A new class of exactly-solvable potentials by means of the hypergeometric equation

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ABSTRACT

We obtained a new class of exactly-solvable potentials by means of the hypergeometric equation for Schrödinger equation, which different from the exactly-solvable potentials introduced by Bose and Natanzon. Using the new class of solvable potentials, we can obtain the corresponding complex PT-invariant potentials. This method can also apply to the other Fuchs equations.
1 Introduction

The exact solutions to the Schrödinger equation play crucial roles in quantum physics. It is well-known that there are several potentials that can be exactly solved, as examples one can site the harmonic oscillator, the Coulomb, the Morse [1], Pöschl-Teller [2], Eckart [3] potentials and so on. The reason why these potentials are exact solvable is that the Schrödinger equation in these potentials can be transformed by into either the hypergeometric or the confluent hypergeometric equation. Here we focus on hypergeometric equation to construct a new class of the solvable potentials within the framework of the non-relativistic Schrödinger equation. Similar works have been carried out [4 [5]]. Using the new class of solvable potentials that we have constructed, we can easily obtain complex PT-invariant potentials, these potentials is a hot issue of recent study [8].

This work is organized as follows. In Section 2, We present the basic method of our argument. In Section 3 as an illustration we take the hypergeometric equation to construct the soluble potentials. In Section 4 we will give some conclusions for this paper.

2 Basic methods

As we know, the stationary Schrödinger equation for a particle of energy $E$ in a potential $V(r)$ has the form

$$\psi''(r) + (E - V(r))\psi(r) = 0.$$  \hspace{1cm} (1)

Through choosing a transformation of the variable $z = z(r)$, the equation become

$$\psi_{zz} + \frac{\rho_z}{\rho} \psi_z + \frac{E - V}{\rho^2} \psi = 0.$$  \hspace{1cm} (2)

Where $\rho = \frac{dz}{dr}$, and apply the transformation $\psi = f(z)u(z)$. we can obtain the following differential equation

$$u_{zz} + \left(2\frac{f_z}{f} + \frac{\rho_z}{\rho}\right)u_z + \left(\frac{f_{zz}}{f} + \frac{f_z \rho_z}{f \rho} + \frac{E - V}{\rho^2}\right)u = 0.$$  \hspace{1cm} (3)

Compared with our target equation

$$u_{zz} + g(z)u_z + h(z)u = 0.$$  \hspace{1cm} (4)

we have

$$g(z) = 2\frac{f_z}{f} + \frac{\rho_z}{\rho}$$  \hspace{1cm} (5)

$$h(z) = \frac{f_{zz}}{f} + \frac{f_z \rho_z}{f \rho} + \frac{E - V}{\rho^2}.$$  \hspace{1cm} (6)
Integrating the equation (5) allows us to obtain

\[ f(z) = \sqrt{c_1} \rho^{-1/2} e^{\int g(z)dz/2}. \]  

(7)

Substitution of this equation into (6) yields

\[ E - V = \rho^2 \left( h - \frac{g_z}{2} - \frac{g^2}{4} \right) + \frac{1}{2} \{ z, r \}. \]  

(8)

Where the Schwarzian derivative given as

\[ \{ z, r \} = \frac{z'''(r)}{z'(r)} - \frac{3}{2} \left( \frac{z''(r)}{z'(r)} \right)^2 = \rho \rho_{zz} - \frac{1}{2} \rho_z^2. \]  

(9)

Therefore, on the basis of the solvable differential equation (4), we are able to construction of the solvable potentials for the original Schrödinger equation. This problem will become how to find that new variable \( z(r) \) and function \( f(z) \) satisfy equation (5), (6).

3 Hypergeometric equation

In this section, we application the methods to hypergeometric equation

\[ z(1 - z)u''(z) + \left[ \gamma - (\alpha + \beta + 1)z \right] u'(z) - \alpha \beta u(z) = 0. \]  

(10)

Obviously, in this case \( g(z) = \frac{\gamma - (\alpha + \beta + 1)z}{z(1 - z)} \) and \( h(z) = -\frac{\alpha \beta}{z(1 - z)}. \)

In order to determine the form of \( f \), we follow the Ishkhanyan et al [6] who discussed the reduction of the Schrödinger equation to a rather large class of target equations by a transformation of \( \rho(z) \), and \( \rho(z) \) is a polynomial, which all the roots should necessarily coincide with the singular points of the target equation to which the Schrödinger equation is reduced. Then, the coordinate transformation of \( f \) need the form \( f = z^p(1 - z)^q \), so the equation (5) have solution

\[ \rho = c_1 z^{\gamma - 2p}(1 - z)^{\alpha + \beta + 1 - \gamma - 2q}. \]  

(11)

Substitution of this equation (11) and \( h(z) = -\frac{\alpha \beta}{z(1 - z)} \) into (6), we can easily figure out that the Schrödinger equation have solvable potentials only if \( a = \gamma - 2p \) and \( b = \alpha + \beta + 1 - \gamma - 2q \) satisfy some of these discrete values \( a = 0, 1, 2, b = 0, 1, 2 \), and in these cases the functional form of \( z(r) \) and \( V(r) \) are likely to be simple.

