Community detection in inhomogeneous random graphs

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September 10, 2019

Abstract

We study the problem of detecting whether an inhomogeneous random graph contains a planted community. Specifically, we observe a single realization of a graph. Under the null hypothesis, this graph is a sample from an inhomogeneous random graph, whereas under the alternative, there exists a small subgraph where the edge probabilities are increased by a multiplicative scaling factor. We present a scan test that is able to detect the presence of such a planted community, even when this community is very small and the underlying graph is inhomogeneous. We also derive the information theoretic lower bound for this problem which shows that in some regimes the scan test is almost asymptotically optimal. We illustrate our results in examples and using simulations.

1 Introduction

Many complex systems can be described by networks of vertices connected by edges. Usually, these systems can be organized in communities, with certain groups of vertices being more densely connected than others. A central topic in the analysis of these systems is that of community detection where the goal is to find these more densely connected groups. This can often reveal interesting properties of the network with important applications in sociology, biology, computer science, and many other areas of science [10].

Much of the community detection literature is concentrated around methods that extract the communities from a given network, see [12, 21, 22]. These methods typically output an estimate of the community structure regardless of whether it really is present. Therefore, it is important to investigate when an estimated community structure is meaningful and when it simply is an artifact of the algorithm.

To answer this question, it has been highly fruitful to analyze the performance of these methods on random graphs with a known community structure. The stochastic block model is arguably the simplest model that still captures the relevant community structure, and the study of this model has led to many interesting results [1, 6, 9, 19, 20]. However, there are significant drawbacks because of this simplicity: the communities are typically assumed to be very large (i.e., linear in the graph size), and the graph is homogeneous within each community (i.e., vertices within a community are exchangeable and, in particular have the same degree distribution).

To overcome these issues, several suggestions have been made. For example, the degree-corrected block model allows for inhomogeneity of vertices within each community [18]. This allows one to model real-world networks more accurately, while remaining tractable enough to obtain results similar to those obtained for the stochastic block model [11, 13, 14, 16, 17]. However, the degree-corrected block model still assumes that the communities are very large. To be able to detect small communities, Arias-Castro and Verzelen consider a hypothesis testing problem where the goal is not to find communities, but instead decide whether or not any communities structure is present in an otherwise homogeneous graph [4, 5].
In this paper, we also focus on the detection of small communities and we investigate when it is possible to detect the presence of a small community in an already inhomogeneous random graph. In particular, we present a scan test and provide conditions under which it is able to detect the presence of a small community. These results are valid under a wide variety of parameter choices, including cases where the underlying graph is inhomogeneous. Furthermore, we show that for some parameter choices the scan test is optimal. Specifically, we show that if the conditions of the scan test are reversed then it is impossible for any test to detect such a community.

2 Model and results

We consider the problem of detecting a planted community inside an inhomogeneous random graph. This is formalized as a hypothesis testing problem, where we observe a single instance of a simple random graph $G = (V, E)$, with vertex set $V$ and edge set $E$. We denote the adjacency matrix of $G$ by $A$, i.e. $A_{ij} = 1_{(i,j) \in E}$. That is, $A_{ij} = 1$ if and only if there is an edge between the vertices $i, j \in V$. Because we only consider simple graphs, we have $A_{ii} = 0$ for all $i \in V$.

Under the null hypothesis, denoted by $H_0$, the observed graph is an inhomogeneous random graph on $|V| = n$ vertices, where an edge between two vertices $i, j \in V$ is present, independently of all other edges, with probability $p_{ij}$. In other words, the entries of the adjacency matrix $A$ are independent Bernoulli random variables such that $\mathbb{P}(A_{ij} = 1) = p_{ij}$. The alternative hypothesis, denoted by $H_1$, is similar, but within a subset of the vertices the connection probabilities are increased. Formally, there is a subset $C \subseteq V$ of size $|C| = r$, called the planted community, for which the edge probabilities are increased by a multiplicative scaling factor $\rho_C \geq 1$. Note that the scaling $\rho_C$ is allowed to depend on the planted community $C \subseteq V$. Concretely, under the alternative hypothesis the edge probabilities are $\mathbb{P}_1(A_{ij} = 1) = \rho_C p_{ij}$ for $i, j \in C$ and $\mathbb{P}_1(A_{ij} = 1) = p_{ij}$ otherwise. Throughout this paper, we assume that the location of the planted community $C \subseteq V$ is unknown, but that we do know its size $|C| = r$. In particular, we focus on the setting where $r$ is much smaller than $n$.

In our analysis we begin by considering the (unrealistic) case where the parameters $p_{ij}$ are all known. This allows us to get a precise characterization of the statistical difficulty of the problem. In Section 2 we relax this assumption and show that it is possible to adapt to unknown parameters under some conditions on the structure of the edge probabilities $p_{ij}$. In particular, there we will assume that the random graph is rank-1, so that $p_{ij} = \theta_i \theta_j$ for some vertex weights $(\theta_i)_{i=1}^n$.

To summarize, our goal is to decide whether a given graph contains a planted community, or equivalently to decide between the hypotheses:

$H_0$: There is no planted community, that is

$$A_{ij} \sim \begin{cases} \text{Bern}(p_{ij}), & \text{if } i \neq j, \\ 0, & \text{otherwise}. \end{cases}$$

$H_1$: There exists a planted community $C \subseteq V$ of size $|C| = r$, and $\rho_C > 1$, such that

$$A_{ij} \sim \begin{cases} \text{Bern}(\rho_C p_{ij}), & \text{if } i \neq j, \text{ and } i, j \in C, \\ \text{Bern}(p_{ij}), & \text{if } i \neq j, \text{ and } i \notin C \text{ or } j \notin C, \\ 0, & \text{otherwise}. \end{cases}$$

Note that in the above definition we are implicitly assuming that $\rho_C$ is not too large, so that $\rho_C p_{ij} \leq 1$ for all $i, j \in C$.

Given a graph we want to determine which of the above models gave rise to the observation. A test $\psi_n$ is any function taking as input a graph $g$ on $n$ vertices, and that outputs either $\psi_n(g) = 0$ to claim that there is reason to believe that the null hypothesis is true (i.e., no community is present) or $\psi_n(g) = 1$ to deem the alternative hypothesis true (i.e., the graph
contains a planted community). The worst-case risk of such a test is defined as

\[ R_n(\psi_n) := \mathbb{P}_0(\psi_n 
eq 0) + \max_{C \subseteq V, |C| = r} \mathbb{P}_C(\psi_n \neq 1), \]

where \( \mathbb{P}_0(\cdot) \) denotes the distribution under the null hypothesis, and \( \mathbb{P}_C(\cdot) \) denotes the distribution under the alternative hypothesis when \( C \subseteq V \) is the planted community. A sequence of tests \( (\psi_n)_{n=1}^\infty \) is called asymptotically powerful when it has vanishing risk, that is \( R_n(\psi_n) \to 0 \), and asymptotically powerless when it has risk tending to 1, that is \( R_n(\psi_n) \to 1 \).

Our primary goal is to characterize the asymptotic distinguishability between the null and alternative hypothesis as the graph size \( n \) increases. Throughout this paper, when limits are unspecified they are taken as the graph size satisfies \( n \to \infty \). The other parameters \( p_{ij}, \rho_C \), and \( r \) are allowed to depend on \( n \), although this dependence is left implicit to avoid notational clutter.

**Notation.** We also use standard asymptotic notation: \( a_n = O(b_n) \) when \( |a_n/b_n| \) is bounded, \( a_n = \Omega(b_n) \) when \( b_n = O(a_n) \), \( a_n = o(b_n) \) when \( a_n/b_n \to 0 \), and \( a_n \asymp b_n \) when \( a_n = (1 + o(1))b_n \). Also, we use the probabilistic versions of these: \( a_n = \mathbb{O}_b(b_n) \) when \( |a_n/b_n| \) is stochastically bounded, \( a_n = \Omega_b(b_n) \) when \( b_n = \mathbb{O}_a(a_n) \), and \( a_n = o_b(b_n) \) when \( a_n/b_n \) converges to 0 in probability.

We write \( e(C) := \sum_{i,j \in C} A_{ij} \) for the number of edges in the subgraph induced by \( C \subseteq V \), and \( e(C, \bar{C}) := \sum_{i \in C, j \in \bar{C}} A_{ij} \) for the number of edges between \( C \) and its complement \( \bar{C} = V \setminus C \). For two numbers \( a, b \in \mathbb{R} \), we write \( a \lor b = \max\{a, b\} \), and \( a \land b = \min\{a, b\} \). Finally, define the entropy function

\[ h(x) := (x + 1) \log(x + 1) - x. \]

This function plays a prominent role in most of the results.

### 2.1 Information theoretic lower bound

We start with a result highlighting conditions under which all tests are asymptotically powerless. Here we assume that the edge probabilities \( p_{ij} \), the scaling parameters \( \rho_C \), and the size of the planted community \( |C| = r \) are all known. When some of these parameters are unknown the problem of detecting a planted community becomes more difficult, hence any test that is asymptotically powerless when these parameters are known remains asymptotically powerless when they are unknown.

In order to obtain the result in this section, we need to make the following somewhat strong assumptions. We note that these assumptions are only necessary to obtain the information theoretic lower bound and are not needed to obtain the results in the subsequent sections. First, we assume that the planted community is not too large:

**Assumption 1.** The planted community size \( |C| = r \) cannot be too large, specifically

\[ \log(r) = o\left( \min_{i,j \in V} \frac{\log(n)}{\log(1/p_{ij})} \right). \]

This assumption requires that \( r \leq n^{o(1)} \) and \( p_{ij} \geq 1/n^{o(1)} \) for all \( i, j \in V \). We also require that the underlying graph is not too dense. This is made precise in the following assumption:

**Assumption 2.** We assume that \( \max_{C \subseteq V, |C| = r} \max_{i,j \in C} \rho_C^2 p_{ij} \to 0 \) as \( n \to \infty \).

This assumption accomplishes two goals. First, since \( \rho_C > 1 \) it forces \( p_{ij} \to 0 \) for every \( i, j \in V \). This ensures that the number of edges in subsets of the vertices are in essence a sufficient statistic for the testing problem. Secondly, at a more technical level, \( p_{ij} \to 0 \) is necessary for the Poisson approximations we use and it ensures that the differences in edge probabilities \( p_{ij} \) are not magnified too much under the alternative. We note that Assumption 2...
is only needed when the underlying graph is inhomogeneous. In homogeneous settings (i.e., when the null hypothesis corresponds to an Erdős-Rényi random graph) this assumption can be omitted, see [4].

This brings us to the main result of this section, providing conditions under which all tests are asymptotically powerless by deriving a minimax lower bound:

**Theorem 1.** Suppose that Assumptions 1 and 2 hold. Let \(0 < \varepsilon < 1\) be fixed. Then all tests are asymptotically powerless if, for all \(C \subseteq V\) of size \(|C| = r\),

\[
\max_{D \subseteq C} \frac{E_0[e(D)]h\left(\rho_C - 1\right)}{|D| \log(n/|D|)} \leq 1 - \varepsilon .
\]

Condition (4) has its counterpart in the work by Arias-Castro and Verzelen [4], who derive a similar result when the underlying graph is an Erdős-Rényi random graph. However, because of the inhomogeneity in our graphs, the maximum in (4) is not necessarily attained at the planted community \(C \subseteq V\) of size \(|C| = r\), but it could be attained at any of its smaller subgraphs \(D \subseteq C\). This is why our condition is more complex.

The proof of Theorem 1 is given in Section 5.5 and follows a common methodology in these cases, by first reducing the composite alternative hypothesis to a simple alternative hypothesis and then characterizing the optimal likelihood ratio test. This is done via a second-moment method, but it requires a highly careful truncation argument to attain the sharp characterization above.

The result in Theorem 1 happens to be tight, even in some scenarios where the edge probabilities \(p_{ij}\) are unknown, as we construct a scan test that is powerful when the inequality in (4) is, roughly speaking, reversed. This is discussed in the following sections.

### 2.2 Scan test for known edge probabilities

In this section we present a scan test that is asymptotically powerful. We first consider the case where all edge probabilities \(p_{ij}\) and the community size \(|C| = r\) are known. Although this case is unrealistic in practice, it allows us to understand the fundamental statistical limits of detection. In a sense, knowing the edge probabilities \(p_{ij}\) is the most optimistic scenario, and so the focus is primarily on whether or not it is possible to detect a planted community. In the subsequent section we relax this assumption by showing how the scan test can be extended when the edge probabilities \(p_{ij}\) are unknown.

