An Application of the Schur Complement to Truncated Matricial Power Moment Problems

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Dedicated to the 80th birthday of M. A. Kaashoek

The main goal of this paper is to reconsider a phenomenon which was treated in earlier work of the authors’ on several truncated matricial moment problems. Using a special kind of Schur complement we obtain a more transparent insight into the nature of this phenomenon. In particular, a concrete general principle to describe it is obtained. This unifies an important aspect connected with truncated matricial moment problems.

Keywords: Truncated matricial Hamburger moment problems, truncated matricial \( \alpha \)-Stieltjes moment problems, Schur complement.

1 Introduction

In this paper, we reconsider a phenomenon which was touched in our joint research with Yu. M. Dyukarev on truncated matricial power moment problems (see [8], for the Hamburger case and [7] for the \( \alpha \)-Stieltjes case).

In this introduction, we restrict our considerations to the description of the Hamburger case. In order to describe more concretely the central topics studied in this paper, we give some notation. Throughout this paper, let \( p \) and \( q \) be positive integers. Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) be the set of all positive integers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers, respectively. For every choice of \( \rho, \kappa \in \mathbb{R} \cup \{-\infty, \infty\} \), let \( \mathbb{Z}_{\rho, \kappa} := \{ k \in \mathbb{Z} : \rho \leq k \leq \kappa \} \). We will write \( \mathbb{C}^{p \times q} \), \( \mathbb{C}_H^{q \times q} \), \( \mathbb{C}_>^{q \times q} \), and \( \mathbb{C}_>^{q \times q} \) for the set of all complex \( p \times q \) matrices, the set of all Hermitian complex \( q \times q \) matrices, the set of all non-negative Hermitian complex \( q \times q \) matrices, and the set of all positive Hermitian complex \( q \times q \) matrices, respectively. We will use \( \mathcal{B}_\mathbb{R} \) to denote the \( \sigma \)-algebra of all Borel subsets of \( \mathbb{R} \). For all \( \Omega \in \mathcal{B}_\mathbb{R} \setminus \{ \emptyset \} \), let \( \mathcal{B}_\Omega := \mathcal{B}_\mathbb{R} \cap \Omega \).
Furthermore, we will write \( \mathcal{M}_q^\geq(\Omega) \) to designate the set of all non-negative Hermitian \( q \times q \) measures defined on \( \mathfrak{B}_\Omega \), i.e., the set of \( \sigma \)-additive mappings \( \mu : \mathfrak{B}_\Omega \to \mathbb{C}_+^{q \times q} \). We will use the integration theory with respect to non-negative Hermitian \( q \times q \) measures which was worked out independently by I. S. Kats \([12]\) and M. Rosenberg \([13]\). For all \( j \in \mathbb{N}_0 \), we will use \( \mathcal{M}_{\geq,j}^q(\Omega) \) to denote the set of all \( \sigma \in \mathcal{M}_q^\geq(\Omega) \) such that the integral

\[
 s_j^{[\sigma]} := \int_{\Omega} x^j \sigma(dx)
\]

exists. Obviously, if \( k, \ell \in \mathbb{N}_0 \) with \( k < \ell \), then it can be verified, as in the scalar case, that the inclusion \( \mathcal{M}_{\geq,k}^q(\Omega) \subseteq \mathcal{M}_{\geq,k}^q(\Omega) \) holds true. Now we formulate two related versions of truncated matricial moment problems. (The Hamburger moment problem corresponds to \( \Omega = \mathbb{R} \).)

**Problem MP[\( \Omega; (s_j)_{j=0}^{m} \leq \)]** Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{m} \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{M}_{\geq,m}^q(\Omega; (s_j)_{j=0}^{m} \leq) \) of all \( \sigma \in \mathcal{M}_q^\geq(\Omega) \) for which \( s_j^{[\sigma]} = s_j \) for all \( j \in \mathbb{Z}_{0,m} \).

The just formulated moment problem is closely related to the following:

**Problem MP[\( \Omega; (s_j)_{j=0}^{m} \geq \)]** Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{m} \) be a sequence of complex \( q \times q \) matrices. Describe the set \( \mathcal{M}_{\geq,m}^q(\Omega; (s_j)_{j=0}^{m} \geq) \) of all \( \sigma \in \mathcal{M}_q^\geq(\Omega) \) for which \( s_m - s_j^{[\sigma]} \) is non-negative Hermitian and, in the case \( m \in \mathbb{N} \) moreover \( s_j^{[\sigma]} = s_j \) for all \( j \in \mathbb{Z}_{0,m-1} \).

**Remark 1.1.** If \( m \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{m} \) is a sequence of complex \( q \times q \) matrices, then \( \mathcal{M}_{\geq,m}^q(\Omega; (s_j)_{j=0}^{m} \geq) \subseteq \mathcal{M}_{\geq,m}^q(\Omega; (s_j)_{j=0}^{m} \leq) \).

**Remark 1.2.** If \( m \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{m} \) and \((t_j)_{j=0}^{m} \) are two sequences of complex \( q \times q \) matrices satisfying \( s_m - t_m \in \mathbb{C}_+^{q \times q} \) and \( s_j = t_j \) for all \( j \in \mathbb{Z}_{0,m-1} \), then \( \mathcal{M}_{\geq,m}^q(\Omega; (t_j)_{j=0}^{m} \leq) \subseteq \mathcal{M}_{\geq,m}^q(\Omega; (s_j)_{j=0}^{m} \leq) \).

In order to state a necessary and sufficient condition for the solvability of each of the above formulated moment problems in the case \( \Omega = \mathbb{R} \), we have to recall the notion of two types of sequences of matrices. If \( n \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{2n} \) is a sequence of complex \( q \times q \) matrices, then \((s_j)_{j=0}^{2n} \) is called **Hankel non-negative definite** if the block Hankel matrix \( H_n := (s_{j+k})_{j,k=0}^{n} \) is non-negative Hermitian. For all \( n \in \mathbb{N}_0 \), we will write \( \mathcal{H}_q^{\geq,n} \) for the set of all Hankel non-negative definite sequences \((s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices. Furthermore, for all \( n \in \mathbb{N}_0 \), let \( \mathcal{H}_q^{\geq,n,e} \) be the set of all sequences \((s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices for which there exist complex \( q \times q \) matrices \( s_{2n+1} \) and \( s_{2n+2} \) such that \((s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_q^{\geq,2(n+1)} \). For each \( n \in \mathbb{N}_0 \), the elements of the set \( \mathcal{H}_q^{\geq,n,e} \) are called **Hankel non-negative definite extendable** sequences. Now we can characterize the situations that the mentioned problems have a solution:

**Theorem 1.3 (\([5\) Theorem 3.2])**. Let \( n \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{2n} \) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_q^{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n} \leq] \neq \emptyset \) if and only if \((s_j)_{j=0}^{2n} \in \mathcal{H}_q^{\geq,2n} \).
In addition to Theorem 1.3, one can show that, in the case $(s_j)_{j=0}^{2n} \in H_{q,2n}^{q}$, a distinguished molecular non-negative Hermitian measure belongs to $M_{\leq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ (see, [8, Theorem 4.16]).

Now we characterize the solvability of Problem $MP[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$.

**Theorem 1.4** (cf. [9, Theorem 6.6]). Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $M_{\leq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in H_{q,2n}^{q}$.

The following result is the starting point of our subsequent considerations:

**Theorem 1.5** ([8, Theorem 7.3]). Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in H_{q,2n}^{q}$. Then there exists a unique sequence $(\tilde{s}_j)_{j=0}^{2n} \in H_{q,2n}^{q}$ such that

$$M_{\leq}^{q}[\mathbb{R}; (\tilde{s}_j)_{j=0}^{2n}, \leq] = M_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq].$$

(1.1)

Theorem 1.4 was very essential for the considerations in [8].

Following [8], we sketch now some essential features of the history of Theorem 1.5. The existence of a sequence $(\tilde{s}_j)_{j=0}^{2n} \in H_{q,2n}^{q}$ satisfying (1.1) was already formulated by V. A. Bolotnikov (see [4, Lemma 2.12] and [2, Lemma 2.12]). However it was shown in [8, pp. 804–805] by constructing a counterexample that the proof in [4] is not correct. However, in the subsequent considerations of [8, Section 7] it was then shown that the result formulated by V. A. Bolotnikov is correct and that the sequence $(\tilde{s}_j)_{j=0}^{2n}$ in Theorem 1.5 is unique. The proof of Theorem 1.5 given in [8] is constructive and does not yield a nice formula. The main goal of this paper is to present a general purely matrix theoretical object which yields applied to a special case the explicit construction of the desired sequence $(\tilde{s}_j)_{j=0}^{2n} \in H_{q,2n}^{q}$. Furthermore, we will see that another application of our construction yields the answer to a similar situation connected with truncated matricial $\alpha$-Stieltjes moment problems (see Section 6).

The discovery of the above mentioned object of matrix theory was inspired by investigations of T. Ando [1] in the context of Schur complements and its applications to matrix inequalities. (It should be mentioned that Ando’s view on the Schur complement is the content of Chapter 5 in the book [14] which is devoted to several aspects of the Schur complement.) Given a non-negative Hermitian $q \times q$ matrix $A$ and a linear subspace $V$ of $\mathbb{C}^q$, we introduce a particular non-negative Hermitian $q \times q$ matrix $G_{A,V}$ which turns out to possess several extremal properties. Appropriate choices of $A$ and $V$ lead to the construction of the desired sequences connected to the truncated matricial $\alpha$-Stieltjes moment problems under consideration.

### 2 On a special kind of Schur complement

Against to the background of application to matrix inequalities T. Ando [1] presents an operator theoretic approach to Schur complements.

In particular, T. Ando generalized the notion of Schur complement of a block matrix by considering block partitions with respect to an arbitrary fixed linear subspace. For
the case of a given non-negative Hermitian $q \times q$ matrix $A$ and a fixed linear subspace $\mathcal{V}$ the construction by T. Ando produces a non-negative Hermitian $q \times q$ matrix $G_{A,\mathcal{V}}$ having several interesting extremal properties.

For our purposes it is more convenient to choose a more matrix theoretical view as used by T. Ando \cite{1}. For this reason we use a different starting point to the main object of this section (see Definition 2.2). This leads us to a self-contained approach to several results due to Ando.

In the sequel, $\mathbb{C}^p$ is short for $\mathbb{C}^{p \times 1}$. Let $O_{p \times q}$ be the zero matrix from $\mathbb{C}^{p \times q}$. Sometimes, if the size of the zero matrix is clear from the context, we will omit the indices and write $O$. We denote by $\mathcal{N}(A):= \{x \in \mathbb{C}^q : Ax = O_{p \times 1}\}$ the null space of a complex $p \times q$ matrix $A$.

**Remark 2.1.** If $M \in \mathbb{C}^{q \times p}$ and $\mathcal{V}$ is a linear subspace of $\mathbb{C}^q$, then $\Phi_M(\mathcal{V}) := \{x \in \mathbb{C}^p : Mx \in \mathcal{V}\}$ is a linear subspace of $\mathbb{C}^p$. Obviously, $\Phi_M(\{O_{q \times 1}\}) = \mathcal{N}(M)$ and $\Phi_M(\mathbb{C}^q) = \mathbb{C}^p$.

We write $A^*$ for the conjugate transpose and $\mathcal{R}(A) := \{Ax : x \in \mathbb{C}^q\}$ for the column space of a complex $p \times q$ matrix $A$, resp. With the Euclidean scalar product $(\cdot, \cdot)_E: \mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}$ given by $(x, y)_E := y^* x$, which is $\mathbb{C}$-linear in its first argument, the vector space $\mathbb{C}^q$ over the field $\mathbb{C}$ becomes a unitary space. Let $\mathcal{U}$ be an arbitrary non-empty subset of $\mathbb{C}^q$. The orthogonal complement $\mathcal{U}^\perp := \{v \in \mathbb{C}^q : (v, u)_E = 0\}$ for all $u \in \mathcal{U}\}$ is a linear subspace of the unitary space $\mathbb{C}^q$. If $\mathcal{U}$ is a linear subspace itself, the unitary space $\mathbb{C}^q$ is the orthogonal sum of $\mathcal{U}$ and $\mathcal{U}^\perp$. In this case, we write $\mathbb{P}_\mathcal{U}$ for the transformation matrix corresponding to the orthogonal projection onto $\mathcal{U}$ with respect to the standard basis of $\mathbb{C}^q$, i.e., $\mathbb{P}_\mathcal{U}$ is the uniquely determined matrix $P \in \mathbb{C}^{q \times q}$ satisfying $P^2 = P = P^*$ and $\mathcal{R}(P) = \mathcal{U}$. For each matrix $A \in \mathbb{C}_\geq^{q \times q}$, there exists a uniquely determined matrix $Q \in \mathbb{C}_\geq^{q \times q}$ with $Q^2 = A$ called the *non-negative Hermitian square root* $Q = \sqrt{A}$ of $A$.

