INTRODUCTION

The first step in the noncommutative dynamics was undertaken by L.C. Biedenharn\(^1\) who considered the quantum noncommutative harmonic oscillator. Recently Aref’eva and Volovich\(^2\) published paper devoted to some nonrelativistic dynamical system in a noncommutative phase-space framework.

Noncommutative analogon of the Galilean particle, as described in Aref’eva and Volovich\(^2\), has two main features:

– Consistency of the formalism demands noncommutativity of the inertial mass. This phenomena holds also in Rembielinski\(^3\) in the relativistic case.
– There is no unitary time development of the system on the quantum level.

In this paper we formulate unitary noncommutative \(q\)-dynamics on the quantum level. To do this let us notice that a possible deformation of the standard quantum mechanics lies in change of the algebra of observables with consequences on the level of dynamics. This is pictured on the Fig. 1. The main observation is the well known statement, that probabilistic interpretation of quantum mechanics causes an unitary time evolution of physical system irrespectively of the choice of the algebra of observables (standard or \(q\)-deformed). As a consequence the Heisenberg equations of motion hold in each case (in the Heisenberg picture). In the following we restrict ourselves to the one degree of freedom systems.

ALGEBRA OF OBSERVABLES—STANDARD QM CASE

Construction of quantum spaces by Manin\(^4\) as quotient of a free algebra by two-
Figure 1. This scheme is showing possible changes in the structure of QM.

A unital associative algebra freely generated by \( I \), \( x \), and \( p \), while \( J(I, x, p) \) is a two-sided ideal in \( A \) defined by the Heisenberg rule

\[
xp = px + i\hbar I.
\]

There is an antilinear anti-involution (star operation) in \( A \) defined on generators as below

\[
x^* = x, \quad p^* = p.
\]

From the above construction it follows that this anti-involution induces in \( \mathcal{H} \) a \(*\)-anti-automorphism defined again by the eqs. (3).

Now, according to the result of Aref'eva & Volovich\(^2\), confirmed in Rembielinski\(^3\) for the relativistic case, some parameters of the considered dynamics, like inertial mass, do not commute with the generators \( x \) and \( p \). This means that these parameters should...
be treated themselves as generators of the algebra. To be more concrete let us consider a conservative system described by the Hamiltonian

\[ H + p^2 \kappa^2 + V(x, \kappa, \lambda). \]  

(4)

Here \( \kappa \) and \( \lambda \) are assumed to be additional hermitean generators of the extended algebra \( \mathcal{H}' \) satisfying the following re-ordering rules

\[
\begin{align*}
xp &= px + i\hbar \lambda^2 \\
x\lambda &= \lambda x \\
p\lambda &= \lambda p \\
x\kappa &= \kappa x \\
p\kappa &= \kappa p \\
\kappa\lambda &= \lambda \kappa.
\end{align*}
\]  

(5)

We observe that the generators \( \kappa \) and \( \lambda \) belong to the center of \( \mathcal{H}' \). Thus the irreducibility condition on the representation level implies that \( \lambda \) and \( \kappa \) are multipliers of the identity \( I \). Consequently they can be chosen as follows

\[
\lambda = I \quad \kappa = \frac{1}{\sqrt{2\mu}} I
\]  

(6)

so the extended algebra \( \mathcal{H}' \) reduces to the homomorphic Heisenberg algebra \( \mathcal{H} \) defined by (1) and (2). Notice that \( \mathcal{H}' \) can be interpreted as a quotient of a free unital, associative and involutive algebra \( A(I, x, p, \kappa, \lambda) \) by the two-sided ideal \( J(I, x, p, \kappa, \lambda) \) defined by eqs. (5) i.e.

\[ \mathcal{H}' = A(I, x, p, \kappa, \lambda) / J(I, x, p, \kappa, \lambda) \]  

(7)

It is remarkable, that eqs. (5) are nothing but the Bethe Ansatz for \( \mathcal{H}' \).

