"Transparent" Boundary Conditions for the Equation of Rod Transverse Vibrations

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Abstract
Local perturbations of an infinitely long rod go away to infinity. On the contrary, in the case of a finite length of the rod the perturbations reach its boundary and are reflected from them. The boundary conditions constructed here for the implicit difference scheme imitate the Cauchy problem and provide almost no reflection. These boundary conditions are non-local in time, and their practical implementation requires additional calculations at every time step. To minimise them, a special rational approximation similar to the Hermite-Padé approximation is used. Numerical experiments confirm the high "transparency" of these boundary conditions.

Keywords: Rod Equation, Boundary Condition, Rational Approximation, Finite-Difference Approximation, Implicit Scheme, Z-transformation

1. Introduction
The equation of transverse vibrations of a rod (beam) with a circular cross section
\[
\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( R^2 \rho \frac{\partial^3 u}{\partial x \partial t^2} \right) + \frac{\partial^2}{\partial x^2} \left( E R^2 \frac{\partial^2 u}{\partial x^2} \right) = f
\]
has many applications, see e.g. [1]. Here \( \rho \) is the density of the rod, \( R \) is the radius of the cross section of the rod, \( E \) is the Young’s modulus of the rod, \( u = u(t, x) \) is the transverse displacement of the rod, \( x \in [-L/2, L/2] \), \( L \) is the length of the rod. The right hand side (forcing) \( f \) describes external force.

Eq. (1) may be obtained from the Principle of Least Action with Lagrangian
\[
\mathcal{L}[u] = \frac{1}{2} \int \int \rho(\partial_x u)^2 - \left[ R^2 \rho(\partial_x \partial_t u)^2 + ER^2(\partial_x^2 u)^2 \right] \, dx \, dt,
\]
where the first term describes kinetic energy of the rod and the term in square brackets is potential energy, [2].

Eq. (1) is supplemented by two initial conditions:
\[
u(0, x) = U_0(x), \quad \partial_t u(0, x) = U_1(x),
\]
and two boundary conditions at each edge of the segment. With proper (according to the Shapiro-Lopatinsky theory) type of these boundary conditions, the mixed initial-boundary value problem is correct.
In many practical problems of mathematical physics the necessary complete set of physically adequate boundary conditions is absent. For example, in the problem of weather forecasting for a limited area, we can set Dirichlet boundary conditions, where the right hand side of the conditions is taken from a larger scale (global) forecasting model. In this case, however, waves coming out of the computational area are reflected from the boundary. If we use the finite-difference spatial approximation of the ideal gas dynamics differential equations, according to the central difference formula, the number of boundary conditions required for the uniqueness of the solution increases in comparison with differential problem. For transport equation with viscosity, phenomena such as boundary layer on the outflow from the region may occur. Such computational difficulties are usually overcome by introducing a special finite-difference operator of the large computational viscosity in a vicinity of the boundary into the algorithm. However, this inevitably leads to an increase in the prediction error.

The construction of boundary conditions such that no reflection of outgoing waves from the boundary occurs for an implicit finite-difference equation (see Sect. 2), which approximates Eq. (1), is the subject of this article. We emphasise that for each finite-difference scheme it is necessary to reform such boundary conditions imitating the Cauchy problem (ICP) with continued background conditions and the right hand side.

The construction of the boundary conditions for the Eq. (1) is practically important, because of the place of this equation in the theory of elasticity and in engineering problems. The algorithm here is significantly more complex than in the case of classical equations and systems of mathematical physics in partial derivatives, considered earlier in [2–8]. Since it is not resolved with respect to the highest (second) derivative with respect to time (i.e. it is not differential equation of the Cauchy–Kovalyevskaya type), and also it has 4-th order with respect to the spatial variable.

The high order of the differential (and, as a consequence, of the finite-difference) equation with respect to space requires two boundary conditions at each edge. ICP conditions for various simpler finite-difference schemes approximating equations of mathematical physics were considered earlier in [2, 8–13].

The construction of the ICP conditions for the finite-difference scheme will be considered in Sect. 3. The construction of the second initial Cauchy condition with high approximation order will be considered in Sect. 4. The description and the results of numerical experiments with ICP boundary conditions, and comparisons with the others conditions, see Sect. 5.

2. Finite-Difference Implicit Scheme

Let us consider Eq. (1) with constant coefficients and with zeroth forcing $f$:

$$\rho \frac{\partial^2 u}{\partial t^2} - R^2 \rho \frac{\partial^4 u}{\partial t^4} + ER^2 \frac{\partial^4 u}{\partial x^4} = 0, \quad (2)$$

and the implicit finite-difference scheme on the five-point stencil of the Crank–Nicolson type that approximates Eq. (2) on the uniform grid with the steps $\tau$ with respect to time $t$ and $h$ with respect to spatial variable $x$:

$$\sigma (u^{n+1}_{m+1} + u^{n+1}_{m-1} + u^{n+1}_{m+1} + u^{n+1}_{m-1}) + \beta (u^{n+1}_{m+1} + u^{n+1}_{m-1} + u^{n+1}_{m+1} + u^{n+1}_{m-1}) + \alpha (u^{n+1}_{m+1} + u^{n+1}_{m+1} + u^{n+1}_{m+1} + u^{n+1}_{m-1}) + \gamma (u^{n+1}_{m+1} + u^{n+1}_{m+1} + u^{n+1}_{m+1} + u^{n+1}_{m-1}) + \delta u^{n+1}_{m} = 0, \quad (3)$$

where $m = 2, 3, \ldots, N - 2$. Thus, we define $N - 3$ linear algebraic equations with $N + 1$ unknown values $\{u^{n+1}_{m}\}_{m=0}^{N}$. To close the linear algebraic system we must add two linear algebraic equations that describe two boundary conditions on the left edge of the segment and, similarly, the same number of equations for the right edge. As a result, we obtain a linear algebraic system with a five-diagonal matrix.

