A DE RHAM MODEL FOR COMPLEX ANALYTIC EQUIVARIANT
ELLIPTIC COHOMOLOGY

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Abstract. We construct a cocycle model for complex analytic equivariant elliptic co-
homology that refines Grojnowski’s theory when the group is connected and Devoto’s
when the group is finite. We then construct Mathai–Quillen type cocycles for equivariant
elliptic Euler and Thom classes, explaining how these are related to positive energy rep-
resentations of loop groups. Finally, we show that these classes give a unique equivariant
refinement of Hopkins’ “theorem of the cube” construction of the MString-orientation
of elliptic cohomology.

1. Introduction

Equivariant K-theory facilitates a rich interplay between representation theory and
topology. For example, universal Thom classes come from representations of spin groups;
power operations are controlled by the representation theory of symmetric groups; and the
equivariant index theorem permits geometric constructions of representations of Lie groups.

Equivariant elliptic cohomology is expected to lead to an even deeper symbiosis be-
tween representation theory and topology. First evidence appears in the visionary work of
Grojnowski [Gro07] and Devoto [Dev99]. Grojnowski’s complex analytic equivariant elliptic
cohomology (defined for connected Lie groups) makes contact with positive energy repre-
sentations of loop groups [And00, Gan14]. Devoto’s construction (defined for finite groups)
interacts with moonshine phenomena [BT99, Gan09, Mort09].

Equivariant elliptic cohomology over the complex numbers is already a deep object.
By analogy, equivariant K-theory with complex coefficients subsumes the character the-
ory of compact Lie groups, which in turn faithfully encodes their representation theory.
Analogously, equivariant elliptic cohomology over the complex numbers should be viewed
as a home for “elliptic character theory,” although the complete picture of what elliptic
representation theory really is remains an open question [Seg88, GKV95, HKR00, BZN15].

This paper gives a cocycle model for complex analytic equivariant elliptic cohomology
that is uniform in the group $G$. When $G$ is connected, we recover a cocycle model for
Grojnowski’s equivariant elliptic cohomology, and when $G$ is finite we recover a cocycle

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model for Devoto’s. One great utility of cocycle models is that they bring new computational tools for applications. The elliptic cocycles presented below are concrete and explicit, namely compatible equivariant differential forms on certain fixed point sets. This parallels the differential form models for delocalized equivariant K-theory of Block–Getzler [BG94, §1], Duflo–Vergne [DV93] and Vergne [Ver94, Definition 23]. These prior descriptions of complexified equivariant K-theory give both conceptual and computational insight into the equivariant index theorem and its connections with representation theory [BGV92, Ch. 7].

We expect the model for equivariant elliptic cohomology below to yield analogous insights and permit computations of interest in elliptic representation theory.

One source of such applications has been long in the making. Indeed, Grojnowski’s original motivation for studying equivariant elliptic cohomology was to construct certain elliptic algebras, e.g., an elliptic analog of the affine Hecke algebra. Crucially, he recognized that such algebras should arise geometrically by applying equivariant elliptic cohomology to certain varieties, such as the Steinberg variety. This is the third step in the program that produces increasingly sophisticated representation-theoretic objects by applying first ordinary equivariant cohomology, then equivariant K-theory, and next equivariant elliptic cohomology to varieties built out of algebraic groups. The cohomological and K-theoretic variants of this paradigm have already met great success, notably in Kazhdan–Lusztig’s K-theoretic construction of the affine Hecke algebra [KL87]. The program has seen further development in recent years with the expectation of new examples from supersymmetric gauge theory [BDGH16, BDG+16]. In the corresponding mathematical theory of symplectic resolutions, the closely related work of Maulik–Okounkov [MO12] constructs representations of generalized quantum groups by applying equivariant cohomology theories to Nakajima quiver varieties. Equivariant elliptic cohomology is starting to play an increasingly important role at this nexus of representation theory, geometry and physics, e.g., in the work of Zhao–Zhong [ZZ15] and Yang–Zhao [YZ17]. The construction by Aganagic–Okounkov of elliptic stable envelopes [AO16] in the (extended) equivariant elliptic cohomology of symplectic resolutions has far-reaching consequences in enumerative geometry and integrable systems. In particular, it interweaves with the recent elliptic Schubert calculus of Rimanyi and Weber [RW19]. We emphasize that these applications are already quite interesting for complex analytic equivariant elliptic cohomology; refinements to objects over \( \mathbb{Z} \) will further deepen the story.

Such refinements are the subject of Lurie’s ongoing work as surveyed in [Lur09] with the state of the art being finite group equivariant elliptic cohomology [Lur19]. The setup is inherently derived: Lurie’s equivariant elliptic cohomology arises as a certain sheaf of \( E_\infty \)-ring spectra. The cocycle model below begins to bridge the gap between Grojnowski and Lurie’s. Indeed, our cocycle model is defined as a sheaf of commutative differential graded algebras on a moduli space of \( G \)-bundles over elliptic curves. The higher derived sections of this sheaf are previously unexplored and further intertwine representation theoretic data with the rich geometry of elliptic curves, e.g., see Remark 3.7 and Example 6.12 below.

A brief comment on the relation to physics. There is an anticipated relationship between elliptic cohomology and 2-dimensional supersymmetric quantum field theory [Wit88, Seg88, ST04], namely a conjectured isomorphism

\[
\{ \text{2–dimensional quantum field theories with } \mathcal{N} = (0, 1) \text{ supersymmetry over } X \} \xrightarrow{\text{deformation}} \text{TMF}(X)
\]

that realizes deformation classes of field theories as classes in the universal elliptic cohomology theory of topological modular forms (TMF). This cohomology theory is constructed as the global sections of a sheaf of \( E_\infty \)-ring spectra over the moduli stack of elliptic curves. One of the great challenges is to relate this sophisticated homotopical object to quantum field theory: at a superficial level, the candidate objects from physics have absolutely nothing to do with the objects in homotopy theory. Lurie suggests [Lur09, §5.5] that an equivariant refinement would go a long way to constructing the isomorphism (1). We explain this
story more fully in the companion paper [BET19], together with the de Rham model of this paper, we prove an equivariant refinement of the conjectural isomorphism (1) over \( \mathbb{C} \).

Outline and overview of results. Let \( G \) be a compact Lie group and \( M \) a manifold with \( G \)-action. From this data we construct a sheaf \( \widehat{Ell}_G(M) \) of commutative differential graded algebras on a moduli stack \( \text{Bun}_G(\mathcal{E}) \) of \( G \)-bundles on elliptic curves over \( \mathbb{C} \). This sheaf is the cocycle model for the \( G \)-equivariant elliptic cohomology of \( M \).

In \( \S 2 \) we define the moduli stack \( \text{Bun}_G(\mathcal{E}) \) and describe some of its basic geometric features. Roughly, \( \text{Bun}_G(\mathcal{E}) \) is the moduli space of flat \( G \)-bundles on elliptic curves. When \( G = T \) is a torus, we identify

\[
\text{Bun}_T(\mathcal{E}) \simeq \mathcal{E}^\vee \times_{\mathcal{M}_\text{ell}} \cdots \times_{\mathcal{M}_\text{ell}} \mathcal{E}^\vee
\]

with the iterated fiber product of (dual) universal elliptic curves over the moduli stack \( \mathcal{M}_\text{ell} \) of elliptic curves. This gives \( \text{Bun}_T(\mathcal{E}) \) a holomorphic structure, and \( \text{Bun}_G(\mathcal{E}) \) has a similar holomorphic structure for general \( G \). Supposing that \( G \) is connected, \( \text{Bun}_G(\mathcal{E}) \) supports holomorphic line bundles called \emph{Looijenga line bundles}. When \( G \) is simply connected, sections are spanned by (super) characters of positive energy representations of the loop group \( LG \), where the level of the representation determines the isomorphism class of the Looijenga line bundle.

In \( \S 3 \) we define the sheaf \( \widehat{Ell}_G(M) \) and indicate some of its basic properties. For example, there is a canonical identification \( \widehat{Ell}_G^0(pt) \simeq \mathcal{O}_{\text{Bun}_G(\mathcal{E})} \) with the sheaf of holomorphic functions on \( \text{Bun}_G(\mathcal{E}) \). This gives \( \widehat{Ell}_G(M) \) the canonical structure of a sheaf of \( \mathcal{O}_{\text{Bun}_G(\mathcal{E})} \)-modules (Proposition 3.8). We also show that restricting \( \widehat{Ell}_G(M) \) to the trivial \( G \)-bundle in \( \text{Bun}_G(\mathcal{E}) \) gives a map to Borel equivariant elliptic cohomology. This constructs an elliptic Atiyah–Segal completion map (Theorem 3.17).

In \( \S 4 \) we prove that \( \widehat{Ell}_G(M) \) is a cocycle refinement of Grojnowski’s complex analytic equivariant elliptic cohomology when \( G \) is connected (Theorem 4.1). In \( \S 5 \) we prove that \( \widehat{Ell}_G(M) \) is a cocycle refinement of Devoto’s equivariant elliptic cohomology over \( \mathbb{C} \) when \( G \) is finite (Theorem 5.1). As the literature on equivariant elliptic cohomology can be both terse and diffuse, in these sections we also briefly review the preexisting definitions.

In \( \S 6 \) we construct cocycle representatives of equivariant elliptic Euler and Thom classes for the groups \( G = \text{SU}(n) \) and \( \text{Spin}(2n) \) (Propositions 6.3 and 6.6, respectively). These cocycles come from products of certain Jacobi theta functions, interpreted as sections of the sheaf \( \widehat{Ell}_G \) twisted by a Looijenga line bundle. This connects the Euler and Thom cocycles with super characters of representations of the loop groups \( L\text{SU}(n) \) and \( L\text{Spin}(2n) \).

Thom classes determine orientations for equivariant elliptic cohomology, leading to equivariant Chern classes of vector bundles, pushforward or wrong-way maps, and fundamental classes of appropriately oriented submanifolds. The \( \text{SU}(n) \)-equivariant Thom cocycle therefore gives an equivariant and cocycle refinement of the MU(6)-orientation, and the \( \text{Spin}(2n) \)-equivariant Thom cocycle gives an equivariant and cocycle refinement of the MString = MO(8)-orientation. We verify compatibility with the corresponding nonequivariant classes in complex analytic elliptic cohomology in Theorem 6.10. These orientations beget equivariant refinements of the pushforwards crucial in the major applications of equivariant elliptic cohomology, and it is precisely the explicit cocycles provided here that will be useful for representation-theoretic computations.

In \( \S 7 \) we compare the equivariant characteristic classes from \( \S 6 \) with the ones studied by Hopkins [Hop94] and Ando–Hopkins–Strickland [AHS01] in their construction of MU(6)- and MString-orientations of elliptic cohomology theories and TMF. First, we make a basic observation: complex orientations of elliptic cohomology theories over \( \mathbb{C} \) do not admit equivariant refinements (Proposition 7.5). However, they do admit a \emph{twisted} equivariant refinement, where the twisting and twisted class are essentially unique (Proposition 7.7). We then show that the characteristic class central to MU(6)- and MString-orientations—whose
existence is a consequence of the theorem of the cube—has a unique equivariant refinement (Theorem 7.9). We also show it is the characteristic class associated with the previously mentioned twisted equivariant complex orientation when applied to the universal bundle with \( U(6) \) structure, i.e., \( (L_1 - 1) \otimes (L_2 - 1) \otimes (L_3 - 1) \) on \([pt/U(1)]^\dagger\) for \( L_i \) the tautological bundles on their respective factors.

**Notation and conventions.** For simplicity we make the technical assumption a \( G \)-manifold \( M \) embeds \( G \)-equivariantly into a finite-dimensional \( G \)-representation. This is automatically satisfied when \( M \) is compact by results of Mostow [Mos57] and Palais [Pal57].

A lightning review of smooth stacks is in \([A.2]\) We use the notation \( M/G \) to denote the Lie groupoid quotient of a \( G \)-action on \( M \), with underlying stack \([M/G]\). Let \( M/\sim G \) denote the coarse quotient, taken in sheaves on the site of smooth manifolds. When the \( G \)-action is free, the stack \( M/G \) is representable and we often identify it with the (coarse) quotient in manifolds. In topology, sometimes \( M/\sim G \) is used to denote the stacky quotient, but we avoid this notation because it conflicts with the standard notation for the GIT quotient. Finally, tensor products of algebras or spaces of sections will always be taken as the projective tensor product of Fréchet spaces. This is a completion of the algebraic tensor product having the key property that \( C^\infty(M \times N) \simeq C^\infty(M; C^\infty(N)) \simeq C^\infty(M) \otimes C^\infty(N) \) for manifolds \( M \) and \( N \).

Finally, we view modular forms as functions on the upper half plane \( \mathbb{H} \) or the space of based lattices \( \text{Lat} \) with properties. The sheaf of holomorphic functions on \( \mathbb{H} \) and \( \text{Lat} \) will always be taken to be the one that imposes meromorphicity at infinity, so that by “modular forms,” we always implicitly mean “weakly holomorphic modular forms.” More precisely, for an open \( U \subset \mathbb{H} \), the sections \( \mathcal{O}(U) \) are the holomorphic functions on \( U \) with at most polynomial growth along any geodesic escaping to \( \partial \mathbb{H} \).

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## 2. G-bundles on elliptic curves

### 2.1. Elliptic curves.

The moduli stack of elliptic curves is presented by the quotient

\[
\mathcal{M}_{\text{ell}} \simeq [\mathbb{H}/\text{SL}_2(\mathbb{Z})],
\]

where \( \text{SL}_2(\mathbb{Z}) \) acts on \( \mathbb{H} \) by fractional linear transformations; see \([A.2]\) for our conventions on stacks. There is a universal family of elliptic curves over the stack \( \mathcal{M}_{\text{ell}} \). We first define

\[
\tilde{E} := (\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2,
\]

where the free \( \mathbb{Z}^2 \)-action is by \( (z, \tau, n, m) \mapsto (z + \tau n + m, \tau) \), for \( z \in \mathbb{C}, \tau \in \mathbb{H} \) and \( (n, m) \in \mathbb{Z}^2 \). There is an action of \( \text{SL}_2(\mathbb{Z}) \) on \( \tilde{E} \) that covers the action on \( \mathbb{H} \) as follows:

\[
((\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2) \times \text{SL}_2(\mathbb{Z}) \rightarrow (\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2,
\]

\[
\left(z, \tau, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \mapsto \left(\frac{z}{c \tau + d}, \frac{a \tau + b}{c \tau + d}\right).
\]

The stack quotient of \( \tilde{E} \) by \( \text{SL}_2(\mathbb{Z}) \) is denoted \( \mathcal{E} \), and it has a map to \( \mathcal{M}_{\text{ell}} \),

\[
\mathcal{E} := [\tilde{E}/\text{SL}_2(\mathbb{Z})] \rightarrow \mathcal{M}_{\text{ell}}.
\]

There is a similarly defined universal dual elliptic curve, \( \mathcal{E}^\vee \), defined as a quotient as in \( \mathcal{M}_{\text{ell}} \) but for the \( \mathbb{Z}^2 \)-action

\[
(z, \tau, n, m) \mapsto (z + n - \tau m, \tau).
\]
Remark 2.1. More geometrically, the dual of an elliptic curve $E$ is the space of degree-zero line bundles on $E$. In the complex analytic setting, this is the space of topologically trivial line bundles endowed with flat, unitary connections. We identify a point in $\mathcal{E}^\vee$ with such a line bundle as follows: $(x - \tau y, \tau) \in \mathcal{E}^\vee$ for $x, y \in \mathbb{R}$ gets sent to the line bundle $L$ corresponding to the one-dimensional representation of the fundamental group

$$\pi_1(\mathbb{C}/\langle \tau, 1 \rangle) \to U(1), \quad \tau m + n \mapsto e^{2\pi i(mx + ny)}.$$ 

2.2. The stack $\text{Bun}_G(\mathcal{E})$ of $G$-bundles. Let

$$\text{Bun}_G(\mathcal{E}) \simeq [\mathbb{H} \times C^2(G)/\text{SL}_2(\mathbb{Z})]$$

denote (a version of) the moduli space of $G$-line bundles endowed with flat, unitary connections. We identify a point in $\mathcal{E}^\vee$ with such a line bundle as follows: $(\mathcal{E}^\vee, \tau, x, y)$ for pairs of commuting elements in $G$, or equivalently, smooth families of homomorphisms $\mathbb{Z}^2 \to G$. Let $C^2(G)$ denote the sheafification of the presheaf that assigns the set of conjugacy classes of such pairs of commuting elements. There is an evident action of $\text{SL}_2(\mathbb{Z})$ on the sheaf $C^2(G)$, and this action descends to $C^2(G)$. Then the right hand side of (5) is the stacky quotient for this action along with the previously defined action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

There is an obvious map $\text{Bun}_G(\mathcal{E}) \to \mathcal{M}_{\text{ell}}$, witnessing $\text{Bun}_G(\mathcal{E})$ as a kind of relative coarse moduli space of $G$-bundles on elliptic curves. Indeed, a pair of commuting elements defines a flat $G$-bundle on an elliptic curve, and a conjugacy class of such is a $G$-bundle up to isomorphism. Hence, if we fix an elliptic curve (specified in terms of $\tau \in \mathbb{H}$), the fiber of $\text{Bun}_G(\mathcal{E})$ at $\tau$ is the moduli space of isomorphism classes of $G$-bundles on that fixed curve.

