THE ARITHMETIC OF A TWIST OF THE FERMAT QUARTIC

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Abstract. We study the arithmetic of the twist of the Fermat quartic defined by
\[ X^4 + Y^4 + Z^4 = 0 \] which has no \( \mathbb{Q} \)-rational point. We calculate the Mordell–Weil group of the Jacobian variety explicitly. We show that the degree 0 part of the Picard group is a free \( \mathbb{Z}/2\mathbb{Z} \)-module of rank 2, whereas the Mordell–Weil group is a free \( \mathbb{Z}/2\mathbb{Z} \)-module of rank 3. Thus the relative Brauer group is non-trivial. We also show that this quartic violates the local-global property for linear determinantal representations.

1. Introduction

The aim of this paper is to study the arithmetic of the smooth plane quartic
\[ C_4 := \{ [X : Y : Z] \in \mathbb{P}^2 \mid X^4 + Y^4 + Z^4 = 0 \}. \]
It is a twist of the Fermat quartic. The quartic \( C_4 \) has no \( \mathbb{Q} \)-rational point because it has no \( \mathbb{R} \)-rational point. We shall explicitly calculate the Picard group \( \text{Pic}(C_4) \) and the Mordell–Weil group of the Jacobian variety \( \text{Jac}(C_4) \). These results rely on our previous calculation of the Mordell–Weil group of the Jacobian variety of the Fermat quartic over \( \mathbb{Q}(\zeta_8) \) summarized in [13].

Define the divisors \( D_i \) (\( 0 \leq i \leq 3 \)) on the quartic \( C_4 \) by
\[
D_0 := [1 : \zeta_3 : \zeta_3^2] + [1 : \zeta_3^2 : \zeta_3] \quad D_1 := [1 : \zeta_3 : \zeta_3^2] + [1 : -\zeta_3 : \zeta_3]
\]
\[
D_2 := [1 : \zeta_3 : -\zeta_3^2] + [1 : \zeta_3^2 : -\zeta_3] \quad D_3 := [1 : -\zeta_3 : -\zeta_3^2] + [1 : -\zeta_3^2 : -\zeta_3]
\]
Here we put \( \zeta_n := \exp(2\pi \sqrt{-1}/n) \). The divisors \( D_0, D_1, D_2, D_3 \) are defined over \( \mathbb{Q} \), and twice of them are divisors cut out by bitangents defined over \( \mathbb{Q} \); see Section 2 for details. The divisor classes of the differences \([D_i - D_0]\) (\( 1 \leq i \leq 3 \)) are elements of \( \text{Pic}^0(C_4) \) killed by 2. They satisfy the following relation; see Lemma 2.1 (3).
\[
[D_1 - D_0] + [D_2 - D_0] = [D_3 - D_0].
\]
We also consider the following divisor defined over \( \mathbb{Q}(\zeta_8) \)
\[
E := 2[1 : 0 : \zeta_8^2] - 2[1 : 0 : \zeta_8^4].
\]
We shall show that \( E \) is not linearly equivalent to a divisor defined over \( \mathbb{Q} \), but the divisor class \([E]\) is invariant under the action of \( \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \). Thus it gives a \( \mathbb{Q} \)-rational point of Jacobian variety \( \text{Jac}(C_4) \).

The following theorem is the main theorem of this paper.

Theorem 1.1. (1) The degree 0 part \( \text{Pic}^0(C_4) \) of the Picard group is a free \( \mathbb{Z}/2\mathbb{Z} \)-module of rank 2 generated by \([D_1 - D_0]\) and \([D_2 - D_0]\):
\[
\text{Pic}^0(C_4) = (\mathbb{Z}/2\mathbb{Z})[D_1 - D_0] \oplus (\mathbb{Z}/2\mathbb{Z})[D_2 - D_0].
\]
(2) The Mordell–Weil group $\text{Jac}(C_4)(\mathbb{Q})$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank 3 generated by $[D_1 - D_0]$, $[D_2 - D_0]$, and $[E]$: 
$$\text{Jac}(C_4)(\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})[D_1 - D_0] \oplus (\mathbb{Z}/2\mathbb{Z})[D_2 - D_0] \oplus (\mathbb{Z}/2\mathbb{Z})[E].$$

Let $\text{Pic}_{C_4/\mathbb{Q}}$ be the Picard scheme of $C_4$ over $\mathbb{Q}$ which represents the relative Picard functor; see [3, Section 8.2]. The degree $0$ part $\text{Pic}_{C_4/\mathbb{Q}}^0$ is the Jacobian variety $\text{Jac}(C_4)$, and the degree $d$ part $\text{Pic}_{C_4/\mathbb{Q}}^d$ is a torsor under $\text{Jac}(C_4)$ for any $d \in \mathbb{Z}$. Since $C_4$ has a divisor of degree 2 over $\mathbb{Q}$ (such as $D_i$ for $0 \leq i \leq 3$), the scheme $\text{Pic}_{C_4/\mathbb{Q}}^{2d}$ has a $\mathbb{Q}$-rational point, and it is a trivial torsor under $\text{Jac}(C_4)$.

We shall show that $\text{Pic}_{C_4/\mathbb{Q}}^{2d+1}$ is a non-trivial torsor under $\text{Jac}(C_4)$ for any $d \in \mathbb{Z}$.

**Theorem 1.2.** Let $d \in \mathbb{Z}$ be an integer. The scheme $\text{Pic}_{C_4/\mathbb{Q}}^{2d+1}$ has no $\mathbb{Q}$-rational point. In other words, $\text{Pic}_{C_4/\mathbb{Q}}^{2d+1}$ is a non-trivial torsor under $\text{Jac}(C_4)$.

**Remark 1.3.** We have the following exact sequence

$$0 \longrightarrow \text{Pic}^0(C_4) \xrightarrow{\iota_{C_4}} \text{Jac}(C_4)(\mathbb{Q}) \xrightarrow{\text{Br}} \text{Br}(C_4/\mathbb{Q}).$$

Here
$$\text{Br}(C_4/\mathbb{Q}) := \text{Ker}(\text{Br}(\mathbb{Q}) \to \text{Br}(C_4))$$

is the relative Brauer group; see [4, Theorem 2.1], [3, Section 8.1, Proposition 4]. Theorem 1.2 shows that $\text{Br}([E])$ is non-trivial in $\text{Br}(C_4/\mathbb{Q})$. Thus $\text{Br}(C_4/\mathbb{Q})$ is non-trivial, and $\iota_{C_4}$ is not surjective. This is a peculiar phenomenon for curves without rational points because, if a curve has a rational point, then the relative Brauer group vanishes. After this work was completed, Brendan Creutz kindly informed that he constructed plane quartics over $\mathbb{Q}$ for which the degree 0 part of the Picard group is strictly smaller than the Mordell–Weil group of the Jacobian variety; see [5, Theorem 6.3, Remark 6.4] for details.

We shall give an application to quadratic points. Once $\text{Pic}^2(C_4)$ is calculated, we can determine all of the rational points on $C_4$ defined over quadratic extensions of $\mathbb{Q}$ by Faddeev’s methods [8]. It turns out that all of them are defined over $\mathbb{Q}(\sqrt{-3})$ and they are the points of tangency of bitangents defined over $\mathbb{Q}$. (The following result also follows from the results in [13, Section 7].)

**Theorem 1.4.** There are exactly 8 rational points on $C_4$ which are defined over quadratic extensions of $\mathbb{Q}$. These are

$$[1 : \pm \zeta_3 : \pm \zeta_3^2] \quad \text{and} \quad [1 : \pm \zeta_3^2 : \pm \zeta_3].$$

Finally, we shall give an application to the linear determinantal representations of the quartic $C_4$. Recall that a *linear determinantal representation* of a quartic $C \subset \mathbb{P}^2$ over a field $K$ is a triple of 4 by 4 matrices $M_0, M_1, M_2 \in \text{Mat}_4(K)$ such that
$$\det(XM_0 + YM_1 + ZM_2) = 0$$

is a defining equation of the curve $C$. The problem to find linear determinantal representations of a given plane curve is motivated by Arithmetic Invariant Theory of Bhargava–Gross [2]. In [4], the first author studied the arithmetic properties of linear determinantal representations of smooth cubics. In [12], the authors of the current paper determined all of the linear determinantal representations of the Fermat quartic and the Klein quartic.
As an application of our explicit calculation of Pic²(C₄), we shall show that C₄ violates the local-global property of linear determinantal representations.

