Conjugated filter approach for solving Burgers’ equation with high Reynolds number

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We propose a conjugated filter oscillation reduction scheme for solving Burgers’ equation with high Reynolds numbers. Computational accuracy is tested at a moderately high Reynolds number for which analytical solution is available. Numerical results at extremely high Reynolds numbers indicate that the proposed scheme is efficient, robust and reliable for shock capturing.

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The fundamental equation for the description of complex fluid flow is the Navier-Stokes equation, for which the full solution is still extremely difficult in the full domain of physical interest. Burgers’ equation [1] is an important simple model for the understanding of physical flows. Simulation of Burgers’ equation is a natural first step towards developing methods for computations of complex flows. It appears customary to test new approaches in computational fluid dynamics by applying them to Burgers’ equation. Jamet and Bonnerot solved Burgers’ equation by using isoparametric rectangular space-time finite elements [2]. Jain and Holla [3] developed a cubic spline approach for coupled Burgers’ equation. Varoglu and Finn [4] proposed an isoparametric space-time finite element method for solving Burgers’ equation, utilizing the hyperbolic differential equation associated with Burgers’ equation. They obtain very high accuracy and numerical stability with a reasonable number of elements and time steps. Their method was compared with a least square weak formulation of the finite element method by Nguyen and Reynen [5]. Caldwell et al [6] further developed the finite element method to allow different sizes of the elements at each stage based on the feed back from the previous step. A generalized boundary element approach was proposed by Kakuda and Tosaka [7]. These authors tabulated their accurate results for moderate Reynolds numbers and compared their results with those of Varoglu and Finn [4] and of Nguyen and Reynen [5]. A bidimensional Tau-element method was developed by Ortiz and Pun for solving Burgers’ equation with accurate results. Bar-Yoseph et al discussed a number of space-time spectral element methods for solving Burgers’ equation [8]. Arina and Canuto [9] treated Burgers’ equation by a self-adaptive, domain decomposition method called the $\chi$-formulation. Various finite difference schemes for Burgers’ equation were compared by Biriring and Saati [10]. Recently, Wei et al [11] have developed an accurate solver for Burgers’ equation in one and two space dimensions. Most recently, Hon and Mao [12] have compared performance of their adaptive multiquadric scheme with many other computational methods. It is not our purpose to exhaust the literature. Despite of much effort, numerical solution of Burgers’ equation is still a nontrivial task especially at very high Reynolds numbers where the nonlinear advection leads to shock waves. In fact, Burgers’ inviscid shocks plague many standard computational algorithms.

The purpose of this communication is to report a novel scheme for solving Burgers’ equation for all possible values of Reynolds numbers. We propose a set of conjugated filters to solve the Burgers’ equation. As the first and second derivatives are approximated by using two high-pass filters, the numerical errors of the high-pass filters at the high frequency region lead to oscillation near the Burgers’ shock wave front. The proposed idea is to effectively eliminate such an oscillation by using a conjugated low-pass filter. This set of high-pass and low-pass filters are conjugated in the sense that they are derived from one generating function and consequently have essentially the same degree of regularity, smoothness, time-frequency localization, effective support and bandwidth. In the present work, all conjugated filters are constructed by using a discrete singular convolution (DSC) algorithm [13], which is a potential approach for dynamical simulation [14,15] and numerical computations of Hilbert transform and Radon transform. The DSC low-pass filter is given by

$$\phi_{\Delta,\sigma,k}(x) = \sin \frac{x}{\Delta} (x-x_k) \exp \left[ -\frac{(x-x_k)^2}{2\sigma^2} \right],$$

and its conjugated $n$th order high-pass filters are obtained by differentiation

$$\phi_{\Delta,\sigma,k}^{(n)}(x) = \left( \frac{d}{dx} \right)^n \sin \frac{x}{\Delta} (x-x_k) \exp \left[ -\frac{(x-x_k)^2}{2\sigma^2} \right], \quad n = 1, 2, ..., \quad (2)$$

