THE $v_1$-PERIODIC REGION OF THE C-MOTIVIC Ext

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ABSTRACT. We establish a $v_1$-periodicity theorem in Ext over the C-motivic Steenrod algebra. The element $h_1$ of Ext, which detects the homotopy class $\eta$ in the motivic Adams spectral sequence, is non-nilpotent and therefore generates $h_1$-towers. Our result is that, apart from these $h_1$-towers, $v_1$-periodicity behaves as it does classically.

1. INTRODUCTION

1.1. Background and Motivation. One of the primary tools for computing stable homotopy groups of spheres is the Adams spectral sequence. The $E_2$-page of the Adams spectral sequence is given by $\text{Ext}^{*,*}_{A^{cl}}(F_2, F_2) = H^{*,*}(A^{cl})$, denoted by $\text{Ext}^{a}$, where $A^{cl}$ is the classical Steenrod algebra. For $\text{Ext}^{a}$, Adams showed that there is a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$, and J. P. May showed there is a periodicity line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, where the periodicity operation is defined by the Massey product $P_r(\cdot) := \langle h_{r+1}, h_0^r, \cdot \rangle$. This result has not been published by May, but can be found in the thesis of Krause:

Theorem 1.1. [Kra, Theorem 5.14] For $r \geq 2$, the Massey product operation $P_r(\cdot) := \langle h_{r+1}, h_0^r, \cdot \rangle$ is uniquely defined on $\text{Ext}^{e,f}_{A^{cl}} = H^{e,f}(A^{cl})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$, where $s$ is the stem, $f$ is the Adams filtration.

Furthermore, for $f > \frac{1}{2}s + \frac{12}{5}$,

$$P_r : H^{e,f}(A^{cl}) \cong H^{s+2^{r+1}f+2^r}(A^{cl})$$

is an isomorphism.

The purpose of this article is to discuss an analog of the theorem above in the C-motivic context. Motivic homotopy theory, also known as $A^1$-homotopy theory, is a way to apply the techniques of algebraic topology, specifically homotopy, to algebraic varieties and, more generally, to schemes. The theory was formulated by Fabien Morel and Vladimir Voevodsky [MV].

In this paper we analyze the case where the base field $F$ is the complex numbers $C$. Let $M_2$ denote the bigraded motivic cohomology ring of Spec $C$, with $F_2 = \mathbb{Z}/2$-coefficients (thus $M_2 \cong F_2[\tau]$ due to Voevodsky [Voe]). Let $A$ be the mod 2 motivic Steenrod algebra over $C$. The motivic Adams spectral sequence is a tri-graded spectral sequence with

$$E_2^{e,w,s} = \text{Ext}^{w,s}_{A}(M_2, M_2)$$

(See more in Dugger and Isaksen’s paper [DI]), the extra grading $w$ is the motivic weight. The analogous vanishing line in this motivic $E_2$-page, denoted by Ext, was computed by Guillou and Isaksen [GI1]. JD. Quigley has a partial result for the motivic periodicity theorem in the case $r = 2$.
There are close connections between the classical Adams spectral sequence and the motivic one. For instance, by inverting $\tau$ in Ext, we obtain $\text{Ext}^f$. Let $C(\eta)$ denote the cofiber of the first Hopf map

$$S^{1,1} \eta \rightarrow S^{0,0}.$$  

Writing $C_\eta$ for the cohomology $H^{s,t}(C(\eta))$, we have the following result:

**Theorem 1.2.** [GI1, Theorem 1.1] The group $\text{Ext}^6,f,iv(\mathcal{M}_2, C_\eta)$ vanishes when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$.

The multiplication by $2$ map $S^{0,0} \rightarrow S^{0,0}$ is detected by $h_0$ and the Hopf map $\eta$ is detected by $h_1$ in Ext. They have degrees $(0, 1, 0)$ and $(1, 1, 1)$ respectively. Since all infinite $h_1$-towers are $\tau$-torsion, one would roughly guess that the motivic Ext groups differ from the classical $\text{Ext}^f$ groups by those infinite $h_1$-towers, which is not literally true, but it still makes sense to try to consider only the $h_1$-torsion part in Ext for our purposes. We will write $h_1$-towers for infinite $h_1$-towers, and a short discussion on the $h_1$-towers can be found in subsection 1.2.

**Remark 1.3.** For Ext, we can work over $\mathcal{A}^V$ instead of $\mathcal{A}$. i.e.

$$E^{s,s}_2 = \text{Ext}^{s,s}_V(\mathcal{M}_2, \mathcal{M}_2)^V.$$  

Here we consider $\mathcal{M}_2$ as the homology of the motivic sphere instead of the cohomology, which is an $\mathcal{A}^V$-comodule.

The goal of this paper is the following theorem:

**Theorem 1.4.** For $r \geq 2$, the Massey product operation $P_r(\cdot) := \langle h_{r+1}, h_0^{2r}, \cdot \rangle$ is uniquely defined on $\text{Ext} = H^s,f,iv(\mathcal{A})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2r$.

Furthermore, for $f > \frac{1}{2}s + \frac{12}{5}$, the restriction to the $h_1$-torsion

$$P_r : [H^s,f,iv(\mathcal{A})]_{h_1-\text{torsion}} \rightarrow [H^{s+2r+1},f+2r,iv+2r(\mathcal{A})]_{h_1-\text{torsion}}$$

is an isomorphism.

There are abundant connections between the $\mathcal{C}$-motivic Ext groups, the $\mathcal{R}$-motivic Ext groups and the $\mathcal{C}_2$-equivariant Ext groups. The $\rho$-Bockstein spectral sequence [Hil] is taking the $\mathcal{C}$-motivic Ext groups as input and lands in the $\mathcal{R}$-motivic Ext groups. The $\mathcal{C}_2$-equivariant Ext groups can then be obtained [GHIR] by calculating $R$-motivic Ext groups for a negative cone. Our periodicity results ought to be relevant for future computations in $\mathcal{R}$-motivic and $\mathcal{C}_2$-equivariant homotopy theory.

1.2. **Further Considerations.** The $h_1$-towers part is considered in [GI2], in which all the "positions" of the $h_1$-towers are computed.

**Theorem 1.5.** [GI2, Theorem 1.1] The $h_1$-inverted algebra $\text{Ext}_A[h_1^{-1}]$ is a polynomial algebra over $\mathbb{F}_2[h_1^{+1}]$ on generators $v_1^4$ and $v_n$ for $n \geq 2$, where:

1. $v_1^4$ is the $8$-stem and has Adams filtration $4$ and weight $4$.
2. $v_n$ is in the $(2^{n+1} - 2)$-stem and has Adams filtration $1$ and weight $2^n - 1$.  

[Qui, Corollary 5.4].
It is straightforward that $P_r$ acts injectively on the $h_1$-inverted Ext, that is to say, $P_r$ will eventually send an $h_1$-tower to another $h_1$-tower. But the starting point might not be sent to the starting point of another $h_1$-tower. As to the surjectivity, there are $h_1$-towers not in the image of $P_r$ (those are not multiples of $\nu_1^4$ in the $h_1$-inverted Ext), for instance the $h_1$-tower on $c_0$. Partial results about where those infinite $h_1$-towers get started can be found in [Tha], and with more information revealed we are expecting to have a full understanding on where multiplying by the $\nu_1$-periodicity element is an isomorphism in $\text{Ext}$.

