Interpolation Properties of Certain Classes of Net Spaces

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Abstract—The paper studies the interpolation properties of net spaces \( N_{p,q}(M) \), when \( M \) is the set of dyadic cubes in \( \mathbb{R}^n \), and also when \( M \) is the family of all cubes with parallel faces to the coordinate axes in \( \mathbb{R}^n \). It is shown that, in the case when \( M \) is the set of dyadic cubes the scale of spaces is closed with respect to the real interpolation method. In the case, when \( M \) is the set of all cubes with parallel faces to the coordinate axes, an analogue of the Marcinkiewicz–Calderon theorem on cones of non-negative functions is given.

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1. INTRODUCTION

Let \( \mu \) be \( n \)-dimensional Lebesgue measure in \( \mathbb{R}^n \) and \( M \) be the set of all cubes in \( \mathbb{R}^n \). We call \( M \) a “net” hereinafter. For a function \( f(x) \), defined and integrable on each \( e \) from \( M \), we define the function

\[
\tilde{f}(t, M) = \sup_{e \in M, \ |e| \geq t} \frac{1}{|e|} \left| \int e f(x) \, dx \right|, \quad t > 0,
\]

where the supremum is taken over all \( e \in M \) of measure \( |e| = \mu e \geq t \). If \( \sup \{|e| : e \in M\} = \alpha < \infty \) and \( t > \alpha \), then we set \( \tilde{f}(t, M) = 0 \).

Let parameters \( p, q \) satisfy the conditions \( 0 < p, q \leq \infty \). We define the net spaces \( N_{p,q}(M) \), as the set of all functions \( f \), such that for \( q < \infty \)

\[
\|f\|_{N_{p,q}(M)} = \left( \int_0^{\infty} \left( t^{\frac{1}{p}} \tilde{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,
\]

and for \( q = \infty \)

\[
\|f\|_{N_{p,\infty}(M)} = \sup_{t > 0} t^{\frac{1}{p}} \tilde{f}(t, M) < \infty.
\]

These spaces were introduced in the work [12].

Net spaces have found important applications in various problems of harmonic analysis, operator theory and the theory of stochastic processes [1–3, 13, 15–18].

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In this paper we study the interpolation properties of these spaces. It should be noted here, that net spaces are in a sense close to the Morrey space

\[
M^\alpha = \left\{ f : \sup_{y \in \mathbb{R}^n, t > 0} t^{-\lambda} \left( \int_{|x+y| \leq t} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.
\]

In the case when \( f(x) \geq 0 \), for \( \frac{1}{p} = 1 - \frac{\lambda}{n} \)

\[
\|f\|_{N_{p,\infty}(M)} \asymp\|f\|_{M^\lambda}.
\]

The question of interpolation of Morrey spaces was considered in the \([8, 9, 11, 19–21]\). It follows from the results of \([19]\) that

\[
(M_{\lambda_0}^\lambda, M_{\lambda_1}^\lambda)_{\theta, \infty} \hookrightarrow M_p^\lambda,
\]

where \( \lambda = (1 - \theta) \lambda_0 + \theta \lambda_1 \). It was established in \([8, 20]\) that this inclusion is strict.

For net spaces \( N_{p,q}(M) \), where \( M \) is an arbitrary system of measurable sets from \( \mathbb{R}^2 \), we also have an embedding (see \([12]\), Theorem 1)

\[
(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} \hookrightarrow N_{p,q}(M),
\]

where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, 0 < \theta < 1, 0 < q \leq \infty \).

From (1) it follows that if the linear operator \( T \) bounded from \( A_i \) to \( N_{p,\infty}(M) \), \( i = 0, 1 \), then the operator \( T \) bounded from \( A_{\theta,q} \) to \( N_{p,q}(M) \), where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \).

