THE MORSE-NOVIKOV THEORY OF CIRCLE-VALUED FUNCTIONS AND NONCOMMUTATIVE LOCALIZATION

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ABSTRACT. We use noncommutative localization to construct a chain complex which counts the critical points of a circle-valued Morse function on a manifold, generalizing the Novikov complex. As a consequence we obtain new topological lower bounds on the minimum number of critical points of a circle-valued Morse function within a homotopy class, generalizing the Novikov inequalities.

1. Introduction.

Let us start by recalling the way in which chain complexes are used to count the critical points of a real-valued Morse function.

A Morse function \( f : M \to \mathbb{R} \) on a compact (differentiable) \( m \)-dimensional manifold \( M \) with \( c_i(f) \) critical points of index \( i \) determines a handlebody decomposition of \( M \) with \( c_i(f) \) \( i \)-handles. The Morse-Smale complex is the cellular chain complex of the corresponding \( CW \) decomposition of the universal cover \( \widetilde{M} \) of \( M \), a based f.g. free \( \mathbb{Z}[\pi_1(M)] \)-module chain complex \( C(\widetilde{M}) \) with

\[
\text{rank}_{\mathbb{Z}[\pi_1(M)]} C_i(\widetilde{M}) = c_i(f) .
\]

For any ring morphism \( \rho : \mathbb{Z}[\pi_1(M)] \to R \) there is induced a based f.g. free \( R \)-module chain complex

\[
C(M; R) = R \otimes_{\mathbb{Z}[\pi_1(M)]} C(\widetilde{M})
\]

with homology \( R \)-modules \( H_*(M; R) = H_*(C(M; R)) \) (which in general depend on \( \rho \) as well as \( R \)). The number \( c_i(f) \) of critical points of index \( i \) is bounded from below by the minimum number \( \mu_i(M; R) \) of generators in degree \( i \) of a finite f.g. free \( R \)-module chain complex which is chain equivalent to \( C(M; R) \)

\[
c_i(f) \geq \mu_i(M; R) .
\]

For a principal ideal domain \( R \) and a ring morphism \( \rho : \mathbb{Z}[\pi_1(M)] \to R \) the \( R \)-coefficient Betti numbers of \( M \) are defined as usual by

\[
b_i(M; R) = \text{rank}_R(H_i(M; R)/T_i(M; R)) ,
\]

\[
q_i(M; R) = \text{minimum number of generators of } T_i(M; R)
\]

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with \( T_i(M; R) \subseteq H_i(M; R) \) the torsion submodule, and
\[
\mu_i(M; R) = b_i(M; R) + q_i(M; R) + q_{i-1}(M; R) .
\]

The Morse inequalities
\[
c_i(f) \geq b_i(M; R) + q_i(M; R) + q_{i-1}(M; R)
\]
are thus an algebraic consequence of the existence of the Morse-Smale complex.

Now for circle-valued functions. Given a Morse function \( f : M \to S^1 \) let \( c_i(f) \) denote the number of critical points of index \( i \). The Novikov complex of \([N]\) is a based f.g. free chain complex \( \mathcal{C}^{Nov}(M, f) \) over the principal ideal domain \( \mathbb{Z}((z)) = \mathbb{Z}[z][z^{-1}] \), such that
\[
(i) \quad \text{rank}_{\mathbb{Z}((z))} \mathcal{C}^{Nov}_i(M, f) = c_i(f),
(ii) \quad \mathcal{C}^{Nov}(M, f) \text{ is chain equivalent to } \mathcal{C}(M; \mathbb{Z}((z))),
\]
with
\[
\rho : \mathbb{Z}[\pi_1(M)] \xrightarrow{f_*} \mathbb{Z}[\pi_1(S^1)] = \mathbb{Z}[z, z^{-1}] \to \mathbb{Z}((z)) .
\]

