On properties of theories which preclude the existence of universal models

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Abstract

In this paper we investigate some properties of first order theories which prevent them from having universal models under certain cardinal arithmetic assumptions. Our results give a new syntactical condition, oak property, which is a sufficient condition for a theory not to have universal models in cardinality $\lambda$ when certain cardinal arithmetic assumptions implying the failure of $GCH$ (and close to the failure of $SCH$) hold.
0 Introduction

The existence of a universal model of a theory has been the object of continuous interest to specialists in various disciplines of mathematics, see for example [ArBe], [FuKo]. We approach this problem from the point of view of model theory, more specifically, classification theory, and we concentrate on first order theories. In a series of papers, Kojman-Shelah [KjSh 409] (see there also for earlier references), [KjSh 447], Kojman [Kj], Shelah [Sh 457], [Sh 500], Džamonja-Shelah [DjSh 614], the thesis claiming the connection between the complexity of a theory and its amenability to the existence of universal models, has been pursued. As it follows from the classical results in model theory (see [ChKe]) that if \( GCH \) holds, then every countable first order theory admits a universal model in every uncountable cardinal, the question we need to ask is what happens when \( GCH \) fails. It is usually “easy” to force a situation in which there are no universal models (by adding Cohen subsets), however assuming that \( GCH \) fails and allowing ourselves a vague use of the words “many” and “often”, we can distinguish between those theories which for many cardinals do not have a universal model in that cardinal whenever \( GCH \) fails, and those for which it is possible to construct a model of set theory in which \( GCH \) fails, yet our theory has a universal model in the cardinality under consideration. This division would suggest that the theories first described, let us call them for the sake of this introduction amenable, are of higher complexity than the latter ones.

In his paper [Sh 500], S. Shelah introduced a hierarchy of complexity for first order theories, and showed that past a certain level on that hierarchy, the inherent properties of any theory on that level, will preclude the exis-
tence of universal models in most cardinalities. The details of this hierarchy are described in the following Definition 0.5, and what S. Shelah proved in [Sh 500], is that SOP$_4$ implies high non-amenability. Here we show that this bound is not sharp, by defining a property of theories which is present in some NSOP$_4$ theories (meaning, not SOP$_4$), yet it precludes the existence of a universal model under certain cardinal arithmetic. This property is called the oak property, as its prototype is the model completion of Th($M_{\lambda,\kappa}$), a theory connected to that of the tree $\alpha \geq \lambda$ (for details see Example 1.3). The oak property cannot be made a part of the SOP$_n$ hierarchy, as we exhibit a theory which has oak, and is NSOP$_3$, while the model completion of the theory of triangle free graphs is an example of a SOP$_3$ theory which does not satisfy the oak property. Our research is a continuation of section §1 of [Sh 457], where the universality spectrum of the theory $T_{\text{eq}}^*$ of infinitely many indexed independent equivalence relations is investigated, and it is proved that under cardinals arithmetic assumptions like the ones in our Theorem 2.1, $T_{\text{eq}}^*$ does not have universal models. We show that $T_{\text{eq}}^*$ has the oak property, and in fact exhibit a close connection between $T_{\text{eq}}^*$ and Th($M_{\lambda,\kappa}$).

We commence by giving some background notions which will be used in the main sections of the paper. First, several classical definitions of model theory.

**Convention 0.1** A theory in this paper means a first order complete theory, unless otherwise stated. Such an object is usually denoted by $T$.

**Notation 0.2** Given a theory $T$, we let $\mathfrak{C} = \mathfrak{C}_T$ stand for “the monster model”, i.e. a saturated enough model of $T$. As is usual, we assume without loss of generality that all our discussion takes place inside some such model, so all expressions to the extent “there is”, “exists” and “|=” are to be relativised to this model, all models are $\prec$ $\mathfrak{C}$, and all subsets of $\mathfrak{C}$ we mention have size less than the saturation number of $\mathfrak{C}$. We let $\bar{\kappa} = \bar{\kappa}(\mathfrak{C}_T)$ be the size of $\mathfrak{C}$, so this cardinal is larger than any other cardinal mentioned in connection with $T$.

**Definition 0.3** (1) The tuple $\bar{b}$ is defined by $\varphi(\bar{x}; \bar{a})$ if $\varphi(\mathfrak{C}; \bar{a}) = \{\bar{b}\}$, i.e. if
$\bar{b}$ is the unique $\bar{x}$ which realizes $\varphi(\bar{x}; \bar{a})$. It is defined by the type $p$ if $\bar{b}$ is the unique tuple which realizes $p$. It is definable over $A$ if $\text{tp}(\bar{b}, A)$ defines it.

(2) The formula $\varphi(\bar{x}; \bar{a})$ is algebraic if $\varphi(C; \bar{a})$ is finite. The type $p$ is algebraic if it is realized by finitely many tuples only. The tuple $\bar{b}$ is algebraic over $A$ if $\text{tp}(\bar{b}, A)$ is.

(3) The definable closure of $A$ is

\[
dcl(A) \overset{\text{def}}{=} \{b : b \text{ is definable over } A\}.
\]

(4) The algebraic closure of $A$ is

\[
acl(A) \overset{\text{def}}{=} \{b : b \text{ is algebraic over } A\}.
\]

(5) If $A = acl(A)$, we say that $A$ is algebraically closed. When $dcl(A)$ and $acl(A)$ coincide, then $cl(A)$ denotes their common value.

**Definition 0.4** (1) For a theory $T$ and a cardinal $\lambda$, models $\{M_i : i < i^*\}$ of $T$, each of size $\lambda$, are jointly universal iff for every $N$ a model of $T$ of size $\lambda$, there is an $i < i^*$ and an isomorphic embedding of $N$ into $M_i$.

(2) For $T$ and $\lambda$ as above,

\[
\text{univ}(T, \lambda) \overset{\text{def}}{=} \min \{|\mathcal{F}| : \mathcal{F} \text{ is a family of jointly universal models of } T \text{ of size } \lambda\}.
\]

(5) If $A = acl(A)$, we say that $A$ is algebraically closed. When $dcl(A)$ and $acl(A)$ coincide, then $cl(A)$ denotes their common value.

The following is the main definition of S. Shelah’s [Sh 500].

**Definition 0.5** (Shelah, [Sh 500]) Let $n \geq 3$.

(1) A formula $\varphi(\bar{x}, \bar{y})$ is said to exemplify the $n$-strong order property, $SOP_n$, if $lg(\bar{x}) = lg(\bar{y})$, and there are $\bar{a}_k$ for $k < \omega$, each of length $lg(\bar{x})$ such that

(a) $\models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$,

(b) $\models \neg(\exists \bar{x}_0, \ldots, \bar{x}_{n-1})[\bigwedge\{\varphi(\bar{x}_l, \bar{x}_k) : l, k < n \text{ and } k = l+1 \mod n\}]$. 4
$T$ has SOP$_n$ if there is a formula $\varphi(\bar{x}, \bar{y})$ exemplifying this.

(2) SOP$_{\leq n}$ is defined similarly, except that in (b) we replace “$n$” by each “$m \leq n$”.

(3) NSOP$_n$ stands for the negation of SOP$_n$.

