DYNNIKOV AND TRAIN TRACK TRANSITION MATRICES OF PSEUDO-ANOSOV BRAIDS

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Abstract. We compare the spectra of Dynnikov matrices with the spectra of the train track transition matrices of a given pseudo-Anosov braid on the finitely punctured disk, and show that these matrices are isospectral up to roots of unity and zeros under some particular conditions.

1. Introduction

Let $D_n$ be a standard model of the $n$-times punctured disk ($n \geq 3$). Let $\text{MCG}(D_n)$ denote the mapping class group of $D_n$ (the group of isotopy classes of homeomorphisms of $D_n$). By Nielsen-Thurston Classification Theorem each homeomorphism of a surface is isotopic to either a finite order or a pseudo-Anosov or a reducible homeomorphism. If some iterate of a homeomorphism is the identity, it is called finite order. If a homeomorphism preserves a transverse pair of measured foliations $(F^u, \mu^u)$ and $(F^s, \mu^s)$ on the surface stretching the invariant unstable measured foliation $(F^u, \mu^u)$ uniformly by a real number $\lambda > 1$ and contracting the invariant stable measured foliation $(F^s, \mu^s)$ uniformly by $1/\lambda$, then it is called pseudo-Anosov. We say that $\lambda$ is the dilatation of the pseudo-Anosov. We recall that a measured foliation $(F, \mu)$ on $D_n$ is a singular foliation equipped with a transverse measure $\mu$ [7, 14]. If a homeomorphism preserves a collection of mutually disjoint essential simple closed curves (reducing curves), it is called reducible. Each isotopy class $[f]$ in $\text{MCG}(D_n)$ is represented by a homeomorphism which is of one of these three types, and the isotopy class is named by the type of the homeomorphism it contains. Let $\mathcal{MF}_n$ denote the space of measured foliations on $D_n$ up to isotopy and Whitehead equivalence and $\mathcal{PMF}_n$ be the corresponding space of projective measured foliations on $D_n$. When $[f]$ is pseudo-Anosov, it has exactly two fixed points which both lie on $\mathcal{PMF}_n$, the projective classes $[F^u, \mu^u]$ and $[F^s, \mu^s]$ of its invariant foliations; and every other point on $\mathcal{PMF}_n$ converges to $[F^u, \mu^u]$ rapidly under the action of $[f]$. The induced action of $[f]$ on $\mathcal{MF}_n$ is piecewise linear and is locally described by integer matrices. The matrix on any piece which contains $[F^u, \mu^u]$ on its closure has an eigenvalue $\lambda > 1$ since $[F^u, \mu^u]$

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is a fixed point on $PMFn$. Therefore, if we can compute the action of $f$ on $MFn$ and find a matrix with an eigenvalue $\lambda > 1$ with associated eigenvector contained in the relevant piece, the eigenvector corresponds to $[F^u, \mu^u]$ and $\lambda$ gives the dilatation. In [6] this idea is realized by coordinatizing $PMFn$ using a particular coordinate system called the *Dynnikov coordinate system* and describing the action of $\text{MCG}(D_n)$ on $PMFn$ using the *update rules* [6]. Let us be more specific about some of the details of this procedure. We first take a particular collection of $3n-5$ arcs embedded in $D_n$ and describe each measured foliation by an element of $\mathbb{R}^{3n-5}$, the associated measures of these arcs. *Dynnikov coordinates* [6] are certain linear combinations of these real numbers which yield a one-to-one correspondence between the set of measured foliations (up to isotopy and Whitehead equivalence) on $D_n$ and $S_n = \mathbb{R}^{2n-4} \setminus \{0\}$. The space of *projective Dynnikov coordinates* $PS_n$ is therefore a $(2n-5)$-dimensional sphere and the action of $\text{MCG}(D_n)$ on $PS_n$ is described by the update rules.

As we shall see in Section 1.2, update rules induce a piecewise linear action on $S_n$. Often $[F^u, \mu^u]$ lies on the boundary of several piecewise linear regions: in such cases, each region containing $[F^u, \mu^u]$ on its boundary is called a *Dynnikov region*, and the associated matrix is called a *Dynnikov matrix*. Each Dynnikov matrix has an eigenvalue $\lambda > 1$ with corresponding eigenvector the Dynnikov coordinates $[a^u, b^u]$ of $[F^u, \mu^u]$ which gives the dilatation of the isotopy class from the discussion above. Therefore, the way to compute the dilatation of a given isotopy class is achieved by finding a Dynnikov region and then computing the associated Dynnikov matrix.

We note that since each isotopy class in $\text{MCG}(D_n)$ is described by a braid in Artin’s braid group $B_n$ [1, 2], the isotopy classes in $\text{MCG}(D_n)$ will be represented by sequences of Artin’s braid generators. Our goal in this paper is to compare the spectrum of a Dynnikov matrix $D$ with the spectrum of a train track transition matrix $T$ of a given pseudo-Anosov braid $\beta \in B_n$.

The main results of this paper is given in Section 3. In Section 3.1 we show that when the unstable invariant measured foliation $(F^u, \mu^u)$ of a given pseudo-Anosov braid $\beta \in B_n$ has only unpunctured 3-pronged and punctured 1-pronged singularities, then there is a unique Dynnikov matrix $D$ which is isospectral to the train track transition matrix $T$. We note that the singularities of a foliation are classified with their number of *prongs* $p \geq 1$ and a *prong* is a piece of a leaf beginning at a singularity. Section 3.2 studies the case when $(F^u, \mu^u)$ has singularities other than unpunctured 3-pronged and punctured 1-pronged singularities. In Section 3.2.1 we prove that if $\beta$ fixes the prongs of $(F^u, \mu^u)$, then every Dynnikov matrix is isospectral to $T$ up to some eigenvalues 1. The case in which $\beta$ permutes the prongs of $(F^u, \mu^u)$ non-trivially is discussed in Section 3.2.2. In this case, we have not established that every Dynnikov matrix is isospectral to $T$ up to roots of unity.
The spectrum of $T$ are important for various reasons. First of all, it defines a Markov shift of finite type which gives a symbolic coding for orbits of each period and hence models the dynamics of $\beta$. Also, in [4] Birman, Brinkman and Kawanu shows that the characteristic polynomial of $T$ factors into three polynomial invariants: the first has the dilatation $\lambda$ of $\beta$ as its largest root; the second relates to the action of $\beta$ on the singularities of the invariant foliations $(F^*, \mu^*)$ and $(F^u, \mu^u)$; and the third relates to the degeneracies of a symplectic form introduced in [12].

Also, finding Dynnikov matrices is much easier than finding train track transition matrices: train tracks are computed using, for example, the Bestvina-Handel algorithm [3]. The drawback in train track methods is that if the braid is complicated that is, if its dilatation is high, then it is far from straightforward to find a train track invariant under the relevant braid since the image edge paths will be too long to track. On the other hand, a Dynnikov region for $\beta$ is easy to find since $[a^u, b^u]$ is a globally attracting fixed point for the action of $\beta$ on $\mathcal{PS}_n$. In addition, Dynnikov approach is more transparent since it relies on algebraic calculations rather than on understanding the image of a train track under the action of $\beta$. Finally, we note that the Dynnikov matrices and train track transition matrices throughout this paper has been computed using Dynnikov and train track programs implemented by Toby Hall both of which can be found at [8]. We begin with the Dynnikov coordinate system [6, 9, 15] which puts global coordinates on $\mathcal{MF}_n$.

1.1. Dynnikov coordinates. Let $A_n$ be the set of arcs in $D_n$ which have each endpoint either on the boundary or a puncture. Consider the arcs $\alpha_i \in A_n \ (1 \leq i \leq 2n - 4)$ and $\beta_i \in A_n \ (1 \leq i \leq n - 1)$ as depicted in Figure 1. Let $(\mathcal{F}, \mu) \in \mathcal{MF}_n$. Note that given $\gamma \in A_n$, then its measure $\mu(\gamma)$ is defined to be

$$\mu(\gamma) = \sup \sum_{i=1}^k \mu(\gamma_i),$$

where the supremum is taken over all finite collections $\gamma_1, \ldots, \gamma_k$ of mutually disjoint subarcs of $\gamma$ which are transverse to $(\mathcal{F}, \mu)$ and the isotopy class $[\gamma]$ (under isotopies through $A_n$), has measure

$$\mu([\gamma]) = \inf_{\delta \in [\gamma]} \mu(\delta).$$

The triangle coordinate function $\tau : \mathcal{MF}_n \to \mathbb{R}^{3n-5}_{\geq 0}$ defined by

$$\tau((\mathcal{F}, \mu)) = (\mu([\alpha_1]), \ldots, \mu([\alpha_{2n-4}]), \mu([\beta_1]), \ldots, \mu([\beta_{n-1}]))$$

is injective but it is not surjective [13]. However, the Dynnikov coordinate function $\rho : \mathcal{MF}_n \to \mathbb{R}^{2n-4} \setminus \{0\}$ defined by

and zeros: this conjectured result has been observed in a wide range of examples, one of which is presented in detail in Section 3.2.2.
\[ \rho(F, \mu) = (a, b) = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2}), \]

where for \(1 \leq i \leq n - 2\)
\[ a_i = \frac{\mu([\alpha_{2i}]) - \mu([\alpha_{2i-1}])}{2} \quad \text{and} \quad b_i = \frac{\mu([\beta_i]) - \mu([\beta_{i+1}])}{2} \]
is a homeomorphism \([6, 9, 15]\). Let \(C_n = \mathbb{R}^{2n-4} \setminus \{0\}\) denote the space of Dynnikov coordinates. Theorem 1.1 describes the inverse \(C_n \rightarrow \mathbb{R}^{3n-5}\) of the Dynnikov coordinate function which sends each \((a, b) \in C_n\) to the triangle coordinates of a measured foliation \((F, \mu)\) with Dynnikov coordinates \((a, b)\). A detailed proof of
Theorem 1.1 can be found in \([15]\). Let \(L_n\) denote the set of integral laminations (disjoint unions of finitely many essential simple closed curves) on \(D_n\). The Dynnikov coordinate function restricts to a bijection \(\rho : L_n \rightarrow \mathbb{Z}^{2n-4} \setminus \{0\}\) \([9, 15]\). The right side of Figure 1 depicts the Dynnikov coordinates of an integral lamination \(L \in L_5\).

**Figure 1.** (a) The arcs \(\alpha_i\) and \(\beta_i\) (b) The Dynnikov coordinates of \(L\) is given by \(\rho(L) = (0, 1, -1, -1, 0, -1)\).

**Theorem 1.1** (Inversion of Dynnikov coordinates). Let \((a, b) \in C_n\). Then \((a, b)\) is the Dynnikov coordinate of exactly one element \((F, \mu)\) of \(\mathcal{MF}_n\), which has
\[ \beta_i = 2 \max_{1 \leq k \leq n-2} \left[ |a_k| + \max(b_k, 0) + \sum_{j=1}^{k-1} b_j \right] - 2 \sum_{j=1}^{i-1} b_j \]
and
\[ \alpha_i = \begin{cases} (-1)^i a_{[i/2]} + \frac{\beta_{i/2}}{2} & \text{if } b_{[i/2]} \geq 0; \\ (-1)^i a_{[i/2]} + \frac{\beta_{i+1/2}}{2} & \text{if } b_{[i/2]} \leq 0 \end{cases} \]
where \([x]\) denotes the smallest integer which is not less than \(x\).