There is another family called $\omega_1$-periodicity in motivic $\text{Ext}$, which does not exist classically. Parallel to the Massey product $P_2(-) := \langle h_3, h_0^4 \rangle$ detected by $P = h_2^1$ at $(8,4,4)$, there is another Massey product $g(-) := \langle h_4, h_1^4 \rangle$ detected by $g = h_2^1$ at $(20,4,12)$. The obstruction is $g$ has a relatively low slope. Thus the method in this paper is not very applicable. Plus one needs to start at least at $\mathcal{A}(2)^\vee$ to compute $g$-periodicity since $g$ restricts to zero in $\text{Ext}_{\mathcal{A}(1)^\vee}$.

1.3. **Organization.** We mainly follow the approach of [Kra]. In Section 2, we briefly introduce the stable (co)module category, in which we can consider the $h_0$ or $h_1$-torsion part of Ext by taking sequential colimits. In Section 3, we establish the existence of a homological self-map $\theta$ and use this to show that $P_r(-)$ is uniquely defined. In Section 4, we explicitly show where $\theta$ is an isomorphism in $\mathcal{A}(1)^\vee$, and get where it is an isomorphism in $\mathcal{A}^\vee$ by moving along the Cartan-Eilenberg spectral sequence. And in Section 5, we put the results in the previous two sections together to get the motivic periodicity theorem 1.4.

2. **Working environment: the Stable (co)module Category $\text{Stab}(\Gamma)$**

To consider only the $h_1$-torsion (also $h_0$-torsion) part, first we would like to choose a suitable working environment: a category with some nice properties that will do the job. Usually Ext is defined in the derived category of $\mathcal{A}^\vee$-comodules, denoted $D(\mathcal{A}^\vee)$. But the coefficient $M_2$ is not compact in $D(\mathcal{A}^\vee)$, which means that it does not interact well with colimits. In order to fix this, we introduce the stable (co)module category. That is a category $\mathcal{C}$ such that:

- If $M$ and $N$ are $\mathcal{A}^\vee$-comodules that are free of finite rank over $M_2$, then $\text{Hom}_{\mathcal{C}}(M, N) \cong \text{Ext}_{\mathcal{A}^\vee}(M, N)$;
- Furthermore $M$ is compact, that is to say for any sequential colimit of $\mathcal{A}^\vee$-comodules in $\mathcal{C}$
  \[
  \operatorname{colim}_i N_i := \operatorname{colim}(N_0 \xrightarrow{f_0} N_1 \to \cdots \to N_i \xrightarrow{f_i} \cdots)
  \]
  we have $\operatorname{colim}_i \text{Ext}_{\mathcal{A}^\vee}(M, N_i) \cong \text{Hom}_{\mathcal{C}}(M, \operatorname{colim}_i N_i)$.

The correct choice of $\mathcal{C}$ is called $\text{Stab}(\mathcal{A}^\vee)$. It can be constructed in various ways (see [Bel, Sec. 2.1] for details). Now let us take a look at the nice properties this category has. The following proposition summarizes some of the discussion in [BHV, Sec. 4]:

**Proposition 2.1.** If $M$ and $N$ are $\mathcal{A}^\vee$-comodules that are free of finite rank over $M_2$, then in $\text{Stab}(\mathcal{A}^\vee)$, 

\[
\text{Ext}_{\mathcal{A}^\vee}(M, N) \cong \text{Hom}_{\text{Stab}(\mathcal{A}^\vee)}(M, N).
\]

Moreover $M$ is compact in $\text{Stab}(\mathcal{A}^\vee)$, i.e.

\[
\operatorname{colim}_i \text{Ext}_{\mathcal{A}^\vee}(M, N_i) \cong \text{Hom}_{\text{Stab}(\mathcal{A}^\vee)}(M, \operatorname{colim}_i N_i).
\]
Namely, for a Hopf algebra $\Gamma$ and comodule $M$ that is free of finite rank, we have a diagram

\[
\begin{array}{ccc}
D(\Gamma) & \xrightarrow{i} & \text{Comod}_\Gamma \\
\downarrow & & \downarrow \text{Stab}(\Gamma) \\
\text{Hom}(\text{Ext}_\Gamma(M,-)) & \rightarrow & \text{Hom}_{\text{Stab}(\Gamma)}(jM,-)
\end{array}
\]

where $i$ is the canonical inclusion and $j$ is an inclusion that is well defined only for comodules that are free of finite rank over $M_2$, and this diagram commutes. Because the stable comodule category plays well with taking colimits, we could compute the colimit of a sequence of $\text{Ext}_\Gamma(M, N)$.

Now we will introduce a new notation that will be used in future sections.

**Notation 2.2.** For a spectrum $M$ having $H_*(M)$ free of finite rank over $M_2$, let $M$ also denote the embedded image of the homology of the spectrum $M$ in the stable comodule category (i.e. $M = j(H_*(M))$). We use $[M, N]^{\Gamma}_{s,f,w}$ to denote $\text{Hom}_{\text{Stab}(\Gamma)}(M, N)$, where $M, N \in \text{Stab}(\Gamma)$. For example, if $M = S^0$, then $H_*(S^0) = M_2$, which we also denote by $S$. Thus $\text{Ext}^{\Gamma}_{\text{Stab}(\Gamma)}(M_2, M_2) = [S, S]_{s,f,w}^{\Gamma}$. When $\Gamma$ is the motivic dual Steenrod algebra, we omit the superscript $\Gamma$. This notation is consistent with [Kra].

The index will mostly show up in triples $(s, f, w)$, where $s$ is the stem, $f$ is the Adams filtration and $w$ is the motivic weight. Notice that $t = s + f$ is the internal degree. Therefore for a "standard" cofiber sequence of $\mathcal{A}^\vee$-comodules with degree $(s_0, f_0, w_0)$ self-map $\theta$: $\Sigma^{s_0, f_0, w_0} M \xrightarrow{\theta} M \rightarrow M/\theta$, the associated long exact sequence will index in the following way:

\[
\cdots \rightarrow [M, N]_{s+s_0+1, f+f_0-1, w+w_0} \rightarrow [M/\theta, N]_{s,f,w} \rightarrow [M, N]_{s,f,w} \rightarrow [M, N]_{s+s_0+f+f_0, w+w_0} \rightarrow \cdots
\]

Sometimes the indexes are omitted when there is no confusion with the layout.

3. **Self-maps and Massey products**

In the strict sense, self-maps are maps of suspensions of an object to itself. For a dualizable object $Y$, self maps $\Sigma^n Y \xrightarrow{\theta} Y$ can also be described as elements of $\pi_*(Y \otimes DY)$, with $DY$ the $\otimes$-dual of $Y$. In this paper we mainly deal with homological self-maps in $\text{Stab}(\mathcal{A}^\vee)$.

