NON-LOCAL EFFECTS IN THE MEAN-FIELD DISC DYNAMO.
I. AN ASYMPTOTIC EXPANSION

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Abstract

We reconsider thin-disc global asymptotics for kinematic, axisymmetric mean-field dynamos with vacuum boundary conditions. Non-local terms arising from a small but finite radial field component at the disc surface are consistently taken into account for quadrupole modes. As in earlier approaches, the solution splits into a local part describing the field distribution along the vertical direction and a radial part describing the radial (global) variation

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of the eigenfunction. However, the radial part of the eigenfunction is now governed by an integro-differential equation whose kernel has a weak (logarithmic) singularity. The integral term arises from non-local interactions of magnetic fields at different radii through vacuum outside the disc. The non-local interaction can have a stronger effect on the solution than the local radial diffusion in a thin disc, however the effect of the integral term is still qualitatively similar to magnetic diffusion.

KEY WORDS: Mean-field dynamos, Thin-disc asymptotics, Boundary conditions, Galactic magnetic fields

1 Introduction

In the standard thin-disc asymptotic approach to mean-field dynamos, the dynamo equation

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B + \alpha B - \beta \nabla \times B),$$  \hspace{1cm} (1)

splits into a local part containing only derivatives with respect to the vertical coordinate $z$ and the horizontal part containing derivatives with respect to the cylindrical radius $r$ and the azimuthal angle $\phi$. Here $B$ is the mean magnetic field, $V$ is the mean velocity, $\alpha$ is the coefficient describing the alpha-effect, and $\beta$ is the turbulent magnetic diffusivity. We use cylindrical polar coordinates $(r, \phi, z)$ and consider axisymmetric solutions of the kinematic dynamo equation, $\partial/\partial \phi = 0$. In terms of the azimuthal components of the magnetic field $B$ and the vector potential $A$, Eq. (1) reduces to

$$\frac{\partial B}{\partial t} = -DG \frac{\partial A}{\partial z} + \frac{\partial^2 B}{\partial z^2} + \lambda^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} rB \right),$$ \hspace{1cm} (2)

$$\frac{\partial A}{\partial t} = \alpha B + \frac{\partial^2 A}{\partial z^2} + \lambda^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} rA \right),$$ \hspace{1cm} (3)

where $D = \alpha_0 G h^3 / \beta^2$ is the dynamo number, $G = r \partial \Omega / \partial r$ (with $\Omega$ the angular velocity), and we have used dimensionless variables defined by the disc semi-thickness $h$ for $z$, characteristic disc radius $R$ for $r$, magnetic diffusion time $h^2 / \beta$ for $t$, and the ratio of units for $B$ and $A$ is $\alpha_0 h^2 / \beta$, with $\alpha_0$ a representative value of the $\alpha$-coefficient (see, e.g., Ruzmaikin et al., 1988 for details). We have neglected the term with $\partial \Omega / \partial z$ in Eq. (2) assuming that $|\partial \Omega / \partial z| \ll |\partial \Omega / \partial r|$ under typical conditions in astrophysical discs. In terms of the dimensionless variables, we consider
the region \(|z| \leq 1, \ 0 < r < \infty\). Astrophysical discs have no well defined radial boundary, and the dynamo active region has a finite size because the intensity of the dynamo action decreases at large \(r\); therefore, the characteristic radius of the dynamo region \(R\), introduced below, is finite. The disc boundary is not sharp at its surface as well, but the turbulent magnetic diffusivity rapidly increases with height (at least, in spiral galaxies), so that vacuum boundary conditions posed at a finite height are a good approximation (Moss et al., 1998).

Equations (2) and (3) are written for the \(\alpha \omega\)-dynamo, but generalization to \(\alpha^2 \omega\)-dynamos is straightforward as long as the radial scale of the solution in not affected much. Our basic result, Eq. (32), remains valid for both \(\alpha \omega\)- and \(\alpha^2 \omega\)-dynamos, but \(\gamma_0(r)\) and \(\eta(r)\) defined below will follow from different local equations.

An asymptotic solution of Eqs. (2) and (3) is represented by Ruzmaikin et al. (1988) in the form
\[
\left( \begin{array}{c} B \\ A \end{array} \right) = \exp (\Gamma t) \left[ Q(\lambda^{-1/2}r) \left( \begin{array}{c} b(z; r) \\ a(z; r) \end{array} \right) + \ldots \right],
\]
where \(\Gamma\) is the growth rate of the field (\(\Gamma\) is known to be real for the dominant modes in a thin disc if \(D < 0\)), \((b, a)\) is the local solution (suitably normalized) depending on \(r\) only parametrically, and \(Q(r)\) is the amplitude of the global solution. The asymptotic parameter is the disc aspect ratio
\[
\lambda = \frac{h}{R} \simeq 10^{-2} - 10^{-1}.
\]

Equation (1) should be supplemented by appropriate boundary conditions (discussed in detail below), and then the lowest-order approximation in \(\lambda\) yields a boundary value problem for \((b, a)\), whereas the solvability condition for the first-order equations leads to an equation for \(Q\) and \(\Gamma\). The procedure for the derivation of the asymptotic equations is reconsidered in detail below.

