Differentiable Fuzzy $\mathcal{ALC}$: A Neural-Symbolic Representation Language for Symbol Grounding

Xuan Wu, Xinhao Zhu, Yizheng Zhao, Xinyu Dai

National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, 210023, China

Abstract

Neural-symbolic computing aims at integrating robust neural learning and sound symbolic reasoning into a single framework, so as to leverage the complementary strengths of both of these, seemingly unrelated (maybe even contradictory) AI paradigms. The central challenge in neural-symbolic computing is to unify the formulation of neural learning and symbolic reasoning into a single framework with common semantics, that is, to seek a joint representation between a neural model and a logical theory that can support the basic grounding learned by the neural model and also stick to the semantics of the logical theory. In this paper, we propose differentiable fuzzy $\mathcal{ALC}$ (DF-$\mathcal{ALC}$) for this role, as a neural-symbolic representation language with the desired semantics. DF-$\mathcal{ALC}$ unifies the description logic $\mathcal{ALC}$ and neural models for symbol grounding; in particular, it infuses an $\mathcal{ALC}$ knowledge base into neural models through differentiable concept and role embeddings. We define a hierarchical loss to the constraint that the grounding learned by neural models must be semantically consistent with $\mathcal{ALC}$ knowledge bases. And we find that capturing the semantics in grounding solely by maximizing satisfiability cannot revise grounding rationally. We further define a rule-based loss for DF adapting to symbol grounding problems. The experiment results show that DF-$\mathcal{ALC}$ with rule-based loss can improve the performance of image object detectors in an unsupervised learning way, even in low-resource situations.

Introduction

For decades, trends in the computational modeling of intelligent behavior have followed a recurring pattern, cycling between a primary focus on symbolic methods and sub-symbolic methods. The symbolic approach believes that the best way to nurture an AI is to feed it human-readable information related to what you think it needs to know to become capable of solving a particular human-level intelligence task. On the other hand, the sub-symbolic approach admits that human-based information formats are not always the best fit for an AI, and encourages feeding raw data into the AI so that it can construct its own implicit knowledge and have its own way to interpret the meaning of the data. To find an in-between solution of symbolic and sub-symbolic methods, neural-symbolic computing (Hammer and Hitzler 2007) has its own way to interpret the meaning of the data. To find an in-between solution of symbolic and sub-symbolic methods, neural-symbolic computing (Hammer and Hitzler 2007) isKindOf i.e., hasKindOf($x_1, A$) ∧ SimilarEntity($x_1, x_2$) ⇒ hasKindOf($x_2, A$). The SimilarEntity predicate is learned from the feature of the entity, under the hypothesis that the similar structure or context the pair of entities have, the higher probability that they are the same entity, which is similar to the distributional hypothesis, which is not extracted in a human-readable way. So the result is monitored by the distribution of data, and the knowledge should be used in a soft way. While declarative knowledge (e.g. armChair($x_1$) ⇒ has($x_1, ChairArm$) ∧ has($x_1, Chair$) should be compulsory for AI system to admit to human society. So the encoding of declarative knowledge should not lose the logical semantics. Compared to past works, we consider two

Copyright © 2022, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.
symbol grounding ways between sub-symbolic methods and two kinds of knowledge separately.

Another bottleneck of symbol grounding is knowledge representation. Existing neural-symbolic works ignore the importance of choosing a logic language that can “optimal” balance the expressive power with computational complexity, as using declarative and knowledge requires the full logic semantics to be maintained in the neural-symbolic representation. Knowledge graph (KG) embedding methods (Lamb et al. 2020; Xie et al. 2019; Zhang et al. 2021), which are generally based on resource description framework (RDF), achieve great success in many applications, especially in link prediction, while deductive/abductive reasoning in the graph is inefficient (as inductive reasoning should be done first) and not fully explainable. In other words, RDF-based KG has not enough expressivity to satisfy the requirement of neural-symbolic computing. While the expressive power of first-order logic (FOL) is too strong, accompanying undecidability. FOL-based neural-symbolic methods are mostly used in experimental tasks (Riegel et al. 2020; Badreddine et al. 2022) and ignore analyzing in what situations the semantics of FOL axiom can be lost in computing. Description logics (DLs) as a family of decidable segments of FOL, balance the expressive power and complexity. Besides, DLs as the core of ontology web language (OWL) are used in many applications, the deductive complexity of which is well-studied.

To adapt description logic to a neuro-symbolic paradigm, we develop a differentiable fuzzy language based on $\text{ALC}$, which can depict knowledge under unstable situations and band between sub-symbolic system and $\text{ALC}$ ontology without losing the logical semantics. Behind the choice of $\text{ALC}$ is a series of deliberations: 1) currently, $\text{EL}^+ +$ is the logic language with the highest expressive power with fully captured logic semantics in existing work (Kulmanov et al. 2019), but when considering the negation operator, all current knowledge embedding models cannot ensure the semantics are not lost; $\text{ALC}$ is with high expressive power as it introduces negation beyond $\text{EL}$, but there is no work banding the logic semantics of $\text{ALC}$ ontology with sub-symbolic methods; fuzzy $\text{ALC}$ based on Gödel semantics is decidable (Baader, Borgwardt, and Penaloza 2017) (Borgwardt, Distel, and Penaloza 2014a,b) without finite model property, which means that traditional tableau-based reasoning algorithm for infinitely valued fuzzy $\text{ALC}$ is not sound.

In this work, we propose differentiable fuzzy $\text{ALC}$ (DF-$\text{ALC}$) for symbol grounding, based on infinitely valued fuzzy $\text{ALC}$. Different from the existing differentiable fuzzy language (van Krieken, Acar, and van Harmelen 2022; Badreddine et al. 2022), we consider knowledge represented in a language that can contain more logical information in neuro-symbolic computing, and introduce two effective interpreting ways when given declarative/procedural knowledge.

The key contributions of this work can be summarized as follows:

- We present a representation language DF-$\text{ALC}$ which facilitates a sound and complete mechanism to revise the probabilistic semantics by a neural model according to a consistent $\text{ALC}$ ontology. This makes us the first to combine neuro-symbolic computing with description logic.
- Rather than ground by maximizing satisfiability. We designed rule-based loss, which helps fuzzy description logic adapt to revise grounding rationally, and improves the performance in semantic image interpretation.
- Experiments show that DF-$\text{ALC}$ can keep the reliable component of the perceptual grounding. Meanwhile, unknown situations are few to affect grounding, this further demonstrates that the semantics of DF-$\text{ALC}$ are solid in terms of crisp $\text{ALC}$ under OWA.
- We are the first to enable $\text{ALC}$ ontology to be compatible with sub-symbolic methods while not losing the logical semantics. The source code, alongside the experimental settings, is publicly accessible at https://anonymous.4open.science/r/DF-ALC.

Related Works

Our work is an intersection branch of ontology representation learning and neuro-symbolic computing. And we give a feasible way out for a symbol grounding problem — semantic image interpretation.

Neuro-symbolic Computing

Neural-symbolic computing aims at computing with both learning and reasoning abilities, to step towards the combination of symbolic and sub-symbolic systems. Current learning ability relies largely on differentiable programming to draw conclusions from observations and apply them, while current reasoning ability relies largely on logical programming to give conclusions inferred from premises and rules through deductive reasoning, give rules according to observations comprising premises and conclusions through inductive reasoning, and give premises that can interpret conclusions according to rules through abductive reasoning. So it comes with challenges in the integration and representation of these two kinds of programming paradigms. From the perspective of integration, research works differ in logical techniques that are mainly consumed. Neural-symbolic inductive logical programming (Wang, Mazaitis, and Cohen 2013) and statistical relational learning (e.g. Markov logic network (Richardson and Domingos 2006), probabilistic soft logic (Bach et al. 2017)) works seek to learn probabilistic logical rules from observations. This requires learning model parameters in a continuous space and the structure in a discrete space. SATNet (Wang et al. 2019) learns rules from labeled data by transforming the learning problem as SAT problem. To combine the ability of deductive reasoning, the first line of research learn to reason by modeling the inference procedure using neural networks or replacing logical computations with differentiable functions (Towell and Shavlik 1994; Hölldobler, Kalinke, 1994).
Hospedales, and Michael 2021; Cai et al. 2021), descriptive learning-based neural-symbolic works are proposed to explain observations according to theories, so abductive explanation can be used directly in knowledge-requiring downstream tasks. Two kinds of methods exist with different input requirements. The first kind focuses on coupling the metadata of an OWL ontology into an efficient graph, and then uses the generated corpus based on the graph as the input to the existing representation learning methods (Smaili, Gao, and Hoehndorf 2018, 2019, Chen et al. 2021). The second kind focuses on modeling the logical semantics of an OWL ontology. EL2Vec (Kulmanov et al. 2019) approximates geometric models for EL + + ontologies and has achieved an interpretable embedding for GO. E2R (Garg et al. 2019) can model the logical operators of intersection, union, negation, and universal quantifier, but fails to capture the distributive law. In all, there often comes a loss of the semantics of an OWL ontology in the transformation of most embedding methods (Smaili, Gao, and Hoehndorf 2018, 2019, Chen et al. 2021). Though the geometric construction method (Kulmanov et al. 2019) can preserve the logical semantics well in EL + +, the embedding may bring unexpected knowledge (unknown becomes true) because of the closed world assumption (CWA), and EL + + is not expressible as ALC studies in this work.

