ACTION OF COXETER GROUPS ON \( m \)-HARMONIC POLYNOMIALS AND KZ EQUATIONS

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Abstract. The Matsuo–Cherednik correspondence is an isomorphism from solutions of Knizhnik–Zamolodchikov equations to eigenfunctions of generalized Calogero–Moser systems associated to Coxeter groups \( G \) and a multiplicity function \( m \) on their root systems. We apply a version of this correspondence to the most degenerate case of zero spectral parameters. The space of eigenfunctions is then the space \( H_m \) of \( m \)-harmonic polynomials, recently introduced in [11]. We compute the Poincaré polynomials for the space \( H_m \) and for its isotypical components corresponding to each irreducible representation of the group \( G \). We also give an explicit formula for \( m \)-harmonic polynomials of lowest positive degree in the \( S_n \) case.

1. Introduction

Let \( G \) be a finite group generated by orthogonal hyperplane reflections of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). We label all reflections \( s_\alpha \) by a set of arbitrarily chosen normal vectors \( \alpha \in A \). We also fix a system of multiplicities \( m_\alpha \in \mathbb{Z}_{\geq 0} \) such that \( m_\alpha = m_\beta \) if \( s_\alpha \) and \( s_\beta \) belong to the same conjugacy class. The group \( G \) also acts on the complexification \( V = \mathbb{C}^n \) of \( \mathbb{R}^n \). We denote by \( \Pi_\alpha \) the complexification of the reflection hyperplane of \( s_\alpha \).

By Chevalley’s theorem, the algebra of invariants \( S^G \) in the graded algebra \( S = S(V) = \bigoplus_{j=0}^\infty S^j \) of polynomial functions on \( V \) is freely generated by homogeneous polynomials \( \sigma_1, \ldots, \sigma_n \), \( \sigma_i \in S^{d_i} \). We identify \( V \) with its dual space by means of the bilinear form \(( \cdot, \cdot)\) induced by the Euclidean inner product. So we identify the symmetric algebra \( S \) both with polynomial functions \( z \mapsto p(z) \) on \( V \) and with differential operators \( p(\partial) \) with constant coefficients. Under this identification, a vector \( \xi \in V \) corresponds to the linear function \( z \mapsto (\xi, z) \) and to the derivative \( \partial_\xi \) in the direction of \( \xi \).

Let us consider the following system of equations

\[
\sigma_i(\partial) \phi = 0, \quad i = 1, \ldots, n,
\]

where \( \sigma_1, \ldots, \sigma_n \) as above are the generators of \( S^G \). The solutions of this system form a \( |G| \)-dimensional space \( H \) and turn out to be polynomial [29]. These polynomials are called harmonic and play an important role in the theory of Coxeter groups and symmetric spaces (see [17]). It is known that they freely generate \( S(V) \) as a module over \( S^G \) and the corresponding Poincaré polynomial

\[
P(H, t) = \sum_k t^k \dim H^k,
\]

where \( H^k \) is the subspace of harmonic polynomials of degree \( k \), has the form

\[
P(H, t) = \prod_{k=1}^n \frac{1 - t^{d_k}}{1 - t},
\]

d_1, \ldots, d_n are the degrees of the basic invariants \( \sigma_1, \ldots, \sigma_n \).
In the present paper, we investigate a generalization of harmonic polynomials — the so-called \( m \)-harmonic polynomials introduced recently in [11].

To define them let us assign to each reflection hyperplane \( \Pi_\alpha \) a nonnegative integer \( m_\alpha \in \mathbb{Z}_{\geq 0} \) such that if \( \Pi_\alpha = g(\Pi_\beta) \), \( g \in G \) then \( m_\alpha = m_\beta \). Such a function \( m \) on the set of the reflections is called multiplicity. Since this function is constant on each orbit of the action of \( G \) on its reflection hyperplanes, the number \( q \) of parameters is equal to the number of such orbits (or equivalently, the number of conjugacy classes \( C_1, \ldots, C_q \) of reflections in \( G \)). For irreducible groups \( G \), \( q \) is actually either 1 or 2.

Corresponding \( m \)-harmonic polynomials are defined as the solutions of the following system generalizing (1):
\[
\mathcal{L}_i \phi = 0, \quad i = 1, \ldots, n,
\]
(2)
where
\[
\mathcal{L}_1 = \Delta - \sum_{\alpha \in A} 2m_\alpha \frac{\partial}{\partial x},
\]
is (up to a gauge transformation) the generalized Calogero–Moser operator and \( \mathcal{L}_i \) are its quantum integrals with the highest terms \( \sigma_i(\partial) \) (see the next section for details). As it was shown in [11] all the solutions of this system are polynomial and form a \( |G| \)-dimensional space which is denoted as \( H_m \). When the multiplicity is zero we have the space \( H_0 \) of usual harmonic polynomials. For dihedral groups and constant multiplicity functions all \( m \)-harmonic polynomials have been described in [11] but for the general groups even the question of the degrees of such polynomials was open.

Our main idea to attack this problem goes back to Matsuo’s and Cherednik’s observations that the Calogero–Moser system
\[
\mathcal{L}_i \phi = \sigma_i(\lambda) \phi, \quad i = 1, \ldots, n.
\]
(3)
for generic \( \lambda \in V \), namely if \( \prod_{\alpha \in A}(\lambda, \alpha) \neq 0 \), is equivalent to a certain version of the Knizhnik–Zamolodchikov (KZ) equations (see [23], [5] and Section 3). These equations form a compatible system of first order equations for a function \( u \) on \( V \) taking values in the group algebra \( \mathbb{C}[G] \). They have the form
\[
\partial_\xi u(x) = \sum_{\alpha \in A} m_\alpha \frac{(\alpha, \xi)}{(\alpha, x)} (s_\alpha + 1) u(x) + \pi(\xi, \lambda) u(x), \quad \xi \in V,
\]
(4)
where \( \pi(\xi, \lambda) \) is the linear endomorphism of \( \mathbb{C}[G] \) so that \( \pi(\xi, \lambda) g = (\xi, g\lambda) g \), for \( g \in G \). Then, for generic \( \lambda \), the composition with the alternating representation \( \mu : u \mapsto \epsilon \circ u \) is an isomorphism from the sheaf of local solutions of the KZ equations to the sheaf of local solutions of the Calogero–Moser system (3).

For non-generic \( \lambda \), in particular for \( \lambda = 0 \) this construction does not work, as the map \( \mu \) fails to be an isomorphism if \( (\lambda, \alpha) = 0 \) for some \( \alpha \in A \).

For this reason, we use a modification, due essentially to Cherednik [5], [6], of the KZ equations and of the isomorphism \( \mu \), which works for all \( \lambda \in V \) including \( \lambda = 0 \). The idea is that, instead of \( \mathbb{C}[G] \), it is more natural to consider the KZ equations for a function with values in the \( G \)-module \( S(V)/I(\lambda) \), where \( I(\lambda) \) is the ideal generated by \( G \)-invariant polynomials vanishing at \( \lambda \). This module is (non-uniquely) isomorphic to \( \mathbb{C}[G] \) for all \( \lambda \in V \). The KZ equations with values in \( S(V)/I(\lambda) \) still have the form (3) with a suitable \( \pi(\xi, \lambda) \), namely the multiplication by \( \xi \in V \subset S(V) \). These equations come with a Matsuo-Cherednik map \( \mu \) from
local solutions to local solutions of the Calogero–Moser system (3), which is an isomorphism for all $\lambda \in V$, see Theorem 3.4. The equations (12) are recovered from the KZ equations with values in $S(V)/I(\lambda)$ via the map $S(V)/I(\lambda) \to \mathbb{C}[G]$ induced from the morphism sending a polynomial $p \in S(V)$ to $\sum_{g \in G} p(g \lambda) g$. This map is an isomorphism only for generic $\lambda$.

This explicit form of the Calogero–Moser system (2) as a holonomic system seems to be quite convenient. We show that it allows one not only to find the degrees of $m$-harmonic polynomials but to compute them explicitly in some special cases.

An application of our construction is a formula for the Poincaré polynomial $P(H_m, t)$ for the general Coxeter group $G$ and multiplicity function $m$.

Let us describe this formula, which is given as a sum of the Poincaré polynomials for each isotypical component. Let $V_1, ..., V_p$ be a list of all inequivalent irreducible representations of $G$. For any representation $V_j$ we can ask what is the multiplicity $p_k(V_j)$ of this representation in the $G$-module of (usual) harmonic polynomials of degree $k$ and define the corresponding Poincaré polynomial

$$P_j(t) = \sum_k p_k(V_j)t^k.$$  

These polynomials are known for all Coxeter groups (see 4.1 below). They obey Poincaré duality

$$P_j^*(t) = t^N P_j(t^{-1}),$$  

where $V_j^*$ is the tensor product of $V_j$ by the alternating representation and $N$ is the total number of the reflections in $G$.

