POISSON SUMMATION FORMULAS
INVOLVING THE SUM-OF-SQUARES FUNCTION*

BY
NIR LEV AND GILAD RETI

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel
e-mail: levnir@math.biu.ac.il, gilad.reti@gmail.com

ABSTRACT
We obtain new Poisson type summation formulas with nodes $\pm \sqrt{n}$ and with weights involving the function $r_k(n)$ that gives the number of representations of a positive integer $n$ as the sum of $k$ squares. Our results extend summation formulas due to Guinand and Meyer that involve the sum-of-three-squares function $r_3(n)$.

1. Introduction

1.1. The classical Poisson summation formula asserts that for any function $f$ on $\mathbb{R}$ satisfying sufficient smoothness and decay assumptions, we have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

where

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) \exp(-2\pi i tx) dx$$

is the Fourier transform of $f$. The equality (1.1) holds in particular for any function $f$ from the Schwartz class. There is an equivalent way to state the Poisson summation formula, by saying that the measure $\mu = \sum_{n \in \mathbb{Z}} \delta_n$ (the sum of unit masses at the integer points) satisfies $\hat{\mu} = \mu$, where $\hat{\mu}$ is the Fourier transform of $\mu$ understood in the sense of temperate distributions.

* Research supported by ISF Grant No. 227/17 and ERC Starting Grant No. 713927.
Received February 6, 2020 and in revised form December 11, 2020
The problem of which other Poisson type formulas may exist was studied by different authors. A recent result proved in [LO13, LO15] states that if
\[
\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda
\]
is a pure point measure on \( \mathbb{R} \) supported on a uniformly discrete set \( \Lambda \), and if the Fourier transform \( \hat{\mu} \) (in the sense of temperate distributions) is also a pure point measure
\[
\hat{\mu} = \sum_{s \in S} b_s \delta_s
\]
supported on a uniformly discrete set \( S \), then the measure \( \mu \) can be obtained from Poisson’s formula using dilation and a finite number of shifts, modulations, and taking linear combinations. (We recall that a set is said to be uniformly discrete if the distance between any two of its points is bounded from below by some positive constant.)

On the other hand, in [LO16] the authors established the existence of pure point measures \( \mu \) whose distributional Fourier transform \( \hat{\mu} \) is also a pure point measure, and such that the supports of both \( \mu \) and \( \hat{\mu} \) are locally finite sets (i.e., they have no finite accumulation points), but the support of \( \mu \) contains only finitely many elements of any arithmetic progression. In particular, such a measure \( \mu \) cannot be obtained from Poisson’s summation formula using the procedures mentioned above (see also [Kol16]).

We note that interest in the subject has been largely inspired by the experimental discovery in the middle of the 80’s of quasicrystalline materials, i.e., nonperiodic atomic structures that have a discrete diffraction pattern; see [Mey95], [Lag00]. For recent related work see, e.g., [Mey16], [Mey17], [LO17], [Fav18] and the references therein.

1.2. There is a remarkable summation formula due to Guinand [Gui59, p. 265] which states that if \( \varphi \) is an odd Schwartz function on \( \mathbb{R} \), and if \( \psi = \hat{\varphi} \) is the Fourier transform of \( \varphi \), then
\[
\varphi'(0) + \sum_{n=1}^{\infty} \frac{r_3(n)}{\sqrt{n}} \varphi(\sqrt{n}) = i\psi'(0) + i \sum_{n=1}^{\infty} \frac{r_3(n)}{\sqrt{n}} \psi(\sqrt{n})
\]
where \( r_k(n) \) is the number of representations of \( n \) as the sum of \( k \) squares, that is,
\[
r_k(n) = \# \{ m \in \mathbb{Z}^k : |m|^2 = n \}.
\]
Notice that while (1.2) holds for odd functions only, one can obtain a summation formula that is valid for an arbitrary Schwartz function $f$ (i.e., not necessarily an odd function) by applying (1.2) to

$$\varphi(t) := f(t) - f(-t) \quad \text{and} \quad \psi(t) := \hat{f}(t) - \hat{f}(-t).$$

This yields a nonclassic Poisson type formula with nodes at the points $\pm \sqrt{n}$ and with weights involving the sum-of-three-squares function $r_3(n)$.

The summation formula (1.2) can be equivalently stated by saying that the Fourier transform of the temperate distribution

$$(1.4) \quad \sigma_3 := -2\delta_0' + \sum_{n=1}^{\infty} \frac{r_3(n)}{\sqrt{n}} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$$

is $-i\sigma_3$. In particular, the distribution $\sigma_3$ is supported on the locally finite set of points $\pm \sqrt{n}$, and its Fourier transform $\hat{\sigma}_3$ is supported on the same set.

Guinand’s formula (1.2) remained little known until it was recently rediscovered by Meyer; see [Mey16]. In that paper, a new proof of Guinand’s formula was given, and moreover new examples of Poisson type summation formulas generalizing Guinand’s one were obtained.

There is a recent result due to Radchenko and Viazovska [RV19] which states that a Schwartz function $f$ is uniquely determined by the values of $f$ and $\hat{f}$ at the points $\pm \sqrt{n}$ and the values $f'(0)$ and $\hat{f}'(0)$. Moreover, there exists a linear summation formula that recovers the function $f$ from these values [RV19, Theorems 1 and 7]. As pointed out in [RV19, pp. 77–78], one can use this result to derive Guinand’s formula (1.2).

Remarkably, the Poisson and Guinand formulas (and their linear combinations) are the only weighted summation formulas that involve only the values of $f$ and $\hat{f}$ at the points $\pm \sqrt{n}$ and the values $f'(0)$ and $\hat{f}'(0)$. In other words, if $\sigma$ is a temperate distribution such that both $\sigma$ and its Fourier transform $\hat{\sigma}$ can be expressed as a weighted sum of terms of the form $\delta_0, \delta_0', \delta_{\sqrt{n}}$ or $\delta_{-\sqrt{n}}$ ($n = 1, 2, 3, \ldots$), then $\sigma$ must be a linear combination of the Poisson measure $\mu = \sum_{n \in \mathbb{Z}} \delta_n$ and the Guinand distribution $\sigma_3$. This is a consequence of [Via18, Theorem 5.2]. We note that the Poisson component of $\sigma$ corresponds to its even part, while the Guinand component is the odd part of $\sigma$. 
2. Results

2.1. In this paper we obtain new Poisson type summation formulas with nodes \( \pm \sqrt{n} \), and with weights involving the function \( r_k(n) \) defined by (1.3) that gives the number of representations of \( n \) as the sum of \( k \) squares.

