On the Uniqueness of the Moyal Structure of Phase-Space Functions

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Abstract:
It is shown that the only associative algebras with a trivial center defined on functions of $\mathbb{R}^N$ by an integral kernel are generalized Moyal algebras, corresponding to some particular operator ordering. Similarly, the only such Lie algebras are generalized Moyal or Poisson Lie algebras. In both cases these structures are isomorphic respectively to the Moyal algebra, the Moyal Lie algebra and the Poisson Lie algebra. These results are independent of $N$, which is necessarily even, and generalize previous work on the subject.

1 Introduction

It is well-known that quantum operators can be mapped to phase-space functions and vice versa, in many different ways, depending on the ordering rule chosen for the operator to which the monomial $q^n p^m$ is mapped (see e.g. [1]). Originally this was done by the Weyl transformation corresponding to a symmetric ordering [2]. The inverse of this mapping, the Wigner transformation, was originally devised in order to formulate quantum expectation values as classical averages on the phase space $\Gamma$ of the system under consideration [3]. Since the work of Moyal [4], who showed the relation between these two mappings, many other ordering rules have been considered, corresponding to some generalization of the Wigner transformation. A large class of such transformations is given by the formula

\footnote{It can be shown that any linear transformation of quantum operators to phase-space functions, which is phase-space translation invariant is the inverse of \cite{1}. The calculations will not be given here, but see \cite{3} eq.(15).}

\[ \int _{\Gamma} \text{d} \Gamma \psi (q, p) = \int _{\mathbb{R}^N} f(x) \text{d} x, \]

\[ \int _{\mathbb{R}^N} f(x) \text{d} x = \int _{\Gamma} \psi (q, p) \text{d} \Gamma. \]
\[ \Omega : A(\vec{q}, \vec{p}) \rightarrow \Omega(A) = \hat{A} := \frac{1}{(2\pi)^n} \int d\sigma \Omega(\sigma) \hat{A}(\sigma) e^{i\sigma \cdot \hat{z}}, \]

(1.1)

where we use the following notation:

- phase-space coordinates: \((\vec{q}, \vec{p}) =: z\)
- corresponding quantum operators: \((\hat{q}, \hat{p} = -i\hbar \frac{\partial}{\partial q}) =: \hat{z}\)
- quantum mechanical operators: \(\hat{A}\)
- phase-space functions: \(A(\vec{q}, \vec{p})\)
- Fourier transform: \(\hat{A}(\vec{\eta}, \vec{\xi}) = \frac{1}{(2\pi)^n} \int d\sigma e^{-i\sigma \cdot \vec{z}} A(z), \quad \sigma = (\vec{\eta}, \vec{\xi}).\)

Here and in what follows, we consider an \(n\)-dimensional configuration space. Expressions like \(\sigma \cdot z\) denote the scalar product in \(\mathbb{R}^{2n}\) and \(d\vec{q}\) etc. is an abbreviation for an \(n\)-dimensional differential.

In (1.1) the kernel \(\Omega(\sigma)\) is an entire analytic function of \(\sigma\). If it has no zeros, then the inverse transformation exists and is given by (1.2)

\[ \Omega^{-1} : \hat{A} \rightarrow A(\vec{q}, \vec{p}) = \frac{1}{(2\pi)^n} \int d\vec{q}'d\vec{p}'d\vec{t} e^{-i\vec{p}' \cdot \vec{t}/\mu} \omega(\vec{q} - \vec{q}', \vec{p} - \vec{p}') \langle \vec{q}' - \vec{t}|\hat{A}|\vec{q}' + \vec{t} \rangle, \]

where

\[ \omega(z) := \frac{1}{(2\pi)^n} \int d\sigma \frac{e^{i\sigma \cdot z}}{\Omega(\sigma)}, \quad \mu := \frac{i\hbar}{2}, \]

(1.3)

and we use Dirac’s notation for the matrix elements of \(\hat{A}\). For \(\Omega = 1\), (1.1), (1.2) reduce to the Weyl and Wigner transformations respectively, hence for \(\Omega \neq 1\) we may call them generalized Weyl and Wigner transformations.

