THE UNCONDITIONAL BASIC SEQUENCE PROBLEM

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Abstract. We construct a Banach space that does not contain any infinite unconditional basic sequence.

§0. Introduction.

A fundamental role in the theory of Banach spaces is played by the notion of a Schauder basis. If $X$ is a Banach space, then a sequence $(x_n)_{n=1}^\infty$ is a Schauder basis (or simply a basis) of $X$ if the closed linear span of $\{x_n\}_{n=1}^\infty$ is $X$ and $\sum_{n=1}^\infty a_n x_n$ is zero only if each $a_n$ is zero. The second condition, asserting a kind of independence, clearly depends very much on the order of the $x_n$, and it is certainly possible for a permutation of a basis to fail to be a basis. On the other hand, many bases that occur naturally, such as the standard basis of $\ell_p$ when $1 \leq p < \infty$, are bases under any permutation. It is therefore natural to give a name to this special kind of basis. As it happens there are several equivalent definitions.

Theorem 0.1. Let $X$ be a (real or complex) Banach space and let $(x_n)_{n=1}^\infty$ be a basis of $X$. Then the following are equivalent.

(i) $(x_{\pi(n)})_{n=1}^\infty$ is a basis of $X$ for every permutation $\pi : \mathbb{N} \to \mathbb{N}$.

(ii) Sums of the form $\sum_{n=1}^\infty a_n x_n$ converge unconditionally whenever they converge.

(iii) There exists a constant $C$ such that, for every sequence of scalars $(a_n)_{n=1}^\infty$ and every sequence of scalars $(\epsilon_n)_{n=1}^\infty$ of modulus at most 1, we have the inequality

$$\left\| \sum_{n=1}^\infty \epsilon_n a_n x_n \right\| \leq C \left\| \sum_{n=1}^\infty a_n x_n \right\| .$$

A basis satisfying these conditions is called an unconditional basis, and a basis satisfying the third condition for some given constant $C$ is called $C$-unconditional. An infinite sequence that is a basis of its closed linear span is called a basic sequence: if it is an unconditional basis of its closed linear span then it is an unconditional basic sequence.

For a long time a major unsolved problem was whether every separable Banach space had a basis. This question was answered negatively by P. Enflo in 1973 [E]. On the other
hand, it is not hard to show that every space contains a basic sequence. Spaces with
unconditional bases have much more structure than general spaces, so examples of spaces
without them are easier to find. Indeed, the spaces $C([0, 1])$ and $L_1$ do not have an uncon-
ditional basis. This leaves the question of whether every space contains an unconditional
basic sequence, or equivalently has an infinite-dimensional subspace with an unconditional
basis. The earliest reference we know for the problem is Bessaga-Pełczyński (1958), where
it appears as problem 5.1; actually we solve the more precise problem 5.11, since our exam-
ple is a reflexive Banach space. The easier related problem 5.12 was solved already many
years ago [MR].

In the summer of 1991, the first named author found a counterexample. A short time
afterwards the second named author independently found a counterexample as well. On
comparing our examples, we discovered that they were almost identical, as were the proofs
that they were indeed counterexamples, so we decided to publish jointly and work together
on further properties of the space. As a result of our collaboration, the proofs of some of
the main lemmas have been simplified and tightened.

After reading our original preprints, W. B. Johnson pointed out that our proof(s)
could be modified to give a much stronger property of the space. J. Lindenstrauss had
asked whether every infinite-dimensional Banach space was decomposable, that is, could
be written as a topological direct sum $Y \oplus Z$ with $Y$ and $Z$ infinite-dimensional. Johnson’s
observation was that our space, which for the remainder of the introduction we shall call $X$,
is not only not decomposable, but does not even have a decomposable subspace. That is, $X$
is hereditarily indecomposable or H.I. Equivalently, if $Y$ and $Z$ are two infinite-dimensional
subspaces of $X$ and $\epsilon > 0$ then there exist $y \in Y$ and $z \in Z$ such that $\|y\| = \|z\| = 1$ and
$\|y - z\| < \epsilon$. This turned out to be a key property of $X$ in that all of the pathological
properties that we know about $X$ can be deduced from the fact that it is H.I. In particular
it is easy to see that a space with this property cannot contain an unconditional basic
sequence.

Another immediate consequence is that either the space is a new prime Banach space
(which means that it is isomorphic to all its complemented subspaces) or it fails to be
isomorphic to a subspace of finite codimension. If the second statement is true then it
must fail to be isomorphic to a subspace of codimension 1, giving a counterexample to a
question of Banach which has come to be known as the hyperplane problem. The first
author modified the construction of $X$ to give such a counterexample, and in fact one with an unconditional basis [G]. Soon afterwards, we managed to use the H.I. property to show that the complex version of $X$ gives another example. Later, we were able to pass to the real case, so $X$ itself is a counterexample to the hyperplane problem by virtue of being a H.I. space.

In fact, the space of operators on $X$ is very small: every bounded linear operator on $X$ can be written as $\lambda Id + S$, where $S$ is a strictly singular operator. A question we have not answered is whether there exists a space on which every bounded linear operator is of the form $\lambda Id + K$ for a compact operator $K$. We do not even know whether our space has that property, though it seems unlikely.

The rest of this paper is divided into five sections. The first concerns the notion of an asymptotic set, which is a definition of great importance for this problem, but which arises most naturally in the context of the distortion problem, about which we shall have more to say later. In particular, we give a criterion for a space to have an equivalent norm in which it contains no $C$-unconditional basic sequence.

The second section is about a remarkable space constructed by T. Schlumprecht, on which our example builds. We show that, for every $C$, his space satisfies our criterion and therefore can be renormed so as not to contain a $C$-unconditional basic sequence.

The third section contains the definition of $X$ and a proof that it is H.I. and therefore contains no unconditional basic sequence, and ends with the (easy) proof that $X$ is reflexive.

The fourth is about consequences of this property, especially the existence of very few operators on a complex space having it. The final section concerns the passage to the real case of the results of the previous one.

We are very grateful to W. B. Johnson for his influence on this paper. As we have mentioned, his observation that our space is H.I. lies at the heart of all its interesting properties. He also explained to us much simpler arguments for some of the consequences of this property. We would also like to thank P. G. Casazza and R. G. Haydon for interesting conversations and suggestions about the problems solved here.

For the rest of this paper we shall use the words “space” and “subspace” to refer to infinite-dimensional spaces and subspaces. Similarly a basis will always be assumed to be infinite.
§1. Asymptotic sets.

Let $X$ be a normed space and let $S(X)$ be its unit sphere. We shall say that a subset $A \subset S(X)$ is asymptotic if $A \cap S(Y) \neq \emptyset$ for every infinite-dimensional (not necessarily closed) subspace $Y \subset X$. A key observation for this paper is that if a space $X$ contains infinitely many asymptotic sets that are all well disjoint from one another, then these can be used to construct an equivalent norm on $X$ such that no sequence is $C$-unconditional in this norm. In this section, we shall make that statement precise and prove it. The technique we use will underlie our later arguments as well.

Let $A_1, A_2, \ldots$ be a sequence of subsets of the unit sphere of a normed space $X$ and let $A^*_1, A^*_2, \ldots$ be a sequence of subsets of the unit ball of $X^*$. (It is slightly more convenient in applications to take the ball rather than the sphere). We shall say that $A_1, A_2, \ldots$ and $A^*_1, A^*_2, \ldots$ are an asymptotic biorthogonal system with constant $\delta$ if the following conditions hold.

(i) For every $n \in \mathbb{N}$, the set $A_n$ is asymptotic.
(ii) For every $n \in \mathbb{N}$ and every $x \in A_n$ there exists $x^* \in A^*_n$ such that $x^*(x) > 1 - \delta$.
(iii) For every $n, m \in \mathbb{N}$ with $n \neq m$, every $x \in A_n$ and every $x^* \in A^*_m$, $|x^*(x)| < \delta$.

Under these circumstances, we shall say that $X$ contains an asymptotic biorthogonal system. The definition is not interesting if $\delta > 1/2$ since one may take $A_n = S(X)$ and $A^*_n = \frac{1}{2}B(X^*)$ for every $n$. On the other hand, if $\delta < 1/2$, it is not at all obvious that any Banach space contains an asymptotic biorthogonal system with constant $\delta$. We shall see later, however, that this is not as rare a phenomenon as it might seem.

Note that the $A_n$ are separated in the following sense. If $n \neq m$, $x \in A_n$ and $y \in A_m$, then there exists $x^* \in A^*_n$ such that $x^*(x) \geq 1 - \delta$ and $|x^*(y)| < \delta$. Since $A^*_n \subset B(X^*)$, it follows that $\|x - y\| \geq 1 - 2\delta$.

The main result of this section is the following theorem.

**Theorem 1-1.** Let $0 < \delta < 1/36$ and let $X$ be a separable normed space containing an asymptotic biorthogonal system with constant $\delta$. Then there is an equivalent norm on $X$ such that no sequence is $1/\sqrt{36\delta}$-unconditional.

**Proof.** Let $\|\cdot\|$ be the original norm on $X$ and let $A_1, A_2, \ldots$ and $A^*_1, A^*_2, \ldots$ be the asymptotic biorthogonal system with constant $\delta$. For each $n$ let $Z^*_n$ be a countable subset of $A^*_n$ such that property (ii) of asymptotic biorthogonal systems holds for some $x^* \in Z^*_n$. 


Let $Z^* = \bigcup_{n=1}^{\infty} Z_{n}^*$. Next, let $\sigma$ be an injection to the natural numbers from the collection of finite sequences of elements of $Z^*$.