Projectivizing the space of Dynnikov coordinates we get a homeomorphism between \(\mathcal{PMF}_n\) and \(\mathcal{PS}_n\). The next section gives the update rules which describe the action of \(B_n\) on \(S_n\) and hence on \(\mathcal{PS}_n\), and introduces Dynnikov matrices.
with illustrative examples. For computational and notational convenience, we will work in the max-plus semiring \((\mathbb{R}, \oplus, \otimes)\) [9]. It will be convenient to use normal additive and multiplicative notation, and to indicate that these are to be interpreted in the max-plus sense by enclosing the formulae in square brackets. Thus, 
\[ [a + b] = \max(a, b), [ab] = a + b, [a/b] = a - b, [1] = 0. \]

1.2. Update Rules and Dynnikov matrices. The action of Artin’s braid generators \(\sigma_i, \sigma_i^{-1}, (1 \leq i \leq n - 1)\) on \(\mathcal{MF}_n\) in terms of Dynnikov coordinates is described by the update rules [3] [9]. Therefore, using the update rules one can compute \(\beta : C_n \rightarrow C_n\) given by,

\[ \beta(a, b) = \rho \circ \beta \circ \rho^{-1}(a, b) \]
for each \(\beta \in B_n\).

**Theorem 1.2.** Let \((a, b) \in C_n\) and \(1 \leq i \leq n - 1\), and write \(\sigma_i(a, b) = (a', b'), \sigma_i^{-1}(a, b) = (a'', b'')\). Then \(a_j' = a_j, b_j' = b_j, a_j'' = a_j\) and \(b_j'' = b_j\) except when \(j = i - 1\) or \(j = i\), and:

- **if** \(i = 1\) **then**
  \[ a_1' = \left[ \frac{a_1b_1}{a_1 + 1 + b_1} \right], \quad b_1' = \left[ \frac{1 + b_1}{a_1} \right] \]
  \[ a_1'' = \left[ \frac{1 + a_1(1 + b_1)}{b_1} \right], \quad b_1'' = [a_1(1 + b_1)] \]

- **if** \(2 \leq i \leq n - 2\) **then**
  \[ a_{i-1}' = \left[ a_{i-1}(1 + b_{i-1}) + a_i b_{i-1} \right], \quad b_{i-1}' = \left[ \frac{a_i b_{i-1} b_i}{a_{i-1}(1 + b_{i-1})(1 + b_i) + a_i b_{i-1}} \right] \]
  \[ a_i' = \left[ \frac{a_{i-1} a_i}{a_{i-1}(1 + b_i) + a_i} \right], \quad b_i' = \left[ \frac{a_{i-1}(1 + b_{i-1})(1 + b_i) + a_i b_{i-1}}{a_i} \right] \]
  \[ a''_{i-1} = \left[ \frac{a_{i-1} a_i}{a_{i-1} b_{i-1} + a_i(1 + b_{i-1})} \right], \quad b_{i-1}'' = \left[ \frac{a_{i-1} b_{i-1} b_i}{a_{i-1} b_{i-1} + a_i(1 + b_{i-1})(1 + b_i)} \right] \]
  \[ a_i'' = \left[ \frac{a_{i-1} + a_i(1 + b_i)}{b_i} \right], \quad b_i'' = \left[ \frac{a_{i-1} b_{i-1} + a_i(1 + b_{i-1})(1 + b_i)}{a_{i-1}} \right] \]

- **if** \(i = n - 1\) **then**
  \[ a_{n-2}' = \left[ a_{n-2}(1 + b_{n-2}) + b_{n-2} \right], \quad b_{n-2}' = \left[ \frac{b_{n-2}}{a_{n-2}(1 + b_{n-2})} \right] \]
  \[ a_{n-2}'' = \left[ \frac{a_{n-2}}{a_{n-2} b_{n-2} + 1 + b_{n-2}} \right], \quad b_{n-2}'' = \left[ \frac{a_{n-2} b_{n-2}}{1 + b_{n-2}} \right] \].
Example 1.3. Let $\mathcal{L}$ be the integral lamination given in Figure 1. Using the update rules we get the Dynnikov coordinates of $\sigma_3^{-1} \sigma_2^{-1} \ell_1 (\mathcal{L})$ in Figure 2 as $\rho(\sigma_3^{-1} \sigma_2^{-1} (\mathcal{L})) = (0, 2, -3, -2, -1, 0)$.

![Figure 2. $\rho(\sigma_3^{-1} \sigma_2^{-1} (\mathcal{L})) = (0, 2, -3, -2, -1, 0)$](image)

Example 1.4. Let $\mathcal{PF}_3 \cong S^1$ be the space of projective measured foliations on $D_3$. Using Theorem 1.2 one can explicitly compute the $2 \times 2$ integer matrices which describe the piecewise linear action of $\sigma_1 \sigma_2^{-1}$ on $\mathcal{PF}_3$. We remark that this example concretely illustrates the action of a pseudo-Anosov braid (which is the simplest possible) on the whole space $\mathcal{PF}_3$.

![Figure 3. The action of $\sigma_1 \sigma_2^{-1}$ on $\mathcal{PF}_n$](image)

The matrix

$$D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
has an eigenvalue $\lambda > 1$ and the eigenvector $p^u = [a^u, b^u]$ corresponding to it belongs to $E$. That is, $a^u \leq 0$ and $b^u \leq 0$. Hence $p^u$ is a fixed point for $\sigma_1 \sigma_2^{-1}$ on $\mathcal{P} \mathcal{S}_3$. Hence, $p^u = [a^u, b^u]$ corresponds to the invariant unstable foliation $[F^u, \mu^u]$. Similarly, the matrix
\[
\begin{bmatrix}
1 & -1 \\
-1 & 2
\end{bmatrix}
\]
has an eigenvalue $1/\lambda$ and the associated eigenvector $p^s$ belongs to the region $a \geq 0$, $b \geq 0$, $b \leq a \leq 2b$. Hence $p^s$ is a fixed point and corresponds to the invariant stable foliation $[F^s, \mu^s]$. In this example, $p^u = -\left(\frac{1+\sqrt{5}}{2}, 1\right)$ and $\lambda = \frac{3+\sqrt{5}}{2}$.

**Definition 1.5.** Let $\beta \in B_n$ be a pseudo-Anosov braid with invariant unstable measured foliation $(F^u, \mu^u)$ given by the Dynnikov coordinates $p^u = (a^u, b^u)$. The action of $\beta$ on $\mathcal{S}_n$ is piecewise linear and each closed piece $\mathcal{R}_i \subset \mathcal{S}_n$ containing $(a^u, b^u)$ is called a *Dynnikov region*. Then a *Dynnikov matrix* $D_i : R_i \to \mathcal{S}_n$, $(1 \leq i \leq k)$ is a $(2n-4) \times (2n-4)$ integer matrix which describes the behaviour of the braid on a Dynnikov region $\mathcal{R}_i$. That is,

\[
\rho(\beta(\mathcal{F}, \mu)) = D_i(a, b) \text{ for } (a, b) \in \mathcal{R}_i.
\]

**Example 1.6.** There is one Dynnikov region for $\sigma_1 \sigma_2^{-1}$ which is

\[
\mathcal{R} = \{[a, b] \in S^1 : a \leq 0, b \leq 0\}.
\]

Hence, the Dynnikov matrix is
\[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]
as computed in Example 1.4.

There can be more than one Dynnikov region for a given pseudo-Anosov braid $\beta$. This happens when $(a^u, b^u)$ is on the boundary of several regions on $\mathcal{S}_n$. We shall illustrate this case with the following example.

**Example 1.7.** Take $\beta = \sigma_1 \sigma_2 \sigma_3 \sigma_4^{-1} \in B_5$. Using Theorem 1.2 we compute that the action of $\sigma_1 \sigma_2 \sigma_3 \sigma_4^{-1}$ on $\mathcal{S}_5$ in a region where $a_i \leq 0$ and $b_i \leq 0$ for all $i$ is given by two matrices $D_1$ (when $b_1 \leq a_2 - a_1$) and $D_2$ (when $b_1 \geq a_2 - a_1$) where $D_1$ and $D_2$ are given as follows.

\[
D_1 = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
D_2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
Both of these matrices have eigenvalue $\frac{3+\sqrt{5}+\sqrt{6}}{2}$, with the corresponding eigenvector $p^v$ having all negative entries and satisfying the equality $a_2 = a_1 + b_1$. Therefore, both $D_1$ and $D_2$ are Dynnikov matrices. In fact $D_1$ and $D_2$ are isospectral, we shall return to this issue later.

1.3. Train track coordinates and transition matrices. The usual way to study the dynamics of pseudo-Anosov braids on $D_n$ is to use Thurston’s train tracks. A train track $\tau$ on $D_n$ is a one dimensional CW complex made up of vertices (switches) and edges (branches) smoothly embedded on $D_n$ such that at each switch there is a unique tangent vector, and every component of $D_n - \tau$ is either a once-punctured $p$-gon with $p \geq 1$ or an unpunctured $k$-gon with $k \geq 3$ (where the boundary of $D_n$ is regarded as a puncture). A train track $\tau$ is called complete if each component of $D_n - \tau$ is either a trigon or a once punctured monogon. A transverse measure on $\tau$ is a function which assigns a measure to each branch of $\tau$ such that these measures satisfy the switch conditions at each switch of $\tau$. That is, for each switch $v$ of $\tau$

$$\sum_{\text{incoming branches at } v} \mu(e) = \sum_{\text{outgoing branches at } v} \mu(e)$$

Train tracks equipped with a transverse measure are called measured train tracks: they provide another way to coordinatize measured foliations and integral laminations. We denote by $W(\tau)$ and $W^+(\tau)$ the space of transverse measures and non-negative transverse measures associated to $\tau$. Given a measured train track $\tau$ define a function $\phi_\tau : W^+(\tau) \to \mathcal{MF}_n$ as follows: Replace each branch $e_i$ of $\tau$ which has non-zero measure with a Euclidean rectangle $R_i$ of length 1 and height $\mu(e_i)$ and endow each $R_i$ with a “horizontal” measured foliation where the transverse measure is induced from the Euclidean metrics on the rectangles. At each switch glue the vertical sides of the rectangles and denote this union of glued rectangles $R^*$. Since $\tau$ satisfies the switch condition at each switch there is a unique measure preserving way to glue together the horizontal leaves, hence there is a well defined transverse measure on $R^*$. A pre-foliation $\mathcal{F}^*$ is the collection of leaves on $R^*$. Collapsing each component of $D_n - \mathcal{F}^*$ which doesn’t contain any branch of zero measure onto a spine yields a measured foliation $\phi_\tau(\mu) = (\mathcal{F}, \mu)$ [3, 12]. We say that $(\mathcal{F}, \mu) \in \mathcal{MF}_n$ is carried by $\tau$ if it arises from some transverse measure $\mu$ on $\tau$ in this way. We remark that the components $D_n - \tau$ correspond to the singularities of the constructed measured foliation and hence $\tau$ is complete if and only if every foliation in $\mathcal{MF}(\tau)$ has only 1-pronged singularities at punctures and 3-pronged singularities elsewhere. We write $\mathcal{MF}(\tau) = \phi_\tau(W^+(\tau))$ for the set of measured foliations carried by $\tau$ and $\mathcal{P}\mathcal{MF}(\tau)$ for the corresponding projective space. Noting that $\mathcal{MF}_n$ and $\mathcal{P}\mathcal{MF}_n$ are homeomorphic to $\mathbb{R}^{2n-4} \setminus \{0\}$ and $S^{2n-5}$ respectively, $\mathcal{MF}(\tau)$ and $\mathcal{P}\mathcal{MF}(\tau)$ have the subspace topology. Furthermore, the
maps \( \phi_r : W^+(\tau) \to \mathcal{MF}(\tau) \) and \( \hat{\phi}_r : \mathcal{PF}W^+(\tau) \to \mathcal{PMF}(\tau) \) are homeomorphisms where \( W^+(\tau) \) has the subspace and \( \mathcal{PF}W^+(\tau) \) has the quotient topology. Let \( \text{rank}(\tau) \) denote the dimension of \( \mathcal{W}(\tau) \). The proofs of the following lemmas can be found in [12].

