Compact families of Jordan curves and convex hulls in three dimensions

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December 2, 2013

Abstract

We prove that for certain families of semi-algebraic convex bodies in $\mathbb{R}^3$, the convex hull of $n$ disjoint bodies has $O(n \lambda_s(n))$ features, where $s$ is a constant depending on the family: $\lambda_s(n)$ is the maximum length of order-$s$ Davenport-Schinzel sequences with $n$ letters. The argument is based on an apparently new idea of ‘compact families’ of convex bodies or discs, and of ‘crossing content’ and ‘footprint width’ among disc intersections.

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1 Introduction

The construction of convex hulls is a well-studied problem, certainly for finite sets of points in any dimension, and for more general sets, such as curved objects in two dimensions [1], quadric surfaces in three dimensions [14], and spheres in any dimension [3]. This paper gives a reasonably straightforward derivation of an $o(n^2 \log^* n)$ upper bound for the feature complexity (descriptive complexity) of the convex hull of $n$ disjoint bodies in three dimensions, granted that the bodies come from a ‘compact family,’ a term defined in this paper.

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In 1995 Hung and Ierardi [8] reported $O(n^{2+\epsilon})$ complexity bounds, together with algorithms for constructing the hull, but their approach is indirect and hard to understand. In this paper we (hopefully) develop a theory sufficient for a convincing proof.

(1.3) $S$ will be a set of $n$ disjoint convex bodies in $\mathbb{R}^3$. $H(S)$ denotes the convex hull of $S$. As in [14] the boundary $\partial H(S)$ is divided into exposed facets, tunnel facets, and planar facets. These, with their separating edges and vertices, constitute the features of $H(S)$. In the case of spherical bodies it is known that $H(S)$ has $O(n^2)$ features, and this is also a lower bound (Figure 1, [10,3]).

Every facet is incident to an edge or vertex of an exposed facet, so the feature complexity can be estimated by counting the edges and/or vertices on the exposed facets. Thus the complexity can be reduced to that of unions of discs.

(1.4) It is necessary to assume some complexity bounds on the bodies. For example, Figure 2 shows how the convex hull of two bodies can have many features. To eliminate this we assume that the bodies are semialgebraic of bounded degree.

Unions of $n$ circular discs have complexity $O(n)$, whereas unions of $n$ thin ellipses can have complexity $\Omega(n^2)$, obviously because they are ‘thin,’ and the analysis of various notions of ‘fatness’ which reduce the complexity, has been of great interest [4,5].

One distinguishes two kinds of disc intersection: overlaps and crossways. Given two (topological) discs $D_1$ and $D_2$, an overlap (respectively, crossway) is a connected component of $D_1 \cap D_2$ whose intersection with the boundaries $\partial D_1$ and $\partial D_2$ is connected (respectively, disconnected): see Figure 3.

1This is how the complexity was stated, though probably an estimate close to ours could have been given.
Given a list of \( n \) discs where any two intersect in at most one component, and that an overlap, the arrangement is termed one of pseudodiscs and the union has \( O(n) \) features \[4\]. We generalise this, slightly: if there is a bound on the number of intersection components between any two discs, then the union has \( O(n) \) overlaps, no matter how many crossways.

On the other hand, \( n \) thin ellipses can have \( \Omega(n^2) \) crossways.

In order to limit the number of crossways, we posit the notion of positive crossing content, where there is a lower bound on the area of any crossway. This requires the disc boundaries to be differentiable (Figure 4).

We achieve positive crossing content using arguments based on compactness and continuity.

\[1.5\] Accordingly, our point of departure is the notion of a compact family of convex bodies, which have twice-differentiable boundaries and have a distance function based on the \( C^2 \) norm. From these we pass to compact families of discs which are \( C^1 \) and have a metric based on the \( C^1 \) norm. We show that the map from bodies to discs — which are hidden regions on the boundaries — is continuous, from which the compactness of the disc family and positive crossing content are derived.

With one further idea, that of footprint width, we are able to show that on any body \( B \) there are \( O(n) \) pairs \((D, E)\) of incident hidden and exposed areas (called discs and holes in the paper), whence the exposed areas on \( B \) have \( O(\lambda_s(n)) \) features, and \( H(S) \) has \( O(n\lambda_s(n)) \) features overall. Here \( \lambda_s(n) \) is the maximum length of \( n \)-letter order-\( s \) Davenport-Schinzel sequences, and \( s \) is a constant depending on the semialgebraic complexity of the bodies.

2 Convex hull

2.1 First assumptions; features.

\[2.1\] Let \( S \) be a set of solid bodies in \( \mathbb{R}^3 \). We make the following assumptions.

- The bodies are closed, bounded, and convex.
Figure 5: Convex hull of five spheres. Exposed facets, tunnel facets, and planar facets are marked e, t, and p, respectively.

- They are in general position: no four bodies possess a common tangent plane.
- They are pairwise disjoint.
- They are rounded: their boundary surfaces have unique tangent planes (or outward unit normals) at all points, and every tangent plane meets the boundary at just one point.

$H(S)$ is the (closed) convex hull of $\bigcup S$, i.e., of $\bigcup\{B : B \in S\}$.

**Structure of $H(S)$**. The features of $H(S)$ are its facets, edges, and vertices, as follows. As discussed in [14], $\partial H(S)$ is naturally divided into connected regions: its exposed facets, tunnel facets, and planar facets. The exposed facets are (path-) connected components of $(\partial H(S)) \cap \bigcup S$, tunnel facets are connected part-surfaces generated by line-segments touching two bodies, and (since the bodies are in general position) planar facets are triangular. Tunnel facets are bounded by two exposed facets and by two planar facets (or are quasi-cylindrical, joining two bodies).

Facets meet along edges, and edges meet at vertices; also, an edge could be a closed loop.

Under the assumption of general position, no facet touches more than three bodies. Figure 5 illustrates these features, except that exposed facets need not be simply connected.

The feature complexity of $H(S)$ is the total number of features, generally proportional to the number of facets.

If $B \in S$, we call $\partial B \cap \partial H(S)$ the exposed part of $B$, whereas $\overline{\partial B \cap H(S)^\circ}$ is its hidden part. (The exposed and hidden parts are both closed and they intersect along their common boundaries).²

### 2.2 Compact families and placements

Our analysis requires further assumptions about the kinds of body occurring in $S$. We require that each is a translated copy of a ‘model’ body. The ‘model’ bodies are to be taken from a restricted

---

² $X^\circ$ is the interior of $X$ and $\overline{X}$ is its closure.
family. For this reason, a model is a convex body subject to various restrictions. One restriction is that its boundary should be twice differentiable.

By the derivative $f'(x)$ of a function $f$ at $x$ we mean the Fréchet derivative [12], i.e., the linear map $h \mapsto f'(x)h$, if it exists, such that

$$f(x + h) = f(x) + f'(x)h + o(\|h\|).$$

A $C^r$-function is one which is $r$ times continuously differentiable.

We assume that each body in $S$ is specified by an inequality

$$f(x - a) \leq 1 : \quad B^{f,a} = \{ x \in \mathbb{R}^3 : f(x - a) \leq 1 \}.$$ 

$B^{f,a}$ is the translation by $a$, or a placement, of a model

$$B^f = B^{f,O} = \{ x : f(x) \leq 1 \}.$$ 

$\mathcal{F}$ is the family of all such functions $f$.

Our notation for open and closed balls in $\mathbb{R}^3$ is

$$N_d(x) = \{ y \in \mathbb{R}^3 : \|y - x\| < d \}$$

$$\overline{N}_d(x) = \{ y \in \mathbb{R}^3 : \|y - x\| \leq d \}$$

(2.2) In addition to the assumptions [2.1] for every $f \in \mathcal{F}$,

- The domain of $f$ is an open set containing $\overline{N}_1(O)$.
  The codomain of $f$ is $\mathbb{R}$.

- $f$ is piecewise algebraic of bounded algebraic degree (involving a bounded number of polynomials in $\mathbb{R}[x, y, z]$ with bounded total degree).

- $f$ is $C^2$. The derivative $f'(x)$ is, properly speaking, a row vector, but we shall work with its transpose, a column vector. Then $f''(x)$ is equivalent to a $3 \times 3$ matrix. The matrix is symmetric since the second derivatives are continuous.

- $f(x) > 1$ if $\|x\| = 1$, so $B^f$ is contained in the open ball $N_1(O)$.

- $f''(x)$ is positive definite, and $f'(x)$ is nonzero, for all $x$ in $\partial B^f$.

- The origin is interior to all models, i.e., $f(O) < 1$ for all $f \in \mathcal{F}$.

- $\mathcal{F}$ is closed under rotation around $O$, i.e., for any $f \in \mathcal{F}$ and $R \in SO(3)$, the group of rotations, $f \circ R \in \mathcal{F}$.

The last two assumptions are for simplicity. The norm $\|x\|$ is the usual Euclidean norm, which may also be used for matrices, and thus for second derivatives.
The $C^2$ norm on parametrisations $f \in \mathcal{F}$ is
\[
|f|_{C^2} = \text{(def)} \sup_{\|x\| \leq 1} \max(\|f(x)\|, \|f'(x)\|, \|f''(x)\|) = 
\max \left( \sup_{\|x\| \leq 1} \|f(x)\|, \sup_{\|x\| \leq 1} \|f'(x)\|, \sup_{\|x\| \leq 1} \|f''(x)\| \right). 
\]
and the $C^2$-distance $d(f, g)$ between two functions is $|f - g|_{C^2}$.

**Lemma** $SO(3)$ acts continuously on $\mathcal{F}$.  

**Sketch proof.** That is, if $A$ and $B$ are rotations, and $\|B - A\|$ is small, then $|f \circ B - f \circ A|_{C^2}$ is small.

If $\|B - A\|$ is small, then for all $x \in \overline{N}_1(O)$, $Bx - Ax$ is small, whence $f(Bx) - f(Ax)$, $f'(Bx) - f'(Ax)$, and $f''(Bx) - f''(Ax)$ are small.

**Definition** A family of models is compact if the parametrising family $\mathcal{F}$ is compact under the $C^2$ metric.

**Definition** Given a body $B = B^{f,a}$ and $p \in \partial B$, the (outward) unit normal $n(p)$ at $p$ is
\[
n(p) = \frac{f'(p - a)}{\|f'(p - a)\|}. 
\]

**Proposition** If $B$ is a rounded compact convex body, then the map
\[
\partial B \to S^2 : \quad p \mapsto n(p) 
\]
is a homeomorphism [6, Lemma 1].

**Compact families of discs**

We shall prove that hidden regions arising from a compact family of models form a compact family of discs. A transformation will be applied to hidden regions so they are topological discs on the unit sphere $S^2$.

Suppose $\phi : [0, 2\pi] \to \mathbb{R}^3$ is a continuous map. By its derivative $\frac{df}{d\phi}$ at $\phi$ is meant a one- or two-sided limit, presuming it exists:
\[
\frac{df}{d\phi} = \begin{cases} 
\lim_{h \to 0} \frac{f(\phi + h) - f(\phi)}{h} & \text{if } 0 < \phi < 2\pi, \\
\lim_{h \to 0+} \frac{f(\phi + h) - f(\phi)}{h} & \text{if } \phi = 0, \\
\lim_{h \to 0-} \frac{f(2\pi + h) - f(2\pi)}{h} & \text{if } \phi = 2\pi.
\end{cases}
\]

**Definition** A (closed) disc is generally taken in the topological sense, i.e., a topological space homeomorphic to the closed unit disc
\[
\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. 
\]

This paper is concerned with discs on the unit sphere $S^2$. An oriented $C^1$ Jordan curve in $S^2$ is the image of a map $f : [0, 2\pi] \to S^2$, satisfying the following conditions.
• The map $f$ is injective, except that $f(0) = f(2\pi)$.