Note 0.6 Using a compactness argument and Ramsey theorem, one can prove that if $T$ is a theory with SOP$_n$ and $\varphi(\bar{x}, \bar{y})$, and $\langle \bar{a}_n : n < \omega \rangle$ exemplify it, without loss of generality $\langle \bar{a}_n : n < \omega \rangle$ is an indiscernible sequence. See [Sh -c], or [GrIoLe] for examples of such arguments.

Example 0.7 The model completion of the theory of triangle-free graphs is a prototypical example of a SOP$_3$ theory, with the formula $\varphi(x, y)$ just stating that $x$ and $y$ are connected. It can be shown that this theory is NSOP$_4$, see [Sh 500].

The following fact indicates that SOP$_n(3 \leq n < \omega)$ form a hierarchy, and the thesis is that this hierarchy is reflected in the complexity of the behavior of the relevant theories under natural constructions in model theory.

Fact 0.8 (Shelah, [Sh 500], §2) SOP$_{n+1} \implies$ SOP$_n$.

1 An NSOP$_3$ theory without universals

Definition 1.1 (1) Let $T_0$ be the following theory in the language

\{Q_0, Q_1, Q_2, F_0, F_1, F_2, F_3\}:

(i) $Q_0, Q_1, Q_2$ are unary predicates which form a partition of the universe,

(ii) $F_0$ is a partial function from $Q_1$ to $Q_0$,

(iii) $F_1$ is a partial two-place function from $Q_2 \times Q_0$ to $Q_1$.

(iv) $F_2$ is a partial function from $Q_0$ to $Q_2$. 

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(v) $F_3$ is a partial function from $Q_2$ to $Q_0$,

(vi) $F_0(F_1(z, x)) = x$ for all $(z, x) \in \text{Dom}(F_1)$, and

(vii) $F_3(F_2(x)) = x$ for all $x$.

(2) Let $T_0^+$ be like $T_0$, but with the requirement that $F_0, F_1, F_2$ and $F_3$ are total functions.

Remark 1.2 It is to be noted that the above definition of $T_0$ uses partial, rather than the more usual, full function symbols. Using partial functions, we have to be careful when we speak about submodels, where we have a choice of deciding whether statements of the form “$F_l(x)$ is undefined” are preserved in the larger model. We choose to request that the fact that $F_l$ is undefined at a certain entry, is not necessarily preserved in the larger model. Functions $F_2$ and $F_3$ are “dummies” whose sole purpose is to assure that models of $T_0^+$ are non-trivial, while keeping $T_0^+$ a universal theory (which is useful when discussing the model completion). Also note that neither $T_0$ nor $T_0^+$ is complete, but every model $M$ of $T_0$ in which $Q_0^M, Q_2^M \neq \emptyset$ and $F_0$ and $F_3$ are onto, can be extended to a model of $T_0^+$ with the same universe (Claim [1.4] (2)), and every model of $T_0$ is a submodel of a model of $T_0^+$ (Claim [1.4] (4)). $T_0^+$ has a complete model completion (Claim [1.5]). This model completion is the main theory we shall work with and, as we shall show, it has the oak property (Claim [1.11]) and is $NSOP_4$ (Claim [1.7]).

Example 1.3 An example which we take as the prototype of a model of $T_0^+$, is a model $M = M_{\lambda, \kappa}$ obtained when for given infinite cardinals $\kappa, \lambda$, we take $Q_0^M$ to be $\kappa$, $Q_1^M$ to be $^{\kappa^+}\lambda$, and $Q_2^M = ^{\kappa}\lambda$. We let $F_0(\eta)$ be the length of $\eta$ for $\eta \in Q_1$, and let $F_1(\nu, \alpha) = \nu \upharpoonright \alpha$. Let $F_3$ be any surjective function from $Q_2^M$ onto $Q_0^M$, and for $\alpha < \kappa$ let $F_2(\alpha) = \nu_\alpha$ for any $\nu_\alpha$ such that $F_3(\nu_\alpha) = \alpha$.

Claim 1.4 (1) If $M$ is a model of $T_0^+$, then $Q_0^M, Q_1^M$ and $Q_2^M$ are all non-empty, and $F_0^M$ and $F_3^M$ are onto.
(2) Every model $M$ of $T_0$ in which $Q_0^M \neq \emptyset$ and $Q_2^M \neq \emptyset$, while $F_0$ and $F_3$ are onto, can be extended to a model of $T_0^+$ with the same universe (and every model of $T_0^+$ is a model of $T_0$).

(3) There are models $M$ of $T_0$ with $Q_0^M \neq \emptyset$ and $Q_2^M \neq \emptyset$ and $F_3^M$ onto, which cannot be extended to a model of $T_0^+$ with the same universe.

(4) Every model of $T_0$ is a submodel of a model of $T_0^+$.

(5) $T_0^+$ has the amalgamation property and the joint amalgamation property $JEFP$.

(6) If $M \models T_0$ and $A \subseteq M$ is finite, then the closure of $B$ of $A$ under $F_0$, $F_1$, $F_2$ and $F_3$ is finite (in fact $|B| \leq 12|A|^2 + 8|A|$), moreover:

(a) $B \cap Q_2^M = (A \cap Q_2^M) \cup \{F_2(a) : a \in A \cap Q_0^M\}$,

(b) $B \cap Q_0^M = (A \cap Q_0^M) \cup \{F_0(b) : b \in A \cap Q_1^M\} \cup \{F_3(c) : c \in A \cap Q_2^M\}$

and

(c) $B \cap Q_1^M = (A \cap Q_1^M) \cup \{F_1(c, a) : c \in B \cap Q_2^M \land a \in B \cap Q_0^M\}$.

In this case, $B \models T_0$ and if $M \models T_0^+$, then $B \models T_0^+$.

**Proof of the Claim.**

(1) As $M$ is a model we have that $M \not= \emptyset$, so at least one among $Q_0^M, Q_1^M, Q_2^M$ is not empty.

If $Q_0^M \neq \emptyset$, then $F_2$ guarantees that $Q_2^M \neq \emptyset$, so $Q_1^M \neq \emptyset$ because of $F_1$. If $Q_2^M \neq \emptyset$, then $Q_0^M \neq \emptyset$ because of $F_1$. Finally, if $Q_2^M \neq \emptyset$, then $Q_0^M \neq \emptyset$ because of $F_3$, and we can again argue as above.

If $a \in Q_0^M$, let $b \in Q_2^M$ be arbitrary. Then $F_1(a, b) \in Q_1^M$ and $F_0(F_1(a, b)) = a$. Hence, $F_0$ is onto. Also, $F_3(F_2(a)) = a$, so $F_3^M$ is onto.

(2) Let $M \models T_0$ and $Q_0^M, Q_2^M \neq \emptyset$. For $x \in Q_0^M$ and $z \in Q_2^M$ let $F_1(z, x) = y$ for any $y \in Q_1^M$ such that $F_0(y) = x$, which exists as $F_0^M$ is already onto. For $x \in Q_0^M$ for which $F_3(x)$ is not already defined, let $F_3(x) = z$ for any $z$ such that $F_3(z) = x$, which exists as $F_3^M$ is onto. Finally, extend $F_0$ and $F_3$ to be total. The described model is a model of $T_0^+$. 