Since the work of Moyal, it is well-known that defining for any phase-space functions \(f, g\), the operation \(*_\Omega\), by

\[ f *_\Omega g := \Omega^{-1}(\Omega(f)\Omega(g)), \]

(1.4)

and

\[ [f, g]_\Omega := \frac{1}{2\mu}(f *_\Omega g - g *_\Omega f), \]

(1.5)

we endow phase-space functions equipped with the ordinary vector space operations, \(F(\Gamma)\), with the structure of an associative, in general non-abelian algebra, eq.(1.4), hence also with a Lie-algebra structure, eq.(1.5). For \(\Omega = 1\) these operations are called the Moyal, product and bracket respectively, denoted by \(*\), \([\, ,\,]\). In the classical limit \(\mu \rightarrow 0\), the latter reduces to the Poisson bracket.
It can be shown from (1.1), (1.2) that ([7], section 3):

\[
(f \ast \Omega g)(z) = \frac{1}{(2\pi)^{2n}} \int d\sigma d\sigma' \tilde{f}(\sigma)\tilde{g}(\sigma') B(\sigma, \sigma') e^{i(\sigma + \sigma') \cdot z}, \tag{1.6}
\]

\[
[f, g]_{\Omega}(z) = \frac{1}{(2\pi)^{2n}} \int d\sigma d\sigma' \tilde{f}(\sigma)\tilde{g}(\sigma') A(\sigma, \sigma') e^{i(\sigma + \sigma') \cdot z}, \tag{1.7}
\]

where

\[
B(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma + \sigma')} e^{\mu \sigma' \wedge \sigma}, \tag{1.8}
\]

\[
A(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma + \sigma')} \sinh(\mu \sigma' \wedge \sigma) \mu, \tag{1.9}
\]

with \(\sigma' \wedge \sigma := J_{ij} \sigma'{}^i \sigma^j\), where \(J_{ij}\) is the canonical symplectic matrix of \(\mathbb{R}^{2n}\), that is \(J = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix}\).

These imply that the mapping \(U\),

\[
f \rightarrow Uf : \quad (Uf)(z) = \int d\sigma \Omega(\sigma) \tilde{f}(\sigma) e^{i\sigma \cdot z}, \tag{1.10}
\]

is an algebra and Lie-algebra isomorphism of \((F(\Gamma), \ast_{\Omega}, [\cdot, \cdot]_{\Omega})\) and \((F(\Gamma), \ast, [\cdot, \cdot])\). From this follows that a necessary and sufficient condition for \([\cdot, \cdot]_{\Omega}\) to reduce to the Poisson bracket, given that \((\hat{q}, \hat{p}) \rightarrow (\hat{q}, \hat{p})\), is that \(\lim_{\mu \to 0} \Omega(\sigma) = 1\) ([7] propositions 3.1, 3.2).

In general, if \(\Omega(\sigma) \rightarrow \Omega_0(\sigma)\) as \(\mu \to 0\) then

\[
\lim_{\mu \to 0} A(\sigma, \sigma') = \frac{\Omega_0(\sigma)\Omega_0(\sigma')}{\Omega_0(\sigma + \sigma')} \sigma' \wedge \sigma, \tag{1.11}
\]

and (1.10) with \(\Omega_0\), replacing \(\Omega\), is a Lie algebra isomorphic with the Poisson Lie algebra. We may call such algebras generalized Poisson Lie algebras, and consider them as singular limits of generalized Moyal algebras in the sense that they cannot be defined through an associative product. We agree to include them in (1.9) for \(\mu = 0\), and for the sake of brevity we will use only the term Moyal Lie algebra.

Therefore, although (1.6), (1.7) define binary operations of functions in classical phase-space that are very useful in various problems of quantum physics, the underlying abstract algebraic structure is independent of \(\Omega\). For a detailed treatment of algebraic and topological questions, as well as applications to physical problems see [3, 4]. Thus the question naturally arises, whether more general binary operations are defined via (1.6), (1.7), which are respectively an associative product and a Lie product and for which the corresponding kernels are not of the form (1.8), (1.9) and if so, to characterize the corresponding algebras.

It is the aim of this paper to study this problem, which is closely related to that considered in [4] as we will show in the next section.
2 The problem

The linear, binary operations on $F(\Gamma)$ defined by (1.6), (1.7), where $B, A$ are not a priori given by (1.8), (1.9), will be denoted by $\star_B, [\cdot, \cdot]_A$ respectively.

Using (1.6), (1.7), a straightforward calculation gives

\[
\int d\sigma d\sigma' e^{i(\sigma + \sigma' + \sigma'')} B(\sigma', \sigma') f(\sigma) = \frac{1}{(2\pi)^3/n} \int d\sigma d\sigma' d\sigma'' e^{i(\sigma + \sigma' + \sigma'')} A(\sigma, \sigma', \sigma'') f(\sigma) g(\sigma') h(\sigma''),
\]

(2.1)

\[
A(\sigma, \sigma') = -A(\sigma', \sigma).
\]

(2.3)

To connect the above results with the previous works (cf. especially (14), (15a) of [7]), Elementary manipulations of these expressions imply that associativity and Jacobi’s identity for (1.6), (1.7) are respectively equivalent to

\[
B(\sigma, \sigma') B(\sigma + \sigma', \sigma'') = B(\sigma, \sigma' + \sigma'') B(\sigma', \sigma''),
\]