For each positive integer \( k \), consider a distribution \( \sigma_k \) on \( \mathbb{R} \) defined by

\[
(2.1) \quad \sigma_k := -2\delta_0' + \sum_{n=1}^{\infty} \frac{r_k(n)}{\sqrt{n}} \left( \delta_{\sqrt{n}} - \delta_{-\sqrt{n}} \right).
\]

We observe that \( \sum_{n=1}^{N} r_k(n)n^{-1/2} \) increases polynomially in \( N \), which implies that \( \sigma_k \) is a temperate distribution. It is obtained from Guinand’s distribution (1.4) by replacing the sum-of-three-squares function \( r_3(n) \) with the sum-of-\( k \)-squares function \( r_k(n) \).

It is clear that \( \sigma_k \) is an odd distribution supported on the set of points \( \pm \sqrt{n} \). Guinand’s result states that for \( k = 3 \), the Fourier transform \( \hat{\sigma}_3 \) is supported on the same set of points, and in fact, \( \hat{\sigma}_3 = -i\sigma_3 \). In the present paper we obtain an extension of this result to all odd values of \( k \) greater than 3. Our result provides an explicit expression for the Fourier transform \( \hat{\sigma}_k \), which implies in particular that it is again supported on the points \( \pm \sqrt{n} \). However we will see that \( \hat{\sigma}_k \) is not \(-i\sigma_k \) if \( k > 3 \).

2.2. For simplicity we first state the result for \( k = 5 \).

**Theorem 2.1:** Let \( \varphi \) be an odd Schwartz function on \( \mathbb{R} \), and let \( \psi = \hat{\varphi} \) be the Fourier transform of \( \varphi \). Then

\[
(2.2) \quad \varphi'(0) + \sum_{n=1}^{\infty} \frac{r_5(n)}{\sqrt{n}} \varphi(\sqrt{n}) = -\frac{i}{6\pi} \psi'''(0) + \frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{r_5(n)}{n^{3/2}} [\psi(\sqrt{n}) - \sqrt{n} \psi'(\sqrt{n})].
\]

The result establishes a new summation formula with nodes \( \pm \sqrt{n} \) and with weights involving the sum-of-five-squares function \( r_5(n) \). We observe that the right hand side of (2.2) involves not only the values of \( \psi \), but also those of its derivative \( \psi' \), at the points \( \sqrt{n} \) (\( n = 1, 2, 3, \ldots \)). This is not the case in Guinand’s formula (1.2).
Theorem 2.1 can be equivalently stated by saying that the Fourier transform of the distribution $\sigma_5$ is given by

$$
\hat{\sigma}_5 = -\frac{i}{3\pi} \delta_0''' - \frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{r_5(n)}{n^{3/2}} \left[ (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) + \sqrt{n} (\delta_0' + \delta_0'') \right].
$$

As a consequence we obtain that $\hat{\sigma}_5$ is also supported on the set of points $\pm \sqrt{n}$.

2.3. Theorem 2.1 is a special case of the next result, which yields a different summation formula for each $k = 3, 5, 7, 9, \ldots$. To state the result, we define the numbers

$$
\alpha_k := \frac{(-1)^{(k-3)/2}}{(k-2)!!} \left( \frac{1}{2\pi} \right)^{(k-3)/2}, \quad \beta_{j,k} := \frac{(-1)^j (k-j-3)!}{j!(k-2j-3)!!} \left( \frac{1}{2\pi} \right)^{(k-3)/2}
$$

where $k \geq 3$ is an odd integer, and $0 \leq j \leq (k-3)/2$.

**Theorem 2.2:** Let $\varphi$ be an odd Schwartz function on $\mathbb{R}$, and let $\psi = \hat{\varphi}$ be the Fourier transform of $\varphi$. For each odd integer $k \geq 3$ we have

$$
\varphi'(0) + \sum_{n=1}^{\infty} \frac{r_k(n)}{\sqrt{n}} \varphi(\sqrt{n})
$$

$$
= i \alpha_k \psi^{(k-2)}(0) + i \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{(k-2)/2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} n^{j/2} \psi^{(j)}(\sqrt{n}) \right],
$$

where the coefficients $\alpha_k$ and $\beta_{j,k}$ are given by (2.4).

We thus obtain, for each $k = 3, 5, 7, 9, \ldots$, a new Poisson type summation formula with weights involving the sum-of-\(k\)-squares function $r_k(n)$. The nodes remain at the points $\pm \sqrt{n}$, but the number of derivatives at each node depends on $k$ and it increases with $k$. We observe that (1.2) and (2.2) are the special cases obtained for $k = 3$ and 5.

The result is equivalent to saying that for each odd integer $k \geq 3$, the Fourier transform of the distribution $\sigma_k$ defined by (2.1) is given by

$$
\hat{\sigma}_k = 2i \alpha_k \delta_0^{(k-2)} - i \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{(k-2)/2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} n^{j/2}((-1)^j \delta_{\sqrt{n}}^j - \delta_{-\sqrt{n}}^j) \right].
$$

In particular, $\sigma_k$ is a temperate distribution supported on the locally finite set of points $\pm \sqrt{n}$, and its Fourier transform $\hat{\sigma}_k$ is supported on the same set.
2.4. Our proof of the results above is inspired by Meyer’s proof in [Mey16] of Guinand’s formula (1.2). We will obtain Theorem 2.2 as a consequence of the following result, which generalizes [Mey16, Theorem 7].

**Theorem 2.3:** Let \( \mu = \sum_{\lambda \in \Lambda} a(\lambda) \delta_{\lambda} \) be a measure on \( \mathbb{R}^k \), \( k \in \{3, 5, 7, 9, \ldots \} \), whose support \( \Lambda \) is a locally finite set. Suppose that \( \mu \) is a temperate distribution, and that its Fourier transform \( \hat{\mu} = \sum_{s \in S} b(s) \delta_{s} \) is also a measure, supported on a locally finite set \( S \). Assume that \( 0 \notin \Lambda \), \( 0 \notin S \), and let \( \sigma \) be a measure on \( \mathbb{R} \) defined by

\[
\sigma := \sum_{\lambda \in \Lambda} \frac{a(\lambda)}{|\lambda|} (\delta_{|\lambda|} - \delta_{-|\lambda|}).
\]

Then \( \sigma \) is a temperate distribution whose one-dimensional Fourier transform is

\[
\hat{\sigma} = -i \sum_{s \in S} \frac{b(s)}{|s|^{k-2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} |s|^j ((-1)^j \delta_{|s|} - \delta_{-|s|}) \right],
\]

where the coefficients \( \beta_{j,k} \) are defined in (2.4).