When talking about the vanishing region as well as the periodicity region, we do not take the first column, which is the only $h_0$-tower, into account. That is to say, we only look at the $h_0$-torsion part. And now we would like to further consider the $h_1$-torsion part inside the $h_0$-torsion. For this purpose, we introduce the following notion.

**Definition 3.1.** Let $F_0$ be the fiber of $S \rightarrow S[h_0^{-1}]$, where $S[h_0^{-1}] := \text{colim}(S^0 \xrightarrow{h_0} S \xrightarrow{h_0} \cdots)$ in $\text{Stab}(\mathcal{A}^\vee)$. Similarly let $F_{01}$ be the fiber of $F_0 \rightarrow F_0[h_1^{-1}]$ with $F_0[h_1^{-1}]$ defined as an analogous colimit.

The group $[S, F_{01}]$ will pick up all the $h_0$ and $h_1$-torsion from $[S, S]$, plus the negative parts of those $h_0$ and $h_1$-towers, but the latter will not interfere with the regions we are considering. We display the corresponding Ext groups in Figure 1 and 2.
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The periodicity operator $P$ corresponds to multiplying by the element $h_{20}^4$ of the May spectral sequence, meaning that for every $x$, $h_{20}^4 x \in \langle h_3, h_{10}^1, x \rangle$. However, $h_{20}^4$ does not survive to Ext. As a result, multiplying by $P$ is not a map from $[S, S]$ to $[S, S]$. Luckily, [GI1, Figure 2] shows that $P$ survives in $[S/h_0, S]$. Similarly, we have the following proposition:

**Proposition 3.2.** $P^{2r-2}$ survives in $[S/h_0^r, S]$ for $k \leq 2^r$, and thus gives a corresponding element in $[S/h_0^r, S/h_0^k]$, i.e. a self-map of $S/h_0^k$.

If $N$ is an $A^\vee$-comodule in $\text{Stab}(A^\vee)$, then $[S/h_0^k, S/h_0^k]$ acts on $[S/h_0^k, N]$. The corresponding element $P$ (or some power of $P$) inside $[S/h_0^k, S/h_0^k]$ induces a map from $[S/h_0^k, N]$ to itself as multiplying by (some power of) $P$. We would like to show that for any $k \leq 2^r$ and $r \geq 2$, multiplying by $P^{2r-2}$ on $[S/h_0^k, S]$ coincides with the Massey product $P_r(-) := \langle h_{r+1}, h_0^2, - \rangle$ in a certain region. In other words, we must show there is zero indeterminacy.

The Massey product is defined on the kernel of $h_0^{2r}$ on $[S, S]$, which we will denote $\ker(h_0^{2r})$, and lands in the cokernel of multiplication by $h_{r+1}$:

$$P_r(-) : \ker(h_0^{2r}) \to [S, S]/h_{r+1}. $$

**Remark 3.3.** Originally one would like to consider the following square and see that it commutes in a certain region

$$\begin{array}{ccc}
[S/h_0^k, S] & \xrightarrow{P_{2r-2}} & [S/h_0^k, S] \\
\downarrow & & \downarrow \\
\ker(h_0^{2r}) & \xrightarrow{P_r(-)} & [S, S]/h_{r+1}
\end{array}
$$

The vertical maps are induced by $S \to S/h_0^k$. However, since we lost the advantage of a vanishing region of $f > \frac{1}{2}s + \frac{1}{4}$ that we need in the classical setting, the region where the vertical maps are isomorphisms is not satisfactory. We would like to consider only the $h_0$ and $h_1$-torsion instead.

To better fit our purposes, consider the Massey product defined on $[S, F_01]$

$$P_r(-) : \ker_{F_01}(h_0^{2r}) \to [S, F_01]/h_{r+1}. $$
This gives the following squares, over which we have more control:

\[
\begin{array}{c}
[S/h_0^1, F_{01}] & \xrightarrow{p_{2r-2}} & [S/h_0^1, F_{01}] \\
\vert & & \vert \\
\ker_{F_{01}} (h_0^2) & \xrightarrow{p_r(-)} & [S,F_{01}]/h_{r+1} \\
\vert & & \vert \\
\ker_{\theta} (h_0^2) & \xrightarrow{p_r(-)} & [S,S]/h_{r+1}
\end{array}
\]

The set \([S,F_{01}]\) maps into \([S,S]\) by embedding all the \(h_0\) and \(h_1\)-torsion parts and sending negative towers to zero, hence the bottom square commutes for \(s > 0\) and \(f > 0\) modulo potential indeterminacy. We would like to show the indeterminacy vanishes under some conditions.

Theorem 1.2 gives us that \([S,C_\eta]_{s,f,w}\) vanishes when \(s > 0\) and \(f > \frac{1}{2}s + \frac{3}{2}\). In other words, there are only \(h_1\)-towers when \(s > 0\) and \(f > \frac{1}{2}s + \frac{3}{2}\) in \([S,S]_{s,f,w}\). And we have the following fact:

**Proposition 3.4.** (Corollary of [GI2, Theorem 1.1]) For \(r \geq 1\), \(h_{r+1}\) does not support an \(h_1\)-tower.

Therefore the indeterminacy \((h_{r+1}[S,S])_{s,f,w}\) must vanish when \(f > \frac{1}{2}s + 3 - 2r\), given that \(h_{r+1}\) has \(s = 2^{r+1} - 1\) and there are only \(h_1\)-towers in \([S,S]_{s,f,w}\) when \(s > 0\) and \(f > \frac{1}{2}s + \frac{3}{2}\), which are \(h_{r+1}\)-torsion.

**Remark 3.5.** It is easy to see that the indeterminacy \((h_{r+1}[S,F_{01}])_{s,f,w}\) also vanishes when \(f > \frac{1}{2}s + 3 - 2r\).

As to the top square, the first row is multiplying by some power of the element \(P\). Now it is time to show when the vertical maps are isomorphisms.

**Lemma 3.6.** (Motivic version of [Kra, Lemma 5.2]) Let \(M, N \in \text{Stab}(A^s)\). Assume that \([M,N]\) vanishes when \(f > as + bw + c\) for some \(a, b, c \in \mathbb{R}\). Let \(\theta : \Sigma^0_{s_0,0,w_0} M \to M\) be a map with \(f_0 > a s_0 + b w_0\) and let \(M/\theta\) denote the cofiber of \(\Sigma^0_{s_0,0,w_0} M \to M\). Then

\[ [M/\theta, N] \to [M,N] \]

is an isomorphism above a vanishing plane parallel with the one in \([M,N]\) but with \(f\)-intercept given by \(c - (f_0 - a s_0 - b w_0)\).