To begin the computational process and calculate the values $\{u^{n+1}_{m}\}_{m=0}^{N}$, we need initial functions $\{u^{0}_{m}\}_{m=0}^{N}$ and $\{u^{0}_{m}\}_{m=0}^{N}$, where $N + 1$ is the number of grid points in the segment $[-L/2, L/2]$. Coefficients of the scheme are deduced using dimensionless parameters $\nu = ER^2u^{-1} \cdot \tau h^{-4}$, $\mu = R^2 \cdot h^{-2}$ and formulae $\alpha = 1 + 3\nu + 2\mu$, $\beta = -2\nu - \mu$, $\gamma = 2\mu$, $\delta = -2 - 4\mu$, $\sigma = \nu/2$. Here the upper index $n$ corresponds to a number of time step, and the lower index corresponds to the spatial variable.
3. ICP Boundary Conditions and Rational Approximations

3.1. General Plan of Approach

The algorithm for constructing ICP boundary conditions at \( x = L/2 \) for Eq. (3) is as follows:

Step 1. Apply the \( \mathcal{Z} \)-transformation (discrete analogue of the Laplace integral transformation) in time to Eq. (3) and obtain a linear ordinary finite-difference equation with respect to the variable \( m \); the equation depending on the parameter \( z \in \mathbb{C} \).

Step 2. Construct for the corresponding homogeneous finite-difference 4-th order equation a fundamental set of solutions \( \{ Y_j(m) \}_{j=1}^4 \), such that solutions \( Y_1, Y_2 \) decrease as \( m \to +\infty \), and solutions \( Y_3, Y_4 \) decrease as \( m \to -\infty \).

Step 3. Decompose the obtained growing (as \( m \to +\infty \)) solutions (functions from the dual variable \( z \) with respect to time \( t \)) into a series in a neighbourhood of \( z = \infty \).

Step 4. Apply the inverse \( \mathcal{Z} \)-transformation to the obtained coefficients of the ICP boundary conditions. In order to calculate the inverse \( \mathcal{Z} \)-transformation, it is necessary (see, e.g. \( [2, 8, 9, 11, 13, 16] \)) to decompose the symbols of the corresponding operators into a Laurent series in the neighbourhood of the point \( z = \infty \). For convenience, we introduce the change of variable \( \omega = 1/z \) and decompose the symbol into a Taylor series in the neighbourhood of the point \( \omega = 0 \).

Step 5. Construct vectorial rational functions (the construction is similar to the Hermite – Padé approximation in the point \( \omega = 0 \); see e.g. \( [2, 8, 12, 13, 14] \)), which are asymptotically orthogonal to two growing solutions of this \( \mathcal{Z} \)-transformed equation. The corresponding polynomials are symbols of the ICP boundary operators.

The growth or decrease of Taylor coefficients of the meromorphic function are associated with the location of the function singularities on the complex plane. It is important to estimate the areas of convergence of the obtained power series, which depend on the features of functions (solutions of the homogeneous finite-difference equation) — points of pole or branch points.

3.2. ICP Boundary Conditions for the Finite-Difference Equation

Let us apply \( \mathcal{Z} \)-transformation with respect to time to Eq. (3). Then we obtain linear ordinary non-homogeneous finite-difference equation

\[
\sigma \left( z^2 + 1 \right) [v(m + 4h) + v(m)] + \left( \beta \left( z^2 + 1 \right) + \gamma z \right) [v(m + 3h) + v(m + h)] + (a \left( z^2 + 1 \right) + \delta z) v(m + 2h) = g(z, m),
\]

where \( z \in \mathbb{C} \) is the variable that is dual to time, \( v(m) = v(m, z) \) is an image of the solution \( u_m^0 \), and \( g(z, m) \) is the right hand side that is obtained by \( \mathcal{Z} \)-transformation from initial functions \( u_m^0 \) and \( u_m^1 \).

The corresponding homogeneous equation after change of variable \( \omega = 1/z \) takes the form

\[
\sigma \left( \omega^2 + 1 \right) [v(m + 4h) + v(m)] + \left( \beta \left( \omega^2 + 1 \right) + \gamma \omega \right) [v(m + 3h) + v(m + h)] + (a \left( \omega^2 + 1 \right) + \delta \omega) v(m + 2h) = 0.
\]

The order of the characteristic equation for ordinary finite-difference Eq. (5) (see \( [2, 8, 11, 13, 16] \))

\[
\sigma \left( 1 + \omega^3 \right) [x^2 + 1] + \left( \beta \left( 1 + \omega^2 \right) + \gamma \omega \right) [x^3 + x] + (a \left( 1 + \omega^2 \right) + \delta \omega) x^2 = 0
\]

at \( \omega \neq \pm i \) is equal to 4 and is reciprocal. Let us divide Eq. (6) by \( x^2 \) and rewrite it in the form

\[
\sigma \left( 1 + \omega^3 \right) [x + x^{-1}]^2 + \left( \beta \left( 1 + \omega^2 \right) + \gamma \omega \right) \left[ 1 + x^{-1} \right] + \delta \omega + (\alpha - 2\alpha \omega^2) (1 + \omega^3) = 0.
\]

Note. The order of algebraic Eq. (6) at \( \omega = \pm i \) is equal to 3:

\[
\gamma \omega (\lambda^3 + \lambda) + \delta \omega \lambda^2 = 0.
\]
Therefore, $\lambda_1 = 0$, and as $\omega \to \pm i$ we obtain $\lambda_3 \to \infty$. Eq. (6) has zero in the origin in this case only. We determine the remaining two roots of the cubic equation from the quadratic reciprocal equation:

$$\lambda^2 + \frac{\delta}{\gamma} \lambda + 1 = 0,$$

where $\delta/\gamma = -2 - 1/\mu$, and therefore

$$\lambda_{2,4} = 1 + \frac{1}{2\mu} \pm \sqrt{\left(1 + \frac{1}{2\mu}\right)^2 - 1}.$$

Thus, both roots are real and positive. According to Vieta theorem, the following inequalities are fulfilled:

$$\lambda_2 > 1 > \lambda_4 > 0.$$

Corollary. Two roots of Eq. (6) as $\omega \to \pm i$ takes absolute values larger than 1, and absolute values of two other roots are smaller.

At other values of $\omega$, in Eq. (7) we change variable: $\eta = \lambda + \lambda^{-1}$ and obtain quadratic equation for the auxiliary variable $\eta$:

$$\sigma (1 + \omega^2) \eta^2 + (\beta (1 + \omega^2) + \gamma \omega) \eta + \delta \omega + (\alpha - 2\sigma)(1 + \omega^2) = 0. \tag{8}$$

Roots of Eq. (8) are

$$\eta_1(\omega) = \frac{-\beta (1 + \omega^2) - \gamma \omega - \sqrt{(\beta (1 + \omega^2) + \gamma \omega)^2 - 4\sigma (1 + \omega^2)(\delta \omega + (\alpha - 2\sigma)(1 + \omega^2))}}{2\sigma (1 + \omega)} \tag{9}$$

$$\eta_2(\omega) = \frac{-\beta (1 + \omega^2) - \gamma \omega + \sqrt{(\beta (1 + \omega^2) + \gamma \omega)^2 - 4\sigma (1 + \omega^2)(\delta \omega + (\alpha - 2\sigma)(1 + \omega^2))}}{2\sigma (1 + \omega)}.$$  

where $\sqrt{7}$ is the complex root with a positive real part of $y \in \mathbb{C} \setminus \mathbb{R}$.