Semantic Image Interpretation

A symbolic grounding application of our work is semantic image interpretation (SII) (Hudelot, Mailot, and Thonnat 2008; Neumann and Möller 2008; Krishna et al. 2017; Donadello, Serafini, and Garcez 2017), which aims to generate a structured and human-readable description of the content of images. Current successful SII researches (Donadello, Serafini, and Garcez 2017) rely on background knowledge of the images. LTN (Donadello, Serafini, and Garcez 2017) models predicates and functions as neural networks and learns the representation through maximizing the satisfiability in a supervised way. The main struggle of these neural-symbolic works in leveraging logical knowledge to adapt to the symbol grounding problem is that the revision signal cannot be properly conveyed. Some early works (Hudelot, Atif, and Bloch 2008; Dasiopoulou, Kompat Matthews, and Strintzis 2009) revise image semantics solely with fuzzy description logic, the idea of which can be summarized as selecting plausible image scene descriptions (assumptions) ignoring disjoint cases, and then handle the inconsistency by removing assertions using methods e.g. reversed tableau expansion procedure. Though these fuzzy description logic-based methods can remain as much reliable parts of perceptual grounding as possible, these works achieve

### Table 1: AI Tasks aims to ground symbol system (relevant concepts) in raw data.

| Tasks                   | Raw Data                                                                 | Symbol System               | Description                                                                                       | Natural-symbolic Works                                      |
|-------------------------|--------------------------------------------------------------------------|------------------------------|--------------------------------------------------------------------------------------------------|-------------------------------------------------------------|
| Semantic Image Interpretation | images                                                                   |                              | Knowledge relates natural and logical relationships between objects and knowledge (relevance)   | Donadello, Serafini, and Garcez 2017                      |
| Named-entity Recognition | natural language sentence                                                |                              | Ground some concepts (relevance) to images, object concepts remain in images                    | Arvori et al., 2019                                       |
| Schema Matching          | two knowledge bases                                                       |                              | Solutions above schema matching                                                                  | Ohlmann et al., 2007                                       |
| Link Prediction          | graph-structured data, e.g., knowledge graph                              |                              | Ground the proposition (relevance) to entities from different sources with the same semantics    | Zhang et al., 2021                                        |

and Störr 1999] Rocktäschel and Riedel 2016, 2017; Diligenti, Gori, and Sacca 2017] Ebrahimi, Eberhart, and Hitzler 2021. But this neglects factual knowledge which bridges the physical world and the conceptual world, so the second line of research aims to find an interpretation (grounding) that satisfies theories which can be a mapping between these two worlds by encoding the satisfiability of theories in the loss function (Badreddine et al. 2022; Seraphi and Garcez 2016; Riegel et al. 2020; Topan, Kolnick, and St 2021; van Krieken, Acar, and van Harmelen 2023). The notable work Logical Tensor Network (LTN) (Badreddine et al. 2022) uses neural networks to represent the fuzzy function and predicates of theories, which is learned from labeled data. To solve the symbol grounding problem, LTN learns the interpretation with trained parameters that can maximize the satisfiability of theories. But these works cannot find explanations of observations according to theories, so abductive learning-based neural-symbolic works are proposed to use the explanations getting through abductive reasoning to promote the interpretability of the computing (Zhou 2019; Huang et al. 2020; Tsamoura, Hospedales, and Michael 2021; Cai et al. 2021). From the view of representation, some works are based on classical logic — propositional logic (Howell and Shavlik 1994; Zhou 2019; Tsamoura, Hospedales, and Michael 2021; Cai et al. 2021), description logic (Bühmann, Lehmann, and Westphal 2016; Eberhart et al. 2019; Ebrahimi, Eberhart, and Hitzler 2021), or first-order logic (Höldobler, Kalinke, and Störr 1999; Wang, Mazzaitis, and Cohen 2013) Rocktäschel and Riedel 2016; Serafini and Garcez 2016; Rocktäschel and Riedel 2017; Yang, Yang, and Cohen 2017; Evans and Grefenstette 2018; Sen et al. 2022), others are based on non-classical logic, such as fuzzy logic (Diligenti, Gori, and Sacca 2017; Riegel et al. 2020; van Krieken, Acar, and van Harmelen 2022), or probabilistic logic (Wang, Mazzaitis, and Cohen 2013) (Manhaeve et al. 2018). Our work tries to combine the deductive reasoning ability in a novel way, which follows the work of LTN. As stated in the introduction, when using declarative knowledge, data bias should be ignored, so different from works that only maximize the satisfiability of the knowledge base, we give a compulsory way to inject knowledge.

### Ontology Representation Learning

The main motivation for embedding OWL ontology into vector space is to transfer the knowledge to vectors so it can be used directly in knowledge-requiring downstream tasks. Two kinds of methods exist with different input requirements. The first kind focuses on coupling the metadata of an OWL ontology into an efficient graph, and then uses the generated corpus based on the graph as the input to the existing representation learning methods (Smaili, Gao, and Hoehndorf 2018, 2019, Chen et al. 2021). The second kind focuses on modeling the logical semantics of an OWL ontology. EL2Vec (Kulmanov et al. 2019) approximates geometric models for EL + + ontologies and has achieved an interpretable embedding for GO. E2R (Garg et al. 2019) can model the logical operators of intersection, union, negation, and universal quantifier, but fails to capture the distributive law. In all, there often comes a loss of the semantics of an OWL ontology in the transformation of most embedding methods (Smaili, Gao, and Hoehndorf 2018, 2019, Chen et al. 2021). Though the geometric construction method (Kulmanov et al. 2019) can preserve the logical semantics well in EL + +, the embedding may bring unexpected knowledge (unknown becomes true) because of the closed world assumption (CWA), and EL + + is not expressible as ALC studies in this work.
most of the expected revisions and are specific to the ontology designed for image scene classification, which lacks generalization ability.

Preliminaries
The Description Logic $\mathcal{ALC}$
Let $N_C$ and $N_R$ be pairwise disjoint and countably infinite sets of concept names and role names, respectively. $\mathcal{ALC}$-concepts are inductively constructed based on the following syntax rule:

$$C, D \to \top \mid \bot \mid A \mid \neg C \mid C \cap D \mid C \cup D \mid \exists r.C \mid \forall r.C,$$

where $A \in N_C$, $r \in N_R$, and $C$ and $D$ range over concepts. A concept of the form $A \in N_C$ is called atomic, otherwise, it is compound. An ontology $\mathcal{O}$ consists of a TBox and an ABox. An $\mathcal{ALC}$-TBox $\mathcal{T}$ is a finite set of axioms of the form:

- $C \sqsubseteq D$ (concept inclusion), and
- $C \equiv D$ (concept equivalence),

where $C$ and $D$ are concepts. The disjointness between $C$, $D$ is $C \cap D \subseteq \bot$. We use the axiom $C \equiv D$ as abbreviation for $C \sqsubseteq D$ and $D \sqsubseteq C$.

Let $N_I$ be disjoint and countably infinite sets of individual names, while an $\mathcal{ALC}$-ABox $\mathcal{A}$ is a finite set of crisp assertions of the form:

- $a : C$ (concept assertion), and
- $(a, b) : r$ (role assertion),

where $C$ is a concept, $r$ is a role name, and $a, b$ are individuals from $N_I$.

An $\mathcal{ALC}$ ontology is comprised of an $\mathcal{T}$ and $\mathcal{A}$, denoted as $\mathcal{O} = (\mathcal{T}, \mathcal{A})$. The signature of $\mathcal{O}$ is $\text{sig}(\mathcal{O}) = N_C \cup N_R \cup N_I$.

The semantics of $\mathcal{O}$ is defined in terms of an interpretation $\mathcal{I} = (\Delta^\mathcal{T}, \mathcal{I})$, where $\Delta^\mathcal{T}$ denotes the domain of the interpretation (a non-empty crisp set), and $\mathcal{I}$ denotes the interpretation function, which assigns to every concept name $A \in N_C$ a set $A^\mathcal{I} \subseteq \Delta^\mathcal{T}$, and to every role name $r \in N_R$ a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{T} \times \Delta^\mathcal{T}$. The interpretation function $\mathcal{I}$ is inductively extended to concepts as follows:

$$\top^\mathcal{I} = \Delta^\mathcal{T}, \bot^\mathcal{I} = \emptyset, \neg(C)^\mathcal{I} = \Delta^\mathcal{T} \setminus C^\mathcal{I},$$

$$(C \cap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}, (C \cup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I},$$

$$\exists r.C)^\mathcal{I} = \{a \in \Delta^\mathcal{T} \mid \exists b. (a, b) \in r^\mathcal{I} \land b \in C^\mathcal{I}\},$$

$$\forall r.C)^\mathcal{I} = \{a \in \Delta^\mathcal{T} \mid \forall b. (a, b) \in r^\mathcal{I} \rightarrow b \in C^\mathcal{I}\}.$$

Let $\mathcal{I}$ be an interpretation. A concept inclusion $C \sqsubseteq D$ is $\text{true} \text{ in } \mathcal{I}$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$. A concept assertion $a : A$ is $\text{true} \text{ in } \mathcal{I}$ iff $a^\mathcal{I} \subseteq A^\mathcal{I}$. A role assertion $(a, b) : r$ is $\text{true} \text{ in } \mathcal{I}$ iff $(a^\mathcal{I}, b^\mathcal{I}) \subseteq r^\mathcal{I}$. $\mathcal{I}$ is a model of an ontology $\mathcal{O}$, write $\mathcal{I} \models \mathcal{O}$, iff every axiom in $\mathcal{O}$ is $\text{true} \text{ in } \mathcal{I}$. An axiom $\beta$ is entailed by an ontology $\mathcal{O}$, write $\mathcal{O} \models \beta$, iff $\beta$ is true in every model $\mathcal{I}$ of $\mathcal{O}$. An ontology $\mathcal{O}$ is consistent (true) if there exists a model $\mathcal{I}$ of $\mathcal{O}$. A concept $C$ is satisfiable w.r.t. $\mathcal{O}$ if there exists a model $\mathcal{I}$ of $\mathcal{O}$ and some $d \in \Delta^\mathcal{T}$ with $d \in C^\mathcal{I}$. A concept assertion $a : C$ is satisfiable in $\mathcal{I}$ iff $a^\mathcal{I} \in C^\mathcal{I}$. A role assertion $(a, b) : r$ is satisfiable in $\mathcal{I}$ iff $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$.

