**SHARP INELASTIC CHARACTER OF SLOWLY VARYING NLS SOLITONS**

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**Abstract.** We consider soliton-like solutions of the variable coefficients, subcritical nonlinear Schrödinger equation (NLS)

\[ iu_t + u_{xx} + a(\varepsilon x)|u|^{m-1}u = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [3,5), \]

where \( a(\cdot) \in (1,2) \) is an increasing, asymptotically flat potential, and \( \varepsilon \) small enough. In [30] we proved the existence of a pure, global-in-time generalized soliton \( u(t) \) of the above equation, satisfying

\[ \lim_{t \to -\infty} \| u(t) - Q_1(-v_0 t) e^{i(1-v_0^2/4)t} \|_{H^1(\mathbb{R})} = 0, \]

provided \( \varepsilon \) is small enough. Here \( Q_1(s) \) is the positive solution of \( Q_1' - cQ_1 + Q_1^m = 0 \), \( Q_1 \in H^1(\mathbb{R}) \). In addition, we proved that there are \( c_\infty > 1 \) and \( v_\infty > 0 \), and functions \( \rho(t), \gamma(t) \in \mathbb{R} \), such that the solution \( u(t) \) satisfies

\[ \sup_{t \geq \frac{1}{\varepsilon}} \| u(t) - 2^{-1/(m-1)} Q_{c_\infty}(-v_\infty t - \rho(t)) e^{i(1-v_\infty^2/2) t} e^{i\gamma(t)} \|_{H^1(\mathbb{R})} + |\rho'(t)| \lesssim \varepsilon^2. \]

In this paper we prove that the soliton is not pure as \( t \to +\infty \). Indeed, we give a sharp lower bound on the defect induced by the potential \( a(\cdot) \). More precisely, one has

\[ \liminf_{t \to +\infty} \| u(t) - 2^{-1/(m-1)} Q_{c_\infty}(-v_\infty t - \rho(t)) e^{i(1-v_\infty^2/2) t} e^{i\gamma(t)} \|_{H^1(\mathbb{R})} \gtrsim \varepsilon^2, \]

for all parameters \( \rho(t) \) and \( \gamma(t) \) satisfying the above upper bound. This result shows the existence of nontrivial dispersive effects acting on generalized solitons of slowly varying NLS equations, and for the first time, the inelasticity of the NLS soliton-potential dynamics.

1. Introduction

This paper deals with the problem of inelastic interaction of soliton-like solutions of some generalized nonlinear Schrödinger equations (NLS), and it is the natural continuation of our previous paper [30]. In that paper, the goal was the study of generalized soliton solutions for the following subcritical, variable coefficients NLS equation:

\[ iu_t + u_{xx} + a(\varepsilon x)|u|^{m-1}u = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [2,5). \]  

(1.1)

Here \( u = u(t,x) \) is a complex-valued function, \( \varepsilon > 0 \) is a small number, and the potential \( a(\cdot) \) a smooth, positive function satisfying some specific properties, see [15] below.

This equation represents, in some sense, a simplified model of weakly nonlinear, narrow band wave packets, which considers large variations in the shape of the solitary wave. A primary physical model, and the dynamics of a generalized soliton-like solution, was described by Kaup-Newell [23], using the Inverse Scattering Transform, and by Grimshaw [15], using asymptotic expansions matched with approximate conservation laws. See e.g. [23] and [34] and references therein for a detailed physical introduction to these problems.

On the other hand, from the mathematical point of view, equation (1.1) is a variable-coefficients version of the standard NLS equation

\[ iu_t + u_{xx} + |u|^{m-1}u = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x; \quad m \in [2,5). \]  

(1.2)

This last equation is well-known because of the existence of exponentially decreasing, smooth solutions called solitons, or solitary waves. Given real numbers \( x_0, v_0, \gamma_0 \) and \( c_0 > 0 \), solitons are

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[^1]: In this paper we will not consider any distinction between soliton and solitary wave.
solutions of (1.2) of the form

\[ u(t, x) := Q_{c_0}(x - x_0 - v_0 t)e^{ix_0/2 e^{i(c_0 - v_0^2 t)/4} t}, \quad \text{with} \quad Q_c(s) := e^{-s^2/4} Q(c^{1/2} s), \quad (1.3) \]

and where \( Q := Q_1 \) is the unique –up to translations– function satisfying the following second order, nonlinear ordinary differential equation

\[ Q'' - Q + Q^m = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}). \]

In this case, \( Q \) belongs to the Schwartz class and it is explicit:

\[ Q(s) = \left[ \frac{m + 1}{2 \cosh^2 \left( \frac{1}{2} (m - 1) s \right) } \right]^{1/2}. \quad (1.4) \]

In particular, \( \text{sol} \) represents a solitary wave of scaling \( c_0 \) and velocity \( v_0 \), defined for all time, traveling without any change in shape, velocity, etc. In other words, a soliton represents a pure, traveling wave solution with invariant profile. Moreover, under certain conditions, solitons and the sum of solitons have been showed to be orbitally and asymptotically stable, see e.g. \[5, 13, 14, 40, 41, 6, 7\] and references therein.

Coming back to (1.1), the corresponding Cauchy problem in \( H^1(\mathbb{R}) \) has been considered in \[30\], where it was proved that, under some conditions on \( a(\cdot) \) to be explained below, solutions are globally well-defined in the \( L^2 \)-subcritical regime \( m \in [2, 5) \). The proof of this result is an adaptation of the fundamental work of Ginibre-Velo \[12\], see also \[4\].

A fundamental question related to (1.1) is how to generalize a soliton-like solution to more general models. In (1), the existence of solitons for NLS equations with autonomous nonlinearities has been considered. However, the understanding is more reduced in the case of an inhomogeneous nonlinearity, such as equation (1.1). In a general situation, no elliptic, time-independent ODE can be associated to the solution, in opposition to the autonomous case studied in (1). Therefore, other methods are needed.

The first mathematically rigorous results in the case of time and space dependent NLS equations were proved by Bronski-Jerrard \[2\]. In addition, Gustafson et al. \[17, 16\], Gang-Sigal \[9\], and Holmer-Zworski \[19\] have considered the dynamics of a small perturbation of a solitary wave, under general potentials, and for not too large times, namely of the order \( t \sim \frac{1}{\varepsilon} \) and \( t \sim \frac{1}{\varepsilon} |\log \varepsilon| \), with \( \varepsilon \) the slowly varying parameter. The best result in that case \[19\] states that for any \( \delta > 0 \) and for all time \( t \sim \delta^{-1} |\log \varepsilon| \), the solution \( u(t) \) of the corresponding Cauchy problem remains close in \( H^1(\mathbb{R}) \) to a modulated solitary wave, up to an error of order \( \varepsilon^{2-\delta} \). In addition, the dynamical parameters of the solitary wave follow a well defined dynamical system.

In \[30\] we described dynamics of a generalized soliton, for all time, in for time-independent, slowly varying NLS equations of the form (1.1). The main novelty was the understanding of the dynamics as a nonlinear interaction, or collision, between the soliton and the potential, in the spirit of the recent works of Holmer-Zworski \[19\], Martel-Merle \[25, 20\], and the author \[31, 32\]. In order to state these results, and our present main results, let us first describe the framework that we have considered for the potential \( a(\cdot) \) in (1.1).

**Setting and hypotheses.** Concerning the function \( a \) in (1.1), we assume that \( a \in C^4(\mathbb{R}) \) and there exist fixed constants \( K, \gamma > 0 \) such that

\[
\begin{cases}
1 < a(r) < 2, & a'(r) > 0, \quad \text{for all } r \in \mathbb{R}, \\
0 < a(r) - 1 \leq Ke^{\gamma r}, & \text{for all } r \leq 0, \quad 0 < 2 - a(r) \leq Ke^{-\gamma r}, \quad \text{for all } r \geq 0, \quad \text{and} \quad (1.5) \\
|a^{(k)}(r)| \leq Ke^{-\gamma |r|}, & \text{for all } r \in \mathbb{R}, \quad k = 1, 2, 3, 4.
\end{cases}
\]

In particular, \( \lim_{r \to -\infty} a(r) = 1 \) and \( \lim_{r \to +\infty} a(r) = 2 \). The limits (1 and 2) do not imply a loss of generality, it just simplifies the computations.

We remark some important facts about (1.1) (see \[30\] for more details). First of all, this equation is not invariant under scaling and spatial translations. Second, the momentum

\[ P[u](t) := \frac{1}{2} \text{Im} \int_{\mathbb{R}} \bar{u} u_x(t, x) \, dx \quad (1.6) \]
satisfies the relation
\[ \partial_t P[u](t) = \frac{\varepsilon}{m+1} \int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1}(t,x)dx \geq 0. \] (1.7)

On the other hand, the mass and energy
\[ M[u](t) := \frac{1}{2} \int_{\mathbb{R}} |u|^2(t,x)dx \] (1.8)
\[ E[u](t) := \frac{1}{2} \int_{\mathbb{R}} |u_x|^2(t,x)dx - \frac{1}{m+1} \int_{\mathbb{R}} |u|^{m+1}(t,x)dx \] (1.9)
remain conserved along the flow. Let us recall that these quantities are conserved for local \( H^1 \)-solutions of (1.2).

Since \( a \sim 1 \) as \( x \to -\infty \), given \( v_0 > 0 \), one should be able to construct a generalized soliton-like solution \( u(t) \), satisfying \( u(t,x) \sim Q(x-v_0 t)e^{i\varepsilon x/v_0/2}e^{i(1-v_0^2/4)t} \) as \( t \to -\infty \). Indeed, this sort of scattering property has been proved in [30], but for the sake of completeness, it is briefly described in the following paragraph.

**Description of the dynamics.** Let us recall the setting of our problem. Consider the equation
\[
\begin{cases}
iu + u_{xx} + a(\varepsilon x)|u|^{m-1}u = 0 & \text{in } \mathbb{R} \times \mathbb{R}_x, \\
m \in [2,5]; & 0 < \varepsilon \leq \varepsilon_0; & a(\varepsilon) \text{ satisfying (1.10)}.
\end{cases}
\tag{1.10}
\]
Here \( \varepsilon_0 > 0 \) is a small parameter. Assuming the validity of (1.10), one has the following generalization of [24].

**Theorem 1.1.** (Existence of solitons for NLS under variable medium, [30].) Suppose \( m \in [2,5) \).
Let \( v_0 > 0 \) be a fixed number. There exists a small constant \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) the following holds. There exists a unique solution \( u \in C(\mathbb{R},H^1(\mathbb{R})) \) of (1.2), global in time, such that
\[
\lim_{t \to -\infty} \| u(t) - Q(-v_0 t)e^{i\varepsilon x/v_0/2}e^{i(1-v_0^2/4)t} \|_{H^1(\mathbb{R})} = 0.
\] (1.11)

Let us remark that (1.11) is just a consequence of the following, more specific property: there exist \( K,\mu > 0 \) such that
\[
\| u(t) - Q(-v_0 t)e^{i\varepsilon x/v_0/2}e^{i(1-v_0^2/4)t} \|_{H^1(\mathbb{R})} \leq Ke^{\mu |t|}, \quad \text{for all } t \leq \varepsilon^{-1/100} \quad (\text{cf. [30]}). \] (1.12)

Next, we have described the dynamics of interaction soliton-potential. Let \( \nu_\infty,\nu_\infty,\lambda_0 \) be the following parameters:
\[
\nu_\infty := (\nu_0^2 + 4\lambda_0(\nu_0 - 1))^{1/2}, \quad \nu_\infty := 2^{4/(5-m)}, \quad \lambda_0 := \frac{5-m}{m+3}.
\] (1.13)

Using the mass (1.8) and energy (1.9), one can guess the behavior of the solution \( u(t) \) as \( t \to +\infty \), assuming the stability of the solution \( u(t) \). Indeed, if for some \( c,v > 0 \), \( \rho(t),\gamma(t) \in \mathbb{R} \), \( \rho'(t) \) small, one has
\[
u(t) = 2^{-\frac{m-1}{10}}Q_c(-vt - \rho(t))e^{i\nu_\infty \gamma(t)} + z(t),
\]
with \( \|z(t)\|_{H^1(\mathbb{R})} \to 0 \) as \( t \to +\infty \), then necessarily \( c = \nu_\infty \) and \( v = \nu_\infty \). Following this idea, we have defined the notion of pure generalized soliton-like solution.

**Definition 1.1.** Let \( v_0 > 0 \) be a fixed number. We say that (1.10) has a pure generalized soliton-like wave solution (of scaling equals 1 and velocity equals \( v_0 \)) if there exist \( C^4 \) real valued functions \( \rho = \rho(t),\gamma = \gamma(t) \) defined for all large times and a global in time \( H^1(\mathbb{R}) \) solution \( u(t) \) of (1.10) such that
\[
\lim_{t \to -\infty} \| u(t) - Q(-v_0 t)e^{i\varepsilon x/v_0/2}e^{i(1-v_0^2/4)t} \|_{H^1(\mathbb{R})} = 0,
\] (1.14)
\[
\lim_{t \to +\infty} \| u(t) - 2^{-\frac{m-1}{10}}Q_{\nu_\infty}(-v_\infty t - \rho(t))e^{i\nu_\infty \gamma(t)} \|_{H^1(\mathbb{R})} = 0,
\] (1.15)
\footnote{Note that, with no loss of generality, we have chosen the scaling parameter equals one.}
\footnote{The factor \( 2^{-1/\nu_\infty} \) in front of \( Q_c \) is required since \( a \to 2 \) as \( x \to +\infty \).}
with $|\rho'(t)| \ll v_0$ for all large times, and where $c_\infty, v_\infty > 0$ are the scaling and velocity suggested by the mass and energy conservation laws, as in (1.13).

The solution $u(t)$ constructed in Theorem 1.1 satisfies (1.14). However, it is believed that, due to deep dispersive effects coming from the interaction between the soliton and the potential, the second condition (1.15) above is never satisfied. Our first approach in that direction is the following stability result.

**Theorem 1.2** (Interaction soliton-potential [30]). Suppose $v_0 > 0$, and $m \in [3, 5]$. There exists $K_0, \epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the following holds. There are smooth $C^1$ parameters $\rho(t), \gamma(t) \in \mathbb{R}$, such that the function

$$w(t, x) := u(t, x) - 2^{-1/(m-1)}Q_{c_\infty}(x - v_\infty t - \rho(t))e^{ixv_\infty/2\epsilon^\gamma(t)}$$

satisfies, for all $t \gg \epsilon^{-1}$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| \leq K_0 \epsilon^2. \quad (1.16)$$

This result is in agreement with our expectations: the generalized soliton is in some sense stable along the positive direction of time and obeys, up to second order in $\epsilon$, the dynamics predicted by the mass and conservation laws. If $m$ belongs to the interval $[2, 3]$, or if we consider the two-dimensional case, then our conclusions are weaker: one has an upper bound of $O(\epsilon)$ (cf. [31]), revealing the dependence of the error on the smoothness of the nonlinearity.

**Main results.** A natural question to be considered is the following: can one obtain a quantitative lower bound on the defect $w(t)$ as the time goes to infinity? In this paper we improve Theorem 1.2 by showing a sharp lower bound on the defect $w(t)$ at infinity. In other words, any perturbation of the constant coefficients NLS equation of the form (1.10) induces non trivial dispersive effects on the soliton, and the solution is not pure anymore. This result clarifies the inelastic character of generalized solitons for perturbations of some dispersive equations, and moreover, it seems to be the general behavior. Moreover, our result can be seen as the first mathematical proof of inelastic behavior in the case of an NLS dynamics. Additionally, one can see this result as a generalization to the case of interaction soliton-potential of the ground-breaking papers by Martel and Merle, concerning the inelastic character of the collision of two solitons for non-integrable gKdV equations [25, 27].

However, in order to obtain such a quantitative bound, and compared with the proofs in [27] or [33], we require a new approach, because the defect is in some sense degenerate. As we will describe below, our lower bounds are related to third order corrections to the dynamical parameters of the soliton solution, propagated to time infinity using the forward stability of the solution, a consequence of (1.7). The first result of this paper is the following.

**Theorem 1.3** (Sharp inelastic character of the soliton-potential interaction). Suppose $m = 3$, or $m \in [4, 5]$). There exist $\bar{v}_0 \geq 0$ (possibly zero), and $K, \epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, and $v_0 \neq \bar{v}_0$, the following holds. For any $\tilde{\rho}(t), \tilde{\gamma}(t) \in \mathbb{R}$ satisfying, for all $t \gg \epsilon^{-1}$,

$$\|\tilde{w}(t)\|_{H^1(\mathbb{R})} + |\tilde{\rho}'(t)| \leq K_0 \epsilon^2, \quad (K_0 \text{ given in (1.16)},$$

where

$$\tilde{w}(t, x) := u(t, x) - 2^{-1/(m-1)}Q_{c_\infty}(x - v_\infty t - \tilde{\rho}(t))e^{ixv_\infty/2\epsilon^{\tilde{\gamma}(t)}},$$

one has

$$\lim_{t \to +\infty} \inf \|\tilde{w}(t)\|_{H^1(\mathbb{R})} \geq \frac{\epsilon^2}{K}. \quad (1.17)$$

Moreover, in the case $m = 3$ one has $\bar{v}_0 = 0$.

**Remark 1.1.** The requisite $m = 3$ or $m \in [4, 5]$ is due to the regularity required to obtain a better description of the interaction, which is this time of third order in $\epsilon$. We believe that the above results hold for $m \in [3, 5]$, but with a harder proof. The two-dimensional case seems even more difficult, since $m = 3$ is the $L^2$-critical nonlinearity.
Remark 1.2. The extra condition \( v_0 \neq \tilde{v}_0 \) is technical but not essential. It is related to the proof of the nonzero character of a defect on the main velocity. Fortunately, we are able to prove that in the case \( m = 3 \) one has \( \tilde{v}_0 = 0 \) and therefore Theorem 1.3 holds for all \( v_0 > 0 \). We believe that the same result holds in the case \( m \in [4, 5] \).

Remark 1.3. Note that from Theorem 1.2 it is not clear if the parameters \( \rho(t) \) and \( \gamma(t) \) are the best choices to satisfy (1.16). Indeed, any perturbation of order less than \( \varepsilon^2 \) satisfies the same inequality. For that reason Theorem 1.3 rules out all possible values of \( \rho(t), \gamma(t) \), and proves inelasticity independently of the choice of parameters.

Remark 1.4. In [30] we have considered the case of a strictly decreasing potential. In that case, the soliton is reflected, provided the initial velocity is small. Our proof does not cover that case, since no evident lack of symmetry is present at the third order in that case. Instead, one should look at the next orders of magnitude of the main solution, in order to find a defect because of the interaction.

Remark 1.5. If we compare with the results obtained for gKdV equations [25, 27, 29, 33], our result is sharp since there is no essential gap between the bounds (1.10) and (1.17). Note that the gap in those papers was related to the emergence of infinite mass tails in an approximate solution (see [31, 32] for a proof in the case of a slowly varying potential), which do not appear in the NLS case because the linearized NLS operators are solvable between localized spaces, unlike the gKdV case.

