DENSE SETS OF INTEGERS WITH PRESCRIBED REPRESENTATION FUNCTIONS

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Abstract. Let \( A \) be a set of integers and let \( h \geq 2 \). For every integer \( n \), let \( r_{A,h}(n) \) denote the number of representations of \( n \) in the form \( n = a_1 + \cdots + a_h \), where \( a_i \in A \) for \( 1 \leq i \leq h \), and \( a_1 \leq \cdots \leq a_h \). The function \( r_{A,h} : \mathbb{Z} \to \mathbb{N} \), where \( \mathbb{N} = \mathbb{Z} \cup \{0, \infty\} \), is the representation function of order \( h \) for \( A \).

We prove that every function \( f : \mathbb{Z} \to \mathbb{N} \) satisfying \( \lim \inf \left| n \right| \to \infty f(n) \geq g \) is the representation function of order \( h \) for a sequence \( A = \{a_k\} \) of integers, and that \( A \) can be constructed so that it increases “almost” as slowly as any given \( B_h[g] \) sequence. In particular, for every \( \varepsilon > 0 \) and \( g \geq g(h, \varepsilon) \), we can construct a sequence \( A \) satisfying \( r_{A,h} = f \) and \( A(x) \gg x^{1/h-\varepsilon} \).

1. Introduction

Let \( A \) be a set of integers and let \( h \geq 2 \). For every integer \( n \), let \( r_{A,h}(n) \) denote the number of representations of \( n \) in the form

\[
n = a_1 + \cdots + a_h
\]

where

\[
a_1 \leq \cdots \leq a_h \quad \text{and} \quad a_i \in A \quad \text{for} \quad 1 \leq i \leq h.
\]

The function \( r_{A,h} : \mathbb{Z} \to \mathbb{N} \) is the representation function of order \( h \) for \( A \), where \( \mathbb{N} = \mathbb{Z} \cup \{0, \infty\} \).

Nathanson proved \([8]\) that any function \( f : \mathbb{Z} \to \mathbb{N} \) satisfying \( \lim \inf_{|n| \to \infty} f(n) \geq 1 \) is the representation function of order \( h \) of a set of integers \( A \) such that

\[
A(x) \gg x^{1/(2h-1)},
\]

where \( A(x) \) counts the number of elements \( a \in A \) with \( |a| \leq x \). It is an open problem to determine how dense the sets \( A \) can be.

In this paper we study the connection between this problem and the problem of finding dense \( B_h[g] \) sequences. We recall that a set \( B \) of nonnegative integers is called a \( B_h[g] \) sequence if

\[
r_{B,h}(n) \leq g
\]

for every nonnegative integer \( n \). It is usual to write \( B_h \) to denote \( B_h[1] \) sequences.

1 The notation \( f(x) \gg g(x) \) means that there exists a constant \( C > 0 \) such that \( f(x) \geq Cg(x) \) for \( x \) large enough.
Luczak and Schoen proved that any $B_h$ sequence satisfying an additional kind of Sidon property (see [7] for the definition of this property, which they call the $S_h$ property) can be enlarged to obtain a sequence with any prescribed representation function given $f$ satisfying that $\liminf_{|x| \to \infty} f(x) \geq 1$. In particular, since they prove that there exists a $B_h$ sequence $A$ satisfying the $S_h$ property with $A(x) \gg x^{1/(2h-1)}$, they recover Nathanson’s result.

In this paper we prove that any $B_h[g]$ sequence can be modified slightly to have any prescribed representation function $f$ satisfying $\liminf_{|x| \to \infty} f(x) \geq g$. Our main theorem is the following.

**Theorem 1.** Let $f : \mathbb{Z} \to \mathbb{N}$ any function such that $\liminf_{|n| \to \infty} f(n) \geq g$ and let $B$ be any $B_h[g]$ sequence. Then, for any decreasing function $\epsilon(x) \to 0$ as $x \to \infty$, there exists a sequence $A$ of integers such that

$$r_{A,h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$A(x) \gg B(x\epsilon(x)).$$

It is difficult problem to construct dense $B_h$ sequences. A trivial counting argument gives

$$B(x) \ll x^{1/h}$$

for these sequences. On the other hand, the greedy algorithm shows that there exists a $B_h$ sequence $B$ such that

$$B(x) \gg x^{1/(2h-1)}. \quad (2)$$

For $B_2$ sequences, also called Sidon sets, Ruzsa proved [11] that there exists a Sidon set $B$ such that

$$B(x) \gg x^{\sqrt{2} - 1 + o(1)} \quad (3)$$

This result and Theorem 1 give the following corollary.