**Theorem 1.5.**  
(1) The quartic C₄ does not admit a linear determinantal representation over Q.
(2) For K = ℝ or a p-adic field Qₚ, the quartic C₄ admits a linear determinantal representation over K.

Here is a brief sketch of the proof of our results. In [13], based on the results of Faddeev and Rohrlich, we explicitly calculated the Mordell–Weil group of the Fermat quartic F₄ ⊂ ℙ² defined by X⁴ + Y⁴ = Z⁴. Since the quartic C₄ is isomorphic to the Fermat quartic F₄ over Q(ζ₈), we have

\[ \text{Jac}(C₄)(Q(ζ₈)) \cong \text{Jac}(F₄)(Q(ζ₈)) \cong (Z/4Z)⊕² ⊕ Z/2Z. \]

Calculating the twist of Galois action explicitly, we calculate Jac(C₄)(Q) and Pic⁰(C₄). Since Pic⁰(C₄) ≅ Pic²(C₄), we determine every Q-rational divisor of degree 2 on C₄. Theorem 1.4 is obtained by Faddeev’s methods. Theorem 1.2 and Theorem 1.5 are consequences of our explicit calculation.

In the appendix, we give a sample source code for Singular (version 4.2.1) which confirms the formulae on divisors used in this paper.

## 2. Bitangents and their points of tangency

Recall that a line L ⊂ ℙ² is called a bitangent of C₄ if the intersection multiplicity at every P ∈ C₄ ∩ L is divisible by 2. The divisor on C₄ cut out by L is divisible by 2. Since C₄ is a smooth quartic, it has 28 bitangents over Q. The defining equations of them are well-known, and can be found in [14, p. 14].

Among the 28 bitangents, the following four bitangents are defined over Q:

\[
L₀ : X + Y + Z = 0, \\
L₁ : X - Y + Z = 0, \\
L₂ : X + Y - Z = 0, \\
L₃ : X - Y - Z = 0.
\]

They give divisors of degree 2 over Q:

\[
D₀ : [1 : 3 : 3^2] + [1 : 5^2 : 3] \\
D₁ : [1 : -3 : 3^2] + [1 : -5^2 : 3] \\
D₂ : [1 : 3 : -5^2] + [1 : 5^2 : -3] \\
D₃ : [1 : -3 : -5^2] + [1 : -5^2 : -3]
\]

**Lemma 2.1.**  
(1) For any i, j with 0 ≤ i < j ≤ 3, the divisor Dᵢ is not linearly equivalent to Dⱼ. In particular, Pic²(C₄) has at least 4 elements.
(2) For any i, j with 0 ≤ i < j ≤ 3, the divisor class [Dᵢ − Dⱼ] is killed by 2 in Pic⁰(C₄).
(3) We have an injective homomorphism

\[(Z/2Z)⊕² \hookrightarrow \text{Pic}⁰(C₄), \quad (c₁, c₂) ↦ c₁[D₁ - D₀] + c₂[D₂ - D₀].\]

Moreover, we have [D₁ − D₀] + [D₂ − D₀] = [D₃ − D₀] in Pic⁰(C₄).

**Proof.**  
(1) Assume that Dᵢ is linearly equivalent to Dⱼ. Then there is a non-zero rational function f on C₄ with \(\text{div}(f) = Dᵢ - Dⱼ\). This implies the morphism \(C₄ \rightarrow \mathbb{P}^1\) induced by f has degree 1, which is absurd because C₄ has genus 3.

(2) By definition of bitangents, the divisors on C₄ cut out by Lᵢ, Lⱼ are 2Dᵢ, 2Dⱼ. Thus 2Dᵢ and 2Dⱼ are linearly equivalent to each other because both are hyperplane sections. Therefore, we have [2Dᵢ − 2Dⱼ] = 0 in Pic⁰(C₄).
(3) Since
\[ \text{div}(X^2 + Y^2 + Z^2) = D_0 + D_1 + D_2 + D_3, \]
\[ \text{div}(X + Y + Z) = 2D_0, \]
we have
\[ D_1 + D_2 + D_3 - 3D_0 = \text{div}\left( \frac{X^2 + Y^2 + Z^2}{(X + Y + Z)^2} \right). \]
Hence we have \([D_1 - D_0] + [D_2 - D_0] = [D_3 - D_0] \) in Pic\(^0\)(\(C_4\)). The injectivity follows from (1).

\[ \square \]

3. Divisors on the Fermat Quartic

Let \(F_4 \subset \mathbb{P}^2\) be the Fermat quartic over \(\mathbb{Q}\) defined by \(X^4 + Y^4 = Z^4\). Rohrlich calculated the subgroup of Jac\((F_4)(\mathbb{Q}(\zeta_8))\) generated by the differences of the cusps; see [17] p. 117, Corollary 1. (This subgroup is denoted by \(\mathcal{D}^{\infty}/\mathcal{D}^\infty\) in [17].) In [13], with the aid of computers, we proved there are no other elements in Jac\((F_4)(\mathbb{Q}(\zeta_8))\).

Thus, we have an isomorphism
\[ \text{Jac}(F_4)(\mathbb{Q}(\zeta_8)) \cong (\mathbb{Z}/4\mathbb{Z})^{\oplus 5} \oplus (\mathbb{Z}/2\mathbb{Z}). \]
Explicit generators are given in [13, Theorem 6.5].

The quartic \(C_4\) is isomorphic to the Fermat quartic \(F_4\) over \(\mathbb{Q}(\zeta_8)\). We fix an isomorphism between \(C_4\) and \(F_4\) over \(\mathbb{Q}(\zeta_8)\) as follows:
\[ \rho: C_4 \xrightarrow{\cong} F_4, \quad [X : Y : Z] \mapsto [X : Y : \zeta_8Z]. \]

Via the above isomorphism \(\rho\), we shall translate the results in [13] into the results on divisor classes on \(C_4\). We define \(A_i, B_i, C_i \in C_4\) \((0 \leq i \leq 3)\) by
\[ A_i := [0 : \zeta_4^i : \zeta_8^4], \quad B_i := [\zeta_4^i : 0 : \zeta_8^4], \quad C_i := [\zeta_8\zeta_4^i : 1 : 0]. \]
There points correspond to the cusps on \(F_4\) via the isomorphism \(\rho\). Then we define
\[ \alpha_i := [A_i - B_0], \quad \beta_i := [B_i - B_0], \quad \gamma_i := [C_i - B_0]. \]
Here the linear equivalence class of a divisor \(D\) is denoted by \([D]\). Moreover, we define
\[ e_1 := \alpha_1, \quad e_2 := \alpha_2, \quad e_3 := \beta_1, \]
\[ e_4 := \beta_2, \quad e_5 := \gamma_1, \quad e_6 := \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2. \]
Note that we use the same notation as in [13, Section 2]. The points \(A_i, B_i, C_i\) \((0 \leq i \leq 3)\) and the divisors \(\alpha_i, \beta_i, \gamma_i\) \((0 \leq i \leq 3)\), \(e_i\) \((1 \leq i \leq 6)\) correspond to the points and the divisors denoted by the same letters in [13, Section 2].

