UPPER BOUNDS FOR THE CRITICAL VALUES OF HOMOLOGY CLASSES OF LOOPS

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Abstract. In this short note we discuss upper bounds for the critical values of homology classes in the based and free loop space of manifolds carrying a Riemannian or Finsler metric of positive Ricci curvature. In particular it follows that a shortest closed geodesic on a simply-connected $n$-dimensional manifold of positive Ricci curvature $\text{Ric} \geq n - 1$ has length $\leq n\pi$.

1. Results

We start with a compact differentiable manifold $M$ equipped with a Riemannian metric $g$ resp. a non-reversible Finsler metric $f$. Then the corresponding norm $\|v\|$ of a tangent vector $v$ is defined by $\|v\|^2 = g(v, v)$ resp. $\|v\| = f(v)$. In the following we use as common notation also for a Finsler metric the letter $g$.

For a non-negative integer $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ let $\text{conj}(k) \in (0, \infty]$ be the infimum of all $L > 0$ such that any geodesic $c$ of length at least $L$ has Morse index $\text{ind}_\Omega(c)$ at least $(k + 1)$. Hence for any geodesic $c : [0, 1] \rightarrow M$ of length $> \text{conj}(k)$ the Morse index is at least $(k + 1)$. By the Morse index theorem it follows that there are $(k + 1)$ conjugate points $c(s)$ with $0 < s < 1$, which are conjugate to $c(0)$ along $c|[0, s]$. Here we count conjugate points with multiplicity, cf. [6, Sec. 2.5]. And we conclude: $\text{conj}(k) \leq (k + 1) \text{conj}(0)$.

Let $P([0, 1], M)$ be the space of $H^1$-curves $\gamma : [0, 1] \rightarrow M$ on the manifold $M$. Let $l, E, F : P([0, 1], M) \rightarrow \mathbb{R}$ denote the following functionals on this space. The length $l(c)$, resp. the energy $E(c)$ is defined as

$$l(\gamma) = \int_0^1 \|\gamma'(t)\| dt ; \quad E(\gamma) = \frac{1}{2} \int_0^1 \|\gamma'(t)\|^2 dt.$$ 

We use instead of $E$ the square root energy functional $F : P([0, 1], M) \rightarrow \mathbb{R}$ with $F(\gamma) = \sqrt{2E(\gamma)}$, cf. [5, Sec. 1]. For a curve parametrized proportional to arc length we have $F(\gamma) = l(\gamma)$. We consider the following subspaces of $P$. The free loop space $\Lambda M$ is the subset of loops $\gamma$ with $\gamma(0) = \gamma(1)$. For points $p, q \in M$ the space $\Omega_{pq}M$ is the subspace of curves $\gamma$ joining $p = \gamma(0)$ and $q = \gamma(1)$. The (based) loop space $\Omega_pM$ equals $\Omega_{pp}M$. As common notation we use $X$, i.e. $X$ denotes $\Lambda M, \Omega_{pq}M$, or $\Omega_pM$. It is well known that the critical points of the square root energy functional $F : X \rightarrow \mathbb{R}$

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are geodesics joining \( p \) and \( q \) for \( X = \Omega_{pq}(M) \), the closed (periodic) geodesics for \( X = \Lambda M \), and the geodesic loops for \( X = \Omega_p(M) \). The index form \( I_c \) can be identified with the hessian \( \partial^2 E(c) \) of the energy functional, for the two cases \( X = \Lambda M \) resp. \( X = \Omega_{pq}M \) (allowing also \( p = q \)) we obtain different indices \( \text{ind}_\Lambda(c) \) resp. \( \text{ind}_\Omega(c) \). If \( c \in \Lambda M \) is a closed geodesic with index \( \text{ind}_\Lambda(c) \) then for \( p = c(0) \) it is at the same time a geodesic loop \( c \in \Omega_pM \) with index \( \text{ind}_\Omega(c) \). The difference \( \text{con}(c) = \text{ind}_\Lambda(c) - \text{ind}_\Omega(c) \) is called concavity. It satisfies \( 0 \leq \text{con}(c) \leq n - 1 \), cf. [6, thm. 2.5.12] for the Riemannian case and [9, Sec. 6] for the Finsler case.

