Relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index

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Abstract. Poly-Bernoulli numbers $B_n^{(k)} \in \mathbb{Q}(n \geq 0, k \in \mathbb{Z})$ are defined by Kaneko in 1997. Multi-Poly-Bernoulli numbers $B_n^{(k_1, k_2, \ldots, k_r)}$, defined by using multiple polylogarithms, are generations of Kaneko’s Poly-Bernoulli numbers $B_n^{(k)}$. We researched relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in particular. In section 2, we introduce a identity for Multi-Poly-Bernoulli numbers of negative index which was proved by Kamano. In section 3, as main results, we introduce some relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in particular.

1. Introduction

For any integer $k$, Kaneko [1] introduced Poly-Bernoulli numbers of index $k$ by the following generating function:

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

where $Li_k(t)$ is the $k$-th polylogarithm defined by

$$Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}.$$ 

Since $Li_1(t) = -\log(1-t)$, the number $B_n^{(1)}$ is the ordinary $n$-th Bernoulli number $B_n$, which is defined by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$ 

It is known that Poly-Bernoulli numbers of negative index are positive integers and we have a closed formula

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ \begin{array}{c} n+1 \\ j+1 \end{array} \right\} \left\{ \begin{array}{c} k+1 \\ j+1 \end{array} \right\}.$$ 

In particular, we have the following duality formula:
Moreover, these numbers have combinatorial applications: see [2] and [3] for details.

As a generalization of Poly-Bernoulli numbers, Multi-Poly-Bernoulli numbers $B^{(k_1,k_2,...,k_r)}_n$ are defined for integers $k_1,\ldots,k_r$ by the generating function

$$Li_{k_1,\ldots,k_r}(1-e^{-t})/((1-e^{-t})^r) = \sum_{n=0}^{\infty} B^{(k_1,\ldots,k_r)}_n \frac{t^n}{n!}.$$

where $Li_{k_1,\ldots,k_r}(t)$ is a multiple polylogarithm defined by

$$Li_{k_1,\ldots,k_r}(t) = \sum_{0<m_1<\cdots<m_r} \frac{t^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

When $r = 1$, the number $B^{(k)}_n$ is Poly-Bernoulli numbers. When $r = 1$ and $k_1 = 1$, the number $B^{(1)}_n$ is the classical Bernoulli numbers. It is also known that we have the following duality formula [4] for Multi-Poly-Bernoulli numbers:

$$B^{(0,\ldots,0,-k)}_n = B^{(0,\ldots,0,-n)}_k.$$

2. Relations of Multi-Poly-Bernoulli numbers of negative index

In this section, we introduce a identity for Multi-Poly-Bernoulli numbers of negative index which was proved by Kamano [4]. If $(k_1,\ldots,k_r) \neq (0,\ldots,0)$, then Multi-Poly-Bernoulli numbers of negative index $B^{(-k_1,-k_2,-\cdots,-k_r)}_n$ have the following expression.

**Theorem 2.1.** Let $r$ be a positive integer and let $k_1,\ldots,k_r$ be non-negative integers with $(k_1,\ldots,k_r) \neq (0,\ldots,0)$. We put $k := k_1 + \cdots + k_r$. Then the following identity holds:

$$B^{(-k_1,\ldots,-k_r)}_n = \sum_{l=1}^{k} \alpha_l^{(k_1,\ldots,k_r)} (l + r)^n \cdot A,$$

where $\alpha_l^{(k_1,\ldots,k_r)}$ $(1 \leq l \leq k)$ are integers depending only on $k_1,\ldots,k_r$, and they are inductively determined by the following recurrence relations:

(i) $\alpha_l^{(k_1)} = (-1)^l + k_1 \cdot \binom{l}{k_1}$,

(ii) $\alpha_l^{(k_1,\ldots,k_{r-1},0)} = \alpha_l^{(k_1,\ldots,k_{r-1})}$,

(iii) $\alpha_l^{(k_1,\ldots,k_{r-1},k_r+1)} = (l + r - 1) \alpha_{l-1}^{(k_1,\ldots,k_r)} - l \alpha_l^{(k_1,\ldots,k_r)}$. 

Here we set
\[ \alpha_0^{(k_1, \ldots, k_r)} = \begin{cases} 1 & \text{if } (k_1, \ldots, k_r) = (0, \ldots, 0), \\ 0 & \text{otherwise}, \end{cases} \]
and \( \alpha_l^{(k_1, \ldots, k_r)} = 0 \) for \( l > k \).

First we give recurrence relation [4] of Multi-Poly-Bernoulli numbers for the proof of Theorem 2.1.

**Lemma 2.2.** For non-negative integers \( n, k_1, \ldots, k_r \), we have
\[ B_n^{(-k_1, \ldots, -k_r-1)} = \sum_{m=0}^{n} \binom{n}{m} B_{m+1}^{(-k_1, \ldots, -k_r)} + r B_n^{(-k_1, \ldots, -k_r)} - B_{n+1}^{(-k_1, \ldots, -k_r)}. \]

Theorem 2.1 is proved by induction on \( r \). The following lemma [4] says that Theorem 2.1 holds for \( r = 1 \).

**Lemma 2.3.** For \( n \geq 0 \) and \( k \geq 1 \), we have
\[ B_n^{(-k)} = \sum_{l=1}^{k} (-1)^{l+k} \binom{k}{l} (l + 1)^n. \]

We note here the Corollary 2.4 has been proved by Hamahata and Masubuchi [5].

**Corollary 2.4.** Let \( r \) be a positive integer and let \( n \) and \( k \) be non-negative integers. Then the following identities hold:

1. \( \overline{B_n^{(0, \ldots, 0)}} = r^n \),
2. (duality) \( B_n^{(0, \ldots, 0, -k)} = B_k^{(0, \ldots, 0, -n)} \),
3. \( B_n^{(-k_1, \ldots, -k_r-1, 0)} = \sum_{i=0}^{n} \binom{n}{i} B_i^{(-k_1, \ldots, -k_r-1)} \) \((r \geq 2)\),
4. \( \sum_{i=0}^{k} \binom{k}{i} B_n^{(-i, -k)} p^i q^{k-i} = \sum_{i=0}^{k} \sum_{j=0}^{n} \binom{k}{i} \binom{n}{j} (p+q)^i q^{k-i} B_j^{(-i)} B_{n-j}^{(i-k)} \) where \( p \) and \( q \) are any real numbers.

We use the following generating function of Multi-Poly-Bernoulli numbers of negative index for the proof of Corollary 2.4. This generating function is a natural generalization of the following function:
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k \frac{1}{n! k!} = \frac{1}{e^{-x} + e^{-y} - 1}.
\]

**Theorem 2.5.** The following identity holds:

\[
\sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k_1, \ldots, -k_r)} x_1^{k_1} \frac{1}{k_1!} \cdots x_r^{k_r} \frac{t^n}{k_r!} \frac{1}{n!} = \frac{1}{(e^{-x_1} - x_2 - \cdots - x_r + e^{-t} - 1)(e^{-x_2} - \cdots - x_r + e^{-t} - 1) \cdots (e^{-x_r} + e^{-t} - 1)}.
\]

We can express Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index in a sum of powers by using Theorem 2.1. We give examples [4], [5] of Theorem 2.1 for \(1 \leq r \leq 3\) and \(1 \leq k \leq 3\) (Table 1).

| \(r = 1\) | \(B_n^{(-1)} = 2^n\) |
| --- | --- |
| \(B_n^{(-2)} = -2^n + 2 \cdot 3^n\) |
| \(B_n^{(-3)} = 2^n - 6 \cdot 3^n + 6 \cdot 4^n\) |

| \(r = 2\) | \(B_n^{(0,-1)} = 2 \cdot 3^n\) |
| --- | --- |
| \(B_n^{(-1,0)} = 3^n\) |
| \(B_n^{(0,-2)} = -2 \cdot 3^n + 6 \cdot 4^n\) |
| \(B_n^{(-2,0)} = -3^n + 2 \cdot 4^n\) |
| \(B_n^{(0,-3)} = 2 \cdot 3^n - 18 \cdot 4^n + 24 \cdot 5^n\) |
| \(B_n^{(-1,-2)} = 3^n - 9 \cdot 4^n + 12 \cdot 5^n\) |
| \(B_n^{(-2,-1)} = 3^n - 7 \cdot 4^n + 8 \cdot 5^n\) |
| \(B_n^{(-3,0)} = 3^n - 6 \cdot 4^n + 6 \cdot 5^n\) |
We found regularities from Table 1 and got the following relations. The proof uses Theorem 2.1.