The boundary conditions traditionally used in conjunction with Eqs. (2) and (3) are the so-called vacuum boundary conditions. If there are no electric currents outside the disc, then \(\nabla \times \mathbf{B} = 0\) and, for axisymmetric solutions, (see, e.g., Zeldovich et al., 1983, p. 151)
\[
B = 0 \quad \text{for} \quad |z| \geq 1.
\]
Further, \(\nabla^2 \mathbf{A} = 0\) outside the disc, so that
\[
\frac{\partial^2 \mathbf{A}}{\partial z^2} + \lambda^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} r \mathbf{A} \right) = 0 \quad \text{for} \quad |z| \geq 1.
\]
In order to impose boundary conditions at the disc surface, we recall that the eigenfunctions of Eq. (4) are of the form $A \propto \exp(-k\lambda|z|)J_1(kr)$ with some radial wave number $k$. Since we require that the scale of the radial variation of $A$ is of order unity, $k = O(1)$, we obtain

$$\frac{\partial A}{\partial z} = -k\lambda A = O(\lambda) \quad \text{for} \quad z \geq 1.$$  (5)

To the lowest order in $\lambda$, the resulting boundary condition is

$$\frac{\partial A}{\partial z} = 0 \quad \text{for} \quad z = 1.$$  (6)

It was noted by Soward (1992a,b) that the term of order $\lambda$ in Eq. (5) must be retained if higher-order terms of the asymptotic solution of Eqs. (2) and (3) are considered. Our goal here is to include these terms in a consistent manner into a regular asymptotic scheme. The physical meaning of these corrections is as follows. Since $-\partial A/\partial z = B_r$, the boundary condition (6), together with $B = 0$, restricts the magnetic field to be purely vertical outside the disc, similarly to an external field of an infinite homogeneous slab. Then magnetic fields at different radii can be connected only via magnetic lines passing through the disc where $B_r \neq 0$. However, the boundary condition (5) implies that the field on and above the disc surface has a weak but finite radial component, and magnetic lines can thus connect different regions in the disc through the vacuum. This leads to non-local effects arising in higher orders of asymptotic expansion, and a description of these effects is our subject here.

A treatment of the non-local asymptotics for mean-field dynamos in a thin disc can be found in Soward (1978, 1992a,b, 2000) where both steady and oscillatory dipolar and quadrupolar modes are considered, but only for marginally stable solutions and in a local, Cartesian geometry. Here we consider exponentially growing (or decaying), non-oscillatory, quadrupolar modes in cylindrical geometry, ready for applications to astrophysical discs, especially those in spiral galaxies. Another new element in our approach is that we do not expand the coefficients of the dynamo equations into power series in $r$, and so we allow for arbitrary radial variation in the dynamo coefficients. In this paper we present the derivation of the asymptotic equations. Their solutions will be discussed elsewhere (Willis et al., 2000).
2 Vacuum boundary conditions for a thin disc

In this section we derive boundary conditions for a thin disc surrounded by vacuum in a form suitable for asymptotic analysis. For this purpose we first perform the Hankel transform of Eq. (4) to obtain:

$$\frac{\partial^2 \hat{A}}{\partial z^2} - \lambda^2 k^2 \hat{A} = 0,$$

(7)

where

$$\hat{A}(z, k) = \int_0^\infty A(z, r) J_1(kr)r \, dr$$

(8)

is the Hankel transform of $A(z, r)$ with respect to $r$. Equation (7) has a simple solution decaying at infinity, $\hat{A}(z, k) = A_0(k) \exp(-k\lambda|z|)$. However, this solution contains an arbitrary function of $k$, $A_0(k)$, so it cannot be used to formulate a closed boundary condition at $|z| = 1$. Instead, we note that, for $z > 0$, solution of Eq. (7) decaying at $z \to \infty$ satisfies

$$\frac{\partial \hat{A}}{\partial z} + \lambda k \hat{A} = 0 \quad \text{for } z \geq 1,$$

which is useful to compare with Eq. (5). We apply the inverse Hankel transform to this equation and set $z = 1$ to obtain

$$\frac{\partial A}{\partial z} \bigg|_{(1, r)} + \lambda \int_0^\infty \hat{A}(1, k) J_1(kr)k^2 \, dk = 0,$$

(9)

and the plus sign must be replaced by minus for $z = -1$.

