal regions everywhere on $\Omega$? The answer is obviously positive if $A^{(\infty)}$ is finite as in the case when $\Omega$ is finite. The following lemma shows that even when $A^{(\infty)}$ is infinite, the recursion will terminate everywhere on $\Omega$ (with probability 1 under the true sampling distribution of the data) under very weak conditions.

**Lemma 3.** Suppose we observe $n$ i.i.d. data points from some distribution $P_0$ such that $P_0 \ll Q_0$. When $A^{(\infty)}$ is finite, the recursion for computing $\Phi(A, s, x)$ will always reach a terminal region everywhere on $\Omega$. When $A^{(\infty)}$ is infinite, the recursion will reach terminal regions everywhere on $\Omega$ with $P_0(n)$ probability 1 if

$$\max_{A \in A^k} Q_0(A) \to 0 \quad \text{as } k \to \infty.$$  

Remark: This condition is easily satisfied by commonly adopted partition rules such as the DDS.

**Proof of Lemma 3.** We just consider the nontrivial case when $A^{(\infty)}$ contains an infinite number of elementary regions. In this case, because $P_0 \ll Q_0$, we know that $\lim_{k \to \infty} \max_{A \in A^k} P_0(A) = 0$. Now if $P_0$ has atoms, then the previous condition ensures that for every $A \subset \Omega$ with $P_0(\tilde{A}) > 0$, the set $\{k: \exists A \in A^k \text{ such that } \tilde{A} \subset A\}$ is finite. In particular, for large enough $k$, no $A$ in $A^k$ shall contain any of the potential atoms among $x_1, x_2, \ldots, x_n$. So without loss of generality, let us consider the case when $P_0$ has no atoms. But then the probability for the event $E_k$ that any two $x_i$ and $x_j$ belong to a common elementary region at level $k$ can be made arbitrarily small as $k$ increases. Thus the event $\bigcap_k E_k$ has $P_0(n)$ probability 0. Therefore, with $P_0(n)$ probability 1, the recursion will eventually reach terminal regions everywhere on $\Omega$. \hfill \Box

Now that we have showed that the mapping $\Phi(\cdot, \cdot, x)$ can be computed recursively, we are now ready to present the main result of this subsection.

**Theorem 4** (Posterior of the APT prior). Suppose $x = (x_1, x_2, \ldots, x_n)$ are i.i.d. samples from a distribution $Q$, which has an APT prior:

$$Q \sim \APT(\mathcal{R}, \rho, \lambda, \alpha, Q_0).$$

Then given $x$, the posterior distribution of $Q$ is again an APT with partition rule $\mathcal{R}$ and base measure $Q_0$, along with the following parameters.

- Posterior state transition probabilities:

$$\rho_{s,t}(A | x) = \frac{\rho_{s,t}(A) \cdot Z(A, t, x)}{\Phi(A, s, x)}$$

for $A \in A^{(\infty)}$ and $s, t \in S$.  

15
• Posterior partition selection probabilities:

\[ \lambda_j(A, t \mid x) = \frac{Z_j(A, t, x)}{Z(A, t, x)} \]

for \( A \in \mathcal{A}(\infty), t \in \mathcal{S} \setminus \{ \infty \} \) and \( j = 1, 2, \ldots, N(A) \).

• Posterior pseudo-count parameters:

\[ \alpha_i^j(A, t \mid x) = \alpha_i^j(A, t) + n(A_i^j) \]

for \( A \in \mathcal{A}(\infty), t \in \mathcal{S} \setminus \{ \infty \}, j = 1, 2, \ldots, N(A) \) and \( i = 1, 2, \ldots, K^j(A) \).

