POINTS OF ORDER TWO ON THETA DIVISORS

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Abstract. We give a bound on the number of points of order two on the theta divisor of a principally polarized abelian variety $A$. When $A$ is the Jacobian of a curve $C$ the result can be applied in estimating the number of effective square roots of a fixed line bundle on $C$.

Introduction

In this paper we give an upper bound on the number of 2-torsion points lying on a theta divisor of a principally polarized abelian variety. Given any principally polarized abelian variety $A$ of dimension $g$ and symmetric theta divisor $\Theta \subset A$, $\Theta$ contains at least $2^{2g-1}(2^g - 1)$ points of order two, the odd theta characteristics. Moreover, in [Mum66] and [Igu72, Chapter IV, Section 5] it is proved that $\Theta$ cannot contain all points of order two on $A$.

In this work we use the projective representation of the theta group to prove the following:

Given a principally polarized abelian variety $A$, any translated $t_a^*\Theta$ of a theta divisor $\Theta \subset A$ contains at most $2^{2g} - 2^g$ points of order 2 ($2^{2g} - (g + 1)2^g$ if $t_a^*\Theta$ is irreducible and not symmetric).

Our bound is far from being sharp and we conjecture that the right estimate should be $2^{2g} - 3^g$ as in the case of a product of elliptic curves.

When $A$ is the Jacobian of a curve $C$ the result can be applied in estimating the number of effective square roots of a fixed line bundle on $C$ (cf. Section 2).

1. Main result

In this section we prove our main result.

Theorem 1.1. Let $A$ be a principally polarized abelian variety of dimension $g$ and let $\Theta$ be a symmetric theta divisor.

1. For each $a \in A$ there are at most $2^{2g} - 2^g$ points of order two lying on $t_a^*\Theta$.
2. Let $a \in A$ and assume that $\Theta$ is irreducible and $t_a^*\Theta$ is not symmetric with respect to the origin. Then there are at most $2^{2g} - (g + 1)2^g$ points of order two lying on $t_a^*\Theta$.

Proof. Denote by $(K, \langle \cdot, \cdot \rangle)$ the group of 2-torsion points on $A$ with the perfect pairing induced by the polarization. Let

\[ \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \]

be a basis of $K$ over the field of order two such that

\[ \langle a_i, b_j \rangle = \delta_{ij}, \quad \langle a_i, a_j \rangle = 0, \quad \langle b_i, b_j \rangle = 0, \]

Date: February 8, 2012

2010 Mathematics Subject Classification. 14K25.

This work has been partially supported by 1) FAR 2010 (PV) "Varietà algebriche, calcolo algebrico, grafi orientati e topologici" 2) INdAM (GNSAGA) 3) PRIN 2009 "Moduli, strutture geometriche e loro applicazioni".

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and let

\[ H := \langle a_1, \ldots, a_g \rangle \]

be the subgroup of \( K \) generated by the elements \( a_1, \ldots, a_g \). Consider the projective morphism \( \varphi : A \to \mathbb{P}^{2^g-1} \) associated to the divisor \( 2\Theta \). By the construction of the projective representation of the theta group \( K(2\Theta) \) (see [Mum66], Chapter II, Section 6, Corollary 4) and [Kem89], we know that the elements of \( \varphi(H) \) are a basis of the projective space. In the same way, the images of the elements of a coset \( H_b \) of \( H \) in \( K \) generate the projective space \( \mathbb{P}^{2^g-1} \).

Suppose by contradiction that there exists a subset \( S \subset K \) such that all points of \( S \) lie on \( t^{*}_a \Theta \) and \( |S| > 2^{2g} - 2^g \). By the previous argument, since \( H_b \subset S \) for some \( b \), the points of \( \varphi(S) \) generate the entire projective space \( \mathbb{P}^{2^g-1} \). On the other hand, by the Theorem of the Square (see [Mum68] Chapter II, Section 6, Corollary 4),

\[ t^{*}_a \Theta + t^{*}_{-a} \Theta \equiv 2\Theta. \]

It follows that the points of \( \varphi(S) \) lie on an hyperplane of \( \mathbb{P}^{2^g-1} \). This proves (1).