**Theorem 2.6.** We have the following relations, and i-th component is \(-1\) and others are 0 in (3).

1. \(B_n^{(-1, 0, \ldots, 0)} = (r + 1)^n = B_n^{(0, \ldots, 0)}\).
2. \(B_n^{r-1, 0, \ldots, 0} = r(r + 1)^n\).
3. \(B_n^{0, \ldots, 0, -1, 0, \ldots, 0} = i(r + 1)^n\) (1 ≤ i ≤ r).
Proof. (1) We use Theorem 2.1(A) and (i),(ii). The second equality obtains from Corollary 1.4(1).

\[
B_n^{(-1,0,\ldots,0)} = \sum_{l=1}^{r-1} \alpha_l^{(1,0,\ldots,0)} (l + r)^n \\
= \alpha_1^{(1,0,\ldots,0)} (1 + r)^n \\
= \alpha_1^{(1)} (1 + r)^n.
\]

Since \( \alpha_1^{(1)} = (-1)^{1+1}! \{1\} = 1 \), we obtain \( B_n^{(-1,0,\ldots,0)} = (r + 1)^n = B_n^{(0,\ldots,0)} \).

(2) We use Theorem 2.1(A) and (iii).

\[
B_n^{(0,\ldots,0,-1)} = \sum_{l=1}^{r-1} \alpha_l^{(0,\ldots,0,1)} (l + r)^n \\
= \alpha_1^{(0,\ldots,0,1)} (1 + r)^n.
\]

Here, \( \alpha_1^{(0,\ldots,0,1)} = (1 + r - 1)\alpha_0^{(0,\ldots,0)} - 1 \cdot \alpha_1^{(0,\ldots,0)} \)

\[
= r\alpha_0^{(0,\ldots,0)} - \alpha_1^{(0,\ldots,0)} = r.
\]

Thus, we obtain \( B_n^{(0,\ldots,0,-1)} = r(r + 1)^n \).

(3) We use Theorem 2.1(A) and (ii).

\[
B_n^{(0,\ldots,0,-1,0,\ldots,0)} = \sum_{l=1}^{r-1} \alpha_l^{(0,\ldots,0,-1,0,\ldots,0)} (l + r)^n \\
= \alpha_1^{(0,\ldots,0,-1,0,\ldots,0)} (r + 1)^n \\
= \alpha_1^{(0,\ldots,0,1)} (r + 1)^n.
\]
Here from (2), we have 

\[ B_{n}^{(0,\ldots,0,-1)} = i(i + 1)^n. \]

Moreover from Theorem 2.1(A), we have

\[ B_{n}^{(0,\ldots,0,-1)} = \sum_{l=1}^{i-1} \alpha_{l}^{(0,\ldots,0,1)} (l + i)^n = \alpha_{1}^{(0,\ldots,0,1)} (i + 1)^n. \]

Since \( \alpha_{1}^{(0,\ldots,0,1)} = i \), we obtain 

\[ B_{n}^{(0,\ldots,0,-1)} = i(r + 1)^n. \]

We have 

\[ B_{n}^{(0,\ldots,0,-1)} = rB_{n}^{(-1,0,\ldots,0)} \]

from (1) and (2).

\[ \Box \]

3. Relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index

In this section, we introduce relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers of negative index. Poly-Bernoulli numbers of negative index can express by using the Stirling numbers of the second kind \( \{ \binom{n}{m} \} \) and Multi-Poly-Bernoulli numbers. Here Stirling numbers of the second kind are the number of ways to divide a set of \( n \) elements into \( m \) nonempty sets.

**Theorem 3.1.** Poly-Bernoulli numbers \( B_{n}^{(-k)} \) \( (n \geq 0, k \geq 1) \) can express as follows;

1. \( B_{n}^{(-k)} = (-1)^{k-1} B_{n}^{(0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \binom{k}{r+1} B_{n}^{(r,0,\ldots,0,-1)}. \)
2. \( B_{n}^{(-k)} = (-1)^{k-1} B_{n}^{(0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \binom{k}{r+1} B_{n}^{(r-1,0,\ldots,0)}. \)
3. \( B_{n}^{(-k)} = (-1)^{k-1} B_{n}^{(0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \binom{r-i+2}{i} B_{n}^{(r-i+2,0,\ldots,0,-1,0,\ldots,0)}. \)

**Example 3.2.** We give examples of Theorem 3.1(1) and (2) for \( 1 \leq k \leq 4 \).

1. \( B_{n}^{(-1)} = B_{n}^{(0,0)} \)  
\[ B_{n}^{(-2)} = B_{n}^{(0,0)} + B_{n}^{(0,-1)} \]  
\[ B_{n}^{(-3)} = B_{n}^{(0,0)} - 3B_{n}^{(0,-1)} + 2B_{n}^{(0,0,-1)} \]
\[B_n^{(-4)} = -B_n^{(0,0)} + 7B_n^{(0,-1)} - 12B_n^{(0,0,-1)} + 6B_n^{(0,0,0,-1)}\]

(2) \[B_n^{(-1)} = B_n^{(0,0)}\]
\[B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}\]
\[B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(-1,0)} + 6B_n^{(-1,0,0)}\]
\[B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(-1,0)} - 36B_n^{(-1,0,0)} + 24B_n^{(-1,0,0,0)}\]

Proof of the Theorem 3.1.

(1) From Lemma 2.3, we have
\[B_n^{(-k)} = \sum_{l=1}^{k} \frac{(-1)^{l+k} l!}{l} \binom{k}{l} (l+1)^n.\]
Here by putting \(l - 1 = r\), we obtain
\[B_n^{(-k)} = \sum_{r=0}^{k-1} (-1)^{r+k-1}(r+1)! \binom{k}{r+1} (r+2)^n\]
\[= (-1)^{k-1} 2^n + \sum_{r=1}^{k-1} (-1)^{r+k-1}(r+1)! \binom{k}{r+1} (r+2)^n.\]
Since \(B_n^{(0,0)} = 2^n\) from Corollary 2.4(1), we have
\[B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} r! \binom{k}{r+1} (r+1)(r+2)^n.\]
Here from Theorem 2.6(2), we have \(B_n^{(0,\ldots,0,-1)} = r(r+1)^n\). Thus we have
\[B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} r! \binom{k}{r+1} B_n^{(0,\ldots,0,-1)},\]
and we obtain the identity of (1).

(2) The proof of (2) uses the proof of (1). In the proof of (1), we have
\[B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1}(r+1)! \binom{k}{r+1} (r+2)^n.\]
Here from Theorem 2.6(1), we have \(B_n^{(-1,0,\ldots,0)} = (r+1)^n\). Hence we have
\[ B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k-1} (r+1)! \binom{k}{r+1} B_n^{(-1,0,\ldots,0)} , \]

and we obtain the identity of (2).

(3) From Lemma 2.3, we have

\[ B_n^{(-k)} = \sum_{l=1}^{k} (-1)^{l+k} l! \binom{k}{l} (l+1)^n. \]

Here by putting \( l - 1 = r - i + 1 \), we obtain

\[ B_n^{(-k)} = \sum_{r=i-1}^{k+i-2} (-1)^{r-i+k}(r+i+2)! \binom{k}{r-i+2}(r+i+3)^n \]

\[ = (-1)^{k-1} 2^n + \sum_{r=i}^{k+i-2} (-1)^{r-i+k}(r+i+2)! \binom{k}{r-i+2} \binom{r+i+2}{r-i+1} (r+i+3)^n. \]

Here from Theorem 2.6(3), we have \( B_n^{(0,0,0,\ldots,0,0)} = i(r+1)^n \). Thus we have

\[ B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r+i+2)!}{i} \binom{k}{r+i+2} i(r+i+3)^n. \]

Futhermore we can also express Poly-Bernoulli numbers of negative index using the Stirling numbers of the first kind \( \left[ \begin{array}{c} n \\ m \end{array} \right] \) and Multi-Poly-Bernoulli numbers.

Corollary 3.3. We have the following relations

(1) \( B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \left[ \begin{array}{c} -r-1 \\ -k \end{array} \right] B_n^{(0,0,0,\ldots,0)}. \)
\[(2) \quad B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \left[ \frac{-r-1}{-k} \right] B_n^{(r-1,0,...,0)}.\]

\[(3) \quad B_n^{(-k)} = (-1)^{k-1} B_n^{(0,0)} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} (r-i+2)! \left[ \frac{-r+i-2}{-k} \right] B_n^{(r-i,0,...,0)} B_n^{(0,0,-1,0,...,0)}.\]

The proof of Corollary 3.3 can be obtained from the following Lemma 3.4 [1].