The question arises whether the following equality will take place

\[
(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} = N_{p,q}(M).
\]

Here, in contrast to the Morrey spaces in the one-dimensional case, when \( M \) is the set of all segments, the answer is positive \([18]\). In this paper we show that, if \( M \) is the set of dyadic cubes in \( \mathbb{R}^n \), then the relation (2) holds. In case, if \( M \) is the set of all cubes, an analogue of the Marcinkiewicz–Calderon theorem on the cones of non-negative functions is obtained.

Given functions \( F \) and \( G \), in this paper \( F \lesssim G \) means that \( F \leq CG \), where \( C \) is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, \( F \asymp G \) means that \( F \lesssim G \) and \( G \lesssim F \).

2. MAIN RESULTS

Let \((A_0, A_1)\) be a compatible pair of Banach spaces \([4]\);

\[
K(t, a; A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \ a \in A_0 + A_1,
\]

be the Petre functional. For \( 0 < q < \infty, 0 < \theta < 1 \) the interpolation space is defined by

\[
(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0, A_1)_{\theta,q}} = \left( \int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},
\]

and for \( q = \infty \) by

\[
(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0, A_1)_{\theta,q}} = \sup_{0 < t < \infty} t^{-\theta} K(t, a) < \infty \right\}.
\]

A family of sets from \( \mathbb{R}^n \) of the form

\[
Q^n_k = \left[ \frac{k_1}{2m}, \frac{k_1+1}{2m} \right) \times \ldots \times \left[ \frac{k_n}{2m}, \frac{k_n+1}{2m} \right),
\]

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where $k \in \mathbb{Z}^n$, $m \in \mathbb{Z}$, is called the family of dyadic cubes and denoted by $M$.

Note that for arbitrary $m \in \mathbb{Z}$ the space $\mathbb{R}^n$ can be represented in the form $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k^m$, and the measure of intersection

$$|Q_k^m \cap Q_r^m| = \begin{cases} 2^{nm}, & k_i = r_i, \; i = 1, \ldots, m \\ 0, & \text{in other cases.} \end{cases}$$

Let $m \in \mathbb{Z}$, cubes $Q_k^m$, $k \in \mathbb{Z}^n$ are called cubes of $m$ order. Note also that if $n \geq m$, then each cube of $n$ order is partitioned into $4^{n-m}$ cubes of $m$ order.

**Theorem 1.** Let $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $M$ be the family of dyadic cubes. Then,

$$\left( N_{p_0, q_0}(M), N_{p_1, q_1}(M) \right)_{\theta, q} = N_{p, q}(M),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0, 1)$.

The following statement is an attempt to answer the question about the interpolation of net spaces, when $M$ is the family of all cubes with parallel faces to the coordinate axes in $\mathbb{R}^n$.

**Theorem 2.** Let $n \leq p_0 < p_1 < \infty$, $0 < q \leq \infty$, $M$ be the family of all cubes with parallel faces to the coordinate axes in $\mathbb{R}^n$. Let $G = \{ f : f(x) \geq 0 \}$, then for any $f \in G \cap N_{p, q}(M)$ it is true

$$\|f\|_{(N_{p_0, q_0}(M), N_{p_1, q_1}(M))_{\theta, q}} \leq \|f\|_{N_{p, q}(M)},$$

where the corresponding constants depend only on $p_1, q_1, \theta, q, i = 0, 1$.

The following corollary holds from Theorem 2.

**Corollary 1.** Let $n \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$, $q_0 \neq q_1$, $0 < \tau$, $\sigma < \infty$; $M$ and $G$ are sets from Theorem 2. If the following inequalities hold for a quasilinear operator

$$\|Tf\|_{N_{p_0, q_0}(M)} \leq F_0\|f\|_{N_{p_0, \sigma}(M)}, \; f \in N_{p_0, \sigma}(M),$$

$$\|Tf\|_{N_{p_1, q_1}(M)} \leq F_1\|f\|_{N_{p_1, \sigma}(M)}, \; f \in N_{p_1, \sigma}(M),$$

then for any $f \in G \cap N_{p, \tau}$ we have

$$\|Tf\|_{N_{q, \tau}(M)} \leq cF_0^{1-\theta}F_1^\theta\|f\|_{N_{p, \tau}(M)},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0, 1)$ and the corresponding constant depends only on $p_1, q_1, \sigma, i = 0, 1$.

