STATIONARY AXISYMMETRIC FIELDS AS TWO-DIMENSIONAL GEODESICS

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ABSTRACT

Einstein’s equations for stationary axisymmetric fields are reformulated as the equations for affine geodesics in a two–dimensional space. The affine collineations of this space are investigated and used to relate explicit solutions of Einstein’s equations with different physical properties. Particularly, the solutions describing the exterior fields of a dyon and a slowly rotating body are discussed.

PACS No. 04.20.–q, 04.20.Fy
1. Introduction

To simplify the structure of Einstein’s equations, it is usual to postulate the existence of one or more Killing vector fields in the spacetime under consideration or, in less technical terms, the independence of certain coordinates. In a more general sense, the omission of the coordinates can be regarded as a special case of the Kaluza–Klein approach. Indeed, to investigate solutions with two Killing vectors in a systematic fashion, we can consider a Kaluza–Klein type reduction of Einstein’s theory to two dimensions [1]. The dimensional reduction just amounts to dropping, for all the fields in the spacetime, the dependence on the coordinates that can be associated with the Killing vectors.

In this work, we are concerned with a different type of dimensional reduction in which the number of fields – in our case, the metric coefficients – is reduced to the minimum necessary for describing the spacetime. This reduction occurs at the level of the Einstein–Hilbert Lagrangian and consists in dropping the terms that can be represented as total divergences, and rearranging the non–ignorable terms so that the Lagrangian becomes two–dimensional. The spacetime coordinates are absorbed into certain differential operators that act on the remaining metric coefficients, i.e., the metric coefficients become the coordinates of

the reduced space. This idea is in the spirit of the construction of the superspace studied by de Witt and others [2], where each point is a space. In our case, each geodesic defines a solution to the Einstein equations.

Neugebauer and Kramer [3] introduced the abstract potential space determined by the Einstein–Hilbert Lagrangian of Einstein–Maxwell fields, and in-
vestigated the field equations which are derivable from a minimal surface problem in the potential space. In a recent work [4], we showed that canonical transformations can be used to reduce the dimensionality of the

potential space, and Einstein’s equations coupled to any matter field are equivalent to the geodesic equations in a two–dimensional space.

As it is known, there exists a great deal of hypersymmetry in bidimensional physics, that is, supposedly unrelated problems happen to be the very same one, once a type of conformal transformation is given [5]. In our geodesic problem,

this is equivalent to relate, or generate, new solutions to the affine

two–dimensional geodesic motion, which, again refrased in the Einstein equations, translates into the generation of solutions.

We will focus attention on stationary axisymmetric solutions to the Einstein equations. In Section 2, the corresponding field equations are derived from a two–dimensional metric Lagrangian and it is shown that they may be interpreted as the equations for an affine geodesic. We then investigate the equation for affine collineations and present the general solution for the special case of a symmetry vector that depends on the coordinates only.

Section 3 contains a solution generated by applying three different transformations on the Chazy–Curzon metric. We study the properties of this solution and show that it may be interpreted as describing the exterior field of a gravitational dyon. Section 4 is devoted to the derivation and study of a solution which contains the parameters necessary to describe the field of a slowly rotating mass.

2. Field equations and affine collineations
Consider the general stationary axisymmetric line element in Weyl canonical coordinates

\[ ds^2 = e^{2\psi}(dt - \omega d\phi)^2 - e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] , \]  

where \( \psi, \omega, \) and \( \gamma \) are functions of \( \rho \) and \( z \) only. If \( \omega = \text{const.} \), Eq.(1) leads to the special case of static axisymmetric fields. The calculation of the corresponding scalar curvature leads to the Lagrangian

\[ L = \frac{e^{4\psi}}{2\rho}(\omega^2 + \omega_z^2) + 2\rho(\psi_{\rho\rho} + \psi_{zz} - \gamma_{\rho\rho} - \gamma_{zz} - \psi_{\rho}^2 - \psi_z^2) + 2\psi_\rho , \]  

which generates the usual Einstein equations. We proceed to give the main steps to construct from Eq.(2) another Lagrangian which generates the equation for affine geodesics.