As a starting point of our subsequent considerations, we choose a matrix which will turn out to coincide with a matrix which was introduced in an alternate way in \cite{1} see Equation (5.1.11) and Theorem 5.8.

**Definition 2.2.** Let $A \in \mathbb{C}_\geq^{q \times q}$ and let $\mathcal{V}$ be a linear subspace of $\mathbb{C}^q$. Then we call the matrix $Q_{A,\mathcal{V}} := \mathbb{P}_{\Phi(\mathcal{V})} \sqrt{A}$ the *orthogonal projection* corresponding to $(A, \mathcal{V})$ and the matrix $G_{A,\mathcal{V}} := \sqrt{A} Q_{A,\mathcal{V}} \sqrt{A}$ is said to be the *Schur complement* associated to $A$ and $\mathcal{V}$.

Let $I_q := \{\delta_{jk}\}_{j,k=1}^{q}$ be the identity matrix from $\mathbb{C}^{q \times q}$, where $\delta_{jk}$ is the Kronecker delta. Sometimes, we will omit the indices and write $I$.

**Remark 2.3.** If $A \in \mathbb{C}_\geq^{q \times q}$, then $Q_{A,\{O_q\}} = \mathbb{P}_{\mathcal{N}(A)}$, $G_{A,\{O_q\}} = O_{q \times q}$, $Q_{A,\mathbb{C}^q} = I_q$, and $G_{A,\mathbb{C}^q} = A$.

**Remark 2.4.** Let $A \in \mathbb{C}_\geq^{q \times q}$ and let $\mathcal{V}$ be a linear subspace of $\mathbb{C}^q$. Then the matrices $Q_{A,\mathcal{V}}$ and $G_{A,\mathcal{V}}$ are both Hermitian.

**Remark 2.5.** If $\mathcal{V}$ is a linear subspace of $\mathbb{C}^q$, then $Q_{I_q,\mathcal{V}} = \mathbb{P}_\mathcal{V}$, $G_{I_q,\mathcal{V}} = \mathbb{P}_\mathcal{V}$, $Q_{\mathbb{P}_\mathcal{V},\mathcal{V}} = I_q$, and $G_{\mathbb{P}_\mathcal{V},\mathcal{V}} = \mathbb{P}_\mathcal{V}$. 

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We write $\text{rank} A$ for the rank of a complex $p \times q$ matrix $A$ and $\mathbb{C}_q^{\times q}$ for the set of positive Hermitian matrices from $\mathbb{C}^{q \times q}$. The set $\mathbb{C}_H^{q \times q}$ of Hermitian matrices from $\mathbb{C}^{q \times q}$ is a partially ordered vector space over the field $\mathbb{R}$ with positive cone $\mathbb{C}_q^+$. For two complex $q \times q$ matrices $A$ and $B$, we write $A \leq B$ or $B \geq A$ if $A, B \in \mathbb{C}_H^{q \times q}$ and $B - A \in \mathbb{C}_q^{\times q}$ are fulfilled. For a complex $q \times q$ matrix $A$, we have obviously $A \geq 0$ if and only if $A \in \mathbb{C}_q^+$. The above mentioned partial order $\leq$ on the set of Hermitian matrices is sometimes called Löwner semi-ordering. Parts of the following proposition coincide with results stated in [14, Theorems 5.3 and 5.6 in combination with Theorem 5.8].

**Proposition 2.6.** If $A \in \mathbb{C}_q^{\times q}$ and $\mathcal{V}$ is a linear subspace of $\mathbb{C}^q$, then:

(a) $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{R}(\sqrt{A}Q_{A,\mathcal{V}})$, $\mathcal{N}(G_{A,\mathcal{V}}) = \mathcal{N}(Q_{A,\mathcal{V}}\sqrt{A})$, and $O_{q \times q} \leq G_{A,\mathcal{V}} \leq A$.

(b) $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{R}(A) \cap \mathcal{V}$, $\mathcal{N}(G_{A,\mathcal{V}}) = \mathcal{N}(A) + \mathcal{V}^\perp$, and in particular $\text{rank} G_{A,\mathcal{V}} \leq \min\{\text{rank} A, \dim \mathcal{V}\}$.

(c) The following statements are equivalent:
   (i) $G_{A,\mathcal{V}} = A$.
   (ii) $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{R}(A)$.
   (iii) $\mathcal{R}(A) \subseteq \mathcal{V}$.

(d) $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{V}$ if and only if $\mathcal{V} \subseteq \mathcal{R}(A)$.

(e) $G_{A,\mathcal{V}} \in \mathbb{C}_q^{\times q}$ if and only if $A \in \mathbb{C}_q^{\times q}$ and $\mathcal{V} = \mathbb{C}^q$.

(f) If $\mathcal{V} \neq \mathbb{C}^q$, then $G_{A,\mathcal{V}} \in \mathbb{C}_q^{\times q} \setminus \mathbb{C}_q^{\times q}$.

**Proof.**

(i) Use $O_{q \times q} \leq Q_{A,\mathcal{V}} \leq I_q$ and $Q_{A,\mathcal{V}}^2 = Q_{A,\mathcal{V}}$.

(ii) First we consider an arbitrary $y \in \mathcal{R}(G_{A,\mathcal{V}})$. According to (i), we have $y = \sqrt{A}Q_{A,\mathcal{V}} z$ with some $z \in \mathbb{C}^q$. Obviously, $x := Q_{A,\mathcal{V}} z$ belongs to $\Phi_A(\mathcal{V})$ and fulfills $y = \sqrt{A}x$. In particular, $y$ belongs to $\mathcal{R}(A) \cap \mathcal{V}$. Conversely, assume that $y$ belongs to $\mathcal{R}(A) \cap \mathcal{V}$. Then $y \in \mathcal{R}(\sqrt{A})$, i.e. there exists an $x \in \mathbb{C}^q$ with $y = \sqrt{A}x$. Consequently, $\sqrt{A}x \in \mathcal{V}$, i.e. $x \in \Phi_A(\mathcal{V})$. This implies $Q_{A,\mathcal{V}} x = x$. Hence, $y \in \mathcal{R}(\sqrt{A}Q_{A,\mathcal{V}})$. Taking (i) into account, then $y \in \mathcal{R}(G_{A,\mathcal{V}})$ follows. Thus, $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{R}(A) \cap \mathcal{V}$ is proved. Therefore, we get

\[
\mathcal{N}(G_{A,\mathcal{V}}) = \mathcal{R}(G_{A,\mathcal{V}})^\perp = \mathcal{R}(G_{A,\mathcal{V}})^\perp = [\mathcal{R}(A) \cap \mathcal{V}]^\perp = \mathcal{R}(A)^\perp + \mathcal{V}^\perp = \mathcal{N}(A^*) + \mathcal{V}^\perp = \mathcal{N}(A) + \mathcal{V}^\perp.
\]

(iii) Condition (i) is obviously sufficient for (ii). According to (ii) statements (ii) and (iii) are equivalent. If (iii) holds true then $\Phi_{\sqrt{A}}(\mathcal{V}) = \mathbb{C}^q$ and, consequently, $Q_{\sqrt{A},\mathcal{V}} = I_q$. This means that (iii) implies (i).

(iv) This equivalence follows from (iii).

(v) This is an immediate consequence of the definition of $G_{A,\mathcal{V}}$.

(vi) This follows from (i) and (v).
Remark 2.7. If $A \in \mathbb{C}^{p \times q}$ and $\mathcal{V}$ is a linear subspace of $\mathbb{C}^q$, then Proposition 2.6 shows that $\mathcal{R}(G_{A,\mathcal{V}}) = \mathcal{V}$.

In the sequel the Moore–Penrose inverse plays an essential role. For this reason, we recall this notion. For each matrix $A \in \mathbb{C}^{p \times q}$, there exists a uniquely determined matrix $X \in \mathbb{C}^{q \times p}$, satisfying the four equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad (XA)^* = XA.$$  

This matrix $X$ is called the Moore–Penrose inverse of $A$ and is denoted by $A^\dagger$.

Given $n \in \mathbb{N}$ and arbitrary rectangular complex matrices $A_1, A_2, \ldots, A_n$, we write $\text{diag} (A_j)_{j=1}^n = \text{diag}(A_1, A_2, \ldots, A_n) := [\delta_{jk} A_j]_{j,k=1}^n$ for the corresponding block diagonal matrix.

The following lemma yields essential insights into the structure of the Schur complement associated to $A$ and $\mathcal{V}$.

**Lemma 2.8.** Assume $q \geq 2$, let $d \in \mathbb{Z}_{1,q-1}$, and let $\mathcal{V}$ be a linear subspace of $\mathbb{C}^q$ with $\dim \mathcal{V} = d$. Let $u_1, u_2, \ldots, u_d$ be an orthonormal basis of $\mathbb{C}^d$ such that $u_1, u_2, \ldots, u_d$ is an orthonormal basis of $\mathcal{V}$ and let $U := [u_1, \ldots, u_q]$. Let $A \in \mathbb{C}^{q \times q}$ and let $B = [B_{jk}]_{j,k=1}^q$ be the block representation of $B := U^* AU$ with $d \times q$ block $B_{11}$. Then

$$U^* G_{A,\mathcal{V}} U = \text{diag}(B_{11} - B_{12} B_{22}^\dagger B_{21}, O_{(q-d) \times (q-d)}).$$  

(2.1)

**Proof.** Let $R := U^* \sqrt{A} U$ and let $R = [F_{G}]$ be the block representation of $R$ with $d \times q$ block $F$. Then $B = RR^* = [F F^* E E^*]$. Consequently,

$$B_{11} - B_{12} B_{22}^\dagger B_{21} = E \left[ I_q - F^* (F F^*)^\dagger F \right] E^* = E (I_q - F F^*) E^*.$$  

(2.2)

We set $U_1 := [u_1, \ldots, u_q]$ and $\mathcal{W} := \Phi_{\sqrt{A}}(\mathcal{V})$. Because of $U^* U = I_q$, we have

$$U^* \mathcal{W} = \{ U^* x : x \in \mathcal{W} \} = \{ y \in \mathbb{C}^q : \sqrt{A} U y \in \mathcal{V} \} = \{ y \in \mathbb{C}^q : \sqrt{A} U y \in \mathcal{R}(U_1) \} = \{ y \in \mathbb{C}^q : \exists z \in \mathbb{C}^d : \sqrt{A} U y = U [O_{(q-d) \times 1}] \} = \{ y \in \mathbb{C}^q : [O_{(q-d) \times d}, I_{q-d}] U^* \sqrt{A} U y = O_{(q-d) \times 1} \} = \mathcal{N}(F).$$

Hence, $\mathbb{P}_{U^* \mathcal{W}} = I_q - F^\dagger F$. Since $U^* \mathbb{P}_{\mathcal{W}} U$ is an idempotent and Hermitian complex matrix fulfilling $\mathcal{R}(U^* \mathbb{P}_{\mathcal{W}} U) = U^* \mathcal{W}$, we get then $U^* \mathbb{P}_{\mathcal{W}} U = I_q - F^\dagger F$. Thus, in view of $\mathbb{P}_{\mathcal{W}} = Q_{A,\mathcal{V}}$, we obtain

$$U^* G_{A,\mathcal{V}} U = (U^* \sqrt{A} U)(U^* \mathbb{P}_{\mathcal{W}} U)(U^* \sqrt{A} U)^*$$

$$= R(I_q - F^\dagger F) R^* = \begin{bmatrix} E(I_q - F^\dagger F) E^* & E(I_q - F^\dagger F) F^* \\ F(I_q - F^\dagger F) E^* & F(I_q - F^\dagger F) F^* \end{bmatrix}.$$  

Using (2.2), $F(I_q - F^\dagger F) = O$, and $F^\dagger F = (F^\dagger F)^*$, then (2.1) follows. \qed
The following observation makes clear why in Definition 2.2 the terminology “Schur complement associated to $A$ and $V$” was chosen.

**Remark 2.9.** Assume $q \geq 2$, let $d \in \mathbb{Z}_{1, q-1}$, let $A = [A_{jk}]_{j,k=1}^2$ be the block representation of a matrix $A \in \mathbb{C}_{\geq 2}^{d \times q}$ with $d \times d$ block $A_{11}$, and let $V := \mathcal{R}(I_d \otimes O_{(q-d) \times d})$. Then $G_{A, V} = \text{diag}(A_{11} - A_{12} A_{22}^* A_{21}, O_{(q-d) \times (q-d)})$.

The following result shows in combination with [14, Theorem 5.1] that the construction introduced in Definition 2.2 coincides with the matrix introduced by Ando [11, Formula (5.1.11)].