**Proof.** The result follows from the long exact sequence associated to the cofiber sequence \(\Sigma^0_{s_0,0,w_0} M \to M \to M/\theta:\)

\[
\cdots \to [M,N]_{s_0+1,f_0+1,w_0+1} \to [M/\theta,N]_{s_0,f_0,w_0} \to [M,N]_{s_0,f_0,w_0} \to [M,N]_{s_0+1,f_0+1,w_0+1} \to \cdots
\]

**Remark 3.7.** This could also work for a vanishing region above several planes or even a surface. So one could rephrase the vanishing condition of Lemma 3.6 as the following:

Assume that \([M,N]_{s,s,s} \) vanishes when \(f > \varphi(s,w)\) where \(\varphi : \mathbb{R}^2 \to \mathbb{R}\) is a smooth function, then the gradient \(v(-,-) = (\frac{\partial \varphi}{\partial w}(-), \frac{\partial \varphi}{\partial s}(-))\) is a vector field. Then let \(d = \text{Max}_{(s_0,w_0)} |v(s_0,w_0)|\) and...
assume both $\frac{d}{s_0}$ and $\frac{d}{w_0} > d$, the other parts would work similarly with the $f$-intercept given by $\text{Max}\{c - (f_0 - d s_0), c - (f_0 - d w_0)\}$.

We have this as a corollary:

**Corollary 3.8.** (the motivic version of [Kra, Lemma 5.9]) Let $k \geq 1$. For $f > \frac{1}{2}s + \frac{3}{2} - k$, the natural map $[S/h^k_{0}, F_0]|_{s,f,w} \to [S,F_0]|_{s,f,w}$ is an isomorphism.

**Proof.** To see this, we just need to check $[S,F_0]|_{s,f,w}$ admits a vanishing region of $f > \frac{1}{2}s + \frac{3}{2}$. The fiber sequence $F_0 \to F_0 \leftarrow F_0[h^{-1}_1]$ gives us an exact sequence in homotopy:

$$\cdots \to [S,F_0]|_{s,f,w} \to [S,F_0]|_{s,f,w} \overset{h^{-1}_1}{\longrightarrow} [S,F_0]|_{s,f,w} \to [S,\Sigma^{-1,0}_0 F_0]|_{s,f,w} \to \cdots$$

Since $[S,F_0]$ differs from $[S,S]$ only in the first column, there are only $h_1$-towers when $f > \frac{1}{2}s + \frac{3}{2}$. And by Theorem 1.2 again, $[S,C_{\eta}]|_{s,f,w}$ vanishes when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$. In other words, above the plane $f = \frac{1}{2}s + \frac{3}{2}$, multiplying by $h_1$, which detects $\eta$, is an isomorphism from $[S,F_0]|_{s,f,w}$ to $[S,F_0]|_{s+1,f+1,w+1}$.

As a result, inverting $h_1$ would be an isomorphism from $[S,F_0]|_{s,f,w}$ to $[S,F_0]|_{s,f,w}$ when $f > \frac{1}{2}s + \frac{3}{2}$. Therefore $[S,F_0]|_{s,f,w}$ vanishes when $f > \frac{1}{2}s + \frac{3}{2}$. Apply Lemma 3.6 we will get this corollary.

Putting the results in 3.2 and 3.5 together, we obtain the region where both squares commute, thus obtaining the first part of Theorem 1.4.

**Theorem 3.9.** (Motivic version of [Kra, Proposition 5.12]) For $k \leq 2^r$ and $r \geq 2$, the cofiber $S/h^k_0$ admits a self-map $P^{2^{r-2}}$ of degree $(2^{r+1}, 2^r, 2^r)$. Let $N \in \text{Stab}(\mathbb{A}^r)$, this self-map induces a map $[S/h^k_0,N] \to [S/h^k_0,N]$ given by multiplication by $P^{2^{r-2}}$.

And when $f > \frac{1}{2}s + 3 - k$, the induced map coincides with the Massey product $P_r(\cdot) := \langle h_{r+1}, h^r_0, \cdot \rangle$ with zero indeterminacy.

4. THE COLIMITS AND THE CARTAN-EILENBERG SPECTRAL SEQUENCE

Consider $F_0/h^\infty_0 := \text{colim}(\Sigma^{-1,1,0}_0 F_0/d \to \Sigma^{-1,1,0}_1 F_0/d \to \cdots)$, a fiber sequence different from $F_0$ in the region we are concerning.

**Proposition 4.1.** When $f > \frac{1}{2}s + \frac{3}{2}$,

$$[S,\Sigma^{-1,1,0}_0 F_0/h^\infty_0]|_{s,f,w} \cong [S,F_0]|_{s,f,w}$$

**Proof.** To see this, the colimit $F_0/h^\infty_0$ is picking up all the $h_1$-torsion in $F_0$, while the fiber $F_0$ is picking up the $h_1$-torsion together with those negative $h_1$-towers.

Analogously, $F_0$ is almost a suspension different from $S/h^\infty_0 := \text{colim}(\Sigma^{0,1,0}_0 S/h_0 \to \cdots \to \Sigma^{0,1,0}_1 S/h_0 \to \cdots)$, if we ignore the negative $h_0$-tower. i.e. $[S,\Sigma^{-1,1,0}_0 S/h^\infty_0]|_{s,f,w} \cong [S,F_0]|_{s,f,w}$ when $f > 0$. 

Remark 4.2. We have shown that the map \([S/h_0^k, F_0/h_1^\infty]_{s,f,w} \to [S, F_0/h_1^\infty]_{s,f,w}\) is an isomorphism when \(f > \frac{1}{2}s + 3 - k\). We consider this colimit because it is better for computation purposes (fibers are harder to deal with than the colimit \(F_0/h_1^\infty\)).

Let \(\theta\) be a self-map of \(S/h_0^k\), and consider the cofiber sequence \(S/h_0^k \xrightarrow{\partial} S/h_0^k \to S/(h_0^k, \theta)\). The vanishing region for \([S/(h_0^k, \theta), F_0/h_1^\infty]_{s,s,s}\) gives the region where

\[
[S/h_0^k, F_0/h_1^\infty]_{s,f,w} \xrightarrow{\theta} [S/h_0^k, F_0/h_1^\infty]_{s+f_0,f+w_0}
\]

is an isomorphism. The goal of this section is to get a vanishing region for \([S/(h_0^k, \theta), F_0/h_1^\infty]_{s,s,s}\) in the case \(k = 1\) and \(\theta = \mathcal{P}\).

The dual Steenrod algebra is too large to work with, so we would like to start with a small one, namely \(A(1)^\vee \cong M_2[t_0, t_1, \zeta_1]/(t_0^\infty = \tau t_1^2, \tau, \zeta_1^2)\). Then for \(A^\vee\)-comodules \(M\) and \(N\) (thus also \(A(1)^\vee\)-comodules), we can recover \([M, N]_{A^\vee}\) from \([M, N]_{A(1)^\vee}\) via infinitely many Cartan-Eilenberg spectral sequences along normal extensions of Hopf algebras.

Here is a brief introduction to the Cartan-Eilenberg spectral sequence (See [CE, Ch.XV] for details). Given an extension of Hopf algebras over \(M_2\)

\[
E \to \Gamma \to C
\]

(so in particular \(E \cong \Gamma \square_C M_2\)), the Cartan-Eilenberg spectral sequence computes \(Cotor_\Gamma(M, N)\) for \(\Gamma\)-comodule \(M\) and \(E\)-comodule \(N\) arises from the double complex (\(\Gamma\)-resolution of \(M\)\(\square\_\Gamma\) (\(E\)-resolution of \(N\)). And we have \(Cotor_\Gamma(M, N) \cong Ext_\Gamma(M, N)\) when \(M\) and \(N\) are \(\tau\)-free.