Introducing the differential operator \( D = (\partial_\rho, \partial_z) \), Eq. (2) can be written as

\[ L = 2D\rho D\gamma + \frac{e^{4\psi}}{2\rho}(D\omega)^2 - 2\rho(D\psi)^2 . \]  

In obtaining Eq.(3), we made the substitution \( \rho D^2 B = D(\rho DB) - D\rho DB \) and neglected the total divergence terms. The Lagrangian (3) may now be interpreted as describing a kinematic system defined in the three-dimensional “space of metric coefficients” with generalized “coordinates” \( \psi, \omega, \) and \( \gamma \) that depend on \( \rho \) and \( z \). Consequently, \( \rho \) and \( z \) may be considered as quantities used to parametrize the coordinates.

Equation (3) shows that in this special case the Lagrangian explicitly depends on the parameter \( \rho \). Now, since \( \gamma \) and \( \omega \) are cyclic coordinates of the Lagrangian (3), it is convenient to use the Routhian \( R \) obtained from the Lagrangian \( L \) by means of Legendre transformation acting on the cyclic
coordinates only, i.e.,

\[ R = \frac{\partial L}{\partial (D\gamma)} D\gamma + \frac{\partial L}{\partial (D\omega)} D\omega - L = \frac{1}{2} \rho e^{-4\psi}\Pi_\omega^2 + 2\rho(D\psi)^2. \]  

(4)

Here \( \Pi_\omega \) is the canonically conjugate “momentum” associated with the generalized coordinate \( \omega \).

Note that the conjugate momentum \( \Pi_\gamma \) (as well as \( \gamma \)) does not enter the Routhian (4) at all. As a consequence, it can be shown that the metric function \( \gamma \) is determined by two first order partial differential equations that can be integrated by quadratures once \( \psi \) and \( \omega \) are known [6].

It follows from Eq.(4) that \( \Pi_\omega \) is a “constant of motion” (i.e., \( D\Pi_\omega = 0 \)) in space of metric coefficients. Using this fact, we can define an additional differential operator \( \tilde{D} = (-\partial_z, \partial_\rho) \) such that \( D\tilde{D} \equiv 0 \). Introducing a function \( \Omega \) by means of the relationship

\[ \Pi_\omega = \rho^{-1}e^{4\psi}D\omega = \tilde{D}\Omega, \]

(5)

the Routhian (4) becomes

\[ R = \frac{1}{2} \rho f^{-2}[(D\Omega)^2 + (Df)^2], \]

(6)

where \( f = \exp(2\psi) \). In this way we have obtained a new Lagrangian, Eq. (6), which corresponds to the squared line element of an abstract space described by \( f \) and \( \Omega \) and endowed with a two-dimensional metric tensor conformal to the Euclidean one, i.e.,

\[ g_{ab} = \frac{1}{2} \rho f^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

(7)

The metric functions \( f \) and \( \Omega \) plays the role of coordinates, and the operator \( D \) may be interpreted as the derivative with respect to the affine parameter(s) used to parametrize
the coordinates $f$ and $\Omega$. Consequently, the spacetime coordinates turn out to be affine parameters on the space of metric coefficients.

Therefore, Eq.(6) determines a metric Lagrangian that explicitly depends on the affine parameter $\rho$.

The geometric non–vanishing quantities associated with the metric (7) are, up to symmetries,

$$\Gamma^{1}_{11} = -\Gamma^{1}_{22} = \Gamma^{2}_{12} = -\frac{1}{f},$$

$$R^{1}_{212} = \frac{1}{f^2},$$

$$R_{11} = R_{22} = \frac{1}{f^2},$$

and the scalar curvature is $4/\rho$.