Before expansion into Taylor series of the functions $\lambda_j(\omega)$, $j = 1, 4$ in a vicinity of the origin $\omega = 0$, we do it for auxiliary functions $\eta_1(\omega)$, $\eta_2(\omega)$.

We represent the radicand in Eq. (9) with a help of dimensionless parameters $\nu$ and $\mu$ and simplify it:

$$\eta_1(\omega) = \frac{-\beta (1 + \omega^2) - \gamma \omega - \sqrt{(1 - \omega^2)[(\mu^2 - 2\nu)\omega^2 - 2\mu^2\omega + \mu^2 - 2\nu]}}{2\sigma (1 + \omega)}.$$  

If $|\omega| \ll 1$, we can take out the multiplier from the quadratic root:

$$\eta_1(\omega) = \frac{-\beta (1 + \omega^2) - \gamma \omega - \sqrt{\mu^2 - 2\nu(1 - \omega)\sqrt{\omega^2 - 2\frac{\mu^2}{\mu^2 - 2\nu}\omega + 1}}}{2\sigma (1 + \omega^2)}.$$  

Let us apply the formula for generating function of the Legendre polynomials (see e.g. [2, 8, 11, 13, 17]):

$$(\omega^2 - 2\varepsilon \omega + 1)^{-1/2} = \sum_{n=0}^{\infty} P_n(\varepsilon)\omega^n,$$

where $P_n(\varepsilon)$ is a Legendre polynomial of degree $n$ at point $\varepsilon$. We obtain

$$\eta_1(\omega) = \frac{1}{2\sigma (1 + \omega^2)} \left[ -\beta (1 + \omega^2) - \gamma \omega - \sqrt{\mu^2 - 2\nu(1 - \omega)\left(\omega^2 - 2\frac{\mu^2}{\mu^2 - 2\nu}\omega + 1\right)} \sum_{n=0}^{\infty} P_n\left(\frac{\mu^2}{\mu^2 - 2\nu}\right)\omega^n \right].$$
Then we use the formula for geometric progression:

\[
\frac{1}{1 + a} = \sum_{k=0}^{\infty} (-1)^k a^k,
\]

where \(a = \omega^2\), and express the values \(\beta, \gamma, \delta\) across \(\mu\) and \(\nu\). We obtain:

\[
\eta_1(\omega) = \frac{1}{\nu} \sum_{k=0}^{\infty} (-1)^k \omega^{2k} \left[ (\mu + 2\nu)(1 + \omega^2) - 2\mu\omega - \sqrt{\mu^2 - 2\nu(1 - \omega)} \left( \omega^2 - 2\frac{\mu^2}{\mu^2 - 2\nu}\omega + 1 \right) \sum_{n=0}^{\infty} \nu \left( \frac{\mu^2}{\mu^2 - 2\nu} \right)^n \omega^n \right].
\]

In the same way we obtain

\[
\eta_2(\omega) = \frac{1}{\nu} \sum_{k=0}^{\infty} (-1)^k \omega^{2k} \left[ (\mu + 2\nu)(1 + \omega^2) + 2\mu\omega + \sqrt{\mu^2 - 2\nu(1 - \omega)} \left( \omega^2 - 2\frac{\mu^2}{\mu^2 - 2\nu}\omega + 1 \right) \sum_{n=0}^{\infty} \nu \left( \frac{\mu^2}{\mu^2 - 2\nu} \right)^n \omega^n \right].
\]

Asymptotic of the functions as \(\omega \to 0\) are described by the formulae

\[
\eta_1(\omega) = \theta_1 + r_1(\omega),
\]
\[
\eta_2(\omega) = \theta_2 + r_2(\omega),
\]

where \(r_j(\omega) \to 0, j = 1, 2\), and

\[
\theta_1 = \frac{1}{2\sigma} \left[ \beta - \sqrt{\beta^2 - 4\sigma(\alpha - 2\sigma)} \right] = 2 + \frac{\mu}{\nu} - \frac{1}{\nu} \sqrt{\mu^2 - 2\nu},
\]
\[
\theta_2 = \frac{1}{2\sigma} \left[ \beta + \sqrt{\beta^2 - 4\sigma(\alpha - 2\sigma)} \right] = 2 + \frac{\mu}{\nu} + \frac{1}{\nu} \sqrt{\mu^2 - 2\nu}.
\]

If the inequality \(\mu^2 > 2\nu\) is fulfilled, i.e. if

\[
\tau < R \sqrt{\frac{\rho}{2E}},
\]

then the radicand in Eq. (13) is positive, and the values \(\theta_1, \theta_2\) are real. According to Vieta theorem, their product is equal to 1, and the inequalities

\[
\theta_1 < 1 < \theta_2
\]

are fulfilled. On the contrary, if inequality (14) is not fulfilled, then the values \(\theta_1, \theta_2\) are complex conjugated, and \(|\theta_1| = |\theta_2| = 1\).

Let us resolve the relation \(\eta = \lambda + \lambda^{-1}\) as a quadratic equation

\[
\lambda^2 - \eta\lambda + 1 = 0.
\]

For both \(\eta_1, \eta_2\) we obtain the following roots of characteristic Eq. (6):

\[
\lambda_1 = \eta_1(\omega)/2 - \sqrt{\eta_1^2(\omega)/4 - 1},
\]
\[
\lambda_2 = \eta_2(\omega)/2 - \sqrt{\eta_2^2(\omega)/4 - 1},
\]
\[
\lambda_3 = \eta_1(\omega)/2 + \sqrt{\eta_1^2(\omega)/4 - 1},
\]
\[
\lambda_4 = \eta_2(\omega)/2 + \sqrt{\eta_2^2(\omega)/4 - 1}.
\]

Note. According to Vieta theorem, absolute values of both roots of Eq. (15) are equal to 1, or one absolute value is smaller than 1 and the other one is greater. The first version takes place iff \(\eta \in [-2, 2] \subset \mathbb{R}\). As \(\omega \to 0 \in \mathbb{C}\) the version is not realised nevermore. Really, if inequality (14) is fulfilled, the square roots
in Eq. (13) are real and \(2 < \theta_1 < \theta_2\). If equality \(\mu^2 = 2\nu\) is fulfilled, then \(2 < \theta_1 = \theta_2\). If inequality, which inverse to (14) is fulfilled, then the values \(\theta_1, \theta_2\) are not real. Thus, at small \(\omega\) we obtain \(\eta_{1,2} \notin [-2, 2]\), i.e. the necessary condition of the mixed initial-boundary problem correctness for the finite-difference scheme (3) is fulfilled.