**Proof of Theorem 4.** For any \( A \in \mathcal{A}(\infty) \), conditional on \( E_1(A) \) and \( E_2,s(A) \) for \( s \in \mathcal{S} \setminus \{ \infty \} \), the process restricted on \( A \) is a mixture of \( T + 1 \) processes, each of which corresponds to the constructive procedure transitioning into one of the \( T + 1 \) states on \( A \), with the \( \rho_{s,t}(A) \)'s being the prior mixing probabilities. The marginal likelihood under the restricted process on \( A \) given \( E_1(A) \) and \( E_2,s(A) \) is \( \Phi(A, s, x) \), while that under each of the \( T + 1 \) mixing components is \( Z(A, t, x) \) for \( t \in \mathcal{S} \). Therefore, the posterior process given \( E_1(A) \) and \( E_2,s(A) \) is a still mixture over the \( T + 1 \) components, with the new mixing probabilities given by Bayes' theorem as \( \rho_{s,t}(A \mid x) = \rho_{s,t}(A)Z(A, t, x)/\Phi(A, s, x) \).

Similarly, now define \( E_3,t(A) \) as the event that the process transitions into state \( t \) on \( A \). Then for \( t \in \mathcal{S} \setminus \{ \infty \} \), conditional on \( E_1(A), E_2,s(A) \) and \( E_3,t(A) \), the process is a mixture of \( N(A) \) components, each of which corresponds to \( A \) being partitioned into one of the available ways under the partition rule, with the \( \lambda_j(A, t) \)'s being the mixing probabilities. The marginal likelihood from the data on \( A \) given \( E_1(A), E_2,s(A) \) and \( E_3,t(A) \) is \( Z(A, t, x) \) while the marginal likelihood given these three events for the \( j \)th component, that is under the \( j \)th way of partitioning is \( Z_j(A, t, x)/\lambda_j(A, t) \). Thus again, \textit{a posteriori} the process conditional on \( E_1(A), E_2,s(A) \) and \( E_3,t(A) \) is still a mixture of \( N(A) \) components with the new mixing probabilities given by Bayes' theorem \( \lambda_j(A, t \mid x) = Z_j(A, t, x)/Z(A, t, x) \).

Continuing this argument, let \( E_{4,t,j}(A) \) denote the event that the process transitions into state \( t \) on \( A \) and \( A \) is divided in the \( j \)th way under \( \mathcal{R} \), then for \( t \in \mathcal{S} \setminus \{ \infty \} \), by the Dirichlet-Multinomial conjugacy, the posterior probability assignment vector \( Y^j(A) \) still has a Dirichlet distribution, whose pseudo-count parameters are updated by the number of data points in each of the children. That is, \( \alpha_i^j(A, t \mid x) = \alpha_i^j(A, t) + n(A_i^j) \). This completes the proof. \( \square \)

Another immediate but important consequence from Lemmas 1 through 3 is a way to compute the overall marginal likelihood under an APT prior.

**Theorem 5 (Marginalization of APT).** Under the same assumptions as those in Theorem 4, the overall marginal likelihood, i.e. the likelihood integrated over the APT prior, is given by

\[ M(x) = \Phi(\Omega, 1, x) = \Phi(\Omega, 2, x) = \cdots = \Phi(\Omega, T, x) = \sum_{t \in \mathcal{S}} \rho_{0,t} \cdot Z(\Omega, t, x), \]

which can be computed through recursion according to Lemma 1.

Remark: Under the default prior specification recommended in Section 2.3, the marginal likelihood is a function of the three hyperparameters \( c, T, \) and \( \gamma \). We can then find the empirical Bayes estimates by maximizing the marginal likelihood.

The following corollary can be used for computing the posterior predictive density of an APT.
COROLLARY 1 (Posterior predictive likelihood). Given the set up in Theorem 4, the posterior predictive likelihood for a new i.i.d. sample $y = (y_1, y_2, \ldots, y_n)$ from $Q$, $M(y \mid x)$, can be computed in two ways:

1. Treating the posterior APT given $x$ as the new prior for $y$, use Lemma 1 and Theorem 5 to compute the marginal likelihood, $M(y \mid x)$ directly.
2. Let $z = (x, y)$, the combined sample, and compute $M(z)$ using Lemma 1 and Theorem 5. Then $M(y \mid x) = M(z)/M(x)$.

In particular, for a single $y \in \Omega$, $M(y \mid x)$ is the posterior predictive density at $y$.

