ON A TOPOLOGICAL SIMPLE WARNE EXTENSION OF A SEMIGROUP

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Abstract. In the paper we introduce topological $\mathbb{Z}$-Bruck-Reilly and topological $\mathbb{Z}$-Bruck extensions of (semi)topological monoids which are generalizations of topological Bruck-Reilly and topological Bruck extensions of (semi)topological monoids and study their topologizations. The sufficient conditions under which the topological $\mathbb{Z}$-Bruck-Reilly ($\mathbb{Z}$-Bruck) extension admits only the direct sum topology and conditions under which the direct sum topology can be coarsened are given. Also, topological characterizations of some classes of $I$-bisimple (semi)topological semigroups are given.

1. Introduction and preliminaries

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [12, 13, 18, 38]. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $\text{cl}_Y(A)$ we shall denote the topological closure of $A$ in $Y$. Later by $\mathbb{N}$ we denote the set of positive integers. Also for a map $\theta : X \to Y$ and positive integer $n$ we denote by $\theta^{-1}(A)$ and $\theta^n(B)$ the full preimage of the set $A \subseteq Y$ and the $n$-power image of the set $B \subseteq X$, respectively, i.e., $\theta^{-1}(A) = \{x \in X : \theta(x) \in A\}$ and $\theta^n(B) = \{(\theta \circ \ldots \circ \theta)(x) : x \in B\}$.

A semigroup $S$ is regular if $x \in xSx$ for every $x \in S$. A semigroup $S$ is called inverse if for any element $x \in S$ there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function $\text{inv} : S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called an inversion. An inverse semigroup $S$ is said to be Clifford if $x \cdot x^{-1} = x^{-1} \cdot x$ for all $x \in S$.

If $S$ is a semigroup, then we shall denote the subset of idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). If the band $E(S)$ is a non-empty subset of $S$, then the semigroup operation on $S$ determines
the following partial order \( \leq \) on \( E(S) \): \( e \leq f \) if and only if \( ef = fe = e \). This order is called the natural partial order on \( E(S) \). A semilattice is a commutative semigroup of idempotents. A semilattice \( E \) is called linearly ordered or a chain if its natural order is a linear order. If \( E \) is a semilattice and \( e \in E \) then we denote \( \downarrow e = \{ f \in E \mid f \leq e \} \) and \( \uparrow e = \{ f \in E \mid e \leq f \} \).

If \( S \) is a semigroup, then by \( R, L, J \) and \( D \) and \( H \) we shall denote the Green relations on \( S \) (see \cite[Section 2.1]{13}). A semigroup \( S \) is called simple if \( S \) does not contain any proper two-sided ideals and bisimple if \( S \) has only one \( D \)-class.

A semitopological (resp., topological) semigroup is a Hausdorff topological space together with a separately (resp., jointly) continuous semigroup operation \cite{12,38}. An inverse topological semigroup with continuous inversion is called a topological inverse semigroup. A topology \( \tau \) on a (inverse) semigroup \( S \) which turns \( S \) into a topological (inverse) semigroup is called a semigroup (inverse) topology on \( S \). A semitopological group is a Hausdorff topological space together with a separately continuous group operation \cite{38} and a topological group is a Hausdorff topological space together with a jointly continuous group operation and inversion \cite{12}.

The bicyclic semigroup \( C(p,q) \) is the semigroup with the identity 1 generated by elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism \( h \) of the bicyclic semigroup is either an isomorphism or the image of \( C(p,q) \) under \( h \) is a cyclic group (see \cite[Corollary 1.32]{13}). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen’s result \cite{6} states that a \((\text{0–})\text{simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology and a topological semigroup} \( S \) can contain the bicyclic semigroup \( C(p,q) \) as a dense subsemigroup only as an open subset \cite{12}. Also Bertman and West in \cite{10} proved that the bicyclic semigroup as a Hausdorff semitopological semigroup admits only the discrete topology. The problem of an embedding of the bicycle semigroup into compact-like topological semigroups was solved in the papers \cite{17,18,19,26,27} and the closure of the bicyclic semigroup in topological semigroups studied in \cite{16}.

The properties of the bicyclic semigroup were extended to the following two directions: bicyclic-like semigroups which are bisimple and bicyclic-like extensions of semigroups. In the first case such are inverse bisimple semigroups with well-ordered subset of idempotents: \( \omega^n \)-bisimple semigroups \cite{28}, \( \omega^\alpha \)-bisimple semigroups \cite{29} and an \( \alpha \)-bicyclic semigroup, and bisimple inverse semigroups with linearly ordered subsets of idempotents which are isomorphic to either \([0, \infty)\) or \((-\infty, \infty)\) as subsets of the real line: \( B^1_{[0,\infty)} \),
Ahre [1, 2, 3, 4, 5] and Korkmaz [33, 34] studied Hausdorff semigroup topologizations of the semigroups $B^1_{[0,\infty)}$, $B^2_{[0,\infty)}$, $B^1_{(-\infty,\infty)}$, and $B^2_{(-\infty,\infty)}$ and their closures in topological semigroups. Annie Selden [42] and Hogan [30] proved that the only locally compact Hausdorff topology making an $\alpha$-bicyclic semigroup into a topological semigroup is the discrete topology. In [31] Hogan studied Hausdorff inverse semigroup topologies on an $\alpha$-bicyclic semigroup and there he constructed non-discrete Hausdorff inverse semigroup topology on an $\alpha$-bicyclic semigroup.

Let $Z$ be the additive group of integers. On the Cartesian product $C_Z = Z \times Z$ we define the semigroup operation as follows:

$$(a, b) \cdot (c, d) = \begin{cases} (a - b + c, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, d - c + b), & \text{if } b > c, \end{cases}$$

for $a, b, c, d \in Z$. The set $C_Z$ with such defined operation is called the extended bicyclic semigroup [44]. It is obvious that the extended bicyclic semigroup is an extension of the bicyclic semigroup. The extended bicyclic semigroup admits only the discrete topology as a semitopological semigroup [19]. Also, the problem of a closure of $C_Z$ in topological semigroup was studied in [19].

The conception of Bruck-Reilly extensions started from the Bruck paper [11] where he proposed the construction of an embedding of semigroups into simple monoids. Reilly in [37] generalized the Bruck construction up to so called in our time Bruck-Reilly construction and using it described the structure of $\omega$-bisimple semigroups. Annie Selden in [39, 40, 41] described the structure of locally compact topological inverse $\omega$-bisimple semigroups and their closure in topological semigroups.

The disquisition of topological Bruck-Reilly extensions of topological and semitopological semigroups was started in the papers [22, 24] and continued in [35, 25]. Using the ideas of the paper [22] Gutik in [23] proposed the construction of embedding of an arbitrary topological (inverse) semigroup into a simple path-connected topological (inverse) monoid.

Let $G$ be a linearly ordered group and $S$ be any semigroup. Let $\alpha : G^+ \to \text{End}(S^1)$ be a homomorphism from the positive cone $G^+$ into the semigroup of all endomorphisms of $S^1$. By $B(S, G, \alpha)$ we denote the set $G \times S^1 \times G$ with the following binary operation

$$(g_1, s_1, h_1) \cdot (g_2, s_2, h_2) = (g_1(h_1 \land g_2)^{-1}g_2, \alpha(e \lor h_1^{-1}g_2)[s_1] \cdot \alpha(e \lor g_2^{-1}h_1)(s_2), h_2(h_1 \land g_2)^{-1}h_1).$$

A binary operation so defined on the set $B(S, G^+, \alpha) = G^+ \times S^1 \times G^+$ with the semigroup operation induced from $B(S, G, \alpha)$ is a subsemigroup of $B(S, G, \alpha)$ [20].

Now we let $G = Z$ be the additive group of integers with usual order $\leq$ and $S$ be any semigroup. Let $\alpha : Z^+ \to \text{End}(S^1)$ be a homomorphisms from
the positive cone \( \mathbb{Z}^+ \) into the semigroup of all endomorphisms of \( S^1 \). Then formula (2) determines the following semigroup operation on \( \mathcal{B}(S, \mathbb{Z}, \alpha) \):

\[
(i, s, j) \cdot (m, t, n) = (i+m-\min\{j, m\}, \alpha[m-\min\{j, m\}](s) \cdot \alpha[j-\min\{j, m\}](t), j+n-\min\{j, m\}),
\]

where \( s, t \in S^1 \) and \( i, j, m, n \in \mathbb{Z} \).

Let \( \theta : S^1 \to H(1_S) \) be a homomorphism from the monoid \( S^1 \) into the group of units \( H(1_S) \) of \( S^1 \). Then we put \( \alpha[n](s) = \theta^n(s) \), for a positive integer \( n \) and \( \theta^0: S^1 \to S^1 \) be an identity map of \( S^1 \). Later the semigroup \( \mathcal{B}(S, \mathbb{Z}, \alpha) \) with such a defined homomorphism \( \alpha \) we shall denote by \( \mathcal{B}(S, \mathbb{Z}, \theta) \), and in the case when the homomorphism \( \theta: S^1 \to H(1_S) \) is defined by the formula

\[
\theta^n(s) = \begin{cases} 1_S, & \text{if } n > 0; \\ s, & \text{if } n = 0, \end{cases}
\]

we shall denote it by \( \mathcal{B}(S, \mathbb{Z}) \). We observe that the semigroup operation on \( \mathcal{B}(S, \mathbb{Z}, \theta) \) is defined by the formula

\[
(i, s, j) \cdot (m, t, n) = \begin{cases} (i-j+m, \theta^{m-j}(s) \cdot t, n), & \text{if } j < m; \\ (i, s \cdot t, n), & \text{if } j = m; \\ (i, s \cdot \theta^{-m}(t), n-m+j), & \text{if } j > m, \end{cases}
\]

for \( i, j, m, n \in \mathbb{Z} \) and \( s, t \in S^1 \). Later we shall call the semigroup \( \mathcal{B}(S, \mathbb{Z}, \theta) \) the \( \mathbb{Z} \)-Bruck-Reilly extension of the semigroup \( S \) and \( \mathcal{B}(S, \mathbb{Z}) \) the \( \mathbb{Z} \)-Bruck extension of the semigroup \( S \), respectively. Also we observe that if \( S \) is a trivial semigroup then the semigroups \( \mathcal{B}(S, \mathbb{Z}, \theta) \) and \( \mathcal{B}(S, \mathbb{Z}) \) are isomorphic to the extended bicyclic semigroup (see [43]).

**Proposition 1.1.** Let \( S^1 \) be a monoid and \( \theta: S^1 \to H(1_S) \) be a homomorphism from \( S^1 \) into the group of units \( H(1_S) \) of \( S^1 \). The following statements hold:

\( (i) \) \( \mathcal{B}(S, \mathbb{Z}, \theta) \) and \( \mathcal{B}(S, \mathbb{Z}) \) are simple semigroups;

\( (ii) \) \( \mathcal{B}(S, \mathbb{Z}, \theta) \) (resp., \( \mathcal{B}(S, \mathbb{Z}) \)) is an inverse semigroup if and only if \( S^1 \) is an inverse semigroup;

\( (iii) \) \( \mathcal{B}(S, \mathbb{Z}, \theta) \) (resp., \( \mathcal{B}(S, \mathbb{Z}) \)) is a regular semigroup if and only if \( S^1 \) is a regular semigroup.

