Non-monotone DR-submodular Maximization: Approximation and Regret Guarantees

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Abstract

Diminishing-returns (DR) submodular optimization is an important field with many real-world applications in machine learning, economics and communication systems. It captures a subclass of non-convex optimization that provides both practical and theoretical guarantees.

In this paper, we study the fundamental problem of maximizing non-monotone DR-submodular functions over down-closed and general convex sets in both offline and online settings. First, we show that for offline maximizing non-monotone DR-submodular functions over a general convex set, the Frank-Wolfe algorithm achieves an approximation guarantee which depends on the convex set. Next, we show that the Stochastic Gradient Ascent algorithm achieves a 1/4-approximation ratio with the regret of $O(1/\sqrt{T})$ for the problem of maximizing non-monotone DR-submodular functions over down-closed convex sets. These are the first approximation guarantees in the corresponding settings. Finally we benchmark these algorithms on problems arising in machine learning domain with the real-world datasets.

1 Introduction

We consider the fundamental problem of optimizing DR-submodular function over a convex set. This problem has recently gained a significant attention in both, machine learning and theoretical computer science communities [1, 3, 4, 10, 21, 29, 34] due to its numerous applications in formulating real-world problems. Some examples of this can be found in [23, 26], and [11].

Previous work on this problem have been focused either on smooth and/or monotone DR-submodular functions or, on unconstrained or down-closed convex sets. Though, the majority of real-world problems can be formulated as non-monotone DR-submodular functions over a constrained convex set. Hence in this paper, we investigate the problem of maximizing constrained non-monotone DR-submodular functions. Our contribution is twofold. First, we provide an approximation algorithm for maximizing smooth non-monotone DR-submodular function over down-closed convex sets. Second we provide an online algorithm for maximizing non-monotone DR-submodular function over down-closed convex sets. Prior to this work, no theoretical guarantees were known for both of these problems.

Without the loss of generality, we assume that the DR-submodular function $F$ is positive, and that at any point $x$ in the convex set $K$, $F(x)$ and its gradient (denoted by $\nabla F(x)$) can be evaluated in the polynomial time. In addition, we assume that the projection of any point $x \in [0,1]^n$ on $K$ can be computed in polynomial time (this implies the availability of a polynomial time membership oracle).

Our offline algorithm is a discrete time local search procedure which produces a solution $x^T$ after $T$ iterations, such that

$$F(x^T) \geq \max_{x^* \in K} \alpha \cdot F(x^*) - \beta$$

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where \( \alpha, \beta \) are some parameters. Such an algorithm is called an \( \alpha \)-approximation with convergence rate \( \beta \).

The online setting consists of discrete time steps \( t = 1, \ldots, T \) for some time horizon \( T \). At each step \( t \) the algorithm first outputs a point \( x^t \in \mathcal{K} \), and then learns a function \( F^t : \mathcal{K} \rightarrow \mathbb{R}^+ \). The value \( F^t(x^t) \) is called its reward and the goal of the algorithm is to maximize the average reward. Hence, the goal is to minimize the regret. Formally we say that an algorithm achieves \( (\alpha, \beta) \)-regret if it produces points \( x^t \) such that

\[
\frac{1}{T} \sum_{t=1}^{T} F^t(x^t) \geq \alpha \cdot \max_{x^* \in \mathcal{K}} \frac{1}{T} \sum_{t=1}^{T} F^t(x^*) - \beta.
\]

Equivalently, we say that the algorithm has \( \alpha \)-regret at most \( \beta \). The factor \( \alpha \) is also called the approximation ratio of the algorithm.

### 1.1 Our contributions

Exploring the underlying properties DR-submodularity, we design algorithms with performance guarantees for each of the above mentioned settings. Our contributions are summarized as follows: (also see Table 1).

|                | Monotone | Non-monotone |
|----------------|----------|--------------|
| Offline        | smooth   | non-smooth   | smooth | non-smooth |
| unconstrained  | \( \frac{1}{2} \)-approx | \( \frac{1}{2} \), \( O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2} \), \( O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2} \), \( O\left(\frac{1}{\sqrt{T}}\right)\) |
| down-closed    | \( 1 - \frac{1}{e}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) |
| general        | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) |

|                | Online   |
|----------------|----------|
| unconstrained  | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) |
| down-closed    | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) |
| general        | \( 1 - \frac{1}{e}, O\left(\frac{1}{\sqrt{T}}\right)\) | \( \frac{1}{2}, O\left(\frac{1}{\sqrt{T}}\right)\) |

Table 1: Summary of results on DR-submodular maximization. Results from the present paper are shown in red. Entries \( (\alpha, \beta) \) refer either to an \( \alpha \)-approximation offline algorithm with convergence ratio \( \beta \) or to an \( (\alpha, \beta) \)-regret online algorithm.

**Offline setting.** First, we consider the problem of maximizing a non-monotone DR-submodular function over general convex sets. This problem has been proved to be hard. Specifically, any constant-approximation algorithm for the problem over a general convex domain must require exponentially many value queries to the function \[35\]. Determining the approximation ratio as a function depending on the problem parameters and characterizing necessary and sufficient regularity conditions that enable efficient approximation algorithm for the problem constitute an important direction.

We show that the celebrated Frank-Wolf algorithm achieves an approximation ratio of \( \left(1 - \min_{x \in \mathcal{K}} \frac{\|x\|_{\infty}}{3\sqrt{3}}\right) \) with the rate of convergence of \( O\left(\frac{n}{\ln^2 T}\right)\), where \( T \) is the number of iterations applied in the algorithm. In particular, if the domain \( \mathcal{K} \) (not necessarily down-closed)\[1\] contains the origin \( 0 \) then for arbitrary constant \( \epsilon > 0 \),

\[1\] We refer to Section 2 for a formal definition.
after $T = O\left(e^{\sqrt{n}/\epsilon}\right)$ (sub-exponential) iterations, the algorithm outputs a solution $x^T$ such that $F(x^T) \geq \left(\frac{1}{3\sqrt{3}}\right)\max_{x^* \in K} F(x^*) - \epsilon$. To the best of our knowledge, this is the first algorithm with an approximation guarantee for maximizing non-monotone DR-submodular function over general convex sets.

