Generalized List Colouring of Graphs

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Abstract
This paper disproves a conjecture in Wang et al. (Graphs Comb. 31:1779–1787, 2015) and answers in the negative a question in Dvořák et al. (Electron J Comb:P26, 2019). In return, we pose five open problems.

Keywords Generalized list colouring · List vertex arboricity · List star arboricity · Choice number

1 Introduction

Assume \( \mathcal{G} \) is a hereditary family of graphs, i.e., if \( G \in \mathcal{G} \) and \( H \) is an induced subgraph of \( G \), then \( H \in \mathcal{G} \). A \( \mathcal{G} \)-colouring of a graph \( G \) is a colouring \( \phi \) of the vertices of \( G \) such that each colour class induces a graph in \( \mathcal{G} \). A \( \mathcal{G} \)-n-colouring of \( G \) is a \( \mathcal{G} \)-colouring \( \phi \) of \( G \) such that \( \phi(v) \in \{1, 2, \ldots, n\} \) for each vertex \( v \). We say \( G \) is \( \mathcal{G} \)-n-colourable if there exists a \( \mathcal{G} \)-n-colouring of \( G \). The \( \mathcal{G} \)-chromatic number of \( G \) is

\[
\chi_{\mathcal{G}}(G) = \min \{ n : G \text{ is } \mathcal{G} \text{-n-colourable} \}.
\]

Assume \( L \) is a list assignment of \( G \). A \( \mathcal{G} \)-L-colouring of \( G \) is a \( \mathcal{G} \)-colouring \( \phi \) of \( G \) such that \( \phi(v) \in L(v) \) for each vertex \( v \). We say \( G \) is \( \mathcal{G} \)-n-choosable if for every \( n \)-list assignment \( L \) of \( G \), there exists a \( \mathcal{G} \)-L-colouring of \( G \). The \( \mathcal{G} \)-choice number of \( G \) is

\[
ch_{\mathcal{G}}(G) = \min \{ n : G \text{ is } \mathcal{G} \text{-n-choosable} \}.
\]

Note that if \( \mathcal{G}_1 \subseteq \mathcal{G}_2 \), then \( ch_{\mathcal{G}_2}(G) \leq ch_{\mathcal{G}_1}(G) \) for all graphs \( G \).

The concept of \( \mathcal{G} \)-colouring of a graph is a slight modification of the concept of generalized colouring of graphs introduced in [1], where the graph class \( \mathcal{G} \) is assumed to be of the form \( \mathcal{G} = \{ G : f(G) \leq d \} \) for some graph parameter \( f \) and constant \( d \). We find that there are some graph families \( \mathcal{G} \), say the family of linear

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forests, star forests, or the family of graphs with no induced subgraph isomorphic to \( H \) for a given graph \( G \), for which the \( G \)-colouring problems are interesting, and yet \( G \) is not easily expressed in such a form.

Many colouring concepts studied in the literature are \( G \)-colourings for special graph families \( G \). We denote by

- \( G_k \) the family of graphs whose connected components are of order at most \( k \);
- \( D_k \) the family of graphs of maximum degree at most \( k \);
- \( F \) the family of forests;
- \( S \) the family of star forests;
- \( L \) the family of linear forests;
- \( C_k \) the family of graphs of colouring number at most \( k \);
- \( M_k \) the family of graphs of maximum average degree at most \( k \).

Note that each of these families is a hereditary family of graphs. Several of the \( G \)-colourings have special names and are studied extensively in the literature. The following is a list of some popular colouring concepts phrased in the above terms.

- A \( G_k \)-colouring of \( G \) is a colouring of \( G \) with clustering \( k \). In particular, a \( G_1 \)-colouring of \( G \) is a proper colouring of \( G \).
- A \( D_k \)-colouring of \( G \) is a \( k \)-defective colouring of \( G \). The parameter \( \chi_{D_k}(G) \) is the \( k \)-defective chromatic number of \( G \). In particular, a \( D_0 \)-colouring of \( G \) is a proper colouring of \( G \).
- An \( F \)-colouring of \( G \) is a vertex arboreal colouring of \( G \). The parameter \( \chi_F(G) \) is the vertex arboricity of \( G \), and \( \chi_L(G) \) is the list vertex arboricity of \( G \).
- The parameter \( \chi_S(G) \) is the star vertex arboricity of \( G \), and \( \chi_L(G) \) is the list vertex arboricity of \( G \).
- The parameter \( \chi_L(G) \) is the linear vertex arboricity of \( G \), and \( \chi_L(G) \) is the list linear vertex arboricity of \( G \).

