Third Hankel Determinant Problem for Certain Subclasses of Analytic Functions Associated with Nephroid Domain

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Abstract

In this research article we consider two well known subclasses of starlike and bounded turning functions associated with nephroid domain. Our aims to find third Hankel determinant for these classes.

1 Introduction and Definitions

Let \( \mathcal{A} \) be the collections of all normalized analytic functions defined in the unit disc \( \mathcal{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{D}).
\]  

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Let us denote the most basic, well known and important subclass of class $A$ by $S$ which consists of all univalent functions in $D$. Since in the early stage of 20th century, researchers have been interested in coefficients of function $f$ in class $A$. In year 1916, Bieberbach see [1] was first to discover a coefficient conjecture for the function $f \in S$ and finally in year 1985 De-Branges [2] solved Bieberbach coefficient conjecture. In era 1916 to 1985 many researcher have tried to prove or disprove this conjecture and they discovered different subclasses of the class $S$ associated with different image domains. The definition of class $S^*$, $C$ and $R$ which are subclasses of class $S$ can be written in terms of subordination

$$S^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} < \frac{1+z}{1-z}, \quad z \in D \right\},$$

$$C = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} < \frac{1+z}{1-z}, \quad z \in D \right\},$$

$$R = \left\{ f \in S : f'(z) < \frac{1+z}{1-z}, \quad z \in D \right\},$$

(1.2)

where “≺” represent subordination. Two analytic functions $f$ and $g$, a function $f$ is subordinate to $g$ symbolically $f \prec g$ if there exist an analytic function $w(z)$ with limitation $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. If $g \in S$, then equivalence conditions

$$f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D).$$

Let $P$ denote class of all analytic functions $p$ such that $\text{Re}(p(z)) > 0$, and of the form

$$p(z) = z + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in D).$$

(1.3)

By changing the function right hand side of subordinations in (1.2), we obtain some subclasses of the class $S$ which have interesting geometric properties, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. From among these subfamilies we recall here

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the families that are associated with trigonometric function as follows:

\[ S^*_N e = \left\{ f \in S : \frac{zf'(z)}{f(z)} \preceq 1 + z - \frac{1}{3} z^3, \quad (z \in D) \right\}, \quad (z \in D). \quad (1.4) \]

\[ S^*_N e = \left\{ f \in S : f'(z) \preceq 1 + z - \frac{1}{3} z^3 \right\}, \quad (z \in D). \quad (1.5) \]

Recently, authors in [18], introduced the class \( S^*_N e \) which are associated with nephroid domain.

The Hankel determinant \( H_{q,n}(f) \) where parameters \( q, n \in \mathbb{N} = \{1, 2, 3, \ldots \} \) for function \( f \in S \) of the form (1.1) was introduced by Pommerenke [14, 15] as;

\[ H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (1.6) \]

The growth of \( H_{q,n}(f) \) has been evaluated for different subcollections of univalent functions. Exceptionally, the sharp bound of the determinant \( H_{2,2}(f) = |a_2a_4 - a_3^2| \) for class \( S^*, \mathcal{C} \) and \( \mathcal{R} \) were found by Janteng et al. [16, 17] while for the family of close-to-convex functions the sharp estimation is still unknown (see, [19]). On the other hand, for the set of Bazilevič functions, the best estimate of \( |H_{2,2}(f)| \) was proved by Krishna et al. [20]. For more work on \( H_{2,2}(f) \), see [21, 22, 23, 24, 25].

\[ H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \quad (1.7) \]

is known as third order Hankel determinant and the estimation of this determinant \( |H_{3,1}(f)| \) is so hard. In 2010, the first article on \( H_{3,1}(f) \) by Babalola [26], in which he obtained the upper bound of \( |H_{3,1}(f)| \) for the groups of \( S^*, \mathcal{K} \) and \( \mathcal{R} \). Later on, a few creators distributed their work regarding \( |H_{3,1}(f)| \) for various subcollections of holomorphic and univalent functions, see [34, 35, 36, 37, 38], which served as a base for the research in this field. Recently various authors explored some interesting classes for the said property of Hankel determinant.
Srivastava et al. [27] discussed this result for a class of Bi-valent functions defined by \(q\)-derivative and gave various interesting properties of it. Then he along with coauthors in [28] investigated the class of close to convex functions associated with lemniscate of Bernoulli and evaluated its Hankel determinant. Continuing the same trend he in [29] incorporated the research on Toeplitz forms and Hankel determinant for some \(q\)-starlike functions associated with a generalized domain. Many other domains were also investigated for its Hankel determinant like a class of starlike functions associated with \(k\)-Fibonacci numbers. Whose third Hankel was evaluated by Shafiq et al. [30]. Further related work on the subject the reader is referred to [31, 32, 33]. Motivated from above discussed work on the topic we investigate \(|H_{3,1}(f)|\) for classes of functions defined in the relations (1.4) and (1.5).