We use the following notation for sublevel sets of \( F : X^{\leq a} = \{ \gamma \in X ; F(\gamma) \leq a \}, X^a = \{ \gamma \in X ; F(\gamma) = a \} \). For a non-trivial homology class \( h \in H_j(X, X^{\leq b}; R) \) we denote by \( \text{cr}_X(h) \) the critical value, i.e. the minimal value \( a \geq b \) such that \( h \) lies in the image of the homomorphism \( H_j(X^{\leq a}, X^{\leq b}; R) \rightarrow H_j(X, X^{\leq b}; R) \) induced by the inclusion, cf. [5, Sec.1]. It follows that for a non-trivial homology class \( h \in H_j(X, X^{\leq b}; R) \) there exists a geodesic in \( X \) with length \( l(c) = \text{cr}_X(h) \). Its index satisfies \( \text{ind}_X(c) \leq j \).

The Morse theory of the functional \( F : X \rightarrow \mathbb{R} \) implies

**Theorem 1.** Let \( M \) be a compact manifold endowed with a Riemannian metric resp. non-reversible Finsler metric \( g \). Let \( h \in H_*(X, X^{\leq b}; R) \) be a non-trivial homology class of degree \( \text{deg}(h) \) for some coefficient field \( R \). Then \( \text{cr}_X(h) \leq \text{conj}(\text{deg}(h)) \leq (1 + \text{deg}(h))\text{conj}(0) \), and the homomorphism

\[
H_j(X^{\leq \text{conj}(\text{deg}(h))}, X^{\leq b}; R) \rightarrow H_j(X, X^{\leq b}; R)
\]

induced by the inclusion is surjective for all \( j \leq \text{deg}(h) \).

For positive Ricci curvature \( \text{Ric} \) and for positive sectional curvature \( K \) (resp. positive flag curvature \( K \) in the case of a Finsler metric) we obtain in Lemma 1 upper bounds for the sequence \( \text{conj}(k), k \in \mathbb{N}_0 \). As a consequence we obtain:

**Theorem 2.** Let \((M, g)\) be a compact \( n \)-dimensional Riemannian or Finsler manifold.

(a) If \( \text{Ric} \geq (n - 1)\delta \) for \( \delta > 0 \) then \( \text{cr}_X(h) \leq \pi(\text{deg}(h) + 1)/\sqrt{\delta} \) for a non-trivial homology class \( h \in H_*(X, X^{\leq b}; R) \) of degree \( \text{deg}(h) \).

(b) If \( K \geq \delta \) for \( \delta > 0 \) then \( \text{cr}_X(h) \leq \pi \{ 1 + \text{deg}(h)/(n - 1) \} /\sqrt{\delta} \) for a non-trivial homology class \( h \in H_*(X, X^{\leq b}; R) \) of degree \( \text{deg}(h) \).

(c) If \( K \leq 1 \) then \( \text{cr}_\Omega(h) \geq \lfloor \text{deg}(h)/(n - 1) \rfloor \pi \) for \( h \in H_*(\Omega_{pq}M; R) \) and \( \text{cr}_\Lambda(h) \geq \lfloor \text{deg}(h)/(n - 1) \rfloor - 1 \rfloor \pi \) for \( h \in H_*(\Lambda M, \Lambda^{\leq b}M; R) \). Here for a real number \( x \) we denote by \( \lfloor x \rfloor \) the largest integer \( \leq x \).

As consequence from Theorem 2(a) we obtain an upper bound for the length of a shortest closed geodesic on a manifold of positive Ricci curvature:
Theorem 3. Let \((M, g)\) be a compact and simply-connected Riemannian or Finsler manifold of dimension \(n\) of positive Ricci curvature \(\text{Ric} \geq (n-1)\delta\) for some \(\delta > 0\). And let \(m\) be the smallest integer with \(1 \leq m \leq n-1\) for which \(M\) is \(m\)-connected and \(\pi_{m+1}(M) \neq 0\). We denote by \(L = L(M, g)\) the length of a (non-trivial) shortest closed geodesic. Then \(L \leq \pi(m+1)/\sqrt{\delta}\), in particular \(L \leq \pi n/\sqrt{\delta}\).

Remark 1. (a) This improves the estimate \(L \leq 8\pi m \leq 8\pi(n-1)\) given in [12, Thm. 1.2].

