Driving Quantum System into Decoherence-free Subspaces by Lyapunov Control

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We present a scheme to drive a finite-dimensional quantum system into the decoherence-free subspaces (DFS) by Lyapunov control. Control fields are established by Lyapunov function. This proposal works well for both closed and open quantum systems, with replacing the DFS by desired subspaces for closed systems. An example which consists of a four-level system with three degenerate states driven by three lasers is presented to gain further insight of the scheme, numerical simulations for the dynamics of the system are performed and the results are good.

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I. INTRODUCTION

Quantum computing and quantum communication have attracted a lot of attention due to their promising applications such as the speedup of classical computations and secure key distributions [1]. Although the physical implementation of basic quantum information processors has been reported recently [2], the realization of powerful and useable devices is still a challenging and as yet unresolved task. A major difficulty arises from the coupling of a quantum system to its environment that leads to decoherence. One promising solution to this problem is provided by the concept of decoherence-free subspaces (DFS) [3]. The decoherence-free subspaces have been defined as collections of states that undergo unitary evolution in the presence of decoherence. Experimental realizations of DFS have been achieved with photons [4] and in nuclear spin systems [5]. A decoherence-free quantum memory for one qubit has been realized experimentally with two trapped ions [4, 7]. As the DFS is a promising candidate to solve the problem of decoherence, it is natural to ask how can we drive a quantum system into the DFS by quantum control?

Several approaches to controlling [8–17] a quantum system have been proposed in the past decade, which can be divided into coherent (unitary) and incoherent (non-unitary) control, according to how the controls enter the dynamics. In the coherent control scheme, the controls enter the dynamics through the system Hamiltonian. It affects the time evolution of the system, but not its spectrum. In the incoherent (but linear) control scheme [18–20], an auxiliary system, called probe, is introduced to manipulate the target system through their mutual interactions. This incoherent control scheme is of relevance whenever the system dynamics cannot be directly accessed, and it provides a non-unitary evolution capable for transferring all initial states (pure or mixed) into an arbitrary (pure or mixed) target state. While the above control strategies render the quantum system a linear dynamics, feedback control can lead the dynamics to both nonlinear and stochastic [21].

In certain special cases, the dynamics can be mapped into a linear classical system driven by Gaussian noise, however, this method is not applicable to most quantum systems under feedback control, and the experience from nonlinear control theory in classical system tells us that it is unlikely to manipulate optimally a quantum system into a specific target state by such a control. Nevertheless, it may be possible to manipulate the quantum system into a collection of states, for example the decoherence-free subspaces.

In this paper, we explore the Lyapunov control to manipulate an open quantum system through driving it into the DFS. The Lyapunov control has been proven to be a sufficient simple control to be analyzed rigorously, in particular, this control can be shown to be highly effective for systems that satisfy certain sufficient conditions, which roughly speaking are equivalent to the controllability of the linearized system. In Lyapunov control, Lyapunov functions which were originally used in feedback control to analyze the stability of the control system, have formed the basis for new control design. By properly choosing the Lyapunov function, we illustrate by an example that quantum systems can be controlled into the DFS.

This paper is organized as follows. In Sec. II we present a general analysis of Lyapunov control for open quantum systems, Lyapunov functions and control fields are given and discussed. To illustrate the general formulism, we exemplify a four-level system with 2-dimensional DFS in Sec. III showing that the system can be driven in the DFS by Lyapunov control. Finally, we conclude our results in Sec. IV.

II. GENERAL FORMULISM

A controlled quantum system can be modeled in different ways, either as a closed system evolving unitarily governed by a Hamiltonian, or as an open system coupling to its environment. In this paper, we restrict our discussion to a \( N \)-dimensional open quantum system, and consider its dynamics as Markovian and therefore the dynamics
obey the Markovian master equation (ℏ = 1, throughout this paper),
\[ \dot{\rho} = -i[H, \rho] + \mathcal{L}(\rho), \]
where \( \mathcal{L}(\rho) = \frac{1}{2} \sum_{m=1}^{M} \lambda_m ([J_m, \rho J_m^\dagger] + [J_m^\dagger, \rho J_m]) \)
and
\[ H = H_0 + \sum_{n=1}^{F} f_n(t) H_n, \quad (1) \]
with \( \lambda_m (m = 1, 2, ..., M) \) are positive and time-independent parameters, which characterize the decoherence. \( J_m (m = 1, 2, ..., M) \) are jump operators. \( H_0 \) is a free Hamiltonian and \( H_n (n = 1, 2, ..., F) \) are control Hamiltonian, while \( f_n(t) (n = 1, 2, ..., F) \) are control fields. Equation (1) is of Lindblad form, this means that the solution to Eq. (1) has all the required properties of a physical density matrix at all times.