Other basic reasoning problems are polynomial-time reducible to the satisfiability problem. A concept inclusion $C \subseteq D$ is true in $\mathcal{I}$ iff the concept $C \cap \neg D$ is unsatisfiable in $\mathcal{I}$. The retrieval problem of computing the instantiation of concept $C$ is polynomial-time reducible to that of checking the satisfiability of $a : C$.

Under the interpretation $\mathcal{I}$, concepts and roles are mapped into crisp sets in $\Delta^\mathcal{T}$, so the vagueness cannot be modeled.

Fuzzy $\mathcal{ALC}$
Fuzzy set theory and fuzzy logic were proposed by Zadeh (Zadeh 1965) to manage imprecise and vague knowledge. Based on fuzzy set theory (Zadeh 1965), a fuzzy set $X$ w.r.t. an universe is characterized by a membership function $\mu_X : U \to [0, 1]$. Each element $u \in U$ is assigned with an $X$-membership degree $\mu_X(u)$. In fuzzy logic, $\mu_X(u)$ is the truth-value of the statement ‘$u$ is $X$’.

Fuzzy $\mathcal{ALC}$ retains the same syntax with $\mathcal{ALC}$, only semantics changes. Here, we follow fuzzy $\mathcal{ALC}$ proposed in (Straccia 2001), which is based on Gödel logic (Dummett 1959).

A fuzzy interpretation (also called grounding here) $\mathcal{I}$ consists of a non-empty domain $\Delta^\mathcal{I}$ and an interpretation function $\mathcal{I}$ defined as: (1) an individual $a$ is interpreted by $\mathcal{I}$ as an element $a^\mathcal{I} \in \Delta^\mathcal{I}$, and; (2) a concept $C$ is interpreted by $\mathcal{I}$ as a fuzzy set $C^\mathcal{I} : \Delta^\mathcal{I} \to [0, 1]$, and; (3) a role $r$ is interpreted by $\mathcal{I}$ as a fuzzy set $r^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0, 1]$.

The fuzzy interpretation function $\mathcal{I}$ is inductively extended to concepts as follows, for all $a \in \Delta^\mathcal{I}$:

$$\top^\mathcal{I}(a) = 1, \bot^\mathcal{I}(a) = 0, (\neg C)^\mathcal{I}(a) = 1 - C^\mathcal{I}(a),$$

$$(C \cap D)^\mathcal{I}(a) = \min\{C^\mathcal{I}(a), D^\mathcal{I}(a)\},$$

$$(C \cup D)^\mathcal{I}(a) = \max\{C^\mathcal{I}(a), D^\mathcal{I}(a)\},$$

$$\exists r.C)^\mathcal{I}(a) = \sup_{b \in \Delta^\mathcal{I}} \{\min\{r^\mathcal{I}(a, b), C^\mathcal{I}(b)\}\},$$

$$\forall r.C)^\mathcal{I}(a) = \inf_{b \in \Delta^\mathcal{I}} \{\max\{1 - r^\mathcal{I}(a, b), C^\mathcal{I}(b)\}\}.$$

A fuzzy $\mathcal{ALC}$ TBox is a finite set of fuzzy inclusion of the form $C \subseteq D$. $C \subseteq D$ is true (i.e., truth-value is 1) in $\mathcal{I}$ (or we say $\mathcal{I}$ satisfies $C \subseteq D$) iff

$$\forall a \in \Delta^\mathcal{I}, C^\mathcal{I}(a) \leq D^\mathcal{I}(a) \tag{6}$$

We say that two concepts $C$ and $D$ are fuzzy equivalent (C $\equiv$ D) when $C^\mathcal{I}(a) = D^\mathcal{I}(a)$ for all $a \in \Delta^\mathcal{I}$.

A fuzzy $\mathcal{ALC}$ ABox is a finite set of fuzzy assertion of the form $(a : C) \ni n$ or $(a, b) : r \ni n$, where $\ni n$ stands for $\geq$, $>$, $\leq$, $\leq$, and $n \in [0, 1]$ is the truth value. Formally, a fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy assertion $(a : C) \ni n$ (resp. $(a, b) : r \ni n$) iff $C^\mathcal{I}(a) \ni n$ (resp. $r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \ni n$). For simplicity, we write a fuzzy assertion as $\varphi = \{\phi \ni n\}$, where $\phi$ is $(a : C)$ or $(a, b) : r$.

A fuzzy interpretation $\mathcal{I}$ is a model of a fuzzy ontology $\mathcal{O}$, write $\mathcal{I} \models \mathcal{O}$, iff $\mathcal{I}$ satisfies each axiom in $\mathcal{O}$. A fuzzy ontology $\mathcal{O}$ fuzzy entails a fuzzy assertion $\varphi$, write $\mathcal{O} \models \varphi$ iff every model of $\mathcal{O}$ also satisfies $\varphi$.

A crisp $\mathcal{ALC}$ ontology is a specialization of fuzzy $\mathcal{ALC}$ ontology, and can easily be extended to a fuzzy ontology by assigning truth value 1 to assertions.
Ontology-based Semantic Image Interpretation

Let \( S = \{s_1, \ldots, s_n\} \) be a set of segments (a segment is a set of contiguous pixels) returned by a low-level analysis (e.g., object detection) of picture \( P \). Given an ontology \( O \), the semantic image interpretation task can be formed as labeling picture \( P \) with an interpretation \( I \) defined in the domain \( S \), which maps each segment \( s \in S \) to a set of values \( \{C^I(s) \mid C \text{ is any concept in } O\} \).

**Differentiable Fuzzy ALC**

DF-\(\text{ALC} \) is an extension of fuzzy \(\text{ALC} \). The semantics of ontology represented in DF-\(\text{ALC} \) can be infused into symbol grounding in a continuous space. To solve a symbol grounding problem, such as the semantic image interpretation problem shown in Figure 1, where the concept symbols are about cat, bird and their components and the relation symbol isPartOf, an input, the neural model does the low-level analysis for the image, which gives the wrong grounding for the image.

Ontology gives background knowledge about the cat and bird. The neural model does the low-level analysis for the image, which gives the wrong grounding for the image.

![Figure 1: An ontology-based semantic image interpretation example utilizing DF-\(\text{ALC} \). Ontology gives background knowledge about the cat and bird. The neural model does the low-level analysis for the image, which gives the wrong grounding for the image.](image)

**Normalization**

Given an \(\text{ALC} \) ontology \( O = \langle T, A \rangle \), concepts in \( T \) are transformed into negation normal forms using De Morgan’s Laws until all concepts have no indirect negation. Then we recursively apply NF1-9 in Figure 2 until all the axioms are in the forms in Figure 3.

| NF 1 | \( D \sqsubseteq \hat{E} \Rightarrow \hat{D} \sqsubseteq A, A \sqsubseteq \hat{E} \) |
| NF 2 | \( \hat{D} \cap C \sqsubseteq B \Rightarrow \hat{D} \sqsubseteq A, A \cap C \sqsubseteq B \) |
| NF 3 | \( C \cup \hat{D} \sqsubseteq B \Rightarrow \hat{D} \sqsubseteq B, C \sqsubseteq B \) |
| NF 4 | \( \exists r. \hat{D} \sqsubseteq B \Rightarrow \hat{D} \sqsubseteq A, \exists r. A \sqsubseteq B \) |
| NF 5 | \( \forall r. \hat{D} \sqsubseteq B \Rightarrow \hat{D} \sqsubseteq A, \forall r. A \sqsubseteq B \) |
| NF 6 | \( B \sqsubseteq D \cap E \Rightarrow B \sqsubseteq D, B \sqsubseteq E \) |
| NF 7 | \( B \sqsubseteq D \sqcup \hat{E} \Rightarrow B \sqsubseteq D \sqcup A, A \sqsubseteq \hat{E} \) |
| NF 8 | \( B \sqsubseteq \exists r. \hat{D} \Rightarrow A \sqsubseteq \hat{D}, B \sqsubseteq \exists r. A \) |
| NF 9 | \( B \sqsubseteq \forall r. \hat{D} \Rightarrow A \sqsubseteq \hat{D}, B \sqsubseteq \forall r. A \) |
| NF 10 (De Morgan’s Laws) | \( \neg(C \cap D) \equiv \neg C \sqcup \neg D, \neg(C \sqcup D) \equiv \neg C \sqcap \neg D \) |
| | \( \neg \exists r. C \equiv \forall r. \neg C, \neg \forall r. C \equiv \exists r. \neg C, \neg \forall r. C \equiv C \) |
| | \( \hat{D}, \hat{E} \) are complex concepts. So they are neither \( \top \), \( \bot \) nor concept names. |
| | \( A \) is a new introduced concept name. |
| | \( B \) is a concept name or a concept name with negation. |
| | \( C, D, E \) are arbitrary concepts. |

![Figure 2: Normalization rules for \(\text{ALC} \)](image)
nominal size in $|O|$ using the normalization described above such that (i) for every model $I$ of $O$, there exists a model $J$ of $O'$ such that $I$ is semantically equivalent to $J$ in $\text{sig}(O)$, denoted as $I \sim_{\text{sig}(O)} J$, and (ii) for every model $J$ of $O'$ there exists a model $I$ of $O$ such that $I \sim_{\text{sig}(O)} J$.

