ABSTRACT: A nonperturbative approach to the vacuum polarization for quantized fermions in external vector potentials is discussed. It is shown that by a suitable choice of counterterms the vacuum polarization phase is both gauge and renormalization independent, within a large class of nonperturbative renormalizations.

1. INTRODUCTION

For Dirac fermions in (non second quantized) external Yang-Mills fields the 1-particle scattering matrix $S$ is a well-defined unitary operator and moreover it has a canonical second quantization $\hat{S}$ operating in the fermionic Fock space [1]; for twice differentiable vector potentials one only needs to assume certain fall-off properties at spatial and time infinity.

On the other hand, if one tries to expand the quantum scattering matrix in ordinary Dyson-Feynman perturbation series one meets the well-known vacuum polarization divergences, which must be taken care by suitable (infinite) subtractions. Actually, it is only the phase of the vacuum expectation value $\langle 0|\hat{S}|0 \rangle$ which is diverging; the absolute value is uniquely defined and finite by canonical quantization. The crucial point is that when passing from $S$ to $\hat{S}$ using the rules of canonical hamiltonian quantization the phase remains ill-defined. It is exactly this quantity which is diverging in perturbation theory.

One of the basic principles of relativistic quantum field theory is locality. The fields are supposed to either commute (bosons) or anticommute (fermions) at space-like separations. This is the case at least in four or higher space-time dimensions; in lower dimensions there are more alternatives. For this reason one prefers field theories in which the action is a local differential polynomial of the fields. On the other hand, the requirement that the action is a function of the fields and their derivatives up to a finite degree at the same space-time point is by no means a necessary condition for the locality in the above sense. In this paper we continue the work on a nonperturbative renormalization which breaks the locality in the narrow sense (the renormalized action contains field derivatives up to infinite degree) but nevertheless is local in the framework of Wightman axioms.
This work was initiated in [2] and further generalized in [3]. In the previous paper [3] the question of gauge and renormalization independence was left open. We fill the gap in the present paper in the case of infinitesimal gauge and renormalization variations.

2. THE RENORMALIZATION

We shall study massless Dirac fermions coupled to a gauge potential \( A \) in Minkowski space. The potential is a smooth 1-form \( A_\mu(x) \, dx^\mu \) in space-time with values in the Lie algebra \( \mathfrak{g} \) of a compact gauge group \( G \). The elements of \( \mathfrak{g} \) are represented by hermitean (according to physics literature convention) matrices in the complex vector space \( \mathbb{C}^N \). The free Dirac operator is then \( i \gamma^\mu \partial_\mu \). The metric is \( x^\mu x_\mu = x_0^2 - x_1^2 - \cdots - x_d^2 \). The Dirac gamma matrices satisfy \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \), \( \gamma_0 \) is hermitean and \( \gamma_k \) is antihermitean for \( k \neq 0 \). We fix a hermitean matrix \( \Gamma \) such that \( \Gamma^2 = 1 \) and \( \Gamma \gamma_\mu = -\gamma_\mu \Gamma \). The chiral projectors are \( P_\pm = \frac{1}{2}(\Gamma \pm 1) \).

The Dirac Hamiltonian in background gauge field \( A \) is

\[
D_A = -\gamma^0 \gamma^k(i \partial_k + A_k) - A_0.
\]

We shall assume that \( A(x) \) and its derivatives vanish faster than \( |x|^{-d/2} \) when \( |x| \to \infty \).

The one-particle scattering operator \( S \) is defined as the limit of the time evolution operator in the interaction picture, \( U_I(t, -t) \to S \) as \( t \to \infty \). The time evolution in the Schrödinger picture is defined by

\[
i \partial_t U(t, t_0) = h(t)U(t, t_0) \text{ with } U(t_0, t_0) = 1,
\]

and in the interaction picture by

\[
i \partial_t U_I(t, t_0) = V_I(t)U_I(t, t_0), \text{ with } U_I(t_0, t_0) = 1.
\]

The interaction is \( V_I(t) = e^{ith_0}V(t)e^{-ith_0} \), the total Hamiltonian being \( h(t) = D_A = h_0 + V(t) \) with \( h_0 = D_0 = -i\gamma_0 \gamma^k \partial_k \). The quantum divergences are related to the fact that when \( V = -\gamma^0 \gamma^k A_k(x, t) - A_0(x, t) \) is the interaction with a Yang-Mills potential then the quantization of \( \hat{U}_I(t, -\infty) \) for intermediate times \( t < \infty \) is not well-defined. Let \( \epsilon = h_0/|h_0| \) be the sign of the free Hamiltonian. In general, a 1-particle operator \( K \) has a canonical quantization in the Fock space of free fermions iff \( [\epsilon, K] \) is Hilbert-Schmidt. For a review of this and other properties of representations of CAR algebra, see [9]. Generically, the Hilbert-Schmidt property is not satisfied when \( K = V(t) \).

