\[ S = T \] FOR SHIMURA VARIETIES AND \( p \)-ADIC SHTUKAS

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Abstract. We prove the \( S = T \) conjecture proposed by Liang Xiao and Xinwen Zhu in [1], making use of Scholze’s theory of diamonds and v-stacks and Fargues-Scholze’s geometric Satake equivalence. We deduce the Eichler-Shimura relation for Shimura varieties of Hodge type.

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1. Introduction
Liang Xiao and Xinwen Zhu have raised a question on \( S = T \) for Shimura varieties in [1], which states that excursion operators on the compactly supported cohomology is the same as Hecke operators, and is an analogue for Vincent Lafforgue’s \( S = T \) theorem in [2]. This is a fundamental question which can be interpreted as a local-global compatibility in a categorical formulation of
Langlands correspondence, see [3]. Moreover, it has consequences on Eichler-Shimura relation for compactly supported cohomology of Shimura varieties of Hodge type.

Strictly speaking, there are two versions of S=T in mixed characteristics (while there is only one in equal characteristic). The first is S=T for mixed characteristic Shtukas, and the second is the one for Shimura varieties. In [1], Xiao-Zhu have proved S=T for Witt vector Shtukas, and zero dimensional Shimura varieties. However, even their version for Shtukas is strictly weaker than Lafforgue’s S=T theorem in equal characteristics, and is again zero dimensional in nature. More precisely, they have proved that the excursion operator associated to a representation of the Langlands dual group acts on the cohomology of zero dimensional Shtukas by the function associated to the representation through classical Satake. While Lafforgue’s S=T theorem states that the Hecke correspondence associated to a representation V is the same as the excursion operator associated to V on any Shtukas.

The reason Xiao-Zhu only prove their results for zero dimensional objects is the same for both Shimura varieties and Shtukas. Namely, they work with Witt vector affine Grassmannian and Shtukas, which is over a characteristic p base, so they can only talk about the special fiber of Shimura varieties and Shtukas. However, Hecke operators at p are only naturally visible (as etale correspondence) in characteristic 0, or at least prime to p. The usual treatment is to take the closure of the Hecke correspondence in an integral model and reduce to characteristic p, this usually creates a lot of degeneration, as the classical example $T_p \equiv \text{Frob} + V$ for modular curves shows. This problem only goes away in zero dimension, where the Hecke operators ”flatten” to characteristic p. Since Xiao-Zhu can only work with the special fiber, they can not even define Hecke operators for general Witt vector Shtukas. Indeed, they only state their conjecture and results for cohomology, which is nothing but passing to zero dimension, where they have the obvious notion of Hecke operators, namely the action of a function.

There is a natural way to proceed. Namely the theory of diamonds and v-stacks developed by Scholze allows us to talk about mixed characteristic Shtukas over generic fibers, so we can define Hecke operators for Shtukas. Further, we have integral models of Shtukas using this framework, whose special fiber is essentially the Witt vector Shtukas considered by Xiao-Zhu, and the proof in this article is in some sense a nearby cycle construction relating cohomology in characteristic 0 to cohomology in characteristic p.

More precisely, the excursion operators on Shtukas are defined by the composition of creation, partial Frobenius and annihilation correspondences, each of which are essentially pullback of cohomological correspondences on Hecke stacks. The key to the construction of these correspondences on the Hecke Stacks is the geometric Satake established by Fargues and Scholze in [4], which can be viewed as a globalization of the Witt vector geometric Satake established by Xinwen Zhu. We can perform this construction on the integral models of Shtukas, whose restriction to the special fiber are the excursion operators for Witt vector Shtuaks constructed by Xiao-Zhu.

On the other hand, we can prove that the excursion operators on the generic fiber is the same as the Hecke operators. We follow the same strategy as Vincent Lafforgue’s proof in [2], which is basically to move legs to reduce to the zero dimensional case that has already been established by Xiao-Zhu.

Now we have the $S = T$ for p-adic Shtukas, and we want to deduce the corresponding $S = T$ for Shimura varieties. Recall that excursion operators for the special fiber of Shimura varieties are
defined as the pullback of the excursion operators for Witt vector Shtukas through a crystalline period map from Shimura varieties to Shtukas, see [1]. We generalise this construction to a period map from integral models of Shimura varieties to (integral) $p$-adic Shtukas. The special fiber of this map is essentially the old crystalline period map, while the generic fiber is closely related to (a quotient of) Hodge-Tate period map in [5] or [6]. The construction of this map makes essential use of the classification of $p$-divisible groups over integral perfectoid rings in terms of Breuil-Kisin-Fargues modules in appendix of lecture 17 in [7]. Now the pullback of the excursion operators on $p$-adic Shtukas along this period map produces the excursion operators on Shimura varieties, and the $S = T$ for Shtukas pullback to the $S = T$ for Shimura varieties.

The integral period map also appears in the recent work [8] of Georgios Pappas and Michael Rapoport. Indeed, they constructed it more generally for Shimura varieties with parahoric reduction, while we work only with the good reduction case. Moreover, they construct it on the full adic space associated to the integral model of Shimura varieties, while we work only with the good reduction locus.

The Eichler-Shimura relation for Shimura varieties of Hodge type has also recently been established by Si Ying Lee in [9]. We provide an extensive remark on the comparison between the two approaches to congruence relations, which we believe to help clarify the ideas underlying this paper.

The strategy of proof in [9] is similar to that of Faltings-Chai ([10]), Torsten Wedhorn ([11]) and many others. The key to all approaches to congruence relations is to detect the degeneration of $p$-isogenies between abelian varieties from characteristic 0 to characteristic $p$ using linear algebra data coming from cohomology theory, and the difference lies in which cohomology theory to use. The approach in [10] is to use étale cohomology to detect the effect of specialization of isogenies. For ordinary reduction, this is easy, since the specialization just adds a canonical filtration on the Tate module, and the effect on isogenies is simply to preserve this filtration. Hence their proof makes crucial use of the ordinary abelian varieties, which restricts the applicability of the method, namely one has to add a condition on the density of ordinary isogeny locus.

To get rid of this condition, it is necessary to look at the specialization of general abelian varieties and their isogenies, and detect them using linear algebra data. This is exactly what $p$-adic Hodge theory could help us. In the ordinary case, $p$-adic Hodge theory is hidden since the comparison already takes places at the rational level, namely the canonical filtration. In general, the period rings manifest themselves. However, it is hard to find a direct linear algebra description of the specialization as in the ordinary case. Indeed, it is well-known that the relation between the Hodge-Tate period of $p$-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ and the Dieudonne module of its special fiber is mysterious. The way we proceed is exactly the way Scholze describes the mysterious relation in [7] using intermediate integral objects, namely $p$-adic Shtukas.

The lack of explicit linear algebra models in the previous approach is remedied by the high power functorial geometric models, which puts us in the framework of geometric Langlands. In particular, the degeneration is described by the comparison theorems in $p$-adic Hodge, in the disguise of the Fargues Fontaine curve (or closely related objects), and the lack of explicit identification is remedied by the functoriality of Fargues Fontaine curve, which significantly mechanizes the situation by putting us in the framework of diamonds and v-stacks, so we can mimic what has been done in geometric Langlands. More precisely, the degeneration is reflected by the behaviour of excursion
operators when the legs of Shtukas collide, somehow the excursion operators "flatten" the situation by allowing extra freedom on adding legs and intermediate modifications, where the magic ultimately boils down to the fusion product in geometric Satake.

We learned from Peter Scholze after initial progress on this project that Xinwen Zhu also has a proof of \( S = T \) conjecture for some time, which in principle should be similar to the proof presented here.

We now briefly describe the content of this paper. In section 2, we provide definitions for all the spaces that we will use. In section 3, we construct the excursion operators for Shtukas using geometric Satake. In section 4, we prove the \( S = T \) for Shtukas. In the last section, we construct the integral period map from Shimura varieties to Shtukas, and prove the \( S = T \) for Shimura varieties. We collect some results on the comparison between cohomology of algebraic stacks and v-stacks, which is used in the article to compare the construction given here with the construction in [1].

**Convention.** We fix a prime \( p \) throughout. We will work with the theory of v-stacks and diamonds as developed in [12], by definition they are stacks over the category of characteristic \( p \) perfectoid spaces with v-topology. Unlike the usual notation, a prestack is meant to be a groupoid valued functor from the category of characteristic \( p \) perfectoid spaces, without separatedness condition.

We follow the notation in [12] to denote \( X^\diamond \) the diamond associated to an adic space \( X \) over \( \mathbb{Z}_p \), which parameterizes untilts \( S^\sharp \) of \( S \) together with a map \( S^\sharp \to X \). When \( X = \text{Spa}(A, A^+) \), we denote as usual \( X^\diamond = \text{Spd}(A, A^+) \). When \( A^+ \) is clear from the situation, we abbreviate the notation by writing \( \text{Spd}(A) = \text{Spd}(A, A^+) \). For example, when \( K \) is a non-archimedean field, we write \( \text{Spd}(K) = \text{Spd}(K, \mathcal{O}_K) \).

Let \( X \) be an algebraic stack or scheme locally of finite type over a discrete valuation ring \( \mathcal{O} \) with perfect residue field, as in appendix A.1, we write \( X^\diamond \) (resp. \( X^\diamond\diamond \)) for the stackification of the prestack sending a characteristic \( p \) perfectoid space \( \text{Spa}(R, R^+) \) to untilts \( \text{Spa}(R^\sharp, R^{\sharp+}) \) over \( \mathcal{O} \) together with a map \( \text{Spec}(R^\sharp) \to X \) (resp. \( \text{Spec}(R^\sharp) \to X \)) of schemes over \( \mathcal{O} \). When \( X \) is a scheme, this deviates from the notation in [12] section 27, where they denote \( X^\diamond \) for our \( X^\diamond\diamond \).

For \( E \) an extension of \( \mathbb{Q}_p \), \( \text{Spd}(E) \) is the diamond parameterizing characteristic 0 untilts with a morphism to \( \text{Spa}(E, \mathcal{O}_E) \), and \( \text{Spd}(\mathcal{O}_E) \) is the v-sheaf parameterizing all untilts with a morphism to \( \text{Spa}(\mathcal{O}_E, \mathcal{O}_E) \).

Following the notation of [7], we denote
\[
\mathcal{Y}_{[0, \infty)}(S) := \text{Spa}(W(R^+), W(R^+)) \setminus V([\omega])
\]
for \( S = \text{Spa}(R, R^+) \) a characteristic \( p \) affinoid perfectoid space with topological nilpotent units \( \omega \), and the general \( S \) by gluing. We have a natural identification
\[
\mathcal{Y}_{[0, \infty)}(S)^\circ = S \times \text{Spd}(\mathbb{Z}_p)
\]
Then an untilt of \( S \) is equivalent to a section of \( \mathcal{Y}_{[0, \infty)}(S)^\circ \to S \), giving rise to an effective Cartier divisor on \( \mathcal{Y}_{[0, \infty)}(S) \).

We denote
\[
\mathcal{X}_{\text{FF}, S} := \mathcal{Y}_{[0, \infty)}(S) / Frob^\mathbb{Z}_S
\]
the relative Fargues-Fontaine curve with respect to $S$, where

$$\mathcal{Y}_{(0,\infty)}(S) := \text{Spa}(W(R^+), W(R^+)) \setminus V(p[\omega])$$

for $S = \text{Spa}(R, R^+)$. We note that $\mathcal{Y}_{(0,\infty)}(S)$ is an open subspace of $\mathcal{Y}_{(0,\infty)}(S)$, the generic fiber of $\mathcal{Y}_{(0,\infty)}(S)$ over $\mathbb{Z}_p$, whose analytification can be identified with $S \times \text{Spd}(\mathbb{Q}_p)$.

Now let $S^\sharp = \text{Spa}(R^\sharp, R^{\sharp,+})$ be an untilt of $S$, we denote

$$\mathcal{O}^\wedge_{\mathcal{Y}_{(0,\infty)}(S), S^\sharp} := W(R^+)[\frac{1}{[\omega]}]^\wedge,$$

i.e. the $\xi$-adic completion of $W(R^+)[\frac{1}{[\omega]}]$, where $\xi \in W(R^+)$ is a generator of the kernel of $W(R^+) \to R^{\sharp,+}$, and $\omega \in R^+$ is a pseudouniformizer. Note that

$$\mathcal{O}^\wedge_{\mathcal{Y}_{(0,\infty)}(S), S^\sharp} = \mathbb{B}^+_\text{dR}(R^\sharp)$$

if $R^\sharp$ is of characteristic 0, and

$$\mathcal{O}^\wedge_{\mathcal{Y}_{(0,\infty)}(S), S^\sharp} = W(R)$$

if $S^\sharp = S$, where $\mathbb{B}^+_\text{dR}(R^\sharp)$ is the $\xi$-adic completion of $W(R^+)[\frac{1}{[\omega]}]$ with $\xi \in W(R^+)$ being a generator of the kernel of the canonical map $W(R^+)[\frac{1}{[\omega]}] \to R^\sharp$.

When we have finite many untilts $S_i^\sharp = \text{Spa}(R_i^\sharp, R_i^{\sharp,+})$ of $S$ (possibly not all distinct), we denote

$$\mathcal{O}^\wedge_{\mathcal{Y}_{(0,\infty)}(S), \sum S_i^\sharp} := W(R^+)[\frac{1}{[\omega]}]^\wedge,$$

where $\xi = \prod \xi_i$ with $\xi_i$ a generator of the kernel of $W(R^+) \to R_i^{\sharp,+}$.

We will use frequently the formal properties of cohomological correspondences. Let $X, Y$ and $Z$ be small $\nu$-stacks which forms a diagram

$$\begin{array}{ccc}
Z & \xleftarrow{q} & Y \\
\downarrow{p} & & \downarrow{q} \\
X & & \\
\end{array}$$

Let $A \in D_{\text{et}}(X, \Lambda)$ and $B \in D_{\text{et}}(Y, \Lambda)$, we denote

$$\mathcal{C} : (X, A) \to (Y, B)$$

a cohomological correspondence supported on $Z$ to be a map

$$\mathcal{C} : p^* A \to q^1 B.$$
Let us given a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{p} & & \downarrow{q} \\
Y & \xleftarrow{p'} & Z \\
\downarrow{f} & & \downarrow{q'} \\
X & \xrightarrow{g} & Y
\end{array}
\]

of small \(v\)-stacks. Let \(C : (X, A) \rightarrow (Y, B)\) be a cohomological correspondence supported on \(Z\), then we can pullback the \(C\) along the diagram to obtain a cohomological correspondence

\[f^*C : (X, h^*A) \rightarrow (Y, g^*B)\]

supported on \(Z\), if either the right square of the diagram is Cartesian or \(f, g\) and \(h\) are cohomologically smooth, see [1] appendix A.2.

On the other hand, let \(C : (X, A) \rightarrow (Y, B)\) be a cohomological correspondence supported on \(Z\), then we can \(!\)-pushforward \(C\) to obtain a cohomological correspondence

\[Rf_!C : (X, Rh_!A) \rightarrow (Y, Rg_!B)\]

supported on \(Z\), if \(p'\) and \(p\) are proper, see [1] appendix A.2.

Moreover, we can compose cohomological correspondence, and pullback and \(!\)-pushforward commute with composition.

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2. Basic Definitions

Let \(G\) be a reductive group over \(\mathbb{Q}_p\), together with a reductive model \(\mathcal{G}\) over \(\mathbb{Z}_p\). Then \(G\) is unramified, so split over \(L := W((\mathbb{F}_p))[[\frac{1}{p}]]\). Let \(\mu\) be a conjugacy class of cocharacters \(\mathbb{G}_m \rightarrow G_{\overline{\mathbb{Q}_p}}\), whose field of definition is \(E\), which is contained in \(L\).

2.1. \(p\)-adic Shtukas and Hecke stacks.

**Definition 2.1.** The moduli space of Shtukas associated to \(\mathcal{G}\) and \(\mu\) is the prestack

\[\text{Sht}_{\mu}\]

over \(\text{Spd}(\mathcal{O}_L)\) whose value at a characteristic \(p\) perfectoid space \(S\) is the groupoid of the data of

- a \(\mathcal{G}\)-torsor \(\mathcal{P}\) over \(\mathcal{Y}_{[0,\infty)}(S)\),
- an (not necessarily of characteristic 0) untilt \(S'^{\dagger} \hookrightarrow \mathcal{Y}_{[0,\infty)}(S)\) of \(S\) defined over \(\mathcal{O}_L\), and
• an isomorphism
\[ \varphi_P : \mathcal{P}\big|_{\mathcal{Y}_{[0,\infty)}(S)} \cong \text{Frob}_S^* \mathcal{P}\big|_{\mathcal{Y}_{[0,\infty)}(S)}, \]
that is meromorphic along the Cartier divisor \( S^2 \rightarrow \mathcal{Y}_{[0,\infty)}(S) \), and bounded by \( \mu \) in the sense that for any geometric rank 1 point \( C \) of \( S \), and trivializations of \( \text{Frob}_C^* \mathcal{P} \) and \( \mathcal{P} \) over \( \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(C) \cup \{ \frac{1}{2} \}} \), the "punctured formal neighborhood of \( C^2 \)", \( \varphi_P \) lies in \( \mathcal{P}(\mathcal{O}_{\mathcal{Y}_{[0,\infty)}(C) \cup \{ \frac{1}{2} \}}) \mu(t) \mathcal{P}(\mathcal{O}_{\mathcal{Y}_{[0,\infty)}(C) \cup \{ \frac{1}{2} \}}), \) where \( t \in \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(C) \cup \{ \frac{1}{2} \}} \) is a uniformizer. Note that this is a pointwise condition on \( S \).

The isomorphism is defined to be an isomorphism of the torsors that preserves \( \varphi_P \).

Remark 2.2. Although we work with the base \( \mathcal{O}_L \), the definition of Shtukas really depends on the group \( \mathcal{G} \) over \( \mathbb{Z}_p \) rather than its base change to \( \mathcal{O}_L \), as otherwise the Frobenius pullback \( \text{Frob}_S^* \mathcal{P} \) does not make sense (it will be a torsor with respect to a Frobenius twist of the structure group over \( \mathcal{O}_L \)).

Remark 2.3. The definition given here is close to the classical function field counterpart, which is slightly different from the ones in [7] in that we do not fix a level structure, nor do we fix the type of the Shtuka in \( B(G) \). The treatment in [7] is to emphasize the way Shtukas generalize Rapoport-Zink spaces, whereas we want to mimic the treatment of Vincent Lafforgue.

We will also make use of Shtukas with several legs.

Definition 2.4. Let \( I \) be a finite set, and \( \mu_* \) be a collection of dominant cocharacters of \( G \) indexed by \( I \). Let \( \{I_1, \cdots, I_k\} \) be a partition of \( I \). The moduli space of Shtukas associated to \( G, \mu_* \) and \( \{I_1, \cdots, I_k\} \) is the prestack
\[ \text{Sht}_{\mu_*}^{(I_1, \cdots, I_k)} \]
over \( \text{Spd}(\mathcal{O}_L)^I \) whose value at a characteristic \( p \) perfectoid space \( S \) is the groupoid of the data of
- \( \mathcal{G} \)-torsors \( \mathcal{P}_1, \cdots, \mathcal{P}_k \) over \( \mathcal{Y}_{[0,\infty)}(S) \),
- (not necessarily of characteristic 0) untilts \( S^2_i \rightarrow \mathcal{Y}_{[0,\infty)}(S) \) of \( S \) defined over \( \mathcal{O}_L \) indexed by \( i \in I \), and
- modifications
\[ \mathcal{P}_1 \rightarrow \cdots \rightarrow \mathcal{P}_2 \rightarrow \cdots \rightarrow \mathcal{P}_k \rightarrow \text{Frob}_S^* \mathcal{P}_1 \]
where the dotted arrows \( \varphi_i \) are isomorphisms
\[ \varphi_i : \mathcal{P}_i\big|_{\mathcal{Y}_{[0,\infty)}(S) \cup \bigcup_{j \in I_i} S^2_j} \cong \mathcal{P}_{i+1}\big|_{\mathcal{Y}_{[0,\infty)}(S) \cup \bigcup_{j \in I_i} S^2_j} \]
with \( \mathcal{P}_{k+1} = \text{Frob}_S^* \mathcal{P}_1 \), which are meromorphic along the Cartier divisor \( \bigcup_{j \in I_i} S^2_j \rightarrow \mathcal{Y}_{[0,\infty)}(S) \), and bounded by \( \sum \mu_j \) at every geometric point of \( S^2 \).

When the partition is trivial, we drop the upper script to simplify the notation, i.e.
\[ \text{Sht}_{\mu_*} := \text{Sht}_{\mu_*}^{(I)} \]
which parametrizes
\[ \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S) \cup_{i \in I} S_i^{\sharp}} \cong \text{Frob}_S^* \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S) \cup_{i \in I} S_i^{\sharp}}. \]

The following variant defines a correspondence between Shtukas, and will be very important for our purpose.

**Definition 2.5.** Let \( \mu, \mu' \) and \( \nu \) be conjugacy classes of cocharacters \( \mathbb{G}_m \to G_{\overline{\mathbb{Q}_p}} \), with \( E \) and \( F \) the field of definition of \( \mu \) and \( \mu' \) respectively. We define
\[ \text{Sht}^\nu_{\mu | \mu'} \]
to be the prestack with value at a characteristic \( p \) perfectoid space \( S \) being the groupoid of the data of
- \( \mathcal{P} \in \text{Sht}_{\mu}(S) \) and \( \mathcal{P}' \in \text{Sht}_{\mu'}(S) \),
- a modification bounded by \( \nu \) (in the sense similar to the above)
\[ \gamma : \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)} \cong \mathcal{P}'|_{\mathcal{Y}_{[0,\infty)}(S)} \]
at the characteristic until \( S \hookrightarrow \mathcal{Y}_{[0,\infty)}(S) \) as Shtukas, i.e. the following diagram is commutative
\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\varphi} & \text{Frob}_S^* \mathcal{P} \\
\downarrow{\gamma} & & \downarrow{\text{Frob}_S \gamma} \\
\mathcal{P}' & \xrightarrow{\varphi'} & \text{Frob}_S^* \mathcal{P}'
\end{array}
\]
with the notation that dashed arrows denote generically defined modifications.

We denote \( \text{Sht}^\nu_{\mu | \mu'} \) the union of \( \text{Sht}^\nu_{\mu | \mu'} \) for all \( \nu \).

Similarly, we have the version for Shtukas with several legs.

**Definition 2.6.** Let \( I, J \) be finite sets, and \( \mu_\bullet, \mu'_\bullet \) be collections of dominant cocharacters of \( G \) indexed by \( I \) and \( J \) respectively. Let \( \{I_1, \ldots, I_k\} \) and \( \{J_1, \ldots, J_l\} \) be partitions of \( I \) and \( J \). Let \( \nu \) be a dominant cocharacter of \( G \). Then we define
\[ \text{Sht}^{(I_1, \ldots, I_k),(J_1, \ldots, J_l)}_{\mu_\bullet | \mu'_\bullet} \]
to be the prestack with value at a characteristic \( p \) perfectoid space \( S \) being the groupoid of the data of
- \( \{\mathcal{P}_1, \ldots, \mathcal{P}_k, \varphi_1, \ldots, \varphi_k\} \in \text{Sht}^{(I_1, \ldots, I_k)}_{\mu_\bullet}(S) \) and \( \{\mathcal{P}'_1, \ldots, \mathcal{P}'_l, \varphi'_1, \ldots, \varphi'_l\} \in \text{Sht}^{(J_1, \ldots, J_l)}_{\mu'_\bullet}(S) \),
- a modification bounded by \( \nu \) (in the sense similar to the above)
\[ \gamma : \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)} \cong \mathcal{P}'|_{\mathcal{Y}_{[0,\infty)}(S)} \]
at the characteristic until \( S \hookrightarrow \mathcal{Y}_{[0,\infty)}(S) \) as Shtukas, i.e. the following diagram is commutative
\[
\begin{array}{cccccccc}
\mathcal{P}_1 & \xrightarrow{\varphi_1} & \mathcal{P}_2 & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_{k-1}} & \mathcal{P}_k & \xrightarrow{\varphi_k} & \text{Frob}_S^* \mathcal{P}_1 \\
\downarrow{\gamma} & & \downarrow{\gamma} & & \cdots & & \downarrow{\gamma} & & \downarrow{\text{Frob}_S \gamma} \\
\mathcal{P}'_1 & \xrightarrow{\varphi'_1} & \mathcal{P}'_2 & \xrightarrow{\varphi'_2} & \cdots & \xrightarrow{\varphi'_{l-1}} & \mathcal{P}'_l & \xrightarrow{\varphi'_l} & \text{Frob}_S^* \mathcal{P}'_1
\end{array}
\]
with the usual notation that dashed arrows denote generically defined modifications.

When the partitions of $I$ and $J$ are trivial, we will simplify the notation by writing

$$\text{Sht}_{\mu|\mu'} := \text{Sht}_{(I)}$$

which parametrizes

$${\mathcal P} \xrightarrow{\phi} \text{Frob}_S^* {\mathcal P}$$

and similarly for

Remark 2.7. Let $E_i$ be the reflex field of $\mu_i$, then $\text{Sht}_{(I_1, \ldots, I_k)}$ is actually defined over $\prod_{i} \text{Spd}(O_{E_i})$, and similarly for $\text{Sht}_{(I_1, \ldots, I_k)}$. The same refinement works for all the spaces defined in this section.

Notation 2.8. Throughout the paper, we use the following notation to denote the restriction to the generic fiber,

$$\text{Sht}_{(I_1, \ldots, I_k)} := \text{Sht}_{(I_1, \ldots, I_k)}|_{\text{Spd}(L)^I},$$

$$\text{Sht}_{(I_1, \ldots, I_k)} := \text{Sht}_{(I_1, \ldots, I_k)}|_{\text{Spd}(O_E)^I \times \text{Spd}(L)^I},$$

and in particular

$$\text{Sht}_{(I_1, \ldots, I_k)} := \text{Sht}_{(I_1, \ldots, I_k)}|_{\text{Spd}(O_E)^I \times \text{Spd}(L)^I}.$$

Note that in the last case, $\mu$ has to equal to $\mu'$.

Similarly, we have the restriction to the special fiber

$$\text{Sht}_{(I_1, \ldots, I_k)} := \text{Sht}_{(I_1, \ldots, I_k)}|_{\text{Spd}(\mathbb{F}_p)^I},$$

$$\text{Sht}_{(I_1, \ldots, I_k)} := \text{Sht}_{(I_1, \ldots, I_k)}|_{\text{Spd}(\mathbb{F}_p)^I \times \text{Spd}(\mathbb{F}_p)^I}.$$

We now introduce the Hecke stacks.

Definition 2.9. Let $I$ be a finite set, and $\mu_*$ be a collection of dominant cocharacters of $G$ indexed by $I$. Let $\{I_1, \ldots, I_k\}$ be a partition of $I$. We define

$$\text{Hecke}_{(I_1, \ldots, I_k)}$$

to be the prestack over $\text{Spd}(O_L)^I$ whose value at $S$ is the groupoid of

- $G$-torsors $\mathcal{P}_1, \ldots, \mathcal{P}_{k+1}$ on $\mathcal{Y}_{[0,\infty)}(S)$
- untilts $S^g_i \hookrightarrow \mathcal{Y}_{[0,\infty)}(S)$ of $S$ defined over $O_L$ indexed by $i \in I$, and
- modifications

$${\mathcal P}_1 \xrightarrow{\phi_1} \mathcal{P}_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{P}_k \xrightarrow{\phi_k} \mathcal{P}_{k+1},$$

where the dotted arrows $\phi_i$ are isomorphisms

$$\phi_i : \mathcal{P}_i|_{\mathcal{Y}_{[0,\infty)}(S)\cup \bigcup_{j \in I_j} S^g_j} \cong \mathcal{P}_{i+1}|_{\mathcal{Y}_{[0,\infty)}(S)\cup \bigcup_{j \in I_j} S^g_j}.$$
that are meromorphic along the Cartier divisor \( \bigcup_{j \in I_i} S_j^\sharp \to \mathcal{Y}_{[0,\infty)}(S) \), and bounded by \( \sum_{S_j^\sharp = S_i^\sharp} \mu_j \) at every geometric point of \( S^\sharp \).

When the partition is trivial, we make the following simplification

\[
\text{Hecke}_{\mu^\bullet} := \text{Hecke}_{\mu^\bullet}^{(I)},
\]

which parametrizes

\[
\varphi : \mathcal{P}_1|_{\mathcal{Y}_{[0,\infty)}(S) \setminus \bigcup_{i \in I} S_i^\bullet} \cong \mathcal{P}_2|_{\mathcal{Y}_{[0,\infty)}(S) \setminus \bigcup_{i \in I} S_i^\bullet}.
\]

Lastly, we denote

\[
\text{Hecke}^{(I_1,\ldots,I_k)} := \bigcup_{\mu^\bullet} \text{Hecke}^{(I_1,\ldots,I_k)}_{\mu^\bullet}.
\]

**Remark 2.10.** We have remarked that \( \text{Hecke}^{(I_1,\ldots,I_k)}_{\mu^\bullet} \) is really defined over the product of the defining fields of \( \mu^\bullet \). The base can be further refined if we do not specify the boundedness condition. Indeed, \( \text{Hecke}^{(I_1,\ldots,I_k)} \) can be naturally defined over \( \text{Spd}(\mathbb{Z}_p)^I \).

Similarly, we have the correspondence version.

