THE VARIATIONAL DISCRETIZATION OF THE CONSTRAINED HIGHER-ORDER LAGRANGE-POINCARE EQUATIONS

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Abstract. In this paper we investigate a variational discretization for the class of mechanical systems in presence of symmetries described by the action of a Lie group which reduces the phase space to a (non-trivial) principal bundle. By introducing a discrete connection we are able to obtain the discrete constrained higher-order Lagrange-Poincaré equations. These equations describe the dynamics of a constrained Lagrangian system when the Lagrangian function and the constraints depend on higher-order derivatives such as the acceleration, jerk or jounces. The equations, under some mild regularity conditions, determine a well defined (local) flow which can be used to define a numerical scheme to integrate the constrained higher-order Lagrange-Poincaré equations.

Optimal control problems for underactuated mechanical systems can be viewed as higher-order constrained variational problems. We study how a variational discretization can be used in the construction of variational integrators for optimal control of underactuated mechanical systems where control inputs act solely on the base manifold of a principal bundle (the shape space). Examples include the energy minimum control of an electron in a magnetic field and two coupled rigid bodies attached at a common center of mass.

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1. Introduction. Reduction theory is one of the fundamental tools in the study of mechanical systems with symmetries and it essentially concerns the removal of certain variables by using the symmetries of the system and the associated conservation laws. Such symmetries arise when one has a Lagrangian which is invariant under a Lie group action $G$, i.e. if the Lagrangian function is invariant under the tangent lift action is expressed by $L$ denote by $\Phi$ tangent lift of the action of the Lie group on the configuration manifold $Q$.

Introduction.

If we denote by $\Phi_g : Q \rightarrow Q$ this (left-) action, for $g \in G$ then the invariance condition under the tangent lift action is expressed by $L \circ T\Phi_g = L$. If such an invariance property holds when the action $\Phi_g$ is given by left translations on the group $G$, that is, $\Phi_g = L_g$ where $L_g : G \rightarrow G$ is given by $L_g(h) = gh$ we say that the Lagrangian $L$ is $G$-invariant. For a symmetric mechanical system, reduction by symmetries eliminates the directions along the group variables and thus provides a system with fewer degrees of freedom.

If the (finite-dimensional) differentiable manifold $Q$ has local coordinates $(q^i)$, $1 \leq i \leq \dim Q$ and we denote by $TQ$ its tangent bundle with induced local coordinates $(q^i, \dot{q}^i)$, given a Lagrangian function $L : TQ \rightarrow \mathbb{R}$, its Euler–Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq \dim Q. \tag{1}$$

As is well-known, when $Q$ is the configuration manifold of a mechanical system, equations (1) determine its dynamics.

A paradigmatic example of reduction is the derivation of the Euler-Poincaré equations from the Euler-Lagrange equations (1) when the configuration manifold is a Lie group, i.e. $Q = G$. Assuming that the Lagrangian $L : TG \rightarrow \mathbb{R}$ is left invariant under the action of $G$ it is possible to reduce the system by introducing the body fixed velocity $\xi \in \mathfrak{g}$ and the reduced Lagrangian $\ell : TG/G \simeq \mathfrak{g} \rightarrow \mathbb{R}$, provided by the invariance condition $\ell(\xi) = L(g^{-1}g, g^{-1}\dot{g}) = L(e, \xi)$. The dynamics of the reduced Lagrangian is governed by the Euler–Poincaré equations (see [3] and [9] for instance) and given by the system of first order ordinary differential equations

$$\frac{d}{dt} \left( \frac{\partial \ell}{\partial \xi} \right) = \text{ad}_\xi^* \left( \frac{\partial \ell}{\partial \xi} \right). \tag{2}$$

This system, together with the reconstruction equation $\xi(t) = g^{-1}(t)\dot{g}(t)$, is equivalent to the Euler-Lagrange equations on $Q$, which are given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \Rightarrow \begin{cases} \dot{g} = g\xi, \\ \frac{d}{dt} \left( \frac{\partial \ell}{\partial \xi} \right) = \text{ad}_\xi^* \left( \frac{\partial \ell}{\partial \xi} \right). \end{cases}$$

Reduction theory for mechanical systems with symmetries can be also developed by using a variational principle formulated on a principal bundle $\pi : Q \rightarrow Q/G$, where the principal connection $\mathcal{A}$ is introduced on $Q$ [18] (see Definition 2.4). The connection yields the bundle isomorphism $\alpha^{(1)}_\mathcal{A} : (TQ)/G \rightarrow T(Q/G) \oplus_{Q/G} \mathfrak{g}$,

$$\alpha^{(1)}_\mathcal{A}([v_q]) := (T\pi(v_q), [q, \mathcal{A}(v_q)]_{\mathfrak{g}}),$$

(see equation (8)) where the bracket is the standard Lie bracket on the Lie algebra $\mathfrak{g}$ and $\mathfrak{g} := \text{Ad}Q$ is the adjoint bundle $\text{Ad}Q := (Q \times \mathfrak{g})/G$. A curve $q(t) \in Q$ induces the two curves $p(t) := \pi(q(t)) \in Q/G$ and $\sigma(t) = [q(t), \mathcal{A}(q(t))]_{\mathfrak{g}} \in \mathfrak{g}$.
Variational Lagrangian reduction [18] states that the Euler-Lagrange equations on $Q$ with a $G$-invariant Lagrangian $L$ are equivalent to the Lagrange-Poincaré equations on $TQ/G \equiv T(Q/G) \oplus_{Q/G} \mathfrak{g}$ with reduced Lagrangian $\mathcal{L} : T(Q/G) \oplus_{Q/G} \mathfrak{g} \rightarrow \mathbb{R}$. The Lagrange-Poincaré equations read

$$\begin{cases}
\frac{D}{D\tau} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} - \text{ad}^* \sigma \frac{\partial \mathcal{L}}{\partial \sigma} = 0, \\
\frac{\partial \mathcal{L}}{\partial p} - \frac{D}{D\tau} \frac{\partial \mathcal{L}}{\partial \dot{p}} = \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}, i_p \mathcal{B} \right\rangle,
\end{cases}
$$

(3)

where $\mathcal{B}$ is the reduced curvature form associated to the principal connection $\mathcal{A}$ and $D/D\tau$ denotes the covariant derivative in the associated bundle (see Definition 2.5).

The derivation of variational integrators for (1) and (2) from the discretization of variational principles has received a lot attention from the Dynamical Systems Geometric Mechanics community in the recent years [46], [47], [49], [51], [52], [53] (and in particular for optimal control of mechanical systems [2], [7], [8], [12], [16], [17], [22], [27], [45], [43], [54]). The preservation of the symplectic form and momentum map are important properties which guarantee the competitive qualitative and quantitative behavior of the proposed methods and mimic the corresponding properties of the continuous problem. That is, these methods allow substantially more accurate simulations at lower cost for higher-order problems with constraints. Moreover, if the system is subject to constraints, then, under a regularity condition, it can be shown that the system also preserves a symplectic form or a Poisson structure in the reduced case ([26] and [27] for instance).

The construction of variational integrators for mechanical systems where the configuration space is a principal bundle has been studied in the geometric framework of Lie groupoids [46] and as a motivation for the construction of a discrete time connection form [42], [31]. This line of research has been further developed in the last decade by T. Lee, M. Leok and H. McClamroch [41]. We focus on systems whose phase space is of higher-order, i.e. $T^{(k)}Q$ [20], [33], [34], and moreover is invariant under the action of symmetries. The Euler-Lagrange and Lagrange-Poincaré equations for these systems were introduced by F. Gay-Balmaz, D. Holm and T. Ratiu in [32]. In this work, we aim to develop their discrete analogue for non-trivial principal bundles and its extension to constrained systems (where the constraints will be as well of the higher-order type). With this in mind we employ the discrete Hamilton’s principle by introducing a discrete connection and using Lagrange multipliers in order to obtain discrete paths that approximately satisfy the dynamics and the constraints. As examples, we will illustrate our theory by applying the obtained discrete equations to the problem of energy minimum control of an electron in a magnetic field and two coupled rigid bodies attached at a common center of mass.

The structure of the work is as follows: Section 2 introduces preliminaries on geometric mechanics, Lagrange-Poincaré equations, higher-order tangent bundles and the derivation of the constrained higher-order Lagrange-Poincaré equations (Theorem 2.6). Section 3 starts by introducing discrete mechanics and the notion of discrete connection. Next, we study the variational discretization of the constrained higher-order Lagrange-Poincaré equations to obtain a discrete time flow that integrates the continuous time constrained higher-order Lagrange-Poincaré equations. Moreover we provide sufficient regularity conditions for the discrete flow to exist. We proceed by treating the second-order case (the discrete constrained Lagrange-Poincaré equations are given in Theorem 3.2 and the regularity conditions
in Proposition 1) as an illustration of our approach. Then we carry out the full higher-order case (the equations are given in Theorem 3.3, while the regularity conditions are in Proposition 2). Finally, in Section 4, we apply the discrete equations to underactuated mechanical systems in two examples of optimal control, showing that they give rise to a meaningful discretization of the continuous systems.

2. Constrained higher-order Lagrange-Poincaré equations. In this section we introduce some preliminaries about geometric mechanics on Lie groups, Lagrange-Poincaré reduction, higher order tangent bundles and we study the constrained variational principle for higher-order mechanical systems on principal bundles.

2.1. Mechanics on Lie groups and Euler-Poincaré equations.

**Definition 2.1.** A Lie group is a smooth manifold \( G \) that is a group and for which the operations of multiplication \( (g,h) \mapsto gh \) for \( g,h \in G \) and inversion, \( g \mapsto g^{-1} \), are smooth.

**Definition 2.2.** A symmetry of a function \( F : G \to \mathbb{R} \) is a map \( \phi : G \to G \) such that \( F \circ \phi = F \). In such a case \( F \) is said to be a \( G \)-invariant function under \( \phi \).

**Definition 2.3.** Let \( G \) be a Lie group with identity element \( e \in G \). A left-action of \( G \) on a manifold \( Q \) is a smooth mapping \( \Phi : G \times Q \to Q \) such that \( \Phi(g, e) = q \) \( \forall q \in Q \), \( \Phi(g, \Phi(h, q)) = \Phi(gh, q) \forall h \in G, q \in Q \) and for every \( g \in G \), \( \Phi_g : Q \to Q \) defined by \( \Phi_g(q) := \Phi(g, q) \) is a diffeomorphism.

\( \Phi : G \times Q \to Q \) is a right-action if it satisfies the same conditions as for a left action except that \( \Phi(g, \Phi(h, q)) = \Phi(h, q) \forall g, h \in G, q \in Q \).

We often use the notation \( gq := \Phi_g(q) = \Phi(g, q) \) and say that \( g \) acts on \( q \). All actions of Lie groups will be assumed to be smooth.

Let \( G \) be a finite dimensional Lie group and let \( \mathfrak{g} \) denote the Lie algebra associated to \( G \) defined as \( \mathfrak{g} := T_eG \), the tangent space at the identity \( e \in G \). Let \( L_g : G \to G \) be the left translation of the element \( g \in G \) given by \( L_g(h) = gh \) for \( h \in G \). Similarly, \( R_g \) denotes the right translation of the element \( g \in G \) given by \( R_g(h) = hg \) for \( h \in G \). \( L_g \) and \( R_g \) are diffeomorphisms on \( G \) and a left-action (respectively right-action) from \( G \) to \( G \) [36]. Their tangent maps (i.e., the linearization or tangent lift) are denoted by \( T_gL_g : T_hG \to T_{gh}G \) and \( T_gR_g : T_hG \to T_{hg}G \), respectively. Similarly, the cotangent maps (cotangent lift) are denoted by \( T^*_gL_g : T^*_hG \to T^*_{gh}G \) and \( T^*_gR_g : T^*_hG \to T^*_{hg}G \), respectively. It is well known that the tangent and cotangent lifts are actions (see [36], Chapter 6).

Let \( \Phi_g : Q \to Q \) for any \( g \in G \) be a left action on \( G \); a function \( f : Q \to \mathbb{R} \) is said to be invariant under the action \( \Phi_g \), if \( f \circ \Phi_g = f \), for any \( g \in G \) (that is, \( \Phi_g \) is a symmetry of \( f \)). The Adjoint action, denoted \( \text{Ad}_g : \mathfrak{g} \to \mathfrak{g} \) is defined by \( \text{Ad}_g \chi := g\chi g^{-1} \) where \( \chi \in \mathfrak{g} \). Note that this action represents a change of basis on the Lie algebra.

If we assume that the Lagrangian \( L : TG \to \mathbb{R} \) is \( G \)-invariant under the tangent lift of left translations, that is \( L \circ T_gL_{g^{-1}} = L \) for all \( g \in G \), then it is possible to obtain a reduced Lagrangian \( \ell : \mathfrak{g} \to \mathbb{R} \), where

\[ \ell(\xi) = L(g^{-1}g, T_gL_{g^{-1}}(\dot{y})) = L(e, \xi). \]
The reduced Euler–Lagrange equations, that is, the Euler–Poincaré equations (see, e.g., [3], [36]), are given by the system of first order ODE’s

\[
\frac{d}{dt} \frac{\partial \ell}{\partial \dot{\xi}} = \text{ad}^* \xi \frac{\partial \ell}{\partial \xi},
\]

(4)

where \( \text{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \), \((\xi, \mu) \mapsto \text{ad}^*_\xi \mu\) is the co-adjoint operator defined by \( \langle \text{ad}^*_\xi \mu, \eta \rangle = \langle \mu, \text{ad}\xi \eta \rangle \) for all \( \eta \in \mathfrak{g} \) with \( \text{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) the adjoint operator given by \( \text{ad}_\xi \eta := [\xi, \eta] \), where \([\cdot, \cdot]\) denotes the Lie bracket of vector fields on the Lie algebra \( \mathfrak{g} \), and where \( \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \) denotes the so-called natural pairing between vectors and co-vectors defined by \( \langle \alpha, \beta \rangle := \langle \alpha, \beta \rangle \) for \( \alpha \in \mathfrak{g}^* \) and \( \beta \in \mathfrak{g} \) where \( \alpha \) is understood as a row vector and \( \beta \) a column vector. For matrix Lie algebras \( \langle \alpha, \beta \rangle = \alpha^T \beta \) (see [36], Section 2.3 pp.72 for details).

Using this pairing between vectors and co-vectors one can write a useful relation between the tangent and cotangent lifts

\[
\langle \alpha, T_h L_g (\beta) \rangle = \langle T_h^* L_{g^*} (\alpha), \beta \rangle
\]

(5)

for \( g, h \in G \), \( \alpha \in \mathfrak{g}^* \) and \( \beta \in \mathfrak{g} \).

The Euler–Poincaré equations together with the reconstruction equation \( \xi = T_g L_{g^{-1}} (g) \) are equivalent to the Euler–Lagrange equations on \( G \).

2.2. Geometry of principal bundles. In this subsection we recall the basic tools for analysis of the geometry of principal bundles that are useful in this paper (for more details see [18] and references therein).

**Definition 2.4.** Let \( G \) be a Lie group and \( \mathfrak{g} \) its Lie algebra. Given a free and proper left Lie group action \( \Phi : G \times Q \to Q \), one can consider the principal bundle \( \pi : Q \to Q/G \). A connection \( \mathcal{A} \) on the principal bundle \( \pi \) is a one-form on \( Q \) taking values on \( \mathfrak{g} \), such that \( \mathcal{A}(\xi_Q(q)) = \xi \), for all \( \xi \in \mathfrak{g}, q \in Q \) and \( \Phi^*_h \mathcal{A} = \text{Ad}_{g^{-1}} \mathcal{A} \) where \( \xi_Q \) is the infinitesimal generator associated with \( \xi \) defined as \( \xi_Q(q) := \left. \frac{d}{dt} \right|_{t=0} q \cdot \exp(t \xi) \).

The associated bundle \( N \) with standard fiber \( M \) (a smooth manifold), is defined as

\[
N = Q \times_G M = (Q \times M)/G,
\]

(6)

where the action of \( G \) on \((Q \times M)\) is diagonal, i.e. given by \( g(q, m) = (gq, gm) \) for \( q \in Q \) and \( m \in M \). The orbit of \((q, m)\) is denoted \( [q, m]_G \) or simply \([q, m]\). The projection \( \pi_N : N \to Q/G \) is given by \( \pi_N([q, m]_G) = \pi(q) \) and it is a surjective submersion. The adjoint bundle is the associated vector bundle with \( M = \mathfrak{g} \) under the adjoint action by the inverse element \( g^{-1} \in G \), \( \xi \mapsto \text{Ad}_{g^{-1}} \xi \), and is denoted

\[
\text{Ad}Q := Q \times_G \mathfrak{g}.
\]

(7)

We will usually employ the short-hand notation \( \tilde{\mathfrak{g}} := \text{Ad}Q \). The orbits in this case are denoted \([q, \eta]_{\mathfrak{g}}\) for \( q \in Q \) and \( \eta \in \mathfrak{g} \). \( \text{Ad}Q \) is a Lie algebra bundle, that is, each fibre is a Lie algebra with the Lie bracket defined by

\[
[[q, \xi], [q, \eta]]_{\mathfrak{g}} = [q, [\xi, \eta]]_{\mathfrak{g}}.
\]

Reduction theory for mechanical systems with symmetries can be performed by a variational principle formulated on a principal bundle \( \pi : Q \to Q/G \), with fixed principal connection \( \mathcal{A} \) on \( Q \) (see [18]). In other words, the reduced Lagrangian will be defined on the reduced space \( TQ/G \), say \( \mathcal{L} : TQ/G \to \mathbb{R} \). The bundle
isomorphism $\alpha^{(1)}_A : TQ/G \to T(Q/G) \times_{Q/G} \mathfrak{g}$, provided by the connection, will facilitate the study of the suitable variations. It is defined by

$$\alpha^{(1)}_A([v_q]) := (T\pi(v_q), [q, A(v_q)])_g,$$

where the bracket is the standard Lie bracket on the Lie algebra $\mathfrak{g}$. $v_q \in T_q Q$ and $[v_q] \in (T_q Q, Q)/G$ with $[qG] \in Q/G$. A curve $q(t) \in Q$ induces the two curves $p(t) := \pi(q(t)) \in Q/G$ and $\sigma(t) := [q(t), A((q(t), \dot{q}(t)))]_g \subset \mathfrak{g}$, where we denote by $(q(t), \dot{q}(t))$ the local coordinates of $v_q(t) \in T_q(q(t))Q$ at each $t$.

