Variants of the Maurey-Rosenthal theorem

for

quasi Köthe function spaces

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Abstract

The Maurey-Rosenthal theorem states that each bounded and linear operator $T$ from a quasi normed space $E$ into some $L_p(\nu)$ ($0 < p < r < \infty$) which satisfies a vector-valued norm inequality

$$
\|
\left(\sum |Tx_k|^r\right)^{1/r}
\|_{L_p} \leq \left(\sum \|x_k\|_E^r\right)^{1/r}
$$

for all $x_1, \ldots, x_n \in E$, even allows a weighted norm inequality: there is a function $0 \leq w \in L_0(\nu)$ such that

$$
\left(\int \frac{|Tx|^r}{w} \, d\nu\right)^{1/r} \leq \|x\|_E
$$

for all $x \in E$.

Continuing the work of Garcia-Cuerva and Rubio de Francia we give several scalar and vector-valued variants of this fundamental result within the framework of quasi Köthe function spaces $X(\nu)$ over measure spaces.

The fundamental Maurey-Rosenthal theorem – due to Maurey [15] with roots in the classical work [19] of Rosenthal – states that each bounded and linear operator $T$ from a quasi normed space $E$ into some $L_p(\nu)$ ($\nu$ a measure and $0 < p < \infty$) satisfies for $0 < p < r < \infty$ a vector-valued norm inequality

$$
\left\|\left(\sum_{k=1}^n |Tx_k|^r\right)^{1/r}\right\|_{L_p} \leq \left(\sum_{k=1}^n \|x_k\|_E^r\right)^{1/r}
$$

(0.1)

if and only if it satisfies a weighted norm inequality

$$
\left(\int \frac{|Tx|^r}{w} \, d\nu\right)^{1/r} \leq \|x\|_E;
$$

(0.2)

more precisely, $T$ fulfills (0.1) for all choices of vectors $x_1, \ldots, x_n \in E$ if and only if there exists a weight $0 \leq w \in L_0(\nu)$ (with an appropriate norm estimate) such that (0.2) holds for all $x \in E$.

This result is of special interest for operators $T : E \rightarrow F$, $F$ a subspace of some $L_1(\nu)$. Dually, the famous Grothendieck-Pietsch domination theorem says that a bounded and
Linear operator $T : E \to F$, $E$ now a subspace of some $L_\infty(\mu)$ and $F$ a Banach space, allows for $1 \leq r < \infty$ a vector-valued norm inequality

$$\left( \sum_{k=1}^{n} \|Tx_k\|_F^r \right)^{1/r} \leq \left\| \left( \sum_{k=1}^{n} |Tx_k|^r \right)^{1/r} \right\|_{L_\infty}$$

($T$ is $r$-summing) if and only if the weighted norm inequality

$$\|Tx\|_F \leq \varphi(|x|^r)^{1/r}$$

holds true; more accurately: $T$ satisfies (0.3) for all $x_1, \ldots, x_n \in E$ iff there is a positive continuous functional $\varphi \in L_\infty(\mu)'$ (with $\|\varphi\| = 1$) such that (0.4) holds for all $x \in E$.

Nowadays such equivalences of vector-valued and weighted norm inequalities are of fundamental importance in different parts of analysis (see [19], [15], [18], [6], [24] and [3] for operator theory and geometry in Banach spaces, and [8], [9], [23] and [24] for harmonic analysis). In particular, Rubio de Francia [21], [20] and later Garcia-Cuerva [8] (see also [8]) used the Maurey-Rosenthal cycle of ideas to create a crucial link between functional analysis and harmonic analysis – Rubio’s $L_p$-boundedness principle for singular integral operators.

Continuing and improving their work, the aim of this paper is to extend the Maurey-Rosenthal theorem within the framework of quasi Köthe and Banach function spaces over measure spaces (examples are Lorentz and Orlicz spaces); we prove a general theorem which combines the Maurey-Rosenthal theorem with the Grothendieck-Pietsch domination theorem, with Krivine’s factorization theory for operators acting between Banach lattices, and includes many scalar and vector-valued variants of the original Maurey-Rosenthal theorem as special cases.

We shall use standard notation and notions from Banach space theory, as presented e.g. in [1] or [13]. If $E$ is a Banach space over the scalars $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $B_E$ denotes its closed unit ball and $E'$ its continuous dual; Banach lattices $X$ by definition are real. The term “operator” stands for a bounded and linear mapping between (quasi) normed spaces. But note that most of our results can be formulated for operators $T : E \to F$ which are only homogeneous (in the sense that $T(\lambda x) = \lambda T(x)$ for all $\lambda \geq 0$, $x \in E$) and not necessarily additive – in such a case the term “homogeneous operator” will be used. Recall that for example in harmonic analysis many nonlinear operators like square or maximal operators naturally arise.

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1 Powers of $r$-convex quasi Banach function spaces

In this first section we fix some terminology and lemmata on so-called quasi Köthe function spaces.

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite and complete measure space, and denote all $\mu$-a.e. equivalence classes of real-valued measurable functions on $\Omega$ by $L_0(\mu)$. A quasi normed space $(X(\mu), \| \cdot \|_X)$ of functions in $L_0(\mu)$ is said to be a quasi Köthe function space if it satisfies the following three conditions:

(I) If $|x| \leq |y|$ on $\Omega$, with $x \in L_0(\mu)$ and $y \in X(\mu)$, then $x \in X(\mu)$ and $\|x\|_X \leq \|y\|_X$. 


(II) There is some $0 < t < \infty$ such that for all $x, y \in X(\mu)$

$$\|(|x|^t + |y|^t)^{1/t}\|_X \leq (\|x\|_X + \|y\|_X)^{1/t}.$$  

(III) The support of $X(\mu)$ (i.e., the smallest set in $\Omega$ which contains $\mu$-a.e. all supports of functions in $X(\mu)$) equals $\Omega$, and moreover $X(\mu)$ satisfies the so-called Fatou property: $\|x_n\|_X \to \|x\|_X$ for non negative functions $x_n, x \in X(\mu)$ such that $x_n \uparrow x$ pointwise $\mu$-a.e.

Complete quasi Köthe function spaces are called quasi Banach function spaces – if $\| \cdot \|_X$ is a (complete) norm or, equivalently, (II) holds for $t=1$, then we shortly speak of (Banach) Köthe function spaces. See e.g. [1], [13, II], [17] and [22] for information on function spaces – but note that in all these references the definitions are slightly different.

Clearly, every (quasi) Köthe function space can be interpreted as a (quasi) normed lattice. We mention that already under weak additional assumptions normed lattices are order isomorphic to Köthe function spaces (see e.g. [1] and [13, I, 1.b.14]).

A quasi Köthe function space $X(\mu)$ is said to be $\sigma$-order continuous if $\|x_n\|_X \to 0$ for each sequence $(x_n)$ in $X(\mu)$ with $0 \leq x_n \downarrow 0 \mu$-a.e.

Define for a Köthe function space $X(\mu)$ the intersection of the order continuous dual and topological dual:

$$X^\times(\mu) := \left\{ y \in L_0(\mu) \mid \|y\|_{X^\times} := \sup_{\|x\|_X \leq 1} |\int xy \, d\mu| < \infty \right\}.$$

The following well-known result will be important (for (1) see e.g. [17, 2.6.4 and 2.4.19], for (2) [17, 2.4.21] and [22, 13.5], and for (3) again [17, 2.6.4]):

**Lemma 1.** Let $X(\mu)$ be a Köthe function space.

1. $(X^\times(\mu), \| \cdot \|_{X^\times})$ is a Banach function space.
2. $\|x\|_X = \sup_{\|y\|_{X^\times} \leq 1} |\int xy \, d\mu|$ for all $x \in X(\mu)$.
3. $X^\times(\mu) = X'(\mu)$ whenever $X(\mu)$ is $\sigma$-order continuous.

For $0 < r < \infty$ the $r$-th power of a quasi Köthe function space $(X(\mu), \| \cdot \|_X)$ is defined to be

$$X^r(\mu) := \left\{ x \in L_0(\mu) \mid \|x|^{1/r}\|_{X} \in X(\mu) \right\};$$

together with the quasi norm

$$\|x\|_{X^r} := \|x|^{1/r}\|_{X^r}, \quad x \in X^r(\mu)$$

this vector space obviously forms a quasi Köthe function space ((II) then holds for $t/r$ instead of $t$). Observe that $X^r(\mu)$ is $\sigma$-order continuous whenever $X(\mu)$ is.

