Hilbert Modules—Square Roots of Positive Maps

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We reflect on notions of positivity and square roots.
More precisely:

- In a good notions of positivity, it should be a theorem that every positive thing has a square root!
- The square root must allow to recover the positive thing in an easy way, making also manifest in that way that the positive thing is positive. (leadsto facilitate proofs of positivity.)
- We prefer unique square roots.
- We wish to compose two positive things to get new ones.

To achieve this:

- We will allow for quite general square roots.
- It turns out that it is good to view positive things as maps.
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**Note:** Complex numbers are excellent square roots of positive numbers!

(Think of the richness of wave functions \( x \mapsto \varphi(x) \in \mathbb{C} \) in QM such that \( p(x) := \overline{\varphi(x)} \varphi(x) \) becomes a probability density over \( \mathbb{R}^3 \). Volkmar: What about complex square roots of RN-derivatives?)
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Note: Suppose \( z' \in \mathbb{C} \) such that \( \overline{z'}z' = \lambda \geq 0 \).
Then \( u := \frac{z'}{z} = e^{i\alpha} \in S^1 \).
In fact, \( u : \lambda \mapsto u\lambda \) is a unitary in \( \mathcal{B}(\mathbb{C}) \) that maps \( z \) to \( z' \).
All square roots of \( \lambda \geq 0 \) are unitarily equivalent in that sense.
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Volkmar: What about complex square roots of RN-derivatives?)

**Note:** Suppose \( z' \in \mathbb{C} \) such that \( \overline{z'}z' = \lambda > 0 \).
Then \( u := \frac{z'}{z} = e^{i\alpha} \in S^1 \).
In fact, \( u : \lambda \mapsto u\lambda \) is a unitary in \( B(\mathbb{C}) \) that maps \( z \) to \( z' \).
All square roots of \( \lambda \geq 0 \) are unitarily equivalent in that sense.

**Note:** Positive numbers \( \lambda, \mu \geq 0 \) can be multiplied.
In fact, if \( z, w \in \mathbb{C} \) are square roots of \( \lambda, \mu \), respectively, then \( \overline{zw}(zw) = (\overline{z}z)(\overline{w}w) = \lambda\mu \), so that \( \lambda\mu \geq 0 \).
Example. \( \mathcal{B} \) a \( C^* \)-algebra, \( b \in \mathcal{B} \). Then
\[ b \geq 0 \iff \exists \beta \in \mathcal{B} \text{ such that } \beta^* \beta = b. \]
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Note: Suppose $\beta' \in \mathcal{B}$ such that $\beta'^* \beta' = b$. Then $\frac{\beta'}{\beta} = ???$. (\leadsto polar decomposition.)
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However, \( u : \beta \mapsto \beta' \) defines a unitary \( \mathcal{B}^a(\beta \mathcal{B}, \beta' \mathcal{B}) \). (\text{Hilbert modules}!)

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This square root depends on the choice (at least of \( \gamma \)) and it is noncommutative.
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Note: In order to compose in that way a fixed \( c \) with any \( b \), we need to
know the whole map \( \gamma^* \bullet \gamma \)! (\( \leadsto \) Hilbert bimodules!)
Example. A kernel $k: S \times S \rightarrow \mathbb{C}$ over a set $S$ is positive definite if
\[ \sum_{i,j} \bar{z}_i \, k^{\sigma_i, \sigma_j} \, z_j \geq 0 \]
for all finite choices of $\sigma_i \in S$ and $z_i \in \mathbb{C}$. 

(4)
Example. A kernel $\kappa: S \times S \to \mathbb{C}$ over a set $S$ is positive definite if
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**Theorem. (Kolmogorov decomposition.)** If $\kappa$ is $\mathbb{C}$–valued PD-kernel over $S$, then there exist a Hilbert space $H$ and a map $i: S \to H$ such that
\[ \langle i(\sigma), i(\sigma') \rangle = \kappa^{\sigma, \sigma'} \]
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Example. A kernel $\tilde{t}: S \times S \to \mathbb{C}$ over a set $S$ is positive definite if
$$\sum_{i,j} \tilde{z}_i \tilde{t}^{\sigma_i, \sigma_j} z_j \geq 0$$
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Proof. On $S_{\mathbb{C}} := \bigoplus_{\sigma \in S} \mathbb{C} = \left\{ (z_{\sigma})_{\sigma \in S} \mid \# \{\sigma : z_{\sigma} \neq 0\} < \infty \right\}$ define the sesquilinear form
$$\left\langle (z_{\sigma})_{\sigma \in S}, (z'_{\sigma})_{\sigma \in S} \right\rangle := \sum_{\sigma, \sigma' \in S} \tilde{z}_{\sigma} \tilde{t}^{\sigma, \sigma'} z'_{\sigma'}.$$
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PD is born to make that positive. Rest: Quotient by \( \mathcal{N} \) and completion, with \( i : \sigma \mapsto e_\sigma + \mathcal{N} \) where \( e_\sigma := (\delta_{\sigma, \sigma'})_{\sigma' \in S} \).
Note: $(H, i)$ is an excellent square root of $f$!
Note: \((H, i)\) is an excellent square root of \(\mathfrak{f}\)!

