Some Statistics on Generalized Motzkin Paths with Vertical Steps

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Abstract
Recently, several authors have considered lattice paths with various steps, including vertical steps permitted. In this paper, we consider a kind of generalized Motzkin paths, called $G$-Motzkin paths for short, that is lattice paths from $(0,0)$ to $(n,0)$ in the first quadrant of the $XY$-plane that consist of up steps $u = (1,1)$, down steps $d = (1,-1)$, horizontal steps $h = (1,0)$ and vertical steps $v = (0,-1)$. The main purpose of this paper is to count the number of $G$-Motzkin paths of length $n$ with given number of $z$-steps for $z \in \{ u, h, v, d \}$, and to enumerate the statistics “number of $z$-steps” at given level in $G$-Motzkin paths for $z \in \{ u, h, v, d \}$. Some explicit formulas and combinatorial identities are given by bijective and algebraic methods, some enumerative results are linked with Riordan arrays according to the structure decompositions of $G$-Motzkin paths. We also discuss the statistics “number of $z_1z_2$-steps” in $G$-Motzkin paths for $z_1, z_2 \in \{ u, h, v, d \}$, the exact counting formulas except for $z_1z_2 = dd$ are obtained by the Lagrange inversion formula and their generating functions.

Keywords Dyck path · G-Motzkin path · Catalan number · Riordan array

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1 Introduction

Lattice paths have been studied by many mathematicians and have produced numerous interesting and important results. Research in this area has resulted in well known classes of lattice paths such as those named after Dyck [14], Motzkin [4, 15], Schröder [9] and Delannoy [3]. They are used in physics [28], computer science [30, 50], biology [7, 16, 24, 38, 42, 52] and probability theory [12, 29, 32, 36, 37, 49]. We refer the reader to the wonderful survey by Humphreys [25] for additional historical information.

A Dyck path of length \(2n\) is a lattice path from \((0, 0)\) to \((2n, 0)\) in the first quadrant of the XY-plane that consists of up steps \(u = (1, 1)\) and down steps \(d = (1, -1)\). Let \(C_n\) be the set of Dyck paths of length \(2n\). It is well known [14, 47] that \(|C_n| = C_n = \frac{1}{n+1} \binom{2n}{n}\), the \(n\)th Catalan number, has the generating function

\[
C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \tag{1.1}
\]

with the relation \(C(x) = 1 + xC(x)^2 = \frac{1}{1-xC(x)}\).

A Motzkin path of length \(n\) is a lattice path from \((0, 0)\) to \((n, 0)\) in the first quadrant of the XY-plane that consists of up steps \(u = (1, 1)\), down steps \(d = (1, -1)\) and horizontal steps \(h = (1, 0)\). Let \(M_n\) be the set of Motzkin paths of length \(n\). It is well known [4, 15, 45] that \(|M_n| = M_n\), the \(n\)th Motzkin number, has the generating function

\[
M(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}
\]

with the relation \(M(x) = 1 + xM(x) + x^2M(x)^2 = \frac{1}{1-x}C\left(\frac{x^2}{(1-x)^2}\right)\). This implies the following relation between the Catalan numbers \(C_n\) and the Motzkin numbers \(M_n\) [15], i.e.,

\[
M_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} C_k.
\]

In fact, the sequence \(M_{n,k} = \binom{n}{2k} C_k\) counts the number of Motzkin paths of length \(n\) with \(k\) \(d\)-steps. The first values of \(M_{n,k} = \binom{n}{2k} C_k\) are illustrated in Table 1.

A Schröder path of length \(2n\) is a path from \((0, 0)\) to \((2n, 0)\) in the first quadrant of the XY-plane that consists of up steps \(u = (1, 1)\), down steps \(d = (1, -1)\) and horizontal steps \(H = (2, 0)\). Let \(S_n\) be the set of Schröder paths of length \(2n\). It is well known [45] that \(|S_n| = R_n\), the \(n\)th large Schröder number, has the generating function

\[
R(x) = \sum_{n \geq 0} R_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}
\]
Table 1  The first values of $M_{n,k}$

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 0     | 1 |   |   |   |   |   |
| 1     | 1 |   |   |   |   |   |
| 2     | 1 | 1 |   |   |   |   |
| 3     | 1 | 3 |   |   |   |   |
| 4     | 1 | 6 | 2 |   |   |   |
| 5     | 1 | 10| 10|   |   |   |
| 6     | 1 | 15| 30| 5 |   |   |
| 7     | 1 | 21| 70| 35|   |   |
| 8     | 1 | 28| 140|140|14 |   |
| 9     | 1 | 36| 252|420|126|   |
| 10    | 1 | 45| 420|1050|630|42 |

with the relation $R(x) = 1 + xR(x) + xR(x)^2 = \frac{1}{1-x}C(\frac{x}{1-x})^2$. This shows the following relation between the Catalan numbers $C_n$ and the large Schröder numbers $R_n$, i.e.,

$$R_n = \sum_{k \geq 0} M_{n+k,k} = \sum_{k=0}^{n} \binom{n + k}{2k} C_k.$$

In fact, the sequence $R_{n,k} = \binom{n+k}{2k} C_k$ counts the number of Schröder paths of length $2n$ with $k$ d-steps.

Recently, several authors [18, 19, 26, 51] have considered lattice paths with various steps, including vertical steps permitted. Irvine, Melczer and Ruskey [26], inspired by a new mathematical model for bobbin lace, encountered lattice paths very similar to the Motzkin paths but with the addition of vertical steps, namely, they considered finite lattice paths formed from the step set $S = \{u = (1, 1), d = (1, -1), h = (1, 0), v_1 = (0, -1), v_2 = (0, 1)\}$ with the restriction that vertical steps $v_i v_j$ can not be consecutive for $i, j \in \{1, 2\}$. The significance of this constraint on vertical steps can be understood by looking at the role of the paths in bobbin lace tessellations. Dziemiańczuk [18] examined lattice paths generated by $S$ without the vertical steps $v_2$. Yan and Zhang [51] proved a bijective relationship between the number of positive lattice paths and $m$-flawed lattice paths generated by $S$ without the vertical steps $v_2$, a result analogous to the Chung–Feller theorems for Dyck paths [8] and for Schröder paths [22]. Dziemiańczuk [19] also studied non-simple directed lattice paths running between two fixed points and for which the set of allowed steps contains vertical step $v_1 = (0, -1)$ and forward steps $s_k = (1, k)$ for some integers $k$. In this paper, we consider a kind of generalized Motzkin paths, called G-Motzkin paths for short. That is, a G-Motzkin path of length $n$ is a lattice path from $(0, 0)$ to $(n, 0)$ in the first quadrant of the XY-plane that consists of up steps $u = (1, 1)$, down steps $d = (1, -1)$, horizontal steps $h = (1, 0)$ and vertical steps $v = (0, -1)$. See Fig. 1 for a G-Motzkin path of length 24.
Fig. 1 A G-Motzkin path of length 24
Let $\varepsilon$ be the empty path, that is a dot path. If $P_1$ and $P_2$ are G-Motzkin paths, then we define $P_1P_2$ as the concatenation of $P_1$ and $P_2$. For example, $P_1 = uuhduuvddh$ and $P_2 = uhuuhvuuudd$, then $P_1P_2 = uhuuuuvvhhuuuhvuuudd$.

A point of a G-Motzkin path with ordinate $\ell$ is said to be at level $\ell$. A step of a G-Motzkin path is said to be at level $\ell$ if the ordinate of its endpoint is $\ell$. A $uv$-peak (uv-peak) in a G-Motzkin path is an occurrence of $ud$ (uv). A $vu$-valley (vu-valley) in a G-Motzkin path is an occurrence of $du$ (vu). A peak (valley) in a G-Motzkin path is a $ud$-peak or $uv$-peak (du-valley or vu-valley). By the height of a peak (valley) we mean the level of the intersection point of its two steps. By a return step we mean a $d$-step or $v$-step at level 0. A matching step of a $u$-step at level $k \geq 1$ in a G-Motzkin path is the leftmost step among all $d$-steps and $v$-steps at level $k - 1$ right to the $u$-step. A G-Motzkin path $P$ is said to be primitive if $P = uP'd$ or $P = uP've$ for certain G-Motzkin path $P'$.

In the present paper, we concentrate on several statistics in G-Motzkin paths. Precisely, the next section mainly counts the number of G-Motzkin paths of length $n$ with given number of $z$-steps for $z \in \{u, h, v, d\}$, some explicit formulas and combinatorial identities are given by bijective and algebraic methods. The third section mainly focuses on the enumeration of statistics “number of $z$-steps” at given level in G-Motzkin paths for $z \in \{u, h, v, d\}$, the enumerative results are linked with Riordan arrays according to the structure decompositions of G-Motzkin paths. The last section discusses the statistics “number of $z_1z_2$-steps” in G-Motzkin paths for $z_1, z_2 \in \{u, h, v, d\}$, the exact counting formulas except for $z_1z_2 = dd$ are provided according to the method of the first return decomposition of G-Motzkin paths and the Lagrange inversion formula.

### 2 The Statistics “Number of $z$-Steps” in G-Motzkin Paths

In this section, we first consider the weighted G-Motzkin paths and then enumerate several statistics “number of $z$-steps” in G-Motzkin paths for $z \in \{u, h, v, d\}$. The weight of each step of a G-Motzkin path $P$ is assigned as follows. The $u$-step, $h$-step, $v$-step and $d$-step are weighted respectively by 1, $a$, $b$ and $c$. The weight of $P$, denoted by $w(P)$, is the product of the weight of each step of $P$. For example, $w(uuhduuvddh) = a^3b^2c^2$. The weight of a subset $A$ of the set $G(a, b, c)$ of all weighted G-Motzkin paths, denoted by $w(A)$, is the sum of the total weights of all paths in $A$. Denoted by $w(G_n(a, b, c)) = G_n(a, b, c)$ the weight of the set $G_n(a, b, c)$ of all weighted G-Motzkin paths of length $n$. When $a = b = c = 1$, we write $G = G(1, 1, 1), G_n = G_n(1, 1, 1), G_n = G_n(1, 1, 1)$ for short.

Let $G(a, b; c; x) = \sum_{n=0}^{\infty} G_n(a, b, c)x^n$ be the generating function. According to the first return decomposition, a G-Motzkin path $P$ can be decomposed as one of the following four forms:

$$P = \varepsilon, \quad P = hQ_1, \quad P = uQ_1vQ_2 \text{ or } P = uQ_1dQ_2,$$
where $Q_1$ and $Q_2$ are (possibly empty) G-Motzkin paths. Then we get the relation

$$G(a, b, c; x) = 1 + axG(a, b, c; x) + bxG(a, b, c; x)^2 + cx^2G(a, b, c; x)^2. \quad (2.1)$$

Solve this, we have

$$G(a, b, c; x) = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4x(b + cx)}}{2x(b + cx)} = \frac{1}{1 - ax} C \left( \frac{x(b + cx)}{(1 - ax)^2} \right). \quad (2.2)$$

By (1.1), taking the coefficient of $x^n$ in $G(a, b, c; x)$, we get the following result.

**Proposition 2.1** For any integer $n \geq 0$, there holds

$$G_n(a, b, c) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-k}{j} \binom{n+k-j}{2k} C_k a^{n-k-j} b^k c^j$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{n+j}{2k} C_k a^{n-2k+j} b^j c^{k-j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{2k+j}{j} \binom{k}{n-k-j} C_k a^j b^{2k+j-n} c^{n-k-j}. \quad (2.1)$$

Set $T = xG(a, b, c; x)$, (2.1) produces

$$T = x \frac{1 + aT + cT^2}{1 - bT},$$

using the Lagrange inversion formula [23], taking the coefficient of $x^{n+1}$ in $T$ in three different ways, we derive the following proposition.

**Proposition 2.2** For any integer $n \geq 0$, there holds

$$G_n(a, b, c) = \frac{1}{n+1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{n-2k} \binom{n+1-k}{k} \binom{n+1-k-j}{n-2k-j} a^j b^{n-2k-j} c^k$$

$$= \frac{1}{n+1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{2n-k-2j} \binom{n+k}{k} \binom{2n-k-j}{n-k-2j} a^k b^{n-k-2j} c^j$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n+1-k}{k} \binom{2n-k-j}{n-k-j} a^k b^{n-k-j} c^j. \quad (2.3)$$
Clearly, $G_n = G_n(1, 1, 1)$ is the number of $G$-Motzkin paths of length $n$ with the generating function

$$G(x) = \sum_{n=0}^{\infty} G_n x^n = \frac{1 - x - \sqrt{1 - 6x - 3x^2}}{2x(1 + x)} = \frac{1}{1 - x} C\left(\frac{x(1 + x)}{(1 - x)^2}\right). \quad (2.4)$$

The explicit form of $G(x)$ was given by Drake [17] by counting lattice paths without regard to area and by Dziemiańczuk [18] by counting special lattice paths with four types of steps. The sequence

$$(G_n)_{n \geq 0} = (1, 2, 7, 29, 133, 650, 3319, 17498, 94525, 520508, 2910895, \ldots)$$

is denoted by A064641 in OEIS [43], and has the formula [18]

$$G_n = \frac{1}{n + 1} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n + 1}{j} \binom{j}{k - j} \binom{2n - k}{n}$$

which obeys the recurrence relation [43]

$$(n + 1)G_n = (5n - 4)G_{n-1} + 9(n - 1)G_{n-2} + 3(n - 2)G_{n-3}.$$ 

If setting $T = xG(1, 1, 1; x)$, (2.1) produces

$$T = x \frac{1 + T + T^2}{1 - T} = x \frac{1 - T^3}{(1 - T)^2},$$

using the Lagrange inversion formula, taking the coefficient of $x^{n+1}$ in $T$, one has another simple formula for $G_n$, namely,

$$G_n = \frac{1}{n + 1} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n + 1}{k} \binom{3n - 3k + 1}{n - 3k}.$$ 

Let $G_{n,k} = \frac{1}{n + 1} \binom{n + 1}{k} \binom{3n - 3k + 1}{n - 3k}$, the first values of $G_{n,k}$ are illustrated in Table 2. The following is an interesting identity related to $G_{n,k}$, that is,

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} G_{n+k,k} = \frac{1}{n + 1} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n + k}{k} \binom{3n + 1}{n - 2k} = 2^n C_n,$$

which can be proved as follows,
Table 2 The first values of \( G_{n,k} \)

| n/k | 0   | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   |     |     |     |     |     |
| 1   | 2   |     |     |     |     |     |
| 2   | 7   |     |     |     |     |     |
| 3   | 30  | 1   |     |     |     |     |
| 4   | 143 | 10  |     |     |     |     |
| 5   | 728 | 78  |     |     |     |     |
| 6   | 3876| 560 | 3   |     |     |     |
| 7   | 21318| 3876| 56  |     |     |     |
| 8   | 120175| 26334| 684 |     |     |     |
| 9   | 690690| 177100| 6930| 12  |     |     |

where \( [x^n] f(x) \) denotes the coefficient of \( x^n \) in \( f(x) \).

As is well known that the statistic enumerations of lattice paths have caught strong attention in the literature such as peaks, valleys [14], “number of ud’s” [48], strings in Dyck paths [39] and others [2, 10, 11, 13, 20, 21, 34]. In the following four subsections, we focus on several statistics “number of \( z \)-steps” in G-Motzkin paths for \( z \in \{u, h, v, d\} \). Let \( Z_{n,i} \) denote the number of G-Motzkin paths of length \( n \) with \( i \) \( z \)-steps for \( Z \in \{U, H, V, D\} \) and \( z \in \{u, h, v, d\} \). We not only derive the explicit formulas for \( Z_{n,i} \) in terms of Catalan numbers \( C_n \) by combinatorial methods, but also obtain several weighted alternating sums by bijective or algebraic methods. Specially, we build a relation between \( D_{n,i} \) and Narayana polynomials.

