Remarks on the geodesic-Einstein metrics of a relative ample line bundle

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Abstract
In this paper, we introduce the associated geodesic-Einstein flow for a relative ample line bundle \( L \) over the total space \( \mathcal{X} \) of a holomorphic fibration and obtain a few properties of that flow. In particular, we prove that the pair \( (\mathcal{X}, L) \) is nonlinear semistable if the associated Donaldson type functional is bounded from below and the geodesic-Einstein flow has long-time existence property. We also define the associated \( S \)-classes and \( C \)-classes for \( (\mathcal{X}, L) \) and obtain two inequalities between them when \( L \) admits a geodesic-Einstein metric. Finally, in the appendix of this paper, we prove that a relative ample line bundle is geodesic-Einstein if and only if an associated infinite rank bundle is Hermitian–Einstein.

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Introduction

Let $E$ be a holomorphic vector bundle over a compact Kähler manifold $(M, \omega)$. A Hermitian metric $h$ on $E$ is called a Hermitian–Einstein metric if $\Lambda_\omega F_h = \lambda \mathrm{Id}$ for some constant $\lambda$, where $F_h = \bar{\partial} (\partial h \cdot h^{-1}) \in A^{1,1}(M, \mathrm{End}(E))$ is the Chern curvature of the Hermitian metric $h$. The famous Donaldson-Uhlenbeck-Yau Theorem reveals the deep relationship between the stability of a holomorphic vector bundle and the existence of Hermitian–Einstein metrics (cf. [12–14,26,34]). On the other hand, in the paper [19], we introduced the notion of the geodesic-Einstein flow (0.1) is reduced to the Hermitian–Yang–Mills flow (see Remark 2.4). We prove the uniqueness of the solutions of (0.1) and establish some estimates in this paper (see Propositions 2.2 and 2.3). For any fixed metric $\psi \in F^+(L)$ on $L$ and any $\phi \in F^+(L)$, we also define a Donaldson type functional

$$L(\phi, \psi) = \int_M \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \land \omega - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \frac{\omega^{n-1}}{(m-1)!},$$

(0.2)

where such that every fiber is a compact complex manifold.

Let $(X, L)$ be a relative ample line bundle over $X$. For any holomorphic vector bundle $E$, there is a canonical associated projective bundle pair $(P(E), \mathcal{O}_{P(E)}(1))$. A geodesic-Einstein metric $\phi$ on $\mathcal{O}_{P(E)}(1)$ is a metric satisfying the geodesic-Einstein equation $tr_{\omega c}(\phi) = \lambda_{X, \mathcal{O}_{P(E)}(1)}$ (cf. Definition 1.2). In this case, a geodesic-Einstein metric is same as a Finsler–Einstein metric by the natural one-to-one correspondence between Finsler metrics on $E$ and the metrics on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ (cf. [23, p. 82], [19]). Combining with [19, Theorem 0.3], it shows that the existence of geodesic-Einstein metrics is equivalent to the existence of Hermitian–Einstein metrics, which is also equivalent to polystability of the holomorphic vector bundle.

More generally, we consider a holomorphic fibration $\pi : \mathcal{X} \to M$ over a compact complex manifold $M$ with compact fibers, that is, $\pi : \mathcal{X} \to M$ is a proper surjective holomorphic mapping between complex manifolds $\mathcal{X} \to M$ whose differential has maximal rank everywhere such that every fiber is a compact complex manifold. Let $L$ be a relative ample line bundle over $\mathcal{X}$ and let $F^+(L)$ denote the space of all metrics $\phi$ such that $\sqrt{-1} \partial \bar{\partial} \phi |_{\pi^{-1}(y)} > 0$ for any $y \in M$. A metric $\phi \in F^+(L)$ is geodesic-Einstein if

$$tr_{\omega c}(\phi) = \lambda_{\mathcal{X}, L}.$$
where $\lambda$ is the constant given by (1.4), $m = \dim M$, $n = \dim X - \dim M$. Here $E(\phi, \psi)$ and $E_1(\phi, \psi)$ are defined by (2.1) and (2.2). We can prove that the critical points of $L(\cdot, \psi)$ are exactly the geodesic-Einstein metrics.

Now we recall the definition of the nonlinear semistable pair $(X, L)$. A fibration $Y \rightarrow M - S$, where $S$ is a closed subvariety in $M$ of codim $S \geq 2$, is called a sub-fibration of the holomorphic fibration $X \rightarrow M$ if for any $p \in M - S$, the fiber $Y_p$ is a closed complex submanifold of the fiber $X_p$. Let $\mathcal{F}$ be the set of sub-fibrations of the holomorphic fibration $X \rightarrow M$. For any $Y \in \mathcal{F}$, we set

$$\lambda_{Y, L} = \frac{2\pi m}{\dim Y/M + 1} \frac{\left( [\omega]^{m-1} c_1(L)^{\dim Y/M+1} \right)[Y]}{\left( [\omega]^m c_1(L)^{\dim Y/M} \right)[Y]}.$$

Note that $\lambda_{Y, L}$ is well-defined and independent of the metrics on $L$ by Stokes’ theorem and codim $S \geq 2$. Similar to the semistability of a holomorphic vector bundle, a pair $(X, L)$ is called nonlinear semistable if $\lambda_{Y, L} \geq \lambda_{X, L}$ for any sub-fibration $Y \in \mathcal{F}$ (see Definition 2.5).

A pair $(X, L)$ admits an approximate geodesic-Einstein structure if for any given $\epsilon > 0$, there exists an metric $\phi_\epsilon$ on $L$ such that

$$\max_M \left| \int_{X/M} \left( tr_{\omega, c}(\phi_\epsilon) - \lambda \right)^2 (\sqrt{-1} \partial \bar{\partial} \phi_\epsilon) \right|^\frac{1}{2} < \epsilon.$$

**Theorem 0.1** Suppose that the geodesic-Einstein flow (0.1) has a smooth solution for $0 \leq t < +\infty$, then we have implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for the following statements:

1. the functional $L$ is bounded from below;
2. $L$ admits an approximate geodesic-Einstein structure;
3. the pair $(X, L)$ is nonlinear semistable.

We also introduce the $S$-class and $C$-class associated to the pair $(X, L)$, which are the generalizations of the Segre class and Chern class. For a general pair $(X, L)$, we define the total $S$-class and the $S$-classes of $L$ by

$$S(L) = \sum_{i=0}^m S_i(L), \quad S_i(L) = \pi_*((c_1(L)^{n+i}) \in H^{2i}(M, \mathbb{Z}), \quad 0 \leq i \leq m,$$

and the total $C$-class and the $C$-classes are defined by

$$C(L) = \frac{1}{S(L)}, \quad C(L) = \sum_{i=0}^m C_i(L), \quad C_i(L) \in H^{2i}(M, \mathbb{Q}).$$

**Theorem 0.2** (1) If $\phi$ is a geodesic-Einstein metric on $L$, i.e., $tr_{\omega, c}(\phi) = \lambda_{X, L}$, then

$$S_2(L, \phi) \wedge \omega^{n-2} \leq \frac{(n+1)(n+2)}{8\pi^2 m^2} \lambda_{X, L}^2 S_0(L) \omega^n,$$

the equality holds if and only if $c(\phi) = \frac{\lambda_{X, L}}{m} \omega$. In particular,

$$\int_M S_2(L) \wedge [\omega]^{n-2} \leq \frac{(n+1)(n+2)}{8\pi^2 m^2} \lambda_{X, L}^2 S_0(L) \int_M [\omega]^n.$$

(2) If $\phi$ is a geodesic-Einstein metric on $L$, then

$$(nC_1(L, \phi)^2 - 2(n+1)C_0(L)C_2(L, \phi)) \wedge \omega^{n-2} \leq 0.$$
the equality holds if and only if $c(\phi) = \frac{2\pi}{(n+1)S_0(L)} S_1(L, \phi)$. In particular,

$$\int_M (nC_1(L)^2 - 2(n + 1)C_0(L)C_2(L)) \wedge [\omega]^{m-2} \leq 0.$$ 

We also consider the Hermitian–Einstein metrics on a quasi-vector bundle. Let $\pi : X \to B$ be a proper holomorphic submersion from a complex manifold $X$ to another complex manifold $B$ (need not compact), each fiber $X_t := \pi^{-1}(t)$ is an $n$-dimensional compact complex manifold; $E$ a holomorphic vector bundle over $X$, $E_t := E|_{X_t}$; $\omega$ a $d$-closed $(1,1)$-form on $X$ and is positive on each fiber, $\omega^t := \omega|_{X_t}; h_E$ is a smooth Hermitian metric on $E$, $h_{E_t} := h_E|_{E_t}$. For each $t \in B$, let us denote by $A^{p,q}(E_t)$ the space of smooth $E_t$-valued $(p,q)$-forms on $X_t$. Denote $A^{p,q} := \{A^{p,q}(E_t)\}_{t \in B},$ and let $(A, \Gamma)$ denote the corresponding quasi-vector bundle (see Sect. 6). There is a natural Chern connection $D^A$ on each $A^{p,q}$ with respect the standard $L^2$-metric. The $L^2$-metric on $A^{p,q}$ is called Hermitian–Einstein with respect to a Hermitian metric $\omega_B = \sqrt{-1}g_{i\bar{j}} dt^i \wedge d\bar{t}^j$ if

$$A_{\omega_B}(D^A)^2 = \lambda \text{Id} \quad (0.3)$$

for some constant $\lambda$ (see Definition 6.3). Now we consider the case $p = q = 0$ and $E$ is the trivial bundle. Let $L$ be a relative ample line bundle over $X$, i.e. there exists a metric $\phi$ on $L$ such that its curvature $\omega := \sqrt{-1}d\bar{\partial}\phi$ is positive along each fiber. The $L^2$-metric on $A^{0,0}$ is given by

$$(u, v) = \int_{X_t} u\bar{v} \omega^n \frac{n!}{n^n}. \quad (0.4)$$

We call a metric $\phi$ on $L$ is weak geodesic-Einstein with respect to $\omega_B$ if $tr_{\omega_B} c(\phi) = \pi^* f (z)$ for some function $f (z)$ on $B$. Then

**Proposition 0.3** $\phi$ is a weak geodesic-Einstein metric on $L$ if and only if the $L^2$-metric (0.4) is a Hermitian–Einstein metric on $A^{0,0}$. In particular, if $B$ is compact, then up to a smooth function on $B$, $\phi$ is a geodesic-Einstein metric on $L$ if and only if the $L^2$-metric (0.4) is a Hermitian–Einstein metric on $A^{0,0}$.

This article is organized as follows. In Sect. 1, we will recall some basic definitions and facts on geodesic-Einstein metrics of a relative ample line bundle over a holomorphic fibration. For more details one may refer to [19,36].

