COMPLEX SYMMETRIC PARTIAL ISOMETRIES

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Abstract. An operator $T \in B(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$. We provide a concrete description of all complex symmetric partial isometries. In particular, we prove that any partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

1. Introduction

The aim of this note is to complete the classification of complex symmetric partial isometries which was started in [10]. In particular, we give a concrete necessary and sufficient condition for a partial isometry to be a complex symmetric operator.

Before proceeding any further, let us first recall a few definitions. In the following, $\mathcal{H}$ denotes a separable, complex Hilbert space and $B(\mathcal{H})$ denotes the collection of all bounded linear operators on $\mathcal{H}$.

Definition. A conjugation is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both involutive (i.e., $C^2 = I$) and isometric (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$).

Definition. We say that $T \in B(\mathcal{H})$ is $C$-symmetric if $T = CT^*C$. We say that $T$ is complex symmetric if there exists a conjugation $C$ with respect to which $T$ is $C$-symmetric.

It is straightforward to show that if $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator. For instance, the unilateral shift is perhaps the most ubiquitous example of a partial isometry which is not complex symmetric (see [7, Prop. 1], [9, Ex. 2.14], [6, Cor. 7]). On the other hand, we have the following theorem from [10]:

Theorem 1. Let $T \in B(\mathcal{H})$ be a partial isometry.

(i) If $\dim \ker T = \dim \ker T^* = 1$, then $T$ is a complex symmetric operator,

(ii) If $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator.

(iii) If $2 \leq \dim \ker T = \dim \ker T^* \leq \infty$, then either possibility can (and does) occur.

Although these results are the sharpest possible statements that can be made given only the data $(\dim \ker T, \dim \ker T^*)$, they are in some sense unsatisfactory. For instance, it is known that partial isometries on $\mathcal{H}$ that are not complex symmetric exist if $\dim \mathcal{H} \geq 5$ and that every partial isometry on $\mathcal{H}$ is complex symmetric if $\dim \mathcal{H} \leq 3$, the authors were unable to answer the corresponding question if...
$\dim \mathcal{H} = 4$. To be more specific, the techniques used in $[10]$ were insufficient to resolve the question in the case where $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$. Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has recently been produced by J. Tener $[13]$.

Suppose that $T$ is a partial isometry on $\mathcal{H}$ and let

$$\mathcal{H}_1 = (\ker T)^\perp = \text{ran} T^*$$

(1)
denote the initial space of $T$ and $\mathcal{H}_2 = (\mathcal{H}_1)^\perp = \ker T$ denote its orthogonal complement (see $[12]$ Pr. 127 or $[2]$ Ch. VIII, Sect. 3 for terminology). With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

(2)

where $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_1 \to \mathcal{H}_2$. Furthermore, the fact that $T^*T$ is the orthogonal projection onto $\mathcal{H}_1$ yields the identity

$$A^*A + B^*B = I,$$

(3)

where $I$ denotes the identity operator on $\mathcal{H}_1$. Finally, observe that the operator $A \in B(\mathcal{H}_1)$ is simply the compression of the partial isometry $T$ to its initial space.

The main result of this note is the following concrete description of complex symmetric partial isometries:

**Theorem 2.** Let $T \in B(\mathcal{H})$ be a partial isometry. If $A$ denotes the compression of $T$ to its initial space, then $T$ is a complex symmetric operator if and only if $A$ is a complex symmetric operator.

Due to its somewhat lengthy and computational proof, we defer the proof of the preceding theorem until Section 3. We remark that Theorem 2 remains true if one instead considers the final space of $T$. Indeed, simply apply the theorem with $T^*$ in place of $T$ and then take adjoints.