**Online setting.** DR-submodular maximization has been studied in online environments but only for monotone functions. However, in numerous applications the functions are intrinsically non-monotone DR-submodular. The quest of algorithms with performance guarantee for online non-monotone DR-submodular maximization is a major research line.

We show that the Online Gradient Ascent algorithm achieves $(1/4, O(1/\sqrt{T}))$-regret. The result holds also if only unbiased estimates of the gradients are available. Prior to our work, no approximation guarantee has been shown even for the simpler setting of maximizing online non-monotone DR-submodular functions over an unconstrained hypercube (i.e., $K = [0, 1]^n$).

**Experiments.** We experimentally demonstrate the efficiency of our algorithms on the problems arising in domain of machine learning. We conduct following three set of experiments.

1. We compare the performance of offline Gradient Ascent algorithm against the previous known algorithms for maximizing DR-submodular function over down-closed polytopes. Note that our theoretical guarantee holds for more general case.
2. We show the performance of offline Gradient Ascent algorithm for revenue maximizing problem on the real-world dataset (Advogato user-user relationship graph) over a general (not down-closed) polytope.
3. We show the performance of online Frank-Wolfe algorithm for revenue maximizing revenue on the real-world dataset (Facebook user-user relationship graph) on a down-closed polytope.

**1.2 Related work**

Submodular optimization has been widely studied for decades [28, 16]. The domain has been investigating even more extensively in recent years due to numerous applications in statistics and machine learning, for example active learning [18], viral marketing [25], network monitoring [19], document summarization [27], crowd teaching [32], feature selection [14], deep neural networks [13], diversity models [12] and recommender systems [20].

**Offline submodular/DR-submodular optimization.** The problem of submodular (set) minimization has been studied in [31, 24]. See [2] for a survey on connections with and applications in machine learning. Submodular (set) maximization is an NP-hard problem. Several approximation algorithms have been given in the offline setting, for example a $1/2$-approximation for unconstrained domains [7, 6], a $(1 - 1/e)$-approximation for monotone smooth submodular functions [8, 9], or a $(1/e)$-approximation for non-monetone submodular functions on down-closed polytopes [15, 9].

Continuous extension of submodular functions play a crucial role in submodular optimization, especially in submodular maximization, including the multilinear relaxation and the softmax extension. These belong to the class of DR-submodular functions. Bian et al. [5] considered the problem of maximizing monotone DR-functions subject to down-closed convex domains and proved that the greedy method proposed by [8], which is a variant of the Frank-Wolfe algorithm, guarantees a $(1 - 1/e)$-approximation. It has been observed by Hassani et al. [21] that the greedy method is not robust in stochastic settings (where only unbiased estimates of gradients are available). Subsequently, they showed that the gradient methods achieve $1/2$-approximations in stochastic settings. Maximizing non-monotone DR-submodular functions is harder. Very recently, Bian et al. [3] and Niazadeh et al. [29] have independently presented algorithms with the same approximation guarantee of $1/2$ for the problem of maximizing non-monotone DR-submodular functions over a hypercube. Both algorithm are inspired by the bi-greedy algorithm in [7, 6]. Bian et al. [4] made a further step by providing an $1/e$-approximation algorithm over down-closed convex sets. Remark that when aiming for approximation algorithms, the restriction to down-closed polytopes is unavoidable. Specifically, Vondrák [35] proved that any algorithm for the problem over a non-down-closed domain that guarantees a constant approximation must require exponentially many value queries to the function.
Online submodular/DR-submodular optimization. Online optimization has been broadly studied for convex/concave functions [22]. An important research agenda is to design algorithms with performance guarantees in terms of regret and approximation for non-convex functions in general and for DR-submodular functions in particular. Chen et al. [10] have considered the online problem of maximizing monotone DR-submodular functions and provided an \((1 - 1/e)\)-approximation with regret \(O(\sqrt{T})\) when the functions are smooth. More generally, if the functions are not necessarily smooth, they proved that the online gradient ascent algorithm achieves a 1/2-approximation with regret \(O(\sqrt{T})\). No guarantee has been shown for online maximizing non-monotone DR-submodular functions. Very recently, Roughgarden and Wang [30] have studied the online problem of maximizing submodular (set) functions over the unconstrained domain \([0, 1]^n\). They gave an optimal \((1/2, O(\sqrt{T}))\)-regret algorithm.

2 Preliminaries and Notations

We introduce some basic notions, concepts and lemmas which will be used throughout the paper. We use boldface letters, e.g., \(x, z\) to represent vectors. We denote \(x_i\) as the \(i\)th entry of \(x\) and \(x^t\) as the decision vector at time step \(t\). For two \(n\)-dimensional vectors \(x, y\), we say that \(x \leq y\) iff \(x_i \leq y_i\) for all \(1 \leq i \leq n\). Moreover, \(x \vee y\) is defined as a vector such that \((x \vee y)_i = \max\{x_i, y_i\}\) and similarly \(x \wedge y\) is a vector such that \((x \wedge y)_i = \min\{x_i, y_i\}\). In the paper, we use the Euclidean norm \(\| \cdot \|\) by default (so the dual norm is itself). The infinity norm \(\|\cdot\|_\infty\) is defined as \(\|x\|_\infty = \max_{i=1}^n |x_i|\).