Now we prove the second part. Suppose by contradiction that there exists a subset \( S \subset K \) such that all points of \( S \) lie on \( t^{*}_a \Theta \) and \( |S| > 2^{2g} - (g + 1)2^g \). We claim that

\[ \text{the points in } \varphi(S) \text{ lie on a } 2^g - g - 2 \text{-plane in } \mathbb{P}^{2^g-1}. \]

Given a point \( \varepsilon \in S \), it holds also \( \varepsilon \in t^{*}_a \Theta \). Thus \( S \subset t^{*}_a \Theta \cap t^{*}_{-a} \Theta \). If \( t^{*}_a \Theta \) is not symmetric and irreducible, \( t^{*}_a \Theta \cap t^{*}_{-a} \Theta \) has codimension 2 in \( A \) and we can consider the natural exact sequence

\[ 0 \to \mathcal{O}_A(-2\Theta) \to \mathcal{O}_A(-t^{*}_{-a} \Theta) \oplus \mathcal{O}_A(-t^{*}_a \Theta) \to I_{t^{*}_a \Theta \cap t^{*}_{-a} \Theta} \to 0; \]

by tensoring it with \( \mathcal{O}_A(2\Theta) \) we get

\[ 0 \to \mathcal{O}_A \to \mathcal{O}_A(t^{*}_a \Theta) \oplus \mathcal{O}_A(t^{*}_{-a} \Theta) \to I_{t^{*}_a \Theta \cap t^{*}_{-a} \Theta} \otimes \mathcal{O}_A(2\Theta) \to 0. \]

Passing to the corresponding sequence on the global sections, we have

\[ 0 \to H^0(A, \mathcal{O}_A) \to H^0(A, \mathcal{O}_A(t^{*}_a \Theta)) \oplus H^0(A, \mathcal{O}_A(t^{*}_{-a} \Theta)) \to H^0(I_{t^{*}_a \Theta \cap t^{*}_{-a} \Theta} \otimes \mathcal{O}_A(2\Theta)) \to H^1(A, \mathcal{O}_A(2\Theta)) \to 0, \]

since, by the Kodaira vanishing theorem (see e.g. [GH94] Chapter 1, Section 2]),

\[ H^1(A, \mathcal{O}_A(t^{*}_a \Theta)) = H^1(A, \mathcal{O}_A(t^{*}_{-a} \Theta)) = 0. \]

It follows that

\[ \dim H^0(I_{t^{*}_a \Theta \cap t^{*}_{-a} \Theta} \otimes \mathcal{O}_A(2\Theta)) \geq g + 1. \]

Thus the points in \( \varphi(t^{*}_a \Theta \cap t^{*}_{-a} \Theta) \) lie on a \( 2^g - g - 2 \)-plane of \( \mathbb{P}^{2^g-1} \) and the claim (1) is proved.

To conclude the proof of (2) we notice that if \( |S| > 2^{2g} - (g+1)2^g \) then \( |S \cap H_b| > 2^g - (g + 1) \) for some coset \( H_b \) of \( H \) (see (1)). Then it follows that \( \varphi(S) \) contains at least \( 2^g - g \) independent points and we get a contradiction. \( \square \)

Remark 1.2. One might expect the right bound to be \( 2^{2g} - 3^g \) and that this is realized only in the case of a product of elliptic curves.

Remark 1.3. The argument of Theorem 1.1 can be also used to obtain a bound on the number of \( n \)-torsion points (with \( n > 2 \)) lying on a theta divisor.
2. Applications

In this section we apply Theorem [1.1] to the case of Jacobians. This gives a generalization of [MP, Proposition 2.5].

