Some extensions of Hardy’s integral inequalities to Hardy type spaces

Shunchao Long

Abstract

In this paper some extensions of Hardy’s integral inequalities to $0 < p \leq 1$ are established.

1. Introduction

Let

$$H f(x) = \frac{1}{x} \int_0^x f(t) \, dt, \quad (x > 0),$$

and (dual form)

$$H^* f(x) = \int_x^\infty f(t) \, dt, \quad (x > 0).$$

The Hardy integral inequality can be stated as (see [3]):

$$\int_0^\infty (H f(x))^p \, dx < p^p \int_0^\infty (f(x))^p \, dx, \quad p > 1, \quad f(x) \geq 0, \quad (1)$$

$$\int_0^\infty (H^* f(x))^p \, dx < p^p \int_0^\infty (xf(x))^p \, dx, \quad p > 1, \quad f(x) \geq 0, \quad (2)$$

unless $f \equiv 0$.

The inequality (1) was firstly proved by Hardy [4], but the constant was not determined; and Landau found out precisely the constant is $p^p$ in [7]; later, Hardy [5] generalized it to the inequality (2) himself. However, if $0 < p < 1$, the reverse direction inequalities hold (see [3]):

$$\int_0^\infty (H f(x))^p \, dx > p^p \int_0^\infty (f(x))^p \, dx, \quad 0 < p < 1, \quad f(x) \geq 0, \quad (3)$$

$$\int_0^\infty (H^* f(x))^p \, dx > p^p \int_0^\infty (xf(x))^p \, dx, \quad 0 < p < 1, \quad f(x) \geq 0. \quad (4)$$

unless $f \equiv 0$. The constants in (1),(2), (3) and (4) are the best possible.

The positive direction inequalities (1) and (2) play an important role in many areas such as harmonic analysis [12], PDE [8], etc. In view of this, much efforts and time have been devoted to their improvement and generalizations over the years.
When $p > 1$, many generalizations include the works in numerous papers, for example, [3,11,6] and some of the references cited therein.

When $p = 1$, in view of the theory of Hardy spaces on $\mathbb{R}^n$ established by Coifman and Weiss in [1] and some others, J. Garcia-Cuerva and J. L. Rubio de Francia extended (1) to Hardy spaces, and established a positive direction inequality of Hardy type for $p = 1$: let $f \in H^1(\mathbb{R})$ be supported in $[0, \infty)$, then

$$\int_0^\infty |Hf(x)|dx < (\log 2) \|f\|_{H^{1,\infty}_a(\mathbb{R})},$$

where $H^{1,\infty}_a$ is the atom Hardy spaces. See [2].

In this paper we extend the positive direction inequalities (1), (2) and (5) to $0 < p \leq 1$, as well as establish some estimates of $H$ and $H^*$ from Hardy type spaces to Hardy type spaces.

Let us introduce some definitions of the Hardy type spaces.

**Definition 1** Let $0 < p \leq 1 \leq q \leq \infty, p < q$ and $s \in \mathbb{N}$ and $w \geq 0$ is a weight function on $\mathbb{R}^+$. 

(a) A function $a(x)$ on $\mathbb{R}^+$ is said to be a $(p, q, s)_w$-atom, if

(i) $\text{supp} \ a \subset (x_0, x_1) \subset \mathbb{R}^+, x_0 > 0$,

(ii) $\|a\|_{L^q_w(\mathbb{R}^+)} \leq \left(\int_{x_0}^{x_1} w(x)|dx|\right)^{1/q-1/p}$,

(iii) $\int_{\mathbb{R}^+} a(x)x^\beta dx = 0, \beta = 0, 1, \ldots, s$,

(b) and $a(x)$ is said to be a $L - (p, q, s)_w$-atom, if it is a $(p, q, s)_w$-atom and satisfies

(iv) $\int_{\mathbb{R}^+} a(x) \ln x dx = 0$.

