Induced nets and Hamiltonicity of claw-free graphs

Shuya Chiba 1 2  Jun Fujisawa 3 4

Abstract
The connected graph of degree sequence $3, 3, 3, 1, 1, 1$ is called a net, and the vertices of degree 1 in a net is called its endvertices. Broersma conjectured in 1993 that a 2-connected graph $G$ with no induced $K_{1,3}$ is hamiltonian if every endvertex of each induced net of $G$ has degree at least $(|V(G)| − 2)/3$. In this paper we prove this conjecture in the affirmative.

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1 Introduction

Hamiltonian cycles in graphs have been extensively studied in the literature (cf., e.g., [8, 9, 10]). Several decades ago, research was mainly focused on their relation to the four-color problem; however, since the approval of Dirac’s and Ore’s theorems, various studies have considered degree conditions. While some of these degree conditions became extremely complicated as this type of research progressed, there remains several unsolved problems that can only be stated in short, easily understandable form, and these problems still engage our interest. One of these is provided in Conjecture 1, which generalizes two classical results obtained by Matthews and Sumner (Theorem 2) and by Duffus, Gould and Jacobson (Theorem 3). The connected graph of degree sequence $3, 3, 3, 1, 1, 1$ is called a net, and the vertices of degree 1 in a net is called its endvertices. Moreover, a graph is called claw-free if it has no induced $K_{1,3}$.

Conjecture 1 (Broersma [1]). Let $G$ be a 2-connected claw-free graph of order $n$. If every endvertex of each induced net in $G$ has degree at least $\frac{n−2}{3}$, then $G$ is hamiltonian.

Theorem 2 ([12]). Let $G$ be a 2-connected claw-free graph of order $n$. If the minimum degree of $G$ is at least $\frac{n−2}{3}$, then $G$ is hamiltonian.

Theorem 3 ([7]). Let $G$ be a 2-connected claw-free graph. If $G$ has no induced net, then $G$ is hamiltonian.
To verify that the degree condition in Conjecture 1 is sharp, we assume that $B_1$, $B_2$ and $B_3$ are complete graphs of the same order with $\{x_i, y_i, z_i\} \subseteq V(B_i)$. Further, let $G$ be a graph obtained from $B_1 \cup B_2 \cup B_3$ by adding six edges $x_ix_j$, $y_iy_j$ ($1 \leq i < j \leq 3$). We can observe that $G$ is non-hamiltonian, and $\{x_i, z_i | 1 \leq i \leq 3\}$ induces a net with endvertices $z_1$, $z_2$ and $z_3$ in $G$. Since each $z_i$ has degree $|V(B_i)| - 1 = \frac{|V(G)| - 3}{3}$, the degree condition is observed to be indeed sharp.

The only partial solution to Conjecture 1 that is known to the authors of this study is a theorem by Čada et al. [4], which states that Conjecture 1 is true if the degree condition is strengthened to $\frac{|V(G)| - 3}{3}$.

In this article, we prove Conjecture 1. Our theorem relies heavily on the closure concept that was introduced by Ryjáček [14]. In Section 2, we introduce the terminology and present the preliminary results related to Ryjáček’s closure, before we introduce some key lemmas in Section 3. In Section 4, we provide the proof of Conjecture 1 for graphs that contain at least 33 vertices. Since the proof for the smaller graphs (at most 32 vertices) comprises a tedious case-by-case analysis, which is not enlightening, we have instead provided a sketch of the theorem in Section 5 with the complete theorem being provided in [5].

2 Preliminaries

For standard terminology and notation, we refer the readers to [6]. In this paper, a graph or a simple graph means a finite undirected graph without loops or multiple edges. A multigraph may contain multiple edges but no loops. For a graph $G$ and $v \in V(G)$, $N_G(v)$ denotes the set of the neighbors of $v$, and $d_G(v) = |N_G(v)|$. Moreover, for $U \subseteq V(G)$, $G[U]$ denotes the induced subgraph of $G$ induced by $U$. For a graph $H$ and $x \in E(H)$, let $N_H^x = \{y \in E(H) \setminus \{x\} | y \text{ is adjacent to } x \in H\}$ and $d_H^x = |N_H^x|$. The set of the endvertices of $x$ is denoted by $V_H(x)$, or simply $V(x)$. We denote the set of vertices of degree one in $H$ by $V_1(H)$, and a pendant edge is an edge in which one endvertex has degree one. For a vertex $v \in V(H)$, the set of all the pendant edges which are incident with $v$ is denoted by $l_H(v)$, or simply $l(v)$. For $X \subseteq V(H)$ and $e \in E(H)$, we say that $e$ is dominated by $X$ if $V(e) \cap X \neq \emptyset$. We often identify a subgraph $H$ of a graph $G$ with its vertex set $V(H)$. The complete bipartite graph $K_{1,3}$ is called a claw, and a clique is a maximal complete subgraph of a graph.

In the rest of this section, we prepare previous studies which are commonly used in hamiltonian graph theory for claw-free graphs. Let $G$ be a claw-free graph. We call a vertex $v$ of $G$ locally connected (resp. locally disconnected) if $G[N_G(v)]$ is connected (resp. disconnected). For a locally connected vertex $v$ of $G$, the operation of joining all pairs of nonadjacent vertices in $N_G(v)$ is called the local completion at $v$. In [14], it is shown that this operation preserves the claw-freeness of the original graph. Iterating local completions, we obtain a graph $G^*$ in which $G^*[N_G^*(v)]$ is a complete graph for every locally connected vertex $v$. We call this graph the closure of $G$, and denote it $cl(G)$. The closure of a claw-free graph has the following properties.

**Theorem 4** (Ryjáček [14]). Let $G$ be a claw-free graph. Then $cl(G)$ is uniquely defined and is the line graph of some triangle-free simple graph. Moreover, $G$ is hamiltonian if and only if $cl(G)$ is hamiltonian.
For a claw-free graph $G$, the set of locally connected vertices and locally disconnected vertices are denoted by $LC(G)$ and $LD(G)$, respectively. If $v \in LC(G)$ and $G[N_G(v)]$ is not a complete graph, then we call $v$ an eligible vertex, and let $EL(G)$ denote the set of eligible vertices of $G$. For $x \in EL(G)$, $G_x$ denotes the graph obtained from $G$ by local completion at $x$. It is shown in [13, Lemma 9] that $LD(G_x) \subseteq LD(G)$ holds for every claw-free graph $G$ and $x \in EL(G)$. This yields the following.

**Proposition 5.** Let $G$ be a claw-free graph and let $v \in LC(G)$, then $v \in LC(cl(G))$.

The following theorem is a basic tool for the study on the hamiltonicity of line graphs. A closed trail $T$ in a graph $H$ is called a dominating closed trail, or a DCT, if every edge of $H$ is dominated by $T$.

**Theorem 6** (Harary and Nash-Williams [11]). Let $H$ be a multigraph with $|E(H)| \geq 3$. Then the line graph $L(H)$ is hamiltonian if and only if $H$ has a DCT.

Let $H$ be a multigraph and $F$ be a subgraph of $H$. Then $H/F$ denotes the multigraph obtained from $H$ by identifying the vertices of $F$ with a new vertex, which is denoted by $v_F$, and by deleting the created loops. A multigraph $H$ with at least two vertices is called collapsible if for every $S \subseteq V(H)$ with $|S|$ even, there exists a spanning connected subgraph $F$ of $H$ such that $v \in S$ if and only if $d_F(v)$ is odd. Collapsible subgraphs have the following property.

**Proposition 7** (Catlin [3]). Let $H$ be a multigraph and $F \subset H$ be a collapsible subgraph.

i) If $H/F$ has a DCT containing $v_F$, then $H$ has a DCT containing all the vertices in $F$.

ii) If $H/F$ is collapsible, then $H$ is collapsible.

By $F$ we denote the class of graphs obtained by taking two vertex-disjoint triangles $a_1a_2a_3$ and $b_1b_2b_3$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path of length at least two or by a triangle.

**Theorem 8** (Brousek [2]). Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph which is in $F$.

3 Lemmas

In this section we prove some lemmas. Among them, Corollary 2 (which is derived from Lemma 1) plays an important role in our proof. We denote an induced net of $G$ with six vertices $x_1, x_2, x_3, y_1, y_2, y_3$ by $N(x_1, x_2, x_3; y_1, y_2, y_3)$ if $\{x_i, x_{i+1}, x_iy_i \mid 1 \leq i \leq 3\} \subseteq E(G)$, where $x_4 = x_1$.

**Lemma 1.** Let $G$ be a claw-free graph and let $G^0, G^1, \ldots, G^{l-1}, G^l$ be a sequence of graphs such that $G^0 = G$, $G^l = cl(G)$ and $G^i$ is obtained from $G^{i-1}$ by the local completion of an eligible vertex of $G^{i-1}$, for $1 \leq i \leq l$. Moreover, let $N(x_1, x_2, x_3; y_1, y_2, y_3)$ be an induced net of $cl(G)$, let $R_0$ be the clique of $cl(G)$ which contains the triangle $x_1x_2x_3$, and let $R_j$ be the clique of $cl(G)$ which contains the edge $x_jy_j$ for $1 \leq j \leq 3$. Then the following holds:
i) For each $i$ with $0 \leq i \leq l$, there exists an induced net $N^i = N(x^i_1, x^i_2, x^i_3; y^i_1, y^i_2, y^i_3)$ of $G^i$ such that $x^i_j \in \{x_j\} \cup (V(R_0) \cap LC(cl(G)))$ and $y^i_j \in \{y_j\} \cup (V(R_j) \cap LC(cl(G)))$ for $j = 1, 2, 3$.

ii) $x_1x_2x_3$ is a triangle in $G$ if and only if $y^i_j \in \{y_j\} \cup (V(R_j) \cap LC(cl(G)))$ for each $j$.

Proof. First we prove i) by reverse induction on $i$. If $i = l$, then $N(x_1, x_2, x_3; y_1, y_2, y_3)$ is the desired induced net. Assume that $N^i = N(x^i_1, x^i_2, x^i_3; y^i_1, y^i_2, y^i_3)$ is the desired induced net of $G^i$ for some $i$ with $1 \leq i \leq l$. Let $F_i = E(N^i) \setminus E(G^{i-1})$. If $F_i = \emptyset$, then $N^i$ is also the desired induced net of $G^{i-1}$, and hence we assume that $F_i \neq \emptyset$. Let $u \in EL(G^{i-1})$ such that $G^{i-1}_u = G^i$. Then, all the edges of $F_i$ are contained in the clique induced by $N(u)$ in $G^i$. Hence either $F_i = \{x^i_jy^i_j\}$ for some $j$ or $F_i \subseteq \{x^i_1x^i_2, x^i_2x^i_3, x^i_3x^i_1\}$.