We shall now define a collection of functionals which we call special functionals. A special sequence of functionals of length $r$ is a sequence of the form $z_1^*, z_2^*, \ldots, z_r^*$, where $z_1^* \in Z_1^*$ and, for $1 \leq i < r$, $z_{i+1}^* \in Z^*_{\sigma(z_1^*, \ldots, z_i^*)}$. A special functional of length $r$ is simply the sum of a special sequence of length $r$. We shall let $\Gamma_r$ stand for the collection of special functionals of length $r$.

We can now define an equivalent norm on $X$. Let $r = \lfloor \delta^{-1/2} \rfloor$ and define a norm $\| \cdot \|$ by

$$
\|x\| = \|x\| \lor r \max \left\{ |z^*(x)| : z^* \in \Gamma_r \right\}.
$$

Let $x_1, x_2, \ldots$ be any sequence of linearly independent vectors in $X$. We shall show that it is not $(r - 1)/4$-unconditional in the norm $\| \cdot \|$. We shall do this by constructing a sequence of vectors $z_1, \ldots, z_r$, generated by $x_1, x_2, \ldots$ and disjointly supported with respect to these vectors, with the property that

$$(r - 1) \sum_{i=1}^{r} (-1)^i z_i \leq 4 \sum_{i=1}^{r} z_i.$$

This will obviously prove the theorem, since $(r - 1)/4 > 1/\sqrt{36\delta}$.

Let $X_1$ be the algebraic subspace generated by $(x_i)_{i=1}^{\infty}$. Since $A_1$ is an asymptotic set, we can find $z_1 \in A_1 \cap X_1$. This implies that $z_1$ has norm 1 and is generated by finitely many of the $x_i$. Next we can find $z_2^* \in Z_1^*$ such that $z_2^*(z_1) > 1 - \delta$. Now let $X_2$ be the algebraic subspace generated by all the $x_i$ not used to generate $z_1$. Since $A_{\sigma(z_1^*)}$ is asymptotic, we can find $z_2 \in A_{\sigma(z_1^*)} \cap X_2$ of norm 1. We can then find $z^*_2 \in Z^*_{\sigma(z_2^*)}$ such that $z^*_2(z_2) > 1 - \delta$.

Continuing this process, we obtain sequences $z_1, \ldots, z_r$ and $z_1^*, \ldots, z_r^*$ with the following properties. First, $\|z_i\| = 1$ for each $i$. Second, $z_{i+1}^* \in Z^*_{\sigma(z_1^*, \ldots, z_i^*)}$ for each $i$ (i.e. $z_1^*, \ldots, z_r^*$ is a special sequence of length $r$). Third, $z_i^*(z_i) > 1 - \delta$ for each $i$. Fourth, since $\sigma$ is an injection, the $z_i^*$ belong to different $A_n^*$s, so $|z_i^*(z_j)| < \delta$ when $i \neq j$.

Let us now estimate the norm of $\sum_{i=1}^{r} z_i$. Since $z_1^*, \ldots, z_r^*$ is a special functional of length $r$, the norm is at least

$$r \left( \sum_{i=1}^{r} z_i^* \right) \left( \sum_{i=1}^{r} z_i \right) > r ((1 - \delta)r - \delta r(r - 1)) \geq r(r - 1).$$
On the other hand, if \((w^*_i)_{i=1}^r\) is any special sequence of length \(r\), let \(t\) be maximal such that \(w^*_i = z^*_i\) (or zero if \(w^*_i \neq z^*_i\)). Then
\[
\left| \sum_{i=1}^r (-1)^i w^*_i(z_i) \right| \leq \left| \sum_{i=1}^t (-1)^i w^*_i(z_i) \right| + \left| w^*_{i+1}(z_{t+1}) \right| + \sum_{i=t+2}^r \left| w^*_i(z_i) \right|.
\]
Since \(\sigma\) is an injection, \(w^*_i\) and \(z^*_j\) are chosen from different sets \(A^*_n\) whenever \(i \neq j\) or \(i = j > t + 1\). By property (iii) this tells us that \(\left| w^*_i(z_i) \right| < \delta\). In particular, \(\sum_{i=t+2}^r \left| w^*_i(z_i) \right| < \delta r\). When \(i < t\) we know that \(1 - \delta < w^*_i(z_i) \leq 1\), so \(\left| \sum_{i=1}^t (-1)^i w^*_i(z_i) \right| \leq 1 + \delta t/2\). It follows that
\[
\left| \sum_{i=1}^r (-1)^i w^*_i(z_i) \right| \leq 1 + \delta r/2 + 1 + \delta r \leq 2(1 + \delta r) .
\]
We also know that \(\sum_{i \neq j} \left| w^*_i(z_j) \right| \leq \delta r(r - 1)\). Finally, by the triangle inequality, \(\left\| \sum_{i=1}^r (-1)^i z_i \right\| \leq r\).

Putting all these estimates together, we find that
\[
\left\| \sum_{i=1}^r (-1)^i z_i \right\| \leq r \left( 2(1 + \delta r) + \delta r(r - 1) \right) < 4r
\]
from which it follows that the basic sequence \(x_1, x_2, \ldots\) was not \((r - 1)/4\)-unconditional in the equivalent norm. \(\square\)

With a little more care one can increase the best unconditional constant from roughly \(\delta^{-1/2}\) to roughly \(\delta^{-1}\), but some of the details of this would obscure the main point of the proof. It also does not seem to be necessary in applications. Indeed, it is not known whether there exists a space containing an asymptotic biorthogonal system for some (non-trivial) \(\delta\) but not for every \(\delta > 0\). In the next section, we examine a space that contains them for every \(\delta\).

§2. Schlumprecht’s space.

A space \((Y, \| \cdot \|)\) is said to be \(\lambda\)-distortable if there exists an equivalent norm \(\| \cdot \|\) on \(Y\) such that, for every subspace \(Z \subset Y\) the quantity \(\sup \{ \| y \| / \| z \| : \| y \| = \| z \| = 1 \}\) is at least \(\lambda\). A space is distortable if it is \(\lambda\)-distortable for some \(\lambda > 1\). A famous open problem, known as the distortion problem, used to be whether \(\ell_2\) was distortable. This is equivalent to asking whether every space isomorphic to \(\ell_2\) contains a subspace almost isometric to \(\ell_2\). A few months after the results in this paper were obtained, the distortion problem was also
solved, by E. Odell and T. Schlumprecht [OS]; actually a stronger statement is proved in
[OS], namely that $\ell_2$ contains an asymptotic biorthogonal system with any constant $\delta > 0$.
This implies that $\ell_2$ can be renormed so as not to contain a $C$-unconditional basic sequence.
However, we shall consider in this section a space constructed by Schlumprecht [S1]. This
space was the first known example of a space that is $\lambda$-distortable for every $\lambda$. The main
result of this section is that it contains an asymptotic biorthogonal system for any $\delta$. In
proving this, we shall use very little more than what was already proved by Schlumprecht
in order to show that it is arbitrarily distortable.

First, let us give the definition of Schlumprecht’s space. He defines a class of functions
$f : [1, \infty) \to [1, \infty)$, which we shall call $\mathcal{F}$, as follows. The function $f$ is a member of $\mathcal{F}$ if
it satisfies the following five conditions.

(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$;
(ii) $f$ is strictly increasing and tends to infinity;
(iii) $\lim_{x \to \infty} x^{-q} f(x) = 0$ for every $q > 0$;
(iv) the function $x / f(x)$ is concave and non-decreasing;
(v) $f(xy) \leq f(x) f(y)$ for every $x, y \geq 1$.

It is easily verified that $f(x) = \log_2 (x + 1)$ satisfies these conditions, as does the
function $\sqrt{f(x)}$.

Schlumprecht’s space is a Tsirelson-type construction, in that it is defined inductively.
As with an earlier construction due to L. Tzafriri, the admissibility condition used in
Tsirelson’s space ([T]) is not needed (see [CS]). Before giving the definition, let us fix some
notation.

Let $c_{00}$ be the space of sequences of real numbers all but finitely many of which
are zero. We shall let $e_1, e_2, \ldots$ stand for the unit vector basis of this vector space. If
$E \subset \mathbb{N}$, then we shall also use the letter $E$ for the projection from $c_{00}$ to $c_{00}$ defined by
$E \left( \sum_{i=1}^{\infty} a_i e_i \right) = \sum_{i \in E} a_i e_i$. If $E, F \subset \mathbb{N}$, then we write $E < F$ to mean that $\max E < \min F$, and if $k \in \mathbb{N}$ and $E \subset \mathbb{N}$, then we write $k < E$ to mean $k < \min E$. The support of
a vector $x = \sum_{i=1}^{\infty} x_i e_i \in c_{00}$ is the set of $i \in \mathbb{N}$ for which $x_i \neq 0$. An interval of integers
is a subset of $\mathbb{N}$ of the form $\{a, a+1, \ldots, b\}$ for some $a, b \in \mathbb{N}$. We shall also define the
range of a vector, written $\text{ran}(x)$, to be the smallest interval containing its support. We
shall write $x < y$ to mean $\text{ran}(x) < \text{ran}(y)$. If $x_1 < \ldots < x_n$ we shall say that $x_1, \ldots, x_n$ are successive.
Now let \( f(x) \) be the function \( \log_2(x+1) \) as above. If \( x \in c_{00} \), its norm in Schlumprecht’s space is defined inductively by

\[
\|x\| = \|x\|_\infty \vee \sup f(N)^{-1} \sum_{i=2}^{N} \|E_i x\|
\]

where the supremum runs over all integers \( N \geq 2 \) and all sequences of sets \( E_1 < \ldots < E_N \).