In the paper, we consider a bounded convex domain \(\mathcal{K}\) and w.l.o.g. assume that \(\mathcal{K} \subseteq [0, 1]^n\). We say that \(\mathcal{K}\) is unconstrained if \(\mathcal{K} = [0, 1]^n\); and \(\mathcal{K}\) is down-closed if for every \(z \in \mathcal{K}\) and \(y \leq z\) then \(y \in \mathcal{K}\); and \(\mathcal{K}\) is general if \(\mathcal{K}\) is simply a convex domain without any particular property. Besides, the diameter of the convex domain \(\mathcal{K}\) (denoted by \(D\)) is defined as \(\sup_{x, y \in \mathcal{K}} \|x - y\|\). The projection of a point \(x\) onto a convex set \(\mathcal{K}\) is a point in \(\mathcal{K}\) that is closest to \(x\); formally defined as follows.

\[
\text{Proj}_{\mathcal{K}}(x) := \arg\min_{y \in \mathcal{K}} \|y - x\|
\]  

A useful property of projections is that they satisfy the Pythagorean inequality, that is for any \(z \in \mathcal{K}\) and for any \(x\),

\[
\|\text{Proj}_{\mathcal{K}}(x) - z\| \leq \|x - z\|.
\]

A function \(f : \{0, 1\}^n \rightarrow \mathbb{R}^+\) is submodular if for all \(x \geq y \in \{0, 1\}^n\),

\[
f(x \vee a) - f(x) \leq f(y \vee a) - f(y) \quad \forall a \in \{0, 1\}^n.
\]  

Submodular functions can be generalized over continuous domains. A function \(F : [0, 1]^n \rightarrow \mathbb{R}^+\) is DR-submodular if for all vectors \(x, y \in [0, 1]^n\) with \(x \geq y\), any basis vector \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) and any constant \(\alpha > 0\) such that \(x + \alpha e_i \in [0, 1]^n\), \(y + \alpha e_i \in [0, 1]^n\), it holds that

\[
F(x + \alpha e_i) - F(x) \leq F(y + \alpha e_i) - F(y).
\]  

Note that if function \(F\) is differentiable then the diminishing-return (DR) property (3) is equivalent to

\[
\nabla F(x) \leq \nabla F(y) \quad \forall x \geq y \in [0, 1]^n.
\]  

Moreover, if \(F\) is twice-differentiable then the DR property is equivalent to all of the entries of its Hessian being non-positive, i.e., \(\frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0\) for all \(1 \leq i, j \leq n\). A differentiable function \(F : \mathcal{K} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}\) is said to be \(\beta\)-smooth if for any \(x, y \in \mathcal{K}\), we have

\[
F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle + \frac{\beta}{2} \|y - x\|^2
\]  

or equivalently,

\[
\|\nabla F(x) - \nabla F(y)\| \leq \beta \|x - y\|.
\]
Properties of DR-submodularity

In the following, we present properties of DR-submodular functions that are crucial in our analyses. The properties have been proved in [21] and [4]. For completeness, we provide their proofs in the appendix.

Lemma 1 ([21]). For every \( x, y \in \mathcal{K} \) and any DR-submodular function \( F : [0, 1]^n \to \mathbb{R}^+ \), it holds that
\[
\langle \nabla F(x), y - x \rangle \geq F(x \lor y) + F(x \land y) - 2F(x).
\]

Lemma 2 ([4]). For any DR-submodular function \( F \) and for all \( x, y, z \in \mathcal{K} \) it holds that
\[
F(x \lor y) + F(x \land y) + F(z^* \lor z) + F(z^* \land z) \geq F(y)
\]

where \( z^* = (x \lor y) - x \).

3 Offline Continuous DR-Submodular Maximization

In this section, we consider the problem of maximizing a DR-submodular function over a general convex set in the offline setting. Approximation algorithms [5, 4] have been presented and all of them are adapted variants of the Frank-Wolfe method. However, those algorithms require that the convex set is down-closed. This structure is crucial in their analyses in order to relate their solution to the optimal solution. Using some property of DR-submodularity (specifically, Lemma 1), we show that beyond the down-closed structure, the Frank-Wolfe algorithm guarantees an approximation solution for general convex sets. Below, we present the pseudocode of our variant of the Frank-Wolfe algorithm.

Algorithm 1 Frank-Wolfe Algorithm
1: Let \( x^1 \leftarrow \arg \min_{x \in \mathcal{K}} \|x\|_\infty \).
2: for \( t = 1 \) to \( T \) do
3: \[ \text{Compute } v^t \leftarrow \arg \max_{v \in \mathcal{K}} \langle \nabla F(x^{t-1}), v \rangle \]
4: \[ \text{Update } x^t \leftarrow (1 - \eta_t)x^{t-1} + \eta_tv^t \text{ with step-size } \eta_t = \frac{\delta}{TH_T} \text{ where } H_T \text{ is the } T\text{-th harmonic number and } \delta \]
5: \[ \text{represents the constant } (\ln 3)/2. \]
6: end for
7: return \( x^T \)

Next, we show that during the execution of the algorithm, the following invariant is maintained.

Lemma 3. It holds that \( 1 - x^t_i \geq e^{-\delta(1+O(1/\ln^2 T))} \cdot (1 - x^1_i) \) for every \( 1 \leq i \leq n \) and every \( 1 \leq t \leq T \).

Proof. Fix a dimension \( i \in [n] \). We first obtain the following recursion on fixed \( x_i \).
\[
1 - x^t_i = 1 - (1 - \eta_t)x^{t-1}_i - \eta_tv^t_i \quad \text{(Using the Update step from Algorithm 1)}
\]
\[
\geq 1 - (1 - \eta_t)x^{t-1}_i - \eta_t \quad (1 \geq v^t_i)
\]
\[
= (1 - \eta_t)(1 - x^{t-1}_i)
\]
\[
\geq e^{-\eta_t - \eta_t^2} \cdot (1 - x^{t-1}_i) \quad (1 - u \geq e^{-u - u^2} \text{ for } 0 \leq u < 1/2)
\]

Using this recursion, we have,
\[
1 - x^t_i \geq e^{-\sum_{t'=2}^t \eta_{t'}} \cdot (1 - x^1_i) \geq e^{-\delta(1+O(1/\ln^2 T))} \cdot (1 - x^1_i)
\]

since \( \sum_{t'=2}^t \eta_{t'}^2 = \sum_{t'=2}^t \frac{\delta^2}{t'H_T} = O(1/\ln^2 T) \).
The following lemma was first observed in [15] and was generalized in [9, Lemma 7] and [4, Lemma 3].