The last result in this subsection is the posterior consistency of inference under the APT prior. This assures us that as we get more and more data, the posterior will eventually concentrate around the truth distribution. For any probability measures $P_0$ on $\Omega$, a weak neighborhood $U$ of $P_0$ is a set of probability measures on $\Omega$ of the form

$$U = \left\{ Q : \left| \int g_i(\cdot) dQ - \int g_i(\cdot) dP_0 \right| < \epsilon_i, \text{ for } i = 1, 2, \ldots, K \right\}$$

for any bounded continuous functions $g_i$'s and non-negative constants $\epsilon_i$’s.

THEOREM 6 (Posterior consistency under weak topology). Let $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. data from a probability distribution $Q$ on $\Omega$, and let $\pi(\cdot)$ and $\pi(\cdot | X_1, X_2, \ldots, X_n)$ be the prior and posterior distribution for $Q$ as given in Theorem 4. Also assume that the APT prior $\pi(\cdot)$ satisfies the conditions in Theorem 2. Then for any $P_0 \ll Q_0$ with bounded $dP_0/dQ_0$, we have with $P_0^{(\infty)}$ probability 1 that

$$\pi(U \mid X_1, X_2, \ldots, X_n) \rightarrow 1 \text{ as } n \rightarrow \infty$$

for all weak neighborhoods $U$ of $P_0$. In particular, if $\Omega$ is a bounded rectangle in $\mathbb{R}^d$ and $Q_0$ is the uniform distribution on $\Omega$ then the APT process is consistent under the weak topology for any $P_0$ with bounded Lebesgue density.

PROOF. Let $p_0 = dP_0/d\mu$, $q_0 = dQ_0/d\mu$, $\tilde{p}_0 = dP_0/dQ_0$, and for any $Q \ll Q_0$, $\tilde{q} = dQ/dQ_0$. Let $M$ be a finite upperbound of $\tilde{p}_0$. Then the Kullback-Leibler (K-L) distance between $p_0$ and $q$ is given by

$$\text{KL}_{\mu}(p_0, q) = \int p_0 \log(p_0/q) d\mu = \int \tilde{p}_0 \log(\tilde{p}_0/\tilde{q}) dQ_0 = \text{KL}_{Q_0}(\tilde{p}_0, \tilde{q}).$$

By Lusin’s theorem we have a compact $E \subset \Omega$ with $Q_0(E^c) < \epsilon'$ such that $\tilde{p}_0$ is continuous on $E$. By the fine partition criterion, this $E$ can be chosen such that for every $\epsilon > 0$, there exists a partition, $\Omega = \cup A_i$ with all $A_i \in A^{(k)}$ for some $k$, such that the diameter of each $A_i \cap E$ is less than $\epsilon$. Then we define

$$\delta_E(\epsilon) = \sup_{x, y \in E, |x - y| < \epsilon} |\tilde{p}_0(x) - \tilde{p}_0(y)| \text{ and } d_i = \max_i \left( \sup_{A_i \cap E} \tilde{p}_0(x) + \delta_E(\epsilon), \epsilon' \right)$$

and let $D_\epsilon(\tilde{p}_0)$ be the collection of step functions $g(x) = \sum_i g_i 1_{A_i}(x)$ with $d_i \leq g_i < d_i + \delta_E(\epsilon)$. For every $g \in D_\epsilon(\tilde{p}_0)$, let $\tilde{g}$ be the normalized version of $g$, that is $\tilde{g} = g/\int gdQ_0$. Then

$$\int_E (g - \tilde{p}_0) dQ_0 - \int_{E^c} |g - \tilde{p}_0| dQ_0 \leq \int (g - \tilde{p}_0) dQ_0 \leq \int_E (g - \tilde{p}_0) dQ_0 + \int_{E^c} |g - \tilde{p}_0| dQ_0,$$
and so
\[ \delta_E(\epsilon) - (2M + \epsilon')\epsilon' \leq \int (g - \tilde{p}_0)dQ_0 \leq 3\delta_E(\epsilon) + (2M + \epsilon')\epsilon'. \]

