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2017 J. Phys.: Conf. Ser. 841 012033
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Optimal estimation of parameters of an entangled quantum state

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Abstract. Two-photon entangled quantum states are a fundamental tool for quantum information and quantum cryptography. A complete description of a generic quantum state is provided by its density matrix: the technique allowing experimental reconstruction of the density matrix is called quantum state tomography. Entangled states density matrix reconstruction requires a large number of measurements on many identical copies of the quantum state. An alternative way of certifying the amount of entanglement in two-photon states is represented by the estimation of specific parameters, e.g., negativity and concurrence. If we have a priori partial knowledge of our state, it’s possible to develop several estimators for these parameters that require lower amount of measurements with respect to full density matrix reconstruction. The aim of this work is to introduce and test different estimators for negativity and concurrence for a specific class of two-photon states.

1. Introduction

The amount of entanglement plays a crucial role in quantum information [1 - 3]. Therefore, characterization and quantification of entanglement in a quantum system is a crucial issue for development of quantum technologies.

There exist two ways for measuring the amount of entanglement: the first one is performing a complete quantum state tomography [2] and evaluating parameters like negativity [4] or concurrence [5] by reconstructed density matrix [1, 2], while the second one is based on estimating such parameters with an algorithm based on optimal measurements [6] exploiting some a priori knowledge of quantum state.

In our case we implemented a quantum optical system that, by means of spontaneous Parametric Down-Conversion (PDC) [7, 8], generates two entangled photons in the singlet state $|\psi_\text{-}\rangle = \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle)$, where $H$ is the horizontal component of polarization and $V$ is the vertical one: the notation $|XY\rangle$ is a shortcut to indicate the tensorial product $|X\rangle \otimes |Y\rangle$. 
With this \textit{a priori} knowledge we can perform the estimation of the above mentioned parameters. We test two estimators for each parameter: an optimal one and a non-optimal one. An estimator is optimal if the smallest statistical uncertainty associated to it coincides with the limit imposed by the Cramér-Rao bound [9], corresponding to the minimum uncertainty available.

2. Procedure

The experimental setup for singlet-state preparation is shown in Fig. 1: it hosts a 9 W CW laser\(^1\) at 532 nm pumping a Ti:Sapphire crystal in an optical cavity. At the exit of this cavity the mode-locked laser generated has a wavelength of 808 nm with a FWHM\(^2\) of 7 nm. Such laser is frequency-doubled by means of second harmonic generation (SHG) [10] and then injected into a BBO\(^3\) crystal where Type-II PDC occurs. After this, the photons make two cones: one with horizontal polarization and the other with vertical polarization. We are interested in photons belonging to the intersections of these two cones, because these photon pairs are in a superposition of singlet state \(|\psi_-\rangle\) and triplet state \(|\psi_+\rangle\).\(^4\) In order to compensate the walk-off between the two polarizations induced by the birefringence of the BBO and select only the singlet state without any unwanted relative phase, we put another BBO crystal in both photon paths. In the second part of the setup there are in both optical branches a quarter wave plate (\(\lambda/4\)), a half wave plate (\(\lambda/2\)), a polarizing beam splitter and a Fiber Coupler (FC) preceded by a bandpass filter: after being projected onto different polarization bases the photons are filtered by the bandpass filters (3 nm and 20 nm FWHM) and fiber coupled.

Fiber-coupled photons are addressed to two Silicon Single-Photon Avalanche Diodes (Si-SPADs), whose outputs are sent to coincidence electronics.

\(^1\) Continuous wave laser
\(^2\) Full Width at Half Maximum
\(^3\) \(\beta\)-Barium borate
\(^4\) \(|\psi_+\rangle = \frac{|HV\rangle + |VH\rangle}{\sqrt{2}}\)
In order to test the fidelity of the generated entangled state with respect to the expected one, we perform a quantum state tomography and we calculate the Uhlmann’s Fidelity [2] between the reconstructed and the theoretical density matrices of the two-photon state.

We also prepare and test the completely decoherent mixture $\rho_{\text{mix}} = \frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|)$, obtained by adding a birefringent crystal on one of the two photon paths as shown in Fig. 2. This birefringent crystal has the optical axis orthogonal to the photon propagation direction, and its thickness is 2.7 mm, much greater than the photons wavelength, in order to introduce only temporal decoherence into the two-photon state.

By using a fraction $p$ of singlet-state data and a $1 - p$ portion of the decoherent-state data, we create a statistical mix of data simulating a generic partially-decoherent state described by the density matrix:

$$\rho = p|\psi_-\rangle\langle \psi_-| + (1 - p)\frac{(|HV\rangle\langle HV| + |VH\rangle\langle VH|)}{2}$$

(1)

where $p \in [0, 1]$.

Quantum tomography validates our a priori knowledge of the state: being aware that our state density matrix is in the form (1), we can implement estimator algorithms for the parameters that we want to measure. We choose the $\{|+, |\rangle\rangle\}}\rangle$ polarization basis for the measurement sets of the estimators, because all the estimators are based on projections onto such basis. With the same setups of Fig. 1 and Fig. 2 we perform coincidences measurements in this polarization basis.

3. Data Analysis

The preliminary results of the density matrix reconstruction with quantum state tomography are shown in Fig. 3 and Fig. 4.