The proof of Theorem 1.3 is actually a consequence of the following deeper result, which reveals the exact nature of the inelasticity for the case of slowly varying NLS dynamics:

**Theorem 1.4.** Suppose \( m = 3 \), or \( m \in [4, 5] \), \( v_0 \neq \tilde{v}_0 \), and \( a(\cdot) \) satisfying (1.3). There exist constants \( K, \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon < \varepsilon_0 \), the following holds. There is a number \( \tilde{v}_\infty > 0 \) and \( C^1 \)-functions \( \rho(t), \gamma(t) \in \mathbb{R} \) such that

\[
w(t, x) := u(t, x) - 2^{-1/(m-1)}Q_{v_\infty}(x - \tilde{v}_\infty t + \rho(t))e^{ix\tilde{v}_\infty/2e^{i\gamma(t)}},
\]

satisfies

\[
\sup_{t \geq T} \|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| \leq Ke^3.
\]

Moreover, there is \( \kappa_0 > 0 \), independent of \( \varepsilon \), such that, for all \( 0 < \varepsilon < \varepsilon_0 \), one has

\[
|\tilde{v}_\infty - v_\infty| \geq \kappa_0 \varepsilon^2.
\]

**About the proofs.** As we have explained before, the proof of the above results are originally based in a recent argument introduced by Martel and Merle in [27], to deal with the interaction of two nearly equal solitons of the quartic gKdV equation. Roughly speaking, Martel and Merle proved that the interaction is inelastic because of a small but not zero lack of symmetry on the soliton trajectories, contrary to the symmetric integrable case. Later, in [33], we improved the foundational Martel-Merle idea in two directions: first, we generalized that argument to the case of the interaction soliton-potential (in the gKdV case), nontrivial since the problem has no evident symmetries to be exploited; and second, we have faced in addition, a somehow degenerate case, the cubic one, where the original Martel-Merle argument is not longer available.

It turns out that the NLS case satisfies the same degeneracy as the cubic gKdV equation, in a sense to be explained in the following lines. In [30], we have considered an approximate solution of (1.10), describing the interaction soliton-potential. The objective was to obtain first and second order corrections on the translation, phase, velocity and scaling parameters \( \rho(t), \gamma(t), v(t), c(t) \) of the soliton solution, as we proceed to explain. Indeed, the solution \( u(t) \) behaves along the interaction as follows,

\[
\|u(t) - a^{-1/(m-1)}(\varepsilon\rho(t))Q_{v(t)}(- \rho(t))e^{i\gamma(t)}/2e^{i\gamma(t)}\|_{H^1(\mathbb{R})} \lesssim \varepsilon^2,
\]

(1.18)
Therefore, after integration on a time interval of size $O(\varepsilon^{-1})$ near $t \sim 0$, this term formally induces a perturbation of order $O(\varepsilon)$ on the trajectory $\rho(t)$, namely a defect on the dynamics in agreement with the conservation laws.

Using this property, one should be tempted to follow the same argument described in [33], but in the NLS case we have several deep issues, that we describe below. The argument in [33] requires the introduction of a sort of opposite solution $v(t)$, pure as $t \to +\infty$, with slightly different dynamical parameters, to be more specific, at the second order in $\varepsilon$. This crucial observation, first noticed by Martel and Merle in [27] for the quartic gKdV model, represents a lack of symmetry in the dynamics, and is the key point of the proof in [33]. It seems that the NLS case does not enjoy this property. Second, the proof in [33] employs a backward stability property for the difference between $v(t)$ and $u(t)$, which is not known in the NLS case. Moreover, probably the most difficult problem to face is the sort of degeneracy of the defect $\varepsilon^2 f_4(\varepsilon t)$, in the sense that it has the same order of magnitude compared with the error in (1.18) and (1.20).

A first answer to the last problem was given in the same paper [33], where we have faced a similar degenerate problem, the cubic case of a slowly varying gKdV equation. The idea in that case is to profit of the existence of a defect (of order $\varepsilon^2$) emerging at the level of the scaling law. Indeed, we improved the approximate solution (32) at the level of the dynamical system, but the global error does not improve, since a dispersive tail appears and destroys the symmetry of the solution, and therefore the accuracy of the approximate solution. In order to avoid that problem, we have used a sharp virial identity to get a bound of order $o(\varepsilon)$ when we integrate the global error over large intervals of time. At that time the defect appears as a concrete obstruction to elasticity. After that point, one can conclude the proof by propagating the defect to time infinity.

In the NLS case, independently of the nonexistence of suitable virial identities, the improvement of the approximate solution at the level of the dynamical system leads to the improvement of the global error (1.18), and vice versa. In fact, we will show (cf. Proposition 2.3) that, after some pages of lengthy computations (Sections 5 and 6, (1.19)-(1.20)) contain now two additional corrections, denoted by $f_3(\varepsilon t)$ and $f_6(\varepsilon t)$, and such that

$$
u(t) = \varepsilon f_1(\varepsilon t) + \varepsilon^3 f_4(\varepsilon t) + O(\varepsilon^4), \quad c(t) = \varepsilon f_2(\varepsilon t) + \varepsilon^3 f_6(\varepsilon t) + O(\varepsilon^4),$$

$$\rho(t) = \varepsilon^2 f_4(\varepsilon t) + O(\varepsilon^3),$$

with $f_j(\varepsilon t)$, $j = 1, \ldots, 6$ not identically zero and for $t \sim \varepsilon^{-1-1/100}$,

$$\|u(t) - a^{-1/(m-1)}(\varepsilon \rho(t))Q_{\varepsilon c(t)}(-\rho(t))e^{i(t)(t/2)}e^{i\gamma(t)}\|_{H^1(\mathbb{R})} \lesssim \varepsilon^3.$$  

Now the defects $f_3$ and $f_6$ are relevant for the dynamics, and induce nontrivial $O(\varepsilon^2)$ corrections to the final scaling and velocity parameters, provided certain nonzero integral conditions are satisfied, similar to (1.21). Indeed, one has, for $t \sim \varepsilon^{-1-1/100}$,

$$c(t) \sim c_\infty, \quad \nu(t) \sim \nu_\infty + \kappa_0 \varepsilon^2, \quad \kappa_0 \neq 0,$$

where $c_\infty, \nu_\infty$ are the scaling and velocity predicted by the mass and energy conservation laws, given in (1.13) (cf. Lemma 2.4 for a detailed proof).

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4We write $f_j = f_j(\varepsilon t)$ in order to emphasize the fact that we are working with slowly varying functions, but in the rigorous proof below we only use the notation $f_j(t)$.

5This key difference between NLS and gKdV emerges at the level of the linearized problem. In the NLS case, the invertibility of the linear operator leads to localized solutions, which is not the case of gKdV, by the presence of an additional derivative. See e.g. [29] for the consequences of this fact when describing the interaction of two solitons.
The purpose for the rest of proof is to exploit this property. The idea is the following: if \( \|u(t)\| \) is not satisfied, then using the stability of \( u(t) \) for large times (cf. \([30]\)) we can propagate the defect \( (1.25) \) with a global error of \( O(\varepsilon^3) \), a bound that contradicts \( (1.17) \). Finally, note that our argument do not require a backward stability result to be proved. In that sense, our proof differs from that of \([33]\).

**Remark 1.6** (The gKdV case). As previously mentioned, the interaction soliton-potential has also been considered in the case of generalized KdV equations with a slowly varying potential, or a soliton-defect interaction. See e.g. Dejak-Sigal \([8]\), Holmer \([18]\), Holmer-Perelman-Zworski \([22]\), and our recent works \([31, 32, 33]\).

**Remark 1.7.** Additionally, one can consider the problem of solitary wave-defect interaction, namely the case where the potential is similar to a Dirac distribution. In this case, one may expect the splitting of the solitary wave, see e.g. \([11, 20, 21, 36]\). Finally, the behavior of perturbations of small solitary waves of NLS equations, and its corresponding dynamics, has been considered in \([10, 39]\).

Let us explain the organization of this paper. First, in Section 2 is devoted to the rigorous proof of \((1.25)\). In Section 3 we prove the main theorems. Finally, in Section 4 we improve the approximate solution associated to the interaction problem, and we find the corrections \( f_5 \) and \( f_6 \) above mentioned.

**Notation.** We follow the notation introduced in \([30]\). In particular, in this paper both \( K, \mu > 0 \) will denote fixed constants, independent of \( \varepsilon \), and possibly changing from one line to another. Additionally, we introduce, for \( \varepsilon > 0 \) small, the time of interaction

\[
T_\varepsilon := \frac{1}{v_0} \varepsilon^{-1 - \frac{5}{4m}} > 0.
\]  

(1.26)

## 2. Existence of a defect

The purpose of this section is to show rigorously the existence of a defect associated to the scaling and velocity parameters of the soliton solution constructed in \([30]\). The main result of this section is contained in a simple computational result, Lemma 2.1

Denote, for \( C > 0, V, P \in \mathbb{R} \) given, and \( m \in [2, 5] \),

\[
f_1(C, V) := \frac{8a'(\varepsilon U)C}{(m + 3)a(\varepsilon U)}, \quad f_2(C, V) := \frac{4a'(\varepsilon U)CV}{(5 - m)a(\varepsilon U)}.
\]

(2.1)

We recall the existence of a unique solution for a dynamical system involving the evolution of the first order scaling, velocity, translation and phase parameters of the soliton solution, denoted by \((C(t), V(t), U(t), H(t))\), in the interaction region. The behavior of this solution is essential to understand the dynamics of the soliton inside this region.

**Lemma 2.1** (\([30]\), Lemma 3.4). Let \( m \in [2, 5] \), and \( v_0 > 0 \). Let \( \lambda_0, a(\cdot) \) and \( f_1, f_2 \) be as in \((1.13), (1.4) \) and \((2.1)\) respectively. There exists \( \varepsilon_0 > 0 \) small such that, for all \( 0 < \varepsilon < \varepsilon_0 \), the following holds.

1. **Existence.** There exists a unique solution \((C(t), V(t), U(t), H(t))\), with \( C(t) \) bounded, monotone increasing and positive, defined for all \( t \geq -T_\varepsilon \), of the following nonlinear ODE system

\[
\begin{align*}
    V'(t) &= \varepsilon f_1(C(t), U(t)), & V(-T_\varepsilon) &= v_0, \\
    C'(t) &= \varepsilon f_2(C(t), V(t), U(t)), & C(-T_\varepsilon) &= 1, \\
    U'(t) &= V(t), & U(-T_\varepsilon) &= -v_0 T_\varepsilon, \\
    H'(t) &= -\frac{1}{2}V'(t)U(t), & H(-T_\varepsilon) &= 0.
\end{align*}
\]

(2.2)

Moreover, \( C(t), V(t) \) satisfy the relation

\[
V^2(t) = v_0^2 + 4\lambda_0(C(t) - 1).
\]

(2.3)
(2) Asymptotic behavior. Let \( v_\infty, c_\infty \) be defined as in (1.15). Then one has \( \lim_{t \to +\infty} C(t) = c_\infty(1 + O(\varepsilon^{10})), \lim_{t \to +\infty} V(t) = v_\infty(1 + O(\varepsilon^{10})), \) and \( \lim_{t \to +\infty} U(t) = +\infty. \)

We remind to the reader some notation introduced in [30]. Let \( t \in [-T_\varepsilon, \tilde{T}_\varepsilon], Q, c \) given in (1.2), \( c(t) > 0 \) and \( v(t), \rho(t), \gamma(t) \in \mathbb{R} \) be bounded functions to be chosen later, and

\[
y := x - \rho(t), \quad \tilde{R}(t, x) := \frac{Q_c(t)(y)}{\bar{a}(\varepsilon \rho(t))}e^{\Theta(t,x)},
\]

where

\[
\bar{a} := a_{\rho}, \quad \Theta(t, x) := \int_0^t c(s) ds + \frac{1}{2} v(t)x - \frac{1}{4} \int_0^t v'(s) ds + \gamma(t).
\]

The parameter \( \bar{a} \) describes the shape variation of the soliton along the interaction. Concerning the parameters \( c(t), v(t), \rho(t) \) and \( \gamma(t) \), it is assumed that, for all \( t \in [-T_\varepsilon, \tilde{T}_\varepsilon] \),

\[
|c(t) - C(t)| + |v(t) - V(t)| + |\gamma'(t) - H'(t)| + |\rho'(t) - P'(t)| \leq \varepsilon^{1/100}.
\]

with \( (C(t), V(t), P(t), H(t)) \) from Lemma 2.1. Let

\[
\tilde{u}(t, x) := \tilde{R}(t, x) + w(t, x),
\]

where the correction is given by

\[
w(t, x) := \sum_{k=1}^3 \varepsilon^k [A_{k,c}(t, y) + iB_{k,c}(t, y)]e^{\Theta(t,x)},
\]

where \( A_{k,c} \) and \( B_{k,c} \) are unknown real valued functions to be determined. More precisely, given \( k = 1, 2 \) or 3, we look for functions \( (A_{k,c}, B_{k,c}) \) such that for all \( t \in [-T_\varepsilon, \tilde{T}_\varepsilon] \) and for some fixed constants \( K, \mu > 0 \),

\[
\|A_{k,c}(t, \cdot)\|_{H^1(\mathbb{R})} + \|B_{k,c}(t, \cdot)\|_{H^1(\mathbb{R})} \leq K e^{-\mu |\rho(t)|}, \quad A_{k,c}(t, \cdot), B_{k,c}(t, \cdot) \in \mathcal{S}(\mathbb{R}).
\]

(here \( \mathcal{S}(\mathbb{R}) \) is the standard Schwartz class). We want to estimate the size of the error obtained by inserting \( \tilde{u} \) as defined in (2.7)-(2.8) in the equation (1.10). For this, we define the residual term

\[
S[\tilde{u}](t, x) := \tilde{u}_t + \tilde{u}_{xx} + a(\varepsilon x) |\tilde{u}|^{m-1} \tilde{u}.
\]

For this quantity one has the following improved decomposition.

**Proposition 2.2.** Let \((c(t), v(t), \rho(t), \gamma(t))\) be satisfying (2.9). There are unique functions \( A_{k,c} = A_{k,c}(t, y) \) and \( B_{k,c} = B_{k,c}(t, y) \), of the form (2.9) such that \( u(t) \) defined in (2.7)-(2.8) satisfies, for every \( t \in [-T_\varepsilon, \tilde{T}_\varepsilon] \), the following:

\[
S[\tilde{u}](t, x) = F_0(t, y) e^{i\Theta} + \tilde{S}[\tilde{u}](t, x),
\]

where

1. \( F_0 \) is an approximate dynamical system:

\[
F_0(t, y) := -\frac{1}{2} (v'(t) - \varepsilon f_1(t) - \varepsilon^3 f_3(t)) y \tilde{u} + \varepsilon (c'(t) - \varepsilon f_2(t) - \varepsilon^3 f_4(t)) \partial_y \tilde{u}
\]

\[
- (\gamma'(t) + \varepsilon^3 v'(t) \rho(t) - \varepsilon^2 f_3(t)) \tilde{u} + \varepsilon (\gamma' \rho - v(t) - \varepsilon^2 f_4(t)) \partial_y \tilde{u},
\]

and \( \partial_y \tilde{u} := \partial_y \tilde{R} = \rho \).

2. The parameters \( f_j, j = 1, \ldots, 6, \) are smooth, time dependent functions, more specifically depending on the parameters \( c(t), \rho(t), v(t) \) and \( \gamma(t) \). Indeed, \( f_1 \) and \( f_2 \) are given by

\[
f_1 := \frac{8a'(\varepsilon \rho)}{(m+3)a(\varepsilon \rho)}, \quad f_2 := \frac{4a'(\varepsilon \rho)v}{(5-m)a(\varepsilon \rho)}, \quad (\text{compare with (2.7))},
\]

and there are unique coefficients \( \alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R} \) such that

\[
f_3 := \left( \alpha_1 + \alpha_2 \frac{v^2}{c} a'(\varepsilon \rho) + (\alpha_3 + \alpha_4 \frac{v^2}{c}) \frac{a''}{a^2}(\varepsilon \rho) \right),
\]

The uniqueness is related to the obtention of estimate (2.18), see Lemma 4.2 for a detailed description.
We define \( f_4 := \frac{v}{c} \left[ \frac{\alpha''(\varepsilon \rho)}{a} + \beta_2 \frac{a''(\varepsilon \rho)}{a^2} \right] \), \( f_5 := \frac{\alpha^{(3)}}{a}(\varepsilon \rho) + \frac{\delta_2 + \delta_3}{a^2} \frac{a'a''(\varepsilon \rho)}{a^2} + \frac{\delta_4}{a^3} \frac{a^3(\varepsilon \rho)}{a^3} \), and
\[
f_6 := \frac{4\varepsilon f_4 a'(\varepsilon \rho)}{(5-m)a(\varepsilon \rho)} + v \left[ (\eta_1 + \frac{\varepsilon^2}{c^{\eta_2}}) \frac{a'a''(\varepsilon \rho)}{a^2} + (\eta_3 + \frac{\varepsilon^2}{c^{\eta_4}}) \frac{a^3(\varepsilon \rho)}{a^3} \right].
\] (2.15)

(3) Finally, \((A_{k,c}, B_{k,c})\) satisfy (2.2) for \( k = 1, 2 \) and 3, and
\[
\| \tilde{S}[\tilde{u}](t) \|_{H^1(\mathbb{R})} \leq K\varepsilon^4 (e^{-\varepsilon^2|\rho(t)|} + \varepsilon),
\] (2.18)
uniformly in time.

Remark 2.1. Some of the coefficients \( \alpha_j, \beta_j \) have been explicitly computed in [30]. Note that our notation slightly differs from that of [30]. Later, in Section 5, we will compute the remaining parameters.

In order to maintain the continuity of the argument, we have preferred to prove Proposition 2.2 in Section 4.

From (2.18), we have an estimate at the fourth order in \( \varepsilon \) for the associated error of the approximate solution \( \tilde{u} \). It turns out that with this new estimate we can prove an improved version of [30] Proposition 3.10], after following step by step the lines of that proof. We claim:

**Proposition 2.3.** Let \( m = 3 \) or \( m \in [4, 5] \). There exist \( K_0, \varepsilon_0 > 0 \) such that the following holds for all \( 0 < \varepsilon < \varepsilon_0 \). There are \( C^{1}-\)functions \( c, v, \rho, \gamma : [-T, \tilde{T}]ightarrow \mathbb{R} \) such that, for all \( t \in [-T, \tilde{T}] \), one has
\[
\| u(t) - \tilde{u}(t; c(t), v(t), \rho(t), \gamma(t)) \|_{H^1(\mathbb{R})} \leq K_0 \varepsilon,
\] (2.19)
\[
|\rho'(t) - v(t) - \varepsilon^2 f_4(t)| + |\gamma'(t) - \frac{1}{2} v'(t)\rho(t) - \varepsilon^2 f_3(t)| \leq K_0 \varepsilon^3,
\] (2.20)
\[
|v'(t) - \varepsilon f_1(t) - \varepsilon^3 f_5(t)| + |c'(t) - \varepsilon f_2(t) - \varepsilon^3 f_6(t)| \leq K_0 \varepsilon^4,
\] (2.21)
and for \( K > 0 \) independent of \( K_0 \),
\[
|c(-T) - 1| + |v(-T) - v_0| \leq K \varepsilon^{10}.
\] (2.22)

Finally,
\[
|c(t) - C(t)| + |v(t) - V(t)| + |\rho'(t) - U'(t)| + |\gamma'(t) - H'(t)| \leq K K_0 \varepsilon^3.
\] (2.23)

Remark 2.2. Note that, compared with estimates (3.54)-(3.55) in [30], now the terms \( \varepsilon^2 f_3(t), \varepsilon^2 f_4(t), \varepsilon^3 f_5(t) \) and \( \varepsilon^3 f_6(t) \) are dynamically nontrivial, compared with the error on the right hand side.

Remark 2.3. Note that estimates (2.23) improve (2.6). In addition, (2.22) are consequences of (1.12) at time \( -T \), and (2.2).

