Effect of charge on the dynamics of an acoustically forced bubble

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Abstract
The effect of charge on the dynamics of a gas bubble undergoing forced oscillations in a liquid due to incidence of an ultrasonic wave is theoretically investigated. The limiting values of the possible charge a bubble may physically carry are obtained. The presence of charge influences the regime in which the bubble’s radial oscillations fall. The extremal compressive and expansive dimensions of the bubble are also studied as a function of the amplitude of the driving pressure. It is shown that the limiting value of the bubble charge is dictated both by the minimal value reachable of the bubble radius as well as the amplitude of the driving ultrasonic pressure wave. A non-dimensional ratio \( \zeta \) is defined that is a comparative measure of the extremal values the bubble can expand or contract to, and we find the existence of an unstable regime for \( \zeta \) as a function of the driving pressure amplitude, \( P_s \). This unstable regime is gradually suppressed with increasing bubble size. The Blake and the upper transient pressure thresholds for the system are then discussed.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The study of bubble dynamics and cavitation has a long and interesting history in the scientific literature. One of the earliest works was that of Rayleigh [1] in his study of cavitation...
phenomena, motivated by the need to understand and minimize the damage to ships’ propellers due to cavitation (the low pressure on the surface of the propeller blades causes the liquid in contact with the surface to spontaneously form unstable bubble clouds which often self-organize into dendritic structures [2]. These bubble clouds implode with enormous force resulting in serious damages to the propeller). Cavitation, bubble formation and dynamics are present in different instances and situations. Analyses and studies of the phenomena have been motivated by and have explained very distinct natural, practical phenomena. Small amplitude oscillations of a gas bubble in a liquid were studied by Minnaert [3]. His work showed the important contribution of radial oscillations of entrained air bubbles in the sound heard from running water.

In nature, the snapping shrimp uses rapid closure of its claws to generate cavitating bubbles which stun its prey [4]. Bubble formation and kinetics contribute to fluid flow in biological systems in blood and cells in the micrometre scale [5]. In the presence of incident ultrasound waves, bubbles can also enhance the rate of chemical reactions [6, 7]. Bubble formation and cavitation can be recreated under controlled conditions in the laboratory by subjecting a liquid in a container to a standing ultrasonic wave, setting up a pressure field within the liquid. When the driving pressure amplitude of the sonic field becomes larger than the ambient pressure, the pressure in the liquid becomes negative. When this pressure exceeds the vapour pressure of the liquid, local evaporation is caused. The liquid ‘breaks’ up forming tiny micrometre size cavitation bubble clouds, which implode violently within a very short period of time. Frenzel and Schultes demonstrated that these cavitation clouds emitted low-intensity visible light [8]. This phenomenon, wherein light is emitted by a gas bubble in a liquid due to its rapid expansion and collapse when ultrasound is incident on it, is known as sonoluminescence. This has also contributed to an extensive study of bubble dynamics in the context of sonoluminescence. Several other studies have followed in the literature, including those of Gaitan and others [9–12]. The radial oscillations of gas bubbles in a liquid that are caused due to incident acoustic waves cannot be described trivially. These are a type of driven nonlinear oscillation that can greatly depend on initial conditions and can be chaotic in nature. This has been shown conclusively (see, for example, [13–17] and references therein). Reviews may be found in, for example, [7, 13, 18].

The presence of electric charge on bubbles in fluids has been reported in the experimental literature [19–23].

In this paper we use the Rayleigh–Plesset equation modified by Parlitz et al., further modified to take into account the presence of charge on the bubble, and study the effects of charge, driving frequency and amplitude of the ultrasonic driving field on the dynamics of the bubble, with the specific heat ratio being taken as 5/3 in our calculations, consistent with the gas in the bubble being a monatomic ideal, inert gas under adiabatic conditions. In section 2, we describe the model used; the natural frequency of oscillation of the bubble is obtained and the phase plots show that the maximal radial velocity for the charged bubble greatly exceeds that of the neutral bubble. In section 3 the Blake radius and threshold for the charged bubble are calculated. This is followed by a discussion of Rayleigh collapse and the influence of driving pressure amplitude in section 4. We show that for a bubble of given ambient radius $R_0$, there exists a cut-off value for the maximum charge it can carry, which is dictated by the van der Waals hard-core radius. In section 5 we introduce a new dimensionless ratio $\zeta$ as a measure of the maximum and minimum radius a bubble can achieve under acoustic driving. This when plotted as a function of pressure amplitude clearly captures the positions of the Blake threshold and the upper critical transient pressure threshold for acoustic cavitation, and their distinct dependence on the pressure amplitude, driving frequency and bubble charge. Bubbles that are present in various systems can carry a non-zero charge. Hence, a proper understanding
of their behaviour and dependence on pressure, driving frequency, ambient radius and other parameters, is essential. This is the subject of investigation of this paper. The results obtained in our work therefore become very useful in the context of medical uses of acoustic cavitation and ultrasound for diagnostic and therapeutic purposes.

2. The model

We shall briefly describe the model and the set of equations being used to describe the radial oscillations of gas bubbles in liquids. Various models have investigated distinct, disparate limits of bubble collapse. The original work of Rayleigh assumed the surrounding liquid to be inviscid and incompressible [1]. Plesset and others [24–26] have included viscosity, surface tension etc. Keller and Kolodner used the same expression but with a modification accounting for acoustic radiation by the bubble by considering the liquid as slightly compressible [27]. Keller and Miksis further included all these modifications—of viscosity, surface tension, the incident sound wave and acoustic radiation—in one model to obtain a modified equation [28].

The model we use to describe the system is based on the earlier equations of Prosperetti, Parlitz and Keller and Miksis and others [17, 28–30], which are modifications of the Rayleigh–Plesset equations [1, 24, 25].

In our work we have made the tacit assumption that the oscillating bubble does not undergo shape deformations. The pressure regime in which our calculations have been made are consistent with the occurrence of purely radial oscillations with negligible shape deformations—see, for example [10,31,32]. The oscillations and bubble wall motion therefore do not account for shape deformations and are as per the Rayleigh–Plesset equations in, for example, Parlitz et al [17], and Brenner et al [13]. In our model, there are no angular oscillation modes; only purely radial oscillations are being considered.

The system we consider is that of a bubble that is suspended in a liquid and has some net charge. The presence of charge on a bubble is not speculative. As mentioned earlier, electric charge has been found on bubbles under acoustic forcing. Various mechanisms could cause charge to be present on the bubble, e.g., the migration of ionic charges in the liquid onto the bubble surface, although the exact mechanism has been debated. See, for example, the work of Alty [20, 21] and Akulichev [22]. That gas bubbles in water can carry a charge has been clearly demonstrated for a long time [20, 21]. Indeed, it had been shown earlier [19] that air and other gas bubbles in water become negatively charged. More recently, this has been shown by Shiran and Watmough [23]. Therein, bubbles placed in an electric field are clearly demonstrated to veer towards one of the electrodes. Other work on the presence of charge on bubbles in different situations include that done by Bunkin and Bunkin [33].