After normalization, we can see that formula in the form of Figure 3 has at most one logical operation except the subclass operator, so any normalized $\mathcal{ALC}$ ontology in Figure 2 can be efficiently used as the input to a neural network.

**Syntax**

The syntax of $\text{DF-ALC}$ mainly differs from fuzzy $\mathcal{ALC}$ in the definition of fuzzy assertion. For each fuzzy assertion $\varphi = \{ \phi \otimes n \}$, we set the $\otimes$ be $= \otimes$ by considering two assertions of the from $\phi \geq n$ and $\phi \leq n$. Then we introduce how to transform the given $\mathcal{ALC}$ ontology $O = (T, A)$ into a $\text{DF-ALC}$ ontology $\Gamma = (\hat{T}, A)$. Firstly, normalize ontology $O$ into $O'$ where only TBox of $O$ and the concept name set changed. Assign $\hat{T}'$ to the TBox of $\Gamma$. Fuzzy extend $A$ to be the $\text{ABox}$ of $\Gamma$ (i.e. transform each assertion $\phi$ in $A$ to be $\phi = 1$). The signature of $\Gamma$ is defined as the signature of $O'$, containing $N_C$, $N_I$, and $N_T$. To enable differentiable operators to transfer gradient information, we reform the fuzzy interpretation as differentiable fuzzy interpretation. The domain of the grounding $\hat{\Gamma}$ is $N_T$. The interpretation function of $\hat{\Gamma}$ is reformed as an embedding function, which embeds each concept name $C \in N_C$ into $|\Delta|$-dimensional vector, $C^\hat{\Gamma} = \mathbb{R}^{|\Delta|}$, and each role name $r \in N_R$ into a $|\Delta| 	imes |\Delta|$-dimensional matrix, $r^\hat{\Gamma} = \mathbb{R}^{|\Delta| \times |\Delta|}$. The ith item of $C^\hat{\Gamma}$ is the truth value of $C_i : C$, and the $(i,j)$th item of $r^\hat{\Gamma}$ is the truth value of $(\Delta_i, \Delta_j) : r$.

**Semantics**

$\text{DF-ALC}$ is a member of differentiable fuzzy logics \cite{vanKrieken2022}, which means the truth values of axioms are continuous, logical operators are interpreted into differentiable fuzzy operators, and the interpretation function is an embedding function. Except for the differentiable property, in semantics, $\text{DF-ALC}$ is the same with fuzzy $\mathcal{ALC}$. Given $\mathcal{ALC}$ ontology $\Gamma$, the properties that $\text{DF-ALC}$ holds are shown in Table 2. Only $C \sqcap \neg C \equiv \bot$ and $C \sqcup \neg C \equiv \top$ do not hold in $\text{DF-ALC}$ and fuzzy $\mathcal{ALC}$. But for any $C$ and $\mathcal{L}$, $(C \sqcup \neg C)^\hat{\Gamma} \geq 0.5$ and $(C \sqcap \neg C)^\hat{\Gamma} \leq 0.5$ hold.

The semantics of $\text{DF-ALC}$ is sound w.r.t crisp semantics under the open-world assumption. This is an extension of the soundness of fuzzy $\mathcal{ALC}$. We define the crisp transformation $\sharp(\cdot)$ of $\text{DF-ALC}$ assertion $\varphi$ into three-valued $\mathcal{ALC}$ assertion.

\begin{equation}
\sharp \varphi = \sharp\{\phi = n\} \mapsto \begin{cases} \text{unknown} & \text{when } 1 - \alpha \leq n \leq \alpha \\
\neg \phi & \text{when } n < 1 - \alpha \end{cases}
\end{equation}

where $\alpha \in [0, 0.5]$ is predefined according to the application, and $\neg \phi$ is $= \neg \phi$ or $(a, b) : = r$. For TBox axioms, $\sharp(\cdot)$ to fuzzy TBox axioms: $\sharp\{\psi \in T\} = \{\psi \in T\}$. So for $O = (T, A)$, $\sharp O = \sharp\{\varphi \in A\} \cup \{\psi \in T\}$.

**Proposition 1** (Soundness of the semantics) Let $\sharp O$ be an $\mathcal{ALC}$ ontology, and $\varphi$ be a fuzzy assertion. $O \models \varphi$ if and only if $\sharp O \models \sharp \varphi$ (i.e. fuzzy entailment is consistent with entailment in $\mathcal{ALC}$).

Proof. 1. $\Rightarrow$ Consider any crisp interpretation $\mathcal{I}$ that is a model of $\sharp O$. $\mathcal{I}$ can also be considered as a fuzzy interpretation that $C^\mathcal{I}(a) \in \{0, 0.5, 1\}$ and $r^\mathcal{I}(a, b) \in \{0, 0.5, 1\}$ hold. By induction on the structure of a concept $C$, $\mathcal{I}$ satisfies $a : C$ iff $C^\mathcal{I}(a) = 1$. $\mathcal{I}$ satisfies $a : C$ is unknown iff $C^\mathcal{I}(a) = 0.5$. And similarly for roles. Therefore, $\mathcal{I}$ is also a model of $\mathcal{O}$. And for every model of $\sharp O$, $\mathcal{I}$ satisfies $\sharp \varphi$, so $\sharp O \models \sharp \varphi$ holds.

2. $\Leftarrow$ If $\sharp O \models \sharp \varphi$, consider the crisp interpretation $\mathcal{I}$ discussed above, it is similar to proof that each model $\mathcal{I}$ of $\sharp O$, is also the model of $\mathcal{O}$, and satisfies $\varphi$, so we have $O \models \varphi$. To sum up, this proposition is proven to be true.

**Learning to Ground Symbols**

Given an $\mathcal{ALC}$ ontology $\mathcal{O}$, we have introduced how to transform it into the $\text{DF-ALC}$ ontology $\Gamma = (\hat{T}', A)$. Then, transform the perceptual grounding $\hat{\Gamma}'$ into a differentiable fuzzy interpretation of $\Gamma$, as the initialization of the grounding of $\Gamma$. Backpropagation on the grounding of $\Gamma$ can learn a model $\hat{\Gamma}''$ of $\Gamma$, which is also a model of $\mathcal{O}$ in the signature of $\mathcal{O}$, according to Theorem 1. The forward process is to compute the truth values of axioms in $\hat{T}'$, by maximizing the satisfiability of $\hat{T}'$ (minimizing the hierarchical loss in Equation 8).
\[ \text{Loss}(I, \Gamma) = \frac{1}{|T'|} \sum_{(C \subseteq D) \in T'} \sum_{a \in \Delta^2} \max(0, C^2(a) - D^2(a)) \]  

(8)

where \( C \) and \( D \) is any concept.

The main idea of Equation 8 is to ensure that the interpretation \( I \) should satisfy every \( C \subseteq D \in T' \), denoting that \( \forall a \in \Delta^2, C^2(a) \leq D^2(a) \). Though \( C \subseteq D \) is also equivalent to \( C \rightarrow D \geq n \) in fuzzy \( \text{ALC} \), where \( (C \rightarrow D)^I = \min_{a \in \Delta^2} \{ \max(1 - C^2(a), D^2(a)) \} \), according to Straccia (2001), it is hard to assign \( n \). Besides, using \( C \rightarrow D \geq n \) as a constraint will lead to \( D^2(a) \geq n \) or \( C^2(a) \leq 1 - n \), which is highly dependent on \( n \) rather than the reliable observations of the neural system. Besides, it is hard to decide whether to compel \( D^2(a) \geq n \) or \( C^2(a) \leq 1 - n \). So we use \( \forall a \in \Delta^2, C^2(a) \leq D^2(a) \) to constraint concept inclusion in \( \Gamma \).

Proposition 2 (Soundness of learning to ground in DF-\( \text{ALC} \)): When the hierarchical loss converges to 0, the learned interpretation \( T'' \) is the model of the given \( \text{ALC} \) ontology \( \mathcal{O} \). For any model \( \mathcal{J} \) of \( \mathcal{O} \), \( T'' \sim_{\text{sig}(\mathcal{O})} \mathcal{J} \).

Proof: When loss converges to 0, the learned \( T'' \) satisfies any \( C \subseteq D \) in the normalized ontology \( \mathcal{O}' \), so \( T'' \) is the model of \( \mathcal{O}' \). And according to Theorem 1, any model of \( \mathcal{O}' \) is semantically equivalent to the model of \( \mathcal{O} \). So this proposition is proved to be true.

Rule-based Learning

But learning grounding by maximizing the satisfiability captured by DF-\( \text{ALC} \) semantics cannot effectively reason to revive the perceptual grounding. Standard reasoning in fuzzy description logic is to decide whether \( \mathcal{O} \models \varphi \) holds and in crisp description logic to decide whether \( \mathcal{O} \models \varphi \) holds, which means to check whether all the models of \( \mathcal{O} \) satisfy \( \varphi \) or \( \varphi \). So the grounding considered in the standard reasoning is the models of knowledge base \( \mathcal{O} \). While reasoning to ground is to revise perceptual grounding as a model of knowledge base \( \mathcal{O} \), where the grounding contains plausible and implausible parts. The hierarchical loss proposed in Equation 8 can lead \( A \subseteq B \) to learn a grounding \( I \) that \( A^2 = B^2 \), so cannot distinguish between \( \subseteq \) and \( \models \). Besides, \( B^2(s) > a \) can be revised as 0.5, which loses information.

