On the integrability of a new lattice equation found by multiple scale analysis

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Received 13 January 2014, revised 22 May 2014
Accepted for publication 27 May 2014
Published 17 June 2014

Abstract

In this paper we discuss the integrability properties of a nonlinear partial difference equation on the square obtained by the multiple scale integrability test from a class of multilinear dispersive equations defined on a four-point lattice.

Keywords: integrability, partial difference equations, symmetries
PACS numbers: 02.30.Ik, 02.30.Ks

1. Introduction

Discrete equations are important in mathematical physics and play a double role. From one side discrete space-time seems to be basic in the description of fundamental phenomena of nature as provided by quantum gravity [10]. From the other, discrete equations provide good numerical schemes for integrating differential equations [6].

A classification of integrable partial difference equations has been given by Adler, Bobenko and Suris [3] (ABS) in the simple case of equations defined on four lattice points using the consistency around the cube condition plus some discrete symmetry constrains and the tetrahedron property, to be able to get definite results. These results were subsequently generalized by Boll [5] by dropping the symmetry constraints.

We define a difference equation as being integrable if it satisfies one of the following conditions: either if it has at least a generalized symmetry depending on more than one lattice point [19] or if it satisfies the algebraic entropy criterion [30] or if we can construct a non trivial Lax pair [8].

In a recent article Heredero, Levi and Scimiterna [13] used the multiple scale expansion to classify integrable real, multilinear, dispersive equations on the square lattice. The result of this classification gives a set of equations that do not belong to the ABS list. The multiple scale expansion of dispersive nonlinear discrete equations carried out above the nonlinear
Schrödinger regime gives necessary, but not sufficient, conditions for the integrability (see [9] and references therein). A few results in this direction are already known in the literature [2, 11, 12, 18, 29, 31], but the discovery of new examples is always welcomed as it can provide new insight in the classification of integrable equations on the square that do not belong to ABS.

The results presented in [13] deal with a class of real, autonomous difference equations in the variable \( u : \mathbb{Z}^2 \to \mathbb{R} \) defined on a \( \mathbb{Z}^2 \) square-lattice

\[
Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \beta_1, \beta_2, \ldots) = 0,
\]

where the \( \beta_i \)’s are real constant parameters. We require that equation (1.1) be rewritable in the form

\[
u_{n+1,m+1} = f^{(1,1)}(u_{n+1,m}, u_{n,m}, u_{n,m+1}),
\]

where

\[
\left( f^{(1,1)}_{u_{n,m}}, f^{(1,1)}_{u_{n,m}} \right) \neq 0,
\]

and the index \((1,1)\) indicates that the function \( f \) is obtained from (1.1) by solving for the function \( u \) at the point \((n+1, m+1)\). Here and in the following, by the lower index \( u_{i,j} \) we mean the partial derivative of the function with respect to \( u_{i,j} \).

The conditions (1.3) are necessary conditions to prevent triviality of the equation (1.1). To be able to propagate through all the \( \mathbb{Z}^2 \) plane, we suppose, as in [3], that (1.1) is a multilinear equation in all its variables, i.e., a polynomial equation in its variables with at most fourth order

\[
Q_{\text{IV}} = f_0 + a_{00} u_{00} + a_{10} u_{10} + a_{11} u_{11} + (\alpha_1 - \alpha_2) u_{00} u_{10}
\]

\[
+ (\beta_1 - \beta_2) u_{00} u_{01} + d_{10} u_{00} u_{11} + d_{20} u_{01} u_{11} + (\beta_1 + \beta_2) u_{10} u_{11}
\]

\[
+ (\alpha_1 + \alpha_2) u_{00} u_{11} + (\tau_1 - \tau_2) u_{00} u_{10} + (\tau_1 + \tau_2) u_{01} u_{10} u_{11}
\]

\[
+ (\tau_2 + \tau_4) u_{00} u_{01} u_{11} + (\tau_2 - \tau_4) u_{10} u_{01} u_{11} + f_1 u_{00} u_{01} u_{11} = 0,
\]

where all coefficients have been taken to be real and independent of \( n \) and \( m \). Heredero et al [13] considered the multiple scale expansion around the dispersive solution

\[
u_{n,m} = e^{i\omega} e^{-i\omega \eta},
\]

of the linearized equation of (1.4) with a dispersion relation \( \omega = \omega(k) \)

\[
\omega_{\eta}(k) = \arctan \left[ \frac{2a_1 a_2 + (a_1^2 + a_2^2) \cos(k)}{(a_1^2 - a_2^2) \sin(k)} \right]
\]

where \( f_0 = 0, a_{00} = a_{11} = 1, a_{01} = a_{10} = 2, a_1 \) and \( a_2 \) cannot be zero and their ratio cannot be equal to \( \pm 1 \), to get a nontrivial dispersion relation. The effective expansion of the dependent variable is

\[
u_{n,m} = \sum_{\ell=1}^{\infty} \varepsilon^{\ell} \sum_{\alpha = -\ell}^{\ell} K^\alpha \Omega^\alpha m^{(\alpha)}_{\ell},
\]

where \( \varepsilon \in \mathcal{R} \) is a perturbative parameter and \( m^{(\alpha)}_{\ell} = m^{(\alpha)}(n_1, \{m_j\}) \) is a bounded, slowly varying function of its arguments with \( m^{(\alpha)}_{\ell} = \bar{m}^{(\alpha)}_{\ell} \), \( \bar{m}^{(\alpha)}_{\ell} \) being the complex conjugate of \( m^{(\alpha)}_{\ell} \), as we are looking for real solutions. Here \( n_1 = \varepsilon n, m_j = \varepsilon/m, j = 1, 2, \ldots \) are the slow-varying lattice variables and \( K = e^{i\alpha}, \Omega = e^{-i\omega} \).

The integrability conditions presented in [13] were determined by comparing the various equations obtained through a multiple scale perturbative expansion with those of the integrable hierarchy whose first element is the nonlinear Schrödinger equation. The results of [13] are
a series of integrability theorems and a table of equations, invariant under restricted Möbius transformations \(v_0 = v_0/\sqrt{Cv_0 + D}\), that pass the very stringent integrability conditions obtained by considering the multiple scale expansion up to \(\epsilon^3\) order. In this way we have obtained the following equations:

\[
v_{n,m} + v_{n+1,m+1} + 2(v_{n+1,m} + v_{n,m+1}) + v_{n+1,m}v_{n,m+1}(1 + \tau) + (v_{n+1,m}v_{n+1,m+1} + v_{n,m}v_{n,m+1})\tau + v_{n,m+1}v_{n+1,m+1} + v_{n,m}v_{n+1,m} + v_{n+1,m}v_{n,m+1}(v_{n,m} + v_{n+1,m+1})\tau = 0;
\]

\[
v_{n,m} + v_{n+1,m+1} + \epsilon(v_{n+1,m} + v_{n,m+1}) + \delta(v_{n+1,m}v_{n,m+1}) = 0,
\]

\[
-1 < \epsilon < 1, \quad \epsilon \neq 0, \quad \delta = \pm 1, \quad \tau \geq 0;
\]

\[
v_{n,m} + v_{n+1,m+1} + \epsilon(v_{n+1,m} + v_{n,m+1}) + n_{m+1}v_{n,m+1}v_{n+1,m+1} = 0,
\]

\[
-1 < \epsilon < 1, \quad \epsilon \neq 0, \quad \delta = \pm 1, \quad \tau \geq 0;
\]

\[
v_{n,m} + v_{n+1,m+1} + \epsilon(v_{n+1,m} + v_{n,m+1}) + v_{n,m}v_{n+1,m+1} - \epsilon(v_{n+1,m} + v_{n,m+1})v_{n+1,m+1} = 0,
\]

\[
-1 < \epsilon < 1, \quad \epsilon \neq 0, \quad \frac{1}{2}.
\]