Let us define now for any conjugacy class of reflections $C_a$ the number

$$d_a^-(V_j) = \frac{2N_a \dim(V_j^-)}{\dim(V_j)}, \quad a = 1, \ldots, q,$$

where $Na$ is the number of elements in $C_a$ and $(V_j^-)$ is the $-1$-eigenspace of the action of any $s_\alpha \in C_a$ on $V_j$. This number can be expressed in terms of the polynomials $P_j(t)$ in the following way. Let $P_{j \otimes \alpha}(t)$ be the Poincaré polynomial corresponding to the representation $V_j \otimes \chi_\alpha$ where $\chi_\alpha$ is the one-dimensional representation of $G$ such that

$$\chi_\alpha(s) = \begin{cases} -1, & \text{if } s \in C_a, \\ 1, & \text{if } s \in C_b, b \neq a, \end{cases}$$

We show that

$$d_a^-(V_j) = N_a + \frac{d}{dt} \bigg|_{t=1} \ln \frac{P_j(t)}{P_{j \otimes \alpha}(t)}, \quad a = 1, \ldots, q.$$

If the group $G$ acts transitively on its reflection hyperplanes (i.e. $q = 1$) the formula simplifies to Solomon’s formula [27]

$$d^-(V_j) = d_a^-(V_j) = 2 \frac{d}{dt} \bigg|_{t=1} \ln P_j(t).$$

**Theorem 1.1.** The Poincaré polynomial of the graded space of $m$-harmonic polynomials has a form

$$P(H_m, t) = \sum_{j=1}^p \dim(V_j) t^{d_j^-(m)} P_j(t),$$
where $d_j^m(m) = \sum_{a=1}^q m_a d_a^m(V_j)$. It has degree $M = \sum_{\alpha \in A} (2m_\alpha + 1) = \sum_{a=1}^q N_a (2m_a + 1)$ and satisfies the palindromic relation $P(H_m, t) = t^M P(H_m, t^{-1})$.

For example, in the case of the symmetric group $G = S_n$, the representations are given by Young diagrams with $n$ boxes. Let the arm length $a_k$ of the $k$th box of a Young diagram be the number of boxes on its right in the same row, and the leg length be the number of boxes below it in the same column. The hook length of the $k$th box is then $h_k = a_k + \ell_k + 1$. Then the formula for $P(H_m, t)$ can be written as

$$P(H_m, t) = \frac{n! t^{mn(n-1)/2}}{\prod_{k=1}^n t^{m(\ell_k-a_k)+\ell_k} \frac{1-t^k}{h_k(1-t^k)}},$$

see 4.5 below.

It seems to be impossible to generalize the product formula for $P(H, t)$ for general $m$ since already the dihedral case showed that different isotypical components corresponding to irreducible representations of $G$ behave in a different way as a function of $m$.

We should mention that the description of the action of the Coxeter group on the space of solutions of (2) can be extracted also from Opdam’s papers [25], [26]. We believe that our derivation is more illuminating and can be used also for effective description of $m$-harmonic polynomials.

The paper is organised as follows. In Section 2 we review the basic facts about the Calogero–Moser systems and $m$-harmonic polynomials. The construction of the system of KZ equations and of the Matsuo–Cherednik isomorphism is given in Section 3. In Section 4 we apply this construction to $m$-harmonic polynomials. We first describe the action of the Coxeter group on the space of solutions of the KZ equations and give the proof of Theorem 1.1. Then we give an explicit construction of $m$-harmonic polynomials for $G = S_n$ of lowest positive degree using an integral representation of solutions of the KZ equations. Finally, in the $S_n$ case, the asymptotic distribution of degrees of harmonic polynomials for large $m$ and $n$ are described using results of Kerov on the statistical properties of large Young diagrams. We conclude our paper with Section 5, where we comment on interesting recent developments and open questions in this subject.

2. Generalized Calogero–Moser systems and $m$-harmonic polynomials

The generalized Calogero–Moser operator related to a Coxeter group $G$ and a multiplicity function $m$ (not necessary integer-valued but $G$-invariant) was introduced by Olshanetsky and Perelomov [24] and has the form

$$L = \Delta - \sum_{\alpha \in A} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}.$$

We will use its gauge equivalent version $\mathcal{L} = \hat{g} L \hat{g}^{-1}$, where $\hat{g}$ is the operator of multiplication by $g = \prod_{\alpha \in A} (\alpha, x)^{m_\alpha}$ which has a form

$$\mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{\alpha, x} \partial_x.$$

This operator is one of $n$ commuting operators $\mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2, ..., \mathcal{L}_n$ with the highest symbols $\sigma_i$. One of the best ways to describe these operators has been discovered

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1We are grateful to I. Cherednik who attracted our attention to these very interesting papers
by G. Heckman \[14\] and uses the following difference-differential Dunkl operators \[8\]

\[ D_\xi u(x) = \partial_\xi u(x) + \sum_{\alpha \in A} m_\alpha (\alpha, x) (u(s_\alpha x) - u(x)), \tag{9} \]

where \( \xi \in V \). A remarkable property of these operators is their commutativity:

\[ [D_\xi, D_\eta] = 0 \]

for all \( \xi, \eta \in V \). This allows us to define the difference-differential operators \( \sigma_i(D) \) corresponding to the basic invariants \( \sigma_i \in S^G \). Now the operators \( L_i \) can be defined as restrictions of the operators \( \sigma_i(D) \) on the space of \( G \)-invariant functions (see \[14\]).

Consider the joint eigenspace of these operators:

\[
\begin{aligned}
L_1 \phi &= \sigma_1(\lambda) \phi \\
\vdots \\
L_n \phi &= \sigma_n(\lambda) \phi.
\end{aligned}
\tag{10}
\]

**Theorem 2.1.** \[11\] For any Coxeter group \( G \) and integer multiplicity function \( m \) all the solutions of the system \( \text{(10)} \) are holomorphic everywhere. They form a space of dimension \( |G| \) where the natural action of \( G \) is its regular representation. When \( \lambda = 0 \) all the solutions are polynomial.

When all the multiplicities are zero we have the usual harmonic polynomials related to a Coxeter group and this result is well-known (see \[29, 17\]). Following \[11\] we will call the solutions of the system

\[
\begin{aligned}
L_1 \phi &= 0 \\
\vdots \\
L_n \phi &= 0.
\end{aligned}
\tag{11}
\]

\( m \)-harmonic polynomials and denote the corresponding space \( H_m \).

The system \( \text{(10)} \) with generic \( m \) has been discussed by Heckman and Opdam \[13\] (see also \[25\]) who showed that it can be represented as a holonomic system of rank \( |G| \) but have never written this system explicitly. In the next section we present such a representation.\[2\]

3. Knizhnik–Zamolodchikov equations and the isomorphism theorem

In this section, we explain the construction of the Matsuo–Cherednik isomorphism. It is essentially an adaptation to the rational Coxeter group case of a construction of Cherednik (see \[1\] and \[3\]), who treated the trigonometric case for Weyl groups. However we still present it here, since in this case the results can be proved more directly, without Hecke algebra theory.

We are going first to introduce a certain version of Knizhnik–Zamolodchikov (KZ) equations on \( V \) with values in \( G \)-modules. By a \( G \)-module we mean a left module over the group algebra \( \mathbb{C}[G] \). The alternating (sign) representation \( \epsilon \) is defined as the homomorphism \( G \to \{1, -1\} \) such that \( \epsilon(s_\alpha) = -1 \) for all reflections \( s_\alpha \). The alternating \( G \)-module \( \mathbb{C}_\epsilon \) is the one-dimensional module with action given by \( \epsilon \).

\[2\] There is a remark in Opdam’s paper \[25\] (see page 335) which suggests a possibility of such representation.
3.1. **KZ equations with values in G-modules.** Let $M$ be a $G$-module, $\pi : V \to \text{End}_G(M)$ be a $G$-equivariant linear map whose image $\pi(V)$ is contained in a commutative subalgebra. The *Krichever–Zamolodchikov connection* is the system of first order differential operators

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in A} m_\alpha \frac{(\alpha, \xi)}{(\alpha, x)} (s_\alpha + 1) - \pi(\xi), \quad \xi \in V,$$

acting on functions $\psi : V - \cup_{\alpha} \Pi_\alpha \to M$.

**Lemma 3.1.** \[3, 9\] Let $M$ be a $G$-module, $\pi \in \text{Hom}_G(V, \text{End}_G(M))$, such that $\pi(\xi) \pi(\eta) = \pi(\eta) \pi(\xi)$ for all $\xi, \eta \in V$. Then

(i) The KZ connection is flat: $\nabla_\xi \nabla_\eta = \nabla_\eta \nabla_\xi$, $\forall \xi, \eta \in V$.

(ii) Let $G$ act on $M$-valued functions on $V$ by $(^g \psi)(x) = g(\psi(g^{-1}x))$, $g \in G$. Then the KZ connection is $G$-equivariant: $\nabla_g^\psi = ^g(\nabla_\xi \psi)$, $g \in G$.

It follows that if $\dim(M) = d$, the solutions of the system of KZ equations with values in $M$,

$$\nabla_\xi \psi(x) = 0, \quad \xi \in V$$

on any connected, simply connected open subset of $V - \cup_{\alpha} \Pi_\alpha$ form a $d$-dimensional vector space. Moreover, global solutions form a $G$-module.

For $\lambda \in V$, let $I(\lambda)$ be the ideal in $S$ generated by invariant polynomials vanishing at $\lambda$. It is clearly invariant under the action of $G$. We consider the KZ equations with values in the $G$-module $M(\lambda) = S/I(\lambda)$, with $\pi(\xi)$ being the operator of multiplication by the linear function $z \mapsto (\xi, z)$:

$$\partial_\xi \psi(x) = \sum_{\alpha \in A} m_\alpha \frac{(\alpha, \xi)}{(\alpha, x)} (s_\alpha + 1) \psi(x) + \pi(\xi) \psi(x), \quad \xi \in V. \tag{12}$$

We now show that the $G$-module $S/I(\lambda)$ is isomorphic to $C[G]$. The isomorphism is not canonical and we choose it in such a way that it is well-behaved as $\lambda \to 0$.