**Definition 2.5.** The connection $A$ also allows to define the curvature form $B$, a 2-form on $Q$ taking values on $\mathfrak{g}$, determined by

$$B(v_q, u_q) := dA(v_q, u_q) - [A(v_q), A(u_q)]_g \in \mathfrak{g},$$

where $u_q, v_q$ are arbitrary vectors in $T_q Q$ such that $T_q \pi(u_q) = u_p$ and $T_q \pi(v_q) = v_p$, with $p = \pi(q)$. The curvature form $B$ induces a $\mathfrak{g}$-valued two-form $\tilde{B}$ on $Q/G$ defined by

$$\tilde{B}(u_p, v_p) = [q, B_q(u_q, v_q)]_g \in \mathfrak{g}, \quad u_p, v_p \in T_p(Q/G),$$

where $u_q, v_q$ and $u_p, v_p$ are related as above. The two-form $\tilde{B}$ is called the reduced curvature form (for more details see [18] and references therein).

2.2.1. The covariant derivative. It is well known that the covariant derivative on a vector bundle induces an associated covariant derivative on its dual bundle. In this work, as in [18] and [32], we use this fact to define the covariant derivative in the dual of the adjoint bundle. If $\tilde{\sigma}(t)$ is a curve on $\mathfrak{g}^*$ the covariant derivative of $\tilde{\sigma}(t)$ is defined in such a way that for some curve $\sigma(t)$ on $\mathfrak{g}$, both, $\tilde{\sigma}(t)$ and $\sigma(t)$ project onto the same curve $p(t)$ on $Q/G$. Then

$$\frac{d}{dt} \langle \tilde{\sigma}(t), \sigma(t) \rangle = \left\langle \frac{D \tilde{\sigma}(t)}{Dt}, \sigma(t) \right\rangle + \left\langle \tilde{\sigma}(t), \frac{D \sigma(t)}{Dt} \right\rangle.$$

In the same way one can define the covariant derivative on $T^*(Q/G)$ and therefore a covariant derivative on $T^*(Q/G) \times_{Q/G} \mathfrak{g}^*$ (see [18] Section 3 for more details).

2.3. Lagrange-Poincaré reduction. Lagrangian reduction by stages ([18], Theorem 3.4.1) states that the Euler-Lagrange equations (1) with a $G$-invariant Lagrangian $L : TQ \to \mathbb{R}$ are equivalent to the Lagrange-Poincaré equations on $TQ/G \cong T(Q/G) \times_{Q/G} \mathfrak{g}$ (under the isomorphism (8)) with reduced Lagrangian $L : T(Q/G) \times_{Q/G} \mathfrak{g} \to \mathbb{R}$. The Lagrange-Poincaré equations read

$$\begin{aligned}
\frac{D}{Dt}\frac{\partial L}{\partial \dot{\sigma}} - \text{ad}^* \frac{\partial L}{\partial \sigma} &= 0, \\
\frac{\partial L}{\partial \dot{p}} - \frac{D}{Dt}\frac{\partial L}{\partial \dot{\sigma}} &= \left\langle \frac{\partial L}{\partial \sigma}, i_p \tilde{B} \right\rangle,
\end{aligned}$$

where $\tilde{B}$ is the reduced curvature form defined in (9) and $D/Dt$ denotes the covariant derivative in the associated bundle. Note that we are employing coordinates $(p, \dot{p}, \sigma)$ for $T(Q/G) \times_{Q/G} \mathfrak{g}$. Moreover, $i_p \tilde{B}$ denotes the $\mathfrak{g}$-valued 1-form on $Q/G$ defined by

$$i_{\dot{p}} \tilde{B}(-) = \tilde{B}(\dot{p}, -).$$

Consider a local trivialization of the principal bundle $\pi : Q \to Q/G$, i.e. a trivial principal bundle $\pi_U : U \times G \to U$ where $U$ is an open subset of $Q/G$ with structure group $G$ acting on the second factor by left multiplication. Denote by $(q^s)$, $s = 1, \ldots, r = \dim(Q) - \dim(G)$ local coordinates on $U$ and define maps $e_b : U \to \mathfrak{g}$
satisfying that for each \( p \in U, \{ \epsilon_b \} \) is a basis of \( \mathfrak{g} \). \( b = 1, \ldots, \dim(G) \). We choose the standard connection on \( U \), that is, at a tangent vector \( (p, g, \dot{p}, \dot{g}) \in T_{(p,g)}(U \times G) \) we have \( \mathcal{A}(p, g, \dot{p}, \dot{g}) = A_d \gamma(A_e \dot{p}) \dot{g} + \xi \) where \( \xi = g^{-1} \dot{g}, \dot{e} \) is the identity of \( G \), and \( A_e : U \to \mathfrak{g} \) is a 1-form given by \( A_e(p) \dot{p} = \mathcal{A}(p, e, \dot{p}, 0) \).

Denote by \( \tilde{e}_b \) a section of \( \tilde{g} \) given by \( \tilde{e}_b(p) = [p, e, e_b(p)]_g \), \( \sigma = \sigma^a \tilde{e}_a \), and \( \tilde{p}_b = \frac{\partial \mathcal{L}}{\partial \sigma}(\tilde{e}_b) \). With this notation the Lagrange-Poincaré equations (10) read (see [48] and [18] Section 4.2 for details)

\[
- \frac{d}{dt} \tilde{p}_b = \tilde{p}_a (C^a_{db} \sigma^d - C^a_{db} A^d \dot{p}^a),
\]

where \( C^a_{db} \) are the structure constants of the Lie algebra of \( \mathfrak{g} \), \( B^a_{db} \) are the coefficients of the curvature in the local trivialization and \( A^a_e(p) \) are the coefficients of \( A_e \), for given local coordinates \( p^a \in U \) determined by \( (A_e(p) \dot{p})^a e_a = A^a_e(p) \dot{p}^a e_a, A_e(p) \dot{p} = \mathcal{A}(p, e, \dot{p}, 0) \).

### 2.4 Higher-order tangent bundles.

It is possible to introduce an equivalence relation on the set \( C^k(\mathbb{R}, Q) \) of \( k \)-differentiable curves from \( \mathbb{R} \) to \( Q \) (see [39] for more details): By definition, two given curves in \( Q \), \( \gamma_1(t) \) and \( \gamma_2(t) \), where \( t \in I \subset \mathbb{R} \), \( (0 \in I) \), have a contact of order \( k \) at \( q_0 = \gamma_1(0) = \gamma_2(0) \), if there is a local chart \( (U, \varphi) \) of \( Q \) such that \( q_0 \in U \) and

\[
\frac{d^s}{dt^s}(\varphi \circ \gamma_1(t)) \bigg|_{t=0} = \frac{d^s}{dt^s}(\varphi \circ \gamma_2(t)) \bigg|_{t=0},
\]

for all \( s = 0, \ldots, k \). This is a well defined equivalence relation on \( C^k(\mathbb{R}, Q) \) and the equivalence class of a curve \( \gamma \) will be denoted by \( [\gamma]_{\gamma_0}^{(k)} \). The set of equivalence classes will be denoted by \( T^kQ \) and it is not hard to show that it has the natural structure of a differentiable manifold. Moreover, \( \tau_Q^k : T^kQ \to Q \) where \( \tau_Q^k ([\gamma]_{\gamma_0}^{(k)}) = \gamma(0) \), is a fiber bundle called the tangent bundle of order \( k \) (or higher-order tangent bundle) of \( Q \). In the sequel we will employ HO as short for higher-order.

Given a differentiable function \( f : Q \to \mathbb{R} \) and \( l \in \{0, \ldots, k\} \), its \( l \)-lift \( f^{(l,k)} \) to \( T^kQ \), \( 0 \leq l \leq k \), is the differentiable function defined as

\[
f^{(l,k)}([\gamma]_{\gamma_0}^{(k)}) = \frac{d^l}{dt^l} (f \circ \gamma(t)) \bigg|_{t=0}.
\]

Of course, these definitions can be applied to functions defined on open sets of \( Q \).

From a local chart \( (q^i) \) on a neighborhood \( U \) of \( Q \), it is possible to induce local coordinates \( (q^{(0)i}, q^{(1)i}, \ldots, q^{(k)i}) \) on \( T^kU = (\tau_Q^k)^{-1}(U) \), where \( q^{(s)i} = (q^i)^{(s,k)} \) if \( 0 \leq s \leq k \). Sometimes, we will use the standard conventions, \( q^{(0)i} \equiv q^i, q^{(1)i} \equiv \dot{q}^i, q^{(2)i} \equiv \ddot{q}^i \), etc.

#### 2.4.1 HO quotient space:

A smooth map \( f : M \to N \) induces a map \( T^k(f) : T^kM \to T^kN \) given by

\[
T^k(f)(([\gamma]_{\gamma_0}^{(k)})) := \left[f \circ \gamma\right]_{f(\gamma_0)}^{(k)}.
\]

The action of a Lie group \( \Phi_g \) is lifted to an action \( \Phi_g^k : T^kQ \to T^kQ \), given by

\[
\Phi_g^k([\gamma]_{\gamma_0}^{(k)}) := T^k(\Phi_g([\gamma]_{\gamma_0}^{(k)})) = \left[\Phi_g \circ \gamma\right]_{\Phi_g(\gamma_0)}^{(k)}.
\]
If $\Phi_g$ is free and proper, we get a principal $G$-bundle $T^{(k)}Q \rightarrow T^{(k)}Q/G$, which is a fiber bundle over $Q/G$. The class of an element $[\gamma]_{q_0}^{(k)} \in T^{(k)}q_0$ in the quotient is denoted $[\gamma]_{q_0}^{(k)}G$. From [18] (see Lemma 2.3.4) we know that the covariant derivative of a curve $\sigma(t) = [q(t),\xi(t)]_g \subset \widehat{g}$ relative to a principal connection $\mathcal{A}$ is given by

$$\frac{D}{Dt}\sigma(t) = [q(t),\dot{\xi}(t) - [\mathcal{A}(q(t),\dot{q}(t)),\xi(t)]]_g.$$  

In the particular case when $\sigma(t) = [q(t),\mathcal{A}(q(t),\dot{q}(t))]_g$ we have

$$\frac{D}{Dt}\sigma(t) = [q(t),\dot{\xi}(t)]_g \quad \text{and} \quad \frac{D^2}{Dt^2}\sigma(t) = [q(t),\ddot{\xi}(t)]_g.$$  

If we denote by $\xi_1(t) = \xi(t)$, $\xi_2(t) = \dot{\xi}(t)$, $\xi_3(t) = \ddot{\xi}(t)$, $\xi_4(t) = [\xi(t),\xi_2(t)],...,$ $\xi_k(t) = \dot{\xi}_{k-1}(t) - [\xi(t),\dot{\xi}_{k-1}(t)]$, operating recursively one obtains

$$\frac{D^{k-1}}{Dt^{k-1}}\sigma(t) = [q(t),\xi_k(t)]_g,$$

where $\xi_k \in \mathfrak{g}$ (see [32] for example).

Taking all these elements into account, the bundle isomorphism that generalizes $\alpha^{(1)}_\mathcal{A}$ (8) to the HO case is given by $\alpha^{(k)}_\mathcal{A} : T^{(k)}Q/G \rightarrow T^{(k)}(Q/G) \times_{Q/G} \kappa \mathfrak{g}$:

$$\alpha^{(k)}_\mathcal{A}([\gamma]_{q_0}^{(k)}) = \left(T^{(k)}\pi([\gamma]_{q_0}^{(k)}),\sigma(0), \frac{D}{Dt}_{t=0}\sigma(t), \frac{D^2}{Dt^2}_{t=0}\sigma(t),...\frac{D^{k-1}}{Dt^{k-1}}_{t=0}\sigma(t)\right).$$

Note that with some abuse of notation we are denoting the class $[\gamma]_{q_0}^{(k)}G$ by $[\gamma]_{q_0}^{(k)}G$. In the expression (13), $\sigma(t) := [q(t),\mathcal{A}(q(t),\dot{q}(t))]_g$, $q(t)$ is any curve representing $[q]_{q_0}^{(k)} \in T^{(k)}Q$ with $q(0) = q_0$, and $\kappa \mathfrak{g} := \mathfrak{g} \times \mathfrak{g} \times \ldots \times \mathfrak{g}$ (see [18] and [42]).

For further purposes, it will be useful to establish a local notation for the reduced variables. We follow [32] in this respect:

$$\alpha^{(k)}_\mathcal{A}([\gamma]_{q_0}^{(k)}) = (p,\dot{p},\ddot{p},...p^{(k)},\sigma,\dot{\sigma},\ddot{\sigma},...\sigma^{(k-1)}),$$

where $(p,\dot{p},\ddot{p},...p^{(k)})$ are local coordinates on $T^{(k)}(Q/G)$ and the dots denote the time derivatives in a local chart; $\sigma,\dot{\sigma},\ddot{\sigma},...\sigma^{(k-1)}$ are independent elements in $\mathfrak{g}$, where we employ the notation $\sigma^{(l)} := \frac{D^l}{Dt^l}\sigma(t)$ for the covariant derivative.

We introduce the notation $M^{(k)} := T^{(k)}(Q/G) \times_{Q/G} \kappa \mathfrak{g} \rightarrow \mathbb{R}$, $M := M^{(1)}$, and $s^{(k,k-1)}$ to denote the elements $(p,\dot{p},\ddot{p},...p^{(k)},\sigma,\dot{\sigma},\ddot{\sigma},...\sigma^{(k-1)}) \in M^{(k)}$, $s^{(k,k-1)}_\sigma = (\sigma,\dot{\sigma},\ddot{\sigma},...\sigma^{(k-1)}) \in \kappa \mathfrak{g}$ and $s^{(k,k-1)}_\sigma = (p,\dot{p},\ddot{p},...p^{(k)}) \in T^{(k)}(Q/G)$.

### 2.5. Constrained Hamilton’s principle.

We derive the constrained HO Lagrange-Poincaré equations using the variational principles studied in [18] and [32] for first order systems and unconstrained HO systems respectively.

The constraint $\phi^\alpha : T^{(k)}Q \rightarrow \mathbb{R}$ is said to be $G$-invariant if it is invariant under the $k$-order tangent lift of left translations, that is,

$$\phi^\alpha \circ T^{(k)}\Phi_g([\gamma]_{q_0}^{(k)}) = \phi^{(k)}([\gamma]_{q_0}^{(k)})$$

where $\Phi_g$ is just the left translation of the Lie group $L_g : G \rightarrow G$, and $T^{(k)}\Phi_g$ as in (12).
Let $L : T^{(k)}Q \rightarrow \mathbb{R}$, and $\phi^\alpha : T^{(k)}Q \rightarrow \mathbb{R}$ be a $G$-invariant HO Lagrangian and $G$-invariant HO (independent) constraints, respectively, $\alpha = 1, \ldots, m$. The $G$-invariance allows to induce the reduced Lagrangian $L : T^{(k)}Q/G \rightarrow \mathbb{R}$ and reduced constraints $\chi^\alpha : T^{(k)}Q/G \rightarrow \mathbb{R}$.

After fixing a connection $\mathcal{A}$ we can employ the isomorphism (13). Then it is possible to write the reduced Lagrangian and the reduced constraints $L : M^{(k)} \rightarrow \mathbb{R}$ and $\chi^\alpha : M^{(k)} \rightarrow \mathbb{R}$, and employ the local coordinates $s^{(k,k-1)}$ as in (14).

**Remark 1.** Note that if $Q$ is the Lie group $G$, the adjoint bundle is identified with $\mathfrak{g}$ via the isomorphism $\alpha_{\mathcal{A}}^k : T^{(k)}G/G \rightarrow k\mathfrak{g} :$

$$\alpha_{\mathcal{A}}^k([[g]^{(k)}_{g_0}]) = \left(g^{-1}(0)\dot{g}(0), \left. \frac{d}{dt} \right|_{t=0} \xi(t), \ldots, \left. \frac{d^{k-1}}{dt^{k-1}} \right|_{t=0} \xi(t) \right),$$

where $\xi(t) = g^{-1}(t)\dot{g}(t)$. If we choose $g_0 = e$, that is, $[[g_0 g]^{(k)}_e]_G = [[g]^{(k)}_{g_0}]_G$, we can define the reduced Lagrangian and the reduced constraints given by $L : k\mathfrak{g} \rightarrow \mathbb{R}$ and $\chi^\alpha : k\mathfrak{g} \rightarrow \mathbb{R}$ (see [18]).

