An implicit matrix-free hydrodynamic model for river and urban drainage system

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Abstract. An implicit matrix-free numerical model for river and urban drainage system is presented in this paper. The four-point implicit Preissmann scheme is used to discretise the governing equations of free surface-pressurised flow. To avoid solving a large sparse matrix, the junction-point water stage prediction and correction (SPC) method after some improvements, which was initially proposed for free-surface flow, is extended to the simulation of pressurised flow and the flow where free surface and pressurised flows coexist. A total of four test cases, namely, an extensive analytical benchmark case, a tree network system case, a looped network system case and a free-surface flow case with from real world, are applied to validate the proposed model and to demonstrate its performance in comparison with the analytical solution, published results and measured data. The proposed model has proven accurate for a number of test cases involving free-surface and pressurised flows.

1 Introduction

One-dimensional (1D) hydrodynamic models are widely accepted for application in unsteady flows in open-channel and urban drainage networks [1-7]. Given their considerable efficiency and good stability, implicit finite difference schemes, particularly the Preissmann scheme, have been widely used in 1D models in the past two decades [8-10]. When an implicit scheme is applied, attention will inevitably turn to the problem of solving large sparse nonlinear systems if a direct method that simultaneously solves the solution of the entire network is adopted, which means that $2(M-1)$ equations will be generated for a river system having $M$ sections. Three-phase algorithm (TSA) was used to reduce the matrix order in several 1D hydrodynamic models. The main idea of TSA is to use end node-state variables to denote the state variables of each section, and by first solving the junction variables and then solving cross-section variables to reduce the number of equations that need to be solved at once. However, for a large system, such as a city-scale drainage network, the number of junction equations can still be very large. In addition, instability may occur if the initial conditions are inappropriate or the time step and space step are not carefully chosen [11].

A junction-point water stage prediction and correction method (SPC) was introduced [12, 13], which is matrix free and suitable for looped and tree networks, to address the above problems. One of the major advantages of SPC is that establishing and solving the global branch or junction matrix are
unnecessary, thereby leading to a reduction of storage requirement and simulation time. SPC was initially proposed for open channel flows, and it couldn’t take full advantage of the method because the matrix of a natural river system is often not great and can be dealt with easily by using TSA. If the method proves equally applicable to pressurised flows, then implicit difference schemes will be able to handle a large-scale urban drainage network where free-surface flows and pressurised flows co-exist while avoiding solving the large matrix.

This study develops a matrix-free model for both open channel flow and pressurised flow by using the SPC method. The paper is organised as follows. First, the 1D governing equations for open channel and pressurised flows and the details of numerical schemes are presented. Then the proposed model is validated against four numerical test problems. The final section summarizes and concludes the work.

2 Method

2.1. Governing equations

Free-surface flows can be described by the Saint-Venant equations, which are given by

\[ \frac{\partial Z}{\partial t} + \frac{1}{B} \frac{\partial Q}{\partial x} = 0 \]  \hspace{1cm} (1)

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial Z}{\partial x} + gS_f = 0 \]  \hspace{1cm} (2)

Where \( Q \) is the discharge, \( A \) is the wet cross-section area, \( Z \) is the water level, \( B \) is the width of free water surface, \( g \) is the gravity acceleration with a value of 9.81 m/s\(^2\) and \( S_f \) is the friction slope.

Through a proper transformation, the same set of governing equations can be used for both free-surface and pressurised flows if we use water level \( Z \) instead of water head \( H \) and introduce a fictitious Preissmann slot. As shown in Figure 1, assume that a slot is present in the top of the closed conduit with a width of \( B \), and ignore the extra area and wetted perimeter from the slot. The width of the slot is given by

\[ B = \frac{gA}{a^2} \]  \hspace{1cm} (3)

Where \( a \) is the wave speed.

![Figure 1 Schematic of Preissmann slot](image)

In practice, the spurious numerical oscillations may be induced while the flow is switching from open channel to pressurized flow, and the method of increasing the slot width is adopted to suppress the numerical oscillations in the present model. The slot width is assumed to be 1.0% of the maximum width of a conduit under surcharging conditions.