First we consider the former case. We may assume without loss of generality that $j = 1$. Let $y^1_1 = u$, $y^1_j = y^i_2$ for $j = 2, 3$ and $x^1_i = x^i_j$ for $1 \leq j \leq 3$. Since $x^1_1y^1_1 \in F_i$, we have $ux^1_1 \in E(G^{i-1})$. Moreover, since the neighbors of $u$ in $G^{i-1}$ induce a complete graph in $G^i$, $uv \notin E(G^{i-1})$ for each $v \in \{x^i_2, x^i_3, y^i_3\}$. Hence $\{x^i_1, x^i_2, x^i_3, y^i_1, y^i_2, y^i_3\}$ induces a net, say $N^{i-1}$, in $G^{i-1}$. Recall that $N^i$ satisfies i). Since $ux^1_1 \in E(G^i)$, it is a triangle in $cl(G)$ as well, and hence $u$ and $y^1_1$ are contained in the same clique $(R_0$ or $R_1)$ of $cl(G)$. Moreover, by Proposition $u \in LC(G^{i-1}) \subseteq LC(cl(G))$. Thus $N^{i-1}$ is the desired induced net of $G^{i-1}$.

Next we consider the case $F_i \subseteq \{x^i_1x^i_2, x^i_2x^i_3, x^i_3x^i_1\}$. If $|F_i| = 3$, then $\{u, x^i_1, x^i_2, x^i_3\}$ induces a claw in $G^{i-1}$, a contradiction. Moreover, if $|F_i| = 1$, say $F_i = \{x^i_1x^i_2\}$, then $\{x^i_1, x^i_2, y^i_3\}$ induces a claw in $G^{i-1}$, a contradiction. Therefore $|F_i| = 2$, say $F_i = \{x^i_1x^i_2, x^i_2x^i_3\}$. Then, since the neighbors of $u$ in $G^{i-1}$ induce a complete graph in $G^i$, we have $ux^i_1 \in E(G^{i-1})$ and $uy^i_2 \notin E(G^{i-1})$ for $j = 1, 2, 3$. Moreover, we have $x^i_1y^i_2 \notin E(G^{i-1})$ for $j = 1, 3$, since $N^i$ is an induced net in $G^i$. Let $x^i_1x^i_2 = u$ and $y^i_2 = x^i_2$. Furthermore, for $j = 1, 3$, let $x^i_j = x^i_3$ and $y^i_j = y^i_1$. Then $\{x^i_1, x^i_2, x^i_3, y^i_1, y^i_2, y^i_3\}$ induces a net, say $N^{i-1}$, in $G^{i-1}$. Recall that $N^i$ satisfies i). Since $ux^i_1x^i_3$ is a triangle in $G^{i-1}$, it is a triangle in $cl(G)$ as well, and hence $u$, $x^i_1$ and $x^i_3$ are contained in $R_0$. Moreover, by Proposition $u \in LC(G^{i-1}) \subseteq LC(cl(G))$, and by the induction hypothesis, $y^i_2 = x^i_3 \in \{x^i_2\} \cup (V(R_0) \cap LC(cl(G)))$. Thus $x^i_2$, $y^i_2$ satisfies i), and hence $N^{i-1}$ is a desired induced net of $G^{i-1}$. Consequently, i) holds for each $i$ with $0 \leq i \leq l$.

By the above procedure, $x_1x_2x_3$ is a triangle in $G$ if and only if $|F_i| \leq 1$ for every $1 \leq i \leq l$, and $y^i_j \in \{y_j\} \cup (V(R_j) \cap LC(cl(G)))$ for each $j$ if and only if $|F_i| \leq 1$ for every $1 \leq i \leq l$. Hence ii) holds. \relax
a path of \( \Lambda \) for \( i = 1, 2, 3 \). Moreover, let \( R_0R_i = x_i \) and \( R_iR_i^+ = y_i \) for \( i = 1, 2, 3 \). Then the following holds.

i) There exists an induced net \( N(x'_1, x'_2, x'_3; y'_1, y'_2, y'_3) \) in \( G \) such that \( x'_i \in \{x_i\} \cup l_H(R_0) \) and \( y'_i \in \{y_i, x_i\} \cup l_H(R_i) \cup l_H(R_0) \) (see Figure 1).

ii) Each \( y'_i \) is heavy. Moreover, if \( y'_i \in \{x_i\} \cup l_H(R_0) \), then \( d_H'(y'_i) \geq \frac{|E(H)| - 2}{3} + 2 + |J| \), where \( J = \{y'_j \mid j \neq i, y'_j \in \{x_j\} \cup l_H(R_0)\} \).

iii) \( x_1x_2x_3 \) is a triangle of \( G \) if and only if \( y'_i \in \{x_i\} \cup l_H(R_i) \) for each \( i \).

Proof. Note that \( x_i, y_i \in E(H) = V(G) \). Since \( L(\Lambda) \) is a net, \( \{x_i, y_i \mid 1 \leq i \leq 3\} \) induces a net \( N(x_1, x_2, x_3; y_1, y_2, y_3) \) in \( cl(G) \). Moreover, for \( 0 \leq i \leq 3 \), \( R_i \) corresponds to a clique, say \( \hat{R}_i \), in \( cl(G) \) and \( V(\hat{R}_i) \cap LC(cl(G)) \) corresponds to \( l_H(R_i) \). Thus by Lemma 1 there exists an induced net \( N' = N(x'_1, x'_2, x'_3; y'_1, y'_2, y'_3) \) in \( G \) such that \( x'_i \in \{x_i\} \cup l_H(R_0) \) and \( y'_i \in \{y_i, x_i\} \cup l_H(R_i) \cup l_H(R_0) \) for \( i = 1, 2, 3 \). Hence i) holds.

Since \( N' \) is an induced net of \( G \), \( d_H'(y'_i) = d_{cl(G)}(y'_i) \geq d_G(y'_i) \geq \frac{|V(G)| - 2}{3} = \frac{|E(H)| - 2}{3} \) for each \( i \). Thus each \( y'_i \) is heavy. Assume that \( y'_i \in \{x_i\} \cup l_H(R_0) \) for some \( i \). Without loss of generality, we may assume that \( i = 1 \). Since \( N' \) is an induced net of \( G \), none of the vertex in \( \{x'_2, x'_3\} \cup J \) is adjacent to \( y'_1 \) in \( G \). On the other hand, since \( \{x_1, x_2, x_3\} \cup l_H(R_0) \) induces a complete graph in \( cl(G) \) and \( \{x'_2, x'_3\} \cup J \subseteq \{x_2, x_3\} \cup l_H(R_0) \), each vertex in \( \{x'_2, x'_3\} \cup J \) is adjacent to \( y'_1 \) in \( cl(G) \). Hence \( d_{cl(G)}(y'_1) - d_G(y'_1) \geq 2 + |J| \), and thus \( d_H'(y'_1) = d_{cl(G)}(y'_1) \geq d_G(y'_1) + 2 + |J| = \frac{|V(G)| - 2}{3} + 2 + |J| \). Therefore ii) holds. Since \( l_H(R_i) \) corresponds to \( V(\hat{R}_i) \cap LC(cl(G)) \), iii) follows from Lemma 1 ii).

A connected multigraph \( H \) is called essentially \( k \)-edge-connected if \( H - F \) has at most one component which contains an edge for every \( F \subseteq E(H) \) with \( |F| < k \). Note that a graph \( H \) is essentially \( k \)-edge-connected if and only if \( L(H) \) is \( k \)-connected or complete.

Lemma 3. Let \( H \) be an essentially \( 2 \)-edge-connected multigraph and let \( x \in V(H) \) such that \( d_H(x) \geq 2 \). If \( |E(H - \{x\})| \leq 3 \), then there exists a DCT of \( H \) containing \( x \).

Proof. Let \( H \) be the minimal counterexample. Assume that there exists a cycle \( C \) of length 2 or 3 containing \( x \). Since \( C \) is not a DCT of \( H \), \( E(H - V(C)) \neq \emptyset \). Let \( H' = H/C \), then \( H' \) is essentially
Figure 2: Possible arrangements of $S$ (circled vertices) in $K_{3,3}^-$ and corresponding spanning subgraphs $F$ (solid edges)

2-edge-connected. Since $H$ is essentially 2-edge-connected and $E(H - V(C)) \neq \emptyset$, we have $d_{H'}(v_C) \geq 2$. Moreover, the assumption $|E(H - \{x\})| \leq 3$ yields $|E(H' - \{v_C\})| \leq 3$. Hence, by the minimality of $H$, there exists a DCT of $H'$ containing $v_C$. Since a cycle of length 2 or 3 is collapsible, by Proposition 7 i), $H$ has a DCT which contains $V(C)$, a contradiction. Hence there exists no cycle of length 2 or 3 containing $x$.

Since $H$ does not have a DCT, $E(H - \{x\}) \neq \emptyset$. Hence $H$ is not a star with center $x$. Since $H$ is essentially 2-edge-connected and $d_H(x) \geq 2$, there exists a cycle $C$ in $H$ which contains $x$. If $|V(C)| \geq 5$, then $|E(C - \{x\})| \geq 3$. Since $|E(H - \{x\})| \leq 3$, we have $E(H - \{x\}) = E(C - \{x\})$. Hence $C$ is a DCT of $H$ containing $x$, a contradiction. Therefore there exists a cycle of length 2 or 3 containing $x$, a contradiction. Hence $H$ is essentially 2-edge-connected and $d_H(x) \geq 2$, there exists a cycle $C$ in $H$ which contains $x$. If $|V(C)| \geq 5$, then $|E(C - \{x\})| \geq 3$. Since $|E(H - \{x\})| \leq 3$, we have $E(H - \{x\}) = E(C - \{x\})$. Hence $C$ is a DCT of $H$ containing $x$, a contradiction. Therefore there exists a cycle of length 2 or 3 containing $x$.

Lemma 3 yields the following corollary.

**Corollary 4.** Let $H$ be an essentially 2-connected multigraph and let $\Xi$ be a collapsible subgraph of $H$. If $|E(H - \Xi)| \leq 3$, then there exists a DCT of $H$.

**Proof.** Let $H' = H/\Xi$, then $H'$ is an essentially 2-edge-connected multigraph. If $v_\Xi$ has degree at most 1 in $H'$, then $H' \simeq K_1$ or $K_2$ since $H$ is essentially 2-edge-connected. Therefore, $v_\Xi$ is a DCT of $H'$. On the other hand, if $v_\Xi$ has degree at least 2 in $H'$, then there exists a DCT of $H'$ containing $v_\Xi$ by Lemma 3. In either case, $H$ has a DCT by Proposition 7 i).

Let $K_{3,3}^-$ be the graph obtained from $K_{3,3}$ by deleting one edge. By a straightforward case analysis we obtain the following lemma (see Figure 2).

**Lemma 5.** Both of $K_{3,3}$ and $K_{3,3}^-$ are collapsible.

**Lemma 6.** Let $H$ be an essentially 2-edge-connected triangle-free simple graph which does not contain a DCT and let $n = |E(H)|$. If $\{e_1, e_2, e_3\}$ is a matching of $H$, then $\sum_{i=1}^3 d_H(e_i) \leq n + 1$. 
Moreover, if $c_l \in F$ the triangle-free graph such that Theorem 9. 