Thus for any fixed \( \epsilon \), when \( \epsilon' \) is small enough, we have \( \int (g - \tilde{p}_0)dQ_0 \geq 0 \), and thus,
\[ \log \left( \int gdQ_0 \right) = \log \left( 1 + \int (g - \tilde{p}_0)dQ_0 \right) \leq 3\delta_E(\epsilon) + (2M + \epsilon')\epsilon'. \]

Now,
\[ 0 \leq \text{KL}_{Q_0}(\tilde{p}_0, \tilde{g}) = \int \tilde{p}_0 \log(\tilde{p}_0/\tilde{g})dQ_0 \]
\[ = \int_E \tilde{p}_0 \log(\tilde{p}_0/g)\,dQ_0 + \int_{E^c} \tilde{p}_0 \log(\tilde{p}_0/g)\,dQ_0 + \log \left( \int gdQ_0 \right) \]
\[ \leq M \log(M/\epsilon')\epsilon' + 3\delta_E(\epsilon) + (2M + \epsilon')\epsilon'. \]

By first choosing \( \epsilon' \) and then \( \epsilon \) small enough, we can make \( \text{KL}_{Q_0}(\tilde{p}_0, \tilde{g}) \) arbitrarily small. So \( p_0 \) lies in the K-L support of \( \pi \). Therefore, by Schwartz’s theorem, we have posterior consistency at \( p_0 \) under the weak topology. \( \square \)

3. Numerical examples. We next illustrate the work of the APT prior through three numerical examples. The first two examples illustrate the performance gain in density estimation when multi-resolution adaptive shrinkage is incorporated. The last illustrates how in multi-dimensional cases the multi-resolution grid and the shrinkage can be jointly inferred under the APT. In all three examples, we follow the recommended default prior specificatoin given in Section 2.3. For \( \lambda \), we always let \( \lambda_j(A, t) = 1/N(A) \). For \( \alpha \), we choose \( l(A, t) = h(k, t) = 10^t - 1 \), and thus \( \alpha_j(A, t) = c \cdot 10^{t-1} \cdot 0.5 \) for \( t = 0, 1, \ldots, T \). For \( \rho \), we use the decelerating specification for \( \rho \) as given in Eq. (2.2) for elementary regions of levels up to 11, and set \( \rho_s,\infty(A) = 1 \) for all \( s \in S \) and \( A \in A^1 \). So we investigate features with resolution up to 1/2,000 of the support. We use empirical Bayes to choose the values of the three hyperparameters \( c, T, \) and \( \gamma \), which we find by maximizing the marginal likelihood over a find grid of hyperparameter values.

Example 1 (Continued). We start from the motivating example given in Introduction. Let us now place the APT prior on the underlying density with \( Q_0 \) being the standard uniform and \( R_U \), the UDDS rule. Because in this case \( \Omega \) is one-dimensional, under the UDDS rule, each elementary region \( A \) has only \( N(A) = 1 \) way of being partitioned, and so \( \lambda_1(A, t) = 1 \). For comparison, we also apply two other priors—the optional Pólya tree (OPT) [20] and the PT. For the PT, we use the standard specification by setting the pseudo-counts to \( k^2 \) as we did earlier. For the OPT, we follow the default recommendation given in [20] by setting all pseudo-counts to 0.5. This leaves one free hyperparameter for the OPT (the stopping probability), which we also choose by empirical Bayes.

To measure performance, we use the \( L_1 \) distance between an estimated density \( \hat{f} \) and the true density, \( f_0 \), as the loss. That is, \( ||\hat{f} - f_0||_1 = \int |\hat{f} - f_0|d\mu \). The expected \( L_1 \) distance under \( f_0 \), \( E_{f_0}||\hat{f} - f_0||_1 \), is a commonly adopted risk for measuring performance of density estimators. It can be estimated by Monte Carlo. Suppose we repeat our simulation of the data set \( K \) times, with \( \hat{f}^{(k)} \) being the estimate for the \( k \)th simulation. Then an estimator for the risk is \( \hat{E}_{f_0}||\hat{f} - f_0||_1 = \frac{1}{K} \sum_{k=1}^{K} ||\hat{f}^{(k)} - f_0||_1 \), and its standard error can be computed accordingly.