The study of the dynamics of charged gas bubbles in fluids is important because such systems find various applications, including in medicine—a widely used one being in the medical application of ultrasound. While it has been overlooked in several models, various investigations of charged bubbles have appeared in the literature in different contexts. These include, for example, [34]. Their work is however of very limited scope since they consider the specific case for which the polytropic index \( \Gamma \) takes the value 4/3 and for this case it turns out that the terms containing the charge mutually cancel out in the Rayleigh–Plesset equation.

We take \( \Gamma = 5/3 \) consistent with taking the heat transfer across the bubble to be an adiabatic process. For the sake of simplicity, we assume the charge to be strictly limited to, and uniformly distributed on, the surface of the bubble. The surface charge density of the bubble thus changes as the bubble expands and contracts along with the acoustic forcing being externally applied to it. As mentioned above, we use a modified form of the Rayleigh–Plesset equation, one introduced by Parlitz et al [17], which
is equivalent in first order in \(1/c\) to that of Keller and Miksis \([28]\) \((c \text{ being the velocity of sound})\) and which includes, approximately, the sound radiation which is the most significant contribution to damping at higher amplitudes of bubble excitation. In the presence of charge, this equation describing the dynamical evolution of the bubble radius \(R\) in time gets further modified.

Considering the charged bubble as a non-conducting charged shell with a constant charge \(Q\), the electrostatic pressure can be calculated as has been done by Akulichev, Atchley and others \([22, 33–35]\). The presence of charge \(Q\) (or surface charge density \(S\)) causes the inclusion of an electrostatic pressure term \(2\pi S^2/\varepsilon = Q^2/(8\pi \varepsilon R^4)\) into the equation (see e.g., \([34]\), also \([36]\)):

\[
\left[\left(1 - \frac{\dot{R}}{c}\right) + \frac{4\eta}{c^2\rho} \right] \frac{\ddot{R}}{\rho} + \frac{\dot{R}}{\rho} \left(\frac{P_0 - P_s + 2\pi}{R_0 - \frac{Q^2}{8\pi \varepsilon R^4}}\right) \\
\times \left(\frac{R_0}{R}\right)^{3G} \left(1 - \frac{\dot{R}}{c}(1 - 3G)\right) - \frac{R^2}{2} \left(\frac{3}{c} - \frac{\dot{R}}{c}\right) \\
+ \frac{Q^2}{8\pi \varepsilon R^4} \left(1 - \frac{3\dot{R}}{c}\right) - \frac{2\sigma}{\rho R} - \frac{4\eta}{\rho} \left(\frac{\dot{R}}{\rho}\right) \\
- \frac{1}{\rho} \left(P_0 - P_s + P_s \sin(\omega t)\right) \left(1 + \frac{\dot{R}}{c}\right) - \frac{R}{\rho c} P_0\omega \cos(\omega t),
\]

(1)

where \(R_0\) is the ambient bubble radius, \(P_0\) is the static pressure of the liquid, \(P_s\) and \(\omega = 2\pi v\) denote respectively the amplitude and angular frequency of the driving sonic field, \(P_s = 2.34 \text{ kPa}\) is the vapour pressure of the gas, \(\sigma\), \(\rho\) and \(\eta\) denote respectively the surface tension, density and viscosity of the liquid surrounding the bubble. \(c\) is the velocity of sound in the liquid. In this work, for the purpose of the numerical results reported, we consider water to be the liquid, with \(\rho = 998 \text{ kg m}^{-3}\), \(\eta = 10^{-3} \text{ N s m}^{-2}\), \(c = 1500 \text{ m s}^{-1}\), \(P_0 = 101 \text{ kPa}\), \(\sigma = 0.0725 \text{ N m}^{-1}\), \(\varepsilon = 85\varepsilon_0\) where \(\varepsilon_0\) is the permittivity of vacuum.

The presence of charge \(Q\) modifies the influence of surface tension, reducing its effective value (the effective surface tension changes from \(\sigma\) for the uncharged case to \(\sigma - Q^2/(16\pi \varepsilon R_0^4)\)) and induces several interesting changes to the dynamics of bubble oscillations.

When a pressure wave is incident on a bubble in a liquid, the difference in pressure can cause expansion and rapid collapse of the bubble, this being followed immediately by further, smaller oscillations which are termed afterbounces. This entire sequence of a maximal expansion of the bubble followed by collapse and afterbounces is repeated in each cycle when we have sinusoidal forcing by ultrasound. When a bubble is in the sonoluminescent regime, light emission occurs shortly after the bubble’s violent contraction to a minimal radius, before the onset of afterbounces and repetition of this sequence. Details of the phenomenon of sonoluminescence may be found in, for example, the review articles \([13, 37]\). We recast equation (1) in dimensionless form for ease of evaluation by redefining the radius \(R\) through \(r = R/R_0\) and time \(t\) through \(\tau = \omega t\), \(r\) and \(\tau\) being the new dimensionless radius and time variables. Using the static pressure \(P_0\) as the reference pressure we also define the dimensionless quantities \(P_{\text{sv}} = P_s/P_0\) and \(P_{\text{as}} = P_a/P_0\). Using an overdot to now denote differentiation with respect to \(\tau\) (the dimensionless time variable) rather than with respect to \(t\), equation (1) can be written in dimensionless form as

\[
\left(1 - \frac{\dot{r}}{c_s}\right) r\ddot{r} + \dot{F}r + \frac{r^2}{2} \left(3 - \frac{\dot{r}}{c_s}\right) = G \left(1 - P_{\text{sv}} + P_{\text{as}}\right) \left(1 - 3G\right) \left(1 + \frac{\dot{r}}{c_s}(1 - 3G)\right)
\]
Table 1. Critical Coulomb radius values for a Rayleigh bubble in water for various values of surface charge.

| $Q$ (pC) | $R_c$ (water) $\mu$m |
|----------|---------------------|
| 0.05     | 0.96                |
| 0.10     | 1.53                |
| 0.15     | 2.01                |
| 0.20     | 2.44                |
| 0.30     | 3.20                |
| 0.40     | 3.87                |
| 0.50     | 4.49                |
| 0.60     | 5.08                |
| 0.80     | 6.15                |
| 1.00     | 7.14                |

\[
+ \frac{C}{r^4} \left(1 - \frac{3\dot{r}}{c_s}ight) - S \frac{1}{r} - F c_s \left(\frac{\dot{r}}{r}\right) - G (1 - P_{sv} + P_{vs} \sin(\tau)) \left(1 + \frac{\dot{r}}{c_s}\right) - G \frac{r P_{ss}}{c_s} \cos(\tau),
\]

where

\[
c_s = \frac{c}{R_0 \omega}; \quad F = \frac{4 \eta}{\rho R_0 c}; \quad G = \frac{P_0}{R_0^2 \omega^2 \rho};
\]

\[
M = \frac{1}{P_0} \left(\frac{2\sigma}{R_0} - \frac{Q^2}{8\pi \epsilon R_0^2}\right);
\]

\[
C = \frac{Q^2}{8\pi \epsilon R_0^2 \omega^2 \rho}; \quad S = \frac{2\sigma}{\rho R_0^3 \omega^2}
\]

are all dimensionless constants.