## 1 Preliminaries

In this section, we will recall some basic definitions and facts on geodesic-Einstein metrics of a relative ample line bundle over holomorphic fibration. For more details one may refer to [19,36].

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Let $\pi : \mathcal{X} \to M$ be a holomorphic fibration over a compact complex manifold $M$ with compact fibers. We denote by $(z, v) = (z^1, \ldots, z^m; v^1, \ldots, v^n)$ a local admissible holomorphic coordinate system of $\mathcal{X}$ with $\pi(z; v) = z$, where $m = \text{dim}_\mathbb{C} M$, $n = \text{dim}_\mathbb{C} \mathcal{X} - \text{dim}_\mathbb{C} M$.

For any smooth function $\phi$ on $\mathcal{X}$, we denote

$$\phi_\alpha := \frac{\partial \phi}{\partial z^\alpha}, \quad \phi_\beta := \frac{\partial \phi}{\partial z^\beta}, \quad \phi_i := \frac{\partial \phi}{\partial v^i}, \quad \phi_j := \frac{\partial \phi}{\partial \bar{v}^j},$$

where $1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq m$.

For any holomorphic line bundle $L$ over $\mathcal{X}$, we denote by $F^+(L)$ the space of smooth metrics $\phi$ on $L$ with $$(\sqrt{-1}\partial \bar{\partial} \phi)|_\mathcal{X} > 0$$
for any point $y \in M$. Now we assume that $L$ is a relative ample line bundle, i.e. $F^+(L) \neq \emptyset$. For any $\phi \in F^+(L)$, set

$$\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - N^k_i \frac{\partial}{\partial v^k},$$

(1.1)

where $N^k_i = \phi_{ij} \phi^{jk}$. By a routine computation, one can show that $\{ \frac{\delta}{\delta z^\alpha} \}_{1 \leq \alpha \leq m}$ spans a well-defined horizontal subbundle of $T\mathcal{X}$ (see [19, Sect. 1]).

Let $\{dz^\alpha; \delta v^k\}$ denote the dual frame of $\left\{ \frac{\delta}{\delta z^\alpha} : \frac{\partial}{\partial v^i} \right\}$. Then

$$\delta v^k = dv^k + \phi^{ij} \phi_{ij} d\bar{z}^\alpha.$$  

Moreover, the differential operators

$$\partial^V = \frac{\partial}{\partial v^i} \otimes \delta v^i, \quad \partial^H = \frac{\delta}{\delta z^\alpha} \otimes dz^\alpha.$$  

(1.2)

are well-defined.

For any $\phi \in F^+(L)$, the geodesic curvature $c(\phi)$ is defined by

$$c(\phi) = \left( \phi_{\alpha \beta} - \phi_{j \beta} \phi^{ij} \phi_{i \beta} \right) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta,$$

which is a horizontal real $(1, 1)$-form on $\mathcal{X}$. Then we have the following decomposition.

**Lemma 1.1** ([19, Lemma 1.1]) The following decomposition holds,

$$\sqrt{-1} \partial \bar{\partial} \phi = c(\phi) + \sqrt{-1} \phi_{ij} \delta v^i \wedge \delta \bar{v}^j.$$  

From [19, Definition 1.2], the geodesic-Einstein metric is defined as follows.

**Definition 1.2** Let $\omega = \sqrt{-1}g_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta$ be a (fixed) Kähler metric on $M$. A metric $\phi \in F^+(L)$ is called a geodesic-Einstein metric on $L$ with respect to $\omega$ if it satisfies that

$$\text{tr}_\omega c(\phi) := g^{\alpha \bar{\beta}} \left( \phi_{\alpha \beta} - \phi_{j \beta} \phi^{ij} \phi_{i \beta} \right) = \lambda,$$

(1.3)

where $\lambda$ is a constant. By [19, Proposition 1.3], if $M$ is compact, $\lambda$ is a topological quantity, which is given by

$$\lambda = \frac{2\pi m}{n + 1} \left( \frac{[\omega]^{m-1} c_1(L)^{n+1} |\mathcal{X}|}{[\omega]^{m} c_1(L)^n |\mathcal{X}|} \right).$$  

(1.4)
2 Geodesic-Einstein flow and nonlinear semistable

For any fixed metric $\psi \in F^+(L)$ on $L$ and any $\phi \in F^+(L)$, one can define the following two functionals $\mathcal{E}, \mathcal{E}_1$:

\[
\mathcal{E}(\phi, \psi) = \frac{1}{n+1} \int_{X/M} (\phi - \psi) \sum_{k=0}^{n} (\sqrt{-1} \bar{\partial} \phi)^k \wedge (\sqrt{-1} \bar{\partial} \psi)^{n-k} \quad (2.1)
\]

and

\[
\mathcal{E}_1(\phi, \psi) = \frac{1}{n+2} \int_{X/M} (\phi - \psi) \sum_{k=0}^{n+1} (\sqrt{-1} \bar{\partial} \phi)^k \wedge (\sqrt{-1} \bar{\partial} \psi)^{n+1-k} \quad (2.2)
\]

Note that $\mathcal{E}(\phi, \psi)$ is a smooth function, while $\mathcal{E}_1(\phi, \psi)$ is a smooth real $(1,1)$-form on $M$.

From [19, (1.14)], the Donaldson type functional $L$ on $F^+(L)$ is defined by

\[
L(\phi, \psi) = \int_{M} \left( \frac{\lambda}{m} \mathcal{E}(\phi, \psi) \wedge \omega - \frac{1}{n+1} \mathcal{E}_1(\phi, \psi) \right) \frac{\omega^{m-1}}{(m-1)!}, \quad (2.3)
\]

where $\lambda$ is the constant given by (1.4). Let $\phi_t$ be a smooth family of metrics depends on $t$, then the first variation of Donaldson type function is given by

\[
\frac{d}{dt} L(\phi_t, \psi) = - \int_{X} \hat{\phi}_t (tr_{\omega} c(\phi_t) - \lambda)(\sqrt{-1} \bar{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!}, \quad (2.4)
\]

(see [19, (1.15)]). So $\phi \in F^+(L)$ is a geodesic-Einstein metric if and only if it is a critical point of $L(\cdot, \psi)$ on $F^+(L)$ (see [19, Proposition 1.4]).

In order to make $\frac{d}{dt} L(\phi_t, \psi) \leq 0$, it is natural to consider the following geodesic-Einstein flow

\[
\left\{ \begin{array}{l}
\frac{\partial \phi}{\partial t} = tr_{\omega} c(\phi) - \lambda \\
\phi \in F^+(L) \\
\phi(0) = \phi_0
\end{array}\right. \quad (2.5)
\]

for an initial metric $\phi_0 \in F^+(L)$. For the convenience, we also denote $\dot{\phi}_t := \frac{\partial \phi}{\partial t}$ and $\ddot{\phi}_t := \frac{\partial^2 \phi}{\partial t^2}$.

2.1 Some properties of the geodesic-Einstein flow

In this subsection, we will study some properties of the geodesic-Einstein flow (2.5) by using the method of studying Kähler-Ricci flow (see e.g. [9,33]).

For any smooth function $f \in C^\infty(X)$, we denote the horizontal and vertical Laplacian by

\[
\Delta_{\omega} f := g^{\alpha \bar{\beta}} (\partial_{\bar{\beta}} f) \left( \frac{\delta}{\delta z^\alpha} \frac{\delta}{\delta \bar{z}^\beta} \right) , \quad \Delta_{\phi} f := \phi^{k\bar{l}} \frac{\partial^2 f}{\partial v^k \partial \bar{v}^l} , \quad (2.6)
\]

respectively. The following proposition is a maximum (minimum) principle for degenerate parabolic elliptic equations. Its proof is the same as [9, Proposition 3.1.7].

Proposition 2.1 ([9, Proposition 3.1.7]) Fix $T$ with $0 < T \leq \infty$. Suppose that $f = f(x, t)$ is a smooth function on $X \times [0, T)$ satisfying

\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega} \right) f \leq 0 \quad (resp. \geq 0). \quad (2.7)
\]
Then \( \sup_{(x,t) \in \mathcal{X} \times [0,T)} f(x, t) \leq \sup_{x \in \mathcal{X}} f(x, 0) \) (resp. \( \inf_{(x,t) \in \mathcal{X} \times [0,T)} f(x, t) \geq \inf_{x \in \mathcal{X}} f(x, 0) \)).

By above proposition, we can prove the uniqueness of the solutions of the flow (2.5).

**Proposition 2.2** If \( \phi(t) \) and \( \psi(t) \) are the two solutions of the flow (2.5), then \( \phi(t) = \psi(t) \).

**Proof** We assume that \( T_{\max} \) is the maximum existence time and let \( \tau < T_{\max} \). Let \( \phi(t) \) and \( \psi(t) \) be the two solutions with the same initial metric \( \phi_0 \). Then

\[
(\phi - \psi)'_t = tr_{\omega}c(\phi) - tr_{\omega}c(\psi), \quad (\phi - \psi)(0) = 0.
\]

We assume that on \( \mathcal{X} \times [0, \tau], \) the maximum of \( \phi - \psi - \epsilon t \) is taken at \( (x, t) \) in \( \mathcal{X} \times [0, \tau], \) \( t > 0 \). Then

\[
0 \leq (\phi - \psi - \epsilon t)'_t = tr_{\omega}c(\phi) - tr_{\omega}c(\psi) - \epsilon \tag{2.8}
\]

and \(-\sqrt{-1} \partial \bar{\partial} (\phi - \psi - \epsilon t) = -\sqrt{-1} \partial \bar{\partial} (\phi - \psi) \) is semi-positive \((1, 1)\)-form at the point \((x, t)\). Denote by \( \frac{\delta \phi}{\delta z^a} \) (resp. \( \frac{\delta \psi}{\delta z^a} \)) the horizontal lifts with respect to \( \phi \) (resp. \( \psi \)), which are given by (1.1). Then at the point \((x, t)\), one has

\[
tr_{\omega}c(\psi) = g^{\bar{a}\bar{b}} (\partial \bar{\partial} \psi) \left( \frac{\delta \psi}{\delta z^a}, \frac{\delta \psi}{\delta z^b} \right) \\
\geq g^{\bar{a}\bar{b}} (\partial \bar{\partial} \phi) \left( \frac{\delta \psi}{\delta z^a}, \frac{\delta \psi}{\delta z^b} \right) \\
= g^{\bar{a}\bar{b}} (\partial \bar{\partial} \phi) \left( \frac{\delta \psi}{\delta z^a} - \frac{\delta \phi}{\delta z^a} \right) + \frac{\delta \phi}{\delta z^a} \left( \frac{\delta \psi}{\delta z^b} - \frac{\delta \phi}{\delta z^b} \right) + \frac{\delta \phi}{\delta z^b} \right) \\
= tr_{\omega}c(\phi) + g^{\bar{a}\bar{b}} (\partial \bar{\partial} \phi) \left( \frac{\delta \psi}{\delta z^a} - \frac{\delta \phi}{\delta z^a}, \frac{\delta \psi}{\delta z^b} - \frac{\delta \phi}{\delta z^b} \right) \geq tr_{\omega}c(\phi), \tag{2.9}
\]

where the fourth equality holds by Lemma 1.1 and noting that \( \frac{\delta \psi}{\delta z^b} - \frac{\delta \phi}{\delta z^b} \) is a vertical vector, the last inequality follows from \( \phi \in F^+(L) \).

