Are ultra-spinning Kerr-Sen-AdS\(_4\) black holes always super-entropic?

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We study thermodynamics of the four-dimensional Kerr-Sen-AdS black hole and its ultra-spinning counterpart, and verify that both black holes fulfill the first law and Bekenstein-Smarr mass formulae of black hole thermodynamics. Furthermore, we derive new Christodoulou-Ruffini-like squared-mass formulae for the usual and ultra-spinning Kerr-Sen-AdS\(_4\) solutions. We show that this ultra-spinning Kerr-Sen-AdS\(_4\) black hole does not always violate the Reverse Isoperimetric Inequality (RII) since the value of the isoperimetric ratio can be larger/smaller than, or equal to unity, depending upon where the solution parameters lie in the parameters space. This property is obviously different from that of the Kerr-Newman-AdS\(_4\) super-entropic black hole, which always strictly violates the RII, although both of them have some similar properties in other aspects, such as horizon geometry and conformal boundary. In addition, it is found that while there exists the same lower bound on mass (\(m_\geq 8l/\sqrt{27}\) with \(l\) being the cosmological scale) both for the extremal ultra-spinning Kerr-Sen-AdS\(_4\) black hole and for the extremal super-entropic Kerr-Newman-AdS\(_4\) case, the former has a maximal horizon radius: \(r_{\text{HP}} = l/\sqrt{3}\) which is the minimum of the latter. Therefore, these two different kinds of four-dimensional ultra-spinning charged AdS black holes exhibit some significant physical differences.

I. INTRODUCTION

Black hole is one of the most remarkable and fascinating objects in nature. It has an event horizon beyond which any event inside has no effect. As for the horizon topology of black holes, Hawking proved that for four-dimensional asymptotically flat, stationary black holes satisfying the dominant energy condition, their two-dimensional event horizon cross sections have a topology of \(S^2\)-sphere \([1]\). To obtain different horizon topologies, one needs to relax some assumptions made in Hawking’s uniqueness theorem. Among various different possibilities, one is to consider higher dimensional spacetimes. For example, in the five-dimensional asymptotically flat spacetimes, apart from the well-known black hole \([2]\) which has the horizon topology of a round \(S^3\)-sphere, the black ring \([3]\) owns the \(S^3 \times S^1\) horizon topology, whilst a rotating black lens solution \([4]\) has the horizon topology of a lens-space \(L(n, 1)\). On the other hand, if the spacetime is considered to be asymptotically non-flat, it has been found that especially for the four-dimensional anti-de Sitter (AdS) spacetime, the Einstein equation also admits topological solutions with their event horizons being Riemann surfaces of any genus, namely, planar, toroidal and hyperbolic horizons \([5–18]\), and rotating black string solutions in all dimensions \([19]\). Higher dimensional spacetimes can have even more rich horizon topologies, for instance, the event horizons of the d-dimensional asymptotically AdS black holes are also possible to be the Einstein manifolds with positive, zero, or negative curvature \([20]\).

Recently, a new class of AdS black holes \([21–23]\) which is considered as ultra-spinning since one of their rotation angular velocities is boosted to the speed of light, has received considerable interest and enthusiasm. This kind of black hole, occasionally called as the “black spindle” spacetime \([24]\) because of its bottle-shaped horizon \([15]\), has a noncompact horizon topology since its spherical horizon has two punctures at the north and south poles, although it has a finite horizon area. The ultra-spinning black hole violates the “RII” \([25, 26]\), which implies that the Schwarzschild-AdS black hole has a maximum upper entropy. Due to the fact that the ultra-spinning black hole can exceed the maximum entropy bound, it is also dubbed “super-entropic”. Remarkably, it has been shown \([21]\) that the super-entropic black hole solution can be alternatively obtained by taking a simple ultra-spinning limit from the usual rotating AdS one. This solution generating procedure is very simple: firstly recast the rotating AdS black hole in the frame rotating at infinity, then boost one rotation angular velocity to the velocity of light, and finally compactify the corresponding azimuthal direction. Up to date, a lot of new super-entropic black hole solutions \([27–31]\) from the known rotating AdS black holes have been obtained so far. Very recently, it has been found that the super-entropic black hole can also be obtained by running a conical deficit from the usual rotating AdS black hole \([32]\). In addition, other aspects of the super-entropic black holes, including thermodynamic properties \([21, 27, 29–31, 33–35]\), horizon geometry \([23, 27, 29]\), geodesic motion \([24]\), Kerr/CFT correspondence \([28–30]\), and so on, have been investigated consequently.

Although there has been much progress in the last few years in constructing super-entropic black hole solutions and studying their physical properties, ultra-spinning black holes in gauged supergravities remain to be the virgin territory and thus need to be explored deeply, which motivates us to conduct the present work. Since the most famous rotating charged black hole in the four-dimensional low energy heterotic string theory is the Kerr-Sen solution \([36]\), we first consider its gen-

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eralization by including a nonzero negative cosmological constant, namely the Kerr-Sen-AdS black hole, and then get its ultra-spinning counterpart. Along the way, we also address their thermodynamical properties and show that the obtained thermodynamical quantities perfectly obey both the extended first law and the Bekenstein-Smarr mass formulae for both black holes.