Let us recall the basic facts about (free) Fock space quantization of the canonical anticommutation relations. To each vector \( u \in H \) in the 1-particle Hilbert space one associates a creation operator \( a^*(u) \) and an annihilation operator \( a(u) \), the latter depending antilinearly on the parameter and the former linearly, with the only nonzero anticommutation relations

\[
a^*(u)a(v) + a(v)a^*(u) = <v, u>.
\]

where \( <v, u> \) is the inner product in \( H \). Let \( H = H_+ \oplus H_- \) be splitting to a pair of closed infinite-dimensional subspaces. In our discussion below \( H_- \) will
be the spectral subspace corresponding to nonnegative energies with respect to
the free Dirac hamiltonian $D_0$. There is a unique (up to equivalence) irreducible
representation of the CAR algebra in a Hilbert space $\mathcal{F}$ such that
\[ a^*(u)|0\rangle = 0 = a(v)|0\rangle \quad \text{for } u \in H_- \text{ and } v \in H_+ \]
where $|0\rangle \in \mathcal{F}$ is a normalized vector, the vacuum.

Let $a_i^*, a_j$ with $i, j \in \mathbb{Z}$, be a complete set of creation and annihi-
ation operators such that the index $i \geq 0$ corresponds to nonnegative energies and $i < 0$ to negative
energies,
\[ a_i^* a_j + a_j a_i^* = \delta_{ij}. \]
The normal ordering is defined by: $a_i^* a_j := a_i^* a_j$ except when $i = j < 0$ and
then: $a_i^* a_j := -a_j a_i^*$. If $X, Y$ is a pair of one-particle operators then the canonical
quantizations
\[ \hat{X} = \sum X_{ij} : a_i^* a_j : \]
satisfy the algebra
\[ [\hat{X}, \hat{Y}] = [X, Y] + \omega(X, Y), \]
and the quantum operators are defined such that $[\hat{X}, a_i^*] = \sum X_{ji} a_j^*$. The 2-cocycle
$\omega$ is defined by, [10],
\[ \omega(X, Y) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y]. \tag{3} \]

Here we meet again the Hilbert-Schmidt condition: The 2-cocycle is defined only
for bounded operators $X$ such that $[\epsilon, X]$ is Hilbert-Schmidt.

The starting point for our discussion here is the following renormalization which
makes $U_1$ quantizable, [2,3]. For each (smooth) potential $A = A_0 dt + A_k dx^k$ one
defines a unitary operator $T_A$ in the 1-particle space with the following property. De-
dine $U_{\text{ren}}(t, t_0) = T_A(t) U_1(t, t_0) T_A(t_0)^{-1}$. Then 1) $[\epsilon, U_{\text{ren}}(t, t_0)]$ is Hilbert-Schmidt,
2) $T_A(t) \to 1$ as $t \to \pm \infty$. The last property guarantees that the renormalization
does not affect the scattering matrix $\hat{S}$ whereas the first condition guarantees that
the renormalized time evolution is quantizable in the free Fock space.

The Hilbert-Schmidt condition on operators $[\epsilon, g]$ defines the restricted unitary
group $U_1$ of unitary operators $g$, [8]. The second quantization of elements in $U_1$
defines a central extension $\hat{U}_1$ as discussed in detail in [8]. Now we have a smooth
path of operators $g(t) = U_{\text{ren}}(t) = U_{\text{ren}}(t, -\infty)$ in $U_1$ with the initial condition
$g(-\infty) = 1$ and $g(+\infty) = S$. The central extension of $U_1$ defines a natural connection
in the circle bundle $\hat{U}_1 \to U_1$. The curvature of this connection has a simple
formula: It is simply given as $\text{curv}(X, Y) = \omega(X, Y)$ where $X, Y$ are tangent vec-
tors at $g \in U_1$, identified through left translation on the group as elements of the
Lie algebra of $U_1$. The phase of $\hat{S}$ is defined through parallel transport, with respect
to the connection above.