Remark 2.2. Categorically-minded readers might find it alarming that we work with the above version of $\text{Bun}_G(\mathcal{E})$ rather than the stack that also records isomorphisms between $G$-bundles. This version of $\text{Bun}_G(\mathcal{E})$ turns out to be the right home for Gromov-Witten’s equivariant elliptic cohomology. Lurie’s construction also takes place over a moduli space of $G$-bundles rather than a moduli stack [Lur09, Remark 5.1]. There does exist a more stacky version of $\text{Bun}_G(\mathcal{E})$ that accommodates cocycles for equivariant elliptic cohomology in the sense below, but this object is much less familiar, being a super stack, i.e., a stack on the site of supermanifolds. This super stack arises naturally when studying 2-dimensional supersymmetric gauge theory; see [BET19].

Example 2.3. Let $T$ be a torus, so that all pairs of elements commute and

$$\text{Bun}_T(\mathcal{E}) \simeq [\mathbb{H} \times T \times T/\text{SL}_2(\mathbb{Z})].$$

Then $\text{Bun}_T(\mathcal{E})$ has a holomorphic atlas such that $\text{Bun}_T(\mathcal{E}) \simeq \mathcal{E}^\vee, \text{rk}(T)$ where

$$\mathcal{E}^\vee, \text{rk}(T) := \mathcal{E}^\vee \times_{\mathcal{M}_{\text{ell}}} \cdots \times_{\mathcal{M}_{\text{ell}}} \mathcal{E}^\vee,$$

is the iterated fibered product of the universal (dual) elliptic curve. Indeed, consider the refinement of the atlas associated with the groupoid presentation (6) by the covering map $u$

$$u: \mathbb{H} \times \mathbb{C} \simeq \mathbb{H} \times t \times t \to \mathbb{H} \times T \times T, \quad (\tau, X_1, X_2) \mapsto (\tau, e^{X_1}, e^{X_2}).$$

One checks that the quotient by the $\mathbb{Z}^{2\text{rk}(T)}$ kernel of $u$ is the dual of the universal elliptic curve to the appropriate power, where $\mathbb{H} \times \mathbb{C}$ has the obvious complex structure. This complex structure is $\text{SL}_2(\mathbb{Z})$-invariant, so $\mathbb{H} \times \mathbb{C}$ gives a holomorphic atlas for $\text{Bun}_T(\mathcal{E})$.

Example 2.4. Let $G$ be connected with torsion-free fundamental group, maximal torus $T < G$, and Weyl group $W = N(T)/T$. Borel [Bor82, Corollary 3.5] shows that in this case any pair of commuting elements can be simultaneously conjugated into $T$, and that pairs of elements in $T$ are conjugate if and only if they are conjugate by an element of $N(T)$. In brief, we have $C^2(G) \simeq (T \times T)/^cW$, where $/^cW$ denote the coarse (non-stacky) quotient, taken in sheaves on the site of manifolds. The previous example gives a holomorphic structure,

$$\text{Bun}_G(\mathcal{E}) \simeq \mathcal{E}^\vee, \text{rk}(T)/^cW.$$
using $W$-invariant open sets on $E^{\vee, rk(T)}$ to define an open cover of $\Bun_G(E)$. In more detail, a locally-defined function on $\Bun_G(E)$ is holomorphic if and only if its pullback along

\[(8) \quad \mathbb{H} \times t_C \cong \mathbb{H} \times t \times t \to \mathbb{H} \times T \times T \to \mathbb{H} \times (T \times T)/\gamma W \to \Bun_G(E)\]

defines a holomorphic function on a (necessarily $W$-invariant) open subset of $\mathbb{H} \times t_C$.

**Example 2.5.** When $G$ is connected (without any additional hypotheses) there is the inclusion of stacks

\[(9) \quad (\mathbb{H} \times (T \times T)/\gamma W)/\SL_2(\mathbb{Z}) \hookrightarrow \Bun_G(E)\]

as the connected component of the identity (or trivial bundle), and the previous example gives a holomorphic atlas for this substack.

**Example 2.6.** When $G$ doesn’t have torsion-free fundamental group, \[\mathbb{H}\] can fail to be surjective. For example, take $G = \SO(3)$. Then pairs of commuting elements are given by either pairs of rotations about a fixed common axis or pairs of reflections about orthogonal axes. In the former case, both elements are in a common maximal torus, whereas there is a unique conjugacy class for the latter pair. Hence $\mathcal{C}^2[G] = (T \times T)/\gamma W \cong \{\text{pt}\}$, where here $T = \SO(2)$ is rotations about a fixed axis and $W = \mathbb{Z}/2$ acts by inversion. So we find

\[(10) \quad \Bun_{SO(3)}(E) = (\mathcal{E}^{\vee})/(\mathbb{Z}/2) \bigcup \mathcal{M}_{\text{ell}}.\]

The forgetful map $\Bun_{SO(3)}(E) \to \mathcal{M}_{\text{ell}}$ is the usual forgetful map on the first component and the identity on the second component.

2.3. **Modular forms and theta functions.** There is a holomorphic line bundle $\omega$ over $\mathcal{M}_{\text{ell}}$ whose fiber at a given elliptic curve $E$ is the vector space of holomorphic 1-forms on $E$. Pulling back along $\mathbb{H} \to \mathcal{M}_{\text{ell}}$, the line $\omega^{\otimes k}$ trivializes with trivializing section determined by the holomorphic 1-form descending from $dz$ on $\mathbb{C}$ along the quotient \[\mathbb{C}/(\mathbb{Z}/k) \cong \mathbb{C}/(\mathbb{Z}/2).\] Sections can be described explicitly as

\[F(\gamma \cdot \tau) = (c\tau + d)^k F(\tau) \quad \text{for} \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \SL_2(\mathbb{Z}).\]

Global sections of $\omega^{\otimes k}$ are then modular forms of weight $k$; we recall our standing convention that we automatically impose meromorphy at the cusp and hence always implicitly mean weakly holomorphic modular forms. For cohomology theories valued in modular forms, it is customary to double the degree and take the dual grading as follows:

**Definition 2.7.** Define the graded commutative algebra of modular forms, $\text{MF}$ whose $2k^{\text{th}}$ graded piece $\text{MF}^{2k}$ is weakly holomorphic modular forms of weight $-k$, and whose $(2k+1)^{\text{st}}$ graded piece $\text{MF}^{2k+1}$ is zero.

Consider now the case of $G$ connected with maximal torus $T < G$ and Weyl group $W = N(T)/T$, and let $X_*(T) = \ker(t \to T)$ be the cocharacter lattice.

**Definition 2.8.** Let $\ell$ be a $W$-invariant positive definite inner product on $t$ satisfying $\ell(n, n) \in 2\mathbb{Z}$ for $n \in X_*(T)$. The level $\ell$ Looijenga line $\ell_L$ is the $W$-equivariant line bundle on $E^{\vee, rk(T)}/W \subset \Bun_G(E)$ whose sections are $W$-invariant holomorphic functions $f$ on $\mathbb{H} \times t \times t$ satisfying the descent conditions

\[f(\tau, \tilde{h}_1 + n, \tilde{h}_2 + m) = f(\tau, \tilde{h}_1, \tilde{h}_2) \exp \left(2\pi i (-\ell(\tilde{h}_1 - \tau \tilde{h}_2, n) - \frac{1}{2} \ell(n, n) \tau) \right)\]

for $n, m \in X_*(T)$ and

\[f(\gamma \cdot \tau, \tilde{h}_1, \tilde{h}_2) = f(\tau, a\tilde{h}_1 + b\tilde{h}_2, c\tilde{h}_1 + d\tilde{h}_2) \exp \left(2\pi ic(\tau + d)^{-1} \ell(\tilde{h}_1 - \tau \tilde{h}_2, \tilde{h}_1 - \tau \tilde{h}_2) \right)\]

for $\gamma \in \SL_2(\mathbb{Z})$. Sections of Looijenga line bundles are called theta functions.
3. A cocycle model for equivariant elliptic cohomology

Throughout this section, \( h = (h_1, h_2) \) is a pair of commuting elements in a compact Lie group \( G \). For \( g \in G \), let \( ghg^{-1} = (gh_1g^{-1}, gh_2g^{-1}) \) denote the conjugate commuting pair. For \( G \) acting on a manifold \( M \), let \( M^h = M^{h_1,h_2} \) denote the submanifold of \( M \) fixed by \( h_1 \) and \( h_2 \). Let \( g \) be the Lie algebra of \( G \), equipped with its adjoint \( G \)-action. The \( h \)-fixed locus \( g^h \) is the Lie algebra of the centralizer \( C(h) = C(h_1, h_2) \). Let \( T < G \) denote a maximal torus of the identity component of \( G \), \( t \) the Lie algebra of \( T \), and \( W = N(T)/T \) the Weyl group. Let \( t_{gh} \) be the Lie algebra of a maximal torus of the identity component of the centralizer \( C(h) \). The Lie algebras \( t_{gh} \) and \( t \) are always conjugate when \( G \) is connected, but \( t_{gh} \) can have strictly smaller dimension than \( t \) for \( G \) disconnected and \( h_1, h_2 \) not both in the identity component.

3.1. The sheaf of equivariant elliptic cocycles on \( \text{Bun}_G(\mathcal{E}) \). For a commutative algebra \( A \) over \( \mathbb{C} \) and an \( H \)-manifold \( X \), let \( \beta \) a formal variable with \( |\beta| = -2 \) and define

\[
\Omega^*_H(X; A[\beta, \beta^{-1}]) := \bigoplus_j \mathcal{O}_0(\mathfrak{h}/C; \Omega^j(X; A[\beta, \beta^{-1}]))^H
\]

as the stalk at \( 0 \in \mathfrak{h}/C \) of \( H \)-invariant holomorphic functions on \( \mathfrak{h}/C \) valued in \( \Omega^j(X; A[\beta, \beta^{-1}]) \) for the \( H \)-action on \( \mathfrak{h}/C \) and the \( H \)-action on \( \Omega^j(X) \) induced by the \( H \)-action on \( X \). Endow this with the total grading from differential forms and the graded ring \( A[\beta, \beta^{-1}] \) and equip \( \Omega^*_H(X; A[\beta, \beta^{-1}]) \) with the Cartan differential \( Q = d - \beta^{-1}d \) (see (50)).

Fix an open subset \( U \subset \mathbb{H} \) and \( h = (h_1, h_2) \) a pair of commuting elements in \( G \). Consider the the following open manifolds \( \{U^*_h\} \)

\[
U^*_h = \{ \tau, h_1 e^{X_1}, h_2 e^{X_2} \mid \tau \in U, (X_1, X_2) \in B_{\varepsilon}(t_{gh}) \times B_{\varepsilon}(t_{gh}) \}
\]

where \( B_{\varepsilon}(t_{gh}) \) is an \( \varepsilon \)-ball about the origin for an \( \text{Ad} \)-invariant metric on \( g \) restricted to \( t_{gh} \).

We observe that for each \( X_1, X_2 \), the elements \( h_1 e^{X_1}, h_2 e^{X_2} \in G \) are generators for a homomorphism \( \mathbb{Z}^2 \to G \), and so there is a canonical map \( U^*_h \to \mathbb{H} \times \mathbb{C}^2[G] \to \text{Bun}_G(\mathcal{E}) \).

**Lemma 3.1.** For all \( h = (h_1, h_2) \) commuting elements in \( G \), there exists an \( \varepsilon \) such that \( U^*_h \) as in (12) has the property that for all \( h' = (h'_1, h'_2) \in U^*_h \)

\[ M^{h'} \subset M^h, \quad C(h') < C(h), \]

i.e., fixed points and centralizers get smaller. Furthermore, for \( h'_1 = h_1 e^{X_1} \) and \( h'_2 = h_2 e^{X_2} \) in \( U^*_h \), the vector fields \( X_1 \) and \( X_2 \) vanish on \( M^{h'} \).

**Proof.** Block and Getzler [BG94, Lemma 1.3] prove a version of the above for fixed points by a single element \( h \) deformed by an element \( X \in \mathfrak{g} \), and we simply apply their lemma twice. Indeed, their lemma provides a ball so that \( X_1 \in U^*_{h_1} \) has \( M^{h_1 e^{X_1}} \subset M^{h_1} \) and another ball so that \( X_2 \in U^*_{h_2} \) has \( M^{h_2 e^{X_2}} \subset M^{h_2} \). Setting \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \) and considering the special case where \( h_1 \) and \( h_2 \) commute and \( X_1, X_2 \in t_{gh} \), we have

\[ M^{h'} = M^{h_1, h_2} = M^{h_1 e^{X_1}, h_2 e^{X_2}} = M^{h_1 e^{X_1}} \cap M^{h_2 e^{X_2}} \subset M^{h_1} \cap M^{h_2} = M^{h_1, h_2} = M^h \]

proving the first part of the lemma. For the second, we observe that \( M^{h'_1, h'_2} = M^{h_1, h_2} \cap M^{e^{X_1}, e^{X_2}} \) for \( h_1, h_2 \) sufficiently small. Furthermore, \( M^{e^{X_1}, e^{X_2}} \) is constant for \( \varepsilon > 0 \) sufficiently small, which follows from our assumption that \( M \) can be equivariantly embedded in a finite-dimensional \( G \)-representation. Hence \( X_1 \) and \( X_2 \) both vanish on \( M^{h'} \). \qed

Given \( U^*_0 \subset \mathbb{H} \) an open subset and \( U \to \mathbb{C}^2[G] \to \text{Bun}_G(\mathcal{E}) \) an open map, consider \( U := U^*_0 \times U^*_1 \to \mathbb{H} \times \mathbb{C}^2[G] \to \text{Bun}_G(\mathcal{E}) \). The following is the main definition of the paper.

**Definition 3.2.** The sheaf of commutative differential graded algebras \( \hat{\text{Ell}}^*_G(M) \) on \( \text{Bun}_G(\mathcal{E}) \) has sections \( \alpha \in \hat{\text{Ell}}^*_G(M)(U) \) over \( U \) given by elements of the Cartan complex (conventions following (11))

\[
\alpha_h \in \Omega^*_C(h)(M^h; \mathcal{O}(U)[\beta, \beta^{-1}])
\]
for all \([h] = [h_1, h_2] \in \mathcal{C}^2[G]\) in the image of \(U_1 \to \mathcal{C}^2[G]\). These data are required to satisfy:

1. **Invariance:** for all \(g \in G\), we have \(\alpha_h = g^* \alpha_{ghg^{-1}}\), where \(g^*\) is the pullback along left multiplication by \(g\), \(M^h \to M^{ghg^{-1}}\).

2. **Analyticity:** for \(h', h_2, X_1, X_2\) in the notation of Lemma 3.1 and \(X \in (\mathfrak{t}_\mathfrak{g})_C\) we have
   \[
   \alpha_{h'}(X) = \text{res}(\alpha_h(X + (X_1 - \tau X_2))) \in \Omega^*(C(h')\mathcal{O}(U_0)[\beta, \beta^{-1}])
   \]
   where \(\text{res}: \Omega^*(\mathcal{O}(U_0)[\beta, \beta^{-1}]) \to \Omega^*(\mathcal{O}(U_0)[\beta, \beta^{-1}])\) is the restriction map of Cartan complexes associated to the inclusions \(M^{h'} \to M^h\) and \(C(h') < C(h)\) from Lemma 3.1. Compatibility with the Cartan differential also follows from Lemma 3.1.

Given an isomorphism \(U_0 \times U_1 = U \to U' = U''_0 \times U''_1\) between opens in the stack \(\text{Bun}_G(\mathcal{E})\) determined by \(\gamma \in \text{SL}_2(\mathbb{Z})\), we obtain a map \(\hat{\mathcal{E}}^k_G(M)(U') \to \hat{\mathcal{E}}^k_G(M)(U)\) induced by

\[
\Omega^*(\gamma^* \mathcal{O}(U'_0)[\beta, \beta^{-1}]) \to \Omega^*(\mathcal{O}(U_0)[\beta, \beta^{-1}])
\]

for each \(h\). The map (14) uses the pullback of functions \(\mathcal{O}(U'_0) \to \mathcal{O}(U_0)\) and the equalities \(M^h = M^{\gamma^* h}\), \(C(h) = C(\gamma \cdot h)\). We then modify this pullback map by rescaling the Lie algebra \(\mathfrak{g}^h\) by \(c\tau + d\) (so \(z \in (\mathfrak{g}^h)^\vee\) is sent to \(\overline{c\tau + d}z\)), and sending \(\beta\) to \(\beta/(c\tau + d)\).

**Remark 3.3.** We observe that \(\hat{\mathcal{E}}^*\) is natural in \(M\) and in \(G\): a map \(M \to M'\) induces a morphism of sheaves of chain complexes \(\hat{\mathcal{E}}^*_G(M') \to \hat{\mathcal{E}}^*_G(M)\) on \(\text{Bun}_G(\mathcal{E})\), and a homomorphism \(G \to H\) induces a map of sheaves of chain complexes \(\hat{\mathcal{E}}^*_G(M) \to \hat{\mathcal{E}}^*_H(M)\) over the map \(\text{Bun}_G(\mathcal{E}) \to \text{Bun}_H(\mathcal{E})\). We also have Mayer–Vietoris sequences: an open cover of \(M\) leads to an exact sequence of chain complexes of sheaves on \(\text{Bun}_G(\mathcal{E})\).