The Euler–Lagrange motion equations obtained from the Routhian (6) are

$$D^2 f - f^{-1}(Df^2 - D\Omega^2) + \rho^{-1} D\rho Df = 0,$$  

$$D^2 \Omega - 2f^{-1} Df D\Omega + \rho^{-1} D\rho D\Omega = 0,$$

which are the same principal equations which follow from $R_{\mu\nu} = 0$. For completeness, we mention that taking $E = f + i\Omega$, the Routhian (6) can be rewritten as

$$R = \frac{2\rho}{(E + E^*)^2} DE DE^* ,$$

where an asterisk represents complex conjugation. The variation of Eq. (10) with respect to $E$ or $E^*$ leads to the Ernst equation [7]

$$(\text{Re} \ E) \Delta E = (DE)^2 , \quad \text{with} \quad \Delta E = D^2 E + \rho^{-1} D\rho DE .$$

Since the starting Routhian (6) depends explicitly on the parameter $\rho$, Eqs.(9) coincide with the equations for an affine geodesic.
$D^2 X^a + \Gamma^a_{bc} DX^b DX^c = \lambda(\rho) DX^a$.

An affine geodesic is therefore a two-dimensional curve $X^a = (f, \Omega)$ with tangent vector $DX^a$ satisfying Eqs.(9) for the real function $\lambda(\rho) = -1/\rho$. The existence and uniqueness of solutions of Eqs.(9) follow from theorems on systems of differential equations. Let $X^a_0$ be a point of the space generated by $g_{ab}$ and $DX^a_0$ the value of the tangent vector at this point. Then, there exists a unique, up to a change of the parameter, maximal affine geodesic $X^a$ such that $X^a(0) = X^a_0$ and $DX^a(0) = DX^a_0$ (see, for instance, Ref. [8]). This result represents an alternative proof of the fact that a stationary axisymmetric solution is uniquely determined by its values on the axis of symmetry [9].

Let $X^a_1$ and $X^a_2$ represent two different affine geodesics. Our goal is to find transformations that relate $X^a_1$ with $X^a_2$ and may be used to generate new solutions from known ones. The existence of this type of transformations cannot be assumed a priori and it depends on the symmetry properties of the underlying equations as well as on the explicit form of $X^a_1$ and $X^a_2$. In the four-dimensional spacetime, transformations generating new solutions have been extensively studied and applied to diverse problems [10]; here, we first carry out the dimensional reduction and then reduce the problem to that of affine geodesics. To begin with the study of the transformations relating two different solutions of Eqs.(9), we consider the simplest case of a linear infinitesimal transformation.

For the general affine geodesic equation,

$$D^2 X^a + \Gamma^a_{bc} DX^b DX^c + g^{ab} D(g_{bc}) DX^c = 0 \ ,$$

an infinitesimal transformation

$$X^a \rightarrow X'^a = X^a + \epsilon \eta^a \ ,$$
is said to be a symmetry transformation, that is, maps solutions into solutions, to order \( \epsilon \), if the symmetry vector \( \eta^a \) satisfies the condition

\[
\bar{D}^2 \eta^a + R^a_{\ bcd} DX^b DX^c \eta^d - (\Gamma^a_{\ bc})_s DX^b \eta^c = 0 ,
\]

where \( \bar{D} \) is the total derivative operator on shell

\[
\bar{D} = \frac{\partial}{\partial s} + DX^a \frac{\partial}{\partial X^a} - [\Gamma^a_{\ bc} DX^b DX^c + g^{ab} D(g_{bc}) DX^c] \frac{\partial}{\partial DX^a} ,
\]

and \( s \) is a parameter along the geodesics. Moreover, \( R^a_{\ bcd} \) is the Riemann tensor of the space of metric coefficients and period stands for partial derivative. If the metric Lagrangian is independent of the parameter \( s \), Eq.(14) reduces to the equation of affine collineations \( \mathcal{L} \Gamma^a_{\ bc} = 0 \), where \( \mathcal{L} \) is the Lie derivative along a curve with tangent vector \( \eta^a \), which is equivalent to the equation of the geodesic deviation for the connecting vector \( \eta^a \). Hence the geometrical basis of our approach becomes plausible. A family of solutions of Einstein’s equations is equivalent to a congruence of geodesics in the space of metric coefficients. If \( \eta^a \) is a vector connecting two neighboring geodesics at a given point, then the condition for \( \eta^a \) to remain a connecting vector at any other point of the space of metric coefficients, at which the geodesics are well-defined, is that it must satisfy the equation of geodesic deviation.