Finally, dynamics defined by the Hamiltonian \( H \) and the Heisenberg equations lead to the Hamilton form of the equations of motion:

\[
\begin{align*}
\dot{\lambda} &= 0 \\
\dot{\kappa} &= 0 \\
\dot{x} &= \frac{1}{\mu} p \\
\dot{p} &= -V'(x).
\end{align*}
\]  

(8)

**ALGEBRA OF OBSERVABLES—q-QM CASE**

Now, the formulation of the standard quantum mechanics by means of the algebra \( \mathcal{H}' \) suggest a natural \( q \)-deformation of the algebra of observables; namely the \( q \)-deformed algebra \( \mathcal{H}_q \) is a quotient algebra

\[ \mathcal{H}_q = a(I, x, p, K, \Lambda) / J(I, x, p, K, \Lambda) \]  

(9)
where the two-sided ideal \( J \) is defined now by the following Bethe Ansatz re-ordering rules

\[
\begin{align*}
   xp &= q^2px + ihqA^2 \\
   xA &= \xi A x \\
   pA &= \xi^{-1}Ap \\
   xK &= \tau^2 K x \\
   pK &= \varepsilon^2 Kp \\
   \Lambda K &= \tau \varepsilon K \Lambda
\end{align*}
\]

where \( K \) and \( \Lambda \) are assumed to be invertible and

\[
x^* = x, \quad p^* = p, \quad K^* = K, \quad \Lambda^* = \Lambda.
\]

A consistency of the system (10) requires

\[
|q| = |\xi| = |\tau| = |\varepsilon| = 1.
\]

The corresponding conservative Hamiltonian has the form

\[
H = p^2 K^2 + V(x, K, \Lambda).
\]

Now, similarly to the standard case, \( \Lambda \) and \( K \) are assumed constant in time:

\[
\begin{align*}
   \dot{\Lambda} &= \frac{i}{\hbar} [H, \Lambda] \equiv 0 \\
   \dot{K} &= \frac{i}{\hbar} [H, K] \equiv 0
\end{align*}
\]

which implies, under the assumption of the proper classical limit (3),

\[
\varepsilon = 1, \quad \tau = \xi^{-1}
\]

and by means of eqs. (16)

\[
\begin{align*}
   V(x, K, \Lambda) &= V(\xi x, \xi K, \Lambda) \quad (17) \\
   V(x, K, \Lambda) &= V(\xi^2 x, K, \Lambda
\end{align*}
\]

Furthermore, taking into account (16)

\[
\dot{x} = \frac{i}{\hbar} [H, x] = K^2 \left[\frac{i}{\hbar} (1 - (\frac{q}{\xi})^4)p^2x + q\xi^{-4}(\frac{q}{\xi})^2 + 1)A^2p\right],
\]

and

\[
\dot{p} = \frac{i}{\hbar} [H, p] = -\frac{i}{\hbar} p[V(x, K, \Lambda) + V(q^2 x, K, \xi A)] +
\]

\[
-\frac{q}{(\frac{q}{\xi})^2 - 1} x [V((\frac{q}{\xi})^2 x, \xi^{-2} K, \xi A) - V(x, \xi^{-2} K, \xi A)] A^2.
\]

Notice that the last term is the quantum (Gauss-Jackson) gradient of \( V(x, \xi^{-2} K, \xi A)A^2 \).

Now, a consistency of the Hamilton form of the equations of motion (14), (15), (18) and (19) with the algebra (10) and with the Leibniz rule confirms (16)–(17) and implies

\[
V(x, K, A) = V((\frac{q}{\xi})^2 x, K, A)
\]
Furthermore, eqs. (17) and (20) implies that in the formula (19) the term linear in $p$ vanish. Consequently

$$\dot{p} = -q \partial_x^{(q/\xi)^2} V(x, \xi^{-2} K, \xi \Lambda) A^2 $$  \hspace{1cm} (21)

where $\partial_x^{(q/\xi)^2}$ is the Gauss-Jackson derivative as defined in the eq. (19).

