DO POTENTIAL FIELDS DEVELOP CURRENT SHEETS UNDER SIMPLE COMPRESSION OR EXPANSION?

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ABSTRACT

The recent demonstration of current singularity formation by Low et al. assumes that potential fields will remain potential under simple expansion or compression. An explicit counterexample to their key assumption is constructed. Our findings suggest that their results may need to be reconsidered.

Key words: MHD – Sun: corona – Sun: magnetic fields

1. INTRODUCTION

The theory of force-free magnetic fields is of great interest in many astrophysical situations (Marsh 1996; Sturrock 1994). For instance, in the quiescent solar corona, the magnetic energy dominates the thermal energy, the kinetic energy, and the gravitational potential energy. Under these conditions, the Lorentz force $\mathbf{J} \times \mathbf{B}$ of the magnetic field is approximately zero; therefore, the field is force-free to the lowest order. A force-free field satisfies

$$\mathbf{J} = \nabla \times \mathbf{B} = \lambda \mathbf{B};$$

that is, the electric current $\mathbf{J}$ is parallel to the magnetic field $\mathbf{B}$ everywhere. From the condition $\nabla \cdot \mathbf{B} = 0$, we obtain $\mathbf{B} \cdot \nabla \lambda = 0$.

The magnetic field lines in the solar coronal loops are deeply anchored in the dense photosphere, which is continually moving due to the convective motions in the solar interior. As a result, the coronal magnetic field is constantly changing in response to the photospheric motions. Since the typical turnover time of photospheric motions ($\sim$ hundreds of seconds) is much longer than the Alfvén transit time ($\sim$ tens of seconds), the solar coronal loops remain approximately force-free for all time, as long as they are quiescent. In other words, to a good approximation, the photospheric motions carry the coronal magnetic field through a sequence of quasi-static, force-free equilibria. It is, of course, possible that the force-free field may become unstable to magnetohydrodynamic (MHD) instabilities. When instabilities occur, the coronal loops will no longer remain quasi-static, and subsequently some violent events, such as coronal mass ejections (CMEs), may ensue.

The solar corona is highly conducting. To a very good approximation, the plasma resistivity is negligible. One of the most important problems in solar physics is to explain how the solar corona could be heated to millions of degrees when the resistive dissipation seems so weak (Klimchuk 2006). Parker proposed a scenario, which he termed “topological dissipation,” as a possible mechanism of coronal heating (Parker 1972, 1983, 1979, 1994). According to Parker, the magnetic field in an ideal plasma has a natural tendency to form tangential discontinuities, that is, current singularities, when the field line topology is rendered sufficiently complicated by the boundary motions of the photosphere. In reality, because of the finite resistivity, the ideal current singularities will be smoothed out and become thin current filaments. The current filaments are so intense that even the tiny resistivity of solar corona can lead to significant dissipation. Parker suggests that the dissipation from ubiquitous thin current filaments could be the energy source of coronal heating.

Parker’s claim has stimulated considerable debate that continues to this day (van Ballegooijen 1985; Zweibel & Li 1987; Antiochos 1987; Longcope & Strauss 1994; Ng & Bhattacharjee 1998; Craig & Sneyd 2005), without apparent consensus. Recently, in a series of papers (Low 2006, 2007; Janse & Low 2009), Low and coauthors try to demonstrate unambiguously the formation of current singularities using what they call “topologically untwisted fields,” which, in the context of force-free field, is synonymous with potential fields. That is, they limit themselves to a special subset of force-free fields with $\lambda = 0$ in Equation (1), which correspond to current-free fields. In this case, the magnetic field can be expressed as $\mathbf{B} = \nabla \chi$ with some potential $\chi$, where $\chi$ satisfies $\nabla^2 \chi = 0$ because $\mathbf{B}$ is divergenceless. In a simply connected, compact domain, $\chi$ is uniquely determined up to an additive constant if the normal derivative of $\chi$ is given on the boundary. Therefore, a potential field is uniquely determined by prescribing the normal component of $\mathbf{B} (= \hat{n} \cdot \nabla \chi)$, where $\hat{n}$ is the unit normal vector) on the boundary. Janse & Low (2009) consider a potential field in a cylinder of finite length as an initial condition; the normal component of the field is nonzero only on the top and the bottom of the cylinder. The cylinder is then compressed to a shorter length. Because the whole process is governed by the ideal induction equation, the normal component of the field remains the same. By making a key assumption that the field will remain potential during the process, they circumvent the complicated problem of solving for the various stages of quasi-static evolution, and simply calculate the magnetic field in the final state as the potential field which satisfies the boundary conditions. They then numerically calculate the magnetic field line mapping from one end to the other, for both the initial and the final states. They find that for a three-dimensional field, the field line mapping, and hence, the topology, is changed. They claim that current singularities must form during the process; furthermore, singularities may form densely due to the ubiquitous change of the field line mapping.