The \(E_1\)-page has the form

\[
E_1^{s,l,s,*} = Cotor_C^{l,*}(M, E^{s,s} \otimes N) \Rightarrow Cotor_\Gamma^{s+l,*}(M, N).
\]

If \(E\) has trivial \(C\)-coaction, then we have \(E_1^{s,l,s,*} \cong Cotor_C^{l,*}(M, N) \otimes E^{s,s}\), and taking cohomology we get the \(E_2\)-page:

\[
E_2^{s,l,s,*} = Cotor_E^{l,*}(M_2, Cotor_C^{l,*}(M, N)) \cong Ext_E^{l,*}(M_2, M_2) \otimes Ext_C^{l,*}(M, N).
\]

Let \(N = F_0/h_1^\infty\), so that we would like to compute a middle step \([S/h_0, F_0/h_1^\infty]_{A(1)^\vee}\) before getting to our goal \([S/(h_0, \theta), F_0/h_1^\infty]_{A(1)^\vee}\). As a starting point, we can compute \([S/h_0, F_0]\) over \(A(1)^\vee\), via the cofiber sequence \(S \xrightarrow{h_0} S \to S/h_0\).

The pattern is \((8, 4, 4)\)-periodic to the right. Since \([S/h_0, F_0/h_1^\infty]_{A(1)^\vee}\) is a colimit, it is essential to know the maps we are taking the colimit over. First let us take a look at the maps induced by multiplying by \(h_1\) (we write \(\Sigma^{-1, -1, -1}\) as \(\Sigma^{-1}\) so there is no confusion):
All rows are exact and the left side picks up $\text{coker}(h^k_1)$, while the right side picks up $\text{ker}(h^k_1)$. As a result, taking the colimit in the middle would merely be taking the colimits of the cokernel part from the left and the kernel part from the right. There are hidden relations between the cokernel and kernel, yet they wouldn’t affect the vanishing region, which is all we concern. A more illuminating diagram looks like the following:

(4.3)
The maps $i$ on the right column are canonical inclusions, and passing to colimits gives $\text{colim}_k (\text{ker}(h^k_1)) \rightarrow [S/h_0, F_0/h^\infty_k] \rightarrow \text{colim}_k (\text{ker}(h^k_1))$. Working over the subalgebra’s dual $\mathcal{A}(1)^\vee$, we can calculate $[S/h_0, \Sigma^{-1,1,0}F_0/h^\infty_k]^{\mathcal{A}(1)^\vee}_{s,s,s}$ directly. Furthermore we have:

**Proposition 4.4.** For any $k \in \mathbb{Z}, k \geq 1$, the maps $[S/h_0, \Sigma^{-k}F_0/h^k_1]^{\mathcal{A}(1)^\vee}_{s,s,s} \rightarrow [S/h_0, \Sigma^{-k-1}F_0/h^{k+1}_1]^{\mathcal{A}(1)^\vee}_{s,s,s}$ are injective.

The calculation result is shown in the following picture.

![Figure 4](image.png)

**Figure 4.** $[S/h_0, \Sigma^{-1,1,0}F_0/h^\infty_k]^{\mathcal{A}(1)^\vee}_{s,s,s}$

The pattern is $(8, 4, 4)$-periodic to the right as well. Note that $[S/h_0, \Sigma^{-1,1,0}F_0/h^\infty_k]^{\mathcal{A}(1)^\vee}_{s,s,s}$ differs from the classical $[S/h_0, S]^{\mathcal{A}(1)^\vee}_{s,s}$ with two extra negative $h_1$-towers associated to each “lighting flash”. The point $(-1, 0, -1)$ in the first pattern is generated by $\tau$ with a shift.
Recall that $\theta$ is the self-map $P$ on $S/h_0$. Then $\theta$ acts injectively as can be seen in Figure 4. Combining this with the long exact sequence:

\[
\cdots \rightarrow [S/(h_0, \theta), F_0/h_1^{\infty}]_{s,f,w}^{A(1)^\vee} \rightarrow [S/h_0, F_0/h_1^{\infty}]_{s,f,w}^{A(1)^\vee} \rightarrow [S/h_0, F_0/h_1^{\infty}]_{s,f,w}^{A(1)^\vee} \rightarrow \cdots
\]

(continue) \[
[S/h_0, F_0/h_1^{\infty}]_{s+8,f+4,w+4}^{A(1)^\vee} \rightarrow [S/(h_0, \theta), F_0/h_1^{\infty}]_{s-1,f+1,w}^{A(1)^\vee} \rightarrow \cdots
\]
gives $[S/(h_0, \theta), \Sigma^{-1,1,0}F_0/h_1^{\infty}]_{s,s,s}^{A(1)^\vee}$ as in Figure 5. The highest point of the pattern is $(-6,0, -2)$.

**Remark 4.5.** Analogously to Proposition 4.4, for any $k \in \mathbb{Z}$, $k \geq 1$, the following maps are also injective:

\[
[S/(h_0, \theta), \Sigma^{-k}F_0/h_1^{k}]^{A(1)^\vee} \rightarrow [S/(h_0, \theta), \Sigma^{-k+1}F_0/h_1^{k+1}]^{A(1)^\vee}.
\]

![Figure 5. $[S/(h_0, \theta), \Sigma^{-1,1,0}F_0/h_1^{\infty}]_{s,s,s}^{A(1)^\vee}$](image)

Now the input is ready, and we would like to borrow the strength of the Cartan-Eilenberg spectral sequence to move from over $A(1)^\vee$ to over $A^\vee$. It is known that the Cartan-Eilenberg spectral sequence converges when the input is a bounded below $A^\vee$-comodule. We are going to obtain a vanishing region of each finite stage $[S/(h_0, \theta), \Sigma^{-k}F_0/h_1^{k}]^{A^\vee}$, and then deduct the vanishing region of $[S/(h_0, \theta), F_0/h_1^{\infty}]^{A^\vee}$ by passing through the colimit.

\[
[S/(h_0, \theta), \Sigma^{-1}F_0/h_1]^{A(1)^\vee} \rightarrow [S/(h_0, \theta), \Sigma^{-2}F_0/h_1^{2}]^{A(1)^\vee} \rightarrow \cdots [S/(h_0, \theta), F_0/h_1^{\infty}]^{A(1)^\vee}
\]

Going from $A(1)^\vee$ to $A^\vee$ is too big of a step, and we would like to stop by

\[
A(2)^\vee = M_2[\tau_0, \tau_1, \tau_2, \xi_1, \xi_2]/(\tau_0^2 = \tau_1^2 = \tau_2^2, \tau_2 = \xi_1^2, \tau_1 = \xi_2^2, \tau_1 = \xi_2^2)
\]
via a sequence of normal extensions first:
\[ \mathcal{A}(2)^\vee \to \mathcal{A}(2)^\vee / \mathcal{A}(2)^\vee / \mathcal{A}(2)^\vee / (\xi_1^2, \xi_2) \to \mathcal{A}(1)^\vee. \]