**Definition 2.11.** Let \( I,J \) be finite sets, and \( \mu^\bullet, \mu'_\bullet \) be collections of dominant cocharacters of \( G \) indexed by \( I \) and \( J \) respectively. Let \( \nu \) be a dominant cocharacter of \( G \). Then we define

\[
\text{Hecke}^{\nu}_{\mu^\bullet | \mu'_\bullet}
\]

to be the prestack with value at a characteristic \( p \) perfectoid space \( S \) being the groupoid of the data of

- \( \{\mathcal{P}_1, \mathcal{P}_2, \varphi\} \in \text{Hecke}_{\mu^\bullet}(S) \) and \( \{\mathcal{P}_1', \mathcal{P}_2', \varphi'\} \in \text{Hecke}_{\mu'_\bullet}(S) \),

- modifications bounded by \( \nu \) (in the sense similar to the above)

\[
\gamma_1 : \mathcal{P}_1|_{\mathcal{Y}_{[0,\infty)}(S)} \cong \mathcal{P}_1'|_{\mathcal{Y}_{[0,\infty)}(S)}
\]

\[
\gamma_2 : \mathcal{P}_2|_{\mathcal{Y}_{[0,\infty)}(S)} \cong \mathcal{P}_2'|_{\mathcal{Y}_{[0,\infty)}(S)}
\]

at the characteristic \( p \) until \( S \to \mathcal{Y}_{[0,\infty)}(S) \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{\varphi} & \mathcal{P}_2 \\
\downarrow \gamma_1 & & \downarrow \gamma_2 \\
\mathcal{P}_1' & \xrightarrow{\varphi'} & \mathcal{P}_2'
\end{array}
\]

with the usual notation that dashed arrows denote generically defined modifications.

We will occasionally use the following more general version.

**Definition 2.12.** Let \( I,J \) be finite sets, and \( \mu^\bullet, \mu'_\bullet \) be collections of dominant cocharacters of \( G \) indexed by \( I \) and \( J \) respectively. Let \( \{I_1,\ldots,I_k\} \) and \( \{J_1,\ldots,J_l\} \) be partitions of \( I \) and \( J \). Let \( \nu \) be a dominant cocharacter of \( G \). Then we define

\[
\text{Hecke}^{(I_1,\ldots,I_k)|J_1,\ldots,J_l}}_{\mu^\bullet | \mu'_\bullet, \nu}
\]
to be the prestack with value at a characteristic $p$ perfectoid space $S$ being the groupoid of the data of

- $\{P_1, \cdots, P_{k+1}, \varphi_1, \cdots, \varphi_k\} \in \text{Hecke}^{(I_1, \cdots, I_k)}_{\mu_\bullet}(S)$ and $\{P_1', \cdots, P_{l+1}', \varphi_1', \cdots, \varphi_l'\} \in \text{Hecke}^{(J_1, \cdots, J_l)}_{\mu_\bullet}(S)$,
- modifications bounded by $\nu$ (in the sense similar to the above)

\[ \gamma_1 : P_1|_{\mathcal{Y}_{(0, \infty)}(S)} \cong P_1'|_{\mathcal{Y}_{(0, \infty)}(S)} \]
\[ \gamma_2 : P_2|_{\mathcal{Y}_{(0, \infty)}(S)} \cong P_2'|_{\mathcal{Y}_{(0, \infty)}(S)} \]

at the characteristic $p$ untilt $S \hookrightarrow \mathcal{Y}_{(0, \infty)}(S)$ such that the following diagram is commutative

\[
\begin{array}{ccccccc}
P_1 \xrightarrow{\varphi_1} P_2 & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_{k-1}} & P_k & \xrightarrow{\varphi_k} & P_{k+1} \\
\downarrow{\gamma_1} & & & & & & \\
\cdots \quad | \quad \cdots & & & & & & \\
\downarrow{\gamma_1} & & & & & & \\
P_1' \xrightarrow{\varphi_1'} P_2' & \xrightarrow{\varphi_2'} & \cdots & \xrightarrow{\varphi_{l-1}'} & P_l' & \xrightarrow{\varphi_l'} & P_{l+1}'
\end{array}
\]

with the usual notation that dashed arrows denote generically defined modifications.

We will also make use of a local version of Hecke stacks, which parameterizes modifications of torsors over the formal neighborhoods of Cartier divisors corresponding to untilts.

**Definition 2.13.** Let $I$ be a finite set, and $\mu_\bullet$ be a collection of dominant cocharacters of $G$ indexed by $I$. Let $\{I_1, \cdots, I_k\}$ be a partition of $I$. We define

\[ \text{Hecke}_{\mu_\bullet}^{\text{loc},(I_1, \cdots, I_k)} \]

to be the prestack over $\text{Spd}(O_L)^I$ whose value at $S$ is the groupoid of

- untilts $S_i^\sharp$ of $S$ defined over $O_L$ indexed by $i \in I$,
- $G$-torsors $P_1, \cdots, P_{k+1}$ on $\text{Spec}(O_{\mathcal{Y}_{[0, \infty)}(S)}^{\otimes} \sum_s S_i^\sharp)$, and,
- modifications

\[ P_1 \xrightarrow{\varphi_1} P_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} P_k \xrightarrow{\varphi_k} P_{k+1}, \]

where the dotted arrows $\varphi_i$ are isomorphisms

\[ \varphi_i : P_i|_{O_{\mathcal{Y}_{[0, \infty)}(S)}^{\otimes} \sum_s S_i^\sharp} \xrightarrow{\otimes_{j \in I_i} \frac{1}{s_j}} P_{i+1}|_{O_{\mathcal{Y}_{[0, \infty)}(S)}^{\otimes} \sum_s S_i^\sharp} \]

that are bounded by $\sum_{S_j^\sharp} \mu_j$ at the point $S^\sharp$, where again we have product over distinct $S_j^\sharp$.

When the partition is trivial, we make the following simplification

\[ \text{Hecke}_{\mu_\bullet}^{\text{loc},(I)} := \text{Hecke}_{\mu_\bullet}^{\text{loc},(I)}(I). \]

Lastly, we denote

\[ \text{Hecke}_{\mu_\bullet}^{\text{loc},(I_1, \cdots, I_k)} := \bigcup_I \text{Hecke}_{\mu_\bullet}^{\text{loc},(I_1, \cdots, I_k)}. \]
Remark 2.14. The drawback of local Hecke stacks is that there is no obvious correspondence between them, due to that the torsors are defined on different domains. However, this problem disappears if we fix the legs at characteristic $p$ untilts. An important example is the Witt vector local Hecke stacks to be defined below, where we have the correspondences.

There is a canonical map
\[ \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k)} \to \text{Hecke}_{\mu_*}^{\text{loc}, (I_1, \cdots, I_k)} \]
being the restriction from global $G$-torsors to local $G$-torsors. We will use it to define the canonical perverse sheaves on $\text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k)}$.

Let $L^+ G$ be the group $v$-sheaf over $\text{Spd}(\mathcal{O}_L)$ whose value at $S$ is
\[ L^+ G(S) = \{ (x, S^\sharp) | S^\sharp \in \text{Spd}(\mathcal{O}_L)(S), x \in G(\mathcal{O}_{\mathcal{Y}_{\lambda, \infty}}(S), S^\sharp) \}, \]
and $\text{Gr}_{G, \text{Spd}(\mathcal{O}_L), \leq \mu}$ be the Beilinson-Drinfeld affine Grassmannian as defined in \cite{7} 20.3.1, which classifies $G$-torsors on the formal neighborhood of $S^\sharp$ together with a trivialization of the generic fiber that is bounded by $\mu$. By \cite{4} 20.5.4, $\text{Gr}_{G, \text{Spd}(\mathcal{O}_L), \leq \mu} \to \text{Spd}(\mathcal{O}_L)$ is representable in spatial $v$-sheaf. By definition, we have
\[ \text{Hecke}_{\mu_*}^{\text{loc}} = L^+ G \setminus \text{Gr}_{G, \text{Spd}(\mathcal{O}_L), \leq \mu} \]
where $L^+ G$ acts on the trivialization, so $\text{Hecke}_{\mu_*}$ is a small $v$-stack.

We have similar expression for the global Hecke stack $\text{Hecke}_{\mu}$, which can be written as the quotient of $\text{Gr}_{G, \text{Spd}(\mathcal{O}_L), \leq \mu}$ by the group $v$-sheaf parameterizing automorphisms of trivial global $G$-torsors on $\mathcal{Y}_{\lambda, \infty}$. Moreover, for Hecke stacks with multiple legs, it can be written again as quotients of twisted products of affine Grassmannians.

If the characteristic of the untilt $S^\sharp$ is $p$, then $S^\sharp = S$, and
\[ \mathcal{O}_{\mathcal{Y}_{\lambda, \infty}}(S), S^\sharp = W(R) \]
if $S = \text{Spa}(R, R^+)$. Then the fiber of the Beilinson-Drinfeld affine Grassmannian at the characteristic $p$ untilt is the Witt vector affine Grassmannian considered in \cite{13} (we embed perfect schemes into the category of $v$-sheaves as explained in lecture 18 of \cite{7}). On the other hand, if $S^\sharp = \text{Spa}(R^2, R^2, +)$ is a characteristic zero untilt, we have
\[ \mathcal{O}_{\mathcal{Y}_{\lambda, \infty}}(S), S^\sharp = \mathbb{B}^+_{\text{dr}}(R^2), \]
then the fiber of $\text{Gr}_{G, \text{Spd}(\mathcal{O}_L), \leq \mu}$ at $S^\sharp$ is the $\mathbb{B}^+_{\text{dr}}$-affine Grassmannian as defined in \cite{7} lecture 19.

Notation 2.15. We have the similar notation for the restriction of Hecke stack to the generic fiber,
\[ \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k)} : = \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k)}|_{\text{Spd}(L)^1} \]
\[ \text{Hecke}_{\mu_*}^{\text{loc}, (I_1, \cdots, I_k)} : = \text{Hecke}_{\mu_*}^{\text{loc}, (I_1, \cdots, I_k)}|_{\text{Spd}(L)^1} \]
\[ \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k), \nu} : = \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k), \nu}|_{\text{Spd}(L)^1 \times \text{Spd}(L)^1}. \]
Similarly for the restriction to the special fiber,
\[ \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k), s} : = \text{Hecke}_{\mu_*}^{(I_1, \cdots, I_k)}|_{\text{Spd}(\mathbb{F}_p)^1} \]
The correspondence (1) is an étale correspondence, i.e. both bounded by finite étale morphisms.

Proof. The proof of [7, 23.3.1] goes without change. Indeed, in loc.cit. they fix the type of Shtuka in B(G), which corresponds to rigidifying E. Dropping this extra condition gives exactly the data as in the statement.

We have the following alternative characterization of Shtukas over the generic fiber.

**Theorem 2.16.** Sht_{μ,η} is equivalent to the prestack over Spd(L) sending a characteristic p perfectoid space S over L to the groupoid of

- G-torsors E_0 and E over the Fargues-Fontaine curve X_{FF,S} such that E_0 is trivial at every geometric point of S, which corresponds to a pro-étale G(ℚ_p)-torsor L_η on S by [14],
- a characteristic 0 untill S^2 ↪ Y_{[0,∞)}(S) of S defined over L,
- an isomorphism

\[ α : E_0|_{X_{FF,S}\backslash S^2} ≅ E|_{X_{FF,S}\backslash S^2} \]

that is meromorphic along S^2 and bounded by μ,

- a sub G(ℤ_p)-torsor L inside L_η.

Proof. The proof of [7, 23.3.1] goes without change. Indeed, in loc.cit. they fix the type of Shtuka in B(G), which corresponds to rigidifying E. Dropping this extra condition gives exactly the data as in the statement.

We have a natural correspondence diagram

\[
\begin{array}{ccc}
\text{Sht}_{μ,η} & \xrightarrow{p_1} & \text{Sht}_{μ,η} \\
\downarrow \text{Sht}^{ν}_{μ|μ,η} & & \downarrow \text{Sht}_{μ,η} \\
\text{Sht}_{μ|μ,η} & \xrightarrow{p_2} & \text{Sht}_{μ,η}
\end{array}
\]

where p_1 (resp. p_2) sends \{P, P', γ, S^2\} to \{P, S^2\} (resp. \{P', S^2\}).

**Proposition 2.17.** The correspondence [1] is an étale correspondence, i.e. both p_1 and p_2 are finite étale morphisms.

Proof. With the characterization of theorem 2.16 we see that Sht^{ν}_{μ|μ,η}(S) parametrizes

- \{E_0, E, α, L, S^2\} ∈ Sht_{μ,η}(S),
- \{E_0', E', α', L', S^2\} ∈ Sht_{μ,η}(S),
- isomorphisms

\[ β_0 : E_0 \cong E_0' \]
\[ β : E \cong E' \]

such that α' ∘ β_0 = β ∘ α, and under the isomorphism

\[ L_η \cong L'_η \]

of pro-étale G(ℚ_p)-torsors on S induced by β_0, the sub-G(ℤ_p)-torsors L and L' have relative position bounded by ν at every rank 1 geometric point of S.

Now for a given S-point of Sht_{μ,η} specified by the data \{E_0, E, α, L, S^2\} ∈ Sht_{μ,η}(S), we see that the fiber of p_1 at this point is the data of sub-G(ℤ_p)-torsors L' of L_η which have relative position with respect to L bounded by ν at every geometric point of S. Since pro-étale locally \L_η is the
trivial torsor $S \times G(\mathbb{Q}_p)$, the relevant sub-$G(\mathbb{Z}_p)$-torsors corresponds to a subset of $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, which is visibly étale over $S$. Thus we see that $p_1$ is étale after base change to a pro-étale cover, hence it is finite étale by [12] 9.11.

The same argument shows that $p_2$ is finite étale. \qed

2.2. Witt vector Shtukas and Hecke stacks. We will need the Witt vector version of Shtukas and Hecke stacks as well.

**Definition 2.18.** Let $I$ be a finite set, and $\mu_\bullet$ be a collection of dominant cocharacters of $G$ indexed by $I$. The moduli space of Witt vector local Shtukas associated to $G$ and $\mu_\bullet$ is the prestack $\mathcal{Sht}_{\mu_\bullet}^W$ over $\text{Spd}(\overline{\mathbb{F}}_p)$ whose value at a characteristic $p$ affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $\overline{\mathbb{F}}_p$ is the groupoid of the data of

- $G$-torsors $\mathcal{P}_1, \cdots, \mathcal{P}_k$ over $\text{Spec}(W(R^+))$, and
- modifications

\[
\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} \mathcal{P}_k \xrightarrow{\varphi_k} \text{Frob}_S^* \mathcal{P}_1
\]

where the dotted arrows $\varphi_i$ are isomorphisms

\[
\varphi_i : \mathcal{P}_i|_{\text{Spec}(W(R^+)[1/p])} \cong \mathcal{P}_{i+1}|_{\text{Spec}(W(R^+)[1/p])}
\]

with $\mathcal{P}_{k+1} = \text{Frob}_S^* \mathcal{P}_1$, which are bounded by $\mu_i$ at every geometric point of $S$.

We have the local version as well.

**Definition 2.19.** Let $I$ be a finite set, and $\mu_\bullet$ be a collection of dominant cocharacters of $G$ indexed by $I$. The moduli space of Witt vector local Shtukas associated to $G$ and $\mu_\bullet$ is the prestack $\mathcal{Sht}_{\mu_\bullet}^{\text{loc}, W}$ over $\text{Spd}(\overline{\mathbb{F}}_p)$ whose value at a characteristic $p$ perfectoid space $S$ over $\overline{\mathbb{F}}_p$ is the groupoid of the data of

- $G$-torsors $\mathcal{P}_1, \cdots, \mathcal{P}_k$ over $\text{Spec}(\mathcal{O}^\wedge_{\mathcal{Y}_{[0,\infty]}(S), S})$, and
- modifications

\[
\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} \mathcal{P}_k \xrightarrow{\varphi_k} \text{Frob}_S^* \mathcal{P}_1
\]

where the dotted arrows $\varphi_i$ are isomorphisms

\[
\varphi_i : \mathcal{P}_i|_{\text{Spec}(\mathcal{O}_{\mathcal{Y}_{[0,\infty]}(S), S})}|_S \cong \mathcal{P}_{i+1}|_{\text{Spec}(\mathcal{O}_{\mathcal{Y}_{[0,\infty]}(S), S})}|_S
\]

with $\mathcal{P}_{k+1} = \text{Frob}_S^* \mathcal{P}_1$, which are bounded by $\mu_i$ at every geometric point of $S$.

The possibility of having such a definition come from that Frobenius fixes the characteristic $p$ legs, which is not the case for the characteristic 0 ones.
Remark 2.20. Note that there is no extra data on the partition of $I$ since we fix legs at the characteristic $p$ untilt, so essentially the only possibility of the partition is $(\{1\}, \ldots, \{k\})$.

When $S = \text{Spa}(R, R^+)$ is affinoid perfectoid, $C^\wedge_{\mathcal{Y}_{[0, \infty)}}(S) = W(R)$ by definition, whence the name.

Again, there is the correspondence version.

Definition 2.21. Let $I, J$ be finite sets, and $\mu_\bullet, \mu'_\bullet$ be collections of dominant cocharacters of $G$ indexed by $I$ and $J$ respectively. Let $\nu$ be a dominant cocharacter of $G$. Then we define

$$Sht^W_{\nu, W_{\mu_\bullet}|_{\mu'_\bullet}}$$

to be the prestack with value at a characteristic $p$ affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $\mathbb{F}_p$ being the groupoid of the data of

- $\{P_1, \cdots, P_k, \varphi_1, \cdots, \varphi_k\} \in Sht^W_{\mu_\bullet}(S)$ and $\{P'_1, \cdots, P'_l, \varphi'_1, \cdots, \varphi'_l\} \in Sht^W_{\mu'_\bullet}(S)$,
- a modification bounded by $\nu$

$$\gamma : P|_{W(R^+)[\frac{1}{p}]} \cong P'|_{W(R^+)[\frac{1}{p}]}$$

as Shtukas, i.e. the following diagram is commutative

$$\begin{array}{cccccc}
P_1 & \xrightarrow{\varphi_1} & P_2 & \cdots & \cdots & \xrightarrow{\varphi_k} & P_k & \xrightarrow{\varphi_{k+1}} & \text{Frob}_S^* P_1 \\
\downarrow{\gamma} & & & & & & & & & \\
P'_1 & \xrightarrow{\varphi'_1} & P'_2 & \cdots & \cdots & \xrightarrow{\varphi'_l} & P'_l & \xrightarrow{\varphi'_{l+1}} & \text{Frob}_S^* P'_1.
\end{array}$$

We now define the Witt vector Hecke stacks.

Definition 2.22. Let $I$ be a finite set, and $\mu_\bullet$ be a collection of dominant cocharacters of $G$ indexed by $I$. We define

$$\text{Hecke}^W_{\mu_\bullet}$$

to be the prestack whose value at $S = \text{Spa}(R, R^+)$ over $\mathbb{F}_p$ is the groupoid of

- $G$-torsors $P_1, \cdots, P_{k+1}$ on $\text{Spec}(W(R^+))$,
- modifications

$$P_1 \xrightarrow{\varphi_1} P_2 \xrightarrow{\varphi_2} \cdots \cdots \xrightarrow{\varphi_k} P_k \xrightarrow{\varphi_{k+1}} P_{k+1},$$

where the dotted arrows $\varphi_i$ are isomorphisms

$$\varphi_i : P_i|_{W(R^+)[\frac{1}{p}]} \cong P_{i+1}|_{W(R^+)[\frac{1}{p}]}$$

that are bounded by $\mu_i$ at every geometric point of $S$.

Definition 2.23. Let $I, J$ be finite sets, and $\mu_\bullet, \mu'_\bullet$ be collections of dominant cocharacters of $G$ indexed by $I$ and $J$ respectively. Let $\nu$ be a dominant cocharacter of $G$. Then we define

$$\text{Hecke}^W_{\nu, W_{\mu_\bullet}|_{\mu'_\bullet}}$$

to be the prestack with value at a characteristic $p$ affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $\mathbb{F}_p$ being the groupoid of the data of
• \( \{P_1, \ldots, P_{k+1}, \varphi_1, \ldots, \varphi_k\} \in \text{Hecke}_W(S) \) and \( \{P'_1, \ldots, P'_{l+1}, \varphi', \ldots, \varphi'_l\} \in \text{Hecke}_W(S) \),

• modifications bounded by \( \nu \)

\[
\gamma_1 : P_1|_{W(R^+)[\frac{1}{p}]} \cong P'_1|_{W(R^+)[\frac{1}{p}]}
\]

\[
\gamma_2 : P_2|_{W(R^+)[\frac{1}{p}]} \cong P'_2|_{W(R^+)[\frac{1}{p}]}
\]

such that the following diagram is commutative

\[
\begin{array}{c}
P_1 \xrightarrow{\varphi_1} P_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} P_k \xrightarrow{\varphi_k} P_{k+1} \\
\downarrow{\gamma_1} \quad \downarrow{\gamma_2} \\
P'_1 \xrightarrow{\varphi'_1} P'_2 \xrightarrow{\varphi'_2} \cdots \xrightarrow{\varphi'_{l-1}} P'_l \xrightarrow{\varphi'_l} P'_{l+1}.
\end{array}
\]

We observe that the Witt vector Shtukas and Hecke stacks are analytifications of perfect stacks.

Let \( Hk_{\mu^*} \) be the perfect Hecke stack as defined in \cite{1} definition 5.1.1, then it is easy to see that

\[
Hk_{\mu^*} = \text{Hecke}_W
\]

and

\[
Hk_{\mu^*}^{\infty} = \text{Hecke}_{\mu^*}^{\loc,(\{1\}, \ldots, \{n\})}
\]

with notation as in \cite{A.1}. On the other hand, we have

\[
\text{Sht}_{\mu^*}^{\perf, \infty} = \text{Sht}_W
\]

and

\[
\text{Sht}_{\mu^*}^{\perf, \infty} = \text{Sht}_{\mu^*}^{\loc, W}
\]

where \( \text{Sht}_{\mu^*}^{\perf} \) is the perfect Shtuka as defined in \cite{1} definition 5.2.1. Similarly for the correspondence version \( \text{Hecke}_W^{\nu, W} \) and \( \text{Sht}_{\mu^*}^{\nu, W} \).

As a sanitary check, all the prestacks defined above are small v-stacks.

**Proposition 2.24.** \( \text{Hecke}_{\mu^*}(I_1, \ldots, I_k) \), \( \text{Hecke}_{\mu^*}^{\loc,(I_1, \ldots, I_k)} \), \( \text{Hecke}_{\mu^*}(I_1, \ldots, I_k)(J_1, \ldots, J_l)^{\nu, W} \), \( \text{Hecke}_{\mu^*}^{\nu, W} \), \( \text{Hecke}_{\mu^*}|_{\mu^{\nu}^*} \), \( \text{Sht}_{\mu^*}(I_1, \ldots, I_k) \), \( \text{Sht}_{\mu^*}^{\loc, W} \), \( \text{Sht}_W^{\nu, W} \) and \( \text{Sht}_{\mu^*}(I_1, \ldots, I_k)(J_1, \ldots, J_l)^{\nu, W} \) are small v-stack.

**Proof.** Being stacks follows from \cite{7} proposition 19.5.3. The smallness follows from that the data we are parameterizing are sets, see the proof of \cite{4} proposition III.1.3. for example.

\[\square\]

3. **Excursion operators**

3.1. **Geometric Satake.** We follow the same notation as in the previous section. Let us first recall the geometric Satake proved by Fargues and Scholze.

**Theorem 3.1.** (Fargues-Scholze) Let \( I \) be a finite set, and \( \mu^* \) be a collection of dominant cocharacters of \( G \) indexed by \( I \). Let \( \{I_1, \ldots, I_k\} \) be a partition of \( I \). There is a natural functor

\[
U \rightarrow S_U^{(I_1, \ldots, I_k)}
\]
from the category of representations of $\hat{G}^I$, the product of the dual group of $G$ (with $\mathbb{Q}_l$-coefficient), to the category $D^b_{\text{et}}(\text{Hecke}^{(I_1,\cdots,I_k)}, \mathbb{Q}_l)$. It has the following properties.

1) If $U$ is irreducible with highest weight $\mu$, then $S^l_U(I_1,\cdots,I_k)$ is supported on $\text{Hecke}^{(I_1,\cdots,I_k)}$.

2) Let $\zeta : I \rightarrow J$ be a map of finite sets, $(J_1, \cdots, J_k)$ be a partition of $J$, and $I_i = \zeta^{-1}(J_i)$, there exists a natural closed immersion

$$i : \text{Hecke}^{(I_1,\cdots,I_k)} \times \text{Spd}(O_L)^J \hookrightarrow \text{Hecke}^{(J_1,\cdots,J_k)},$$

where we abuse the notation by writing $\zeta : \text{Spd}(O_L)^J \rightarrow \text{Spd}(O_L)^I$ the permutation map corresponding to $\zeta$, i.e. sending $(x_j)_{j \in J}$ to $(x_{\zeta(j)})_{j \in I}$. Let

$$p : \text{Hecke}^{(I_1,\cdots,I_k)} \times \text{Spd}(O_L)^J \rightarrow \text{Hecke}^{(I_1,\cdots,I_k)}$$

be the projection on the first factor, then there exists a canonical isomorphism

$$i_*p^*S^l_U(I_1,\cdots,I_k) \cong S^l_{\Delta^\zeta U}(I_1,\cdots,I_k)$$

where $\Delta^\zeta : \hat{G}^J \rightarrow \hat{G}^I$ is the map sending $(g_j)_{j \in J}$ to $(g_{\zeta(j)})_{j \in I}$.

3) Let

$$\pi(I_1,\cdots,I_k) : \text{Hecke}^{(I_1,\cdots,I_k)} \rightarrow \text{Hecke}(I)$$

be the convolution map, i.e. it takes

$$\mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \cdots \longrightarrow \mathcal{P}_k \longrightarrow \mathcal{P}_{k+1}$$

to

$$\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \longrightarrow \cdots \longrightarrow \mathcal{P}_k \longrightarrow \mathcal{P}_{k+1},$$

then there is a canonical isomorphism

$$(\pi(I_1,\cdots,I_k))_!S^l_U(I_1,\cdots,I_k) \cong S^l_U(I).$$

4) Let $(J_1, \cdots, J_l)$ be a partition of $I$ such that $J_j = I_{i_j} \cup \cdots \cup I_{i_{j+1}}$ for some partition $1 = i_1 < i_2 < \cdots < i_{k+1} = k + 1$ of $k + 1$. Let

$$\kappa(I_1,\cdots,I_k) : \text{Hecke}^{(I_1,\cdots,I_k)} \rightarrow \prod_j \text{Hecke}^{(I_{i_j}\cup\cdots\cup I_{i_{j+1}})}$$

be the forgetting map that sends

$$\mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \cdots \longrightarrow \mathcal{P}_k \longrightarrow \mathcal{P}_{k+1}$$

to

$$\prod_j (\mathcal{P}_{i_j}, \cdots, \mathcal{P}_{i_{j+1}}, \varphi_{i_j}, \cdots, \varphi_{i_{j+1}}),$$

then we have a canonical isomorphism

$$(\kappa(I_1,\cdots,I_k))^* \boxtimes_S^l U_j^{(I_1,\cdots,I_{i_{j+1}}-1)} \cong S^l_U(I_1,\cdots,I_k).$$

where $U_j$ is a representation of $\hat{G}^{I_j}$. 
5) Let $U$ be irreducible with hightest weight $\mu_s$, then the pullback of $S_U^{(1),\ldots,(n)}$ restricted on $\text{Hecke}_{\mu_s}^{W}$ through the canonical map

$$\text{Hecke}_{\mu_s}^{W} \rightarrow \text{Hecke}_{\mu_s}^{(1),\ldots,(n)}$$

is identified with the analytification of the intersection sheaves $IC_{\mu_s}$ of the Witt vector affine Grassmannian $Gr_{\mu_s}$, see [1] section 3.4.1.

More precisely, $IC_{\mu_s}$ being $L^+G$-equivariant descends to the truncated Witt vector Shtukas $\text{Hecke}_{\mu_s}^{\text{loc}(m)}$, see appendix [A.3] or [1] section 5.1 for definition, and we denote it by $A_{\mu_s}$. As in appendix [A.1], we have the analytification $a^*_X c^*_X A_{\mu_s}$ defined on $\text{Hecke}_{\mu_s}^{\text{loc}(m)\circ}$ with $X = \text{Hecke}_{\mu_s}^{\text{loc}(m)}$, which can be further viewed as sheaves on $\text{Hecke}_{\mu_s}^{W}$ through the identification in [A.3], and this is canonically identified with the pullback of $S_U^{(1),\ldots,(n)}$ along

$$\text{Hecke}_{\mu_s}^{W} \rightarrow \text{Hecke}_{\mu_s}^{(1),\ldots,(n)}$$

6) Lastly, there is a natural functor

$$U \rightarrow S_U^{\text{loc}(I_1,\ldots,I_k)}$$

from the category of representations of $L^+G$ to the category $D_{\text{et}}^b(\text{Hecke}_{\mu_s}^{\text{loc}(I_1,\ldots,I_k)}, \mathbb{Q}_l)$ which satisfies the local version of 1), 2), 3) and 4). For the comparison with Witt vector Hecke stacks, we have a natural restriction map

$$\text{Hecke}_{\mu_s}^{\text{loc}(I_1,\ldots,I_k)} \rightarrow \text{Hecke}_{\mu_s}^{\text{loc}(m)\circ}$$

along which the pullback of $c^*_X A_{\mu_s}$ is identified with $S_U^{\text{loc}(I_1,\ldots,I_k)}$ with $U$ irreducible with highest weight $\mu_s$.