4 Proof for the large graphs

In this section we prove Conjecture for the graphs with at least 33 vertices. As we will see at the end of this section, the proof immediately follows from the following theorem. We call a matching heavy if each edge of the matching is heavy.

**Theorem 9.** Let $G$ be a graph of order $n$ which satisfies the assumption of Conjecture 7 and let $H$ be the triangle-free graph such that $L(H) = cl(G)$. Then there exists either a DCT or a heavy matching of size 4 in $H$.

**Proof.** Suppose that $H$ has neither a DCT nor a heavy matching of size 4. Then, by Theorem 8, $cl(G)$ is not hamiltonian, and hence it follows from Theorem 9 that $cl(G)$ contains an induced subgraph $F \in F$. Let $\Theta$ be the subgraph of $H$ such that $L(\Theta) = F$. Then there exist two vertices $A, B$ of degree 3 and three internally vertex-disjoint paths $P_i$ $(1 \leq i \leq 3)$ of length at least two joining $A$ and $B$ in $\Theta$. Moreover, if $|P_i| = 3$, then the middle vertex of $P_i$ is joined to one pendant edge in $\Theta$ (that is, the middle vertex of $P_i$ has degree 3 in $\Theta$). We denote $p_i = |P_i| - 2$. Let $V(\Theta) \setminus \{A, B\} = \{C_{i,j} \mid 1 \leq i \leq 3, 1 \leq j \leq p_i\} \cup \{D_i \mid 1 \leq i \leq 3, p_i = 1\}$, where $P_i = AC_{i,1}C_{i,2}\ldots C_{i,p_i}B$ for $1 \leq i \leq 3$ and $C_{i,1}D_i$ is the pendant edge if $p_i = 1$. For $1 \leq i \leq 3$, let $a_i = AC_{i,1}$ and $b_i = C_{i,p_i}B$. Moreover, if $p_i \neq 1$, then let $c_{i,j} = C_{i,j}C_{i,j+1}$ for $1 \leq j < p_i - 1$, and if $p_i = 1$, then let $c_{i,1} = C_{i,1}D_i$. Since $F = L(\Theta)$ is an induced subgraph of $cl(G)$, $D_i \neq D_j$ and $D_i \notin V(P_j)$ hold for $i \neq j$. Let $X = \{\{C_{i,j} \mid 1 \leq i \leq 3, j = 1, 2\} \cap V(H)\} \cup \{A, B\}$ (see Figure 3).

For $1 \leq i \leq 3$, let $b'_i = C_{i,p_i - 1}C_{i,1}$, if $p_i \neq 1$ and let $b'_i = c_{i,1}$ if $p_i = 1$. Since $\{a_i, c_{i,1} \mid 1 \leq i \leq 3\}$ and $\{b_i, b'_i \mid 1 \leq i \leq 3\}$ induces a subdivided claw in $H$, it follows from Corollary 8 that there exists
a heavy edge $c_i^* \in \{c_{i,1}, a_i\} \cup l(C_{i,1}) \cup l(A)$ and a heavy edge $\bar{c}_i^* \in \{b_i, b'_i\} \cup l(C_{i,p_i}) \cup l(B)$ for each $i$.

**Case 1.** $c_i^* \in \{c_{i,1}\} \cup l(C_{i,1})$ holds for $i = 1, 2, 3$ or $\bar{c}_i^* \in \{b'_i\} \cup l(C_{i,p_i})$ holds for $i = 1, 2, 3$.

Without loss of generality, we may assume that $c_i^* \in \{c_{i,1}\} \cup l(C_{i,1})$ holds for $i = 1, 2, 3$. Then by Corollary(2)i), $a_1a_2a_3$ is a triangle in $G$.

**Claim 1.**

i) For each $i$ with $p_i = 1$, there exists $b_i^* \in \{b_i\} \cup l(C_{i,1})$ such that $b_i^*$ is heavy in $H$.

ii) If there exists $u \in E(H - X)$ such that $u$ and $c_i^*$ are adjacent in $H$, then $p_i = 1$ and $c_i^* = c_{i,1}$.

**Proof.** If $p_i = 1$, then $\{a_i, b_i\} \cup \{a_j, c_{j,1} : 1 \leq j \leq 3, j \neq i\}$ induces a subdivided claw in $H$. Hence by Corollary(2)i) and ii) there exists a heavy edge $b_i^*$ in $\{b_i, a_i\} \cup l(C_{i,1}) \cup l(A)$. Since $a_1a_2a_3$ is a triangle in $G$, by Corollary(2)i), we have $b_i^* \in \{b_i\} \cup l(C_{i,1})$. Therefore i) holds.

If there exists $u$ as in ii), then $u$ is incident with $D_i$, since $u$ is incident with neither $C_{i,1}$ nor $C_{i,2}$. Hence $p_i = 1$ and $c_i^* = c_{i,1}D_i = c_{i,1}$. □

**Claim 2.** $p_i \leq 3$ for any $i$ with $1 \leq i \leq 3$.

**Proof.** Assume to the contrary that $p_i \geq 4$ for some $i$. Recall that there exists a heavy edge $\bar{c}_i^* \in \{b'_i, b_i\} \cup l(C_{i,p_i}) \cup l(B)$. For each $j$ with $1 \leq j \leq 3$, $\bar{c}_j^*$ and $c_j^*$ are not adjacent in $H$ since $c_j^* \in \{c_{j,1}\} \cup l(C_{j,1})$. Therefore $\{\bar{c}_i^*, c_i^*, c_2^*, c_3^*\}$ is a heavy matching of size 4, a contradiction. □

**Claim 3.** Let $UW \in E(H)$ such that $U$ is adjacent to $A$ or $B$. Then $UW$ is dominated by $X$.

**Proof.** Assume to the contrary that the edge $u = UW$ is not dominated by $X$. Then $U,W \notin \{C_{i,1}, C_{i,2}, A, B\}$ for any $i$. Let $R = A$ if $U$ is adjacent to $A$, and let $R = B$ otherwise. Moreover, let $r$ be the edge of $H$ joining $U$ and $R$. We shall prove that there exists a subdivided claw containing both $u$ and $r$ with center $R$.

Since $D_l$ ($p_l = 1$) are distinct vertices, we can take $i,j$ so that $W \neq D_i,D_j$. Moreover, since $H$ is triangle-free, we have $U \neq D_l$ for each $l$. Hence, if $U \in N_H(A)$, then $\{r,u,a_i,a_j,c_{j,1}\}$ is a desired subdivided claw. Thus we assume $U \in N_H(B)$. Let $k = \{1,2,3\} \setminus \{i,j\}$. If $U = C_{i,3}$, either $\{r,u,b_j,b'_j,b_k,b'_k\}$ (in the case $W \neq D_k$) or $\{r,u,b_j,b'_j,b_k,a_k\}$ (in the case $W = D_k$) induces a desired subdivided claw. The case $U = C_{j,3}$ is similar, and hence we may suppose $U \neq C_{i,3},C_{j,3}$. Since $H$ is triangle-free, we have $W \neq C_{i,3}$ for each $l$. Hence $u$ and $b_l$ are not adjacent in $H$ for $l = i,j$ and, since $u$ is not dominated by $X$, $u$ and $b'_l$ are not adjacent in $H$ for $l = i,j$. Therefore $\{b_i,b'_i,b_j,b'_j,r,u\}$ induces a desired subdivided claw.

By Corollary(2)i) and ii), there exists $u^* \in \{u,r\} \cup l(U) \cup l(R)$ which is heavy. Since there exists no heavy matching of size 4 in $H$, we may assume that $u^*$ and $c_i^*$ are adjacent. Since both of the endvertices of $c_i^*$ are contained in $\{C_{i,1}, C_{i,2}, D_i\} \cup V_i(H)$ and $U,W \notin \{C_{i,1},C_{i,2}\}$, we have $W = D_i$, which implies $p_i = 1$. Then by Claim(2)i), there exists $b_i^* \in \{b_i\} \cup l(C_{i,1})$ which is heavy. Since both of the endvertices of $b_i^*$ are contained in $\{C_{i,1}, B\} \cup V_1(H)$ and both of the endvertices of $c_j^*$ are contained in $\{C_{i,1}, C_{i,2}, D_i\} \cup V_1(H)$ for $i = 2,3$, $\{b_1^*, c_2^*, c_3^*, u^*\}$ is a heavy matching of size 4, a contradiction. □
Claim 4. For any \( i \) with \( p_i = 3 \), \( N_H(C_{1,3}) \subseteq X \).

Proof. Assume to the contrary that there exists \( U \in N_H(C_{1,3}) \setminus X \). Without loss of generality, we may assume that \( i = 1 \). Let \( u = UC_{1,3} \). Since \( b'_2 \) and \( b'_3 \) have no common endvertex, we may assume without loss of generality that \( b'_2 \) is not incident with \( U \). Note that \( U \neq C_{1,3} \) in the case \( p_3 = 3 \), since \( H \) is triangle-free. Hence, if \( b'_2 \) is incident with \( U \), then \( p_3 = 1 \) and \( U = D_3 \), because \( U \not\in X \). Therefore, if \( p_3 \neq 1 \), then \( \{b_1, u, b'_2, b'_3, b'_3\} \) induces a subdivided claw, and if \( p_3 = 1 \), then \( \{b_1, u, b'_2, b'_3, a_3\} \) induces a subdivided claw. In either case, by Corollary 2 i) and ii), there exists a heavy edge \( b'_3 \in \{u, b_1\} \cup l(C_{1,3}) \cup l(B) \). Since \( U \not\in X \), \( u \) and \( c'_3 \) are not adjacent for \( i = 1, 2 \). Hence we can deduce that \( \{b'_3, c'_1, c'_3\} \) is a heavy matching of \( H \). Since there exists no heavy matching of size \( 4, b'_1 \) and \( c'_3 \) are adjacent. This implies that \( p_3 = 1, b'_1 = u \) and \( U = D_3 \). Hence \( \{b_1, u, b'_2, b'_3, a_3\} \) induces a subdivided claw. By Corollary 2 ii) and ii), there exists a heavy edge \( a'_3 \in \{a_3, b_3\} \cup l(C_{3,1}) \cup l(B) \). Now \( b'_3 = u = D_3C_{1,3} \) yields \( \{b'_3, c'_1, c'_2, a'_3\} \) is a heavy matching of \( H \), a contradiction. \( \square \)

Claim 5. If \( u \in E(H) \) is not dominated by \( X \), then \( u \) is not heavy in \( H \).