(b) If \((M, g)\) is not simply-connected and \(\text{Ric} \geq (n-1)\delta\) for some positive \(\delta\) then there is a shortest closed curve \(c\) which is homotopically non-trivial. This closed curve is a closed geodesic and \(\text{ind}_\Lambda (c) = \text{ind}_\Omega (c) = 0\). From Lemma 1 we obtain \(L(c) \leq \pi/\sqrt{\delta}\). On the other hand choose \(k \in \mathbb{N}\) such that \(l(c^k) = kl(c) > \pi/\sqrt{\delta}\), here \(c^k(t) = c(kt)\) denotes the \(k\)-th iterate of the closed geodesic \(c\). Then we conclude from Remark 3(a) that \(\text{ind}_\Lambda (c^k) \geq \text{ind}_\Omega (c) \geq 1\), hence the closed geodesic \(c\) is not hyperbolic, cf. [6, Thm. 3.3.9].

(c) For a compact and simply-connected Riemannian manifold \((M, g)\) of positive sectional curvature \(K \geq \delta\) it follows from the estimate \(\text{conj}(n-1) \leq 2\pi/\sqrt{\delta}\) that the length \(L\) of a shortest closed geodesic satisfies \(L \leq 2\pi/\sqrt{\delta}\). In the limiting case \(L = 2\pi/\sqrt{\delta}\) the metric is of constant sectional curvature, cf. [10, Cor. 1].

Theorem 4. Let \((M, g)\) be a compact Riemannian or Finsler manifold of dimension \(n\) with \(\text{Ric} \geq (n-1)\delta\) (resp. \(K \geq \delta\)) for some positive \(\delta\). For any pair \(p, q \in M\) of points (also allowing \(p = q\)) and \(k \in \mathbb{N}\) there exist at least \(k\) geodesics joining \(p\) and \(q\) (i.e. geodesic loops for \(p = q\)) with length \(\leq (2(n-1)k+1)\pi/\sqrt{\delta}\), (resp. \(\leq (2k+1)\pi/\sqrt{\delta}\)).

Remark 2. (a) This result improves the bounds \(16\pi(n-1)k\) resp. \((16(n-1)k+1)\pi\) given in [12, Thm. 1.3] for \(\delta = 1\).

(b) Here two geodesics \(c_1, c_2 \in \Omega_{pq}M\) are called distinct if their lengths \(l(c_1) \neq l(c_2)\) are distinct. From a geometric point of view this is not very satisfactory. If we choose distinct points \(p, q \in S^n\) on the sphere with the standard metric of constant sectional curvature \(K = 1\), which are not antipodal points, then any geodesic joining \(p\) and \(q\) is part of the unique great circle \(c : \mathbb{R} \rightarrow S^n\) through \(p\) and \(q\). So in this case the geodesics whose existence is claimed in Theorem 4 all come from a single closed geodesic, cf. [6, p.181]. Closed geodesics are called geometrically distinct if they are different as subsets of \(M\) (or in the case of a non-reversible Finsler metric if their orientations are different when they agree as subsets of \(M\)). If the metric \(g\) is bumpy then there are only finitely many geometrically distinct closed geodesics below a fixed length. Hence for a bumpy metric for almost all pairs of points \(p, q\) on \(M\) there is no
closed geodesics through these points. Hence in this case the geodesics constructed in Theorem 4 do not come from a single closed geodesic.

(c) There are related curvaturefree estimates depending only on the diameter due to Nabutovsky and Rotman. In [8] they show that for any pair \( p, q \) of points in a compact \( n \)-dimensional Riemannian manifold with diameter \( d \) and for every \( k \in \mathbb{N} \) there are at least \( k \) distinct geodesics joining \( p \) and \( q \) of length \( \leq 4nk^2d \).