**Lemma 1.8.** Let \( \tau \) be a train track on \( D_n \) with \( k \) branches and \( s \) switches. The switch conditions on \( \tau \) are linearly independent and hence \( \text{rank}(\tau) = k - s \). Therefore, \( \mathcal{W}(\tau) \cong \mathbb{R}^{k-s} \setminus \{0\} \). \( \tau \) is complete if and only if \( \text{rank}(\tau) = 2n - 4 \). That is, \( \tau \) is complete if and only if \( \text{rank}(\tau) \) is the same as the dimension of \( \mathcal{MF}_n \).

**Lemma 1.9.** The complete train tracks on \( D_n \) give an atlas for the piecewise integral linear structure of \( \mathcal{MF}_n \) and \( \mathcal{PMF}_n \). That is, the transition functions between charts are piecewise linear with integer coefficients.

**Definitions 1.10.** Endowing a regular neighborhood \( N_\tau \) of \( \tau \) with fibres of the retraction \( \tau : N_\tau \setminus \tau \), we obtain a fibred neighbourhood \( N_\tau \) of \( \tau \). Let \( \tau \) and \( \tau' \) be two train tracks on \( D_n \). We say that \( \tau \) is **carried** by \( \tau' \) and write \( \tau < \tau' \) if there is a homeomorphism \( \psi : D_n \to D_n \) isotopic to the identity such that

- \( \psi(\tau) \subseteq N_{\tau'} \),
- Each branch of \( \psi(\tau) \) is transverse to the fibers in \( N_{\tau'} \),
- for each branch \( e_i \) of \( \tau \) the end points of \( \psi(e_i) \) are contained in singular leaves of \( N_{\tau'} \).

Let \{\( e_i \}_{1 \leq i \leq k} \) and \{\( f_i \)\}_{1 \leq i \leq k'} be the oriented branches of \( \tau \) and \( \tau' \) respectively. Let \( \psi' : N_{\tau'} \to \tau' \) be the retraction. For each \( 1 \leq i \leq k \), \( \psi'(\psi(e_i)) \) is an edge path in \( \tau' \):

\[
\psi'(\psi(e_i)) = f^{e_1}_{i_1} f^{e_2}_{i_2} \ldots f^{e_{k'}}_{i_{k'}} , \quad \epsilon_j = \pm 1 .
\]

The **incidence matrix** associated to \( \tau \) and \( \tau' \) is the \( k' \times k \) matrix \( G : W(\tau) \to W(\tau') \) whose \( ij \)th entry \( G_{ij} \) is given by the number of occurences of \( f^\epsilon_{i_1} f^\epsilon_{i_2} \ldots f^\epsilon_{i_{k'}} \) in \( \psi'(\psi(e_j)) \).

**Lemma 1.11.** Let \( \tau < \tau' \). Then \( \mathcal{MF}(\tau) \subset \mathcal{MF}(\tau') \) and the following diagram commutes:

\[
\begin{array}{ccc}
W^+(\tau) & \xrightarrow{G} & W^+(\tau') \\
\downarrow\phi_r & \quad \quad & \downarrow\phi_{r'} \\
\mathcal{MF}(\tau) & \xrightarrow{} & \mathcal{MF}(\tau') .
\end{array}
\]

**Definition 1.12.** We say that a train track \( \tau \) is **invariant** under \( \beta \in B_n \), if \( \beta(\tau) \) is carried by \( \tau \). Let \( \tau \) be an invariant train track of \( \beta \) and \( e_1, \ldots, e_k \) be the oriented branches of \( \tau \). Then, for each \( 1 \leq i \leq k \), \( r(\psi(\beta(e_i))) \) is of the form \( r(\psi(\beta(e_i))) = e^{\epsilon_1}_{i_1} e^{\epsilon_2}_{i_2} \ldots e^{\epsilon_{k'}}_{i_{k'}} , \epsilon_j = \pm 1 \). The **transition matrix** \( T \) associated to \( \tau \) is the \( k \times k \) incidence matrix \( T : W(\beta(\tau)) \to W(\tau) \) described as in Definitions 1.10.

**Example 1.13.** An invariant train track \( \tau \) for the 5-braid \( \beta = \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_4 \) and its image under \( \beta \) are depicted in Figure 4. Therefore the transition matrix \( T' \)
Figure 4. An invariant train track for the 5-braid $\beta = \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_4$ and its image under $\beta$.

associated to $\tau$ is of the form $T' = \begin{pmatrix} T & 0 \\ A & P \end{pmatrix}$ where

$$T = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

gives the action on the edges $a, b, c, d$ and $e$; and $P$ is a permutation matrix giving the action on the other edges.

**Theorem 1.14.** Every pseudo-Anosov braid $\beta \in B_n$ has an invariant train track $\tau$. This train track $\tau$ can be chosen so that

- The branches which bound interior $p$-gons (i.e. those which are disjoint from $\partial D_n$) are permuted by $\beta$
- The transition matrix is of the form

$$T' = \begin{pmatrix} T & 0 \\ A & P \end{pmatrix}$$

where $P$ is a permutation matrix giving the action on the permuted branches and $T$ is the matrix that gives the action on the other branches.

- For each $p$, there are the same number of unpunctured (resp. punctured) $p$-gons in $\tau$ as there are unpunctured (resp. punctured) $p$-pronged singularities in $(F^u, \mu^u)$ (this includes the “exterior” punctured $p$-gon and the singularity at infinity).

The final point will be clarified in Lemma 1.15. A train track $\tau$ of the type in Theorem 1.14 is called a regular train track. A branch of a regular train track is called infinitesimal if it bounds an interior $p$-gon, it is called main otherwise.
The proof of the following Lemma can be found in [10].

**Lemma 1.15.** Let \( \beta \in B_n \) be a pseudo-Anosov braid with dilatation \( \lambda \), unstable foliation \( (F^u, \mu^u) \) and invariant regular train track \( \tau \) with associated transition matrix \( T' \). The largest eigenvalue of \( T' \) equals \( \lambda \) and the entries of the unique associated column eigenvector \( v^u \) (up to scale) are strictly positive. \( v^u \) defines a transverse measure on \( \tau \) which yields a pre-foliation \( F^* \) as described above whose prongs do not join and from which \( (F^u, \mu^u) \) is constructed. That is, \( \phi_\tau(v^u) = (F^u, \mu^u) \).

**Lemma 1.16.** Let \( \tau \) be an invariant train track for \( \beta \in B_n \). If all components of \( D_n - \tau \) are odd-gons (in particular, if \( \tau \) is complete), then there is a basis for \( \mathcal{W}(\tau) \) consisting of transverse measures \( \mu \) such that \( \mu(e) \neq 0 \) for exactly one main branch \( e \).

**Proof.** If \( D_n - \tau \) consists only of odd-gons, the switch conditions give a unique solution for the measures on the infinitesimal branches given measures on the main branches and a basis for \( \mathcal{W}(\tau) \) can be obtained by assigning 1 to one main branch and 0 to all other main branches of \( \tau \) since negative measures are allowed on the infinitesimal branches (Figure 5). See also [4]. \( \square \)

**Remark 1.17.** Lemma 1.11 gives the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{W}^+(\tau) & \xrightarrow{T'} & \mathcal{W}^+(\tau) \\
\downarrow \phi_\tau & & \downarrow \phi_\tau \\
\mathcal{MF}(\tau) & \xrightarrow{\beta} & \mathcal{MF}(\tau)
\end{array}
\]

Hence the action on \( \mathcal{W}(\tau) \subseteq \mathbb{R}^k \) is given by the \( k \times k \) transition matrix

\[
T' = \begin{pmatrix}
T & 0 \\
A & P
\end{pmatrix}
\]

where \( T \) is the \( m \times m \) matrix which gives the action on the main branches of \( \tau \) by Lemma 1.14. When all components of \( D_n - \tau \) are odd-gons, a basis for \( \mathcal{W}(\tau) \cong \mathbb{R}^m \).
can be constructed as described in Lemma 1.16 and hence the action on $W(\tau) \cong \mathbb{R}^n$ is given by the $m \times m$ transition matrix $T$.

We note that such a basis can not be taken if $D_n - \tau$ has an even-gon since the switch conditions are satisfied only when the alternating sum of the incoming measures on the switches of each even-gon is zero. However, there is still a basis consisting of weights on edges which may include infinitesimal ones, see [3].

2. Train track coordinates and Dynnikov coordinates

In this section we show that, for any train track $\tau$ on $D_n$, the change of coordinate function $L : W^+(\tau) \to S_n$ between train track coordinates and Dynnikov coordinates is piecewise linear.

Definitions 2.1. Suppose $\tau$ is a train track with oriented branches $e_1, \ldots, e_k$. A train path $p = e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_m^{\epsilon_m}$, $\epsilon_j = \pm 1$ is a smooth oriented edge-path in $\tau$. Given a train path $p$ on $\tau$ we define $\hat{p} : W^+(\tau) \to \mathbb{R}_{\geq 0}$ as follows: for each $\mu \in W^+(\tau)$, $\hat{p}(\mu)$ is the total measure of leaves of $\phi_\tau(\mu)$ following the train path $p$.

Lemma 2.2. For each train path $p$ in $\tau$, the map $\hat{p} : W^+(\tau) \subseteq \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ is piecewise linear.

Proof. Let $R_1, \ldots R_k$ be the rectangles used in the construction of $\phi_\tau(\mu)$ as described above. Associate a copy of $R_i$ to each branch $e_i^{\epsilon_i}$ in $p$, and glue $R_i$ to $R_{i-1}$ and $R_{i+1}$ using the described identifications. Denote the identification space $K$. Then $\hat{p}(\mu)$ is the width of the largest rectangle which fits in $K$ with edges parallel to the edges of each $R_i$. This is clearly a piecewise linear function of the widths of the rectangles (observe that $\hat{p}(\mu)$ is the measure of leaves that pass along the shaded rectangle in Figure 6). □

Remark 2.3. Note that Lemma 2.2 implies that $\hat{p} : W^+(\tau) \subseteq \mathbb{R}^k_+ \to \mathbb{R}_{\geq 0}$ is linear in a neighbourhood of any measure for which the prongs of pre-foliation $F^*$ are not connected.