**Definition 2** Let $p, q, s$ and $w$ as in Definition 1. Some Hardy spaces on $\mathbb{R}^+$ are defined by

$$H_{w}^{p,q,s}(\mathbb{R}^+) = \{f : f = \sum_{k=1}^\infty \lambda_k a_k, \text{where each } a_k \text{ is a } (p, q, s)_w\text{-atom, } \sum_{k=1}^\infty |\lambda_k|^p < +\infty, \text{and the series converges in the sense of distributions}\},$$

and

$$LH_{w}^{p,q,s}(\mathbb{R}^+) = \{f : f = \sum_{k=1}^\infty \lambda_k a_k, \text{where each } a_k \text{ is a } L - (p, q, s)_w\text{-atom, } \sum_{k=1}^\infty |\lambda_k|^p < +\infty, \text{and the series converges in the sense of distributions}\}.$$

And define the quasinorms of a function of $H_{w}^{p,q,s}(\mathbb{R}^+)$ or $LH_{w}^{p,q,s}(\mathbb{R}^+)$ by

$$\|f\|_{H_{w}^{p,q,s}(\mathbb{R}^+)} = \inf \left(\sum_{k=1}^\infty |\lambda_k|^p\right)^{1/p}, \text{ or } \|f\|_{LH_{w}^{p,q,s}(\mathbb{R}^+)} = \inf \left(\sum_{k=1}^\infty |\lambda_k|^p\right)^{1/p}$$

respectively, where the infimum is taken over all the decompositions of $f$ as above.

Simply, we denote $H_{p,q,0}^{p,q,s}(\mathbb{R}^+)$ and $LH_{p,q,0}^{p,q,s}(\mathbb{R}^+)$ by $H^{p,q,s}(\mathbb{R}^+)$ and $LH^{p,q,s}(\mathbb{R}^+)$ respectively.

$H^{p,q,0}(\mathbb{R}^+)$ is an example of the Hardy spaces on spaces of homogeneous type studied by Coifman and Weiss [1], and A.Marcias and C. Segovia [9,10].

Throughout this paper, we denote $p' = p/(p-1)$ for $1 < p < \infty, \infty' = 1$, and

$1' = \infty$, and suppose all functions are not identical to zero.

We extend (1) and (5) to the case of $0 < p \leq 1$:

**Theorem 1** Let $0 < p \leq 1 \leq q \leq \infty$ and $p < q$. If $f$ is in $H^{p,q,0}(\mathbb{R}^+)$, then

$$\|Hf\|_{L^p(\mathbb{R}^+)} < \frac{1}{(1 - p/q)^{1/p}}\|f\|_{H^{p,q,0}(\mathbb{R}^+)}.$$
We extend (2) as well:

**Theorem 2** Let $0 < p \leq 1$. If $f$ is in $H_{xp,0}^{p,q}(\mathbb{R}^+)$, then,

\[
\| Hf \|_{L^p(\mathbb{R}^+)} < \begin{cases} 
\| f \|_{H_{xp,0}^{p,q}(\mathbb{R}^+)}, & \text{if } q = \infty, \\
\frac{1}{(1-p)^{1/p}(1+p)} \| f \|_{H_{x}^{p,q}(\mathbb{R}^+)}, & \text{if } q = 1, p \neq 1, \\
\frac{1}{(1-pq^p/(q-1/p))^{1/p}} \| f \|_{H_{xp,0}^{p,q}(\mathbb{R}^+)}, & \text{if } 1 < q < \infty, p < q - 1.
\end{cases}
\]  

(7)

We also extend these results to the estimates of $H$ and $H^*$ from Hardy type spaces to Hardy type spaces.

**Theorem 3** Let $0 < p \leq 1 < q \leq \infty$ and $s \in \mathbb{N}$. If $f$ is in $H_{xp,0}^{p,q,s}(\mathbb{R}^+)$, then,

\[
\| Hf \|_{H_{xp,0}^{p,q,s}(\mathbb{R}^+)} \leq q' \| f \|_{LH_{xp,0}^{p,q,s}(\mathbb{R}^+)}. 
\]  

(8)

**Theorem 4** Let $0 < p \leq 1 \leq q \leq \infty$ and $s \in \mathbb{N}$ and $s - 1 > 0$. If $f$ is in $H_{xp,0}^{p,q,s}(\mathbb{R}^+)$, then,

\[
\| H^* f \|_{H_{xp,0}^{p,q,s-1}(\mathbb{R}^+)} \leq \begin{cases} 
(1+p)^{1/p} \| f \|_{H_{xp,0}^{p,q,s}(\mathbb{R}^+)}, & \text{if } q = \infty, \\
\frac{1}{(1-p(1+p))^{1/p}} \| f \|_{H_{xp,0}^{p,q,s}(\mathbb{R}^+)}, & \text{if } q = 1, p \neq 1, \\
q \| f \|_{H_{xp,0}^{p,q,s}(\mathbb{R}^+)}, & \text{if } 1 < q < \infty, 0 < p \leq 1.
\end{cases}
\]  

(9)

2. Proof of Theorems

Firstly, let us introduce two inequalities. By Minkowski inequalities [3], it is easy to see that

\[
(x_1 - x_0)^{p+1} < x_1^{p+1} - x_0^{p+1};
\]  

(10)

when $x_1 > x_0 > 0$ and $p > 0$, and

\[
(x_1 - x_0)^{1-p} > x_1^{1-p} - x_0^{1-p}.
\]  

(11)

when $x_1 > x_0 > 0$ and $1 > p > 0$.