**Corollary 1.** Every partial isometry of rank $\leq 2$ is complex symmetric.

**Proof.** Let $T \in B(\mathcal{H})$ be a partial isometry such that $\text{rank} T \leq 2$. If $\text{rank} T = 0$, then $T = 0$ and there is nothing to prove. If $\text{rank} T = 1$, then this is handled in $[10]$. In the case $\text{rank} T = 2$, we may write

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where $A$ is an operator on a two-dimensional space. Since every operator on a two-dimensional Hilbert space is complex symmetric (see $[1]$ Cor. 3, $[3]$ Cor. 3.3, $[7]$ Ex. 6, $[10]$ Cor. 1, $[12]$ Cor. 3)), the desired conclusion follows from Theorem 2. □

**Corollary 2.** Every partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

**Proof.** As mentioned earlier, the results of $[10]$ indicate that only the case $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$ requires resolution. The corollary is now immediate consequence of Theorem 2 and the fact that every operator on a two-dimensional Hilbert space is complex symmetric. □
We conclude this section with the following theorem, which asserts that each C-symmetric partial isometry can be extended to a C-symmetric unitary operator on the whole space (the significance lies in the fact that the corresponding conjugations for these two operators are the same).

**Theorem 3.** If $T$ is a C-symmetric partial isometry, then there exists a C-symmetric unitary operator $U$ and an orthogonal projection $P$ such that $T = UP$.

**Proof.** Since $T$ is a C-symmetric partial isometry, it follows that $|T| = P$ is an orthogonal projection and that $T = CJP$ where $J$ is a conjugation supported on $\text{ran } P$ which commutes with $P$ \[8\] Sect. 2.2. We may extend $J$ to a conjugation $\tilde{J}$ on all of $\mathcal{H}$ by forming the internal direct sum $J \oplus J'$ where $J'$ is a partial conjugation supported on $\text{ker } P$. The operator $U = CJ$ is a C-symmetric unitary operator. \[\square\]

2. Partial isometries and the norm closure problem

Partial isometries on infinite-dimensional spaces often provide examples of note. For instance, one can give a simple example of a partial isometry $T$ satisfying $\dim \ker T = \dim \ker T^* = \infty$ which is not a complex symmetric operator:

**Example 1.** Let $S$ denote the unilateral shift on $l^2(\mathbb{N})$. Although $S$ is certainly not a complex symmetric operator (by (ii) of Theorem 1, see also \[9\] Ex. 2.14), or \[6\] Cor. 7), part (i) of Theorem 1 does ensure that the partial isometry $S \oplus S^*$ is complex symmetric. Indeed, simply take $N$ to be the bilateral shift on $l^2(\mathbb{Z})$ and note that $S \oplus S^*$ is unitarily equivalent to $N - Ne_0 \otimes e_0$. That $S \oplus S^*$ is complex symmetric can also be verified by a direct computation \[8\] Ex. 5]. On the other hand, the partial isometry $T = S \oplus 0$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ is not a complex symmetric operator by Lemma 1.

Let $S(\mathcal{H})$ denote the subset of $B(\mathcal{H})$ consisting of all bounded complex symmetric operators on $\mathcal{H}$. There are several ways to think about $S(\mathcal{H})$. By definition, we have

$$S(\mathcal{H}) = \{T \in B(\mathcal{H}) : \exists \text{ a conjugation } C \text{ s.t. } T = CT^*C\}.$$

If $C$ is a fixed conjugation on $\mathcal{H}$, then we also have

$$S(\mathcal{H}) = \{UTU^* : T = CT^*C, U \text{ unitary}\}.$$

Thus if we identify $\mathcal{H}$ with $l^2(\mathbb{N})$ and $C$ denotes the canonical conjugation on $l^2(\mathbb{N})$ (i.e., entry-by-entry complex conjugation), we can think of $S(\mathcal{H})$ as being the *unitary orbit* of the set of all bounded (infinite) complex symmetric matrices.