**Comment.** In the case when $M$ is the set of parallelepipeds in $\mathbb{R}^n$ for the scale of spaces $N_{p, q}(M)$ equality (3) does not hold. Here, apparently, it is necessary to use interpolation methods for spaces with mixed metrics [10, 13, 14], and also to use the ideas of works [5–7].

3. PROOF OF THEOREM 1

**Proof.** Let $M$ be the set of dyadic cubes, $1 < p_0 < \infty$. Let us prove first

$$(N_{1, \infty}(M), N_{\infty, \infty}(M))_{\theta, q} = N_{p, q}(M),$$

where $\frac{1}{p} = 1 - \theta$, $\theta \in (0, 1)$.

Let $m \in \mathbb{Z}$, then the Euclidean space $\mathbb{R}^n$ is partitioned into disjoint cubes of order $m$ from $M$

$$\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k^m.$$

Let $f \in N_{p, q}(M)$, define the function

$$\varphi_0(x) = \frac{1}{|Q_k^m|} \int_{Q_k^m} f(x) dx, \; x \in Q_k^m, \; k \in \mathbb{Z}^n.$$

Then, taking into account that the measure $|Q_k^m| = 2^{nm}$ we have

$$|\varphi_0(x)| \leq \tilde{f}(2^{nm}, M), \; x \in \mathbb{R}^n,$$

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and

\[ \int_{Q_k^n} (f(x) - \varphi_0(x)) dx = 0. \]

For the Petre functional, we have the following

\[ K(t, f; N_{p_0, \infty}(M), N_{\infty, \infty}(M)) = \inf_{f = f_0 + f_1} \left( ||f_0||_{N_{p_0, \infty}(M)} + t ||f_1||_{N_{\infty, \infty}(M)} \right) \]

\[ \leq \sup_{s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) + t \sup_{s > 0} \varphi_0(s, M) = \sup_{s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) + t \bar{f}(2^{nm}, M). \]

Consider the first term

\[ \sup_{s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) \geq \sup_{2^{nm} \geq s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) + \sup_{s \geq 2^{nm}} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M). \]

Let \( I \) be an arbitrary cube from \( M \) such that \(|I| \geq 2^{nm}\). Hence, \( I \) is some cube of order \( n \), where \( n \geq m \). Taking into account that each dyadic cube of \( n \) order is partitioned into mutually disjoint cubes of \( m \) order, we obtain

\[ \left| \int_I (f - \varphi_0)(x) dx \right| = \left| \sum_{Q_k^n \subseteq I} \int_{Q_k^n} (f(x) - \varphi_0(x)) dx \right| = 0. \]

Hence,

\[ \sup_{s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) = \sup_{2^{nm} \geq s > 0} s^{\frac{1}{p_0}} (f - \varphi_0)(s, M) \leq \sup_{2^{nm} \geq s > 0} s^{\frac{1}{p_0}} \bar{f}(s, M) \]

\[ + \sup_{2^{nm} \geq s > 0} s^{\frac{1}{p_0}} \varphi_0(s, M) \leq \sup_{2^{nm} \geq s > 0} s^{\frac{1}{p_0}} \int_0^s \frac{1}{p_0} \bar{f}(t, M) \frac{dt}{t} + \sup_{s \geq 2^{nm}} s^{\frac{1}{p_0}} \bar{f}(2^{nm}, M) \]

\[ \leq \sup_{2^{nm} \geq s > 0} \frac{1}{p_0} \int_0^s \frac{1}{p_0} \bar{f}(t, M) \frac{dt}{t} + 2^{nm} \bar{f}(2^{nm}, M) \]

\[ = \frac{1}{p_0} \int_0^{2^{nm}} \frac{1}{p_0} \bar{f}(t, M) \frac{dt}{t} + \frac{1}{p_0} \int_0^{2^{nm}} \frac{1}{p_0} \bar{f}(t, M) \frac{dt}{t} = \frac{2}{p_0} \bar{f}(2^{nm}, M). \]