Let us introduce new limiting scaling and velocities, which will differ from the expected ones. We define
\[
\tilde{c}_\infty := c(\tilde{T}), \quad \text{and} \quad \tilde{v}_\infty := v(\tilde{T}).
\] (2.24)

The key result of this paper is the following

**Lemma 2.4** (Existence of a defect). There are \( \kappa_0, \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon < \varepsilon_0 \), one has
\[
\tilde{c}_\infty = c_\infty + O(\varepsilon^{10}), \quad |\tilde{v}_\infty - v_\infty| \geq \kappa_0 \varepsilon^2,
\] (2.25)
provided \( v_0 \neq \tilde{v}_0 \), for some \( \tilde{v}_0 \geq 0 \).
Proof. 1. It is not to difficult to visualize that $|c(t) - C(t)| \leq K\varepsilon^3$ (see (2.23), (2.21) and Lemma 2.4) imply the positivity of $c(t)$, uniformly in $\varepsilon$. Additionally, from (2.20), (2.21), (2.13) and the boundedness properties of the functions $f_1(t)$, one has
\[c'(t) = \varepsilon f_2(t) + \varepsilon^3 f_6(t) + O(\varepsilon^4) = \frac{4\varepsilon a'(\varepsilon\rho(t))c(t)(\rho'(t) + \varepsilon^2 f_4(t) + O(\varepsilon^3)) + \varepsilon^3 f_6(t) + O(\varepsilon^4)}{(5 - m)\alpha(\varepsilon\rho(t))}.
\]
uniformly on $[-T_\varepsilon, T_\varepsilon]$. Dividing by $c(t)$ and integrating, we obtain
\[
\log \frac{c(t)}{c(-T_\varepsilon)} = \log \left(\frac{a^{4/(5-m)}(\varepsilon\rho(t))}{a^{4/(5-m)}(\varepsilon\rho(-T_\varepsilon))}\right) + \varepsilon^3 \int_{-T_\varepsilon}^{T_\varepsilon} \left[\frac{4a'(\varepsilon\rho(s))f_4(s)}{(5 - m)\alpha(\varepsilon\rho(s))} + \frac{f_6(s)}{c(s)}\right] ds + O(\varepsilon^{3-1/100}).
\]
From (2.22) and (1.5), we have
\[c(t) = a^{4/(5-m)}(\varepsilon\rho(t)) + \varepsilon^2 h(t) + O(\varepsilon^{3-1/100}),
\]
where
\[h(t) := \varepsilon a^{4/(5-m)}(\varepsilon\rho(t)) \int_{-T_\varepsilon}^{t} \left[\frac{4a'(\varepsilon\rho(s))f_4(s)}{(5 - m)\alpha(\varepsilon\rho(s))} + \frac{f_6(s)}{c(s)}\right] ds.
\]
In Section 6 we will prove that
\[h(T_\varepsilon) \sim 0.
\]
In particular,
\[\bar{c}_\infty = c(T_\varepsilon) = a^{4/(5-m)}(\varepsilon\rho(T_\varepsilon)) + \varepsilon^2 h(T_\varepsilon) + O(\varepsilon^{3-1/100}) = c_\infty + o(\varepsilon^2),
\]
which proves the first case in (2.24).

2. In order to prove the second identity, note that
\[v'(t)v(t) = \varepsilon f_1(t)v(t) + \varepsilon^3 f_5(t)v(t) + O(\varepsilon^4) = \varepsilon f_1(t)(\rho'(t) - \varepsilon^2 f_4(t) + O(\varepsilon^3)) + \varepsilon^3 f_5(t)v(t) + O(\varepsilon^4) = \frac{8\varepsilon a'(\varepsilon\rho(t))c(t)}{(m + 3)\alpha(\varepsilon\rho(t))}\rho'(t) + \varepsilon^3(-f_1(t)f_4(t) + f_5(t)v(t)) + O(\varepsilon^4)
\]
uniformly on $[-T_\varepsilon, T_\varepsilon]$. Replacing (2.26) in the above identity, we get
\[
\frac{1}{2}(v^2(t))' = \frac{8\varepsilon a'(\varepsilon\rho(t))}{m + 3} a^{m/(m-1)}(\varepsilon\rho(t))\rho'(t)
\]
\[+ \varepsilon^3 \left[\frac{8\varepsilon a'(\varepsilon\rho(t))}{(m + 3)\alpha(\varepsilon\rho(t))} h(t)v(t) - f_1(t)f_4(t) + f_5(t)v(t)\right] + O(\varepsilon^{4-1/100}).
\]
Therefore, after integration
\[v^2(t) = v_0^2 + \frac{4(5-m)}{m+3}(a^{4/(5-m)}(\varepsilon\rho(t)) - 1) + \varepsilon^2 k(t) + O(\varepsilon^{3-2/100})
\]
\[= v_0^2 + 4\lambda_0(c(t) - 1) + \varepsilon^2 k(t) + O(\varepsilon^{3-2/100}),
\]
with
\[k(t) := 2\varepsilon \int_{-T_\varepsilon}^{t} \left[\frac{8\varepsilon a'(\varepsilon\rho(t))}{(m + 3)\alpha(\varepsilon\rho(t))} h(s)v(s) - f_1(s)f_4(s) + f_5(s)v(s)\right] ds.
\]
Now we claim that, there is $\tilde{v}_0 \geq 0$ (equals zero if $m = 3$), such that, for all $v_0 \neq \tilde{v}_0$,
\[k(T_\varepsilon) \sim \kappa_1 \neq 0,
\]
and then
\[\bar{v}_\infty^2 = v^2(T_\varepsilon) = v_0^2 + 4\lambda_0(c_\infty - 1) + \varepsilon^2 k(T_\varepsilon) = v_\infty^2 + \varepsilon^2 \kappa_1 + o(\varepsilon^2),
\]
which proves the second assertion. The proof of the non degeneracy condition $\kappa_1 \neq 0$ is carried out in Section 6. \qed
Finally, in this last section we prove Theorems 1.3 and 1.4. In order to simplify our arguments, we split the proof into several steps.

**Step 1. Behavior at** \( t = \hat{T}_\varepsilon \). Following the argument described in subsection 3.7.4 of [30], it is not difficult to conclude that
\[
\| \tilde{u}(\hat{T}_\varepsilon; c(\hat{T}_\varepsilon), v(\hat{T}_\varepsilon), \rho(\hat{T}_\varepsilon), \gamma(\hat{T}_\varepsilon)) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho_\varepsilon)\xi(\cdot)\bar{v}_\infty e^{i\gamma_\varepsilon(t)} \|_{H^1(\mathbb{R})} \leq K\varepsilon^3,
\]
for some fixed \( \gamma_\varepsilon \in \mathbb{R} \), \( \rho_\varepsilon := \rho(\hat{T}_\varepsilon) \), and \( K > 0 \) independent of \( \varepsilon \). Note that we have used [29, 2.24], the composition of \( \tilde{u} \) given in (2.7), and the fact that \( \rho_\varepsilon \) satisfies
\[
\frac{99}{100}v_0T_\varepsilon \leq \rho_\varepsilon \leq \frac{101}{100}(2v_\infty - v_0)T_\varepsilon.
\]
Therefore, from (2.19) and the previous estimate, one has
\[
\| u(\hat{T}_\varepsilon) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho_\varepsilon)\xi(\cdot)\bar{v}_\infty e^{i\gamma_\varepsilon(t)} \|_{H^1(\mathbb{R})} \leq K\varepsilon^3,
\]
(3.2)
Moreover, from Lemma 2.4 there is \( \kappa_0 > 0 \), independent of \( \varepsilon \), such that
\[
|\bar{v}_\infty - v_\infty| > \kappa_0\varepsilon^2.
\]
(3.3)
We recall that this identity and (3.2) imply that \( \bar{v}_\infty \) are different from \( v_\infty \) by a quantity larger than the global error associated to the approximate solution.

**Step 2. Propagation of the defect.** Now we prove that the defect is still present as \( t \to +\infty \). The key idea is to use the stability of \( Q \) to propagate the error [33]. Indeed, from (3.2), and using [30, Proposition 2.3] with \( p_m = 3 \) provided \( \varepsilon_0 \) is taken smaller if necessary, we get the existence of a constant \( K > 0 \) and \( C^1 \) modulation parameters \( \rho(t), \gamma(t) \in \mathbb{R} \), defined in \( [\hat{T}_\varepsilon, +\infty) \), and such that
\[
w(t) = u(t) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \bar{v}_\infty t - \rho(t))\xi(\cdot)\bar{v}_\infty e^{i\gamma(t)}
\]
satisfies, for all \( t \geq \hat{T}_\varepsilon \),
\[
\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| \leq K\varepsilon^3.
\]
From (3.3) and (3.4), Theorem 1.4 is proved.

**Step 3. Conclusion.** Let us prove Theorem 1.3. By contradiction, let us assume that, for \( \alpha > 0 \) small, there exist \( T > T_\varepsilon \) very large, and parameters \( \bar{\rho}(t), \bar{\gamma}(t) \in \mathbb{R} \), defined for all \( t \geq \hat{T}_\varepsilon \) large, such that
\[
\tilde{w}(t, x) := u(t, x) - 2^{-1/(m-1)}Q_{c_\infty}(x - v_\infty t - \tilde{\rho}(t))\xi(\cdot)\bar{v}_\infty e^{i\tilde{\gamma}(t)}
\]
satisfies
\[
\|\tilde{w}(T)\|_{H^1(\mathbb{R})} \leq \alpha\varepsilon^2.
\]
(3.5)
Therefore, from (3.3) and (3.5), and the triangle inequality,
\[
\|Q_{c_\infty}(\cdot - v_\infty T - \tilde{\rho}(T))\xi(\cdot)\bar{v}_\infty e^{i\tilde{\gamma}(T)} - Q_{c_\infty}(\cdot - \tilde{v}_\infty T - \rho(T))\xi(\cdot)\bar{v}_\infty e^{i\gamma(T)}\|_{H^1(\mathbb{R})} \leq K\varepsilon^2(\alpha + \varepsilon),
\]
or
\[
\|Q_{c_\infty}(\cdot - (v_\infty - \tilde{v}_\infty)T - (\tilde{\rho}(T) - \rho(T)))\xi(\cdot)\bar{v}_\infty e^{i(\tilde{\gamma}(T) - \gamma(T))} - Q_{c_\infty}\xi(\cdot)\bar{v}_\infty e^{i\gamma(T)}\|_{H^1(\mathbb{R})} \leq K\varepsilon^2(\alpha + \varepsilon).
\]
A simple argument shows that, for some constant \( K > 0 \) independent of \( \varepsilon \), one has
\[
|\tilde{v}_\infty - v_\infty| + |(v_\infty - \tilde{v}_\infty)T + \tilde{\rho}(T) - \rho(T)| + |\tilde{\gamma}(T) - \gamma(T)| \leq K\varepsilon^2(\alpha + \varepsilon),
\]
otherwise the previous inequality does not hold. By taking \( \alpha > 0 \) smaller and \( T \) larger if necessary, this result is in contradiction with (3.3).

\textsuperscript{7}A careful study of the proof of Proposition 2.3 in [30] shows that we can take \( p_m = 3 \) with similar conclusion.
4. Approximate solution revisited

This section is devoted to the proof of Proposition 2.2.

**Proof.** We revisit the proof of [30] Proposition 3.3 and the resolution of the corresponding linear systems carried out in that paper. In what follows, we state the necessary modifications. First of all, one has

\[ S[u] = S[R] + L[w] + \tilde{N}[w], \]

where \( S[R] = i\hat{R}_t + \hat{R}_{xx} + a(\varepsilon x)|\hat{R}|^{m-1}\hat{R}, \)

\[ L[w] := iw_t + w_{xx} + \frac{a(\varepsilon x)}{2a(\varepsilon p)}Q^{m-1}_c((m+1)w + e^{2i\theta}(m-1)\bar{w}), \quad (4.1) \]

and

\[ \tilde{N}[w] := a(\varepsilon x) \left\{ |\hat{R} + w|^{m-1}(\hat{R} + w) - |\hat{R}|^{m-1}\hat{R} - \frac{Q^{m-1}_c((m+1)w + e^{2i\theta}(m-1)\bar{w})}{2a(\varepsilon p)} \right\}. \quad (4.2) \]

In what follows, we compute these terms this time up to third order in \( \varepsilon. \)

**Step 1.** The first modification comes at the level of Claim 2, where a Taylor expansion up to fifth order gives us

\[ S[R] = \left[ F^R_0 + \varepsilon F^1_1 + \varepsilon^2 F^2_2 + \varepsilon^3 F^3_3 + \varepsilon^4 F^4_4 \right] (t, y)e^{i\theta} + \varepsilon^5 f^R(t)F^c_5(y)e^{i\theta}, \quad (4.3) \]

where \( F^R_0 \) is given now by the expression

\[ F^R_0 := -\frac{1}{2}(v' - \varepsilon f_1 - \varepsilon^3 f_3)\frac{yQ_c}{a(\varepsilon p)} + \frac{i(c' - \varepsilon f_2 - \varepsilon^3 f_6)\Lambda Q_c}{a(\varepsilon p)} \]

\[ -\left( \gamma + \frac{1}{2}v' \rho - \varepsilon^2 f_3 \right) \frac{Q_c}{a(\varepsilon p)} - i(\rho' - \varepsilon^2 f_4)\left[ \frac{Q'_c}{a(\varepsilon p)} - \frac{\varepsilon^2}{a^2(\varepsilon p)}Q_c \right]. \quad (4.4) \]

\( F^R_0 \) and \( F^R_2 \) do not change. They are given by

\[ F^R_1(t, y) := \frac{a'(\varepsilon p)}{a^m(\varepsilon p)}yQ_c[Q^{m-1}_c - \frac{4c}{m + 3}] + i\frac{a'(\varepsilon p)}{a^m(\varepsilon p)}v \left[ \frac{4c}{m - 1} - \frac{1}{m - 1}Q_c \right], \quad (4.5) \]

\[ F^R_2(t, y) := \frac{a''(\varepsilon p)}{2a^m(\varepsilon p)}y^3Q^{m-1}_c - \frac{f_3}{a(\varepsilon p)}Q_c - \frac{f_4}{a(\varepsilon p)}Q'_c. \quad (4.6) \]

The novelty is the term \( F^R_3 \), given by the expression

\[ F^R_3(t, y) := \frac{a^{(3)}(\varepsilon p)}{6a^m(\varepsilon p)}y^3 Q^{m-1}_c - \frac{f_5}{2a(\varepsilon p)}yQ_c + i\frac{f_6}{a(\varepsilon p)}\Lambda Q_c + i\frac{f_4}{a^2(\varepsilon p)}Q_c. \quad (4.7) \]

It is not difficult to see that \( \|F^R_4(t)\|_{H^1(\mathbb{R})} \leq K e^{-\varepsilon |t|}. \) Finally, \( |f^R(t)| \leq K, F^R_e \in \mathcal{S}. \) Therefore, for every \( t \in [-T_\varepsilon, T_\varepsilon], \)

\[ \|e^{\varepsilon^k F^R_4(t, y) + \varepsilon^5 f(t)F^c_5(y)}\|_{H^1(\mathbb{R})} \leq K e^{A e^{-\varepsilon |t|} + \varepsilon}. \]

**Step 2.** Now we consider the computation, up to third order in \( \varepsilon, \) of [30] Claim 3, which deals with \( L[w], \) previously introduced in [11]. The computations are very similar. We get in this opportunity

\[ L[w] = -3 \sum_{k=1}^{3} \varepsilon^k \left[ \mathcal{L}_+(A_{k,c}) + i\mathcal{L}_-(B_{k,c}) \right] e^{i\theta} - \frac{1}{2}(v' - \varepsilon f_1 - \varepsilon^3 f_3)yw \]

\[ + i(c' - \varepsilon f_2 - \varepsilon^3 f_6)v\partial_y w - \left( \gamma + \frac{1}{2}v' \rho - \varepsilon^2 f_3 \right)w - i(\rho' - \varepsilon^2 f_4)w_y \]

\[ + \varepsilon^2[F^L_2(t, y) + iG^L_2(t, y)]e^{i\theta} + \varepsilon^3[F^L_3(t, y) + iG^L_3(t, y)]e^{i\theta} + \varepsilon^4 f^L(t)F^L_c(y)e^{i\theta}. \]

Here, as already computed in [30],

\[ F^L_2(t, y) := m\frac{a'}{a}Q^{m-1}_c - \frac{1}{2}f_1 y A_{1,c} - \left[ \frac{1}{\varepsilon}(B_{1,c})_t + f_2 \Lambda B_{1,c} \right], \quad (4.8) \]
with the previously known second order terms and the new, third order terms:
\[
\varepsilon w_1 := 1
\]
and between real and imaginary parts. We perform this computation in several steps.

2\widehat{\varepsilon} + 1) \left( \frac{1}{\varepsilon} (A_1,c)_t + f_2 \Lambda A_{1,c} + \frac{a'}{a} Q_c^{m-1} y B_{1,c} - \frac{1}{2} f_1 y B_{1,c} \right)
\leq \kappa \varepsilon^4 \varepsilon^{-\mu(\rho(t))} + \varepsilon.
\]

Step 3. Finally, we consider the improvement of [30, Claim 4], namely the term \( \tilde{N}[w] \) defined in [4,5]. The computations here need more care, since several new terms appear. Following the proof in [30], one has now the improved decomposition
\[
\tilde{N}[w] = \varepsilon^2 (N^{2,1} + iN^{2,2}) e^{i\theta} + \varepsilon^3 (N^{3,1} + iN^{3,2}) e^{i\theta} + O_{H^1(\mathbb{R})} (\varepsilon^4 e^{-\varepsilon\mu(\rho(t))}),
\]
with the previously known second order terms
\[
N^{2,1} := \frac{1}{2} (m-1) \widehat{\mu}(\varepsilon \rho) Q_c^{m-2} (m A_{1,c}^2 + B_{1,c}^2), \quad N^{2,2} := (m-1) \widehat{\mu}(\varepsilon \rho) Q_c^{m-2} A_{1,c} B_{1,c},
\]
and the new, third order terms:
\[
N^{3,1} := (m-1) Q_c^{m-3} \left[ \widehat{\mu}(\varepsilon \rho) Q_c (m A_{1,c} A_{2,c} + B_{1,c} B_{2,c}) + \frac{1}{2} \frac{a'(\varepsilon \rho)}{a} y Q_c (m A_{1,c}^2 + B_{1,c}^2) \right]
\]
\[
\frac{1}{6} \widehat{a}^2(\varepsilon \rho) \left[ m(m-2) A_{1,c}^4 + 3(m-2) A_{1,c} B_{1,c}^2 \right] \bigg];
\]
\[
N^{3,2} := (m-1) Q_c^{m-3} \left[ \widehat{\mu}(\varepsilon \rho) Q_c (A_{1,c} B_{2,c} + A_{2,c} B_{1,c}) \right]
\]
\[
+ \frac{a'(\varepsilon \rho)}{a-m^{-2}(\varepsilon \rho)} y Q_c A_{1,c} B_{1,c} + \frac{1}{2} \widehat{a}^2(\varepsilon \rho) \left[ (m-2) A_{1,c}^2 B_{1,c} + B_{1,c}^3 \right].
\]
Indeed, since \( m = 3 \) or \( m \in \{4,5\} \), one has the following third order expansion of \( \tilde{N}[w] \):
\[
\tilde{N}[w] = \frac{(m-1)a(\varepsilon x)}{2a^{m-2}} Q_c^{m-3} \left\{ e^{i\theta} |w|^2 + 2 \Re(e^{i\theta} \bar{w})w + (m-3)e^{i\theta}(\Re(e^{i\theta} \bar{w}))^2 \right\}
\]
\[
+ (m+1)(m-1)(m-3) \frac{a(\varepsilon x)}{48} Q_c^{m-3} \left[ 3 e^{2i\theta} |w|^2 \bar{w} + 3 |w|^2 w + e^{-2i\theta} w^3 + \frac{(m-5)}{(m-1)} e^{4i\theta} \bar{w}^3 \right]
\]
\[
+ \frac{1}{8} (m+1)(m-1)a(\varepsilon x) Q_c^{m-3} |w|^2 w + O_{H^1(\mathbb{R})} (\varepsilon^4 e^{-\varepsilon\mu(\rho(t))}).
\]
It is easy to check that this term simplifies enormously when \( m = 3 \). However, we will consider the general case.