We can rewrite the functions \(\lambda_{1,3}(\omega)\) in the form

\[
\lambda_{1,3}(\omega) = \frac{\eta_1(\omega)}{2} \times \sqrt{\frac{1}{4} (\theta_1 + r_1(\omega))^2 - 1},
\]

and factor the radicand:

\[
\lambda_{1,3}(\omega) = \frac{\eta_1(\omega)}{2} \times \sqrt{\frac{\theta_1^2}{4} - 1} \cdot \sqrt{\frac{1 + r_1(\omega)}{\theta_1 + 2} \cdot \sqrt{1 + \frac{r_1(\omega)}{\theta_1 - 2}}},
\]

Then we represent the radicand as a product

\[
\lambda_{1,3}(\omega) = \frac{\eta_1(\omega)}{2} \times \sqrt{\frac{\theta_1^2}{4} - 1} \cdot \frac{1}{\sqrt{1 + \frac{r_1(\omega)}{\theta_1 + 2} \cdot \sqrt{1 + \frac{r_1(\omega)}{\theta_1 - 2}}}},
\]

and use the formula for Taylor series expansion of a square root:

\[
\sqrt{1 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1 - 2n)n!4^n} x^n
\]

to obtain Taylor series for characteristic roots in a vicinity of \(\omega = 0\)

\[
\lambda_{1,3}(\omega) = \frac{\eta_1(\omega)}{2} \times \sqrt{\frac{\theta_1^2}{4} - 1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! r_1^n(\omega)}{(1 - 2n)n!4^n (\theta_1 + 2)^n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! r_1^n(\omega)}{(1 - 2n)n!4^n (\theta_1 - 2)^n},
\]

where \(\eta_1(\omega)\) and \(r_1(\omega)\) are taken from Eq. (10) and Eq. (12) correspondingly.

In the same way we obtain the Taylor series expansion for other characteristic roots:

\[
\lambda_{2,4}(\omega) = \frac{\eta_2(\omega)}{2} \times \sqrt{\frac{\theta_2^2}{4} - 1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! r_2^n(\omega)}{(1 - 2n)n!4^n (\theta_2 + 2)^n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! r_2^n(\omega)}{(1 - 2n)n!4^n (\theta_2 - 2)^n},
\]

where functions \(\eta_2(\omega)\) and \(r_2(\omega)\) are taken from Eq. (11) and Eq. (12) correspondingly.

The following inequalities are fulfilled as \(\omega \to 0\)

\[|\lambda_1|, |\lambda_2| < |\lambda_3|, |\lambda_4|.
\]

Therefore, as \(m \to +\infty\) it is possible to derive decreasing \(\lambda_1^m, \lambda_2^m\) and increasing \(\lambda_3^m, \lambda_4^m\) solutions of finite-difference Eq. (5), which form the fundamental set of solutions for Eq. (5).

3.3. ICP Boundary Conditions and Rational Approximations

As with differential Eq. (2), for correctness of mixed initial-boundary value problem for finite-difference Eq. (3) two boundary conditions at each edge of the rod are required. Thus, the values of the solution \(u\) at boundary and preboundary grid points \(x_0 = -L/2\) and \(x_1 = -L/2 + h\) should be calculated at every time step. We start by construction of \(Z\)-image of the boundary conditions for the left edge in the form

\[
P_1(\omega) v(0) + Q_1(\omega) v(h) + R_1(\omega) v(2h) + S_1(\omega) v(3h) = 0,\]

\[
P_2(\omega) v(0) + Q_2(\omega) v(h) + R_2(\omega) v(2h) + S_2(\omega) v(3h) = 0,
\]

which (after inverse \(Z\)-transformation) corresponds to the relations
\[
\sum_{j=0}^{\infty} p_{kj} u_0^{n-j} + \sum_{j=0}^{\infty} q_{kj} u_1^{n-j} + \sum_{j=0}^{\infty} r_{kj} u_2^{n-j} + \sum_{j=0}^{\infty} s_{kj} u_3^{n-j} = 0, \quad k = 1, 2.
\]

(18)

where values \( p_{kj}, q_{kj}, r_{kj} \) and \( s_{kj} \) are the Taylor series coefficients before the term \( \omega^j \) of the functions \( P_k(\omega), Q_k(\omega), R_k(\omega), S_k(\omega) \) correspondingly in a vicinity of the point \( \omega = 0 \in \mathbb{C} \).

Note. Here we compose stencils for the boundary conditions from 4 points: \(-L/2, -L/2 + h, -L/2 + 2h, -L/2 + 3h\). However, more space steps could be considered.

Two (with numbers \( k = 1, 2 \)) linearly independent boundary conditions will provide ICP property, if for the increasing Cauchy problem solutions \( \nu(m) = \lambda_m^0 \) and \( \nu(m) = \lambda_m^1 \) the symbols of the boundary conditions \( \langle P_1, Q_1, R_1, S_1 \rangle \) and \( \langle P_2, Q_2, R_2, S_2 \rangle \) fulfil the following equations:

\[
\begin{align*}
P_k + Q_k \lambda_1 + R_k \lambda_2^2 + S_k \lambda_3^3 &= 0, \\
P_k + Q_k \lambda_1 + R_k \lambda_2^2 + S_k \lambda_3^3 &= 0.
\end{align*}
\]

For any given values \( \omega \) the subspace of solutions of Syst. (19) is two-dimensional and two boundary conditions on every edge can provide a boundary problem correctness for finite-difference Eq. (4). However, the subspace of solutions of Syst. (19) in the space of analytic vector-functions of \( \omega \) is infinity-dimensional.