The proofs of the statements of Proposition 1.1 are similar to corresponding theorems of Section 8.5 of [13] and Theorem 5.6.6 of [32].

Also, we remark that the semigroups \( \mathcal{B}(S, \mathbb{Z}, \theta) \) and \( \mathcal{B}(S, \mathbb{Z}) \) have similar descriptions of Green’s relations to Bruck-Reilly and Bruck extensions of \( S^1 \) (see Lemma 8.46 of [13] and Theorem 5.6.6(2) of [32]), and hence the semigroup \( \mathcal{B}(S, \mathbb{Z}, \theta) \) (resp., \( \mathcal{B}(S, \mathbb{Z}) \)) is bisimple if and only if \( S^1 \) is bisimple.

**Remark 1.2.** Formula (3) implies that if \( (i, s, j) \cdot (m, t, n) = (k, d, l) \) in the semigroup \( \mathcal{B}(S, \mathbb{Z}, \theta) \) then \( k-l = i-j + m-n \).
For every \( m, n \in \mathbb{Z} \) and \( A \subseteq S \) we define \( S_{m,n} = \{(m, s, n): s \in S\} \) and \( A_{m,n} = \{(m, s, n): s \in A\} \).

In this paper we introduce topological the \( \mathbb{Z} \)-Bruck-Reilly and topological \( \mathbb{Z} \)-Bruck extensions of (semi)topological monoids which are generalizations of topological Bruck-Reilly and topological Bruck extensions of (semi)topological monoids and study their topologizations. The sufficient conditions under which the topological \( \mathbb{Z} \)-Bruck-Reilly \((\mathbb{Z} \)-Bruck\) extension admits only the direct sum topology and conditions under which the direct sum topology can be coarsened are given. Also, topological characterizations of some classes of \( I \)-bisimple (semi)topological semigroups are given.

2. On topological \( \mathbb{Z} \)-Bruck-Reilly extensions

Let \( S \) be a monoid with a group of units \( H(1_S) \). Obviously if one of the following conditions holds:

1) \( H(1_S) \) is a trivial group;
2) \( S \) is congruence-free and \( S \) is not a group;
3) \( S \) has zero,

then every homomorphism \( \theta: S^1 \to H(1_S) \) is annihilating. Also, many topological properties of a (semi)topological semigroup \( S \) guarantee the triviality of \( \theta \). For example, such is the following: \( H(1_S) \) is a discrete subgroup of \( S \) and \( S \) has a minimal ideal \( K(S) \) which is a connected subgroup of \( S \).

On the other side there exist many conditions on a (semitopological, topological) semigroup \( S \) which ensure the existence of a non-annihilating (continuous) homomorphism \( \theta: S^1 \to H(1_S) \) from \( S \) into non-trivial group of units \( H(1_S) \). For example, such conditions are the following:

1) the (semitopological, topological) semigroup \( S \) has a minimal ideal \( K(S) \) which is a non-trivial group and there exists a non-annihilating (continuous) homomorphism \( h: K(S) \to H(1_S) \);
2) \( S \) is an inverse semigroup and there exists a non-annihilating homomorphism \( h: S/\sigma \to H(1_S) \), where \( \sigma \) is the least group congruence on \( S \) (see [36, Section III.5]).

Let \( (S, \tau) \) be a semitopological monoid and \( 1_S \) be a unit of \( S \). If \( S \) does not contain a unit then without loss of generality we can assume that \( S \) is a semigroup with an isolated adjoined unit. Also we shall assume that the homomorphism \( \theta: S^1 \to H(1_S) \) is continuous.

Let \( \mathcal{B} \) be a base of the topology \( \tau \) on \( S \). According to [22] the topology \( \tau_{\text{BR}} \) on \( \mathcal{B}(S, \mathbb{Z}, \theta) \) generated by the base

\[
\mathcal{B}_{\text{BR}} = \{(i, U, j): U \in \mathcal{B}, i, j \in \mathbb{Z}\}
\]

is called a direct sum topology on \( \mathcal{B}(S, \mathbb{Z}, \theta) \) and we shall denote it by \( \tau_{\text{ds}}^{\text{BR}} \). We observe that the topology \( \tau_{\text{ds}}^{\text{BR}} \) is the product topology on \( \mathcal{B}(S, \mathbb{Z}, \theta) = \mathbb{Z} \times S \times \mathbb{Z} \).
Proposition 2.1. Let \((S, \tau)\) be a semitopological (resp., topological, topological inverse) semigroup and \(\theta: S^1 \to H(1_S)\) be a continuous homomorphism from \(S\) into the group of units \(H(1_S)\) of \(S\). Then \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) is a semitopological (resp., topological, topological inverse) semigroup.

The proof of Proposition 2.1 is similar to Theorem 1 from [22].

Definition 2.2. Let \(\mathcal{S}\) be some class of semitopological semigroups and \((S, \tau) \in \mathcal{S}\). If \(\tau^*_\text{BR}\) is a topology on \(B(S, \mathbb{Z}, \theta)\) such that the homomorphism \(\theta: S^1 \to H(1_S)\) is a continuous map, \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR}) \in \mathcal{S}\) and \(\tau^*_\text{BR}|_{S_m, m} = \tau\) for some \(m \in \mathbb{Z}\), then the semigroup \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) is called a topological Z-Bruck-Reilly extension of the semitopological semigroup \((S, \tau)\) in the class \(\mathcal{S}\). In the case when \(\theta(s) = 1_S\) for all \(s \in S^1\), the semigroup \((B(S, \mathbb{Z}, \tau^*_\text{BR})\) is called a topological Z-Bruck extension of the semitopological semigroup \((S, \tau)\) in the class \(\mathcal{S}\).

Proposition 2.1 implies that for every semitopological (resp., topological, topological inverse) semigroup \((S, \tau)\) there exists a topological Z-Bruck-Reilly extension \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) of the semitopological (resp., topological, topological inverse) semigroup \((S, \tau)\) in the class of semitopological (resp., topological, topological inverse) semigroups. It is natural to ask: when is \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) unique for the semigroup \((S, \tau)\)?

Proposition 2.3. Let \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) be a semitopological semigroup. Then the following conditions hold:

(i) for every \(i, j, k, l \in \mathbb{Z}\) the topological subspaces \(S_{i,j}\) and \(S_{k,l}\) are homeomorphic and moreover \(S_{i,i}\) and \(S_{k,k}\) are topologically isomorphic subsemigroups in \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\);

(ii) for every \((i, s, j) \in (B(S, \mathbb{Z}, \theta))\) there exists an open neighbourhood \(U_{(i, s, j)}\) of the point \((i, s, j)\) in \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\) such that

\[
U_{(i, s, j)} \subseteq \bigcup \{S_{i-k,j-k}: k = 0, 1, 2, 3, \ldots \}.
\]

Proof. (i) For every \(i, j, k, l \in \mathbb{Z}\) the map \(\phi_{k,l}^{i,j}: B(S, \mathbb{Z}, \theta) \to B(S, \mathbb{Z}, \theta)\) defined by the formula \(\phi_{k,l}^{i,j}(x) = (k, 1_S, i) \cdot x \cdot (j, 1_S, l)\) is continuous as a composition of left and right translations in the semitopological semigroup \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\). Since \(\phi_{k,l}^{i,j}(\phi_{k,l}^{i,j}(s)) = s\) and \(\phi_{k,l}^{i,j}(\phi_{k,l}^{i,j}(t)) = t\) for all \(s \in S_{i,j}\) and \(t \in S_{k,l}\) we conclude that the restriction \(\phi_{k,l}^{i,j}|_{S_{i,j}}\) is the inverse map of the restriction \(\phi_{k,l}^{i,j}|_{S_{k,l}}\). Then the continuity of the map \(\phi_{k,l}^{i,j}\) implies that the restriction \(\phi_{k,l}^{i,j}|_{S_{i,j}}\) is a homeomorphism which maps elements of the subspace \(S_{i,j}\) onto elements of the subspace \(S_{k,l}\) in \((B(S, \mathbb{Z}, \theta), \tau^*_\text{BR})\). Now the definition of the map \(\phi_{k,l}^{i,j}\) implies that the restriction \(\phi_{k,l}^{i,j}|_{S_{i,j}}: S_{i,j} \to S_{k,l}\) is a topological isomorphism of semitopological subsemigroups \(S_{i,j}\) and \(S_{k,l}\).
Since the left and right translations in a semitopological semigroup are continuous maps and left and right translations by an idempotent are retractions, Exercise 1.5.C from [13] implies that \((i+1, 1S, i+1)\) \(\mathcal{B}(S, Z, \theta)\) and \(\mathcal{B}(S, Z, \theta)(j+1, 1S, j+1)\) are closed subsets in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\). Hence there exists an open neighbourhood \(W_{i(s,j)}\) of the point \((i, s, j)\) in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) such that

\[
W_{i(s,j)} \subseteq \mathcal{B}(S, Z, \theta) \setminus ((i+1, 1S, i+1)\mathcal{B}(S, Z, \theta) \cup \mathcal{B}(S, Z, \theta)(j+1, 1S, j+1)) .
\]

Since the semigroup operation in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) is separately continuous we conclude that there exists an open neighbourhood \(U_{i(s,j)}\) of the point \((i, s, j)\) in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) such that

\[
U_{i(s,j)} \subseteq W_{i(s,j)} - (i, 1S, i) \cdot U_{i(s,j)} \subseteq W_{i(s,j)} \quad \text{and} \quad U_{i(s,j)} \cdot (j, 1S, j) \subseteq W_{i(s,j)} .
\]

Next we shall show that \(U_{i(s,j)} \subseteq \bigcup \{S_{i-k,j-k} : k = 0, 1, 2, 3, \ldots\}\). Suppose the contrary: there exists \((m, a, n) \in U_{i(s,j)}\) such that \((m, a, n) \notin \bigcup \{S_{i-k,j-k} : k = 0, 1, 2, 3, \ldots\}\). Then we have that \(m \leq i, n \leq j\) and \(m - n \neq i - j\). If \(m - n > i - j\) then we have that

\[
(m, a, n) \cdot (j, 1S, j) = (m - n + j, 0^{i-m}(a), j) \notin \mathcal{B}(S, Z, \theta) \setminus ((i+1, 1S, i+1)\mathcal{B}(S, Z, \theta), \theta)
\]

because \(m - n + j > i - j + j = i\), and hence \((m, a, n) \cdot (j, 1S, j) \notin W_{i(s,j)}\).