• It is continuously differentiable, i.e., $\frac{df}{d\phi}$ is defined and continuous everywhere and $\frac{df}{d\phi}(0) = \frac{df}{d\phi}(2\pi)$.

• Its derivative is nowhere zero: $\frac{df}{d\phi} \neq \vec{O}$.

The Jordan-Schönflies Theorem (an extension of the Jordan Curve Theorem) [13], adapted to $S^2$, implies that every Jordan curve $J$ defines a unique closed disc in $S^2$: the curve may be oriented in the direction of increasing $\phi$, and $S^2 \setminus J$ is the union of two disjoint open topological discs of which $J$ is the boundary of both; the one meeting the oriented curve from its left-hand side is the interior $D^\circ$ of the disc, and $D = D^\circ \cup J$ is the closed disc. This gives a way of parametrising closed discs in $S^2$ with differentiable boundary, by $C^1$ maps.

(2.9) The $C^1$ norm on parametrisations $f$ is

$$\sup_{0 \leq \phi \leq 2\pi} \max(\|f(\phi)\|, \|df/d\phi\|) = \max \left( \sup_{0 \leq \phi \leq 2\pi} \|f(\phi)\|, \sup_{0 \leq \phi \leq 2\pi} \|df/d\phi\| \right).$$

leading to a metric on the space of all such closed discs in $S^2$. A compact family of discs is a compact set of parametrisations, under this metric.

2.4 The theorem on pre-seams

Suppose that $B_0$ and $B_1$ are disjoint copies of ‘model’ bodies. The $B_0, B_1$-seam is the set of points on $\partial B_0$ at which the tangent plane is also a (supporting) tangent plane to $B_1$. Since the model bodies are rounded, the seam is homeomorphic to the circle $S^1$ [6, Lemma 5].

The normal map from $\partial B_0$ is as follows. Explicitly, if $B_0 = \{x : f_0(x - a_0) \leq 1\}$, and $p \in \partial B_0$ (i.e., $f_0(p - a_0) = 1$), then the outward unit normal to $B_0$ at $p$ is

$$\frac{f'(p)}{\|f'(p)\|}.$$

It is known to map $\partial B_0$ homeomorphically onto the 2-sphere $S^2$ [6, Lemma 1].

(2.10) Definition The $B_0, B_1$ pre-seam is the image of the $B_0, B_1$-seam under the normal map to $\partial B_0$.

(2.11) Lemma Both the $B_0, B_1$-seam, and the $B_0, B_1$ pre-seam, are semi-algebraic of bounded degree.

Proof. (See [6 Theorem 3]). $B_0, B_1$ are semi-algebraic, so $H = H(B_0, B_1)$ is semialgebraic; also, so is $\partial H$, and $D = H \cap B_0^\circ$, and $\overline{D}$, and $E = (\partial H) \cap \partial B_0$, and $\overline{E}$, which is the $B_0, B_1$-seam, call it $S$. Suppose $B_0 = B_{f_0, a_0}$.

The $B_0, B_1$ pre-seam is the set of all unit vectors $\omega$ which are positive multiples of $f'(x - a_0)$ where $x$ is on the seam, i.e., given $S$ is the $B_0, B_1$-seam,

$$\{\omega \in \mathbb{R}^3 : \|\omega\|^2 = 1 \land (\exists t \in \mathbb{R})(\exists x \in S)(\omega = t^2 f'(x - a_0))\}.$$
It is therefore semi-algebraic of bounded degree.

Since the family of models is closed under rotations, we can assume for convenience that the bodies $B_0, B_1$ form a ‘balanced pair’:

(2.12) Definition A balanced pair of bodies $B_0 = B^{f,a}, B_1 = B^{g,b}$ is a pair of bodies such that

- $B_0^* \cap B_1^* = \emptyset$ (the bodies can touch but not overlap properly).

- Let $c_0 \in B_0, c_1 \in B_1$ be the points, unique since the bodies are rounded, such that $\|c_1 - c_0\|$ is minimal. Then they are opposite points on the $z$-axis, with $c_0$ left of $c_1$, i.e., $c_0 = (-s, 0, 0)$ and $c_1 = (s, 0, 0)$, where $s \geq 0$.

Given a compact family $\mathcal{F}$ of convex models,

$$\text{BP}(\mathcal{F})$$

denotes the set of balanced pairs.

Based on the $C^1$ metric on Jordan curves (§2.9), $\text{BP}(\mathcal{F})$ is given the metric

$$d((B^{f_0,a_0}, B^{f_1,a_1}), (B^{g_0,b_0}, B^{g_1,b_1})) = (\text{def}) \max (\|g_0 - f_0\|_{C^2}, \|b_0 - a_0\|, \|g_1 - f_1\|_{C^2}, \|b_1 - a_1\|).$$

The space of balanced pairs is unbounded and therefore not compact.

(2.13) Theorem For any balanced pair $B_0, B_1$, the $B_0, B_1$ pre-seam is an oriented $C^1$ Jordan curve in $S^2$, and the pre-seam map from $\text{BP}(\mathcal{F})$ to oriented $C^1$ Jordan curves on $S^2$, is continuous, and the image of $\text{BP}(\mathcal{F})$ is a compact family of discs.

Proof deferred to Appendix A

3 Discs, overlaps, and crossways

The complexity of unions of discs has been widely studied [5]. In this paper the discs correspond to hidden regions but they are on $S^2$, bounded by pre-seams.

(3.1) Definition In this paper, discs will be subspaces of $S^2$, and be parametrised by $C^1$ maps as described in [2.8].

Two discs $D, E$ are in general position if $(\partial D) \cap \partial E$ is finite, and at any point in the intersection, the boundary tangents intersect transversally.

A list of discs is in general position if every two are in general position (transverse intersections), and for every three, $D_i, D_j, D_k$, $\partial D_i \cap \partial D_j \cap \partial D_k = \emptyset$.

A list of discs $D_i$ (in general position) has bounded intersection number if any two discs intersects in at most $\kappa$ points. The uniform bound $\kappa$ is usually left implicit.

In a list $D_i$ of discs, a disc $D_i$ is redundant if $D_i \subseteq \bigcup_{j \neq i} D_j$. A list is irredundant if no disc is redundant.
Clearly, omission of redundant discs leaves $\bigcup D_i$ unchanged.

The complexity of $\bigcup D_i$ is the total number of edges in the boundary $\partial (\bigcup D_i)$. This could be $\Omega(n^2)$, as with $n$ pairwise intersecting thin ellipses, whose complement has $\Omega(n^2)$ components. Under suitable assumptions related to convex hulls in $\mathbb{R}^3$, there are $O(n)$ exposed regions. Firstly, we distinguish two kinds of intersection, as mentioned in the introduction.

**3.2 Definition** Let $D$ and $E$ be two discs in general position. An intersection component is a connected component of $D \cap E$. We distinguish overlaps from crossways as follows.

Let $K$ be an intersection component. The boundary $\partial K$ is composed of edges alternately from $D$ and from $E$. If $K$ is bounded by just two edges (and vertices), it is an overlap. Otherwise, it is bounded by four or more, and is termed a crossway (see Figure 3).

Put more briefly: $K$ is an overlap iff $K \cap \partial D$ is connected, so $K \cap \partial E$ is connected. Similar ideas occur in [4] in connection with ‘pseudodiscs’ which are sets of discs whose intersection, if nonempty, is a single overlap. Also, [4] discusses discs with polygonal boundaries. It shows that unions of $n$ pseudodiscs have $O(n)$ features.

We could generalise [4] slightly by showing that the union of $n$ discs in general position, with no crossways, and bounded intersection, has $O(n)$ features.

A family (of Jordan curves or the discs they enclose) has positive crossing content if there is a positive lower bound $\kappa$ such that, for every two discs $D, E$ in general position drawn from the family, every crossway $K$ from $D \cap E$ has measure $\geq \kappa$.

## 4 Crossing content

In this section it is proved (Corollary 4.16) that every compact family of discs has positive crossing content, as defined above. One generally expects measure to be a continuous function of sets under various metrics. The curious fact is that measure is discontinuous under the Hausdorff metric, as observed in [2]. The reason is very simple. If $K$ is a bounded set of positive measure, then it contains a countable dense subset, and hence there is an increasing sequence $F_n$ of finite subsets whose union is dense in $K$: $d(F_n, K) \to 0$ in the Hausdorff metric, whereas $\mu(K) > 0$ and $\mu(F_n) = 0$ for all $n$. Measure may or may not be continuous for closed discs under the Hausdorff metric: we show that it is continuous for closed discs under the $C^1$ metric.

**4.1 Successor convention.** If $i$ is an index in finite range $1 \ldots n$, we interpret $i \pm 1$ cyclically:

$$i + 1 \quad \text{means } (i \mod n) + 1; \quad i - 1 \quad \text{means } ((i - 2) \mod n) + 1$$

Our interest in compactness is largely because of the following well-known proposition, which is an easy consequence of the finite intersection property.

**4.2 Proposition** Let $X$ be a nonempty compact space and $f : X \to \mathbb{R}$ a continuous map. Then $f(X)$ is bounded, and there exist points $x_0, x_1$ in $X$ such that

$$f(x_0) = \inf_X f(x) \quad \text{and} \quad f(x_1) = \sup_X f(x).$$

---

3 An edge could be a disc boundary. Otherwise (in general position) it is incident to two vertices.
(4.3) Definition Given two bounded nonempty sets \(X\) and \(Y\) in any metric space, the Hausdorff distance between \(X\) and \(Y\) is

\[
d(X, Y) = (\text{def}) \max \left( \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right). \]

If we take (in any metric space) the \(\epsilon\)-ball around any point \(x\) to mean

\[
N_\epsilon(x) = \{y : d(x, y) < \epsilon\} \quad \text{and} \quad \overline{N}_\epsilon(x) = \{y : d(x, y) \leq \epsilon\},
\]

then in a vector space with bounded nonempty set \(X\), the \(\epsilon\)-neighbourhood of \(X\) with respect to Hausdorff distance is

\[
N_\epsilon(X) = \{Y : X \subseteq Y + N_\epsilon(O) \quad \text{and} \quad Y \subseteq X + N_\epsilon(O)\}. \]

If the metric space is complete, then the set of closed bounded nonempty subsets is complete under the Hausdorff metric \([7]\).

(4.4) Definition Recall that an intersection component is a connected component of \(D \cap E\), where \(D\) and \(E\) are discs in general position (Definition [2.2]). Given a compact family \(F\), an intersection component from \(F\) is an intersection component of two discs, in general position, from \(F\).

The space \(\text{IC}(F)\), the space of intersection components, is the completion of the space \(T\), where \(T\) is the metric space consisting of all triples

\[
D, E, K
\]

where \(D\) and \(E\) are in general position\(^4\) \(K \neq \emptyset\), and \(K\) is a component of \(D \cap E\).\(^5\) The metric on \(T\) is

\[
d((D, E, K), (\hat{D}, \hat{E}, \hat{K})) = \max \left( |f - \hat{f}|_{C^1}, |g - \hat{g}|_{C^1}, d(K, \hat{K}) \right), \]

where \(f, \hat{f}, g, \hat{g}\) parametrise \(D, \hat{D}, E, \hat{E}\), and \(d(K, \hat{K})\) is Hausdorff distance.