The result in [Mey16, Theorem 7] corresponds to the case \( k = 3 \) in Theorem 2.3.

We observe that the measure \( \sigma \) in (2.6) is supported on the locally finite subset \( \{ \pm |\lambda| : \lambda \in \Lambda \} \) of the real line. It follows from Theorem 2.3 that the Fourier transform \( \hat{\sigma} \) is a distribution supported on the (also locally finite) set \( \{ \pm |s| : s \in S \} \).

In Section 5 we will give a more general version of Theorem 2.3, in which the origin is allowed to belong to the supports \( \Lambda, S \) of the measures \( \mu \) and \( \hat{\mu} \) (Theorem 5.1).

2.5. As an example, Theorem 2.3 applies to the measure

\[
\mu(\eta, \xi) = \sum_{m \in \mathbb{Z}^k} e^{2\pi i (m, \xi)} \delta_{m+\eta}
\]

where \( \eta \) and \( \xi \) are two vectors in \( \mathbb{R}^k \setminus \mathbb{Z}^k \). It follows from Poisson’s formula that the Fourier transform of \( \mu(\eta, \xi) \) is given by \( \hat{\mu}(\eta, \xi) = e^{-2\pi i (\eta, \xi)} \mu(\xi, -\eta) \). Hence applying Theorem 2.3 to this measure yields the following generalization of [Mey16, Theorem 5].
Corollary 2.4: Let \( \eta \) and \( \xi \) be two vectors in \( \mathbb{R}^k \setminus \mathbb{Z}^k \), \( k \in \{3, 5, 7, 9, \ldots \} \). Then the one-dimensional Fourier transform of the measure

\[
\sigma := \sum_{m \in \mathbb{Z}^k} \frac{e^{2\pi i \langle m, \xi \rangle}}{|m + \eta|} (\delta_{|m+\eta|} - \delta_{-|m+\eta|})
\]

is the distribution

\[
\widehat{\sigma} = -ie^{-2\pi i \langle \eta, \xi \rangle} \sum_{m \in \mathbb{Z}^k} \frac{e^{-2\pi i \langle m, \eta \rangle}}{|m + \xi|^{k-2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} |m + \xi|^j \left( (-1)^j \delta_{|m+\eta|} - \delta_{-|m+\eta|} \right) \right].
\]

2.6. In order to prove Theorem 2.2, we will give in Section 5 a more general version of Theorem 2.3 (Theorem 5.1), in which the supports \( \Lambda, S \) of the measures \( \mu \) and \( \widehat{\mu} \) are allowed to contain the origin. Theorem 2.2 will then be deduced by applying this result to the Poisson measure \( \mu = \sum_{m \in \mathbb{Z}^k} \delta_m \) which satisfies \( \widehat{\mu} = \mu \).

We observe that (in the same spirit as Meyer’s proof of Guinand’s formula) there exist distributional limits for (2.8) and (2.9) as \( \eta \) and \( \xi \) tend to zero, and these limits coincide with (2.1) and (2.5) respectively.

The rest of the paper is devoted to the proofs of the results stated above.

3. Preliminaries

In this section we briefly recall some preliminary background in the theory of Schwartz distributions (see [Rud91] for more details) and obtain a few basic results that will be used later on.

3.1. We denote by \( \mathcal{D}(\mathbb{R}^k) \) the space of all infinitely smooth, compactly supported functions on \( \mathbb{R}^k \). A sequence of functions \( \varphi_j \) in \( \mathcal{D}(\mathbb{R}^k) \) is said to converge in the space \( \mathcal{D}(\mathbb{R}^k) \) if (i) there exists a compact set \( K \subset \mathbb{R}^k \) such that \( \text{supp}(\varphi_j) \subset K \) for all \( j \); and (ii) for each multi-index \( m = (m_1, \ldots, m_k) \), the sequence of partial derivatives \( \partial^m \varphi_j \) converges uniformly on \( K \) as \( j \to \infty \). A distribution is a linear functional on \( \mathcal{D}(\mathbb{R}^k) \) which is continuous with respect to the convergence in the space \( \mathcal{D}(\mathbb{R}^k) \).

We denote by \( \langle \sigma, \varphi \rangle \) the action of a distribution \( \sigma \) on a function \( \varphi \) belonging to \( \mathcal{D}(\mathbb{R}^k) \).
The **Schwartz space** \( S(\mathbb{R}^k) \) consists of all infinitely smooth functions \( \varphi \) on \( \mathbb{R}^k \) such that for each \( n \geq 0 \) and each multi-index \( m = (m_1, \ldots, m_k) \), the norm
\[
\|\varphi\|_{n,m} := \sup_{x \in \mathbb{R}^k} |x|^n |\partial^m \varphi(x)|
\]
is finite. A sequence \( \varphi_j \) is said to converge in the space \( S(\mathbb{R}^k) \) if it converges with respect to each one of the norms \( \|\cdot\|_{n,m} \). A **temperate distribution** is a distribution that can be extended to a linear functional on the space \( S(\mathbb{R}^k) \) in such a way that the extension is continuous with respect to the convergence in the space \( S(\mathbb{R}^k) \).

If \( \sigma \) is a temperate distribution, then the notation \( \langle \sigma, \varphi \rangle \) can be extended to denote the action of \( \sigma \) on any function \( \varphi \) from the Schwartz space \( S(\mathbb{R}^k) \).

If \( \varphi \) is a Schwartz function on \( \mathbb{R}^k \) then its Fourier transform is defined by
\[
\hat{\varphi}(\xi) = \int_{\mathbb{R}^k} \varphi(x) e^{-2\pi i \langle \xi, x \rangle} dx.
\]
The Fourier transform \( \hat{\sigma} \) of a temperate distribution \( \sigma \) is defined by
\[
\langle \hat{\sigma}, \varphi \rangle = \langle \sigma, \hat{\varphi} \rangle.
\]

A distribution \( \sigma \) on \( \mathbb{R} \) is said to be **even** (respectively **odd**) if we have \( \langle \sigma, \varphi \rangle = 0 \) for every odd (respectively even) function \( \varphi \in D(\mathbb{R}) \).