Our test statistic is inspired by Bennett’s inequality (see [7, Theorem 2.9]), which ensures that, for any \(t > 0\),

\[
P_0(e(D) - E_0[e(D)] \geq t) \leq \exp\left(-E_0[e(D)]h\left(\frac{t}{E_0[e(D)]}\right)\right),
\]

where we recall that \(h(x) = (x + 1) \log(x + 1) - x\). Note that this inequality is also valid when we are under the alternative hypothesis (by simply changing the subscripts 0 to \(C\)). Plugging in \(t = E_0[e(D)]h^{-1}(s/E_0[e(D)])\) yields the bound

\[
P_0\left(\left[\frac{e(D)}{E_0[e(D)]} - 1\right]_{+} \geq s\right) \leq e^{-s}.
\]

This result motivates the use of the statistic

\[
T^k_D := \frac{E_0[e(D)]h\left[\frac{e(D)}{E_0[e(D)]} - 1\right]_{+}}{|D| \log(n/|D|)},
\]

where the superscript \(k\) is used to differentiate between the setting with known edge probabilities, and the setting with unknown edge probabilities in the next section. Note that the statistic \(T^k_D\) can be computed because \(E_0[e(D)]\) is a function of the known edge probabilities \(p_{ij}\).
To construct our test, we simply scan over the whole graph, rejecting the null hypothesis when there exists a subgraph \( D \subseteq V \) of size \(|D| \leq r\) with an unusually high value for \( T_D^k \). To be precise, fix \( \varepsilon > 0 \), then the scan test rejects the null hypothesis when

\[
T^k := \max_{D \subseteq V, |D| \leq r} T_D^k \geq 1 + \frac{\varepsilon}{2}.
\]

(8)

Note that we also scan over subgraphs that are smaller than the size of the planted community \(|C| = r\). This is necessary because of the possible inhomogeneity in the underlying graph; some edges carry little information and therefore it can be beneficial to ignore some of the edges and simply scan over smaller subgraphs. Note that the proposed test is not computationally practical due to the very large number of sets one must consider in the scan (unless \( r \) is very small). However, in this paper we are primarily interested in characterizing the statistical limits of possible tests, apart from computational considerations. See also the discussion in Section 4.

In order for the scan test to be powerful under the alternative we need that \( \mathbb{E}_C[e(D)] \to \infty \) for the most informative subgraph \( D \subseteq C \), because otherwise there is a non-vanishing probability that \( e(D) \) contains no edges under the alternative (by standard Poisson approximation), making it impossible for the scan test to detect the planted community. This subgraph is characterized in the following definition:

**Definition 1.** For every subgraph \( C \) of size \(|C| = r\), the most informative subgraph is

\[
D^*(C) := \arg \max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)]}{|D| \log(n/|D|)}.
\]

(9)

Given \( C \subseteq V \), the subgraph \( D^*(C) \) in the definition above is essentially the densest subgraph under the null hypothesis.

Using the above definition we can state the main result of this section, which provides conditions under which the scan test in (8) is asymptotically powerful:

**Theorem 2.** Suppose that all edge probabilities \( p_{ij} \) and the community size \( r \) are known. Then the scan test (8) is asymptotically powerful when \( r = o(n) \), \( \mathbb{E}_C[e(D^*(C))] \to \infty \) for all \( C \subseteq V \) of size \(|C| = r\), and

\[
\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)] h(\rho_C - 1)}{|D| \log(n/|D|)} \geq 1 + \varepsilon,
\]

(10)

where \( \varepsilon > 0 \) comes from the definition of the scan test in (8).

This result is more widely applicable than the lower bound from Theorem 1. The condition \( \mathbb{E}_C[e(D^*(C))] \to \infty \) is less stringent than Assumption 1. Also, there is no need for a condition like Assumption 2. This is because we can use the upper bound provided by Bennett’s inequality and therefore do not need the Poisson approximations necessary in deriving the lower bounds.

To make this precise and to make the result in Theorem 2 directly comparable to Theorem 1 we provide the following corollary:

**Corollary 1.** Suppose that all edge probabilities \( p_{ij} \) and the community size \( r \) are known, and that Assumption 1 holds. Then the scan test in (8) is asymptotically powerful when for all \( C \subseteq V \) of size \(|C| = r\),

\[
\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)] h(\rho_C - 1)}{|D| \log(n/|D|)} \geq 1 + \varepsilon,
\]

(11)

where \( \varepsilon > 0 \) comes from the definition of the scan test in (8).

To show that Theorem 2 applies in a broader setting than the lower bound from Theorem 1 we also provide the following corollary. This shows that the scan test is able to detect relatively large communities (of size polynomial in the graph size \( n \)), even when the edge probabilities are very small.
Corollary 2. Suppose that all edge probabilities $p_{ij}$ and the community size $r$ are known. Define $p_{\text{max}} := \max_{i,j \in V} p_{ij}$ and $p_{\text{min}} := \min_{i,j \notin V} p_{ij}$. If $r \geq n^a$, $p_{\text{min}} \geq n^{-2b}$, and $p_{\text{max}}/p_{\text{min}} = o(n^{a-b})$ for $0 < b < a < 1$, then the scan test in (8) is asymptotically powerful when for all $C \subseteq V$ of size $|C| = r$,

$$\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)]h(p_C - 1)}{|D| \log(n/|D|)} \geq 1 + \varepsilon,$$

where $\varepsilon > 0$ comes from the definition of the scan test in (8).

Moreover, both $a$ and $b$ above may to depend on the graph size $n$. In particular, if $p_{\text{max}}/p_{\text{min}} = O(1)$ then it is possible that $a = b = o(1)$, provided that $(a - b) \log(n) \rightarrow \infty$.

A downside of the scan test presented in this section is that it requires knowledge of all edge probabilities $p_{ij}$. In practice, these are often unavailable to a statistician. The next section is devoted to extending the scan test to cope with unknown edge probabilities, assuming that the edge probabilities have a rank-1 structure.

2.3 Scan test for unknown rank-1 edge probabilities

In this section we show how the scan test from the previous section can be extended to the setting where the edge probabilities $p_{ij}$ are unknown. We do still assume that the community size $|C| = r$ is known. As can be seen in (7), the scan statistic depends on the edge probabilities $p_{ij}$ only through $\mathbb{E}_0[e(D)] = \sum_{i,j \in D} p_{ij}$. Therefore, a natural way to approach the situation where the edge probabilities $p_{ij}$ are unknown is to devise a good surrogate for $\mathbb{E}_0[e(D)]$ that can be computed solely based on the observed graph (which could be a sample from either the null hypothesis or alternative hypothesis). Clearly, this is not possible in full generality, but if the edge probabilities have some additional structure then this become possible.

Here we consider the scenario where, under the null hypothesis, the edge probabilities $p_{ij}$ have a so-called rank-1 structure. That is, we assume that each vertex $i \in V$ is assigned a weight $\theta_i \in (0,1)$ and that the edge probabilities are given by $p_{ij} = \theta_i \theta_j$. Furthermore, to make it possible to estimate $\mathbb{E}_0[e(D)]$ we need to assume that the graph is not too inhomogeneous and not too sparse, as formulated in the following assumption:

Assumption 3. Let $\theta_{\text{max}} = \max_{i \in V} \theta_i$ and $\theta_{\text{min}} = \min_{i \in V} \theta_i$, then the maximum allowed inhomogeneity is

$$\left(\frac{\theta_{\text{max}}}{\theta_{\text{min}}}\right)^2 \leq o(\frac{n^2/3}{\theta_{\text{min}}^2 r}).$$

Using the above assumption, we will show that it is possible to estimate $\mathbb{E}_0[e(D)]$ by using the observed edges going from $D$ to the rest of the graph $-D = V \setminus D$. When $C \subseteq V$ is the planted community, and we estimate $\mathbb{E}_0[e(D)]$ for a large enough subgraph $D \subseteq C$ using this approach, we will obtain an almost unbiased estimate both under $H_0$ as well as under $H_1$. This is because enough of the edges used in this estimate have the same distribution under the null and alternative hypothesis. Our estimator is based on the identity

$$\mathbb{E}_0[e(D)] = \left(\sqrt{\mathbb{E}_0[e(V)]} + \frac{1}{2} \sum_{i \in V} \theta_i^2 - \sqrt{\mathbb{E}_0[e(V)]} + \frac{1}{2} \sum_{i \in V} \theta_i^2 - 2\mathbb{E}_0[e(D, -D)]\right)^2 - \frac{1}{2} \sum_{i \in D} \theta_i^2. \quad (14)$$

Note that both $\mathbb{E}_0[e(V)]$ and $\mathbb{E}_0[e(D, -D)]$ are the sum of a large number of edge probabilities $p_{ij} = \theta_i \theta_j$, and most of these remain unaffected under the alternative hypothesis. Because of this, and since $\sum_{i \in V} \theta_i^2$ will generally be negligible, we will estimate $\mathbb{E}_0[e(D)]$ by

$$\hat{e}(D) := \left(\sqrt{e(V)} - \sqrt{e(V) - 2e(D, -D)}\right)^2. \quad (15)$$
Here we used the fact that \((\theta_{\text{max}}/\theta_{\text{min}})^2 \leq r^{2/3}\) by Assumption 3 which ensures that the term \(\sum_{e \in D} \theta_e^2/2\) in (14) is negligible, and therefore that our estimator \(e(D)\) is a good surrogate for \(E[e(D)]\). This also explains why the exponent 2/3 appears in Assumption 3 as this is the largest exponent that still guarantees that the term \(\sum_{e \in D} \theta_e^2/2\) is negligible. This is discussed in more detail in Section 5.2.

In most cases, the estimator in (15) can essentially be used as a plugin for the scan test of the previous section. However, this estimator might not concentrate very well when \(E[e(D)]\) becomes too small. To remedy this, we use a thresholded version of the estimator given by

\[
e(D)^u := \left( e(D) \sqrt{ \frac{|D|^2}{n} \log^4(n/\delta) } \right). \tag{16}
\]

Using the thresholded estimator in (16), we can consider the same scan test as in the previous section but with \(E[e(D)]\) replaced by the estimator \(e(D)^u\). This leads to the definition of the scan test for unknown edge probabilities as

\[
T_D^u := \frac{ e(D)^u h\left( \frac{e(D)}{e(D)^u} - 1 \right) }{|D| \log(n/|D|)} , \tag{17}
\]

where the superscript \(u\) is used to indicate that we consider the setting with unknown rank-1 edge probabilities.

As in the previous section, we scan over subgraphs and reject the null hypothesis when \(T_D^u\) becomes too large. However, as explained above, when scanning over subgraphs \(D \subseteq V\) whose size \(|D|\) is much smaller than \(|C| = r\) we run into a problem because of the bias in \(e(D)^u\). Luckily this is not a problem because Assumption 3 ensures that asymptotically the maximum of \(T_D^u\) will always be attained at a subgraph of size \(|D| \geq r^{1/3}\), see the proof of Lemma 1 in Section 5.2. Therefore, for \(\varepsilon > 0\) fixed, the scan test for unknown edge probabilities rejects the null hypothesis when

\[
T^u := \max_{D \subseteq V, r^{1/3} \leq |D| \leq r} T_D^u \geq 1 + \frac{\varepsilon}{3}. \tag{18}
\]

This brings us to the main result of this section, which provides conditions for the scan test in (18) to be asymptotically powerful:

**Theorem 3.** Suppose that the community size \(r\) is known and that Assumption 3 holds. Then the scan test (18) is asymptotically powerful when \(r \leq o(n)\), \(E_C[e(D^*(C))] \to \infty\) for all \(C \subseteq V\) of size \(|C| = r\), and

\[
\max_{D \subseteq C} \frac{E_D[e(D)] h \rho_C - 1}{|D| \log(n/|D|)} \geq 1 + \varepsilon, \tag{19}
\]

where \(\varepsilon > 0\) comes from the definition of the scan test in (18).