The remaining part of this paper is organized as follows. In Sec. II, we make a recapitulation about the Kerr-Sen black hole, and then turn to the Kerr-Sen-AdS black hole solution in the four-dimensional gauged Einstein-Maxwell-dilaton-axion (EMDA) theory and investigate its thermodynamics. In Sec. III, after the ultra-spinning Kerr-Sen-AdS black hole solution is constructed, its horizon topology and conformal boundary, thermodynamical properties, bounds on the mass and horizon radius of extremal ultra-spinning charged AdS solutions, and the RII are subsequently discussed. In doing so, we establish novel Christodoulou-Ruffini-like squared-solutions, and the RII are subsequently discussed. In doing so, we establish novel Christodoulou-Ruffini-like squared-solutions. The remaining part of this paper is organized as follows. In Sec. II, we make a recapitulation about the Kerr-Sen black hole solution and the RII are subsequently discussed. In doing so, we establish novel Christodoulou-Ruffini-like squared-solutions. Finally, we end up with our summaries in Sec. IV.

II. KERR-SEN BLACK HOLE AND ITS ADS VERSION

A. A brief review of Kerr-Sen black hole

By using a solution generating technique with the Kerr black hole as the seed solution, Sen [36] obtained a new solution of the four-dimensional rotating charged black hole, which is named as the Kerr-Sen black hole. It is an exact solution to the four-dimensional low-energy heterotic string theory, also known as the Einstein-Maxwell-Dilaton-Axion (EMDA) theory, whose Lagrangian has two different but completely equivalent forms:

\[ \mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - e^{-\phi} F^2 - \frac{1}{12} e^{-2\phi} H^2 \right] \]

\[ = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2 - e^{-\phi} F^2 \right] + \frac{\chi}{2} e^{\mu \nu \rho \lambda} F_{\mu \nu} F_{\rho \lambda}, \]

where \( R \) is the Ricci scalar, \( \phi \) is the dilaton scalar field, \( F_{\mu \nu} \) is the Faraday-Maxwell electro-magnetic tensor and \( F^2 = F_{\mu \nu} F^{\mu \nu} \). \( \chi \) is the axion pseudoscalar field dual to the three-form antisymmetric tensor: \( H = -e^{2\phi} * d \chi \), and \( H^2 = H_{\mu \nu \rho} H^{\mu \nu \rho} \) is the four-dimensional Levi-Civita antisymmetric tensor density.

The Kerr-Sen black hole solution can be expressed in the Boyer-Lindquist coordinates as \([37, 38]\)

\[ ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta \, d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [a dt - (r^2 + 2br + a^2) d\phi]^2, \]

\[ \Lambda = \frac{qr}{\Sigma} (dt - a \sin^2 \theta \, d\phi), \]

\[ \phi = \ln \left( \frac{r^2 + a^2 \cos^2 \theta}{\Sigma} \right), \quad \chi = \frac{2ba \cos \theta}{r^2 + a^2 \cos^2 \theta}, \]

where

\[ \Delta = r^2 + 2(b - m)r + a^2, \quad \Sigma = r^2 + 2br + a^2 \cos^2 \theta, \]

in which \( b = q^2/2m \) is the dilatonic scalar charge, the parameters \( m \) and \( q \) are the mass and electric charge of the black hole, respectively, and its angular momentum is \( J = ma \).

Although the metric and gauge field of the Kerr-Sen black hole have almost the same forms as those of the Kerr-Newman black hole (therefore they own many very similar physical properties, such as geometric feature \([37]\), quantum thermal property and thermodynamical four laws \([38]\), instability of the bound state of the charged mass scalar field and \( \text{CFT}_2 \) holographic duality of the scattering process, etc.), there are some significant differences between them. For example, the Kerr-Sen black hole is a non-vacuum, non-algebraically special solution in the four-dimensional low energy heterotic string theory. In addition to the metric and an Abelian vector field, the Kerr-Sen solution contains another two non-gravitational fields: an antisymmetric third-order tensor field (or a dual axion pseudoscalar field), and a dilaton scalar field, which are absent from the Kerr-Newman solution. For the Petrov-Pirani classification, the Kerr-Newman black hole, which is an exact electric vacuum solution to the Einstein-Maxwell theory, belongs to the family of type-D, whilst the Kerr-Sen solution is of type-I \([39]\). While the electro-static potentials of the stringy left- and right-movers of the Kerr-Sen black hole are identical, they are unequal in the Kerr-Newman case \([40]\). Of course, there are still many other salient different aspects between them, for instance the capture region of scattered photons, the emission probability of black hole evaporation process, the magnetic induction ratio, and so on (See Refs. \([41, 42]\) and references therein).