We could now substitute Eq. (8) into Eq. (9) in order to obtain a closed boundary condition for $A$ at $z = 1$ in an integro-differential form, but the resulting operator contains a strongly divergent kernel $\int_0^\infty J_1(kr) J_1(kr')k^2 \, dk$. Therefore, we develop a slightly more elaborate procedure devised to isolate the singularity in the integral kernel. Firstly, we improve the convergence of the kernel by eliminating the factor $k^2$ in it. This can be done by rewriting the integral operator in terms of $\partial^2 A/\partial r^2$.

For this purpose, we use the general relation (Sneddon, 1951, Sect. 10)

$$k^2 \hat{A} = -\int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} r A \right) J_1(kr) r \, dr,$$
written at \(|z| = 1\), to reduce Eq. (9) to the form

\[
\frac{\partial A}{\partial z} - \frac{\lambda}{r} \int_0^\infty \frac{\partial}{\partial r'} \left( \frac{1}{r'} \frac{\partial}{\partial r'} r' A \right) W(r, r') \, dr' = 0 \quad \text{at } z = 1 ,
\]

where

\[
W(r, r') = rr' \int_0^\infty J_1(kr)J_1(kr') \, dk ;
\]

note that we have isolated a factor \(rr'\) in \(W(r, r')\) to ensure that it does not diverge at \(r, r' \to 0\) (see below).

We can prove that \(W(r, r')\) is square integrable and has only a weak singularity at \(r - r' = 0\). For this purpose we explicitly isolate the singularity of \(W(r, r')\) which occurs at \(r - r' = 0\). We shall consider consecutively the cases prone to a singular behaviour.

We start with \(r, r' \neq 0\) when a singularity at \(r = r'\) can be expected. We introduce a new variable \(\sigma = kr\) for \(r' > r\) or \(\sigma = kr'\) for \(r' < r\) in Eq. (11) and use Eq. (A1) of Appendix A to see that

\[
W(r, r') = -\frac{1}{\pi} (rr')^{1/2} \ln |r - r'| + \ldots ;
\]

here and henceforth dots stand for a nonsingular part. In a similar way, the case when \(r, r' \to 0\) can be shown to lead to the singularity of Eq. (12) because terms like \((rr')^{1/2} \ln r\) are not singular.

Next consider \(r = \text{const} \neq 0, r' \to 0\). The new variable is now \(\sigma = kr'\), so that

\[
W(r, r') = -\frac{1}{\pi} (rr')^{1/2} \ln |r - r'| + \frac{1}{\pi} (rr')^{1/2} \ln r' + \ldots .
\]

Both logarithmic terms are finite due to the factor \(rr'\) in \(W(r, r')\), so in this case the kernel is free of singularities. As \(r\) and \(r'\) enter \(W\) in a symmetrical manner, we conclude that there is no singularity at \(r \to 0, r' = \text{const} \neq 0\), as well.

Altogether, Eq. (12) shows that the integral kernel has an integrable (logarithmic) singularity, and thus it can be treated as a finite kernel (see, e.g., Kolmogorov and Fomin, 1957). As a result Eq. (10) provides a viable exact boundary condition for a slab surrounded by vacuum.
3 Representation of the Asymptotic Solution

In this section we consider asymptotic solutions of Eqs. (2) and (3) which are free of boundary layers. Such layers may arise at the disc surface and also at certain radii (e.g., contrast structures — Belyanin et al., 1994).

With the definition \( Y = \left( \begin{array}{c} B(z, r) \\ A(z, r) \end{array} \right) \), the eigenvalue problem for Eqs. (2) and (3) can be conveniently rewritten in the form

\[
\Gamma Y = \mathcal{L}_z Y + \lambda^2 \mathcal{L}_r Y ,
\]

with the boundary conditions

\[
P_{z,0}Y = 0 \quad \text{at} \quad z = 0 ; \quad P_{z,1}Y - \frac{\lambda}{r} K Y = 0 \quad \text{at} \quad z = 1 ,
\]

where the following operators have been introduced:

\[
\mathcal{L}_z = \left( \begin{array}{cc} \frac{\partial^2 / \partial z^2}{\alpha} & -DG \partial / \partial z \\ \partial^2 / \partial z^2 \end{array} \right) , \quad \mathcal{L}_r X = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} rX \right) ,
\]

\[
P_{z,0} = \left( \begin{array}{cc} \partial / \partial z & 0 \\ 0 & 1 \end{array} \right) , \quad P_{z,1} = \left( \begin{array}{cc} 1 & 0 \\ 0 & \partial / \partial z \end{array} \right) ,
\]

and

\[
K \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} 0 \\ \int_0^\infty dr' W(r, r') \mathcal{L}_r x_2 \end{array} \right) .
\]

The operators \( P_{z,1} \) and \( K \) are defined at \( z = 1 \) and \( P_{z,0} \) at \( z = 0 \).