\(^4\) or rather, parametrisations of these discs, being \(C^1\) oriented Jordan curves with the disc interiors to their left.

\(^5\) On the other hand, \(K\) is taken at face value, as a set.
(4.5) Remarks. Suppose \((D_n, E_n, K_n)\) is a Cauchy sequence, converging to \((D, E, K)\). \(D\) and \(E\) are (parametrised by maps) in \(\mathcal{F}\), and \(K\) consists of all limit points of all sequences \(x_n, x_n \in K_n\). Theorem 3-3. \(K\) is a (path)-connected subset of \(D \cap E\), but not necessarily a connected component of \(D \cap E\), and not necessarily a closed disc (Figure 6). It can be a contractible union of closed discs which are connected by touching or linked by edges. Each linking edge is a closed segment of \(\partial D \cap \partial E\) where the ‘inside’ of \(D\) and of \(E\) are on opposite sides of this segment.

(4.6) Lemma The above set \(K = \lim_n K_n\) is connected.

Proof. Again from [7]: limits of connected sets are connected.

(4.7) Definition If \(A\) is a 2-dimensional set in \(S^2\), its natural metric measure (spherical area) will be written \(\mu(A)\).

If \(J\) is a rectifiable curve in \(S^2\), then its length will be written \(\lambda(J)\).

(4.8) Lemma For any disc \(D\) parametrised by a function in \(\mathcal{F}\), and any \(\epsilon > 0\),
\[
\mu((S^2 \cap (\partial D + N_\epsilon(O)))) = O(\epsilon \lambda(\partial D)).
\]

Proof. For each \(x \in \partial D\), let \(I_x\) be the connected component of \(N_\epsilon(x) \cap \partial D\) which contains \(x\). Because \(\partial D\) is compact, we can choose \(x_1, \ldots, x_n\), in cyclic order around \(\partial D\), so that \(\partial D \subseteq \bigcup I_{x_j}\). We can assume that \(n\) is minimal, which implies that for \(1 \leq j \leq n\),
\[
I_{x_j} \setminus \bigcup_{i \neq j} I_{x_i} \neq \emptyset.
\]
Let
\[
V = \bigcup_j N_{2\epsilon}(x_j)
\]
Next, \(\partial D + N_\epsilon(0) \subseteq V\).

Given \(x \in \partial D + N_\epsilon(0)\), choose \(y \in \partial D\) so that \(x \in y + N_\epsilon(0)\). Then choose \(j\) so that \(y \in x_j + N_\epsilon(0)\). Then \(x \in x_j + N_{2\epsilon}(0)\). Thus \(\partial D + N_\epsilon(0) \subseteq V\) as claimed.

For any \(j\),
\[
S^2 \cap N_{2\epsilon}(x_j)
\]
is a circular region on \(S^2\) subtending an angle \(\theta = 2 \sin^{-1}(\epsilon)\) at \(O\). Projection from \(O\) onto the tangent plane at \(x_j\) is an area-increasing map, so the area of the circular region
\[
\mu(S^2 \cap N_{2\epsilon}(x_j)) \leq \pi \tan^2(2 \sin^{-1}(\epsilon))
\]
is \(O(\epsilon^2)\).

\(^\text{6}\) Recall that all these discs are in the unit sphere \(S^2\).
Consider the subsequence \( x_1, x_3, x_5, \ldots \). Note that \( I_{x_1} \cap I_{x_3} = \emptyset \), since otherwise \( I_{x_1} \cap I_{x_3} \) would be an interval containing \( I_{x_2} \), so \( x_2 \) would be redundant. In general, successive intervals \( I_{x_{2j-1}} \) and \( I_{x_{2j+1}} \) are disjoint. If \( n \) is odd then \( I_n \cap I_1 \) is nonempty and we need to discard \( I_n \), but in any case \( n/2 \) intervals are retained, and

\[
\sum_{j=1,3,\ldots} \lambda(I_j) \leq \lambda(\partial D)
\]

Each interval \( I_j \) is the connected component containing \( x_j \) of \( N_\epsilon(x_j) \cap \partial D \). Its endpoints are on the boundary of that region, and it passes through the centre, so its length is at least \( 2\epsilon \). Therefore

\[2\epsilon n/2 \leq \lambda(\partial D).\]

Therefore

\[n\epsilon^2 \quad \text{is} \quad O(\epsilon \lambda(\partial D)).\]

Also,

\[\mu(V) \leq \sum_j \mu(S^2 \cap N_{2\epsilon}(x_j))\]

which is \( O(n\epsilon^2) \), so \( \mu(V) \) is \( O(\epsilon \lambda(\partial D)) \).

\[\mu(K_n) \leq \mu(K) + O(\epsilon(\lambda(\partial D) + \lambda(\partial E)))\]

\[\mu(K_n) \leq \mu(K) + O(U\epsilon).\]

**Lemma** The map

\[\text{IC}(\mathcal{F}) \rightarrow \mathbb{R}; \quad (D, E, K) \rightarrow \mu(K)\]

is continuous on \( \text{IC}(\mathcal{F}) \).

**Proof.** Suppose \( (D_n, E_n, K_n) \rightarrow (D, E, K) \) where \( D_n \) and \( E_n \) are in general position, so \( \partial K_n \) is a Jordan curve. (This cannot be assumed for \( \partial K \)).

Since \( \lim D_n = D \) under the \( C^1 \) norm on parametrisations, \( \partial D \) is rectifiable and \( \lambda(\partial D_n) \rightarrow \lambda(\partial D) \). Similarly for \( E_n \) and \( E \). Therefore there exists an upper bound, call it \( U \), on all these lengths:

\[U = \max \left( \lambda(\partial D), \sup_n \lambda(\partial D_n), \lambda(\partial E), \sup_n \lambda(\partial E_n) \right).\]

Fix \( \epsilon > 0 \). For sufficiently large \( n \),

\[K_n \subseteq S^2 \cap (K + N_\epsilon(O)) \quad \text{and} \quad K \subseteq S^2 \cap (K_n + N_\epsilon(O)).\]

Given

\[x \in S^2 \cap (K + N_\epsilon(O)) \setminus K,\]

let \( y \) be a point in \( K \) closest to \( x \). Clearly \( y \) is not interior to \( K \), so \( y \in \partial K \) and therefore \( y \in \partial D \cup \partial E \). Therefore

\[x \in (\partial D \cup \partial E) + N_\epsilon(0).\]

Therefore

\[K_n \subseteq K \cup ((\partial D \cup \partial E) + N_\epsilon(O))\]

\[\mu(K_n) \leq \mu(K) + O(\epsilon(\lambda(\partial D) + \lambda(\partial E)))\]

\[\mu(K_n) \leq \mu(K) + O(U\epsilon).\]
Similarly $\mu(K) \leq \mu(K_n) + O(U\epsilon)$. Therefore

$$\lim_{n} \mu(K_n) = \mu(K).$$

(4.10) **Lemma** Taking discs $D$ from a compact family, $\mu D$ is a continuous function of $D$ (with respect to the $C^1$ norm). (Proof similar to above lemma, but easier.)

(4.11) **Corollary** Taking discs $D$ (parametrised) from a compact family $\mathcal{F}$,

$$\inf_{D \in \mathcal{F}} \mu D > 0.$$

**Proof.** Measure is continuous on $\mathcal{F}$, and $\mathcal{F}$ is compact and therefore complete, so there exists a disc $D$ of minimal measure, and that measure must be positive.

(4.12) **Definition** A list $D_1, \ldots, D_n$ of discs has positive crossing content if there is a positive constant $\kappa$, left implicit, such that every crossway occurring among the discs has crossing content $\geq \kappa$.

(4.13) **Definition** Suppose that $f : [0, 2\pi]$ parametrises a $C^1$ Jordan curve on $S^2$, enclosing a disc $D$ on $S^2$. The normal at any point $x$ on $S^2$ is just $x$ itself.

The second outward normal to $\partial D$ at $f(\theta)$ is

$$\frac{df/d\theta \times f(\theta)}{\|df/d\theta \times f(\theta)\|}$$

The lemma below is a form of mean-value theorem.

(4.14) **Lemma** Let $e$ be a differentiable curve-segment in $S^2$ joining two points $A$ and $B$. Let $L$ be the line through $O$ in $\mathbb{R}^3$ parallel to $AB$. Then there exists a plane through $L$ tangent to $e$. 
Proof (sketch). See Figure 8. Let $P$ be the plane $OAB$. $S^2 \cap P$ is the great circle containing $AB$. If $e$ is entirely within $P$ then the statement holds. Otherwise $e \setminus P$ consists of a union of nonempty open intervals. Let $I$ be one of them: it is contained in one of the open hemispheres whose union is $S^2 \setminus P$. The plane $P$ can be rotated around $L$, maintaining nonempty intersection with $I$, until it becomes tangent to $I$ and thus to $e$. \[\]

(4.15) Theorem. Let $(D_i, E_i, K_i)$ be an infinite sequence of triples in $IC(F)$ converging to $(D, E, K)$ (Definition 4.4). Then $\mu(K) > 0$.

Proof. Suppose $\mu(K) = 0$.

Referring to the remarks in [4.3] and Figure 6 it must be that $K^\circ = \emptyset$ and $K = \partial K$ is entirely contained in $\partial D \cap \partial E$. Also, $D \neq E$ since otherwise $K = D = E$ would have positive measure.

By the Jordan Curve Theorem (on $S^2$) $\partial D \neq \partial E$, and $\partial D \cap \partial E$ is a union of closed curve-segments and/or points. Thus $K$ is a closed curve-segment contained in $\partial D \cap \partial E$. Possibly $K$ is a single point.

Implicitly $D_i$ is parametrised by a $C^1$ map $f_i : [0, 2\pi] \to S^2$ and $D$ by a map $f$, where $f_i \to f$ in the $C^1$ metric. Similarly $E_i$ and $E$ are parametrised by maps $g_i$ and $g$. We write $\theta$ for the argument of $f_i$ and $f$ and $\phi$ for that of $g$.

By choosing a subsequence if necessary, it may be assumed that all $K_i$ have the same number of vertices. Suppose they have $\nu$ vertices. The vertices $V_{ij}$ can be written as

$$V_{ij} = f_i(\theta_{ij}) = g_i(\phi_{ij}), \quad 1 \leq j \leq \nu.$$  

By choosing a subsequence, it may be assumed that for each $j$, $\theta_{ij}$ and $\phi_{ij}$ converge, so the vertices converge,

$$V_{ij} \to V_j, \quad 1 \leq j \leq \nu,$$

and the second outward normals to $D_i$ and $E_i$ at those vertices also converge.

$K$ is a point or a simple closed curve-segment common to $\partial D$ and $\partial E$. Let $X$ and $Y$ be the endpoints of $K$.

Claim: $V_j \neq V_{j+1}$ (4.1). For suppose that $V_{i,j}$ and $V_{i,j+1}$ converge to the same point. Let $d_i$ and $e_i$ be the edges joining these vertices along $\partial D_i$ and $\partial E_i$ respectively. One of these edges is an edge of $K_i$, and the other is outside $K_i$ except at its endpoints. Certainly one or the other happens infinitely often and we can take a subsequence so that, without loss of generality, $d_i$ is an edge of $K_i$ and $e_i$ is outside $K_i$, for each $i$.