The first step is throwing in the element \( h_{30} \), which corresponds to \( h_{30} \) in the May spectral sequence. It has degree \( (6, 1, 3) \) and the associated extension is:
\[ E(h_2) \to \mathcal{A}(2)^\vee / (\xi_1^2, \xi_2) \to \mathcal{A}(1)^\vee. \]

The \( \mathcal{A}(1)^\vee \)-coaction on \( E(h_2) \) is trivial by degree reason. So we can start with the \( E_1 = E_2 \)-page.

\[
\begin{array}{c}
[S/(h_0, \theta), \Sigma^{-1}F_0/h_1]^{A(1)^\vee} \otimes M_2[h_{30}] \\
\vdots \\
[S/(h_0, \theta), \Sigma^{-1}F_0/h_1]^{A(2)^\vee / (\xi_1^2, \xi_2)} \\
\vdots \\
[S/(h_0, \theta), \Sigma^{-1}F_0/h_1]^{A(2)^\vee / (\xi_1^2, \xi_2)}
\end{array}
\]

For the normal extension \( E(\beta) \to \Gamma \to C \) of Hopf algebras we state a motivic version of [Kra, Lemma 4.10], which gives a relation between the vanishing condition of \( [M, N]^\Gamma \) and the vanishing condition of \( [M, N]^C \) together with the two "slopes" of \( \beta = (s_0, f_0, w_0) \), \( \frac{f_0}{w_0} \) and \( \frac{w_0}{f_0} \). i.e. we are basically looking at the projections onto the plane \( w = 0 \) and the plane \( s = 0 \).

**Theorem 4.6.** Let \( E(\alpha) \to \Gamma \to C \) be a normal extension of Hopf algebras and \( M, N \in \text{Stab}(\Gamma) \). Suppose \( \beta \) is an element in \( [S, S]^\Gamma \) that has coordinate \( (s_0, f_0, w_0) \) with \( s_0, f_0, w_0 \) being positive, its image in \( [S, S]^\Gamma \) (which we also call \( \beta \)) acts on \( [M, N]^\Gamma \). And suppose for some \( a, b, c, m, c_0 \in \mathbb{R} \) with \( a, b > 0 \) and \( m \geq \frac{f_0}{w_0} > 0 \), \( [q_4(M), q_4(N)]^C \) vanishes when \( f > as + bw + c \) and also vanishes when \( f > ms + c_0 \). Then

1. if \( f_0 \leq as_0 + bw_0 \), or \( \beta \) acts nilpotently on \( [M, N]^\Gamma \), then \( [M, N]^\Gamma \) has a parallel vanishing region.
   i.e. it vanishes when \( f > as + bw + c' \) and also vanishes when \( f > ms + c_0 \).

2. if otherwise, then \( [M, N]^\Gamma \) vanishes when \( f > \frac{mbw_0 - f_0(m-a)}{bw_0 - s_0(m-a)} s + \frac{bf_0 - mbw_0}{bw_0 - s_0(m-a)} w + c' \) and vanishes when \( f > ms + c_0 \).

**Remark 4.7.** One may wonder why there is another vanishing plane \( f > ms + c_0 \). It is to generalize the bounded below condition. In the classical setting we have \( [M, N]^\Gamma \) vanishes when \( s < c_0 \), but due to those negative \( h_1 \)-towers we do not have a vertical vanishing plane. So we adjust the "\(-\infty\)-slope" plane to be \( f = ms + c_0 \) to fulfill our purpose. This bound should not bring any concerns with the periodicity region we care about, so we omit it in the following contexts.

**Proof of Theorem 4.6.** If \( \beta \) has \( f_0 \leq as_0 + bw_0 \), then \( [M, N]^C \) multiplied by \( \beta \) will lie under the existing vanishing planes.

If \( f_0 > as_0 + bw_0 \), then \( \beta \) will eventually "poke" out of the existing vanishing plane \( f > as + bw + c \). If it acts nilpotently, we can still get a parallel vanishing plane \( f > as + bw + c' \) on \( [M, N]^\Gamma \) by moving up the \( f \)-intercept sufficiently far.

Now for case two, if \( f_0 > as_0 + bw_0 \) and \( \beta \) acts non-nilpotently, then there needs to be an element \( x \in [M, N]^\Gamma \) for which the \( \beta^k x \) are not zero on the \( E_n \) page of the Cartan-Eilenberg spectral sequence for every \( k \). Thus no matter how we move up the existing vanishing plane \( f > as + bw + c \), \( \beta \) will stick out of it. As a result we would like to adjust the vanishing boundary to block \( \beta \). Let \( a', b', c' \in \mathbb{R} \), the new vanishing region \( f > a's + b'w + c' \) needs to satisfy the condition \( f_0 > a's_0 + b'w_0 + c' \). From multi-variable calculus, this plane is spanned by the direction of \( \beta \) and
the intersecting line of the two existing vanishing plane. Hence we can solve $a' = \frac{mbw_0 - f_0(m-a)}{bw_0 - s_0(m-a)}$ and $b' = \frac{b_0 - mbs_0}{bw_0 - s_0(m-a)}$ to fulfill the condition.

**Remark 4.8.** In the cases we concern about, the starting vanishing regions will have $b = 0$. One can think of these cases as 2-dimensional stated in 3-dimensional language.

We rewrite the conditions and the results of Theorem 4.6 as the following: Suppose for some $a, c, m, c_0 \in \mathbb{R}$ with $a > 0$ and $m \geq \frac{b_0}{s_0} > 0$, $[q_s(M), q_s(N)]^C$ vanishes when $f > as + c$ and also vanishes when $f > ms + c_0$. Then

(1) if $f_0 \leq as_0$, or $\beta$ acts nilpotently on $[M, N]^F$, then $[M, N]^F$ has a parallel vanishing region.

i.e. it vanishes when $f > as + c'$ and also vanishes when $f > ms + c_0$.

(2) if otherwise, then $[M, N]^F$ vanishes when $f > \frac{b_0}{s_0}s + c'$ and vanishes when $f > ms + c_0$.

**Remark 4.9.** Similarly, we could generalize the case to $[q_s(M), q_s(N)]^C$ vanishes when $f > \varphi(s, w)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, then the gradient $\nabla(\varphi, -\varphi) = \left(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial w}\right)$ is a vector field. Now we would like to consider $g = \min (\varphi(s_0, w_0))$ and compare $g$ with $\frac{b_0}{s_0}$ and $\frac{b_0}{w_0}$. The conditions rewrite as following:

(1) if $\frac{b_0}{s_0} \leq g$ or $\frac{b_0}{w_0} \leq g$, or $\beta$ acts nilpotently, then $[M, N]^F$ has a parallel vanishing region.

(2) if both $\frac{b_0}{s_0}$ and $\frac{b_0}{w_0}$ > $g$, and $\beta$ acts non-nilpotently, then we need to modify the vanishing region of $[M, N]^F$. However it takes a little bit work to write down a precise modification, we would omit it here.