In order to establish the variational principle, we must derive the variations on $T^{(k)}(Q/G) \oplus k\mathfrak{g}$ induced by variations on $Q$, i.e. $\delta q(t) = \left. \frac{d}{ds} \right|_{s=0} q(t, s) \in T_{q(t)}Q$ at each $t$. Consider an arbitrary deformation $p(t, s) \oplus \sigma(t, s)$ with $p(t, 0) \oplus \sigma(t, 0) = p(t) \oplus \sigma(t)$, the corresponding covariant variation is

$$\delta p(t) \oplus \delta \sigma(t) := \left. \frac{\partial}{\partial s} \right|_{s=0} p(t, s) \oplus \left. \frac{D}{Ds} \right|_{s=0} \sigma(t, s).$$

Since $s_p^{(k)} = T^{(k)}\pi([q]^{(k)}_q) := [\pi \circ \eta]^{(k)}_p$, the variations $\delta p$ of $p(t)$ are arbitrary except at the extremes; that is, $\delta p^{(l)}(0) = \delta p^{(l)}(T) = 0$ for $l = 1, \ldots, k-1$; $t \in [0, T]$. Then, locally we have that

$$\delta s_p^{(k)} := (\delta p, \delta \dot{p}, \ldots, \delta p^{(k)}).$$ (15)

On the other hand, the covariant variations of $\sigma$ are given by,

$$\delta \sigma(t) = \left. \frac{D}{Dt} \right|_{t} \left[ q(t), \mathcal{A}(q(t), \delta q(t)) \right]_{\mathfrak{g}} + \left[ q(t), [\mathcal{B}(\delta q(t), \dot{q}(t))]_{\mathfrak{g}} + [q(t), [\mathcal{A}(q(t), \dot{q}(t)), \mathcal{A}(q(t), \delta q(t))]_{\mathfrak{g}}.\right.$$  

In general, (see [32]) for $\tilde{B}$ the reduced curvature (9), it follows that

$$\delta \sigma^{(j)}(t) = \left. \frac{D^i}{Dt^i} \right|_{t} \delta \sigma(s, t) + \sum_{j=0}^{i-1} \left. \frac{D^j}{Dt^j} \right|_{t} \left[ \tilde{B}(p)(\dot{p}(t), \delta p(t)), \sigma^{(i-1-j)}(t) \right].$$ (16)

Note that in the expression above, $[\cdot, \cdot]$ denotes the usual Lie algebra bracket in $\tilde{\mathfrak{g}}$.

Consider the augmented Lagrangian $\tilde{L} : M^{(k)} \times \mathbb{R}^m \rightarrow \mathbb{R}$ given by

$$\tilde{L}(s^{(k,k-1)}, \lambda) = L(s^{(k,k-1)}) + \lambda \chi^\alpha (s^{(k,k-1)})$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$.

A curve on $\gamma(t) \in C^\infty(\mathbb{R}, M^{(k)} \times \mathbb{R}^m)$ is locally represented by

$$\gamma(t) = (s^{(k,k-1)}(t), \lambda_\alpha(t)).$$

Constrained HO Lagrange-Poincaré equations are derived by considering the constrained variational principle for the action $S : C^\infty(\mathbb{R}, M^{(k)} \times \mathbb{R}^m) \rightarrow \mathbb{R}$ given by

$$S(\gamma) = \int_0^T \tilde{L}(\gamma(t)) \, dt.$$
for variations \( \delta s^{(k,k-1)} = (\delta p, \delta \dot{p}, \ldots, \delta p^{(k)}, \delta \sigma, \delta \dot{\sigma}, \ldots, \delta \sigma^{(k-1)}) \) such that

(I) \( \delta \dot{p}^{(j)}(0) = \delta p^{(j)}(T) = 0 \), for \( j = 1, \ldots, k-1 \)

(II) Variations \( \delta \sigma^{(j)} \) are of the form \( (16) \), for \( j = 0, \ldots, k-1 \),

(III) \( \delta \sigma = \delta \sigma_\Xi + [\sigma, \Xi] + \tilde{B}(\dot{\gamma}, \dot{p}) \) where \( \Xi \) is an arbitrary curve in \( \tilde{g} \) with \( \frac{D^j}{D\gamma^k} \Xi \) vanishing at the endpoints.

**Theorem 2.6.** Let \( L : T^{(k)}Q \to \mathbb{R} \) be a \( G \)-invariant Lagrangian and \( \chi^\alpha : T^{(k)}Q \to \mathbb{R} \) \( G \)-invariant constraints, \( \alpha = 1, \ldots, m \). Consider the principal \( \mathcal{G} \)-bundle \( \pi : Q \to Q/G \) and let \( \mathcal{L} : M^{(k)} \to \mathbb{R} \) and \( \chi^\alpha : M^{(k)} \to \mathbb{R} \) be the reduced HO Lagrangian and the reduced HO constraints, respectively, associated with \( \mathcal{A} \).

The curve \( \gamma(t) \in \mathcal{M} \) satisfies \( \delta S(\gamma) = 0 \) with respect to the variations \( \delta s^{(k,k-1)} \) satisfying (I)-(III) if and only if \( \gamma(t) \) is a solution of the constrained HO Lagrange-Poincaré equations given by

\[
0 = \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \mathcal{L}}{\partial p^{(i)}} + \lambda_{\alpha} \frac{\partial \chi^\alpha}{\partial \sigma^{(i)}} \right) - \sum_{i=0}^{k-1} \left( (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \mathcal{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \chi^\alpha}{\partial \sigma^{(i)}} \right) \right) - \sum_{i=0}^{k-1} (-1)^i \alpha d^i \sigma_{(i-1)} \frac{D^{(i)}}{D\sigma^{(i)}} \left( \frac{\partial \mathcal{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \chi^\alpha}{\partial \sigma^{(i)}} \right) \right)
\]

\[
0 = \left( \frac{D}{Dt} - \alpha d^k \sigma \right) \sum_{i=0}^{k-1} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \mathcal{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \chi^\alpha}{\partial \sigma^{(i)}} \right),
\]

\[
0 = \chi^\alpha(\gamma(t)),
\]

(17)

where \( i_{\gamma} \tilde{B} \) denotes the \( \text{Ad}Q \)-valued 1-form on \( Q/G \) defined by \( i_{\gamma} \tilde{B}(\cdot) = \tilde{B}(\dot{\gamma}(\cdot)) \), given in (9).

**Proof.** The proof follows in a straightforward way by replacing the Lagrangian in the proof of Theorem 4.1 of [32] by the extended Lagrangian \( \tilde{\mathcal{L}} \).

**Remark 2.** If \( Q \) is a Lie group \( G \) then the constrained HO Lagrange-Poincaré equations (17) become the constrained HO Euler-Poincaré equations, i.e:

\[
\begin{align*}
0 &= \left( \frac{d}{dt} - \alpha d^k \sigma \right) \sum_{i=0}^{k-1} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \mathcal{L}}{\partial \sigma^{(i)}} + \lambda_{\alpha} \frac{\partial \chi^\alpha}{\partial \sigma^{(i)}} \right), \\
0 &= \chi^\alpha(\gamma(t)),
\end{align*}
\]

(18)

where \( \gamma(t) = (s_{\sigma}^{(k-1)}(t), \lambda_{\alpha}(t)) \in C^\infty(\mathbb{R}, kg \times \mathbb{R}^m) \) (see [25] and [33]).

**Remark 3.** In the examples, we will be interested in the case \( k = 2 \). In that case, the second-order Lagrange-Poincaré equations are locally written as:

\[
\begin{align*}
\frac{\partial \tilde{\mathcal{L}}}{\partial p^s} - \frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{p}^s} \right) + \frac{d^2}{dt^2} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \ddot{p}^s} \right) &= \left( \frac{d}{dt} \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma^a} \right) \left( B_{i\alpha}^{a} \dot{p}^i + C^a_{db} A^b \dot{p}^d \right), \\
\frac{d^2}{dt^2} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma^b} \right) - \frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\sigma}^b} \right) &= \left( \frac{d}{dt} \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma^a} - \frac{\partial \tilde{\mathcal{L}}}{\partial \sigma^a} \right) \left( C^a_{db} \dot{\sigma}^d - C^a_{db} A^d \dot{p}^s \right), \\
\chi^\alpha(s^{(2,1)}) &= 0,
\end{align*}
\]

where \( \tilde{\mathcal{L}}(s^{(2,1)}, \lambda_{\alpha}) = \mathcal{L}(s^{(2,1)}) + \lambda_{\alpha} \chi^\alpha(s^{(2,1)}) \).
3. Discrete constrained higher-order Lagrange-Poincaré equations.

3.1. Discrete mechanics and variational integrators. Variational integrators are a class of geometric integrators which are determined by a discretization of a variational principle. As a consequence, some of the main geometric properties of continuous system, such as symplecticity and momentum conservation, are present in these numerical methods (see [35],[49] and [53] and references therein). In the following we will summarize the main features of this type of geometric integrator.

A discrete Lagrangian is a map $L_d : Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the action integral defined by a continuous Lagrangian $L : TQ \to \mathbb{R}$, that is

$$L_d(q_0, q_1) \approx \int_{t_0}^{t_0+h} L(q(t), \dot{q}(t)) \, dt,$$

where $q(t)$ is a solution of the Euler-Lagrange equations (1) joining $q(t_0) = q_0$ and $q(t_0 + h) = q_1$ for small enough $h > 0$, where $h$ is viewed as the step size of the integrator.

Define the action sum $S_d : Q^{N+1} \to \mathbb{R}$ corresponding to the Lagrangian $L_d$ by

$$S_d = \sum_{n=0}^{N-1} L_d(q_n, q_{n+1}),$$

where $q_n \in Q$ for $0 \leq n \leq N$, $N$ is the number of discretization steps. The discrete variational principle states that the solutions of the discrete system determined by $L_d$ must extremize the action sum given fixed endpoints $q_0$ and $q_N$. By extremizing $S_d$ over $q_n$, $1 \leq n \leq N - 1$, it is possible to derive the system of difference equations

$$D_1 L_d(q_n, q_{n+1}) + D_2 L_d(q_{n-1}, q_n) = 0. \tag{21}$$

These equations are usually called the discrete Euler–Lagrange equations. If the matrix $D_{12} L_d(q_n, q_{n+1})$ is regular, it is possible to define from (21) a (local) discrete flow map $F_{L_d} : Q \times Q \to Q \times Q$, by $F_{L_d}(q_{n-1}, q_n) = (q_n, q_{n+1})$. We will refer to the $F_{L_d}$ flow, and also (with some abuse of notation) to the equations (21), as a variational integrator. Using the discrete Legendre transformations $\mathbb{F}^\pm L_d : Q \times Q \to T^*Q$ (which we assume regular), one can construct the discrete Hamiltonian flow $\tilde{F}_{L_d} : T^*Q \to T^*Q$ out of the discrete Lagrangian one, namely $\tilde{F}_{L_d} = \mathbb{F}^+ L_d \circ F_{L_d} \circ (\mathbb{F}^- L_d)^{-1}$, see [49].

The choice of discrete Lagrangian (19) is crucial in the discrete variational procedure, since it determines the order of local truncation error of the discrete flows with respect to the continuous ones. The optimal approximation is given by the so-called exact discrete Lagrangian [35, 49], say

$$L_d^E(q_0, q_1) = \int_{t_0}^{t_0+h} L(q(t), \dot{q}(t)) \, dt,$$

which provides the exact continuous flow in one time step $h$ via the discrete Euler-Lagrange equations (21). Nevertheless, the choice of $L_d^E$ is not practical since it involves the analytic solution of the continuous Euler-Lagrange equations; thus we need to take approximations. It was proven in [49] and [58] that, if $||L_d^E - L_d|| \sim$
$O(h^{r+1})$, for $r \in \mathbb{N}$, then the local truncation error of the variational integrator is of the same order, i.e. $||\tilde{F}_{L_d}(q_0, p_0) - (q(h), p(h))|| \sim O(h^{r+1})$, where $(q, p)$ are the coordinates of $T^*Q$, we define by $(q(t), p(t))$ the continuous flow and we set $(q(0), p(0)) = (q_0, p_0)$ (equivalent results can be established at a Lagrangian level). Furthermore, the symplecticity of $\tilde{F}_{L_d}$ ensures its stability in the long-term performance when $h \to 0$, as proven in [35] (Backward Error Analysis). In other words, if $H : T^*Q \to \mathbb{R}$ determines the energy of the system, then for a $r$-th order consistent $\tilde{F}_{L_d}$, we have that $\|H(\tilde{F}^N_{L_d}(q_0, p_0)) - H(q_0, p_0)\| \sim O(h^{r+1})$.

Since $L^F_d$ is normally not available, what one can pick is the order of the approximation of $L_d$. This is done by the interpolation of the continuous curves $q(t)$ and $\dot{q}(t)$ in the right hand side of (19). First order interpolations lead to the well-known midpoint rule, leapfrog, RATTLE and Störmer-Verlet methods [49]. High-order interpolations lead to higher-order approximations of $L^F_d$ and consequently to higher-order variational integrators, see for instance [17, 55] (note that with high-order we refer here to the local truncation error of the numerical methods).

In the following sections we are going to concentrate on discrete HO problems with symmetry, which are the main topic of this work. Our focus is on the discretization procedure and the mathematical tools that it involves, whereas the numerical behavior is planned to be explored in further works. However, regarding the local truncation error and stability (which will be still ensured thanks to the symplecticity of the HO variational integrators), discussed in the last paragraphs, the construction of $L^F_d$ for HO systems has been developed in [21]. In this reference the reader can find some numerical tests and more details on numerical aspects.

3.2. The discrete connection. The discretization of the reduced HO tangent bundle $T^{(k)}Q/G \cong M^{(k)}$ is based on the decomposition of the space $(Q \times Q)/G$ by means of the so-called discrete connection [42] (see also [31]):

$$\mathcal{A}_d : Q \times Q \to G,$$

which is defined to account for a reasonable discretization of the properties of the continuous connection $\mathcal{A}$ and, moreover, is $G$-equivariant (see [31, 42, 46] for more details).

Important properties that characterize the discrete connection are [42]:

1. $\mathcal{A}_d(q_0, g q_0) = g$,
2. $\mathcal{A}_d(g q_0, h q_1) = h \mathcal{A}_d(q_0, q_1) g^{-1}$,
3. Consider a local trivialization of the principal bundle $\pi : Q \to Q/G$; namely, for any open neighborhood $V \subset Q$ we have

$$V \cong U \times G,$$

where $U \subset Q/G$. In other words, for any $U \subset Q/G$, $\pi^{-1}(U) \cong U \times G$. In such a case, we have:

$$\mathcal{A}_d((p_0, g q_0), (p_1, g_0)) = g_1 \mathcal{A}_d((p_0, e), (p_1, e)) g^{-1}_0,$$  

where locally $\pi(g q_0) = \pi((p_0, g q_0)) = p_0$. This defines the local expression of $\mathcal{A}_d$, say $\mathcal{A}_d((p_0, e), (p_1, e)) = A(p_0, p_1) \in G$, which according to (24) leads to

$$\mathcal{A}_d((p_0, g q_0), (p_1, g g_0)) = g_1 A(p_0, p_1) g^{-1}_0.$$  

In particular, if $Q = G$, this leads to $\mathcal{A}_d((e, e), (e, e)) = e$ and consequently $\mathcal{A}_d(g q_0, g_1) = g_1 g_0^{-1}$. 

Remark 4. Sometimes (see [46]) the discrete connection is defined as the application \( A_d : Q \times Q \to G \) satisfying the properties (1) and (2) listed above.

In particular, given a discrete connection \( A_d \) the following isomorphism between bundles is well-defined (see [42] for the proof):

\[
\alpha^{(1)}_{A_d} : (Q \times Q)/G \to (((Q/G) \times (Q/G)) \oplus \tilde{G}),
\]

\[
[q_0, q_1]_G \mapsto (\pi(q_0), \pi(q_1)) \oplus [q_0, A_d(q_0, q_1)]_G,
\]

where we denote \( \tilde{G} := (Q \times G)/G \) in analogy with the adjoint bundle \( \tilde{g} \). We note that (26) is the discrete counterpart of the isomorphism (8).

Remark 5. In the case \( Q = G \), the isomorphism (26) is given by \( A_d(q_0, q_1) = g_0^{-1} g_1 \) in view of property (3), which leads to the usual Euler-Poincaré discrete reduction as in [51].

We consider the following extension of (26) in the case of HO tangent bundles, which is local for non-trivial bundles (see [31, 42] for more details):

\[
\alpha^{(k)}_{A_d} : \left\{ Q^{(k+1)} \right\}/G \to (Q/G)^{(k+1)} \times_{Q/G} \tilde{G}^{(k)},
\]

\[
[q_0, ..., q_k]_G \mapsto (\pi(q_0), ..., \pi(q_k)) \times_{Q/G} \bigoplus_{n=0}^{k-1} [q_0, A_d(q_n, q_{n+1})]_G
\]

where \( Q^{(k+1)} \) denotes the Cartesian product of \((k+1)\)-copies of \( Q \), \((Q/G)^{(k+1)}\) denotes the Cartesian product of \((k+1)\)-copies of \((Q/G)\) and \( \tilde{G}^{(k)} \) denotes the sum of \( k \)-copies of \( \tilde{G} \). Consequently, we consider \( H^{(k+1,k)} := (Q/G)^{(k+1)} \times_{Q/G} \tilde{G}^{(k)} \) as the discretization of the space \( M^{(k)} \), which is natural according to [2, 42, 49].

### 3.3. Variational discretization of constrained HO Lagrange-Poincaré equations

In the following, we aim to derive the variational discrete flow obtained from a discretization of \( \mathcal{L} \) and \( \chi^\alpha \). Therefore, we shall work in local coordinates, particularly in the local trivialization of the principal bundle (23).

The first task consists of obtaining the variational discretization of equations (17). For this, we must fix the discrete connection (22) and the discrete isomorphism (27). Next, we can induce through \( A_d \) and \( \alpha^{(k)}_{A_d} \) the discrete reduced HO Lagrangian and the discrete reduced HO constraints,

\[
\mathcal{L}_d : H^{(k+1,k)} \to \mathbb{R}, \text{ and } \chi^\alpha_d : H^{(k+1,k)} \to \mathbb{R},
\]

for \( \alpha = 1, ..., m \).