There is a natural restriction map

$$\text{Hecke}^{(I_1,\ldots,I_k)} \rightarrow \text{Hecke}_{\mu_s}^{\text{loc}(I_1,\ldots,I_k)},$$

along which the pullback of $S_U^{\text{loc}(I_1,\ldots,I_k)}$ is identified with $S_U^{(I_1,\ldots,I_k)}$, and 1), 2), 3), 4) and 5) is the pullback of the corresponding local version.

**Remark 3.2.** By the derived category of a small $v$-stack $X$ with $\mathbb{Q}_l$-coefficients, we take the naive definition

$$D^b_{\text{et}}(X, \mathbb{Q}_l) := \text{colim}_E D^b_{\text{et}}(X, E),$$

where $E$ ranges over finite extensions of $\mathbb{Q}_l$, $D^b_{\text{et}}(X, E)$ is the idempotent completion of the $D^b_{\text{et}}(X, \mathcal{O}_E)_{[1]}^\perp$ and the colimit is taken in the $\infty$-categorical sense. Recall that the derived category $D^b_{\text{et}}(X, \mathcal{O}_E)$ with adic coefficients is defined in [12] section 26. The six functor formalism generalises directly to this setting. Moreover, the constructions in this paper works already with the coefficient ring $\mathcal{O}_{\mathbb{Q}_l(\sqrt{q})}$, and we can take that instead.

The definition given here compares favorably with the constructible derived categories of schemes, see [15] section 6.8. or [16], which suffices for our use in this paper.

**Remark 3.3.** We can state 3) for more general partitions, at the cost of further complicating the notation. The special cases as stated above suffices for our purpose.
Remark 3.4. We can refine the theorem using Langlands L-group. More precisely, let $\text{Hecke}_{\text{Spd}(\mathbb{Z}_p)^I}^{(I_1,\cdots,I_k)}$ be the Hecke stack over $\text{Spd}(\mathbb{Z}_p)^I$, see remark 2.10. Then there is a functor from representations $U$ of the L-group $^LG$ to the complexes of sheaves $S^I_U$ on $\text{Hecke}_{\text{Spd}(\mathbb{Z}_p)^I}^{(I_1,\cdots,I_k)}$ with similar properties as in the theorem, which refines the functor in the theorem.

Proof. Using the pullback from the canonical map

$$\text{Hecke}_{\text{loc}}^{(I_1,\cdots,I_k)} \to \text{Hecke}_{\text{loc}}^{(I_1,\cdots,I_k)}$$

it is enough to prove the local version. Moreover, it is enough to work with coefficients with ring of integers of $\mathbb{Q}(\sqrt{q})$, which is treated in [4].

It is enough to consider irreducible $U$’s. From the geometric Satake proved in [4] theorem VI.11.1, there exists a canonical perverse sheaf $S^I_{\text{loc},(I)}$ on $\text{Hecke}_{\text{loc}}^{(I)}$ for each representation $U$ of $\hat{G}^I$. Now let $\{I_1,\cdots,I_k\}$ be a partition of $I$, and $U$ be an irreducible representation of $\hat{G}^I$. Then $U = \bigotimes_j U_j$, where $U_j$ is a representation of $\hat{G}^{I_j}$. We define

$$S^I_{\text{loc}}^{(I_1,\cdots,I_k)} := (\kappa^{(I_1,\cdots,I_k)}_{(I_1,\cdots,I_k)}* \bigotimes_j S^I_{\text{loc},(I_j)})_{\text{loc}},$$

(so $S^I_{\text{loc}}^{(I_1,\cdots,I_k)}$ on global Hecke stacks is the pullback of $S^I_{\text{loc}}^{(I_1,\cdots,I_k)}$ along $\text{Hecke}_{\text{loc}}^{(I_1,\cdots,I_k)} \to \text{Hecke}_{\text{loc}}^{(I_1,\cdots,I_k)})$

then 4) is automatic, using the identification $\kappa^{(I_1,\cdots,I_k)}_{(I_1,\cdots,I_k)} = (\prod_j \kappa^{(I_1,\cdots,I_{ij-1})}_{(I_1,\cdots,I_{ij-1})}) \circ \kappa^{(I_1,\cdots,I_{ij})}_{(I_1,\cdots,I_{ij})}$.

By the proof of [4] proposition VI.9.4, $(\pi^{(I_1,\cdots,I_k)})S^I_{\text{loc},(I_1,\cdots,I_k)}$ is the fusion product of $S^I_{\text{loc},(I_j)}$, then [4] proposition VI.10.3 tells us that the corresponding representation is $\bigotimes_j U_j = U$. In other words,

$$(\pi^{(I_1,\cdots,I_k)})S^I_{\text{loc},(I_1,\cdots,I_k)} \cong S^I_{\text{loc},(I)},$$

which proves 3).

For 1), we observe that

$$(\kappa^{(I_1,\cdots,I_k)}_{(I_1,\cdots,I_k)})^{-1}\prod_j \text{Hecke}_{\mu_j}^{\text{loc},(I_j)} = \text{Hecke}_{\mu}^{\text{loc},(I_1,\cdots,I_k)}.$$
We check \( \{1,2\} \to \{1\} \) first. In this case, \( i \) is an isomorphism and \( p \) is the closed immersion

\[
p^\text{loc} : \text{Hecke}_{\text{loc},\{1\}} \hookrightarrow \text{Hecke}_{\text{loc},\{1,2\}}
\]

identifying \( \text{Hecke}_{\text{loc},\{1\}} \) with the substack of \( \text{Hecke}_{\text{loc},\{1,2\}} \) where the two legs are the same. We can assume that \( U = U_1 \boxtimes U_2 \) is irreducible, then there is a canonical identification

\[
p^\text{loc} \cdot S^\text{loc}_{\{1,2\}} \cong S^\text{loc}_{\{1\}}
\]

where \( \Delta : \hat{G} \to \hat{G}^2 \) is the diagonal map and \( U \) is a representation of \( \hat{G}^2 \). This is because \( \Delta^* U = U_1 \boxtimes U_2 \) and \( p^* S^\text{loc}_{\{1,2\}} \) is the convolution product of \( S^\text{loc}_{U_1} \) and \( S^\text{loc}_{U_2} \) (using \( S^\text{loc}_{U_1} \), \( S^\text{loc}_{U_2} \)), see the paragraph following the proof of [4] proposition VI.9.4. Then the identification follows from the geometric Satake ([4] theorem VI.11.1).

Next, we assume that \( \zeta = \emptyset \to \{1\} \), then \( p \) is

\[
p^\text{loc} : \text{Spd}(O_L) \to \ast,
\]

the unique map to the final object, and

\[
i^\text{loc} : \text{Spd}(O_L) \hookrightarrow \text{Hecke}_{\text{loc},\{1\}}
\]

is the section of the structure map corresponding to the trivial modification, i.e. the modification is an isomorphism. Then

\[
i^\text{loc} \circ p^\text{loc} \cdot S^\text{loc}_{\emptyset} \cong S^\text{loc}_{\{1\}}
\]

where \( 1 \) is the trivial representation (of either \( \hat{G} \) or the trivial group \( \hat{G}^\emptyset \)). This follows from \( S^\text{loc}_{\emptyset} \) being the constant local system, and \( S^\text{loc}_{\{1\}} \) being the constant local system supported on \( \text{Hecke}_{\emptyset,\{1\}} \) (by 1). The general \( U \) are direct sums of 1, so the identification follows.

Now for general partition \((J_1, \ldots, J_k)\), we have a Cartesian diagram

\[
\begin{array}{ccc}
\text{Hecke}_{\text{loc},\{I_1,\ldots,I_k\}}^\text{loc} & \xrightarrow{p^\text{loc}} & \text{Hecke}_{\text{loc},\{I_1,\ldots,I_k\}} \times \text{Spd}(O_L)^{J} \\
\downarrow^{\pi^\text{loc}(\{I_1,\ldots,I_k\})} & & \downarrow^{\pi^\text{loc}(\{I_1,\ldots,I_k\}) \times \text{id}} \\
\prod_{j} \text{Hecke}_{\text{loc},\{I_j\}}^\text{loc} & \xrightarrow{\prod_{j} p^\text{loc}} & \prod_{j} \text{Hecke}_{\text{loc},\{I_j\}} \times \text{Spd}(O_L)^{J_j} \\
\end{array}
\]

and we can pullback the identification for trivial partition to obtain the desired one.

For the comparison with Witt vector Hecke stacks. We note that when \( I = \{1\} \), it follows from the paragraph before [12] proposition VI.7.5 and the proof of loc.cit. that under the inclusion \( D_{\text{et}}(\text{Hecke}_{\mu,s}^{\text{loc},\{1\}}, \Lambda) \subset D_{\text{et}}(\text{Gr}_{\mu}^{W,\infty}, \Lambda) \) with \( \Lambda = O_{Q_{l}(\sqrt{q})} \) (recall that \( \text{Hecke}_{\mu,s}^{\text{loc},\{1\}} = L^+ G \setminus \text{Gr}_{\mu}^{W,\infty} \)),

\[
c_X IC_{\mu} = S^\text{loc}_{U}^{\{1\}}
\]

with \( X = \text{Gr}_{\mu}^{W} \) the Witt vector affine Grassmannian. Now we work with the general \( IC_{\mu_\bullet} \), recall that the construction of \( IC_{\mu_\bullet} \) is by taking the twisted product of \( IC_{\mu_1} \) on \( \text{Gr}_{\mu_1} = \tilde{\text{Gr}}_{\mu_1} \), see
section 3.4.1. By passing to Hecke stacks, this corresponds precisely to $(\kappa_{\text{loc},\{1\}^{n}})^* \boxtimes \mathcal{E}_{U_j}^{\text{loc},\{1\}}$.

### 3.2. Hecke Correspondence.

We have a canonical map

$$\epsilon_{\mu}^{(1)} : \text{Sh}_{\mu, \eta} \to \text{Hecke}_{\mu, \eta}^{\text{loc}}$$

that sends a Shtuka $\{\mathcal{P}, \varphi_{\mathcal{P}}, S^2\}$ to $\{\mathcal{P}, \text{Frob}_L^{\mu}\varphi_{\mathcal{P}}, S^2\}$, and we can use it to pullback the canonical sheaves in geometric Satake to sheaves on Shtukas.

Let $U$ be an irreducible representation of $\hat{G}$ with highest weight $\mu$. We have a diagram

$$
\begin{array}{ccc}
\text{Sh}_{\mu} \quad & \quad \text{Sh}_{\mu}^{\nu} \quad & \quad \text{Sh}_{\mu} \\
p_1 & \quad \quad & \quad \quad p_2 \\
\text{Hecke}_{\mu, \eta} & \quad \quad & \quad \quad \text{Hecke}_{\mu, \eta}^{\text{loc}} \\
r & \quad \quad & \quad \quad \\
\end{array}
$$

(2)

where the last vertical map is the canonical restriction map. The square in the diagram is not commutative, but they become commutative after composing with vertical map. Indeed, recall from notation $\text{notation } 2.8$, the legs of Shtukas in the diagram are in characteristic zero, while the modification bounded by $\nu$ takes place at characteristic $p$, hence it does not affect the local type of $\varphi_{\mathcal{P}}$, i.e. the diagram is commutative after composing with the vertical map.

Now we have a cohomological correspondence $\mathcal{C}_\nu$ from $(\text{Sh}_{\mu, \eta}^{\nu}\epsilon_{\mu}^{(1)}, S_{U}^{\{1\}})$ to itself supported on $\text{Sh}_{\mu, \eta}^{\nu}$, namely

$$
\mathcal{C}_\nu : p_1^*\epsilon_{\mu}^{(1)}, s_{U}^{\{1\}} \cong p_1^*\epsilon_{\mu}^{(1)}, s_{U}^{\text{loc},\{1\}} \cong p_2^*\epsilon_{\mu}^{(1)}, r^*s_{U}^{\text{loc},\{1\}} \cong p_2^*\epsilon_{\mu}^{(1)}, s_{U}^{\{1\}} \cong p_2^*\epsilon_{\mu}^{(1)}, s_{U}^{\text{loc},\{1\}}, r^*s_{U}^{\text{loc},\{1\}}, s_{U}^{\{1\}}, r^*s_{U}^{\text{loc},\{1\}}, s_{U}^{\{1\}},
$$

where we use $r^*s_{U}^{\text{loc},\{1\}} = s_{U}^{\{1\}}$ (see 6) of theorem $\text{3.1}$, and the étaleness of $p_2$ for the last isomorphism.

Now let $V$ be a representation of $\hat{G}$, and $h_V \in C_c(G(L)/\mathcal{G}(\mathcal{O}_L), \mathcal{O}_L)$ be the function corresponding to $V$ through classical Satake. We know that $C_c(G(L)/\mathcal{G}(\mathcal{O}_L), \mathcal{O}_L)$ has a basis indexed by the dominant cocharacters $\nu$ ($G$ is unramified, being reductive over $\mathbb{Z}_p$), i.e. $1_{\mathcal{G}(\mathcal{O}_L)\nu(p)\mathcal{G}(\mathcal{O}_L)}$ forms a basis of $C_c(G(L)/\mathcal{G}(\mathcal{O}_L), \mathcal{O}_L)$. Let $\Gamma_V$ be the union of $\text{Sh}_{\mu, \eta}^{\nu}$ with $\nu$ showing up in $h_V$. Then we define the Hecke operator $T_V$ as the cohomological correspondence supported on $\Gamma_V$, and is $h_V(\nu(p))$ times the cohomological correspondence $[\mathcal{B}]$ on $\text{Sh}_{\mu, \eta}^{\nu}$. More precisely, we define

$$
T_V := \sum_{h_V(\nu(p)) \neq 0} h_V(\nu(p)) \cdot \mathcal{C}_\nu
$$
as a cohomological correspondence from \((Sht_{\mu,\eta}, c_\mu^*S_U)\) to itself supported on 
\[ \Gamma_V := \bigcup_{h_V(\nu(p)) \neq 0} Sht^\nu_{\mu|\mu,\eta}. \]

Note that when \(V\) is irreducible with highest weight \(\delta\), then \(h_V(\nu(p)) \neq 0\) precisely when \(\nu \leq \delta\), hence \(\Gamma_V = Sht^\nu_{\mu|\mu,\eta}\). Since \(Sht^\nu_{\mu|\mu,\eta} \subset Sht^\delta_{\mu|\mu,\eta}\) as closed substack, we can view \(c_\nu\) as supported on \(Sht^\delta_{\mu|\mu,\eta}\) whence the sum in the definition of \(T_V\) makes sense. In general \(\Gamma_V\) is the union of \(Sht^\delta_{\mu|\mu,\eta}\) where \(\delta\) ranges over highest weights of irreducible summands of \(V\).

We observe that when \(\mu\) is minuscule, \(S_U = \overline{\mathcal{O}_\Gamma}\) is the constant sheaf and the cohomological correspondence \(c_\nu\) is the canonical isomorphism
\[ p_1^*\overline{\mathcal{O}_\Gamma} \cong p_2^*\overline{\mathcal{O}_\Gamma} \cong p_3^*\overline{\mathcal{O}_\Gamma}. \]

We assume from now on that \(V\) is a representation of \(LG\), in other words, \(V\), as a representation of \(\hat{G}\), is canonically isomorphic to its Frobenius twist \(\sigma V\), where \(\sigma V\) is the representation of \(\hat{G}\) obtained by twisting the action of \(G\) on \(V\) with the inverse Frobenius action on \(\hat{G}\), see [1] section 3.4.5. Then the (unramified) classical Satake gives \(h_V \in C_c(G(\mathbb{Q}_p)/G(\mathbb{Z}_p), \overline{\mathcal{O}_\Gamma})\).

### 3.3. Creation correspondence

We now start to define the excursion operators. First we define the creation correspondence. We follow the same notation as above.

**Notation 3.5.** We use subscript \(V\) and \(V^*\) to mean that we restrict to the special fiber at the legs they correspond. For instance,
\[
\text{Hecke}_{V^*}^{(\{1\},\{2\})} := \text{Hecke}_{(\delta,\delta)}^{(\{1\},\{2\})}|_{\text{Spd}(\mathbb{F}_p) \times \text{Spd}(\mathbb{F}_p)},
\]
\[
\text{Hecke}_{V^*}^{(\{1\},\{2\},\{3\})} := \text{Hecke}_{(\delta,\delta,\mu)}^{(\{1\},\{2\},\{3\})}|_{\text{Spd}(\mathbb{F}_p) \times \text{Spd}(\mathbb{F}_p) \times \text{Spd}(\mathbb{O}_L)},
\]
\[
\text{Hecke}_{V^*}^{(\{1\},\{2\},\{3\})} := \text{Hecke}_{(\delta,\delta,\mu)}^{(\{1\},\{2\},\{3\})}|_{\text{Spd}(\mathbb{F}_p) \times \text{Spd}(\mathbb{F}_p) \times \text{Spd}(L)}
\]
and similarly for others.

Let \(\delta\) and \(\delta^*\) be the highest weight of \(V\) and the dual representation \(V^*\) respectively. We denote
\[
Sht^{(\{1\},\{2\},\{3\})}_{V^*} := Sht^{(\{1\},\{2\},\{3\})}_{(\delta,\delta,\mu)}|_{\text{Spd}(\mathbb{F}_p) \times \text{Spd}(\mathbb{F}_p) \times \text{Spd}(L)}
\]
in other words, \(Sht^{(\{1\},\{2\},\{3\})}_{V^*}\) parameterizes
\[
\mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_3 \longrightarrow \text{Frob}_p^*\mathcal{P}_1
\]
where the two modifications takes place at the characteristic \(p\) until that are bounded by \(\delta^*\) and \(\delta\) respectively, while the third modification takes place at the characteristic 0 until and is bounded by \(\mu\).

Similarly, let
\[
Sht^{0}_{\mu|V^*} \subset Sht^{(\{1\},\{2\},\{3\})}_{(\delta,\delta,\mu),\eta}
\]
be the $v$-substack parametrizes modifications

\[ \mathcal{P} \longrightarrow \text{Frob}_S^* \mathcal{P} \]

\[ \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_3 \longrightarrow \text{Frob}_S^* \mathcal{P}_1 \]

with the first line in $\text{Sht}_{\mu,\eta}$ and the second line in $\text{Sht}_{V^* \hat{G} \otimes \mathbb{Z}_p, \mu, \eta}$. By our convention, $\text{Sht}_{\mu,\eta}$ has the leg in characteristic zero, while the first two modifications of the second line takes place in characteristic $p$, hence the third modification has the same as leg as the first line, and the diagram is equivalent to

\[ \mathcal{P} \sim \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \text{Frob}_S^* \mathcal{P} \]

(5)

where the first two dashed arrows are modifications at characteristic $p$ untilt bounded by $\delta^*$ and $\delta$ respectively whose composition is an isomorphism, and the last dashed arrow is a modification at characteristic zero until bounded by $\mu$. It means that $\text{Sht}_{\mu[V^* \hat{G} \otimes \mathbb{Z}_p, \mu, \eta]}^0$ is a closed $v$-substack of $\text{Sht}_{(\delta^*, \delta, \mu), \eta}$. More precisely, it lies in $\text{Sht}_{V^* \hat{G} \otimes \mathbb{Z}_p, \mu, \eta}$, which parameterizes diagrams as in (5) without the isomorphism condition.

From geometric Satake, we have a morphism

\[ S_1^{\{(1)\}} \longrightarrow S_{V^* \hat{G}}^{\{(1)\}} \]

(6)

corresponding to the canonical morphism $1 \rightarrow V^* \otimes V$. We note that this defines a cohomological correspondence supported on

\[ \text{Hecke}^0_{V^* \hat{G}} \]

\[ \text{Hecke}_{V^* \hat{G}}^{\{(1)\}} \]

where we adopt the convention that for a representation $W$ of $\hat{G}$,

\[ \text{Hecke}_U := \bigcup_{\mu^*} \text{Hecke}_{\mu^*} \]

where the index $\mu^*$ ranges over the highest weight of irreducible summand of $U$, which are exterior products of highest weights of $\hat{G}$. Similarly for $\text{Hecke}_{U[U']}$.

Indeed, we have

\[ \text{Lemma 3.6. } \text{Let } I \text{ be a finite set, } U \text{ and } U' \text{ be two representations of } \hat{G}^I, \text{ then we have a natural identification} \]

\[ \text{Hom}_{\hat{G}^I}(U, U') \cong \text{Corr}_{\text{Hecke}_{U[U']}}(S^{(I)}_U, S^{(I)}_{U'}) \]
where the right hand side denote the cohomological correspondence from \((\text{Hecke}_U, S_U^{(1)})\) to \((\text{Hecke}_U, S_U^{(1)})\) supported on \(\text{Hecke}_U^{0}\). Moreover, the identification is compatible with composition on both sides.

**Proof.** It suffices to assume that \(W\) and \(W'\) are irreducible with highest weight \(\mu\) and \(\mu'\) respectively. We have a Cartesian diagram

\[
\begin{array}{ccc}
\text{Hecke}_U^{0} & \xrightarrow{p_2} & \text{Hecke}_U' \\
\downarrow{p_1} & & \downarrow{i'} \\
\text{Hecke}_U' & \xleftarrow{i} & \text{Hecke}_U^{(1)}
\end{array}
\]

hence by proper base change (12 22.19, \(p_2\) is proper as it is the base change of a closed immersion, so \(p_2,1\) is defined, similarly for \(p_1\))

\[
\text{Hom}(p_1^*S_U^{(1)}, p_2^!S_U^{(1)}) = \text{Hom}(p_2^!p_1^*S_U^{(1)}, S_U^{(1)}) = \text{Hom}(p_2^!p_1^*S_U^{(1)}, S_U^{(1)}) = \text{Hom}(i'^*i_1^*S_U^{(1)}, S_U^{(1)})
\]

\[
= \text{Hom}(i_1^*S_U^{(1)}, i_1^*S_U^{(1)}) = \text{Hom}(S_U^{(1)}, S_U^{(1)}) = \text{Hom}_{\text{Gr}}(U, U')
\]

where the last identification is geometric Satake. The compatibility with composition is easy to see from the naturality of the above chain of identifications. \(\square\)

Further, 2) of theorem 3.1 gives us a natural identification

\[
S_V^{((1)} \cong \Delta^*S_V^{((1,2))}
\]

under the canonical injection

\[
\Delta : \text{Hecke}^{((1)} \hookrightarrow \text{Hecke}^{((1,2))},
\]

while 3) of theorem 3.1 provides a canonical isomorphsim

\[
(\pi^{((1),(2))})_1S_V^{((1),(2))} \cong S_V^{((1,2))}
\]

from the convolution map

\[
\pi^{((1),(2))} : \text{Hecke}^{((1),(2))} \rightarrow \text{Hecke}^{((1,2))}.
\]

We can view (8) and (9) as cohomological correspondences. Combining (6), (8) and (9), we obtain

\[
S_V^{((1)} \rightarrow S_V^{((1)}, \cong \Delta^*S_V^{((1,2))} \cong \Delta^*(\pi^{((1),(2))})_1S_V^{((1),(2))}
\]

which can be view as a cohomological correspondence

\[
\varepsilon^{+,0} : \pi^*S_1^{((1)} \rightarrow i^!S_V^{((1,2))}
\]

from \((S_1^{((1)}), \text{Hecke}_0,s)\) to \((S_V^{((1),(2))}, \text{Hecke}_V^{((1),(2))})\) supported on

\[
\begin{array}{ccc}
\text{Hecke}_V^{((1,2))} & \xrightarrow{i} & \text{Hecke}_V^{((1,2))} \\
\pi & & \\
\text{Hecke}_V^{((1)} & \xleftarrow{s} & \text{Hecke}_V^{((1)}
\end{array}
\]
where

\[ \text{Hecke}_{V^* \otimes V}^{(1),(2)},0 \subset \text{Hecke}_{V^* \otimes V}^{(1),(2)} \]

is defined to be the closed substack parametrizing

\[ \begin{array}{ccc}
P_1 & \xrightarrow{\sim} & P_3 \\
\downarrow & & \downarrow \\
P_2 & & \\
\end{array} \]

i.e. it is the closed substack of \( \text{Hecke}_{V^* \otimes V}^{(1),(2)} \) defined by the condition that \( \varphi_2 \circ \varphi_1 \) is an isomorphism, in other words, it is \( \text{Hecke}_{V^* \otimes V}^{((1))((1),(2))},0 \) and the correspondence diagram is exactly the defining diagram of \( \text{Hecke}_{V^* \otimes V}^{((1))((1),(2))},0 \). Indeed, we can write down the support of (6), (8) and (9), being

\[ \text{Hecke}_{V^* \otimes V}^{(1),(2)},0 \]

\[ \xrightarrow{\sim} \text{Hecke}_{V^* \otimes V}^{0} \]

and \( \text{Hecke}_{V^* \otimes V}^{((1),(2))},0 \) is precisely the fiber product of \( \pi^{((1),(2))} \) and \( \Delta \circ p_2 \).

We can repeat the last paragraph for local Hecke stacks to obtain a cohomological correspondence

\[ C_{\ast}^{\text{loc},+,0} : \pi^{\text{loc},+}_{11} \otimes_{\text{loc},+} \to \pi^{\text{loc},+}_{22} \]

supported on

\[ \text{Hecke}_{V^* \otimes V}^{\text{loc},(1),(2)},0 \]

\[ \xrightarrow{\pi^{\text{loc}}} \text{Hecke}_{V^* \otimes V}^{\text{loc},(1),(2)},0 \]

Moreover, by 6) of theorem [3.1] \( C_{\ast}^{+,0} \) is the pullback of \( C_{\ast}^{\text{loc},+,0} \) along the diagram
which clearly have Cartesian squares. Note that it is important that we restrict to legs at the characteristic $p$ until, as otherwise the correspondence Hecke stacks $\text{Hecke}^0_{0|V^* \otimes V}$ need not exist, see remark 2.14.

We observe that we can repeat the construction of $\mathcal{C}^{\text{loc},+,0}_x$ using truncated Witt Hecke stacks, and obtain $\mathcal{C}^{\text{loc}(m),+,0}_x$, and it follows essentially from 5) of theorem 3.1 that

$$\mathcal{C}^{\text{loc},+,0}_x = \mathcal{C}^{\text{loc}(m,n),+,0,\infty}_x$$

under the identification in appendix A.3 between sheaves on truncated and untruncated Shtukas, i.e. $\mathcal{C}^{\text{loc},+,0}_x$ is the $\otimes$-analytification of $\mathcal{C}^{\text{loc}(m,n),+,0}_x$, see appendix A.2 for the notation.

We now consider the correspondence from definition 2.12

$$\begin{array}{ccc}
\text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta) & \overset{p}{\longrightarrow} & \text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta) \\
(\text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta) & \overset{\varphi_1}{\longrightarrow} & \mathcal{P}_3 \\
\mathcal{P}_3 & \overset{\varphi_2}{\longrightarrow} & \mathcal{P}_4
\end{array} \quad \quad \text{(13)}$$

Observe that when the leg of $\mu$ is different from the legs of $V$ and $V'$,

$$\text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta) = \text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta)$$

where

$$\text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta) \overset{i}{\longrightarrow} \text{Hecke}^{\{1\}{\{2\}}}|(V^*,V,\mu,\eta)$$

is the substack parametrizing

$$\mathcal{P}_1 \overset{\sim}{\longrightarrow} \mathcal{P}_3 \longrightarrow \mathcal{P}_4$$

with $\varphi_1$ and $\varphi_2$ bounded by (highest weights of irreducible summand of) $V^*$ and $V$ respectively, and $\varphi_3$ bounded by $\mu$. The morphism $p$ sends the above diagram to the first line

$$\mathcal{P}_1 \overset{\sim}{\longrightarrow} \mathcal{P}_3 \longrightarrow \mathcal{P}_4.$$
We have a commutative diagram

\[
\begin{array}{ccc}
\text{Hecke}_{((1),(2))} & \xrightarrow{\kappa_{(1,2)}} & \text{Hecke}_{V^{*}\otimes V_{\mu,\eta}} \\
\downarrow{\pi \times \text{id}} & & \downarrow{i \times \text{id}} \\
\text{Hecke}_{((1),(2))} \times \text{Hecke}_{\mu,\eta} & \xrightarrow{\kappa_{(1,2),(3)}} & \text{Hecke}_{V^{*}\otimes V_{\mu,\eta}} \\
& & \downarrow{i \times \text{id}} \\
\text{Hecke}_{0} \times \text{Hecke}_{\mu,\eta} & \xrightarrow{\kappa_{(1),(2),(3)}} & \text{Hecke}_{((1),(2),(3))} \\
\end{array}
\]

It is clear that the two squares are Cartesian, thus we can pullback (see [1] A.2.13, which is true in any six functor formalism) the cohomological correspondence \[^{(10)}\] times identity correspondence from \((S_{U}^{((1))}, \text{Hecke}_{\mu}^{((1))})\) to itself, i.e. we pullback

\[
\mathcal{C}_{+}^{*}: S_{U}^{((1))} \xrightarrow{i} (i \times \text{id})^{*} S_{V^{*}\otimes V_{\mu,\eta}}^{((1))}
\]

along \(\kappa_{(1),(2),(3)}^{((1))}\) to obtain the cohomological correspondence

\[
\mathcal{C}_{+}^{*}: p^{*}S_{U}^{((1))} \xrightarrow{14} i^{*} S_{V^{*}\otimes V_{\mu,\eta}}^{((1))}
\]

supported on \(\text{Hecke}_{V^{*}\otimes V_{\mu,\eta}}^{((1),(2),(3))}\). More precisely, it is the composition

\[
\mathcal{C}_{+}^{*}: p^{*}S_{U}^{((1),(2))} \cong p^{*} \kappa_{(1),(2)}^{((1),(2))}(S_{U}^{((1))} \otimes S_{U}^{((2))}) \cong (\kappa_{(1),(2),(3)})^{*}(\pi \times \text{id})^{*}(S_{V^{*}\otimes V_{\mu,\eta}}^{((1))} \otimes S_{U}^{((2))})
\]

\[
\xrightarrow{(\kappa_{(1),(2),(3)})^{*} \otimes \mathcal{C}_{+}^{*} \otimes \text{id}} (\kappa_{(1),(2),(3)})^{*}(i \times \text{id})^{*}(S_{V^{*}\otimes V_{\mu,\eta}}^{((1),(2))} \otimes S_{U}^{((3))})
\]

\[
\xrightarrow{\text{BC}} i^{*} \kappa_{(1),(2),(3)}^{*}(S_{V^{*}\otimes V_{\mu,\eta}}^{((1),(2))} \otimes S_{U}^{((3))}) \cong i^{*} S_{V^{*}\otimes V_{\mu,\eta}}^{((1),(2),(3))}
\]

We have a canonical map

\[
\epsilon_{\mu_{i},\ldots,\mu_{k}}: \text{Sh}_{d_{i},\ldots,d_{k}} \rightarrow \text{Hecke}_{\mu_{i},\ldots,\mu_{k}}
\]

which is simply viewing \(\text{Frob}_{S}^{*} \mathcal{P}_{1}^{\mu_{i}} \otimes_{\mathcal{O}_{\mathcal{P}_{k+1}}}^{j_{0,\infty}(S), \Sigma \mathcal{P}_{\mu_{i}}^{S}} \mathcal{P}_{k+1}\). Similarly we have

\[
\epsilon_{\mu_{i},\mu_{j}}: \text{Sh}_{d_{i},d_{j}}^{0} \rightarrow \text{Hecke}_{\mu_{i},\mu_{j}}^{0}
\]
which forms a commutative diagram

with the two squares Cartesian.