**Lemma 3.4.** For any integers \(n\) and \(m\), we have

\[
\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} -m \\ -n \end{bmatrix}.
\]

Next we see the sum of coefficients on Multi-Poly-Bernoulli numbers of the identity which hold on Theorem 3.1. Therefore we revisit Example 3.2.

**Example 3.2** (Example 3.2 revisited).

We give examples of Theorem 3.1(1) and (2) for \(1 \leq k \leq 4\).

(1) \(B_n^{(-2)} = -B_n^{(0,0)} + B_n^{(0,-1)}\)

\[
B_n^{(-3)} = B_n^{(0,0)} - 3B_n^{(0,-1)} + 2B_n^{(0,0,-1)}
\]

\[
B_n^{(-4)} = -B_n^{(0,0)} + 7B_n^{(0,-1)} - 12B_n^{(0,0,-1)} + 6B_n^{(0,0,0,-1)}
\]

(2) \(B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)}\)

\[
B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(-1,0)} + 6B_n^{(-1,0,0)}
\]

\[
B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(-1,0)} - 36B_n^{(-1,0,0)} + 24B_n^{(-1,0,0,0)}
\]

In the case of (1), the sum of coefficients on Multi-Poly-Bernoulli numbers are 0 \((k \geq 2)\). In the case of (2), the sum of coefficients on Multi-Poly-Bernoulli numbers are 1 \((k \geq 2)\). From here we can be considered the following relations.

**Theorem 3.5.** We have the following relations for \(k \geq 2\)

(1) \((-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \left\{ \begin{array}{c} k \\ r + 1 \end{array} \right\} = 0.\)

(2) \((-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \left\{ \begin{array}{c} k \\ r + 1 \end{array} \right\} = 1.\)
\[(3) \ (-1)^{k-1} + \sum_{r=i}^{k+i-2} (-1)^{r-i+k} \frac{(r-i+2)!}{i!} \binom{k}{r-i+2} = \begin{cases} 1 & (k: \text{odd}) \\ \frac{1}{2} & (k: \text{even}) \end{cases} \]

We regard the sums of coefficients as 1 for \( k = 1 \).

Proof. (1) We have

\[ (-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} r! \binom{k}{r+1} = (-1)^{k-1} \left( 1 + \sum_{r=0}^{k-1} (-1)^r r! \binom{k}{r+1} \right) \]

Here, \((-1)^{r} r! \binom{k}{r+1} = 1\) for \( r = 0 \) and \((-1)^{r} r! \binom{k}{r+1} = 0\) for \( r = k \). Thus we have

\[ (-1)^{k-1} \left( 1 + \sum_{r=1}^{k-1} (-1)^{r} r! \binom{k}{r+1} \right) = (-1)^{k-1} \left( 1 + \sum_{r=0}^{k-1} (-1)^r r! \binom{k}{r+1} - 1 \right) \]

\[ = (-1)^{k-1} \sum_{r=0}^{k} (-1)^{r} r! \binom{k}{r+1} \]

\[ = (-1)^{k-1} \sum_{r=0}^{k} (-1)^{r} \left( r+1 \right) ! \binom{k}{r+1} \cdot \binom{k}{r+1} \]

Furthermore, since \( \sum_{l=0}^{n} (-1)^{l} \binom{n}{l} \binom{l}{m} = (-1)^{m} \delta_{m,n} ([1]) \), we have

\[ (-1)^{k} \sum_{r=0}^{k} (-1)^{r} \binom{r+1}{1} \binom{k}{r+1} = (-1)^{k} \delta_{1,k} \]

\[ = 0, \]

and we obtain the results.

(2) Considering in the same way with (1), we have

\[ (-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1} (r+1)! \binom{k}{r+1} = (-1)^{k-1} \left( 1 + \sum_{r=1}^{k-1} (-1)^{r} (r+1)! \binom{k}{r+1} \right) \]
\[
= (-1)^{k-1} \sum_{r=0}^{k-1} (-1)^r (r+1)! \binom{k}{r+1}
\]
\[
= (-1)^{k-1} \sum_{r=0}^{k-1} (-1)^r \sum_{l=0}^{r+1} \binom{r+1}{l} \binom{k}{r+1}
\]
\[
= (-1)^{k-1} \sum_{r=0}^{k} (-1)^r \sum_{l=0}^{k} \binom{r+1}{l} \binom{k}{r+1}
\]
\[
= (-1)^k \sum_{r=0}^{k} \sum_{l=0}^{k} (-1)^{r+1} \binom{k}{r+1} \binom{r+1}{l}.
\]

Here we use the aforesaid formula again; \[
\sum_{l=0}^{n} (-1)^l \binom{m}{l} \binom{n}{m} = (-1)^m \delta_{m,n} \] ([1]).

Then we have
\[
(-1)^k \sum_{l=0}^{k} \sum_{r=0}^{k} (-1)^{r+1} \binom{k}{r+1} \binom{r+1}{l} = (-1)^k \sum_{l=0}^{k} (-1)^l \delta_{l,k}
\]
\[
= (-1)^k \cdot (-1)^k
\]
\[
= 1.
\]

(3) (i) If \( k \) is odd, we put \( k = 2m + 1 \). Then we have
\[
(-1)^{2m} + \sum_{r=1}^{2m+1-i} (-1)^{r-i+2m+1} \binom{r-i+2m+1}{i} \binom{2m+1}{r-i+2} \]
\[
= 1 + \sum_{r=1}^{2m+1-i} (-1)^{r-i+2m+1} \binom{r-i+2m+1}{i} \binom{2m+1}{r-i+2}.
\]

Hence it suffices to show the following identity;
\[
\sum_{r=i}^{2m+1-i} (-1)^{2m+r-i+1} \binom{r-i+2}{i} \binom{2m+1}{r-i+2} = 0.
\]
\[
\sum_{r=i}^{2m+1-i} (-1)^{2m+r-i+1} \binom{r-i+2}{i} \binom{2m+1}{r-i+2}
\]
\[
= \sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} \binom{r-i+2}{i} \binom{2m+1}{r-i+2} - \frac{1}{i} \binom{2m+1}{1}
\]
\[
= \frac{1}{i} \sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1} \binom{r-i+2}{i} \binom{2m+1}{r-i+2} - \frac{1}{i}.
\]
Here from Theorem 3.5(2), since \((-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{r+k+1}(r+1)! \binom{k}{r+1} = 1\), we put \(k = 2m + 1\). Then we obtain

\[
(-1)^{2m} + \sum_{r=1}^{2m} (-1)^{r+2m}(r+1)! \binom{2m+1}{r+1} = 1
\]

\[
\sum_{r=1}^{2m} (-1)^{r+2m}(r+1)! \binom{2m+1}{r+1} = 0
\]

\[
\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1}(r-i+2)! \binom{2m+1}{r-i+2} - 1 = 0
\]

\[
\sum_{r=i-2}^{2m+i-1} (-1)^{2m+r-i+1}(r-i+2)! \binom{2m+1}{r-i+2} = 1.
\]

Hence, we have \(\sum_{r=i}^{2m+i-1} (-1)^{2m+r-i+1}(r-i+2)! \binom{2m+1}{r-i+2} = 0\) and, the sum of the coefficients are 1.

(ii) If \(k\) is even, we put \(k = 2m\). Then we have

\[
(-1)^{2m-1} + \sum_{r=1}^{2m+i-2} (-1)^{r-i+2m}(r-i+2)! \binom{2m}{r-i+2} = -1 + \sum_{r=i}^{2m+i-2} (-1)^{r-i+2m}(r-i+2)! \binom{2m}{r-i+2}.
\]

Hence, it suffices to show the following identity;

\[
\sum_{r=i}^{2m+i-2} (-1)^{2m+r-i}(r-i+2)! \binom{2m}{r-i+2} = \frac{2}{i}.
\]
\[
= \frac{1}{i} \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i}(r - i + 2)! \left\{ \frac{2m}{r - i + 2} \right\} + \frac{1}{i}.
\]

Here from Theorem 3.5(2), we put \( k = 2m \). Then we have

\[
(-1)^{2m-1} + \sum_{r=1}^{2m-1} (-1)^{r+2m+1}(r + 1)! \left\{ \frac{2m}{r + 1} \right\} = 1
\]

\[
2m+i-2 \sum_{r=i}^{2m-1} (-1)^{r+2m+1}(r + 1)! \left\{ \frac{2m}{r + 1} \right\} = 2
\]

\[
2m+i-2 \sum_{r=i}^{2m+i-2} (-1)^{2m+r-i}(r - i + 2)! \left\{ \frac{2m}{r - i + 2} \right\} = 2
\]

\[
2m+i-2 \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i}(r - i + 2)! \left\{ \frac{2m}{r - i + 2} \right\} + 1 = 2
\]

\[
2m+i-2 \sum_{r=i-2}^{2m+i-2} (-1)^{2m+r-i}(r - i + 2)! \left\{ \frac{2m}{r - i + 2} \right\} = 1.
\]

Hence, since \( \sum_{r=1}^{2m+i-2} (-1)^{2m+r-i}(r - i + 2)! \left\{ \frac{2m}{r - i + 2} \right\} = \frac{2}{i} \), the sum of coefficients are \( \frac{2}{i} - 1 \).