**Lemma 4 ([15, 9, 4]).** For every \( x, y \in K \), it holds that \( F(x \lor y) \geq (1 - \|x\|_\infty)F(y) \).

**Theorem 1.** Let \( K \subseteq [0, 1]^n \) be a convex set and let \( F : K \to \mathbb{R} \) is a non-monotone \( \beta \)-smooth DR-submodular function. Let \( D \) be the diameter of \( K \). Then Algorithm 1 yields a solution \( x^T \in K \) such that the following inequality holds:

\[
F(x^T) \geq \left( \frac{1}{3\sqrt{3}} \right) (1 - \min_{x \in K} \|x\|_\infty) \cdot \max_{x \in K} F(x^*) - O \left( \frac{\beta D^2}{\ln^2 T} \right).
\]

**Proof.** Let \( x^* \in K \) be the maximum solution of \( F \). Let \( r = e^{-\delta(1 + O(1/\ln^2 T))} \cdot (1 - \max_i x_i^*) \). Note that from Lemma 3 it follows that \( (1 - \|x^t\|_\infty) \geq r \) for every \( t \). Next we present a recursive formula in terms of \( F(x^t) \) and \( F(x) \):

\[
2F(x^{t+1}) - rF(x^*)
= 2F((1 - \eta_{t+1})x^t + \eta_{t+1}v^{t+1}) - rF(x^*) \quad \text{(Using the Update step from Algorithm 1)}
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\langle \nabla F((1 - \eta_{t+1})x^t + \eta_{t+1}v^{t+1}), (v^{t+1} - x^t) \rangle
- \beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(\( \beta \)-smoothness as defined in Inequality 5)}
\]

\[
= 2F(x^t) - rF(x^*) + 2\eta_{t+1}\langle \nabla F(x^t), (v^{t+1} - x^t) \rangle
+ 2\eta_{t+1}\langle \nabla F((1 - \eta_{t+1})x^t + \eta_{t+1}v^{t+1}) - \nabla F(x^t), (v^{t+1} - x^t) \rangle
- \beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\langle \nabla F(x^t), (v^{t+1} - x^t) \rangle
- 2\eta_{t+1}\|\nabla F((1 - \eta_{t+1})x^t + \eta_{t+1}v^{t+1}) - \nabla F(x^t)\|\|v^{t+1} - x^t\|
- \beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(Cauchy-Schwarz)}
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\langle \nabla F(x^t), (v^{t+1} - x^t) \rangle - 3\beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(\( \beta \)-smoothness as defined in Inequality 5)}
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\langle \nabla F(x^t), x^* - x^t \rangle - 3\beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(definition of \( v^{t+1} \))}
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\left(F(x^t \lor x^t) - 2F(x^t)\right) - 3\beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(Lemma 1)}
\]

\[
\geq 2F(x^t) - rF(x^*) + 2\eta_{t+1}\left(rF(x^*) - 2F(x^t)\right) - 3\beta(\eta_{t+1})^2\|v^{t+1} - x^t\|^2
\quad \text{(Lemma 3)}
\]

\[
\geq (1 - 2\eta_{t+1}) \left(2F(x^t) - rF(x^*)\right) - 3\beta(\eta_{t+1})^2D^2,
\]

where \( D \) is the diameter of \( K \).

Let \( h^t = 2F(x^t) - rF(x^*) \). By the previous inequality and the choice of \( \eta_t \), we have

\[
h^{t+1} \geq (1 - 2\eta_{t+1}) h^t - 3\beta(\eta_{t})^2D^2 = (1 - 2\eta_{t+1}) h^t - O \left( \frac{\beta^2D^2}{t^2 \ln^2 T} \right).
\]
where we used the facts that \( H_T = O(\ln T) \). Therefore,
\[
\begin{align*}
    h^T &\geq \prod_{t=2}^{T} (1 - 2\eta_t) h^1 - O\left( \frac{\beta \delta^2 D^2}{\ln^2 T} \right) \sum_{t=1}^{T} \frac{1}{T^2} \\
    &\geq e^{-2 \sum_{t=2}^{T} \eta_t} \eta_2 \cdot h^1 - O\left( \frac{\beta \delta^2 D^2}{t^2 \ln^2 T} \right) \\
    &= e^{-2\delta(1+O(1/\ln^2 T))} \cdot h^1 - O\left( \frac{\beta \delta^2 D^2}{\ln^2 T} \right) \\
&\quad \text{since } (1 - u) \geq e^{-u-u^2} \text{ for } 0 \leq u < 1/2
\end{align*}
\]
Hence,
\[
2F(x^T) - rF(x^*) \geq e^{-2\delta(1+O(1/\ln^2 T))} \left( 2F(x^1) - rF(x^*) \right) - O\left( \frac{\beta \delta^2 D^2}{\ln^2 T} \right)
\]
which implies,
\[
F(x^T) \geq \frac{r}{2} \left( 1 - e^{-2\delta(1+O(1/\ln^2 T))} \right) F(x^*) - \frac{6\beta \delta^2 D^2}{\ln^2 T} \\
= \frac{e^{-\delta(1+O(1/\ln^2 T))} \cdot \left( 1 - e^{-2\delta(1+O(1/\ln^2 T))} \right)}{2} (1 - \max_i x^1_i) F(x^*) - O\left( \frac{\beta \delta^2 D^2}{\ln^2 T} \right).
\]
Note that for \( T \) sufficiently large, \( O(1/\ln^2 T) \ll 1 \). By the choice \( \delta = \left( \frac{\ln^3}{2} \right) \), we get
\[
F(x^T) \geq \left( \frac{1}{3\sqrt{3}} \right) \left( 1 - \max_i x^1_i \right) F(x^*) - O\left( \frac{\beta D^2}{\ln^2 T} \right)
\]
and the theorem follows.

**Corollary 1.** If \( 0 \in K \), then the guarantee in Theorem 1 can be written as:
\[
F(x^T) \geq \left( \frac{1}{3\sqrt{3}} \right) \max_{x^* \in K} F(x^*) - O\left( \frac{\beta n}{\ln^2 T} \right).
\]
where the starting point \( x^1 = 0 \) and the diameter \( D \leq \sqrt{n} \).

Note that inclusion of \( 0 \) in \( K \) does not necessarily imply that \( K \) is a down-closed polytope.

### 4 Online Continuous DR-Submodular Maximization

We consider the DR-submodular maximization problem over down-closed convex sets in the online setting. It has been observed that Stochastic Gradient Ascent performs well in practice for DR-submodular maximization (e.g., see Section 5). In this section, we establish a provable guarantee of the Gradient Ascent method by exploring useful properties of DR-submodularity. The result can be seen as a theoretical evidence of the performance of the method. Note that the algorithm requires only the stochastic gradient. Below, we present the pseudocode of our variant of the Stochastic Gradient Ascent algorithm.