In this way,

\[ K(a^m, f; N_{p_0, \infty}(M), N_{\infty, \infty}(M)) \leq c \int_0^{2^{nm}} \frac{1}{p_0} \bar{f}(y, M) \frac{dy}{y} + a^m \bar{f}(2^{nm}, M), \]

where \( a = 2^{\frac{m}{p_0}} > 1 \). Then, we have

\[ ||f||_{(N_{p_0, \infty}(M), N_{\infty, \infty}(M))_{\theta, q}} \leq \left( \sum_{m \in \mathbb{Z}} \left( a^{-\theta m} K(a^m, f) \right)^q \right)^{\frac{1}{q}} \]

\[ \leq \left( \sum_{m \in \mathbb{Z}} \left( a^{-\theta m} \left( c \int_0^{2^{nm}} \frac{1}{p_0} \bar{f}(y, M) \frac{dy}{y} + a^m \bar{f}(2^{nm}, M) \right) \right)^q \right)^{\frac{1}{q}}. \]
Applying the Minkowski inequality, we obtain the following
\[
\|f\|_{(N_{p_0,\infty}(M),N_{\infty,\infty}(M))_{\theta,q}} \leq \left( \sum_{m \in \mathbb{Z}} \left( \sum_{k=-\infty}^{m} \frac{1}{p_0} f(2^n, M) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]

Further, taking into account that \(a = 2^{\frac{m}{p_0}}\) and applying Hardy’s inequality for the first term, we obtain
\[
\|f\|_{(N_{p_0,\infty}(M),N_{\infty,\infty}(M))_{\theta,q}} \leq \left( \sum_{m \in \mathbb{Z}} \left( \sum_{k=-\infty}^{m} \frac{1}{p_0} f(2^n, M) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]
\[
\text{where } \frac{1}{p} = \frac{1-\theta}{p_0}, \theta \in (0, 1).
\]

So we got the embedding
\[
N_{p,q} \hookrightarrow (N_{1,\infty}(M),N_{\infty,\infty}(M))_{\theta,q},
\]
where \(\frac{1}{p} = 1 - \theta, \theta \in (0, 1)\). The reverse embedding follows from (1). Hence the relation (7) holds. To prove the general case, we use the reiteration theorem ([4], Theorem 3.5.3).

Let \(1 < p_0 < p_1 < \infty\). From (7) it follows that, there are \(\theta_0, \theta_1 \in (0, 1)\), such that
\[
(N_{1,\infty}(M),N_{\infty,\infty}(M))_{\theta_0,q_0} = N_{p_0,q_0}(M), \quad (N_{1,\infty}(M),N_{\infty,\infty}(M))_{\theta_1,q_1} = N_{p_1,q_1}(M),
\]
then by the reiteration theorem it follows that
\[
(N_{p_0,q_0}(M),N_{p_1,q_1}(M))_{\theta,q} = (N_{1,\infty}(M),N_{\infty,\infty}(M))_{\eta,q} = N_{p,q}(M).
\]
In the last equality, we took into account that \(\eta = (1-\theta)\theta_0 + \theta \theta_1\). \(\square\)