(2.1)

\[
A(\sigma, \sigma') A(\sigma', \sigma'') + A(\sigma', \sigma'' + \sigma') A(\sigma'', \sigma) + A(\sigma'', \sigma + \sigma') A(\sigma, \sigma') = 0,
\]

(2.2)

\[
\frac{\partial}{\partial q} f = \frac{\partial f}{\partial q} etc.
\]

Equation (2.4) is identical with the Lie product of a 2-index infinite Lie algebra considered in [10], provided that the summation in $r$ and $s$ starts from $r = s = 1$, which is easily seen to be equivalent to the requirement

\[
[f, 1]_A = 0, \quad \text{for all } f \in F(\Gamma).
\]

(2.6)

Conversely, for any such algebra, (2.3) defines the derivatives at zero of an entire function $A$, hence (2.4) reduces to (1.7), provided of course that

\[
\sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} \sum_{j=0}^{r} \sum_{k=0}^{s} b_{rjsk}(\partial_q^r \partial_p^j f) (\partial_q^k \partial_p^s g) = \frac{1}{(2\pi)^3/n} \int d\sigma d\sigma' e^{i(\sigma + \sigma' + \sigma'')} A(\sigma, \sigma', \sigma'') f(\sigma) g(\sigma') h(\sigma'').
\]

(2.4)

\[
b_{rjsk} = \binom{r}{j} \binom{s}{k} (-i)^{r+s} \partial_q^r \partial_p^j \partial_q^k \partial_p^s A|_{(0,0,0,0)},
\]

(2.5)
converges absolutely.

In view of the above discussion, the result of [10] can be formulated as follows:

Proposition: If $\Gamma$ is 2-dimensional, a nontrivial $[\cdot,\cdot]_A$-Lie algebra is isomorphic either to the Moyal Lie algebra or the Poisson Lie algebra.

The method followed consists essentially in substituting (2.4) in (2.2) and solving the resulting recurrent relations for the coefficients $b_{rj,sk}$. It should be stressed however, that the proof has been explicited only for the case $b_{rj,sk} = \delta_{rs} b_{rjk}$. In addition this method, besides leading to cumbersome calculations, becomes unhelpfully complicated when applied to the case of a multidimensional configuration space. Here we adopt a different approach, namely, we try to solve the functional equations (2.1)–(2.3) without using series expansions.

Specifically, in the next section we show that any $\star_B$-algebra with a trivial center, is a generalized Moyal algebra, and in the last section we show that any $[\cdot,\cdot]_A$-Lie algebra with a trivial center is a generalized Moyal Lie algebra. In all cases (1.10) shows that, up to isomorphism the Moyal and Poisson structures are unique.

3 The characterization of the $\star_B$-algebras

In this section we will show that (2.4) essentially implies (1.8) — the converse is trivial. This will be done in several steps:

(i) Putting $\sigma = \sigma' = 0$ and $\sigma'' = 0$ in (2.1) we get $B(0, \sigma'') = B(0,0)$, $B(\sigma, 0) = B(0, \sigma'')$ so that

$$B(0, \sigma) = B(\sigma, 0) = B(0,0). \quad (3.1)$$

(ii) Making a cyclic permutation of $\sigma, \sigma', \sigma''$ in (2.1) and multiplying the resulting three expressions we get

$$B(\sigma + \sigma', \sigma'')B(\sigma' + \sigma'', \sigma)B(\sigma'' + \sigma, \sigma') = B(\sigma, \sigma' + \sigma'')B(\sigma', \sigma'' + \sigma)B(\sigma'', \sigma + \sigma'). \quad (3.2)$$

(iii) Since $B(\sigma, \sigma')$ is an entire function without zeros, we may write it as (see e.g. [11], theorem 1 in section 1, extended by induction on the number of variables)

$$B(\sigma, \sigma') = e^{b(\sigma,\sigma')} = e^{b_s(\sigma,\sigma')}e^{b_a(\sigma,\sigma')}, \quad (3.3)$$

where $b$ is an entire function and $b_s$, $b_a$ its symmetric and antisymmetric part respectively. Substituting (3.3) in (2.1) we get

$$b_s(\sigma, \sigma') - b_s(\sigma', \sigma'') + b_s(\sigma + \sigma', \sigma'') - b_s(\sigma, \sigma' + \sigma'') = b_a(\sigma', \sigma'') - b_a(\sigma, \sigma') + b_a(\sigma, \sigma' + \sigma'') - b_a(\sigma + \sigma', \sigma'').$$
Since the l.h.s. and the r.h.s. are respectively antisymmetric and symmetric in $\sigma$, $\sigma''$, both are zero,

$$b_a(\sigma, \sigma') - b_\sigma(\sigma', \sigma'') + b_a(\sigma + \sigma', \sigma'') - b_a(\sigma, \sigma' + \sigma'') = 0, \quad (3.4)$$

$$b_\sigma(\sigma, \sigma') - b_\sigma(\sigma', \sigma'') + b_\sigma(\sigma + \sigma', \sigma'') - b_\sigma(\sigma, \sigma' + \sigma'') = 0. \quad (3.5)$$

