Finding The Sign Of A Function Value By Binary Cellular Automaton

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Given a continuous function \( f(x) \), suppose that the sign of \( f \) only has finitely many discontinuous points in the interval \([0, 1]\). We show how to use a sequence of one dimensional deterministic binary cellular automata to determine the sign of \( f(\rho) \) where \( \rho \) is the (number) density of 1s in an arbitrarily given bit string of finite length provided that \( f \) satisfies certain technical conditions.

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I. INTRODUCTION

Cellular Automaton (CA) is a simple local interaction model of natural systems used extensively in various fields of physics [1]. Besides, CA can be regarded as a computation model motivated by biological phenomenon. Various inequivalent definitions of CA exist in the community. In this paper, we restrict ourselves to consider the time evolution of CA to be governed by a local synchronous deterministic uniform rule. In other words, the state of each site in the next time step depends deterministically only on the states of its finite neighborhood, the states of all sites are updated in parallel, and the transformation CA rule table is covariant under translation of the background lattice. In this respect, CA can be regarded as a decentralized deterministic parallel computation model without central memory storage. It is useful to study its power and limitation. In fact, a recent renewal interest in CA computing [2] makes such an investigation timely.

Since CA rules are local in nature, it is instructive to see if CA can perform tasks that involve global quantities. Perhaps the most well-known example is the so-called density classification problem. Considering a bit string of finite length \( N \) with periodic boundary conditions, the problem is to change all bits in the string to 0 if the number of 0s is greater than the number of 1s in the input bit string, and to change all bits to 1 if the number of 1s is greater than the number of 0s. Clearly, the global quantity involved in this problem is the density of 1s which is defined as the number of 1s in the string divided by the string length \( N \).

Various CA rules have been proposed to solve the density classification problem both by human [3] and by genetic algorithm [4]. But they only provide approximate solutions. In other words, these rules work for most but not all randomly chosen initial configurations. In fact, Land and Belew showed that density classification cannot be performed perfectly using a single one dimensional CA rule [5]. Later on, Capcarrere and his collaborators proved that a single CA rule can solve the density classification and other related problems exactly if we modify either the required output of the automaton or the boundary conditions [6]. However, in their approach, it has to scan through the states of all sites in the final configuration, in general, before knowing the answer. This requires global memories in the read out process. In contrast, read out in the original density classification problem can clearly be done by looking at the states of a few local sites, and hence can be done with an additional finite rule table whose size is independent of \( N \).

Recently, F ukš pointed out that the density classification problem can be solved exactly if we apply two CA rules in succession [7]. More precisely, he showed that by applying a CA rule a fixed number of times depending only on the lattice size and then followed by applying another CA rule a fixed number of times depending again only on the lattice size, the density classification problem can be solved exactly. F ukš further asked if it is possible to classify a rational density. That is, he questioned if it is possible to determine, using succession of CA rules, whether the density of 1s in an arbitrary one dimensional array with periodic boundary conditions is less than, equal to or greater than a prescribed rational number \( p/q \), called the critical density. Chau et al. answered his question affirmatively by showing that two CA rules are necessary and sufficient in solving the rational density classification problem exactly [8]. His group further generalized their algorithm to find the majority state of an one dimensional array when each site may only have finitely many discontinuous points [9]. Their group has also been working on the rational density classification problem with periodic boundary conditions as well as the generalization of the rational density classification problem [10]. In Section II, we shall spell out the details of the problem that we are interested in. Then after introducing some technical results in Section III, we shall solve the problem using a set of binary CAs with local synchronous deterministic uniform rules in Section IV. The general solution reported in Section IV can be simplified in a

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II. STATEMENT OF THE PROBLEM

In this paper, we propose another generalization of the density classification problem, called the generalized density classification problem (GDCP) and solve it using a sequence of CA rules. Before we discuss the significance of our generalization, let us first state our problem precisely.

Suppose we are given an input bit string \( \alpha \) of length \( N \) in periodic boundary conditions, as well as a continuous function \( f(x) \) with finitely many discontinuous points of \( \text{sgn}(f(x)) \) in the interval \([0, 1]\), where

\[
\text{sgn}(y) = \begin{cases} 
1 & \text{if } y > 0, \\
-1 & \text{if } y < 0, \\
0 & \text{if } y = 0.
\end{cases}
\]

Moreover, those discontinuous points are rational and shall be denoted by \( p_i/q \) where \( p_i, q \in \mathbb{Z}^+ \cup \{0\} \), and \( i \) is the index for the discontinuous points. In addition, we have the freedom to choose the smallest positive \( q \) for the denominators of all the expressions \( p_i/q \). With this choice, \( p_i \) and \( q \) need not be relatively prime. Finally, we require that \( f(\frac{1}{2q}) \) and \( f(\frac{2q-1}{2q}) \) are non-zero.

We define the density of 1s of the bit string \( \alpha \) as the total number of 1s in the bit string divided by the string length \( N \) and is denoted by the symbol \( \rho(\alpha) \). We may simply write \( \rho \) instead of \( \rho(\alpha) \) when it is clear from the content which bit string we are referring to.

The task of GDCP is to evolve the bit string \( \alpha \) to all 1 (or 0) if \( \text{sgn}(f(\rho(\alpha))) = 1 \) (or \( \text{sgn}(f(\rho(\alpha))) = -1 \)). Besides, we have to preserve the density of 1s of the bit string if \( \text{sgn}(f(\rho(\alpha))) = 0 \). In addition, by inspecting a fixed portion of the final output string, we are able to distinguish between the above three cases with certainty. (We shall consider a slightly relaxed variant to this definition later on in Section III.) To solve the GDCP by a sequence of binary CA rules, we mean that every initial configuration can be correctly classified by applying some binary CA rule \( R_1 \) \( t_1 \) times followed by another binary CA rule \( R_2 \) \( t_2 \) times and so on up to the \( k \)th binary CA rule \( R_k \) \( t_k \) times, where \( k \) is fixed and independent of the string length \( N \). Besides, each CA rule \( R_i \) must be independent of the input \( \alpha \) and has a bounded rule table size that is also independent of \( N \). Finally, the number of iterations \( t_i \) is independent of the initial configuration, but may depend on \( N \).

Clearly, the GDCP reduced to the rational density classification problem by choosing \( f(x) = x - \rho_c \) where \( \rho_c \) is a rational number. Moreover, the GDCP can be solved trivially if we have a global counter.