The sheaves \(\hat{\mathcal{E}}^*_G(M)\) exhibit a twisted form of 2-periodicity:

**Proposition 3.4** (Twisted Bott periodicity). There is a natural isomorphism of chain complexes

\[
\hat{\mathcal{E}}^*_G(M) \otimes \omega \to \hat{\mathcal{E}}^*_G(M)
\]

where \(\omega\) denotes the holomorphic line on \(\text{Bun}_G(\mathcal{E})\) obtained via pullback from \(\omega\) on \(M_{\ell}^\text{ell}\) along \(\text{Bun}_G(\mathcal{E}) \to M_{\ell}^\text{ell}\).

Indeed, one may view \(\beta\) as a local trivialization of the Hodge bundle over \(U_0\), as the transformation properties under \(\text{SL}_2(\mathbb{Z})\) are precisely the same as for a section of the Hodge bundle. For a fixed family of elliptic curves for \(U_0 \subset \mathbb{H}\), the Hodge bundle trivializes and so \(\Gamma(U_0; \omega^\otimes j) \simeq \mathcal{O}(U_0)\). As \(\beta\) does not define a global section of \(\hat{\mathcal{E}}_G(\text{pt})\) over \(\text{Bun}_G(\mathcal{E})\), the global sections are no longer 2-periodic precisely because the Hodge bundle is not trivializable globally. However, \(\Delta^{-1} \beta^{-12}\) is a globally defined invertible element of degree 24 where \(\Delta \in \mathcal{O}(\mathbb{H})\) is the discriminant (an invertible weight 12 modular form). This gives the global sections of \(\hat{\mathcal{E}}^*_G(M)\) a 24-periodicity.

**Definition 3.5.** Define the sheaves \(\hat{\mathcal{E}}^*_G(M)\) on \(\text{Bun}_G(\mathcal{E})\) as the cohomology sheaves of the chain complex of sheaves \(\hat{\mathcal{E}}^*_G(M)\).

As a corollary to Proposition 3.4, we also have twisted Bott periodicity at the level of cohomology sheaves.

**Corollary 3.6** (Twisted Bott periodicity). There are natural isomorphisms

\[
\hat{\mathcal{E}}^*_G(M) \otimes \omega \to \hat{\mathcal{E}}^*_G(M).
\]
Remark 3.7. The chain complex of sheaves $\widehat{Ell}^\bullet_G(M)$ allows one to consider spaces of derived global sections over $\text{Bun}_G(\mathcal{E})$, i.e., the hypercohomology groups $\mathbb{H}^i(\text{Bun}_G(\mathcal{E}); \widehat{Ell}^\bullet_G(M))$. The applications considered in this paper either concern the non-derived sections (i.e., global sections over $G$ observe that the higher cohomology is often nontrivial. For example, when $G = \text{pt}$, the nonvanishing cohomology groups are

$$\mathbb{H}^0(\text{Bun}_{U(1)}(\mathcal{E}); \widehat{Ell}^\bullet_{U(1)}(\text{pt})) \simeq \mathbb{H}^0(\mathcal{E}'; \omega^{*}/2) = MG^*$$

$$\mathbb{H}^1(\text{Bun}_{U(1)}(\mathcal{E}); \widehat{Ell}^\bullet_{U(1)}(\text{pt})) \simeq \mathbb{H}^1(\mathcal{E}'; \omega^{*}/2) = MG^{*+2}.$$  

More generally, the higher derived global sections for $G = T$ and $M = \text{pt}$ are shifted copies of modular forms.

3.2. Holomorphicity, twistings, and loop group representations. Next we show that the sheaves $\widehat{Ell}^\bullet_G(M)$ have the canonical structure of sheaves of $\mathcal{O}_{\text{Bun}_G(\mathcal{E})}$-modules. We begin in the case that $G$ is connected with torsion-free fundamental group. The analytic condition in Definition 3.2 implies the following.

Proposition 3.8. Let $G$ be connected with torsion-free fundamental group. Then there is a canonical isomorphism of sheaves

$$(15) \quad \hat{\mathcal{E}}^0_G(\text{pt}) \simeq \mathcal{O}_{\text{Bun}_G(\mathcal{E})}$$

For a $G$-manifold $M$, this implies $\hat{\mathcal{E}}^\bullet_G(M)$ is canonically a sheaf of $\mathcal{O}_{\text{Bun}_G(\mathcal{E})}$-modules.

Proof. We recall from [Bor62, Corollary 3.5] that for $G$ with the above hypothesis, centralizers of pairs of commuting elements are connected. Let $W^h$ denote the Weyl group of the centralizer $C(h)$; since $G$ is connected, $C(h)$ and $G$ share a maximal torus $T$.

We will define a morphism of sheaves $\hat{Ell}_0^G(\text{pt}) \to \mathcal{O}_{\text{Bun}_G(\mathcal{E})}$ thought of as $\text{SL}_2(\mathbb{Z})$-equivariant sheaves on $\mathbb{H} \times (T \times T)/W$. So let $U_0 \subset \mathbb{H}$ be an open subset and $U_1 \subset T \times T$ a $W$-invariant open subset. Then the value of $\hat{Ell}_0^G(\text{pt})$ on $U = U_0 \times U_1$ is given by

$$\alpha_h \in \mathcal{O}_0(t^h; \mathcal{O}(U_0))^{C(h)} \simeq \mathcal{O}_0(t^0; \mathcal{O}(U_0))^{W^h}$$

for all $h \in U_1$ where the $\alpha_h$ satisfy the conjugation invariance and analytic properties. The conjugation invariance in this case implies Weyl invariance, so that $\alpha_h$ is determined by

$$\alpha'_h \in \mathcal{O}_0(t^0; \mathcal{O}(U_0))^W.$$  

We observe the element $\alpha'_h$ of this stalk determines a $\mathcal{O}(U_0)$-valued holomorphic function

$$\hat{\alpha}'_h \in \mathcal{O}(B_\epsilon(0); \mathcal{O}(U_0))^W \simeq \mathcal{O}(U_0 \times B_\epsilon(0))^W$$

on an $\epsilon$-ball $B_\epsilon(0) \subset t^h$ for some $\epsilon > 0$. Such $\epsilon$-balls cover $U = U_0 \times U_1$, so we obtain an element of $\mathcal{O}_{\text{Bun}_G(\mathcal{E})}(U)$ by the description in the preceding if these locally defined sections are compatible. We claim that analyticity is precisely this compatibility. To see this, we pull $U_1$ back along the exponential map $\exp : t \to T \times T$. Then the holomorphic structure on $U_0 \times U_1$ for $\text{Bun}_G(\mathcal{E})$ is inherited from the holomorphic structure on $U_0 \times \exp^{-1}(U_1)$ using (7). The analytic structure in this case exactly states that $\hat{\alpha}'_h$ on a connected component of $U_0 \times \exp^{-1}(B_\epsilon(0))$ is the Taylor expansion at the preimage of $0 \in B_\epsilon(0)$ of a holomorphic function defined on the entire connected component $U_0 \times \exp^{-1}(U_1)$, with the relationship between the values on different $\epsilon$-balls coming from changing the point at which the Taylor expansion is taken. Finally, we observe that the assignment is $\text{SL}_2(\mathbb{Z})$-equivariant as this $\text{SL}_2(\mathbb{Z})$-action can be lifted to $\mathbb{H} \times t^h$. Therefore we have defined a morphism of sheaves $\hat{Ell}_0^G(\text{pt}) \to \mathcal{O}_{\text{Bun}_G(\mathcal{E})}$ on $\text{Bun}_E(\mathcal{E})$.

To check that this morphism of sheaves is an isomorphism, it suffices to check that it induces an isomorphism on stalks. But this is clear from the identifications above.  \qed
For simplicity, we use the idea of the above proposition to simply define the holomorphic structure on $\text{Bun}_G(E)$ for general compact Lie group $G$; a similar analysis to the above indicates that it agrees with any other “obvious” holomorphic structure one may have used.

**Definition 3.9.** Define the sheaf of holomorphic functions on $\text{Bun}_G(E)$ by

$$O_{\text{Bun}_G(E)} := \hat{\text{Ell}}^0_G(\text{pt}).$$

A holomorphic line bundle on $\text{Bun}_G(E)$ is a locally free rank one sheaf over $O_{\text{Bun}_G(E)}$. 

**Example 3.10.** Applying the above to Example 2.6 where $G = \text{SO}(3)$, we observe that the holomorphic structure in Definition 3.9 coincides with the obvious one in (10).

Recall that $\omega \otimes k$ denotes the sheaf on $\text{Bun}_G(E)$ that is the pullback of $\omega \otimes k$ in $O$-modules under the forgetful morphism $\text{Bun}_G(E) \to \text{Mell}$, i.e., pulled back and tensored up with $O_{\text{Bun}_G(E)}$. We now compute the $G$-equivariant elliptic cocycles for $G$ acting on a point.

**Proposition 3.11.** For $G$ acting on pt, the $G$-equivariant elliptic cocycles are the sheaves

$$\hat{\text{Ell}}^* G(\text{pt}) = \begin{cases} \omega \otimes -n/2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

(16)

equipped with the zero differential.

**Proof.** This follows from Proposition 3.8, the twisted Bott periodicity of Proposition 3.4, and the observation that there are no nonzero cocycles in odd degrees. □

When $G$ is connected with torsion-free fundamental group, Proposition 3.8 shows that we shouldn’t expect $\hat{\text{Ell}}^0_G(\text{pt})$ to have many interesting global sections. Indeed, the only global holomorphic functions on an elliptic curve are constant, and so (for example) global sections of $\hat{\text{Ell}}^0_G(\text{pt})$ pull back from functions on $\text{Mell}$: the group plays no role. More generally, if $G$ acts on $M$ so that the stabilizers are connected with torsion-free fundamental group, the global sections of $\hat{\text{Ell}}^* G(M)$ are just the ordinary de Rham complex valued in modular forms. Global sections are more interesting for twisted versions of equivariant elliptic cohomology.

**Definition 3.12.** Let $L$ be a holomorphic line bundle on $\text{Bun}_G(E)$. The $L$-twisted equivariant elliptic cocycles of a $G$ manifold $M$ is the sheaf of chain complexes $\hat{\text{Ell}}^* G(M) \otimes L$ on $\text{Bun}_G(E)$.

An important class of twists for $G$ connected arise from the Looijenga line bundles (Definition 2.8). The $L$-twisted $G$-equivariant elliptic cohomology of a point for the Looijenga twist is the sheaf whose sections are (nonabelian) theta functions.

**Proposition 3.13.** Let $G$ be a simple, simply connected compact Lie group and $L_\ell$ be the level $\ell$ Looijenga line bundle over $\text{Bun}_G(E)$. Then global sections of the twisted equivariant elliptic cohomology sheaf

$$\Gamma(\text{Bun}_G(E); \hat{\text{Ell}}^* G(\text{pt}) \otimes L_\ell) \simeq \text{Rep}^\ell (LG) \otimes \text{MF}^0$$

is the free module over the ring of modular forms generated by super characters of positive energy representations of $LG$ at level $\ell$, i.e., the vector space underlying the Verlinde algebra.

**Proof.** This follows immediately from the fact that sections of the Looijenga line bundle are spanned by the characters of loop group representations at the relevant level (e.g., see [And00 Corollary 10.9]), and super characters are differences of ordinary characters. □
3.3. Non-equivariant complex analytic elliptic cohomology. When $G = \{e\}$ is the trivial group, $\hat{\text{Ell}}(M) = \hat{\text{Ell}}_{\{e\}}^*(M)$ is a sheaf on $\text{Bun}_{\{e\}}(\mathcal{E}) = \mathcal{M}_{\text{ell}}$ whose value on $U \subset \mathbb{H} \to \mathcal{M}_{\text{ell}}$ is the 2-periodic de Rham complex

$$\hat{\text{Ell}}(M)(U) = \Omega^*(M; \mathcal{O}(U)[\beta, \beta^{-1}]).$$

The global sections of $\hat{\text{Ell}}(M)$ are given by

$$\Gamma(\mathcal{M}_{\text{ell}}; \hat{\text{Ell}}^*(M)) \simeq \bigoplus_{j+k=\bullet} \Omega^j(M; \text{MF}^k)$$

i.e., the de Rham complex of differential forms valued in modular forms. Recall again the grading convention on MF: the total degree of a cocycle is the differential form degree minus twice the modular form weight. The above complex is a cocycle model for $\text{TMF} \otimes \mathbb{C}$, the complexification of topological modular forms.

Remark 3.14. When $G = \{e\}$, the map from global sections to derived global sections of $\hat{\text{Ell}}(M)$ is a quasi-isomorphism.

3.4. Elliptic Atiyah–Segal completion. Given $h \in \mathcal{C}^2(G)$, we obtain a map

$$(17) \quad j_h : \mathbb{H}/\Gamma \to \text{Bun}_G(\mathcal{E})$$

that on objects includes at $[h] \in \mathcal{C}^2[G]$, and where $\Gamma < \text{SL}_2(\mathbb{Z})$ is the stabilizer of $[h]$ for the $\text{SL}_2(\mathbb{Z})$-action on $\mathcal{C}^2[G]$.

Proposition 3.15. There is an isomorphism of $\Gamma$-equivariant sheaves on $\mathbb{H}$ that on $U \subset \mathbb{H}$ is given by

$$j^*_h \hat{\text{Ell}}^*_G(M)(U) \simeq \Omega^*_c(h^!(M^h; \mathcal{O}(U)[\beta, \beta^{-1}])).$$

Proof. This follows from the analyticity property in Definition 3.2, Proposition A.15, and Lemma A.17. \qed

Corollary 3.16. For $M$ compact, there is an isomorphism of $\Gamma$-equivariant sheaves on $\mathbb{H}$ that on $U \subset \mathbb{H}$ is given by

$$j^*_h \hat{\text{Ell}}^*_G(M)(U) \simeq \begin{cases} 
\mathcal{H}^{ev}_{c}(h^!(M^h; \mathcal{O}(U))) & k \text{ even} \\
\mathcal{H}^{odd}_{c}(h^!(M^h; \mathcal{O}(U))) & k \text{ odd}
\end{cases}$$

i.e., the 2-periodic Borel equivariant cohomology of $M^h$ with its $C(h)$ action.

A special case of the above takes $h = (e, e)$, where (17) is the map $j_e : \mathcal{M}_{\text{ell}} \to \text{Bun}_G(\mathcal{E})$ that assigns to each elliptic curve the trivial $G$-bundle over that curve. This allows us to compare complex analytic equivariant elliptic cohomology to the Borel equivariant refinement of $\hat{\text{Ell}}(M)$ as follows. For a $G$-manifold $M$, the Borel equivariant refinement is the sheaf $\hat{\text{Ell}}^*_{G, \text{Bor}}$ on $\mathcal{M}_{\text{ell}}$ whose value on $U \subset \mathbb{H} \to \mathcal{M}_{\text{ell}}$ is

$$(18) \quad \hat{\text{Ell}}^*_{G, \text{Bor}}(M)(U) = \Omega^*_G(M; \mathcal{O}(U)[\beta, \beta^{-1}]),$$

i.e., a chain complex that computes the 2-periodic Borel equivariant cohomology of $M$.

Theorem 3.17 (Atiyah–Segal completion). The restriction of the sheaf $\hat{\text{Ell}}^*_{G}(M)$ along the section $j_e : \mathcal{M}_{\text{ell}} \to \text{Bun}_G(\mathcal{E})$ associated with the trivial $G$-bundle gives an isomorphism of sheaves of commutative differential graded algebras on $\mathcal{M}_{\text{ell}}$

$$j^*_e \hat{\text{Ell}}^*_{G}(M) \simeq \hat{\text{Ell}}^*_{G, \text{Bor}}(M).$$

Proof. This is a special case of Proposition 3.15. \qed
3.5. An example. Consider the sheaf $\hat{\mathcal{E}}_{U(1)}(S^2)$ on $\text{Bun}_{U(1)}(E)$ for the $U(1)$-action on $S^2 = \mathbb{C}P^1$ that rotates the sphere about an axis. We observe that for $h \in U(1) \times U(1)$ not equal to $(e, e)$, the fixed points are the poles $(S^2)^h = \{\text{poles}\}$. Since $\Omega^*_{U(1)}(\{\text{poles}\}) \simeq \mathbb{C} \oplus \mathbb{C}$ with the zero differential, we have the isomorphism of sheaves

$$\hat{\mathcal{E}}^k_{U(1)}(S^2)|_{E^\vee \setminus \{0 \text{ section}\}} \simeq \omega^{-k} \oplus \omega^{-k}.$$  

(19)

Next, a section defined in a small neighborhood of the zero section in $E^\vee \simeq \text{Bun}_{U(1)}(E)$ is determined by an element of the stalk $\Omega^*_{U(1)}(S^2; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])$. Extending this section to a larger neighborhood demands a compatibility with (19) given by the restriction map

$$\Omega^*_{U(1)}(\{\text{poles}\}; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}]) \simeq \left(\mathcal{O}_0(\mathcal{C}; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])\right)^{\otimes^2} \rightarrow \left(\mathcal{O}_0(\mathbb{C} \setminus \{0\}; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])\right)^{\otimes^2},$$

where the last map restricts a germ of a holomorphic function at the origin in $\mathbb{C}$ to one in a punctured neighborhood of the origin. We then identify this neighborhood in $\mathbb{C}$ with a neighborhood of zero in $E^\vee$ which we then identify with a function on a punctured neighborhood of $0$ in $E^\vee$ (which is uniquely specified by the analytic condition in Definition 3.2) and $\beta^n$ with a section of $\omega^{\otimes n}$. A global section of $\hat{\mathcal{E}}_{U(1)}(S^2)$ is therefore given by an element of $\Omega^*_{U(1)}(S^2; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])^{SL_2(\mathbb{Z})} = \Omega^*_{U(1)}(S^2; \mathcal{E})$ (i.e., a Borel equivariant cocycle) whose restriction to the poles is a constant function on the Lie algebra of $U(1)$.