**Remark 4.10.** From the cofiber sequence $S \xrightarrow{h_0} S \to S/\pi h_0^k$ we can derive the fiber sequence by taking tensor dual $D(S/\pi h_0^k) \to S \to S$. Thus, since $D(S/\pi h_0^k) \simeq \Sigma^{-1,1,0}_S S/\pi h_0^k$, we have

$$[S/\pi h_0^k, S]_{s,f,u} = [S, D(S/\pi h_0^k)]_{s,f,u} = [S, S/\pi h_0^k]_{s+1, f+k-1,u}.$$ 

Because $S/\pi h_0^k$ is compact in $\text{Stab}(\mathcal{A}^\vee)$, smashing the second spot with some $N \in \text{Stab}(\mathcal{A}^\vee)$, we get

$$[S/\pi h_0^k, N]_{s,f,u} = [S, D(S/\pi h_0^k) \wedge N]_{s,f,u} = [S, S/\pi h_0^k \wedge N]_{s+1, f+k-1,u}.$$ 

As a result $\beta \in [S, S]_F$ acts on $[M, N]^F$ for compact $M \in \text{Stab}(\mathcal{A}^\vee)$, since $\beta$ acts on $[S, DM \wedge N]^F$.

The group $[S/\pi (h_0, \theta), \Sigma^{-1,1,0}_S \mathcal{F}_0/\pi h_0^k]_{s, s, s}^{A(1)\vee}$ has a single pattern of “lighting flash” along with two negative $h_1$-towers (see Figure 5), so the vanishing region to start off with is $f > c$ (We obtain the same vanishing region of $[S/\pi (h_0, \theta), \Sigma^{-1,1,0}_S \mathcal{F}_0/\pi h_0^k]_{s, s, s}^{A(1)\vee}$ for each $k$, since the maps we are taking colimit over are injections by remark 4.5). With $M$ and $N$ being what they should be respectively, we just need to focus on the $\beta$ we are throwing in. It is $\tau_2 = (6, 1, 3)$, and will be $\tau_2 = (5, 1, 3)$ and $\tau_2 = (3, 1, 2)$ in order.

Recall we are working with the Cartan-Eilenberg spectral sequence

$$[S/\pi (h_0, \theta), \Sigma^{-1,1,0}_S \mathcal{F}_0/\pi h_0^k]_{s, s, s}^{A(1)\vee} \otimes \mathbb{M}_2[h_{30}] \Rightarrow [S/\pi (h_0, \theta), \Sigma^{-1,1,0}_S \mathcal{F}_0/\pi h_0^k]_{s, s, s}^{A(2)\vee}/(\tau_1, \tau_2).$$

There won’t be any differentials for degree reasons. By Theorem 4.6 the element $h_{30}$ will bring us a vanishing region $f > \frac{1}{s} s + constant$ for each $k$ (We do obtain the same constant for all $k$). Passing
through the colimit we conclude that $[S/(h_0, \theta), \Sigma^{-1, 1, 0} F_0 / h_1^\infty] \cdot A(2)^V / (\xi_1^2, \xi_2)$ shares the same vanishing region $f > \frac{1}{2}s + \text{constant}$.

The second step would be moving along the normal extension, in which we throw in $\xi_2$, corresponding to the class $h_2$:

$$E(\xi_2) \to A(2)^V / \xi_1^2 \to A(2)^V / (\xi_1^2, \xi_2).$$

The $A(2)^V / (\xi_1^2, \xi_2)$-coaction on $E(\xi_2)$ is trivial. We have $E_2$-pages as the first row:

$$[S/(h_0, \theta), \Sigma^{-1} F_0 / h_1] A(2)^V / (\xi_1^2, \xi_2) \otimes M_2[h_2] \quad \cdots \quad [S/(h_0, \theta), F_0 / h_1^\infty] A(2)^V / (\xi_1^2, \xi_2) \otimes M_2[h_2]$$

The spectral sequence collapses at $E_2$-page, due to the fact that the May differential $d_1(h_{30}) = h_0 h_{12} + h_2 h_{20}$ yet neither $h_0$ nor $h_2$ (corresponds to $\xi_1^2$) is there. As a result $h_{21}$ is also non-nilpotent, bringing a vanishing region $f > \frac{1}{2}s + \text{constant}$ of $[S/(h_0, \theta), \Sigma^{-1, 1, 0}(\Sigma^{-k} F_0 / h_1^\infty)] A(2)^V / \xi_1^2$ for each $k$ according to Theorem 4.6, and the colimit $[S/(h_0, \theta), \Sigma^{-1, 1, 0} F_0 / h_1^\infty] A(2)^V / \xi_1^2$ as well.

The third step is throwing in $\xi_1^2$, corresponding to the class $h_2$:

$$E(\xi_1^2) \to A(2)^V \to A(2)^V / \xi_1^2.$$

The $A(2)^V / \xi_1^2$-coaction on $E(\xi_1^2)$ is trivial as well. We have $E_2$-pages as the first row:

$$[S/(h_0, \theta), \Sigma^{-1} F_0 / h_1] A(2)^V / \xi_1^2 \otimes M_2[h_2] \quad \cdots \quad [S/(h_0, \theta), F_0 / h_1^\infty] A(2)^V / \xi_1^2 \otimes M_2[h_2]$$

We do get some differentials this time. In previous steps by throwing in $[\tau_2] = (6, 1, 3)$ and $[\xi_2] = (5, 1, 3)$, which are not nilpotent, we end up with a vanishing region of $f > \frac{1}{2}s + \text{constant}$. However $[\xi_1^2] = (3, 1, 2)$ is nilpotent since $h_2^2 = 0$ in Ext over $A(2)^V$ and over $A^V$.

Moving from $A(2)^V$ to $A^V$, we have many more elements to throw in. However those elements won’t have $f > \frac{1}{2}$, Theorem 4.6 (or the handy Remark 4.8) tells us we stay with the plane $f = \frac{1}{2}s + \text{constant}$ for the vanishing region of $[S/(h_0, \theta), \Sigma^{-1, 1, 0}(\Sigma^{-k} F_0 / h_1^\infty)] A$ for each $k$. Since the vanishing plane passes through the point $(-6, 0, -1) + 3 \cdot (3, 1, 2) = (3, 3, 5)$, the constant is $\frac{12}{3}$ and the region $f > \frac{1}{2}s + \frac{12}{3}$ would be carried through till $A^V$. We conclude that

**Proposition 4.11.** The group $[S/(h_0, \theta), \Sigma^{-1, 1, 0} F_0 / h_1^\infty]_{s,f, w}$ has a vanishing region of $f > \frac{1}{2}s + \frac{12}{3}$.