The assumption that the field will remain potential during the compression, however, is not proven in their work. Intuitively it looks plausible, as parallel current is related to the relative twist between neighboring field lines. It would appear to require a nonzero vorticity on the boundary, which is absent in a simple expansion or compression, to twist up the field.
However, parallel current is related to the relative twist between neighboring field lines only in an average sense; the relative twist between a specific pair of field lines is actually independent of the parallel current. Although evolutions following the ideal induction equation preserve the footprint mapping of each individual field line, it does not follow that parallel current will not be generated during a simple expansion or compression. In this Letter an explicit counterexample is constructed in a two-dimensional slab geometry; therefore, the demonstration of current singularity formation by Low et al. is put in doubt.

2. TWO-DIMENSIONAL COUNTEREXAMPLE

Consider a two-dimensional configuration in Cartesian geometry, with \( x \) the direction of symmetry. A general magnetic field may be expressed as

\[
\mathbf{B} = B_x \mathbf{\hat{x}} + \mathbf{\hat{z}} \times \nabla \psi,
\]

where \( B_x = B_x(y, z) \) and \( \psi = \psi(y, z) \). For a force-free field, \( B_x = B_x(\psi) \) and \( \psi \) must satisfy the Grad–Shafranov equation:

\[
\nabla^2 \psi = -B_x \frac{dB_x}{d\psi}
\]

(3)

The \( \lambda \) for a force-free field in Equation (1) is equal to \(-d B_x/d \psi\) in this representation. If the field is potential, then \( B_x = \text{const} \) and \( \nabla^2 \psi = 0 \).

Let us consider a force-free field enclosed by two conducting boundaries at \( z = 0 \) and \( z = L \), with all field lines connecting one end plate to the other. The footpoint mapping from one end to the other is characterized by \( \psi \) on the boundaries, as well as the axial displacement along \( x \) when following a field line from the bottom to the top:

\[
h(\psi) = B_x(\psi) \left( \int_{z=0}^{z=L} \frac{1}{|\nabla \psi|} \, ds \right)_{\psi}.
\]

(4)

Here \( ds \) is the line element on the \( y - z \) plane, and the subscript \( \psi \) indicates that the integration is done along a constant \( \psi \) contour. Note that \( h(\psi) = d \Phi/d \psi \), with \( \Phi(\psi) \) the axial magnetic flux function up to an additive constant. The displacement function \( h(\psi) \) resembles the safety factor \( q(\psi) \) in toroidal geometry.

Let the initial condition be \( (B_{x0}, \psi_0) \) with a system length \( L_0 \). Suppose the system has undergone a simple expansion or compression such that the system length changes to some \( L \neq L_0 \). Since ideal evolution preserves the footpoint mapping, the flux function \( \psi \) on the boundaries and the axial displacement \( h(\psi) \) must remain unchanged. To determine the final force-free equilibrium, we have to solve the two coupled equations

\[
\nabla^2 \psi = -B_x \frac{dB_x}{d\psi}
\]

(5)

\[
B_x(\psi) = h(\psi) \left( \int_{z=0}^{z=L} \frac{1}{|\nabla \psi|} \, ds \right)^{-1}_{\psi}
\]

(6)

subject to \( \psi(y, 0) = \psi_0(y, 0), \psi(y, L) = \psi_0(y, L_0) \), where the axial displacement \( h(\psi) \) is determine from the initial condition. The set of coupled Equations (5) and (6) is called a generalized differential equation, which in general requires numerical solutions (Grad et al. 1975).