In addition to naturally generalize the density classification problem, GDCP is interesting on its own right. First, by choosing

\[
f(x) = \begin{cases} 
x - \rho_c & \text{if } x < \rho_{c1}, \\
0 & \text{if } \rho_{c1} \leq x \leq \rho_{c2}, \\
x - \rho_c & \text{if } x > \rho_{c2},
\end{cases}
\]

where \( \rho_{c1} < \rho_{c2} \) both being rational, it determines if the density of 1s is in the interval \((\rho_{c1}, \rho_{c2})\) from a single copy of input bit string. (Note that it is more efficient to ask if \( \rho \) is between 1/10 and 4/39 rather than to ask if \( \rho \) equals 0.10215 partly because of our result in Section IV that the former question requires smaller CA rule tables.) Thus, GDCP can be used to determine the coarse-grained density of 1s efficiently, while in the rational density classification problem, the resolution is infinite in the sense that we can only determine whether \( \rho > \rho_c \), \( \rho = \rho_c \) or \( \rho < \rho_c \), no matter how closed \( \rho \) is to \( \rho_c \). This reason alone is good enough to investigate the GDCP. Second, the GDCP has a more interesting flow diagram. While the rational density classification automata has two stable and one unstable fixed points in the density of 1s, the GDCP automata in general has at most two stable fixed points and one unstable fixed points in the density of 1s, \( \rho \). The GDCP automata has two stable fixed points \((\rho = 1 \text{ and } \rho = 0)\), finitely many unstable fixed points \((\text{at where } f(\rho) = 0 \text{ but } f \text{ is not identical zero locally near } \rho)\) and infinitely many neutral fixed points \((\text{at where } f(\rho) = 0 \text{ and } f \text{ is identical zero locally near } \rho)\) in the limit of \( N \to \infty \).

III. SOME USEFUL CELLULAR AUTOMATON BUILDING BLOCKS

Let us first report several essential building blocks of our solution to the GDCP.

A. Car Hopping Automaton

The first step in solving the problem is to make the density of 1s uniform so as to make the local density of 1s a good estimation of the global density of 1s in the bit string. To measure the local density, we introduce the concept of \( k \)th order local number. It is just the number of 1s around the site we are interested. The CA rule \( H_k \) will move the 1s to the right according to the local density gradient. If we repeatedly apply \( H_k \) to an arbitrary bit string, eventually the density of 1s of the resulting string will be more or less uniform. Of course we cannot expect that all local numbers are equal, but we can prove that the difference between local numbers of different sites can only be less than or equal to one. This CA rule is the most important CA rule in this paper.

Definition 1 We denote a configuration in our bit string of length \( N \), which can be identified with an one dimensional array of sites in periodic boundary conditions, by \( \alpha \) and the state of the \( i \)th site by \( \alpha(i) \). That is, \( \alpha(i) \) is
the value of the $i$th bit in the string counting from left to right. We define the $k$th order local number of the $i$th site in our one dimensional array by $n_k(i) = \sum_{j=0}^{k-1} \alpha(i+j)$ where the sum $i+j$ in the index here and elsewhere in this paper is understood to be modulo $N$.

In other words, the $k$th order local number is the number of sites with state $i$. This CA rule can be rewritten in another form as follows:

$$\text{if density equivalently, empty sites are driven to move from right to left}$$

We expect that for any bit string $\alpha$, the density of $0$ and $1$ in the above CA rule is the value of the $i$th local number gives the local number of cars near a site. Thus, the density of $1$s of a bit string can also be referred to as the car density.

In what follows, we shall write $n_k(i)$ instead of $n_k(\alpha, i)$ when there is no confusion on which bit string we are referring to.

**Rule 1 (kth order car hopping rule)** Let us denote this CA rule by $H_k$. The state of site $i$ in the next time step $H_k(\alpha)(i)$ is given by:

$$H_k(\alpha)(i) = \begin{cases} 0 & \text{if } \alpha(i) = 1, \alpha(i+1) = 0 \text{ and } n_k(i-k+1) > n_k(i+1), \\ 1 & \text{if } \alpha(i) = 0, \alpha(i-1) = 1 \text{ and } n_k(i-k) > n_k(i), \\ \alpha(i) & \text{otherwise.} \end{cases}$$

This CA rule can be rewritten in another form as follows: we interchange the states in site $i$ and $i+1$ if and only if $\alpha(i) = 1, \alpha(i+1) = 0$ and $n_k(i-k+1) > n_k(i+1)$. All other states remain unchanged. All updates are taken in parallel. In what follows, we shall sometimes refer to such an interchange as hopping of a car.

For example, $H_1(01100101010110) = 010100101010$.

Note that the role of $0$ and $1$ in the above CA rule is symmetric in the sense that it is invariant under the interchange of state $0$ and $1$ plus the relabeling of $i$ as $-i$. Following the spirit of particle physics, we shall refer to this property as the CP symmetry. Furthermore, the car density $\rho$ is conserved under this CA rule. In fact one way to interpret this CA rule is that cars are driven to move from left to right by local car density gradient. Or equivalently, empty sites are driven to move from right to left by local empty site density gradient. Therefore, we expect that the $k$th order local numbers in all recurrent states under the repeated iteration of $H_k$ is evenly distributed. Nonetheless, a rigorous proof turns out to be rather involved partly because a car at site $i$ cannot hop whenever another car is occupying site $i+1$ even in the presence of a car density gradient. We now begin rigorous proof by first introducing a few technical lemmas.

We expect that for any bit string $\alpha$, the density of $\beta = H_k(\alpha)$ will be uniform if $l$ is large enough. The first lemma shows that this is indeed the case when $\beta$ is a fixed point of $H_k$.

**Lemma 1** The $k$th order local number $n_k(i)$ is equal for all $i$ if and only if $H_k(\alpha) = \alpha$ (that is, no car hops under the action of the $k$th order car hopping rule).

**Proof:** If $H_k(\alpha) \neq \alpha$, then some car in the bit string $\alpha$ hops under the action of $H_k$. So clearly $n_k(i) > n_k(i+k)$ for some $i$.

Conversely, if $H_k(\alpha) = \alpha$, then we may assume that $\alpha$ does not equal to all $0$ or all $1$, for otherwise the Lemma is trivially true. Now locally we have any one of the following three situations.

Case (a) $\alpha(i-1) = 1$ and $\alpha(i) = 0$: in this case we clearly require $n_k(i-k) \leq n_k(i)$ in order to prevent the car located at site $i-1$ from hopping.

Case (b) $\alpha(i) = \cdots = \alpha(i+j-1) = 1$ and $\alpha(i+j) = 0$ for some $j > 0$: in this case we again conclude that $n_k(i-k) \leq n_k(i)$ for otherwise we have $n_k(i+j-k) \geq n_k(i+j-k-1) \geq \cdots \geq n_k(i-k) > n_k(i) \geq n_k(i+1) \geq \cdots \geq n_k(i+j)$ contradicting the assumption that the car located at site $i+j-1$ does not move.

Case (c) $\alpha(i) = \cdots = \alpha(i-1) = 1$ and $\alpha(i-j-1) = 1$ for some $j > 0$: the CP symmetry of the CA rule $H_k$ implies that we can use the same trick as in the proof of case (b) to show that $n_k(i-k) \leq n_k(i)$.

In summary, we conclude that $n_k(i-k) \leq n_k(i)$ for all $i$; and since we are working in a finite bit string with periodic boundary, this is possible only when $n_k(i-k) = n_k(i)$ for all $i$.