**Theorem 2.** Let \( K \subseteq [0, 1]^n \) be a down-closed convex body and assume that \( F^t : K \to \mathbb{R} \) are DR-submodular functions for \( t = 1, 2, 3, \ldots, T \). Let \( D \) be the diameter of the convex set \( K \) and and \( G = \sup_{1 \leq t \leq T} \| g^t \| \). Then for \( \eta_t = \frac{D}{\sqrt{T}} \), we have
\[
\frac{1}{T} \cdot \frac{1}{4} \cdot \sum_{t=1}^{T} F^t(x^*) - \frac{1}{T} \cdot \sum_{t=1}^{T} \mathbb{E} [F^t(x^t)] \leq O\left( \frac{DG}{\sqrt{T}} \right).
\]
The last inequality is due to Lemma 1; and rearranging the last inequality and note that Algorithm 2
end for

Proof: Let z be some arbitrary point in $\mathcal{K}$. Then for every $t$ we have that

$$\|x^{t+1} - z\|^2 = \|\text{Proj}_K(x^t + \eta_t g^t) - z\|^2$$

(Using the Update step in Algorithm 2)

$$\leq \|x^t + \eta_t g^t - z\|^2$$

(by Pythagorean inequality)

$$= \|x^t - z\|^2 + \eta_t^2 \|g^t\|^2 - 2\eta_t \langle g^t, z - x^t \rangle.$$  

Rearranging the last inequality and note that $\|g^t\|^2 \leq G^2$, we have

$$\langle g^t, z - x^t \rangle \leq \|x^t - z\|^2 - \|x^{t+1} - z\|^2 + \eta_t^2 G^2$$

$$\frac{2\eta_t}{2}$$

Define $x^t := (x^* \lor x^t) - x^t$ for every $t$. As $\mathcal{K}$ is down-closed, $z^t \in \mathcal{K}$. We have

$$\frac{1}{4} \sum_{t=1}^T F_t(x^t) - \sum_{t=1}^T E[F_t(x^t)]$$

$$\leq \frac{1}{4} \sum_{t=1}^T \left[ F_t(x^* \lor x^t) + F_t(x^* \land x^t) + F_t(z^t \lor x^t) + F_t(z^t \land x^t) - 4 \sum_{t=1}^T F_t(x^t) \right]$$

(Using Lemma 2)

$$= \frac{1}{4} \sum_{t=1}^T \left[ F_t(x^* \lor x^t) + F_t(x^* \land x^t) + F_t(z^t \lor x^t) + F_t(z^t \land x^t) - 4F_t(x^t) \right]$$

$$= \frac{1}{4} \sum_{t=1}^T \left[ F_t(x^* + x^t) + F_t(x^* \land x^t) - 2F_t(x^t) + F_t(z^t \lor x^t) + F_t(z^t \land x^t) - 2F_t(x^t) \right]$$

$$\leq \frac{1}{4} \sum_{t=1}^T \left[ \langle g^t, z^t - x^t \rangle \right].$$

The last inequality is due to Lemma 1 and $\mathbb{E}[g^t|x^t] = \nabla F_t(x^t)$; and linearity of expectation. We bound the first term in (8). By Inequality (4), we have

$$\sum_{t=1}^T \langle g^t, x^t - x^* \rangle \leq \sum_{t=1}^T \|x^t - x^*\|^2 - \|x^{t+1} - x^*\|^2 + \eta_t^2 G^2$$

$$\leq \sum_{t=2}^T \|x^t - x^*\|^2 \left( \frac{1}{2\eta_t} - \frac{1}{2\eta_{t-1}} \right) + \frac{1}{2\eta_1} \|x^1 - x^*\|^2 + \sum_{t=1}^{T-1} \eta_t^2 G^2$$

$$\leq \|x^t - x^*\|^2 \left( \frac{G}{D} \right) + \sum_{t=2}^T \left( \frac{\sqrt{t} - \sqrt{t-1}}{2} \right) + \frac{1}{2\eta_1} \|x^1 - x^*\|^2 + \sum_{t=2}^T \eta_t^2 G^2$$

(replacing $\eta_t = \frac{D}{G^2}$)

$$\leq DG \sum_{t=2}^T \frac{1}{4\sqrt{t-1}} + DG \frac{1}{2} + DG \sum_{t=2}^T \frac{1}{\sqrt{t}}$$
\[ = O(DG \sqrt{T}). \] (9)

We are now bounding the second term in (8). Observe that
\[
\|z^t - z^{t-1}\|^2 = \| (x^* \lor x^t) - x^t - (x^* \lor x^{t-1}) + x^{t-1}\|^2 \\
= \| (x^* \lor x^t) - (x^* \lor x^{t-1}) - (x^t - x^{t-1})\|^2 \\
\leq 2 \|x^t - x^{t-1}\|^2 \leq 2\eta_{t-1}^2 \|g^{t-1}\|^2 \leq 2\eta_{t-1}^2 G^2, \tag{10}
\]
where the last inequality follows from the algorithm and the Cauchy-Schwarz inequality. Now we have
\[
\sum_{t=1}^{T} (g^t, z^t - x^t) \leq \sum_{t=1}^{T} \|x^t - z^t\|^2 - \|x^{t+1} - z^t\|^2 + \frac{\eta_t^2 G^2}{2\eta_t} \\
\leq \sum_{t=1}^{T} \|x^t - z^t - x^{t-1}\|^2 + \|z^t - z^{t-1}\|^2 - \|x^{t+1} - z^t\|^2 + \eta_t^2 G^2 \\
\leq \sum_{t=2}^{T} \left(\frac{1}{2\eta_t} - \frac{1}{2\eta_{t-1}}\right) \|x^t - z^{t-1}\|^2 + \frac{\|x^1 - z^1\|^2}{2\eta_1} + \sum_{t=1}^{T} \|z^t - z^{t-1}\|^2 + \eta_t \sum_{t=2}^{T} \eta_t G^2 \\
\leq \sum_{t=2}^{T} \left(\frac{1}{2\eta_t} - \frac{1}{2\eta_{t-1}}\right) D^2 + \frac{1}{2\eta_t} D^2 + \sum_{t=2}^{T} \left(\frac{\eta_{t-1}}{\eta_t} + \frac{\eta_t}{2}\right) G^2 \\
\leq DG \sum_{t=2}^{T} \left(\frac{\sqrt{t}}{2} - \frac{\sqrt{t-1}}{2}\right) + DG \frac{D}{2} + 3 \sum_{t=2}^{T} \eta_t G^2 \\
\leq DG \frac{\sqrt{T}}{2} + DG \frac{D}{2} + 3DG \sum_{t=2}^{T} \frac{1}{\sqrt{t}} \\
= O(DG \sqrt{T}). \tag{11}
\]
The first inequality follows from Inequality (7). The fourth inequality is due to Inequality (10). The fifth inequality holds since \(\eta_{t-1}^2 \leq 2\eta_t^2\). The theorem follows from the Inequalities (8), (9) and (11).