**Lemma 3.2.** The composition $H_0 \to S \to M(\lambda)$ of the inclusion with the canonical projection is an isomorphism of $G$-modules. In particular, $M(\lambda)$ is isomorphic as a $G$-module to $C[G]$ with left action of $G$.

**Proof:** For $\lambda = 0$ this is well-known fact about harmonic polynomials (see \[7\]).

Let $h_1, \ldots, h_{|G|}$ be a basis of $H_0$. Then, by Chevalley’s theorem \[5\] (see also \[17\]), any polynomial in $S$ may be uniquely written as $q = \sum_{i=1}^{|G|} p_i h_i$, with $p_i \in S^G$. We claim that $q \in I(\lambda)$ if and only if $p_i(\lambda) = 0$ for all $i$. Indeed, if $p_i(\lambda) = 0$ for all $i$ then it is clear that $q \in I(\lambda)$. Conversely, suppose that $q = \sum p_i h_i \in I(\lambda)$. Then, by definition, $q = \sum p'_k q_k$ with $p'_k \in S^G$ vanishing at $\lambda$. By expressing each $q_k$ as a linear combination of $h_i$ with coefficients is $S^G$, get that $q = \sum p''_i h_i$ with $p''_i \in S^G$ and $p''_i(\lambda) = 0$. By uniqueness we deduce that $p''_i = p_i$ and the claim follows.

We thus can define a map of $G$-modules from $I(0)$ to $I(\lambda)$ by $\phi_\lambda : \sum p_i h_i \mapsto \sum (p_i - p_i(\lambda)) h_i$. It is an isomorphism with inverse map $\sum p_i h_i \mapsto \sum (p_i - p_i(0)) h_i$. This map sends polynomials of degree $\leq k$ to polynomials of degree $\leq k$. In particular, if we denote by $F^k(S) = \bigoplus_{j=0}^k S^j$ the polynomials of $S$ of degree $\leq k$, then $\dim(F^k(S)/(F^k(S) \cap I(\lambda)))$ is independent of $\lambda$ for all $k$, and thus $\dim(S/I(\lambda)) = \dim(S/I(0))$. Moreover, $\phi_\lambda$ maps a homogeneous polynomial $p \in I(0)$ to a polynomial of the form $p + q$ with $\deg(q) < \deg(p)$. As a consequence, if $h \neq 0$ is in the
3.2. The Matsuo–Cherednik map. The Matsuo–Cherednik map is defined in terms of the function

\[ w_{\lambda}(y) = \frac{\sum_{g \in G} \epsilon(g) e^{\langle g, \lambda \rangle}}{\prod_{\alpha \in A} (\alpha, \lambda)^{|\epsilon(g)|}}. \]

Lemma 3.3.

(i) The function \( w_{\lambda}(y) \) is holomorphic in \( \lambda, y \in V \). It obeys \( w_{\lambda}(y) = \epsilon(g) w_{\lambda}(gy) = w_{g\lambda}(y) \), \( g \in G \).

(ii) For any fixed \( \lambda \), the function

\[ y \mapsto \frac{w_{\lambda}(y)}{\prod_{\alpha \in A} (\alpha, y)} \]

is holomorphic on \( V \) and is not zero at \( y = 0 \).

(iii) \( w_0(y) = C \prod_{\alpha \in A} (\alpha, y) \) for some non-zero number \( C \in \mathbb{R} \).

(iv) Let \( \lambda \in V \), \( p \in S \). Then \( p(\partial) w_{\lambda} = 0 \) if and only if \( p \in I(\lambda) \).

Proof: (i) Since the numerator obeys \( n(s_{\alpha} \lambda, y) = -n(\lambda, y) \), \( \alpha \in A \), it is divisible by \( \prod_{\alpha \in A} (\alpha, \lambda) \) in the ring of holomorphic functions. The behavior under the \( G \)-action follows from the fact that the numerator is skew-invariant as a function of both \( \lambda \) and \( y \), and the denominator is a skew-invariant function of \( \lambda \). (ii) By the same argument, \( w_{\lambda}(y) \) is also divisible by \( \prod_{\alpha \in A} (\alpha, y) \). Let \( C(\lambda) \) be the value of the quotient at \( y = 0 \). Then the Taylor series at 0 of \( w_{\lambda}(y) = C(\lambda) \prod_{\alpha \in A} (\alpha, y) + \cdots \). To show that \( C(\lambda) \neq 0 \) it is sufficient to find a polynomial \( w \in S^{N}, N = |A| \), such that \( w(\partial) w_{\lambda}(y) \) does not vanish at \( y = 0 \). Let \( w(z) = \prod_{\alpha \in A} (\alpha, z) \). Then we have

\[ w(\partial) e^{\langle g, \lambda \rangle} = w(g\lambda) e^{\langle g, \lambda \rangle} = (g) w(\lambda) e^{\langle g, \lambda \rangle}. \]

It follows that \( w(\partial) w_{\lambda}(y) |_{y=0} = |G| \neq 0 \). (iii) It follows from the homogeneity relation \( w_{\lambda}(ay) = a^{N} w_{\lambda}(y) \), that \( w_{0} \) is a homogeneous polynomial of degree \( N \).

With (ii), this implies the claim. (iv) If \( p \in S^{G} \), \( p(\partial) w_{\lambda} = p(\lambda) w_{\lambda} \). Thus \( p(\partial) w_{\lambda} \) vanishes if \( p \in I(\lambda) \). Conversely, suppose that \( p(\partial) w_{\lambda}(y) = 0 \) and let \( p_{h} \) be the homogeneous term of highest degree in \( p \). Then \( p_{h}(\partial) \) vanishes on the term of lowest degree of \( w_{\lambda} \), namely, by (ii), \( p_{h}(\partial) \prod_{\alpha \in A} (\alpha, y) = 0 \). It follows, see [17] or Lemma 3.2, that \( p_{h} \in I(0) \), i.e., \( p_{h}(z) = \sum q_{i}(z) r_{i}(z) \) with \( r_{i} \in S^{G} \) homogeneous of positive degree. Let \( \bar{p}_{h}(z) = \sum q_{i}(z) (r_{i}(z) - r_{i}(\lambda)) \). Clearly, \( \bar{p}_{h} \in I(\lambda) \) and has leading term \( p_{h}(z) \). Thus \( p \) can be written as

\[ p(z) = \bar{p}_{h}(z) + q(z) \equiv q(z) \mod I(\lambda), \]

where \( q \) has lower degree and \( q(\partial) w_{\lambda} = 0 \). By induction on the degree, we thus see that \( p \in I(\lambda) \). \( \square \)

Let \( \mu : S \to \mathbb{C} \) be the \( G \)-module homomorphism

\[ p \mapsto p(\partial) w_{\lambda} |_{y=0}. \]

By Lemma 3.3 (iv), \( \mu \) induces a map, also denoted by \( \mu \), from \( S/I(\lambda) \) to \( \mathbb{C} \).

We first formulate the local version of the isomorphism theorem.
Theorem 3.4. Let \( \lambda \in V, U \) a connected, simply connected open subset of \( V - \cap_{\alpha \in A} \Pi_\alpha \). The Matsuo–Cherednik map

\[ \psi \mapsto \phi = \mu \circ \psi \]

is an isomorphism from the space of solutions \( KZ(\lambda; U) \) of the KZ equations (12) on \( U \) with values in \( S/I(\lambda) \) to the space \( \mathrm{CM}(\lambda; U) \) of solutions of the system of differential equations

(13) \[ L_i \phi = \sigma_i(\lambda) \phi, \quad i = 1, \ldots, n. \]

of the Calogero–Moser system (10).

This theorem is proven in 3.4 below.

The global version of the isomorphism theorem follows from the fact that all local solutions of (13) extend to global solutions (see Section 2).

Theorem 3.5. Let \( \lambda \in V \). The map

\[ \psi \mapsto \phi = \mu \circ \psi \]

is an isomorphism of \( G \)-modules from the space of global solutions \( KZ(\lambda) \) of the KZ equations (12) with values in \( S/I(\lambda) \) to the space \( \mathrm{CM}(\lambda) \otimes \mathbb{C}_\epsilon \) of global solutions of the system of differential equations (13) of the Calogero–Moser system, with \( G \)-action twisted by \( \epsilon \).

Finally, by Theorem 2.1, all solutions of (13) extend to holomorphic functions on all of \( V \). This has the following consequence.

Corollary 3.6. All solutions of the KZ equations (12) extend to holomorphic functions on \( V \). If \( \lambda = 0 \), all solutions of the KZ equations (12) are polynomial.

Theorem 3.5 and Corollary 3.6 are proven in 3.5.

Remark. Matsuo [23] considered the KZ equations with values in the left regular \( G \)-module. He considered similar maps from the solution space of the KZ equations to \( \mathrm{CM}(\lambda) \), which are isomorphisms as long as \( \lambda \) does not belong to the discriminant locus \( \cup_{\alpha \in A} \Pi_\alpha \). Cherednik [3, 4] put the construction in the setting of Hecke algebras, and considered more generally KZ equations with values in representations of Hecke algebras of Weyl groups (in the trigonometric case). In particular he studied the KZ equations with values in an induced Hecke algebra representation and constructed a variant of the Matsuo map which is an isomorphism for all values of the spectral parameters \( \lambda \). In the rational limit, Cherednik’s construction reduces to ours (for Weyl groups).