Proof. Assume, to the contrary, that \( u \) is heavy. Since \( H \) does not contain a heavy matching of size \( 4, u \) and \( c'_i \) are adjacent in \( H \) for some \( i \). Without loss of generality, we may assume that \( i = 1 \). Then by Claim 4 ii) we have \( p_1 = 1 \) and \( c'_1 = c_{1,1} \), and then we can take \( b'_1 \) as in Claim 4 i). Since both of the endvertices of \( b'_1 \) are contained in \( \{C_{1,1}, B\} \cup V_1(H) \), \( u \) and \( b'_1 \) are not adjacent in \( H \).

Since \( \{b'_1, c'_2, c'_3\} \) is a heavy matching of \( H \) and there exists no heavy matching of size \( 4, u \) and \( c'_2 \) are adjacent in \( H \) for \( i = 2 \) or \( 3 \). Without loss of generality, we may assume that \( u \) and \( c'_2 \) are adjacent. Then by Claim 4 ii), \( p_2 = 1 \) and \( c'_2 = c_{2,1} \), and hence there exists \( b'_2 \) as in Claim 4 i).

Recall that both of \( u, c_{1,1} \) and \( u, c_{2,1} \) are adjacent in \( H \). Since \( u \) is not dominated by \( X \), we have \( u = D_1D_2 \). Thus \( \{a_1, a_2, c'_1, u, b_1, b_3\} \) induces a subdivided claw in \( H \). By Corollary 2 i) and ii), there exists \( a'^* \in \{a_2, a_1\} \cup l(A) \cup l(C_{1,1}) \) which is heavy. If \( a'^* \in \{a_1\} \cup l(C_{1,1}) \), then \( \{a'^*, u, b'_2, c'_1\} \) is a heavy matching of \( H \), a contradiction. On the other hand, if \( a'^* \in \{a_2\} \cup l(A) \), then \( \{a'^*, u, b'_1, c'_3\} \) is a heavy matching of \( H \), a contradiction. \( \square \)

Let \( Q_1 \) be the set of paths in \( H - E(P_1 \cup P_2 \cup P_3) \) joining \( C_{i,1} \) and \( C_{j,1} \) for some \( i, j \) with \( i \neq j \) and \( p_i = p_j = 1 \), and let \( Q_2 \) be the set of paths in \( H - E(P_1 \cup P_2 \cup P_3) \) joining \( C_{i,1} \) and \( C_{j,2} \) (\( i \neq j \)) or \( C_{i,1} \) and \( C_{j,1} \) (\( i \neq j \), \( p_i = 1 \) and \( p_j \geq 2 \)). Note that a path in \( Q_1 \cup Q_2 \) may contain the edge \( C_{i,1}D_i \) for some \( i \) with \( p_i = 1 \).

Claim 6. \( Q_1 \cup Q_2 \neq \emptyset \).

Proof. Assume, to the contrary, that \( Q_1 = Q_2 = \emptyset \). Let \( i, j \in \{1, 2, 3\} \) with \( i \neq j \). If \( p_i = p_j = 1 \), then \( C_{i,1}D_i, D_iD_j \notin E(H) \) since \( Q_1 = \emptyset \). If \( p_i = 1 \) and \( p_j \geq 2 \), then \( C_{i,1}C_{j,2}, D_iC_{j,1}, D_jC_{j,2} \notin E(H) \) since \( Q_2 = \emptyset \). If \( p_i \geq 2 \) and \( p_j = 1 \), then \( C_{i,1}C_{j,2} \notin E(H) \) since \( Q_2 = \emptyset \). Moreover, in either case, \( C_{i,1}C_{j,2} \notin E(H) \) since \( H \) is triangle-free.

By the above argument, it follows that \( N_H^c(1) \subseteq \{C_{1,2}C_{j,2}\} \) for any \( i, j \) with \( i \neq j \). Note that if \( p_i \geq 3 \), then \( b_i \notin N_H^c(2) \) for any \( j \). Let \( E_0 = E(H) \setminus (N_H^c(1) \cup N_H^c(2) \cup N_H^c(3)) \cup \{c'_1, c'_2, c'_3\} \).
\[ n = |E(H)| \]
\[ \geq |N^r_H(c_1^*) \cup N^r_H(c_2^*) \cup N^r_H(c_3^*)| + |\{c_1^*, c_2^*, c_3^*\}| + |b_i| p_i \geq 3| + |E_0| \]
\[ \geq |N^r_H(c_1^*)| + |N^r_H(c_2^*)| + |N^r_H(c_3^*)| - |\{C_{1,2}C_{2,3} | 1 \leq i < j \leq 3 \cap E(H)\}| + 3 + |\{b_i| p_i \geq 3\}| + |E_0| \]
\[ \geq 3 \times \frac{n - 2}{3} + 3 - |\{C_{1,2}C_{2,3} | 1 \leq i < j \leq 3 \cap E(H)\}| + |\{b_i| p_i \geq 3\}| + |E_0|, \]
and hence
\[ |\{b_i| p_i \geq 3\}| + |E_0| \leq |\{C_{1,2}C_{2,3} | 1 \leq i < j \leq 3 \cap E(H)\}| - 1. \quad (1) \]

Without loss of generality, we may assume that \( p_1 \geq p_2 \geq p_3 \). Let \( t = |\{i| p_i \geq 3\}| = |\{b_i| p_i \geq 3\}|. \)

If \( t = 0 \), then \(|\{C_{1,2}C_{2,3} | 1 \leq i < j \leq 3 \cap E(H)\}| = 0 \) since \( H \) is triangle-free. Then the right hand side of (1) is \(-1\), a contradiction. If \( t \geq 2 \), then by (1), \(|\{C_{1,2}C_{2,3} | 1 \leq i < j \leq 3 \cap E(H)\}| \geq 3 \). This implies that \( C_{1,2}C_{2,3} \) is a triangle of \( H \), a contradiction. Hence \( t = 1 \), which yields \( p_3 = 3 \) and \( p_2, p_3 \leq 2 \). Then we have \( C_{2,2}C_{3,2} \not\in E(H) \), since otherwise \( C_{2,2}C_{3,2} B \) is a triangle of \( H \). Hence, by (1), \( E_0 = \emptyset \) and \( C_{1,2}C_{2,2}, C_{1,2}C_{3,2} \in E(H) \). This yields \( p_2 = p_3 = 2 \) and \( c_i^* \in \{C_{1,1}C_{i,2}\} \cup l(C_{i,1}) \) for each \( i \). In the case \( C_{1,3}C_{2,1} \in E(H) \), let \( T = AC_{1,1}C_{2,1}C_{3,2}C_{2,1}BC_{3,2} \in E(H) \). Then, since \( T \) contains \( \{C_{1,1}, C_{2,1}\} \) for each \( i \), \( T \) dominates every edge of \( N^r_H(c_i^*) \cup \{c_i^*\} \). Moreover, since \( V(b_1) \subseteq V(T) \) and \( E_0 = \emptyset \), \( T \) dominates \( E(H) \), a contradiction. Hence we have \( C_{1,3}C_{2,1} \not\in E(H) \). By symmetry we have \( C_{1,3}C_{3,1} \not\in E(H) \).

Recall that there exists a heavy edge \( c_i^* \in \{b_1, b_2\} \cup l(C_{1,3}) \cup l(B) \) if \( c_i^* \not\in c_i^* \), then \( \{c_1^*, c_2^*, c_3^*, c_i^*\} \) is a heavy matching of \( H \), a contradiction. Hence \( c_i^* = b_i^* \). Since \( H \) is triangle-free and \( p_2 = p_3 = 2 \), we have \( C_{1,3}C_{2,1} \not\in E(H) \) for \( i = 2, 3 \). Hence \( N^r_H(c_i^*) \cap N^r_H(c_i^*) \subseteq \{C_{1,2}C_{1,2}\} \) for \( i = 2, 3 \). Moreover, since \( C_{2,2}C_{3,2} \not\in E(H) \), \( N^r_H(c_2^*) \cap N^r_H(c_3^*) = \emptyset \).

Let \( E_1 = E(H) \setminus (N^r_H(c_1^*) \cup N^r_H(c_2^*) \cup N^r_H(c_3^*) \cup \{c_1^*, c_2^*, c_3^*\}) \), then
\[ n = |E(H)| \]
\[ \geq |N^r_H(c_1^*) \cup N^r_H(c_2^*) \cup N^r_H(c_3^*)| + |\{c_1^*, c_2^*, c_3^*\}| + |E_1| \]
\[ \geq |N^r_H(c_1^*)| + |N^r_H(c_2^*)| + |N^r_H(c_3^*)| - |\{C_{1,2}C_{2,2}, C_{1,2}C_{3,2}\}| + 3 + |\{a_1\}| \]
\[ \geq 3 \times \frac{n - 2}{3} - 2 + 3 + 1 = n, \]
and hence \( E_1 = \{a_1\} \). Let \( T' = AC_{2,1}C_{2,2}C_{1,3}BC_{3,2}C_{3,1}A \). Then, since \( \{C_{1,2}, C_{1,3}, C_{2,1}, C_{2,2}, C_{3,1}, C_{3,2}\} \subseteq V(T'), T' \) dominates every edge of \( N^r_H(c_1^*) \cup N^r_H(c_2^*) \cup N^r_H(c_3^*) \cup \{c_1^*, c_2^*, c_3^*\} \). Since \( a_1 \) is dominated by \( T' \), \( T' \) is a DCT of \( H \), a contradiction.

In the case where \( Q_1 \neq \emptyset \), take \( Q_1, Q_2, \ldots, Q_m \in Q_1 \) so that \( |V(Q_1)| + \ldots + |V(Q_m)| \) is as large as possible, subject to the condition that \( Q_1, \ldots, Q_m \) are internally vertex-disjoint, and let \( Q = \{Q_1, Q_2, \ldots, Q_m\} \). In the case where \( Q_1 = \emptyset \), take \( Q = Q_2 \) and let \( Q = \{Q\} \).

**Claim 7.** There exists a closed trail \( T \) of \( H \) such that \( X \cup V(P_i) \cup V(P_j) \cup \bigcup_{Q \subseteq Q} V(Q) \subseteq V(T) \subseteq \left( \bigcup_{i=1}^3 V(P_i) \right) \cup \left( \bigcup_{Q \subseteq Q} V(Q) \right) \) for some \( i, j \) with \( 1 \leq i < j \leq 3 \).
Proof. Assume $Q_1 = \emptyset$ and let $Q$ be the (unique) path in $Q$. Then without loss of generality, we may assume that either $Q$ joins $C_{1,1}$ and $C_{2,2}$ or $p_1 = 1$, $p_2 \geq 2$ and $Q$ joins $C_{1,1}$ and $C_{2,1}$. In the former (resp. latter) case, $Ac_{2,2}Q_c_{C_{1,1}}C_{1,2} \cdots C_{1,p_1}BP_{3}A$ (resp. $Ac_{1,1}Qc_{2,1}C_{2,2} \cdots C_{2,p_2}BP_{3}A$) is a required closed trail, where $i = 1$ and $j = 3$. Hence we may assume that $Q_1 \neq \emptyset$.