5 Experiments

In this section, we validate offline and online algorithms for non-monotone DR submodular optimization on both, the real-world and the synthetic datasets. Our experiments are broadly classified into following three categories:

1. We compare our offline algorithm (from Section 3) against two previous known algorithms for maximizing non-monotone DR-submodular function over down-closed polytopes mentioned in 4. Recall that our algorithm applies to a more general setting where the convex set is not required to be down-closed.

2. Next, we show the performance of our offline algorithm (from Section 3) for maximizing non-monotone DR-submodular function over general polytopes. Recall that no previous algorithm was known to have performance guarantees for this problem.

3. Finally, we show the performance of our online algorithm (from Section 4) for maximizing non-montone DR-submodular function over down-closed polytopes.

All experiments are performed in MATLAB using CPLEX optimization tool on MAC OS version 10.14.
5.1 Offline Algorithm over Down-closed Polytopes

Here, we benchmark the performance of our variant of the Frank-Wolfe algorithm from Section 3 against the previous known two algorithms for maximizing continuous DR submodular function over down-closed polytopes mentioned in [4]. We considered QUADPROGIP, which is a global solver for non-convex quadratic programming, as a baseline. We run all the algorithms for 100 iterations. All the results are the average of 20 repeated experiments. For the sake of completion, we describe below the problem and different settings used. We follow closely the experimental settings from [4], and adapted their source codes to our algorithms.

5.1.1 Quadratic Programming

As a state-of-the-art global solver, we used QUADPROGIP to find the global optimum which is used to calculate the approximation ratios. Our problem instances are synthetic quadratic objectives with down-closed polytope constraints, i.e.,

\[ f(x) = \frac{1}{2} x^\top H x + h^\top x + c \]

and

\[ \mathcal{K} = \{ x \in \mathbb{R}_+^n | Ax \leq b, x \leq u, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \} \]

Note that in previous sections, we have assumed w.l.o.g that \( \mathcal{K} \subseteq [0, 1]^n \). By scaling our results hold as well for the general box constraint \( x \leq u \), provided the entries of \( u \) are upper bounded by a constant.

Both objective and constraints were randomly generated, using the following two ways:

**Uniform Distribution**: \( H \in \mathbb{R}^{n \times n} \) is a symmetric matrix with uniformly distributed entries in \([-1, 0] \); \( A \in \mathbb{R}^{m \times n} \) has uniformly distributed entries in \([\mu, \mu + 1] \), where \( \mu = 0.01 \) is a small positive constant in order to make entries of \( A \) strictly positive for down-closed polytope.

**Exponential Distribution**: Here, the entries of \( -H \) and \( A \) are sampled from exponential distributions \( Exp(\lambda) \) where given a random variable \( y \geq 0 \), the probability density function of \( y \) is defined by \( \lambda e^{-\lambda y} \), and for \( y < 0 \), its density is fixed to be 0. Specifically, each entry of \( H \) is sampled from \( Exp(1) \), then the matrix \( -H \) is made to be symmetric. Each entry of \( A \) is sampled from \( Exp(0.25) + \mu \), where \( \mu = 0.01 \) is a small positive constant.

We set \( b = 1^m \), and \( u \) to be the tightest upper bound of \( \mathcal{K} \) by \( u_j = \min_{i \in [m]} \frac{b_i}{A_{ij}}, \forall j \in [n] \). In order to make \( f \) non-monotone, we set \( h = -0.2 * H^\top u \). To make sure that \( f \) is non-negative, we first of all solve the problem \( \min_{x \in \mathbb{R}} \frac{1}{2} x^\top H x + h^\top x \) using QUADPROGIP. Let the solution to be \( \tilde{x} \), then we set \( c = -f(\tilde{x}) + 0.1 * |f(\tilde{x})| \).

The approximation ratios w.r.t. dimensionalities (n) are plotted in Figures 1 and 2 for the two distributions. In each figure, we set the number of constraints to be \( m = \lfloor 0.5n \rfloor, m = n \) and \( m = \lfloor 1.5n \rfloor \), respectively.

We can observe that our version of Frank-Wolfe (denoted our-frank-wolfe) and gradient ascent algorithm (denoted by proj-gradient) have comparable performance with the state-of-the-art algorithms when optimizing submodular functions over down-closed convex sets. Note that the performance is clearly consistent with the proven approximation guarantee of \( 1/e \) shown in [4]. We also show that the performance of our algorithms are consistent with the proven approximation guarantee of \( 1/3\sqrt{3} \) for down-closed convex sets.
5.1.2 Maximizing Softmax Extentions

Determinantal point processes (DPPs) are probabilistic models of repulsion, that have been used to model diversity in machine learning [25]. The constrained MAP (maximum a posteriori) inference problem of a DPP is an NP-hard combinatorial problem. One of the current methods with the best known approximation guarantee is based on the softmax extension [17], which is a DR-submodular function. Let $L$ be the positive semidefinite kernel matrix of a DPP, its softmax extension is:

$$f(x) = \log \det(\text{diag}(x)(L-I)+I), \; x \in [0,1]^n,$$

where $I$ is the identity matrix, $\text{diag}(x)$ is the diagonal matrix with diagonal elements set as $x$. The problem of MAP inference in DPPs corresponds to the problem of maximizing $f$ over a convex polytope $K$.