Comparing this result with Theorem 2, we see that for rank-1 random graphs, Assumption 3 is the only extra condition necessary when the edge probabilities are unknown. Furthermore, by the same argument as in the previous section it can be shown that Assumption 1 is sufficient to ensure that \(E_C[e(D^*(C))] \to \infty\). Therefore, to make the result in Theorem 3 directly comparable to Theorem 1 we provide the following corollary:

**Corollary 3.** Suppose that the community size \(r\) is known and that Assumptions 1 and 3 hold. Then the scan test (18) is asymptotically powerful when, for all \(C \subseteq V\) of size \(|C| = r\),

\[
\max_{D \subseteq C} \frac{E_D[e(D)] h \rho_C - 1}{|D| \log(n/|D|)} \geq 1 + \varepsilon, \tag{20}
\]

where \(\varepsilon > 0\) comes from the definition of the scan test in (18).
Moreover, a result similar to Corollary 2 also applies in the setting with unknown edge probabilities. This is shown in the following corollary, which shows that Theorem 3 is more widely applicable than the lower bound from Theorem 1, even when the vertex weights \( \theta_i \) are very small:

**Corollary 4.** Suppose the community size \( r \) is known and that Assumption 3 holds. If \( r \geq n^a \), \( \theta_{\min} \geq n^{-b} \), and \( (\theta_{\max}/\theta_{\min})^2 = o(n^{a-b}) \) for \( 0 < b < a < 1 \), then the scan test in (8) is asymptotically powerful when for all \( C \subseteq V \) of size \( |C| = r \),

\[
\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)]h(\rho_C - 1)}{|D| \log(n/|D|)} \geq 1 + \varepsilon,
\]

where \( \varepsilon > 0 \) comes from the definition of the scan test in (18).

### 3 Examples

The results in the previous section provide conditions for when it is possible to detect a planted community \( C \subseteq V \). When the scaling \( \rho_C \) is large enough it is asymptotically possible to detect a planted community using the scan test, and when the scaling \( \rho_C \) is too small it is impossible for any test to detect a planted community. To understand at which scaling \( \rho_C \) this change happens, we need to characterize the behavior of

\[
\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)]h(\rho_C - 1)}{|D| \log(n/|D|)}.
\]

The subgraph that attains the maximum above will be denoted by \( D^* = D^*(C) \) and was defined in Definition 1. In this section, we present several examples of different random graph models and illustrate how (22) depends on the inhomogeneity structure.

#### 3.1 Erdős-Rényi random graph

The arguably simplest setting where we can apply our results is that of an Erdős-Rényi random graph, where all edge probabilities \( p_{ij} = p \) are equal, so that the graph is completely homogeneous. In this case, the subgraph \( D^* \) that attains the maximum in (22) is always the complete planted community \( C \subseteq V \). Let \( r = o(n) \), \( r \to \infty \) and \( p \to 0 \) such that \( r^2 p \to \infty \). One easily sees that (22) becomes

\[
\max_{D \subseteq C} \frac{\mathbb{E}_0[e(D)]h(\rho_C - 1)}{|D| \log(n/|D|)} = \frac{\mathbb{E}_0[e(C)]h(\rho_C - 1)}{|C| \log(n/|C|)} \approx \frac{r \rho_C - 1}{2 \log(n/r)} \approx \frac{r H_p(\rho_C p)}{2 \log(n/r)}.
\]

where \( H_p(\rho_C p) \) is the Kullback-Leibler divergence between \( \text{Bern}(p) \) and \( \text{Bern}(\rho_C p) \). Note that this is the same condition found by Arias-Castro and Verzelen, who considered the problem of detecting a planted community in an Erdős-Rényi random graph [4].

#### 3.2 Rank-1 random graph with 2 weights

A slightly more complex setting is where the underlying graph has a rank-1 structure with two different weights. Some of the vertices have large weight \( \theta_{\max} \), and the remaining vertices have small weight \( \theta_{\min} \). Therefore, there are three different edge probabilities in the underlying graph: \( p_{ij} = \theta_{\max}^2 \) when both endpoints have large weight, \( p_{ij} = \theta_{\min}^2 \) when both endpoints have small weight, and \( p_{ij} = \theta_{\max} \theta_{\min} \) when one of the endpoints has large weight and the other small weight.

The subgraph \( D^*(C) \) that attains the maximum in (22) depends crucially on the amount of inhomogeneity in \( C \subseteq V \), and because we only have two different weights this translates to the
ratio of vertices with large weight $\theta_{\text{max}}$ and vertices with small weight $\theta_{\text{min}}$ in $C$. Moreover, it can be checked that the maximum in (22) is attained either on the whole subgraph $C$, or on the subgraph $C_{\text{max}} \subseteq C$ consisting of only the large-weight vertices in $C$. Specifically, assuming $\log(n/|C|) \asymp \log(n)$, the maximum in (22) is attained at $C_{\text{max}}$ when

$$|C_{\text{max}}| > (1 + o(1)) \frac{|C| - 1 + (\theta_{\text{max}}/\theta_{\text{min}})^2}{(\theta_{\text{max}}/\theta_{\text{min}} - 1)^2},$$

and otherwise it is attained at $C$. Here we can see that the amount of inhomogeneity plays an important role in determining the maximum in (22), and therefore in determining whether a planted community can be detected or not.

In Figure 1 we give two examples of the minimal scaling $\rho_C$ required for the scan test to be asymptotically powerful. When, for every $C \subseteq V$, the scaling $\rho_C$ is chosen above the blue curve then the scan test is asymptotically powerful by Theorem 3 and when it is chosen below the blue curve then all tests are asymptotically powerless by Theorem 4. Here we can clearly see a sharp bend in the blue curve at the point where $|C_{\text{max}}|$ crosses the threshold in (24). This happens because when $|C_{\text{max}}|$ is large there are many vertices with large weight and it is optimal to only use these vertices when trying to detect a planted community. However, when $|C_{\text{max}}|$ becomes too small there no longer are enough vertices with large weight and it becomes more beneficial to also use the vertices with small weight.

3.3 Rank-1 random graph with 3 weights

Extending the setting in the previous section, we can consider a rank-1 random graph with three different weights. Some vertices have large weight $\theta_{\text{max}}$, some vertices have medium weight $\theta_{\text{med}}$, and the remaining vertices have small weight $\theta_{\text{min}}$. In this setting the situation becomes more complex, and the subgraph $D^*$ that attains the maximum in (22) depends on the amount of vertices of each type in $C \subseteq V$.

In Figure 2 we give an example of the minimal scaling $\rho_C$ required for the scan test to be asymptotically powerful in the setting with three weights. We can see that when there are enough vertices with large weight $\theta_{\text{max}}$ then it is optimal to only use these large-weight vertices, but as the number of large-weight vertices decreases it becomes beneficial to include also medium-weight vertices or even small-weight vertices. Note that the cross-section with no medium-weight vertices is the same as Figure 1(a) and the cross-section with no large-weight vertices is the same as Figure 1(b).
Figure 2: Example of the minimal $\rho_C$ required for detecting a planted community when using the optimal subgraph $D'(C)$. In the blue region $D'(C)$ consists of all vertices, in the orange region $D'(C)$ consists of both large and medium-weight vertices, and in the green region $D'(C)$ consists only of large-weight vertices. The parameters used are $r = \lfloor \log(n)^4 \rfloor$, $\theta_{\max} = 1 \log(n)$, $\theta_{\med} = \frac{1}{3 \log(n)}$, $\theta_{\min} = \frac{1}{6 \log(n)}$. These values are chosen for ease of comparison with Figure 1.

### 3.4 Rank-1 random graph with an arbitrary number of weights

In this section we consider the setting where the underlying graph contains several different vertex weights. In this case it is more difficult to characterize the subgraph $D'(C)$ that is required to detect a planted community.

Given a subgraph $C \subseteq V$, finding the subgraph $D \subseteq C$ that maximizes (22) is a simple optimization problem, because for a given size $|D|$ we only need to consider the subgraph $D$ consisting of the $|D|$ largest weights in $C$. Using this insight we can approximate (22). Let $\hat{F}_C(x)$ be the the empirical distribution function of the weights in $C$, then

$$
\max_{D \subseteq C} \frac{\mathbb{E}[e(D)|h(\rho_C - 1)}{|D| \log(n/|D|)} \approx \max_{k \in \{1, \ldots, r\}} \frac{\binom{k}{2} \int_{\alpha \rho_C}^{\hat{F}_C(y)} \hat{F}_C^{-1}(y) dy^2 h(\rho_C - 1)}{k \log(n/k)} \label{eq:approx}
$$

where $\hat{F}_C^{-1}(y) = \inf \{x \in \mathbb{R} : y \leq \hat{F}_C(x) \}$ is the quantile function of $\hat{F}_C(x)$, and we have assumed that $r = n^{o(1)}$ to ensure that $\log(n/r) \approx \log(n)$ in the second approximation above.

To apply (22) we need to know $\hat{F}_C(x)$, which is different for every subgraph $C \subseteq V$. However, instead of characterizing the minimal scaling $\rho_C$ for every subgraph $C$, we can instead consider a uniformly chosen subgraph $C$. In this way, if the vertex weights are sampled from a distribution $W$ with distribution function $F(x)$, then we know from the Glivenko-Cantelli theorem that $\hat{F}_C(x)$ will eventually be close to $F(x)$ uniformly in $x$. With this in mind, we can consider the minimal required scaling $\rho_C$ when $C$ is a uniformly chosen subgraph and the vertex weights are sampled from a distribution $W$.

In Table 1 this is done for a community of size $r = \lfloor \log(n)^4 \rfloor$ and weight distribution $W = (s + X)/\log(n)^{3/2}$, where we consider several different distributions $X$. We add a small constant $s$ to ensure that none of the vertex weights can become too small and we have normalized the weights by $\log(n)^{3/2}$ to ensure that in each example the maximum weight is less than $1$ with high probability. Furthermore, these choices ensure that $r\mathbb{E}[W]^2/\log(n/r) = O(1)$, and by (22) this guarantees that $\rho_C = O(1)$, so we obtain a numerical value for $\rho_C$ that is independent of $n$. 

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Table 1: Minimal scaling $\rho_C$ required to detect a planted community $C$ that is planted uniformly at random. We provide the analytic results together with a numerical example where $E[X] = 1$. The planted community size is $|C| = r = \lceil \log(n) \rceil$.

| $W$ | Minimal $\rho_C$ | $|D^*|$ |
|-----|-----------------|-------|
| $\frac{s+X}{\log(n)^{3/2}}$, $X \sim \text{Deg}n(\delta)$, $\delta = 1$, $s = 0.1$ | $3.311$ | $1.000 \cdot r$ |
| $\frac{s+X}{\log(n)^{3/2}}$, $X \sim \text{Bern}(q)$, $q = 0.5$, $t = 2$, $s = 0.1$ | $2.624$ | $0.500 \cdot r$ |
| $\frac{s+X}{\log(n)^{3/2}}$, $X \sim \text{Unif}(a,b)$, $a = 0$, $b = 2$, $s = 0.1$ | $3.144$ | $0.700 \cdot r$ |
| $\frac{s+X}{\log(n)^{3/2}}$, $X \sim \text{Exp}(\lambda)$, $\lambda = 1$, $s = 0.1$ | $2.939$ | $0.407 \cdot r$ |

4 Discussion

In this section we remark on our results and discuss some possibilities for future work.

Alternatives to the scan test. In the setting where Assumptions 1 and 2 hold, the result of Theorem 2 provides conditions where the scan test is asymptotically powerful, and Theorem 1 shows that this is essentially optimal. However, Theorem 2 is still applicable when these assumptions do not hold, although the scan test might no longer be optimal. Here we discuss this and examine possible alternatives to the scan test when Assumptions 1 or 2 do not hold.

When the community size $|C| = r$ becomes much larger than allowed by Assumption 1, the scan test is no longer optimal. This was considered by Arias-Castro and Verzelen for an Erdős-Rényi random graph [4], where they show that for large communities, a statistic based on simply counting the total number of edges is optimal. A similar idea can also be applied in the inhomogeneous settings. This suggests that such a test is asymptotically powerful, if for all $C \subseteq V$ of size $|C| = r$,

$$\frac{E_C[e(C)] - E_0[e(C)]}{\sqrt{E_0[e(V)]}} \to \infty. \quad (26)$$

Another setting where the scan test is no longer optimal is where the underlying graph is very sparse. In this case, one could consider tests similar to those considered by Arias-Castro and Verzelen [5].

Unknown community size. When presenting our results, we have always assumed that the size of the planted community is known. In practice, this is often not the case and it would be necessary to estimate the community size before testing. In our case, the scan test can easily be extended to the setting of unknown community size. To see this, note that the scan test can detect any planted community provided that it is not larger than $r$. Hence, one can simply use the scan test with a large enough value for $r$ and it will detect a planted community of size at most $r$.

