Real-Value Power-Voltage Formulations of, and Bounds for, Three-Wire Unbalanced Optimal Power Flow

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Abstract

Unbalanced optimal power flow refers to a class of optimization problems subject to the steady state physics of three-phase power grids with nonnegligible phase unbalance. Significant progress on this problem has been made on the mathematical modeling side of unbalanced OPF, however there is a lack of information on implementation aspects as well as data sets for benchmarking. One of the key problems is the lack of definitions of current and voltage bounds across different classes of representations of the power flow equations. Therefore, this tutorial-style paper summarizes the structural features of the unbalanced (optimal) power problem for three-phase systems. The resulting nonlinear complex-value matrix formulations are presented for both the bus injection and branch flow formulation frameworks, which typically cannot be implemented as-is in optimization toolboxes. Therefore, this paper also derives the equivalent real-value formulations, and discusses challenges related to the implementation in optimization modeling toolboxes. The derived formulations can be re-used easily for continuous and discrete optimization problems in distribution networks for a variety of operational and planning problems. Finally, bounds are derived for all variables involved, to further the development of benchmarks for unbalanced optimal power flow, where consensus on bound semantics is a pressing need. We believe benchmarks remain a cornerstone for the development and validation of scalable and reproducible optimization models and tools. The soundness of the derivations is confirmed through numerical experiments, validated w.r.t. OpenDSS for IEEE test feeders with $3 \times 3$ impedance matrices.

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Nomenclature

This article depends on the definition of a variety of scalar, vector and matrix parameters and variables related to grid buses and branches (see Tables 1-5). The core variables are current, voltage and power, whereas parameters are mainly impedance and variable bounds. Fig. 1 summarizes the variables and parameters defined in the fundamental 3 × 3 branch model for which the well known The Π-equivalent model used. With respect to balance networks, both series and shunt elements are represented by full complex-valued matrices including the mutual impedance coupling between the conductors. Using the Π-equivalent model, the branch current $I_{ij}$ can be split into a series component $I_{ij}^s$ and a shunt component $I_{ij}^h$, respectively. All circuit element voltages are defined w.r.t. (local) ground voltage $U_{i,g} = 0V$.

Table 1 illustrates typography and mathematical notation used throughout; Table 2 defines sets and indices; Table 3 defines parameters; Table 4 defines typical engineering variables; Table 5 defines lifted variables. Black and red colors indicate real-valued variables and parameters, respectively. Blue and brown colors are used for complex-valued variables and parameters, respectively.

1 Introduction

Driven by the increased rollout of distributed energy resources (DERs) such as PV, battery storage as well as electric vehicles, power distribution grids are facing a number of challenges associated with the large scale integration of these technologies. In low-voltage grids specifically, one can observe phase unbalance readily, due to the presence of a significant amount of single-phase loads and DERs. Many of the current electric vehicles use single phase charging and small rooftop solar systems are connected via single phase inverters to the low voltage grid, further increasing the unbalance. Furthermore, phase unbalance can stem from insufficient conductor transposition in radial distribution networks. Unbalance implies an underutilization of the grid’s transfer capacity, as it leads to higher losses and to faster-than-expected congestion. Simulation techniques, i.e.
Table 1: Typography and mathematical notation

| Symbol | Description             |
|--------|-------------------------|
| $x$    | real scalar variable    |
| $X$    | real vector or matrix variable |
| $x$    | complex scalar variable |
| $X$    | complex vector or matrix variable |
| $x$    | real scalar parameter   |
| $X$    | real vector or matrix parameter |
| $x$    | complex scalar parameter |
| $X$    | complex vector or matrix parameter |
| $\mathcal{X}$ | set |
| $X^T$  | transpose of $X$         |
| $X^*$  | conjugate of $X$         |
| $X^H$  | conjugate transpose of $X$ |
| $\odot$ | element-wise multiplication |
| $\oslash$ | element-wise division |
| $j$    | imaginary unit, satisfies $j^2 = -1$ |
| $a \angle b$ | polar notation of complex number $a \cdot e^{jb}$ |
| $\mathbb{R}^{n \times m}$ | set of real $n \times m$ matrices |
| $\mathbb{C}^{n \times m}$ | set of complex $n \times m$ matrices |
| $\mathbb{H}^n \subset \mathbb{C}^{n \times n}$ | set of Hermitian $n \times n$ matrices |
| diag($X$) | extract diagonal of $X$, $\text{diag} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{1 \times n}$ |

Table 2: Sets and indices

| Set                  | Description                                              |
|----------------------|----------------------------------------------------------|
| Phases               | $p, q \in \mathcal{P} = \{a, b, c\}$                     |
| Branches             | $l \in \mathcal{J}$                                     |
| Buses                | $i \in \mathcal{I}$                                     |
| Topology (forward)   | $lij \in \mathcal{T} \subseteq \mathcal{J} \times \mathcal{I} \times \mathcal{I}$ |
| Topology (reverse)   | $lij \in \mathcal{T} = \{ lji \mid lij \in \mathcal{T} \}$ |
| Topology             | $lij \in \mathcal{T} = \mathcal{T} \cup \mathcal{T}^*$  |
| Bus pairs            | $ij \in \mathcal{B} = \{ ij \mid lij \in \mathcal{T} \} \subseteq \mathcal{I} \times \mathcal{I}$ |
| Units                | $u \in \mathcal{U}$                                     |
| Unit connectivity    | $ui \in \mathcal{T}^{\text{units}} \subseteq \mathcal{U} \times \mathcal{I}$ |
| Shunt connectivity   | $hi \in \mathcal{T}^{\text{shunts}} \subseteq \mathcal{U} \times \mathcal{I}$ |
### Table 3: Parameters

| Parameter                                      | Expression                  |
|------------------------------------------------|----------------------------|
| Bus voltage magnitude min./max. (V)            | $U_{i}^\text{min}, U_{i}^\text{max} \in \mathbb{R}^{|P|\times1}$ |
| Bus phase angle diff. min./max. (rad)          | $\Theta_{i}^\text{min}, \Theta_{i}^\text{max} \in \mathbb{R}^{|P|\times1}$ |
| Branch current rating (A)                      | $I_{ij}^\text{rated} \in \mathbb{R}^{|P|\times1}$ |
| Branch apparent power rating (VA)              | $S_{ij}^\text{rated} \in \mathbb{R}^{|P|\times1}$ |
| Branch series impedance (Ω)                    | $Z_{ij}^s \in \mathbb{C}^{|P|\times|P|}$ |
| Branch series admittance (S)                   | $Y_{ij}^s \in \mathbb{C}^{|P|\times|P|}$ |
| Branch from/to shunt admittance (S)            | $Y_{ij}^\text{sh}, Y_{ij}^\text{sh} \in \mathbb{C}^{|P|\times|P|}$ |
| Bus pair angle diff. min./max. (rad)           | $\Theta_{ij}^\text{min}, \Theta_{ij}^\text{max} \in \mathbb{R}^{|P|\times1}$ |
| Bus shunt admittance (S)                       | $Y_{h} \in \mathbb{C}^{|P|\times|P|}$ |
| Unit current rating (A)                        | $I_{u}^\text{rated} \in \mathbb{R}^{|P|\times1}$ |
| Unit active power bounds (W)                   | $P_{u}^\text{min}, P_{u}^\text{max} \in \mathbb{R}^{|P|\times1}$ |
| Unit reactive power bounds (var)               | $Q_{u}^\text{min}, Q_{u}^\text{max} \in \mathbb{R}^{|P|\times1}$ |

### Table 4: Optimization variables

| Parameter                                      | Expression                  |
|------------------------------------------------|----------------------------|
| Bus voltage (V)                                | $U_{i}, U_{j} \in \mathbb{C}^{|P|\times1}$ |
| Branch current (A)                             | $I_{ij}, I_{ij}^s \in \mathbb{C}^{|P|\times1}$ |
| Branch series current (A)                      | $I_{ij}, I_{ij}^s \in \mathbb{C}^{|P|\times1}$ |
| Branch shunt current (A)                       | $I_{ij}^h, I_{ij}^h \in \mathbb{C}^{|P|\times1}$ |
| Branch power flow (W)                          | $S_{ij}, S_{ij}^s \in \mathbb{C}^{|P|\times|P|}$ |
| Branch series power flow (W)                   | $S_{ij}^s, S_{ij}^s \in \mathbb{C}^{|P|\times|P|}$ |
| Unit current (A)                               | $I_{u} \in \mathbb{C}^{|P|\times1}$ |
| Unit power (W)                                 | $S_{u} \in \mathbb{C}^{|P|\times|P|}$ |

### Table 5: Lifted optimization variables

| Parameter                                      | Expression                  |
|------------------------------------------------|----------------------------|
| Bus voltage product (V$^2$)                    | $W_{i}, W_{j} \in \mathbb{H}^{|P|}$ |
| Bus pair voltage product (V$^2$)               | $W_{ij}, W_{ji} \in \mathbb{C}^{|P|\times|P|}$ |
| Branch current product (A$^2$)                 | $L_{ij}, L_{ij} \in \mathbb{H}^{|P|}$ |
| Branch series current product (A$^2$)          | $L_{ij}^s \in \mathbb{H}^{|P|}$ |
| System voltage product (V$^2$)                 | $M_{i} \in \mathbb{H}^{|R|\times|P|}$ |
| Bus pair voltage product (V$^2$)               | $M_{ij} \in \mathbb{H}^{2|P|}$ |
| Branch voltage-current product (mix)           | $M_{ij} \in \mathbb{H}^{2|P|}$ |
deriving solutions to the power flow (PF) equations in unbalanced networks has long been a topic of interest [1, 2, 3], and is used in deriving hosting capacity by means of scenario analysis. The accuracy of modeling of low-voltage grids has been studied in-depth by Urquhart [4].