- \(\mathfrak{f}\) is easily computable in terms of \((H, i)\).

Try to do the same with the collection of numbers
\[\sqrt{\sum_{i,j=1}^{n} z_i t^{\sigma_i;\sigma_j} z_j}\]
or with the collection of matrices
\[\sqrt{(t^{\sigma_i;\sigma_j})_{i,j=1,...,n}}.\]
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- \((H, i)\) is unique in a very specific sense.

In fact, if also \((K, j)\) fulfills \(\text{span}\ j(S) = K\), then \(v\) becomes a unitary.

Also, compare positive numbers: \(S = \{\omega\}\), \(t^{\omega, \omega} := \lambda \geq 0\).
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Try to do the same with the collection of numbers
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\sqrt{\sum_{i, j=1}^{n} z_i \mathfrak{f}^{\sigma_i} i z_j} \mathfrak{f}^{\sigma_j} j
\]
or with the collection of matrices
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\sqrt{(\mathfrak{f}^{\sigma_i} i \mathfrak{f}^{\sigma_j})}_{i, j=1, \ldots, n}.
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- Composition of PD-kernels is reflected by tensor products.

\[(\mathfrak{f}^{\sigma, \sigma'}) := 1^{\sigma, \sigma'} \mathfrak{f}^{\sigma, \sigma'} \text{ (Schur prod.)}\]
\(\mathfrak{f} \leadsto i : S \rightarrow H, \quad 1 \leadsto j : S \rightarrow K\).
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- \(\mathfrak{f}\) is easily computable in terms of \((H, i)\).
  - Try to do the same with the collection of numbers \(\sqrt{\sum_{i,j=1}^{n} z_i \mathfrak{f}^{\sigma_i, \sigma_j} z_j}\)
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- Composition of PD-kernels is reflected by tensor products.
  \((\mathfrak{f})^{\sigma, \sigma'} := l^{\sigma, \sigma'} \mathfrak{f}^{\sigma, \sigma'}\). \(\mathfrak{f} \sim i: S \to H, \ l \sim j: S \to K\)
  \((i \otimes j)(\sigma) := i(\sigma) \otimes j(\sigma) \in H \otimes K\). Note: \(\text{span}(i \otimes j)(S) \subseteq H \otimes K\)!
Example. A kernel $\kappa: S \times S \to B$ over a set $S$ is positive definite if
\[
\sum_{i,j} b_i^* \kappa_{\sigma_i,\sigma_j} b_j \geq 0 \quad \text{for all finite choices of } \sigma_i \in S \text{ and } b_i \in B.
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Example. A kernel \( \mathfrak{k} : S \times S \to \mathcal{B} \) over a set \( S \) is positive definite if
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Theorem. (Kolmogorov decomposition.) If \( \mathfrak{k} \) is \( \mathcal{B} \)–valued PD-kernel over \( S \), then there exist a Hilbert \( \mathcal{B} \)–module \( E \) and a map \( i : S \to E \) such that
\[
\langle i(\sigma), i(\sigma') \rangle = \mathfrak{k}^{\sigma,\sigma'}
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and \( E = \text{span} i(S)\mathcal{B} \). Moreover, if \( j : S \to F \) fulfills \( \langle j(\sigma), j(\sigma') \rangle = \mathfrak{k}^{\sigma,\sigma'} \), then \( v : i(\sigma) \mapsto j(\sigma) \) extends to a unique isometry \( E \to F \).
**Example.** A kernel $\check{\kappa}: S \times S \to \mathcal{B}$ over a set $S$ is positive definite if
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**Theorem.** (Kolmogorov decomposition.) If $\check{\kappa}$ is $\mathcal{B}$–valued PD-kernel over $S$, then there exist a Hilbert $\mathcal{B}$–module $E$ and a map $i: S \to E$ such that
$$\langle i(\sigma), i(\sigma') \rangle = \check{\kappa}_{\sigma,\sigma'}$$
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**Proof.** On $S \otimes \mathcal{B}$ define the $\mathcal{B}$–valued sesquilinear map
$$\langle e_\sigma \otimes b, e_{\sigma'} \otimes b' \rangle := b^* \check{\kappa}_{\sigma,\sigma'} b'.$$
PD is born to make that positive.
**Example.** A kernel $\mathfrak{t}: S \times S \to \mathcal{B}$ over a set $S$ is positive definite if