Table 1 presents the estimated \( L_1 \) risk for the three PPDs along with the standard errors for three different sample sizes 250, 500, and 750. For each sample size, we carried out \( K = 500 \) simulations.
The reduction in risk achieved by the APT, in comparison to the OPT and PT, is substantial, especially for the two larger sample sizes. To have a sense of sampling variability of the reduction in $L_1$ loss over repeated experiments, we compute for each simulation the percentage change in $L_1$ loss for APT relative to the other two priors

$$\frac{||\hat{f}_{APT}^{(k)} - f_0||_1 - ||\hat{f}_{OPT}^{(k)} - f_0||_1}{||\hat{f}_{OPT}^{(k)} - f_0||_1} \times 100\%$$

and

$$\frac{||\hat{f}_{APT}^{(k)} - f_0||_1 - ||\hat{f}_{PT}^{(k)} - f_0||_1}{||\hat{f}_{PT}^{(k)} - f_0||_1} \times 100\%$$

where $\hat{f}_{APT}^{(k)}$, $\hat{f}_{OPT}^{(k)}$, and $\hat{f}_{PT}^{(k)}$ are the PPDs of the three priors under the $k$th simulation. Figure 2 gives the histograms of these two quantities over three different sample sizes. We see that the APT rarely does substantially worse than either of the other two methods but frequently reduces the loss by over 20% for sample sizes 500 and 750.

To understand the nature in which the APT outperforms the other two priors, we plot in Figure 3 the three PPDs along with the true underlying density for a typical simulation. We see that while the OPT captures the high-resolution feature well, it over-smooths the low-resolution feature. This is not surprising. The OPT can be considered a special case of APT with only one non-stopping state, corresponding to a constant pseudo-count of 0.5. Thus OPT in a sense also allows scale-location specific adaptive shrinkage, but in a very crude way—either no shrinkage or complete shrinkage (i.e. stopping) is allowed at each scale-location combination. Consequently, it is capable of capturing the highly spiky feature through the no shrinkage state, but is not flexible enough to capture the relatively smooth feature. By incorporating more shrinkage states and Markov dependence among

\begin{table}[h]
\centering
\begin{tabular}{|c|ccc|}
\hline
Sample size & APT & OPT & PT \\
\hline
250 & 0.241 (0.002) & 0.256 (0.001) & 0.325 (0.001) \\
500 & 0.173 (0.001) & 0.211 (0.001) & 0.258 (0.001) \\
750 & 0.146 (0.001) & 0.176 (0.001) & 0.225 (0.001) \\
\hline
\end{tabular}
\caption{Estimated $\hat{E}_{f_0}||\hat{f} - f_0||_1$ for the PPD estimators of three priors. Standard errors are given in the parentheses.}
\end{table}
these states, the APT effectively captures both the high and low resolution features. Note that it produces appropriate smoothing in higher resolutions within the large-scale feature (middle plot of Figure 3), in contrast to the PT, which overfits, and the OPT, which underfits.

Because we can marginalize out APT analytically following Theorem 5, one can also use the Bayes factor, or ratio of marginal likelihood, to judge whether the data support a more complex and flexible model such as the APT over its simple special case like the OPT. In Figure 4 we present histograms of the ratio of the maximized marginal likelihoods under these two models for the three sample sizes. Under these sample sizes, this ratio can often be substantial. (Because these involved maximized likelihood, however, one cannot interpret them as a usual Bayes factor, but they still heuristically support the need for the more flexible APT.)

In the previous example, the underlying distribution has two features of different scales, and the multi-resolution adaptive feature pays off handsomely. What if the underlying distribution does not have any special local features but is globally smooth? Will there be a hefty price to pay in such cases for being scale-location adaptive? The next example illustrates that even in globally smooth cases, the multi-resolution adaptive feature is robust, in fact still improves performance.
EXAMPLE 2. In this example, we simulate i.i.d. data from a simple Beta(5,5) distribution from three sample sizes 250, 500, and 750. We repeat exactly the same evaluation for the three priors—APT, OPT, and PT—as in the previous example, adopting exactly the same prior specification and the $L_1$ loss and risk. Table 2 presents the estimated $L_1$ risks from 500 simulations.

| Sample size | APT       | OPT       | PT        |
|-------------|-----------|-----------|-----------|
| 250         | 0.195 (0.002) | 0.218 (0.002) | 0.211 (0.001) |
| 500         | 0.145 (0.001) | 0.163 (0.001) | 0.178 (0.001) |
| 750         | 0.125 (0.001) | 0.139 (0.001) | 0.162 (0.001) |

Table 2 $\hat{E}_{f_0}||\hat{f} - f_0||_1$ for the PPD estimators of three priors for Example 2. Standard errors are given in the parentheses.