**Proposition 2.10.** Let $A \in \mathbb{C}_{\geq 2}^{d \times q}$ and let $V$ be a linear subspace of $\mathbb{C}^q$. For all $x \in \mathbb{C}^q$, then

$$x^* G_{A, V} x = \min_{y \in V^\perp} (x - y)^* A(x - y).$$

(2.3)

**Proof.** Set $d := \dim V$. We consider an arbitrary $x \in \mathbb{C}^q$. If $d = 0$, then Proposition 2.6 yields $G_{A, V} = O_{q \times q}$ and, because of $A \in \mathbb{C}_{\geq 2}^{d \times q}$, therefore (2.3). If $d = q$, then $V^\perp = \{0\}$ and Proposition 2.6 yields $G_{A, V} = A$, which implies (2.3). Now we consider the case $1 \leq d \leq q - 1$. Let $u_1, u_2, \ldots, u_d$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_d$ is an orthonormal basis of $V$. Let $U_1 := [u_1, \ldots, u_d]$ and let $U_2 := [u_{d+1}, \ldots, u_q]$. Then $U := [U_1, U_2]$ is unitary. Let $B := [B_{jk}]_{j,k=1}^2$ be the block representation of $B := U^* A U$ with $d \times d$ block $B_{11}$. Setting $S := B_{11} - B_{12} B_{22}^* B_{21}$, from Lemma 2.2 we get $U^* G_{A, V} U = \text{diag}(S, O_{(q-d) \times (q-d)})$. Let $f_1 := U_1^* x$ and let $f_2 := U_2^* x$. Then $f := U^* x$ admits the block representation $f = [f_1 \ f_2]$. Thus, we get

$$x^* G_{A, V} x = f^* \left[ \text{diag}(S, O_{(q-d) \times (q-d)}) \right] f = f_1^* S f_1.$$  

(2.4)

We consider an arbitrary $y \in V^\perp$. Setting $g_1 := U_1^* y$ and $g_2 := U_2^* y$, we see that $g := U^* y$ admits the block representation $g = [g_1 \ g_2]$ and that $g_1 = O_{d \times 1}$. Thus, $h := f - g$ can be represented via $h = [h_1 \ h_2]$, where $h_1 := f_1$ and $h_2 := f_2 - g_2$. The matrices $B$ and $B_{22}$ are obviously non-negative Hermitian. Therefore, using a well-known factorization formula (see, e.g. [6, Lemmata 1.1.9 and 1.1.7]), we have $B = R^* \text{diag}(S, B_{22}) R$ where $R := \begin{bmatrix} I_d & O_{d \times (q-d)} \\ B_{22}^{\dagger} B_{21} & I_{q-d} \end{bmatrix}$. It is easily checked that $R h = [f_1 \ r_2 - g_2]$ where $r_2 := B_{22}^{\dagger} B_{21} f_1 + f_2$.

Applying $x - y = U h$ and (2.4) we conclude then

$$(x - y)^* A(x - y) = h^* B h = \begin{bmatrix} f_1 \\ r_2 - g_2 \end{bmatrix}^* \begin{bmatrix} f_1 \\ r_2 - g_2 \end{bmatrix} = f_1^* S f_1 - (r_2 - g_2)^* B_{22} (r_2 - g_2) = x^* G_{A, V} x - (r_2 - g_2)^* B_{22} (r_2 - g_2).$$

Consequently, $(x - y)^* A(x - y) \geq x^* G_{A, V} x$ for all $y \in V^\perp$ with equality if and only if $r_2 - g_2 \in \mathcal{N}(B_{22})$. Obviously, the particular vector $y := U_2 r_2$ belongs to $V^\perp$ and we have $g_2 = r_2$. Thus, $y := U_2 r_2$ fulfills $B_{22}(r_2 - g_2) = O$. 

$\square$
Proposition 2.10 establishes the coincidence of the matrix introduced in Definition 2.2 with the construction used by T. Ando [1, Formula (5.1.11)]. This leads us to interesting insights about the main object of this section.

For two Hermitian \( q \times q \) matrices \( A \) and \( B \) with \( A \preceq B \), the (closed) matricial interval \([A, B] := \{ X \in \mathbb{C}_H^{q \times q} : A \preceq X \preceq B \}\) is non-empty.

**Notation 2.11.** If \( A \in \mathbb{C}_H^{q \times q} \) and \( V \) is a linear subspace of \( \mathbb{C}^q \), then \( \mathcal{H}_{A,V} := \{ X \in [O_{q,q}, A] : \mathcal{R}(X) \subseteq V \} \).

**Theorem 2.12** (cf. [14, Theorem 5.3]). If \( A \in \mathbb{C}_H^{q \times q} \) and \( V \) is a linear subspace of \( \mathbb{C}^q \), then \( G_{A,V} \in \mathcal{H}_{A,V} \) and \( G_{A,V} \geq X \) for all \( X \in \mathcal{H}_{A,V} \).

**Proof.** For the convenience of the reader, we reproduce the proof given in [14, Theorem 5.3]: From parts (i) and (ii) of Proposition 2.6 we infer \( G_{A,V} \in \mathcal{H}_{A,V} \). Now consider an arbitrary \( X \in \mathcal{H}_{A,V} \). Furthermore, consider an arbitrary \( x \in \mathbb{C} \) and an arbitrary \( y_0 \in V^\perp \). Because of \( \mathcal{R}(X) \subseteq V \) and \( X \in \mathbb{C}_H^{q \times q} \), then \( y_0 \in \mathcal{R}(X) \perp = \mathcal{N}(X^*) = \mathcal{N}(X) \), implying \((x - y_0)^*X(x - y_0) = x^*Xx \). Because of \( A \geq X \), moreover \((x - y_0)^*A(x - y_0) \geq (x - y_0)^*X(x - y_0) \). Taking additionally into account Proposition 2.11 we thus obtain (2.23) and, hence, \( x^*G_{A,V}x \geq x^*Xx \). Consequently, \( G_{A,V} \geq X \).

**Remark 2.13.** Let \( A, B \in \mathbb{C}_H^{q \times q} \). Then \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) if and only if \( B \preceq \gamma A \) for some \( \gamma \in (0, \infty) \). In this case, \( \gamma_0 := \text{sup}\{ (x^*Bx)(x^*Ax)^{-1} : x \in \mathbb{C} \setminus \mathcal{N}(A) \} \) fulfills \( \gamma_0 \in [0, \infty) \) and \( B \preceq \gamma_0 A \).

**Lemma 2.14** (cf. [14, Equivalence (5.0.7)]). If \( A, B \in \mathbb{C}_H^{q \times q} \), then \( \mathcal{R}(A) \cap \mathcal{R}(B) = \{ O_{q \times 1} \} \) if and only if \( [O_{q,q}, A] \cap [O_{q,q}, B] = \{ O_{q \times q} \} \).

**Proof.** Let \( A, B \in \mathbb{C}_H^{q \times q} \). Then \( O_{q,q} \in [O_{q,q}, A] \cap [O_{q,q}, B] \).

First assume \( \mathcal{R}(A) \cap \mathcal{R}(B) = \{ O_{q \times 1} \} \). We consider an arbitrary \( X \in [O_{q,q}, A] \cap [O_{q,q}, B] \). Then \( \mathcal{R}(X) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B) \), by virtue of Remark 2.13. Consequently, \( X = O_{q \times q} \).

Conversely, assume that \( \mathcal{R}(A) \cap \mathcal{R}(B) \neq \{ O_{q \times 1} \} \). Then \( P := \mathbb{P}_{\mathcal{R}(A) \cap \mathcal{R}(B)} \) fulfills \( P \in \mathbb{C}_H^{q \times q} \setminus \{ O_{q \times q} \} \) and \( \mathcal{R}(P) = \mathcal{R}(A) \cap \mathcal{R}(B) \). From Remark 2.13 we can conclude then the existence \( \alpha, \beta \in (0, \infty) \) with \( P \preceq \alpha A \) and \( P \preceq \beta B \). Then \( \gamma := \text{min}\{ 1/\alpha, 1/\beta \} \) fulfills \( \gamma \in (0, \infty) \) and \( \gamma P \in ([O_{q,q}, A] \cap [O_{q,q}, B]) \setminus \{ O_{q \times q} \} \).

**Theorem 2.15** (cf. [14, Theorem 5.7]). Let \( A \in \mathbb{C}_H^{q \times q} \) and let \( V \) be a linear subspace of \( \mathbb{C}^q \). Then:

(a) \( \mathcal{R}(A - G_{A,V}) \cap V = \{ O_{q \times 1} \} \).

(b) Let \( X, Y \in \mathbb{C}_H^{q \times q} \) be such that \( X + Y = A \). Then the following statements are equivalent:

(i) \( \mathcal{R}(X) \subseteq V \) and \( \mathcal{R}(Y) \subseteq V = \{ O_{q \times 1} \} \).

(ii) \( X = G_{A,V} \) and \( Y = A - G_{A,V} \).

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Proof. For the convenience of the reader, we reproduce the proof given in [13, Theorem 5.7]: First observe that \( O_{q \times q} \leq G_{A, V} \leq A \) and \( R(G_{A, V}) \subseteq V \), by virtue of parts (a) and (b) of Proposition 2.6.

(a) We have \( \{ A - G_{A, V}, \mathbb{F}_V \} \subseteq \mathbb{C}_{q \times q}^{\geq} \). Hence, \( O_{q \times q} \in \{ A - G_{A, V}, \mathbb{F}_V \} \cap \{ O_{q \times q}, \mathbb{F}_V \} \). Consider an arbitrary \( X \in \{ O_{q \times q}, A - G_{A, V} \} \cap \{ O_{q \times q}, \mathbb{F}_V \} \). Then \( X \in \mathbb{C}_{q \times q}^{\geq} \) and, moreover, \( X \leq A - G_{A, V} \) and \( X \leq \mathbb{F}_V \). In particular, \( R(X) \subseteq R(\mathbb{F}_V) = V \), by virtue of Remark 2.13.

Taking additionally into account \( G_{A, V} \in \mathbb{C}_{q \times q}^\geq \) and \( R(G_{A, V}) \subseteq V \), we thus obtain \( G_{A, V} + X \in H_{A, V} \). Hence, Theorem 2.12 yields \( O_{q \times q} \leq G_{A, V} + X \leq G_{A, V} \). This implies \( X = O_{q \times q} \). Consequently, \( \{ O_{q \times q}, A - G_{A, V} \} \cap \{ O_{q \times q}, \mathbb{F}_V \} = \{ O_{q \times q} \} \). According to Lemma 2.14, then \( R(A - G_{A, V}) \cap R(\mathbb{F}_V) = \{ O_{q \times 1} \} \) follows. In view of \( R(\mathbb{F}_V) = V \), the proof of part (a) is complete.

(b) Obviously \( O_{q \times q} \leq X \leq A \).

First suppose (i). Then \( X \in H_{A, V} \). Theorem 2.12 yields then \( G_{A, V} \geq X \). Taking into account \( A - G_{A, V} \in \mathbb{C}_{q \times q}^{\geq} \), thus \( O_{q \times q} \leq G_{A, V} - X \leq A - X \). According to Remark 2.13 and \( Y = A - X \), then \( R(G_{A, V} - X) \subseteq R(Y) \). Because of \( X \in \mathbb{C}_{q \times q}^{\geq} \), we can furthermore conclude \( O_{q \times q} \leq G_{A, V} - X \leq G_{A, V} \). By virtue of Remark 2.13 and \( R(G_{A, V}) \subseteq V \), then \( R(G_{A, V} - X) \subseteq R(G_{A, V}) \subseteq V \). Taking additionally into account (i), we thus obtain \( R(G_{A, V} - X) \subseteq R(Y) \cap V = \{ O_{q \times 1} \} \). Consequently, \( G_{A, V} = X \). Thus, (ii) holds true.

If we conversely suppose (ii), then (i) follows from \( R(G_{A, V}) \subseteq V \) and (a). \( \square \)

3 On a restricted extension problem for a finite Hankel non-negative definite extendable sequence

This section is written against to the background of applying the results of Section 2 to the truncated matricial Hamburger moment problems formulated in the introduction for \( \Omega = \mathbb{R} \). In the heart of our strategy lies the treatment of a special restricted extension problem for matrices. The complete answer to this problem is contained in Theorem 3.17 which is the central result of this section.

If \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \) and \( (s_j)_{j=0}^\kappa \) is a sequence of complex \( p \times q \) matrices, then let \( H_n := [s_{j+k}]_{j,k=0}^n \) for all \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \).

For each \( n \in \mathbb{N}_0 \), by \( H_{q,2n}^\infty \) we denote the set of all sequences \( (s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices for which the corresponding block Hankel matrix \( H_n \) is non-negative Hermitian. Furthermore, denote by \( H_{q,2n}^\kappa \) the set of all sequences \( (s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices satisfying \( (s_j)_{j=0}^{2n} \in H_{q,2n}^\infty \) for all \( n \in \mathbb{N}_0 \). The sequences belonging to \( H_{q,2n}^\kappa \) or \( H_{q,2n}^\infty \) are said to be Hankel non-negative definite. Using [11, Lemma 3.2], we can conclude:

Remark 3.1. If \( \kappa \in \mathbb{N}_0 \cup \{ \infty \} \) and if \( (s_j)_{j=0}^{2n} \in H_{q,2n}^\infty \), then \( s_j \in \mathbb{C}_{q \times q} \) for all \( j \in \mathbb{Z}_{0,2n} \) and \( s_{2k} \in \mathbb{C}_{q \times q}^\geq \) for all \( k \in \mathbb{Z}_{0,\kappa} \).