3.3. A non-degenerate bilinear form on \( S/I(\lambda) \). A convenient technical tool in the proofs is a non-degenerate bilinear form. Let \( O(V) \) be the space of holomorphic functions on \( V \). The bilinear form \( \langle p, q \rangle = p(\bar{\partial})q(y)\big|_{y=0} \) on \( S \) extends to a pairing \( S \times O(V) \to \mathbb{C} \) defined by the same formula. For \( \lambda \in V \), let \( \langle \ , \ \rangle_\lambda \) be the symmetric bilinear form on \( S \) defined by

\[ \langle p, q \rangle_\lambda = p(\bar{\partial})q(\bar{\partial})w_\lambda(y)\big|_{y=0} = \langle p, q(\bar{\partial})w_\lambda \rangle. \]

Lemma 3.7. The bilinear form \( \langle \ , \ \rangle_\lambda \) induces a well-defined symmetric non-degenerate bilinear form on \( S/I(\lambda) \).
Proof: By Lemma 3.3 (iv), \((p, q)\lambda\) vanishes if one of the arguments is in \(I(\lambda)\) so it is well-defined on the quotient. It is clearly symmetric. Suppose \((p, q)\lambda = 0\) for all \(p \in S\). This means all Taylor coefficients of \(q(\partial)w_\lambda\) vanish. Since it is a holomorphic function on \(V\), this function vanishes identically. By Lemma 3.3 \(q \in I(\lambda)\). This shows that the induced bilinear form on \(S/I(\lambda)\) is non-degenerate. \(\square\)

Note that the map \(\mu\) is \(p \mapsto (p, 1)_\lambda = \langle p, w_\lambda \rangle\).

3.4. Proof of Theorem 3.4. We adapt Matsuo’s construction to this case. We first show that the map sends solutions of the KZ equations to eigenfunctions, following his proof word by word.

We need a description of the polynomial algebra \(D = \mathbb{C}[\mathcal{L}_1, \ldots, \mathcal{L}_n]\) generated by the commuting operators \(\mathcal{L}_i\). These differential operators are defined through Dunkl operators \([8], [14]\), by the condition that \(L_\lambda\) acts on invariant functions as the restriction of the differential-difference operators \(\sigma_i(D)\). An other system of generators is more convenient here \([10]\). Define for each \(\xi \in V\), differential operators \(D^{(d)}\) recursively by \(D^{(0)} = 1\) and

\[
D^{(d)} = \partial_\xi D^{(d-1)} + \sum_{\alpha \in A} m_\alpha (\alpha, \xi) (D_{s_\alpha \xi}^{(d-1)} - D_{\xi}^{(d-1)}).
\]

Alternatively, \(D^{(d)}\) is the unique differential operator whose restriction to \(S^G\) coincides with the restriction to \(S^G\) of the \(d\)th power of the Dunkl differential-difference operator \([8]\). The algebra \(D\) is generated by the operators \(D_{\xi,d} = \sum_{g \in G} D_{g\xi}\), \(\xi \in V\), \(d \in \mathbb{Z}_{\geq 0}\). Indeed, \(D_{\xi,d} = p_{\xi,d}(D)\) on invariant functions, where \(p_{\xi,d}(x) = \sum_{g \in G} (g\xi, x)\); since any invariant polynomial can be written as a linear combination of polynomials \(p_{\xi,d}\), it is clear that \(D\) is spanned by the operators \(D_{\xi,d}\). Thus the space \(CM(\lambda)\) consists of solutions of the differential equations

\[
D_{\xi,d} \phi(x) = \sum_{g \in G} (g\xi, \lambda)^d \phi(x)
\]

Let \(O(V)\) be the space of holomorphic functions on \(V\).

Lemma 3.8. Let \(\psi \in KZ(\lambda; U)\), \(\phi(x) = \mu \circ \psi(x) = \langle \psi(x), w_\lambda \rangle\). Then

\[
D^{(d)} \phi(x) = \mu(\pi(\xi)^d \psi(x)).
\]

Proof: Clearly this holds for \(d = 0\). Assume inductively that the claim holds for \(D_{\xi}^{(d-1)}\) and all \(\xi \in V\). Then

\[
D^{(d)} \langle \psi(x), w_\lambda \rangle = \partial_\xi D^{(d-1)} \langle \psi(x), w_\lambda \rangle \]

\[
+ \sum_{\alpha \in A} m_\alpha (\alpha, \xi) (D_{s_\alpha \xi}^{(d-1)} - D_{\xi}^{(d-1)}) \langle \psi(x), w_\lambda \rangle
\]

\[
= \partial_\xi (\pi(\xi)^{d-1} \psi(x), w_\lambda) \]

\[
+ \sum_{\alpha \in A} m_\alpha (\alpha, \xi) ((\pi(s_\alpha \xi)^{d-1} - \pi(\xi)^{d-1}) \psi(x), w_\lambda).
\]
Since \( \psi \) is a solution of the KZ equations, we have
\[
\partial_\xi \langle \pi(\xi)^d \psi(x), w_\lambda \rangle = \sum_{\alpha \in A} m_\alpha \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} \langle \pi(\xi)^d (s_\alpha + 1) \psi(x), w_\lambda \rangle \\
+ \langle \pi(\xi)^d \psi(x), w_\lambda \rangle \\
= \sum_{\alpha \in A} m_\alpha \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} \langle (-\pi(s_\alpha \xi)^d - 1 + \pi(\xi)^d - 1) \psi(x), w_\lambda \rangle \\
+ \langle \pi(\xi)^d \psi(x), w_\lambda \rangle,
\]
and it follows that \( D^{(d)}(\phi)(x) = \langle \pi(\xi)^d \psi(x), w_\lambda \rangle \). \( \square \)

**Lemma 3.9.** If \( \psi \in KZ(\lambda; U) \) then \( \phi(x) = \mu \circ \psi(x) \in CM(\lambda; U) \).

**Proof:** By Lemma 3.8,
\[
D_{\xi, \mu}(\phi)(x) = \langle \sum_{g \in G} (\pi(g \xi))^d \psi(x), w_\lambda \rangle \\
= \langle \psi(x), \sum_{g \in G} \partial_{g \xi}^d w_\lambda \rangle.
\]
For any invariant polynomial \( p \in S^G \), we have \( p(\partial) w_\lambda = p(\lambda) w_\lambda \). In particular,
\[
\sum_{g \in G} \partial_{g \xi}^d w_\lambda = \sum_{g \in G} (g \xi, \lambda)^d w_\lambda,
\]
which implies the claim. \( \square \)

It remains to prove that the Matsuo–Cherednik map \( \psi \to \mu \circ \psi \) is an isomorphism between solution spaces on \( U \). By Theorem 2.4, \( CM(\lambda; U) \) is a \( |G| \)-dimensional space. Since the KZ equations are a holonomic system with local solution space of the same dimension, it suffices to show that the kernel of the Matsuo–Cherednik map is trivial. Let \( \psi \) be in the kernel, i.e., \( \langle \psi(x), w_\lambda \rangle = 0, \forall x \in U \). By using the KZ equations, we may express \( \langle \psi, \partial_\xi u \rangle = \langle \pi(\xi) \psi(x), u \rangle \) in terms of \( \langle \psi(x), u \rangle \) and \( \langle \psi(x), s_\alpha u \rangle \). It follows inductively that \( \langle \psi(x), p(\partial) w_\lambda \rangle = 0 \), for any \( p \in S \). Thus, by Lemma 3.7, \( \psi(x) \) vanishes identically.

### 3.5. Proof of Theorem 3.3 and of Corollary 3.4

We have \( \mu \circ g \psi(x) = \mu(g \psi(g^{-1}x)) = \epsilon(g) \mu \circ \psi(g^{-1}x) \). This means that the Matsuo–Cherednik map is a morphism of \( G \)-modules if we twist the \( G \)-action on \( CM(\lambda) \) by \( \epsilon \). Local solutions of (13) extend to holomorphic functions on \( V \), see Theorem 2.3. Therefore, if \( \psi \) is a local solution of the KZ equations, then \( \langle \psi(x), w_\lambda \rangle \) extends to a holomorphic function on \( V - \cup_\alpha \Pi_\alpha \). It follows as above, by using the equations, that all coordinates \( \langle \psi(x), p(\partial) w_\lambda \rangle \) of \( \psi \) are meromorphic functions with possible poles on the reflection hyperplanes. We now show that these functions are holomorphic on \( V \). By Hartogs’ theorem it is sufficient to show that they are holomorphic at generic points of the hyperplanes.

In the vicinity of a point of the hyperplane \( \Pi_\alpha \) which does not lie on any other hyperplane, a nontrivial solution \( \psi \) has a Laurent expansion \( \psi(x) = (\alpha, x)^k \psi_k + (\alpha, x)^{k+1} \psi_{k+1} + \cdots \) for some functions \( \psi_k, \psi_{k+1}, \ldots \) of the transversal coordinates and \( \psi_k \neq 0 \). The characteristic exponent \( k \) is an eigenvalue of \( m_\alpha(s_\alpha + 1) \), and is therefore 0 or \( 2m_\alpha \geq 0 \). This proves that \( \psi \) is regular.
Moreover, we know from Theorem 2.1 that if \( \lambda = 0 \), eigenfunctions \( \phi \) are polynomials, so solutions of the KZ equations are regular rational functions, i.e., polynomials.