**3.2.** The following result is a version of the well-known Hadamard Lemma, which states that if \( f \) is a function of the class \( C^r(\mathbb{R}) \) (that is, \( f \) is \( r \)-times continuously differentiable) such that \( f(0) = 0 \), then \( f(t) = tg(t) \) where \( g \) is a function of the class \( C^{r-1}(\mathbb{R}) \).

**Lemma 3.1:** Let \( f \) be a function in the Schwartz space \( S(\mathbb{R}) \), \( f(0) = 0 \). Then
\[
f(t) = tg(t), \quad t \in \mathbb{R},
\]
where \( g \) is a function in \( S(\mathbb{R}) \). Moreover, the mapping which takes \( f \) to \( g \) is a continuous linear mapping from the space of Schwartz functions vanishing at the origin into \( S(\mathbb{R}) \).

**Proof.** Define
\[
g(t) := \int_0^1 f'(tu) du;
\]
then \( g \) is an infinitely smooth function and satisfies condition (3.1). The fact that \( g(t) = f(t)/t \) \( (t \neq 0) \) implies that \( \|g\|_{n,m} \) is finite for every \( n \) and \( m \), and thus \( g \in S(\mathbb{R}) \). The mapping which takes \( f \) to \( g \) is thus a linear mapping...
from the space of Schwartz functions vanishing at the origin into $S(\mathbb{R})$. Now suppose that $f_j \to f$ in $S(\mathbb{R})$. If we denote by $g_j$ and $g$ the functions corresponding to $f_j$ and $f$ respectively by the mapping, then $g_j \to g$ pointwise. The continuity of the mapping thus follows from the closed graph theorem (see [Rud91, Theorem 2.15]).

3.3.

**Lemma 3.2:** Let $f$ be an even function in the Schwartz space $S(\mathbb{R})$, and let $F$ be a radial function on $\mathbb{R}^k$ defined by

$$F(x) := f(|x|), \quad x \in \mathbb{R}^k.$$ 

Then $F$ belongs to the Schwartz space $S(\mathbb{R}^k)$. Moreover, the mapping which takes $f$ to $F$ is a continuous linear mapping from the space of even functions in $S(\mathbb{R})$ into $S(\mathbb{R}^k)$.

**Proof.** Since $f$ is an even function we have $f'(0) = 0$. Hence by Lemma 3.1 there is a function $g \in S(\mathbb{R})$ such that $f'(t) = tg(t), \ t \in \mathbb{R}$. We claim that

$$\frac{\partial F}{\partial x_i}(x) = g(|x|)x_i, \quad x = (x_1, \ldots, x_k)$$

for each $1 \leq i \leq k$. Indeed, this follows from the chain rule for $x \neq 0$, while for $x = 0$ we simply observe that both sides of (3.2) vanish. Since the right-hand side of (3.2) is a continuous function for each $i$, we obtain that $F$ is a function in the class $C^1(\mathbb{R}^k)$.

Applying the same considerations to the (even) function $g$ in place of $f$, we obtain that the function $G(x) := g(|x|)$ is also in the class $C^1(\mathbb{R}^k)$. This implies that the right-hand side of (3.2) belongs to $C^1(\mathbb{R}^k)$ for each $i$, and it follows that $F \in C^2(\mathbb{R}^k)$. Continuing in this way we obtain that $F \in C^r(\mathbb{R}^k)$ for every $r$, hence $F$ is infinitely smooth.

It is straightforward to check that $\|F\|_{n,m}$ is finite for every $n$ and every multi-index $m = (m_1, \ldots, m_k)$, which implies that $F \in S(\mathbb{R}^k)$. The mapping which takes $f$ to $F$ is thus a linear mapping from the space of even functions in $S(\mathbb{R})$ into $S(\mathbb{R}^k)$.

Finally, suppose that $f_j \to f$ in $S(\mathbb{R})$. If $F_j$ and $F$ are the functions corresponding to $f_j$ and $f$ respectively by the mapping, then $F_j \to F$ pointwise. This establishes the continuity of the mapping again as a consequence of the closed graph theorem.
4. Fourier transform of a radial function in odd dimensions

4.1. Let $f$ be an even Schwartz function on $\mathbb{R}$, and suppose that we use $f$ to construct a radial function $F_k$ on $\mathbb{R}^k$, defined by $F_k(x) := f(|x|)$. The $k$-dimensional Fourier transform $\hat{F}_k$ of the function $F_k$ is again a radial function, so there is an even function $f_k$ on $\mathbb{R}$ such that

$$ \hat{F}_k(\xi) = f_k(|\xi|). $$

It turns out that if the dimension $k$ is an odd integer, then there exists an explicit expression for the new function $f_k$ in terms of the one-dimensional Fourier transform $\hat{f}$ and its derivatives. This expression is given in the following result:

**Theorem 4.1:** Let $f$ be an even Schwartz function on $\mathbb{R}$, and let $k \geq 3$ be an odd integer. Define a radial function $F_k$ on $\mathbb{R}^k$ by $F_k(x) := f(|x|)$. Then the $k$-dimensional Fourier transform of $F_k$ is given by

$$ \hat{F}_k(\xi) = -\frac{1}{2\pi |\xi|^{k-1}} \sum_{j=0}^{(k-3)/2} \beta_{j,k} |\xi|^{j+1} \hat{f}^{(j+1)}(|\xi|), \quad \xi \in \mathbb{R}^k \setminus \{0\} $$

where the coefficients $\beta_{j,k}$ are defined as in (2.4), and where $\hat{f}$ is the one-dimensional Fourier transform of the function $f$.

This result was obtained in [GT13, Corollary 1.2]. In this paper the authors also give an expression for $\hat{F}_k$ for even dimensions $k$, but in this case the expression is not “explicit”, i.e., it is not given in terms of the one-dimensional Fourier transform $\hat{f}$. Below we give a proof of Theorem 4.1 that is different from the one in [GT13] (our method can also be used to establish the corresponding result for even dimensions $k$).