We found that Poly-Bernoulli numbers of negative index can express using the Stirling numbers of the second kind \( \{^n_m\} \) and the sum of Multi-Poly-Bernoulli numbers. This time, we introduce that special values of Multi-Poly-Bernoulli numbers which hold on Theorem 2.6 can express by using the sum of Poly-Bernoulli numbers.

**Theorem 3.6** \((r \geq 1)\). We have the following relations

\[(1) \quad B_n^{(-1,0,\ldots,0)} = \frac{1}{r!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)}.
\]

\[(2) \quad B_n^{(0,\ldots,0,-1)} = \frac{1}{(r - 1)!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)}.
\]

\[(3) \quad B_n^{(0,\ldots,0,-1,0,\ldots,0)} = \frac{i}{r!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)}.
\]
Proof of Theorem 3.6.

(1) \( \frac{1}{r!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)} = \frac{1}{r!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] \sum_{l=1}^{k} (-1)^{l+k} l! \left\{ \begin{array}{c} k \\ l \end{array} \right\} (l + 1)^n \)

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] \sum_{l=0}^{k} (-1)^{l+k} l! \left\{ \begin{array}{c} k \\ l \end{array} \right\} (l + 1)^n \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] \sum_{l=0}^{r} (-1)^{l+k} l! \left\{ \begin{array}{c} k \\ l \end{array} \right\} (l + 1)^n \]

\[ = \frac{1}{r!} \sum_{l=0}^{r} (-1)^l l! (l + 1)^n \sum_{k=l}^{r} (-1)^k \left[ \begin{array}{c} r \\ k \end{array} \right] \left\{ \begin{array}{c} k \\ l \end{array} \right\} \]

\[ = \frac{1}{r!} \sum_{l=0}^{r} (-1)^l l! (l + 1)^n (1 - l) \delta_{l,r} \]

\[ = \frac{1}{r!} \sum_{l=0}^{r} l! (l + 1)^n \delta_{l,r} \]

\[ = \frac{1}{r!} \cdot r! (r + 1)^n \]

\[ = (r + 1)^n = B_n^{(-1,0,...,0)} \]

(2) We consider in the same way as (1), and we obtain

\[ \frac{1}{(r - 1)!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)} = r(r + 1)^n = B_n^{(0,...,0,-1)}. \]

(3) We consider in the same way as (1), and we obtain

\[ \frac{i}{r!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_n^{(-k)} = i(r + 1)^n = B_n^{(0,...,0,-1,0,...,0)}. \]

This completes the proof. \( \square \)

The following Collorary 3.7 can be obtained by using Lemma 3.4 in the identity of Theorem 3.6. Therefore we omit the proof.

**Collorary 3.7** \((r \geq 1)\). We have the following relations

(1) \( B_n^{(-1,0,...,0)} = \frac{1}{r!} \sum_{k=1}^{r} \left\{ \begin{array}{c} -k \\ -r \end{array} \right\} B_n^{(-k)}. \)

(2) \( B_n^{(0,...,0,-1)} = \frac{1}{(r - 1)!} \sum_{k=1}^{r} \left\{ \begin{array}{c} -k \\ -r \end{array} \right\} B_n^{(-k)}. \)
We consider $B_{n}^{(0,\ldots,0,-1,0,\ldots,0)}$ which are the generalizations of Theorem 3.6(1).

Moreover, we consider $B_{n}^{(m,0,\ldots,0)}$, which are the generalizations of Theorem 3.6(1).

First, we consider in the case of $m = 2$, that is, $B_{n}^{(-2,0,\ldots,0)}$. We give examples for $1 \leq r \leq 4$ and $m = 2;

\begin{align*}
B_{n}^{(-2)} &= -2^n + 2 \cdot 3^n \\
B_{n}^{(-2,0)} &= -3^n + 2 \cdot 4^n \\
B_{n}^{(-2,0,0)} &= -4^n + 2 \cdot 5^n \\
B_{n}^{(-2,0,0,0)} &= -5^n + 2 \cdot 6^n \\
&\quad \vdots
\end{align*}

We use Theorem 2.6(1) ($B_{n}^{(r-1)} = (r+1)^n$) and Theorem 3.6(1), and we have

\begin{align*}
B_{n}^{(-2,0,\ldots,0)} &= \frac{2}{(r+1)!} \sum_{k=1}^{r+1} \binom{r+1}{k} B_{n}^{(-k)} - \frac{1}{r!} \sum_{k=1}^{r} \binom{r}{k} B_{n}^{(-k)}.
\end{align*}

We consider similarly in the case of $m = 3$, that is, $B_{n}^{(-3,0,\ldots,0)}$. Then we have
From here we can be considered the generalizations, that is, \( B_{n}^{(-m,0,\ldots,0)} \) as follows.

**Theorem 3.9.** We have the following relations on \( B_{n}^{(-m,0,\ldots,0)} \)

\[
B_{n}^{(-m,0,\ldots,0)} = \sum_{l=1}^{m} \frac{(-1)^{l+m} l! \{ m \}_{l}}{(r + l - 1)!} \sum_{k=1}^{r + l - 1} \begin{bmatrix} r + l - 1 \ k \end{bmatrix} B_{n}^{(-k)}.
\]

Proof. Using Theorem 2.6(1) and Theorem 3.6(1), we have

\[
\frac{1}{r!} \sum_{k=1}^{r} \begin{bmatrix} r \ k \end{bmatrix} B_{n}^{(-k)} = (r + 1)^{n}.
\]

Here, we replace \( r \rightarrow r + l - 1 \), and we have

\[
\frac{1}{(r + l - 1)!} \sum_{k=1}^{r + l - 1} \begin{bmatrix} r + l - 1 \ k \end{bmatrix} B_{n}^{(-k)} = (r + l)^{n}.
\]

Hence we obtain

\[
\sum_{l=1}^{m} \frac{(-1)^{l+m} l! \{ m \}_{l}}{(r + l - 1)!} \sum_{k=1}^{r + l - 1} \begin{bmatrix} r + l - 1 \ k \end{bmatrix} B_{n}^{(-k)} = \sum_{l=1}^{m} (-1)^{l+m} l! \{ m \}_{l} (r + l)^{n}.
\]

Here the right hand of the last equality can be obtained by putting \( k_{1} = m, \ k_{2} = \cdots = k_{r} = 0 \) in Theorem 2.1. Thus it equals to \( B_{n}^{(-m,0,\ldots,0)} \) and we obtain Theorem 3.9. \( \square \)

By using Lemma 3.4, we can also express Theorem 3.9 by using the Stirling numbers of the second kind.

Next, we consider \( B_{n}^{(0,\ldots,0,-m)} \) which are the generalizations of Theorem 3.6(2) in the same way. We consider the small values on \( m \), and we have

\[
B_{n}^{(0,\ldots,0,-1)} = \{ \overline{1} \}_{1} r (r + 1)^{n}
\]
\[
B_{n}^{(0, \ldots, 0, -2)} = -\binom{2}{1} r(r+1)^n + \binom{2}{2} r(r+1)(r+2)^n
\]
\[
B_{n}^{(0, \ldots, 0, -3)} = \binom{3}{1} r(r+1)^n - \binom{3}{2} r(r+1)(r+2)^n + \binom{3}{3} r(r+1)(r+2)(r+3)^n
\]
\[\ldots\]

From here we can be considered the generalizations, that is, \(B_{n}^{(0, \ldots, 0, -m)}\) as follows.

**Theorem 3.10.** We have the following relations on \(B_{n}^{(0, \ldots, 0, -m)}\)
\[
B_{n}^{(0, \ldots, 0, -m)} = \sum_{l=1}^{m} \frac{(-1)^{l+m}(r)_{l-1}\binom{m}{l}}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{array}{c} r+l-1 \\ k \end{array} \right] B_{n}^{(-k)}.
\]

Here, we define \((r)_l = r(r+1) \cdots (r+l-1)\) and \((r)_0 = 1\).