Similarly if \(m - n < i - j\) then we have that

\[
(i, 1S, i) \cdot (m, a, n) = (i, \theta^{i-m}(a), n-m+i) \notin \mathcal{B}(S, Z, \theta) \setminus \mathcal{B}(S, Z, \theta)(j+1, 1S, j+1),
\]

because \(n - m + i > j - i + i = j\) and hence \((i, 1S, i) \cdot (m, a, n) \notin W_{i(s,j)}\). This completes the proof of our statement. □

**Theorem 2.4.** Let \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) be a topological \(\mathbb{Z}\)-Bruck-Reill extension of the semitopological semigroup \((S, \tau)\). If \(S\) contains a left (right or two-sided) compact ideal, then \(\tau_{\text{BR}}\) is the direct sum topology on \(\mathcal{B}(S, Z, \theta)\).

**Proof.** We consider the case when the semitopological semigroup \(S\) has a left compact ideal. In other cases the proof is similar. Let \(L\) be a left compact ideal in \(S\). Then by Definition 2.2 there exists an integer \(n\) such that the subsemigroup \(S_{n,n}\) in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) is topologically isomorphic to the semitopological semigroup \((S, \tau)\). Hence Proposition 2.3 implies that \(L_{i,j}\) is a compact subset of \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) for all \(i, j \in \mathbb{Z}\).

We fix an arbitrary element \((i, s, j)\) of the semigroup \(\mathcal{B}(S, Z, \theta)\), \(i, j \in \mathbb{Z}\) and \(s \in S^1\). Now we fix an element \((i - 1, t, j - 1)\) from \(L_{i-1,j-1}\) and define a map \(h: \mathcal{B}(S, Z, \theta) \rightarrow \mathcal{B}(S, Z, \theta)\) by the formula \(h(x) = x \cdot (j - 1, t, j - 1)\). Then by Proposition 2.3(ii) there exists an open neighbourhood \(U_{i(s,j)}\) of the point \((i, s, j)\) in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) such that \(U_{i(s,j)} \subseteq \bigcup \{S_{i-k,j-k} : k = 0, 1, 2, 3, \ldots\}\). Since left translations in \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) are continuous we conclude that the full pre-image \(h^{-1}(L_{i-1,j-1})\) is a closed subset of the topological space \((\mathcal{B}(S, Z, \theta), \tau_{\text{BR}})\) and Remark 1.2 implies that \(h^{-1}(L_{i-1,j-1}) = \bigcup \{S_{i-k,j-k} : k = 1, 2, 3, \ldots\}\). This implies that an arbitrary
element \((i, s, j)\) of the semigroup \(\mathcal{B}(S, Z, \theta)\), where \(i, j \in Z\) and \(s \in S^1\), has an open neighbourhood \(U_{(i,s,j)}\) such that \(U_{(i,s,j)} \subseteq S_{i,j}\). \(\square\)

Theorem 2.4 implies the following corollary:

**Corollary 2.5** ([18]). Let \(\tau\) be a Hausdorff topology on the extended bicyclic semigroup \(C_Z\). If \((C_Z, \tau)\) is a semitopological semigroup then \((C_Z, \tau)\) is the discrete space.

**Theorem 2.6.** Let \((\mathcal{B}(S, Z, \theta), \tau_{BR})\) be a topological \(\mathbb{Z}\)-Bruck-Reilly extension of the topological inverse semigroup \((S, \tau)\) in the class of topological inverse semigroups. If the band \(E(S)\) contains a minimal idempotent, then \(\tau_{BR}\) is the direct sum topology on \(\mathcal{B}(S, Z, \theta)\).

**Proof.** Let \(e_0\) be a minimal element of the band \(E(S)\). Then \((i, e_0, i)\) is a minimal idempotent in the band of the subsemigroup \(S_{i,i}\) for every integer \(i\).

Since the semigroup operation in \((\mathcal{B}(S, Z, \theta), \tau_{BR})\) is continuous we conclude that for every idempotent \(i\) from the semigroup \(\mathcal{B}(S, Z, \theta)\) the set \(\uparrow i = \{e \in E(\mathcal{B}(S, Z, \theta)) : e \cdot i = i \cdot e = i\}\) is a closed subset in \(E(\mathcal{B}(S, Z, \theta))\) with the topology induced from \((\mathcal{B}(S, Z, \theta), \tau_{BR})\). We define the maps \(l : \mathcal{B}(S, Z, \theta) \to E(\mathcal{B}(S, Z, \theta))\) and \(r : \mathcal{B}(S, Z, \theta) \to E(\mathcal{B}(S, Z, \theta))\) by the formulae \(l(x) = x \cdot x^{-1}\) and \(r(x) = x^{-1} \cdot x\). We fix any element \((i, s, j) \in \mathcal{B}(S, Z, \theta)\). Since the semigroup operation and inversion are continuous in \((\mathcal{B}(S, Z, \theta), \tau_{BR})\) we conclude that the sets \(\uparrow^{-1}(\uparrow(i - 1, e_0, i - 1))\) and \(r^{-1}(\uparrow(j - 1, e_0, j - 1))\) are closed in \((\mathcal{B}(S, Z, \theta), \tau_{BR})\). Then by Proposition 2.3(ii) there exists an open neighbourhood \(U_{(i,s,j)}\) of the point \((i, s, j)\) in \((\mathcal{B}(S, Z, \theta), \tau_{BR})\) such that \(U_{(i,s,j)} \subseteq \bigcup \{ S_{i-k,j-k} : k = 0, 1, 2, 3, \ldots \}\). Now elementary calculations show that

\[ W_{(i,s,j)} = U_{(i,s,j)} \setminus \left( \Gamma^{-1}(\uparrow(i - 1, e_0, i - 1)) \cup r^{-1}(\uparrow(j - 1, e_0, j - 1)) \right) \subseteq S_{i,j}. \]

This completes the proof of our theorem. \(\square\)

The following examples show that the arguments stated in Theorems 2.4 and 2.6 are important.

**Example 2.7.** Let \(N_+ = \{0, 1, 2, 3, \ldots \}\) be the discrete topological space with the usual operation of addition of integers. We define a topology \(\tau_{BR}\) on \(\mathcal{B}(N_+, Z)\) as follows:

(i) for every point \(x \in N_+ \setminus \{0\}\) the base of the topology \(\tau_{BR}\) at \((i, x, j)\) coincides with the base of the direct sum topology \(\tau_{BR}^{ds}\) at \((i, x, j)\) for all \(i, j \in Z\);

(ii) for any \(i, j \in Z\) the family \(\mathcal{B}_{(i,0,j)} = \{U_{i,j}^k : k = 1, 2, 3, \ldots \}\), where

\[ U_{i,j}^k = \{(i, 0, j)\} \cup \{(i - 1, s, j - 1) : s = k, k + 1, k + 2, k + 3, \ldots \}, \]

is the base of the topology \(\tau_{BR}\) at the point \((i, 0, j)\).
Simple verifications show that $(\mathcal{B}(N_+, \mathbb{Z}), \tau_{BR})$ is a Hausdorff topological semigroup.

**Example 2.8.** Let $N_m = \{0, 1, 2, 3, \ldots\}$ be the discrete topological space with the semigroup operation $x \cdot y = \max\{x, y\}$. Now we identify the set $\mathcal{B}(N_m, \mathbb{Z})$ with $\mathcal{B}(N_+, \mathbb{Z})$. Let $\tau_{BR}$ be the topology on $\mathcal{B}(N_+, \mathbb{Z})$ defined as in Example 2.7. Then simple verifications show that $(\mathcal{B}(N_m, \mathbb{Z}), \tau_{BR})$ is a Hausdorff topological inverse semigroup.

**Definition 2.9.** We shall say that a semitopological semigroup $S$ has an open ideal property (or shortly, $S$ is an OIP-semigroup) if there exist a family $\mathcal{I} = \{I_\alpha\}_{\alpha \in A}$ of open ideals in $S$ such that for every $x \in S$ there exist an open ideal $I_\alpha \in \mathcal{I}$ and open neighbourhood $U(x)$ of the point $x$ in $S$ such that $U(x) \cap I_\alpha = \emptyset$.

We observe that Definition 2.9 implies that the family $\mathcal{I} = \{I_\alpha\}_{\alpha \in A}$ of open ideals in $S$ satisfies the finite intersection property and every semitopological OIP-semigroup does not contain a minimal ideal.

**Theorem 2.10.** Let $(S, \tau)$ be a Hausdorff semitopological OIP-semigroup and $\theta : S^1 \to H(1S)$ be a continuous homomorphism. Then there exists a topological $\mathbb{Z}$-Bruck-Reilly extension $(\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})$ of $(S, \tau)$ in the class of semitopological semigroups such that the topology $\tau_{BR}$ is strictly coarser than the direct sum topology $\tau_{ds}^{BR}$ on $\mathcal{B}(S, \mathbb{Z}, \theta)$.

**Proof.** Let $\mathcal{I} = \{I_\alpha\}_{\alpha \in A}$ be a family of open ideals in $(S, \tau)$ such that for every $x \in S$ there exists $I_\alpha \in \mathcal{I}$ and open neighbourhood $U(x)$ of the point $x$ in $(S, \tau)$ such that $U(x) \cap I_\alpha = \emptyset$.

We shall define a base of the topology $\tau_{BR}$ on $\mathcal{B}(S, \mathbb{Z}, \theta)$ in the following way:

1. for every $s \in S \setminus H(1S)$ and $i, j \in \mathbb{Z}$ the base of the topology $\tau_{BR}$ at the point $(i, s, j)$ coincides with a base of the direct sum topology $\tau_{ds}^{BR}$ at $(i, s, j)$; and
2. the family

$$\mathcal{B}(i, a, j) = \{(U_a)_{i,j} = (U_a)_{i,j} \cup (\theta^{-1}(U_a) \cap I_{\alpha})_{i-1,j-1} : U_a \in \mathcal{B}_a, I_{\alpha} \in \mathcal{I}\},$$

where $\mathcal{B}_a$ is a base of the topology $\tau$ at the point $a$ in $S$, is a base of the topology $\tau_{BR}$ at the point $(i, a, j)$, for every $a \in H(1S)$ and all $i, j \in \mathbb{Z}$.

Since $(S, \tau)$ is a Hausdorff semitopological OIP-semigroup we conclude that $\tau_{BR}$ is a Hausdorff topology on $\mathcal{B}(S, \mathbb{Z}, \theta)$ and moreover $\tau_{BR}$ is a proper subfamily of $\tau_{ds}^{BR}$. Hence $\tau_{BR}$ is a coarser topology on $\mathcal{B}(S, \mathbb{Z}, \theta)$ than $\tau_{ds}^{BR}$.

Now we shall show that the semigroup operation on $(\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})$ is separately continuous. Since by Proposition 2.1 the semigroup operation on $(\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{ds}^{BR})$ is separately continuous we conclude that the definition
of the topology $\tau_{\mathcal{BR}}$ on $\mathcal{BR}(S, \mathbb{Z}, \theta)$ implies that it is sufficient to show that the semigroup operation in $(\mathcal{BR}(S, \mathbb{Z}, \theta), \tau_{\mathcal{BR}})$ is separately continuous in the following three cases:

1) $(i, h, j) \cdot (m, g, n)$; 2) $(i, h, j) \cdot (m, s, n)$; and 3) $(m, s, n) \cdot (i, h, j)$,

where $s \in S \setminus H(1_S)$, $g, h \in H(1_S)$ and $i, j, m, n \in \mathbb{Z}$.