The meromorphic functions \( \lambda_1(\omega), \lambda_2(\omega) \) are not rational, and usually solutions of Syst. (19) in the subspace of polynomials do not exist. Therefore ICP boundary conditions are strongly non-local with respect to time. They include values of the solution of the boundary problem for Eq. (3) in the infinite number of previous time moments. For such realisation of the ICP boundary conditions the number of arithmetic operations, as well as a necessary computer memory, is proportional to the number of the temporal steps \( n \).

That is why we relax the requirements to symbols of the operators of ICP boundary conditions, and exchange analytic functions in Syst. (19) by polynomials and exact equalities by asymptotic (as \( \omega \to 0 \)) equalities:

\[
\begin{align*}
P_k(\omega) + Q_k(\omega) \lambda_1(\omega) + R_k(\omega) \lambda_2^2(\omega) + S_k(\omega) \lambda_3^3(\omega) &= O(\omega^{K_k}), \\
P_k(\omega) + Q_k(\omega) \lambda_1(\omega) + R_k(\omega) \lambda_2^2(\omega) + S_k(\omega) \lambda_3^3(\omega) &= O(\omega^{K_k}), \quad k = 1, 2.
\end{align*}
\]

(20)

"Physical" interpretation of the exchange: we neglect the impact of solution’s values at the long time in the past, assuming the resulting opacity is small.

We fix degrees of the polynomials \( P_k, Q_k, R_k, \) and \( S_k \), i.e. stencils for the ICP boundary conditions. The number of degrees of freedom in these four polynomials is equal to \( M_k = \deg P_k + \deg Q_k + \deg R_k + \deg S_k + 4 \), and the number of linear algebraic equations, which are obtained from Syst. (20) is equal to \( 2K_k \). Together with two normalisation conditions, which will be considered below, we obtain \( 2K_k + 2 \) linear algebraic equations. Thus, if \( M_k = 2K_k + 2 \) and the system of linear algebraic equations is non-degenerated, we determine unique set of the polynomials \( P_k, Q_k, R_k, \) and \( S_k \) with given degrees.

If we choose normalisation condition at \( k = 1 \):

\[
P_1(0) = p_{1,0} = 1, \quad Q_1(0) = q_{1,0} = 0,
\]

then we are able to compute the value \( u_0^n \) for every temporal step \( n \) using the values in the inner points \( u_{2h}^j, u_{3h}^j \) at \( j \leq n \) and in the border and preborder points at previous time moments: \( u_0^j, u_1^j \) at \( j < n \). We obtain the first boundary condition in the form:

\[
\sum_{j=1}^{\deg P_1} p_{1j} u_0^{n-j} + \sum_{j=1}^{\deg Q_1} q_{1j} u_1^{n-j} + \sum_{j=0}^{\deg R_1} r_{1j} u_2^{n-j} + \sum_{j=0}^{\deg S_1} s_{1j} u_3^{n-j} = 0.
\]

(21)

If we choose normalisation condition at \( k = 2 \):

\[
P_2(0) = p_{2,0} = 0, \quad Q_2(0) = q_{2,0} = 1,
\]

7
then we are able to compute the value $u_n^j$ for every temporal step $n$ using the values in the inner points $u_n^j$, $u_n^j$ at $j \leq n$ and in the border and preborder points at previous time moments: $u_n^j$, $u_n^j$ at $j < n$. We obtain the second ICP boundary condition:

$$u_n^j + \sum_{j=1}^{\deg P_2} p_{j} u_{n-j} + \sum_{j=1}^{\deg Q_2} q_{j} u_{n-j} + \sum_{j=0}^{\deg P_2} r_{j} u_{n-j} + \sum_{j=0}^{\deg S_2} s_{j} u_{n-j} = 0. \quad (22)$$

Note. Conditions in Syst. (20) are similar to the famous Hermite – Padé rational approximation of meromorphic functions. However, it is identical in the case of unique ICP boundary conditions (see [4, 8, 12, 13]), but for two ICP boundary conditions it is more general vector rational approximation. Similar constructions, which can be considered as a generalisation of the Hermite – Padé rational approximation, were studied in [13].

The coefficients of Syst. (20) are not real, because the functions $\lambda_j(\omega)$ and $\lambda_j(\omega)$ are, in general, complex. As a result, the characteristic values are complex conjugated, and we can transform the systems to real form:

$$\begin{align*}
\begin{cases}
P_1(\omega) + Q_1(\omega) \frac{\lambda_1(\omega) + \lambda_1(\omega)}{2} + R_1(\omega) \frac{\lambda_1(\omega) + \lambda_1(\omega)}{2} + S_1(\omega) \frac{\lambda_1(\omega) + \lambda_1(\omega)}{2} = O(\omega^{K_1}), \\
Q_1(\omega) \frac{\lambda_1(\omega) - \lambda_1(\omega)}{2} + R_1(\omega) \frac{\lambda_1(\omega) - \lambda_1(\omega)}{2} + S_1(\omega) \frac{\lambda_1(\omega) - \lambda_1(\omega)}{2} = O(\omega^{K_1}), \\
P_1(0) = 1, \\
Q_1(0) = 0,
\end{cases}
\end{align*} \quad (23)$$

at $k = 1$, and

$$\begin{align*}
\begin{cases}
P_2(\omega) + Q_2(\omega) \frac{\lambda_2(\omega) + \lambda_2(\omega)}{2} + R_2(\omega) \frac{\lambda_2(\omega) + \lambda_2(\omega)}{2} + S_2(\omega) \frac{\lambda_2(\omega) + \lambda_2(\omega)}{2} = O(\omega^{K_1}), \\
Q_2(\omega) \frac{\lambda_2(\omega) - \lambda_2(\omega)}{2} + R_2(\omega) \frac{\lambda_2(\omega) - \lambda_2(\omega)}{2} + S_2(\omega) \frac{\lambda_2(\omega) - \lambda_2(\omega)}{2} = O(\omega^{K_1}), \\
P_2(0) = 0, \\
Q_2(0) = 1,
\end{cases}
\end{align*} \quad (24)$$

at $k = 2$.

We solve Syst. (23) (24) and find the coefficients of the approximate ICP boundary conditions on the left edge of the rod. Using the same approach we find coefficients for the approximate ICP boundary conditions on the right edge. The results of numerical experiments with such ICP boundary conditions are presented in Sect. (5).