**Proof.** Using Theorem 2.6(2) and Theorem 3.6(2), we have
\[
\frac{1}{(r-1)!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] B_{n}^{(-k)} = r(r+1)^n.
\]
Here we replace \(r \rightarrow r + l - 1\), and we have
\[
\frac{1}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{array}{c} r+l-1 \\ k \end{array} \right] B_{n}^{(-k)} = (r + l - 1)(r + l)^n.
\]
Hence we obtain
\[
\sum_{l=1}^{m} \frac{(-1)^{l+m}(r)_{l-1}\binom{m}{l}}{(r+l-2)!} \sum_{k=1}^{r+l-1} \left[ \begin{array}{c} r+l-1 \\ k \end{array} \right] B_{n}^{(-k)} = \sum_{l=1}^{m} (-1)^{l+m}(r)_{l-1}\binom{m}{l} (r + l - 1)(r + l)^n
\]
\[
= \sum_{l=1}^{m} (-1)^{l+m}(r)_{l}\binom{m}{l} (r + l)^n.
\]

Here from Theorem 2.1, since we have \(B_{n}^{(0, \ldots, 0, -m)} = \sum_{l=1}^{m} \alpha_{l}^{(0, \ldots, 0, m)} (l + r)^n\), we prove by induction on \(m\)

and \(l\) that we have \(\alpha_{l}^{(0, \ldots, 0, m)} = (-1)^{l+m}(r)_{l}\binom{m}{l}\).

First, we prove that we have \(\alpha_{1}^{(0, \ldots, 0, m)} = (-1)^{m+1} r \cdots (B)\).
From Theorem 2.1(iii), since we have \( \alpha_1^{(0, \ldots, 0, m)} = r \alpha_0^{(0, \ldots, 0, m-1)} - \alpha_1^{(0, \ldots, 0, m-1)} \), we have (B) for \( m = 1 \).

We assume that we have \( \alpha_1^{(0, \ldots, 0, k)} = (-1)^{k+1} r \) for \( m = k \) \((k \geq 1)\).

For \( m = k + 1 \), we have
\[
\alpha_1^{(0, \ldots, 0, k+1)} = r \alpha_0^{(0, \ldots, 0, k)} - \alpha_1^{(0, \ldots, 0, k)} = -\alpha_1^{(0, \ldots, 0, k)} = (-1)^{k+1} r = (-1)^{k+2} r.
\]

Since this shows that (B) is true for \( m = k + 1 \), we have (B) for all integers \( m \).

Next, we prove that we have \( \alpha_l^{(0, \ldots, 0, m)} = (-1)^{l+m}(r)_l \binom{m}{l} \cdots (C) \).

From (B), we have (C) for \( m = 1 \).

We assume that we have \( \alpha_k^{(0, \ldots, 0, m)} = (-1)^{k+m}(r)_k \binom{m}{k} \) for \( l = k \) \((k \geq 1)\).

For \( l = k + 1 \), we have
\[
\alpha_{k+1}^{(0, \ldots, 0, m)} = (k + r) \alpha_k^{(0, \ldots, 0, m-1)} - (k + 1) \alpha_{k+1}^{(0, \ldots, 0, m-1)}
= (k + r) \alpha_k^{(0, \ldots, 0, m-1)} - (k + 1) \{ (k + r) \alpha_k^{(0, \ldots, 0, m-2)} - (k + 1) \alpha_{k+1}^{(0, \ldots, 0, m-2)} \}
= (k + r) \alpha_k^{(0, \ldots, 0, m-1)} - (k + 1)(k + r) \alpha_k^{(0, \ldots, 0, m-2)} + (k + 1)^2 \alpha_{k+1}^{(0, \ldots, 0, m-2)}
= (-1)^{k+m-1}(r)_k(k + r) \binom{m-1}{k} - (-1)^{k+m-2}(r)_k(k + 1)(k + r) \binom{m-2}{k}
+ (k + 1)^2 \alpha_{k+1}^{(0, \ldots, 0, m-2)}
\]
\[
= (-1)^{k+m-1}(r)_{k+1} \left[ \binom{m-1}{k} + (k + 1) \binom{m-2}{k} + \cdots + (k + 1)^{m-k-1} \binom{m-k}{k} \right].
\]

Since the Stirling numbers of the second kind satisfy the recurrence formula
\[
\binom{n+1}{m+1} = \binom{n}{m} + (m + 1) \binom{n}{m+1},
\]
we have
\[
\binom{m-1}{k} + (k + 1) \binom{m-2}{k} + (k + 1)^2 \binom{m-3}{k} + \cdots + (k + 1)^{m-k-1} \binom{m-(m-k)}{k}
= \sum_{i=1}^{m-k} (k + 1)^{i-1} \binom{m-i}{k}
\]
\begin{align*}
\sum_{i=1}^{m-k} (k+1)^{i-1} \left( \sum_{k+1}^{m-i+1} - (k+1)^{(m-i)} \binom{m-i}{k+1} \right) \\
\sum_{i=1}^{m-k} \left( (k+1)^{i-1} \sum_{k+1}^{m-i+1} - (k+1)^{i} \binom{m-i}{k+1} \right) \\
= \binom{m}{k+1} - (k+1)\binom{m-1}{k+1} + (k+1)\binom{m-1}{k+1} - (k+1)^{2}\binom{m-2}{k+1} + \cdots \\
+ (k+1)^{m-k-1}\binom{k+1}{k+1} - (k+1)^{m-k}\binom{k}{k+1} \\
= \binom{m}{k+1}.
\end{align*}

Therefore we have $\alpha_{k+1}^{(0,\ldots,0,m)}=(-1)^{k+m-1}(r)_{k+1}\binom{m}{k+1}$ and this shows that (C) is true for $l = k+1$.

Hence, we have (C) for all integers $l, m$, and this completes the proof.

By using Lemma 3.4, we can also express Theorem 3.10 by using the Stirling numbers of the second kind.

Finally, we consider the case of $B_{n}^{(0,\ldots,0,-m,0,\ldots,0)}$ (where i-th component is $-m$ and others are 0) which are the extension of Theorem 3.9 and Theorem 3.10. For example, we fluctuate the value of 2-th. Then we obtain

\begin{align*}
B_{n}^{(0,-1,0,\ldots,0)} &= 2(r+1)^{n} \\
B_{n}^{(0,-2,0,\ldots,0)} &= -2(r+1)^{n} + 6(r+2)^{n} \\
B_{n}^{(0,-3,0,\ldots,0)} &= 2(r+1)^{n} - 6(r+2)^{n} + 12(r+3)^{n} \\
B_{n}^{(0,-4,0,\ldots,0)} &= -2(r+1)^{n} + 6(r+2)^{n} - 12(r+3)^{n} + 24(r+4)^{n} \\
&\cdots.
\end{align*}

By using this relations and the recurrence formula on Theorem 2.1, we can be considered the following relations.

**Theorem 3.11.** We have the following relations on $B_{n}^{(0,\ldots,0,-m,0,\ldots,0)}$, and i-th component is $-m$ and others are 0;
\[ B_n^{(0,\ldots,0,-m,0,\ldots,0)} = \sum_{l=1}^{m} \frac{(-1)^{l-m} (i) l \{ m \}}{(r + l - 1)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)}. \]

Proof. In the proof of Theorem 3.9, we have
\[ \frac{1}{(r+l-1)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)} = (r + l)^n. \]

Furthermore, using Theorem 2.1 and \( \alpha_l^{(0,\ldots,0,m)} = (-1)^{l+m} (r) l \{ m \} \) in the proof of Theorem 3.10, we have
\[
\sum_{l=1}^{m} \frac{(-1)^{l-m} (i) l \{ m \}}{(r + l - 1)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)} = \sum_{l=1}^{m} (-1)^{l-m} (i) l \{ m \} (r + l)^n \]
\[
= \sum_{l=1}^{m} \alpha_l^{(0,\ldots,0,m)} (r + l)^n \]
\[
= \sum_{l=1}^{m} \alpha_l^{(0,\ldots,0,m,0,\ldots,0)} (r + l)^n \]
\[
= B_n^{(0,\ldots,0,-m,0,\ldots,0)}. \]

Hence we obtain
\[ B_n^{(0,\ldots,0,-m,0,\ldots,0)} = \sum_{l=1}^{m} \frac{(-1)^{l-m} (i) l \{ m \}}{(r + l - 1)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)}, \]
and this completes the proof. \( \square \)

By using Lemma 3.4, we can also express Theorem 3.11 by using the Stirling numbers of the second kind.