where $\Delta$ is the grid spacing and $\sigma$ is a regularization parameter. FIG. 1 shows the frequency responses of the conjugated DSC low-pass filter, 1st and 2nd order high-pass filters at $\sigma = 3.2\Delta$. It is noted that all conjugated filters have essentially the same effective bandwidth. Wavelet multiscale analysis is utilized for adaptive oscillation control. A detailed theoretical analysis and justification of this scheme is accounted elsewhere [16].
Burgers’ equation is given by
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial x^2},
\]
where \( u(x,t) \) is the dependent variable resembling the flow velocity and \( \text{Re} \) is the Reynolds number characterizing the size of viscosity. The competition between the nonlinear advection and the viscous diffusion is controlled by the value of \( \text{Re} \) in Burgers’ equation, and thus determines the behavior of the solution. We consider Eq. (3) using the following initial-boundary conditions
\[
\begin{align*}
  u(x,0) &= \sin(\pi x), \\
  u(0,t) &= u(1,t) = 0.
\end{align*}
\]
Cole has provided exact solution \[17\] for this problem in terms of a series expansion which is readily computable roughly for the parameter \( \text{Re} \leq 100 \). For the parameter \( \text{Re} = 100 \), the present calculations use 41 grid points in the interval \([0,1]\). The fourth order Runge-Kutta is used for the time integration with a time increment of 0.01.

In the present DSC treatment, we use the DSC high-pass filters \[9\] for solving Burgers’ equation \[13\]. The DSC kernel parameters are chosen as \( W = 35 \) and \( \sigma/\Delta = 4.5 \) in these calculations. Both \( L_\infty \) and \( L_1 \) errors at 9 different times are listed in TABLE I. In an earlier work, Kakuda and Tosaka \[7\] tested their generalized boundary element method by using 100 elements, up to 6 iterations and the same time increment as ours \( (\Delta t = 0.01) \). Their results are also listed in TABLE I for a comparison. The errors in both methods are very small. The DSC results are from ten to 10\(^5\) times more accurate than those of Kakuda and Tosaka \( \text{(K-T)} \), while they were obtained using much fewer grid points. Note that at a late time, the accuracy of the present DSC algorithm reaches 15 significant figures.

The numerical solution of Burgers’ equation at a high Reynolds number \( (\text{Re} = 10^5) \) is very difficult due to the presence of shock \[2\]. A direct application of the DSC algorithm using 64 grid points \( (N = 64) \) and a small time increment \( (\Delta t = 0.001) \) leads to a highly oscillatory results as shown in FIG. 2(a). The time integration is shown up to 0.5 time units and eventually collapses at a later time. The plot is given in the spatial interval of \([0,2]\), which is generated by an antisymmetric extension of the original numerical results in the spatial interval of \([0,1]\). The oscillation starts near 0.3 time units and is accumulated and amplified in late integrations. The solid line in FIG. 1 shows the Fourier image of the result at \( t = 0.5 \). The image has two large peaks, one near the zero and another near the Nyquist frequency \( \pi/a \).

To analyze FIG. 2(a) further, a three-scale wavelet transform is performed and the result is depicted in FIG. 2(b). Daubechies’ biorthogonal wavelets \( \text{(D7/9)} \) \[18\] are employed for the wavelet transform. At the first scale, a response extended over a large domain is recorded over the southeast quarter of the quadrangle which corresponds to high frequency oscillations in FIG. 2(a). Note that high frequency oscillations reside exclusively at the southeast quarter of quadrangle because the oscillations only occur at a special frequency range. The peak in the middle of the high frequency response region is due to the shock front, which produces similar two other narrow high frequency responses at the second and third scales respectively. The response of the highest amplitude at the southwest corner is enlarged in FIG. 2(c). Surprisingly, this part seems containing the desired solution to Burgers’ equation. However, there is a kink in the late part of the low pass solution which does not belong to the desired solution. Obviously, had the high frequency oscillations been controlled in the course of integration, such a kink would not have appeared. From this analysis we conclude that the peak near the Nyquist frequency in the Fourier image in FIG 1 is due the undesired oscillations.