The chain complex \( \mathcal{C}^{Nov}(M, f) \) is constructed geometrically using the gradient flow. The Novikov inequalities
\[
c_i(f) \geq \mu_i(M; \mathbb{Z}((z))) = b_i(M; \mathbb{Z}((z))) + q_i(M; \mathbb{Z}((z))) + q_{i-1}(M; \mathbb{Z}((z)))
\]
are an algebraic consequence of the existence of a chain complex \( \mathcal{C}^{Nov}(M, f) \) satisfying (i) and (ii). The \( \mathbb{Z}((z)) \)-coefficient Betti numbers are called the Novikov numbers of \( M \). The Novikov numbers depend only on the cohomology class \( \xi = f^*(1) \in H^1(M) \), and so may be denoted by
\[
b_i(M; \mathbb{Z}((z))) = b_i(\xi) , \quad q_i(M; \mathbb{Z}((z))) = q_i(\xi) .
\]

A map \( f : M \to S^1 \) classifies an infinite cyclic cover \( \overline{M} = f^*\mathbb{R} \) of \( M \). We shall assume that \( M \) and \( \overline{M} \) are connected, so that there is defined a short exact sequence
\[
0 \to \pi \to \pi_1(M) \xrightarrow{f_*} \pi_1(S^1) = \mathbb{Z} \to 0
\]
with \( \pi = \pi_1(\overline{M}) \). Let \( z \in \pi_1(M) \) be such that \( f_*(z) = 1 \), so that
\[
\pi_1(M) = \pi \times_\alpha \mathbb{Z} , \quad \mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi]_\alpha[z, z^{-1}] .
\]

Let \( \Sigma \) denote the set of square matrices with entries in \( \mathbb{Z}[\pi_1(M)] \) having the form \( 1 - ze \) where \( e \) is a square matrix with entries in \( \mathbb{Z}[\pi] \). A ring morphism \( \mathbb{Z}[\pi_1(M)] \to R \) is called \( \Sigma \)-inverting if it sends matrices in \( \Sigma \) to invertible matrices over the ring \( R \). There exists a noncommutative localization in the sense of P. M. Cohn \( [C] \), a
ring $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ together with a ring morphism $\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ which has the universal property that every $\Sigma$-inverting homomorphism $\mathbb{Z}[\pi_1(M)] \to R$ has a unique factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to R.$$ 

In particular, the inclusion of the group ring $\mathbb{Z}[\pi_1(M)]$ in the Novikov completion

$$\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi][[z]]_\alpha[z^{-1}]$$

is $\Sigma$-inverting, so that there is a factorization

$$\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)].$$

Pazhitnov [P1] extended the geometric construction of the Novikov complex to a based f.g. free chain complex $C^{Nov}(M, f)$ over $\mathbb{Z}[\pi_1(M)]$, such that

(i) $\text{rank}_{\mathbb{Z}[\pi_1(M)]} C_i^{Nov}(M, f) = c_i(f),$ 

(ii) $C^{Nov}(M, f)$ is chain equivalent to $C(M; \mathbb{Z}[\pi_1(M)])$, with

$$\rho = \text{inclusion} : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)].$$

Moreover, Pazhitnov [P2],[P3] showed that the geometric construction of $C^{Nov}(M, f)$ can be adjusted so that the differentials are rational, in the sense that the entries of their matrices belong to $\text{im}(\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)])$. If $\pi_1(M)$ is abelian and $\alpha = 1$ the localization is a subring of the completion

$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] = (1 + z\mathbb{Z}[\pi])^{-1}\mathbb{Z}[\pi][z, z^{-1}] \subset \mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi][[z]][z^{-1}]$$

so that $C^{Nov}(M, f)$ is induced from a chain complex defined over $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$.

Our main result is a direct construction of such a rational lift of the Novikov complex, which is valid for arbitrary $\pi_1(M)$ (not necessary abelian) and arbitrary $\alpha$ :

**Main Theorem.** For every Morse function $f : M \to S^1$ there exists a based f.g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complex $\hat{C}(M, f)$ such that

(i) $\text{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} \hat{C}_i(M, f) = c_i(f),$ 

(ii) $\hat{C}(M, f)$ is chain equivalent to $\Sigma^{-1}C(\tilde{M}).$

The Main Theorem is proved in section 3 by cutting $M$ at the inverse image $N = f^{-1}(x) \subset M$ of a regular value $x \in S^1$ of $f$, to obtain a fundamental domain $(M_N; N, zN)$ for $\overline{M}$, and considering the chain homotopy theoretic properties of the handle decomposition of the cobordism, with $c_i(f)$ $i$-handles.