4. Applications to \( m \)-harmonic polynomials

In this section, we apply the Matsuo–Cherednik isomorphism to the most degenerate case \( \lambda = 0 \), the case of \( m \)-harmonic polynomials. In particular, we study the action of the Coxeter group on the space \( H_m \) of \( m \)-harmonic polynomials. As \( H_m \) is, as a \( G \)-module, isomorphic to the left regular module, we know that the multiplicity of any simple \( G \)-module in \( H_m \) is equal to its dimension. A more detailed information is given by the Poincaré polynomial of a simple module, which is the generating function of the multiplicities of the simple module in the space of \( m \)-harmonic polynomials of given degree. Let \( V_1, \ldots, V_p \) be a list of all inequivalent simple \( G \)-modules up to isomorphism. Let

\[
\mu(V_j, H^d_m) = \dim \text{Hom}_G(V_j, H^d_m)
\]

be the multiplicity of \( V_j \) in the space \( H^d_m \) of homogeneous \( m \)-harmonic polynomials of degree \( d \). We set

\[
P_j(H_m, t) = \sum_{d \geq 0} \mu(V_j, H^d_m) t^d.
\]

Since \( H_m \) is isomorphic to the left regular \( G \)-module, the Poincaré polynomial of \( V_j \) obeys \( P_j(H_m, 1) = \dim(V_j) \).

Our formula for this polynomial is given in terms of the corresponding polynomial for \( m = 0 \). Thus we first review what is known for \( m = 0 \).

4.1. The action of \( G \) on harmonic polynomials. By Chevalley’s theorems, the ring of invariants \( S^G \) is freely generated by homogeneous polynomials \( \sigma_j \) of degree \( d_j \), \( j = 1, \ldots, n \), and \( S \) is a free module over \( S^G \). Moreover, as a basis of \( S \) over \( S^G \) we may take any basis of the space \( H_0 \) of harmonic polynomials. It follows that the Poincaré polynomial \( P_j(H_0, t) \) can be expressed in terms of the generating series \( P_j(S, t) \) of the multiplicities of \( V_j \) in the space of all polynomials of given degree:

\[
P_j(H_0, t) = \prod_{k=1}^{n} (1 - t^{d_k}) P_j(S, t).
\]

These generating series are known for all simple modules of all Coxeter groups: the case of Weyl groups of classical root systems is treated by Lusztig in [22], §2, (in the \( A_k \) his formula is deduced from results of Steinberg [28]). Simpler formulae for the classical cases were discovered by Kirillov [23]. Formulae for \( F_4 \), due to Macdonald, and tables for the coefficients of \( P_j(H_0, t) \) for exceptional Weyl groups are given in [2]. As for general Coxeter groups, the case of the dihedral groups is easy to work out by hand, see the first example below. The case of \( H_4 \) was treated in [1]. For a comprehensive overview and a computer program see [12].

Examples.

1. The dihedral group \( G = I_2(N) \) of symmetries of a regular \( N \)-gon has exponents \( d_1 = 2 \) and \( d_2 = N \). If \( N \) is odd, there are two one-dimensional modules, the trivial module \( V_0 \) and the alternating module \( V_N \). There are
$N-1$ two-dimensional simple $G$-modules $V_1 = V, V_2, \ldots, V_{N-1}$. The $2\pi/N$-rotation has eigenvalues $\exp(\pm 2\pi ij/N)$ in $V_j$. We then have $P_j(S, t) = P_j(H_0, t)/(1-t^j)(1-t^N)$, and, if $N$ is odd,

$$P_0(H_0, t) = 1, \quad P_N(H_0, t) = t^N, \quad P_j(H_0, t) = t^j + t^{N-j},$$

for $1 \leq j \leq N-1$. If $N = 2k$ is even, $V_k$ is replaced by two one-dimensional modules $V_k^\pm$ with $P_k^\pm(H_0, t) = t^k$. The other formulae hold without modification.

2. Let $G = S_n$ be the group of permutations of $n$ letters, generated by reflections with respect to the hyperplanes $x_j = x_k$ ($j < k$) of $\mathbb{R}^n$. Simple $G$-modules are enumerated by Young diagrams with $n$ boxes. A simple product formula for $P_j(S, t)$ was given by Kirillov [20]. The hook length of a box with coordinates $(i, j)$ in a Young diagram is the number of boxes with coordinates $(i, j+p)$ or $(i+p, j)$, $p \geq 0$. Its leg length is the number of boxes with coordinates $(i+p, j)$, $p > 0$. If $h_k, \ell_k$ denote the hook length and the leg length of the $k$th box of the Young diagram corresponding to $V_j$, we have

$$P_j(S, t) = \prod_{k=1}^n \frac{t^{\ell_k}}{(1-t^{h_k})}.$$ 

Since the exponents of $S_n$ are $d_j = j$, $j = 1, \ldots, n$, it follows that

$$P_j(H_0, t) = \prod_{k=1}^n \frac{t^{\ell_k}(1-t^k)}{(1-t^{h_k})}.$$ 

Remark. If $G$ is the Weyl group of a semisimple complex Lie algebra, $P_j(H_0, t)$ is the generating function of the multiplicity of $V_j$ in the cohomology groups of the corresponding flag variety. Thus $\sum_j \dim(V_j) P_j(H_0, t)$ is the topological Poincaré polynomial of the flag variety.

4.2. Coxeter groups and Hecke algebras. The calculation of the action of $G$ on the space of solutions of the Knizhnik–Zamolodchikov equations uses the fact (discovered in a special case in [30], and generalized in [1], see also [15] for the result in the context of Calogero–Moser systems) that the monodromy representation of this system of differential equation factors through a Hecke algebra.

Let $(u_{\alpha})_{\alpha \in A}$ be indeterminates associated to reflections of a Coxeter group $G$ so that $u_{\alpha} = u_{\beta}$ if the corresponding reflections are conjugated in $G$. Fix a presentation of $G$ with set of generators $\Sigma$ and relations $s^2 = 1, (st)^{m(s,t)} = 1$, $s, t \in \Sigma$. We write $u_s = u_{\alpha}$ if $s = s_{\alpha} \in \Sigma$. Accordingly, we write $m_s = m_{\alpha}$ if $s = s_{\alpha} \in \Sigma$. The length $\ell(g)$ of $g$ is the minimal number of factors needed to write $g$ as a product of generators. The Iwahori–Hecke algebra $H(G)$ of the Coxeter group $G$ is defined as the algebra over $R = \mathbb{Z}[u_s, s \in \Sigma]$ spanned by elements $T_g, g \in G$ and multiplication rules $T_s T_g = T_{sg}$ if $\ell(sg) > \ell(g)$ and $T_s T_g = u_s T_{sg} + (u_s - 1)T_g$ if $\ell(sg) < \ell(g)$, $s \in \Sigma$.

If we specialise the parameters $u_s$ to 1 we recover the group algebra of $G$. The algebra $H(G)^K = H(G) \otimes_R K$ over the field $K = \mathbb{C}(\sqrt{m_s}, s \in \Sigma)$ of rational functions of $\sqrt{m_s}, s \in \Sigma$, is split semisimple and isomorphic to the group algebra $KG$. Moreover every irreducible character $\chi_j$ of $G$ is the specialization at $\sqrt{m_s} = 1$ of an irreducible character $\tilde{\chi}_j(\sqrt{m_s})$ of $H(G)^K$ with values in $\mathbb{C}(\sqrt{m_s})$ (Benson–Curtis, Lusztig, Alvis–Lusztig, Digne–Michel, see [12], Theorem 9.3.5).
The function \( \mu \mapsto \hat{\chi}_j(\exp(\pi i \mu_\alpha)) \) defined for \( G \)-invariant collections \( \mu = (\mu_\alpha)_{\alpha \in A} \) of complex numbers is a holomorphic function whose values at integer points is the character of an irreducible \( G \)-module and we have for \( \mu_s = m_s \in \mathbb{Z} \),

\[
\hat{\chi}_j(\exp(\pi i m_s)) = \chi_{\pi_m(j)},
\]

for some permutation \( \pi_m \) of the set of equivalence classes of irreducible \( G \)-modules. Clearly \( \pi_{m+m'} = \pi_m \circ \pi_{m'} \), \( \pi_0 = \text{id} \) and \( \pi_m \) depends only on the class of the integers \( m_\alpha \) in \( \mathbb{Z}/2\mathbb{Z} \). In particular it is an involution. We will need two simple properties of the homomorphism \( \pi \):

(A) Let \( j \mapsto j^* \) be the tensor multiplication by the alternating representation.

Then \( \pi_m(j^*) = \pi_m(j^*) \).

(B) \( \chi_{\pi_m(j)}(s) = \chi_j(s) \) for reflections \( s \).

The first property follows from the fact that the automorphism \( \tau \) of \( H(G) \) such that \( \tau(T_s) = -T_s + (1 - u_s)1 \) for reflections \( s \), induces an involution \( j \mapsto j^* \) on the set of irreducible modules over \( H(G)^K \), which specialises to the tensor multiplication by the alternating representation at \( u_s = 1 \). The character \( \chi_{j^*} \) is then the specialisation at \( \sqrt{u_s} = 1 \) of a character \( \chi_{j^*} \) with values in \( \mathbb{C}(\sqrt{u_s}) \). By using the fact that, for any \( g \in G \), \( \tau(T_g) \in H(G) \) has coefficients which are polynomials in \( u_s \), we have

\[
\chi_{\pi_m(j)}(g) = \hat{\chi}_j(\sqrt{u_s})(\tau(T_g)) \bigg|_{\sqrt{u_s} = \epsilon \pi_m s} = \hat{\chi}_j^*(\sqrt{u_s})(T_g) \bigg|_{\sqrt{u_s} = \epsilon \pi_m s} = \chi_{\pi_m(j^*)}(g).
\]

This proves the first property.