If we put \( i = 1, i = r \) in Theorem 3.11, we obtain the following
\[
B_n^{(-m,0,\ldots,0)} = \sum_{l=1}^{m} \frac{(-1)^{l+m} l \{ m \}}{(r + l - 1)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)}, \]
\[
B_n^{(0,\ldots,0,-m)} = \sum_{l=1}^{m} \frac{(-1)^{l+m} (r) l \{ m \}}{(r + l - 2)!} \sum_{k=1}^{r+l-1} \binom{r + l - 1}{k} B_n^{(-k)}. \]

Hence we find that Theorem 3.11 is the extension of Theorem 3.9 and Theorem 3.10.
Here we represent $B_n^{(-m, 0, \ldots, 0)}$, $B_n^{(0, \ldots, 0, -m)}$, $B_n^{(0, \ldots, 0, -m, 0, \ldots, 0)}$ in the form of powers on $r + l$ (1 ≤ l ≤ m), and we see the sum of coefficients. For example, we put $m = 1$. Then we have the following relations from Theorem 2.6:

\[
B_n^{(-1, 0, \ldots, 0)} = (r + 1)^n,
\]
\[
B_n^{(0, \ldots, 0, -1)} = r(r + 1)^n,
\]
\[
B_n^{(0, \ldots, 0, -1, 0, \ldots, 0)} = i(r + 1)^n.
\]

Hence each coefficients are 1, $r$, and $i$. From this results, we can be considered the following.

**Theorem 3.12.** We have the following relations on the sum of coefficients

1. The sum of coefficients on $B_n^{(-m, 0, \ldots, 0)}$ are 1.
2. The sum of coefficients on $B_n^{(0, \ldots, 0, -m)}$ are $r^m$.
3. The sum of coefficients on $B_n^{(0, \ldots, 0, -m, 0, \ldots, 0)}$ are $i^m$.

**Proof.** (1) In the proof of Theorem 3.9, we have

\[
B_n^{(-m, 0, \ldots, 0)} = \sum_{l=1}^{m} (-1)^{l+m} \binom{m}{l} (r + l)^n.
\]

Hence it suffices to show that $\sum_{l=1}^{m} (-1)^{l+m} \binom{m}{l} = 1$. We have

\[
\sum_{l=1}^{m} (-1)^{l+m} \binom{m}{l} = (-1)^m \sum_{l=1}^{m} (-1)^l \sum_{k=0}^{l} \binom{l}{k} \binom{m}{l}.
\]

\[
= (-1)^m \sum_{l=0}^{m} (-1)^l \sum_{k=0}^{l} \binom{l}{k} \binom{m}{l}.
\]

\[
= (-1)^m \sum_{k=0}^{m} \sum_{l=0}^{m} (-1)^l \binom{m}{l} \binom{l}{k}.
\]
Here since $\sum_{l=0}^{m}(-1)^{l}\binom{n}{l} \left[ \frac{l}{m} \right] = (-1)^{m}\delta_{m,n}$ ([1]), we obtain

$$\sum_{l=1}^{m}(-1)^{l+m}l!\binom{m}{l} = (-1)^{m}\sum_{k=0}^{m}(-1)^{k}\delta_{k,m} = (-1)^{m}(-1)^{m}\delta_{m,m} = 1.$$ 

(2) In the proof of Theorem 3.10, we have

$$B_{n}^{(0,...,0,-m)}(r)^{n} = \sum_{l=1}^{m}(-1)^{l+m}(r)_{l}\left\{ \frac{m}{l} \right\} (r+l)^{n}.$$ 

Hence it suffices to show that $\sum_{l=1}^{m}(-1)^{l+m}(r)_{l}\left\{ \frac{m}{l} \right\} = r^{m}$.

Since $x^{n} = \sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}(x)_{k}$ ($n \geq 0$) ([6]), $\sum_{l=1}^{m}(-1)^{l+m}(r)_{l}\left\{ \frac{m}{l} \right\} = r^{m}$.

(3) In the proof of Theorem 3.11, we have

$$B_{n}^{(0,...,0,-m,0,...,0)}(r)^{n} = \sum_{l=1}^{m}(-1)^{l+m}(i)_{l}\left\{ \frac{m}{l} \right\} (r+l)^{n}.$$ 

Hence it suffices to show that $\sum_{l=1}^{m}(-1)^{l+m}(i)_{l}\left\{ \frac{m}{l} \right\} = i^{m}$. This identity can be obtained by putting $r = i$ in (2), and we obtain the result. \(\square\)

For $m = 1$, since each coefficients are $1, r,$ and $i$, we find that Theorem 3.12 is the generalizations.

We introduced several relations between Multi-Poly-Bernoulli numbers and Poly-Bernoulli numbers up to here. We obtained Theorem 3.1 by fluctuating values of $r$ which represent numbers of $0$. Here we consider that fluctuating values of $m$ which represent numbers except for $0$.

**Theorem 3.13.** We have the following relations

(1) $B_{n}^{(-k)} = \sum_{m=0}^{k-1}(-1)^{k-m-1}\binom{k}{m}B_{n}^{(-m,0)}$ ($k \geq 1$).

Where the sum of coefficients on $B_{n}^{(-m,0)}$ are 1.
(2) \[ B_n^{(-k)} = \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} B_n^{(0,-m)} \quad (k \geq 1). \]

Where the sum of coefficients on \( B_n^{(0,-m)} \) are 0 for \( k \geq 2 \). we regard the sum on the right hand as 1 if \( k = 1 \).

Proof. (1) R.H.S. = \[ \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k}{m} \sum_{l=1}^{m} (-1)^{l+1+k} \frac{m!}{l!} \binom{m}{l} (l+2)^n \]
\[ = \sum_{m=0}^{k-1} \binom{k}{m} \sum_{l=1}^{m} (-1)^{l+1+k} \frac{m!}{l!} \binom{m}{l} (l+2)^n \]
\[ = \sum_{m=0}^{k-1} \binom{k}{m} \sum_{l=0}^{k-1} (-1)^{l+1+k} \frac{m!}{l!} \binom{m}{l} (l+2)^n \]
\[ = \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+2)^n \sum_{m=0}^{k-1} \binom{k}{m} \frac{m!}{l!} \binom{m}{l} - \binom{k}{l} \]
\[ = \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+2)^n \left[ \binom{k+1}{l+1} - \binom{k}{l} \right] \]
\[ = \sum_{l=0}^{k-1} (-1)^{l+1+k} (l+2)^n (l+1) \binom{k}{l+1} \]
\[ = \sum_{l=1}^{k} (-1)^{l+1+k} (l+1)! \binom{k}{l} (l+2)^n \]
\[ = \sum_{l=1}^{k} (-1)^{l+k} (l+1)! \binom{k}{l} (l+1)^n \]
\[ = B_n^{(-k)}. \]

Hence we obtain the result. Furthermore, the sum of coefficients on \( B_n^{(-m,0)} \) are
\[ \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k}{m} = \sum_{m=0}^{k} (-1)^{k-m-1} \binom{k}{m} + 1 \]
\[ -\sum_{m=0}^{k} (\frac{1}{1^{k-m}} \binom{k}{m}) + 1 = -(1 - 1)^{k} + 1 = 1. \]

This completes the proof and we obtain (1).

(2) We put \( r = 2 \) in Theorem 3.10, and we use Theorem 3.6 and Theorem 2.6. Then we have

\[ B_{n}^{(0,-m)} = \sum_{l=1}^{m} \frac{(-1)^{l+m}(2l-1)!}{l!} \binom{m}{l} \sum_{k=1}^{l+1} \binom{l+1}{k} B_{n}^{(-k)} \]

\[ = \sum_{l=1}^{m} (-1)^{l+m} l! \binom{m}{l} \sum_{k=1}^{l+1} \binom{l+1}{k} B_{n}^{(-k)} \]

\[ = \sum_{l=1}^{m} (-1)^{l+m} l! \binom{m}{l} \sum_{k=1}^{l+1} \binom{l+1}{k} \]

\[ = \sum_{l=1}^{m} (-1)^{l+m} l! \binom{m}{l} B_{n}^{(0,\ldots,0,1)} \]

\[ = \sum_{l=1}^{m} (-1)^{l+m} l! \binom{m}{l} (l+1)(l+2)^n. \]

Hence we substitute this identity on the right of Theorem 3.13, and we have

R.H.S. = \[ \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} \sum_{l=1}^{m} (-1)^{l+1} l! \binom{m}{l} (l+1)(l+2)^n \]

\[ = \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=0}^{m} (-1)^{l+1} (l+1)! \binom{m}{l} (l+2)^n \]

\[ = \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=0}^{m} (-1)^{l+1} (l+1)! \binom{m}{l} (l+2)^n \]

\[ = \sum_{m=0}^{k-1} (-1)^{l+1} (l+1)! (l+2)^n \sum_{m=0}^{k-1} \binom{k-1}{m} \binom{m}{l} \]

\[ = \sum_{l=0}^{k-1} (-1)^{l+1} (l+1)! (l+2)^n \binom{k}{l+1} \]

\[ = \sum_{l=0}^{k-1} (-1)^{l+1} (l+1)! (l+2)^n \binom{k}{l+1} \]
\[
\begin{align*}
&= \sum_{l=0}^{k-1} (-1)^{l+1+k(l+1)} \binom{k}{l+1} (l+2)^n \\
&= \sum_{l=1}^{k} (-1)^{l+k} \binom{k}{l} (l+1)^n \\
&= B_n^{(-k)}.
\end{align*}
\]

Therefore we obtain the result. Furthermore, the sum of coefficients on \(B_n^{(0,-m)}\) are
\[
\sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} = (1-1)^{k-1} = \begin{cases} 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases}.
\]

This completes the proof and we obtain (2). \[\square\]

We extend Theorem 3.13, and we can write \(B_n^{(-k)}\) by using the sum of \(B_n^{(0,...,0)}\) \((l \geq 1)\) and them of \(B_n^{(-m,0,...,0)}\), or \(B_n^{(0,...,0,-m)}\). First, we write \(B_n^{(-k)}\) by using the sum of \(B_n^{(0,...,0)}\) \((l \geq 1)\) and them of \(B_n^{(-m,0,...,0)}\).