The integrability of equations (1.8b), (1.8c) with \(\tau = 0\) and of (1.8d) was proved in [13, 29]. Here we will consider in detail equation (1.8a) which depends on one free parameter \(\tau\). If, when \(\tau = 0\) in (1.8a), we apply the transformation \(v_{n,m} = \sqrt{3}u_{n,m} - 1\), we obtain

\[
u_{n,m}u_{n+1,m} + u_{n+1,m}u_{n,m+1} + u_{n,m+1}u_{n+1,m+1} = 0.
\]

Equation (1.9) admits a point symmetry of generator \(\tilde{\chi} = (-1)^{\mu}w_{n,m}\partial_{w_{n,m}}\). We look for the generalized symmetries of this equation, which demonstrate its integrability. The existence of five-point generalized symmetries is an integrability condition for it.

If, when \(\tau = 1\), we apply to (1.8a) the transformation \(v_{n,m} = -(2^{1/3}u_{n,m} + 1)\), we obtain

\[
u_{n+1,m}u_{n,m+1}(v_{n,m} + u_{n+1,m+1}) + 1 = 0,
\]

an integrable equation considered in [27], which possesses a 3 \(\times\) 3 Lax pair and which is a degeneration of the discrete integrable Tzitzéica equation proposed by Adler in [2].

Finally, if we choose \(\tau \neq 0, 1\) in (1.8a), we can apply the transformation \(v_{n,m} = \frac{1-\tau}{1-\tau}w_{n,m} - 1\) and we obtain

\[
u_{n+1,m}u_{n,m+1} + u_{n+1,m}u_{n,m+1} + u_{n,m+1}u_{n+1,m+1}(1 + u_{n,m} + u_{n+1,m+1}) + \chi = 0,
\]

where \(\chi = \frac{(1-\tau)^2}{(1-\tau)^2}\).

In section 2 we present a review of the essential notions necessary to derive generalized symmetries for equations on a lattice. In section 3 we apply the results of section 2 to (1.9) and then a sequence of results on (1.8a) are obtained. In section 3 we also provide evidence of the integrability of (1.11) by calculating its algebraic entropy and further evidence on the integrability of (1.9) by presenting a Lax pair associated with the equation. Section 4 is devoted to some concluding remarks.
2. Generalized symmetries for quasi-graph equations

It is possible to use the generalized symmetry method as a classification tool on a given class of nonlinear partial differential equations, or to test the integrability of a particular equation [23, 24]. This method has been automatized in the form of a computer package called DELiA, which can be applied to systems of partial differential equations in 1 + 1 dimensions [4]. This symmetry approach has also been adapted to suit discrete equations, both for systems with one discrete and one continuous independent variable [1, 32], and more recently for partial difference equations with two discrete independent variables [19, 21, 22]. For the latter, similar results have been obtained requiring the existence of a recursive operator for the symmetries of the partial difference equation [25–27].

Given equation (1.1), following [19], we use as an integrability criterion the existence of an autonomous generalized symmetry, which, since (1.1) is autonomous, we can write at the point \( n = m = 0 \) without loss of generality by applying translation invariance. Thus we have:

\[
\frac{d}{dt} u_{\ell,0} = g_{0,0} = G(u_{\ell,0}, u_{\ell-1,0}, \ldots, u_{\ell-k,0}, u_{0,0}, u_{0,1}, \ldots, u_{0,k}).
\]  

(2.1)

with \( \ell > 0 \), \( \ell' < \ell \), \( k > 0 \), \( k' < k \) and \( t \) denotes the group parameter. The order of the symmetry is defined by \( \ell \) and \( k \). As the symmetries of an integrable partial difference equation themselves have to be integrable differential difference equations, they too must satisfy the theorem presented in section 2.4.1 of [32, 17], i.e. they must be symmetric, \( \ell' = -\ell \) and \( k' = -k \). If \( \ell' \neq -\ell \) and \( k' \neq -k \) then the equation will be linearizable in general and will not have conservation laws of high order. It is straightforward to check [11, 22, 25] that, if

\[
G = G(u_{\ell,0}, u_{\ell-1,0}, \ldots, u_{\ell-\ell,0}, u_{0,0}, u_{0,1}, \ldots, u_{0,k}).
\]

(2.2)

Equation (2.1) is a generalized symmetry of equation (1.2) if the following compatibility condition is satisfied:

\[
\frac{d(u_{1,1} - f(1,1))}{dt}\bigg|_{u_{1,1}=f(1,1), u_{0,0}=g_{0,0}} = 0.
\]

(2.3)

Explicitly the compatibility condition reads:

\[
[g_{1,1} - (g_{1,0} \delta_{u_{1,0}} + g_{0,0} \delta_{u_{0,0}} + g_{0,1} \delta_{u_{1,0}}) \mathcal{f}(1,1)]|_{u_{1,1}=f(1,1), u_{0,0}=g_{0,0}} = 0,
\]

(2.4)

where \( g_{r,j} = T_1^{r}T_2^{j}g_{0,0} \), and \( T_1, T_2 \) are the shift operators acting on the first and second indexes, respectively, i.e. \( T_1g_{0,0} = g_{1,0} \) and \( T_2g_{0,0} = g_{0,1} \). To be able to check the compatibility condition between (1.1) and (2.1) we need to define the set of independent variables in terms of which (2.3) can be split into an overdetermined system of independent equations. In this article we choose the functions

\[
u_{i,0}, u_{0,j}, \mathcal{V}(i, j)
\]

(2.4)

as independent variables for any fixed \( n \) and \( m \). Then, using (1.2), all the other functions \( u_{i,j} \) can be explicitly written in terms of the independent variables (2.4). So we require that equation (2.3) is satisfied identically for all values of the independent variables. In [19] the following theorem has been proven.