This form is used when solving the equation numerically. In what follows, we will use the dimensional form of the equation everywhere. It should be understood though, that data points shown in all the graphs have been obtained after numerical evaluation of the corresponding dimensionless quantities, followed by rescaling by the appropriate multiplicative factors to obtain the variables in physically realizable units.

The original critical Rayleigh Coulomb-radius is for a charged drop, obtained by balancing surface-tension to Coulomb force. The analogous case for a charged bubble does not consider the effects of gas pressure within a bubble and is in the absence of any external forcing. The critical value $R_c$ so obtained is thus solely dependent on charge $Q$ and surface tension $\sigma$. It also depends on the dielectric constant of the liquid in which the bubble is placed. Thus, for a Rayleigh bubble with say charge $Q = 0.1$ pC suspended in water, the critical radius would be around 1.53 $\mu$m, while for the same bubble in liquid helium, $R_c = 38.93$ $\mu$m. It is the latter bubble that is the subject of studies of multi-electron bubbles (MEBs) in liquid helium by several authors, for example, Tempere et al [38, 39], and Salomaa and Williams [40] who investigate their time-evolution and dynamics. Effects of external forcing pressure, driving frequency, inclusion of an adiabatic equation of state for the bubble, etc., are not considered by these authors in their models.

To give an idea of the values attainable by the bubble radius of a charged, vacuum bubble at equilibrium, we include table 1 showing a comparison of values of $R_c$ for various values of charge $Q$ for a bubble in water.
Figure 1 shows a plot of the radius of the bubble as a function of time, obtained by solving equation (2), for charge present on, as well as absent from, the bubble surface.

For a given driving frequency of the pressure wave, the maximum radius attainable by the bubble increases with charge present. Conversely, the minimum radius achievable reduces with increasing charge. These are as expected.

The difference in behaviour can be seen more emphatically in a plot of the bubble’s radial velocity as a function of time (in figure 1). In the plot shown, in the presence of charge, the maximum velocity increases to more than five quarters of its uncharged value. This gives us a picture of the dynamics consistent with the time-series of the bubble radius, namely, that the bubble oscillations become more violent in the presence of charge, causing the bubble to expand to a greater extent and contract to a lesser minimal radius than in the absence of any surface charge.

The phase portrait for the system is shown in figure 2 for both a charged as well as uncharged bubble. The phase plot for the charged bubble is larger than that of the uncharged case, reaching
larger values of the magnitudes of both radius and velocity. One would intuitively expect this to be so. During the expansion part of the cycle, the charges present on the bubble surface move away from each other as the bubble expands, decreasing the surface charge density of the bubble wall. The change in the value of the maximal radius of the bubble due to the presence of charge, though present, is small. Due to the decrease in surface tension, we would expect the radius of the bubble to grow slightly more than in the case when there is no charge present. On the other hand, during the collapse, the surface charge density increases rapidly. It is in this regime that we would expect to see most clearly the effects of charge on the dynamics of the bubble. Since the maximum radius attained is larger, the collapse would be expected to be more violent resulting in a higher peak velocity of the bubble wall at the moment of collapse and lower minimum radius attained.

Since a charged bubble attaining smaller minimum radius during collapse may actually seem counter-intuitive, it would be well to put some more thought into what would be happening in such a system. Consider the case of radial oscillations of a bubble, such that a maximal radial velocity of $c_1$ is attained. This may be attainable by an uncharged bubble at lower forcing frequencies. If, however, the frequency increases so that the bubble’s expansion is not completed before it is forced to contract again, its collapse occurs with a smaller radial velocity than if it had been permitted to expand to a greater size. When a charge is present on the bubble, the effective surface tension is reduced and expansion to a larger radial dimension is facilitated, which in turn leads to a more violent collapse to a smaller radius.

The natural frequency of bubble oscillations of small amplitude may be found by assuming that the sound field can be introduced through a perturbation of amplitude $\alpha$ which is small [29]. Then by assuming that the bubble oscillates about its equilibrium radius $R_0$, one can express its radius at time $t$ as

$$R = R_0(1 + x(t)), \quad (3)$$

where $x(t)$ is a small quantity of order $\alpha$. Substituting equation (3) in the unforced Rayleigh–Plesset equation and linearizing it, we obtain:

$$\ddot{x} + \beta \dot{x} + \omega_0^2 x = 0, \quad (4)$$

\[\]
where the damping coefficient $\beta$ and the natural frequency of the oscillator are given by

$$\beta = \frac{1}{\rho c R_0} \left( \frac{4 \phi / R_0^5}{1 + 4q} + \frac{3Q^2}{8\pi \epsilon R_0^4} + \frac{4\eta \epsilon}{R_0} \right),$$

$$\omega_0^2 = \frac{1}{\rho R_0^2} \left( \frac{5\phi / R_0^5}{1 + 4q} - \frac{2\sigma}{R_0} + \frac{4Q^2}{8\pi \epsilon R_0^4} \right),$$

(5)

where only terms linear in $x$ and its derivatives have been retained. In equations (5), the quantity $\phi$ defined as

$$\phi / R_0^5 = \left( P_0 - P_v + \frac{2\sigma}{R_0} - \frac{Q^2}{8\pi \epsilon R_0^4} \right),$$

(6)

is the equilibrium gas pressure in the bubble.

The solutions to the damped equation equation (4) represent small amplitude oscillations. Substituting the values of the various parameters in equations (5) yields a natural frequency $\omega_0$ for a micrometre-sized bubble that is in the MHz range, in conformity with observations in the literature. From equations (5) and (6), we see that the dependence of $\omega_0$ on $Q$ is of the form $\omega_0^2 = A - BQ^2$, where $A$ and $B$ denote the rest of the terms and coefficients in the equations. Since $A$ contains the dominant pressure terms, $A \gg BQ^2$. Hence $\omega_0 \approx A^{1/2} - B^{1/2}Q^2$.

In our model, there is no shape-deformation of the bubble occurring and bubble oscillations are purely radial—i.e., there are no angular oscillation modes. Since we are considering purely radial oscillations, the breathing mode frequency is therefore coincident with $\omega_0$ calculated in equation (5).

Details on the frequency of forced bubble oscillations can be found at greater length in [36], wherein approximate scaling relations between the charge $Q$ and the frequency are also obtained.

3. The Blake threshold

The growth of a bubble can be determined by various threshold conditions [41]. One is the well-known Blake threshold for mechanical growth of a gas bubble. The Blake threshold corresponds to the minimum acoustic pressure $P_s = P_{\text{Blake}}$ exceeding which will result in the explosive growth of the bubble, culminating in cavitation. Blake threshold calculations are made under the assumption that the pressure fields are quasistatic in nature, and the surface tension dominates over viscous and inertial contributions.

Another threshold that is important is the transient cavitation threshold. This is the minimum acoustic pressure required for the forced oscillating bubble to collapse violently after its maximal expansion at radial velocities at least equalling the speed of sound.

In this section we calculate and discuss the Blake cavitation threshold for the bubble. Further discussion and results related to the upper transient cavitation threshold will be presented in section 5.

