Threshold Production of the Hypertriton

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Abstract

The cross section for threshold production of the hypertriton in the reaction $pd \rightarrow ^3\!\!H K^+$ is calculated in a two-step model and compared to the break-up process $pd \rightarrow d\Lambda K^+$. The latter process is shown to be dominant already at 2 MeV above threshold. The amplitude squared at threshold for the $pd \rightarrow ^3\!\!H K^+$ reaction is $|f|^2 = 1.0 \text{ nb/sr}$. 

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1 Introduction

One way to gain information about the hyperon-nucleon interaction is to study bound states of the hyperon, i.e., hypernuclei. The most suitable candidate for such a study would be the lightest of the hypernuclei, $^3\Lambda$H, since the theoretical treatment becomes considerably easier for few-body systems (no collective effects that screens the perturbation caused by the hyperon). However, the hypertriton is very weakly bound ($E_B = 0.13$ MeV), which makes the distinction between the bound and the scattering state difficult. Because of the weak binding, the decay properties of the hypertriton are essentially those of the hyperon itself.

The purpose of the present study is to calculate the cross section for the process

$$pd \rightarrow ^3\Lambda HK^+$$

(1)

near threshold ($T_{lab}^p = 1126.517$ MeV). In the similar reaction $pd \rightarrow ^3\text{He}\eta$, the two-step model of [1] has been successful in explaining the experimental data. Hence, the same mechanism is employed here for the hypertriton production. The fundamental advantage of the two-step model is that it allows the exchanged momentum to be shared between the two nucleons of the deuteron, thereby increasing the reaction probability. In addition, the kinematics is miraculous in the sense that the subprocesses can be considered as real, which simplifies the calculations.

The concept of a transition matrix, with a definition close to that of Kondratyuk et al. [2] and Komarov et al. [3], gives a somewhat different formulation than the one of [4]. In [3], the cross section for hypertriton production is calculated for several angles and preferentially at energies far from threshold, while this study is devoted to the behavior near to threshold and the relation to the corresponding break-up reaction.

The index $\tau$ will in this paper stand for the $(\Lambda\text{-})$hypertriton.

2 Bound final state

For the case of a bound state between the deuteron and the lambda hyperon, a two-step model like that of [1] is used. The Feynman diagram for this reaction is shown in Fig. 1. The trick used in calculating the cross section for this diagram is to consider the processes of the $A$ and $C$ vertices to be almost real. This is accomplished by putting the intermediate deuteron and neutron on their mass shells.
Figure 1: Feynman diagram of the two-step model for the $pd \to ^3\Lambda H K^+$ reaction, defining the various momenta. The partition of the hypertriton momenta is weighted by $\gamma = m_{\Lambda}/(m_d + m_{\Lambda}) = 0.373$. The diagram for the $\pi^0$ case follows upon exchanging $p'$ and $n$.

2.1 Cross section for bound final state

With the normalization of Bjorken and Drell the cross section in the overall c.m. system can be written

$$
\frac{d\sigma}{d\Omega}(pd \to ^3\Lambda H K^+) = \frac{m_r m_p}{4(2\pi)^2 s_{\tau K}} \frac{|\mathbf{p}_K|}{|\mathbf{p}_d|} N_I \frac{1}{6} \sum_{\text{spins}} |M|^2;
$$

where $m_i$ and $p_j$ are the masses and momenta of the indicated particles and $s_{\tau K}$ is the c.m. energy squared. An isospin correction factor $N_I = 9/4$ is included because of the contribution from $\pi^0$-exchange.

The matrix element is the integral over the Fermi momenta

$$
\mathcal{M} = \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} M,
$$

after the $q_0$ and $k_0$ integrations have been performed (giving a factor $m/E \approx 1$ for each of the on-shell propagators when the Lorentz boost is neglected).

The expression for $M$ in the low energy limit and S-wave approximation becomes

$$
M = M_D M_C M_B M_A \frac{i}{p_{\pi}^2 - m_{\pi}^2 + i\epsilon},
$$
where the respective vertex matrices are given by

\[ M_D = \eta^\dagger \left( (2\pi)^{\frac{3}{2}} \sqrt{2m_d} \left( \frac{1}{\sqrt{3}} \sigma \cdot \epsilon_{d'} \right) \Psi^\dagger(k) \right) \xi_A \]  

(5)

\[ M_C = \xi^\dagger_n [G] \xi_n \]  

(6)

\[ M_B = \xi^\dagger_n \left( (2\pi)^{\frac{3}{2}} \sqrt{2m_d} \left( \frac{-1}{\sqrt{2}} \sigma \cdot \epsilon_{d'} \right) \Phi_{d'}(q) \right) \xi_{p^c} \]  

(7)

\[ M_A = \xi^\dagger_{p^c} [A \hat{p} \cdot \epsilon_{d'}^+ + iB \sigma \cdot \epsilon_{d'}^+ \times \hat{p}] \eta_p. \]  

(8)

The parametrizations for the B and D vertices stem from vertex functions in the pole approximation, while for A, A and B are invariant functions of the total energy only and for C, G is an invariant function of energy and angle. Only S-wave contributions are considered for the deuteron and the hypertriton.