Consider case 1). Then we have that

$$(i, h, j) \cdot (m, g, n) = \begin{cases} (i - j + m, \theta^{m-j}(h) \cdot g, n), & \text{if } j < m; \\ (i, h \cdot g, n), & \text{if } j = m; \\ (i, h, \theta^{j-m}(g) \cdot n - m + j), & \text{if } j > m. \end{cases}$$

Suppose that $j < m$. Then the separate continuity of the semigroup operation in $(S, \tau)$ and the continuity of the homomorphism $\theta: S \to H(1_S)$ imply that for every open neighbourhood $U_{g_{n-1}(h) \cdot g}$ of the point $\theta^{m-j}(h) \cdot g$ in $(S, \tau)$ there exist open neighbourhoods $V_h$ and $W_g$ of the points $h$ and $g$ in $(S, \tau)$, respectively, such that

$$\theta^{m-j}(h) \cdot W_g \subseteq U_{g_{n-1}(h) \cdot g} \quad \text{and} \quad \theta^{m-j}(V_h) \cdot g \subseteq U_{g_{n-1}(h) \cdot g}.$$ 

Hence for every $I_{\alpha} \in \mathcal{I}$ we have that

$$(i, h, j) \cdot (W_g)_{m,n} \subseteq (i, h, j) \cdot \left( (W_g)_{m,n} \cup (\theta^{-1}(W_g) \cap I_{\alpha})_{m-1,n-1} \right) \subseteq \left( (W_g)_{m,n} \cup (i, h, j) \cdot (\theta^{-1}(W_g) \cap I_{\alpha})_{m-1,n-1} \right) \subseteq \left\{ \begin{array}{ll} (\theta^{m-j}(h) \cdot W_g)_{i-j+m,n} \cup (\theta^{m-1-j}(h) \cdot (\theta^{-1}(W_g) \cap I_{\alpha})_{i-j+m-1,n-1}), & \text{if } j < m-1; \\ (\theta(h) \cdot W_g)_{i-j+m,n} \cup (h \cdot (\theta^{-1}(W_g) \cap I_{\alpha})_{i,n-1}), & \text{if } j = m-1 \end{array} \right.$$ 

because $\theta \left( (\theta^{m-1-j}(h) \cdot (\theta^{-1}(W_g) \cap I_{\alpha})_{i,j}) \right) \subseteq \theta^{m-j}(h) \cdot W_g \subseteq U_{g_{n-1}(h) \cdot g}$, and

$$(V_h)_{i,j} \cdot (m, g, n) \subseteq \left( (V_h)_{i,j} \cup (\theta^{-1}(V_h) \cap I_{\alpha})_{i-1,j-1} \right) \cdot (m, g, n) \subseteq \left( (V_h)_{i,j} \cup (\theta^{-1}(V_h) \cap I_{\alpha})_{i-1,j-1} \cdot (m, g, n) \right) \subseteq \left( \theta^{m-j}(V_h) \cdot g \right)_{i-j+m,n} \cup (\theta^{m-1-j}(V_h) \cap I_{\alpha})_{i,j-m,n} \subseteq \left( \theta^{m-j}(V_h) \cdot g \right)_{i-j+m,n} \cup (\theta^{m-j}(V_h) \cdot g)_{i-j+m,n} \subseteq \left( \theta^{m-j}(V_h) \cdot g \right)_{i-j+m,n} \subseteq \left( U_{g_{n-1}(h) \cdot g} \right)_{i-j+m,n} \subseteq \left( U_{g_{n-1}(h) \cdot g}^{\alpha} \right)_{i-j+m,n}.$$ 

Suppose that $j = m$. Then the separate continuity of the semigroup operation in $(S, \tau)$ implies that for every open neighbourhood $U_{h \cdot g}$ of the point $h \cdot g$ in $(S, \tau)$ there exist open neighbourhoods $V_h$ and $W_g$ of the points $h$ and $g$ in $(S, \tau)$, respectively, such that

$$V_h \cdot g \subseteq U_{h \cdot g} \quad \text{and} \quad h \cdot W_g \subseteq U_{h \cdot g}.$$
Then for every \( I_\alpha \in \mathcal{I} \) we have that
\[
(V_h)_{i,j}^\alpha (m, g, n) \subseteq \left( (V_h)_{i,j} \cdot (m, g, n) \right) \cup \left( (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (m, g, n) \right) \subseteq
\]
\[
(V_h \cdot g)_{i,n} \cup \left( \theta \left( \theta^{-1}(V_h) \cap I_\alpha \right) \cdot g \right)_{i,n} \subseteq (V_h \cdot g)_{i,n} \cup (V_h \cdot g)_{i,n} =
\]
\[
= (V_h \cdot g)_{i,n} \subseteq (U_{h \cdot g})_{i,n}^\alpha,
\]
and
\[
(i, h, j) \cdot (W_g)_{m,n} \subseteq \left( (i, h, j) \cdot (W_g)_{m,n} \right) \cup \left( (i, h, j) \cdot (\theta^{-1}(W_g) \cap I_\alpha)_{m-1,n-1} \right) \subseteq
\]
\[
\subseteq \left( h \cdot \theta^j \cdot (W_g) \right)_{i,n-m+j} \cup \left( h \cdot \theta^{j-1} \left( \theta^{-1}(W_g) \cap I_\alpha \right) \right)_{i,n-m+j} \subseteq
\]
\[
= \left( h \cdot \theta^j \cdot (W_g) \right)_{i,n-m+j} \subseteq (U_{h \cdot \theta^j \cdot (W_g)})_{i,n-m+j}^\alpha,
\]
Suppose that \( j > m \). Then the separate continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \( \theta : S \to H(1_S) \) imply that for every open neighbourhood \( U_{h \cdot \theta^j \cdot m} \cdot (g) \) of the point \( h \cdot \theta^j \cdot m \cdot (g) \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_g \) of the points \( h \) and \( g \) in \((S, \tau)\), respectively, such that
\[
h \cdot \theta^j \cdot m \cdot (W_g) \subseteq U_{h \cdot \theta^j \cdot m} \cdot (g) \quad \text{and} \quad V_h \cdot \theta^j \cdot m \cdot (g) \subseteq U_{h \cdot \theta^j \cdot m} \cdot (g).
\]
Hence for every \( I_\alpha \in \mathcal{I} \) we have that
\[
(i, h, j) \cdot (W_g)_{m,n} \subseteq \left( (i, h, j) \cdot (W_g)_{m,n} \right) \cup \left( (i, h, j) \cdot (\theta^{-1}(W_g) \cap I_\alpha)_{m-1,n-1} \right) \subseteq
\]
\[
\subseteq \left( h \cdot \theta^j \cdot (W_g) \right)_{i,n-m+j} \cup \left( h \cdot \theta^{j-1} \left( \theta^{-1}(W_g) \cap I_\alpha \right) \right)_{i,n-m+j} \subseteq
\]
\[
= \left( h \cdot \theta^j \cdot (W_g) \right)_{i,n-m+j} \subseteq (U_{h \cdot \theta^j \cdot m} \cdot (g))_{i,n-m+j}^\alpha,
\]
and
\[
(V_h)_{i,j}^\alpha (m, g, n) \subseteq \left( (V_h)_{i,j} \cdot (m, g, n) \right) \cup \left( (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (m, g, n) \right) \subseteq
\]
\[
\left\{ \begin{array}{ll}
(V_h \cdot \theta^j \cdot m \cdot (g))_{i,n-m+j} \cup \left( (\theta^{-1}(V_h) \cap I_\alpha) \cdot g \right)_{i-1,n-1}, & \text{if } j-1=m; \\
(V_h \cdot \theta^j \cdot m \cdot (g))_{i,n-m+j} \cup \left( (\theta^{-1}(V_h) \cap I_\alpha) \cdot \theta^{j-1} \cdot m \cdot (g) \right)_{i-1,n-m+j}, & \text{if } j-1=m,
\end{array} \right.
\]
\[
\subseteq \left( U_{h \cdot \theta^j \cdot m} \cdot (g) \right)_{i,n-m+j}^\alpha,
\]
because \( \theta \left( \theta^{-1}(V_h) \cap I_\alpha \right) \cdot \theta^{j-1} \cdot m \cdot (g) = V_h \cdot \theta^j \cdot m \cdot (g) \subseteq U_{h \cdot \theta^j \cdot m} \cdot (g) \).

We observe that if \( g \cdot s \in H(1_S) \) and \( s \in S \setminus H(1_S) \) then \( g \cdot s \cdot s \in S \setminus H(1_S) \). Otherwise, if \( g \cdot s \in H(1_S) \) then we have that \( g^{-1} \cdot g \cdot s = 1_S \cdot s = s \in H(1_S) \), which contradicts that every translation on an element of the group of units of \( S \) is a bijective map (see [12] Vol. 1, p. 18).

Consider case 2). Then we have that
\[
(i, h, j) \cdot (m, s, n) = \left\{ \begin{array}{ll}
(i - j + m, \theta^m \cdot (h) \cdot s, n), & \text{if } j < m; \\
(i, h \cdot s, n), & \text{if } j = m; \\
(i, h \cdot \theta^m \cdot (s), n - m + j), & \text{if } j > m.
\end{array} \right.
\]
Suppose that \( j < m \). Then the separate continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \( \theta : S \to H(1_S) \) imply that for every open neighbourhood \( U_{\theta^{m-j}(h) \cdot s} \) of the point \( \theta^{m-j}(h) \cdot s \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( g \) in \((S, \tau)\), respectively, such that

\[
\theta^{m-j}(h) \cdot W_s \subseteq U_{\theta^{m-j}(h) \cdot s} \quad \text{and} \quad \theta^{m-j}(V_h) \cdot s \subseteq U_{\theta^{m-j}(h) \cdot s}.
\]

Hence for every \( I_\alpha \in I \) we have that

\[
(i, h, j) \cdot (W_s)_{m,n} \subseteq (\theta^{m-j}(h) \cdot W_s)_{i-j+m,n} \subseteq (U_{\theta^{m-j}(h) \cdot s})_{i-j+m,n}
\]

and

\[
(V_h)_{i,j} \cdot (m, s, n) \subseteq ((V_h)_{i,j} \cdot (m, s, n)) \cup ((\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (m, s, n)) \subseteq
\]

\[
\subseteq (\theta^{m-j}(V_h) \cdot s)_{i-j+m,n} \cup (\theta^{m-j-1}(\theta^{-1}(V_h) \cap I_\alpha) \cdot s)_{i-j+m,n} \subseteq
\]

\[
\subseteq (\theta^{m-j}(V_h) \cdot s)_{i-j+m,n} \cup (\theta^{m-j}(V_h) \cdot s)_{i-j+m,n} \subseteq
\]

\[
(V_h)_{i,j} \cdot s \subseteq (U_{\theta^{m-j}(h) \cdot s})_{i-j+m,n}.
\]

