Validity of the lowest-Landau-level approximation for rotating Bose gases

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The energy spectrum for an ultracold rotating Bose gas in a harmonic trap is calculated exactly for small systems, allowing the atoms to occupy several Landau levels. Two vortex-like states and two strongly correlated states (the Pfaffian and Laughlin) are considered in detail. In particular, their critical rotation frequencies and energy gaps are determined as a function of particle number, interaction strength, and the number of Landau levels occupied (up to three). For the vortex-like states, the lowest-Landau-level (LLL) approximation is justified only if the interaction strength decreases with the number of particles; nevertheless, the constant of proportionality increases rapidly with the angular momentum per particle. For the strongly correlated states, however, the interaction strength can increase with particle number without violating the LLL condition. The results suggest that in large systems, the Pfaffian and Laughlin states might be stabilized at rotation frequencies below the centrifugal limit for sufficiently large interaction strengths, with energy gaps a significant fraction of the trap energy.

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I. INTRODUCTION

Over the past few years, there has been much interest in the possibility of coaxing ultracold atomic gases into the neutral analog of a quantum Hall state. Various approaches have been proposed, which include rapidly rotating the confinement potential [1–14], and generating effective magnetic fields for particles confined in optical lattice potentials [15–18] and harmonic oscillator traps [19, 20].

One of the many intriguing prospects for these systems is the possibility of producing the bosonic analog of the Pfaffian [2, 7, 10, 14], also known as the Moore-Read state [21], which is believed to give rise to the ν = 5/2 fractional quantum Hall plateau in two-dimensional electron gases [22]. The quasiparticle excitations of this state are believed to be non-Abelian anyons obeying fractional statistics [23–25]; in principle, non-Abelian anyons could be physically wound around one another to operate an intrinsically fault-tolerant quantum computer, protected from errors by the topological properties of the underlying state [23, 26, 27].

One of the fundamental sources of error in these topological approaches to computing is the thermal excitation of unwanted quasiparticles that contribute to the winding: the error rate is intrinsically related to the size of the energy gap separating the ground state from the next excited one. One might expect that strengthening the particle interactions would monotonically increase the splitting between sub-bands in the Landau levels; this would also allow the system to access a larger number of quantum Hall states. Indeed, for ultracold atomic systems the size (and sign) of the scattering length (characterizing the interaction strength) can be varied arbitrarily through the use of Feshbach resonances [28]. It is not known, however, how large interaction strengths affect the quantum Hall states and the excitation gaps, except right on resonance [10].

The difficulty is that calculations used to locate the various quantum Hall ground states usually rely on the Lowest Landau Level (LLL) approximation, which is valid only in the weakly interacting regime. This is turn requires that the particle densities are low, the interaction strength is small, or the rotation frequencies are exceedingly close to the harmonic oscillator confinement frequency. But robust quantum Hall states with large excitation gaps require strong interactions.

In the present work, we exactly calculate the ground states for rotating bosons for small systems without employing the LLL approximation. The central goals are (1) to determine an upper bound to the interaction strength consistent with the LLL approximation; (2) to obtain the physical parameters (number of bosons, scattering length, rotation frequency) to unambiguously select a given quantum Hall state; and (3) to show that negative (attractive) interactions always lead to instability for large densities, contrary to prior claims [29, 30]. The main result of this paper is that the widely accepted criterion for the LLL approximation, which requires that the inter-particle interaction energy be smaller than the Landau level spacing, is generally too restrictive: we expect that the most interesting quantum Hall states can be obtained for relatively large particle numbers and rotation frequencies within reach of current experiments.

In Sec. II, we describe the Bose gas and our approach to calculating the boson energy spectrum. Limits on the validity of the LLL approximation are investigated in Sec. III by considering the effects of higher Landau levels on the boson energy spectrum. The case of attractive Bose gases is discussed in Sec. IV, and concluding remarks are provided in Sec. V.