**Theorem 2.1.** If (1.2) possesses a generalized symmetry of the form (2.1), with \( \ell = 1 \) and \( k = 1 \), then its solutions must satisfy the following conservation laws

\[
(T_1 - 1)p_{0,0}^{(k)} = (T_2 - 1)q_{0,0}^{(k)}.
\]

(2.5)
where
if $G_{\kappa,0} \neq 0$, then
\[ p^{(1)}_{0,0} = \log f_{u_{0,1}}^{(1,1)}, \quad q^{(1)}_{0,0} = Q_{0,0}^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}); \] (2.6)
if $G_{u_{-,1}} \neq 0$, then
\[ p^{(2)}_{0,0} = \log \frac{f_{u_{0,1}}^{(1,1)}}{f_{u_{1,0}}^{(1,1)}}, \quad q^{(2)}_{0,0} = Q_{0,0}^{(2)}(u_{2,0}, u_{1,0}, u_{0,0}); \] (2.7)
if $G_{u_{0,1}} \neq 0$, then
\[ q^{(3)}_{0,0} = \log f_{u_{0,1}}^{(1,1)}, \quad p^{(3)}_{0,0} = P_{0,0}^{(3)}(u_{0,2}, u_{0,1}, u_{0,0}); \] (2.8)
if $G_{u_{0,-1}} \neq 0$, then
\[ q^{(4)}_{0,0} = \log \frac{f_{u_{0,1}}^{(1,1)}}{f_{u_{1,-1}}^{(1,1)}}, \quad p^{(4)}_{0,0} = P_{0,0}^{(4)}(u_{0,2}, u_{0,1}, u_{0,0}). \] (2.9)

Theorem 2.1 provides four necessary conditions for the existence of a five-point generalized symmetry, with $\ell$ and $k$ equal unity, which are written in the form of conservation laws which, in the case of a non-degenerate symmetry (2.1), must be all satisfied. This method has been used to confirm the integrability of the equations on the ABS list [3, 16, 19, 22], in addition to some newer examples that fall outside that classification [12, 20].

For the sake of clarity, we repeat here all steps necessary to apply the integrality test in the case of $\ell$ and $k$ equal unity. First we consider the integrability conditions (2.5) with $\kappa = 1, 2$. The unknown functions in the right-hand side of (2.5), given by (2.6), (2.7), contain the dependent variable $u_{2,1}$ which, from (1.2), depends on $u_{2,0}, u_{1,0}, u_{1,1}$ and thus it is not immediately expressed in terms of independent variables, but gives rise to extremely complicated functional expressions of the independent variables. We can avoid this problem by applying the operators $T_{1}^{-1}$ and $T_{2}^{-1}$ to (2.5). In this case (2.5) is replaced by:
\[ p^{(\kappa)}_{0,0} - p^{(\kappa)}_{0,-1} = Q^{(\kappa)}_{1,0}(u_{1,1}, u_{0,1}, u_{-1,1}) - Q^{(\kappa)}_{1,0}(u_{1,0}, u_{0,0}, u_{-1,0}), \] (2.10)
\[ p^{(\kappa)}_{0,-1} - p^{(\kappa)}_{0,-1} = Q^{(\kappa)}_{1,0}(u_{1,0}, u_{0,0}, u_{-1,0}) - Q^{(\kappa)}_{1,0}(u_{1,-1}, u_{0,-1}, u_{-1,-1}). \] (2.11)
Here $p^{(\kappa)}_{0,j}$ are known functions expressed in terms of (1.2). The functions $Q^{(\kappa)}_{1,0}$ are unknown, and $(Q^{(\kappa)}_{1,0}, Q^{(\kappa)}_{1,0})$ contain the dependent variables $u_{1,1}, u_{1,1}, u_{1,-1}, u_{1,-1}$ which are expressed in terms of the independent functions through (1.1). Our aim is to derive from (2.10), (2.11) a set of equations for the unknown function, $Q^{(\kappa)}_{1,0}$.

To do so let us extract from (1.1) three further expressions of the form of (1.2) for the dependent variables contained in (2.10), (2.11):
\[ u_{-1,1} = f^{(-1,1)}(u_{1,0}, u_{0,0}, u_{0,0}), \quad u_{1,-1} = f^{(-1,1)}(u_{1,0}, u_{0,0}, u_{0,0}), \quad u_{-1,-1} = f^{(-1,1)}(u_{1,0}, u_{0,0}, u_{0,0}). \] (2.12)
All functions $f^{(i,j)}$ have a nontrivial dependence on the independent variables, as is the case with $f^{(1,1)}$.

Let us introduce the two differential operators [14, 19, 22, 28]:
\[ A = \partial_{u_{0,0}} \frac{f_{u_{0,1}}^{(1,1)}}{f_{u_{1,0}}^{(1,1)}} \partial_{u_{0,0}} - \frac{f^{(1,1)}_{u_{0,0}}}{f^{(-1,1)}_{u_{1,0}}} \partial_{u_{-1,0}}, \] (2.13)
\[ B = \partial_{u_{0,0}} \frac{f_{u_{0,1}}^{(1,1)}}{f_{u_{1,-1}}^{(1,1)}} \partial_{u_{0,0}} - \frac{f^{(-1,1)}_{u_{0,0}}}{f^{(-1,1)}_{u_{1,0}}} \partial_{u_{-1,0}}, \] (2.14)
chosen in such a way to annihilate the functions \( Q^{(\kappa)}_{-1,1} \) and \( Q^{(\kappa)}_{-1,-1} \), namely \( AQ^{(\kappa)}_{-1,1} = 0, BQ^{(\kappa)}_{-1,-1} = 0 \). Applying \( A \) to (2.10) and \( B \) to (2.11), we obtain two equations for the unknown \( Q^{(\kappa)}_{-1,0} \):

\[
AQ^{(\kappa)}_{-1,0} = r^{(\kappa,1)}, \quad BQ^{(\kappa)}_{-1,0} = r^{(\kappa,2)},
\]

(2.15)

where \( r^{(\kappa,1)}, r^{(\kappa,2)} \) are some explicitly known functions of (1.2). Considering the standard commutator of \( A \) and \( B \), \([A, B] = AB - BA\), we can add a further equation

\[
[A, B]Q^{(\kappa)}_{-1,0} = r^{(\kappa,3)}.
\]

(2.16)

Equations (2.15), (2.16) represent a linear partial differential system of three equations for the three derivatives of the unknown functions \( q^{(\kappa)}_{-1,0} = Q^{(\kappa)}_{-1,0}(u_1, \ldots, u_n) \). For the three partial derivatives of \( q^{(\kappa)}_{-1,0} \), this is just a linear algebraic system of three equations in three unknowns. When this system is non-degenerate it provides one and only one solution for the three derivatives of \( q^{(\kappa)}_{-1,0} \). In these cases we can find the partial derivatives of \( q^{(\kappa)}_{0,0} \) uniquely.

Then we can check the consistency of the partial derivatives and, if satisfied, find \( q^{(\kappa)}_{0,0} \) up to an arbitrary constant. Finally we fix the constant by checking the integrability condition (2.5) with \( \kappa = 1, 2 \) in either of the equivalent forms (2.10) or (2.11).

The non-degeneracy of the system (2.15), (2.16) depends on (1.2) only. So, if we have checked the non-degeneracy for \( k = 1 \), we know that this is also true for \( k = 2 \) and vice versa. So both functions \( q^{(\kappa)}_{n,n}, q^{(\kappa)}_{n,m} \) are found in unique way up to a constant of integration.