**Proof of Theorem 1** It suffices to prove the following propositions.

**Proposition 1** Let $0 < p \leq 1 \leq q \leq \infty$ and $p < q$. We have

\[
\int_0^\infty |Ha(x)|^p dx < \frac{1}{1 - p/q}
\]  

for all $(p,q,0)$-atom $a$ on $\mathbb{R}^+$.

Let $FH_{xp,0}^{p,q,0}(\mathbb{R}^+)$ be the set of all finite linear combination of $(p,q,0)$-atoms.

From Proposition 1 it is easy to get that

**Proposition 2** Let $0 < p \leq 1 \leq q \leq \infty$ and $p < q$. Then (6) holds for $f$ in $FH_{xp,0}^{p,q,0}(\mathbb{R}^+)$.

**Proposition 3** Let $0 < p \leq 1 \leq q \leq \infty$ and $p < q$. Then

\[
Hf(x) = \sum_{k=1}^\infty \lambda_k Ha_k(x) \quad \text{a.e.}
\]
for all $f = \sum_{k=1}^{\infty} \lambda_k a_k \in H^{p,q,0}(\mathbf{R}^+)$, where each $a_k$ is a $(p, q, 0)$-atom and $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$.

In fact, once Proposition 1 and Proposition 3 have been proved, then, for $f \in H^{p,q,0}(\mathbf{R}^+)$, i.e. $f = \sum_{k=1}^{\infty} \lambda_k a_k$, we have

$$
\int_0^\infty |Hf(x)|^p dx = \int_0^\infty \left| \sum_{k=1}^{\infty} \lambda_k Ha_k(x) \right|^p dx \\
\leq \int_0^\infty \sum_{k=1}^{\infty} \left| \lambda_k \right|^p \left| Ha_k(x) \right|^p dx \\
\leq \sum_{k=1}^{\infty} \left| \lambda_k \right|^p \int_0^\infty \left| Ha_k(x) \right|^p dx \\
\leq \frac{1}{1 - p/q} \sum_{k=1}^{\infty} \left| \lambda_k \right|^p,
$$

from this, (6) follows easily. Thus, we finish the proof of Theorem 1.

Proof of Proposition 1 Let $a$ be a $(p, q, 0)$-atom on $\mathbf{R}^+$, i.e. supp $a \subset (x_0, x_1) \subset \mathbf{R}^+, (x_0 > 0)$, $\|a\|_{L^q(\mathbf{R}^+)} \leq (x_1 - x_0)^{1/q - 1/p}$, and $\int_0^\infty a(x) dx = 0$. Let us prove (12).

From the vanishing property of $a$, it is easy to see that supp $Ha \subset (x_0, x_1)$. Thus, noticing that $x_0 > 0$, we have

$$
\int_0^\infty |Ha(x)|^p dx = \int_{x_0}^{x_1} |Ha(x)|^p dx \\
= \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^{x} a(t) dt \right|^p dx \\
\leq \int_{x_0}^{x_1} \left( \frac{1}{x} \int_{x_0}^{x} |a(t)|^q dt \right)^{1/q} (x - x_0)^{1/q} dx \\
\leq \|a\|_{L^q(\mathbf{R}^+)}^{p} \int_{x_0}^{x_1} \frac{1}{x} (x - x_0)^{1/q} dx \\
\leq (x_1 - x_0)^{p/q - 1} \int_{x_0}^{x_1} \frac{x^p}{x^p} (x - x_0)^{-p/q} dx \\
< (x_1 - x_0)^{p/q - 1} \int_{x_0}^{x_1} (x - x_0)^{-p/q} dx \\
= 1/(1 - p/q)
$$

when $1 < q < \infty$;

$$
\int_0^\infty |Ha(x)|^p dx = \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^{x} a(t) dt \right|^p dx \\
\leq \|a\|_{L^q(\mathbf{R}^+)}^{p} \int_{x_0}^{x_1} \frac{1}{x} dx \\
\leq (x_1 - x_0)^{p-1} \int_{x_0}^{x_1} \frac{(x - x_0)^p}{x^p} (x - x_0)^{-p} dx \\
< (x_1 - x_0)^{p-1} \int_{x_0}^{x_1} (x - x_0)^{-p} dx \\
= 1/(1 - p)
$$