Our definition of phase of $\hat{S}$ is causal: A scattering process in an external field
$A$ followed by the scattering in $A'$ defines the same phase as the scattering for
$A'' = A \cup A'$, when both $A, A'$ have finite nonoverlapping support in time and
$A''$ denotes a field $A''(t) = A(t)$ for $t < t_0$ and $A''(t) = A'(t)$ for $t > t_0$; here $t_0$
serves as the supports of $A, A'$. The operator $T_{\text{ren}}$ is not uniquely defined. It is
more convenient to define the transformation first in the Schrödinger picture (1). A simple formula which works is (here $\mathcal{A} = \gamma^0 \gamma^k A_k$ and $E = \partial_t \mathcal{A} - \gamma^i A_0 + [A_0, \mathcal{A}]$)

$T_A = \exp \left( \frac{1}{4} \left[ \frac{1}{D_0}, \mathcal{A} \right] - \frac{1}{8} \left[ \frac{1}{D_0}, \mathcal{A} \right] - \frac{i}{4} \frac{1}{D_0} E\frac{1}{D_0} \right)$.

where it is understood that the singularity at the zero modes of $D_0$ is taken care of by an infrared regularization, for example $\frac{1}{D_0} \to \frac{D_0}{D_0^2 + \alpha^2}$ for some nonzero real number $\alpha$. In the interaction picture one uses the operator $\exp(ith_0) T_A \exp(-ith_0)$. Note that this choice commutes with the chiral projection operators $P_\pm$.

For the proof of validity of the choice (4) it is convenient to use the symbol calculus for pseudodifferential operators. A PSDO $B$ acting on vector valued functions in $\mathbb{R}^d$ is represented by a smooth matrix valued function $b(x,p)$ of coordinates $x_i$ and momenta $p_i$. The operators which we need have an asymptotic expansion

$\sum b_j(x,p) \sim \sum b_{k-1} + b_{k-2} + \ldots$

where each $b_j$ is a smooth function of $x,p$ for $|p| > 1$, homogeneous of degree $j$ in the momenta. The asymptotic expansion for the product of two operators $A, B$ is obtained from the formula

$a * b = \sum \frac{(-i)^{|m|}}{m!} \frac{\partial^m a}{\partial x^m} \frac{\partial^m b}{\partial p^m}$

where the multi-index notation $m = (m_1, m_2, \ldots, m_d)$ is used; $|m| = m_1 + m_2 + \ldots m_d$, $m! = m_1! m_2! \ldots m_d!$ etc. In particular, the degree of the highest term in the product is the sum of degrees of the factors. Each differentiation in momentum space lowers the degree of the operator by one.

If $A$ is an operator with symbol $a$ such that $a(x,p) \to 0$ faster than $|x|^{-d}$ as $|x| \to \infty$ and $\deg a < -d$ then $A$ is a trace-class operator and

$\text{tr} A = \frac{1}{(2\pi)^d} \int \text{tr} (a(x,p)) dxdp.$

We shall now assume that the components $A_\mu$ of the vector potential satisfy the above boundary condition at $|x| \to \infty$.

The time evolution equation for $U_{\text{ren}}(t, t_0) = T_A(t) U(t, t_0) T_A(t_0)^{-1}$ in the Schrödinger picture is

$i \partial_t U_{\text{ren}}(t, t_0) = (\hbar_0 + W(t)) U_{\text{ren}}(t, t_0)$

with

$W(t) = (i \partial_t T_A) T_A^{-1} + T_A (\hbar + V(t)) T_A^{-1}$.