If the symmetry vector \( \eta^a \) is just a function of the parameter \( s \) and the coordinates, then the symmetry equation (14) can be rewritten as

\[
\eta^a_{ss} + 2(\eta^a_\ , s) DX^b + (\eta^a_\ , bc + R^a_{\ bcd} \eta^d) DX^b DX^c = 0 ,
\]

where a semicolon represents the covariant derivative associated with the metric \( g_{ab} \) given in Eq.(7). Notice that in the case that \( \eta^a \) is just a function of the coordinates, even for a
metric depending on the non–affine parameter, the symmetry equation reduces to that of affine collineations. Consider this last case, \( \eta^a = \eta^a(X^b) \). Introducing the metric (7) into the symmetry equation (16), we get

\[
D^2 \eta^1 - 2 f^{-1} (Df D\eta^1 - D\Omega D\eta^2) + \eta^1 f^{-2} [(Df)^2 - (D\Omega)^2] + \rho^{-1} D\rho D\eta^1 = 0 ,
\]

\[
D^2 \eta^2 - 2 f^{-1} (Df D\eta^2 + D\Omega D\eta^1) + 2 \eta^1 f^{-2} Df D\Omega + \rho^{-1} D\rho D\eta^2 = 0 .
\]

(17)

A detailed investigation of Eq.(17) shows that it possesses three independent solutions:

\[
\eta^a_1 = (0, 1) ,
\]

(18)

\[
\eta^a_2 = (f, \Omega) ,
\]

(19)

\[
\eta^a_3 = (f \Omega, \Omega^2 - f^2 / 2) .
\]

(20)

Moreover, it can be shown that there are no affine eigencollineations, that is, the solutions (18–20) coincide with the Killing vectors of the metric (7). To find more general symmetry vectors of the potential space, it is necessary to consider the most general ansatz \( \eta^a = \eta^a(s, X^b, DX^b) \). In this work, however, we want to focus attention on the symmetry vectors (18–20) and to show that even these simple vectors can be used to connect classes of solutions with different physical properties.

We will now consider the type of solutions which can be generated by means of the vectors (18–20). Let \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \) be the parameters introduced by the symmetry vectors \( \eta^a_1, \eta^a_2, \) and \( \eta^a_3, \) respectively, according to Eq.(13). Acting on a seed solution \((f, \Omega)\), the vector \( \eta^a_1 \) leads to the new affine geodesic \( f' = f \) and \( \Omega' = \Omega + \epsilon_1 \). According to Eq.(5), this is equivalent to adding a constant \( \omega_0 \) to the metric function \( \omega \). Obviously, this symmetry transformation is trivial since a coordinate transformation of the form \( t' = t - \omega_0 \phi \) in the line element (1) absorbs the new term. Physically, this is equivalent to the introduction
of a rotating frame for the line element (1). Similarly, it is possible to show that the parameter $\epsilon_2$ associated with the symmetry vector $\eta^a_2$ can be absorbed by means of a rescaling of coordinates. The only non–trivial symmetry vector is $\eta^a_3$ and it can be used to generate new solutions of the form

$$f' = f(1 + \epsilon_3\Omega), \quad \Omega' = \Omega + \frac{\epsilon_3}{2}(\Omega^2 - f^2).$$

(21)

Although, when acting alone, the symmetry vectors $\eta^a_1$ and $\eta^a_2$ are trivial, we will see below that they are helpful when used together with $\eta^a_3$ to generate non–trivial solutions.

Note, moreover, that the corresponding parameters $\epsilon_1$ and $\epsilon_2$ can take any real value because they do not enter the symmetry equations at all. That is,

putting the infinitesimal transformation (13) with $\eta^a_1$ and $\eta^a_2$, one sees that the resulting equation is identically satisfied regardless of the values of the parameters $\epsilon_1$ and $\epsilon_2$, respectively.