Summation over internal spins and polarizations gives

\[ \mathcal{M} = \frac{2m_d}{\sqrt{6}} G \eta^\dagger \left[ (A - 2B) \hat{p} \cdot \epsilon_d + iA \sigma \cdot \epsilon_{d'}^+ \right] \eta_p G_B, \]  

(9)

where the transition matrix \( G_B \) is defined as

\[ G_B = i \int \frac{d^3q d^3k}{(2\pi)^6 (p^0_\pi)^2 - p^2_\pi - m^2_\pi + i\epsilon} \Psi^\dagger(k) \Phi_{d'}(q). \]  

(10)

The zeroth component of the virtual pion four-momentum is defined by the on-shell energies: \( p^0_\pi = E_d(p_d) + E_p(-p_d) - E_d(-1-\gamma)p_K - k - E_n(p_d + k). \) Since \( B \) is much smaller than \( A \), it is possible to factorize, approximately, the total cross section in the forward direction as

\[ \left[ \frac{d\sigma}{d\Omega} \left( pd \to ^3\Lambda K^+ \right) \right]_{CM} \left[ \frac{d\sigma}{d\Omega} \left( pp' \to \pi^+ d \right) \right]_{CM} \left[ \frac{d\sigma}{d\Omega} \left( \pi^+ n \to \Lambda K^+ \right) \right]_{CM} \]  

(11)

The cross section for the vertex processes are given by

\[ \left[ \frac{d\sigma}{d\Omega} \left( pp' \to d\pi^+ \right) \right]_{CM} = \frac{m_p^2}{8(2\pi)^2 s_{\pi d}} \frac{|P_p|}{|P_d|} |A|^2 \]  

(12)

\[ \left[ \frac{d\sigma}{d\Omega} \left( \pi^+ n \to \Lambda K^+ \right) \right]_{CM} = \frac{m_\Lambda m_n}{4(2\pi)^2 s_{\Lambda K}} \frac{|P_K|}{|P_\pi|} |G|^2. \]  

(13)
3 Break-up

Because of the small binding energy of the hypertriton, there is a large probability that the final deuteron and hyperon are scattered instead of forming a bound state. This demands a calculation of the scattering cross section to be compared with the bound state cross section. The Feynman diagram for the break-up reaction is shown in Fig. 2 where the kinematics is the same as before.

Figure 2: Feynman diagram of the break-up reaction, \( pd \rightarrow d\Lambda K^+ \), defining the various momenta. The partition of the d and \( \Lambda \) momenta is weighted by \( \gamma = m_\Lambda / (m_d + m_\Lambda) = 0.373 \). The diagram for the \( \pi^0 \) case follows upon exchanging \( p' \) and n.

3.1 Cross section

The kinematic part of the cross section is calculated in the low energy (non-relativistic) limit which gives

\[
\frac{d\sigma}{d\Omega} = \frac{m_p m_d}{8(2\pi)^5 m_d s_{\tau K}} \frac{N_f}{|P_d|} \int d^3P |p_K| \frac{1}{6_{\text{spins}}} \sum |\mathcal{M}|^2, \tag{14}
\]

where \( P \) is the relative momentum of the lambda-deuteron system and

\[
|p_K| = \sqrt{2\mu_{\Lambda d}(Q - \frac{P^2}{2\mu_{\Lambda d}})} \tag{15}
\]
is the momentum of the kaon in the final state.