Given \( n \in \mathbb{N} \) arbitrary rectangular complex matrices \( A_1, A_2, \ldots, A_n \), we write \( \text{col} (A_j)_{j=1}^n = \text{col}(A_1, A_2, \ldots, A_n) \) (resp., \( \text{row} (A_j)_{j=1}^n := [A_1, A_2, \ldots, A_n] \)) for the block column (resp., block row) build from the matrices \( A_1, A_2, \ldots, A_n \) if their numbers of columns (resp., rows) are all equal.
Notation 3.2. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)^n_{j=0}$ be a sequence of complex $p \times q$ matrices.

(a) Let $y_{\ell,m} := \text{col} (s_j)^m_{j=\ell}$ and $z_{\ell,m} := \text{row} (s_j)^m_{j=\ell}$ for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m \leq \kappa$.

(b) Let $\Theta_0 := O_{p \times q}$ and let $\Theta_n := z_{n,2n-1}H_{n-1}^T y_{n,2n-1}$ for all $n \in \mathbb{N}$ with $2n-1 \leq \kappa$.

(c) Let $L_n := s_{2n} - \Theta_n$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

Remark 3.3. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2\kappa}$, then [8, Remark 2.1(a)] shows that $(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2\kappa}$ and $L_n \in \mathbb{C}^{q \times q}$ for all $n \in \mathbb{Z}_{0,\kappa}$.

Let $n \in \mathbb{N}_0$. Denote by $\mathcal{H}_{q,2n}^{e}$ the set of all sequences $(s_j)^{2n}_{j=0}$ of complex $q \times q$ matrices for which there exists a pair $(s_{2n+1}, s_{2n+2})$ of complex $q \times q$ matrices such that the sequence $(s_j)^{2n+2}_{j=0}$ belongs to $\mathcal{H}_{q,2n+2}^{e}$. Denote by $\mathcal{H}_{q,2n+1}^{e}$ the set of all sequences $(s_j)^{2n+1}_{j=0}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix $s_{2n+2}$ such that the sequence $(s_j)^{2n+2}_{j=0}$ belongs to $\mathcal{H}_{q,2n+2}^{e}$. Furthermore, let $\mathcal{H}_{q,\infty}^{e} := \mathcal{H}_{q,\infty}^{e}$. The sequences belonging to $\mathcal{H}_{q,2n}^{e}$, $\mathcal{H}_{q,2n+1}^{e}$, or $\mathcal{H}_{q,\infty}^{e}$ are said to be Hankel non-negative definite extendable.

Remark 3.4. Remark 3.3 shows that $\mathcal{H}_{q,2n}^{e} \subseteq \mathcal{H}_{q,2\kappa}^{e}$ for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$.

Remark 3.5. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)^{n}_{j=0} \in \mathcal{H}_{q,\kappa}^{e}$, then, because of Remarks 3.3 and 3.1, we have $\Theta_n \in \mathbb{C}^{q \times q}$ for all $n \in \mathbb{Z}_{0,\kappa}$.

Notation 3.6. Let $n \in \mathbb{N}$ and let $(s_j)^{2n-1}_{j=0}$ is a sequence of complex $q \times q$ matrices. Then let $\mathcal{H}_{q,2n-1}^{e}$ be the set of all $s_{2n} \in \mathbb{C}^{q \times q}$ for which $(s_j)^{2n}_{j=0}$ belongs to $\mathcal{H}_{q,2n}^{e}$. Obviously, $\mathcal{H}_{q,2n-1}^{e} \neq \emptyset$ if and only if $(s_j)^{2n-1}_{j=0} \in \mathcal{H}_{q,2n-1}^{e}$.

Proposition 3.7 ([8 Proposition 2.22(a)]). Let $n \in \mathbb{N}$, let $(s_j)^{2n-1}_{j=0} \in \mathcal{H}_{q,2n-1}^{e}$, and let $s_{2n} \in \mathbb{C}^{q \times q}$. Then $s_{2n} \in \mathcal{H}_{q,2n-1}^{e}$ if and only if $L_n \in \mathbb{C}^{q \times q}$.

Notation 3.8. Let $n \in \mathbb{N}$, let $(s_j)^{2n-1}_{j=0}$ be a sequence of complex $q \times q$ matrices, and let $Y \in \mathbb{C}^{q \times q}$. Then denote by $\mathcal{H}_{q,2n-1}^{e}$ the set of all $X \in \mathcal{H}_{q,2n-1}^{e}$ satisfying $Y - X \in \mathbb{C}^{q \times q}$.

Lemma 3.9. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2\kappa}^{e}$. Then $O_{q \times q} \leq \Theta_n \leq s_{2n}$ for all $n \in \mathbb{Z}_{0,\kappa}$.

Proof. Remark 3.1 shows that $O_{q \times q} \leq \Theta_0 \leq s_0$. Now assume $\kappa \geq 1$ and consider an arbitrary $n \in \mathbb{Z}_{1,\kappa}$. Remark 3.1 yields $s_{2n} \in \mathbb{C}^{q \times q}$. According to Remark 3.3, furthermore $L_n \in \mathbb{C}^{q \times q}$ and $(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^{e}$. In particular, $(s_j)^{2n-1}_{j=0} \in \mathcal{H}_{q,2n-1}^{e}$. Thus, Remark 3.3 and Notation 3.2(1) yield $O_{q \times q} \leq \Theta_n \leq s_{2n}$.

To indicate that a certain (block) matrix $X$ is built from a sequence $(s_j)^{n}_{j=0}$, we sometimes write $X(s)$ for $X$.

Proposition 3.10. If $n \in \mathbb{N}$ and $(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^{e}$, then $O_{q \times q} \leq \Theta_n \leq s_{2n}$ and $\mathcal{H}_{q,2n}^{e} = [\Theta_n, s_{2n}]$. 
Proof. By virtue of Lemma 3.9, we have $O_{q \times q} \leq \Theta_n \leq s_{2n}$. In particular, the matrices $\Theta_n$ and $s_{2n}$ are Hermitian. Let the sequence $(t_j)_{j=0}^{2n-1}$ be given by $t_j := s_j$ for each $j \in \mathbb{Z}_{0,2n-1}$. Obviously, then $\Theta_n^{(t)} = \Theta_n$ and $(s_j)_{j=0}^{2n-1}, (t_j)_{j=0}^{2n-1} \subseteq \mathcal{H}_{q,2n-1}^{\geq, e}$. Obviously, then

$$t_{2n} - \Theta_n = t_{2n} - \Theta_n^{(t)} = L_n^{(t)}.$$  (3.1)

We first consider an arbitrary $t_{2n} \in \mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}, s_{2n}]$. Then $(t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq, e}$ and $s_{2n} - t_{2n} \in \mathbb{C}^{q \times q}_e$. In particular, $t_{2n} \in \mathbb{C}^{q \times q}_e$. Since Remark 3.3 shows that $L_n^{(t)}$ is non-negative Hermitian, (3.1) yields $\Theta_n \leq t_{2n} \leq s_n$.

Conversely, we consider now an arbitrary $t_{2n} \in \{\Theta_n, s_n\}$. Then $s_{2n} - t_{2n} \in \mathbb{C}^{q \times q}_e$ and $t_{2n} - \Theta_n \in \mathbb{C}^{q \times q}_e$. Thus, (3.1) yields $L_n^{(t)} \in \mathbb{C}^{q \times q}_e$. According to Proposition 3.7, thus $(t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq, e}$. Consequently, $t_{2n} \in \mathcal{H}_{q,2n}^{\geq, e}\{\Theta_n, s_n\}$. \qed

**Corollary 3.11.** If $n \in \mathbb{N}_0$ and $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq, e}$, then $\{s_{2n}^2; s \in \mathcal{M}_{q,2n}^{\geq, e}\} \subseteq [\Theta_n, s_{2n}]$.

**Proof.** In view of $\Theta_0 = O_{q \times q}$, the case $n = 0$ is obvious. If $n \geq 1$, combine Theorem 1.4 Remark 3.9 and Proposition 3.10. \qed

**Notation 3.12.** If $n \in \mathbb{N}$ and $(s_j)_{j=0}^{2n-1}$ is a sequence of complex $q \times q$ matrices. Then let $\mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}]$ be the set of all complex $q \times q$ matrices $s_{2n}$ such that $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq, e}$.

From [S] Proposition 2.22(a), (b) we know that $\mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq, e}$.

**Proposition 3.13** (cf. [S] Proposition 2.22(b)). Let $n \in \mathbb{N}$, let $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n}^{\geq, e}$, and let $s_{2n} \in \mathbb{C}^{q \times q}_e$. Then $s_{2n} \in \mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}]$ if and only if $L_n \in \mathbb{C}^{q \times q}_e$ and $\mathcal{R}(L_n) \subseteq \mathcal{R}(L_{n-1})$.

**Notation 3.14.** Let $n \in \mathbb{N}$, let $(s_j)_{j=0}^{2n-1}$ be a sequence of complex $q \times q$ matrices, and let $Y \in \mathbb{C}^{q \times q}_e$. Then denote by $\mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}, Y]$ the set of all $X \in \mathcal{H}_{q,2n}^{\geq, e}[(s_j)_{j=0}^{2n-1}]$ satisfying $Y - X \in \mathbb{C}^{q \times q}_e$.

Observe that the following construction is well defined, due to Remark 3.3 and Definition 2.2.

**Notation 3.15.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\geq, e}$. Then let $\Xi_0 := s_0$. If $n \in \mathbb{N}_1, \kappa$, then let $\Xi_n := \Theta_n + G_{L_n, \mathcal{R}(L_{n-1})}$.

**Lemma 3.16.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\geq, e}$. For all $n \in \mathbb{N}_0, \kappa$, then

$$O_{q \times q} \leq \Theta_n \leq \Xi_n \leq s_{2n}.$$  (3.2)

**Proof.** According to Lemma 3.9, we have $O_{q \times q} \leq \Theta_n \leq s_{2n}$ for all $n \in \mathbb{N}_0, \kappa$. In particular, the matrices $\Theta_n$ and $s_{2n}$ are Hermitian for all $n \in \mathbb{N}_0, \kappa$. By virtue of $\Xi_0 = s_0$, we get (3.2) for $n = 0$. Now assume $\kappa \geq 1$ and $n \in \mathbb{N}_1, \kappa$. According to Remark 3.3, we have $L_n \in \mathbb{C}^{q \times q}_e$. From Proposition 2.6 we obtain $O_{q \times q} \leq G_{L_n, \mathcal{R}(L_{n-1})} \leq L_n$, implying, by virtue of Notation 3.15 and Notation 3.2, then $O_{q \times q} \leq \Theta_n \leq \Xi_n \leq s_{2n}$. \qed
Theorem 3.17. Let \( n \in \mathbb{N} \) and let \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \). Then:

(a) \( \Theta_n, \Xi_n \) and \( L_n \in \mathbb{C}^{q \times q} \). With \( A := L_n \) and \( \mathcal{V} := \mathcal{R}(L_{n-1}) \) we have \( \Xi_n = \Theta_n + G_{A,V} \) and \( s_{2n} = \Theta_n + A \). Let the sequence \( (t_j)_{j=0}^{2n-1} \) be given by \( t_j := s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Then \( L_n = L_{n-1} \) and \( \mathcal{V} = \mathcal{R}(L_{n-1}) \). Furthermore, \( \Xi_n = s_{2n} \) if and only if \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \).

(b) \( \Xi_n = s_{2n} \) if and only if \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \).

Proof. From Lemma 3.16 we get \( \Xi_n \). In particular, the matrices \( \Theta_n, \Xi_n, \) and \( s_{2n} \) are Hermitian. Remark 3.3 yields \( L_n \in \mathbb{C}^{q \times q} \). With \( A := L_n \) and \( \mathcal{V} := \mathcal{R}(L_{n-1}) \) we have \( \Xi_n = \Theta_n + G_{A,V} \) and \( s_{2n} = \Theta_n + A \). Let the sequence \( (t_j)_{j=0}^{2n-1} \) be given by \( t_j := s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Then \( L_n = L_{n-1} \) and \( \mathcal{V} = \mathcal{R}(L_{n-1}) \). Furthermore, \( \Xi_n = s_{2n} \) if and only if \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \).

First consider now an arbitrary \( t_{2n} \in \mathcal{H}_{q,2n}^e \). Then \( (t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \). In particular, \( t_{2n} \in \mathbb{C}^{q \times q} \). Let \( X = t_{2n} - \Theta_n \). Hence, \( X \leq s_{2n} - \Theta_n = A \). Consequently, \( X \in \mathcal{H}_{A,V} \). Theorem 3.13 yields then \( G_{A,V} \leq X \). In view of Notation 3.13, we have \( \Xi_n = s_{2n} \).