B. Kerr-Sen-AdS\(_4\) black hole solution

We now turn to include a nonzero negative cosmological constant into the Kerr-Sen solution, and present a simple form of the Kerr-Sen-AdS\(_4\) black hole, which is an exact solution to the gauged EMDA theory, whose Lagrangian has the following form

\[ \mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2 - e^{-\phi} F^2 \right. \]

\[ + \frac{1}{12} \left[ 4 + e^{-\phi} + e^{\phi} (1 + \chi^2) \right] \right] + \frac{\chi}{2} e^{\mu \nu \rho \lambda} F_{\mu \nu} F_{\rho \lambda}, \]

\( \chi = \frac{2ba \cos \theta}{r^2 + a^2 \cos^2 \theta}, \]

\( \Delta = r^2 + 2(b - m)r + a^2, \quad \Sigma = r^2 + 2br + a^2 \cos^2 \theta, \)

in which \( b = q^2/2m \) is the dilatonic scalar charge, the parameters \( m \) and \( q \) are the mass and electric charge of the black hole, respectively, and its angular momentum is \( J = ma \).

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with \( l \) being the cosmological scale. Distinct from the un-gauged case, now the above Lagrangian receives a potential term contributed from the dilaton and axion fields, so it is impossible to reexpress it in the dualized version in terms of the three-form field that appeared in the ungauged Lagrangian.

Written in terms of the Boyer-Lindquist coordinates and adapted to the frame rotating at infinity, the Kerr-Sen-AdS\(_4\) black hole solution can be given by the following  

\[
dS^2 = \frac{\Delta_r}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Sigma} d\varphi \right)^2 + \frac{\Sigma}{\Delta_r'} dr^2 + \frac{\Sigma}{\Delta_0} d\theta^2 + \frac{\Delta_0 \sin^2 \theta}{\Sigma} \left( a dt - r^2 + 2br + a^2 \right)^2 ,
\]

\[
\tilde{A} = \frac{qr}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Sigma} d\varphi \right) ,
\]

\[
\tilde{\phi} = \ln \left( \frac{r^2 + a^2 \cos^2 \theta}{\Sigma} \right) , \quad \tilde{\Xi} = \frac{2ba \cos \theta}{r^2 + a^2 \cos^2 \theta} ,
\]

where \( \Sigma = r^2 + 2br + a^2 \cos^2 \theta \) as before, and now we have

\[
\Delta_r = \left( 1 + \frac{r^2 + 2br}{l^2} \right) (r^2 + 2br + a^2) - 2mr ,
\]

\[
\Delta_0 = 1 - \frac{a^2}{l^2} \cos^2 \theta , \quad \Xi = 1 - \frac{a^2}{l^2} .
\]

Obviously, the above solution (5) consistently reduces to the Kerr-Sen black hole solution (3) when the AdS radius \( l \) tends to infinity.

It should be pointed out that more general solutions have been already constructed [43, 44] in the special case of the pair-wise equal charge parameters of the four-dimensional gauged STU supergravity theory. As the gauged EMDA theory is a more special case of that theory, therefore the above solution can be included as a special case obtained in [43, 44], however, here we present it in a slight different and more suitable form.

C. Thermodynamics

Now we are in a position to investigate thermodynamics of the Kerr-Sen-AdS\(_4\) black hole. In the framework of the extended phase space [45, 46], thermodynamic quantities associated with the above solution (5) can be computed through the standard method and have the following expressions:

\[
\tilde{M} = \frac{m}{\Xi} , \quad \tilde{J} = \frac{ma}{\Xi^2} , \quad \tilde{Q} = \frac{q}{\Xi} ,
\]

\[
\tilde{T} = \frac{(r_+ + b)(2r_+^2 + 4br_+ + l^2 + a^2) - ml^2}{2\pi (r_+^2 + 2br_+ + a^2)l^2} ,
\]

\[
\tilde{S} = \frac{\pi (r_+^2 + 2br_+ + a^2)}{\Xi} , \quad \tilde{\Omega} = \frac{a\Xi}{r_+^2 + 2br_+ + a^2} ,
\]

\[
\tilde{\Phi} = \frac{qr_+}{r_+^2 + 2br_+ + a^2} .
\]

It is easy to verify that these thermodynamic quantities (6) satisfy the Bekenstein-Smarr mass formulas

\[
\tilde{M} = 2\tilde{T} \tilde{S} + 2\tilde{\Omega} \tilde{J} + \tilde{\Phi} \tilde{Q} - 2\tilde{V} \tilde{P} .
\]

Here \( \tilde{V} \) is the thermodynamic volume

\[
\tilde{V} = \frac{4\pi}{3\Xi} (r_+ + b)(r_+^2 + 2br_+ + a^2) ,
\]

which is conjugate to the pressure \( \tilde{P} = 3/(8\pi l^2) \). Unfortunately, the first law, however, boils down to a differential identity only

\[
d\tilde{M} = \tilde{T} d\tilde{S} + \tilde{\Omega} d\tilde{J} + \tilde{\Phi} d\tilde{Q} + \tilde{V} d\tilde{P} + \tilde{J} d\Xi / (2a) .
\]

The reason for this is simply because we have just adopted the rotating frame at infinity, not the rest frame at infinity.