2.2. Numerical schemes
2.2.1. Discretisation of the governing equations
Preissmann’s four-point implicit difference scheme [14] was applied to discretise the Saint-Venant equations in the individual channels. On the basis of the discretised equations, a recursive method is then adopted, and the water level and flow discharge at each section of a branch can be denoted by using the discharge and water level at the river network junctions (two ends of the branch)

\[
\begin{align*}
Q_j &= \alpha_j^e + \beta_j^e Z_j + \varsigma_j^e Z^e \\
Q_j &= \alpha_j^d + \beta_j^d Z_j + \varsigma_j^d Z^d \\
Z_j &= \frac{\alpha_j^e - \alpha_j^d + \varsigma_j^e Z^e - \varsigma_j^d Z^d}{\beta_j^d - \beta_j^e}
\end{align*}
\]

(4)

Where \( Q_j \) and \( Z_j \) is the discharge and water level of section \( j \) respectively, \( Z^e \) and \( Z^d \) are the water level of upstream and downstream junction and \( \alpha, \beta \) and \( \varsigma \) are coefficients that can be obtained by the solutions of the previous time step.

According to the law of conservation of energy and mass, the hydraulic conditions at river junctions can be expressed as

\[
\sum_{r=1}^{m} Q_r = \frac{(Z_{i+1} - Z_i)}{\Delta t} A_{node}
\]

(6)

Where \( \Delta t \) is the time step, \( n \) is the index of time level, \( Q_r \) is the inflow or outflow of the \( r \)th branch around the junction, \( Z_i \) is the water level of the \( r \)th branch around the junction, \( m \) is the number of the branches around the junction, \( Z_i \) is the water level of junction \( i \) and \( A_{node} \) is the area of the junction.

Based on the junction balance equations and boundary conditions, the water level of each junction can be solved by using a variety of methods including traditional method of solving the simultaneous equations and the SPC method adopted in this study, and then the section state variables of branches can be updated after solving the water levels of all junctions.

2.2.2. SPC method
The SPC method computes each individual branch independently within one round of iteration [12], and the basic steps of the SPC method are as follows:

(1) The net inflow of the junction is computed. If the net inflow is less than a specific value (10^{-6} is adopted in this study), then the iteration terminates. The net inflow that considers the water storage capacity of the junction is given by

\[
\delta Q = \sum_{r=1}^{m} Q_r = \frac{(Z^k - Z^{k-1})}{\gamma} A_{node}
\]

(7)

Where \( Q_r \) is the discharge of the \( r \)th branch around the junction after \( k \) iterations, \( \delta Q \) is the net inflow of the junction and \( Z^k \) is the water level of the junction after \( k \) iterations.

(2) The revised water level of the junction can be obtained according to the net inflow of the junction and the current water level

\[
Z^{k+1} = Z^k + \frac{\delta Q}{\gamma}
\]

(8)

Where \( A_r \) is the iterative parameter, which is given by

\[
A_r = \alpha_r \left( \sqrt{gA_r B_r} - \frac{Q_r B_r}{A_r} \right) (1) d
\]

Where \( d \) is the direction factor with a value of 1 for the outflow branch and 2 for the inflow branch, \( \alpha_r \) is a stability coefficient and \( B_r \) is the water width.
The value of water width \( B_r \) in Equation 9 is very small based on the assumption of Preissmann slot for the pressurised flows, thereby leading to a high probability of model instability. Two methods were applied in an attempt to solve this problem in this study; one method is to provide a minimum value for the iterative parameter \( A_i \) directly, and the other method is to estimate the width \( B \) on the basis of the maximum width of the section. The latter performs better in numerical tests and comparison. Therefore, the water width in Equation 9 for the pressurised flows was obtained by using the latter method, which is given by

\[
B_r = \beta W_r,
\]  
(10)

Where \( \beta_r \) is a positive width factor less than 1.0 and \( W_r \) is the maximum width of the section.