Expanding the exponential and arranging terms according to powers of the inverse of momentum (i.e., of $D_0$,) one gets $[\epsilon, W] = R_1 + R_2 + \ldots$, where the dots denote terms which behave explicitly as $|D_0|^k$ with $k \leq -2$ for high momenta, and

$R_1 = \frac{1}{2} D_0 A - \frac{1}{2} D_0 A D_0 - \frac{1}{2} |D_0| A \frac{1}{D_0} - \frac{1}{2} [D_0, A] + \frac{1}{2} [D_0, A] D_0 - \frac{1}{2} D_0 [A, D_0]$. 

and the second term, which is quadratic in $A$, is

$$R_2 = \frac{1}{4} D_0 \left[ |D_0|, A^2 \right] \frac{1}{D_0} + \frac{1}{4} \left[ \frac{1}{|D_0|}, A^2 \right] + \frac{1}{8} D_0 \left[ A \frac{1}{D_0} A, \frac{1}{|D_0|} \right] +$$

$$+ \frac{1}{8} \left[ A \frac{1}{D_0} A, \frac{1}{|D_0|} \right] D_0 + \frac{1}{16} \left[ A \frac{1}{D_0} A, |D_0| \right] + \frac{1}{8} \left[ \frac{1}{D_0} A \frac{1}{D_0}, |D_0| \right] \frac{1}{D_0}.$$

Since the commutator $[|D_0|^k, A]$ is of order $|D_0|^{k-1}$ in momenta, all terms in $R_2$ are actually of order $-2$ or less. In order to estimate $R_1$ we write it in equivalent form

$$R_1 = \frac{1}{4} D_0 \left[ |A|, |D_0| \right] \frac{1}{|D_0|} - \frac{1}{4} \left[ |A|, |D_0| \right] \frac{1}{|D_0|}$$

and observe both terms are of order $-2$ for the same reason as in the case of $R_2$. We have disregarded all the low order terms because in three space dimensions the condition that a PSDO is Hilbert-Schmidt (which was required for canonical quantization) is precisely the requirement that the operator vanishes for high momenta faster than $|p| = |D_0|$ raised to power $-3/2$. Thus after our renormalization (= conjugation by the time-dependent unitary operator $T_A$) the gauge interaction can be lifted to a finite operator in the fermionic Fock space.

The total renormalized hamiltonian is now a sum of the free (unbounded) self-adjoint hamiltonian and the bounded self-adjoint interaction. By the Kato-Rellich theorem [11] the total hamiltonian is self-adjoint and according to Stone’s theorem it defines the time evolution as a strongly continuous unitary one-parameter group in the Fock space.

This method can be extended to other interactions as well, under very general conditions [3]. It has been also used to derive the chiral anomaly in the hamiltonian framework [2], [3].

The curvature formula (3) is nonlocal. The computation of the trace involves space derivatives up to arbitrary high power because of the Green’s function $1/D_0$ in the renormalization prescription. However, the curvature is equivalent in cohomology to a local formula. We use the residue formula, [7],

$$\text{Res } B = \frac{1}{(2\pi)^d} \int_{|p|=1} \text{tr } b \cdot d(x, p) d^d x d^{d-1} p.$$ 

It behaves like trace, $\text{Res } [A, B] = 0$. Let $g$ be the Lie algebra of bounded pseudodifferential operators satisfying the conditions 1) that the degree of the commutator $[\epsilon, X]$ is strictly less than $-d/2$ (here the space dimension $d = 3$), 2) the symbol of $X$ and its derivatives decrease faster than $|x|^{-d/2}$ as $|x| \to \infty$. We have then

**Theorem 1.** Define the following 2-cocycle on $g$ :

$$\omega_{\text{loc}}(X, Y) = \text{Res } [\log |p|, X] Y,$$

where $|p|$, the length of three momentum, is the symbol of $|D_0|$. Then the difference $\omega - \omega_{\text{loc}}$ is a coboundary.

For a proof see [4] (for the special case of renormalized current operators [2]). Actually, in [4] it was assumed that the manifold is compact. This was needed
for the convergence of the integral over $x$ in the trace formula for PSDO’s. Our boundary condition does the same job.

Without the sign $\epsilon$ this formula would give the Radul cocycle, which is a 2-cocycle on the algebra of all PSDO’s on a compact manifold [5]. In quantum field theory it is important to keep the sign because this is the cocycle arising from normal ordering. The equivalence of the two cocycles means simply that if $\hat{X} = \hat{X} + \lambda(X)$ for a suitable complex linear form on the algebra of one-particle operators then

$$\left[\tilde{X}, \tilde{Y}\right] = \left[X, Y\right] + \omega_{\text{loc}}(X, Y).$$

The great advantage of the residue formula is that it only depends on the term in the appropriate PSDO which is precisely of asymptotic degree $-d$ (here $-3$) when expanded asymptotically in powers of $1/|p|$. In operator products each derivative in configuration space is associated with a derivative in momentum space. Since a differentiation in momentum space decreases the homogeneous degree of the operator it follows that a calculation of the residue can involve derivatives with respect to $x_i$’s only up to a finite order.