The transformation of the above Kerr-Sen-AdS\(_4\) solution into the frame rest at infinity can be easily done by taking a simple coordinate transformation: \( \varphi \to \varphi - at/l^2 \). After a tedious calculation of the thermodynamic quantities in this rest frame, it is not difficult to find that only the mass, the angular velocity and the thermodynamic volume are different from those given in Eq. (6) and can be written as follows:

\[
\tilde{M} = \tilde{M} + \frac{a}{l^2} \tilde{J} , \quad \tilde{\Omega} = \tilde{\Omega} + \frac{a}{l^2} , \quad \tilde{V} = \tilde{V} + \frac{4\pi}{3} a \tilde{J} .
\]

In this situation, now thermodynamic quantities can indeed satisfy both the standard forms of the first law and the Bekenstein-Smarr mass formula simultaneously:

\[
d\tilde{M} = \tilde{T} d\tilde{S} + \tilde{\Omega} d\tilde{J} + \tilde{\Phi} d\tilde{Q} + \tilde{V} d\tilde{P} ,
\]

\[
\tilde{M} = 2\tilde{T} \tilde{S} + 2\tilde{\Omega} \tilde{J} + \tilde{\Phi} \tilde{Q} - 2\tilde{V} \tilde{P} .
\]

It is not difficult to check that the above differential and integral mass formulae can indeed satisfy both the standard forms of the first law and the Bekenstein-Smarr mass formula simultaneously.

\[
\tilde{M}^2 = \left( 1 + \frac{8PS}{3} \right) \left[ \left( 1 + \frac{8PS}{3} \right) \frac{S}{4\pi} + \frac{\pi l^2}{S} + \frac{\tilde{Q}^2}{2} \right] .
\]

III. ULTRA-SPINNING KERR-SEN-ADS\(_4\) BLACK HOLE

A. The ultra-spinning solution

To construct the ultra-spinning version from the above Kerr-Sen-AdS\(_4\) black hole solution (5), we just need to perform three steps: (i) redefine the angle coordinate \( \varphi \) by multiplying it with a factor \( \Xi \); (ii) take the \( a \to l \) limit; (iii) then compactify the \( \varphi \) direction with a period of the dimensionless parameter \( \mu \). Having finished these steps, we then obtain the ultra-spinning Kerr-Sen-AdS\(_4\) black hole solution:

\[
d\tilde{S}^2 = \frac{\Delta_r}{\Sigma} \left( dt - l \sin^2 \theta d\varphi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\xi}{\sin^2 \theta} d\theta^2 + \frac{\sin^4 \theta}{\Sigma} \left[ l dt - (r^2 + 2br + l^2) d\varphi \right]^2 ,
\]

\[
\tilde{A} = \frac{qr}{\Sigma} \left( dt - l \sin^2 \theta d\varphi \right) ,
\]

\[
\tilde{\phi} = \ln \left( \frac{r^2 + l^2 \cos^2 \theta}{\Sigma} \right) , \quad \tilde{\xi} = \frac{2bl \cos \theta}{r^2 + l^2 \cos^2 \theta} ,
\]

where

\[
\Delta_r = (r^2 + 2br + l^2)^2 l^{-2} - 2mr , \quad \Xi = r^2 + 2br + l^2 \cos^2 \theta .
\]
B. Horizon geometry and conformal boundary

In this subsection, we first focus on other basic properties, such as the horizon geometry and conformal boundary of the ultra-spinning Kerr-Sen-AdS$_4$ black hole. To ensure that the geometry is free of any closed timelike curve (CTC), one needs to check whether the inequality $g_{\varphi \varphi} \geq 0$ satisfies or not. From its expression: $g_{\varphi \varphi} = \frac{2mr^2}{\Sigma^2} \sin^4 \theta$, we find that $g_{\varphi \varphi}$ is always positive (due to $m \geq 0$, $r \geq 0$ and $\Sigma \geq 0$) in the entire spacetime and thus the spacetime is free of CTC.

To investigate the geometry of the event horizon, we consider the constant $(t, r)$ surface on which the induced metric reads

$$d\hat{s}_h^2 = \frac{\hat{\Sigma}_+}{\sin^2 \theta} d\theta^2 + \frac{2mr^2}{\hat{\Sigma}_+} \sin^4 \theta d\varphi^2,$$

where $\hat{\Sigma}_+ = \hat{\Sigma}|_{r_+}$. This metric appears to be singular at $\theta = 0$ and $\theta = \pi$. To examine whether the metric is ill-defined at $\theta = 0$ and $\theta = \pi$, one can analyze it in two limits: $\theta \rightarrow 0$ and $\theta \rightarrow \pi$. As an example, let us consider the small angle case $\theta \sim 0$, (similarly for the $\theta \sim \pi$ case). By introducing a new variable: $k = l(1 - \cos \theta)$ for a small angle $\theta$, and noting that: $\sin^2 \theta \sim 2k/l$, the two-dimensional cross section (14) for small $k$ becomes

$$d\hat{s}_h^2 = (r_+^2 + 2br_+ + l^2) \left( \frac{dk^2}{4k^2} + \frac{4k^2}{l^2} d\varphi^2 \right),$$