(3) Let $\kappa_1 < \kappa_2 < \lambda$ and let $Q^M_0 = \kappa_2$, $Q^M_1 = \kappa_2^{>\lambda}$, while $Q^M_2 = \kappa_1^{\lambda}$. For $\alpha < \kappa_2$ let $F_2(\alpha)$ be the function in $\kappa_1^{\lambda}$ which is constantly $\alpha$, and for $\nu \in \kappa_1^{\lambda}$ let $F_2(\nu) = \min(\text{Rang}(\nu_2))$ if this value is $< \kappa_2$, and 0 otherwise. Also, let $F_0(\eta) = \lg(\eta)$ and $F_1(\nu, \alpha) = \nu \upharpoonright \alpha$ be defined for $\nu \in \kappa_1^{\lambda}$ and $\alpha < \kappa_1$.

This is a model of $T_0$, but not of $T_0^+$ because $F_1$ is not total. If this model were to be extended to a model of $T_0^+$ with the same universe, we would have that for every $\nu \in \kappa_1^{\lambda}$

$$F_0(F_1(\nu, \kappa_1)) = \kappa_1 \& F_1(\nu, \kappa_1) = \eta$$

for some $\eta \in \kappa_2^{>\lambda}$. As $F_0(\eta)$ is already defined, $F_0(\eta) = \lg(\eta) < \kappa_1$, which is a contradiction.

(4) Given a model $M$ of $T_0$. First assure that $Q^M_0, Q^M_1, Q^M_2 \neq \emptyset$ by adding new elements if necessary. Then make sure that $F_0$ and $F_3$ are total and onto, again by adding new elements if needed. Now define $F_1(z, x) = y$ if $F_0(y) = x$, which is possible. Finally, declare $F_2(x) = z$ for any $z$ such that $F_3(z) = x$.

(5) Suppose that $M_0, M_1$ and $M_2$ are models of $T_0^+$ with $|M_1| \cap |M_2| = |M_0|$, and $M_0 \subseteq M_1, M_2$. We define $M_3$ as follows. Let $|M_3| = |M_1| \cup |M_2|$, and for $m \in \{0, 2, 3\}$ let $F^{M_3}_m(x) = F^{M_l}_m(x)$ if $x \in M_l$ for some $l$. This is well defined, because $M_1$ and $M_2$ agree on $M_0$. Also, the identity $F_3(F_2(x)) = x$ is satisfied in $M_3$.

For $(z, x) \in Q_2 \times Q^M_0$ such that for some $l$ we have $x \in M_l$ and $z \notin M_l$ choose $y_{z, x} \in M_l$ such that $F^{M_l}_0(y) = x$, which is possible by part (1) of this Claim. Now we define $F_1$ by letting or $(z, x) \in Q_2 \times Q^M_0$

$$F_1(z, x) = \begin{cases} F^{M_l}_1(z, x) & \text{if } z, x \in M_l, \\ y_{z, x} & \text{otherwise.} \end{cases}$$

Now it can be easily seen that $M_3$ is a model of $T_0^+$ and that both $M_1$ and $M_2$ are submodels. This proves the amalgamation property for $T_0^+$. 

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To see that JEP holds, suppose that we are given two models $M_1, M_2$ of $T_0^+$. We let $M$ be their disjoint union and define the functions $F_m$ for $m \in \{0, 1, 2, 3\}$.

(6) Clearly $B$ is contained in the closure of $A$ and the size of $B$ is as claimed. It can be checked directly that $B$ is closed, using the equations of $T_0$, and it also easily follows that $B$ is a model of $T_0$, or of $T_0^+$ if $M$ is.

\[ \star \]

**Claim 1.5** $T_0^+$ has a complete model completion $T^*$ which admits elimination of quantifiers, and is $\aleph_0$-categorical. In this theory, the closure and the algebraic closure coincide.

**Proof of the Claim.** We can construct $T^*$ directly. $T^*$ admits elimination of quantifiers because $T_0^+$ has the amalgamation property ([ChKe] 3.5.19). It can be seen from the construction of $T^*$ that it is complete, or alternatively, it can be seen that $T^*$ has JEP and so by [ChKe] 3.5.11, it is complete. To see that the theory is $\aleph_0$-categorical, observe that Claim 1.4(6) implies that for every $n$ there are only finitely many $T_0$-types in $n$-variables. Then by the Characterisation of complete $\aleph_0$-categorical theories ([ChKe] 2.3.13), $T^*$ is $\aleph_0$-categorical. Using the elimination of quantifiers and the fact that all relational symbols of the language of $T^*$ have infinite domains in every model of $T^*$, we can see that the algebraic closure and the definable closure coincide in $T^*$. \[ \star \]

**Observation 1.6** If $A, B \subseteq \mathcal{C}_{T^*}$ are closed and $c \in \text{cl}(A \cup B) \setminus A \setminus B$, then $c \in Q^\mathfrak{c}_{T^*}$.

**Proof.** Notice that
\[
\text{cl}(A \cup B) = A \cup B \cup \{F_1(c, a) : c \in (A \cup B) \cap Q_2 \& a \in (A \cup B) \cap Q_0 \\
\& \{c, a\} \notin A \& \{c, a\} \notin B \}
\]
by Claim 1.4(6).
Claim 1.7 \( T^* \) is NSOP\(_3\), consequently NSOP\(_4\).

**Proof of the Claim.** Suppose that \( T^* \) is SOP\(_3\) and let \( \varphi(\bar{x}, \bar{y}) \), and \( \langle \bar{a}_n : n < \omega \rangle \) exemplify this in a model \( M \) (see Definition 1.3(1)). Without loss of generality, each \( \bar{a}_n \) is without repetition and is closed (recall Claim 1.4(6)). By the Ramsey theorem and compactness, we can assume that the given sequence is a part of an indiscernible sequence. By the Ramsey theorem and compactness, we can assume that the given sequence is a part of an indiscernible sequence \( \langle \bar{a}_k : k \in \mathbb{Z} \rangle \). Hence, by indiscernibility, we have that \( |X_k| = n^* \) for some fixed \( n^* \) not depending on \( k \). Let \( \bar{a}_k^+ \) list \( X_k \) with no repetition. By Observation 1.6, Claim 1.4(6), indiscernibility and the fact that each \( \bar{a}_k \) is closed under \( F_0 \), we have that

\[
X_k \cap Q^e_0 \subseteq \text{Rang}(\bar{a}_k) \quad \text{and} \quad X_k \cap Q^e_2 \subseteq \text{Rang}(\bar{a}_k).
\]

Applying Ramsey theorem again, without loss of generality we have that \( \langle \bar{a}_k^+ : k \in \mathbb{Z} \rangle \) are indiscernible. Let

\[
w_0^* \overset{\text{def}}{=} \{ l : \bar{a}_k^+(l) = \bar{a}_k^+(l) \text{ for some (equivalently all) } k_1 \neq k_2 \}.
\]

If \( \bar{a}_k^+(l_1) = \bar{a}_k^+(l_2) \) for some \( k_1 \neq k_2 \), without loss of generality \( k_1 < k_2 \), by indiscernibility. By transitivity, using \( k_1 < k_2 < k_3 \), we get \( l_1 = l_2 \in w_0^* \). Let

\[
w_1^* \overset{\text{def}}{=} n^* \setminus w_0^*, \quad \text{and} \quad \bar{a} = \bar{a}_k^+ \upharpoonright w_0^* \text{ and } \bar{a}_k^+ \upharpoonright w_1^*.
\]

Hence, \( \langle \bar{a} \bar{a}_k^+ : k \in \mathbb{Z} \rangle \) is an indiscernible sequence, and \( \text{Rang}(\bar{a}) \cap \text{Rang}(\bar{a}_k^+) = \emptyset \) for all \( k \). In addition, and for \( k_1 \neq k_2 \) we have \( \text{Rang}(\bar{a}_k^+) \cap \text{Rang}(\bar{a}_k^+ \upharpoonright w_1^*) = \emptyset \) and \( \text{Rang}(\bar{a} \bar{a}_k^+) \).