3. GAUGE INDEPENDENCE

The prescription for the vacuum polarization phase in section 2 is not gauge independent. Furthermore, if we change the renormalization operator $T_A$ the phase will be changed, too. Of course, in the case of chiral fermions the gauge dependence is to be expected because of the chiral anomaly; this was discussed in [3].

In the case of Dirac fermions we expect on topological grounds that one should be able to construct counterterms in the Hamiltonian such that the gauge variance of the counterterms exactly balances the gauge variation of the phase, so that the total phase would be gauge invariant. We shall confirm this explicitly below in the case of infinitesimal variations. At the same time we shall also get rid of the renormalization dependent part of the phase.

Suppose that we have constructed another renormalization operator $T'_A$ which commutes with the chiral projection operators and with $\gamma_0$, such that each term in the asymptotic expansion

$$T'_A = 1 + t_{-1}(A) + t_{-2}(A) + \ldots,$$

in homogeneous terms $t_k$ of degree $k$ in momenta, is a local differential polynomial in the components of the vector potential $A$. The condition that the renormalization commutes with $\gamma_0$ means that the action of $T'_A$ on left chiral sector is equivalent to the action on the right-hand spinors.

The change of phase of the quantum scattering is given by a parallel transport around a loop of time evolution operators obtained by first travelling the path $g(t)$ from the point $g(-\infty) = 1$ to $g(\infty) = S$ defined by the renormalized time evolution due to $T_A$ and then following backwards in time the time evolution defined by the renormalization $T'_A$. But the logarithm of the parallel transport phase is equal to the integral of the curvature over a surface bounded by the loop.

In the following we shall use the local curvature formula (6). This has the advantage that the parallel transport around a closed loop is given by a space
integral of a local differential expression in the vector potential. This is in the spirit of local quantum field theory: A change in the renormalization corresponds to a local counterterm in the Lagrangian.

Since all our operators, the (renormalized) Hamiltonian, the renormalization operator, and gauge transformations, are composed of a pair of operators \((K_L, K_R)\) acting on the two chiral sectors we can split the curvature into chiral constituents as follows. The left and right spinors are identified through the unitary transformation \(\gamma_0\) so that we can think of \(K_L\) and \(K_R\) as acting in the same space. Thus we may define \(K_{\pm} = K_L \pm K_R\). Since the sign \(\epsilon\) is odd, \(\epsilon_L = -\epsilon_R\), we can write

\[
2\omega(X, Y) = \omega_L(X_-, Y_+) + \omega_L(X_+, Y_-).
\]

Here \(\ell = \log |p|\) is the symbol of \(h_0\) and \(\omega_L(X; Y) = \text{Res} \, \epsilon_L[\ell, X_L]Y_L\). We want to show that this is equivalent to a 2-form \(\omega_-\) such that \(\omega_{-}(X, Y) = 0\) if either \(X_- = 0\) or \(Y_- = 0\). We denote \(\text{Res}' X = \text{Res} \, \epsilon_L X\).

Let \(GL_1\) denote the group of all invertible bounded PSDO’s \(g\) in \(H\) such that the degree of \([\epsilon, g]\) is strictly less than \(-d/2\). The Lie algebra of \(GL_1\) is then equal to \(g\). Let \(GL_{1,c}\) be the subgroup of \(GL_1\) consisting of the chirally diagonal operators \(f = (f_L, f_R)\).

The time evolution operator in Schrödinger picture does not have a good asymptotic expansion in inverse powers of momenta and therefore it is not immediately clear that one can define the Wodzicki residue for a product involving \(U(t)\) as a factor. For example, the free time evolution \(\exp(it\mathcal{H})\) contains arbitrary high powers of momenta and therefore it is not a classical PSDO. However, the time evolution \(U_I(t) = U_I(t, -\infty)\) in the interaction picture has a well behaved symbol. The leading terms (in powers of momenta) are obtained from the Dyson expansion in which the individual terms are composed of products of the Green’s function of the Dirac operator and the interaction Hamiltonian. This gives directly a classical asymptotic expansion for the PSDO \(U_I(t)\).