4. PROOF OF THEOREM 2

Proof. Let \(\tau > 0\), Euclidean space \(\mathbb{R}^n = \bigcup_{k=1}^{\infty} I_k\) partitions into non-intersecting half-open cubes \(\{I_k\}_{k=1}^{\infty}\) with faces parallel to the coordinate axes and such that \(|I_k| = \tau\). Let \(f \in G \cap N_{p,q}(M)\), define the function
\[
\varphi_0(x) = \frac{1}{|I_k|} \int_{I_k} f(x) dx, \quad x \in I_k, \quad k \in \mathbb{N}.
\]
Then, it is obvious that
\[
\varphi_0(x) \leq \bar{f}(\tau), \quad x \in \mathbb{R}^n,
\]
and
\[
\int_{I_k} (f(x) - \varphi_0(x)) dx = 0.
\]
Let $0 < \sigma < \min \{ q_0, q_1, q \}$, then
\[
K(t; f; N_{p_0,\sigma}(M), N_{p_1,\sigma}(M)) = \inf_{f = f_0 + f_1} (\|f_0\|_{N_{p_0,\sigma}(M)} + t\|f_1\|_{N_{p_1,\sigma}(M)})
\leq \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \frac{(f - \varphi_0)(s)}{s} \right)^\frac{\sigma}{p} \frac{ds}{s} \right)^\frac{1}{\sigma} + t \left( \int_0^\infty \left( \frac{1}{s^{p_1}} \varphi_0(s) \right)^\frac{\sigma}{p} \frac{ds}{s} \right)^\frac{1}{\sigma}.
\]

Estimate the first term, taking into account the inequality (8), we have
\[
\left( \int_0^\infty \left( \frac{1}{s^{p_0}} \frac{(f - \varphi_0)(s)}{s} \right)^\frac{\sigma}{p} \frac{ds}{s} \right)^\frac{1}{\sigma} \leq \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \frac{f(s)}{s} \right)^\frac{\sigma}{p} \frac{ds}{s} \right)^\frac{1}{\sigma} + \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \varphi_0(s) \right)^\frac{\sigma}{p} \frac{ds}{s} \right)^\frac{1}{\sigma}.
\]

Let $I$ be an arbitrary cube from $M$, $\partial I$ – its boundary, then
\[
\left| \int_I (f - \varphi_0)(x)dx \right| = \left| \sum_{I_k \subset I} \int_{I_k} (f(x) - \varphi_0(x))dx \right| + \sum_{|I_k \cap \partial I| \neq 0} \int_{I_k} (f(x) - \varphi_0(x))dx \right|
\]

Note that the first sum is equal to zero, and the second contains at most $2^n \left( \left( \frac{H}{\tau} \right) \frac{1}{n} + 1 \right)^{n-1}$ terms. Moreover, taking into account the non-negativity of the function $f$, the second sum is estimated as follows
\[
\left| \sum_{|I_k \cap \partial I| \neq 0} \int_{I_k} (f(x) - \varphi_0(x))dx \right| \leq 2 \sum_{|I_k \cap \partial I| \neq 0} \int_{I_k} f(x)dx \leq 2 \sum_{|I_k \cap \partial I| \neq 0} \tau \bar{f}(\tau) \leq 4^n \left( \frac{n+1}{n} \right) \frac{1}{\tau^n} \bar{f}(\tau).
\]