For a clear exposition, first we develop the first-order case, i.e. \( k = 1 \), in the next subsection, where the main objects employed in the HO case shall be introduced.

#### 3.3.1. Discrete constrained Lagrange-Poincaré equations

We shall consider the discrete reduced Lagrangian and discrete reduced constraints:

\[
\mathcal{L}_d : H^{(2,1)} \to \mathbb{R} \text{ and } \chi^\alpha_d : H^{(2,1)} \to \mathbb{R},
\]

where \( H^{(2,1)} = (((Q/G) \times (Q/G)) \times_{Q/G} \tilde{G} \) according to the notation introduced above. Moreover, we will employ the trivialization (23) to fix a local representation of \( \tilde{G} \), and consequently of \([q_0, q_1]_G \in (Q \times Q)/G \). Indeed, employing the discrete connection \( A_d \) and the isomorphism (26), we can make the following identification

\[
\frac{(\pi^{-1}(U) \times \pi^{-1}(U))}{G} \cong \frac{(U \times G) \times (U \times G)}{G} \cong U \times U \times G.
\]
Moreover, one can prove that the map
\[
((U \times G) \times (U \times G))/G \longrightarrow U \times U \times G,
\]
\[
[(p_0, g_0), (p_1, g_1)] \longmapsto (p_0, p_1, A_d((p_0, e), (p_1, g_0^{-1}g_1)))
\]
\[
= (p_0, p_1, g_0^{-1}g_1A(p_0, p_1)),
\]
is a bijection (see [46]), where
\[
A : U \times U \rightarrow G
\]
is the local representation of the discrete connection as established in (25). Therefore, in this trivialization we can define the local coordinates
\[
a_n := [q_n, q_{n+1}]_G = (p_n, p_{n+1}, g_n^{-1}g_{n+1}A(p_n, p_{n+1})),
\]
where \(n\) is 0 or a positive integer.

**Lemma 3.1.** The variations for an element \(a_n \in U \times U \times G\) defined in (32) are determined by
\[
\delta a_n := \delta[q_n, q_{n+1}]_G = (\delta p_n, \delta p_{n+1}, -\eta_n W_n A(p_n, p_{n+1}) + W_n \eta_{n+1} A(p_n, p_{n+1}) + W_n (D_1 A(p_n, p_{n+1}), \delta p_n) + W_n (D_2 A(p_n, p_{n+1}), \delta p_{n+1})),
\]
where \(W_n = g_n^{-1}g_{n+1} \in G\) and \(\delta g_n := g_n \eta_n\), with \(\eta_n \in g\).

**Proof.** We observe that
\[
\delta[q_n, q_{n+1}]_G = (\delta p_n, \delta p_{n+1}, -g_n^{-1}\delta g_n g_n^{-1}A(p_n, p_{n+1}) + g_n^{-1}\delta g_n A(p_n, p_{n+1}) + g_n^{-1}g_{n+1}A(p_n, p_{n+1}) + g_n^{-1}g_{n+1}(D_1 A(p_n, p_{n+1}), \delta p_n) + g_n^{-1}g_{n+1}(D_2 A(p_n, p_{n+1}), \delta p_{n+1}))
\]
where, for \(i = 1, 2; D_i A(p_n, p_{n+1})\) is a one form on \(T_p U\) taking values on \(T A(p_n, p_{n+1})\) \(G\) for \(j = n\) if \(i = 1\) and \(j = n+1\) if \(i = 2\), according to (31). Using \(W_n = g_n^{-1}g_{n+1} \in G\) and \(\delta g_n := g_n \eta_n\), with \(\eta_n \in g\), then (34) can be rewritten as (33).

Given the grid \(\{t_n = nh \mid n = 0, \ldots, N\}\), with \(Nh = T\), define the discrete path space \(C_d(U \times U \times G) := \{\gamma_d : \{t_n\}_{n=0}^N \rightarrow U \times U \times G\}\). This discrete path space is isomorphic to the smooth product manifold which consists of \(N + 1\) copies of \(U \times U \times G\) (which is locally isomorphic to \(N + 1\) copies of \(((Q/G \times Q/G) \times Q/G)\)). The discrete trajectory \(\gamma_d \in C_d(U \times U \times G)\) will be identified with its image, i.e. \(\gamma_d(t_n) = \{a_n\}_{n=0}^N\) where \(a_n = (p_n, p_{n+1}, g_n^{-1}g_{n+1}A(p_n, p_{n+1}))\). Let us consider the reduced discrete Lagrangian \(L_d\) in (29). Define the discrete action sum, \(S_d : C_d(U \times U \times G) \rightarrow \mathbb{R}\), by
\[
S_d(\gamma_d) = \sum_{n=0}^{N-1} L_d([\gamma_n, \gamma_{n+1}]_G) = \sum_{n=0}^{N-1} L_d(p_n, p_{n+1}, g_n^{-1}g_{n+1}A(p_n, p_{n+1})),
\]
where the equality is established at a local level. From now on, we use the notation \(A_n := A(p_n, p_{n+1})\) and \(S_d(\gamma_d) = \sum_{n=0}^{N-1} L_d(a_n)\).

The **discrete constrained variational problem** associated with (29), consists of finding a discrete path \(\gamma_d \in C_d(U \times U \times G)\), given fixed boundary conditions, which extremizes the discrete action sum (35) subject to the discrete constraints \(\chi_d^a\).
This constrained optimization problem is equivalent to studying the (unconstrained) optimization problem for the augmented Lagrangian $\tilde{L}_d : H^{(2,1)} \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\tilde{L}_d([q_n, q_{n+1}]_G, \lambda_n^a) = L_d([q_n, q_{n+1}]_G) + \lambda_n^a \lambda_n^a([q_n, q_{n+1}]_G)$$

(36)

where $\lambda_n^a = (\lambda_1^n, ..., \lambda_m^n) \in \mathbb{R}^m$ are Lagrange multipliers. The associated action sum is given by

$$S_d(\tilde{\gamma}_d) = \sum_{n=0}^{N-1} \tilde{L}_d([q_n, q_{n+1}]_G, \lambda_n^a) = \sum_{n=0}^{N-1} \tilde{L}_d(p_n, p_{n+1}, g_n^{-1} g_{n+1} A(p_n, p_{n+1}), \lambda_n^a),$$

(37)

where again the equality is given at a local level and $\tilde{\gamma}_d \in C_d(U \times U \times G \times \mathbb{R}^m) := \{\tilde{\gamma}_d : [t_n]_N \to U \times U \times G \times \mathbb{R}^m\}$ is the discrete augmented path space. We establish the result in the following theorem, where the discrete constrained Lagrange-Poincaré equations are obtained.

**Theorem 3.2.** A discrete sequence $\{a_n, \lambda_n^a\}_{n=0}^N \in C_d(U \times U \times G \times \mathbb{R}^m)$ is an extremum of the action sum (37), with respect to variations $\delta[q_n, q_{n+1}]_G$ set in (33) and endpoint conditions $\delta q_0 = \delta q_N = 0$ where $q_j = (p_j, q_j)$ (while the Lagrange multipliers are free), if it is a solution of the discrete constrained Lagrange-Poincaré equations (42).

**Proof.** The proof will be divided into two parts. The first one consists on studying the variations of the action sum (35) associated with $L_d$. After that, our result follows by the incorporation of the constraints and Lagrange multipliers by considering $\tilde{L}_d$ instead of $L_d$ and (37) instead of (35).

Taking variations on the discrete action sum (35) with $q_0 = (p_0, q_0)$ and $q_N = (p_N, q_N)$ fixed, which in terms of variations implies $\delta p_0 = \delta p_N = 0$ and $\delta q_0 = \delta q_N = 0$, the latter leading to $\eta_0 = \eta_N = 0$, and using the Lemma 3.1, we obtain

$$\delta \sum_{n=0}^{N-1} L_d(p_n, p_{n+1}, W_n A_n)$$

$$= \sum_{n=1}^{N-1} \langle D_1 L_d(a_n) + D_2 L_d(a_{n-1}), \delta p_n \rangle$$

$$+ \sum_{n=1}^{N-1} \langle T_{W_n}^* L_{W_n}^{-1} (T_{W_n}^* A_{n-1} R_{A_{n-1}}^{-1} D_3 L_d(a_{n-1})) , \eta_n \rangle$$

$$- \sum_{n=1}^{N-1} \langle T_{W_n}^* R_{W_n}^{-1} (T_{W_n}^* A_n R_{A_n}^{-1} D_3 L_d(a_n)) , \eta_n \rangle$$

$$+ \sum_{n=1}^{N-1} \langle T_{W_n}^* A_n, L_{W_n}^{-1} D_3 L_d(a_n); \langle D_1 A(p_n, p_{n+1}), \delta p_n \rangle \rangle$$

$$+ \sum_{n=1}^{N-1} \langle T_{W_n}^* A_{n-1}, L_{W_n}^{-1} D_3 L_d(a_{n-1}); \mathcal{D} \rangle,$$

where $\mathcal{D} = \langle D_2 A(p_{n-1}, p_n), \delta p_n \rangle$, $D_i$ denotes the partial derivative with respect to the $i$-th variable, $R_q, L_g : G \to G$ are the left and right translations by the group variables, while $T_h R_q : T_h G \to T_h q G$, $T_h L_g : T_h G \to T_{gh} G$ are their cotangent
action. Therefore, \( \delta S_d = 0 \) for arbitrary variations implies
\[
\begin{align*}
0 &= D_1 \mathcal{L}_d(a_n) + D_2 \mathcal{L}_d(a_{n-1}) + T^* \hat{L}_{(W,A)}(n) D_3 \mathcal{L}_d(a_n) \\
&\quad + T^* \hat{L}_{(W,A)}(n-1) D_3 \mathcal{L}_d(a_{n-1}), \\
0 &= T_{W_{n-1}}^* L_{W_{n-1}}^* (T_{W_{n-1}A_{n-1}}^* R_{A_{n-1}}^{-1} D_3 \mathcal{L}_d(a_{n-1})) \\
&\quad - T_{W_n}^* R_{W_n}^{-1} (T_{W_n A_n}^* R_{A_n}^{-1} D_3 \mathcal{L}_d(a_n)),
\end{align*}
\]
for \( n = 1, \ldots, N - 1 \), where we define locally the operator \( T^* \hat{L}_{(W,A)} \) by its action on \( T^* G \). Namely, \( T^* \hat{L}_{(W,A)}(j) : T_{g^*}^* G \to T_p U \) for \( U \subset (Q/G) \) is defined by
\[
(T^* \hat{L}_{(g,A)}(j) D_3 \mathcal{L}_d(a), \delta p_j) := (T_{g^*}^* g^{-1} D_3 \mathcal{L}_d(a), \langle D_1 A, \delta p_j \rangle),
\]
where \( a \in U \times U \times G, a := (p_0, p_1, g A(p_0, p_1)) \), \( i = \{1, 2\} \) and \( j = i - 1 \) for each \( i \).

Let us define \( \mu_n := T_{W_n}^* R_{W_n}^{-1} (T_{W_n A_n}^* R_{A_n}^{-1} D_3 \mathcal{L}_d(a_n)) \in g^* \). It is easy to see that (39b) can be rewritten in its dual version as
\[
\mu_n = Ad_{W_{n-1}}^* \mu_{n-1}.
\]

Next, we introduce constraints in our picture by considering the augmented Lagrangian (36) instead of \( \mathcal{L}_d \), which inserted into (38) leads to
\[
\begin{align*}
0 &= D_1 \mathcal{L}_d(a_n) + D_2 \mathcal{L}_d(a_{n-1}) + T^* \hat{L}_{(W,A)}(n) D_3 \mathcal{L}_d(a_n) \\
&\quad + T^* \hat{L}_{(W,A)}(n-1) D_3 \mathcal{L}_d(a_{n-1}) + \lambda_n^\alpha \left\{ D_1 \chi^\alpha_d(a_n) + T^* \hat{L}_{(W,A)}(n) D_3 \chi^\alpha_d(a_n) \right\} \\
&\quad + \lambda_n^{-1} \left\{ D_2 \chi^\alpha_d(a_n-1) + T^* \hat{L}_{(W,A)}(n-1) D_3 \chi^\alpha_d(a_{n-1}) \right\},
\end{align*}
\]
(42)

\[
\begin{align*}
\mu_n &= Ad_{W_{n-1}}^* \mu_{n-1} - \lambda_n^\alpha \varepsilon^\alpha_n + \lambda_n^{-1} Ad_{W_{n-1}}^* \varepsilon^\alpha_n, \\
0 &= \chi^\alpha_d(a_n),
\end{align*}
\]
for \( n = 1, \ldots, N - 1 \), where we denote \( \varepsilon^\alpha_n := T_{W_n}^* R_{W_n}^{-1} (T_{W_n A_n}^* R_{A_n}^{-1} D_3 \chi^\alpha_d(a_n)) \in g^* \).

To obtain the discrete-time equations (42) we used the approach studied in [46]. That is, by using a discrete connection instead of deriving the local description of the curvature terms as in [31]. This approach automatically gives preservation of momentum and symplecticity since we employ a variational approach (see [46] for further details).

Note that the first equation in (42) represents a discrete-time version of the second equation in (10) (or equivalently (11) in a local description) where the curvature terms are included in the terms that come from (40). The second equation represents the (constrained) Euler-Poincaré part (first equation in (42)) in (10), or (11) in the local representation.

Next, define \( \mathcal{M}_{1,3}(a_n) := D_1 \chi^\alpha_d(a_n) + T^* \hat{L}_{(W,A)}(n) D_3 \chi^\alpha_d(a_n) \), where \( D_i \) denotes the partial derivative with respect to the \( i \)-th component, while \( D_{ij} = D_i D_j = D_j D_i \). Equations (42) determine a numerical integrator giving rise to a unique (local) variational flow given an initial value on \( U \times U \times G \times \mathbb{R}^m \) under the following algebraic conditions:

**Proposition 1.** Let \( \mathcal{M}_d \) be a regular submanifold of \( (U \times U \times G) \) given by
\[
\mathcal{M}_d = \{ a_n \in (U \times U \times G) \mid \chi^\alpha_d(a_n) = 0 \},
\]
where \( a_n \) is defined in (32). If the matrix
\[
\begin{bmatrix}
D_1\tilde{L}_d(a_n, \lambda^n) & D_3 \left(\tilde{T}_n^\top \tilde{L}_d(W_{A_1})(n)D_3\tilde{L}_d(a_n, \lambda^n)\right) & M_{(1,3)}(a_n) \\
D_2\mu_n(a_n) & D_3\mu_n(a_n) & e_n(a_n) \\
(D_2\chi^1_n(a_n))^T & (D_3\chi^1_n(a_n))^T & 0 \\
\end{bmatrix}
\]
is non singular for all \( a_n \in M_d \), there exists a neighborhood \( U_k \subset M_d \times \mathbb{R}^m \) of \( (a_n^*, \lambda^0_n) \) satisfying equations (42), and an unique (local) application \( \Upsilon_{L_d} : U_k \subset M_d \times \mathbb{R}^m \rightarrow M_d \times \mathbb{R}^m \) such that
\[
\Upsilon_{L_d}(a_n, \lambda^0_n) = (a_{n+1}, \lambda^1_n).
\]

Proof. It is a direct consequence of the implicit function theorem applied to equations (42).

**Remark 6.** Note that the regularity condition given in Proposition 1 represents a first order discretization of the regularity condition for first order vakonomic systems presented, for instance, in \([3]\) (Section 7.3, Equation 7.3.5), \([1]\), \([19]\), \([29]\), \([30]\) and \([37]\). In general such a condition for continuous time systems is expressed as a
\[
2 \times 2 \text{ block-matrix } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ where } A \text{ corresponds to the matrix giving the classical hyper-regularity condition for the equivalence between Lagrangian and Hamiltonian formalism in mechanics by means of the Legendre transform (this corresponds with the first two entries in the first row of the matrix in Proposition 1). The sub-matrix } C \text{ corresponds to the partial derivative of the constraints with respect to the velocities, as does the sub-matrix } B, \text{ and } D \text{ is the null sub-matrix. Taking into account the split into vertical and horizontal variables, it is easy to see the similarities of the regularity condition of the continuous-time and discrete-time systems. } \diamond
\]

**Remark 7.** In the case \( Q = G, (41) \) reduces to the usual discrete Euler-Poincaré equations [51]. In this case \( A = e \), and therefore \( a_n = W_n \). Thus, (39b) becomes
\[
T^\ast_{W_{n-1}}L^\prime_{W_{n-1}}L^\ast_d(W_{n-1}) - T^\ast_{W_{n}}R^\ast_{W_{n-1}}L^\prime_d(W_n) = 0,
\]
where \( \prime \) denotes the derivative with respect to \( W \). Setting \( \mu_n := T^\ast_{W_{n}}R^\ast_{W_{n-1}}L^\prime_d(W_n) \), \( \mu_n \in \mathfrak{g}^* \), we arrive at the discrete Lie-Poisson equations \( \mu_n = A^\ast_{W_{n-1}}\mu_{n-1} \).