As a simple example, consider the following initial condition:

\[
\psi_0 = y + \epsilon \sin(y) \cosh(z - 1/2), B_{x0} = 1, \text{ and } L_0 = 1,
\]

where \( \epsilon \ll 1 \) is a small parameter. This initial field is potential. Now the question is, will the final field remain potential when the system is expanded or compressed to another length \( L \)? We may first assume that it will, and see if that leads to a contradiction. Assuming that, we have to solve \( \nabla^2 \psi = 0 \) subject to boundary conditions \( \psi(y, 0) = \psi(y, L) = y + \epsilon \sin(y) \cosh(1/2) \), and \( B_x = \text{const} \). The solution for \( \psi \) is

\[
\psi = y + \epsilon a \sin(y) \cosh(z - L/2),
\]

(7)

where

\[
a = \frac{\cosh(1/2)}{\cosh(L/2)}.
\]

(8)

The constant \( B_x \) may be determined by the conservation of axial flux, which yields \( B_x = 1/L \). To be consistent with the initial footpoint mapping, the final state \( (\psi, B_x) \) must give the same axial displacement as \( (\psi_0, B_{x0}) \). We examine this along the field line \( y = 0 \), corresponding to \( \psi_0 = \psi = 0 \). For the initial field,

\[
h(\psi_0 = 0) = \int_{0}^{1} \frac{1}{1 + \epsilon \cosh(z - 1/2)} \, dz
\]

\[
= \int_{-1/2}^{1/2} \frac{1}{1 + \epsilon \cosh z'} \, dz';
\]

(9)

and for the final field,

\[
h(\psi = 0) = \frac{1}{L} \int_{0}^{L} \frac{1}{1 + \epsilon a \cosh(z - L/2)} \, dz
\]

\[
= \int_{-1/2}^{1/2} \frac{1}{1 + \epsilon a \cosh(Lz')} \, dz';
\]

(10)

The condition that the axial displacement remains unchanged implies

\[
\int_{-1/2}^{1/2} \frac{a \cosh(Lz') - \cosh z'}{(1 + \epsilon \cosh z')(1 + \epsilon a \cosh(Lz'))} \, dz' = 0.
\]

(11)

However, it may be shown that the integrand of Equation (11) for \( z' \in (-1/2, 1/2) \) is positive when \( L < 1 \) and negative when \( L > 1 \); therefore the condition (11) cannot be satisfied except for the trivial case \( L = L_0 = 1 \) and we have a contradiction.

Having shown that a potential field cannot be the final force-free equilibrium, the next question is whether a smooth, nonpotential equilibrium exists. To demonstrate the existence of a smooth solution, let us consider a perturbative solution

\[
\psi = y + \epsilon \psi_1 + O(\epsilon^2),
\]

(12)

with the following ansatz

\[
\psi_1 = \sin(y) g(z).
\]

(13)

From the initial condition, we have

\[
h(\psi) = 1 - 2\epsilon \sinh \frac{1}{2} \cos(\psi) + O(\epsilon^2).
\]

(14)

And Equation (6), to \( O(\epsilon) \), yields

\[
B_x(\psi) = h(\psi) \left( \int_{z=0}^{z=L} \frac{1}{|\nabla \psi|} \, ds \right)^{-1}_{\psi}
\]

\[
= h(\psi) \left( L - \epsilon \cos(\psi) \int_{0}^{L} g(z) \, dz + O(\epsilon^2) \right)^{-1}_{\psi}
\]

\[
= \frac{1}{L} + \epsilon \cos(\psi) \left[ \frac{1}{L} \int_{0}^{L} g(z) \, dz - 2 \sinh \frac{1}{2} \right] + O(\epsilon^2).
\]

(15)
The right-hand side of Equation (5) is
\[-B_x \frac{dB_x}{d\psi} = \frac{\epsilon}{L^2} \sin \psi \left( \frac{1}{L} \int_0^L g(z)dz - 2 \sinh \left( \frac{1}{2} \right) \right) + O(\epsilon^2)\]
\[= \frac{\epsilon}{L^2} \sin \psi \left( \frac{1}{L} \int_0^L g(z)dz - 2 \sinh \left( \frac{1}{2} \right) \right) + O(\epsilon^2).\]
(16)