More generally, $\text{Bun}_{U(1)}(E)$ admits Looijenga line bundles $\mathcal{L}_\ell$ parametrized by levels $\ell \in H^1(BU(1); \mathbb{Z}) \simeq \mathbb{Z}$ and we have the twisted equivariant elliptic cohomology $\hat{\mathcal{E}}_{U(1)}(S^2) \otimes \mathcal{L}_\ell$. We recall that global sections of $\hat{\mathcal{E}}_{U(1)}(pt) \otimes \mathcal{L}_\ell$ over $\text{Bun}_{U(1)}(E)$ are theta-functions (or Jacobi forms) of index $\ell$, $\Gamma(\text{Bun}_{U(1)}(E), \hat{\mathcal{E}}_{U(1)}(pt) \otimes \mathcal{L}_\ell) \simeq \bigoplus_k \mathcal{JF}_{k,\ell},$

where $\mathcal{JF}_{k,\ell}$ is the space of (weakly holomorphic) Jacobi forms of weight $k$ and index $\ell$ sitting in degree $-2k$. Global sections of $\hat{\mathcal{E}}_{U(1)}(S^2) \otimes \mathcal{L}_\ell$ are then given by

$$\Gamma(\text{Bun}_{U(1)}(E), \hat{\mathcal{E}}_{U(1)}(S^2) \otimes \mathcal{L}_\ell) \simeq \Omega^*_{U(1)}(S^2; \mathcal{E}) \times_{\Omega^*_{U(1)}(\{\text{poles}\}; \mathcal{JF}_{*,\ell})} \Omega^*_{U(1)}(\{\text{poles}\}; \mathcal{JF}_{*,\ell}).$$

In words, these are $U(1)$-equivariant, modular form-valued differential forms on $S^2$ whose restriction to the poles are germs of Jacobi forms of index $\ell$.

4. Comparing with Grojnowski’s equivariant elliptic cohomology

Let $G$ be a connected Lie group, $M$ a $G$-space, and $\tau \in \mathbb{H}$ a point defining an elliptic curve $E = E_\tau$. Grojnowski [Gro07] constructs a $\mathbb{Z}/2$-graded sheaf $\mathcal{E}ll^{\mathcal{E}ll}(M)$ of $\mathcal{O}_{E^{\vee}, \text{rk}(T)/\mathcal{L}_W}$-modules on $E^{\vee, \text{rk}(T)}/\mathcal{L}_W \subset \text{Bun}_G(E)$. We compare this with the cocycle model from the previous section. We recall $\mathcal{E}ll^{\mathcal{E}ll}_{G}(M)$ are the cohomology sheaves of $\hat{\mathcal{E}}_{U(1)}(S^2)$; see Definition 3.5.

**Theorem 4.1.** The pullback in $\mathcal{O}$-modules of $\mathcal{E}ll^{\mathcal{E}ll}_{G}(M)$ to $\{\tau\} \times (T \times T)/\mathcal{L}_W \hookrightarrow \text{Bun}_G(E)$ is naturally isomorphic to the 2-periodic version of $\mathcal{E}ll^{\mathcal{E}ll}_{G}(M)$.

4.1. A review of Grojnowski’s equivariant elliptic cohomology. Our presentation below hews closely to the original source [Gro07], though we also refer to [Ros01, §3] for an accounting when $G = U(1)$. To begin, let $G = T$ be a torus. As we have fixed a curve $E = E_\tau$, the identification (5) specializes to $T \times T \simeq E^{\vee, \text{rk}(T)} = \text{Bun}_T(E)$. For $h = (h_1, h_2) \in T \times T \simeq E^{\vee, \text{rk}(T)}$, let $L_h: E^{\vee, \text{rk}(T)} \rightarrow E^{\vee, \text{rk}(T)}$ denote left multiplication by $h$, and let $U_h^c \subset T \times T \simeq E^{\vee, \text{rk}(T)}$ be an open subset diffeomorphic to an open ball in $t_c$ (specified as in (12)) with the property that for all $h' = (h_1', h_2') \in U_h^c$, we have $M^{h'} \subset M^h$ and $C(h') < C(h)$. Existence of this open is guaranteed by Lemma 3.1.
**Definition 4.2** (Grojnowski). For a $T$-manifold $M$, define a sheaf $\text{Ell}^{\text{Groj}}_T(M)$ on $E^{\vee, \text{rk}(T)}$ that assigns to each $U_h^\circ \subset E^{\vee, \text{rk}(T)}$ the $\mathbb{Z}/2$-graded $\mathcal{O}_{U_h^\circ}$-module

$$
\Gamma(U_h; \text{Ell}^{\text{Groj}}_T(M)) := L_h^*(H_T(M^h) \otimes_{H_T(pt)} L_{h^{-1}}^1 \mathcal{O}(U_h^\circ))
$$

where we have identified $H_T(pt) \simeq S(t_C^\vee)$ with the polynomial algebra on $t_C^\vee$, which is a subalgebra of holomorphic functions $\mathcal{O}(U_h^\circ)$. Define restriction maps on open subsets $U_h'^\circ \subset U_h^\circ$ with $h \notin U_h'^\circ$ by

$$
i^*: H_T(M^h) \otimes_{H_T(pt)} L_{h^{-1}}^1 \mathcal{O}(U_h^\circ) \to H_T(M^{h'}) \otimes_{H_T(pt)} L_{h'^{-1}}^1 \mathcal{O}(U_{h'}^\circ)
$$

induced by pulling back along the inclusions $M^{h'} \hookrightarrow M^h$ and the isomorphism from pulling back along left multiplication

$$
L_{h'^{-1}h}^*: L_{h^{-1}}^1 \mathcal{O}(U_h^\circ) \xrightarrow{\sim} L_{h'^{-1}}^1 \mathcal{O}(U_{h'}^\circ).
$$

By Atiyah–Bott localization [AB84, Theorem 3.5], (20) is an isomorphism, and so this data on opens defines a sheaf without a need for further sheafification.

Let $T < G$ be a maximal torus, and $h = (h_1, h_2) \in T \times T \subset G \times G$. Observe that $T < C(h)$ is a maximal torus for the connected component of the identity of $C(h)$. Define $W^h = (C(h) \cap N(T))/T$. Let $C(h)^0 < C(h)$ be the connected component of the identity. The Weyl group $W(C(h)^0)$ and $W^h$ are related by the exact sequence

$$
1 \to W(C(h)^0) \to W^h \to C(h)/C(h)^0 \to 1,
$$

Note that with this definition, we have $H_{C(h)(pt)} \simeq (H_T(pt))^W$.

Let $U_h \subset T \times T$ be an open subset as above that also satisfies $W^h U_h = U_h$ and $W^h \cap U_h = \emptyset$ if $w \in W$ but $w \notin W^h$. Let $U_h^w$ denote the orbit of $U_h$ under the action of the Weyl group, so that $U^w_h$ is a $W$-invariant open subset of $E^{\vee, \text{rk}(T)}$.

**Definition 4.3** (Grojnowski). For $G$ connected, define a sheaf $\text{Ell}^{\text{Groj}}_G(M)$ on $E^{\vee, \text{rk}(T)}/G$ that assigns to each $W$-invariant open $U_h^w \subset E^{\vee, \text{rk}(T)}$ the $\mathbb{Z}/2$-graded $\mathcal{O}_{U_h^w}$-module

$$
\Gamma(U_h^w, \text{Ell}^{\text{Groj}}_G(M)) := L_h^*(H_{C(h)}(M^h) \otimes_{H_{C(h)}(pt)} L_{h^{-1}}^1 \mathcal{O}(U_h^w)^{W^h})
$$

where we use the isomorphism $H_{C(h)(pt)} \simeq H_{C(h)^0}(pt)$ to define the tensor product. The transition maps are defined identically to in the case that $G = T$.

4.2. The comparison map.

**Proof of Theorem 4.4.** Let $U \subset T \times T$ be a $W$-invariant open subset. Then the value of $\text{Ell}_U^\bullet(M)$ on $\{\tau\} \times U$ is given by

$$
\alpha_h \in \mathcal{O}_0(t_C^\circ; \Omega^\bullet(M^h; \mathbb{C}[[\beta, \beta^{-1}]])^{C(h)} \simeq \mathcal{O}_0(t_C; \Omega^\bullet(M^h; \mathbb{C}[[\beta, \beta^{-1}]])^{W^h}
$$

for all $h \in U$ satisfying the conjugation invariance and analyticity properties. The conjugation invariance in this case implies Weyl invariance, so that $\alpha_h$ is determined by

$$
\alpha_h' \in \left( \bigoplus_{w \in W/W^h} \mathcal{O}_0(t_C; \Omega^\bullet(M^{w-h}; \mathbb{C}[[\beta, \beta^{-1}]]) \right)^W.
$$

We observe the above element $\alpha_h'$ determines $\Omega^\bullet(M^{w-h}; \mathbb{C}[[\beta, \beta^{-1}])$-valued holomorphic functions on some $B_\epsilon(0) \subset t_C$ for $\epsilon > 0$

$$
\tilde{\alpha}_h^\epsilon \in \left( \bigoplus_{w \in W/W^h} \mathcal{O}(B_\epsilon(0); \Omega^\bullet(M^{w-h}; \mathbb{C}[[\beta, \beta^{-1}]]) \right)^W.
$$
Such \( \epsilon \)-balls cover \( U \). If each \( \tilde{a}_h^a \) is also closed under the Cartan differential we obtain classes

\[
[\tilde{a}_h^a] \in \left( \bigoplus_{w \in W/W^h} H(B_{\epsilon}(0); \Omega^*(M^{w-h}; \mathbb{C}[\beta, \beta^{-1}]), Q) \right)^W
\]

\[
\simeq \left( \bigoplus_{w \in W/W^h} H_T(M^{w-h}) \right)^W \otimes_{H_T(pt)^W} \mathcal{O}(B_\epsilon(0); \mathbb{C}[\beta, \beta^{-1}]). \]

The correspondence between 2-periodic cohomology and \( \mathbb{Z}/2 \)-graded cohomology yields a class corresponding to \( \tilde{a}_h^a \) in Grojnowski’s equivariant elliptic cohomology sheaf on each open ball. The analyticity condition guarantees that these classes glue to give a section of \( \text{Ell}^{\text{Groj}}_G(M) \) over \( U \): the translations in Grojnowski’s formulas are precisely the translations appearing in the analytic condition. Finally one must check that this is an isomorphism on stalks, but this is clear from the maps defined on each \( B_\epsilon(0) \).

\[ \square \]

5. Comparing with Devoto’s equivariant elliptic cohomology

In this section we compare our model with previous ones for \( G \)-equivariant elliptic cohomology where \( G \) is finite. The definition of Devoto’s equivariant elliptic cohomology we adopt is used by Ganter [Gan09] and Morava [Mor09] in their studies of generalized moonshine; it is also the complexification of a version of equivariant elliptic cohomology appearing in the work of Baker and Thomas [BT99]. These definitions are based on the early work of Devoto [Dev96, Dev98], simplifying his construction over \( \mathbb{Z}[1/2, 1/3] \) to one over \( \mathbb{C} \), and replacing the congruence subgroup \( \Gamma_0(2) \) by the full modular group \( \text{SL}_2(\mathbb{Z}) \). As such, we refer to this finite group version of equivariant elliptic cohomology as Devoto’s equivariant elliptic cohomology, \( \text{Ell}^G_{\text{Dev}}(M) \), to be defined shortly; we first state the main theorem of the section.

**Theorem 5.1.** For \( G \) finite, the space of global sections of \( \text{Ell}^G_*(M) \) over \( \text{Bun}_G(\mathcal{E}) \) is Devoto’s equivariant elliptic cohomology over \( \mathbb{C} \), i.e.,

\[
\Gamma(\text{Bun}_G(\mathcal{E}), \text{Ell}^G_*(M)) \simeq \text{Ell}^{\text{Dev}}_G(M).
\]

5.1. A review of Devoto’s equivariant elliptic cohomology. Consider

\[
\text{SL}_2(\mathbb{Z}) \subset \left( \bigoplus_{h \in C^2(G)} H^*(M^h; \mathcal{O}(\mathbb{H})) \right)^G
\]

where \( \text{SL}_2(\mathbb{Z}) \) acts through the indexing set \( C^2(G) = \text{Hom}(\mathbb{Z}^2, G) \) by precomposition and on \( H \) through the usual fractional linear transformations. The \( G \)-invariants in \( \text{[22]} \) are taken with respect to the \( G \)-action by conjugation on \( C^2(G) \) and left multiplication \( L_g : M^h \rightarrow M^{gbg^{-1}} \) on fixed point sets. The following is an adaptation of [Dev98, Definition 3.2] to complex coefficients and the full modular group \( \text{SL}_2(\mathbb{Z}) \).

**Definition 5.2.** Let \( G \) be a finite group and \( M \) a \( G \)-manifold \( M \). Define Devoto’s \( G \)-equivariant elliptic cohomology of \( M \) as a subspace

\[
\text{Ell}^G_{\text{Dev}, k}(M) \subset \bigoplus_j \left( \bigoplus_{h \in C^2(G)} H^j(M^h; \mathcal{O}(\mathbb{H})) \right)^G
\]

whose \( j \)-th summand consists of functions that transform under the \( \text{SL}_2(\mathbb{Z}) \)-action \( \text{[22]} \) with weight \( (j-k)/2 \) (so in particular, \( j-k \) must be even for the \( j \)-th summand to be nonzero).

**Remark 5.3.** We recall that our definition of \( \mathcal{O}_\mathbb{H} \) takes sections on any unbounded open to be holomorphic functions with the additional condition of polynomial growth along any geodesics that escape to the boundary, or equivalently, the usual meromorphicity condition at the cusps so that global sections are weakly holomorphic modular forms. We re-emphasize this point now as the modularity condition for classes in Devoto’s equivariant elliptic cohomology will typically be for finite-index subgroups of \( \text{SL}_2(\mathbb{Z}) \), in which case we remark that our convention agrees with the usual notion of weakly holomorphic modular forms of
higher level (in terms of imposing meromorphicity at all cusps). Devoto imposes this same condition in terms of Fourier expansions in $e^{2\pi i \tau |G|} = q^{1/|G|}$ for $\tau \in \mathbb{H}$.

5.2. The comparison map.

Proof of Theorem 5.1. We evaluate $\text{Ell}_G^*(M)$ on the cover $\mathbb{H} \times C^2(G)$ of $\text{Bun}_G(E)$, and then compute the action of $G \times \text{SL}_2(\mathbb{Z})$. On this cover, a section is the data of

$$[\alpha_h] \in H^*(M^h; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])^{C(h)}$$

for each $h \in C^2(G)$, satisfying a conjugation invariance property and an $\text{SL}_2(\mathbb{Z})$-equivariance property; the analytic property in this case is trivially satisfied because the Lie algebra is the zero vector space (and $C^2(G)$ is discrete). Conjugation invariance implies that these $[\alpha_h]$ assemble into a class

$$[\alpha] \in \left( \bigoplus_h H^*(M^h; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}]) \right)^G.$$

Finally, the $\text{SL}_2(\mathbb{Z})$-invariance extracts Devoto’s $\text{Ell}_{G}^{\text{Dev}, k}(M)$: invariant classes come with a power of $\beta$ that reads off the weight of the $\text{SL}_2(\mathbb{Z})$-action.

6. Loop group representations and cocycle representatives of Thom classes

In this section we construct cocycle representatives of universal equivariant Euler and Thom classes in complex analytic equivariant elliptic cohomology. We connect these refinements with the representation theory of loop groups. This structure can be thought of as an elliptic version of Chern–Weil theory: characteristic classes in (non-equivariant) complex analytic elliptic cohomology are determined by universal equivariant classes, which in turn are constructed out of Lie-theoretic data. The approach applies to both real and complex vector bundles, recovering universal characteristic classes for the complexifications of the MString- and $\text{MU}(6)$-orientations of TMF, respectively. Some structural aspects of this story were previously known to Ando in a somewhat different language [And00, And03]. The cocycle-level description is new, which leads to a more explicit treatment.

6.1. Review from K-theory and ordinary cohomology. Let $V \to M$ be a real $d$-dimensional vector bundle. The Thom class of $V$ in ordinary cohomology $[\text{Th}_V] \in H^d_s(V)$ has compact vertical support and the property that the exterior product map

$$H^*(M) \sim H^*_{cs}^{d}(V) \quad [\alpha] \mapsto [\alpha] \boxtimes [\text{Th}_V]$$

is an isomorphism, called the Thom isomorphism. The Euler class $E_{U} \in H^d(M)$ is the pullback of $[\text{Th}_V]$ along the zero section $0 \to V$. The Euler class for line bundles encodes the additive formal group law, and the Thom class determines pushforwards in cohomology using the Pontrjagin–Thom collapse map. The Euler and Thom class are both natural for the vector bundle $V$, and so they are determined by universal Euler and Thom classes for the universal bundle over $\text{BSO}(n)$. An analogous story for complex vector bundles again yields universal Euler and Thom classes for the universal bundle over $\text{BU}(n)$. The cohomology of these classifying spaces is the equivariant cohomology of a point

$$H(\text{BU}(n)) = H_{U(n)}(pt), \quad H(\text{BSO}(n)) = H_{SO(n)}(pt)$$

so that universal Euler and Thom classes are (canonically) classes in equivariant cohomology.