2 Sets of Lemma

The following lemmas are important as they help in our main results.

Lemma 1. If \(p(z) \in \mathcal{P}\) and it is of the form (1.3), then

\[
|c_n| \leq 2 \text{ for } n \geq 1, \quad (2.1)
\]

\[
|c_{n+k} - \delta c_n c_k| \leq \begin{cases} 2 & \text{for } 0 \leq \delta \leq 1, \\ 2|2\delta - 1| & \text{elsewhere.} \end{cases}, \quad (2.2)
\]

\[
|c_n c_m - c_l c_k| \leq 4 \text{ for } n + m = l + k, \quad (2.3)
\]

\[
|c_{n+2k} - \delta c_n c_k^2| \leq 2(1 + 2\delta) \text{ for } \delta \in \mathbb{R}, \quad (2.4)
\]

\[
|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}, \quad (2.5)
\]

and for \(\xi \in \mathbb{C}\)

\[
|c_2 - \xi c_1^2| \leq 2 \max \{1; |2\xi - 1|\}. \quad (2.6)
\]

For the results in (2.1), (2.2), (2.3), (2.4), (2.5) see [44]. Also see [43] for (2.6).
Lemma 2. \([46]\). If \(p(z) \in \mathcal{P}\) and is represented by \((1.3)\), then

\[
\left| c_2 - \nu c_1^2 \right| \leq \begin{cases} 
-4\nu + 2 & (\nu \leq 0), \\
2 & (0 \leq \nu \leq 1), \\
4\nu - 2 & (\nu \geq 1).
\end{cases}
\]

Lemma 3. \([47]\). If \(p(z) \in \mathcal{P}\) and is represented by \((1.3)\), then

\[
\left| ac_3^3 - bc_1 c_2 + dc_3 \right| \leq 2 |a| + 2 |b - 2a| + 2 |a - b + d|.
\]

3 Bounds of \(|H_{3,1}(f)|\) for class \(\mathcal{S}_{Ne}^*\)

Theorem 1. Let \(f \in \mathcal{S}_{Ne}^*\) of the form \((1.1)\). Then

\[
\begin{align*}
|a_2| & \leq 1, \\
|a_3| & \leq \frac{1}{2}, \\
|a_4| & \leq \frac{7}{18}, \\
|a_5| & \leq \frac{5}{12}. \\
\end{align*}
\]

The first two bounds are sharp.

Proof. Since \(f \in \mathcal{S}_{Ne}^*\), there exists an analytic function \(s(z), |s(z)| \leq 1\) and \(s(0) = 0\), such that

\[
\frac{zf'(z)}{f(z)} = 1 + w(z) - \frac{1}{3} (w(z))^3. \tag{3.1}
\]

Denote

\[
\Psi (s(z)) = 1 + w(z) - \frac{1}{3} (w(z))^3,
\]

and

\[
k(z) = 1 + c_1 z + c_2 z^2 + \cdots = \frac{1 + w(z)}{1 - w(z)}. \tag{3.2}
\]

Obviously, the function \(k(z) \in \mathcal{P}\), and \(w(z) = \frac{k(z) - 1}{k(z) + 1}\). This gives

\[
w(z) = \frac{k(z) - 1}{k(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.
\]
And
\[
1 + (w(z)) - \frac{1}{3} (w(z))^3
= 1 + \frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{12} c_1^3 - \frac{1}{2} c_2 c_1 + \frac{1}{2} c_3\right) z^3 \\
+ \left(\frac{1}{4} c_1^2 c_2 - \frac{1}{2} c_3 c_1 - \frac{1}{4} c_2^2 + \frac{1}{2} c_4\right) z^4 + \cdots. \tag{3.3}
\]