The following example shows that the set $S(\mathcal{H})$ is not closed in the strong operator topology (SOT):

**Example 2.** We maintain the notation of Example 1. For $n \in \mathbb{N}$, let $P_n$ denote the orthogonal projection onto the span of the basis vectors $\{e_i : i \geq n\}$ of $l^2(\mathbb{N})$. Now observe that each operator $T_n = P_nS \oplus S^*$ is unitarily equivalent to $S \oplus 0_n \oplus S^*$ where $0_n$ denotes the zero operator on an $n$-dimensional Hilbert space. Each $T_n$ is complex symmetric since $S \oplus S^*$ is complex symmetric (by Lemma 1). On the other hand, since $P_nS$ is SOT-convergent to 0, it follows that the SOT-limit of the sequence $T_n$ is $0 \oplus S^*$, which is not a complex symmetric operator (by Lemma 1).
The preceding example demonstrates that the set of all complex symmetric operators (on a fixed, infinite-dimensional Hilbert space $\mathcal{H}$) is not SOT-closed. We also remark that the conjugations corresponding to the operators $T_n$ from Example 2 depend on $n$. In contrast, if we fix a conjugation $C$, then it is elementary to see that the set of $C$-symmetric operators is a SOT-closed subspace of $B(\mathcal{H})$.

We conclude with a related question, which we have been unable to resolve:

**Question.** Is $S(\mathcal{H})$ norm closed?

3. **Proof of Theorem 2**

This entire section is devoted to the proof of Theorem 2. We first require the following lemma:

**Lemma 1.** If $\mathcal{H}, \mathcal{K}$ are separable complex Hilbert spaces, then $T \in B(\mathcal{H})$ is a complex symmetric operator if and only if $T \oplus 0 \in B(\mathcal{H} \oplus \mathcal{K})$ is a complex symmetric operator.

**Proof.** If $T$ is a $C$-symmetric operator on $\mathcal{H}$, then it is easily verified that $T \oplus 0$ is $(C \oplus J)$-symmetric on $\mathcal{H} \oplus \mathcal{K}$ for any conjugation $J$ on $\mathcal{K}$. The other direction is slightly more difficult to prove.

Suppose that $S = T \oplus 0$ is a complex symmetric operator on $\mathcal{H} \oplus \mathcal{K}$. Before proceeding any further, let us remark that it suffices to consider the case where

$$\mathcal{H} = \text{ran}T + \text{ran}T^*.$$  \hspace{1cm} (4)

Otherwise let $\mathcal{H}_1 = \text{ran}T + \text{ran}T^*$ and note that $\mathcal{H}_1$ is a reducing subspace of $\mathcal{H}$. If $\mathcal{H}_2$ denotes the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$, then with respect to the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K}$, the operator $S$ has the form $T' \oplus 0 \oplus 0$, where $T'$ denotes the restriction of $T'$ to $\mathcal{H}_1$. By now considering $S$ with respect to the orthogonal decomposition $\mathcal{H} \oplus \mathcal{K} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{K})$, it follows that we need only consider the case where (4) holds.

Suppose now that (4) holds and that $S$ is $C$-symmetric where $C$ denotes a conjugation on $\mathcal{H} \oplus \mathcal{K}$. Writing the equations $CS = S^*C$ and $CS^* = SC$ in terms of the $2 \times 2$ block matrices

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$  \hspace{1cm} (5)

(the entries $C_{ij}$ of $C$ are conjugate-linear operators), we find that

$$C_{11}T = T^*C_{11},$$  \hspace{1cm} (6)

$$C_{21}T = C_{21}T^* = 0,$$  \hspace{1cm} (7)

$$T^*C_{12} = TC_{12} = 0.$$  \hspace{1cm} (8)

Since $C_{21}T = C_{21}T^* = 0$, it follows that $C_{21}$ vanishes on $\text{ran}T + \text{ran}T^*$ and hence on $\mathcal{H}$ itself by (4). On the other hand, (8) implies that $C_{12}$ vanishes on the orthogonal complements of $\ker T$ and $\ker T^*$ in $\mathcal{H}$. By (4), this implies that $C_{12}$ vanishes identically.