For K-theory one again finds Euler and Thom classes living in equivariant refinements. However, the existence of refinements is a more interesting question if one considers the non-Borel version of equivariant K-theory coming from equivariant vector bundles. For example, we recall that the Euler class of a complex vector bundle $V$ in K-theory is the class underlying the virtual vector bundle $\Lambda^\text{ev} V - \Lambda^\text{odd} V$. By universal properties, the Euler class is determined by the corresponding virtual vector bundle on $\text{BU}(n)$. It admits an equivariant refinement,

$$\text{Rep}(U(n)) = K_{U(n)}(pt) \xrightarrow{\text{completion}} K(\text{BU}(n))$$
as the virtual representation \( \Lambda^v R - \Lambda^{\text{odd}} R \) where \( R \) is the defining representation of \( U(n) \) and \( \text{[23]} \) is the Atiyah–Segal completion map. There is a similar story for equivariant refinements of Thom classes, as well as analogous constructions in KO-theory.

Below we construct refinements of Euler and Thom classes in elliptic cohomology with two goals that run in analogy to \( \text{[23]} \): (i) refine pre-existing non-equivariant classes, and (ii) give representation-theoretic meaning to the refinements.

6.2. The Weierstrass sigma function and loop group representations. At the heart of the construction is the \textit{Weierstrass sigma function}

\[
\sigma(\tau, z) = \left( y^{1/2} - y^{-1/2} \right) \prod_{n>0} \frac{(1 - q^n y)(1 - q^n y^{-1})}{(1 - q^n)^2}
\]

\((\text{24})\)

where \( \tau \in \mathbb{H}, q = \exp(2\pi i \tau), z \in \mathbb{C}, y = \exp(2\pi iz) \) and \( y^{1/2} = \exp(\pi iz) \), and \( E_{2k}(\tau) \in \mathcal{O}(\mathbb{H}) \) is the \( 2k \)-th Eisenstein series

\[
E_{2k}(\tau) = \sum_{n,m \in \mathbb{Z}^2} \frac{1}{(n\tau + m)^{2k}}, \quad \mathbb{Z}_*^2 = \{(n, m) \in \mathbb{Z}^2 \mid (n, m) \neq (0, 0)\}
\]

for \( k > 1 \), and we take \( E_2 \) to be the standard holomorphic version of the 2nd Eisenstein series (the above sum is conditionally convergent when \( k = 1 \)). The equality \((\text{24})\) is proved (for example) in \[\text{AHR10} \text{ Proposition 10.9}\]. The sigma function is a Jacobi form of weight \(-1\) and index \(1/2\). We also consider the closely related function

\[
v(q, z) = e^{-\pi iz} \sigma(q, z) = \left( 1 - e^{-2\pi iz} \right) \prod_{n>0} \frac{(1 - q^n y)(1 - q^n y^{-1})}{(1 - q^n)^2}.
\]

The relevance of the \( \sigma \)-function in elliptic cohomology originally came by way of the \textit{Witten genus}, the Hirzebruch genus associated with the power series

\[
\text{Wit}(z) = \frac{z}{\sigma(q, z)} = \exp \left( \sum_{k>0} \frac{E_{2k} z^k}{2k} \right) \in \mathbb{C}[z, q].
\]

The families refinement of the genus leads to the \( \text{MU}(6) \) and MString orientations of topological modular forms reviewed in the next subsection \[\text{Hop94} \text{ AHS01} \text{ AHR10}\].

The Weierstrass sigma function is also closely related to (super) characters of loop group representations. Recall that there are two irreducible representations of \( \text{Spin}(2n) \), usually denoted \( S^+ \) and \( S^- \), and the character of the \( \mathbb{Z}/2 \)-graded representation \( S^+ - S^- \) is the product

\[
\prod_{j=1}^{n} 2 \sinh(2\pi iz_j) = \prod_{j=1}^{n} (\exp(\pi iz_j) - \exp(-\pi iz_j))
\]

where \( z_j \) are the Chern roots. Following \[\text{Liu96} \text{ §1.2}\], there are 4 irreducible representation of \( \text{LSpin}(2n) \) at level 1. One lifts the super character \((\text{26})\) of \( \text{Spin}(2n) \) to

\[
\chi_{\text{LSpin}(2n)} = \eta(\tau)^{2n} \prod_{j=1}^{n} \sigma(\tau, z_j) \in \mathcal{O}(\mathbb{H} \times \mathfrak{t}_j).
\]

where \( \eta(\tau) \) is the Dedekind \( \eta \)-function. For \( U(n) \), there is a similar story: starting with the standard representation of \( U(n) \) on \( \mathbb{C}^n \), consider the \( \mathbb{Z}/2 \)-graded representation on \( \Lambda^v \mathbb{C}^n \simeq \Lambda^v \mathbb{C}^n - \Lambda^{\text{odd}} \mathbb{C}^n \) whose character is \( \prod_{j=1}^{n} (1 - \exp(-2\pi iz_j)) \). This has a lift to
a loop group representation via the vacuum representation of $LU(n)$ associated with the fermionic Fock space construction, whose super character is given by

$$\chi_{LU(n)} = \eta(\tau)2^n \prod_{j=1}^n \psi(\tau, z_j) \in \mathcal{O}(\mathbb{H} \times t_C).$$

For example, see [And00, Equation 11.10]. It will be convenient to consider characters normalized by powers of the $\eta$-function, namely $\chi_{LSpin(2n)}/\eta^{2n}$ and $\chi_{LU(n)}/\eta^{2n}$; below, we will simply refer to these functions as loop group characters.

### 6.3. Characteristic classes in (non-equivariant) elliptic cohomology over $\mathbb{C}$.

The universal elliptic cohomology theory of topological modular forms has Thom and Euler classes for $BU(\langle 6 \rangle)$ and $BO(\langle 8 \rangle)$ bundles. We recall that $BU(\langle 6 \rangle)$ is the classifying space for (stable) complex vector bundles with $c_1 = c_2 = 0$; $BO(\langle 8 \rangle) = BString$ classifies (stable) real vector bundles with $w_1 = w_2 = \frac{p_1}{2} = 0$. These classifying spaces sit in the diagram

$$
\begin{array}{ccc}
BU(\langle 6 \rangle) & \rightarrow & BSU \\
| & & | \\
BString & \rightarrow & BSpin \\
\end{array}
\rightarrow
\begin{array}{ccc}
BU & \rightarrow & BU \\
| & & | \\
BSO & \rightarrow & BSO.
\end{array}
$$

Remark 6.1. The notation $BU(\langle 6 \rangle)$ and $BO(\langle 8 \rangle)$ comes from canonical maps $BU(\langle 6 \rangle) \rightarrow BU$ and $BO(\langle 8 \rangle) \rightarrow BO$ giving the 5-connected cover and the 7-connected cover in the Whitehead towers of $BU$ and $BO$, respectively.

Let $MString = MO(\langle 8 \rangle)$ and $MU(\langle 6 \rangle)$ denote the Thom spectrum associated with the universal bundle on $BString = BO(\langle 8 \rangle)$ and $BU(\langle 6 \rangle)$, respectively. The $\sigma$-orientation of $TMF$ is a map

$$\sigma: MString \rightarrow TMF$$

that assigns a (vertically) compactly supported Thom class $Th_V \in TMF_{\mathbb{C}}^m(V)$ to an $m$-dimensional real vector bundle $V \rightarrow M$ with string structure [AHS01, AHR10]. The Chern–Dold character is a map

$$ch: TMF(M) \rightarrow H(M; TMF(pt) \otimes \mathbb{C}) \simeq H(M; MF)$$

from $TMF$ to ordinary cohomology with coefficients in the graded ring of modular forms $MF$. The Riemann–Roch theorem compares the Thom class in $TMF$ with the Thom class $u_V$ in ordinary cohomology by means of the commuting square

$$
\begin{array}{ccc}
TMF^\bullet(M) & \xrightarrow{ch} & H^\bullet(M; MF) \\
\downarrow Th_V & & \downarrow [u_V \cdot \text{Wit}(V)^{-1}] \\
TMF_{cs}^{+ \infty}(V) & \xrightarrow{ch} & H_{cs}^{+ \infty}(V; MF).
\end{array}
$$

where the vertical arrows are exterior multiplication with the indicated class, $\text{Wit}(V)$ is the characteristic class associated with the power series [25], and the cohomology groups and $TMF_{cs}^{+ \infty}(V)$ and $H_{cs}^{+ \infty}(V; MF)$ are with compact vertical support. This defines the elliptic Thom class in $H_{cs}^{+ \infty}(V; MF)$ as the class $[u_V \cdot \text{Wit}(V)^{-1}]$. The elliptic Euler class is gotten by pulling back along the zero section, and is just $[e_V \cdot \text{Wit}(V)^{-1}]$ where $e_V$ is the ordinary Euler class of $V$. We have two flavors of these classes, depending on whether $V$ is real (as was assumed above) or complex, corresponding to the $MU(\langle 6 \rangle)$ or $MString$ orientation respectively. These orientations are of course related by precomposing (28) with the map $MU(\langle 6 \rangle) \rightarrow MO(\langle 8 \rangle)$ coming from taking Thom spectra of universal bundles in [27].
6.4. Descending equivariant Euler cocycles from an atlas. We define universal Euler classes for $U(n)$ and $SO(2n)$ in terms of functions on $\mathbb{H} \times t_C$ and then study descent to the stack along the map (see Example 2.5)

\[ \mathbb{H} \times t_C \to \text{Bun}_G(\mathcal{E}), \quad G = U(n), SO(2n). \]

**Definition 6.2.** For \( \{z_1, \ldots, z_n\} \) coordinates on \( t_C \), and \( q = \exp(2\pi i \tau) \) for \( \tau \in \mathbb{H} \), consider the holomorphic functions on \( \mathbb{H} \times t_C \) given by

\[ \sigma_{U(n)}(q, z_1, \ldots, z_n) = \prod_{j=1}^n \tau_j = \prod_{j=1}^n e^{-\pi i z_j} \sigma(\tau, z_j) \in \mathcal{O}(\mathbb{H} \times t_C) \]

\[ \sigma_{SO(2n)}(q, z_1, \ldots, z_n) = \prod_{j=1}^n \tau_j \in \mathcal{O}(\mathbb{H} \times t_C). \]

It turns out that neither \( \sigma_{U(n)} \) nor \( \sigma_{SO(2n)} \) descend to the (neutral component of the) stacks \( \text{Bun}_{U(n)}(\mathcal{E}) \) and \( \text{Bun}_{SO(2n)}(\mathcal{E}) \), respectively. In the case of \( \sigma_{U(n)} \), the problem is that the factor \( \prod e^{\pi i z_j} \) does not transform well under the action of \( SL_2(\mathbb{Z}) \). This problematic factor vanishes if we require \( c_1 = \sum z_j = 0 \), i.e., we restrict to the subgroup \( SU(n) \subset U(n) \).

For \( \sigma_{SO(2n)} \), the problem is that the factor \( e^{\pi i z_j} - e^{-\pi i z_j} \) requires a square root of the Chern roots, which requires we pass from \( SO(2n) \) to its double cover, \( Spin(2n) \). These necessary adjustments to the above universal formulae turn out to be sufficient for descent to the corresponding stacks. We recall that for the simply-connected simple Lie groups \( SU(n) \) and \( Spin(2n) \), the possible Looijenga line bundles are labeled by an integer \( \ell \in \mathbb{Z} \), called the level; e.g., see [And00, 310].

**Proposition 6.3.** The functions \( \sigma_{U(n)} \) and \( \sigma_{SO(2n)} \) given by the formulas (31) and (32) respectively define global sections

\[ E_{U(n)} := \sigma_{U(n)} \cdot \beta^{-n} \in \Gamma(\text{Bun}_{U(n)}(\mathcal{E}); \mathcal{E}\mathbb{H}^2_{SU(n)}(pt) \otimes \mathcal{L}) \]

\[ E_{Spin(2n)} := \sigma_{SO(2n)} \cdot \beta^{-n} \in \Gamma(\text{Bun}_{Spin(2n)}(\mathcal{E}); \mathcal{E}\mathbb{H}^2_{Spin(2n)}(pt) \otimes \mathcal{L}) \]

twisted by the level 1 Looijenga line bundle \( \mathcal{L} \) for \( G = SU(n) \) or \( Spin(2n) \), respectively. We identify the sections \( E_{U(n)} \) and \( E_{Spin(2n)} \) of the Looijenga line with the super character of the level 1 vacuum representation of \( LSU(n) \) and \( LSpin(2n) \), respectively.

**Proof.** The expressions for the super characters of loop group representations are well-known, given by \( \sigma_{U(n)} \) and \( \sigma_{SO(2n)} \) above. Identifying these characters with sections of the Looijenga line bundle is also well-known, essentially being a consequence of the Weyl–Kac character formula; for example, see [And00, Corollary 10.9]. These functions transform with weight \( -n \) under the action of \( SL_2(\mathbb{Z}) \), so multiplying by \( \beta^{-n} \) puts us in the correct degree and determines an invariant section of \( \tilde{E}\mathbb{H}^2_{G}(pt) \otimes \mathcal{L} \).

**Remark 6.4.** Invariance of \( \sigma_{SO(2n)} \) and \( \sigma_{U(n)} \) under the action of the Weyl groups of \( Spin(2n) \) and \( SU(n) \), respectively, can be checked explicitly, but it also follows from the Weyl–Kac character formula.

6.5. Constructing equivariant Thom classes. We shall again define the Thom classes in terms of descent along (30), but in this case we define a section \( Th_T \) of the Looijenga line bundle in terms of compatible values at stalks (\( Th_T \)) on the pullback to the cover where \( h \in t \times t \).

For \( G = U(1) \) or \( Spin(2) \), recall that the ordinary (non-elliptic) equivariant Mathai–Quillen Thom form on \( V = \mathbb{R}^2 \cong \mathbb{C} \) is given by

\[ u_V = \frac{1}{\pi} e^{-|z|^2} (\beta^{-1} z + d \text{vol}) \in \Omega^2_G(V) \]

\[ 1 \text{Note that Spin}(2n) \text{ is not actually simply-connected for } n = 1, \text{ but its Looijenga lines are still classified by a level } \ell \in H^4(BSpin(2); \mathbb{Z}) \cong \mathbb{Z} \text{ and we hence need make no disclaimers for this special case.} \]
where \( z \in \mathbb{C} \) is the generator (the occurrence of \( \beta \) is from our grading conventions; see [3.1] and \( \text{dvol} \in \Omega^2(V) \) is the orientation 2-form. Also recall that the ordinary equivariant Euler class is just given by \( \beta^{-1}z \in \Omega^2_G(\text{pt}) \).

Let \( V = \mathbb{C}^n \) be the standard representation of \( \text{SU}(n) \), and \( V = \mathbb{R}^{2n} \) denote the representation of \( \text{Spin}(2n) \) factoring through \( \text{SO}(2n) \). For \( h = (h_1, h_2) \in T \times T \), let \( V^h_+ \subset V \) be the orthogonal complement of the fixed point subspace \( V^h \subset V \). Choose logarithms \( h_1, h_2 \in \mathfrak{t} \) so that \( h_i = \exp(h_i) \) and a basis of \( \mathfrak{t} \) so that
\[
\tilde{h}_1 = \text{diag}(0, \ldots, 0, \tilde{h}_1^{k+1}, \ldots, \tilde{h}_1^n), \quad \tilde{h}_2 = \text{diag}(0, \ldots, 0, \tilde{h}_2^{k+1}, \ldots, \tilde{h}_2^n), \quad \tilde{h}_j^i \in \mathbb{R}.
\]

**Definition 6.5.** The \( \text{SU}(n) \)-equivariant elliptic Thom form at \( \tilde{h} = (\tilde{h}_1, \tilde{h}_2) \) is defined as
\[
(\text{Th}_{\text{SU}(n)})_{\tilde{h}} = \left( \prod_{j=1}^{k} \frac{u_j}{z_j} \right) \left( \prod_{j=1}^{n} v(\tau, z_j + \tilde{h}_1^j - \tau \tilde{h}_2^j) \right) \beta^{k-n} \in \Omega^2_G(V^h; \mathcal{O}(\mathbb{H}))^{W^h}
\]
where \( u_j = e^{-|z_j|^2}(\beta^{-1}z_j + \text{dvol}_j) \) is the Mathai–Quillen Thom form associated with the Chern root \( z_j \). Similarly, the \( \text{Spin}(2n) \)-equivariant elliptic Thom form is defined as
\[
(\text{Th}_{\text{Spin}(2n)})_{\tilde{h}} = \left( \prod_{j=1}^{k} \frac{u_j}{z_j} \right) \left( \prod_{j=1}^{n} \sigma(\tau, z_j + \tilde{h}_1^j - \tau \tilde{h}_2^j) \right) \beta^{k-n} \in \Omega^2_T(V^h; \mathcal{O}(\mathbb{H}))^{W^h}.
\]

**Proposition 6.6.** For \( G = \text{SU}(n) \) or \( \text{Spin}(2n) \), the values \( (\text{Th}_T)_{\tilde{h}} \) assemble to give a cocycle in \( \bar{\text{Ell}}^G_2(V) \otimes \mathcal{L} \) that implements the universal twisted Thom isomorphism in equivariant elliptic cohomology twisted by the level 1 Looijenga line bundle \( \mathcal{L} \)
\[
\bar{\text{Ell}}^*_G(\text{pt}) \sim \bar{\text{Ell}}^*_G(V) \otimes \mathcal{L} \quad \alpha \mapsto \alpha \boxtimes \text{Th}_G
\]
as a quasi-isomorphism of sheaves of chain complexes over \( \text{Bun}_G(\mathcal{E}) \).