If the system (2.15), (2.16) is degenerate, the functions \( q^{(\kappa)}_{n,n}, q^{(\kappa)}_{n,m} \) are defined up to some arbitrary functions. In this case checking the integrability conditions (2.5) may be more difficult.

Let us consider now the conditions (2.5) with \( \kappa = 3, 4 \). We have a similar situation as when \( \kappa = 1, 2 \). By appropriate shifts we rewrite (2.5) in the two equivalent forms:

\[
q^{(\kappa)}_{1,-1} - q^{(\kappa)}_{0,-1} = P^{(\kappa)}_{1,-1}(u_1, \ldots, u_n) - P^{(\kappa)}_{0,-1}(u_0, \ldots, u_{n-1}),
\]

(2.17)

\[
q^{(\kappa)}_{0,-1} - q^{(\kappa)}_{-1,-1} = P^{(\kappa)}_{0,-1}(u_0, \ldots, u_{n-1}) - P^{(\kappa)}_{-1,-1}(u_{-1}, \ldots, u_{n-1}).
\]

(2.18)

We can introduce the operators

\[
\hat{A} = \partial_{u_0} - \frac{\partial_{u_1}}{f^{(1,1)}_{h_{-1}}} \partial_{u_1} - \frac{\partial_{u_{n-1}}}{f^{(n-1,1)}_{h_{-1}}} \partial_{u_{n-1}},
\]

(2.19)

\[
\hat{B} = \partial_{u_0} - \frac{\partial_{u_1}}{f^{(-1,1)}_{h_{-1}}} \partial_{u_1} - \frac{\partial_{u_{n-1}}}{f^{(-n+1,1)}_{h_{-1}}} \partial_{u_{n-1}},
\]

(2.20)

such that \( \hat{A}P^{(\kappa)}_{-1,-1} = 0 \) and \( \hat{B}P^{(\kappa)}_{-1,-1} = 0 \). Then we are led to the system

\[
\hat{A}P^{(\kappa)}_{0,-1} = \hat{r}^{(\kappa,1)}, \quad \hat{B}P^{(\kappa)}_{0,-1} = \hat{r}^{(\kappa,2)}, \quad [\hat{A}, \hat{B}]P^{(\kappa)}_{0,-1} = \hat{r}^{(\kappa,3)}.
\]

(2.21)

for the function \( P^{(\kappa)}_{0,-1} \) depending on \( u_0, \ldots, u_{n-1} \), where \( \hat{r}^{(\kappa,j)} \) are known functions expressed in terms of \( r^{(\kappa,j)} \), whose solution is obtained in the same way as for the system (2.15), (2.16).

After we have solved (2.5)–(2.9) we can construct a generalized symmetry. When both systems (2.15), (2.16) and (2.21) are non-degenerate, we find \( \Phi \) and \( \Psi \), given by (2.2), up to at most four arbitrary constants which are specified using the compatibility condition (2.3).

In the general case when \( k \) and \( \ell \) are greater than one, the situation is more complicated. By requiring the existence of a symmetry of order \( \ell \) and \( k \) greater than one, in [11] one can find the conditions contained in theorem 2 of [19] complemented by further integrability conditions. This set of conditions, contained in proposition 1 of [11], reads as follows using our notation.

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**Proposition 2.1.** If an equation of the form (1.2) admits a higher symmetry of sufficiently great order then the following conditions must hold for some entire \( \ell \) and \( \ell^\prime \):

1. \((T_{\ell}^{-1} - T_{\ell^\prime}^{-1}) \log f^{1,1}_{m_0,0} \in \text{Im}(I - T_2)\) for \( \ell \geq 1 \) and \( \ell^\prime \leq -1 \);
2. \((T_{\ell} - T_{\ell^\prime}^{+1}) \log f^{-1,1}_{m_0,0} \in \text{Im}(I - T_2)\) for \( \ell \geq 1 \) and \( \ell^\prime \leq -1 \);
3. \(R_{0,0}(u_{0,0}, u_{+1,0}, \ldots, u_{\ell^\prime,0}) \in \text{Im}(I - T_2)\) for \( \ell \geq 2 \) and \( \ell^\prime \leq -2 \);
4. \(R_{0,0}(u_{0,1}, u_{+1,1}, \ldots, u_{\ell^\prime,1}) \in \text{Im}(I - T_2)\) for \( \ell \geq 2 \) and \( \ell^\prime \leq -2 \), where \( I \) is the identity operator.

Conditions (1) and (2) are already contained in [19] while (3) and (4) are higher order and

\[
R_{0,0} = \frac{1}{T_1^{-\ell-2}(f_{m_0,0}^{1,1})} \cdot \left( T_1^{-1} \left\{ f_{m_0,0}^{1,1} T_2(g_{m_0,0}) T_{1}^{-\ell-1}(f_{m_0,0}^{1,1}) + f_{m_0,0}^{1,1} g_{m_0,0} \right\} - T_1 T_2(g_{m_0,0}) T_{1}^{-\ell-1}(f_{m_0,0}^{1,1}) \right).
\]

\[
r = \frac{T_1^{-1}(f_{m_0,0}^{1,1})}{T_1^{-\ell-2}(f_{m_0,0}^{1,1})}, \quad \tilde{r} = \frac{T_1(f_{m_0,0}^{1,1})}{T_1^{\ell+2}(f_{m_0,0}^{1,1})}.
\]

\[
\tilde{R}_{0,0} = \frac{1}{T_1^{\ell+2}(f_{m_0,0}^{1,1})} \cdot \left( T_1 \left\{ f_{m_0,1}^{-1,1} T_2(g_{m_0,1}) T_{1}^{\ell+1}(f_{m_0,1}^{-1,1}) + f_{m_0,1}^{-1,1} g_{m_0,1} \right\} - T_1 T_2(g_{m_0,1}) T_{1}^{\ell+1}(f_{m_0,1}^{-1,1}) \right).
\]

Conditions (1)–(4) of proposition 2.1 provide a sequence of integrability conditions for a given nonlinear partial difference equation in the form of functional equations. These integrability conditions have also been obtained, with a different approach, in [25]. Extending the approach used above, which reduced the functional equation to a system of differential equations for the unknown symmetry function \( \Phi \), we can derive differential consequences of these equations in the form of a system of first order partial differential equations, which can be solved on characteristics. Let us remember, moreover, that if the starting equation is to be integrable then \( \ell^\prime = -\ell \).