One other threshold condition is that determining the diffusion of a gas into the liquid causing bubble expansion, which is determined by factors including the natural resonance frequency of the bubble, the acoustic forcing frequency, the gas saturation coefficient, surface tension, specific heat ratio and the ambient radius of the bubble [41]. We do not discuss this threshold, known as the rectified diffusion threshold, in this work.

The pressure of the liquid on the outer surface of the bubble wall $p_L$ may be written down:

$$p_L(R(t)) = p_i(t) - \frac{2\sigma}{R} + \frac{Q^2}{8\pi \epsilon R^2} - \frac{4\eta \dot{R}}{R},$$

(7)
where the pressure inside the bubble $p_i(t)$ comprises the pressure of the gas and the vapour pressure $P_v$:

$$p_i = \left( P_0 - P_v + \frac{2\sigma}{R_0} + \frac{Q^2}{8\pi \epsilon R^4} \right) \left( \frac{R_0}{R} \right)^{3\Gamma} + P_v. \quad (8)$$

The net pressure on the bubble wall from the surrounding liquid is [34]:

$$P = \left( P_0 - P_v + \frac{2\sigma}{R_0} - \frac{Q^2}{8\pi \epsilon R^4} \right) \left( \frac{R_0}{R} \right)^{3\Gamma} - \frac{2\sigma}{R} + \frac{Q^2}{8\pi \epsilon R^4} - 4\eta \frac{\dot{R}}{R} + P_v - P_{\text{ext}}, \quad (9)$$

where $P_{\text{ext}} = P_0 + p(t)$, $p(t)$ being the ultrasound driving pressure.

For our choice of the polytropic index $\Gamma = \frac{2}{3}$, the change in the bubble radius resulting from quasistatic changes in the pressure of the liquid $p_L$ (that is, very slow pressure changes of $p_i$ with inertial and viscous effects being assumed negligible during bubble expansion and contraction) outside the bubble may be determined from the equation:

$$p_L = \left( P_0 - P_v + \frac{2\sigma}{R_0} - \frac{Q^2}{8\pi \epsilon R^4} \right) \left( \frac{R_0}{R} \right)^{5} + P_v - \frac{2\sigma}{R} + \frac{Q^2}{8\pi \epsilon R^4}. \quad (10)$$

The charge term completely changes the liquid pressure profile: in particular for small bubbles, the charge term dominates over surface tension, reducing its effect.

For instance for a 5 μm bubble in water at atmospheric pressure ($\sigma \approx 0.0725$ N m$^{-1}$, $P_0 = 101$ kPa), while the surface tension contribution to the pressure in the bubble is roughly $2.8 \times 10^4$ Pa, the effect of introducing a small charge of about 0.415 pC would be to reduce this by half. Clearly, this has a significant effect on the radial mechanical stability of the bubble and the Blake threshold that determines the nature of the bubble’s radial oscillations. The behaviour of equation (9) is depicted in figure 3 for purpose of illustration. In the absence of charge, there exists no equilibrium radius below a critical value $R_{\text{crit}}$ of the pressure—the bubble radius at this point undergoes explosive expansion. The presence of even a small amount of charge on the bubble surface produces a drastic change of behaviour—there exists no equilibrium radius for pressures larger than a critical value $P_{C_{\text{max}}}$; the pressure region $P_C < P \leq P_{C_{\text{max}}}$ is a metastable region. Our study in this paper is restricted to physically realistic regimes, with applied pressures roughly in the range 0.4–1.5 bar.

To obtain the Blake radius for the charged bubble, we adapt the procedure for the uncharged case (see for example Harkin et al [42]) to our situation. We first minimize equation (9) with respect to $R$, $R > 0$. This leads to the quartic equation:

$$R^4 - \frac{Q^2}{4\pi \epsilon \sigma} R - \frac{5}{2\sigma} \phi = 0, \quad (11)$$

The Blake radius $R_{\text{crit}}$ is given by the real and positive root of this equation. We find that:

$$R_{\text{crit}} = \frac{1}{2\sqrt{18a}} \left[ \sqrt{\frac{5}{2\sigma}} \sqrt[3]{\left( \frac{10\sqrt{52} \phi}{\sigma} \right)} + \frac{6\sqrt{52} \phi}{\sigma} \sqrt[3]{\frac{10\sqrt{12} \phi}{\sigma}} + \frac{10\sqrt{12} \phi}{\sigma} \right]^{1/2}, \quad (12)$$

where

$$a = \frac{9 Q^4}{(4\pi \epsilon \sigma)^2} + \sqrt{\frac{27}{2\pi}} \left( \frac{27 Q^8}{(4\pi \epsilon \sigma)^3} + \frac{4000}{\sigma^3} \right)^{1/2}. \quad (13)$$

The liquid pressure $p_{L_{\text{crit}}}$ corresponding to this critical value of the radius is obtained by substituting equation (9) back into equation (4):

$$p_{L_{\text{crit}}} = P_v + \frac{\phi}{R_{\text{crit}}} - \frac{2\sigma}{R_{\text{crit}}} + \frac{Q^2}{8\pi \epsilon R_{\text{crit}}^4}. \quad (14)$$
The Blake threshold pressure may be obtained from the standard definition \[42\]:

\[ p_{\text{Blake}} = P_0 - p_{\text{crit}}. \] (15)

A rough estimate of the Blake radius and threshold can be made for sub-micrometre sized bubbles for which the contributions from the static pressure is negligible in comparison with the charge-corrected terms. In this approximation, we find that

\[ R_{\text{crit}} \approx \frac{1}{2}(1 + \sqrt{18}) \left( \frac{Q^2}{4\pi \epsilon \sigma} \right)^{1/3} = 1.81 \left( \frac{Q^2}{4\pi \epsilon \sigma} \right)^{1/3}. \] (16)

Substituting this in equation (8) and using equation (9), we find the following approximate expression for the Blake threshold:

\[
\begin{align*}
    p_{\text{Blake}} &= P_0 + \frac{Q^2}{8\pi \epsilon R_{\text{crit}}^4} \left( \frac{R_0}{R_{\text{crit}}} - 1 \right) + \frac{2\sigma}{R_{\text{crit}}} \\
    &= P_0 + \left( \frac{4\pi \epsilon \sigma^4}{Q^2} \right)^{1/2} \left( -6.95 + 4.416 R_0 \left( \frac{4\pi \epsilon \sigma}{Q^2} \right)^{1/2} \right).
\end{align*}
\] (17)

We should mention here that the work of [38] which deals with a MEB encapsulating a void in liquid helium, gives an expression for the potential energy of the MEB system, which they use to anticipate the modification to the Blake threshold. Their expression, however, differs from ours and does not capture the region of inflection in the pressure–radius plot.
4. Rayleigh collapse and the influence of driving pressure amplitude

After the bubble attains its maximum radius \( R_{\text{max}} \), it proceeds to the main collapse. Its dynamics during this phase is described by the Rayleigh equation:

\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 = 0.
\]  

(18)