3. Exterior field of a gravitational dyon

The interest in monopole structures has rapidly increased during the past few years due to their discovery in generalizations of the standard model of particle physics. Magnetic monopoles were first introduced by Dirac [11] in electrodynamics to symmetrize Maxwell’s equation in a direct way. Certainly, the most important consequence of the existence of magnetic monopoles is the quantization of electric charge. Most grand unified theories possess t’Hooft–Polyakov monopoles [12]. In general relativity there exist two different sorts of monopole structures: a magnetically charged black hole and a gravitational dyon. In fact, the magnetic black hole is the magnetic counterpart of the electrically charged black hole.
described by the Reissner–Nordstrom metric, and is related to it by a duality rotation. A magnetic black hole can also be interpreted as a magnetic monopole with mass greater than a determined critical value [13].

A gravitational dyon is a hypothetical object the existence of which follows from the relativistic character of gravitation. In Newtonian theory, the only source of gravitation is the mass. In contrast, general relativity predicts that mass as well as rotation are stationary sources of gravitational interaction. This leads to the well-known analogy between relativistic gravity and electromagnetism. The gravitational field generated by a distribution of mass turns out to be analogous to the electric field, and the field of an angular momentum current presents characteristics similar to those of a pure magnetic field. For this reason, the field generated by an angular momentum current is called “gravitomagnetic” field. For this analogy to be complete, it is necessary to require the existence of a “gravitomagnetic monopole” as the counterpart of the magnetic Dirac monopole of electrodynamics. A gravitational dyon is thus a mass endowed with a gravitomagnetic monopole. In this section, we will investigate solutions that can be generated from a static seed metric by means of a combination of symmetry transformations, and may be used to describe the exterior field of a gravitational dyon.

To give a correct interpretation of the solutions presented here, we will use a coordinate–invariant method based upon the investigation of the relativistic multipole moments for asymptotically flat solutions, according to the definition proposed by Geroch and Hansen [14]. We now proceed to derive the solution for a gravitational dyon. If we consider a static asymptotically flat solution \((f, \Omega = 0)\) as seed metric and apply to it the symmetry transformation associ-
ated with the vector $\eta^a_3$, we obtain a stationary solution with $f' = f$ and $\Omega' = -\epsilon_3 f'^2 / 2$. It can be shown that for any given asymptotically flat $f$ the new solution does not satisfy the condition of asymptotic flatness à la Geroch–Hansen [15]. Consequently, it is not possible to covariantly interpret the solutions generated by this type of transformation. To avoid this difficulty, we use a combination of three different symmetry transformations (18–20). To the seed static solution $f$ we first apply the symmetry vector $\eta^a_1$ with parameter $\epsilon_1$. The resulting solution is then used as seed solution for a transformation with the vector $\eta^a_2$ and parameter $\epsilon_2$, and, finally, we apply the symmetry vector $\eta^a_3$. The new solution can be written as

$$f' = (1 + \epsilon_2) f [1 + \epsilon_1 \epsilon_3 (1 + \epsilon_2)] ,$$  

(22)

and

$$\Omega' = (1 + \epsilon_2) \left[ \epsilon_1 - \frac{\epsilon_3}{2} (1 + \epsilon_2) (f'^2 - \epsilon_1^2) \right] .$$  

(23)

It is now necessary to choose the parameters introduced by the symmetry transformations such that the new solution becomes asymptotically flat. This condition leads to the relationships

$$\epsilon_1^2 = -\frac{\epsilon_2}{2 + \epsilon_2} , \quad \text{and} \quad \epsilon_3 = -\frac{\epsilon_2}{\epsilon_1(1 + \epsilon_2)^2} ,$$  