While the performance difference is not as large as in the presence of features of different scales, APT still outperforms the other two priors. This may seem surprising, but the reason is actually suggested in the previous example when the three priors try to capture the global feature (middle plot of Figure 3). The APT provides more effective smoothing through resolution-specific adaptiveness. In contrast, OPT tends to oversmooth by enforcing “stopping” too early at higher resolutions while the PT (with the $k^2$ pseudo-counts) tends to undersmooth, by enforcing too little smoothing at higher resolutions.

We use our last example to illustrate how the APT prior achieves adaptive shrinkage on a data-adaptive multi-resolution grid in multivariate cases. In particular, we aim to visualize a representative multi-resolution grid inferred from the data, along with the corresponding shrinkage states on each region in the grid.

EXAMPLE 3. We simulate 3,000 i.i.d. data points from the following mixture of bivariate normals

$$0.9 \mathcal{N}_2\left(\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.08^2 & 0 \\ 0 & 0.08^2 \end{pmatrix}\right) + 0.1 \mathcal{N}_2\left(\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}, \begin{pmatrix} 0.02^2 & 0 \\ 0 & 0.02^2 \end{pmatrix}\right)$$

truncated to the unit square. The two mixing components again represent two features of different scales—low-resolution and high-resolution. We place a APT($\mathcal{R}_U$, $\rho$, $\lambda$, $\alpha$, $U$) prior on the underlying distribution with the same specification as in the previous examples, the only change being that now with $\mathcal{N}(A) = 2$ for each elementary region $A$, we have $\lambda_j(A, t) = 1/2$ for $j = 1, 2$.

To visualize the adaptively inferred multi-resolution grid and shrinkage states, we compute the hierarchical maximum a posteriori (hMAP) sequence of partitions on the sample space. (The hMAP is introduced in [20] as a representative summary statistic, preferred to the MAP, for posteriors on recursive partitions.) For each elementary region that arises in the hMAP sequence of partitions, we find the shrinkage state that receives the highest posterior probability.

In Figure 5 we plot the hMAP sequence of partitions by scale (or level) for a typical simulation up to level 11, along with the corresponding a posteriori most probable shrinkage states for regions in the partition. The shrinkage states are visually represented using different gray-scale colors with darker color indicating stronger shrinkage. The partition sequence (i.e. the multi-resolution grid) adapts to the structure of the underlying distribution, and adaptive shrinkage is reflected in the different shrinkage states across the sample space at each level. In particular, while most of the sample space corresponding to the low-resolution feature feature of the distribution have mostly entered high shrinkage states by level 7, the region covering the high-resolution feature in the lower right corner is still mostly in low shrinkage states on levels 9 and 10.
Fig 5. Posterior most probable shrinkage states on the elementary regions (by level) in the hMAP sequence of partitions. The shrinkage states are represented using different gray-scale colors with darker colors indicating stronger shrinkage. White: state 1, light gray: state 2, medium gray: state 3, dark gray: stopping state. The empirical Bayes estimates for the hyperparameters in this instance are $\hat{\varphi}^{EB} = 0.22$, $\hat{\gamma}^{EB} = 1.6$, and $T^{EB} = 3$. 

4. Concluding remarks. We have introduced a nonparametric Bayesian process that allows multi-resolution adaptive shrinkage. We established its basic theoretical properties and illustrated its work in the context of density estimation. Inference using this process is computationally efficient in relatively low (≤10) dimensional problems. For example, computing the posterior APT in each of our three numerical examples takes about 1/10 second on a single Intel Core-i7 3.6Ghz CPU core. Application to higher dimensional problems requires effective ways for approximating recursions, which is currently under development.