which is clearly a metric of constant, negative curvature on a quotient of the hyperbolic space $\mathbb{H}^2$. This result is very similar to that of the Kerr-Newman-AdS$_4$ super-entropic black hole [27]. Due to the symmetry, the $\theta = \pi$ limit gives rise to the same result. Apparently, the space is free from pathologies near the north and south poles. Topologically, the event horizon is a sphere with two punctures, and occasionally is to the same result. Apparently, the space is free from pathologies near the north and south poles. Topologically, the event horizon is a sphere with two punctures, and occasionally is called as the black spindle. This implies that the above ultra-spinning Kerr-Sen-AdS$_4$ black hole enjoys a finite area but noncompact horizon.

Next, we want to investigate the conformal boundary of the ultra-spinning Kerr-Sen-AdS$_4$ black hole. Multiplying the metric (13) with the conformal factor $l^2/r^2$ and taking the $r \rightarrow \infty$ limit, we find that the boundary metric has the form

$$d\hat{s}_{bdry}^2 = -dt^2 + 2l \sin^2 \theta dt d\varphi + l^2 d\theta^2 / \sin^2 \theta,$$

which is the same one as that of the super-entropic Kerr-Newman-AdS$_4$ black hole [27]. It is easy to see that the coordinate $\varphi$ is null on the conformal boundary. In the small $k = l(1 - \cos \theta)$ limit, we can reexpress the conformal boundary metric (16) as

$$d\hat{s}_{bdry}^2 = -dt^2 + 4k dt d\varphi + dk^2 / (4k^2),$$

which can be interpreted as an AdS$_3$ written as a Hopf-like fibration over $\mathbb{H}^2$. It means again that the metric has nothing pathological near two poles $\theta = 0$ and $\theta = \pi$.

C. Mass formulae

Now we shall investigate thermodynamics of the ultra-spinning Kerr-Sen-AdS$_4$ black hole. Its fundamental thermodynamic quantities can be obtained through the standard method and are given below

$$M = \frac{\mu m}{2\pi}, \quad J = \frac{\mu ml}{2\pi}, \quad Q = \frac{\mu q}{2\pi},$$

$$T = \frac{r_+ + b}{\pi l^2} - \frac{m}{2\pi(r_+^2 + 2br_+ + l^2)} = \frac{3r_+^2 + 2br_+ - l^2}{4\pi r_+ l^2},$$

$$S = \frac{1}{2} \mu (r_+^2 + 2br_+ + l^2), \quad \Omega = \frac{l}{r_+^2 + 2br_+ + l^2},$$

$$\Phi = \frac{q\varphi_+}{r_+^2 + 2br_+ + l^2},$$

Note that the angular momentum and the mass satisfy a chirality condition: $J = Ml$. Usually, the mass and angular momentum are computed by the conformal completion method [47]. However, the angular momentum can also be evaluated correctly by the Komar method and are given below

$$V = \frac{4}{3}(r_+ + b)S = \frac{2}{3} \mu (r_+ + b)(r_+^2 + 2br_+ + l^2),$$

$$K = \frac{m(r_+^2 + 2br_+ + l^2)}{4\pi(r_+^2 + 2br_+ + l^2)} = \frac{l^2 - r_+^2 (r_+ + 2b)^2}{8\pi r_+ l^2},$$

which are conjugate to the pressure $P = 3/(8\pi l^2)$ and the dimensionless parameter $\mu$, respectively.

As was done in a previous work [34], here we propose to establish the following simple relations

$$M = \frac{\mu \overline{\Xi} M}{2\pi}, \quad J = \frac{\mu \overline{\Xi} J}{2\pi}, \quad Q = \frac{\mu \overline{\Xi} Q}{2\pi},$$

$$\Omega = \frac{\overline{\Omega}}{2\pi}, \quad S = \frac{\mu \overline{\Xi} S}{2\pi}, \quad V = \frac{\mu \overline{\Xi} V}{2\pi},$$

$$T = \overline{T}, \quad \Phi = \overline{\Phi}, \quad P = \overline{P},$$

and take the ultra-spinning limit: $a \rightarrow l$. Then we can see that the above thermodynamical quantities given in Eq. (18) for the ultra-spinning Kerr-Sen-AdS$_4$ black hole can also be obtained directly from those of its corresponding usual black hole. This further confirms that our previous method advised
Similarly for the pressure $P$ which is the expected Christodoulou-Ruffini-like squared-sous that the thermodynamical quantities $S$ relation $\frac{\partial}{\partial T} = \frac{\partial}{\partial S}$ be regarded as independent thermodynamical variables for the ultra-spinning Kerr-Sen-AdS black holes. Hence we hope to seek a similar one for the ultra-spinning Kerr-Sen-AdS black hole. Since the event horizon equation ($\Delta r_+ = 0$) can be rewritten as

$$S^2 / (\pi l^2) = \mu M r_+,$$  

(24)

then after using $3/l^2 = 8\pi P$, we get: $r_+ = 8PS^2 / (3\mu M)$. Now, we can substitute it into the entropy: $S = \mu (r_+^2 + 2br_+ l^2) / 2$ and use the chirality condition ($J = Ml$) as well as $b = \pi Q^2 / (\mu M)$ to arrive at an identity:

$$M^2 = \frac{8PS}{3\mu\lambda} \left( \frac{4\lambda}{3} S^2 + \pi Q^2 \right),$$

(25)

which is the expected Christodoulou-Ruffini-like squared-mass formula for the ultra-spinning Kerr-Sen-AdS black hole. It is interesting to note that using this squared-mass formula, it is very convenient to study black hole chemistry and possible thermodynamical phase transition of this ultra-spinning Kerr-Sen-AdS black hole.