So the algorithm for obtaining the symmetries is similar to the one introduced by Levi and Yamilov for the case of three–point symmetries in the \( n \) and \( m \) directions, but the operators \( A \) and \( B \) are replaced by the characteristic vector fields \( Y_k = T_2^{-k} \frac{\partial}{\partial u_{-k}} T_2^{k}, Y_{-k} = T_2^{k} \frac{\partial}{\partial u_{-k}} T_2^{-k}, k \in \mathbb{Z}^+ \), introduced in [11] and references therein. \( Y_1 \) and \( Y_{-1} \) read:

\[
Y_1 = \frac{\partial}{\partial u_{0,0}} + x_{0,0} \frac{\partial}{\partial u_{1,0}} + \frac{1}{x_{1,0}} \frac{\partial}{\partial u_{-1,0}} + x_{0,0} x_{1,0} \frac{\partial}{\partial u_{2,0}} + \frac{1}{x_{-1,0} x_{-2,0}} \frac{\partial}{\partial u_{-2,0}} + \cdots, \quad \text{(2.22)}
\]

\[
Y_{-1} = \frac{\partial}{\partial u_{0,0}} + y_{0,0} \frac{\partial}{\partial u_{1,0}} + \frac{1}{y_{1,0}} \frac{\partial}{\partial u_{-1,0}} + y_{0,0} y_{1,0} \frac{\partial}{\partial u_{2,0}} + \frac{1}{y_{-1,0} y_{-2,0}} \frac{\partial}{\partial u_{-2,0}} + \cdots, \quad \text{(2.23)}
\]

where

\[
x_{0,0} = T_2^{-1} \left( \frac{\partial f^{1,1}(u_{0,0}, u_{1,0}, u_{0,0})}{\partial u_{0,1}} \right) = \frac{\partial f^{1,1}(u_{0,0}, u_{1,0}, u_{0,0})}{\partial u_{0,1}} \bigg|_{u_{0,1}}
\]

and

\[
y_{0,0} = T_2 \left( \frac{\partial f^{-1,1}(u_{0,0}, u_{1,0}, u_{-1,0})}{\partial u_{0,-1}} \right) = \frac{\partial f^{-1,1}(u_{0,0}, u_{1,0}, u_{-1,0})}{\partial u_{0,-1}} \bigg|_{u_{0,-1}}.
\]
The characteristic vector fields, \( Y_1 \) and \( Y_{-1} \), reduce to \( A \) and \( B \) when acting on a function that depends only on \( u_{0,m}, u_{0\pm 1,m} \) and \( u_{0-1,m} \).

If the result of the test, as stated above for a five–point symmetry, with three points in the \( n \) direction and three in the \( m \) direction, is negative, we have to proceed tentatively. We must look for higher order symmetries following proposition 1, i.e. look for a five-point symmetry in the \( n \) direction or a five-point symmetry in the \( m \) direction using the operators \( Y_1, Y_{-1} \), its differential consequences obtained by applying its commutators and, if necessary, its higher order expression \( Y_2 \) and \( Y_{-2} \), etc.

3. Integrability properties of (1.9) and (1.11)

In this section we use the results presented in the previous section to find the generalized symmetries for (1.9), thus proving its integrability. We then study its transformation properties, the relation to other known integrable equations on the square and its relation to (1.11).

Equation (1.9) satisfies the two necessary conditions in the \( n \)-direction given in [22] but does not admit a three-point generalized symmetry, either autonomous or not, while the necessary conditions in the \( m \)-direction given in [22] are not satisfied. Equation (1.11) is the sum of (1.9), (1.10) and an arbitrary constant and does not satisfy the integrability conditions given in [22] for three-point generalized symmetries either autonomous or not, in either the \( n \) or \( m \)-direction.

To find the generalized symmetries for (1.9) we need to consider higher order symmetries. Applying the theory sketched in the previous section we find, after a long and tedious but straightforward calculation, that (1.9) has two five–point symmetries in the \( n \) and \( m \) direction given by:

\[
\begin{align*}
\mathcal{G}_{0,0} \epsilon &= w_{0,0}(w_{-1,0}w_{0,0} - 1)(w_{0,0}w_{1,0} - 1)(w_{2,0}w_{1,0} - w_{-1,0}w_{-2,0}), \\
\mathcal{G}_{0,0} \check{\epsilon} &= \frac{w_{0,0}(w_{0,-1} + w_{0,0})(w_{0,0} + w_{0,1})(w_{0,2} + w_{0,1} - w_{0,-1} - w_{0,-2})}{(w_{0,-2} + w_{0,-1} + w_{0,0})(w_{0,-1} + w_{0,0} + w_{0,1})(w_{0,0} + w_{0,1} + w_{0,2})},
\end{align*}
\]

where \( \epsilon \) and \( \check{\epsilon} \) are group parameters. We notice that (1.9) and both its generalized symmetries are invariant under the discrete transformation \( \tilde{w}_{0,0} = -w_{0,0} \). While these two generalized symmetries are not compatible in general, that is for a generic evolution of \( w_{0,0} \) in \( n, m \), they do become compatible under the evolution (1.9) and its difference consequences. The first symmetry (3.1a) is polynomial and of the form of a Bogoyavlenskyi lattice but of higher polynomial order. The second symmetry (3.1b) is given by a rational expression where the sum of the order of the numerator and the denominator is equal to the polynomial order of (3.1a).

Analyzing the structure of (3.1a) we easily see that we can reduce its polynomial order by introducing a new dependent variable \( t_{0,0} \), written in the term of the variable \( w_{0,0} \):

\[
t_{0,0} = w_{0,0}w_{1,0} - 1.
\]

The relation between \( t \) and \( w \) corresponds to a potentiation as, by going over to the variables \( \tau_{n,m} = -\log(t_{n,m}) + 1 \) and \( \nu_{n,m} = (-1)^n \log w_{n,m} \), (3.2) can be rewritten as:

\[
\tau_{n,m} = \nu_{n+1,m} - \nu_{n,m}.
\]

We thus can easily invert (3.2) and we obtain

\[
w_{n,m} = \left\{ v_{0} e^{-\sum_{i=1}^{n+1} \log \tau_{i,i+1}} \right\}^{(-1)^n}.
\]

By applying this transformation to the generalized symmetry (3.1a) and (3.1b) we find the five-point generalized symmetries of (1.9) in the \( n \) and \( m \) directions, respectively, written in the variable \( t \):

\[
t_{0,0,\epsilon} = t_{0,0}(t_{0,0} + 1)(t_{2,0}t_{1,0} - t_{-1,0}t_{-2,0}).
\]
Applying the change of variable (3.3a) turns out to be a Bogoyavlensky lattice. Analyzing the structure of (3.1b) we see that we can also transform it to a Bogoyavlensky lattice. In fact, defining

\[
\theta_{n,m} = \frac{w_{n,m+1}}{w_{n,m}},
\]

(3.4)

\[
u_{n,m} = 1 + \theta_{n,m}(\theta_{n,m+1} + 1),
\]

(3.5)

\[
u_{n,m} = \frac{1}{\nu_{n,m}}.
\]

(3.6)

i.e. by a potential transformation (3.4), a Miura transformation (3.5) and a point transformation (3.6), we get the transformation

\[
\tilde{t}_{n,m} = -\frac{w_{n,m+2} + w_{n,m+1}}{w_{n,m} + w_{n,m+2} + w_{n,m+1}},
\]

(3.7)

which transforms (3.1b) into the following Bogoyavlensky lattice equation

\[
\tilde{t}_{0,0} = \tilde{t}_{0,0} (\tilde{t}_{0,0} + 1)(\tilde{t}_{0,0} - \tilde{t}_{0,-1}\tilde{t}_{0,-2}).
\]

(3.8)

Applying the change of variable (3.2) we can transform (1.9) to an integrable equation in the variable \(t_{0,0}\), given in the following theorem.