The second property follows from the relation \( (T_s + 1)(T_s - u_s) = 0 \) for reflections \( s \) (take \( g = s \) in the multiplication rule). This relation implies that the character of any simple module evaluated at \( T_s \) has the form \( a_s u_s - b_s \), where \( a_s, b_s \) are the multiplicities of the eigenvalues \( u_s, -1 \). So there are no square roots here and \( \pi_m \) acts trivially.

The explicit description of the homomorphism \( m \mapsto \pi_m \) can be extracted from the literature on Hecke algebras and was given in [26]. In fact, \( \pi_m \) is trivial for \( A_n, B_n, D_n, E_6, F_4 \) and the dihedral groups of order \( 2N \), with \( N \) odd. For dihedral groups of order \( 4n \), \( n \geq 3 \), (this includes \( G_2 \)), \( \pi_m \) is trivial on one-dimensional modules; on two-dimensional modules it is the tensor multiplication by the one-dimensional representation \( \chi_m \) such that \( \chi_m(s_\alpha) = (-1)^{m_\alpha} \). In the remaining cases \( E_7, E_8, H_3, H_4 \), for which the reflections are all in the same conjugacy class, \( \pi_{m=1} \) exchanges one or two pairs of modules (see [26] for details) and is the identity on all others.

4.3. The action of \( G \) on solutions of the KZ equations and on \( m \)-harmonic polynomials. Let \( C_1, \ldots, C_q \) be the conjugacy classes of reflections in \( G \). The multiplicity function \( m \) may be regarded as a function on the set of reflections which is constant on each conjugacy class. We denote accordingly by \( m_a \) the value of \( m \) on \( C_a \). We also let \( N_a \) be the number of elements of \( C_a \).

**Theorem 4.1.** Let \( V_{j,\alpha}^\pm = \{ v \in V_j \mid s_\alpha v = \pm v \} \) be the \((\pm 1)\)-eigenspace of \( s_\alpha \) acting on the simple \( G \)-module \( V_j \) and set

\[
d_a^\pm(V_j) = \frac{2N_a \dim(V_{j,\alpha}^\pm)}{\dim(V_j)}, \quad a = 1, \ldots, q,
\]
for any $s_a \in C_a$. Then

(16) \[ P_{\pi_m(j)}(H_m, t) = t^{\sum_{j=1}^{M} m_a d^a_a(V_j)} P_j(H_0, t). \]

(17) \[ P_j(H_m, t) = t^{\sum_{a=1}^{M} m_a d_a^a(V_j)} P_{\pi_m(j)}(H_0, t). \]

Note that (17) is an immediate consequence of (16), the fact that $\pi_m$ is an involution and property (B) of $\pi_m$.

This Theorem can also be obtained as a special case of a result of Opdam [25, 26], see in particular Theorem 7 in [26]. We give a more transparent and effective proof in 4.4 below, based on the computation of the action of $G$ on the space of solutions of the KZ equations.

From the formula (17) we obtain a formula for the total Poincaré polynomial $P(H_m, t) = \sum_{d \geq 0} \dim(H_m^d)t^d$. It is obtained by summing the contribution of the individual $G$-modules, by noticing that $\pi_m$ is a permutation preserving the dimension of the modules:

\[ P(H_m, t) = \sum_j \dim(V_j) t^{\sum_{a=1}^{M} m_a d_a^a(V_j)} P_j(H_0, t). \]

**Corollary 4.2.** We have Poincaré duality:

\[ P_j^*(H_m, t) = t^M P_j(H_m, t^{-1}). \]

The total Poincaré polynomial $P(H_m, t)$ has degree $M = \sum_{a \in A}(2m_a + 1)$ and obeys

\[ P(H_m, t) = t^M P(H_m, t^{-1}). \]

**Proof:** The first formula follows from Poincaré duality for $H_0$, see (3), the fact that $\pi_m(j^* \chi) = \pi_m(j)^* \chi$ (Property (A)), and the identity $d_a^a(V_j^* \chi) = 2N_a - d_a^a(V_j)$. The formula for the total Poincaré polynomial is then an easy consequence. From this formula, we deduce that $P(H_m, t)$ is a polynomial of degree $\leq M$. The alternating module (and only this module) gives a term $t^M$. Thus the polynomial is of degree $M$. $\square$

The proof of Theorem 4.3 is complete.

We now show that $d_a^a$ may also be calculated in terms of $P_j(H_0, t)$, generalizing a result of Solomon [27]. The group Hom$(G, \mathbb{C}^\times)$ of one-dimensional representations of $G$ is generated by representations $\chi_a : G \to \{1, -1\}, a = 1, \ldots, q$ such that

\[ \chi_a(s) = \begin{cases} -1, & \text{if } s \in C_a, \\ 1, & \text{if } s \in C_b, b \neq a, \end{cases} \]

The group Hom$(G, \mathbb{C}^\times)$ acts on the set of simple $G$-modules by tensor multiplication. In particular, $\chi_a$ acting on $V_j$ gives a module isomorphic to $V_j \otimes a$ for some permutation $j \mapsto j \otimes a$ of $\{1, \ldots, p\}$.

**Proposition 4.3.**

\[ d_a^a(V_j) = N_a + \left. \frac{d}{dt} \ln \frac{P_j(H_0, t)}{P_{j \otimes a}(H_0, t)} \right|_{t=1}, \quad a = 1, \ldots, q. \]

If $q = 1$ the formula simplifies to

\[ d^-(V_j) = d^+(V_j) = 2 \left. \frac{d}{dt} \ln P_j(H_0, t) \right|_{t=1}. \]
Remark. The second formula in Theorem 4.1 is due to Solomon [27]. It can be stated, following his formulation, in the following equivalent way: if $V_\lambda$ appears in the decomposition of $H^0_\pm$ into simple $G$-modules for $q = q_1, \ldots, q_{\dim(V_\lambda)}$ then $d^-(V_\lambda)/2$ is the average of the $q_i$. From our result follows the non-obvious fact that this average is always half an integer.

4.4. Action of $G$ on $KZ(0)$ and proof of Theorem 4.1. The proof is based on the $G$-module isomorphism between the space of $m$-harmonic polynomials and the space $KZ = KZ(0)$ of solutions of the KZ equations with values in $M = M(0) = S/I(0)$. The first observation is that the space $M$ of values has a filtration preserved by the KZ connection: let $F_d(S) = \oplus_{j \geq d} S^j$ be the subspace of polynomials of degree $\geq d$ and set $F_d(M) = F_d(S)/(F_d(S) \cap I(0))$. Since $I(0)$ is the direct sum of its homogeneous components, and the action of $G$ preserves degrees, we have natural inclusion maps of $G$-modules,

$$\mathbb{C} w_0(z) = F_N(M) \subset F_{N-1}(M) \subset \cdots \subset F_0(M) = M.$$ 

Here, as usual, $N = \deg w_0(z)$ is the number of reflections of $G$. The KZ connection preserves functions with values in each $F_j(M)$, and we thus have a corresponding $G$-module filtration of the space $KZ$:

$$\mathbb{C} w_0(z) = KZ_N \subset KZ_{N-1} \subset \cdots \subset KZ_0 = KZ.$$ 

The space $KZ_N$ is spanned by the constant solution $x \rightarrow w_0(z) \in M$, which is mapped to a constant eigenfunction under the Matsuo–Cherednik map.

Additionally to this filtration, we also have a grading with respect to the total degree: if we represent a polynomial function $x \rightarrow \psi(x,z)$ on $V$ with values in $M$ by a scalar function $\psi(x,z)$ of two variables, we say that $\psi$ has total degree $\delta$ if $\psi(x,\lambda^{-1}z) = \lambda^\delta \psi(x,z)$. Then the KZ connection decreases the total degree by 1. Therefore we may decompose the space of solutions in homogeneous components with respect to the total degree:

$$KZ_d = \oplus_{d \geq -N} KZ_d^\delta.$$ 

Lemma 4.4. The Matsuo–Cherednik map sends solutions of total degree $\delta$ to homogeneous $m$-harmonic polynomials of degree $\delta + N$.

Proof: The Matsuo–Cherednik map sends $\psi(x,z)$ to $\psi(x,\partial_y)w_0(y)|_{y=0}$, which is the value at $y = 0$ of a homogeneous polynomial in $x,y$ of degree $\delta + N$. □

We are thus left to describe solutions in $KZ_d^\delta$. Suppose $\psi \in KZ_d$ has a nontrivial projection in $KZ_d^\delta/KZ_{d+1}^\delta$. Then $\psi = \psi^d + \psi^{d+1} + \cdots + \psi^N$ with $\psi^d$ taking values in $M^d = S^d/(S^d \cap I(0))$. Then the lowest component $\psi^d$ is a nontrivial solution of the KZ equations with values in $M^d$ without $\pi$-term:

$$\partial_\xi \psi^d(x) = \sum_{\alpha \in A} m_{\alpha} \binom{\alpha, \xi}{\alpha, x} (s_\alpha + 1) \psi^d(x), \quad \xi \in V.$$ 

Lemma 4.5. The map $\psi \rightarrow \psi^d$ is an isomorphism of $G$-modules from $KZ_d^\delta/KZ_{d+1}^\delta$ onto the space of homogeneous polynomial solutions of (18) of degree $\delta + d$ with values in $M^d = S^d/(S^d \cap I(0))$, homogeneous of degree $\delta + d$. 