Consider asymptotic solutions of Eqs. (2) and (3) for \( \lambda \ll 1 \). We denote the asymptotic parameter \( \epsilon \), a function of \( \lambda \) to be determined below, and represent the asymptotic solution in the form

\[
Y(z, r) = Q(\epsilon^{-1}r)Y_0(z; r) + \epsilon^2 Y_1(z, \epsilon^{-1}r; r) + \ldots ,
\]

with unknown functions

\[
Q(\epsilon^{-1}r) , \quad Y_0 = \left( \begin{array}{c} b(z; r) \\ a(z; r) \end{array} \right) , \quad Y_1 = \left( \begin{array}{c} B_1(z, \epsilon^{-1}r; r) \\ A_1(z, \epsilon^{-1}r; r) \end{array} \right) .
\]
Our motivation for the particular choice of this form will immediately become clear. We only note here that Eq. (16) features two types of dependence on $r$. One of them is connected with the presence of the radial derivatives in Eq. (13), this involves the argument $\epsilon^{-1}r$. Another kind of $r$-dependence arises from the radial variation of the coefficients of the local operator, $L_z$; this is a slow parametric variation of the solution with $r$ unrelated to any differential operator in $r$. The asymptotic expansion of the eigenvalue $\Gamma$ will emerge later.

In order to preserve a similarity to earlier asymptotics developed for galactic dynamos, we do not introduce scaled variables, e.g., $\epsilon^{-1}r$. Therefore, some of our asymptotic equations, e.g., Eq. (32), will formally contain terms with $\epsilon$ or $\lambda$, but all terms in such equations will always be of the same order of magnitude in $\lambda$.

### 3.1 Zeroth-order equations

Substituting Eq. (16) into Eqs. (13) and (14) and combining terms of equal orders in $\epsilon$ we obtain in the leading order in $\lambda$

\[
\left(\Gamma Y_0 - L_z Y_0\right) Q = 0 ,
\]

\[
Q P_{z,0} Y_0 = 0 , \quad Q P_{z,1} Y_0 = 0 ,
\]

where we have deliberately retained the factor $Q$, a function of $\epsilon^{-1}r$. If $Q$ were cancelled, Eq. (17) would become intrinsically contradictory as $\Gamma$ must be independent of $r$ whereas the coefficients of $L_z$ depend on $r$. To overcome this difficulty, we consider separately two radial ranges. At those $r$ where $Q$ is small, Eqs. (17) and (18) are satisfied, to the required accuracy of $O(\epsilon^2)$, just because $Q$ is small, of order $\epsilon^2$ or less. At radii where $Q = O(1)$ we replace Eqs. (17) and (18) with the following equations:

\[
\gamma_0(r) Y_0 - L_z Y_0 = 0 ,
\]

\[
P_{z,0} Y_0 = 0 , \quad P_{z,1} Y_0 = 0 ,
\]

where $\gamma_0(r)$ is the local growth rate (more precisely, its zeroth-order term), a function of radius. With this definition of $\gamma_0(r)$, we have

\[
\Gamma - \gamma_0(r) = O(\epsilon^2) \quad \text{if } Q(\epsilon^{-1}r) = O(1) .
\]

Outside the radial range where $Q(\epsilon^{-1}r) = O(1)$, $\gamma_0(r)$ still can be defined as the leading eigenvalue of Eqs. (19) and (20).
Equation (21) in fact restricts the radial range $\Delta r$ where $Q = O(1)$ to be $\Delta r = O(\varepsilon)$ provided $\gamma_0(r)$ can be approximated by a quadratic function of $r$ near its maximum, $\gamma_0(r + \Delta r) \simeq \gamma_0(r) + C\Delta r^2$ with a certain constant $C$. The radial variation of $\gamma_0(r)$ is controlled by that of $\alpha$ and $G$, the coefficients of $L_z$. The weaker is the radial variation of $\alpha$ and $G$, the wider is the radial range where Eq. (21) can be satisfied.

Earlier experience with solutions of Eqs. (19) and (20) (Ruzmaikin et al., 1988) shows that the leading local eigenvalue is real for $D < 0$ and, furthermore, this eigenvalue is isolated, i.e., there are no other positive eigenvalues for moderate values of $|D|$ (in other words, the difference of the eigenvalues is of order unity). We restrict our attention to the mode with the largest real eigenvalue $\gamma_0(r)$.

Since the boundary value problem (19) and (20) has been extensively studied earlier (e.g., Ruzmaikin et al., 1988 and references therein), we do not discuss it further, but assume that its solution, $Y_0(z;r)$ and $\gamma_0(r)$, is given.