For two graph families \( G \) and \( G' \) and a graph \( G \), it follows easily from the definition that

\[
\chi_{G'}(G) \leq \left( \max_{H \in G'} \chi_{G'}(H) \right) \chi_{G'}(G),
\]

and this upper bound is tight. For example,

\[
\chi(G) \leq 2\chi_F(G), \quad \chi_S(G) \leq 2\chi_F(G), \quad \text{and} \quad \chi(G) \leq (k + 1)\chi_{M_k}(G),
\]

and for integers \( k, k' \),

\[
\chi_{G_k}(G) \leq \left\lceil \frac{k'}{k} \right\rceil \chi_{G_{k'}}(G).
\]

Note that in each of the inequalities above, equality holds for appropriately sized complete graphs or paths, as well as for many other graphs.
It is natural to ask if the same or similar inequalities hold for the corresponding choice number. Some of such inequalities are posed as conjectures or questions in the literature. For example, the following conjecture was proposed in [2]:

**Conjecture 1.1**  For every graph $G$,

$$\text{ch}(G) \leq 2 \text{ch}_F(G).$$

Also, the following question was asked in [1]:

**Question 1.2**  Is it true that for every graph $G$ and every positive integer $k$,

$$\text{ch}(G) \leq (k + 1)\text{ch}_{M_k}(G)?$$

In this note, we disprove Conjecture 1.1 and give a negative answer to Question 1.2.

## 2 The Proofs

**Lemma 2.1**  Assume $k \geq 2$ is an integer and let $m = k(k + 1) - 1$. For every positive integer $n$, $\text{ch}_S(K_{m,n}) \leq k$.

**Proof**  When $n = 1$, $K_{m,n}$ belongs to $S$, so $\text{ch}_S(K_{m,n}) = 1$. Assume $k, n \geq 2$ are integers and $m = k(k + 1) - 1$. Let $G$ be the complete bipartite graph with partite sets $A, B$ where $|A| = m$ and $|B| = n$. Now we show that $\text{ch}_S(G) \leq k$.

Let $L$ be a $k$-list assignment of $G$. Let $H$ be an auxiliary bipartite graph with partite sets $A$ and $C = \cup_{v \in A} L(v)$, where $vc$ is an edge if and only if $c \in L(v)$. Note that each vertex $v \in A$ has degree $k$ in $H$.

A subset $C'$ of $C$ is **heavy** if $|N_H(C')| \geq (k + 1)|C'|$. In particular, $\emptyset$ is a heavy subset of $C$. Let $C'$ be a maximal heavy subset of $C$. Let $A' = N_H(C')$ and $H' = H - (A' \cup C')$.

Then each vertex $v \in A \setminus A'$ has degree $k$ in $H'$. If there is a colour $c$ for which $d_{H'}(c) \geq k + 1$, then let

$$C'' = C' \cup \{c\}.$$  

Then $|N_H(C'')| = |N_H(C')| + d_{H'}(c) \geq (k + 1)|C''|$. So $C''$ is heavy, contrary to our assumption that $C'$ is a maximal heavy subset of $C$.

So each vertex $c \in C \setminus C'$ has degree at most $k$ in $H'$. For every $A'' \subseteq A \setminus A'$, we have $|A''| \leq k|N_H(A'')|$, by counting the number of edges between $A''$ and $C \setminus C'$.