It should be noted that the positive width factor \( \beta_r \) in Equation 10 is only used for the calculation of the iterative parameter in Equation 9. The ratio that used to determine the width of Preissmann slot for the purpose of suppressing the numerical oscillations can be different from the positive width factor.

(3) Back substitution is used to solve the discharge and water level of branches by using the last revised water level of each junction, and the first step is reentered.

SPC method is proposed for subcritical flows, and not valid for supercritical flows. To overcome this limitation, the supercritical flow is treated as subcritical flow by reducing the influence of the convective momentum term in the momentum equation. The convective momentum term can be expanded to two parts:

\[
\frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) = Q \frac{\partial u}{\partial x} + u \frac{\partial Q}{\partial x}
\]  
(11)

A reduction coefficient based on Froude number is then introduced to gradually reduce the first part of the convective momentum term [10, 15].

3 Application and Discussion
In this section, the capability of the model was validated by four extensive test case applications, namely, a steady pressurised flow case, a tree networks system case, a looped network system case and a natural river flow case.

3.1 Steady pressurised flow case
The steady pressurised flow is a classic problem with exact solutions, and this test aims to investigate the ability of the present model to handle the pressurised flow and converge to a steady state. A looped network system shown in Figure 2(c) was used for the test and the basic characteristics of each conduits are given in Table 1. The inflows of Nodes 1 to 6 are 220, −60, −40, −30, −50 and −40 L/s, respectively.

Under the given conditions, the hydraulic state of the conduits and nodes will tend to be stable after a period of time. All conduits were pressurised flow after the results stabilised, and the analytical solution can be solved by Hardy-Cross method [16, 17]. To determine whether the steady state was reached, the relative discharge difference \( E_q \) was used as a criterion. It was considered that the steady state was achieved once the \( E_q \) of all conduits were less than \( 10^{-6} \), and the \( E_q \) is given by

\[
E_q = \left| \frac{Q_e - Q_s}{Q_e} \right|
\]  
(12)

Where \( Q_s \) and \( Q_e \) are the discharges of the two ends of the conduit.

| Conduit | Length (m) | Manning’s coefficient | Diameter (m) | Elevation difference (m) |
|---------|------------|-----------------------|--------------|-------------------------|


A space distance of 50 m was used to discrete the conduits in the simulation while the time step was 1 s. Table 2 shows the comparison of numerical discharge of conduits and head difference of two ends of conduits with analytical solutions. As shown in Table 2, the relative errors of discharge and head difference are controlled within 1.0%, thereby indicating that the proposed model performs well in handling pressurised flow with a satisfactory accuracy.

| Conduit | Discharge (L/s) | Head difference (m) |
|---------|----------------|---------------------|
|         | Analytical     | Numerical           | Relative error (%) | Analytical | Numerical | Relative error (%) |
| 1       | 131.53         | 131.52              | -0.01              | 34.10      | 34.04     | -0.18          |
| 2       | 46.75          | 46.75               | 0.00               | 65.66      | 65.58     | -0.11          |
| 3       | 6.76           | 6.75                | -0.15              | 3.97       | 3.97      | -0.09          |
| 4       | 24.77          | 24.77               | 0.00               | 53.40      | 53.34     | -0.11          |
| 5       | 88.47          | 88.48               | 0.01               | 16.89      | 16.88     | -0.07          |
| 6       | 48.47          | 48.48               | 0.02               | 70.56      | 70.50     | -0.08          |
| 7       | 23.24          | 23.25               | 0.04               | 16.23      | 16.21     | -0.11          |

To assess the effects of width factor on the stability and efficiency of the present model in simulating pressurised flows, various width factors (0.001, 0.005, 0.01, 0.02 and 0.05) were used in this test. The model cannot converge when the 0.001 was adopted, and the number of convergence steps were 2039, 1933, 2571 and 5577, respectively, thereby indicating that a larger width factor is conducive to model stability but requires additional computation effort. A similar conclusion for the free-surface flows was obtained by Zhu et al [12], that is, a larger stability coefficient can improve the
stability of the model, but it will reduce efficiency.