Suppose that $n_k(i-k+1) > n_k(i-k+2)$ for some $i$, then $\alpha(i-k+1) = 1$ and $\alpha(i-1) = 0$. However, this also implies that $n_k(i+1) > n_k(i+2)$ and hence $\alpha(i-1) = 1$ and $\alpha(i+k+1) = 0$ which is a contradiction. By the same token, it is absurd to have $n_k(i-k+1) < n_k(i-k+2)$. Thus, the only possibility is that $n_k(i) = n_k(i+1)$ for all $i$ and hence $n_k(i) = n_k(j)$ for all $i, j$. \qed

**Lemma 2** If $n_k(\alpha, i-k+1) > n_k(\alpha, i+1)$, then there exists $j$ with $i-k+1 \leq j < i+k$ with $n_k(\alpha, j-k+1) \geq n_k(\alpha, i-k+1)$ such that there is a car located at site $j$ and this car hops by applying the rule $H_k$. Moreover, a car will hop from site $i$ to $i+1$ in no more than $k$ time steps. That is to say, there exists $r \leq k$ such that $H_k^{-r}[\alpha](i) = 1$, $H_k^{-r}[\alpha](i+1) = 0$ and $H_k[\alpha](i) = 0$.

**Proof:** We divide the proof into the following three cases.

Case (a) $\alpha(i) = 1$ and $\alpha(i+1) = 0$: we choose $j = i$ as the car located at site $i$ hops by applying $H_k$.

Case (b) $\alpha(i+1) = 1$: since $n_k(i-k+1) > n_k(i+1)$, we can always find $1 \leq m < k$ such that $\alpha(i+1) = \alpha(i+2) = \cdots = \alpha(i+m) = 1$ and $\alpha(i+m+1) = 0$. Then it is easy to see that $n_k(i+m-k+1) \geq n_k(i-k+1) >
\(n_k(i+1) \geq n_k(i + m + 1)\). So the car located at site \(i + m\) hops to \(i + m + 1\) when we apply \(H_k\). Hence, we choose \(j = i + m\). Inductively, in the next time step, we may choose \(j = i + m - 1\) and so on. Consequently, in no more than \(k\) time steps, a car must hop from site \(i\) to \(i + 1\).

Case (c) \(\alpha(i) = \alpha(i + 1) = 0\): exploiting the CP symmetry of the \(k\)th order car hopping rule, this case can be proved in the same way as in case (b).

We now introduce the fluctuation amplitude, which is the difference of the local numbers in different sites. It will be shown that the maximum fluctuation amplitude will not increase by the repeated application of \(H_k\).

**Definition 2** Let \(\alpha\) be a bit string of length \(N\). We denote \(\Delta_k(\alpha) = \max\{n_k(\alpha, i) - n_k(\alpha, j) : 1 \leq i, j \leq N\}\) the maximum fluctuation amplitude of the \(k\)th order local number. We write the maximum and minimum \(k\)th order local numbers of a bit string by \(n_k^{\max}(\alpha) = \max\{n_k(\alpha, i) : 1 \leq i \leq N\}\) and \(n_k^{\min}(\alpha) = \min\{n_k(\alpha, i) : 1 \leq i \leq N\}\), respectively. In addition, we denote the number of sites with \(k\)th order local number equals \(m\) by \(M_k(\alpha, m) = \{1 \leq i \leq N : n_k(\alpha, i) = m\}\).

For simplicity, we shall drop the label \(\alpha\) in \(\Delta_k, n_k^{\max}, n_k^{\min}\) and \(M_k\) when it is clear which bit string we are referring to in the text.

Now, we are ready to show that repeated application of \(H_k\) decreases the fluctuation amplitude of the \(k\)th order local number to its minimum possible value in \(O(N)\) time steps. More precisely, we are going to prove the following lemma:

**Lemma 3** Let \(\alpha\) be a bit string of length \(N\). Then \(\Delta_k(\alpha) \leq 1\) where \(\ell \equiv \ell(k) = 2k(k-2)\left\lfloor \frac{1}{2} \frac{1}{k}\right\rfloor\). Besides, \(\Delta_k(H_k^{(k)}(\alpha)) = 0\) if and only if \(\rho(\alpha) = r/k\) for some \(r \in \mathbb{Z}\).

**Proof:** First of all, we show that if \(\Delta_k(\beta) \leq 1\), then \(\Delta_k(\beta) = 0\) if and only if \(\rho(\beta) = r/k\) for some \(r \in \mathbb{Z}\).

Since the fluctuation of \(n_k(\beta, i)\) is less than 2, \(n_k(\beta, i)\) can either be \([k\rho(\beta)]\) or \([k\rho(\beta)]\). So, if \(\rho(\beta) = r/k\) for some \(r \in \mathbb{Z}\), then \(\Delta_k(\beta) = 0\). Conversely, if \(\rho(\beta)\) is not in the form \(r/k\) for some \(r \in \mathbb{Z}\), then there exist \(i\) and \(j\) such that \(n_k(\beta, i) = [k\rho(\beta)]\) and \(n_k(\beta, j) = [k\rho(\beta)]\). Hence, \(\Delta_k(\beta) = 1\).

Since \(H_k\) conserves car density, the last statement of this Lemma is proved by setting \(\beta = H_k^{(k)}(\alpha)\).

To prove the first part of this Lemma, we may further assume that (a) \(H_k^{(i)}(\alpha) \neq H_k^{(i-1)}(\alpha)\) for all \(i < \ell\) and (b) \(\Delta_k(\alpha) > 1\). If for (a) does not hold, then \(H_k^{(i)}(\alpha)\) is a fixed point of \(H_k\) and hence Lemma 2 tells us that \(\Delta_k^{(i)}(\alpha) = 0\). In addition, Eq. (3) tells us that it is not possible to increase the \(k\)th order local number \(n_k(\beta, i)\) of a site \(i\) in a bit string \(\beta\) if \(n_k(\beta, i)\) equals \(n_k^{\max}(\beta)\). Hence, \(n_k^{\min}(H_k^{(k)}(\alpha))\) is a non-increasing function of \(i\); and by CP symmetry of the \(k\)th order car hopping rule, \(n_k^{\min}(H_k^{(k)}(\alpha))\) is a non-decreasing function of \(i\). As a result, the first part of this Lemma is trivially true if \(\Delta_k(\alpha) \leq 1\).

Eq. (3) also tells us that for any bit string \(\beta\), \(n_k(\alpha) = n_k(\beta, i) + 1\) if and only if a car in \(\beta\) hops from site \(i - 1\) to site \(i\) while no car in \(\beta\) hops from \(i - k - 1\) to \(i + k\) under the action of \(H_k\). Besides, \(n_k(\alpha) = n_k(\beta, i) - 1\) if and only if a car in \(\beta\) hops from site \(i + k - 1\) to \(i + k\) while no car in \(\beta\) hops from \(i - 1\) to \(i\) under \(H_k\). Finally, \(n_k(\beta, i) = n_k(\beta, i)\) if and only if either no car in \(\beta\) hops from sites \(i - 1\) and \(i + k - 1\) or cars in \(\beta\) hop from both sites \(i - 1\) and \(i + k - 1\). We deduce from this observation together with Lemma 2 that at least one site with \(k\)th order local number \(n_k^{\max}\) hops under \(H_k\). Consequently, the number of cars with maximum value of \(k\)th order local number does not increase with time. In other words,

\[M_k(H_k(\beta), n_k^{\max}(\beta)) \leq M_k(\beta, n_k^{\max}(\beta))\]

and the equality holds if and only if the number of sites \(i\) with \(n_k(\beta, i) = n_k^{\max}(\beta)\) and \(n_k(H_k(\beta), i) = n_k^{\max}(\beta) - 1\) equals the number of sites \(j\) with \(n_k(\beta, j) = n_k^{\max}(\beta) - 1\) and \(n_k(H_k(\beta), j) = n_k^{\max}(\beta)\). Eq. (3) demands that \(n_k(\beta, j - k) = n_k^{\max}(\beta) - 1\) and hence no car can hop to the site \(j - k\), so we also have \(n_k(H_k(\beta), j - k) = n_k^{\max}(\beta) - 1\). Therefore, the strict inequality holds in Eq. (3) if there exists a site \(i\) such that \(n_k(\beta, i - k + 1) = n_k^{\max}(\beta)\), \(n_k(\beta, i + 1) < n_k^{\max}(\beta) - 1\), \(\beta(i + 1) = 0\).