Translating [13, Theorem 6.5] via \(\rho\), we have an isomorphism
\[ (\mathbb{Z}/4\mathbb{Z})^{\oplus 5} \oplus (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} \text{Jac}(C_4)(\mathbb{Q}(\zeta_8)), \quad (c_1, c_2, c_3, c_4, c_5, c_6) \leftrightarrow \sum_{i=1}^{6} c_ie_i \]
Remark 3.1. It can be checked by Singular that we have the following relations between divisors.

\[
\begin{align*}
\alpha_0 &= 2e_1 + e_2 + 2e_3 + e_4, & \alpha_1 &= e_1, \\
\alpha_2 &= e_2, & \alpha_3 &= e_1 + 2e_2 + 2e_3 + 3e_4, \\
\beta_0 &= 0, & \beta_1 &= e_3, \\
\beta_2 &= e_4, & \beta_3 &= 3e_3 + 3e_4, \\
\gamma_0 &= 3e_1 + 3e_2 + e_3 + e_5 + e_6, & \gamma_1 &= e_5, \\
\gamma_2 &= 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + e_6, & \gamma_3 &= 2e_1 + 2e_2 + e_4 + 3e_5.
\end{align*}
\]

**4. Calculation of the Galois Action**

Here we shall calculate the action of Galois group on \(\text{Jac}(C_4)(\mathbb{Q}(\zeta_8))\). Since

\[
\text{Jac}(C_4)(\mathbb{Q}) = \text{Jac}(C_4)(\mathbb{Q}(\zeta_8))^{\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})},
\]

we can calculate \(\text{Jac}(C_4)(\mathbb{Q})\) explicitly and prove Theorem 1.1 (1).

The Galois group \(\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times\) is generated by \(\sigma_3, \sigma_5\) defined by

\[
\sigma_3 : \zeta_8 \mapsto \zeta_8^3, \quad \sigma_5 : \zeta_8 \mapsto \zeta_8^5.
\]

**Proposition 4.1.** The Mordell–Weil group \(\text{Jac}(C_4)(\mathbb{Q})\) is a free \(\mathbb{Z}/2\mathbb{Z}\)-module of rank 3 generated by \([D_1 - D_0]\), \([D_2 - D_0]\), \([E]\).

**Proof:** The actions of \(\sigma_3, \sigma_5\) on \(A_i, B_i, C_i\) \((0 \leq i \leq 3)\) are summarized in the following table:

| \(\sigma_3\) | \(A_0\) | \(A_1\) | \(A_2\) | \(A_3\) | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(C_0\) | \(C_1\) | \(C_2\) | \(C_3\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(\sigma_3\) | \(A_1\) | \(A_0\) | \(A_3\) | \(A_2\) | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(C_0\) | \(C_1\) | \(C_2\) | \(C_3\) |
| \(\sigma_5\) | \(A_2\) | \(A_3\) | \(A_0\) | \(A_1\) | \(B_0\) | \(B_1\) | \(B_3\) | \(B_2\) | \(C_0\) | \(C_3\) | \(C_0\) | \(C_1\) |
Using the formulae in Section 3 (see also [13, Section 3]), the actions of $\sigma_3, \sigma_5$ on the basis $e_i$ ($1 \leq i \leq 6$) of $\text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$ are calculated as follows:

|   | $e_1$     | $e_2$     | $e_3$     | $e_4$     |
|---|-----------|-----------|-----------|-----------|
| $\sigma_3$ | $2e_1 + e_2 + e_3 + e_4$ | $e_1 + 2e_2 + e_3 + 3e_4$ | $3e_3$ | $2e_3 + 3e_4$ |
| $\sigma_5$ | $e_1 + 2e_2 + 2e_3 + 2e_4$ | $2e_1 + e_2 + 2e_3$ | $3e_3 + 2e_4$ | $3e_4$ |

|   | $e_5$     | $e_6$     |
|---|-----------|-----------|
| $\sigma_3$ | $3e_1 + 3e_2 + e_5 + e_6$ | $2e_3 + e_6$ |
| $\sigma_5$ | $2e_1 + 2e_2 + 3e_5$ | $2e_4 + e_6$ |

In other words, we have the following matrix representation of $\sigma_3, \sigma_5$ on $\text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$:

\[
s_3 = \begin{pmatrix}
2 & 1 & 0 & 0 & 3 & 0 \\
1 & 2 & 0 & 0 & 3 & 0 \\
1 & 1 & 3 & 2 & 0 & 2 \\
1 & 3 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad
s_5 = \begin{pmatrix}
1 & 2 & 0 & 0 & 2 & 0 \\
2 & 1 & 0 & 0 & 2 & 0 \\
2 & 2 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The $\mathbb{Q}$-rational points of $\text{Jac}(C_4)$ are the fixed points by $\sigma_3, \sigma_5$. We have to study the kernel of $s_3 - 1$ and $s_5 - 1$. It is straightforward to check that the kernel is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank 3 generated by $2e_1 + 2e_2, 2e_3, 2e_4$. Thus, the assertion follows from Remark 3.[1]

Next, we shall calculate the action of Galois group on $\text{Pic}^1(C_4)(\mathbb{Q}(\zeta_8))$. Although $\text{Pic}^1(C_4)(\mathbb{Q}(\zeta_8))$ is isomorphic to $\text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$, the Galois actions are different.

**Proposition 4.2.**  
(1) Each automorphism $\sigma_3, \sigma_5, \sigma_3\sigma_5 \in \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ does not have a fixed point in $\text{Pic}^1_{C_4/\mathbb{Q}}(\mathbb{Q}(\zeta_8))$.

(2) For $K = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$, the scheme $\text{Pic}^1_{C_4/\mathbb{Q}}$ has no $K$-rational point.

**Proof.** (1) By the matrix representations $s_3, s_5$ of the automorphisms $\sigma_3, \sigma_5 \in \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ in the proof of Proposition [4.1], we see that

(4.1)

$$(\sigma_5 - 1) \text{Jac}(C_4)(\mathbb{Q}(\zeta_8)) = 2 \text{Jac}(C_4)(\mathbb{Q}(\zeta_8)).$$

Moreover, if $a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6$ ($a_i \in \mathbb{Z}$) is in $(\sigma_3 - 1) \text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$, then we have

(4.2)

$$a_2 + a_3 + a_6 \equiv 0 \pmod{2}.$$ 

Since $s_3s_5 - 1 = s_3(s_3 - 1) + s_3 - 1 \equiv s_3 - 1 \pmod{2}$, we have the same congruence on $(\sigma_3\sigma_5 - 1) \text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$.

Since $A_0 = [0 : 1 : \zeta_8^5]$ is a $\mathbb{Q}(\zeta_8)$-rational point, we have the following bijection over $\mathbb{Q}(\zeta_8)$

$$\text{Jac}(C_4)(\mathbb{Q}(\zeta_8)) \overset{\sim}{\rightarrow} \text{Pic}^1_{C_4/\mathbb{Q}}(\mathbb{Q}(\zeta_8)), \quad [D] \mapsto [D + A_0].$$

If a divisor class $[D + A_0]$ is fixed by $\sigma_5$, we have $\sigma_5([D + A_0]) = [D + A_0]$. In particular, we have $(\sigma_5 - 1)[A_0] = -(\sigma_5 - 1)[D]$. Using the formulae in Section 3 (see also [13, Section 3]), we see that

$$(\sigma_5 - 1)[A_0] = [A_2 - A_0] = \alpha_2 - \alpha_0 = 2e_1 + 2e_3 + 3e_4.$$
is not divisible by 2 in $\text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$. On the other hand, $-(\sigma_5 - 1)[D]$ is an element of $2 \text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$ by \((\ref{prop1})\). The contradiction shows $\sigma_5$ does not have a fixed point in $\text{Pic}_{C_4/\mathbb{Q}}^1(\mathbb{Q}(\zeta_8))$. Since two divisor classes

$$\begin{align*}
(\sigma_3 - 1)[A_0] &= [A_1 - A_0] = \alpha_1 - \alpha_0 = 3e_1 + 3e_2 + 2e_3 + 3e_4, \\
(\sigma_3\sigma_5 - 1)[A_0] &= [A_3 - A_0] = \alpha_3 - \alpha_0 = 3e_1 + e_2 + 2e_4
\end{align*}$$

do not satisfy the congruence \((\ref{prop2})\), by similar arguments, we see that neither $\sigma_3$ nor $\sigma_3\sigma_5$ has a fixed point in $\text{Pic}_{C_4/\mathbb{Q}}^1(\mathbb{Q}(\zeta_8))$.