4
when \( q = 1 \) and \( p \neq 1 \); and

\[
\int_0^\infty |H a(x)|^p \, dx = \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^{x} a(t) \, dt \right|^p \, dx \\
\leq \|a\|_{L^\infty(\mathbb{R}^+)}^p \int_{x_0}^{x_1} \left( \frac{1}{x} (x - x_0) \right)^p \, dx \\
\leq (x_1 - x_0)^{-1} \int_{x_0}^{x_1} \frac{(x - x_0)^p}{x^p} \, dx \leq \|a\|_{L^\infty(\mathbb{R}^+)}^p < 1.
\]

when \( q = \infty \). Thus, (12) has been proved for \( 1 \leq q \leq \infty \), and \( p < q \). Thus, we finish the proof of Proposition 1.

**Proof of Proposition 3** Let \( H^{p,q,0}(\mathbb{R}^+) \), then \( f = \sum \lambda_i a_i \) where \( a_i \) are \((p, q, 0)\)-atoms and \( \sum |\lambda_i|^p < +\infty \). We know that \( H a_i \) is well defined for every \( i \) since \( a_i \in L^q(\mathbb{R}^+) \) with \( 1 \leq q < \infty \) and (12) holds by Proposition 1. Then

\[
\| \sum_i \lambda_i H a_i \|_{L^p(\mathbb{R}^+)}^p \leq \sum_i |\lambda_i|^p \| H a_i \|_{L^p(\mathbb{R}^+)}^p \leq C \sum_i |\lambda_i|^p \leq C \| f \|_{H^{p,q,0}(\mathbb{R}^+)}^p < \infty,
\]

it follows \( | \sum_i \lambda_i H a_i(x) | < \infty \text{ a.e.} \). Let

\[
f = \sum_i \lambda_i^{(1)} a_i^{(1)} = \sum_i \lambda_i^{(2)} a_i^{(2)}
\]

(13)

with

\[
\sum_i |\lambda_i^{(1)}|^p < +\infty \quad \text{and} \quad \sum_i |\lambda_i^{(2)}|^p < +\infty
\]

(14)

and \( a_i^{(1)} \) and \( a_i^{(2)} \) are \((p, s, \alpha)\)-atoms. Once it is proved that

\[
\sum_i \lambda_i^{(1)} H a_i^{(1)} = \sum_i \lambda_i^{(2)} H a_i^{(2)} \quad \text{a.e.},
\]

then,

\[
H f(x) = \sum_i \lambda_i H a_i(x) \text{ a.e.}
\]

is well defined for all \( f = \sum_{k=1}^\infty \lambda_k a_k \in H^{p,q,0}(\mathbb{R}^+) \). Thus, Proposition 2 holds.

It is remained to prove (13). For any \( \delta > 0 \), by (14), there exists \( i_0 \) such that

\[
\sum_{i=i_0}^{\infty} |\lambda_i^{(1)}|^p < \delta^p \quad \text{and} \quad \sum_{i=i_0}^{\infty} |\lambda_i^{(2)}|^p < \delta^p.
\]

(15)

From (13), we see that

\[
\sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) = \sum_{i=i_0}^{\infty} \lambda_i^{(2)} a_i^{(2)} - \sum_{i=i_0}^{\infty} \lambda_i^{(1)} a_i^{(1)}.
\]
then,
\[
\| \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \|_{H_{p,q},o(R^+)}^p \leq \sum_{i=i_0}^{\infty} |\lambda_i^{(1)}|^p + \sum_{i=i_0}^{\infty} |\lambda_i^{(2)}|^p < 2\delta^p.
\] (16)

By the linearity of $H$, we have
\[
\sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)} = H \left( \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right) + \sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)}.
\] (17)

By Proposition 2 and (16), we see that
\[
\| H \left( \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right) \|_{L^p(R^+)}^p \leq \| \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \|_{H_{p,q},o(R^+)}^p < 2\delta^p.
\] (18)

From (17), (18), (12) and (15), we have
\[
\| \sum_{i=1}^{\infty} \lambda_i^{(1)} T_\varepsilon a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} T_\varepsilon a_i^{(2)} \|_{L^p(R^+)}^p < 4\delta^p.
\]

Let $\delta \to 0$, we get that $\| \sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)} \|_{L^p(R^+)}^p = 0$, it follows that
\[
\sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} = \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)}
\]
a.e.. Thus, we finish the proof of Proposition 3.