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Note: \((E, i)\) is a square root of \(\xi\), that fulfills:

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Note: \((E, i)\) is a square root of \(t\), that fulfills:

- \(t\) is easily computable in terms of \((E, i)\).
- \((H, i)\) is unique in a very specific sense.

In fact, if also \((F, j)\) fulfills \(\text{span} j(S) \mathcal{B} = F\), then \(v\) becomes a unitary.
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  In fact, if also \((F, j)\) fulfills \(\text{span} \ j(S)\mathcal{B} = F\), then \(v\) becomes a unitary.

However:

- It does NOT help composing PD-kernels.
  
  There is no reasonable tensor product of right Hilbert \(\mathcal{B}\)–modules that recovers what we did for the one-point set \(S = \{\omega\}\).

In fact, how could it?

Our composed square root \(\beta \gamma\) depends on the choice of \(\gamma\)!
Example. A kernel $\mathcal{K}: S \times S \rightarrow \mathcal{B}(\mathcal{A}, \mathcal{B})$ over a set $S$ is completely positive definite (CPD) if

$$\sum_{i,j} b_i^* \mathcal{K}^{\sigma_i, \sigma_j}(a_i^* a_j) b_j \geq 0$$

for all finite choices of $\sigma_i \in S$, $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. 
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- Heo [Heo99]: $S = \{1, \ldots, n\}$. (Completely multi-positive map.) No composition considered. In particular, no semigroups.
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- Accardi and Koyzyrev [AK01]: Special case \( \mathcal{B}(H) \) for \( S = \{0, 1\} \). However, semigroups! (The technique of the four semigroups)
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- Barreto, Bhat, Liebscher, and MS [BBLS04]: General case. In particular, CPD-semigroups.
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- Possibly Speicher [Spe98] (Habilitation thesis 1994)?
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for all finite choices of $\sigma_i \in S$, $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$.

**Theorem. (Kolmogorov decomposition.)** \(\mathcal{A} \ni 1\). If $\mathcal{K}$ is a CPD-kernel over $S$ from $\mathcal{A}$ to $\mathcal{B}$, then there exist an $\mathcal{A}–\mathcal{B}$–correspondence $E$ and a map $i: S \rightarrow E$ such that

$$\langle i(\sigma), ai(\sigma') \rangle = \mathcal{K}^{\sigma,\sigma'}(a)$$

and $E = \overline{\text{span} \mathcal{A}i(S)\mathcal{B}}$. 
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**Theorem. (Kolmogorov decomposition.)** \( \mathcal{A} \ni 1 \). If $\mathcal{K}$ is a CPD-kernel over $S$ from $\mathcal{A}$ to $\mathcal{B}$, then there exist an $\mathcal{A}$–$\mathcal{B}$–correspondence $E$ and a map $i: S \to E$ such that

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and $E = \text{span}_{\mathcal{A}i(S)\mathcal{B}}$. Moreover, if $j: S \to F$ fulfills $\langle j(\sigma), j(\sigma') \rangle = \mathcal{K}^{\sigma, \sigma'}(a)$, then $v: i(\sigma) \mapsto j(\sigma)$ extends to a unique bilinear isometry $E \to F$. 
Example. A kernel $\mathcal{K} : S \times S \to \mathcal{B}(\mathcal{A}, \mathcal{B})$ over a set $S$ is completely positive definite (CPD) if