(2) The assertion for $\sigma_5$ follows from

$$\mathbb{Q}(\zeta_8)^{\sigma_5} := \{x \in \mathbb{Q}(\zeta_8) \mid \sigma_5(x) = x\} = \mathbb{Q}(\sqrt{-1}).$$

Other parts follow from $\mathbb{Q}(\zeta_8)^{\sigma_3} = \mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\zeta_8)^{\sigma_3\sigma_5} = \mathbb{Q}(\sqrt{2})$. \(\Box\)

**Remark 4.3.** The class $[\text{Pic}^1_{C_4}/\mathbb{Q}] \in H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Jac}(C_4)(\overline{\mathbb{Q}}))$ is a non-trivial element killed by 2. The class $[\text{Pic}^2_{C_4}/\mathbb{Q}] \in H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Jac}(C_4)(\overline{\mathbb{Q}}))$ is trivial because $C_4$ has a divisor of degree 2 over $\mathbb{Q}$. In fact, we have $2[\text{Pic}^1_{C_4/\mathbb{Q}}] = [\text{Pic}^2_{C_4/\mathbb{Q}}] = 0$ in $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Jac}(C_4)(\overline{\mathbb{Q}}))$.

Finally, we shall calculate the action of Galois group on the divisor $E := 2[1 : 0 : \zeta_8^3] - 2[1 : 0 : \zeta_8^5]$.

**Proposition 4.4.** (1) The divisor class $[E] \in \text{Jac}(C_4)(\mathbb{Q}(\zeta_8))$ is invariant under the action of $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$.

(2) The divisor $E$ is not linearly equivalent to a divisor defined over $\mathbb{Q}$. In other words, the image of $[E]$ in the relative Brauer group $\text{Br}(C_4/\mathbb{Q})$ is non-trivial.

**Proof.** (1) Since $[E] = 2e_4$, we have

$$\sigma_3([E]) = 2\sigma_3(e_4) = 2e_4 = [E], \quad \sigma_5([E]) = 2\sigma_5(e_4) = 2e_4 = [E].$$

(See Remark \((\ref{prop1})\) and the table in the proof of Proposition \((\ref{prop2})\)). Hence $[E]$ is invariant under the action of $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$.

(2) By (1), we see that the divisors $E - \sigma_3(E), E - \sigma_5(E), 2[E]$ are linearly equivalent to 0. To calculate the obstruction in the relative Brauer group explicitly, we shall give the rational functions which give linear equivalences. We have

$$\begin{align*}
\sigma_3(E) &= 2[1 : 0 : \zeta_8] - 2[1 : 0 : \zeta_8^5], \\
\sigma_5(E) &= 2[1 : 0 : \zeta_8^3] - 2[1 : 0 : \zeta_8^5] = -E,
\end{align*}$$

$$\sigma_3\sigma_5(E) = 2[1 : 0 : \zeta_8^3] - 2[1 : 0 : \zeta_8] = -\sigma_3(E).$$

Hence we have

$$\begin{align*}
2E &= \text{div} \left( \frac{X - \zeta_8^5 Z}{X - \zeta_8 Z} \right), \\
E + \sigma_3(E) &= \text{div} \left( \frac{Y^2}{(X - \zeta_8 Z)(X - \zeta_8^5 Z)} \right), \\
E - \sigma_3(E) &= \text{div} \left( \frac{Y^2}{(X - \zeta_8 Z)(X - \zeta_8^5 Z)} \right).
\end{align*}$$

To prove (2), it is enough to show that the image of $\text{Br}([E])$ in $\text{Br}(\mathbb{R}) = H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \{\pm 1\}$
is non-trivial. We shall calculate the cocycle explicitly. Let $\tau$ denote the complex conjugation. We have $\tau = \sigma_3\sigma_5$ in $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$. We have $\tau(E) = \sigma_3(E) = -\sigma_3(E)$. Let $u_1 = 1$ and

$$u_\tau = \frac{Y^2}{(X - \zeta_8 Z)(X - \zeta_8^3 Z)}.$$ 

We have $\text{div}(u_\tau) = E + \sigma_3(E) = E - \tau(E)$. Thus a 2-cocycle $a: \{1, \tau\}^2 \to \{\pm 1\}$ corresponding to $\text{Br}([E])$ is given by

$$a_{s,t} = \frac{u_s \cdot s(u_t)}{u_{st}} \quad (s, t \in \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \tau\}).$$

Since $u_1 = 1$, we have $a_{1,1} = a_{1,\tau} = a_{\tau,1} = 1$. When $s = t = \tau$, we have

$$a_{\tau,\tau} = \frac{u_\tau \cdot \tau(u_\tau)}{u_{\tau^2}} = \frac{Y^2}{(X - \zeta_8 Z)(X - \zeta_8^3 Z)} \cdot \frac{Y^2}{(X - \zeta_8^3 Z)(X - \zeta_8^5 Z)} = \frac{Y^4}{X^4 + Z^4} = -1.$$

Therefore, the cocycle $a$ gives a non-trivial element of $\text{Br}(\mathbb{R}) \cong \{\pm 1\}$ corresponding to Hamilton’s quaternion algebra $\mathbb{H}$. \hfill \Box

5. PROOF OF THEOREM 1.1 AND THEOREM 1.2

Proof of Theorem 1.1. We shall complete the proof of Theorem 1.1. We consider the following exact sequence

$$0 \longrightarrow \text{Pic}^0(C_4) \longrightarrow \text{Jac}(C_4)(\mathbb{Q}) \longrightarrow \text{Br}(C_4/\mathbb{Q}).$$

We have $|\text{Pic}^0(C_4)| \geq 4$ by Lemma 2.1 (3).

On the other hand, we have $|\text{Jac}(C_4)(\mathbb{Q})| = 8$ by Proposition 4.1. (See also Remark 5.1.) Since $[E] \in \text{Jac}(C_4)(\mathbb{Q})$ has a non-trivial image in $\text{Br}(C_4/\mathbb{Q})$ by Proposition 4.1 (2), we have $|\text{Pic}^0(C_4)| < |\text{Jac}(C_4)(\mathbb{Q})| = 8$.

Theorem 1.1 follows from these results.

Proof of Theorem 1.2. Since the quartic $C_4$ has a $\mathbb{Q}$-rational divisor of degree 2 (such as $D_i$ ($0 \leq i \leq 3$) in Section 2), we have an isomorphism $\text{Pic}^{2d+1}_{C_4/\mathbb{Q}} \cong \text{Pic}^1_{C_4/\mathbb{Q}}$ over $\mathbb{Q}$. Hence it is enough to show $\text{Pic}^1_{C_4/\mathbb{Q}}$ has no $\mathbb{Q}$-rational point. This follows from Proposition 4.2 (2).

Remark 5.1. Our calculation shows that, for $K = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$, $\text{Pic}^1_{C_4/K}$ is a non-trivial torsor under $\text{Jac}(C_4 \otimes_{\mathbb{Q}} K)$.

6. DIVISORS OF DEGREE 2 AND QUADRATIC POINTS

Recall that we have divisors $D_0, D_1, D_2, D_3$ of degree 2 over $\mathbb{Q}$; see Section 2. We shall show that these are the only divisors of degree 2 on $C_4$ over $\mathbb{Q}$.

Proposition 6.1. The degree 2 part $\text{Pic}^2(C_4)$ of the Picard group consists of the four elements $[D_i]$ ($0 \leq i \leq 3$), i.e.,

$$\text{Pic}^2(C_4) = \{[D_0], [D_1], [D_2], [D_3]\}$$

In particular, every $\mathbb{Q}$-rational divisor of degree 2 on $C_4$ is linearly equivalent to an effective divisor.
Proof. We have a bijection

\[ \text{Pic}^2(C_4) \cong \text{Pic}^0(C_4), \quad [D] \mapsto [D - D_0]. \]

Hence \(|\text{Pic}^2(C_4)| = |\text{Pic}^0(C_4)| = 4\) by Theorem 1.1 (1). It remains to show that the four divisor classes \([D_i]|\ (0 \leq i \leq 3)\) are distinct from each other. This follows from Lemma 2.1 (1).