**Corollary 1.** Let $f : \mathbb{Z} \to \mathbb{N}$ any function such that $\liminf_{|n| \to \infty} f(n) \geq 1$. Then there exists a sequence of integers $A$ such that

$$r_{A,2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$A(x) \gg x^{\sqrt{2} - 1 + o(1)}.$$
**Corollary 2.** Given \( h \geq 2 \), for any \( \epsilon > 0 \), there exists \( g = g(h, \epsilon) \) such that, for any function \( f : \mathbb{Z} \to \mathbb{N} \) satisfying \( \liminf_{|n| \to \infty} f(n) \geq g \), there exists a sequence \( \mathcal{A} \) of integers such that

\[
 r_{\mathcal{A}, h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}
\]

and

\[
 \mathcal{A}(x) \gg x^{1-\epsilon}.
\]

The construction in [3] for the set \( \mathcal{A} \) satisfying the growth condition (1) was based on the greedy algorithm. In this paper we construct the set \( \mathcal{A} \) by adjoining a very sparse sequence \( \mathcal{U} = \{u_k\} \) to a suitable \( \mathcal{B}_h[g] \) sequence \( \mathcal{B} \). This idea was used in [2], but in a simpler way, to construct dense perfect difference sets, which are sets such that every nonzero integer has a unique representation as a difference of two elements of \( \mathcal{A} \). The proof of the main theorem in [2] can be adapted easily to our problem in the simplest case \( h = 2 \).

**Theorem 2.** Let \( f : \mathbb{Z} \to \mathbb{N} \) be a function such that \( \liminf_{|n| \to \infty} f(n) \geq g \), and let \( \mathcal{B} \) be a \( \mathcal{B}_2[g] \) sequence. Then there exists a sequence of integers \( \mathcal{A} \) such that

\[
 r_{\mathcal{A}, 2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}
\]

and

\[
 \mathcal{A}(x) \gg \mathcal{B}(x/3).
\]

We omit the proof because it is very close to the proof of the main theorem in [2]. Unfortunately, that proof cannot be adapted to the case \( h \geq 3 \). We need another definition of a “suitable” \( \mathcal{B}_h \) set. In section §2 we shall show how to modify a \( \mathcal{B}_h[g] \) sequence \( \mathcal{B} \) so that it becomes “suitable.” We do this by applying the “Inserting Zeros Transformation” to an arbitrary \( \mathcal{B}_h[g] \) set. This is the main ingredient in the proof of Theorem 1.

Chen [1] has proved that for any \( \epsilon > 0 \) there exists a unique representation basis \( \mathcal{A} \) (that is, a set \( \mathcal{A} \) with \( r_{\mathcal{A}, 2}(k) = 1 \) for all integers \( k \neq 0 \)) such that \( \limsup_{x \to \infty} \mathcal{A}(x)/x^{1/2-\epsilon} > 1 \). J. Lee [6] has improved this result by proving that for any increasing function \( \omega \) tending to infinity there exists a unique representation basis \( \mathcal{A} \) such that \( \limsup_{x \to \infty} \mathcal{A}(x)\omega(x)/\sqrt{x} > 0 \).

Theorem 2 and the classical constructions of Erdős [12] and Kruckéberg [5] of infinite Sidon sets \( \mathcal{B} \) such that \( \limsup_{x \to \infty} \mathcal{B}(x)/\sqrt{x} > 0 \) provide a unique representation basis \( \mathcal{A} \) such that \( \limsup_{x \to \infty} \mathcal{A}(x)/\sqrt{x} > 0 \). Indeed, we can easily adapt the proof of Theorem 1.3 in [2] to the case of the additive representation function \( r(n) \) (instead of the subtractive representation function \( d(n) = \#\{n = a-a', a, a' \in \mathcal{A}\} \)).