Note that this definition, although apparently circular, in fact determines a unique norm. Note also that the standard basis of \( c_{00} \) is 1-unconditional in this norm, so there is no difference if we assume that all the sequences \( E_1 < \ldots < E_N \) are sequences of intervals. Later in the paper it will make a great difference, and we now adopt the convention that all such sequences mentioned are sequences of intervals.

We now prove various lemmas about this space. As we have already said, they are essentially due to Schlumprecht [S1][S2], but are stated here in slightly greater generality so that they can be applied in the main space of this paper.

Let \( \mathcal{X} \) be the set of normed spaces of the form \( X = (c_{00}, \|\cdot\|) \) such that \( (e_i)_{i=1}^\infty \) is a normalized monotone basis of \( X \). If \( f \in \mathcal{F}, X \in \mathcal{X} \) and every \( x \in X \) satisfies the inequality

\[
\|x\| \geq \sup \left\{ f(N)^{-1} \sum_{i=1}^{N} \|E_i x\| : N \in \mathbb{N}, E_1 < \ldots < E_N \right\}
\]

then we shall say that \( X \) satisfies a lower \( f \)-estimate. (It is important that, in the supremum above, the \( E_i \) are intervals). Note that this implies that \( \|Ex\| \leq \|x\| \) for every interval \( E \) and vector \( x \), so the standard basis of a space with a lower \( f \)-estimate is automatically bimonotone.

Given a space \( X \in \mathcal{X} \) and a vector \( x \in X \), we shall say that \( x \) is an \( \ell_1^n \)-average with constant \( C \) if \( \|x\| = 1 \) and \( x = \sum_{i=1}^{n} x_i \) for some sequence \( x_1 < \ldots < x_n \) of non-zero elements of \( X \) such that \( \|x_i\| \leq Cn^{-1} \) for each \( i \). An \( \ell_1^n \)-vector is any positive multiple of an \( \ell_1^n \)-average. In other words, a vector \( x \) is an \( \ell_1^n \)-vector with constant \( C \) if it can be written \( x = x_1 + \ldots + x_n \), where \( x_1 < \ldots < x_n \), the \( x_i \) are non-zero and \( \|x_i\| \leq Cn^{-1} \|x\| \) for every \( i \).

Finally, by a block basis in a space \( X \in \mathcal{X} \) we mean a sequence \( x_1, x_2, \ldots \) of successive non-zero vectors in \( X \) (note that such a sequence must be a basic sequence) and by a block subspace of a space \( X \in \mathcal{X} \) we mean a subspace generated by a block basis.
Lemma 1. Let $f \in \mathcal{F}$ and let $X \in \mathcal{X}$ satisfy a lower $f$-estimate. Then, for every $n \in \mathbb{N}$ and every $C > 1$, every block subspace $Y$ of $X$ contains an $\ell_{1+}^n$-average with constant $C$.

Proof. Suppose the result is false. Let $k$ be an integer such that $k \log C > \log f(n^k)$ (such an integer exists because of property (iii) in the definition of $\mathcal{F}$), let $N = n^k$, let $x_1 < \ldots < x_N$ be any sequence of successive norm-1 vectors in $Y$ and let $x = \sum_{i=1}^{N} x_i$. For every $0 \leq i \leq k$ and every $1 \leq j \leq n^{k-i}$, let $x(i, j) = \sum_{t=(j-1)n^{i}+1}^{jn^{i}} x_t$. Thus $x(0, j) = x_j$, $x(k, 1) = x$ and, for $1 \leq i \leq k$, each $x(i, j)$ is a sum of $n$ successive $x(i-1, j)$s. By our assumption, no $x(i, j)$ is an $\ell_{1+}^n$-vector with constant $C$. It follows easily by induction that $\|x(i, j)\| \leq C^{-i}n^i$, and, in particular, that $\|x\| \leq C^{-k}n^k = C^{-k}N$. However, it follows from the fact that $X$ satisfies a lower $f$-estimate that $\|x\| \geq Nf(N)^{-1}$. This is a contradiction, by choice of $k$. \hfill \square

Lemma 2. Let $M, N \in \mathbb{N}$ and $C \geq 1$, let $x$ be an $\ell_{1+}^N$-vector with constant $C$ and let $E_1 < \ldots < E_M$ be a sequence of intervals. Then

$$\sum_{j=1}^{M} \|E_j x\| \leq C(1 + 2M/N) \|x\|.$$ 

Proof. For convenience, let us normalize so that $\|x\| = N$ and $x = \sum_{i=1}^{N} x_i$, where $x_1 < \ldots < x_N$ and $\|x_i\| \leq C$ for each $i$. Given $j$, let $A_j$ be the set of $i$ such that $\text{supp}(x_i) \subset E_j$ and let $B_j$ be the set of $i$ such that $E_j(x_i) \neq 0$. By the triangle inequality and the fact that the basis is bimonotone,

$$\|E_j x\| \leq \left\| \sum_{i \in B_j} x_i \right\| \leq C(|A_j| + 2).$$

Since $\sum_{j=1}^{M} |A_j| \leq N$, we get

$$\sum_{j=1}^{M} \|E_j x\| \leq C(N + 2M)$$

which gives the result, because of our normalization. \hfill \square

In order to state the next lemma, we shall need some more definitions. The first is a technicality. If $f \in \mathcal{F}$, let $M_f : \mathbb{R} \to \mathbb{R}$ be defined by $M_f(x) = f^{-1}(36x^2)$.
The next definition is of great importance in this paper. We shall say that a sequence $x_1 < \ldots < x_N$ is a rapidly increasing sequence of $\ell_{1+}$-averages, or R.I.S., for $f$ of length $N$ with constant $1 + \epsilon$ if $x_k$ is an $\ell_{1+}^n$-average with constant $1 + \epsilon$ for each $k$, $n_1 \geq 2(1 + \epsilon)M_f(N/\epsilon')/\epsilon' f'(1)$ and

$$\frac{\epsilon'}{2} f(n_k)^{1/2} \geq |\text{supp}(x_{k-1})|$$

for $k = 2, \ldots, N$. Here $f'(1)$ is the right derivative of $f$ at 1 and $\epsilon'$ is a useful notation for $\min\{\epsilon, 1\}$ which we shall use throughout the section. Obviously there is nothing magic about the exact conditions in this definition. The important point is that the $n_k$s increase fast enough, the speed depending on the sizes of the supports of the earlier $x_j$s.

We make one further definition. A functional $x^*$ is an $(M, g)$-form if $\|x^*\| \leq 1$ and $x^* = \sum_{j=1}^M x_j^*$ for some sequence $x_1^* < \ldots < x_M^*$ of successive functionals such that $\|x_j^*\| \leq g(M)^{-1}$ for each $j$.

**Lemma 3.** Let $f, g \in F$, let $g \geq f^{1/2}$ and let $X \in \mathcal{X}$ satisfy a lower $f$-estimate. Let $\epsilon > 0$, let $x_1, \ldots, x_N$ be a R.I.S. in $X$ for $f$ with constant $1 + \epsilon$ and let $x = \sum_{i=1}^N x_i$. Let $M \geq M_f(N/\epsilon')$, let $x^*$ be an $(M, g)$-form and let $E$ be any interval. Then $|x^*(Ex)| \leq 1 + \epsilon + \epsilon'$.

**Proof.** If $x^*$ is an $(M, g)$-form then so is $Ex^*$ for any interval $E$. Since $x^*(Ex) = (Ex^*)(x)$, we can forget about the interval $E$ in the statement of the lemma. For each $i$, let $n_i$ be maximal such that $x_i$ is an $\ell_{1+}^n$-average with constant $1 + \epsilon$. Let us also write $x^* = \sum_{j=1}^M x_j^*$ in the obvious way and set $E_j = \text{ran}(x_j^*)$. We first obtain three easy estimates for $|x^*(x_i)|$. Since $\|x^*\| \leq 1$, we obviously have $|x^*(x_i)| \leq 1$. Then, since $\|x_j^*\| \leq g(M)^{-1} \leq f(M)^{-1/2}$, we have $|x^*(x_i)| \leq f(M)^{-1/2} \sum_{j=1}^M \|E_j x_i\|$. By our assumption about $X$, this is at most $f(M)^{-1/2} f(|\text{supp}(x_i)|)$ and by Lemma 2 it is at most $(1 + \epsilon)(1 + 2Mn_i^{-1})f(M)^{-1/2}$.