Unbalanced optimal power flow (OPF) refers to the mathematical optimization of problems subject to the physics of unbalanced grids, and serves as the core for a variety of problem classes such as benchmarking centralized or distributed optimal control solutions [5], determination of expansion options and hosting capacity analysis considering control actions.

1.1 State of the Art on Unbalanced (O)PF

Similar to the balanced (positive sequence) modeling, one can distinguish between the bus injection model (BIM) and the branch flow model (BFM) formulations of the unbalanced PF equations. BIM forms eliminate all current variables, which leads to active and reactive power flows being expressed purely as a function of the voltage differences between connected buses. The series impedance is consequently used in admittance form, which makes it impossible to represent zero-impedance branches. Conversely, the BFM forms keep (a representation of) the current variable through the series impedance. In this case, series impedance is represented in impedance form, therefore the edge case of zero series impedance remains representable.

Most OPF problems are developed in the complex power-voltage variable space, instead of the current-voltage variable space common in PF solvers. Table 6 maps a number of published formulations to BIM/BFM categories and the variable spaces in which they are defined. We refer to [6, 7] for recent in-depth reviews of mathematical formulations for the OPF problem.

A nonlinear programming (NLP) unbalanced OPF formulation for branches is presented in [8], in which the shunt impedances have been neglected. The unbalanced current-voltage form, with generation power dispatch constraints, is derived in [9]. The nonconvex power-lifted voltage form is presented in [10]. The first rank-constrained semi-definite programming (SDP) of unbalanced OPF, and its SDP relaxation, are proposed by Dall’Anese et al. in [11]. Gan et al. developed the BFM variant, and defined both BFM and BIM in a consistent notation in [12], without the provision of the real-value form. Extensions to these SDP relaxations have been proposed: Zhao et al. develop models for delta-connected loads [13]; Bazrafshan et al. propose extensions for voltage regulators [14]; Usman et al. discuss neutral conductor modeling [15]; Claesys et al. present detailed transformer models for unbalanced OPF [16]; Claesys et al. also present convex relaxation of delta/wye ZIP unbalanced loads [17]; Vanin et al. explore further relaxation to SOC problems [18].

It is noted that the formulations so far published are not easily compared due to the inherent complexity of the notation, the variety of notations used, and lacking details of approximations applied during implementation in modeling software. For instance, the branch shunt impedance is often neglected, assumed to be diagonal, or simply modeled as bus shunts. This leads to inaccuracies in the power and current flow values and in the enforcement of the proper branch flow limits, e.g. when only the series current is bounded, not the total including the shunt current. This in turn makes it hard to validate the feasibility and correctness of results on published data sets which include such components without modification. Examples include the IEEE PES distribution test feeders [19]; the IEEE123 bus system specifically contains non-diagonal branch shunt matrices.
1.2 Bound definitions and data

In an OPF problem, authors typically assume \[6, 7, 20\] the power flow is subject to a subset of:

- apparent power, upper bound;
- current magnitude, upper bound;
- voltage magnitude, lower and upper bound;
- voltage angle difference between adjacent buses, lower and upper bound.

These bounds can be generalized to the case with phase unbalance. Furthermore, in the context of phase unbalance, additional limits have been discussed \[21\]. The most obvious missing bound is that of phase angle differences at a certain bus. Note that bounds represent a key aspect of optimization problems, however currently there are few unbalanced power flow data sets that include all of these parameters. This makes benchmarking \textit{optimality} of different unbalanced OPF engines challenging.

1.3 Implementation of Complex-Value Optimization Problems

Figure 2 illustrates the typical sequence of formulating, implementing and solving optimization problems as commonly applied to engineering problems. First, a mathematical model of the optimization problem is formulated. The problem is formulated so that it can be handled by the chosen modeling language and optimization solver. Eventually, the modeling toolbox translates the mathematical model into the appropriate form for the solver interface, fills out parameter values, and dispatches the optimization solver. Finally, the mathematical solver uses one or a variety of different algorithms to solve the optimization problem, and returns a solution.

In the literature, real-value formulations of the convex relaxations of unbalanced OPF are rarely discussed. CVX has been used as an optimization modeling tool in \[11\]. It is one of the very few optimization packages with support for complex optimization variable primitives, as well as matrix variables. CVX automates the reformulation process from a complex-value matrix to a set of real-value scalar variables and constraints. Functionality to automate the real-value reformulation
process was initially in a JuMP extension [22], but has recently become a built-in JuMP feature [23].

Gilbert and Josz [24] illustrate with a limited-scope solver that there are potential advantages of solving SDP problems directly in the complex domain. Nevertheless, a mature implementation of such a solver has not been developed so far. Overall, the interest in real-value forms remains high.
| Formulation                     | Ref. | Variable space                      | Exact?                      | Section |
|--------------------------------|------|-------------------------------------|----------------------------|---------|
| NLP Kirchhoff unbalanced       | [1, 9] | $I_{lij}, U_i$                      | by definition, discussion: \[20\] | 3       |
| NLP BIM unbalanced scalar polar | [8]  | $\text{diag}(S_{lij}), U_i$        | if $U_{i}^{\text{min}} \neq 0$ | 5       |
| NLP BIM unbalanced scalar rect. |       | $\text{diag}(S_{lij}), U_i$        | if $U_{i}^{\text{min}} \neq 0$ | 5       |
| SDP BIM unbalanced             | [11, 26, 27] | $S_{lij}, W_{i} \succeq 0, W_{ij}, M \succeq 0$ | $\text{rank}(M) = 1$ | 6       |
| SDP BIM unbalanced radial      | [12]  | $S_{lij}, W_{i} \succeq 0, W_{ij}, M_{ij} \succeq 0$ | $\text{rank}(M_{ij}) = 1$, only if radial | 6       |
| SDP BFM unbalanced             | [12]  | $S_{lij}, W_{i} \succeq 0, L_{l}^{2} \succeq 0, M_{lij} \succeq 0$ | $\text{rank}(M_{lij}) = 1$, only if radial | 7       |
1.4 Scope, Contributions and Report Structure

In summary, we observe the following gaps in the literature:

- polar and rectangular real forms in the power-voltage variable space for branches with shunts (Π-model);
- real form of the power-lifted voltage BIM and BFM SDP relaxations;
- definitions of current, power and voltage angle difference limits in both the power-voltage and power-lifted voltage variable spaces.

This work therefore provides consistent derivations of real-value formulations for variants of unbalanced optimal power flow in different variable spaces, to enable straight-forward implementation in optimization modeling toolboxes and efficient comparison on published data sets.

We limit ourselves here to the three-wire case, and note that recent works [28, 29] have explored OPF models with explicit representation of the neutral.

First, the basic relationship between the different variables and parameters is defined in §2, which builds the mathematical foundation of this report; it also contains a generalized model to represent loads and generators in the system. Section §3 provides Kirchhoff’s and Ohm’s laws in the multiconductor form. Next, §4 develops a formulation of nonlinear BFM unbalanced power flow and §5 the equivalent BIM. Furthermore, §6 derives the BIM and §7 the BFM in the lifted variable space which is the basis various convex relaxations. Moreover §8 presents the feasible sets of the derived real-value formulations and discusses implementation aspects. Finally, §9 presents the conclusions.

2 Notation and Basic Relationships

2.1 Scalar and Matrix Variables

The voltage $U_i$ of a bus $i$ is a complex value, vector variable, encapsulating variables for each conductor:

$$U_i = \begin{bmatrix} U_{i,a} \\ U_{i,b} \\ U_{i,c} \end{bmatrix} = \begin{bmatrix} U_{i,\text{mag}} \angle \theta_{i,a} \\ U_{i,\text{mag}} \angle \theta_{i,b} \\ U_{i,\text{mag}} \angle \theta_{i,c} \end{bmatrix} = \begin{bmatrix} U_{i,\text{re}} + jU_{i,\text{im}} \\ U_{i,\text{re}} + jU_{i,\text{im}} \\ U_{i,\text{re}} + jU_{i,\text{im}} \end{bmatrix} = \begin{bmatrix} U_{i,\text{re}} \\ U_{i,\text{im}} \end{bmatrix}.$$  \hspace{1cm} (1)

Similarly, the total current of a branch $l$ connecting a pair of buses $i$ and $j$ is a complex value vector variable,

$$I_{lij} = \begin{bmatrix} I_{lij,a} \\ I_{lij,b} \\ I_{lij,c} \end{bmatrix}.$$

The sending-side current ($i \rightarrow j$) through the series element in the Π-model is defined as $I_{lij}^s$, the sending-side shunt current as $I_{lij}^h$, respectively. The complex power flow in branch $l$ from bus $i$ to $j$ depends on the bus voltage of the sending-side $U_i$ and the conjugate transpose (indicated with superscript H) of the current $I_{lij}$,

$$S_{lij} = U_i (I_{lij}^H)^H = \begin{bmatrix} S_{lij,aa} & S_{lij,ab} & S_{lij,ac} \\ S_{lij,ba} & S_{lij,bb} & S_{lij,bc} \\ S_{lij,ca} & S_{lij,cb} & S_{lij,cc} \end{bmatrix}.$$  \hspace{1cm} (2)
We observe that \( \text{diag} (S_{lij}) = U_i \odot (I_{lij})^* \), where \( \odot \) is the element-wise (Hadamard) product. It is noted that \( \text{rank} (S_{lij}) = 1 \), as it is defined as the outer product of two vectors. The off-diagonals relate to the diagonal elements according to,

\[
\begin{align*}
\frac{S_{lij,aa}}{U_{i,a}} &= \frac{S_{lij,ba}}{U_{i,b}} = \frac{S_{lij,ca}}{U_{i,c}} = (I_{lij,a})^*, \\
\frac{S_{lij,bb}}{U_{i,b}} &= \frac{S_{lij,ab}}{U_{i,a}} = \frac{S_{lij,cb}}{U_{i,c}} = (I_{lij,b})^*, \\
\frac{S_{lij,cc}}{U_{i,c}} &= \frac{S_{lij,ac}}{U_{i,a}} = \frac{S_{lij,bc}}{U_{i,b}} = (I_{lij,c})^*,
\end{align*}
\]

which means that the off-diagonals of \( S_{lij} \) are scaled and rotated versions of the more easily interpretable diagonal elements.