The control of oscillations can be accomplished in a number of ways. For example, Godunov algorithms \[19\], up-wind schemes, essentially non-oscillatory (ENO) schemes \[20\] and weighted ENO schemes \[22\] are standard methods for handling oscillations. In the present work, we propose an alternative approach which makes use of a conjugated low-pass filter, Eq. (4). Since the peak of undesired oscillations is resided outside the effective bandwidth of the conjugated low-pass filter as shown in FIG.1, it can be removed by the filtering of the conjugated low-pass filter. To eliminate oscillations and preserve the true solution effectively, we design the following conjugated filter oscillation reduction (CFOR) scheme. We define a wavelet high-pass measure via a multiscale wavelet transform of the results of conjugated high-pass filter \( \{u(x_k,t_n)\}_{k=1}^N \) (i.e. the numerical solution of Burgers’ equation) at time \( t_n \) as
\[
\| \mathcal{W}^m \| = \sum_m \| \mathcal{W}_m^m \|,
\]
where \( \| \mathcal{W}_m^m \| \) is given by a convolution with a wavelets \( \psi_{mj} \) of scale \( m \)
\[
\| \mathcal{W}_m^m \| = \sum_k \left| \sum_j \psi_{mj}(x_k)u^n(x_j) \right|.
\]
The CFOR is adaptively implemented whenever the high pass measure accesses an appropriate positive alarm threshold \( \eta \)

\[
\| W^{n+1} \| - \| W^n \| \geq \eta.
\]  

(7)

The choice of \( \eta \) depends on the time increment \( \Delta t \) and the grid size \( \Delta x \). FIG. 2(d) shows the results under the same conditions as those of FIG. 2(a), obtained by using the CFOR scheme. Note that the oscillations are eliminated and meanwhile, a sharp shock profile is resolved. The dots in FIG. 1 shows the Fourier image of the solution at time 0.5. It is noted that there is little change to the image inside the effective bandwidth of conjugated filters. However, peaks of high frequency oscillations are effectively eliminated.

We test this scheme for the case of \( Re=10^3 \) and \( Re=10^5 \) at \( \Delta x = 0.01, \Delta t = 0.001 \). DSC kernel parameters are the same as stated earlier. As shown in FIG. 3(b) and FIG. 3(c), our results are excellent. Clearly, all oscillations are effective removed and the shock front is very sharp. In a dramatic case, we consider inviscid Burgers’ equation (\( Re=\infty \)). As depicted in FIG. 3(d), our CFOR scheme works extremely well for this case too. The results for \( Re=100 \) are shown in FIG. 3(a) for a comparison.

In conclusion, a novel approach, the conjugated filter oscillation reduction (CFOR) scheme is introduced for solving Burgers’ equation with a wide range of Reynolds numbers. The essence of the CFOR scheme is to adaptively implement a conjugated low-pass filter to effectively remove the accumulated numerical errors produced by a set of high-pass filters. The conjugated low-pass and high-pass filters have essentially the same degree of regularity, smoothness, time-frequency localization, effective support and bandwidth. In this work, all conjugated filters are constructed by using discrete singular convolution kernels [13].

The accuracy of the approach is tested at a moderately high Reynolds numbers (\( Re=100 \)) for which analytical solution is available. While using much fewer grid points, our results are about 10 to \( 10^5 \) times more accurate than a previous finite element approach [7]. For extremely high Reynolds numbers (\( Re \geq 10^5 \)), the DSC algorithm develops severe oscillations near the shock front. The CFOR scheme is proposed to minimize error accumulations and to resolve the shock front. It is found that the present scheme is very accurate and robust for integrating Burgers’ equation over all possible Reynolds numbers.

We note that the present approach is very general. It can be applied to the numerical solution of other partial differential equations, particularly, compressible flows and hyperbolic conservation laws. Moreover, the CFOR scheme can be implemented along with any other standard computational methods, such as high-order central difference schemes, finite element methods and spectral approximations. A theoretical analysis of the present approach and an evaluation of various wavelets and filter banks for the CFOR scheme will be reported elsewhere [16].

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1. J. Burgers, A mathematical model illustrating the theory of turbulence, (Advances in Applied Mechanics, Academic Press, 1948).

2. P. Jamet and R. Bonnerot, J. Comput. Phys. 18, 21 (1975).

3. D. H. Jain and D. N. Holla, Int. J. Non-linear Mech. 13, 213 (1978).

4. E. Varoglu and W. D. L. Finn, Int. J. Numer. Methods Engrg. 16, 171 (1980).

5. H. Nguyen and J. Reynen, in Numerical Methods for Non-Linear Problems, Vol. 2, Proc. Int. Conf., ed. C. Taylor et al, (Universidad Politecnica de Barcelona, Spain, Pineridge Press, Swansea, U.K., 1984).