Since we assume that \(\Delta_k(\alpha) \geq 2\), we claim that we can always find \(1 \leq i \leq N\) and \(j \geq 0\) such that \(n_k^{\max}(\beta) = n_k(\beta, i - k + 1) > n_k^{\max}(\beta) - 1\) and \(n_k(\beta, i + 1) = n_k(\beta, i + k + 1) = \cdots = n_k(\beta, i + jk + 1) = n_k(\beta, i + (j + 1)k + 1)\). Otherwise, suppose that for some \(i, j, n_k(\beta, i + mk + 1) = n_k^{\max}(\beta)\) is equal to either \(n_k^{\max}(\beta)\) or \(n_k^{\max}(\beta) - 1\) for all \(m \in \mathbb{Z}\). Hence, the density \(\rho\) satisfies \(n_k^{\max}(\beta) - 1 \leq kp \leq n_k^{\max}(\beta)\). By the assumption that \(\Delta_k(\alpha) \geq 2\), there must be some \(i'\) such that \(n_k(\beta, i' + 1) < n_k^{\max}(\beta) - 2\). The conclusion on the density forces that for some \(m \in \mathbb{Z}\), \(n_k(\beta, i' + mk + 1) = n_k^{\max}(\beta)\). That is the claim.

Lemma 2 tells us that a car will hop from site \(i\) to \(i + 1\) and hence \(n_k(i - k + 1)\) is reduced in no more than \(k\) time steps. That is, there exists \(r\) with \(1 \leq r \leq k\) such that \(n_k(H_k^r(\beta), i - k + 1) < n_k(H_k^r(\beta), i - k + 1)\). Besides, \(n_k(H_k^r(\beta), i + 1) = n_k(\beta, i + 1)\) if \(j \geq 2\). Inductively, in no more than \(a \equiv \ell \left\lfloor \frac{i + 1}{2}\right\rfloor\) time steps, number of sites with local number equals \(n_k^{\max}(\beta)\) must be reduced by at least one. In other words, \(M_k(H_k^r(\beta), n_k^{\max}(\beta)) < M_k(\beta, n_k^{\max}(\beta))\). In the above discussion, it is clear that a packet of cars with \(k\)th order local number \(n_k^{\max}(\beta)\) initially located around site \(i\) is moving from left to right at a speed of at least 1 site per time step until it
hops into a region with kth order local number less than \( n_k^{(\text{max})}(\beta) - 1 \). Thus, this packet of cars may prevent another packet of kth order local number \( n_k^{(\text{max})} \) cars in its left hand side from moving at most k times. Inductively, if \( n_k(\beta, i - rk + 1) = n_k^{(\text{max})}(\beta) \), then a car will begin to hop from site \( i - rk \) to \( i - rk + 1 \) in no more than \( rk \) time steps. After that, this packet of cars will move at a speed of at least 1 site per time step from left to right provided that \( \Delta_k \) of the configuration is still greater than 1. By CP symmetry, a similar conclusion can be drawn for sites with kth order local number equals to \( n_k^{(\text{min})} \).

Since a site with kth order local number \( n_k^{(\text{max})} \) and another site with kth order local number \( n_k^{(\text{min})} \) separate by at most \( N - 1 \) sites, so the above discussions imply that in at most \( 2k \left\lceil \frac{1}{2} \left\lceil \frac{N}{k} \right\rceil \right\rceil \) time steps, \( \Delta_k \) is reduced by at least one until \( \Delta_k \leq 1 \). Hence, within \( \ell \equiv 2k(k - 1) \left\lceil \frac{1}{2} \left\lceil \frac{N}{k} \right\rceil \right\rceil \) time steps, \( \Delta_k \leq 2 \).

Combining Lemmas 3 and 4, we know that the maximum fluctuation amplitude in car density evolves towards the minimum possible value under the repeated action of \( H_k \). We remark that relaxation time estimate \( \ell \) in the above Lemma is rather conservative. It is not difficult to reduce this estimation by a factor of two or more. Nevertheless, for the purpose of solving the GDCP, the present estimation which states that \( \ell = O(kN) \) for \( k \leq N \) is already enough.

Our investigation so far can be summarized in the following theorem.

**Theorem 1** Let \( \alpha \) be a bit string of length \( N \) and \( \ell \equiv \ell(k) \equiv 2k(k - 2) \left\lceil \frac{1}{2} \left\lceil \frac{N}{k} \right\rceil \right\rceil \). Then the bit string \( \beta \equiv H_k(\alpha) \) has the following properties:

(a) The kth order local number \( n_k(\beta, i) \) is equal to either \( \lfloor k\rho \rfloor \) or \( \lceil k\rho \rceil \);

(b) Suppose that \( H_k(\beta) \neq \beta \). If \( \rho(\alpha) \leq 1/2 \), then \( \beta \) does not contain two consecutive 1s as its substring. Similarly, if \( \rho(\alpha) \geq 1/2 \), then \( \beta \) does not contain two consecutive 0s as its substring. Thus, if \( \rho(\alpha) \leq 1/2 \), a car at site \( i \) in the string \( \beta \) hops to the right under \( H_k \) if and only if \( \lfloor k\rho \rfloor = n_k(i - k + 1) > n_k(i + 1) = \lceil k\rho \rceil \). Similarly, if \( \rho \geq 1/2 \), an empty site \( i \) in the string \( \beta \) hops to the left if and only if \( \lceil k\rho \rfloor = n_k(i - k) > n_k(i) = \lfloor k\rho \rfloor \).

**Proof:** We only need to prove (b) as (a) is already contained in the proof of Lemma 3.

To prove (b), we use part (a) of this Theorem. It tells us that \( n_k(\beta, i) \) equals either \( \lfloor k\rho \rfloor \) or \( \lceil k\rho \rceil \). We denote the minimum distance between two successive cars in the bit string \( \beta \) by \( d \), that is, \( d = \min\{i - j : \beta(i) = 0, \beta(j) = 1 \} \). Since \( \Delta_k = 1 \), Lemmas 3 and 4 imply that \( k\rho \notin Z \). Hence, the maximum distance between \( \lfloor k\rho \rfloor \) cars in the string \( \beta \) is greater than or equals to \( k + 1 \). Thus, \( d \geq k + 1 \). For \( \rho \leq 1/2 \),

\[
d \geq \frac{k + 1}{\lfloor k\rho \rfloor} \geq \frac{k + 1}{k/2 + 1/2} = 2. \tag{5}
\]

As a result, we conclude that \( \beta \) does not contain the substring 11 if \( \rho \leq 1/2 \). By CP symmetry, \( \alpha \) does not contain the substring 00 whenever \( \rho \geq 1/2 \).