4. Initial Data Construction

Let us decompose solution $u(t, x)$ into Taylor series with respect to time in a vicinity of $t = 0$:

$$u(t, x) = U_0(x) + U_1(x)t + U_2(x)t^2 + \ldots, \quad (25)$$

where $U_t(x) = \frac{\partial u}{\partial t}(0, x)$. The functions $U_0$, $U_1$ compose initial conditions for Eq. (1). To determine the left hand side in Eq. (25) with a small error $O(t^3)$, we need to compute the function $U_2$.

We differentiate Eq. (25):

$$\frac{\partial^2 u}{\partial t^2}(t, x) = \sum_{k=2}^{\infty} U_k(x) \frac{\tau^{k-2}}{(k-2)!},$$

$$\frac{\partial^4 u}{\partial t^4}(t, x) = \sum_{k=2}^{\infty} U_k^\prime(x) \frac{\tau^{k-2}}{(k-2)!},$$

$$\frac{\partial^4 u}{\partial t^4}(t, x) = \sum_{k=2}^{\infty} U_k^{(4)}(x) \frac{\tau^k}{k!}.$$
and substitute the series into Eq. (2):

\[
\sum_{k=2}^{\infty} U_k(x) \frac{x^{k-2}}{(k-2)!} - R^2 \sum_{k=2}^{\infty} U_k'(x) \frac{x^{k-2}}{(k-2)!} + \frac{ER^2}{\rho} \sum_{k=0}^{\infty} U_k^{iv}(x) \frac{x^k}{k!} = 0.
\]

Collecting the similar terms at zeroth degree of \( \tau \) we obtain the following linear ordinary differential equation for \( U_2 \) with constant coefficients:

\[
\left[ D\frac{d^2}{dx^2} - 1 \right] U_2(x) = C U_0^{iv}(x),
\]

where \( D = R^2 \) and \( C = ER^2\rho^{-1} \).

We assume here that both initial function \( U_0^{iv}(x) \) and auxiliary function \( U_2(x) \) decrease at infinity. In this case, the principal term of the asymptotic at infinity of a limited solution of non-homogeneous differential equation can be determined from homogeneous one:

\[
U_2(x) \sim \exp \left( \frac{-x}{\sqrt{D}} \right) \text{ as } x \to +\infty, \quad U_2(x) \sim \exp \left( \frac{x}{\sqrt{D}} \right) \text{ as } x \to -\infty.
\]

We approximate the Eq. (26) by compact finite-difference scheme (for details see, e.g. [2, 19]):

\[
aU_2(x_{j-1}) + U_2 + aU_2(x_{j+1}) = pU_0(x_{j-2}) + qU_0(x_{j-1}) + rU_0(x_j) + qU_0(x_{j+1}) + pU_0(x_{j+2}),
\]

\[j \in [-J + 1, J - 1].\]

To determine the unknown four values \( a, p, q, r \) we substitute into Eq. (28) four pairs of the test functions \( \langle u_k, f_k \rangle, k = 0, 2, 4, 6 \), which satisfy Eq. (26) and are listed in Table 1.

| No | \( u_k \) | \( f_k \) | Algebraic equation for the coefficients of Eq. (28) |
|----|-----------|-----------|--------------------------------------------------|
| 0  | 0         | 1         | \( 2p + 2q + r = 0 \) |
| 1  | 0         | \( x^2 \) | \( 8p + 2q = 0 \) |
| 2  | \(-24C\)  | \( x^4 \) | \(-24C(2a + 1) = (32p + 2q)h^4 \) |
| 3  | \(-360C(x^2 + 2D)\) | \( x^6 \) | \(-720DC(2a + 1) - 720Cah^2 = 2p(2h)^6 + 2q(2h)^6 \) |

Table 1: Pairs of test functions and the corresponding linear algebraic equations for coefficients of compact finite-difference scheme (28)

Solution of the system of these four linear algebraic equations is

\[
a = \frac{h^2 - 6D}{12D + 4h^2}, \quad p = \frac{-3C}{2h^2(3D + h^2)}, \quad q = \frac{6C}{h^2(3D + h^2)}, \quad r = \frac{-9C}{h^2(3D + h^2)}.
\]

Three-diagonal system of \( 2J-1 \) linear algebraic Eq. (28) and finite-difference approximations of Eq. (27): \( U_2(x_{j-1}) = \exp(-h/\sqrt{D})U_2(x_{j-1}) \) and \( U_2(x_j) = \exp(-h/\sqrt{D})U_2(x_{j-1}) \) is closed and non-degenerated. It can be solved by the classical double-sweep method.

Note: If it is necessary to increase the order of accuracy by \( \tau \) of the initial function \( u(\tau, x) \), we should similarly define a function \( U_3(x) \) and substitute it into the Eq. (2), etc.

5. Results of Numerical Experiments

5.1. Rod’s and Scheme’s Parameters, Initial Conditions, and Reference Solution

For our experiments we choose rod’s parameters that are similar to steel: \( \rho = 7860 \text{ kg m}^{-3}, \ E = 210 \times 10^9 \text{ Pa} \). Radius of the rod is \( R = 10^{-3} \text{ m} \). The step \( h \) with respect to \( x \) is equal to 0.02 m, and step \( \tau \) with respect to \( t \) is equal to 1.6 \times 10^{-4} \text{ s} \). The length of the rod is \( L = 1 \text{ m} \), integration time is \( T = 0.04 \text{ s} \). In this case, dimensionless parameters are \( v = 4.274809 \) and \( \mu = 0.0025 \).
We set initial conditions for Eq. (2) as $u(0, x) = x \exp \left(-\frac{x^2}{0.02}\right)$ and $\frac{\partial u}{\partial t}(0, x) = 0$ for $x \in [-L/2, L/2]$. Therefore, for Eq. (3) we have initial conditions $u_0(x_i) = u(0, x_i)$ and $u_\tau(x_i) = u(\tau, x_i)$ is calculated using the algorithm proposed in Sect. 4 (see Fig. 1). Note that the initial condition $u_0(x)$ is close to zero at the ends of the segment $[-L/2, L/2]$. We define a reference solution $u^*(t, x)$ on the extended segment $[-15L, 15L]$ using the same formulae (see Fig. 2). We use the simplest (Dirichlet + Neumann) homogeneous boundary conditions for the extended segment. The initial rod’s perturbation goes away from the small segment. However, the plot Fig. 2 confirms that the boundary conditions cannot significantly influence the solution at the segment $[-L/2, L/2]$ during the period $[0, T]$.