2. Proofs

Proof of Theorem 1. (a) We first give the proof for the case \( X = \Omega_{pq}M \) for points \( p, q \) and for a homology class \( h \in H_k(\Omega_{pq}M, \Omega_{pq}^bM; R) \). Here we also allow the case \( p = q \). We denote by \( d : M \times M \rightarrow \mathbb{R} \) the distance induced by the metric \( g \). We choose a sequence \( \left( q_j \right)_{j \geq 1} \subset M \) such that \( \lim_{j \rightarrow \infty} d(p, q_j) = 0 \) and such that along any geodesic joining \( p \) and \( q_j \) the point \( q_j \) is not a conjugate point to \( p \). This is possible as a consequence of Sard’s theorem, cf. [7, Cor. 18.2] for the Riemannian case and [9, Cor. 8.3] for the Finsler case. As a consequence the square root energy functional \( F_j = F : \Omega_{pq}^jM \rightarrow \mathbb{R} \) is a Morse function. There is a homotopy equivalence \( \zeta_{pqj} : \Omega_{pq}^jM \rightarrow \Omega_{pq}^jM \) between loops spaces with \( F(\gamma) = \lim_{j \rightarrow \infty} F(\zeta_{pqj}(\gamma)) \) for all \( \gamma \in \Omega_{pq}^jM \), cf. [10, Lem.1]. Let \( h \in H_k(\Omega_{pq}^jM, \Omega_{pq}^b^jM; R) \), then it follows from Morse theory for the functional \( F_j \) that there is a geodesic \( c_j \) joining \( p \) and \( q_j \) whose length \( l(c_j) \) equals the critical value \( \text{cr}(\zeta_{pqj}(h)) \) of the homology class \( \zeta_{pqj}(h) \in H_k(\Omega_{pq}^jM, \Omega_{pq}^b^jM; R) \). The Morse index \( \text{ind}_\Omega(c_j) \) as critical point of \( F_j \) equals the degree of the homology class by the Morse lemma, cf. [9, Sec. 8]. By definition of \( \text{conj}(k) \) we obtain \( l(c_j) \leq \text{conj}(k) \). But since \( \text{cr}_\Omega(h) = \lim_{j \rightarrow \infty} l(c_j) \leq \text{conj}(k) \) we finally arrive at the claim \( \text{cr}_\Omega(h) \leq \text{conj}(k) \).

(b) Now we assume \( X = \Lambda M \). Then we use a sequence \( g_j \) of bumpy Riemannian or Finsler metrics converging to the metric \( g \) with respect to the strong \( C^r \) topology for \( r \geq 2 \), resp. \( r \geq 4 \) in the Finsler case. We can choose such a sequence by the bumpy metrics theorem for Riemannian metrics due to Abraham [1] and Anosov [2], and by the generalization to the Finsler case, cf. [11]. The square root energy functional \( F_j : \Lambda M \rightarrow \mathbb{R} \) is then a Morse-Bott function, the critical set equals the set of closed geodesics which is the union of disjoint and non-degenerate critical \( S^1 \)-orbits. Hence all closed geodesics are non-degenerate, i.e. there is no periodic Jacobi field orthogonal to the geodesic. Then for any \( j \) there is a closed geodesic of \( g_j \) such that the length \( l(c_j) \) with respect to \( g_j \) equals the critical value \( \text{cr}_{\Lambda,j}(h) \) with respect to \( g_j \). Hence Morse theory implies that the index \( \text{ind}_{\Lambda}(c_j) \in \{ k, k - 1 \} \), since the critical submanifold is 1-dimensional. Then \( \text{ind}_{\Lambda,j}(c_j) \leq k \) which implies that the length \( l_j(c_j) \) of \( c_j \) with respect to the metric \( g_j \) satisfies \( \text{cr}_{\Lambda,j}(h) = l_j(c_j) \leq \text{conj}(k) \). Here \( \text{conj}(k) \) is defined
with respect to the metric $g_j$. Then $\text{cr}_\Lambda(h) = \lim_{j \to \infty} \text{cr}_{\Lambda,j}(h) \leq \lim_{j \to \infty} \text{conj}_j(k) = \text{conj}(k)$.

□

Remark 3. The Morse-Schoenberg comparison result [6, Thm. 2.6.2], [9, Lem. 3] implies: Let $c : [0,1] \longrightarrow M$ be a geodesic of length $l(c)$, and $k \in \mathbb{N}$.

(a) If $\text{Ric} \geq (n-1)\delta$ for $\delta > 0$ and if $l(c) > \pi k/\sqrt{\delta}$, then $\text{ind}_\Omega(c) \geq k$.

(b) If $K \geq \delta$ for a positive $\delta$ and if $l(c) > \pi k/\sqrt{\delta}$, then $\text{ind}_\Omega(c) \geq k(n-1)$.

(c) If $K \leq 1$ and if $l(c) \leq \pi k$ then $\text{ind}_\Omega(c) \leq (k-1)(n-1)$.