Let $t$ be maximal such that $n_t \leq M$. Then, if $i < t$, we have $f(|\text{supp}(x_i)|) \leq 2^{i-t+1} f(|\text{supp}(x_{t-1})|)$, and also $f(|\text{supp}(x_{t-1})|) \leq (\epsilon'/2) f(n_t)^{1/2} \leq (\epsilon'/2) f(M)^{1/2}$. Using this and the other two estimates above, we obtain

$$|x^*(x)| \leq \sum_{i=1}^N |x^*(x_i)| \leq \epsilon' + 1 + 3(1 + \epsilon)(N - t) f(M)^{-1/2}$$

$$\leq 1 + \epsilon' + 3(1 + \epsilon)N(\epsilon'/6N)$$

$$= 1 + \epsilon' + (\epsilon'/2)(1 + \epsilon) \leq 1 + \epsilon + \epsilon'$$

as stated. \qed

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Corollary 4. Let \( f, X, \epsilon, M, x_1, \ldots, x_N \) and \( x \) be as in Lemma 3, let \( E_1 < \ldots < E_M \) and let \( E \) be any interval. Then

\[
f(M)^{-1} \sum_{i=1}^{M} \|E_i E x\| \leq 1 + \epsilon + \epsilon'.
\]

Proof. Let \( x_j^* \) be a support functional of \( E_j x \) and let \( x^* = f(M)^{-1} \sum_{i=1}^{M} x_i^* \). Then \( \|x^*\| \leq 1 \) because \( X \) satisfies a lower \( f \)-estimate. It follows that \( x^* \) is an \((M, f)\)-form, so we can apply Lemma 3 with \( g = f \). □

We now introduce a further convenient definition. Let \( x_1 < \ldots < x_N \) be a R.I.S. for \( f \) with constant \( 1 + \epsilon \), for some \( f \in F \) and some \( \epsilon > 0 \). For each \( i \), let \( n_i \) be maximal such that \( x_i \) is an \( \ell_{1+\epsilon} \)-average with constant \( 1 + \epsilon \) and let us write it out as \( x_i = x_{i1} + \ldots + x_{in_i} \), where \( \|x_{ij}\| \leq (1 + \epsilon)n_i^{-1} \) for each \( j \). Given an interval \( E \subset \mathbb{N} \), let \( i = i_E \) and \( j = j_E \) be respectively minimal and maximal such that \( E x_i \) and \( E x_j \) are non-zero, and let \( r = r_E \) and \( s = s_E \) be respectively minimal and maximal such that \( E x_{ir} \) and \( E x_{js} \) are non-zero.

Define the length \( \lambda(E) \) of the interval \( E \) to be \( j_E - i_E + (s_E/n_j - (r_E/n_i)) \). Thus the length of \( E \) is the number of \( x_i \)'s contained in \( E \), allowing for fractional parts. It is easy to check that if \( E_1 < \ldots < E_M \) and \( E = \bigcup E_i \) then \( \sum \lambda(E_i) \leq \lambda(E) \). Obviously this definition depends completely on the R.I.S. but it will always be clear from the context which R.I.S. is being considered.

The next lemma is the most important one for our purposes.

Lemma 5. Let \( f, g \in F \) with \( g \geq \sqrt{f} \), let \( X \in X \) satisfy a lower \( f \)-estimate, let \( \epsilon > 0 \), let \( x_1 < \ldots < x_N \) be a R.I.S. in \( X \) for \( f \) with constant \( 1 + \epsilon \) and let \( x = \sum_{i=1}^{N} x_i \). Suppose that

\[
\|E x\| \leq \sup \{ \|x^*(E x)\| : M \geq 2, x^* \text{ is an } (M, g)\text{-form} \}
\]

for every interval \( E \) of length at least 1. Then \( \|x\| \leq (1 + \epsilon + \epsilon') Ng(N)^{-1} \).

Proof. It follows from the triangle inequality that \( \|E x\| \leq (1 + \epsilon)(\lambda(E) + n_1^{-1}) \). If \( \lambda(E) \geq (1 + \epsilon)/\epsilon' n_1 \) then we get \( \|E x\| \leq (1 + \epsilon + \epsilon') \lambda(E) \). Let \( G \) be defined by \( G(x) = x \) when \( 0 \leq x \leq 1 \) and \( G(x) = xg(x)^{-1} \) when \( x \geq 1 \). Recall that, because of the properties of the set \( F \), \( G \) is concave and increasing on \([1, \infty)\) and satisfies \( G(xy) \geq G(x)G(y) \) for every \( x, y \) in the same interval. It is easy to check that these properties are still true on the
whole of \( \mathbb{R}_+ \). We shall show that if \( \lambda(E) \geq (1 + \epsilon)/\epsilon'n_1 \), then \( \|Ex\| \leq (1 + \epsilon + \epsilon')G(\lambda(E)) \).

The remarks we have just made show this when \( \lambda(E) \leq 1 \).

Let us suppose then that \( E \) is a minimal interval of length at least \((1 + \epsilon)/\epsilon'n_1 \) for which the inequality fails. We know that \( \lambda(E) \geq 1 \). We also know that there exists some \((M, g)\)-form \( x^* = \sum_{i=1}^{M} x_i^* \) such that \( \|Ex\| \leq |x^*(Ex)| \). By Lemma 3, we must have \( M \leq M_f(N/\epsilon') \) or the inequality would not fail for \( E \). Letting \( E_i = E \cap \text{ran}(x_i^*) \), we have

\[
\|Ex\| \leq g(M)^{-1} \sum_{i=1}^{M} \|E_i x\|
\]

by the definition of an \((M, g)\)-form.

Let \( \lambda_i = \lambda(E_i) \) for each \( i \). For each \( i \) we either have \( \lambda_i \leq (1 + \epsilon)/\epsilon'n_1 \) or, by the minimality of \( E \), that \( \|E_i x\| \leq (1 + \epsilon + \epsilon')G(\lambda_i) \). Let \( A \) be the set of \( i \) with the first property and let \( B \) be the complement of \( A \). Let \( k \) be the cardinality of \( A \).

Since \( G \) is a concave and non-decreasing function and \( \sum \lambda_i \leq \lambda \), Jensen’s inequality gives us that

\[
\sum_{i \in B} \|E_i x\| \leq (1 + \epsilon + \epsilon') \sum_{i \in B} G(\lambda_i) \\
\leq (1 + \epsilon + \epsilon')(M - k)G(\lambda/(M - k)) .
\]

It follows that

\[
\|Ex\| \leq M^{-1}G(M)\left[(1 + \epsilon + \epsilon')(M - k)G(\lambda/(M - k)) + (1 + \epsilon)(1 + \epsilon + \epsilon')k/\epsilon'n_1\right] \\
\leq (1 + \epsilon + \epsilon')\left[(1 - k/M)G(M)G(\lambda/(M - k)) + (1 + \epsilon)k/\epsilon'n_1\right] \\
\leq (1 + \epsilon + \epsilon')\left[(1 - k/M)G((1 - k/M)^{-1}\lambda) + (1 + \epsilon)k/\epsilon'n_1\right] .
\]

Let \( G'(1) \) be the right derivative of \( G \) at 1. Since \( G \) is a concave function we have the easy inequality

\[
(1 - t)G\left(\frac{\lambda}{1 - t}\right) + t(G(1) - G'(1)) \leq G(\lambda)
\]

for every \( 0 \leq t < 1 \) and \( \lambda \geq 1 \). Also, \( G(1) - G'(1) = 1 - G'(1) = g'(1) \), and since \( g \geq \sqrt{f} \) we have \( g'(1) \geq f'(1)/2 > 0 \).

By the definition of R.I.S. we have \( n_1 \geq 2(1 + \epsilon)M_f(N/\epsilon'')/\epsilon'f'(1) \). It follows that

\[
(1 + \epsilon)k/\epsilon'n_1 \leq (k/M)((1 + \epsilon)M_f(N/\epsilon')/\epsilon'f'(1)) \leq (k/M)(f'(1)/2) \leq (k/M)g'(1) .
\]
Hence, by the inequality above with \( t = k/M \), we have
\[
(1 + \epsilon + \epsilon') \left[ (1 - k/M)G((1 - k/M)^{-1}\lambda) + (1 + \epsilon)k/\epsilon' n_1 \right] \\
\leq (1 + \epsilon + \epsilon') \left( (1 - t)G\left(\frac{\lambda}{1 - t}\right) + tg'(1) \right) \\
\leq (1 + \epsilon + \epsilon')G(\lambda) ,
\]
contradicting our assumption about the interval \( E \), and proving the lemma. \( \square \)

It is now easy to construct an asymptotic biorthogonal system in Schlumprecht’s space. Let \( \delta \in (0, 1) \), let \( N_1 < N_2 < \ldots \) be a sequence of integers satisfying \( f(N_j)/N_1 < \delta/2 \), \( f(N_j) > 8\delta^{-1} \) and \( N_j > M_f(2N_{j-1}) \) for all \( j > 1 \). Let \( A_k \) be the set of norm-1 vectors of the form \( x = \sum_{i=1}^{N_k} x_i \) where \( x_1, \ldots, x_{N_k} \) is a multiple of a R.I.S. with constant \( 1 + \delta/2 \). Because Schlumprecht’s space satisfies a lower \( f \)-estimate, we know that the multiple is at most \( f(N_k)N_k^{-1} \). Let \( A_k^* \) be the set of functionals of the form \( f(N_k)^{-1} \sum_{i=1}^{N_k} x_i^* \) where \( x_1^* < \ldots < x_{N_k}^* \) and \( \|x_i^*\| \leq 1 \) for each \( i \). It is clear that the sets \( A_k \) are asymptotic for every \( k \). If \( j > k \) then using the fact that \( N_j > M_f(2N_k) \), we may apply Lemma 3 with \( \epsilon = 1/2 \) and \( M = N_j \), since \( y^* \) is clearly an \((M, f)\)-form whenever \( y^* \in A_j^* \). Because of the normalization of the R.I.S., this gives us \( |y^*(x)| \leq 2f(N_k)/N_k < \delta \) for every \( y^* \in A_j^* \) and \( x \in A_k \).