The matrix element is now given, in the low energy limit, by

$$\mathcal{M} = \int \frac{d^3q}{(2\pi)^3} M, \quad (16)$$

where the $q_0$-integration over the neutron propagator already is performed and

$$M = M_C M_B M_A \frac{i}{p^2_\pi - m^2_\pi + i\epsilon}, \quad (17)$$

where the vertex matrices are given by Eqs (3)–(8). Summing over internal spins and expanding the products of the Pauli matrices now gives

$$\mathcal{M} = -G\sqrt{m_d} G_S \eta_\Lambda^\dagger [ -B \hat{p} \cdot \epsilon_d \sigma \cdot \epsilon_{d'}^\dagger + \mathcal{Q} \cdot \epsilon_{d'}^\dagger ] \eta_\pi, \quad (18)$$

where $G_S$ is the transition matrix element

$$G_S = i \int \frac{d^3q}{(2\pi)^3/2} \frac{\Phi_d(q)}{(p^0_\pi)^2 - p^2_\pi - m^2_\pi + i\epsilon} \quad (19)$$

and the vector $\mathcal{Q}$ is defined by

$$\mathcal{Q} = \hat{p} [ A(\sigma \cdot \epsilon_d) ] + (\hat{p} \times \epsilon_d)[iB] + \epsilon_d [B(\sigma \cdot \hat{p})]. \quad (20)$$

The spinor part of Eq. (18) can be separated into two orthogonal parts with the total spin of the deuteron-lambda system equal to 1/2 and 3/2 respectively. This is accomplished by using the projection operators

$$P_{\frac{1}{2}} = \frac{1}{\sqrt{3}} \sigma \cdot \epsilon_{d'}^\dagger \quad (21)$$

$$P_{\frac{3}{2}} = \frac{1}{3} [\mathcal{Q} \cdot \epsilon_{d'}^\dagger - i\sigma \cdot \mathcal{Q} \times \epsilon_{d'}^\dagger] \quad (22)$$

$$P_{\frac{3}{2}} = \frac{1}{3} [2\mathcal{Q} \cdot \epsilon_{d'}^\dagger + i\sigma \cdot \mathcal{Q} \times \epsilon_{d'}^\dagger], \quad (23)$$

where the two spin-1/2 projections are orthogonal (after squaring and summing over external spins) to the spin-3/2 projection, but not to each other. If the $B$ terms are neglected, the final form for the matrix element squared is

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 6m_d |A G|^2 \left( \frac{1}{3} |G_{S,\frac{1}{2}}|^2 + \frac{2}{3} |G_{S,\frac{3}{2}}|^2 \right). \quad (24)$$

The total cross section in the forward direction becomes

$$\left[ \frac{d\sigma}{d\Omega} (pd \to d\Lambda K^+) \right]_{CM,s} = \frac{3m_r s_{pd}}{2\pi m_n^2 m_A s_{\pi K}} \left[ \frac{|p_p|}{|p_\pi|} \frac{d\sigma}{d\Omega} (pp' \to \pi^+ d) \right]_{CM} \times \int_0^{\sqrt{2m_d Q}} d^3P |p_K| |G_{S,s}|^2 s_{\pi K} \left[ \frac{|p_\pi|}{|p_K|} \frac{d\sigma}{d\Omega} (\pi^+ n \to \Lambda K^+) \right]_{CM}. \quad (25)$$
where the index $s$ denotes the different spin states $1/2$ and $3/2$ and $a_s$ the corresponding fractions $1/3$ and $2/3$.

4 Approximation of transition matrices

Introducing a new parameter $\kappa^2 = (\kappa_0)^2 - m_\pi^2$, where $\kappa_0 = p_\pi^0(k = q = 0)$, the pion propagator can be rewritten with the aid of the integral

$$
\int d^3x e^{ip_\pi \cdot x} \frac{e^{i(\kappa + i\epsilon)r}}{r} = \frac{-4\pi}{\kappa^2 - p_\pi^2 + i\epsilon}.
$$

(26)

Substituting this for the propagator in the transition matrices $G_B$ and $G_{S,s}$ they become (in configuration space)

$$
G_B = -i\frac{1}{4\pi} \int d^3x \psi_\uparrow^\dagger(x) \frac{e^{i\kappa r}}{r} e^{i p_\pi \cdot x} \varphi_d(x)
$$

(27)

$$
G_{S,s} = -i\frac{1}{4\pi} \int d^3x 1 \frac{e^{i\kappa r}}{r} e^{i p_\pi \cdot x} \varphi_d(x),
$$

(28)

where $p_\pi$ is the pion momentum when Fermi momenta are neglected. The similarity of the expressions (27) and (28) suggests the way to include a scattering wave function (not equal to unity) for the unbound lambda-deuteron system. In order to do this properly, a bound state wave function related to the scattering state wave function is needed. A good and fairly simple candidate for the bound state is the wave function of Congleton [7], which, unfortunately, is not easily transformed to a scattering wave function. A simpler alternative would be a two-term Hulthén wave function, but its behaviour close to the origin is like $a - br$, while the Congleton function goes like $a - cr^2$. This discrepancy could be remedied by using a modified Hulthén wave function (the result of using a squared Yamaguchi form factor in a separable potential) of the type

$$
\psi_\tau(r) = N \frac{1}{r} \left[ e^{-\alpha r} - e^{-\beta r} - \frac{\beta^2 + k^2}{2\beta} r e^{-\beta r} \right]
$$

(29)

and then adjust the parameter $\beta_\tau$ to give the same rms radius as Congleton.