II. NUMERICAL METHOD

The system under study consists of ultracold interacting bosons subject to a cylindrically symmetric harmonic trapping potential, which is rotated around the z-axis at a frequency Ω. We assume a tight harmonic confinement along the axis of rotation such that the axial ground state energy far exceeds any other transverse energy scale, yielding a quasi-2D system. The Hamiltonian
in the co-rotating frame is then \( H = H_0 + H_{\text{int}} \), where

\[
H_0 = \sum_i \left( \frac{p_i^2}{2M} + \frac{1}{2} M \omega^2 r_i^2 - \tilde{\Omega} L_i \right)
\]

and

\[
H_{\text{int}} = \hat{g} \sum_{i<j}^N \delta(\mathbf{r}_i - \mathbf{r}_j).
\]

Here, \( M \) is the particle mass, \( N \) the number of bosons, \( \omega \) the radial trap frequency and \( \hat{g} = \sqrt{8\pi\hbar\omega^3 a/\ell_z} \) is the 2D-interaction strength where variables \( \ell = \sqrt{\hbar/M\omega} \) and \( \ell_z = \sqrt{\hbar/M\omega_z} \) are the characteristic oscillator lengths along the radial and axial directions respectively, and \( a \) is the three-dimensional scattering length [31].

We proceed to calculate the energy spectrum of the rotating Bose gas by exact diagonalization of the Hamiltonians \( H_{\text{int}} \) and \( H_0 \) in blocks of definite total angular momentum. To do so, we choose a Fock basis of the form

\[
|N_1, N_2, \ldots \rangle = \prod_k \left( \frac{\hat{b}^\dagger_0^{N_k}}{\sqrt{N_k!}} \right),
\]

where \( \hat{b}^\dagger_k \) creates a boson in state \( k \) (in our case the index \( k \) represents a distinct pair of numbers \((n, m)\), the principle quantum number and the projection of the angular momentum) and \( N_k \) is the occupation number of level \( k \). The Hamiltonians are then written in terms of the Bose field operators where \( \hat{H}_0 = \int \psi^\dagger(\mathbf{r}) H_0 \psi(\mathbf{r}) d\mathbf{r} \) and \( \hat{H}_{\text{int}} = \frac{1}{2} \int \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') H_{\text{int}} \psi(\mathbf{r}) \psi(\mathbf{r}') d\mathbf{r} d\mathbf{r}' \). Expanding the field operators in terms of 2D harmonic oscillator basis states, \( \psi^\dagger(\mathbf{r}) = \sum_k \hat{b}^\dagger_k \Phi_k(\mathbf{r}) \), one obtains

\[
\hat{H}_0 = \sum_k \hat{b}^\dagger_k \hat{b}_k \epsilon_k
\]

and

\[
\hat{H}_{\text{int}} = \hat{g} \sum_{ijkl} \hat{b}^\dagger_i \hat{b}^\dagger_j \hat{b}_k \hat{b}_l \int \Phi_i^* \Phi_k^* \Phi_j \Phi_l d\mathbf{r} d\mathbf{r}',
\]

where the \( \epsilon_k = \hbar \omega (2n + |m| + 1 - \epsilon_k) \) are the single particle 2D harmonic oscillator eigenvalues, and \( \Omega = \tilde{\Omega}/\omega \) is the dimensionless rotation frequency.

In our Fock basis, \( \hat{H}_0 \) is diagonal with eigenvalues of \( E_n/\hbar\omega = 2N + \sum_n n_i + \sum_m |m_i| + N - \Omega L \), where \( L = \sum_i n_i \) is the total angular momentum of the system in units of \( \hbar \). Thus, in the purely non-interacting case, the energy spectrum consists solely of degenerate levels separated by an energy gap \( 2\hbar\omega \). We will adopt the usual terminology where these different levels are called Landau Levels, in reference to the similarities between the rotating Bose gas and the 2D electron gas subjected to a strong perpendicular magnetic field. The LLL approximation corresponds to enforcing that all particles have \( m_i \geq 0 \) and \( n_i = 0 \), which greatly reduces the Hilbert space dimension. The interaction Hamiltonian, on the other hand, is not diagonal in our basis and so once it is included the degeneracy of the Landau levels is lifted and each Landau level is split into a multitude of distinct states. The interaction strength is parameterized by the dimensionless coupling constant \( g = \hat{g}/\hbar \omega \ell^2 = \sqrt{8\pi a/\ell_z} \).