By Lemma 4.14 if $L$ is the line through $O$ parallel to $V_{ij} V_{i,j+1}$, then there are planes through $L$, one tangent to $d_i$, at $x_i$, say, one through $V_{ij} V_{i,j+1}$, and one tangent to $e_i$, at $y_i$, say.

For each $i$, the outward normal to $\partial D_i$, call it $n(d_i)$, at $x_i$, is normal to the first of the three planes, and that to $\partial E_i$ at $y_i$, call it $n(e_i)$, is normal to the third. If $\eta_i$ is the angle between the first and third planes, then rotation through $\eta_i$ about $L$ takes the first plane to the third, and takes $n(d_i)$ to $\pm n(e_i)$.

Furthermore, rotation through $\eta_i$ is in the general direction of $n(d_i)$, away from $D_i$ at $x_i$, so in fact rotation takes $n(d_i)$ to $n(e_i)$.

As $i \to \infty$, the points $x_i$ and $y_i$ become arbitrarily close to $V_{ij}$, and the angle separating the two planes decreases to zero. This implies that the outward normals to $\partial D_i$ and to $\partial E_i$ at $V_{ij}$ converge to the same vector. But, in the limit, the outward normals must add to zero, since $D$ and $E$ are externally tangent along $K$ (otherwise $K$ would have nonempty interior). This contradiction shows that the points $V_j$ and $V_{j+1}$ are different, as claimed. As a consequence, $X \neq Y$.  

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Again, claim that no vertex $V_j$ can differ from $X$ and $Y$. For suppose that the edges incident to $V_{ij}$ are $d_i$ and $e_i$. Their outward normals $n(d_i)$ and $n(e_i)$ converge to vectors which are either equal or complementary. But they cannot be complementary, because the angle of separation must decrease to zero (Figure 10), proving the claim.

Summing up: all vertices $V_j$ are distinct, and they only be at $X$ or $Y$. Therefore $\nu = 2$, and the $K_i$ are overlaps: a contradiction.

(4.16) **Corollary** Every compact family has positive crossing content.

**Proof.** Crossing content is continuous on IC$(\mathcal{F})$ (Lemma 4.9). Since the family is compact, there exists a pair $D, E$ of discs with a limiting crossway $K$ such that $\mu(K)$ minimises the crossing content. But $\mu(K) > 0$.

5 **Footprint width**

From positive crossing content it is possible to deduce that $S^2 \setminus \bigcup D_i$ has $O(n)$ components, which we shall call ‘holes.’ However, we need a stronger bound, an $O(n)$ bound on the number of pairs $(D, H)$ where $D$ is a disc and $H$ a hole and $D \cap H \neq \emptyset$. It is not clear whether this bound is a consequence of bounded intersection number. But the ‘positive footprint width’ property discussed here will make it clear (assuming that all intersections are crossways, not overlaps).

Suppose that $D$ is a disc with disjoint crossways $K_1$ and $K_2$. Each side of $K_1$ not lying in $\partial D$ is incident to a unique component of $D \setminus K_1$, and exactly one of these components contains $K_2$. The point is that there is a lower bound on the length of that side, or, more simply, on the separation of its endpoints.

(5.1) **Definition** Consider the family of tuples $(D, E, s, E')$ where the discs $D, E, E'$ are in general position, $s$ is a side of a crossway from of $D \cap E$, and the component of $D \setminus E$ incident to $s$ contains a crossway from $E \cap E'$. 
The edge $s$ we call a footprint on $\partial D$.

The distance separating the endpoints of $s$ we call a footprint width from $E$ on $D$.

(5.2) Lemma If $\mathcal{G}$ is a compact family of discs, then there is a strictly positive lower bound on the set of possible footprint widths.

Proof. We parametrise the given family by the three discs and the two vertices (on $\partial D \cap \partial D_1$) which are endpoints of $s$. Thus we represent configurations as quintuples. We give it a metric in the usual way: the distance between $D_1, E_1, x_1, y_1, E'_1$ and $D_2, E_2, x_2, y_2, E'_2$ is the maximum distance between corresponding components. Consider the completion of this space as a compact metric space.

Given a convergent sequence $(D_n, E_n, x_n, y_n, E'_n)$, since the component of $D_n \setminus E_n$ incident to $s_n$ contains $E'_n$, its area is bounded from below (by positive content), and therefore the endpoints of $s_n$ cannot converge to the same point. But the distance $\|x_n - y_n\|$ is a continuous function of the quintuples, so the distance cannot decrease to zero, and the footprint width is bounded below.  

6 $O(n)$ overlaps

In this section, and the next, we consider a list $D_1, \ldots, D_n$ of discs in general position, irredundant (Definition 3.1), with bounded intersection number, and positive crossing content.

(6.1) Definition An o-vertex is an external vertex in $\bigcup D_i$ which is incident to an overlap between two discs.

In this section, the goal is to prove that $\bigcup D_i$ has $O(n)$ o-vertices.

(6.2) Definition A proper hub is a connected component of the union of all crossways. A hub is either a proper hub or a disc without crossways, i.e., a disc $D_i$ whose only intersection components with other discs $D_j$, if any, are overlaps.

The result below depends crucially on crossing content.
(6.3) **Lemma** There are $O(1)$ proper hubs.

**Proof.** For each proper hub $H$, choose a crossway $K_H$ contained in $H$. These crossways $K_H$ are disjoint. The discs have positive crossing content, so the measure of crossways, and hence of proper hubs, is bounded below, and $S^2$ has area $4\pi$, so there are $O(1)$ proper hubs.

(6.4) **Lemma** There are $O(n)$ (external) vertices incident to overlaps.

**Proof.** In other words, there are $O(n)$ o-vertices. Given a disc $D_i$, let $V_i$ be the set of all overlaps between $D_i$ and other discs, i.e., connected components $L$ of $D_i \cap D_j$ such that $L \cap \partial D_i$ and $L \cap \partial D_j$ are connected. For the rest of this proof, let ‘overlap’ mean a maximal connected union of such overlaps $L$, i.e., a connected component of $\bigcup_{E \in V_i} E$.

The overlaps can be made arbitrarily thin by retracting without changing the external features of $\bigcup D_i$.

Once they are sufficiently thin the overlaps around $D_i$ become disjoint$^7$ and

$$C_i = \text{ (definition) } D_i \setminus \bigcup_{E \in V_i} E$$

will be simply connected. Whether $C_i$ intersects any crossways is irrelevant: it is only overlaps which are being counted. Choose an internal vertex $x_i$ in each $C_i$.

Choose paths joining $x_i$ to all the o-vertices on $\partial C_i$. Since $C_i$ is path-connected, these paths can be made disjoint except at the internal vertices $x_i$. The union of these paths define a planar multigraph whose vertices are the $n$ vertices $x_i$. Let $k$ be the maximum number of edges joining any two vertices $x_i$ and $x_j$, so there are $\leq 3nk$ edges. By assumption the discs have bounded intersection number, so $k$ is bounded, and there are $O(n)$ such edges and $O(n)$ o-vertices.

One can continue the retraction a little further to actually eliminate the overlaps and leave only crossways. This will of course remove $O(n)$ vertices and probably introduce some new ones; we may conclude

(6.5) **Corollary** Given a collection of $n$ discs $D_i$ in general position, it is possible to shrink the discs so as to remove the $O(n)$ overlaps while leaving the crossways unchanged.

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$^7$This distinguishes overlaps from crossways.
7 The complexity bounds

(7.1) Lemma Let \( D_1, \ldots, D_n \) be a collection of discs in \( S^2 \), in general position.

In establishing upper bounds for the union \( \bigcup D_i \), we can assume that the union is connected and every component of the complement \( S^2 \setminus \bigcup D_i \) is simply connected.

Proof. Let \( C_1, \ldots, C_k \) be the connected components of \( \bigcup D_i \). Every (boundary) feature on \( \bigcup D_i \) is on one of these components, so the feature complexity is the total of the component complexities. It is enough, therefore, to estimate the complexity of a component \( C_j \) in terms of the number of discs forming the component.

This allows us to assume \( \bigcup D_i \) is connected, in which case every component of its complement is simply connected.  

(7.2) Removing overlaps. For the rest of this section we shall assume that \( D_1, \ldots, D_n \) are in general position, that \( \bigcup D_i \) is connected, and, using Corollary 6.5 that there are no overlaps between discs, only crossways.

(7.3) Definition A hole is a connected component of

\[
S^2 \setminus \bigcup D_i
\]

Since \( \bigcup D_i \) is connected, the holes themselves are homeomorphic to closed discs.

(7.4) Lemma For each disc \( D_k \), the number of holes incident to \( D_k \) is uniformly bounded.

Proof. Follow \( \partial D_k \) anticlockwise, noting the holes to which it is incident. Suppose that \( H_1 \) and \( H_2 \) meet \( D_k \) in consecutive order, and \( H_1 \not= H_2 \).

Let \( H'_i, i = 1, 2, \) be the holes in

\[
\bigcup_{i \not= k} D_i
\]

containing \( H_i \). (Possibly the smaller union is disconnected.)

Suppose first that \( H'_1 = H'_2 \). Then, as illustrated in Figure 13, \( D_k \) intersects some other disc between successive intersections of \( \partial D_k \) with this hole, and by the positive crossing content property (Corollary 4.16), this can happen \( O(1) \) times.

Figure 13: \( H'_1, H'_2 \) merge into one ‘hole’ when \( D_k \) is omitted. The shaded region contains a crossway, so it has positive measure.
Otherwise $H'_1 \neq H'_2$. Let $x$ and $y$ be successive intersections of $\partial D_k$ with $H'_1$ and $H'_2$, and consider the segment of $\partial D_k$ joining $x$ and $y$. It meets other discs $D_i$, such as the disc $E$ illustrated in Figure 14.

Claim that at least one such disc intersects another disc to the right of the segment $xy$. For otherwise there would be a chain of exposed edges joining $x$ to $y$ and they would belong to the same hole in $S^2 \setminus \bigcup_{i \neq k} D_i$.

By this claim, the disc $E$, say, intersects another disc $E'$ to the right of $xy$ and has a positive footprint width on $\partial D_k$ (Lemma 5.2). So this, too, happens $O(1)$ times.

Therefore there are $O(n)$ pairs $D_i, H_j$ where disc $D_i$ intersects hole $H_j$. If $i_j$ is the number of discs meeting $H_j$, then $H_j$ has at most $\lambda_s(i_j)$ features, where $s + 1$ bounds the number of disc boundary intersections, and we deduce

(7.5) Corollary Suppose $D_1, \ldots, D_n$ are drawn from a compact family $G$ of discs on $S^2$. Then $\bigcup D_i$ has feature complexity $\leq \lambda_s(dn)$ where $s$ and $d$ are constants.

(7.6) Corollary Let $\mathcal{F}$ be a compact family of convex bodies in $\mathbb{R}^3$. Given a set $B_1, \ldots, B_n$ of bodies in general position drawn from $\mathcal{F}$, $H(\cup B_i)$ has complexity $O(n \lambda_s(n))$ where $s$ is a constant depending on $\mathcal{F}$.

From [11] section 3.4] estimates for $\lambda_s(n)$ are given of the form $n \cdot 2^{p(\alpha(n))}$ where $p$ is a polynomial and $\alpha(n)$ is the inverse Ackermann function. For any fixed $s$, $n \log^* n$ is a simple overestimate. So we have

(7.7) Corollary Under the conditions of the above corollaries, the union of $n$ discs has $O(n \log^* n)$ features and the convex hull of $n$ bodies has $O(n^2 \log^* n)$ features.