We replace \( w \) in the above expression and we arrange the obtained terms according to the powers of \( \varepsilon \) and between real and imaginary parts. We perform this computation in several steps. First, note that
\[
a(\varepsilon x) = a(\varepsilon \rho) + \varepsilon a'(\varepsilon \rho) y + O(\varepsilon^2 y^2).
\]
On the other hand, from (2.8),
\[
|w|^2 = \varepsilon^2 (A_{1,c}^2 + B_{1,c}^2) + 2 \varepsilon^3 (A_{1,c} A_{2,c} + B_{1,c} B_{2,c}) + O_{H^1(\mathbb{R})} (\varepsilon^4 e^{-\varepsilon\mu(\rho(t))}).
\]
Similarly \( \text{Re}(e^{i\Theta} \tilde{w}) = \varepsilon A_{1,c} + \varepsilon^2 A_{2,c} + \varepsilon^3 A_{3,c} \). Therefore
\[
\text{Re}(e^{i\Theta} \tilde{w})w = \varepsilon^2 (A_{1,c}^2 + iA_{1,c}B_{1,c})e^{i\Theta} + \varepsilon^3 (2A_{1,c}A_{2,c} + i(A_{1,c}B_{2,c} + A_{2,c}B_{1,c}))e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^4 e^{-\varepsilon\mu|t|}),
\]
and
\[
e^{i\Theta}(\text{Re}(e^{i\Theta} \tilde{w}))^2 = \varepsilon^2 A_{1,c}^2 e^{i\Theta} + 2\varepsilon^3 A_{1,c}A_{2,c}e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^4 e^{-\varepsilon\mu|t|}).
\]
On the other hand,
\[
|w|^2 w = \varepsilon^3 [A_{1,c}^3 + A_{1,c}B_{1,c}^2 + i(A_{1,c}^2 B_{1,c} + B_{1,c}^3)]e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^4 e^{-\varepsilon\mu|t|}),
\]
and
\[
w^3 = \varepsilon^3 [A_{1,c}^3 - 3A_{1,c}B_{1,c}^2 + i(3A_{1,c}^2 B_{1,c} - B_{1,c}^3)]e^{3i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^4 e^{-\varepsilon\mu|t|}).
\]
Collecting these expansions and replacing in (4.17) we obtain, after some simplifications,
\[
\tilde{N}[w] = \frac{1}{2} \varepsilon^2 (m-1)\tilde{\alpha}(\varepsilon\rho)Q_c^{m-2} \left\{ mA_{1,c}^2 + B_{1,c}^2 + 2iA_{1,c}B_{1,c} \right\} e^{i\Theta} + \varepsilon^3 (m-1)\tilde{\alpha}(\varepsilon\rho)Q_c^{m-3} \left\{ mA_{1,c}^2 A_{2,c} + B_{1,c}^2 + i(A_{1,c}B_{2,c} + A_{2,c}B_{1,c}) \right\} e^{i\Theta} + \frac{1}{2} \varepsilon^3 (m-1) \tilde{\alpha}'(\varepsilon\rho) yQ_c^{m-2} \left\{ mA_{1,c}^2 + B_{1,c}^2 + 2iA_{1,c}B_{1,c} \right\} e^{i\Theta} + \frac{1}{24} \varepsilon^3 (m-1)(m-3)\tilde{\alpha}'(\varepsilon\rho) Q_c^{m-3} (4m+1)A_{1,c}^3 + 9A_{1,c}B_{2,c} + 3i(3A_{1,c}^2 B_{1,c} - B_{1,c}^3) e^{i\Theta} + \frac{1}{8} \varepsilon^3 (m+1)(m-1)\tilde{\alpha}'(\varepsilon\rho) Q_c^{m-3} \left\{ A_{1,c}^3 + A_{1,c}A_{2,c}B_{1,c} + i(A_{1,c}^2 B_{1,c} + B_{1,c}^3) \right\} e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^4 e^{-\varepsilon\mu|t|}).
\]
From the decomposition into real and imaginary parts, and additional simplifications, we get (4.18)-(4.19), and finally (4.13).

**Step 4. First conclusion.** Collecting the previous estimates, we get
\[
S[\tilde{u}](t,x) = [F_0(t,y) + \varepsilon F_1(t,y) + \varepsilon^2 F_2(t,y) + \varepsilon^3 F_3(t,y) + \varepsilon^4 F_4(t,y) + \varepsilon^5 f(t)F_c(y)]e^{i\Theta}. \tag{4.18}
\]
The term \( F_0 \) is defined in (2.12). In addition,
\[
F_k(t,y) := F_k(t,y) + iG_k(t,y) - [\mathcal{L}_+(A_{k,c}) + i\mathcal{L}_-(B_{k,c})], \quad k = 1, 2, 3; \tag{4.19}
\]
with \( F_1, G_1 \) given by (A.1, Q := \partial_t Q_c)
\[
F_1 := \frac{\partial'(\varepsilon\rho)}{\partial \varepsilon(\varepsilon\rho)} yQ_c [Q_c^{m-1} - \frac{4c}{m+3}], \quad G_1 := \frac{\partial'(\varepsilon\rho)v}{\partial \varepsilon(\varepsilon\rho)} \left[ \frac{4c}{5-m} \Lambda Q_c - \frac{1}{m-1} Q_c \right], \tag{4.20}
\]
\[
F_2 := \text{Re} \{ F_2^R \} + F_2^L + N^{2,1} \quad \text{(cf. (1.0), (1.8) and (1.1))}
\]
\[
= \frac{\partial''}{2\partial \varepsilon^2} y^2 Q_c^m + \frac{m\partial'}{\partial \varepsilon^2} Q_c^{m-1} yA_{1,c} - \frac{1}{2} f_1 y A_{1,c} - \frac{1}{\varepsilon} (B_{1,c})_t - f_2 A_{1,c} + \frac{1}{2} \frac{1}{(m-1)} \partial Q_c^{m-2} (mA_{1,c}^2 + B_{1,c}^2) - \frac{f_3(t)}{\tilde{\alpha}} Q_c, \quad (4.21)
\]
\((\partial A_{1,c} := \partial_t A_{1,c} \text{ and so on})\) and from (4.6)-(4.9) and (4.14),
\[
G_2 := \text{Im} \{ F_2^R \} + G_2^L + N^{2,2}
\]
\[
= \frac{1}{\varepsilon} (A_{1,c})_t + f_2 \Lambda A_{1,c} + \frac{\partial'}{\partial \varepsilon} Q_c^{m-1} y B_{1,c} - \frac{1}{2} f_1 y B_{1,c} + (m-1) \partial Q_c^{m-2} A_{1,c} B_{1,c} - \frac{f_4}{\tilde{\alpha}} Q_c'. \tag{4.22}
\]
The third order terms are new in the decomposition. They are given by the expressions

\[ F_3 := \text{Re}(F_3^R) + F_3^L + N^{3.1} \quad (\text{cf. (4.7), (4.10) and (4.15))} \]

\[ = \frac{a(3)(\varepsilon)'}{6a_m(\varepsilon)} y^3 Q_c^m - \frac{f_5}{2a(\varepsilon)} yQ_c + \frac{ma''}{2a} y^2 Q_c^{m-1} A_{1,c} - f_3 A_{1,c} + f_4 (B_{1,c})_y \]

\[ + \frac{ma'}{a} Q_c^{m-1} y A_{2,c} - \frac{1}{2} f_1 y A_{2,c} - \frac{1}{\varepsilon} (B_{2,c})_t + f_2 A B_{2,c} \]

\[ + (m-1) \tilde{a}(\varepsilon) Q_c^{m-2} (m A_{1,c} A_{2,c} + B_{1,c} B_{2,c}) + \frac{1}{2} (m-1) \frac{a'(\varepsilon)}{a^{m-2}(\varepsilon)} y Q_c^{m-2} (m A_{1,c}^2 + B_{1,c}^2) \]

\[ + \frac{1}{6} (m-1) \tilde{a}^2(\varepsilon) Q_c^{m-3} \{ m(m-2) A_{1,c}^3 + 3(m-2) A_{1,c} B_{1,c}^2 \}, \quad (4.23) \]

and

\[ G_3 := \text{Im}(F_3^R) + G_3^L + N^{3.2} \quad (\text{cf. (4.7), (4.11) and (4.10))} \]

\[ = \frac{f_6}{a(\varepsilon)} \Lambda Q_c + f_4 \frac{\tilde{a}'(\varepsilon)}{a^2(\varepsilon)} Q_c + \frac{a''}{2a} y^2 Q_c^{m-1} B_{1,c} - f_3 B_{1,c} - f_4 (A_{1,c})_y + \frac{1}{\varepsilon} (A_{2,c})_t + f_2 A A_{2,c} \]

\[ + \frac{a'}{a} Q_c^{m-1} y B_{2,c} - \frac{1}{2} f_1 y B_{2,c} + (m-1) \tilde{a}(\varepsilon) Q_c^{m-2} (A_{1,c} B_{2,c} + A_{2,c} B_{1,c}) \]

\[ + (m-1) \left[ \frac{a'(\varepsilon)}{a^{m-2}(\varepsilon)} y Q_c^{m-2} A_{1,c} B_{1,c} + \frac{1}{2} \tilde{a}^2(\varepsilon) Q_c^{m-3} (m - 2) A_{1,c}^2 B_{1,c} + B_{1,c}^3 \right]. \quad (4.24) \]

Moreover, suppose that \((A_{k,c}, B_{k,c})\) satisfy \((2.9)\) for \(k = 1, 2\) and \(3\). Then

\[ \| \varepsilon^4(F_k(t, \cdot) + \varepsilon f(t) F_c) e^{i \theta} \|_{H^1(\mathbb{R})} \leq K \varepsilon^4 C \left( \varepsilon \right) \| f \|_{L^2(\mathbb{R})}, \quad (4.25) \]

uniformly in time. The objective now is to set \(F_k \equiv 0\) for \(k = 1, 2\) and \(3\), which amounts to solve, for \(t \in [-T, T]\) fixed, the linear systems in the \(y\) variable

\[ (\Omega_k) \quad \mathcal{L}_+(A_{k,c}) = F_k; \quad \mathcal{L}_-(B_{k,c}) = G_k. \]

The cases \(k = 1, 2\) were solved in \([30]\); for the sake of completeness we state these results without proofs. The case \(k = 3\) is one of the novelties of this paper. In the next step, the following results will be required.

**Spectral properties of linear NLS operators.** Fix \(c > 0, m \in [2, 5]\), and let

\[ \mathcal{L}_+ w := -w_{yy} + c w - m Q_c^{m-1} w; \quad \mathcal{L}_- w := -w_{yy} + c w - Q_c^{m-1} w; \quad (4.26) \]

where \(w = w(y)\). Then one has

**Lemma 4.1.** The linear operators \(\mathcal{L}_\pm\), defined on \(L^2(\mathbb{R})\) by \((4.26)\), have domain \(H^2(\mathbb{R})\). In addition, they are self-adjoint and satisfy the following properties:

1. The kernel of \(\mathcal{L}_+\) and \(\mathcal{L}_-\) is spanned by \(Q_c\) and \(Q_c\) respectively. Moreover,

\[ \Lambda Q_c := \partial_c Q_c \big|_{c = c} = \frac{1}{c} \left[ \frac{1}{m-1} Q_c + \frac{1}{2} y Q_c' \right], \quad (4.27) \]

satisfies \(\mathcal{L}_+(\Lambda Q_c) = -Q_c\). Finally, the continuous spectrum of \(\mathcal{L}_\pm\) is given by \(\sigma_{\text{cont}}(\mathcal{L}_\pm) = [c, +\infty)\).

2. (Inverse.) For all \(h = h(y) \in L^2(\mathbb{R})\) such that \(\int_R h Q_c' = 0\) (resp. \(\int_R h Q_c = 0\)), there exists a unique \(h_+ \in H^2(\mathbb{R})\) (resp. \(h_- \in H^2(\mathbb{R})\)) such that \(\int_R h_+ Q_c' = 0\) (resp. \(\int_R h_- Q_c = 0\)) and \(\mathcal{L}_+ h_+ = h\) (resp. \(\mathcal{L}_- h_- = h\)). Moreover, if \(h\) is even (resp. odd), then \(h_+\) is even (resp. odd).

3. (Regularity in the Schwartz space \(S(\mathbb{R})\).) For \(h \in H^2(\mathbb{R})\), \(\mathcal{L}_\pm h \in S(\mathbb{R})\) implies \(h \in S(\mathbb{R})\).

For the proof of these properties see e.g. Weinstein \([40]\), and Martel-Merle \([25]\).

**Step 5. Resolution of linear systems.** The next step of the proof is to look at the linear systems appearing in \([30]\) Subsection 3.4. The first system, \((\Omega_1)\), does not suffer any modification, and is given by

\[ (\Omega_1) \quad \mathcal{L}_+ A_{1,c} = F_1; \quad \mathcal{L}_- B_{1,c} = G_1, \]
with $F_1, G_1$ given in (4.20). It turns out that $\int_R Q'_c F_1 = \int_R Q_c G_1 = 0$, therefore Lemma 4.1 applies. We have existence and uniqueness of a solution in the Schwartz class for this linear system. Moreover, the solution of this system is given by (see Remark 3.3 in [30]),

$$A_{1,c}(t,y) = \frac{a'}{\bar{a}m}e^{\frac{1}{\bar{a}m} \frac{a'}{a} y} A_1(\sqrt{c}y), \quad B_{1,c}(t,y) = \frac{a''}{\bar{a}m}e^{\frac{1}{\bar{a}m} \frac{a''}{a} y} B_1(\sqrt{c}y),$$

(4.28)

with

$$A_1(y) := \frac{1}{m+3}(y(yQ' - Q) + \xi Q'), \quad B_1(y) := -\frac{1}{2(5-m)}(y^2 + \chi)Q,$$

(4.29)

and $\xi, \chi$ given by

$$\xi := -\frac{\int_R \left(y^2Q^2 + y^2Q'^2\right)}{\int_R Q^2} = -\frac{m+7}{2(m-1)} + \chi, \quad \chi := -\frac{\int_R y^2Q^2}{\int_R Q^2}.$$  

(4.30)

In addition, $A_{1,c}$ and $B_{1,c}$ satisfy (2.9), and the following orthogonality conditions

$$\int_R A_{1,c}Q_c = \int_R A_{1,c}'Q_c = \int_R B_{1,c}Q_c = \int_R B_{1,c}'Q_c = 0.$$  

(4.31)

Finally, note that $A_{1,c}$ is odd and $B_{1,c}$ is even.

**Remark 4.1.** Note that from $(\Omega_1)$, (4.25) and (4.20) we can conclude that

$$\mathcal{L}_+ A_1 = yQ(Q^{m-1} - \frac{4}{m+3}), \quad \mathcal{L}_- B_1 = \frac{4}{5-m} \Lambda Q - \frac{1}{m-1} Q.$$  

(4.32)

This property will be useful to prove the Main Theorem.

**Step 6. Second order linear system.** Now we consider the linear system, $(\Omega_2)$, given by

$$(\Omega_2) \quad \mathcal{L}_+ A_{2,c} = F_2, \quad \mathcal{L}_- B_{2,c} = G_2,$$

with $F_2, G_2$ are given in (4.21) and (4.22). It turns out that both terms contain a time derivative of the shape parameter (cf. Claim 1 in [30] for more details). Indeed, $F_2$ and $G_2$ possess the terms $\frac{1}{\varepsilon}(B_{1,c})$, and $\frac{1}{\varepsilon}(A_{1,c})$, which, thanks to (4.28), can be written as follows (here $\rho'_1(t) := \rho'(t) - v - \varepsilon^2 f_4(t)$)

$$\frac{1}{\varepsilon}(A_{1,c}) = \frac{\rho'}{\bar{a}m}[a'' - \frac{ma'^2}{(m-1)a}]c^{\frac{m-1}{2}} A_1(\sqrt{c}y)$$

$$= \left[\frac{v + \varepsilon^2 f_4}{\bar{a}m}[a'' - \frac{ma'^2}{(m-1)a}]c^{\frac{m-1}{2}} - \frac{1}{\varepsilon} A_1(\sqrt{c}y) + O_H(1)|\rho'_1(t)|e^{-\varepsilon\rho(t)}|\right] + O_H(1)|\rho'_1(t)|e^{-\varepsilon\rho(t)}|.$$  

The term with the coefficient $\varepsilon^2 f_4$ is too small to be considered (it leads to an error $O(\varepsilon^4e^{-\mu t}|\rho(t)|))$, and the terms with coefficient $|\rho'_1(t)|$ can be added to the dynamical system (4.12). Then, we discard such terms. Similarly, using (2.13),

$$\frac{1}{\varepsilon}(B_{1,c}) = \frac{1}{\bar{a}m}[a''v^2 + a'f_1 - \frac{m}{m-1} \frac{a'^2v^2}{a}]c^{\frac{m-1}{2}} B_1(\sqrt{c}y) + O((\varepsilon^4|f_4| + \varepsilon^3|f_5| + |\rho'_1(t)|)e^{-\varepsilon\rho(t)})$$

$$= \frac{1}{\bar{a}m}[a''v^2 + \frac{8a'^2c}{(m+3)a} - \frac{m}{m-1} \frac{a'^2v^2}{a}]c^{\frac{m-1}{2}} B_1(\sqrt{c}y),$$

where we have discarded the error terms following the same analysis as above. In addition, we replace (4.25) in $F_2, G_2$, and compute the term $\Lambda B_{1,c} = \partial_c B_{1,c}$ using (4.28). We finally obtain 30 simplified source terms, $\hat{F}_2$ and $\hat{G}_2$, given by

$$\hat{F}_2(t,y) = \frac{a''}{\bar{a}m}(\varepsilon \rho(t))[F^I_{2,c}(y) + \frac{v^2(t)}{c(t)}F^{II}_{2,c}(y)]$$

$$+ \frac{a'^2}{\bar{a}m}(\varepsilon \rho(t))[F^{II}_{2,c}(y) + \frac{a^2(t)}{c(t)}F^{IV}_{2,c}(y)] - \frac{f_3(t)}{\bar{a}(\varepsilon \rho(t))}Q_c(y),$$

where $F^{I}_{2,c}(y) = c^{\frac{m-1}{2}} F^I_{2,c}(\sqrt{c}y)$, and with

$$F^I_2(t) := \frac{1}{2} y^2 Q^{m}(y), \quad F^I_{2,c}(y) := -B_1(y),$$

(4.33)
Now we solve the last linear system. From Proposition (4.37) and (2.14), one has
\begin{equation}
\begin{aligned}
F_2^{IV}(y) := & \frac{1}{2} (m-1) Q^{m-2}(y) B_2^I(y) - \frac{2}{5m} y B'_2(y) - \frac{m - 8}{5m} B_1(y), \\
& - \frac{8}{(m + 3)} B_1(y);
\end{aligned}
\end{equation}
(4.35)

Note that each term above is even and thus orthogonal to $Q'$. On the other hand,
\begin{equation}
\begin{aligned}
\dot{G}_2(t) := & v(t) \left[ \frac{a''}{a_m} (\varepsilon \rho(t)) G_{2,c}^I(y) + \frac{a'^2}{a_m} (\varepsilon \rho(t)) G_{2,c}^{II}(y) \right] - \frac{f_1(t)}{\alpha(\varepsilon \rho(t))} Q'_c(t);
\end{aligned}
\end{equation}
with $G_{2,c}^{(1)}(y) = c^{m-r} \dot{G}_2^I(\sqrt{cy})$ and $G_{2}^I(y) := A_1(y)$,
\begin{equation}
G_{2}^{II}(y) := \frac{m - 6}{5m} A_1(y) + (Q^{m-1} - \frac{4}{m + 3}) y B_1(y) + \frac{2}{5m} y A'_1(y) + (m - 1) Q^{m-2} A_1 B_1(y).
\end{equation}
(4.36)

Now (Ω2) is replaced by the linear system $(\dot{\Omega}_2)$:
\begin{equation}
(\dot{\Omega}_2) \quad \mathcal{L}_+ A_{2,c} = \dot{F}_2, \quad \mathcal{L}_- B_{2,c} = \dot{G}_2.
\end{equation}

Lemma 3.6 in [30] ensures that there exist unique solution to $(\dot{\Omega}_2)$ satisfying $A_{2,c}$ even and $B_{2,c}$ odd, the estimates (2.9), and the following decomposition:
\begin{equation}
A_{2,c}(t, y) = \frac{a''}{a_m} (A_{2,c}^I(y) + \frac{v^2}{c} A_{2,c}^{II}(y)) + \frac{a'^2}{a_m} (A_{2,c}^{III}(y) + \frac{v^2}{c} A_{2,c}^{IV}(y)) + \frac{f_3}{a} \Lambda Q_c,
\end{equation}
with $A_{2,c}^{(1)}(y) = c^{m-r} A_{2,c}^{(1)}(\sqrt{cy})$, $A_{2,c}^{(2)}$ even, and
\begin{equation}
B_{2,c}(t, y) = \frac{a''}{a_m} B_{2,c}^I(y) + \frac{a'^2}{a_m} B_{2,c}^{II}(y) + \frac{f_4}{2a} y Q_c,
\end{equation}
with $B_{2,c}^{(1)}(y) = c^{m-r} \frac{1}{2} B_{2,c}^{(1)}(\sqrt{cy})$ and $B_{2,c}^{(2)}$ odd. Finally, $A_{2,c}$ and $B_{2,c}$ satisfy the orthogonality conditions
\begin{equation}
\int_{\mathbb{R}} A_{2,c} Q_c = \int_{\mathbb{R}} B_{2,c} Q_c' = \int_{\mathbb{R}} B_{2,c} Q_c' = 0,
\end{equation}
(4.39)
provided $f_3, f_4$ are of the form (2.14)-(2.15), with
\begin{equation}
\alpha(\cdot) := \frac{1}{20 M Q} \int_{\mathbb{R}} A Q F_2^{(1)}, \quad \beta(\cdot) := - \frac{1}{20 M Q} \int_{\mathbb{R}} y Q G_2^{(1)}.
\end{equation}
(4.40)

Later, in Lemma 5.2 we will give explicit expressions for these parameters.