Conversely, let \( t_{2n} \in [\Theta_n, \Xi_n] \). Then \( X = t_{2n} - \Theta_n \) is Hermitian and fulfills \( O_{q \times q} \leq X \leq G_{A,V} \). Hence, \( \mathcal{R}(X) \subseteq \mathcal{R}(G_{A,V}) \) and, according to Proposition 3.13, \( \mathcal{R}(G_{A,V}) = \mathcal{R}(A) \). Consequently, \( \mathcal{R}(X) \subseteq \mathcal{R}(A) \). Therefore, \( L_n = L_{n-1} \) and \( \mathcal{V} = \mathcal{R}(L_{n-1}) \). According to Proposition 3.13, \( (t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \). Moreover, because of \( s_{2n} - t_{2n} = A - X \), we get \( t_{2n} \leq s_{2n} \). Consequently, \( t_{2n} \in \mathcal{H}_{q,2n}^e \).

In view of \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \), we have \( (s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^e \) and \( s_{2n} \in \mathcal{H}_{q,2n}^e \). Proposition 3.7 yields then \( L_n \in \mathbb{C}^{q \times q} \). By virtue of Notation 3.15 and Notation 3.2 we have \( \Xi_n = s_{2n} \) if and only if \( L_n \in \mathcal{R}(L_{n-1}) \). According to Proposition 3.13, \( \mathcal{R}(L_{n}) \subseteq \mathcal{R}(L_{n-1}) \). In view of \( (s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^e \) and \( L_n \in \mathbb{C}^{q \times q} \), Proposition 3.13 shows the equivalence of \( \mathcal{R}(L_{n}) \subseteq \mathcal{R}(L_{n-1}) \). Hence, \( \Xi_n = s_{2n} \).

Theorem 3.17 leads us now quickly in an alternative way to one of the main results of [3].

Theorem 3.18 (cf. [3] Theorem 7.8). If \( n \in \mathbb{N}_0 \) and \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \), then \( \{ s_{2n}^{[\sigma]} : \sigma \in \mathcal{M}_{q,2n}^{[\mathbb{R}] (s_j)_{j=0}^{2n}, \leq} \} = [\Theta_n, \Xi_n] \).

Proof. In view of \( \Theta_0 = O_{q \times q} \) and \( \Xi_0 = s_0 \), the case \( n = 0 \) is obvious. If \( n \geq 1 \), combine Theorems 3.3 and 3.17.
4 On equivalence classes of truncated matricial moment problems of type \( MP[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] \)

This section contains an aspect of our considerations in [8]. We are striving for a natural classification of the set of truncated matricial Hamburger moment problems of type \( \leq \). From Theorem 1.3 we see that these problems have a solution if and only if the sequence of data is Hankel non-negative definite. This leads us to the following relation in the set \( \mathcal{H}_{q,2n}^\geq \).

**Notation 4.1.** If \( n \in \mathbb{N}_0 \) and \( \{(s_j)^{2n}_{j=0}, (t_j)^{2n}_{j=0}\} \subseteq \mathcal{H}_{q,2n}^\geq \), then we write \((s_j)^{2n}_{j=0} \sim_\mathbb{R} (t_j)^{2n}_{j=0}\) if \( M^q_\geq[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] = M^q_\geq[\mathbb{R}; (t_j)^{2n}_{j=0}, \leq] \).

**Remark 4.2.** Let \( n \in \mathbb{N}_0 \). Then the relation \( \sim_\mathbb{R} \) is an equivalence relation on the set \( \mathcal{H}_{q,2n}^\geq \).

Let \( n \in \mathbb{N}_0 \). If \((s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^\geq\), then let \( \langle(s_j)^{2n}_{j=0}\rangle_\mathbb{R} := \{(t_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^\geq: (t_j)^{2n}_{j=0} \sim_\mathbb{R} (s_j)^{2n}_{j=0}\} \). Furthermore, if \( \mathcal{S} \) is a subset of \( \mathcal{H}_{q,2n}^\geq\), then let \( \langle\mathcal{S}\rangle_\mathbb{R} := \{\langle(s_j)^{2n}_{j=0}\rangle_\mathbb{R}: (s_j)^{2n}_{j=0} \in \mathcal{S}\} \).

Looking back to Theorem 1.5 we see that each equivalence class contains a unique representative belonging to \( \mathcal{H}_{q,2n}^\geq \). The considerations of Section 3 provide us now not only detailed insights into the explicit structure of this distinguished representative but even an alternative approach. The following notion is the central object of this section.

**Definition 4.3.** If \( n \in \mathbb{N}_0 \) and \((s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^\geq\), then the sequence \((\tilde{s}_j)^{2n}_{j=0}\) given by \( \tilde{s}_2n := \Xi_n \), where \( \Xi_n \) is given in Notation 3.15, and by \( \tilde{s}_j := s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \) is called the Hankel non-negative definite extendable sequence equivalent to \((s_j)^{2n}_{j=0}\).

Now we derive the announced sharpened version of Theorem 1.5.

In the following, we will use the notation given in Definition 4.3.

**Proposition 4.4.** Let \( n \in \mathbb{N}_0 \) and let \((s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^\geq\). Then \((\tilde{s}_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^{\geq, e}\) and \((\tilde{s}_j)^{2n}_{j=0} \sim_\mathbb{R} (s_j)^{2n}_{j=0}\).

**Proof.** In the case \( n = 0 \), we have \( \tilde{s}_0 = \Xi_0 = s_0 \) and \( \mathcal{H}_{q,0}^\geq = \mathcal{H}_{q,0}^{\geq, e} \). Now assume \( n \geq 1 \). According to Theorem 3.17 we have \( \tilde{s}_{2n} \leq s_{2n} \) and, in view of Definition 4.3, furthermore, \( \tilde{s}_j = s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). From Remark 1.2 we get then \( M^q_\geq[\mathbb{R}; (\tilde{s}_j)^{2n}_{j=0}, \leq] \subseteq M^q_\geq[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] \). Conversely, we consider now an arbitrary \( \sigma \in M^q_\geq[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] \). Let the sequence \( (u_j)^{2n}_{j=0} \) be given by \( u_j := \int_{\mathbb{R}} x^j \sigma(dx) \). Then \( \tilde{s}_{2n} - u_{2n} \in \mathbb{C}^{2n\times q} \) and \( u_j = s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Furthermore, \( \sigma \in M^q_\geq[\mathbb{R}; (u_j)^{2n}_{j=0}, \leq] \), implying \((u_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^\geq\), by virtue of Theorem 1.3. Consequently, \( u_{2n} \in \mathcal{H}_{q,0, e}(s_j)^{2n}_{j=0}, s_{2n}\). According to Theorem 3.17 thus \( u_{2n} \in [\Theta_n, \tilde{s}_{2n}] \). In particular, \( \tilde{s}_{2n} - u_{2n} \in \mathbb{C}^{2n\times q} \). Since \( u_j = s_j = \tilde{s}_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \), then Remark 1.2 yields \( M^q_\geq[\mathbb{R}; (u_j)^{2n}_{j=0}, \leq] \subseteq M^q_\geq[\mathbb{R}; (\tilde{s}_j)^{2n}_{j=0}, \leq] \). Taking additionally into account Remark 1.1 we can conclude \( \sigma \in M^q_\geq[\mathbb{R}; (\tilde{s}_j)^{2n}_{j=0}, \leq] \). Hence, \( M^q_\geq[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] \subseteq M^q_\geq[\mathbb{R}; (\tilde{s}_j)^{2n}_{j=0}, \leq] \). Consequently, \( M^q_\geq[\mathbb{R}; (\tilde{s}_j)^{2n}_{j=0}, \leq] = M^q_\geq[\mathbb{R}; (s_j)^{2n}_{j=0}, \leq] \), implying \( \tilde{s}_j^{2n}_{j=0} \sim_\mathbb{R} s_j^{2n}_{j=0} \). \( \square \)
Proposition 4.5. Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \). If \( (t_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \) satisfies \( (t_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \), then \( (t_j)_{j=0}^{2n} \) coincides with the Hankel non-negative definite extendable sequence equivalent to \( (s_j)_{j=0}^{2n} \).

Proof. Let \( (t_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \), be such that \( (t_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \). Observe that \( (t_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \), by virtue of Remark 4.2. In view of Remark 4.2, we infer from Proposition 4.3 then \( (t_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \), i.e., \( \mathcal{M}^\geq_2 [\mathbb{R}; (t_j)_{j=0}^{2n}, \leq] = \mathcal{M}^\geq_2 [\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \). According to Theorem 4.4, we have chosen a measure \( \tau \in \mathcal{M}_2^\geq [\mathbb{R}; (t_j)_{j=0}^{2n}, \leq] \). By virtue of Remark 4.2, \( \tau \in \mathcal{M}^\geq_2 [\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \). Consequently, we have \( t_{2n} = \int_{\mathbb{R}} \tau (dx) \leq \tilde{s}_{2n} \) and \( t_j = \int_{\mathbb{R}} \tau (dx) = \tilde{s}_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Since \( (s_j)_{j=0}^{2n} \) belongs to \( \mathcal{H}^ \geq_{q,2n} \), according to Proposition 4.4, we conclude in a similar way \( \tilde{s}_{2n} \leq t_{2n} \). Hence, \( t_{2n} = \tilde{s}_{2n} \) follows.

Now we state the main result of this section, which sharpens Theorem 1.5.

Theorem 4.6. Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \). Then \( \{(s_j)_{j=0}^{2n}\}_R \cap \mathcal{H}^ \geq_{q,2n} = \{(s_j)_{j=0}^{2n}\}_R \).

Proof. Combine Propositions 4.1 and 4.5.

Our next aim can be described as follows. Let \( n \in \mathbb{N} \) and let \( (s_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \). Then an appropriate application of Theorem 2.15 leads us to the determination of all sequences \( (r_j)_{j=0}^{2n} \) which are contained in the equivalence class \( \{(s_j)_{j=0}^{2n}\}_R \).

Proposition 4.7. Let \( n \in \mathbb{N} \) and let \( (s_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \). Then \( \{(s_j)_{j=0}^{2n}\}_R \) coincides with the set of all sequences \( (r_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices fulfilling \( \mathcal{R}(t_{2n} - \Xi_n) \cap \mathcal{R}(L_{n-1}) = \{O_{q\times 1}\}, r_{2n} - \Xi_n \in \mathbb{C}^{q\times q}, \) and \( r_j = s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \).

Proof. Let \( (t_j)_{j=0}^{2n} \) be the Hankel non-negative definite extendable sequence equivalent to \( (s_j)_{j=0}^{2n} \). By virtue of Proposition 4.4 we have \( (t_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \) and \( (t_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \). According to Definition 4.3 furthermore \( t_{2n} = \Xi_n \) and \( t_j = s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). In particular, \( L_{n-1} = L_{n-1} \) and \( \Theta_{n} = \Theta_{n} \).

Consider now an arbitrary \( (r_j)_{j=0}^{2n} \in \{(s_j)_{j=0}^{2n}\}_R \), i.e., \( (r_j)_{j=0}^{2n} \in \mathcal{H}^ \geq_{q,2n} \) with \( (r_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \). In particular, \( (t_j)_{j=0}^{2n} \sim \mathbb{R} \ (s_j)_{j=0}^{2n} \), by virtue of Remark 4.2. From Proposition 4.3 we can conclude then that \( (t_j)_{j=0}^{2n} \) coincides with the Hankel non-negative definite extendable sequence equivalent to \( (r_j)_{j=0}^{2n} \). In view of Definition 4.3, consequently \( t_{2n} = \Xi_n \) and \( t_j = r_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Hence, \( \Xi_n = \Xi_n \) and \( s_j = r_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). In particular, \( L_{n-1} = L_{n-1} \) and \( \Theta_{n} = \Theta_{n} \). From Remark 3.3 we know that \( L_{n} \in \mathbb{C}^{q\times q} \).

Setting \( A := L_{n} \) and \( V := \mathcal{R}(L_{n-1}) \), we obtain, in view of Notation 3.15 and Notation 3.2, then

\[
 r_{2n} - \Xi_n = r_{2n} - \Xi_n = r_{2n} - \Theta_{n} - G_{A,V} = L_{n} - G_{A,V} = A - G_{A,V}.
\]

Because of \( \mathcal{R}(L_{n-1}) = V \) and Theorem 4.4, thus \( \mathcal{R}(r_{2n} - \Xi_n) \cap \mathcal{R}(L_{n-1}) = \{O_{q\times 1}\} \).