Moreover, under the assumption of the proper classical limit, eq. (20) implies that

$$\xi = q $$  \hspace{1cm} (22)

and $V$ depends only on the variable $xK^{-1}\Lambda^{-2}$ or $V$ does not depend on $x$, so taking into account (17) we obtain in this case

$$V = 0.$$  \hspace{1cm} (23)

Therefore we have two cases.

Case I

$$H = p^2 K^2 $$

$$\dot{x} = \left[ \frac{i}{\hbar} \left( \xi^4 - q^4 \right) + q \left( \xi^2 + q^2 \right) pA^2 \right] K^2 $$  \hspace{1cm} (24)

$$\dot{p} = 0 $$

and

$$xp = q^2 px + i\hbar q A^2 $$

$$xA = \xi A x $$

$$pA = \xi^{-1} Ap $$  \hspace{1cm} (25)

$$xK = \xi^{-2} Kx $$

$$pK = Kp $$

$$\Lambda K = \xi^{-1} \Lambda A .$$  \hspace{1cm} (26)

Case II

$$H = p^2 K^2 + V \left( (2m)^{-1/2} q^{-1} x K^{-1} \Lambda^{-2} \right) $$

$$\dot{x} = 2(\Lambda K)^2 p $$  \hspace{1cm} (27)

$$\dot{p} = -q (\partial_x V) A^2 .$$

and the algebra (23) holds under the condition (24) $\xi = q$. The meaning of the normalisation factor $\sqrt{2m}$, $m > 0$, will be evident later. Notice that from the eqs. (27) we can identify the inertial mass $M$ as

$$M = \frac{1}{2} q (K \Lambda)^{-2} ,$$  \hspace{1cm} (28)
so

\[ xM = q^2 M x \]
\[ pM = q^2 M p \]
\[ \Lambda M = q^2 M \Lambda. \]  

(29)

Now, let us consider the dynamical models by Aref’eva & Volovich\textsuperscript{2}.

**Free particle**

We choose the potential \( V = 0 \) so \( H = p^2 K^2 \) and consequently

\[ \dot{x} = q^{-1} M^{-1} p \quad \text{(30)} \]
\[ \dot{p} = 0. \]

Notice that eqs. (30) do not contain \( \Lambda \). The equations (30) and the algebra (29) are the same as in Aref’eva & Volovich\textsuperscript{2}. However it is impossible to fulfil the unitarity condition without of the operator \( \Lambda \) (rest of the algebra is defined by eqs. (25), (27)). Therefore the lacking of the unitarity in Aref’eva & Volovich\textsuperscript{2} is caused by the choice \( \Lambda = I \) which contradicts the reordering rules (25).

**Harmonic oscillator**

We start with the Hamiltonian:

\[ H = p^2 K^2 + \frac{\omega^2}{2} (q^{-1} K^{-1} \Lambda^{-2})^2. \]  

(31)

Consequently

\[ \dot{x} = q^{-1} M^{-1} p \quad \text{(32)} \]
\[ \dot{p} = -\frac{\omega^2}{2} xM. \]

Eqs. (32) still do not contain \( \Lambda \). The reason of the lacking unitarity in the Aref’eva & Volovich\textsuperscript{2} is the same as in the free-particle case.

**REPARAMETRISATION**

The dependence of the potential \( V \) on the element \( q^{-1}(2m)^{-1/2} x K^{-1} \Lambda^{-2} \) and the form of the kinetic term in Hamiltonian (27) suggest the following non-canonical reparametrisation of the q-QM dynamics in the Case II:

\[ X := q^{-1}(2m)^{-1/2} x K^{-1} \Lambda^{-2} \]
\[ P := (2m)^{1/2} pK. \]  

(33)

By means of the eqs. (33), (25) and (22) we obtain the following form of the reordering rules (in terms of \( X, P, K, \) and \( \lambda \))

\[ X P = PX + i\hbar I \]
\[ K \Lambda = q \Lambda K \]
\[ [\Lambda, X] = [\Lambda, P] = [K, X] = [K, P] = 0. \]  