Substituting (2.9) into (2.8) we get a contradiction. So the maximum of \( \phi - \psi - \epsilon t \) is taken at \( t = 0 \), i.e.

\[
\max_{x \in \mathcal{X} \times [0, \tau]} (\phi - \psi - \epsilon t) = \max_{x \in \mathcal{X}} (\phi(0) - \psi(0)) = 0.
\]

Thus

\[
\phi - \psi \leq \epsilon t \leq \epsilon \tau.
\]

It follows that

\[
\phi \leq \psi
\]

for any \([0, \tau], \tau < T_{\max} \). Thus \( \phi \leq \psi \) on \([0, T_{\max}] \). Similarly, we have

\[
\psi \leq \phi.
\]

Therefore, \( \psi(t) = \phi(t) \).

\[ \square \]

Also by Proposition 2.1, we obtain the following estimates.
Proposition 2.3 Along the flow (2.5), one has
\[ |tr_\omega c(\phi)| < C \quad \text{and} \quad |\phi(t) - \phi_0| \leq Ct \] (2.10)
for some constant \( C > 0. \)

Proof By Lemma 1.1 and (2.5), one has
\[
(\phi_t + \lambda)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n = tr_\omega c(\phi)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n = \frac{m}{n+1} \omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^{n+1}.
\] (2.11)

Taking derivative on \( t \) to the both sides of above equation, then
\[
\frac{d}{dt}((\phi_t + \lambda)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n)
= \phi_t \omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n + (\phi_t + \lambda)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi_t)
= (\phi_t + tr_\omega c(\phi))\phi_t \omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n
\] (2.12)
and
\[
\frac{m}{n+1} \frac{d}{dt} \omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^{n+1}
= m\omega^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n \wedge (\sqrt{-1}\partial\bar{\partial}\phi_t)
= m\omega^{m-1} \wedge (c(\phi) + \sqrt{-1}\phi_i \delta v^i \wedge \delta \bar{v}^j)^n \wedge (\sqrt{-1}\partial\bar{\partial}\phi_t)
= (\Delta \omega \phi_t + tr_\omega c(\phi)\phi_t)\omega^m \wedge (\sqrt{-1}\partial\bar{\partial}\phi)^n.
\] (2.13)

From (2.12) and (2.13), one has
\[
\left( \frac{\partial}{\partial t} - \Delta_\omega \right) \phi_t = 0.
\] (2.14)

From Proposition 2.1, one has
\[ |\phi_t| < C_1 \] (2.15)
for some constant \( C_1 > 0. \) It follows that
\[ |tr_\omega c(\phi)| = |\phi_t + \lambda| \leq |\phi_t| + |\lambda| \leq C_1 + |\lambda| =: C. \] (2.16)

By (2.15), one has
\[
|\phi(t) - \phi_0| = \left| \int_0^t \phi_t dt \right| \leq \int_0^t |\phi_t| dt \leq C_1 t \leq Ct.
\] (2.17)

\( \square \)

Remark 2.4 If the total space \( X \) is a projective bundle and \( L \) is a hyperplane line bundle over \( X \), then this flow (2.5) has been studied in [35]. More precisely, let \( E \to M \) be a holomorphic vector bundle over \( M \), \( X := P(E) \) denotes the projective bundle of \( E \), and \( L = O_{P(E)}(1) \) the hyperplane line bundle. Let \( h \) be a Hermitian metric on \( E \), then it induces a metric on the line bundle \( L = O_{P(E)}(1) \) by
\[
\phi' = -\log \frac{|v'|^2}{G}
\]
where $G = \sum_{i,j=1}^{r} h_{ij} \bar{v}^i \bar{v}^j$. So $G_{ij} := \frac{\partial^2 G}{\partial v^i \partial v^j} = h_{ij}$ is independent of the fibers. Suppose that the initial metric $\phi_0$ is induced from a Hermitian metric $h_0$ on $E$, then the flow (2.5) is equivalent to

$$0 = \frac{\partial \phi}{\partial t} - tr_{\omega_{(\phi)}}(\phi) + \lambda$$

$$= \frac{1}{G} \frac{\partial G}{\partial t} - g^{\alpha\beta}(\phi_{\alpha\beta} - \phi_{kl} \phi_{\alpha l}) + \lambda$$

$$= \frac{\bar{v}^i \bar{v}^j}{G} \left( \frac{\partial G_{ij}}{\partial t} - g^{\alpha\beta}(G_{ij})_{\alpha\beta} - G_{kl} G_{\alpha l} + \lambda G_{ij} \right),$$

(2.18)

By the argument of [35, Remark 2.1] or by the uniqueness of solution (Proposition 2.2 and [13, Corollary 1.4]), (2.18) is reduced to

$$G^{-1} \cdot \frac{\partial G}{\partial t} : = G^{ij} \frac{\partial G}{\partial t} = -g^{\alpha\beta} F^j_{\alpha\beta} + \lambda \delta^j_i = -\Lambda F + \lambda I,$$

(2.19)

which is exactly the Hermitian–Yang–Mills flow (cf. [2,13]), where the curvature operator $F$ is defined by

$$F := \sqrt{-1} \partial (\partial G \cdot G^{-1}) \in A^{1,1}(M, \text{End}(E)).$$

So the flow (2.5) is indeed a natural generalization of Hermitian–Yang–Mills flow. By [13, Proposition 20], the Hermitian–Yang–Mills flow (2.19) has a unique smooth solution for $0 \leq t < +\infty$. Immediately, one natural and interesting problem is that whether the geodesic-Einstein flow (2.5) has a smooth solution for $0 \leq t < +\infty$.

### 2.2 Nonlinear semistable

In this subsection, we will assume that the geodesic-Einstein flow (2.5) has a smooth solution for $0 \leq t < +\infty$ and consider nonlinear semistability of a pair $(\mathcal{X}, L)$ (see Definition 2.7).

Firstly, we recall the definition of nonlinear semistable. A fibration $\mathcal{Y} \to M - S$, with $S$ a closed subvariety in $M$ of codim $S \geq 2$, is called a sub-fibration of the holomorphic fibration $\mathcal{X} \to M$ if for any $p \in M - S$, the fiber $\mathcal{Y}_p$ is a closed complex submanifold of the fiber $\mathcal{X}_p$. Let $\mathcal{F}$ be the set of sub-fibrations of the holomorphic fibration $\mathcal{X} \to M$. For any $\mathcal{Y} \in \mathcal{F}$, we set

$$\lambda_{\mathcal{Y},L} = \frac{2\pi m}{\dim \mathcal{Y}/M + 1} \frac{([\omega]^{m-1} c_1(L)^{\dim \mathcal{Y}/M+1})[\mathcal{Y}]}{([\omega]^{m} c_1(L)^{\dim \mathcal{Y}/M})[\mathcal{Y}]}.$$  

(2.20)

Note that $\lambda_{\mathcal{Y},L}$ is well-defined and independent of the metrics on $L$ by the Stoke’s theorem and codim $S \geq 2$. Similar to the semistability of a holomorphic vector bundle, the nonlinear semistable of a pair $(\mathcal{X}, L)$ is given by the following.

**Definition 2.5** ([19, Definition 2.1]) A pair $(\mathcal{X}, L)$ is called nonlinear semistable if $\lambda_{\mathcal{Y},L} \geq \lambda_{\mathcal{X},L}$ for any sub-fibration $\mathcal{Y} \in \mathcal{F}$.

Now we assume that the flow (2.5) has a smooth solution for $0 \leq t < +\infty$, we obtain

**Proposition 2.6** Suppose that the geodesic-Einstein flow (2.5) has a smooth solution $\phi_t = \phi(t)$ for $0 \leq t < +\infty$, then
The Donaldson type functional is monotone decreasing function of $t$; in fact,
\[ \frac{d}{dt} \mathcal{L}(\phi(t), \phi_0) = - \int_{\mathcal{X}} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} \leq 0. \tag{2.21} \]

(2) Max $\chi(t)(tr_{\omega}c(\phi) - \lambda)^2$ is a monotone decreasing function of $t$.
(3) If $\mathcal{L}(\phi(t), \phi_0)$ is bounded below, i.e., $\mathcal{L}(\phi(t), \phi_0) \geq A > -\infty$ for $0 \leq t < +\infty$, then
\[ \max_M \int_{\mathcal{X}/M} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \rightarrow 0 \]
as $t \rightarrow +\infty$.

**Proof** (1) Substituting (2.5) into the first variation (2.4) of the Donaldson type functional, one has
\[ \frac{d}{dt} \mathcal{L}(\phi(t), \phi_0) = - \int_{\mathcal{X}} (\partial_t (tr_{\omega}c(\phi_t) - \lambda)(\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} ) \]
\[ = - \int_{\mathcal{X}} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} \leq 0. \]

(2) By a direct calculation, one has
\[ \frac{1}{2} \left( \frac{\partial}{\partial t} - \Delta_\omega \right) (tr_{\omega}c(\phi) - \lambda)^2 = \frac{1}{2} \left( \frac{\partial}{\partial t} - \Delta_\omega \right) \phi_t^2 \]
\[ = \phi_t \phi_{tt} - \frac{1}{2} g^{\alpha \beta} (\partial_t \phi_t^2) \left( \frac{\delta}{\delta z^\alpha} \phi_t \phi_{t\bar{\beta}} \right) \]
\[ = \phi_t \phi_{tt} - \phi_t \phi_t \Delta_\omega \phi_t - |\partial H \phi_t|^2 \]
\[ = \phi_t \left( \frac{\partial}{\partial t} - \Delta_\omega \right) \phi_t - |\partial H \phi_t|^2 \]
\[ = -|\partial H \phi_t|^2 \leq 0, \]
where the last equality holds by (2.14). By Proposition 2.1, we complete the proof.

