A simple test for the stability of a black hole by S-deformation

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Abstract

We study a sufficient condition for proving the stability of a black hole when the master equation for linear perturbation takes the form of the Schrödinger equation. If the potential contains a small negative region, the S-deformation method is usually used to show the non-existence of an unstable mode. However, in some cases, it is hard to find an appropriate deformation function analytically because the only way found so far to find it is by trial-and-error. In this paper, we show that it is easy to find a regular deformation function by numerically solving the differential equation such that the deformed potential vanishes everywhere, when the spacetime is stable. Even if the spacetime is almost marginally stable, our method still works. We also discuss a simple toy model which can be solved analytically, and show that the condition for the non-existence of a bound state is the same as that for the existence of a regular solution for the differential equation in our method. From these results, we conjecture that our criteria is also a necessary condition.

Keywords: black hole, stability analysis, S-deformation

(Some figures may appear in colour only in the online journal)

1. Introduction

It is known that the equations for gravitational perturbation around a highly symmetric black hole spacetime usually reduce to decoupled master equations [1–15] in the form

\[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x)\hat{\Phi} = 0.\] (1)

Using the Fourier transformation with respect to the time coordinate, \(\hat{\Phi}(t, x) = e^{-i\omega t}\Phi(x)\), the master equation takes the form of the Schrödinger equation.
\[ -\left[ -\frac{d^2}{dx^2} + V \right] \Phi = \omega^2 \Phi. \quad (2) \]

When we wish to prove the stability of the black hole spacetime, we need to show the non-existence of an \( \omega^2 < 0 \) solution, i.e. non-existence of an exponentially growing solution, under the boundary conditions\(^1\), \( \Phi \to 0, \frac{d\Phi}{dx} \to 0 \) at \( x \to \pm \infty \), and the conditions: \( \Phi \) and \( \frac{d\Phi}{dx} \) are continuous and bounded everywhere. From equation (2), we obtain

\[ - \lim_{x \to \mp \infty} \Phi \frac{d\Phi}{dx} + \int dx \left[ \frac{d\Phi}{dx}^2 + V |\Phi|^2 \right] = \omega^2 \int dx |\Phi|^2, \quad (3) \]

where \( \Phi^* \) is the complex conjugate of \( \Phi \). The first term in the lhs vanishes from the above boundary conditions. If the effective potential \( V \) is non-negative everywhere, this implies \( \omega^2 \geq 0 \).

In some cases, even if the effective potential contains a small negative region, the spacetime can still be shown to be stable against linear perturbation in the following way. From equation (2), we can also show

\[ - \frac{d}{dx} \left[ \Phi \frac{d\Phi}{dx} + S|\Phi|^2 \right] + \frac{d\Phi}{dx}^2 + S \Phi^* \Phi + \left( V + \frac{dS}{dx} - S^2 \right) |\Phi|^2 = \omega^2 |\Phi|^2, \quad (4) \]

where \( S \) is an arbitrary function of \( x \). If we impose continuity on \( S \) everywhere, then the integral

\[ \int dx \frac{d}{dx} (S|\Phi|^2), \quad (5) \]

becomes a surface term. Integrating equation (4), we obtain

\[ - \lim_{x \to \mp \infty} \Phi \frac{d\Phi}{dx} + \int dx \left[ \frac{d\Phi}{dx}^2 + S \Phi^* \Phi + \left( V + \frac{dS}{dx} - S^2 \right) |\Phi|^2 \right] = \omega^2 \int dx |\Phi|^2, \quad (6) \]

Also, we assume that \( S \) is not divergent at \( x \to \pm \infty \) so that the above surface term vanishes only from the boundary condition for \( \Phi \). The potential term is deformed as

\[ \tilde{V} := V + \frac{dS}{dx} - S^2. \quad (7) \]

This is called the \( S \)-deformation of the potential \( V \). If the deformed effective potential \( \tilde{V} \) is non-negative everywhere by choosing an appropriate function \( S \), we can also say \( \omega^2 \geq 0 \). In [5–10, 13, 16, 17], the stability of spacetime was shown analytically by using this \( S \)-deformation method. However, the only way known so far to find an appropriate function \( S \) analytically is via a trial-and-error approach. When it is hard to find an appropriate \( S \) deformation, a numerical study is needed. In [18–23], stability and instability were investigated by numerically solving the two-dimensional partial differential equation (1) in time domain.

In this paper, we propose a simple way to show the stability by showing the existence of the \( S \)-deformation such that the deformed potential \( \tilde{V} \) vanishes. For this purpose, we need to solve the equation

\(^1\) In this paper, since we mainly focus on asymptotically flat (or de Sitter) black holes, we assume that the range of \( x \), which is the tortoise coordinate, is \(-\infty < x < \infty \). Here, \( x \to -\infty \) and \( x \to \infty \) correspond to the black hole horizon and the spatial infinity (or cosmological horizon), respectively. If we consider asymptotically anti-de Sitter black holes, the range of \( x \) becomes \(-\infty < x < x_{\text{max}} \) with the finite value \( x_{\text{max}} \).
\[ V + \frac{dS}{dx} - S^2 = 0. \] (8)

The approximate solution near \( x \to \pm \infty \) is \( S \simeq -1/(c_\pm - x) \) with constants \( c_\pm \) if the potential rapidly decays to zero. Since equation (8) is a first-order ordinary differential equation, all solutions near \( x \to \pm \infty \) should behave like this approximate solution. To obtain a solution with finite \( S \) at \( x \to \pm \infty \), we need to find a boundary condition so that \( S \) is positive near \( x \to -\infty \) and \( S \) is negative near \( x \to \infty \), then \( S \) behaves \(-1/x\) and finite at \( x \to \pm \infty \). While \( S \) diverges at some point for an inappropriate boundary condition, we can find the continuous range of appropriate boundary conditions, which corresponds to bounded \( S \), for typical examples.