### 2.1 The Statistic “Number of v-Steps”

Let \( V_{n,i} \) denote the number of G-Motzkin paths of length \( n \) with \( i \) v-steps, the first values of \( V_{n,i} \) are illustrated in Table 3.

This shows that there is a close relation between \( V_{n,i} \) and \( C_n \). Exactly, each Dyck path \( P \) of length \( 2k \) can be extended to a G-Motzkin paths of length \( n \) with \( i \) v-steps for...
Table 3 The first values of $V_{n,i}$

| $n/i$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-------|-----|-----|-----|-----|-----|-----|-----|
| 0     | 1   |     |     |     |     |     |     |
| 1     | 1   | 1   |     |     |     |     |     |
| 2     | 2   | 3   | 2   |     |     |     |     |
| 3     | 4   | 10  | 10  | 5   |     |     |     |
| 4     | 9   | 30  | 45  | 35  | 14  |     |     |
| 5     | 21  | 90  | 175 | 196 | 126 | 42  |     |
| 6     | 51  | 266 | 644 | 924 | 840 | 462 | 132 |

Table 4 The first values of $H_{n,i}$

| $n/i$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-------|-----|-----|-----|-----|-----|-----|-----|
| 0     | 1   |     |     |     |     |     |     |
| 1     | 1   | 1   |     |     |     |     |     |
| 2     | 3   | 3   | 1   |     |     |     |     |
| 3     | 9   | 13  | 6   | 1   |     |     |     |
| 4     | 31  | 55  | 36  | 10  | 1   |     |     |
| 5     | 113 | 241 | 200 | 80  | 15  | 1   |     |
| 6     | 431 | 1071| 1080| 560 | 155 | 21  | 1   |

Let $H_{n,i}$ denote the number of G-Motzkin paths of length $n$ with $i$ h-steps, the first values of $H_{n,i}$ are illustrated in Table 4.

Note that there exist $2k+1$ points and $k$ d-steps in $P$, so there are $\binom{k}{i}$ ways to replace $i$ d-steps by v-steps and there are $\binom{2k+1}{n-2k+i} = \binom{n+i}{2k}$ ways to insert repeatedly $n-2k+i$ h-steps into $2k+1$ points of $P$ to form a G-Motzkin path of length $n$ with $i$ v-steps. Summing over $k$, we have the following result.

**Theorem 2.3** For any integers $n \geq i \geq 0$, there holds

$$V_{n,i} = \sum_{k=i}^{n} \binom{k}{i} \binom{n+i}{2k} C_k.$$

Note that $V_{n-i,i}$ also counts the set of G-Motzkin paths of length $n-i$ with $n$ steps and $V_{n,i}$ is also the coefficient of $b^i$ in $G_n(1, b, 1)$ in (2.3) which has another expression

$$V_{n,i} = \frac{1}{n+1} \binom{n+i}{i} \sum_{k=0}^{n-i} \binom{n+1}{k} \binom{k}{n-k-i}.$$

2.2 The Statistic “Number of h-Steps”

Let $H_{n,i}$ denote the number of G-Motzkin paths of length $n$ with $i$ h-steps, the first values of $H_{n,i}$ are illustrated in Table 4.

Note that each Dyck path $P$ of length $2k$ can be extended to a G-Motzkin path of length $n$ with $i$ h-steps for $\left\lfloor \frac{n-i}{2} \right\rfloor \leq k \leq n-i$. Similarly, here are $\binom{k}{n-i-k}$ ways to
replace \( n - i - k \) d-steps by v-steps and there are \( \binom{2k + 1}{i} = \binom{2k + i}{2k} \) ways to insert repeatedly \( i \) h-steps into \( 2k + 1 \) points of \( P \) to form a G-Motzkin path of length \( n \) with \( i \) h-steps. Summing over \( k \), we have the following result.

**Theorem 2.4** For any integers \( n \geq i \geq 0 \), there holds

\[
H_{n,i} = \sum_{k=\lceil \frac{n-i}{2} \rceil}^{n-i} \binom{2k + i}{2k} \binom{k}{n-i-k} C_k.
\]

Note that \( H_{n,i} \) is also the coefficient of \( a^i \) in \( G_n(a, 1, 1) \) in (2.3) which has another expression

\[
H_{n,i} = \frac{1}{n+1} \binom{n+1}{i} \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \binom{n+1-i}{j} \binom{2n-i-2j}{n-i-2j},
\]

and the special case \( G(-2, 1, 1; x) = \frac{1}{1+x} \) by (2.2) deduces the following identity whose combinatorial proof is also provided.

**Theorem 2.5** For any integer \( n \geq 0 \), there holds

\[
\sum_{i=0}^{n} (-2)^i H_{n,i} = (-1)^n.
\]

**Proof** Let \( \mathcal{H}_n^e (\mathcal{H}_n^o) \) denote the set of weighted G-Motzkin paths of length \( n \) with even (odd) number of h-steps such that each h-step is weighted by 2 (regarded as \( h_1 \) and \( h_2 \) for convenience) and other steps are weighted by 1. Clearly,

\[
w(\mathcal{H}_n^e) = \sum_{i \text{ even}} 2^i H_{n,i} \quad \text{and} \quad w(\mathcal{H}_n^o) = \sum_{i \text{ odd}} 2^i H_{n,i}.
\]

So it is sufficient to give a bijection \( \phi \) between \( \mathcal{H}_n^e/\{h_1^n\} \) and \( \mathcal{H}_n^o/\{h_1^n\} \) for \( n \) even and between \( \mathcal{H}_n^e \) and \( \mathcal{H}_n^o/\{h_1^n\} \) for \( n \) odd. When \( n \geq 2 \), any \( P \in \mathcal{H}_n^e/\{h_1^n\} \) for \( n \) even or \( P \in \mathcal{H}_n^e \) for \( n \) odd has at least one of the four subpaths, \( \tilde{d}, h_1v, h_2 \) and \( uv \), find the last one, say \( x \), \( P \) can be partitioned uniquely into \( P = P_1x^k1P_2 \), where \( P_2 = h_1^{k_2} \) for certain \( k_1 \geq 0 \) and \( 0 \leq k_2 \leq n-1 \). Then define \( \phi(P) = P_1x^{k_1'}P_2 \), where

\[
x' = \begin{cases} h_1v, & \text{if } x = d, \\ d, & \text{if } x = h_1v, \\ uv, & \text{if } x = h_2, \\ h_2, & \text{if } x = uv. \end{cases}
\]

This way ensures that the number of h-steps in \( \phi(P) \) is one more or less than that in \( P \in \mathcal{H}_n^e/\{h_1^n\} \) for \( n \) even or in \( P \in \mathcal{H}_n^e \) for \( n \) odd, so \( \phi(P) \in \mathcal{H}_n^o \) for \( n \) even and

\( \leq \) Springer
$\phi(P) \in \mathcal{H}_n^o/\{h_1^o\}$ for $n$ odd. Moreover, $x'$ in $\phi(P)$ is also the last one of the four subpaths, $d, h_1v, h_2$ and $uv$. The reverse procedure can be handled similarly. Hence, $\phi$ is a bijection between $\mathcal{H}_n^o/\{h_1^o\}$ and $\mathcal{H}_n^o$ for $n$ even and between $\mathcal{H}_n^o$ and $\mathcal{H}_n^o/\{h_1^o\}$ for $n$ odd. This completes the proof.

**Theorem 2.6** For any integers $n, m \geq 0$, there holds

$$
\sum_{i=0}^{2n} (-1)^i \binom{2n}{i} H_{n+m+i, m+i} = C_n.
$$

**Proof** By Theorem 2.4, we have

$$
\sum_{i=0}^{2n} (-1)^i \binom{2n}{i} H_{n+m+i, m+i} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \sum_{k=[\frac{n}{2}]}^{n} \frac{2k + m + i}{2k} \binom{k}{n-k} C_k
$$

$$
= \sum_{k=[\frac{n}{2}]}^{n} \binom{k}{n-k} C_k \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \frac{2k + m + i}{2k}
$$

$$
= (-1)^m \sum_{k=[\frac{n}{2}]}^{n} \binom{k}{n-k} C_k \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \frac{2k + m + i}{2k}
$$

$$
= (-1)^m \sum_{k=[\frac{n}{2}]}^{n} \binom{k}{n-k} C_k \binom{2n - 2k - 1}{2n + m}
$$

$$
= (-1)^m \binom{n}{0} C_n \binom{-1}{2n + m} (k = n)
$$

$$
= C_n,
$$

where the fourth equality follows by the Chu–Vandermonde identity. This completes the proof. □

**2.3 The Statistic “Number of d-Steps”**

Let $D_{n,i}$ denote the number of G-Motzkin paths of length $n$ with $i$ d-steps, the first values of $D_{n,i}$ are illustrated in Table 5.

Similarly, each Dyck path $P$ of length $2k$ can be extended to a G-Motzkin path of length $n$ with $i$ d-steps for $i \leq k \leq n - i$. Note that there are $\binom{k}{i}$ ways to replace $k - i$ d-steps by v-steps and there are $\binom{2k + 1}{n - k - i}$ ways to insert repeatedly $n - k - i$ h-steps into $2k + 1$ points of $P$ to form a G-Motzkin path of length $n$ with $i$ d-steps. Summing over $k$, we have the following result.
Table 5  The first values of $D_{n,i}$

|   | 0  | 1  | 2  | 3  | 4  | 5  |
|---|----|----|----|----|----|----|
| 0 | 1  |    |    |    |    |    |
| 1 | 2  |    |    |    |    |    |
| 2 | 6  | 1  |    |    |    |    |
| 3 | 22 | 7  |    |    |    |    |
| 4 | 90 | 41 | 2  |    |    |    |
| 5 | 394| 231| 25 |    |    |    |
| 6 | 1806|1289|219|5  |    |    |
| 7 | 8558|7183|1666|91 |    |    |
| 8 | 41586|40081|11780|1064|14 |    |

Theorem 2.7  For any integers $n \geq i \geq 0$, there holds

$$D_{n,i} = \sum_{k=i}^{n-i} \binom{k}{i} \binom{n-i+k}{2k} C_k.$$  

Note that $D_{n,i}$ is also the coefficient of $c^i$ in $G_n(1, 1, c)$ in (2.3) which has another expression

$$D_{n,i} = \frac{1}{n+1} \binom{n+1}{i} \sum_{k=i}^{n-i} \binom{n+1-i}{k-i} \binom{2n-i-k}{n-i-k}.$$  

Notice that $D_{2n,n}$ is the $n$th Catalan number $C_n$ and $D_{n,0}$ is the $n$th large Schröder number $R_n$. Since any G-Motzkin path of length $n$ with no $d$-steps can generate a Schröder path of length $2n$ by replacing each $h$-step by an $H$-step and each $v$-step by a $d$-step, and vice versa. The special cases $G(1, 1, -2; x) = \frac{1}{1-2x}$ and $G(1, 1, -1; x) = \frac{1}{1-x} C\left(\frac{x}{1-x}\right)$ by (2.2) deduce the following identities whose combinatorial proofs are also provided.

Theorem 2.8  For any integer $n \geq 0$, there holds

$$\sum_{i=0}^{n} (-2)^i D_{n,i} = 2^n,$$

$$\sum_{i=0}^{n} (-1)^i D_{n,i} = \sum_{k=0}^{n} \binom{n}{k} C_k.$$  

Proof  Let $D_n^e (D_n^o)$ denote the set of weighted G-Motzkin paths of length $n$ with even (odd) number of $d$-steps such that each $d$-step is weighted by 2 (regarded as $d_1$ and $d_2$ for convenience) and other steps are weighted by 1. Let $D_n^*$ be the subset of $D_n^e$ such that each path in $D_n^*$ has no $d$-steps and only consists of $h$-steps and $uv$-peaks.
Clearly,

\[ w(\mathcal{D}_n^e) = \sum_{i \text{ even}} 2^i D_{n,i}, \quad w(\mathcal{D}_n^o) = \sum_{i \text{ odd}} 2^i D_{n,i} \quad \text{and} \quad w(\mathcal{D}_n^w) = 2^n. \]

To prove (2.5), it is sufficient to give a bijection \( \tau \) between \( \mathcal{D}_n^e / \mathcal{D}_n^w \) and \( \mathcal{D}_n^o \). It is trivial for \( n = 0, 1 \). For \( n \geq 2 \), any \( \mathbf{P} \in \mathcal{D}_n^e / \mathcal{D}_n^w \) has at least one of the four subpaths, \( \mathbf{d}_1, \mathbf{d}_2, \mathbf{uvv} \) and \( \mathbf{hv} \), find the last one, say \( \mathbf{z}, \mathbf{P} \) can be partitioned uniquely into \( \mathbf{P} = \mathbf{P}_1 z^k \mathbf{P}_2 \) for \( k_1 \geq 0 \), where \( \mathbf{P}_2 \in \mathcal{D}_m^+ \) for certain \( 0 \leq m \leq n-2 \). Then define \( \tau(\mathbf{P}) = \mathbf{P}_1 z' z_1^{k_1} \mathbf{P}_2 \), where

\[
\begin{align*}
z' &= \left\{ \begin{array}{l}
\mathbf{uvv}, \text{ if } z = \mathbf{d}_1, \\
\mathbf{hv}, \text{ if } z = \mathbf{d}_2, \\
\mathbf{d}_1, \text{ if } z = \mathbf{uvv}, \\
\mathbf{d}_2, \text{ if } z = \mathbf{hv}.
\end{array} \right.
\end{align*}
\]

This way ensures that the number of \( \mathbf{d} \)-steps in \( \tau(\mathbf{P}) \) is one more or less than that in \( \mathbf{P} \in \mathcal{D}_n^e / \mathcal{D}_n^w \), so \( \tau(\mathbf{P}) \in \mathcal{D}_n^o \) and \( z' \) in \( \tau(\mathbf{P}) \) is also the last one of the four subpaths, \( \mathbf{d}_1, \mathbf{d}_2, \mathbf{uvv} \) and \( \mathbf{hv} \). The reverse procedure can be handled similarly. Hence, \( \tau \) is a bijection between \( \mathcal{D}_n^e / \mathcal{D}_n^w \) and \( \mathcal{D}_n^o \). This completes the proof of (2.5).