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### A Appendix: nature and continuity of the pre-seam map

This appendix contains a proof of Theorem 2.13.

Let $\mathcal{F}$ be a compact family of convex bodies in $\mathbb{R}^3$. For any balanced pair $B_0, B_1$, the $B_0, B_1$ pre-seam is an oriented $C^1$ Jordan curve in $S^2$, and the pre-seam map from $\text{BP}(\mathcal{F})$ to oriented $C^1$ Jordan curves on $S^2$, is continuous, and its image is a compact family of discs.
A.1 Silhouettes

The convention that $[0, 2\pi]$ is the domain of parametrisations can complicate the notation. To remedy this, we introduce the following ad-hoc

(A.1.1) Notation Given $\theta \in [0, 2\pi]$ and $0 < \kappa < \pi$, we use $(\theta \pm \kappa)$ as an abbreviation for $(\theta - \kappa, \theta + \kappa)$, and $((\theta \mp \kappa))$ for an equivalent open subset of $[0, 2\pi]$:

$((\theta \mp \kappa)) = ([0, 2\pi] \cap (\theta \mp \kappa)) \cup ([0, 2\pi] \cap (\theta + 2\pi \mp \kappa))$.

If $\theta \neq 0$ and $\theta \neq 2\pi$ and $\kappa \leq |\theta|$ then $((\theta \mp \kappa)) = (\theta \mp \kappa)$. Also,

$((0 \mp \kappa)) = ((2\pi \mp \kappa)) = [0, \kappa) \cup (2\pi - \kappa, 2\pi]$.

We call $((\theta_0 \mp \kappa))$ a generalised interval.

Silhouettes enable one to relate tangent planes in $\mathbb{R}^3$ to tangent lines in $\mathbb{R}^2$: let $P$ be a plane in $\mathbb{R}^3$. If a tangent plane $T$ to $B$ intersects $P$ perpendicularly, then its projection in $P$ is its intersection $T \cap P$, a line tangent to the silhouette of $B$. This simplifies the construction of tangent planes.

(A.1.2) Definition The silhouette of a (translated) convex body $B$ in a plane $P \subseteq \mathbb{R}^3$ is the image of $B$ under orthogonal projection from $\mathbb{R}^3$ onto $P$. It is a convex two-dimensional set.

(A.1.3) In discussing the silhouette of a body $B$,

(i) $B = B^{f,O}$

will be a model centred at the origin, (ii) $P$ will be

$P = \{(x, y, 0) : x, y \in \mathbb{R}\}$,

i.e., the $xy$-plane; until further notice, (iii) $\pi$ will be vertical projection onto $P$, or (by abuse of language) onto $\mathbb{R}^2$,

$\pi : (x, y, z) \mapsto (x, y, 0) \in \mathbb{R}^3 \equiv (x, y) \in \mathbb{R}^2$,
(iv) $S$ will be the silhouette in $P$ (the $xy$-plane) or $\mathbb{R}^2$,

$$S = \pi B$$

and (v) $X$ or $X_P$ will be the inverse image in $B$ of the silhouette boundary

$$X = X_P = B \cap \pi^{-1} \partial S.$$ 

Note

- $S = \{(x, y) \text{ or } (x, y, 0) : \exists z ((x, y, z) \in B)\}$ so $S$ is semi-algebraic of bounded degree.

- $X \subseteq \partial B$, because projection is an open map, taking the interior of $B$ to the (relative) interior of $S$.

- By Proposition 2.7, $\partial B \to S^2 : p \mapsto n(p)$ is a homeomorphism. It takes $X$ to the equator, so $X$ is homeomorphic to $S^1$ and to $\partial S$.

(A.1.4) Lemma $p \in X_P \iff \text{(p \in \partial B and the tangent plane to \partial B at p is vertical)}$.

**Proof.** Suppose $p \in \partial B$ and $T$ is vertical, where $T$ is the tangent plane to $\partial B$ at $p$. $B$ is entirely on one side of $T$, i.e., $B \subseteq H$ where $H$ is one of the two closed half-spaces bounded by $T$. Let $L = P \cap T$ and $h = P \cap H$, so $L$ is a line in $P$ and $h$ is a half-plane in $P$ bounded by $L$. Note that $x \in H \iff \pi(x) \in h$. Since $B \subseteq H$, $S = \pi B \subseteq h$. Also, $\pi(p) \in L$, so $L$ is tangent to $S$ at $\pi(p)$, $\pi(p) \in \partial S$, and $p \in \pi^{-1} \partial S$. Therefore $p \in X_P$.

Suppose $p \in X_P$, so $\pi p \in \partial S$. Let $L$ be the line tangent to $\partial S$ at $\pi p$ and let $T = \pi^{-1} L$. Let $h$ be the half-plane bounded by $L$ which contains $S$. Then $\pi^{-1}(h)$ is a half-space bounded by $T$ and containing $\pi^{-1} S$, so it contains $B$. Since $\pi p \in L$, $p \in T$, so $p \in \partial B$, and $T$ is the plane tangent to $\partial B$ at $p$, so the tangent plane is vertical.

Next we shall use the Implicit Function Theorem [12] to provide $X$, and hence $\partial S$, with local $C^1$ coordinate systems. Given $B = B^{f,0}$, that is,

$$B = \{x : f(x) \leq 1\} \text{ and } \partial B = \{x : f(x) = 1\},$$

we define

$$F(p) = \begin{bmatrix} f(p) \\ \frac{\partial f}{\partial z} |_p \end{bmatrix}.$$ 

Since $f'(p)$ is normal to the tangent plane to $\partial B$ at $p$, the plane is vertical if and only if $f'(p)$ is horizontal, i.e., $\partial f / \partial z = 0$. Therefore $p \in X \iff f(p) = 1 \land \partial f / \partial z = 0$, so

(A.1.5) Lemma

$$X = F^{-1}(1, 0).$$
Since $F$ involves only $f$ and its derivative, and $f$ is $C^2$, $F$ is $C^1$. Differentiating $F$, the top row is $f'(p)^T$ and the bottom row is as shown below. The bottom row is $(\vec{k})^T f''(p)$ where the column-vector $\vec{k} = [0 \ 0 \ 1]^T$ is the unit vector in the $z$-direction.

$$F'(p) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} f'(p)^T \\ (\vec{k})^T f''(p) \end{bmatrix}$$

To apply the Implicit Function Theorem we need only show

**(A.1.6) Lemma** $F(p)$ has rank 2 for all $p \in X$.

**Proof.** With $\vec{k}$ as above, for any $p \in X$,

$$(\vec{k})^T f'(p) = 0$$

and

$$(\vec{k})^T f''(p) \vec{k} \neq 0$$

since $f''$ is positive definite (§2.2). Also, $f''(p)$ is symmetric since $f$ is $C^2$. Therefore $f'(p)$ is nonzero and horizontal, and $f''(p)\vec{k}$ is nonzero and not horizontal, so the two column vectors are linearly independent. Transposing, we get the rows of $F'(p)$, so the latter are linearly independent.

From the Implicit Function Theorem it follows that

**(A.1.7) Lemma** At any point $p \in X$, projection onto one of the three coordinate axes is a local $C^1$ coordinate system.

Moreover, in the case of this map $F$,

**(A.1.8) Lemma** At any point $p \in X$, either the $x$- or the $y$-coordinate is a local $C^1$ coordinate system.

**Proof.** One could use the $z$-coordinate near $p$ if

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} \end{bmatrix}$$

has rank 2. In order to rule out the other two coordinates, the other two square submatrices would need to be singular. Looking at the third column of $F''(p)$,

$$\begin{bmatrix} \frac{\partial f}{\partial z} \\ \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

its top entry is zero and its bottom entry, $(\vec{k})^T f''(p) \vec{k}$, is nonzero. For the other two columns to depend on it, their top entry would be zero; that is, $f'(p) = \vec{0}$, which is false. Clearly the projection map $(x, y, z) \mapsto (x, y, 0) \equiv (x, y)$ is $C^\infty$, so

**(A.1.9) Lemma** Whenever the $y$-coordinate (respectively, $x$-) is a local $(C^1)$ coordinate system for $X$ near $p$, it is also a local coordinate system for $\pi X = \partial S$ near $\pi p$. Therefore $\partial S$ is a $C^1$ manifold.
Having established that $\partial S$ is a $C^1$ manifold, we consider a particular parametrisation.

Let $\rho : P \rightarrow P$ be radial projection from $P \setminus \{O\}$ onto the unit circle in $P$, and thence to the unit circle $S^1$ in $\mathbb{R}^2$. Explicitly

\[
\rho : (x, y, 0) \mapsto \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).
\]

Take the usual parametrisation of $S^1$

\[
[0, 2\pi] \rightarrow S^1; \quad \theta \mapsto (\cos \theta, \sin \theta).
\]

Compose it with the inverse of $\rho$ (restricted to $\partial S$) to get a continuous parametrisation of $\partial S$, denoted $b^{f, O}$ (where $B = \{x : f(x) \leq 1\}$).

\[
b^{f, O}(\theta) = \rho|_{\partial S}^{-1}(\cos \theta, \sin \theta).
\]

By construction,

(A.1.10) Lemma $b^{f, O}$ is a continuous parametrisation of $\partial S$. (See Definition 2.8).

(A.1.11) Lemma $b^{f, O}$ is $C^1$.

Proof. Fix $\theta_0 \in [0, 2\pi]$ and let $q_0 = b^{f, O}(\theta_0)$.

By Lemma A.1.8, there exists an open interval containing $q_0$ in $\partial S$ on which either the $y$-coordinate or the $x$-coordinate map, i.e., the map $(x, y) \rightarrow y$ or $(x, y) \rightarrow x$, is a homeomorphism with $C^1$ inverse. Say the $y$- or $x$-coordinate is 'suitable.' Write $\sigma_y$ or $\sigma_x$ for the inverse of this map, which parametrises $\partial S$ near $q_0$.

First assume $\theta_0$ is different from $0$ and $2\pi$, and that the $y$-coordinate is suitable. There exists an open interval $I \subseteq (0, 2\pi)$ containing $\theta_0$ and a composition of maps from $I$ onto an open interval $I' \subseteq \partial S$ and then onto $J \subseteq \mathbb{R}$:

\[
\theta \mapsto (\cos \theta, \sin \theta) \xrightarrow{\rho|_{\partial S}^{-1}} (x, y) \mapsto y.
\]

If we invert this chain of maps, the first (from $J$ to $I'$) is $C^1$ and the others ($\rho$, $(\cos \theta, \sin \theta) \mapsto \theta$) are $C^\infty$. Therefore the composite inverse from $J$ to $I$ is $C^1$. By the Inverse Function theorem [12], the original composite from $I$ to $J$ is $C^1$. If we extend the composition by $\sigma_y$, noting of course that $\sigma_y \circ y$ is the identity,

\[
\theta \mapsto (\cos \theta, \sin \theta) \xrightarrow{\rho|_{\partial S}^{-1}} (x, y)
\]

is a $C^1$ homeomorphism. But this is just the restriction of $b^{f, O}$ to an open interval containing $\theta_0$.

If the $y$-coordinate is unsuitable, we use the $x$-coordinate, and reach the same conclusion.