Consider the commutative diagram
Clearly the squares are Cartesian, so we can pullback cohomological correspondence $\mathcal{C}_\mathfrak{s}^+$ supported on $\text{Hecke}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2),(3)),0}$ to a cohomological correspondence $\mathcal{C}_\mathfrak{s} : p_1^*\mathcal{C}_\mathfrak{U}^{((1))} \rightarrow p_2^*\mathcal{C}_\mathfrak{U}^{((1),(2),(3)),0}$ supported on $\text{Sh}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2),(3)),0}$. As $\mathcal{C}_\mathfrak{s}^+$ is by definition the pullback along $\kappa^{((1),(2),(3)),0}_{((1),(2),(3))}$ of $\mathcal{C}_\mathfrak{s}^+ \boxtimes \text{id}$ supported on $\text{Hecke}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2)),0} \times \text{Hecke}_{\mathfrak{H}}^{(1)}$, we have that $\mathcal{C}_\mathfrak{s}$ is the pullback of $\mathcal{C}_\mathfrak{s}^+ \boxtimes \text{id}$ along $\kappa^{((1),(2),(3)),0}_{((1),(2),(3))} \circ \epsilon^{((1),(2),(3)),0}_{V^* SV\mathfrak{Z}_\mathfrak{U}}$, i.e.

$$\kappa^{((1),(2),(3)),0}_{((1),(2),(3))} \circ \epsilon^{((1),(2),(3)),0}_{V^* SV\mathfrak{Z}_\mathfrak{U}} \ast \mathcal{C}_\mathfrak{s}^+ \boxtimes \text{id} = \mathcal{C}_\mathfrak{s}.$$

Further, we know that $\mathcal{C}_\mathfrak{s}^{+,0}$ is the pullback of $\mathcal{C}_\mathfrak{s}^{\text{loc},+}$ along the canonical restriction map, so $\mathcal{C}_\mathfrak{s}$ is the pullback of $\mathcal{C}_\mathfrak{s}^{\text{loc},+} \boxtimes \text{id}$ all the way from the bottom of the diagram.

The key observation for us is that the above diagram factorizes through local Shtukas at the special fiber, namely we have the following commutative diagram whose composition is the same as that of the above diagram.

\[
\begin{array}{ccc}
\text{Sh}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2),(3)),0} & \xrightarrow{p_1} & \text{Sh}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2),(3))} \\
\downarrow & & \downarrow

\text{Hecke}_{\mathfrak{H}}^{(1),\text{loc},(1)} & \xrightarrow{p_2 \times \text{id}} & \text{Hecke}_{\mathfrak{H}}^{(1),\text{loc},(1)} \\
\downarrow & & \downarrow

\text{Hecke}_{0}^{\text{loc},(1)} \times \text{Hecke}_{\mathfrak{H}}^{\text{loc},(1)} & \xrightarrow{p_2 \times \text{id}} & \text{Hecke}_{0}^{\text{loc},(1)} \times \text{Hecke}_{\mathfrak{H}}^{\text{loc},(1)}
\end{array}
\]

where the vertical maps along the first lines are restriction to the local Shtukas at the special fiber.

The restriction map

$$\text{Sh}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{((1),(2),(3))} \rightarrow \text{Sh}_{V^* SV\mathfrak{Z}_\mathfrak{U}}^{\text{loc},(1)}$$

in the diagram is a Shtuka version of $\kappa^{((1),(2),(3))}_{((1),(2),(3))}$, namely sending

$$\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \mathcal{P}_3 \xrightarrow{\varphi_3} \text{Frob}_S \mathcal{P}_1$$
to
\[ \mathcal{P}_1|_{\text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S})} \xrightarrow{\varphi_1} \mathcal{P}_2|_{\text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S})} \xrightarrow{\varphi_3 \circ \varphi_2} \text{Frob}_S^* \mathcal{P}_1|_{\text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S})} \]
and
\[ \mathcal{P}_3|_{\text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S^S})} \xrightarrow{\varphi_3} \text{Frob}_S^* \mathcal{P}_1|_{\text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S^S})}, \]
and similarly for the other two. Since the modification \( \varphi_3 \) has leg at characteristic \( 0 \) until \( t \), it is an isomorphism after restricting to \( \text{Spec}(\mathcal{O}^{\wedge}_{Y_{[0,\infty]}(S),S}) \).

Similar to the \( p \)-adic Shtuka case, we can pullback \( C^{\text{loc},+,0}_z \) in (12) along

\[ \begin{array}{c}
\text{Sht}_{V \star \mathbb{G}_V}^{\text{loc},W,0} \\
\text{Sht}_{0 \text{loc}}^{\text{loc},W} \\
\text{Hecke}_{0 \text{loc}}^{\text{loc},(\{1\})} \\
\end{array} \xleftarrow{p_1} \xrightarrow{p_2} \begin{array}{c}
\text{Sht}_{V \star \mathbb{G}_V}^{\text{loc},W} \\
\text{Sht}_{0 \text{loc}}^{\text{loc},W} \\
\text{Hecke}_{0 \text{loc}}^{\text{loc},(\{1\})} \\
\end{array} \]

(18)

to obtain the Witt vector creation cohomological correspondence
\[ C^{\text{loc},0,W}_z : p_1^{\text{loc}} \circ \epsilon_0^{\text{loc},W} : \epsilon_1^{\text{loc},(\{1\})} \xrightarrow{p_2^{\text{loc}}} \epsilon_2^{\text{loc},W} : \epsilon_{V \star \mathbb{G}_V}^{\text{loc},(\{1\}),0} \]
supported on \( \text{Sht}_{V \star \mathbb{G}_V}^{\text{loc},W,0} \).

One important thing to notice is that the truncated version of diagram (18) is really the \( \infty \)-analytification of the diagram below of perfect stacks
which defines $C_{\gamma}^{\text{loc}(m,n),0}$ by pullback of $C_{\gamma}^{\text{loc}(m),+0}$. We know that $C_{\gamma}^{\text{loc}(m),+0,\infty} = C_{\gamma}^{\text{loc}+,0}$ so it follows from proposition [A.4] that

$$C_{\gamma}^{\text{loc},0,W} = C_{\gamma}^{\text{loc}(m,n),0,\infty}$$

under the identification in appendix [A.3].

Then by what we have just seen, we have

**Proposition 3.7.** The creation cohomological correspondence $C_{\gamma}$ of (17) is naturally identified with the pullback of $C_{\gamma}^{\text{loc},0,W} \boxtimes \text{id}$ along

$$\text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\},0} \xrightarrow{p_{1}} \text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}}$$

Moreover, $C_{\gamma}^{\text{loc}}$ is the $\infty$-analytification of the corresponding truncated Witt vector version as considered in [1].

We have defined the creation correspondence on $\text{Sht}_{\mu,\eta}$, we next make the observation that it can be extended to $\text{Sht}_{\mu,\eta}^{\{1\}}$. Let

$$\text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}} := \text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}} \big|_{\text{Spd}(\bar{F}_{p}) \times \text{Spd}(O_{L}) \times \text{Spd}(\bar{F}_{p})},$$

i.e. it parameterizes

$$\mathcal{P}_{1} \longrightarrow \mathcal{P}_{2} \longrightarrow \text{Frob}_{\gamma}^{*} \mathcal{P}_{1}$$

where the second modification takes place at the characteristic $p$ untilt and is bounded by $\delta$. The first modification has two poles, the first is fixed at the characteristic $p$ untilt where the modification is bounded by $\delta^{*}$, while the second varies along $\text{Spd}(O_{L})$ over which the modification is bounded by $\mu$.

We first observe that $\text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}}$ extends $\text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}}$.

**Lemma 3.8.** There is a natural identification

$$\text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}} \big|_{\text{Spd}(L)} \cong \text{Sht}_{V^{*}O_{L}^{\infty}O_{L}^{\infty}}^{\{1\},\{2\},\{3\}}$$

**Proof.** When the two legs of $\mathcal{P}_{1} \longrightarrow \mathcal{P}_{2}$ are different, it is equivalent to

$$\mathcal{P}_{1} \longrightarrow \mathcal{P}_{3} \longrightarrow \mathcal{P}_{2}$$
where the first modification takes place at characteristic \( p \) untilt, and the second is at \( Spd(L) \). Now as the two legs of
\[
\mathcal{P} \rightarrow \mathcal{P} \rightarrow \text{Frob}_S^* \mathcal{P}
\]
are different, it is equivalent to
\[
\mathcal{P} \rightarrow \mathcal{P} \rightarrow \text{Frob}_S^* \mathcal{P}
\]
where the first modification takes place at the characteristic \( p \) untilt, and the second takes place at characteristic zero untilt. \(\Box\)

Let
\[
Sht^0_{\mu|V^* \boxtimes \mu \boxtimes V} \subset Sht^{\{1\},\{2\},\{3\}}_{\mu|\delta^* \mu \delta}
\]
be the \( v \)-substack parameterizes modifications
\[
\mathcal{P} \rightarrow \text{Frob}_S^* \mathcal{P}
\]
\[
\mathcal{P} \rightarrow \mathcal{P} \rightarrow \text{Frob}_S^* \mathcal{P}
\]
with the first line in \( Sht_{\mu} \) and the second line in \( Sht^{\{1\},\{2\},\{3\}}_{V^* \boxtimes \mu \boxtimes V} \).

We have a commutative diagram
\[
\begin{array}{ccc}
Sht^0_{\mu|V^* \boxtimes \mu \boxtimes V} & \rightarrow & Sht^{\{1\},\{2\},\{3\}}_{\mu|\delta^* \mu \delta} \\
\downarrow & & \downarrow \\
\text{Hecke}^0_{\mu|V^* \boxtimes \mu \boxtimes V} & \rightarrow & \text{Hecke}^{\{1\},\{2\},\{3\}}_{V^* \boxtimes \mu \boxtimes V}
\end{array}
\]
(20)

We identify \( \text{Hecke}^0_{\mu} \) with \( \text{Hecke}^{\{1\},\{2\}}_{(0,\mu)} \), then we have a cartesian diagram
where we abuse the notation by writing
\[ \pi^{(1,2),(3)} : \text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V} \rightarrow \text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V} \subset \text{Hecke}_{(1,2)} \]
where we can view \( \text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V} \) as a subspace of \( \text{Hecke}_{(1,2)} \) because the legs indexed by \( V \) and \( V^* \) are fixed at the characteristic \( p \) until. Then
\[
\text{Corr}_{\text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V}}(S_{1,2}^{(1,2)}, \pi^{1,2,3}) = \text{Hom}(p_1^*S_{1,2}^{(1,2)}, p_2^*S_{1,2}^{(1,2,3)})
\]
\[
= \text{Hom}(p_2, p_1^*S_{1,2}^{(1,2)}, S_{1,2}^{(1,2,3)}) = \text{Hom}(\pi^{1,2,3}, (\pi^{1,2,3})) = S_{1,2}^{(1,2,3)}
\]
where we use 2) and 3) of theorem 3.1 in the second last equality, and geometric Satake of [1] theorem VI.11.1 in the last equality. More precisely, 3) of theorem 3.1 tells us that
\[
(\pi^{1,2,3})_{V \otimes \mathbb{Q}_p \otimes V} \simeq S_{1,2}^{(1,2,3)}
\]
on \( \text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V} \), then 2) of theorem 3.1 gives us
\[
S_{1,2}^{(1,2,3)} \simeq S_{1,2}^{(1,2)}
\]
under the inclusion \( \text{Hecke}_{V \otimes \mathbb{Q}_p \otimes V} \subset \text{Hecke}_{(1,2)} \).

Now under the above identification, the canonical morphism \( 1 \rightarrow V \otimes V^* \) times the identity of \( U \) gives us a cohomological correspondence
\[
\mathcal{C}_+^* : (\text{Hecke}^{(1,2)}_{0,\mu}, S_{1,2}^{(1,2)}) \rightarrow (\text{Hecke}^{(1,2,3)}_{V \otimes \mathbb{Q}_p \otimes V}, S_{1,2}^{(1,2,3)})
\]
supported on \( \text{Hecke}^0_{\mu V \otimes \mathbb{Q}_p \otimes V} \). Since the squares in diagram (20) are cartesian, we can pullback \( \mathcal{C}_+^* \) along \( \epsilon_{\mu V \otimes \mathbb{Q}_p \otimes V} \) to obtain the desired creation correspondence
\[
\mathcal{C}_+^* = (Sht_{\mu}^{(1,2)}, \epsilon_{\mu}^{(1,2)}, S^{U}_{1,2}^{(1,2)}) \rightarrow (Sht_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)}, \epsilon_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)}, S^{U}_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)})
\]
supported on \( Sht^0_{\mu V \otimes \mathbb{Q}_p \otimes V} \). To summarize, we have the proposition.

**Proposition 3.9.** The creation correspondence \((\mathcal{C}_+^*)\) extends naturally to a cohomological correspondence
\[
\mathcal{C}_+^* : (Sht_{\mu}^{(1,2)}, \epsilon_{\mu}^{(1,2)}, S^{U}_{1,2}^{(1,2)}) \rightarrow (Sht_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)}, \epsilon_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)}, S^{U}_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)})
\]
supported on \( Sht^0_{\mu V \otimes \mathbb{Q}_p \otimes V} \).

**Remark 3.10.** The reason we use \( Sht_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2,3)} \) instead of \( Sht_{V \otimes \mathbb{Q}_p \otimes V}^{(1,2)} \) is explained in the paragraph before [1] remark 6.1.10, namely, we want the corresponding (integral) excursion operators have the same support as the Hecke operators, so we can pullback the cohomological correspondence to Shimura varieties.
Lastly, we identify the creation correspondence at the special fiber. As usual, let

$$Sht_{V^* \otimes \mathbb{G}_m \otimes V,s}^{((1,2),(3))} := Sht_{V^* \otimes \mathbb{G}_m \otimes V}^{((1,2),(3))} \mid_{Spd(\mathbb{F}_p)};$$

and

$$Sht^{0}_{\mu | V^* \otimes \mathbb{G}_m \otimes V,s} := Sht^{0}_{\mu | V^* \otimes \mathbb{G}_m \otimes V} \mid_{Spd(\mathbb{F}_p)};$$

i.e. it parameterizes

\[ \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \text{Frob}_{\psi}^* \mathcal{P}_1, \]

and

\[ \mathcal{P}_1 \longrightarrow \text{Frob}_{\psi}^* \mathcal{P}_1 \]

respectively, where the modifications all take place at the characteristic $p$ untilt, $\mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is bounded by $V^* + \mu$ (the highest weights of irreducible components of $V^* + \mu$), $\mathcal{P}_2 \longrightarrow \text{Frob}_{\psi}^* \mathcal{P}_1$ is bounded by $V$, and $\mathcal{P}_1 \longrightarrow \text{Frob}_{\psi}^* \mathcal{P}_1$ is bounded by $\mu$.

As before, there is the diagram

where the vertical maps are identifying $\mathcal{P}_{k+1}$ as $\text{Frob}_{\psi}^* \mathcal{P}_1$.

On the other hand, we have the map

$$Sht^{W}_{\mu^*} \longrightarrow Sht^{((1),(2),\cdots,(k))}_{\mu^*,s}.$$
which is the restriction of Shtukas from $W(R^+)$ to $\mathcal{Y}_{[0,\infty)}(S)$, and similarly for $Sht^0_{\mu|\mu}^W$, which forms a commutative diagram

\[
\begin{array}{ccc}
Sht^0_{\mu|\mu}^W & \xrightarrow{Sht^W_{\mu|\mu}} & Sht^W_{\mu|\mu} \\
\downarrow & & \downarrow \\
Sht^{(1),(2),\ldots,(k)}_{\mu|\mu|s} & \xrightarrow{Sht^{(1),(2),\ldots,(k)|((1),\ldots,l)|0}_{\mu|\mu|s}} & Sht^{(1),(2),\ldots,(l)}_{\mu|\mu|s} \\
\end{array}
\]

The diagram (20) restricted on the special fiber is

\[
\begin{array}{ccc}
Sht^0_{\mu|V^*\otimes\mu\otimes V|s} & \xrightarrow{Sht^{0}} & Sht^0_{V^*\otimes\mu\otimes V|s} \\
\downarrow & & \downarrow \\
\epsilon^{(1)}_{\mu|V^*\otimes\mu\otimes V} & \xrightarrow{\epsilon^0_{V^*\otimes\mu\otimes V}} & \epsilon^{(1)}_{V^*\otimes\mu\otimes V} \\
\downarrow & & \downarrow \\
\text{Hecke}^{(1)}_{\mu|s} & \xrightarrow{\text{Hecke}^0_{V^*\otimes\mu\otimes V|s}} & \text{Hecke}^{(1)}_{V^*\otimes\mu\otimes V|s} \\
\end{array}
\]
We can compose it with the previous one to obtain

\[
\begin{align*}
\text{Hecke}_{\mu,s}^{\{1\}} & \quad \text{Hecke}_{V^*\oplus\mu\mathbb{C}V,s}^{\{1,2\},\{3\}} \\
\text{Hecke}^{0}_{\mu|V^*\oplus\mu\mathbb{C}V,s} & \quad \text{Hecke}^{0}_{V^*\oplus\mu\mathbb{C}V,s}
\end{align*}
\]

and we want to understand the pullback of (restriction to the special fiber of) creation correspondence (22) along the upper half of the diagram. We note that the diagram factorizes as

\[
\begin{align*}
\text{Hecke}_{\mu,s}^{\{1\}} & \quad \text{Hecke}_{V^*\oplus\mu\mathbb{C}V,s}^{\{1,2\},\{3\}} \\
\text{Hecke}^{0}_{\mu|V^*\oplus\mu\mathbb{C}V,s} & \quad \text{Hecke}^{0}_{V^*\oplus\mu\mathbb{C}V,s}
\end{align*}
\]
where the vertical maps of the second row are restriction of torsors from $W(R^+)$ to $Y_{[0,\infty)}(R, R^+)$. We abuse the notation as usual by writing $V^* + \mu$ the highest weights of $V^*$ plus $\mu$.

We can pullback the cohomological correspondence \[21\] (restricting to the special fiber) along the last diagram, which defines the creation correspondence on the global Witt vector shtuka

\[
C^W_\sharp : (\text{Sh}_{V^*+\mu, V}^W, (e_{V^*+\mu, V})^*) \to (\text{Sh}_{V^*+\mu, V}^W, (e_{V^*+\mu, V})^*)^S_{(V^* \otimes U) \boxtimes V}.
\] (24)

Moreover, the factorization tells us that $C^W_\sharp$ is the pullback of creation correspondence \[22\] along

The importance of $C^W_\sharp$ is that it is analytification of the creation correspondence for Witt vector Shtukas considered in [1]. Indeed, we first observe that the restriction of $C^W_\sharp$, as defined in \[21\], to the special fiber has a local counter part $C^\text{loc}_\sharp$, and $C^\sharp_\sharp$ is the pullback of $C^\text{loc}_\sharp$ along the canonical restriction diagram

The restriction to the special fiber is important here, as otherwise $Hecke^\text{loc},0_{\mu|V^* \boxtimes \mu \boxtimes V, s}$ does not exist, see remark 2.14. As always, $C^\sharp_\sharp$ being the pullback of $C^\text{loc},\sharp$ follows ultimately from 6) of theorem
3.1 Now we can compose the diagram (23) with the above restriction diagram, and obtain

\[
\begin{align*}
Sht^0_{\mu[[V^*+\mu,V)}} \quad & \quad Sht^W_{\mu[[V^*+\mu,V)} \\
Sht^W_{\mu} \quad & \quad Sht^W_{V^*+\mu,V} \\
Hecke^0_{\mu[[V^*+\mu,V)} \quad & \quad Hecke^W_{V^*+\mu,V} \\
Hecke^W_{\mu} \quad & \quad Hecke^W_{V^*+\mu,V} \\
Hecke^0_{\mu[V^*\ominus\mu\ominus V,s]} \quad & \quad Hecke^0_{\mu[V^*\ominus\mu\ominus V,s]} \\
Hecke^0_{\mu,[\{1\}]} \quad & \quad Hecke^0_{\mu,[\{1,2\},\{3\}]} \\
\end{align*}
\]

and \( C^W_\mu \) is the pullback of \( C^0_{\mu,+} \) along the diagram. Up to truncation, this is precisely

\[
\begin{align*}
Sht^0_{\mu[[V^*+\mu,V)}} \quad & \quad Sht^0_{\mu[[V^*+\mu,V)} \\
Sht^0_{\mu[[V^*+\mu,V)} \quad & \quad Sht^0_{\mu[[V^*+\mu,V)} \\
Hecke^0_{\mu[[V^*+\mu,V)} \quad & \quad Hecke^0_{\mu[[V^*+\mu,V)} \\
Hecke^0_{\mu[[V^*+\mu,V)} \quad & \quad Hecke^0_{\mu[[V^*+\mu,V)} \\
Hecke^0_{\mu[[V^*+\mu,V)} \quad & \quad Hecke^0_{\mu[[V^*+\mu,V)} \\
Hecke^0_{\mu[[V^*+\mu,V)} \quad & \quad Hecke^0_{\mu[[V^*+\mu,V)} \\
\end{align*}
\]
see appendix [A.1] for the notation of \( a_7 \), and [A.3] for the truncated Witt vector Hecke stacks and Shtukas. We know from appendix [A.3] that the truncated version does not change cohomology theory, so we can naturally view the cohomological correspondences on the (analytification of) truncated Hecke stacks or Shtukas as being defined on the untruncated ones.

We can perform the same construction as of \( C_\mu^{+} \) (in the theory of perfect stacks) to obtain

\[
C^{\text{loc}(m),+}_{\mu} : (\text{Hecke}_{\mu}^{\text{loc}(m)}, IC_{\mu}) \rightarrow (\text{Hecke}_{V^*+\mu,V}^{\text{loc}(m)}, IC_{V^*+\mu,V})
\]

supported on

\[
\text{Hecke}_{\mu}^{\text{loc}(m)} \rightarrow \text{Hecke}_{V^*+\mu,V}^{\text{loc}(m)},
\]

which is precisely the creation correspondence considered in [1]. Now 6) of theorem 3.1 (and the comparison results in appendices A.1 and A.2) tells us that

\[
C^{\text{loc}(m),+}_{\mu} = C^{\text{loc}+}_{\mu}
\]

under the identification of sheaves on truncated and untruncated Hecke stacks in [A.3]. By definition of \( C^{\text{loc}(m),+}_{\mu} \), see appendix A.2, and \( C^{+}_{\mu} \) being the pullback of \( C^{\text{loc}+}_{\mu} \), we have

\[
C^{\text{loc}(m),+}_{\mu} = C^{+}_{\mu}.
\]

Then the commutation between the analytification and pullback of cohomological correspondence (proposition A.4) tells us that

\[
C^{\text{loc}(m,n),\circ}_{\mu} = C^{W}_{\mu}
\]

again under the identification of sheaves truncated and untruncated Hecke Shtukas, where \( C^{\text{loc}(m,n)}_{\mu} \) is the pullback of \( C^{\text{loc}(m)}_{\mu} \) along

\[
\text{Sht}_{\mu}^{0,\text{loc}(m,n)}_{V^*+\mu,V} \rightarrow \text{Hecke}_{\mu}^{0,\text{loc}(m)}_{(V^*+\mu,V)} \rightarrow \text{Sht}_{\mu}^{\text{loc}(m,n)}_{\mu(V^*+\mu,V)} \rightarrow \text{Hecke}_{\mu}^{\text{loc}(m)}_{(V^*+\mu,V)} \rightarrow \text{Sht}_{V^*+\mu,V}^{\text{loc}(m,n)} \rightarrow \text{Hecke}_{V^*+\mu,V}^{\text{loc}(m)}
\]
As usual, we use $\epsilon_?^\mu$ to denote the (in the sense of perfect stack) forgetting map (viewing the Frobenius twist of $P_1$ as $P_2$). $S_{\mu}^{\text{loc}(m,n)}$ is exactly the construction of creation correspondence on the Witt vector Shtukas in [1], and we have proved that $C^W_\mu$ is the analytification of the creation correspondence in [1].

In summary, we have

**Proposition 3.11.** $C^W_\mu$ is the pullback of the restriction of the creation correspondence $C^+_\mu$ in (22) to $\text{Spd}(\mathbb{F}_p)$ along

Moreover, $C^W_\mu = C^{\text{loc}(m,n)}_\mu$ being the analytification of the creation correspondence in [1].

### 3.4. Annihilation correspondence.

Dually, we have the annihilation cohomological correspondence

$$\mathfrak{c}_\eta: p^*_1 \epsilon_{V\otimes V^*}^{(1\{1\},\{2\},\{3\})} \mathcal{S}_{V\otimes V^*}^{(1\{1\})} \rightarrow p^!_2 \epsilon_{\mu}^{(1\{1\})} \mathcal{S}_{U}^{(1\{1\})}$$

(25)
supported on

$$\text{Sh}_V^{(1\{1\},\{2\},\{3\})} \rightarrow \text{Sh}_U^{(1\{1\})},$$

together with the local Witt version

$$\mathfrak{c}^{\text{loc},W}_\eta: p^*_1 \epsilon_{V\otimes V^*}^{\text{loc}} \mathcal{S}_{V\otimes V^*}^{\text{loc}} \rightarrow p^!_2 \epsilon_{\mu} \mathcal{S}_1^{\text{loc}}(1\{1\})$$

supported on

$$\text{Sh}_{V\otimes V^*}^{\text{loc},W} \rightarrow \text{Sh}_0^{\text{loc},W}$$

which come from the evaluation map

$$V \otimes V^* \rightarrow 1.$$
We have the similar proposition

**Proposition 3.12.** The creation cohomological correspondence \( C_b \) of \((25)\) is naturally identified with the pullback of

\[
C_{loc}^{0, W} \boxtimes id : (\text{Sh}_V^{loc, W} \times \text{Hecke}_{\mu, \eta}^{loc, \{1\}}, (\epsilon_{V, V}^{loc, \{1\}, \{2\}})_1 \boxtimes \mathcal{S}_U^{loc, \{1\}}) \rightarrow
(\text{Sh}_V^{loc, W} \times \text{Hecke}_{\mu, \eta}^{loc, \{1\}}, (\epsilon_{V, V}^{loc, \{1\}, \{2\}})_1 \boxtimes \mathcal{S}_U^{loc, \{1\}})
\]

along the restriction diagram

Moreover, \( C_{loc}^{0, W} \) is the \( \infty \)-analytification of the corresponding truncated Witt vector version \( C_{b (m,n)}^{0} \) as considered in \([1]\).

Moreover, the annihilation correspondence extends integrally. Let

\[
\text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\})} := \text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\})} \subset \text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\}) \times \{1\}},
\]

and

\[
\text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\}) \times \{1\}}
\]

be the \( v \)-substack parameterizes modifications

\[
\mathcal{P}_1 \longrightarrow \mathcal{P}_2 \longrightarrow \text{Frob}_S^* \mathcal{P}_1
\]

\[
\mathcal{P}_1 \longrightarrow \text{Frob}_S^* \mathcal{P}_1
\]

with the first line in \( \text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\})} \), and the second line in \( \text{Sh}_V^{\{1\}} \).