### 2.2 Variable Bounds

Voltage magnitudes have minimum and maximum operational limits, which are specific to each bus and phase,

\[
0 \leq \begin{bmatrix} \frac{U_{i,a}}{U_i^{\text{min}}} \\ \frac{U_{i,b}}{U_i^{\text{min}}} \\ \frac{U_{i,c}}{U_i^{\text{min}}} \end{bmatrix} \leq \begin{bmatrix} U_{i,a} \\ U_{i,b} \\ U_{i,c} \end{bmatrix} \leq \begin{bmatrix} \frac{U_{i,a}}{U_i^{\text{max}}} \\ \frac{U_{i,b}}{U_i^{\text{max}}} \\ \frac{U_{i,c}}{U_i^{\text{max}}} \end{bmatrix}.
\]

In this report, we overload ‘\( \leq \)’ for ‘\( \geq \)’ for vectors and matrices to indicate element-wise inequality (conversely, ‘\( \geq \)’ is used as the symbol for matrix positive semidefiniteness). Recognizing we obtain the magnitude squared by multiplying complex numbers with their own conjugates, the nodal voltage bounds can also be presented quadratically as:

\[
0 \leq U_i^{\text{min}} \odot U_i^{\text{min}} \leq U_i \odot (U_i)^* \leq U_i^{\text{max}} \odot U_i^{\text{max}}.
\]

Note that the upper bound constraints are convex (space inside a circle), but the lower bound constraints are nonconvex (space outside of a circle). Next, apparent power limits are defined using the absolute value of the diagonals of the branch flow matrix,

\[
0 \leq \begin{bmatrix} |S_{lij,aa}| \\ |S_{lij,bb}| \\ |S_{lij,cc}| \end{bmatrix} \leq \begin{bmatrix} S_{\text{rated},ija} \\ S_{\text{rated},ijb} \\ S_{\text{rated},ijc} \end{bmatrix} = S_{\text{rated},ij},
\]

which is equivalent to second order cone (SOC) constraints,

\[
\begin{bmatrix} |S_{lij,aa}|^2 \\ |S_{lij,bb}|^2 \\ |S_{lij,cc}|^2 \end{bmatrix} = \begin{bmatrix} (P_{lij,aa})^2 + (Q_{lij,aa})^2 \\ (P_{lij,bb})^2 + (Q_{lij,bb})^2 \\ (P_{lij,cc})^2 + (Q_{lij,cc})^2 \end{bmatrix} \leq \begin{bmatrix} (S_{\text{rated},ija})^2 \\ (S_{\text{rated},ijb})^2 \\ (S_{\text{rated},ijc})^2 \end{bmatrix},
\]

and can succinctly be written as,

\[
0 \leq \text{diag} (S_{lij}) \odot \text{diag} (S_{lij})^* \leq S_{\text{rated},ij} \odot S_{\text{rated},ij}.
\]
The magnitudes of the branch current should stay below rated values and are bounded for the diagonals of the branch current matrix,

$$0 \leq \begin{bmatrix} I_{lij,a} & I_{lij,b} & I_{lij,c} \end{bmatrix} \leq \begin{bmatrix} I_{\text{rated},a} & I_{\text{rated},b} & I_{\text{rated},c} \end{bmatrix} = I_{\text{rij}}^{\text{rated}},$$

which again can be written as a set of SOC constraints,

$$0 \leq I_{lij} \circ (I_{lij}^*)^T \leq I_{\text{rated},a,b,c} \circ I_{\text{rated},a,b,c}^{\text{rated}}.$$  

(9)

A valid SOC representation of current magnitude limits (9) that does not require explicit current variables $I_{lij}$ is useful for BIM forms. Using the nodal voltage magnitudes, the branch current limits can directly be enforced on the power flows,

$$\begin{bmatrix} |S_{lij,aa}|^2 & |S_{lij,bb}|^2 & |S_{lij,cc}|^2 \end{bmatrix} \leq \begin{bmatrix} (I_{\text{rated},a})^2 & (I_{\text{rated},b})^2 & (I_{\text{rated},c})^2 \end{bmatrix} \circ \begin{bmatrix} |U_{i,a}|^2 & |U_{i,b}|^2 & |U_{i,c}|^2 \end{bmatrix},$$

(10)

which can also be developed in matrix notation as,

$$\text{diag}(S_{lij}) \circ \text{diag}(S_{lij}^*) \leq I_{\text{rated},a,b,c} \circ I_{\text{rated},a,b,c} \circ U \circ (U^*).$$

(11)

Valid bounds on all elements of the power flow matrix $S_{lij} = P_{lij} + jQ_{lij}$ are,

$$-U_{\text{max},i} (I_{\text{rated}})_{lj} \leq P_{lij} \leq U_{\text{max},i}^{\text{max}} (I_{\text{rij}}^{\text{rated}})_{lj}^T, \quad Q_{lij} \leq U_{\text{max},i} (I_{\text{rated}})_{lj}^T,$$

(12)

both for active and reactive power components, respectively. The voltage angle differences between connected buses $i$ and $j$ are bounded,

$$\begin{bmatrix} \theta_{\text{min},ij,aa} & \theta_{\text{min},ij,bb} & \theta_{\text{min},ij,cc} \\ \theta_{\text{min},ij,ba} & \theta_{\text{min},ij,bb} & \theta_{\text{min},ij,cc} \\ \theta_{\text{min},ij,cj} & \theta_{\text{min},ij,bj} & \theta_{\text{min},ij,cc} \end{bmatrix} \leq \begin{bmatrix} \theta_{ij,aa} - \theta_{ij,ba} - \frac{2\pi}{3} \\ \theta_{ij,bb} - \theta_{ij,bj} - \frac{2\pi}{3} \\ \theta_{ij,cc} - \theta_{ij,cj} - \frac{2\pi}{3} \end{bmatrix} \leq \begin{bmatrix} \theta_{\text{max},ij,aa} & \theta_{\text{max},ij,bb} & \theta_{\text{max},ij,cc} \\ \theta_{\text{max},ij,ba} & \theta_{\text{max},ij,bb} & \theta_{\text{max},ij,cc} \\ \theta_{\text{max},ij,cj} & \theta_{\text{max},ij,bj} & \theta_{\text{max},ij,cc} \end{bmatrix}.$$

(13)

(14)

For a relatively balanced voltage phasor, we expect \( \theta_{i,a} - \theta_{i,b} \approx \theta_{i,b} - \theta_{i,c} \approx \theta_{i,c} - \theta_{i,a} \approx \frac{2\pi}{3} \). The voltage angle differences between phases on buses $i$ can be bounded to enforce angle balance relative to the expected 120 degrees,

$$\begin{bmatrix} \theta_{\text{min},i,aa} & \theta_{\text{min},i,bb} & \theta_{\text{min},i,cc} \\ \theta_{\text{min},i,ba} & \theta_{\text{min},i,bb} & \theta_{\text{min},i,cc} \\ \theta_{\text{min},i,cj} & \theta_{\text{min},i,bj} & \theta_{\text{min},i,cc} \end{bmatrix} \leq \begin{bmatrix} \theta_{i,aa} - \theta_{i,ba} - \frac{2\pi}{3} \\ \theta_{i,bb} - \theta_{i,bj} - \frac{2\pi}{3} \\ \theta_{i,cc} - \theta_{i,cj} - \frac{2\pi}{3} \end{bmatrix} \leq \begin{bmatrix} \theta_{\text{max},i,aa} & \theta_{\text{max},i,bb} & \theta_{\text{max},i,cc} \\ \theta_{\text{max},i,ba} & \theta_{\text{max},i,bb} & \theta_{\text{max},i,cc} \\ \theta_{\text{max},i,cj} & \theta_{\text{max},i,bj} & \theta_{\text{max},i,cc} \end{bmatrix}.$$

(15)

The voltage phasor in reference buses $i \in \mathcal{I}_{\text{ref}} \subset \mathcal{I}$ is assumed fixed, e.g.,

$$U_i = U_i^{\text{ref}} = \begin{bmatrix} U_{i,a}^{\text{ref}} & U_{i,b}^{\text{ref}} & U_{i,c}^{\text{ref}} \end{bmatrix}.$$