And other side,
\[
f'(z) = f(z) = 1 + a_2 z + (2a_3 - a_2^2) z^2 + (a_2^3 - 3a_2 a_3 + 3a_4) z^3 + \\
(-a_4^2 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5) z^4 + \cdots. \tag{3.4}
\]

On equating coefficients of (3.3), and (3.4), we get
\[
a_2 = \frac{1}{2} c_1, \tag{3.5}
a_3 = \frac{1}{4} c_2, \tag{3.6}
a_4 = -\frac{1}{72} c_1^3 - \frac{1}{24} c_2 c_1 + \frac{1}{6} c_3, \tag{3.7}
a_5 = \frac{5}{576} c_1^4 - \frac{1}{48} c_1^2 c_2 - \frac{1}{24} c_3 c_1 - \frac{1}{32} c_2^2 + \frac{1}{8} c_4. \tag{3.8}
\]

Now using (2.1) in (3.5) and (3.6), we get
\[
|a_2| \leq 1 \text{ and } |a_3| \leq \frac{1}{2}.
\]

Rearrange the equation (3.7), we may write
\[
|a_4| = \left| \frac{1}{12} \left( c_3 - \frac{1}{2} c_1 c_2 \right) + \frac{1}{12} \left( c_3 - \frac{1}{6} c_1^3 \right) \right|. \tag{3.7}
\]

Using triangle inequality along with (2.2) and (2.4), we get
\[
|a_4| \leq \frac{7}{18}.
\]

Now rearrange the (3.8), we may write
\[
|a_5| = \left| \frac{1}{16} \left( c_4 - \frac{2}{3} c_1 c_3 \right) + \frac{1}{16} \left( c_4 - \frac{1}{2} c_2^2 \right) - \frac{c_1^2}{48} \left( c_2 - \frac{5}{12} c_1^2 \right) \right|.
\]

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Application of triangle inequality along with (2.1) and (2.2), we get

\[ |a_5| \leq \frac{5}{12}. \]

The first two bounds are sharp for function define as

\[ f(z) = z \exp \left( z - \frac{z^3}{9} \right) = z + z^2 + \frac{z^3}{2} + \cdots. \]

**Theorem 2.** Let \( f(z) \in S_{N_e}^* \) be of the form (1.1). Then

\[ |a_3 - \lambda a_2^2| \leq \begin{cases} 
\frac{1 - 2\lambda}{2}, & \lambda \leq 0 \\
\frac{1}{2}, & 0 \leq \lambda \leq 1 \\
\frac{2\lambda - 1}{2}, & \lambda \geq 1.
\end{cases} \]

**Proof.** Since using (3.5) and (3.6), we get

\[ |a_3 - \lambda a_2^2| = \frac{1}{4} |c_2 - \lambda c_1^2|, \]

applying Lemma 2 we get the required results. \(\square\)

**Theorem 3.** Let \( f(z) \in S_{N_e}^* \) be of the form (1.1). Then for \( \xi \in \mathbb{C} \), we have

\[ |a_3 - \xi a_2^2| \leq \frac{1}{2} \max \{1, |2\xi - 1|\}. \]

**Proof.** Since using (3.5) and (3.6), we get

\[ |a_3 - \xi a_2^2| = \frac{1}{4} |c_2 - \xi c_1^2|, \]

applying Lemma 2.6 we get the required results. \(\square\)

If we put \( \xi = 1 \), the above result become as:

**Theorem 4.** Let \( f(z) \in S_{N_e}^* \) be of the form (1.1). Then

\[ |a_3 - a_2^2| \leq \frac{1}{2}. \]

This result is sharp.
Theorem 5. Let $f(z) \in S_{N_{c}}^{*}$ be of the form (1.1). Then

$$|a_{2a_{3} - a_{4}}| \leq \frac{7}{18}.$$  

Proof. Since using (3.5), (3.6) and (3.7) also rearrangement term, we get

$$|a_{2a_{3} - a_{4}}| = \left| \frac{1}{12} (c_{3} - c_{2}c_{1}) + \frac{1}{12} \left( c_{3} - \frac{1}{6} c_{1}^{3} \right) \right|,$$

Using triangle inequality along with (2.2) and (2.4), we get

$$|a_{2a_{3} - a_{4}}| \leq \frac{7}{18}.$$  

\[ \square \]