**Theorem 3.** There exists a unique representation basis \( \mathcal{A} \) such that

\[
 \limsup_{x \to \infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.
\]

Again we omit the proof because it is very close to the proof of Theorem 1.3 in [2].

Theorem above answers affirmatively the first open problem in [1]. Note also that if \( \mathcal{A} \) is an infinite Sidon set of integers, then the set

\[
 \mathcal{A}' = \{4a : a \geq 0\} \cup \{-4a + 1 : a < 0\}
\]

is also a Sidon set and, in this case, \( \liminf |\mathcal{A} \cap (-x, x)|/\sqrt{x} = \liminf \mathcal{A}'(4x)/\sqrt{x} \). A well known result of Erdős states that \( \liminf \mathcal{B}(x)/\sqrt{x} = 0 \) for any Sidon set \( \mathcal{B} \).
Then the above limit is zero, so it answers negatively the second open problem in [1].

We do not know if it is possible to obtain a similar result for \( h \geq 3 \), because it is open problem to determine if there exists an infinite \( B_h \) sequence \( B \) with \( \limsup_{x \to \infty} B(x)/x^{1/h} > 0 \). It is easy, however, to prove that for any function \( \omega \) tending to infinity there exists a unique representation basis of order \( h \) such that \( \limsup_{x \to \infty} B(x)\omega(x)/x^{1/h} > 1 \). We can construct the set \( B \) as follows: Let \( x_1, \ldots, x_k, \ldots \) be a sequence of positive integers such that \( \omega(x_k) > (hx_{k-1})^{1/h} \) and consider, for each \( k \), a \( B_h \) sequence \( B_k \subset [1, x_k/(hx_{k-1})] \) with \( |B_k| > (x_k/(hx_{k-1}))^{1/h} \). The set \( B = \cup_k (hx_{k-1}) \ast B_k \) satisfies the conditions, where we use the notation \( t \ast A = \{ ta, \ a \in A \} \).

The construction above and Theorem 1 yield the following Corollary, which extends Theorem 6 in [6] in several ways.

**Corollary 3.** Let \( f : \mathbb{Z} \to \mathbb{N} \) any function such that \( \liminf_{|n| \to \infty} f(n) \geq 1 \). For any increasing function \( \omega \) tending to infinity there exists a set \( A \) such that \( r_{A,h}(n) = f(n) \) for all integers \( n \), and

\[
\limsup_{x \to \infty} A(x)\omega(x)/x^{1/h} > 0.
\]

**2. The Inserting Zeros Transformation**

Consider the binary expansion of the elements of a set \( B \) of positive integers. We will modify these integers by inserting strings of zeros at fixed places. We will see that this transformation of the set \( B \) preserves certain additive properties.

In this paper we denote by \( \gamma \) any strictly increasing function \( \gamma : \mathbb{N}_0 \to \mathbb{N}_0 \) with \( \gamma(0) = 0 \). For every positive integer \( r \), we define the “Inserting Zeros Transformation” \( T^r_\gamma \) by

\[
T^r_\gamma \left( \sum_{i \geq 0} \varepsilon_i 2^i \right) = \sum_{k \geq 0} 2^{2r} \sum_{i = \gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i.
\]

In other words, if the integer \( b \) has the binary expansion

\[
b = \varepsilon_0 \cdots \varepsilon_{\gamma(1)-1} \varepsilon_{\gamma(1)} \cdots \varepsilon_{\gamma(2)-1} \varepsilon_{\gamma(2)} \cdots \varepsilon_{\gamma(k)-1} \varepsilon_{\gamma(k)} \cdots,
\]

then

\[
T^r_\gamma(b) = \varepsilon_0 \cdots \varepsilon_{\gamma(1)-1} 0 \cdots 0 \varepsilon_{\gamma(1)} \cdots \varepsilon_{\gamma(2)-1} 0 \cdots 0 \varepsilon_{\gamma(2)} \cdots \varepsilon_{\gamma(k)-1} 0 \cdots 0 \varepsilon_{\gamma(k)} \cdots,
\]

Note that if \( b < b' \), then \( T^r_\gamma(b) < T^r_\gamma(b') \). We define the set

\[
T^r_\gamma(B) = \{ T^r_\gamma(b) : b \in B \}.
\]

The next proposition proves that the function \( T^r_\gamma \) preserves some Sidon properties.