The existence of the chain complex $\hat{C}(M, f)$ has as immediate consequence:
Corollary (Generalized Novikov inequalities). For any Morse function \( f : M \to S^1 \) and any \( \Sigma \)-inverting ring morphism \( \rho : \mathbb{Z}[\pi_1(M)] \to R \)

\[
c_i(f) \geq \mu_i(M; R).
\]

Recall that the number \( \mu_i(M; R) \) is defined as the minimum number of generators in degree \( i \) of any f.g. free \( R \)-module chain complex, which is chain homotopy equivalent to \( C(M; R) \). The numbers \( \mu_i(M; R) \) are homotopy invariants of \( M \) as a space over \( S^1 \) (via \( f \)); they depend on the homotopy class of \( f \) and on the ring homomorphism \( \rho \).

As an example consider the following ring of rational functions

\[
\mathcal{R} = (1 + z\mathbb{Z}[z])^{-1}\mathbb{Z}[z, z^{-1}]
\]

introduced in [F]. This ring is a principal ideal domain [F]. The induced homomorphism \( f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z} \) determines a ring homomorphism

\[
\rho : \mathbb{Z}[\pi_1(M)] \to \mathcal{R} ; \ g \mapsto z^{f_*(g)} \quad (g \in \pi_1(M)).
\]

It is easy to see that \( \rho \) is \( \Sigma \)-inverting. The corollary implies the inequalities

\[
c_i(f) \geq \mu_i(M; R), \quad i = 0, 1, 2, \ldots
\]

These inequalities coincide with the classical Novikov inequalities since

\[
\mu_i(M; \mathcal{R}) = \mu_i(M; \mathbb{Z}((z))) = b_i(\xi) + q_i(\xi) + q_{i-1}(\xi).
\]

The equivalence between this approach (using the ring \( \mathcal{R} \) of rational functions) and the original approach of Novikov (which used the formal power series ring \( \mathbb{Z}((z)) \)) was proved in [F].

2. The endomorphism localization.

This section describes the endomorphism localization of a Laurent polynomial extension, and provides the algebraic machinery required for the construction of the chain complex \( \hat{C}(M, f) \) in section 3.

Given a ring \( A \) and an automorphism \( \alpha : A \to A \) let \( A_\alpha[z], A_\alpha[[z]], A_\alpha[z, z^{-1}], A_\alpha((z)) \) be the \( \alpha \)-twisted polynomial extension rings of \( A \), with \( z \) an indeterminate over \( A \) with \( az = z\alpha(a) \) (\( a \in A \)); for the record, \( A_\alpha[z] \) is the ring of finite polynomials \( \sum_{j=0}^\infty a_j z^j \), \( A_\alpha[[z]] \) is the ring of power series \( \sum_{j=0}^\infty a_j z^j \), \( A_\alpha[z, z^{-1}] \) is the ring of finite Laurent polynomials \( \sum_{j=-\infty}^\infty a_j z^j \), and \( A_\alpha((z)) = A_\alpha[[z]][z^{-1}] \) is the Novikov ring of power series \( \sum_{j=-\infty}^\infty a_j z^j \) with only a finite number of non-zero coefficients \( a_j \in A \) for \( j < 0 \).

Definition 2.1 Let \( \Sigma \) be the set of square matrices in \( A_\alpha[z, z^{-1}] \) of the form \( 1 - ze \) with \( e \) a square matrix in \( A \). The endomorphism localization \( \Sigma^{-1}A_\alpha[z, z^{-1}] \) is the noncommutative localization of \( A_\alpha[z, z^{-1}] \) inverting \( \Sigma \) in the sense of Cohn [C]. □
By construction, $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ is the ring obtained from $A_{\alpha}[z, z^{-1}]$ by adjoining generators corresponding to the entries in formal inverses $(1 - ze)^{-1}$ of elements $1 - ze \in \Sigma$, and the relations given by the matrix equations

$$(1 - ze)^{-1}(1 - ze) = (1 - ze)(1 - ze)^{-1} = 1.$$ 

Given an $A$-module $B$ let $zB$ be the $A$-module with elements $zx$ ($x \in B$) and $A$ acting by