**Case:1** consider \( a = 0, b = 1 \), so \( \rho = \frac{dz}{dr} = c_1(1 - z) \), therefore

\[ z = 1 + c_2 e^{-c_1 r}. \]  

(12)
From equation (8), we can get

\[ E - V = C - A \frac{1}{(c_2 + e^{c_1 r})^2} - B \frac{1}{c_2 + e^{c_1 r}}. \]  

(13)

Where the coefficients are given

\[ A = \frac{c_1^2}{4} \gamma (\gamma - 2) \]
\[ B = \frac{c_1^2}{2} (\alpha \gamma + \beta \gamma + \gamma - \gamma^2 - 2\alpha \beta) \]
\[ C = -\frac{c_1^2}{4} (\alpha + \beta - \gamma)^2. \]  

(14)

**Case: 2** consider \( a = 1/2, b = 1/2, \rho = c_1 z^{1/2}(1 - z)^{1/2} \), therefore

\[ z = 1 - \sin(1/2(c_1 r + c_2))^2. \]  

(15)

So equation (8) tells us that

\[ E - V = C - A \csc(c_1 r + c_2)^2 - B \cot(c_1 r + c_2) \csc(c_1 r + c_2). \]  

(16)

Where the coefficients are given

\[ A = \frac{c_1^2}{4} (2(\alpha + \beta)^2 + 1) - c_1^2 (\alpha + \beta - \gamma + 1) \gamma \]
\[ B = \frac{c_1^2}{2} (\alpha + \beta - 1)(\alpha + \beta + 1 - 2\gamma) \]
\[ C = \frac{c_1^2}{4} (\alpha - \beta)^2. \]  

(17)

**Case: 3** consider \( a = 1/2, b = 1, \rho = c_1 z^{1/2}(1 - z) \), therefore

\[ z = \tanh^2(\frac{c_1 r - c_2}{2}). \]  

(18)

So we have

\[ E - V = C - A \sech^2[(c_1 r - c_2)/2] - B \csch^2[(c_1 r - c_2)/2]. \]  

(19)

Where the coefficients are given

\[ A = -\frac{c_1^2}{16} (4(\alpha - \beta)^2 - 1) \]
\[ B = \frac{c_1^2}{16} (4\gamma^2 - 8\gamma + 3) \]
\[ C = -\frac{c_1^2}{4} (\alpha + \beta - \gamma)^2. \]  

(20)
Case: 4 consider \( a = 1, b = 0, \rho = c_1z \), therefore

\[
z = c_2 e^{c_1r}.
\] (21)

we have

\[
E - V = C - A \frac{1}{(c_2 e^{c_1r} - 1)^2} - B \frac{1}{c_2 e^{c_1r} - 1}.
\] (22)

Where the coefficients are given

\[
A = -\frac{c_1^2}{4}((\alpha + \beta - \gamma)^2 + 1)
\]
\[
B = \frac{c_1^2}{2}(\alpha^2 + \beta^2 + \gamma(1 - \alpha - \beta) - 1)
\]
\[
C = -\frac{c_2^2}{4}(\alpha - \beta)^2.
\] (23)

Case: 5 consider \( a = 1, b = 1/2, \rho = c_1z(1 - z)^{1/2} \), therefore

\[
z = \text{sech}^2(\frac{c_1r + c_2}{2}).
\] (24)

we have

\[
E - V = C - \frac{1}{2} \cosh^2(c_1r + c_2) - B \coth(c_1r + c_2) \csch(c_1r + c_2).
\] (25)

Where the coefficients are given

\[
A = \frac{c_1^2}{4}[4(\alpha^2 + \beta^2) - 4(\alpha + \beta) + 2\gamma - 1]
\]
\[
B = \frac{c_1^2}{2}(2\alpha - \gamma)(2\beta - \gamma)
\]
\[
C = -\frac{c_2^2}{4}(\gamma - 1)^2.
\] (26)

Case: 6 consider \( a = 1, b = 1, \rho = c_1z(1 - z) \), therefore

\[
z = 1 - \frac{c_2}{e^{c_1r} + c_2}.
\] (27)

we have

\[
E - V = C - \frac{e^{2c_1r}}{4(e^{c_1r} + c_2)^2} - B \frac{e^{c_1r}}{4(e^{c_1r} + c_2)}.
\] (28)

Where the coefficients are given

\[
A = \frac{c_1^2}{4}((\alpha - \beta)^2 - 1]
\]
\[
B = \frac{c_1^2}{2}(2\alpha\beta + \gamma - \alpha\gamma - \beta\gamma)
\]
\[
C = -\frac{c_2^2}{4}(\gamma - 1)^2.
\] (29)
So far we have discussed six cases, which represent six possible types of potential energy. The other three possible cases $a = 0, b = 0; a = 0, b = 1/2$ and $a = 1/2, b = 0$ cannot be considered as solvable potential due to the lack of an energy terms.