One may think that solving the equation \( V + \frac{dS}{dx} - S^2 = W \) for a positive function \( W(>0) \) is easier than solving equation (9). However, since this equation can be written in the form \((V - W) + \frac{dS}{dx} - S^2 = 0\), the problem is to find a solution of equation (9) for a deeper potential \( V - W \), i.e. a more dangerous case. This suggests that solving equation (9) is the most efficient way to find an appropriate function \( S \).

This paper is organized as follows. In the next section, we discuss the existence of the solution of equation (8) in the case of positive potential and that in a simple toy model which can be solved analytically. We also discuss the relation between the solution of equation (8) in a marginally stable case and the solution of equation (2) with \( \omega^2 = 0 \). In section 3, we numerically solve equation (8) for higher-dimensional spherically symmetric black holes, and construct appropriate deformation functions from various boundary conditions. Section 4 is devoted to the summary and discussion.

2. Existence of the solution

2.1. Local existence

Let us consider the differential equation

\[ V(x) + \frac{dS(x)}{dx} - S(x)^2 = 0. \] (9)

We assume that \( V \) is continuous and bounded in \(-\infty < x < \infty\). From the uniqueness theorem of the ordinary differential equation, there exists a solution of the above equation locally at least.

We can see this by considering the Taylor expansion if \( S \) and \( V \) are analytic functions. The series around some point \( x = x_0 \) are

\[ S = \sum_{n=0}^{\infty} s_n(x - x_0)^n, \quad V = \sum_{n=0}^{\infty} v_n(x - x_0)^n. \] (10)

From the differential equation, we obtain the coefficients of \( S \) as

\[ s_{n+1} = \frac{1}{n+1} \left[ -v_n + \sum_{m=0}^{n} s_m s_{n-m} \right], \] (11)

where \( s_0 \) is an arbitrary constant, which corresponds to the integration constant (or boundary condition). This shows that we can find a solution \( S \) locally.

In general, while \( dS/dx - S^2 \) is finite, \( S \) might be divergent at a finite coordinate value \( x \). The problem is finding a function \( S \) which is continuous and bounded everywhere. In some cases, we can show the existence of such a regular function \( S \).
2.2. Positive potential

First, we consider that the potential is positive and bounded above in \(-\infty < x < \infty\). While this corresponds to a manifestly stable case, showing the existence of a continuous and bounded solution of equation (9) is not trivial. We would like to show the following proposition.

**Proposition 1.** If the potential \(V\) is positive and bounded above in \(-\infty < x < \infty\), there exists a continuous and bounded solution of equation (9) in \(-\infty < x < \infty\).

The proof is given in appendix A. We can understand this proposition as follows. Since we have \(dS/dx = (S - \sqrt{V})(S + \sqrt{V})\), the relation among \(\pm \sqrt{V}\), the value of \(S\) and the sign of \(dS/dx\) becomes like figure 1. If we choose the value of \(S\) in \(-\sqrt{V} < S < \sqrt{V}\) at some point \(x = x_0\) as a boundary condition, we can say that the solution of equation (9) satisfies \(-\sqrt{V}_{\text{max}} < S < \sqrt{V}_{\text{max}}\). Thus, \(S\) is bounded above and below in the region \(-\infty < x < \infty\).

2.3. Toy model

To obtain a qualitative understanding, we consider a toy model of the potential, which can be solved analytically,

\[
V = \begin{cases} 
0 & (-\infty < x \leq x_1) \\
-h_1^2 & (x_1 < x \leq x_2) \\
h_2^2 & (x_2 < x \leq x_3) \\
0 & (x_3 < x < \infty) 
\end{cases}
\]  

with constants \(h_1 > 0, h_2 > 0\). The local solutions of equation (9) are\(^2\)

\[
S = \begin{cases} 
\frac{1}{c_1 - \alpha}, & (-\infty < x \leq x_1) \\
h_1 \tan(h_1 x + c_2), & (x_1 < x \leq x_2) \\
-h_2 \tanh(h_2 x + c_3), & (x_2 < x \leq x_3) \\
\frac{1}{c_4 - \alpha}, & (x_3 < x < \infty) 
\end{cases}
\]