For small number of particles it is feasible to calculate the entire spectrum, but for larger \( N \) and greater number of Landau Levels the Hilbert space becomes too large. In these cases, only the lowest-lying eigenvalues were calculated through the use of a Lanczos diagonalization algorithm. Even so, computational constraints have limited our calculations to the first two or three Landau Levels, which are obtained by including states where \( \sum_i \left| n_i + (|m_i| - m_i)/2 \right| \leq 1 \) or 2, respectively. The 5 particle spectrum with \( g = 1 \) and \( \Omega = 1 \) is shown in Fig. 1, including the three lowest Landau levels.

### III. Repulsive Interactions

We begin by considering a repulsive Bose gas. As the interaction energy is increased, mixing between the different Landau Levels becomes more important and at some critical interaction strength \( g_{\max} \), the LLL approximation can no longer accurately describe the rotating gas. The standard criterion for the validity of the LLL approximation is that the interaction energy should be smaller than the spacing between Landau Levels, or \( \tilde{g} \rho < 2\hbar\omega \) where \( \rho \) is the particle density [1, 32–35]. In unitless form, the crossover between the weakly and strongly interacting regime occurs when

\[
g_{\max} \sim N^{-1}.
\]

This scaling does not provide an absolute estimate on \( g_{\max} \); however, the coefficient of proportionality might be large enough that interesting quantum Hall states...
could be achieved for experimentally accessible rotation frequencies and interaction strengths. Most important though, the $1/N$ scaling is found to be invalid for the most interesting ground states, as discussed below.

For a non-rotating trap, the stationary Bose gas with no angular momentum has the lowest energy. However, as the rotation rate is increased, states with higher $L$ experience a greater Doppler shift in energy than those with lower $L$ and, as a result, at specific rotation frequencies $\Omega_c$ ground state transitions are observed. Almost all stable ground states can be described by a relationship of the form $L = a(N - b)$, where $a$ and $b$ are integers [2]. We will be concentrating on the $L = N$ state (sometimes called the single vortex state), the $L = 2(N - 1)$ state, the Pfaffian (where $L = (N - 1)^2/2$ for odd $N$ and $L = N(N - 1)/2$ for even $N$), and the Laughlin state for which $L = N(N - 1)$. 

To characterize the transition from the weakly interacting to strongly interacting regime, we consider changes in the critical trap rotation frequency $\Omega_c$ of these four ground states as higher Landau levels are included in the eigenvalue calculations, as shown in Fig. 2. When the difference between $\Omega_c$ obtained with the LLL approximation and that obtained with higher Landau levels in the spectrum become apparent, we say that the LLL approximation is no longer applicable. Specifically, we define the crossover from weakly to strongly interacting regimes to occur when the relative error between the results of the LLL approximation and those with two or three Landau levels exceeds a threshold around 10%:

$$\frac{|\Omega_c(g) - \Omega_c(g)_{LLL}|}{\Omega_c(g)_{LLL}} = 0.1.$$ 

The interaction strength for which this occurs is defined as $g_{max}$, and depends on the number of particles $N$ and the particular transition considered. The criterion of 10% was chosen to reflect typical experimental uncertainty; we explicitly considered error thresholds of 5% and 15%, and the results were not significantly changed. We also repeated the entire calculation using the gap $\Delta E$ between the ground and first excited state as the marker; however, this method proved to be unreliable. While $g_{max}$ values found using $\Delta E$ tend to be smaller than those obtained with the $\Omega_c$ criterion, there is too much scatter to obtain a useful relation between $g_{max}$ and $N$. This is a consequence of the small number of particles considered: as $N$ is increased, the energy gap undergoes considerable fluctuations as new states are interjected into the spectrum. Furthermore, it is more valuable from an experimental perspective to consider variations in $\Omega_c$ since the trap rotation frequency can be directly controlled, whereas $\Delta E$ is indirectly controlled through the choice of $g$ and $\Omega$.