The remaining assertion of part (b) follows directly from Eq. (3). \( \square \)

**B. Separation Automaton**

With the GDCP as stated, we need to distinguish the two cases, namely, (1) \( \alpha = 0^N \) and \( f(0) \neq 0 \), and (2) \( 0 < \rho(\alpha) \) and \( f(\rho(\alpha)) > 0 \). In order to do so, we need to tell if the bit string \( \alpha \) is equal to \( 0^N \) or not. It can be done using the following automaton \( S_k \) together with \( H_k \). In a similar way, we also need to distinguish the case of \( \alpha = 1^N \) and \( \alpha \neq 1^N \). It can be done using a conjugate automaton \( S_k^c \). These two automata \( S_k \) and \( S_k^c \) are collectively known as the kth order separation automaton.

**Rule 2 (kth order separation rule)** We denote this CA rule by \( S_k \). The state of site \( i \) in the next time step \( S_k[\alpha](i) \) is given by:

\[
S_k[\alpha](i) = \begin{cases} 
1 & \text{if } \alpha(i - k) = 1 \text{ and } n_k(i - k) = 1, \\
\alpha(i) & \text{otherwise.}
\end{cases}
\tag{6a}
\]

We denote its conjugate CA rule by \( S_k^c \). That is,

\[
S_k[\alpha](i) = \begin{cases} 
0 & \text{if } \alpha(i) = 0 \text{ and } n_k(i - k) = k - 1, \\
\alpha(i) & \text{otherwise.}
\end{cases}
\tag{6b}
\]

**Theorem 2** Let \( \alpha \) be a bit string of length \( N \). Then

\[
\begin{align*}
\rho(\beta) &= 0 & \text{if } \rho(\alpha) = 0, \\
\frac{1}{k} &\leq \rho(\beta) < \frac{2}{k} & \text{if } 0 < \rho(\alpha) < \frac{1}{k}, \\
\beta &= \gamma & \text{and } \rho(\beta) = \rho(\alpha) \text{ otherwise,}
\end{align*}
\tag{7a}
\]

where \( \beta = S_k^{\lfloor N/k \rfloor} \circ H_k(\alpha) \) and \( \gamma = H_k(\alpha) \). Similarly,

\[
\begin{align*}
\rho(\delta) &= 1 & \text{if } \rho(\alpha) = 1, \\
\frac{k - 2}{k} &\leq \rho(\delta) < \frac{k - 1}{k} & \text{if } \frac{k - 1}{k} < \rho(\alpha) < 1, \\
\delta &= \gamma & \text{and } \rho(\delta) = \rho(\alpha) \text{ otherwise,}
\end{align*}
\tag{7b}
\]

where \( \delta = S_k^{\lfloor N/k \rfloor} \circ H_k(\alpha) \).
Proof: If $\rho(\alpha) = 0$, then $\alpha = 0^N$ (this notation denotes $N$ consecutive 0s and we shall use similar notations for a bit string) and hence $\beta = 0^N$ and $\rho(\beta) = 0$.

If $\rho(\alpha) \geq 1/k$, then Theorem 3 tells us that $n_k(\gamma,i) > 0$ for all $i$. Hence, from Eq. (8a), $\beta = \gamma$ and $\rho(\beta) = \rho(\alpha)$.

Finally, if $0 < \rho(\alpha) < 1/k$, then Theorem 3 implies that $n_k(\gamma,i)$ is equal to either 0 or 1 for all $i$. Moreover, at least one bit in the string $\gamma$ is 1. Thus, it is straightforward to check that applying $S_k^{(N/k)}$ to $\gamma$ makes $1 \leq n_k(\beta,i) \leq 2$ for all $i$ and $1/k \leq \rho(\beta) < 2/k$.

The proof for the conjugate separation rule $S_k$ is similar. \hfill \Box

Thus, $S_k^{(N/k)} \circ H_k$ separates the string $0^N$ from the rest in the sense that no string in the form $S_k^{(N/k)} \circ H_k(\alpha)$ has a density of $1$s in the interval $(0, 1/k)$. This is the reason why we call $S_k$ the $k$th order separation automaton.

C. Inversion Automaton

This CA rule and the next one (the exchange automaton) are two technical CA rules to transform the bit string to the form we desired. They are needed to correctly solve the GDCP for input strings $0^N$ and $1^N$.

Rule 3 (kth order inversion rule) We denote this CA rule by $I_k$. The state of site $i$ in the next time step $I_k[\alpha](i)$ is given by:

$$I_k[\alpha](i) = \begin{cases} 1 & \text{if } n_k(i) = 0, \\ \alpha(i) & \text{otherwise.} \end{cases} \quad (8a)$$

We denote its conjugate CA rule by $I_k$. In other words,

$$I_k[\alpha](i) = \begin{cases} 0 & \text{if } n_k(i) = k, \\ \alpha(i) & \text{otherwise.} \end{cases} \quad (8b)$$

A direct consequence of Theorem 3 and Eqs. (8a)–(8b) is:

Theorem 3 Let $\alpha$ be a bit string of length $N$ and $\beta = I_k \circ H_k(\alpha)$. Then $\rho(\beta) = 1$ if $\rho(\alpha) = 0$, and $\rho(\beta) = \rho(\alpha)$ if $\rho(\alpha) \geq 1/k$. In fact, the bit string $H_k(\alpha)$ is a fixed point of $I_k$ provided that $\rho(\alpha) \geq 1/k$. Similarly, if $\gamma = I_k \circ H_k(\gamma)$, then $\rho(\gamma) = 0$ if $\rho(\alpha) = 1$, and $H_k(\alpha)$ is a fixed point of $I_k$ if $\rho(\alpha) \leq (k-1)/k$.

D. Exchange Automaton

Rule 4 (kth order exchange rule) We denote this CA rule by $E_k$. The state of site $i$ in the next time step $E_k[\alpha](i)$ is given by:

$$E_k[\alpha](i) = \begin{cases} 1 & \text{if } n_k(i) = 0, \\ 0 & \text{if } n_k(i) = k, \\ \alpha(i) & \text{otherwise.} \end{cases} \quad (9)$$

The following theorem is a direct consequence of Theorem 3 and Eq. (8b):

Theorem 4 Let $\alpha$ be a bit string of length $N$. Then $\beta = E_k \circ H_k(\alpha) = 1^N$ if $\alpha = 0^N$, $\beta = 0^N$ if $\alpha = 1^N$. Moreover, $H_k(\alpha)$ is a fixed point of $E_k$ if $1/k \leq \rho \leq (k-1)/k$.

E. Function Automaton

The two automata introduced in this subsection are derived from the continuous function in question. What we want it to achieve in the first automaton $F_f$ is that after applying it once, there will be a substring of $0^{2q}$ or $1^{2q}$ according to the sign$(f(\rho))$ provided that $\rho$ is not in the form $r/q$ for some $r \in \mathbb{Z}$. This substring will be further manipulated by the next automaton. The second automaton $F_f$ is used to classify a bit string with $\rho$ in the form $r/q$ for some $r \in \mathbb{Z}$.