We apply induction on $|Q|$, and we find the desired closed trail without using the assumption that $H$ is essentially 2-edge-connected. In the case $|Q| = 1$, we may assume without loss of generality that $Q$ joins $C_{1,1}$ and $C_{2,1}$ and $p_1 = p_2 = 1$. Then $Ac_{1,1}Qc_{2,1}BP_{3}A$ is a required closed trail. Suppose that $|Q| = 2$. If $Q_1$ and $Q_2$ have the same endvertices, say $C_{1,1}$ and $C_{2,1}$, then $Ac_{1,1}Q_1C_{2,1}Q_2C_{1,1}BP_{3}A$ is a required closed trail. Otherwise, without loss of generality we may assume that $Q_1$ joins $C_{1,1}$ and $C_{2,1}$ and $Q_2$ joins $C_{2,1}$ and $C_{3,1}$. Then $Ac_{1,1}Q_1C_{2,1}BC_{3,1}Q_2C_{2,1}A$ is a required closed trail.

Assume that $|Q| \geq 3$. If $|Q| = 3$ and $Q_1$ joins $C_{1,1}$ and $C_{2,1}$, $Q_2$ joins $C_{2,1}$ and $C_{3,1}$ and $Q_3$ joins $C_{2,1}$ and $C_{i,1}$ for some $i, j, k$ with $\{i, j, k\} = \{1, 2, 3\}$, then $Ac_{i,1}BC_{j,1}Q_2C_{k,1}Q_3C_{i,1}Q_1C_{j,1}A$ is a required closed trail. Otherwise, there exist $Q_a, Q_b \in Q$ such that $Q_a$ and $Q_b$ have the same endvertices. Let $Q' = Q \setminus \{Q_a, Q_b\}$, then by the induction hypothesis, there exists a closed trail $T$ in $H - E(Q_a \cup Q_b)$ such that $X \cup V(P_i) \cup V(P_j) \cup (\bigcup_{Q \in Q'} V(Q)) \subseteq V(T) \subseteq (\bigcup_{Q \in Q'} V(P_i)) \cup (\bigcup_{Q \in Q'} V(Q))$ for some $i, j$, and then $T \cup Q_a \cup Q_b$ is a required closed trail.

Claim 8. Let $P_0 = RUW$ be the path of length two in $H$ such that $R \in V(T)$ and the edge $UW$ is not dominated by $T$. Then $R = C_{i,1}$ or $C_{i,2}$ for some $i$.

Proof. Assume to the contrary that $R \neq C_{i,1}$, $C_{i,2}$ for any $i$. By Claim 3 we have $R \neq A, B$, and by Claim 4 we have $R \neq C_{i,3}$ for any $i$. Hence $R$ is an internal vertex of a path $Q \in Q$. Without loss of generality, we may assume that $Q$ joins either $C_{1,1}$ and $C_{2,1}$ or $C_{1,1}$ and $C_{2,2}$.

Let $Q^1$ be the path in $P_1 \cup Q$ which joins $R$ and $B$, and let $Q^2$ be the path in $P_2 \cup Q$ which joins $R$ and $A$. Moreover, let $\tilde{Q}^1$ (resp. $\tilde{Q}^2$) be the subpath of $Q^1$ (resp. $Q^2$) of length two which contains $R$. Then both of $\tilde{Q}^1$ and $\tilde{Q}^2$ are contained in $P_1 \cup P_2 \cup Q$. Since $UW$ is not dominated by $T$, $E(\tilde{Q}^1 \cup \tilde{Q}^2 \cup P_a)$ induces a subdivided claw. By Corollary 2 and ii), there exists a heavy edge $u^* \in \{WU, UR\} \cup l(U) \cup l(R)$. Since $R \notin X$, $u^*$ is not dominated by $X$, which contradicts Claim 5.

Claim 9. Let $v_a, v_b \in E(H)$ such that $v_a \in \{a_i\} \cup l(C_{i,1}) \cup l(A)$ and $v_b \in \{b_i\} \cup l(C_{i,p_i}) \cup l(B)$ for some $i$ with $p_i \geq 2$. Then $\{v_a, v_b\}$ is not a heavy matching.

Proof. If $\{v_a, v_b\}$ is a heavy matching, then $\{v_a, v_b, c^1, c^2\}$ is a heavy matching, where $\{j, k\} = \{1, 2, 3\}$ \setminus $\{i\}$. This is a contradiction.

Since $H$ does not have a DCT, there exists $u \in E(H)$ which is not dominated by $T$. Since $H$ is essentially 2-edge-connected, there exist two edge-disjoint paths $Q^1_u$, $Q^2_u$ each of which joins an endvertex of $u$ and a vertex in $T$. For $i = 1, 2$, take such $Q^i_u$ so that $|V(Q^i_u) \cap V(T)| = 1$, and let $R^i$ be the vertex in $V(T) \cap V(Q^i_u)$ (see Figure 4). Let $r_i$ be the edge of $Q^i_u$ which is incident with $R^i$, then we can take the edge $s_i \in E(Q^i_u) \cup \{u\}$ so that $s_i$ and $r_i$ are adjacent in $H$ and $s_i$ is not dominated by $T$. Hence it follows from Claim 3 that

$$R^i = C_{j,1} \text{ or } C_{j,2} \text{ for some } j \in \{1, 2, 3\}. \quad (2)$$
By Claim 4, $C_{j,3} \notin V(Q_u)$ for any $j$ with $p_j = 3$. Since $s_i$ is not dominated by $X$, $s_i$ is not dominated by $\bigcup_{i=1}^{3} P_i$ as well.

In the case $R^1 \neq R^2$, let $Q_u$ be the path joining $R^1$ and $R^2$ which is contained in $Q_u^1 \cup Q_u^2 \cup \{u\}$, and in the case $R^1 = R^2$, let $Q_u$ be the maximal closed trail which is contained in $Q_u^1 \cup Q_u^2 \cup \{u\}$. Moreover, let $S^i$ be the common endvertex of the two edges $r_i$ and $s_i$.

**Claim 10.** For $i = 1, 2$, there exists $r_i^* \in \{r_i\} \cup l(R^i)$ and $d_H^e(r_i^*) \geq \frac{n-2}{3} + 2$.

**Proof.** Let $i \in \{1, 2\}$. By 2, we may assume $R^i \in V(P_1)$ without loss of generality. Recall that $s_i$ is not dominated by $\bigcup_{i=1}^{3} P_i$. If $R^i = C_{1,1}$ and $p_1 = 1$, then there exists a subdivided claw induced by $\{r_i, s_i, a_1, a_2, b_1, b_3\}$. If $R^i = C_{1,1}$ and $p_1 \geq 2$, then there exists a subdivided claw induced by $\{r_i, s_i, a_1, a_2\}$ and two edges in $E(P_1) \setminus \{a_1\}$. If $R^i = C_{1,2}$, then there exists a subdivided claw induced by $\{r_i, s_i, c_{1,1}, a_1, b_1, b_2\}$ or $\{r_i, s_i, c_{1,1}, a_1, c_{1,2}, b_1\}$. In either case, the assertion follows from Corollary 2) and ii), since Claim 5 implies that each edge in $\{s_i\} \cup l(S^i)$ is not heavy. □

**Claim 11.** $R^1 = R^2$.

**Proof.** By 2, $R^1 \in \{C_{1,1}, C_{1,2}\}$ and $R^2 \in \{C_{1,1}, C_{1,2}\}$ for some $i, j \in \{1, 2, 3\}$. Recall that $V(Q_u) \cap V(T) = \{R^1, R^2\}$. Hence $Q_u$ and any path in $Q$ are internally vertex-disjoint. Suppose that $p_i = p_j = 1$. If $i \neq j$, then $Q_u \in Q_1$, which contradicts the maximality of $Q$. On the other hand, if $i = j$, then we have $R^1 = R^2 = C_{1,1}$, and hence the assertion holds. Thus it suffices to consider the case $\max\{p_i, p_j\} \geq 2$.

Without loss of generality, we may assume that $R^1 \in \{C_{1,1}, C_{1,2}\}$, $p_1 \geq 2$ and $R^2 \in \{C_{1,1}, C_{1,2}, C_{2,1}, C_{2,2}\}$ with $R^1 \neq R^2$. Take $r^*_1$ and $r^*_2$ as in Claim 10. Suppose that $r^*_1$ and $r^*_2$ are not adjacent, then, if $p_3 \geq 2$, \{r^*_1, r^*_2, c^*_3\} is a matching with $d_H^e(r^*_1) + d_H^e(r^*_2) + d_H^e(c^*_3) \geq n + 2$, and if $p_3 = 1$, \{r^*_1, r^*_2, b^*_3\} is a matching with $d_H^e(r^*_1) + d_H^e(r^*_2) + d_H^e(b^*_3) \geq n + 2$. This contradicts Lemma 6 and hence $r^*_1$ and $r^*_2$ are adjacent. Since $R^1 \neq R^2$, we have $S^1 = S^2$. If $R^2 \in \{C_{1,1}, C_{1,2}\}$, then $R^1 S^1 R^2$ is a triangle of $H$, a contradiction. Hence we may assume that $R^2 \in \{C_{2,1}, C_{2,2}\}$.

Suppose $S^1 = S^2 = D_j$ for some $j \in \{2, 3\}$ with $p_j = 1$. Then it follows from the fact $p_1 \geq 2$ that \{c_{j,1}, s_1, a_j, a_1, b_j, b_1\} induces a subdivided claw. By Corollary 2) and ii), there exist heavy edges $a^*$ and $b^*$ such that $a^* \in \{a_1, a_j\} \cup l(A) \cup l(C_{j,1})$ and $b^* \in \{b_1, b_j\} \cup l(B) \cup l(C_{j,1})$. By Claim 2 we have either $a^* \notin \{a_1\} \cup l(A)$ or $b^* \notin \{b_1\} \cup l(B)$, and hence $a^* \in \{a_j\} \cup l(C_{j,1})$ or $b^* \in \{b_j\} \cup l(C_{j,1})$. Let $c^*_j = a^*$ (resp. $b^*$) in the former (resp. latter) case, then it follows from Corollary 2) and ii) that $d_H^e(c^*_j) \geq \frac{n-2}{3} + 2$. Hence \{r^*_1, c^*_j, c^*_k\} is a matching with $d_H^e(r^*_1) + d_H^e(c^*_j) + d_H^e(c^*_k) \geq n + 2$, where $k \in \{2, 3\} \setminus \{j\}$. This contradicts Lemma 6 and hence we obtain $S^1 = S^2$ for $j = 2, 3$ with $p_j = 1$. This implies that $c^*_2$ and $r^*_1$ are not adjacent and $c^*_3$ and $r^*_1$ are not adjacent for $i = 1, 2$. 12
Let 
\[ \hat{E} = \{a_1, a_3\}, \hat{c}_{1,2} = b_1 \text{ and } \hat{C}_1 = C_{1,2} \text{ in the case } R^1 = C_{1,1} \text{ and } p_1 = 2, \]
\[ \hat{E} = \{a_1, a_3\}, \hat{c}_{1,2} = b_1' \text{ and } \hat{C}_1 = C_{1,2} \text{ in the case } R^1 = C_{1,1} \text{ and } p_1 = 3, \]
\[ \hat{E} = \{b_1, b_3\}, \hat{c}_{1,2} = a_1 \text{ and } \hat{C}_1 = C_{1,1} \text{ in the case } R^1 = C_{1,2} \text{ and } p_1 = 2 \text{ and } \]
\[ \hat{E} = \{b_1', b_1\}, \hat{c}_{1,2} = a_1 \text{ and } \hat{C}_1 = C_{1,1} \text{ in the case } R^1 = C_{1,2} \text{ and } p_1 = 3. \]