We now proceed to determine $g_{max}(N)$ for various $N$ values, for each of the four ground states defined in the previous paragraph. From Fig. 3, it is apparent that, for the $L = N$ state at least, 3 Landau levels (3LL’s) or more are required to accurately describe the $g_{max}(N)$ for $N \geq 10$ since the 2LL curve slightly deviates from the power law for these large $N$’s. While it would have been advantageous to examine the effect of even higher Landau levels, computational constraints have prevented us from doing so. Nevertheless, we are confident that 3LL’s are sufficient for our purposes since we do not observe any significant deviation from a power law scaling.
for the small number of particles that we have considered. Both the $L = N$ and $L = 2(N - 1)$ states give a scaling relation that is consistent with Eq. (3) since we find that $g_{\text{max}} \propto N^{-1}$. Most important, the prefactor for the $L = 2(N - 1)$ transition is more than an order of magnitude larger than that for the single vortex state. The numerics indicate that this prefactor continues to increase for all the weakly correlated ground states (those with angular momentum $L \propto N$), though we don’t have enough data to give quantitative predictions.

The results for $g_{\text{max}}(N)$ are markedly different for the Pfaffian and Laughlin states. A cursory glance at Fig. 4 shows that in fact $g_{\text{max}}$ increases with particle number, which is a drastic departure from the scaling relation (3). Again, the prefactor increases as the angular momentum of these strongly correlated ground states (where $L \propto N^2$) increases. The ground state with $L = N = 4$ is believed to be close to the Pfaffian [11] and is plotted with the other Pfaffian data, though numerically it appears to be mixed with the single-vortex state. Our calculations are restricted to $N \leq 10$ for the Pfaffian and $N \leq 7$ for the Laughlin state, and thus it is difficult to make quantitative predictions based on finite-size scaling with such few atoms. Despite these limitations, however, the numerics clearly indicate that the standard criteria for the validity of the LLL approximation does not apply to highly correlated ground states.

Now that we have established the limits on the interaction strength required to ensure the validity of the LLL approximation (for each value of $N$ considered), we turn our attention to the values of $\Omega_c$ and the energy gap $\Delta E$ between the ground and first excited state as the number of particles is increased. The central motivation is to use finite-size scaling to make predictions for the critical rotation frequencies and gap sizes for values of $N$ approaching those relevant to future experiments.

Using the $g_{\text{max}}(N) \propto N^{-1.046}$ and $N^{-1.01}$ obtained from the best fits to the data shown in Fig. 3, the scaling of $\Omega_c$ and $\Delta E$ with particle number is shown in Fig. 5 for the $L = N$ and $L = 2(N - 1)$ states. The results for the $L = N$ state are mostly consistent with expectations in the sense that in the thermodynamic limit ($N \to \infty$) the critical frequency does approach a constant value, albeit larger than the mean-field result ($\Omega_c \approx 0.85$ versus $\Omega_c = 1/\sqrt{2}$ [36, 37]), while the gap goes to a constant $\Delta E \approx 0.2$ in units of $\hbar \omega$. The asymptotic behavior of the $L = 2(N - 1)$ state is more interesting, with the large-$N$ value of $\Omega_c$ appearing to approach unity while the gap appears to close.

The results for $\Omega_c$ and $\Delta E$ associated with the Pfaffian and Laughlin states are shown in Figs. 6 and 7 respectively. The results for both $g = \text{const.}$ and the best fit values of $g = g_{\text{max}}(N)$ are shown. The constant $g$ is chosen to be the maximum value satisfying the LLL criterion for all the $N$ in the series (which corresponds to that for the smallest $N$). The varying $g$ values are the best fits to the data shown in Fig. 4.

The data shown in Fig. 6 clearly indicate that when $g$ is kept fixed the Laughlin gap approaches a constant in the large-$N$ limit; this result is consistent with previous numerical studies of the neutral-atom Laughlin state on the surface of a sphere [7]. For $g = g_{\text{max}}(N = 2) = 3.12$, the asymptotic gap is found to be $\Delta E(N \gg 1) \approx 0.2$ in units of $\hbar \omega$. Unfortunately, when keeping $g$ fixed the critical rotation frequency approaches the experimentally challenging limit $\Omega_c \to 1$. On the other hand, if $g$ increases with $N$ according to its maximum value $g_{\text{max}}(N)$, it appears that the critical frequency levels off slightly before $\Omega_c = 1$ for large $N$ while $\Delta E$ is larger than in the LLL case, as shown in the inset of Fig. 6. Although an exact value is impossible to obtain, it seems like the gap is approximately $0.4\hbar \omega$ for large $N$. This implies that with a
judicious choice for $g \gg 1$, the Laughlin state could be stabilized at lower rotation frequencies without violating the LLL condition.