**Proof.** The Thom isomorphism statement follows immediately from the claim that the elliptic Thom form is a nowhere vanishing section of the claimed line bundle. Indeed, the statement can be checked locally, and the definitions [33] and [34] have the stalks of \( \text{Th}_G \) as an invertible element of the stalk multiplied by the usual Mathai–Quillen Thom form. The claim then immediately follows.

Showing that the stalk-level definition lifts to a global section is a bit more delicate. First we observe that \( (\text{Th}_G)_{\tilde{h}} \) is defined as a product of Weyl-invariant quantities, namely the ordinary Mathai–Quillen Thom and Euler forms and the product of sigma functions. Then the fact that the Looijenga line is a tensor product of lines coming from summands in the maximal torus of \( G \), it suffices to compute values for \( G = \text{Spin}(2) \) or \( U(1) \).

Note that for the stalk datum, we have two cases, that for \( \tilde{h} = 0 \) and the generic case \( \tilde{h} \neq 0 \); in the former case, \( V^h = V^{\exp(\tilde{h})} = V \) while generically \( V^h = \{0\} \). The analyticity condition from Definition [3.2] necessitates we consider deformations of the group elements \( h = \exp(\tilde{h}) = (\exp(h_1), \exp(h_2)) \) together with a translation in the Lie algebra dependence of the equivariant differential form. In the present case, either \( h \) and its deformation \( h' \) are both generic and the fixed locus is simply \( \{0\} \) for both, or we have \( h = (h_1, h_2) = (1, 1) \) and \( h' \) is generic so that \( M' = \{0\} \).

The previous paragraph shows that all the relevant restrictions to check involve compatibility with the forms \( (\text{Th}_G)_{\tilde{h}} \in \Omega^*_G(\{0\}; \mathcal{O}(\mathbb{H})) \). But the restriction of the Thom form to \( \{0\} \) is the Euler form of the previous subsection. Therefore, these stalks glue together to give a section of the Looijenga line. Hence, we have shown that the stalk-level data of the Thom form glues together correctly as per the compatibility conditions for \( \bar{\text{Ell}}^*_G(V) \otimes \mathcal{L} \).

**Remark 6.7.** We sketch an alternative formula for the equivariant Thom classes that more closely resembles the formulas that occur in K-theory. Since \( \tilde{h}_j^i = 0 \) for \( j \leq k \), we have
\[
z_j = z_j + \tilde{h}_1^j - \tau \tilde{h}_2^j \quad j \leq k.
\]
We will now construct elliptic versions of the Chern–Weil maps. Theorem 3.17 on the $SL$-algebras for level 1 positive energy representations.

...The elliptic Chern–Weil map.

6.6. The elliptic Chern–Weil map. Let $A$ be a graded commutative $\mathbb{C}$-algebra and $E \to M$ a real or complex vector bundle classified by a map $f : M \to BG$ for $G = U(n)$ or $O(n)$, respectively. The Chern–Weil form in ordinary cohomology is

\[
\text{Poly}(g; A)^G \simeq H_G(pt; A) \simeq H(BG; A) \xrightarrow{f^*} H(M; A).
\]

To any invariant polynomial on the Lie algebra, this map associates a characteristic class of $E$ in the cohomology of $M$. When $E$ is equipped with a connection, (35) refines to a map of chain complexes, $\text{Poly}(g; A)^G \to (\Omega^\bullet(M; A), d)$, where the source has trivial differential. We will now construct elliptic versions of the Chern–Weil maps

\[
\begin{align*}
\text{Rep}(\text{SU}(n)) \otimes_{\text{MF}_0} \text{MF} & \to (\Omega^\bullet(M; \text{MF}), d), \\
\text{Rep}(\text{Spin}(2n)) \otimes_{\text{MF}_0} \text{MF} & \to (\Omega^\bullet(M; \text{MF}), d),
\end{align*}
\]

that send the vacuum representations of loop groups to cocycle representatives of characteristic classes for complex vector bundles with $U(6)$-structure or real vector bundles with $O(8)$-structure. Above, $\text{Rep}(\text{SU}(n))$ and $\text{Rep}(\text{Spin}(2n))$ are the respective Verlinde algebras for level 1 positive energy representations.

The first step in constructing (36) is an $L$-twisted version of the completion map from Theorem 3.17 on the $SL_2(Z)$-cover $\mathbb{H}$ of $\mathcal{M}_{\text{ell}}$. Consider the composition $e : \mathbb{H} \to \mathbb{H} \times t_C \to \mathbb{H} \times (T \times T) \to W \simeq \mathbb{H} \times C^2[G] \to \text{Bun}_G(\mathcal{E})$ where the first arrow includes at $0 \in t_C$, the second map is induced by the exponential map from the Lie algebra to the Lie group, and the remaining maps are from Example 2.4. By definition of the Looijenga line bundle $\mathcal{L}$, its pullback to $\mathbb{H} \times t_C$ has a preferred trivialization. This yields the following.

**Construction 6.8.** The restriction of the sheaf $\widehat{\text{Ell}}^*_G(pt) \otimes \mathcal{L}$ along the map $e : \mathbb{H} \to \text{Bun}_G(\mathcal{E})$ together with the trivialization of $\mathcal{L}$ specified above gives an isomorphism of commutative differential graded algebras on $\mathbb{H}$

\[
\Gamma(\mathbb{H}; e^*\widehat{\text{Ell}}^*_G(pt) \otimes \mathcal{L}) \xrightarrow{\sim} \Omega^\bullet_G(pt; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}]),
\]
for $G = \text{SU}(n)$ or $\text{Spin}(2n)$, where the target is the Cartan model for Borel equivariant cohomology of the point with coefficients in $O(\mathbb{H})[\beta, \beta^{-1}]$.

**Definition 6.9.** Define the elliptic Chern–Weil map as the composition

$$
\text{Rep}(LG) \otimes_{\text{MF}} \text{MF} \simeq \Gamma(\text{Bun}_G(E); \text{Ell}_G^m(\text{pt}) \otimes \mathcal{L})
$$

(38)

$$
\rightarrow \Omega^*_{G}(\text{pt}; o(\mathbb{H})[\beta, \beta^{-1}]) \rightarrow (\Omega^*(M; O(\mathbb{H})[\beta, \beta^{-1}]), d)
$$

where the isomorphism is from Theorem 6.3, the middle map is restriction to the vacuum representation of the appropriate loop group at level 1. We observe that

$$
\text{MU} \text{coming from the } [G] \text{ followed by (37) and the final map is the usual Chern–Weil map determined by a vector bundle with } G\text{-structure and } G\text{-invariant connection for } G = \text{SU}(n) \text{ or } \text{Spin}(2n).
$$

The nontriviality of the line bundle $\mathcal{L}$ manifests itself above as the possible non-modularity of the image of the elliptic Chern–Weil map, i.e., a possible failure of invariance under the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. We analyze this question of descent from $\mathbb{H}$ to $[\mathbb{H}/\text{SL}_2(\mathbb{Z})] \simeq \mathcal{M}_\text{ell}$ for the equivariant elliptic Euler class, which we recall corresponds to the vacuum representation of the appropriate loop group at level 1. We observe that

$$
(\Omega^*(M; O(\mathbb{H})[\beta, \beta^{-1}])^{\text{SL}_2(\mathbb{Z})}, d) \simeq (\Omega^*(M; \text{MF}), d)
$$

is a cochain model for $\text{TMF}(M) \otimes \mathbb{C} \simeq H(M; \text{MF})$, i.e., cohomology with coefficients in modular forms.

**Theorem 6.10.** For $G = \text{SU}(n)$ and $E$ a complex vector bundle with $c_2(V) = 0$, the image of $\text{Ell}_G$ under (38) is a cocycle representative for the elliptic Euler class in $\text{TMF}(M) \otimes \mathbb{C}$ coming from the $\text{MU}(6)$ orientation of $\text{TMF} \otimes \mathbb{C}$.

Similarly, for $G = \text{Spin}(2n)$ and $E$ a real vector bundle with spin structure and $\frac{p_1}{2}(V) = 0$, the image of $\text{Ell}_G$ along (38) is a cocycle representative for the elliptic Euler class coming from the $\text{MString}$ orientation of $\text{TMF} \otimes \mathbb{C}$.

**Proof.** The image of the section $\text{Ell}_G$ under (38) is the Borel equivariant characteristic class defined in terms of Chern roots via the formulas (31) and (32). The image in the cohomology of $M$ under the elliptic Chern–Weil map sends the Lie algebra dependence to traces of powers of curvature. The obstruction to the resulting class in $H(M; O[\beta, \beta^{-1}])$ being $\text{SL}_2(\mathbb{Z})$-invariant is the coefficient of the 2nd Eisenstein series for the description of the Witten class as in (25). We observe that this coefficient is precisely the Chern–Weil representative for $c_2(E)$ or $p_1(E)$ in the complex and real cases, respectively. \hfill \square

**Remark 6.11.** A completely analogous result to the above holds for the images of elliptic Thom classes under the elliptic Chern–Weil map: the equivariant refinement does indeed recover the standard non-equivariant Thom class.

6.7. **Some examples.** To give a flavor for how to compute with Euler and Thom classes, we spell out a couple simple examples.

**Example 6.12.** This is a continuation of Example 3.5 for $S^1 = U(1) = \text{Spin}(2)$ acting on $S^2$ by rotation about an axis. Using the Thom isomorphism in $\text{Spin}(2)$-equivariant elliptic cohomology, we have the isomorphism of sheaves on $\text{Bun}_{\text{Spin}(2)}(E) \simeq E^\vee$

$$
\text{Ell}^2_{\text{Spin}(2)}(S^2, \infty) \simeq \text{Ell}_0^0(\text{Spin}(2))(\text{pt}) \otimes \mathcal{L}^{-1} \simeq \mathcal{L}^{-1}.
$$

Here, $\infty$ refers to the “point at $\infty$” in the Riemann sphere $\mathbb{CP}^1 \simeq S^2$, and $\text{Ell}_0^0(\text{Spin}(2))(S^2, \infty)$ is the relative equivariant cohomology, i.e., the cohomology sheaf of the subsheaf of sections of $\text{Ell}_{\text{Spin}(2)}(S^2)$ that vanish on restriction to $\infty \mapsto S^2$. We have a long exact sequence for the pair $(S^2, \infty)$ in sheaves of chain complexes on $\text{Bun}_{\text{Spin}(2)}(E) \simeq E^\vee$, which gives

$$
\text{Ell}_0^0(\text{Spin}(2))(S^2) \simeq \text{Ell}^0_0(\text{Spin}(2))(\text{pt}) \oplus \text{Ell}^0_0(\text{Spin}(2))(S^2, \infty)
$$

$$
\simeq \mathcal{O} \oplus (\text{Ell}^2_0(\text{Spin}(2))(S^2, \infty) \otimes \omega)
$$

$$
\simeq \mathcal{O} \oplus (\mathcal{L}^{-1} \otimes \omega).
$$
as a consequence of the \( \mathcal{L} \)-twisted Thom isomorphism and \( \omega \)-twisted Bott periodicity. As a sanity check, we observe that \( \mathcal{L}^{-1} \otimes \omega \) is indeed a trivial bundle away from the zero section in \( \mathcal{E}^\vee \) (with trivializing section given by \( \text{Spin}(2) \)-equivariant elliptic Euler cocycle) and this conforms with the computations in Example 3.5 under the identification \( \text{Ell}_{\text{Spin}(2)}^0(S^2) \cong \text{Ell}_{U(1)}^0(S^2) \). More generally, we have the isomorphism of sheaves

\[
\text{Ell}_{\text{Spin}(2)}^0(S^2) \otimes \mathcal{L}^m \cong \mathcal{L}^m \oplus \left( \mathcal{L}^{m-1} \otimes \omega \right).
\]

We observe this sheaf has global sections if and only if \( m \) is nonnegative. However, there are nontrivial \emph{derived} global sections for any \( m \), e.g., by Serre duality on \( \text{Bun}_{\text{Spin}(2)}(E) \cong \mathcal{E}^\vee \).

In the literature, authors often identify the (quasi-coherent) sheaf \( \text{Ell}_{U(1)}^0(M) \) with a scheme by taking the relative \( \text{Spec} \) over \( \text{Bun}_G(E) \), especially when the cohomology is concentrated in even degree. We now explain this perspective for \( \text{Ell}_{U(1)}^0(S^2) \), i.e., for \( \text{Spec}_G\left( \mathcal{O} \oplus (\mathcal{L}^{-1} \otimes \omega) \right) \cong \text{Spec}_G\left( \mathcal{O} \oplus \mathcal{O}(-0) \right) \), where we freely use the isomorphism (via \( \sigma \)) of \( \mathcal{L} \otimes \omega^{-1} \cong \mathcal{O}(0) \). We determine the algebra structure on \( \mathcal{O} \oplus \mathcal{O}(-0) \) by the Mayer–Vietoris sequence for the standard \( U(1) \)-equivariant cover of \( S^2 \) by the upper and lower hemispheres. One finds

\[
\text{Ell}_{U(1)}^0(S^2) \cong \ker \left( \text{Ell}_{U(1)}^0(\text{pt}) \oplus \text{Ell}_{U(1)}^0(\text{pt}) \to \text{Ell}_{U(1)}^0(U(1)) \right)
\]

where the above computations are in the category of sheaves on \( \mathcal{E}^\vee \) and \( \mathcal{O}_0 \) is the structure sheaf of the zero section \( 0: M_{\text{ell}} \to \text{Bun}_{U(1)}(E) \). Indeed, the above description makes it clear that as a coherent sheaf, the above kernel is isomorphic to \( \mathcal{O} \oplus \mathcal{O}(-0) \), but the Mayer–Vietoris description has the additional property of making manifest the algebra structure. Pullbacks of sheaves of algebras become pushouts of schemes under (relative) Spec, so we have that \( \text{Spec}_G(\text{Ell}_{U(1)}^0(S^2)) \) is simply two copies of the (universal) elliptic curve \( \mathcal{E}^\vee \) glued along their zero sections.

**Example 6.13.** More generally, consider \( U(1) \) acting on \( S^2 \) by \( n \) times the rotation action: to emphasize the dependence of the equivariant structure on \( n \), we denote this representation sphere as \( S^2[n] \).

We repeat the computation of \( \text{Ell}_{\text{Spin}(2)}^0(S^2[n]) \) from the previous example using the Thom isomorphism, only now we use the “charge \( n \)” representation of \( \text{Spin}(2) \) on \( V = \mathbb{R}^2 \). By naturality, the Thom class of this representation is the pullback of the universal Thom class from \( [\text{pt}/\text{Spin}(2)] \) along the multiplication by \( n \) map \( \text{Spin}(2) \to \text{Spin}(2) \). Hence, the induced twisting bundle on \( \mathcal{E}^\vee \) is given by the pullback of the universal Looijenga (level 1) twisting bundle \( \mathcal{L} \) under the map \( \mathcal{E}^\vee \to \mathcal{E}^\vee \) and if we apply the Thom isomorphism as before, we find

\[
\text{Ell}_{\text{Spin}(2)}^0(S^2[n]) \cong \mathcal{O} \oplus \left( n^* \mathcal{L}^{-1} \otimes \omega \right),
\]

where we denote \( n^* \mathcal{L} \) as the pullback of the level 1 Looijenga line under the multiplication by \( n \) morphism.

Next we repeat the Mayer–Vietoris computation from the previous example, using the same cover to find

\[
\text{Ell}_{U(1)}^0(S^2[n]) \cong \ker \left( \text{Ell}_{U(1)}^0(\text{pt}) \oplus \text{Ell}_{U(1)}^0(\text{pt}) \to \text{Ell}_{U(1)}^0(S^1[n]) \right)
\]

where we use similar notation for \( S^1[n] \) as an \( S^1 \) with its \( U(1) \)-equivariant structure given as \( n \) times the usual. The above description makes it clear that \( \text{Spec}_G(\text{Ell}_{U(1)}^0(S^2[n])) \) is now two copies of \( \mathcal{E}^\vee \) glued along their \( n \)-torsion subschemes, while as a sheaf, one may rewrite \( \text{Ell}_{U(1)}^0(S^2[n]) \) as \( \mathcal{O} \oplus \mathcal{O}(-\{n \text{-torsion}\}) \). Indeed, as \( \mathcal{L} \otimes \omega^{-1} \cong \mathcal{O}(0) \), we have \( n^* \mathcal{L} \otimes \omega^{-1} \cong n^* \left( \mathcal{L} \otimes \omega^{-1} \right) \cong n^* \mathcal{O}(0) \cong \mathcal{O}(\{n \text{-torsion}\}) \) and so our two descriptions indeed agree.
7. Equivariant orientations and the theorem of the cube

This section studies a more derived algebro-geometric point of view on the string orientation following the constructions in [Hop94, AHS01] that rely on the theorem of the cube. We show that this geometry refines essentially uniquely to the equivariant picture.