In considering cavitation in this limit, one is essentially considering collapse of a void, ignoring all terms such as viscosity, surface tension, etc. The solution for this is found to be

\[
R(t) = R_r \left( t_c - t \right)^{2/5},
\]  

(19)

where at \( t = t_c \), the bubble collapses to a point \( R = 0 \). \( R_r \) is a characteristic radius, and \( T \) is the time period of oscillation of the bubble. It may be noted that the characteristic radius \( R_r \) in this scaling law is different from that reported in [43] where the polytropic constant was taken to be unity, corresponding to an isothermal process. Here we find an estimate for \( R_r \) for \( \Gamma = 5/3 \), using a similar energy argument as in [43]. Converting the potential energy \( E_{\text{pot}} \) of the bubble at \( R_{\text{max}} \) to kinetic energy at \( R_0 \) we get

\[
\dot{R} = - \left( \frac{2P_0}{\rho R_0} \right)^{1/2} \left( \frac{4\pi}{3} \Gamma^{-1} \right)^{1/5} R_{\text{max}}^{3\Gamma/5},
\]  

(20)

Using equation (19) in (20) we obtain

\[
R_r = \left( \frac{25T^2 P_0}{2\rho} \left( \frac{4\pi}{3} \right)^{\Gamma-1} \right)^{1/5} R_{\text{max}}^{3\Gamma/5}.
\]  

(21)

For \( \Gamma = 5/3 \), we get

\[
R_r = \left( \frac{25T^2 P_0}{2\rho} \left( \frac{4\pi}{3} \right)^{2/3} \right)^{1/5} R_{\text{max}} \approx 2.006 \left( \frac{P_0 T^2}{\rho} \right)^{1/5} R_{\text{max}}.
\]  

(22)

The extremely simplified expression, equation (22), is none the less useful for making some physically relevant approximations. This approximation would hold best during the bubble’s collapse to a minimum radius \( R_{\text{min}} \). It would also be less inaccurate when the dimensions of the collapsing bubble are very small, that is, when \( R_{\text{min}} \) is very small. This would tend to match more closely those cases where the charge on the bubble is high, so that the reduction in values of \( R_{\text{min}} \) is correspondingly more. We can see that in the presence of a driving frequency \( \omega \) for the system, \( R(t) \) would show a frequency dependence \( R(t) \sim \omega^{2/5} \) in the regime near bubble collapse, at higher driving pressure amplitudes, so that we have

\[
R_{\text{min}} \sim a_1 \omega^{2/5},
\]  

(23)

\( a_1 \) being a prefactor with appropriate dimensions.

At higher driving frequencies, the bubble typically has larger values for its minimal radius, there not being sufficient time for complete collapse to occur before the expanding regime sets in. Increasing the charge present on the bubble enables it to reach smaller dimensions.

The minimum radius scaled by the driving frequency, through \( R_{\text{min}}/\omega^{2/5} \) is shown in the plot of figure 4(a). As expected from the discussion above, best agreement of equation (23), as evidenced through a superposition of all the curves for different frequencies, is seen at higher charge values and lower \( R_{\text{min}} \) values.

It is to be expected however, that in reality a stable bubble cannot carry an indefinite magnitude of charge \( Q \). This can also be seen on plotting the minimum radius \( R_{\text{min}} \) as a
function of charge, where, for a given driving pressure, the minimum radius of the bubble for every driving frequency converges to one value of the charge. This, however, needs to be modified as a further physical constraint to the system exists in that bubble contraction cannot also indiscriminately progress indefinitely. The smallest dimensions that the bubble could take, that is, the least value of the minimum radius $R_{\text{min}}$ that could be possibly reached during the bubble’s compressive regime, given suitable conditions and parameter values, is that bounded by the value of the van der Waals hard-core radius $h$. The value of $h$ is determined by the gas enclosed within the bubble. For example, for argon, $h$ has the value $h = R_0/8.86$. For $R_0 = 5 \mu m$, this equals a value of 0.564 $\mu m$.

$R_{\text{min}}$ with this constraint is shown in figure 4(b).

Under different forcing pressure amplitudes and driving frequency conditions, the bubble will naturally collapse to different values of minimum radius. It should be clearly understood here that $R_{\text{min}}$ will not be routinely equal to $h$, the hard-core radius, during the course of bubble oscillations. Indeed, such a value would not be reached usually. However, the absolute limit on the maximum value of charge a bubble could carry, in the presence of forcing and in the presence of a gas within the bubble, would be dictated by the value of $h$.

The system is not an ideal Rayleigh bubble, since the bubble now has other parameters influencing the force-balance equation—both the driving frequency as well as the amplitude of the imposed pressure (acoustic) wave, as well as the initial ambient radius of the bubble $R_0$, influence the extent to which the bubble can contract. The charge that would be dictated by the force-balance condition for the Rayleigh bubble for a given ambient radius $R_0$ is thus not very useful in this forced system, since it dictates only the condition of equilibrium for an unforced, vacuum-enclosing bubble.

There is a maximal value for the charge a bubble could possibly carry for attainment to $R_{\text{min}} = h$ to be possible, that depends on the driving amplitude of the pressure $P_s$, as well as the driving frequency, for a given $R_0$. This limiting value of the charge we denote by $Q_h$. This is shown in figure 4(b), where the minimum radius of collapse, $R_{\text{min}}$, has been plotted as a function of charge $Q$, for a bubble of initial radius $R_0 = 5 \mu m$, and pressure amplitude $P_s = 1.35P_0$. The minimal bound of $R_{\text{min}} = h$ has been shown by a dotted line. The point of intersection of the curve for a particular driving frequency with this line gives the value of $Q_h$.

It should be appreciated here that $Q_h$ is distinct from the value of $Q$ permitted by the force-balance criterion for a static, unforced, Rayleigh bubble with charge.

For example, for a 5 $\mu m$ bubble enclosing a void, the value of the charge it could carry as dictated by the Rayleigh Coulomb critical radius can be estimated from table 1 to be near 0.6 pC (which is the value for a 5.08 $\mu m$ bubble). However, as can be seen from figure 4(b), for a 5 $\mu m$ bubble, at $P_s = 1.35P_0$ with a driving frequency of 20 kHz, any value greater than about 0.38 pC or so would be unphysical, since that would take the bubble below the van der Waals radius. Clearly, therefore, the charge dictated by the static, unforced Rayleigh Coulomb bubble would not be permitted as it would be unphysical.

On the other hand, for a driving frequency of 30 kHz, we observe that $Q_h \approx 0.9$ pC, which is larger than the Rayleigh value near 0.6 pC.

It should be clearly understood therefore, that the figures we have shown serve to explicitly show what possible combination of values of $Q$, $R_0$ would be physically permissible for given values of driving frequency $\nu$ and pressure amplitude $P_s$.

Correspondingly, there is an upper bound on the maximum radial velocity of the bubble $V_{\text{max}}$. The presence of charge on the bubble serves to reduce the effective magnitude of the surface tension. With increasing charge, the bubble is thus able to reach a smaller radius and a greater velocity. $V_{\text{max}}$ and $R_{\text{max}}$ as a function of charge $Q$ are shown in figure 5 for high
driving pressure of $P_s = 1.35P_0$. The limits to the curves in the plot are due to the limit in the maximal value $Q_h$ that $Q$ can take for each frequency.