Definitions 2.4. Let $\tau$ be a train track on $D_n$ and $\mathcal{A}_n \subset D_n$ be the set of Dynnikov arcs $\alpha_i$ ($1 \leq i \leq 2n-4$) and $\beta_i$ ($1 \leq i \leq n-1$). A standard embedding of $\tau$ in $D_n$ with respect to $\mathcal{A}_n$ satisfies the following:

- each branch $e_i$ of $\tau$ is tight (that is $e_i$ doesn’t bound any unpunctured disk with any Dynnikov arc)
- the arcs $\alpha_i$ ($1 \leq i \leq 2n-4$) and $\beta_i$ ($1 \leq i \leq n-1$) do not pass through the switches of $\tau$.

We shall always take a standard embedding of a given train track $\tau$ as described in Definitions 2.4 throughout the text.
We say that a train path \( p \) in \( \tau \) is \textit{non-tight} with respect to a Dynnikov arc \( \gamma \) if some subarc of \( p \) together with some subarc of \( \gamma \) bounds a disk containing no punctures. For each Dynnikov arc \( \gamma \) write \( \Pi_\gamma \) for the set of all train paths which are non-tight with respect to \( \gamma \). There is a partial order \( \leq \) on \( \Pi_\gamma \) defined as follows: \( p_1 \leq p_2 \) if \( p_1 \) is a subpath of \( p_2 \). We define \( \Pi'_\gamma \subseteq \Pi_\gamma \) as the subset of minimal train paths with respect to the relation \( \leq \). Then any minimal non-tight train path \( p \in \Pi'_\gamma \) is the concatenation \( p = \delta_1\delta_2\delta_3 \) of three paths (not train paths) where \( \delta_2 \) is the subarc bounding a disk with some subarc of \( \gamma \) and \( \delta_1 \) and \( \delta_3 \) are contained in single branches of \( \tau \) (Figure 7).

**Lemma 2.5.** Let \( \tau \) be a train track. Then, the change of coordinate function \( L : W^+(\tau) \to S_n \) is piecewise linear.

**Proof.** Given a Dynnikov arc \( \gamma \) and the branches \( \{e_1, \ldots, e_k\} \) of \( \tau \) (standardly embedded) let \( n_i \) be the number of intersections of \( e_i \) with \( \gamma \). To compute \( \mu(\gamma) \) we need to subtract the measure on all independent train paths which form a loop with \( \gamma \). The only condition is that a train path should not be a subpath of another.
(two train paths neither of which is a subpath of the other define disjoint packets of leaves except perhaps their boundary leaves).

Thus the measure $\mu(\gamma)$ of $\gamma$ is given by

$$\mu(\gamma) = \sum_{i=1}^{k} n_i \mu(e_i) - 2 \sum_{p \in \Pi'_{\gamma}} \hat{p}(\mu).$$

We know that any $p \in \Pi'_{\gamma}$ is of the form $e_{i_1} \hat{p} e_{i_2}$, where $e_{i_1}$ and $e_{i_2}$ cross $\gamma$ ($e_{i_1}$ contains $\delta_1$ and $e_{i_2}$ contains $\delta_3$). Note that $\hat{p}$ cannot contain the same branch with the same orientation twice since then it would contain a non-trivial loop which is impossible (a non-tight train path which contains a non-trivial loop is not minimal). Hence $\mu(\gamma)$ is piecewise linear since $\Pi'_{\gamma}$ is finite and for each of these train paths $\hat{p}(\mu)$ is piecewise linear by Lemma 2.2. Therefore the map $W^+(\tau) \to S_n$ is piecewise linear. □

3. The spectrum of a Dynnikov matrix and a train track transition matrix

Let $\beta \in B_n$ be a pseudo-Anosov braid with unstable invariant foliation $(F^u, \mu^u)$ and an invariant train track $\tau$ with transition matrix $T$. The aim of this section is to compare the spectra of Dynnikov matrices with the spectra of the train track transition matrices of $\beta \in B_n$. In Section 3.1 we study the case when $(F^u, \mu^u)$ has only unpunctured 3-pronged and punctured 1-pronged singularities and show that in this case there is a unique Dynnikov matrix $D$ which is isospectral to $T$. In Section 3.2 we study the case when $(F^u, \mu^u)$ has singularities other than unpunctured 3-pronged and punctured 1-pronged singularities. This second case is divided into two subcases. In Section 3.2.1 we show that if $\beta$ fixes the prongs of $(F^u, \mu^u)$, then every Dynnikov matrix is isospectral to $T$ up to some eigenvalues 1. The subcase in which $\beta$ permutes the prongs of $(F^u, \mu^u)$ non-trivially is discussed in Section 3.2.2 a conjectured result, which has been observed in a wide range of examples, is illustrated with an example.

3.1. The spectrum of a Dynnikov matrix when $\tau$ is complete. This section focuses on the case where $(F^u, \mu^u)$ has only unpunctured 3-pronged and punctured 1-pronged singularities. In this case, we shall show that there is a unique Dynnikov region $\mathcal{R}$ (and hence a unique Dynnikov matrix $D$) and the transition matrix $T$ associated with a (complete) invariant regular train track of $\beta$ is isospectral to $D$. We shall then prove that the same result holds for any invariant train track of $\beta$ up to roots of unity and zeros. This will follow from a result by Rykken 13 given in Theorem 3.2 below.

**Theorem 3.1.** Let $\beta \in B_n$ be a pseudo-Anosov braid with unstable invariant foliation $(F^u, \mu^u)$ and dilatation $\lambda > 1$. Let $\tau$ be a regular invariant train track
with associated transition matrix $T$. If $(F^u,\mu^u)$ has only unpunctured 3-pronged and punctured 1-pronged singularities, then $\beta$ has a unique Dynnikov matrix $D$, and $D$ and $T$ are isospectral.

**Proof.** Since $(F^u,\mu^u)$ has only unpunctured 3-pronged and punctured 1-pronged singularities, $\tau$ is complete. That is, $W^+(\tau)$ has dimension $2n - 4$. Therefore, $\mathcal{MF}(\tau)$ is a chart on $\mathcal{MF}_n$ by Lemma 1.9. By Lemma 1.15, the eigenvector $v$ associated with the dilatation $\lambda > 1$ is a transverse measure on $\tau$ with $(F^u,\mu^u) = \phi_\tau(v)$. Furthermore, the entries of $v$ are strictly positive. Therefore, $W^+(\tau)$ and $\mathcal{MF}(\tau)$ are neighbourhoods of $v$ and $(F^u,\mu^u)$ respectively. Construct the pre-foliation $F^*$ from $\tau$ as described above. Because none of the prongs of the pre-foliation $F^*$ are connected by Lemma 1.15, it follows from Remark 2.3 that there is a neighbourhood $U$ of $v \in W^+(\tau)$ on which the change of coordinate function $L = \rho \circ \phi_\tau$ from train track coordinates to Dynnikov coordinates is linear. Write $R = L(U) \subseteq S_n$ which is a neighbourhood of $L(v)$. We have the following commutative diagram:

\[
\begin{array}{ccc}
W^+(\tau) & \xrightarrow{T} & W^+(\tau) \\
\phi_\tau \downarrow & & \phi_\tau \downarrow \\
\mathcal{MF}(\tau) & \xrightarrow{\beta} & \mathcal{MF}(\tau) \\
\rho \downarrow & & \rho \downarrow \\
R \subseteq S_n & \xrightarrow{F} & S_n \\
\end{array}
\]

Then $F|_R = D = L \circ T \circ L^{-1}$ is linear and isospectral to $T$. $\square$

In the above we considered the transition matrix for a regular complete train track. A similar result follows for general train tracks from a result of Rykken [13].

**Theorem 3.2** (Rykken). Let $f : M \to M$ be a pseudo-Anosov homeomorphism on an orientable surface $M$ of genus $g$ with oriented unstable manifolds. Let $T$ be a train track transition matrix for $f$. If $f$ preserves the orientation of unstable manifolds, then the eigenvalues of $f_{1*} : H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ are the same as those of $T$, including multiplicity, up to roots of unity and zeros.

**Lemma 3.3.** Let $\beta$ be a pseudo-Anosov isotopy class on $D_n$ with invariant train track $\tau$ and associated transition matrix $T$. Let $\tilde{f}$ be the lift of $f$ to the orientation double cover $\tilde{M}$. Let $\tilde{\tau}$ and $\tilde{T}$ be the lifted invariant train track and transition matrix associated to $[\tilde{f}]$. Then $T$ and $\tilde{T}$ are isospectral up to roots of unity.

**Proof.** Let $\{e_i\}_{1 \leq i \leq N}$ be the oriented branches of $\tau$. Take a copy $e_i'$ of each $e_i$ and endow it with the opposite orientation. The lifted train track $\tilde{\tau}$ is obtained by gluing together the branches $e_i$ and $e_i'$ following the pattern of the original train track $\tau$. Then $\tilde{T}$ is isospectral to $T$. $\square$
track $\tau$, but in such a way that the orientations of all of the branches at each switch are consistent. By construction, the edge path $\tilde{f}(e_i)$ is obtained from the edge path $f(e_i)$ by replacing each occurrence of $e_j'$ with $e_j'$; and similarly, the edge path $\tilde{f}(e_i')$ is obtained from the edge path $\overline{f}(e_i)$ by replacing each occurrence of $\overline{e_j}$ with $e_j'$.

Let $A_{ij}$ be the number of occurrences of $e_i$ in $\tilde{f}(e_j)$ (that is, the number of occurrences of $e_i$ in $f(e_j)$), which by construction is equal to the number of occurrences of $e_i'$ in $\tilde{f}(e_j')$; and let $B_{ij}$ be the number of occurrences of $e_i'$ in $\tilde{f}(e_j)$ (that is, the number of occurrences of $\overline{e_j}$ in $f(e_j)$), which by construction is equal to the number of occurrences of $e_i$ in $\tilde{f}(e_j')$. Hence the lifted transition matrix $\tilde{T}$ is of the form

$$\tilde{T} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A + B = T$ (and we have restricted to the main branches $e_i$ ($1 \leq i \leq k$) and their copies $e_i'$). Hence we have

$$\chi(\tilde{T}) = |xI_{2k} - \tilde{T}| = \begin{vmatrix} xI_k - A & -B \\ -B & xI_k - A \end{vmatrix} = \begin{vmatrix} xI_k - A & xI_k - T \\ -B & xI_k - T \end{vmatrix} = \begin{vmatrix} xI_k - A + B & 0_k \\ -B & xI_k - T \end{vmatrix} = |xI_k - A + B| |xI_k - T|$$

That is, the set of eigenvalues of $\tilde{T}$ is the union of the set of eigenvalues of $T$ and the set of eigenvalues of $A - B$. It remains to show that the eigenvalues of $A - B$ are roots of unity.