Note that it is possible that the vanishing region we have found is not minimal. First, we could consider the "slope" of the motivic weight side $\frac{f}{s}$ instead of $\frac{f}{s}$ under certain bounded below conditions. Second, after throwing in more elements, more differentials would occur. Thus the vanishing region would potentially collapse down, but to make sure whether this will happen, more calculation is required.
5. The Motivic Periodicity Theorem

Let \( F_0 \) and \( F_{01} \) still be the same as in Definition 3.1, so that \([S, \Sigma^{-1,1,0}F_0/h_1^n]_{s,f,w} \cong [S, F_{01}]_{s,f,w}\) when \( f > \frac{1}{2}s + 3 \). Given self-map \( \theta \) let us recall the diagram where the first row is exact:

\[
\begin{array}{cc}
[S/(h_0^k \theta), \Sigma^{-1,1,0}F_0/h_1^n] & \longrightarrow [S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^n] \\
\downarrow & \\
[S, \Sigma^{-1,1,0}F_0/h_1^n] & \longrightarrow [S/(h_0^k, \theta), \Sigma^{-1,1,0}F_0/h_1^n]
\end{array}
\]

The vertical maps are isomorphisms whenever \( f > \frac{1}{2}s + \frac{3}{2} - k \) due to Corollary 3.8. We would like to further restrict the condition to \( f > \frac{1}{2}s + 3 - k \) in order to eliminate the indeterminacy. And whether \( \theta \) is an isomorphism is given by the vanishing condition on \([S/(h_0^k, \theta), \Sigma^{-1,1,0}F_0/h_1^n]\), which is the same as the vanishing condition on \([S/(h_0^k, \theta), F_{01}]_{s,f,w}\).

In the previous section we figured out the case when \( k = 1 \), that is Proposition 4.11. We show in Picture 6 the \((2^r+1, 2^r, 2^r)\)-periodic pattern for \([S/(h_0^k, \Sigma^{-1,1,0}F_0/h_1^n)]_{A(1)^\vee}\), where \( k \leq 2^r \). By an analogous computation process, one can see that for a general positive integer \( k \leq 2^r \), \([S/(h_0^k, p^{2^r-2}), F_{01}]_{s,f,w}\) admits a parallel vanishing region as the \( k = 1 \) case.

**Figure 6.** \([S/(h_0^k, \Sigma^{-1,1,0}F_0/h_1^n)]_{A(1)^\vee}\)

As to the \( f \)-intercept, we have the following lemma:

**Lemma 5.1.** (as a corollary of [Kra, Lemma 5.4]) Let \( M, N \in \text{Stable}(A^\vee) \) with \( M \) compact. Let \( M_1 = M/\theta_1 \) be the cofiber of the self-maps \( \Sigma^{-1,1,0}F_0/h_1^n \) \( \theta_1 \rightarrow M \), and let \( M_2 = M/(\theta_1, \theta_2) \) be the cofiber of the
self-maps \( \Sigma^2, f; \omega_2 M / \theta_1 \rightarrow M / \theta_1 \). Define \( M'_1 \) and \( M'_2 \) with respect to the self-maps \( \Sigma^2_i, f'_i, \omega'_1 M \rightarrow M \) and \( \Sigma^2_i, f; \omega_2 M / \theta'_1 \rightarrow M / \theta'_1 \) in the same way. Suppose \( \theta_i \) and \( \theta'_i \) parallel, i.e. \((s_i, f_i, \omega_i) = \lambda_i(s'_i, f'_i, \omega'_i)\) where \( \lambda_i \) are non-zero real numbers and \( i = 1, 2 \).

Further let \( a, b \in \mathbb{R} \) and suppose \( f_i > a s_i + b w_i \) and \( f'_i > a s'_i + b w'_i \) for \( i = 1, 2 \). Then the minimal \( f \)-intercepts of the vanishing planes (we make the convention that the \( f \)-intercept is \( \infty \) if there is no such vanishing plane) parallel to \( f = as + bw \) on \([M_2, N]\) and \([M'_2, N]\) agree.

**Proof of Lemma 5.1.** We would like to construct the iterated cofiber \( L_1 = M / (\theta_1, \theta'_1) \) and \( L_2 = M / (\theta_2, \theta'_1, \theta'_2) \). Since \( f_i > a s_i + b w_i \) and \( f'_i > a s'_i + b w'_i \) for \( i = 1, 2 \), the minimal \( f \)-intercepts for the vanishing planes parallel to \( f = as + bw \) agree on \([M_1, N], [M'_1, N] \) and \([L_1, N]\) by inductively applying Lemma 3.6.

Note that \( L_1 \) and \( L_2 \) is an intuitive but not correct notation. It is not indicating that \( M / \theta_1 \) should admit a \( \theta'_1 \) self-map or vice versa. Because of the uniqueness of (homological) self-maps that Krause has shown in [Kra, Sec. 4], there is a self-map \( \theta''_1 \) compatible with both \( \theta_1 \) and \( \theta'_1 \), which acts on \( M \) by a power of \( \theta_1 \), and by a power of \( \theta_2 \). We will take \( L_1 \) to be the cofiber of the self-map \( \theta''_1 \). Similarly, there exists a self-map \( \theta''_2 \) on \( L_1 \) such that it acts on \( M'_1 \) by a power of \( \theta_2 \), and on \( M'_2 \) by a power of \( \theta'_2 \). So we can set \( L_2 \) as the cofiber of the self-map \( \theta''_2 \).

**Remark 5.2.** Krause’s proof of the uniqueness of self-maps is in the classical setting, yet for the \( C \)-motivic case the proof is analogous.

**Remark 5.3.** An alternative way to issue the vanishing condition of \([S / (h^k_0, P^{2r-2}), F_{01}]_{s, f, \omega}\) is via the cofiber sequences arise from the Verdier’s axiom and the \( 3 \times 3 \) lemma. Let \( m, n, l, l' \in \mathbb{N} \) be positive with \( m \leq 4l \) and \( m + n \leq 4(l + l') \), we have the following cofiber sequences:

\[
S / (h^m_0, p^{l'+l'}) \rightarrow S / (h^{m+n}_0, p^{l+l'}) \rightarrow S / (h^n_0, p^{l+l'})
\]

Passing through the induced long exact sequences in homology, we conclude that for \( k \leq 2' \), \([S / (h^k_0, P^{2r-2}), F_{01}]_{s, f, \omega}\) admits the same vanishing condition as \([S / (h^0_0, P), F_{01}]_{s, f, \omega}\). It follows that for any \( k \leq 2' \) and any self-map \( \theta = P^{2r-2} \) of \( S / h^k_0 \), the corresponding \( [S / (h^k_0, \theta), F_{01}] \) has a vanishing region of \( f > 1/2s + 12/\sigma \). In particular we have the previous results Theorem 3.9, for \( k \leq 2', r \geq 2, S / h^k \omega_0 \) admits a self-map \( P^{2r-2} \) of degree \( (2r+1, 2') \), corresponding to the Massey product \( P_r(\cdot) = \langle h_{r+1}, h^r_0, \cdot \rangle \).

Putting all the things together, we have the motivic version of Theorem 1.1:

**Theorem 5.4** (Another way of stating Theorem 1.4). For \( r \geq 2 \), the Massey product operation \( P_r(\cdot) := \langle h_{r+1}, h^r_0, \cdot \rangle \) is uniquely defined on \( \text{Ext} = H^{s, f, \omega}(A) \) when \( s > 0 \) and \( f > 1/2s + 3 - 2' \).

Furthermore, for \( f > 1/2s + 12/\sigma \),

\[
P_r : [S, F_{01}]_{s, f, \omega} \xrightarrow{P_r(\cdot)} [S, F_{01}]_{s+2^{r+1}, f+2', \omega+2'}
\]

is an isomorphism between \( h_0 \) and \( h_1 \)-torsions.
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