**Proposition 1.** Let \( 2r > \log_2 h \). If \( b_1, \ldots, b_h, b'_1, \ldots, b'_h \) are positive integers such that

\[
T^r_\gamma(b_1) + \cdots + T^r_\gamma(b_h) = T^r_\gamma(b'_1) + \cdots + T^r_\gamma(b'_h),
\]

then

\[
b_1 + \cdots + b_h = b'_1 + \cdots + b'_h.
\]

In particular, if \( B \) is a \( B_h[\gamma] \) set and \( 2r \geq \log_2 h \), then \( T^r_\gamma(B) \) is also a \( B_h[\gamma] \) set.
Proposition 2. For $k \geq 1$ and for any positive integer $b$

$$\|T_x^r(b)\|_{m_k} < \frac{m_k}{2^{2r}},$$

where $m_k$ is defined in (7).

Proof. Let $b = \varepsilon_0 \varepsilon_1 \varepsilon_2 \ldots$ be the binary expansion of $b$. Then

$$T_x^r(b) \equiv \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \pmod{m_k}$$

and

$$0 \leq \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \leq 2^{r(k-1)+\gamma(k)-1} 2^i < \frac{m_k}{2^{2r}}.$$
This completes the proof.

3. Proof of Theorem

3.1. Two auxiliary sequences. Consider the sequence \( \{z_j\}_{j=1}^\infty \) defined by

\[
(8) \quad z_j = j - [\sqrt{j}][\sqrt{j}] + 1. 
\]

For every positive integer \( j \) there is a unique positive integer \( s \) such that \( s^2 \leq j < (s+1)^2 \). Then \( j = s^2 + s + i \) for some \( i \in [-s, s] \) and \( z_j = i \). It follows that for every integer \( i \) there are infinitely many positive integers \( j \) such that \( z_j = i \). Moreover, \( |z_j| \leq s \leq \sqrt{j} \) for all \( j \geq 1 \).

Let \( f : \mathbb{Z} \to \mathbb{N} \) any function such that \( \liminf_{|n| \to \infty} f(n) \geq g \). Let \( n_0 \) be the least positive integer such that \( f(n) \geq g \) for all \( |n| \geq n_0 \). Choose an integer \( r > 1 + \log_2(h^2 + n_0) \). Then

\[
(9) \quad h^2 < 2^{r-1} \quad \text{and} \quad n_0 < 2^{r-1}.
\]

Let \( \gamma : \mathbb{N}_0 \to \mathbb{N}_0 \) be a strictly increasing function such that \( \gamma(0) = 0 \).

Consider the sequence \( \mathcal{U} = \{u_i\}_{i=1}^\infty \) defined by

\[
(10) \quad \left\{ \begin{array}{ll}
    u_{2k-1} & = -m_k2^{-r}, \\
    u_{2k} & = (h-1)m_k2^{-r} + z_k
    \end{array} \right.
\]

where \( m_k = 2^{r+k+\gamma(k)} \). We write

\[
(11) \quad \mathcal{U}_k = \{u_{2k-1}, u_{2k}\} \quad \text{and} \quad \mathcal{U}_{<k} = \bigcup_{s<k} \mathcal{U}_s.
\]

Note that for all \( j \leq k \) we have

\[
(12) \quad |z_j| \leq \sqrt{k} < 2^k < 2^{\gamma(k)} < 2^{2^{r(k-1)+\gamma(k)} = m_k2^{-2r}}.
\]

3.2. The recursive construction. For any \( B_{h}[g] \)-sequence \( B \) we consider the set \( T_\gamma^r(B) \) defined in \( [5] \). Let \( f : \mathbb{Z} \to \mathbb{N} \) be a function such that \( f(n) \geq g \) for \( |n| \geq n_0 \). We construct an increasing sequence \( \{A_k\}_{k=0}^\infty \) of sets of integers as follows:

\[
(13) \quad A_0 = \{a \in T_\gamma^r(B) : a \geq n_0\}
\]

and, for \( k \geq 1 \),

\[
A_k = \begin{cases} 
    A_{k-1} \cup \mathcal{U}_k & \text{if } r_{A_{k-1}, h}(z_k) < f(z_k) \\
    A_{k-1} & \text{otherwise}
\end{cases}
\]

where \( z_k \) and \( \mathcal{U}_k \) are defined in \( [8] \) and \( [11] \).