The data shown in Fig. 7 for the Pfaffian are less clear than those for the Laughlin state. For fixed $g = g_{\text{max}}(N = 4) = 1.62$, the critical frequency approaches the high rotation limit $\Omega_c = 1$ while the gap in fact appears to close for large $N$. On the other hand, increasing $g$ proportional to $N^{1.6}$ according to the trend observed in Fig. 4 yields what appears to be a constant value for the gap and a critical frequency slightly lower than the centrifugal limit (though there is too much scatter in the data to make definite statements). Thus, like the Laughlin state, the Pfaffian can in principle be stabilized at experimentally accessible rotation frequencies without violating the LLL condition.

The striking contrast between the weakly and strongly correlated states in the behavior of $g_{\text{max}}(N)$ is the result of a decreased sensitivity of energy levels to changes in interaction strength and number of Landau levels, as the total angular momentum increases. In the LLL approximation, the energy spectrum is invariant (up to an overall scale factor) under changes in the interaction strength. The Laughlin state by definition remains a zero-energy eigenstate of the many-body Hamiltonian. Thus, the shift with $g$ of all other levels must be directly proportional to their energies relative to that of the Laughlin state. In short, the sensitivity to interactions as a function of angular momentum must be proportional to the $\Omega = 1$ ‘yrast’ line [32].

Numerically, we find that the relative ground-state energies for a given value of $L > N$ decrease exponentially with angular momentum, $E_{\text{min}} \propto \exp(-\gamma L)$, where $\gamma = 0.24$, 0.16, and 0.08 for $N = 4$, 5, and 6, respectively. For $2 \leq L \leq N$ the relationship is linear [33, 38, 39]. Small angular momentum states have thus an inherently increased sensitivity to changes in the interaction strength.

Low angular momentum states also have an increased sensitivity to the presence of higher Landau levels. We quantify this sensitivity by the resulting shift $\delta E$ in ground-state energy. Like $E_{\text{min}}, \delta E$ decreases with $L$, although the functional form is different: we find that $\delta E(L) \propto L^x$ where $x = -1.0, -1.2, -1.3$ for 4, 5, and 6 particles, respectively. Most relevant though is the scaling of $\delta E$ with $N$ for the weakly and strongly correlated states. Recall that for each $N$, $g_{\text{max}}$ was obtained by setting a 10% limit on variations of $\Omega_c$ resulting from the encroaching presence of higher Landau levels. Since $\Omega_c$ characterizes the transition between two states, any variation in $\Omega_c$ will directly result from a mismatch in $\delta E$ between these two states. We observe the $L = N$ state to always transition from $L = 0$, and the Laughlin state from $L = N(N-2)$. After numerically evaluating $\delta E(N)$ with $g = 1$ and $\Omega = 1$, we find that $\delta E_{L=0} - \delta E_{L=N} \propto N^{2.8}$, while $\delta E_{L=N(N-2)} - \delta E_{\text{Laughlin}} \propto N^{-0.054}$. Let the difference of $\delta E$ between two states be called $\Gamma$. Should $\Gamma = 0$, then both state energies are equally shifted, and as a result there is no change in the critical transition rotation frequency $\Omega_c$. In other words, $\Gamma$ characterizes the offset in $\Omega_c$ compared to its LLL value. As $N$ is increased, any increase in $\Gamma$ can be countered by a concomitant decrease in the interaction strength, and vice versa, such that a constant higher Landau level influence is maintained. For the $L = 0$ to $L = N$ transition $g$ must be decreased with $N$ to keep $\Gamma(N)$ unchanged while for the Laughlin transition the reverse is true. The contrasting behavior of $g_{\text{max}}(N)$ for the weakly and strongly correlated states is thus recovered.