**Proposition 3.13.** The annihilation correspondence \((25)\) extends to a cohomological correspondence

\[
\mathcal{E}_b : (\text{Sh}_{V_{/V}}^{(\{1\}, \{2\}, \{3\})}, (\epsilon_{V, V}^{(\{1\}, \{2\}, \{3\}) \times S_{/V_{/V}}^{(\{1\}, \{2\}, \{3\})}}) \rightarrow (\text{Sh}_V^{\{1\}}, \epsilon_{V, V}^{(\{1\}) \times S_{/V_{/V}}^{(\{1\})}})
\]

\((26)\)
supported on $\text{Sht}_{\mu|V^*V^*\otimes\mu}^0$. More precisely, $C_\nu$ is the pullback along

\[
\begin{array}{ccc}
\text{Sht}_{(\{1\},\{2,3\})}^0_{V^*V^*\otimes\mu} & \xrightarrow{\mu|V^*V^*\otimes\mu} & \text{Sht}_\mu^{\{1\}} \\
\downarrow & & \downarrow \\
\text{Hecke}_{V^*V^*\otimes\mu}^{\{1\},\{2,3\}} & \xrightarrow{c_{\mu|V^*V^*\otimes\mu}} & \text{Hecke}_\mu^{\{1\}} \\
\end{array}
\]

of the correspondence

\[
C_\nu^+ : (\text{Hecke}_{V^*V^*\otimes\mu}^{\{1\},\{2,3\}}, S_{V^*V^*\otimes U}^{\{1\},\{2,3\}}) \rightarrow (\text{Hecke}_\mu^{\{1\}}, S_U^{\{1\}})
\]

supported on $\text{Hecke}_\mu^{\{1\}}$ corresponding to the evaluation map

\[
(V \otimes V^*) \boxtimes U \rightarrow 1 \boxtimes U.
\]

We can identify the annihilation correspondence at the special fiber as well.

**Proposition 3.14.** $C_\nu^{W}$ is the pullback of the restriction of the annihilation correspondence (26) to $\text{Spd}(\overline{F}_p)$ along the diagram

\[
\begin{array}{ccc}
\text{Sht}_{V^*V^*+\mu}^W & \xrightarrow{\nu|V^*V^*+\mu} & \text{Sht}_\mu^W \\
\downarrow & & \downarrow \\
\text{Sht}_{V^*V^*\otimes\mu,s}^0 & \xrightarrow{\nu|V^*V^*\otimes\mu,s} & \text{Sht}_\mu^{\{1\}} \\
\end{array}
\]

where

\[
C_\nu^{W} : (\text{Sht}_{V^*V^*+\mu}^W, (\nu|V^*V^*+\mu)^* r_{V^*V^*+\mu}) S_{V^*V^*(V^*\otimes U)}^{\{1\},\{2\}} \rightarrow (\text{Sht}_\mu^W, (\nu|\mu)^* r_{\mu U} S_U^{\{1\}})
\]
is the pullback of (27) (restricted to $Spd(F_p)$) along

Moreover, $C^W_\flat$ is the analytification of the annihilation correspondence in [1], i.e.

under the identification in A.2 between sheaves on truncated and untruncated Shtukas.

3.5. Partial Frobenius. Lastly, we have the partial Frobenius morphism

sending

\[ \mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \mathcal{P}_3 \xrightarrow{\varphi_3} \text{Frob}_V^* \mathcal{P}_1 \]

to

\[ \mathcal{P}_2 \xrightarrow{\varphi'_2} \mathcal{P}_3 \xrightarrow{\varphi'_3} \text{Frob}_V^* \mathcal{P}_1 \xrightarrow{\text{Frob}_V^* \varphi_1} \text{Frob}_V^* \mathcal{P}_2, \]

which is the same as

\[ \mathcal{P}_2 \xrightarrow{\varphi'_2} \mathcal{P}_3 \xrightarrow{\varphi'_3} \text{Frob}_V^* \mathcal{P}_1 \xrightarrow{\text{Frob}_V^* \varphi_1} \text{Frob}_V^* \mathcal{P}_2, \]

where $\varphi'_3$ is bounded by $\mu$ at a characteristic 0 leg, and $\varphi'_2$ is bounded by $V^*$ at a characteristic $p$ leg, i.e. it defines an element of $Sh_{V^{*\otimes \mu}}^{((1),(2),(3))}$. Note that $\text{Frob}_V^* \varphi_1$ has modification type $^gV$, which is equal to $V$ by our assumption that $V$ is a representation of $^LG$. 
Now we have a commutative diagram
\[
\begin{array}{ccc}
\text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta} & \xrightarrow{F} & \text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta} \\
\downarrow_{\rho^{((1),(2),(3))}_V} & & \downarrow_{\rho^{((1),(2),(3))}_V} \\
\text{Hecke}^{((1))}_V \times \text{Hecke}^{((1),(2))}_{V^*{\otimes}_\mu V,\eta} & \xrightarrow{\text{Frob} \times \text{id}} & \text{Hecke}^{((1))}_V \times \text{Hecke}^{((1),(2))}_{V^*{\otimes}_\mu V,\eta}
\end{array}
\]
(29)

using the canonical isomorphism
\[
\mathcal{C}_F^+ : \text{Frob}^* S_V^{((1))} \cong S_V^{((1))}
\]
we obtain
\[
F^* \epsilon^{((1),(2),(3))}_V \circ S_V^{((1),(2),(3))} \cong F^* \epsilon^{((1),(2),(3))}_V \circ S_V^{((1),(2),(3))} \circ (S_V^* \boxtimes S_V^* U)
\]
\[
\cong \epsilon^{((1),(2),(3))}_V \circ \rho^{((1),(2),(3))}_V \circ (\text{Frob} \times \text{id})^* (S_V^{((1))} \boxtimes S_V^{((1))})
\]
\[
\mathcal{C}_{F,\eta} : (\text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta}, \epsilon^{((1),(2),(3))}_V \circ S_V^{((1),(2),(3))}) \to (\text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta}, \epsilon^{((1),(2),(3))}_V \circ S_V^{((1),(2),(3))})
\]
supported on
\[
\begin{array}{ccc}
\text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta} & \xrightarrow{F} & \text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta} \\
\downarrow & & \downarrow \\
\text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta} & \xrightarrow{\text{Frob}^* \epsilon^{((1),(2),(3))}_V} & \text{Sh}^{((1),(2),(3))}_{V^*{\otimes}_\mu V,\eta}
\end{array}
\]

Similarly, we have the local Witt vector partial Frobenius
\[
F_{\text{loc}, W} : \text{Sh}^{\text{loc}, W}_{V^*{\otimes} V, \star} \to \text{Sh}^{\text{loc}, W}_{V^*{\otimes} V, \star}
\]
defined by sending
\[
\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^* \mathcal{P}_1
\]
to
\[
\mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^* \mathcal{P}_1 \xrightarrow{\text{Frob}^* \varphi_1} \text{Frob}^* \mathcal{P}_2,
\]
and there is a commutative diagram
\[
\begin{array}{ccc}
\text{Sh}^{\text{loc}, W}_{V^*{\otimes} V, \star} & \xrightarrow{F^{\text{loc}, W}} & \text{Sh}^{\text{loc}, W}_{V^*{\otimes} V, \star} \\
\downarrow_{\rho^{\text{loc}, (1),(2)}_V} & & \downarrow_{\rho^{\text{loc}, (1),(2)}_V} \\
\text{Hecke}^{\text{loc}, (1)_V} \times \text{Hecke}^{\text{loc}, (1),(2)_V} & \xrightarrow{\text{Frob} \times \text{id}} & \text{Hecke}^{\text{loc}, (1)_V} \times \text{Hecke}^{\text{loc}, (1),(2)_V}
\end{array}
\]
which induces a pullback cohomological correspondence
\[
\mathcal{C}_{F,\eta}^{\text{loc}, 0, W} : F^{\text{loc}, W} \circ \rho^{\text{loc}, (1),(2)}_V \circ \epsilon^{\text{loc}, (1),(2)}_V \circ S_V^{\text{loc}, (1),(2)} \cong \epsilon^{\text{loc}, (1),(2)}_V \circ S_V^{\text{loc}, (1),(2)}
\]
supported on

\[ Sht^\text{loc,}W_{V^*V^*} \]
\[ \xrightarrow{F^\text{loc,}W} \]

\[ Sht^\text{loc,}W_{V^*V^*} \]
\[ \xrightarrow{\kappa^\text{loc,}W_{V^*V^*}} Sht^\text{loc,}W_{V^*V^*} \]

As usual, this is the \( \otimes \)-analytification of the corresponding truncated Witt vector version \( \xi_F^\text{loc,(m,n),0} \).

Moreover, we have the commutative diagram

\[ Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]
\[ \xrightarrow{F} \]

\[ Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]
\[ \xrightarrow{\kappa^\text{loc,}W_{V^*V^*\mu,\eta}} Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]

\[ Sht^\text{loc,}W_{V^*V^*} \times Hecke^\text{loc,}(1)_{\mu,\eta} \]
\[ \xrightarrow{\kappa^\text{loc,}W_{V^*V^*} \times \text{id}} \]

\[ Sht^\text{loc,}W_{V^*V^*} \times Hecke^\text{loc,}(1)_{\mu,\eta} \]

where the vertical arrows are as before restriction maps. The two squares are Cartesian, so we can pullback the cohomological correspondence \( \xi_F^{\text{loc,0,W}} \boxtimes \text{id} \), and this is exactly \( \xi_{F,\eta} \). Indeed, the composition of (30) with \( \text{id} \times \kappa^{\text{(1),(2)}}_{(1),(2)} \) factorizes through

\[ Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]
\[ \xrightarrow{F} \]

\[ Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]
\[ \xrightarrow{\kappa^\text{loc,}W_{V^*V^*\mu,\eta}} Sht^{\text{(1),(2),(3)}}_{V^*V^*\mu,\eta} \]

\[ Hecke^\text{loc,}(1)_{V^*} \times Hecke^\text{loc,}(1)_{V^*} \times Hecke^\text{loc,}(1)_{\mu,\eta} \]
\[ \xrightarrow{\kappa^\text{loc,}W_{V^*V^*} \times \text{id}} \]

\[ Hecke^\text{loc,}(1)_{V^*} \times Hecke^\text{loc,}(1)_{V^*} \times Hecke^\text{loc,}(1)_{\mu,\eta} \]

from which it is easy to deduce the identification of the cohomological correspondence. To summarize, we have
**Proposition 3.15.** The cohomological correspondence $\mathcal{C}_{F,\eta}$ is the same as the pullback of $\mathcal{C}_{\mathcal{F}^0,W}^{\text{id}}$ along the diagram

\[
\begin{array}{ccc}
\text{Sh}_{V^*\boxtimes \mathbb{V}_\mathcal{E},\eta}^{((1),(2),(3))} & \overset{F}{\longrightarrow} & \text{Sh}_{V^*\boxtimes \mathbb{V}_\mathcal{E},\eta}^{((1),(2),(3))} \\
\downarrow \epsilon_{((1),(2),(3))}^{((1),(2),(3))} & & \downarrow \epsilon_{((1),(2),(3))}^{((1),(2),(3))}
\end{array}
\]

Moreover, $\mathcal{C}_{F}^{\text{id},0,W} = \mathcal{C}_{F}^{\text{loc}(m,n),0,\infty}$, being the $\infty$-analytification of the truncated Witt vector version considered in $\mathbb{F}$.

The partial Frobenius can also be extended to the integral model, namely we have

$F : \text{Sh}_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \longrightarrow \text{Sh}_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))}$

sending

$\mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^*_\mathcal{S} \mathcal{P}_1$

to

$\mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^*_\mathcal{S} \mathcal{P}_1 \xrightarrow{\text{Frob}^*_\mathcal{S} \varphi_1} \text{Frob}^*_\mathcal{S} \mathcal{P}_2$.

We have a commutative diagram

\[
\begin{array}{ccc}
\text{Hecke}_{V}^{((1),(2),(3))} & \overset{\text{Frob} \times \text{id}}{\longrightarrow} & \text{Hecke}_{V^*\boxtimes \mathcal{E},\mu}^{((1),(2),(3))} \\
\downarrow \epsilon_{((1),(2),(3))}^{((1),(2),(3))} & & \downarrow \epsilon_{((1),(2),(3))}^{((1),(2),(3))}
\end{array}
\]

using the canonical isomorphism

$\mathcal{C}_{F}^+ : \text{Frob}^* \mathcal{S}_V^{((1),(2),(3))} \cong \mathcal{S}_V^{((1),(2),(3))}$

we obtain

$F^* \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))} \cong F^* \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))}$

$\cong \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))}$

$\cong \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))}$

$\cong \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))}$

$\cong \epsilon_{V^*\boxtimes \mathbb{V}_\mathcal{E},\mu}^{((1),(2),(3))} \mathcal{S}_V^{((1),(2),(3))}$
which defines a pullback cohomological correspondence
\[ \mathcal{E}_F : \left( \text{Sht}_{V^*+\mu} \otimes_{V} \text{Sht}^{(1),(2,3)} \right) \to \left( \text{Sht}_{V^*+\mu} \otimes_{V} \text{Sht}^{(1),(2,3)} \right) \]
supported on
\[ Sht_{V^*+\mu}^{(1),(2,3)} \]

Lastly, we want to study its restriction to the special fiber. As usual, there is a partial Frobenius for global Witt vector Shtukas
\[ F^W : \text{Sht}_{V^*+\mu}^W \to \text{Sht}_{V^*+\mu,V}^W \]
sending
\[ \mathcal{P}_1 \xrightarrow{\varphi_1} \mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^*_s \mathcal{P}_1 \]
to
\[ \mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}^*_s \mathcal{P}_1 \xrightarrow{\text{Frob}^*_s \varphi_1} \text{Frob}^*_s \mathcal{P}_2. \]
We have the commutative diagram
\[ Sht_{V^*+\mu}^W \xrightarrow{F^W} Sht_{V^*+\mu,V}^W \]
\[ \xrightarrow{\text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_{V^*+\mu,s} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V} \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V} \]
which factorizes as
\[ Sht_{V^*+\mu}^W \xrightarrow{F^W} Sht_{V^*+\mu,V}^W \]
\[ \xrightarrow{\text{Hecke}^W_V \times \text{Hecke}^W_{V^*+\mu} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^W_V \times \text{Hecke}^W_V} \text{Hecke}^W_V \times \text{Hecke}^W_V} \]
The last diagram defines the global Witt vector partial Frobenius correspondence
\[ \mathcal{E}_F^W : F^W \circ (\text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_{V^*+\mu,s} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V} \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V) \]
which factorizes as
\[ \xrightarrow{\text{Hecke}^W_V \times \text{Hecke}^W_{V^*+\mu} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^W_V \times \text{Hecke}^W_V} \text{Hecke}^W_V \times \text{Hecke}^W_V} \]

The last diagram defines the global Witt vector partial Frobenius correspondence
\[ \mathcal{E}_F^W : F^W \circ (\text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_{V^*+\mu,s} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V} \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V) \]
which factorizes as
\[ \xrightarrow{\text{Hecke}^W_V \times \text{Hecke}^W_{V^*+\mu} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^W_V \times \text{Hecke}^W_V} \text{Hecke}^W_V \times \text{Hecke}^W_V} \]

The last diagram defines the global Witt vector partial Frobenius correspondence
\[ \mathcal{E}_F^W : F^W \circ (\text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_{V^*+\mu,s} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V} \text{Hecke}^{(1)}_V \times \text{Hecke}^{(1)}_V) \]
which factorizes as
\[ \xrightarrow{\text{Hecke}^W_V \times \text{Hecke}^W_{V^*+\mu} \xrightarrow{\text{Frob} \times \text{id} \times \text{Hecke}^W_V \times \text{Hecke}^W_V} \text{Hecke}^W_V \times \text{Hecke}^W_V} \]
which is the analytification of the partial Frobenius correspondence studied by Xiao-Zhu in [1], namely,
\[ C^w_F = C^\text{loc}(m,n) \circ \]
under the identification in [A.2] between sheaves on truncated and untruncated Shtukas, where \( C^\text{loc}(m,n) \) is the partial Frobenius cohomological correspondence considered in [1]. Note that the inverse of the partial Frobenius is used in [1] as it is better suited with the truncated Witt vector Shtukas, after passing to the analytic untruncated version it agrees with our construction.

Now the diagram (32) tells us that \( C^w_F \) is the pullback of \( C^s_F \) along the first square of (32). In summary, we have

**Proposition 3.16.** \( C^w_F \) is the pullback of the restriction of correspondence (31) to \( \text{Spd}(\mathbb{F}_p) \) along the diagram

Moreover, \( C^w_F \) is the analytification of the partial Frobenius cohomological correspondence considered in [1].

3.6. **Excursion Operators.** We can now define the excursion operator associated to \( V \) as the cohomological correspondence
\[ S_V := c \circ c_F \circ c_z \]
from \( (\text{Sht}_\mu, e^{(1)} \otimes e^{(1)}) \) to itself supported on
\[ \text{Sht}_\mu^{V_{|\mu}} := \bigcup_{\nu} \text{Sht}_\nu^{V_{|\mu}} \]
where \( \nu \) ranges over highest weights of irreducible components of \( V \). Indeed, we have

**Lemma 3.17.** The support of \( S_V \) is \( \text{Sht}^{V_{|\mu}}_{\mu} \).
Proof. By definition, the support of $S_V$ is the fiber product of

\[
\begin{array}{ccc}
Sht^0_{V \otimes \mu|\mu} & \xleftarrow{F} & Sht^{(1),\{2,3\}}_{V \otimes \mu|\mu} \\
\downarrow & & \downarrow \\
Sht^{(1,2),\{3\}}_{\otimes \mu|\mu} & \xrightarrow{\phi_1} & Sht^0_{\otimes \mu|\mu}
\end{array}
\]

which by definition parameterizes

\[
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{\text{Frob}_S^*} & \mathcal{P}_1 \\
\downarrow & & \downarrow \\
\mathcal{P}_2 & \xleftarrow{\text{Frob}_S^*} & \mathcal{P}_2
\end{array}
\]

where the upper left triangle lies in $Sht^0_{V \otimes \mu|\mu}$, and the lower right triangle lies in $Sht^0_{\otimes \mu|\mu}$. In other words, $\phi_1$ has leg at the characteristic $p$ untilt and is bounded by $V$, the two horizontal modifications have legs varying in $Spd(O_L)$ and is bounded by $\mu$, while the diagonal one has two legs, with one fixed at characteristic $p$ untilt bounded by $V^*$, and the other varying in $Spd(O_L)$ bounded by $\mu$. This is then the same as

\[
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{\text{Frob}_S^*} & \mathcal{P}_1 \\
\downarrow & & \downarrow \\
\mathcal{P}_2 & \xleftarrow{\text{Frob}_S^*} & \mathcal{P}_2
\end{array}
\]

with the two lines lying in $Sht_\mu$ and $\phi_1$ bounded by $V$, which is nothing but what $Sht^V_{\mu|\mu}$ parameterizes. \hfill \square

Note that we have

\[Sht^V_{\mu|\mu,\eta} = \Gamma_V,\]

thus $S_V$ restricted on $Sht_{\mu,\eta}$ is supported on $\Gamma_V$, the same as the support of the Hecke correspondence.

Similarly, we have the local Witt vector version cohomological correspondence

\[S_{V,\mu|\mu,\eta}^{loc,0,W} := \mathcal{C}^{loc,0,W}_g \circ \mathcal{C}^{loc,0,W}_F \circ \mathcal{C}^{loc,0,W}_\delta\]

from $(\text{Sh}_{0|\mu}^{loc,W}, \xi^{loc,\{1\}}, \mathcal{C}^{loc,\{1\}}_g)$ to itself supported on $\Gamma^{loc,W}_V$, which can be identified with $Sht_{\mu|\mu,\eta}^{loc,V,W}$. Indeed, similar to the proof of previous lemma, we have that the support of $S_{V,\mu|\mu,\eta}^{loc,0,W}$...
parametrizes diagrams

$$\begin{array}{c}\mathcal{P}_1 \xrightarrow{\sim} \text{Frob}_S^* \mathcal{P}_1 \\
\downarrow \varphi_2 \downarrow \text{Frob}_S^* \varphi_1 \\
\mathcal{P}_2 \xrightarrow{\sim} \text{Frob}_S^* \mathcal{P}_2 \end{array}$$

which is nothing but

$$\begin{array}{c}\mathcal{P}_1 \xrightarrow{\varphi_2 \varphi_1} \text{Frob}_S^* \mathcal{P}_1 \\
\downarrow \varphi_1 \downarrow \text{Frob}_S^* \varphi_1 \\
\mathcal{P}_2 \xrightarrow{\sim} \text{Frob}_S^* \mathcal{P}_2 \end{array}$$

whence $\Gamma_{V}^{\text{loc},W} = \text{Sh}_{0,0}^{\text{loc},V,W}$. Now proposition 3.7, 3.12 and 3.15 together imply that (pullback of cohomological correspondences respects composition)

**Theorem 3.18.** The cohomological correspondence $S_{V,\eta}$, the restriction of $S_V$ on $\text{Sh}_{\mu,\eta}$, is the pullback of

$$S_{V}^{\text{loc},0,W} \boxtimes \text{id} : (\text{Sh}_{0}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)}), (\epsilon_0^{\text{loc},W,*} S_1^{\text{loc},(1)}) \boxtimes S_0^{\text{loc},(1)}) \rightarrow (\text{Sh}_{0}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)}), (\epsilon_0^{\text{loc},W,*} S_1^{\text{loc},(1)}) \boxtimes S_0^{\text{loc},(1)})$$

along the diagram

$$\begin{array}{c}Sht_{\mu,\eta} \xrightarrow{p_1} \Gamma_V \xrightarrow{p_2} Sht_{\mu,\eta} \\
\downarrow \Gamma_{V}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)} \xrightarrow{p_1^{\text{loc},x\text{id}}} Sht_{0}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)} \\
\downarrow \left(Sht_{0}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)} \right) \xrightarrow{p_2^{\text{loc},x\text{id}}} Sht_{0}^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},(1)} \end{array}$$

where the vertical arrows are restriction maps, in particular, the middle one sends

$$\begin{array}{c}\mathcal{P}_1 \xrightarrow{\varphi_1} \text{Frob}_S^* \mathcal{P}_1 \\
\downarrow \gamma \downarrow \text{Frob}_S^* \gamma \\
\mathcal{P}_2 \xrightarrow{\varphi_2} \text{Frob}_S^* \mathcal{P}_2 \end{array}$$

to

$$\begin{array}{c}\mathcal{P}_1\big|_{\text{Spec}(\mathcal{O}_{\gamma_{[0,\infty]}(S),S})} \xrightarrow{\gamma} \mathcal{P}_2\big|_{\text{Spec}(\mathcal{O}_{\gamma_{[0,\infty]}(S),S})} \xrightarrow{\sim} \mathcal{P}_1\big|_{\text{Spec}(\mathcal{O}_{\gamma_{[0,\infty]}(S),S})} \xrightarrow{\gamma} \mathcal{P}_2\big|_{\text{Spec}(\mathcal{O}_{\gamma_{[0,\infty]}(S),S})} \end{array}$$
\[ S = T \text{ FOR SHIMURA VARIETIES AND } p\text{-ADIC SHTUKAS} \]

and

\[ \mathcal{P}_1|_{\text{Spec}(\mathcal{O}_{\mathcal{Y}_{[0,\infty]}(S),st})} \xrightarrow{\varphi_1} \text{Frob}_S^*\mathcal{P}_1|_{\text{Spec}(\mathcal{O}_{\mathcal{Y}_{[0,\infty]}(S),st})} \]

Moreover, \( S_V^{\text{loc},0,W} \) is the \( \infty \)-analytification of the

\[ S_V^{\text{loc}(m,n),0} := \mathfrak{g}_{\omega}^{\text{loc}(m,n,0)} \circ \mathcal{C}_F^{\text{loc}(m,n,0)} \circ \mathfrak{g}_{\omega}^{\text{loc}(m,n,0)}, \]

the truncated Witt vector excursion operator considered in [1].

**Remark 3.19.** The theorem holds with \( \text{Sht}_{\mu,\eta} \) replaced by arbitrary \( \text{Sht}_{(I_1,\ldots,I_k)}^{(\mu_1,\ldots,\mu_k)} \) (remember the notation means that the legs are in characteristic 0). The proof goes without change. We state it only for Shtukas with a single leg to simplify the notation and that is the only case matters for Shimura varieties.

Lastly, we want to study the excursion operators at the special fiber. We have the global Witt vector excursion operator, which is the composition of (24), (33) and (28), i.e.

\[ S_V^W := \mathfrak{c}_{\omega}^W \circ \mathcal{C}_F^{\omega} \circ \mathfrak{c}_{\omega}^{\omega} \]

is the cohomological correspondence from \( (\text{Sht}_\mu^W,\epsilon_\mu^W,\tau_\mu^S(\mathbb{G})) \) to itself supported on \( \text{Sht}_{\mu|\mu}^V \). As \( \mathfrak{c}_{\omega}^W \), \( \mathfrak{c}_{\omega}^W \) and \( \mathfrak{c}_{\omega}^{\omega} \) are all analytification of the truncated Witt vector counter part considered in [1], so does the composite \( S_V^W \) by proposition A.4. Then proposition 3.16, proposition 3.14 and proposition 3.11 tells us that \( S_V^W \) is the pullback of \( S_{V,s} \), the restriction of \( S_V \) to the special fiber. More precisely, we have

**Proposition 3.20.** \( S_V^W \) is the pullback of \( S_{V,s} \) along

\[ \text{Sh}_{V,\mu|\mu}^V \xrightarrow{\text{Sh}_{V,\mu}|\mu} \text{Sh}_{V,\mu} \xrightarrow{\text{Sh}_{V,s}} \text{Sh}_{\mu,s}. \]

Moreover, \( S_V^W \) is the analytification of the truncated Witt vector excursion operator considered in [1], under the equivalence between sheaves on truncated and untruncated Shtukas in [A.3]

4. S=T FOR SHTUKAS

Let us first look more closely at \( S_V^{\text{loc},W} \). Recall that it is supported on \( \Gamma_\text{loc}^W = \text{Hecke}_{\text{loc}}^W = L_{\mu}^W \mathcal{G} \setminus \text{Gr}_{\mu}^W \mathcal{G}_{\leq V} \), where \( L_{\mu}^W \mathcal{G} \) is the \( v \)-sheaf whose value at a characteristic \( p \) affinoid perfectoid space \( \text{Spa}(R,R^+) \) is \( \mathcal{G}(W(R)) \), and \( \text{Gr}_{\mu}^W \mathcal{G}_{\leq V} \) is the analytification of Witt vector Schubert variety.
Similar to Hecke operators on $\text{Sht}_{\mu, \eta}$, we also have them on $\text{Sht}_{0,0}^{\text{loc}, W}$. There is the commutative diagram

\[
\begin{array}{ccc}
\text{Sht}_{0,0}^{\text{loc}, W} & \xleftarrow{p_1} & \text{Sht}_{0,0}^{\text{loc}, W} \\
\downarrow{\epsilon_0^{(1)}} & & \downarrow{\epsilon_0^{(1)}} \\
\text{Hecke}_{\text{loc}}^{\nu} & \xrightarrow{\gamma} & \text{Hecke}_{\text{loc}}^{\nu}
\end{array}
\]

We observe that $p_1$ and $p_2$ are étale. This is because given a morphism $\text{Spa}(R, R^+) \to \text{Sht}_{0,0}^{\text{loc}, W}$ from an affinoid characteristic $p$ perfectoid space into $\text{Sht}_{0,0}^{\text{loc}, W}$, which is determined by the data $\varphi_2 : \mathcal{P}_2 \cong \text{Frob}_S^* \mathcal{P}_2$,

the fiber product with $p_2$ is the (analytification of) refined Witt vector affine Deligne-Lusztig variety $X_{0,\nu}^W(1)$ (base change to $R$), which parametrizes

\[
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{\varphi_1} & \text{Frob}_S^* \mathcal{P}_1 \\
\downarrow{\gamma} & & \downarrow{\text{Frob}_S^* \gamma} \\
\mathcal{P}_2 & \xrightarrow{\varphi_2} & \text{Frob}_S^* \mathcal{P}_2
\end{array}
\]

with $\gamma$ bounded by $\nu$ (the second line is fixed). This can be also characterized as the fiber product

\[
\begin{array}{ccc}
X_{0,\nu}^W(1) & \xrightarrow{a} & G_\nu^W \\
\downarrow{b} & & \downarrow{\Delta} \\
G_\nu^W \times G_\nu^W & \xrightarrow{\text{id} \times \varphi_2^{-1} \text{Frob}} & G_\nu^W \times G_\nu^W
\end{array}
\]

where $a$ sends the above diagram to $\gamma \varphi_1^{-1}$, $b$ maps it to $\gamma$ and $\varphi_2^{-1} \text{Frob}$ is the map sending $\gamma : \mathcal{P}_1 \dashrightarrow \mathcal{P}_2$ to $\varphi_2^{-1} \text{Frob}_S^* \gamma$ (fixing a trivialization of $\mathcal{P}_2$), as $\varphi_2$ is an isomorphism, we see that $X_{0,\nu}^W(1) \cong G_\nu^W(\mathbb{F}_p)$ as zero dimensional varieties, which is obviously étale. Thus $p_2$ is étale, and similarly for $p_1$. Moreover, we see that

\[
\text{Sht}_{0,0}^{\text{loc}, W} = L^{+, W} \mathcal{G} \setminus X_{0,\nu}^W(1) \cong L^{+, W} \mathcal{G} \setminus G_\nu^W(\mathbb{F}_p)
\]

which can be identified with the substack

$\Gamma_{\nu, \text{loc}, W} : = \text{Hecke}_{\nu}(\mathbb{F}_p) \subset G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$

defined by the obvious bounded by $\nu$ condition. Moreover, we have $\text{Sht}_{0,0}^{\text{loc}, W} = L^{+, W} \mathcal{G} \setminus \bullet$ and $S_{\nu,0,1}^{(1)} = \mathbb{Q}_l$, so any function $f \in C(\Gamma_{\nu, \text{loc}, W})$ defines a cohomological correspondence $\nu_{\text{loc}, W}$ from $(\text{Sht}_{0,0}^{\text{loc}, W}, \epsilon_0^{(1)}, s_{\nu,0,1}^{(1)})$ to itself supported on $\Gamma_{\nu, \text{loc}, W}$ which is multiplication by $f(x)$ at $x \in \Gamma_{\nu, \text{loc}, W}$. See [1] 5.4.4 for more details.
We can do the same construction with truncated Witt vector Shtukas as in [1] to obtain $\mathcal{C}_f^{\text{loc},(m,n)}$, and we have
$$C_f^{\text{loc},(m,n),\infty} = C_f^{\text{loc},W}$$
under the identification between sheaves on truncated and untruncated Shtukas.