(3) Integrating (2.21) from 0 to $s$, we obtain
\[ \mathcal{L}(\phi(s), \phi_0) - \mathcal{L}(\phi(0), \phi_0) = - \int_0^s \int_{\mathcal{X}} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} ds \]
Since the $\mathcal{L}$ is bounded below by a constant independent of $s$, we have
\[ \int_0^\infty \int_{\mathcal{X}} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} ds < \infty. \]
In particular,
\[ \int_{\mathcal{X}} (tr_{\omega}c(\phi_t) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi_t)^n \wedge \frac{\omega^m}{m!} \rightarrow 0 \quad t \rightarrow \infty. \tag{2.22} \]

Let $H(z, w, t)$ be the heat kernel for $\partial_t - \Delta_\omega$ when acting on $C^\infty(M)$. Set
\[ F(z, t) = \left( \int_{\mathcal{X}/M} (tr_{\omega}c(\phi) - \lambda)^2 (\sqrt{-1} \partial \overline{\partial} \phi)^n \right)(z), \quad (z, t) \in M \times [0, \infty). \]
Fix $t_0 \in [0, \infty)$ and set
\[ u(z, t) = \int_M H(z, w, t - t_0) F(w, t_0) dw, \quad dw = \frac{\omega^m}{m!}. \]

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Then \( u(z, t) \) is of class \( C^\infty \) on \( M \times (t_0, \infty) \) and extends to a continuous function on \( M \times [t_0, \infty) \). It satisfies
\[
\begin{cases}
(\partial_t - \Delta_{\omega})u(z, t) = 0 & (x, t) \in M \times (t_0, \infty), \\
u(z, t_0) = F(z, t_0) & z \in M.
\end{cases}
\]
And we have
\[
\frac{\partial F(z, t)}{\partial t} = \pi_*(\left( \frac{\partial}{\partial t}(\text{tr}_{\omega} c(\phi_t) - \lambda)^2 + \Delta_{\phi}\dot{\phi}(\text{tr}_{\omega} c(\phi_t) - \lambda)^2 \right) (\sqrt{-1} \partial \bar{\partial} \phi)^n)
\]
and
\[
\Delta_{\omega} F(z, t) = \pi_*(((\Delta_{\omega}(\text{tr}_{\omega} c(\phi_t) - \lambda)^2 + \Delta_{\phi}(\text{tr}_{\omega} c(\phi_t) - \lambda)^2 \text{tr}_{\omega} c(\phi)))(\sqrt{-1} \partial \bar{\partial} \phi)^n).
\]
By (2.14) and Stoke’s Theorem, one has
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega} \right) F(z, t) - u(z, t) \leq 0, \quad (z, t) \in M \times (t_0, \infty).
\]

By Proposition 2.1, one has
\[
\max_{z \in M} (F(z, t) - u(z, t)) \leq \max_{z \in M} (F(z, t_0) - u(z, t_0)) = 0, \quad t \geq t_0.
\]
It follows that
\[
\max_{z \in M} F(z, t_0 + a) \leq \max_{z \in M} u(z, t_0 + a) \leq \max_{z \in M} \int_M H(z, w, a) F(w, t_0) dw \leq C_a \int_M F(w, t_0) dw,
\]
where \( C_a = \max_{M \times M} H(z, w, a) \). Fix \( a \) and let \( t_0 \to \infty \), we conclude
\[
\max_{z \in M} F(z, t) \to 0 \quad t \to \infty,
\]
which competes the proof.

Inspired by Proposition 2.6 (3), we give the following definition of approximate geodesic-Einstein structure, which is similar as the approximate Hermitian–Einstein structure for a holomorphic vector bundle (see e.g. [23, Sect. 4.5]).
Definition 2.7  We say that $L$ admits an approximate geodesic-Einstein structure, if for any given $\epsilon > 0$, there exists an metric $\phi_\epsilon$ on $L$ such that
\[
\max_M \left| \int_{\mathcal{X}/M} (tr_\omega c(\phi_\epsilon) - \lambda)^2(\sqrt{-1}\partial\bar\partial \phi_\epsilon)^n \right|^\frac{1}{2} < \epsilon.
\]

By Proposition 2.6 (3) and above definition, we obtain the following theorem.

Theorem 2.8  Suppose that the geodesic-Einstein flow has a smooth solution for $0 \leq t < +\infty$, then we have implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for the following statements:

(1) the functional $\mathcal{L}$ is bounded from below;
(2) $L$ admits an approximate geodesic-Einstein structure;
(3) the pair $(\mathcal{X}, L)$ is nonlinear semistable.

Proof  By Proposition 2.6 (3) and Definition 2.7, then $\mathcal{L}$ admits an approximate geodesic-Einstein structure. Now we begin to prove that a pair $(\mathcal{X}, L)$ is nonlinear semistable if $L$ admits an approximate geodesic-Einstein structure. In fact,
\[
\lambda_{\mathcal{X}, L} \geq \frac{(tr_\omega c(\phi)[\omega]^m c_1(L)^n)^{[\mathcal{Y}]} - (tr_\omega c(\phi)[\omega]^m c_1(L)^n)^{[\mathcal{Y}]}}{(\omega)^m c_1(L)^n)^{[\mathcal{Y}]}}
\]
\[
= \lambda_{\mathcal{X}, L} + \frac{(tr_\omega c(\phi)[\omega]^m c_1(L)^n)^{[\mathcal{Y}]}}{(\omega)^m c_1(L)^n)^{[\mathcal{Y}]}} \epsilon
\]
\[
\geq \lambda_{\mathcal{X}, L} - \epsilon (c_1(L)^n)^{[\mathcal{Y}/M]} \right)^{1/2},
\]

where the last inequality holds by
\[
\left| \int_{\mathcal{Y}/M} (tr_\omega c(\phi)[\omega][\omega]^m c_1(L)^n)^{[\mathcal{Y}]}
\right| \leq \left| \int_{\mathcal{Y}/M} (tr_\omega c(\phi)[\omega][\omega]^m c_1(L)^n)^{[\mathcal{Y}]}
\right| \leq \epsilon (c_1(L)^n)^{[\mathcal{Y}/M]} \right)^{1/2}.
\]

By taking $\epsilon \to 0$, we have $\lambda_{\mathcal{Y}, L} \geq \lambda_{\mathcal{X}, L}$, which completes the proof by Definition 2.7.

Remark 2.9  For the case of holomorphic vector bundle over compact Kähler manifold $M$, Proposition 2.6 and Theorem 2.8 were proved in [23, Proposition 6.9.1, Theorem 6.10.13]. In particular, if $M$ is projective, then (1), (2), (3) are equivalent (see [23, Theorem 6.10.13]), and he also conjectured that they should be equivalent in general whether $M$ is algebraic or not. Later, (3) $\Rightarrow$ (2) was proved in [22,24] if $M$ is Kähler. For a general pair $(\mathcal{X}, L)$, one may ask that whether (3) $\Rightarrow$ (1) if $M$ is projective, and whether (3) $\Rightarrow$ (2) if $M$ is Kähler.

3 The case of $tr_\omega c(\phi) \geq 0$

In this section, we assume that there exists a metric $\phi$ on $L$ such that
\[
tr_\omega c(\phi) \geq 0.
\]
For any given \( \phi \in F^+(L) \) and the natural frame \( \{ \frac{\partial}{\partial \sigma}, 1 \leq \alpha \leq m \} \) of \( TM \), one sees that there is canonical liftings \( \{ \frac{\partial}{\partial \zeta}, 1 \leq \alpha \leq m \} \). Thus for any vector \( X = X^\alpha \frac{\partial}{\partial \sigma} \big|_y \in T_y M \) at a point \( y \in M \), there is a canonical lifting

\[
\tilde{X} = X^\alpha \frac{\delta}{\delta \zeta^\alpha},
\]

which is a vector field on \( \mathcal{X}_y = \pi^{-1}(y) \). We call the canonical lifting \( \tilde{X} \) is holomorphic if

\[
\bar{\partial}^V \tilde{X} = \bar{\partial}^V \left( X^\alpha \frac{\delta}{\delta \zeta^\alpha} \right) = X^\alpha \bar{\partial}^V \left( \frac{\delta}{\delta \zeta^\alpha} \right) = X^\alpha \frac{\partial}{\partial \theta^i} (-\phi_{\alpha i} \phi^i j) \delta^j = 0. \tag{3.2}
\]

For any holomorphic vector bundle \( E \) over \( M \), the degree of \( E \) is denoted

\[
deg_{\omega} E = \int_M c_1(E) \wedge [\omega]^{m-1}. \tag{3.3}
\]

By using Berndtsson’s curvature formula of direct image bundle, we obtain

**Theorem 3.1** If there exists a metric \( \phi \in F^+(L) \) such that \( tr_{\omega} c(\phi) \geq 0 \), then \( \deg_{\omega} \pi_*(L + K_{\mathcal{X}/M}) = 0 \) if and only if \( \lambda_{\mathcal{X}/L} = 0 \) and the canonical lifting of any vector is holomorphic. In particular, \( \phi \) is a geodesic-Einstein metric on \( L \).

**Proof** We first prove the last argument. If \( tr_{\omega} c(\phi) \geq 0 \), by the definition of \( \lambda_{\mathcal{X}/L} \) (see (1.4)), then

\[
\lambda_{\mathcal{X}/L} = \frac{2\pi m}{n + 1} \frac{(\omega)^m c_1(L)^{n+1}[\mathcal{X}]}{\omega^n} = \frac{\int_{\mathcal{X}/M} tr_{\omega} c(\phi) \omega^n}{\int_{\mathcal{X}/M} \omega^n} \geq 0, \tag{3.4}
\]

the equality holds if and only if \( tr_{\omega} c(\phi) = 0 \). Thus \( \phi \) is a geodesic-Einstein metric on \( L \) if \( tr_{\omega} c(\phi) \geq 0 \) and \( \lambda_{\mathcal{X}/L} = 0 \).

Denote \( E = \pi_*(L + K_{\mathcal{X}/M}) \). By [4, Theorem 1.2] and taking trace with respect to \( \omega \), one has

\[
\langle tr_{\omega} \Theta^E u, u \rangle = \int_{\mathcal{X}/y} tr_{\omega} c(\phi)[u]^2 e^{-\phi} + g^{\alpha \beta} (1 + \Delta)^{-1} i_{\bar{\partial}^V (\frac{1}{\pi^j})} u, i_{\bar{\partial}^V (\frac{1}{\pi^j})} u \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta. \tag{3.5}
\]

Combining the assumption \( tr_{\omega} c(\phi) \geq 0 \) shows that

\[
\langle tr_{\omega} \Theta^E u, u \rangle \geq 0. \tag{3.6}
\]

By taking trace to \( tr_{\omega} \Theta^E \) and using (3.6), one gets

\[
tr_E(tr_{\omega} \Theta^E) \geq 0. \tag{3.7}
\]

Therefore,

\[
\deg_{\omega} E = \int_M c_1(E) \wedge \omega^{m-1} = \frac{1}{m} \int_M tr_{\omega} c_1(E) \omega^m \geq 0. \tag{3.8}
\]

Moreover, by (3.4), (3.5) and (3.6), \( \deg_{\omega} E = 0 \) if and only if \( tr_{\omega} c(\phi) = 0 \) and \( \bar{\partial}^V \left( \frac{\delta}{\delta \zeta^\alpha} \right) = 0 \), which is equivalent to

\[
\lambda_{\mathcal{X}/L} = \frac{\int_{\mathcal{X}/M} tr_{\omega} c(\phi) \omega^n}{\int_{\mathcal{X}/M} \omega^n} = 0. \tag{3.9}
\]
and for any canonical lifting $\tilde{X}$, one has
\[
\bar{\partial} V \tilde{X} = X^\alpha \bar{\partial} V \left( \frac{\delta}{\delta \bar{z}^\alpha} \right) = 0,
\]
(3.10)
i.e., $\tilde{X}$ is holomorphic, which completes the proof. \(\square\)

**Remark 3.2**

(1) The above theorem is inspired by [3, Theorem 2.4] for the case of semipositive line bundle $L$ over a local holomorphic fibration.