Computational Complexity. In general, the computational complexity of scan tests is not polynomial in the graph size $n$. In the homogeneous settings, it has been conjectured that
polynomial time algorithms are not able to achieve the minimax rate \[3\]. Inhomogeneity in the graphs can make computations easier - for instance in very inhomogeneous cases it is possible to recover the largest clique of a graph in polynomial time \[2\]. It thus remains an interesting avenue for future work to thoroughly characterize the statistical limits of tests under computational constrains.

5 Proofs

In this section we prove our results. We start with the proof of Theorem \[2\] because it is the simplest and it sets the stage for some of the arguments in the proof of Theorem \[3\]. We end this section with the proof of Theorem \[1\] which shows that the results obtained in Theorem \[2\] and Theorem \[3\] are, roughly speaking, the best possible.

5.1 Proof of Theorem \[2\]

\textbf{Scan test for known edge probabilities is powerful}

In this section we prove that the scan test in \[8\] is asymptotically powerful. That is, under the conditions of the theorem, both type-I and type-II errors vanish.

\textbf{Type-I error.} We will show that \(P_0(T^k \geq 1 + \varepsilon/2) \to 0\). This is done using a relatively straightforward use of Bennett’s inequality and the union bound. Using \(\binom{n}{r} \leq \left(\frac{en}{r}\right)^r\), it follows that

\[
P_0 \left( T^k \geq 1 + \frac{\varepsilon}{2} \right) = P_0 \left( \max_{D \subseteq V, |D| \leq r} T^k_D \geq 1 + \frac{\varepsilon}{2} \right) \\
= P_0 \left( \max_{1 \leq k \leq r} \max_{D \subseteq V, |D| = k} \frac{E_0[\varepsilon(D)] h \left( \frac{\varepsilon(D)/E_0[\varepsilon(D)] - 1}{k \log(n/k)} \right)}{k \log(n/k)} \geq 1 + \frac{\varepsilon}{2} \right) \\
\leq \sum_{1 \leq k \leq r} \sum_{D \subseteq V, |D| = k} P_0 \left( \frac{E_0[\varepsilon(D)] h \left( \frac{\varepsilon(D)/E_0[\varepsilon(D)] - 1}{k \log(n/k)} \right)}{k \log(n/k)} \geq 1 + \frac{\varepsilon}{2} \right) \\
\leq \sum_{1 \leq k \leq r} \binom{n}{k} \exp \left( - \left( 1 + \frac{\varepsilon}{2} \right) k \log \left( \frac{n}{k} \right) \right) \\
\leq \sum_{1 \leq k \leq r} \left( e \left( \frac{k}{n} \right)^{\varepsilon/2} \right)^k \leq \frac{e \left( \frac{\varepsilon}{2} \right)^{\varepsilon/2}}{1 - e \left( \frac{\varepsilon}{2} \right)^{\varepsilon/2}} \to 0.
\]

The first and second inequality follow from simple union bound and Bennett’s inequality given in \[9\]. The final step relies on the fact that \(k/n \leq r/n\) and \(r \leq o(n)\). Therefore we conclude that the scan test \[8\] has vanishing type-I error.

\textbf{Type-II error.} Showing that we have vanishing type-II error entails first showing that \(P_C(T^k \geq 1 + \varepsilon/2) \geq P_C(T^k_{D^*(C)} \geq 1 + \varepsilon/2)\), for every \(C \subseteq V\) of size \(|C| = r\), where \(D^*(C)\) was introduced in Definition \[1\]. The rest of the proof entails showing that for every \(C \subseteq V\) of size \(|C| = r\),

\[
T^k_{D^*(C)} \geq (1 + o_C(1)) \frac{E_0[\varepsilon(D^*(C))] h(\rho_C - 1)}{|D^*(C)| \log(n/|D^*(C)|)}.
\] (27)

Together with \[10\] this implies that, for every \(C\), we have \(P_C(T^k_{D^*(C)} \geq 1 + \varepsilon/2) \to 1\).

Let \(C \subseteq V\) be an arbitrary subgraph of size \(|C| = r\) and recall \(D^* := D^*(C)\) from Definition \[1\] (we drop the explicit dependence of \(D^*\) on \(C\) to avoid notational clutter). To prove \[(27)\] it suffices
to show that
\[ E_0[e(D^*)] h \left( \frac{e(D^*)}{E_0[e(D^*)]} - 1 \right)_+ \geq (1 + o_\epsilon(1)) E_0[e(D^*)] h(\rho_C - 1). \]  \hfill (28)

To see this, note that \( h(x-1) \) is convex, with derivative \( h'(x-1) = \log(x) \) and therefore \( h(x-1) \geq h(y-1) + (x-y) \log(y) \). Using this with \( x = e(D^*)/E_0[e(D^*)] \) and \( y = E_C[e(D^*)]/E_0[e(D^*)] = \rho_C > 1 \), we obtain the lower bound

\[
\begin{align*}
E_0[e(D^*)] h \left( \left[ \frac{e(D^*)}{E_0[e(D^*)]} - 1 \right]_+ \right) - E_0[e(D^*)] h(\rho_C - 1) \\
\geq (e(D^*) - E_C[e(D^*)]) \log \left( \frac{E_C[e(D^*)]}{E_0[e(D^*)]} \right) \\
= (e(D^*) - E_C[e(D^*)]) \log(\rho_C).
\end{align*}
\]

It follows by Chebyshev’s inequality that

\[(e(D^*) - E_C[e(D^*)]) \log(\rho_C) = O_{\nu_C} \left( \sqrt{E_C[e(D^*)]} \log(\rho_C) \right).\]

Therefore, the inequality in (28) holds when

\[
\frac{\sqrt{E_C[e(D^*)]} \log(\rho_C)}{E_0[e(D^*)] h(\rho_C - 1)} = o(1).
\hfill (29)
\]

To show this, we consider three cases depending on the asymptotic behavior of \( \rho_C \). Although these three cases do not cover all possibilities, they suffice, by the argument in Remark 1 below.

**Case 1 (\( \rho_C \to 1 \))**: Using \( \sqrt{x} \log(x) \asymp (x-1) \) as \( x \to 1 \), and \( h(x-1) \asymp (x-1)^2/2 \) as \( x \to 1 \) gives

\[
\begin{align*}
\sqrt{E_C[e(D^*)]} \log(\rho_C) &= (1 + o(1)) \sqrt{E_0[e(D^*)]}/(\rho_C - 1), \quad \text{and} \\
E_0[e(D^*)] h(\rho_C - 1) &= (1 + o(1)) E_0[e(D^*)]/(\rho_C - 1)^2/2.
\end{align*}
\]

Hence, by (10) we have \( E_0[e(D^*)] / (\rho_C - 1)^2 \asymp 2 E_0[e(D^*)] h(\rho_C - 1) > 2 |D^*| \log(n/|D^*|) \to \infty \). Combining the above gives

\[
\frac{\sqrt{E_C[e(D^*)]} \log(\rho_C)}{E_0[e(D^*)] h(\rho_C - 1)} = (1 + o(1)) \frac{2}{\sqrt{E_0[e(D^*)]}/(\rho_C - 1)^2} = o(1).
\]

This shows that (29) holds when \( \rho_C \to 1 \).

**Case 2 (\( \rho_C \to \alpha \in (1, \infty) \))**: In this case \( \sqrt{\rho_C} \log(\rho_C) = O(h(\rho_C - 1)) \), and by (10) we have \( E_0[e(D^*)] h(\rho_C - 1) \geq |D^*| \log(n/|D^*|) \to \infty \). Therefore

\[
\sqrt{E_C[e(D^*)]} \log(\rho_C) = \sqrt{E_0[e(D^*)]} \sqrt{\rho_C} \log(\rho_C) = o(\sqrt{E_0[e(D^*)]} h(\rho_C - 1)).
\]

This shows that (29) holds when \( \rho_C \to \alpha \in (1, \infty) \).

**Case 3 (\( \rho_C \to \infty \))**: Using \( h(x-1) \asymp x \log(x) \) as \( x \to \infty \) and because \( E_C[e(D^*)] \to \infty \) we have

\[
\frac{\sqrt{E_C[e(D^*)]} \log(\rho_C)}{E_0[e(D^*)] h(\rho_C - 1)} = \frac{1}{\sqrt{E_C[e(D^*)]}} = o(1).
\hfill (30)
\]

This shows that (29) holds when \( \rho_C \to \infty \), and therefore that (28) holds in all the three cases.
Remark 1. Note that \( \rho_C \) might not fit one of the above cases, but may rather oscillate between a combination of the three. However, this is not a problem. For every subsequence of \( \rho_C \), there exists a further subsequence along which the scaling \( \rho_C \) satisfies one of the three cases. Hence, \( \rho_p \) holds along this (further) subsequence, which implies that \( \rho_p \) also holds along the full sequence. This type of argument will be used in several more places in the proofs. \( \triangle \)

The proof of Theorem 2 is now easily completed using (27) together with (10). For every \( C \subseteq V \) of size \( |C| = r \),

\[
T^k \geq T^k_D = \frac{\mathbb{E}_0[e(D^*) ] h\left( \left[ e(D^*)/\mathbb{E}_0[e(D^*)] - 1 \right]^+ \right)}{|D^*| \log(n/|D^*|)} \geq (1 + o_C(1)) \frac{\mathbb{E}_0[e(D^*) ] h(\rho_C - 1)^+)}{|D^*| \log(n/|D^*|)} \geq (1 + o_C(1))(1 + \varepsilon).
\]

Hence, \( \mathbb{P}_C(T^k \geq 1 + \varepsilon/2) \rightarrow 1 \). This shows that the type-II error vanishes, completing the proof.

5.2 Proof of Theorem 3
Scan test for unknown rank-1 edge probabilities is powerful

In this section we prove that the scan test in (18) is asymptotically powerful, but we first derive some auxiliary results. The first of these shows that if a planted community can be detected then it can be detected based on the evidence of the subgraph \( D^*(C) \) from Definition 1. Moreover, by Assumption 3 it follows that \( D^*(C) \) must be relatively large. Specifically, we show that \( |D^*(C)| \geq r^{1/3} \). This explains why the scan test in (18) is defined to only scan over subgraphs larger than \( r^{1/3} \).

Lemma 1. For any \( C \subseteq V \) of size \( |C| = r \), let \( D^*(C) \) be as given in Definition 1. When Assumption 3 holds then \( |D^*(C)| \geq r^{1/3} \).

Proof. We use a proof by contradiction. For any \( D \subseteq V \) of size \( |D| \leq r^{1/3} \), it follows by Assumption 4 that

\[
\frac{\mathbb{E}_0[e(D)]}{|D| \log(n/|D|)} \leq \frac{|D| - 1}{2} \frac{\theta_{\max}^2}{\log(n/|D|)} \leq o(r) \frac{\theta_{\min}^2}{2 \log(n/r^{1/3})} \leq \frac{\mathbb{E}_0[e(C)]}{|C| \log(n/|C|)}.
\]

Hence, a subset \( D \subseteq V \) of size \( |D| \leq r^{1/3} \) does not maximize the right-hand side of (9), and therefore \( |D^*(C)| \geq r^{1/3} \). \( \square \)

In the second auxiliary result we quantify the deviations of \( \hat{e}(D) \) around \( \mathbb{E}_0[e(D)] \). We note that the lemma below remains true when all \( (1 + o_o(1)) \) terms are replaced by \( (1 + o_C(1)) \) terms.