Though it is somewhat silly to extrapolate these small-$N$ results to experimentally relevant values of $N \sim 10^3$, it is nevertheless useful to approximately determine the
relevant experimental parameters required to stabilize the Laughlin and Pfaffian states. First, it is important that the excitation gap be larger than the temperature; for a typical temperature $T \sim 10$ mK one requires that $\Delta E/h \gtrsim 200$ Hz. Thus, stabilizing the Pfaffian state with an asymptotic gap of $0.1\hbar\omega$ would therefore require a large radial trap frequency on the order of kHz unless the temperature were strongly reduced.

The Laughlin state with $\Delta E \approx 0.4$ is more accessible. This gap size corresponds to $N = 7$ and $g \approx 10$ in the data presented in Fig. 4. Assuming that the $N = 7$ result is already close to the large-$N$ asymptotic gap, which seems to be the case according to the inset of Fig. 6, then $g = 10$ corresponds to a scattering length of approximately $8\mu m \approx 1500a_{\text{Rb}}$, assuming $\omega/2\pi = 200\text{Hz}$ and $\omega_z = 1\text{kHz}$, and where $a_{\text{Rb}}$ is the s-wave scattering length for $^{87}\text{Rb}$ [40]. Such a large scattering length can be obtained through the use of Feshbach resonances [41], but then the pseudopotential employed in the present calculations to describe the low-energy atomic collisions would have to be modified to explicitly take into account the presence of atomic pairs [10].

\section*{IV. ATTRACTIVE INTERACTIONS}

We also have investigated the effect of having attractive interactions between bosons. In the thermodynamic limit, it is known that gases of untrapped bosons are unstable against collapse [42, 43] when $g < 0$. However, with the addition of a trapping potential, such gases can stably exist for a limited number of particles [44] because of the finite zero-point energy of the trap. It has been suggested recently [29, 30] that for the special case of a rotating, attractive gas of 2D harmonically trapped bosons, there might not be an upper limit to the number of particles in which a stable gas could exist. The claim is that the statistical pressure of the anyons comprising the quantum Hall states can stabilize the cloud against collapse. The argument, however, has recently been disputed [45].

The main result of Ref. [29] is that the bare interaction is replaced by an effective constant:

$$g_{\text{eff}} = g + \frac{4\pi}{\kappa\hbar\rho}$$

Here, $\kappa$ is the Chern-Simons coefficient that relates the (statistical) magnetic field $B$ to the particle density $\rho$ through $-\kappa B = 4\pi\rho$. In the rotating system, $B = -2m\Omega/\hbar$ and the effective interaction strength is then

$$g_{\text{eff}} = g + \frac{4\pi\hbar\Omega}{\rho}, \quad (4)$$

and thus for a specific rotation frequency $\Omega$ there exists a critical negative interaction strength $g = g_c = -4\pi\hbar\Omega/\rho$ above which the gas can be stabilized by the so-called anyonic pressure. To consider the validity of this claim, we examine the two distinct cases where $\Omega = \omega$ and $\Omega < \omega$.

Let us begin by considering the high rotation frequency limit $\Omega = \omega$. In this case, the energy spectrum depends solely on the interaction Hamiltonian (1). A sample spectrum of the Bose gas when $g < 0$ is shown in Fig. 8 for $N = 5$ and $g = -1.0$. The spectrum is identical to that for positive interactions, except that it is inverted about the Laughlin state. The ground state energy is now degenerate with that of the $L = 0$ state, which can be obtained by directly evaluating Eq. (2) with all particles having $n, m = 0$:

$$E_{L=0} = N + g \frac{N(N-1)}{4\pi}.$$  \hfill (5)

It is important to emphasize that when $\Omega = \omega$, no matter how much $L$ is imparted to the system, the ground state energy will always be the same as for the $L = 0$ state. Thus, we see that from Eq. 5 the gas is unstable ($E < 0$) when $N > 1 - (4\pi/g)$. Accordingly, in the thermody-
dynamic limit, the attractive gas will never be stable even for the weakest interaction parameter: for large enough \(N\) the system always crosses over into an unstable regime, as seen in Fig. 9. This result is intuitively obvious: with attractive interactions, the ground state has \(L = 0\), which obviously contains no anyons because it is non-rotating. Thus, the stabilization observed when \(N\) is smaller than its critical value is due solely to the zero point energy.