This leads to $\beta_\tau = 23.77 \alpha_\tau$ ($\alpha_\tau = \sqrt{2\mu_{d\Lambda}E_B} = 0.068$ fm$^{-1}$). For the deuteron the S-wave part of the parametrized Paris wave function [8] is employed.

4.1 Scattering wave functions

In scattering theory it is possible to relate the scattering wave function to the bound state wave function. The expression for this is deduced by Fälldt
and Wilkin and is given by

$$
\psi(r) = -\left[ \sqrt{\frac{\alpha(\alpha^2 + k^2)}{2\pi}} \bar{\psi}^{(-)}(k, r) \right]_{k=\imath \alpha}, \quad (30)
$$

The scattering state wave function for the spin-1/2 Λd-state has a form reminding of Eq. (29):

$$
\bar{\psi}^{(-)}_{\tau\frac{1}{2}}(k, r) = \frac{1}{kr} \left[ \cos \delta \sin kr + \sin \delta (\cos kr - e^{-\beta r} - \frac{\beta^2 + k^2}{2\beta r} re^{-\beta r}) \right], \quad (31)
$$

where

$$
\tan \delta = \frac{-16\beta^5 k(\alpha + \beta)^4}{(5\beta^6 - 15\beta^4 k^2 - 5\beta^2 k^4 - k^6)(\alpha + \beta)^4 - (\beta^2 + k^2)^4((2\beta + \alpha)^2 + \beta^2)^4}. \quad (32)
$$

For the spin-3/2 Λd-state, the same formulae are used but with another value for \(\alpha_\tau\). The new value is estimated from the expression for the potential strength

$$
\lambda = -\frac{1}{4\pi N^2} \frac{\beta^5 (\alpha + \beta)^4}{(2\beta + \alpha)^2 + \beta^2}, \quad (33)
$$

where \(N\) is a suitable normalization constant (independent of \(\alpha\)). It is stated in [10] that the spin-3/2 case is unbound up to 1.32 times the strength parameter. Keeping \(\beta\) fixed, this gives an estimate for the strength parameter for a bound state \((\alpha = 0)\) which can be divided by 1.32 and used to calculate the new \(\alpha'_\tau\) for the unbound state. The calculations indicated above result in the values:

$$
\alpha_\tau = 0.068 \, \text{fm}^{-1}, \quad \beta_\tau = 1.616 \, \text{fm}^{-1}, \quad \alpha'_\tau = -0.1334 \, \text{fm}^{-1}.
$$

The wave functions for the bound state and the scattering states are plotted in Fig. 3.

### 4.2 Final expression for the transition matrices

The angular integrations in Eqs (27) and (28) can be performed in the S-wave approximation, giving (to first order),

$$
G_B = -\int r dr \psi^+_\tau(r) \frac{e^{ikr}}{r} j_0((1 - \gamma)p_K r) j_0(\frac{1}{2} p_d r) \varphi_d(r) \quad (34)
$$

$$
G_{S,s} = -\int r dr \psi^{(-)}_{\tau, s}(r) \frac{e^{ikr}}{r} j_0((1 - \gamma)p_K r) j_0(\frac{1}{2} p_d r) \varphi_d(r). \quad (35)
$$
5 Parametrization of vertices

The cross sections for the vertices A and C, which turn up on the right hand sides of Eqs (11) and (25), are parametrized empirically.

5.1 Empirical fit of \( pp \rightarrow \pi^+d \) data

Fäldt and Wilkin use cross section data for the reaction \( \pi^+d \rightarrow pp \) and fit them to an exponential form. By the principle of detailed balance the \( pp \rightarrow \pi^+d \) and \( \pi^+d \rightarrow pp \) reactions have a simple relationship:

\[
\left[ \frac{|p_p|}{|p_\pi|} \frac{d\sigma}{d\Omega} (pp \rightarrow \pi d) \right]_{cm}^{\theta=0} = \frac{3}{4} \left[ \frac{|p_\pi|}{|p_p|} \frac{d\sigma}{d\Omega} (\pi d \rightarrow pp) \right]_{cm}^{\theta=0} = \frac{3}{4} \left[ \frac{|p_\pi|}{|p_p|} \right]_{cm} \left\{ e^{2.5914 - 0.011115 T_\pi} + 0.000065 (T_\pi - 500) \right\},
\]