Furthermore, Proposition 2.6 yields \( r_{2n} - \Xi_n \in \mathbb{C}^{q\times q} \).
Conversely, we consider now an arbitrary sequence \( (r_j)^{2n}_{j=0} \) of complex \( q \times q \) matrices fulfilling \( R(2n - \Xi) \cap R(L_{n-1}) = \{ O_{q \times 1} \} \), \( r_{2n} - \Xi \in \mathbb{C}_{\geq}^{q \times q} \), and \( r_j = s_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Then \( L_{n-1}^{(r)} = L_{n-1} \) and \( \Theta_{n \times q}^{(r)} = \Theta_n \). Because of \( (s_j)^{2n}_{j=0} \in \mathbb{H}_{\geq}^{q \times 2n} \), we have \( (r_j)^{2n}_{j=0} \in \mathbb{H}_{\geq}^{q \times 2n} \). From Lemma 3.16 we infer that \( \Xi_n = \Xi_n \) with \( \Theta_n \leq \Xi_n \). Consequently, \( r_{2n} \in \mathbb{C}_{\geq}^{q \times q} \) and \( \Theta_n \leq \Xi_n \leq r_{2n} \). Hence, \( \Theta_n^{(r)} \leq r_{2n} \), i.e., \( L_{n}^{(r)} \in \mathbb{C}_{\geq}^{q \times q} \). By virtue of Proposition 3.7 we obtain then \( (r_j)^{2n}_{j=0} \in \mathbb{H}_{\geq}^{q \times 2n} \). Denote by \( \tilde{(r_j)^{2n}_{j=0}} \) the Hankel non-negative definite extendable sequence equivalent to \( (r_j)^{2n}_{j=0} \) and let \( A := L_n^{(r)} \) and \( V := R(L_{n-1}^{(r)}) \). By Definition 3.3 and Notation 3.15, \( \tilde{r}_{2n} = \Theta_n^{(r)} + G_{A,V} \) and \( \tilde{r}_j = r_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Consequently, \( \tilde{r}_j = r_j = s_j = t_j \) for all \( j \in \mathbb{Z}_{0,2n-1} \). Setting \( X := t_{2n} - \Theta_n^{(r)} \) and \( Y := r_{2n} - t_{2n} \), we have, by virtue of Notation 4.2(c), then \( X + Y = r_{2n} - \Theta_n^{(r)} = L_n^{(r)} = A \) and \( X = t_{2n} - \Theta_n = t_{2n} - \Theta_n^{(t)} = L_n^{(t)} \) and, furthermore, \( Y = r_{2n} - \Xi_n \). By assumption, then \( R(Y) \cap R(L_{n-1}) = \{ O_{q \times 1} \} \) and \( Y \in \mathbb{C}_{\geq}^{q \times q} \). From Remark 3.4 we infer \( (t_j)^{2n}_{j=0} \in \mathbb{H}_{\geq}^{q \times 2n} \). In particular, \( (t_j)^{2n}_{j=0} \in \mathbb{H}_{\geq}^{q \times 2n} \) and \( t_{2n} \in \mathbb{H}_{\geq}^{(s_j)^{2n}_{j=0}} \). Proposition 3.13 yields then \( L_n^{(t)} \in \mathbb{C}_{\geq}^{q \times q} \) and \( R(L_n^{(t)}) \subseteq R(L_{n-1}^{(t)}) \). Hence, \( X \in \mathbb{C}_{\geq}^{q \times q} \). Taking into account \( L_{n-1}^{(t)} = L_{n-1} = L_n^{(t)} \), we see furthermore \( R(X) \subseteq V \) and \( R(Y) \cap V = \{ O_{q \times 1} \} \). From Theorem 2.16(b) we get then \( X = G_{A,V} \). Hence, \( \tilde{r}_{2n} = \Theta_n^{(r)} + G_{A,V} = t_{2n} \). Thus, the sequences \( (\tilde{r}_j)^{2n}_{j=0} \) and \( (t_j)^{2n}_{j=0} \) coincide. Using Proposition 4.4 and Remark 4.2 we get then \( (r_j)^{2n}_{j=0} \sim \mathbb{R} (\tilde{r}_j)^{2n}_{j=0} \sim \mathbb{R} (t_j)^{2n}_{j=0} \). Consequently, \( (r_j)^{2n}_{j=0} \in \langle (s_j)^{2n}_{j=0} \rangle_{\mathbb{R}} \).

5 On truncated matricial \([\alpha, \infty)\)-Stieltjes moment problems

In our following considerations, let \( \alpha \) be a real number. In order to state a necessary and sufficient condition for the solvability of each of the moment problems

\[
\text{MP}[[\alpha, \infty); (s_j)^{2n}_{j=0} ; \leq] \text{ and } \text{MP}[[\alpha, \infty); (s_j)^{2n}_{j=0} ; =],
\]

we have to recall the notion of two types of sequences of matrices.

Let \( \kappa \in \mathbb{N} \cup \{ \infty \} \) and let \( (s_j)^{\kappa}_{j=0} \) is a sequence of complex \( p \times q \) matrices. For each \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let the block Hankel matrix \( K_n \) be given by \( K_n := [s_j + k + 1]_{j,k=0}^{n} \). Furthermore, let the sequence \( (a_j)^{\kappa-1}_{j=0} \) be given by \( a_j := -\alpha s_j + s_{j+1} \). For each matrix \( X_k = X_k^{(s)} \) built from the sequence \( (s_j)^{\kappa}_{j=0} \), denote (if possible) by \( X_{\alpha,k} := X_k^{(a)} \) the corresponding matrix built from the sequence \( (a_j)^{\kappa-1}_{j=0} \) instead of \( (s_j)^{\kappa}_{j=0} \). In particular, we have then \( H_{\alpha,n} = -\alpha H_n + K_n \) for all \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \). In the classical case \( \alpha = 0 \), we see that \( a_j = s_{j+1} \) for all \( j \in \mathbb{Z}_{0,\kappa-1} \).

Let \( K_{\geq,0,\alpha} := \mathbb{H}_{\geq,0} \). For each \( n \in \mathbb{N} \), denote by \( K_{\geq,2n,\alpha} \) the set of all sequences \( (s_j)^{2n}_{j=0} \) of complex \( q \times q \) matrices for which the block Hankel matrices \( H_n \) and \( H_{\alpha,n} \) are both non-negative Hermitian. For each \( n \in \mathbb{N}_0 \), denote by \( K_{\geq,2n+1,\alpha} \) the set of all sequences \( (s_j)^{2n+1}_{j=0} \) of complex \( q \times q \) matrices for which the block Hankel matrices \( H_n \) and \( H_{\alpha,n} \) are both non-negative Hermitian. Furthermore, denote by \( K_{\geq,\infty,\alpha} \) the set of all sequences \( (s_j)^{\kappa}_{j=0} \) of complex \( q \times q \) matrices satisfying \( (s_j)^{\kappa}_{j=0} \in K_{\geq,\kappa,m,\alpha} \) for all \( m \in \mathbb{N}_0 \). The
sequences belonging to $\mathcal{K}_{q,0,\alpha}^{\geq}, \mathcal{K}_{q,2n,\alpha}^{\geq}, \mathcal{K}_{q,2n+1,\alpha}^{\geq}, \text{ or } \mathcal{K}_{q,\infty,\alpha}^{\geq}$ are said to be $\alpha$-Stieltjes right-sided non-negative definite.

Now we can characterize the situations that the mentioned problems have a solution:

**Theorem 5.1** ([7, Theorem 1.4]). Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_q^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$.

Now we characterize the solvability of Problem $\text{MP}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$.

**Theorem 5.2** ([7, Theorem 1.3]). Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_q^{q^2}[[\alpha, \infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha,\alpha}^{\geq}$.

The following result is the starting point of our subsequent considerations:

**Theorem 5.3** ([7, Theorem 5.2]). Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$.

Then there exists a unique sequence $(\hat{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ such that

$$\mathcal{M}_q^q[[\alpha, \infty); (\hat{s}_j)_{j=0}^m, \leq] = \mathcal{M}_q^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq].$$  \hspace{1cm} (5.1)

**Theorem 5.3** was very essential for the considerations in [7].

The main goal of the rest of this paper is to derive this result by use of an appropriate application of the machinery developed in Section 2. This will lead us to an explicit formula for the desired sequence $(\hat{s}_j)_{j=0}^m$.

Following [7] we sketch now some essential features of the history of Theorem 5.3.

In the case $\alpha = 0$ the existence of a sequence $(\hat{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ satisfying (5.1) was already formulated by V. A. Bolotnikov [3, Theorem 1.5, Lemma 1.6]. This result is true. However, it was shown in [7, Example 5.1] that the concrete sequence $(\hat{s}_j)_{j=0}^m$ constructed in [3, Lemmata 2.7 and 6.3] does not produce a moment problem equivalent to $\text{MP}[[0, \infty); (s_j)_{j=0}^m, \leq]$.

6 On a restricted extension problem for a finite $\alpha$-Stieltjes right-sided non-negative definite extendable sequence

This section is written against to the background of applying the results of Section 2 to the truncated matricial $[\alpha, \infty)$-Stieltjes moment problems formulated in the introduction for $\Omega = [\alpha, \infty)$. In the heart of our strategy lies the treatment of a special restricted extension problem for matrices. The complete answer to this problem is contained in Theorem 6.20 which is the central result of this section.

Using Remark 1.1 we can conclude:

**Remark 6.1.** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. Then $s_j \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{N}_0$ and $s_{2k} \in \mathbb{C}_H^{q \times q}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$. If $\kappa \geq 1$, furthermore $a_j \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{N}_0, j \neq 0$ and $a_{2k} \in \mathbb{C}_H^{q \times q}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa - 1$.

**Definition 6.2** ([10, Definition 4.2]). If $(s_j)_{j=0}^\kappa$ is a sequence of complex $p \times q$ matrices, then the sequence $(Q_j)_{j=0}^\kappa$ given by $Q_{2k} := s_{2k} - \Theta_k$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and $Q_{2k+1} := a_{2k} - \Theta_{\alpha,k}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$ is called the right-sided $\alpha$-Stieltjes parametrization of $(s_j)_{j=0}^\kappa$.  

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Remark 6.3. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^n \in \mathcal{K}_{q,k}^{\geq}$, then one can easily see from Theorem 4.12(b) that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$ and $Q_m \in \mathbb{C}^{q \times q}$ for all $m \in \mathbb{Z}_{0,k}$.

Notation 6.4. If $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m$ is a sequence of complex $q \times q$ matrices, then denote by $\mathcal{K}_{\geq}[m]((s_j)_{j=0}^m)$ the set of all complex $q \times q$ matrices $s_{m+1}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1}^{\geq}$. For each $m \in \mathbb{N}_0$, denote by $\mathcal{K}_{\geq}^{\geq}[m]((s_j)_{j=0}^m)$ the set of all sequences $(s_j)_{j=0}^m$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix $s_{m+1}$ such that the sequence $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1}^{\geq}$. Furthermore, let $\mathcal{K}_{\geq}^{\geq} := \mathcal{K}_{q,\infty}^{\geq}$. The sequences belonging to $\mathcal{K}_{q,m}^{\geq}$ or $\mathcal{K}_{q,\infty}^{\geq}$ are said to be $\alpha$-Stieltjes right-sided non-negative definite extendable. Obviously, if $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m$ is a sequence of complex $q \times q$ matrices, then $\mathcal{K}_{\geq}(s_j)_{j=0}^m \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$. The following result shows why the class $\mathcal{K}_{q,m}^{\geq}$ is important.

Theorem 6.5 (Theorem 1.6)). Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^n$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}^{[\geq]}((\alpha, \infty);(s_j)_{j=0}^n, \geq) \neq \emptyset$ if and only if $(s_j)_{j=0}^n \in \mathcal{K}_{q,\infty}^{\geq}$.

Remark 6.6. From Remark 6.3 one can easily see that $\mathcal{K}_{\geq}^{\geq} \subseteq \mathcal{K}_{q,\infty}^{\geq}$ for all $k \in \mathbb{N}_0 \cup \{\infty\}$.

Proposition 6.7 (cf. Theorem 4.12(b), (c)). Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$, and let $s_{m+1} \in \mathbb{C}^{q \times q}$. Then $s_{m+1} \in \mathcal{K}_{\geq}(s_j)_{j=0}^m$ if and only if $Q_{m+1} \in \mathbb{C}_{\geq}^{q \times q}$.

Notation 6.8. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^n$ is a sequence of complex $p \times q$ matrices, then let $a_{2k-1} := \Theta_k$ for all $k \in \mathbb{N}_0$ with $2k-1 \leq k$ and let $a_{2k} := \alpha s_{2k} + \Theta_{\alpha,k}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$.

Remark 6.9. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^n$ is a sequence of complex $p \times q$ matrices, then Definition 6.2 shows that $Q_j = s_j - a_{j-1}$ for all $j \in \mathbb{Z}_{0,k}$.

Remark 6.10. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^n \in \mathcal{K}_{q,k}^{\geq}$, then Remarks 6.3 and 6.1 show that $a_m = a_m$ for all $m \in \mathbb{Z}_{-1,k}$.

Notation 6.11. Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices, and let $Y \in \mathbb{C}^{q \times q}$. Then denote by $\mathcal{K}_{\geq}(s_j)_{j=0}^m \cap Y$ the set of all matrices satisfying $Y - X \in \mathbb{C}^{q \times q}$.