Now we define a model \( N \). Its universe is \( \bigcup_{0 \leq t < 3} \{ \text{cl}_M(\bar{a} \bar{a}_t^+ \bar{a}_t^+) \} \), and \( Q^N_i = Q^M_i \cap N \), \( F^N_j = \bigcup \{ F_{j,l} : l < 3 \} \), where \( F_{j,l} = F^M_j \upharpoonright \text{cl}_M(\bar{a} \bar{a}_t^+ \bar{a}_t^+) \), or \( F_{j,l} = F^M_j \upharpoonright (\text{cl}_M(\bar{a} \bar{a}_t^+ \bar{a}_t^+))^2 \), as appropriate. Note that \( N \) is well defined, and that it is a model of \( T_0^+ \). \( N \) is not necessarily a model of \( T_0^+ \), as the function \( F_1 \) may be only partial. Notice that \( X_l \subseteq N \) for \( l \in [0, 3] \). We wish to define \( N' \) like \( N \), but identifying \( \bar{a}_0^+ \) and \( \bar{a}_3^+ \) coordinatwise. We shall now
check that this will give a well defined model of $T_0$. Note that by the proof of Observation 1.6 we have

$$N' = \bigcup_{0 \leq t < 3} X_t \cup \bigcup_{0 \leq t < 3} \{ F_3^t(c, d) : c, d \in X_t \cup X_{t+1} \}$$

\& \{ c, d \} \notin X_t \& \{ c, d \} \notin X_{t+1} \& F_3^t(c, d) \notin X_t \cup X_{t+1} \}.

The possible problem is that $F_i^{N'}$ might not be well defined, i.e. there could perhaps be a case defined in two distinct ways. We verify that this does not happen, by discussing various possibilities.

**Case 1.** For some $b \in \text{Rang}(a_0^+)$, say $b = a_0^+(t)$, $b' = a_3^+(t)$ and $j \in \{0, 2, 3\}$ we have $F_j(b) \neq F_j(b')$ after the identification of $a_0^+$ with $a_3^+$. As $a_k^+$’s are closed, we have $F_j(b) = a_0^+(s)$ and $F_j(b') = a_3^+(s')$ for some $s, s'$. By indiscernibility, we have $s = s'$, hence the identification will make $F_j(b) = F_j(b')$.

**Case 2.** For some $s, t$ we have that $F_1(a_0^+(s), a_0^+(t))$ and $F_1(a_3^+(s), a_3^+(t))$ are well defined, but not the same after the identification of $a_0^+$ and $a_3^+$. This case cannot happen, as can be seen similarly as in the Case 1.

**Case 3.** For some $\tau(x, y) \in \{ F_1(x, y), F_1(y, x) \}$ and $d_1 = a_0^+(s), d_2 = a_3^+(s)$ and some $e \in N$ we have that $\tau^N(d_1, e), \tau^N(d_2, e)$ are well defined but do not get identified when $N'$ is defined.

By Case 2, we have that $e \notin \bar{a}$ and $s \notin w_0^\ast$. As $\tau(e, d_1)$ is well defined and $d_1 \in X_0 \setminus \bar{a}$, necessarily $e \in \text{cl}_M(X_0 \cup X_1)$. Similarly, as $\tau(e, d_2)$ is well defined and $d_2 \in X_3 \setminus \bar{a}$, we have $e \in \text{cl}_M(X_2 \cup X_3)$. But, as $F_1(e, d_1)$ is well defined, we have $e \in Q_2 \cup Q_0$. Hence $e \in \text{cl}_M(X_0 \cup X_1) \setminus Q_1 \subseteq X_0 \cup X_1$ and similarly $e \in X_2 \cup X_3$. But this implies $e \in \bar{a}$, a contradiction.

As $M$ is a model of $T_0$, $F_0^M$ is onto (Claim 1.4(1)). Suppose $y \in Q_0^N$, then for some $l \in [0, 3)$ we have that $y \in \text{cl}_M(X_l \cup X_{l+1})$, so by Observation 1.6, we have $y \in X_l \cup X_{l+1}$. As each $X_l$ is closed in $M$, by Claim 1.4(6) each $X_l$ is a model of $T_0^+$, so $y \in \text{Rang}(F_0^M)$, hence $y \in \text{Rang}(F_0^N)$ and $y \in \text{Rang}(F_0^{N'})$. We can similarly prove that $F_3^N$ is onto, and as each $X_l$ is a model of $T_0^+$ we have by Claim 1.4(1) that $Q_0^N, Q_1^N$ and $Q_2^N$ are all non-empty. By Claim 1.4(2), $N'$ can be extended to a model of $T_0^+$.

By the choice of $\phi$ and the fact that $T^*$ is complete we have that

$$T^* \models (\forall \bar{x}_0, \bar{x}_1, \bar{x}_2) - [\phi(\bar{x}_0, \bar{x}_1) \land \phi(\bar{x}_1, \bar{x}_2) \land \phi(\bar{x}_2, \bar{x}_0)].$$
As $T^*$ is the model completion of $T_0^+$, in particular $T^*$ and $T_0^+$ are cotheories, so we have that

$$T^* \models (\forall \bar{x}_0, \bar{x}_1, \bar{x}_2)[\varphi(\bar{x}_0, \bar{x}_1) \land \varphi(\bar{x}_1, \bar{x}_2) \land \varphi(\bar{x}_2, \bar{x}_0)],$$

yet in $N'$ we have

$$N' \models \varphi(\bar{a}_0, \bar{a}_1) \land \varphi(\bar{a}_1, \bar{a}_2) \land \varphi(\bar{a}_2, \bar{a}_0),$$

by the identification of $\bar{a}_0$ and $\bar{a}_3$. This is a contradiction. ⋆

### Definition 1.8

1. A theory $T$ is said to satisfy the oak property as exhibited by a formula $\varphi(\bar{z}, \bar{y}, \bar{x})$ iff for any $\lambda, \kappa$ there are $\bar{b}_\eta (\eta \in ^{\kappa > \lambda})$ and $\bar{c}_\nu (\nu \in ^{\kappa \lambda})$ and $\bar{a}_i (i < \kappa)$ such that
   
   (a) $[\eta < \nu & \nu \in ^{\kappa \lambda}] \implies \varphi[\bar{c}_\nu, \bar{b}_\eta, \bar{a}_{lg(\eta)}],$
   
   (b) If $\eta \in ^{\kappa > \lambda}$ and $\eta \hat{\langle} \alpha \hat{\rangle} < \nu_1 \in ^{\kappa \lambda}$ and $\eta \hat{\langle} \beta \hat{\rangle} < \nu_2 \in ^{\kappa \lambda}$, while $\alpha \neq \beta$ and $i > lg(\eta)$, then $\exists \bar{y} [\varphi(\bar{c}_{\nu_1}, \bar{y}, \bar{a}_i) \land \varphi(\bar{c}_{\nu_2}, \bar{y}, \bar{a}_i)],$

   and in addition $\varphi$ satisfies

   (c) $\varphi(\bar{z}, \bar{y}_1, \bar{x}) \land \varphi(\bar{z}, \bar{y}_2, \bar{x}) \implies \bar{y}_1 = \bar{y}_2.$

   We allow for the replacement of $\mathfrak{C}_T$ by $\mathfrak{C}_T^{eq}$ (i.e. allow $\bar{y}$ to be a definable equivalence class).