It follows immediately from (5) that $C_{11}$ and $C_{22}$ must be conjugations on $\mathcal{H}$ and $\mathcal{K}$, respectively, whence $T$ is $C_{11}$-symmetric by (6). This concludes the proof of the lemma. \hfill $\square$
Now let us suppose that $T$ is a partial isometry on $\mathcal{H}$ and let
\[ \mathcal{H}_1 = (\ker T)^\perp = \text{ran } T^*. \]
and $\mathcal{H}_2 = \ker T$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, it follows that
\[ T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \]
where $A : \mathcal{H}_1 \to \mathcal{H}_1$, $B : \mathcal{H}_1 \to \mathcal{H}_2$, and
\[ A^*A + B^*B = I. \quad (9) \]

($\Rightarrow$) Suppose that $T$ is a complex symmetric operator. For an operator with polar decomposition $T = U|T|$ (i.e., $U$ is the unique partial isometry satisfying $\ker U = \ker T$ and $|T|$ denotes the positive operator $\sqrt{T^*T}$), the Aluthge transform of $T$ is defined to be the operator $\tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}$. Noting that $T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, we find that
\[ \tilde{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \]

By $[5]$ Thm. 1, we know that the Aluthge transform of a complex symmetric operator is complex symmetric. Applying Lemma 1 to $\tilde{T}$, we conclude that $A$ is complex symmetric, as desired.

($\Leftarrow$) Let us now consider the more difficult implication of Theorem 2, namely that if $A$ is a complex symmetric operator, then $T$ is as well. We claim that it suffices to consider the case where $\text{ran } B = \mathcal{H}_2$. In other words, we argue that if $K = \text{ran } T + \text{ran } T^*$, then we may suppose that $K = \mathcal{H}$. Indeed, $K$ is a reducing subspace for $T$ and $T = 0$ on $K^\perp$. By Lemma 1 if $T|_K$ is a complex symmetric operator, then so is $T$.

Write $B = V|B|$ where $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a partial isometry with initial space $(\ker B)^\perp \subseteq \mathcal{H}_1$ and final space $\mathcal{H}_2$ (since $\text{ran } B = \mathcal{H}_2$). In particular, we have the relations
\[ V^*B = |B| = B^*V, \quad |B| = \sqrt{I - A^*A}. \quad (10) \]
By hypothesis, the operator $A \in B(\mathcal{H}_1)$ is complex symmetric. Therefore suppose that $K$ is a conjugation on $\mathcal{H}_1$ such that $KA = A^*K$ and observe that the equations
\[ A\sqrt{I - A^*A} = \sqrt{I - AA^*A}, \]
\[ A^*\sqrt{I - AA^*} = \sqrt{I - A^*AA^*}, \]
\[ K\sqrt{I - A^*A} = \sqrt{I - AA^*K}, \]
\[ K\sqrt{I - AA^*} = \sqrt{I - A^*AK}, \]
follow from a standard polynomial approximation argument (i.e., if $p(x) \in \mathbb{R}[x]$, then $Ap(A^*A) = p(AA^*)A$ and $Kp(A^*A) = p(AA^*)K$ hold whence the desired identities follow upon passage to the strong operator limit). In particular, it follows from the preceding that
\[ (KA)\sqrt{I - A^*A} = \sqrt{I - A^*A(KA)}, \]
that is
\[
KA|B| = |B|KA, \quad A^*K|B| = |B|A^*K. \tag{11}
\]

Let us now define a conjugate-linear operator \(C\) on \(H\) by the formula
\[
C = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}.
\tag{12}
\]

Assuming for the moment that \(C\) is a conjugation on \(H\), we observe that
\[
\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}^T = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & |T| \end{pmatrix}.
\]

Since it is clear that \(J\) is a partial conjugation which is supported on the range of \(|T|\) and which commutes with \(|T|\), it follows immediately that \(T\) is a \(C\)-symmetric operator (see [8, Thm. 2]).

To complete the proof of Theorem 2, we must therefore show that \(C\) is a conjugation on \(H\). In other words, we must check that \(C^2\) is the identity operator on \(H\) and that \(C\) is isometric. Since these computations are somewhat lengthy, we perform them separately:

**Claim:** \(C^2 = I\).

**Pf. of Claim.** We first expand out \(C^2\) as a \(2 \times 2\) block matrix:
\[
C^2 = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}
= \begin{pmatrix} AKAK + KB^*BK & AKKB^* - KB^*VA^*KV^* \\ BKAK - VA^*KV^*BK & BKKB^* + VA^*KV^*VA^*KV^* \end{pmatrix}
= \begin{pmatrix} AA^* + KB^*BK & AB^* - KB^*VA^*KV^* \\ BA^* - VA^*KV^*BK & BB^* + VA^*KV^*VA^*KV^* \end{pmatrix}.
\]