We say (in analogy to the theory of Banach lattices) that a quasi Köthe function space $X(\mu)$ is $r$-convex $(0 < r < \infty)$ if there is a constant $c \geq 0$ such that for all $x_1, \ldots, x_n \in X(\mu)$

$$\|(|x_k|^r)^{1/r}\|_X \leq c(\|x_k\|_X)^{1/r},$$

and $r$-concave if there is a $c \geq 0$ such that for all $x_1, \ldots, x_n \in X(\mu)$

$$(\sum\|x_k\|_X^{1/r})^{1/r} \leq c(\sum|x_k|^r)^{1/r} \|x\|_X;$$

$M^{(r)}(X)$ and $M_{(r)}(X)$ stand for the best constants, respectively (and, as usual, we define these constants to be $\infty$ if $X$ does not have the corresponding property).
Lemma 2.

(1) For $0 < t, r < \infty$ and every quasi Köthe function space $X(\mu)$

$$M^{(r/t)}(X^t) = M^{(r)}(X)^t \quad \text{and} \quad M^{(r/t)}(X^t) = M^{(r)}(X)^t.$$ 

(2) For $1 \leq r < \infty$ and every Köthe function space $X(\mu)$

$$M^{(r)}(X^\infty) = M^{(r)}(X) \quad \text{and} \quad M^{(r)}(X^\infty) = M^{(r)}(X).$$

Statement (1) is an immediate consequence of the definitions. The proof of (2) follows from the formulas

$$\left(\sum ||x_k||_X^r\right)^{1/r} = \sup\left\{ \frac{1}{t} \left(\sum ||y_k||_{X^t}^{r/t}\right) \right\}$$

$$\left(\sum ||x_k||_X^r\right)^{1/r} = \sup\left\{ \frac{1}{t} \left(\sum ||y_k||_{X^t}^{r/t}\right) \right\}$$

which are consequences of the isometric embeddings

$$\ell^n_r(X) \hookrightarrow \ell^n_r(X^\infty) \hookrightarrow \ell^n_r((X^\infty)^\prime) = (\ell^n_r(X^\infty))^\prime$$

$$X(\ell^n_r) \hookrightarrow X^\infty(\ell^n_r) \hookrightarrow (X^\infty)^\prime(\ell^n_r) = (X^\infty(\ell^n_r))^\prime$$

(see Lemma [2] and [13, II, p. 47], in particular for the notation). We now clarify when the quasi norm $\| \cdot \|_{X^r}$ on $X^r(\mu)$ is equivalent to a norm. If $X(\mu)$ is $r$-convex, then for $x \in X^r(\mu)$ and $x_1, \ldots, x_n \in X^r(\mu)$ with $\sum_{i=1}^n |x_i|$

$$\|x\|_{X^r} = \|x\|^{1/r} \leq \left(\sum (|x_i|^{1/r})^r\right)^{1/r} \leq \left(\sum (|x_i|)^r\right)^{1/r}$$

$$\leq M^{(r)}(X^r) \sum ||x_i||_{X^r} = M^{(r)}(X^r) \sum ||x_i||_{X^r},$$

hence for the lattice norm

$$|||x|||_{X^r} := \inf\left\{ \sum_{i=1}^n ||x_i||_{X^r} \left| n \in \mathbb{N}, \sum_{i=1}^n |x_i| \right. \right\}, \quad x \in X^r(\mu)$$

the following result holds true:

**Lemma 3.** Let $(X(\mu), \| \cdot \|_X)$ be a quasi Köthe function space which for $0 < r < \infty$ is $r$-convex. Then $(X^r(\mu), \| \cdot \|_{X^r})$ is a (normed) Köthe function space, and on $X^r(\mu)$

$$M^{(r)}(X)^{-r}\| \cdot \|_{X^r} \leq \| \cdot \|_{X^r} \leq \| \cdot \|_{X^r}.$$ \quad (1.1)

Note that every quasi Köthe function space $X(\mu)$ by condition (II) and induction is $t$-convex with constant 1 for some $0 < t < \infty$, hence by the preceding lemma $(X^t(\mu), \| \cdot \|_{X^t})$ is a (normed) Köthe function space and

$$(X(\mu), \| \cdot \|_X) = ((X^t(\mu), \| \cdot \|_{X^t})^{1/t}, \| \cdot \|_{(X^t)^{1/t}})$$

holds isometrically – the quasi Köthe function spaces are the powers of (normed) Köthe function spaces.

Recall that every Banach lattice is 1-convex and $\infty$-concave and that the properties “$r$-convexity” and “$r$-concavity” for $1 \leq r \leq \infty$ are “decreasing and increasing in $r$”, respectively ([13, II, 1.d.5]). The argument usually given is strongly based on duality, hence the following lemma needs an alternative proof.
Lemma 4. Let $0 < t < r < \infty$. Then each $r$-convex quasi Köthe function space $X(\mu)$ is $t$-convex, and
\[ M^{(t)}(X) \leq M^{(r)}(X). \]
In particular, Köthe function spaces $X(\mu)$ being 1-convex, are $t$-convex for each $0 < t < 1$.

Proof. Consider the norm $\| \cdot \|_{X^r}$ on $X^r(\mu)$. We will show that on $X^t(\mu)$
\[ p_t(x) := \| |x|^{r/t}\|_{X^r}^{t/r} \]
satisfies the triangle inequality. Since by (1.3) on $X^t(\mu)$
\[ M^{(r)}(X)^{-t} \cdot \| X^t \| \leq p_t(\cdot) \leq \| \| X^t \| , \]
the conclusion then is a simple consequence of Lemma 2:
\[ M^{(t)}(X) = M^{(t)}(((X^t, \| \cdot \|_{X^t})^{1/t}, \| \cdot \|_{(X^t)^1/r})) \]
\[ \leq M^{(1)}(((X^t, \| \cdot \|_{X^t}))^{1/t} \leq M^{(r)}(X). \]

For the proof of the triangle inequality for $p_t$ show – in a first step and in complete analogy to the proof of the usual Hölder inequality – that for each Köthe (!) function space $Y(\nu)$ and $0 < u, v \leq 1$ with $u + v = 1$
\[ Y^u \cdot Y^v \subset Y, \quad \| xy \|_Y \leq \| x \|_{Y^u} \| y \|_{Y^v} \]
(see also [3]). In particular, we obtain with $\frac{t}{r} + \frac{r-t}{r} = 1$ that
\[ (X^r)^{r/t} \cdot (X^r)^{r/(r-t)/r} \subset X^r \]
\[ \| xy \|_{X^r} \leq \| \| x\|_{X^r}^{r/t} \| \| y\|_{X^r}^{r/(r-t)/r} \|_{X^r}^{(r-t)/r} \quad \text{for } x \in (X^r)^{r/t}, y \in (X^r)^{r/(r-t)/r}. \]
Now simulate for $x, y \in X^t$ the proof of the Minkowski inequality:
\[ p_t(x + y)^{r/t} = \| |x + y|^{r/t}\|_{X^r} = \| |x + y|^{(r-t)/t}\|_{X^r} \]
\[ \leq \| |x| \| x + y|^{(r-t)/t}\|_{X^r} + \| |y| \| x + y|^{(r-t)/t}\|_{X^r} \]
\[ \leq \| |x|^{r/t}\|_{X^r} \| x + y|^{r/t}\|_{X^r} \| + \cdots \]
\[ = p_t(x) p_t(x + y)^{r/t - 1} + p_t(y) p_t(x + y)^{r/t - 1}. \]

This obviously completes the proof. \hfill \Box

The last lemma needed is

Lemma 5. For $0 < r < \infty$ let $X(\mu)$ be an $r$-convex quasi Köthe function space. Then
\[ (X^{r/2}(\mu), \| \cdot \|_{X^{r/2}})^X \]
is $\sigma$-order continuous.