The proof of Theorem 1 is finished.

**Proof of Theorem 2** As the proof of Theorem 1, it suffices to prove the following propositions.

**Proposition 4** Let $0 < p \leq 1$. We have
\[
\int_0^{\infty} |H^* a(x)|^p dx < \begin{cases} 
1, & \text{if } q = \infty, \\
\frac{1}{(1-p)(p+1)}^p, & \text{if } q = 1, p \neq 1, \\
\frac{1}{(1-pq/q)^p/q'(1/q'-p/q)p+1}, & \text{if } 1 < q < \infty, p < q - 1,
\end{cases}
\] (19)

for all $(p, q, 0)_{xp}$-atom on $R^+$.

Let $FH_{x,p}^{p,q,0}(R^+)$ be the set of all finite linear combination of $(p, q, 0)_{xp}$-atoms.

From Proposition 4 it is easy to get that

**Proposition 5** Let $p$ and $q$ as in Theorem 2. Then (7) holds for all $f \in FH_{x,p}^{p,q,0}(R^+)$.

**Proposition 6** Let $0 < p \leq 1, 1 \leq q \leq \infty, p < q - 1, p \neq 1$ and $p < q$. Then
\[
H f (x) = \sum_{k=1}^{\infty} \lambda_k H a_k (x) \quad \text{a.e.}
\]

for all $f = \sum_{k=1}^{\infty} \lambda_k a_k \in H_{x,p}^{p,q,0}(R^+)$, where each $a_k$ is a $(p, q, 0)_{xp}$-atom on $R^+$ and $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$.

**Proof of Proposition 4** Let $a$ be a $(p, q, 0)_{xp}$-atom on $R^+$ with $\text{supp } a \subset (x_0, x_1) \subset R^+, (x_0 > 0)$. Let us prove (19).
As those discussions in the proof of Theorem 1 we have supp $H^*a \subset (x_0, x_1)$. Thus, when $q = \infty$, noticing that $\|a\|_{L^\infty(\mathbb{R}^+)} = \|a\|_{L^\infty_{\mathbb{R}^+}}$, by (10), we have

$$
\int_0^\infty |H^*a(x)|^p dx = \int_{x_0}^{x_1} \left| \int_x^{x_1} |a(t)| dt \right|^p dx
$$

$$
= \int_{x_0}^{x_1} \left| \int_x^{x_1} a(t) dt \right|^p dx
$$

$$
\leq \|a\|^p_{L^\infty_{\mathbb{R}^+}} \int_{x_0}^{x_1} (x_1 - x)^p dx
$$

$$
\leq \left( \int_{x_0}^{x_1} t^p dt \right)^{-1} \int_{x_0}^{x_1} (x_1 - x)^p dx
$$

$$
\leq \frac{(x_1 - x_0)^{p+1}}{x_1^{p+1} - x_0^{p+1}}
$$

$$
< 1;
$$

when $q = 1, 0 < p < 1$, by (11), we have

$$
\int_0^\infty |H^*a(x)|^p dx \leq \int_{x_0}^{x_1} \left| \int_x^{x_1} |a(t)| t^p dt \right|^p dx
$$

$$
\leq \|a\|^p_{L^p_{\mathbb{R}^+}} \int_{x_0}^{x_1} \frac{1}{x^{p^2}} dx
$$

$$
\leq \left( \int_{x_0}^{x_1} t^p dt \right)^{-1} \int_{x_0}^{x_1} \frac{1}{x^{p^2}} dx
$$

$$
= \frac{(1 + p)^{-1-p}}{1 - p^2} \frac{x_1^{-p^2} - x_0^{-p^2}}{(x_1^{1+p} - x_0^{1+p})^{1-p}}
$$

$$
= \frac{1}{(1 - p)(1 + p)^p};
$$

and when $1 < q < \infty$, $0 < p < q - 1$, we see that $(p + 1)/q < 1$, and then $pq'/q < 1$, by (10) and (11), we have

$$
\int_0^\infty |H^*a(x)|^p dx = \int_{x_0}^{x_1} \left| \int_x^{x_1} a(t) dt \right|^p dx
$$

$$
\leq \int_{x_0}^{x_1} \left( \int_x^{x_1} |a(t)|^q t^{pq'} dt \right)^{1/q} \left( \int_{x_0}^{x_1} \frac{1}{t^{pq'/q}} dt \right)^{1/q'} dx
$$