Lemma 2. Let \( D \) be a set of subsets of the vertices \( V \), such that for all \( D \in D \) we have \( r^{1/3} \leq |D| \leq r \). Under Assumption 3 and

\[
e(V) = (1 + o_o(1))\mathbb{E}_0[e(V)],
\]

\[
e(D,-D) = (1 + o_o(1))\mathbb{E}_0[e(D,-D)],
\]

the deviations of \( \hat{e}(D) \) around \( \mathbb{E}_0[e(D)] \) are given by

\[
\frac{e(D)}{\mathbb{E}_0[e(D)]} = 1 + o_o(1), \quad \text{uniformly over all } D \in D.
\]
Proof. Define \( f(x_1, x_2) := (\sqrt{x_1} - \sqrt{x_1 - 2x_2})^2 \) for \( x_1 \geq 2x_2 \). Then the partial derivatives of \( f(x_1, x_2) \) are given by

\[
\frac{\partial f}{\partial x_1}(x_1, x_2) = -\frac{(\sqrt{x_1} - \sqrt{x_1 - 2x_2})^2}{\sqrt{x_1} \sqrt{x_1 - 2x_2}} = -\frac{f(x_1, x_2)}{\sqrt{x_1} \sqrt{x_1 - 2x_2}},
\]

\[
\frac{\partial f}{\partial x_2}(x_1, x_2) = 2 \frac{x_1 - \sqrt{x_1 - 2x_2}}{\sqrt{x_1 - 2x_2}} = \frac{2 f(x_1, x_2)}{\sqrt{x_1 - 2x_2} (\sqrt{x_1} - \sqrt{x_1 - 2x_2})}.
\]

We use a Taylor expansion of \( f(x_1, x_2) \) around \((a_1, a_2)\) with \( a_1 > 2a_2 \). Specifically, there exists \((\xi_1, \xi_2)\) with \( \xi_1 \) in between \( x_1 \) and \( a_1 \), and \( \xi_2 \) in between \( x_2 \) and \( a_2 \), such that

\[
f(x_1, x_2) = f(a_1, a_2) + \frac{\partial f}{\partial x_1}(\xi_1, \xi_2)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\xi_1, \xi_2)(x_2 - a_2).
\]

We will apply (31) using \((x_1, x_2) = (e(V), e(D, -D))\) and \((a_1, a_2) = (E_0[e(V)], E_0[e(D, -D)])\).

Now, because \( e(V) = (1 + o_\theta(1))E_0[e(V)] \) it follows that for any \( \xi_1 \) in between \( e(V) \) and \( E_0[e(V)] \) we also have \( \xi_1 = (1 + o_\theta(1))E_0[e(V)]. \) Similarly, \( e(D, -D) = (1 + o_\theta(1))E_0[e(D, -D)] \) uniformly over all \( D \in D \), and therefore \( \xi_2 = (1 + o_\theta(1))E_0[e(D, -D)] \). Hence,

\[
f(e(V), e(D, -D)) = T_{00} + T_{10} + T_{01},
\]

where

\[
T_{00} = f(E_0[e(V)], E_0[e(D, -D)]),
\]

\[
T_{10} = \frac{-(1 + o_\theta(1)) f(E_0[e(V)], E_0[e(D, -D)])}{\sqrt{E_0[e(V)]} \sqrt{E_0[e(V)]} - 2E_0[e(D, -D)]} (e(V) - E_0[e(V)]),
\]

\[
T_{01} = \frac{(2 + o_\theta(1)) f(E_0[e(V)], E_0[e(D, -D)])}{\sqrt{E_0[e(V)]} - 2E_0[e(D, -D)]} (e(D, -D) - E_0[e(D, -D)]),
\]

where we have used that \( E_0[e(D, -D)] = o(E_0[e(V)]) \) and \( E_0[e(V)] \to \infty \) in the second equality of both \( T_{10} \) and \( T_{01} \), which is ensured by Assumption 3. To see this, note that \((\theta_{\text{max}}/\theta_{\text{min}})^2 \leq o(\frac{r}{\theta_{\text{min}}^2}) \leq o(\frac{1}{\theta_{\text{min}}}) \) because \( \theta_{\text{min}} \leq 1 \), hence

\[
\frac{E_0[e(D, -D)]}{E_0[e(V)]} \leq (1 + o(1)) \frac{|D| n \theta_{\text{max}}^2}{n \theta_{\text{min}}^2} \leq \frac{|D|}{n} o\left(\frac{n}{r}\right) = o\left(\frac{|D|}{r}\right) = o(1).
\]

To continue, observe that

\[
e(V) = f(e(V), e(D, -D)),
\]

\[
E_0[e(D)] = (1 + o(1)) \frac{f(E_0[e(V)], E_0[e(D, -D)])}{4} - \frac{1}{2} \sum_{i \in D} \theta_i^2,
\]

where in (33) we have used Assumption 3 to ensure that \( E_0[e(V)] + \frac{1}{2} \sum_{i \in V} \theta_i^2 = (1 + o(1))E_0[e(V)]. \) This last statement is easily shown since

\[
\frac{E_0[e(V)] + \sum_{i \in V} \theta_i^2}{E_0[e(\theta)]} = 1 + \frac{\sum_{i \in V} \theta_i^2}{\theta_i^2} \leq 1 + \frac{n \theta_{\text{max}}^2}{n \theta_{\text{min}}^2} \leq 1 + \frac{n r^{2/3} \theta_{\text{min}}^2}{n (\theta_{\text{min}}^2)} = 1 + \frac{2 r^{2/3}}{n - 1} = 1 + o(1).
\]

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Using \( \hat{e}(D) \), \( \hat{\mathbb{E}}_0[\hat{e}(D)] \), and \( \hat{e}(D) \), we obtain, uniformly over \( D \in \mathcal{D} \),

\[
\frac{\hat{e}(D)}{\mathbb{E}_0[\hat{e}(D)]} = \frac{1}{\mathbb{E}_0[\hat{e}(D)]} f(e(V), e(D, -D)) = (1 + o_n(1)) \frac{\mathbb{E}_0[e(D)] + \frac{1}{2} \sum_{i \in D} \theta_i^2}{\mathbb{E}_0[e(D)]} \left( 1 - \frac{e(V) - \mathbb{E}_0[e(V)]}{\mathbb{E}_0[e(V)]} + 2 \frac{e(D, -D) - \mathbb{E}_0[e(D, -D)]}{\mathbb{E}_0[e(D, -D)]} \right)
\]

To complete the proof we need to show \( \sum_{i \in D} \theta_i^2 / \mathbb{E}_0[e(D)] = o(1) \). To this end, we first show that

\[
\frac{\sum_{i \in D} \theta_i^2}{\sum_{i \in D} \theta_i^2} \leq \frac{1}{4|D|} (\frac{\theta_{\min} + \theta_{\max}}{|D|})^2.
\]

To see this, note that \( \sum_{i \in D} \theta_i^2 / \left( \sum_{i \in D} \theta_i^2 \right) \) is maximized when a fraction \( \alpha = \theta_{\min} / (\theta_{\min} + \theta_{\max}) \) of the vertices in \( D \) has weight \( \theta_{\max} \) and the remaining \( 1 - \alpha \) vertices have weight \( \theta_{\min} \). Plugging this in we obtain (36).

Then, by Assumption 3 we have \( \theta_{\max} / \theta_{\min} = o(r^{1/3}) \) and using that \( |D| \geq r^{1/3} \) together with (36), we obtain

\[
\frac{\sum_{i \in D} \theta_i^2}{\sum_{i \in D} \theta_i^2} \leq \frac{1}{4|D|} \left( \frac{\theta_{\min} + \theta_{\max}}{|D|} \right)^2 \leq \frac{1}{4|D|} \left( \frac{\theta_{\max}}{|D|} \right)^2 = o(r^{1/3}) = o(1).
\]

Finally, we conclude that \( \sum_{i \in D} \theta_i^2 / \mathbb{E}_0[e(D)] \leq o(1) \). Plugging this into (35) completes the proof.

We are now ready to prove Theorem 3 which shows that the scan test in (18) is still asymptotically powerful even when the edge probabilities are not known. To this end, we again show that both the type-I and the type-II error vanish, which we do separately below.

**Type-I error.** Here we show \( P_0(T^u \geq 1 + \varepsilon / 3) \to 0 \). To this end, we show that using the truncated estimator \( \hat{e}(D) \) from (16) is asymptotically as good as using \( \mathbb{E}_0[e(D)] \). Specifically, we show that uniformly over all subgraphs \( D \subseteq V \) of size \( r^{1/3} \leq |D| \leq r \),

\[
\max_{D \subseteq V : r^{1/3} \leq |D| \leq r} \frac{\mathbb{E}_0[e(D)]}{e(D)} \leq 1 + o_n(1).
\]

To show this, define the random set \( \mathcal{D} := \{ D \subseteq V : r^{1/3} \leq |D| \leq r, e(D) \leq \mathbb{E}_0[e(D)] \} \) and rewrite (37) as

\[
\max_{D \subseteq V : r^{1/3} \leq |D| \leq r} \left( \frac{\mathbb{E}_0[e(D)]}{e(D)} \mathbb{1}_{\{D \in \mathcal{D}\}} + \frac{\mathbb{E}_0[e(D)]}{e(D)} \mathbb{1}_{\{D \notin \mathcal{D}\}} \right).
\]

In the second term above we have \( D \notin \mathcal{D} \), so this term is trivially less than or equal to 1. Therefore we will focus on the first term in (38). For any \( D \in \mathcal{D} \), it follows by definition of the thresholded estimator \( e(D) \) in (16) that

\[
\frac{|D|^2}{n \log^2 \left( \frac{n}{|D|} \right)} \leq e(D) \leq \mathbb{E}_0[e(D)] \leq \left( \sum_{i \in D} \theta_i \right)^2, \quad \text{hence} \quad \frac{|D|}{\sqrt{n}} \log^2 \left( \frac{n}{|D|} \right) \leq \sum_{i \in D} \theta_i.
\]

Now, by the second part of Assumption 3 we have \( \left( \frac{\theta_{\max}}{|D|} \right)^2 \leq \frac{\alpha}{r} \), and therefore \( \theta_{\min} \geq \sqrt{r/n} \geq 1/\sqrt{n} \). Using this we obtain

\[
\mathbb{E}_0[e(D, -D)] = \left( \sum_{i \in D} \theta_i \right) \left( \sum_{i \notin D} \theta_j \right) \geq \frac{|D|}{\sqrt{n}} \log^2 \left( \frac{n}{|D|} \right), \quad \frac{n - |D|}{\sqrt{n}}
\]
\[ = (1 + o(1))|D| \log^2 \left( \frac{n}{|D|} \right). \]

Recall that Bennett’s inequality ensures that, for \( t > 0 \),
\[
\mathbb{P}_0(e(D, -D) - \mathbb{E}_0[e(D, -D)] \leq -t) \leq \exp \left( -\mathbb{E}_0[e(D, -D)] h \left( \frac{t}{\mathbb{E}_0[e(D, -D)]} \right) \right).
\]

To get a uniform bound over all subgraphs \( D \in \mathcal{D} \), we use a union bound together with (39). For any \( \delta > 0 \) and \( n \) large enough, this gives
\[
\mathbb{P}\left( \min_{D \in \mathcal{D}} e(D, -D) - \mathbb{E}_0[e(D, -D)] \leq -(1 + \delta) \sqrt{2 \mathbb{E}_0[e(D, -D)] |D| \log(n/|D|)} \right)
\leq \sum_{1 \leq k \leq r} \sum_{D \subseteq V, |D| = k} \mathbb{P}\left( |\mathbb{E}_0[e(D, -D)] - |D| \log(n/|D|)| \right)
\leq \sum_{1 \leq k \leq r} \sum_{D \subseteq V, |D| = k} \mathbb{P} \left( |\mathbb{E}_0[e(D, -D)] - |D| \log(n/|D|)| \right)
\leq \sum_{1 \leq k \leq r} \binom{n}{k} \exp \left( -(1 + \delta) k \log \left( \frac{n}{k} \right) \right)
\leq \sum_{1 \leq k \leq r} \left( \frac{k}{n} \right)^{\delta k} \frac{e \left( \frac{k}{n} \right)^{\delta} \log \left( \frac{n}{k} \right)}{1 - e \left( \frac{k}{n} \right)^{\delta} \log \left( \frac{n}{k} \right)} \to 0.
\]

For the step in (40) we used the result in (39) together with \( h(x) \approx x^2/2 \) as \( x \to 0 \), and the final step relies on the fact that \( k/n \leq r/n \) and \( r \leq o(n) \).

Then, using the above together with (39), it follows that uniformly over \( D \in \mathcal{D} \),
\[
e(D, -D) - \mathbb{E}_0[e(D, -D)] \mathbb{E}_0[e(D, -D)] = \mathcal{O}_n \left( \frac{|D| \log(n/|D|)}{\mathbb{E}_0[e(D, -D)]} \right) = o_n(1). \]

To bound the deviations of \( e(V) \) we use Chebyshev’s inequality,
\[
e(V) - \mathbb{E}_0[e(V)] \mathbb{E}_0[e(V)] = \mathcal{O}_n \left( \frac{1}{\mathbb{E}_0[e(V)]} \right) = o_n(1). \]

Using (41) and (42), it follows by Lemma 2 that uniformly over \( D \in \mathcal{D} \),
\[
\frac{\mathbb{E\left( e(D) \right)}}{\mathbb{E}_0[e(D)]} \mathbb{E}_0[e(D)] \geq 1 + o_n(1).
\]

This shows that the first term in (38) is less than or equal to \( 1 + o_n(1) \), and therefore that (37) holds.