Let \( \mathcal{D}_n^e (\mathcal{D}_n^o) \) denote the set of weighted G-Motzkin paths of length \( n \) with even (odd) number of \( \mathbf{d} \)-steps such that each step is weighted by 1. Clearly,

\[
w(\mathcal{D}_n^e) = \sum_{i \text{ even}} D_{n,i} \quad \text{and} \quad w(\mathcal{D}_n^o) = \sum_{i \text{ odd}} D_{n,i}.
\]

Let \( \mathcal{D}_n^+ \) be the subset of \( \mathcal{D}_n^e \) such that each path \( \mathbf{Q} \in \mathcal{D}_n^+ \) has no \( \mathbf{d} \)-steps and no \( \mathbf{hv} \)-steps. Note that any \( \mathbf{Q} \in \mathcal{D}_n^+ \) with \( k \) \( \mathbf{u} \)-steps (with \( k \) \( \mathbf{v} \)-steps naturally) and \( n-k \) \( \mathbf{h} \)-steps can be obtained from Dyck paths \( \mathbf{Q}' \) of length \( 2k \) for \( 0 \leq k \leq n \) as follows. First replace each \( \mathbf{d} \)-step of \( \mathbf{Q}' \) by a \( \mathbf{v} \)-step to get a G-Motzkin path \( \mathbf{Q}'' \) with no \( \mathbf{h} \)-steps and no \( \mathbf{d} \)-steps, and there are \( \binom{k+1}{n-k} \) \( \binom{n}{k} \) ways to insert \( n-k \) \( \mathbf{h} \)-steps repeatedly into the \( k+1 \) positions exactly before \( k \) \( \mathbf{u} \)-steps and at the endpoint of the path \( \mathbf{Q}'' \) to get \( \mathbf{Q} \). This way can not produce \( \mathbf{d} \)-steps and \( \mathbf{hv} \)-steps in \( \mathbf{Q} \). Summing over \( k \), one has

\[
w(\mathcal{D}_n^+) = \sum_{k=0}^{n} \binom{n}{k} C_k.
\]

To prove (2.6), it is sufficient to give a bijection \( \bar{\tau} \) between \( \mathcal{D}_n^e / \mathcal{D}_n^+ \) and \( \mathcal{D}_n^o \). It is trivial for \( n = 0 \). For \( n \geq 1 \), any \( \mathbf{Q} \in \mathcal{D}_n^e / \mathcal{D}_n^+ \) has at least one of the two subpaths, \( \mathbf{d} \) and \( \mathbf{hv} \), find the last one, say \( \mathbf{z}, \mathbf{Q} \) can be partitioned uniquely into \( \mathbf{Q} = \mathbf{Q}_1 z \mathbf{Q}_2 \). Then define \( \bar{\tau}(\mathbf{Q}) = \mathbf{Q}_1 z' \mathbf{Q}_2 \), where

\[
z' = \left\{ \begin{array}{l}
\mathbf{hv}, \text{ if } z = \mathbf{d}, \\
\mathbf{d}, \text{ if } z = \mathbf{hv}.
\end{array} \right.
\]
This way ensures that the number of \( d \)-steps in \( \tau(Q) \) is one more or less than that in \( Q \in \bar{D}_n^e/\bar{D}_n^o \), so \( \tau(Q) \in \bar{D}_n^e \) and \( z' \) in \( \tau(Q) \) is also the last one of the two subpaths, \( d \) and \( hv \). The reverse procedure can be handled similarly. Hence, \( \bar{\tau} \) is a bijection between \( \bar{D}_n^e/\bar{D}_n^o \) and \( \bar{D}_n^o \). This completes the proof (2.6).

\[ \square \]

**Theorem 2.9** For any integer \( n \geq 0 \), there hold

\[ \sum_{i=0}^{n} (-1)^i D_{n+i,i} = 1, \quad (2.7) \]
\[ \sum_{i=0}^{n} (-2)^i D_{n+i,i} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise}. \end{cases} \quad (2.8) \]

**Proof** Let \( D_{n+i,i} \) denote the set of weighted G-Motzkin paths of length \( n+i \) with \( i \) \( d \)-steps such that all steps are weighted by 1. Set

\[ \bar{A}_n^e = \bigcup_{i=0, i \text{ even}}^n D_{n+i,i}, \quad \bar{A}_n^o = \bigcup_{i=0, i \text{ odd}}^n D_{n+i,i}. \]

Clearly,

\[ w(\bar{A}_n^e) = \sum_{i \text{ even}} D_{n+i,i} \quad \text{and} \quad w(\bar{A}_n^o) = \sum_{i \text{ odd}} D_{n+i,i}. \]

To prove (2.7), it is sufficient to give a bijection \( \phi \) between \( \bar{A}_n^e/\{h^n\} \) and \( \bar{A}_n^o \). It is trivial for \( n = 0 \). For \( n \geq 1 \), any \( P \in \bar{A}_n^e/\{h^n\} \) has at least a \( u \)-step, so there exist \( d \)-steps or \( v \)-steps in \( P \). Find the last return step \( z \), \( P \) can be partitioned uniquely into \( P = P_1zh^k \) for certain \( 0 \leq k < n \). Then define \( \phi(P) \) as follows:

\[ \phi(P) = \begin{cases} P_1dh^k, & \text{if } z = v, \\ P_1vh^k, & \text{if } z = d. \end{cases} \]

This way ensures that the number of \( d \)-steps in \( \phi(P) \) is one more or less than that in \( P \in \bar{A}_n^e/\{h^n\} \), so \( \phi(P) \in \bar{A}_n^o \). The reverse procedure can be handled similarly. Hence, \( \phi \) is a bijection between \( \bar{A}_n^e/\{h^n\} \) and \( \bar{A}_n^o \). This completes the proof of (2.7).

Let \( \bar{D}_{n+i,i} \) denote the set of weighted G-Motzkin paths of length \( n+i \) with \( i \) \( d \)-steps such that each \( d \)-step is weighted by 2 (regarded as \( d_1 \) and \( d_2 \) for convenience) and other steps are weighted by 1. Set

\[ \bar{\bar{A}}_n^e = \bigcup_{i=0, i \text{ even}}^n \bar{D}_{n+i,i}, \quad \bar{\bar{A}}_n^o = \bigcup_{i=0, i \text{ odd}}^n \bar{D}_{n+i,i}. \]

Clearly,

\[ w(\bar{\bar{A}}_n^e) = \sum_{i \text{ even}} 2^i D_{n+i,i} \quad \text{and} \quad w(\bar{\bar{A}}_n^o) = \sum_{i \text{ odd}} 2^i D_{n+i,i}. \]
It is trivial for \( n = 0 \) in (2.8). To prove (2.8), it is sufficient to give a bijection \( \tilde{\varphi} \) between \( \hat{A}_n^c \) and \( \hat{A}_n^o \) for \( n \geq 1 \). Note that any \( Q \in \hat{A}_n^c \) for \( n \geq 1 \) has at least one of the four subpaths, \( h, ud_1, d_2 \) and \( v \), find the last one, say \( z \), \( Q \) can be partitioned uniquely into \( Q = Q_1 z d_1^k \) for certain \( 0 \leq k < n \). Then define \( \tilde{\varphi}(Q) = Q_1 z d_1^k \), where

\[
Q = \begin{cases}
    ud_1, & \text{if } z = h, \\
    u, & \text{if } z = ud_1, \\
    v, & \text{if } z = d_2, \\
    d_2, & \text{if } z = v.
\end{cases}
\]

This way ensures that the number of \( d \)-steps in \( \tilde{\varphi}(Q) \) is one more or less than that in \( Q \in \hat{A}_n^c \), so \( \tilde{\varphi}(Q) \in \hat{A}_n^o \) and \( z' \) in \( \tilde{\varphi}(Q) \) is also the last one of the four subpaths, \( h, ud_1, d_2 \) and \( v \). The reverse procedure can be handled similarly. Hence, \( \tilde{\varphi} \) is a bijection between \( \hat{A}_n^c \) and \( \hat{A}_n^o \). This completes the proof of (2.8). \( \square \)

**Theorem 2.10** For any integer \( n \geq 0 \), there hold

\[
\sum_{i=0}^{n} y^i D_{n+i,i} = (y + 1)^n N_n \left( \frac{y + 2}{y + 1} \right), \tag{2.9}
\]

where \( N_n(y) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} y^k = y^{n+1} N_n(y^{-1}) \) with \( N_0(y) = 1 \) is the Narayana polynomial.

**Proof** Let \( \hat{D}_{n+i,i} \) denote the set of weighted G-Motzkin paths of length \( n + i \) with \( i \) \( d \)-steps such that each \( d \)-step is weighted by \( y \) (regarded as \( d_y \) for convenience) and other steps are weighted by 1. Let \( \hat{C}_{n,k} \) denote the set of weighted Dyck paths of length \( 2n \) with \( k \) \( ud \)-peaks such that each \( ud \)-step is weighted by \( y + 2 \) (regarded as \( d_y \) and \( d_2 \) for convenience) and other \( d \)-steps are weighted by \( y + 1 \) (regarded as \( d_y \) and \( d_1 \) for convenience). Set \( \hat{D}_n = \bigcup_{i=0}^{n} \hat{D}_{n+i,i} \) and \( \hat{C}_n = \bigcup_{k=0}^{n} \hat{C}_{n,k} \). Since the Narayana number \( N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k} \) counts the number of Dyck paths of length \( 2n \) with \( k \) \( ud \)-peaks, it is clear that

\[
|\hat{C}_{n,k}| = N_{n,k}(y + 2)^k (y + 1)^{n-k}.
\]

So it is sufficient to give a bijection \( \hat{\varphi} \) between \( \hat{C}_n \) and \( \hat{D}_n \) for \( n \geq 1 \). For any \( Q \in \hat{C}_n \) for \( n \geq 1 \) with \( k_0 d_1 \)-steps, \( k_1 d_1 \)-steps and \( k_2 d_2 \)-steps, note that \( k_0 + k_1 + k_2 = n \) and each \( d_2 \)-step must in a \( ud_2 \)-peak, replace each \( d_1 \)-step by \( v \)-step and each \( ud_2 \)-peak by \( h \)-step, we obtain a weighted G-Motzkin path \( Q' \in \hat{D}_{n+k_0,k_0} \). Then define \( \hat{\varphi}(Q) = Q' \). It is not difficult to verify that \( \hat{\varphi} \) is a bijection \( \hat{C}_n \) and \( \hat{D}_n \). This completes the proof of (2.9). \( \square \)

In order to give a more intuitive view on the bijection \( \hat{\varphi} \), a pictorial description of \( \hat{\varphi} \) is presented for \( Q = ud_y uud_2 uuud, d_1 d_y uuuud_2 ud_1 d_y d_1 d_1 d_1 uud_2 uuud, d_1 d_1 \), we have

\[
\hat{\varphi}(Q) = ud_y uhuuuud_y vd_y uuuhuvd_y dvvvhuuud_y vv.
\]
See Fig. 2 for detailed illustrations.

Note that the cases $y = -1$ and $y = -2$ in (2.9) lead to (2.7) and (2.8) respectively. When $y = -3$ in (2.9), by the relation $R_{n} = N_{n}(2) = 2r_{n}$ for $n \geq 1$, one can derive that the following identity which is asked for a direct combinatorial proof similar to that of Theorem 2.9.

**Corollary 2.11** For any integer $n \geq 0$, there holds

$$\sum_{i=0}^{n} (-3)^{i} D_{n+i,i} = (-1)^{n} r_{n},$$

where $r_{n}$ is the little Schröder number.

### 2.4 The Statistic “Number of u-Steps”

Let $U_{n,i}$ denote the number of G-Motzkin paths of length $n$ with $i$ u-steps, the first values of $U_{n,i}$ are illustrated in Table 6.

Note that each Dyck path $P$ of length $2i$ can be extended to a G-Motzkin path of length $n$ with $i$ u-steps for $0 \leq i \leq n$. For $0 \leq k \leq i$ there are $\binom{i}{k}$ ways to replace $k$ d-steps by v-steps and $\left(\binom{2i+1}{n-2i+k}\right) = \binom{n+k}{2i}$ ways to insert repeatedly $n-2i+k$ h-steps into $2i+1$ points of $P$ to form a G-Motzkin path of length $n$ with $i$ u-steps. Summing over $k$, we have the following result.

**Theorem 2.12** For any integers $n \geq i \geq 0$, there holds

$$U_{n,i} = \sum_{k=0}^{i} \binom{i}{k} \binom{n+k}{2i} C_{i}.$$

Note that $U_{n,i}$ is also the coefficient of $b^i$ in $G_n(1, b, b)$ in (2.3) which has another expression

$$U_{n,i} = \frac{1}{n+1} \binom{n+1}{i+1} \sum_{j=0}^{i} \binom{n-i}{j} \binom{n+i-j}{i-j},$$

and $U_{n,n}$ is the $n$th Catalan number $C_{n}$. By (2.2), the special case $G(1, -2, -2; x) = \frac{1}{1+x}$ deduces the following identity. We provide a combinatorial proof for this identity.

**Theorem 2.13** For any integer $n \geq 0$, there holds

$$\sum_{i=0}^{n} (-2)^{i} U_{n,i} = (-1)^{n}.$$
Fig. 2 An example of the bijection \( \hat{\phi} \) described in the proof of Theorem 2.10.

\[
Q = ud_yuudd_yuuud_yd_yuuudd_yud_1d_yd_yd_1ud_2uuud_yd_1d_1
\]

\[
\hat{\phi}(Q) = ud_yuhuuudd_yvd_yuuuhuvd_yd_yvvhuudd_yvv
\]
Table 6  The first values of $U_{n,i}$

| $n/i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| 0     | 1 |   |   |   |   |   |   |
| 1     | 1 | 1 |   |   |   |   |   |
| 2     | 1 | 4 | 2 |   |   |   |   |
| 3     | 1 | 9 | 14| 5 |   |   |   |
| 4     | 1 | 16| 52| 50| 14|   |   |
| 5     | 1 | 25| 140|260|182|42 |   |
| 6     | 1 | 36| 310|950|1218|462|132|

Proof  Let $U^e_n$ ($U^o_n$) denote the set of weighted G-Motzkin paths of length $n$ with even (odd) number of $u$-steps such that the matching step of each $u$-step, that is each of $d$-steps and $v$-steps is weighted by 2 (regarded respectively as $d_1$ and $d_2$, $v_1$ and $v_2$ for convenience) and other steps are weighted by 1. Clearly,

$$w(U^e_n) = \sum_{i \text{ even}} 2^i U_{n,i} \quad \text{and} \quad w(U^o_n) = \sum_{i \text{ odd}} 2^i U_{n,i}.$$

So it is sufficient to give a bijection $\theta$ between $U^e_n/\{(uv_1)^n\}$ and $U^o_n$ for $n$ even and between $U^e_n$ and $U^o_n/\{(uv_1)^n\}$ for $n$ odd. When $n \geq 1$, any $P \in U^e_n/\{(uv_1)^n\}$ for $n$ even or $P \in U^e_n$ for $n$ odd has at least one of the six subpaths, $h$, $uv_2$, $d_1$, $uv_1^2$, $d_2$ and $uv_1v_2$, find the last one, say $s$, $P$ can be partitioned uniquely into $P = P_1sP_2$, where $P_2 = t_1 \cdots t_k$ and $t_j \in \{v_1, v_2\}$ for $0 \leq j \leq k < n$. Then define $\theta(P) = P_1s'P_2$, where

$$s' = \begin{cases} 
  uv_2, & \text{if } s = h, \\
  h, & \text{if } s = uv_2, \\
  uv_1^2, & \text{if } s = d_1, \\
  d_1, & \text{if } s = uv_1^2, \\
  uv_1v_2, & \text{if } s = d_2, \\
  d_2, & \text{if } s = uv_1v_2.
\end{cases}$$

This way ensures that the number of $u$-steps in $\theta(P)$ is one more or less than that in $P \in U^e_n/\{(uv_1)^n\}$ for $n$ even or in $P \in U^e_n$ for $n$ odd, so $\theta(P) \in U^o_n$ for $n$ even and $\theta(P) \in U^o_n/\{(uv_1)^n\}$ for $n$ odd. Moreover, $s'$ is also the last one of the six subpaths, $h$, $uv_2$, $d_1$, $uv_1^2$, $d_2$ and $uv_1v_2$. The reverse procedure can be handled similarly. Hence, $\theta$ is a bijection between $U^e_n/\{(uv_1)^n\}$ and $U^o_n$ for $n$ even and between $U^e_n$ and $U^o_n/\{(uv_1)^n\}$ for $n$ odd. This completes the proof.  \hfill $\square$
Table 7 The first values of $B_{n,i}$

| $n/i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| 0     | 1  |    |    |    |    |    |    |
| 1     | 3  | 1  |    |    |    |    |    |
| 2     | 9  | 5  | 1  |    |    |    |    |
| 3     | 28 | 20 | 7  | 1  |    |    |    |
| 4     | 90 | 75 | 35 | 9  | 1  |    |    |
| 5     | 297| 275|154| 54|11|1|    |
| 6     | 1001|1001|637|273|77|13|1  |

Table 8 The first values of $C_{n,i}$

| $n/i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| 0     | 1  |    |    |    |    |    |    |
| 1     | 2  | 1  |    |    |    |    |    |
| 2     | 5  | 4  | 1  |    |    |    |    |
| 3     | 14 | 14 | 6  | 1  |    |    |    |
| 4     | 42 | 48 | 27 | 8  | 1  |    |    |
| 5     | 132|165 |110|44 |10|1|    |
| 6     | 429|572|429|208|65|12|1  |

3 The Statistics “Number of z-Steps” at Given Level in G-Motzkin Paths

In the literature, there are several statistics considered at given level in Dyck paths, such as “number of $u$-steps”, “number of $ud$-peaks” and “number of points” [6], strings at height $j$ [33] and Dyck paths with peaks and valleys avoiding an arbitrary set of heights [31]. Precisely, let $B_{n,i}$ be the number of $u$-steps” at level $i+1$ in all Dyck paths in $C_{n+1}$ and $C_{n,i}$ be the number of points at level $i$ in all Dyck paths in $C_n$. It is known that $B_{n,i} = \frac{2i+3}{2n+3} \binom{2n+3}{n-i}$ is the bollat number and $C_{n,i} = \frac{i+1}{n+1} \binom{2n+1}{n-i}$ also enumerates the number of $ud$-peaks at level $i+1$ in all Dyck paths in $C_{n+1}$. The first values of $B_{n,i}$ and $C_{n,i}$ are illustrated respectively in Tables 7 and 8.