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Note: $S = \{\omega\} \sim$ CP-maps. (Do NOT use $n$–positive for all $n$!) Kolmogorov $\sim$ Paschke’s GNS-construction [Pas73].
1st proof. The $\mathcal{B}$–valued kernel $t^{(a,\sigma),(a',\sigma')} := \mathcal{K}^{\sigma,\sigma'}(a^*a')$ over $\mathcal{A} \times S$ is, clearly, PD. On its Kolmogorov decomposition $(E, \tilde{i})$ check that $a\tilde{i}(a',\sigma) := \tilde{i}(aa',\sigma)$ defines a left action of $\mathcal{A}$ on $E$. Put $i(\sigma) := \tilde{i}(1,\sigma).$
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2nd proof. On $\mathcal{A} \otimes \mathcal{S}_{\mathbb{C}} \otimes \mathcal{B}$ define the $\mathcal{B}$–valued sesquilinear map

$$\left\langle a \otimes e_{\sigma} \otimes b, a' \otimes e_{\sigma'} \otimes b' \right\rangle := b^*\mathcal{K}^{\sigma,\sigma'}(a^*a')b'.$$

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1st proof.  The $\mathcal{B}$–valued kernel $\tilde{k}^{(a,\sigma), (a',\sigma')} := \mathcal{K}^{\sigma,\sigma'}(a^*a')$ over $\mathcal{A} \times S$ is, clearly, PD. On its Kolmogorov decomposition $(E, \tilde{i})$ check that $\tilde{a}(a', \sigma) := \tilde{i}(aa', \sigma)$ defines a left action of $\mathcal{A}$ on $E$. Put $i(\sigma) := \tilde{i}(1, \sigma)$. ■

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The first proof is “classical”:
Guess a PD-kernel, do Kolmogorov, show its algebraic properties.
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The first proof is “classical”:
Guess a PD-kernel, do Kolmogorov, show its algebraic properties.

The second proof is “modern”: Start with a bimodule, define the only reasonable inner product that emerges from CPD. (The algebraic properties are general theory of correspondences.)
Example: The Stinespring construction.
Let $\varphi: \mathcal{A} \to \mathcal{B}$ be a CP-map.
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Guess that the kernel $t_{(a, g), (a', g')} := \langle g, \varphi(a^*a')g \rangle$ over $(\mathcal{A}, G)$ is PD.
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Let $\varphi: \mathcal{A} \to \mathcal{B}$ be a CP-map.
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Prove it!
Do the Kolmogorov decomposition $(H, i)$ for $\tau$.
Show that $ai(a', g) := i(aa', g)$ defines an action of $\mathcal{A}$.
Show that $v: g \mapsto i(1, g)$ defines a bounded operator $G \to H$. 
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Do the Kolmogorov decomposition $(H, i)$ for $\tilde{t}$.
Show that $ai(a', g) := i(aa', g)$ defines an action of $\mathcal{A}$.
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Verify $v^*av = \varphi(a)$. 
Example: The Stinespring construction.

Let \( \varphi : \mathcal{A} \to \mathcal{B} \) be a CP-map.

Represent \( \mathcal{B} \subset \mathcal{B}(G) \) faithfully on a Hilbert space \( G \).

Guess that the kernel \( \xi(a,g),(a',g') := \langle g, \varphi(a^*a')g \rangle \) over \( (\mathcal{A}, G) \) is PD.

Prove it!

Do the Kolmogorov decomposition \((H, i)\) for \( \xi \).

Show that \( ai(a', g) := i(aa', g) \) defines an action of \( \mathcal{A} \).

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Prove it!
Do the Kolmogorov decomposition $(H, i)$ for $\hat{t}$.
Show that $ai(a', g) := i(aa', g)$ defines an action of $\mathcal{A}$.
Show that $v: g \mapsto i(1, g)$ defines a bounded operator $G \to H$.
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Paschke: $\varphi \sim (E, \xi)$ such that $\langle \xi, a\xi \rangle = \varphi(a)$ and $\overline{\text{span}} \mathcal{A}\xi\mathcal{B} = E$.
If $\mathcal{B} \subset \mathcal{B}(G)$, then $H := E \odot G$ (tensor product of correspondences) $\sim \mathcal{B}(H) \ni (a \odot \text{id}_G): x \odot g \mapsto ax \odot g$ and $\mathcal{B}(G, H) \ni (\xi \odot \text{id}_G): g \mapsto \xi \odot g$. 
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Then $(\xi \odot \text{id}_G)^*(a \odot \text{id}_G)(\xi \odot \text{id}_G) = \varphi(a) \odot \text{id}_G = \varphi(a)$.
A recent example: (Ramesh)
A map $T : E_A \to F_B$ is a $\varphi$–map if $\langle T(x), T(x') \rangle = \varphi(\langle x, x' \rangle)$. 
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$\rho(a) := a \odot \text{id}_G \in \mathcal{B}(H)$, and $v := \xi \odot \text{id}_G \in \mathcal{B}(G, H)$. (~ Stinespring.)
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($\sim w\Psi(x)v = (\zeta(x \otimes \xi)) \otimes \text{id}_G = T(x) \otimes \text{id}_G \in \mathcal{B}(G, F \otimes G) = \mathcal{B}(G, L)$.)
Note: \((E, i)\) is an excellent square root of \(\mathbb{R}\)!
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Note: $(E, i)$ is an excellent square root of $\mathcal{R}$!