Then, using (37) it becomes relatively straightforward to show that the type-I error vanishes. Note that\( a h \left( \left\lfloor \frac{a}{b} \right\rfloor + 1 \right) \leq b h \left( \left\lfloor \frac{a}{b} \right\rfloor + 1 \right) \) for \( a > b \), and therefore
\[
\mathbb{P}_0 \left( T^w \geq 1 + \frac{\varepsilon}{3} \right)
\]

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Here we have Case 1. We continue by considering the two cases in the maximum of (43) separately. Therefore by definition of the thresholded estimator in (16),

\[
\rho = \max_{D \subseteq V, r/3 \leq |D| \leq r} \frac{\left( e(D)/e(D)^\dagger - 1 \right)_+}{|D| \log(n/|D|)} \geq 1 + \frac{\varepsilon}{3}.
\]

Then using the same reasoning as in the proof of Theorem 2 it follows that the type-I error vanishes.

**Type-II error.** Here we show that \( \mathbb{P}_C(T^n \geq 1 + \varepsilon/3) \geq \mathbb{P}_C(T^n \geq 1 + \varepsilon/3) \rightarrow 1 \), for every \( C \subseteq V \) of size \( |C| = r \), where \( D^* = D^*(C) \) is defined as in [9]. To this end, we start by quantifying the deviation of \( e(D^*)/E_0[e(D^*)] \) under the alternative.

By Chebyshev’s inequality,

\[
e(D^*, -D^*) - E_C[e(D^*, -D^*)]/E_D[e(D^*, -D^*)] = \mathcal{O}_C \left( \frac{1}{E_D[e(D^*, -D^*)]} \right) = o_c(1),
\]

\[
e(V) - E_C[e(V)] = \mathcal{O}_C \left( \frac{1}{E_D[e(V)]} \right) = o_c(1).
\]

Moreover, \( \rho_C \theta_{min}^2 \leq 1 \) and therefore \( \frac{\theta_{max}^2}{\theta_{min}^2} = o(\frac{\theta_{max}^2}{\theta_{min}^2}) \) by Assumption 3. Hence \( \rho_C \leq o \left( \frac{n \theta_{max}^2}{\theta_{min}^2} \right) \). Therefore

\[
1 \leq \frac{E_C[e(D^*, -D^*)]}{E_D[e(D^*, -D^*)]} \leq 1 + \frac{E_C[e(D^*, C \setminus D^*)]}{E_D[e(D^*, V \setminus D^*)]} = 1 + \rho_C \frac{E_D[e(D^*, C \setminus D^*)]}{E_D[e(D^*, V \setminus D^*)]} \leq 1 + \rho_C \frac{D^* | \theta_{max}^2 / |D^*| \theta_{min}^2} {n \theta_{min}^2} \leq 1 + o(1).
\]

By the above it follows that \( E_C[e(D^*, -D^*)] = (1 + o(1))E_D[e(D^*, -D^*)] \), and similarly \( E_C[e(V)] = (1 + o(1))E_D[e(V)] \). Therefore

\[
e(D^*, -D^*) = (1 + o_c(1))E_D[e(D^*, -D^*)], \quad \text{and} \quad e(V) = (1 + o_c(1))E_D[e(V)].
\]

Then, applying Lemma 2 (using the set \( D = \{D^*\} \)), we obtain

\[
\frac{e(D^*)}{E_D[e(D^*)]} = 1 + o_c(1).
\]

Therefore by definition of the thresholded estimator in (43),

\[
e(D^*)^\dagger = \left( e(D^*) \sqrt{\frac{|D^*|^2}{n} \log^4 \left( \frac{n}{|D^*|^2} \right)} \right) \geq \left( 1 + o_c(1) \right) E_D[e(D^*)] \sqrt{\frac{|D^*|^2}{n} \log^4 \left( \frac{n}{|D^*|^2} \right)}.
\]

We continue by considering the two cases in the maximum of (43) separately.

**Case 1:** Here we have \( e(D^*)^\dagger = (1 + o_c(1))E_D[e(D^*)] \). Plugging this into the definition of the test statistic we obtain

\[
T^*_D = \frac{e(D^*)^\dagger h \left( \left( \frac{e(D^*)}{e(D^*)^\dagger} - 1 \right)_+ \right)}{|D^*| \log(n/|D^*|)}.
\]
\[
\begin{aligned}
&= \frac{(1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]h\left((1 + \alpha_c(1))\frac{\epsilon(D^*)}{\mathbb{E}[\epsilon(D^*)]} - 1\right)}{|D^*|\log(n/|D^*|)}. \\
\end{aligned}
\]

The proof can then be completed by using the same reasoning as in the proof of Theorem 2 from \cite{27} to \cite{30}. Here the additional \(\alpha_c(1)\) terms do not make any difference.

**Case 2:** Here we have \(\epsilon(D^*)^\dagger = (|D^*|^2/n)\log^4(n/|D^*|)\). This corresponds to the case where the underlying graph is very sparse, and therefore a very large signal \(\rho_C\) is required to detect a planted community.

We start by deriving a lower bound on \(\rho_C\). Using condition \(\dagger\) together with \(\mathbb{E}[\epsilon(D^*)] \leq (|D^*|^2/n)\log^4(n/|D^*|)\) and \(h^{-1}(x) \geq \sqrt{x}\), we obtain

\[
\rho_C \geq h^{-1}\left(\frac{|D^*|\log(n/|D^*|)}{\mathbb{E}[\epsilon(D^*)]}\right) \geq h^{-1}\left(\frac{n}{|D^*|\log^4(n/|D^*|)}\right) \geq (1 + o(1))\sqrt{n/|D^*|}. \quad (44)
\]

Moreover, by the second part of Assumption 3 we have \(1 \leq \left(\frac{\theta_{\min}}{\theta_{\max}}\right)^2 \leq \frac{n}{r} \theta_{\min}^2\), and therefore \(\theta_{\min} \geq \sqrt{r/n} \geq 1/\sqrt{n}\). Using this together with (44) gives

\[
\frac{\mathbb{E}[\epsilon(D^*)]}{\epsilon(D^*)^\dagger} \geq \frac{\rho_C|D^*|^2\theta_{\min}^2}{n\log^4(n/|D^*|)} \geq \frac{\rho_C}{\log^4(n/|D^*|)} \to \infty.
\]

Then, using that \(\epsilon(D^*) = (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]\) by Chebyshev’s inequality and \(h(x - 1) \approx x \log(x)\) as \(x \to \infty\), we obtain

\[
\begin{aligned}
\epsilon(D^*)^\dagger h\left(\frac{\epsilon(D^*)}{\epsilon(D^*)^\dagger} - 1\right) \geq \epsilon(D^*)^\dagger h\left(1 + \alpha_c(1)\frac{\mathbb{E}[\epsilon(D^*)]}{\epsilon(D^*)^\dagger} - 1\right) \\
= (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]\log\left(\frac{\mathbb{E}[\epsilon(D^*)]}{\epsilon(D^*)^\dagger}\right) \\
= (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]\log\left(\frac{\rho_C\mathbb{E}[\epsilon(D^*)]}{\epsilon(D^*)^\dagger}\right).
\end{aligned}
\]

Now, by the same argument as above we have \(\theta_{\min} \geq 1/\sqrt{n}\). Hence, it follows that \(\mathbb{E}[\epsilon(D^*)] \geq |D^*|^2\theta_{\min}^2 \geq |D^*|^2/n\), so that \(\mathbb{E}[\epsilon(D^*)]/|D^*|^2 \geq \log^{-4}(n/|D^*|)\). Then, using (44),

\[
\begin{aligned}
\frac{\log(\rho_C \mathbb{E}[\epsilon(D^*)]/\epsilon(D^*)^\dagger)}{\log(\rho_C)} \geq \frac{\log(\rho_C/\log^4(n/|D^*|))}{\log(\rho_C)} = 1 - 4\frac{\log\log(n/|D^*|)}{\log(\rho_C)} = 1 + o(1).
\end{aligned}
\]

Plugging this into (45), we obtain

\[
\begin{aligned}
\epsilon(D^*)^\dagger h\left(\frac{\epsilon(D^*)}{\epsilon(D^*)^\dagger} - 1\right) = (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]\log\left(\frac{\rho_C\mathbb{E}[\epsilon(D^*)]}{\epsilon(D^*)^\dagger}\right) \\
\geq (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]\log(\rho_C) \\
= (1 + \alpha_c(1))\mathbb{E}[\epsilon(D^*)]h(\rho_C - 1) \\
\geq (1 + \alpha_c(1))(1 + \varepsilon)|D^*|\log\left(\frac{n}{|D^*|}\right),
\end{aligned}
\]

where the final step follows from \cite{19}. Therefore, \(T_{D^*}^\varepsilon \geq 1 + \varepsilon/3\) with high probability, completing the proof. \(\Box\)
5.3 Proof of Corollaries 1 and 3

To prove Corollaries 1 and 3 we need to show that Assumption 1 is sufficient to ensure that \( E_C[e(D^*)] \rightarrow \infty \) for every \( C \subseteq V \) of size \(|C| = r\). When \( p_C = O(1) \) this is a direct consequence of conditions (10) and (19), therefore we will consider the case where \( p_C \rightarrow \infty \).

Using \( h(x - 1) \approx x \log(x) \) as \( x \to \infty \) and \( 10 \) or \( 19 \) we obtain

\[
E_C[e(D^*)] = (1 + o(1)) \frac{E_0[e(D^*)]h(p_C - 1)}{\log(p_C)} \geq (1 + o(1)) \frac{|D^*| \log(n/|D^*|)}{\log(p_C)}.
\]

Now, for every \( C \subseteq V \) and \( i, j \in C \) we have \( p_C p_{ij} \leq 1 \), and therefore \( \log(p_C) \leq \log(1/p_{ij}) \leq o(\log(n)) \) by Assumption 1. Therefore

\[
E_C[e(D^*)] \geq (1 + o(1)) \frac{|D^*| \log(n/|D^*|)}{\log(p_C)} \geq (1 + o(1)) \frac{|D^*| \log(n)}{o(\log(n))} \rightarrow \infty.
\]

This shows that Assumption 1 is sufficient to ensure that \( E_C[e(D^*)] \rightarrow \infty \) for every \( C \subseteq V \) of size \(|C| = r\), proving Corollaries 1 and 3.

5.4 Proof of Corollaries 2 and 4

Proof. Begin by noting that conditions in Corollary 4 imply the conditions on \( p_{\text{max}} \) and \( p_{\text{min}} \) that are stated in Corollary 2. To prove Corollaries 2 and 3 we need to show that \( E_C[e(D^*)] \rightarrow \infty \) for every \( C \subseteq V \) of size \(|C| = r\). Because \( p_{\text{max}}/p_{\text{min}} = \omega(n^{a-b}) \), there exists a sequence \( x_n \to \infty \) such that \( p_{\text{max}}/p_{\text{min}} = n^{a-b}/x_n \). We will first show that \( |D^*| \geq b \sqrt{x_n} \), which we will do by a similar argument as in Lemma 1. Suppose \( |D^*| \leq n^b \sqrt{x_n} \), then because \( r \geq n^{a} \),

\[
\frac{E_0[e(D^*)]}{|D^*| \log(n/|D^*|)} \leq \frac{|D^*| - 1}{2} \frac{p_{\text{max}}}{\log(n/|D^*|)} = \frac{|D^*| - 1}{2} \frac{n^{a-b} x_n}{p_{\text{min}} \log(n/|D^*|)} \leq \mathcal{O}(1) \frac{n^a}{\sqrt{x_n}} \log(n/r) \leq \frac{r - 1}{2} \frac{p_{\text{min}}}{\log(n/r)} \leq \frac{E_0[e(C)]}{|C| \log(n/|C|)}.
\]

Hence, \( D^* \) cannot be the maximizer in (9) when \( |D^*| \leq n^b \sqrt{x_n} \), and therefore we must have \( |D^*| \geq n^b \sqrt{x_n} \). Therefore,

\[
E_C[e(D^*)] \geq E_0[e(D^*)] \geq \frac{|D^*|^2}{2} p_{\text{min}} \geq \frac{n^{2b} x_n}{2} n^{-2b} \rightarrow \infty.
\]

The proof of Corollary 2 is then completed by applying Theorem 2 and similarly the proof of Corollary 3 is completed by applying Theorem 3.