(by Holder's inequality)

$$
\leq \|a\|^p_{L^p_{\mathbb{R}^+}} \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \int_{x_0}^{x_1} (x_1^{1-pq'/q} - x^{1-pq'/q})^{p/q'} dx
$$

$$
\leq \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \left( \int_{x_0}^{x_1} t^p dt \right)^{-1-p/q} \int_{x_0}^{x_1} ((x_1 - x)^{1-pq'/q})^{p/q'} dx
$$

$$
= \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \frac{(1 + p)^{p/q - 1}}{(1/q' - p/q)p + 1} \frac{(x_1 - x_0)^{(1/q' - p/q)p + 1}}{(x_1^{p+1} - x_0^{p+1})^{1-p/q}}
Thus, (19) has been proved for $1 \leq q \leq \infty$, and $p < q - 1$.

Proposition 5 follows easily from Proposition 4. Proof of Proposition 6 is same as that of Proposition 3.

Thus, we finish the proof of Theorem 2.

**Proof of Theorem 3** Let $a$ be a $L - (p, q, s)$-atom on $\mathbb{R}^+$ with supp $a \subset (x_0, x_1)$, $(x_0 > 0)$. Let us prove that $\frac{1}{q'}Ha$ is a $(p, q, s)$-atom on $\mathbb{R}^+$. In fact,

(i) we have already proved supp $Ha \subset (x_0, x_1)$,

(ii) when $q = \infty$, noticing that $|Ha(x)| = \frac{1}{x} \int_{x_0}^x a(t) dt \leq \|a\|_{L^\infty(\mathbb{R}^+)} \times \frac{x - x_0}{x} < \|a\|_{L^\infty(\mathbb{R}^+)}$, we have

\[ \|Ha\|_{L^\infty(\mathbb{R}^+)} < \|a\|_{L^\infty(\mathbb{R}^+)} \leq (x_1 - x_0)^{-1/p}; \]

and when $1 < q < \infty$, by the $L^q(\mathbb{R}^+)$ boundedness of $H$, we have

\[ \|\frac{1}{q'}Ha\|_{L^q(\mathbb{R}^+)} < \|a\|_{L^q(\mathbb{R}^+)} \leq (x_1 - x_0)^{1/q - 1/p}, \]

(iii) by the vanishing property of $a$, we have

\[
\int_0^{+\infty} x^\beta Ha(x)dx = \int_{x_0}^{x_1} x^\beta Ha(x)dx \\
= \int_{x_0}^{x_1} x^{\beta - 1} \int_{x_0}^x a(t) dt dx \\
= \int_{x_0}^{x_1} a(t) \int_{t}^{x_1} x^{\beta - 1} dx dt \\
= \begin{cases} 
\frac{1}{\beta} \int_{x_0}^{x_1} a(t) (x^{\beta} - t^{\beta}) dt, & \text{if } \beta = 1, 2, \ldots, s, \\
\int_{x_0}^{x_1} a(t) (\ln x_1 - \ln t) dt, & \text{if } \beta = 0 \\
0, & \text{if } \beta = 0, 1, 2, \ldots, s.
\end{cases}
\]

Thus, combining (i), (ii) and (iii), we see that $\frac{1}{q'}Ha$ is a $(p, q, s)$-atom on $\mathbb{R}^+$. Let $f \in LH^{p,q,s}(\mathbb{R}^+)$, then $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where each $a_j$ is a $L - (p, q, s)$-atom on $\mathbb{R}^+$. So we have $\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \lambda_j \frac{1}{q'}Ha_j$, where each $\frac{1}{q'}Ha_j$ is a $(p, q, s)$-atom on $\mathbb{R}^+$. And by the definition: $\|\frac{1}{q'}Hf\|_{H^{p,q,s}(\mathbb{R}^+)} = \inf \left( \sum_{j=1}^{\infty} |\mu_j|^p \right)^{1/p}$, where the infimum is taken over all the decompositions $\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \mu_j b_j$, and each $b_j$ is a $(p, q, s)$-atom, we have