Actually, the matrices $(C_{n,i})_{n\geq i \geq 0}$ and $(B_{n,i})_{n\geq i \geq 0}$ form Riordan arrays $(C(x)^2, xC(x)^2)$ and $(C(x)^3, xC(x)^2)$ respectively. Recall that a Riordan array [40, 41, 44] is an infinite lower triangular matrix $\mathcal{D} = (d_{n,i})_{n,i\in\mathbb{N}}$ such that its $i$-th column has generating function $d(x)h(x)^i$, where $d(x)$ and $h(x)$ are formal power series with $d(0) = 1$ and $h(0) = 0$. That is, the general term of $\mathcal{D}$ is $d_{n,i} = [x^n]d(x)h(x)^i$, where $[x^n]$ is the coefficient operator. The matrix $\mathcal{D}$ corresponding to the pair $d(x)$ and $h(x)$ is denoted by $(d(x), h(x))$. The product of a Riordan array $(d(x), h(x))$ and a formal power series $A(x) = \sum_{n\geq 0} a_n x^n$ is given by $(d(x), h(x))A(x) = d(x)A(h(x))$, this implies that

$$\sum_{i=0}^{n} d_{n,i}a_i = [x^n]d(x)A(h(x)).$$  \hfill (3.1)
Table 9 The first values of $\alpha_{n,i}$

| $n/i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| 0     | 1 |   |   |   |   |   |   |
| 1     | 7 | 1 |   |   |   |   |   |
| 2     | 39| 12| 1 |   |   |   |   |
| 3     | 212| 96| 17| 1 |   |   |   |
| 4     | 1157| 665| 178| 22| 1 |   |   |
| 5     | 6384| 4320| 1513| 285| 27| 1 |   |
| 6     | 35,647| 27,177| 11,522| 2881| 417| 32| 1 |

In the following four subsections, we focus on the enumeration of statistics “number of $z$-steps” for $z \in \{u, h, v, d\}$, “number of return steps” and “number of points” at given level in G-Motzkin paths. Some counting results are linked with Riordan arrays, and some weighted alternating sum identities related to these counting formulas are also derived by using Riordan arrays.

3.1 The Statistic “Number of u-Steps” at Level $i + 1$

Let $\alpha_{n,i}$ denote the number of $u$-steps at level $i + 1$ in all G-Motzkin paths of length $n + 1$, the first values of $\alpha_{n,i}$ are illustrated in Table 9.

Theorem 3.1 For any integers $n \geq i \geq 0$, there holds

$$
\alpha_{n,i} = \sum_{j=i}^{n} B_{j,i} \sum_{k=0}^{n-j} \binom{j+1}{k} \binom{n+j-k+2}{n-j-k}.
$$

Moreover, $\alpha_{n,i}$ is the $(n, i)$-entry of the Riordan array

$$
\left(\frac{1 + x}{(1 - x)^3} C \left(\frac{x(1+x)}{(1-x)^2}\right)^3, \frac{x(1+x)}{(1-x)^2} C \left(\frac{x(1+x)}{(1-x)^2}\right)^2\right).
$$

Proof For each Dyck path $P$ of length $2j + 2$, it can be extended to G-Motzkin paths $Q$ of length $n + 1$ for $i \leq j \leq n$ such that $P$ and $Q$ have the same number of $u$-steps at level $i + 1$. First replace $j + 1 - k$ $d$-steps in $P$ by $v$-steps to get $P'$, there are $\binom{j+1}{k}$ ways, and insert repeatedly $n - j - k$ $h$-steps into $2j + 3$ points of $P'$ to form G-Motzkin paths $Q$ of length $n + 1$, there $\binom{2j+3}{n-j-k}$ ways. Note that there are totally $B_{j,i}$ $u$-steps at level $i + 1$ in all $P \in C_{j+1}$, summing over $k$ and $j$, we obtain the desired result.

On the other hand, for any G-Motzkin path $G \in G$ with at least one $u$-step at level $i + 1$, given such a $u$-step, marked as $u^*$, $G$ can be partitioned uniquely into

$$
G = G_0 uG_1 \ldots uG_{i+1}z_1G_1z_2G_2 \ldots z_{i+1}G_{i+1},
$$

$\Box$ Springer
where $G_0, \ldots, G_{i+1}, \tilde{G}_0, \ldots, \tilde{G}_{i+1} \in \mathcal{G}$ and $z_1, \ldots, z_{i+1} \in [d, v]$. Since each of $G_k$ and $\tilde{G}_k$ has the generating function $G(x)$, each of $u$ and $d$ produces an $x$ and each $v$ leads to a 1, this makes $z_1 z_2 \ldots z_{i+1}$ generate $(1 + x)^{i+1}$, so according to the length of $G$, all $G \in \mathcal{G}$ produce the generating function $x^{i+1} (1 + x)^{i+1} G(x)^{2i+3}$. Hence, the total number $\alpha_{n,i}$ of $u^*$-steps in all G-Motzkin paths $G \in \mathcal{G}_{n+1}$ is the coefficient of $x^{n+1}$ in $x^{i+1} (1 + x)^{i+1} G(x)^{2i+3}$, namely,

$$\alpha_{n,i} = [x^{n+1}] x^{i+1} (1 + x)^{i+1} G(x)^{2i+3} = [x^n] (1 + x) G(x)^3 \left( x (1 + x) G(x)^2 \right)^i.$$

By (2.4), $\alpha_{n,i}$ is the $(n, i)$-entry of the Riordan array

$$((1 + x) G(x)^3, x(1 + x) G(x))^2 = \left( \frac{1 + x}{1 - x} \right)^3 C \left( \frac{x(1 + x)}{(1 - x)^2} \right)^3, \frac{x(1 + x)}{(1 - x)^2} \right)^2 C \left( \frac{x(1 + x)}{(1 - x)^2} \right)^2.$$

This completes the proof of Theorem 3.1. $\square$

**Theorem 3.2** For any integers $n, m \geq 0$, there holds

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \alpha_{n+m+i, m+i} = 5^n. \quad (3.2)$$

**Proof** By Theorem 3.1, using the relation $G(x) = 1 + x G(x) + x G(x)^2 + x^2 G(x)^2$, we have

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \alpha_{n+m+i, m+i}$$

$$= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} [x^{n+m+i}] (1 + x) G(x)^3 \left( x (1 + x) G(x)^2 \right)^m$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (1 + x) G(x)^2^i$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \left( (1 + x) G(x)^2 - 1 \right)^n$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \left( \frac{G(x) - x G(x) - 1}{x} \right)^n$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \left( \sum_{k=0}^{\infty} (G_{k+1} - G_k) x^k - 1 \right)^n$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \left( 5 x + \sum_{k=2}^{\infty} (G_{k+1} - G_k) x^k \right)^n$$

$$= [x^n] (1 + x)^{m+1} G(x)^{2m+3} \left( 5 + \sum_{k=2}^{\infty} (G_{k+1} - G_k) x^{k-1} \right)^n$$

$$= 5^n.$$
Table 10  The first values of $\beta_{n,i}$ and $\gamma_{n,i}$

| n/i | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   |     |     |     |     |     |     |
| 1   | 6   | 1   |     |     |     |     |     |
| 2   | 33  | 11  | 1   |     |     |     |     |
| 3   | 179 | 85  | 16  | 1   |     |     |     |
| 4   | 978 | 580 | 162 | 21  | 1   |     |     |
| 5   | 5406| 3740| 1351| 264 | 26  | 1   |     |
| 6   | 30,241| 23,437| 10,171| 2617| 391 | 31  | 1   |

This completes the proof.  

**Theorem 3.3**  For any integer $n \geq 0$, there holds

$$\sum_{i=0}^{n} (-1)^i \binom{i+2}{2} \alpha_{n,i} = (n+1)^2.$$  

**(3.3)**

**Proof**  By (3.1) and Theorem 3.2, together with the relation $C(x) = 1 + xC(x)^2$, we have

$$\sum_{i=0}^{n} (-1)^i \binom{i+2}{2} \alpha_{n,i} = [x^n] \frac{1+x}{(1-x)^3} \frac{\left(\frac{x(1+x)}{1-x}\right)^3}{\left(\frac{x(1+x)}{1-x}\right)^2} \frac{1}{1+x}.$$

$$= [x^n] \frac{1+x}{(1-x)^3} = (n+1)^2.$$

This completes the proof.  

One can be asked for combinatorial proofs of these two identities (3.2) and (3.3).

### 3.2 The Statistics “Number of v-Steps” and “Number of d-Steps” at Level $i$

Let $\beta_{n,i}$ denote the number of $v$-steps at level $i$ in all G-Motzkin paths of length $n + 1$ and let $\gamma_{n,i}$ denote the number of $d$-steps at level $i$ in all G-Motzkin paths of length $n + 2$, the first values of $\beta_{n,i}$ and $\gamma_{n,i}$ are illustrated in Table 10.

**Lemma 3.4**  For any integers $n \geq i \geq 0$, there holds

$$\beta_{n,i} = \gamma_{n,i}.$$  

**Proof**  Given a $v$-step counted at level $i$ in a G-Motzkin path $P$ of length $n + 1$, replace it by a $d$-step, one get a G-Motzkin path $P'$ of length $n + 2$ with a $d$-step counted at level $i$ and vice versa. This implies that $\beta_{n,i} = \gamma_{n,i}$.  

$\square$
Theorem 3.5  For any integers \( n \geq i \geq 0 \), there holds

\[
\beta_{n,i} = \gamma_{n,i} = \sum_{j=i}^{n} B_{j,i} \sum_{k=0}^{j} \binom{j}{k} \left( \frac{n + j + 2 - k}{n - j - k} \right).
\]

Moreover, \( \beta_{n,i} \) is the \((n, i)\)-entry of the Riordan array

\[
\left( \frac{1}{(1-x)^3} C \left( \frac{x(1+x)}{(1-x)^2} \right)^3, \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right).
\]

Proof For any G-Motzkin path \( G \in \mathcal{G} \) with at least one \( v \)-step at level \( i \), given such a \( v \)-step, marked as \( v^* \), \( G \) can be partitioned uniquely into

\[
G = G_0 u G_1 \ldots u G_i u G_{i+1} v^* G_0 \bar{G}_1 \ldots \bar{G}_i.
\]

where \( G_0, \ldots, G_{i+1}, \bar{G}_0, \ldots, \bar{G}_i \in \mathcal{G} \) and \( z_1, \ldots, z_i \in \{d, v\} \). Similar to the proof of Theorem 3.1, according to the length of \( G \), all \( G \in \mathcal{G} \) produce the generating function \( x^{i+1} (1+x)^i G(x)^{2i+3} \). Hence, the total number \( \beta_{n,i} \) of \( v^* \)-steps in all G-Motzkin paths \( G \in \mathcal{G}_{n+1} \) is the coefficient of \( x^{n+1} \) in \( x^{i+1} (1+x)^i G(x)^{2i+3} \), namely,

\[
\beta_{n,i} = [x^{n+1}] x^{i+1} (1+x)^i G(x)^{2i+3} = [x^n] G(x)^3 (x(1+x)G(x))^2 \big|_{x^i}.
\]

By (2.4), \( \beta_{n,i} \) is the \((n, i)\)-entry of the Riordan array

\[
\left( G(x)^3, x(1+x)G(x)^2 \right) = \left( \frac{1}{(1-x)^3} C \left( \frac{x(1+x)}{(1-x)^2} \right)^3, \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right).
\]

By the relation \( B_{j,i} = [x^j] C(x)^3 (x C(x)^2)^i \) and by Lemma 3.4, we have

\[
\gamma_{n,i} = \beta_{n,i} = x^n \frac{1}{(1-x)^3} C \left( \frac{x(1+x)}{(1-x)^2} \right)^3 \left( \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right)^i
\]

\[
= x^n \sum_{j=0}^{\infty} B_{j,i} x^j (1+x)^j (1-x)^{2j} = x^n \sum_{j=i}^{\infty} B_{j,i} x^j (1+x)^j (1-x)^{2j+3}
\]

\[
= \sum_{j=i}^{n} B_{j,i} \sum_{k=0}^{n-j} \binom{j}{k} \left( \frac{n + j + 2 - k}{n - j - k} \right).
\]

This completes the proof of Theorem 3.5. \( \square \)

By Theorem 3.5, similar to the proofs of Theorem 3.2 and 3.3, we have the following result.
Theorem 3.6 For any integers \( n, m \geq 0 \), there holds

\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \beta_{n+m+i, m+i} = 5^n, \tag{3.4}
\]

\[
\sum_{i=0}^{n} (-1)^i \binom{i+2}{2} \beta_{n,i} = \binom{n+2}{2}. \tag{3.5}
\]

One can be asked for combinatorial proofs for these two identities (3.4) and (3.5).

Note that any \( u \)-step at level \( i + 1 \) in a G-Motzkin path \( P \) of length \( n + 1 \) has a matching step, it is a \( v \)-step or \( d \)-step at level \( i \), together with Lemma 3.4, we have the following result.