2. We say that oak holds for $T$ if this is true for some $\varphi$.

### Observation 1.9

If some $\lambda, \kappa$ exemplify that oak($\varphi$) holds, then so do all $\lambda, \kappa$. (This holds by the compactness theorem).

### Remark 1.10

We shall not need to use this, but let us remark that witnesses $\bar{a}, \bar{b}, \bar{c}$ to oak($\varphi$) can be chosen to be indiscernible along an appropriate index set (a tree). This can be proved using the technique as in [Sh -c], Chapter VII, which uses the compactness argument and an appropriate partition theorem.
Claim 1.11 $T^*$ has oak.

Proof of the Claim. Let

$$\varphi(z, y, x) \overset{\text{def}}{=} Q_2(z) \land Q_1(y) \land Q_0(x) \land F_0(y) = x \land F_1(z, x) = y.$$ 

Clearly, (c) of Definition 1.8 (1) is satisfied. Given $\lambda, \kappa$, we shall define a model $N = N_{\lambda, \kappa}$ of $T^*_0$. This will be a submodel of $C = C_{T^*}$ such that its universe consists of $Q_N^0 \overset{\text{def}}{=} \{a_i : i < \kappa\}$ with no repetitions, $Q_N^1 \overset{\text{def}}{=} \{b_\eta : \eta \in ^\kappa \lambda\}$ with no repetitions and $Q_N^2 \overset{\text{def}}{=} \{c_\nu : \nu \in ^\kappa \lambda\}$ with no repetitions, while $Q_0, Q_1, Q_2$ are pairwise disjoint. We also require that the following are satisfied in $C = C_{T^*}$:

$$F_0(b_\eta) = a_{lg(\eta)}, F_1(c_\nu, a_i) = b_\nu | i$$

and that $N$ is closed under $F_2$ and $F_3$. That such a choice is possible can be seen by writing the corresponding type and using the saturativity of $C$.

We can check that $N \models T^*_0$, and that $N$ is a submodel of $C$ when understood as a model of $T^*_0$. Clearly, (a) from Definition 1.8(1) is satisfied for $\varphi$ and $a_i, b_\eta, c_\nu$ in place of $\bar{a}_i, \bar{b}_\eta, \bar{c}_\nu$ respectively. To see (b), suppose that $\eta, \alpha, \beta, \nu_1, \nu_2$ and $i$ are as there, but $d$ is such that $\varphi(c_{\nu_1}, d, a_i) \land \varphi(c_{\nu_2}, d, a_i)$. Hence $F_1(c_{\nu_1}, a_i) = F_1(c_{\nu_2}, a_i)$, so $\nu_1 \upharpoonright i = \nu_2 \upharpoonright i$, a contradiction. This shows that $\varphi$ is a witness for $T^*$ having oak. \(\star\)

Finally, a remark showing why this research continues [Sh 457]. The readers unfamiliar with $T^*_\text{feq}$ can skip to the next section without loss of generality. We use the notation for $T^*_\text{feq}$ which was used in [DjSh 692], while the fact that this is equivalent to the notation in [Sh 457] was explained in [DjSh 692].

Remark 1.12 After renaming, $C^\text{eq}_{T^*_\text{feq}}$ and $C^\text{eq}_{T^*}$ are isomorphically embeddable into each other. To see this suppose that $M$ is a model of $T^*_\text{feq}$. Let $A = \{x_\alpha : \alpha < \alpha^*\}$ be a set of representatives of $E^M$-equivalence classes. By the construction of $T^*_\text{feq}$, for every finite $F \subseteq \alpha^*$, there is $z$ such that $\land_{\alpha \in F} F(x_\alpha, z) = x_\alpha$. By the saturativity of $C_{T^*}$, there is $z_A \in C_{T^*}$
such that $\land_{\alpha<\alpha} F(x_{\alpha}, z_{A}) = x_{\alpha}$. By the axioms of $T_{eq}$ it follows that $A \neq A' \implies z_{A} \neq z_{A'}$.

Now we define a model $N = N_{0}[M]$ of $T^{*}$. Its universe is

$$|M| \cup (P^{M}/E^{M}) \cup \{z_{A} : A \text{ is a set of representatives of } E^{M}\text{-equivalence classes}\}.$$  

We let $Q_{0} = P^{M}/E^{M}$, $Q_{1}^{N} = P^{M}$ and 

$$Q_{2}^{N} = Q_{1}^{M} \cup \{z_{A} : A \text{ is a set of representatives of } E^{M}\text{-equivalence classes}\}.$$  

The functions of $N$ are defined as follows. We firstly let $F_{0}(x) = x/E^{M}$ and $F_{1}(z, x/E) = F(x, z)$. Notice that $F_{1}$ is well defined, because if $z \in M$ then certainly $F(x, z) \in M$ for all $x \in Q_{1}^{N}$ and if $z = z_{A}$ for some $A$ then the definition of $A$ guarantees that $F(x, z_{A}) \in M$ for every $x \in M$. Also, we have

$$F_{0}(F_{1}(z, x/E^{M})) = F_{0}(F(x, z)) = x/E^{M}.$$  

It remains to define $F_{2}$ and $F_{3}$. Let us first see that $|Q_{2}| \geq |Q_{0}|$. By the definition of $T^{*}_{eq}$, each equivalence class of $M$ is infinite. Hence, the number of distinct sets of representatives of the $E^{M}$-equivalence classes is at least $|Q_{0}|^{|\mathbb{N}|} \geq |Q_{0}|$, and by the definition of $Q_{2}$ we have $|Q_{2}| \geq |Q_{0}|$. We can choose $F_{3}$ as any onto function from $Q_{2}$ to $Q_{0}$, and apply Claim 1.4(2). Hence $N$ is a model of $T_{0}^{+}$ and can be seen as a submodel of $\mathcal{C}^{eq}_{T}$.  

Conversely, given $M$ a model of $T^{*}$, we define $N = N_{1}[M]$ by letting its universe be $Q_{1}^{M} \cup Q_{2}^{M}$ and $P^{N} = Q_{1}^{M}$, while $Q^{N} = Q_{2}^{M}$. We let

$$y Ez \iff F^{M}_{0}(y) = F^{M}_{0}(z) \text{ and } F^{N}(x, z) = F_{1}(z, F_{0}(x)).$$

We also let $x R z \iff F^{N}(x, z) = x$. It is easily seen that $N \models T_{eq}$. Using this equivalence and the fact that oak and $NSOP_{3}$ are preserved up to isomorphism of $\mathcal{C}^{eq}$, we obtain:

**Corollary 1.13** (1) $T^{*}_{eq}$ has oak.  
(2) $T^{*}_{eq}$ has $NSOP_{3}$.  

Part (2) of Corollary 1.13 was stated without proof in [Sh 500].
2 The theorems

In this section we present two general theorems showing that under certain cardinal arithmetic assumptions oak theories do not admit universal models.