**Step 7. Third order linear system.** Now we solve the last linear system. From Proposition 2.22 more precisely 4.19, we seek for a solution of the following system,
\begin{equation}
(\Omega_3) \quad \mathcal{L}_+ A_{3,c} = F_3, \quad \mathcal{L}_- B_{3,c} = G_3,
\end{equation}
(4.41)
where $F_3$ and $G_3$ were defined in (4.23)-(4.24). As in the previous linear system, $F_3$ and $G_3$ contain terms with time derivatives, that we proceed to simplify. Indeed, from the decomposition (4.37) and (2.14), one has
\begin{equation}
A_{2,c} = \frac{a''}{a_m} \left[ (A_{2,c}^I + \alpha_1 \Lambda Q_c) + \frac{v^2}{c} (A_{2,c}^{II} + \alpha_2 \Lambda Q_c) \right] + \frac{a'^2}{a_m} \left[ (A_{2,c}^{III} + \alpha_3 \Lambda Q_c) + \frac{v^2}{c} (A_{2,c}^{IV} + \alpha_4 \Lambda Q_c) \right],
\end{equation}
(4.42)

Therefore, since $\rho_1 = \rho - \varepsilon f_4$ and $v' = \varepsilon f_1 + \varepsilon^3 f_5$,
\begin{equation}
\begin{aligned}
\frac{1}{\varepsilon} (A_{2,c})_t &= \left( \frac{a''}{a_m} \right) v \left[ (A_{2,c}^I + \alpha_1 \Lambda Q_c) + \frac{v^2}{c} (A_{2,c}^{II} + \alpha_2 \Lambda Q_c) \right] \\
&+ \left( \frac{a'^2}{a_m} \right) v \left[ (A_{2,c}^{III} + \alpha_3 \Lambda Q_c) + \frac{v^2}{c} (A_{2,c}^{IV} + \alpha_4 \Lambda Q_c) \right] \\
&+ \frac{a''}{a_m} f_1 (A_{2,c}^I + \alpha_2 \Lambda Q_c) + \frac{a'^2}{a_m} f_1 (A_{2,c}^{II} + \alpha_2 \Lambda Q_c) \\
&+ O ((\rho_1 | + \varepsilon^{-1}) | v' - \varepsilon f_1 - \varepsilon^3 f_5 | + \varepsilon^2 f_1) e^{-\varepsilon |\rho(t)|}).
\end{aligned}
\end{equation}
(4.43)
The error term in the last row above can be neglected either by putting it on the dynamical system \( \mathcal{F}_0 \), or in the error term \( \tilde{S}[\tilde{u}](t) \). Similarly, from (4.38) and (2.15),

\[
B_{2,c} = \frac{a''}{a} v(B_{2,c}^I + \frac{\beta_1}{2c} y Q_c) + \frac{a''}{a^{2m-1}} v(B_{2,c}^{II} + \frac{\beta_2}{2c} y Q_c),
\]

and then

\[
\frac{1}{\varepsilon}(B_{2,c})_t = \left( \frac{a''}{a} v^2(B_{2,c}^I + \frac{\beta_1}{2c} y Q_c) + \left( \frac{a''}{a^{2m-1}} \right) v^2(B_{2,c}^{II} + \frac{\beta_2}{2c} y Q_c) \right)
\]

\[
+ \frac{a''}{a} f_1(B_{2,c}^I + \frac{\beta_1}{2c} y Q_c) + \frac{a''}{a^{2m-1}} f_1(B_{2,c}^{II} + \frac{\beta_2}{2c} y Q_c)
\]

\[
+ O((|\rho_t| + \varepsilon^{-1})|v' - \varepsilon f_1 - \varepsilon^3 f_3 | + \varepsilon^2 |f_3| e^{-\mu |\rho(t)|})).
\]

(4.45)

**Remark 4.2.** Note that, from \((\Omega_2), \ (4.42), \ (4.44)\) and \((4.39)-(4.36)\), we can conclude that

\[
\begin{cases}
\mathcal{L}_+ A^{I}_{1} = F_{1}^{I} - \alpha_1 Q, \quad \mathcal{L}_+ A^{II}_{1} = F_{1}^{II} - \alpha_2 Q, \quad \mathcal{L}_+ A^{III}_{1} = F_{1}^{III} - \alpha_3 Q, \\
\mathcal{L}_+ A^{IV}_{1} = F_{1}^{IV} - \alpha_4 Q, \quad \mathcal{L}_- B^{I}_{1} = G_{1}^{I} - \beta_1 Q', \quad \mathcal{L}_- B^{II}_{1} = G_{1}^{II} - \beta_2 Q'.
\end{cases}
\]

(4.46)

Now we replace \( \breve{A}_{2,c} \) and \( \breve{B}_{2,c} \) in \((4.23)\) and \((4.24)\). A simple remark that will be useful later is the fact that \((4.42), \ (4.44)\) and \((4.39)\) imply that

\[
\begin{align*}
\int_R (A^{I}_{2,c} + \alpha_1 \Lambda Q_c)Q_c &= \int_R (A^{II}_{2,c} + \alpha_2 \Lambda Q_c)Q_c = 0, \\
\int_R (A^{III}_{2,c} + \alpha_3 \Lambda Q_c)Q_c &= \int_R (A^{IV}_{2,c} + \alpha_4 \Lambda Q_c)Q_c = 0, \quad \text{and} \\
\int_R (B^{I}_{2,c} + \frac{\beta_1}{2c} y Q_c)Q_c &= \int_R (B^{II}_{2,c} + \frac{\beta_2}{2c} y Q_c)Q_c = 0.
\end{align*}
\]

(4.47) (4.48) (4.49)

Let us come back to our problem. We get then new, simplified terms \( \breve{F}_3 \) and \( \breve{G}_3 \). Note that \( \breve{F}_3 \) and \( \breve{G}_3 \) are odd and even functions in the \( y \) variable, respectively. In what follows, we consider the modified linear system

\[
(\breve{\Omega}_3) \quad \mathcal{L}_+(A_{3,c}) = \breve{F}_3, \quad \mathcal{L}_-(B_{3,c}) = \breve{G}_3.
\]

According to Lemma 4.11, this system has unique solutions provided the two orthogonality conditions

\[
\int_R \breve{F}_3 Q_c = \int R \breve{G}_3 Q_c = 0,
\]

(4.50) are satisfied, for all \( t \in [-T_z, T_z] \). In particular, we claim the following

**Lemma 4.2.** There are unique parameters \( \delta_j, \nu_j \in \mathbb{R} \) such that, for \( f_5(t) \) and \( f_6(t) \) given in (2.10)- (2.17), the orthogonality conditions (4.50) are satisfied.

The proof of this result is a long, tedious but straightforward computation, that we carry out in Section 5. Note finally that we can choose \( A_{3,c} \) odd and \( B_{3,c} \) even, and moreover, they satisfy (2.4).

**Step 8. Final conclusion.** Having solved three linear systems in the decomposition (4.18), the error term is now given by the quantity

\[
S[\tilde{u}](t, x) = \mathcal{F}_0(t, y) e^{i\theta} + \tilde{S}[\tilde{u}](t, x),
\]

with \( \mathcal{F}_0 \) in (2.12), and \( \tilde{S}[\tilde{u}](t) = \varepsilon^4 [\mathcal{F}_4(t, y) + \varepsilon f(t) \mathcal{F}_c(y)] e^{i\theta} \). Moreover, from (4.25), we have, for some constants \( K, \mu > 0 \),

\[
\| \tilde{S}[\tilde{u}](t) \|_{H^1(\mathbb{R})} \leq K \varepsilon^4 (\varepsilon + \varepsilon^{-\mu |s(t)|}), \quad t \in [-T_z, T_z],
\]

(4.51) as required in (2.18). The proof is complete, provided Lemma 4.2 is satisfied. The next section is devoted to the proof of this result. □
5. PROOF OF LEMMA 4.2

Step 1. Recall that the soliton is given by \( Q_c(y) = c^{1/(m-1)}Q(\sqrt{c}y) \). We prove the left hand identity in (4.30). From (4.23), and (4.35), we get

\[
\int R F^\prime \tilde{Q}_c - \frac{f_5}{2a} \int y^3 Q_c Q''_c + \frac{ma''}{2a} \int y^2 Q_c^{m-1} Q'_c A_1,c - f_3 \int A_1,c Q'_c \\
+ f_4 \int (B_{1,c}) y Q'_c + \frac{ma''}{a} \int Q'_c Q'_c y A_{2,c} - \frac{1}{2} f_1 \int y Q'_c A_{2,c} - \int Q'_c \tilde{B}_{2,c} \\
- f_2 \int Q'_c A B_{2,c} + (m-1) \tilde{a} \int Q'_c Q_c^{m-2}(m A_{1,c} A_{2,c} + B_{1,c} B_{2,c}) \\
+ \frac{1}{2} (m-1) \frac{a'}{a^{m-2}} \int y Q'_c - \int Q'_c Q_c^{m-3}(m(m-2)A_{1,c}^3 + 3(m-2)A_{1,c} B_{1,c}) = \sum_{j=1}^{12} I_j.
\]

First of all, note that from (4.31) one has \( I_4 \equiv 0 \). Let us note that \( \theta = \frac{1}{m-1} - \frac{1}{4} \). We have

\[
I_1 = -\frac{a(3)}{2(m+1) \tilde{a}^m} \int y^2 Q_c^{m+1} = -\frac{a(3) c^{2\theta}}{2(m+1) \tilde{a}^m} \int y^2 Q_c^{m+1}.
\]

On the other hand, from (4.28) and (4.29),

\[
I_2 = \frac{f_5}{4a} \int R^2 = \frac{f_5 c^{2\theta}}{4a} \int R^2, \quad I_3 = \frac{ma'' a' c^{2\theta}}{2a^{2m-1}} \int y^2 Q_c^{m-1} Q'_c A_1.
\]

From the identity \( Q''_c = c Q_c - Q_c^{m-1} \), (4.31), (4.28), (4.29) and (2.15),

\[
I_5 = f_4 \int R B_{1,c} Q_c^{m} = c^{2\theta} \left( \frac{v^2}{c} + \frac{\beta_1 a'' a''}{a^{2m-1}} + \frac{\beta_2 a'' a}{a^{3m-2}} \right) \int B_1 Q_c^{m}.
\]

From (4.24)

\[
I_6 = \frac{ma' a'' c^{2\theta}}{a^{2m-1}} \int R y Q_c^{m-1} Q'_c [(A_2 + c A_1) + \frac{v^2}{c} (A_2 + c A_1) ] \\
+ \frac{ma'' a' c^{2\theta}}{a^{3m-2}} \int R y Q_c^{m-1} Q'_c [(A_2 + c A_1) + \frac{v^2}{c} (A_2 + c A_1) ]
\]

Using (2.1) and (4.32),

\[
I_7 = -\frac{4a' a'' c^{2\theta}}{(m+3) a^{2m-1}} \int R y Q'_c [(A_2 + c A_1) + \frac{v^2}{c} (A_2 + c A_1) ] \\
- \frac{4a'' a c^{2\theta}}{(m+3) a^{3m-2}} \int R y Q'_c [(A_2 + c A_1) + \frac{v^2}{c} (A_2 + c A_1) ]
\]

Now, from (4.35) and (4.49),

\[
I_8 = -\int R B_{2,c} Q'_c = -c^{2\theta} \left( \frac{a''}{a} \right)^2 \cdot \int R Q'_c (B_2 + \frac{\beta_1}{2} y Q) + c^{2\theta} \left( \frac{a''}{a} \right)^2 \cdot \int R Q'_c (B_2 + \frac{\beta_2}{2} y Q) \\
+ \frac{8a' a'' c^{2\theta}}{(m+3) a^{2m-1}} \int R Q'_c (B_2 + \frac{\beta_1}{2} y Q) + \frac{8a'' a c^{2\theta}}{(m+3) a^{3m-2}} \int R Q'_c (B_2 + \frac{\beta_2}{2} y Q)
\]

\( = 0. \)
Using (4.39), (2.13) and (4.44),

\[ I_9 = -f_2\partial_c \int_{\mathbb{R}} Q'_c B_{2,c} + f_2 \int_{\mathbb{R}} B_{2,c} AQ'_c \]
\[ = e^{2\theta} v^2 \left[ \frac{4a''}{c} \int_{\mathbb{R}} \Lambda Q'(B_2' + \frac{\beta_1}{2} yQ) + \frac{4a'^3}{(5-m)\alpha^{2m-2}} \int_{\mathbb{R}} \Lambda Q'(B_2'' + \frac{\beta_2}{2} yQ) \right] \]
\[ = e^{2\theta} v^2 \left[ \frac{2a''}{c} \int_{\mathbb{R}} yQ''(B_2' + \frac{\beta_1}{2} yQ) + \frac{2a'^3}{(5-m)\alpha^{2m-2}} \int_{\mathbb{R}} yQ''(B_2'' + \frac{\beta_2}{2} yQ) \right]. \]

From (4.28), (4.42) and (4.44),

\[ I_{10} = m(m-1)\partial_a \int_{\mathbb{R}} Q_{c}^{m-2} Q'_c A_{1,c} A_{2,c} + (m-1)\partial_a \int_{\mathbb{R}} Q_{c}^{m-2} Q'_c B_{1,c} B_{2,c} \]
\[ = m(m-1)\frac{a''a''^c 2^\theta}{\alpha^{2m-2}} \int_{\mathbb{R}} Q^{m-2} Q'A_1 [(A_1^I + \alpha_1 \Lambda Q) + \frac{v^2}{c}(A_1^{II} + \alpha_2 \Lambda Q)] \]
\[ + m(m-1)\frac{a'^3 2^\theta}{\alpha^{3m-2}} \int_{\mathbb{R}} Q^{m-2} Q'A_2 [(A_2^{II} + \alpha_3 \Lambda Q) + \frac{v^2}{c}(A_1^{IV} + \alpha_4 \Lambda Q)] \]
\[ + (m-1)\frac{a''a''^c 2^\theta}{\alpha^{2m-2}} \int_{\mathbb{R}} Q^{m-2} Q'B_1 (B_2' + \frac{1}{2} \beta_1 yQ) \]
\[ + (m-1)\frac{a'^3 2^\theta}{\alpha^{3m-2}} \int_{\mathbb{R}} Q^{m-2} Q'B_2 (B_2'' + \frac{1}{2} \beta_2 yQ) \].

From (4.28),

\[ I_{11} = \frac{(m-1)a'^3 2^\theta}{\alpha^{3m-2}} \int_{\mathbb{R}} yQ^{m-2} Q'(mA_1^I + \frac{v^2}{c} B_1^2) \]
\[ I_{12} = \frac{(m-1)a'^3 2^\theta}{6\alpha^{3m-2}} \int_{\mathbb{R}} Q^{m-3} Q'[m(m-2)A_1^I + 3(m-2)\frac{v^2}{c} A_1 B_1^2] \]

Collecting all the previous computations, we get

\[ \int_{\mathbb{R}} \tilde{f}_3 Q'_c = \frac{2^\theta}{2a} M[Q] \left[ f_5 - \delta_1 \frac{a'(3)}{a} - (\delta_2 + \delta_3 \frac{v^2}{a^2} \frac{a''}{a^2} - (\delta_4 + \delta_5 \frac{v^2}{a^2} \frac{a'^3}{a^3} \right] = 0, \]

provided the parameters \( \delta_j \) are defined as follows:

\[ \delta_1 := \frac{1}{(m+1)M[Q]} \int_{\mathbb{R}} y^2 Q^{m+1} > 0; \] (5.1)

\[ \delta_2 := -\frac{2}{M[Q]} \left[ \frac{m}{2} \int_{\mathbb{R}} y^2 Q^{m-1} Q'A_1 + m \int_{\mathbb{R}} yQ^{m-1} Q'(A_1^I + \alpha_1 \Lambda Q) - \frac{4}{(m+3)} \int_{\mathbb{R}} yQ'(A_1^{II} + \alpha_1 \Lambda Q) \right] \]
\[ + m(m-1) \int_{\mathbb{R}} Q^{m-2} Q'A_1 (A_1^{III} + \alpha_2 \Lambda Q) \]; (5.2)

\[ \delta_3 := -\frac{2}{M[Q]} \left[ \beta_1 \int_{\mathbb{R}} B_1 Q^{m-1} + m \int_{\mathbb{R}} yQ^{m-1} Q'(A_1^{II} + \alpha_2 \Lambda Q) - \frac{4}{(m+3)} \int_{\mathbb{R}} yQ'(A_1^{III} + \alpha_2 \Lambda Q) \right] \]
\[ + \frac{2}{(5-m)} \int_{\mathbb{R}} yQ''(B_2' + \frac{\beta_1}{2} yQ) + m(m-1) \int_{\mathbb{R}} Q^{m-2} Q'A_2 (A_2^{III} + \alpha_3 \Lambda Q) \]
\[ + (m-1) \int_{\mathbb{R}} Q^{m-2} Q'B_1 (B_2' + \frac{1}{2} \beta_1 yQ) \]; (5.3)
\[
\delta_4 := -\frac{2}{M(Q)} \left[ m \int_R yQ^{m-1}Q'(A_2^{III} + \alpha_3 \Lambda Q) - \frac{4}{(m+3)} \int_R yQ'(A_2^{IV} + \alpha_3 \Lambda Q) \right. \\
\left. + m(m-1) \int_R Q^{-m-2}Q'A_1(A_2^{III} + \alpha_3 \Lambda Q) + m(m-1) \int_R yQ^{-m-2}Q'A_1 \right] \\
+ \frac{1}{6} m(m-1)(m-2) \int_R Q^{-m-3}Q' \Lambda A_1^3; \quad (5.4)
\]

and

\[
\delta_5 := -\frac{2}{M(Q)} \left[ \beta_2 \int_R B_1 Q^m + m \int_R yQ^{m-1}Q'(A_2^{IV} + \alpha_4 \Lambda Q) - \frac{4}{(m+3)} \int_R yQ'(A_2^{IV} + \alpha_4 \Lambda Q) \right. \\
\left. + \frac{2}{(5-m)} \int_R yQ'Q'_{B_2} + \frac{\beta_2}{2} yQ) + m(m-1) \int_R Q^{-m-2}Q'A_1(A_2^{IV} + \alpha_4 \Lambda Q) \right] \\
+ (m-1) \int_R Q^{-m-2}Q'B_1B_1 + \frac{1}{2} \beta_2 yQ) + (m-1) \int_R yQ^{-m-2}Q'B_1^2 \\
+ \frac{1}{2} m(m-1)(m-2) \int_R Q^{-m-3}Q'\Lambda A_1 B_1^2. \quad (5.5)
\]