![Figure 2: Exponential distribution with down-closed polytope and (a) $m = \lfloor 0.5n \rfloor$, (b) $m = n$, (c) $m = \lfloor 1.5n \rfloor$](image)

Figure 3: Softmax extension in uniform distribution with down-closed polytope and (a) $m = \lfloor 0.5n \rfloor$, (b) $m = n$, (c) $m = \lfloor 1.5n \rfloor$.

![Figure 4: Softmax extension in exponential distribution with down-closed polytope and (a) $m = \lfloor 0.5n \rfloor$ (b) $m = n$](image)

We generate the softmax objectives in the following way: first generate the $n$ eigenvalues $d \in \mathbb{R}^n_+$, each randomly distributed in $[0,1.5]$, and set $D = \text{diag}(d)$. After generating a random unitary matrix $U$, we set $L = UDU^\top$. One can verify that $L$ is positive semidefinite and has eigenvalues as the entries of $d$. We generate polytope constraints.
in the same form and same way as that for DR submodular quadratic and exponential functions, except for setting $b = 2 \cdot 1^m$. Function values returned by different solvers w.r.t. $n$ are shown in Figures 3 and 4.

We can observe that our version of Frank-Wolfe performs at least as good as the two-phase Frank-Wolfe algorithm mentioned in [4] for the down-closed polytope are generated using uniform distribution. In case of exponential distributions, our algorithms have comparable performance with the state-of-the-art algorithms.

5.1.3 Offline Algorithm over General Polytopes

Here, we consider the problem of revenue maximization on a (undirected) social network graph $G = (V, W)$, where $w_{ij} \in W$ represents the weight of the edge between vertex $i$ and vertex $j$. The goal is to offer for free or advertise a product to users so that the revenue increases through their “word-of-mouth” effect on others. If one invests $x$ unit of cost on a user $i \in V$, the user $i$ becomes an advocate of the product (independently from other users) with probability $1 - (1 - p)^x$ where $p \in (0, 1)$ is a parameter. Intuitively, it signifies that for investing a unit cost to $i$, we have an extra chance that the user $i$ becomes an advocate with probability $p$.

Let $S \subset V$ be a set of users who advocate for the product. Note that $S$ is random set. Then the revenue with respect to $S$ is defined as $\sum_{i \in S} \sum_{j \in V \setminus S} w_{ij}$. Let $f: \mathbb{Z}_+^E \rightarrow \mathbb{R}$ be the expected revenue obtained in this model, that is

$$f(x) = \mathbb{E}_S \left[ \sum_{i \in S} \sum_{j \in V \setminus S} w_{ij} \right] = \sum_i \sum_{j \neq i} w_{ij} (1 - (1 - p)x_i) (1 - p)x_j$$

It has been shown that $f$ is a non-monotone DR-submodular function [33]. In our experiments, we used the Advogato network with 6.5K users (vertices) and 61K weighted relationship (edges). We set $p = 0.0001$. We imposed a minimum and a maximum investment constraint on the problem such that $0.25 \leq \sum_i x_i \leq 1$. This, in addition with $x_i \geq 0$ constitutes a general feasible polytope.

In Figure 5(a), we show the performance of the Frank-Wolfe algorithm (as mentioned in Section 3) and commonly used Gradient Ascent algorithm. It is imperative to note that no performance guarantee is known for the Gradient Ascent algorithm for maximizing a non-monotone DR-submodular function over a general constraint polytope. We can clearly observe that the Frank-Wolfe algorithm performs at least as good as the commonly used Gradient Ascent algorithm.

5.1.4 Online Algorithm over Down-closed Polytopes

In this subsection, we consider the online variant of the revenue maximization on a (undirected) social network where at time $t$ the weight of an edge is given $w_{ij}^t \in \{0, 1\}$. The experiments are performed on the Facebook dataset that
contains 64K users (vertices) and 1M relationships (edges). We choose the number of time steps to be $T = 50,000$. At each time $t \in 1, \ldots, T$, we randomly uniformly select 2000 vertices $V^t \subset V$, independently of $V^1, \ldots, V^{t-1}$, and construct a batch $B_t$ with edge-weights $w^t_{ij} = 1$ if and only if $i, j \in V^t$ and edge $(i, j)$ exists in the Facebook dataset. In case if $i$ or $j$ do not belong to $V^t$, $w^t_{ij} = 0$.

We again set $p = 0.0001$ and impose a maximum investment constraint on the problem such that $\sum_{i \in V^t} x^t_i \leq 1$. This, in addition to $x_i \geq 0, \forall i \in V$ constitutes a down-closed feasible polytope.

For comparison purposes, we chose the (offline) Frank-Wolfe algorithm that is shown to be $\frac{1}{e}$-approximation for maximizing non-monotone DR-submodular function over down-closed polytopes \cite{3}. Using this algorithm, we first computed $x^*$ such that $x^*$ approximately maximizes $\sum_t F^t(x)$ and then computed $\sum_{t \leq t'} F^t(x^*)$ for every $t' \in 1, \ldots, T$. In Figure 5(b), we show how the function $\sum_{t \leq t'} F^t(x)$ evolves with time $t'$ for the Online Gradient Ascent algorithm (as mentioned in Section 4) in comparison to $\sum_{t \leq t'} F^t(x^*)$. In Figure 5(c), we show the ratio of between the objective value achieved by the Online Ascent algorithm and $\sum_{t \leq t'} F^t(x^*)$. The gradual reduction in this ratio (over time) conforms with the fact that the additive term in the theoretical guarantee reduces with time.