Suppose that \( j = m \). Then the separate continuity of the semigroup operation in \((S, \tau)\) implies that for every open neighbourhood \( U_{h \cdot s} \) of the point \( h \cdot s \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( s \) in \((S, \tau)\), respectively, such that

\[
V_h \cdot s \subseteq U_{h \cdot s} \quad \text{and} \quad h \cdot W_s \subseteq U_{h \cdot s}.
\]

Then for every \( I_\alpha \in I \) we have that

\[
(i, h, j) \cdot (W_s)_{m,n} \subseteq (h \cdot W_s)_{i,n} \subseteq (U_{h \cdot s})_{i,n}
\]

and

\[
(V_h)_{i,j} \cdot (m, s, n) \subseteq ((V_h)_{i,j} \cdot (m, s, n)) \cup ((\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (m, s, n)) \subseteq
\]

\[
\subseteq (V_h \cdot s)_{i,n} \cup (\theta^{-1}(V_h) \cap I_\alpha) \cdot s)_{i,n} \subseteq (V_h \cdot s)_{i,n} \cup (V_h \cdot s)_{i,n} =
\]

\[
= (V_h \cdot s)_{i,n} \subseteq (U_{h \cdot s})_{i,n}.
\]

If \( j > m \) then the separate continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \( \theta : S \to H(1_S) \) imply that for every open neighbourhood \( U_{h \cdot \theta^{j-m}(s)} \) of the point \( h \cdot \theta^{j-m}(s) \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( s \) in \((S, \tau)\), respectively, such that

\[
h \cdot \theta^{j-m}(W_s) \subseteq U_{h \cdot \theta^{j-m}(s)} \quad \text{and} \quad V_h \cdot \theta^{j-m}(s) \subseteq U_{h \cdot \theta^{j-m}(s)}.
\]

Hence for every \( I_\alpha \in I \) we have that

\[
(i, h, j) \cdot (W_s)_{m,n} \subseteq (h \cdot \theta^{j-m}(W_s))_{i,n-m+j} \subseteq (U_{h \cdot \theta^{j-m}(s)})_{i,n-m+j}^\alpha
\]
and
\[(V_h i, j^\alpha \cdot (m, s, n) \subseteq (V_h i, j^\alpha \cdot (m, s, n) \cup (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (m, s, n)) \subseteq \]
\[
\begin{cases}
(V_h \cdot \theta^j - m(s))_{i,n,m} \cup ((\theta^{-1}(V_h) \cap I_\alpha) \cdot s)_{i-1,n}, & \text{if } j-1=m; \\
(V_h \cdot \theta^j - m(s))_{i,n,m} \cup ((\theta^{-1}(V_h) \cap I_\alpha) \cdot \theta^{-1} - m(s))_{i-1,n,m} \cup (V_h \cdot \theta^j - m(s))_{i,n,m} \cup (V_h \cdot \theta^j - m(s))_{i,n,m}, & \text{if } j-1>m \end{cases}
\]

because \( \theta ((\theta^{-1}(V_h) \cap I_\alpha) \cdot \theta^{-1} - m(s)) \subseteq V_h \cdot \theta^{-1} - m(s) \subseteq U_{h \cdot \theta^{-1} - m(s)} \).

In case 3) we have that
\[(m, s, n) \cdot (i, g, j) = \begin{cases} 
(m - n + i, \theta^{i-n}(s) \cdot g, j), & \text{if } n < i; \\
(m, s \cdot g, j), & \text{if } n = i; \\
(m, s \cdot \theta^{i-n}(g), j - i + n), & \text{if } n > i.
\end{cases}
\]

and in this case the proof of separate continuity of the semigroup operation in \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})\) is similar to case 2.\[\square\]

We observe that in the case when \(\theta(s) = 1_S\) for all \(s \in S^1\) then a base of the topology \(\tau_{BR}\) on \(\mathcal{B}(S, \mathbb{Z})\) is determined in the following way:

(1) for every \(s \in S^1 \setminus \{1_S\}\) and \(i, j \in \mathbb{Z}\) the base of the topology \(\tau_{BR}\) at the point \((i, s, j)\) coincides with a base of the direct sum topology \(\tau_{ds}^{BR}\) at \((i, s, j)\);

(2) the family \(\mathcal{B}_{i, s, j} = \{U_{i, j}^\alpha = U_{i, j} \cup (I_\alpha)_{i-1,j-1}; U \in \mathcal{B}_{i, s, j}, I_\alpha \in \mathcal{A}\}\), where \(\mathcal{B}_{i, s, j}\) is a base of the topology \(\tau\) at the point \(1_S\) in \(S\), is a base of the topology \(\tau_{BR}\) at the point \((i, 1_S, j)\), for all \(i, j \in \mathbb{Z}\).

Then Theorem 2.10 implies the following:

**Theorem 2.11.** Let \((S, \tau)\) be a Hausdorff semitopological OIP-semigroup. Then there exists a topological \(\mathbb{Z}\)-Bruck extension \((\mathcal{B}(S, \mathbb{Z}), \tau_{BR})\) of \((S, \tau)\) in the class of semitopological semigroups such that the topology \(\tau_{BR}\) is strictly coarser than the direct sum topology \(\tau_{ds}^{BR}\) on \(\mathcal{B}(S, \mathbb{Z})\).

Later we need the following:

**Proposition 2.12.** Let \((S, \tau)\) be a topological (inverse) OIP-semigroup. Let \(\tau_{BR}\) be a topology on the semigroup \(\mathcal{B}(S, \mathbb{Z}, \theta)\) which is determined in the proof of Theorem 2.10 Then \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})\) is a topological (inverse) semigroup.

**Proof.** If \((S, \tau)\) is a topological semigroup then Proposition 2.1 implies that the semigroup operation is continuous in \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR}^{ds})\). Similarly, if inversion in an inverse topological semigroup \((S, \tau)\) is continuous then Proposition 2.1 implies that the inversion in \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR}^{ds})\) is continuous too. Therefore it is sufficient to show that the semigroup operation is jointly continuous in \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})\) in the following three cases:

1) \((i, h, j) \cdot (m, g, n); 2) (i, h, j) \cdot (m, s, n); 3) (m, s, n) \cdot (i, g, j),\)
and in the case when \((S, \tau)\) is an topological inverse semigroup it is sufficient to show that inversion is continuous at the point \((i, h, j)\), for all \(h, g \in H(1_S)\), 
\(s \in S \setminus H(1_S)\) and \(i, j, m, n \in \mathbb{Z}\).

Consider case 1). Then we have that

\[
(i, h, j) \cdot (m, g, n) = \begin{cases}
(i - j + m, \theta^{m-j}(h) \cdot g, n), & \text{if } j < m; \\
(i, h \cdot g, n), & \text{if } j = m; \\
(i, h \cdot \theta^{j-m}(g), n - m + j), & \text{if } j > m.
\end{cases}
\]

If \(j < m\) then the continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \(\theta: S \to H(1_S)\) imply that for every open neighbourhood \(U_{\theta^{m-j}(h), g}\) of the point \(\theta^{m-j}(h) \cdot g\) in \((S, \tau)\) there exist open neighbourhoods \(V_h\) and \(W_g\) of the points \(h\) and \(g\) in \((S, \tau)\), respectively, such that \(\theta^{m-j}(V_h) \cdot W_g \subseteq U_{\theta^{m-j}(h), g}\). Hence for every \(I_\alpha \in \mathcal{I}\) we have that

\[
(V_h)_{i,j}^\alpha \cdot (W_g)_{m,n}^\alpha \subseteq (V_h)_{i,j} \cdot (W_g)_{m,n} \cup ((V_h)_{i,j} \cdot (\theta^{-1}(W_g) \cap I_\alpha)_{m-1,n-1}) \cup
((\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (W_g)_{m,n}) \cup ((\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (\theta^{-1}(W_g) \cap I_\alpha)_{m-1,n-1}) \subseteq
(\theta^{m-j}(V_h) \cdot W_g)_{i-j+m,n} \cup (\theta^{m-j+1}(\theta^{-1}(V_h) \cap I_\alpha) \cdot W_g)_{i-j+m,n} \cup
(\theta^{m-j}(\theta^{-1}(V_h) \cap I_\alpha) \cdot (\theta^{-1}(W_g) \cap I_\alpha))_{i-j+m-1,n-1} \subseteq (U_{\theta^{m-j}(h), g})_{i-j+m,n}^\alpha,
\]

where

\[
A = \begin{cases}
(V_h \cdot (\theta^{-1}(W_g) \cap I_\alpha))_{i,n-1}, & \text{if } j = m - 1; \\
(\theta^{m-j}(V_h) \cdot (\theta^{-1}(W_g) \cap I_\alpha))_{i-j+m-1,n-1}, & \text{if } j < m - 1,
\end{cases}
\]

because

\[
\theta^{m-j+1}(\theta^{-1}(V_h) \cap I_\alpha) \cdot W_g \subseteq \theta^{m-j}(V_h) \cdot W_g \subseteq U_{\theta^{m-j}(h), g},
\]

\[
\theta(\theta^{m-j}(\theta^{-1}(V_h) \cap I_\alpha) \cdot (\theta^{-1}(W_g) \cap I_\alpha)) \subseteq \theta^{m-j}(V_h) \cdot W_g \subseteq U_{\theta^{m-j}(h), g},
\]

and

\[
\theta(A) = \begin{cases}
\theta(V_h) \cdot W_g, & \text{if } j = m - 1; \\
\theta^{m-j}(V_h) \cdot W_g, & \text{if } j < m - 1,
\end{cases} \subseteq U_{\theta^{m-j}(h), g}.
\]

The proof of the continuity of the semigroup operation in \((\mathcal{B}(S, \mathbb{Z}, \theta), \tau_{BR})\) in the case when \(j > m\) is similar to the previous case.

If \(j = m\) then the continuity of the semigroup operation in \((S, \tau)\) implies that for every open neighbourhood \(U_{h,g}\) of the point \(h \cdot g\) in \((S, \tau)\) there exist open neighbourhoods \(V_h\) and \(W_g\) of the points \(h\) and \(g\) in \((S, \tau)\), respectively, such that \(V_h \cdot W_g \subseteq U_{h,g}\). Then for every \(I_\alpha \in \mathcal{I}\) we get that

\[
(V_h)_{i,j}^\alpha \cdot (W_g)_{m,n}^\alpha \subseteq (V_h \cdot W_g)_{i,n}^\alpha \subseteq (U_{h,g})_{i,n}^\alpha.
\]

In case 2) we have that

\[
(i, h, j) \cdot (m, s, n) = \begin{cases}
(i - j + m, \theta^{m-j}(h) \cdot s, n), & \text{if } j < m; \\
(i, h \cdot s, n), & \text{if } j = m; \\
(i, h \cdot \theta^{j-m}(s), n - m + j), & \text{if } j > m,
\end{cases}
\]
where \( \theta^{m-j}(h) \cdot s, h \cdot s \in S \setminus H(I_S) \) and \( h \cdot \theta^{j-m}(s) \in H(I_S) \).