- $\mathcal{R}$ is easily computable in terms of $(E, i)$.
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In fact, if also $(F, j)$ fulfills $\text{span} \mathcal{A} j(S) \mathcal{B} = F$, then $v$ becomes a bilinear unitary.
Note: \((E, i)\) is an excellent square root of \(K\)!

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- Tensor product shows that composition of CPD-kernels is CPD.

\[
(\mathcal{L} \circ \mathcal{K})^{\sigma,\sigma'} := \mathcal{L}^{\sigma,\sigma'} \circ \mathcal{K}^{\sigma,\sigma'}. \text{ (Schur product.)}
\]

\(\mathcal{K} \sim i: S \to E, \quad \mathcal{L} \sim j: S \to F\)
Note: \((E, i)\) is an excellent square root of \(\mathcal{R}\)!

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\(\mathcal{R} \leadsto i: S \to E, \quad \mathcal{L} \leadsto j: S \to F\)

\[(\mathcal{L} \circ \mathcal{R}) \leadsto (i \circ j)(\sigma) := i(\sigma) \odot j(\sigma) \in E \odot F.\]
Note: \((E, i)\) is an excellent square root of \(\mathcal{R}\)!

- \(\mathcal{R}\) is easily computable in terms of \((E, i)\).
- \((E, i)\) is unique in a very specific sense.
  In fact, if also \((F, j)\) fulfills \(\text{span} \mathcal{A} j(S) \mathcal{B} = F\), then \(v\) becomes a bilinear unitary.
- Tensor product shows that composition of CPD-kernels is CPD.

\[(\mathcal{L} \circ \mathcal{R})^{\sigma, \sigma'} := \mathcal{L}^{\sigma, \sigma'} \circ \mathcal{R}^{\sigma, \sigma'}. \quad (\text{Schur product.})\]
\[\mathcal{R} \sim i: S \rightarrow E, \quad \mathcal{L} \sim j: S \rightarrow F\]

\[(\mathcal{L} \circ \mathcal{R}) \sim (i \odot j)(\sigma) := i(\sigma) \odot j(\sigma) \in E \odot F.\]

Here for \(\mathcal{A}E \mathcal{B}\) and \(\mathcal{B}F \mathcal{C}\), the internal tensor product \(E \odot F\) is the unique \(\mathcal{A}–C–\)correspondence that is spanned by elementary tensors \(x \odot y\) fulfilling

\[\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle \text{ and } a(x \odot y) = (ax) \odot y.\]
Construction: Start with $E \otimes F$. 
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Positivity:  

Observe: $\langle x \otimes y, x \otimes y \rangle = \langle y, \langle x, x \rangle y \rangle = \langle y, \beta^* \beta y \rangle = \langle \beta y, \beta y \rangle \geq 0$. 

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- Put $xy^* : z \mapsto x\langle y, z \rangle$ and $E^* := \{ x^* : x \in E \}$.  
  Then $\langle x'^*, x^* \rangle := x'x^*$ and $bx^*a := (a^*xb^*)^*$ turns $E^*$ into a $\mathcal{B}$–$\mathcal{B}^a(E)$–correspondence.  
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Define the Hilbert $M_n(\mathbb{B})$–module $E_n := ((E^*)^n)^*$. Check that $\langle X_n, X'_n \rangle = (\langle x_i, x'_j \rangle)_{i,j}$ and $(X_nB)_i = \sum j x_j b_{ji}$.  

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- Then $\langle \sum_i x_i \otimes y_i, \sum_i x_i \otimes y_i \rangle = \langle X_n \otimes Y^n, X_n \otimes Y^n \rangle \geq 0$. 
Note:

- A CPD-kernel $\mathcal{A}$ from $\mathcal{A}$ to $\mathcal{B}$ and a CPD-kernel $\mathcal{L}$ from $\mathcal{B}$ to $\mathcal{C}$ can be composed to form a CPD-kernel $\mathcal{L} \circ \mathcal{A}$ from $\mathcal{A}$ to $\mathcal{C}$.