Theorem 6. Let $f(z) \in S_{N_{c}}^{*}$ be of the form (1.1). Then

$$|a_{2a_{4} - a_{3}^{2}}| \leq \frac{4}{9}.$$  

Proof. Since using (3.5), (3.6) and (3.7), we get

$$|a_{2a_{4} - a_{3}^{2}}| = \left| - \frac{1}{144} c_{1}^{4} - \frac{1}{48} c_{2}^{2} c_{2} + \frac{1}{12} c_{3} c_{1} - \frac{1}{16} c_{2}^{2} \right|$$

$$= \left| \frac{1}{16} (c_{1} c_{3} - c_{2}^{2}) + \frac{1}{48} c_{1} (c_{3} - c_{1} c_{2}) - \frac{1}{144} c_{1}^{4} \right|,$$

applying (2.1), (2.2) and (2.3), we get the required result.  

\[ \square \]

Theorem 7. Let $f(z) \in S_{N_{c}}^{*}$ be of the form (1.1). Then

$$|H_{3,1} (f)| \leq \frac{377}{648} \simeq 0.58179.$$  

Proof. Since

$$H_{3,1} (f) = a_{3} (a_{2} a_{4} - a_{3}^{2}) - a_{4} (a_{4} - a_{2} a_{3}) + a_{5} (a_{3} - a_{2}^{2}),$$

by applying triangle inequality, we obtain

$$|H_{3,1} (f)| \leq |a_{3}| \left| a_{2} a_{4} - a_{3}^{2} \right| + |a_{4}| \left| a_{4} - a_{2} a_{3} \right| + |a_{5}| \left| a_{3} - a_{2}^{2} \right|.$$  

Next, from Theorems 4, 5 and 6 we get the required result.  

\[ \square \]
4 Bounds of $|H_{3,1}(f)|$ for class $\mathcal{R}_{Ne}$

**Theorem 8.** Let $f \in \mathcal{R}_{Ne}$ of the form \[1.1\]. Then

\[
\begin{align*}
|a_2| & \leq \frac{1}{2}, \\
|a_3| & \leq \frac{1}{3}, \\
|a_4| & \leq \frac{1}{4}, \\
|a_5| & \leq \frac{3}{5}.
\end{align*}
\]

**Proof.** Since $f \in \mathcal{R}_{Ne}$, there exists an analytic function $w(z)$, $|w(z)| \leq 1$ and $w(0) = 0$, such that

\[f'(z) = 1 + w(z) - \frac{1}{3} (w(z))^3.\]

And

\[f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 3a_4 z^3 + 4a_5 z^4 + \cdots. \tag{4.1}\]

On equating coefficients of \[3.3\] and \[4.1\], we get

\[
\begin{align*}
a_2 &= \frac{1}{4} c_1, \tag{4.2} \\
a_3 &= \frac{1}{6} c_2 - \frac{1}{12} c_1^2, \tag{4.3} \\
a_4 &= \frac{1}{48} c_3^2 - \frac{1}{8} c_2 c_1 + \frac{1}{8} c_3, \tag{4.4} \\
a_5 &= \frac{1}{20} c_1^2 c_2 - \frac{1}{10} c_3 c_1 - \frac{1}{20} c_2^2 + \frac{1}{10} c_4. \tag{4.5}
\end{align*}
\]

Now using \[2.1\] to \[4.2\], we get

\[|a_2| \leq \frac{1}{2}.\]

Using \[2.5\] to \[4.3\], we obtain

\[|a_3| \leq \frac{1}{3}.\]
Application of Lemma 3 to (4.4), lead us to

\[ |a_4| \leq \frac{1}{4}. \]

Rearranging the (4.5), we have

\[ |a_5| = \left| \frac{1}{10}\left(c_4 - \frac{1}{2}c_2^2\right) - \frac{c_1}{10}\left(c_3 - \frac{1}{2}c_1c_2\right) \right|, \]

applying (2.1) and (2.2), we get

\[ |a_5| \leq \frac{3}{5}. \]

\[ \square \]

**Theorem 9.** Let \( f(z) \in \mathcal{R}_N \) be of the form (1.1). Then for \( \xi \in \mathbb{C} \), we have

\[ |a_3 - \xi a_2^2| \leq \frac{1}{3} \max\left\{1, \frac{3|\xi|}{4}\right\}. \]