To obtain the preceding line, we used the fact that \(K\) is a conjugation and \(A\) is \(K\)-symmetric. Letting \(E_{ij}\) denote the entries of the preceding block matrix we find that
\[
E_{11} = AA^* + KB^*BK
= AA^* + K(I - A^*A)K
= AA^* + (I - AA^*)
= I.
\]
\[
E_{12} = AB^* - KB^*VA^*KV^*
= AB^* - K|B|A^*KV^* \tag{11}
= AB^* - KA^*K|B|V^* \tag{11}
= AB^* - A|B|V^*
= AB^* - AB^* \quad \text{since } B^* = |B|V
= 0.
\]
\[
E_{21} = BA^* - VA^*KV^*BK
\]
\[ BA^* - VA^*K|B|K \quad \text{since } V^*B = |B| \]
\[ BA^* - V|B|A^*KK \quad \text{by } (11) \]
\[ BA^* - VA^* \quad \text{since } V = V|B| \]
\[ = 0. \]

As for \( E_{22} \), it suffices to show that \( E_{22} \) agrees with \( I \) (the identity operator on \( \mathcal{H}_2 \)) on the range of \( B \), which is dense in \( \mathcal{H}_2 \). In other words, we wish to show that
\[ E_{22}Bx = Bx \quad \text{for all } x \in \mathcal{H}_2, \]
which is equivalent to showing that
\[ E_{22}Bx = BB^*Bx + VA^*KV^*VA^*KV^*Bx = Bx \quad \text{(13)} \]
for all \( x \in \mathcal{H}_2 \). Let us investigate the second term of (13):
\[ VA^*KV^*VA^*KV^*Bx = VA^*KV^*VA^*K|B|x \quad \text{by } (10) \]
\[ = VA^*KV^*|B|A^*Kx \quad \text{by } (11) \]
\[ = VA^*K|B|A^*Kx \quad \text{since } V^*V = P_{\text{ran}}|B| \]
\[ = V|B|A^*KA^*Kx \quad \text{by } (11) \]
\[ = BA^*KA^*Kx \quad \text{since } B = V|B| \]
\[ = BA^*Ax \]
\[ = B(I - B^*B)x \quad \text{since } A^*A + B^*B = I \]
\[ = Bx - BB^*Bx. \]

Putting this together with (13), we find that \( E_{22}Bx = Bx \) for all \( x \in \mathcal{H}_2 \) whence \( E_{22} = I \), as claimed. \( \square \)

**Claim:** \( C \) is isometric.

**Pf. of Claim.** The proof requires three steps:

(i) Show that \( C \) is isometric on \( \mathcal{H}_1 \),

(ii) Show that \( C \) is isometric on \( B\mathcal{H}_1 \), which is dense in \( \mathcal{H}_2 \),

(iii) Show that \( C\mathcal{H}_1 \perp C(B\mathcal{H}_1) \).

For the first portion, observe that
\[
\left\| C\begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} AK \\ BK \end{pmatrix} - VA^*KV^* \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\
= \left\| \begin{pmatrix} AKx \\ BKx \end{pmatrix} \right\|^2 \\
= (AKx, AKx) + (BKx, BKx) \\
= (A^*AKx, Kx) + (B^*BKx, Kx) \\
= ((A^*A + B^*B)Kx, Kx) \\
= (Kx, Kx) \\
= \|Kx\|^2 \\
= \|x\|^2.
\]

Thus (i) holds.
Now for (ii):

\[
\left\| C \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 \\
= \left\| \begin{pmatrix} KB^*Bx \\ -VA^*KV^*Bx \end{pmatrix} \right\|^2 \\
= \|KB^*Bx\|^2 + \|VA^*KV^*Bx\|^2 \\
= \|B^*Bx\|^2 + \|VA^*K[B]x\|^2 \\
= \|B^*Bx\|^2 + \|VA^*Kx\|^2 \\
= \|B^*Bx\|^2 + \|KA^*Kx\|^2 \\
= \langle (I - A^*A)x, A^*Kx \rangle \\
= \langle x, (I - A^*A)x \rangle - \langle A^*Ax, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
= \langle x, (I - A^*A)x \rangle \\
= \langle x, B^*Bx \rangle \\
= \langle Bx, Bx \rangle \\
= \|Bx\|^2.
\]

Thus (ii) holds.