Proof: The map \( \psi \mapsto \psi^d \) is an injective \( G \)-module homomorphism from \( KZ_d/KZ_{d+1} \) to the space of polynomial solutions of (18) with values in \( M^d \). Taking the direct sum over \( d \) and comparing dimensions we see that all solutions of (18) are polynomial and that the map is an isomorphism. It maps \( KZ_d \rightarrow KZ_{d+1} \) onto the space of solutions \( \psi \) of (18) with \( \psi(\lambda x, z) = \lambda^d \psi(\lambda x, \lambda^{-1} z) = \lambda^{d+r} \psi(x, z) \).

To describe solutions of (18) with values in \( M^d \) we fix a decomposition of \( M^d \) as a direct sum of simple modules and notice that both sides of (i) contain operators preserving this decomposition.

**Proposition 4.6.** Let \( KZ(V_j) \) be the space of polynomial solutions of (18) with values in a simple \( G \)-module \( V_j \). Then

(i) Solutions in \( KZ(V_j) \) are homogeneous polynomials of degree \( \sum_{a=1}^g m_a d_a^+ (V_j) \), see (15).

(ii) \( KZ(V_j) \) with action \( \psi \mapsto g \psi, g \psi(x) = g \cdot \psi(g^{-1}x), g \in G, \) is isomorphic to \( V_{\pi_{\alpha,j}} \) as a \( G \)-module (see (13)).

Proof: (i) Let \( E\psi(x) = \frac{d}{dx} \psi(x)|_{x=1} = \sum_{j=1}^n x_j \partial \psi(x)/\partial x_j \) be the Euler vector field. The KZ equation implies

\[
E \psi(x) = \sum_{\alpha \in A} m_\alpha (s_\alpha + 1) \psi(x).
\]

The operator on the right-hand side is a central element of \( \mathbb{C}[G] \) (it clearly commutes with all reflections) and is thus, by Schur’s lemma, a scalar \( a \) times the identity. To compute \( a \) we take the trace and obtain

\[
a = \frac{1}{\dim(V_j)} \sum_{\alpha \in A} m_\alpha \text{tr}_{V_j}(s_\alpha + 1)
= \frac{1}{\dim(V_j)} \sum_{\alpha \in A} m_\alpha 2 \dim \text{Ker}(s_\alpha - 1 : V_j \rightarrow V_j).
\]

Since the terms of the sum with \( s_\alpha \) in the same conjugacy class are equal to each other, we obtain the claimed formula.

(ii) We apply the method of (15), which consists of considering the KZ equation with complex \( G \)-invariant multiplicities \( \mu_\alpha \). Let \( \psi_\mu \) be the solution of the KZ equation

\[
\partial \psi_\mu(x) = \sum_{\alpha \in A} \mu_\alpha \frac{\alpha(x)}{(\alpha, x)} (s_\alpha + 1) \psi_\mu(x), \quad \xi \in V.
\]

on some neighborhood of some regular point \( x_0 \) with values in \( \text{End}(V_j) \) and initial condition \( \psi_\mu(x_0) = \text{id} \). If \( g \in G \) and \( \gamma \) is a path in \( V - \cup_a \Pi_a \) from \( x_0 \) to \( g^{-1}x_0 \), we set

\[
g \psi_\mu(x) = g \psi_\mu(g^{-1}x),
\]

where the value at \( g^{-1} \) is obtained by analytic continuation along \( \gamma \). Then \( g \psi_\mu = \psi_\mu \circ T_g \) for some monodromy matrix \( T_g \). By the results of (12), the \( T_g \) obey, for generic \( \mu \), the relations of the Iwahori–Hecke algebra with \( u_\alpha = \exp(2\pi i \mu_\alpha) \). At \( \mu = 0 \), \( T_g \) is given by the action of the group on \( V_j \). More generally, for all integer \( \mu \), \( g \mapsto T_g \) is a representation of \( G \), since we know that all solutions are polynomial. It follows that the character of the Iwahori–Hecke algebra corresponding to the representation \( g \mapsto T_g \) at generic \( \mu \) coincides with the character \( \chi_j(\exp(\pi i \mu_\alpha, \alpha \in A)) \) described in (12) and the claim follows.

We are ready to complete the proof. Solutions of (18) with values in a simple module \( V_j \) occurring in the decomposition of \( M^d \) give rise to a subspace of solutions...
in $KZ_\delta^G$ of total degree $\delta = \sum a_m a^d_n (V_j) - d$ and isomorphic to $V_{\pi_m(j)}$. The Matsuo-Cherednik map sends these solutions to $m$-harmonic polynomials of degree $\delta + N$ that form a $G$-module isomorphic to $V_{\pi_m(j)'} = V_{\pi_m(j)} \otimes \mathbb{C}$. On the other hand, $M^d$ is isomorphic to the space of harmonic polynomials of degree $d$. Thus

$$P_{\pi_m(j)}(H_m, t) = t^{\sum a_m d_n^m (V_j) + N} P_j(H_0, t^{-1})$$

by Poincaré duality for $H_0$ and the fact that $d^+_m(V_j) = d^m_n(V_j')$.

The proof of Theorem 4.1 is complete.

4.5. Example: Lowest degree $m$-harmonic polynomials for $G = S_n$. If $G = S_n$, Kirillov’s formula gives

$$d^-(V_j) = \sum_{k=1}^n (\ell_k - a_k) + n(n - 1)/2$$

where $a_k = h_k - \ell_k - 1$ is the “arm length” of the $k$th box of the Young diagram corresponding to $V_j$. Therefore,

$$P_j(H_m, t) = t^{\sum a_m d_n^m (V_j)} t^{\sum a_m d_n^m (V_j')} P_j(H_0, t),$$

Thus, a part from the trivial module, which occurs in degree $0$, the lowest degree $m$-harmonic polynomials appear in degree $n m + 1$. They form an $(n - 1)$-dimensional simple $G$-module corresponding to the minimal leg-length, maximal arm-length partition $(n - 1, 1, 0, \ldots, 0)$. It is isomorphic to the module $\{ x \in \mathbb{C}^n | \sum x_i = 0 \}$ with permutation action of $S_n$.

These $m$-harmonic polynomials are associated to solutions of the KZ equations with values in $F_{N-1}(M) = M^N \oplus M^{N-1}$, $M = S/I(0)$, $N = n(n - 1)/2$. Let $\psi$ be such a solution, $\phi(x) = \langle \psi(x), \varrho_0 \rangle$ the corresponding $m$-harmonic polynomial, and $\psi_k(x) = \langle \psi(x), \partial_k \varrho_0 \rangle$, where $\partial_k$ is the derivative with respect to the $k$th coordinate of $V = \mathbb{C}^n$. Then $\psi$ is uniquely determined by the components $\phi, \psi_1, \ldots, \psi_n$. They obey

$$\psi_1 + \cdots + \psi_n = 0.$$  \hfill (19)

In these terms, the KZ equations are equivalent to the system

$$\frac{\partial \phi}{\partial x_i} = \psi_i, \quad (20)$$

$$\frac{\partial \psi_j}{\partial x_i} = -m \frac{\psi_i - \psi_j}{x_i - x_j}, \quad i \neq j, \quad (21)$$

(the equations for $\partial_i \psi_i$ are redundant in view of (19)).

The simplest solutions of these equations are the ones with $\psi_i = 0$. They give rise to constant $m$-harmonic polynomials $\phi = \text{const}$, which form a trivial $G$-module.

We next describe the solutions corresponding to $m$-harmonic polynomials of degree $mn + 1$. We assume that $m \geq 1$ (the case $m = 0$ is left as an exercise). It is easy to check that for any family of cycles $\gamma(x)$ in the relative integral homology $H_1(\mathbb{C}, \{ x_1, \ldots, x_n \})$, the following formula gives a solution of the equations (21).

$$\psi_j(x) = \int_{\gamma(x)} \frac{dt}{t - x_j} \prod_{k=1}^n (t - x_k)^m.$$
Indeed the integrand obeys the equations, and the derivative with respect to the endpoints appearing in $\partial_i \psi_j$ with $i \neq j$ vanishes since the integrand vanishes there for $m \geq 1$. Moreover, we have

$$\sum_{j=1}^{n} \psi_j(x) = \frac{1}{m} \int_{\gamma(x)} d \prod_{k=1}^{n} (t - x_k)^m = 0,$$

since $m \geq 1$. The corresponding $m$-harmonic polynomials of degree $mn + 1$ are obtained by integrating (20):

$$\phi(x) = \frac{1}{1 + nm} \sum_{i=1}^{n} x_i \psi_i(x) = \frac{1}{1 + nm} \int_{\gamma(x)} \sum_{i=1}^{n} x_i dt \prod_{k=1}^{n} (t - x_k)^m.$$

It remains to show that we get $n - 1$ linearly independent solutions in this way. A basis of $H_j(C, \{x_1, \ldots, x_k\})$ for generic $x$ is given by paths $\gamma_j$ joining $x_1$ to $x_j$, $2 \leq j \leq n$. Let $\phi^{(j)}$ be the corresponding $m$-harmonic polynomials:

$$\phi^{(j)}(x) = \frac{1}{1 + nm} \int_{x_1}^{x_j} \sum_{i=1}^{n} x_i dt \prod_{k=1}^{n} (t - x_k)^m, \quad j = 2, \ldots, n$$