Theorem 2.1 Assume that

1. \(\text{cf}(\kappa) = \kappa < \mu < \mu^+ < \lambda = \text{cf}(\lambda)\),
2. \(\lambda < \mu^\kappa\),
3. \(\kappa \leq \sigma \leq \lambda\),
4. There are families \(P_1 \subseteq [\lambda]^\kappa\) and \(P_2 \subseteq [\sigma]^\kappa\) such that
   1. for every \(g : \sigma \to \lambda\) there is \(X \in P_2\) with \(\{g(i) : i \in X\} \in P_1\),
   2. \(|P_1| < \mu^\kappa, |P_2| \leq \lambda\),
5. \(T\) is a theory of size \(< \lambda\) which satisfies \(\text{oak}(\varphi(\bar{z}, \bar{y}, \bar{x}))\).

Then

\[\text{univ}(T, \lambda) \geq \mu^\kappa.\]

Definition 2.2 For cardinals \(\kappa, \mu\) we define

\[U_{\mu^\kappa}(\mu) \overset{\text{def}}{=} \min\{|P| : P \subseteq [\mu]^\kappa \& (\forall b \in [\mu]^\kappa)(\exists a \in P)(|a \cap b| = \kappa)\}.

Theorem 2.3 Assume that

1. \(\text{cf}(\kappa) = \kappa < \mu < \mu^+ < \lambda = \text{cf}(\lambda)\),
2. \(\lambda < U_{\mu^\kappa}(\mu)\),
3. \(\kappa \leq \sigma \leq \lambda\),
4. There are families \(P_1 \subseteq [\lambda]^\kappa\) and \(P_2 \subseteq [\sigma]^\kappa\) such that
(i) for every $g : \sigma \to \lambda$ there is $X \in \mathcal{P}_2$ such that $|\{g(i) : i \in X\} \cap Y| = \kappa$
for some $Y \in \mathcal{P}_1$,
(ii) $|\mathcal{P}_1| < \mathcal{U}_{\mathcal{J}^b}(\mu)$, $|\mathcal{P}_2| \leq \lambda$,

(5) $T$ has oak($\varphi(\bar{z}, \bar{y}, \bar{x})$).

Then

$$\text{univ}(T, \lambda) \geq \mathcal{U}_{\mathcal{J}^b}(\mu).$$

Proof. We shall use the same proof for both Theorem 2.1 and Theorem 2.3.

The two main Lemmas are the same for both theorems, and we shall indicate the differences which occur toward the end of the proof. Let $a_i (i < \kappa)$, $b_\eta (\eta \in {}^{<\kappa} \lambda)$ and $c_\nu (\nu \in {}^{<\lambda} \lambda)$ exemplify the oak property of $\varphi(\bar{z}, \bar{y}, \bar{x})$ for $\lambda$ and $\kappa$.

For notational simplicity, let us assume that $\lg(\bar{x}) = \lg(\bar{y}) = \lg(\bar{z}) = 1$.

Let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ for some $S \subseteq \mathcal{S}_\lambda^{<\kappa}$ with $\text{otp}(C_\delta) = \mu$ and $C_\delta$ a closed subset of $\delta$, be a club guessing sequence, (i.e. for every $E$ a club of $\lambda$, there is $\delta \in S$ with $C_\delta \subseteq E$) such that

$$\alpha < \lambda \implies |\{C_\delta \cap \alpha : \delta \in S, \& \alpha \in \text{nacc}(C_\delta)\}| < \lambda.$$

Such a sequence exists by S. Shelah’s [Sh 420] (section §1). For each $\delta$, let $\langle \alpha_{\delta, \zeta} : \zeta < \mu \rangle$ be the increasing enumeration of $C_\delta$. Let $\mathcal{C}_T^+$ be a (saturated enough) expansion of $\mathcal{C}_T$ by Skolem functions for $\mathcal{C}_T$.

Definition 2.4 (1) For $\bar{N} = \langle N_\gamma : \gamma \leq \lambda \rangle$ an $\prec$-increasing continuous sequence of models of $T$ of size $\leq \lambda$, and for $c, a \in \bigcup_{\gamma < \lambda} N_\gamma$, and $\delta \in S$, we let

$$\text{inv}_N(c, C_\delta, a) \overset{\text{def}}{=} \{\zeta < \mu : [(\exists b \in N_{\alpha_{\delta, \zeta} + 1})(N_\lambda \models \varphi[c, b, a]) \&$$

$$\neg(\exists b \in N_{\alpha_{\delta, \zeta}})(N_\lambda \models \varphi[c, b, a])\}.$$ 

(2) For a set $A$ and $\delta$, $\bar{N}$ as above, let

$$\text{inv}_{\bar{N}}^A(c, C_\delta) \overset{\text{def}}{=} \bigcup \{\text{inv}_N(c, C_\delta, a) : a \in A\}.$$
Note 2.5 Notice that \( \text{inv}_N(c, C_\delta, a) \) is always a singleton or empty and that 
\[ \text{inv}^A_N(c, C_\delta) \in [\mu]^{\leq |A|}. \]

Construction Lemma 2.6 For every unbounded \( A^* \in [\mu]^\kappa \) of order type \( \kappa \), there is an \( \prec \)-increasing continuous sequence \( \bar{N}_{A^*} = (N^A_\gamma : \gamma < \lambda) \) of models of \( T \) of size \( < \lambda \) and a set \( \{ \hat{a}_i : i < \sigma \} \) of elements of \( N_0^A \) such that for every \( X \in \mathcal{P}_2 \), for every \( \delta \in S \) with \( \min(C_\delta) \) large enough, there is 
\[ c \in N_{A^*} \overset{\text{def}}{=} \bigcup_{\gamma < \lambda} N^A_\gamma \] such that 
\[ \text{inv}_{\bar{N}_{A^*}}(c, C_\delta) = A_{\delta}. \]

Proof of the Lemma. Let \( \mathcal{P}_2 = \{ X_\alpha : \alpha < \alpha^* \leq \lambda \}. \)

Given \( A^* \). Let \( f = f_{A^*} \) be the increasing enumeration of \( A^* \), so \( f : \kappa \to \mu \).
For \( \delta \in S \) let \( \nu_\delta \overset{\text{def}}{=} (\alpha_\delta : \zeta \in A^*) \) be an increasing enumeration, hence \( c_{\nu_\delta} \) is well defined, as is \( b_\eta \) for \( \eta \prec \nu_\delta \). For \( X \in \mathcal{P}_2 \), let \( \rho_X \) be an increasing surjection from the successor ordinals \( < \kappa \) onto \( X \). By a compactness argument, we can see that there are 
\[ \{ \hat{a}_i : i < \sigma \} \] and for \( X \in \mathcal{P}_2 \), sequences 
\[ \{ c_{\nu_\delta}^X : \delta \in S \}, \]
\[ \{ b_\eta^X : \eta \prec \nu_\delta \text{ & } \lg(\eta) \text{ a successor} \} \] 
for \( \gamma < \lambda \) the model \( N^A_\gamma \) be the reduction to \( L(T) \) of the Skolem hull in \( \mathcal{C}^+ \) of

\[ \{ \hat{a}_i : i \in \bigcup_{\alpha < \gamma} X_\alpha \} \cup \{ c_{\nu_\delta}^X : \alpha < \gamma \text{ & } \delta \in S \cap \gamma \text{ & } \sup(\text{Rang}(\nu_\delta)) < \gamma \} \]

\[ \cup \{ b_\eta^X : \alpha < \gamma \text{ & } \eta \prec \nu_\delta \text{ for some } \delta \in S \cap \gamma \text{ & } \sup(\text{Rang}(\eta)) < \gamma \text{ & } \lg(\eta) \text{ a successor} \}. \]