5.5 Proof of Theorem 1

Information theoretic lower bound

To prove Theorem 1 we need to show that \( R_n(\psi_n) \to 1 \), where \( R_n \) is the worst-case risk given in (1) and \( \psi_n \to \{0, 1\} \) is any test deciding between the null and alternative hypothesis. The first step is a reduction from the worst-case risk to the average risk

\[
R_n(\psi_n) := \mathbb{P}_0(\psi_n(G) = 1) + \left( \frac{n}{r} \right)^{-1} \sum_{C \subseteq V, |C| = r} \mathbb{P}_C(\psi_n(G) = 0).
\]

Note that the average risk is a lower bound for the worst-case risk, that is \( R_n(\psi_n) \geq \bar{R}_n(\psi_n) \). This average risk corresponds to a hypothesis test between two simple hypotheses, because the alternative hypothesis is now simple. The means that the likelihood ratio test is optimal (by the Neyman-Pearson lemma). In particular the test \( \psi_n^{\text{LR}} = \mathbb{I}(L > 1) \) minimizes the average risk,
where $L$ is the likelihood ratio, given in (46) below. Furthermore, the risk of this test is given by

$$
\hat{R}_n(\psi_n^{LR}) = \mathbb{P}_0(L > 1) + \mathbb{E}_0[\mathbb{I}(L \leq 1)] = 1 - \frac{1}{2} \mathbb{E}_0[|L - 1|].
$$

Therefore, to prove Theorem 1, it suffices to show that $\mathbb{E}_0[|L - 1|] \to 0$.

Given a graph $g$, the likelihood ratio $L$ is given by

$$
L(g) = \binom{n}{r}^{-1} \sum_{C \subseteq V, |C| = r} \frac{\mathbb{P}_C(G = g)}{\mathbb{P}_0(G = g)} = \binom{n}{r}^{-1} \sum_{C \subseteq V, |C| = r} L_C(g) = \bar{\mathbb{E}}[L_C(g)],
$$

(46)

where $\bar{\mathbb{E}}[\cdot]$ denotes expectation with respect to a uniformly chosen subgraph $C \subseteq V$ of size $|C| = r$, and

$$
L_C(g) := \prod_{i < j \in C} \left( \frac{\rho C p_{ij}}{p_{ij}} \right)^{A_{ij}} \left( \frac{1 - \rho C p_{ij}}{1 - p_{ij}} \right)^{1 - A_{ij}}
$$

(47)

$$
= \exp \left( \sum_{i < j \in C} A_{ij} \log \left( \frac{\rho C p_{ij}}{p_{ij}} \right) + (1 - A_{ij}) \log \left( \frac{1 - \rho C p_{ij}}{1 - p_{ij}} \right) \right)
$$

$$
= \exp \left( \sum_{i < j \in C} A_{ij} \theta_{ij}(\rho C p_{ij}) - \Lambda_{ij}(\theta_{ij}(\rho C p_{ij})) \right),
$$

with

$$
\theta_{ij}(q) := \log \left( \frac{q(1 - p_{ij})}{p_{ij}(1 - q)} \right), \quad \text{and} \quad \Lambda_{ij}(\theta) := \log \left( 1 - p_{ij} + p_{ij}e^{\theta} \right).
$$

Note that, $\Lambda_{ij}(\theta)$ is the cumulant generating function of Bern$(p_{ij})$, with Fenchel-Legendre transform given by

$$
H_{p_{ij}}(q) = \sup_{x \geq 0} \{ qx - \Lambda_{ij}(x) \} = q \theta_{ij}(q) - \Lambda_{ij}(\theta_{ij}(q)), \quad \text{for } q \in (p_{ij}, 1),
$$

(48)

where $H_p(q) := q \log \left( \frac{q}{p} \right) + (1 - q) \log \left( \frac{1 - q}{1 - p} \right)$ is the Kullback-Leibler divergence between Bern$(p)$ and Bern$(q)$.

To bound $\mathbb{E}_0[|L - 1|]$ one generally resorts to the Cauchy-Schwarz inequality to control instead the second moment of $L$ and obtain $\mathbb{E}_0[|L - 1|] \leq \mathbb{E}_0[L^2] - 1$. However, in our setting this bound is too crude, and the variance of $L$ will be rather large, compared to the first moment. Instead, we use a more refined approach suggested by Ingster [15] and later used by Butucea and Ingster [8] and Arias-Castro and Verzelen [4]. This approach relies on a truncation of the likelihood ratio

$$
\bar{L} = \binom{n}{r}^{-1} \sum_{C \subseteq V, |C| = r} \mathbb{I}_{\Gamma C} L_C = \bar{\mathbb{E}}[\mathbb{I}_{\Gamma C} L_C],
$$

where the events $\Gamma_C$ are specified below in (50), and $L_C$ is as given by (47). Using $\bar{L} \leq L$, the triangle inequality, and the Cauchy-Schwarz inequality, we obtain the upper bound

$$
\mathbb{E}_0[|L - 1|] \leq \mathbb{E}_0[|\bar{L} - 1|] + \mathbb{E}_0[L - \bar{L}]
$$

$$
\leq \sqrt{\mathbb{E}_0[\bar{L}^2] - 2\mathbb{E}_0[\bar{L}] + 1} + 1 - \mathbb{E}_0[\bar{L}].
$$

Therefore, $\hat{R}_n(\psi_n^{LR}) \to 1$ when both $\mathbb{E}_0[\bar{L}^2] \to 1$ and $\mathbb{E}_0[\bar{L}] \to 1$, which we consider separately below. The ideal truncation event will lower the variance of $L$ while still ensuring that the first moment of $L$ is asymptotically the same as that of $\bar{L}$.
Loosely speaking \( \theta \) truncation of the likelihood, it follows from Fubini’s theorem that here we show that moreover, \( \zeta \) inequality allows us to control \( \rho \) \( i, j \) \( \theta \) uniformly over all \( \rho \) \( i, j \) ≤ \( \rho \) \( i, j \) \( \rho \) \( i, j \). Using Assumption 2 we see that if \( \rho \) \( i, j \) \( \rho \) \( i, j \). Hence, it suffices to show that \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \). Let Assumptions 1 and 2 hold and let \( \rho \) \( i, j \) \( \rho \) \( i, j \). Loosely speaking \( \theta \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \). The definition of the truncation event is somewhat involved. Begin by defining

\[
E_C := \left\{ D \subseteq C : (\rho_C - 1)^2 E_0[\epsilon(D)] > (1 - \epsilon/2)|D| \left( \log \left( \frac{n|D|}{r^2} \right) - b_n \right) \right\}, \tag{49}
\]

where \( b_n \to \infty \) very slowly, for concreteness we will take \( b_n = \log \log(n) \). Using this, we define the numbers \( \zeta \) \( D \) using the lemma below, the proof of which is mainly technical and therefore deferred to the appendix:

**Lemma 3.** Let Assumptions 1 and 2 hold and let \( n \) be large enough. Then for any \( C \subseteq V \) of size \( |C| = r \), \( D \in E_C \) there exists a unique number \( \zeta_D \geq 1 \), such that

\[
(1 + \epsilon)E_0[\epsilon(D)]h(\zeta_D - 1) = |D| \log \left( \frac{n}{|D|} \right). \tag{50}
\]

Moreover, \( \zeta_D \) satisfies \( \theta_{ij}(\zeta_D p_{ij}) \leq 2\theta_{ij}(\rho_C p_{ij}) \) for every \( i, j \in D \).

Using the numbers \( \zeta_D \) and \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \). Below we will show the slightly stronger result that \( \min_{C \subseteq V, |C| = r} P_C(\Gamma_C) \to 1 \), which together with \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \). Hence, it suffices to show that \( P_C(\Gamma_C) \to 1 \) for most \( C \). Below we will show the slightly stronger result that \( \min_{C \subseteq V, |C| = r} P_C(\Gamma_C) \to 1 \), which together with \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \) \( \rho \) \( i, j \). Begin by noting that

\[
\max_{C \subseteq V, |C| = r} \max_{i, j \in C} \left| \frac{\theta_{ij}(\rho_C p_{ij})}{\log(\rho_C)} - 1 \right| \to 0, \quad \text{as } n \to \infty. \tag{51}
\]

To see this, note that

\[
\frac{\theta_{ij}(\rho_C p_{ij})}{\log(\rho_C)} - 1 = \frac{\log \left( \frac{1 - p_{ij}}{1 - \rho_C p_{ij}} \right)}{\log(\rho_C)}. \tag{52}
\]

Using Assumption 2 we see that if \( \rho_C \) is bounded away from 1 then the above converges to 0 uniformly over all \( i, j \in C \), since \( p_{ij} = o(1) \). For the case \( \rho_C \to 1 \) we can simply use L’Hospital’s rule. Any case in between is handled as in Remark 1. Loosely speaking, this means that \( \theta_{ij}(\rho_C p_{ij}) \simeq \log(\rho_C) \) for all \( C \) and \( i, j \in C \). This, together with a union bound and Bennett’s inequality allows us to control \( P_C(\Gamma_C) \). Indeed,

\[
1 - P_C(\Gamma_C) \leq \sum_{D \in E_C} P_C \left( \sum_{i < j \in D} A_{ij} \theta_{ij}(\rho_C p_{ij}) > \sum_{i < j \in D} p_{ij} \zeta_D \theta_{ij}(\rho_C p_{ij}) \right)
\]

\[
\leq \sum_{D \in E_C} P_C \left( \sum_{i < j \in D} A_{ij} > (1 + o(1)) \zeta_D \sum_{i < j \in D} p_{ij} \right)
\]

\[
\leq \sum_{D \in E_C} \exp \left( -E_C[\epsilon(D)] h \left( (1 + o(1)) \left( \frac{\zeta_D}{\rho_C} - 1 \right) \right) \right).
\]
\[ = \sum_{D \in \mathcal{C}} \exp\left( -(1 + o(1))\mathbb{E}_C[e(D)] h\left( \frac{\xi_D}{\rho_C} - 1 \right) \right), \]

where the last step uses a property of the \( h \) function, which ensures that for \( t \geq 1, x \geq 0 \) we have \( \sqrt{th(x)} \leq h(tx) \leq t^2 h(x) \).

To show that this vanishes we need the following lemma, the proof of which is mainly technical and therefore deferred to the appendix.

**Lemma 4.** Define the sequence \( a_n \) as

\[ a_n := \min_{C \subseteq V, |C| = r} \min_{D \in \mathcal{E}_C} \left( (1 - \varepsilon) \frac{\mathbb{E}_C[e(D)]}{|D|} h\left( \frac{\xi_D}{\rho_C} - 1 \right) - \log\left( \frac{r}{|D|} \right) \right). \]

Assume that [4] and Assumptions [1] and [2] hold. Then \( a_n \to \infty \) as \( n \to \infty \).

Using Lemma 4 and grouping the sets \( D \in \mathcal{E}_C \) by their size \( |D| \), together with the bound on binomial coefficient \( \left( \begin{array}{c} \xi \end{array} \right) \leq \left( \frac{\xi}{k} \right)^k \) we conclude that, for \( n \) large enough,

\[ 1 - \min_{C \subseteq V, |C| = r} \mathbb{P}_C(\Gamma_C) \leq \max_{C \subseteq V, |C| = r} \sum_{D \in \mathcal{E}_C} \exp\left( -(1 + o(1))\mathbb{E}_C[e(D)] h\left( \frac{\xi_D}{\rho_C} - 1 \right) \right) \]

\[ = \max_{C \subseteq V, |C| = r} \sum_{k=1}^{r} \sum_{D \in \mathcal{E}_C, |D| = k} \exp\left( -(1 + o(1))\mathbb{E}_C[e(D)] h\left( \frac{\xi_D}{\rho_C} - 1 \right) \right) \]

\[ = \max_{C \subseteq V, |C| = r} \sum_{k=1}^{r} \sum_{D \in \mathcal{E}_C, |D| = k} \exp\left( \left(1 + o(1)\right)\mathbb{E}_C[e(D)] h\left( \frac{\xi_D}{\rho_C} - 1 \right) \right) \]

\[ - \log\left( \frac{r}{|D|} \right) \]

\[ \leq \max_{C \subseteq V, |C| = r} \sum_{k=1}^{r} \left( \begin{array}{c} r \end{array} \right)^{k-1} \sum_{D \in \mathcal{E}_C, |D| = k} \exp\left( -(1 + o(1))\mathbb{E}_C[e(D)] h\left( \frac{\xi_D}{\rho_C} - 1 \right) \right) \]

\[ - \log\left( \frac{r}{|D|} \right) \]

\[ \leq \sum_{k=1}^{r} \left( \begin{array}{c} r \end{array} \right)^{k-1} \sum_{D \subseteq C, |D| = k} \exp(-k(a_n - 1)) \]

\[ = \sum_{k=1}^{r} \exp(-k(a_n - 1)) \leq \frac{\exp(-a_n)}{1 - \exp(-a_n)} \to 0, \]

where the final step follows because \( a_n \to \infty \) by Lemma 4. Hence, from [51] we see that \( \mathbb{E}_0[\tilde{L}] \to 1 \).