The assumption in Theorem 4.1 that $f$ is a Schwartz function is not essential for the conclusion. The result remains valid (with the same proof), e.g., for any $f$ such that $(1 + |t|)^{k-1} f(t)$ is in $L^1(\mathbb{R})$, a condition which ensures that $F_k$ is a function in $L^1(\mathbb{R}^k)$.

4.2. We now turn to the proof of Theorem 4.1. We begin with a simple claim about a recurrence relation satisfied by the Fourier transforms of the surface measures on the unit spheres in $\mathbb{R}^k$. 
Lemma 4.2: Let $\mu_k$ denote the surface measure on the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$. The Fourier transform $\hat{\mu}_k$ is a radial function, hence there is an even function $s_k$ satisfying $\hat{\mu}_k(\xi) = s_k(|\xi|)$, $\xi \in \mathbb{R}^k$. Then the recurrence relation

$$s_k(t) = (2\pi t^2)^{-1}((k-4)s_{k-2}(t) - 2\pi s_{k-4}(t)), \quad t \neq 0,$$

is valid for each (even or odd) integer $k \geq 5$.

This follows from the recurrence relation satisfied by the Bessel functions:

Proof of Lemma 4.2. It is well-known (see for instance [SW71, p. 154]) that the Fourier transform of the surface measure $\mu_k$ satisfies

$$\hat{\mu}_k(\xi) = 2\pi|\xi|^{-(k-2)/2} J_{(k-2)/2}(2\pi|\xi|),$$

where $J_\nu$ is the Bessel function of order $\nu$. It is also known that the Bessel functions satisfy the recurrence relation $2\nu z^{-1}J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$; see [Leb72, Section 5.3]. For $\nu = (k-4)/2$ this relation yields (4.2), as one can verify in a straightforward manner. ■

4.3. Equipped with Lemma 4.2, we can now give the proof of Theorem 4.1.

Proof of Theorem 4.1. Recall that given an even Schwartz function $f$ on $\mathbb{R}$ we define a radial function $F_k$ on $\mathbb{R}^k$ by $F_k(x) := f(|x|)$. We observe that $F_k$ is in the Schwartz space, by Lemma 3.2. Since the Fourier transform of a radial function is also radial, there is an even function $f_k$ such that $\hat{F}_k(\xi) = f_k(|\xi|)$, $\xi \in \mathbb{R}^k$. This allows us to define a linear operator $A_k$ on the space of even Schwartz functions on $\mathbb{R}$, given by $A_k f := f_k$.

Using $k$-dimensional spherical integration we have

$$\hat{F}_k(\xi) = \int_{\mathbb{R}^k} f(|x|) e^{-2\pi i \langle \xi, x \rangle} \, dx = \int_0^\infty f(r)\hat{\mu}_k(r\xi) r^{k-1} \, dr,$$

where $\mu_k$ is the surface measure on the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$. If we let $s_k$ be the function defined in Lemma 4.2, then this yields the formula

$$A_k f(t) = \int_0^\infty f(r)s_k(rt) r^{k-1} \, dr.$$

We note that this formula is well-known and can be found, e.g., in [SW71, p. 155].
Next, we define another family of operators $B_k$ ($k = 1, 3, 5, 7, \ldots$) on the space of even Schwartz functions on $\mathbb{R}$. We let
\[
B_1 f(t) := \hat{f}(t),
\]
while for $k = 3, 5, 7, \ldots$ we define
\[
(4.5) \quad B_k f(t) := -\frac{1}{2\pi t^{k-1}} \sum_{j=0}^{(k-3)/2} \beta_{j,k} t^{j+1} \hat{f}^{(j+1)}(t), \quad t \neq 0
\]
(we do not specify the value of $B_k f$ at the point $t = 0$). The assertion in Theorem 4.1 may thus be reformulated by saying that $A_k f(t) = B_k f(t)$, $t \neq 0$, for every even function $f$ in $\mathcal{S}(\mathbb{R})$. A straightforward calculation whose details are omitted shows that the operators $\{B_k\}$, $k = 1, 3, 5, 7, \ldots$, satisfy the recurrence relation
\[
(4.6) \quad B_k f(t) = \frac{k - 4}{2\pi t^2} B_{k-2} f(t) - \frac{1}{t^2} B_{k-4} (t^2 f(t)).
\]
In order to prove our claim, we will show that the sequence $\{A_k\}$, $k = 1, 3, 5, 7, \ldots$ satisfies the same recurrence relation as (4.6) with the same initial conditions, and the conclusion will thus follow by induction on $k$.

Indeed, using (4.4) together with the recurrence relation (4.2) we get
\[
A_k f(t) = \frac{k - 4}{2\pi t^2} \int_0^\infty f(r) r^{k-3} s_{k-2}(rt) \, dr - \frac{1}{t^2} \int_0^\infty (r^2 f(r)) r^{k-5} s_{k-4}(rt) \, dr,
\]
and thus we see that
\[
(4.7) \quad A_k f(t) = \frac{k - 4}{2\pi t^2} A_{k-2} f(t) - \frac{1}{t^2} A_{k-4} (t^2 f(t)).
\]

It remains to check the base cases $k = 1$ and $3$. If $k = 1$ then $F_1(x) = f(x)$ and so $A_1 f(t) = \hat{f}(t) = B_1 f(t)$. For $k = 3$ we have $s_3(t) = 2 \sin(2\pi t)/t$, and (4.4) implies that
\[
A_3 f(t) = \frac{2}{t} \int_0^\infty r f(r) \sin(2\pi rt) \, dr.
\]
But since $f$ is an even function, this yields
\[
A_3 f(t) = \frac{1}{t} \int_\mathbb{R} r f(r) \sin(2\pi rt) \, dr = -\frac{1}{2\pi t} \hat{f}'(t) = B_3 f(t),
\]
as required. This concludes the proof of Theorem 4.1. ■
Remark 4.3: Theorem 4.1 provides an expression for $\hat{F}_k(\xi)$ for all $\xi \in \mathbb{R}^k \setminus \{0\}$, that is, except for the value at $\xi = 0$. This value may be not easy to find directly from (4.1) by continuity. Fortunately, using a simple argument one can obtain also an explicit expression for $\hat{F}_k(0)$. Indeed, since $f$ is an even function, it follows from (4.3) that

$$\hat{F}_k(0) = \hat{\mu}_k(0) \int_0^{\infty} f(r)r^{k-1} dr = \frac{1}{2}\hat{\mu}_k(0) \int_{\mathbb{R}} f(r)r^{k-1} dr.$$

We observe that $\hat{\mu}_k(0)$ is the total surface area of the unit sphere in $\mathbb{R}^k$, which (since $k$ is odd) is known to be equal to $2(2\pi)^{(k-1)/2}/(k-2)!$. This yields the expression

$$\hat{F}_k(0) = -\frac{\alpha_k}{2\pi} \hat{f}^{(k-1)}(0),$$

where $\alpha_k$ is defined as in (2.4).