(16)
2.3 Branch Impedance

The circuit series impedance matrix is defined as a full matrix with no assumption on the particular structure:

\[
\begin{bmatrix}
  z_{s,aa} & z_{s,ab} & z_{s,ac} \\
  z_{s,ba} & z_{s,bb} & z_{s,bc} \\
  z_{s,ca} & z_{s,cb} & z_{s,cc}
\end{bmatrix},
\]

where each element consists of a series resistive and reactive impedance. It is noted that in physical systems, we expect \( r_{s,l} \geq 0, x_{s,l} \geq 0 \). The impedance matrix can be rewritten in the corresponding admittance form,

\[
y_{l} = (z_{l})^{-1} = g_{l} + j b_{l}.
\]

where \((z_{l})^{-1}\) is the matrix inverse of \(z_{l}\). In case of missing conductors, e.g. single, or two-conductor connections, \(z_{l}\) is not invertible but it is valid to use the Moore-Penrose inverse instead. The shunt admittances at the sending and receiving sides, respectively \(y_{lij}, y_{lij}^{sh}\), are defined,

\[
y_{lij}^{sh} = g_{lij}^{sh} + j b_{lij}^{sh} = \begin{bmatrix}
  y_{lij,aa}^{sh} & y_{lij,ab}^{sh} & y_{lij,ac}^{sh} \\
  y_{lij,ba}^{sh} & y_{lij,bb}^{sh} & y_{lij,bc}^{sh} \\
  y_{lij,ca}^{sh} & y_{lij,cb}^{sh} & y_{lij,cc}^{sh}
\end{bmatrix}.
\]

Although for typical distribution lines and cables the shunt admittances are diagonal and have equal values for both sides, in this generalized derivation they can be considered as a two different and full matrices. This also allows for re-use of the representation for other elements, such as transformers.

2.4 Loads and Generators as Units

We define units \( u \in U \) to generalize loads, generators and storage elements. The current flowing from the connected bus into the unit is

\[
I_u = \begin{bmatrix}
  I_{u,a} \\
  I_{u,b} \\
  I_{u,c}
\end{bmatrix}.
\]

The unit current is bounded by the corresponding current rating \(I_{\text{rated}}_u\) similarly to (9). Tuples of units and the buses they are connected to are defined in the connectivity set \(u \in F_{\text{units}}\). The active and reactive power consumed by a unit is defined \(S_u = P_u + j Q_u = U_i(I_{u})^H\). We define bounds on active/reactive power dispatch separately,

\[
\begin{align*}
  P_{\min u} & \leq \text{diag}(P_u) \leq P_{\max u}, \\
  Q_{\min u} & \leq \text{diag}(Q_u) \leq Q_{\max u}.
\end{align*}
\]

We do not provide any further detail on the specific modeling of delta or wye-connected units and refer to [13, 17].
2.5 Shunts

A shunt element, e.g. shunt capacitance, inductance or resistance, with index $h$ has an admittance $y_h = g_h + j b_h$. Tuples of shunts and the bus they are connected to are defined in the connectivity set $h \in T_{\text{shunts}}$. The current from the bus to the shunt is,

$$I_h = \begin{bmatrix} I_{h,a} \\ I_{h,b} \\ I_{h,c} \end{bmatrix}. \quad (21)$$

and is bounded by the rated current $I_{h}^{\text{rated}}$.

3 Application of Circuit Laws

This section illustrates how Kirchhoff’s and Ohm’s laws are used in the context of multi-conductor branches with matrix impedances for the representation of the circuit physics, providing the relationship between the voltage, current and power variables.

3.1 Branch Model

Ohm’s law for branch $lij$ representing the voltage drop along the branch is formulated in matrix form using nodal voltages, the branch current and the impedance matrix,

$$U_j = U_i - z_s I_{lij}. \quad (22)$$

Kirchhoff’s current law (KCL) is used to split up the series and shunt (to ground) currents in the Π-section,

$$I_{lij} = I_{s,lij} + I_{sh,lij}, \quad I_{lji} = I_{s,lji} + I_{sh,lji}, \quad I_{s,lij} + I_{s,lji} = 0. \quad (23)(24)$$

Fig. 3 defines the harmonized single-wire equivalent vector/matrix variables in both natural and lifted (defined in upcoming sections) variable spaces for a clear presentation of the basic relationships. The scalar representation of all variables and parameters is defined in Fig. 1.
3.2 Shunts and Units at Nodes

The unit model can be adapted for a variety of optimization problems, e.g., optimal dispatch of distributed generation, optimal scheduling of storage and electric vehicle charging and optimal demand management. Such extensions essentially define feasible sets dependent on \( \text{diag}(P_u), \text{diag}(Q_u) \).

Without loss of generality, in this work, we focus on the feasible sets defined by (20a)-(20b).

Similarly, the shunt power \( S_h \) is defined as

\[
S_h = P_h + jQ_h = U_i(I_h)^H. \tag{25}
\]

The shunt current relates to the voltage through Ohm’s law,

\[
I_h = y_h U_i. \tag{26}
\]

Substituting (26) into (25), the shunt power consumption can be expressed depending on the nodal voltage only,

\[
S_h = U_i(U_i)^H(y_h)^H. \tag{27}
\]

3.3 Bus Model: Kirchhoff’s Current Law

Kirchhoff’s Current Law (KCL) is conventionally expressed in current variables, but can also be lifted to the complex power variable space. KCL in current variables at each bus \( i \) is,

\[
\sum_{lj \in T} I_{lj} + \sum_{ui \in T_{\text{units}}} I_u + \sum_{bi \in T_{\text{shunts}}} I_h = 0. \tag{28}
\]

KCL in the complex power variable space is obtained by taking the conjugate transpose of (28) and element-wise multiplying with \( U_i \neq 0 \) on the left,

\[
\sum_{lj \in T} \text{diag}(S_{lij}) + \sum_{ui \in T_{\text{units}}} \text{diag}(S_u) + \sum_{bi \in T_{\text{shunts}}} \text{diag}(S_h) = 0, \tag{29}
\]

which means that the diagonal elements of the apparent power matrices of the connected branch flows, units and shunt need to sum to zero for each bus. Commonly, the shunt power expression (27) is substituted into this equation. The real-value equivalent forms are obtained,

\[
\sum_{lj \in T} \text{diag}(P_{lij}) + \sum_{ui \in T_{\text{units}}} \text{diag}(P_u) + \sum_{bi \in T_{\text{shunts}}} \text{diag}(P_h) = 0, \tag{30a}
\]

\[
\sum_{lj \in T} \text{diag}(Q_{lij}) + \sum_{ui \in T_{\text{units}}} \text{diag}(Q_u) + \sum_{bi \in T_{\text{shunts}}} \text{diag}(Q_h) = 0. \tag{30b}
\]

4 Unbalanced Branch Flow Model

This section illustrates how the unbalanced BFM is derived. First, we define a variable for the complex power flow \( S'_{lij} \) through the series element of the \( \Pi \)-section,

\[
S'_{lij} = P'_{lij} + jQ'_{lij} = U_i(I_{lij})^H. \tag{31}
\]
4.1 Power Flow Model

Passive components in electrical circuits cause losses. In the BFM we distinguish between the losses associated with the from-side shunt, i.e. \( S_{loss,sh}^{i} \), the series impedance, i.e. \( S_{loss,s}^{i} \), and the to-side shunt, i.e. \( S_{loss,sh}^{j} \). The series losses depend on the voltage drop over the impedance and the current flow through it,

\[
S_{loss,s}^{i} = (\mathbf{U}_i - \mathbf{U}_j)(\mathbf{f}_{ij})^H. \tag{32}
\]

The voltage drop itself is can be derived from Ohm’s law,

\[
(\mathbf{U}_i - \mathbf{U}_j) = z_l f_{ij}. \tag{33}
\]

Substituting (33) into (32) we obtain the series loss

\[
S_{loss,s}^{i} = z_l f_{ij}(f_{ij})^H = z_l f_{ij}(f_{ji})^H, \tag{34}
\]

which is symmetric because of (24). The shunt losses \( S_{loss,sh}^{i,j} \) are derived from the current through the shunt, i.e. \( f_{ij}^{sh} \), and the voltage at the shunt, i.e. \( \mathbf{U}_i \),

\[
S_{loss,sh}^{i,j} = \mathbf{U}_i (f_{ij}^{sh})^H, \tag{35}
\]

The shunt current relates to the voltage through Ohm’s law,

\[
f_{ij}^{sh} = y_{ij}^{sh} \mathbf{U}_i. \tag{36}
\]

Substituting (36) into (35), the shunt losses are,

\[
S_{loss,sh}^{i,j} = \mathbf{U}_i (y_{ij}^{sh})^H, S_{loss,sh}^{j,i} = \mathbf{U}_j (y_{ji}^{sh})^H. \tag{37}
\]

As the the sum of the sending and receiving side power flows need to equal the branch losses, the branch loss balance (see Fig. 3) can be written using the different loss components (34) and (37),

\[
S_{ij} + S_{ji} = S_{loss,s}^{i} + S_{loss,s}^{j} + S_{loss,sh}^{i} + S_{loss,sh}^{j} = \mathbf{U}_i (y_{ij}^{sh})^H + z_l f_{ij}(f_{ij})^H + \mathbf{U}_j (y_{ji}^{sh})^H. \tag{38}
\]

5 Unbalanced Bus Injection Model

This section illustrates how the unbalanced BIM is obtained for which the branch flow current is substituted by nodal voltage variables.