### 3.3.2. Variational integrators for constrained HO Lagrange-Poincaré equations:

Next, we consider the HO case (28). According to (27) and (30), we can find local coordinates \([q_0, q_1, \ldots, q_k]_G \in U^{(k+1)} \times G^k \) given by
\[
(p_0, \ldots, p_k, \bar{g}_0, \bar{g}_1, \ldots, \bar{g}_{k-1}),
\]
where \( \bar{g}_i := g_i^{-1}g_{i+1}A(p_i, p_{i+1}) \) we denote the element of the \( i \)-th copy of \( \bar{G} \), for \( i = 0, \ldots, k - 1 \) and \( U^{(k+1)} \) denotes \( (k + 1) \)-copies of the neighborhood \( U \subset Q/G \). The variation of the \( i \)-th copy of \( Q/G \) is given as before by \( \delta p_i, \) for \( i = 0, \ldots, k \); while the variation of \( \bar{g}_i \) is given by
\[
\delta \bar{g}_i = -\eta_i W_i A_i + W_i \eta_{i+1} A_i + W_i \langle D_1 A_i, \delta p_i \rangle + W_i \langle D_2 A_i, \delta p_{i+1} \rangle,
\]
where we have set \( W_i = g_i^{-1}g_{i+1} \in G, \) \( \eta_i = g_i^{-1}\delta g_i \in \mathfrak{g} \) and \( A_i := A(p_i, p_{i+1}) \).

In the HO case, given the grid \( \{t_n = nh : n = 0, \ldots, N\} \), with \( Nh = T \), the discrete path space is determined by
\[
\mathcal{C}_d \left(U^{(k+1)} \times G^k\right) := \left\{ \tilde{z}_d : \{t_n\}^N_{n=0} \rightarrow U^{(k+1)} \times G^k \right\}.
\]
The discrete space will be identified with its image, i.e. \( \tilde{\gamma}_d(t_n) = \{ \tilde{a}_n \}_{n=0}^N \), where we employ the notation

\[
\tilde{a}_n := (p_n, p_{n+1}, \ldots, p_{n+k}, \tilde{g}_n, \tilde{g}_{n+1}, \ldots, \tilde{g}_{n+k-1}).
\]

We see that \( \tilde{a}_n \) is a \((2k + 1)\)-tuple with \( 2k + 1 \) elements. This discrete path space is isomorphic to the smooth product manifold which consists of \( N + 1 \) copies of \( U^{(k+1)} \times \tilde{G}^k \) (which locally is isomorphic to \( N + 1 \) copies of \( (Q/G)^{(k+1)} \times Q/G \times \tilde{G}^k \)).

Let us define the discrete action sum associated with the HO Lagrangian \( \mathcal{L}_d \) as \( S_d: \mathcal{C}_d \left( U^{(k+1)} \times \tilde{G}^k \right) \rightarrow \mathbb{R} \) given by

\[
S_d(\tilde{\gamma}_d) = \sum_{n=0}^{N-k} \mathcal{L}_d([q_n, q_{n+1}, \ldots, q_{n+k}]_G) = \sum_{n=0}^{N-k} \mathcal{L}_d(\tilde{a}_n)
\]

where the second equality is established at a local level.

The \emph{discrete constrained HO variational problem} associated with (28), consists of finding a discrete path \( \tilde{\gamma}_d \in \mathcal{C}_d(U^{(k+1)} \times \tilde{G}^k) \), given fixed boundary conditions, which extremizes the discrete action sum (46) subject to the discrete constraints \( \chi^2_\beta \).

This constrained optimization problem is equivalent to studying the \emph{(unconstrained) optimization problem} for the \emph{augmented Lagrangian} \( \tilde{\mathcal{L}}_d: H^{(k+1)} \times \mathbb{R}^m \rightarrow \mathbb{R} \) given by

\[
\tilde{\mathcal{L}}_d([q_n, \ldots, q_{n+k}]_G, \lambda_n) := \mathcal{L}_d([q_n, \ldots, q_{n+k}]_G) + \lambda^0_n \chi^0_d([q_n, \ldots, q_{n+k}]_G),
\]

where \( \lambda^0_n = (\lambda^0_1, \ldots, \lambda^0_m) \in \mathbb{R}^m \) are Lagrange multipliers, and its associated action sum is given by

\[
S_d(\tilde{\gamma}_d) = \sum_{n=0}^{N-k} \tilde{\mathcal{L}}_d([q_n, \ldots, q_{n+k}]_G, \lambda_n) = \sum_{n=0}^{N-k} \tilde{\mathcal{L}}_d(\tilde{a}_n, \lambda^0_n),
\]

where again the second equality is given at a local level and \( \tilde{\gamma}_d \in \mathcal{C}_d(U^{(k+1)} \times \tilde{G}^k \times \mathbb{R}^m) := \{ \tilde{\gamma}_d : \{ t_n \}_{n=0}^N \rightarrow U^{(k+1)} \times \tilde{G}^k \times \mathbb{R}^m \} \) is the discrete augmented path space.

Regarding the endpoint conditions, we shall consider \( q_{(0,k-1)} = (p_{(0,k-1)}, g_{(0,k-1)}) \) and \( q_{(N-k+1,N)} = (p_{(N-k+1,N)}, g_{(N-k+1,N)}) \) fixed, where \( q_{(0,k-1)} = (q_0, q_1, \ldots, q_{k-1}) \), \( q_{(N-k+1,N)} = (q_{N-k+1}, q_{N-k+2}, \ldots, q_N) \), and analogously for any sequence. In terms of variations this implies \( \delta p_{(0,k-1)} = \delta q_{(N-k+1,N)} = 0 \) and \( \delta g_{(0,k-1)} = \delta g_{(N-k+1,N)} = 0 \), the latter leading to \( \eta_{(0,k-1)} = \eta_{(N-k+1,N)} = 0 \). Furthermore, the Lagrange multipliers are set freely as in the first order case.

We establish the result in the following theorem, where the discrete constrained HO Lagrange-Poincaré equations are obtained. As in the case of Theorem 3.2, our proof strategy consists in studying the unconstrained problem (46), and afterwards adding the constraints (47).

**Theorem 3.3.** A discrete sequence \( \{ \tilde{a}_n, \lambda^0_n \}_{n=0}^N \in \mathcal{C}_d(U^{(k+1)} \times \tilde{G}^k \times \mathbb{R}^m) \) is an extremum of the action sum (48), with respect to variations \( \delta [q_n, \ldots, q_{n+k}]_G \) defined in (44) and endpoint conditions expressed above, if it is a solution of the discrete constrained HO Lagrange-Poincaré equations (50).

**Proof.** In the proof we will employ the index \( i \) for the \( k + 1 \) first elements, i.e. the \( p \) coordinates, and the index \( z \) for the last \( k \), i.e. the \( \tilde{g} \) coordinates. Taking variations in (46), according to the endpoint conditions detailed above and the variations (44)
we obtain:

\[
\begin{align*}
\delta \sum_{n=0}^{N-k} \mathcal{L}_d(\tilde{a}_n) &= \sum_{n=0}^{N-k} \left( \sum_{i=1}^{k+1} \langle D_i \mathcal{L}_d(\tilde{a}_n), \delta p_i \rangle + \sum_{z=k+2}^{2k+1} \langle D_z \mathcal{L}_d(\tilde{a}_n), \delta g_z \rangle \right) \\
&= \sum_{n=0}^{N-k} \sum_{i=1}^{k+1} \langle D_i \mathcal{L}_d(\tilde{a}_n), \delta p_i \rangle \\
&\quad + \sum_{n=0}^{N-k} \sum_{z=k+2}^{2k+1} \left( \langle T_{W_{n}}^{*} R_{W_{n}}^{-1} (T_{W_{n}}^{*} A_{n} R_{A_{n}^{-1}} D_z \mathcal{L}_d(\tilde{a}_n)), \eta_z \rangle + \langle T_{W_{n}}^{*} L_{W_{n}}^{-1} (T_{W_{n}}^{*} A_{n} R_{A_{n}^{-1}} D_z \mathcal{L}_d(\tilde{a}_n)), \eta_{z+1} \rangle \right.
\end{align*}
\]

where we have employed (44). Next, we assume that the \( z \)-th component, for \( z = k + 2, \ldots, 2k + 1 \), is labeled by \( n + z - k - 2 \) and rearranging the sum above after taking into account the endpoint conditions we obtain:

\[
\begin{align*}
\delta \sum_{n=0}^{N-k} \mathcal{L}_d(\tilde{a}_n) &= \sum_{n=k}^{N-k} \left( \sum_{i=1}^{2k+1} \langle T_{W_{n}}^{*} R_{W_{n}}^{-1} (T_{W_{n}}^{*} A_{n} R_{A_{n}^{-1}} D_z \mathcal{L}_d(\tilde{a}_n)), \eta_z \rangle + \langle T_{W_{n}}^{*} L_{W_{n}}^{-1} (T_{W_{n}}^{*} A_{n} R_{A_{n}^{-1}} D_z \mathcal{L}_d(\tilde{a}_n)), \eta_{z+1} \rangle \right)
\end{align*}
\]

where the operator \( T_{W_{n}}^{*} \hat{L}_{W_{A_{n}}} (n) \) is defined in (40). Equating this variation to zero and considering that \( \delta p_n \) and \( \eta_n \) are free for \( k \leq n \leq N-k \), we arrive at the discrete equations of motion:

\[
\begin{align*}
0 &= \sum_{i=1}^{k+1} D_i \mathcal{L}_d(\tilde{a}_{n-i+1}) \\
&\quad + \sum_{z=k+2}^{2k+1} \left( T_{W_{n}}^{*} \hat{L}_{W_{A_{1}}} (n) D_z \mathcal{L}_d(\tilde{a}_{n-z+k+2}) + T_{W_{n}}^{*} \hat{L}_{W_{A_{2}}} (n-1) D_z \mathcal{L}_d(\tilde{a}_{n-z+k+1}) \right),
\end{align*}
\]

\[
\begin{align*}
0 &= \sum_{z=k+2}^{2k+1} \left( T_{W_{n}}^{*} R_{W_{n}}^{-1} (T_{W_{n}}^{*} A_{n} R_{A_{n}^{-1}} D_z \mathcal{L}_d(\tilde{a}_{n-z+k+2})) - T_{W_{n-1}}^{*} L_{W_{n-1}}^{-1} (T_{W_{n-1}}^{*} A_{n-1} R_{A_{n-1}^{-1}} D_z \mathcal{L}_d(\tilde{a}_{n-z+k+1})) \right),
\end{align*}
\]

for \( k \leq n \leq N-k \). The second equation may be rewritten in a more compact way in its dual version by making the following identifications

\[
\hat{\rho}_n := D_z \mathcal{L}_d(\tilde{a}_{n-z+k+2}) \in T_{W_{n}}^{*} A_{n} G \text{ for } k + 2 \leq z \leq 2k + 1,
\]
- $\tilde{M}_n := \sum_{z=k+2}^{2k+1} \tilde{\mu}_{n_z}^z \in T_{W_{n,A_n}} G$,
- $M_n := T_{W_{n,A_n}} R_{W_{n,A_n}} (T_{W_{n,A_n}} R_{A_n}^{-1} \tilde{M}_n) \in g^*$, which leads to the equation $M_n = \text{Ad}_{W_{n-1}} M_{n-1}, \ k \leq n \leq N - k$.

Next, introducing constraints into our picture by considering the augmented Lagrangian (47) we find the discrete constrained HO Lagrange-Poincaré equations

$$\begin{align*}
0 &= \sum_{i=1}^{k+1} D_i \mathcal{L}_d(\tilde{a}_{n-i+1}) + T^* \tilde{L}_{(W A_i)} (n) \sum_{z=k+2}^{2k+1} D_z \mathcal{L}_d(\tilde{a}_{n-z+k+2}) + T^* \tilde{L}_{W A_2} (n-1) \sum_{z=k+2}^{2k+1} D_z \mathcal{L}_d(\tilde{a}_{n-z+k+2}) + T^* \tilde{L}_{W A_1} (n) \sum_{z=k+2}^{2k+1} \lambda_{\alpha}^{n-z+k+2} D_z \chi^*_d(\tilde{a}_{n-z+k+2}) + T^* \tilde{L}_{W A_2} (n-1) \sum_{z=k+2}^{2k+1} \lambda_{\alpha}^{n-z+k+1} D_z \chi^*_d(\tilde{a}_{n-z+k+1}),
\end{align*}$$

(50)

$$M_n = \text{Ad}_{W_{n-1}} M_{n-1} - \sum_{z=k+2}^{2k+1} \lambda_{\alpha}^{n-z+k+2} \varepsilon_{(n,z)}^\alpha + \text{Ad}_{W_{n-1}}^{*} \sum_{z=k+2}^{2k+1} \lambda_{\alpha}^{n-z+k+1} \varepsilon_{(n-1,z)}^\alpha,$$

$$0 = \chi^*_d(\tilde{a}_n),$$

for $k \leq n \leq N - k$, where we define

$$\varepsilon_{(n,z)}^\alpha := T_{W_{n-1}} R_{W_{n}}^{-1} (T_{W_{n,A_n}} R_{A_n}^{-1} D_z \chi^*_d(\tilde{a}_{n-z+k+2})) \in g^*.$$

As in the first order case, a direct consequence of the implicit function theorem applied to (50) is the existence of the (local) variational flow for the numerical method.

Denoting $M_{(1,k+2)}(\tilde{a}_n) := D_1 \chi^*_d(\tilde{a}_n) + T^* \tilde{L}_{(W A_1)} (n) D_{k+2} \chi^*_d(\tilde{a}_n)$ we arrive at the following proposition.

**Proposition 2.** Let $\tilde{M}_d$ be a regular submanifold of $U^{(k+1)} \times G^k$ given by

$$\tilde{M}_d = \{ \tilde{a}_n \in U^{(k+1)} \times G^k \mid \chi^*_d(\tilde{a}_{n-j}) = 0 \text{ for all } j = 0, \ldots, k \}$$

where $\tilde{a}_n$ is of the form (45).

If the matrix

$$\begin{bmatrix}
D_{(1,k+1)} \tilde{L}_d(\tilde{a}_n, \lambda^n) & D_{2k+1} \left( T^* \tilde{L}_{(W A_1)} (n) D_{k+2} \tilde{L}_d(\tilde{a}_n, \lambda^n) \right) & M_{(1,k+2)}(\tilde{a}_n) \\
D_{k+1} M_n(\tilde{a}_n) & D_{2k+1} M_n(\tilde{a}_n) & \varepsilon_{(n,k+2)}^\alpha(\tilde{a}_n) \\
(D_{k+1} \chi_d^*(\tilde{a}_n))^T & (D_{2k+1} \chi_d^*(\tilde{a}_n))^T & 0
\end{bmatrix}$$

is non-singular for all $\tilde{a}_n \in \tilde{M}_d$, there exists a neighborhood $\mathcal{V}_k \subset \tilde{M}_d \times k \mathbb{R}^m$ of $\gamma^* = (\tilde{a}_{n-k}, \tilde{a}_{n-1}, \lambda_{\alpha}^{n-k}, \ldots, \lambda_{\alpha}^{n-1})$ satisfying equations (50), and an unique
(local) application \( \tilde{\Upsilon}^{\mathcal{L}_d} : \mathcal{V}_k \subset \tilde{\mathcal{M}}_d \times k\mathbb{R}^m \to \tilde{\mathcal{M}}_d \times k\mathbb{R}^m \) such that

\[
\tilde{\Upsilon}^{\mathcal{L}_d}(\tilde{a}_{n-k}, \ldots, \tilde{a}_{n-1}, \lambda_{n-k}^{n-1}) = (\tilde{a}_{n-k+1}, \ldots, \tilde{a}_n, \lambda_{n-k+1}^{n}).
\]

Observe that when \( k = 1 \) equations (50) are the discrete constrained Lagrange-Poincaré equations (42) and the regularity condition given in Proposition 2 is the one obtained in Proposition 1.

**Remark 8.** In [26] it has been shown that under a regularity condition equivalent to the one given in Proposition 2, the discrete constrained system preserves the symplectic 2-form (see Remark 3.4 in [26]). Therefore the methods that we are deriving in this work are automatically symplectic methods. Moreover, under a group of symmetries preserving the discrete Lagrangian and the constraints, we additionally obtain momentum preservation. In the case when the principal bundle is a trivial bundle, and therefore the terms associated with the connection and curvature are zero, we obtain the same results as [22].

4. Application to optimal control of underactuated systems. Underactuated mechanical system are controlled mechanical systems where the number of the control inputs is strictly less than the dimension of the configuration space. In this section we consider dynamical optimal control problems for a class of underactuated mechanical systems determined by Lagrangian systems on principal bundles.

We assume that we are only allowed to have control systems that are controllable, that is, for any two points \( q_0 \) and \( q_T \) in the configuration space, there exists an admissible control defined on some interval \([0, T]\) such that the system with initial condition \( q_0 \) reaches the point \( q_T \) in time \( T \) (see [3] for more details).

Let \( L : TQ \to \mathbb{R} \) be a \( G \)-invariant Lagrangian inducing a reduced Lagrangian \( \mathcal{L} : M \to \mathbb{R} \) where \( M := T(Q/G) \times_{Q/G} \tilde{g}^* \) and \((p, \dot{p}, \sigma)\) are local coordinates on an open set \( \Omega \subset M \). Consider the control manifold \( \mathcal{U} \subset \mathbb{R}^r \) where \( r < \dim Q \) and \( u \in \mathcal{U} \) is the control input (control parameter) which in coordinates reads \( u = (u_1, \ldots, u_r) \in \mathbb{R}^r \).

We denote by \( \Gamma(M^*) \) the space of sections of a smooth manifold

\[
M^* := T^*(Q/G) \times_{Q/G} \tilde{g}^*
\]

and consider a set of linearly independent sections \( B^a = \{(\eta^a, \bar{\eta}^a)\} \in \Gamma(M^*) \), such that \( \eta^a([q]_G) \in T_{[q]_G}(Q/G) \); \( \bar{\eta}^a([q]_G) \in \tilde{g}^* \) for \( a = 1, \ldots, r \) and \( [q]_G \in \tau(\Omega) \subset Q/G \), where \( \tau : M \to Q/G \). Therefore \( \eta^a \oplus \bar{\eta}^a \in \Gamma(M^*) \).