**Proof.** Since using (4.2) and (4.3), we get

\[ |a_3 - \xi a_2^2| = \frac{1}{6}\left|c_2 - 4 + 3\xi c_1^2\right|, \]

application of relation (2.6), we get the required results. \( \square \)

If we put \( \xi = 1 \), the above result become as:

**Theorem 10.** Let \( f(z) \in \mathcal{R}_N \) be of the form (1.1). Then

\[ |a_3 - a_2^2| \leq \frac{1}{3}. \]

**Theorem 11.** Let \( f(z) \in \mathcal{R}_N \) be of the form (1.1). Then

\[ |a_2a_3 - a_4| \leq \frac{1}{4}. \]

**Proof.** Since using (4.2), (4.3) and (4.4), we get

\[ |a_2a_3 - a_4| = \left|\frac{1}{24}c_1^3 - \frac{1}{6}c_2c_1 + \frac{1}{8}c_3\right|, \]

using Lemma 3, we have

\[ |a_2a_3 - a_4| \leq \frac{1}{4}. \]

\( \square \)
Theorem 12. Let \( f(z) \in \mathcal{R}_N \) be of the form (1.1). Then
\[
|a_2a_4 - a_3^2| \leq \frac{11}{72}.
\]

Proof. Since using (4.2), (4.3) and (4.4), we get
\[
|a_2a_4 - a_3^2| = \left| -\frac{1}{576} c_1^4 - \frac{1}{288} c_1^2 c_2 + \frac{1}{32} c_3 c_1 - \frac{1}{36} c_2^2 \right|
= \left| \frac{1}{36} (c_1^2 - c_2^2) + \frac{1}{288} c_1 (c_3 - c_1 c_2) - \frac{1}{576} c_4^4 \right|,
\]
an application of triangle inequality along with (2.1), (2.2) and (2.3), we get the required result.

Theorem 13. Let \( f(z) \in \mathcal{R}_N \) be of the form (1.1). Then
\[
|H_{3,1}(f)| \leq \frac{677}{2160} \approx 0.31343.
\]

Proof. Since
\[
H_{3,1}(f) = a_3 \left( a_2a_4 - a_3^2 \right) - a_4 \left( a_4 - a_2a_3 \right) + a_5 \left( a_3 - a_2^2 \right),
\]
by applying triangle inequality, we obtain
\[
|H_{3,1}(f)| \leq |a_3| |a_2a_4 - a_3^2| + |a_4| |a_4 - a_2a_3| + |a_5| |a_3 - a_2^2|.
\]
Next, from Theorems 8, 10, 11 and 12 we get the required result.

Authors contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Conflict of Interest
The authors declare that there is no conflict of interests.
References

[1] L. Bieberbach, Uber die koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermittein, Sitzungsberichte Preussische Akademie der Wissenschaften 138 (1916), 940-955.

[2] L. De Branges, A proof of the Bieberbach conjecture, *Acta Math.* 154 (1985), 137-152. [https://doi.org/10.1007/BF02392821](https://doi.org/10.1007/BF02392821)

[3] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* 23 (1970), 159-177. [https://doi.org/10.4064/ap-23-2-159-177](https://doi.org/10.4064/ap-23-2-159-177)

[4] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* 19 (1996), 101-105.

[5] R.K. Raina and J. Sokół, Some properties related to a certain class of starlike functions, *C. R. Math. Acad. Sci. Paris* 353(11) (2015), 973-978. [https://doi.org/10.1016/j.crma.2015.09.011](https://doi.org/10.1016/j.crma.2015.09.011)

[6] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, *Bull. Malays. Math. Sci. Soc.* 38 (2015), 365-386. [https://doi.org/10.1007/s40840-014-0026-8](https://doi.org/10.1007/s40840-014-0026-8)

[7] S. Kumar and V. Ravichandran, A subclass of starlike functions associated with a rational function, *Southeast Asian Bull. Math.* 40 (2016), 199-212.