**Theorem 3.1.** If (1.9) is satisfied, using the definition (3.2), we have:

\[
\begin{align*}
w_{1,0} &= \frac{t_{0,0} + 1}{w_{0,0}}, \\
w_{0,1} &= -\frac{t_{0,1} + t_{0,0} + 1}{t_{0,0} + 1} w_{0,0}, \\
(t_{1,1} + t_{1,0} + 1)(t_{0,1} + t_{0,0} + 1) - (t_{1,0} + 1)(t_{0,1} + 1) &= 0.
\end{align*}
\]

(3.9)

**Proof.** To prove (3.9a) solve (3.2) for \(w_{1,0}\); to prove (3.9b) it is sufficient to substitute (3.9a) and the same equation shifted once in the \(m\) direction, into (1.9) and solve for \(w_{0,1}\); to prove (3.9c) just substitute the expressions for \(t_{0,0}\) and all its shifts as given by (3.2) into its left-hand side, then impose (1.9) and its shift obtained by shifting once in the \(n\) direction.

Theorem 3.1 shows that, through the potentiation (3.2), we can transform (1.9) into (3.9c), which is an inhomogeneous polynomial equation on the square. Equation (3.9b) is also a potentiation relation, however in the \(m\) direction, which can be solved for either \(w\) or \(t\) in terms of a summation of a function of the other variable.

Thus the two relations (3.9a), (3.9b) constitute a potential transformation between (1.9) and (3.9c); the compatibility between the \(w\)-variables implies equation (3.9c) while the compatibility between the \(t\)-variables implies (1.9). Equation (3.9c) and its generalized symmetries (3.3a), (3.3b) have been introduced for the first time in [27].

Applying the change of variable (3.7), we can transform (1.9) to an integrable equation in the variable \(\tilde{t}_{n,m}\), as stated in the following theorem.
Theorem 3.2. If equation (1.9) is satisfied, given (3.7), then
\[ u_{0,1} = -\tilde{t}_{1,0} + \tilde{t}_{0,1} + 1 \frac{w_{0,0}}{\tilde{t}_{0,1} + 1}. \]  
\[ (3.10a) \]
\[ u_{0,2} = \tilde{t}_{1,0} + 1 \frac{w_{0,0}}{\tilde{t}_{0,1} + 1}. \]  
\[ (3.10b) \]
\[ u_{1,0} = \left[ \tilde{t}_{0,0} + 1 \right] \left[ \tilde{t}_{0,1} + 1 \right] \frac{(\tilde{t}_{0,1} + 1)(\tilde{t}_{1,0} + 1)}{(1 + \tilde{t}_{0,1} + 1)(\tilde{t}_{0,0} + 1)} w_{0,0}. \]  
\[ (3.10c) \]
\[ (\tilde{t}_{1,1} + \tilde{t}_{0,1} + 1)(\tilde{t}_{1,0} + \tilde{t}_{0,0} + 1) - (\tilde{t}_{1,0} + 1)(\tilde{t}_{0,1} + 1) = 0. \]  
\[ (3.10d) \]

Proof. Equation (3.10a) is obtained by solving (3.7) for \( w_{0,2} \) and substituting the obtained expression, as well as its shift in the \( n \) direction, into (1.9) shifted once in the \( m \) direction, then solving the resulting expression for \( w_{1,1} \). Then we substitute the resulting expression into (1.9) and solve for \( w_{0,1} \).

Equation (3.10b) is obtained by substituting (3.10a) into the expression found previously for \( w_{0,2} \).

Equation (3.10c) is obtained by substituting (3.10a) into the expression found previously for \( w_{1,1} \) and solving the compatibility condition between this expression and (3.10a) for \( w_{1,0} \).

The proof that \( \tilde{t}_{0,0} \) satisfies (3.10d) is obtained by substituting the expressions for \( \tilde{t}_{0,0} \) given by (3.7), and any required shifts, into the left-hand side of (3.10d), then imposing (1.9) as well as its first and second shifts in the \( m \) direction.

The three relations (3.10a)–(3.10c) constitute a Miura transformation between (1.9) and (3.10d). The compatibility between the \( v \)–variables implies (3.10d) while the compatibility between the \( t \)–variables implies (1.9).

From (3.9c) we can infer the integrability of two non-trivial, non-autonomous extensions of (1.9). The first extension is provided by the following theorem.

Theorem 3.3. If (3.9c) is satisfied, given (3.9a) as a definition for the variable \( w_{0,0} \), then \( w_{0,0} \) satisfies the non autonomous equation
\[ u_{0,0}^2 u_{1,0} + u_{1,0} w_{0,0}^2 \left( 1 - \frac{2}{\gamma_m(-1)^n + 1} \right) + w_{0,0} w_{1,1} - 1 = 0, \]  
\[ (3.11) \]
where \( \gamma_m \) is an arbitrary function of \( m \). The solution \( \tilde{w}_{0,0} \) of (3.11) and the solution \( w_{0,0} \) of (1.9) are related by the following point wise transformation:
\[ \tilde{w}_{0,0} = \kappa_m(-1)^{n+1} w_{0,0}, \quad \kappa_m \neq 0, \]  
\[ (3.12) \]
\[ \kappa_{m+1} = \frac{\gamma_m + 1}{\gamma_m - 1} \kappa_m, \]  
\[ (3.13) \]
where \( \kappa_m \) is an arbitrary function of \( m \), solution of (3.13).

Proof. Substitution of (3.9a) and its shifts into (3.9c), gives the equation
\[ \alpha_{0,0} + \alpha_{1,0} - \alpha_{0,0} \alpha_{1,0} = 0, \quad \alpha_{0,0} = \frac{\xi_{0,0}}{w_{1,0} w_{0,1}}, \]  
\[ (3.14) \]
which, if \( \alpha_{0,0} = 0 \), implies \( \xi_{0,0} = w_{0,0} w_{1,0} + w_{1,0} w_{0,1} + w_{0,1} w_{1,1} - 1 = 0 \), which is nothing but equation (1.9). If \( \alpha_{0,0} \neq 0 \) and we introduce the variable \( z_{0,0} = 1/\alpha_{0,0} \), (3.14) linearizes to
\[ z_{1,0} + z_{0,0} - 1 = 0, \] which gives (3.11) when integrated. One could prove that the converse is also true by straightforward calculation.

As relation (3.9a) is covariant under (3.12), given (3.9a), (3.9c), we can always fix a gauge for \( w_{0,0} \) so that (1.9) is satisfied. \( \square \)

The second extension is provided by the following theorem.