\(^2\)Defining \(\alpha := e^{2\beta}\), then \(S\) in \(x_2 < x \leq x_3\) becomes \(S = -h_2((-\alpha + e^{2\beta})/(\alpha + e^{2\beta}))\). We can see that the \(\alpha < 0\) case corresponds to a complex value of \(c_3\), and \(S\) might be divergent in that case. The solutions with \(\alpha < 0\) satisfy \(S^2 > h_2^2\). To construct continuous \(S\) in this section, we need to consider \(S^2 < h_2^2\) (i.e. real \(c_3\)) so that \(S\) can take both negative and positive values in \(x_2 < x \leq x_3\).
where $c_1, c_2, c_3, c_4$ are integration constants. From the continuity of $S$ at $x = x_1, x_2, x_3$, we obtain the conditions

$$\frac{1}{c_1 - x_1} - h_1 \tan(c_2 + h_1 x_1) = 0, \quad (14)$$

$$h_1 \tan(c_2 + h_1 x_2) + h_2 \tanh(c_3 + h_2 x_2) = 0, \quad (15)$$

$$\frac{1}{-c_4 + x_3} - h_2 \tanh(c_3 + h_2 x_3) = 0. \quad (16)$$

From the conditions $S|_{x=x_1} \geq 0, S|_{x=x_2} > 0, S|_{x=x_3} \leq 0$ and the finiteness of $S$, we obtain the inequalities

$$0 < c_1 - x_1, \quad 0 \leq h_1 x_1 + c_2 < \frac{\pi}{2}, \quad (17)$$

$$0 < h_1 x_2 + c_2 < \frac{\pi}{2}, \quad h_2 x_2 + c_3 < 0, \quad (18)$$

$$0 \leq h_2 x_3 + c_3, \quad c_4 - x_3 < 0. \quad (19)$$

If the above equations and inequalities are satisfied, then $S$ is continuous and bounded everywhere. We plot the typical profile of $V$ and $S$ in figure 2. Note that it is not necessary for the derivative of $S$ to be continuous, because we only impose the condition such that equation (5) becomes a surface term.

We would like to derive an inequality between the areas of the potential in negative and positive regions. From the above matching conditions and inequalities, we can show\(^3\)

$$h_1 \tan(h_1 (x_2 - x_1)) \leq h_1 \tan(c_2 + h_1 x_1 + h_1 (x_2 - x_1)) \quad (\because 0 \leq c_2 + h_1 x_1)$$

$$= h_1 \tan(c_2 + h_1 x_3)$$

$$= -h_2 \tanh(c_3 + h_2 x_2) \quad (\because \text{equation (15)})$$

$$= h_2 \tanh(-c_3 - h_2 x_2)$$

$$\leq h_2 \tanh(h_2 x_3 - h_2 x_2) \quad (\because 0 \leq h_2 x_3 + c_3)$$

\(^3\)The derivation is as follows:
\[ h_1 \tan(h_1(x_2 - x_1)) \leq h_2 \tanh(h_2(x_3 - x_2)). \] (20)

Since we have \( X < \tan X \) for \( 0 < X < \pi/2 \), and \( \tanh Y < Y \) for \( 0 < Y \), we obtain an inequality
\[ h_1^2(x_2 - x_1) < h_2^2(x_3 - x_2). \] (21)

The area of the negative region is smaller than that of the positive region. This is consistent with the result in [24] where the existence of an unstable mode is shown if \( \int_{-\infty}^{\infty} dx V < 0 \).

### 2.4. Relation between existence of \( S \) and non-existence of the bound state in the toy model

To check the relation between the existence of regular \( S \) and the non-existence of the bound state, we further study the toy model in the previous section and the dependence of their existence on the ratio of the areas
\[ \Gamma = \frac{h_2^2(x_3 - x_2)}{h_1^2(x_2 - x_1)}. \] (22)

First, for simplicity, we consider the case
\[ x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad h_1 = 1. \] (23)

If \( \Gamma = 0 \), this is just a single-well problem. In that case, there is only a bound state in the solution of the Schrödinger equation whose energy is \( \omega^2_{\Gamma=0} \simeq -0.43 \). After some calculation, we obtain the condition for the existence of the bound state, which can be calculated from \( \omega^2 < 0 \), as
\[ \Gamma < \Gamma_{cr} \] (24)

where \( \Gamma_{cr} \simeq 2.79 \) is defined by
\[ \tan(1) = \sqrt{\Gamma_{cr}} \tanh(\sqrt{\Gamma_{cr}}). \] (25)

Also, we can calculate the condition for the existence of continuous and bounded solution \( S \) of equation (9). That condition, which is derived from \( S|_{x=x_1} \geq 0 \) with \( S|_{x=x_3} = 0 \), becomes \( \Gamma \geq \Gamma_{cr} \) with the same critical value in equation (25).

For general parameters, we can derive the same relations. The critical value \( \Gamma_{cr} \) in the general case is defined by
\[ \frac{\sqrt{x_3-x_2}}{\sqrt{x_2-x_1}} \tan(h_1(x_2 - x_1)) = \sqrt{\Gamma_{cr}} \tanh(h_1 \sqrt{\Gamma_{cr} \sqrt{x_2-x_1} \sqrt{x_3-x_2}}). \] (26)

Note that if \( h_1(x_2 - x_1) \geq \pi/2 \), there exists at least one bound state regardless of the value of \( h_2 \), in that case, \( \Gamma_{cr} \) is not defined. Thus, in this toy model the condition for the non-existence of the bound state coincides with that for the existence of a continuous and bounded solution \( S \).