(2) Theorem 3.1 gives a sufficient condition for the existence of geodesic-Einstein metric for the case $\lambda_{X,L} = 0$. For a geodesic-Einstein equation $tr_\omega c(\phi) = \lambda_{X,L}$ with $\lambda_{X,L} \neq 0$, we can reduce it to the case of $tr_\omega c(\phi) = 0$. In fact, for any line bundle $L' \to M$ with $\deg_\omega L' \neq 0$, there exists a Hermitian–Einstein metric $\phi'$ on $M$ (see [23, Proposition 4.1.4 and Proposition 4.2.4] or [32, Chapter 1, (1.4) Remark (i)]), i.e. $tr_\omega c_1(L', \phi') = \lambda' \neq 0$. Since both $\lambda_{X,L}$ and $\lambda'$ are rational, so there exist integers $a > 0, b$ such that $a \lambda_{X,L} + b \lambda' = 0$, thus
\[
tr_\omega c(\psi) = a \lambda_{X,L} + b \lambda' = 0.
\]
Here $\psi = a \phi + b \pi^* \phi'$ is the weight of the line bundle $aL + b \pi^* L'$.

(3) By a direct calculation, a homogenous geodesic-Einstein equation $tr_\omega c(\phi) = 0$ is equivalent to
\[
(\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} \wedge \omega^{m-1} = 0.
\]
Moreover, if one considers the homogenous geodesic curvature equation $c(\phi) = 0$, by Lemma 1.1, it is equivalent to
\[
(\sqrt{-1} \partial \bar{\partial} \phi)^{n+1} = 0.
\]

In our next paper, we will try to study the above two Eqs. (3.11) and (3.12).

### 4 S-class and C-class

In this section, we will define the total $S$-class $S(L)$ and the total $C$-class $C(L)$ for a relative ample line bundle $L$ and discuss some inequalities and positivity.

Let $\pi : E \to M$ be a holomorphic vector bundle of rank $r$ over $M$, then there is a canonical pair $(P(E), O_{P(E)}(1))$. The Segre classes are defined by
\[
s_i(E) = \pi_* ((c_1(O_{P(E)}(1)))^{r-1+i}) \in H^{2i}(M, \mathbb{Z}), \quad 0 \leq i \leq m = \dim M.
\]
(4.1)
Then the total Segre class is given by
\[
s(E) = \sum_{i=0}^{m} s_i(E).
\]
(4.2)
The total Chern class and Chern classes of $E$ can be defined by
\[
c(E) = \frac{1}{s(E)}, \quad c_i(E) = \sum_{i=0}^{m} c_i(E), \quad c_i(E) \in H^{2i}(M, \mathbb{Z}),
\]
(4.3)
(see e.g. [20, Chapter 3] or [7, Sect. 20]).

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Inspired by the above construction, for a general pair \((X, L), \pi : X \to M\), we define the total \(S\)-class and the \(S\)-classes of \(L\) by

\[
S(L) = \sum_{i=0}^{m} S_i(L), \quad S_i(L) = \pi_*((c_1(L))^{n+i}) \in H^{2i}(M, \mathbb{Z}), \quad 0 \leq i \leq m,
\]

and the total \(C\)-class and the \(C\)-classes are defined by

\[
C(L) = \frac{1}{S(L)}, \quad C(L) = \sum_{i=0}^{m} C_i(L), \quad C_i(L) \in H^{2i}(M, \mathbb{Q}).
\]

By above definition, one has

\[
C_0(L) = \frac{1}{S_0(L)}, \quad C_1(L) = -\frac{S_1(L)}{S_0(L)^2}, \quad C_2(L) = \frac{S_1(L)^2 - S_0(L)S_2(L)}{S_0(L)^3},
\]

where \(S_0(L) = \int_{X/M} c_1(L)^n \in \mathbb{N}_+\).

**4.1 Some inequalities**

In this subsection, we assume that there exists a geodesic-Einstein metric on \(L\) and discuss some inequalities in terms of \(S\)-class and \(C\)-class.

For any smooth metric \(\phi\) on \(L\), it induces a natural representation of \(S_i(L)\) by

\[
S_i(L, \phi) = \pi_*((c_1(L, \phi))^{n+i}) = \int_{X/M} \left(\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \phi\right)^{n+i}
\]

and set \(S(L, \phi) = \sum_{i=0}^{m} S_i(L, \phi)\). By the relation (4.5), we obtain the representations of \(C(L, \phi)\) by

\[
C(L, \phi) = \frac{1}{S(L, \phi)} = \sum_{i=1}^{m} C_i(L, \phi).
\]

We call \(S_i(L, \phi)\) and \(C_i(L, \phi)\) the \(S\)-form and \(C\)-form, respectively. If \(L\) admits a geodesic-Einstein metric, then we obtain

**Theorem 4.1** If \(\phi\) is a geodesic-Einstein metric on \(L\), i.e., \(tr_\omega c(\phi) = \lambda_{X,L}\), then

\[
S_2(L, \phi) \wedge \omega^{m-2} \leq \frac{(n+1)(n+2)}{8\pi^2 m^2} \lambda_{X,L}^2 S_0(L) \omega^m,
\]

the equality holds if and only if \(c(\phi) = \frac{\lambda_{X,L}}{m} \omega\). In particular,

\[
\int_M S_2(L) \wedge [\omega]^{m-2} \leq \frac{(n+1)(n+2)}{8\pi^2 m^2} \lambda_{X,L}^2 S_0(L) \int_M [\omega]^m.
\]

**Proof** By Lemma 1.1, the first Chern class of \(L\) is represented by

\[
\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \phi = \frac{1}{2\pi} c(\phi) + \frac{\sqrt{-1}}{2\pi} \phi_{ij} \delta v^i \wedge \delta \bar{v}^j.
\]
Denote \( \omega_F := \frac{\sqrt{-1}}{2\pi} \phi_i \bar{\phi}^i \wedge \delta \bar{v}^j \) and by (4.7), then

\[
S_2(L, \phi) \wedge \omega^{m-2} = \pi_*(\left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^{n+2} \wedge \omega^{m-2})
\]

\[
= \frac{(n+1)(n+2)}{8\pi^2} \pi_*(\omega_F^2) \wedge \omega^{m-2}
\]

\[
= \frac{(n+1)(n+2)}{8\pi^2 m(m-1)} \pi_*(\omega_F^n) \wedge \omega^m (4.11)
\]

\[
\leq (n+1)(n+2) \pi_*(\omega_F^n) \wedge \omega^m
\]

\[
= \frac{(n+1)(n+2)}{8\pi^2 m^2} \lambda_{X, L}^2 S_0(L) \omega^m,
\]

where the third equality follows from the following formula,

\[
m(m-1)\alpha \wedge \alpha \wedge \omega^{m-2} = (\langle tr_{\omega} \alpha \rangle^2 - |a_{\omega}|^2) \omega^m (4.12)
\]

for any real \((1,1)\)-form \(\alpha\) (see e.g. [29, Lemma 4.7]). The fourth equality in (4.11) holds since

\[
|c(\phi)|_{\omega_F}^2 := c(\phi)_{\alpha \beta} c(\phi)_{\gamma \delta} g^{\alpha \beta} g^{\gamma \delta} (4.13)
\]

and \(|c(\phi)|_{\omega_F}^2 \geq \frac{1}{m} \langle tr_{\omega} c(\phi) \rangle^2 \). Moreover, the equality holds if and only if

\[
c(\phi)_{\alpha \beta} = \frac{\lambda_{X, L}}{m} \omega_{\alpha \beta}, (4.14)
\]

that is, \(c(\phi) = \frac{\lambda_{X, L}}{m} \omega\). By integrating the both sides of (4.9), we conclude (4.10). \(\square\)

In terms of \(C\)-classes, we have the following Kobayashi–Lübke type inequality.

**Theorem 4.2** If \(\phi\) is a geodesic-Einstein metric on \(L\), then

\[
(n C_1(L, \phi)^2 - 2(n+1) C_0(L) C_2(L, \phi)) \wedge \omega^{m-2} \leq 0, (4.15)
\]

the equality holds if and only if \(c(\phi) = \frac{2\pi}{(n+1) S_0(L)} S_1(L, \phi)\). In particular,

\[
\int_M \langle n C_1(L)^2 - 2(n+1) C_0(L) C_2(L) \rangle \wedge [\omega]^{m-2} \leq 0. (4.16)
\]

**Proof** By (4.6) and (4.12), one has

\[
C_1(L, \phi)^2 \wedge \omega^{m-2} = \frac{1}{S_0(L)^4} S_1(L, \phi)^2 \wedge \omega^{m-2}
\]

\[
= \frac{1}{S_0(L)^4} \frac{1}{m(m-1)} \langle (tr_{\omega} S_1(L, \phi))^2 \rangle \omega^m
\]

\[
= \frac{1}{S_0(L)^4} \frac{1}{m(m-1)} \langle (n+1) \lambda_{X, L} S_0(L)^2 \rangle \omega^m.
\]

(4.17)
and
\[
C_2(L, \phi) \wedge \omega^{m-2} = \frac{S_1(L, \phi)^2 - S_0(L)S_2(L, \phi)}{S_0(L)^3} \wedge \omega^{m-2}
\]
\[
= S_0(L)C_1(L, \phi)^2 \wedge \omega^{m-2} - \frac{1}{S_0(L)^2} S_2(L, \phi) \wedge \omega^{m-2}
\]
\[
= S_0(L)C_1(L, \phi)^2 \wedge \omega^{m-2} - \frac{1}{S_0(L)^2} \frac{(n+1)(n+2)}{8\pi^2 m(m-1)} (\lambda_{\nabla, L}^2 S_0(L)
\]
\[
- \pi_*(|c(\phi)|^2 \omega_F^n)) \omega^m.
\]