- Viewing $w \in \mathcal{C}$ as map $z \mapsto zw$ on $\mathcal{C}$
  $\mathcal{C}$–valued PD-kernels correspond 1-1 with CPD-kernel from $\mathcal{C}$ to $\mathcal{C}$. Schur product of PD-kernels = compositions of CPD-kernels.

- Viewing $b \in \mathcal{B}$ as map $z \mapsto zb$ from $\mathcal{C}$ to $\mathcal{B}$
  $\mathcal{B}$–valued PD-kernels correspond 1-1 with CPD-kernel from $\mathcal{C}$ to $\mathcal{B}$. Usually, no composition! (Codomain and domain match only in the $\mathcal{C}$–valued case.)
Recall: $\mathcal{K} \leadsto (E, i)$, $\mathcal{L} \leadsto (F, j)$, then $\mathcal{L} \circ \mathcal{K} \leadsto$

$$\overline{\text{span}\{ai(\sigma) \circ j(\sigma)c : a \in \mathcal{A}, c \in \mathcal{C}, \sigma \in \mathcal{S}\}}$$

with embedding $i \circ j : \sigma \mapsto i(\sigma) \circ j(\sigma)$. This is (usually much!) smaller than

$$E \circ F = (\overline{\text{span} \mathcal{A}i(S)\mathcal{B}}) \circ (\overline{\text{span} \mathcal{B}j(S)\mathcal{C}})$$

$$= \overline{\text{span} \{ ai(\sigma) \circ bj(\sigma')c : a \in \mathcal{A}; b \in \mathcal{B}; c \in \mathcal{C}; \sigma, \sigma' \in \mathcal{S}\}}.$$ 

So, $E \circ F$ does not coincide but at least contains the GNS-correspondence of $\mathcal{L} \circ \mathcal{K}$.

The GNS-correspondences for $\mathcal{K}$ and $\mathcal{L}$ allow easily to compute GNS-correspondence for $\mathcal{L} \circ \mathcal{K}$.

Nothing like this is true for Stinespring constructions!
Recall: (For simplicity for CP-maps.)

\[ T : \mathcal{A} \to \mathcal{B} \subset \mathcal{B}(G) \leadsto H = E \otimes G, \quad v = \xi \otimes \text{id}_G, \quad \rho(a) = a \otimes \text{id}_G. \]

\[ S : \mathcal{B} \to C \subset \mathcal{B}(K) \leadsto L = F \otimes K, \quad w = \zeta \otimes \text{id}_K, \quad \pi(b) = b \otimes \text{id}_K. \]

By no means does the Stinespring representation \( \rho \) for \( T \) help to construct the Stinespring representation for \( S \circ T \)!

(One needs to “tensor” \( E \) with the representation space \( L = F \otimes G \) of the Stinespring representation \( \pi \) for \( S \), not with \( G \)!) 

The GNS-correspondences \( E \) and \( F \), on the other hand, are universal! (For each CP-map they need to be computed only once.)
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The GNS-correspondences \( E \) and \( F \), on the other hand, are universal! (For each CP-map they need to be computed only once.)

Doing Stinespring representations for the individual members of a CP-semigroup on \( \mathcal{B} \subset \mathcal{B}(G) \), is like considering a \( 2 \times 2 \)–system of complex linear equations as a real \( 4 \times 4 \)–system (ignoring all the structure hidden in the fact that certain \( 2 \times 2 \)–submatrices are very special) and applying the Gauß algorithm to the \( 4 \times 4 \)–system instead of trivially resolving the \( 2 \times 2 \)–system by hand.
\( \mathcal{I} = (\mathcal{I}_t)_{t \geq 0} \) a CPD-semigroup over \( S \) on \( \mathcal{B} \ni 1 \).

Then the GNS-correspondences \( \mathcal{E}_t \) of the \( \mathcal{I}_t \) fulfill \( \mathcal{E}_s \circ \mathcal{E}_t \supset \mathcal{E}_{s+t} \), so

\[
(\mathcal{E}_{s_{mn}} \circ \ldots \circ \mathcal{E}_{s_1}) \circ \ldots \circ (\mathcal{E}_{s_{m_1}} \circ \ldots \circ \mathcal{E}_{s_1}) \supset \mathcal{E}_{s_{mn}+\ldots+s_1} \circ \ldots \circ \mathcal{E}_{s_{m_1}+\ldots+s_1}
\]

Fix \( t > 0 \), \( \leadsto \) inductive limit over

\[ t = (t_n, \ldots, t_1) \in (0, \infty)^n \] with

\[ t_n + \ldots + t_1 = t. \]