### 2.5. Relation between regular \( S \) in a marginally stable case and the onset of an unstable mode

Since equation (9) is the Riccati equation, defining
\[ \frac{1}{\phi} \frac{d\phi}{dx} := -S, \] (27)

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we can write equation (9) in a second-order linear ordinary differential equation

$$-\frac{d^2\phi}{dx^2} + V\phi = 0.$$  \hspace{1cm} (28)$$

This is the master equation (2) with $\omega^2 = 0$. If this equation has a non-trivial regular solution, it probably corresponds to the onset of an unstable mode. We can expect that there is some relation between the solutions of equation (9) in the marginally stable case and the onset of an unstable mode. In this section, we briefly discuss this.

Suppose that the potential contains a parameter $\alpha$, and the master equation has a property such that there is no unstable mode if $\alpha \leq \alpha_{cr}$ and there are unstable modes if $\alpha > \alpha_{cr}$. In the case of $\alpha > \alpha_{cr}$, we define $\Phi_0(x; \omega(\alpha))$ as the ground state of the Schrödinger equation (2), then the energy takes a negative value $\omega^2 < 0$. If we assume that the potential rapidly decays in $x \to \pm \infty$, then $\Phi_0 \sim e^{\pm \sqrt{-\omega^2} x}$ at $x \to \pm \infty$. In that case, $\Phi_0^{-1}d\Phi_0/dx$ takes finite values at the boundary $x \to \pm \infty$. Since it is known that the ground state for the one-dimensional Schrödinger equation has no nodes (for example, see [25, 26]), $\Phi_0$ cannot become zero except at the boundary. So, we can define a function

$$S := -\frac{1}{\Phi_0(x; \omega(\alpha))} \frac{d\Phi_0(x; \omega(\alpha))}{dx},$$

and this is regular in $-\infty < x < \infty$. We can show that $S$ satisfies

$$V + \frac{dS}{dx} - S^2 = \omega^2.$$  \hspace{1cm} (30)$$

If we assume that the function $S$ has a smooth limit in $\alpha \to \alpha_{cr} + 0$, this becomes a regular solution of equation (9) since $\omega^2 \to 0$ in this limit.

In fact, the toy model in the previous sections satisfies the above assumptions. This is the reason why the critical values are the same in the toy model. In general, the validity of the above assumption is not clear, but it seems to be reasonable.

3. Numerical calculation

If the effective potential $V$ rapidly decays as $x \to \pm \infty$, the approximate solution of equation (9) becomes $-1/(c_\pm - x)$ with constants $c_\pm$. To obtain a solution with finite $S$ at $x \to \pm \infty$, we need to find a boundary condition so that $S$ is positive near $x \to -\infty$ and $S$ is negative near $x \to \infty$. Then, $S$ behaves as $-1/x$ and is finite at $x \to \pm \infty$. A bounded $S$ usually changes the value from positive to negative at some point in the $V > 0$ region so that $S$ takes a negative value near $x \to \infty$ like $S$ in figure 2. We can expect that the appropriate boundary conditions can be found by setting $S = 0$ at a point where $V$ is positive. In this section, we study the stability of ten-dimensional spherically symmetric black holes and the five-dimensional Schwarzschild black string, as typical examples. The explicit form of the metrics and the master equations, which was derived in [5–7, 21, 28, 29], is given in the appendices B and C.

In the following examples, we used the function NDSolve in Mathematica to solve the equation numerically. In a stable case, for an appropriate boundary condition, we could obtain a bounded function $S$ without using a special technique, as far as we confirmed. We estimated

5 We should note that we cannot construct a regular solution of equation (28) from a regular solution of equation (9) except for the marginally stable case. This is because the asymptotic behavior of the regular solution of equation (9) is $S \sim -1/x$, but the corresponding $\phi$ behaves $\phi \sim cx$ with a constant $c$, which is not a regular solution of equation (28).
the numerical error by \( \epsilon := (V + dS/dx - S^2)/(|V| + |dS/dx| + |S^2|) \) and confirmed \( \epsilon < 10^{-6} \) in the following calculations.

### 3.1. Ten-dimensional Schwarzschild black hole

Let us consider the \( \ell = 2 \) vector perturbation in the ten-dimensional Schwarzschild black hole. In fact, in this case, the existence of the \( S \)-deformation such that the deformed potential is positive is already known [6], but this is still a good example for checking that our method works well. In figure 3, we plot the effective potential, which contains a small negative region near the horizon, and the numerical solution of equation (9) for various boundary conditions. Note that we use the radial coordinate \( r \) in the black hole spacetime. The relation between \( r \) and \( x \) is in appendix B. If we plot \(-1/x\) as a function of \( r \), it seems to rapidly decrease to zero from a finite value near the horizon since \(-1/x \sim -1/\ln(r - r_H)\). There are two attractors of solutions in the \( V > 0 \) region, and they are almost \( \pm \sqrt{V} \). This is because \( dS/dx = (S + \sqrt{V})(S - \sqrt{V}) \) in the \( V > 0 \) region (see also figure 1). We can see that the bounded solutions can be found by choosing the boundary condition as \( S = 0 \) at the points where \( V > 0 \) (solid blue lines in figure 3). If we choose the boundary condition of \( S \) to be larger than \( \sqrt{V} \) in \( V > 0 \) region, then \( S \) is divergent at some point (dashed red lines in figure 3).