### 3.2 First-order equations

Having isolated zeroth-order terms, we can represent the remaining terms in Eq. (13) as

$$\Gamma Y_1 = L_z Y_1 - \frac{\Gamma - \gamma_0(r)}{\varepsilon^2} Y_0 Q + \frac{\lambda^2}{\varepsilon^2} Y_0 L_r Q + \ldots ,$$

(22)

where dots denote terms of order $\varepsilon^2$ and higher. We temporarily retain the term of the order $\lambda^2/\varepsilon^2$ as the connection of $\varepsilon$ and $\lambda$ still has not been determined; we shall show that this term must be omitted in the first-order equations.

Similarly the terms remaining in the boundary conditions are

$$\mathcal{P}_{z,0} Y_1 + \ldots = 0 ,$$

$$\mathcal{P}_{z,1} Y_1 = \frac{\lambda}{\varepsilon^2 r} \mathcal{K}(Q Y_0) + \ldots .$$

We determine $\varepsilon$ from the requirement that the two terms in the second boundary condition are of equal orders of magnitude in $\lambda$. As shown in Eq. (16), the eigenfunction is expected to be of order unity in a region whose radial extent $\Delta r$ is of order $\varepsilon$, so that $dQ/dr \simeq Q/\Delta r \simeq Q/\varepsilon$. This is true for the leading eigenfunctions $Q$ that do not rapidly oscillate with $r$. Since $L_r A = O(\Delta r^{-2}) = O(\varepsilon^{-2})$, we obtain $\mathcal{K}(Q Y_0) \simeq \varepsilon^{-2} \Delta r \simeq \varepsilon^{-1}$ (see Appendix B). Thus, we require that

$$\frac{\lambda}{\varepsilon^2} \varepsilon^{-1} = 1 ,$$
or
\[ \epsilon = \lambda^{1/3}. \]

Then Eq. (21) implies that
\[ \Gamma - [\gamma_0(r)]_{\text{max}} = O(\lambda^{2/3}). \]

Now we can see that \(\lambda/\epsilon = o(1)\), so that \(\lambda^2\epsilon^{-2}L_r(Q) = O(\lambda^{2/3})\) and the desired first-order boundary value problem based on Eq. (22) reduces to
\[
\begin{align*}
\Gamma Y_1 &= L_z Y_1 - \frac{\Gamma - \gamma_0(r)}{\lambda^{2/3}} Y_0 Q, \\
P_{z,0} Y_1 &= 0, \quad P_{z,1} Y_1 = \frac{\lambda^{1/3}}{r} K(Q Y_0).
\end{align*}
\]

(23)

(24)

However, this form of the first-order boundary value problem is inconvenient for further analysis. We rewrite it to obtain a boundary value problem with homogeneous boundary conditions. Let us represent the solution of Eqs. (23) and (24) in the form
\[
Y_1 = X_1 + X_2,
\]

where \(X_1\) solves the following boundary value problem:
\[
L_z X_1 = 0,
\]
\[
P_{z,0} X_1 = 0, \quad P_{z,1} X_1 = \frac{\lambda^{1/3}}{r} K(Q Y_0).
\]

Then \(X_2\) is the solution of the following boundary value problem
\[
\begin{align*}
\Gamma X_2 - L_z X_2 &= \frac{\gamma_0(r) - \Gamma}{\lambda^{2/3}} Y_0 Q - \Gamma X_1, \\
P_{z,0} X_2 &= 0, \quad P_{z,1} X_2 = 0.
\end{align*}
\]

(25)

(26)

Here \(X_1\) is in fact a first-order correction to the eigenfunction \(Y_0\) of Eqs. (17) and (18) arising from the term with \(\lambda^{1/3}\) in the boundary condition (24) at \(z = 1\). Note that \(|X_1| = O(Q)\). The boundary value problem for \(X_1\) can be solved straightforwardly upon solving Eq. (32) for \(q(r) = Q(r) a'(1, r)\).

It is useful to represent \(Y_1\) as a sum of the two terms, \(X_1\) and \(X_2\), because this results in a universal form of the left-hand sides in the local boundary value problems
in all asymptotic orders. An alternative approach, without this representation, is discussed by Soward (2000).

We expect that \( X_2 \) has the same functional form as the zeroth-order term in the asymptotic expansion (16), so we take \( X_2(z; r) = Q_1(\lambda^{-1/3} r) X_{2,0}(z; r) \). Then Eqs. (25) and (26) reduce to a form similar to Eqs. (17) and (18):

\[
\Gamma X_{2,0} - (L_z X_{2,0}) Q_1 = \frac{\gamma_0(r) - \Gamma}{\lambda^{2/3}} Y_0 Q - \Gamma X_1 ,
\]

where, as above, we retain the factor \( Q_1 \) in the boundary conditions and consider again two radial ranges. At those radii where \( \Gamma - \gamma_0(r) = O(\lambda^{2/3}) \) and \( Q = O(1) \), we expect that \( Q_1 = O(1) \). In this radial range we obtain the following first-order equations:

\[
\Gamma X_{2,0} - L_z X_{2,0} = \left[ \frac{\gamma_0(r) - \Gamma}{\lambda^{2/3}} Y_0 Q - \Gamma X_1 \right] Q_1^{-1} ,
\]

\[
P_{z,0} X_{2,0} = 0 , \quad Q_1 P_{z,1} X_{2,0} = 0 ,
\]

These equations have the same operators on the left-hand sides as Eqs. (19) and (20). As for the radial range where \( Q_1 = o(1) \), the form of \( Q_1 \) is unimportant there and we do not discuss it further. Equations (29) and (30) represent the desired form of the first-order boundary value problem which we use below to obtain an equation for \( Q \).