We have $\Gamma_{V,\text{loc}}^{\text{loc},W} = \bigcup_{\nu} \Gamma_{V,\text{loc}}^{\text{loc},W}$ where $\nu$ ranges over highest weights of irreducible summands of $V$, and we can similarly define $C_f^{\text{loc},W}$ for $f \in C(\Gamma_{V,\text{loc}}^{\text{loc},W}, \mathcal{Q}_l)$ and
$$T_{V,\text{loc}}^{\text{loc},W} := C_{h_V}^{\text{loc},W}$$
as cohomological correspondence from $(\text{Sht}^{\text{loc},W}_0, \epsilon_0^{\text{loc},\{1\}}, s_1^{\text{loc},\{1\}})$ to itself supported on $\Gamma_{V,\text{loc}}^{\text{loc},W}$, where $h_V$ is the function corresponding to $V$ under classical Satake isomorphism. Similarly, we can define $T_V^{\text{loc}(m,n)} = C_{h_V}^{\text{loc}(m,n)}$, and we have
$$T_{V,\text{loc}}^{\text{loc},W} = T_V^{\text{loc}(m,n),\infty}$$
under the canonical identification in A.3 between sheaves on truncated and untruncated Witt vector Shtukas.

**Theorem 4.1.** (Xiao-Zhu) We have an equality
$$T_{V,\text{loc}}^{\text{loc},W} = S_{V,\text{η}}^{\text{loc},0,W}$$
as cohomological correspondence from $(\text{Sht}^{\text{loc},W}_0, \epsilon_0^{\text{loc},\{1\}}, s_1^{\text{loc},\{1\}})$ to itself supported on $\Gamma_{V,\text{loc}}^{\text{loc},W}$.

**Proof.** This is the analytification of [1] theorem 6.0.1 (2). \qed

**Theorem 4.2.** There is a canonical identification
$$S_{V,\eta} = T_V$$
as cohomological correspondence from $(\text{Sht}_{\mu,\eta}, \epsilon_\mu^{\{1\}}, s_\mu^{\{1\}})$ to itself supported on $\Gamma_V$.

**Remark 4.3.** Again, there is no difficulty generalizing the theorem to Shtukas with several legs (in characteristic zero).

**Proof.** Theorem 3.18 and 4.1 together imply that $S_{V,\eta}$ is the pullback of $T_V^{\text{loc},W} \boxtimes \text{id}$ along

$$\begin{array}{cccc}
\text{Sht}_{\mu,\eta} & \xleftarrow{p_1} & \Gamma_V & \xrightarrow{p_2} & \text{Sht}_{\mu,\eta} \\
\downarrow & & & & \downarrow \\
\text{Sht}_0^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},\{1\}} & \xleftarrow{p_1^{\text{loc} \times \text{id}}} & \Gamma_V^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},\{1\}} & \xrightarrow{p_2^{\text{loc} \times \text{id}}} & \text{Sht}_0^{\text{loc},W} \times \text{Hecke}_{\mu,\eta}^{\text{loc},\{1\}}
\end{array}$$
which is the same as \( T_V \). Indeed, we have a commutative diagram with the two squares Cartesian

\[
\begin{array}{ccc}
Sht_{\mu,\eta} & \xrightarrow{p_1} & Sht'_{\mu,\eta} \\
\downarrow & & \downarrow \\
Sht^\text{loc,}(1) \times Hecke^\text{loc,}\{(1)\} & \xrightarrow{\pi} & Sht^\text{loc,}\{(1)\} \\
\downarrow & & \downarrow \\
Sht^\text{loc,}(1) & \xrightarrow{p_2} & Sht^\text{loc,}(1) \\
\end{array}
\]

and under the canonical identification \( Hecke^\text{loc,}(1) \times Hecke^\text{loc,}\{(1)\} = Hecke^\text{loc,}\{(1)\} \), the composition is precisely \( \mathcal{C}^\text{loc,} \). Let \( \mathcal{C}^\text{loc,} = \mathcal{C}^\text{loc,} \) with \( \Gamma^\text{loc,} \) the characteristic function of \( \Gamma^\text{loc,} \). Then the pullback of \( \mathcal{C}^\text{loc,} \otimes \text{id} \) is

\[
p_1^* e^\mu_{\{1\}} \circ S^\text{loc,} U = \pi^* (p_1^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

\[
\cong \pi^* (p_2^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

\[
\cong \pi^* (p_2^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

Using that \( p_2 \) and \( p_2^* \times \text{id} \) are étale, we see that

\[
\pi^* (p_2^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}} = \pi^* (p_2^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

is nothing but

\[
\pi^* (p_2^* \times \text{id})^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}} \cong p_2^* \pi^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

\[
\cong p_2^* \pi^* (e^\text{loc,}_{\{1\}} \times \text{id})^* S^\text{loc,}_{\{1\}} \otimes S^\text{loc,}_{\{1\}}
\]

thus the pullback of \( \mathcal{C}^\text{loc,} \otimes \text{id} \) is simply \( \mathcal{C} \), hence the pullback of \( T^\text{loc,} \otimes \text{id} \) is \( T_V \) by adding them together.
5. $S=T$ for Shimura varieties

In this section, we assume that $G/Q$ is a reductive group equipped with a $G_\mathbb{R}$-conjugacy class $X$ of homomorphisms $h : \text{Res}_{\mathbb{C}/\mathbb{R}}\, G_m \to G_\mathbb{R}$ that gives rise to a Hodge type Shimura variety. Namely, there is a closed embedding $G \hookrightarrow \text{GSp}(Q, \psi)$ such that $X$ is mapped to the canonical Siegel conjugacy classes, where $Q$ is a finite dimensional $\mathbb{Q}$-vector space, $\psi$ is a non-degenerate alternating form on $Q$, and $\text{GSp}(Q, \psi)$ is the similitude sympletic group with respect to $(Q, \psi)$, i.e. the automorphism of $Q$ preserving $\psi$ up to a constant. We fix such an embedding throughout, then by [17], there is a set of tensors $(s_\alpha)$ in

$$Q^\otimes := \bigoplus_{s,t \in \mathbb{N}} Q^{\otimes s} \otimes (Q^\vee)^{\otimes t}$$

such that $G$ is the stabilizer of $(s_\alpha)$, which we also fix from now on.

We can base change $h$ to $\mathbb{C}$ to obtain $h_\mathbb{C} : G_m \times G_m \to G_\mathbb{C}$, where the two factors $G_m$ correspond to the canonical embedding of $\mathbb{C}$ into $\mathbb{C}$ and the conjugation embedding respectively. Let $\mu : G_m \to G_\mathbb{C}$ be the restriction of $h_\mathbb{C}$ to the first factor, whose conjugacy class has field of definition $E$, which is a number field.

Moreover, we fix a prime $p$ such that $G$ is hyperspecial at $p$, i.e. there is a reductive group $\mathcal{G}$ over $\mathbb{Z}_{(p)}$ with generic fiber $G$. Let $K = K^p K_p \subset G(\mathbb{A}_f)$ be a compact open subgroup where $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$ is compact open. We assume that $K^p$ is sufficiently small for convenience. By enlarging $Q$, we may assume that $Q$ has a lattice $Q_{\mathbb{Z}(p)}$ over $\mathbb{Z}_p$ on which $\psi$ induces a perfect pairing $Q_{\mathbb{Z}(p)} \times Q_{\mathbb{Z}(p)} \to \mathbb{Z}(p)$. Moreover, we can assume that the embedding $G \hookrightarrow \text{GSp}(Q, \psi)$ induces an embedding

$$\mathcal{G} \hookrightarrow \text{GSp}(Q_{\mathbb{Z}(p)}, \psi)$$

as reductive groups over $\mathbb{Z}_p$, and the tensors $(s_\alpha)$ extends to tensors of $Q_{\mathbb{Z}(p)}$ whose stabilizer group is $\mathcal{G}$, see [18] 2.3.2 for details.

By [18] 2.1.2, there is a compact open subgroup $\tilde{K}_p \subset \text{GSp}(Q, \psi)(\mathbb{A}_f^p)$ such that $K^p = \tilde{K}_p \cap G(\mathbb{A}_f^p)$, and the injection of $G$ into $\text{GSp}(Q, \psi)$ induces a closed embedding

$$S_K \hookrightarrow S_{\tilde{K}_p \tilde{K}_p} \times_Q E$$

of Shimura varieties, where $\tilde{K}_p := \text{GSp}(Q_{\mathbb{Z}(p)}, \psi)(\mathbb{Z}_p)$, $S_K$ is the canonical model over $E$ of the Shimura variety associated to $(G, X)$ with level $K$, and $S_{\tilde{K}_p \tilde{K}_p}$ is the Siegel variety associated to $\text{GSp}(Q, \psi)$ with level $\tilde{K}_p \tilde{K}_p$ over $\mathbb{Q}$.

By [18], and also the earlier work of Vaisu and Moonen, $S_K$ has a canonical smooth integral model $\mathfrak{S}_K$ over $\mathcal{O}_L$. It is defined as the normalization of the closure of $S_K$ in $\mathfrak{A}_{\tilde{K}_p \tilde{K}_p}$, where $\mathfrak{A}_{\tilde{K}_p \tilde{K}_p}$ is the canonical integral model of $S_{\tilde{K}_p \tilde{K}_p}$ as moduli space of polarized abelian varieties (over $\mathcal{O}_L$). By construction, there is a canonical abelian scheme $A$ over $\mathfrak{S}_K$, being the pullback of the universal one on $\mathfrak{A}_{\tilde{K}_p \tilde{K}_p}$. We denote $\mathfrak{X}_K$ the $p$-adic completion of $\mathfrak{S}_K$, and $\mathfrak{X}_K$ the adic generic fiber of $\mathfrak{X}_K$, i.e. the good reduction locus in $(S_K \times E L)^{\text{ad}}$. Moreover, we denote $\mathfrak{X}_K := \mathfrak{S}_{K, \overline{\mathbb{Q}}}$ the special fiber of $\mathfrak{S}_K$. 
Remark 5.1. The integral models $\mathfrak{S}_K$ is really defined over $\mathcal{O}_{E_v}$, where $v$ is a place of $E$ lying over $p$, and similarly for the other spaces in the previous paragraph. We can upgrade everything below to this situation.

By construction, the representation $Q$ defines a Betti local system

$$\mathcal{V}_B := Q \times^{G(\mathbb{Q})} (X \times G(h_f)/K)$$

on

$$S_K(\mathbb{C}) \cong G(\mathbb{Q}) \setminus (X \times G(h_f)/K)$$

which can be identified with the first homology of $A$ over $S_K(\mathbb{C})$, i.e. $\mathcal{V}_B = (R^1h_{B,*}Q)^\vee$ where $h : A \to S_K$ is the structure map of $A$, and $h_{B,*}$ means that we take pushforward in the classical topology.

We can similarly define

$$\mathcal{V}_B^\otimes := \bigoplus_{s,t \in \mathbb{N}} \mathcal{V}_B^\otimes_s \otimes (\mathcal{V}_B^\vee)^\otimes_t$$

with duals and tensor products in the category of Betti $\mathbb{Q}$-local systems over $S_K$. Similar notation will apply to other categories of local systems, namely étale, de Rham (filtered vector bundles with flat connections) and crystalline ($F$-crystals). Now the tensors $s_\alpha$ define global sections

$$s_{\alpha,B} : 1 \to \mathcal{V}_B^\otimes$$

of $\mathcal{V}_B^\otimes$ since they are $G(\mathbb{Q})$-invariant, where $1$ denotes the unit object in the rigid Tannakian category, i.e. the 1 dimensional constant local system.

By [18] 2.2.1, $s_{\alpha,B}$ determines global sections

$$s_{\alpha,p} : 1 \to \mathcal{V}_p^\otimes$$

with $\mathcal{V}_p := (R^1h_{\acute{e}t,*}Q_p)^\vee$, characterized by $s_{\alpha,p}$ and $s_{\alpha,B}$ matching under the comparison isomorphism between Betti and étale cohomology, where $h_{\acute{e}t,*}$ denotes pushforward in the étale topology of $h : A \to S_K$ (over $E$). Note that the real content of the statement is that the sections defined by comparison isomorphism descend to $E$.

We also have the de Rham tensors

$$s_{\alpha,\text{dR}} : 1 \to \mathcal{V}_{\text{dR}}^\otimes$$

where $\mathcal{V}_{\text{dR}} := (R^1h_*\Omega_{A/\mathbb{S}_K})^\vee$ is the relative de Rham homology, which is a filtered vector bundle with flat connection. Over $\mathbb{C}$, this is defined by the comparison between the de Rham and Betti cohomology, which descends to $E$ by Deligne’s theory of absolute Hodge and the equivalence of analytic vector bundles with a flat connection and algebraic vector bundles with a flat connection with regular singularities, see [18] 2.2.2 for details. Moreover, it is proved in [18] corollary 2.2.2 that $s_{\alpha,\text{dR}}$ extends to the integral model.

We can restrict (the analytification of) $\mathcal{V}_p$ to the good reduction locus $X_K$, then it is a de Rham local system in the sense of [19] with associated filtered vector bundle with flat connection $\mathcal{V}_{\text{dR}}$ (restricted on $X_K$), i.e. there is an isomorphism

$$\mathcal{V}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{dR}} \cong \mathcal{V}_{\text{dR}} \otimes_{\mathcal{O}_{X_K}} \mathcal{O}_{\text{dR}},$$ (34)
an isomorphism \( S = Spd \) proved in [18] section 6 for the definition of other period sheaves used in the following. We also have similar comparison isomorphism between \( \mathcal{V}_p^{\otimes} \) and \( \mathcal{V}_{dR}^{\otimes} \).

By the result of Blasius and Wintenberger ([20]), the isomorphism takes \( s_{a,p} \otimes 1 \) to \( s_{a,dR} \otimes 1 \) at points defined over number fields, so they are matched globally by the flatness of these sections.

We have two \( \mathbb{B}_{dR}^+ \)-local systems
\[
\mathcal{M} := \mathcal{V}_p \otimes X_{p} \mathbb{B}_{dR}^+
\]
and
\[
\mathcal{M}_0 := (\mathcal{V}_{dR} \otimes \mathcal{O}_{X_{K}} \mathcal{O}_{\mathbb{B}_{dR}^+})^{\nabla=0}
\]
such that
\[
\mathcal{M} \otimes \mathbb{B}_{dR}^+ \mathbb{B}_{dR} \cong \mathcal{M}_0 \otimes \mathbb{B}_{dR}^+ \mathbb{B}_{dR}
\]
which is taking \( \nabla = 0 \) on both sides of (34) and \( \mathcal{M} \subset \mathcal{M}_0 \), see [19] proposition 7.9. Note that we have used homology instead of cohomology, which corresponds to a reverse of the inclusion. Since the identification (34) matches \( s_{a,p} \otimes 1 \) with \( s_{a,dR} \otimes 1 \), we see that the same holds for the identification (35).

Moreover, we can take 0th graded piece of (34) to obtain the Hodge Tate sequence
\[
0 \longrightarrow (\text{Lie } A) \otimes \mathcal{O}_{X_{K}} \mathcal{O}_{X_{K}}(1) \longrightarrow \mathcal{V}_p \otimes \mathcal{O}_{X_{K}} \mathcal{O}_{X_{K}} \longrightarrow (\text{Lie } A)^{\vee} \otimes \mathcal{O}_{X_{K}} \mathcal{O}_{X_{K}} \longrightarrow 0
\]
see [5] corollary 2.2.4.

Let \( D(A) \) be the covariant \( F \)-crystal associated to \( A \) on the special fiber \( \bar{X}_K \). As explained in [1] section 7.1.5, the de Rham tensors give rise to
\[
s_{a,0} : 1 \longrightarrow D(A)^{\otimes}.
\]
Indeed, as \( \mathcal{S}_K \) is smooth over \( \mathcal{O}_L \) (and \( \mathcal{O}_L \) is unramified), the crystal \( D(A) \) is equivalent to the vector bundle with flat connections \( \mathcal{V}_{dR} \), so \( s_{a,dR} \) gives rise to \( s_{a,0} \) as arrows of crystals. Then it is proved in [18] that pointwise \( s_{a,0} \) is an arrow of \( F \)-crystals, which gives what we want.

We now want to construct a map
\[
\text{loc}_p : \mathcal{S}_K^{\otimes} \longrightarrow \text{Sh}_\mu
\]
over \( Spd(\mathcal{O}_L) \). Recall that \( \mathcal{S}_K^{\otimes} \) is the sheafification of the presheaf sending an affinoid perfectoid \( S = Spa(R, R^+) \) over \( \mathbb{F}_p \) to the set of untilts \( S^\diamond = Spa(R^\diamond, R^\diamond) \) over \( Spa(\mathcal{O}_L, \mathcal{O}_L) \) together with a map \( Spec(R^\diamond) \rightarrow \mathcal{S}_K \) of schemes over \( \mathcal{O}_L \).

Given \( f : Spec(R^\diamond) \rightarrow \mathcal{S}_K \), we have the \( p \)-divisible group \( f^* A[p^\infty] \) over \( Spec(R^\diamond) \), which by [7] theorem 17.5.2 is equivalent to a finite projective \( A_{inf}(R^\diamond) \)-module \( \mathcal{M} \) over \( R^\diamond \) together with an isomorphism
\[
\varphi_{\mathcal{M}} : \varphi^* \mathcal{M}[\frac{1}{\varphi(\xi)}] \cong \mathcal{M}[\frac{1}{\varphi(\xi)}]
\]
such that \( \mathcal{M} \subset \varphi_{\mathcal{M}}(\varphi^* \mathcal{M}) \subset \frac{1}{\varphi(\xi)} \mathcal{M} \), where \( \xi \in A_{inf}(R^\diamond) \) is a generator of the kernel of \( A_{inf}(R^\diamond) \rightarrow R^\diamond, \) and \( \varphi \) is the canonical lift of Frobenius on \( A_{inf}(R^\diamond) \). It gives rise to a minuscule \( GL_n \)-Shukha with one leg over \( Spa(R^\diamond, R^\diamond) \) by restricting on \( \mathcal{V}_{[0, \infty]}(\bar{R}, R^\diamond) \) (and twist by Frobenius).
Let us first recall how \( \mathcal{M} \) is constructed in [7] theorem 17.5.2. Let \( G \) be a \( p \)-divisible group over an integral perfectoid ring \( S \). If \( p = 0 \) in \( S \), then \( \mathcal{M}(G) \) is given by Dieudonné theory. If \( p \neq 0 \) in \( S \), then Dieudonné theory applied to the restriction of \( G \) to \( S/p \) gives us an \( A_{crys}(S) \)-module \( \mathcal{M}_{crys}(G) \) together with an isomorphism

\[
\varphi^* \mathcal{M}_{crys}(G)[\frac{1}{p}] \cong \mathcal{M}_{crys}(G)[\frac{1}{p}],
\]

where \( A_{crys}(S) \) is the universal \( p \)-adically complete divided power thickening of \( S/p \). Now \( \mathcal{M}(G) \subset \mathcal{M}_{crys}(G) \) is characterized by being the largest submodule of \( \mathcal{M}_{crys}(G) \) such that for every map \( S \to V \) with \( V \) an integral perfectoid valuation ring with algebraically closed fraction field, its image in \( \mathcal{M}_{crys}(G_V) \) lies in \( \mathcal{M}(G_V) \), where \( \mathcal{M}(G_V) \) is the Dieudonné module if \( V \) is of characteristic \( p \).

If \( p \neq 0 \) in \( V \) and \( V = \mathcal{O}_C \) with \( C \) the fraction field of \( V \), \( \mathcal{M}(G_V) \) is constructed as in [7] theorem 14.4.1. For general \( V \) not of characteristic \( p \), let \( k \) be the residue field of \( \mathcal{O}_C \) with \( C \) the fraction field of \( V \), and \( V \subset k \) the image of \( V \) in \( k \), then \( \mathcal{M}(G_V) \) is defined as \( \mathcal{M}(G_{\mathcal{O}_C}) \times_{\mathcal{M}(G_k)} \mathcal{M}(G_V) \), in other words, \( \mathcal{M}(G_V) \) is the largest submodule of \( \mathcal{M}(G_{\mathcal{O}_C}) \) whose image in \( \mathcal{M}(G_k) \) lies in \( \mathcal{M}(G_V) \).

In summary, the construction of \( \mathcal{M}(G) \) is to use \( v \)-descent to reduce to Dieudonné theory and the classification over \( \mathcal{O}_C \). The key input is then the classification in terms of Hodge-Tate periods as in [21], and Fargues’ theorem ([7] theorem 14.4.1) that establishes the equivalence between de Rham pairs and Breuil-Kisin-Fargues modules.

We now prove that \( s_\alpha \) give rise to tensors on \( \mathcal{M}^\otimes \).

**Proposition 5.2.** Let \( R^{\frac{p^\infty}{p}} \) be an integral perfectoid ring over \( \mathcal{O}_L \), \( f : \text{Spec}(R^{\frac{p^\infty}{p}}) \to \mathcal{S}_K \) a map over \( \mathcal{O}_L \), and \( G := f^*A[p^\infty] \) the induced \( p \)-divisible group over \( R^{\frac{p^\infty}{p}} \). Let \( \mathcal{M} \) be the \( \varphi \)-inf\( \mathcal{M}(R^{\frac{p^\infty}{p}}) \)-module corresponding to \( G \), then there exists

\[
s_{\alpha,\Delta} : 1 \to \mathcal{M}^\otimes
\]

such that under the identification (see [7] theorem 17.5.2)

\[
\mathcal{M} \otimes_{\text{inf}(R^{\frac{p^\infty}{p}})} A_{crys}(R^{\frac{p^\infty}{p}}) \cong \mathcal{M}_{crys}
\]

\( s_{\alpha,\Delta} \otimes 1 \) is mapped to \( s_{\alpha,0} \) (restricted from \( \mathfrak{X}_K \) to \( R^{\frac{p^\infty}{p}}/p \)), where \( \mathcal{M}_{crys} \) is the Dieudonné crystal of \( G \) restricted to \( R^{\frac{p^\infty}{p}}/p \). Moreover, when \( R^{\frac{p^\infty}{p}} = \mathcal{O}_C \), then under the identification

\[
\mathcal{M} \otimes_{\text{inf}(\mathcal{O}_C)} A_{\text{inf}}(\mathcal{O}_C)[\frac{1}{\mu}] \cong T_p G \otimes_{\mathbb{Z}_p} A_{\text{inf}}(\mathcal{O}_C)[\frac{1}{\mu}]
\]

from (the dual of) [7] theorem 14.9.1 (2), \( s_{\alpha,\Delta} \otimes 1 \) is sent to \( s_{\alpha,p} \).

**Proof.** We have

\[
s_{\alpha,0} : 1 \to \mathcal{M}^\otimes_{crys}
\]

by pulling back

\[
s_{\alpha,0} : 1 \to D(A)^\otimes
\]

along \( \text{Spec}(R^{\frac{p^\infty}{p}}) \to \mathfrak{X}_K \), the restriction of \( f \) to characteristic \( p \) fibers, and we are aiming to show that \( s_{\alpha,0} \) factorizes through \( \mathcal{M}^\otimes \). By construction of \( \mathcal{M} \) that we have recalled above, we need
to show that for every map $R^{d,+} \to V$ with $V$ an integral perfectoid valuation ring with algebraically closed fraction field,

$$s_{\alpha,0}|_V : 1 \to \mathcal{M}_{\text{crys}}(G_V)^\otimes$$

factorizes through $\mathcal{M}(G_V)^\otimes$. If $V$ has characteristic $p$, then $\mathcal{M}(G_V) = \mathcal{M}_{\text{crys}}(G_V)$ and there is nothing to prove.

If $V$ has mixed characteristic and $V = \mathcal{O}_C$ with $C$ the fraction field of $V$, then $\mathcal{M}(G_V)$ is the Breuil-Kisin-Fargues module corresponding to the Hodge-Tate periods of $G_V$. Let us recall its construction first. We have the Hodge-Tate exact sequence

$$G^\text{Breuil-Kisin-Fargues} \text{ module corresponding to the Hodge-Tate periods of } G_V$$

which gives rise to a $\mathbb{B}_{dR}^+(\mathbb{C})$-submodule $\Xi \subset T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}^+(\mathbb{C})$ characterized by $\Xi \mod t \subset T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{C}$ being $\text{Lie } G_V \otimes_{\mathcal{O}_C} \mathbb{C}(1) \subset T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{C}$, where $t \in \mathbb{B}_{dR}^+(\mathbb{C})$ is a generator of the kernel $\mathbb{B}_{dR}^+(\mathbb{C}) \to \mathbb{C}$. We have now a de Rham pair $(\Xi, T_p G_V)$, which by [7] theorem 14.4.1 is equivalent to a minuscule Breuil-Kisin-Fargues module $\mathcal{M}(G_V)$. Moreover, we have by proposition 14.8.1 of [7] that

$$\Xi \subset T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C})$$

is the same as

$$\mathcal{M}_{\text{crys}}(G_V) \otimes_{\mathcal{A}_{\text{crys}(V)}} \mathbb{B}_{dR}^+(\mathbb{C}) \subset T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C})$$

under a canonical identification

$$\mathcal{M}_{\text{crys}}(G_V) \otimes_{\mathcal{A}_{\text{crys}(V)}} \mathbb{B}_{dR}(\mathbb{C}) \cong T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C}),$$

where the identification actually holds over a smaller period ring $\mathcal{B}_{\text{crys}}$.

Now as $G_V$ is the pullback of $A[p^\infty]$ along a map $\text{Spec}(V) \to S_K$, we have

$$s_{\alpha,p} : 1 \to (T_p G_V)^\otimes$$

which by the following lemma matches with

$$s_{\alpha,0} : 1 \to \mathcal{M}_{\text{crys}}(G_V)^\otimes$$

under the identification

$$\mathcal{M}_{\text{crys}}(G_V) \otimes_{\mathcal{A}_{\text{crys}(V)}} \mathbb{B}_{dR}(\mathbb{C}) \cong T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C}).$$

It follows that $s_{\alpha,0}$ and $s_{\alpha,p}$ glue to an arrow of de Rham pairs

$$s_{\alpha,\text{dR}} : 1 \to (\Xi, T_p G_V)^\otimes,$$

which by [7] theorem 14.4.1 is equivalent to

$$s_{\alpha,\Delta} : 1 \to \mathcal{M}(G_V)^\otimes.$$
for \( M(G_{O_C}) \), \( M(G_V) \) and \( M(G_k) \) individually, it remains to see that they glue to \( M(G_V) \). This is clear as they are pullback from the commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(O_C) & \xrightarrow{f} & \mathfrak{S}_K \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{g} & \mathfrak{S}_K \\
\downarrow & & \downarrow \\
\text{Spec}(\bar{V}) & \xrightarrow{h} & \mathfrak{S}_K \\
\end{array}
\]

and we have the identification

\[
M(G_{O_C}) \otimes_{A_{\inf}(O_C)} W(k) \cong (M(G_{O_C}) \otimes_{A_{\inf}(O_C)} A_{\text{crys}}(O_C)) \otimes_{A_{\text{crys}}} W(k) \\
\cong M_{\text{crys}}(G_{O_C/p}) \otimes_{A_{\text{crys}}} W(k) \cong M_{\text{crys}}(G_k).
\]

\[\square\]

**Lemma 5.3.** Let \( f : \text{Spec}(O_C) \rightarrow \mathfrak{S}_K \) be a map over \( O_L \), and \( G_{O_C} \) be the pullback of \( A[p^\infty] \) along \( f \), then (pullback of) \( s_{\alpha,0} \otimes 1 \) is mapped to \( s_{\alpha,p} \otimes 1 \) under the canonical isomorphism

\[
M_{\text{crys}}(G_V) \otimes_{A_{\text{crys}}(V)} \mathbb{B}_{dR}(\mathbb{C}) \cong T_p G_V \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C}).
\]

**Proof.** Using the canonical duality between crystalline cohomology (resp. étale cohomology) of abelian varieties and Dieudonné modules (resp. Tate modules) of the associated \( p \)-divisible group (see [7] proposition 14.9.3), it is enough to prove the corresponding statements for abelian varieties. Since \( O_C \) is local, \( f \) factorizes through an affine open \( \text{Spec}(T) \) of \( \mathfrak{S}_K \), which is smooth over \( O_L \). By smoothness, we can lift \( f \) to \( g : \text{Spec}(A_{\text{crys}}(O_C)) \rightarrow \text{Spec}(T) \), then

\[
H^1_{\text{crys}}(A_{O_C/p}/A_{\text{crys}}(O_C)) \cong H^1_{\text{dR}}(g^* A) \cong g^*(V_{\text{dR}})^\vee
\]

where \( A_{O_C/p} \) is the restriction of \( f^* A \) to \( \text{Spec}(O_C/p) \).