From (4.17) and (4.18), we have
\[
(nC_1(L, \phi)^2 - 2(n+1)C_0(L)C_2(L, \phi)) \wedge \omega^{m-2}
\]
\[
= \left( \frac{1}{S_0(L)^4} \frac{n+2}{m(m-1)} |S_1(L, \phi)|^2_{\omega} - \frac{1}{S_0(L)^2} \frac{(n+2)(n+1)^2}{4\pi^2 m(m-1)} \pi_*(|c(\phi)|^2 \omega_F^n) \right) \omega^m
\]
\[
= \frac{1}{S_0(L)^4} \frac{(n+2)(n+1)^2}{4\pi^2 m(m-1)} \left( |\pi_*(c(\phi)\omega_F^n)|^2_{\omega} - S_0(L)\pi_*(|c(\phi)|^2 \omega_F^n) \right) \omega^m.
\]

For any \(p \in M\) and taking the normal coordinate system near \(p\) with \(g_{\alpha\beta} = \delta_{\alpha\beta}\), then
\[
|\pi_*(c(\phi)\omega_F^n)|^2_{\omega} - S_0(L)\pi_*(|c(\phi)|^2 \omega_F^n)
\]
\[
= \sum_{\alpha, \beta} \pi_*(c(\phi)_{\alpha\beta} \omega_F^n)\pi_*(c(\phi)_{\alpha\beta} \omega_F^n) - \pi_*(\omega_F^n)\pi_*(\sum_{\alpha, \beta} |c(\phi)_{\alpha\beta}|^2 \omega_F^n)
\]
\[
\leq \sum_{\alpha, \beta} \left( (\pi_*(|c(\phi)_{\alpha\beta}| \omega_F^n))^2 - \pi_*(\omega_F^n)\pi_*(|c(\phi)_{\alpha\beta}|^2 \omega_F^n) \right) \leq 0,
\]
where the last inequality follows from Cauchy–Schwarz inequality. Thus,
\[
(nC_1(L, \phi)^2 - 2(n+1)C_0(L)C_2(L, \phi)) \wedge \omega^{m-2} \leq 0,
\]
which proves (4.15). Moreover, the equality holds if and only if \((c(\phi)_{\alpha\beta})\) is constant along each fiber by Cauchy–Schwarz inequality, which is equivalent to
\[
c(\phi) = \frac{\int_{\nabla / M} c(\phi) \omega_F^n}{\int_{\nabla / M} \omega_F^n} = \frac{2\pi}{(n+1)S_0(L)} S_1(L, \phi).
\]

By taking integrate the both sides of (4.15) over \(M\), we obtain the topological inequality (4.16).

**Remark 4.3** For the case of holomorphic vector bundle \(E \to M\), the geodesic-Einstein metric is equivalent to Finsler–Einstein metric (see [19, Lemma 3.6]). With the assumption of existence of Finsler–Einstein metric, Theorems 4.1 and 4.2 were proved in [18, Theorem 3.7, 3.8]. If \(h\) is a Hermitian–Einstein metric on \(E\), then Theorem 4.2 is reduced to the classical Kobayashi–Lübke inequality [23, Theorem 4.4.7] (see also [32, Chapter 1, (1.8)]), while Theorem 4.1 is exactly the [11, Theorem 1.2].
4.2 Positivity of classes

In this subsection, we will discuss the positivity of $S$-classes and $C$-classes.

Recall a smooth $(p, p)$-form $\Phi$ on a complex manifold $M$ is positive if for any $y \in M$ and any linearly independent $(1, 0)$-type tangent vectors $v_1, v_2, \ldots, v_p$ at $y$, it holds that

$$(-\sqrt{-1})^{p^2} \Phi(v_1, v_2, \ldots, v_p, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p) > 0.$$  \hfill (4.21)

**Proposition 4.4** If $\sqrt{-1}\partial\bar{\partial} \phi > 0$, then the $k$-th $S$-form $S_k(L, \phi)$ is a positive $(k, k)$-form for any $0 \leq k \leq m$. In particular, the class $S_k(L)$ can be represented by a positive $(k, k)$-form if $L$ is ample.

**Proof** By Lemma 1.1, $\sqrt{-1}\partial\bar{\partial} \phi > 0$ is equivalent to

$$c(\phi) > 0$$ \hfill (4.22)

on horizontal directions. For any $(z_0, v_0) \in \mathcal{X}$, there exists a basis $\{\psi^1, \ldots, \psi^n\}$ such that

$$c(\phi) = \sqrt{-1} \sum_{\alpha=1}^{n} \psi^\alpha \wedge \bar{\psi}^\alpha,$$ \hfill (4.23)

and so for $k \geq 1$,

$$(-\sqrt{-1})^{k^2} c(\phi)^k = k! \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \psi^{\alpha_1} \wedge \cdots \wedge \psi^{\alpha_k} \wedge \bar{\psi}^{\alpha_1} \wedge \cdots \wedge \bar{\psi}^{\alpha_k}.$$ \hfill (4.24)

Hence for any independent horizontal vectors $X_1, \ldots, X_k$ at $(z_0, v_0)$, one has

$$(-\sqrt{-1})^{k^2} c(\phi)^k (X_1, \ldots, X_k, \bar{X}_1, \ldots, \bar{X}_k)$$

$$= k! \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \psi^{\alpha_1} \wedge \cdots \wedge \psi^{\alpha_k} \wedge \bar{\psi}^{\alpha_1} \wedge \cdots \wedge \bar{\psi}^{\alpha_k} (X_1, \ldots, X_k, \bar{X}_1, \ldots, \bar{X}_k)$$

$$= k! \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} |\psi^{\alpha_1} \wedge \cdots \wedge \psi^{\alpha_k} (X_1, \ldots, X_k)|^2 > 0.$$ \hfill (4.25)

On the other hand, by Lemma 1.1, by (4.25), one has

$$S_k(L, \phi) = \int_{\mathcal{X}/M} (\sqrt{-1}\partial\bar{\partial} \phi)^{n+k} = \frac{1}{(2\pi)^k} \binom{n+k}{k} \int_{\mathcal{X}/M} c(\phi)^k \omega_F^n.$$ \hfill (4.26)

For any $z_0 \in M$ and any linearly independent $(1, 0)$-type tangent vectors $Y_1, \ldots, Y_k$ in $T_{z_0} M$, one has

$$(-\sqrt{-1})^{k^2} S_k(L, \phi) (Y_1, \ldots, Y_k, \bar{Y}_1, \ldots, \bar{Y}_k)$$

$$= \frac{1}{(2\pi)^k} \binom{n+k}{k} \int_{\mathcal{X}_{z_0}} (-\sqrt{-1})^{k^2} c(\phi)^k (Y_1^h, \ldots, Y_k^h, \bar{Y}_1^h, \ldots, \bar{Y}_k^h) \omega_F^n > 0,$$

where $Y^h$ denote the horizontal lifting along the fibre $P(E_{z_0})$ of a vector $Y \in T_{z_0} M$. Thus we conclude $S_k(L, \phi)$ is a positive $(k, k)$-form. Since $[S_k(L, \phi)] = S_k(L)$, so the class $S_k(L)$ can be represented by a positive $(k, k)$-form if $L$ is ample. \hfill \Box
Corollary 4.5 Let $M$ be a compact complex surface and $L$ be an ample line bundle over $\mathcal{X}$, $\pi : \mathcal{X} \to M$. If moreover, $L$ admits a geodesic-Einstein metric, then

$$\int_M C_2(L) > 0.$$ \hspace{1cm}

Proof By Theorem 4.2, (4.6) and noting $\dim M = 2$, then

$$\int_M C_2(L) \geq \int_M \frac{nC_1(L)^2}{2(n+1)C_0(L)} = \frac{n}{2(n+1)S_0(L)^3} \int_M S_1(L)^2 > 0,$$ \hspace{1cm} (4.27)

where the last inequality follows from Theorem 4.4. \hfill \Box

Remark 4.6 In terms of complex Finsler vector bundles, Theorem 4.4 was proved in [18, Theorem 2.8] with the assumption of positive or negative Kobayashi curvature. On the other hand, if one considers an ample vector bundle $E$, which is equivalent to the pair $(P(E^*), \mathcal{O}_{P(E^*)}(1))$ with positive line bundle $\mathcal{O}_{P(E^*)}(1)$, by [8, Theorem 2.5], one has $\int_M c_m(E) > 0$. Since the $C$-classes $C_1(L)$ can be viewed as a generalization of Chern classes of a holomorphic vector bundle, so it is natural to ask whether $(-1)^m \int_M c_m(L) > 0$ if $L$ is ample.

5 Some examples

In this section, we will discuss some examples on the geodesic-Einstein metrics.

Example 5.1 (Product manifolds) Let $X$ and $M$ be two compact complex manifolds and consider the holomorphic fibration $\pi_1 : M \times X \to M$. For any line bundle $L_1$ over $M$ and any ample line bundle $L_2$ over $X$, then the line bundle $L := \pi_1^*L_1 + \pi_2^*L_2$ over $X \times M$ admits a geodesic-Einstein metric with respect to any given Kähler metric $\omega$ on $M$, where $\pi_2 : M \times X \to X$. In fact, for any Kähler metric $\omega$ on $M$, we take a Hermitian–Einstein metric $\varphi_1$ on $L_1$, i.e. $tr_\omega(\sqrt{-1}\partial\bar{\partial}\varphi_1) = \text{constant}$, and take a metric $\varphi_2$ on $L_2$ such that $\sqrt{-1}\partial\bar{\partial}\varphi_2 > 0$. Therefore, $\varphi = \pi_1^*\varphi_1 + \pi_2^*\varphi_2$ is a metric on $L$ and its curvature is $\sqrt{-1}\partial\bar{\partial}\varphi = \pi_1^*\partial\bar{\partial}\varphi_1 + \pi_2^*\partial\bar{\partial}\varphi_2$. So $L$ is a relative ample line bundle and

$$tr_\omega c(\phi) = tr_\omega(\sqrt{-1}\partial\bar{\partial}\varphi_1) = \text{constant}.$$ \hspace{1cm}

Example 5.2 (Ruled manifolds) An algebraic manifold $\mathcal{X}$ is said to be a ruled manifold if $\mathcal{X}$ is a holomorphic $\mathbb{P}^r$-bundle with structure manifold $PGL(r+1, \mathbb{C}) = GL(r+1, \mathbb{C})/\mathbb{C}^*$ (see e.g. [1, Section 4.2]). By [1, Proposition 4.3], every ruled manifold $\mathcal{X}$ over a compact Riemann surface $M$ is holomorphically isomorphic to $P(E)$ for some holomorphic vector bundle $E \to M$ of rank $(E) = r + 1$. Such a bundle $E$ is uniquely determined up to tensor product with a holomorphic line bundle. Since $\text{Pic}(P(E)) = \text{Pic}(M) \oplus \mathbb{Z}\mathcal{O}_{P(E)}(1)$, so any line bundle $L$ over $P(E)$ is the form $L := \pi^*L_1 + k\mathcal{O}_{P(E)}(1)$ for some line bundle $L_1$ over $M$ and $k \in \mathbb{Z}$. Moreover, $L$ is relative ample if and only if $k > 0$. If there exists a Hermitian–Einstein metric on $E$, then $L$ can admit a geodesic-Einstein metric. In fact, by a direct calculation, any Hermitian–Einstein metric on $E$ induces a natural geodesic-Einstein metric on $\mathcal{O}_{P(E)}(1)$, so is $k\mathcal{O}_{P(E)}(1)$. Similar as Example 5.1, the geodesic-Einstein metric on $k\mathcal{O}_{P(E)}(1)$ and the Hermitian–Einstein metric on $L_1$ give a geodesic-Einstein metric on $L$. 