Now for each $m \geq 1$, let $A_{ij}^{(m)}$ denote the number of occurrences of $e_i$ in $f^m(e_j)$, and $B_{ij}^{(m)}$ denote the number of occurrences of $\overline{e_i}$ in $f^m(e_j)$. A straightforward induction shows that the matrix $A^{(m)}$ is the sum of all products of $m$ copies of $A$ and $B$ having an even number of $B$s, and $B^{(m)}$ is the sum of all products of $m$ copies of $A$ and $B$ having an odd number of $B$s: therefore $A^{(m)} - B^{(m)} = (A - B)^m$.

Let $m$ be such that $f^m$ fixes all of the prongs of $\tau$. Then for each $e_i$, the initial and terminal points of $e_i$ and of $f^m(e_i)$ are the same. Since each real branch disconnects $\tau$, it follows that $A_{ij}^{(m)} = B_{ij}^{(m)}$ for all $i \neq j$, and $A_{ii}^{(m)} = B_{ii}^{(m)} + 1$ for all $i$ (the number of times that $f^m(e_i)$ crosses $e_i$ in the positive direction is one more than the number of times it crosses in the negative direction). That is

$$(A - B)^m = A^{(m)} - B^{(m)} = Id,$$

so that all of the eigenvalues of $A - B$ are roots of unity as required. \qed
Corollary 3.4. Let \([f] \in \text{MCG}(D_n)\) be a pseudo-Anosov isotopy class with unstable invariant foliation \((\mathcal{F}^u, \mu^u)\) and dilatation \(\lambda > 1\). Let \(\tau\) be any complete invariant train track with associated transition matrix \(T\). Then \(T\) and \(D\) are isospectral up to roots of unity and zeros.

Proof. If \(f : D_n \to D_n\) is a pseudo-Anosov homeomorphism it lifts to a pseudo-Anosov homeomorphism \(\tilde{f} : M \to M\) where \(M\) is the orientation double cover \([?]\). Pick a regular invariant train track \(\tau_r\) and an arbitrary invariant train track \(\tau\) of \(f : D_n \to D_n\) with associated transition matrices \(T_r\) and \(T\). Given two matrices \(A\) and \(B\), write \(A \sim B\) if \(A\) and \(B\) are isospectral up to roots of unity and zeros. Then, \(D \sim T_r\) by Theorem 3.1, \(T_r \sim \tilde{T}_r\) by Lemma 3.3, \(\tilde{T}_r \sim \tilde{T}\) by Theorem 3.2 and \(\tilde{T} \sim T\) by Lemma 3.3. Therefore, \(D \sim T\). \(\square\)

Example 3.5. The 4-braid \(\beta = \sigma_1\sigma_2^{-1}\sigma_3^3\sigma_2\sigma_1\sigma_2^{-1}\) has an invariant train track as depicted in Figure 8 with associated transition matrix

\[
T = \begin{bmatrix}
2 & 0 & 2 & 1 \\
2 & 0 & 3 & 1 \\
1 & 1 & 2 & 0 \\
1 & 0 & 4 & 0 \\
\end{bmatrix},
\]

and the coordinates of the eigenvector of \(T\) corresponding to the Perron-Frobenius eigenvalue \(\lambda = 4.61158\) are given by

\((0.50135, 0.59215, 0.41871, 0.47190)\).

\((\mathcal{F}^u, \mu^u)\) is in the interior of a Dynnikov region \(\mathcal{R}\) and the action on this region is given by the Dynnikov matrix

\[
D = \begin{bmatrix}
5 & -2 & 3 & 1 \\
3 & 0 & 1 & -2 \\
1 & -1 & 1 & 1 \\
1 & 1 & 0 & -2 \\
\end{bmatrix},
\]

Both \(D\) and \(T\) have spectrum

\(\{1 + \sqrt{2} \pm \sqrt{2 + 2\sqrt{2}}, 1 - \sqrt{2} \pm i\sqrt{2\sqrt{2} - 2}\}\).
3.2. The spectrum of a Dynnikov matrix when $\tau$ is not complete. This section studies the case where the invariant unstable measured foliation $(F^u, \mu^u)$ has other than punctured 1-pronged and unpunctured 3-pronged singularities. In this case $\tau$ is not complete and since $\text{rank}(\tau) < 2n - 4$, $MF(\tau)$ does not define a chart on $MF_n$. We shall study this case considering the two possibilities: first, where the prongs of the invariant foliations are fixed by $\beta$; and second, where they are permuted non-trivially.

If $\beta$ fixes the prongs, we shall see that every Dynnikov matrix is isospectral to $T$ up to some eigenvalues 1. If $\beta$ permutes the prongs non-trivially, then for some power $m$, $\beta^m$ fixes the prongs and it follows that every Dynnikov matrix for $\beta^m$ is isospectral to $T^m$ up to some eigenvalues 1 and zeros. However, since the induced action of $\beta^m$ on $PS_n$ is a product of several Dynnikov matrices, we cannot conclude in the permuted prongs case that a Dynnikov matrix $D_i$ and $T$ are isospectral up to roots of unity. This point will be clarified in Section 3.2.2.

The main tool to prove our results will be to extend non-complete train tracks to those which are complete. We shall use two basic moves pinching and diagonal extension [11, 12] as described as follows.

**Definition 3.6 (Pinching unpunctured $t$-gons).** Let $\tau$ be a train track with an unpunctured $t$-gon $P$, where $t \geq 4$. Let $e_1, \ldots, e_t$ denote the (infinitesimal) edges of $P$. Pinching across $e_i$ is a move which constructs a new train track $\tau' > \tau$ by pinching together the two edges $e_{i-1}$ and $e_{i+1}$ adjacent to $e_i$. See Figure 9. The train track $\tau'$ has three additional edges denoted $e'_{i-1}$, $e'_{i+1}$ and $\epsilon$: in place of the $t$-gon $P$ it has a $(t - 1)$-gon and a trigon. The function $\psi_{e_i} : W(\tau) \to W(\tau')$ is defined as follows.

If $w = (w_1, \ldots, w_i, w_{i+1}, \ldots, w_k) \in W(\tau)$, then $\psi_{e_i}(w)$ gives weights $w_{i-1}$ to $e'_{i-1}$, $w_{i+1}$ to $e'_{i+1}$, $w_{i-1} + w_{i+1}$ to $\epsilon$ and $w_j$ to $e_j$ for $1 \leq j \leq k$. We remark that if every component of $w$ is positive, then the same is true for $\psi_{e_i}(w)$.

![Figure 9. Pinching move (across $e_2$) on an unpunctured 5-gon](image)

**Definition 3.7 (Pinching punctured $t$-gons).** Let $\tau$ be a train track with a punctured $t$-gon $P$ where $t \geq 2$. Let $e_1, \ldots, e_t$ denote the (infinitesimal) edges of $P$. Pinching of $e_i$ is a move which constructs a new train track $\tau' > \tau$ by pinching $e_i$ to itself around the puncture as depicted in Figure 10. The train track $\tau'$ has

---

1Here and in what follows, indices are taken modulo $t$
three additional edges denoted $\varepsilon, \varepsilon_i', \varepsilon_i'':$ in place of the punctured $t$-gon, it has an unpunctured $(t + 1)$-gon and a punctured monogon.

The function $\psi_{\varepsilon_i} : \mathcal{W}(\tau) \to \mathcal{W}(\tau')$ is given as follows.

If $w = (w_1, \ldots, w_t, w_{t+1}, \ldots, w_k) \in \mathcal{W}(\tau)$, $\psi_{\varepsilon_i}(w)$ gives weights $2w_i$ to $\varepsilon$, $w_i$ to $\varepsilon_i'$ and $\varepsilon_i''$, and $w_j$ to $e_j$ for $1 \leq j \leq k$. We remark again that if every component of $w$ is positive, then the same is true for $\psi_{\varepsilon_i}(w)$.

**Definition 3.8.** We say that a complete train track $\tau_p$ on $D_n$ is a pinching of $\tau$ if it is constructed from $\tau$ by a sequence of pinching moves.

**Remark 3.9.** Given a train track $\tau$, pinching each punctured $t$-gon with $t \geq 2$ yields a train track with only punctured monogons and unpunctured $t$-gons for $t \geq 3$. Pinching each unpunctured $t$-gon $t - 3$ times then yields a pinching of $\tau$. Observe that there are many different pinchings of $\tau$ (Figure 11). The main result about pinched train tracks in Lemma 3.14 doesn’t depend on the choice of pinching.

**Figure 10.** Pinching move (of $e_1$) on a punctured bigon and a punctured trigon

**Figure 11.** Two different pinchings of an unpunctured 4-gon
Therefore, pinching constructs a complete train track $\tau_p$ from a non-complete one $\tau$ in such a way that $\tau < \tau_p$, with the important feature that a strictly positive measure on $\tau$ induces a strictly positive measure on $\tau_p$. Hence, $\mathcal{MF}(\tau_p)$ defines a chart on $\mathcal{MF}_n$ which contains $(\mathcal{F}^u, \mu^u)$ in its interior. However, it should be noted that if $\tau$ is an invariant train track for $\beta$, $\tau_p$ will not be invariant unless relevant prongs of $(\mathcal{F}^u, \mu^u)$ are fixed by $\beta$. Therefore, we need a set of charts that fit nicely in $\mathcal{MF}(\tau_p)$ with the property that the action in each of them is described explicitly.

We shall use the diagonal extension move to describe such charts. Diagonal extension gives a collection of diagonally extended train tracks $\tau_i$ in such a way that $\tau < \tau_i$. The disadvantage of diagonal extension is that a strictly positive measure on $\tau$ induces zero measure on the additional branches of $\tau_i$ and hence $(\mathcal{F}^u, \mu^u)$ is on the boundary of each $\mathcal{MF}(\tau_i)$. However, Lemma 3.14 gives that the charts $\mathcal{MF}(\tau_i)$ fit together nicely and have union $\mathcal{MF}(\tau_p)$: moreover for each $i$ there is some $j$ such that $\beta(\mathcal{MF}(\tau_i)) = \mathcal{MF}(\tau_j)$, and this action can be simply described with respect to appropriate bases.

**Definition 3.10 (Diagonal extension on unpunctured t-gons).** Let $\tau$ be a train track with an unpunctured $t$-gon $P$ where $t \geq 4$. Let $v_1, \ldots, v_t$ denote the vertices of $P$. Diagonal extension of $P$ is a move which constructs a new train track $\tau' > \tau$ by adding $t - 3$ branches (with disjoint interiors) inside $P$ such that each additional branch joins two non-consecutive vertices $v_i$ and $v_j$ and is tangent to the (infinitesimal) edges of $P$ at these vertices. See Figure 12. The train track $\tau'$ has $t - 3$ additional edges denoted $\epsilon_{ij}$ for appropriate choices of $i$ and $j$ with $|i - j| > 1$: in place of the $t$-gon, it has $t - 2$ unpunctured trigons. The function $\psi : \mathcal{W}(\tau) \to \mathcal{W}(\tau')$ is given as follows.