4.4. As an application of Theorem 4.1 we consider the Fourier transform of the surface measure on the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$. It is well-known that for odd dimensions $k$, the Fourier transform of this measure is expressible as a finite sum using powers, sines and cosines (this can be inferred, e.g., from the recurrence relation given in Lemma 4.2). Theorem 4.1 allows us to derive a closed-form expression for the Fourier transform of the surface measure (such formulas are in fact known; see, e.g., [KF49]).

**Theorem 4.4:** Let $\mu_k$ be the surface measure on the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$. If $k$ is an odd integer, $k \geq 3$, then the Fourier transform of the measure $\mu_k$ is

$$\hat{\mu}_k(\xi) = \frac{2}{|\xi|^{k-2}} \sum_{j=0}^{(k-3)/2} \beta_{j,k} (2\pi|\xi|)^j \sin(2\pi|\xi| + \frac{\pi}{2}j), \quad \xi \in \mathbb{R}^k \setminus \{0\},$$

where the coefficients $\beta_{j,k}$ are as in (2.4).

**Proof.** Choose a non-negative, infinitely smooth, compactly supported, even function $\varphi$ on $\mathbb{R}$, such that $\int_{\mathbb{R}} \varphi(t) dt = 1$. For each $\varepsilon > 0$ define the function

$$f_\varepsilon(t) := \varepsilon^{-1}(\varphi((t-1)/\varepsilon) + \varphi((t+1)/\varepsilon));$$

then $f_\varepsilon$ is an even function and $f_\varepsilon \to \delta_1 + \delta_{-1}$ as $\varepsilon \to 0$ in the sense of distributions. In particular it follows from (4.4) that $A_k f_\varepsilon(t)$ tends to $s_k(t)$ pointwise as $\varepsilon \to 0$, where $s_k$ is the even function on $\mathbb{R}$ which satisfies $\hat{\mu}_k(\xi) = s_k(|\xi|)$, $\xi \in \mathbb{R}^k$. 
Recall that in the proof of Theorem 4.1 we have shown that $A_k f(t) = B_k f(t)$, $t \neq 0$, for every even function $f$ in the Schwartz space, where $B_k f(t)$ is defined as in (4.5). We can therefore obtain the value of $s_k(t)$ for $t \neq 0$ as the limit of $B_k f(\varepsilon t)$ as $\varepsilon \to 0$.

First we observe that $\tilde{f}_\varepsilon^{(j+1)}(t) = 2 \tilde{f}_\varepsilon(\varepsilon t) \cos(2\pi t)$. Then by the Leibniz rule we have

$$\tilde{f}_\varepsilon^{(j+1)}(t) = 2 \tilde{f}_\varepsilon(\varepsilon t) \{\cos(2\pi t)\}^{(j+1)} + \cdots,$$

where the omitted terms tend to zero as $\varepsilon \to 0$. This implies that

$$\lim_{\varepsilon \to 0} \tilde{f}_\varepsilon^{(j+1)}(t) = 2\{\cos(2\pi t)\}^{(j+1)} = -2(2\pi)^{j+1} \sin(2\pi t + \frac{\pi}{2} j).$$

Finally, combining (4.5) and (4.10) yields

$$s_k(t) = \lim_{\varepsilon \to 0} B_k f(\varepsilon t) = \frac{2}{t^{k-2}} \sum_{j=0}^{(k-3)/2} \beta_{j,k} (2\pi t)^j \sin(2\pi t + \frac{\pi}{2} j), \quad t \neq 0,$$

which is equivalent to the assertion in (4.9).

Remark 4.5: We could have proved Theorem 4.1 and Theorem 4.4 also the other way around, that is, at the first stage we could have proved Theorem 4.4 by induction on $k$, showing that (4.9) is valid using the recurrence relation (4.2) and checking the base cases $k = 3$ and 5, while at the second stage we could have derived Theorem 4.1 based on (4.3) and (4.9). We have opted for giving a direct proof of Theorem 4.1, since it is the result that will be used in the next section.

Remark 4.6: The formula (4.9) for the Fourier transform of the surface measure $\mu_k$ is closely related to the Bessel polynomials $\theta_n(z)$. This is a sequence of polynomials satisfying the recurrence relation

$$\theta_n(z) = (2n-1)\theta_{n-1}(z) + z^2\theta_{n-2}(z)$$

with the initial conditions $\theta_0(z) = 1$ and $\theta_1(z) = z + 1$. The $n$’th polynomial in the sequence can be written explicitly as

$$\theta_n(z) = (2\pi)^n \sum_{j=0}^n \beta_{j,k}(-z)^j,$$
where \( k = 2n + 3 \) and where the coefficients \( \beta_{j,k} \) are the same as in (2.4). We can therefore rewrite (4.9) in the form

\[
\hat{\mu}_k(\xi) = \frac{2}{|\xi|^{k-2}} \text{Im} \left\{ \frac{\theta_n(-2\pi i|\xi|)}{(2\pi)^n} \exp(2\pi i|\xi|) \right\}, \quad n = \frac{k - 3}{2},
\]

which clarifies the relation between the Bessel polynomials and the Fourier transform of the surface measure \( \mu_k \) for odd dimensions \( k \). We refer to [Gro78] for a comprehensive survey on the Bessel polynomials (see also [KF49]).

5. New Poisson type summation formulas

In this section we prove the results stated in Section 2. We also state and prove Theorem 5.1 that generalizes Theorem 2.3 to the case where the origin is allowed to belong to the supports \( \Lambda, S \) of the measures \( \mu \) and \( \hat{\mu} \).

5.1. First we will prove Theorem 2.3 which generalizes [Mey16, Theorem 7] to all odd values of \( k \) greater than 3. Meyer’s proof in the case \( k = 3 \) was based on the following observation: Let a radial function \( G \) on \( \mathbb{R}^3 \) be defined by \( G(x) := f(|x|)/|x| \), where \( f \) is an odd Schwartz function on \( \mathbb{R} \). Then the three-dimensional Fourier transform of the function \( G \) is given by \( \hat{G}(\xi) = i\hat{f}(|\xi|)/|\xi| \).