Corollary 3.7 For any integers \( n \geq i \geq 0 \), there holds

\[
\alpha_{n,i} = \beta_{n,i} + \beta_{n-1,i}.
\]

3.3 The Statistics “Number of h-Steps” and “Number of Points” at Level \( i \)

Let \( \mu_{n,i} \) denote the number of \( h \)-steps at level \( i \) in all G-Motzkin paths of length \( n + 1 \) and let \( \lambda_{n,i} \) denote the number of points at level \( i \) in all G-Motzkin paths of length \( n \), the first values of \( \mu_{n,i} \) and \( \lambda_{n,i} \) are illustrated in Table 11.

| \( n/i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 4 | 1 | | | | | |
| 2 | 18 | 9 | 1 | | | | |
| 3 | 86 | 60 | 14 | 1 | | | |
| 4 | 431 | 368 | 127 | 19 | 1 | | |
| 5 | 2238 | 2190 | 970 | 219 | 24 | 1 | |
| 6 | 11,941 | 12,894 | 6803 | 2017 | 336 | 29 | 1 |

Lemma 3.8 For any integers \( n \geq i \geq 0 \), there holds

\[
\lambda_{n,i} = \mu_{n,i}.
\]

Proof Given a point at level \( i \) in a G-Motzkin path \( P \) of length \( n \), insert an \( h \)-step into the point, one get a G-Motzkin path \( P' \) of length \( n + 1 \) with an \( h \)-step counted at level \( i \). Conversely, given an \( h \)-step at level \( i \) in a G-Motzkin path \( P' \) of length \( n + 1 \), remove the \( h \)-step, one get a G-Motzkin path \( P \) of length \( n \) with a point counted at level \( i \). This one-to-one mapping implies that \( \lambda_{n,i} = \mu_{n,i} \). \qed
Theorem 3.9  For any integers \( n \geq i \geq 0 \), there holds

\[
\mu_{n,i} = \lambda_{n,i} = \sum_{j=i}^{n} C_{j,i} \sum_{k=0}^{n-j} \binom{n+j-k+1}{n-j-k}.
\]

Moreover, \( \mu_{n,i} \) is the \((n, i)\)-entry of the Riordan array

\[
\left( \frac{1}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2, \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right).
\]

**Proof** For any G-Motzkin path \( G \in \mathcal{G} \) with at least one \( h \)-step at level \( i \), given such an \( h \)-step, marked as \( h^* \), \( G \) can be partitioned uniquely into

\[ G = G_0 u G_1 \ldots u G_i h^* G_0 z_1 \bar{G}_1 \ldots z_i \bar{G}_i, \]

where \( G_0, \ldots, G_i, \bar{G}_0, \ldots, \bar{G}_i \in \mathcal{G} \) and \( z_1, \ldots, z_i \in \{ d, v \} \). Similar to the proof of Theorem 3.1, according to the length of \( G \), all \( G \in \mathcal{G} \) produce the generating function \( x^{i+1} (1+x)^i G(x)^{2i+2} \). Hence, the total number \( \mu_{n,i} \) of \( h^* \)-steps in all G-Motzkin paths \( G \in \mathcal{G}_{n+1} \) is the coefficient of \( x^{n+1} \) in \( x^{i+1} (1+x)^i G(x)^{2i+2} \), namely,

\[ \mu_{n,i} = [x^{n+1}] x^{i+1} (1+x)^i G(x)^{2i+2} = [x^n] G(x)^2 \left( x(1+x) G(x)^2 \right)^i. \]

By (2.4), \( \mu_{n,i} \) is the \((n, i)\)-entry of the Riordan array

\[
\left( G(x)^2, x(1+x) G(x)^2 \right) = \left( \frac{1}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2, \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right).
\]

By the relation \( C_{j,i} = [x^j] C(x)^2 (x C(x)^2)^i \) and by Lemma 3.8, we have

\[
\lambda_{n,i} = \mu_{n,i} = [x^n] \frac{1}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \left( \frac{x(1+x)}{(1-x)^2} C \left( \frac{x(1+x)}{(1-x)^2} \right)^2 \right)^i \]

\[
= [x^n] \frac{1}{(1-x)^2} \sum_{j=0}^{\infty} C_{j,i} \frac{x^j (1+x)^j}{(1-x)^{2j}} = [x^n] \sum_{j=0}^{\infty} C_{j,i} \frac{x^j (1+x)^j}{(1-x)^{2j+2}} \]

\[
= \sum_{j=i}^{n} C_{j,i} \sum_{k=0}^{n-j} \binom{n+j-k+1}{n-j-k}.
\]

This completes the proof of Theorem 3.9. \(\square\)

By Theorem 3.9, similar to the proofs of Theorem 3.2 and 3.3, we have the following result.
Theorem 3.10  For any integers $n, m \geq 0$, there holds

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \mu_{n+m+i,m+i} = \delta^n, \quad (3.6)$$

$$\sum_{i=0}^{n} (-1)^{i}(i + 1)\mu_{n,i} = n + 1. \quad (3.7)$$

One can be asked for combinatorial proofs of these two identities (3.6) and (3.7).

Remark 3.11  Note that if replacing the marked $h$-step $h^*$ by marked $uv$-peak, one can derive that the number of $uv$-peaks at level $i + 1$ in all G-Motzkin paths of length $n + 1$ is $\mu_{n,i}$. Similarly, if replacing the marked $h$-step $h^*$ by marked $ud$-peak, one can derive that the number of $ud$-peaks at level $i + 1$ in all G-Motzkin paths of length $n + 2$ is also $\mu_{n,i}$.  

3.4 The Statistic “Number of Return Steps”

Let $r_{n,i}$ denote the number of G-Motzkin paths of length $n$ with $i$ return steps, the first values of $r_{n,i}$ are illustrated in Table 12.

| $n/i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| 0     | 1 |   |   |   |   |   |   |
| 1     | 1 | 1 |   |   |   |   |   |
| 2     | 1 | 5 | 1 |   |   |   |   |
| 3     | 1 | 18| 9 | 1 |   |   |   |
| 4     | 1 | 67| 51| 13| 1 |   |   |
| 5     | 1 | 278| 253| 100| 17| 1 |   |
| 6     | 1 | 1272| 1236| 623| 165| 21| 1 |

Theorem 3.12  For any integers $n \geq i \geq 0$, there holds

$$r_{n,i} = \sum_{j=i}^{n} \frac{i}{2j-i} \binom{2j-i}{j} \sum_{k=0}^{n-j} \binom{j}{k} \binom{n+j-k}{n-j-k}.$$

Moreover, $r_{n,i}$ is the $(n, i)$-entry of the Riordan array

$$\left(\frac{1}{1-x}, \frac{x(1+x)}{(1-x)^2} C\left(\frac{x(1+x)}{(1-x)^2}\right)\right).$$

Proof  For any G-Motzkin path $G \in \mathcal{G}$ with $i$ return steps, $G$ can be partitioned uniquely into

$$G = h^k_0 uG_1 z_1 h^k_1 uG_2 z_2 h^k_2 \ldots uG_i z_i h^k_i.$$
where \( G_1, \ldots, G_i \in \mathcal{G}, z_1, \ldots, z_i \in \{d, v\} \) and \( k_0, k_1, \ldots, k_i \geq 0 \). Similar to the proof of Theorem 3.1, according to the length of \( G \), all \( G \in \mathcal{G} \) produce the generating function \( \frac{1}{(1-x)^{i+1}} x^i (1+x)^i G(x)^i \). Hence, the total number \( r_{n,i} \) of return steps in all G-Motzkin paths \( G \in \mathcal{G}_n \) is the coefficient of \( x^n \) in \( \frac{1}{(1-x)^{i+1}} x^i (1+x)^i G(x)^i \), namely,

\[
r_{n,i} = [x^n] \frac{1}{(1-x)^{i+1}} x^i (1+x)^i G(x)^i = [x^n] \frac{1}{1-x} \left( \frac{x(1+x)}{1-x} G(x) \right)^i.
\]

By (2.4), \( r_{n,i} \) is the \((n, i)\)-entry of the Riordan array

\[
\left( \frac{1}{1-x}, \frac{x(1+x)}{1-x} G(x) \right) = \left( \frac{1}{1-x}, \frac{x(1+x)}{1-x} C \left( \frac{x(1+x)}{(1-x)^2} \right) \right).
\]

By the relation \( \frac{i}{2j-i} \binom{2j-i}{j} = [x^j](xC(x))^i \), we have

\[
r_{n,i} = [x^n] \frac{1}{1-x} \left( \frac{x(1+x)}{1-x} C \left( \frac{x(1+x)}{(1-x)^2} \right) \right)^i
= [x^n] \frac{1}{1-x} \sum_{j=i}^{\infty} \frac{i}{2j-i} \binom{2j-i}{j} x^j (1+x)^j \left( \frac{x(1+x)}{(1-x)^2} \right)^j
= \sum_{j=i}^{n} \frac{i}{2j-i} \binom{2j-i}{j} \sum_{k=0}^{n-j} \binom{j}{k} \frac{n-j-k}{n-j-k}.
\]

This completes the proof of Theorem 3.12. \( \square \)

By Theorem 3.12, similar to the proof of Theorem 3.2, we have the following result.

**Theorem 3.13** For any integers \( n, m \geq 0 \), there holds

\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} r_{n+m+i, m+i} = 4^n.
\]  \hspace{1cm} (3.8)

One can be asked for a combinatorial proof of this identity (3.8).

### 4 The Statistics “Number of \( z_1z_2\)-Steps” in G-Motzkin Paths

In this section, similar to the second section, we discuss the statistics “number of \( z_1z_2\)-steps” in G-Motzkin paths for \( z_1, z_2 \in \{u, h, v, d\} \). Despite that there are 16 cases to be considered, but in fact it only needs to study 10 cases in the set \( \{ud, uh, uu, hh, hd, vu, vv, du, dd, dv\} \). Let \( L_{n,i}^{z_1z_2} \) denote the number of G-Motzkin paths of length \( n \) with \( i \) \( z_1z_2 \)-steps, it is not difficult to verify that

\[
L_{n,i}^{uv} = H_{n,i}, \quad L_{n,i}^{hv} = L_{n,i}^{vh} = D_{n,i}, \quad L_{n,i}^{hu} = L_{n,i}^{uh} = L_{n,i}^{hd} = L_{n,i}^{dh}, \quad L_{n,i}^{dv} = L_{n,i}^{vd}.
\]
The results, $L_{n,i}^{uv} = H_{n,i}, L_{n,i}^{hv} = L_{n,i}^{vh} = D_{n,i},$ can be easily obtained respectively by replacing all the $uv$-steps ($hv$-steps, $vh$-steps) by new $h$-steps ($d$-steps) and replacing the original $h$-steps ($d$-steps) by $uv$-steps ($hv$-steps, $vh$-steps) in each G-Motzkin paths of length $n$ and vice versa. The results, $L_{n,i}^{uh} = L_{n,i}^{du} = L_{n,i}^{hd} = L_{n,i}^{dv} = L_{n,i}^{vd},$ can be easily obtained respectively by replacing all the $uh$-steps ($hd$-steps, $dv$-steps) by new $uh$-steps ($hd$-steps, $dv$-steps) and replacing the original $uh$-steps ($hd$-steps, $dv$-steps) (if exist) by $hu$-steps ( $hd$-steps, $dv$-steps) in each G-Motzkin paths of length $n$ and vice versa.

In the following 10 subsections, we derive the explicit formulas for $L_{n,i}^{z_1z_2}$ in terms of Catalan numbers $C_n$ by generating functions, and obtain several difference identities and weighted sum identities in the cases $z_1z_2 \in \{ud, vu, vv\}.$ What’s more interesting is that we provide combinatorial interpretations for weighted sums related to $L_{n,i}^{vu}$ by G-Motzkin paths.

#### 4.1 The Statistic “Number of ud-Peaks”

Let $L_{n,i}^{ud}(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_{n,i}^{ud} x^i y^j$ be the generating function of $L_{n,i}^{ud},$ the number of G-Motzkin paths of length $n$ with $i$ $ud$-peaks. According to the first return decomposition, a G-Motzkin path $P$ can be decomposed as one of the following five forms:

$$P = \varepsilon, \ P = hQ_1, \ P = uQ_1vQ_2, \ P = udQ_2, \ or \ P = uP_1dQ_2,$$

where $Q_1$ and $Q_2$ are (possibly empty) G-Motzkin paths and $P_1$ are nonempty. Then we get the relation

$$L_{n,i}^{ud}(x, y) = 1 + xL_{n,i}^{ud}(x, y) + xL_{n,i}^{ud}(x, y)^2 + x^2yL_{n,i}^{ud}(x, y) + x^2(L_{n,i}^{ud}(x, y) - 1)L_{n,i}^{ud}(x, y)$$

$$= 1 + x(1-x+xy)L_{n,i}^{ud}(x, y) + x(1+x)L_{n,i}^{ud}(x, y)^2.$$

Solve this, we have

$$L_{n,i}^{ud}(x, y) = \frac{1 - x + x^2 - x^2y - \sqrt{(1-x+x^2-x^2y)^2 - 4x(1+x)}}{2x(1+x)}$$

$$= \frac{1}{1 - x + x^2 - x^2y} \frac{x(1+x)}{(1-x+x^2-x^2y)^2}. \quad (4.1)$$

By (1.1) and (4.1), taking the coefficient of $x^ny^j$ in $L_{n,i}^{ud}(x, y),$ we derive the result as follows.

**Theorem 4.1** For any integers $n \geq i \geq 0,$ there holds

$$L_{n,i}^{ud} = \sum_{k=0}^{n-2i} \sum_{j=0}^{\left\lfloor \frac{n-k-2i}{2} \right\rfloor} (-1)^j \binom{2k+i}{i} \binom{2k+i+j}{j} \binom{3k+i+1}{n-k-2i-3j} C_k.$$
The first values of $L_{n,i}^{ud}$ are illustrated in Table 13.

Theorem 4.2 For any integers $n, m \geq 0$, there holds

$$\sum_{i=0}^{2n} (-1)^i \binom{2n}{i} L_{n+2m+2i,m+i}^{ud} = C_n.$$ 

**Proof** Note that

$$L_{n+2m+2i,m+i}^{ud} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} (-1)^j \binom{2k+m+i}{2k} \binom{2k+m+i+j}{j} \binom{3k+m+i+1}{n-k-3j} C_k,$$

each inner term in $L_{n+2m+2i,m+i}^{ud}$, denoted by $h_{n,m,k,j}(i)$, is a polynomial on $i$ with degree

$$\partial h_{n,m,k,j}(i) = 2k + j + (n - k - 3j) = n + k - 2j \leq 2n$$

such that $\partial h_{n,m,k,j}(i) = 2n$ if and only if $k = n$ and $j = 0$. Clearly, the leading term in

$$h_{n,m,n,0}(i) = \binom{2n+m+i}{2n} C_n$$

is $\frac{C_n}{(2n)!} i^{2n}$. So $L_{n+2m+2i,m+i}^{ud}$ is also a polynomial on $i$ with degree $2n$ such that the leading term is $\frac{C_n}{(2n)!} i^{2n}$. By the Euler difference identity

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^r = \begin{cases} 0, & \text{if } 0 \leq r < n, \\ (-1)^n n!, & \text{if } r = n, \end{cases}$$

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we have
\[
  \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} L_{n+2m+2i,m+i}^{ud} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} 2n \frac{C_n}{(2n)!} = C_n.
\]

This completes the proof. \qed

By (1.1) and (4.1), taking the coefficient of \(x^n\) in \(L_{n}^{ud}(x, -\frac{3}{x})\), we derive the following result.