Acknowledgment. The author thanks Jacopo Soriano for helpful comments.

REFERENCES

[1] Chipman, H. A., George, E. I., and McCulloch, R. E. (1998). Bayesian CART model search. *Journal of the American Statistical Association* 93, 443, 935–948.

[2] Chipman, H. A., Kolaczyk, E. D., and McCulloch, R. E. (1997). Adaptive Bayesian wavelet shrinkage. *Journal of The American Statistical Association* 92, 1413–1421.

[3] Clyde, M. and George, E. I. (2000). Flexible empirical bayes estimation for wavelets. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 62, 4, 681–698. http://dx.doi.org/10.1111/1467-9868.00257.

[4] Clyde, M., Parmigiani, G., and Vidakovic, B. (1998). Multiple shrinkage and subset selection in wavelets. *Biometrika* 85, 2, 391–401. http://biomet.oxfordjournals.org/content/85/2/391.abstract.

[5] Crouse, M. S., Nowak, R. D., and Baraniuk, R. G. (1998). Wavelet-based statistical signal processing using hidden Markov models. *Signal Processing, IEEE Transactions on* 46, 4 (Apr.), 886–902. http://dx.doi.org/10.1109/78.668544.

[6] Denison, D. G. T., Mallick, B. K., and Smith, A. F. M. (1998). A Bayesian CART algorithm. *Biometrika* 85, 2, 363–377.

[7] Donoho, D. L., Johnstone, I. M., Kerkyacharian, G., and Picard, D. (1996). Density estimation by wavelet thresholding. *The Annals of Statistics* 24, 2, pp. 508–539. http://www.jstor.org/stable/2242660.

[8] Ferguson, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* 1, 209–230. MR0350949 (50 #3441)

[9] Ferguson, T. S. (1974). Prior distributions on spaces of probability measures. *Ann. Statist.* 2, 615–629. MR0438568 (55 #11479)

[10] Hanson, T. E. (2006). Inference for mixtures of finite Pólya tree models. *J. Amer. Statist. Assoc.* 101, 476, 1548–1565. MR2279479

[11] Herrick, D. R. M., Nason, G. P., and Silverman, B. W. (2001). Some New Methods for Wavelet Density Estimation. *Sankhya: The Indian Journal of Statistics, Series A* 63, 3, 394–411. http://www.jstor.org/stable/25051366.

[12] Koo, J.-Y. and Kooperberg, C. (2000). Logspline density estimation for binned data. *Statistics & Probability Letters* 46, 2, 133 – 147. http://www.sciencedirect.com/science/article/pii/S0167715299000978.

[13] Lavine, M. (1992). Some aspects of Pólya tree distributions for statistical modelling. *Ann. Statist.* 20, 3, 1222–1235. MR1186248 (93k:62006b)

[14] Lavine, M. (1994). More aspects of Pólya tree distributions for statistical modelling. *Ann. Statist.* 22, 3, 1161–1176. MR1311107 (96e:62004)

[15] Ma, L. and Wong, W. H. (2011). Coupling optional Pólya trees and the two sample problem. *Journal of the American Statistical Association* 106, 496, 1553–1565.

[16] Mauldin, D. R., Sudderth, W. D., and Williams, S. C. (1992). Pólya trees and random distributions. *Ann. Statist.* 20, 3, 1203–1221.

[17] Vannucci, M. (1995). Nonparametric density estimation using wavelets. Discussion paper 95-26, Duke University.

[18] Vidakovic, B. (1998). Nonlinear wavelet shrinkage with bayes rules and bayes factors. *Journal of the American Statistical Association* 93, 173–179.

[19] Vidakovic, B. (1999). *Statistical Modeling by Wavelets*, 1st ed. John Wiley & Sons, Inc., Hoboken, NJ, USA.

[20] Wong, W. H. and Ma, L. (2010). Optional Pólya tree and Bayesian inference. *Annals of Statistics* 38, 3, 1433–1459. http://projecteuclid.org/euclid.aos/1268056622.