where the exponential fit is inserted in the last step. In this expression \( T_\pi \) is the laboratory kinetic energy of the pion (in MeV) in the reaction \( \pi^+d \rightarrow pp \), giving the cross section in mb/sr. The threshold for the pd \( \rightarrow 3\Lambda K^+ \) reaction corresponds to \( T_\pi = 419.5 \) MeV, \( |p_\pi| = 428 \) MeV/c and \( |p_p| = 727 \) MeV/c, yielding a cross section of 53.3 \( \mu b/sr \) according to Eq. (36). The Fermi momenta are neglected in this calculation.

5.2 Empirical fit of \( \pi^+n \rightarrow \Lambda K^+ \) data

There is a recent fit of the cross section for the reaction \( \pi^+n \rightarrow \Lambda K^+ \) to a simple formula, but this does not have the correct energy dependence near threshold, which is crucial for the present application. Instead the threshold fit of Jones et al. is used. This gives

\[
\frac{d\sigma}{d\Omega} (\pi^+n \rightarrow \Lambda K^+) = \frac{1}{4\pi} \left[ A p_K + \frac{B p_K^2}{1 + (R p_K)^2} \right] \quad p_{\pi}^{lab} < 970
\]

where \( A = 122 \cdot 10^{-6} \) fm\(^2\)/MeV, \( B = 12 \cdot 10^{-9} \) fm\(^2\)/MeV and \( R = 7.165 \cdot 10^{-3} \) (MeV)\(^{-1} \) (the Compton wavelength of the pion). At threshold the amplitude squared to use in Eqs (11) and (25) is 50.3 \( \mu b/sr \).
6 Results and conclusions

The differential cross sections for the two processes $pd \rightarrow ^3_HK^+$ and $pd \rightarrow d\Lambda K^+$ have been calculated in the forward direction for energies up to $T_{p_{lab}} = 1137$ MeV, i.e., 10 MeV above threshold. The wave functions employed are discussed in Sec. 4. The result of the calculation is plotted in Fig. 4.

In addition the threshold amplitude squared ($|f|^2 = p_{fin}/p_{ini}d\sigma/d\Omega$) for the reaction leading to a bound final state has been computed and found to be

$$|f(pd \rightarrow ^3_HK^+)|^2 = 1.03 \text{ nb/sr}.$$  

The cross section for the $pd \rightarrow ^3_HK^+$ reaction is small and in addition almost completely drowned by the break-up reaction $pd \rightarrow d\Lambda K^+$. Thus, if one has the ambition to study a bound final state, it is of utmost importance to identify the hypertriton. The ratio of the cross sections for the bound and unbound final states in the $\Lambda d$ spin-1/2 channel is given by

$$\frac{\sigma_{\frac{1}{2}}(d\Lambda K)}{\sigma(^3_HK)} \approx \frac{1}{4} \left( \frac{Q}{E_B} \right)^{\frac{3}{2}} \left( 1 + \sqrt{1 + Q/E_B} \right)^{-2},$$  

where the $Q$-value refers to the reaction $pd \rightarrow d\Lambda K^+$. It is evident from this formula that the break-up reaction will dominate when $Q$ is greater than $E_B$ and that this will happen at small $Q$ because of the small binding energy ($E_B = 0.13$ MeV). The ratio of the cross sections calculated from Eqs (25) and (11) for the $\Lambda d$ spin-1/2 case agrees with Eq. (38). The cross section for the unbound $\Lambda d$ spin-3/2 channel is larger than the cross section for the unbound $\Lambda d$ spin-1/2 channel, by a factor 0.7 near threshold and increasing to 1.9 at $T_{p_{lab}} = 1137$ MeV.

Some future improvements should include more accurate wave functions (like the one of Miyagawa et al. [10]), higher energies and angular dependence. The deuteron wave function should be corrected for the Lorentz boost. It would also be valuable to have a better experimental determination of the $\pi^+n \rightarrow \Lambda K^+$ vertex.

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Figure 3: Comparison of the Congleton hypertriton wave function (solid line) with the spin-1/2 (dashed line) and spin-3/2 (dotted line) $Λ_d$ scattering wave functions. The scattering wave functions are normalized according to Eq. (30), which is approximatively valid even for $k \neq iα$ (actually $k = 0$ in this plot).
Figure 4: The differential cross sections for the reactions \( pd \rightarrow ^3\Lambda HK^+ \) (solid line) and \( pd \rightarrow d\Lambda K^+ \) (dashed line) in the forward direction.