3.2. Tree network system case

The tree network case, which was first reported by Akan and Yen [18], is widely used as a benchmark test case to assess the ability and accuracy of the 1D model in simulating the flows in a tree network [19, 20]. As shown in Figure 2(a), the network is formed by six branches and seven nodes, which constitutes a typical tree network system. The branches are numbered and the characteristics are listed in Table 3. All sections of these conduits are open rectangular, and the flow in the entire network system is open channel flow. Thus, the ability to simulate the free surface flow can also be assessed by this test case.

The initial condition is a steady state that corresponds to a discharge of 3.0 m³/s in Conduits 1 and 4, 2 m³/s in Conduits 2 and 3, and a constant water level of 0.6 m in Conduit 6. In the above given conditions, after each pipeline hydraulic conditions stabilised, the depth of the pipe sections was 0.6 m. Under the above given initial conditions, the depth of each conduit was 0.6 m after the steady state was reached. The boundary conditions were specified at inlets of four upstream conduits, as illustrated in Figure 3(a).

Figure 3(b) to Figure 3(e) show the historical change processes of the discharge at the outlet of Conduits 1, 2, 4 and 6, and the results of Akan and Yen [18] are also plotted in Figure 3 for comparison with this study’s model. The numerical results computed by this model are similar to those of the model of Akan and Yen [18], thereby indicating that the model can effectively deal with the tree network system. The result near the peak of Conduit 6 is a little lower than the result computed by Akan and Yen [18]. A similar phenomenon occurred in simulations by other models [19, 20].
Table 3 Conduit characteristics of the tree network

| Conduit | Length (m) | Slope (%) | Section width (m) | Manning’s coefficient |
|---------|------------|-----------|-------------------|----------------------|
| 1       | 600        | 0.05      | 5                 | 0.0138               |
| 2       | 600        | 0.05      | 5                 | 0.0207               |
| 3       | 600        | 0.05      | 5                 | 0.0207               |
| 4       | 600        | 0.05      | 5                 | 0.0138               |
| 5       | 600        | 0.10      | 8                 | 0.0141               |
| 6       | 600        | 0.10      | 10                | 0.0125               |

3.3. Looped network system case

This test aims to investigate the performance of the present model in the case of looped networks. The trial network system reported by Ji [21] was used in this test. The network, shown in Figure 2(b), was formed by six branches and six nodes, in which Conduits 2, 4 and 6 form a closed loop. The characteristics of all conduits are listed in Table 4. As shown in Figure 4(a) and 4(b), Nodes 1 and 3 were given the flow boundary condition while Nodes 4 and 6 were given the water level boundary condition, and no base flow was imposed on the system. The transition from free surface to
pressurised flow occurred in part of conduits under the given boundary conditions. Thus, the case can also verify the ability of the model to deal with the free-surface-pressurised flow.

Table 4 Conduit characteristics of the looped network

| Conduit | Length (m) | Slope (%) | Diameter (m) | Manning’s coefficient | Elevation of start point (m) |
|---------|------------|-----------|--------------|----------------------|----------------------------|
| 1       | 300        | 0.1000    | 0.8          | 0.01429              | 10.7                       |
| 2       | 300        | 0.1300    | 0.5          | 0.01429              | 10.4                       |
| 3       | 300        | 0.1700    | 0.5          | 0.01429              | 10.0                       |
| 4       | 500        | 0.0600    | 0.5          | 0.01429              | 10.4                       |
| 5       | 410        | 0.2000    | 0.5          | 0.01429              | 10.1                       |
| 6       | 310        | 0.0325    | 0.5          | 0.01429              | 10.1                       |

Figure 4 Boundary conditions and computed results: (a) & (b) boundary conditions at four nodes; (c) & (d) Computed discharge hydrographs of Conduit 3 and Conduit 5, respectively

Figure 4(c) and Figure 4(d) show the simulated discharge process of Conduits 3 and 5, and the results computed by the model of Ji [21] are also plotted in the figure for comparison. As seen in Figure 4(c) and Figure 4(d), because of the rising stage at the downstream node, the discharges of Conduits 3 and 5 are negative and are known as reversal flow. The simulated results showed good agreement with the results computed by the model of Ji [21], thereby indicating the results of the present model are reliable. Overall, the proposed model showed good performance in handling the flow in the looped network system and the phenomenon of reversal and free-surface-pressurised flows.