If \( j < k \) then we know from Lemma 5 that \( \|\sum_{i \in A} x_i\| \leq 2|A|f(N_k)/N_k f(|A|) \) for every subset \( A \) of \( \{1, 2, \ldots, N_k\} \). If \( |A| \geq \sqrt{N_k} \) then this is at most \( 4|A|/N_k \). By splitting into \( \sqrt{N_k} \) successive pieces of this form, we find that \( x \) is an \( \ell_{1+\sqrt{N_k}}^1 \)-average with constant \( 4 \). By Lemma 2 we obtain that \( |y^*(x)| \leq f(N_j)^{-1} A(1 + 2N_j/\sqrt{N_k}) \leq 8f(N_j)^{-1} < \delta \).

Finally, we know that \( \|x\| \leq (1 + \delta)N_k f(N_k)^{-1} \|x_i\| \) for each \( i \), so if we let \( x_i^* \) be a support functional of \( x_i \) then \( x^* = f(N_k)^{-1} \sum_{i=1}^{N_k} x_i^* \) is an element of \( A_k^* \) and \( x^*(x) \geq (1 + \delta)^{-1} > 1 - \delta \). It follows that \( A_1, A_2, \ldots \) and \( A_1^*, A_2^*, \ldots \) form an asymptotic biorthogonal system with constant \( \delta \).

This together with the result of the last section shows that, for every \( C \), Schlumprecht’s space can be renormed so as not to contain a \( C \)-unconditional basic sequence. Since Schlumprecht’s space itself has a 1-unconditional basis, it follows that it is arbitrarily distortable. This is also an easy direct consequence of the existence of an asymptotic biorthogonal system, or indeed from Lemma 5, which is what Schlumprecht used.
§3. A space containing no unconditional basic sequence.

We now come to the main result of the paper, namely the construction of a Banach space $X$ containing no unconditional basic sequence. As we mentioned in the introduction, it was observed by W. B. Johnson that our original arguments could be modified to show that $X$ was actually H.I. This is what we shall actually present in this section.

The definition of the space resembles that of Schlumprecht’s space, or at least can do. We shall actually give three equivalent definitions, for which we shall need a certain amount of preliminary notation.

First, let $Q$ be the set of real sequences with finite support, rational coordinates and maximum at most 1 in modulus. Let $J \subset \mathbb{N}$ be a set such that, if $m < n$ and $m, n \in J$, then $\log \log \log n \geq 2m$. Let us write $J$ in increasing order as \{\( j_1, j_2, \ldots \)\}. We shall also assume that $f(j_1) \geq 36$. (Recall that $f(x)$ is the function $\log_2(x + 1)$.) Now let $K \subset J$ be the set \{\( j_2, j_4, j_6, \ldots \)\} and let $L \subset \mathbb{N}$ be the set of integers $j_1, j_3, j_5, \ldots$.

Let $\sigma$ be an injection from the collection of finite sequences of successive elements of $Q$ to $L$ such that, if $z_1, \ldots, z_s$ is such a sequence, $S = \sigma(z_1, \ldots, z_s)$ and $z = \sum_{i=1}^{s} z_i$ then $(1/20)f\left(S^{1/40}\right)^{1/2} \geq |\text{supp}(z)|$.

We shall use the injection $\sigma$ to define special functionals in an arbitrary normed space of the form $(c_00, \| \cdot \|)$ in much the same way that we defined them in Section 1. (Of course, for most spaces they are not terribly interesting).

If $X = (c_00, \| \cdot \|)$ is a normed space on the finitely supported sequences, and $m \in \mathbb{N}$, let $A^*_m(X)$ be the set of functionals of the form $f(m)^{-1} \sum_{i=1}^{m} f_i$ such that $f_1 < \ldots < f_m$ and $\|f_i\| \leq 1$ for each $i$. If $k \in \mathbb{N}$, let $\Gamma^X_k$ be the set of sequences $g_1, \ldots, g_k$ such that $g_i \in Q$ for each $i$, $g_1 \in A^*_{j_{2k-1}}(X)$ and $g_{i+1} \in A^*_{\sigma(g_1, \ldots, g_i)}(X)$ for each $1 \leq i \leq k - 1$. We call these special sequences. Let $B^*_k(X)$ be the set of functionals of the form $f(k)^{-1/2} \sum_{j=1}^{k} g_j$ such that $(g_1, \ldots, g_k) \in \Gamma^X_k$. These are special functionals.

Dually, if a convex set $D \subset c_00$ is given, we define $A_m(D)$ to be the set of vectors of the form $f(m)^{-1} \sum_{i=1}^{m} x_i$ where $x_1 < \ldots < x_m$ and $x_i \in D$ for each $i$. Then special sequences of vectors are defined using $\sigma$ in the obvious corresponding way, and this gives us sets $B_k(D)$.

Our first definition of the norm is geometrical, and goes via the dual space. Let $D_0$ be the intersection of $c_00$ with the unit ball of $\ell_1$. Once we have defined $D_N$, let $D'_N$ be the set of vectors of the form $f(N)^{-1} \sum_{i=1}^{N} x_i$, where $x_1, \ldots, x_N$ are successive vectors in $D_N$. Let
\( D'_N \) be the set of special vectors for \( D_N \) with lengths in \( K \), that is, \( D'_N = \bigcup_{k \in K} B_k(D_N) \). Let \( D''_N \) be the set of vectors \( Ex \) where \( x \in D_N \). Then let \( D_{N+1} \) be the convex hull of the union of \( D'_N, D''_N \) and \( D'''_N \).

Now let \( D = \bigcup_{N=0}^{\infty} D_N \). It is easy to see that \( D \) is the smallest convex set closed under taking sums of the form \( f(N)^{-1} \sum_{i=1}^{N} x_i \), taking special vectors with lengths in \( K \) and under interval projections. Our space is defined by

\[
\|x\| = \sup\{|\langle x, y \rangle| : y \in D\}.
\]

The second definition of the norm is as the limit of a sequence of norms. Define \( X_0 = (c_0, \|\cdot\|_0) \) by \( \|x\|_0 = \|x\|_\infty \), and, for \( n \geq 0 \), let

\[
\|x\|_{X_{n+1}} = \sup\left\{ f(n)^{-1} \sum_{i=1}^{n} \|E_ix\|_{X_n} : n \in \mathbb{N}, E_1 < \ldots < E_n \right\}
\]

\[
\vee \sup\left\{ |g(Ex)| : k \in K, g \in B_k^*(X_n), E \subset \mathbb{N} \right\}.
\]

Note that this is an increasing sequence of norms, because the sets \( B_k^*(X_n) \) increase as \( n \) increases (and more generally, if \( \|x\|_Y \leq \|x\|_Z \) for every \( x \in c_0 \), then \( B_k^*(Y) \subset B_k^*(Z) \)). They are also all bounded above by the \( \ell_1 \)-norm. Define \( \|\cdot\| \) by \( \|x\| = \lim_{n \to \infty} \|x\|_{X_n} \).

Finally, we give an implicit definition of the norm in the obvious way. Set

\[
\|x\| = \|x\|_c \vee \sup\left\{ f(n)^{-1} \sum_{i=1}^{n} \|E_ix\| : 2 \leq n \in \mathbb{N}, E_1 < \ldots < E_n \right\}
\]

\[
\vee \sup\left\{ |g(Ex)| : k \in K, g \in B_k^*(X), E \subset \mathbb{N} \right\}.
\]

Recall that \( E \subset \mathbb{N} \) is always an interval in these definitions. Its role is to ensure that \( (e_i)_{i=1}^{\infty} \) is a (bimonotone) normalized Schauder basis for the completion of \( X \). Note also that if we did not insist that the \( E_i \) were intervals then the unit vector basis of this space would trivially be unconditional. It is not hard to check that the norm given by the third definition is indeed well-defined and agrees with both the previous ones.

Before getting down to analysing the space, we shall need a few simple facts about functions in the class \( \mathcal{F} \) defined earlier.

We shall now introduce some convenient definitions. Let \( f : [1, \infty) \to [1, \infty) \) be a function. The (increasing) submultiplicative hull of \( f \) is the function \( F \) defined by

\[
F(x) = \inf\left\{ f(x_1)f(x_2)\ldots f(x_k) : k \in \mathbb{N}, x_i \geq 1, x_1 \ldots x_k \geq x \right\}.
\]
The following facts are trivial. First, \( F \leq f \). Second, \( F(xy) \leq F(x)F(y) \). Third, if \( g : [1, \infty) \to [1, \infty) \) is any non-decreasing submultiplicative function dominated by \( f \), then \( g \) is dominated by \( F \). (That is, \( F \) is the largest non-decreasing submultiplicative function dominated by \( f \)).

Now let \( g : [1, \infty) \to [1, \infty) \) be any function. The concave envelope of \( g \) is of course the smallest concave function \( G : [1, \infty) \to [1, \infty) \) dominating \( g \), that is,
\[
G(x) = \sup \left\{ \lambda g(y) + (1-\lambda)g(z) : 0 \leq \lambda \leq 1, \lambda y + (1-\lambda)z = x \right\}.
\]

We now prove an easy lemma.

**Lemma 6.** If \( g : [1, \infty) \to [1, \infty) \) is a supermultiplicative function, then its concave envelope is also supermultiplicative.

**Proof.** Let \( \epsilon > 0 \) and let \( x_1, x_2 \geq 1 \). We shall show that \( (G(x_1) - \epsilon)(G(x_2) - \epsilon) \leq G(x_1)G(x_2) \), which will prove the result. First, for \( i = 1, 2 \), pick \( \lambda_i, \mu_i, y_i \) and \( z_i \) such that \( 0 \leq \lambda_i \leq 1, \lambda_i + \mu_i = 1, \lambda_i y_i + \mu_i z_i = x_i \) and \( \lambda_i g(y_i) + \mu_i g(z_i) \geq G(x_i) - \epsilon \).