A relaxed revision is to reduce \( A^2(s) \) or improve less than \( B^2(s) \) to satisfy \( A^2 \leq B^2 \). Here is the rule-based loss for axioms in the normal form 1-3 shown in Figure 3.

\[ \text{Loss}_{\text{NF1-NF3}}(A^2, B^2; I, \Gamma) = \sum_{A \subseteq B} \sum_{a \in \Delta^2} ((1 - B^2(s)) \ast G(A^2(s), B^2(s))) \]  

(9)

, where \( A \) and \( B \) is any concept, \( A^2 \) and \( B^2 \) are in the representation of DF-\( \text{ALC} \) calculated according to the semantics of \( \text{ALC} \) based on Gödel logic. \( G(v, w) = \text{ReLU}(v - w) \), and \( G(v, w) \) does not take part in the gradient descent.

For axioms in the normal form 4-7, the semantics of \( \exists \) and \( \forall \) in fuzzy \( \text{ALC} \) based on Gödel logic can not be reasoned in a proper way with hierarchical loss. So different losses are designed for axioms in normal form 4-7 respectively.

Consider four ontologies \( \mathcal{O}_1 = \{\exists \text{r.A} \sqsubseteq \text{B}\}, \mathcal{O}_2 = \{\forall \text{r.A} \sqsubseteq \text{B}\}, \mathcal{O}_3 = \{\text{B} \sqsubseteq \exists \text{r.A}\}, \mathcal{O}_4 = \{\text{B} \sqsubseteq \forall \text{r.A}\} \), in the following three examples, the performance of DF-\( \text{ALC} \) with hierarchical loss and the revision calculus are shown in Table 3.

Example 1 Given a perceptual grounding \( I \) in the domain \( \{s_1, s_2\}, A^2(s_1) = 0, A^2(s_2) = 0, B^2(s_1) = 0.9, B^2(s_2) = 0, r^2(s_1, s_2) = 0.9, r^2(s_1, s_1) = r^2(s_2, s_1) = r^2(s_2, s_2) = 0 \). Notated as vectors \( A^2 = [0, 0], B^2 = [0.9, 0] \), and matrix

\[
\begin{bmatrix}
0 & 0.9 \\
0 & 0
\end{bmatrix}
\]

According to the semantics of fuzzy \( \text{ALC} \), in \( \mathcal{O}_1 \), \( (\exists \text{r.A})^2 = [0, 0] \), which satisfies \( (\exists \text{r.A})^2 \leq B^2 \), so hierarchical loss is 0, and no revision is executed. But this is not what we want. As we know that \( s_1 \) is likely to be \( \text{B} \), and \( r(s_1, s_2) \) is likely to be true, so \( s_2 \) is likely to be a membership of \( \text{A} \). In \( \mathcal{O}_2 \), \( (\forall \text{r.A})^2 = [0.1, 1] \), which does not satisfy \( (\forall \text{r.A})^2 \leq B^2 \), and hierarchical loss is 1.1. Through gradient decent, until loss becomes 0, \( A^2 = [0.24, 0], B^2 = [0.4, 1], r^2(s_2, s_1) \) will be 0.7 and \( r^2(s_1, s_2) \) will be 1. In \( \mathcal{O}_3 \), with hierarchical loss, \( A^2 \) will be \([0.38, 0]\), \( B^2 \) will be \([0.35, 0]\) and \( r^2(s_1, s_1) \) will be \([0.36, 0]\). In \( \mathcal{O}_4 \), with hierarchical loss, \( B^2 = [0, 0] \) and \( r^2(s_1, s_2) = 0 \).

Example 2 Given a perceptual grounding \( I \) in the domain \( \{s_1, s_2\}, A^2 = [0, 0.9], B^2 = [0, 0] \), \( r^2 \) is the same as in Example 1.

According to the semantics of fuzzy \( \text{ALC} \), in \( \mathcal{O}_1 \), \( (\exists \text{r.A})^2 = [0, 0.9] \), which does not satisfy \( (\exists \text{r.A})^2 \leq B^2 \), so hierarchical loss is 0.9. Through gradient decent, until loss becomes 0, \( A^2 \) is decreased as \([0, 0]\), and \( B^2 \) is increased as \([0, 0.9]\). \( A^2 \) is not expected to be changed and \( B^2 \) is expected to be increased as \([0, 0.9]\). In \( \mathcal{O}_2 \), \( (\forall \text{r.A})^2 = [0.9, 1] \), \( A^2 \) will be revised as \([0, 0]\), \( B^2 \) will be revised as \([0.5, 1]\), and \( r^2(s_1, s_2) = 1 \). In \( \mathcal{O}_3 \) and \( \mathcal{O}_4 \), there is no revision.

Example 3 Given a perceptual grounding \( I \) in the domain \( \{s_1, s_2\}, A^2 = [0, 0.9] \) and \( B^2 = [0, 0] \), \( r^2 \) is the same as in Example 1.

According to the semantics of fuzzy \( \text{ALC} \), \( (\exists \text{r.A})^2 = (\forall \text{r.A})^2 = [0, 0] \), which satisfies \( (\exists \text{r.A})^2 \leq B^2 \), so no revision is executed. In \( \mathcal{O}_3 \), \( B^2 \) and \( r^2 \) is revised if there are other \( s_n \), that \( A^2 \) is not zero. In \( \mathcal{O}_4 \), there is no revision.

We introduce the following rule-based loss to solve the problem meets in Table 3.

\[ \text{Loss}_{\text{Rule}}(A^2, B^2; I, \Gamma) = \sum_{\text{B} \subseteq \exists \text{r.A}} \sum_{a \in \Delta^2} ((1 - A^2(s)) \ast G(A^2(s), B^2(s))) \]

\[ G(\sum_{a \in \Delta^2} (B^2(\alpha) \otimes r^2(\alpha, s)) + (1 - r^2(s, a))) \ast G(B^2(s) \otimes A^2(\alpha), r^2(s, a))) \]

(10)
where $\otimes$ is the t-norm. With the rule-based loss, the parts of interpretation that we believed to be true are not determined by a threshold, but by the specialty of the task and dataset.

In this paper, we use the product t-norm as $\otimes$ in the rule-based loss validated by the evaluation in the experiments. $\alpha' \in [0.5, 1]$ is the threshold for the truth value.

$$\text{Loss}_{\text{HF}}(A^T, r^T; \mathcal{T}, \Gamma) = \sum_{\forall r.A \subseteq B} \sum_{s \in \Delta^T} (\{1 - A^T(s)\} * G(\alpha, A^T(s)))$$

$$G(\{1 - B^T(a) \otimes r^T(s, a, \alpha')\})$$

(11)

$$\text{Loss}_{\text{HF}}(B^T; \mathcal{T}, \Gamma) = \sum_{\exists r.A \subseteq B} \sum_{s \in \Delta^T} (\{1 - B^T(s)\} * G(\alpha', B^T(s)))$$

$$G(\{\alpha^T(a) \otimes r^T(s, a, \alpha')\})$$

(12)

$$\text{Loss}_{\text{HF}}(A^T; \mathcal{T}, \Gamma) = \sum_{\forall r.A \subseteq B} \sum_{s \in \Delta^T} (\{1 - A^T(s)\} * G(\alpha', A^T(s))) +$$

$$G(\{\alpha^T(a) \otimes r^T(s, a, \alpha')\} + (1 - A^T(s)) * G(\alpha', A^T(s)) *$$

$$G(\{\alpha^T(a) \otimes r^T(s, a, \alpha')\})$$

(13)

We only consider the situations when an assertion is larger than $\alpha'$ here, w.l.o.g., the opposite situations (less than $1 - \alpha'$) are dual and can be added to the loss according to the distribution of perceptual grounding, e.g. the opposite situations are more plausible.

### Experiments

**Performance Evaluation**

We design two experiments to verify the efficiency of DF-\textit{ALC}, and answer the following research questions (RQ):

- **RQ1**: Can hierarchical loss always converge to zero? If not, what can the result be in these cases?
- **RQ2**: How successful the learning in DF-\textit{ALC} is in keeping reliable observations while revising erroneous ones?
- **RQ3**: How does rule-based loss perform compared to hierarchical loss?

To answer these questions, learning grounding for DF-\textit{ALC} ontologies should be evaluated in different neural networks under various situations. But the distribution of observations and the properties of a specific neural network gives bias (in a way of having the same pattern of errors) to the perceptual grounding. Therefore, we design an experimental task — masked ABox revision, for evaluation in various observation distributions. This task is not oriented to tackle a concrete symbol grounding problem. Given an \textit{ALC} ontology $O = (T, A)$, where $A$ is completed by a logical reasoner, then fuzzy extended in DF-\textit{ALC}. We assign the ideal grounding $\mathcal{T}$ with the processed $A$. Mask the random part of grounding $\mathcal{T}$ into a random truth value in an unknown region, and reformulate it into a differentiable fuzzy interpretation $\mathcal{T}'$ as the imitation of a perceptual grounding. Then transform ontology $O$ into DF-\textit{ALC} ontology $\Gamma$. Use $\mathcal{T}'$ as the initialization to learn the revised grounding $\mathcal{T}''$ based on $\Gamma$. Using the crisp transformation defined in Formula 7 with $\alpha = 0.5$ to transform the revised grounding into crisp grounding, evaluate it with the satisfiability calculated in the crisp mode. However, solely evaluating the satisfiability of $O$ with $\mathcal{T}''$ cannot show the constancy (in keeping the reliable parts) between $\mathcal{T}'$ and $\mathcal{T}''$, as one ontology is satisfiable in multiple groundings. So we design another task called conjunctive query answering, which is a kind of ontology-mediated query answering. The target is to retrieve individuals for complex concepts based on the revised grounding.

**Settings**

In the masked ABox revision task, we used 6 ontologies (“OntoDm” and “Nif Dys” are not consistent), while in the conjunctive query answering task, we used 4 consistent ontologies.