Unlike the toy model in section 2.3, the potential is negative near the horizon. Since the potential behaves as \( V \sim -a^2(r - r_H) \sim -a^2 e^{b^2 x} \) with constants \( a > 0, b > 0 \) near the horizon, it rapidly decays in \( x \) coordinate. In that case, if the \( S \) takes a positive value near the horizon, \( S \) will be bounded above and below as \( x \) decreases (see appendix D).

### 3.2. Ten-dimensional Schwarzschild–de Sitter black hole

As another example, we consider the \( \ell = 2 \) scalar perturbation in the ten-dimensional Schwarzschild–de Sitter black hole. In this case, the existence of the \( S \)-deformation such that the deformed potential is positive is not known, but there is a numerical proof of stability based on the quasi normal mode [18]. In figure 4, we plot the effective potential and the numerical solution of equation (9) for various boundary conditions when the cosmological constant is \( r_H^2 \lambda = 0.05 \). The effective potential also contains a small negative region near the
horizon. We can see that the solutions are bounded above and below if we set the boundary condition as $S = 0$ at the points where $V > 0$ (blue lines in figure 3). Since our method is different from the previous work [18], this is a complementary result which also supports the stability of the spacetime.

3.3. Ten-dimensional Reissner–Nordström–de Sitter black hole

We consider $\ell = 2$ scalar gravitational perturbation in the ten-dimensional Reissner–Nordström–de Sitter black hole. In this case, it is known that there exists an unstable mode for large values of electric charge and cosmological constant [22, 23, 27]. According to figure 4 in [22], for the parameter $r_H = 1, r_{ds} = 5/4$, if the electric charge is larger than the critical value around $Q/Q_{\text{extremal}} \approx 0.75$, the spacetime is unstable. In figure 5, we plot the effective potential and the numerical solution of equation (9) for $r_H = 1, r_{ds} = 5/4, Q/Q_{\text{extremal}} = 0.74$. We can find a continuous and bounded $S$ which is positive at $r \approx r_H$ and negative at $r \approx r_{ds}$ by setting the boundary condition $S|_{r=r_{ds}^{-1}}=0$. We can see that our method still works even in the almost marginally stable case.
Finally, we discuss the case of five-dimensional Schwarzschild black string spacetime. This spacetime has an unstable mode (Gregory–Laflamme instability) for a long wave perturbation $k < k_c \simeq 0.876$ [28]. The master equation for the scalar perturbation also takes the form of a Schrödinger equation (2) as shown in [21, 29]. We should note that the effective potential equation (C.2) is positive in $r > r_H$ if $k > (1 + \sqrt{3})/\sqrt{2} \simeq 1.93$. In figure 6, we plot the effective potential and the numerical solution of equation (9) for $r_H k = 0.877$ with the boundary condition $S|_{r=5r_H} = 0$. In this case, the value of the potential asymptotes to a positive constant at large distance. From the same discussion in the proof of the proposition 1 (see appendix A), it is guaranteed that the solution is continuous and bounded in $r > r_{\text{ini}} = 5r_H$. Also, our method works for the almost marginally stable mode.

4. Summary and discussion

We have studied a sufficient condition to prove the stability of a black hole when the linear perturbative equation takes the Schrödinger form by showing the $S$-deformation such that the deformed potential vanishes everywhere. While our method is just a sufficient condition for the non-existence of bound state $\omega^2 < 0$, it also becomes a necessary condition in a simple toy model. We have also found the numerical solution $S$ for the vector and scalar perturbation on ten-dimensional spherically symmetric black holes, and the scalar perturbation on the five-dimensional Schwarzschild black string. While $S$ diverges at some point for an inappropriate boundary condition, we found the continuous range of appropriate boundary conditions, which corresponds to bounded $S$. Furthermore, as shown in sections 3.3 and 3.4, our method can work even in almost marginally stable cases. From these results, we conjecture the following:

**Conjecture 1.** A continuous and bounded solution of equation (9) exists in $-\infty < x < \infty$ if and only if there is no bound state ($\omega^2 < 0$ mode) in the Schrödinger equation.

At the very least, the present results suggest that our method is a good test for the stability of black holes. As for the existence of the solution, we gave a proof for the positive and bounded potential which corresponds to a manifestly stable case. If the potential $V$ contains negative regions, we can still show the existence from a continuous and bounded solution for a different potential $V_0 (\leq V)$, under some assumptions (see appendix E). The case of a marginally (un)stable parameter was discussed in section 2.5.
If there exists a regular solution of equation (9), the Schrödinger equation (2) becomes
\[
\left( -\frac{d}{dx} + S \right) \left( \frac{d}{dx} + S \right) \Phi = \omega^2 \Phi.
\] (31)
This is known as the supersymmetric quantum mechanics system where the energy \( \omega^2 \) is manifestly non-negative. If the above conjecture is correct, the quantum mechanics system which does not have a bound state with negative energy becomes supersymmetric.