Proof. $X(\mu)$ by Lemma 4 is $r/2$-convex, hence $Y := (X^{r/2}, \| \cdot \|_{X^{r/2}})$ by Lemma 3 is a Köthe function space. Moreover, $Y$ and
\[ Z := ((X^r, \| \cdot \|_{X^r})^{1/2}, \| \cdot \|_{(X^r, \| \cdot \|_{X^r})^{1/2}}) \]
by (1.1) are isomorphic Köthe function spaces, and $Z$ by Lemma 3 (1) is 2-convex (since the space $(X^r, \| \cdot \|_{X^r})$ is 1-convex). Assume now that $Y^X$ is not $\sigma$-order continuous – then it contains $\ell \infty$ as a topological subspace (note first that Banach function spaces are $\sigma$-complete and see e.g. [3, II, 1.a.7] or [22, 3.7]). But this contradicts the fact that $Y^X$ by Lemma 2, (2) is 2-concave. \hfill \Box
Our basic examples are:

1. For \(0 < p_1, p_2 \leq \infty\) denote by \(L_{(p_1,p_2)}(\mu_1 \otimes \mu_2)\) the space of all (equivalence classes of) measurable real-valued functions \(f\) on \(\Omega_1 \times \Omega_2\) such that

\[
\|f\|_{(p_1,p_2)} := \left( \int \left( \int |f(w_1, w_2)|^{p_2} \, d\mu_2(w_2) \right)^{p_1/p_2} \, d\mu_1(w_1) \right)^{1/p_1}
\]

(with the obvious modification if \(p_1 \) or \(p_2 = \infty\)). This gives a quasi Banach function space over \(\mu_1 \otimes \mu_2\) which is normed whenever \(1 \leq p_1, p_2 \leq \infty\), and by the continuous triangle inequality it is \(\min(p_1, p_2)\)-convex and \(\max(q_1, q_2)\)-concave with constants \(1\). For \(p_1 = p_2\) one gets the usual \(L_p\)'s. \(L_\infty(\mu)\) being \(\infty\)-convex, is \(r\)-convex for all \(0 < r < \infty\). Obviously,

\[
L_{(p_1,p_2)}(\mu_1 \otimes \mu_2)^r = L_{(p_1/r, p_2/r)}(\mu_1 \otimes \mu_2) \quad \text{for} \quad 0 < r < \infty, \quad 0 < p_1, p_2 \leq \infty.
\]

2. For \(0 < p_1 < \infty\) and \(0 < p_2 \leq \infty\) the Lorentz function spaces \(L_{p_1,p_2}(\mu)\) (see e.g. [4] or [10] for the definition) form quasi Banach function spaces over \(\mu\); recall that these spaces are normable whenever \(1 < p_1 < \infty, \ 1 \leq p_2 \leq \infty\). Again it is straightforward to check that

\[
L_{p_1,p_2}(\mu)^r = L_{p_1/r, p_2/r}(\mu) \quad \text{for} \quad 0 < p_1, r < \infty, \quad 0 < p_2 \leq \infty.
\]

Convexity and concavity of Lorentz spaces for \(1 < p_1 < \infty, \ 0 < p_2 \leq \infty\) and non-atomic \(\mu\) were studied in Creekmore [3]. The following arguments handle the general case and seem to be easier: let \(0 < r < p_1 < \infty\) and \(0 < r \leq p_2 \leq \infty\), and choose \(0 < s < \infty\) such that \(1 < sr < \min(sp_1, sp_2)\). Then by Lemma 2

\[
M^{(r)}(L_{p_1,p_2}) = M^{(sr/s)}(L_{sp_1,sp_2}^s)
\]

\[
\leq M^{(sr)}(L_{sp_1,sp_2})^s
\]

\[
= M^{(1/1/sr)}(L_{sp_1,sp_2}^{1/sr})^s
\]

\[
\leq M^{(1)}(L_{sp_1,sp_2}^{1/sr})^{1/r} < \infty
\]

(since the latter space is normable). Similarly,

\[
M^{(r)}(L_{p_1,p_2}) < \infty \quad \text{for} \quad 0 < p_1 < r < \infty, \quad 0 < p_2 \leq r;
\]

for \(1 < p_1, p_2\) use the result on convexity together with duality, and for arbitrary \(0 < p_1, p_2\) Lemma 2 as above.

3. Orlicz function spaces \(L_\varphi(\mu)\) are Banach function spaces in the above sense, and convexity and concavity can be characterized in terms of \(\varphi\) (see [13, II]). Clearly, for \(0 < r < \infty\)

\[
L_\varphi(\mu)^r = L_{\varphi(\varphi^{-})}(\mu) := \left\{ f : \exists \lambda > 0 : \int \varphi\left(\frac{|f|^{1/r}}{\lambda^{1/r}}\right) \, d\mu < \infty \right\}.
\]

It can easily be seen that \(L_p\), \(L_{(p_1,p_2)}\) and \(L_{p_1,p_2}\) are \(\sigma\)-order continuous whenever \(0 < p, p_1, p_2 < \infty\); order continuity of Orlicz spaces \(L_\varphi\) can be characterized in terms of so-called \(\Delta_2\)-conditions of \(\varphi\) (see [13, II]).
2 Vector-valued norm inequalities and weighted norm inequalities for homogeneous forms

We call a set $U$ homogeneous whenever it carries a multiplication with positive scalars: $U \times [0, \infty[ \to U$, $(x, \lambda) \mapsto \lambda x$. If there is a homogeneous set $U$, a quasi Köthe function space $X(\mu)$ and a homogeneous mapping $\phi : U \to X(\mu)$ (i.e., $\phi(\lambda x) = \lambda \phi(x)$ for $\lambda \geq 0$, $x \in U$), then we say that $\phi$ represents $U$ in $X(\mu)$ homogeneously. For two homogeneous sets $U_1, U_2$ a form $u : U_1 \times U_2 \to \mathbb{K}$ is said to be homogeneous if $u(\lambda x, y) = \lambda u(x, y)$ for all $\lambda \geq 0$, $x \in U_1$, $y \in U_2$, and a mapping $T : U_1 \to U_2$ is homogeneous if $T(\lambda x) = \lambda T(x)$ for $\lambda \geq 0$, $x \in U_1$.

The following result allows to transform vector-valued norm inequalities for forms on homogeneous sets which are homogeneously representable in quasi Köthe function spaces, into weighted norm inequalities (and vice versa).

**Theorem 1.** For $\ell = 1, 2$ let $0 < r_\ell < \infty$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Let $u : U_1 \times U_2 \to \mathbb{K}$ be a homogeneous form on homogeneous sets such that each $U_\ell$ via $\phi_\ell$ can be represented homogeneously in an $r_\ell$-convex quasi Köthe function space $X_\ell(\mu_\ell)$. If $u$ satisfies

$$
\left( \sum_{k=1}^{n} |u(x_k, y_k)|^{r_1}\right)^{1/r_1} \leq \left( \sum_{k=1}^{n} |\phi_1(x_k)|^{r_1}\right)^{1/r_1} \left( \sum_{k=1}^{n} |\phi_2(y_k)|^{r_2}\right)^{1/r_2} \leq M^{(r_1)}(X_\ell)
$$

for all $x_1, \ldots, x_n \in U_1$ and $y_1, \ldots, y_n \in U_2$, then there are two positive linear functionals

$$
\varphi_\ell : X^{r_\ell}(\mu_\ell) \to \mathbb{R} \quad \text{with} \quad \varphi_\ell(|x|^r)^{1/r} \leq M^{(r_\ell)}(X_\ell)
$$

such that for all $x \in U_1$ and $y \in U_2$

$$
|u(x, y)| \leq \varphi_1(|\phi_1(x)|^{r_1})^{1/r_1} \varphi_2(|\phi_2(y)|^{r_2})^{1/r_2}.
$$

If $X_\ell(\mu_\ell)$ is $\sigma$-order continuous, then $\varphi_\ell$ can be chosen to be a function in $(X^{r_\ell}(\mu_\ell))^\times$.