[8] N.E. Cho, V. Kumar, S.S. Kumar and V. Ravichandran, Radius problems for starlike functions associated with the Sine function, *Bull. Iran. Math. Soc.* 45 (2019), 213-232. [https://doi.org/10.1007/s41980-018-0127-5](https://doi.org/10.1007/s41980-018-0127-5)

[9] R. Kargar, A. Ebadian and J. Sokół, Radius problems for some subclasses of analytic functions, *Complex Anal. Oper. Theory* 11 (2017), 1639-1649. [https://doi.org/10.1007/s11785-016-0584-x](https://doi.org/10.1007/s11785-016-0584-x)

[10] R. Kargar, A. Ebadian and J. Sokół, On Booth lemniscate and starlike functions, *Anal. Math. Phys.* 9 (2019), 143-154. [https://doi.org/10.1007/s13324-017-0187-3](https://doi.org/10.1007/s13324-017-0187-3)

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On Analytic Functions Associated with Nephroid Domain

[11] J. Dziok, R.K. Raina and J. Sokół, On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers, *Math. Comput. Model.* 57 (2013), 1203-1211. [https://doi.org/10.1016/j.mcm.2012.10.023](https://doi.org/10.1016/j.mcm.2012.10.023)

[12] N.E. Cho, S. Kumar, V. Kumar, V. Ravichandran and H.M. Srivastava, Starlike functions related to the Bell numbers, *Symmetry* 11 (2019), 219. [https://doi.org/10.3390/sym11020219](https://doi.org/10.3390/sym11020219)

[13] S. Mahmood, H.M. Srivastava and S.N. Malik, Some subclasses of uniformly univalent functions with respect to symmetric points, *Symmetry* 11 (2019), 287. [https://doi.org/10.3390/sym11020287](https://doi.org/10.3390/sym11020287)

[14] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.* 41 (1966), 111-122. [https://doi.org/10.1112/jlms/s1-41.1.111](https://doi.org/10.1112/jlms/s1-41.1.111)

[15] Ch. Pommerenke, On the Hankel determinants of univalent functions, *Mathematika* 14 (1967), 108-112. [https://doi.org/10.1112/S002557930000807X](https://doi.org/10.1112/S002557930000807X)

[16] A. Jangteng, S.A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Ineq. Pure Appl. Math.* 7 (2006), 1-5.

[17] A. Jangteng, S.A. Halim and M. Darus, Coefficient inequality for starlike and convex functions, *Int. J. Ineq. Math. Anal.* 1 (2007), 619-625.

[18] L.A. Wani and A. Swaminathan, Starlike and convex functions associated with a Nephroid domain having cusps on the real axis, 2019. arXiv.1912.05767v1

[19] D. Răducanu and P. Zaprawa, Second Hankel determinant for close-to-convex functions, *C. R. Math. Acad. Sci. Paris* 355(10) (2017), 1063-1071. [https://doi.org/10.1016/j.crma.2017.09.006](https://doi.org/10.1016/j.crma.2017.09.006)

[20] D. V. Krishna and T. RamReddy, Second Hankel determinant for the class of Bazilevic functions, *Stud. Univ. Babes-Bolyai Math.* 60 (2015), 413-420.

[21] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, *Appl. Math. Lett.* 23 (2013), 103-107. [https://doi.org/10.1016/j.aml.2012.04.002](https://doi.org/10.1016/j.aml.2012.04.002)
[22] S.K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Inequal. Appl. 2013, 2013:281, 17 pp. [https://doi.org/10.1186/1029-242X-2013-281]

[23] M. S. Liu, J. F. Xu and M. Yang, Upper bound of second Hankel determinant for certain subclasses of analytic functions, Abstr. Appl Anal. 2014, Art. ID 603180, 10 pp. [https://doi.org/10.1155/2014/603180]

[24] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, Trans. Amer. Math. Soc. 223 (1976), 337-346. [https://doi.org/10.2307/1997533]

[25] H. Orhan, N. Magesh and J. Yamini, Bounds for the second Hankel determinant of certain bi-univalent functions, Turk. J. Math. 40 (2016), 679-687. [https://doi.org/10.3906/mat-1505-3]

[26] K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequal. Theory Appl. 6 (2010), 1-7.