Lemma 6.12. If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^n \in \mathcal{K}_{q,k}^{\geq}$, then $a_{m-1} \leq s_m$ for all $m \in \mathbb{Z}_{0,k}$.

Proof. Obviously, $a_{-1} = Q_{q \times q} \leq s_0$ by virtue of Remark 6.1. Now assume $k \geq 1$ and we consider an arbitrary $m \in \mathbb{Z}_{1,k}$. Remark 6.1 yields $s_m \in \mathbb{C}_{q \times q}^{q \times q}$. Remark 6.3 shows that $Q_m \in \mathbb{C}_{\geq}^{q \times q}$. In view of Remark 6.3, then $a_{m-1} \in \mathbb{C}_{\geq}^{q \times q}$ and $a_{m-1} \leq s_m$. \qed

Theorem 6.13. If $m \in \mathbb{N}$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$, then $a_{m-1} \leq s_m$ and $\mathcal{K}_{\geq}(s_j)_{j=0}^{m-1} \cap s_m ] = [a_{m-1}, s_m]$.

Proof. By virtue of Lemma 6.12, we have $[a_{m-1}, s_m] \subseteq \mathbb{C}_{\geq}^{q \times q}$ and $a_{m-1} \leq s_m$. Let the sequence $(t_j)_{j=0}^{m-1}$ be given by $t_j := s_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Obviously, then $a_{m-1} = a_{m-1}$. Furthermore, $\{(s_j)_{j=0}^{m-1}, (t_j)_{j=0}^{m-1}\} \subseteq \mathcal{K}_{q,m-1,\alpha}^{\geq}$.\}
First consider now an arbitrary \( t_m \in \mathcal{K}_{\geq,\alpha}[(s_j)_{j=0}^{m-1}, s_m] \). Then \((t_j)_{j=0}^{m} \in \mathcal{K}_{\geq,\alpha}^{\geq,q}\) and \( s_m - t_m \in \mathcal{C}_{\geq}^{\geq,q} \). In particular, \( t_m \in \mathcal{C}_{\geq}^{\geq,q} \). In view of Remark 6.9, we get then \( a_{m-1} = t_m - a_{m-1} = Q_m^{(t)} \), and Remark 6.3, we get then \( a_{m-1} \leq t_m \leq s_m \).

Conversely, let \( t_m \in [a_{m-1}, s_m] \). Then \( s_m - t_m \in \mathcal{C}_{\geq}^{\geq,q} \) and, in view of Remark 6.9, furthermore \( Q_m^{(t)} = t_m - a_{m-1} = t_m - a_{m-1} \). In particular, \( Q_m^{(t)} \) is non-negative Hermitian. According to Proposition 6.7, thus \( (t_j)_{j=0}^{m} \in \mathcal{K}_{\geq,\alpha}^{\geq,q} \). Consequently, \( t_m \in \mathcal{K}_{\geq,\alpha}[(s_j)_{j=0}^{m-1}, s_m] \).

\[ \square \]

**Corollary 6.14.** If \( m \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{m} \in \mathcal{K}_{\geq,\alpha}^{\geq,q}\), then \( \{ s_{[\cdot]}^m : \sigma \in \mathcal{M}_{\geq}^{\geq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq] \} \subseteq [a_{m-1}, s_m] \).

**Proof.** In view of \( a_{-1} = \Theta_0 = O_{q \times q} \), the case \( m = 0 \) is obvious. If \( m \geq 1 \), combine Theorem 6.5, Remark 6.6 and Theorem 6.13.

**Notation 6.15.** If \( m \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{m} \) is a sequence of complex \( q \times q \) matrices, then denote by \( \mathcal{K}_{\geq,\alpha}^{\geq,e}[(s_j)_{j=0}^{m}] \) the set of all complex \( q \times q \) matrices \( s_{m+1} \) such that \((s_j)_{j=0}^{m+1} \in \mathcal{K}_{\geq,\alpha}^{\geq,q}\).

The following result is essential for the realization of our concept of a new approach to Theorem 5.3.

**Theorem 6.16 (cf. [10] Theorem 4.12(c)).** Let \( m \in \mathbb{N}_0 \), let \((s_j)_{j=0}^{m} \in \mathcal{K}_{\geq,\alpha}^{\geq,q}\), and let \( s_{m+1} \in \mathcal{C}_{\geq}^{\geq,q} \). Then \( s_{m+1} \in \mathcal{K}_{\geq,\alpha}^{\geq}(s_j)_{j=0}^{m} \) if and only if \( Q_{m+1} \in \mathcal{C}_{\geq}^{\geq,q} \) and \( R(Q_{m+1}) \subseteq R(Q_m) \).

**Notation 6.17.** Let \( m \in \mathbb{N}_0 \), let \((s_j)_{j=0}^{m} \) be a sequence of complex \( q \times q \) matrices, and let \( Y \in \mathcal{C}_{\geq}^{\geq,q} \). Then denote by \( \mathcal{K}_{\geq,\alpha}^{\geq,e}(s_j)_{j=0}^{m}, Y \) the set of all \( X \in \mathcal{K}_{\geq,\alpha}^{\geq,q}[(s_j)_{j=0}^{m}] \) satisfying \( Y - X \in \mathcal{C}_{\geq}^{\geq,q} \).

Observe that the following construction is well defined due to Remark 6.3.

**Notation 6.18.** If \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and \((s_j)_{j=0}^{\kappa} \in \mathcal{K}_{\geq,\alpha}^{\geq,q}\), then let \( \Gamma_0 := s_0 \) and let \( \Gamma_m := a_{m-1} + G_{Q_m, R(Q_{m-1})} \) for all \( m \in \mathbb{Z}_{1,\kappa} \).

**Lemma 6.19.** Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \((s_j)_{j=0}^{\kappa} \in \mathcal{K}_{\geq,\alpha}^{\geq,q} \). Then \( \Gamma_\kappa = \Gamma_m = a_{m-1} \leq \Gamma_m \leq s_m \) for all \( m \in \mathbb{Z}_{0,\kappa} \).

**Proof.** Let \( m \in \mathbb{Z}_{0,\kappa} \). From Lemma 6.12, we see that the matrices \( a_{m-1} \) and \( s_m \) are Hermitian and that \( a_{m-1} \leq s_m \). Remark 2.4 shows that \( \Gamma_\kappa = \Gamma_m \). By virtue of \( \Gamma_0 = s_0 \), we get in particular \( a_{-1} \leq \Gamma_0 \leq s_0 \). Now assume \( \kappa \geq 1 \) and consider an arbitrary \( m \in \mathbb{Z}_{1,\kappa} \). According to Remark 6.3, we have \( Q_m \in \mathcal{C}_{\geq}^{\geq,q} \). From Remark 2.3 and Proposition 2.6, we obtain \( O_{q \times q} \leq G_{Q_m, R(Q_{m-1})} \leq Q_m \), implying, by virtue of Notation 6.18 and Remark 6.9, then \( a_{m-1} \leq \Gamma_m \leq s_m \).

**Theorem 6.20.** Let \( m \in \mathbb{N} \) and let \((s_j)_{j=0}^{m} \in \mathcal{K}_{\geq,\alpha}^{\geq,q} \). Then:

(a) \( a_{m-1} \leq \Gamma_m \leq s_m \) and \( \mathcal{K}_{\geq,\alpha}^{\geq,q}[(s_j)_{j=0}^{m-1}, s_m] = [a_{m-1}, \Gamma_m] \).

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Theorem 6.21
This section contains an aspect of our considerations in [7]. We are striving for a natural
classification of the set of truncated matricial [α, ∞)-Stieltjes moment problems of type

\[ \text{MP}\{[\alpha, \infty); (s_j)_{j=0}^m; \leq\} \]

This section contains an aspect of our considerations in [7]. We are striving for a natural
classification of the set of truncated matricial [α, ∞)-Stieltjes moment problems of type
“≤”. From Theorem 5.1 we see that these problems have a solution if and only if the sequence of data is \( \alpha \)-Stieltjes right-sided non-negative definite. This leads us to the following relation in the set \( \mathcal{K}_{q,m,\alpha}^{\geq} \).

If \( m \in \mathbb{N}_0 \) and \((s_j)^m_{j=0}, (t_j)^m_{j=0} \) are two sequences belonging to \( \mathcal{K}_{q,m,\alpha}^{\geq} \), then we write \((s_j)^m_{j=0} \sim_{[\alpha, \infty)} (t_j)^m_{j=0} \) if \( \mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)^m_{j=0}; \leq] = \mathcal{M}_{\geq}^q[[\alpha, \infty); (t_j)^m_{j=0}; \leq] \) holds true.

**Remark 7.1.** It is readily checked that \( \sim_{[\alpha, \infty)} \) is an equivalence relation on the set \( \mathcal{K}_{q,m,\alpha}^{\geq} \).

**Notation 7.2.** Let \( m \in \mathbb{N}_0 \). If \((s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq} \), then let \( \langle(s_j)^m_{j=0};[\alpha, \infty) \rangle := \{(t_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq}; (t_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\} \). Furthermore, if \( S \) is a subset of \( \mathcal{K}_{q,m,\alpha}^{\geq} \), then let \( \langle S \rangle_{[\alpha, \infty)} := \{(s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq}; (s_j)^m_{j=0} \in S\} \).

Regarding [7] Theorem 5.2 we see that each equivalence class contains a unique representative belonging to \( \mathcal{K}_{q,m,\alpha}^{\geq} \). The considerations of Section 6 provide us now not only detailed insights into the explicit structure of this distinguished representative but even an alternative approach. The following notion is the central object of this section.

**Definition 7.3.** If \( m \in \mathbb{N}_0 \) and \((s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq} \), then the sequence \((\hat{s}_j)^m_{j=0} \) given by \( \hat{s}_m := \Gamma_m \), where \( \Gamma_m \) is given in Notation 6.1.3 and by \( \hat{s}_j := s_j \) for all \( j \in \mathbb{Z}_0,m-1 \) is called the \( \alpha \)-Stieltjes right-sided non-negative definite extendable sequence equivalent to \((s_j)^m_{j=0}\).

Now we derive a sharpened version of [7] Theorem 5.2.

**Proposition 7.4.** Let \( m \in \mathbb{N}_0 \) and let \((s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq} \). Then the \( \alpha \)-Stieltjes right-sided non-negative definite extendable sequence \((\hat{s}_j)^m_{j=0}\) equivalent to \((s_j)^m_{j=0}\) belongs to \( \mathcal{K}_{q,m,\alpha}^{\geq,e} \) and \((\hat{s}_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\).

**Proof.** Let \( m \in \mathbb{N}_0 \) and let \((s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq} \). First assume \( m = 0 \). We have \( \hat{s}_0 = \Gamma_0 = s_0 \). In view of \( \mathcal{K}_{q,0,\alpha}^{\geq} = \mathcal{K}_{q,0,\alpha}^{\geq,e} \), then the assertions follow. Now assume \( m \geq 1 \). According to Theorem 6.20 we have \( \hat{s}_m \leq s_m \) and, in view of Definition 7.3, furthermore \( \hat{s}_j = s_j \) for all \( j \in \mathbb{Z}_0,m-1 \). From Remark 1.2 we get then \( \mathcal{M}_{\geq}^q[[\alpha, \infty); (\hat{s}_j)^m_{j=0}; \leq] \subseteq \mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)^m_{j=0}; \leq] \). Conversely, let \( \sigma \in \mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)^m_{j=0}; \leq] \). Let the sequence \((u_j)^m_{j=0}\) be given by \( u_j := \int_{[\alpha, \infty)} \sigma(dz) \). Then \( u_m - u_j \in \mathbb{C}^{q \times q} \) and \( u_j = s_j \) for all \( j \in \mathbb{Z}_0,m-1 \). Furthermore, \( \sigma \in \mathcal{M}_{\geq}^q[[\alpha, \infty); (u_j)^m_{j=0}; =] \), implying \( (u_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq,e} \), by virtue of Theorem 6.6. Consequently, \( u_m \in \mathcal{K}_{q,\alpha}^{\geq,e}(s_j)^m_{j=0}; s_m \), by Theorem 6.20. Thus \( u_m \in \mathcal{K}_{q,\alpha}^{\geq,e}(s_j)^m_{j=0}; s_m \). Hence, we have shown \( \mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)^m_{j=0}; \leq] \subseteq \mathcal{M}_{\geq}^q[[\alpha, \infty); (\hat{s}_j)^m_{j=0}; \leq] \). Taking additionally into account Remark 1.1 we can conclude \( \sigma \in \mathcal{M}_{\geq}^q[[\alpha, \infty); (\hat{s}_j)^m_{j=0}; \leq] \).