Rule 5 (function rule) Let $f(x)$ be the continuous function specified in GDCP. We denote the discontinuous points in $[0, 1]$ by $p_i/q$, where $p_i, q \in \mathbb{Z}^+ \cup \{0\}$. (Here we choose $q$ to be the smallest positive integral denominators of all the expressions $p_i/q$.) Recall in Section 7 that we require $f\left(\frac{2k+1}{4q}\right)$ and $f\left(\frac{2k+1}{4q}\right)$ to be non-zero. We denote the function CA rule associated with $f(x)$ by $F_f$. The state of the site $i$ in the next time step $F_f[\alpha](i)$ is given by:

$$F_f[\alpha](i) = \begin{cases} \theta(f(\frac{2k+1}{4q})) & \text{if } n_{2q}(i-j) = 2k + 1 \\ \alpha(i) & \text{for some } 0 \leq j \leq 2q - 1, k \in \mathbb{Z} \text{ and } f(\frac{2k+1}{4q}) \neq 0, \\ \text{otherwise,} \end{cases} \quad (10a)$$

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \quad (10b)$$

In addition, we denote the associated function CA rule of $f(x)$ by $F_f$. The state of the site $i$ in the next time step is given by:

$$F_f[\alpha](i) = \begin{cases} \theta(f(\frac{x}{2q})) & \text{if } n_{2q}(i,j) = k, 0 < k < 2q, \\ \alpha(i) & \text{and } f(\frac{k}{2q}) \neq 0, \quad (10c) \\ \text{otherwise.} \end{cases}$$

Theorem 5 We use the notations in the function rule. Let $\alpha$ be a bit string of length $N$. Then $\beta = F_f \circ H_{2q}(\alpha)$ equals $0^N$ if $\alpha = 0^N$, $\beta = 1^N$ if $\alpha = 1^N$. More importantly, if $1/2q \leq \rho(\alpha) \leq (2q-1)/2q$ and $\rho(\alpha)$ is not in the form $r/2q$ for some integer $r$, then $\beta$ contains the substring $0^{2q}$ if and only if $f(\rho(\alpha)) < 0$ while $\beta$ contains the substring $1^{2q}$ if and only if $f(\rho(\alpha)) > 0$. Furthermore, $H_{2q}(\alpha)$ is a fixed point of $F_f$ provided that either $f(\rho(\alpha)) = 0$ or $\rho(\alpha)$ is in the form $r/2q$ for some $r = 1, 2, \ldots, 2q - 1$.
That is to say, the function rule maps a portion of the substring to everywhere.

Theorem 7 We use the notations of the function rule. Let \( \alpha \) be a bit string of length \( N \) and \( \beta \equiv \tilde{F}_f \circ \mathbf{P}_f^{N-2q} \circ F_f \circ \mathcal{H}_{2q}^{(2q)}(\alpha) \).

(a) The strings \( 0^N \) and \( 1^N \) are fixed points of \( \tilde{F}_f \circ \mathbf{P}_f^{N-2q} \circ F_f \circ \mathcal{H}_{2q}^{(2q)} \). Moreover, \( \beta = 0^N \) provided that \( 0 < \rho(\alpha) < 1/2q \) and \( f(0) < 0 \). Similarly, \( \beta = 1^N \) provided that \( 2q - 1/2q < \rho(\alpha) < 1 \) and \( f(1/2q) > 0 \).

(b) Suppose \( 1/2q \leq \rho(\alpha) \leq (2q - 1)/2q \). Then \( \beta = 0^N \) if and only if \( f(\rho(\alpha)) < 0 \). Moreover, \( \beta = 1^N \) if and only if \( f(\rho(\alpha)) > 0 \). Finally, \( \beta = \mathcal{H}_{2q}^{(2q)}(\alpha) \) and hence \( \rho(\beta) = \rho(\alpha) \) if and only if \( f(\rho(\alpha)) = 0 \).

Proof: To show that \( 0^N \) and \( 1^N \) are fixed points of \( \tilde{F}_f \circ \mathbf{P}_f^{N-2q} \circ F_f \circ \mathcal{H}_{2q}^{(2q)} \) is straightforward. Suppose \( 0 < \rho(\alpha) < 1/2q \) and \( f(1/2q) < 0 \). Then, Theorem 1 implies that \( n_{2q}(\mathcal{H}_{2q}^{(2q)}(\alpha), i) \leq 1 \) with equality holds for some \( i \). Since \( f(1/2q) < 0 \), by continuity of \( f \) and the distribution of discontinuous points of \( \text{sgn}(f) \), we know that \( f(x) < 0 \) whenever \( 0 < x < 1/2q \). Hence \( F_f \circ \mathcal{H}_{2q}^{(2q)}(\alpha) \) contains the substring \( 0^{2q} \) and does not contain the substring \( 1^{2q} \). Thus by Eq. (11) and Theorem 1, we conclude that \( \beta = 0^N \). The proof of \( \beta = 1^N \) provided that \( (2q - 1)/2q < \rho(\alpha) < 1 \) and \( f(2q-1) > 0 \) is similar. So, we have proved the validity of part (a).

We know from Theorem 1 that \( \mathcal{H}_{2q}^{(2q)}(\alpha) \) is a fixed point of \( \mathbf{P}_f \) provided that \( 1/2q \leq \rho(\alpha) \leq (2q - 1)/2q \). Therefore, part (b) of this Theorem follows directly from Theorems 3, 4, and Eq. (11).

IV. GENERALIZED DENSITY CLASSIFICATION AUTOMATA

After introducing all the necessary building blocks in the last section, we are ready to report our solution of the GDCP.

Theorem 8 Let \( \alpha \) be a bit string of length \( N \). Let \( f(x) \) be a continuous function. Denote the rational discontinuous points of \( \text{sgn}(f) \) by \( p_i/q \) with \( p_i, q \in \mathbb{Z}^+ \cup \{0\} \).

(Here \( q \) is chosen to be the smallest positive integral denominator for the expressions \( p_i/q \).) Suppose further that \( f(1/2q) \) and \( f(2q-1)/2q \) are both non-zero, then the following sequence of CA rules solves the GDCP for the function \( f(x) \):

\[
C_f = F_f \circ \mathbf{P}_f^{N-2q} \circ F_f \circ (E_{2q})^{a_5} \circ (T_{2q})^{a_4} \circ (I_{2q})^{a_3} \circ \left( \mathcal{H}_{2q}^{(2q)} \circ S_{2q}^{(N/2q)} \right)^{a_2} \circ \left( \mathcal{H}_{2q}^{(2q)} \circ S_{2q}^{(N/2q)} \right)^{a_1} \circ \mathcal{H}_{2q}^{(2q)},
\]

(12)

where \( a_1 = 1 \) if \( f(1/2q) > 0 \), \( a_1 = 0 \) otherwise; \( a_2 = 1 \) if \( f(2q-1)/2q) < 0 \), \( a_2 = 0 \) otherwise; \( a_3 = 1 \) if \( f(0) > 0 \).
and \( f(1) \geq 0, a3 = 0 \) otherwise; \( a4 = 1 \) if \( f(1) < 0 \) and \( f(0) \leq 0, a4 = 0 \) otherwise; \( a5 = 1 \) if \( f(0) > 0 \) and \( f(1) < 0 \) and \( a5 = 0 \) otherwise.