**Theorem 3.14.** We have the following relations
\[
B_n^{(-k)} = \sum_{l=1}^{r} (-1)^{k-l} \binom{k}{l} B_n^{(0,...,0)} + \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \binom{k-m}{r} B_n^{(-m,0,...,0)}.
\]

**Proof.** \((1)\) \(\sum_{l=1}^{r} (-1)^{k-l} \binom{k}{l} B_n^{(0,...,0)} = \sum_{l=1}^{r} (-1)^{k-l} \binom{k}{l} (l+1)^n.\)

Furthermore, we have
\[
\begin{align*}
&= \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \binom{k-m}{r} B_n^{(-m,0,...,0)} \\
&= \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r! \binom{k-m}{r} \sum_{l=1}^{m} (-1)^{l+m} \binom{m}{l} (l+r+1)^n \\
&= \sum_{m=r+1}^{k-r} (-1)^{k-m} \binom{k}{m-r} r! \binom{k-m+r}{r} \sum_{l=1}^{m-r} (-1)^{l+m-r} \binom{m-r}{l} (l+r+1)^n \\
&= \sum_{m=r+1}^{k-r} \binom{k}{m-r} r! \binom{k-m+r}{r} \sum_{l=1}^{k-1} (-1)^{k-l-r} \binom{m-r}{l} (l+r+1)^n
\end{align*}
\]
\[
= \sum_{m=r+1}^{k} \binom{k}{m} (m-r)^{l} \binom{k-m+r}{r} \sum_{l=1}^{k} (-1)^{k-l} \binom{m-r}{l} (l+r+1)^{n}
\]

\[
= \sum_{l=1}^{k} (-1)^{k-l} r! (l+r+1)^{n} \sum_{m=r+1}^{k} \binom{m-r}{l} \binom{k-m+r}{r} \binom{k}{m-r}
\]

\[
= \sum_{l=1}^{k} (-1)^{k-l} r! (l+r+1)^{n} \binom{k}{l+r} \binom{l+r}{r}
\]

\[
= \sum_{l=1}^{k-r} (-1)^{k-l} r! (l+r+1)^{n} \binom{k}{l+r} \frac{(l+r)!}{l! r!}
\]

\[
= \sum_{l=1}^{k-r} (-1)^{k-l} (l+r)! \binom{k}{l+r} (l+r+1)^{n}
\]

\[
= \sum_{l=r+1}^{k} (-1)^{k-l} \binom{k}{l} (l+1)^{n} \quad (l \to l-r).
\]

Hence the right hand of Theorem 3.14 can be expressed as follows, and we obtain the result.

R.H.S. = \[ \sum_{l=1}^{r} (-1)^{k-l} \binom{k}{l} (l+1)^{n} + \sum_{l=r+1}^{k} (-1)^{k-l} \binom{k}{l} (l+1)^{n} \]

\[ = \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} (l+1)^{n} \]

\[ = \sum_{l=1}^{k-r} (-1)^{k-l} \binom{k}{l} (l+1)^{n} \]

\[ = B^{(-k)}_{n}. \quad \square \]

**Example 3.15.**

We give examples of Theorem 3.14 for 1 ≤ r ≤ 2, 1 ≤ k ≤ 6.

(i) For \( r = 1 \)

\[
B^{(-1)}_{n} = B^{(0,0)}_{n}
\]

\[
B^{(-2)}_{n} = -B^{(0,0)}_{n} + 2B^{(-1,0)}_{n}
\]

\[
B^{(-3)}_{n} = B^{(0,0)}_{n} - 3B^{(-1,0)}_{n} + 3B^{(-2,0)}_{n}
\]

\[
B^{(-4)}_{n} = -B^{(0,0)}_{n} + 4B^{(-1,0)}_{n} - 6B^{(-2,0)}_{n} + 4B^{(-3,0)}_{n}
\]
\[ B_n^{(-5)} = B_n^{(0,0)} - 5B_n^{(-1,0)} + 10B_n^{(-2,0)} - 10B_n^{(-3,0)} + 5B_n^{(-4,0)} \]
\[ B_n^{(-6)} = -B_n^{(0,0)} + 6B_n^{(-1,0)} - 15B_n^{(-2,0)} + 20B_n^{(-3,0)} - 15B_n^{(-4,0)} + 6B_n^{(-5,0)} \]

(ii) For \( r = 2 \)
\[ B_n^{(-1)} = B_n^{(0,0)} \]
\[ B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)} \]
\[ B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(-1,0,0)} \]
\[ B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 24B_n^{(-1,0,0)} + 12B_n^{(-2,0,0)} \]
\[ B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 70B_n^{(-1,0,0)} - 60B_n^{(-2,0,0)} + 20B_n^{(-3,0,0)} \]
\[ B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 180B_n^{(-1,0,0)} + 210B_n^{(-2,0,0)} - 120B_n^{(-3,0,0)} + 30B_n^{(-4,0,0)} \]

Next we write \( B_n^{(-k)} \) by using the sum of \( B_n^{(0, \ldots, 0)} \) \((l \geq 1)\) and them of \( B_n^{(0, \ldots, 0, -m)} \). First of all, we give concrete examples for \( 1 \leq r \leq 3, 1 \leq k \leq 6 \) such that Example 3.15.

**Example 3.16.**

We give examples for \( 1 \leq r \leq 3, 1 \leq k \leq 6 \).

(i) For \( r = 1 \)
\[ B_n^{(-1)} = B_n^{(0,0)} \]
\[ B_n^{(-2)} = -B_n^{(0,0)} + B_n^{(0,-1)} \]
\[ B_n^{(-3)} = B_n^{(0,0)} - 2B_n^{(0,-1)} + B_n^{(0,-2)} \]
\[ B_n^{(-4)} = -B_n^{(0,0)} + 3B_n^{(0,-1)} - 3B_n^{(0,-2)} + B_n^{(0,-3)} \]
\[ B_n^{(-5)} = B_n^{(0,0)} - 4B_n^{(0,-1)} + 6B_n^{(0,-2)} - 4B_n^{(0,-3)} + B_n^{(0,-4)} \]
\[ B_n^{(-6)} = -B_n^{(0,0)} + 5B_n^{(0,-1)} - 10B_n^{(0,-2)} + 10B_n^{(0,-3)} - 5B_n^{(0,-4)} + B_n^{(0,-5)} \]

(ii) For \( r = 2 \)
\[ B_n^{(-1)} = B_n^{(0,0)} \]
\[ B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)} \]
\[ B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 2B_n^{(0,0,-1)} \]
\[ B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 10B_n^{(0,0,-1)} + 2B_n^{(0,0,-2)} \]
\[ B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 34B_n^{(0,0,-1)} - 14B_n^{(0,0,-2)} + 2B_n^{(0,0,-3)} \]
$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 98B_n^{(0,0,-1)} + 62B_n^{(0,0,-2)} - 18B_n^{(0,0,-3)} + 2B_n^{(0,0,-4)}$$

(iii) For $r = 3$

$$B_n^{(-1)} = B_n^{(0,0)}$$

$$B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)}$$

$$B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(0,0,0,0)}$$

$$B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 36B_n^{(0,0,0,0)} + 6B_n^{(0,0,0,-1)}$$

$$B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 150B_n^{(0,0,0,0)} - 54B_n^{(0,0,0,-1)} + 6B_n^{(0,0,0,-2)}$$

$$B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 540B_n^{(0,0,0,0)} + 312B_n^{(0,0,0,-1)} - 72B_n^{(0,0,0,-2)}$$

$$+ 6B_n^{(0,0,0,-3)}$$

By Example 3.16, we see the parts on coefficients except for plus or minus sign on Multi-Poly-Bernoulli numbers of the right hand. Then we obtain Pascal circles for $r = 1$, and from here we can be considered the case of the generalizations as follows.