By its definition, DFS is composed of states that undergo unitary evolution. Considering the fact that there are many ways for a quantum system to evolve unitarily, we focus in this paper on the DFS for which the dissipative part \( \mathcal{L}(\rho) \) of the master equation is zero, leading to the following conditions for DFS\([22]\). A space spanned by \( \mathcal{H}_{DFS} = \{ |\psi_1\rangle, |\psi_2\rangle, ..., |\psi_D\rangle \} \) is a decoherence-free subspace for all time \( t \) if and only if (1) \( \mathcal{H}_{DFS} \) is invariant under \( H_0 \); (2) \( J_m |\psi_n\rangle = c_m |\psi_n\rangle \) and (3) \( \Gamma |\psi_n\rangle \) is a state at time \( t \) evolving into the DFS and stays there forever. In contrast to the conventional control problem\([24]\), we here do not specify the target state, instead the resulting state being in DFS is desired. Since the free Hamiltonian \( H_0 \) cannot be turned off in general, we assume that the resulting state \( \rho_D = \rho_D(t) \in \mathcal{H}_{DFS} \) is time-dependent and satisfies
\[ \dot{\rho}_D(t) = -i[H_0, \rho_D(t)]. \]

The control fields \( \{ f_n(t) \} \) can be established by Lyapunov function. Define a function \( V(\rho_D, \rho) \) as
\[ V(\rho_D, \rho) = \text{Tr}[\rho_D^2] - \text{Tr}(\rho_D \rho), \]
we find \( V \geq 0 \) and
\[ \dot{V} = -\sum_n f_n(t) \text{Tr}[\rho_D [-iH_n, \rho]] - \text{Tr}[\rho_D \mathcal{L}(\rho)]. \]

For \( V \) to be a Lyapunov function, it requires \( \dot{V} \leq 0 \) and \( V \geq 0 \). If we choose a \( n_0 \) such that
\[ f_{n_0}(t) \text{Tr}[\rho_D [-iH_{n_0}, \rho]] + \text{Tr}[\rho_D \mathcal{L}(\rho)] = 0 \quad (24), \]
and \( f_{n}(t) = \text{Tr}[\rho_D [-iH_n, \rho]] \) for \( n \neq n_0 \), then \( \dot{V} \leq 0 \). With these choices, \( V \) is a Lyapunov function. Therefore, the evolution of the open system with Lyapunov feedback control described by the following nonlinear equations
\[ \dot{\rho}(t) = -i[H_0 + \sum_n f_n(t) H_n, \rho(t)] + \mathcal{L}(\rho), \]
\[ f_n(t) = \text{Tr}[\rho_D [-iH_n, \rho]] \quad (n \neq n_0), \]
\[ f_{n_0}(t) = -\frac{\text{Tr}[\rho_D \mathcal{L}(\rho)]}{\text{Tr}[\rho_D \rho(iH_{n_0})]} \quad \text{and} \]
\[ \dot{\rho}_D(t) = -i[H_0, \rho_D(t)] \quad (5) \]
is stable in Lyapunov sense at least. Notice that the choice of Lyapunov function is not unique, we may define the other Lyapunov function via the bases of DFS as,
\[ V_b (\{ |\psi_j\rangle \}, \rho) = \frac{1}{D} \left( 1 - \sum_{j=1}^{D} \langle \psi_j | \rho | \psi_j \rangle \right), \quad (6) \]
where \( |\psi_j\rangle = |\psi_j(t)\rangle \) \( (j = 1, 2, ..., D) \) satisfy \( i \frac{\partial}{\partial t} |\psi_j\rangle = H_0 |\psi_j\rangle \), i.e., \( |\psi_j\rangle \) is a state at time \( t \) evolving from \( |\psi_j\rangle \) driven by \( H_0 \). Clearly, \( V_b \geq 0 \) with equality only when \( \rho \in \mathcal{H}_{DFS} \). Taking the derivative of \( V_b \) with respect to time, we have
\[ \dot{V}_b = -\frac{1}{D} \left[ \sum_n \sum_{j=1}^{D} f_n(t) |\psi_j\rangle \text{Tr}[\rho_D [-iH_n, \rho]| \psi_j \rangle] + \sum_{j=1}^{D} \langle \psi_j | \mathcal{L}(\rho) | \psi_j \rangle \right]. \]