This implies

Lemma 1. Let $(M, g)$ be a manifold with Riemannian metric resp. Finsler metric $g$.

(a) If $\text{Ric} \geq (n-1)\delta$ for $\delta > 0$ then $\text{conj}(k) \leq (k+1)\pi/\sqrt{\delta}$ for $k \in \mathbb{N}_0$.

(b) If $K \geq \delta$ for $\delta > 0$ we have $\text{conj}(k(n-1)) \leq (k+1)\pi/\sqrt{\delta}$ for $k \in \mathbb{N}_0$.

Proof of Theorem 2. From Theorem 1 and Lemma 1 we immediately obtain the statements (a) and (b). Statement (c) follows analogously to the arguments in the proof of Theorem 1 together with Remark 3(c) and the estimate $\text{con}(c) \leq n-1$. □

Proof of Theorem 3. By assumption there is a homotopically non-trivial map $\phi : S^{m+1} \longrightarrow M$, which also defines a homotopically non-trivial map $\tilde{\phi} : (D^m, S^{m-1}) \longrightarrow (\Lambda M, \Lambda^0 M)$, cf. [6, Thm. 2.4.20]. This defines a non-trivial homology class $h \in H_m(\Lambda M, \Lambda^0 M; R)$ for some coefficient field $R$. Then there exists a closed geodesic $c$ with length $l(c) = \text{cr}_\Lambda(h)$. We conclude from Theorem 2(a) that $l(c) = \text{cr}_\Lambda(h) \leq \pi(m+1)/\sqrt{\delta}$. □

Proof of Theorem 4. Since $M$ is simply-connected we conclude from a minimal model for the rational homotopy type of $\Omega M$ : There exists a non-trivial cohomology class $\omega \in H^d(\Omega_{pq}; \mathbb{Q})$ of even degree $2l$ for some $1 \leq l \leq n-1$, which is not a torsion class with respect to the cup product, i.e. $\omega^k \neq 0$ for all $k \geq 1$. There is a sequence $h_k \in H_* (\Omega_{pq}M, \Omega_{pq}^\leq k M; R), k \geq 1$ of non-trivial homology classes with $h_k = \omega \cap h_{k+1}, \deg(h_k) = 2lk, k \geq 1$. Here $\cap$ denotes the cap product.

Then we use the principle of subordinated homology classes, cf. [3, p.225-226] and conclude: $\text{cr}_\Omega(h_k) \leq \text{cr}_\Omega(h_{k+1})$ for all $k \geq 1$. Here equality only holds if there are infinitely many distinct geodesics in $\Omega_{pq}(M)$ of equal length $l(c) = \text{cr}_\Omega(h_k) = \text{cr}_\Omega(h_{k+1})$. Hence we can assume that $\text{cr}_\Omega(h_k) < \text{cr}_\Omega(h_{k+1})$ and obtain a sequence $c_k \in \Omega_{pq}M$ of geodesics with $l(c_k) = \text{cr}_\Omega(h_k)$. Since $\deg(h_k) = 2lk \leq 2(n-1)k$ we obtain the claim from Theorem 2. □

Remark 4. (a) If $M$ is simply-connected and compact then it was shown by Gromov [4, Thm. 7.3] that there exist positive constants $C_1 = C_1(g), C_2 = C_2(g)$ depending on the
metric $g$ such that for all homology classes $h \in H_*(\Lambda M; R)$ the following inequalities hold:

$$C_1 \text{cr}_A(h) < \text{deg}(h) < C_2 \text{cr}_A(h).$$

(b) If $M = S^n$ is a sphere of dimension $n \geq 3$ it is shown in [5, Thm. 1.1] that there are positive numbers $\overline{\alpha} = \overline{\alpha}(g), \beta = \beta(g)$, depending on $g$ such that

$$\overline{\alpha} \text{cr}_A(h) - \beta < \text{deg}(h) < \overline{\alpha} \text{cr}_A(h) + \beta$$

holds for all $h \in H_*(\Lambda S^n)$. The number $\overline{\alpha}$ is called \textit{global mean frequency}. In case of positive Ricci curvature $\text{Ric} \geq (n-1)\delta$ we conclude from Theorem 2(a): $\sqrt{\delta}/\pi \leq \overline{\alpha}$. If $K \leq 1$ then $\overline{\alpha} \leq (n-1)/\pi$.

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