In either case, \( \{r_1, s_1, c_{1,1}, c_{1,2}\} \cup \hat{E} \) induces a subdivided claw. By Corollary 2 i) and ii), there exist a heavy edge \( c_1^{**} \in \{c_{1,2}, c_{1,1}\} \cup l(\hat{C}_1) \cup l(R^1). \) If \( c_1^{**} \in \{c_{1,2}\} \cup l(\hat{C}_1), \) then \( \{c_1^{**}, r_1^*, c_2^*, c_3^*\} \) is a heavy matching, a contradiction. Hence \( c_1^{**} \in \{c_{1,1}\} \cup l(R^1). \) By Corollary 2 ii), \( d_{R^1}(c_1^{**}) = \frac{n-2}{3} + 2. \) Then \( \{c_1^{**}, r_2^*, c_3^*\} \) is a matching with \( d_{R^1}(c_1^{**}) + d_{R^1}(r_2^*) + d_{R^1}(c_3^*) \geq n + 2, \) which contradicts Lemma 3 \( \square \)

Without loss of generality, we may assume that \( R^1 \in \{C_{1,1}, C_{1,2}\}. \) Note that Claim 10 yields

\[ V(Q_u') \cap V(T) = \{R^1\} \text{ for any path } Q_u' \text{ which joins an endvertex of } u \text{ and a vertex in } T. \]  

(3)

Take \( r_i^* \) as in Claim 10 Since \( Q_u \) is a closed trail and \( H \) is a triangle-free simple graph, \( |E(Q_u)| \geq 4. \) Hence we can take \( r_3 \in E(Q) \) so that \( r_3 \) and \( r_i^* \) are not adjacent. By 3, \( r_i \) and \( c_j^* \) are not adjacent for each \( i \in \{1, 3\} \) and \( j \in \{2, 3\}. \) This implies that \( \{r_1^*, c_2^*, c_3^*\} \) is a matching in \( H. \)

Again by 3, for each \( i \in \{2, 3\}, \) neither of the two endvertices of \( c_i^* \) is adjacent to \( S_1. \) Moreover, since \( H \) is triangle-free, \( R^1 \) is adjacent to at most one of the endvertices of \( c_i^*. \) Hence

\[ |N_{H}^{\pm}(r_i^*) \cap N_{H}^{\pm}(c_i^*)| \leq 1 \text{ for } i = 2, 3. \]  

(4)

Let \( \gamma = \sum_{\{e_1, e_2\} \subseteq \{r_i^*, c_i^*\}} |N_{H}^{\pm}(e_1) \cap N_{H}^{\pm}(e_2)|. \) Since \( r_3 \notin N_{H}^{\pm}(r_i^*) \cup N_{H}^{\pm}(c_2^*) \cup N_{H}^{\pm}(c_3^*), \)

\[ n = |E(H)| \geq |N_{H}^{\pm}(r_i^*)| + |N_{H}^{\pm}(c_2^*)| + |N_{H}^{\pm}(c_3^*)| = |\{r_1^*, c_2^*, c_3^*\}| - \gamma + |\{r_3\}| \geq \frac{n-2}{3} + 2 + \frac{n-2}{3} + \frac{n-2}{3} + 3 - \gamma + 1 = n + 4 - \gamma, \]

and hence \( \gamma \geq 4. \) Since \( H \) is triangle-free, \( |N_{H}^{\pm}(e_1) \cap N_{H}^{\pm}(e_2)| \leq 2 \) for every pair of non-adjacent edges \( e_1, e_2 \in E(H). \) Hence by 4, we have \( |N_{H}^{\pm}(c_2^*) \cap N_{H}^{\pm}(c_3^*)| = 2, |N_{H}^{\pm}(r_1^*) \cap N_{H}^{\pm}(c_i^*)| = 1 \text{ for } i = 2, 3, \) and \( E(H) = N_{H}^{\pm}(r_1^*) \cup N_{H}^{\pm}(c_2^*) \cup N_{H}^{\pm}(c_3^*) \cup \{r_3\}. \) This yields \( p_1 = 1, \) since otherwise \( a_1 \) or \( b_1 \) is not contained in \( N_{H}^{\pm}(r_1^*) \cup N_{H}^{\pm}(c_2^*) \cup N_{H}^{\pm}(c_3^*) \cup \{r_3\}. \)

Let \( i \in \{2, 3\}. \) If \( p_i = 3, \) then \( b_1 \) is not contained in \( N_{H}^{\pm}(r_1^*) \cup N_{H}^{\pm}(c_2^*) \cup N_{H}^{\pm}(c_3^*) \cup \{r_3\}, \) a contradiction. Hence \( p_i \leq 2. \) Recall that neither of the endvertices of \( c_i^* \) is adjacent to \( S_1. \) Since \( |N_{H}^{\pm}(r_i^*) \cap N_{H}^{\pm}(c_i^*)| = 1, \) \( R^1 \) has a neighbor in an endvertex of \( c_i^*. \) On the other hand, \( p_1 = 1 \) yields \( R^1 = C_{1,1}, \) and since \( H \) is triangle-free, \( C_{1,1}C_{1,1}C_{1,1}C_{1,2} \notin E(H). \) Thus we can deduce that \( p_i = 1, c_i^* = c_i, \) and \( C_{1,1}D_i \in E(H) \) for \( i = 2, 3. \)

Let \( Q' \) be the path \( C_{2,1}D_2C_{1,1}D_3C_{1,3}, \) then \( T' = P_1 \cup P_2 \cup P_3 \cup Q_u \cup Q' - \{a_2, b_3\} \) is a closed trail passing through \( c_2^*, c_3^* \) and \( r_3. \) Moreover, \( T' \) contains \( V(r_1). \) Since \( E(H) = N_{H}^{\pm}(r_1^*) \cup N_{H}^{\pm}(c_2^*) \cup N_{H}^{\pm}(c_3^*) \cup \{r_3\} \) and \( r_1^* \in \{r_1\} \cup l(R_1), \) \( T' \) is a DCT of \( H, \) a contradiction. This completes the proof of Case 1.

Case 2. \( c_i^* \in \{a_i\} \cup l(A) \) for some \( i \) and \( c_j^* \in \{b_j\} \cup l(B) \) for some \( j. \)

Recall that, by Corollary 2 i) and ii), \( d_{R^1}(c_i^*), d_{R^1}(c_j^*) \geq \frac{n-2}{3} + 2. \)

Subcase 2.1. \( c_k^* \in \{c_k, k\} \cup l(C_{k, p_k}) \) or \( c_k^* \in \{b_k^*\} \cup l(C_{k, p_k}) \) holds for some \( k \in \{1, 2, 3\} \) \( \setminus \{i, j\}. \)
Without loss of generality, we may assume that $c_k^* \in \{c_{k,1}\} \cup l(C_{k,1})$. If $i \neq j$, then $\{c_i^*, c_j^*, c_k^*\}$ is a heavy matching with $d_H^i(c_i^*) + d_H^j(c_j^*) + d_H^k(c_k^*) \geq 2 \left( \frac{n-2}{3} + 2 \right) + \frac{n-2}{3} = n + 2$, which contradicts Lemma 6. Hence we have $i = j$. Without loss of generality, we may assume that $i = 1$ and $k = 3$. Then we can deduce that $c_3^* \in \{c_{3,1}\} \cup l(C_{3,1})$, for otherwise Corollary 2 ii) implies $d_H^i(c_2^*) \geq \frac{n-2}{3} + 2$, and so $\{c_1^*, c_2^*, c_3^*\}$ is a heavy matching with $d_H^i(c_1^*) + d_H^j(c_2^*) + d_H^k(c_3^*) \geq n + 2$. Moreover, we have $p_1 = 1$, for otherwise $\{c_1^*, c_2^*, c_3^*\}$ is a heavy matching with $d_H^i(c_1^*) + d_H^j(c_2^*) + d_H^k(c_3^*) \geq n + 2$.

Let $\Xi = H[\bigcup_{1 \leq i \leq 3} V(c_i^*)]$, $E_0 = E(H - V(\Xi))$ and $\Gamma = \bigcup_{1 \leq i \leq 3} (N_H(c_i^*) \cap N_H(c_j^*))$. Then $n = |E(H)| \geq |N_H(c_1^*)| + |N_H(c_2^*)| + |N_H(c_3^*)| + |\{c_1^*, c_2^*, c_3^*\}| - |\Gamma| + |E_0| \geq \frac{n-2}{3} + 2 + \frac{n-2}{3} + \frac{n-2}{3} + 3 - |\Gamma| + |E_0| = n + 3 - |\Gamma| + |E_0|$, thus

$$|\Gamma| \geq |E_0| + 3.$$ \hfill (5)

Since $H$ is triangle-free, we have $|\Gamma| \leq 6$, and hence we have $|E_0| \leq 3$. If $\Xi$ is collapsible, then we obtain a DCT of $H$ by Corollary 1 a contradiction. Hence $\Xi$ is not collapsible. By Lemma 5 and the fact that $H$ is triangle-free, we have $|\Gamma| \leq 4$. Hence it follows from 6 that $|E_0| \leq 1$.

Let $C_i = D_i$ in the case $p_i = 1$ and $C_i = C_{i,2}$ in the case $p_i \geq 2$. Since $c_2^* \in \{a_1\} \cup l(A)$ and $c_i^* \in \{c_{i,1}\} \cup l(C_{i,1})$ for $i = 2, 3$, $\Gamma \subseteq E(H[V(a_1) \cup V(c_{2,1}) \cup V(c_{3,1})])$. Thus each edge of $\Gamma$ joins two vertices of $\{A, C_{1,1}, C_{2,1}, C_{3,1}, C_{3,1}\}$. On the other hand, since $H$ is triangle-free, $AC_2, AC_3, C_{1,1}C_{2,1}, C_{2,1}C_{3,1}, C_{3,1}C_{3,1} \notin E(H)$. Hence, for every $e \in \Gamma \setminus \{AC_{2,1}, AC_{3,1}\}$, $e = C_iC_{j,1}$ or $C_iC_{j,2}$ with $i \neq j, j$. \hfill Claim 12. $p_i \leq 2$ for $i = 2, 3$.