If \( j < m \) then the continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \( \theta: S \to H(I_S) \) imply that for every open neighbourhood \( U_{\theta^{m-j}(h),s} \) of the point \( \theta^{m-j}(h) \cdot s \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( s \) in \((S, \tau)\), respectively, such that \( \theta^{m-j}(V_h) \cdot W_s \subseteq U_{\theta^{m-j}(h),s} \). Hence for every \( I_\alpha \in \mathcal{I} \) we have that

\[
(V_h)_{i,j}^\alpha \cdot (W_s)_{m,n} \subseteq \left( (V_h)_{i,j} \cdot (W_s)_{m,n} \right) \cup \left( (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (W_s)_{m,n} \right) \subseteq \left( \theta^{m-j}(V_h) \cdot W_s \right)_{i-j+m,n} \subseteq (U_{\theta^{m-j}(h),s})_{i-j+m,n}.
\]

If \( j = m \) then the continuity of the semigroup operation in \((S, \tau)\) implies that for every open neighbourhood \( U_{h,s} \) of the point \( h \cdot s \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( s \) in \((S, \tau)\), respectively, such that \( V_h \cdot W_s \subseteq U_{h,s} \). Then for every \( I_\alpha \in \mathcal{I} \) we get that

\[
(V_h)_{i,j}^\alpha \cdot (W_s)_{m,n} \subseteq \left( (V_h)_{i,j} \cdot (W_s)_{m,n} \right) \cup \left( (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (W_s)_{m,n} \right) \subseteq (V_h \cdot W_s)_{i,n} \cup (\theta (\theta^{-1}(V_h) \cap I_\alpha) \cdot W_s)_{i,n} \subseteq (U_{h,s})_{i,n}.
\]

If \( j > m \) then the continuity of the semigroup operation in \((S, \tau)\) and the continuity of the homomorphism \( \theta: S \to H(I_S) \) imply that for every open neighbourhood \( U_{h,\theta^{j-m}(s)} \) of the point \( h \cdot \theta^{j-m}(s) \) in \((S, \tau)\) there exist open neighbourhoods \( V_h \) and \( W_s \) of the points \( h \) and \( s \) in \((S, \tau)\), respectively, such that \( V_h \cdot \theta^{j-m}(W_s) \subseteq U_{h,\theta^{j-m}(s)} \). Hence for every \( I_\alpha \in \mathcal{I} \) we have that

\[
(V_h)_{i,j}^\alpha \cdot (W_s)_{m,n} \subseteq \left( (V_h)_{i,j} \cdot (W_s)_{m,n} \right) \cup \left( (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,j-1} \cdot (W_s)_{m,n} \right) \subseteq \begin{cases}
(V_h, \theta^{j-m}(W_s))_{i-n-m+j} \cup (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,n} & \text{if } j-1=m; \\
(V_h, \theta^{j-m}(W_s))_{i-n-m+j} \cup (\theta^{-1}(V_h) \cap I_\alpha)_{i-1,n} \cdot \theta^{j-1-m}(W_s)_{i-1,n-m-j-1} & \text{if } j-1>m
\end{cases}
\subseteq (U_{h,\theta^{j-m}(s)})_{i,n} \cup (\theta^{-1}(U_{h,\theta^{j-m}(s)}) \cap I_\alpha)_{i-1,n-m-j-1} \subseteq (U_{h,\theta^{j-m}(s)})_{i,n}.
\]

because

\[
\theta \left( (\theta^{-1}(V_h) \cap I_\alpha) \cdot \theta^{j-1-m}(W_s) \right) \subseteq V_h \cdot \theta^{j-m}(W_s) \subseteq U_{h,\theta^{j-m}(s)}.
\]

The proof of the continuity of the semigroup operation in \((\mathcal{B}(S, Z, \theta), \tau_{BR})\) in case 3) is similar to case 2).

If \((S, \tau)\) is a topological inverse semigroup then for every ideal \( I \) in \( S \) we have that \( I^{-1} = I \) and for every open neighbourhoods \( V_h \) and \( U_{s^{-1}} \) of the points \( s \) and \( s^{-1} \) in \((S, \tau)\), respectively, such that \( (V_h)^{-1} \subseteq U_{s^{-1}} \) we have that

\[
( (V_h)_{i,j} )^{-1} \subseteq (U_{s^{-1}})_{j,i}, \text{ for } s \in S \setminus H(I_S) \quad \text{and}
\]
for all $I_a \in \mathcal{J}$, and hence $(\mathcal{B}(S,\mathbb{Z},\theta),\tau_{BR})$ is a topological inverse semi-group. This completes the proof of the proposition. \qed

Theorem 2.10 and Proposition 2.12 imply the following:

**Theorem 2.13.** Let $(S,\tau)$ be a topological (inverse) OIP-semigroup. Then there exists a topological Z-Bruck-Reilly extension $(\mathcal{B}(S,\mathbb{Z},\theta),\tau_{BR})$ of $(S,\tau)$ in the class of topological (inverse) semigroups such that the topology $\tau_{BR}$ is strictly coarser than the direct sum topology $\tau_{BR}^{ds}$ on $\mathcal{B}(S,\mathbb{Z},\theta)$.

Theorem 2.13 implies the following:

**Corollary 2.14.** Let $(S,\tau)$ be a topological (inverse) OIP-semigroup. Then there exists a topological Z-Bruck extension $(\mathcal{B}(S,\mathbb{Z}),\tau_{BR})$ of $(S,\tau)$ in the class of topological (inverse) semigroups such that the topology $\tau_{BR}$ is strictly coarser than the direct sum topology $\tau_{BR}^{ds}$ on $\mathcal{B}(S,\mathbb{Z})$.

Recall [12], a topological semilattice $E$ is said to be a U-semilattice if for every $x \in E$ and every open neighbourhood $U = \uparrow U$ of $x$ in $E$, there exists $y \in U$ such that $x \in \text{Int}_E(y)$.

**Remark 2.15.** Let $S$ be a Clifford inverse semigroup. We define a map $\varphi : S \to E(S)$ by the formula $\varphi(x) = x \cdot x^{-1}$. Theorem 4.11 from [13] implies that if $I$ is an ideal of $E(S)$ then $\varphi^{-1}(I)$ is an ideal of $S$.

The following theorem gives examples of topological OIP-semigroups.

**Theorem 2.16.** Let $(S,\tau)$ be a topological inverse Clifford semigroup. If the band $E(S)$ of $S$ has no a smallest idempotent and satisfies one of the following conditions:

1. for every $x \in E(S)$ there exists $y \in \downarrow x$ such that there is an open neighbourhood $U_y$ of $y$ with the compact closure $\text{cl}_{E(S)}(U_y)$;
2. $E(S)$ is locally compact;
3. $E(S)$ is a U-semilattice,

then $(S,\tau)$ is an OIP-semigroup.

**Proof.** Suppose condition (1) holds. We fix an arbitrary $x \in E(S)$. By Proposition VI-1.14 of [21], the partial order on the topological semilattice $E(S)$ is closed, and hence the compact set $K = \text{cl}_{E(S)}(U_y)$ has a minimal element $e$, which must also be a minimal element of $\uparrow K$. If $\uparrow K = E(S)$, then $e$ is a minimal element of $E(S)$ and hence $e$ is a least element of $E(S)$, because $ef \leq e$ for any $f \in E(S)$ implies $e = ef$, i.e., $e \leq f$. This contradicts that $E(S)$ hasn’t a least element.

Then the set $I_x = E(S) \setminus \uparrow (\text{cl}_{E(S)}(U_y))$ is an open ideal in $E(S)$ and by Proposition VI-1.13(iii) from [21] the set $U_x = \uparrow U_y$ is an open neighbourhood of the point $x$ in $E(S)$ such that $I_x \cap U_x = \emptyset$. Therefore for every $x \in E(S)$
we constructed an open neighbourhood $U_x$ of the point $x$ in $E(S)$ and an
open ideal $I_x$ in $E(S)$ such that $I_x \cap U_x = \varnothing$, and hence the topological
semilattice $E(S)$ is an OIP-semigroup. Now we apply Remark 2.15 and get
that $(S, \tau)$ is an OIP-semigroup.

We observe that every locally compact semilattice satisfies condition (1).

Suppose condition (3) holds. We fix an arbitrary $x \in E(S)$. Since the
semilattice $E(S)$ does not contain a minimal idempotent we conclude that
there exists an idempotent $e \in \downarrow x \setminus \{x\}$. Then by Proposition VI-1.13(i)
from [21] the set $U_x = E(S) \setminus \downarrow e$ is open in $E(S)$ and it is obvious that
$x \in U_x = \uparrow U_x$. Let $y_{[x,e]} \in U_x$ be such that $x \in \text{Int}_{E(S)}(\uparrow y_{[x,e]})$. We
put $V_x = \text{Int}_{E(S)}(\uparrow y_{[x,e]})$ and $I_{[x,e]} = E(S) \setminus \uparrow y_{[x,e]}$. Then $V_x$ is an open
neighbourhood of $x$ in $E(S)$ and $I_{[x,e]}$ is an open ideal in $E(S)$. Hence
similar arguments as in case (1) show that $(S, \tau)$ is an OIP-semigroup. □

3. On $I$-bisimple topological inverse semigroups

A bisimple semigroup $S$ is called an $I$-bisimple semigroup if and only if
$E(S)$ is order isomorphic to $\mathbb{Z}$ under the reverse of the usual order.

In [44] Warne proved the following theorem:

Theorem 3.1 ([44] Theorem 1.3). A regular semigroup $S$ is $I$-bisimple if
and only if $S$ is isomorphic to $\mathcal{B}_W = \mathbb{Z} \times G \times \mathbb{Z}$, where $G$ is a group, under
the multiplication

$$(a, g, b) \cdot (c, h, d) = \begin{cases} (a, g \cdot f_{b-c,c}^{-1} \cdot \theta^{b-c}(h) \cdot f_{b-c,d}, d - c + b), & \text{if } b \geq c; \\ (a - b + c, f_{c-b,a}^{-1} \cdot \theta^{c-b}(g) \cdot f_{c-b,d}, h, d), & \text{if } b \leq c, \end{cases}$$

(4)

where $\theta$ is an endomorphism of $G$, $\theta^0$ denoting the identity automorphism
of $G$, and for $m \in \mathbb{N}$, $n \in \mathbb{Z}$,

(1) $f_{0,n} = e$ is the identity of $G$; and
(2) $f_{m,n} = \theta^{m-1}(u_{n+1}) \cdot \theta^{m-2}(u_{n+2}) \cdot \ldots \cdot \theta(u_{n+(m-1)}) \cdot u_{n+m}$, where

$\{u_n : n \in \mathbb{Z}\}$ is a collection of elements of $G$ with $u_n = e$ if $n \in \mathbb{N}$.

For arbitrary $i, j \in \mathbb{Z}$ we denote $G_{i,j} = \{(i, g, j) \in \mathcal{B}_W : g \in G\}$.