Furthermore, as the frequency of the driving ultrasonic acoustic wave is increased, the maximum radius attained by the bubble, $R_{\text{max}}$, reduces. This is understandable by recalling that the higher the frequency, the shorter is the period of negative pressure shear and the bubble is driven to cavitation collapse in a shorter time span resulting in shorter expansion time of the bubble. This also causes the value of the minimum radius to become larger with increasing frequency of the driving ultrasound wave. The larger the maximum radius reached, the more violent the collapse is, resulting in smaller minimum radius.

The system is very sensitive to changes in the pressure conditions. The amplitude $P_s$ of the driving pressure determines the physically viable minimal radius attainable by the bubble for a given charge. The maximal, bounding value of the charge, $Q_h$, reduces progressively with increasing amplitude of pressure $P_s$, until beyond a critical pressure $P_m$, it is no longer
Figure 5. $R_{\text{max}}$ on the left, and $v_{\text{max}}$ on the right, plotted as a function of charge, $Q$, for various driving frequencies ($\nu = 20, 22, 24, 26, 28, 30$ kHz), lower frequency curves on the top. $R_0 = 5 \, \mu m$, $P_s = 1.35 P_0$.

Figure 6. Plot of $Q_h$ as a function of driving pressure amplitude $P_s$, for bubbles of ambient radius $R_0 = 2$ and $5 \, \mu m$. The area below each curve corresponds to the domain where $R_{\text{min}} > h$. The region above each curve corresponds to $R_{\text{min}} < h$ and is physically unreachable for the bubble of that particular $R_0$. $R_0 = 5 \, \mu m$, $\nu = 20, 25, 30$ kHz; $R_0 = 2 \, \mu m$, $\nu = 20, 25, 30$ kHz.

physically possible for the bubble radius to contract to such a small value. In figure 6 we plot $Q_h$ as a function of $P_s$ for three different frequencies (20, 25 and 30 kHz). We show plots for two values of ambient bubble radius $R_0 = 2$ and $R_0 = 5 \, \mu m$. Each curve demarcates two regions—the space below (or to the left of) the curve corresponds to the physically permissible region of $R_{\text{min}} > h$. The region above (or to the right of) each curve corresponds to $R_{\text{min}} < h$, which cannot be reached in practice by a bubble of that corresponding ambient radius.

It can be seen that the value of the critical pressure $P_m$ increases with driving frequency, and reduces with ambient radius $R_0$. At very low values of $P_s$, $Q_h$ becomes essentially independent of frequency for a given ambient radius. We denote the pressure where this frequency-independence first sets in (approached from above) by $P_{fi}$. It can be seen that $P_{fi}$
occurs at a lower value \( P_{bl} = 1.12 P_0 \) (with corresponding \( Q_{bl} \approx 1.3 \) pC) for the larger bubble \((R_0 = 5 \, \mu \text{m})\) than for the smaller bubble \((P_{bl} = 1.2 P_0 \) for \( R_0 = 2 \, \mu \text{m}\)). Moreover there exists a brief cross-over region for the \( R_0 = 5 \, \mu \text{m} \) bubble where the frequency-dependence of \( Q_{bl} \) reverses, before true frequency-independence of \( Q_{bl} \) occurs at \( P_{b_{5\mu \text{m}}} = 0.9 P_0 \) (for \( P_0 > P_{bl} \), \( Q_{bl} \) for higher frequencies are greater than those for lower frequency values, while for \( P_{b_{5\mu \text{m}}} < P_0 < P_{bl} \), this is reversed and \( Q_{bl} \) for higher frequencies have smaller values than those for lower frequencies for the \( 5 \, \mu \text{m} \) bubble).

To understand the effect of the driving pressure, we briefly paraphrase below the arguments given in [43] for the isothermal case adapting it to our adiabatic system.

Combining the driving sound field with the static pressure [29] the total external field \( P_{ext} \) can be expressed as

\[
P_{ext} = P_0(1 - \alpha \cos \omega t). \tag{24}
\]

Substituting this in the Rayleigh–Plesset equation under quasistatic conditions:

\[
\left(P_0 - P_v + \frac{2\sigma}{R_0} - \frac{Q^2}{8\pi \epsilon R_0^4}\right) \left(\frac{R_0}{R}\right)^3 + \frac{Q^2}{8\pi \epsilon R^3} - \frac{2\sigma}{R} - P_0(1 - \alpha \cos \omega t) = 0 \tag{25}
\]

we obtain the quintic equation

\[
R^5 - \frac{2\sigma}{(\alpha - 1) P_0} R^4 + \frac{Q^2}{8\pi \epsilon (\alpha - 1) P_0} R + \frac{R_0^5}{\alpha - 1} \left(1 + \frac{2\sigma}{R_0 P_0} - \frac{Q^2}{8\pi \epsilon R_0^4 P_0}\right) = 0. \tag{26}
\]

The behaviour of the equation is completely determined by the quantity \( \alpha - 1 \). For \( P_{ext} > 0 \), we have a a stable, single solution for \( R \). For negative, small amplitude \( P_{ext} \) there are two solutions with that at lower \( R \) being the stable one.

These two merge only at a critical value \( P_{ext} = P_{blake} \), \( P_{blake} < 0 \). For this, \( P_{gas} > P_{ext} + P_0 \) is always the case and leaves the equation without a solution, where \( P_{gas} = \phi / R^3 \) and \( \phi = 2\sigma / R \).

Liquid pressure becoming negative, opposes the confinement effect of the surface-tension contribution, \( P_\sigma \). Once the bubble is larger than a critical radius \( R_c \), pressure balance at the bubble wall cannot be maintained, and explosive growth sets in. The bubble is unstable at this stage, and further growth leads to increased instability and still more expansion, and quasistatic conditions no longer hold.

With the time period of oscillation \( T = 2\pi / \omega \) being larger than the time scale of the bubble’s oscillations, oscillations of the external pressure can be considered as being quasistatic. For crossing the Blake threshold \( P_{ext} < 0 \) is required so that \( \alpha > 1 \). \( t = 0 \) gives \( P_{ext} = (\alpha - 1) P_0 \) to be negative. Thus the behaviour shown by bubbles for \( \alpha < 1 \) will be very different from that for \( \alpha > 1 \); for values of \( \alpha \) less than 1, bubble oscillations will tend to be less violently expansive and the effect of surface tension dominate the dynamics.