**Definition 4.1.** The reduced controlled Euler-Lagrange equations or controlled Lagrange-Poincaré equations are

\[
\frac{D}{Dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{p}} \right) - \frac{\partial \mathcal{L}}{\partial p} + \left( \frac{\partial \mathcal{L}}{\partial \sigma}, i_p \bar{B} \right) = u_a \eta^a([q]_G), \tag{51a}
\]

\[
\left( \frac{D}{Dt} - \text{ad}_{\sigma}^* \right) \frac{\partial \mathcal{L}}{\partial \sigma} = u_a \bar{\eta}^a([q]_G). \tag{51b}
\]

A controlled Lagrange-Poincaré system is a controlled mechanical systems whose dynamics is given by the controlled Lagrange-Poincaré equations (51).
We refer to a controlled decoupled Lagrange-Poincaré system when equations (51a)-(51b) can be written as a system of equations of the form

\[ \left\langle \frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{p}} \right) - \frac{\partial L}{\partial p} - \langle \frac{\partial L}{\partial \sigma}, i_p \tilde{B} \rangle, \eta_a \rangle + \left\langle \left( \frac{D}{Dt} - \text{ad}^* \right) \frac{\partial L}{\partial \sigma}, \eta_a \right\rangle = u_a, \quad (52a) \]

\[ \left\langle \frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{p}} \right) - \frac{\partial L}{\partial p} - \langle \frac{\partial L}{\partial \sigma}, i_p \tilde{B} \rangle, \eta_a \rangle + \left\langle \left( \frac{D}{Dt} - \text{ad}^* \right) \frac{\partial L}{\partial \sigma}, \eta_a \right\rangle = 0, \quad (52b) \]

that is, a controlled Lagrange-Poincaré system is written as a control system showing which configurations are actuated and which ones unactuated.

The next Lemma shows that a controlled Lagrange-Poincaré system always permits a description for the controlled dynamics as a controlled decoupled Lagrange-Poincaré system.

**Lemma 4.2.** A controlled Lagrange-Poincaré system defined by (51) is equivalent to the controlled decoupled Lagrange-Poincaré system described by (52).

**Proof.** Given that \( B^a = \{ (\eta^a, \tilde{\eta}^a) \} \), are independent elements of \( \Gamma(M^*) \) we complete \( B^a \) to be a basis of \( \Gamma(M^*) \), i.e. \( \{ B^a, B^\alpha \} \), and take its dual basis \( \{ B_a, B_\alpha \} \) on \( \Gamma(M) \). If we set \( B_a = \{ (\eta_a, \tilde{\eta}_a) \} \) and \( B_\alpha = \{ (\eta_\alpha, \tilde{\eta}_\alpha) \} \), where \( \eta_a, \eta_\alpha \in \mathfrak{X}(Q/G) \) and \( \tilde{\eta}_a, \tilde{\eta}_\alpha \in \Gamma(\tilde{g}) \), we obtain the relationships

\[ \langle \eta^a, \eta_b \rangle = \delta^a_b, \quad \langle \eta^a, \tilde{\eta}_b \rangle = \langle \eta^a, \eta_\beta \rangle = \langle \eta^a, \tilde{\eta}_\beta \rangle = 0, \]

\[ \langle \tilde{\eta}^a, \tilde{\eta}_b \rangle = \delta^a_b, \quad \langle \tilde{\eta}^a, \eta_\beta \rangle = \langle \tilde{\eta}^a, \tilde{\eta}_\beta \rangle = 0, \]

\[ \langle \tilde{\eta}^a, \eta_\beta \rangle = \delta^a_\beta, \quad \langle \tilde{\eta}^a, \tilde{\eta}_\beta \rangle = 0. \]

Coupling (51a) to \( \eta_a \) and (51b) to \( \tilde{\eta}_a \), and adding up the results we obtain (52a). Equivalently, if we couple (51a) to \( \eta_\alpha \) and (51b) to \( \tilde{\eta}_\alpha \), and add up the resultants we obtain (52b).

\( \square \)

**Remark 9.** Observe that (52a) provides an expression of the control inputs as a function on the second-order tangent bundle \( M^{(2)} \) locally described by coordinates \( (p, \dot{p}, \ddot{p}, \sigma, \dot{\sigma}) \),

\[ u_a = F_a(p, \dot{p}, \ddot{p}, \sigma, \dot{\sigma}) = \left\langle \frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{p}} \right) - \frac{\partial L}{\partial p} - \langle \frac{\partial L}{\partial \sigma}, i_p \tilde{B} \rangle, \eta_a \rangle + \left\langle \left( \frac{D}{Dt} - \text{ad}^* \right) \frac{\partial L}{\partial \sigma}, \eta_a \right\rangle. \]

Next we consider an optimal control problem.

**Definition 4.3** (Optimal control problem). Find a trajectory \( \gamma(t) = (p(t), \sigma(t), u(t)) \) of the state variables and control inputs satisfying (51), subject to boundary conditions \( (p(0), \dot{p}(0), \sigma(0)) \) and \( (p(T), \dot{p}(T), \sigma(T)) \), and minimizing the cost functional

\[ J(s^{(2,1)}, u) = \int_0^T C(s^{(2,1)}(t), u(t)) \, dt \]

for a cost function \( C : M \times \mathfrak{U} \to \mathbb{R} \).

Solving the optimal control problem is equivalent to solving a constrained second-order variational problem [6], with Lagrangian \( \tilde{L} : M^{(2)} \to \mathbb{R} \) locally described by

\[ \tilde{L}(s^{(2,1)}) := C \left( s^{(1,0)}, F_a(s^{(2,1)}) \right), \quad (54) \]
where \( C \) is the cost function and \( F_a \) is defined in (53); and subject to the constraints

\[
\chi^\alpha(s^{(2,1)}) = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial p} \right) - \frac{\partial L}{\partial p} - \left( \frac{\partial L}{\partial \eta} \right) \right) + \left( \frac{D}{Dt} - ad^*_\eta \right) \frac{\partial L}{\partial \eta} = 0,
\]

equivalent to equation (52b).

Then, given boundary conditions, necessary optimality conditions for the optimal control problem are determined by the solutions of the constrained second-order Lagrange-Poincaré equations for the Lagrangian (54) subject to (55). The resulting equations of motion are a set of combined third order and fourth order ordinary differential equations.

Motivated by the examples that we study in the next section, we restrict our self to a particular class of these control problems where we assume full controls in the base manifold \( Q/G \), that is, using Lemma 4.2, we consider the controlled Lagrange-Poincaré equations, in a local trivialization \( \pi_U : U \times G \to U \) of the principal bundle \( \pi : Q \to Q/G \), i.e.

\[
\frac{d}{dt} \frac{\partial L}{\partial \sigma^\beta} (C^\delta_{\gamma \beta} \sigma^\gamma - C^\delta_{\gamma \beta} A^\gamma_i \dot{p}^\gamma_i) = 0,
\]

(56a)

\[
\frac{\partial L}{\partial \sigma^a} (B^b_{ca} \dot{p}^c + C^b_{de} \sigma^d A^e_a) = u_a,
\]

(56b)

In this context, the optimal control problem consists of finding a solution of the state variables and control inputs for the previous equations (56) given boundary conditions and minimizing the cost functional

\[
J(s^{(2,1)}) = \int_0^T C \left( p^a, \dot{p}^a, \sigma^a, \frac{\partial L}{\partial \sigma^a} - \frac{d}{dt} \frac{\partial L}{\partial \sigma^a} \right) \left( B^b_{ca} \dot{p}^c + C^b_{de} \sigma^d A^e_a \right) dt.
\]

Necessary conditions for optimality in the optimal control problem are characterized by the constrained second-order variational problem determined by the second-order Lagrangian

\[
\bar{L}(s^{(2,1)}) = C \left( p^a, \dot{p}^a, \sigma^a, \frac{\partial L}{\partial \sigma^a} - \frac{d}{dt} \frac{\partial L}{\partial \sigma^a} \right) \left( B^b_{ca} \dot{p}^c + C^b_{de} \sigma^d A^e_a \right)
\]

subject to the second-order constraints

\[
\chi^\alpha(s^{(2,1)}) = \frac{d}{dt} \frac{\partial L}{\partial \sigma^\beta} (C^\delta_{\gamma \beta} \sigma^\gamma - C^\delta_{\gamma \beta} A^\gamma_i \dot{p}^\gamma_i)
\]

(58)

whose solutions satisfy the constrained second-order Lagrange-Poincaré equations for \( \bar{L}(s^{(2,1)}, \lambda) = \bar{L}(s^{(2,1)}) + \lambda_\alpha \chi^\alpha(s^{(2,1)}) \) with \( \lambda_\alpha \in \mathbb{R}^m \) the Lagrange multipliers.

Those equations are in general given by a set of fourth order nonlinear ordinary differential equations which are very difficult to solve explicitly. Thus, constructing numerical methods is in order, a task for which the results in the previous sections must be implemented.

**Remark 10.** It is well known that, under some mild regularity conditions, necessary conditions for optimality obtained through a constrained variational principle, are equivalent to the ones given by Pontryagin Maximum Principle (see [3], section 7.3, Theorem 7.3.3 for the proof).

For higher-order systems, the same result can be proved. In particular, in [28] for unconstrained higher-order mechanical system without symmetries the equivalence between higher-order Euler-Lagrange equations and higher-order Hamilton equations was shown. It would be interesting to study such equivalence for constrained
systems and the relationship with necessary conditions for optimality in optimal control problems of underactuated mechanical systems. Such results were demonstrate for nonholonomic systems in [4], where the equivalence between conditions for optimal solutions obtained by the Pontryagin Maximum Principle and as a constrained variational problem for this particular class of constraints was established. Moreover, once such equivalence for constrained systems can be understood, by using the results of [32] and [56] the relation between optimality conditions obtained by a constrained variational principle and the ones obtained by Pontryagin Maximum principle can be extended for the class of higher-order systems with symmetries studied in this work.

Given discretizations of (57) and (58), denoted \( \mathcal{L}_d \) and \( \chi_d^\alpha \) respectively, defined on \( 3U \times 2G \), with local coordinates \( \tilde{a}_n = (p_{n-2}, p_{n-1}, p_n, \tilde{g}_n, \tilde{g}_{n-1}) \), \( \tilde{g}_i := g_i^{-1}g_{i+1}A(p_i, p_{i+1}) \), \( 2 \leq n \leq N - 2 \), the associated discrete optimal control problem consist of obtaining the sequences \{ \( p_n \) \}_{0:N}, \{ \tilde{g}_n \}_{0:N} \, \text{and} \, \{ \lambda_n \}_{0:N} \, \text{from the second-order constrained discrete Lagrange-Poincaré equations, i.e. (50) for} \, k = 2 \). By Theorem 3.3, the discrete constrained second-order Lagrange-Poincaré equations are given by

\[
0 = D_1\mathcal{L}_d(\tilde{a}_n) + D_2\mathcal{L}_d(\tilde{a}_{n-1}) + D_3\mathcal{L}_d(\tilde{a}_{n-2}) + T^*\tilde{L}_{W_A}(n)(D_4\mathcal{L}_d(\tilde{a}_n)) + \lambda_n^0 D_1\chi_d^\alpha(\tilde{a}_n)
\]

\[
+ \lambda_n^{n-2}D_3\chi_d^\alpha(\tilde{a}_{n-2}) + T^*\tilde{L}_{W_A}(n)(\lambda_n^0 D_4\chi_d^\alpha(\tilde{a}_n) + \lambda_n^{n-1}D_5\chi_d^\alpha(\tilde{a}_{n-1}))
\]

\[
+ T^*\tilde{L}_{W_A}(n-1)(\lambda_n^{n-1}D_4\chi_d^\alpha(\tilde{a}_{n-1}) + \lambda_n^{n-2}D_5\chi_d^\alpha(\tilde{a}_{n-2})) + \lambda_n^{n-1}D_2\chi_d^\alpha(\tilde{a}_{n-1}),
\]

(59)

\[
0 = M_n - Ad_{W_{\alpha-1}}^\alpha M_{n-1} + \lambda_n^{n-1}\varepsilon_{(n,4)}^\alpha + \lambda_n^{n-2}\varepsilon_{(n,5)}^\alpha
\]

\[- Ad_{W_{\alpha-1}}^\alpha (\lambda_n^{n-1}\varepsilon_{(n-1,4)}^\alpha + \lambda_n^{n-2}\varepsilon_{(n-1,5)}^\alpha),
\]

(60)

\[
0 = \chi_d^\alpha(\tilde{a}_n), \quad 0 = \chi_d^\alpha(\tilde{a}_{n-1}), \quad 0 = \chi_d^\alpha(\tilde{a}_{n-2}),
\]

(61)

for \( 2 \leq n \leq N - 2 \), and where

\[
M_n = T^*_W R_{W_{\alpha-1}}(T^*_W A_{\alpha^{-1}} R_{A_{\alpha^{-1}}}(D_4\mathcal{L}_d(\tilde{a}_n) + D_5\mathcal{L}_d(\tilde{a}_{n-1}))),
\]

\[
\varepsilon_{(n,4)}^\alpha = T^*_W R_{W_{\alpha-1}}(T^*_W A_{\alpha^{-1}} D_4\chi_d^\alpha(\tilde{a}_n)),
\]

\[
\varepsilon_{(n,5)}^\alpha = T^*_W R_{W_{\alpha-1}}(T^*_W A_{\alpha^{-1}} D_5\chi_d^\alpha(\tilde{a}_n)),
\]

\[
\varepsilon_{(n-1,4)}^\alpha = T^*_W R_{W_{\alpha-1}}(T^*_W A_{\alpha^{-1}} R_{A_{\alpha^{-1}}} D_4\chi_d^\alpha(\tilde{a}_{n-1})),
\]

\[
\varepsilon_{(n-1,5)}^\alpha = T^*_W R_{W_{\alpha-1}}(T^*_W A_{\alpha^{-1}} R_{A_{\alpha^{-1}}} D_5\chi_d^\alpha(\tilde{a}_{n-1})).
\]

By Proposition 2 the equations given above determine (locally) the flow map for the numerical method: they indicate how to obtain \( \tilde{a}_n \) and \( \lambda_n \) given \( \tilde{a}_{n-1}, \tilde{a}_{n-2}, \lambda_{n-1}, \lambda_{n-2} \) if the matrix

\[
\begin{bmatrix}
D_3\mathcal{L}_d(\tilde{a}_n, \lambda^n) & D_5(T^*\tilde{L}_{W_A}(n) D_4\mathcal{L}_d(\tilde{a}_n, \lambda^n)) & M_{(1,4)}(\tilde{a}_n) \\
D_3 M_n(\tilde{a}_n) & D_5 M_n(\tilde{a}_n) & \varepsilon_{(n,4)}^\alpha(\tilde{a}_n) \\
(D_3\chi_d^\alpha(\tilde{a}_n))^T & (D_5\chi_d^\alpha(\tilde{a}_n))^T & 0
\end{bmatrix}
\]

is non singular, where \( M_{(1,4)}(\tilde{a}_n) := D_1\chi_d^\alpha(\tilde{a}_n) + T^*\tilde{L}_{W_A}(n) D_4\chi_d^\alpha(\tilde{a}_n) \).
4.1. Examples.

4.1.1. Optimal control of an electron in a magnetic field. We study the optimal control problem for the linear momentum and charge of an electron of mass \(m\) in a given magnetic field (see [3] Section 3.9).

One of the motivations for constructing structure preserving variational integrators for this example is that the charge is a conserved quantity and our method, since it is variational, preserves the momentum map associated with a Lie group of symmetries.

Let \(\mathcal{M}\) be a 3 dimensional Riemannian manifold and \(\pi : Q \to \mathcal{M}\) be a circle bundle (that is, \(\mathbb{S}^1\) acts on \(Q\) on the left and then \(\pi : Q \to \mathcal{M}\) is a principal bundle where \(\mathcal{M} = Q/\mathbb{S}^1\)) with respect to a left \(SO(2)\) action. We will use the isomorphism (as Lie group) of \(SO(2)\) and \(\mathbb{S}^1\) to make our analysis consistent with the theory.

Let \(\mathcal{A} : TQ \to \mathfrak{so}(2)\) be a principal connection on \(Q\) and consider the Lagrangian on \(TQ\) given by

\[
L(q, \dot{q}) = \frac{m}{2} \|T\pi(q, \dot{q})\|_M^2 + \frac{e}{c} \|\mathcal{A}(q, \dot{q})\|_{\mathfrak{so}(2)} - \phi(q),
\]

where \(e\) is the charge of the electron, \(c\) is the speed of light, \(\|\cdot\|_{\mathfrak{so}(2)} : \mathfrak{so}(2) \to \mathbb{R}\) the norm on \(\mathfrak{so}(2)\), given by \(\|\xi\|_{\mathfrak{so}(2)} = \langle\langle \xi, \xi\rangle\rangle^{1/2} = \sqrt{\text{tr}(\xi^T \xi)}\), for any \(\xi \in \mathfrak{so}(2)\) where the inner product on \(\mathfrak{so}(2)\) is given by \(\langle\langle \xi, \xi\rangle\rangle = \text{tr}(\xi^T \xi)\). \(\phi : \mathcal{M} \to \mathbb{R}\) represents the potential energy and \(\cdot\) denotes the left-action of \(\mathbb{S}^1\) on \(Q\). Note that in the absence of potential, \(L\) is a Kaluza-Klein Lagrangian type (see [18] for instance).