If $w = (w_1, \ldots, w_k) \in \mathcal{W}(\tau)$, $\psi(w)$ gives zero weights to each $\epsilon_{ij}$, and weight $w_i$ to $\epsilon_i$ for $1 \leq i \leq k$.

![Figure 12. Diagonal extension on an unpunctured 5-gon](image)

**Definition 3.11 (Diagonal extension on punctured t-gons).** Let $\tau$ be a train track with a punctured $t$-gon $P$ where $t \geq 2$. Let $v_1, \ldots, v_t$ denote the vertices of $P$. Diagonal extension of $P$ is a move which constructs a new train track $\tau' > \tau$ by first adding a branch $\epsilon_{ii}$ which encircles the puncture with both end points at a
single vertex $v_i$ (so that $P$ is divided into a punctured monogon and an unpunctured $(t + 1)$-gon); and then adding $t - 2$ additional branches to divide the $(t + 1)$-gon into $t - 1$ trigons as in the unpunctured case. See Figure 13. The train track $\tau'$ therefore has a punctured monogon and $t - 1$ unpunctured trigons in place of the punctured $t$-gon $P$. The function $\psi : W(\tau) \to W(\tau')$ is given as follows.

If $w = (w_1, \ldots, w_k) \in W(\tau)$, $\psi(w)$ gives weight $w_j$ to $e_j$ for $1 \leq j \leq k$, and weight zero to the other edges.

**Definition 3.12.** We say that a complete train track $\tau'$ on $D_n$ is a diagonal extension of $\tau$ if it is constructed from $\tau$ by a sequence of diagonal extensions. We write $\tau_1, \tau_2, \ldots, \tau_\xi$ to denote the different diagonal extensions of $\tau$.

**Remark 3.13.** Note that the number of diagonal extensions of an unpunctured $t$-gon is $\xi = c_{t-2}$ where

$$c_t = \binom{2t}{t} - \binom{2t}{t-1}$$

is the $t$th Catalan number, since the Catalan number gives the number of different ways to divide a polygon into triangles by joining its vertices with additional edges.

Similarly, the number of diagonal extensions of a punctured $t$-gon is $\xi = t \cdot c_{t-1}$: after adding the encircling branch $\epsilon_{ii}$ at vertex $v_i$ ($t$ choices), $P$ is divided into an unpunctured $(t + 1)$-gon and a punctured monogon. Since there are $c_{t-1}$ different ways to divide an unpunctured $(t + 1)$-gon into triangles the result follows.

Given a train track $\tau$, let $G$ denote the set of unpunctured polygons of $\tau$, and given $P \in G$ write $n_P$ for the number of vertices of $P$. Similarly, let $G'$ denote the set of punctured polygons of $\tau$, and given $P \in G'$ write $n_P$ for the number of
vertices of $P$. Then the number of diagonal extensions of $\tau$ is given by

$$\xi = \left( \prod_{P \in G} c_{nP} \right) \cdot \left( \prod_{P \in G'} nP \cdot c_{nP} \right).$$

$\xi$ can be large for relatively simple train tracks and hence there can be many Dynnikov regions for braids on relatively few strings.

It will later be seen from the proof of Theorem 3.15 that there is a unique Dynnikov region that corresponds to a diagonal extension $\tau_i$ of $\tau$. Therefore, it follows that the number of Dynnikov regions is bounded above by the number of diagonal extensions of $\tau$ (which is given by the formula in Remark 3.13).

Since each pinching $\tau_p$ and diagonal extension $\tau_i$ of $\tau$ is complete, they define charts on $\mathcal{MF}_n$ by Lemma 3.13. The following key lemma describes how these charts fit together.

**Lemma 3.14.** Let $\tau$ be a regular invariant train track for $\beta$ with associated matrix $T : \mathcal{W}(\tau) \to \mathcal{W}(\tau)$. Let $\tau_p$ be a pinching of $\tau$, and let $\tau_1, \ldots, \tau_\xi$ denote the diagonal extensions of $\tau$. Then,

i. $\bigcup_{1 \leq i \leq \xi} \mathcal{MF}(\tau_i) = \mathcal{MF}(\tau_p)$.

ii. If $i \neq j$, then $\mathcal{MF}(\tau_i)$ and $\mathcal{MF}(\tau_j)$ intersect only on their boundaries.

iii. For each $i$ there is some $j$ such that $\beta(\mathcal{MF}(\tau_i)) = \mathcal{MF}(\tau_j)$, and the induced action of $\beta : \mathcal{W}(\tau_i) \to \mathcal{W}(\tau_j)$ is given by a matrix of the form

$$\tilde{T} = \begin{bmatrix} T & X \\ 0 & \text{Id} \end{bmatrix}$$

with respect to an appropriate choice of bases of $\mathcal{W}(\tau_i)$ and $\mathcal{W}(\tau_j)$.

iv. For each $i$, the change of coordinate function $\rho \circ \phi_{\tau_i} : \mathcal{W}^+(\tau_i) \to \mathcal{S}_n$ is linear in a neighbourhood in $\mathcal{W}^+(\tau_i)$ of $v^u = \phi_{\tau_i}^{-1}(F^u, \mu^u)$.

**Proof.** Assume first that every component of $D_n - \tau$ is a punctured monogon or unpunctured trigon, except for one unpunctured $t$-gon $P$ ($t \geq 4$). Let $v_1, v_2, \ldots, v_t$ denote the vertices of $P$. Let $\tau_p$ be a pinching of $\tau$ and $N$ denote a regular neighbourhood of the pinched $t$-gon. Let $a_1, \ldots, a_t$ denote the gates of $N$ (that is, the components of the subset of $\partial N$ which is not comprised of leaves). See Figure 14.

To each $(\mathcal{F}, \mu) \in \mathcal{MF}(\tau_p)$ we associate the collection of two element sets $\{j, k\}$ ($|j - k| > 1$) such that $(\mathcal{F}, \mu)$ has a leaf which enters $N$ through $a_j$ and exits through $a_k$. Denote this label set $\Gamma(\mathcal{F}, \mu)$ and observe that the cardinality $|\Gamma(\mathcal{F}, \mu)| \leq t - 3$ since leaves don’t cross.
Figure 14. The regular neighbourhood $N$ of a pinched unpunctured 5-gon.

Similarly, to each diagonal extension $\tau_i$ of $\tau$, we associate the two element sets $\{j, k\}$ such that $\tau_i$ has a branch joining $v_j$ to $v_k$. Denote this label set $\Delta(\tau_i)$. It is clear that $(F, \mu) \in \mathcal{MF}(\tau_i)$ if and only if $\Gamma(F, \mu) \subseteq \Delta(\tau_i)$.

If $|\Gamma(F, \mu)| = t - 3$, then there is a unique $\tau_i$ with $\Gamma(F, \mu) = \Delta(\tau_i)$; while if $|\Gamma(F, \mu)| < t - 3$ then there are several $\tau_i$ with $\Gamma(F, \mu) \subset \Delta(\tau_i)$ and for each of these $\tau_i$, $\phi_{\tau_i}^{-1}(F, \mu)$ has some zero coordinates (see Figure 15). This establishes that

i. $\mathcal{MF}(\tau_p) \subseteq \bigcup_{i=1}^{\xi} \mathcal{MF}(\tau_i)$; and

ii. If $\tau_i \neq \tau_j$, then $\mathcal{MF}(\tau_i)$ and $\mathcal{MF}(\tau_j)$ can only intersect along their boundary.

To show that $\bigcup_{i=1}^{\xi} \mathcal{MF}(\tau_i) \subseteq \mathcal{MF}(\tau_p)$, we observe that any two vertices of the pinched polygon of $\tau_p$ can be connected by a smooth path in $\tau_p$ and hence if $(F, \mu)$ is carried by any $\tau_i$, it is also carried by $\tau_p$.

Next assume that every component of $D_n - \tau$ is a punctured monogon or unpunctured trigon, except for one punctured $t$-gon $P$ ($t \geq 2$). Let $v_1, v_2, \ldots, v_t$ denote the vertices of $P$. Let $\tau_p$ be a pinching of $\tau$ and $N$ denote a regular neighbourhood.
of the pinched $t$-gon. Label the gates $a_1, \ldots, a_t$ of $N$ in anticlockwise cyclic order and let $l_i$ ($1 \leq i \leq t$) denote the leaf of $(F, \mu)$ in $\partial N$ which joins $a_{i-1}$ to $a_i$. See Figure 16.

![Figure 16: The regular neighbourhood $N$ of a pinched punctured trigon](image)

The label set $\Gamma(F, \mu)$ of a measured foliation $(F, \mu) \in \mathcal{MF}(\tau_p)$ will consist of pairs $(j, k) \in \{1, \ldots, t\} \times \{1, \ldots, t\}$. This contrasts with the case for unpunctured $t$-gons, since there are two possible paths for leaves joining the gates $a_j$ and $a_k$, one on each side of the puncture. To describe this label set, first orient each gate $a_i$ and each leaf $l_i$ anticlockwise around $\partial N$. Then $(j, k) \in \Gamma(F, \mu)$ if and only if

- there is a leaf segment $L$ of $(F, \mu)$ in $N$ which joins $a_j$ to $a_k$;
- when $L$ is oriented from $a_j$ to $a_k$, the oriented loop consisting of $L$ and a subset of $\partial N$ bounds a disk containing the puncture in its interior; and
- $k \neq j + 1$ (we don’t include leaves which must necessarily be part of $N$).

See Figure 17. Notice that $(j, j) \in \Gamma(F, \mu)$ if and only if the leaf from the 1-pronged singularity exits $N$ through $a_j$. Of course, it is possible that this leaf doesn’t exit $N$ (e.g. if $(F, \mu) = (F^u, \mu^u)$). Also, observe that the cardinality $|\Gamma(F, \mu)| \leq t - 1$ since leaves don’t cross.

![Figure 17: The label set $\Gamma(F, \mu)$ is given by $\{(1, 1), (1, 5), (1, 4), (2, 4)\}$](image)

To describe the label set $\Delta(\tau_i)$ for a diagonal extension $\tau_i$ of $\tau$, we label the vertices $v_1, \ldots, v_t$ of $P$ in the anticlockwise cyclic order and put arrows on the edges of $P$ pointing from $v_j$ to $v_{j+1}$. For each additional branch, we place an arrow on the branch so that the loop composed of the branch and of edges of $P$ which encloses the puncture is oriented consistently. Then $\Delta(\tau_i)$ is the set of pairs $(j, k)$
such that there is an additional branch from $v_j$ to $v_k$. See Figure 18. It is clear
that $(\mathcal{F}, \mu) \in \mathcal{MF}(\tau_i)$ if and only if $\Gamma(\mathcal{F}, \mu) \subseteq \Delta(\tau_i)$.

**Figure 18.** The label set $\Delta(\tau_i)$ is given by $\{(1,1), (1,5), (1,4), (2,4)\}$

If $|\Gamma(\mathcal{F}, \mu)| = t - 1$, then there is a unique $\tau_i$ with $\Gamma(\mathcal{F}, \mu) = \Delta(\tau_i)$, while if $|\Gamma(\mathcal{F}, \mu)| < t - 1$ then there are several $\tau_i$ with $\Gamma(\mathcal{F}, \mu) \subset \Delta(\tau_i)$ and for each of these $\tau_i$, $\phi_{\tau_i}^{-1}(\mathcal{F}, \mu)$ has some zero coordinates. See Figure 19.