In the following we deal with the ‘Maurer-Cartan’ form \(\theta = [\ell, f]f^{-1}\). This is a classical PSDO when \(f = U_I(t)\). In the case of the Schrödinger picture it is replaced by \(\exp(it\mathcal{H}_0)\theta \exp(-it\mathcal{H}_0)\). The computation of residues for this type of operators involves oscillatory integrals. These can be handled by Guillemin’s method for Fourier integral operators, [12]. For this reason the curvature formula extends to the bigger group \(GL_F\) which is the smallest group containing \(GL_1\) and the free time evolution operators \(\exp(it\mathcal{H}_0)\).

**Lemma.** The restriction of the curvature \(\omega_{\text{loc}}\) to the chirally diagonal subgroup \(GL_{F,c} \subset GL_F\) is equivalent in de Rham cohomology to the 2-form

\[
\omega'(X, Y) = \omega_{-}(X_-, Y_-) = -\text{Res}'[\ell, X_-]Y_- + \text{Res}'[\ell, f_L] f_L^{-1}[X_-, Y_-].
\]

**Proof.** Define the 1-form

\[
\phi(f; X) = \text{Res}'[\ell, f_L] f_L^{-1} X_-
\]

on the group \(GL_{F,c}\). The exterior derivative of this is computed through

\[
(d\phi)(X, Y) = f \cdot \phi(f; X) - f \cdot \phi(f; X) - \phi(f; [X, Y]).
\]
using the definition of the left action, $L X F(f) = \frac{\partial}{\partial t} F(e^{-tX}f)|_{t=0}$. The result is
\[(d\phi)(X,Y) = -\text{Res}[\ell, X_]Y_ - \text{Res}[\ell, X_L]Y_L + \text{Res}[\ell, X_R]Y_R + \text{Res}[\ell, f_L]f_L^{-1}[X_-, Y_-].\]
Adding the right-hand-side to $\omega(X,Y)$ gives $\omega_-(X,Y)$.

We denote by $q(R)$ the canonical second quantization of an operator $R$ and $W = W(A,T)$ is the renormalized interaction (5).

**Theorem 2.** The modified quantum hamiltonian

$$q'(h_0 + W) = q(h_0 + W) + \phi(U(A;t); h_0 + W)$$

has the property that the phase of the quantum scattering matrix is invariant under infinitesimal gauge transformations. Moreover, the phase is invariant with respect to all infinitesimal variations $X$ of $T_A$ such that 1) $X$ is a PSDO in the Lie algebra of $U_1$, 2) $X$ is chirally even, i.e. it commutes with $\gamma_0$ and $\Gamma$.

**Proof.** The second term on the right-hand-side means that we compute the quantum phase using a modified connection, corresponding to adding to the old connection (with curvature $\omega_{\text{loc}}$) the 1-form $\phi$. In general, a variation of the parallel transport phase defined by a connection $\theta$ on any manifold is given by

$$L X \int \theta(\gamma'(t))dt = \int \Omega_{g(t)}(\gamma'(t), X)dt$$

where $\Omega$ is the curvature form and $X(t)$ is a vector field along the curve $\gamma(t)$. Applying this to the case when $\gamma(t) = U(t)$, (the tangent vector $\gamma'(t)$ is replaced by the element $h_0 + W = iU'(t)U(t^{-1})$) and $X$ is an infinitesimal gauge transformation shows that at time $t$ the infinitesimal gauge variation of the phase is

$$L X (\text{phase}) = \omega'(h_0 + W, X).$$

But now $X_-$ is 0 and therefore this expression vanishes and thus also the total gauge variation of the quantum phase as an integral with respect to $t$ of (8).

**Remark** We started from the standard minimal coupling of fermions to external gauge fields. We performed a renormalization by the field dependent unitary operator $T_A$. The new interaction hamiltonian, although nonlocal, is still unitarily equivalent to the original local interaction. There is a very natural generalization of our discussion in the spirit of noncommutative geometry [6]. The original interaction $\gamma^\mu A_\mu$ could be replaced by any bounded pseudodifferential operator without affecting our results (or even by a larger class of operators, see [3]). This would lead to a hamiltonian counterpart of the noncommutative Yang-Mills action functional. It would be an intermediate step between the ordinary Yang-Mills theory and the abstract universal Yang-Mills theory proposed by Rajeev, [13].

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