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**Appendix A. Proof of proposition 1**

We give proof of proposition 1 from section 2.2.

**Proof.** We consider solving equation (9) from the boundary condition \( S|_{x=x_0} = 0 \) at a point \( x = x_0 \). We already know the local existence of the equation (9), so we only need to exclude the possibility that \( S \) is divergent at some point. At \( x = x_0 \), we have \( S|_{x=x_0} = 0 \) and \( dS/dx|_{x=x_0} = -V|_{x=x_0} < 0 \), so \( S \) takes a positive value at \( x_0 x_0 < x < x_0 \), and a negative value at \( x_0 < x < x + \delta_0 \) for a small constant \( \delta_0 > 0 \).

First, we consider the region \( x_0 < x \). We can say that once \( S \) becomes negative, \( S \) cannot be zero as \( x \) increases in \( x_0 < x \). If \( S = 0 \) at some point \( x_1 (> x_0) \) and \( S < 0 \) in \( x_0 < x < x_1 \), then \( dS/dx|_{x=x_1} = -V|_{x=x_1} < 0 \). However, this is a contradiction, because \( dS/dx|_{x=x_1} < 0 \) implies that \( S \) is already positive in \( x_1 - \delta_1 < x < x_1 \) for a small constant \( \delta_1 > 0 \). Thus, \( S \) is bounded above in the region \( x_0 < x \).

We also have \( dS/dx = -V + S^2 > -(V_{\text{max}} + \delta_2) + S^2 \) for a small constant \( \delta_2 > 0 \). If the value of \( S \) changes from the value larger than \( -\sqrt{V_{\text{max}} + \delta_2} \) into the value \( -\sqrt{V_{\text{max}} + \delta_2} \) at \( x = x_2 (> x_0) \), \( dS/dx \) becomes positive at \( x = x_2 \). However, this is a contradiction because, \( S|_{x=x_2} = -\sqrt{V_{\text{max}} + \delta_2} \) and \( dS/dx|_{x=x_2} > 0 \) implies \( S > -\sqrt{V_{\text{max}} + \delta_2} \) in \( x_2 < x < x_3 - \delta_3 \) for a small constant \( \delta_3 > 0 \). Since \( \delta_3 \) is an arbitrary constant, \( S \) cannot be smaller than \( -\sqrt{V_{\text{max}}} \).

This shows that \( S \) is bounded below in the region \( x_0 < x \).

Next, we consider the region \( x < x_0 \). Defining \( \bar{S} := -S, \bar{x} := -x \), then equation (9) becomes
\[
\frac{d\bar{S}}{dx} = -V + \bar{S}^2, \tag{A.1}
\]
then \( x = x_0 \) corresponds to \( \tilde{x} = -x_0 =: \tilde{x}_0 \). We need to show that \( \bar{S} \) is bounded above and below in \( x_0 < \tilde{x} \), but this is formally the same problem as in the case of \( x_0 < x \). Thus, \( S \) is bounded above and below in the region \( -\infty < x < \infty \). \( \square \)

**Appendix B. Effective potential for the \( D \)-dimensional Reissner–Nordström–de Sitter black hole**

The metric for the \( D \)-dimensional Reissner–Nordström–de Sitter black hole is

\[
ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2d\Omega_n^2, \tag{B.1}\]

\[
f = 1 - \frac{2M}{r^{D-1}} + \frac{Q^2}{r^{2n-2}} - \lambda r^2, \tag{B.2}\]

where \( n = D - 2 \), \( \lambda = 2\Lambda/(n(n+1)) \) with a cosmological constant \( \Lambda \), and \( d\Omega_n^2 \) denotes the metric of the unit \( n \)-dimensional sphere. As shown in [5–7], the linear gravitational and electromagnetic perturbation reduces to decoupled single master equations of the same form as equation (1). The effective potentials for the vector and scalar gravitational modes\(^7\) are

\[
V_{\text{vector}} = \frac{f}{r^2} \left[ \ell (\ell + n - 1) + \frac{n^2 - 2n}{4} \right] \Lambda r^2 + \frac{n(5n - 2)Q^2}{4r^{2n-2}} \left[ (n^2 - 1)M^2 + 2n(n - 1)(\ell + n)(\ell - 1)Q^2 \right], \tag{B.3}\]

\[
V_{\text{scalar}} = \frac{f}{64r^2H^2}, \tag{B.4}\]

with

\[
U = \left[ -4n^3(n + 2)(n + 1)^2(1 + m\delta)^2X^2 + 48n^2(n + 1)(n - 2)m(1 + m\delta)X \\
-16(n - 2)(n - 4)m^3 \right] \Lambda r^2 + n^3(3n - 2)(n + 1)^4\delta(1 + m\delta)^3X^4 \\
-4n^2(n + 1)^2(1 + m\delta)^2 \left[ (n + 1)(3n - 2)m\delta - n^2 \right] X^3 \\
+4(n + 1)(1 + m\delta) \left[ m(n - 2)(n - 4)(n + 1)(m + n^2)\delta + 4n(2n^2 - 3n + 4)m \\
+ n^2(n - 2)(n - 4)(n + 1) \right] X^2 - 16m(n + 1)m(-4m + 3n^2(n - 2))\delta \\
+3n(n - 4)m + 3n^2(n + 1)(n - 2)X + 64m^3 + 16n(n + 2)m^2, \tag{B.5}\]

\[
H = m + \frac{n(n + 1)}{2}(1 + m\delta)X, \tag{B.6}\]

\[
X = \frac{2M}{r^{D-1}}. \tag{B.7}\]