**Figure 19.** If $\Gamma(\mathcal{F}, \mu) = \{(3,3)\}$, then $(\mathcal{F}, \mu) \in \mathcal{MF}(\tau_1) \cap \mathcal{MF}(\tau_2)$.

This establishes that

i. $\mathcal{MF}(\tau_p) \subseteq \bigcup_{i=1}^{\xi} \mathcal{MF}(\tau_i)$; and

ii. If $\tau_i \neq \tau_j$, then $\mathcal{MF}(\tau_i)$ and $\mathcal{MF}(\tau_j)$ can only intersect along their boundary.

To show that $\bigcup_{i=1}^{\xi} \mathcal{MF}(\tau_i) \subseteq \mathcal{MF}(\tau_p)$, we observe that any two vertices of the
pinched polygon of $\tau_p$ can be connected by a smooth path in $\tau_p$ and hence if $(\mathcal{F}, \mu)$
is carried by any $\tau_i$, it is also carried by $\tau_p$.

Therefore, we have proved the first two statements of the lemma in the case where
$\tau$ has only one polygon which is not a punctured monogon or an unpunctured trigon.
For the general case, we argue for each punctured and unpunctured polygon of $\tau$ in the same way as above and observe that if $(F, \mu) \in MF(\tau_p)$, then there is a diagonal extension $\tau_i$ of $\tau$ so that $(F, \mu) \in MF(\tau_i)$. Conversely, if $(F, \mu) \in MF(\tau_i)$ for some diagonal extension $\tau_i$ of $\tau$, then $(F, \mu) \in MF(\tau_p)$ since any two vertices of each pinched polygon of $\tau_p$ can be connected by a smooth path. Also, if $(F, \mu)$ is carried by two diagonal extensions $\tau_i$ and $\tau_j$, then $\phi_{\tau_i}^{-1}(F, \mu)$ has some zero coordinates from the argument above and hence $MF(\tau_i)$ and $MF(\tau_j)$ can only intersect along their boundary.

For the proof of the third statement, we first note that $\beta$ permutes the vertices of $\tau$. Hence, given a diagonal extension $\tau_i$ of $\tau$, the permutation on the vertices of $\tau$ sends each additional branch of $\tau_i$ onto another additional branch, and so gives another diagonal extension $\tau_j$ of $\tau$. Therefore, we have $\beta(MF(\tau_i)) = MF(\tau_j)$. Then, $\beta : W^+(\tau_i) \to W^+(\tau_j)$ is described by the matrix

$$
\tilde{T} = \begin{bmatrix}
    T & X \\
    0 & \text{Id}
\end{bmatrix}
$$

with respect to the natural coherent choice of bases of $W^+(\tau_i)$ and $W^+(\tau_j)$. We remark that if all components of $D_n - \tau$ are odd-gons then $W^+(\tau_i)$ and $W^+(\tau_j)$ have bases consisting of weights on the main branches of $\tau$ and the additional branches and hence $X$ is zero. If $D_n - \tau$ has an even-gon then $X$ can be non-zero since the bases of $W^+(\tau_i)$ and $W^+(\tau_j)$ consists of weights on edges which includes infinitesimal and additional ones. For some main branch $e_k$ of $\tau_i$, the corresponding weight $w_k$ is the sum of weights on some infinitesimal and additional branches and $f(e_k)$ may cover some basis elements.

For the fourth statement, we recall from (the proof of) Lemma 2.5, that the fact that each change of coordinates from train track coordinates $W^+(\tau_i)$ to Dynnikov coordinates is only piecewise linear is a consequence of the piecewise linearity of the function $\hat{p} : W^+(\tau_i) \to \mathbb{R}^+$, where $\hat{p}$ is a minimal non-tight train path with respect to some Dynnikov arc. The function $\hat{p}$ makes a transition from one linear region to another at measures for which some leaf which follows the train path $p$ connects two singularities (see the proof of Lemma 2.2).

At $v^\mu = \phi_{\tau_i}^{-1}(F^\mu, \mu^\nu) \in MF(\tau_i)$ there are several such leaves connecting singularities but the choice of diagonal extension $\tau_i$ is precisely a choice of the relative configurations of these leaves when the connection between singularities are broken, and therefore $L : W^+(\tau_i) \to S_n$ is linear near $v^\mu$. \hfill \Box

### 3.2.1. The spectrum of the Dynnikov matrices when $\beta$ fixes the prongs of $\tau$

In this section we shall prove that when $\beta$ fixes the prongs of $\tau$, then every Dynnikov matrix is isospectral to $T$ up to some eigenvalues 1.
Theorem 3.15. Let $\beta \in B_n$ be a pseudo-Anosov braid with unstable invariant measured foliation $(F^u, \mu^u)$ and dilatation $\lambda > 1$. Let $\tau$ be a regular invariant train track of $\beta$ with associated transition matrix $T$. If $\beta$ fixes the prongs at all singularities other than unpunctured 3-pronged and punctured 1-pronged singularities, then any Dynnikov matrix $D_i$ is isospectral to $T$ up to some eigenvalues 1.

Proof. Let $\tau_p$ be a pinching of $\tau$ and $\tau_i$ ($1 \leq i \leq \xi$) be the diagonal extensions of $\tau$. Since each strictly positive measure on $\tau_p, W^+(\tau_p)$ and $MF(\tau_p)$ are neighbourhoods of $v^u$ and $(F^u, \mu^u)$ respectively. Furthermore, by Lemma 3.14, $\bigcup_{1 \leq i \leq \xi} MF(\tau_i) = MF(\tau_p)$ and for $i \neq j$, $MF(\tau_i)$ and $MF(\tau_j)$ intersect only on their boundaries. Since $\beta$ fixes the prongs at all singularities other than unpunctured 3-pronged and punctured 1-pronged singularities, for each $\tau_i$ we have $\beta(MF(\tau_i)) = MF(\tau_i)$ and the induced action $\beta : W^+(\tau_i) \rightarrow W^+(\tau_i)$ is given by the matrix

$$\tilde{T} = \begin{bmatrix} T & X \\ 0 & Id \end{bmatrix}.$$ 

Using the fourth statement of Lemma 3.14 the change of coordinate function $\rho \circ \phi_{\tau_i} : W^+(\tau_i) \rightarrow S_n$ is linear in a neighbourhood in $W^+(\tau_i)$ of $v^u = \phi_{\tau_i}^{-1}(F, \mu)$. Therefore, for each $\tau_i$ we have the following commutative diagram:

$$\begin{array}{ccc}
W^+(\tau_i) & \xrightarrow{\tilde{T}} & W^+(\tau_i) \\
\phi_{\tau_i} \downarrow & & \phi_{\tau_i} \downarrow \\
MF(\tau_i) & \xrightarrow{\beta} & MF(\tau_i) \\
\rho \downarrow & & \rho \downarrow \\
S_n & \xrightarrow{F} & S_n
\end{array}$$

where $L_i = \rho \circ \phi_{\tau_i}$ is linear in a neighbourhood $U_i \subseteq W^+(\tau_i)$ of $v^u$. Let $(a^u, b^u)$ denote the Dynnikov coordinates of $(F^u, \mu^u)$. For each $1 \leq i \leq \xi$, we write $R_i = L_i(U_i)$: by the above, $\bigcup_{1 \leq i \leq \xi} R_i$ is a neighbourhood of $(a^u, b^u)$. Then in $R_i$, $D_i = F|_{R_i} = L_i \circ \tilde{T} \circ L_i^{-1}$ is linear and isospectral to $T$ up to some eigenvalues 1. These matrices $D_i$ ($1 \leq i \leq \xi$) are precisely the Dynnikov matrices for $\beta \in B_n$. □

In fact, the next theorem shows that if $D_n - \tau$ has only odd-gons all of the Dynnikov matrices are equal and hence there is only one Dynnikov region in the fixed-pronged case.
Theorem 3.16. Let $\beta \in B_n$ be a pseudo-Anosov braid with unstable invariant measured foliation $(F^u, \mu^u)$ and dilatation $\lambda > 1$. Let $\tau$ be a regular invariant train track of $\beta$ with associated transition matrix $T$. If all components of $D_n - \tau$ are odd-gons and $\beta$ fixes the prongs at all singularities other than unpunctured 3-pronged and punctured 1-pronged singularities, then there is a unique Dynnikov region.

Proof. We use the notation in the proof of Theorem 3.15. Let $k = \text{rank}(\tau)$ and $N = 2n - 4$ be the dimension of $S_n$. We first note that each $L_j$ is of the form $(L_i | X_i)$ for some fixed $N \times k$ matrix $L$, where $L$ is the change of coordinates from $\mathcal{MF}(\tau)$ to $S_n$ on the hyperplane $\mathcal{MF}(\tau)$. Then each $L_i^{-1}$ is of the form $(A \ Y_i)$ for some fixed $k \times N$ matrix $A$ which gives the change of coordinates from the $k$-dimensional subspace of $S_n$ corresponding to $\mathcal{MF}(\tau)$ to $\mathcal{W}(\tau)$. Therefore we have,

$$L_i^{-1}L_i = (A \ (L_i X_i))$$

Since $L_i^{-1}L_i = Id$, $AX_i$ and $Y_iL$ are zero matrices for all $i$. It follows that for any $i, j$ we have

$$L_j^{-1}L_i = \begin{bmatrix} Id_k & 0 \\ 0 & P_{ij} \end{bmatrix}.$$ 

for some $(N - k) \times (N - k)$ matrix $P_{ij}$. In particular, $L_j^{-1}L_i$ commutes with

$$\tilde{T} = \begin{bmatrix} T & 0 \\ 0 & Id \end{bmatrix}.$$ 

Hence, $D_i = L_i \tilde{T} L_i^{-1} = L_j \tilde{T} L_j^{-1} = D_j$ for all $i$ and $j$. \hfill \Box

In the above we considered the transition matrix for a regular non-complete train track. A similar result follows for general train tracks from Rykken’s result Theorem 3.2 and Lemma 3.3.

Corollary 3.17. Let $\beta \in \text{MCG}(D_n)$ be a pseudo-Anosov braid with unstable invariant foliation $(F^u, \mu^u)$ and dilatation $\lambda > 1$. Let $\tau$ be any invariant train track with associated transition matrix $T$. If $\beta$ fixes the prongs at all singularities other than unpunctured 3-pronged and punctured 1-pronged singularities, then any Dynnikov matrix is isospectral to $T$ up to roots of unity and zeros. Furthermore, if $D_n - \tau$ consists of only odd-gons then there is a unique Dynnikov matrix.

3.2.2. The spectrum of Dynnikov matrices when $\beta$ permutes the prongs of $\tau$ non-trivially. In this section we discuss the case when $\beta$ permutes the prongs of $(F^u, \mu^u)$ non-trivially. We would like to prove that any Dynnikov matrix $D_i$ is isospectral to $T$ up to roots of unity: this has been confirmed with a wide range
of examples. The problems which arise in this case, and some approaches to their solutions are illustrated in the following example.