By the same procedure, the control fields can be established,
\[ f_{n_0}(t) = \sum_{j=1}^{D} \langle \psi_j | \mathcal{L}(\rho) | \psi_j \rangle, \quad (7) \]
\[ f_{n}(t) = \sum_{m} (|\psi_j\rangle \text{Tr}[\rho_D [-iH_n, \rho]| \psi_j \rangle), \quad (8) \]

We observe from Eq. (3) and Eq. (7) that \( f_n(t) \) and \( f_{n_0}(t) \) are equivalent by setting \( \rho_D = 1/D \sum_{j=1}^{D} |\psi_j\rangle \langle \psi_j | \), which can be understood as a density matrix in the DFS.

Discussions on this set of nonlinear equations Eq. (5) are in order as follows. By the LaSalle’s invariant principle\([25]\), the autonomous dynamical system Eq. (5) converges to an invariant set defined by \( \mathcal{E} = \{ \dot{V} = 0 \} \). This set is in general not empty and of finite dimension, indicating that it is difficult to control a quantum system from an arbitrary initial state to a given target state. Nevertheless, by elaborately designing the control Hamiltonian, we can control a quantum system to evolve into a desired subspace by Lyapunov control. From the deviation, we find that for our dynamical system, the invariant set \( \mathcal{E} \) is equivalent to \( f_{n}(t) = 0 \) (for any \( n \neq n_0 \)), namely,
\[ \text{Tr}[H_n \rho_D] = \text{Tr}[H_n \rho_D \rho], \quad (n \neq n_0) \]
gives necessary conditions for the invariant set \( \mathcal{E} \). Eq. (9) shows that the invariant set \( \mathcal{E} \) depends on both the Hamiltonian and the target state \( \rho_D \). So, the invariant set in principle can be designed by both \( \rho_D \) and the control fields \( \{ H_n \} \).
FIG. 1: Atomic configurations. Three degenerate stable states are coupled to an excited state $|e\rangle$ by three lasers with coupling constants $\Omega_i$ ($i = 1, 2, 3$). $\Delta$ denotes the detuning.

III. EXAMPLE

As an illustration, we show in this section that the proposal works in the setup detailed in Fig. 1, where three degenerate stable Zeeman ground states are coupled to an excited state through three separate external lasers.

The Hamiltonian of such a system has the form

$$H_0 = \Delta |e\rangle \langle e| + \sum_{j=1}^{3} \Omega_j |e\rangle \langle j| + h.c. \tag{10}$$

in the rotating frame, where $\Omega_j$ ($j = 1, 2, 3$) are coupling constants. Without loss of generality, in the following the coupling constants are parameterized as $\Omega_1 = \Omega \sin \theta \cos \phi$, $\Omega_2 = \Omega \sin \theta \sin \phi$ and $\Omega_3 = \Omega \cos \theta$, with $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}$. The excited state $|e\rangle$ is not stable, it decays to the three degenerate ground states with rates $\gamma_1$, $\gamma_2$ and $\gamma_3$, respectively. We assume this process is Markovian and can be described by the Liouvilian,

$$\mathcal{L}(\rho) = \sum_{j=1}^{3} \gamma_j \left[ \sigma_j^{-} \rho \sigma_j^{+} - \frac{1}{2} \sigma_j^{+} \sigma_j^{-} \rho - \frac{1}{2} \rho \sigma_j^{+} \sigma_j^{-} \right] \tag{11}$$