**Proof.** Assume not. By symmetry, we may assume that $p_3 \geq 3$. Then the fact $|E_0| \leq 1$ yields $p_2 \leq 2$, $p_3 = 3$ and $E_0 = \{b_3\}$.

Assume that $C_{1,1}C_{3,2} \in E(H)$. Then $\{C_{1,1}C_{3,2}, c_{1,1}, c_{3,2}, a_3, c_{3,2}, b_3\}$ induces a subdivided claw, and hence there exists a heavy edge $b_3^* \in \{b_3, c_{3,2}\} \cup l(C_{3,2}) \cup l(C_{3,2})$ by Corollary 2 i) and ii). If $b_3^* \in \{b_3\} \cup l(C_{3,2})$, then $\{c_1^*, c_2^*, c_3^*, b_3^*\}$ is a heavy matching of size 4, a contradiction. Moreover, if $b_3^* \in \{c_{3,2}\} \cup l(C_{3,2})$, then since Corollary 2 ii) yields $d_H^i(b_3^*) \geq \frac{n-2}{3} + 2$, $\{c_1^*, c_2^*, b_3^*\}$ is a heavy matching with $d_H^i(c_1^*) + d_H^j(c_2^*) + d_H^k(b_3^*) = n + 2$, which contradicts Lemma 6. Hence $C_{1,1}C_{3,2} \notin E(H)$.

If $C_2C_{i,1} \in E(H)$ for $i = 1$ or 3, then $T = AC_{2,1}C_2C_{i,1} \cup (P_i - a_i) \cup P_j$, where $j \in \{1, 2, 3\} \setminus \{i, 2\}$, is a closed trail containing all the vertices in $\{A, C_{1,1}, C_{2,1}, C_{2,1}, C_{3,1}, C_{3,2}, B\} = V(a_1) \cup V(c_{2,1}) \cup V(c_{3,1}) \cup B$. Since $b_3$ is dominated by $B$, $T$ is a DCT of $H$, a contradiction. Hence $C_{2,1}C_{i,1} \notin E(H)$ for $i = 1$ and 3.

By 6, we have $|\Gamma| = 4$. Since $C_{1,1}C_{3,2}, C_2C_{3,1}, C_2C_{3,1} \notin E(H), \Gamma = \{AC_{2,1}, AC_{3,1}, C_{2,1}C_{3,2}, C_{2,1}C_{3,2}\}$.

Then $C_{2,1}C_{2,3,2}$ is a triangle, a contradiction. \hfill \Box

By Claim 12, we obtain $X = V(P_1 \cup P_2 \cup P_3)$.

**Claim 13.** There exists a closed trail $T$ of $H$ such that $X \subseteq V(T) \subseteq X \cup \{D_i \mid p_i = 1\}$.

**Proof.** Since 5 yields $|\Gamma| \geq 3$, there exists an edge $e \in \Gamma \setminus \{AC_{2,1}, AC_{3,1}\}$. If $C_iC_{j,1} \in E(H)$ for some $i \neq j$, then $AC_iC_jC_{j,1} \cup (P_j - a_j) \cup P_i$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$, is a required closed trail. Hence we may assume that $e = C_2C_3$. Then by Claim 12 either $p_2 = 1$ or $p_3 = 1$ holds since $H$ is triangle-free.
Without loss of generality, we may assume that $p_2 = 1$. Then $AC_{3,1}C_3C_{2,1}B \cup P_1$ is a required closed trail.

\[ \square \]

**Claim 14.** If $u \in E(H - X) \setminus \{D_2D_3\}$, then $u$ is not heavy.

**Proof.** Assume to the contrary that $u$ is heavy. Since $u \in E(H - X)$, $u$ and $c'_2$ are not adjacent in $H$. Moreover, if $u$ and $c'_2$ are adjacent in $H$ for $i = 2$ or $3$, then $p_i = 1$ and $D_i$ is the common endvertex of $u$ and $c'_2$. Since $u \neq D_2D_3$, either $c'_2$ or $c'_3$ is not adjacent to $u$. If $u$ is adjacent to neither $c'_2$ nor $c'_3$, then $\{c'_1, c'_2, c'_3, u\}$ is a heavy matching of size 4, a contradiction. Hence, without loss of generality, we may assume that $u$ is adjacent to $c'_2$ but not to $c'_3$. Then $p_2 = 1$ and $D_2$ is the common endvertex of $u$ and $c'_2$.

Since $\{a_1, c_{1,1}, a_2, b_2, a_3, c_{3,1}\}$ induces a subdivided claw in $H$, by Corollary 2(i) and ii), there exists a heavy edge $b'_2 \in \{a_2, b_2\} \cup \{C_{2,1}, l\}(A)$. If $b'_2 \in \{b_2\} \cup \{C_{2,1}, l\}(A)$, then $\{c'_1, b'_2, c'_3, u\}$ is a heavy matching of size 4, and if $b'_2 \in \{a_2\} \cup \{A\}$, then $\{c'_1, b'_2, c'_3, u\}$ is a heavy matching of size 4, a contradiction. \[ \square \]

It follows from Claim 13 that $E(H - X) \neq \emptyset$. Moreover, in the case $p_2 = p_3 = 1$ and $D_2D_3 \in E(G)$, we can deduce that $E(H - X) \setminus \{D_2D_3\} \neq \emptyset$, since otherwise $AC_{2,1}D_2D_3C_{3,1}B_{C_{1,1}A}$ is a DCT of $H$. Since $H$ is connected, we can take $u \in E(H - X) \setminus \{D_2D_3\}$ so that an endvertex $S$ of $u$ is adjacent to a vertex $R \in X$ (possibly $S = D_i$ for some $i$; in this case let $R = C_{i,1}$). Let $S'$ be the other endvertex of $u$ and let $r = SR$.

We shall prove that there exist two paths $\Lambda_1, \Lambda_2$ of length two such that $\{u, r\} \cup E(\Lambda_1) \cup E(\Lambda_2)$ induces a subdivided claw. If $p_i \geq 2$ for some $i$, then $R$ is contained in a cycle of length at least 5 in $P_1 \cup P_2 \cup P_3$. Since $S, S' \notin X$, we can find $\Lambda_1$ and $\Lambda_2$ in this cycle. Hence we consider the case where $p_i = 1$ for each $i$. If $R = C_{i,1}$ for some $i$, then we can find $\Lambda_1$ and $\Lambda_2$ from $P_1 \cup P_2 \cup P_3 - \{a_j, b_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$. If $R = A$ or $B$, then we can take $j, k$ so that $S' \neq D_j, D_k$, and then $\Lambda_1 = RC_{j,1}D_j$ and $\Lambda_2 = RC_{k,1}D_k$ are the desired paths.

Since $\{u, r\} \cup E(\Lambda_1) \cup E(\Lambda_2)$ induces a subdivided claw, by Corollary 2(i) and ii), there exists a heavy edge $u^* \in \{u, r\} \cup \{l(S) \cup \{l(R)\}$. By Claim 14 we obtain $u^* \in \{r\} \cup l(R)$, and hence $d_H^*(u^*) \geq \frac{4 - 2}{2} + 2$. If $R = A$ (resp. $B$), then $\{u^*, c'_1, c'_2, c'_3\}$ (resp. $\{u^*, c'_2, c'_3, c'_4\}$) is a heavy matching of size 4, a contradiction. If $R \in V(P_i) \setminus \{A, B\}$ for $i = 2$ or $3$, then $S \neq D_j$ follows from the choice of $R$, where $j \in \{2, 3\} \setminus \{i\}$. Hence $\{u^*, c'_1, c'_3\}$ is a matching with $d_H^*(u^*) + d_H^*(c'_3) + d_H^*(c'_2) \geq n + 2$, which contradicts Lemma 5. Therefore we have $R = C_{i,1}$. Note that, by the above argument, we can deduce that

\[ V(Q_u) \cap X = \{C_{i,1}\} \]

for any path $Q_u$ which joins an endvertex of $u$ and a vertex in $X$. (6)

Since $H$ is essentially 2-edge-connected, we can take two edge-disjoint paths $Q_u^1$ and $Q_u^2$ each of which joins $C_{i,1}$ and an endvertex of $u$. By (6), we obtain $V(Q_u^i) \cap V(\Xi) = \{C_{i,1}\}$ for $i = 1, 2$. Since $H$ is triangle-free simple graph, we can find two edges in $E(Q_u^1) \cup E(Q_u^2) \cup \{u\}$ which are not dominated by any vertex in $V(\Xi)$. Hence $|E_0| \geq 2$, a contradiction.

**Subcase 2.2.** $c_{h}^i \in \{a_k\} \cup \{l(A)\}$ and $c_{h}^k \in \{b_k\} \cup \{l(B)\}$ holds for any $k \in \{1, 2, 3\} \setminus \{i, j\}$.

**Claim 15.** $c_{h}^i \in \{a_k\} \cup \{l(A)\}$ and $c_{h}^k \in \{b_k\} \cup \{l(B)\}$ holds for any $h \in \{1, 2, 3\}$.
Proof. Recall that \( c^*_i \in \{a_i\} \cup l(A) \) for some \( i \) and \( c^*_j \in \{b_j\} \cup l(B) \) for some \( j \) by the assumption of Case 2. If \( i = j \), then the claim follows from the assumption of Subcase 2.2. Hence we assume \( i \neq j \). By Corollary 2(ii) and ii), \( d^*_H(c^*_k) \geq \frac{n+2}{3} \) for \( k \in \{1, 2, 3\} \setminus \{i, j\} \). If \( c^*_i \in \{b'_i\} \cup l(C_{i,p}) \) or \( c^*_j \in \{c_{j,1}\} \cup l(C_{j,1}) \), then in the former case \( \{c^*_1, c^*_j, c^*_k\} \) is a heavy matching with \( d^*_H(c^*_i) + d^*_H(c^*_j) + d^*_H(c^*_k) \geq n + 2 \), and in the latter case \( \{c^*_1, c^*_j, c^*_k\} \) is a heavy matching with \( d^*_H(c^*_i) + d^*_H(c^*_j) + d^*_H(c^*_k) \geq n + 2 \). This contradicts Lemma 8 and thus \( c^*_i \in \{b_i\} \cup l(B) \) and \( c^*_j \in \{a_j\} \cup l(A) \). Hence the claim holds. \( \Box \)

Now we turn our attention to the graphs \( G \) and \( cl(G) \). For \( U \in V(H) \), let \( E_U = \{e \in E(H) \mid e \) is incident with \( U\} \). Then \( E_U \subset V(G) \) and \( E_U \) induces a clique in \( cl(G) \).