**Example 3.18.** Consider the 6-braid \( \beta_6 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5^{-1} \) on \( D_6 \). A regular invariant train track \( \tau \) and its image under \( \beta_6 \) are depicted in Figure 20 and Figure 21.

![Figure 20. Invariant train track of \( \beta_6 = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5^{-1} \)](image)

The transition matrix \( T \) associated to \( \tau \) is given by

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

Working to three decimal places, the largest eigenvalue of \( T \) is \( \lambda = 2.081 \) and the associated eigenvector \( v^u \) corresponding to \((F^u, \mu^u)\) is given by \( v^u = (a, b, c, d, e, f) \approx (0.057, 0.370, 0.521, 0.250, 0.120, 0.714) \). Using the switch conditions we find that

\[
x_1 = 0.018, \ x_2 = 0.352, \ x_3 = 0.169, \\
x_4 = 0.081, \ x_5 = 0.039, \ m_1 = 0.028, \ m_2 = 0.06, \\
m_3 = 0.125, \ m_4 = 0.260, \ m_5 = 0.542, \ m_6 = 0.357.
\]

![Figure 21. The image train track of \( \tau \) under \( \beta \)](image)
In order to find the change of coordinate function \( L : W^+(\tau) \to S_8 \) in a neighbourhood of \( v^u \) we use a standard embedding of \( \tau \) with respect to the Dynnikov arcs as depicted in Figure 22.

![Figure 22. A standard embedding of \( \tau \) with respect to Dynnikov arcs](image)

Observe that \( \hat{p}_1(\mu) = \min(x_3, d/2) \) and \( \hat{p}_2(\mu) = \min(x_2, b/2) \) where \( p_1 \) and \( p_2 \) denote the minimal non-tight train paths depicted in Figure 22. Since \( x_3 \leq d/2 \) and \( x_2 \leq b/2 \) at \( v^u \), we have in some neighbourhood of \( v^u \) that \( \hat{p}_1(\mu) = x_3 \) and \( \hat{p}_2(\mu) = x_2 \). We compute that,

\[
\begin{align*}
\beta_1 &= a, \quad \beta_2 = b + d - c, \quad \beta_3 = a + d - e, \quad \beta_4 = b, \quad \beta_5 = f \\
\alpha_1 &= \frac{e}{2}, \quad \alpha_3 = \frac{d}{2}, \quad \alpha_5 = \frac{e}{2}, \quad \alpha_7 = \frac{b + f}{2} \\
\alpha_2 &= \max(a, b + d - c) - \frac{e}{2}, \quad \alpha_4 = \max(b + d - c, a + d - e) - \frac{d}{2} \\
\alpha_6 &= \max(a + d - c, b) - \frac{c}{2}, \quad \alpha_8 = \max(b, f) - \frac{b + f}{2}.
\end{align*}
\]

Since \( b + d - c > a, \quad a + d - e > b + d - c, \quad b > a + d - e \), and \( f > b \) at \( v^u \simeq (0.057, 0.370, 0.521, 0.250, 0.120, 0.714) \) we have,

\[
\begin{align*}
\alpha_2 &= b + d - c - \frac{e}{2}, \quad \alpha_4 = a + d - e - \frac{d}{2}, \quad \alpha_6 = b - \frac{c}{2}, \quad \alpha_8 = f - \frac{b + f}{2}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
a_1 &= \frac{1}{2}(b - c + d - e), \quad a_2 = \frac{1}{2}(a - e), \quad a_3 = \frac{1}{2}(b - c), \quad a_4 = -\frac{b}{2}, \\
b_1 &= \frac{1}{2}(a - b + c - d), \quad b_2 = \frac{1}{2}(-a + b - c + e), \\
b_3 &= \frac{1}{2}(a - b + d - e), \quad b_4 = \frac{1}{2}(b - f).
\end{align*}
\]
That is, the change of coordinate function $L : \mathcal{W}^+ (\tau) \to S_8$ is given by

$$L = \frac{1}{2} \begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}$$

in a neighbourhood of $v^u$. Now we shall construct the diagonal extensions of $\tau$ and observe the connection between the associated change of coordinate functions and $L$.

![Figure 23. 5 different ways to diagonalize an unpunctured 5-gon](image)

Figure 23 depicts the five different ways of diagonalizing an unpunctured 5-gon. Hence, there are five diagonal extensions of $\tau$. We compute the change of coordinate function for the diagonal extension $\tau_1$ as depicted in Figure 24. The matrices associated to the other diagonal extensions are calculated exactly the same way. We first observe that the measures on the infinitesimal branches of $\tau_1$ are determined by $a, b, c, d, e, f, \epsilon_1, \epsilon_2$ and

$$\beta_1 = a, \quad \beta_2 = b + d - c + 2\epsilon_1, \quad \beta_3 = a + d - e - 2\epsilon_2, \quad \beta_4 = b, \quad \beta_5 = f$$

$$\alpha_1 = \frac{e}{2}, \quad \alpha_3 = \frac{d}{2}, \quad \alpha_5 = \frac{c}{2}, \quad \alpha_7 = \frac{b + f}{2}.$$ 

Since $\alpha_{2i} = \max(\beta_i, \beta_{i+1}) - \alpha_{2i-1}$ we have,

$$\alpha_2 = \max(a, b + d - c + 2\epsilon_1) - \frac{e}{2}, \quad \alpha_4 = \max(b + d - c + 2\epsilon_1, a + d - e - 2\epsilon_2) - \frac{d}{2}$$

$$\alpha_6 = \max(a + d - e - 2\epsilon_2, b) - \frac{c}{2}, \quad \alpha_8 = \max(b, f) - \frac{b + f}{2}.$$ 

Therefore, the change of coordinate function $L_1 : \mathcal{W}^+ (\tau_1) \to S_8$ is given by

$$L_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & -2 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 2 & 2 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.$$
in a neighbourhood of $v^u$ in $W^+(\tau_1)$. Similar computations for the change of coordinate matrices $L_2, L_3, L_4, L_5$ associated to the other diagonal extensions $\tau_2, \tau_3, \tau_4$ and $\tau_5$ give:

$$L_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 2 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & -2 & -2 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & -2 & -2 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L_4 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 2 \\ -1 & 1 & -1 & 0 & 1 & 0 & 2 & -2 \\ 1 & -1 & 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_5 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & -1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & -2 \\ -1 & 1 & -1 & 0 & 1 & 0 & -2 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$
where $L_i = \rho \circ \phi_{\tau_i}$ is linear in a neighbourhood $U_i$ of $v^u$ in $W^+(\tau_i)$. For $1 \leq i \leq 5$, write $R_i = L_i(U_i)$. Let $(a^u, b^u)$ be the Dynnikov coordinates of $(F^u, \mu^u)$. Then, \[ \bigcup_{1 \leq i \leq 5} R_i \] is linear. These matrices $D_i (1 \leq i \leq 5)$ are precisely the Dynnikov matrices for $\beta \in B_6$. Let us calculate the Dynnikov matrices in our example and see how it is possible to determine the corresponding Dynnikov regions. We first observe that

\[
L_1^{-1} = \begin{pmatrix}
0 & 0 & 0 & -2 & 2 & 2 & 2 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & -2 & 0 & 0 & 2 & 0 \\
-2 & 0 & -2 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

and from the bottom two rows of this matrix we can see that the relevant Dynnikov region is determined by the inequalities $\epsilon_1 = a_2 - a_3 + b_2 \geq 0$ and $\epsilon_2 = a_1 - a_2 + b_1 \geq 0$. In $\mathcal{M}F(\tau_1)$, the Dynnikov matrix $D_1$ is given by

\[
D_1 = L_2 \hat{T} L_1^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Similarly, we compute the other Dynnikov matrices for $\beta$, and find that

\[
D_2 = L_3 \hat{T} L_2^{-1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
is the Dynnikov matrix in the region $-a_1 + a_2 - b_1 \geq 0$ and $-a_2 + a_3 - b_2 \geq 0$,

$$D_3 = L_4 \tilde{T} L_3^{-1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

is the Dynnikov matrix in the region $-a_1 + a_2 - b_1 \geq 0$ and $a_1 - a_3 + b_1 + b_2 \geq 0$,

$$D_4 = L_5 \tilde{T} L_4^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

is the Dynnikov matrix in the region $a_1 - a_2 + b_1 \geq 0$ and $-a_2 + a_3 - b_2 \geq 0$

$$D_5 = L_1 \tilde{T} L_5^{-1} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 & -1 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

is the Dynnikov matrix in the region $-a_1 + a_3 - b_1 - b_2 \geq 0$ and $a_2 - a_3 + b_2 \geq 0$.

We observe that $D_2$ and $D_5$ are the same and hence there are four Dynnikov matrices, and all these Dynnikov matrices are isospectral to $T$ up to roots of unity.

**Question 3.19.** Let $\beta \in B_n$ be a pseudo-Anosov braid with unstable invariant foliation $(\mathcal{F}_u, \mu_u)$, dilatation $\lambda > 1$ and regular invariant train track $\tau$ having transition matrix $T$. If $\beta$ permutes the prongs of $(\mathcal{F}_u, \mu_u)$ non-trivially, is every Dynnikov matrix $D_i$ isospectral to $T$ up to roots of unity?

**Remark 3.20.** Note that when $\beta$ permutes the prongs of $(\mathcal{F}_u, \mu_u)$ non-trivially, then for some $m \in \mathbb{Z}^+$, $\beta^m$ fixes the prongs. The transition matrix for $\beta^m$ on a diagonal extension of $\tau$ is of the form

$$T' = \begin{bmatrix} T^m & 0 \\ 0 & Id \end{bmatrix}.$$
By Theorems 3.15 and 3.16 every Dynnikov matrix for $\beta^n$ is isospectral to $T^n$ up to some eigenvalues 1. In the above example we have

$$\mathcal{MF}(\tau_1) \rightarrow \mathcal{MF}(\tau_2) \rightarrow \mathcal{MF}(\tau_3) \rightarrow \mathcal{MF}(\tau_4) \rightarrow \mathcal{MF}(\tau_5) \rightarrow \mathcal{MF}(\tau_1)$$

and $D_i = L_{i+1}T_iL_i^{-1}$. Hence, for $i = 1, 2, 3, 4, 5$ the Dynnikov matrix for $\beta^i$ on $\mathcal{MF}(\tau_i)$ is given by

$$D_i = L_iT^5L_i^{-1}$$

which is clearly isospectral to $T^5$ up to two eigenvalues 1. By Theorem 3.16 the five Dynnikov matrices $D_i = L_iT^5L_i^{-1}$ are all equal, and given by

$$D = \begin{pmatrix}
1 & 1 & 2 & 4 & 1 & 1 & 2 & 4 \\
0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
2 & 6 & 14 & 33 & 4 & 8 & 16 & 31 \\
-1 & -2 & -4 & -8 & 0 & -2 & -4 & -8 \\
-1 & -3 & -6 & -12 & -2 & -6 & -12 & -12 \\
-1 & -3 & -7 & -14 & -2 & -6 & -14 & -14 \\
1 & 3 & 7 & 17 & 2 & 4 & 8 & 16
\end{pmatrix}$$

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