(iv) Using (3.3) in (3.2) we obtain

$$b_a(\sigma + \sigma', \sigma'') + b_a(\sigma' + \sigma'', \sigma) + b_a(\sigma'' + \sigma, \sigma') = 0. \quad (3.6)$$

Adding (3.4), (3.6) we get

$$b_a(\sigma'', \sigma') = b_a(\sigma', \sigma'') + b_a(\sigma, \sigma'). \quad (3.7)$$

From this and the continuity of $b_a$, using standard arguments we conclude that $b_a$ is a bilinear, antisymmetric function, i.e. it is an exterior 2-form on the even dimensional manifold $\Gamma$. Therefore, if it is nondegenerate then there is a canonical basis such that (cf. (1.8))

$$b_a(\sigma, \sigma') = \mu \sigma' \wedge \sigma, \quad \mu \in \mathbb{F}. \quad (3.8)$$

(v) Acting on (3.5) with $\partial^2_\sigma \partial_{\sigma'} - \partial^2_{\sigma'} \partial_{\sigma}$ we readily obtain

$$(\partial^2_\sigma \partial_{\sigma'} - \partial^2_{\sigma'} \partial_{\sigma}) b_\sigma(\sigma, \sigma') = (\partial^2_\sigma \partial_{\sigma'} - \partial^2_{\sigma'} \partial_{\sigma}) b_\sigma(\sigma, \sigma' + \sigma''). \quad (3.9)$$

Therefore the r.h.s. is independent of $\sigma''$, hence of $\sigma'$ as well. Thus

$$(\partial^2_\sigma \partial_{\sigma'} - \partial^2_{\sigma'} \partial_{\sigma}) b_\sigma(\sigma, \sigma') =: f(\sigma). \quad (3.10)$$

Interchanging $\sigma$, $\sigma'$ we get $f(\sigma) = -f(\sigma')$ by the symmetry of $b_\sigma$, hence $f(\sigma) = f(0) = 0$. Therefore

$$(\partial_{\sigma} - \partial_{\sigma'}) \partial_{\sigma'} \partial_{\sigma} b_\sigma(\sigma, \sigma') = 0,$$

so that

$$b_\sigma(\sigma, \sigma') = \chi(\sigma + \sigma') + g(\sigma) + h(\sigma'). \quad (3.11)$$

By the symmetry of $b_\sigma$, $g(\sigma) = h(\sigma) + c$. However (3.3) with $\sigma' + \sigma'' = 0$ and with the aid of (3.1), (3.3) gives

$$b_\sigma(\sigma + \sigma', -\sigma') + b_\sigma(\sigma, \sigma') = b_\sigma(0, 0) + b_\sigma(\sigma', -\sigma').$$

Substituting (3.11) to this, we get

$$\chi(\sigma) + \chi(\sigma + \sigma') + 2\chi(0) = h(\sigma) + h(\sigma + \sigma') + 2h(0).$$

\[2\text{Here partial derivations denote } 2n\text{-dimensional gradients, i.e. (3.9) is a 3-index tensor relation.}\]
Putting $\sigma' = 0$ we finally obtain $h(\sigma) + h(0) = \chi(\sigma) + \chi(0)$, hence by properly absorbing a constant

$$b_s(\sigma, \sigma') = -\chi(\sigma + \sigma') + \chi(\sigma) + \chi(\sigma').$$

(3.12)

This together with \[3.3\], \[3.8\] shows that $B(\sigma, \sigma')$ is of the form \[1.8\], which was to be proved. When $b_a$ is degenerate, \[3.8\] holds on $\Gamma / \text{Ker} b_a$.