Proof of Theorem 1.4. Since \(\text{Pic}^2(C_4)\) is explicitly calculated, we can determine points on \(C_4\) defined over a quadratic extension of \(\mathbb{Q}\) by Faddeev’s method [8]. (See [13] Lemma 7.1 for a precise statement.)

Since the proof is essentially same as [13] Section 7], we briefly give an outline of the proof. Since \(C_4\) has no \(\mathbb{R}\)-rational point, it has no \(\mathbb{Q}\)-rational point. Let \(P \in C_4\) be a point defined over a quadratic extension of \(\mathbb{Q}\), and \(\overline{P}\) be the conjugate of \(P\). By Proposition 6.1, the effective divisor \(P - \overline{P}\) is linearly equivalent to \(D_i\) for some \(0 \leq i \leq 3\). To prove Theorem 1.4, it is enough to show \(P + \overline{P}\) is equal to \(D_i\) as divisors. If \(P + \overline{P} \neq D_i\), there is a non-zero rational function \(f\) on \(C_4\) with \(\text{div}(f) = P + \overline{P} - D_i\). The morphism \(C \to \mathbb{P}^1\) induced by \(f\) has degree 2. This is absurd since \(C_4\) is non-hyperelliptic.

7. LINEAR DETERMINANTAL REPRESENTATIONS

In this final section, we give an application to the local–global property of linear determinantal representations.

Recall that a linear determinantal representation of a plane curve \(C \subset \mathbb{P}^2\) over \(K\) of degree \(d\) is a triple of \(d\) by \(d\) matrices \(M = (M_0, M_1, M_2)\) \((M_0, M_1, M_2 \in \text{Mat}_d(K))\) such that

\[ \det(XM_0 + YM_1 + ZM_2) = 0 \]

is a defining equation of the curve \(C \subset \mathbb{P}^2\). Two linear determinantal representations \(M = (M_0, M_1, M_2), M' = (M'_0, M'_1, M'_2)\) are said to be equivalent if there are invertible matrices \(P, Q \in \text{GL}_d(K)\) such that \(M'_i = PM_iQ\) for \(i = 0, 1, 2\). It is well-known that there is a natural bijection between the following sets:

- the set of equivalence classes of linear determinantal representations of \(C\) over \(K\), and
- the set of linear equivalence classes of \(K\)-rational divisors \(D\) on \(C\) of degree \(g(C) - 1 = d(d - 3)/2\) which are not linearly equivalent to effective divisors.

(Here \(g(C) = (d - 1)(d - 2)/2\) is the genus of \(C\).)

(See [12] Theorem 1]. See also [1], [6].)

Proof of Theorem 1.5. To prove Theorem 1.5 (1), it is enough to prove that every divisor of degree 2 on \(C_4\) over \(\mathbb{Q}\) is linearly equivalent to an effective divisor. This follows from Proposition 6.1.

Theorem 1.5 (2) follows from the following general results on linear determinantal representations over large fields. Recall the a field \(K\) is called a large field if any smooth algebraic curve over \(K\) with a \(K\)-rational point has infinitely many \(K\)-rational points [16]. (Large fields are also called ample fields in [15].) Note that \(\mathbb{R}\) and \(p\)-adic fields \(\mathbb{Q}_p\) are large fields.

Proposition 7.1. Let \(K\) be a large field. Let \(C \subset \mathbb{P}^2\) be a smooth plane curve of degree \(d \geq 3\) defined over \(K\). Assume that there exists a \(K\)-rational divisor of degree
By Proposition 7.1, we conclude that \( C \) admits infinitely many equivalence classes of linear determinantal representations over \( K \).

**Proof.** For smooth cubics, this result is proved by the first author in [9, Theorem 8.2]. Essentially the same proof applies to smooth curves of higher degree. There exists an exact sequence

\[
0 \longrightarrow \text{Pic}(C) \longrightarrow \text{Pic}_{C/K}(K) \xrightarrow{\text{Br}} \text{Br}(C/K) \longrightarrow 0.
\]

(See [4, Theorem 2.1], [3, Section 8.1, Proposition 4].) Take a finite separable extension \( L/K \) such that \( \text{Pic}^{g-1}(C) \) has an \( L \)-rational point. We put \( n := [L : K] \). Then the relative Brauer group \( \text{Br}(C/K) \) is killed by \( n \) by [9, Lemma 4.2].

Let \( D_0 \) be a \( K \)-rational divisor of degree \( g(C) - 1 \) on \( C \). Since \( [D_0] \in \text{Pic}^{g-1}(C) \) comes from a \( K \)-rational divisor, we have \( \text{Br}([D_0]) = 0 \).

We have an isomorphism \( \text{Pic}^{g-1}(C) \cong \text{Jac}(C) \) sending \([D] \) to \([D] - [D_0] \). Let

\[
W \subset \text{Pic}^{g-1}(C)
\]

be the closed subscheme whose geometric points corresponds to divisors of degree \( g(C) - 1 \) which are linearly equivalent to effective divisors. Then we have \( \dim W = g(C) - 1 \) because \( W \) coincides with the image of the map

\[
C \times \cdots \times C \twoheadrightarrow \text{Pic}^{g-1}(C).
\]

Let \([n] : \text{Jac}(C) \to \text{Jac}(C)\) be the multiplication-by-\( n \) isogeny and consider the subvariety \( Z \subset \text{Jac}(C) \) defined by

\[
Z := [n]^{-1}(W - [D_0]).
\]

It is a subvariety of \( \text{Jac}(C) \) of dimension \( g(C) - 1 \) defined over \( K \).

Since \( K \) is a large field, there exist infinitely many \( K \)-rational points \( P \in \text{Jac}(C)(K) \) outside \( Z \). (To see this, take a smooth curve \( C' \subset \text{Jac}(C) \) passing through 0 with \( C' \not\subset Z \). Since \( K \) is a large field, the curve \( C' \) contains infinitely many \( K \)-rational points.) Since \([n](P) \notin W - [D_0]\), we see that \([n](P) + [D_0]\) is a \( K \)-rational point of \( \text{Pic}^{g-1}(C) \) which does not sit in \( W \). Since

\[
\text{Br}([n](P) + [D_0]) = n \text{Br}(P) + \text{Br}([D_0]) = 0,
\]

the \( K \)-rational point \([n](P) + [D_0] \in \text{Pic}^{g-1}(C) \) comes from a \( K \)-rational divisor on \( C \) which is not linearly equivalent to an effective divisor.

Therefore, the plane curve \( C \subset \mathbb{P}^2 \) admits infinitely many equivalence classes of linear determinantal representations over \( K \).

The quartic \( C_4 \) has a \( \mathbb{Q} \)-rational divisor of degree 2 such as \( D_i \) \((0 \leq i \leq 3)\) in Section 2. By Proposition 7.1, we conclude that \( C_4 \) admits linear determinantal representations over \( \mathbb{R} \) and \( p \)-adic fields \( \mathbb{Q}_p \). The proof of Theorem 1.5 is complete.

**Remark 7.2.** A linear determinantal representation \( M = (M_0, M_1, M_2) \) is called symmetric if the matrices \( M_0, M_1, M_2 \) are symmetric matrices. We may ask similar questions for the existence of symmetric determinantal representations. Symmetric determinantal representations are classical objects related to bitangents and theta characteristics studied by algebraic geometers in the 19th century. (For historical account,
see \[1, \[3\] .) The local–global property of symmetric determinantal representations is studied in \[10, \[11, \[14\] . Recently, we constructed a smooth plane quartic over \( \mathbb{Q} \) which violates the local–global property of symmetric determinantal representations; see \[14, \text{Section 7}\] for details.