We shall prove that the set

\[
\mathcal{A} = \bigcup_{k=0}^\infty A_k
\]

satisfies \( r_{\mathcal{A},h}(n) = f(n) \) for all integers \( n \).

Lemma 1. Let \( k \geq 1 \). For nonnegative integers \( s \) and \( t \) with \( s + t \leq h \), let

\[
A_k^{(s,t)} = (h-s-t)A_{k-1} + su_{2k-1} + tu_{2k}.
\]

The sets \( A_k^{(s,t)} \) are pairwise disjoint, except possibly the sets \( A^{(0,0)} \) and \( A^{(h-1,1)} \).
Proof. If \( n \in \mathcal{A}_k^{(s,t)} \) then
\[
n = a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} = a_1 + \cdots + a_{h-s-t} + ((h-1) - s)m_k2^{-r} + tz_k.
\]
If \( a_i \in A_0 \), then \( \|a_i\|_{m_k} \leq m_k2^{-2r} \) by Proposition \( [2] \). If \( a_i \in U_{\leq k} \) then we use \( [10] \) and \( [12] \) to obtain
\[
\|a_i\|_{m_k} \leq |a_i| \leq (h-1)m_{k-1}2^{-r} + m_{k-1}2^{-2r} < hm_k2^{-2r}.
\]
Therefore,
\[
\|a_1 + \cdots + a_{h-s-t} + t\zeta_k\|_{m_k} \leq \|a_1\|_{m_k} + \cdots + \|a_{h-s-t}\|_{m_k} + \|t\zeta_k\|_{m_k} \\
\leq (h-s-t)m_k2^{-2r} + tm_k2^{-2r} \\
\leq h^2m_k2^{-2r}.
\]

Now suppose that \( n \in \mathcal{A}_k^{(s',t')} \) for some \((s',t') \neq (s,t)\). If \( \{(s,t), (s',t')\} \neq \{(0,0), (h-1,1)\} \), then
\[
t(h-1) - s \neq t'(h-1) - s'
\]
and
\[
m_k2^{-r} \leq \|(t(h-1) - s) - (t'(h-1) - s')\|_{m_k} \\
= \|(t(h-1) - s)m_k2^{-r} - (t'(h-1) - s')m_k2^{-r}\|_{m_k} \\
= \|(n - (t(h-1) - s)m_k2^{-r}) - (n - (t'(h-1) - s')m_k2^{-r})\|_{m_k} \\
\leq \|a_1 + \cdots + a_{h-s-t} + t\zeta_k\|_{m_k} + \|a'_1 + \cdots + a'_{h-s-t'} + t'\zeta_k\|_{m_k} \\
\leq 2h^2m_k2^{-2r}.
\]
It follows that \( h^2 \geq 2^{r-1} \), which contradicts \( [9] \). This completes the proof. \( \square \)

Lemma 2. If \( n \in \mathcal{A}_k^{(s,t)} \) for some \( k \geq 1 \) and \((s,t) \notin \{(0,0), (h-1,1)\}\), then \(|n| > n_0\).

Proof. If \( n \in \mathcal{A}_k^{(s,t)} \), then
\[
n = a_1 + \cdots + a_{h-s-t} + ((h-1) - s)m_k2^{-r} + tz_k
\]
and
\[
|n| \geq \|n\|_{m_k} \\
= \|a_1 + \cdots + a_{h-s-t} + t\zeta_k + ((h-1)t - s)m_k2^{-r}\|_{m_k} \\
\geq \|((h-1)t - s)m_k2^{-r}\|_{m_k} - \|a_1 + \cdots + a_{h-s-t} + t\zeta_k\|_{m_k} \\
\geq \|((h-1)t - s)m_k2^{-r}\|_{m_k} - h^2m_k2^{-2r} \\
\geq m_k2^{-r} - h^2m_k2^{-2r} \geq m_k2^{-r-1} \geq 2^{2r-2r-1} \\
\geq 2^{r-1} > n_0.
\]
We have used that if \( \|(h-1)t - s)m_k2^{-r}\| < m_k/2 \), then
\[
\|((h-1)t - s)m_k2^{-r}\|_{m_k} = \|(h-1)t - s)m_k2^{-r}\| \geq m_k2^{-r}.
\]
Also we have used \((h-1)t - s \neq 0\) and \( [10] \) in the last inequalities. \( \square \)
Lemma 3. For any $k \geq 0$, for any $h' < h$ and for any integer $m$ we have that