4 The radial dynamo equation

In this section we derive a closed equation for the radial part of the zeroth-order eigenfunction, \( Q \), which follows from the solvability condition for the first-order boundary value problem (29) and (30). By Fredholm’s solvability condition, the r.h.s. of Eq. (29) must be orthogonal to the eigenvector \( X^\dagger \) of the corresponding adjoint homogeneous problem (this problem is formulated and discussed in Appendix C), i.e.,

\[
\left\langle \left[ \frac{\Gamma - \gamma_0(r)}{\lambda^{2/3}} Y_0 Q + \Gamma X_1 \right] Q_1^{-1} , X^\dagger \right\rangle = 0 ,
\]

where \( \left\langle F, G \right\rangle = \int_0^1 F \cdot G \, dz \) denotes the scalar product in the space of continuous vector functions on the interval \( 0 \leq z \leq 1 \). This yields the following equation for \( Q \).
and $\Gamma$:

$$\Gamma - \gamma_0(r) \frac{\lambda^{2/3}}{Q} Q\langle Y_0, X^\dagger \rangle + \Gamma\langle X_1, X^\dagger \rangle = 0.$$  \hspace{1cm} (31)$$

We recall that $\Gamma - \gamma_0(r) = O(\lambda^{2/3})$ and $|X_1| = O(1)$ in the radial range where $Q = O(1)$ and $|X_1| = o(1)$ outside this range. Therefore, to the accuracy $O(\lambda^{2/3})$ we can replace $\Gamma$ by $\gamma_0(r)$ to obtain

$$\Gamma\langle X_1, X^\dagger \rangle = \langle X_1, \gamma_0(r) X^\dagger \rangle = \langle X_1, L^\dagger_{z} X^\dagger \rangle = -a^\dagger(1, r) \frac{\lambda^{1/3}}{r} \mathcal{K}(aQ),$$

where, in order to obtain the last equality, we performed two integrations by parts and took into account the boundary conditions (20) and (C3).

Now Eq. (31) can be rewritten as

$$\Gamma Q = \gamma_0(r) Q + \frac{\lambda a^\dagger(1; r)}{r\langle Y_0, X^\dagger \rangle} \int_0^\infty dr' W(r, r') L_{r'}(Qa(1, r')).$$

This is the desired equation for magnetic field amplitude $Q$ and global growth rate $\Gamma$. We stress that all terms in this equation are of the same order in $\lambda$ because $dQ/dr = O(\lambda^{-1/3})$.

To reduce this equation to a final form, we introduce

$$q(r) = Q(r)a^\dagger(1, r),$$

to obtain

$$\Gamma q = \gamma_0(r) q + \frac{\lambda}{r} \eta(r) \int_0^\infty dr' W(r, r') L_{r'}(q),$$  \hspace{1cm} (32)$$

where

$$\eta(r) = \frac{a(1, r)a^\dagger(1, r)}{\langle Y_0, X^\dagger \rangle}$$

is a function of radius which is obtained from the zeroth-order equations,

$$Y_0 = \begin{pmatrix} b(z; r) \\ a(z; r) \end{pmatrix}, \quad X = \begin{pmatrix} b^\dagger(z; r) \\ a^\dagger(z; r) \end{pmatrix}.$$  \hspace{1cm} (33)$$

It is important to note that both $q(r)$ and $\eta(r)$ are independent of the normalization of the local solutions.
5 Discussion

It is useful to compare the main result of this paper, Eq. (32), with the corresponding equation resulting from using the boundary condition (6) in all orders (see, e.g., Ruzmaikin et al., 1988):

$$\Gamma Q = \gamma_0(r)Q + \lambda^2 \mathcal{L}_r Q. \quad (33)$$

The last term on the right-hand side describes magnetic diffusion in the radial direction. The new feature of the theory discussed here is that this term is replaced by an integro-differential operator. Numerical analysis of Eq. (32) presented elsewhere (Willis et al., 2000) shows that the integral term results in enhanced magnetic diffusion, so that Eqs. (32) and (33) have similar physical significance. This is understandable since the kernel $W(r, r')$ has a singularity. If this singularity were extreme, say, $W(r, r') \approx \delta(r - r')$, the functional form of Eq. (33) would be recovered precisely, albeit with additional factor $\eta(r)$ which can hardly be of major importance. However, the actual kernel has a wider radial profile (and weaker singularity), so the magnitude of this term is enhanced by non-local coupling, but its nature still remains basically diffusive.