(34)
Therefore

\[ \mathcal{H}_q = \mathcal{H} \oplus \mathcal{M}_q^2 \]  

i.e. \( \mathcal{H}_q \) is the direct sum of the Heisenberg algebra generated by \( X, P \) and of the real Manin’s plane \( \mathcal{M}_q^2 \) (generated by \( K \) and \( \Lambda \)). Moreover the Hamilton equations take the standard form

\[ \begin{align*}
\dot{X} &= \frac{1}{m} P \\
\dot{P} &= -V'(X)
\end{align*} \]  

with

\[ H = p^2 K^2 + V(q^{-1}(2m)^{-1/2}xK^{-1}\Lambda^{-2}) = P^2 \frac{1}{2m} + V(X). \]  

It is evident that energy spectra of both dynamics (defined by \( x \) and \( p \) or by \( X \) and \( P \)) are the same. However both theories are unitary nonequivalent so its physical content (identification of observables) is rather different. In the Case I for \( \xi = q \) an analogous reparametrisation is impossible. It is remarkable, that a similar analysis given in Brzezinski & al.\(^5\) for a quantum particle on a \( q \)-circle leads to quite analogous conclusions.

**QUANTUM DE RHAM COMPLEX**

Now, we observe that the Hamiltonian equations of motion (8) in the standard quantum mechanics can be written as

\[ \begin{align*}
dx &\equiv \dot{x} dt = \frac{1}{\mu} p dt \\
dp &\equiv \dot{p} dt = -V'(x) dt.
\end{align*} \]  

By means of the Heisenberg reordering rule (3) it is easy to calculate that

\[ \begin{align*}
x dx &= dx (x + i\hbar p^{-1}) \\
p dx &= dx p \\
x dp &= dp x \\
p dp &= dp (p - i\hbar V''(x)/V'(x))
\end{align*} \]  

or in a more symmetric form

\[ \begin{align*}
px dx &= dx xp \\
p dx &= dx p \\
x dp &= dp x \\
V'(x)p dp &= dp pV'(x).
\end{align*} \]  

Assuming that \( dx \) and \( dp \) are obtained from \( x \) and \( p \) respectively as an effect of the application of external differential \( d \) satisfying usual conditions (linearity, nilpotency and the graded Leibniz rule) we can complete the differential algebra with a two-form sector. It is matter of simple calculations to show that

\[ \begin{align*}
dx dp &= -dp dx \\
p^2(dx)^2 &= (px - i\hbar/2)dx dp \\
(dp)^2 &= -\frac{i\hbar}{2} dx dp D_x \left( \frac{V''(x)}{V'(x)} \right),
\end{align*} \]
where $D_x$ is the partial $\hbar$-derivative with respect to $x$, defined via $df(x,p) = dx \, D_x f + dp \, D_p f$.

As a consequence

$$(dx)^3 = (dp)^3 = dx \, (dp)^2 = (dx)^2 \, dp = 0. \tag{42}$$

Therefore we have defined a $Z_2$ graded $\mathcal{H}$-bi-module with $\dim(\mathcal{H}) = 1 + 2 + 1 = 4$—a quantum analogon of the deRham complex.

Now, the above quantum deRham complex can be $q$-deformed according to the deformation of the Heisenberg algebra $\mathcal{H}$. The resulting first order differential calculus reads

$$\begin{align*}
px \, dx &= q^{-4} dx \, xp \\
x \, dp &= q^2 dp \, x \\
dx \, p &= q^2 p \, dx \\
\partial_x V(X) \, dp &= q^{-4} dp \, p \partial_x V(X) \\
dK &= dA = 0,
\end{align*} \tag{43}$$

where $X$ is given in (33) while the derivative $\partial_x$ is with respect to $x$.

It can be verified that the Hamilton equations (27) can be reconstructed from (43) by means of the eqs. (22), (25) under substitution

$$\begin{align*}
dx &= \dot{x}(x,p,K,A) dt \\
dp &= \dot{p}(x,p,K,A) dt.
\end{align*} \tag{44}$$

Therefore the quantum deRham complex contain all information about the algebra of observables and dynamics of the theory.

Recently Dimakis et al.\textsuperscript{6} also applied some differential geometric methods to the Heisenberg algebra but from another point of view.

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