These functions are linearly independent, as can be seen by comparing their behaviour at infinity:

$$\lim_{x_j \to \infty} x_j^{-mn-1} \phi^{(k)}(x) = (-1)^{m-1} \frac{(m-1)!^2}{(1+nm)(2m-1)!} \delta_{jk}.$$ 

### 4.6. Proof of Proposition 4.3

Let $g^k$ denote the action of $g \in G \subset GL(V)$ on the symmetric power $S^k$ of the reflection module $V$. Then the character of $S^k$ is $\chi_{S^k}(g) = \text{tr}(g^k)$ and $\sum_{k=0}^{\infty} \text{tr}(g^k)t^k = \det(1 - gt)$ is the generating series of these characters. If $\chi_j$ is the character of $V_j$, the orthogonality relations of characters implies that

$$P_j(S,t) = \frac{1}{|G|} \sum_{g \in G} \chi_j(g^{-1}) \frac{1}{\det(1 - gt)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_j(g^{-1}) \prod_{i=1}^{n} \frac{1}{1 - \lambda_i(g)t}.$$

Here $\lambda_1(g), \ldots, \lambda_n(g)$ are the eigenvalues of $g \in GL(V)$. It follows from (14) that

$$P_j(H_0,t) = \frac{1}{|G|} \sum_{g \in G} \chi_j(g^{-1}) \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - \lambda_i(g)t}$$

$$= \frac{1}{|G|} R(t) \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t},$$

$$R(t) = \sum_{g \in G} \chi_j(g^{-1}) \prod_{i=1}^{n} \frac{1 - t}{1 - \lambda_i(g)t}.$$

If we take the derivative of $R(t)$ at $t = 1$, the only terms surviving in the sum over $G$ are the reflections, since they are the only group elements with multiplicity of
the eigenvalue 1 equal to $n - 1$. On the other hand, $R(1) = \dim(V_j)$ since only the identity survives in this case. We then have

$$\frac{d}{dt} \mid_{t=1} \ln P_j(H_0, t) = \sum_{i=1}^{n} \frac{d}{dt} \mid_{t=1} \ln \frac{1 - t^{d_i}}{1 - t} + \frac{1}{\dim(V_j)} \sum_{\alpha \in A} \frac{d}{dt} \mid_{t=1} \frac{1 - t}{1 + t} \chi_j(s_\alpha)$$

$$= \sum_{i=1}^{n} \frac{d_i - 1}{2} + \frac{1}{\dim(V_j)} \left(-\frac{1}{2}\right) \sum_{\alpha \in A} \chi_j(s_\alpha)$$

Let $\chi_a$ be the one-dimensional representation of $G$ so that $\chi_a(s) = -1$ only on reflections $s \in C_a$. Since $\chi_j \otimes a(s_\alpha) = \chi_a(s_\alpha) \chi_j(s_\alpha)$, we obtain

$$\frac{d}{dt} \mid_{t=1} \ln \frac{P_j(H_0, t)}{P_j \otimes a(H_0, t)} = -\frac{1}{\dim(V_j)} \sum_{s_\alpha \in C_a} \chi_j(s_\alpha).$$

Since $\chi_j(s_\alpha) = \dim(V_j) - 2 \dim(V_j^-)$, the result follows. The second formula in Theorem 4.3 valid for groups with one conjugacy class of reflections, is obtained from the first formula by applying Poincaré duality $P_j \otimes e(H_0, t) = t^N P_j(H_0, t^{-1})$. In this case the only one-dimensional representation is the sign $\epsilon$.

4.7. Distribution of the degrees of $m$-harmonic polynomials for large $m$.

Let us consider again the case of the symmetric group $G = S_n$. Choose any basis in the corresponding space of $m$-harmonic polynomials $H_m$. Let $d_1^{(m,n)} = 0, d_2^{(m,n)} = d_3^{(m,n)} = ... = d_n^{(m,n)} = mn + 1, ..., d_n^{(m,n)} = (2m+1)n(n-1)/2$ be the degrees of the $m$-harmonic polynomials of the basis written in increasing order (with repetitions). We are interested in the distribution of these degrees for large $m$.

As we have shown above all the degrees $d_j^{(m,n)}$ are growing linearly as a function of $m$ so it is natural to consider the limits of the scaled degrees $d_j^{n} = \lim_{m \to \infty} d_j^{(m,n)}/m$. From our Theorem 4.3 it follows that the set of $d_j^{n}$ coincide with the set of the numbers $d^-(U), U \in \hat{S}_n$ where $\hat{S}_n$ is the set of all irreducible representations of $S_n$. Moreover each number $d^-(U)$ appears $(\dim U)^2$ times since the action of $G$ on $H_m$ is regular and therefore each representation $U$ appears with multiplicity equal to its dimension.

On the set $\hat{S}_n$, which can be represented as the set of all Young diagrams $Y_n$ of size $n$, there exists a natural probability measure called Plancherel measure: each representation $U$ has a weight equal to $(\dim U)^2/n!$ (see e.g. 31). The function $d^-(U)$ can be considered as a random variable on this set. Let us consider the degree $d_j^{(m,n)}$ of $m$-harmonic polynomial as a random variable on the discrete set of $j \in 1, 2, ..., n!$ where each $j$ has the same probability $1/n!$ (in other words we assume that each of the basic $m$-harmonic polynomials has the same weight). We have the following

**Proposition 4.7.** The distribution of the scaled degrees $d_j^{(m,n)}/m$ of $m$-harmonic polynomials as $m$ goes to infinity converges to the distribution of the random variable $d^-$ on the set of Young diagrams of size $n$ considered with Plancherel measure.

In 1977 Vershik and Kerov 32 and independently Logan and Shepp 21 have found remarkable results on the random Young diagrams of size $n$ for large $n$. They have shown that as $n$ goes to infinity the random shape of such a diagram after proper rescaling concentrates near some universal curve. The Gaussian fluctuations
around this limiting shape have been described later by Kerov \[19\] (a detailed proof of Kerov’s result can be found in recent paper by V. Ivanov and G. Olshanski \[18\]).

We can use Kerov’s following result to describe the distribution of the random variable \(d^-(U)\) on \(Y_n\) for a large \(n\). Let us introduce following \[18\] the normalised characters

\[p_2(n)(U) = \frac{n(n-1)\chi_U(s)}{\dim U} = \frac{n(n-1)(\dim U^+ - \dim U^-)}{\dim U},\]

where \(\chi_U\) is the character of \(U\), \(s = s_{ij} \in S_n\) is any transposition and \(U^+, U^-\) are the \(\pm\)-eigenspaces of its action on \(U\).

Kerov’s theorem \[18, 19\]. When \(n\) goes to infinity the distribution of the random variables \(\frac{p_2(n)}{\sqrt{2n}}\) converges to the standard normal distribution \(N(0, 1)\).

Since \(p_2(n)(U) = \frac{n(n-1)(\dim U^+ - \dim U^-)}{\dim U} = n(n-1) - 2d^-(U)\) we have as a corollary that when \(n\) goes to infinity the distribution of \(\frac{1}{n}(d^- - \frac{n(n-1)}{2})\) converges to \(N(0, 1/2)\), where \(N(0, 1/2)\) is the normal distribution with mean 0 and variance 1/2.

This gives some idea of how the degrees of \(m\)-harmonic polynomials of the symmetric group \(S_n\) are distributed for large \(m\) and \(n\). We should note that Proposition 4.7 has an obvious generalization for any Coxeter group but Kerov’s result is known only for symmetric groups.

5. Concluding remarks

The space \(H_m\) of \(m\)-harmonic polynomials has been introduced in \[11\] in relation with the algebra of quasiinvariants \(Q_m\) of the Coxeter group \(G\). In \[11\] it was shown that if \(G\) is any dihedral group and the multiplicity \(m\) is a constant function then \(Q_m\) is freely generated by any basis in \(H_m\) as a module over the subring \(S^G\) of \(G\)-invariant polynomials and it was conjectured that the same is true in general situation. However this turned out not to be true as it was shown in the recent very interesting paper \[10\] by P. Etingof and V. Ginzburg. They have shown that for the group \(G\) of type \(B_6\) and the multiplicity function \(m\) equal to 0 on the long roots and 1 on the short roots \[10\] \(m\)-harmonic polynomials are linearly dependent over \(S^G\). This raises the question of how exceptional this situation is. Probably this never happens if the multiplicity \(m\) is constant (in particular for all Coxeter groups with only one conjugacy class of reflections).

Etingof and Ginzburg showed also that \(Q_m\) is freely generated over \(S^G\) by some homogeneous polynomials which have the same degrees as \(m\)-harmonic polynomials. This means that one can use our results to compute the Poincaré series for the algebra of quasiinvariants. The fact that the Poincaré polynomials \(P(H_m, t)\) are palindromic corresponds to the Gorenstein property of \(Q_m\) (see \[11\]).

It is interesting to compare the theory of \(m\)-harmonic polynomials with the classical case \(m = 0\). Probably the main novelty is a more explicit role of the action of the group: as we have seen each isotypical component of \(H_m\) behaves differently when \(m\) changes. In the classical case one can write down an explicit formula for the Poincaré polynomial of \(H_0\) without any reference to representation theory (see Introduction) but for general \(m\) this seems to be impossible.

We should mention that the problem of explicit description of \(m\)-harmonic polynomials is still open. As we have shown, an essential step in its solution is finding
the solutions of the KZ equations with values in the irreducible representations of $G$, but this is another interesting open problem.

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