6. J. Caldwell, P. Wanless and A. E. Cook, Appl. Math. Modeling 5, 189 (1981).

7. K. Kakuda and N. Tosaka, Int. J. Numer. Methods Engrg. 29, 245 (1990).

8. P. Bar-Yoseph, E. Moses, U. Zrahia and A. L. Yarin, J. Comput. Phys. 119, 62 (1995).

9. R. Arina and C. Canuto, J. Comput. Phys. 105, 290 (1993).

10. S. Biringen and A. Saati, J. Aircraft. 27, 90 (1990).

11. G. W. Wei, D. S. Zhang, D. J. Kouri and D. K. Hoffman, Comput. Phys. Commun. 111, 93 (1998).

12. Y. C. Hon and X. Z. Mao, Appl. Math. Comput. 95, 37 (1998).

13. G. W. Wei, J. Chem. Phys., 110, 8930 (1999).
14. G. W. Wei, Physica D, 137, 247 (2000).
15. G. W. Wei, J. Phys. A, Mathematics and General, 33, 4935 (2000).
16. Y. Gu and G. W. Wei, Shock capturing by conjugated filter oscillation reduction, to be published.
17. J. D. Cole, Quart. Appl. Math. 9, 225 (1951).
18. I. Daubechies, Ten Lectures on Wavelets, (Society for Industrial and Applied Math., Philadelphia 1992).
19. S. K. Godunov, Mathematichesi Sbornik, 47, 271 (1959).
20. A. Harten, B. Engquist, S. Osher, and S. Chakravarthy, J. Comput. Phys. 71, 231 (1987).
21. C. -W. Shu and S. Osher, J. Comput. Phys. 83, 32 (1989).
22. G.-S. Jiang and C.-W. Shu, J. Comput. Phys. 126, 202 (1996).
TABLE I. Comparison of errors for solving Burgers’ equation

| Time | K-T $L_\infty$ | DSC $L_\infty$ | DSC $L_1$ |
|------|----------------|----------------|-----------|
| 0.4  | 2.6(-02)       | 2.4(-03)       | 2.2(-04)  |
| 0.8  | 2.9(-02)       | 3.3(-03)       | 2.9(-04)  |
| 1.2  | 1.8(-02)       | 4.7(-04)       | 1.1(-05)  |
| 3.0  | 6.9(-03)       | 7.6(-08)       | 1.1(-08)  |
| 10.0 | 3.1(-11)       | 1.2(-11)       |           |
| 30.0 | 1.4(-12)       | 8.6(-13)       |           |
| 60.0 | 6.9(-14)       | 4.4(-14)       |           |
| 90.0 | 3.6(-15)       | 2.3(-15)       |           |
**Figure Captions**

**FIG. 1.** Graph of the frequency responses of the conjugated DSC filters (in the unit of $\pi/\Delta$). The maximum amplitude is normalized to the unit. Stars: conjugated low-pass filter; Dashed line: 1st order high-pass filter; Dash-dots: 2nd order high-pass filter. Solid line: Fourier image of the numerical solution of Burgers’ equation ($t = 0.5$, $\text{Re}= 10^5$) with oscillations; Dots: Fourier image of the numerical solution obtained by using the CFOR scheme.

**FIG. 2.** (a) The oscillatory numerical solution of Burgers’ equation ($\text{Re}= 10^5$, $\Delta t = 0.001$, $N = 64$, $t = 0 \sim 0.5$); (b) Three scale wavelet analysis of FIG. 2(a); (c) The last scale low pass response of FIG. 2(b); (d) The CFOR solution of Burgers’ equation ($\text{Re}= 10^5$, $\Delta t = 0.001$, $N = 64$, $t = 0 \sim 0.5$).

**FIG. 3.** The CFOR solutions of Burger’s equation at $t = 0.2$ (i), 0.5 (ii), 1.0 (iii), 1.5 (iv) and 2.0 (v) ($\Delta t = 0.001, N = 101$). (a) $\text{Re}= 100$; (b) $\text{Re}= 10^3$; (c) $\text{Re}= 10^5$; (d) $\text{Re}= \infty$. 
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