\[ \|\frac{1}{q'}Hf\|_{H^{p,q,s}(\mathbb{R}^+)} = \inf_{\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \mu_j b_j, \text{ each } b_j \text{ is a } (p, q, s)\text{-atom}} \left( \sum_{j=1}^{\infty} |\mu_j|^p \right)^{1/p}. \]
\[
\inf \frac{1}{q} Hf = \sum_{j=1}^{\infty} \lambda_j H a_j \leq \left( \sum_{j=1}^{\infty} \lambda_j^p \right)^{1/p} \]
each \(a_j\) is a \(L-(p, q, s)\)-atom (since each \(\frac{1}{q} H a_j\) is a \((p, q, s)\)-atom)

\[
= \inf \frac{1}{q} Hf = \sum_{j=1}^{\infty} \lambda_j H a_j \leq \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \]
each \(a_j\) is a \(L-(p, q, s)\)-atom

\[
= \|f\|_{L H^{p, q, s}(R^+)}. \]

Then

\[
\|Hf\|_{H^{p, q, s}(R^+)} \leq q^{'} \|f\|_{L H^{p, q, s}(R^+)}. \]

Thus, we finish the proof of Theorem 3.

**Proof of Theorem 4**

Let \(a\) be a \((p, q, s)_{x^p}\)-atom on \(R^+\) with \(\text{supp } a \subset (x_0, x_1) \subset R^+, (x_0 > 0)\). Let us prove that

\[
(1 + p)^{-1/p} H^* a(x), \quad \text{if } q = \infty,
\]

\[
(1 - p)(1 + p)^{-1/p} H^* a(x), \quad \text{if } q = 1, p \neq 1,
\]

and \(\frac{1}{q} H^* a(x), \quad \text{if } 1 < q < \infty, 0 < p \leq 1,\)

are the \((p, q, s - 1)\)-atoms.

(i) Clearly, \(\text{supp } H^* a \subset (x_0, x_1)\).

(ii) When \(q = \infty\), and \(x \in (x_0, x_1)\), noticing that \(\|a\|_{L^\infty_{x^p}(R^+)} = \|a\|_{L^\infty(R^+)}\), and by (10), we have

\[
|H^* a(x)| \leq (x_1 - x) \|a\|_{L^\infty(R^+)}
\]

\[
= (x_1 - x) \|a\|_{L^\infty_{x^p}(R^+)}
\]

\[
< (x_1 - x_0) \left( \int_{x_0}^{x_1} x^p dx \right)^{-1/p}
\]

\[
= (1 + p)^{1/p} \frac{(x_1 - x_0)}{(x_1^{p+1} - x_0^{p+1})^{1/p}}
\]

\[
< (1 + p)^{1/p} \frac{(x_1 - x_0)}{((x_1 - x_0)^{p+1})^{1/p}}
\]

\[
= (1 + p)^{1/p} (x_1 - x_0)^{-1/p}. \quad (22)
\]

When \(q = 1, 0 < p < 1\), by (10) and (11), we have,

\[
\int_{x_0}^{x_1} |H^* a(x)| dx \leq \int_{x_0}^{x_1} \left| \int_{x_0}^{x_1} |a(t)| t^p \frac{1}{t^p} dt \right| dx
\]

\[
\leq \|a\|_{L^{1}_{x^p}(R^+)} \int_{x_0}^{x_1} \frac{1}{x^p} dx
\]
\[ \leq \left( \int_{x_0}^{x_1} t^p \, dt \right)^{1-1/p} \int_{x_0}^{x_1} \frac{1}{x^p} \, dx \]
\[ = \frac{1}{(1-p)(1+p)^{1-1/p}} \frac{x_1^{1-p} - x_0^{1-p}}{(x_1^{1+p} - x_0^{1+p})^{1/p-1}} \]
\[ < \frac{1}{(1-p)(1+p)^{1-1/p}} \frac{(x_1 - x_0)^{1-p}}{(x_1 - x_0)^{(1+p)(1-p)/p}} \]
\[ = \frac{1}{(1-p)(1+p)^{1-1/p}}(x_1 - x_0)^{1-1/p}. \]

When \(1 < q < \infty\), by (2), we have
\[ \| H^*a \|_{L^q(R^+)} < \|a\|_{L^q(R^+,t^q)} \leq (x_1 - x_0)^{1/q-1/p}. \] (23)

(iii) By the vanishing property of \(a\), we have
\[ \int_0^{+\infty} x^\beta H^*a(x) \, dx = \int_{x_0}^{x_1} x^\beta H^*a(x) \, dx \]
\[ = \int_{x_0}^{x_1} x^\beta \int_x^{x_1} a(t) \, dt \, dx \]
\[ = \int_{x_0}^{x_1} a(t) \int_t^{x_0} x^\beta \, dx \, dt \]
\[ = \frac{1}{\beta + 1} \int_{x_0}^{x_1} a(t) t^{\beta+1} \, dt \]
\[ = 0, \quad \text{if } \beta = 0, 1, 2, \ldots, s - 1, \]
when \(0 \leq \beta \leq s - 1\). Thus, (21) has been proved. Therefore, Theorem 4 follows from this by analogous arguments to those in the proof of Theorem 3.