Summarizing our results, we have that there exist, not uniquely determined, coordinates, such that $\sigma = (\rho, \tau)$, with $(\rho, 0) \in \text{Ker} b_a$ and

$$B(\sigma, \sigma') = \Omega(\sigma) \Omega(\sigma') e^{b_a(\sigma, \sigma')} , \quad \Omega(\sigma) = e^{\chi(\sigma)} ,$$

(3.13)

where $b_a(\sigma, \sigma') = \mu \tau' \wedge \tau$ (cf. \[1.8\]). If accordingly, the dual splitting of the phase-space coordinates is $z = (x, y)$, then it is easily seen via the Fourier transformation that functions of $x$ only — in other words functions which satisfy $b^{ij}_a(\partial_i f) = 0$ — belong to the center\footnote{Notice that conversely we have already shown that nondegeneracy of $b_a$ implies that the algebra is a generalized Moyal algebra, hence by \[1.4\] it has a trivial center.} of $(F(\Gamma), \star_B)$. Therefore our results can be restated as

**Theorem 1:** Any $\star_B$-associative algebra, having a trivial center, and for which $B(\sigma, \sigma')$ is an entire analytic function without zeros, is a generalized Moyal algebra \[1.4\], \[1.8\], hence by \[1.10\] it is isomorphic to the Moyal algebra defined by \[1.4\], \[1.8\] with $\Omega = 1$.

We may notice here that the following considerations yield a deeper insight to the $\star_B$ algebras: If $\star$ is any binary operation that makes $F(\Gamma)$ an associative algebra over $\mathbb{C}$, then

$$(f \star g)(z) = \frac{1}{(2\pi)^{2n}} \int d\sigma d\sigma' \hat{f}(\sigma) \hat{g}(\sigma') e^{i\sigma \cdot z} * e^{i\sigma' \cdot z} .$$

A $\star_B$-algebra corresponds to the case

$$e^{i\sigma \cdot z} * e^{i\sigma' \cdot z} = B(\sigma, \sigma') e^{i(\sigma + \sigma') \cdot z} .$$

(3.14)

A necessary and sufficient condition for \[3.14\] is readily seen to be that $\partial / \partial z^i$ are derivations of the $*$-operation as well. By \[3.14\], this operation is somehow related to the additive group structure of $\mathbb{R}^{2n}$. In fact if we consider the extension $E$ of the translation group $(\mathbb{R}^{2n}, +)$ by the multiplicative group $\mathbb{C}^* = \mathbb{C} - \{0\}$, i.e. a short exact sequence

$$1 \to \mathbb{C}^* \to E \to \mathbb{R}^{2n} \to 0 ,$$

then by writing $(\sigma, \zeta)$ for an element of $E$, and defining

$$(\sigma, \zeta)(\sigma', \zeta') := (\sigma + \sigma', B(\sigma, \sigma') \zeta \zeta') ,$$

(3.15)

we can show, that the action of $E$ on $\mathbb{C}^*$ is trivial, i.e. $\sigma \zeta = \zeta$ and that the associativity of \[3.13\] is equivalent to our conditions \[2.1\] (see \[12\] section 6.10, eq.(61)). Thus, determination of $B$, or for that matter, $b$, is equivalent to the determination of all equivalence
classes of extensions of \((\mathbb{R}^{2n}, +)\) by \(\mathcal{C}^*\), which is known to be in 1-1 correspondence with the second cohomology group \(H^2(\mathbb{R}^{2n}, \mathcal{C}^*)\) ([12] theorem 6.15).

Let \(\delta\) be the coboundary operator of the complex \(C(\mathbb{R}^{2n}, \mathcal{C}^*)\), i.e. of the functions from \((\mathbb{R}^{2n})^k\) to \(\mathcal{C}^*\), \(k = 0, 1, 2, \ldots\). Since the action of \(\mathbb{R}^{2n}\) on \(\mathcal{C}^*\) is trivial, we immediately get from the definition of \(\delta\) that for \(b \in \text{Ker} \delta^{(2)}\)
\[
(\delta b)(\sigma, \sigma', \sigma'') = b(\sigma', \sigma'') - b(\sigma + \sigma', \sigma'') + b(\sigma, \sigma + \sigma'' - b(\sigma, \sigma') = 0, \tag{3.16}
\]
which in view of (3.3) is identical to (2.1). On the other hand, for any \(\chi \in C^1(\mathbb{R}^{2n}, \mathcal{C}^*)\) we have
\[
(\delta \chi)(\sigma, \sigma') = \chi(\sigma') - \chi(\sigma + \sigma') + \chi(\sigma). \tag{3.17}
\]
Since \(\delta^2 = 0\), elements \(b\) of \(H^2(\mathbb{R}^{2n}, \mathcal{C}^*)\) are determined up to \(\delta \chi\), a fact equivalent to the isomorphism (1.10) of generalized Moyal algebras.

In the light of the above remarks, our result can be restated by saying that the equivalence classes of \(\star_B\)-algebras (cf. (1.10)) are in 1-1 correspondence with the elements of \(H^2(\mathbb{R}^{2n}, \mathcal{C}^*)\) depending on the rank of \(b_a\) (cf. (3.13)).