Hence \( \langle \bar{N}^A_\gamma : \gamma < \lambda \rangle \) is \( \prec \)-increasing continuous and for \( \gamma < \lambda \) we have \( |N^A_\gamma| < \lambda \). The latter is true because in the last clause

\[ |\{ C_\delta \cap \alpha : \delta \in S, \alpha < \gamma, \alpha \in \text{nacc}(C_\delta) \}| < \lambda \]

by the choice of \( C \). Given \( \alpha < \alpha^* \), \( X = X_\alpha \) and \( \delta \in S \) with \( \min(C_\delta) \geq \alpha + 1 \) we shall show that with 
\[ I \overset{\text{def}}{=} \text{inv}^{\bar{N}_{A^*}}(c, C_\delta) \]
we have $I = A^*$. Notice that that $\varepsilon < \kappa \implies \alpha_{\delta, f(\varepsilon)} > \alpha$. Let $i \in X$ and 
let $\eta = \langle \alpha_{\delta, f(\varepsilon)} : \varepsilon \leq \beta \rangle$, where $\beta + 1 = \rho^{-1}_X(i)$. We have that $\eta < \nu_{\delta}$ and 
i = \rho_X(\lg(\eta))$. Hence $\varphi[c^X_{\nu_\delta}, b^X_\eta, \hat{a}_i]$ holds. Let $\zeta = f(\beta)$. We then have that 
b^X_\eta \in N_{\alpha_{\delta, \zeta + 1}} \subseteq N_{\alpha_{\delta, \zeta + 1}}^*$ (as $\alpha_{\delta, \zeta + 1} > \gamma$), but $b^X_\eta \notin N_{\alpha_{\delta, \zeta}}^*$. It follows from the 
property (c) of Definition $\|L.8$ that $b^X_\eta$ is the only $b$ for which $\models \varphi[c^X_{\nu_\delta}, b, \hat{a}_i]$. Hence $\zeta = f(\beta) \in I$. So $A^* \subseteq I$ because every element of $A^*$ is $f(\beta)$ for some $\beta$ as above.

In the other direction, suppose $\zeta \in I$ and let $i \in X$ be such that $\zeta$ is in 
$\mathrm{inv}_N(c^X_{\nu_\delta}, \hat{C}_\delta, \hat{a}_i)$. Hence for some $b \in N_{\alpha_{\delta, \zeta + 1}}^* \setminus N_{\alpha_{\delta, \zeta}}^*$ we have $\models \varphi[c^X_{\nu_\delta}, b, \hat{a}_i]$. Constructing $\eta$ as in the previous paragraph, we have $\models \varphi[c^X_{\nu_\delta}, b^X_\eta, \hat{a}_i]$. In 
collection, using property (c) of Definition $\|L.8$ again, we see that $b = B^X_\eta$ so 
$\zeta = f(\beta)$ for some $\beta$. So $A^* = I$.

$\star$\|L.4

**Preservation Lemma 2.7** Suppose that $N$ and $N^*$ are models of $T$ both 
with universe $\lambda$, and $f : N \to N^*$ is an elementary embedding, while 
$\langle N_\gamma : \gamma < \lambda \rangle$ and $\langle N^*_\gamma : \gamma < \lambda \rangle$ are continuous increasing sequences of 
models of $T$ of cardinality $< \lambda$ with $\bigcup_{\gamma < \lambda} N_\gamma = N$ and $\bigcup_{\gamma < \lambda} N^*_\gamma = N^*$. 
Further suppose that $\{\hat{a}_\alpha : \alpha < \kappa\} \subseteq N$ is given. Let 

$$E \overset{\text{def}}{=} \{\gamma : (N, N^*, f) \not\models \gamma < (N, N^*, f) \& \sup(\{f(\hat{a}_\alpha) : \alpha < \kappa\}) < \gamma\};$$

hence a club of $\lambda$. 

Then for every $c \in N$ and $\delta$ with $C_\delta \subseteq E$ we have 

$$\mathrm{inv}_N^\{\hat{a}_\alpha : \alpha < \kappa\}(c, C_\delta) = \mathrm{inv}_N^\{f(\hat{a}_\alpha) : \alpha < \kappa\}(f(c), C_\delta).$$

**Proof of the Lemma.** Fix $c \in N$ and $\delta \in S$ as required, and let $a = \hat{a}_\alpha$ for 
some $\alpha < \kappa$. We shall see that $\mathrm{inv}_N(c, C_\delta, a) = \mathrm{inv}_{N^*}(f(c), C_\delta, f(a))$.

Suppose $\zeta < \mu$ is an element of $\mathrm{inv}_N(c, C_\delta, a)$, so there is $b \in N_{\alpha_{\delta, \zeta + 1}}$ 
with $N \models \varphi[c, b, a]$, while there is no such $b \in N_{\alpha_{\delta, \zeta}}$. We have that $N^*$ 
satisfies $\varphi[f(c), f(b), f(a)]$. As $C_\delta \subseteq E$ we have that $\alpha_{\delta, \zeta + 1} \in E$, and as 
b \in N_{\alpha_{\delta, \zeta + 1}}$, clearly $f(b) \in N^*_{\alpha_{\delta, \zeta + 1}}$. Similarly, by the definition of $E$ again, we 
have $f(b) \notin N^*_{\alpha_{\delta, \zeta}}$. By the assumptions on $\varphi$ we have 

$$N^* \models "(\forall y)[\varphi(f(c), y, f(a)) \implies y = f(b)]", \quad 18$$
so \( \zeta \in \text{inv}_{\bar{N}}(f(c), C_\delta, f(a)) \).

In the other direction, suppose \( \zeta < \mu \) is an element of \( \text{inv}_{\bar{N}}(f(c), C_\delta, f(a)) \), so there is \( b^* \in N_{\alpha_\delta+1}^* \) with \( N^* \models \varphi(\bar{f}(c), b^*, f(a)) \), while there is no such \( b^* \in N_{\alpha_\delta+1}^* \). Hence \( N^* \models \exists y (\varphi(\bar{f}(c), y, f(a))) \), so \( N \models \exists y (\varphi(c, y, a)) \). Let \( b \in N \) be such that \( N \models \varphi[c, b, a] \). Hence \( N^* \models \varphi(\bar{f}(c), f(b), f(a)) \). Again by (c) of Definition 1.8, we have \( f(b) = b^* \), so \( b \in N_{\alpha_\delta+1} \setminus N_{\alpha_\delta} \) by elementarity. As this \( b \) is unique (by (c) of Definition 1.8), we have \( \zeta \in \text{inv}_{\bar{N}}(c, C_\delta, a) \). ★

Proof of the Theorems continued. s Theorem 2.1 [Theorem 2.3]. To conclude the proof of the theorems, given \( \theta < \mu^\kappa [\theta < \mathcal{U}_{\theta_\text{bd}}(\mu)] \), we shall see that \( \text{univ}(T, \lambda) > \theta \). Without loss of generality, we can assume that \( \theta \geq \lambda + |\mathcal{P}_1| \).