The mask rate of ABox ranges from $\{20\%, 40\%, 60\%, 80\%\}$. We set the unknown region as $[0.2, 0.8]$. Meanwhile, the truth values (greater) less than $\alpha = 0.8 (1 - \alpha = 0.2)$ were assumed to be true (false). We used the Logical Tensor Network (LTN) as the comparison model. LTN is a differentiable fuzzy logic model in product real logic based on first-order logic. But LTN needs to pre-train predicates and functions with labeled data. To adapt LTN to this task, we removed the parameters of predicates and functions and used the hierarchical loss to train LTN. Besides, we trained DF-\textit{ALC} with rule-based loss ($\alpha' = 0.8$). These two compared models share the same masked grounding in different settings.

We used success rate (S.R.) to evaluate soundness. The success rate is the percentage of the TBox axiom in original ontology (without normalization) that is satisfied w.r.t. the

| Example | Description | Axiom | Expected | Performance of Hierarchical Loss |
|---------|-------------|------|----------|---------------------------------|
| 1       | $(\exists^2(s_1, s_2) > a) \land (\exists^2(s_1) > a) \implies (\exists^2(s_2) > a)$ | $\forall r.A \subseteq B, \exists r.A \subseteq B$ | Unknown | do not revise, as expected |
| 2       | $(\exists^2(s_1, s_2) > a) \land (\exists^2(s_1) > a) \implies (\exists^2(s_2) > a)$ | $B \subseteq \forall r.A, B \subseteq \exists r.A$ | True | can revise, but not in an expected way |
| 3       | $(\exists^2(s_1, s_2) > a) \land (\exists^2(s_1) > a) \implies (\exists^2(s_2) > a)$ | $\forall r.A \subseteq B, \exists r.A \subseteq B$ | True | can revise, but not in an expected way |
| 4       | $(\exists^2(s_1, s_2) > a) \land (\exists^2(s_1) > a) \implies (\exists^2(s_2) > a)$ | $B \subseteq \exists r.A$ | Unknown | do not revise, as expected |
| 5       | $(\exists^2(s_1, s_2) > a) \land (\exists^2(s_1) > a) \implies (\exists^2(s_2) > a)$ | $\exists r.A \subseteq B, \forall r.A \subseteq B$ | Unknown | do not revise, as expected |

Table 3: Performance of hierarchical loss based on four kinds of axioms given the perceptual grounding in three examples. Unknown situations expect no revision. True situations expect revision executed in the implication.
Table 4: Success rate (%) in four mask rate settings with $\alpha = 0.8$. M is the masked grounding. D is the DF-ALC revised grounding based on M (w/HL is with hierarchical loss, w/RL is with rule-based loss). L is the LTN revised grounding based on M.

|        | 0.2 | 0.4 | 0.6 | 0.8 |
|--------|-----|-----|-----|-----|
| M      |     |     |     |     |
| D w/HL |     |     |     |     |
| D w/RL |     |     |     |     |
| L      |     |     |     |     |
| Family | 0.0 | 100 | 100 | 100 |
| Family2| 0.0 | 100 | 73  | 92  |
| GlycoRDF| 4.1 | 100 | 91  | 98  |
| Nifdys | 7.3 | 97  | 89  | 95  |
| Nihss  | 16.1| 100 | 56  | 48  |
| Ontodm | 5.4 | 91  | 41  | 37  |
| So     | 0.0 | 100 | 100 | 100 |

Figure 4: Conjunctive query answering results.

Table 5: Ontology information

|        | Family | Family2 | GlycoRDF | Nifdys | Nihss | Ontodm | So |
|--------|--------|---------|----------|--------|-------|--------|----|
| TBox axioms | 2032   | 2054    | 1453     | 6435   | 318   | 3476   | 2050|
| ABox axioms  | 224    | 224     | 518      | 2920   | 146   | 1113   | 366 |
| Concepts    | 19     | 19      | 113      | 2751   | 18    | 838    | 176 |
| Roles       | 4      | 4       | 91       | 68     | 16    | 78     | 22  |
| Individuals | 202    | 202     | 219      | 102    | 106   | 187    | 158 |
| Expressivity| ~      | ~       | ~        | ~      | ~     | ~      | ~   |

The ontologies used for the experiments are taken from Bioportal[^1], which, currently, includes more than 700 biomedical ontologies from different sources. We require the ontologies to have at least the logical operator of negation, disjunction, or universal quantifier, as well as 100 ABox assertions. Five ontologies fall into this set, with two of them (“Ontodm” and “Nifdys”) not consistent in some assertions; it remains to see whether DF-ALC would revise these errors. A taxonomy ontology (“Sso”) is also added for comparison. We also test a terseness ontology “Family”, which contains multiple instantiated families but its knowledge is incomplete. Based on “Family”, we augment it into “Family2” by adding some knowledge that can bridge with the instantiation. The information about these ontologies is shown in Table 5. Adam optimizer was used with a learning rate of 2e-4 to learn the grounding. Early stopping with 10 epochs tolerance was used to limit the running time.

Results From the results shown in Table 4, where we can see that DF-ALC (w/HL) and LTN succeeded in most cases. Not surprisingly, the success rate is low for masked

[^1]: http://bioportal.bioontology.org/ontologies
grounding, since any small fault in the grounding can dis-
satisfy an axiom in the ontology. DF-\(\text{ALC}\) (w/HL) does
not perfectly ground “Ontodm” and “Nifdys”, as these
two ontologies are not consistent. In “Ontodm”, DF-\(\text{ALC}\)
(w/HL) predicts wrongly disjoint concepts, and these con-
cepts are incompletely asserted. The same problem occurs
in “Nifdys”. We further study the failures in “Family2”, and
find that the failures are caused by unknown cases. More
specifically, we can see that an individual “F6M80” is
asserted as a Male, but his parents are not asserted, therefore
the values of “Son(F6M80)” and “Child(F6M80)” are un-
known. In learned grounding of \(\Gamma\), though they are all in
unknown region \((0.2, 0.8)\), \(\text{Son}^{\text{I}}(\text{F6M80}) = 0.5490\)
\(>\) \(\text{Child}^{\text{I}}(\text{F6M80}) = 0.5489\) can still lead to \(\text{Son}^{\text{I}} \not\subset \text{Child}^{\text{I}}\). For LTN, almost all of the axioms in the form of
\(\exists r. T \subset C\) is not learned well. For DF-\(\text{ALC}\), several ax-
ioms fail in the complex forms (e.g. \(\exists r_1.(\exists r_2.B) \subset C\))
when the masked rate gets higher. The success rate of DF-
\(\text{ALC}\) (w/RL) is low in most cases, as in most of the ontolo-
gies, the rule-based loss cannot achieve zero. Besides, with
rule-based loss, the semantics of quantifiers are not follow-
ing fuzzy \(\text{ALC}\).

To answer RQ1, it is worth noting that learning in “Fam-
ily2”, “GlycoRDF”, “Nifdys”, and “Ontodm” cannot get the
hierarchical loss to converge to 0 in finite time in the four
settings. But we still get the success rate of “Family2” and
“GlycoRDF” being 100\% which is due to the crisp transfor-
mation for masked grounding. So if the given \(\text{ALC}\) ontology
\(\mathcal{O}\) is consistent, though the learning loss cannot con-
verge to 0 in some cases, the crisp transformed grounding is
the model of \(\mathcal{O}\).

To answer RQ2, from the results shown in Figure 4, we
find that DF-\(\text{ALC}\) and LTN cannot do well in this task as
expected. Because the masked semantics loses much informa-
tion, DF-\(\text{ALC}\) can revise the grounding in a shifted direc-
tion. High precision (i.e. remaining the reasoning properties
well) and relatively high recall (i.e. being constancy with reliable
part of observation) can be expected with DF-\(\text{ALC}\)
compared to LTN when the mask rate is low (e.g. 0.2). While
both DF-\(\text{ALC}\) and LTN have problems in revising role inter-
pretation function.

To answer RQ3, DF-\(\text{ALC}\) (w/RL) performs better than
DF-\(\text{ALC}\) (w/HL) in the CQs masking task, while ground-
ings trained by DF-\(\text{ALC}\) (w/RL) have less interpretability
than those trained by DF-\(\text{ALC}\) (w/HL) when knowledge
base contains axioms formed in NF 4-7. As rule-based loss
deduces based on the assumption that grounding where the
truth value is larger than \(\alpha^\prime\) is reliable.

Overall, DF-\(\text{ALC}\) outperforms LTN in most observation
revision cases. The common and significant problem for
both of them is to avoid the disturbance of unknown cases to satis-
fiability to knowledge.

Semantic Image Interpretation

In this experiment, we apply DF-\(\text{ALC}\) (w/RL) to solve the
SI problem, as DF-\(\text{ALC}\) (w/HL) cannot revise the ground-
ning properly which has been explained in Section.

We use PASCAL-PART Dataset [Chen et al. 2014].
The dataset consists of images annotated with bounding boxes
denoting distinct objects. The simple semantics between
these objects like part-of relation can be detected by comput-
ing the pixel cover rate between bounding boxes, construct-
ing the role interpretation function of perceptual interpreta-
tion \(\mathcal{I}'\). Objects are then grounded by object detector Fast
R-CNN (FRCNN) [Girshick 2015], which gives each object
\(b\) the label \(C\) with \(\text{score}(C, b)\), constructing the concept
interpretation function of \(\mathcal{I}'\). To revise \(\mathcal{I}'\), we intro-
duced an OWL ontology \(\mathcal{O}_\text{partOf}\) with two kinds of axioms, which
is similar to the ontology introduced in Figure 6. The first
kind of axiom depicts the part-of relation between types, e.g.
\(\exists \text{isPartOf}.\text{Chair} \equiv \text{Name} \cup \text{Leg}\). The second kind of
axiom asserts the disjointness between different types, e.g.
\(\text{Chair} \not\equiv \text{isPartOf}.\text{Chair} \subseteq \uparrow, \text{Chair} \not\equiv \text{Table} \subseteq \downarrow\).
Then we used the rule-based loss to revise \(\mathcal{I}'\) according to
ontology \(\mathcal{O}_\text{partOf}\). LTN was trained with constraints fol-
lowing the settings proposed in [Donadello, Serafini, and
Garcez 2017].