As a final remark we notice that for \(b_a\) nondegenerate (cf. (3.8)), the extension (3.15) of \(\mathbb{R}^{2n}\) by \(\mathcal{C}^*\) is the direct product of \(\mathbb{R}^*\) with the Heisenberg group, a fact following from (3.13) in view (3.13) (see [13] section 15, particularly eq.(15.2)).

4 The characterization of the \([,]_A\)-Lie algebras

We next turn to the study of the Lie algebras defined by (1.7), i.e. to the study of (2.2), (2.3), assuming that constants annihilate the Lie product, i.e (2.6) holds. We readily see that this is equivalent to
\[
A(0, \sigma) = 0. \tag{4.1}
\]
(i) Differentiating (2.2) with respect to \(\sigma^i\) and putting \(\sigma = 0\) we get
\[
[\partial^1_i A(0, \sigma' + \sigma'') - \partial^1_i A(0, \sigma') - \partial^1_i A(0, \sigma')] \sigma(\sigma', \sigma'') = 0,
\]
where in this section we write \(\partial^a_i A\) for the the \(i\)-th component of the gradient of \(A\) in the \(a\)-th argument \((a = 1, 2)\). This leads to
\[
\partial^1_i A(0, \sigma' + \sigma'') = \partial^1_i A(0, \sigma') + \partial^1_i A(0, \sigma''), \tag{4.2}
\]

hence to the linearity of \(\partial^1_i A(0, \sigma)\). Accordingly setting
\[
\partial^1_i A(0, \sigma) = \omega_{ij} \sigma^j, \tag{4.3}
\]

\(^4\)Notice that \(A\) cannot be identically zero in an open region of \(\mathbb{R}^{2n} \times \mathbb{R}^{2n}\), by its analyticity (see e.g. [14] theorem 3.13). Hence the zeros of \(A\) are accumulation points of nonzero points, and then (4.2) follows by continuity.
(the summation convention always assumed) and differentiating (2.3) with respect to \( \sigma^i \), \( \sigma'^j \) at \( \sigma = \sigma' = 0 \) we find from (1.3) that

\[
\omega_{ij} + \omega_{ji} = 0 .
\]  

(4.4)

(ii) Differentiating (2.2) with respect to \( \sigma^i \) and \( \sigma^j \) at \( \sigma = 0 \) we obtain after some reductions that

\[
(X_{ij}(\sigma') + X_{ij}(\sigma''))A(\sigma', \sigma'') = (a_{ij}(\sigma') + a_{ij}(\sigma'') - a_{ij}(\sigma' + \sigma''))A(\sigma', \sigma'') ,
\]  

(4.5)

where

\[
a_{ij}(\sigma) := \partial_i^1 \partial_j^1 A(0, \sigma) ,
\]  

(4.5a)

\[
X_{ij}(\sigma) := \sigma^k \omega_{k(i} \partial_{j)} \partial_{\sigma} ,
\]  

(4.5b)

To simplify the notation by supressing indices whenever it is necessary, we introduce symmetric parameters \( \alpha_{ij} \), \( \beta_{ij} \) and set

\[
X_{\alpha} := \frac{1}{2} \alpha_{ij} X_{ij}(\sigma) = \sigma^k \omega_{ki} \alpha_{ij} \frac{\partial}{\partial \sigma^j} ,
\]  

(4.6a)

\[
Z_{\alpha}(\sigma, \sigma') := X_{\alpha}(\sigma) + X_{\alpha}(\sigma') ,
\]  

(4.6b)

\[
a_{\alpha} := \frac{1}{2} \alpha_{ij} a_{ij}(\sigma) ,
\]  

(4.6c)

\[
\hat{a}_{\alpha}(\sigma, \sigma') = a_{\alpha}(\sigma) + a_{\alpha}(\sigma') - a_{\alpha}(\sigma + \sigma') .
\]  

(4.6d)

In this notation, (1.5) takes the equivalent compact form

\[
(Z_{\alpha} A)(\sigma', \sigma'') = (\hat{a}_{\alpha} A)(\sigma', \sigma'') .
\]  

(4.7)

(iii) The crucial next step is to notice that by (1.6a), (1.5b) implies

\[
[X_{\alpha}, X_{\beta}] = X_{\gamma} ,
\]  

(4.8a)

hence

\[
[Z_{\alpha}, Z_{\beta}] = Z_{\gamma} ,
\]  

(4.8b)

with

\[
\gamma^{ij} = \alpha^{ik} \omega_{k\ell} \beta^{ij} - \beta^{ik} \omega_{k\ell} \alpha^{ij} ,
\]  