with $\sigma_j^{-} = |e\rangle \langle j|$ and $\sigma_j^{+} = (\sigma_j^{-})^\dagger$. It is not difficult to find that the two degenerate dark states of the free Hamiltonian $H_0$

$$|D_1\rangle = \cos \phi |2\rangle - \sin \phi |1\rangle,$$
$$|D_2\rangle = \cos \theta (\cos \phi |1\rangle + \sin \phi |2\rangle) - \sin \theta |3\rangle \tag{12}$$

form a DFS. Now we show how to drive the system into the DFS. For this purpose, we choose the control Hamiltonian $H' = \sum_{j=1}^{3} f_j(t) H_j$ with $H_j = (|e\rangle \langle j| + |j\rangle \langle e|)$. We shall use Eq. (8) to determine the control fields $\{f_n(t)\}$, and choose

$$|\Psi\rangle = \sin \beta_1 \cos \beta_2 |e\rangle + \cos \beta_1 \cos \beta_3 |1\rangle + \cos \beta_1 \sin \beta_3 |2\rangle + \sin \beta_1 \sin \beta_3 |3\rangle \tag{13}$$

as initial states, where $\beta_1$, $\beta_2$ and $\beta_3$ are allowed to change independently. We should stress that for a four-dimensional system, 15 independent real parameters are needed to describe a state, making numerical simulations to exhaust all possible initial states difficult. We have performed extensive numerical simulation with some initial states that can be written in the form of Eq. (13). Two types of numerical simulations are presented. Firstly, we numerically simulate the dynamical system without atomic decay, i.e., $\gamma_1 = \gamma_2 = \gamma_3 = 0$, selected results are presented in Fig. 2. Secondly, we simulate the effects of atomic decay on the convergence of the dynamics, the results are plotted in Fig. 3. From Fig. 2 we find that all initial states in the form of Eq. (13) convert to the DFS due to the Lyapunov feedback control (see Fig. 2(a) and (c)), and the probability of the system in $|D_1\rangle$ depends on the initial state (Fig. 2(b) and (d)). Fig. 3 shows us that the atomic decay does not always speed up the convergence (Fig. 3(a)), and it also affects the probability of the system in the dark state $|D_1\rangle$. These simulations suggest that the quantum system can be driven into the DFS, at least from the initial states Eq. (13).

Noticing that the open quantum system will evolve into the DFS driven by the atomic system alone,
one may ask the following questions. How fast does this control scheme move a state into the DFS? Is it faster than the atomic decay? To answer these questions, we calculate the convergence time $T_{\text{conv}}$ for the system without the Lyapunov control but with atomic decay, the result is shown in Fig. 4(b). In contrast, we compute the convergence time $T_{\text{conv}}$ for the case when the atomic decay is zero, with the Lyapunov control (see Fig. 4(a)). By comparing these results, we find that the Lyapunov control for this problem is indeed effective. To shed further light on the numerical simulations, we now show that $\{ |D_1\rangle, |D_2\rangle \}$ spans LaSalle’s invariant subspace. Recall that $\rho_D = 1/2(|D_1\rangle\langle D_1| + |D_2\rangle\langle D_2|)$, $f_n(t) = 0$ requires $\sum_{i=1}^2 \langle D_i| [\rho, iH_n] |D_i\rangle = 0$ for any $n \neq n_0$. Clearly, $\rho = |D_1\rangle\langle D_1|$ or $\rho = |D_2\rangle\langle D_2|$ meets this requirement, therefore, $\{ |D_1\rangle, |D_2\rangle \}$ spans LaSalle’s invariant set.

IV. CONCLUSION

In summary, we have proposed a scheme to drive an open quantum system into the decoherence-free subspaces. This scheme works also for closed quantum systems, by replacing the DFS with a desired subspace. This study was motivated by the fact that for a nonlinear system, it is usually difficult to optimally control the system from an arbitrary initial state to a given target state. Our present case study suggests that it is possible to drive a quantum system to a set of states. To demonstrate the proposal we exemplify a four-level system and numerically simulate the controlled dynamics. The dependence of the convergence time as well as the distribution of the system in the DFS are calculated and discussed. A comparison between the cases with and without the Lyapunov control is also presented. These results suggest that by designing control Hamiltonian and target state, we can move a quantum open system into a desired subspace.

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