5. Expansion–compression ratio and the transient threshold

The surface tension greatly influences bubble dynamics and can give rise to very distinct behaviours for bubbles of different ambient radii \( R_0 \), even when all other conditions are identical. In smaller bubbles, surface tension is a very dominant term. Looking at the \( R_{min} \) versus \( P_s \) plots (figure 7) for \( R_0 = 2 \) and \( R_0 = 5 \, \mu \text{m} \), we at once see a striking difference between the two. For the larger, \( 5 \, \mu \text{m} \) bubble, we see that on lowering the pressure \( P_s \), at a value corresponding to \( P_s = P_h = 1.12 P_0 \), the \( R_{min} \) curves all converge to a point. Lowering \( P_s \) further brings the bubble to a cross-over regime, until reaching a lower pressure
Figure 7. $R_{\text{min}}$ versus $P_s$ for different values of $\omega$ ($\nu = 20, 25, 30, 35, 40 \text{ kHz}$) and $Q$. $Q = 0 \text{ C}, 0.1 \text{ pC}, 0.14 \text{ pC}$ for (a) $R_0 = 2 \mu\text{m}$ and (b) $Q = 0 \text{ C}, 0.4 \text{ pC}, 0.58 \text{ pC}$ for $R_0 = 5 \mu\text{m}$.

$P_s = P_{\text{lim}} \approx 0.9P_0$, below which pressure, the curves largely show frequency- and charge-independent behaviour. As will be seen in the ensuing paragraphs, $P_s = P_{\text{h}}$ is actually the transition pressure $P_{\text{tr}}$ beyond which violent bubble collapse occurs.

For the smaller, $2 \mu\text{m}$ bubble, we do not see any cross-over regime, and the pressure $P_s = P_{\text{h}} = 1.2P_0$ where all curves converge such that for $P_s < P_{\text{h}}$ charge or frequency-dependence of the curves is suppressed, is actually less than the transition pressure $P_{\text{tr}} = 1.3P_0$ for $R_0 = 2 \mu\text{m}$.

Introduction of charge on the bubble serves to dramatically move $P_{\text{h}}$ to a lower value, with the effective surface tension being reduced due to electrostatic interaction, and its effect being enhanced due to the bubble’s smaller dimensions.

One obvious measure of the relative extremal values of the bubble dimensions is $R_{\text{max}}/R_{\text{min}}$. Other measures used to quantify the bubble’s dimensions are the expansion ratio $E \equiv R_{\text{max}}/R_0$, and the compression ratio $C \equiv R_{\text{min}}/R_0$. Figure 8 shows plots of the relative extremal bubble radius measure, $R_{\text{max}}/R_{\text{min}}$ as a function of the amplitude $P_s$ of the driving pressure wave for two different bubble radii: $R_0 = 2 \mu\text{m}$ and $R_0 = 5 \mu\text{m}$. We
introduce yet another useful and significant measure of the relative extent of bubble expansion to compression, which we term the expansion–compression ratio, 

$$\zeta \equiv (E - 1)/(1 - C) = (R_{\text{max}} - R_0)/(R_0 - R_{\text{min}}).$$

(27)

Investigating the dependence of this EC ratio \(\zeta\) on the amplitude of applied pressure \(P_s\) yields some delightful results and clearly shows the great utility of this measure of bubble expansion/contraction. Figure 9 is a plot of \(\zeta\) as a function of \(P_s\) for two different ambient radius values, \(R_0 = 2\) and \(R_0 = 5\) \(\mu\)m. For bubbles with not too small ambient radius \(R_0\) (for example, for \(R_0 = 5\) \(\mu\)m), at low values of \(P_s\), the expansion–compression ratio \(\zeta\) becomes independent of charge and frequency below some \(P_s = P_{s1}\), and \(\zeta\) curves for various frequencies and charges all superimpose (figure 9(b)). This behaviour is not shown by smaller bubbles (for example, for \(R_0 = 2\) \(\mu\)m), whose \(\zeta\) curves instead show distinct charge and frequency dependence even at very low amplitudes of driving pressure (figure 9(a)).

In general for all bubble sizes, the following generic behaviour of the EC ratio \(\zeta\) is shown: at lower pressures, till a certain pressure \(P_s = P_{s1}\), \(\zeta\) shows only very weak dependence on charge and driving frequency. In figure 10(a) the dashed and solid curves are shown as representative
Figure 9. Expansion–contraction ratio $\zeta$ for (a) $R_0 = 2\mu m$ ($Q = 0, 0.1, 0.14\ pC$) and (b) $R_0 = 5\mu m$ ($Q = 0, 0.4, 0.58\ pC$) as a function of $P_s$. Curves labelled as 1,2,3,4, and 5 correspond to $\nu = 20, 25, 30, 35, 40\ kHz$, respectively. Increasing $Q$ shifts curves downwards and to the left.

of different charge and frequency values, which are coincident at pressures below $P_{sl}$. With increasing $P_s$, the EC ratio increases to a peak at a critical pressure value $P_b$, followed by a short, steep dip up to a second critical pressure value $P_{tr}$. This is followed further by a regime of an ever-increasing $\zeta$ with increasing $P_s$ (figure 10(a)). Between $P_b$ and $P_{tr}$ lies a region of negative slope, which is essentially an unstable, transient region.

Surface tension is overcome at a critical radius corresponding to the pressure $P_b$ after which significant bubble expansion occurs and the motion is transient until pressure $P_{tr}$. For $R_0$ being sufficiently large, bubble collapse cannot be completed fully during the compression part of the cycle of applied pressure and bubble motion is then stable. This explanation accounts for the presence of two thresholds, enclosing a transient regime with stable regions on either side [2]. At the lower threshold, it can be seen from figures 9 and 10 the transition from stable to transient conditions occurs very steeply.

The lower transient threshold pressure $P_b$ delineates a pressure value above which the bubble expansion occurs dramatically, and this in fact equals the Blake threshold pressure, $P_b = P_{Blake}$. 

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It is important to recall at this point that while the Blake threshold is a measure of the onset of rapid bubble expansion, it gives us no information at all about bubble implosion [44]. The upper transient threshold pressure, $P_{ts}$, on the other hand, is the value of $P_s$ above which violent bubble contraction begins. It can be seen, by inspecting the $R_{\text{min}}$ versus $P_s$ curve (figure 7) that at $P_s = P_{ts}$, $R_{\text{min}} = R_0/2$. This, in fact, identifies the transition to a strong collapse regime from weaker oscillations [43]. This can be verified by incorporating the radial velocity into the EC plot. As can be seen in figure 10(b), where a scale representing the magnitude of maximum radial velocity has been included, it is only at driving pressures above the upper transient threshold $P_{ts}$ that velocities rise dramatically to high values, whereas at lower $P_s$, the radial oscillations occur more slowly.

In figures 11(a) and (b) we show the frequency response diagrams for the minimum radius and maximum velocity attained by a 5 $\mu$m bubble in water at the low driving pressure of 0.4 $P_0$ and carrying charges $Q = 0, 0.4, 1.0$ and 1.24 pC. It is clear that increasing the magnitude of charge present on the bubble increases the magnitude of the response and advances it to lower frequencies. The peaks in the response diagram appear much earlier, at lower frequencies, for higher charges and their enhanced magnitudes (smaller minimum radius and larger maximal velocity) point to more violent collapse.

We can make a further, very rough estimate of the maximal bubble expansion limits at this transient pressure. We first consider the following vastly simplifying assumptions: that at $R \to R_{\text{min}}$, $R \to c$, where $c$ is the speed of sound in the liquid. We further assume, for purpose of this estimation, that as $R \to R_{\text{max}}$, $R \to 0$.