**Step 2.** Now we prove the second identity in (4.50). From (4.24) and (4.18), we get

\[
\int_R \hat{g}_3 Q_c = \frac{f_5}{a} \int_R Q_c \Lambda Q_c + f_4 \frac{\tilde{a}}{a} \int_R \tilde{Q}_c^2 + \frac{a''}{a} \int_R y^2 Q_c B_{1,c} - f_3 \int_R B_{1,c} Q_c - f_4 \int_R Q_c (A_{1,c})_y \\
+ \int_R \hat{A}_{2,c} Q_c + f_2 \int_R Q_c \Lambda A_{2,c} + \frac{a'}{a} \int_R y^2 Q_c^2 B_{2,c} - \frac{1}{2} f_1 \int_R y Q_c B_{2,c} \\
+ (m-1) \intR Q^{-m-1}(A_{1,c} B_{2,c} + A_{2,c} B_{1,c}) + (m-1) \frac{a'}{a} \intR y Q^{-m-1} A_{1,c} B_{1,c} \\
+ \frac{1}{2} (m-1) \intR Q^{-m-2}(m-2) A_{1,c} B_{1,c} + B_{1,c} = \sum_{i=1}^{12} J_i.
\]

It is easy to see that from (4.31), one has \( J_4 \equiv J_5 \equiv 0. \) On the other hand,

\[
J_1 + J_2 = \frac{f_5}{a} \partial_c \intR \tilde{Q}_c^2 + \frac{f_4 a^{-1} a' c^{20}}{a} \intR \tilde{Q}_c^2 + \frac{f_5}{c} \intR \tilde{Q}_c^2 + \frac{4 f_4 a'}{(5-m) a}.
\]

From (4.28),

\[
J_3 = \frac{a''}{a} \partial_c \intR y^2 Q^m B_1.
\]

Similarly to the proof that \( J_8 = 0 \), one has from (4.47)-(4.48),

\[
J_6 = c^{20-1} v \intR Q \left[ \frac{a''}{a} \right] \left[ (A_2^4 + \alpha_1 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_2 \Lambda Q) \right] \\
+ \left( \frac{a^2}{a^{2m-1}} \right) \left[ A_2^{II} + \alpha_3 \Lambda Q \right] + \left( \frac{v^2}{c} \right) \left( A_2^{IV} + \alpha_4 \Lambda Q \right) \\
+ \left( \frac{16 a^3}{(m+3) a^{3m-2}} \right) \left[ A_2^{IV} + \alpha_4 \Lambda Q \right] = 0.
\]

Following the same argument as in the computation of \( I_9 \),

\[
J_7 = -f_2 \intR \Lambda Q_c A_{2,c} = \frac{4 c^{20-1} v}{(5-m)} \intR \frac{a''}{a} \partial_{2m-1} \Lambda Q (A_2^4 + \alpha_1 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_2 \Lambda Q)) \\
+ \frac{a^3}{a^{3m-2}} \intR \Lambda Q (A_2^{III} + \alpha_3 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_4 \Lambda Q)) \\
= -\frac{2 c^{20-1} v}{(5-m)} \intR \frac{a''}{a} \partial_{2m-1} \Lambda Q (A_2^4 + \alpha_1 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_2 \Lambda Q)) \\
+ \frac{a^3}{a^{3m-2}} \intR y Q' (A_2^{III} + \alpha_3 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_4 \Lambda Q)).
\]
Replacing (4.44), we get
\[ J_8 = \frac{a^3 c^{2\theta - 1} v}{a^{2m - 1}} \int_{\mathbb{R}} yQ^m(B_2^I + \frac{1}{2} \beta_1 yQ) + \frac{a^3 c^{2\theta} v}{a^{3m - 2}} \int_{\mathbb{R}} yQ^m(B_2^{II} + \frac{1}{2} \beta_2 yQ), \]
and using (2.13),
\[ J_9 = -\frac{4c^{2\theta - 1} v}{(m + 3)} \left[ \frac{a^3 c^{2\theta - 1} v}{a^{2m - 1}} \int_{\mathbb{R}} yQ(B_2^I + \frac{1}{2} \beta_1 yQ) + \frac{a^3 c^{2\theta} v}{a^{3m - 2}} \int_{\mathbb{R}} yQ(B_2^{II} + \frac{1}{2} \beta_2 yQ) \right]. \]
From (4.28), (4.42) and (4.44),
\[ J_{10} = (m - 1) \tilde{a} \int_{\mathbb{R}} Q_c^{m-1} B_{1,c} A_{2,c} + (m - 1) \tilde{a} \int_{\mathbb{R}} Q_c^{m-1} A_{1,c} B_{2,c} \]
\[ = (m - 1) \frac{a^3 c^{2\theta - 1} v}{a^{2m - 1}} \int_{\mathbb{R}} Q^m B_1 \left[ (A_2^I + \alpha_1 \Lambda Q) + \frac{v^2}{c} (A_2^{II} + \alpha_2 \Lambda Q) \right] \]
\[ + (m - 1) \frac{a^3 c^{2\theta} v}{a^{3m - 2}} \int_{\mathbb{R}} Q^m B_1 \left[ (A_2^{III} + \alpha_3 \Lambda Q) + \frac{v^2}{c} (A_2^{IV} + \alpha_4 \Lambda Q) \right] \]
\[ + (m - 1) \left[ \frac{a^3 c^{2\theta - 1} v}{a^{2m - 1}} \int_{\mathbb{R}} Q^m A_1 (B_2^I + \frac{1}{2} \beta_1 yQ) + \frac{a^3 c^{2\theta} v}{a^{3m - 2}} \int_{\mathbb{R}} Q^m A_1 (B_2^{II} + \frac{1}{2} \beta_2 yQ) \right]. \]
Finally, from (4.28) and scaling properties,
\[ J_{11} = (m - 1) \frac{a^3 c^{2\theta - 1} v}{a^{3m - 2}} \int_{\mathbb{R}} yQ^{m-1} A_1 B_1. \]
Similarly,
\[ J_{12} = \frac{1}{2} (m - 1) \frac{a^3 c^{2\theta - 1} v}{a^{3m - 2}} \int_{\mathbb{R}} Q^{m-2} [(m - 2) A_1^2 + \frac{v^2}{c} B_1^2] B_1. \]
Collecting the above estimates, we get
\[ \int_{\mathbb{R}} C_3 Q_c = \frac{2c^{2\theta}}{a} M[Q] \left[ \frac{f_0}{c} + \frac{4f_1 a'}{(5 - m)a} - \frac{v^2}{c} \left( \eta_1 + \eta_2 \frac{v^2}{c} a^2 - \frac{v^2}{c} \eta_3 + \frac{v^2}{c} \frac{a^3}{a^3} \right) \right] = 0, \]
provided the parameters \( \eta_j \) are defined as follows:
\[ \eta_1 := -\frac{1}{2M[Q]} \left[ \frac{1}{2} \int_{\mathbb{R}} y^2 Q^m B_1 - \frac{2}{(5 - m)} \int_{\mathbb{R}} yQ'(A_2^I + \alpha_1 \Lambda Q) + \int_{\mathbb{R}} yQ^m(B_2^I + \frac{1}{2} \beta_1 yQ) \right. \]
\[ - \frac{1}{(m + 3)} \int_{\mathbb{R}} yQ(B_2^I + \frac{1}{2} \beta_1 yQ) + (m - 1) \int_{\mathbb{R}} Q^m B_1 (A_2^I + \alpha_1 \Lambda Q) \]
\[ + (m - 1) \int_{\mathbb{R}} Q^m A_1 (B_2^I + \frac{1}{2} \beta_1 yQ); \]  \hfill (5.6)
\[ \eta_2 := -\frac{1}{2M[Q]} \left[ - \frac{2}{(5 - m)} \int_{\mathbb{R}} yQ'(A_2^{II} + \alpha_2 \Lambda Q) + (m - 1) \int_{\mathbb{R}} Q^m B_1 (A_2^{II} + \alpha_2 \Lambda Q) \right]; \]  \hfill (5.7)
\[ \eta_3 := -\frac{1}{2M[Q]} \left[ \frac{1}{(5 - m)} \int_{\mathbb{R}} yQ'(A_2^{III} + \alpha_3 \Lambda Q) + \int_{\mathbb{R}} yQ^m(B_2^{II} + \frac{1}{2} \beta_2 yQ) \right. \]
\[ - \frac{1}{(m + 3)} \int_{\mathbb{R}} yQ(B_2^{II} + \frac{1}{2} \beta_2 yQ) + (m - 1) \int_{\mathbb{R}} Q^m B_1 (A_2^{III} + \alpha_3 \Lambda Q) \]
\[ + (m - 1) \int_{\mathbb{R}} Q^m A_1 (B_2^{II} + \frac{1}{2} \beta_2 yQ) + (m - 1) \int_{\mathbb{R}} yQ^m A_1 B_1 \]
\[ + \frac{1}{2} (m - 1)(m - 2) \int_{\mathbb{R}} Q^{m-2} A_1^2 B_1; \]  \hfill (5.8)
and
\[ \eta_i := -\frac{1}{2\eta M[Q]} \left[ -\frac{2}{(5-m)} \int_{\mathbb{R}} yQ'(A_2^{IV} + \alpha_4 \Lambda Q) + (m-1) \int_{\mathbb{R}} Q^{m-1}B_1(A_2^{IV} + \alpha_4 \Lambda Q) \right] + \frac{1}{2}(m-1) \int_{\mathbb{R}} Q^{m-2}B_1 \].

### Step 3. Auxiliary functions.
We want to simplify the expressions for \(\delta_j\) and \(\eta_j\). Let \(\hat{Y}_j, \hat{Z}_j \in \mathcal{S}(\mathbb{R})\), \(j = 1, 2\), be the following functions:
\[
\begin{cases}
\hat{Y}_1 := myQ^{m-1}Q' - \frac{4}{m+3}yQ' + m(m-1)Q^{m-2}Q'A_1, \\
\hat{Y}_2 := -\frac{2}{5-m}yQ' + (m-1)Q^{m-1}B_1;
\end{cases}
\]
and
\[
\begin{cases}
\hat{Z}_1 := \frac{2}{5-m}yQ'' + (m-1)Q^{m-2}Q'B_1, \\
\hat{Z}_2 := yQ^m - \frac{4}{m+3}yQ + (m-1)Q^{m-1}A_1.
\end{cases}
\]

Notice that \(\hat{Y}_j\) is even and \(\hat{Z}_j\) is odd.

**Lemma 5.1 (Inverse functions).** There are unique even functions \(Y_j \in \mathcal{S}(\mathbb{R})\) and odd functions \(Z_j \in \mathcal{S}(\mathbb{R})\) such that
\[ \mathcal{L}_+Y_j = \hat{Y}_j, \quad \text{and} \quad \mathcal{L}_-Z_j = \hat{Z}_j, \quad j = 1, 2; \]
and
\[ \int_{\mathbb{R}} Y_jQ' = \int_{\mathbb{R}} Z_jQ = 0. \]

Moreover, one has,
\[ Y_1 = A_1' + \frac{1}{m-1}Q - \frac{4}{m+3}\Lambda Q; \quad Y_2 = -B_1 - \frac{1}{5-m}\Lambda Q; \]
and
\[ Z_1 = -\frac{1}{2(5-m)}(y^2Q' - yQ + \chi Q') = B_1' + \frac{3yQ}{2(5-m)}, \quad Z_2 = A_1, \]
as well-defined Schwartz functions with the corresponding parity properties.

**Proof.** The existence and uniqueness of such functions are consequence of Lemma [5.12]. Let us prove [5.12] and [5.13]. By simple inspection, \(\mathcal{L}_+Q = 0\), and \(\mathcal{L}_-Q = 0\). By simple inspection, \(\mathcal{L}_-Q = 0\), and \(\mathcal{L}_-(yQ) = -2Q'\).

On the other hand, since
\[ \mathcal{L}_+A_1 = yQ(Q^{m-1} - \frac{4}{m+3}) \quad \text{and} \quad \mathcal{L}_-B_1 = \frac{4}{5-m}\Lambda Q - \frac{1}{m-1}Q, \]
(c.f. [5.12]), taking derivative in both equations we get
\[ \mathcal{L}_+A_1' = (yQ)'(Q^{m-1} - \frac{4}{m+3}) + yQ(Q^{m-1})' + m(Q^{m-1})'A_1 = Q^m - \frac{4}{m+3}Q + myQ^{m-1}Q' - \frac{4}{m+3}yQ' + m(m-1)Q^{m-2}Q'A_1. \]
and
\[ \mathcal{L}_-B_1' = \frac{4}{5-m}\Lambda Q' - \frac{1}{m-1}Q' + (Q^{m-1})'B_1 = \frac{2}{5-m}yQ'' + \left[ \frac{4}{(m-1)(5-m)} + \frac{2}{5-m} - \frac{1}{m-1} \right]Q' + (m-1)Q^{m-2}Q'B_1 \]
and
\[ \frac{2}{5-m}yQ'' + \frac{3}{5-m}Q' + (m-1)Q^{m-2}Q'B_1. \]
Now we prove the first assertion in (5.12). From (5.14) and (5.16),
\[
\mathcal{L}_+ Y_1 = \mathcal{L}_+ A_1' + \frac{1}{m-1} \mathcal{L}_+ Q - \frac{4}{m+3} \mathcal{L}_+ \Lambda Q
\]
\[
= Q'' + \frac{4}{m+3} Q + m Q^{m-1} Q' - \frac{4}{m+3} y Q' + m(m-1) Q^{m-2} Q' A_1 - Q'' + \frac{4}{m+3} Q
\]
\[
= m Q^{m-1} Q' - \frac{4}{m+3} y Q' + m(m-1) Q^{m-2} Q' A_1 = \tilde{Y}_1.
\]
Similarly, from (5.15)
\[
\mathcal{L}_+ Y_2 = -\mathcal{L}_+ B_1 - \frac{1}{5-m} \mathcal{L}_+ \Lambda Q
\]
\[
= -\mathcal{L}_- B_1 + (m-1) Q^{m-1} B_1 + \frac{1}{5-m} Q
\]
\[
= -\frac{4}{5-m} \Lambda Q + \frac{1}{m-1} Q + (m-1) Q^{m-1} B_1 + \frac{1}{5-m} Q
\]
\[
= -\frac{2}{5-m} y Q' + (m-1) Q^{m-1} B_1 = \tilde{Y}_2.
\]

On the other hand, using (5.17),
\[
\mathcal{L}_- Z_1 = \mathcal{L}_- B_1' + \frac{3}{2(5-m)} \mathcal{L}_-(y Q)
\]
\[
= \frac{2}{5-m} y Q'' + \frac{3}{5-m} Q' + (m-1) Q^{m-2} Q' B_1 - \frac{3}{5-m} Q' = \tilde{Z}_1.
\]
Finally, for \( Z_2 = A_1 \),
\[
\mathcal{L}_- A_1 = \mathcal{L}_- A_1 + (m-1) Q^{m-1} A_1
\]
\[
= y Q^{(m-1) - \frac{4}{m+3}} (m-1) Q^{m-1} A_1 = \tilde{Z}_2.
\]

\( \square \)

**Step 4.** Now, we prove the following

**Lemma 5.2.** Let \( \alpha_j, \beta_j \in \mathbb{R} \) be the parameters defined in (4.40). Then
\[
\alpha_1 = \frac{1}{2(m+1)M(Q)} \int_{\mathbb{R}} y^2 Q^{m+1} > 0, \quad \alpha_2 = -\frac{1}{4M^2(Q)} \int_{\mathbb{R}} y Q' A_1, \quad (5.18)
\]
\[
\alpha_3 = \frac{1}{2M^2(Q)} \left[ \frac{m}{m-1} \int_{\mathbb{R}} y^2 Q^{m} A_1 - \frac{4}{(m-1)(m+3)} \int_{\mathbb{R}} y Q A_1 + \frac{m}{2} \int_{\mathbb{R}} y Q' A_1 \right]
\]
\[
- \frac{2}{m+3} \int_{\mathbb{R}} y^2 Q' A_1 + \frac{m}{2} \int_{\mathbb{R}} Q^{-1} A_1^2 + \frac{m}{4} (m-1) \int_{\mathbb{R}} Q^{m-2} Q' A_1^2 - \frac{4}{m+3} \int_{\mathbb{R}} y Q' B_1, \quad (5.19)
\]
and
\[
\alpha_4 = \frac{1}{2M^2(Q)} \left[ \frac{1}{2} \int_{\mathbb{R}} Q^{m-1} B_1^2 + \frac{1}{4} (m-1) \int_{\mathbb{R}} y Q^{m-2} Q' B_1^2 + \frac{13m-8-m^2}{2(5-m)(m-1)} \int_{\mathbb{R}} y Q' B_1 \right]
\]
\[
+ \frac{1}{5-m} \int_{\mathbb{R}} y^2 Q'' B_1 \]. \quad (5.20)

One the other hand,
\[
\beta_1 = -\frac{1}{M(Q)} \int_{\mathbb{R}} y Q A_1, \quad \beta_2 = \frac{1}{M(Q)} \left[ 2 \int_{\mathbb{R}} A_1 B_1' + \frac{5}{5-m} \int_{\mathbb{R}} y Q A_1 \right], \quad (5.21)
\]
for \( m \in [3, 5) \).
Step 5. Simplifications. Coming back to the definition of $\delta_j$ given in (5.11) - (5.13), we use now Lemmas 5.1 and 5.2 (4.31), and the equation $Q'' = Q - Q^m$ in several opportunities, in order to
obtain simplified expressions. Let us consider $\delta_2$ in (5.2). It is direct to check that

$$\delta_2 = -\frac{2}{M}[Q]\left[\frac{m}{2} \int y^2Q^{-1}Q'A_1 + \int \tilde{Y}_1(A_2^I + \alpha_1 A Q)\right].$$

Using Lemma 5.1 the self-adjointedness of $L_+$, and (4.40),

$$\delta_2 = -\frac{2}{M}[Q]\left[\frac{m}{2} \int y^2Q^{-1}Q'A_1 + \int Y_1 L_+(A_2^I + \alpha_1 A Q)\right]
= -\frac{2}{M}[Q]\left[\frac{m}{2} \int y^2Q^{-1}Q'A_1 + \int Y_1 (F_2^I - \alpha_1 Q)\right].$$