3.4 Natural river case

Bei River, which is located in northern Guangdong, is the northern tributary of the Pearl River in southern China. The main stream of Bei River is 573 km long with a water collection area of 52068 km², accounting for 10.3% of the total area of the Pearl River basin. The reach of Bei River from Feilaixia to Shijiao section is the key reach for flood control of downstream Bei River and is an
important part of peripheral flood control project in Guangzhou City, capital of Guangdong Province. Bei River was chosen as this study’s actual case for its irregular topography to help verify the proposed model. The downstream region of Bei River has numerous rivers, five of which were selected to establish a river network model; these rivers are Bei River, Pa River, Dayan River, Bin River and Zheng River. As shown in Figure 5, three hydrological stations with long-time observation data are located in the upper, middle and lower reaches, and they can provide boundary conditions for the model and to verify the model results.

The observed discharge and stage hydrograph at Feilaixia Station and Shijiao Station were adopted as the upper and lower boundary of Bei River respectively, whereas the inflow boundaries, which were calculated by the Xinanjiang hydrological model [22] based on observed rainfall data and water collection area, were adopted for all other branches, including Bin River and Pa River. The proportion of these two branches is small relative to the total flow of Bei River. A fixed time step of 30 s was adopted in this case.

Two measured floods labelled with 20050620 and 20080610 were calculated by the model, and the observed stage hydrograph at Qinyuan Station were used to verify the accuracy of the model. Figure 6(a) and Figure 6(b) illustrate the observed discharge hydrographs of the selected two floods at the Feilaixia Station, and Figure 6(c) and Figure 6(d) illustrate the observed stage hydrographs at Shijiao Station. A comparison of the observed water levels at the Qinyuan Station with the numerical results is shown in Figure 6(e) and Figure 6(f). The overall agreement between the numerical results of the present model and the observations shown in Figure 6(e) and Figure 6(f) was satisfactory. Several small discrepancies existed between the numerical results and the measurements. These discrepancies may be attributed to the measuring uncertainties and the difference between the section data and the actual terrain when the floods occurred.
Figure 6 Measured and simulated hydrographs: (a) & (b) Measured discharge hydrographs at Feilaixia Station; (c) & (d) Measured water level hydrographs at Shijiao Station; (e) & (f) simulated water levels at Qinyuan Station

4 Conclusions

In this paper, a matrix-free implicit numerical model for river and urban drainage system was presented and analysed. The following conclusions can be drawn:

The junction-point water stage prediction and correction method was applied to the implicit 1D model to avoid solving the large sparse matrix. Results showed that the method is applicable to pressurised flow and free-surface-pressurised flow, thereby implying good prospects in large-scale drainage networks.

Four examples were used to test the effectiveness and feasibility of the proposed model, and they showed the good performance and the capability of the model to accurately simulate free-surface flows and pressurised flows and to handle complex flows over irregular terrains.

The original method of computing iterative parameters is not applicable to pressurised flows in SPC. The width factor is introduced to address this problem in this study. Numerical tests showed that the width factor had a significant influence on model efficiency and stability, and the model becomes unstable if a too-small width factor is chosen, whereas a larger width factor is conducive to model
stability but requires additional computation effort.