\(^7\)The effective potentials for other modes, tensor perturbation and electromagnetic vector and scalar perturbation, can be seen in [5–7].
\[
\delta = \frac{\mu - M}{2mM}, \quad \mu = \sqrt{M^2 + \frac{4MQ^2}{(n+1)^2}}, \quad m = \ell(n + 1) - n, \tag{B.8}
\]

where \( \ell \) is a positive integer, \( \ell \geq 1 \) for vector mode, \( \ell \geq 0 \) for scalar mode. The relation between \( x \) and \( r \) is given by \( \frac{dx}{d\ell} = \frac{dr}{f(r)} \). \( x \) behaves \( x \sim \ln(r - r_H) \) near the horizon. In the coordinate \( r \), equation (9) becomes

\[
V + f \frac{dS}{dr} - S^2 = 0. \tag{B.9}
\]

To normalize all the quantities by the radius of the black hole horizon \( r_H \), we set the mass parameter as

\[
M = \frac{1}{2} (1 + Q^2 - \lambda), \tag{B.10}
\]

then the black hole horizon locates at \( r = r_H = 1 \). Also, setting

\[
\lambda = \frac{1 - \rho^n + Q^2(1 - \rho^{n-1})}{1 - \rho^{n-1}}, \tag{B.11}
\]

the location of the de Sitter horizon becomes \( r = r_{\text{dS}} = 1/\rho \). The electric charge for the extremal black hole is given by

\[
Q^2_{\text{extremal}} = \frac{(n - 1)(1 - \rho^{n+1}) - \rho^2(n + 1)(1 - \rho^{n-1})}{(n - 1)(1 - \rho^{n+1}) - \rho^{n+1}(n + 1)(1 - \rho^{n-1})}. \tag{B.12}
\]

This useful normalization was used in [18].

**Appendix C. Effective potential for the five-dimensional black string**

The metric of the five-dimensional black string is

\[
ds^2 = -f \, dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + dz^2, \tag{C.1}
\]

with \( f = 1 - r_H/r \). The effective potential for the scalar perturbation is given by [21]

\[
V = f \frac{k^2 r^9 + 3k^2 r^6(2r - 3r_H) + r_H^3 + 3k^2 r^3 r_H(-4r + 3r_H)}{(k^2 r^9 + r_H^2)^2}, \tag{C.2}
\]

where \( k \) is the wave number along \( z \) direction. In this case, equation (9) also becomes

\[
V + f \frac{dS}{dr} - S^2 = 0. \tag{C.3}
\]

**Appendix D. \( V = -a^2 \exp(b^2 x) \) case**

In some cases, the potential behaves as \( V \simeq -a^2(r - r_H) \) with a constant \( a > 0 \) near the horizon. Since \( r - r_H \simeq e^{b^2 x} \) with a constant \( b > 0 \) near the horizon, the potential in the \( x \) coordinate becomes \( V \simeq -a^2 \exp(b^2 x) \). For this potential, we can solve equation (9) by using the Bessel functions.
\[ S = b^2 X \left( \frac{C_1 J_1(X) + 2Y_1(X)}{2C_1 J_0(X) + 4Y_0(X)} \right), \]  
(D.1)

where \( X = 2ae^{-b^2/2b^2} \) and \( C_1 \) is a constant. For \( X \ll 1 \), the Bessel functions behave as
\[
J_0(X) = 1 + \mathcal{O}(X^2),
\]
(D.2)
\[
J_1(X) = \frac{X}{2} + \mathcal{O}(X^2),
\]
(D.3)
\[
Y_0(X) = \frac{2}{\pi} \ln X + 2 \frac{\gamma - \ln 2}{\pi} + \mathcal{O}(X^2),
\]
(D.4)
\[
Y_1(X) = -\frac{2}{\pi} X + \frac{X}{\pi} \ln X - \frac{2\gamma + 2\ln 2}{2\pi} X + \mathcal{O}(X^2),
\]
(D.5)

where \( \gamma \approx 0.577 \) is Euler’s constant. So, the approximate solution of \( S \) with the integral constant becomes
\[ S \simeq -\frac{b^2}{2} \frac{1}{C_2 \pm \ln X}, \]
(D.6)

where \( C_2 = \gamma - \ln 2 + C_1 \pi/4 \). Once \( S \) takes a positive value in the \( X \ll 1 \) regime, \( S \) will be bounded above and below as \( X \) decreases towards zero. In that case, near \( X \simeq 0 \), \( S \) behaves \( S \sim -1/\ln X \sim -1/x \) like the \( V = 0 \) case.