Hence,
\[
\sup_{|I| \geq s} \frac{1}{|I|} \left| \int_I (f - \varphi_0)(x)dx \right| \leq \sup_{|I| \geq s} \frac{1}{|I|} 4^n |I| \left( \frac{n+1}{n} \right) \frac{1}{\tau^n} \bar{f}(\tau) = 4^n \frac{1}{\tau^n} \bar{f}(\tau) \sup_{|I| \geq s} \frac{1}{|I|} \leq 4^n \frac{\tau^n}{s^n} \bar{f}(\tau).
\]
Further, taking into account that $p_0 \geq n$ we have
\[
\left( \int_0^\infty \left( \frac{1}{s^{p_0}} (f - \varphi_0)(s) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} = \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \sup_{|I| \geq s} \left| \int_I (f - \varphi_0)(x) \, dx \right| \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \\
\leq 4^n \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \frac{1}{\sigma} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} = 4^n \int_0^\infty \left( \frac{1}{s^{p_0}} \frac{1}{\sigma} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \\
\times \tau^n \frac{1}{\sigma} \int_0^\infty \left( \frac{1}{s^{p_0}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds = \tau^n \int_0^\infty \left( \frac{1}{s^{p_0}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds.
\]
Hence,
\[
\left( \int_0^\infty \left( \frac{1}{s^{p_0}} (f - \varphi_0)(s) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \leq c \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma},
\]
where $c$ depends only on parameters $p_0, \sigma$. To estimate the second term, we first show that
\[
\varphi_0(s) \leq \begin{cases} \bar{f}(\tau), & \text{at } s \leq \tau, \\ 4\bar{f}(s), & \text{at } s > \tau. \end{cases} \tag{9}
\]
For $s \leq \tau$ from (8) we have $\varphi_0(s) \leq \bar{f}(\tau)$. Let $s > \tau$, $Q \in M$, $|Q| = s$ and by $Q'$ we denote a cube, whose center coincides with the center of $Q$ and has an edge twice as large, then we have
\[
\varphi_0(s) = \sup_{|Q| \geq s} \frac{1}{|Q|} \sum_{I_k \cap Q \neq 0} \int_{I_k} f(x) \, dx \\
\leq \sup_{|Q| \geq s} \frac{1}{|Q|} \sum_{I_k \cap Q \neq 0} \int_{I_k \cap Q} f(x) \, dx = \sup_{|Q| \geq s} \frac{1}{|Q|} \sum_{I_k \cap Q \neq 0} \int_{I_k \cap Q} f(x) \, dx \\
\leq \sup_{|Q| \geq s} \frac{1}{|Q|} \int_{Q'} f(x) \, dx \leq 4\bar{f}(s).
\]
Further, using the relation (9), we obtain
\[
\left( \int_0^\infty \left( \frac{1}{s^{p_1}} \varphi_0(s) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} = \left( \int_0^\infty \left( \frac{1}{s^{p_1}} \varphi_0(s) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \\
+ \left( \int_0^\infty \left( \frac{1}{s^{p_1}} \varphi_0(s) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \leq \tau^n \int_0^\infty \left( \frac{1}{s^{p_1}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} \\
+ 4 \left( \int_0^\infty \left( \frac{1}{s^{p_1}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma} = c\tau^n \int_0^\infty \left( \frac{1}{s^{p_1}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma},
\]
where $c$ depends only on parameters $p_1, \sigma$. Hence,
\[
K(t, f; N_{p_0, \sigma}(M), N_{p_1, \sigma}(M)) \leq c \left( \int_0^\infty \left( \frac{1}{s^{p_0}} \bar{f}(\tau) \right) \frac{\sigma}{s} \, ds \right)^\frac{1}{\sigma}.
\]
Taking into account the monotonicity of \( f(s) \), for \( \tau = t \left( \frac{1}{p_0} - \frac{1}{p_1} \right)^{-1} \) we obtain

\[
\left\| f \right\|_{(N_{p_0, \sigma}(M), N_{p_1, \sigma}(M))_{\theta, q}} \leq \left( \int_0^\infty t^{-\theta} \left( \int_0^\infty \left( \frac{1}{s^{p_0}} f(s) \right)^{\frac{\sigma}{s}} ds \right)^{\frac{1}{\sigma}} dt \right)^{\frac{q}{\theta} \frac{1}{q}} + \left( \int_0^\infty \gamma \left( \gamma^{1-\theta} \left( \frac{1}{s^{p_0}} f(s) \right)^{\frac{\sigma}{s}} ds \right)^{\frac{1}{\sigma}} d\gamma \right)^{\frac{q}{\theta} \frac{1}{q}}.
\]

Making the replacement \( \gamma = t \left( \frac{1}{p_0} - \frac{1}{p_1} \right)^{-1} \) and applying the Minkowski inequality, we arrive at the following

\[
\left\| f \right\|_{(N_{p_0, \sigma}(M), N_{p_1, \sigma}(M))_{\theta, q}} \leq \left( \int_0^\infty \left( \gamma^{-\theta} \left( \frac{1}{s^{p_0}} f(s) \right)^{\frac{\sigma}{s}} ds \right)^{\frac{1}{\sigma}} \frac{d\gamma}{\gamma} \right)^{\frac{q}{\theta} \frac{1}{q}} + \left( \int_0^\infty \left( \gamma \left( \gamma^{1-\theta} \left( \frac{1}{s^{p_0}} f(s) \right)^{\frac{\sigma}{s}} ds \right)^{\frac{1}{\sigma}} \frac{d\gamma}{\gamma} \right)^{\frac{q}{\theta} \frac{1}{q}} \right).
\]