**Theorem 4.3** For any integer \(n \geq 0\), there holds
\[
  \sum_{i=0}^{n} (-3)^i L_{n+i,i}^{ud} = \sum_{i=0}^{n} (-1)^{n-k} \binom{n+2k+1}{n-k} C_k.
\]

### 4.2 The Statistic "Number of uh-Steps"

Let \(L_{n,i}^{uh}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{uh} x^n y^i\) be the generating function of \(L_{n,i}^{uh}\), the number of G-Motzkin paths of length \(n\) with \(i\) uh-steps. According to the first return decomposition, a G-Motzkin path \(P\) can be decomposed as one of the following four forms:
\[
P = \varepsilon, \quad P = hQ, \quad P = uP_1vQ, \quad \text{or} \quad P = uP_1dQ_2,
\]
where \(P_1\) and \(Q_2\) are (possibly empty) G-Motzkin paths. If \(P_1\) begins with an \(h\)-step, \(uP_1\) contributes at least one uh-step. Then we get the relation
\[
L_{n,i}^{uh}(x, y) = 1 + xL_{n,i}^{uh}(x, y) + x(1 - x)L_{n,i}^{uh}(x, y)^2 + x^2 yL_{n,i}^{uh}(x, y)^2 + x^2(1 - x)L_{n,i}^{uh}(x, y)^2 + x^3 yL_{n,i}^{uh}(x, y)^2
\]
\[
= 1 + xL_{n,i}^{uh}(x, y) + x(1 + x)(1 - x + xy)L_{n,i}^{uh}(x, y)^2.
\]

Solve this, we have
\[
L_{n,i}^{uh}(x, y) = \frac{1 - x - \sqrt{(1 - x)^2 - 4x(1 + x)(1 - x + xy)}}{2x(1 + x)(1 - x + xy)}
\]
\[
= \frac{1}{1 - x} C \left( \frac{x(1 + x)(1 - x + xy)}{(1 - x)^2} \right). \quad (4.2)
\]

By (1.1) and (4.2), taking the coefficient of \(x^n y^i\) in \(L_{n,i}^{uh}(x, y)\), we derive the following result.

**Theorem 4.4** For any integers \(n \geq i \geq 0\), there holds
\[
L_{n,i}^{uh} = \sum_{k=i}^{n-i} \sum_{j=0}^{n-k-i} \binom{k}{i} \binom{k}{j} \binom{n-j}{n-k-i-j} C_k.
\]
The first values of $L_{uh}^{n,i}$ are illustrated in Table 14. Note that $L_{2n,n}^{uh} = C_n$, since any Dyck paths of length $2n$ can be obtained by replacing each $uh$-step in any G-Motzkin paths of length $2n$ with $n$ $uh$-steps (no $d$-steps implied) by a $u$-step and each $v$-step by a $d$-step, and vice versa.

### 4.3 The Statistic “Number of uu-Steps”

Let $L^{uu}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{uu} x^n y^i$ be the generating function of $L_{n,i}^{uu}$, the number of G-Motzkin paths of length $n$ with $i$ $uu$-steps. According to the first return decomposition, a G-Motzkin path $P$ can be decomposed as one of the following four forms:

$$P = \epsilon, \quad P = hP_0, \quad P = u^k hP_0 z_1 P_1 z_2 P_2 \ldots z_k P_k, \text{ or } P = u^k z_1 P_1 z_2 P_2 \ldots z_k P_k,$$

where $P_0, P_1, \ldots, P_k$ are (possibly empty) G-Motzkin paths for certain $k \geq 1$ and $z_1, \ldots, z_k \in \{d, v\}$. Note that the last two cases contribute at least $k - 1$ $uu$-steps. Then we get the relation

$$L^{uu}(x, y) = 1 + xL^{uu}(x, y) + \sum_{k=1}^{\infty} x^{k+1}(1 + x)^k y^{k-1} L^{uu}(x, y)^{k+1}$$

$$+ \sum_{k=1}^{\infty} x^k (1 + x)^k y^{k-1} L^{uu}(x, y)^k$$

$$= 1 + xL^{uu}(x, y) + \frac{x^2(1 + x)L^{uu}(x, y)^2}{1 - x(1 + x)yL^{uu}(x, y)} + \frac{x(1 + x)L^{uu}(x, y)}{1 - x(1 + x)yL^{uu}(x, y)}$$

$$= (1 + xL^{uu}(x, y)) \left(1 + \frac{x(1 + x)L^{uu}(x, y)}{1 - x(1 + x)yL^{uu}(x, y)}\right).$$
Solve this, we have

\[
L^{uu}(x, y) = \frac{1 - 2x - x^2 + x(1 + x)y - \sqrt{(1 - 2x - x^2 + x(1 + x)y)^2 - 4x(1 + x)(x + y - xy)}}{2x(1 + x)(x + y - xy)}
\]

By (1.1) and (4.3), taking the coefficient of \(x^n y^i\) in \(L^{uu}(x, y)\) by two different means, we have the following result.

**Theorem 4.5** For any integers \(n \geq i \geq 0\), there holds

\[
L_{n,i}^{uu} = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{r=0}^{n-k-j} \sum_{\ell=0}^{k+r} (-1)^{i+j-k} \binom{k}{j} \binom{2k+r}{r} \binom{j+r}{i+j-k} \binom{k+r}{r} \binom{n+k-j-\ell}{n-k-j-r-\ell} C_k
\]

The first values of \(L_{n,i}^{uu}\) are illustrated in Table 15.

Specially, when \(i = n \geq 1\), it implies that \(j = 0, r = n-k, \ell = 0\), so \(L_{0,0}^{uu} = 1\) and \(L_{n,0}^{uu} = 0\) for \(n \geq 1\) produce that

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} C_k = \begin{cases} 0, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}
\]
which has been derived by Chen and Pang [5] and is a special case of the identity [5, 35]

\[ \sum_{k=0}^{n} \binom{n+k}{n-k} C_k (x-1)^{n-k} = \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} x^k. \]

### 4.4 The Statistic “Number of hh-Steps”

Let \( L_{hh}(x, y) = \sum_{n=0}^\infty \sum_{i=0}^\infty L_{n,i} x^n y^i \) be the generating function of \( L_{n,i} \), the number of G-Motzkin paths of length \( n \) with \( i \) hh-steps. According to the first return decomposition, a G-Motzkin path \( P \) can be decomposed as one of the following six forms:

\[
P = \varepsilon, \quad P = h_k, \quad P = h_k uP_1 vP_2, \quad P = h_k uP_1 dP_2, \quad P = uP_1 vP_2, \quad \text{or} \quad P = uP_1 dP_2
\]

for certain \( k \geq 1 \), where \( P_1 \) and \( P_2 \) are (possibly empty) G-Motzkin paths. Note that the \( h_k u \) part contributes \( k - 1 \) hh-steps. Then we get the relation

\[
L_{hh}(x, y) = 1 + \sum_{k=1}^{\infty} x^k y^{k-1} + \sum_{k=1}^{\infty} x^{k+1} y^{k-1} L_{hh}(x, y)^2 + \sum_{k=1}^{\infty} x^{k+2} y^{k-1} L_{hh}(x, y)^2 \\
+ x L_{hh}(x, y)^2 + x^2 L_{hh}(x, y)^2 \\
= \left( 1 + \frac{x}{1-xy} \right) \left( 1 + x(1+x) L_{hh}(x, y)^2 \right).
\]

Solve this, we have

\[
L_{hh}(x, y) = \frac{1 - xy - \sqrt{(1-xy)^2 - 4x(1+x)(1+x-xy)^2}}{2x(1+x)(1+x-xy)} \\
= \frac{1 - x - xy}{1 - xy} C \left( \frac{x(1+x)(1+x-xy)^2}{(1-xy)^2} \right) \\
= \left( 1 + \frac{x}{1-xy} \right) C \left( x(1+x)(1 + \frac{x}{1-xy})^2 \right). \tag{4.4}
\]

By (1.1) and (4.4), taking the coefficient of \( x^n y^i \) in \( L_{hh}(x, y) \), we derive the following result.

**Theorem 4.6** For any integers \( n \geq i \geq 0 \), there holds

\[
L_{hh}(x, y) = \sum_{k=0}^{n-i} \sum_{j=0}^{n-k-i} \binom{2k+1}{j} \binom{i+j-1}{i} \binom{k}{n-k-i-j} C_k.
\]

The first values of \( L_{n,i}^{hh} \) are illustrated in Table 16.
Table 16  The first values of $L_{n,i}^{hh}$

| n/i | 0   | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   |     |     |     |     |     |
| 1   |     | 2   |     |     |     |     |
| 2   |     | 6   | 1   |     |     |     |
| 3   |     | 25  | 3   | 1   |     |     |
| 4   |     | 110 | 19  | 3   | 1   |     |
| 5   |     | 520 | 104 | 22  | 3   | 1   |
| 6   |     | 2566| 594 | 130 | 25  | 3   |

4.5 The Statistic “Number of hd-Steps”

Let $L_{hd}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{hd} x^n y^i$ be the generating function of $L_{n,i}^{hd}$, the number of G-Motzkin paths of length $n$ with $i$ hd-steps. According to the first return decomposition, a G-Motzkin path $P$ can be decomposed as one of the following four forms:

\[ P = \varepsilon, \quad P = hP_2, \quad P = uP_1 vP_2, \quad \text{or} \quad P = uP_1 dP_2, \]

where $P_1$ and $P_2$ are (possibly empty) G-Motzkin paths. If $P_1$ ends with an $h$-step, $P_1 d$ contributes at least one hd-step. Then we get the relation

\[
L_{hd}(x, y) = 1 + xL_{hd}(x, y) + xL_{hd}(x, y)^2 + x^2 (1 - x)L_{hd}(x, y)^2 + x^3 yL_{hd}(x, y)^2
= 1 + xL_{hd}(x, y) + x(1 + x - x^2 + x^2 y)L_{hd}(x, y)^2.
\]

Solve this, we have

\[
L_{hd}(x, y) = \frac{1 - x - \sqrt{(1 - x)^2 - 4x(1 + x - x^2 + x^2 y)}}{2x(1 + x - x^2 + x^2 y)}
= \frac{1}{1 - x} C \left( \frac{x(1 + x - x^2 + x^2 y)}{(1 - x)^2} \right).
\] (4.5)

By (1.1) and (4.5), taking the coefficient of $x^n y^i$ in $L_{hd}(x, y)$, we have the following result.

**Theorem 4.7** For any integers $n \geq i \geq 0$, there holds

\[ L_{n,i}^{hd} = \sum_{k=i}^{n-2i} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \binom{n+k-2i-2j}{n-k-2i-j} C_k. \]

The first values of $L_{n,i}^{hd}$ are illustrated in Table 17. Note that $L_{3n,n}^{hd} = C_n$, since any Dyck paths of length $2n$ can be obtained by replacing each hd-step in any G-Motzkin paths of length $3n$ with $n$ hd-steps ($n$ u-steps and no v-steps implied) by a d-step, and vice versa.
Table 17  The first values of \( L_{n,i}^{hd} \)

| \( n/i \) | 0   | 1   | 2   | 3   | 4   | 5   |
|---------|-----|-----|-----|-----|-----|-----|
| 0       | 0   | 1   |     |     |     |     |
| 1       | 1   | 2   |     |     |     |     |
| 2       | 2   | 7   |     |     |     |     |
| 3       | 28  | 1   |     |     |     |     |
| 4       | 126 | 7   |     |     |     |     |
| 5       | 605 | 45  |     |     |     |     |
| 6       | 3040| 277 | 2   |     |     |     |
| 7       | 15,781| 1692| 25  |     |     |     |
| 8       | 83,971| 10,320| 234 |     |     |     |
| 9       | 455,553| 63,026| 1924| 5   |     |     |

4.6 The Statistic “Number of \( vu \)-Valleys”

Let \( L^{vu}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{vu} x^n y^i \) be the generating function of \( L_{n,i}^{vu} \), the number of G-Motzkin paths of length \( n \) with \( i \) \( vu \)-valleys. According to the first return decomposition, a G-Motzkin path \( P \) can be decomposed as one of the following four forms:

\[
P = \varepsilon, \quad P = hP_1, \quad P = uP_1v \ldots uP_kvP_{k+1}dP_{k+2}, \quad \text{or} \quad P = uP_1v \ldots uP_kvP_{k+1}vQ
\]

for certain \( k \geq 0 \), where \( P_1, \ldots, P_{k+1} \) are (possibly empty) G-Motzkin paths and \( Q \) is empty or begins with an \( h \)-step. Note that the last two cases contribute at least \( k \) \( vu \)-valleys. Then we get the relation

\[
L^{vu}(x, y) = 1 + xL^{vu}(x, y) + \sum_{k=0}^{\infty} x^{k+2} y^k L^{vu}(x, y) L^{vu}(x, y) L^{vu}(x, y) + \sum_{k=0}^{\infty} x^{k+1} y^k L^{vu}(x, y) L^{vu}(x, y) + \sum_{k=0}^{\infty} x^{k} y^{k+1} L^{vu}(x, y) L^{vu}(x, y)
\]

\[
= 1 + xL^{vu}(x, y) + \frac{xL^{vu}(x, y)^2}{1 - xyL^{vu}(x, y)} + \frac{xL^{vu}(x, y)(1 + xL^{vu}(x, y))}{1 - xyL^{vu}(x, y)}. \tag{4.6}
\]

Solve this, we have

\[
L^{vu}(x, y) = \frac{1 - 2x + xy - \sqrt{(1 - 2x + xy)^2 - 4x(2x + y - xy)}}{2x(2x + y - xy)} = \frac{1}{1 - 2x + xy} C \left( \frac{x(2x + y - xy)}{(1 - 2x + xy)^2} \right). \tag{4.7}
\]
Let $T = xL^{vu}(x, y)$, using the Lagrange inversion formula in (4.6), taking the coefficient of $x^{n+1}y^i$ in $xL^{vu}(x, y)$, we derive that

$$L_{n,i}^{vu} = \binom{n+1}{i} T = \binom{y^i}{n+1} T = \binom{\frac{1}{n+1}[T^n]}{y^i} \left(1 + T + \frac{T(1 + 2T)}{1 - yT}\right)^{n+1}$$

$$= \frac{1}{n+1}[T^n][y^i] \left(1 + T + \frac{T(1 + 2T)}{1 - yT}\right)^{n+1}$$

$$= \frac{1}{n+1}[T^n][y^i] \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(1 - yT)^k} T^k (1 + 2T)^k (1 + T)^{n+1-k}$$

$$= \frac{1}{n+1}[T^n] \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{k+i-1}{i} T^{k+i} (1 + 2T)^k (1 + T)^{n+1-k}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n-i} \sum_{j=0}^{k} \binom{n+1}{k} \binom{k+i-1}{i} \binom{k}{j} \binom{n+1-k}{n-k-i-j} 2^j. \quad (4.8)$$

Hence, we obtain the following result.

**Theorem 4.8** For any integers $n \geq i \geq 0$, there holds

$$L_{n,i}^{vu} = \frac{1}{n+1} \sum_{k=0}^{n-i} \sum_{j=0}^{k} \binom{n+1}{k} \binom{k+i-1}{i} \binom{k}{j} \binom{n+1-k}{n-k-i-j} 2^j.$$

**Remark 4.9** If expanding $(1 + 2T)^k = \sum_{j=0}^{k} \binom{k}{j} T^j (1 + T)^{k-j} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{k-j}$ in (4.8), we have another two formulas for $L_{n,i}^{vu}$, i.e.,

$$L_{n,i}^{vu} = \frac{1}{n+1} \sum_{k=0}^{n-i} \sum_{j=0}^{k} (-1)^j \binom{n+1}{k} \binom{k+i-1}{i} \binom{k}{j} \binom{n+1-j}{n-k-i-j} 2^{k-j}.$$

The first values of $L_{n,i}^{vu}$ are illustrated in Table 18.

By (1.1) and (4.7), taking the coefficient of $x^n y^i$ in $L^{vu}(x, 2y)$, we derive the following identity and provide a combinatorial proof for it.