Hence, Equation (5), to \(O(\epsilon)\), is
\[g''(z) - g(z) = \frac{1}{L^2} \frac{1}{L} \int_0^L g(z)dz - 2 \sinh \left( \frac{1}{2} \right).\]
(17)

And the boundary condition for \(\psi\) requires \(g(0) = g(L) = \cosh(1/2)\). The solution can be readily found as
\[g(z) = c_1 \cosh \left( \frac{z - L}{2} \right) + c_2,\]
where
\[c_1 = \frac{L \left( L^2 + 1 \right) \cosh(1/2) - 2L \sinh(1/2)}{L \left( L^2 + 1 \right) \cosh(L/2) - 2 \sinh(L/2)}\]
(19)
and
\[c_2 = \cosh \frac{1}{2} - c_1 \cosh \frac{L}{2}.\]
(20)

Therefore, we have shown that a smooth solution could indeed be found, at least to \(O(\epsilon)\). From Equation (15), the axial magnetic field is
\[B_x(\psi) = \frac{1}{L} + \frac{\epsilon \cos \psi}{L} \left( \frac{2}{L} c_1 \sinh \frac{L}{2} + c_2 - 2 \sinh \frac{1}{2} \right) + O(\epsilon^2).\]
(21)

And finally,
\[
\lambda = -\frac{d B_x}{d \psi} = \frac{\epsilon \sin \psi}{L} \left( \frac{2}{L} c_1 \sinh \frac{L}{2} + c_2 - 2 \sinh \frac{1}{2} \right) + O(\epsilon^2).\]
(22)

Since \(\lambda \neq 0\) almost everywhere when \(L \neq 1\), a smooth electric current is generated throughout the whole domain.

We have checked the perturbative solution by the numerical solution of the generalized differential equation. We find good agreement for small \(\epsilon\). For larger \(\epsilon\) the numerical solution does not quite agree with the solution, as expected. Furthermore, we have been able to find smooth numerical solutions for a wide range of \(L\) and \(\epsilon\). Figure 1 shows \(LB_x(\psi)\) as a function of \(\psi\) for two representative length \(L = 0.1\) and \(L = 0.9\), with \(\epsilon = 0.1\) and \(\epsilon = 0.5\). Solid lines are numerical solutions and dashed lines are perturbative solutions. The \(\epsilon = 0.1\) case shows good agreement between the perturbative and numerical solutions. For \(\epsilon = 0.5\) the agreement is only qualitative.

3. SUMMARY AND DISCUSSION

In summary, we have constructed an explicit counterexample to the assumption that potential fields remain potential under simple expansion or compression. For our two-dimensional system, the reason that in general the field will not remain potential is quite simple. A potential field has to satisfy \(\nabla^2 \psi = 0\), which uniquely determines \(\psi\) for the given boundary value of \(\psi\). This \(\psi\), however, has to satisfy an extra constraint to be self-consistent. That is, when substituting \(\psi\) into the right-hand side of (6), we should get a constant. This is a very stringent constraint and in general will not be satisfied, as we have demonstrated with a simple example.

It should be pointed out that the domain under consideration is not compact; therefore specifying the normal component of the magnetic field on the boundary does not uniquely determine the potential field. One can, for example, add a constant \(B_z\) or \(B_y\) to the field without changing the normal component of \(B\). In this regard, our configuration does not belong to the same class of configurations as considered by Low et al. However, the purpose of this work is to demonstrate that a potential field can evolve into a nonpotential field by simply changing the system length; therefore it should not be taken for granted that the final field has to remain potential. Since this is the key assumption in the demonstration of current singularity formation by Low et al., our findings suggest that their conclusion needs to be revisited. Furthermore, Equation (22) shows that a parallel current is generated almost everywhere. This may account for why Low and coauthors find ubiquitous change in the final current. Thus, it is not strictly valid.
The question posed by the title “do potential fields develop current sheets under simple compression or expansion?” remains open at this point. Indeed, we cannot preclude the possibility that current sheets may form during the process. However, even if that is the case, a smooth current density is likely to precede the formation of current sheets. This raises the overall question as whether limiting to potential fields is a viable option in solving Parker’s problem. The answer is probably no.

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