**Theorem 3.4.** If (3.9c) is satisfied, then the variable \( w_{0,0} \) defined by (3.9b) satisfies the non-autonomous equation

\[
w_{0,0} w_{1,0} + w_{1,0} w_{0,1} + w_{0,1} w_{1,1} - (1 + \beta_n) = 0, \quad \beta_n \neq -1, \tag{3.15}
\]

where \( \beta_n \) is an arbitrary function of \( n \).

The solution \( \tilde{w}_{0,0} \) of (3.15) and the solution \( w_{0,0} \) of (1.9) are related by the following pointwise transformation:

\[
\tilde{w}_{0,0} = \epsilon_n^{-1} w_{0,0}, \quad \epsilon_n \neq 0, \quad \beta_n = \epsilon_{n+1} \epsilon_n - 1, \tag{3.16}
\]

where \( \epsilon_n \) is an arbitrary function of \( n \).

**Proof.** Inserting into (3.9c) the expression for \( t_{0,1} \) derived from (3.9b) and its shift in the \( n \) direction, we obtain

\[
t_{0,0} = -\frac{\theta_{0,0}(\theta_{1,0} + 1)}{\theta_{0,0}(\theta_{1,0} + 1) + 1}, \quad \theta_{0,0} = \frac{w_{0,1}}{w_{1,0}}, \tag{3.17}
\]

where \( \theta_{0,0}(\theta_{1,0} + 1) + 1 \neq 0 \) because \( \mathcal{E}_{0,0} \neq -1 \). The compatibility between (3.17) and (3.9b) gives

\[
1 + \theta_{0,0} - \theta_{0,0} \theta_{1,0} (1 + \theta_{1,1}) = 0 \Rightarrow \mathcal{E}_{0,0} = \mathcal{E}_{0,1}, \tag{3.18}
\]

which, when integrated, gives equation (3.15) (the condition \( \beta_n \neq -1 \) follows directly from \( \mathcal{E}_{0,0} \neq -1 \)).

As relation (3.9b) is covariant under (3.16), given (3.9b), (3.9c), we can always fix a gauge for \( w_{0,0} \) so that (1.9) is satisfied. \( \square \)

In [27] the authors present the nonlinear partial difference equation on the square

\[
(u_{0,0} + u_{1,1}) u_{1,0} u_{0,1} + 1 = 0 \tag{3.19}
\]

whose lowest generalized symmetry involves just five points in each direction and is given by the Bogoyavlenskiy lattice (3.3a) in the variable \( t_{0,0} \), defined by

\[
t_{0,0} = \frac{1}{u_{-1,0} u_{0,0} u_{1,0} - 1}. \tag{3.20}
\]

This leads to the following theorem [27] which we present here for completeness.

**Theorem 3.5.** If (3.19) is satisfied, given (3.20), then

\[
u_{1,0} = \frac{t_{0,0} + 1}{t_{0,0} u_{1,0} - u_{0,0}}, \tag{3.21a}
\]

\[
u_{0,1} = \frac{t_{0,1} + t_{0,0} + 1}{t_{0,0} + 1 - u_{-1,0}}, \tag{3.21b}
\]

\[
u_{-1,1} = \frac{t_{0,0} + 1}{t_{0,1} u_{-1,0} - u_{0,0}}, \tag{3.21c}
\]

and \( t_{0,0} \) satisfies (3.9c).
Proof. Equation (3.21a) follows directly from (3.20).
To prove (3.21b) one has to: a) substitute (3.21a) and its $m$-shift into (3.19) and solve for $u_{-1,1}$; b) substitute the resulting expression into (3.19) back-shifted once in the $n$-direction.
To prove (3.21c) just substitute (3.21b) into the previous expression for $u_{-1,1}$.

The three relations (3.21a)–(3.21c) constitute a potential transformation between (3.9c) (for $t_{0,0}$) and (3.19): the compatibility between the $t$–variables implies (3.19) while the compatibility between the $u$–variables implies (3.9c) (for $t_{0,0}$).

When $t_{0,0} \neq 0$, we can define the variable

$$z_{0,0} = \frac{1}{t_{0,0}},$$

and we have the following theorem which is proved by straightforward calculation.

**Theorem 3.6.** If (3.9c) is satisfied, given (3.22), then

$$z_{0,0}z_{1,0} + z_{0,1}z_{1,1} + z_{0,0}z_{1,1}(z_{1,1} + z_{0,0} + 1) = 0.$$  

Equation (3.23) is (1.11) when $\chi = 0$. The calculation of the symmetries of (1.11) when $\chi \neq 0$ turn out to be so large that our computer runs out of memory even though it should be strictly related by the transformation (3.22) to the Bogoyavlensky lattice, the symmetry of (3.9c). So to show its integrability we rely on the calculation of its algebraic entropy.

### 3.1. Algebraic entropy

Equation (1.11) is an equation for the dependent variable $w_{n,m}$ that depends on two discrete independent variables, $n$ and $m$, its shifts and on an arbitrary parameter $\chi$. In the previous subsection we have shown by constructing its generalized symmetries the integrability of this equation for the case where the parameter is set to $\chi = 0$, but it is shown here also to occur in the general case for arbitrary $\chi$ as the entropy for this equation is zero and the growth of its degrees is quadratic.

To be self-contained, we include the following brief description of algebraic entropy based on [30]. Finding the algebraic entropy of an equation requires us to begin with an initial condition and then use the equation to find values of the field over the other points on the lattice. The initial condition is usually chosen to be a sequence of arbitrary values along a staircase of points through the lattice. Because of the form of the equation, the field values will always be rational functions of the initial conditions. The degree at any point, denoted $d^{(\ell)}$, is the greatest polynomial degree of either the numerator or denominator of the field value, after any possible cancellations have been performed. Only one discrete index, $\ell$, is needed, this represents the distance of the point in question from the staircase of initial conditions. The algebraic entropy is then given by:

$$\epsilon = \lim_{\ell \to \infty} \frac{1}{\ell} \log d^{(\ell)}.$$ 

There are four fundamental directions of evolution related to the direction chosen for the staircase of initial conditions (see section 4 of [30]). Thus there are four sequences of degrees for an equation on a lattice in general, and in this case, i.e. for (1.11), we find the following pairs of degree sequences:

$$d_{-+}^{(\ell)} = d_{+}^{(\ell)} = \{1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, \ldots\}.$$
and
\[ d_{+}^{(l)} = \{ 1, 2, 5, 10, 16, 24, 35, 46, 59, 76, 92, 110, 133, 154, 177, 206 \ldots \}. \]

where the signs in the subscript indicate the direction of the associated staircase of initial conditions.