The results are shown in Figure 5. Indoor objects have the
simplest relationships and animal objects have the most
complex relationships, which interprets why DF-\(\text{ALC}\) per-

Figure 5: Object types classification results. The baseline model is the classification results of FRCNN. For evaluation, the
performance is classified into three categories: vehicle, indoor, and animal. The metrics are macro averaged.
forms the best in indoor objects. Animal objects can have many common types of objects, e.g. ear, head, and eye, but DF-\textit{ALC} can still improve recall. LTN fails in improving the object types classification performance upon the baseline because the fuzzy logical operators cannot convey the proper information by maximizing the satisfiability. But LTN can do link prediction (e.g. revise the part-of-relation interpretation in this case), while current DF-\textit{ALC} does not perform well in link prediction when the role grounding or concept grounding does terribly. Theoretically, DF-\textit{ALC} (w/RL) can guide the revision of role interpretation with the knowledge formed as $A \sqsubseteq \exists \forall B$, bridged by the similarities between individuals. On the whole, DF-\textit{ALC} provides an unsupervised way to improve symbol grounding by utilizing logical knowledge compared to the existing methods which need supervised training before symbol grounding.

We also evaluated DF-\textit{ALC} and LTN \cite{donadello2017} in the low-resource SII task, shown in Table 6. As LTN cannot promote performance, the results run with LTN are not shown here. We can see that DF-\textit{ALC} can promote the performance of baseline even when in low-resource cases.

\textbf{Conclusion and Future Work}

In this work, we have presented DF-\textit{ALC}, a differentiable fuzzy description logic language for symbol grounding, which is also the first representation learning method for \textit{ALC} ontologies. We have proved the soundness of the semantics of DF-\textit{ALC} under OWA and the soundness of learning to ground. And we pointed out the limitations of directly using the semantics of fuzzy \textit{ALC} in DF-\textit{ALC}, so we also presented a rule-based loss for symbol grounding, which is effective in the semantic image interpretation task and CQs answering task. If a given \textit{ALC} ontology is complete, directed by a perceptual neural network, learning in the transformed DF-\textit{ALC} ontology can learn an interpretable grounding the crisp transformed of which is the model the \textit{ALC} ontology and can remain much valuable information from the perceptual neural network. Compared with the most related differentiable fuzzy logic model, LTN, we find that DF-\textit{ALC} is better at retaining the reliable part of probability. Besides, DF-\textit{ALC} is under the open-world assumption (OWA) which is more robust and close to realistic situations.

However, current DF-\textit{ALC} does not do well in part-of-relation recognition revision, which remains to be tested and updated for other link prediction tasks. Besides, \textit{ALC} cannot express cardinality constraints, which is important in situations where the definition of concepts is highly sensitive to quantity. So to extend expressive power, we will explore the best differentiable fuzzy model for \textit{ALC}. Our next step is to construct common sense knowledge in \textit{ALC} and apply DF-\textit{ALC} in combining knowledge into a dialogue state tracker.

\textbf{References}

Arvor, D.; Belgiu, M.; Falomir, Z.; Mougenot, I.; and Durieux, L. 2019. Ontologies to interpret remote sensing images: why do we need them? GIScience & remote sensing, 56(6): 911–939.

Baader, F.; Borgwardt, S.; and Penaloza, R. 2017. decidability and complexity of fuzzy description logics. \textit{KI-Künstliche Intelligenz}, 31(1): 85–90.

Bach, S. H.; Broecheler, M.; Huang, B.; and Getoor, L. 2017. Hinge-loss markov random fields and probabilistic soft logic.

Badreddine, S.; Garcez, A. d.; Serafini, L.; and Spranger, M. 2022. Logic tensor networks. \textit{Artificial Intelligence}, 303: 103649.

Borgwardt, S.; Distel, F.; and Peñaloza, R. 2014a. Decidable Gödel description logics without the finitely-valued model property. In \textit{Fourteenth International Conference on the Principles of Knowledge Representation and Reasoning}.

Borgwardt, S.; Distel, F.; and Peñaloza, R. 2014b. Gödel Description Logics with General Models. In \textit{Description Logics}, 391–403.

Bühmann, L.; Lehmann, J.; and Westphal, P. 2016. DL-Learner—A framework for inductive learning on the Semantic Web. \textit{Journal of Web Semantics}, 39: 15–24.

Cai, L.-W.; Dai, W.-Z.; Huang, Y.-X.; Li, Y.-F.; Muggleton, S.; and Jiang, Y. 2021. Abductive learning with ground knowledge base. In \textit{Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI), 1815–1821.}

Cangelosi, A. 2011. Solutions and open challenges for the symbol grounding problem. \textit{International Journal of Signs and Semiotic Systems (IJSSS)}, 1(1): 49–54.

Charest, M.; and Delisle, S. 2006. Ontology-guided intelligent data mining assistance: Combining declarative and procedural knowledge. In \textit{Artificial Intelligence and Soft Computing}, volume 2006, 9–14.

Chen, J.; Hu, P.; Jimenez-Ruiz, E.; Holter, O. M.; Antonyrajah, D.; and Horrocks, I. 2021. OWL2Vec*: Embedding of OWL ontologies. \textit{Machine Learning}, 1–33.

Chen, X.; Mottaghi, R.; Liu, X.; Fidler, S.; Urtasun, R.; and Yuille, A. 2014. Detect what you can: Detecting and representing objects using holistic models and body parts. In \textit{Proceedings of the IEEE conference on computer vision and pattern recognition}, 1971–1978.

Coradeschi, S.; Loutfi, A.; and Wrede, B. 2013. A short review of symbol grounding in robotic and intelligent systems. \textit{KI-Künstliche Intelligenz}, 27(2): 129–136.

Dasipouloou, S.; Kompatsiaris, I.; and Strintzis, M. G. 2009. Applying fuzzy DLs in the extraction of image semantics. \textit{Journal on data semantics XIV}, 105–132. Springer.

d’Avila Garcez, A. S.; Gori, M.; Lamb, L. C.; Serafini, L.; Spranger, M.; and Tran, S. N. 2019. Neural-symbolic Computing: An Effective Methodology for Principled Integration of Machine Learning and Reasoning. \textit{FLAP}, 6(4): 611–632.

Diligenti, M.; Gori, M.; and Saccà, C. 2017. Semantic-based regularization for learning and inference. \textit{Artif. Intell.}, 244: 143–165.
Diligenti, M.; Gori, M.; and Sacca, C. 2017. Semantic-based regularization for learning and inference. Artificial Intelligence, 244: 143–165.

Donadello, I.; Serafini, L.; and Garcez, A. D. 2017. Logic tensor networks for semantic image interpretation. arXiv preprint arXiv:1705.08968.

Dummett, M. 1959. Wittgenstein’s philosophy of mathematics. The Philosophical Review, 68(3): 324–348.

Eberhart, A.; Ebrahimi, M.; Zhou, L.; Shimizu, C.; and Hitzler, P. 2019. Completion reasoning emulation for the description logic EL+. arXiv preprint arXiv:1912.05063.

Ebrahimi, M.; Eberhart, A.; and Hitzler, P. 2021. On the Capabilities of Pointer Networks for Deep Deductive Reasoning. arXiv preprint arXiv:2106.09225.

Evans, R.; and Grefenstette, E. 2018. Learning explanatory rules from noisy data. Journal of Artificial Intelligence Research, 61: 1–64.

Garcez, A.; Gori, M.; Lamb, L.; Serafini, L.; Spranger, M.; and Tran, S. 2019. Neural-symbolic computing: An effective methodology for principled integration of machine learning and reasoning. Journal of Applied Logics, 6(4): 611–631.

Garcez, A. d.; Besold, T. R.; De Raedt, L.; Földiak, P.; Hitzler, P.; Icard, T.; Kühnberger, K.-U.; Lamb, L. C.; Miikkulainen, R.; and Silver, D. L. 2015. Neural-symbolic learning and reasoning: contributions and challenges. In 2015 AAAI Spring Symposium Series.

Garcez, A. d.; and Lamb, L. C. 2020. Neurosymbolic AI: the 3rd wave. arXiv preprint arXiv:2012.05876.

Garg, D.; Ikbal, S.; Srivastava, S. K.; Vishwakarma, H.; Karanam, H.; and Subramaniam, L. V. 2019. Quantum embedding of knowledge for reasoning. Advances in Neural Information Processing Systems, 32: 5594–5604.

Girshick, R. 2015. Fast r-cnn. In Proceedings of the IEEE international conference on computer vision, 1440–1448.

Goldstein, I.; and Papert, S. 1977. Artificial intelligence, language, and the study of knowledge. Cognitive science, 1(1): 84–123.

Hammer, B.; and Hitzler, P. 2007. Perspectives of neural-symbolic integration, volume 77. Springer.

Harnad, S. 1990. The symbol grounding problem. Physica D: Nonlinear Phenomena, 42(1-3): 335–346.

Harnad, S. 1993. Symbol grounding is an empirical problem: Neural nets are just a candidate component.