Now for (iii):

\[
\langle C \begin{pmatrix} x \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ By \end{pmatrix} \rangle = \langle \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ By \end{pmatrix} \rangle \\
= \langle \begin{pmatrix} AKx \\ BKx \end{pmatrix}, \begin{pmatrix} KB^*By \\ -VA^*KV^*By \end{pmatrix} \rangle \\
= \langle AKx, KB^*By \rangle - \langle BKx, VA^*KV^*By \rangle \\
= \langle B^*By, KAx \rangle - \langle BKx, VA^*KBy \rangle \\
= \langle B^*By, A^*x \rangle - \langle BKx, V|B|A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle BKx, BA^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle B^*BKx, A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle K(I - AA^*)x, A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle KA^*Ky, (I - AA^*)x \rangle \\
= \langle AB^*By, x \rangle - \langle Ay, (I - AA^*)x \rangle 
\]
\[= \langle AB^*By, x \rangle - \langle (I - AA^*)Ay, x \rangle = \langle AB^*By, x \rangle - \langle A(I - A^*A)y, x \rangle = \langle AB^*By, x \rangle - \langle AB^*By, x \rangle = 0. \]

By the polarization identity, it follows that
\[\left\langle C \begin{pmatrix} x_1 \\ Bx_2 \end{pmatrix}, C \begin{pmatrix} y_1 \\ By_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_2 \\ By_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ By_1 \end{pmatrix} \right\rangle\]
holds for all \(x_1, x_2, y_1, y_2 \in \mathcal{H}_1\) whence \(C\) is isometric on \(\mathcal{H}\). \(\square\)

**References**

[1] Balayan, L., Garcia, S.R., *Unitary equivalence to a complex symmetric matrix: geometric criteria*, (preprint).
[2] Conway, J.B., *A Course in Functional Analysis* (second edition), Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990.
[3] Chevrot, N., Fricain, E., Timotin, D., *The characteristic function of a complex symmetric contraction*, Proc. Amer. Math. Soc. 135 (2007), 2877–2886. MR2317964 (2008c:47025)
[4] Garcia, S.R., *Approximate antilinear eigenvalue problems and related inequalities*, Proc. Amer. Math. Soc. 136 (2008), no. 1, 171–179. MR2350402
[5] Garcia, S.R., *Aluthge transforms of complex symmetric operators*, Integral Equations Operator Theory 60 (2008), no. 3, 357–367. MR2392831
[6] Garcia, S.R., *Means of unitaries, conjugations, and the Friedrichs operator*, J. Math. Anal. Appl. 335 (2007), 941–947. MR2445511 (2008i:47070)
[7] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. 358 (2006), 1285–1315. MR2187654 (2006j:47036)
[8] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. 359 (2007), 3913–3931. MR2302518 (2008b:47005)
[9] Garcia, S.R., *Conjugation and Clark Operators*, Contemp. Math. 393 (2006), 67-112. MR2198373 (2007b:47073)
[10] Garcia, S.R., Wogen, W.R., *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. (to appear).
[11] Gilbreath, T.M., Wogen, W.R., *Remarks on the structure of complex symmetric operators*, Integral Equations Operator Theory 59 (2007), no. 4, 585–590. MR2370050
[12] Halmos, P.R., *A Hilbert Space Problem Book* (Second Edition), Springer-Verlag, New York, 1982.
[13] Tener, J.E., *Unitary equivalence to a complex symmetric matrix: an algorithm*, J. Math. Anal. Appl. 341 (2008), no. 1, 640–648. MR2394112 (2008m:15062)
[14] Sarason, D., *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices 1 (2007), no. 4, 491–526. MR2363975 (2008i:47060)

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