APPENDIX A. CALCULATION OF DIVISORS

Here is a sample source code for Singular (version 4.2.1) which confirms the formulae on divisors used in this paper.

```plaintext
LIB "divisors.lib";
ring r=(0,d1),(x,y,z),dp; minpoly = d1^8 - d1^4 + 1;
number x8 = d1^3; number y4 = x8^2; number z3 = d1^8;
divisor A0 = makeDivisor(ideal(x, x8^7 * y - y^4^0 * z), ideal(1));
divisor A1 = makeDivisor(ideal(x, x8^7 * y - y^4^1 * z), ideal(1));
divisor A2 = makeDivisor(ideal(x, x8^7 * y - y^4^2 * z), ideal(1));
divisor A3 = makeDivisor(ideal(x, x8^7 * y - y^4^3 * z), ideal(1));
divisor B0 = makeDivisor(ideal(y, x8^7 * x - x^4^0 * z), ideal(1));
divisor B1 = makeDivisor(ideal(y, x8^7 * x - x^4^1 * z), ideal(1));
divisor B2 = makeDivisor(ideal(y, x8^7 * x - x^4^2 * z), ideal(1));
divisor B3 = makeDivisor(ideal(y, x8^7 * x - x^4^3 * z), ideal(1));
divisor C0 = makeDivisor(ideal(z, x - z8 * y^4^0 * y), ideal(1));
divisor C1 = makeDivisor(ideal(z, x - z8 * y^4^1 * y), ideal(1));
divisor C2 = makeDivisor(ideal(z, x - z8 * y^4^2 * y), ideal(1));
divisor C3 = makeDivisor(ideal(z, x - z8 * y^4^3 * y), ideal(1));
divisor D0 = makeDivisor(ideal(z3 * x - y, z3^2 * x - z), ideal(1));
divisor D1 = makeDivisor(ideal(-z3^2 * x + y, z3 * x - z), ideal(1));
divisor D2 = makeDivisor(ideal(-z3^2 * x + y, -z3^2 * x - z), ideal(1));
divisor D3 = makeDivisor(ideal(-z3 * x + y, -z3^2 * x - z), ideal(1));
divisor E1 = A1 + multdivisor(-1, B0);
divisor E2 = A2 + multdivisor(-1, B0);
divisor E3 = B1 + multdivisor(-1, B0);
divisor E4 = B2 + multdivisor(-1, B0);
divisor E5 = C1 + multdivisor(-1, B0);
divisor E6 = A1 + C1 + A2 + B2 + C2 + multdivisor(-6, B0);
divisor Lin = multdivisor(4, makeDivisor(ideal(x-z, y), ideal(1)));
divisor E =
  multdivisor (2, makeDivisor(ideal(z8^3 * x - z, y), ideal(1))
  + multdivisor(-2, makeDivisor(ideal(z8^7 * x - z, y), ideal(1)));
proc lineqcheck(divisor D1,divisor D2){
  if(linearlyEquivalent(D1,D2)[1]!=0){ return(1);}
  else{ return(0);};}
proc divcheck(string S,divisor D1,divisor D2){
  if(lineeqcheck(D1,D2)==1){printf("%s \& OK", S);}
  else{printf("%s \& x", S);};}
proc prdivcheck(string S, divisor D, poly F1, poly F2){
  if(isEqualDivisor(D,makeDivisor(ideal(F1), ideal(F2))==1)
    printf("%s \& OK", S);
  else{printf("%s \& x", S);};
}
// Check linear equivalences of divisors
print("/");
```
print("Check linear equivalences of divisors");
divcheck("alpha_0 = 2e_1 + e_2 + 2e_3 + e_4",
        A0 + multdivisor(-1,B0), E1+E1 + E2 + E3+E3 + E4);
divcheck("alpha_1 = e_1",
        A1 + multdivisor(-1,B0), E1);
divcheck("alpha_2 = e_2",
        A2 + multdivisor(-1,B0), E2);
divcheck("alpha_3 = e_1 + 2e_2 + 2e_3 + 4e_4",
        A3 + multdivisor(-1,B0), E1 + E2+E2 + E3+E3 + E4+E4+E4);
divcheck("beta_1 = e_3",
        B1 + multdivisor(-1,B0), E3);
divcheck("beta_2 = e_4",
        B2 + multdivisor(-1,B0), E4);
divcheck("beta_3 = 3e_3 + 3e_4",
        B3 + multdivisor(-1,B0), E3+E3+E3 + E4+E4+E4);
divcheck("gamma_0 = 3e_1 + 3e_2 + e_3 + e_5 + e_6",
        C0 + multdivisor(-1,B0), E1+E1+E1 + E2+E2+E2 + E3 + E5 + E6);
divcheck("gamma_1 = e_5",
        C1 + multdivisor(-1,B0), E5);
divcheck("gamma_2 = 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + e_6",
        C2 + multdivisor(-1,B0), E1+E1+E1 + E2+E2+E2 + E3 + E5 + E6 + E6);
divcheck("gamma_3 = 2e_1 + 2e_2 + e_4 + 3e_5",
        C3 + multdivisor(-1,B0), E1+E1 + E2+E2 + E4 + E5+E5+E5);

// Points of tangency of bitangents
print("Points of tangency of bitangents");
prdivcheck("2 D_0 = div(x + y + z)", D0 + D0, x + y + z, 1);
prdivcheck("2 D_1 = div(x - y + z)", D1 + D1, x - y + z, 1);
prdivcheck("2 D_2 = div(x + y - z)", D2 + D2, x + y - z, 1);
prdivcheck("2 D_3 = div(x - y - z)", D3 + D3, x - y - z, 1);
D0 + D1 + D2 + D3, x^2 + y^2 + z^2, 1); 
prdivcheck("D_0 + D_1 + D_2 + D_3 = div(X^2 + Y^2 + Z^2)",
        D0 + D1 + D2 + D3, x^2 + y^2 + z^2, 1);
prdivcheck("D_1 - D_0 = 2B_1 + 2B_2 - 4B_0 + div(...)",
        D1 + multdivisor(-1,D0) + multdivisor(-2,B1+B2) + multdivisor(4,B0),
        (x - z^8*z) * (x - y + z),
        (x^2 + y^2 + z^2) * (d^5 - d^3 - d) * (y^2 - x*z));
prdivcheck("D_2 - D_0 = 2A_1 + 2A_2 + 2B_1 + 2B_2 - 4B_0 + div(...)",
        D2 + multdivisor(-1,D0) + multdivisor(-2,A1+A2+B1+B2) + multdivisor(8,B0),
        (x - z^8*z) * (x + y - z),
        (x^2 + y^2 + z^2) * (d^5 - d^3 - d) * (x^2 + x*y + y^2));
prdivcheck("D_3 - D_0 = 2A_1 + 2A_2 - 4B_0 + div(...)",
        D3 + multdivisor(-1,D0), (x - z^8*z) * (x - y + z),
        (x^2 - y^2 - 2*x*y - z^2) * (d^5 - d^3 - d) * (y^2 + x*y + z^2));
prdivcheck("E = 2B_2 - 2B_0", E,
        (x - z^8*z) * (x - y + z),
        (x^2 + y^2 + z^2) + (d^5 - d^3 - d) * (y^2 + x*y + z^2));
K + multdivisor(-2,B2) + B0+B0, 1, 1);

divcheck("D_1 - D_0 = 2e_3 + 2e_4",
        D1 + multdivisor(-1,D0), E3+E3 + E4+E4);
divcheck("D_2 - D_0 = 2e_1 + 2e_2 + 2e_3 + 2e_4",
        D2 + multdivisor(-1,D0), E1+E1 + E2+E2 + E3+E3 + E4+E4+E4);
divcheck("D_3 - D_0 = 2e_1 + 2e_2",
        D3 + multdivisor(-1,D0), E1+E1 + E2+E2);
divcheck("E = 2e_4", E, E4+E4);