The proof of Theorem 2.3 for an arbitrary odd integer \( k \geq 3 \) will follow a similar line, where in this case we will use the connection given in Theorem 4.1 between the one-dimensional Fourier transform of an even Schwartz function on \( \mathbb{R} \) and the \( k \)-dimensional Fourier transform of a radial function on \( \mathbb{R}^k \).

**Proof of Theorem 2.3.** First we show that the distribution \( \sigma \) defined by (2.6) is indeed a temperate distribution. If \( f \) is a function in the Schwartz space \( S(\mathbb{R}) \), then by Lemma 3.1 we have \( f(t) - f(-t) = tg(t) \) where \( g \) is also in \( S(\mathbb{R}) \). Then \( g \) is an even function and so by Lemma 3.2 the function \( G(x) := g(|x|), x \in \mathbb{R}^k \), is in the Schwartz space \( S(\mathbb{R}^k) \). Moreover, the mapping which takes \( f \) to \( G \) is a continuous linear mapping from the space \( S(\mathbb{R}) \) into \( S(\mathbb{R}^k) \). Observe that if \( f \) is compactly supported then we have

\[
\langle \sigma, f \rangle = \sum_{\lambda \in \Lambda} \frac{a(\lambda)}{|\lambda|} (f(|\lambda|) - f(-|\lambda|)) = \sum_{\lambda \in \Lambda} a(\lambda)G(\lambda) = \langle \mu, G \rangle.
\]

Since the measure \( \mu \) is assumed to be a temperate distribution, the mapping \( f \mapsto \langle \mu, G \rangle \) thus defines a continuous linear functional on the Schwartz space \( S(\mathbb{R}) \) which agrees with \( \sigma \) on the space \( D(\mathbb{R}) \) of infinitely smooth, compactly supported functions. This means that \( \sigma \) is a temperate distribution.
Next we show that the Fourier transform of $\sigma$ is given by (2.7). Since $\sigma$ is an odd distribution, the Fourier transform of $\hat{\sigma}$ is $-\sigma$. Hence (2.7) is equivalent to saying that

\begin{equation}
\langle \sigma, f \rangle = i \sum_{s \in S} \frac{b(s)}{|s|^{k-2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} |s|^j (\hat{f}^{(j)}(|s|) - (-1)^j \hat{f}^{(j)}(-|s|)) \right]
\end{equation}

for every Schwartz function $f$ such that $\hat{f}$ is compactly supported. If we keep using $g$ and $G$ to denote the two functions related to $f$ as above, then by Theorem 4.1 we have

\begin{equation}
\hat{G}(\xi) = -\frac{1}{2\pi |\xi|^{k-1}} \sum_{j=0}^{(k-3)/2} \beta_{j,k} |\xi|^{j+1} \hat{g}^{(j+1)}(|\xi|), \quad \xi \neq 0,
\end{equation}

while the property $tg(t) = f(t) - f(-t)$ implies that

\begin{equation}
\hat{g}^{(j+1)}(t) = -2\pi i (\hat{f}^{(j)}(t) - (-1)^j \hat{f}^{(j)}(-t)).
\end{equation}

It now follows using (5.3) and (5.4) that the right-hand side of (5.2) is equal to

\begin{equation}
\sum_{s \in S} b(s) \hat{G}(s) = \langle \hat{\mu}, \hat{G} \rangle = \langle \mu, G \rangle = \langle \sigma, f \rangle
\end{equation}

as we had to show. (We note that in the second equality in (5.5) we used the fact that the Fourier transform of $\hat{G}$ is $G$, which follows from the property $G(-x) = G(x)$.)

5.2. Corollary 2.4 now follows easily as a special case of Theorem 2.3.

**Proof of Corollary 2.4.** We apply Theorem 2.3 to the measure

$$
\mu_{(\eta, \xi)} = \sum_{m \in \mathbb{Z}^k} e^{2\pi i (m, \xi)} \delta_{m+\eta}
$$

where the vectors $\eta$ and $\xi$ are in $\mathbb{R}^k \setminus \mathbb{Z}^k$. By Poisson’s summation formula, the Fourier transform of $\mu_{(\eta, \xi)}$ is the measure $\hat{\mu}_{(\eta, \xi)} = e^{-2\pi i (\eta, \xi)} \mu_{(\xi, -\eta)}$. Since neither $\eta$ nor $\xi$ belongs to $\mathbb{Z}^k$, the origin lies in neither the support of the measure $\mu_{(\eta, \xi)}$ nor the support of its Fourier transform. Hence all the assumptions in Theorem 2.3 are satisfied and the assertion in Corollary 2.4 follows. ■
5.3. Now we give a more general version of Theorem 2.3, in which the origin is allowed to lie in the support $\Lambda$ of the measure $\mu$ or the support $S$ of its Fourier transform $\hat{\mu}$ (or both). The result can be stated as follows:

**Theorem 5.1:** Let $\mu$ be a measure satisfying the same assumptions as in Theorem 2.3 except that $\Lambda$ and $S$ may contain the origin. Then

\[
\sigma := -2a(0)\delta'_0 + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{a(\lambda)}{|\lambda|} (\delta|\lambda| - \delta_{-|\lambda|})
\]

is a temperate distribution on $\mathbb{R}$ whose one-dimensional Fourier transform is

\[
\hat{\sigma} = 2ib(0)\alpha_k\delta_0^{(k-2)} - i \sum_{s \in S \setminus \{0\}} \frac{b(s)}{|s|^{k-2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} |s|^j ((-1)^j \delta_{|s|}^{(j)} - \delta_{-|s|}^{(j)}) \right],
\]

where the coefficients $\alpha_k$ and $\beta_{j,k}$ are defined as in (2.4).

If the support $\Lambda$ of the measure $\mu$ does not contain the origin, then we understand the coefficient $a(0)$ in (5.6) to be zero; and similarly, if the origin does not belong to the support $S$ of the measure $\hat{\mu}$, then the coefficient $b(0)$ in (5.7) is zero. If the origin lies in neither $\Lambda$ nor $S$, the assertion in Theorem 5.1 reduces to that of Theorem 2.3.