**Proof.** The proof is based on a standard separation argument. Define the weak$^*$-compact and convex set

$$
K_\ell := \left\{ \varphi \in (X^{r_\ell}_\ell, \| \cdot \|_{X^{r_\ell}_\ell})' \left| \varphi \geq 0, \| \varphi \| \leq 1 \right. \right\},
$$

and for $\ell = 1, 2$ and $x_1^{(\ell)}, \ldots, x_n^{(\ell)} \in U_\ell$ the affine and continuous function

$$
\phi_{x_k^{(1)}, x_k^{(2)}} : K_1 \times K_2 \to \mathbb{K}
$$

$$
\phi_{x_k^{(1)}, x_k^{(2)}}(\varphi_1, \varphi_2) := \sum_{\ell=1}^{2} \frac{t}{r_\ell} M^{r_\ell}_\ell \varphi_\ell \left( \sum_{k=1}^{n} |\phi_\ell(x_k^{(\ell)})|^{r_\ell} \right)^{1/r_\ell} - \sum_{k=1}^{n} |u(x_k^{(1)}, x_k^{(2)})|^{t}
$$

(put $M_\ell := M^{(r_\ell)}(X_\ell)$). Note first that the set $K$ of all these functions is convex: the sum of two such functions belongs to $K$, and for $\alpha \geq 0$ and $x_1^{(\ell)}, \ldots, x_n^{(\ell)} \in U_\ell$

$$
\alpha \phi_{x_k^{(1)}, x_k^{(2)}} = \phi_{\alpha^{1/r_1}x_k^{(1)}, \alpha^{1/r_2}x_k^{(2)}}.
$$

We will now show that for each $\phi_{x_k^{(1)}, x_k^{(2)}}$ there is $(\varphi_1, \varphi_2) \in K_1 \times K_2$ with

$$
\phi_{x_k^{(1)}, x_k^{(2)}}(\varphi_1, \varphi_2) \geq 0;
$$
indeed, by the Hahn-Banach theorem and Lemma 3 there are \( \varphi_\ell \in K_\ell \) such that
\[
\varphi_\ell \left( \sum_k |\phi_\ell (x_k^{(\ell)})|^{r_\ell} \right) = \left\| \sum_k |\phi_\ell (x_k^{(\ell)})|^{r_\ell} \right\|_{X^{r_\ell}} \geq \frac{1}{M^{r_\ell}} \left\| \sum_k |\phi_\ell (x_k^{(\ell)})|^{r_\ell} \right\|_{X^r}^{1/r_\ell},
\]
hence (recall that \( a/s + b/s' \geq a^{1/s} b^{1/s'} \) for \( s > 1 \) and \( a, b \geq 0 \))
\[
\phi_{x^{(1)},x^{(2)}}(\varphi_1, \varphi_2) = 2 \prod_{\ell=1}^{2} M_{\ell}^{r_\ell} \left( \sum_k |\phi_\ell (x_k^{(\ell)})|^{r_\ell} \right)^{1/r_\ell} - \sum_k |u(x_1^{(\ell)}, x_2^{(\ell)})|^{t}
\geq 2 \prod_{\ell=1}^{2} \left\| \sum_k |\phi_\ell (x_k^{(\ell)})|^{r_\ell} \right\|_{X^r}^{1/r_\ell} - \sum_k |u(x_1^{(\ell)}, x_2^{(\ell)})|^{t} \geq 0.
\]
By Ky Fan’s lemma (see e.g. [3, 9.10]) there is \((\varphi_1, \varphi_2) \in K_1 \times K_2\) such that
\[
\phi_{x^{(1)},x^{(2)}}(\varphi_1, \varphi_2) \geq 0 \text{ for all } n \text{ and } x_1^{(\ell)}, \ldots, x_n^{(\ell)} \in U_\ell.
\]
This easily gives the conclusion: define for \( x^{(1)} \in U_1, x^{(2)} \in U_2 \)
\[
a_\ell := M_\ell \varphi_\ell \left( |\phi_\ell (x^{(\ell)})|^{r_\ell} \right)^{1/r_\ell}.
\]
Then
\[
|u(x^{(1)}, x^{(2)})|^t = (a_1 a_2)^t \left| u \left( \frac{x^{(1)}}{a_1}, \frac{x^{(2)}}{a_2} \right) \right|^t
\leq (a_1 a_2)^t 2 \prod_{\ell=1}^{2} \frac{t}{r_\ell} M_\ell^{r_\ell} \varphi_\ell \left( |\phi_\ell (x^{(\ell)})|^{r_\ell} \right) = (a_1 a_2)^t;
\]
if \( a_1 = 0 \), then \( |u(nx^{(1)}, x^{(2)})|^t \leq \frac{t}{r_2} M_2^{r_2} \varphi_2 \left( |\phi_2 (x^{(2)})|^{r_2} \right) \) for all \( n \), hence \( u(x^{(1)}, x^{(2)}) = 0 \).

\[\square\]

**Remark 1.** (1) An easy calculation shows that (2.2) implies (2.1) with an additional constant \( M^{(r_1)}(X_1) M^{(r_2)}(X_2) \). (2) Recall from Lemma 4 that Köthe function spaces are \( r \)-convex for each \( 0 < r \leq 1 \). (3) A short look at the proof shows that an analogous result holds for forms \( u : U_1 \times \ldots \times U_\ell \rightarrow \mathbb{K} \) which are homogeneous in each coordinate. (4) \( \sigma \)-finiteness of the measures is only needed for the last statement of the theorem. (5) If \( K_\ell \) is a compact space and \( \phi_\ell : U_\ell \rightarrow \ell_\infty(K_\ell) \) has values in \( C(K_\ell) \), then by restriction \( \phi_\ell : \ell_\infty(K_\ell) \rightarrow \mathbb{K} \) defines a Borel measure on \( K_\ell \) (the proof shows that in this case it is even possible to obtain a Borel probability measure).

Together with Lemma 3 we obtain the following interesting

**Corollary 1.** For two Köthe function spaces \( X(\mu) \) and \( Y(\nu) \), and \( 1 \leq r < \infty \) let
\[
u : X^{1/r}(\mu) \times Y^{1/r'}(\nu) \rightarrow \mathbb{K}
\]
be a homogeneous form such that for all \( x_1, \ldots, x_n \in X^{1/r}(\mu) \), \( y_1, \ldots, y_n \in Y^{1/r'}(\nu) \)
\[
\sum |u(x_k, y_k)| \leq \left\| \left( \sum |x_k|^r \right)^{1/r} \right\|_{X^{1/r}} \left\| \left( \sum |y_k|^{r'} \right)^{1/r'} \right\|_{Y^{1/r'}}.
\]  
(2.3)
Then there exist positive functionals
\[ \varphi \in X'(\mu) \text{ with } \|\varphi\|_{X'} \leq 1 \]
\[ \psi \in Y'(\nu) \text{ with } \|\psi\|_{Y'} \leq 1 \]
such that for all \( x \in X^{1/r}(\mu) \), \( y \in Y^{1/r'}(\nu) \)
\[ |u(x, y)| \leq \varphi(|x|^r)^{1/r} \psi(|y|^{r'})^{1/r'}. \]

If \( X(\mu) \) and \( Y(\nu) \), respectively, are \( \sigma \)-order continuous, then \( \varphi \) and \( \psi \) are functions in \( X^\times(\mu) \) and \( Y^\times(\nu) \), respectively.

**Proof.** Take \( U_1 = X^{1/r} \), \( U_2 = Y^{1/r'} \) and \( \phi_1, \phi_2 \) the identities, and recall from Lemma 2, (1) that \( X^{1/r} \) is \( r \)-convex and \( Y^{1/r'} \) \( r' \)-convex with constants 1.

For \( r = 2 \) every continuous and bilinear \( u : X^{1/2} \times Y^{1/2} \to \mathbb{K} \) satisfies (2.1) – this is an easy consequence of the Grothendieck-Krivine theorem from [11] (see also [13, II, 1.f.14]). Moreover, it can be shown that each positive continuous bilinear form \( u : X^{1/r} \times X^{1/r'} \to \mathbb{K} \), i.e. \( u(x, y) \geq 0 \) for \( x, y \geq 0 \), satisfies (2.3) (use [13, II, 1.d.9]).