5.1 Power Flow Model

Therefore, we write (33) in admittance form,

\[
f_{ij} = y_{ij}^{i} (\mathbf{U}_i - \mathbf{U}_j). \tag{39}
\]

The sending end apparent power flow can be calculated using the sending end voltage and the sum of the series current and the sending end shunt current, respectively,

\[
S_{ij} = \mathbf{U}_i (f_{ij}^{s} + f_{ij}^{sh})^H. \tag{40}
\]
By substituting (36) and (39) into (40) we derive,

$$S_{lij} = U_i(U_j)^H (y_{lij}^{sh})^H + U_i(U_j - U_j)^H (y_{lij})^H,$$

which is the nonlinear complex matrix form of the BIM. Note that this constraint is defined for both the sending end $S_{lij}$ and the receiving end $S_{lij}$, respectively. We now choose rectangular coordinates for the voltage and power variables and obtain,

$$P_{lij} = (U_i^{re}(U_i^{re})^T + U_i^{im}(U_i^{im})^T) (b_{lij}^{sh})^T + (U_i^{re}(U_i^{re})^T + U_i^{im}(U_i^{im})^T) (b_{lij})^T$$

$$+ (U_i^{im}(U_i^{im})^T - U_i^{im}(U_i^{im})^T) (g_{lij}^{sh})^T$$

$$+ (U_i^{im}(U_i^{im})^T - U_i^{im}(U_i^{im})^T) (g_{lij})^T,$$

$$Q_{lij} = -(U_i^{re}(U_i^{re})^T + U_i^{im}(U_i^{im})^T) (b_{lij}^{sh})^T + (U_i^{im}(U_i^{im})^T - U_i^{im}(U_i^{im})^T) (g_{lij}^{sh})^T$$

$$+ (U_i^{im}(U_i^{im})^T - U_i^{im}(U_i^{im})^T) (g_{lij})^T,$$

In the coming subsections we derive the diagonalized and scalarized real-valued formulation of the BIM in both the polar and rectangular voltage coordinate systems.

### 5.2 Unbalanced BIM Rectangular Scalar Form

Using $U_i = U_i^{re} + jU_i^{im}$, the expressions for active power of the diagonal elements of $S_{lij}$ can be parameterized, $p, q \in P$, as,

$$P_{lij,pp} = \sum_{q \in P} (U_i^{re}U_{i,q}^{re} + U_i^{im}U_{i,q}^{im}) (g_{lij,pp} + g_{lij,pp}^{sh})$$

$$+ \sum_{q \in P} (U_{i,q}^{im}U_i^{re} - U_{i,q}^{re}U_i^{im}) (b_{lij,pp} + b_{lij,pp}^{sh})$$

$$- \sum_{q \in P} (U_i^{im}U_{i,q}^{re} + U_i^{re}U_{i,q}^{im}) g_{lij,pp}^{sh}$$

$$- \sum_{q \in P} (U_i^{im}U_{i,q}^{re} - U_i^{re}U_{i,q}^{im}) b_{lij,pp}^{sh},$$

and

$$Q_{lij,pp} = - \sum_{q \in P} (U_i^{re}U_{i,q}^{re} + U_i^{im}U_{i,q}^{im}) (b_{lij,pp} + b_{lij,pp}^{sh})$$

$$+ \sum_{q \in P} (U_{i,q}^{im}U_i^{re} - U_{i,q}^{re}U_i^{im}) (g_{lij,pp} + g_{lij,pp}^{sh})$$

$$+ \sum_{q \in P} (U_i^{im}U_{i,q}^{re} + U_i^{re}U_{i,q}^{im}) b_{lij,pp}^{sh}$$

$$- \sum_{q \in P} (U_i^{im}U_{i,q}^{re} - U_i^{re}U_{i,q}^{im}) g_{lij,pp}^{sh}.$$

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The bus shunt expressions are,
\[ P_{h,p} = \sum_{q \in P} (U_{i,p}^{\text{re}} v_{i,q}^{\text{re}} + U_{i,p}^{\text{im}} v_{i,q}^{\text{im}}) g_{h,pq} \]
\[ + \sum_{q \in P} (U_{i,p}^{\text{im}} v_{i,q}^{\text{re}} - U_{i,p}^{\text{re}} v_{i,q}^{\text{im}}) b_{h,pq}, \]
(46)
\[ Q_{h,p} = - \sum_{q \in P} (U_{i,p}^{\text{re}} U_{i,q}^{\text{re}} + U_{i,p}^{\text{im}} U_{i,q}^{\text{im}}) b_{h,pq} \]
\[ + \sum_{q \in P} (U_{i,p}^{\text{im}} U_{i,q}^{\text{re}} - U_{i,p}^{\text{re}} U_{i,q}^{\text{im}}) g_{h,pq}. \]
(47)

In rectangular coordinates, the nodal voltage bounds as defined in [5] become
\[
\begin{bmatrix}
(U_{i,a}^{\text{min}})^2 \\
(U_{i,b}^{\text{min}})^2 \\
(U_{i,c}^{\text{min}})^2
\end{bmatrix}
\leq
\begin{bmatrix}
(U_{i,a}^{\text{max}})^2 \\
(U_{i,b}^{\text{max}})^2 \\
(U_{i,c}^{\text{max}})^2
\end{bmatrix},
\]
(48)
which are nonconvex for strictly positive voltage lower bounds.

The voltage angle difference constraint between buses \([14]\) needs to be reformulated. We derive the unbalanced tangent identity \([49]\),
\[ \tan \begin{bmatrix} \theta_{i,a} \\ \theta_{i,b} \\ \theta_{i,c} \end{bmatrix} - \begin{bmatrix} \theta_{j,a} \\ \theta_{j,b} \\ \theta_{j,c} \end{bmatrix} = \begin{bmatrix} U_{i,a}^{\text{im}} v_{i,a}^{\text{re}} - U_{i,a}^{\text{re}} v_{i,a}^{\text{im}} \\
U_{i,b}^{\text{im}} v_{i,b}^{\text{re}} - U_{i,b}^{\text{re}} v_{i,b}^{\text{im}} \\
U_{i,c}^{\text{im}} v_{i,c}^{\text{re}} - U_{i,c}^{\text{re}} v_{i,c}^{\text{im}} \end{bmatrix} \circ \begin{bmatrix} U_{i,a}^{\text{re}} U_{j,a}^{\text{re}} + U_{i,a}^{\text{im}} U_{j,a}^{\text{im}} \\
U_{i,b}^{\text{re}} U_{j,b}^{\text{re}} + U_{i,b}^{\text{im}} U_{j,b}^{\text{im}} \\
U_{i,c}^{\text{re}} U_{j,c}^{\text{re}} + U_{i,c}^{\text{im}} U_{j,c}^{\text{im}} \end{bmatrix}. \]
(49)

Because the angle difference is bounded by \([-\pi/2, \pi/2]\) for voltage stability, we can rewrite this as quadratic nonconvex constraint \([50]\), where the tangent function is applied element-wise to the components of the vector containing the bounds on the angle difference,
\[ \tan \circ (\Theta_{ij}^{\text{min}}) \circ \begin{bmatrix} U_{i,a}^{\text{re}} + U_{i,a}^{\text{im}} v_{i,a}^{\text{re}} \\
U_{i,b}^{\text{re}} + U_{i,b}^{\text{im}} v_{i,b}^{\text{re}} \\
U_{i,c}^{\text{re}} + U_{i,c}^{\text{im}} v_{i,c}^{\text{re}} \end{bmatrix} \leq \begin{bmatrix} U_{i,a}^{\text{re}} - U_{i,a}^{\text{im}} v_{i,a}^{\text{re}} \\
U_{i,b}^{\text{re}} - U_{i,b}^{\text{im}} v_{i,b}^{\text{re}} \\
U_{i,c}^{\text{re}} - U_{i,c}^{\text{im}} v_{i,c}^{\text{re}} \end{bmatrix} \leq \tan \circ (\Theta_{ij}^{\text{max}}) \circ \begin{bmatrix} U_{i,a}^{\text{re}} + U_{i,a}^{\text{im}} v_{i,a}^{\text{re}} \\
U_{i,b}^{\text{re}} + U_{i,b}^{\text{im}} v_{i,b}^{\text{re}} \\
U_{i,c}^{\text{re}} + U_{i,c}^{\text{im}} v_{i,c}^{\text{re}} \end{bmatrix}. \]
(50)

We derive the equivalent expression for the angle differences between phases on the same bus, The phase angle difference constraint \([15]\) is equivalent to,
\[ \frac{2\pi}{3} + \Theta_{i,a}^{\text{min}} \leq \begin{bmatrix} \theta_{i,a} - \theta_{i,b} \\
\theta_{i,b} - \theta_{i,c} \\
\theta_{i,c} - \theta_{i,a} \end{bmatrix} \leq \Theta_{i,a}^{\text{max}} \leq \frac{2\pi}{3} + \Theta_{i,a}^{\text{max}}. \]
(51)