Now we replace $Y_1, F_2^I$ and $\alpha_1$ using (5.12), (4.33) and (5.18), we get

$$\delta_2 = -\frac{2}{M}[Q]\left[\frac{m}{2} \int y^2Q^{-1}Q'A_1 + \int A_1 F_2^I + \frac{1}{m-1} \int Q(F_2^I - \alpha_1 Q)\right]
= -\frac{2}{M}[Q]\left[- \int yQ^m A_1 + \frac{1}{m-1} \int Q^2 \frac{1}{2} y^2Q^{-1} - \alpha_1 Q\right]
= -\frac{2}{M}[Q]\left[- \int yQ^m A_1 + \frac{1}{2(m+1)} \int y^2Q^{-1}\right]. \quad (5.22)$$

On the other hand, from (5.3) and (5.10)-(5.11) one has

$$\delta_4 = -\frac{2}{M}[Q]\left[\beta_1 \int R_1 Q^m + \int \tilde{Y}_1(A_2^{II} + \alpha_2 A \lambda Q) + \int R_1 \tilde{Z}_1(B_2^I + \beta_1 yQ)\right].$$

Using Lemma 2.29 and (4.42) with $c = 1$, we get

$$\delta_4 = -\frac{2}{M}[Q]\left[\beta_1 \int R_1 Q^m + \int Y_1 L_+(A_2^{II} + \alpha_2 A \lambda Q) + \int Z_1 L_-(B_2^I + \beta_1 yQ)\right]
= -\frac{2}{M}[Q]\left[\beta_1 \int R_1 Q^m + \int Y_1 (F_2^{II} - \alpha_2 Q) + \int Z_1 (G_2^I - \beta_1 Q)\right].$$

Now we replace $Y_1$ and use (4.40) to cancel out the terms with $\lambda Q$ and $yQ$. Indeed,

$$\delta_3 = -\frac{2}{M}[Q]\left[\beta_1 \int R_1 Q^m + \int A_1 F_2^{II} + \frac{1}{m-1} \int Q(F_2^{II} - \alpha_2 Q) + \int B_1(G_2^I - \beta_1 Q)\right]
= -\frac{2}{M}[Q]\left[\int A_1 F_2^{II} + \frac{1}{m-1} \int Q(F_2^{II} - \alpha_2 Q) + \int B_1 G_2^I\right].$$

In the last identity we have used the identity $Q'' = Q - Q^m$ and (4.33) to cancel out the term with $\beta_1$. Finally, replacing $F_2^{II}$ and $\alpha_2$ (cf. (4.33) and (5.18)),

$$\delta_3 = -\frac{2}{M}[Q]\left[\int A_1 F_2^{II} - \alpha_2 \frac{1}{m-1} \int Q + \int B_1 G_2^I\right]
= -\frac{2}{M}[Q]\left[2 \int A_1 B_1^I - \frac{\alpha_2}{m-1} \int Q^2\right] = -\frac{2}{M}[Q] \left[2 \int A_1 B_1^I + \frac{2}{5-m} \int yQ^2 B_1\right]. \quad (5.23)$$

We repeat the same analysis with $\delta_4$:

$$\delta_4 = -\frac{2}{M}[Q]\left[\int Y_1(F_2^{II} - \alpha_3 Q) + m(m-1) \int yQ^{m-2}Q'A_2^I\right]
+ \frac{1}{6} m(m-1)(m-2) \int Q^{m-3}Q'A_2^I\right]
= -\frac{2}{M}[Q]\left[\int A_1 F_2^{II} + \frac{1}{m-1} \int Q(F_2^{II} - \alpha_3 Q) + m(m-1) \int yQ^{m-2}Q'A_2^I\right]
+ \frac{1}{6} m(m-1)(m-2) \int Q^{m-3}Q'A_2^I\right].$$
Now we replace $F_{2}^{I I}$ and $\alpha_3$ (cf. (4.34) and (5.19)). After some computations, we get

$$\delta_4 = -\frac{2}{M(Q)} \left[ \frac{2}{m+3} \int A^2_1 + \frac{8}{m+3} \int A_1 B'_1 + \frac{1}{m-1} \int R y A_1 (m Q^{-m} - \frac{4}{m+3} Q) - \frac{\alpha_3}{m-1} \int R Q^2 \right]$$

$$+ \frac{1}{2} m (m-1) \int R y Q^{m-2} Q' A^2_1 \right]$$

$$= -\frac{2}{M(Q)} \left[ \frac{2}{m+3} \int A^2_1 + \frac{8}{m+3} \int A_1 B'_1 - \frac{m}{5-m} \int R y Q^m A_1 + \frac{4}{(5-m)(m+3)} \int R y Q A_1 \right.$$  

$$- \frac{2m}{5-m} \int R y^2 Q' Q^{-m-1} A_1 + \frac{8}{(5-m)(m+3)} \int R y^2 Q' A_1 - \frac{2m}{5-m} \int R Q^{-m-1} A^2_1$$

$$- \frac{m(m-1)(m-3)}{2(5-m)} \int R y Q^{m-2} Q' A^2_1 + \frac{16}{(5-m)(m+3)} \int R y Q B'_1 \right].$$

Finally,

$$\delta_5 = -\frac{2}{M(Q)} \left[ \beta_2 \int R B_1 Q^m + \int R Y_1 \mathcal{L} + (A_{1}'V + \alpha_4 \Lambda Q) + \int R Z_1 \mathcal{L} - (B_{1}^{I I} + \frac{\beta_2}{2} y Q)$$

$$+ (m-1) \int R y Q^{m-2} Q' B^2_1 + \frac{1}{2} m (m-1)(m-2) \int R Q^{m-3} Q' A^2_1 B^2_1 \right]$$

$$= -\frac{2}{M(Q)} \left[ \beta_2 \int R B_1 Q^m + \int R Y_1 (F_2^{I V} - \alpha_4 Q) + \int R Z_1 (G_2^{I I} - \beta_2 Q')$$

$$+ (m-1) \int R y Q^{m-2} Q' B^2_1 + \frac{1}{2} m (m-1)(m-2) \int R Q^{m-3} Q' A^2_1 B^2_1 \right]$$

$$= -\frac{2}{M(Q)} \left[ \beta_2 \int R B_1 Q^m + \int R A_1 F_1^{I V} + \frac{1}{m-1} \int R Q(F_2^{I V} - \alpha_4 Q) + \int R B'_1 (G_2^{I I} - \beta_2 Q')$$

$$+ (m-1) \int R y Q^{m-2} Q' B^2_1 + \frac{1}{2} m (m-1)(m-2) \int R Q^{m-3} Q' A^2_1 B^2_1 \right]$$

$$= -\frac{2}{M(Q)} \left[ \frac{1}{2} m-1 \int R Q(F_2^{I V} - \alpha_4 Q) + \int R B'_1 (G_2^{I I} - \beta_2 Q')$$

$$+ (m-1) \int R y Q^{m-2} Q' B^2_1 + \frac{1}{2} m (m-1)(m-2) \int R Q^{m-3} Q' A^2_1 B^2_1 \right]$$

Now we replace $F_2^{I V}$ and $G_2^{I I}$, using (4.35) and (4.36) respectively, to obtain, after some simplifications,

$$\delta_5 = -\frac{2}{M(Q)} \left[ \frac{1}{2} m-1 \int R A_1 B'_1 + \frac{1}{2} m-3 \int R y Q^{m-2} Q' B^2_1 + \frac{2}{m+3} \int R B^2_1$$

$$+ \frac{2}{(m-1)(5-m)} \int R y Q B'_1 - \frac{\alpha_4}{m-1} \int R Q^2 \right].$$

The last step is to replace $\alpha_4$, using (5.20). The final result is

$$\delta_5 = -\frac{2}{M(Q)} \left[ \frac{1}{2} m-7 \int R A_1 B'_1 - \frac{m-1}{2} (m-3) \int R y Q^{m-2} Q' B^2_1 + \frac{2}{m+3} \int R B^2_1$$

$$- \frac{2}{5-m} \int R Q^{m-1} B^2_1 - \frac{4}{(5-m)^2} \int R y^2 Q'' B_1 + \frac{2(m-13)}{(5-m)^2} \int R y Q B'_1 \right].$$

Now we consider the case of $\eta_j$’s, defined in (5.39). Using the same arguments as before,

$$\eta_1 = -\frac{1}{2 \theta M(Q)} \left[ \frac{1}{2} \int R y^2 Q'' B_1 + \int R Y_2 \mathcal{L} + (A_{2}^j + \alpha_1 \Lambda Q) + \int R A_1 \mathcal{L} - (B_{2}^j + \frac{1}{2} \beta_1 y Q) \right]$$

$$= -\frac{1}{2 \theta M(Q)} \left[ \frac{1}{2} \int R y^2 Q'' B_1 + \int R Y_2 (F_2^{I V} - \alpha_1 Q) + \int R A_1 (G_2^{I I} - \beta_1 Q') \right]$$

$$= -\frac{1}{2 \theta M(Q)} \left[ \frac{1}{2} \int R y^2 Q'' B_1 - \frac{1}{2} \int R B_1 y^2 Q^m + \int R A^2_1 \right] = -\frac{1}{2 \theta M(Q)} \int R A^2_1.$$
On the other hand,
\[ \eta_2 = - \frac{1}{2 \theta M(Q)} \int_{\mathbb{R}} Y_2 L_+ (A_{12}^{IV} + \alpha_2 \Lambda Q) = - \frac{1}{2 \theta M(Q)} \int_{\mathbb{R}} Y_2 (F_{12}^{IV} - \alpha_2 Q) = - \frac{1}{2 \theta M(Q)} \int_{\mathbb{R}} B_1^2. \]

Concerning \( \eta_3 \),
\[ \eta_3 = - \frac{1}{2 \theta M(Q)} \left[ \int_{\mathbb{R}} Y_2 L_+ (A_{12}^{IV} + \alpha_4 \Lambda Q) + \int_{\mathbb{R}} A_1 L_-(B_{12}^{IV} + \frac{1}{2} \beta_2 y Q) + (m - 1) \int_{\mathbb{R}} y Q^{m-1} A_1 B_1 
+ \frac{1}{2} (m - 1)(m - 2) \int_{\mathbb{R}} Q^{m-2} A_1^2 B_1 \right] \]
\[ = - \frac{1}{2 \theta M(Q)} \left[ \int_{\mathbb{R}} Y_2 (F_{12}^{IV} - \alpha_3 Q) + \int_{\mathbb{R}} A_1 G_{12}^{IV} + (m - 1) \int_{\mathbb{R}} y Q^{m-1} A_1 B_1 
+ \frac{1}{2} (m - 1)(m - 2) \int_{\mathbb{R}} Q^{m-2} A_1^2 B_1 \right] \]
\[ = - \frac{1}{2 \theta M(Q)} \left[ \frac{8}{m + 3} \int_{\mathbb{R}} B_1^2 + \frac{m - 7}{5 - m} \int_{\mathbb{R}} A_1^2 \right] \]
and
\[ \eta_4 = - \frac{1}{2 \theta M(Q)} \left[ \int_{\mathbb{R}} Y_2 L_+ (A_{12}^{IV} + \alpha_4 \Lambda Q) + \frac{1}{2} (m - 1) \int_{\mathbb{R}} Q^{m-2} B_1 \right] \]
\[ = - \frac{1}{2 \theta M(Q)} \left[ - \int_{\mathbb{R}} B_1 F_{12}^{IV} + \frac{1}{2} (m - 1) \int_{\mathbb{R}} Q^{m-2} B_1^3 \right] = \frac{(9 - m)}{2(5 - m) \theta M(Q)} \int_{\mathbb{R}} B_1^2. \]

In conclusion, we have
\[ \eta_1 = - \frac{1}{2 \theta M(Q)} \int_{\mathbb{R}} A_1^2, \quad \eta_2 = - \frac{1}{2 \theta M(Q)} \int_{\mathbb{R}} B_1^2, \quad \eta_3 = - \frac{1}{2 \theta M(Q)} \left[ \frac{8}{m + 3} \int_{\mathbb{R}} B_1^2 + \frac{m - 7}{5 - m} \int_{\mathbb{R}} A_1^2 \right], \quad \eta_4 = \frac{(9 - m)}{2(5 - m) \theta M(Q)} \int_{\mathbb{R}} B_1^2. \] (5.26) (5.27)

6. PROOF OF (2.28) AND (2.30)

We start with the following

Claim 1. Let \( q > 0 \). Then
\[ \int_{\mathbb{R}} \frac{a^q a'^q}{a^{q+1}}(s)ds = \frac{1}{2}(q + 1) \int_{\mathbb{R}} \frac{a^3}{a^q}(s)ds, \quad \int_{\mathbb{R}} \frac{1}{a^q}(s)ds = \frac{1}{2}q(q + 1) \int_{\mathbb{R}} \frac{a^3}{a^{q+2}}(s)ds. \]

Proof. A direct consequence of integration by parts and (1.5).

Proof of (2.28). From (2.27) it is enough to prove that
\[ \varepsilon \int_{-T_x}^{T_x} \left[ \frac{4a'(\varepsilon f_1)}{(5 - m)a(\varepsilon \rho)} + \frac{f_6}{c} \right] ds = O(\varepsilon). \] (6.1)

Replacing \( f_6 \) from (2.41), we get
\[ \text{l.h.s. of (6.1)} = \varepsilon \int_{-T_x}^{T_x} \left[ \eta_1 + \frac{v^2}{c} \eta_2 \right] \frac{a'^q a''(\varepsilon \rho)}{a^q}(s)ds + \left( \eta_3 + \frac{v^2}{c} \eta_4 \right) \frac{a'^3}{a^q}(\varepsilon \rho)ds. \]
Additionally, using (2.20) and (2.21), we get

\[ \text{l.h.s. of (6.1)} = \int_{\varepsilon^{-1/100}}^{\varepsilon} \left[ (\eta_1 + \frac{v^2}{c} \eta_2) a'' a'' a'' a'' + (\eta_3 + 4 \lambda_0 \eta_4) \frac{a''}{a'' + 3} (s) \right] ds + O(\varepsilon). \]

where \( p := \frac{4}{5-m} \). Applying the previous Lemma and the identity (2.20),

\[ \text{l.h.s. of (6.1)} = \int_{\varepsilon^{-1/100}}^{\varepsilon} \left[ (\eta_1 + 4 \lambda_0 \eta_2) a'' a'' + (\eta_3 + 4 \lambda_0 \eta_4) \frac{a''}{a'' + 3} (s) \right] ds + O(\varepsilon) \]

From the previous section, more specifically (5.26)-(5.27), we have

\[ (p + 2) \eta_1 + 2 \eta_3 = \frac{8}{\theta(m + 3) M/Q} \int_R B_1^2, \]

and \( (p + 1) \eta_2 + \eta_4 = 0 \). Since

\[ (p + 2) \eta_2 + 2 \eta_4 = \frac{2}{\theta(5 - m) M/Q} \int_R B_1^2, \]

we finally get \( \lambda_0 = (5 - m)/(m + 3) \)

\[ (p + 2) \eta_1 + 2 \eta_3 + 4 \lambda_0 ((p + 2) \eta_2 + 2 \eta_4) = 0, \]

as desired.

**Proof of (2.30).** We have to compute the following number:

\[ k(T) = 2 \varepsilon \int_{-T}^{T} \left[ \frac{8 a''(\varepsilon \rho(t))}{m + 3 a(\varepsilon \rho(t))} h(s) v(s) - f_1(s) f_4(s) + f_5(s) v(s) \right] ds. \]

Replacing \( h(t) \) using (2.24), and integrating by parts, we get

\[ k(T) = \frac{16 h(T)}{p(m + 3)} + 2 \varepsilon \int_{-T}^{T} \left[ \frac{-8 a''(\varepsilon \rho(t))}{p(m + 3)} \frac{a''(\varepsilon \rho(t))}{a''(\varepsilon \rho(t))} f_4(s) + f_5(s) v(s) \right] ds. \]

Note that from the previous computation \( h(T) \sim 0 \) at higher order in \( \varepsilon \); therefore, replacing \( f_4, f_5 \) and \( f_6 \) from (2.13)-(2.17), we obtain

\[ k(T) = 2 \varepsilon \int_{-T}^{T} \left[ \frac{a''(\varepsilon \rho(t))}{a'' + 3} \left( \frac{2 \eta_2 + 2 \eta_3}{a'' + 3} \right) (\varepsilon \rho(t)) v(s) ds, \right. \]

where

\[ \bar{\delta}_2 = \hat{\delta}_2 - 2 \lambda_0 \eta_1 - \frac{8 \beta_1}{m + 3}, \quad \bar{\delta}_3 = \hat{\delta}_3 - 2 \lambda_0 \eta_2; \]

and

\[ \bar{\delta}_4 = \hat{\delta}_4 - 2 \lambda_0 \eta_3 - \frac{8 \beta_2}{m + 3}, \quad \bar{\delta}_5 = \hat{\delta}_5 - 2 \lambda_0 \eta_4. \]
We apply the same argument as in the proof of (5.28). Changing variables, using Claim 1 and the identity (6.20) we get

\[
k(\tilde{T}_\varepsilon) = 2 \int_\mathbb{R} \left[ \delta_1 \frac{a^{(3)}}{a} + (\tilde{\delta}_2 + 4\lambda_0 \tilde{\delta}_3) \frac{a' a''}{a^2} + (\tilde{\delta}_4 + 4\lambda_0 \tilde{\delta}_5) \frac{a'^3}{a^3} \right] ds + (\tilde{\delta}_2 + 4\lambda_0 \tilde{\delta}_3) \int_\mathbb{R} \frac{a'^3}{a^3} ds + (\tilde{\delta}_4 + 4\lambda_0 \tilde{\delta}_5) \int_\mathbb{R} \frac{a'^3}{a^3} ds + o_\varepsilon(1)
\]

where \( o_\varepsilon(1) \to 0 \) as \( \varepsilon \to 0 \). We compute now the coefficient \( \tilde{\delta} \). One has

\[
\tilde{\delta} = (p + 2)\delta_3 + 2\tilde{\delta}_5 - 2\lambda_0((p + 2)\eta_2 + 2\eta_4).
\]

From (6.3) we have

\[
\tilde{\delta} = (p + 2)\delta_3 + 2\tilde{\delta}_5 - \frac{16(m - 1)}{(5 - m)(m + 3)}M[Q] \int_\mathbb{R} B_1^2.
\]

Replacing \( \tilde{\delta}_3 \) and \( \tilde{\delta}_5 \) from (5.23) and (5.25), we get

\[
\tilde{\delta} = -\frac{4}{M[Q]} \left[ \frac{7 - m}{5 - m} \int_\mathbb{R} A_1 B_1^2 + \frac{2}{5 - m} \int_\mathbb{R} yQ'B_1 + \frac{(m - 7)}{5 - m} \int_\mathbb{R} A_1 B_1' \right.
\]

\[
- \frac{(m - 1)(m - 3)}{2(5 - m)} \int_\mathbb{R} yQ^{-2}Q'B_1^2 + \frac{2}{m + 3} \int_\mathbb{R} B_1^2 - \frac{2}{5 - m} \int_\mathbb{R} Q^{m - 1}B_1^2
\]

\[
- \frac{4}{(5 - m)^2} \int_\mathbb{R} y^2Q'B_1 + \frac{2}{5 - m} \int_\mathbb{R} yQ'B_1 + \frac{4}{(m - 1)(m + 3)} \int_\mathbb{R} B_1^2
\]

\[
= -\frac{4}{(5 - m)M[Q]} \left[ -\frac{12}{5 - m} \int_\mathbb{R} yQ'B_1 - \frac{4}{5 - m} \int_\mathbb{R} y^2Q'B_1 + \frac{1}{5 - m} \int_\mathbb{R} yQ'^{-2}Q'B_1^2 \right].
\]

Let us deal with the term \( \hat{\delta} \). From the definition, one has

\[
\hat{\delta} = \delta_1 + \delta_2 + \delta_4 - 2\lambda_0(\eta_1 + \eta_3) - \frac{8}{m + 3} (\beta_1 + \beta_2) + 4\lambda_0(\delta_3 + \delta_5) + 8\lambda_0^2(\eta_2 + \eta_4).
\]