Hölldobler, S.; Kalinke, Y.; and Störr, H.-P. 1999. Approximating the semantics of logic programs by recurrent neural networks. Applied Intelligence, 11(1): 45–58.

Hu, Z.; Ma, X.; Liu, Z.; Hovy, E.; and Xing, E. 2016. Harnessing Deep Neural Networks with Logic Rules. In Proceedings of the 54th Annual Meeting of the Association for Computational Linguistics (Volume 1: Long Papers), 2410–2420.

Huang, Y.-X.; Dai, W.-Z.; Yang, J.; Cai, L.-W.; Cheng, S.; Huang, R.; Li, Y.-F.; and Zhou, Z.-H. 2020. Semi-supervised abductive learning and its application to theft judicial sentencing. In 2020 IEEE international conference on data mining (ICDM), 1070–1075. IEEE.

Hudelot, C.; Atif, J.; and Bloch, I. 2008. Fuzzy spatial relation ontology for image interpretation. Fuzzy Sets and Systems, 159(15): 1929–1951.

Hudelot, C.; Maillot, N.; and Thonnat, M. 2005. Symbol grounding for semantic image interpretation: from image data to semantics. In Tenth IEEE International Conference on Computer Vision Workshops (ICCVW’05), 1875–1875. IEEE.

Jiang, H.; Gurajada, S.; Lu, Q.; Neelam, S.; Popa, L.; Sen, P.; Li, Y.; and Gray, A. 2021. LNN-EL: A Neuro-Symbolic Approach to Short-text Entity Linking. In Proceedings of the 59th Annual Meeting of the Association for Computational Linguistics and the 11th International Joint Conference on Natural Language Processing (Volume 1: Long Papers), 775–787.

Kahneuman, D.; Rossi, F.; Hinton, G.; Bengio, Y.; and LeCun, Y. 2020. AAAI20 fireside chat with Daniel Kahneman.

Khanmam, R.; Zhu, Y.; Groth, O.; Johnson, J.; Hata, K.; Kravitz, J.; Chen, S.; Kalantidis, Y.; Li, L.-J.; Shamma, D. A.; et al. 2017. Visual genome: Connecting language and vision using crowdsourced dense image annotations. International journal of computer vision, 123(1): 32–73.

Kulmanov, M.; Liu-Wei, W.; Yan, Y.; and Hoehndorf, R. 2019. El embeddings: geometric construction of knowledge embeddings Deep Neural Networks with Logic Rules. In Proceedings of the 59th Annual Meeting of the Association for Computational Linguistics and the 11th International Joint Conference on Natural Language Processing (Volume 1: Long Papers), 775–787.

Kahenman, D.; Rossi, F.; Hinton, G.; Bengio, Y.; and LeCun, Y. 2020. AAAI20 fireside chat with Daniel Kahneman.

Krishna, R.; Zhu, Y.; Groth, O.; Johnson, J.; Hata, K.; Kravitz, J.; Chen, S.; Kalantidis, Y.; Li, L.-J.; Shamma, D. A.; et al. 2017. Visual genome: Connecting language and vision using crowdsourced dense image annotations. International journal of computer vision, 123(1): 32–73.

Kulmanov, M.; Liu-Wei, W.; Yan, Y.; and Hoehndorf, R. 2019. El embeddings: geometric construction of models for the descriptive logic el++. arXiv preprint arXiv:1902.10499.

Lamb, L.; Garcez, A.; Gori, M.; Prates, M.; Avelar, P.; and Vardi, M. 2020. Graph Neural Networks Meet Neural-Symbolic Computing: A Survey and Perspective. In IJCAI-PRICAI 2020-29th International Joint Conference on Artificial Intelligence-Pacific Rim International Conference on Artificial Intelligence.
Lample, G.; and Charton, F. 2020. Deep Learning For Symbolic Mathematics. In Proc. ICLR’20. OpenReview.net.

Liang, C.; Berant, J.; Le, Q.; Forbus, K.; and Lao, N. 2017. Neural Symbolic Machines: Learning Semantic Parsers on Freebase with Weak Supervision. In Proceedings of the 55th Annual Meeting of the Association for Computational Linguistics (Volume 1: Long Papers), 23–33.

Manhaeve, R.; Dumancic, S.; Kimmig, A.; Demeester, T.; and De Raedt, L. 2018. Deepproblog: Neural probabilistic logic programming. Advances in Neural Information Processing Systems, 31.

Marcus, G. 2020. The next decade in AI: four steps towards robust artificial intelligence. arXiv preprint arXiv:2002.06177.

Neumann, B.; and Müller, R. 2008. On scene interpretation with description logics. Image and Vision Computing, 26(1): 82–101.

Oltramari, A.; Francis, J.; Henson, C.; Ma, K.; and Wickramachchi, R. 2020. Neuro-symbolic architectures for context understanding. arXiv preprint arXiv:2003.04707.

Richardson, M.; and Domingos, P. 2006. Markov logic networks. Machine learning, 62(1): 107–136.

Riegel, R.; Gray, A.; Luus, F.; Khan, N.; Makondo, N.; Akhalwaya, I. Y.; Qian, H.; Fagin, R.; Barahona, F.; Sharma, U.; et al. 2020. Logical neural networks. arXiv preprint arXiv:2006.13155.

Rocktäschel, T.; and Riedel, S. 2016. Learning knowledge base inference with neural theorem provers. In Proceedings of the 5th workshop on automated knowledge base construction, 45–50.

Rocktäschel, T.; and Riedel, S. 2017. End-to-end differentiable proving. Advances in neural information processing systems, 30.

Sen, P.; de Carvalho, B. W.; Riegel, R.; and Gray, A. 2022. Neuro-symbolic inductive logic programming with logical neural networks. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 36, 8212–8219.

Serafini, L.; and Garcez, A. d. 2016. Logic tensor networks: Deep learning and logical reasoning from data and knowledge. arXiv preprint arXiv:1606.04422.

Seyler, D.; Dembelova, T.; Del Corro, L.; Hoffart, J.; and Weikum, G. 2018. A study of the importance of external knowledge in the named entity recognition task. In Proceedings of the 56th Annual Meeting of the Association for Computational Linguistics (Volume 2: Short Papers), 241–246.

Smaili, F. Z.; Gao, X.; and Hoehndorf, R. 2018. Onto2vec: joint vector-based representation of biological entities and their ontology-based annotations. Bioinformatics, 34(13): i52–i60.

Smaili, F. Z.; Gao, X.; and Hoehndorf, R. 2019. Opa2vec: combining formal and informal content of biomedical ontologies to improve similarity-based prediction. Bioinformatics, 35(12): 2133–2140.

Straccia, U. 2001. Reasoning within fuzzy description logics. Journal of artificial intelligence research, 14: 137–166.

Topan, S.; Rolnick, D.; and Si, X. 2021. Techniques for Symbol Grounding with SATNet. Advances in Neural Information Processing Systems, 34: 20733–20744.

Torisawa, K.; et al. 2007. Exploiting Wikipedia as external knowledge for named entity recognition. In Proceedings of the 2007 joint conference on empirical methods in natural language processing and computational natural language learning (EMNLP-CoNLL), 698–707.

Towell, G. G.; and Shavlik, J. W. 1994. Knowledge-based artificial neural networks. Artificial intelligence, 70(1-2): 119–165.

Trouillon, T.; Welbl, J.; Riedel, S.; Gaussier, É.; and Bouchard, G. 2016. Complex embeddings for simple link prediction. In International conference on machine learning, 2071–2080. PMLR.

Tsamoura, E.; Hospedales, T.; and Michael, L. 2021. Neural-symbolic integration: A compositional perspective. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 35, 5051–5060.

Unal, O.; Afsarmanesh, H.; et al. 2006. Using linguistic techniques for schema matching. In ICSOFT (2), 115–120.

van Harmelen, F.; and ten Teije, A. 2019. A Boxology of Design Patterns forHybrid Learning and Reasoning Systems. J. Web Eng., 18(1-3): 97–124.

van Krieken, E.; Acar, E.; and van Harmelen, F. 2022. Analyzing differentiable fuzzy logic operators. Artificial Intelligence, 302: 103602.

Wang, P.-W.; Dotti, P.; Wilder, B.; and Kolter, Z. 2019. Satnet: Bridging deep learning and logical reasoning using a differentiable satisfiability solver. In International Conference on Machine Learning, 6545–6554. PMLR.

Wang, W. Y.; Mazaitis, K.; and Cohen, W. W. 2013. Programming with personalized pagerank: a locally groundable first-order probabilistic logic. In Proceedings of the 22nd ACM international conference on Information & Knowledge Management, 2129–2138.

Xie, Y.; Xu, Z.; Kankanhalli, M. S.; Meel, K. S.; and Soh, H. 2019. Embedding symbolic knowledge into deep networks. Advances in neural information processing systems, 32.

Yang, F.; Yang, Z.; and Cohen, W. W. 2017. Differentiable learning of logical rules for knowledge base reasoning. Advances in neural information processing systems, 30.

Zadeh, L. A. 1996. Fuzzy sets. In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh, 3065–3072. World Scientific.

Zhang, J.; Chen, B.; Zhang, L.; Ke, X.; and Ding, H. 2021. Neural, symbolic and neural-symbolic reasoning on knowledge graphs. AI Open, 2: 14–35.

Zhang, Z.; Cai, J.; Zhang, Y.; and Wang, J. 2020. Learning hierarchy-aware knowledge graph embeddings for link prediction. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 34, 3065–3072.

Zhou, Z.-H. 2019. Abductive learning: towards bridging machine learning and logical reasoning. Science China Information Sciences, 62(7): 1–3.