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References
[1] SANDERS B F. High-resolution and non-oscillatory solution of the St. Venant equations in non-rectangular and non-prismatic channels[J]. Journal of Hydraulic Research, 2010, 39(3): 321-330.
[2] DELENNE C, FINAUD-GUYOT P, GUINOT V, et al. Sensitivity of the 1D shallow water equations with source terms: Solution method for discontinuous flows[J]. International Journal for Numerical Methods in Fluids, 2011, 67(8): 981-1003.
[3] MASOOD M, TAKEUCHI K. Assessment of flood hazard, vulnerability and risk of mid-eastern Dhaka using DEM and 1D hydrodynamic model[J]. Natural Hazards, 2012, 61(2): 757-770.
[4] ROELVINK J A, CROSATO A, WRIGHT N G, et al. Water resource assessment along the Blue Nile River, north Africa with a one-dimensional model[J]. Proceedings of the Institution of Civil Engineers - Water Management, 2014, 167(7): 394-413.
[5] THAKUR P K, AGGARWAL S, AGGARWAL S P, et al. One-dimensional hydrodynamic modeling of GLOF and impact on hydropower projects in Dhauliganga River using remote sensing and GIS applications[J]. Natural Hazards, 2016, 83(2): 1057-1075.
[6] LEANDRO J, MARTINS R. A methodology for linking 2D overland flow models with the sewer network model SWMM 5.1 based on dynamic link libraries[J]. Water Science and Technology, 2016, 73(12): 3017-3026.
[7] PARK I H, LEE J Y, LEE J H, et al. Evaluation of the causes of inundation in a repeatedly flooded zone in the city of Cheongju, Korea, using a 1D/2D model[J]. Water Science and Technology, 2014, 69(11): 2175-2183.
[8] SART C, BAUME J, MALATERRE P, et al. Adaptation of Preissmann's scheme for transcritical open channel flows[J]. Journal of Hydraulic Research, 2010, 48(4): 428-440.
[9] CHEN Y, WANG Z, LIU Z, et al. 1D–2D Coupled Numerical Model for Shallow-Water Flows[J]. Journal of Hydraulic Engineering, 2011, 138(2): 122-132.
[10] YU H, HUANG G. A coupled 1D and 2D hydrodynamic model for free-surface flows[J]. Proceedings of the Institution of Civil Engineers - Water Management, 2014, 167(9): 523-531.
[11] LV M Y, JIANG W, ZHAN J M. Study of treatment for linking conditions at junction points of river networks[J]. Yellow River, 2007, 29(3): 31-32.
[12] ZHU D J, CHEN Y C, WANG Z Y, et al. Simple, Robust, and Efficient Algorithm for Gradually Varied Subcritical Flow Simulation in General Channel Networks[J]. Journal of Hydraulic Engineering, 2011, 137(7): 766-774.
[13] ZHU D, CHEN Y, LIU Z. One-dimensional hydrodynamic-water quality model for large complex river networks[J]. Journal of Hydroelectric Engineering, 2012, 31(3): 83-87.
[14] PREISSMANN A. Propagation des intumescences dans les canaux et rivieres[A]. Proceedings of 1st Congress of the French Association for Computation[C]. Grenoble, France,1961: 433-442.
[15] KUTIJA V. On the numerical modelling of supercritical flow[J]. Journal of Hydraulic Research, 1993, 31(6): 841-858.
[16] CROSS H. Analysis of flow in networks of conduits or conductors[J]. Revista Odonto Ciência, 1936, 25(1): 42-47.
[17] LOPES, A. M G. Implementation of the Hardy-Cross method for the solution of piping networks[J]. Computer Applications in Engineering Education, 2004, 12(2): 117-125.
[18] AKAN A O, YEN B C. Diffusion-Wave Flood Routing in Channel Networks[J].
Hydraulics Division, 1981, 107(6): 719-732.

[19] NOTO L, TUCCIARELLI T. DORA algorithm for network flow models with improved stability and convergence properties[J]. Journal of Hydraulic Engineering, 2001, 127(5): 380-391.

[20] SHE Y, MAO Z. Flow Simulation of Urban Sewer Networks[J]. Tsinghua Science & Technology, 2003, 8(6): 719-725.

[21] JI Z. General hydrodynamic model for sewer/channel network systems[J]. Journal of hydraulic engineering, 1998, 124(3): 307-315.

[22] ZHAO R J, ZHANG Y L, FANG L R, et al. The Xinanjiang model [C]; proceedings of the Hydrological Forecasting Proceedings Oxford Symposium, Oxford, UK, F, 1980. IASH.