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Example 5.3 (Geodesic curve) From Definition 1.2, one can allow $M$ to be a non-compact manifold. Now we assume that $M$ is a Riemann surface with boundary and $\mathcal{X} = X \times M$ for some compact complex manifold $X$, the line bundle $L \to \mathcal{X}$ is taken to be the pullback of some ample line bundle over $X$. Then $tr_\omega c(\phi) = 0$ is equivalent to $c(\phi) = 0$, which is also equivalent to $(\sqrt{-1}\partial \bar{\partial} \phi)^{n+1} = 0$ (see [15,30]). As seen in [10, Sect. 2.3] or [25, Sect. 3], one can define a Riemannian metric on the space of Kähler potentials in a fixed Kähler class, which is an infinite dimensional manifold and equipped a $L^2$-norm, then its geodesic equation is exactly the equation $c(\phi) = 0$. By [10, Theorem 3], the following equation

$$
\begin{cases}
(\sqrt{-1}\partial \bar{\partial} \phi)^{n+1} = 0 \text{ in } \mathcal{X} \\
\phi = \phi_0 \text{ in } \partial \mathcal{X}
\end{cases}
$$

has a $C^{1,1}$-solution for any metric $\phi_0$ in $F^+(L|_{\partial \mathcal{X}})$.

Example 5.4 (Complex quotient equation) From (2.11), the geodesic-Einstein equation $tr_\omega c(\phi) = \lambda$ is equivalent to

$$
c^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n = \omega^{m-1} \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1},
$$

where $c = \frac{\lambda(n+1)}{m}$. If moreover, we assume that $L$ is an ample line bundle, and one may consider the limits of the solutions of (5.2), (5.4) as $\alpha$-solution for any metric $\omega_0 = \omega + \epsilon \sqrt{-1}\partial \bar{\partial} \phi_0$ is a Kähler metric on $\mathcal{X}$. For any $\epsilon > 0$, we consider the following equation

$$
c_\epsilon \omega_\epsilon^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n = \omega_\epsilon^{m-1} \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1},
$$

where $c_\epsilon = \int_\mathcal{X} \omega_\epsilon^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1} / \int_\mathcal{X} \omega_\epsilon^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n$. The above equation is studied in [27,31] by using the following parabolic flow

$$
\frac{\partial u}{\partial t} = \log \frac{\omega_\epsilon^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n}{\omega_\epsilon^{m-1} \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1}} + \log c_\epsilon,
$$

where $u = \phi - \phi_0$. From [31, Theorem 1.2] or [27, Corollary 7], if there is a $C$-subsolution to (5.3) (see [31, Definition 1.1]), then there exists a long time solution $u$ to (5.3). Moreover, the normalization $\hat{u}$ (see [31, (2.34)]) of $u$ is $C^\infty$ convergent to a smooth solution $\hat{u}_\infty$. Thus $\sqrt{-1}\partial \bar{\partial} \phi = \sqrt{-1}\partial \bar{\partial} (\phi_0 + \hat{u}_\infty)$ solves (5.2). In particular, if $M$ is a Riemann surface, then (5.2) becomes

$$
c_\epsilon \omega_\epsilon \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n = (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1},
$$

which is the Euler equation of $J$-functional (see [16]). From [28, Theorem 1.1], there exists a solution to (5.4) if and only if there exists a metric $\sqrt{-1}\partial \bar{\partial} \phi' > 0$ such that

$$
((n+1)\sqrt{-1}\partial \bar{\partial} \phi' - n c_\epsilon \omega_\epsilon) \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n-1} > 0.
$$

Comparing with the Eqs. (5.2) and (5.4), the metric $\omega$ in geodesic-Einstein equation (5.1) is degenerate along the fibers of $\mathcal{X}$. Thus, in order to solve the geodesic-Einstein equation (5.1), one may consider the limits of the solutions of (5.2), (5.4) as $\epsilon \to 0$.

Example 5.5 (Calabi–Yau family) From (5.1), one can define a geodesic-Einstein metric for any holomorphic line bundle (need not relative ample). More precisely, for any line bundle $L$ over $\mathcal{X}$, a smooth metric $\phi$ on $L$ is called geodesic-Einstein if

$$
c^m \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^n = \omega^{m-1} \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n+1}
$$

as a solution to (5.2) if and only if there exists a metric $\sqrt{-1}\partial \bar{\partial} \phi > 0$ such that

$$
((n+1)\sqrt{-1}\partial \bar{\partial} \phi' - n c_\epsilon \omega_\epsilon) \wedge (\sqrt{-1}\partial \bar{\partial} \phi)^{n-1} > 0.
$$

Comparing with the Eqs. (5.2) and (5.4), the metric $\omega$ in geodesic-Einstein equation (5.1) is degenerate along the fibers of $\mathcal{X}$. Thus, in order to solve the geodesic-Einstein equation (5.1), one may consider the limits of the solutions of (5.2), (5.4) as $\epsilon \to 0$. 

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for some constant $c$.

Following [6], let $\pi : X \to M$ be a holomorphic, polarized family of Calabi-Yau manifolds $X_s = \pi^{-1}(s), s \in M$, i.e. compact manifolds with $c_1(X_s) = 0$, equipped with Ricci flat Kähler forms $\omega_{X_s}$. The relative volume form $\omega_{X/M}^n = g dV$ induces a hermitian metric $g^{-1}$ on the relative canonical bundle $K_{X/M}$. By [6, Formula (1)], one has

$$2\pi c_1(K_{X/M}, g^{-1}) = \sqrt{-1} \partial \bar{\partial} \log g = \frac{1}{\text{Vol}(X)} \bar{\partial}^* \omega^{WP}. \quad (5.7)$$

Now we take $L = K_{X/M}, \phi = \log g$ and for any Kähler metric $\omega$ on $M$. By (5.7), one easily sees that the metric $\phi$ satisfies the Eq. (5.6), i.e., $\phi$ is a geodesic-Einstein-metric on $L$.

### 6 Appendix: Hermitian–Einstein metrics on quasi-vector bundles (By Xu Wang)

In this section, we will prove that the geodesic-Einstein metric is equivalent to a Hermitian–Einstein metric on a quasi-vector bundle. For more details on quasi-vector bundle, one can refer to [5,37].

Following [5,37], we shall use the following setup:

1. $\pi : X \to B$ is a proper holomorphic submersion from a complex manifold $X$ to another complex manifold $B$, each fiber $X_t := \pi^{-1}(t)$ is an $n$-dimensional compact complex manifold;
2. $E$ is a holomorphic vector bundle over $X$, $E_t := E|_{X_t}$;
3. $\omega$ is a $d$-closed $(1,1)$-form on $X$ and is positive on each fiber, $\omega^1 := \omega|_{X_t}$;
4. $h_E$ is a smooth Hermitian metric on $E$, $h_{E_t} := h_E|_{E_t}$.

**Definition 6.1** [5] Let $V := \{V_t\}_{t \in B}$ be a family of $\mathbb{C}$-vector spaces over $B$. Let $\Gamma$ be a $C^\infty(B)$-submodule of the space of all sections of $V$. We call $\Gamma$ a smooth quasi-vector bundle structure on $V$ if each vector of the fiber $V_t$ extends to a section in $\Gamma$ locally near $t$.

For each $t \in B$, let us denote by $A^{p,q}(E_t)$ the space of smooth $E_t$-valued $(p,q)$-forms on $X_t$. Consider

$$A^{p,q} := \{A^{p,q}(E_t)\}_{t \in B},$$

and denote by $A^{p,q}(E)$ the space of smooth $E$-valued $(p,q)$-forms on $X$. Let us define

$$\Gamma^{p,q} := \{u : t \mapsto u^t \in A^{p,q}(E_t) : \exists u \in A^{p,q}(E), u|_{X_t} = u^t, \forall t \in B\}.$$

We call $u$ above a smooth representative of $u \in \Gamma^{p,q}$, each $\Gamma^{p,q}$ defines a quasi-vector bundle structure on $A^{p,q}$. Denote

$$(A, \Gamma) := \bigoplus_{k=0}^{\infty}(A^k, \Gamma^k), \quad (A^k, \Gamma^k) := \bigoplus_{p+q=k}(A^{p,q}, \Gamma^{p,q}).$$

**Definition 6.2** The Lie-derivative connection, say $\nabla^A$, on $(A, \Gamma)$ is defined as follows:

$$\nabla_A u := \sum dt^j \otimes [d^E, \delta_{V_j}]u + \sum dt^j \otimes [d^E, \delta_{\bar{V}_j}]u, \quad u \in \Gamma,$$

where $d^E := \bar{\partial} + \partial^E$ denotes the Chern connection on $(E, h_E)$ and each $V_j$ is the horizontal lift of $\partial/\partial t^j$ with respect to $\omega$. 

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Let $D^A$ denote the connection on each $(V^{p,q}, \Gamma^{p,q})$ induced by $\nabla^A$,
\[
D^A u := \sum dt^j \otimes [\partial E, \delta V_j] u + \sum d\bar{t}^j \otimes [\bar{\partial}, \delta \bar{V}_j] u, \quad u \in \Gamma^{p,q}
\]
\[
= \nabla^A - \left( \sum dt^j \otimes \kappa_j + \sum d\bar{t}^j \otimes \kappa_{\bar{j}} \right),
\]
where the non-cohomological Kodaira-Spencer actions (see [37, Definition 5.6])
\[
\kappa_j u := [\bar{\partial}, \delta V_j] u, \quad \kappa_{\bar{j}} u := [\partial E, \delta \bar{V}_j] u.
\]
Then $D^A$ defines a Chern connection on each $(\mathcal{A}^{p,q}, \mathcal{A}^{p,q})$ (see e.g. [37, Theorem 5.6]). The curvature of the Chern connection $D^A$ is given by
\[
(D^A)^2 = (\nabla^A)^2 - \sum ([\kappa_j, \kappa_{\bar{k}}]) dt^j \wedge d\bar{t}^k,
\]
see [37, (5.2)]. Here the curvature of Lie-derivative connection is
\[
(\nabla^A)^2 = \sum ([dE, \delta V_j], [dE, \delta \bar{V}_k]) dt^j \wedge d\bar{t}^k.
\]
The $L^2$-metric on each $\mathcal{A}^{p,q}$ is defined by
\[
(u, v) = \int_{X_t} \langle u, v \rangle \omega^{n/2}.
\]
Here $\langle \cdot, \cdot \rangle$ denotes the point-wise inner on $\mathcal{A}^{p,q}$ with respect to $\omega|_{X_t}$ and $h_{E_t}$.