$$A \times zB \to zB ; (a, zx) \mapsto azx = z\alpha(a)x.$$ 

If $B$ is a f.g. free $A$-module with basis $\{b_1, b_2, \ldots, b_r\}$ then $zB$ is a f.g. free $A$-module with basis $\{zb_1, zb_2, \ldots, zb_r\}$. For f.g. free $A$-modules $B, B'$ an $A_{\alpha}[z, z^{-1}]$-module morphism $f : B_{\alpha}[z, z^{-1}] \to B'_{\alpha}[z, z^{-1}]$ is a finite Laurent polynomial

$$f = \sum_{j=-\infty}^{\infty} z^j f_j : B_{\alpha}[z, z^{-1}] \to B'_{\alpha}[z, z^{-1}]$$ 

with coefficients $A$-module morphisms $f_j : z^j B \to B'$. Note that in the special case $\alpha = 1 : A \to A$ there is defined a natural $A$-module isomorphism

$$B \to zB ; x \mapsto zx.$$ 

**Proposition 2.2.** (i) A ring morphism $A_{\alpha}[z, z^{-1}] \to R$ which sends every $1 - ze \in \Sigma$ to an invertible matrix in $R$ has a unique factorization

$$A_{\alpha}[z, z^{-1}] \to \Sigma^{-1}A_{\alpha}[z, z^{-1}] \to R.$$ 

(ii) If $E$ is a f.g. free $A$-module then for any $A$-module morphism $e : zE \to E$ there is defined an automorphism of a f.g. free $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$-module

$$1 - ze : \Sigma^{-1}E_{\alpha}[z, z^{-1}] \to \Sigma^{-1}E_{\alpha}[z, z^{-1}].$$ 

(iii) The inclusion $A_{\alpha}[z, z^{-1}] \to A_{\alpha}((z))$ factorizes through $\Sigma^{-1}A_{\alpha}[z, z^{-1}]$ 

$$A_{\alpha}[z, z^{-1}] \to \Sigma^{-1}A_{\alpha}[z, z^{-1}] \to A_{\alpha}((z))$$ 

with $A_{\alpha}[z, z^{-1}] \to \Sigma^{-1}A_{\alpha}[z, z^{-1}]$ an injection.

**Proof.** (i) This is the universal property of noncommutative localization.

(ii) By construction.

(iii) Every matrix of the type $1 - ze$ is invertible in $A_{\alpha}((z))$, with

$$(1 - ze)^{-1} = 1 + ze + z^2e^2 + \ldots.$$ 

so that (i) applies. □

For commutative $A$ and $\alpha = 1 : A \to A$

$$\Sigma^{-1}A[z, z^{-1}] = (1 + zA[z])^{-1}A[z, z^{-1}]$$ 

is just the usual commutative localization inverting the multiplicative subset

$$1 + zA[z] \subset A[z, z^{-1}].$$ 

We shall need the following
2.3. Deformation Lemma. Let $C$ be an $A$-module chain complex of the form

\[
d_C = \begin{pmatrix} d_D & a & c \\ 0 & d_F & b \\ 0 & 0 & d_{D'} \end{pmatrix} : C_i = D_i \oplus F_i \oplus D_i' \to C_{i-1} = D_{i-1} \oplus F_{i-1} \oplus D_{i-1}'
\]

where $a : F_i \to D_{i-1}$, $b : D_i' \to F_{i-1}$ and $c : D_i' \to D_{i-1}$. Suppose that the morphism $c : D_i' \to D_{i-1}$ is an isomorphism for all $i$. The formula

\[
\hat{d}_C = d_F - bc^{-1}a : F_i \to F_{i-1}
\]

defines a "deformed differential" on $F_i$ (i.e. $(\hat{d}_C)^2 = 0$), and the chain complex $\hat{C}$ defined by

\[
\hat{d}_C : \hat{C}_i = F_i \to \hat{C}_{i-1} = F_{i-1}
\]
is chain equivalent to $C$.