**Proposition 2.1.** Let \( C \) be a curve of genus \( g \) and \( M \) be a line bundle of degree \( d \leq g - 1 \). Given an integer \( k \leq g - 1 - d \), for each \( L \in \text{Pic}^{2k}(C) \) there are at least \( 2^g \) line bundles \( \eta \in \text{Pic}^k(C) \) such that \( \eta^2 \simeq L \) and \( h^0(\eta \otimes M) = 0 \).

**Proof.** We prove the statement for \( M = \mathcal{O}_C \) and \( k = g - 1 \). The general case follows from this by replacing \( L \) with \( M^2 \otimes L \otimes \mathcal{O}_C(p)^{2n} \), where \( p \) is an arbitrary point of \( C \) and \( n := g - 1 - k - d \). Denote by \( \Theta \) the divisor of effective line bundles of degree \( g - 1 \) in \( \text{Pic}^{g-1}(C) \). Given the morphism

\[
m_2: \text{Pic}^{g-1}(C) \to \text{Pic}^{2g-2}(C)
\eta \mapsto \eta^2,
\]

we want to prove that \( |m_2^{-1}(L) \cap \Theta| \leq 2^{2g} - 2^g \). Let \( \alpha \in m_2^{-1}(L) \), we have

\[
m_2^{-1}(L) = \{ \alpha \otimes \sigma \text{ s.t. } \sigma^2 = \mathcal{O}_C \}.
\]

If \( |m_2^{-1}(L) \cap \Theta| > 2^{2g} - 2^g \), then there are more than \( 2^{2g} - 2^g \) points of order two lying on a translated of a symmetric theta divisor of \( J(C) \) and, by [11] of Theorem [11.1] we get a contradiction. \( \square \)

**Remark 2.2.** If we apply Proposition [2.1] to \( M = \mathcal{O}_C, L = \omega_C \), we get that on a curve of genus \( g \) there are at most \( 2^{2g} - 2^g \) effective theta characteristics. We notice that when \( g = 2 \) they are the 6 line bundles of type \( \mathcal{O}_C(p) \) where \( p \) is a Weierstrass point. When \( g = 3 \) and \( C \) is not hyperelliptic, they correspond to the 28 bi-tangent lines to the canonical curve.

**Corollary 2.3.** Let \( C \) be a curve of genus \( g \) and \( M_1, \ldots, M_N \) be a finite number of line bundles of degree \( d \leq g - 1 \). Given an integer \( k \leq g - 1 - d \), if \( \eta \) is a generic line bundle of degree \( k \) such that \( h^0(\eta^2) > 0 \), then

\[
h^0(\eta \otimes M_i) = 0 \quad \forall i = 1, \ldots, N.
\]

**Proof.** Let

\[
\Lambda := \left\{ \eta \in \text{Pic}^k(C) : h^0(\eta^2) > 0 \right\},
\]

and, for each \( i = 1, \ldots, N \), consider its closed subset

\[
\Lambda_i := \left\{ \eta \in \Lambda : h^0(M_i \otimes \eta) > 0 \right\}.
\]

We remark that \( \Lambda \) is a connected \( 2^{2g} \)-étale covering of the image of the \( 2k \)-th symmetric product of \( C \) in \( \text{Pic}^{2k}(C) \). By Proposition [2.1] for each effective \( L \in \text{Pic}^{2k}(C) \) there exists \( \eta \in \Lambda \setminus \Lambda_i \) such that \( \eta^2 \simeq L \). It follows that \( \Lambda_i \) is a proper subset of \( \Lambda \). Since \( \Lambda \) is irreducible, also the set

\[
\bigcup_{i=1}^N \Lambda_i = \left\{ \eta \in \text{Pic}^k(C) : h^0(M_i \otimes \eta) > 0 \text{ for some } i \right\}
\]

is a proper closed subset of \( \Lambda \). \( \square \)

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