**Proof of Theorem 5.1.** We keep using the same notation as in the proof of Theorem 2.3. We recall that if $f$ is a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ then there is $g \in \mathcal{S}(\mathbb{R})$ such that $f(t) - f(-t) = tg(t)$. Then by the continuity of $g$ we have

\[
g(0) = 2f'(0),
\]

and the equality $\langle \sigma, f \rangle = \langle \mu, G \rangle$ used in (5.1) remains valid also in the case when $0 \in \Lambda$.

To obtain the Fourier transform of $\sigma$ we argue as before. If $f$ is a Schwartz function such that $\hat{f}$ is compactly supported, then

\[
\langle \sigma, f \rangle = \langle \mu, G \rangle = \langle \hat{\mu}, \hat{G} \rangle = b(0)\hat{G}(0) + \sum_{s \neq 0} b(s)\hat{G}(s).
\]

If we apply Remark 4.3 to the function $G$ and use (5.4), then we obtain

\[
\hat{G}(0) = -\frac{\alpha_k}{2\pi} \hat{g}^{(k-1)}(0) = 2i\alpha_k \hat{f}^{(k-2)}(0).
\]
It thus follows from (5.8) that

\[
\langle \sigma, f \rangle = 2ib(0)\alpha_k\hat{f}^{(k-2)}(0) + i\sum_{s \neq 0} \frac{b(s)}{|s|^{k-2}} \left[ \cdots \right],
\]

where the omitted terms in the square brackets are as in (5.2). This confirms (5.7).

5.4. Finally we prove Theorem 2.2 that extends Guinand’s formula (1.2) to all odd values of \( k \) greater than 3. Guinand’s formula is equivalent to the assertion that \( \hat{\sigma}_3 = -i\sigma_3 \), where \( \sigma_k \) is the distribution defined in (2.1) that involves the sum-of-\( k \)-squares function \( r_k(n) \). Theorem 2.2 is a consequence of the next result which provides an expression for the Fourier transform \( \hat{\sigma}_k \) for all odd values of \( k \) greater than 3.

**Theorem 5.2:** Let \( k \) be an odd integer, \( k \geq 3 \). Then the Fourier transform of the temperate distribution

\[
\sigma_k := -2\delta_0' + \sum_{n=1}^{\infty} \frac{r_k(n)}{\sqrt{n}} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})
\]

is

\[
\hat{\sigma}_k = 2i\alpha_k\delta_0^{(k-2)} - i\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{(k-2)/2}} \left[ \sum_{j=0}^{(k-3)/2} \beta_{j,k} n^{j/2} (-1)^{j} \delta_{\frac{j}{\sqrt{n}}} - \delta_{-\frac{j}{\sqrt{n}}} \right].
\]

**Proof.** We apply Theorem 5.1 to the measure

\[
\mu = \sum_{m \in \mathbb{Z}^k} \delta_m
\]

which satisfies \( \hat{\mu} = \mu \).

6. Remark

It is natural to ask what can be said about the Fourier transform of the distribution \( \sigma_k \) defined in (2.1) when \( k \) is a positive even integer (i.e., \( k = 2, 4, 6, 8, \ldots \)). We expect that for these values of \( k \) the Fourier transform \( \hat{\sigma}_k \) does not have a discrete support. It would be interesting to have a proof of this claim, but we do not address this problem in the present work.
References

[Fav18] S. Yu. Favorov, *Tempered distributions with discrete support and spectrum*, Bulletin of the Hellenic Mathematical Society 62 (2018), 66–79.

[GT13] L. Grafakos and G. Teschl, *On Fourier transforms of radial functions and distributions*, Journal Fourier Analysis and Applications 19 (2013), 167–179.

[Gro78] E. Grosswald, *Bessel Polynomials*, Lecture Notes in Mathematics, Vol. 698, Springer, Berlin, 1978.

[Gui59] A. P. Guinand, *Concordance and the harmonic analysis of sequences*, Acta Mathematica 101 (1959), 235–271.

[Kol16] M. N. Kolountzakis, *Fourier pairs of discrete support with little structure*, Journal of Fourier Analysis and Applications 22 (2016), 1–5.

[KF49] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Transactions of the American Mathematical Society 65 (1949), 100–115.

[Lag00] J. C. Lagarias, *Mathematical quasicrystals and the problem of diffraction*, in Directions in Mathematical Quasicrystals, CRM Monograph Series, Vol. 13, American Mathematical Society, Providence, RI, 2000, pp. 61–93.

[Leb72] N. N. Lebedev, *Special Functions and their Applications*, Dover Publications, New York, 1972.

[LO13] N. Lev and A. Olevskii, *Measures with uniformly discrete support and spectrum*, Comptes Rendus Mathématiques. Académie des Sciences. Paris 351 (2013), 599–603.

[LO15] N. Lev and A. Olevskii, *Quasicrystals and Poisson’s summation formula*, Inventiones Mathematicae 200 (2015), 585–606.

[LO16] N. Lev and A. Olevskii, *Quasicrystals with discrete support and spectrum*, Revista Matemática Iberoamericana 32 (2016), 1341–1352.

[LO17] N. Lev and A. Olevskii, *Fourier quasicrystals and discreteness of the diffraction spectrum*, Advances in Mathematics 315 (2017), 1–26.

[Mey95] Y. Meyer, *Quasicrystals, Diophantine approximation and algebraic numbers*, in Beyond Quasicrystals (Les Houches, 1994), Springer, Berlin, 1995, pp. 3–16.

[Mey16] Y. Meyer, *Measures with locally finite support and spectrum*, Proceedings of the National Academy of Sciences of the United States of America 113 (2016), 3152–3158.

[Mey17] Y. Meyer, *Measures with locally finite support and spectrum*, Revista Matemática Iberoamericana 33 (2017), 1025–1036.

[RV19] D. Radchenko and M. Viazovska, *Fourier interpolation on the real line*, Publications Mathématiques. l’Institut de Hautes Études Scientifiques 129 (2019), 51–81.

[Rud91] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1991.

[SW71] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, Vol. 32, Princeton University Press, Princeton, NJ, 1971.

[Via18] M. Viazovska, *Sharp sphere packings*, in Proceedings of the International Congress of Mathematicians–Rio de Janeiro 2018. Vol. II, World Scientific, Hackensack, NJ, 2018, pp. 455–466.