The motivation for including a potential function in our analysis is twofold. Firstly, it is inspired by possible further applications including static obstacles in the workspace. We use \(\phi\) as a artificial potential function (for instance a Coloumb potential) to avoid the obstacle. Secondly, it is motivated by use of this example in the theory of controlled Lagrangians and potential shaping for systems with breaking symmetries. Note that here \(V\) is not invariant under the symmetry group (see [3] Section 4.7) for more details.

Note also that \(\pi(\theta \cdot q) = \pi(q)\) for all \(q \in Q\) and \(\theta \in \mathbb{S}^1\). Thus

\[
L(\theta \cdot (q, \dot{q})) = \frac{m}{2} \|T\pi(\theta \cdot (q, \dot{q}))\|_M^2 + \frac{e}{c} \|\mathcal{A}(\theta \cdot (q, \dot{q}))\|_{\mathfrak{so}(2)} - \phi(\pi(\theta \cdot q))
\]

\[
= \frac{m}{2} \|T\pi(q, \dot{q})\|_M^2 + \frac{e}{c} \|\text{Ad}_\theta \cdot \mathcal{A}(q, \dot{q})\|_{\mathfrak{so}(2)} - \phi(\pi(q))
\]

\[
= \frac{m}{2} \|T\pi(q, \dot{q})\|_M^2 + \frac{e}{c} \|\mathcal{A}(q, \dot{q})\|_{\mathfrak{so}(2)} - \phi(\pi(q))
\]

\[
= L(q, \dot{q})
\]

where \(\text{Ad}_\theta = \text{Id}_{\mathfrak{so}(2)}\) because \(SO(2)\) is Abelian. That is, \(L\) is \(SO(2)\)-invariant and we may perform Lagrange-Poincaré reduction by symmetries to get the equations of motion on the principal bundle \(TQ/\mathcal{Q}(2)\).

Fixing the connection \(\mathcal{A}\) on \(Q\), we can use the principal connection \(\mathcal{A}\) to get an isomorphism \(\alpha_\mathcal{A} : TQ/\mathcal{Q}(2) \to TM \oplus \mathfrak{so}(2)\) which permits us to define the reduced Lagrangian

\[
\mathcal{L}(x, \dot{x}, \xi) = \frac{m}{2} \|\dot{x}\|_M^2 + \frac{e}{c} \|\xi\|_{\mathfrak{so}(2)} - \phi(x).
\]
For the reduced Lagrangian $\ell$, the dynamics is determined by the Lagrange-Poincaré equations (10), in this particular case

$$\frac{D\dot{x}^b}{Dt} + d\phi = \langle \mu, \bar{\mathcal{B}}(\dot{x}(t), \cdot) \rangle$$

where $\mu = \frac{\partial \mathcal{L}}{\partial \xi}$ is the charge of the particle. Here, $\bar{\mathcal{B}} : TM \wedge TM \to \mathfrak{so}(2)$ is the reduced curvature tensor associated with the connection form $A$, $d$ is the exterior differential and $\bar{\cdot} : g \to g^*$ is the associated isomorphisms to the inner product defined by the metric (see [3] and [11] for instance). Note that this equation corresponds with Wong’s equations [18].

In the case where $Q = \mathbb{R}^3 \times S^1$ the Lagrangian is

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{m}{2} \dot{x}^2 + \frac{e}{c}(A(x, \dot{x}) \cdot \dot{x}) - \phi(x).$$

In this case, we have that $TQ/\text{SO}(2) \simeq \mathbb{R}^3 \times \mathbb{R}$ where $\text{Ad}Q = \mathbb{R}$ and the reduced Lagrangian is

$$\mathcal{L}(x, \dot{x}, \xi) = \frac{m}{2} \dot{x}^2 + \frac{e}{c} \xi - \phi(x).$$

The above equations reduce to type of Lorentz force law describing the motion of a charged particle of mass $m$ in a magnetic field under the influence of a potential function

$$m\ddot{x} + \nabla \phi(x) = \frac{e}{c}(\dot{x} \times \mathbf{B}), \quad \dot{\mu} = 0,$$

where $\mu = \frac{\partial \mathcal{L}}{\partial \xi} = \frac{\xi}{c}$ and $\mathbf{B} = (B_x, B_y, B_z) \in \mathfrak{X}(\mathbb{R}^3)$.

Next, we introduce controls in our picture. Let $U \subset \mathbb{R}^3$, where $u = (u_1, u_2, u_3) \in U$ are the control inputs. Then, given $u(t) \in U$, the controlled decoupled Lagrange-Poincaré system (52) is given by

$$m \frac{D\dot{x}^b}{Dt} + d\phi - \langle \mu, \bar{\mathcal{B}}(\dot{x}(t), \cdot) \rangle = u(t),$$

$$\frac{D\dot{\mu}}{Dt} = 0.$$

If $Q = \mathbb{R}^3 \times S^1$ then the above system becomes the controlled decoupled Lagrange-Poincaré system describing the controlled dynamics of a charged particle of mass $m$ in a magnetic field under the influence of a potential function:

$$m\ddot{x} + \nabla \phi(x) - \frac{e}{c}(\dot{x} \times \mathbf{B}) = u(t)$$

$$\dot{\mu} = 0.$$

The optimal control problem consists of finding trajectories of the state variables and controls inputs, satisfying the previous equations subject to given initial and final conditions and minimizing the cost functional,

$$\min_{(x, \dot{x}, \xi, u)} \int_0^T C(x, \dot{x}, \xi, u)dt = \min_{(x, \dot{x}, \xi, u)} \frac{1}{2} \int_0^T ||u||^2 dt$$

where the norm $|| \cdot ||$ represents the Euclidean norm on $\mathbb{R}^3$. 
This optimal control problem is equivalent to solving the following constrained second-order variational problem given by
\[
\min_{(x, \dot{x}, \ddot{x}, \xi)} \mathcal{L}(x, \dot{x}, \ddot{x}, \xi) = \frac{1}{2} \left\| m \ddot{x} + \nabla \phi(x) - \frac{e}{c} (\dot{x} \times \mathbf{B}) \right\|^2, \quad (62)
\]
such that the constraint \( \chi(x, \dot{x}, \ddot{x}, \xi) = \xi \) arising from \( \dot{\mu} = 0 \), with \( \mathcal{L} : 3\mathbb{R}^3 \times 2\mathbb{R} \rightarrow \mathbb{R} \) and \( \chi : 3\mathbb{R}^3 \times 2\mathbb{R} \rightarrow \mathbb{R} \) (note that \( TQ/SO(2) \simeq \mathbb{R}^3 \times \mathbb{R} \) where \( \text{Ad} \mathcal{P} = \mathbb{R} \)).

For a simple exposition of the resulting equations describing necessary conditions for optimality in the optimal control problem, we restrict our analysis to the particular case when the magnetic field is aligned with the \( x_3 \)-direction and orthogonal to the \( x_1 - x_2 \) plane, that is, \( \mathbf{B} = (0, 0, B_z) \) with \( B_z \) constant, and the potential field is quadratic \( \phi = (x_1^2 + x_2^2 + x_3^2) \).

The constrained second-order Lagrange-Poincaré equations are
\[
\begin{align*}
x_1^{(iv)} &= 2\omega \dddot{x}_1 + \dddot{x}_1 \left( \omega^2 - \frac{4}{m} \right) + \frac{4\omega}{m} \dddot{x}_2 - \frac{4x_1}{m^2}, \quad (63a) \\
x_2^{(iv)} &= -2\omega \dddot{x}_1 + \dddot{x}_2 \left( \omega^2 - \frac{4}{m} \right) - \frac{4\omega}{m} \dddot{x}_1 - \frac{4}{m^2} x_2, \quad (63b) \\
x_3^{(iv)} &= -\frac{4\dddot{x}_3}{m} - \frac{4x_4}{m^2}, \quad (63c)
\end{align*}
\]
where \( \omega = \frac{eB_z}{mc} \), \( \lambda(t) \) is constant and \( \xi(t) = \xi \). \( \xi(t) \) comes from the the constraint given by preservation of the charge and \( \lambda(t) \) is obtained from \( \dddot{x} \lambda = 0 \), the Lagrange-Poincaré equation arising from \( \xi(t) \) (note that we obtain the same result for the multiplier as in [3] Section 7.5.)

In terms of the discretization of this system as presented in Section 3.2, we need to define the discrete connection (22), but given that the bundle is trivial, the connection vanishes. Denoting by \((x_n, \xi_n) = (x_n^1, x_n^2, x_n^3, \xi_n, \xi_{n+1})\), the discrete second order Lagrangian for the reduced optimal control problem corresponding to (62), is given by
\[
\tilde{\mathcal{L}}_d(x_n, x_{n+1}, x_{n+2}, \xi_n) = \frac{h}{2} \left\| x_{n+2} - 2x_{n+1} + x_n \right\|^2 + \nabla \phi(x_n) - \frac{e}{c} \frac{x_{n+1} - x_n}{h} \times \hat{\mathbf{B}}(x_n) \right\|
\]
\[
\chi_d(x_n, x_{n+1}, x_{n+2}, \xi_n) = \frac{e}{c},
\]
where \( \hat{\mathbf{B}}(x_n) := (B_1(x_n^1), B_2(x_n^2), B_3(x_n^3)) \).

By Theorem 3.3 the discrete second-order constrained Lagrange-Poincaré equations giving rise to the integrator which approximates the necessary conditions for optimality in the optimal control problem are given by
\[
\begin{align*}
\frac{x_n^{1}}{h^4} - 4x_n^{1} + 6x_n^{1} - 4x_n^{1} - x_n^{1} &= 2\omega \frac{x_n^{1} - 3x_n^{1} + 3x_n^{1} - x_n^{1}}{h^4} + \left( \frac{\omega^2}{m} \right) \frac{x_n^{1} - 2x_n^{1} + x_n^{1}}{h^2} + \frac{4\omega}{m} \frac{x_n^{2} - x_n^{2}}{h} - \frac{4}{m^2} x_n^{1}, \quad (64a)
\end{align*}
\]
\[
\begin{align*}
\frac{x_n^{1}}{h^4} - 4x_n^{1} + 6x_n^{1} - 4x_n^{1} - x_n^{1} &= 2\omega \frac{x_n^{1} - 3x_n^{1} + 3x_n^{1} - x_n^{1}}{h^4} + \left( \frac{\omega^2}{m} \right) \frac{x_n^{1} - 2x_n^{1} + x_n^{1}}{h^2} + \frac{4\omega}{m} \frac{x_n^{2} - x_n^{2}}{h} - \frac{4}{m^2} x_n^{1}, \quad (64a)
\end{align*}
\]
\[
\begin{align*}
\frac{x_{n+2}^2 - 4x_{n+1}^2 + 6x_n^2 - 4x_{n-1}^2 + x_{n-2}^2}{h^4} &= -2\omega^2 \frac{x_{n+1}^1 - 3x_n^1 + x_{n-1}^1 - x_{n-2}^1}{h^3} \\
&\quad + \left(\frac{\omega^2}{m} - \frac{4}{m} \right) \frac{x_{n+1}^2 - 2x_n^2 + x_{n-1}^2}{h^2} \\
&\quad - \frac{4\omega}{m} \frac{x_{n+1}^1 - x_{n-1}^1}{h} - \frac{4}{m^2} x_n^2, \quad (64b) \\
\frac{x_{n+2}^3 - 4x_{n+1}^3 + 6x_n^3 - 4x_{n-1}^3 + x_{n-2}^3}{h^4} &= -\frac{4}{m} \frac{x_{n+2}^3 - 2x_n^3 + x_{n-1}^3 - x_{n-2}^3}{h^2} - \frac{4}{m^2} x_n^3, \quad (64c)
\end{align*}
\]

together with \(\xi_n = \xi_{n-1} = \xi_{n-2} = \xi,\) and \(\lambda_n = \lambda_{n-1}\) for \(n = 2, \ldots, N - 2.\) We observe that (64a), (64b) and (64c) are a discretization in finite differences of (63a), (63b) and (63c), respectively.\(^1\)

4.1.2. Energy minimum control of two coupled rigid bodies: We consider a discretization of the energy minimum control for the motion planning of an underactuated system composed by two planar rigid bodies attached at their center of mass and moving freely in the plane, also known in the literature as Elroy’s beanie (see [44], [57] for details) which is an example of a dynamical system with a non-Abelian Lie group of symmetries.

The configuration space is \(Q = SE(2) \times \mathbb{S}^1\) with local coordinates denoted by \((x, y, \theta, \psi)\). The Lagrangian function \(L : TQ \to \mathbb{R}\) is given by

\[
L(x, y, \theta, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 (\dot{\theta} + \dot{\psi})^2 - V(\psi),
\]

(65)

where \(m\) denotes the mass of the system, \(I_1\) and \(I_2\) are the inertias of the first and the second body, respectively, and \(V\) is the potential energy. Note that the system is invariant under \(SE(2)\). After choosing a decomposition determined by the metric on \(Q\) which describes the kinetic energy of the Lagrangian (65) (see [44] and [57] for details) one can fix a connection \(A : T(SE(2) \times \mathbb{S}^1) \to \mathfrak{se}(2),\) with local expression

\[
A = \begin{bmatrix} 1 & 0 & y \frac{I_2}{I_1 + I_2} y & 0 & 1 & -x \frac{I_2}{I_1 + I_2} x & 0 \end{bmatrix}, \quad (66)
\]

and vanishing curvature.

Consider the base of \(\mathfrak{se}(2)\) denoted by \(\{\bar{e}_a\}\) with \(a = 1, 2, 3\) and given by

\[
\bar{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{e}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{e}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

\(^1\)Considering the forward difference \(\frac{x_{n+1} - x_n}{h}\) as a first order discretization of the velocity \(\dot{x},\) it is straightforward to check that

\[
\begin{align*}
\frac{x_{n+2} - 2x_{n+1} + x_n}{h^2}, \\
\frac{x_{n+1} - 3x_{n+1} + 3x_{n+1} - x_n}{h^3}, \\
\frac{x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n}{h^4},
\end{align*}
\]

are discretization of \(\dot{x},\) \(\ddot{x}\) and \(x^{(iv)},\) respectively. The shift of the \(n\) index present in equations (64) comes from the particular expression of the discrete constrained HO Lagrange-Poincaré equations provided in Theorem 3.3.
In terms of this basis, \( \xi \in \mathfrak{s}(2) \) can be written as \( \xi = \xi^1 \varepsilon_1 + \xi^2 \varepsilon_2 + \xi^3 \varepsilon_3 \) with \( \xi^1 = \cos \theta \dot{x} + \sin \theta \dot{y}, \xi^2 = \sin \theta \dot{x} + \cos \theta \dot{y} \) and \( \xi^3 = -\dot{\theta} - \frac{I_2}{I_1 + I_2} \). Moreover, since \([\varepsilon_1, \varepsilon_2] = 0, [\varepsilon_1, \varepsilon_3] = \varepsilon_2 \) and \([\varepsilon_2, \varepsilon_3] = -\varepsilon_1 \), then the non-vanishing constant structures of the Lie algebra \( \mathfrak{s}(2) \) are \( C^1_{13} = C^2_{13} = 1 \) and \( C^3_{13} = C^3_{23} = -1 \).

The isomorphism (8), \( \alpha_A : T(SE(2) \times S^1)/SE(2) \rightarrow T\mathbb{S}^1 \times \mathfrak{s}(2) \) is

\[
\alpha_A(\psi, \dot{\psi}, \xi) = (\psi, \dot{\psi}, \xi + A_c(\psi) \psi),
\]

with \( A_c(\psi) = (0, 0, \frac{I_2}{I_1 + I_2})^T \) \((A_c : U \subset \mathbb{S}^1 \rightarrow \mathfrak{s}(2) \) is a 1-form determined by \( A_c(\psi) \psi = A(\psi, e, \dot{\psi}, 0) \)). \( A_c \) is locally prescribed by the coefficients \( A^1_1 = A^2_1 = 0 \) and \( A^3_1 = \frac{I_2}{I_1 + I_2} \).