// Action of sigma_3
print("Action of sigma_3");
print("Action of sigma_3");

number z8_3 = (d^3)^3; number z4_3 = z8_3^2;
divisor A1_3 = makeDivisor(ideal(x, z8_3^7 * y - z4_3^0 * z), ideal(1));
divisor A2_3 = makeDivisor(ideal(x, z8_3^7 * y - z4_3^1 * z), ideal(1));
divisor A3_3 = makeDivisor(ideal(x, z8_3^7 * y - z4_3^2 * z), ideal(1));
divisor $B_0_3 = \text{makeDivisor}(\text{ideal}(y, z8_3^7 \cdot x - z4_3^0 \cdot z), \text{ideal}(1))$

divisor $B_1_3 = \text{makeDivisor}(\text{ideal}(y, z8_3^7 \cdot x - z4_3^1 \cdot z), \text{ideal}(1))$

divisor $B_2_3 = \text{makeDivisor}(\text{ideal}(y, z8_3^7 \cdot x - z4_3^2 \cdot z), \text{ideal}(1))$

divisor $B_3_3 = \text{makeDivisor}(\text{ideal}(y, z8_3^7 \cdot x - z4_3^3 \cdot z), \text{ideal}(1))$

divisor $C_0_3 = \text{makeDivisor}(\text{ideal}(z, x - z8_3 \cdot z4_3^0 \cdot y), \text{ideal}(1))$

divisor $C_1_3 = \text{makeDivisor}(\text{ideal}(z, x - z8_3 \cdot z4_3^1 \cdot y), \text{ideal}(1))$

divisor $C_2_3 = \text{makeDivisor}(\text{ideal}(z, x - z8_3 \cdot z4_3^2 \cdot y), \text{ideal}(1))$

divisor $C_3_3 = \text{makeDivisor}(\text{ideal}(z, x - z8_3 \cdot z4_3^3 \cdot y), \text{ideal}(1))$

divisor $E_1_3 = A_1_3 + \text{multdivisor}(-1, B_0_3)$

divisor $E_2_3 = A_2_3 + \text{multdivisor}(-1, B_0_3)$

divisor $E_3_3 = B_1_3 + \text{multdivisor}(-1, B_0_3)$

divisor $E_4_3 = B_2_3 + \text{multdivisor}(-1, B_0_3)$

divisor $E_5_3 = C_1_3 + \text{multdivisor}(-1, B_0_3)$

divisor $E_6_3 = A_1_3 + B_1_3 + C_1_3 + A_2_3 + B_2_3 + C_2_3 + \text{multdivisor}(-6, B_0_3)$

divisor $E_3 = \text{multdivisor}(2, \text{makeDivisor}(\text{ideal}(z8_3^3 \cdot x - z, y), \text{ideal}(1))) + \text{multdivisor}(-2, \text{makeDivisor}(\text{ideal}(z8_3^7 \cdot x - z, y), \text{ideal}(1)))$

divcheck("sigma_3(A_0) = A_1", A0_3, A1);

divcheck("sigma_3(A_1) = A_0", A1_3, A0);

divcheck("sigma_3(A_2) = A_3", A2_3, A3);

divcheck("sigma_3(A_3) = A_2", A3_3, A2);

divcheck("sigma_3(B_0) = B_1", B0_3, B1);

divcheck("sigma_3(B_1) = B_0", B1_3, B0);

divcheck("sigma_3(B_2) = B_3", B2_3, B3);

divcheck("sigma_3(B_3) = B_2", B3_3, B2);

divcheck("sigma_3(C_0) = C_1", C0_3, C1);

divcheck("sigma_3(C_1) = C_0", C1_3, C0);

divcheck("sigma_3(C_2) = C_3", C2_3, C3);

divcheck("sigma_3(C_3) = C_2", C3_3, C2);

divcheck("sigma_3(e1) = 2e_1 + e_2 + e_3 + e_4", E1_3, E1 + E1 + E2 + E3 + E4);

divcheck("sigma_3(e2) = e_1 + 2e_2 + e_3 + 3e_4", E2_3, E1 + E2 + E2 + E3 + E4 + E4);

divcheck("sigma_3(e3) = 3e_3", E3_3, E3 + E3 + E3);

divcheck("sigma_3(e4) = e_1 + e_2 + 3e_4", E4_3, E3 + E4 + E4 + E4);

divcheck("sigma_3(e5) = e_1 + e_2 + e_3 + e_4", E5_3, E1 + E1 + E2 + E2 + E2 + E5 + E6);

// Action of sigma_5

print("\n");

print("Action of sigma_5");

number z8_5 = (d^3)^5; number z4_5 = z8_5^2;

divisor $A_0_5 = \text{makeDivisor}(\text{ideal}(x, z8_5^7 \cdot y - z4_5^0 \cdot z), \text{ideal}(1))$

divisor $A_1_5 = \text{makeDivisor}(\text{ideal}(x, z8_5^7 \cdot y - z4_5^1 \cdot z), \text{ideal}(1))$

divisor $A_2_5 = \text{makeDivisor}(\text{ideal}(x, z8_5^7 \cdot y - z4_5^2 \cdot z), \text{ideal}(1))$

divisor $A_3_5 = \text{makeDivisor}(\text{ideal}(x, z8_5^7 \cdot y - z4_5^3 \cdot z), \text{ideal}(1))$

divisor $B_0_5 = \text{makeDivisor}(\text{ideal}(y, z8_5^7 \cdot x - z4_5^0 \cdot z), \text{ideal}(1))$

divisor $B_1_5 = \text{makeDivisor}(\text{ideal}(y, z8_5^7 \cdot x - z4_5^1 \cdot z), \text{ideal}(1))$

divisor $B_2_5 = \text{makeDivisor}(\text{ideal}(y, z8_5^7 \cdot x - z4_5^2 \cdot z), \text{ideal}(1))$

divisor $B_3_5 = \text{makeDivisor}(\text{ideal}(y, z8_5^7 \cdot x - z4_5^3 \cdot z), \text{ideal}(1))$

divisor $C_0_5 = \text{makeDivisor}(\text{ideal}(z, x - z8_5 \cdot z4_5^0 \cdot y), \text{ideal}(1))$

divisor $C_1_5 = \text{makeDivisor}(\text{ideal}(z, x - z8_5 \cdot z4_5^1 \cdot y), \text{ideal}(1))$

divisor $C_2_5 = \text{makeDivisor}(\text{ideal}(z, x - z8_5 \cdot z4_5^2 \cdot y), \text{ideal}(1))$

divisor $C_3_5 = \text{makeDivisor}(\text{ideal}(z, x - z8_5 \cdot z4_5^3 \cdot y), \text{ideal}(1))$

divisor $E_1_5 = A_1_5 + \text{multdivisor}(-1, B_0_5)$

divisor $E_2_5 = A_2_5 + \text{multdivisor}(-1, B_0_5)$

divisor $E_3_5 = B_1_5 + \text{multdivisor}(-1, B_0_5)$

divisor $E_4_5 = B_2_5 + \text{multdivisor}(-1, B_0_5)$

divisor $E_5_5 = C_1_5 + \text{multdivisor}(-1, B_0_5)$

divisor $E_6_5 = A_1_5 + B_1_5 + C_1_5 + A_2_5 + B_2_5 + C_2_5 + \text{multdivisor}(-6, B_0_5)$

divisor $E_5 = \text{multdivisor}(2, \text{makeDivisor}(\text{ideal}(z8_5^3 \cdot x - z, y), \text{ideal}(1))) + \text{multdivisor}(-2, \text{makeDivisor}(\text{ideal}(z8_5^7 \cdot x - z, y), \text{ideal}(1)))$
divcheck("\sigma_5(A_2) = A_0", A2_5, A0);
divcheck("\sigma_5(A_3) = A_1", A3_5, A1);
divcheck("\sigma_5(B_0) = B_2", B0_5, B2);
divcheck("\sigma_5(B_1) = B_3", B1_5, B3);
divcheck("\sigma_5(B_2) = B_0", B2_5, B0);
divcheck("\sigma_5(B_3) = B_1", B3_5, B1);
divcheck("\sigma_5(C_0) = C_2", C0_5, C2);
divcheck("\sigma_5(C_1) = C_3", C1_5, C3);
divcheck("\sigma_5(C_2) = C_0", C2_5, C0);
divcheck("\sigma_5(C_3) = C_1", C3_5, C1);
divcheck("\sigma_5(e_1) = e_1 + 2e_2 + 2e_3 + 2e_4", E1_5, E1 + E2 + E2 + E3 + E4 + E4);
divcheck("\sigma_5(e_2) = 2e_1 + e_2 + 2e_3", E2_5, E1 + E2 + E2 + E3 + E4);
divcheck("\sigma_5(e_3) = 3e_3 + 2e_4", E3_5, E3 + E3 + E4 + E4 + E4);
divcheck("\sigma_5(e_4) = 3e_4", E4_5, E4 + E4 + E4);
divcheck("\sigma_5(e_5) = 2e_1 + 2e_2 + 3e_5", E5_5, E1 + E2 + E2 + E3 + E4 + E4 + E4 + E4);
divcheck("\sigma_5(e_6) = 2e_4 + e_6", E6_5, E4 + E4 + E6);