(24)

where $\epsilon_2$ is a negative constant defined in the interval $\epsilon_2 \in (-2, 0) \setminus \{-1\}$. As we mentioned at the end of section 2, the parameters $\epsilon_1$ and $\epsilon_2$ do not need to be infinitesimally small. Consequently, they can be chosen such that Eq.(24) is satisfied and $\epsilon_3$ becomes infinitesimally small as required by the transformation law (13). In fact, even for very large values of $\epsilon_1$, $\epsilon_3$ remains infinitesimal and $\epsilon_2$ remains in its domain of definition.
To analyze a concrete solution, we have to specify the asymptotically flat seed metric. Consider the Chazy–Curzon metric [16]

\[ f = \exp(-2m/r) , \quad r^2 = \rho^2 + z^2 , \quad (25) \]

where \( m \) is a positive constant. The new solution is then given by Eqs. (22), (23) and (25). Choosing the new parameters according to Eq. (24), we calculate the corresponding Geroch–Hansen multipole moments and obtain

\[ M_0 = m , \quad J_0 = -m\epsilon_3 . \quad (26) \]

There are higher mass multipole moments \( M_n \) which corresponds to the axisymmetric mass distribution of the source, and higher moments for the angular momentum current \( J_n \) which, however, can be neglected since they are proportional to \( \epsilon_3^2 \). Equation (26) shows that this solution represents the gravitational field of a body with mass \( m \) and gravitomagnetic monopole \(-m\epsilon_3\). Hence, the new parameter \( \epsilon_3 \) may be interpreted as the specific “gravitomagnetic” mass which may be positive as well as negative. The total “gravitoelectric” mass of the seed solution has not been affected by the action of symmetry transformations. For the sake of completeness, we present the metric functions of the new solution:

\[ f' = \exp(-2m/r) , \quad \omega' = -2m\epsilon_3(1 + \epsilon_2)^2 z/r , \quad \gamma' = -m^2 \rho^2/r^4 . \quad (27) \]

Finally, we would like to mention that using the Schwarzschild metric as starting solution, it is possible to generate the linearized Taub–NUT (Newman–Unti–Tamburino) solution which is also a candidate for describing the exterior field of a gravitational dyon. In general, it should be possible to find other solutions which, being different from the
Taub–NUT metric or the one presented here, present similar properties and hence might be used to describe a dyon. They all could differ only in the set of multipole moments higher than the monopole one; that is, there may exist different distributions of mass possessing the same gravitomagnetic monopole structure.

4. Field of a slowly rotating mass

For the study of the gravitational field of astrophysical bodies like stars and planets, it is necessary to investigate solutions which possess a set of mass multipole moments as well as a set of gravitomagnetic moments representing the rotation of the source. In contrast to the solution presented in the last section, a solution with realistic rotational properties may have only gravitomagnetic multipoles higher than or equal to the dipole one. In this section we derive a solution which satisfies this condition.

Consider any stationary seed solution \((f, \Omega)\) satisfying the conditions of asymptotic flatness. As we have done in section 3, we apply three consecutive symmetry transformations according to Eqs.(13) and (18–20). The new solution is then given by

\[ f' = (1 + \epsilon_2) f[1 + \epsilon_3(1 + \epsilon_2)(\Omega + \epsilon_1)] , \]
\[ \Omega' = (1 + \epsilon_2) \left[ \Omega + \epsilon_1 - \frac{\epsilon_3}{2}(1 + \epsilon_2)(f^2 - \epsilon_1^2 - 2\epsilon_1\Omega - \Omega^2) \right] . \]

In general, this new solution is not asymptotically flat. However, if we demand that the parameters \(\epsilon_1\) and \(\epsilon_2\) satisfy the relationships (24), asymptotic flatness is conserved and the resulting solution can be written as

\[ f' = f[1 + \epsilon_3(1 + \epsilon_2)^2 \Omega] , \]
\[ \Omega' = \Omega + \frac{\epsilon_3}{2}(1 + \epsilon_2)^2(1 + \Omega^2 - f^2) . \]
The calculation of new solutions does not present any difficulties. We will present here only one solution which illustrates our approach and can easily be interpreted. Consider the seed solution [17]