For the tangent function to be increasing and invertible, we want to further restrict ourselves to,
\[ \frac{\pi}{2} \leq \frac{2\pi}{3} + \Theta_{i,a}^{\text{min}} \leq \frac{2\pi}{3} + \Theta_{i,a}^{\text{max}} \leq \pi, \]
(52)
which implies,
\[ -\frac{\pi}{6} \leq \Theta_{i,a}^{\text{min}} \leq \Theta_{i,a}^{\text{max}} \leq \frac{\pi}{3}. \]
(53)
With this restriction we derive the quadratic nonconvex constraint (54) from (51), using an identity similar to (49),

\[
\tan \left( \frac{2 \pi}{3} + \Theta_{i}^{\min} \right) \leq \tan \left( \frac{2 \pi}{3} + \Theta_{i}^{\max} \right)
\]

\[
\left( \begin{array}{c}
U_{i,a}^{\text{re}} U_{i,b}^{\text{re}} + U_{i,c}^{\text{re}}
U_{i,a}^{\text{im}} U_{i,b}^{\text{im}} + U_{i,c}^{\text{im}}
U_{i,a}^{\text{re}} U_{i,b}^{\text{im}} + U_{i,c}^{\text{re}}
U_{i,a}^{\text{im}} U_{i,b}^{\text{re}} + U_{i,c}^{\text{im}}
\end{array} \right) \leq \left( \begin{array}{c}
U_{i,a}^{\text{re}} U_{i,b}^{\text{re}} - U_{i,a}^{\text{im}} U_{i,b}^{\text{im}}
U_{i,b}^{\text{re}} U_{i,c}^{\text{re}} - U_{i,b}^{\text{im}} U_{i,c}^{\text{im}}
U_{i,c}^{\text{re}} U_{i,a}^{\text{re}} - U_{i,c}^{\text{im}} U_{i,a}^{\text{im}}
\end{array} \right).
\]  

(54)

5.3 Unbalanced BIM Polar Scalar Form

The active and reactive power flow in each conductor can be written in a parameterized way, using nodal voltages per conductor \( U_{i,p} = U_{i,p}^{\text{mag}} e^{j \theta_{i,p}} \). The expressions of the diagonal elements of \( S_{lij} \) can be parameterized, \( p, q \in \mathcal{P} = \{ a, b, c \} \) and the real-value formulation of active and reactive power is obtained using trigonometric functions,

\[
P_{ij,pp} = \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{i,q}^{\text{mag}} \cos(\theta_{i,p} - \theta_{i,q}) \left( g_{l,pp} + g_{l,pp}^{\text{sh}} \right)
\]

\[
+ \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{i,q}^{\text{mag}} \sin(\theta_{i,p} - \theta_{i,q}) \left( b_{l,pp} + b_{l,pp}^{\text{sh}} \right)
\]

\[
- \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{j,q}^{\text{mag}} \cos(\theta_{i,p} - \theta_{j,q}) g_{l,pp}^{\text{sh}}
\]

\[
- \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{j,q}^{\text{mag}} \sin(\theta_{i,p} - \theta_{j,q}) b_{l,pp}^{\text{sh}}.
\]  

(55)

and

\[
Q_{ij,pp} = - \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{i,q}^{\text{mag}} \cos(\theta_{i,p} - \theta_{i,q}) \left( t_{l,pp}^{\text{sh}} + b_{l,pp}^{\text{sh}} \right)
\]

\[
+ \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{i,q}^{\text{mag}} \sin(\theta_{i,p} - \theta_{i,q}) \left( g_{l,pp}^{\text{sh}} + g_{l,pp}^{\text{sh}} \right)
\]

\[
+ \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{j,q}^{\text{mag}} \cos(\theta_{i,p} - \theta_{j,q}) b_{l,pp}^{\text{sh}}
\]

\[
- \sum_{q \in \mathcal{P}} U_{i,p}^{\text{mag}} U_{j,q}^{\text{mag}} \sin(\theta_{i,p} - \theta_{j,q}) g_{l,pp}^{\text{sh}}.
\]  

(56)
Thus, the active and reactive power flow through each conductor is obtained using the mutual
 coupling of the nodal voltages and branch impedances. The bus shunt expressions are,

\[ P_{h,pp} = \sum_{q \in P} U^\text{mag}_{i,p} U^\text{mag}_{i,q} \cos(\theta_{i,p} - \theta_{i,q}) g_{h,pq} \]

\[ + \sum_{q \in P} U^\text{mag}_{i,p} U^\text{mag}_{i,q} \sin(\theta_{i,p} - \theta_{i,q}) b_{h,pq}, \]  

(57)

\[ Q_{h,pp} = -\sum_{q \in P} U^\text{mag}_{i,p} U^\text{mag}_{i,q} \cos(\theta_{i,p} - \theta_{i,q}) b_{h,pq} \]

\[ + \sum_{q \in P} U^\text{mag}_{i,p} U^\text{mag}_{i,q} \sin(\theta_{i,p} - \theta_{i,q}) g_{h,pq}. \]  

(58)

The bus pair voltage angle difference constraint (14) and phase angle difference constraint (15)
directly apply in this variable space.

6 Lifting of Bus Injection Model

This section defines new variables to represent products of voltages and currents for the BIM, to
enable a lift-and-project approach. This approach is commonly used for SOC and SDP relaxations
of the nonlinear power-voltage formulation.

6.1 Lifted Variables

6.1.1 Bus Voltage

We define an auxiliary variable for the voltage products, \( W_i \), satisfying

\[ W_i = W^\text{re}_i + j W^\text{im}_i = U_i(U_i^*)^H, \]

(59a)

\[ W_i \succeq 0, \text{rank}(W_i) = 1. \]  

(59b)

Note that (59a) is a quadratic nonconvex constraint, and (59b) is the well-known rank-constrained
SDP equivalent form. We illustrate the structure of \( W_i = U_i(U_i)^H \) as a real-valued matrix in
rectangular coordinates,

\[ W_i = \begin{bmatrix}
W^\text{re}_{i,aa} & W^\text{re}_{i,ab} & W^\text{re}_{i,ac} \\
W^\text{re}_{i,ba} & W^\text{re}_{i,bb} & W^\text{re}_{i,bc} \\
W^\text{re}_{i,ca} & W^\text{re}_{i,cb} & W^\text{re}_{i,cc}
\end{bmatrix} + j \begin{bmatrix}
0 & W^\text{im}_{i,ab} & W^\text{im}_{i,ac} \\
-W^\text{im}_{i,ba} & 0 & W^\text{im}_{i,bc} \\
-W^\text{im}_{i,ca} & -W^\text{im}_{i,cb} & 0
\end{bmatrix}. \]  

(60)

Note that this representation requires 9 unique scalar variables (underlined) and that the diagonal
is real-valued,

\[ \text{diag}(W_i) = U_i \odot U_i^* = \begin{bmatrix}
|U_{i,a}|^2 \\
|U_{i,b}|^2 \\
|U_{i,c}|^2
\end{bmatrix} = \begin{bmatrix}
W^\text{re}_{i,aa} \\
W^\text{re}_{i,bb} \\
W^\text{re}_{i,cc}
\end{bmatrix}. \]  

(61)
One can define bounds on the matrix entries as,

\[
U^\min_i \circ U^\min_i \leq \text{diag}(W^\text{re}_{ij}) \leq U^\max_i \circ U^\max_i,
\]

\[
- U^\max_i (U^\max_j)^T \leq W^\text{re}_{ij} \leq U^\max_i (U^\max_j)^T.
\]

(62a)

(62b)

In this variable space, the phase voltage angle difference constraint (15) becomes,

\[
\tan \left( \frac{2\pi}{3} + \Theta^\min_i \right) \circ \begin{bmatrix}
W^\text{re}_{i,ab} \\
W^\text{re}_{i,ac} \\
W^\text{re}_{i,bc}
\end{bmatrix} \leq \begin{bmatrix}
W^\text{im}_{i,ab} \\
W^\text{im}_{i,ac} \\
W^\text{im}_{i,bc}
\end{bmatrix} \leq \tan \left( \frac{2\pi}{3} + \Theta^\max_i \right) \circ \begin{bmatrix}
W^\text{re}_{i,ab} \\
W^\text{re}_{i,ac} \\
W^\text{re}_{i,bc}
\end{bmatrix}.
\]

(63)

The reference bus phasor is fixed,

\[
W_i = U_i^\text{ref}(U_i^\text{ref})^H.
\]

(64)

### 6.1.2 Bus Voltage Cross Product

We define a variable \(W_{ij}\) for the cross-product of the voltages \(U_i\) and \(U_j\) of the buses associated with a branch (i.e. all bus-pairs) as used in equations (55-56) and (44-45),

\[
W_{ij} = U_i(U_j)^H.
\]

(65)

Although \(W_{ij}\) is rank-1 by construction, it is not Hermitian. Note that this definition implies,

\[
W_{ij} = (W_{ji})^H.
\]

(66)

One can define bounds on the matrix entries of \(W_{ij}\) as,

\[
- U^\max_i (U^\max_j)^T \leq W^\text{re}_{ij} \leq U^\max_i (U^\max_j)^T.
\]

(67)

### 6.2 Power Flow Model

Now power flow equation (41) can be written using the lifted variables,

\[
S_{lij} = W_i(y^\text{sh}_{lij})^H + (W_i - W_{ij})(y^\text{e}_i)^H.
\]

(68)

The real-value equivalents are,

\[
P_{lij} = W^\text{re}_{ij}(g^\text{sh}_{lij} + g^e_i)^T + W^\text{im}_{ij}(b^\text{sh}_{lij} + b^e_i)^T - W^\text{re}_{ij}(g_i^e)^T - W^\text{im}_{ij}(b_i^e)^T.
\]

(69a)

\[
Q_{lij} = W^\text{im}_{ij}(g^\text{sh}_{lij} + g^e_i)^T - W^\text{re}_{ij}(b^\text{sh}_{lij} + b^e_i)^T - W^\text{im}_{ij}(g_i^e)^T + W^\text{re}_{ij}(b_i^e)^T.
\]

(69b)

Similarly, the bus shunt power with lifted variables is,

\[
S_h = W_i(y_h)^H.
\]

(70)
and its real-value equivalents are,

\[ P_h = W_i^r (g_h)^T + W_i^i (b_h)^T, \]  
\[ Q_h = W_i^i (g_h)^T - W_i^r (b_h)^T. \]  

(71a)

(71b)

Using (61), we lift the branch current limit (12) to the \( W_i \) variable space as

\[ \text{diag}(S_{lij}) \circ \text{diag}(S_{lij})^* \leq I_{lij}^{\text{rated}} \circ I_{lij}^{\text{rated}} \circ \text{diag}(W_i), \]  

which is a convex SOC constraint.