Although it is usually impossible to calculate an infinite sequence of degrees and use the definition of algebraic entropy directly, if a generating function can be found for the sequence of degrees, we can use it to calculate the entropy. Obtaining generating functions for the above sequences, we find
\[ g_{-}(t) = g_{-}(t) = \frac{1 + t^2}{(1 - t)^3}, \]
and
\[ g_{+}(t) = g_{-}(t) = \frac{1 + t + 3t^2 + 3t^3 + 4t^4 + 2t^5 + 2t^6}{(1 - t)^3(1 + t + t^2)^2}. \]

Generating functions are closely related to the \( Z \)-transform, which is a discrete Laplace transform. The variable \( t \), in \( g(t) \), is the transform variable. The entropy is given by the logarithm of the inverse of the modulus of the smallest pole of the generating function. This shows that the entropy vanishes, the growth of degrees is quadratic in each direction and the equation (1.11) should be integrable. We remark finally that the integrability of equations (1.9)–(1.11) implies the integrability of equation (1.8a).

### 3.2. The Lax pair representation.

Let us consider the Lax pair representation of (3.19) \[27\] given by

\[
\begin{align*}
\Psi_{1,0} & = M_{0,0} \Psi_{0,0}, & \Psi_{0,1} & = N_{0,0} \Psi_{0,0}, \\
M_{0,0} & = \begin{pmatrix} 0 & 1 & 0 \\ -u_{0,0} & -u_{0,0} & \lambda \\ -1 & 0 & \frac{1}{u_{0,0}} \end{pmatrix}, & N_{0,0} & = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & \frac{1}{u_{0,0}} \\ \frac{1}{u_{0,0}} & -1 & 0 \end{pmatrix},
\end{align*}
\]

and perform the gauge transformation

\[
\begin{align*}
\Psi_{0,0} & = V_{0,0} \Phi_{1,0}, & V_{0,0} & = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{0,0} & 0 \\ 0 & 0 & u_{0,0} \end{pmatrix},
\end{align*}
\]

so that

\[
\begin{align*}
\Phi_{1,0} & = \frac{1}{u_{0,0}} \tilde{M}_{0,0} \Phi_{0,0}, & \Phi_{0,1} & = u_{-1,0} \tilde{N}_{0,0} \Phi_{0,0}.
\end{align*}
\]

\[
\tilde{M}_{0,0} = u_{0,0} V_{0,0}^{-1} \cdot M_{-1,0} \cdot V_{-1,0}, & \tilde{N}_{0,0} = \frac{1}{u_{-1,0}} V_{-1,0}^{-1} \cdot N_{-1,0} \cdot V_{-1,0}.
\]

Using (3.9c), (3.21a)–(3.21c) (the last three written for \( t_{0,0} \)) and their shifts, we get

\[
\begin{align*}
\tilde{M}_{0,0} & = \begin{pmatrix} 0 & 1 + \frac{1}{t_{-1,0}} & 0 \\ -1 & -1 - \frac{1}{t_{-1,0}} & \lambda \\ -1 & 0 & 1 \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\tilde{N}_{0,0} & = \begin{pmatrix} 1 + \frac{u_{-1,0}(t_{-1,0} + 1)}{t_{-1,0}(t_{-1,0} + 1)} & 0 & 0 \\ \frac{u_{-1,0}(t_{-1,0} + 1)}{t_{-1,0}(u_{-1,0} + 1)} & 0 & -1 - \frac{u_{-1,0}(t_{-1,0} + 1)}{t_{-1,0}(u_{-1,0} + 1)} \end{pmatrix}.
\end{align*}
\]
The compatibility condition between (3.25) reads
\[ M_{0,1} \tilde{N}_{0,0} = \frac{u_{0,1}}{u_{-1,0}} \tilde{N}_{1,0} \tilde{M}_{0,0}, \] (3.27)
so that, removing \( u_{0,0} \) from (3.27) and using (3.21b), we have that (3.25), (3.26a), (3.26b) is a nonlocal Lax pair representation of equation (3.9c). Alternatively from (3.25) we have
\[ \Phi_{3,0} = Y_{0,0} \Phi_{0,0}, \quad \Phi_{0,1} = u_{-1,0} \tilde{N}_{0,0} \Phi_{0,0}, \quad Y_{0,0} = \frac{1}{u_{2,0} u_{1,0} u_{0,0}} \tilde{M}_{2,0} \tilde{M}_{1,0} \tilde{M}_{0,0}, \] (3.28)
whose compatibility gives
\[ Y_{0,1} \tilde{N}_{0,0} = \frac{u_{2,0}}{u_{-1,0}} \tilde{N}_{3,0} Y_{0,0}. \] (3.29)
Using (3.21a), which implies that \( u_{2,0} - \frac{n_{0} n_{0}^2 + 1}{n_{0} n_{0}^2 + 1} u_{-1,0} = 0, \) to remove \( u_{0,0} \) from the compatibility condition above, we have that (3.28), (3.26a), (3.26b) is a (local in the \( m \)-direction) Lax pair representation of (3.9c). Finally from (3.28) we have
\[ \Phi_{3,0} = Y_{0,0} \Phi_{0,0}, \quad \Phi_{0,3} = X_{0,0} \Phi_{0,0}, \quad X_{0,0} = u_{-1,2} u_{-1,1} u_{-1,0} \tilde{N}_{0,2} \tilde{N}_{0,1} \tilde{N}_{0,0} \] (3.30)
Using (3.21b) and (3.21c), which imply \( u_{-1,2} u_{-1,1} u_{-1,0} = \frac{(n_{0} + 1)(n_{0} + 1)}{n_{0} n_{0}^2 + 1}, \) we have that (3.30), (3.26a), (3.26b) is a local Lax pair representation of equation (3.9c). Using (1.9), (3.9a), (3.9b) and their shifts, we obtain the corresponding Lax pair representations of equation (1.9) while, using (3.22), we obtain the corresponding Lax pair representations of equation (3.23).

4. Conclusions

In this article we have shown that the equations (1.8a), (1.9) and (1.11), obtained from the multiple scale expansion of a quite general class of dispersive multilinear equations on a four-point lattice at the order \( \epsilon^8 \), are integrable. As far as we know, equations (1.8a), (1.9) and (1.11) are new. In the case of (1.9) we do so by calculating its generalized symmetries and by finding its Lax pair. In doing so we are able to show that (1.9) is related respectively by potentiation and Miura transformation to the integrable equations (3.9c), and (3.19) introduced recently by Mikhailov and Xenitidis in [27]. Moreover, from the connection between (1.9) and (3.9c), we are able to show that two non autonomous extensions of (1.9), (3.11) and (3.15) are also integrable.

When we have the Lax pair of a nonlinear equation we can always construct algorithmically both the Bäcklund transformations and the recursive operator, using, for example, the Lax technique [7, 15]. The approach to integrability contained in [25–27] is based on the existence of a formal recursion operator for the symmetries of the difference equation.

The proof of the existence of generalized symmetries for the equation (1.11) as well as the derivation of the recursion operator for (1.9) and for its symmetries is left for future work as its integrability is shown only by algebraic entropy. We also leave for future work the intergrability of the equations (1.8b), (1.8c) when \( \tau \neq 0 \).

Acknowledgments

DL and CS have been partly supported by the Italian Ministry of Education and Research, 2010 PRIN Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps. We thank RI Yamilov and J Hietarinta for many enlightening discussions. Moreover we thank the referees for their useful comments and for pointing out to us the printed version of [27].
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