### 3 Vector-valued norm inequalities and weighted norm inequalities for homogeneous operators

The following theorem is our main result – it is a sort of reformulation of Theorem 1 for homogeneous operators instead of forms. Before we formulate it, let us sketch how to reprove (part of) the original Maurey-Rosenthal theorem on the basis of Theorem 1 in order to motivate the proof of our extension which at first glance may look strange: take an operator \( T : E \to L_p(\nu) \) (\( E \) a quasi normed space and \( 1 \leq p < \infty \)) which for \( 1 < r < \infty \) satisfies for all \( x_1, \ldots, x_n \in E \)
\[ \left\| \left( \sum |Tx_k|^r \right)^{1/r} \right\|_{L_p(\nu)} \leq \left( \sum \|x_k\|^r \right)^{1/r}. \]

Then \( T \) defines a bilinear form
\[ u : E \times L_p(\nu)^\times \to \mathbb{K}, u(x, y) := \int Tx \cdot y d\nu. \]

If we consider the “trivial representations”
\[ \phi_1 : E \to L_r(\mu), \phi_1 x := \|x\|_{E}1 \quad (\mu \text{ some Dirac measure}) \]
\[ \phi_2 : L_p(\nu)^\times \to L_p(\nu)^\times, \phi_2 x := x, \]
then, by Hölder’s inequality, \( u \) satisfies (2.1). Hence, if \( 1 \leq p < r < \infty \), then \( L_p(\nu)^\times \) is \( r' \)-convex, and Theorem 1 gives a functional
\[ 0 \leq \varphi \in (L_p(\nu)^\times)^{r'} \text{ with } \|\varphi\| \leq 1 \]
such that for all \( x \in E, y \in L_p(\nu)^\times \)
\[ |u(x, y)| \leq \|x\|_E \varphi(|y|^{r'})^{1/r'}. \]
For \( p > 1 \) this functional \( \varphi \) is a function in \( L_{\left( p'/r' \right)'}(\nu) \), and it is not difficult to show (for details see the next proof) that with \( \omega := \varphi^{r'/r} \) for all \( x \in E \)

\[
\int \frac{|Tx|^r}{\omega} d\nu \leq \|x\|_E.
\]

Hence, for \( 1 < p < r < \infty \) we have reproved the original Maurey-Rosenthal theorem; for \( p = 1 \) we essentially do the same – but we need some scaling trick in order to get a function \( \varphi \in L_1(\nu) \) instead of only a measure in \( (\left( L_1(\nu)^{r'} \right)' = L_{\infty}(\nu)' \). This trick will even cover the general case \( 0 < p < r < \infty \) although the above argument is strongly based on duality.

**Theorem 2.** Let \( T : U \rightarrow V \) be a homogeneous operator where \( U \) and \( V \) are vector spaces (or only homogeneous sets) which via \( \phi \) and \( \psi \) are represented homogeneously in quasi Köthe function spaces \( X(\mu) \) and \( Y(\nu) \), respectively. For \( 0 < r < \infty \) let \( X(\mu) \) be \( r \)-convex, and \( Y(\nu) \) \( r \)-concave. If \( T \) satisfies

\[
\|\left( \sum |\psi(Tx_k)|^r \right)^{1/r} \|_Y \leq \|\left( \sum |\phi(x_k)|^r \right)^{1/r} \|_X
\]

for all \( x_1, \ldots, x_n \in U \), then there are

\[
0 \leq \varphi : X^r \rightarrow \mathbb{R} \quad \text{linear with} \quad \sup_{\|x\|_X \leq 1} \varphi(\|x\|^r)^{1/r} \leq M(r)(X)
\]

\[
0 \leq \omega_2 \in L_0(\nu) \quad \text{with} \quad \sup_{\|y\|_{L_r(\nu)} \leq 1} \left\| \omega_2^{1/r} y \right\|_Y \leq M(r)(Y)
\]

such that for all \( x \in U \)

\[
\int \frac{|\psi(Tx)|^r}{\omega_2} d\nu \leq \varphi(\|\phi(x)\|^r).
\]

If \( X(\mu) \) is \( \sigma \)-order continuous, then there is even a function

\[
0 \leq \omega_1 \in L_0(\mu) \quad \text{with} \quad \sup_{\|x\|_X \leq 1} \left\| \omega_1^{1/r} x \right\|_{L_r(\mu)} \leq M(r)(X)
\]

such that for all \( x \in U \)

\[
\int \frac{|\psi(Tx)|^r}{\omega_2} d\nu \leq \int |\phi(x)|^r \omega_1 d\mu.
\]

Recall that Köthe function spaces by Lemma \( \mathbb{3} \) are \( r \)-convex for all \( 0 < r \leq 1 \).

**Proof.** We may assume without loss of generality that \( V = Y \) and \( \psi \) is the identity (otherwise replace \( T \) by \( \psi \circ T \)). By assumption (property (II)) \( Y(\nu) \) is \( t \)-convex with constant 1 for some \( 0 < t < \infty \), and by Lemma \( \mathbb{3} \) we may assume that \( t < r \). Consider for \( s = t/2 \) a new scalar multiplication on \( U \): for \( \lambda \geq 0 \) and \( x \in U \) define

\[
\lambda \circ x := \lambda^{1/s} x;
\]

then it is easy to check that the mappings

\[
\phi^s : U \rightarrow X^s, \quad \phi^s(x) := |\phi(x)|^s
\]

\[
T^s : U \rightarrow Y^s, \quad T^s x := |Tx|^s
\]
with respect to this multiplication are homogeneous, and for \( x_1, \ldots, x_n \in U \)
\[
\left\| \left( \sum |T^s x_k |^{r/s} \right)^{s/r} \right\|_{Y^*} \leq \left\| \left( \sum |\phi^s(x_k)|^{r/s} \right)^{s/r} \right\|_{X^*}.
\]
(3.5)

From now on \( Y^* \) will always be endowed with its natural quasi norm \( \| \cdot \|_{Y^*} \) which by Lemma 5 is even a norm since \( M^{(s)}(Y) \leq M^{(l)}(Y) \leq 1 \) by Lemma 1. Hence, the homogeneous form

\[
u^s : U \times (Y^*)^\infty \to \mathbb{R}, \quad \nu^s(x, y) := \int T^s x \cdot y \, d\nu
\]
is defined, and by Hölder’s inequality, the inequality (3.3) on \( T^s \), and the definition of the norm on \( (Y^*)^\infty \) it satisfies

\[
\sum |\nu^s(x_k, y_k)| \leq \int |T^s x_k| |y_k| \, d\nu
\]
\[
\leq \int \left( \sum |T^s x_k|^{r/s} \right)^{s/r} \left( \sum |y_k|^{(r/s)'} \right)^{1/(r/s)'} \, d\nu
\]
\[
\leq \left\| \left( \sum |T^s x_k|^{r/s} \right)^{s/r} \right\|_{Y^*} \left\| \left( \sum |y_k|^{(r/s)'} \right)^{1/(r/s)'} \right\|_{(Y^*)^\infty}
\]
for all \( x_1, \ldots, x_n \in U \) and \( y_1, \ldots, y_n \in (Y^*)^\infty \). By Lemma 3

\[
M^{(r/s)}(X^*) = M^{(r)}(X)^s < \infty
\]

\[
M^{(r/s)'}((Y^*)^\infty) = M^{(r/s)}(Y^*) = M^{(r)}(Y)^s < \infty,
\]

hence it follows from Theorem 3 that there are two functionals

\[
0 \leq \varphi_1 : (X^*, \| \cdot \|_{X^*})^{r/s} \to \mathbb{R}, \quad \sup_{\|x\|_{X^*} \leq 1} |\varphi_1(|x|^{r/s})|^{s/r} \leq M^{(r)}(X)^s
\]

\[
0 \leq \varphi_2 : ((Y^*)^\infty, \| \cdot \|_{(Y^*)^\infty})^{(r/s)'} \to \mathbb{R}, \quad \sup_{\|y\|_{(Y^*)^\infty} \leq 1} |\varphi_2(|y|^{(r/s)'})|^{1/(r/s)'} \leq M^{(r)}(Y)^s
\]
such that for all \( x \in U, y \in (Y^*)^\infty \)
\[
|\nu^s(x, y)| \leq \varphi_1(|\phi^s(x)|^{r/s})^{s/r} \varphi_2(|y|^{(r/s)'})^{1/(r/s)'}.
\]
(3.6)

Clearly, for \( \varphi := M^{(r)}(X)^{-r} \varphi_1 \)
\[
0 \leq \varphi : X \to \mathbb{R}, \quad \sup_{\|x\|_X \leq 1} \varphi(|x|)^{1/r} \leq 1.
\]
(3.7)

On the other hand (Lemma 3)

\[
\varphi_2 \in \left( \left( (Y^*)^\infty, \| \cdot \|_{(Y^*)^\infty} \right)^{(r/s)'}, \| \cdot \|_{\ldots} \right)',
\]

but since \( (Y^*)^\infty \) and hence also its \( (r/s)' \)-th power by Lemma 5 are \( \sigma \)-order continuous, we obtain from Lemma 1 (3)
\[
g := M^{(r)}(Y)^{-s(r/s)'} \varphi_2 \in L_0(\nu), \quad \sup_{\|y\|_{(Y^*)^\infty} \leq 1} \int g|y|^{(r/s)'} \, d\nu \leq 1.
\]
(3.8)
Moreover, by (3.3) for all $x \in U$, $y \in (Y^s)^\times$

$$\int |Tx|^s y \, d\nu \leq M(r)(X)^s M(r)(Y)^s \varphi(\|\phi(x)\|^r)^{s/r} \left( \int g |y|^{(r/s)'} \, d\nu \right)^{1/(r/s)'}.$$ 