If we now turn our attention to the situation where \(\Omega < \omega\), states with \(L \neq 0\) have energies that are no longer degenerate with the \(L = 0\) state due to the non-zero term \((1 - \Omega/\omega)L\) in the \(H_0\) eigenvalue equation. In this case, as \(L\) is increased, so does its associated ground state energy. It is therefore plausible that states with large \(L\) might have positive energy. According to this scheme, it would be preferable to have low \(\Omega\) since this would "lift" the high \(L\) states further away from the bottom of the spectrum—larger attractive interactions would be stabilized by lower rotation frequencies. However, this is contradictory to Eq. (4) which states that for fixed \(\rho\) (which is the case when \(L\) is kept constant) increasing \(\Omega\) increases the maximum attractive interaction strength \(g_c\) that can be stabilized by the anyonic pressure. In other words, suppose that we decrease \(\Omega\) while maintaining a constant amount of \(L\). This will ensure that \(\rho\) remains constant and so, according to Eq. (4), \(|g_c|\) should increase with \(\Omega\). However, at fixed \(L\), the \(H_0\) energy is larger for smaller \(\Omega\) meaning that the maximum attractive interaction strength for which the gas is stable should also be larger. This contradicts the predictions of Eq. (4), and thus we have one more reason to refute the claims brought forth in Ref. [29]. While it may be possible to stabilize an attractive boson gas consisting of a finite number particles in a high \(L\) state, the system will always become instable if \(N\) is allowed to increase without bounds. Other objections have been raised in the comment [45].

V. CONCLUSIONS

In summary, we have determined the maximum interaction strength that can be used before the LLL approximation becomes invalid. For ‘mean-field’ states with \(L \propto N\), we obtain a \(g_{\text{max}}\) which decreases with \(N\). With this scaling, in the limit of large \(N\) both the excitation gap and critical rotation frequency for the \(L = N\) ‘single-vortex’ state approach limiting values of around 0.2\(h\omega\) and 0.85\(h\omega\), respectively. In the same limit for the \(L = 2(N - 1)\) state, however, the gap closes and the critical frequency approaches the radial trap frequency. For the ‘strongly correlated’ states where \(L \propto N^2\), we find the remarkable result that the maximum interaction strength consistent with the LLL approximation actually increases: \(g_{\text{max}} \propto N^x\), where \(x\) is a positive number (\(= 1.6\) for the Pfaffian and 0.91 for the Laughlin states). This in turn implies that the critical rotation frequencies required to stabilize these states could be noticeably smaller than the radial trap frequency. The gaps for the Laughlin and Pfaffian approach values of 0.4\(h\omega\) and 0.1\(h\omega\) respectively. On other hand, if the interaction strength is chosen to be a constant for all \(N\), the numerics show that in the large-\(N\) limit the critical frequency for the ‘strongly correlated’ states approaches the trap frequency, and that for the Laughlin state the energy gap approaches a non-zero value of 0.2, while for the Pfaffian it appears to close.

We also have investigated the regime of negative interaction strengths, and conclude that in the thermodynamic limit, the trapped Bose gas is unstable against collapse for any rotation frequency, contrary to previous claims.