Before we prove this theorem, let us first informally explain how it works. The \( H_{2q}^{(2q)} \) in the right-most part of Eq. (12) makes the density of car uniform. Then the \( \left( H_{2q}^{(2q)} \circ S_{2q}^{(N/2q)} \right)^{a1} \) term first separates \( \rho = 0 \) from \( 0 < \rho < 1/2q \) and then makes the resultant bit string uniform in car density whenever \( f(\frac{2q - 1}{2q}) \) and hence also \( f(x) > 0 \) for all \( 0 < x < 1/2q \). Similarly, the \( \left( H_{2q}^{(2q)} \circ S_{2q}^{(N/2q)} \right)^{a2} \) term first separates \( \rho = 1 \) from \( (2q - 1)/2q < \rho < 1 \) and then makes the resultant bit string uniform in car density whenever \( f(\frac{2q - 1}{2q}) \) and hence also \( f(x) < 0 \) for all \( (2q - 1)/2q < x < 1 \). Next, notice that \( a3 \) and \( a5 \) will not simultaneously equal to one, and similarly \( a4 \) and \( a5 \) will not simultaneously equal to one, the \( (E_{2q})^{a5} \circ (T_{2q})^{a4} \circ (I_{2q})^{a3} \) term correctly deals with the GDCP for \( \rho = 0 \) and \( 1 \). This is followed by the term \( P_{N/2q}^{(N/2q)} \circ F_f \) which correctly classifies those \( \rho \) not in the form \( r/2q \). Finally, the term \( F_f \) settles the remaining case of \( \rho \) in the form \( r/2q \) with \( r \neq 0 \) or \( 2q \).

**Proof:** We divide the proof into the following four cases:

Case (a): \( \rho(a) = 0 \) or \( 1 \), that is, \( a = 0^N \) or \( 1^N \); in this case, \( C_f = F_f \circ P_{N}^{N-2q} \circ F_f \circ (E_{2q})^{a5} \circ (T_{2q})^{a4} \circ (I_{2q})^{a3} \).

So by Theorems 8 and 8, \( C_f(0^N) = F_f \circ P_{N}^{N-2q} \circ F_f \circ (E_{2q})^{a5} \circ (T_{2q})^{a4} \circ (I_{2q})^{a3} \).

Case (b): \( 1/2q \leq \rho(a) \leq (2q - 1)/2q \): in this case, Theorems 8, 8 and 8 tell us that \( C_f = F_f \circ P_{N}^{N-2q} \circ F_f \circ H_{2q}^{(2q)} \).

Hence, this case is settled by applying Theorem 8.

Case (c): \( 0 < \rho(a) < 1/2q \): in this case, the continuity of \( f \) together with our assumptions that \( f(1/2q) \neq 0 \) and discontinuous points of \( \text{sgn}(f(x)) \) are in the form \( r/2q \) for some \( r \in \mathbb{Z} \) demand that \( f \) can be further divided into the following two subcases, namely, \( f(x) > 0 \) for all \( 0 < x < 1/2q \) and \( f(x) < 0 \) for all \( 0 < x < 1/2q \).

In subcase (1), \( a1 = 1 \). Theorem 8 tells us that \( 1/2q \leq \rho(S_{2q}^{(N/2q)} \circ H_{2q}^{(2q)}) < 1/2q \). From Theorem 8, we know that \( C_f(\alpha) = F_f \circ P_{N}^{N-2q} \circ F_f \circ H_{2q}^{(2q)} \circ S_{2q}^{(N/2q)} \circ H_{2q}^{(2q)}(\alpha) \).

Combining with Theorem 8, \( C_f(\alpha) = 1^N = (\text{sgn}(f(0)))^N \).

In subcase (2), the continuity of \( f \) implies that \( a1 = a3 = a5 = 0 \) and hence \( C_f(\alpha) = F_f \circ P_{N}^{N-2q} \circ F_f \circ H_{2q}^{(2q)}(\alpha) \). So, applying Theorem 8 gives \( C_f(\alpha) = 0^N = (\text{sgn}(f(0)))^N \).

Therefore, Eq. (12) correctly classifies bit string with \( 0 < \rho < 1/2q \).

Case (d): \( (2q - 1)/2q < \rho(\alpha) < 1 \): The proof of this case is similar to that of case (c) and we are not going to write the details here.

In summary, \( C_f(\alpha) = 0^N \) if \( f(\rho) < 0 \), \( C_f(\alpha) = 1^N \) if \( f(\rho) > 0 \), and \( C_f(\alpha) = H_{2q}^{(2q)}(\alpha) \) if \( f(\rho) = 0 \). Consequently, the result of the GDCP can be read out from any \( 2q \) consecutive sites of the final output bit string. More precisely, if we find that such substring equals \( 0^q \), then either \( f(\rho) < 0 \) or \( f(0) = 0 \) if such substring is \( 1^q \), then either \( f(\rho) > 0 \) or \( f(1) = 0 \). Lastly, if such substring contains both 0 and 1, then \( \rho(\alpha) = 0 \).

\[ f(\rho) = 0. \]

\[ \square \]

V. SIMPLE SOLUTIONS TO CERTAIN SPECIAL CASES

Theorem 8 provides a solution to the GDCP involving all the nine automata introduced in Section III. Nonetheless, it does not mean that solution of any GDCP have to be that complicated. In this section, we report simple solutions to certain useful GDCPs. Comparing to the GDCP, solutions to the following problems are rather straightforward. So the presentation in this section is brief.

A. Cases Related To Rational Density Classification

The simplest non-trivial case of GDCP is the rational density problem which chooses \( f(x) = x - \rho_c \) for a fixed rational number \( 0 < \rho_c < 1 \). Chau et al. solved this problem by the following two automata 8.

**Rule 7 (modified traffic rule)** Let \( \alpha \) be a bit string of length \( N \) and \( \rho_c = p/q \) where \( p, q \in \mathbb{Z}^+ \) with \( p \) and \( q \) are relatively prime. Then the modified traffic rule is give by

\[
T_{p}^{(\alpha)}(i) = \begin{cases} 
1 & \text{if } \alpha(i) = 0, \alpha(i-1) = 1, \text{ and } n_{q-1}(i) \leq p - 1, \\
0 & \text{if } \alpha(i) = 1, \alpha(i+1) = 0, \text{ and } n_{q-1}(i+1) \leq p - 1, \\
\alpha(i) & \text{otherwise.}
\end{cases}
\]

**Rule 8 (Modified Majority Rule)** Let \( \alpha \) be a bit string of length \( N \) and \( \rho_c = p/q \) where \( p, q \in \mathbb{Z}^+ \) with \( p \) and \( q \) are relatively prime. Then the modified majority rule is give by

\[
M_{p}^{(\alpha)}(i) = \begin{cases} 
1 & \text{if } n_{2q+1}(i - q) \geq 2p + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

More precisely, they showed that \( M_{p}^{(\alpha)}(i) = 0 \) solves the rational density classification problem with \( f(x) = x - \rho_c \).
By applying $I_q$, $\tilde{I}_q$ or $E_q$ once to the resultant state of the rational density classification automata, it is clear that we can solve a number of related problems using three CAs, including $f(x) = \rho_c - x$, $f(x) = (x - \rho_c)^2$ and $f(x) = -(x - \rho_c)^2$.