$$r_{A_{k}, h'}(m) \leq g$$

Proof. By induction on $k$. Proposition 1 implies that $T'_r(B)$ and consequently $A_0$ are $B_r[g]$-sequences. In particular, $A_0$ is a $B_r[g]$ sequence. Then $r_{A_0, h'}(m) \leq g$ for any integer $m$.

Suppose that it is true that for any $h' < h$, and for any integer $m$ we have that $r_{A_{k-1}, h'}(m) \leq g$.

Consider $m \in h'A_k$.

- Suppose $m \notin (h' - s - t)A_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s, t) \neq (0, 0)$. Then $r_{A_{k}, h'}(m) = r_{A_{k-1}, h'}(m) \leq g$ by induction hypothesis.
- Suppose that $m \in (h' - s - t)A_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s, t) \neq (0, 0)$. Consider an element $m \in A_{k-1}$. Then

$$m + (h' - h')a \in A_{k-1}^{(s,t)} = (h' - s - t)A_{k-1} + su_{2k-1} + tu_{2k}.$$ 

We apply lemma 1 and since $(s, t) \neq (h - 1, 1)$ (because $h' < h$) we have that

$$r_{A_{k}, h'}(m) \leq r_{A_{k-1}, h}((h' - h')a) = r_{A_{k-1}, h - s - t}(m + (h' - h')a - su_{2k-1} - tu_{2k}),$$

and we can apply induction hypothesis because $h - s - t < h$.

\[\Box\]

Proposition 3. The sequence $\mathcal{A}$ defined above satisfies $r_{\mathcal{A}, h}(n) = f(n)$ for all integers $n$.

Proof. Since

$$\underbrace{u_{2k-1} + \cdots + u_{2k-1} + u_{2k}}_{h-1} = z_k$$

it follows that if $r_{A_{k-1}, h}(z_k) < f(z_k)$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k$ and

$$r_{A_{k}, h}(z_k) \geq r_{A_{k-1}, h}(z_k) + 1.$$ 

For every integer $n$ there are infinitely many integers $k$ such that $z_k = n$ and so $r_{A_{k}, h}(n) \geq f(n)$ for some $k$.

Next we show that, for every integer $k$, the sequence $\mathcal{A}_k$ satisfies $r_{A_{k}, h}(n) \leq f(n)$ for all $n$. The proof is by induction on $k$.

Let $k = 0$. Since $\mathcal{A}_0$ is a $B_h[g]$-sequences, we have $r_{A_{0}, h}(n) \leq g \leq f(n)$ for $n \geq n_0$. If $n < n_0$, then $r_{A_{0}, h}(n) = 0 \leq f(n)$.

Now, suppose that it is true for $k - 1$. In particular $r_{A_{k-1}, h}(z_k) \leq f(z_k)$. If $r_{A_{k-1}, h}(z_k) = f(z_k)$ there is nothing to prove because in that case $\mathcal{A}_k = \mathcal{A}_{k-1}$. But if $r_{A_{k-1}, h}(z_k) \leq f(z_k) - 1$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}\} \cup \{u_{2k}\}$. We will assume that until the end of the proof.

If $n \notin h\mathcal{A}_k$ then $r_{A_{k}, h}(n) = 0 \leq f(n)$.