### 3. Remarks

**Remark 1** If the functions \(f\) in Theorems 3 and 4 are finite linear combinations of corresponding atoms, i.e. \(f = \sum_{j=1}^{k_0} \lambda_j a_j\), then

the “\(\leq\)” in (8), (9) will be changed to “\(<\)”.

In fact, for Theorem 3, suppose that there is \(k_0 < \infty\) such that

\[ f = \sum_{j=1}^{k_0} \lambda_j a_j, \text{ where each } a_j \text{ is a } L - (p, q, s) - \text{atom on } R^+, \]

then, by the proof of theorem 3, for each \(a_j, j = 1, 2, \ldots, k_0\), we have

\[ \| H a_j \|_{L^q(R^+)} < q' (x_1 - x_0)^{1/q-1/p}, \]

and it follows that there is \(0 < \epsilon_j < 1\) such that

\[ \| H a_j \|_{L^q(R^+)} < (q' - \epsilon_j) (x_1 - x_0)^{1/q-1/p}. \]
Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{k_0}\}$, then

$$\|H a_j\|_{L^q(\mathbb{R}^+)} < (q' - \epsilon)(x_1 - x_0)^{1/q - 1/p}.$$ 

As the arguments in the proof of Theorem 3, we have that each $H a_j$ is a $(p, q, s)$-atom on $\mathbb{R}^+$, and

$$\|H f\|_{H^{p,q,s}(\mathbb{R}^+)} \leq (q' - \epsilon)\|f\|^p_{L^{p,q,s}(\mathbb{R}^+)} < q'\|f\|^p_{L^{p,q,s}(\mathbb{R}^+)}.$$

Similar arguments above are suitable for the case of Theorem 4.

**Remark 2** If we drop the restriction $x_0 > 0$ in the definitions of Hardy spaces (in Definition 1), then

the "<" in (6), (7) will be changed to "\leq".

This is because of that the "<" in (12) and (19) will be changed to "\leq" if $x_0 = 0$ in the proofs of Theorems 1 and 2.

**Remark 3** If the functions $f$ in Theorems 3 and 4 are finite linear combinations of corresponding atoms, i.e. $f = \sum_{j=1}^{k_0} \lambda_j a_j$, then, even if dropping the restriction $x_0 > 0$ in the definitions of Hardy spaces (in Definition 1), we have that:

the "\leq" in (8) will be changed to "<" when $1 < q < \infty$, and

the "\leq" in (9) will be changed to "<" when $1 < q \leq \infty$. These follow from the analogous arguments to those of Remark 1, since the "<" in (20),(23), and the first "<" in (22) hold still when $x_0 = 0$.

**References**

[1] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977), 569-645.

[2] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland-Amsterdam,New York, Oxford, (1985).

[3] G. H. Hardy, J. E. Littlewood, and G. Polya , Inequalities, Cambridge Univ. Press, Cambridge, UK, (1959).

[4] G. H. Hardy, Note on a theorem of Hilbert, Math. Z. (7) (1920), 314-317.

[5] G. H. Hardy, Note on some points in the integral calculus, Messenger of Math. (57) (1928), 12-16.

[6] A. Kufner and L.-E. Persson, Weighted inequalities of Hardy type. World Scientific Publishing Co. Inc., River Edge, NJ (2003).
[7] E. Landau, A note on a theorem concerning series of positive terms, *J. London. Math. Soc.* (1) (1926), 38-39.

[8] V. G. Maz’ya, ‘Sobolev Spaces’, Berlin, Springer-Verlag (1985).

[9] A. Marcias and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv.inMath.* 33(1979), 257-271.

[10] A. Marcias and C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, *Adv.inMath.* 33(1979), 271-309.

[11] B. Opic and A. Kufner, Hardy-type inequalities, Longman Scientific and Technical, Harlow (1990).

[12] E.M.Stein and G.Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, N. J., 1971.

Shunchao Long  
Mathematics Department,  
Xiangtan University,  
Xiangtan , 411105, China  
E-mail address: sclong@xtu.edu.cn