7.1. Background: Elliptic cohomology and the theorem of the cube. Let $h$ be a multiplicative cohomology theory and $\hat{h}$ the associated reduced cohomology theory. The isomorphism $\hat{h}^2(S^2) \simeq h^0(\text{pt})$ identifies a canonical generator of $\hat{h}^2(S^2)$ as an $h^0(\text{pt})$-module.

Definition 7.1. A complex orientation (or MU-orientation) of a cohomology theory $h$ is an element $c \in \hat{h}^2(\mathbb{CP}^\infty)$ whose restriction to $\hat{h}^2(S^2)$ is the canonical generator.

A complex orientation defines a Chern class for line bundles valued in $h$-cohomology, where $c = c(\mathcal{O}(1))$ is defined to be the Chern class of the tautological line on $\mathbb{CP}^\infty$. From this class one can build $h$-valued Chern classes for all (virtual) vector bundles using the splitting principle and the Whitney sum formula.

Now suppose that $h$ is even ($h^* = 0$ for odd) and 2-periodic (there exists an invertible element $\beta \in h^{-2}(\text{pt})$). Then the Atiyah–Hirzebruch spectral sequence can be used to show that a complex orientation for $h$ exists. The class $\beta$ allows one to put a choice of complex orientation $\mathfrak{c}$ in degree zero, $\mathfrak{c} = c\beta \in h^0(\mathbb{CP}^\infty)$. Pulling $\mathfrak{c}$ back along the three maps
\[
\mathfrak{c} \in h^0(\mathbb{CP}^\infty) \xrightarrow{p_1,p_2,m} h^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty), \quad p_1,p_2,m: \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty.
\]
gives a formula, $m^*\mathfrak{c} = F(p_1^*\mathfrak{c},p_2^*\mathfrak{c})$ where $F$ is a formal power series in two variables satisfying some properties codifying (homotopy) associativity and unitality of the multiplication map $m$. Quillen observed that these properties make $F$ into a formal group law over $h^0(\text{pt})$ [Qui69].

Example 7.2. Complex K-theory is 2-periodic and is complex oriented with $\mathfrak{c} = 1-\mathcal{O}(1)] \in K^0(\mathbb{CP}^\infty)$ where $\mathcal{O}(1)$ is the tautological line bundle. The associated formal group law is the multiplicative formal group law, i.e., for line bundles $L$ and $L'$ we have
\[
\mathfrak{c}(L \otimes L') = \mathfrak{c}(L) + \mathfrak{c}(L') - \mathfrak{c}(L) \cdot \mathfrak{c}(L').
\]

Recall that a formal group law is equivalent to the data of a formal group with a choice of coordinate, i.e., a function on the formal group that vanishes to first order at the identity. Hence, for an even, 2-periodic, complex oriented cohomology theory $h$, forgetting the choice of $\mathfrak{c}$ leaves the formal spectrum $\text{Spf}(h^0(\mathbb{CP}^\infty))$ with the structure of a formal group.

Definition 7.3. An elliptic cohomology theory is (i) an elliptic curve $E$ defined over a commutative ring $R$, (ii) an even, 2-periodic cohomology theory $h$ with $h^0(\text{pt}) \simeq R$, and (iii) an isomorphism of formal groups $\text{Spf}(h^0(\mathbb{CP}^\infty)) \simeq \hat{E}$ where $\hat{E}$ is the formal completion of $E$ at its identity section.

Consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. Viewing $\mathbb{C}$ as a complex analytic group under addition, the quotient map $\mathbb{C} \to E_\tau$ is a homomorphism with discrete kernel, and so determines an isomorphism of formal groups over $\mathbb{C}$
\[
\widehat{G}_a \simeq \hat{E}_\tau
\]
where $\widehat{G}_a$ is the additive formal group. Consider $H(\cdot; \mathbb{C}[\beta, \beta^{-1}])$, ordinary cohomology with values in the graded ring $\mathbb{C}[\beta, \beta^{-1}]$ where $|\beta| = -2$. The formal group associated with ordinary cohomology is the additive formal group, so the isomorphism gives an elliptic cohomology theory $h_{E_\tau}$ whose underlying cohomology theory is $H(\cdot; \mathbb{C}[\beta, \beta^{-1}])$. Below, Ell will denote the (complex analytic) sheaf of elliptic cohomology theories on $\mathcal{M}_{ell}$ determined by $H(\cdot; \mathcal{O}(\mathbb{H})[\beta, \beta^{-1}])$, consistent with the notation from 3.3. Specializing this sheaf to some $\tau \in \mathbb{H}$ recovers the elliptic cohomology theories $h_{E_\tau}$.

The standard coordinate $z$ on $\mathbb{C}$ determines a coordinate on $\hat{E}_\tau$ giving a complex orientation of $h_{E_\tau}$ associated with the additive formal group law,
\[
\mathfrak{c}(L \otimes L') = \mathfrak{c}(L) + \mathfrak{c}(L').
\]
This choice of coordinate makes the identification
\[
h_{E_*}(\mathbb{C}P^\infty) := H(\mathbb{C}P^\infty; \mathbb{C}\langle \beta, \beta^{-1} \rangle) \simeq \mathbb{C}[c][\beta, \beta^{-1}], \quad |\beta| = -2, |c| = 2
\]
where the cohomology class c is the standard degree 2 generator of the cohomology of \(\mathbb{C}P^\infty\).

The above orientation can be made consistently for all elliptic curves \(E\), and so descends to the moduli stack, \(\mathcal{M}_{\text{ell}}\), as we now explain. The Chern class \(c = \beta^{-1} - 1\) pulls back to itself under isomorphisms \(E \to E'\) associated with elements of \(\text{SL}_2(\mathbb{Z})\) since \(z \mapsto z/(ct + d)\) (therefore \(\tau \mapsto \tau/(ct + d)\)) and \(\beta^{-1} \mapsto (ct + d)^{-1}\). Letting \(\tau\) vary gives the sheaf of cohomology theories \(\text{Ell}\) on \(\mathcal{M}_{\text{ell}}\), e.g., viewed as an \(\text{SL}_2(\mathbb{Z})\)-equivariant on \(\mathcal{H}\). The \(\text{SL}_2(\mathbb{Z})\)-invariant sections of this sheaf are the cohomology theory \(H(\_; \text{MF})\), which we can identify with \(\text{TMF} \otimes \mathbb{C}\). The Chern class \(c\) is \(\text{SL}_2(\mathbb{Z})\)-invariant, so determines a complex orientation of \(\text{TMF} \otimes \mathbb{C}\).

Although the coordinate \(z\) might appear quite natural, there is a huge amount of freedom in choosing a complex orientations of the \(h_{E_*}\), and \(\text{Ell}\). Indeed, any holomorphic function on \(\mathbb{C}\) that vanishes at first order at \(0 \in \mathbb{C}\) defines a different orientation of \(h_{E_*}\). Such a function can be expressed as a power series in \(z\) whose lowest order nonvanishing term is \(z\). In the language of formal group laws, this is the statement that all coordinates are related to the coordinate \(z\) via an isomorphism of formal group laws. We consider two such choices
\[
\sigma(\tau, z) = \frac{z}{\text{Wit}^E(z)}, \quad \nu(\tau, z) = \frac{z}{\text{Wit}^E(z)}.
\]
These coordinates lead to different tensor product formulas for Chern classes than \([41]\). Furthermore, the orientations from \([42]\) are not invariant under the \(\text{SL}_2(\mathbb{Z})\)-action on \(E\), and hence fail to descend to a consistent complex orientation of the sheaf \(\text{Ell}\) or its global sections, \(\text{TMF} \otimes \mathbb{C}\).

From the above discussion, choosing an \(\text{MU}\)-orientation of elliptic cohomology over \(\mathbb{C}\) is an under-constrained problem. As described by Hopkins \([\text{Hop94}]\), if we instead ask for an \textit{a priori} weaker structure, namely an \(\text{MU}(6)\)- or \(\text{MO}(8)\)-orientation, there is a more canonical choice. Just as one can define Chern classes for all complex vector bundles from the data of the top Chern class of the universal line bundle, there is a similar type of splitting principle for characteristic classes of \(\text{U}(6)\)-bundles. Namely, all \(\text{U}(6)\)-bundles formally split into direct sums of trivial bundles and virtual bundles pulled back from
\[
V_3 = (L_1 - 1) \otimes (L_2 - 1) \otimes (L_3 - 1)
\]
over \(\text{BU}(1)^{\times 3} \simeq (\mathbb{C}P^\infty)^{\times 3}\). Hence, a theory of \(\text{MU}(6)\) characteristic classes is determined by a universal characteristic class \([\text{Hop94} \ \S 4-6]\)
\[
s \in h(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty).
\]
This class is required to satisfy the additional consistency conditions
(rigid) \(e^*s = 1 \in h^0(\text{pt})\) where \(e\) is inclusion of the basepoint \(e: \text{pt} \hookrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty\); (symmetric) \(s\) pulls back to itself along the maps \(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty\) that permute the factors; and
(cocycle) \((m_{12}^*s)(p_{13}^*s) = (m_{23}^*s)(p_{23}^*s)\) where \(m_{i(i+1)}: (\mathbb{C}P^\infty)^{\times 4} \to (\mathbb{C}P^\infty)^{\times 3}\) is multiplication on the \(i\) and \((i+1)\)st factors, and \(p_{ijk}: (\mathbb{C}P^\infty)^{\times 4} \to (\mathbb{C}P^\infty)^{\times 3}\) is the projection to the \(i, j\) and \(k\)th factors.

When \(h = h_{E_*}\) is an elliptic cohomology theory, Ando–Hopkins–Strickland \([\text{AHS01}]\) show that a class \([41]\) satisfying these consistency conditions may be produced from a cubical structure on the line bundle \(\mathcal{O}(-0)\) on \(E\), as we review presently. Recall that sections of \(\mathcal{O}(-0)\) are functions that vanish to first order at \(0 \in E\). A cubical structure for a line bundle \(\mathcal{L}\) on \(E\) is the data of a section \(s\) of a line bundle \(\Theta^3(\mathcal{L})\) on \(E \times E \times E\) whose fiber at \((x, y, z) \in E \times E \times E\) is
\[
\Theta^3(\mathcal{L})(x, y, z) = L_{x+y+z} \otimes L_x \otimes L_y \otimes L_z \otimes L_{x+y} \otimes L_{y+z} \otimes L_{x+z} \otimes L_{y+z} \otimes c.
\]
This section is required to satisfy analogous properties to the three above:
(rigid) $s(e,e,e) = 1$;
(symmetric) $s(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) = s(z_1, z_2, z_3)$ for any permutation $\sigma$;
(cocycle) $s(w + x, y, z)s(w, x, z) = s(w, x + y, z)(x, y, z)$

where there are implicit (canonical) isomorphisms between line bundles used in the equalities. The theorem of the cube (or Abel's theorem) shows that there is a canonical cubical structure on $O(-0)$, thereby determining a canonical $MU(6)$-orientation of the elliptic cohomology theory $h_{E}$. Further work of Hopkins shows that if the line bundle $L$ has the additional structure of an isomorphism $L_z \simeq L_{-x}$ and the section $s$ satisfies $s(x, y, -x - y) = 1$, then the $MU(6)$-orientation extends to an $MO(8)$-orientation. Under certain conditions on the elliptic cohomology theory [Hop94, Theorem 6.2], this additional condition is guaranteed, giving a canonical $MO(8) = MString$-orientation of such elliptic cohomology theories.

The construction of $MU(6)$- and $MO(8)$-orientations from a cubical structure can be made completely explicit for elliptic curves over $\mathbb{C}$. In this case, the coordinate $z$ on $\mathbb{C}$ from [40] allows one to express the cubical structure $E_x \times E_y \times E_z$ in terms of a function on the universal cover $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \to E_x \times E_y \times E_z$. One can check explicitly that the (necessarily unique) cubical structure in these coordinates is given by

\begin{equation}
(45) \quad s = \frac{\sigma(\tau, x + y)\sigma(\tau, x + z)\sigma(\tau, y + z)\sigma(\tau, 0)}{\sigma(\tau, x + y + z)\sigma(\tau, x)\sigma(\tau, y)\sigma(\tau, z)} = \frac{u(\tau, x + y)u(\tau, x + z)u(\tau, y + z)u(\tau, 0)}{u(\tau, x + y + z)u(\tau, x)u(\tau, y)u(\tau, z)},
\end{equation}

which we interpret as a class in $s \in h^0_{E_6}(\mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \mathbb{C}[x, y, z]$ expressed in terms of the complex orientation coming from the coordinate $z$ on $\mathbb{C}$. We observe further that $s$ is $SL_2(\mathbb{Z})$-invariant; this follows from the standard transformation properties of the $\sigma$-function. Hence, [45] determines a compatible family of $MU(6)$-orientations for the sheaf of cohomology theories $Ell$ as well as the global sections $TMF \simeq \mathbb{C}$. We observe that the pullback of $O(-0)$ under inversion on $E$ is canonically isomorphic to $O(-0)$, so that we can ask for the additional condition on $s$ to obtain an $MO(8)$-structure. By inspection (e.g., because $\sigma$ is odd) the cubical structure $s$ satisfies this additional requirement and hence gives an $MO(8)$-orientation.

We further observe that the class $s \in h^0_{E_6}(\mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ is the top Chern class of $V_3$ in $h_{E_3}$ relative to the complex orientations given by [42]. Indeed, the value of the $MU(6)$-orientation on any complex vector bundle can be computed using the splitting principle and the complex orientation associated with [42]. To summarize, although these complex orientations fail to descend to $M_{Ell}$, they determine the canonical $MU(6)$-orientation that does descend.

7.2. Equivariant refinements of orientations. We start with a motivating example.

Example 7.4. This is a continuation of Example 7.2. We can ask for an equivariant refinement of the complex orientation of $K$-theory relative to the Atiyah-Segal completion map,

\begin{equation}
\text{Rep}(U(1)) = K(U(1)(pt)) \xrightarrow{\text{completion}} K(BU(1)) = K("\mathbb{CP}^\infty")
\end{equation}

i.e., a virtual representation that maps to the chosen complex orientation. There is indeed a unique such virtual representation, namely $\tau_{U(1)} = 1 - R$ where $R$ is the defining representation of $U(1)$.

In light of the elliptic Atiyah-Segal completion map from [3.4] we can ask for a similar equivariant refinement of a complex orientation of elliptic cohomology over $\mathbb{C}$,

\begin{equation}
\Gamma(\mathcal{O}_{E^\vee}) = \Gamma(\text{Ell}_U(pt)) \xrightarrow{\text{completion}} \text{Ell}(BU(1)) = \text{Ell}(\mathbb{CP}^\infty).
\end{equation}

However, one immediately finds that no such class can exist, even for elliptic cohomology for a single elliptic curve: the class $\tau$ defines a function on a formal neighborhood of $0 \in E^\vee$ that vanishes to first order at zero, and since globally defined functions on $E^\vee$ are constant,
any putative class $\xi_{U(1)}$ is the zero class. Stated in more algebro-geometric language, a lift \([47]\) is asking for a global section of $\mathcal{O}(-0)$ on $E^\vee$, \[
abla \xi_{U(1)} \xrightarrow{\text{completion}} \text{Ell}(BU(1)) = \text{Ell}(\mathbb{CP}^\infty).
\]
and the only such global section $\xi_{U(1)}$ is the zero section. Under completion this is sent to the zero class in $\text{Ell}(\mathbb{CP}^\infty)$, which does not define a complex orientation. We summarize this observation as follows:

**Proposition 7.5.** No MU-orientation of a complex analytic elliptic cohomology theory may be refined to an equivariant MU-orientation of the corresponding complex analytic equivariant elliptic cohomology theory.

Although this result is easy, we believe it is worth emphasizing: Chern classes in elliptic cohomology—even for a single elliptic curve—do not admit equivariant refinements.

We can relax the setup in \([47]\), asking instead for a *twisted equivariant refinement*, \begin{equation}
\Gamma(\mathcal{O}(-0) \otimes \mathcal{L} \otimes \omega^{-1}) \xrightarrow{\text{completion}} \text{Ell}^2(BU(1)) = \text{Ell}^2(\mathbb{CP}^\infty)
\end{equation}
where $\mathcal{L}$ is a line bundle on $E^\vee$, and a section of $\mathcal{O}(-0) \otimes \mathcal{L}$ is one that vanishes to first order at $0 \in E^\vee$. For convenience we have changed points of view, putting the Chern classes $c$ and $c_{U(1)}$ in degree 2. The twisted completion map \([48]\) requires additional data, namely a trivialization of $\mathcal{L}$ near $0 \in E^\vee$ to identify the section $\xi_{U(1)}$ with a class in $\text{Ell}(BU(1))$.