By Hall’s Theorem, there is a matching $M$ in $H'$ that covers all the vertices of $A \setminus A'$. Let $\phi$ be the $L$-colouring of $A \setminus A'$ defined as $\phi(v) = c$ if $vc \in M$. So all vertices of $A \setminus A'$ are coloured by distinct colours from $C \setminus C'$. Extend $\phi$ to an $L$-colouring of $H$ as follows:
• Since $k(k + 1) > |A| \geq |N_H(C')| \geq |C'| (k + 1)$, we know that $|C'| \leq k - 1$. For each vertex $v \in B$, we have $L(v) \setminus C' \neq \emptyset$. Let $\phi(v)$ be a colour in $L(v) \setminus C'$.  
• For each vertex $v \in A'$, as $A' = N_H(C')$, $L(v) \cap C' \neq \emptyset$. Let $\phi(v)$ be a colour in $L(v) \cap C'$.

Then $\phi$ an $S$-$L$-colouring of $G$, as each connected monochromatic subgraph of $G$ contains at most one vertex of $A$, and hence is a star. This completes the proof of Lemma 2.1.

It is known that if $n \geq m^m$, then $ch_{D_d}(K_m, n) = ch(K_m, n) = m + 1$.

The following lemma shows that for a constant $d$, if $n$ is sufficiently large, then $ch_{D_d}(K_m, n) = m + 1$.

**Lemma 2.2** Assume $d$ is a non-negative integer. If $n \geq (dm + 1)m^m$, then $ch_{D_d}(K_m, n) = m + 1$.

**Proof** For an integer $n \geq (dm + 1)m^m$, let $G$ be a complete bipartite graph with partite sets $A, B$, where $|A| = m$ and $|B| = n$. As $G$ is $m$-degenerate, we have $ch_{D_d}(G) \leq ch(G) \leq m + 1$. Now we show that $ch_{D_d}(G) > m$.

Let $L$ be the $m$-assignment that assigns to vertices $v \in A$ pairwise disjoint $m$-sets $\{L(v) : v \in A\}$. Let $\Phi$ be the set of all $L$-colourings $\phi$ of $A$. Thus $|\Phi| = m^m$. For each $\phi \in \Phi$, assign a $(dm + 1)$-subset $B_\phi$ of $B$ so that for distinct $\phi, \phi' \in \Phi$, $B_\phi \cap B_{\phi'} = \emptyset$. Since $|B| \geq (dm + 1)m^m$, such an assignment exists. Extend $L$ to an $m$-assignment of $G$ by letting $L(v) = \phi(A)$ for $v \in B_\phi$. Assign arbitrary $m$ colours to $v$ if $v \in B$ is not contained in a subset $B_\phi$.

Now we show that $G$ is not $D_d$-$L$-colourable. Assume to the contrary that $\phi$ is a $D_d$-$L$-colouring of $G$. Let $\phi|_A$ be the restriction of $\phi$ to $A$. For $v \in B_{\phi|_A}$, $\phi(v) \in L(v) = \phi(A)$. As $|\phi(A)| = m$ and $|B_{\phi|_A}| = (dm + 1)$, there exists a colour $c \in \phi(A)$ such that $|\phi^{-1}(c) \cap B_{\phi|_A}| \geq d + 1$. Assume $u \in A$ and $c = \phi(u)$. Then $u$ has at least $d + 1$ neighbours that are coloured the same colour as $u$ itself. So $\phi$ is not a $D_d$-$L$-colouring of $G$.

This completes the proof of Lemma 2.2.

As a direct consequence of Lemmas 2.1 and 2.2, we obtain the following theorem.

**Theorem 2.3** For integers $k, d$ with $k \geq 2$, there exists a graph $G$ with $ch_S(G) \leq k$ and $ch_{D_d}(G) = k(k + 1)$. In particular, for every positive integer $p$, there exists a graph $G$ such that $ch(G) \geq p \cdot ch_S(G) \geq p \cdot ch_F(G) \geq p \cdot ch_{M_2}(G)$.

**Proof** Given integers $k$ and $d$, let $m$ and $n$ be integers such that $m = k(k + 1) - 1$ and $n \geq (dm + 1)m^m$. By Lemmas 2.1 and 2.2, we have $ch_S(K_m, n) \leq k$ and $ch_{D_d}(K_{m, n}) = m + 1 = k(k + 1)$.
In particular, when \( d = 0 \), we obtain \( ch(K_{m,n}) = ch_{D_0}(K_{m,n}) = k(k + 1) \geq (k + 1)ch(S(K_{m,n})) \). So, for every positive integer \( p \), there exists a graph \( G \) with \( ch(G) \geq p \cdot ch_S(G) \). The other inequalities hold since \( S \subseteq F \subseteq M_2 \).