**Conjecture 3.17.** We will have the following relations

$$B_n^{(-k)} = \sum_{l=1}^{r} (-1)^{k-l} \binom{k}{l} B_n^{(l)} + \sum_{m=1}^{k-r} (-1)^{k-m-r} a_{k-r-l,m} B_n^{(0,\ldots,0,-m)}.$$

We define that $a_{k-r-l,m} = a_{k-r-2,m-1} + ra_{k-r-2,m}$ ($0 \leq k - r - 2, 1 \leq m \leq k - r$), $a_{k-r-1,0} = r! \binom{k}{r}$, and $a_{k-r-1,k-r} = r!$.

For example, for $k = 4$ and $r = 2$,

$$B_n^{(-4)} = \sum_{l=1}^{2} (-1)^{4-l} \binom{4}{l} B_n^{(0,\ldots,0)} + \sum_{m=1}^{2} (-1)^{2-m} a_{1,m} B_n^{(0,0,-m)}$$

$$= -\binom{4}{1} B_n^{(0,0)} + 2! \binom{4}{2} B_n^{(0,0,0)} - a_{1,1} B_n^{(0,0,-1)} + a_{1,2} B_n^{(0,0,-2)}$$

$$= -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 10B_n^{(0,0,-1)} + 2B_n^{(0,0,-2)}.$$

$$a_{1,1} = a_{0,0} + 2 \cdot a_{0,1} = 6 + 2 \cdot 2 = 10$$

Moreover, we see the parts of coefficients on Multi-Poly-Bernoulli numbers of the identity which hold on Theorem 3.14. Therefore we revisit Example 3.15.

**Example 3.15 (Example 3.15 revisited).**

We give examples of Theorem 3.14 for $1 \leq r \leq 2, 1 \leq k \leq 6$.

(i) For $r = 1$
Thus, the left hand of the equality equals Theorem 3.12, we have Theorem 3.18.

\[ B_n^{(-1)} = B_n^{(0,0)} \]
\[ B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(-1,0)} \]
\[ B_n^{(-3)} = B_n^{(0,0)} - 3B_n^{(-1,0)} + 3B_n^{(-2,0)} \]
\[ B_n^{(-4)} = -B_n^{(0,0)} + 4B_n^{(-1,0)} - 6B_n^{(-2,0)} + 4B_n^{(-3,0)} \]
\[ B_n^{(-5)} = B_n^{(0,0)} - 5B_n^{(-1,0)} + 10B_n^{(-2,0)} - 10B_n^{(-3,0)} + 5B_n^{(-4,0)} \]
\[ B_n^{(-6)} = -B_n^{(0,0)} + 6B_n^{(-1,0)} - 15B_n^{(-2,0)} + 20B_n^{(-3,0)} - 15B_n^{(-4,0)} + 6B_n^{(-5,0)} \]

(ii) For \( r = 2 \)
\[ B_n^{(-1)} = B_n^{(0,0)} \]
\[ B_n^{(-2)} = -B_n^{(0,0)} + 2B_n^{(0,0,0)} \]
\[ B_n^{(-3)} = B_n^{(0,0)} - 6B_n^{(0,0,0)} + 6B_n^{(-1,0,0)} \]
\[ B_n^{(-4)} = -B_n^{(0,0)} + 14B_n^{(0,0,0)} - 24B_n^{(-1,0,0)} + 12B_n^{(-2,0,0)} \]
\[ B_n^{(-5)} = B_n^{(0,0)} - 30B_n^{(0,0,0)} + 70B_n^{(-1,0,0)} - 60B_n^{(-2,0,0)} + 20B_n^{(-3,0,0)} \]
\[ B_n^{(-6)} = -B_n^{(0,0)} + 62B_n^{(0,0,0)} - 180B_n^{(-1,0,0)} + 210B_n^{(-2,0,0)} - 120B_n^{(-3,0,0)} + 30B_n^{(-4,0,0)} \]

Here by Example 3.15, the sum of coefficients on Multi-Poly-Bernoulli numbers are all 1 for \( r = 1 \) and \( r = 2 \). From this, we can be considered the following Theorem 3.18.

**Theorem 3.18.** We have the following relations on the sum of coefficients.

\[ \sum_{l=1}^{r} (-1)^{k-l}l!\binom{k}{l} + \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r!\binom{k-m}{r} = 1. \]

**Proof.**
\[ \sum_{m=1}^{k-r} (-1)^{k-m-r} \binom{k}{m} r!\binom{k-m}{r} = \sum_{m=r+1}^{k} (-1)^{k-m} \binom{k}{m-r} r!\binom{k-m+r}{r} \]
\[ = \sum_{l=r+1}^{k} (-1)^{k-l}l!\binom{k}{l}. \]

Thus, the left hand of the equality equals \( \sum_{l=1}^{k} (-1)^{k-l}l!\binom{k}{l} \), and from the proof of Theorem 3.12, we have \( \sum_{l=1}^{k} (-1)^{k-l}l!\binom{k}{l} = 1. \)

Therefore we obtain the result. \( \square \)
Similary, we see the parts of coefficients on Multi-Poly-Bernoulli numbers where
represents $B_n^{(-k)}$ by using the sum of $B_n^{(0,...,0)} (l \geq 1)$ and them of $B_n^{(0,...,0,-m)}$.
But we don’t find regularities, therefore we don’t see the relations yet.

We give tables which show the values of $B_n^{(k)} (-5 \leq k \leq 5, 0 \leq n \leq 7)$ and $B_n^{(k_1,k_2)}$ for small $n, k_i$.

Table 2. [1] $B_n^{(k)} (-5 \leq k \leq 5, 0 \leq n \leq 7)$

| k \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| -5    | 1 | 32 | 454 | 4718 | 41506 | 329462 | 2441314 | 17234438 |
| -4    | 1 | 16 | 146 | 1066 | 6902 | 41506 | 237686 | 1315666 |
| -3    | 1 | 8  | 46  | 230  | 1066 | 4718 | 20266 | 85310 |
| -2    | 1 | 4  | 14  | 46   | 146  | 454  | 1394  | 4246  |
| -1    | 1 | 2  | 4   | 8    | 16   | 32   | 64    | 128   |
| 0     | 1 | 1  | 1   | 1    | 1    | 1    | 1     | 1     |
| 1     | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ | 0 |
| 2     | 1 | $\frac{1}{4}$ | $\frac{1}{36}$ | $\frac{1}{24}$ | $\frac{1}{150}$ | $\frac{1}{30}$ | $-\frac{35}{2268}$ | $-\frac{5}{128}$ |
| 3     | 1 | $\frac{1}{8}$ | $\frac{1}{216}$ | $\frac{1}{288}$ | $\frac{1}{5400}$ | $\frac{1}{7200}$ | $-216$ | $\frac{7}{375}$ |
| 4     | 1 | $\frac{1}{16}$ | $\frac{1}{1296}$ | $\frac{1}{3456}$ | $\frac{1}{32400}$ | $\frac{1}{14400}$ | $-81$ | $\frac{9}{35}$ |
| 5     | 1 | $\frac{1}{32}$ | $\frac{1}{7776}$ | $\frac{1}{41472}$ | $\frac{1}{19440000}$ | $\frac{1}{25920000}$ | $-31$ | $\frac{1}{15}$ |

Table 3. [5] $B_n^{(k_1,k_2)} (0 \leq n \leq 7, k_1, k_2$: small values)

| \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $B_n^{(1,1)}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{20}$ | $-\frac{1}{17}$ | $\frac{5}{81}$ | $\frac{1}{12}$ |
| $B_n^{(1,0)}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{20}$ | $\frac{1}{30}$ | $\frac{253}{12}$ | $\frac{7}{12}$ |
| $B_n^{(0,1)}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{20}$ | $\frac{1}{30}$ | $\frac{1}{42}$ | $\frac{29}{12}$ |
| $B_n^{(0,0)}$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 |
| $B_n^{(0,-1)}$ | 2 | 6 | 18 | 54 | 162 | 486 | 1458 | 4374 |
| $B_n^{(-1,0)}$ | 1 | 3 | 9 | 27 | 81 | 243 | 729 | 2187 |
| $B_n^{(-1,-1)}$ | 2 | 9 | 39 | 165 | 687 | 2829 | 11505 | 46965 |

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