It remains to give this inequality the form of (3.3): note first that for $h := g^{1/(r/s)'}$ the multiplication operator

$$M_h : (Y^s)^\times \longrightarrow L(r/s)',(\nu|_{h>0})$$

by (3.8) is defined, and for all $x \in U$, $y \in (Y^s)^\times$

$$\int \frac{|Tx|^s}{h} M_h y \, d\nu \leq M(r)(X)^s M(r)(Y)^s \varphi(\|\phi(x)\|^r)^{s/r} \|M_h y\|_{L(r/s)'}$$

(obviously, $\|h = 0\| \subset [Tx = 0]$ $\nu$-a.e. for all $x \in U$). Since $M_h$ has dense range (use e.g. the Hahn-Banach theorem), for all $x \in U$

$$\int \frac{|Tx|^r}{h^{r/s}} \, d\nu \leq M(r)(X)^r M(r)(Y)^r \varphi(\|\phi(x)\|^r).$$

Finally, it remains to prove the norm estimates (3.2) and (3.3) for the weights

$$\omega_2 := M(r)(X)^r h^{r/s} \in L_0(\nu)$$

$$\omega_1 := M(r)(X)^r \varphi \in L_0(\mu)$$

(provided $X$ is $\sigma$-order continuous).

The estimate in (3.8) and the fact that $B_{(Y^s)^\times}$ is norming in $Y^s$ (see Lemma 1, (2)) give

$$\sup_{\|y\|_{L^r} \leq 1} \|h^{1/s} y\|_{Y} = \sup_{\|y\|_{L^r,s} \leq 1} \|h^{1/s} y\|_{Y}^{1/s}$$

$$= \sup_{\|y\|_{L^r,s} \leq 1} \|hy\|_{Y}^{1/s}$$

$$= \sup_{\|y\|_{L^r,s} \leq 1} \sup_{\|f\|_{(Y^s)^\times} \leq 1} |\int hyf \, d\nu|^{1/s}$$

$$= \sup_{\|f\|_{(Y^s)^\times} \leq 1} \|hf\|_{L(r/s)'} = \sup_{\|f\|_{(Y^s)^\times} \leq 1} \|g|f|^{(r/s)'}(1/(r/s)') \leq 1,$$

and if $X$ is $\sigma$-order continuous, then (3.7) gives

$$\sup_{\|x\|_X \leq 1} \|\varphi^{1/r} x\|_{L^r} = \sup_{\|x\|_X \leq 1} \varphi(|x|^r)^{1/r} \leq 1.$$ 

This completes the proof. \qed

**Remark 2.** All assumptions of the preceding result are necessary in a sense: (1) A straightforward computation shows that (3.3) implies (3.1) with the additional constant $M(r)(Y)M(r)(X)$. (2) In which sense the convexity assumption on $X$ and concavity assumption on $Y$ are necessary will be seen in 4.2. (3) That the $\sigma$-order continuity of $X$ is unavoidable in order to obtain a function $\omega_1$, will be shown in Remark 4 in 4.3. (4) Clearly, a remark analogous to Remark 4, (5) is possible.
4 Variants of the Maurey-Rosenthal theorem for quasi Banach function spaces

In this section we want to illustrate that the preceding theorem starts living if one looks at the following natural representations \( \phi : U \to X(\mu) \) (for simplicity we will always consider complete spaces):

(A) \( U = E \) a Banach space, \( X = \ell_\infty(B_{E'}) \) and
\[
\phi : E \to \ell_\infty(B_{E'}), \quad (\phi x)(x') := x'(x).
\]

(B) \( U = E \) a quasi Banach space, \( X = L_r(\mu) \) (with \( 0 < r \leq \infty \), \( \mu \) a Dirac measure) and
\[
\phi : E \to L_r(\mu), \quad \phi x := \|x\|_{E^1}.
\]

(C) \( U = X \) a quasi Banach function space and the identity
\[
\phi : X \to X, \quad \phi x := x.
\]

(D) \( U = L_r(\mu, E) \) a space of Bochner integrable functions with values in a quasi Banach space \( E, X = L_r(\mu) \) and
\[
\phi : L_r(\mu, E) \to L_r(\mu), \quad \phi x := \|x\|_{E(\cdot)}.
\]

(E) More generally, take a quasi Banach function space \( X(\mu) \) and a quasi Banach space \( E \). Then the vector-valued quasi Banach function space
\[
X(\mu, E) := \{ x : \Omega \to E \mid x \mu\text{-measurable}, \|x\|_{E(\cdot)} \in X(\mu) \}
\]
\[
\|x\|_{X(\mu, E)} := \|\|x\|_{E(\cdot)}\|_X
\]
is a quasi Banach space, and
\[
\phi : X(\mu, E) \to X(\mu), \quad \phi x := \|x\|_{E(\cdot)}
\]
represents \( X(\mu, E) \) in \( X(\mu) \) homogeneously.

4.1 The Grothendieck-Pietsch domination theorem revisited

It can easily be seen that the fundamental Grothendieck-Pietsch domination/factorization theorem for summing operators is a straightforward consequence of Theorem 2. Let \( T : E \to F \) be an \( r \)-summing (\( 1 \leq r < \infty \)) operator between Banach spaces, i.e., there is some constant \( c \geq 0 \) such that for all \( x_1, \ldots, x_n \in E \)
\[
\left( \sum \|Tx_k\|_F \right)^{1/r} \leq c \sup_{\|x\|_{E'} \leq 1} \left( \sum |x'(x_k)|^r \right)^{1/r};
\]
as usual \( \pi_r(T) := \inf c \) denotes the \( r \)-summing norm of \( T \). With the representations
\[
\phi : E \to \ell_\infty(B_{E'}) \quad \text{as in (A)}
\]
\[
\psi : F \to L_r(\nu) \quad \text{as in (B)}
\]
this inequality transfers into

\[ \left\| \left( \sum |\psi(Tx_k)|^r \right)^{1/r} \right\|_{L_r(\nu)} \leq \pi_r(T) \left\| \left( \sum |\phi(x_k)|^r \right)^{1/r} \right\|_{\ell_\infty(B_{E'})}. \]

Since $\ell_\infty(B_{E'})$ is $r$-convex and $L_r(\nu)$ $r$-concave (with constants 1), Theorem 2 (see also Remark 1, (4)) gives a positive $\varphi \in \ell_\infty(B_{E'})'$ of norm $\leq 1$ such that

\[ \|Tx\|^r \leq \pi_r(T)' \varphi\left( |\langle \cdot, x \rangle|^r \right); \]

this reproves the Grothendieck-Pietsch domination theorem (clearly, $\varphi$ defines a Borel measure on the weak*-compact set $B_{E'}$).

4.2 The Maurey-Rosenthal theorem for $r$-convex homogeneous operators with values in $r$-concave quasi Banach function spaces

Let $T : E \rightarrow Y(\nu)$ be a homogeneous operator, where $E$ is a quasi Banach space and $Y(\nu)$ a quasi Banach function space. As in the linear and normed case, we call $T$ $r$-convex ($0 < r < \infty$) if there is a $c \geq 0$ such that for all $x_1, \ldots, x_n \in E$

\[ \left\| \left( \sum |Tx_k|^r \right)^{1/r} \right\|_Y \leq c \left( \sum \|x_k\|^r_{E} \right)^{1/r}; \]

the best constant $c$ is denoted by $M^{(r)}(T)$.

A theorem of Krivine [11] (see [13, II,1.d.12]) states that for $1 \leq r < \infty$ every $r$-convex operator $T$ from a Banach space $E$ into an $r$-concave Banach lattice $Y$ allows a factorization

\[ E \xrightarrow{T} Y \xleftarrow{R} L_r(\nu) \]

\[ \nu \text{ some measure} \]

\[ R, S \text{ operators} \]

\[ S \text{ positive.} \]

For a smaller class of $Y$'s the Maurey-Rosenthal theorem shows that $S$ can be chosen to be better: every $r$-convex operator $T : E \rightarrow L_p(\nu)$ ($1 \leq p < r < \infty$) has a factorization $T = SR$ as above, where $S : L_r(\nu) \rightarrow Y$ is a positive multiplication operator.

In view of the fact that every Banach lattice under weak additional assumptions is isomorphic to a Banach function space (see Section 1) the following result in a sense combines these two theorems.