**Proposition 7.5.** Let \( m \in \mathbb{N}_0 \) and let \((s_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq} \). If \((t_j)^m_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\geq,e} \) satisfies \((t_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\), then \((t_j)^m_{j=0}\) coincides with the \( \alpha \)-Stieltjes right-sided non-negative definite extendable sequence equivalent to \((s_j)^m_{j=0}\).
Proof. Let \((t_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\) be such that \((t_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\). Observe that \((t_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\) by virtue of Remark 6.6. In view of Remark 7.1, we infer from Proposition 7.3 then \((t_j)^m_{j=0} \sim_{[\alpha, \infty)} (\tilde{s}_j)^m_{j=0}\), i.e., \(\mathcal{M}_{\geq 0}^2([\alpha, \infty); (t_j)^m_{j=0}] \leq \mathcal{M}_{\geq 0}^2([\alpha, \infty); (\tilde{s}_j)^m_{j=0}]\). According to Theorem 6.8, we can choose a measure \(\tau \in \mathcal{M}_{\geq 0}^2([\alpha, \infty); (t_j)^m_{j=0}]\). By virtue of Remark 1.1 then \(\tau \in \mathcal{M}_{\geq 0}^2([\alpha, \infty); (t_j)^m_{j=0}]\). Thus, \(\tau \in \mathcal{M}_{\geq 0}^2([\alpha, \infty); (\tilde{s}_j)^m_{j=0}]\). Consequently, we have \(r_m = \int_{[\alpha, \infty)} x^m \tau(dx) \leq \hat{s}_m\) and \(t_j = \int_{[\alpha, \infty)} x^j \tau(dx) = \hat{s}_j\) for all \(j \in \mathbb{Z}_{0, m-1}\). Since \((\tilde{s}_j)^m_{j=0}\) belongs to \(\mathcal{K}_{\geq 0, m, \alpha}\), according to Proposition 7.4, we can conclude in a similar way \(\hat{s}_m \leq t_m\). Hence, \(t_m = \hat{s}_m\) follows. \(\square\)

Theorem 7.6. If \(m \in \mathbb{N}_0\) and \((s_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\), then \(\langle (s_j)^m_{j=0} \rangle_{[\alpha, \infty)} \cap \mathcal{K}_{\geq 0, m, \alpha} = \{(\hat{s}_j)^m_{j=0}\}\).

Proof. Combine Propositions 7.4 and 7.5. \(\square\)

Our next aim can be described as follows. Let \(m \in \mathbb{N}\) and let \((s_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\). Then an appropriate application of Theorem 2.15 leads us to the determination of all sequences \((r_j)^m_{j=0}\) which are contained in the equivalence class \(\langle (s_j)^m_{j=0} \rangle_{[\alpha, \infty)}\).

Proposition 7.7. Let \(m \in \mathbb{N}\) and let \((s_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\). Then \(\langle (s_j)^m_{j=0} \rangle_{[\alpha, \infty)}\) coincides with the set of all sequences \((r_j)^m_{j=0}\) of \(q \times q\) matrices fulfilling \(\mathcal{R}(r_m - \Gamma_m) \cap \mathcal{R}(Q_{m-1}) = \{Q_{q,1}\}\). By virtue of Proposition 7.4, we have then \((t_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\) and \((t_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\). According to Definition 7.3, furthermore \(t_m = \Gamma_m\) and \(t_j = s_j\) for all \(j \in \mathbb{Z}_{0, m-1}\). In particular, \(Q_{m-1}^r = \Gamma_m = a_m^r\) and \(a_m^r = a_m^r\). Consider an arbitrary \((r_j)^m_{j=0} \in \langle (s_j)^m_{j=0} \rangle_{[\alpha, \infty)}\), i.e., \((r_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\) with \((r_j)^m_{j=0} \sim_{[\alpha, \infty)} (s_j)^m_{j=0}\). In particular, \((t_j)^m_{j=0} \sim_{[\alpha, \infty)} (r_j)^m_{j=0}\). By virtue of Remark 7.1, from Proposition 7.4 we can conclude then \((t_j)^m_{j=0}\) coincides with the \(\alpha\)-Stieltjes right-sided non-negative definite extendable sequence equivalent to \((r_j)^m_{j=0}\). View of Definition 7.5, consequently \(t_m = \Gamma_m^r\) and \(t_j = r_j\) for all \(j \in \mathbb{Z}_{0, m-1}\). Hence, \(\Gamma_m = \Gamma_m^r\) and \(s_j = r_j\) for all \(j \in \mathbb{Z}_{0, m-1}\) follow. In particular, \(Q_{m-1}^r = Q_{m-1}^r\) and \(a_m^r = a_m^r\). From Remark 6.3, we see furthermore \(Q_{m-1}^r \in \mathbb{C}_{\geq q}^{q \times q}\). Setting \(A := Q_{m-1}^r\) and \(\mathcal{V} := \mathcal{R}(Q_{m-1}^r)\) we obtain, in view of Notations 6.18 and Remark 6.9, then

\[
\begin{align*}
\mathcal{R}(r_m - \Gamma_m) & \cap \mathcal{R}(Q_{m-1}) = \{Q_{q,1}\}. \\
\mathcal{R}(Q_{m-1}) & = \mathcal{R}(Q_{m-1})^r. \\
\mathcal{R}(r_m - \Gamma_m) & = \mathcal{R}(Q_{m-1}) - \mathcal{R}(Q_{m-1}^r).
\end{align*}
\]

Because of \(\mathcal{R}(Q_{m-1}) = \mathcal{V}\) and Theorem 2.15, thus \(\mathcal{R}(r_m - \Gamma_m) \cap \mathcal{R}(Q_{m-1}) = \{Q_{q,1}\}\). Furthermore, Proposition 2.6 yields \(r_m - \Gamma_m \in \mathbb{C}_{\geq q}^{q \times q}\).

Conversely, consider an arbitrary sequence \((r_j)^m_{j=0}\) of complex \(q \times q\) matrices fulfilling \(\mathcal{R}(r_m - \Gamma_m) \cap \mathcal{R}(Q_{m-1}) = \{Q_{q,1}\}\), \(r_m - \Gamma_m \in \mathbb{C}_{\geq q}^{q \times q}\), and \(r_j = s_j\) for all \(j \in \mathbb{Z}_{0, m-1}\). In particular, \(Q_{m-1}^r = Q_{m-1}\) and \(a_m^r = a_m^r\). Because of \((s_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m, \alpha}\), we have \((s_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m-1, \alpha}\). Thus, \((r_j)^m_{j=0} \in \mathcal{K}_{\geq 0, m-1, \alpha}\). From Lemma 6.10, we infer that \(\Gamma_m \in \mathbb{C}_{\geq q}^{q \times q}\) and \(a_m^r \leq \Gamma_m\). Consequently, we can conclude \(r_m \in \mathbb{C}_{\geq q}^{q \times q}\) and \(a_m^r \leq \Gamma_m\).
\[ \Gamma_m \leq r_m. \] Hence, \( a_{m-1}^{(r)} \leq r_m, \) implying \( Q_m^{(r)} \in \mathbb{C}^{q \times q}_+ \), by virtue of Remark \[7].\ Using Proposition \[6,7\] we obtain then \( r_m \in K_{\geq,\alpha}[\{(r_j)_{j=0}^{m-1}\}], \) i.e., \( (r_j)_{j=0}^{m} \in K_{q,m,\alpha}. \) Denote by \( (\hat{r}_j)_{j=0}^{m} \) the \( \alpha \)-Stieltjes right-sided non-negative definite extendable sequence equivalent to \( (r_j)_{j=0}^{m} \) and let \( A := Q_m^{(r)} \) and \( V := R(Q_m^{(r)}) \). By Definition \[23\] then \( \hat{r}_m = a_{m-1}^{(r)} + G_{A,V} \) and \( \hat{r}_j = r_j \) for all \( j \in \mathbb{Z}_{0,m-1} \). Consequently, \( \hat{r}_j = s_j = t_j \) for all \( j \in \mathbb{Z}_{0,m-1} \). Setting \( X := t_m - a_{m-1}^{(r)} \) and \( Y := r_m - t_m, \) we have, by virtue of Remark \[23\] then \( X + Y = r_m - a_{m-1}^{(r)} = Q_m^{(r)} = A \) and \( X = t_m - a_{m-1} = t_m - a_{m-1}^{(t)} = Q_m^{(t)} \) and furthermore \( Y = r_m - \Gamma_m. \) In particular, \( R(Y) \cap R(Q_m^{(t)}) = \{O_{q \times 1}\} \) and \( Y \in \mathbb{C}^{q \times q}_+ \) by assumption. From Remark \[6,8\] we infer \( (t_j)_{j=0}^{m} \in K_{q,m,\alpha} \) and, consequently, \( t_m \in K_{\geq,\alpha}[(t_j)_{j=0}^{m-1}] \). In particular, \( (t_j)_{j=0}^{m-1} \in K_{q,m-1,\alpha}. \) Theorem \[6,10\] yields then \( Q_m^{(t)} \in \mathbb{C}^{q \times q}_+ \) and \( R(Q_m^{(t)}) \subseteq R(Q_m^{(t)}). \) Hence, \( X \in \mathbb{C}^{q \times q}_+. \) Taking into account \( Q_m^{(t)} = Q_m^{(t)} = Q_m^{(r)} \), we see furthermore \( R(X) \subseteq V \) and \( R(Y) \cap V = \{O_{q \times 1}\}. \) From Theorem \[2,15,16\] we get then \( X = G_{A,V}. \) Hence, \( \hat{r}_m = a_{m-1}^{(r)} + G_{A,V} = t_m. \) follows. Thus, the sequences \( (\hat{r}_j)_{j=0}^{m} \) and \( (t_j)_{j=0}^{m} \) coincide. Using Proposition \[23\] and Remark \[7,11\] we get then \( (r_j)_{j=0}^{m} \sim_{[a,\infty)} (t_j)_{j=0}^{m} \sim_{[a,\infty)} (s_j)_{j=0}^{m}. \) Consequently, \( (r_j)_{j=0}^{m} \in \{(s_j)_{j=0}^{m}\}_{[a,\infty)}. \) \( \square \)

References

[1] Ando, T.: Schur complements and matrix inequalities. In Zhang, F. (ed.): The Schur complement and its applications, vol. 4 of Numerical Methods and Algorithms, ch. 5, pp. 137–162. Springer-Verlag, New York, 2005.

[2] Bolotnikov, V.A. \url{http://www.math.wm.edu/~vladi/dhmp.pdf} Revised version of \[4\].

[3] Bolotnikov, V.A.: Degenerate Stieltjes moment problem and associated J-inner polynomials. Z. Anal. Anwendungen, 14(3):441–468, 1995.

[4] Bolotnikov, V.A.: On degenerate Hamburger moment problem and extensions of nonnegative Hankel block matrices. Integral Equations Operator Theory, 25(3):253–276, 1996.

[5] Chen, G.N. and Y.J. Hu: The truncated Hamburger matrix moment problems in the nondegenerate and degenerate cases, and matrix continued fractions. Linear Algebra Appl., 277(1-3):199–236, 1998.

[6] Dubovoj, V.K., B. Fritzsche, and B. Kirstein: Matricial version of the classical Schur problem, vol. 129 of Teubner-Texte zur Mathematik. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992. With German, French and Russian summaries.

[7] Dyukarev, Yu.M., B. Fritzsche, B. Kirstein, and C. Mäder: On truncated matricial Stieltjes type moment problems. Complex Anal. Oper. Theory, 4(4):905–951, 2010.
[8] Dyukarev, Yu.M., B. Fritzsche, B. Kirstein, C. Mäddler, and H.C. Thiele: On distinguished solutions of truncated matricial Hamburger moment problems. Complex Anal. Oper. Theory, 3(4):759–834, 2009.

[9] Fritzsche, B., B. Kirstein, and C. Mäddler: On Hankel nonnegative definite sequences, the canonical Hankel parametrization, and orthogonal matrix polynomials. Complex Anal. Oper. Theory, 5(2):447–511, 2011.

[10] Fritzsche, B., B. Kirstein, and C. Mäddler: On a special parametrization of matricial $\alpha$-Stieltjes one-sided non-negative definite sequences. In Interpolation, Schur functions and moment problems. II, vol. 226 of Oper. Theory Adv. Appl., pp. 211–250. Birkhäuser/Springer Basel AG, Basel, 2012.

[11] Fritzsche, B., B. Kirstein, C. Mäddler, and T. Schwarz: On a Schur-type algorithm for sequences of complex $p \times q$-matrices and its interrelations with the canonical Hankel parametrization. In Interpolation, Schur functions and moment problems. II, vol. 226 of Oper. Theory Adv. Appl., pp. 117–192. Birkhäuser/Springer Basel AG, Basel, 2012.

[12] Kats, I.S.: On Hilbert spaces generated by monotone Hermitian matrix-functions. Har’kov Gos. Univ. Uč. Zap. 34 = Zap. Mat. Otd. Fiz.-Mat. Fak. i Har’kov. Mat. Obšč. (4), 22:95–113 (1951), 1950.

[13] Rosenberg, M.: The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure. Duke Math. J., 31:291–298, 1964.

[14] Zhang, F. (ed.): The Schur complement and its applications, vol. 4 of Numerical Methods and Algorithms. Springer-Verlag, New York, 2005.