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[1] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, Phys. Rev. Lett. 80, 2265 (1998).
[2] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. 84, 6 (2000).
[3] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. 87, 120401 (2001).
[4] B. Paredes, P. Zoller, and J. I. Cirac, Phys. Rev. A 66, 033609 (2002).
[5] J. W. Reijnders, F. J. M. van Lankvelt, K. Schoutens, and N. Read, Phys. Rev. Lett. 89, 120401 (2002).
[6] T.-L. Ho and E. J. Mueller, Phys. Rev. Lett. 89, 050401 (2002).
[7] N. Regnault and T. Jolicoeur, Phys. Rev. Lett. 91, 030402 (2003).
[8] J. W. Reijnders, F. J. M. van Lankvelt, K. Schoutens, and N. Read, Phys. Rev. A 69, 023612 (2003).
[9] N. Regnault and T. Jolicoeur, Phys. Rev. B 69, 235309 (2004).
[10] N. R. Cooper, Phys. Rev. Lett. 92, 220405 (2004).
[11] M. Popp, B. Paredes, and J. I. Cirac, Phys. Rev. A 70, 053612 (2004).
[12] N. R. Cooper, F. J. M. van Lankvelt, J. W. Reijnders, and K. Schoutens, Phys. Rev. A 72, 063622 (2005).
[13] M. A. Baranov, K. Osterloh, and M. Lewenstein, Phys. Rev. Lett. 94, 070404 (2005).
[14] M. A. Cazalilla, N. Barberan, and N. R. Cooper, Phys. Rev. B 71, 121303(R) (2005).
[15] D. Jaksch and P. Zoller, New J. Phys. 5, 56 (2003).
[16] E. J. Mueller, Phys. Rev. A 70, 041603(R) (2004).
[17] A. S. Sorensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. 94, 086803 (2005).
[18] K. Osterloh, M. Baig, L. Santos, P. Zoller, and M. Lewen-
stein, Phys. Rev. Lett. 95, 010403 (2005).

[19] J. Ruseckas, G. Juzeliunas, P. Ohberg, and M. Fleischhauer, Phys. Rev. Lett. 95, 010404 (2005).

[20] P. Ohberg, G. Juzeliunas, J. Ruseckas, and M. Fleischhauer, Phys. Rev. A 72, 053632 (2005).

[21] G. Moore and N. Read, Nuclear Physics B 360, 362 (1991).

[22] W. Pan et al., Phys. Rev. Lett. 83, 3530 (1999).

[23] S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. 94, 166802 (2005).

[24] P. Bonderson, A. Y. Kitaev, and C. Nayak, Phys. Rev. Lett. 96, 016401 (2006).

[25] P. Bonderson, K. Shtengel, and J. K. Slingerland, Phys. Rev. Lett. 97, 016401 (2006).

[26] A. Y. Kitaev, Ann. Phys. 303, 2 (2003).

[27] N. E. Bonesteel, L. Hormozi, G. Zikos, and S. H. Simon, Phys. Rev. Lett. 95, 140503 (2005).

[28] S. L. Cornish, N. R. Claussen, J. L. Roberts, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. 85, 1795 (2000).

[29] U. R. Fischer, Phys. Rev. Lett. 93, 160403 (2004).

[30] A. Lakhoua, M. Lassaut, T. Masson, and J. C. Wallet, Phys. Rev. A 73, 023614 (2006).

[31] This assumes the parameters are chosen so that the scattering states are far from resonance with the virtual bound state. See D. S. Petrov and G. V. Shlyapnikov, Phys. Rev. A 64, 012706 (2001).

[32] B. Mottelson, Phys. Rev. Lett. 83, 2695 (1999).

[33] A. D. Jackson and G. M. Kavoulakis, Phys. Rev. Lett. 85, 2854 (2000).

[34] X.-J. Liu et al., Phys. Rev. Lett. 87, 030404 (2001).

[35] S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, and J. Dalibard, Laser Phys. Lett. 2, 275 (2005).

[36] S. Sinha and Y. Castin, Phys. Rev. Lett. 87, 190402 (2001).

[37] K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A 67, 033610 (2003).

[38] R. A. Smith and N. K. Wilkin, Phys. Rev. A 62, 061602 (2000).

[39] T. Papenbrock and G. F. Bertsch, Phys. Rev. A 63, 023616 (2001).

[40] E. G. M. van Kempen, S. J. J. M. F. Kokkelmans, D. J. Heinzen, and B. J. Verhaar, Phys. Rev. Lett. 88, 093201 (2002).

[41] E. Tiesinga, B. J. Verhaar, and H. T. C. Stoof, Phys. Rev. A 47, 4114 (1993).

[42] P. Nozières and D. Saint James, Journal de Physique 43, 1133 (1982).

[43] H. T. C. Stoof, Phys. Rev. A 49, 3824 (1993).

[44] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. 78, 985 (1997).

[45] P. K. Ghosh, Phys. Rev. Lett. 94, 208903 (2005).