Then
\[
(G(x_1) - \epsilon)(G(x_2) - \epsilon) \leq (\lambda_1 g(y_1) + \mu_1 g(z_1))(\lambda_2 g(y_2) + \mu_2 g(z_2))
\]
\[
\leq \lambda_1 \lambda_2 g(y_1 y_2) + \lambda_1 \mu_2 g(y_1 z_2) + \mu_1 \lambda_2 g(z_1 y_2) + \mu_1 \mu_2 g(z_1 z_2)
\]
\[
\leq \lambda_1 G(y_1 x_2) + \mu_1 G(z_1 x_2) \leq G(x_1 x_2)
\]
as we wanted. \( \square \)

Now let us define a function \( \phi : [1, \infty) \to [1, \infty) \) as follows.
\[
\phi(x) = \begin{cases} (\log_2(x+1))^{1/2} & \text{if } x \in K \\ \log_2(x+1) & \text{otherwise.} \end{cases}
\]

Let \( h \) be the submultiplicative hull of \( \phi \), let \( H \) be the function given by \( H(x) = x/h(x) \) and let \( G \) be the concave envelope of \( H \). Since \( h \) is submultiplicative, \( H \) is supermultiplicative, so \( G \) is also supermultiplicative. Now let \( g(x) = x/G(x) \). Then \( g \) is submultiplicative. As before, let \( f \) be the function \( \log_2(x+1) \). The easy facts about submultiplicative hulls and concave envelopes and the fact that \( \sqrt{f} \in F \) give that \( (\log_2(x+1))^{1/2} \leq g(x) \leq \phi(x) \leq \log_2(x+1) \). It will also be useful to extend the definition of \( G \) to the whole of \( \mathbb{R}_+ \) by setting \( G(x) = x \) when \( 0 \leq x \leq 1 \). It is easy to check that \( G \) thus extended is still supermultiplicative and concave, as we commented in the proof of Lemma 5.

We now need to calculate \( G(N) \) when \( N \in L \). In fact we shall want slightly more than this, as is suggested by the statement of the next lemma.
Lemma 7. If \( N \in L \) then \( G(x) = xf(x)^{-1} \) for every \( x \) in the interval \([\log N, \exp N]\).

Proof. Let \( k, l \in K \) be maximal and minimal respectively such that \( k < N \) and \( l > N \), and let \((k!)^4 < x < f^{-1}(f(l)^{1/2})\). We shall show first that \( h(x) = f(x) \). \( h(x) \) is defined to be \( \inf \{ \phi(x_1) \ldots \phi(x_m) : x_i \geq 1, x_1x_2 \ldots x_m \geq x \} \). We know that \( \phi(x_j) = f(x_j)^{1/2} \) if \( x_j \in K \) and \( f(x_j) \) otherwise. By the submultiplicativity of \( f \), there can be at most one \( j \) such that \( x_j > 1 \) and \( x_j \notin K \). Because \( f(l)^{1/2} > f(x) \) we also know that if \( x_j \in K \) then \( x_j \leq k \). Thirdly, it is not possible to find \( r, s, t \) such that \( x_r = x_s = x_t \in K \) since, for every \( p \in K \), \( f(p)^{3/2} > f(p^3) \) by our choice for \( \min(K) \). Since \( x > (k!)^4 \) it is clear that at least one, and hence exactly one \( x_j \) is not in \( K \). Let it be \( x_1 \) and assume that \( m > 1 \). Now we know that \( x_2x_3 \ldots x_m \leq (k!)^2 \leq x^{1/2} \). It follows that \( x_1 \geq x^{1/2} \) and hence that \( \phi(x_1) \ldots \phi(x_m) \geq f(x^{1/2})f(\min(K))^{1/2} \). Since \( f \) is the function \( \log_2(x+1) \) and \( f(\min(K)) \geq 36 \), this is greater than \( f(x) \). This contradiction shows that \( m = 1 \), hence \( h(x) = f(x) \).

This shows that \( H(x) = xf(x)^{-1} \) whenever \((k!)^4 < x < f^{-1}(f(l)^{-1/2})\), and in particular for all \( x \) in the interval \([\log \log N, \exp \exp N]\). It is easy to deduce from this the conclusion of the lemma. Indeed, given \( x_0 \) in the interval \([\log N, \exp N]\), we will certainly know that \( G(x_0) = x_0f(x_0)^{-1} \) if the function given by the tangent to \( xf(x)^{-1} \) at \( x_0 \) is at least \( xf(x)^{-1/2} \) for all positive \( x \) outside the interval \([\log \log N, \exp \exp N]\).

The equation of the tangent at \( x_0 \) is

\[
y = \frac{x_0}{f(x_0)} + \frac{1}{f(x_0)} \left( 1 - \frac{x_0}{(x_0 + 1) \log(x_0 + 1)} \right) (x - x_0).
\]

When \( x \geq 0 \) this is certainly at least \( x_0^2 \log 2/(x_0 + 1)(\log(x_0 + 1))^2 \) which is at least \( x_0^2/(2f(x_0)^2) \). For \( x_0 \geq \log N \) and \( x \leq \log \log N \) this exceeds \( xf(x)^{-1/2} \). When \( x \geq 2x_0 \) we also know that \( y \geq x/4f(x_0) \). When \( x \geq \exp \exp N \) the condition \( x_0 \leq \exp N \) is enough to guarantee that this is at least as big as \( xf(x)^{-1/2} \). \( \square \)

We shall now prove a crucial lemma about \( X \). It is an easy consequence of Lemmas 5 and 7.

Lemma 8. Let \( N \in L \), let \( n \in [\log N, \exp N] \), let \( \epsilon > 0 \) and let \( x_1, \ldots, x_n \) be a R.I.S. with constant \( 1 + \epsilon \). Then \( \| \sum_{i=1}^n x_i \| \leq (1 + \epsilon + \epsilon')nf(n)^{-1} \).

Proof. It is obvious from the implicit definition of the norm in \( X \) that it satisfies a lower \( f \)-estimate. Let \( g \) be the function defined before the last lemma. As usual let \( x = \sum_{i=1}^n x_i \).
Since $g \leq \phi$, it is clear that every vector in $X$ is either normed by an $(M, g)$-form or has the supremum norm. It is also clear that the second possibility does not happen in the case of vectors of the form $Ex$ when $\lambda(E) \geq 1$. Since $g \in F$ and, as we commented above, $g \geq f^{1/2}$, all the hypotheses of Lemma 5 are satisfied. It follows that $\|\sum_{i=1}^{n} x_i\| \leq (1 + \epsilon + \epsilon')G(n)$. By Lemma 7, $G(n) = nf(n)^{-1}$, so the lemma is proved. □

**Lemma 9.** Let $N \in L$, let $0 < \epsilon < 1/4$, let $M = N^\epsilon$ and let $x_1, \ldots, x_N$ be a R.I.S. with constant $1 + \epsilon$. Then $\sum_{i=1}^{N} x_i$ is an $\ell_{1+}^M$-vector with constant $(1 + 4\epsilon)$.

**Proof.** Let $m = N/M$, let $x = \sum_{i=1}^{N} x_i$ and for $1 \leq j \leq M$ let $y_j = \sum_{i=(j-1)m+1}^{jm} x_i$. Then each $y_j$ is the sum of a R.I.S. of length $m$ with constant $(1 + \epsilon)$. By Lemma 8 we have $\|y_j\| \leq (1 + 2\epsilon)mf(m)^{-1}$ for every $j$ while $\|\sum_{j=1}^{m} y_j\| = \|x\| \geq Nf(N)^{-1}$. It follows that $x$ is an $\ell_{1+}^M$-vector with constant at most $(1 + 2\epsilon)f(N)/f(m)$. But $m = N^{1-\epsilon}$ so $f(N)/f(m) \leq (1 - \epsilon)^{-1}$. The result follows. □

We shall now prove that $X$ is H.I. As we noted earlier, this implies that $X$ contains no unconditional basic sequence, but in proving that $X$ is H.I. we shall more or less have proved that directly anyway.

Let $Y, Z$ be two infinite-dimensional subspaces of $X$ such that $Y \cap Z = \{0\}$. Our aim is now to show that the projection from $Y + Z$ to $Y$ given by $y + z \mapsto y$ is not continuous. To do this, we shall construct, for every $\delta > 0$, vectors $y \in Y$ and $z \in Z$ of norm at least 1 such that $\|y - z\| < \delta$. This implies that the above projection has norm at least $\delta^{-1}$, proving the result. So let us now choose $\delta > 0$ and let $k \in K$ be an integer such that $f(k)^{-1/2} < \delta$.

By standard arguments, we may assume that both $Y$ and $Z$ are spanned by block bases. Since $X$ satisfies a lower $f$-estimate, Lemma 1 tells us that every block subspace of $X$ contains, for every $\epsilon > 0$ and $N \in \mathbb{N}$, an $\ell_{1+}^N$-average with constant $1 + \epsilon$. It is also immediate from the definition of the norm that every vector either has the supremum norm or satisfies the inequality

$$\|Ex\| \leq \sup\{|x^*(Ex)| : M \geq 2, x^* \text{ is an } (M, g)\text{-form}\}$$

where $g$ is the function obtained from $\phi$ after the proof of Lemma 6. This allows us to make the following construction.
Let \( x_1 \in Y \) be a normalized R.I.S. vector of length \( M_1 = j_{2k-1} \in L \) and constant \( (1 + \epsilon/4) \), where \( \epsilon = 1/10 \) and \( M_1^{1/4} \geq N_1 = 4M_f(k/\epsilon)/\epsilon f'(1) \).