5.2. Basic Version of ICP Boundary Conditions

There are infinitely many choices of degrees of polynomials $\langle P_1, Q_1, R_1, S_1 \rangle$ and $\langle P_2, Q_2, R_2, S_2 \rangle$, as well as an infinite number of corresponding approximate ICP boundary conditions. Moreover, it is essential to control the solvability of Syst. (23, 24). If at least one of the systems does not have the unique solution for the specific set of polynomials degrees, then for this set of degrees we can not construct the ICP boundary condition.
Table 2: The coefficients of the ICP boundary conditions are obtained from Eq. (21) and Eq. (22) for sets of polynomial degrees \(\langle P_k, Q_k, R_k, S_k \rangle = \langle 4, 4, 8, 8 \rangle\) at \(k = 1, 2\).

| \(k\) | \(P_k\) | \(Q_k\) | \(R_k\) | \(S_k\) | \(P_k\) | \(Q_k\) | \(R_k\) | \(S_k\) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 1   | 0   | 0.555979 | 0.278657 | 0   | 1   | -0.925737 | 0.301010 |
| \(\omega\) | -1.039354 | -1.064260 | -0.925512 | -0.300505 | -1.498177 | 0.962232 | -0.272787 |
| \(\omega^2\) | 1.040798 | 0.175892 | -0.343658 | 0.205584 | -0.057023 | 1.346122 | -0.918314 |
| \(\omega^3\) | -0.484423 | -0.688193 | 1.007943 | -0.361839 | 0.240692 | -1.187154 | 0.993006 |
| \(\omega^4\) | 0.217631 | -0.187829 | 0.258996 | -0.095354 | -0.007746 | 0.054903 | -0.020261 |
| \(\omega^5\) | 0.101158 | -0.063710 | -0.095354 | -0.007746 | -0.002124 | 0.000827 |
| \(\omega^6\) | 0.008250 | -0.016540 | 0.002764 | -0.000070 | 0.000827 |
| \(\omega^7\) | -0.014938 | 0.002764 | -0.005037 | 0.000709 |
| \(\omega^8\) | -0.005839 | 0.002373 | -0.002124 |

Syst. (23) for \(k = 1\), and Syst. (24) for \(k = 2\) consist of two equally righted equations for real and imaginary parts with the smallness order \(K\). Also we have two normalisation conditions on coefficients. Thus, the total number of unknown coefficients in Syst. (23) and Syst. (24) should be even.

As an example, let us consider a set of polynomial degrees

\[
\begin{align*}
\deg P_k = \deg Q_k &= 4, \\
\deg R_k = \deg S_k &= 8, \\
&k = 1, 2,
\end{align*}
\]

(29)

Here and further we consider equal sets of polynomial degrees for border and preborder boundary conditions. Coefficients that correspond to the solutions of Syst. (23, 24) for degree sets (29) are presented in Table 2.

To evaluate the dynamics of error of obtained solution \(u\) of Eq. (3) with ICP boundary conditions we use common logarithm of the Chebyshev norm \(C[-L/2, L/2]\) of solutions' difference, i.e.

\[
\lg \max_{j=0,1,...,N} |u(t, x_j) - u'(t, x_j)|.
\]

Results of numeric experiment with ICP boundary sets \(\langle 4, 4, 8, 8 \rangle\) are presented in Fig. 3.

Note. We submit the values in Tables with 6 decimal digits here. Our numerical experiments showed that without 6-th decimal digit the error increases just slightly, whereas without 5-th decimal digit the increase is significant.

Figure 3: Solid and dotted lines — logarithm of C-norm of the difference between the reference solution \(u'\) and solution with ICP boundary conditions. Dotted line corresponds to ICP boundary conditions with additional requirement Eq. (30) taken into account in the polynomial coefficients calculation. Dashed line — common logarithm of C-norm of the reference solution.
Table 3: The coefficients of the ICP boundary conditions are obtained from Eq. (21), Eq. (22), and additional condition Eq. (30) for sets of polynomial degrees \((P_k, Q_k, R_k, S_k) = (4, 5, 8, 8)\) at \(k = 1, 2\).

| \(P_1\) | \(Q_1\) | \(R_1\) | \(S_1\) | \(P_2\) | \(Q_2\) | \(R_2\) | \(S_2\) |
|------|------|------|------|------|------|------|------|
| 1    | 1    | 0    | -0.555979 | 0.278657 | 0    | 1    | -0.925737 | 0.30101 |
| \(\omega\)  | -2.554692 | 2.432054 | -1.468661 | 0.329664 | -0.491692 | -0.272165 | 0.412516 | -0.146529 |
| \(\omega^2\) | 2.067232 | -1.792876 | 0.313255 | 0.091235 | 0.249452 | 0.758283 | -0.722172 | 0.255585 |
| \(\omega^3\) | -2.170815 | 2.376252 | -0.936209 | 0.136683 | -0.262835 | -0.272165 | 0.412516 | -0.146529 |
| \(\omega^4\) | -1.38585 | 0.900316 | -0.202468 | 0.32292 | -0.3036 | 0.17184 | -0.063135 |
| \(\omega^5\) | 0.54543 | -0.195266 | -1.55023 | 0.009158 | -0.011746 | 0.004913 | -0.005389 |
| \(\omega^6\) | -0.007229 | 0.000655 | -0.002735 | 0.000079 |

5.3. Modification of the ICP Boundary Conditions and Various Versions’ Comparison

The function \(u(t, x) \equiv \text{const}\) is a solution of differential Eq. (2) and the finite-difference Eq. (3). We can additionally require ICP boundary conditions to satisfy this solution. In other words, we introduce an additional linear condition on the polynomial coefficients in Syst. (23, 24) – their sums are equal to zero:

\[
P_k(1) + Q_k(1) + R_k(1) + S_k(1) = 0, \quad k = 1, 2.\tag{30}
\]

As mentioned in Subs. 5.1, the even number of unknown coefficients is required to construct ICP boundary conditions. To maintain the same approximation order of Eq. (23, 24) with additional condition Eq. (30), one extra coefficient is required and, thus, the number becomes odd.

As a new example, modify previous ICP boundary condition sets:

\[
\text{deg } P_k = 4, \ \text{deg } Q_k = 5, \ \text{deg } R_k = 8, \ \text{deg } S_k = 8, \quad k = 1, 2.\tag{31}
\]

Coefficients of these sets that are derived from Eq. (23, 24) with additional condition Eq. (30) are presented in Table 3.