Theorem 3.2. Let $S$ be a regular $I$-bisimple semitopological semigroup.
Then there exist a group $G$ with the identity element $e$, an endomorphism
$\theta : G \to G$, a collection $\{u_n : n \in \mathbb{Z}\}$ of elements of $G$ with the property
$u_n = e$ if $n \in \mathbb{N}$ and a topology on the semigroup $\mathcal{B}_W$ such that the following
assertions hold:

(i) $S$ is topologically isomorphic to a semitopological semigroup $\mathcal{B}_W$ (not
necessarily with the product topology);
(ii) $G_{i,j}$ and $G_{k,l}$ are homeomorphic subspaces of $\mathcal{B}_W$ for all $i, j, k, l \in \mathbb{Z}$;
(iii) $G_{i,i}$ and $G_{k,k}$ are topologically isomorphic semitopological subgroups
of $\mathcal{B}_W$ with the induced topology from $\mathcal{B}_W$ for all $i, k \in \mathbb{Z}$;
are continuous maps and left and right translations by an idempotent are continuous endomorphisms of the semitopological group $G = G_{i,i}$ with the induced from $B_W$ topology, for an arbitrary integer $i$;

(v) for every element $(i, g, j) \in B_W$ there exists an open neighbourhood $U_{(i,g,j)}$ of the point $(i, g, j)$ in $B_W$ such that $U_{(i,g,j)} \subseteq \bigcup\{G_{i-k,j-k}: k = 0, 1, 2, 3, \ldots\}$;

(vi) $E(S)$ is a discrete subspace of $S$.

Proof. The first part of the theorem and assertion (i) follow from Theorem 3.1.

(ii) We fix arbitrary $i, j, k, l \in \mathbb{Z}$ and define the map $\varphi_{i,j}^{k,l}: B_W \to B_W$ by the formula $\varphi_{i,j}^{k,l}(x) = (k, e, i) \cdot x \cdot (j, e, l)$. Then formula (1) implies that the restriction $\varphi_{i,j}^{k,l}|_{G_{i,j}}: G_{i,j} \to G_{k,l}$ is a bijective map. Now the compositions $\varphi_{i,j}^{k,l}|_{G_{i,j}} \circ \varphi_{k,l}|_{G_{k,l}}$ and $\varphi_{k,l}|_{G_{k,l}} \circ \varphi_{i,j}^{k,l}|_{G_{i,j}}$ are identity maps of the sets $G_{i,j}$ and $G_{k,l}$, respectively, and hence we have that the map $\varphi_{i,j}^{k,l}|_{G_{i,j}}: G_{i,j} \to G_{k,l}$ is invertible to $\varphi_{k,l}|_{G_{k,l}}: G_{k,l} \to G_{i,j}$. Since $B_W$ is a semitopological semigroup we conclude that $\varphi_{i,j}^{k,l}|_{G_{i,j}}: G_{i,j} \to G_{k,l}$ and $\varphi_{k,l}|_{G_{k,l}}: G_{k,l} \to G_{i,j}$ are continuous maps and hence the map $\varphi_{i,j}^{k,l}|_{G_{i,j}}: G_{i,j} \to G_{k,l}$ is a homeomorphism.

(iii) Formula (1) implies that $G_{i,i}$ and $G_{k,k}$ are semitopological subgroups of $B_W$ with the induced topology from $B_W$ for all $i, k \in \mathbb{Z}$. Simple verifications show that the map $\varphi_{i,i}^{k,k}|_{G_{i,i}}: G_{i,i} \to G_{k,k}$ is a topological isomorphism.

(iv) Assertion (iii) implies that for arbitrary $i, k \in \mathbb{Z}$ the subspaces $G_{i,i}$ and $G_{k,k}$ with the induced semigroup operation are topologically isomorphic subgroups of $B_W$ and hence the semitopological group $G$ is correctly determined. Next we consider the map $f: G = G_{0,0} \to G = G_{1,1}$ defined by the formula $f(x) = x \cdot (1, e, 1)$. Then by formula (1) we have that

$$(0, g, 0) \cdot (1, e, 1) = (1, f_{1,0}^{-1}(\theta(g)) \cdot f_{1,0}^{-1} \cdot e, 1) = (1, e^{-1} \cdot \theta(g) \cdot e, 1) = (1, \theta(g), 1),$$

and since the translations in $B_W$ are continuous we conclude that $\theta$ is a continuous endomorphism of the semitopological group $G$.

(v) Since the left and right translations in a semitopological semigroup are continuous maps and left and right translations by an idempotent are retractions, Exercise 1.5.C from [18] implies that $(i + 1, e, i + 1)B_W$ and $B_W(j + 1, e, j + 1)$ are closed subsets in $B_W$. Hence there exists an open neighbourhood $W_{(i,g,j)}$ of the point $(i, g, j)$ in $B_W$ such that

$$W_{(i,g,j)} \subseteq B_W \setminus ((i + 1, e, i + 1)B_W \cup B_W(j + 1, e, j + 1)).$$

Since the semigroup operation in $B_W$ is separately continuous we conclude that there exists an open neighbourhood $U_{(i,g,j)}$ of the point $(i, g, j)$ in $B_W$ such that

$$U_{(i,g,j)} \subseteq W_{(i,g,j)}, \quad (i, e, i) \cdot U_{(i,g,j)} \subseteq W_{(i,g,j)} \quad \text{and} \quad U_{(i,g,j)} \cdot (j, e, j) \subseteq W_{(i,g,j)}.$$
Next we shall show that $U_{i,g,j} \subseteq \bigcup \{G_{i-k,j-k} : k = 0, 1, 2, 3, \ldots \}$. Suppose the contrary: there exists $(m, a, n) \in U_{i,g,j}$ such that $(m, a, n) \notin \bigcup \{G_{i-k,j-k} : k = 0, 1, 2, 3, \ldots \}$. Then we have that $m \leq i, n \leq j$ and $m - n \neq i - j$. If $m - n > i - j$ then formula (1) implies that there exists $u \in G$ such that

$$(m, a, n) \cdot (j, e, j) = (m - n + j, u, j) \notin B_W \setminus (i + 1, e, i + 1)B_W,$$

because $m - n + j > i - j + j = i$, and hence $(m, a, n) \cdot (j, e, j) \notin W_{i,s,j}$. Similarly, if $m - n < i - j$ then formula (1) implies that there exists $v \in G$ such that

$$(i, e, i) \cdot (m, a, n) = (i, v, n - m + i) \notin B_W \setminus B_W(j + 1, e, j + 1),$$

because $n - m + i > j - i + i = j$, and hence $(i, e, i) \cdot (m, a, n) \notin W_{i,s,j}$. This completes the proof of our assertion.

(vi) The definition of an $I$-bisimple semigroup implies that $E(S)$ is order isomorphic to $\mathbb{Z}$ under the reverse of the usual order and hence $E(S)$ is a subsemigroup of $S$. Then we have that $E(S) = \{(n, e, n) : n \in \mathbb{Z}\}$ (see [18]). We fix an arbitrary $(i, e, i) \in E(S)$. Since translations on $(i, e, i)$ in $S$ are continuous retractions Theorem 1.4.1 of [18] implies that the set $\{x \in S : x \cdot (i - 1, e, i - 1) = (i - 1, e, i - 1)\}$ is closed in $S$, and Exercise 1.5.C from [18] implies that $(i + 1, e, i + 1)S$ is a closed subset in $S$ too. This implies that $(i, e, i)$ is an isolated point of $E(S)$ with the induced from $S$ topology. This completes the proof of our assertion.

Theorem 3.3. Let $S$ be a regular $I$-bisimple semitopological semigroup. If $S$ has a maximal compact subgroup then the following statements hold:

(i) $S$ is topologically isomorphic to $B_W = \mathbb{Z} \times G \times \mathbb{Z}$ with the product topology;

(ii) $S$ is a locally compact topological inverse semigroup.

Proof. (i) By item (i) of Theorem 3.2 we have that the semitopological semigroup $S$ is topologically isomorphic to a semitopological semigroup $B_W = \mathbb{Z} \times G \times \mathbb{Z}$. It is obvious to show that for arbitrary $i, j \in \mathbb{Z}$ the $H$-class $G_{i,j}$ of $B_W$ is an open subset in $B_W$. We fix an arbitrary $(i, g, j) \in G_{i,j}$. Then by Theorem 3.2(v) there exists an open neighbourhood $U_{i,g,j}$ of the point $(i, g, j)$ in $B_W$ such that $U_{i,g,j} \subseteq \bigcup \{G_{i-k,j-k} : k = 0, 1, 2, 3, \ldots \}$. Since the semitopological semigroup $S$ has a maximal compact subgroup, Theorem 3.2(ii) implies that every $H$-class $G_{m,n}$ of $B_W$ is a compact subset in $B_W$. Then the separate continuity of the semigroup operation in $B_W$ and Theorem 1.4.1 of [18] imply that $\{x \in B_W : x \cdot (i - 1, e, i - 1) \in G_{i-1,i-1}\}$ is a closed set in $B_W$. Therefore there exists an open neighbourhood $V_{i,g,j} \subseteq U_{i,g,j}$ of the point $(i, g, j)$ in $B_W$ such that $V_{i,g,j} \subseteq G_{i,j}$. This completes the proof of the statement.
(ii) Statement (i), Theorem 3.2(ii) and Theorem 3.3.13 of [18] imply that $S$ is a locally compact space. Then statement (i), Corollary 3.3.10 from [18] and Ellis Theorem (see Theorem 2 of [17] or Theorem 1.18 of [12, Vol. 1]) imply that every maximal subgroup $G_{n,n}$ of $B$ is a topological group. We put $G = G_{n,n}$ for some $n \in \mathbb{Z}$ with the induced topology from $B$. Assertion (iii) of Theorem 3.2 implies that the topological group $G$ is correctly defined. Let $\mathfrak{B}_G$ be a base of the topology of the topological group $G$. Then statement (i) and assertion (ii) of Theorem 3.2 imply that the family

$$\mathfrak{B}_{\mathfrak{B}_W} = \{ U_{i,j} : U \in \mathfrak{B}_G \text{ and } i, j \in \mathbb{Z} \},$$

where $U_{i,j} = \{(i, x, j) : x \in U \} \subseteq G_{i,j}$, determines a base of the topology of the semitopological semigroup $B$.