Let the normalized R.I.S. whose sum is \( x_1 = x_{11}, \ldots, x_{1M_1} \). By Lemma 9, \( x_1 \) is an \( \ell_{1+}^{N_1} \)-average with constant \( 1+\epsilon \). By Lemma 5, we have \( \|x_1\| \leq (1+\epsilon)M_1 g(M_1)^{-1} \|x_1\| \). For each \( j \) between 1 and \( M_1 \) let \( x^*_{1j} \) be a support functional for \( x_{1j} \) and let \( x^*_{1j} \) be the \((M_1, g)\)-form \( g(M_1)^{-1} \sum_{j=1}^{M_1} x_{1j} \). The \( x^*_{1} \) is \((1 + \epsilon)^{-1} \|x_1\| \). By continuity and the density of \( Q \) it follows that there exists an \((M_1, g)\)-form \( x^*_1 \in Q \) such that \( |x^*_1(x_1) - 1/2| \leq k^{-1} \) and \( \text{ran}(x^*_1) = \text{ran}(x_1) \). Also, note that by Lemma 7 there is no difference between an \((M_1, g)\)-form and an \((M_1, f)\)-form.

Now let \( M_2 = \sigma(x^*_1) \) and pick a normalized R.I.S. vector \( x_2 \in Z \) of length \( M_2 \) with constant \( 1+\epsilon/4 \) such that \( x_1 < x_2 \). Then \( x_2 \) is an \( \ell_{1+}^{N_2} \)-average with constant \( 1+\epsilon \), where \( N_2 = M_2^{1/4} \). As above, we can find an \((M_2, g)\)-form \( x^*_2 \) such that \( |x^*_2(x_2) - 1/2| \leq k^{-1} \) and \( \text{ran}(x^*_2) = \text{ran}(x_2) \).

Continuing in this manner, we obtain a pair of sequences \( x_1, \ldots, x_k \) and \( x^*_1, \ldots, x^*_k \) with various properties we shall need. First, \( x_i \in Y \) when \( i \) is odd and \( Z \) when \( i \) is even. Second, \( \|x_i\| = 1 \) for every \( i \) and \( \|x^*_i\| \leq 1 \). We also know that \( |x^*_i(x_i) - 1/2| \leq 1/k \) for each \( i \). Recall that \( \sigma \) was chosen so that if \( z_1, \ldots, z_s \) is a sequence of successive vectors in \( Q \), \( S = \sigma(z_1, \ldots, z_s) \) and \( z = \sum_{i=1}^{s} z_i \) then \((1/20) f\left(S^{1/40}\right)^{1/2} \geq |\text{supp}(z)| \).

This and the lower bound for \( N_1 \) ensure that \( x_1, \ldots, x_k \) is a R.I.S. of length \( k \) with constant \( 1+\epsilon \). It will also be important that the sequence \( x_1, -x_2, x_3, \ldots, (-1)^{k-1}x_k \) is a R.I.S. which is obviously true as well. Finally, and perhaps most importantly, the sequence \( x^*_1, \ldots, x^*_k \) has been carefully chosen to be a special sequence of length \( k \). It follows immediately from the implicit definition of the norm and the fact that \( \text{ran}(x^*_i) \subset \text{ran}(x_i) \) for each \( i \) that

\[
\left\| \sum_{i=1}^{k} x_i \right\| \geq f(k)^{-1/2} \sum_{i=1}^{k} x^*_i(x_i) \geq f(k)^{-1/2}(k/2 - 1).
\]

The proof will be complete if we can find a suitable upper bound for \( \left\| \sum_{i=1}^{k} (-1)^{i-1}x_i \right\| \).

To do this we shall apply Lemma 5 one final time. First, we shall show that if \( z^*_1, \ldots, z^*_k \) is any special sequence of functionals of length \( k \) and \( E \) is an interval of length at least 1 with respect to the R.I.S. \( x_1, -x_2, \ldots, (-1)^{k-1}x_k \), then \( |z^*(Ex)| \leq 1/2 \), where \( z^* \) is the \((M, g)\)-form \( f(k)^{-1/2} \sum_{i=1}^{k} z^*_i \) and \( x = \sum_{i=1}^{k} (-1)^{i-1}x_i \).
Indeed, let \( t \) be maximal such that \( z_i^* = x_i^* \) or zero if no such \( t \) exists. We know that if \( i \leq t \) then \( |z_i^*(x_i) - 1/2| \leq k^{-1} \). Suppose \( i \neq j \) or one of \( i, j \) is greater than \( t \). Then since \( \sigma \) is an injection, we can find \( L_1 \neq L_2 \in L \) such that \( z_i^* \) is an \((L_1, f)\)-form and \( x_j \) is the normalized sum of a R.I.S. of length \( L_2 \) and also an \( \ell_{1/2}^{L_2/2} \)-average with constant \( 1 + \epsilon \), where \( L_2' = L_2^{\epsilon/4} \). Just as at the end of section 2, we can now use Lemmas 2 and 5 to show that \( |z_i^*(x_j)| < k^{-2} \).

It follows that

\[
\left| \left( \sum_{i=1}^{k} z_i^* \right)(x) \right| \leq k^2.k^{-2} + \left[ \sum_{i=1}^{t} (-1)^{i+1} z_i^*(x_i) \right] + |z_t^*(x_t)| + \sum_{i=t+2}^{k} |z_i^*(x_i)| \\
\leq 1 + (1 + k.k^{-1}) + 1 + k^2.k^{-2} \leq 5 .
\]

The interval \( E \) is easily seen to increase this to at most 6. It follows that \( |z^*(Ex)| \leq 6f(k)^{-1/2} < 1/2 \) as claimed.

Now let \( \phi' \) be the function

\[
\phi'(x) = \begin{cases} 
\frac{1}{2}(\log_2(x + 1))^{1/2} & \text{if } x \in K, x \neq k \\
\log_2(x + 1) & \text{otherwise}.
\end{cases}
\]

Let \( g' \) be the function obtained from \( \phi' \) just as \( g \) was obtained from \( \phi \). It follows from our remarks about special sequences of length \( k \) that

\[
\|Ex\| \leq \sup \left\{ |x^*(Ex)| : M \geq 2, x^* \text{ is an } (M, g')\text{-form} \right\}
\]

whenever \( E \) is an interval of length at least 1. Since \( x \) is the sum of a R.I.S. it follows that we can use Lemma 5 to show that \( \|x\| \leq (1 + 2\epsilon)kg'(k)^{-1} \). Finally, Lemma 7 gives us that \( g'(k) = f(k) \), since \( \phi \leq \phi' \) yields \( g \leq g' \leq f \).

We have now constructed two vectors \( y \in Y \), the sum of the odd-numbered \( x_i \)s, and \( z \in Z \), the sum of the even-numbered \( x_i \)s, such that \( \|y + z\| \geq (1/3)f(k)^{1/2}\|y - z\| \). Hence \( Y \) and \( Z \) do not form a topological direct sum, so \( X \) is H.I. If \( X \) contained an unconditional basic sequence \( x_1, x_2, \ldots \) then the subspace generated by this sequence would split into a direct sum of the subspaces generated by \( \{x_{2n-1} : n \in \mathbb{N} \} \) and \( \{x_{2n} : n \in \mathbb{N} \} \). It follows that \( X \) does not contain an unconditional basic sequence. The reader will observe that it is easy to use the preceding argument to show this directly. In the next section, we shall examine some of the other consequences of a space being H.I., but first we shall observe
that $X$ is reflexive. For the definitions of the terms “shrinking” and “boundedly complete” see [LT, section 1.b].

First, it follows immediately from the fact that $X$ satisfies a lower $f$-estimate that the standard basis $e_1, e_2, \ldots$ is boundedly complete. Now suppose that it is not a shrinking basis. Then we can find $\epsilon > 0$, a norm-1 functional $x^* \in X^*$ and a sequence of normalized blocks $x_1, x_2, \ldots$ such that $x^*(x_n) \geq \epsilon$ for every $n$. It follows that $\sum_{x \in A} x_n$ is an $\ell_{1+}^{|A|}$-vector with constant $\epsilon^{-1}$ for every $A \subset \mathbb{N}$. Given $N \in L$ we may construct a R.I.S. $y_1, \ldots, y_N$ with constant $\epsilon^{-1}$ where $y_i$ is of the form $\lambda_i \sum_{j \in A_i} x_j$, with $\lambda_i \geq |A_i|^{-1}$. Then $x^*(y_1 + \ldots + y_N) \geq \epsilon N$. For $N$ sufficiently large, this contradicts Lemma 8.

§4. Operators on H.I. spaces.

In this section, we shall prove some results about H.I. spaces over $\mathbb{C}$. This is because we shall need to use a little spectral theory. In the next section we shall show that some of the results carry over to the real case. We do not know of a direct proof.

Let $X$ be a complex Banach space and let $T$ be a bounded linear operator from $X$ into itself. We say that $\lambda \in \mathbb{C}$ is infinitely singular for $T$ if, for every $\epsilon > 0$, there exists an infinite-dimensional subspace $Y_\epsilon$ of $X$ such that the restriction of $T - \lambda I$ to $Y_\epsilon$ has norm at most $\epsilon$.