Leaving aside the chirality condition ($J = Ml$), it is obvious that the thermodynamical quantities $S, J, Q, P$ and $\mu$ can be regarded as independent thermodynamical variables formally and constitute a whole set of extensive variables for the fundamental functional relation $M = M(S, J, Q, P, \mu)$. Differentiating the above squared-mass formula (25) with respect to $S, J, Q, P$ and $\mu$, respectively, yields their corresponding conjugate variables as expected. In doing so, one can arrive at the differential first law (19) and the Bekenstein-Smarr relation (20), with the conjugate thermodynamic potentials correctly given by the common Maxwell relations as follows.

Differentiation of the squared-mass formula (25) with respect to the entropy $S$ leads to the conjugate Hawking temperature:

$$T = \frac{\partial M}{\partial S} = \frac{M}{2S} + \frac{8P}{3\mu M} \left( \frac{4\lambda}{3} S^2 + \pi Q^2 \right),$$

(26)

and the corrected angular velocity and the electrostatic potential, which are conjugate to $J$ and $Q$, respectively, are given by

$$\Omega = \frac{\partial M}{\partial J} = \frac{\mu J}{2SM} = \frac{1}{r_+^2 + 2br_+ + l^2},$$

(27)

$$\Phi = \frac{\partial M}{\partial Q} = \frac{8\pi PQ}{3\mu M} S = \frac{q r_+}{r_+^2 + 2br_+ + l^2}.$$  

(28)

Similarly for the pressure $P$ and the dimensionless quantity $\mu$, one can get the thermodynamical volume and a new chemical potential

$$V = \frac{\partial M}{\partial P} = \frac{4\lambda}{3\mu M} \left( \frac{8P}{3} S^2 + \pi Q^2 \right),$$

$$K = \frac{\partial M}{\partial \mu} = \frac{M}{2} - \frac{8PS}{3\mu M} \left( \frac{4\lambda}{3} S^2 + \pi Q^2 \right),$$

(29)

$$l^2 - r_+^2 (r_+ + 2b)^2 / 8\pi r_+ l^2.$$  

(30)

All the above results reproduce those expressions previously given in Eqs. (18), (21) and (22). Anyway, with all these conjugate variables derived from the squared-mass formula (25), the differential first law (19) is trivially satisfied while the integral mass formula (20) is easily checked to be completely obeyed too.

### D. Chirality condition and reduced mass formulae

Now let us make a careful discussion about the impact of the chirality condition ($J = Ml$) on the thermodynamical relations of the ultra-spinning Kerr-Sen-AdS black hole. Due to the existence of the chirality condition, three thermodynamical quantities ($M, J, P$) are not completely independent, there exists a constraint relation among them

$$J^2 = 3M^2 / (8\pi P),$$

(31)

which means that the ultra-spinning Kerr-Sen-AdS black hole is actually a degenerate thermodynamical system. After taking into account the chirality condition physically, the first law (19) and the Bekenstein-Smarr relation (20) should be constrained by the condition (31), and actually depict a degenerate thermodynamical system.

Considering $J$ as a redundant variable (although it is a real measureable quantity) and eliminating $J$ from the differential and integral mass formulae in favor of $l^2 = 3/(8\pi P)$, the first law (19) and the Bekenstein-Smarr relation (20) now reduce to the following nonstandard forms (so named for their thermodynamic quantities cannot constitute the ordinary canonical conjugate pairs due to the existence of a factor $(1 - \Omega)$ in front of $dM$ and $M$):

$$\left(1 - \frac{\Omega}{\lambda\Omega}\right) dM = T dS + V' dP + \Phi dQ + K d\mu,$$

$$\left(1 - \frac{\Omega}{\lambda}\right) M = 2(TS - V'P) + \Phi Q,$$

(32)

where

$$V' = V - \frac{J\Omega}{2P} = V - \frac{4\pi}{3\mu M} r_+ l^3.$$  

In the same way, the squared-mass formula (25) degenerates to

$$M^2 \left(1 - \frac{\mu}{16\pi PS}\right) = \frac{8PS}{3\mu M} \left( \frac{4\lambda}{3} S^2 + \pi Q^2 \right).$$

(33)

In doing so, one actually views the enthalpy $M$ as the fundamental functional relation $M = M(S, Q, P, \mu)$. Similar to
the strategy adopted before, the above nonstandard differential and integral mass formulae can be derived from Eq. (33) by exploiting the standard Maxwell rule. Alternately, one perhaps prefers to eliminating $P$ instead of $J$ via Eq. (31). As the resulted expressions are rather complicated, we will not present them here.