The integral, non-local character of magnetic diffusion affects the nature of the asymptotic solution for $Q(r)$. From Eq. (33), the radial width of the eigenfunction is of the order of $\epsilon = \lambda^{1/2}$ and $\Gamma - \gamma_0 = O(\lambda)$ (Ruzmaikin et al., 1988), whereas Eq. (32) corresponds to $\epsilon = \lambda^{1/3}$ and $\Gamma - \gamma_0 = O(\lambda^{2/3})$. We should note, however, that the difference is hardly important for most applications to astrophysical discs where, for $\lambda \approx 10^{-2}$, the difference amounts to a factor of $\lambda^{1/6} \approx 1/2$. A wider radial distribution of the magnetic field resulting from Eq. (32) (Soward, 2000; Willis et al., 2000) can help in accelerating the amplification of magnetic fields in the outer regions of young galaxies along the lines discussed by Moss et al. (1998).

Using heuristic arguments, Poezd et al. (1993) generalized Eq. (33) to nonlinear regimes arising from alpha-quenching. The resulting nonlinear equation has $\Gamma$ replaced by $\partial/\partial t$ and $\gamma_0(r)$ by $\gamma_0(r)[1 - Q^2/B^2_0(r)]$, where $B_0(r)$ defines the dynamo saturation level appearing in the alpha-quenching model. It is plausible that a similar generalization is possible with Eq. (32). Another problem is to generalize the theory to nonaxisymmetric solutions where the boundary conditions should take a more complicated form.
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Appendix A. The singularity of \( \int_{0}^{\infty} J_1(k)J_1(k\sigma) \, dk \)

In this section we isolate the singularity of the integral

\[
I(\sigma) = \int_{0}^{\infty} J_1(k)J_1(k\sigma) \, dk,
\]

where \( \sigma \) is a constant. It is a special case of the Weber and Schafheitlin integral discussed at length by Watson (1922, Sect. 13.4), which can expressed as

\[
I(\sigma) = \frac{1}{2} \sigma \, _2F_1\left( \frac{3}{2} , \frac{1}{2} ; 2 , \sigma^2 \right).
\]

We can restrict ourselves to the case \( \sigma > 1 \) because Eq. (11) reduces to the above form with \( \sigma = r/r' \) for \( r > r' \) and \( \sigma = r'/r \) for \( r' > r \).

Asymptotics of this integral for \( \sigma - 1 \ll 1 \) can be obtained by expressing \( I(\sigma) \) in terms of the complete elliptic integral of the first kind, which yields \( I(\sigma) \approx -\pi^{-1} \ln |1 - \sigma| \) (see Willis et al., 2000 for details).

This asymptotic form can also be obtained from the following arguments which make clear the nature of the singularity at \( \sigma \to 1 \). Introduce

\[
\Delta(x) = J_1(x) - \left( \frac{2}{\pi x} \right)^{1/2} \cos \left( x - \frac{3}{4} \pi \right),
\]

so that \( \Delta(x) = O(x^{-1}) \) for \( x \gg 1 \). Then

\[
I(\sigma) = \int_{0}^{1} J_1(k)J_1(k\sigma) \, dk + \int_{1}^{\infty} \Delta(k)\Delta(k\sigma) \, dk
+ \int_{1}^{\infty} \Delta(k) \cos(k\sigma - \frac{3}{4} \pi) \left( \frac{2}{\pi k\sigma} \right)^{1/2} + \int_{1}^{\infty} \Delta(k\sigma) \cos(k - \frac{3}{4} \pi) \left( \frac{2}{\pi k} \right)^{1/2} \, dk
+ 2 \int_{1}^{\infty} \cos(k - \frac{3}{4} \pi) \cos(k\sigma - \frac{3}{4} \pi) \frac{dk}{\pi k\sigma^{1/2}},
\]

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where the only divergent term is the last one. For $\sigma > 1$, the latter integral can be expressed in terms of the sine integral $\text{Si}(x) = -\int_0^x u^{-1} \sin u \, du$ and the cosine integral $\text{Ci}(x) = -\int_\infty^x u^{-1} \cos u \, du$ (which has a singular component $\ln x$) as

$$2 \int_1^\infty \cos(k - \frac{3}{4} \pi) \cos(k\sigma - \frac{3}{4} \pi) \frac{dk}{\pi k\sigma^{1/2}} = \frac{1}{\pi\sigma^{1/2}} \left[-\frac{1}{2}\pi - \text{Si}(\sigma + 1) - \text{Ci}(\sigma - 1)\right]$$

$$= f(\sigma) - \frac{1}{\pi\sigma^{1/2}} \ln (\sigma - 1),$$

where $f(\sigma)$ is a finite function.