**Definition 6.3** The $L^2$-metric (6.4) on $\mathcal{A}^{p,q}$ is called Hermitian–Einstein with respect to a Hermitian metric $\omega_B = \sqrt{-1} 
\begin{pmatrix} g_{ij} \\ d^i \bar{d}^j \end{pmatrix} dt^i \wedge d\bar{t}^j$ if
\[
\Lambda_{\omega_B} (D^A)^2 = \lambda \operatorname{Id}
\]
for some constant $\lambda$.

Now we take $p = q = 0$ and $E$ is the trivial bundle. Let $L$ be a relative ample line bundle over $\mathcal{X}$, i.e. there exists a metric $\phi$ on $L$ such that its curvature $\omega := \sqrt{-1} \partial \bar{\partial} \phi$ is positive along each fiber. The $L^2$-metric (6.4) on $\mathcal{A}^{0,0}$ is
\[
(u, v) = \int_{X_t} uv \omega_{n/2}.
\]
We call a metric $\phi$ on $L$ is weak geodesic-Einstein with respect to $\omega_B$ if $\operatorname{tr}_{\omega_B} c(\phi) = \pi^* f(z)$ for some function on $B$.

**Proposition 6.4** $\phi$ is a weak geodesic-Einstein metric on $L$ if and only if the metric $L^2$-metric (6.6) is a Hermitian–Einstein metric on $\mathcal{A}^{0,0}$.

**Proof** By (6.2) and noting that $p = q = 0$, $E$ is trivial, one has
\[
(D^A)^2 u = (\nabla^A)^2 u = \sum ([dE, \delta V_j], [dE, \delta \bar{V}_k]) dt^j \wedge d\bar{t}^k
\]
\[
= \sum [V_j, V_{\bar{k}}] u dt^j \wedge d\bar{t}^k
\]
for any $u \in \mathcal{A}^{0,0}$. By [38, Lemma 6.1], we have
\[
[V_j, V_{\bar{k}}] = (c(\phi)_{j\bar{k}})^{\prime} \phi^{\prime} \frac{\partial}{\partial v^j} - (c(\phi)_{j\bar{k}}) \phi^\prime \frac{\partial}{\partial \bar{v}^k},
\]
where $c(\phi)_{jk} = \langle V_j, V_k \rangle_\omega$ is the coefficient of geodesic curvature $c(\phi)$. Let $\omega_B = \sqrt{-1} g_{ij} dt^i \wedge d\bar{t}^j$ be a Hermitian metric on $B$, then

$$
\Lambda_{\omega_B}(D^A)^2 u = (g^{jk} c(\phi)_{jk}) \phi^{\bar{\nu}} \frac{\partial}{\partial \bar{v}^\nu} u - (g^{jk} c(\phi)_{jk})_\nu \phi^{\bar{\nu}} \frac{\partial}{\partial v^\lambda} u 
$$

\begin{equation}
= (tr_{\omega_B} c(\phi)) \phi^{\bar{\nu}} \frac{\partial}{\partial \bar{v}^\nu} u - ((tr_{\omega_B} c(\phi))_\nu \phi^{\bar{\nu}} \frac{\partial}{\partial v^\lambda} u).
\end{equation}

Thus, for any $u \in A^{0,0}$, $\Lambda_{\omega_B}(D^A)^2 u = \lambda u$ if and only if $\lambda = 0$ and $tr_{\omega_B} c(\phi) = \pi^* f(t)$ for some function $f(z)$ on $B$, which completes the proof. \hfill \Box

Now we assume that $B$ is compact, then for any smooth function $f$, there is a smooth solution $\tilde{f}$ solves

$$\Delta \tilde{f} + f(t) = \int_B f(t) \frac{\omega_B^m}{m!}$$

where $\Delta := g^{ij} \partial_i \partial_j$ and $m = \dim B$. Denote $\tilde{\phi} = \phi + \tilde{f}$. If $tr_{\omega_B} c(\phi) = \pi^* f(z)$, then $tr_{\omega_B} c(\tilde{\phi}) = \Delta \tilde{f} + f(z) = \int_B f(t) \frac{\omega_B^m}{m!}$,

\begin{equation}
\end{equation}

which implies that $\tilde{\phi}$ is a geodesic-Einstein metric on $L$. Combining with Proposition 6.4, we have

**Corollary 6.5** If $B$ is compact, then up to a smooth function on $B$, $\phi$ is a geodesic-Einstein metric on $L$ if and only if the metric $L^2$-metric (6.6) is a Hermitian–Einstein metric on $A^{0,0}$.

**References**

1. Aikou, T.: Finsler Geometry on complex vector bundles. Riemann–Finsler Geometry. MSRI Publication, Vol. 50, pp. 83–105 (2004)
2. Atiyah, M., Bott, R.: The Yang–Mills equations over Riemann surfaces. Phil. Trans. R. Soc. Lond. A 308, 524–615 (1982)
3. Berndtsson, B.: Positivity of direct image bundles and convexity on the space of Kähler metrics. J. Diff. Geom. 81(3), 457–482 (2009)
4. Berndtsson, B.: Strict and non strict positivity of direct image bundles. Math. Z. 269, 1201–1218 (2011)
5. Berndtsson, B., Paun, M., Wang, X.: Algebraic fiber spaces and curvature of higher direct images. arXiv:1704.02279
6. Bott, R., Tu, L.: Differential Forms in Algebraic Topology. Springer, New York, Heidelberg, Berlin (1982)
7. Bloch, S., Gieseker, D.: The positivity of the Chern classes of an ample vector bundle. Invent. Math. 12, 112–117 (1971)
8. Boucksom, S., Eyssidieux, P., Guedj, V.: An Introduction to the Kähler–Ricci Flow, Vol. 2086. Springer (2013)
9. Chen, X.: The space of Kähler metrics. J. Diff. Geom. 56, 189–234 (2000)
10. Diveriot, S.: Segre forms and Kobayashi–Lübke inequality. Math. Z. 283(3–4), 1033–1047 (2016)
11. Donaldson, S.K.: A new proof of a theorem of Narasimhan and Seshadri. J. Diff. Geom. 18, 269–278 (1983)
12. Donaldson, S.K.: Anti-self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. Proc. Lond. Math. Soc. 50, 1–26 (1985)
13. Donaldson, S.K.: Infinite determinates, stable bundles and curvature. Duke. J. Math. 54, 231–247 (1987)
14. Donaldson, S. K.: Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar, 13-33, Amer. Math. Soc. Transl. Ser. 2, 196, Adv. Math. Sci., 45, Amer. Math. Soc., Providence, RI, (1999)
16. Donaldson, S.K.: Moment maps and diffeomorphisms. Asian J. Math. 3, 1–15 (1999)
17. Feng, H., Liu, K., Wan, X.: Chern forms of holomorphic Finsler vector bundles and some applications. Int. J. Math. 27(4), 1650030 (2016)
18. Feng, H., Liu, K., Wan, X.: A Donaldson type functional on a holomorphic Finsler vector bundle. Math. Ann. 369(3–4), 997–1019 (2017)
19. Feng, H., Liu, K., Wan, X.: Geodesic-Einstein metrics and nonlinear stabilities. Trans. Am. Math. Soc. 371(11), 8029–8049 (2019)
20. Fulton, W.: Intersection Theory, 2nd edn. Springer, Berlin (1998)
21. Griffiths, P., Harris, P.: Principles of Algebraic Geometry. Wiley, New York (1994)
22. Jacob, A.: Existence of approximate Hermitian–Einstein structures on semi-stable bundles. Asian J. Math. 18(5), 859–883 (2014)
23. Kobayashi, S.: Differential Geometry of Complex Vector Bundles. Iwanami-Princeton Univ, Press (1987)
24. Li, J., Zhang, X.: Existence of approximate Hermitian–Einstein structures on semi-stable Higgs bundles. Calc. Var. Part. Diff. Equ. 52(3–4), 783–795 (2015)
25. Mabuchi, T.: Some Symplectic geometry on compact Kähler manifolds. I. Osaka, J. Math. 24, 227–252 (1987)
26. Narasimhan, M., Seshadri, C.: Stable and unitary vector bundles on compact Riemann surfaces. Ann. Math. 82, 540–567 (1965)
27. Phong, D.H., Tô, D.T.: Fully non-linear parabolic equations on compact Hermitian manifolds. arXiv:1711.10697 (2017)
28. Song, J., Weinkove, B.: On the convergence and singularities of the \( J \)-flow with applications to the Mabuchi energy. Commun. Pure Appl. Math. LXI, 210–229 (2008)
29. Székelyhidi, G.: An Introduction to Extremal Kähler Metrics, Graduate Studies in Mathematics, vol. 152. American Mathematical Society, New York (2014)
30. Semmes, S.: Complex Monge–Ampere and symplectic manifolds. Am. J. Math. 114, 495–550 (1992)
31. Sun, W.: The parabolic flows for complex quotient equations. arXiv:1712.00748v1 (2017)
32. Siu, Y.-T.: Lectures on Hermitian–Einstein Metrics for Stable Bundles and Kähler–Einstein Metrics. Birkhäuser Verlag, Basel, Boston (1987)
33. Tosatti, V.: KAWA Lecture notes on the Kähler–Ricci flow. arXiv:1508.04823v1 (2015)
34. Uhlenbeck, K.K., Yau, S.T.: On existence of Hermitian–Yang–Mills connection in stable vector bundles. Commun. Pure Appl. Math. 39, 257–293 (1986)
35. Wan, X.: Positivity preserving along a flow over projective bundle. arxiv:1801.09886 (2017)
36. Wan, X., Zhang, G.: The asymptotic of curvature of direct image bundle associated with higher powers of a relative ample line bundle. arXiv:1708.05922v1 (2017)
37. Wang, X.: Notes on variation of Lefshetz star operaotr and \( T \)-Hodge theory. arXiv:1708.07332v1
38. Wang, X.: A curvature formula associated to a family of pseudoconvex domains. Annales de l’institut Fourier 67(1), 269–313 (2017)

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