Let \( I_A = \{c^*_1, c^*_2, c^*_3\} \) and \( I_B = \{\bar{c}^*_1, \bar{c}^*_2, \bar{c}^*_3\} \). By Lemma 11 Corollary 2, ii) and Claim 15 there exist induced nets \( N_A \) and \( N_B \) of \( G \) such that the vertices in \( I_A \) are the endvertices of \( N_A \) and the vertices in \( I_B \) are the endvertices of \( N_B \). Hence \( d_G(v) \geq \frac{n-2}{3} \) for each \( v \in I_A \cup I_B \). By Claim 15 \( I_A \subseteq E_A \) and \( I_B \subseteq E_B \). Note that, since there is no eligible vertex in \( cl(G) \),

\[
|N_{cl(G)}(y) \cap V(Z)| \leq 1 \text{ if } Z \text{ is a clique of } cl(G) \text{ and } y \notin V(Z). \tag{7}
\]

If there exists \( z \in E_A \cap E_B \), then \( z \) is an edge of \( H \) joining \( A \) and \( B \). Since \( c^*_i \in \{a_i\} \cup l(A) \) and \( \bar{c}^*_i \in \{b_i\} \cup l(B) \), we obtain \( v \notin E_A \cap E_B \) for every \( v \in I_A \cup I_B \).

Assume that there exists \( v \in V(G) \) such that \( |N_G(v) \cap (I_A \cup I_B)| \geq 3 \). Since \( G \) is claw-free and both of \( I_A \) and \( I_B \) are independent sets in \( G \), \( |N_G(v) \cap I_A|, |N_G(v) \cap I_B| \leq 2 \). Hence, without loss of generality, we may assume that \( |N_G(v) \cap I_A| = 2 \). Then \( |N_G(v) \cap I_B| \geq 1 \). Since \( E(G) \subseteq E(cl(G)) \), it follows from (7) that \( v \in E_A \). Again by the claw-freeness of \( G \), there exists \( v_A \in N_G(v) \cap I_A \) and \( v_B \in N_G(v) \cap I_B \) such that \( v_A v_B \in E(G) \). Then \( v_A, v \in N_{cl(G)}(v_B) \). Since \( v_A, v \in E_A \), \( 7 \) yields \( v_B \in E_A \), which contradicts the fact that \( v_B \notin E_A \cap E_B \). Therefore we have \( |N_G(v) \cap (I_A \cup I_B)| \leq 2 \) for each \( v \in V(G) \). Furthermore, \( 11 \) yields \( |N_G(v) \cap I_B| \leq 1 \) for each \( v \in I_A \) and \( |N_G(v) \cap I_A| \leq 1 \) for each \( v \in I_B \), and hence \( |N_G(v) \cap (I_A \cup I_B)| \leq 1 \) for each \( v \in I_A \cup I_B \). Therefore

\[
\sum_{v \in I_A \cup I_B} d_G(v) = \sum_{v \in V(G)} |N_G(v) \cap (I_A \cup I_B)| \leq 1 \times 6 + 2 \times (n - 6) = 2n - 6,
\]

which contradicts the fact that \( d_G(v) \geq \frac{n-2}{3} \) for each \( v \in I_A \cup I_B \). \( \Box \)

Theorem 10. Conjecture 7 is true for graphs with at least 33 vertices.

Proof. Let \( G \) be a graph of order at least 33 which satisfies the assumption of Conjecture 11 and let \( H \) be the triangle-free graph such that \( L(H) = cl(G) \). Assume that there exists a heavy matching of size 4, say \( \{e_1, e_2, e_3, e_4\} \), in \( H \). Since \( H \) is triangle-free, \( |N_H^c(e_i) \cap N_H^c(e_j)| \leq 2 \) for each \( i \neq j \). Then

\[
|E(H)| \geq \sum_{i=1}^{4} d_H^c(e_i) + |\{e_1, e_2, e_3, e_4\}| - \sum_{1 \leq i < j \leq 4} |N_H^c(e_i) \cap N_H^c(e_j)|
\geq 4 \cdot \frac{|E(H)| - 2}{3} + 4 - 6 \times 2 = \frac{4|E(H)| - 32}{3},
\]

which yields \( |V(G)| = |E(H)| \leq 32 \), a contradiction. Hence there exists no heavy matching of size 4 in \( H \). Therefore, by Theorem 9 there exists a DCT of \( H \), and by Theorems 4 and 9 \( G \) is hamiltonian. \( \Box \)
5 Sketch of the proof for the small graphs

In this section we provide a sketch of the proof of Conjecture \( \text{4} \) for graphs of order at most 32. For a detailed proof, we refer the readers to \( \text{3} \).

Let \( H \) be the triangle-free graph such that \( L(H) = cl(G) \). By Theorems \( \text{4} \) and \( \text{8} \) it suffices to prove that \( H \) has a DCT.

First consider the case \( n \geq 15 \). By Theorem \( \text{9} \) we may assume that there exists a heavy matching \( M \) of size 4 in \( H \). Let \( \Xi^* = H[V(M)] \) and \( E_0 = E(H - V(\Xi^*)) \). Since \( \Xi^* \) is triangle-free, we obtain \(|E(\Xi^*)| \leq 16\). Moreover, since \( n = \sum_{c \in M} d^*_H(c) + |M| - |E(\Xi^*)| \leq n + 4 \cdot \frac{|M|}{2} + 4 - (|E(\Xi^*)| - 4) + |E_0| \), we have

\[
|E_0| \leq |E(\Xi^*)| - \frac{n + 16}{3}. \tag{8}
\]

Since \( H \) is essentially 2-edge-connected, we have \(|E(H) \setminus (E(\Xi^*) \cup E_0)| \geq 2\) if \( E_0 \neq \emptyset \) (consider two edge-disjoint paths joining an edge of \( E_0 \) and \( \Xi^* \)). This implies

\[
|E_0| \leq \max\{0, n - |E(\Xi^*)| - 2\}. \tag{9}
\]

If \(|E(\Xi^*)| \geq 12\), then by examining all the possible cases (note that \( \Xi^* \) may not be bipartite in the case \(|E(\Xi^*)| = 12\), we can deduce that either \( \Xi^* \) is collapsible or there exists a vertex \( x \) of degree one in \( \Xi^* \). In the former case, since \( \Xi \) and \( \Xi' \) yield \(|E_0| \leq 3\), we obtain a DCT of \( H \) by Corollary \( \text{4} \). In the latter case, we have \(|E(\Xi^*)| \leq 13\), which yield \(|E_0| \leq 2\). Moreover, since \(|E(\Xi^*)| \geq 12\), \( \Xi^* - \{x\} \) is collapsible. Then by the similar argument as in Corollary \( \text{4} \) we obtain a DCT of \( H \). Thus we assume \(|E(\Xi^*)| \leq 11\).

Then \( \Xi \) and the fact \( n \geq 15 \) yield \(|E(\Xi^*)| = 11\) and \( E_0 = \emptyset \), and hence it suffices to find a spanning closed trail of \( \Xi^* \). If \( \Xi^* \) is bipartite and \( \Xi^* \) has no spanning closed trail, there must exist a vertex \( x \) with \( d_{\Xi^*}(x) = 1 \) and \( \Xi' \subseteq \Xi^* - \{x\} \) which is isomorphic to \( K_{3,3} \) or \( K_{3,3}^{-} \). Then we can find a DCT of \( H/\Xi' \) by using the fact that \( E_0 = \emptyset \), which yields a DCT of \( H \). If \( \Xi^* \) is non-bipartite, then since \( H \) is triangle-free and \(|E(\Xi^*)| = 11\), \( \Xi^* \) contains an induced cycle \( C \) of length 5. Let \( W = \Xi^* - V(C) \), then we have \(|E(W)| \leq 2\) and the number of edges between \( W \) and \( C \) is \( 6 - |E(W)| \). By enumerating all the possible structure of \( \Xi^* \) and by examining each case carefully, we can deduce that either \( \Xi^* \) has a spanning closed trail, \( H \) has a DCT or \( \Xi^* \) is isomorphic to the graph which is induced by the black vertices in Figure \( \text{5} \) without using Corollary \( \text{2} \). In the latter case, since \( \Xi^* - \{w_i\} \) has a spanning closed trail for each \( i = 1, 2 \), we may assume that each \( w_i \) has a neighbor \( x_i \) in \( H - V(\Xi^*) \). Then, for each \( i \), we obtain a subdivided claw containing \( x_i w_i u_i \). By Corollary \( \text{2} \) and \( \text{2} \), we obtain an edge joining a white vertex and a black vertex in Figure \( \text{5} \) which yields a DCT of \( H \).

Next consider the case \( n \leq 14 \). Assume that \( H \) does not have a DCT, and take a closed trail \( T \subseteq H \) so that \( T \) dominates as many edges as possible. Then we can take a component \( S \) of \( H - V(T) \) containing an edge of \( E(H - V(T)) \). Let \( X = \{x_1, x_2, \ldots, x_k\} \) be the set of the vertices of \( T \) which have a neighbor in \( S \), where \( x_1, x_2, \ldots \) appear in this order along \( T \), and let \( x_{k+1} = x_1 \). Let \( U = V(H) \setminus (V(T) \cup V(S)) \), let \( T_i \) be the segment of \( T \) between \( x_i \) and \( x_{i+1} \) and let \( P_i \) be a path of \( H \) joining \( x_i \) and \( x_{i+1} \) whose internal vertices are contained in \( S \). Moreover, let \( F_i \) be the set of edges in \( H \) joining a vertex in \( T_i - \{x_i, x_{i+1}\} \) and a vertex in \( T_i \cup U \) and let \( S_i \) be the set of edges in \( H \) which has at least one
endvertex in \( P_i - \{x_i, x_{i+1}\} \). In the case where \( T \) is a cycle (that is, each vertex appears exactly once on \( T \)), then \( F_1, \ldots, F_k, S_i \) are edge-disjoint for each \( i \). Moreover, by the maximality of the number of edges that \( T \) dominates, we have \( |F_i| \geq |S_i| \) for each \( i \). Since \( S \) is a non-trivial component, \( |S_i| \geq 3 \) for each \( i \), and hence

\[
14 \geq n \geq \sum_{i=1}^{k} |F_i| + |S_i| \geq 3k + 3, \tag{10}
\]

which yields \( k \leq 3 \). Note that, in the last inequality, each of \( |F_i| \) and \( |S_i| \) is estimated at 3, and the set of edges joining \( X \) and \( U \) is estimated to be empty. We derive a contradiction by showing that \( n \geq 15 \).

In both cases \( k = 2 \) and \( 3 \), we can find a subdivided claw with center \( x_i \) containing two edges of \( S_i \), two edges of \( F_i \) and two edges of \( F_{i+1} \) for each \( i \). By Corollary [2] i) and ii) and close examination of \( |F_i| \) and \( |S_i| \), we obtain many edges which is not counted in the last inequality of (10) enough to show that \( n \geq 15 \). The case where \( T \) is not a cycle is basically similar to the above. By observing the structure of \( H \) thoroughly, we can find an induced net with center \( x_i \) for some \( i \). Then by Corollary [2] i) and ii) we obtain \( n \geq 15 \).

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