Moreover, let

\[
h : \text{Spec}(\mathbb{B}_{dR}(\mathbb{C})) \rightarrow \text{Spec}(T)
\]

be the composition of \( g \) with the canonical map

\[
r : \text{Spec}(\mathbb{B}_{dR}^+(\mathbb{C})) \rightarrow \text{Spec}(A_{\text{crys}}(O_C)),
\]

which is a lift of (restriction of) \( f \). It follows from the proof of [22] theorem 13.8 that

\[
H^1_{\text{crys}}(A_C/\mathbb{B}_{dR}^+(\mathbb{C})) \cong H^1_{\text{dR}}(h^* A) \cong h^*(V_{\text{dR}})^\vee
\]

where \( A_C := (f^* A)_C \), and \( H^1_{\text{crys}}(A_C/\mathbb{B}_{dR}^+(\mathbb{C})) \) is defined as in [22] theorem 13.1.

Now under the identification [37] and [36], the canonical isomorphism

\[
H^1_{\text{crys}}(A_C/\mathbb{B}_{dR}^+(\mathbb{C})) \cong H^1_{\text{crys}}(A_{O_C/p}/A_{\text{crys}}(O_C)) \otimes_{A_{\text{crys}}(O_C)} \mathbb{B}_{dR}^+(\mathbb{C})
\]

from [22] lemma 13.11 is nothing but

\[
r^* g^*(V_{\text{dR}})^\vee \cong h^*(V_{\text{dR}})^\vee.
\]
We have
\[ H^1_{\text{crys}}(A_C / \mathbb{B}^+_{dR}(\mathbb{C})) \cong f^* M_0', \]
under which \( f^*(s_{a,dR})^\vee \) is mapped to \( h^*(s_{a,dR})^\vee \) using the identification \( \text{(37)} \). We now abuse the notation by writing \( (s_{a,dR})^\vee \) the corresponding tensors on \( H^1_{\text{crys}}(A_C / \mathbb{B}^+_{dR}(\mathbb{C})) \). Since \( (s_{a,0})^\vee \) is \( g^*(s_{a,dR})^\vee \), it follows that the isomorphism \( \text{(38)} \) takes \( (s_{a,dR})^\vee \) to \( (s_{a,0})^\vee \otimes 1 \).

On the other hand, we have the canonical isomorphism
\[ H^1_{et}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C}) \cong H^1_{\text{crys}}(A_C / \mathbb{B}^+_{dR}(\mathbb{C})) \otimes_{\mathbb{B}^+_{dR}(\mathbb{C})} \mathbb{B}_{dR}(\mathbb{C}) \] (39)
from \( \text{[22]} \) theorem 13.1, which is nothing but the pullback of isomorphism \( \text{(35)} \) along \( f \). Hence \( (s_{a,p})^\vee \otimes 1 \) is mapped to \( (s_{a,dR})^\vee \otimes 1 \) through \( \text{(39)} \). Now the composition of isomorphisms \( \text{(39)} \) and \( \text{(38)} \) is
\[ H^1_{et}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR}(\mathbb{C}) \cong H^1_{\text{crys}}(A_C / \mathbb{B}^+_{dR}(\mathbb{C})) \otimes_{\mathbb{B}^+_{dR}(\mathbb{C})} \mathbb{B}_{dR}(\mathbb{C}) \]
which is exactly the crystalline comparison isomorphism, see \( \text{[7]} \) proposition 14.7.2. It matches \( (s_{a,p})^\vee \otimes 1 \) with \( (s_{a,0})^\vee \otimes 1 \) as the two intermediate isomorphisms do. \( \square \)

We now have a \( G \times_{\mathbb{Z}_p} A_{\text{inf}}(R^\vee) \)-quasisortor \( \mathcal{E} \) over \( \text{Spec}(A_{\text{inf}}(R^\vee)) \) that parametrizes isomorphisms between \( Q_{Z(p)} \otimes_{\mathbb{Z}_p} A_{\text{inf}}(R^\vee) \) and \( \mathcal{M} \) that preserves the Hodge tensors, i.e. for any \( A_{\text{inf}}(R^\vee) \)-algebra \( T \),
\[ \mathcal{E}(T) = \{ \beta : Q_{Z(p)} \otimes_{\mathbb{Z}_p} T \cong \mathcal{M} \otimes_{A_{\text{inf}}(R^\vee)} T \mid \beta(s_{a}) = s_{a,\text{tri}} \otimes 1 \}. \]

**Proposition 5.4.** \( \mathcal{E} \) is a \( G \times_{\mathbb{Z}_p} A_{\text{inf}}(R^\vee) \)-torsor over \( \text{Spec}(A_{\text{inf}}(R^\vee)) \).

**Proof.** We write \( \mathcal{E}_S \) the base change of \( \mathcal{E} \) to \( \text{Spec}(S) \) for any \( A_{\text{inf}}(R^\vee) \)-algebra \( S \).

By \( \text{[22]} \) theorem 4.1, vector bundles over \( W(S) \) is a stack with respect to the \( v \)-topology on perfect \( R \)-algebras \( S \), so we can recover \( \mathcal{E} \) from its pullback to \( W(S) \) (and its descent) for a \( v \)-cover \( S \) of \( R \). On the other hand, it follows quickly from Tannakian formalism and \( \text{[23]} \) theorem 4.1 again that \( G \times_{\mathbb{Z}_p} A_{\text{inf}}(R^\vee) \)-torsors over \( W(S) \) is also a stack with respect to the \( v \)-topology on perfect \( R \)-algebras \( S \). Thus it is enough to prove \( \mathcal{E}_{W(S)} \) is a torsor for a \( v \)-cover \( S \) of \( R \), which we can choose to be a (possibly infinite) product of perfect valuation rings with algebraically closed fraction fields, and we quickly reduce to \( S = V \) a perfect valuation ring with algebraically closed fraction field.

Let \( \xi \in A_{\text{inf}}(R^\vee) \) be a generator of the kernel of \( A_{\text{inf}}(R^\vee) \rightarrow R^\vee \), then \( \xi \in W(V) \) using the map \( R \rightarrow V \). Taking the completion if necessary, we can assume that \( V \) is \( \xi \) mod \( p \) adically complete, so \( V^\sharp = W(V) / \xi \) is integral perfectoid, which is also a valuation ring with algebraically closed fraction field. We have \( R^\vee \rightarrow V^\sharp \), and let \( G_{V^\sharp} \) be the pullback of \( A[p^\infty] \) along
\[ \text{Spec}(V^\sharp) \rightarrow \text{Spec}(R^\vee) \rightarrow S_K, \]
then
\[ \mathcal{E}_{W(V)} = \text{Isom}_{s_{a}}(Q_{Z(p)} \otimes_{\mathbb{Z}_p} W(V), \mathcal{M}(G_{V^\sharp})). \]

If the characteristic of \( V^\sharp \) is \( p \), then \( \mathcal{E}_{W(V)} \) is the pullback of (the homology version and not mesmerizing trivilization of Tate twist) \( \mathcal{E}_{\text{crys}} \) as defined in \( \text{[1]} \) corollary 7.1.14, which is proved in \textit{loc.cit.} to be a torsor.
If $V^\sharp$ is of mixed characteristic, and $V^\sharp = \mathcal{O}_C$ with $\mathbb{C}$ the fraction field of $V^\sharp$, then we have
\[ \mathcal{M}(G_{V^\sharp}) \otimes_{A_{\inf}(V^\sharp)} A_{\inf}(V^\sharp)[\frac{1}{\mu}] \cong T_p G_{V^\sharp} \otimes_{\mathbb{Z}_p} A_{\inf}(V^\sharp)[\frac{1}{\mu}] \]
that matches $s_{\alpha,\Delta} \otimes 1$ with $s_{\alpha,p} \otimes 1$, where $\mu = [\epsilon] - 1 \in A_{\inf}(V^\sharp)$ with $\epsilon = (1, \zeta_p, \zeta_p^2, \cdots) \in \mathcal{O}_C^\times$ for a chosen compatible $p$-power roots of unity $\zeta_p$. But we already know that $s_{\alpha,p}$ matches with $s_{\alpha,B}$ under Betti-étale comparison, from which we easily deduce an isomorphism
\[ T_p G_{V^\sharp} \cong \mathcal{O}_{Z(p)} \otimes \mathbb{Z}_p \]
that matches $s_{\alpha,p}$ with $s_{\alpha} \otimes 1$. Thus $\mathcal{E}_{A_{\inf}(V^\sharp)[\frac{1}{p}]}$ is a torsor. On the other hand, we have another canonical isomorphism
\[ \mathcal{M}(G_{V^\sharp}) \otimes_{A_{\inf}(V^\sharp)} A_{\cris}(V^\sharp) \cong \mathcal{M}_{\cris}(G_{V^\sharp}), \]
matching $s_{\alpha,\Delta} \otimes 1$ with $s_{\alpha,0}$, so $\mathcal{E}_{A_{\cris}(V^\sharp)}$ is a torsor as it is again the pullback of torsor $\mathcal{E}_{\cris}$ in \cite{3} corollary 7.1.14. We know from the proof of \cite{22} lemma 4.19 that $A_{\inf}(V^\sharp) \to A_{\inf}(V^\sharp)[\frac{1}{p}]$ factors through $A_{\cris}(V^\sharp)$, so $\mathcal{E}_{A_{\inf}(V^\sharp)[\frac{1}{p}]}$ is a torsor. Now Beauville-Laszlo (see \cite{24} lemma 7.2 for example) applied to $A_{\inf}(V^\sharp)[\frac{1}{p}]$ tells us that $\mathcal{E}_{A_{\inf}(V^\sharp)[\frac{1}{p}]}$ is a torsor.

We have seen that $\mathcal{E}_{A_{\inf}(V^\sharp)}$ is a torsor over the open subset
\[ U := \text{Spec}(A_{\inf}(V^\sharp)[\frac{1}{p}]) \bigcup \text{Spec}(A_{\inf}(V^\sharp)[\frac{1}{\mu}]), \]
whose complement is the unique closed point of $\text{Spec}(A_{\inf}(V^\sharp))$ since $A_{\inf}(V^\sharp)/(p, \mu) \cong \mathcal{O}_C/(\epsilon - 1)$ where $\epsilon - 1$ is a pseudouniformizer in $\mathcal{O}_C^\times$. Now it follows from \cite{24} proposition 8.5 and \cite{7} lemma 14.2.3 that torsors (with respect to a reductive group) on $U$ are equivalent to torsors on $\text{Spec}(A_{\inf}(V^\sharp))$. Alternatively, we can use the Tannakian formalism, \cite{7} lemma 14.2.3 and the proof of \cite{24} proposition 7.3 (which shows the exactness of equivalence in \cite{7} lemma 14.2.3) to obtain the equivalence between torsors (without the reductive hypothesis) on $U$ and $\text{Spec}(A_{\inf}(V^\sharp))$.

Lastly, if $V^\sharp$ is of mixed characteristic without being equal to $\mathcal{O}_C$, then $V^\sharp$ is the pullback of $\overline{V^\sharp} \to k$ along $\mathcal{O}_C \to k$, where $k$ is the residue field of $\mathcal{O}_C$ and $\overline{V^\sharp}$ is the image of $V^\sharp$ in $k$. Now $\mathcal{E}_{A_{\inf}(V^\sharp)}$ is equivalent to $\mathcal{E}_{W(\overline{V^\sharp})}$ and $\mathcal{E}_{A_{\inf}(\mathcal{O}_C)}$ together with an identification $(\mathcal{E}_{W(\overline{V^\sharp})})|_{W(k)} \cong (\mathcal{E}_{A_{\inf}(\mathcal{O}_C)})|_{W(k)}$, which is a torsor as we have already seen $\mathcal{E}_{A_{\inf}(\mathcal{O}_C)}$ and $\mathcal{E}_{W(\overline{V^\sharp})}$ are torsors.

We now come back to our construction of the map
\[ \text{loc}_{\mu} : \mathcal{G}_K^\circ \to \text{Sh}_{\mathcal{H}}. \]
Given a characteristic $p$ affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $\mathcal{O}_L$, and a untilt $S^a = \text{Spec}(R^a, R^{a+})$ together with a map $\text{Spec}(R^{a+}) \to \mathcal{G}_K$ over $\mathcal{O}_L$, we have constructed a $\mathcal{G} \times_{\mathbb{Z}_p} A_{\inf}(R^{a+})$-torsor over $\text{Spec}(A_{\inf}(R^{a+}))$ parametrizing isomorphisms between $Q_{Z(p)} \otimes_{\mathbb{Z}_p} A_{\inf}(R^{a+})$ and $\mathcal{M}$ preserving Hodge tensors. Now the $\varphi$-module structure on $\mathcal{M}$ (and $s_{\alpha,\Delta}$ are compatible with the $\varphi$-structure) gives us a modification
\[ \varphi_{\mathcal{E}} : \varphi^* \mathcal{E} \longrightarrow \mathcal{E}. \]
at $\phi(\xi)$, where $\xi \in A_{\inf}(R^{\ell^{+}})$ is a generator of the kernel $A_{\inf}(R^{\ell^{+}}) \to R^{\ell^{+}}$. In other words, $\phi(E)$ is an isomorphism on $A_{\inf}(R^{\ell^{+}})[1/\phi(\xi)]$ (meromorphicity at $\xi$ is automatic at the algebraic setting).

We then obtain a $G_{\mathbb{Z}_{p}}$-shtuka by twisting $E$ by $\text{Frob}_{S}^{-1}$ and restrict to $\mathcal{Y}_{[0,\infty)}(S)$. More precisely, let

$$\mathcal{P} := (\text{Frob}_{S}^{-1})^{*}\mathcal{E}|_{\mathcal{Y}_{[0,\infty)}(S)},$$

then $\phi(E)$ gives rise to a modification

$$\phi : \text{Frob}_{S}^{*}(\mathcal{P}) \to \mathcal{P}$$

at the untilt $S^{\sharp}$, i.e. a Shtuka with one leg at $S^{\sharp}$.

**Lemma 5.5.** The modification $\phi$ is bounded by $\mu$.

**Proof.** Since boundedness is a pointwise condition on $R^{\ell^{+}}$, we can assume that $R^{\ell^{+}} = \mathcal{O}_{C}$ for an algebraically closed field $C$. If $C$ is of mixed characteristic, then we know from [7] proposition 12.4.6 and remark 12.4.7 that the relative position of $\phi(E)$ is governed by the Hodge-Tate period $\text{Lie } G_{\mathcal{O}_{C}} \otimes \mathbb{C}(1) \subset \mathcal{T}_{\mathcal{O}_{C}} \otimes \mathbb{C}$ (and the corresponding Hodge tensors), but $\text{Lie } G_{\mathcal{O}_{C}}$ corresponds to the Hodge filtration on the Betti homology of $\mathcal{A}_{C}$, and there are isomorphisms respecting Hodge tensors and filtrations, see [5] lemma 2.3.6 for example.

If $C$ is of equal characteristic, then $\mathcal{M}$ is the (covariant) Dieudonné crystal associated to $G_{\mathcal{O}_{C}}$ evaluated at the initial PD-thickening $W(\mathcal{O}_{C})$ of $\mathcal{O}_{C}$, which is the de Rham homology of $A_{W(\mathcal{O}_{C})}$. We can further reduce to geometric points $k$ of $\mathcal{O}_{C}$, then it is well-known that the relative position of the $F$-structure on $\mathcal{M}_{W(k)}$ is governed by the Hodge Filtration on the de Rham cohomology of $A_{W(k)}$, which again can be compared to the Hodge Filtration on Betti-cohomology respecting Hodge tensors. \hfill $\square$

We have now associated a $G_{\mathbb{Z}_{p}}$-Shtuka $\mathcal{P}$ with one leg at $S^{\sharp}$ bounded by $\mu$ to each element of $\mathcal{S}_{K}(R^{\ell^{+}})$. This defines a functor from the presheaf on affinoid perfectoid spaces over $\overline{\mathbb{F}}_{p}$, which sends $S$ to untilts $S^{\sharp} = \text{Spa}(R^{\sharp}, R^{\ell^{+}})$ together with elements in $\mathcal{S}_{K}(R^{\ell^{+}})$, to the v-stack of $G_{\mathbb{Z}_{p}}$-Shtukas over $\mathcal{O}_{L}$. Since $\mathcal{S}_{K}^{\circ}$ is the sheafification of the presheaf, we have the desired

$$\text{loc}_{\mathcal{P}} : \mathcal{S}_{K}^{\circ} \to \text{Sht}_{\mu}.$$

Let $p$-Isog be the presheaf on schemes over $\mathcal{O}_{L}$ sending a scheme $T$ to the set of triples $(x, y, f)$ with two points $x, y \in \mathcal{S}_{K}(T)$, and $f : A_{x} \to A_{y}$ is a $p$-quasi-isogeny from $A_{x}$ to $A_{y}$, which are the pullback of $A$ along $x$ and $y$. Moreover, we requires that at every geometric point $s$, $f$ preserves the level structure, $s_{\alpha, \text{dir}}$ and $s_{\alpha, \text{t}}$ with $l \neq p$. Further, it preserves $s_{\alpha, 0}$ (resp. $s_{\alpha, p}$) if $s$ is of characteristic $p$ (resp. 0). Then $p$-Isog is represented by a scheme, and we have a diagram

$$\begin{array}{ccc}
\mathcal{S}_{K}^{\circ} & \xleftarrow{s} & \mathcal{S}_{K}^{\circ} \\
& \overset{t}{\searrow} & \\
& & \mathcal{S}_{K}
\end{array}$$

where $s$ (resp. $t$) remembers $x$ (resp. $y$). Moreover, $s$ and $t$ are proper surjective, and étale on the generic fiber. See [9] section 6 for example.
Similar to \( \text{loc}_p \), we can construct a map

\[
\text{loc}_p^H : p - \text{Isog}^\circ \to \text{Sht}_{p|\mu}.
\]

Given \( \text{Spa}(R, R^+) \) affinoid perfectoid of characteristic \( p \), \( \text{Spa}(R^\sharp, R^\sharp+) \) an untilt and \((x, y, f)\) a point of \( p - \text{Isog}(R^\sharp+) \), let \( \mathcal{M}_x \) (resp. \( \mathcal{M}_y \)) be the \( \varphi - \text{A}_{\text{inf}}(R^\sharp+) \)-module associated to \( A_x[p^\infty] \) (resp. \( A_y[p^\infty] \)). Then \( f \) induces an isomorphism \( \mathcal{M}_x[\frac{1}{p}] \xrightarrow{\sim} \mathcal{M}_y[\frac{1}{p}] \) (as \( f \) is a \( p \)-quasi-isogeny), which preserves \( s_{\alpha, \Delta} \) by proposition \ref{5.2} and \( f \) preserving \( s_{\alpha, 0} \), so it induces a modification

\[
f_P : \mathcal{P}_x \to \mathcal{P}_y
\]

at the characteristic \( p \) untilt, where \( \mathcal{P}_x \) and \( \mathcal{P}_y \) denotes the image of \( x \) and \( y \) under \( \text{loc}_p \). Now \( \text{loc}_p^H \) is defined by (sheafification of) sending the data of \((x, y, f)\) to \( f_P \).

Let \( \nu \) be a cocharacter of \( G \). We note that if we restrict to the subscheme \( p - \text{Isog}_\nu \) parameterizing quasi-isogenies of type bounded by \( \nu \), then \( \text{loc}_p^H \) restricts to

\[
\text{loc}_p^H : p - \text{Isog}_\nu^\circ \to \text{Sht}^\nu_{p|\mu}.
\]

Now we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G}_K^\circ & \xrightarrow{p - \text{Isog}_\nu^\circ} & \mathcal{G}_K^\circ \\
\downarrow{\text{loc}_p} & & \downarrow{\text{loc}_p^H} \\
\text{Sht}_\mu & \xrightarrow{t} & \text{Sht}_{p|\mu} \\
\downarrow{p_1} & & \downarrow{p_2} \\
\text{Sht}_\mu & & \text{Sht}_{p|\mu}.
\end{array}
\]

By the following lemma, we can pullback cohomological correspondence along the diagram, so we have cohomological correspondence

\[
\text{loc}_p^* S_U : (\mathcal{G}_K^\circ, \mathbb{Q}_l) \to (\mathcal{G}_K^\circ, \mathbb{Q}_l)
\]

supported on \( p - \text{Isog}_\nu^\circ \), where we have used that \( \mu \) is minuscule so that the structure sheaf \( S_U \) on \( \text{Sht}_\mu \) is \( \mathbb{Q}_l \).
Lemma 5.6. The diagram factorizes through

\[
\begin{array}{ccc}
\mathbb{G}_K^\circ & \xrightarrow{p - \text{Isog}^\circ} & \mathbb{G}_K \\
\downarrow & & \downarrow \\
\mathbb{G}_K^{\text{Spd}(\mathcal{O}_L)} & \xrightarrow{s} & \mathbb{G}_K^{\text{Spd}(\mathcal{O}_L)} \\
\downarrow & & \downarrow \\
\text{Sht}_\mu & \xrightarrow{p_1} & \text{Sht}_\mu \\
\downarrow & & \downarrow \\
\text{Sht}_{\mu|\mu} & \xrightarrow{p_2} & \text{Sht}_{\mu|\mu} \\
\end{array}
\]

where \( \mathbb{G}_K^{\text{Spd}(\mathcal{O}_L)} \) and \( p - \text{Isog}^\circ_{\text{Spd}(\mathcal{O}_L)} \) are the canonical compactifications of \( \mathbb{G}_K^\circ \) and \( p - \text{Isog}^\circ \) respectively, and the vertical maps along the first row are the canonical maps into the compactification, see [12] proposition 18.6. Moreover, the vertical maps along the first line are open immersions, and the two lower squares in the diagram are Cartesian, so we can pullback cohomological correspondences (we can pullback along smooth maps).

**Proof.** Recall that \( \mathbb{G}_K^{\text{Spd}(\mathcal{O}_L)} \) is the \( v \)-sheaf sending \( \text{Spa}(R, R^+) \) to

\[
\mathbb{G}_K^{\circ}(\text{Spa}(R, R^0)) \times_{\text{Spd}(\mathcal{O}_L)(\text{Spa}(R, R^0))} \text{Spd}(\mathcal{O}_L)(\text{Spa}(R, R^+)),
\]

in other words, it is the sheafification of the presheaf sending \( \text{Spa}(R, R^+) \) to the set of untilts \( \text{Spa}(R^0, R_\mathbb{Z}^+) \) of \( \text{Spa}(R, R^+) \) over \( \mathcal{O}_L \) together with morphism \( \text{Spec}(R^{\mathbb{Z},0}) \to \mathbb{G}_K \) of schemes over \( \mathcal{O}_L \), and similarly for \( p - \text{Isog}^\circ_{\text{Spd}(\mathcal{O}_L)} \). The canonical map \( \mathbb{G}_K^{\circ} \to \mathbb{G}_K^{\text{Spd}(\mathcal{O}_L)} \) is simply the composition of the restriction map from \( \text{Spec}(R^{\mathbb{Z},0}) \to \text{Spec}(R_\mathbb{Z}^+) \). Since the \( v \)-stacks \( \text{Sht}_\mu \) and \( \text{Sht}_{\mu|\mu} \) are independent of \( R^+ \), the factorization follows.

It follows from [12] proposition 27.5 that \( \mathbb{G}_K^\circ \) and \( p - \text{Isog}^\circ \) are compactifiable over \( \text{Spd}(\mathcal{O}_L) \), so proposition 22.3 (1) of loc.cit. tells us that the natural maps into their canonical compactifications are open immersions, i.e. the vertical maps along the first row are open immersions.

Next, we prove the Cartesianness of the two lower squares. We know that \( p \)-quasi-isogenies from an abelian variety \( A_x \) preserving Hodge tensors and level structures are the same as \( p \)-quasi-isogenies from \( A_x[p^\infty] \) preserving Hodge tensors, since \( p \)-quasi-isogenies does not affect level structures and \( s_{\alpha,l} \) while the rest Hodge tensors are all detected by the corresponding \( p \)-divisible groups. Moreover, we know from [7] theorem 17.5.2 that \( p \)-quasi-isogenies between \( p \)-divisible groups preserving Hodge tensors are the same as \( p \)-quasi-isogenies between \( M_x \) and \( M_y \) preserving \( s_{\alpha,\Delta} \), so it remains to
show this is the same as the corresponding $p$-quasi-isogenies between $\mathcal{E}_x$ and $\mathcal{E}_y$, whence $\mathcal{P}_x$ and $\mathcal{P}_y$. In other words, we want to know that the restriction functor from Breuil-Kisin-Fargues modules $\mathcal{M}$ to $GL_n$-Shtukas is fully faithful up to quasi-isogeny.

We can use $v$-descent to reduce to the case of $S^2 = \text{Spa}(R^2, R^2)$, where $C_i$ are algebraically closed perfectoid fields, from which we quickly reduce to $S^2 = \text{Spa}(C_i, C_i^+)$. Since both $\text{Sh}(\mu|\mu)$ and $p - \text{Isog}_p/\text{Spd}(\mathcal{O}_L)$ are independent of $R^+$, we can further assume that $C_i^+ = \mathcal{O}_{C_i}$, in which case the fully faithfulness follows from \cite{7} theorem 13.2.1 and theorem 14.2.1. This proves the left square is Cartesian, the right one is similar. 

**Remark 5.7.** The reason we use the factorization as in the lemma is because the diagram \((40)\) is not Cartesian, and the reason for that is we do not have fully faithfulness of the restriction functor from Breuil-Kisin-Fargues modules over $W(R^+)$ to Shtukas over $\mathcal{Y}_{(0, \infty)}(\text{Spa}(R, R^+))$ in general, disproving the expectation in \cite{25} remark 4.5.13. Indeed, Breuil-Kisin-Fargues modules over $W(R^+)$ depends on $R^+$ while Shtukas over $\mathcal{Y}_{(0, \infty)}(\text{Spa}(R, R^+))$ does not, and there are examples where the base change of Breuil-Kisin-Fargues modules (even those coming from $p$-divisible groups) from $W(R^+)$ to $W(R^0)$ is not fully faithful. This is closely related to the failure of Tate’s theorem on homomorphism between $p$-divisible groups (the characteristic $p$ case is due to de Jong) in the case of non-discrete valuation rings.

We now restrict to the generic fiber to have

\[
\begin{array}{ccc}
p - \text{Isog}_p^p & \xrightarrow{s} & \mathcal{X}_K^p \\
\text{loc}_{p, \eta} & \downarrow & \text{loc}_{p, \eta} \\
\text{Sh}(\mu|\mu, \eta) & \xrightarrow{p_1} & \text{Sh}(\mu, \eta),
\end{array}
\]

along which we can pullback the Hecke correspondence $\mathcal{C}_\mu$ in \((3)\) to obtain

$\mathcal{C}_\mu^S : (\mathcal{X}_K^p, \mathcal{U}_p) \to (\mathcal{X}_K^p, \mathcal{U}_p)$,

where we use that $\mu$ is minuscule so that the canonical sheaves are constant. Moreover, it follows from the étaleness of the two correspondences that $\mathcal{C}_\mu^S$ is nothing but

$\mathcal{C}_\mu^S : s^*\mathcal{U}_p \simeq \mathcal{U}_p \simeq t^*\mathcal{U}_p \simeq t^*\mathcal{U}_p$.

Now as in section \(3.2\), let $V$ be a representation of $\hat{G}_{\mathbb{Q}_p}$, and $h_V$ be the function corresponding to $V$ through classical Satake, we define

$\Gamma_V^S := \bigcup_{h_V(\nu(p)) \neq 0} p - \text{Isog}_p^p$.
and
\[ T^S_V := \sum_{h_V(\nu(p)) \neq 0} h_V(\nu(p))S^S_\nu \]
which is a cohomological correspondence from \( (X_K^\circ, \mathcal{Q}_l) \) to itself supported on \( \Gamma^S_\nu \). Note that \( T^S_V \) is exactly the (analytification of) Hecke correspondence of \( X_K \), so we have proved

**Theorem 5.8.** The pullback of \( S_{V,\eta} \) along (42) is exactly \( T^S_V \).

**Proof.** This follows from the above discussion and theorem 4.2. \( \square \)

Next we want to compare the Hecke operator and excursion operator on the compactly supported cohomology of \( \bar{X}_K \), and deduce the S=T conjecture of Xiao-Zhu. We begin with a well-known observation.

**Lemma 5.9.** Let \( \pi : S_K \longrightarrow Spec(O_L) \) be the structure map of \( S_K \), then \( R^i\pi!Q_l \) is a lisse sheaf on \( Spec(O_L) \) for every \( i \).

**Proof.** It follows from the existence of integral toroidal compactification that \( R^j\pi_*Q_l \) is lisse (using the Leray spectral sequence of the open immersion into toroidal compactification and purity of normal crossing divisors). Now Poincare duality gives the result for \( R^i\pi!Q_l \). \( \square \)

**Lemma 5.10.** There is a canonical isomorphism
\[ R^i((\pi^\circ)!Q_l) \cong c^*_{O_L} R^i\pi!Q_l, \]
with notations as in appendix A.1.

**Proof.** It is enough to prove the statement with \( Q_l \) replaced by \( \Lambda \), a finite extension of \( Z_l \). It is enough to prove the identification on the special fiber and generic fiber. Over the special fiber, this is proposition A.3.

Let \( \bar{\eta} \) be a completed algebraically closure of \( L \), then we have
\[ (R^i(\pi^\circ)!\Lambda)_{\bar{\eta}} \cong R\Gamma_c((S_K^\circ)_{\bar{\eta}}, \Lambda) \cong R\Gamma_c(X_K^\circ, \Lambda) \cong R\Gamma_c(X_K^\circ, R\psi(\Lambda)) \cong R\Gamma_c(\bar{X}_K, \Lambda) \]
where \( R\psi \) is the nearby cycle functor with respect to the integral model \( S_K/O_L \). The first isomorphism is proper base change (12 proposition 22.15), the second is (12) lemma 15.6, the third is 26 theorem 5.7.8, and the last follows from \( S_K \) being smooth over \( O_L \).