6.3 Voltage Angle Difference Bound

We note the ‘tangent inequality’ [30] can be extended to the three-phase case in the following way:

\[ \tan \left( \begin{bmatrix} \theta_{i,a} \\ \theta_{i,b} \\ \theta_{i,c} \end{bmatrix} - \begin{bmatrix} \theta_{j,a} \\ \theta_{j,b} \\ \theta_{j,c} \end{bmatrix} \right) = \text{diag}(W_{ij}^i) \circ \text{diag}(W_{ij}^r) \]  

(73)

The voltage angle difference bounds therefore are,

\[ \tan \left( \begin{bmatrix} \Theta_{ij}^{\text{min}} \end{bmatrix} \right) \circ \text{diag}(W_{ij}^r) \leq \text{diag}(W_{ij}^i) \leq \tan \left( \begin{bmatrix} \Theta_{ij}^{\text{max}} \end{bmatrix} \right) \circ \text{diag}(W_{ij}^r). \]  

(74)

6.4 Rank-Constrained SDP Model

6.4.1 Meshed Grids

Note that (65) can be generalized (to support meshed grids), by shaping the matrices into a block matrix \( M \),

\[ M = \begin{bmatrix} U_i & U_j & \cdots & U_z \\ U_j & U_j & \cdots & U_z \\ \vdots & \vdots & \ddots & \vdots \\ U_z & U_z & \cdots & U_z \end{bmatrix}^H = \begin{bmatrix} W_i & W_{ij} & \cdots & W_{iz} \\ W_{ji} & W_j & \cdots & W_{jz} \\ \vdots & \vdots & \ddots & \vdots \\ W_{zi} & W_{zj} & \cdots & W_z \end{bmatrix}, \]  

(75a)

\[ M \succeq 0, \quad \text{rank}(M) = 1. \]  

(75b)

Note that \( M \in \mathbb{H}^{|P||B|} \) with \( |B| \) the number of unique bus pairs in the topology. If there is no branch between buses \( i \) and \( j \), the corresponding \( W_{ij} = 0 \). The real-value equivalent SDP constraint [31], for \( M = M^r + jM^i \), is,

\[ \begin{bmatrix} M^r & M^i \\ -M^i & M^r \end{bmatrix} \succeq 0, \quad \text{rank}\left( \begin{bmatrix} M^r & M^i \\ -M^i & M^r \end{bmatrix} \right) = 1. \]  

(76)

Chordal relaxation can be employed to replace \( M \) with a set of smaller matrices [26, 27].
6.4.2 Radial Grids

In case of radial grids, a decomposition of $M$ is easily derived. Only constraints of type (77) are retained [27]:

$$\forall ij \in B : M_{ij} = \begin{bmatrix} U_i & U_i \\ U_j & U_j \end{bmatrix}^H \begin{bmatrix} W_i & W_{ij} \\ W_{ji} & W_j \end{bmatrix},$$

$$M_{ij} \succeq 0, \text{rank}(M_{ij}) = 1.$$  (77)

Note that $M_{ij} \in \mathbb{H}^{2|P|}$. Note that $M_{ij} \succeq 0$ implies both $W_i \succeq 0$ and $W_j \succeq 0$. The real-value equivalent form [31] is,

$$M^{2re}_{ij} = \begin{bmatrix} W_{ij}^{re} & W_{ij}^{re} & W_{ij}^{im} & W_{ij}^{im} \\ (W_{ij}^{re})^T & W_{ij}^{re} & (-W_{ij}^{im})^T & W_{ij}^{im} \\ -(W_{ij}^{im})^T & -W_{ij}^{im} & W_{ij}^{re} & W_{ij}^{re} \\ (W_{ij}^{im})^T & -W_{ij}^{im} & (W_{ij}^{re})^T & W_{ij}^{re} \end{bmatrix} \succeq 0.$$  (78a)

$$\text{rank}(M^{2re}_{ij}) = 1.$$  (78b)

7 Lifting of Branch Flow Model

This section defines new variables to represent products of voltages and currents specific to the BFM and then details the lift-and-project approach taken.

7.1 Lifted Variables

7.1.1 Series Current

The auxiliary variable for the current products, $L^s$ satisfies,

$$L^s_{ij} = L^s_{ij}(I_{ij})^H = L^s_{ij}(I_{ij})^H = L^s_{ij} \succeq 0, \text{rank}(L^s_{ij}) = 1,$$  (79a)

$$L^s_{ij} \succeq 0, \text{rank}(L^s_{ij}) = 1.$$  (79b)

Which has the following representation in scalar variables:

$$L^s_{ij} = \begin{bmatrix} L^{s,re}_{ij,aa} & L^{s,re}_{ij,ab} & L^{s,re}_{ij,ac} \\ L^{s,im}_{ij,ab} & L^{s,im}_{ij,bb} & L^{s,im}_{ij,cc} \end{bmatrix} + j \begin{bmatrix} 0 & L^{s,im}_{ij,ab} & L^{s,im}_{ij,ac} \\ -L^{s,im}_{ij,ab} & 0 & L^{s,im}_{ij,bc} \\ L^{s,im}_{ij,ac} & -L^{s,im}_{ij,bc} & 0 \end{bmatrix}. $$  (80)

This representation requires 9 unique variables (underlined).

7.1.2 Total Current

The total current $I_{ij}$ is lifted as,

$$I_{ij} = L^{re}_{ij} + jL^{im}_{ij} = I_{ij}(I_{ij})^H$$

$$I_{ij} \succeq 0, \text{rank}(I_{ij}) = 1.$$  (81a)

$$I_{ij} \succeq 0, \text{rank}(I_{ij}) = 1.$$  (81b)
We start from the definition $I^H_{ij} = (y^h_{ij} U_i)^H$ and multiply both sides with their conjugate transpose,

$$I^H_{ij}(I^H_{ij}) = (y^h_{ij} U_i)((I^H_{ij})^H + (U_i)^H(y^h_{ij}^H)^H).$$

Substituting in the lifted variables we obtain,

$$L_{ij} = L^v_i + (y^h_{ij} W_i(y^h_{ij})^H + (y^h_{ij})^H S^s_i + (S^s_{ij})^H(y^h_{ij})^H, \tag{82}$$

proving the lifted total current variable is a linear combination of $L^v_i$, $W_i$ and $S^s_{ij}$. In the real domain this becomes,

$$L^{\text{re}}_{ij} = L^{\text{re}}_i + g_{ij}^h W^{\text{re}}_i (g_{ij}^h)^T + g_{ij}^h W^{\text{im}}_i (b_{ij}^h)^T + b_{ij}^h P_{ij} - b_{ij}^h Q_{ij} + (g_{ij}^h P_{ij})^T - (b_{ij}^h Q_{ij})^T, \tag{83a}$$

and

$$L^{\text{im}}_{ij} = L^{\text{im}}_i + b_{ij}^h P_{ij}^T + b_{ij}^h W^{\text{im}}_i (b_{ij}^h)^T - g_{ij}^h W^{\text{re}}_i (b_{ij}^h)^T + g_{ij}^h P_{ij} - b_{ij}^h Q_{ij}^T - (g_{ij}^h Q_{ij})^T. \tag{83b}$$

One can define current bounds \([10]\) on the matrix entries of $L_{ij}$ as,

$$0 \leq \text{diag}(L^{\text{re}}_{ij}) \leq I^{\text{rated}}_{ij} \circ I^{\text{rated}}_{ij}, \quad -I^{\text{rated}}_{ij} \leq L^{\text{re}}_{ij}, L^{\text{im}}_{ij} \leq I^{\text{rated}}_{ij} \circ I^{\text{rated}}_{ij}. \tag{84a}$$

Note that $(84a)-(84b)$ is equivalent to $(72)$, however the former is linear due to $(83a) - (83b)$ and the latter is quadratic-convex (SOC). There are no direct bounds on the auxiliary variable for the series current $L^v_i$, however we can derive valid bounds through the known bounds total current and voltage,

$$- (I^{\text{rated}}_{ij})^T + |y^h_{ij}^T| U^{\text{max}}_i (U^{\text{max}}_i)^T |y^h_{ij}|^T \leq L^{\text{re}}_i, L^{\text{im}}_i \leq (I^{\text{rated}}_{ij})^T, \tag{85}$$

where $|y^h_{ij}|$ indicates the element-wise application of the absolute value operation to obtain the magnitude.