**Definition 7.6.** A *twisted equivariant refinement* of a complex orientation of an elliptic cohomology theory defined over $\mathbb{C}$ is a line bundle $\mathcal{L}$ on $E^\vee$ together with a nowhere vanishing section $c_{U(1)} \in \Gamma(\mathcal{O}(-0) \otimes \mathcal{L} \otimes \omega^{-1})$ and a choice of trivialization of $\mathcal{L}$ near $0 \in E^\vee$ that identifies the restriction of $c_{U(1)}$ with the non-equivariant Chern class $c$.

With respect to a fixed elliptic cohomology theory, the freedom to choose a complex orientation is absorbed by the many ways to trivialize a fixed line bundle—in the notation of the previous definition, the line bundle $\mathcal{L}$ and its section $c_{U(1)}$ are essentially unique:

**Proposition 7.7.** Any complex orientation of an elliptic cohomology theory over $\mathbb{C}$ admits a twisted equivariant refinement. The line bundle $\mathcal{L}$ has a unique isomorphism to the Looijenga line for $U(1)$ at level 1. The section $c_{U(1)}$ is unique up to scale.

**Proof.** We work on the $\text{SL}_2(\mathbb{Z})$ cover $\tilde{E} = (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$ of the universal curve. We tackle the uniqueness question first. Given two line bundles $\mathcal{L}$ and $\mathcal{L}'$ with sections $c_{U(1)}$ and $\xi'_{U(1)}$ satisfying the requirements, $\xi'_{U(1)}/c_{U(1)}$ is a nowhere vanishing section of $\mathcal{L}' \otimes \mathcal{L}'$ and so determines an isomorphism $\mathcal{L}' \simeq \mathcal{L}$. Hence $\mathcal{L}$ is unique up to unique isomorphism. By construction, $\mathcal{L}' \simeq \mathcal{L}$ sends the section $c_{U(1)}$ to the section $\xi'_{U(1)}$. Any other section of $\mathcal{L}$ satisfying the requirements differs from a given $c_{U(1)}$ by a nonvanishing holomorphic function on $E^\vee$. But a globally defined holomorphic function is constant. This proves the claim of uniqueness.

Next we construct an equivariant refinement where $\mathcal{L}$ is the Looijenga line for $U(1)$ at level 1 with section $c_{U(1)}$ determined by the function $v(\tau, z)$ defined in \([6,2]\). The construction of the Looijenga line bundle in terms of a function on $\mathbb{H} \times \mathbb{C}$ with transformation properties specifies a preferred trivialization near $0 \in E^\vee$: view a section $\xi_{U(1)}$ as the function $v(\tau, z)$ on $\mathbb{C}$ and restrict to a neighborhood of $0 \in \mathbb{C}$, which is identified with a section of a trivialization of $\mathcal{L}$ in a neighborhood of $0 \in E^\vee$. This recovers the complex orientation specified by the coordinate $v(\tau, z)$, as described near \([12]\). All other complex orientations arise from changing the coordinate for the corresponding formal group law, but changes of coordinate exactly correspond to change of trivialization of $\mathcal{L}$ near $0 \in E^\vee$, so all coordinates can be recovered this way. \(\square\)
One can similarly ask for an equivariant refinement of the MU(6)-orientation and MString-orientation, namely as a class

\[ \Theta^3(\mathcal{O}(-0)) \xrightarrow{\text{completion}} \text{Ell}(BU(1) \times BU(1) \times BU(1)) \]

lifting the class \( s \) defined by \( 45 \) to a section of \( \Theta^3(\mathcal{O}(-0)) \) on \( (E^\vee)^{\times 3} \approx \text{Bun}_{U(1)^{\times 3}}(E) \).

**Definition 7.8.** An equivariant refinement of the MO(8)-orientation is a \( \Theta^3(\mathcal{O}(-0)) \)-twisted \( U(1)^{\times 3} \)-equivariant elliptic cohomology class whose image under \( 49 \) is the Ando–Hopkins–Strickland characteristic class for the canonical MO(8)-orientation.

**Theorem 7.9.** There exists a unique equivariant refinement of the MO(8)-orientation. Furthermore, the equivariant refinement is the twisted equivariant Chern class of the virtual vector bundle \( \text{Ell} \). This equivariant extension is defined globally on the stack \( \text{Bun}_{U(1)^{\times 3}}(E) \).

**Proof.** By inspection, the formulas \( 45 \) for the non-equivariant cubical structure have a unique equivariant extension given by the same formulas: when considered as a function on \( \mathbb{H} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \), the formulas \( 45 \) give sections of \( \Theta^3(\mathcal{O}(-0)) \) on \( \tilde{E}^\vee \times_{M_{\ell}} \tilde{E}^\vee \times_{M_{\ell}} \tilde{E}^\vee \). This gives the equivariant characteristic class \( 49 \) for \( V_3 \). Finally, we observe that this cubical structure is invariant under the action of \( \text{SL}_2(\mathbb{Z}) \), and so descends to \( \tilde{E}^\vee \times_{M_{\ell}} \tilde{E}^\vee \times_{M_{\ell}} \tilde{E}^\vee \), and therefore is a global class for \( \text{Ell}_G(U(1)^{\times 3}U(1)\times U(1)) \). \( \square \)

**Remark 7.10.** The uniqueness of a cubical structure for \( O(-0) \) on the elliptic curve produces a canonical MU(6)-orientation of non-equivariant elliptic cohomology. However, there are possibly more cubical structures for \( O(-0) \) on the formal group. But the cubical structure produces a unique equivariant MU(6)-orientation: there is no ambiguity in the equivariant setup.

## Appendix A. Background

**A.1. The Weil and Cartan models for equivariant cohomology.** The equivariant cohomology of a manifold with \( G \)-action is defined by the Borel construction,

\[ H_G(M) := H(M \times_G E_G), \]

where above \( H(\_\_) \) denotes ordinary cohomology with complex coefficients. By naturality, \( H_G(M) \) is a module over \( H_G(pt) = H(BG) \). The following standard facts will be useful.

**Lemma A.1.** For \( G \) connected there is a natural isomorphism \( H_G(M) \simeq H_T(M)^W \).

**Lemma A.2.** For \( H < G \) a subgroup of finite index, there is a natural isomorphism \( H_G(M) \simeq H_{H}(M)^{G/H} \).

The Cartan model for equivariant cohomology starts with the graded algebra \( \Omega_G^{\bullet, \text{pol}}(M) := \text{Sym}(\mathfrak{g}_C^\vee \Omega^*(M))^G \), where the generators in \( \mathfrak{g}_C^\vee \subset \text{Sym}(\mathfrak{g}_C^\vee) \) have degree 2. Identifying elements of this graded algebra with \( G \)-invariant \( \Omega^*(M) \)-valued polynomial functions on \( \mathfrak{g}_C \), define a differential \( Q \) on such an invariant function \( \alpha \) by

\[ (Q\alpha)(X) = d(\alpha(X)) - \iota_X \alpha(X), \quad X \in \mathfrak{g}_C \]

extended complex-linearly, where \( d \) is the ordinary de Rham differential on forms, and \( \iota_X \) denotes contraction with the vector field on \( M \) associated to \( X \) under the infinitesimal action of \( G \) on \( M \). The chain complex \( (\Omega_G^{\bullet, \text{pol}}(M), Q) \) is the Cartan model for equivariant cohomology, and we have an isomorphism

\[ H(\Omega_G^{\bullet, \text{pol}}(M), Q) \simeq H_G(M). \]
A.2. Groupoids and stacks.

Definition A.3. A Lie groupoid is a groupoid object in manifolds.

In a bit more detail, a Lie groupoid, denoted \( \{G_1 \rightrightarrows G_0\} \) consists of a manifold of objects, \( G_0 \), a manifold of morphisms, \( G_1 \), source and target maps, \( s,t: G_1 \to G_0 \), a unit map \( G_0 \to G_1 \), and a composition map \( c: G_1 \times_{G_0} G_1 \to G_1 \). We require that \( s,t \) are submersions so that the fibered product \( G_1 \times_{G_0} G_1 \) exists in manifolds. These data are required to satisfy the usual axioms of a groupoid.

A presheaf is a functor \( \text{Mfld}^{\text{op}} \to \text{Sets} \). A presheaf is representable when its values are determined by the set of maps to a fixed smooth manifold.

Definition A.4. A generalized Lie groupoid, \( \{G_1 \rightrightarrows G_0\} \), is a pair of presheaves \( G_1, G_0 \) on the site of manifolds with the source, target, unit, and composition maps as above, which together define a functor \( \text{Mfld}^{\text{op}} \to \text{Grpd} \) to groupoids, given by \( S \mapsto \{G_1(S) \rightrightarrows G_0(S)\} \).

Example A.5. Let a Lie group \( G \) act on a manifold \( M \). The action groupoid, denoted \( M/G \), has \( M \) as objects and \( G \times M \) as morphisms. The source map \( s: G \times M \to M \) is the projection, and the target map \( t: G \times M \to M \) is the action map. The unit \( M \to G \times M \) is the inclusion along the identity element \( e \in G \).

A stack is basically determined by a generalized Lie groupoid, where maps into the stack satisfying a local to global condition are defined in terms of open covers.

Definition A.6. A stack on the site of manifolds is a category fibered in groupoids over manifolds satisfying descent with respect to open covers.

In particular, for each \( S \) a stack assigns a groupoid, and to each map \( S \to S' \), a stack assigns a functor between the associated groupoids. These data can be assembled into a weak 2-functor from manifolds to groupoids that doesn’t necessarily satisfy descent is called a prestack.

Example A.7. The \( S \)-points of a generalized Lie groupoid \( G = \{G_1 \rightrightarrows G_0\} \) define a prestack whose value on \( S \) is the groupoid of functors from \( \{S \rightrightarrows S\} \) to \( \{G_1 \rightrightarrows G_0\} \).

Stackification is the left adjoint to the forgetful functor from stacks to prestacks. All the stacks in this paper come from applying stackification to prestacks defined by Lie groupoids. In this case, let \( [G_1 \rightrightarrows G_0] \) or \( [G_0/G_1] \) denote the stackification of the prestack \( \{G_1 \rightrightarrows G_0\} \).

Definition A.8. A groupoid presentation of a stack \( \mathcal{X} \) is a Lie groupoid \( \{G_1 \rightrightarrows G_0\} \) whose underlying stack is equivalent to \( \mathcal{X} \), i.e., \( \mathcal{X} \simeq [G_1 \rightrightarrows G_0] \).

Definition A.9. An atlas for a stack \( \mathcal{X} \) is a map \( p: U \to \mathcal{X} \) with source a manifold \( U \) so that for any other map \( q: V \to \mathcal{X} \) with source a manifold \( V \), the 2-fibered product \( U \times_{\mathcal{X}} V \) is representable (by a manifold), and the map \( U \times_{\mathcal{X}} V \to V \) is a submersion.

An atlas defines a groupoid presentation, \( \{U \times_{\mathcal{X}} U \rightrightarrows U\} \), and so a stack has a Lie groupoid presentation if and only if it admits an atlas.

Definition A.10. A holomorphic atlas is an atlas \( U \to \mathcal{X} \) where \( U \) and \( U \times_{\mathcal{X}} U \) are given the structure of a complex manifold and all the structure maps in the groupoid \( \{U \times_{\mathcal{X}} U \rightrightarrows U\} \) are holomorphic.

A.3. Some Lie theory. A great reference for the following (and many other) facts is \[\text{Seg68}\].

Lemma A.11. Let \( T < G \) be a maximal torus for a connected compact Lie group with normalizer \( N(T) < G \). If \( t_1, t_2 \in T \) are conjugate in \( G \), they are conjugate by an element of \( N(T) \).

Proof. For an element \( h \in G \), let \( C(h) < G \) denote the centralizer subgroup and \( C(h)^0 < C(h) \) denote the connected component of the identity. Suppose \( g \in G \) is such that \( g t_1 g^{-1} = t_2 \). Note that \( t_2 \in g T g^{-1} \cap T \). Furthermore, \( g T g^{-1} \) and \( T \) are both maximal tori in the connected compact group \( C(t_2)^0 \), so they are conjugate by some element \( g' \in C(t_2)^0 \). But then \( g'^{-1} g \in N(T) \) also conjugates \( t_1 \) to \( t_2 \).
Let $\mathfrak{t}$ be the Lie algebra of a maximal torus $T$ of a compact connected Lie group $G$, and $W = N(T)/T$ be the Weyl group. The following is proved in the same way as the above.

**Corollary A.12.** If $X_1, X_2 \in \mathfrak{t}$ are conjugate by the adjoint action of $G$ on $\mathfrak{t}$, then they are conjugate by an element of $N(T)$.

**Proposition A.13.** The ring of $W$-invariant holomorphic functions on $\mathfrak{t}_C$ is equivalent to the ring of $G$-invariant holomorphic functions on $\mathfrak{g}_C$.

**Proof.** Any conjugation-invariant function on $\mathfrak{g}_C$ clearly restricts to a $W$-invariant function on $\mathfrak{t}_C$; the interesting direction is to extend a $W$-invariant function on $\mathfrak{t}_C$ to a $G$-invariant function on $\mathfrak{g}_C$. On the (Zariski) open sublocus $\mathfrak{g}_C^C$ of regular semisimple elements, any element by definition may be conjugated into $\mathfrak{t}_C$, so that a holomorphic function on $\mathfrak{t}_C$ can automatically be extended to a holomorphic function on $\mathfrak{g}_C$. By Corollary A.12 the extension is conjugation invariant if the original function is $W$-invariant. It remains to extend further to $\mathfrak{g}_C$ (which would automatically continue to be conjugation-invariant). But we may approximate a holomorphic $W$-invariant function on $\mathfrak{t}_C$ by a $W$-invariant polynomial on $\mathfrak{t}_C$ and instead simply have to extend a polynomial from $\mathfrak{g}_C^C$ to all of $\mathfrak{g}_C$. By Algebraic Hartogs’ Lemma, the polar locus is a closed subset of pure codimension one. However, the closures of all codimension one points of $\mathfrak{g}_C \setminus \mathfrak{g}_C^C$ contain 0, where our polynomial is clearly well-defined, and so the polar locus must be empty and we have a polynomial extension, as desired. 

**Remark A.14.** The same result holds, with the same proof, for germs of holomorphic functions.

**Proposition A.15.** Let $G$ be a compact Lie group, not necessarily connected. Given $h \in G$ and $X, X' \in \mathfrak{g}^h$ sufficiently small, the set of elements which conjugates $he^X$ to $he^{X'}$ is contained in $C(h)$.

**Proof.** Let $S = \{ g \in G | ghe^X g^{-1} = he^{X'} \}$; by construction, it is a coset of $C(he^X)$. By [BG94, Lemma 1.3], we may assume $X$ is sufficiently small such that $C(he^X) \subset C(h)$ (compare Lemma 3.1 above). Hence either $S \subset C(h)$, as desired, or $S$ is entirely disjoint from $C(h)$. Choose a faithful representation $G \to U(n)$ and assume for now the result for $U(n)$. Then $S \subset C_{U(n)}(h)$, where $C_{U(n)}(h) \subset U(n)$ is the subgroup of $U(n)$ which centralizes $h$. But then $S \subset G \cap C_{U(n)}(h) = C_G(h)$, as desired. Hence, it suffices to show the result for $G = U(n)$.

The statement is clearly invariant under conjugation, so we may assume $h$ is diagonal and of some block-form for a partition $n = n_1 + \cdots + n_k$, where $h$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with each eigenvalue $\lambda_i$ occurring with multiplicity $n_i$. Then $C(h)$ is the corresponding group of block-diagonal matrices. Pick disjoint open intervals $U_i$ centered at the $\lambda_i$ and interpret “sufficiently small” to mean that the eigenvalues of the $i^{th}$ block of $he^X, he^{X'}$ remain within $U_i$. Then one may show directly any element conjugating $he^X$ to $he^{X'}$ must be block-diagonal, i.e., lie in $C(h)$, as desired.

**Lemma A.16.** Given a (not necessarily connected) compact Lie group and $g \in G$, consider $Ad_g : \mathfrak{g} \to \mathfrak{g}$ and denote $T = Ad_g - \text{id}$, such that $\ker T = \text{Lie}(C(g))$. Then $\text{Im } T \cap \ker T = 0$.

**Proof.** We wish to show $\text{ker } T^2 = \ker T$, i.e., the generalized eigenspace of $Ad_g$ with eigenvalue 1 is in fact just a usual eigenspace. But this follows from $Ad_g$ being self-adjoint with respect to the nondegenerate Killing form, so that all generalized eigenspaces of $Ad_g$ are usual eigenspaces.

**Lemma A.17.** Given $G$ as above and $g \in G$, for any element $X \in \mathfrak{g}$ sufficiently small, there exists some small $Y \in \mathfrak{g}^g$ such that $ge^X$ is conjugate to $ge^Y$.

**Proof.** It suffices to prove the above infinitesimally, i.e., to show that on the tangent space $T_g G \simeq \mathfrak{g}$ as identified with the Lie algebra by left-translation under $g^{-1}$, the orbit of $\mathfrak{g}^g$
under the \((g\text{-twisted})\) adjoint action spans the full tangent space. But indeed, the centralizer \(g^g\) is exactly \(\ker T\) as above, while the infinitesimal adjoint action under conjugacy spans \(\text{Im} \ T\). The prior lemma plus a simple dimension count yields that \(g \cong \ker T \oplus \text{Im} \ T\), i.e., the full tangent space is spanned by the centralizer and infinitesimal deformations under conjugacy, which is what we wished to show. \(\square\)

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