**Theorem 4.10** For any integer $n \geq 0$, there holds

$$\sum_{i=0}^{n} 2^i L_{n,i}^{vu} = 2^n C_n.$$

**Proof** Let $L_{n,i}^{vu}$ denote the set of weighted G-Motzkin paths of length $n$ with $i$ number of $vu$-steps such that each $vu$-step is weighted by 2 (regarded as $v_1 u$ and $v_2 u$ for

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Table 18 The first values of $L_{n,i}^{\text{vu}}$

| n/i | 0  | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|----|
| 0   | 1  |    |    |    |    |    |
| 1   |    | 2  |    |    |    |    |
| 2   | 6  | 1  |    |    |    |    |
| 3   | 20 | 8  | 1  |    |    |    |
| 4   | 72 | 48 | 12 | 1  |    |    |
| 5   | 272| 260| 100| 17 | 1  |    |
| 6   | 1064|1340|706|185|23 |1  |

convenience) and other steps are weighted by 1. Let $L_{n,i}^{\text{h}}$ denote the set of weighted G-Motzkin paths of length $n$ with $i$ number of $h$-steps at high level (level not less than 1) such that each $h$-step at high level is weighted by 2 (regarded as $h_1$ and $h_2$ for convenience) and other steps are weighted by 1. For convenience, the $h$-steps at level zero of the weighted G-Motzkin paths in $L_{n,i}^{\text{h}}$ are also written as $h$. Let $C_{n}^{\text{d}}$ denote the weighted Dyck paths of length $2n$ such that each $u$-step is weighted by 1 and each $d$-step is weighted by 2 (regarded as $d_1$ and $d_2$ for convenience). Set $L_{n,i}^{\text{vu}} = \bigcup_{i=0}^{n} L_{n,i}^{\text{vu}}$ and $L_{n}^{\text{h}} = \bigcup_{n=0}^{\infty} L_{n,i}^{\text{h}}$.

Clearly,

$$w(L_{n}^{\text{vu}}) = \sum_{i=0}^{n} w(L_{n,i}^{\text{vu}}) = \sum_{i=0}^{n} 2^i L_{n,i}^{\text{vu}}$$

and $w(C_{n}^{\text{d}}) = 2^n C_{n}$.

Firstly, there exists a simple bijection $\chi_1$ between $L_{n,i}^{\text{h}}$ and $L_{n,i}^{\text{vu}}$ for $n \geq i \geq 0$. For any $P \in L_{n,i}^{\text{h}}$, $\chi_1$ can be easily obtained by replacing all the $h$-steps (at high level) weighted by 2 in $P$ by new $\text{vu}$-steps with weight 2 (regarded as $h_1 \leftrightarrow v_1 u$ and $h_2 \leftrightarrow v_2 u$ for convenience) and replacing the old $\text{vu}$-steps in $P$ by $h$-steps (at high level), then $\chi_1(P) \in L_{n,i}^{\text{vu}}$. The inverse procedure can be handled similarly.

Secondly, there exists a recursive bijection $\chi_2$ between $C_{n}^{\text{d}}$ and $L_{n}^{\text{h}}$. For $n = 0, 1$, we define

$$\chi_2(\varepsilon) = \varepsilon, \quad \chi_2(ud_1) = h_1, \quad \chi_2(ud_2) = uv.$$

For any $Q \in C_{n}^{\text{d}}$, $Q$ can be uniquely partitioned into $Q = Q_1 Q_2 \ldots Q_k$ for $n \geq 2$ and certain $k \geq 1$, where $Q_1, Q_2, \ldots, Q_k$ are primitive. Then $\chi_2$ can be recursively defined by

$$\chi_2(Q) = \chi_2(Q_1) \chi_2(Q_2) \ldots \chi_2(Q_k).$$

So it suffices to discuss the cases when $Q \in C_{n}^{\text{d}}$ are primitive. There are four cases for such $Q$ to be considered.
Case 1.

When \( Q = uQ'd_2 \), we define \( \chi_2(Q) = u\chi_2(Q')v \).

Case 2.

When \( Q = uQ'd_1 \), where \( Q' = uQ_1d_1uQ_2d_1 \ldots uQ_kd_1 \) for certain \( k \geq 1 \), namely, each return step of \( Q' \) is a \( d_1 \)-step, we define

\[
\chi_2(Q) = u\chi_2(Q_1)h_2\chi_2(Q_2) \ldots h_2\chi_2(Q_k)d.
\]

Case 3.

When \( Q = uQ''Q'd_1 \), where the last return step of \( Q'' \) is a \( d_2 \)-step and \( Q' = uQ_1d_1uQ_2d_1 \ldots uQ_kd_1 \) for certain \( k \geq 1 \), i.e., each return step of \( Q' \) is a \( d_1 \)-step, we define

\[
\chi_2(Q) = u\chi_2(Q'')h_2\chi_2(Q_1)h_2\chi_2(Q_2) \ldots h_2\chi_2(Q_k)v.
\]

Case 4.

When \( Q = uQ''Q'd_1 \), where \( Q'' \) is empty or the last return step of \( Q'' \) is a \( d_1 \)-step, and \( Q' = uQ_1d_2uQ_2d_2 \ldots uQ_kd_2 \) for certain \( k \geq 1 \), i.e., each return step of \( Q' \) is a \( d_2 \)-step, we define

\[
\chi_2(Q) = u\chi_2(Q'')h_2\chi_2(Q_1)h_2\chi_2(Q_2) \ldots h_2\chi_2(Q_k)v.
\]

Note that according to the definition of \( \chi_2 \), \( \chi_2(Q) \) has no \( h_2 \)-steps at level zero for any \( Q \in C_n^d \) with \( n \geq 0 \), and \( \chi_2(Q) \) in the above four cases are always primitive for \( Q = ud_2 \) or \( Q \) being primitive with length at least 4. Moreover, in the first case \( \chi_2(Q) \) has no \( h_2 \)-steps at level one; In the third case \( \chi_2(Q'') \) must proceed exactly to the leftmost \( h_2 \)-step at level one in \( \chi_2(Q) \) and end with a \( v \)-step, and the last primitive part of \( \chi_2(Q'') \) has no \( h_2 \)-steps at level one by the first case. In the forth case \( \chi_2(Q'') \) must proceed exactly to the leftmost \( h_2 \)-step at level one in \( \chi_2(Q) \), once \( \chi_2(Q'') \) ends with a \( v \)-step, there must exist \( h_2 \)-steps at level one in the last primitive part of \( \chi_2(Q'') \) recursively by the third or the fourth cases; In the second case \( \chi_2(Q) \) is obviously distinguished from the other three cases in the last step.

From the above observation, one can handle the inverse procedure as follows. For any \( P \in \mathcal{L}_n^h \), since \( P \) has no \( h_2 \)-steps at level zero, \( P \) can be uniquely partitioned into \( P = h_1^jP_1h_1^jP_2 \ldots h_1^jP_kh_1^j \) for \( n \geq k \geq 1 \) and \( j_1, \ldots, j_{k+1} \geq 0 \), where \( P_1, P_2, \ldots, P_k \in \mathcal{L}_n^h \) are primitive. Then \( \chi_2^{-1}(P) \) can be recursively defined by

\[
\chi_2^{-1}(P) = (ud_1)^{j_1}\chi_2^{-1}(P_1)(ud_1)^{j_2}\chi_2^{-1}(P_2) \ldots (ud_1)^{j_k}\chi_2^{-1}(P_k)(ud_1)^{j_{k+1}}
\]
such that $\chi^{-1}_2(\epsilon) = \epsilon$, $\chi^{-1}_2(h_1) = u d_1$ and $\chi^{-1}_2(u v) = u d_2$. Naturally, it suffices to consider the cases when $P \in \mathcal{L}_n^h$ are primitive. There are four cases for such $P$ to be considered.

**Case 1.**

When $P = u P' v$ and $P$ has no $h_2$-steps at level one, i.e., $P' \in \mathcal{L}_{n-1}^h$ has no $h_2$-steps at level zero, in this case we define $\chi^{-1}_2(P) = u \chi^{-1}_2(P') d_2$.

**Case 2.**

When $P = u P' d$ and $P$ has exactly $k - 1$ $h_2$-steps at level one, where $P' = P_1 h_2 P_2 h_2 \ldots h_2 P_k$ with $P_1, \ldots, P_k \in \mathcal{L}^h$ for $k \geq 1$, in this case we define

$$\chi^{-1}_2(P) = uu \chi^{-1}_2(P_1) d_1 u \chi^{-1}_2(P_2) d_1 \ldots u \chi^{-1}_2(P_k) d_1 d_1.$$

**Case 3.**

When $P = u P'' P' v$ and $P$ has exactly $k$ $h_2$-steps at level one for $k \geq 1$, where $P' = h_2 P_1 h_2 P_2 \ldots h_2 P_k$ with $P_1, \ldots, P_k \in \mathcal{L}^h$, and $P'' \in \mathcal{L}^h$ is nonempty and ends with a $v$-step such that the last primitive part of $P''$ also has no $h_2$-steps at level one, in this case we define

$$\chi^{-1}_2(P) = u \chi^{-1}_2(P'') u \chi^{-1}_2(P_1) d_1 u \chi^{-1}_2(P_2) d_1 \ldots u \chi^{-1}_2(P_k) d_1 d_1.$$

**Case 4.**

When $P = u P'' P' v$ and $P$ has exactly $k$ $h_2$-steps at level one for $k \geq 1$, where $P' = h_2 P_1 h_2 P_2 \ldots h_2 P_k$ with $P_1, \ldots, P_k \in \mathcal{L}^h$, $P'' \in \mathcal{L}^h$ is possibly empty or ends with a $z$-step for $z \in \{h_1, h_2, d, v\}$ such that once $P''$ ends with a $v$-step, then there exist $h_2$-steps at level one in the last primitive part of $P''$, in this case we define

$$\chi^{-1}_2(P) = u \chi^{-1}_2(P'') u \chi^{-1}_2(P_1) d_2 u \chi^{-1}_2(P_2) d_2 \ldots u \chi^{-1}_2(P_k) d_2 d_1.$$

The above two procedures verify that $\chi_2$ is indeed a bijection between $\mathcal{C}_n^d$ and $\mathcal{L}_n^h$. Hence $\chi_1 \chi_2$ is a desired bijection between $\mathcal{C}_n^d$ and $\mathcal{L}_n^{vu}$.

In order to give a more intuitive view on the bijection $\chi_1$ and $\chi_2$, a pictorial description of $\chi_1$ and $\chi_2$ is presented for $Q = u d_2 u u d_2 u u d_1 d_1 u u u d_2 u d_1 d_1 d_1 u d_1 u u d_1 d_1 d_2 \in \mathcal{C}_n^d$, we have

$$\chi_2(Q) = uvuv(h_2) u(h_2) v(h_2) uvv(h_1) dvh_1 u d v \in \mathcal{L}_{15}^h$$

and

$$\chi_1 \chi_2(Q) = uhuv(v_2 u) u(v_2 u) v(v_2 u) u u v(v_1 u) dvhuudv \in \mathcal{L}_{15}^{vu}.$$

See Fig. 3 for detailed illustrations.
Fig. 3 An example of the bijections \( \chi_1 \) and \( \chi_2 \) described in the proof of Theorem 4.10
By (1.1) and (4.7), taking the coefficient of $x^n$ in $L^{vu}(x, -1)$, we deduce the following result.

**Theorem 4.11** For any integer $n \geq 0$, there holds

$$
\sum_{i=0}^{n} (-1)^i L_{n,i}^{vu} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} 3^{n-i} C_i.
$$

By (4.7), taking the coefficient of $x^n$ in $L^{vu}(x, -2) = \frac{1}{1-2x}$, we derive the following identity whose combinatorial proof is also provided.

**Theorem 4.12** For any integer $n \geq 0$, there holds

$$
\sum_{i=0}^{n} (-2)^i L_{n,i}^{vu} = 2^n.
$$

**Proof** Set $\mathcal{L}_n^{h,e} = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^{h,i}$ and $\mathcal{L}_n^{h,o} = \bigcup_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{L}_n^{h,i+1}$. Note that $\chi_1$ in Theorem 4.10 is a bijection between $\mathcal{L}_n^{h,i}$ and $\mathcal{L}_n^{vu,i}$ for $n \geq i \geq 0$, together with the definition of $\mathcal{D}_n^*$ in the proof of (2.5), in order to prove Theorem 4.12, it is sufficient to establish a bijection between $\mathcal{L}_n^{h,e}/\mathcal{D}_n^*$ and $\mathcal{L}_n^{h,o}$ for $n \geq 0$.

It is trivial for $n = 0, 1$. For $n \geq 2$, any $P \in \mathcal{L}_n^{h,e}/\mathcal{D}_n^*$ has at least one of the four subpaths, $h_1 v, h_2 v, d$ and $uvv$, find the last one, say $z$, $P$ can be partitioned uniquely into $P = P_1 z P_2$, where $P_2 \in \mathcal{D}_k^*$ for certain $0 \leq k \leq n - 2$. Then define $\varphi(P) = P_1 z' P_2$, where

$$
z' = \begin{cases} 
    uvv, & \text{if } z = h_1 v, \\
    d, & \text{if } z = h_2 v, \\
    h_2 v, & \text{if } z = d, \\
    h_1 v, & \text{if } z = uvv.
\end{cases}
$$

This way ensures that the number of $h$-steps at height level in $\varphi(P)$ is one more or less than that in $P \in \mathcal{L}_n^{h,e}/\mathcal{D}_n^*$, so $\varphi(P) \in \mathcal{L}_n^{h,o}$ and $z'$ in $\varphi(P)$ is also the last one of the four subpaths, $h_1 v, h_2 v, d$ and $uvv$. The reverse procedure can be handled similarly. Hence, $\varphi$ is a bijection between $\mathcal{L}_n^{h,e}/\mathcal{D}_n^*$ and $\mathcal{L}_n^{h,o}$. This completes the proof. \(\square\)

**Theorem 4.13** For any integers $n, m \geq 0$, there holds

$$
\sum_{i=0}^{2n} (-1)^i \binom{2n}{i} L_{n+m+i+2, n+i+1}^{vu} = C_n.
$$

**Proof** Set $N = n + m + i + 3$, one has

$$
L_{n+m+i+2, n+i+1}^{vu} = \sum_{k=0}^{n+1} \sum_{j=0}^{k} \frac{1}{N} \binom{N}{k} \binom{k+m+i}{j} \binom{k}{k-1} \binom{N-k-j}{n+1-k-j} 2^j.
$$
each inner term in $L_{n+m+i+2,m+i+1}^{vu}$, denoted by $g_{n,m,k,j}(i)$, is a polynomial on $i$ with degree

$$\partial g_{n,m,k,j}(i) = -1 + k + (k - 1) + (n + 1 - k - j) = n + k - 1 - j \leq 2n$$

such that $\partial g_{n,m,k,j}(i) = 2n$ if and only if $k = n + 1$ and $j = 0$. Clearly, the leading term in

$$g_{n,m,n+1,0}(i) = \frac{1}{N} \binom{N}{n+1} \binom{n+m+i+1}{n} = \frac{1}{n+1} \binom{n+m+i+2}{n} \binom{n+m+i+1}{n}$$

is $\frac{i^{2n}}{(n+1)!n!}$. So $L_{n+m+i+2,m+i+1}^{vu}$ is also a polynomial on $i$ with degree $2n$ such that the leading term is $\frac{i^{2n}}{(n+1)!n!}$. Similar to the proof of Theorem 4.2, one can have the result. \hfill \Box