$|+\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|H\rangle - |V\rangle}{\sqrt{2}}$
The Uhlmann’s Fidelity is defined as:

\[ F = \text{Tr} \left( \sqrt{\sqrt{\rho_{\text{exp}}}} \rho_{\text{th}} \sqrt{\rho_{\text{exp}}} \right) \]  

and the values\(^6\) for \(\rho_{\psi^-}\) and \(\rho_{\text{mix}}\) are respectively \(F = 0.975\) and \(F = 0.985\).

After these tomographies, because the values of the two fidelities are close enough to 1, we are sure that we have a priori knowledge of our state, so we perform measurements in \(\{|+\rangle, |-\rangle\}\) basis in order to estimate negativity and concurrence both for maximally-entangled state and decoherent state. For each parameter we use two estimator algorithms: a non-optimal one and an optimal one. Optimal means that the theoretical variance of the estimated parameter saturates the Quantum Cramér-Rao bound:

\[ \text{Variance} \geq \frac{1}{\text{Quantum Fisher Information}} \]

Here we introduce the parameters that we want to estimate, both with the corresponding optimal and non-optimal estimators.

**Negativity** is defined by:

\[ N(\rho) = \|\rho^{T_A}\|_1 - 1 \]  

\(^6\) The more experimental density matrix tends to the theoretical one, the more Uhlmann’s Fidelity tends to 1.
where: $\rho^{T_A}$ is the partial transpose of $\rho$ with respect to the subsystem $A$ and $\|X\|_1 = Tr \sqrt{X^\dagger X}$ is the trace norm of the operator $X$. When $\rho$ describes a completely separable state like the one in Fig. 4.c, $\mathcal{N}(\rho)$ is equal to 0, while for a maximally entangled state the negativity is 1.

We define the non-optimal estimator $\epsilon \mathcal{N}_1$:

$$\epsilon \mathcal{N}_1 = -4 \left( P(|+\rangle) - \frac{1}{4} \right)$$

(4)

when $P(\bullet)$ is the probability of the event $\bullet$, and the optimal estimator $\epsilon \mathcal{N}_2$:

$$\epsilon \mathcal{N}_2 = (P(|+\rangle) + P(|-\rangle) - P(|+\rangle) - P(|-\rangle))$$

(5)

**Concurrence** is defined by:

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

(6)

where $\lambda_i$ are eigenvalues of the matrix $R = \sqrt{\rho} (\sigma_y \otimes \sigma_y) \rho^\dagger (\sigma_y \otimes \sigma_y) \sqrt{\rho}$, $\sigma_y$ is the y Pauli’s matrix.

Because Concurrence and Negativity have the same theoretical value, we can use the same estimators previously introduced in Eqns (4) and (5).

The results for estimation of $\epsilon \mathcal{N}_1$ are still in evaluation, while the ones for $\epsilon \mathcal{N}_2$ are shown in Fig. 5. The parameter $p$ is defined by Eq. (1): at $p = 0$ we have the mixing state, while at $p = 1$ we have the maximally entangled singlet state. This last state is not properly at $p = 1$ in our graphics, because of small decoherence due to experimental imperfections, but, as previously explained, we can ignore this discrepancy because the Uhlmann’s fidelity of $\rho^{\text{exp}}$ is close enough to 1.

We are now working in order to evaluate intermediate points with $0 < p < 1$, when we have a partially-decoherent two-photon state with their uncertainties.
4. Conclusions

We performed an experiment comparing two different parameters (Negativity and Concurrence) able to quantify the amount of entanglement in specific two-photon states. Both Negativity and Concurrence have the same estimators and we computed two of these: an optimal one and a non-optimal one. In all cases, with our preliminary data, the optimal estimator values and their experimental uncertainties are in good agreement with the theoretical predictions. The optimal estimator shows an uncertainty that is compatible with the minimum uncertainties allowed by the Cramér-Rao bound in the case of the maximally-entangled state, while the uncertainty appears to be slightly bigger for the separable one. The uncertainties on the non-optimal estimator are currently being evaluated. Furthermore, we are evaluating estimators and uncertainties for statistical mixtures of the singlet state and the decoherent two-photon state, in order to have a good sampling of states with density matrix of the form in Eq. (1). We are working to apply the same technique to other parameters like Log-Negativity [4] and Quantum Geometric Discord [11], and also to extend it to the case of non-maximally-entangled states, i.e. states described by:

$$\rho = p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2 \theta & -\cos \theta \sin \theta & 0 \\ 0 & -\cos \theta \sin \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (1-p) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2 \theta & 0 & 0 \\ 0 & 0 & \sin^2 \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

Actually we are able to prepare experimentally this non-maximally-entangled state:

$$\rho_{th} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{\sqrt{3}}{5\sqrt{2}} & 0 \\ 0 & -\frac{\sqrt{3}}{5\sqrt{2}} & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

with Uhlmann’s fidelity $F = 0.935$.

Even though the work is still in progress, we believe that our technique, being able to discriminate between optimal and non-optimal entanglement parameters estimators, will be of widespread use in the quantum communication and computation frameworks, as well as for quantum metrology and, in general, all entanglement-based quantum technologies.

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