**Appendix E. A potential with compact support**

Let us consider a continuous and bounded potential with compact support

\[
V = \begin{cases} 
0 & (-\infty < x \leq x_1) \\
V_- (< 0) & (x_1 < x < x_2) \\
0 & (x = x_2) \\
V_+ (> 0) & (x_2 < x < x_3) \\
0 & (x_3 \leq x < \infty) 
\end{cases}.
\]

(E.1)

We assume that \( V \) is smooth in \( x_1 < x < x_3 \) and \( \lim_{x \to x_1^-} dV/dx > 0 \). Also, we define another potential which is not necessarily continuous

\[
V_0 = \begin{cases} 
0 & (-\infty < x \leq x_1) \\
v_- (v_- < V_- < 0) & (x_1 < x < x_2) \\
v_+ (0 \leq v_+ < V_+) & (x_2 \leq x < x_3) \\
0 & (x_3 \leq x < \infty) 
\end{cases}.
\]

(E.2)

where \( v_\pm \) are functions and we assume \( V > V_0 \) in \( x_1 < x < x_3 \). In this setup, we would like to prove the following proposition.

**Proposition 2.** If there exists a continuous and bounded solution of equation (9) for the potential equation (E.2) in \( -\infty < x < \infty \), there also exists a continuous and bounded solution of equation (9) for the potential equation (E.1) in \( -\infty < x < \infty \).

**Proof.** Let \( S_0 \) be a continuous and bounded solution of \( V_0 + dS_0/dx - S_0^2 = 0 \). Since the behaviors in \( x \leq x_1 \) and \( x_3 \leq x \) are the same as those in the toy model in section 2.3, we can
say \( S_0|_{x=x_1} > 0 \) and \( S_0|_{x=x_2} < 0 \). In the region \( x_1 < x < x_2 \), since \(-V_- < -v_-\), we obtain an inequality

\[
(S_0 - S)(S_0 - S + 2S) < \frac{d(S_0 - S)}{dx}.
\]

(E.3)

We assume \( S_1|_{x=x_1} = \eta S_0|_{x=x_1} \) with a positive constant \( \eta (< 1)^8 \). Since \( dS/dx = -V_- + S^2 > 0 \), \( S \) is a monotonically increasing function in \( x_1 < x < x_2 \). From this and the condition \( S_1|_{x=x_1} = \eta S_0 > 0 \), we can say \( S > 0 \) in \( x_1 < x < x_2 \). In the limit \( x \rightarrow x_1 + 0 \), the above inequality becomes

\[
(1 - \eta^2)S_0^2 < \frac{d(S_0 - S)}{dx},
\]

(E.4)

and the lhs is positive. Thus \( S_0 - S \) is an increasing function near \( x = x_1 \). Once \( S_0 - S \) becomes positive, it cannot be zero in \( x_1 < x < x_2 \) because \( d(S_0 - S)/dx > 0 \) for positive \( S_0 - S \) in \( x_1 < x < x_2 \), i.e. \( S_0 - S \) cannot decrease as \( x \) increases. So, we can say \( S < S_0 \) in \( x_1 < x < x_2 \). Since \( S_0 \) is bounded above, \( S \) cannot be divergent in \( x_1 < x < x_2 \).

In the region \( x_2 < x < x_3 \), since \(-V_+ < -v_+\) we also have the same inequality

\[
(S_0 - S)(S_0 - S + 2S) < \frac{d(S_0 - S)}{dx}.
\]

(E.5)

If we assume \( S \geq 0 \) in \( x_2 < r < x_3 \), we can also say \( S < S_0 \) from the same discussion above. However, now we also assumed \( S_0 < 0 \) at some point because \( S_0 \) should be connected with \( 1/(c_4 - r) \), which is negative, at \( x = x_3 \), then this contradicts with the first assumption \( S > 0 \). Thus, we can say \( S < 0 \) at some point \( x = y_0 \) which satisfies \( x_2 < y_0 < x_3 \). In \( y_0 < x < x_3 \), from the same discussion in the proof of proposition 1, we can show that \( S \) is a continuous and bounded negative function.

There is still a possibility that \( S = 0 \) at \( x = x_3 \) where \( V_+ = 0 \). In that case, \( S_1|_{x=x_3} = 0 \), \( dS/|_{x=x_3} = 0 \), and \( d^2S/dx^2|_{x=x_3} = \lim_{x 
rightarrow x_3} 0 \) \( V/|_{x=x_3} = \chi \beta^2 > 0 \). So, the approximate behavior becomes \( S \simeq \beta^2(x - x_3)^2 \), but this means \( S > 0 \) in \( x_3 - \delta < x < x_3 \) with a small constant \( \delta > 0 \). This contradicts \( S < 0 \) in \( y_0 < x < x_3 \). Thus, \( S_1|_{x=x_3} = 0 \) and \( S \) can be matched with \( 1/(c_4 - r) \) at \( x = x_3 \) with a negative value. This shows the existence of continuous and bounded \( S \) in \(-\infty < x < \infty \).

\[\square\]

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8 Note that this is an appropriate boundary condition for \( S \).
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