Saying that $\lambda$ is not infinitely singular for $T$ is equivalent to saying that $T - \lambda I$ is an isomorphism on some finite-codimensional subspace of $X$. Since this property is clearly unaffected by a small enough perturbation, it follows that

$$F_T = \{ \lambda \in \mathbb{C} : \lambda \text{ not infinitely singular for } T \}$$

is an open subset of $\mathbb{C}$. Notice that $\ker(T - \lambda I)$ is finite dimensional when $\lambda \in F_T$. We shall now prove some lemmas about $F_T$.

**Lemma A.** If $\lambda \in F_T$ and if $(x_n)$ is a bounded sequence such that $(T - \lambda I)x_n$ is norm-convergent, then $(x_n)$ has a norm-convergent subsequence; furthermore, the image by $T - \lambda I$ of any closed subspace of $X$ is closed.

**Proof.** Let $S = T - \lambda I$, let $Y$ be a finite-codimensional subspace on which $S$ is an isomorphism and let $X = Y \oplus Z$. Let $x_n = y_n + z_n$ with $y_n \in Y$ and $z_n \in Z$. Then $Sx_n = Sy_n + S z_n$. Since $Z$ is finite-dimensional and $(x_n)$ is bounded, we can pass to a
subsequence such that $S z_n$ converges. Since $S x_n$ converges this gives us that $S y_n$ converges (relabelling the subsequence as $S y_n$). Since $S$ is an isomorphism on $Y$ it follows that $y_n$ converges. Finally pass to a further subsequence on which $z_n$ converges. To prove the second assertion, note that if $F$ is a closed subspace of $X$, then $F = F \cap Y + G$, for some finite-dimensional $G$, and hence $T(F) = T(F \cap Y) + T(G)$ is closed. \qed

**Lemma B.** If $\lambda \in \partial Sp(T) \cap F_T$, then $\lambda$ is an eigenvalue of $T$ with finite multiplicity.

**Proof.** Since $\lambda \in \partial Sp(T)$ it is an approximate eigenvalue of $T$. Hence, there exists a sequence $x_n$ of norm-one vectors with $T x_i - \lambda x_i \to 0$. By the previous lemma it has a convergent subsequence. But then the limit of the subsequence is an eigenvector with eigenvalue $\lambda$. \qed

The next lemma follows easily from well known facts in Fredholm theory. The argument here is elementary. It was shown to us by W. B. Johnson, as was the proof of Lemma D.

**Lemma C.** If $\lambda \in \partial Sp(T) \cap F_T$, then $\lambda$ is an isolated point of $Sp(T)$.

**Proof.** Since $F_T$ is open it is enough to show that $\lambda$ is an isolated point of $\partial Sp(T) \cap F_T$. Suppose that this is not the case. Then there exists a sequence $(\lambda_n)$ in $\partial Sp(T) \cap F_T$ converging to $\lambda$, with $\lambda_n \neq \lambda$ for every $n$. Since $\lambda_n \in F_T$, $\lambda_n$ is an eigenvalue, by Lemma B. Let $x_n$ be a norm one eigenvector with eigenvalue $\lambda_n$. By Lemma A, since $(T - \lambda I)x_n$ tends to 0, we may assume that $(x_n)$ is norm-convergent to some (norm one) vector $x$ such that $T x = \lambda x$. Let $Y$ be the closed subspace of $X$ generated by the sequence $(x_n)$. Let $U$ be the restriction of $T - \lambda I$ to $Y$. It is clear that $Y$ is invariant under $U$ and that $UY$ is dense in $Y$. Furthermore, since $(T - \lambda I)Y = UY$ and $\lambda \in F_T$, it follows from Lemma A that $UY$ is closed, and hence that $UY = Y$. Since $x \in Y$, we know that $Y_0 = \ker U$ is not $\{0\}$, and that it is finite-dimensional. We can therefore write $Y$ as a direct sum $Y_0 + Y_1$. We have that $UY_1 = Y$, so for small $\epsilon$ it is still true that $(U - \epsilon I)Y_1 = Y$. But since $(U - \epsilon I)Y_0 = Y_0$ when $\epsilon \neq 0$, this yields that $\ker(U - \epsilon I) \neq \{0\}$, for every small $\epsilon$, contradicting the fact that $\lambda \in \partial Sp(T)$. \qed

**Lemma D.** Let $Y$ be a subspace invariant under $T$, let $S$ be the restriction of $T$ to $Y$, and suppose that $Sp(S) = \{\lambda\}$. If $\lambda$ is not infinitely singular for $S$, then $Y$ is finite-dimensional.
Proof. Suppose that $\lambda \in F_S$ but that $Y$ is infinite-dimensional. Then $U = S - \lambda I$ is an isomorphism on some finite-codimensional subspace $Z$ of $Y$, and $Sp(U) = \{0\}$. Replacing $U$ by an appropriate multiple, we may assume that $\|Uz\| \geq \|z\|$ for every $z \in Z$. Define $Z_0 = Z$, $Z_1 = Z \cap UZ, \ldots, Z_{k+1} = Z \cap UZ_k$. All these subspaces of $Y$ are infinite-dimensional. If $z$ is a non-zero element of $Z_k$, we see that $z = U^k z_0$ for some $z_0 \in Z$ and $0 < \|z_0\| \leq \|U^k z_0\|$. This shows that $\|U^k\| \geq 1$ for every $k$, contradicting the fact that the spectral radius of $U$ is 0. $\square$

Suppose now that $X$ is a complex H.I. Banach space. Let $T$ be a bounded linear operator from $X$ into itself. It follows easily from the H.I. property that there exists at most one value $\lambda_0$ that is infinitely singular for $T$. If $\lambda_0$ is infinitely singular for $T$, the H.I. property implies that $T - \lambda_0 I$ is not an isomorphism on any infinite-dimensional subspace of $X$. In other words, $T - \lambda_0 I$ is strictly singular.

It follows from Lemma C that the spectrum of $T$ is finite, or consists of a sequence of eigenvalues converging to $\lambda_0$. In the second case, it is clear that $\lambda_0$ is infinitely singular for $T$. We must check that some $\lambda$ is infinitely singular for $T$ in the case of a finite spectrum.

Assume then that $\lambda \in \partial Sp(T) \cap F_T$. Then $\lambda$ is isolated in $Sp(T)$, by Lemma C. If $Q$ is the spectral projection associated with $\lambda$, then $Y = QX$ is finite-dimensional, by the spectral mapping theorem and Lemma D. It follows that, in the case of a finite spectrum, there must be a value $\lambda \in Sp(T)$ that is infinitely singular for $T$.

We have therefore proved the following theorem.

**Theorem.** If $X$ is a complex H.I. Banach space, then every bounded linear operator $T$ from $X$ into $X$ can be written $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and $S$ is strictly singular. The spectrum of $T$ is finite, or consists of $\lambda$ and a sequence $(\lambda_n)$ of eigenvalues with finite multiplicity converging to $\lambda$.

**Corollary.** A complex H.I. space $X$ is not isomorphic to any proper subspace, and in particular is not isomorphic to its hyperplanes.

§5. Further properties.

We shall now show how to pass from the complex case back to the real case. The following lemma will be useful; it was shown to us by R. Haydon.
Lemma. Suppose $X$ is a real HI-space and $T$ a bounded linear operator from $X$ into itself. If we denote by $S$ the natural extension of $T$ to the complexification of $X$, then the spectrum of $S$ is invariant by conjugation, and the part in the upper complex plane is finite or consists of a converging sequence.

As before, this lemma implies that there exists no isomorphism from $X$ onto a proper subspace.

Proof. If $\lambda \notin F_S$ is real, there exists for every $\epsilon > 0$ a (real) infinite dimensional subspace $Y_\epsilon$ of $X$ such that $\|T - \lambda \text{Id}_{Y_\epsilon}\| < \epsilon$ on $Y_\epsilon$. Since $X$ is HI, it follows that $\mathbb{C} \setminus F_S$ contains at most one real element. Let now $\lambda, \mu \notin F_S$, and $\mu \notin \{\lambda, \overline{\lambda}\}$. We may assume that $\lambda$ is not real. Let

$$T_\lambda = T^2 - 2\text{Re}\lambda T + |\lambda|^2 \text{Id}.$$  

Then $(S - \lambda \text{Id})(S - \lambda \text{Id})(x + iy) = T_\lambda x + i T_\lambda y$; for every $\epsilon > 0$ it is thus possible to find an infinite dimensional subspace $Y_\epsilon$ of $X$ such that $\|T_\lambda\| < \epsilon$ on $Y_\epsilon$. Since $X$ is HI, we may assume the same for $T_\mu$ on the same $Y_\epsilon$. Now, $T_\lambda - T_\mu = aT + b \text{Id}$ for some $a, b \in \mathbb{R}$, not both 0, and it has norm less than $2\epsilon$ on $Y_\epsilon$. Thus $a \neq 0$. We obtain that $T$ is nearly equal to $(-b/a) \text{Id}$ on $Y_\epsilon$; since $T_\lambda$ is nearly 0 on $Y_\epsilon$, we get easily that $-b/a$ must be a root of the polynomial $(X - \overline{\lambda})(X - \lambda)$, which is of course impossible.

We know therefore that $\mathbb{C} \setminus F_S$ contains at most a pair $(\lambda, \overline{\lambda})$, and the rest of the proof is as in section IV.

Theorem V.1. If $X$ is a real HI space (for example, $X$ could be the real version of the space from section III), then $X$ is not isomorphic to any proper subspace. In particular, $X$ is not isomorphic to its hyperplanes.

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