### 4.7 The Statistic “Number of vv-Steps”

Let $L_{n,i}^{vy}(x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{vy} x^n y^i$ be the generating function of $L_{n,i}^{vy}$, the number of G-Motzkin paths of length $n$ with $i$ $vy$-steps. According to the first return decomposition, a G-Motzkin path $P$ can be decomposed as one of the following six forms:

$$P = \varepsilon, \ P = hP_1, \ P = uP_1dP_2, \ P = uP_1uP_2 \ldots uP_kuP_{k+1}dv^kP_{k+2}, \ P = uP_1uP_2 \ldots uP_khv^kP_{k+1} \text{ or } P = uP_1uP_2 \ldots uP_{k-1}uv^kP_k$$

for certain $k \geq 1$, where $P_1, \ldots, P_{k+2}$ are (possibly empty) G-Motzkin paths. Note that the last three cases contribute at least $k - 1$ $vy$-steps. Then we get the relation

$$L_{n,i}^{vy}(x,y) = 1 + xL_{n,i}^{vy}(x,y) + x^2L_{n,i}^{vy}(x,y)^2 + \sum_{k=1}^{\infty} x^{k+2}y^{k-1}L_{n,i}^{vy}(x,y)^{k+2}$$

$$+ \sum_{k=1}^{\infty} x^{k+1}y^{k-1}L_{n,i}^{vy}(x,y)^{k+1} + \sum_{k=1}^{\infty} x^ky^{k-1}L_{n,i}^{vy}(x,y)^k$$

$$= \left(1 + xL_{n,i}^{vy}(x,y) + x^2L_{n,i}^{vy}(x,y)^2\right) \left(1 + \frac{xL_{n,i}^{vy}(x,y)}{1 - xyL_{n,i}^{vy}(x,y)}\right). \quad (4.9)$$

Let $T = xL_{n,i}^{vy}(x,y)$, using the Lagrange inversion formula in (4.9), taking the coefficient of $x^{n+1}y^i$ in $xL_{n,i}^{vy}(x,y)$, we get the following result.
Table 19  The first values of $L_{n,i}^{vv}$

| n/i | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|----|----|----|----|----|----|
| 0   | 1  |    |    |    |    |    |
| 1   | 2  |    |    |    |    |    |
| 2   | 6  | 1  |    |    |    |    |
| 3   | 21 | 7  | 1  |    |    |    |
| 4   | 80 | 41 | 11 | 1  |    |    |
| 5   | 322| 225| 86 | 16 | 1  |    |
| 6   | 1347|1198|589|162|22|1 |

Theorem 4.14  For any integers $n \geq i \geq 0$, there holds

$$L_{n,i}^{vv} = \frac{1}{n+1} \sum_{k=0}^{n-i} \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \binom{n+1}{k} \binom{n+1}{j} \binom{k+i}{i} \binom{n-j+1}{n-k-i-2j}.$$ 

The first values of $L_{n,i}^{vv}$ are illustrated in Table 19.

Theorem 4.15  For any integers $n, m \geq 0$, there holds

$$\sum_{i=0}^{2n} (-1)^i \binom{2n}{i} L_{n+m+i+2,m+i+1}^{vv} = C_n.$$ 

Proof  Set $N = n + m + i + 3$, one has

$$L_{n+m+i+2,m+i+1}^{vv} = \sum_{k=0}^{n+1} \sum_{j=0}^{\lfloor \frac{n-k+1}{2} \rfloor} \frac{1}{N} \binom{N}{k} \binom{N}{j} \binom{k+m+i}{k-1} \binom{N-j}{n-1-k-2j};$$

each inner term in $L_{n+m+i+2,m+i+1}^{vv}$, denoted by $f_{n,m,k,j}(i)$, is a polynomial on $i$ with degree

$$\partial f_{n,m,k,j}(i) = -1 + k + j + (k-1) + (n + 1 - k - 2j) = n + k - 1 - j \leq 2n$$

such that $\partial f_{n,m,k,j}(i) = 2n$ if and only if $k = n + 1$ and $j = 0$. Clearly, the leading term in

$$f_{n,m,n+1,0}(i) = \frac{1}{n+1} \binom{N}{n+1} \binom{n+m+i+1}{n}$$

$$= \frac{1}{n+1} \binom{n+m+i+2}{n} \binom{n+m+i+1}{n}$$
\[
\binom{i+2n}{(n+1)i} \cdot i^{2n} \quad \text{for } n \geq i \geq 0.
\]
So \(L_{n+m+i+2,m+i+1}^{yy} \) is also a polynomial on \(i\) with degree \(2n\) such that the leading term is \(\frac{i^{2n}}{(n+1)i!} \cdot i^{2n} \). Similar to the proof of Theorem 4.2, one can have the result.

\[\square\]

### 4.8 The Statistic “Number of du-Valleys”

Let \(L_{du}^{x,y} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{du} x^n y^i\) be the generating function of \(L_{n,i}^{du}\), the number of G-Motzkin paths of length \(n\) with \(i\) du-valleys. According to the first return decomposition, a G-Motzkin path \(P\) can be decomposed as one of the following four forms:

\[
P = \varepsilon, \quad P = hP_1, \quad P = uP_1 duP_2 d \ldots \quad uP_k duP_{k+1} vP_{k+2} \quad \text{or} \quad P = uP_1 duP_2 d \ldots \quad uP_k duP_{k+1} dQ
\]

for certain \(k \geq 0\), where \(P_1, \ldots, P_{k+2}\) are (possibly empty) G-Motzkin paths and \(Q\) is empty or begins with an \(h\)-step. Note that the last two cases contribute at least \(k\) du-valleys. Then we get the relation

\[
L_{du}^{x,y} = 1 + xL_{du}^{x,y} + \sum_{k=0}^{\infty} x^{2k+1} y^k L_{du}^{x,y} + \sum_{k=0}^{\infty} x^{2k+2} y^k L_{du}^{x,y} (1 + xL_{du}^{x,y})
\]

\[
= 1 + xL_{du}^{x,y} + \frac{xL_{du}^{x,y}}{1 - x^2 y} \frac{1 + xL_{du}^{x,y}}{1 - x^2 y L_{du}^{x,y}}.
\]

Solve this, we have

\[
L_{du}^{x,y} = \frac{1 - x - x^2 + x^2 y - \sqrt{(1 - x - x^2 + x^2 y)^2 - 4x(1 + x^2 + xy - x^2 y)}}{2x(1 + x^2 + xy - x^2 y)}
\]

\[
= \frac{1}{1 - x - x^2 + x^2 y} C \left( \frac{x(1 + x^2 + xy - x^2 y)}{(1 - x - x^2 + x^2 y)^2} \right).
\]

By (1.1) and (4.10), taking the coefficient of \(x^n y^i\) in \(L_{du}^{x,y}\), we obtain the following result.

**Theorem 4.16** For any integers \(n \geq i \geq 0\), there holds

\[
L_{n,i}^{du} = \sum_{k=0}^{n} \sum_{r=0}^{i} \sum_{j=0}^{k-r} \sum_{\ell=0}^{[n+r-k]-i-j} (-1)^{i-r} \binom{2k + i - r}{i - r} \binom{k}{r} \binom{k - r}{j}.
\]
\[
\binom{2k + i + \ell - r}{\ell} \binom{n + k - r - i - 2j - \ell}{n + r - k - 2i - 2j - 2\ell} C_k.
\]

Specially,
\[
L_{n,0}^{du} = \sum_{k=0}^{n} \sum_{j=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \sum_{\ell=0}^{j} \binom{2k + \ell}{j} \binom{n + k - 2j - \ell}{n - k - 2j - 2\ell} C_k.
\]

The first values of \(L_{n,i}^{du}\) are illustrated in Table 20.

### 4.9 The Statistic “Number of dd-Steps”

Let \(L_{dd}^{dd}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L_{n,i}^{dd} x^n y^i\) be the generating function of \(L_{n,i}^{dd}\), the number of G-Motzkin paths of length \(n\) with \(i\) \(dd\)-steps. According to the first return decomposition, a G-Motzkin path \(P\) can be decomposed as one of the following six forms:

\[
P = \varepsilon, P = hP_1, \quad P = uP_1vP_2, \quad P = uP_1uP_2 \ldots uP_kuP_{k+1}vP_{k+2},
\]

\[
P = uP_1uP_2 \ldots uP_khdP_{k+1} \text{ or } P = uP_1uP_2 \ldots uP_{k-1}udP_k
\]

for certain \(k \geq 1\), where \(P_1, \ldots, P_{k+2}\) are (possibly empty) G-Motzkin paths. Note that the last three cases contribute at least \(k - 1\) \(dd\)-steps. Then we get the relation

\[
L_{dd}^{dd}(x, y) = 1 + xL_{dd}^{dd}(x, y) + xL_{dd}^{dd}(x, y)^2 + \sum_{k=1}^{\infty} x^{2k+1} y^{k-1} L_{dd}^{dd}(x, y)^{k+2}
\]

\[
+ \sum_{k=1}^{\infty} x^{2k+1} y^{k-1} L_{dd}^{dd}(x, y)^{k+1} + \sum_{k=1}^{\infty} x^{2k} y^{k-1} L_{dd}^{dd}(x, y)^k
\]

| \(n/i\) | 0   | 1   | 2   | 3   | 4   | 5   |
|---------|-----|-----|-----|-----|-----|-----|
| 0       | 1   |     |     |     |     |     |
| 1       | 2   |     |     |     |     |     |
| 2       | 7   |     |     |     |     |     |
| 3       | 28  | 1   |     |     |     |     |
| 4       | 123 | 10  |     |     |     |     |
| 5       | 576 | 73  | 1   |     |     |     |
| 6       | 2819| 485 | 15  |     |     |     |
| 7       | 14, 250 | 3093 | 154 | 1   |     |     |
| 8       | 73, 833 | 19, 325 | 1346 | 21 |     |     |
| 9       | 390, 048 | 119, 418 | 10, 758 | 283 | 1   |     |

Table 20  The first values of \(L_{n,i}^{du}\)
\[= 1 + xL^{dd}(x, y) + xL^{dd}(x, y)^2 + x^2 L^{dd}(x, y) \left( \frac{1 + xL^{dd}(x, y) + xL^{dd}(x, y)^2}{1 - x^2yL^{dd}(x, y)} \right).\]

However, the exact formula for \(L^{dd}_{n,i}\) is still unknown. Here we give the array \(L^{dd}_{n,i}\) for \(0 \leq n \leq 9\) and \(0 \leq i \leq 5\), see Table 21.

### 4.10 The Statistic “Number of dv-Steps”

Let \(L^{dv}(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} L^{dv}_{n,i} x^n y^i\) be the generating function of \(L^{dv}_{n,i}\), the number of G-Motzkin paths of length \(n\) with \(i\) dv-steps. According to the first return decomposition, a G-Motzkin path \(P\) can be decomposed as one of the following six forms:

\[
P = \varepsilon, \ P = hP_1, \ P = uP_1dP_2, \ P = uP_1uP_2 \ldots uP_{k-1}uv^kP_k, \ P = uP_1uP_2 \ldots uP_khv^kP_{k+1} \text{ or } P = uP_1uP_2 \ldots uP_kuP_{k+1}dv^kP_{k+2}
\]

for certain \(k \geq 1\), where \(P_1, \ldots, P_{k+2}\) are (possibly empty) G-Motzkin paths. Note that the last case contributes at least one dv-step. Then we get the relation

\[
L^{dv}(x, y) = 1 + xL^{dv}(x, y) + x^2 L^{dv}(x, y)^2 + \sum_{k=1}^{\infty} x^k L^{dv}(x, y)^k + \sum_{k=1}^{\infty} x^{k+1} L^{dv}(x, y)^{k+1} + y \sum_{k=1}^{\infty} x^{k+2} L^{dv}(x, y)^{k+2}
= 1 + xL^{dv}(x, y) + x^2 L^{dv}(x, y)^2 + \frac{xL^{dv}(x, y)(1 + xL^{dv}(x, y) + yx^2 L^{dv}(x, y)^2)}{1 - xL^{dv}(x, y)}.
\]

(4.11)

| Table 21: The first values of \(L^{dd}_{n,i}\) |
|---|---|---|---|---|---|---|
| \(n/i\) | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | | | | | |
| 1 | 2 | | | | | |
| 2 | 7 | | | | | |
| 3 | 29 | | | | | |
| 4 | 132 | 1 | | | | |
| 5 | 641 | 9 | | | | |
| 6 | 3254 | 64 | 1 | | | |
| 7 | 17,060 | 427 | 11 | | | |
| 8 | 91,663 | 2770 | 91 | 1 | | |
| 9 | 499,569 | 20,219 | 707 | 13 | | |
Let \( T = x L_{n}^{d}(x, y) \), using the Lagrange inversion formula in (4.11), taking the coefficient of \( x^{n+1} y^{i} \) in \( x L_{n}^{d}(x, y) \) in two different ways, we have the following result.

**Theorem 4.17** For any integers \( n \geq i \geq 0 \), there holds

\[
L_{n,i}^{d} = \frac{1}{n+1} \left( \begin{array}{c} n+1 \\ i \end{array} \right) \sum_{k=0}^{[n/3] - i} \sum_{j=0}^{[n/3] - i - k} (-1)^{k+j} \left( \begin{array}{c} n+1-i \\ k \end{array} \right) \left( \begin{array}{c} n+1-i-k \\ j \end{array} \right) \left( \begin{array}{c} 2n-3i-3k-j \\ n-3i-2k-j \end{array} \right)
\]

The first values of \( L_{n,i}^{d} \) are illustrated in Table 22. Clearly, \( L_{3n,n}^{d} = \frac{1}{3n+1} \binom{3n+1}{n} \) is the Fuss–Catalan numbers \( C_{k}(n) = \frac{1}{kn+1} \binom{kn+1}{n} \) of the third order \( (k = 3) \) which also counts the ternary trees with \( n \) internal vertices and counts the number of lattice paths from \((0, 0)\) to \((2n, 0)\) in the first quadrant of the XY-plane using an up-steps \( u = (1, 1) \) and a down-steps \( d_{2} = (0, -2) \) [43, A001764]. This kind of lattice paths of length \( 2n \) can be easily corresponded bijectively to G-Motzkin paths of length \( 3n \) with \( n \) \( d_{2} \)-steps by replacing \( d_{2} \)-steps by \( d_{1} \)-steps and vice versa.

## 5 Concluding Remarks and Further Works

The main objective of this paper has been achieved in Sects. 2, 3 and 4, where we enumerate “number of \( z \)-steps”, “number of \( z \)-steps” at given level, “number of \( z_1 z_2 \)-...
steps” in G-Motzkin paths for $z, z_1, z_2 \in \{u, h, v, d\}$. Some explicit formulas are obtained by bijective and algebraic methods, including generating functions and the Lagrange inversion formula.

Despite that several identities are proved by algebraic methods, we naturally expect their combinatorial proofs, especially for Theorem 2.6, Theorem 3.2, Theorem 3.3, Theorem 3.6, Theorem 3.10, Theorem 3.13, Theorem 4.2, Theorem 4.11, Theorem 4.13 and Theorem 4.15.

In two forthcoming papers, we further consider the statistics “number of occurrences of $\Gamma$” for an arbitrary string $\Gamma$ with at least three steps and the statistics “number of occurrences of $\Gamma$” at given height, including even or odd height, just as done in [33, 39, 48]. Moreover, we also discuss the generalized Shröder paths with steps in $\{H, u, v_i (i \geq 1), d\}$ analogous to G-Motzkin paths studied in this paper, where $v_i = (0, -i)$.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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