\[ f = \frac{x^2 - 1 + \alpha_1^2(y^2 - 1)}{(x + 1)^2 + \alpha_1^2(y - 1)^2}, \quad \Omega = \frac{2\alpha_1(x + y)}{(x + 1)^2 + \alpha_1^2(y - 1)^2}, \tag{32} \]

with

\[ x = \frac{1}{2m}(r_+ + r_-), \quad y = \frac{1}{2m}(r_+ - r_-), \quad r_\pm^2 = \rho^2 + (z \pm m)^2, \]

where \( m \) and \( \alpha_1 \) are constants. To illustrate the effect of symmetry transformations, we first analyze the seed solution (32). An investigation of the corresponding multipoles show that there are gravitoelectric as well as gravitomagnetic monopole and dipole moments. Due to the presence of the gravitomagnetic monopole and gravitoelectric dipole, this solution cannot be considered as a candidate for the description of the gravitational field of any astrophysical object. Hence solution (32) is of no interest from a physical point of view. However, if we apply three different symmetry transformations to solution (32), its physical meaning can totally be changed. In fact, putting Eq.(32) into Eqs.(30) and (31), and calculating the relativistic multipole moments of the resulting solution, we see that all undesirable multipole moments vanish if \( \alpha_1 \) is assumed to take the value

\[ \alpha_1 = -\epsilon_3(1 + \epsilon_2)^2. \tag{33} \]

Then, the only nonvanishing multipoles are

\[ M_0 = m, \quad \text{and} \quad J_1 = \epsilon_3(1 + \epsilon_2)^2m. \tag{34} \]

The last equation shows that the total mass of the body is given by \( m \) and that only the gravitomagnetic dipole moment survives in accordance with the dipole character of
rotation. The angular momentum per unit mass is given by $\epsilon_3(1+\epsilon_2)^2$ and can be positive as well as negative, corresponding to the two possible directions of rotation of the source with respect to the symmetry axis. Consequently, the new solution may be interpreted as describing the exterior field of a slowly rotating mass. Using Eqs.(30)–(33) and (5), the calculation of the metric components leads to

$$f' = \frac{x - 1}{x + 1}, \quad \omega' = 2m\epsilon_3(1 + \epsilon_2)^2 \frac{1 - y^2}{x - 1}, \quad \gamma' = \frac{1}{2} \ln \frac{x^2 - 1}{x^2 - y^2}. \quad (35)$$

This is equivalent to the Lense–Thirring metric [18], the physical meaning of which has been investigated by using other approaches and coincides with that we have obtained above by just analyzing the corresponding multipole moments.
5. Conclusions

We have presented a different way to view to Einstein’s equations, mainly as geodesic motions in a space where the metric coefficients of the spacetime play the role of coordinates. The approach presented here for the axisymmetric stationary case can be generalized to any spacetime and even to the non-vacuum cases. We have reasons to believe that a dimensional reduction, via canonical transformations of the Hamiltonian, can always be made in this space of metric coefficients, and field potentials, such that the dynamical problem reduces to study the geodesic motion in a two dimensional manifold.

As an application of this point of view we studied the symmetries of the geodesic motion for the space associated with axisymmetric stationary gravitational fields, and were able to generate some solutions, whose implications are currently under study. Nevertheless, we want to stress the fact that the idea presented here is not only a method for generating solutions but more than that, a different point of view to work with Einstein’s equations.

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[15] It can be shown that a stationary axisymmetric solution is asymptotically flat (see Ref. [14]) if for \( \rho = 0 \) and \( z \to \infty \) the metric functions behave like \( f \to 1 + O(z^{-1}) \) and \( \Omega \to O(z^{-1}) \).

[16] J. Chazy, Bull. Soc. Math. France, 52, 17 (1924); H. E. J. Curzon, Proc. Math. Soc. London, 23, 477 (1924).

[17] This is a special case of a more general solution presented in: H. Quevedo, Phys. Rev. D 39, 2904 (1989).

[18] H. Thirring and J. Lense, Phys. Z. 19, 156 (1918); see also B. Mashhoon, F. W. Hehl, and D. S. Theiss, Gen. Rel. Grav. 16, 711 (1984).