**Projection on $K$.** The typical implementation of the projection operator $\text{Proj}^\eta_x(y)$ would consist of solving the quadratic program, where we want to minimize $\|x - y\|$ under the constraint $y \in K$. However using the particular structure of our convex space $K$ a more efficient projection operator is possible. Recall that $K$ is defined as the set of all points $y$ with non negative coordinates and $\sum_{i=1}^n y_i \leq 1$. Without loss of generality the entries of $x$ are sorted in non increasing order $x_1 \geq \ldots \geq x_n$. If $x \not\in K$, then its projection is a vector $y$ of the form

$$y = (x_1 - \delta_j, \ldots, x_j - \delta_j, 0, \ldots, 0),$$

for some index $j$ and $\delta_j = (x_1 + \ldots + x_j - 1)/j$. In the degenerate case when all entries of $x$ are non positive, the projection is the vector 0. The distance between $x$ and its projection $y$ is then

$$\|x - y\|^2 = \delta^2 \cdot j + x^2_{j+1} + \ldots + x^2_n.$$ 

By optimality of the projection, we have that $j$ is the maximal index satisfying $\delta_j \geq x_j$. Hence the projection can be computed in time $O(n \log n)$, by first sorting the entries of $x$ in time $O(n \log n)$ and then by iterating over $j$, maintaining in constant time the sum $x_1 + \ldots + x_j$. However we observed better performance of another projection algorithm with complexity $O(n^2)$ which exploits better the possibilities of MATLAB. It consists of an iterative procedure. While $x \not\in K$ we project $x$ to the positive sub-space (i.e. set all its negative entries to 0) and then remove $\delta$ from all non zero entries, where $\delta$ is the average of all non zero entries in $x$. While this procedure can iterate $n$ times in the worst case, in practice it often iterates only a constant number of times.

6 Conclusion

In this paper, we have provided performance guarantees for the problems of maximizing non-monotone submodular/DR-submodular functions over convex sets in offline and online environments. These results are completed by experiments in different contexts. Moreover, the results give raise to the question of designing online algorithms for non-monotone DR-submodular maximization over a general convex set. Characterizing necessary and sufficient regularity conditions/structures that enable efficient algorithm with approximation and regret guarantees is an interesting direction to pursue.

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A Properties of DR-submodularity

We provide the proofs of the properties of DR-submodular functions mentioned in Section 2.

Lemma 1 ([21]). For every $x, y \in K$ and any DR-submodular function $F : [0, 1]^n \to \mathbb{R}^+$, it holds that

$$\langle \nabla F(x), y - x \rangle \geq F(x \lor y) + F(x \land y) - 2F(x).$$

Proof. For any vectors $x \leq z$, using Inequality (4), we have

$$F(z) - F(x) = \int_0^1 \langle z - x, \nabla F(x + t(z - x)) \rangle dt$$

$$\leq \int_0^1 \langle z - x, \nabla F(x) \rangle dt = \langle z - x, \nabla F(x) \rangle.$$

Therefore,

$$F(x \lor y) - F(x) \leq \langle x \lor y - x, \nabla F(x) \rangle. \tag{12}$$

Similarly for vectors $x \leq z$, we have

$$F(z) - F(x) = \int_0^1 \langle z - x, \nabla F(x + t(z - x)) \rangle dt$$

$$\geq \int_0^1 \langle z - x, \nabla F(z) \rangle dt = \langle z - x, \nabla F(z) \rangle.$$

Therefore,

$$F(x \land y) - F(x) \leq \langle x \land y - x, \nabla F(x) \rangle \tag{13}$$

Summing (12) and (13) and using the fact $(x \lor y) + (x \land y) = x + y$, we obtain

$$F(x \lor y) + F(x \land y) - 2F(x) \leq \langle y - x, \nabla F(x) \rangle.$$

□

Lemma 2 ([4]). For any DR-submodular function $F$ and for all $x, y, z \in K$ it holds that

$$F(x \lor y) + F(x \land y) + F(z^* \lor z) + F(z^* \land z) \geq F(y),$$

where $z^* = (x \lor y) - x$.

Proof. First, we claim the following two inequalities:

$$F(x \lor y) + F(z \lor z^*) \geq F(z^*) + F((x + z) \lor y) \tag{14}$$

$$F(z^*) + F(x \land y) \geq F(y) + F(0). \tag{15}$$

Assuming (14) and (15) holds, we get:

$$F(x \lor y) + F(z \lor z^*) + F(x \land y) + F(z \lor z^*) \geq F(y) + F(0) + F((x + z) \lor y) + F(z \lor z^*) \geq F(y)$$

and the lemma follows. In the remaining, we prove the above two inequalities.

First, we establish the following identity.

$$x \lor y - z^* = (x + z) \lor y - z \lor z^*$$  \tag{16}

For this purpose, we will show that both the RHS and LHS of (16) are equal to $x$. For the LHS we can write $x \lor y - z^* = x \lor y - (x \land y - x) = x$. For the RHS, let us consider any coordinate $i \in [n]$, and show that the following expression equals $x_i$:

$$(x_i + z_i) \lor y_i - z_i \lor z_i^* = (x_i + z_i) \lor y_i - ((x_i + z_i) - x_i) \lor ((x_i \lor y_i) - x_i).$$
Case $(x_i + z_i) \geq y_i$. So $(x_i + z_i)$ is larger than both $y_i$ and $x_i$. Therefore,

$$(x_i + z_i) \vee y_i - ((x_i + z_i) - x_i) \vee ((x_i \vee y_i) - x_i) = (x_i + z_i) - ((x_i + z_i) - x_i) = x_i.$$ 

Case $(x_i + z_i) < y_i$. So $(x_i \vee y_i) = y_i$. Therefore,

$$(x_i + z_i) \vee y_i - ((x_i + z_i) - x_i) \vee ((x_i \vee y_i) - x_i) = y_i - ((x_i \vee y_i) - x_i) = x_i.$$ 

Hence, the RHS of (16) is equal to $x$. So the identity (16) holds.

We are now proving Inequality (14), i.e.,

$$F(x \vee y) - F(z^*) \geq F((x + z) \vee y) - F(z \vee z^*).$$

The above inequality holds due to (16), the fact $z^* \leq z \vee z^*$ and the diminishing return property of $F$.

Now we prove Inequality (15), i.e.,

$$F(z^*) - F(0) \geq F(y) - F(x \wedge y).$$

The above inequality holds by the diminishing return property and

$$y - x \wedge y = x \vee y - x = z^* - 0 \quad \text{and} \quad 0 \leq x \wedge y.$$ 

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