For \( E_t = \lim \text{ind}_t \mathcal{E}_t \supset \mathcal{E}_t \)

\[
\mathcal{E}_s \circ \mathcal{E}_t \supset \mathcal{E}_{s+t} \quad \text{becomes equality} \quad E_s \circ E_t = E_{s+t},
\]

so \( E^\circ = (E_t)_{t \in \mathbb{R}_+} \) is a product system. The \( \xi_\sigma^t := i_t(\sigma) \in \mathcal{E}_t \subset E_t \) fulfill

\[
\xi_\sigma \circ \xi_\sigma^t = \xi_{s+t}^\circ \quad \text{that is, for each } \sigma \in S \text{ the family } \xi_\sigma^\circ = (\xi_\sigma^t)_{t \geq 0} \text{ is a unit, such that } \langle \xi_\sigma^t, \cdot \xi_\sigma^{t'} \rangle = \mathcal{I}_t^{\sigma,\sigma'} \text{ for all } \sigma, \sigma' \in S, \text{ and the set } \{ \xi_\sigma^\circ : \sigma \in S \}\}
\]

of units generates \( E^\circ \) as a product system. We see:

The square root of a CPD-semigroup (in particular, of a CP-semigroup) is a product system with generating set of units; Bhat and MS [BS00].
Other examples

• The product system of a PD-semigroup consists of symmetric Fock spaces. Applications:
  Classical Lévy processes (Parthasarathy and Schmidt [PS72].)
  Quantum Lévy processes (Schürmann, MS, and Volkwardt [SSV07].)

• The product system of uniformly continuous normal CPD-semigroups on von Neumann algebras consists of time ordered Fock modules (Barreto, Bhat, Liebscher, and MS [BBLS04]). For $C^*$-algebras this may fail (Bhat, Liebscher, and MS [BLS10])!

• The Markov semigroups that admit dilations by cocycle perturbations of “noises” are precisely the “spatial” Markov semigroups (MS [Ske09a]). Proof: Via “spatial” product systems (MS [Ske06] (preprint 2001))!
**CP-semigroups on** $\mathcal{B}^a(E)$

Let $\vartheta$ be a semigroup of (unital, for simplicity) endomorphisms $\vartheta_t$ of $\mathcal{B}$. Then $\mathcal{B}_t := \mathcal{B}$ with $b.x_t := \vartheta_t(b)x_t$ is its GNS-system with unit $(1)_{t \in \mathbb{R}_+}$.

It is not a good idea to tensor with $G$ when $\mathcal{B} \subset \mathcal{B}(G)$. (Unless vN-alg.) This **changes** when $\mathcal{B} = \mathcal{B}(G)$ — or better $\mathcal{B} = \mathcal{B}^a(E)$.

But only, if we tensor “from both sides”!

**General:** $T : \mathcal{B}^a(E_\mathcal{B}) \to \mathcal{B}^a(F_\mathcal{C})$ and $S : \mathcal{B}^a(F_\mathcal{C}) \to \mathcal{B}^a(G_\mathcal{D})$ CP-maps. Their GNS-correspondences $\mathcal{E}$ and $\mathcal{F}$.

Require $\overline{\text{span}} \mathcal{K}(E)\mathcal{E} = \mathcal{E}$ and $\overline{\text{span}} \mathcal{K}(F)\mathcal{F} = \mathcal{F}$ (strictness!). Then

$$\left( E^* \circ \mathcal{E} \circ F \right) \circ \left( F^* \circ \mathcal{F} \circ G \right) = E^* \circ \mathcal{E} \circ \left( F \circ F^* \right) \circ \mathcal{F} \circ G$$

$$= E^* \circ \mathcal{E} \circ \mathcal{K}(F) \circ \mathcal{F} \circ G = E^* \circ (\mathcal{E} \circ \mathcal{F}) \circ G.$$

So “sandwiching” between the representation modules (or spaces) preserves tensor products! ($\leadsto$ Morita equivalence.)
Applications:

- $\vartheta$ a strict $E_0$–semigroup on $\mathcal{B}^a(E)$ with GNS-systems $(\mathcal{B}^a(E)_t)_{t \in \mathbb{R}^+}$.
  
  $\leadsto \quad E_t := E^* \circ \mathcal{B}^a(E)_t \circ E = E^* \circ_t E$ is product system via

  $$ (x^* \circ_s x') \circ (y^* \circ_t y') \mapsto x^* \circ_{s+t} \vartheta_t(x'y^*)y'. $$

  (With “unit vector” MS [Ske02]. General [Ske09b] (preprint 2004).