**Corollary 2.** For $0 < r < \infty$ let $T$ be an $r$-convex homogeneous operator from a quasi Banach space $E$ into an $r$-concave quasi Banach function space $Y(\nu)$. Then there is a weight

\[ 0 \leq \omega \in L_0(\nu) \text{ with } \sup_{\|y\|_{L_r(\nu)} \leq 1} \|\omega^{1/r}y\|_Y \leq M^{(r)}(T) M_{(r)}(Y) \]

such that for all $x \in E$

\[ \left( \int \frac{|Tx|^r}{\omega} \ d\nu \right)^{1/r} \leq \|x\|_E. \]

If $T$ is moreover linear and $1 \leq r < \infty$, then it factorizes as follows:
Under the additional assumptions that $T$ is sublinear, $1 < r$ and $(Y^\times)^{\prime\prime}$ is reflexive, this result was proved by Garcia-Cuerva in [8, Theorem 2.9].

**Proof.** Clearly, the proof of the first statement is an immediate consequence of Theorem 2 (represent $E$ as in (B) and $Y$ as in (C)), and for the factorization define

$$g := \omega^{1/r} \quad \text{and} \quad Rx := \frac{Tx}{g}, \quad x \in E.$$ 

We only sketch the

**Corollary 3.** Let $1 \leq r < \infty$ and $E$ be a Banach space. Then the following are equivalent:

1. Every $r$-summing operator on $E'$ is 1-summing ($=: \text{the identity on } E'$ is $(r, 1)$-mixing).

2. Every operator $T : E \to Y$, $Y$ a Banach lattice, is $r$-convex.

3. For every operator $T : E \to Y(\nu)$, $Y(\nu)$ an $r$-concave Banach function space, there is some $0 \leq \omega \in L_0(\nu)$ with $\sup_{\|y\|_{L_r(\nu)}} \|\omega^{1/r}y\|_Y < \infty$ such that for all $x \in E$

$$\left( \int \frac{|Tx|^r}{\omega} d\nu \right)^{1/r} \leq \|x\|_E.$$

We only sketch the

**Proof.** If one replaces the $Y$’s in (3) by the class of all $L_p$’s ($1 \leq p < r < \infty$), then the equivalence (1) $\Leftrightarrow$ (3) is a well-known result of [15] (see also [3, 32.6 and 32.5]). Since (2) $\Leftrightarrow$ (3) follows from Corollary 2, it remains to check the implication (1) $\Leftrightarrow$ (2): note first that by Maurey’s result (just mentioned) there is a $c \geq 0$ such that $M(r)(T) \leq c \|T\|$ for all operators $T : E \to \ell_1$. The use of conditional expectation operators shows that this inequality also holds for all $T : E \to L_1(\nu)$, $\nu$ an arbitrary measure. But then Krivine’s localization technique from [11] (see also [3, II, the proof of 1.f.4]) assures that every $T : E \to Y$, $Y$ an arbitrary Banach lattice, is $r$-convex.

**Remark 3.** Here we collect some well-known statements on the class of all Banach spaces $E$ satisfying (1):

(a) Each $E$ such that $E'$ has cotype 2 for $r = 2$ satisfies (1). Conversely, if $E$ for $r = 2$ satisfies (1), then $E'$ has cotype $2 + \varepsilon$ for all $\varepsilon > 0$. Moreover, for $1 < q < 2$ each $E$ such that $E'$ has cotype $q'$ satisfies (1) for $r = q + \varepsilon$, $\varepsilon > 0$. See [13] and [14].

(b) For $1 \leq r < 2$ each $E$ with stable type $r$ satisfies (1) (see again [13]). Recall that $L_q(\mu)$ for $1 \leq r < q \leq 2$ has stable type $r$. 

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(c) Each Banach lattice $E$ with (1) is $r$-convex: by trace duality (1) holds if and only if every $T : \ell_\infty \to E'$ is $r$-summing (see e.g. [3, 32.2]) which by a result of Maurey from [14] (see also [13, II, 1.d.10]) implies that $E'$ is $r'$-concave, hence $E$ is $r$-convex. For $r = 2$ a Banach lattice satisfies (1) if and only if $E'$ has cotype 2 if and only if $E$ is 2-convex (see [14] and [13, II, 1.f.16]).

Another important result of [11] (see also [13, II, 1.d.9]) shows that for all positive operators $T : E \to Y$ between two Banach lattices $E$ and $Y$

$$
\|\left(\sum |Tx_k|^r\right)^{1/r}\|_Y \leq \|T\|\|\left(\sum |x_k|^r\right)^{1/r}\|_E \text{ for all } x_1, \ldots, x_n \in E.
$$

Hence, if $E$ is $r$-convex, then $M^{(r)}(T) \leq M^{(r)}(E)\|T\|$ which together with Corollary 2 proves the implications (1) $\rightsquigarrow$ (2) $\rightsquigarrow$ (3) of the following result – the remaining implication (3) $\rightsquigarrow$ (1) will be shown in a more general context in [5].

Corollary 4. Let $1 \leq r < \infty$ and $E$ be a Banach lattice. Then the following are equivalent:

1. $E$ is $r$-convex.
2. Every positive operator $T : E \to Y$, $Y$ a Banach lattice, is $r$-convex.
3. Each positive operator $T : E \to Y(\nu)$, $Y(\nu)$ an $r$-concave Banach function space, satisfies (3) of Corollary 2.

4.3 The Maurey-Rosenthal theorem for $r$-concave homogeneous operators on $r$-convex quasi Banach function spaces

Let us now consider the dual result of Corollary 1. A homogeneous operator $T : X(\mu) \to F$, $X(\mu)$ a quasi Banach function space and $F$ a quasi Banach space, is said to be $r$-concave ($0 < r < \infty$) if there is a $c \geq 0$ such that for all $x_1, \ldots, x_n \in X$

$$
\left(\sum \|Tx_k\|_F^r\right)^{1/r} \leq c \left(\sum |x_k|^r\right)^{1/r} \|x\|_X,
$$

and the best $c$ in this inequality is denoted by $M^{(r)}(T)$.

Corollary 5. For $0 < r < \infty$ let $T$ be an $r$-concave homogeneous operator from an $r$-convex quasi Banach function space $X(\mu)$ into a quasi Banach space $F$. Then there is a linear functional $0 \leq \varphi : X'^*(\mu) \to \mathbb{R}$ with $\sup_{\|x\|_X \leq 1} \varphi(|x|^r)^{1/r} \leq M^{(r)}(T)M^{(r)}(X)$ and such that for all $x \in X(\mu)$

$$
\|Tx\|_F \leq \varphi(|x|^r)^{1/r}.
$$

If $X$ is moreover $\sigma$-order continuous, $T$ linear and $1 \leq r < \infty$, then $T$ factorizes as follows:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & F \\
L_r(\mu) \searrow & & \nearrow R \\
& M_f & \\
\end{array}
$$

$M_f$ a positive multiplication operator

$R$ an operator

$\|M_f\|\|R\| \leq M^{(r)}(T)M^{(r)}(X)$.
Proof. Again, represent $X$ as in (C), $F$ as in (B) and use Theorem 2. It remains to prove that for $\sigma$-order continuous $X$ linear $T$ factorize as indicated: with the function $\omega$ representing $\varphi \in (X^r)' = (X^r)\times$ define $f := \omega^{1/r}$, and

$$R_0 : (\text{range } M_f, \| \cdot \|_{L_r}) \rightarrow F, R_0(f) := Tx.$$  

Then $\| M_f \| \leq M_r(T) M^{(r)}(X)$, and since $\text{range } M_f = \{ x \in L_r(\mu) \mid x_{|f=0} = 0 \}$ and $L_r(\mu) = \text{range } M_f \oplus_r \{ x \mid x_{|f>0} = 0 \}$, the operator $R_0$ has an extension $R$ to all of $L_r(\mu)$ with $\| R \| \leq 1$. Clearly, $T = RM_f$. \hfill \square

Remark 4. For $1 \leq r < \infty$ and sublinear $T$ between Banach spaces this result is due to Garcia-Cuerva [8, Theorem 2.3]; for $r = 2$ and linear $T$ the second part is mentioned without proof in [21] – but note that there the assumption on the $\sigma$-order continuity of $X$ is missing which makes the statement false: assume that each continuous functional $\varphi : X \rightarrow \mathbb{K}$ factorizes as above: $\varphi = RM_f$. Represent $R$ by a function $g \in L_r'((\mu)).$ Then for all $x \in X$

$$\varphi(x) = \int gfx \, d\mu,$$

hence $\varphi = gf \in X^\times$. But $X' = X^\times$ implies that $X$ is $\sigma$-order continuous (see e.g. [13, II,p.29]).