At its local minimum at $P_s = P_{tr}$, $\zeta = \zeta^u$ will satisfy $(\frac{\partial \zeta}{\partial P_s})_{P_s = P_{tr}} = 0$, so that we get

$$\frac{\partial R_{\text{max}}}{\partial P_s} = -\frac{R_{\text{max}} - R_0}{R_0 - R_{\text{min}}} \frac{\partial R_{\text{min}}}{\partial P_s} = -\zeta^u \frac{\partial R_{\text{min}}}{\partial P_s}.$$

(28)
Now using
\[
\frac{\partial R_{\text{min}}}{\partial P_s} \approx \left(-5 P_s + \frac{2\sigma}{R_0} + \frac{Q^2}{8\pi\epsilon R^6_0} \right) \left( R^5_0 - R^6_{\text{min}} + \frac{2\sigma}{R^2_{\text{min}}} ight)
\]
\[
- \frac{4Q^2}{8\pi\epsilon R^5_{\text{min}}} + 4\eta c R^2_{\text{min}}
\],
\[
\frac{\partial R_{\text{max}}}{\partial P_s} \approx \left(-5 P_s + \frac{2\sigma}{R_0} + \frac{Q^2}{8\pi\epsilon R^6_0} \right) \left( \frac{R^5_0}{R^6_{\text{max}}} - \frac{2\sigma}{R^2_{\text{max}}} \right)
\]
\[
- \frac{4Q^2}{8\pi\epsilon R^5_{\text{max}}} - 4\eta c R^2_{\text{min}}
\] ,

in equation (28), we get
\[
\zeta'' = -\left( \frac{5\phi}{R^6_{\text{max}}} - \frac{2\sigma}{R^2_{\text{max}}} + \frac{4Q^2}{8\pi\epsilon R^5_{\text{max}}} \right) \left( \frac{5\phi}{R^6_{\text{min}}} - \frac{2\sigma}{R^2_{\text{min}}} + \frac{4Q^2}{8\pi\epsilon R^5_{\text{min}}} - 4\eta c \right),
\]

where \( \phi/R_0^5 \) denotes the equilibrium pressure of the gas in the bubble (equation (6)).
If we further consider the extremal case of $R_{\text{min}} \approx h$, this becomes

$$\zeta_{tr} = -\frac{5\phi}{h^6} - \frac{2\sigma}{h^2} + \frac{4Q^2}{8\pi\epsilon R_{\text{max}}^5} - \frac{4\eta c}{h^2}.$$  \hspace{1cm} (31)

Now extremizing equation (31) with respect to $R_{\text{max}}^{\text{tr}}$ yields a quartic equation for $R_{\text{max}}^{\text{tr}} = R_{tr}^{\text{max}}$ at $P_s = P_{tr}$

$$R_{\text{max}}^4 - \frac{5Q^2}{8\pi\epsilon\sigma} R_{\text{max}} - \frac{15\phi}{2\sigma} = 0.$$ \hspace{1cm} (32)

In the uncharged case, $Q = 0$, this simplifies to

$$R_{\text{max}}^{tr} = \left( \frac{15\phi}{2\sigma} \right)^{1/4},$$ \hspace{1cm} (33)

which is the value of $R_{\text{max}}$ at the transition point at $P_s = P_{tr}$, made under all the simplifying assumptions mentioned above. It will be noted that for mid-sized microbubbles, for example for $R_0 = 5 \mu m$ (plots for which have been shown), the point of transition $P_{tr}$ is approximately a constant and is largely independent of the driving frequency or the charge.

A comparison of the estimate of $R_{\text{max}}$ so obtained from the above equation for the uncharged case to the value obtained numerically is given in table 2 below. Though the estimated values are far from accurate in many cases, they do, nonetheless, give a quick and useful estimation of $R_{\text{max}}$ at a driving pressure equalling the upper transient threshold of pressure $P_{tr}$.

These estimates though rather crude, might provide useful measures for avoiding undesirable regimes involving violent bubble collapses in medical diagnostics and applications.

### 6. Conclusions

In this work we have investigated the dynamics of a bubble forced by an ultrasound field, and seen how charge influences its behaviour. Our calculations are for a system where an adiabatic equation of state prevails, with $\Gamma = 5/3$. We make several interesting observations. Charge serves to reduce the effective surface tension of the bubble. This causes a charged bubble to not only expand to a larger radius as compared to a neutral bubble but also collapse to a smaller minimum radius, the lower bound of which is given by the van der Waals hard-core radius. The charged bubble’s collapse is also more violent, with radial velocities being reached being greater. Charge influences and modifies the liquid pressure profile. The effects are more marked for bubbles of smaller dimensions, where surface tension has a predominant influence. Studies of the effect of the amplitude of the forcing pressure wave show that

### Table 2. $R_{\text{max}}^{tr}$ obtained from equation (32) and graphically.

| $R_0$ ($\mu m$) | $R_{\text{max}}^{\text{tr}}$ (equation) | $R_{\text{max}}^{\text{tr}}$ (graph) ($\mu m$) |
|-----------------|-----------------------------------------|-------------------------------------------|
| 2               | 4.87                                    | 7.14                                      |
| 3               | 7.78                                    | 10.65                                     |
| 4               | 10.92                                   | 12.64                                     |
| 5               | 14.23                                   | 14.3                                      |
| 7               | 21.3                                    | 17.1                                      |

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introduction of charge serves to lower the Blake threshold so that the transition to violent collapses and oscillations occur at lower levels of pressure, especially for microbubbles of smaller dimensions and submicrometre bubbles. We have obtained expressions for the Blake threshold in the presence of charge.

We introduce a measure of the extremal dimensions reached by a bubble, which is a quantity \( \zeta = (E - 1)/(1 - C) \) where \( E \) and \( C \) are the expansion and compression ratios, respectively. The advantage of plotting \( \zeta \) as a function of \( P_s \) is that it captures the distinct positions and behaviours of both the Blake threshold pressure as well as the upper transient threshold pressure, between which points lies a regime of instability for the bubble.

The maximum magnitude of charge a bubble can carry in the system tends to converge to a single asymptotic value, \( Q_{\text{max}} \), and likewise the minimum radius converges to a single value for all frequencies, which is however modulated by the radial length scale cut-off provided by the van der Waals hard-core radius. Exceeding the magnitude of charge \( Q_h \) causes the bubble to contract to values of \( R_{\text{min}} \) that are less than \( \hbar \) which cannot be physically reached and hence provide a physical upper bound for the system’s charge. We also investigated Rayleigh collapse for the bubble, obtaining an expression for the characteristic Rayleigh radius \( R_r \). Frequency-dependence of the minimum radius is also captured. We also obtain approximate relations for the maximal radius at the critical transition point. Most of the numerical results presented in this work are for high pressures, corresponding to regimes of violent bubble collapse. We have demonstrated the importance of including charge in investigating the expansion and contraction of a bubble under forcing. We have also found scaling relations for extremal radial dimensions. Other results, including an investigation of the bifurcation structure as a function of charge, scaling relations for the maximal charge, etc are being reported elsewhere [36].

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