In the case where $\theta$ is $0$ or $2\pi$, the above argument shows that there is a $C^1$ homeomorphism

\[
(-\kappa, \kappa) \rightarrow I'
\]

\[
\theta \mapsto (\cos \theta, \sin \theta) \mapsto \ldots (x, y)
\]
Figure 16: frustum.

Take
\[ g : [0, \kappa) \cup (2\pi - \kappa, 2\pi] \to (-\kappa, \kappa); \]
\[ \theta \mapsto \theta \quad \text{if} \quad \theta < \kappa \]
\[ \theta \mapsto \theta - 2\pi \quad \text{if} \quad \theta > \kappa. \]

Clearly \( g \) is surjective and \( C^\infty \) with derivative 1, and by composition we get a map
\[ [0, \kappa) \cup (2\pi - \kappa, 2\pi] \to I' \]
which is differentiable and whose derivative is everywhere nonzero, and the value and derivative at 0 and \( 2\pi \) are equal (See Definition 2.8).

\textbf{(A.1.12)} Needless to say, Lemma \textbf{A.1.11} applies to silhouettes in \textit{any} plane \( P \) and to \textit{any} translated body \( B^{f,a} \).

It remains to prove that the map
\[ \mathcal{F} \times \mathbb{R}^3 : (f, a) \mapsto b^{f,a} \]
is continuous, in the sense that if \((f, a)\) is close to \((g, b)\) under the metric
\[ \max(|g - f|_{C^2}, \|b - a\|) \]
then \( b^{f,a} \) is close to \( b^{g,b} \) under the metric
\[ |b^{g,b} - b^{f,a}|_{C^1}. \]

We prove it in two stages: first, continuity of \( b^{f,a} \) under the sup norm; second, continuity of \( db/d\theta \) under the sup norm.

\textbf{(A.1.13) Lemma} The map \((f, a) \mapsto b^{f,a}\) is continuous under the sup norm.

\textbf{Proof.} Fix \((f_0, a_0)\). If \( c, d \) are points (not on the same vertical line) and \( \theta \) an angle, we denote by
\[ (c, d), A(c, d), A(c, \theta) \]
respectively the open line-segment from \( c \) to \( d \), the vertical plane containing \( c \) and \( d \), and the vertical plane containing \( c \) and forming the angle \( \theta \) with the positive \( x \)-axis.
Fix $\theta_0$. Let $p_0$ be the unique point in $\partial B^{f_0, a_0}$ whose vertical projection $\pi(p_0)$ in $P$ is $b^{f_0, a_0}(\theta_0)$. Since $\pi(p_0) \in \partial S^{f_0, a_0}$, $p_0$ is in $X^{f_0, a_0}$. Let $T_0$ be the tangent plane to $B^{f_0, a_0}$ at $p_0$. $T_0$ is vertical.

Given $\delta > 0$, let $Q^\delta$ be the square region in $T_0$ whose side-length is $2\delta$, which is centred at $p_0$, and whose bottom and top sides are horizontal, i.e., parallel to the $xy$ plane, so its left and right sides are parallel to the $z$-axis. There is an infinite solid cone of square oblique cross-section formed by rays from $a_0$ passing through $Q^\delta$. It has four faces: a top face, a bottom face, a left face, and a right face. It is bisected by $A(a_0, p_0)$, and its left and right faces are vertical.

There are two planes parallel to $T_0$ and at distance $\delta$ from $T_0$: $T_1$ and $T_2$ where $T_2 \cap B^{f_0, a_0} = \emptyset$ and, assuming $\delta$ is small enough, $T_1 \cap B^{f_0, a_0} \neq \emptyset$.

We assume that $\delta$ is small enough so that $T_1$ intersects the open line-segment $(a_0, p_0)$.

Intersecting the solid cone with the space between $T_1$ and $T_2$, we get a frustum denoted

$$R^{f_0, a_0, \delta}$$

It has six sides: bottom, top, left, right, far (from $a_0$; in $T_2$) and near in $T_1$.

For points $x$ along the ray from $a_0$ through $p_0$,

$$f_0(x - a_0) - 1$$

is negative on $(a_0, p_0)$, zero at $p_0$, and positive beyond $p_0$. Since $f_0$ is continuous, if $\delta$ is small enough,

$$f_0(x - a_0) - 1$$

is negative on the near face and positive on the far face.

For points $x$ along the curve-segment

$$A(a_0, \theta_0) \cap R^{f_0, a_0, \delta} \cap \partial B^{f_0, a_0}$$

the function

$$(\vec{k})^T f_0''(x - a_0)\vec{k}$$

is negative at the bottom point and positive at the top point. Since $f_0$ is $C^2$, this is continuous in $x$, so if $\delta$ is small enough then it is negative on the bottom face and positive on the top.

There exists a small neighbourhood $N(a_0)$ of $a_0$ and a small open set $I_{\theta_0}$ in $[0, 2\pi]$ containing $\theta_0$ such that for every $a \in N(a_0), \theta \in I_{\theta_0}$, $A(a, \theta)$ passes between the left and right sides of $R^{f_0, a_0, \delta}$.

There exists a small neighbourhood $N(f_0)$ of $f_0$ in $\mathcal{F}$ (under the $C^2$ norm) such that for every $f \in N(f_0)$,

(a) $f(x - a) - 1$
is negative on the near face of $R_{f_0,a_0,\delta}$ and positive on the far face, and

$$(b) \quad (\vec{k})^T f''(x - a) \vec{k}$$

is negative on the bottom face and positive on the top.

Given $a$ near $a_0$ and $\theta$ near $\theta_0$, let

$$E = A(a, \theta) \cap R_{f_0,a_0,\delta}.$$  

$E$ is a trapezium with two vertical sides (near and far). Given $f$ close to $f_0$, $f(x - a) - 1$ is negative on the near side and positive on the far side, so $E$ intersects $\partial B^{f,a}$ in a curve-segment passing through its bottom and top sides. Along this curve,

$$(\vec{k})^T f''(x - a) \vec{k}$$

is negative at the bottom and positive at the top, so it is zero at an intermediate point $p$, and $p \in X_{P}^{f,a}$. Therefore $\pi p = b^{f,a}(\theta)$. That is, $b^{f,a}(\theta) \in \pi R_{f_0,a_0,\delta}$. If $\delta$ is small enough,

(A.0) $$\|b^{f,a}(\theta) - b^{f_0,a_0}(\theta)\| < \epsilon.$$  

To summarise: given $\theta_0$, there is an open set $I_{\theta_0}$ containing $\theta_0$ and an open neighbourhood $V_{\theta_0}$ of $(f_0, a_0)$ in $\mathcal{F} \times \mathbb{R}^3$ such that for all $\theta \in I_{\theta_0}$ and $(f, a) \in V_{\theta_0}$, (A.0) holds.

Since $[0, 2\pi]$ is compact, we can choose a finite set $\theta_j^0$, $1 \leq j \leq k$, and a finite subcover $I_{\theta_j^0}$ satisfying the obvious variants of inequality (A.0). Let

$$V = V_{\theta_1^0} \cap \ldots \cap V_{\theta_k^0}.$$  

Then for all $(f, a) \in V$ and $\theta \in [0, 2\pi]$,

$$\|b^{f,a}(\theta) - b^{f_0,a_0}(\theta)\| < \epsilon.$$  

The following lemma will complete the proof that the map $f, a \mapsto b^{f,a}$ is continuous with respect to the $C^1$ norm. The proof cites Lemma A.1.13 without needing to retrace the steps in that lemma, loosely speaking because $db^{f,a}/d\theta$ can be expressed as a continuous function of $b^{f,a}(\theta)$.

(A.1.15) Lemma The map

$$f, a \mapsto \frac{db^{f,a}}{d\theta}$$

is continuous under the sup norm.

Proof. As before, $\pi$ is projection onto $P$. Fix $f_0, a_0$, and $\epsilon > 0$. For each $\theta$ there is a unique point $(x, y, z) = (x(\theta), y(\theta), z(\theta))$ such that $\pi(x, y, z) = b^{f,a}(\theta)$. Recalling that $X_P = X^{f,a} = F^{-1}(1, 0)$ where $F(p) = (f(p), \frac{\partial f}{\partial y}(p))$, $F(p)$ is constant along the curve $X_P$, so its derivative with respect to $\theta$ vanishes.

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} \begin{bmatrix} \frac{dx}{d\theta} \\ \frac{dy}{d\theta} \\ \frac{dz}{d\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$
Note $\partial f / \partial z = 0$. Recall that the first and third, or the second and third, columns are linearly independent, and that $\partial^2 f / \partial z^2 \neq 0$ (Lemma [A.1.6]). Therefore the first and third columns are linearly independent if and only if $\partial f / \partial x \neq 0$.

If the first and third are, the $y$-coordinate can be used to parametrise the curve, and

$$\frac{\partial f}{\partial x} \frac{dx}{dy} + \frac{\partial f}{\partial y} = 0,$$

so

$$\frac{dx}{dy} = -\frac{\partial f / \partial y}{\partial f / \partial x} \quad \text{and} \quad \frac{dx}{dy} = -\frac{\partial f / \partial x}{\partial f / \partial y} \frac{dy}{d\theta}.$$

If the second and third columns are linearly independent then we get a similar expression for $dy/d\theta$ in terms of $dx/d\theta$.

Now, given $a = (\alpha, \beta, \gamma)$, and $b^{f,a}(\theta) = (x, y)$, if $x - \alpha \neq 0$, i.e., $(x, y)$ is not directly above or below $\alpha$ in $S$, and $\partial f / \partial x \neq 0$, so $y$ gives a local coordinate system,

$$\tan \theta = \frac{y - \beta}{x - \alpha},$$

$$\frac{d}{dy} \left( \frac{y - \beta}{x - \alpha} \right) \frac{dy}{d\theta} = \sec^2 \theta = \frac{(x - \alpha)^2 + (y - \beta)^2}{(x - \alpha)^2},$$

$$\frac{dy}{d\theta} = \frac{(x - \alpha)^2 + (y - \beta)^2}{x - \alpha - (y - \beta) \frac{dx}{dy}} = \frac{(x - \alpha)^2 + (y - \beta)^2}{x - \alpha + (y - \beta) \frac{\partial f / \partial y}{\partial f / \partial x}}.$$

Let us write $G_{f,a}(x, y, z)$ for

$$\frac{\partial f / \partial y}{\partial f / \partial x}.$$

Fix $\theta_0$ and let $p_0 = X_{f,a}^{f_0,a_0}(\theta_0)$.

There exists a neighbourhood $N(p_0)$ of $p_0$ on which $G_{f_0,a_0}$ is continuous and bounded, and if we write $b$ for $(x, y)$ and $a$ for $(\alpha, \beta, \gamma)$, we can write

$$H(b, a, t) = \frac{(x - \alpha)^2 + (y - \beta)^2}{x - \alpha + (y - \beta)t} (-t, 1),$$

bounded and continuous in $b, a, t$ except where the denominator vanishes.