This theorem refutes Conjecture 1.1 and gives a negative answer to Question 1.2. We remark that Conjecture 1.1, posed at the end of [2], is not the conjecture referred to in the title of that paper. The main conjecture studied in [2] is the following conjecture, which was posed in [3]:

**Conjecture 2.4** If \( |V(G)| \leq 3\chi_F(G) \), then \( ch_F(G) = \chi_F(G) \).

The above conjecture remains open.

It is known [1] that \( ch(G) \) is bounded from above by a function of \( ch_G(G) \), provided that graphs in \( \mathcal{G} \) have bounded maximum average degree. Or equivalently, graphs in \( \mathcal{G} \) have bounded choice number. In particular, \( ch(G) \leq f(ch_F(G)) \) for some function \( f \). The function \( f \) found in [1] is exponential. Theorem 2.3 shows that \( f \) cannot be a linear function. It would be interesting to determine if there is a polynomial function \( f \) such that \( ch(G) \leq f(ch_F(G)) \).

**Question 2.5** Are there constant integers \( a, b \) such that for every graph \( G \),

\[
ch(G) \leq a(ch_F(G))^b? 
\]

If so, then what is the minimum such integer \( b \)?

It would also be interesting to determine if the bound given in Theorem 2.3 is tight. In other words, is it true that \( ch(G) \leq ch_G(G) (ch_F(G) + 1) \) for all graphs \( G \)?

As observed in the introduction, for two graph classes \( \mathcal{G} \) and \( \mathcal{G}' \),

\[
\chi_{\mathcal{G}}(G) \leq \left( \max_{H \in \mathcal{G}'} \chi_{\mathcal{G}'}(H) \right) \chi_{\mathcal{G}'}(G). 
\]

We are interested in the question whether the same inequality holds for the corresponding choice number. If \( \mathcal{G}' \subseteq \mathcal{G} \), then trivially, the inequality \( ch_{\mathcal{G}'}(G) \leq \left( \max_{H \in \mathcal{G}'} ch_{\mathcal{G}'}(H) \right) ch_{\mathcal{G}'}(G) = ch_{\mathcal{G}'}(G) \) holds. We do not know of any non-trivial case where the inequality \( ch_{\mathcal{G}}(G) \leq \left( \max_{H \in \mathcal{G}'} ch_{\mathcal{G}'}(H) \right) ch_{\mathcal{G}'}(G) \) holds. As remarked in [1], the following question may have a positive answer.

**Question 2.6** [1] Is it true that for every graph \( G \) and every positive integer \( k \),

\[
ch(G) \leq kch_{G_k}(G)? 
\]

The above question is very interesting and challenging even when \( k = 2 \). More generally, the following question seems to be natural and intriguing:

**Question 2.7** Is it true that for every graph \( G \) and positive integers \( k, k' \),

\[
ch_{G_k}(G) \leq \left[ \frac{k}{k'} \right] ch_{G_k}(G)? 
\]
The relation between $\text{ch}_L(G)$ and $\text{ch}_S(G)$ is also worth investigating. By Theorem 2.3, there are graphs $G$ for which

$$\text{ch}_L(G) \geq \text{ch}_{D_2}(G) \geq \text{ch}_S(G)(\text{ch}_S(G) + 1).$$

It follows from (1) that

$$\chi_S(G) \leq 2\chi_L(G).$$

The following questions remain open.

**Question 2.8** Is it true that for every graph $G$,

$$\text{ch}_S(G) \leq 2\text{ch}_L(G)?$$

**Question 2.9** Is it true that for every graph $G$,

$$\text{ch}_L(G) \leq \text{ch}_S(G)(\text{ch}_S(G) + 1)?$$

Or is there an integer $a$ such that for every graph $G$,

$$\text{ch}_L(G) \leq (\text{ch}_S(G))^a?$$

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