Error of ICP boundary condition Eq. (31) with additional requirement Eq. (30) is much smaller, see Fig. 3. Therefore, introduction of this condition on coefficients results in decrease of error for any time moment \(t\).

Figure 4: Logarithm of C-norm of the difference between reference solution \(u^*\) and solutions with ICP boundary conditions under additional requirement Eq. (30). Narrow dashed line — logarithm of C-norm of the reference solution \(u^*\)

As mentioned earlier, there are infinitely many approximate ICP boundary conditions since it is possible to choose any set of polynomial degrees in Syst. (23, 24). The errors for some options with additional requirement Eq. (30) are presented on Fig. 4.
5.4. Comparison of ICP Boundary Conditions with Various Versions of “Usual” Homogeneous Ones

In practice simple homogeneous boundary conditions (i.e., Dirichlet, Neumann) are usually used when there is no information about physical processes on the border. These conditions lead to partial or complete reflection of outgoing waves back into calculation area (sometimes with increased amplitude). On the contrary, ICP boundary conditions that are calculated using vectorial Hermite – Padé approximation techniques have low level of reflection.

Fig. 5 shows the dynamics of solutions errors that are calculated using various “usual” boundary conditions:

i) \( u|_{\Gamma} = 0, \ \partial u / \partial x \bigg|_{\Gamma} = 0 \Rightarrow u^n_0 = u^n_1 = 0 \),

ii) \( u|_{\Gamma} = 0, \ \partial^2 u / \partial x^2 \bigg|_{\Gamma} = 0 \Rightarrow u^n_0 = 0, u^n_1 = u^n_2 / 2 \),

iii) \( \partial^3 u / \partial x^3 \bigg|_{\Gamma} = 0, \ \partial^2 u / \partial x^2 \bigg|_{\Gamma} = 0 \Rightarrow u^n_0 = 3u^n_2 - 2u^n_3, u^n_1 = 2u^n_2 - u^n_3 \),

iv) ICP boundary condition with polynomials degrees Eq. (31) under additional requirement Eq. (30).

All these homogeneous boundary conditions i) - iii) lead to a significant reflection of outgoing waves back into computational area. The obtained solutions almost immediately differ from the reference solution, whereas the solution obtained with ICP boundary condition iv) stays close to \( u^* \) during all integration time.

Evolution of the solution with ICP boundary conditions with degrees \( 4, 5, 8, 8 \) (Eq. (31) with normalisation Eq. (30)) and of the reference solution is shown on Fig. 6.

![Figure 5: Dashed, dotted and dash-dotted lines — logarithm of C-norm of the difference between reference solution \( u^* \) and solution with “usual” boundary conditions. Solid line corresponds to ICP boundary conditions with additional requirement Eq. (30). Narrow dashed line — logarithm of C-norm of the reference solution \( u^* \)](image-url)
Figure 6: Dash-dotted line – reference solution $u^*$, solid line – obtained solution using ICP boundary conditions with polynomial degrees $(P_k, Q_k, R_k, S_k) = (4, 5, 8, 8)$ for border and preborder points with additional requirement Eq. (30) at different time moments: (a) $t = 0$, (b) $t = 5\tau$, (c) $t = 10\tau$, (d) $t = 15\tau$, (e) $t = 20\tau$, (f) $t = 30\tau$, (g) $t = 40\tau$, (h) $t = 50\tau$, (i) $t = 75\tau$, (j) $t = 125\tau$, (k) $t = 175\tau$, (l) $t = 225\tau$.
6. Conclusion

The Imitating Cauchy Problem (ICP) boundary conditions were constructed here for the finite-difference Crank – Nicolson implicit approximation of the transverse vibrations equation of a rod (beam) with a circular cross section. ICP boundary conditions provide solutions of mixed initial-boundary value problem on a segment that is close to the solution on the infinite domain. It is shown that “usual” homogeneous boundary conditions do not have this "transparency" property. The need of such ICP boundary conditions is seen in many scientific and technical applications. The approach may be applied when we need to imitate a non zeroth background solution (at a large area) and a forcing $f$. In these cases ICP boundary conditions will be non-homogeneous.

ICP boundary conditions might be used in mathematical modelling of processes on a limited area, when it is certain that external processes do not have any essential impact on the interior. On the other hand, ICP boundary conditions could suspend all fluctuations in the limited area without using high viscosity on the border.

Previously such boundary conditions were obtained for simpler differential equations of mathematical physics (transport, diffusion, wave equations) and its various finite-difference approximations. One boundary condition on every edge is needed for correctness of these mixed initial-boundary problems.

In this paper we construct ICP boundary conditions for the implicit finite-difference scheme that approximates the fourth order differential equation with respect to space. Both equations (differential and finite-difference) require two boundary conditions on each end. The considered differential equation is more sophisticated than many classic mathematical physics equations, because it is not resolved with respect to the highest derivative of a solution with respect to time (i.e., it does not belong to Cauchy – Kovalevskaya type).

If the coefficients of Eq. (1) are not constant far from the segment’s ends, then the ICP conditions will not change.

If a multidimensional problems is considered in an area $V$, then the boundary $\partial V$ is a sub-manifold (in the simplest case it is a plane). ICP boundary conditions for the problems are non-local and include a convolutions not only with respect to time, but by spatial variables that are tangent to $\partial V$.

All ICP boundary conditions are characterised by lots of numerical parameters, which should be defined with at least five decimal digits. We describe the algorithm of parameters’ determination (symbolic computations were used), which is the main result of this paper.

Proposed algorithms of ICP boundary conditions construction can be used with various parameters, as well as for other evolutionary linear equations and systems and its finite-difference approximations. However, recalculations of all parameters of the ICP boundary conditions are required if other finite-difference approximation scheme is used.

Characteristic feature of ICP boundary conditions is non-locality in time. Values of the solution in the vicinity of a border at previous time steps are required. Theory of functions of a complex variable (generating function, rational approximation that is similar to Hermite – Padé approximation) allow us to reduce this number of time steps. This is especially substantial in the case of spatially multidimensional models (differential and finite-difference), where the corresponding ICP boundary conditions are non-local with respect to variables, that are tangential to the area’s boundary, see [2, 11, 12].

We present also the algorithm (based on compact finite-difference scheme) of initial functions calculation that provide high-order of approximation with respect to time for the implicit finite-difference equation.

Acknowledgements

The article was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2018 — 2019 (grant № 18-05-0011) and by the Russian Academic Excellence Project «5-100».
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