The dual statement of Corollary 3 is

**Corollary 6.** Let $1 \leq r < \infty$ and $F$ be a Banach space. Then the following are equivalent:

1. Every $r'$-summing operator on $F$ is 1-summing (=: the identity on $F$ is $(r',1)$-mixing).

2. Every operator $T : X \rightarrow F$, $X$ a Banach lattice, is $r$-concave.

3. For every operator $T : X(\mu) \rightarrow F$, $X(\mu)$ an $r$-convex quasi Banach function space, there is some linear $0 \leq \varphi : X' \rightarrow \mathbb{R}$ with $\sup_{\| x \|_X \leq 1} \varphi(|x|^r)^{1/r} < \infty$ such that for all $x \in X$

$$\|Tx\|_F \leq \varphi(|x|^r)^{1/r}.$$

Again we only sketch the

Proof. As mentioned in Remark 3 (c), statement (1) by trace duality is equivalent to the fact that each operator $T : \ell_\infty \rightarrow F$ is $r$-summing ([3, 32.2]), and by the definitions an operator $T : \ell_\infty \rightarrow F$ is $r$-summing if and only if it is $r$-concave. On the other hand, by Krivine’s localization technique from [11] (see again the proof of [13, II,1.f.14]) each $T : \ell_\infty \rightarrow F$ is $r$-concave if and only if (2) holds. This shows that (1) $\Leftrightarrow$ (2). Clearly, (2) implies (3) by Corollary 3. In order to prove the implication (3) $\Leftarrow$ (2) we only have to check that each $T : \ell_\infty \rightarrow F$ is $r$-concave. But since $\ell_\infty$ is $r$-convex, this follows by an easy calculation from the inequality in (3). \hfill \square

Finally, we mention
Corollary 7. Let \( 1 \leq r < \infty \) and \( F \) be a Banach lattice. Then the following are equivalent:

(1) \( F \) is \( r \)-concave.

(2) Every positive operator \( T : X \rightarrow F \), \( X \) a Banach lattice, is \( r \)-concave.

(3) Each positive operator \( T : X(\mu) \rightarrow F \), \( X(\mu) \) an \( r \)-convex Banach function space over \( \mu \), satisfies (3) of Corollary 6.

The implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) follow as in the proof of Corollary 2 and by Theorem 2; for (3) \( \Rightarrow \) (1) note first that each positive \( T \) : \( \ell_\infty \rightarrow F \) by assumption is \( r \)-summing (= \( r \)-concave) which by Maurey’s result already mentioned in Remark 3, (c) gives (1).

4.4 The Maurey-Rosenthal theorem for operators between vector-valued quasi Banach function spaces

Finally, we state a variant of Theorem 2 for vector-valued quasi Banach spaces \( X(\mu, E) \) as described at the beginning of this section – the following consequence of Theorem 2 and the representation given in (E) includes Corollary 2 and 5 as special cases.

Corollary 8. Let \( 0 < r < \infty \). Assume that \( X(\mu, E) \) and \( Y(\nu, F) \) are vector-valued quasi Banach spaces, \( X(\mu) \) \( r \)-convex and \( \sigma \)-order continuous, and \( Y(\nu) \) \( r \)-concave. Then for each homogeneous operator \( T : X(\mu, E) \rightarrow Y(\nu, F) \) such that for all \( x_1, \ldots, x_n \in X(\mu, E) \)

\[
\left\| \left( \sum \|Tx_k\|_F^r \right)^{1/r} \right\|_Y \leq \left\| \left( \sum \|x_k\|_E^r \right)^{1/r} \right\|_X
\]

(4.1)

there are weights

\[
0 \leq \omega_1 \in L_0(\mu) \quad \text{with} \quad \sup_{\|x\|_X \leq 1} \|\omega_1^{1/r} x\|_{L_r(\mu)} \leq M^r(X)
\]

\[
0 \leq \omega_2 \in L_0(\nu) \quad \text{with} \quad \sup_{\|y\|_{L_r(\nu)} \leq 1} \|\omega_2^{1/r} y\|_Y \leq M^r(Y)
\]

such that for all \( x \in X \)

\[
\int \frac{\|Tx\|_F^r}{\omega_2} \, d\nu \leq \int \|x\|_E^r \omega_1 \, d\mu.
\]

Clearly, under appropriate additional assumptions this result can also be formulated as a factorization theorem.

Let us give an example in the setting of Lorentz spaces: assume that \( 0 < p_1 < r < q_1 < \infty \), \( 0 < p_2 \leq r \leq q_2 \leq \infty \). Then each homogeneous operator \( T : L_{q_1,q_2}(\mu, E) \rightarrow L_{p_1,p_2}(\nu, F) \) (\( E \) and \( F \) two quasi Banach spaces) which satisfies a vector-valued norm inequality

\[
\left\| \left( \sum \|Tx_k\|_F^r \right)^{1/r} \right\|_{L_{p_1,p_2}} \leq \left\| \left( \sum \|x_k\|_E^r \right)^{1/r} \right\|_{L_{q_1,q_2}},
\]

satisfies a weighted norm inequality

\[
\int \frac{\|Tx\|_F^r}{\omega_2} \, d\nu \leq \int \|x\|_E^r \omega_1 \, d\mu
\]

(and vice versa).
As already mentioned, in the scalar valued case positive operators \( T : X \rightarrow Y \) between Banach lattices satisfy for all \( 1 \leq r < \infty \) a vector-valued norm inequality
\[
\| (\sum |Tx_k|^r)^{1/r} \|_Y \leq \| T \| \| (\sum |x_k|^r)^{1/r} \|_X ;
\]
for \( r = 2 \) the famous Krivine-Grothendieck inequality states that this is even true for all operators (see [11] and [13, II, 1.f.14]).

**Remark 5.** For which pairs \((E, F)\) of Banach spaces is (4.1) satisfied for all operators \( T : L_q(\mu, E) \rightarrow L_p(\nu, F) \) acting between spaces of Bochner integrable functions? For which pairs of Banach lattices \((E, F)\) does this hold for all positive \( T \)?

In [5] it is shown that whenever \( E \) or \( F \) has the approximation property (or is \( K \)-convex or a Banach lattice), then the following are equivalent:

1. \( E' \) and \( F \) have cotype 2.

2. There is \( c \geq 0 \) such that for all \( n \) and all operators \( T : \ell^\infty(E) \rightarrow \ell^1(F) \)
\[
\| (\sum \|Tx_k\|_E^2)^{1/2} \|_{\ell^1} \leq c \| T \| \| (\sum \|x_k\|_E^2)^{1/2} \|_{\ell^\infty} \text{ for all } x_1, \ldots, x_m \in \ell^1_n(E).
\]

Moreover, we know from a monotonicity argument in [3] that (1) implies (2) if one replaces \( \ell^1_n(F) \) by \( \ell^\infty_n(F) \) and \( \ell^\infty_n(E) \) by \( \ell^p_n(E) \) \((1 \leq p, q \leq \infty)\); using conditional expectation operators, it hence can be checked that if \( E \) or \( F \) has the approximation property (or is \( K \)-convex or a Banach lattice) and \( E', F \) have cotype 2, then there is a constant \( c \geq 0 \) such that for all \( 1 \leq p, q < \infty \) and all operators \( T : L_q(\mu, E) \rightarrow L_p(\nu, F) \)
\[
\| (\sum \|Tx_k\|_E^2)^{1/2} \|_{L_p(\nu)} \leq c \| T \| \| (\sum \|x_k\|_E^2)^{1/2} \|_{L_q(\mu)} \text{ for all } x_1, \ldots, x_n \in L_q(\mu, E).
\]
Hence, if moreover \( 1 \leq p < r = 2 < q < \infty \), then all such \( T \) allow a factorization as in Corollary 3.

A result which will be shown in [6] states that for two Banach lattices \( E, F \) and \( 1 \leq p < r < q < \infty \) all positive operators \( T : L_q(\mu, E) \rightarrow L_p(\nu, F) \) satisfy (4.1) (up to some constant) if and only if \( E \) is \( r \)-convex and \( F \) \( r \)-concave.

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