(A.0) \begin{equation}
\frac{db^{f,a}}{d\theta} = H(b^{f,a}(\theta), a, G_{f,a}(X_{f,a}^{f_0,a_0}(\theta))).
\end{equation}

Let $b_0 = b^{f_0,a_0}(\theta_0)$ and $t_0 = G_{f_0,a_0}(X_{f_0,a_0}^{f_0,a_0}(\theta_0))$. Choose $\delta > 0$ so if $\|b - b_0\| < \delta$, $\|a - a_0\| < \delta$, and $|t - t_0| < \delta$,

$$\|H(b, a, t) - H(b_0, a_0, t_0)\| < \epsilon.$$

Choose neighbourhoods $N(f_0), N(a_0), N(p_0)$, and $N(\theta_0)$ so that for all $\theta \in N(\theta_0)$, $f \in N(f_0)$, and $a \in N(a_0)$,

$$b^{f,a}(\theta) \in N(p_0)$$

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and
\[ \| H(b, a, t) - H(b_0, a_0, t_0) \| < \epsilon, \]
where \( b = b^{f,a}(\theta) \), \( t = G_{f,a}(\theta) \), and \( t_0 = G_{f_0,a_0}(\theta) \).

Choose a finite set \( \theta^*_0 \) from which we get a finite cover \( \bigcup N(\theta^*_0) \) of \( [0, 2\pi] \), with associated open sets \( N_i(f_0) \) and \( N_i(a_0) \). Let
\[ V = \bigcap_i N_i(f_0) \times \bigcap_i N_i(a_0). \]
Then for all \((f, a) \in V\),
\[ \sup_\theta \left\| \frac{db^{f,a}}{d\theta} - \frac{db^{f_0,a_0}}{d\theta} \right\| < \epsilon. \]

A.2 Upper common tangent

(A.2.1) Definition If \( B_0, B_1 \) are a balanced pair in \( \mathbb{R}^3 \) (Definition 2.12), and \( H \) is a half-plane bounded by the \( x \)-axis, extending to a plane \( P \), with silhouettes \( S_0, S_1 \) in \( P \), there are four tangent lines common to their silhouettes, two separating, two supporting.

The upper tangent to \( S_0, S_1 \) is that supporting tangent line which touches \( S_0 \) and \( S_1 \) within \( H \).

(A.2.2) Lemma The upper common tangent exists.

Sketch proof. At any point \( p \) on \( \partial S_0 \) above the \( x \)-axis, let \( H(p) \) be the closed half-plane bounded by the tangent at \( p \) and whose interior is disjoint from \( S_0 \). If \( p \) is rightmost in \( S_0 \) then \( H(p) \cap S_1 = S_1 \) and if \( p \) is leftmost then \( H(p) \cap S_1 = \emptyset \). By the finite intersection property (\( S_1 \) being compact) there exists a point \( p \in \partial S_0 \) such that \( H(p) \) is tangent to \( S_1 \): unique, since \( S_0 \) and \( S_1 \) are rounded; the tangent at \( p \) is the upper common tangent.

(A.2.3) Lemma Given a fixed half-plane bounded by the \( x \)-axis, let \( P \) be the plane containing it. The point \( x_0(B_0, B_1, P) \), where the upper common tangent to the silhouettes \( S_0 \) and \( S_1 \) in \( P \) of \( B_0 \) and \( B_1 \) touches \( S_0 \), depends continuously on the balanced pair \( B_0, B_1 \) of convex bodies in \( \mathbb{R}^3 \).
Sketch Proof. Suppose that \( B_i = B^{f_i,a_i} \), \( i = 0, 1 \).

Suppose \( x_0(B_0, B_1, P) = b^{f_0,a_0}(\theta_0) \). As in Lemma A.2.2 given \( p \) in the ‘upper part’ of \( \partial S_0 \), i.e., \( p \) is on the silhouette boundary within the half-plane, let \( H(p) \) be a half-plane coplanar with \( P \) and tangent to \( S_0 \) at \( p \) (\( p \in P \)). Let \( p_0 = b^{f_0,a_0}(\theta_0) \) where \( \theta \) parametrises \( \partial S_0 \). \( H(p_0) \) touches \( S_0 \). Given small positive \( \delta \), \( H(b^{f_0,a_0}(\theta_0 + \delta)) \cap S_1 = \emptyset \) and \( H(b^{f_0,a_0}(\theta_0 - \delta)) \cap S_1 \neq \emptyset \).

For all pairs \( g_i, b_i \) sufficiently close to \( f_i, a_i \), the upper common tangent to the silhouettes of \( B^{g_i,b_i} \) touches \( b^{g_0,b_0}(\theta) \) at some point \( b^{g_0,b_0}(\theta) \) where \( \theta \in ((\theta_0 \pm \delta)) \). Since \( b^{g_0,b_0}(\theta) \) is close to \( b^{f_0,a_0}(\theta) \), the upper common tangent touches \( S^{g_0,b_0} \) at a point close to \( x_0(B_0, B_1, P) \).

A.3 The pre-seam

(A.3.1) Definition Let \( B_0, B_1 \) be a balanced pair of convex bodies in \( \mathbb{R}^3 \). That is, \( B_i = B^{f_i,a_i} \) for \( i = 0, 1 \). For any \( \phi \in [0, 2\pi] \), let \( P_\phi \) be the half-plane bounded by the \( x \)-axis and containing the point \((0, \cos \phi, \sin \phi)\). There is a unique upper common tangent \( U \) to the silhouettes in \( P_\phi \). Define

\[
s(B_0, B_1, \phi) \quad \text{or} \quad s(f_0, a_0, f_1, a_1, \phi)
\]

to be the outward normal to \( U \): i.e., the unit vector in \( P_\phi \) normal to \( U \) and directed away from the two silhouettes.

(A.3.2) Lemma Let \( \pi_\phi \) be orthogonal projection onto the unique plane containing \( P_\phi \). The map

\[
\mathcal{F} \times \mathbb{R}^3 \times [0, 2\pi] \to S^2 : \quad (f, a, \phi) \mapsto \pi_\phi B^{f,a}
\]

is jointly continuous in \( f, a, \phi \).

Proof. If \( R_\phi \) represents rotation around the \( x \)-axis, and \( B \) is any body then

\[
\pi_\phi(B) = R_\phi \pi R_{-\phi}(B)
\]

where \( \pi \) is orthogonal projection onto the \( yz \)-plane. The map \( B \mapsto R_{-\phi}(B) \) is continuous (Lemma 2.4) and \( B \mapsto \pi B \) is continuous and \( \pi B \mapsto R_{-\phi} \pi B \) is an isometry, so the composite map is continuous.

Combining this with Lemma A.2.3

(A.3.3) Lemma The map

\[
\phi \mapsto s(B_0, B_1, \phi)
\]

is jointly continuous in \( B_0, B_1, \) and \( \phi \).

(A.3.4) Corollary For fixed \( B_0, B_1 \),

\[
[0, 2\pi] \to S^2 : \quad \phi \mapsto s(B_0, B_1, \phi)
\]

is continuous and bijective except that \( s(B_0, B_1, 0) = s(B_0, B_1, 2\pi) \).

Passing from upper common tangent line \( U \) to its inverse image, a plane \( T = \pi_\phi^{-1} U \), the latter is the unique tangent plane common to \( B_0 \) and \( B_1 \) whose normal is within the half-plane \( P_\phi \), and that normal is \( s(B_0, B_1, \phi) \). Hence

(A.3.5) Lemma \( \{s(B_0, B_1, \phi) : 0 \leq \phi \leq 2\pi\} \) is a parametrisation of the \( B_0, B_1 \) pre-seam, and the pre-seam is a continuous Jordan curve.
Given

\[ B = B^{f,a} \]

and \( n \) is its normal, we define a right inverse to \( n, p : \mathbb{R}^3 \setminus \{ O \} \rightarrow \partial B \):

\[ p(y) = n^{-1} \left( \frac{y}{\|y\|} \right). \]

It is well-defined and continuous because \( n \) is a homeomorphism from \( \partial B \) onto \( S^2 \).

(A.3.7) **Lemma** For any \( \omega \in \mathbb{R}^3 \setminus \{ O \} \),

\[ \omega^T p'(\omega) = \vec{O}_{1 \times 3}. \]

**Proof.** Let \( x = p(\omega) \) so \( \omega/\|\omega\| = f'(x)/\|f'(x)\| \).

If \( \omega + h \neq 0 \), \( f(p(\omega + h)) = f(p(\omega)) = 1 \), so

\[
\begin{align*}
    f(p(\omega + h)) - f(p(\omega)) &= 0 \\
    (f'(p(\omega)))^T p'(\omega)h &= o(\|h\|) \\
    (f'(x))^T p'(\omega) &= \vec{O} \\
    \omega^T p'(\omega) &= \vec{O}
\end{align*}
\]

since \( \omega \propto f'(x) \). \( \blacksquare \)

(A.3.8) **Lemma** The pre-seam is a \( C^1 \) manifold.

**Proof.** We shall define a \( C^1 \) map \( F : \mathbb{R}^3 \setminus \{ O \} \rightarrow \mathbb{R}^2 \) and show that its derivative has rank 2 along the pre-seam. It then follows from the Implicit Function Theorem [12] that for any point \( \omega \) on the pre-seam, projection onto one of the three coordinate axes is a local \( C^1 \) diffeomorphism near \( \omega \).

Given \( B_0 \) and \( B_1 \), we associate with them \( f_0, n_0, p_0 \) and \( f_1, n_1, p_1 \) (see §A.3.6). Let \( \omega \) be a point in \( S^2 \). It is the outward unit normal at exactly one point in \( \partial B_0 \) and one in \( \partial B_1 \), namely, \( p_0(\omega) \) and \( p_1(\omega) \) respectively. Let

\[ q(\omega) = p_1(\omega) - p_0(\omega). \]

This is the displacement connecting a point in \( \partial B_0 \) with a point in \( \partial B_1 \). The (oriented) tangent plane at \( p_0(\omega) \) is the unique plane tangent to \( B_0 \) with outward normal \( \omega \). The other point \( p_1(\omega) \) is in that plane if and only if the plane is a common (supporting) tangent plane to \( B_0 \) and \( B_1 \). Equivalently,

(A.3.1)

\[ \omega^T q(\omega) = 0. \]

Therefore the pre-seam is the set of all \( \omega \in S^2 \) such that \( \omega^T q(\omega) = 0 \).

The map \( F \) will be

\[ F : \omega \mapsto (\omega^T \omega, \omega^T q(\omega)). \]

Using Equation [A.3.1] the pre-seam is \( F^{-1}(1, 0) \).
Since \((\omega + h)^T(\omega + h) - \omega^T\omega = 2h^T\omega + o(\|h\|)\), the derivative of \(\omega^T\omega\) (as a column vector) is \(2\omega\). The derivative of \(\omega^Tq(\omega)\) is \(q(\omega) + \omega^Tq'(\omega)\) and \(\omega^Tq'(\omega) = \omega^Tp'_1(\omega) - \omega^Tp'_0(\omega) = O\) (Lemma A.3.7). Writing \(F'\) as a \(2 \times 3\) matrix, which is the correct format,

\[
F'(\omega) = \begin{bmatrix} 2\omega^T & q^T(\omega) \end{bmatrix}
\]

All points in the pre-seam have unit length, so near the pre-seam, \(\omega\) is nonzero, and \(q(\omega)\) is nonzero since \(B_0\) and \(B_1\) can touch at one point at most, and at that point the outward normals are opposite. Also, if \(\omega\) is on the pre-seam then \(\omega\) and \(q(\omega)\) are orthogonal (Equation A.3.1). Therefore \(F'(\omega)\) has rank 2 near the pre-seam. By the Implicit Function Theorem [12], the pre-seam is a \(C^1\) manifold with local coordinate systems provided by projection onto the coordinate axes.

For this application we can say more.

(A.3.9) Lemma At any point \(\omega\) in the pre-seam, either the y- or the z-coordinate is a local \(C^1\) coordinate system.