// Calculation of fixed points
print("Calculation of fixed points");
divcheck("(\sigma_5 - 1)[A_0] = 2e_1 + 2e_3 + 3e_4", A2 + multdivisor(-1, A0),
E1 + E1 + E3 + E4 + E4 + E4);
divcheck("(\sigma_3 \sigma_5 - 1)[A_0] = 3e_1 + 3e_2 + 2e_3 + 3e_4", A3 + multdivisor(-1, A0),
E1 + E1 + E2 + E2 + E3 + E4 + E4 + E4);
divcheck("(\sigma_3 - 1)[A_0] = 3e_1 + 2e_2 + 2e_3 + 3e_4", A3 + multdivisor(-1, A0),
E1 + E1 + E2 + E2 + E3 + E4 + E4 + E4);

// Calculation of Brauer obstruction
print("Calculation of Brauer obstruction");
prdivcheck("2E = div((X - z_8^5 * Z)/(X - z_8 * Z))", E + E, x - z_8^5 * z, x - z_8 * z);
prdivcheck("E + \sigma_3(E) = div(Y^2/((X - z_8 * Z) * (X - z_8^3 * Z)))", E + E_3, y^2, (x - z_8 * z) * (x - z_8^3 * z));
prdivcheck("E - \sigma_3(E) = div(Y^2/((X - z_8 * Z) * (X - z_8^7 * Z)))", E + multdivisor(-1, E_3), y^2, (x - z_8 * z) * (x - z_8^7 * z));
quit;

If the above program is executed successfully, it outputs the following message. In the output, “OK” means the formula in this paper is correct.

Check linear equivalences of divisors
alpha_0 = 2e_1 + e_2 + 2e_3 + e_4 : OK
alpha_1 = e_1 : OK

SINGULAR / Development
A Computer Algebra System for Polynomial Computations / version 4.2.1
by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann / May 2021
FB Mathematik der Universität, D-67653 Kaiserslautern
*/
** loaded /usr/local/bin/../share/singular/LIB/divisors.lib (4.2.0.1,Jan_2021)
** loaded (builtin) customstd.so
** loaded (builtin) gfanlib.so
** loaded /usr/local/bin/../share/singular/LIB/primdec.lib (4.2.0.0,Mar_2021)
** loaded /usr/local/bin/../share/singular/LIB/ring.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/absfact.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/triang.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/matrix.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/nctools.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/inout.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/random.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/poly.lib (4.2.0.0,Dec_2020)
** loaded /usr/local/bin/../share/singular/LIB/elim.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/general.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/divisors.lib (4.2.0.1,Jan_2021)
** loaded (builtin) customstd.so
** loaded (builtin) gfanlib.so
** loaded /usr/local/bin/../share/singular/LIB/primdec.lib (4.2.0.0,Mar_2021)
** loaded /usr/local/bin/../share/singular/LIB/ring.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/absfact.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/triang.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/matrix.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/nctools.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/inout.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/random.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/poly.lib (4.2.0.0,Dec_2020)
** loaded /usr/local/bin/../share/singular/LIB/elim.lib (4.1.2.0,Feb_2019)
** loaded /usr/local/bin/../share/singular/LIB/general.lib (4.1.2.0,Feb_2019)

Check linear equivalences of divisors
alpha_0 = 2e_1 + e_2 + 2e_3 + e_4 : OK
alpha_1 = e_1 : OK
alpha_2 = e_2 : OK
alpha_3 = e_1 + 2e_2 + 2e_3 + 4e_4 : OK
beta_1 = e_6 : OK
beta_2 = e_6 : OK
beta_3 = 3e_3 + 3e_4 : OK
gamma_0 = 3e_1 + 3e_2 + e_3 + e_5 + e_6 : OK
gamma_1 = e_5 : OK
gamma_2 = 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + e_6 : OK
gamma_3 = 2e_1 + 2e_2 + e_4 + 3e_5 : OK

Points of tangency of bitangents
D_0 = div(x + y + z) : OK
D_1 = div(x - y + z) : OK
D_2 = div(x + y - z) : OK
D_3 = div(x - y - z) : OK
D_0 + D_1 + D_2 + D_3 = div(X^2 + Y^2 + Z^2) : OK
D_1 - D_0 = 2B_1 + 2B_2 - 4B_0 + div(...) : OK
D_2 - D_0 = 2A_1 + 2A_2 + 2B_1 + 2B_2 - 8B_0 + div(...) : OK
D_3 - D_0 = 2A_1 + 2A_2 - 4B_0 + div(...) : OK
E = 2B_2 - 2B_0 : OK

Action of sigma_3
sigma_3(A_0) = A_1 : OK
sigma_3(A_1) = A_0 : OK
sigma_3(A_2) = A_3 : OK
sigma_3(A_3) = A_2 : OK
sigma_3(B_0) = B_1 : OK
sigma_3(B_1) = B_0 : OK
sigma_3(B_2) = B_3 : OK
sigma_3(B_3) = B_2 : OK
sigma_3(C_0) = C_1 : OK
sigma_3(C_1) = C_0 : OK
sigma_3(C_2) = C_3 : OK
sigma_3(C_3) = C_2 : OK
sigma_3(e_1) = 2e_1 + e_2 + e_3 + e_4 : OK
sigma_3(e_2) = 2e_1 + e_2 + e_3 + 3e_4 : OK
sigma_3(e_3) = 3e_3 : OK
sigma_3(e_4) = 2e_3 + 3e_4 : OK
sigma_3(e_5) = 3e_1 + 3e_2 + e_5 + e_6 : OK
sigma_3(e_6) = 2e_3 + e_6 : OK

Action of sigma_5
sigma_5(A_0) = A_2 : OK
sigma_5(A_1) = A_3 : OK
sigma_5(A_2) = A_0 : OK
sigma_5(A_3) = A_1 : OK
sigma_5(B_0) = B_2 : OK
sigma_5(B_1) = B_3 : OK
sigma_5(B_2) = B_0 : OK
sigma_5(B_3) = B_1 : OK
sigma_5(C_0) = C_2 : OK
sigma_5(C_1) = C_3 : OK
sigma_5(C_2) = C_0 : OK
sigma_5(C_3) = C_1 : OK
sigma_5(e_1) = e_1 + 2e_2 + 2e_3 + 2e_4 : OK
sigma_5(e_2) = e_1 + 2e_2 + 2e_3 : OK
sigma_5(e_3) = 3e_3 + 2e_4 : OK
sigma_5(e_4) = 3e_4 : OK
sigma_5(e_5) = 2e_1 + 2e_2 + 3e_5 : OK
sigma_5(e_6) = 2e_4 + e_6 : OK

Calculation of fixed points
Calculation of Brauer obstruction

\[ 2E = \text{div}(\frac{X - z^5}{X - z^6}) : \text{OK} \]
\[ E + \sigma_3(E) = \text{div}(\frac{Y^2}{(X - z^6)(X - z^9)}) : \text{OK} \]
\[ E - \sigma_3(E) = \text{div}(\frac{Y^2}{(X - z^6)(X - z^9)}) : \text{OK} \]

Auf Wiedersehen.

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