B. Cases Related To Coarse-Grained Rational Density Classification

Another interesting and useful problem we have briefly mentioned in Section I is the so-called approximate rational density classification. Let us recall that if $\rho_c = p/q$ is a rational number, then solution to the rational density classification problem either by the method reported in Section IV or in Ref. [9] requires a rule table whose range scales linearly with $q$. Thus, it is more cost effective to classify a coarse-grained rational density. To achieve this task, we are required to solve the GDCP with the function $f$ given by Eq. (4) with $\rho_{c1} < \rho_c < \rho_{c2}$. An effective way to choose $\rho_{c1}$ and $\rho_{c2}$ is to use the continued fraction approximation of $\rho_c$. If the output string is $1^N$, we know that $\rho$ is greater than $\rho_{c2}$ and hence also $\rho_c$. Similarly, we know that $\rho < \rho_c$ if the output is $0^N$. If $\rho_{c1} < \rho < \rho_{c2}$, then we do not know if the $\rho$ is greater than $\rho_c$ or not. Fortunately, $\rho$ is preserved by the automata in this case and hence we may feed our output bit string to say the full rational density classification CAs for fine-grained density determination. In this way, we can efficiently solve the rational density classification problem with small rule table size with high probability for a randomly given input bit string. Performing a CA which is a function of output of another CA leads us to the notion of CA programming. We shall explore the power and weakness of CA programming elsewhere [12].

After pointing out the significance of the coarse-grained rational density classification problem, we report a solution involving only two CA rules. We write $\rho_{c1} = p_1/q$ and $\rho_{c2} = p_2/q$ as usual. The first CA rule is our car hopping automaton $H_q$. (Clearly we cannot use $T$ as we have two critical densities here.) The second CA rule is a variation of the propagation rule, as stated below.

Rule 9 (modified propagation rule) Let $\alpha$ be a bit string. Then the modified propagation rule is given by

$$\tilde{P}_{p_1,p_2,q}[\alpha](i) = \begin{cases} 1 & \text{if } n_q(i + j) > p_2 \text{ for some } 1 \leq j \leq q, \\ 0 & \text{if } n_q(i + 1) < p_1 \text{ for some } 1 \leq j \leq q, \\ \alpha(i) & \text{otherwise.} \end{cases}$$

Clearly, if $\rho > \rho_{c2}$, there is a site $i$ such that $n_q(H_q^1(\alpha), i) > p_2$. Thus, $\rho$ increases to 1 under the repeated application of $\tilde{P}_{p_1,p_2,q}$. Similarly, $\rho$ decreases to 0 under the repeated application of $\tilde{P}_{p_1,p_2,q}$ if $\rho < \rho_{c1}$. In summary, we know that $H_q^N \circ H_q^{(l)}$ solves the coarse-grained rational density classification problem.

In a similar way, problems with $f(x) = (x - \rho_{c1})(x - \rho_{c2})$, $f(x) = (x - \rho_{c1})(x - \rho_{c2})^2$ and so on can be solved using four CAs.

C. Variation Of The Theme

Because of the difficulties in distinguishing between the strings $0^N$ and $0^{N-1}1$, we introduce the separation automaton $S_k$. Unfortunately, $S_k$ is not car density conserving. This is precisely the reason why we impose the technical conditions that $f(1/2q) \neq 0$ and $f((2q - 1)/2q) \neq 0$. We remark that the above technical conditions can be waived provided that we relax the GDCP a bit. Instead of requiring that the density of 1s of the output string $\beta$ equals to that of the input string $\alpha$ if $f(\rho(\alpha)) = 0$, we replace it by requiring that $|f(\rho(\beta)) - f(\rho(\alpha))| < 1/q$. By doing so, it is clear that the GDCP can be solved even when $f((1/2q))$ or $f((2q - 1)/2q) = 0$ by the following sequence of automata:

$$\tilde{F}_f \circ P_f^{N-2q} \circ F_f \circ (E_{2q})^a \circ (I_{2q})^b \circ (I_{2q})^c \circ H_{2q}^{q(2q)} \circ S_{2q}^{(N/2q)} \circ H_{2q}^{(N/2q)} \circ H_{2q}^{(N/2q)},$$

where $a3 = 1$ if $f(0) > 0$ and $f(1) > 0$, $a3 = 0$ otherwise; $a4 = 1$ if $f(1) < 0$ and $f(0) \leq 0$, $a4 = 0$ otherwise; $a5 = 1$ if $f(0) > 0$ and $f(1) < 0$, $a5 = 0$ otherwise.

However, we do not encourage the use of this relaxed definition of GDCP for the density of 1s is not preserved in case $f(\rho) = 0$.

VI. DISCUSSIONS

A few remarks are in order. First, Theorem 8 provides a CA solution to the GDCP with at most eight CA rules each with rule table of range $\leq 4q$. (Note that the number eight comes from the observation that we can combine $F_f \circ (E_{2q})^a \circ (I_{2q})^b \circ (I_{2q})^c$ together to form one single CA rule whose rule table size is still independent of the string length $N$.) Moreover, the total run time required scales as $O(qN)$ whenever $q < N$, making it asymptotically optimal up to a constant factor. The main ingredient used to solve the GDCP is the $k$th order traffic rule $H_k$ whose repeated application leads to a uniformly distributed 1s in the bit string in the sense that fluctuation of the $k$th order local number $n_k(i)$ does not exceed 1.

We stress that the solution to the GDCP using a sequence of CAs is not unique. We have also found that a slightly different traffic rule involving $2k + 1$ sites together with a generalized majority vote rule based on the
work in Ref. [9] can also do the job. Nevertheless, this alternative method requires in general more than eight CA rules in succession [13].

We are not sure if the GDCP can be solved using fewer than eight CA rules. What we know from the result of Fukš is that density classification, being a special case of GDCP, cannot be solved using a single CA rule [8]. Although certain special cases of the GDCP, such as the original density classification problem can be solved using two CA rules [8, 9], we feel that it is highly unlikely to solve the full GDCP using just two CA rules because of the difficulties involved in separating the cases with \( \rho \) close to 0 and \( \rho \) equals 0 although we have provided solutions of the rational density classification and coarse-grained rational density classification problems using two CA rules in Section V.

Finally, we remark that the CA solution to the GDCP shows a rich flow diagram. The flow of \( \rho \) under the action of \( C_f \) exhibits at most two stable fixed points at \( \rho = 0 \) and 1 corresponding to \( f(\rho) \neq 0 \), finitely many fixed points corresponding to the rational isolated roots of the equation \( f(x) = 0 \), together with possibly infinitely many neutral fixed points corresponding to the remaining non-isolated zeros of \( f(x) = 0 \) in the limit of \( N \to \infty \). In this respect, sequence of CAs may lead to very interesting flow diagrams that are distinctive from those resulting from conventional continuous dynamical systems.

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