E. Bounds on the mass and horizon radius of extremal ultra-spinning black holes

In the following, we would like to establish some new inequalities on the mass and horizon radius of the extremal ultra-spinning black holes. We begin with the extremal super-entropic Kerr-Newman-AdS black hole for which $\Delta_{e} = (r^{2} + l^{2})/l^{2} - 2mr + q^{2}$. Without loss of generality, here and hereafter, we shall assume that both the mass parameter and the AdS scale are positive.

The location of the event horizon of the extremal super-entropic Kerr-Newman-AdS black hole is determined by $\Delta_{e} = \Delta_{e}^{e} = 0$, which gives

$$m_{e} = \frac{2r_{e}(r_{e}^{2} + l^{2})}{l^{2}}, \quad q_{e}^{2} = \frac{(r_{e}^{2} + l^{2})(3r_{e}^{2} - l^{2})}{l^{2}}.$$  

(34)

By virtue of positiveness of $q_{e}^{2}$, it is evident that the following inequalities hold

$$r_{e} \geq \frac{l}{\sqrt{3}}, \quad m_{e} \geq \frac{8l}{3\sqrt{3}},$$  

(35)

which means that the scale of Hawking-Page phase transition: $r_{\text{HP}} = l/\sqrt{3}$ is the minimum radius of the extremal super-entropic Kerr-Newman-AdS black hole, whose mass is bounded from the lower limit: $8l/\sqrt{27}$.

Now we turn to consider the extremal case of a ultra-spinning Kerr-Sen-AdS black hole. Its horizon is determined by $\Delta_{e} = \Delta_{e}^{e} = 0$, which yields

$$m_{e} = \frac{2(r_{e} + b_{e})(r_{e}^{2} + 2b_{e}r_{e} + l^{2})}{l^{2}}, \quad b_{e} = \frac{l^{2} - 3r_{e}^{2}}{2r_{e}}.$$  

(36)

By the requirement: $b_{e} = q_{e}^{2}/(2m_{e}) \geq 0$ and also $r_{e} \geq 0$, it is clear that we must have a distinct inequality:

$$0 \leq r_{e} \leq \frac{l}{\sqrt{3}},$$  

(37)

which means that the scale of Hawking-Page phase transition: $r_{\text{HP}} = l/\sqrt{3}$ now becomes the maximum radius of the extremal ultra-spinning Kerr-Sen-AdS black hole. Substituting the inequality (37) into Eq. (36), one can find that the extremal mass still has the same lower bound:

$$m_{e} = \frac{2(l^{2} - r_{e}^{2})^{2}}{l^{2}r_{e}} \geq \frac{8l}{3\sqrt{3}}.$$  

(38)

Therefore, although the extremal Kerr-Newman-AdS super-entropic black hole and the extremal ultra-spinning Kerr-Sen-AdS black hole share the same lower mass bound, the Hawking-Page phase transition scale: $r_{\text{HP}} = l/\sqrt{3}$ marks the dividing crest of their horizon radii. This is a remarkable signature to distinguish these two ultra-spinning charged AdS$_{4}$ black holes.

F. RII

Almost a decade ago, it is conjectured [25] that the AdS black hole satisfies the following RII:

$$R_{II} = \frac{r_{e}^{3}V}{\mathcal{A}} \geq 1,$$  

(39)

where $\mathcal{A}_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of the unit $(D-2)$-sphere and $A = 4S$ is the horizon area. Equality is attained for the Schwarzschild-AdS black hole, which implies that the Schwarzschild-AdS black hole has the maximum entropy. In other words, it indicates that for a given entropy, the Schwarzschild-AdS black hole occupies the least volume, and hence is most efficient in storing information.

It is straightforward to check whether the ultra-spinning Kerr-Sen-AdS black hole satisfies this RII or not. It is readily known that the area of the unit two-dimensional sphere is: $\mathcal{A}_{2} = 2\mu$, the thermodynamic volume is: $V = 4(r_{e} + b_{e})S/3$, and the horizon area is: $A = 4S = 2\mu(r_{e}^{2} + 2br_{e} + l^{2})$. Consequently, the isoperimetric ratio now reads

$$R_{II} = \left(\frac{r_{e} + b_{e}}{2\mu}A\right)^{\frac{1}{3}} \left(\frac{2\mu}{A}\right)^{\frac{1}{2}} = \left(\frac{r_{e}^{2} + 2br_{e} + l^{2}}{r_{e}^{2} + 2br_{e} + l^{2}}\right)^{\frac{1}{3}}.$$  

(40)