### 7.2 Power Flow Model

Given $S^{\text{loss,s}}_i = z_i^s L^s_i$, $S^{\text{loss,sh}}_ij = W_i (y^h_{ij})^H$, and $S^{\text{loss,sh}}_ij = W_j (y^h_{ij})^H$, the power balance $(85)$ of a branch becomes,

$$S_{ij} + S_{ji} = W_i (y^h_{ij})^H + z_i^s L^s_i + W_j (y^h_{ij})^H. \tag{86}$$

The equivalent active and reactive power expressions are,

$$P_{ij} + P_{ji} = W^{\text{re}}_i (g_{ij}^h)^T + W^{\text{im}}_i (b_{ij}^h)^T + r_i^s L^{\text{re}}_i - x_i^s L^{\text{im}}_i,$$

and

$$Q_{ij} + Q_{ji} = W^{\text{im}}_i (g_{ij}^h)^T - W^{\text{re}}_i (b_{ij}^h)^T + x_i^s L^{\text{re}}_i + r_i^s L^{\text{im}}_i.$$

\[ \tag{87a} \]
Either $S_{ij}$ or $S_{ij}^*$ can be substituted out through,

$$P_{ij} = P_{ij}^* + W_{ij}^{re}(S_{ij}^b)^T, \quad Q_{ij} = Q_{ij}^* - W_{ij}^{re} (b_{ij}^h)^T. \quad (88)$$

### 7.3 Ohm’s Law

Ohm’s law [22] is reformulated by multiplying both sides with their Hermitian adjoint,

$$U_j(U_j)^H = (U_i - z_i)^H (U_i - z_i)^H, \quad \Rightarrow U_i(U_i)^H = z_i(U_i)^H + z_i^*(U_i)^H. \quad (89)$$

Substituting in the lifted variables into (89), we obtain,

$$W_j = W_i - S_{ij}^*(z_i)^H - z_i^*(S_{ij}^*)^H + z_i^* L_i^T (z_i)^H. \quad (90)$$

The equivalent real expressions are,

$$W_j^{re} = W_i^{re} - P_{ij}^*(r_i^T)^T - Q_{ij}^*(x_i^T)^T - r_i^T(P_{ij}^*)^T - x_i^T(Q_{ij}^*)^T + r_i^T L_i^{s,im}(r_i^T)^T + x_i^T L_i^{s,im}(x_i^T)^T, \quad (91a)$$

$$W_j^{im} = W_i^{im} - Q_{ij}^*(r_i^T)^T + P_{ij}^*(x_i^T)^T - x_i^T(P_{ij}^*)^T - r_i^T(Q_{ij}^*)^T - r_i^T L_i^{s,im}(r_i^T)^T + x_i^T L_i^{s,im}(x_i^T)^T. \quad (91b)$$

These equations are symmetrical, so it is sufficient to generate the scalar constraints only on the upper triangle for $W_j^{im}$; upper triangle and diagonal for $W_j^{re}$ (generating all scalar constraints is redundant).

### 7.4 Voltage Angle Difference Bound

We can derive $W_{ij}$ as a function of $W_i$ and $S_{ij}$,

$$W_{ij} = W_i - S_{ij}^*(z_i)^H, \quad (92)$$

which in the reals becomes,

$$W_{ij}^{re} = W_i^{re} - P_{ij}^*(r_i^T)^T - Q_{ij}^*(x_i^T)^T, \quad (93a)$$

$$W_{ij}^{im} = W_i^{im} - Q_{ij}^*(r_i^T)^T + P_{ij}^*(x_i^T)^T. \quad (93b)$$

Therefore, to avoid introducing the variable $W_{ij}$ in the BFM, we substitute this into (74),

$$\tan (\Theta_{ij}^{min}) \circ \text{diag}(W_{ij}^{re} - P_{ij}^*(r_i^T)^T - Q_{ij}^*(x_i^T)^T) \leq \text{diag}(W_i^{im} - Q_{ij}^*(r_i^T)^T + P_{ij}^*(x_i^T)^T) \leq \tan (\Theta_{ij}^{max}) \circ \text{diag}(W_i^{re} - P_{ij}^*(r_i^T)^T - Q_{ij}^*(x_i^T)^T). \quad (94)$$
7.5 Rank-Constrained SDP Model

The product (31) formulated in the lifted variable space is,

\[ \forall \ell ij \in \mathcal{T}^* : M_{\ell ij} = \begin{bmatrix} W_i & S_{iij} \\ (S_{iij})^H & L_i^e \end{bmatrix}, \]  
\[(95a)\]
\[M_{\ell ij} \succeq 0, \quad \text{rank}(M_{\ell ij}) = 1. \]  
\[(95b)\]

Note that \(M_{\ell ij} \in \mathbb{H}_2^{|P|}\). Furthermore \(M_{\ell ij} \succeq 0\) necessitates \(W_i \succeq 0\) and \(L_i \succeq 0\) but not \(W_j \succeq 0\). So for leaf buses, an explicit \(W_j \succeq 0\) is needed. The real-value equivalent form is,

\[ M_{\ell ij}^{2re} = \begin{bmatrix} W_i^{re} & P_{iij}^s & W_i^{im} & Q_{iij}^s \\ (P_{iij}^s)^T & L_i^{re} & -(Q_{iij}^s)^T & L_i^{im} \\ -W_i^{im} & -Q_{iij}^s & W_i^{re} & P_{iij}^s \\ (Q_{iij}^s)^T & -L_i^{im} & (P_{iij}^s)^T & L_i^{re} \end{bmatrix} \succeq 0, \]  
\[(96a)\]
\[\text{rank}(M_{\ell ij}^{2re}) = 1. \]  
\[(96b)\]

8 Implementation of Real-Value Feasible Sets

Table 7 section summarizes the feasible sets in the real domain. The shunt \(P_h, Q_h\) feasible sets can be substituted into KCL and is therefore not listed explicitly.
Table 7: Formulation feasible sets: variable spaces, bounds, constraints and substitution options (NCNL = nonconvex nonlinear, NCQ = nonconvex quadratic, QCP = quadratically constrained programming)

| Overall complexity | \( S_{ij} - U_i \) | \( S_{ij} - W_i \) |
|--------------------|----------------|----------------|
| Voltage variables  | \( U_i \) | \( W_i \) |
| Power flow variables | \( S_{ij} \) | \( W_{ij} \) |
| Current variables  | \( I_{ij} \) | - |
| Voltage bounds     | \( U_{\min}^i \), \( U_{\max}^i \) | \( U_{\min}^i \), \( U_{\max}^i \) |
| Phase angle difference bounds | \( \Theta_{\min}^i \), \( \Theta_{\max}^i \) | \( \Theta_{\min}^i \), \( \Theta_{\max}^i \) |
| Power flow bounds  | \( S_{\text{rated}}^{ij} \) | \( S_{\text{rated}}^{ij} \) |
| Current bounds     | \( I_{\text{rated}}^{ij} \) | \( I_{\text{rated}}^{ij} \) |
| Kirchhoff’s current law | LP | LP |
| Power equations (II-model) | NCNL | LP |
| Kirchhoff’s voltage law | implicit | LP |
| Ohm’s law | implicit | LP |
| Complex power definition | implicit | LP |
| Reference bus fixed phasor | LP | LP |

Substitution: -
The AC polar and rectangular forms can be implemented in optimization toolboxes with support for nonlinear programming. The core power flow equations (55), (56) or (44), (45) are nonlinear, however only the rectangular form is quadratically representable.

Solving the lifted BIM and BFM forms requires dropping the rank constraints (78b) or (96b) to obtain a SDP formulation. It is noted that further SOC relaxation can be performed [32, 18]. Furthermore, for the tightness of the SOC relaxation of the SDP constraints, it is best to work in the complex domain as long as possible. The process for obtaining the original (nonlifted) current and voltage variables is detailed in [33].

Implementations of the real-value formulations are available in the package PowerModelsDistribution.jl [34]. This package extends PowerModels.jl [35] and is built on top of JuMP [35], a Julia package for mathematical programming. In the implementation, where possible, the matrix forms are used, and the scalarization of the equations is performed by JuMP. The discussed substitution and elimination of $L_{rij}^{re}$, $L_{lij}^{im}$, $P_{rij}^{re}$, $Q_{rij}^{re}$, and $W_{rij}^{re}$, $W_{rij}^{im}$ in the BFM are performed through JuMP expressions, which improves the readability of the mathematical model in the code. We note that nondimensionalization (per unit scaling) is performed in the implementation, in an effort to improve numerical conditioning.

Using PowerModelsDistribution [34], one can read OpenDSS power flow case files and run power flow and optimal power flow case studies. OpenDSS is used to validate the accuracy of the power flow formulations. The worst relative voltage error for IEEE 13, 34, 123 and LV test case [19] is 1.4E-7 [34].

9 Conclusions

In this report, the electrical physics of three-phase grids are derived in complex-power matrix variables and different voltage variable spaces, i.e. polar, rectangular and lifted. Using the different variable spaces, the unbalanced power flow formulations are derived for generic Π-model branches with asymmetric shunt impedance (full matrices, no structure assumed). The real-value matrix and scalar equivalent formulations are derived, and implemented in an open-source software package. The derived mathematical framework can easily be extended and applied to a variety of continuous and discrete optimization problems in distribution networks. Expressions are presented to enforce voltage magnitude, angle, power and current bounds across the formulations in an exact manner, without needing to resort to auxiliary variables. Nevertheless, including auxiliary variables instead of eliminating them does not by default increase computational effort. Comparing these variants numerically requires the development of an extensive unbalanced OPF test case library, which includes values for the different bounds discussed, not just voltage magnitudes. Future work includes adding more bounds, e.g. for sequence voltage magnitudes and unbalance metrics [21].

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