The reduced Lagrangian (see [44, 57] for details) \( \mathcal{L} : T\mathbb{S}^1 \oplus \mathfrak{s}(2) \rightarrow \mathbb{R} \) is given by

\[
\mathcal{L}(\psi, \dot{\psi}, \Omega) = \frac{1}{2} m(\Omega^2_2 + \Omega^2_1) + \frac{1}{2} (I_1 + I_2) \Omega^2_3 + \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} \dot{\psi}^2 - V(\psi),
\]

where \((\psi, \dot{\psi})\) are local coordinates for \( T\mathbb{S}^1 \) and \( \Omega \) for \( \mathfrak{s}(2) \), such that \( \Omega^1 = \xi^1 \), \( \Omega^2 = \xi^2 \) and \( \Omega^3 = \xi^1 + \frac{I_2}{I_1 + I_2} \dot{\psi} \), for \( \xi \in \mathfrak{s}(2) \), and consequently we observe that

\[
\begin{align*}
\Omega^1 &= \cos \theta \dot{x} + \sin \theta \dot{y}, \\
\Omega^2 &= -\sin \theta \dot{x} + \cos \theta \dot{y}, \\
\Omega^3 &= -\dot{\theta}.
\end{align*}
\]

The local Lagrange-Poincaré equations (11) in the \((\psi, \dot{\psi}, \Omega)\) coordinates read

\[
\begin{align*}
\dot{\Omega}^1 &= \Omega^2 \left( \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \right), \\
\dot{\Omega}^2 &= -\Omega^1 \left( \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \right), \\
\dot{\Omega}^3 &= 0, \\
\frac{I_1 I_2}{I_1 + I_2} \dot{\psi} &= -\frac{\partial V}{\partial \psi}.
\end{align*}
\]

Now, we introduce a control input in the equation corresponding to \( \mathbb{S}^1 \), i.e. (69d), namely

\[
\frac{I_1 I_2}{I_1 + I_2} \dot{\psi} = -\partial_\psi V(\psi) + u.
\]

As discussed in the previous subsection, the optimal control problem consists of finding trajectories of the state variables and control inputs, satisfying the equations (69a), (69b), (69c) and (70), subject to boundary conditions and minimizing the cost functional \( \int_0^T C(\psi, \dot{\psi}, \Omega, u) dt \). In particular we are interested in energy-minimum problems, where the cost function is of the form \( C(\psi, \dot{\psi}, \Omega, u) = \frac{1}{2} u^2 \). This optimal control problem is equivalent to solving the constrained second-order variational problem defined by the Lagrangian \( \hat{\mathcal{L}} : T^2 \mathbb{S}^1 \oplus 2\mathfrak{s}(2) \rightarrow \mathbb{R} \) and the constraints \( \chi^\alpha : T^2 \mathbb{S}^1 \oplus 2\mathfrak{s}(2) \rightarrow \mathbb{R}, \alpha = 1, 2, 3 \), defined by

\[
\hat{\mathcal{L}}(\gamma) = \frac{1}{2} \left( \frac{I_1 I_2}{I_1 + I_2} \dot{\psi} + \partial_\psi V(\psi) \right)^2, \quad \chi^1(\gamma) = \hat{\Omega}^1 - \Omega^2 \Omega^3 + \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^2, \quad \chi^2(\gamma) = \hat{\Omega}^2 + \Omega^1 \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^1 \quad \text{and} \quad \chi^3(\gamma) = \hat{\Omega}^3,
\]

where \( \gamma = (\psi, \dot{\psi}, \Omega, \dot{\Omega}), (\psi, \dot{\psi}, \ddot{\psi}) \) are
Lagrangian

\[ \mathcal{L}(\psi, \dot{\psi}, \dot{\psi}, \Omega, \dot{\Omega}, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2} \left( \frac{I_1 I_2}{I_1 + I_2} \dot{\psi} + \partial_\psi V(\psi) \right)^2 + \lambda_1 \left( \dot{\Omega}_1 - \Omega^2 \Omega^3 + \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^2 \right) \]
\[ + \lambda_2 \left( \Omega^2 + \Omega^4 \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^1 \right) + \lambda_3 \dot{\Omega}. \]

(71)

(72)

The constrained second-order Lagrange-Poincaré equations determining necessary conditions for the optimal control problem (given in Remark 3) are the following fourth-order nonlinear system of equations:

\[ \frac{I_1 I_2}{I_1 + I_2} \dot{\psi}^{(iv)} = - \frac{d^2}{d\psi^2} \left( \frac{\partial V}{\partial \psi} \right) - \frac{(I_1 + I_2)}{I_2} \frac{\partial^2 V}{\partial \psi^2} \left( \frac{I_1 I_2}{I_1 + I_2} \dot{\psi} + \partial_\psi V(\psi) \right), \]

\[ \lambda_1 = \frac{\dot{\Omega}_1 - \frac{I_2 \dot{\psi}}{I_1 + I_2}}{I_1 + I_2} + \frac{\lambda_1 I_2}{I_1 + I_2} \left( 2 \dot{\psi} \Omega^3 - (\Omega^3)^2 - \frac{\dot{\psi} I_2}{I_1 + I_2} \right), \]

\[ \lambda_2 = \frac{\dot{\psi} I_2}{I_1 + I_2} - \frac{\dot{\Omega}_3}{I_1 + I_2} + \frac{\lambda_1 I_2 (1 - \lambda_2)}{I_1 + I_2} - \lambda_2 \left( \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^1 \right), \]

\[ \lambda_3 = \Omega^2 (\lambda_2 - 2 \lambda_1) - \lambda_1 \dot{\Omega}^2 + \lambda_1 \Omega^3 + \lambda_2 \Omega^4 + (\lambda_1 \Omega^1^2 \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^2), \]

\[ \dot{\Omega}_1 = \Omega^2 \Omega^3 - \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^2, \]

\[ \dot{\Omega}_2 = - \Omega^1 \Omega^3 + \frac{I_2}{I_1 + I_2} \dot{\psi} \Omega^1, \]

\[ \dot{\Omega}_3 = 0. \]

(73a)

(73b)

(73c)

(73d)

(73e)

(73f)

(73g)

In terms of the discretization of this system as presented in Section 3.2, we need to define the discrete connection (22) \( A_d : (SE(2) \times S^1) \times (SE(2) \times S^1) \rightarrow SE(2) \), which should satisfy \( A_d((g_n, \psi_n), (g_{n+1}, \psi_{n+1})) = g_{n+1} A(\psi_n, \psi_{n+1}) g_n^{-1} \) according to (24) for \( g_n, g_{n+1} \in SE(2) \) and \( \psi_n, \psi_{n+1} \in U \subset (SE(2) \times S^1)/SE(2) \cong S^1 \). The local expression of the discrete connection is given by

\[ A(\psi_n, \psi_{n+1}) = \begin{bmatrix} \cos \left( \frac{I_2}{I_1 + I_2} \Delta \psi_n \right) & -\sin \left( \frac{I_2}{I_1 + I_2} \Delta \psi_n \right) & 0 \\ \sin \left( \frac{I_2}{I_1 + I_2} \Delta \psi_n \right) & \cos \left( \frac{I_2}{I_1 + I_2} \Delta \psi_n \right) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

(74)
with $\Delta \psi_n = \psi_{n+1} - \psi_n$. We denote $A_n = A(\psi_n, \psi_{n+1})$. The reduced discrete Lagrangian $L_d : \mathbb{S}^1 \times \mathbb{S}^1 \times SE(2) \to \mathbb{R}$ is locally defined by the coordinates $(\psi_n, \psi_{n+1}, \bar{g}_n)$, where $\bar{g}_n = W_n A_n$, with $W_n = g_n^{-1} g_{n+1}$. According to (74)

$$
\bar{g}_n = \begin{bmatrix}
\cos \Delta \varphi_n & -\sin \Delta \varphi_n & \cos \theta_n \Delta x_n + \sin \theta_n \Delta y_n \\
\sin \Delta \varphi_n & \cos \Delta \varphi_n & -\sin \theta_n \Delta x_n + \cos \theta_n \Delta y_n \\
0 & 0 & 1
\end{bmatrix},
$$

where $\Delta \varphi_n = \Delta \theta_n + \frac{I_2}{I_1+I_2} \Delta \psi_n$, $\Delta \theta_n = \theta_{n+1} - \theta_n$, $\Delta x_n = x_{n+1} - x_n$ and $\Delta y_n = y_{n+1} - y_n$. Establishing

$$
\Omega_1^n = \cos \theta_n (\Delta x_n / h) + \sin \theta_n (\Delta y_n / h),
\Omega_2^n = -\sin \theta_n (\Delta x_n / h) + \cos \theta_n (\Delta y_n / h),
\Omega_3^n = - (\Delta \theta_n / h)
$$

which represent a discretization of (68) where $h$ is the time-step of the integrator, we obtain that

$$
\bar{g}_n = \begin{bmatrix}
\cos \left( h(\Omega_1^n - \frac{I_2}{I_1+I_2} \Delta \psi_n) \right) & \sin \left( h(\Omega_2^n - \frac{I_2}{I_1+I_2} \Delta \psi_n) \right) & h \Omega_1^n \\
-\sin \left( h(\Omega_2^n - \frac{I_2}{I_1+I_2} \Delta \psi_n) \right) & \cos \left( h(\Omega_2^n - \frac{I_2}{I_1+I_2} \Delta \psi_n) \right) & h \Omega_2^n \\
0 & 0 & 1
\end{bmatrix},
$$

and it follows that $\bar{g}_n$ is completely determined by $(\Omega_1^n, \Omega_2^n, \Omega_3^n)$ after fixing $\Delta \psi_n$. Therefore, the discrete Lagrangian $L_d : \mathbb{S}^1 \times \mathbb{S}^1 \times SE(2) \to \mathbb{R}$ is given by

$$
L_d(a_n) = \frac{1}{2} m h (\dot{\psi}_n^2 + (\dot{\theta}_n)^2) + \frac{1}{2} (I_1 + I_2) h (\Omega_3^n)^2 + \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} (\Delta \psi_n)^2 - h V(\psi_{n+1}),
$$

where $a_n = (\psi_n, \psi_{n+1}, \Omega_1^n, \Omega_2^n, \Omega_3^n)$.

Considering a discretization of the action integral $\int_0^h L \, dt$ determined by (67) with local truncation error of first order, the discrete equations of motion (39) read

$$
\Omega_1^n = \Omega_1^{n-1} + h \Omega_2^{n-1} \left( \Omega_3^{n-1} - \frac{I_2}{I_1+I_2} \Delta \psi_{n-1} \right),
\Omega_2^n = \Omega_2^{n-1} - h \Omega_1^{n-1} \left( \Omega_3^{n-1} - \frac{I_2}{I_1+I_2} \Delta \psi_{n-1} \right),
\Omega_3^n = \Omega_3^{n-1},
$$

$$
\frac{I_1 I_2}{I_1 + I_2} \Delta \psi_{n+1} - 2 \psi_n + \psi_{n-1} = -h \partial_\psi V(\psi_n),
$$

for the discrete Lagrangian (76), where we have used the expressions for the local discrete connection (74), $\bar{g}_n$ (75), and we neglected higher-order terms of the time step $O(h^2)$ (equations (77a),(77b) and (77c)) follow from the second equation in (39), while (77d) follows from the first one. It is easy to check that (77) is a discretization in finite differences of (69), (see footnote 1).

**Remark 11.** A different discretization of the potential $V$ in (76), for instance

$$
h V \left( \frac{\psi_{n+1} + \psi_n}{2} \right) \text{ or } \frac{h}{2} V(\psi_{n+1}) + \frac{h}{2} V(\psi_n),
$$

would lead to a second-order discretization of (77d) with respect to (69d). However, the local truncation error of (77) with respect to (69) does not change, since the
order of the $\Omega$ equations remains the same. It seems strange to use asymmetric, $O(h)$ approximations when this is not necessary as we used in the previous example. This is a phenomenon due to the “decoupling” of the variables $\psi$ and $\Omega$ in equation (77d), which allows to enhance its local truncation error via the appropriate discretization of the discrete Lagrangian. However, the overall $L_d$ is $O(h)$, and therefore one expects the same order of the integrator.

The discrete second-order augmented Lagrangian $\tilde{L}_d : S^1 \times S^1 \times S^1 \times 2\tilde{SE}(2) \times \mathbb{R}^3 \to \mathbb{R}$ for the constrained higher-order variational problem is given by

$$
\tilde{L}_d(\tilde{a}_n, \lambda^n) = \frac{h}{2} \left( \frac{I_1 I_2}{I_1 + I_2} \frac{\psi_{n+2} - 2\psi_{n+1} + \psi_n}{h^2} + \partial_\psi V(\psi_{n+1}) \right)^2 
+ \lambda^n_1 \Delta \Omega^n_3 + \lambda^n_1 \left( \Delta \Omega^n_1 - h\Omega^n_2 \Omega^n_3 + \frac{I_2}{I_1 + I_2} \Delta \psi_n \Omega^n_2 \right) 
+ \lambda^n_2 \left( \Delta \Omega^n_2 + h\Omega^n_1 \Omega^n_3 - \frac{I_2}{I_1 + I_2} \Delta \psi_n \Omega^n_1 \right),
$$

(78)

with $\tilde{a}_n = (\psi_n, \psi_{n+1}, \psi_{n+2}, \Omega^n, \Omega^{n+1})$, $\Omega^n = (\Omega^n_1, \Omega^n_2, \Omega^n_3)$, $\lambda^n = (\lambda^n_1, \lambda^n_2, \lambda^n_3)$, and with $\Delta \Omega^n_i = \Omega^{n+1}_i - \Omega^n_i$. Here, to define $\tilde{L}_d$ we used

$$
\chi^1_1(\tilde{a}_n) = \Delta \Omega^n_1 - h\Omega^n_2 \Omega^n_3 + \frac{I_2}{I_1 + I_2} \Delta \psi_n \Omega^n_2,
$$

(80)

$$
\chi^2_2(\tilde{a}_n) = \Delta \Omega^n_2 + h\Omega^n_1 \Omega^n_3 - \frac{I_2}{I_1 + I_2} \Delta \psi_n \Omega^n_1,
$$

(81)

$$
\chi^3_3(\tilde{a}_n) = \Delta \Omega^n_3
$$

(82)

The discrete constrained second-order Lagrange-Poincare equations giving rise to the variational integrator to approximate the necessary conditions for optimality in the optimal control problem are given by equations (59), (60), (61) applied to the discrete second-order augmented Lagrangian $\tilde{L}_d$ (78) where the partial derivatives of $\tilde{L}_d$ and $\chi^\alpha_d$, $\alpha = 1, 2, 3$ follows easily from equations (79)-(82) and are understood as row vectors. The operators $T^*\tilde{L}_{(W.A1)}$ and $T^*\tilde{L}_{(W.A2)}$ can be computed using the tangent lift of left translations as in equation (40), $M_n = [0, 0, 0]$, and the quantities $e^\alpha_{(n, 4)}$, $e^\alpha_{(n, 5)}$, $e^\alpha_{(n-1, 4)}$, $e^\alpha_{(n-1, 5)}$ are given as follow

$$
e^1_{(n, 4)} = [-\cos(h\vartheta_n) + h\vartheta_n \sin(h\vartheta_n), \sin(h\vartheta_n) + h\vartheta_n \cos(h\vartheta_n), h\Omega^n_2 \cos(h\vartheta_n) + h\Omega^n_2 \sin(h\vartheta_n) - h\Omega^n_2 \cos(h\vartheta_n) - h\Omega^n_2 \sin(h\vartheta_n)],$$

$$
e^2_{(n, 4)} = [-\sin(h\vartheta_n) + h\vartheta_n \cos(h\vartheta_n), -\cos(h\vartheta_n) + h\vartheta_n \sin(h\vartheta_n), h\Omega^n_2 \sin(h\vartheta_n) + h\Omega^n_2 \cos(h\vartheta_n) + h\Omega^n_2 \sin(h\vartheta_n) - h\Omega^n_2 \cos(h\vartheta_n) + h\Omega^n_2 \sin(h\vartheta_n)],$$

$$
e^3_{(n, 4)} = e^2_{(n-1, 4)} = [0, 0, 1],$$

$$
e^1_{(n, 5)} = [\cos(h\vartheta_n), -\sin(h\vartheta_n), -h\Omega^n_1 \cos(h\vartheta_n) + h\Omega^n_1 \sin(h\vartheta_n)],$$

$$
e^2_{(n, 5)} = [\sin(h\vartheta_n), \cos(h\vartheta_n), -h\Omega^n_1 \sin(h\vartheta_n) - h\Omega^n_2 \cos(h\vartheta_n)],$$
where we have used that, \( \dot{\vartheta}_n = \Omega_0^n - \frac{I_1 + I_2}{2} \Delta \varphi_n \),
\[
\dot{y}_n^{-1} = A_n^{-1} W_n^{-1} = \begin{bmatrix} R_{-h\vartheta_n} & -R_{-h\vartheta_n} h \varphi_n \\ 0 & 1 \end{bmatrix},
\]
\[
v_n = [\Omega_1^n, \Omega_2^n]^T \quad \text{and} \quad R_{h\vartheta_n} = \begin{bmatrix} \cos(h\vartheta_n) & \sin(h\vartheta_n) \\ -\sin(h\vartheta_n) & \cos(h\vartheta_n) \end{bmatrix}.
\]

Equations (59)-(61) are used to update the current state \((\tilde{a}_{n-1}, \tilde{a}_n, \lambda_{n-2}, \lambda_{n-1})\) to obtain the next state \((\tilde{a}_{n}, \tilde{a}_{n-1}, \lambda_{n}, \lambda_{n-1})\). This is accomplished by solving the dynamics (59)-(61) with boundary conditions satisfying the constraints (61) using a root-finding algorithm such as Newton’s method in terms of the unknowns \((\tilde{a}_n, \lambda_{n-1})\) to obtain the next configuration. Note that as in the example in Section 4.1.1, the discrete constrained second-order Lagrange-Poincaré equations (59)- (61) applied to (78) represent a discretization in finite differences of (73) (a)-(e).

It would be interesting to study how indirect optimization methods for optimal control can be developed to implement the equations of motion. One of the main challenges here is the use of a shooting method to solve the two points boundary value problem. We believe that, due to the complexity of the equations, one should use multiple shooting instead of a single one, and/or add a final state to our cost functional in order to achieve the desired final configurations. In terms of the integration schemes for Lagrange-Poincaré equations, it would be interesting if the variational integrators are employed on particular examples. For instance, in the case of an electron in the magnetic field, the potential function can be used to partially break the symmetry. Then, the extension of the variational integrators presented in this work for symmetry-breaking Lagrange-Poincaré systems can be studied independently and applied to a concrete example of interest in physics.

Note also that a particular construction and study for the exact discrete Lagrangian associated to higher-order systems on principal bundles deserve attention and can be an interesting topic to study, based on the previous results obtained in [21].

We intend in a future work to explore the role of high-order integrators [16], [17], [54] in this class of constrained variational problems for optimal control. As we commented in Section 3.1, higher-order interpolations of the continuous curves lead to more accurate approximations of the exact discrete Lagrangian, and therefore to high-order numerical methods (where here, high-order refers to the local truncation error). This problem, in the context of principal bundles and integration of Lagrange-Poincaré equations, is a promising line of investigation, in particular how to relate higher-order constrained variational problems on principal bundles with higher-order integrators, such as Galerkin variational integrators and modified symplectic Runge-Kutta methods, using the results for first-order systems given in [17] and [55].

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