Hence, for $\sigma > 1$ we have

$$\int_0^\infty J_1(k) J_1(k\sigma) \, dk = F(\sigma) - \frac{1}{\pi\sigma^{1/2}} \ln (\sigma - 1), \quad (A1)$$

where $F(\sigma)$ is finite.

**Appendix B. The order of magnitude of the integral term**

Here we estimate the order of magnitude of $K(Qa) = \int_0^\infty dr' W(r, r') L_{\nu'}\left(Q(\epsilon^{-1} r') a(z; r')\right)$. Since $a$ is a slowly varying function of $r'$, its radial derivatives are of order unity. The dominant term in $L_{\nu'} Q$ arises from $d^2 Q/dr'^2 = O(\epsilon^{-2})$. Since the magnitude of the integral is controlled by the singularity in $W(r, r')$, Eq. (12), we have

$$K(Qa) \sim \int_0^\infty \ln |r - r'| \frac{d^2 Q}{dr'^2} \, dr'.$$

It may seem that the order of magnitude of this integral is $\epsilon^{-2} \Delta r \ln |\Delta r| \sim \epsilon^{-1} \ln \epsilon$ because the main contribution to the integral arises from a neighbourhood of $r = r'$ whose width is $\Delta r \approx \epsilon$.

However, this estimate is wrong because $d^2 Q/dr'^2$ changes rapidly and even changes sign within the $\epsilon$-neighbourhood of $r - r'$. The correct calculation is as follows. We introduce $\delta \to 0$ and consider

$$\int_0^\infty \ln |r - r'| \frac{d^2 Q}{dr'^2} \, dr' \approx \int_0^{r-\delta} \ln |r - r'| \frac{d^2 Q}{dr'^2} \, dr' + \int_{r+\delta}^\infty \ln |r - r'| \frac{d^2 Q}{dr'^2} \, dr',$$
so that now we can integrate by parts to obtain
\[
K(Qa) = \left[ \frac{dQ}{dr} \ln |r - r'| \right]_0^{r-\delta} + \left[ \frac{dQ}{dr} \ln |r - r'| \right]_{r+\delta}^{\infty} - \text{v.p.} \int_0^{\infty} \frac{dQ}{dr} \frac{1}{r - r'} dr'.
\]
Since the \( \ln |r - r'| \) is symmetric with respect to \( r, r' \) and \( dQ/dr = 0 \) for \( r = 0 \) and \( \infty \), the logarithmic terms cancel and
\[
K(Qa) = -\text{v.p.} \int_0^{\infty} \frac{dQ}{dr} \frac{1}{r - r'} dr' \simeq \epsilon^{-1}.
\]

Appendix C. The adjoint problem

Consider the homogeneous counterpart of the boundary value problem (29) and (30), i.e.,
\[
L_z X = \gamma_0 X, \quad \text{and} \quad P_z, 0 X = 0, \quad P_z, 1 X = 0 . \tag{C1} \tag{C2}
\]
This boundary value problem coincides with the zeroth-order problem (19) and (20). Our aim here is to formulate the adjoint problem and to discuss its properties.

We introduce the eigenvectors of the original and adjoint problems, respectively:
\[
X = \begin{pmatrix} b(z; r) \\ a(z; r) \end{pmatrix}, \quad X^\dagger = \begin{pmatrix} b^\dagger(z; r) \\ a^\dagger(z; r) \end{pmatrix}.
\]

By the definition of the adjoint operator, we have
\[
\langle L_z X, X^\dagger \rangle = \int_0^1 \left[ (-D Ga' + b') b^\dagger + (ab + a'') a^\dagger \right] dz
\]
\[
= \int_0^1 \left[ -D G a b + b' b - b b^\dagger \right]_0^1 + \left[ a' a^\dagger - a a^\dagger \right]_0^1 +
\]
\[
+ \int_0^1 \left[ b \left( b'' + aa^\dagger \right) + a \left( D G b^\dagger + a'' \right) \right] dz
\]
\[
= \langle X, L^\dagger_z X \rangle .
\]

Therefore, the problem adjoint to (19), (20) is
\[
\gamma_0^\dagger(r) b^\dagger = b'' + a a^\dagger ,
\gamma_0^\dagger(r) a^\dagger = D G b^\dagger + a'' ,
\]
\[
a^\dagger(0) = 0 , \quad b^\dagger(0) = 0 , \quad b^\dagger(1) = 0 , \quad a^\dagger(1) = 0 . \tag{C3}
\]
According to Fredholm’s theory, the adjoint problem has the same spectrum as the problem (19) and (20), i.e., \( \gamma_0^\dagger(r) = \gamma_0 \).
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