[27] H. M. Srivastava, S. Altinkaya and S. Yalçin, Hankel determinant for a subclass of bi-valent functions defined by using a symmetric $q$-derivative operator, Filomat 32 (2018), 503-516. [https://doi.org/10.2298/FIL1802503S]

[28] H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman and H. H. Shah, Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with Lemniscate of Bernouli, Mathematics 7(9) (2019), 848. [https://doi.org/10.3390/math7090848]

[29] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan and B. Khan, Hankel and Toeplitz determinant for a subclass of $q$-starlike functions associated with a general conic domain, Mathematics 7(2) (2019), 181. [https://doi.org/10.3390/math7020181]

[30] M. Shafiq, H. M. Srivastava, N. Khan, Q. Z. Ahmad, M. Darus and S. Kiran, An upper bound of the third Hankel determinant for a subclass of $q$-starlike functions associated with $k$-Fibonacci numbers, Symmetry 12(6) (2020), 1043. [https://doi.org/10.3390/sym12061043]

[31] L. Shi, M. G. Khan and B. Ahmad, Some geometric properties of a family of analytic functions involving a generalized q-operator, Symmetry 12(2) (2020), 291. [https://doi.org/10.3390/sym12020291]
[32] M.G. Khan, B. Ahmad, G. Muraugusundaramoorthy, R. Chinram and W. K. Mashwani, Applications of modified sigmoid functions to a class of starlike functions, *J. Funct. Spaces* 2020, Art. ID 8844814, 8 pp.  
https://doi.org/10.1155/2020/8844814

[33] M.G. Khan, B. Ahmad, J. Sokol and Z. Muhammad, Coefficient problems in a class of functions with bounded turning associated with Sine function, *European Journal of Pure and Applied Mathematics* 14 (2021), 53-64.  
https://doi.org/10.29020/nybg.ejpam.v14i1.3902

[34] S. Altinkaya and S. Yalçın, Third Hankel determinant for Bazilević functions, *Advances in Math.* 5 (2016), 91-96.

[35] D. Bansal, S. Maharana and J. K. Prajapat, Third order Hankel determinant for certain univalent functions, *J. Korean Math. Soc.* 52 (2015), 1139-1148.  
https://doi.org/10.4134/JKMS.2015.52.6.1139

[36] N.E. Cho, B. Kowalczyk, O.S. Kwon, A. Lecko and J. Sim, Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha, *J. Math. Inequal.* 11 (2017), 429-439.  
https://doi.org/10.7153/jmi-2017-11-36

[37] D.V. Krishna, B. Venkateswarhua and T. RamReddy, Third Hankel determinant for bounded turning functions of order alpha, *J. Niger. Math. Soc.* 34 (2015), 121-127.  
https://doi.org/10.1016/j.jnmms.2015.03.001

[38] G. Shanmugam, B.A. Stephen and K.O. Babalola, Third Hankel determinant for α-starlike functions, *Gulf J. Math.* 2 (2014), 107-113.

[39] P. Zaprawa, Third Hankel determinants for subclasses of univalent functions, *Mediterr. J. Math.* 14(1) (2017), Paper No. 19, 10 pp.  
https://doi.org/10.1007/s00009-016-0829-y

[40] B. Kowalczyk, A. Lecko and Y.J. Sim, The sharp bound of the Hankel determinant of the third kind for convex functions, *Bull. Australian Math. Soc.* 97 (2018), 435-445.  
https://doi.org/10.1017/S0004972717001125

[41] A. Lecko, Y.J. Sim and B. Śmiarowska, The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2, *Complex Anal. Oper. Theory* 13(5) (2019), 2231-2238.  
https://doi.org/10.1007/s11785-018-0819-0
[42] O.S. Kwon, A. Lecko and Y.J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, *Bull. Malays. Math. Sci. Soc.* 42 (2018), 1-14. 
https://doi.org/10.1007/s40840-018-0683-0

[43] C. Pommerenke, *Univalent Functions*, Math. Lehrbucher, Band XXV, Vandenhoeck Ruprecht, Gottingen, 1975.

[44] F. Keogh and E. Merkes, A coefficient inequality for certain subclasses of analytic functions, *Proc. Amer. Math. Soc.* 20 (1969), 8-12.
https://doi.org/10.1090/S0002-9939-1969-0232926-9

[45] V. Ravichandran and S. Verma, Bound for the fifth coefficient of certain starlike functions, *C. R. Math. Acad. Sci. Paris* 353(6) (2015), 505-510.
https://doi.org/10.1016/j.crma.2015.03.003

[46] W.C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), 157-169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[47] M. Arif, M. Raza, H. Tang, S. Hussain and H. Khan, Hankel determinant of order three for familiar subsets of analytic functions related with sine functions, *Open Mathematics* 17 (2019), 1615-1630. https://doi.org/10.1515/math-2019-0132

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