Since $G$ is a topological group and $\theta : G \rightarrow G$ is a continuous homomorphism, we conclude that for arbitrary integers $a, b, c, d$ with $b \geq c$, arbitrary $g, h \in G$ and any open neighbourhood $W$ of the point $g \cdot f_{b, c, c}^{-1} \cdot \theta^{b, e}(h) \cdot f_{b, c, c} \cdot f_{b, c, d}$ in the topological space $G$ there exist open neighbourhoods $W_g$ and $W_h$ of the points $g$ and $h$ in $G$, respectively, such that

$$W_g \cdot f_{b, c, c}^{-1} \cdot \theta^{b, e}(W_h) \cdot f_{b, c, c} \subseteq W.$$

Then in the case when $b \geq c$ we have that

$$(a, W_g, b) \cdot (c, W_h, d) \subseteq (a, W_g) \cdot f_{b, c, c}^{-1} \cdot \theta^{b, c}(W_h) \cdot f_{b, c, c}, d - c + b) \subseteq (a, W, d - c + b).$$

Similarly, the continuity of the group operation in $G$ and the continuity of the homomorphisms $\theta$ imply that for arbitrary integers $a, b, c, d$ with $b \leq c$, arbitrary $g, h \in G$ and any open neighbourhood $U$ of $f_{c, b, a}^{-1} \cdot \theta^{c, b}(g) \cdot f_{c, b, b} \cdot h$ in the topological space $G$ there exist open neighbourhoods $U_g$ and $U_h$ of the points $g$ and $h$ in $G$, respectively, such that

$$f_{c, b, a}^{-1} \cdot \theta^{c, b}(U_g) \cdot f_{c, b, b} \cdot U_h \subseteq U.$$

Then in the case when $b \leq c$ we have that

$$(a, U_g, b) \cdot (c, U_h, d) \subseteq (a - b + c, f_{c, b, a}^{-1} \cdot \theta^{c, b}(U_g) \cdot f_{c, b, b} \cdot U_h, d) \subseteq (a - b + c, U, d).$$

Hence the semigroup operation is continuous in $B$.

Also, since the inversion in $G$ is continuous we have that for every element $g$ of $G$ and any open neighbourhood $W_{g^{-1}}$ of its inverse $g^{-1}$ in $G$ there exists open neighbourhood $U_g$ of $g$ in $G$ such that $(U_g)^{-1} \subseteq W_{g^{-1}}$. Then we get that $(a, U_g, b)^{-1} \subseteq (b, W_{g^{-1}}, a)$, for arbitrary integers $a$ and $b$. This completes the proof that $B$ is a topological inverse semigroup. \qed

If $S$ is a topological inverse semigroup then the maps $l : S \rightarrow E(S)$ and $r : S \rightarrow E(S)$ defined by the formulae $l(x) = x \cdot x^{-1}$ and $r(x) = x^{-1} \cdot x$ are continuous. Hence Theorem 3.2 implies the following corollary:

**Corollary 3.4.** Let $S$ be a regular $I$-bisimple topological inverse semigroup. Then every $\mathcal{H}$-class of $S$ is a closed-and-open subset of $S$.  

A topological space $X$ is called Baire if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of $X$ the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of $X$ [18].

Since every Hausdorff Baire topology on a countable topological group is discrete, Corollary 3.4 implies the following:

**Corollary 3.5.** Every regular $I$-bisimple countable Hausdorff Baire topological inverse semigroup is discrete.

A Tychonoff space $X$ is called Čech complete if for every compactification $cX$ of $X$ the remainder $cX \setminus c(X)$ is an $F_\sigma$-set in $cX$ [18]. Since every Čech complete space (and hence every locally compact space) is Baire, Corollary 3.5 implies the following:

**Corollary 3.6.** Every regular $I$-bisimple countable Hausdorff Čech complete (locally compact) topological inverse semigroup is discrete.

The following example implies that there exists a Hausdorff locally compact zero-dimensional $I$-bisimple topological semigroup $S$ with locally compact (discrete) maximal subgroup $G$ such that $S$ is not topologically isomorphic to $B_W = \mathbb{Z} \times G \times \mathbb{Z}$ with the product topology and hence $S$ is not a topological inverse semigroup.

**Example 3.7.** Let $\mathbb{Z}$ be the additive group of integers and $\theta: \mathbb{Z} \to \mathbb{Z}$ be an annihilating homomorphism, i.e., $\theta(m) = e$ is the unity of $\mathbb{Z}$ for every $m \in \mathbb{Z}$. We put $B(Z, \mathbb{Z})$ be the $\mathbb{Z}$-Bruck extension of the group $\mathbb{Z}$. Then Theorem 3.1 implies that $B(Z, \mathbb{Z})$ is an $I$-bisimple semigroup.

We determine the topology $\tau$ on $B(Z, \mathbb{Z})$ in the following way:

(i) all non-idempotent elements of the semigroup $B(Z, \mathbb{Z})$ are isolated points in $(B(Z, \mathbb{Z}), \tau)$; and

(ii) the family $\mathfrak{B}_{(i, e, j)} = \{U^n_{i, j} : i, j \in \mathbb{Z}, n \in \mathbb{Z}\}$, where $U^n_{i, j} = \{(i, e, j)\} \cup \{(i - 1, k, j - 1) : k \geq n\}$, is a base of the topology $\tau$ at the point $(i, e, j) \in B(Z, \mathbb{Z})$, $i, j \in \mathbb{Z}$.

Simple verifications show that $\tau$ is a Hausdorff locally compact zero-dimensional topology on $B(Z, \mathbb{Z})$. Later we shall prove that $\tau$ is a semigroup topology on $B(Z, \mathbb{Z})$.

We remark that the semigroup operation on $B(Z, \mathbb{Z})$ is defined by the formula

$$(i, g, j) \cdot (m, h, n) = \begin{cases} (i - j + m, h, n), & \text{if } j < m; \\ (i, g \cdot h, n), & \text{if } j = m; \\ (i, g, n - m + j), & \text{if } j > m, \end{cases}$$

for arbitrary $i, j, m, n \in \mathbb{Z}$ and $g, h \in \mathbb{Z}$. Since all non-idempotent elements of the semigroup $B(Z, \mathbb{Z})$ are isolated points in $(B(Z, \mathbb{Z}), \tau)$, it is sufficient
to show that the semigroup operation on \((\mathcal{B}(\mathbb{Z}, \mathbb{Z}), \tau)\) is continuous in the following cases:

**a)** \((i, g, j) \cdot (m, e, n); \quad **b)** \((i, e, j) \cdot (m, g, n); \quad **c)** \((i, e, j) \cdot (m, e, n),\)

where \(e\) is the unity of \(G\) and \(g \in G \setminus \{e\} \).

Then we have that in case **a**):

1. if \(j < m - 1\) then \((i, g, j) \cdot (m, e, n) = (i - j + m, e, n)\) and \(\{(i, g, j)\} \cdot U_{i,j}^k \subseteq U_{i-j+m,n}^k\;
2. if \(j = m - 1\) then \((i, g, j) \cdot (m, e, n) = (i + 1, e, n)\) and \(\{(i, g, j)\} \cdot U_{i,j}^k \subseteq U_{i+1,n}^{k+g}\;
3. if \(j \geq m\) then \((i, g, j) \cdot (m, e, n) = (i, g, n - m + j)\) and \(\{(i, g, j)\} \cdot U_{i,j}^k \subseteq \{(i, g, n - m + j)\},\)

in case **b**):

1. if \(j \leq m\) then \((i, e, j) \cdot (m, g, n) = (i - j + m, g, n)\) and \(U_{i,j}^k \cdot \{(m, g, n)\} \subseteq \{(i - j + m, g, n)\};
2. if \(j = m + 1\) then \((i, e, j) \cdot (m, g, n) = (i, e, n+1)\) and \(U_{i,j}^k \cdot \{(m, g, n)\} \subseteq U_{i,n+1}^{k+g};
3. if \(j > m + 1\) then \((i, e, j) \cdot (m, g, n) = (i, e, n - m + j)\) and \(U_{i,j}^k \cdot \{(m, g, n)\} \subseteq U_{i,n-m+j}^k\)

and in case **c**):

1. if \(j < m\) then \((i, e, j) \cdot (m, e, n) = (i - j + m, e, n)\) and \(U_{i,j}^k \cdot U_{m,n}^l \subseteq U_{i-j+m,n}^k;
2. if \(j = m\) then \((i, e, j) \cdot (m, e, n) = (i, e, n)\) and \(U_{i,j}^k \cdot U_{m,n}^l \subseteq U_{i,n}^{k+l};
3. if \(j > m\) then \((i, e, j) \cdot (m, e, n) = (i, e, n - m + j)\) and \(U_{i,j}^k \cdot U_{m,n}^l \subseteq U_{i,n-m+j}^k\)

for arbitrary integers \(k\) and \(l\). Hence \((\mathcal{B}(\mathbb{Z}, \mathbb{Z}), \tau)\) is a topological semigroup. It is obvious that the inversion in \((\mathcal{B}(\mathbb{Z}, \mathbb{Z}), \tau)\) is not continuous.

**Remark 3.8.** (1) We observe that the similar propositions to Theorems 3.2 and 3.3 Corollaries 3.4, 3.5 and 3.6 hold for \(\omega\)-bisimple (semi)topological semigroups as topological Bruck-Reilly extensions.

(2) Also Example 3.7 shows that there exists a Hausdorff locally compact zero-dimensional \(\omega\)-bisimple topological semigroup \(S\) with a locally compact (discrete) maximal subgroup \(G\) such that \(S\) is not topologically isomorphic to the Bruck-Reilly extension with the product topology and hence \(S\) is not a topological inverse semigroup.

(3) The statement of Theorem 3.3 is true in the case when the subsemigroup \(C(S) = \{(i, g, i) : i \in \mathbb{Z} \text{ and } g \in G\}\) is weakly uniform (the definition of a weakly uniform topological semigroup see in [43]). In this case we have that inversion in \(C(S)\) is continuous (see [14] and
and hence by Proposition 2.3 we get that every \( H \)-class of \( S \) is an open-and-closed subset of \( S \). This implies that the inversion in \( S \) is continuous, too.

The following example implies that there exists a Hausdorff locally compact zero-dimensional \( I \)-bisimple semitopological semigroup \( S \) with continuous inversion and locally compact (discrete) maximal subgroup \( G \) such that \( S \) is not topologically isomorphic to \( B_W = \mathbb{Z} \times G \times \mathbb{Z} \) with the product topology and hence \( S \) is not a topological inverse semigroup.

**Example 3.9.** Let \( Z \) be the additive group of integers and \( \theta: Z \to Z \) be an annihilating homomorphism.

We determine the topology \( \tau \) on \( B(Z, Z) \) in the following way:

(i) all non-idempotent elements of the semigroup \( B(Z, Z) \) are isolated points in \( (B(Z, Z), \tau) \); and

(ii) the family \( B(i, e, j) = \{U_{i,j}^{m,n}: i, j \in \mathbb{Z}, m, n \in \mathbb{Z}\} \), where

\[
U_{i,j}^{m,n} = \{(i, e, j)\} \cup \{(i - 1, k, j - 1): k \leq -n\} \cup \{(i - 1, k, j - 1): k \geq n\},
\]

is a base of the topology \( \tau \) at the point \( (i, e, j) \in B(Z, Z) \), \( i, j \in \mathbb{Z} \).

Simple verifications show that \( \tau \) is a Hausdorff locally compact zero-dimensional topology on \( B(Z, Z) \). The proof of the separate continuity of semigroup operation and the continuity of inversion in \( (B(Z, Z), \tau) \) is similar to Example 3.7.

**Remark 3.10.** Example 3.9 shows that there exists a Hausdorff locally compact zero-dimensional \( \omega \)-bisimple semitopological semigroup \( S \) with continuous inversion and a locally compact (discrete) maximal subgroup \( G \) such that \( S \) is not topologically isomorphic to the Bruck-Reilly extension with the product topology and hence \( S \) is not a topological inverse semigroup.

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