Obviously, the value range of $R_{II}$ is uncertain. If $b_{e}^{2} < l^{2}$ (namely, $q_{e}^{2} < 2ml$), then $R_{II} < 1$. In this case, the ultra-spinning Kerr-Sen-AdS black hole violates the RII, and is super-entropic. Otherwise if $b_{e}^{2} \geq l^{2}$, one then has $R_{II} \geq 1$. In this situation, the ultra-spinning Kerr-Sen-AdS black hole obeys the RII, and is sub-entropic. Since the ratio of $R_{II}$ depends upon the values of the solution parameters ($q$, $m$ and $l$), thus one can see that the ultra-spinning Kerr-Sen-AdS black hole is not always super-entropic. Only when the parameters satisfy $q_{e}^{2} < 2ml$ does it violate the RII, while the Kerr-Newman-AdS super-entropic black hole always violates the RII [21]. As far as this point is concerned, the ultra-spinning Kerr-Sen-AdS black hole and Kerr-Newman-AdS super-entropic black hole exhibit yet another markedly different property. This is one of the main results that we have obtained in this article.

IV. CONCLUSIONS

In this paper, we have studied some interesting properties of the Kerr-Sen-AdS black hole and in particular, its ultra-spinning cousin in the four-dimensional gauged EMDA theory. After a brief review of the famous Kerr-Sen black hole solution, we presented its exquisite generalization to include a nonzero negative cosmological constant, namely the Kerr-Sen-AdS black hole. Then its ultra-spinning cousin
is constructed via employing a simple $a \rightarrow l$ limit procedure. The expressions of these solutions, namely their metric, the Abelian gauge potential, the dilaton scalar and the axion pseudoscalar fields are very convenient for investigating their thermodynamical properties. With these solutions at hand, all thermodynamic quantities that can be computed through the standard method are verified to fulfil both the differential and integral mass formulae. Moreover, new Christodoulou-Ruffini-like squared-mass formulae are displayed for these four-dimensional black holes, from which all expected conjugate pairs are derived via differentiating them with respect to their corresponding thermodynamic variables and are shown to consist of the ordinary canonical conjugate pairs that appear in the standard forms of black hole thermodynamics.

Furthermore, we adopted the method proposed in Ref. [34] to demonstrate that all thermodynamical quantities of the ultra-spinning Kerr-Sen-AdS$_4$ black hole can be attained via applying the same ultra-spinning limit to those of their corresponding predecessors. After that, we have made a detailed discussion about the impact of the chirality condition on the actual thermodynamics of this ultra-spinning black hole. To some extent, these aspects are very similar to those of the Kerr-Newman-AdS$_4$ super-entropic black hole.

What attracts us the most in this work is to peer whether there are some other properties peculiar to the ultra-spinning Kerr-Sen-AdS$_4$ black hole. After investigating its horizon geometry and conformal boundary, we arrived at the conclusion that both of them are still similar to those of the Kerr-Newman-AdS$_4$ super-entropic black hole.

However, when turning to investigate the extremal ultra-spinning black holes and the RII, we indeed discovered that the ultra-spinning Kerr-Sen-AdS$_4$ and super-entropic Kerr-Newman-AdS$_4$ black holes exhibit some significant physical differences.

A summary of three novel consequences obtained in this paper are listed in order:

1. New Christodoulou-Ruffini-like squared-mass formulae are presented both for the usual Kerr-Sen-AdS$_4$ black hole and for its ultra-spinning counterpart. To the best of our knowledge, they do not appear in the literature before, and are useful to study black hole chemistry and possible thermodynamical phase transition of these AdS$_4$ black holes.

2. Remarkably, it have been found that the ultra-spinning Kerr-Sen-AdS$_4$ black hole is not always super-entropic, since the RII is violated only in the space of the solution parameters that satisfy the condition: $q^2 < 2ml$. Once $q^2 \geq 2ml$, the ultra-spinning Kerr-Sen-AdS$_4$ black hole becomes sub-entropic. This property is in sharp contrast with the Kerr-Newman-AdS$_4$ super-entropic black hole which always violates the RII [21].

3. We have established some new inequalities on the mass and horizon radius of the extremal ultra-spinning Kerr-Sen-AdS$_4$ and extremal super-entropic Kerr-Newman-AdS$_4$ black holes. It is observed that while both extremal black holes share the same mass minimum bound: $m_e \geq 8l/\sqrt{27}$, the scale of Hawking-Page phase transition: $r_{HP} = l/\sqrt{3}$ signals the remarkably different sizes of their extremal event horizons. That is, the horizon radius of the extremal ultra-spinning Kerr-Sen-AdS$_4$ black hole never exceeds the Hawking-Page scale $r_{HP}$, while that of the the extremal super-entropic Kerr-Newman-AdS$_4$ black hole always exceeds the same scale $r_{HP}$. This might be taken as a direct signature to distinguish these two extremal ultra-spinning charged AdS$_4$ black holes, which hints that it is possible to judge them via the observation of their shadow sizes.

There are two promising further topics to be pursued in the future. As mentioned above, one intriguing topic is to explore the black hole shadows of two ultra-spinning charged AdS$_4$ solutions, and this might shed light on our knowledge of black holes in gauged supergravity theories. And the other is to extend the present work to the more general dyonic case. We hope to report the related progress along these directions soon.

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