On the other hand, we have
\[ (R\pi!\Lambda)_{\bar{\eta}} \cong R\Gamma_c(S_K^\circ, \Lambda) \cong R\Gamma_c(\bar{X}_K, R\psi(\Lambda)) \cong R\Gamma_c(\bar{X}_K, \Lambda) \]
where the first isomorphism is proper base change, the second follows from the existence of toroidal compactifications, see 27 for example, and the last follows from the smoothness of \( S_K \). We have thus proved the identification over the generic fiber. \( \square \)

It follows from the last two lemmas that \( R^i\pi^\circ!\mathcal{Q}_l \) is a locally constant sheaf whose stalks are identified with \( H^i_c(\bar{X}_K, \mathcal{Q}_l) \).

Next, recall that we can pushforward cohomological correspondence (1 A.2.6), so we can push-forward (41) along \( R(\pi^\circ)! \) to obtain
\[ R^i(\pi^\circ)!\text{loc}^*S_V : R^i(\pi^\circ)!\mathcal{Q}_l \longrightarrow R^i(\pi^\circ)!\mathcal{Q}_l \]
which is a global section of the internal hom $\mathcal{H}om(R^i(\pi^\circ)_! \mathbb{Q}_l, R^i(\pi^\circ)_! \mathbb{Q}_l)$. Since $R^i(\pi^\circ)_! \mathbb{Q}_l$ is locally constant, so is $\mathcal{H}om(R^i(\pi^\circ)_! \mathbb{Q}_l, R^i(\pi^\circ)_! \mathbb{Q}_l)$, and we see that $R^i(\pi^\circ)_! \text{loc}^\ast p_S V$ is constant. In other words, for any geometric point $\bar{\eta}$ of $\text{Spd}(L)$, we have that the restriction of $R^i(\pi^\circ)_! \text{loc}^\ast p_S V$ to $\bar{\eta}$, which is nothing but

$$R^i(\pi^\circ)_! \text{loc}^\ast p_{\bar{\eta}} S_{V,\bar{\eta}} : H^i_c(\mathcal{X}_K, \mathbb{Q}_l) \rightarrow H^i_c(\mathcal{X}_{K,\bar{\eta}}, \mathbb{Q}_l),$$

is identified with its restriction to the special fiber

$$R^i(\pi^\circ)_! \text{loc}^\ast p_{s,\bar{\eta}} S_{V,s,\bar{\eta}} : H^i_c(\mathcal{X}_K, \mathbb{Q}_l) \rightarrow H^i_c(\mathcal{X}_K, \mathbb{Q}_l)$$

under the canonical identification between $H^i_c(\mathcal{X}_K, \mathbb{Q}_l)$ and $H^i_c(\mathcal{X}_{K,\bar{\eta}}, \mathbb{Q}_l)$, where $s := \mathbb{F}_p$.

Now theorem 5.8 tells that $\text{loc}^\ast p_{\bar{\eta}} S_{V,\bar{\eta}} = T^\mathbb{F}_p$ is the Hecke operator, so the action $R^i(\pi^\circ)_! \text{loc}^\ast p_{s,\bar{\eta}} S_{V,s,\bar{\eta}}$ is identified with the Hecke operator on $H^i_c(\mathcal{X}_K, \mathbb{Q}_l)$. Recall that Hecke operators on $H^i_c(\mathcal{X}_K, \mathbb{Q}_l)$ is defined through the identification of $H^i_c(\mathcal{X}_K, \mathbb{Q}_l)$ with $H^i_c(\mathcal{X}_{K,\bar{\eta}}, \mathbb{Q}_l)$ and the Hecke action on the latter, which is exactly the way we identify the excursion operators.

It remains to identify $R^i(\pi^\circ)_! \text{loc}^\ast p_{s,\bar{\eta}} S_{V,s,\bar{\eta}}$ with excursion operator defined by Xiao-Zhu. As usual, we let

$$p - \text{Isog}_V := \bigcup_{\nu} p - \text{Isog}_\nu$$

where $\nu$ varies from heighest weights of irreducible summands of $V$.

We first observe that
factorizes through

\[
\begin{array}{ccc}
\mathcal{X}_K & \xrightarrow{\text{loc}_p^W} & \mathcal{X}_K \\
\downarrow & & \downarrow \\
\text{Sh}_{\mu}^W & \xrightarrow{\text{loc}_p^W} & \text{Sh}_{\mu}^W \\
\downarrow & & \downarrow \\
\text{Sh}_{\mu|\mu,s} & \xrightarrow{p_1} & \text{Sh}_{\mu,s} \\
\end{array}
\]

where \(\text{loc}_p^W\) sends \(A\) to (the corresponding torsor) the Dieudonné module associated to \(A[p^\infty]\) together with Hodge tensors, and similarly for \(\text{loc}_p^{H,W}\). Up to truncation, the upper half diagram is precisely the analytification of the pullback diagram considered in \cite{ref1} section 7.3.11. Then proposition \ref{prop:3.20} tells us that \(\text{loc}_{p,s}^* S_{V,s}\) is the same as the pullback of \( S_{V}^W\) along the upper half of the diagram. Since \( S_{V}^W\) is the analytification of the truncated one (proposition \ref{prop:3.20}), and \(\diamond\)-analytification of cohomological correspondences commutes with pullback (proposition \ref{prop:A.4}), we have that \(\text{loc}_{p,s}^* S_{V,s}\) is exactly the analytification of the excursion operator on the special fiber of Shimura varieties constructed in \cite{ref1} (being defined as the pullback of the truncated version of \(\text{loc}_p^W\) of the truncated \( S_{V}^W\) in the category of perfect schemes). Therefore, we have proved that the Hecke operator \( T_S^c\) on \( H_c^i(\mathcal{X}_K, \mathbb{Q}_l)\) is the same as the action of \(\text{loc}_p^W S_{V}^W\). In other words, we have proved the \(S = T\) conjecture in \cite{ref1}.

**Theorem 5.11.** As operators on \( H_c^i(\mathcal{X}_K, \mathbb{Q}_l)\), the Hecke correspondence \( T_S^c\) is the same as the excursion correspondence \(\text{loc}_p^W S_{V}^W\), the analytification of the excursion operator in \cite{ref1} (whose action on \( H_c^i(\mathcal{X}_K, \mathbb{Q}_l)\) is the same as the action as defined in \cite{ref1} through comparison of cohomology of \(\nu\)-sheaves and perfect schemes in \cite{ref12} section 27).

**Corollary 5.12.** (\ref{cor:28}) The action of \( T_S^c\) on \( H_c^i(\mathcal{X}_K, \mathbb{Q}_l)\) satisfies Eichler-Shimura relation.

**Appendix A. Comparison between algebraic stacks and \(\nu\)-stacks**

**A.1. Comparison of cohomology.** By \cite{ref12} section 27, we can associate a \(\nu\)-sheaf \(X^\infty\) to any scheme \(X\) locally of finite type over a complete discrete valuation ring \(O\) with perfect residue field of characteristic \(p\). \(X^\infty\) is defined by sending characteristic \(p\) affinoid perfectoid \(S = \text{Spa}(R, R^+)\) to the set of untilts \(S^\circ\) over \(\text{Spa}(O, O)\) together with a map \(S^\circ \rightarrow X\) of locally ringed spaces.
over \( \text{Spec}(\mathcal{O}) \). Another way to view \( X^\infty \) is that \( X^\infty \) is the sheafification of the presheaf sending \( S = \text{Spa}(R, R^+) \) to the set of untilts \( S^\sharp = \text{Spa}(R^\sharp, R^{\sharp,+}) \) over \( \text{Spa}(\mathcal{O}, \mathcal{O}) \) together with a map \( \text{Spec}(R^\sharp) \to X \) of schemes over \( \mathcal{O} \).

On the other hand, there is another \( v \)-sheaf \( X^o \) we can associate to any scheme \( X \) locally of finite type over \( \mathcal{O} \). \( X^o \) is defined to be the sheafification of the presheaf sending a characteristic \( p \) affinoid perfectoid space \( S = \text{Spa}(R, R^+) \) to the set of untilts \( S^\sharp = \text{Spa}(R^\sharp, R^{\sharp,+}) \) together with morphisms of stacks \( \text{Spec}(R^{\sharp,+}) \to X \) of schemes over \( \mathcal{O} \). In other words, it is the diamond associated to the formal scheme \( X^\wedge \), the completion of \( X \) along the special fiber of \( \mathcal{O} \). We note that there is a canonical map

\[
a_X : X^o \to X^\infty
\]

sending the map \( \text{Spec}(R^+) \to X \) to its restriction on \( \text{Spec}(R) \), which is an open immersion. It is an isomorphism when \( X \) is proper over \( \mathcal{O} \).

We extend the construction to algebraic stacks locally of finite type over \( \mathcal{O} \). For clarity, we specify our convention on algebraic stacks to be fppf sheaves of groupoids with diagonals representable by algebraic spaces, and admits a representable smooth surjective map from a scheme. Let \( X \) be such an algebraic stack, we define \( X^o \) to be the stackification of the presheaf sending \( S = \text{Spa}(R, R^+) \) to the groupoid of untilts \( S^\sharp = \text{Spa}(R^\sharp, R^{\sharp,+}) \) of \( S \) together with morphisms of stacks \( \text{Spec}(R^{\sharp,+}) \to X \) over \( \mathcal{O} \). Similarly, we define \( X^\infty \) to be the stackification of the presheaf sending \( S = \text{Spa}(R, R^+) \) to the groupoid of untilts \( S^\sharp = \text{Spa}(R^\sharp, R^{\sharp,+}) \) of \( S \) together with morphisms of stacks \( \text{Spec}(R^\sharp) \to X \) over \( \mathcal{O} \). Then as usual, there is a canonical map

\[
a_X : X^o \to X^\infty.
\]

We recall that when \( X \) is a scheme locally of finite type over \( \mathcal{O} \), there is a natural morphism

\[
c_X : X^\infty_u \to X_{\text{ét}}
\]

from the \( v \)-site of \( X^\infty \) to the étale site of \( X \), which is defined by sending a scheme \( U \) together with a smooth morphism \( U \to X \) to \( U^\infty \to X^\infty \), see [12] section 27. Let \( \Lambda \) be a finite ring killed by \( p \) not dividing \( N \), then \( c_X \) induces a functor

\[
c_X^* : D(X_{\text{ét}}, \Lambda) \to D_{\text{ét}}(X^\infty, \Lambda),
\]

and for any \( f : X \to Y \) separated, there is a canonical identification

\[
c_{Y}^* Rf_! \cong Rf_!^\infty c_X^*,
\]

see [12] proposition 27.5. Moreover, we have a canonical identification

\[
f^{\infty,*} c_Y^* \cong c_X^* f^*.
\]

For a general algebraic stack \( X \) locally of finite type over \( \mathcal{O} \), we can choose a smooth surjective map \( U \to X \) with \( U \) being a scheme, then it is well-known that we have a canonical identification

\[
\mathcal{D}(X, \Lambda) \cong \lim \mathcal{D}(U^\bullet, \Lambda)
\]

where \( U^\bullet \) is the Čech nerve of \( U \), \( \mathcal{D}(X, \Lambda) \) is the \( \infty \)-categorical enhancement of the derived category of the lisse-étale site of \( X \) and similarly for the right hand side. We now can extend the comparison morphism to algebraic stacks

\[
c_X^* : \mathcal{D}(X, \Lambda) \cong \lim \mathcal{D}(U^\bullet, \Lambda) \lim c_Y^* \mathcal{D}_{\text{ét}}(U^\bullet, \Lambda) \cong \mathcal{D}_{\text{ét}}(X^\infty, \Lambda),
\]
where the last isomorphism is \([12]\) proposition 17.3 and remark 17.4 (and the commutation of \(\diamond\) with fiber products). It is not hard to see that \(c_X^*\) is independent of the choice of \(U\).

Moreover, given a separated representable map \(f : X \to Y\) between algebraic stacks locally of finite type over \(\mathcal{O}\), we choose a smooth surjective maps \(U \to Y\) from a scheme \(U\), and let \(V\) be the pullback of \(U\) along \(f\) with induced map \(g : V \to U\). Then \(g\) extends to \(g^* : V^* \to U^*\), and the limit of \(c_U^* Rg^* \cong R(g^* \diamond) c_Y^*\) tells us that

\[
c_Y^* Rf^! \cong Rf^! \diamond c_X^*.
\]

Similarly, we have

\[
f^! \diamond c_Y^* \cong c_X^* f^!.
\]

In summary, we have the following theorem.

**Theorem A.1.** Let \(\Lambda\) be a finite ring killed by \(N\) with \(p\) not dividing \(N\), \(X\) an algebraic stack locally of finite type over \(\mathcal{O}\), then we have a canonical comparison map

\[
c_X^* : \mathcal{D}(X, \Lambda) \to \mathcal{D}_{\text{et}}(X^\infty, \Lambda).
\]

Moreover, for any \(f : X \to Y\) representable separated map, there are canonical identifications

\[
c_Y^* Rf^! \cong Rf^! \diamond c_X^*
\]

and

\[
f^! \diamond c_Y^* \cong c_X^* f^!.
\]

Lastly, we pass to adic sheaves. Let \(l\) be a prime different from \(p\), and \(\Lambda\) be a finite extension of \(\mathbb{Z}_l\), we know by \([29]\) that

\[
\mathcal{D}(X, \Lambda) = \lim_n \mathcal{D}(X, \Lambda/l^n),
\]

where the limit is taken as \(\infty\)-categories. On the other hand, we also have

\[
\mathcal{D}_{\text{et}}(X^\infty, \Lambda) = \lim_n \mathcal{D}_{\text{et}}(X^\infty, \Lambda/l^n)
\]

by \([12]\) proposition 26.2. The analogue of theorem A.1 for such \(\Lambda\) follows directly from the limit expression, using that pullback and !-pushforward are compatible with the limit expression, see \([12]\) remark 26.3. Further, we can (idempotently completely) invert \(l\) and taking the limit over finite extensions of \(\mathbb{Z}_l\) to obtain the comparison with coefficient \(\mathbb{Q}_l\).

**Remark A.2.** If the algebraic stack \(X\) is locally of finite type over a perfect field, then \(X^\infty\) depends only on the perfection \(X^{\text{perf}}\) of \(X\), and theorem A.1 holds for \(X^{\text{perf}}\) since perfection does not change cohomology. This is an important case in this article.

A natural question is whether we have a similar comparison result for the \(\diamond\)-analytification. The answer is no in general, but it holds in the following special case.

**Proposition A.3.** Let \(X\) be a scheme locally of finite type over a perfect field \(k\), and \(\Lambda\) be either a finite extension of \(\mathbb{Z}_l\) or \(\mathbb{Q}_l\) then we have a canonical identification

\[
R\pi_!^* \Lambda \cong c_k^* R\pi_! \Lambda
\]

where \(\pi : X \to \text{Spec}(k)\) is the structure map.
Proof. It is enough to show the result after base change to the algebraically closure of $k$, so we assume that $k$ is algebraically closed, and we are aiming to prove

$$R\Gamma_c(X^\circ, \Lambda) = R\Gamma_c(X, \Lambda).$$

By a standard Čech cohomology argument, we can assume that $X = \text{Spec}(R)$ is affine. Moreover, it is enough to prove the case when $\Lambda$ is finite, so we assume that.

Let $\mathbb{C}$ be a completed algebraically closure of $k((t))$, and by proper base change ([12] proposition 22.15), we have

$$R\Gamma_c(X^\circ, \Lambda) = R\Gamma_c(X^\circ_k, \Lambda)$$

where $X^\circ_k := X^\circ \times_{\text{Spd}(k)} \text{Spd}(\mathbb{C})$. We observe that

$$X^\circ_k(t) = \text{Spd}(R[[t]], k[[t]])$$

which visibly is the analytification of analytic adic space $X^\text{ad}_k(t) := \text{Spa}(R((t)), R[[t]])$. Then by [12] lemma 15.6,

$$R\Gamma_c(X^\circ_k, \Lambda) = R\Gamma_c((X^\text{ad}_k(t))_{\mathbb{C}}, \Lambda).$$

To compute $R\Gamma_c((X^\text{ad}_k(t))_{\mathbb{C}}, \Lambda)$, we observe that

$$\text{Spa}(R((t)), R[[t]]) = \text{Spa}(R[[t]], R[[t]]) \eta := \text{Spa}(R[[t]], R[[t]]) \times_{\text{Spa}(k[[t]], k[[t]])} \text{Spa}(k((t)), k[[t]])$$

namely, $X^\text{ad}_k(t)$ is the generic fiber of the formal scheme $X := \text{Spa}(R[[t]], R[[t]])$ over $\text{Spa}(k[[t]], k[[t]])$. Note that $X$ is the $t$-adic completion of the scheme $\text{Spec}(R) \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$ over $\text{Spec}(k[[t]])$, which has special fiber $X$. Then [26] theorem 5.7.8 tells us that

$$R\Gamma_c((X^\text{ad}_k(t))_{\mathbb{C}}, \Lambda) = R\Gamma_c(X, R\psi(\Lambda))$$

where $R\psi$ is the nearby cycle functor with respect to $\text{Spec}(R[[t]])$ over $\text{Spec}(k[[t]])$. Since $\text{Spec}(R) \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$ is a trivial fibration over $\text{Spec}(k[[t]])$, we have $R\psi(\Lambda) = \Lambda$, finishing the proof.

\[\square\]

A.2. Comparison of cohomological correspondences. We have established the comparison of cohomology, and now we want to use it to compare the cohomological correspondences. Let

$$\mathcal{C} : p^* \mathcal{F} \longrightarrow q^! \mathcal{G}$$

be a cohomological correspondence supported on

$$\xymatrix{ & Z \\ X \ar[ru]^p & & \ar[ll]^q Y}$$

where $X, Z$ and $Y$ are algebraic stacks locally of finite type over $\mathcal{O}$, or perfections of such stacks over a perfect field, $p$ and $q$ are representable separated morphisms, and $\mathcal{F} \in D^b(X_{\text{ét}}, \mathbb{Q}_l)$ (resp. $\mathcal{G} \in D^b(Y_{\text{ét}}, \mathbb{Q}_l)$). There is a natural analytification of $\mathcal{C}$,

$$\mathcal{C}^\infty : (X^\infty, c_X^* \mathcal{F}) \longrightarrow (Y^\infty, c_Y^* \mathcal{G})$$
supported on

\[
\begin{array}{ccc}
Z^\infty & \xrightarrow{q^\infty} & Y^\infty,
\end{array}
\]

which is defined by

\[
\mathcal{C}^\infty : p^\infty c_X^* \mathcal{F} \cong c_Z^* p^* \mathcal{F} \xrightarrow{c_Z^* c_G} c_Y^* q^! \mathcal{G} \to q^\infty c_Y^* \mathcal{G},
\]

where the last map is the adjoint of the canonical map

\[
q^\infty c_Y^* q^! \mathcal{G} \cong c_Y^* q^! \mathcal{G} \to c_Y^* \mathcal{G}
\]

with the first isomorphism being theorem \[A.1\] and the second the counit map.

This gives the \( \infty \)-analytification of the cohomological correspondence, but what we need in this article is the analytification with respect to \( \circ \). We can construct the desired cohomological correspondence from \( \mathcal{C}^\infty \) under additional hypothesis. We have naturally a commutative diagram

\[
\begin{array}{ccc}
X^\circ & \xrightarrow{a_X} & Z^\circ & \xrightarrow{a_Z} & Y^\circ
\end{array}
\]

assume the right square is Cartesian, then we can pullback \( \mathcal{C}^\infty \) along it to have the cohomological correspondence

\[
\mathcal{C}^\circ : (X^\circ, a_X^* c_X^* \mathcal{F}) \to (Y^\circ, a_Y^* c_Y^* \mathcal{G})
\]

supported on \( Z^\circ \).

We record some formal properties of \( \mathcal{C}^\circ \).

**Proposition A.4.** \( \mathcal{C}^\circ \) commutes with composition and pullbacks. More precisely, if \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are composable, and assume that the right squares involving \( a_Z \) and \( a_Y \) of both \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are Cartesian, then so does \( \mathcal{C}_1 \circ \mathcal{C}_2 \), and we have

\[
\mathcal{C}_1^\circ \circ \mathcal{C}_2^\circ = (\mathcal{C}_1 \circ \mathcal{C}_2)^\circ.
\]
Given a commutative diagram

\[
\begin{array}{ccc}
Z_1 & \overset{q_1}{\longrightarrow} & Y_1 \\
\downarrow^{p_1} & b & \downarrow^{c} \\
X_1 & \overset{a}{\longrightarrow} & Y_1 \\
\downarrow^{p_2} & \downarrow^{q_2} \\
X_2 & \longrightarrow & Y_2 \\
\end{array}
\]

and a cohomological correspondence
\[
\mathcal{C}_2 : (X_2, \mathcal{F}) \longrightarrow (Y_2, \mathcal{G})
\]
supported on $Z_2$, suppose that either the vertical arrows are smooth or the right square is Cartesian, so we can pullback $\mathcal{C}_2$ along the diagram to obtain $\mathcal{C}_1$ supported on $Z_1$. Assume that both $\mathcal{C}_1^\circ$ and $\mathcal{C}_2^\circ$ exists, i.e. the square corresponding to $a_{Y_1}$ and $a_{Z_1}$ (resp. $a_{Y_2}$ and $a_{Z_2}$) is Cartesian, then $\mathcal{C}_1^\circ$ is the pullback of $\mathcal{C}_2^\circ$ along the diagram

A.3. Truncated Witt vector Shtukas. We would like to compare the cohomological correspondences in [1] between Witt vector Shtukas with the ones between v-stacks. More precisely, only the truncated Witt vector Shtukas are considered in [1] since the untruncated ones are not algebraic stacks and the cohomological formalism does not directly apply to them. However, the problem disappears by passing to v-stacks as the formalism developed in [12] is very general and does apply to stacks with infinite dimensional automorphisms.

We recall the definition of truncated Hecke stacks and Shtukas in [1]. We omit the superscript $W$ for Witt vector Shtukas, as we will only use the truncated version for Witt vector ones. More importantly, we use the superscript $W$ for the untruncated Witt vector Shtukas or Hecke stacks viewed as v-stacks, whereas the truncated ones below are only defined as perfect stacks.

Let $L^mG$ be the perfect scheme defined by $L^mG(R) = G(W_m(R))$ for any perfect algebra $R$ over $\mathbb{F}_q$, and $\text{Hecke}^{\text{loc}(m)}_{\mu^*} := L^mG \setminus Gr^W_{\mu^*}$, where $Gr^W_{\mu^*}$ is the (twisted products of) Witt vector affine Grassmannian. Let $(m, n)$ be a pair of non-negative integers such that $m - n$ is $\mu^*$-large, see [1]
definition 3.1.6, and \( \text{Sh}_{\mu_*}^{\text{loc}(m,n)} \) be defined as in \([1]\) definition 5.3.1, and similarly for \( \text{Sh}_{\mu_*|\nu_*}^{0,\text{loc}(m,n)} \).

By loc.cit. section 5.3.2, both \( \text{Sh}_{\mu_*}^{\text{loc}(m,n)} \) and \( \text{Sh}_{\mu_*|\nu_*}^{0,\text{loc}(m,n)} \) can be written as quotients by \( L^nG \) of perfect schemes perfectly of finite type, so they fit into the cohomological formalism of the previous section by remark \([\Delta\,2]\). Now by \([4]\) proposition VI.4.1, we have canonical identifications

\[
D_\text{et}(\text{Sh}_{\mu_*}^{W(m,n)}, \Lambda) \cong D_\text{et}(\text{Sh}_{\mu_*}^{\text{loc}(m,n), \circ}, \Lambda),
\]

\[
D_\text{et}(\text{Sh}_{\mu_*|\nu_*}^{0,W}, \Lambda) \cong D_\text{et}(\text{Sh}_{\mu_*|\nu_*}^{0,\text{loc}(m,n), \circ}, \Lambda)
\]

and

\[
D_\text{et}(\text{Hecke}_{\mu_*}^{W}, \Lambda) \cong D_\text{et}(\text{Hecke}_{\mu_*}^{\text{loc}(m), \circ}, \Lambda).
\]

More precisely, the conclusion of \([4]\) proposition VI.4.1 holds if we change the assumption to that the group \( H \) has a filtration with graded pieces being extensions of closed unit discs (instead of \( \mathbb{A}^1 \)) \( v \)-locally, which the analytification \( L^nG \) does as, and the proof in cit.loc. goes without change. In other words, the truncated Witt vector Shtukas and Hecke stacks have the same cohomology with the untruncated ones. Similar identifications holds between local Witt vector Shtukas or stacks and \( \varpi \)-analytification of truncated ones, for example

\[
D_\text{et}(\text{Sh}_{\mu_*}^{\text{loc}(m,n), \circ}, \Lambda) \cong D_\text{et}(\text{Sh}_{\mu_*}^{\text{loc}(m,n), \varpi}, \Lambda).
\]

**Remark A.5.** The introduction of truncated Shtukas or Hecke stacks in \([1]\) is because the untruncated ones does not directly fit into the cohomological formalism of Artin stacks. Using the new cohomology theory of v-stacks developed in \([12]\), we believe large parts of \([1]\) can be simplified using the untruncated objects, with some caution on the arguments using fixed points results.

**References**

[1] Liang Xiao and Xinwen Zhu. Cycles on Shimura varieties via geometric Satake. *arXiv e-prints*, page arXiv:1707.05700, July 2017.

[2] Vincent Lafforgue. Chhtoucas pour les groupes r´eductifs et param´etrisation de langlands globale. *Journal of the American Mathematical Society*, 31(3):719–891, February 2018.

[3] Xinwen Zhu. Coherent sheaves on the stack of Langlands parameters. *arXiv e-prints*, page arXiv:2008.02998, August 2020.

[4] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *arXiv e-prints*, page arXiv:2102.13459, February 2021.

[5] Ana Caraiani and Peter Scholze. On the generic part of the cohomology of compact unitary shimura varieties. *Annals of Mathematics*, 186(3):649–766, nov 2017.

[6] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Annals of Mathematics*, pages 945–1066, November 2015.

[7] Jared Weinstein Peter Scholze. *Berkeley Lectures on p-adic Geometry*. Annals of Mathematics Studies. Princeton University Press, 2020.

[8] Georgios Pappas and Michael Rapoport. p-adic shtukas and the theory of global and local Shimura varieties. *arXiv e-prints*, page arXiv:2106.08270, June 2021.

[9] Si Ying Lee. Eichler-Shimura Relations for Shimura Varieties of Hodge Type. *arXiv e-prints*, page arXiv:2006.11745, June 2020.

[10] Gerd Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer, Berlin, Germany, December 2010.

[11] Torstein Wedhorn. Congruence relations on some shimura varieties. 2000(524):43–71, 2000.
[12] Peter Scholze. Etale cohomology of diamonds. *arXiv e-prints*, page arXiv:1709.07343, September 2017.
[13] Xinwen Zhu. Affine grassmannians and the geometric satake in mixed characteristic. *Annals of Mathematics*, 185(2):403–492, March 2017.
[14] Kiran Kedlaya and Ruochuan Liu. Relative p-adic hodge theory: Foundations. *Astérisque*, 2015.
[15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, 2015, 09 2013.
[16] Tamir Hemo, Timo Richarz, and Jakob Scholbach. Constructible sheaves on schemes and a categorical Küneth formula. *arXiv e-prints*, page arXiv:2012.02853, December 2020.
[17] P. Deligne. Hodge cycles on abelian varieties. In *Lecture Notes in Mathematics*, pages 9–100. Springer Berlin Heidelberg, 1982.
[18] Mark Kisin. Integral models for shimura varieties of abelian type. *Journal of the American Mathematical Society*, 23(4):967–1012, April 2010.
[19] Peter Scholze. p-adic Hodge theory for rigid-analytic varieties. *Forum of Mathematics, Pi*, 1, 2013.
[20] D. Blasius. A p-adic property of hodge classes on abelian varieties. In *Motives*, page 293–308. Amer. Math. Soc, 1994.
[21] Peter Scholze and Jared Weinstein. Moduli of p-divisible groups. *Cambridge Journal of Mathematics*, 1(2):145–237, 2013.
[22] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic hodge theory. *Publications mathématiques de l'Institut des Hautes Études Scientifiques*, 128(1):219–397, November 2018.
[23] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine grassmannian. *Inventiones mathematicae*, 209(2):329–423, December 2016.
[24] Johannes Anschütz. Extending torsors on the punctured Spec(A_inf). *arXiv e-prints*, page arXiv:1804.06356, April 2018.
[25] Kiran S. Kedlaya. Sheaves, stacks, and shtukas. In *Perfectoid Spaces: Lectures from the 2017 Arizona Winter School*. American Mathematical Society, 2019.
[26] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics. Vieweg+Teubner Verlag, Wiesbaden, Germany, April 2013.
[27] Kai-Wen Lan and Benoît Stroh. Nearby cycles of automorphic étale sheaves. *Compositio Mathematica*, 154(1):80–119, 2018.
[28] Liang Xiao and Xinwen Zhu. On vector-valued twisted conjugate invariant functions on a group. *arXiv e-prints*, page arXiv:1802.05299, February 2018.
[29] Yifeng Liu and Weizhe Zheng. Enhanced adic formalism and perverse t-structures for higher Artin stacks. *arXiv e-prints*, page arXiv:1404.1128, April 2014.