Further, by applying the relationship between the hypergeometric function and the Riemannian $P$-function and its transformation formula, we can obtain other solutions of different forms.

It is worth noting that in case 2 if we set $c_1 = i, c_2 = \pi/2$, then

$$z = (1 - i \sinh r)/2.$$  \hspace{1cm} (30)

And

$$E - V = C - A \sech^2 r + iB \sech r \tanh r$$

$$A = -\frac{1}{4}(2(\alpha + \beta)^2 + 1) + (\alpha + \beta - \gamma + 1)\gamma$$

$$B = -\frac{1}{2}(\alpha + \beta - 1)(\alpha + \beta + 1 - 2\gamma)$$

$$C = -\frac{1}{4}(\alpha - \beta)^2.$$ \hspace{1cm} (31)

This is nothing but the complex PT-invariant potential energy discussed by Ahmed et al \[8, 9\]. Similarly, we can construct others solvable PT-invariant potentials. Set $c_1$ is pure imaginary number and $c_2$ is real number in all sex cases, we will find these potentials are all complex PT-invariant potentials.

If set $g(z) = 0$ in the equation (5), then we can solve $f(z) = \sqrt{E_1} \rho^{-1/2}$, and also assume that $f(z)$ have the form $f = z^p(1 - z)^q$, then we will arrive at the Bose solvable potentials \[4\]. These potentials have been studied generally \[6, 5, 12, 13\]. In a similar way, we can also construct the corresponding complex PT-invariant potentials.

4 Conclusion

In this work, we construct a new class of solvable potentials by means of the hypergeometric equation, where the Schrödinger equation is rewritten as a hypergeometric equation through taking a similarity transformation. We consider $g(r) \neq 0$, and suppose $f = z^p(1 - z)^q$, such that $p, q$ dependent $\alpha, \beta, \gamma$, this method provides us a new perspective to construct solvable potentials for Schrödinger equation. Our method unlike previously let $g(r) = 0$ or used the invariant identity of the target equation, this method need $f(z) = \sqrt{E_1} \rho^{-1/2}$, cause $f(z)$ to be a fixed function. In a similar way, computing the other Fuchs equations with our method, such as the Heun equation, can also construct corresponding solvable potentials \[14\].
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References

[1] Morse, Philip M.. “Diatom Molecules According to the Wave Mechanics. II. Vibrational Levels.” Physical Review 34, 57-64 (1929).

[2] G. Pöschll and E. Teller. Z. Physik, 83, 143-151 (1933).

[3] C. Eckart, “The Penetration of a Potential Barrier by Electrons,” Phys. Rev. 35, 1303-1309 (1930)

[4] A.K. Bose, “SOLVABLE POTENTIALS.” Phys. Lett. 7, 245 (1963).

[5] Natanzon, G. A.. “General properties of potentials for which the Schrödinger equation can be solved by means of hypergeometric functions.” Theoretical and Mathematical Physics 38 (1979): 146-153.

[6] Ishkhanyan, Artur M. and Vladimir P. Krainov. “Discretization of Natanzon potentials.” The European Physical Journal Plus 131 (2016): 1-11.

[7] Dong, Shishan et al. “Constructions of the Soluble Potentials for the Nonrelativistic Quantum System by Means of the Heun Functions.” Advances in High Energy Physics (2018).

[8] Z. Ahmed, “Real and complex discrete eigenvalues in an exactly solvable one-dimensional complex PT invariant potential,” Phys. Lett. A 282, 343-348 (2001)

[9] Kumari, Nisha et al. “Scattering amplitudes for the rationally extended PT symmetric complex potentials.” Annals of Physics 373 (2016): 163-177.

[10] M. Hasan and B. P. Mandal, “New scattering features in non-Hermitian space fractional quantum mechanics,” Annals Phys. 396, 371-385 (2018)

[11] R. K. Yadav, A. Khare, B. Bagchi, N. Kumari and B. P. Mandal, “Parametric symmetries in exactly solvable real and PT symmetric complex potentials,” J. Math. Phys. 57, no.6, 062106 (2016)
[12] R. Milson, “On the Liouville transformation and exactly solvable Schrodinger equations,” Int. J. Theor. Phys. 37, 1735-1752 (1998)

[13] Morales J, J García-Martínez, J García-Ravelo, et al. “Exactly Solvable Schrodinger Equation with Hypergeometric Wavefunctions.” Journal of Applied Mathematics and Physics, 3 11-18 (2015)

[14] Filipuk, Galina, Artur M. Ishkhanyan and Jan Dereziński. “On the Derivatives of the Heun Functions.” Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) 55 (2019): 200-207.