- Special case: $E$ a Hilbert spaces gives Bhat’s construction [Bha96] of the (anti-)Arveson system [Arv89] of $\vartheta$. (“Reverse” difficult!)

- $\mathcal{E}^\circ = (\mathcal{E}_t)_{t \in \mathbb{R}^+}$ the GNS-system of a strict CP-semigroup $T$ on $\mathcal{B}^a(E)$. Then $E_t := E^* \circ \mathcal{E}_t \circ E$ gives a product system $E^\circ = (E_t)_{t \in \mathbb{R}^+}$ of $\mathcal{B}$–correspondences.

- Special case: $E$ a Hilbert spaces gives Bhat’s Arveson system of $T$ [Bha96] without dilating $T$ first to an endomorphism semigroup.
Only briefly: Positivity in $\ast$–algebras

• For instance: $b$ in a pre-$C^*$–algebra is positive when positive in $\overline{\mathcal{B}}$. $b$ has a square root $\beta \in \overline{\mathcal{B}}$.

• For instance: $b \in \mathcal{L}^a(G)$ ($G$ a pre-Hilbert space) is positive if $\langle g, bg \rangle \geq 0$ for every $g \in G$. By an application of Friedrich’s theorem, $b \in \mathcal{B}$ has a square root $\beta \in \mathcal{L}^a(G, G')$ where $G \subset G' \subset \overline{G}$.

• New: Let $\mathcal{B}$ be a unital $\ast$–algebra and $\mathcal{S}$ a set of positive linear functionals on $\mathcal{B}$. $b \in \mathcal{B}$ is $\mathcal{S}$–positive if $\varphi(c^*bc) \geq 0$ for all $\varphi \in \mathcal{S}$ and $c \in \mathcal{B}$. $\mathcal{B}$ is $\mathcal{S}$–separated if $\varphi(cbc') = 0 \forall \varphi \in \mathcal{S}; c, c' \in \mathcal{B}$ implies $b = 0$. 
Example: Let $\mathcal{B} = \mathbb{C}\langle x \rangle$. Let $Z \subset \mathbb{C}$. Put $\mathcal{S} = \{\varphi_w: p \mapsto p(w), w \in Z\}$.

- $Z = \mathbb{R}$ or $Z = S^1$. Then $p \geq 0 \iff \exists q \in \mathcal{B}: \overline{qq} = p$.

- $Z = \mathbb{C}$. Then $p \geq 0 \implies p = 0$. (Liouville.)

- $Z \subset \mathbb{C}$ compact and $Z \setminus \partial Z \neq \emptyset$. Then $\mathcal{B} \subset C(Z) = \overline{\mathcal{B}}$ and $p \geq 0 \iff \exists f \in C(Z): \overline{ff} = p$.

For instance, $Z = [-1, 0], \ p = -x$

$\leadsto p = \overline{ff} \geq 0$ where $f = \sqrt{-x} \in C[-1, 0]$.
Denote by $G$ the direct sum of the GNS-pre-Hilbert spaces of all $\phi \in S$. Identify $B \subset L_a(G)$.

**Theorem 1.** Let $A$ be a unital $\ast$-algebra. Let $K : S \times S \to L(A, B)$ be a kernel over $S$ from $A$ to $B$. If $K$ is CPD in the sense that $\sum_{i,j} b_{ij}^* K(\sigma_i, \sigma_j) (a_i^* a_j) b_{ij}$ is $S$-positive for all finite choices, then there exists a pre-Hilbert space $H$ with a left action of $A$, and a map $i : S \to L_a(G, H)$ such that $K(\sigma, \sigma') (a) = i(\sigma)^* a i(\sigma')$ for all $\sigma, \sigma' \in S$ and $a \in A$.

We refer to $(\text{span}_A I_{d(G)})$ as the Kolmogorov decomposition of $\mathcal{A}$. That is $S$-positive for all finite choices, then there exists a pre-Hilbert space $H$ with a left action of $A$, and a map $i : S \to L_a(G, H)$ such that $K(\sigma, \sigma') (a) = i(\sigma)^* a i(\sigma')$ for all $\sigma, \sigma' \in S$ and $a \in A$. Let $\mathcal{A} \leftarrow S \times S : S \times S$.

Denote by $G$ the direct sum of the GNS-pre-Hilbert spaces...
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