(4.9)

which is again symmetric and constant. Here \([,]\) denotes the commutator of two operators. Therefore (1.8) says that the \( X_{ij} \) generate a Lie algebra. Moreover, by (1.4)

\[
(X_{ij}(\sigma') + X_{ij}(\sigma''))\omega_{k\ell} \sigma^k \sigma^\ell = 0 .
\]  

(4.10)
Assuming that $\omega_{ij}$ is nondegenerate, (i.e. by (1.4) it is a symplectic form on $\mathbb{R}^{2n}$), (4.10) implies that this Lie algebra is a subalgebra of the $(2n^2 + n)$-dimensional Lie algebra of the symplectic group of $\omega_{ij}$. Since this is exactly the number of the independent generators $X_{ij}$, the latter span the whole symplectic Lie algebra (For general information on the symplectic group see e.g. [15], Part III, ch.16,17).

(iv) Since $Z_\alpha, Z_\beta$ are derivations, applying $[Z_\alpha, Z_\beta]$ to $A(\sigma', \sigma'')$ and using (4.8b), (4.7), we get

$$Z_\alpha \hat{a}_\beta - Z_\beta \hat{a}_\alpha = \hat{a}_\gamma.$$ (4.11)

But, from (4.5b), (4.6b) we have

$$Z_\alpha(\sigma', \sigma'') a_\beta(\sigma') = (X_\alpha a_\beta)(\sigma'),$$
$$Z_\alpha(\sigma', \sigma'') a_\beta(\sigma' + \sigma'') = (X_\alpha a_\beta)(\sigma' + \sigma''),$$

and consequently (4.11) is rewritten as

$$(X_\alpha a_\beta - X_\beta a_\alpha - a_\gamma)(\sigma') + (X_\alpha a_\beta - X_\beta a_\alpha - a_\gamma)(\sigma'') = (X_\alpha a_\beta - X_\beta a_\alpha - a_\gamma)(\sigma' + \sigma'').$$

Therefore

$$(X_\alpha a_\beta - X_\beta a_\alpha - a_\gamma)(\sigma) = \sigma^i c_i(\alpha, \beta).$$ (4.12)

Differentiating (4.12) with respect to $\sigma^i$ at $\sigma = 0$, we get

$$\sigma^i c_i(\alpha, \beta) = (X_\alpha k_\beta - X_\beta k_\alpha - k_\gamma)(\sigma),$$

where

$$k_\alpha(\sigma) := \sigma^i \partial_i a_\alpha(0).$$

Setting

$$\tilde{a}_\alpha(\sigma) := a_\alpha(\sigma) - k_\alpha(\sigma),$$

equation (4.12) becomes

$$X_\alpha \tilde{a}_\beta - X_\beta \tilde{a}_\alpha = \tilde{a}_\gamma.$$ (4.13)

As a consequence of (4.13), (4.8b) the system of first order differential equations (cf. (1.6a))

$$X_{ij} \chi(\sigma) = \tilde{a}_{ij}(\sigma),$$ (4.14)
is locally integrable. Going back to (4.3) and using (4.1) we may rewrite it locally as

\[ [X_{ij}(\sigma') + X_{ij}(\sigma'')] \left( \frac{\Omega(\sigma' + \sigma'')}{\Omega(\sigma')\Omega(\sigma'')} A(\sigma', \sigma'') \right) = 0, \tag{4.15} \]

where \( \Omega(\sigma) := e^{\chi(\sigma)} \). This means that \( \Omega(\sigma' + \sigma'')A(\sigma', \sigma'')/\Omega(\sigma')\Omega(\sigma'') \) is invariant under the symplectic group of \( \omega_{ij} \) (cf. (4.10)), and therefore it is a function of \( \omega(\sigma', \sigma'') \), since this is the only bilinear invariant of the group on \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) (cf. [15], Appendix F). Therefore

\[ A(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma + \sigma')} h(\omega(\sigma, \sigma')), \tag{4.16} \]

for some function \( h \). Evidently \( h \) satisfies the Jacobi condition (2.2). Then by the lemma in the appendix, \( h(x) = c \sinh \mu x \) or \( h(x) = cx \) and therefore locally \( A \) has the form (1.3) or (1.11), in the sense that for any specified point of \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \), \( \chi \) in (4.14) and \( c, \mu \) above are defined in some neighborhood of it.