Proof. Note that $(d_C)^2 = 0$ implies

\[
(d_D)^2 = 0, \quad (d_F)^2 = 0, \quad (d_{D'})^2 = 0,
\]

and also

\[
d_D a + ad_F = 0,
\]

\[
d_F b + bd_{D'} = 0,
\]

\[
d_D c + cd_{D'} + ab = 0.
\]

Using (1) we obtain

\[
(\hat{d}_C)^2 = (d_F - bc^{-1}a) \cdot (d_F - bc^{-1}a)
\]

\[
= -d_Fbc^{-1}a - bc^{-1}ad_F + bc^{-1}a \cdot bc^{-1}a
\]

\[
= -d_Fbc^{-1}a - bc^{-1}ad_F - bc^{-1}[d_Dc + cd_{D'}]c^{-1}a
\]

\[
= 0.
\]

Now we define two chain maps:

\[
u = (-bc^{-1} 1 0) : C \to \hat{C}, \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 1 \\ -c^{-1}a \end{pmatrix} : \hat{C} \to C.
\]

One checks that

\[ud_C = \hat{d}_C u, \quad d_C v = v \hat{d}_C, \quad uv = 1_{\hat{C}}
\]

and

\[vu = 1_C - d_C w - wd_C, \quad \text{where} \quad w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-1} & 0 & 0 \end{pmatrix} : C \to C.
\]

Hence $u$ and $v$ are mutually inverse chain equivalences. □
\textbf{Theorem 2.4.} Let $C$ be a finite based f.g. free $A_\alpha[z, z^{-1}]$-module chain complex of the form
\[ C = C(g - zh : D_\alpha[z, z^{-1}] \to E_\alpha[z, z^{-1}]) \]
where $g : D \to E$, $h : zD \to E$ are chain maps of finite based f.g. free $A$-module chain complexes, and $C$ denotes the algebraic mapping cone. If each $g : D_i \to E_i$ is a split injection sending basis elements to basis elements, then there is defined a $\Sigma^{-1}A_\alpha[z, z^{-1}]$-module chain complex $\hat{C}$ such that

(i) each $\hat{C}_i$ is a based f.g. free $\Sigma^{-1}A_\alpha[z, z^{-1}]$-module with
\[ \text{rank}_{\Sigma^{-1}A_\alpha[z, z^{-1}]} \hat{C}_i = \text{rank}_AE_i - \text{rank}_AD_i , \]

(ii) there is defined a chain equivalence $\Sigma^{-1}C \to \hat{C}$.

\textbf{Proof.} Let $F_i \subseteq E_i$ be the submodule generated by the basis elements not coming from $D_i$, so that
\[ g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \to E_i = D_i \oplus F_i , \]
\[ h = \begin{pmatrix} e \\ f \end{pmatrix} : zD_i \to E_i = D_i \oplus F_i , \]
\[ d_E = \begin{pmatrix} d_D & a \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \to E_{i-1} = D_{i-1} \oplus F_{i-1} , \]
\[ d_C = \begin{pmatrix} d_D & a & 1-ze \\ 0 & d_F & -zf \\ 0 & 0 & d_D \end{pmatrix} : \]
\[ C_i = (D_i \oplus F_i \oplus D_{i-1})_\alpha[z, z^{-1}] \to C_{i-1} = (D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_\alpha[z, z^{-1}] . \]

Now apply Lemma 2.3 to the induced chain complex $\Sigma^{-1}C$ over $\Sigma^{-1}A_\alpha[z, z^{-1}]$ with
\[ d_{\Sigma^{-1}C} = \begin{pmatrix} d_D & a & 1-ze \\ 0 & d_F & -zf \\ 0 & 0 & d_D \end{pmatrix} : \]
\[ \Sigma^{-1}C_i = \Sigma^{-1}(D_i \oplus F_i \oplus D_{i-1})_\alpha[z, z^{-1}] \to \Sigma^{-1}C_{i-1} = \Sigma^{-1}(D_{i-1} \oplus F_{i-1} \oplus D_{i-2})_\alpha[z, z^{-1}] \]

where each
\[ c = 1 - ze : \Sigma^{-1}(D_{i-1})_\alpha[z, z^{-1}] \to \Sigma^{-1}(D_{i-1})_\alpha[z, z^{-1}] \]
is an automorphism. Explicitly, Lemma 2.3 gives a based f.g. free $\Sigma^{-1}A_\alpha[z, z^{-1}]$-module chain complex $\hat{C}$ with
\[ d_{\hat{C}} = d_F + (zf)(1-ze)^{-1}a : \hat{C}_i = \Sigma^{-1}(F_i)_\alpha[z, z^{-1}] \to \hat{C}_{i-1} = \Sigma^{-1}(F_{i-1})_\alpha[z, z^{-1}] \]
which satisfies (i) and (ii). \(\square\)
3. The chain complex $\hat{C}(M, f)$.