Proof. Suppose \(\omega\) is written with coordinates \((x, y, z)\), and \(q = (q_1, q_2, q_3)\). The coordinates of \(F'(\omega)\) are

\[
\begin{bmatrix}
2x & 2y & 2z \\
q_1 & q_2 & q_3
\end{bmatrix}.
\]

We would be obliged to coordinatise by the \(x\)-coordinate if the only choice of columns with rank 2 were the second and third.

But the first column is nonzero \((x < 0\) and \(q_1 > 0)\), so it could be exchanged with one of the other two to produce a linearly independent pair of columns. Therefore the pre-seam can be coordinatised by projection onto the \(y\) or \(z\)-axis, as claimed.

Recall (Definition A.3.1, Lemma A.3.5), the continuous parametrisation \(\phi \mapsto s(B_0, B_1, \phi)\) of the \(B_0, B_1\) pre-seam.

(A.3.10) Lemma The map \(\phi \mapsto s(B_0, B_1, \phi)\) is a \(C^1\) parametrisation of the pre-seam.

Proof. We have seen that the pre-seam is a \(C^1\) manifold, and projection onto the \(y\)- or \(z\)-axis will serve for local coordinate systems.

Let \(q_0 = (x_0, y_0, z_0) = s(B_0, B_1, \phi_0)\). Suppose that the \(z\)-coordinate gives a local coordinate system for the pre-seam near \(q_0\). Reversing the maps, and taking another parametrisation \(\eta \mapsto (\cos(\eta + \phi_0), \sin(\eta + \phi_0))\),

\[
z \mapsto (x, y, z) \mapsto \left(\frac{y}{\sqrt{y^2 + z^2}}, \frac{z}{\sqrt{y^2 + z^2}}\right) = (\cos(\eta + \phi_0), \sin(\eta + \phi_0)) \mapsto \eta
\]

which is a composition of invertible \(C^1\) maps, so, by the Inverse Function Theorem [12] its inverse is a \(C^1\) map taking \(\eta\) to \(z\). Attaching the map \(z \mapsto (x, y, z)\), the latter in the pre-seam, we get a \(C^1\) map.
from a subinterval of \((-\pi/2, \pi/2)\) to the pre-seam. By the same arguments as in Lemma A.1.11, we deduce that the map \(\phi \mapsto s(B_0, B_1, \phi)\) is \(C^1\).

The pre-seam given by a balanced pair \(B_0, B_1\) is a differentiable Jordan curve on \(S^2\), orientated so that it is anticlockwise with respect to the positive \(x\)-direction (the closest points in \(B_0, B_1\) are on the \(x\)-axis). This defines a unique closed disc, that to the left of the pre-seam in \(S^2\), which corresponds to that part of \(\partial B_0\) hidden by \(B_1\) (Figure 19).

(A.3.11) Lemma The map \(B_0, B_1 \mapsto s(B_0, B_1, \cdots)\) is continuous, in the sup norm on parametrisations of Jordan curves in \(S^2\).

Proof. Again \(P_\phi\) is the half-plane bounded by the \(x\)-axis at angle \(\phi\) with the positive \(y\)-axis. By Lemma A.2.3 for any \(\phi_0 \in [0, 2\pi]\), the point \(x_0(B_0, B_1, \phi_0)\), where the upper common tangent to the two silhouettes in \(P_{\phi_0}\) touches that of \(B_0\), depends continuously on \(B_0\) and \(B_1\).

Formally, \(B_i = B^{f_i, a_i}, i = 0, 1\), and we are considering \(x_0(f_0, a_0, f_1, a_1, \phi_0)\). If \(R_\eta\) denotes rotation through angle \(\eta\) around the \(x\)-axis, then
\[
x \in R_{-\eta}B_i \iff R_\eta(x) \in B_i \iff f_i(R_\eta(x - R_{-\eta}(a_i)) \leq 1.
\]
But the map \(\eta \mapsto f \circ R_\eta\) is continuous (Lemma 2.4), as is the map \(\eta \mapsto (x \mapsto x - R_{-\eta}(a_i))\). Also, the silhouette of \(B_i\) in \(P_{\phi_0 + \eta}\) is \(R_\eta S_i\), where \(S_i\) is the silhouette of \(R_{-\eta}B_i\) in \(P_{\phi_0}\).

It follows that \(x_0(B_0, B_1, \phi_0 + \eta)\) depends continuously on \(B_0, B_1,\) and \(\eta\).

The image of \(x_0(B_0, B_1, \phi)\) under the normal map is \(s(B_0, B_1, \phi)\):
\[
s(B_0, B_1, \phi) = \frac{c}{\|c\|}, \quad \text{where} \quad c = f_0'(x_0(B_0, B_1, \phi) - a_0),
\]
and it depends continuously on \(B_0, B_1, \phi\).

Fix \(\epsilon > 0\). For each \(\phi_0\) there is a neighbourhood \(V_{f_0, a_0, f_1, a_1, \phi_0}\) of \(B_0, B_1\), and an open interval \(I_{\phi_0}\) containing \(\phi_0\), such that for all \(C_0, C_1 \in V_{f_0, a_0, f_1, a_1, \phi_0}\) and \(\phi \in I_{\phi_0}\),
\[
\|s(B_0, B_1, \phi) - s(C_0, C_1, \phi)\| < \epsilon
\]

Take a finite cover of \([0, 2\pi]\) by such intervals \(I_{\phi_0}\) and let \(V\) be the intersection of the neighbourhoods \(V_{f_0, a_0, f_1, a_1, \phi_0}\). For any \(\phi \in [0, 2\pi]\), and \(C_0, C_1 \in V\), \(\phi\) belongs to one of the intervals \(I_{\phi_0}\), so
\[
\|s(B_0, B_1, \phi) - s(C_0, C_1, \phi)\| < \epsilon.
\]
Therefore
\[
\sup_{\phi} \|s(B_0, B_1, \phi) - s(C_0, C_1, \phi)\| < \epsilon. \]

Figure 19: left of pre-seam corresponds to hidden region.
(A.3.12) Lemma Continuing the conditions of the above lemma, the map \( B_0, B_1 \mapsto ds/d\phi \in \mathbb{R}^3 \) is continuous under the sup norm.

Proof. Fix \( B_i = B^{f_i,a_i} \). The pre-seam is \( F^{-1}(1,0) \) where

\[
F(\omega) = (\omega^T \omega, \omega^T q(\omega)), \quad F'(\omega) = \begin{bmatrix}
2\omega^T \\
n^T(\omega)
\end{bmatrix},
\]

(these depend implicitly on \( f_i \) and \( a_i \)) and it can be parametrised locally by projection onto the \( y \) or the \( z \)-axis. Let \( s \) or \( s(\phi) \) abbreviate \( s(B_0, B_1, \phi) \), and \( (x(\phi), y(\phi), z(\phi)) \), or \( (x, y, z) \) for short, a point on the pre-seam. \( F \) is constant on the pre-seam, so \( F'(s(\phi))ds/d\phi = 0 \):

\[
\begin{bmatrix}
2x \\
2y \\
2z
\end{bmatrix}
\begin{bmatrix}
dx/d\phi \\
dy/d\phi \\
dz/d\phi
\end{bmatrix} = \vec{0}.
\]

Recall that \( x_0(B_0, B_1, \phi) \) is the point on \( \partial B_0 \) where the tangent plane with outer normal \( s(\phi) \) touches \( B_0 \). Let \( x_1(B_0, B_1, \phi) \) be the corresponding point on \( \partial B_1 \). Then \( q = x_1 - x_0 \), and \( q \) depends continuously on \( B_0, B_1, \phi \) and \( \phi \). Also, \( (x, y, z) = f_0'(x_0(\phi))/\|f_0'(x_0(\phi))\| \).

Let \( C_0, C_1 \) be a balanced pair (near \( B_0, B_1 \)): \( C_i = B^{a_i,b_i} \), and \( x_j = x_j(C_0, C_1, \phi), j = 0,1 \). Since \( g'(x_0)/\|g'(x_0)\| \) is on the \( C_0, C_1 \) pre-seam,

\[
\begin{bmatrix}
g_0'(x_0) \\
\phi - x_0
\end{bmatrix}
\begin{bmatrix}
dx/d\phi \\
dy/d\phi \\
dz/d\phi
\end{bmatrix} = \vec{0}.
\]

If projection on the \( z \)-axis is a local coordinate system near \( s(B_0, B_1, \phi_0) \), then by this formula, \( dx/d\phi \) and \( dy/d\phi \) depend continuously on \( C_0, C_1, \phi \), where \( C_0, C_1 \) are near \( B_0, B_1 \).

Suppose \( \phi_0 \neq \pi/2, 3\pi/2 \) so \( \tan \phi \) is finite and \( C^\infty \) near \( \phi_0 \).

\[
\tan \phi = \frac{z}{y}, \quad \sec^2 \phi = \frac{y^2 + z^2}{y^2} = \frac{dz}{d\phi} \frac{d}{dz} \left( \frac{z}{y} \right) = \left( \frac{z}{y} \right) \frac{dz}{d\phi}, \quad \frac{dz}{d\phi} = \frac{y^2 + z^2}{z} \frac{dy}{dz} - y.
\]

If \( \phi_0 = \pi/2 \) or \( 3\pi/2 \) then a similar formula can be derived by differentiating \( \cot \phi \).

This shows that \( ds(C_0, C_1, \phi)/d\phi \) depends continuously on \( C_0, C_1, \) and \( \phi \) in a suitable neighbourhood \( V_{f_0,a_0,f_1,a_0} \) and interval \( I_{\phi_0} \). We can conclude, in the same way as in the previous lemma, that the map

\[
B_0, B_1 \mapsto \frac{ds}{d\phi}
\]

is continuous in \( B_0, B_1 \).}

The last part of Theorem 2.13 is
(A.3.13) **Corollary** The family $G$ of pre-seams, arising from a compact family $F$ of convex bodies, is compact.

**Proof.** It is enough to show $G$ is sequentially compact. Given a sequence $J_n \to J$ where $J_n$ are pre-seams, taken from a family of balanced pairs $B_0, B_1$, we must show that $J$ is a pre-seam. We choose balanced pairs $B^0_n, B^1_n$ defining $J_n$. We may re-label as follows

$$B^0_n + a_n, B^1_n + b_n$$

where $B^0_n, B^1_n$ are convex bodies and $a_n, b_n$ are displacements. We may translate by $-a_n$ and assume that all the $a_n$ are $O$; of course, the balancing property is lost, but the pre-seams are unchanged. Also, by taking a subsequence if necessary, we can assume that $B^0_n$ and $B^1_n$ converge (with respect to the distance function on $F$) to $B^0$ and $B^1$ respectively.

Given that all the $a_n$ are at the origin, if $b_n$ are bounded then we can take a further subsequence so they converge to some $b$. In this case, since the map $B_0, B_1 \mapsto s(B_0, B_1)$ is continuous, the seams $J_n$ converge to $J$.

If the $b_j$ are unbounded, then by passing to a subsequence if necessary, we can assume that $b_n/\|b_n\|$ converge — i.e., that the $b_n$ go to infinity, but along the $x$-axis. The limiting pre-seam is the circle $\{(0, y, z) : y^2 + z^2 = 1\}$. But this circle itself belongs to $G$, as it is the pre-seam of any balanced pair $B_0, B_1$ where $B_0$ and $B_1$ are copies of the same body. 

This concludes the proof of Theorem 2.13.