To get (1.10) globally, we notice that the symplectic group of \( \omega_{ij} \) acts transitively on \( \mathbb{R}^{2n} - \{0\} \); in other words, \( X_{jk}(\sigma) \) as a \( 2n \times (2n^2 + n) \)-matrix has rank \( 2n \) for \( \sigma \neq 0 \), and consequently constants are the only invariants it has. Therefore if \( \chi_a, \chi_b \) are local solutions of (4.14) in some open subsets \( U_a, U_b \) of \( \mathbb{R}^{2n} \), with \( U_a \cap U_b \neq \emptyset \), then by (4.14), in \( U_a \cap U_b \)

\[ X_{ij}(\chi_a - \chi_b) = 0, \]

hence \( \chi_a = \chi_b + \kappa_{ab} \) for some constant \( \kappa_{ab} \). Now since \( \mathbb{R}^{2n} \) is simply connected we can write in a consistent way \( \kappa_{ab} = \kappa_b - \kappa_a \) for all pairs of open sets \( U_a, U_b \) with \( U_a \cap U_b \neq \emptyset \), of an open cover of \( \mathbb{R}^{2n} \) and thus constants are the only invariants it has. Therefore if \( \chi_a, \chi_b \) are local solutions of (4.14), hence \( h \) in (1.10) is globally defined as well.

Evidently from our results above, follows that if \( (F(\Gamma), [\cdot, \cdot]) \) has a nontrivial center, then \( \omega \) is degenerate (cf. last footnote of the previous section). Conversely, suppose \( \omega \) has a \( d \)-dimensional kernel. Then as in the previous section, we may write \( \sigma = (\rho, \tau) \) with \( (\rho, 0) \in \text{Ker} \omega \), and similarly \( z = (x, y) \). Take any function \( f \) of \( x \), then its Fourier transform is \( \tilde{f}(\rho)\delta(\tau) \) and a direct calculation shows that for any \( (\rho_0, 0) \in \text{Ker} \omega \)

\[ F(x) := \tilde{f}(0)(d - i\rho_0 \cdot x) - (\partial_k\tilde{f})(0)\rho_0^k = \left. \frac{\partial}{\partial \rho^k} [\tilde{f}(\rho)(\rho^k - \rho_0^k)e^{ip\cdot x}] \right|_{\rho = 0}, \]

is in the center of the Lie algebra.

Therefore we summarize the results of this section in

**Theorem 2:** A \([\cdot, \cdot]_A\)-Lie algebra, for which \( A \) is an entire analytic function and which has a trivial center, is a generalized Moyal Lie algebra (1.3), (1.11) and is isomorphic to the Moyal Lie algebra, via (1.10).
Appendix

Here we prove the following lemma used in section 4 with the notation introduced there (cf. [5] theorem 5).

**Lemma:** If $A(\sigma, \sigma') = h(\omega(\sigma, \sigma'))$ where $\omega$ is an antisymmetric 2-form, is an entire analytic function and it satisfies (2.2), (2.3), then if $A$ is not identically zero, $h(x)$ is either $c \sinh \mu x$ or $cx$, $\mu, c$ being constants.

**Proof:** Putting $\omega(\sigma, \sigma') = x$, $\omega(\sigma', \sigma'') = y$, $\omega(\sigma'', \sigma) = z$, (2.2) becomes

$$h(x - z)h(y) + h(y - x)h(z) + h(z - y)h(x) = 0 .$$

Differentiating with respect to $z$ at $z = 0$, we find

$$h'(0)h(y - x) = h(y)h'(x) - h'(y)h(x) , \quad (A1)$$

where we used that by (2.3) $h$ is odd, hence $h'$ is even and an accent denotes the derivative of $h$ with respect to its argument. If $h'(0) = 0$ then $h'(x)/h(x) = h'(y)/h(y)$ hence $h(x) = ce^{\lambda x}$ and since $h'(0) = 0$ either $c = 0$ or $\lambda = 0$. In both cases $h(x) = 0$ since $h$ is odd in $x$. Consequently $h'(0) = \lambda \neq 0$. Putting $\tilde{h}(x) := h(x)/\lambda$, (A1) becomes

$$\tilde{h}(x + y) = \tilde{h}(y)\tilde{h}(x) + \tilde{h}'(y)\tilde{h}(x) . \quad (A2)$$

Differentiating (A2) successively with respect to $x$ and $y$ and equating the results, we get

$$\tilde{h}(y)\tilde{h}''(x) = \tilde{h}''(y)\tilde{h}(x) .$$

If $h''(x)$ is not identically zero in any open region, then

$$\frac{\tilde{h}''(x)}{h(x)} = \frac{\tilde{h}''(y)}{h(y)} =: \mu = \text{constant} ,$$

everywhere and the result follows (cf. footnote 4).

If $\tilde{h}''(x) = 0$ in an open region, then by (2.3) $h(x) = cx$ and by the analyticity of $A$ this holds everywhere.

Q.E.D.

**Remark:** From (1.7), we immediately see that $h(x) = x$, or $h(x) = \sinh \mu x$ implies that $[,]_A$ is essentially the Poisson bracket or the Moyal Lie bracket.

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