If $n \in h\mathcal{A}_k$, since $\mathcal{A}_k = \mathcal{A}_{k-1} \cap \mathcal{U}_k$ we can write

$$h\mathcal{A}_k = \bigcup_{s+t \leq h} \mathcal{A}_{k-1} = (h - s - t)A_{k-1} + su_{2k-1} + tu_{2k}.$$ 

Then

$$n = a_1 + \cdots + a_{h - s - t} + su_{2k-1} + tu_{2k}$$

(14)
for some \( s, t \), satisfying \( 0 \leq s, t, s + t \leq h \) and for some \( a_1, \ldots, a_{k-s-t} \in A_{k-1} \).

For short we write \( r_{s,t}(n) \) for the number of solutions of (14).

- If \( n \in (h-s-t)A_{k-1} + su_{2k-1} + tu_{2k} \) for some \( (s, t) \neq (0, 0), (s, t) \neq (h-1, 1) \) then, due to lemma 4, we have that \( r_{A_k, h}(n) = r_{s,t}(n) \).
  - For \( 0 \leq n \leq n_0 \) we have that \( r_{s,t}(n) = 0 \leq f(n) \) (due to lemma 2).
  - For \( n > n_0 \) we apply lemma 3 in the first inequality below with \( h' = h - s - t \) and \( m = n - su_{2k-1} - tu_{2k} \),

\[
r_{s,t}(n) = r_{A_{k-1}, h-s-t}(n - su_{2k-1} - tu_{2k}) \leq g \leq f(n)
\]

- If \( n \notin (h-s-t)A_{k-1} + su_{2k-1} + tu_{2k} \) for any \( (s, t) \neq (0, 0), (s, t) \neq (h-1, 1) \), then \( r_{A_k, h}(n) = r_{0,0}(n) + r_{h-1,1}(n) \). Notice that \( r_{0,0}(n) = r_{A_{k-1}, h}(n) \) and that \( r_{h-1,1}(n) = 1 \) if \( n = z_k \) and \( r_{h-1,1}(n) = 0 \) otherwise.
  - If \( n = z_k \), then \( r_{A_k, h}(n) = r_{A_{k-1}, h}(z_k) + r_{h-1,1}(z_k) \leq f(z_k) + 1 = f(n) \).

\[ \square \]

### 3.3. The density of \( A \).

Recall that \( \gamma : N_0 \to N_0 \) is a strictly increasing function with \( \gamma(0) = 0 \). Let \( \mathbb{R}_{\geq 0} = \{ x \in \mathbb{R} : x \geq 0 \} \). We extend \( \gamma \) to a strictly increasing function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \). (For example, define \( \gamma(x) = \gamma(k+1)(x-k) + \gamma(k)(k+1-x) \) for \( k \leq x \leq k+1 \).

We have

\[
A(x) \geq A_0(x) \geq T_\gamma^\ell(\mathcal{B})(x) - n_0.
\]

Thus, to find a lower bound for \( A(x) \) it suffices to find a lower bound for the density of \( T_\gamma^\ell(\mathcal{B}) \).

**Lemma 4.** \( T_\gamma^\ell(\mathcal{B})(x) > B(x2^{-2r\gamma^{-1}(\log_2 x)}) \).

**Proof.** Let \( b \) be a positive integer such that

\[
b \leq x2^{-2r\gamma^{-1}(\log_2 x)}.
\]

Let \( \ell \) be such that \( 2^\gamma(\ell) \leq b < 2^\gamma(\ell+1) \). Then we can write

\[
b = \sum_{k=0}^{\ell} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i.
\]

It follows from the definition 4 of the Zeros Inserting Transformation that

\[
T_\gamma^\ell(b) \leq 2^{2r\ell} b
\]

\[
\leq 2^{2r\gamma^{-1}(\log_2 b)} b
\]

\[
\leq 2^{2r(\gamma^{-1}(\log_2 b) - \gamma^{-1}(\log_2 x))} x
\]

\[
\leq x.
\]

\[ \square \]
Recall that $\epsilon$ is a decreasing positive function defined on $[1, \infty)$ such that $\lim_{x \to \infty} \epsilon(x) = 0$. We complete the proof of Theorem 1 by choosing a function $\gamma$ that satisfies the inequality

$$2^{2 - 2r \gamma^{-1}(\log_2 x)} \geq \epsilon(x).$$

It suffices to take $\gamma(x) > \log_2 (\epsilon^{-1}(2^{-2rx}))$.

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