Given a Morse function $f : M \to S^1$ we now construct a chain complex $\hat{C}(M, f)$ over $\Sigma^{-1}Z[\pi_1(M)]$ satisfying the conditions of the Main Theorem.

Choose a regular value $x \in S^1$ for $f : M \to S^1$, and cut $M$ along the codimension 1 framed submanifold

$$N^{m-1} = f^{-1}(x) \subset M^m$$

to obtain a fundamental domain $(M_N; N, zN)$ for the infinite cyclic cover

$$\overline{M} = \bigcup_{j=-\infty}^{\infty} z^j M_N$$

of $M$, with a Morse function

$$f_N : (M_N; N, zN) \to ([0,1]; \{0\}, \{1\})$$

such that $f_N$ has exactly as many critical points of index $i$ as $f$

$$c_i(f_N) = c_i(f).$$

Let $\tilde{N}, \tilde{M}_N$ be the covers of $N, M_N$ obtained from the universal cover $\tilde{M}$ of $M$ by pullback along the inclusions $N \to \overline{M}, M_N \to \overline{M}$. The cobordism $(M_N; N, zN)$ has a handle decomposition

$$M_N = N \times I \cup \bigcup_{i=0}^{m} \bigcup_{i=0}^{c_i(f)} h^i$$

with $c_i(f)$ $i$-handles $h^i = D^i \times D^{m-i}$. Let $\pi_1(\overline{M}) = \pi$ with monodromy automorphism $\alpha : \pi \to \pi$, so that $\pi_1(M) = \pi \times_\alpha \mathbb{Z}, \mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi][z, z^{-1}]$ as in the Introduction. The relative cellular chain complex $C(\tilde{M}_N, \tilde{N})$ is a based f.g. free $\mathbb{Z}[\pi]$-module chain complex with

$$\text{rank}_{\mathbb{Z}[\pi]} C_i(\tilde{M}_N, \tilde{N}) = c_i(f).$$

Choose an arbitrary $CW$ structure for $N$, and let $c_i(N)$ be the number of $i$-cells. Let $M_N$ have the $CW$ structure with $c_i(N) + c_i(f)$ $i$-cells: for each $i$-cell $e^i \subset N$ there is defined an $i$-cell $e^i \times I \subset M_N$, and for each $i$-handle $h^i$ there is defined an $i$-cell $h^i \subset M_N$. Then $M = M_N/(N = zN)$ has a $CW$ complex structure with $c_i(f) + c_i(N) + c_{i-1}(N)$ $i$-cells. The universal cover $\tilde{M}$ has cellular $\mathbb{Z}[\pi_1(M)]$-module chain complex

$$C(\tilde{M}) = C(g - zh : C(\tilde{N})_\alpha[z, z^{-1}] \to C(\tilde{M}_N)_\alpha[z, z^{-1}])$$

where

$$g : C(\tilde{N}) \to C(\tilde{M}_N), \ h : zC(\tilde{N}) \to C(\tilde{M}_N)$$
are the $\mathbb{Z}[\pi]$-module chain maps induced by the inclusions

$$g : N \to M_N, \ h : zN \to M_N.$$  

Since $g : N \to M_N$ is the inclusion of a subcomplex

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_i(\tilde{N}) \to C_i(\tilde{M}_N) = C_i(\tilde{N}) \oplus C_i(\tilde{M}_N, \tilde{N})$$

is a split injection. Now apply Theorem 2.4 to the based f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex

$$C(\tilde{M}) = C(g - zh : C(\tilde{N})_\alpha[z, z^{-1}] \to C(\tilde{M}_N)_\alpha[z, z^{-1}])$$

with $D = C(\tilde{N})$, $E = C(\tilde{M}_N)$. □

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