Further, for the first and third terms, we apply the following variants of Hardy inequalities: if \( \mu > 0, -\infty < \nu < \infty \) and \( 0 < \sigma, \tau \leq \infty \), then

\[
\left( \int_0^\infty \left( y^{-\mu} \left( \int_0^y (r^{-\nu}|g(r)|)^{\sigma} \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dy}{y} \right)^{\tau} \right)^{\frac{1}{\tau}} \leq (\mu\sigma)^{\frac{1}{\tau}} \left( \int_0^\infty (y^{-\mu-\nu}|g(y)|)^{\tau} \frac{dy}{y} \right)^{\frac{1}{\tau}}
\]

and

\[
\left( \int_0^\infty \left( y^{\mu} \left( \int_0^y (r^{-\nu}|g(r)|)^{\sigma} \frac{dr}{r} \right)^{\frac{1}{\sigma}} \frac{dy}{y} \right)^{\tau} \right)^{\frac{1}{\tau}} \leq (\mu\sigma)^{\frac{1}{\tau}} \left( \int_0^\infty (y^{\mu-\nu}|g(y)|)^{\tau} \frac{dy}{y} \right)^{\frac{1}{\tau}}.
\]

According to these inequalities, we have

\[
\left\| f \right\|_{(N_{p_0, \sigma}(M), N_{p_1, \sigma}(M))_{\theta, q}} \leq \left( \int_0^\infty \left( \gamma^{-\theta} f(\gamma) \right) q \frac{d\gamma}{\gamma} \right)^{\frac{1}{q} \frac{q}{\theta}} = \left\| f \right\|_{N_{p,q}},
\]

and hence,

\[
\left\| f \right\|_{(N_{p_0, q_0}(M), N_{p_1, q_1}(M))_{\theta, q}} \leq \left\| f \right\|_{(N_{p_0, \sigma}(M), N_{p_1, \sigma}(M))_{\theta, q}} \leq \left\| f \right\|_{N_{p,q}(M)},
\]

where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \). From the theorem in ([12], Theorem 1) we know that the following embedding holds

\[
(N_{p_0, q_0}(M), N_{p_1, q_1}(M))_{\theta, q} \hookrightarrow N_{p,q}(M),
\]

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From the relation (1) we have

\[ \|f\|_{N_{p,q}(M)} \lesssim \|f\|_{(N_{p_0,q_0}(M),N_{p_1,q_1}(M))_{\theta,q}}. \]

This proves the equivalence (4). \( \square \)

5. PROOF OF COROLLARY 1

**Proof.** According to the real interpolation method ([4], Theorem 3.1.2) and the inequalities (5) and (6) it follows

\[ \|Tf\|_{(N_{p_0,\infty}(M),N_{p_1,\infty}(M))_{\theta,\tau}} \leq F_0^{1-\theta} F_1^\theta \|f\|_{(N_{p_0,\sigma}(M),N_{p_1,\sigma}(M))_{\theta,\tau}}. \]

From the relation (1) we have

\[ \|Tf\|_{N_{q,\tau}(M)} \leq c \|Tf\|_{(N_{p_0,\infty}(M),N_{p_1,\infty}(M))_{\theta,\tau}}. \]

From Theorem 2, taking into account, that \( f \geq 0 \), we obtain

\[ \|f\|_{N_{p,\tau}(M)} \asymp \|f\|_{(N_{p_0,\sigma}(M),N_{p_1,\sigma}(M))_{\theta,q}}. \]

\( \square \)

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