First of all, from (5.21), (5.22) and (5.24), we have

\[
\delta_1 + \delta_2 + \delta_4 =
\]

\[
= -\frac{1}{M[Q]} \left[ 2(m + 1) \int_\mathbb{R} y^2Q'^{m+1} - \int_\mathbb{R} yQ^m A_1 + \frac{1}{2(m + 1)} \int_\mathbb{R} y^2Q'^{m+1} + \frac{2}{m + 3} \int_\mathbb{R} A_1^2 + \frac{8}{m + 3} \int_\mathbb{R} A_1 B_1' - \frac{m}{5 - m} \int_\mathbb{R} yQ^m A_1 + \frac{4}{(5 - m)(m + 3)} \int_\mathbb{R} yQ A_1
\]

\[
- \frac{2m}{5 - m} \int_\mathbb{R} y^2Q'^{m - 1} A_1 + \frac{8}{(5 - m)(m + 3)} \int_\mathbb{R} yQ^m A_1 - \frac{2m}{5 - m} \int_\mathbb{R} Q^{m - 1}A_1^2
\]

\[
- \frac{m(m - 1)(m - 3)}{2(5 - m)} \int_\mathbb{R} yQ^{-2}A_1^2 + \frac{16}{(5 - m)(m + 3)} \int_\mathbb{R} yQ A_1
\]

\[
= -\frac{1}{M[Q]} \left[ 2(m + 1) \int_\mathbb{R} A_1^2 + \frac{8}{m + 3} \int_\mathbb{R} A_1 B_1' - \frac{5}{5 - m} \int_\mathbb{R} yQ^m A_1 + \frac{4}{(5 - m)(m + 3)} \int_\mathbb{R} yQ A_1
\]

\[
- \frac{2m}{5 - m} \int_\mathbb{R} y^2Q'^{m - 1} A_1 + \frac{8}{(5 - m)(m + 3)} \int_\mathbb{R} yQ^m A_1 - \frac{2m}{5 - m} \int_\mathbb{R} Q^{m - 1}A_1^2
\]

\[
- \frac{m(m - 1)(m - 3)}{2(5 - m)} \int_\mathbb{R} yQ^{-2}A_1^2 + \frac{16}{(5 - m)(m + 3)} \int_\mathbb{R} yQ A_1
\].
On the one hand, using (5.21),
\[-2\lambda_0(\eta_1 + \eta_3) = \frac{4(m-1)}{(m+3)M(Q)} \left[ \frac{8}{m+3} \int_{\mathbb{R}} B_1^2 - \frac{2}{5 - m} \int_{\mathbb{R}} A_1^2 \right].\]

Similarly, using (5.21),
\[-\frac{8}{m+3} (\beta_1 + \beta_2) = -\frac{8}{(m+3)M(Q)} \left[ 2 \int_{\mathbb{R}} A_1 B_1' + \frac{m}{5 - m} \int_{\mathbb{R}} y Q A_1 \right].\]

Now,
\[4\lambda_0(\delta_3 + \delta_5) = -\frac{4}{5 - m} \int_{\mathbb{R}} A_1 B_1' - \frac{16}{(5 - m)^2} \int_{\mathbb{R}} y Q' B_1 + \frac{2}{m+3} \int_{\mathbb{R}} B_1^2 - \frac{4}{2(5 - m)} \int_{\mathbb{R}} y Q^{m-2} Q' B_1^2 - \frac{4}{(5 - m)^2} \int_{\mathbb{R}} y^2 Q'' B_1 - \frac{2}{5 - m} \int_{\mathbb{R}} Q^{m-1} B_1^2.\]

Finally,
\[8\lambda_0^2(\eta_2 + \eta_4) = \frac{64(m-1)}{(m+3)^2M(Q)} \int_{\mathbb{R}} B_1^2.\]

Collecting the above identities, we get
\[-2\lambda_0(\eta_1 + \eta_3) + 8\lambda_0^2(\eta_2 + \eta_4) = \frac{8(m-1)}{(m+3)M(Q)} \left[ \frac{12}{m+3} \int_{\mathbb{R}} B_1^2 - \frac{1}{5 - m} \int_{\mathbb{R}} A_1^2 \right],\]

and finally, the terms of the form \( \int_{\mathbb{R}} A_1 B_1' \) cancel each other, to obtain
\[
\begin{aligned}
\hat{\delta} &= -\frac{2}{(m-1)M(Q)} \left[ \int_{\mathbb{R}} y Q'' A_1 + \frac{4(m+1)}{m+3} \int_{\mathbb{R}} y Q A_1 - 2m \int_{\mathbb{R}} y^2 Q^{m-1} Q' A_1 \\
&\quad + \frac{2}{5 - m} \int_{\mathbb{R}} y^2 Q' A_1 - 2m \int_{\mathbb{R}} y^2 Q'' A_1 - \frac{1}{2} 2(m-1)(m-3) \int_{\mathbb{R}} y Q^{m-2} Q' A_1^2 \right] \\
&\quad - \frac{8}{(m+3)M(Q)} \left[ \int_{\mathbb{R}} y^2 Q B_1 - \frac{4}{5 - m} \int_{\mathbb{R}} y^2 Q'' B_1 + \frac{2}{m+3} \int_{\mathbb{R}} B_1^2 \\
&\quad - 2 \int_{\mathbb{R}} Q^{m-1} B_1 - \frac{1}{2} (m-1)(m-3) \int_{\mathbb{R}} y Q^{m-2} Q' B_1^2 \right] \\
&= -\frac{2}{(m-1)M(Q)} \left[ \int_{\mathbb{R}} y Q'' A_1 + \frac{4(m+1)}{m+3} \int_{\mathbb{R}} y Q A_1 - 2m \int_{\mathbb{R}} y^2 Q^{m-1} Q' A_1 \\
&\quad + \frac{8}{m+3} \int_{\mathbb{R}} y^2 Q' A_1 - 2m \int_{\mathbb{R}} y^2 Q'' A_1 - \frac{1}{2} 2(m-1)(m-3) \int_{\mathbb{R}} y Q^{m-2} Q' A_1^2 \right] \\
&\quad + 2\lambda_0 \hat{\delta} + \frac{128(m-1)}{(m+3)^2M(Q)} \int_{\mathbb{R}} B_1^2. \quad (6.8)
\end{aligned}
\]

Now we state some well-known identities satisfied by the soliton \( Q \). For the proofs, see \[25\] and \[30\].

**Lemma 6.1.** Suppose \( m > 1 \) and denote by \( Q_c(x) := e^{\frac{x}{m-1}} Q(\sqrt{e}x) \) the scaled soliton, with \( Q \) solution of \(-Q'' + Q - Q^m = 0 \) in \( \mathbb{R} \). Then

1. Integrals. Let \( \theta := \frac{1}{m-1} - \frac{1}{4} \). Then
\[
\int_{\mathbb{R}} Q_c = e^{\theta - \frac{1}{4}} \int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} Q_c^2 = e^{2\theta} \int_{\mathbb{R}} Q^2. \quad (6.9)
\]

and finally
\[
\int_{\mathbb{R}} Q_c^{m+1} = \frac{2(m+1)c^{2m+1}}{m+3} \int_{\mathbb{R}} Q^2, \quad \int_{\mathbb{R}} \Lambda Q_c = (\theta - \frac{1}{4}) e^{\theta - \frac{1}{4}} \int_{\mathbb{R}} Q, \quad (6.10)
\]
\[
\int_{\mathbb{R}} \Lambda Q_c Q_c = \theta c^{2\theta - 1} \int_{\mathbb{R}} Q^2. \quad (6.11)
\]
(2) Integrals with powers.
\[
\int_{\mathbb{R}} Q^2 = \frac{m - 1}{m + 3} \int_{\mathbb{R}} Q^2, \quad \int_{\mathbb{R}} y^2 Q^{m+1} = \frac{m + 1}{m + 3} \left[ 2 \int_{\mathbb{R}} y^2 Q^2 - \int_{\mathbb{R}} Q^2 \right],
\]
and
\[
\int_{\mathbb{R}} y^4 Q^{m+1} = \frac{m + 1}{m + 3} \left[ 2 \int_{\mathbb{R}} y^4 Q^2 - 6 \int_{\mathbb{R}} y^2 Q^2 \right],
\]
\[
\int_{\mathbb{R}} y^2 Q^2 = \frac{1}{m + 3} \left[ 2 \int_{\mathbb{R}} Q^2 + (m - 1) \int_{\mathbb{R}} y^2 Q^2 \right],
\]
\[
\int_{\mathbb{R}} y^4 Q^2 = \frac{1}{m + 3} \left[ 12 \int_{\mathbb{R}} y^2 Q^2 + (m - 1) \int_{\mathbb{R}} y^4 Q^2 \right].
\]

We use these identities to give a simplified expression for the term \( \delta \) in (6.16). Indeed, we claim that
\[
\delta = \frac{(m - 1)}{(5 - m)^2(m + 3)M[Q]} \left[ 5 \int_{\mathbb{R}} y^4 Q^2 - 5 \chi^2 \int_{\mathbb{R}} Q^2 - 12 \int_{\mathbb{R}} y^2 Q^2 \right].
\]

Proof of (6.16). Using (4.29) and (4.30),
\[
- \frac{12}{(5 - m)} \int_{\mathbb{R}} yQ'B_1 = \frac{6}{(5 - m)^2} \int_{\mathbb{R}} y(y^2 + \chi)Q'Q = \frac{6}{(5 - m)^2} \left[ - \frac{3}{2} \int_{\mathbb{R}} y^2 Q^2 - \frac{1}{2} \chi \int_{\mathbb{R}} Q^2 \right] = - \frac{6}{(5 - m)^2} \int_{\mathbb{R}} y^2 Q^2;
\]
\[- \frac{1}{2} (m - 1)(m - 3) \int_{\mathbb{R}} yQ^{m-2}Q'B_1^2 = - \frac{(m - 1)(m - 3)}{8(5 - m)^2} \int_{\mathbb{R}} yQ^{m-2}Q'(y^4 + 2 \chi y^2 + \chi^2) = \frac{(m - 1)(m - 3)}{8(5 - m)^2(m + 1)} \left[ 5 \int_{\mathbb{R}} y^4 Q^{m+1} + 6 \chi \int_{\mathbb{R}} y^2 Q^{m+1} + \chi^2 \int_{\mathbb{R}} Q^{m+1} \right];
\]
\[
2 \int_{\mathbb{R}} B_1^2 = \frac{1}{2(5 - m)^2} \left[ \int_{\mathbb{R}} y^4 Q^2 + 2 \chi \int_{\mathbb{R}} y^2 Q^2 + \chi^2 \int_{\mathbb{R}} Q^2 \right];
\]
\[- 2 \int_{\mathbb{R}} Q^{m-1}B_1^2 = \frac{1}{2(5 - m)^2} \left[ \int_{\mathbb{R}} y^4 Q^{m+1} + 2 \chi \int_{\mathbb{R}} y^2 Q^{m+1} + \chi^2 \int_{\mathbb{R}} Q^{m+1} \right];
\]
and from the identity \( Q'' = Q - Q^m \),
\[- \frac{4}{5 - m} \int_{\mathbb{R}} y^2 Q''B_1 = \frac{2}{(5 - m)^2} \int_{\mathbb{R}} y^2(y^2 + \chi)(Q^2 - Q^{m+1}) = \frac{2}{(5 - m)^2} \left[ \int_{\mathbb{R}} y^4 Q^2 + \chi \int_{\mathbb{R}} y^2 Q^2 \right] - \frac{2}{(5 - m)^2} \left[ \int_{\mathbb{R}} y^4 Q^{m+1} + \chi \int_{\mathbb{R}} y^2 Q^{m+1} \right].
\]
Replacing these identities in (6.16), we get
\[
\delta = - \frac{4}{(5 - m)^3 M[Q]} \left[ - 6 \int_{\mathbb{R}} y^2 Q^2 + \frac{5(m - 1)(m - 3)}{8(m + 1)} \int_{\mathbb{R}} y^4 Q^{m+1} + \frac{5}{2} \int_{\mathbb{R}} y^4 Q^2 - \frac{5}{2} \int_{\mathbb{R}} y^4 Q^{m+1} + 3 \chi \left\{ \frac{(m - 1)(m - 3)}{4(m + 1)} \int_{\mathbb{R}} y^2 Q^{m+1} + \int_{\mathbb{R}} y^2 Q^2 - \int_{\mathbb{R}} y^2 Q^{m+1} \right\} 
\]
\[
+ \frac{1}{2} \chi \left\{ \frac{(m - 1)(m - 3)}{4(m + 1)} \int_{\mathbb{R}} y^2 Q^{m+1} + \int_{\mathbb{R}} y^2 Q^2 - \int_{\mathbb{R}} y^2 Q^{m+1} \right\} \right] \right].
\]

Using (6.10), (6.12) and (6.13), we get
\[
\delta = - \frac{4}{(5 - m)^3 M[Q]} \left[ - \frac{5(5 - m)(m - 1)}{4(m + 3)} \int_{\mathbb{R}} y^4 Q^2 - 6 \int_{\mathbb{R}} y^2 Q^2 - \frac{15(m^2 - 8m - 1)}{4(m + 3)} \int_{\mathbb{R}} y^2 Q^2 
\]
\[
- \frac{3}{4} \chi \left\{ \frac{2(5 - m)(m - 1)}{(m + 3)} \int_{\mathbb{R}} y^2 Q^2 + \frac{(m^2 - 8m - 1)}{(m + 3)} \int_{\mathbb{R}} Q^2 \right\}
\]
\[
- \frac{(5 - m)(m - 1)}{4(m + 3)} \chi^2 \int_{\mathbb{R}} Q^2 \right].
\]
Using the definition of $\chi$ (cf. (4.30)), we obtain
\[
\delta = \frac{(m-1)}{(5-m)^2(m+3)M[Q]} \left[ 5 \int_{\mathbb{R}} y^4 Q^2 - 5\chi^2 \int_{\mathbb{R}} Q^2 - 12 \int_{\mathbb{R}} y^2 Q^2 \right],
\]
as desired. We are then reduced to check that the above integral is not zero. Indeed, in the most important case $m = 3$ we have, thanks to Mathematica,
\[
M[Q] = \frac{1}{2} \int_{\mathbb{R}} Q^2 = 2, \quad \chi = -\frac{\pi^2}{12}, \quad \int_{\mathbb{R}} y^4 Q^2 = \frac{7\pi^4}{60}, \quad \text{and} \quad \delta = \frac{\pi^2}{6} \left( \frac{\pi^2}{9} - 1 \right) \sim 0.159 > 0.
\]
Note that we have used the explicit expression for the soliton given in (1.4). When $m \in [4,5)$, the resulting expressions are no longer in a closed form. From the definition of $Q$ in (1.4) we have computed explicit approximate values for $\delta$:
\[
\delta(m = 4) \sim 0.286; \quad \delta(m = 4.1) \sim 0.335; \quad \delta(m = 4.3) \sim 0.507;
\]
\[
\delta(m = 4.5) \sim 0.925; \quad \delta(m = 4.7) \sim 2.437; \quad \delta(m = 4.9) \sim 21.096\ldots
\]
It seems clear that $\delta$ is an increasing, positive function of $m$. Note that the computed expressions increase as the exponent approaches the critical case $m = 5$.

We will prove, by a using a very different argument to the previous ones, that $\delta > 0$ for all $m \in [4,5]$. Indeed, we will obtain a positive lower bound on the quantity
\[
5 \int_{\mathbb{R}} y^4 Q^2 - 5\chi^2 \int_{\mathbb{R}} Q^2 - 12 \int_{\mathbb{R}} y^2 Q^2,
\]
in terms of its corresponding value in the critical case, $m = 5$. Indeed, replacing $Q = Q(y;m)$ we have
\[
5 \int_{\mathbb{R}} y^4 Q^2 - 5\chi^2 \int_{\mathbb{R}} Q^2 - 12 \int_{\mathbb{R}} y^2 Q^2 = \frac{m+1}{2} \frac{m^2 - 1}{m-1}^5 \left( \int_{\mathbb{R}} \frac{y^4}{\cosh^{m-1}(y)} \right) - 5 \left( \int_{\mathbb{R}} \frac{1}{\cosh^{m-1}(y)} \right) - 3 \left( \int_{\mathbb{R}} \frac{y^2}{\cosh^m(y)} \right) - 48 \int_{\mathbb{R}} \frac{y^2}{\cosh(y)}.
\]
Now we use the following inequalities, valid for all $m \in [4,5]$ and all $y \in \mathbb{R}$,
\[
\cosh^{-\frac{4}{3}}(y) \leq \cosh^{-\frac{2}{m-1}}(y) \leq \cosh^{-1}(y),
\]
to obtain, for $c_m > 0$,
\[
5 \int_{\mathbb{R}} y^4 Q^2 - 5 \left( \int_{\mathbb{R}} \frac{1}{\cosh^{m-1}(y)} \right) - 3 \left( \int_{\mathbb{R}} \frac{y^2}{\cosh^m(y)} \right) - 48 \int_{\mathbb{R}} \frac{y^2}{\cosh(y)} \geq c_m \times \left[ \int_{\mathbb{R}} \frac{y^4}{\cosh^{m-1}(y)} \right] - 5 \left( \int_{\mathbb{R}} \frac{1}{\cosh^{m-1}(y)} \right) - 3 \left( \int_{\mathbb{R}} \frac{y^2}{\cosh^m(y)} \right) - 48 \int_{\mathbb{R}} \frac{y^2}{\cosh(y)}.
\]
Now we evaluate the integrals above, using Mathematica. We obtain
\[
5 \int_{\mathbb{R}} y^4 Q^2 - 5 \left( \int_{\mathbb{R}} \frac{1}{\cosh^{m-1}(y)} \right) - 3 \left( \int_{\mathbb{R}} \frac{y^2}{\cosh^m(y)} \right) - 48 \int_{\mathbb{R}} \frac{y^2}{\cosh(y)} \geq c_m \times \left[ \int_{\mathbb{R}} \frac{y^4}{\cosh^{m-1}(y)} \right] \geq 57.5135 c_m > 0.
\]
Therefore, we have proved that $\delta > 0$ is positive in the whole range $m \in [4,5]$.

Similarly, we can also compute an explicit expression for the complicated term $\tilde{\delta}$ given in (6.5), in the case $m = 3$, and using Mathematica. We have
\[
\tilde{\delta} = \frac{4298\pi^4 - 17475\pi^2 + 1935}{109350} \sim 2.269.
\]
Finally, replacing in (6.5), we obtain
\[
k(T_\varepsilon) = \tilde{\delta} \int_R \frac{a^3}{a_\varepsilon} ds + (\nu_0^2 - \frac{4}{3} \tilde{\delta}) \int_R \frac{a^3}{a_\varepsilon} ds + o_\varepsilon(1)
\geq (\tilde{\delta} - \frac{4}{3} \tilde{\delta}) \int_R \frac{a^3}{a_\varepsilon} ds + o_\varepsilon(1) \sim 2.05 \int_R \frac{a^3}{a_\varepsilon} ds.
\]
A rigorous bound on $\tilde{\delta}$, in the cases $m \in [4,5]$, has escaped to us.
Final conclusion. Since $\delta > 0$ and the integral $\int_{\mathbb{R}} \frac{a^4}{a^{p+3}}$ is positive, there is at most one $\tilde{v}_0 \geq 0$ such that (6.5) is zero. Therefore, for all $v_0 \neq \tilde{v}_0$, the integral (2.20) is proved and then Theorem 1.3 holds. Moreover, thanks to (6.18), in the case $m = 3$ we have $\tilde{v}_0 = 0$, and Theorem 1.3 is valid for all $v_0 > 0$. This finishes the proof.

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