Given \( \langle N_j^* : j < \theta \rangle \) a sequence of models of \( T \) each of size \( \lambda \), we show that these models are not jointly universal. So suppose they were. Without loss of generality, the universe of each \( N_j^* \) is \( \lambda \). Let \( \bar{N}_j^* = \langle N_{\gamma,j}^* : \gamma < \lambda \rangle \) be an increasing continuous sequence of models of \( T \) of size \( \lambda \) such that \( N_j^* = \bigcup_{\gamma < \lambda} N_{\gamma,j}^* \), for \( j < \theta \). For each \( A \in \mathcal{P}_1 \) (so \( A \in [\lambda]^\kappa \)), \( \delta \in S \), \( j < \theta \) and \( d \in N_j^* \), we compute \( \text{inv}_{N_j^*}^A(d, C_\delta) \), each time obtaining an element of \( [\mu]^\leq\kappa \).

The number of elements of \( [\mu]^\leq\kappa \) obtained in this way is

\[
\leq |\mathcal{P}_1| \cdot |S| \cdot \theta \cdot \lambda \leq \theta. 
\]

By the choice of \( \theta \) [and the definition of \( \mathcal{U}_{\theta_\text{bd}}(\mu) \)], we can choose \( A^* \in [\mu]^\kappa \) such that \( A^* \) is not equal to any of these sets [is almost disjoint (i.e. has intersection of size \( \leq \kappa \)] to all these sets]. Let \( N \overset{\text{def}}{=} N_{A^*} \) be as guaranteed to exist by the Construction Lemma, and let \( \{ \hat{a}_i : i < \sigma \} \) and \( \bar{N}_{A^*} \overset{\text{def}}{=} \langle N_{\gamma,j}^{A^*} : \gamma \leq \lambda \rangle \) be as in that Lemma. Without loss of generality, by taking an isomorphic copy if necessary, the universe of \( N \) is \( \lambda \). Suppose that \( j < \theta \) and \( f : N \rightarrow N_j^* \) is an embedding, and let

\[
E \overset{\text{def}}{=} \{ \delta : (N, N_j^*, f) \models \delta \prec (N, N_j^*, f) \}.
\]

Let \( g : \sigma \rightarrow \lambda \) be given by \( g(i) = f(\hat{a}_i) \). Let \( X = X_\alpha \in \mathcal{P}_2 \) be such that \( \{ f(\hat{a}_i) : i \in X \} \in \mathcal{P}_1 \), [for some \( Y \in \mathcal{P}_1 \) we have \( |\{ f(\hat{a}_i) : i \in X \} \cap Y| = \kappa \)], and let \( c \in N \) be such that \( \text{inv}_{N_j^*}^A(c, C_\delta) = A^* \). By the Preservation Lemma, we have \( \text{inv}_{N_j^*}^A(f(c), C_\delta) = A^* \) [by \( \text{inv}_{N_j^*}^A(f(c), C_\delta) \cap A^* \)
includes \( \text{inv}\{f(\hat{a}_i) : i \in X\} \cap Y \) \( f(c), C \) \( \cap a^* \), which has cardinality \( \kappa \). This is a contradiction with the choice of \( A^* \).

\*\*\*

**Remark 2.8** We comment on the assumptions used in Theorems 2.1 and 2.3. Although the theorems do not use the assumption \( \text{cf}(\mu) = \kappa \), this situation is the natural one for the assumptions given. If \( \text{cf}(\mu) \leq \kappa < \mu \) we have \( \text{pp}_{J_\kappa}^{\text{bd}}(\mu) \leq \mathcal{U}_{J_\kappa}^{\text{bd}}(\mu) \). For example, we have the following

**Corollary 2.9** Let \( T \) be a theory with the oak property. Suppose that \( \text{cf}(\mu) = \kappa < \mu < \mu^+ < \lambda = \text{cf}(\lambda) \) and \( \lambda < \mathcal{U}_{J_\kappa}^{\text{bd}}(\mu) \) (e.g., \( \text{pp}_{J_\kappa}^{\text{bd}}(\mu) > \lambda \)) while \( 2^\kappa \leq \lambda \), and

\[
\text{for some } n, \text{cov}(\lambda, \kappa^{n+1}, \kappa^{+n}, \kappa^{+n}) = \lambda
\] 

\[
(*_{\lambda, \kappa})
\]

then \( \text{univ}(T, \lambda) \geq \mathcal{U}_{J_\kappa}^{\text{bd}}(\mu) \).

**Proof.** We use Theorem 2.3 with \( \sigma = \kappa^{+n+1} \) for \( n \) as in \((*_{\lambda, \kappa})\). By the choice of \( n \), there are \( P_1, P_2 \) as required and of cardinality \( \lambda \). \*\*\*

Note that the consistency of the failure of \((*_{\lambda, \kappa})\) for any \( \lambda \geq \kappa^{+\omega}, \kappa = \text{cf}(\kappa) \) is not known, and that for our purposes even weaker statements suffice. See \[\text{Sh } 460\].

If \( \aleph_0 < \kappa = \text{cf}(\mu) \) and for all \( \theta < \mu \) we have \( \theta^\kappa < \mu \), then

\[
\text{pp}_{J_\kappa}^{\text{bd}}(\mu) = \mu^\kappa = \mathcal{U}_{J_\kappa}^{\text{bd}}(\mu)
\]

(by \[\text{Sh -g}, \text{Chapter VII, } \S 1\]).

If \( \lambda > \kappa = \text{cf}(\kappa) \) and \( \sigma = \lambda \), if we cannot find \( P_1 \) and \( P_2 \) as in Theorem 2.3(i) with \( |P_1| + |P_2| \leq \lambda \), then for every \( P \subseteq [\lambda]^\kappa \) with \( |P| \leq \lambda \), we can find \( X \in [\lambda]^\lambda \) such that \( (\forall a \in P)(|a \cap X| < \kappa) \), which is a rather strong requirement.

Another comment is the necessity of introducing the cardinal \( \sigma \) at the outset of Theorem 2.1 and Theorem 2.3. In most instances of cardinal arithmetic, assuming that the other requirements are satisfied, requirement (4) cannot be fulfilled with \( \kappa = \sigma \). But if for example \( \lambda = \lambda^{[\sigma]} \) (for a definition
see [Sh 460]: the equality holds e.g. if \( \lambda < \aleph_\sigma \), and \( \kappa < \sigma \) is such that \( \sigma^\kappa < \mu \), and for some sequence \( \langle \lambda_i : i < \kappa \rangle \) of regulars increasing to \( \mu \) the reduced product \( (\prod_{i<\kappa} \lambda_i/J_{\kappa^+}) \) is \( \lambda^+ \)-directed, the assumptions of Theorem 2.1 will hold. In fact, by [Sh 460] we have

**Corollary 2.10** In Theorem 2.1, if \( (1)+(2) \) hold, while \( \kappa = \text{cf}(\mu) < \beth_\omega \leq \mu \), then for every large enough \( \sigma \in (\kappa, \beth_\omega) \), parts (3) and (4) of the assumptions of Theorem 2.1 hold as well, so \( \text{univ}(T, \lambda) \geq \mu^\kappa \).
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