**Second truncated moment.** Here we show that \( \mathbb{E}_0[\tilde{L}^2] \to 1 \). In other words

\[ \mathbb{E}_0[\tilde{L}^2] = \mathbb{E}^{\otimes 2} \left[ \mathbb{E}_0[L_{C_1}L_{C_2}{\|}\Gamma_{C_1}\|\Gamma_{C_2}] \right] \leq 1 + o(1), \]

where \( \mathbb{E}^{\otimes 2}[] \) denotes expectation with respect to two \( C_1, C_2 \subseteq V \) of size \( |C_1| = |C_2| = r \) chosen independently and uniformly at random. Let \( D = C_1 \cap C_2 \), then using [47] this becomes

\[ \mathbb{E}_0[\tilde{L}^2] = \mathbb{E}^{\otimes 2} \left[ \mathbb{E}_0[L_{C_1}L_{C_2}{\|}\Gamma_{C_1}\|\Gamma_{C_2}] \right] \]

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Using the Cauchy-Schwarz inequality inside the expectation $\tilde{E}^{\otimes 2}[]$, and that the sets $C_1$ and $C_2$ are chosen independently, we obtain

$$E_0[\tilde{L}^2] = \tilde{E}^{\otimes 2} \left[ E_0 \left[ \exp \left( \sum_{i,j \in D} A_{ij} \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) - A_{ij} \left( \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) \right) \right) \right] \right] \ll_{C_1, C_2}^1$$

Next, we split this expectation into two parts based on whether $D \notin C_1$ or $D \in C_1$. Thus we have the partition

$$E_0[\tilde{L}^2] \leq P_1 + P_2,$$

where

$$P_1 := \tilde{E}^{\otimes 2} \left[ \mathbb{1}_{D \notin C_1} E_0 \left[ \exp \left( \sum_{i,j \in D} 2 A_{ij} \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) - 2 A_{ij} \left( \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) \right) \right) \right] \mathbb{1}_{C_1} \right],$$

$$P_2 := \tilde{E}^{\otimes 2} \left[ \mathbb{1}_{D \in C_1} E_0 \left[ \exp \left( \sum_{i,j \in D} 2 A_{ij} \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) - 2 A_{ij} \left( \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) \right) \right) \right] \mathbb{1}_{C_1} \right].$$

Using this split, we first show that $P_1 \leq 1 + o(1)$ and then show that $P_2 \leq o(1)$.

**Part 1:** Here we show that $P_1 \leq 1 + o(1)$. In this part we can simply ignore the truncation events $\Gamma_{C_1}$ and obtain the bound

$$P_1 \leq \tilde{E}^{\otimes 2} \left[ \mathbb{1}_{D \notin C_1} E_0 \left[ \exp \left( \sum_{i,j \in D} 2 A_{ij} \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) - 2 A_{ij} \left( \theta_{ij} \left( \rho_{C_i,p_{ij}} \right) \right) \right) \right] \mathbb{1}_{C_1} \right],$$

$$\leq \tilde{E}^{\otimes 2} \left[ \mathbb{1}_{D \notin C_1} \exp \left( \sum_{i,j \in D} \Delta_{ij}^{(1)} \right) \right],$$

where

$$\Delta_{ij}^{(1)} := \log \left( 1 + \frac{(\rho_{C_i,p_{ij}} - p_{ij})^2}{p_{ij}(1 - p_{ij})} \right).$$
Then using $\log(1 + x) \leq x$ and by Assumption 2 uniformly over all $i, j \in D$,
\[
\Delta_{ij}^{(1)} \leq \log(1 + (1 + o(1))(\rho_{C_i} - 1)^2 p_{ij}) \leq (1 + o(1))(\rho_{C_i} - 1)^2 p_{ij}.
\]
Now, by definition of $E$, it follows that $(1 + o(1))(\rho_{C_i} - 1)^2 \mathbb{E}_0[e(D)] \leq |D|(\log(n^{D}/r^2) - b_n)$ for every $D \notin E$. Therefore
\[
\begin{align*}
P_1 & \leq \mathbb{E} \mathbb{1}_{D \notin E} \exp\left((1 + o(1))(\rho_{C_i} - 1)^2 \mathbb{E}_0[e(D)]\right) \\
& \leq \mathbb{E} \mathbb{1}_{|D| \leq 1} + \mathbb{E} \mathbb{1}_{|D| > 1} \exp\left(|D|(\log(n^{D}/r^2) - b_n)\right) \\
& \leq \mathbb{P}(|D| = 1) + \sum_{k=2}^{r} \exp\left(k(\log(nk/r^2) - b_n)\right) \mathbb{P}(|D| = k) \\
& \leq 1 + \sum_{k=2}^{r} \exp\left(k(\log(nk/r^2) - b_n)\right) \mathbb{P}(|D| = k).
\end{align*}
\]
(53)

Note that $|D| = |C_1 \cap C_2|$ has a hypergeometric distribution under $\mathbb{P}$, hence
\[
\begin{align*}
\mathbb{P}(|D| = k) &= \binom{l}{k} \binom{n-l}{r-k} \binom{n}{r} = \left(1 + o(1)\right) \frac{re - k}{k} \frac{n - r}{n - k} \\
& \leq \exp\left(-k(\log(nk/r^2) + O(1))\right).
\end{align*}
\]
(54)

Plugging this into (53), we obtain
\[
\begin{align*}
P_1 & \leq 1 + \sum_{k=2}^{r} \exp\left(k(\log(nk/r^2) - b_n - \log(nk/r^2) + O(1))\right) \\
& \leq 1 + \sum_{k=2}^{r} \exp(k(O(1) - b_n)) \leq 1 + o(1),
\end{align*}
\]
where the final step follows because $b_n = \log \log(n) \rightarrow \infty$.

**Part 2:** Here we show that $P_2 \leq o(1)$. First, define
\[
\xi := \frac{\log(\zeta_D)}{2 \log(\rho_{C_i})},
\]
(55)

where $\zeta_D$ was defined in Lemma 3. Then, by the same reasoning as in (52)
\[
\begin{align*}
\max_{C_i \subseteq \mathcal{V}, |C_i| = r} \max_{D \in E_{C_i}} \max_{i, j \in D} \left| \frac{\log(\zeta_D)/\log(\rho_{C_i})}{\theta_{ij}(\zeta_D p_{ij})/\theta_{ij}(\rho_{C_i} p_{ij})} - 1 \right| & \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{align*}
\]
(56)

Loosely speaking, this means that, uniformly over $i, j \in D$, we have $\xi \approx \frac{\theta_{ij}(\zeta_D p_{ij})}{\theta_{ij}(\rho_{C_i} p_{ij})} \leq 1$.

By definition of the truncation event $\Gamma_{C_i}$ in (50), for any $D \in E_{C_i}$
\[
\sum_{i < j \in D} A_{ij} \theta_{ij}(\rho_{C_i} p_{ij}) \leq \sum_{i < j \in D} p_{ij} \zeta_D \theta_{ij}(\rho_{C_i} p_{ij}).
\]

Then for $x \in [0, 1]$, we obtain the bound
\[
\begin{align*}
P_2 & = \mathbb{E} \mathbb{1}_{\Gamma_{C_i}} \left[ \mathbb{E}_0 \left[ \exp\left( \sum_{i < j \in D} 2A_{ij} \theta_{ij}(\rho_{C_i} p_{ij}) - 2A_{ij} \left(\theta_{ij}(\rho_{C_i} p_{ij})\right) \right) \mathbb{1}_{D \notin E_{C_i}} \right] \right] \\
& \leq \mathbb{E} \mathbb{1}_{\Gamma_{C_i}} \left[ \mathbb{E}_0 \left[ \exp\left( \sum_{i < j \in D} 2A_{ij} \theta_{ij}(\rho_{C_i} p_{ij}) - 2A_{ij} \zeta_D \theta_{ij}(\rho_{C_i} p_{ij}) \right) \mathbb{1}_{D \notin E_{C_i}} \right] \right].
\end{align*}
\]
\[ \begin{align*}
&\leq \mathbb{E}^{\xi_2}[E_0 \left[ \exp \left( \sum_{i<j\in D} 2\theta_{ij}(\rho C_i p_{ij}) [x A_{ij} + (1-x)\zeta_D p_{ij}] - 2A_{ij}(\theta_{ij}(\rho C_i p_{ij})) \right) \right] \\
&= \mathbb{E}^{\xi_2}[\exp \left( \sum_{i<j\in D} \Lambda_{ij}(2\theta_{ij}(\rho C_i p_{ij}) x) + (2\theta_{ij}(\rho C_i p_{ij}) - 2\theta_{ij}(\rho C_i p_{ij}) x) \zeta_D p_{ij} - 2\Lambda_{ij}(\theta_{ij}(\rho C_i p_{ij})) \right)].
\end{align*}\]

To obtain the best possible bound, we may optimize the above with respect to \( x \). Here it can be seen from (48) that each individual term in the sum is minimal when \( x = \frac{\theta_{ij}(\rho C_i p_{ij})}{\theta_{ij}(\rho C_i p_{ij})} \). Therefore, by (56) it follows that the overall optimum is attained at \( x = (1 + o(1))\xi \), where \( \xi \) was defined in (55). Plugging this in, and using (56), gives

\[ P_2 \leq \mathbb{E}^{\xi_2}[\exp \left( (1 + o(1)) \sum_{i<j\in D} \Delta^{(2)}_{ij} \right)], \]

where

\[ \Delta^{(2)}_{ij} := \left( \Lambda_{ij}(\theta_{ij}(\zeta_D p_{ij})) - \zeta_D p_{ij} \theta_{ij}(\zeta_D p_{ij}) \right) - 2\left( \Lambda_{ij}(\theta_{ij}(\rho C_i p_{ij})) - \zeta_D p_{ij} \theta_{ij}(\rho C_i p_{ij}) \right) \]

\[ = -H_{p_{ij}}(\zeta_D p_{ij}) - 2(H_{\rho C_i p_{ij}}(\zeta_D p_{ij}) - H_{p_{ij}}(\zeta_D p_{ij})) \]

\[ = H_{p_{ij}}(\zeta_D p_{ij}) - 2H_{\rho C_i p_{ij}}(\zeta_D p_{ij}), \tag{57} \]

where we have used (48) in the second equality. To relate the Kullback-Leibler divergence \( H_p(q) \), appearing in (57), to the function \( h(x) \) from (2) we need the following lemma, the proof of which is deferred to the appendix:

**Lemma 5.** For any \( 0 < p < q < 1/2 \) (possibly depending on \( n \)) it follows that,

\[ \left| \frac{H_p(q)}{ph\left(\frac{2}{p-1}\right)} - 1 \right| \leq O(p + q), \]

where \( H_p(q) \) is the Kullback-Leibler divergence between \( \text{Bern}(p) \) and \( \text{Bern}(q) \), and \( h(x) \) is given in (2).

Recall that \( \zeta_D \leq \rho_C^2 \) by Lemma (2) and therefore \( \max_{i,j\in D} p_{ij} \zeta_D = o(1) \) by Assumption (2). Similarly, it follows that \( \max_{i,j\in D} p_{ij} \rho_C = o(1) \) and \( \max_{i,j\in D} p_{ij} = o(1) \). Then, using Lemma 5, we obtain the bounds, uniformly over \( i, j, D \),

\[ \left| \frac{H_{p_{ij}}(p_{ij}\zeta_D)}{p_{ij} h\left(\frac{\zeta_D}{\rho_C} - 1\right)} - 1 \right| \leq \max_{i,j\in D} O(p_{ij}(\zeta_D + 1)) = o(1), \]

\[ \left| \frac{H_{\rho_{ij}\rho_C}(p_{ij}\zeta_D)}{p_{ij} \rho_C h\left(\frac{\zeta_D}{\rho_C} - 1\right)} - 1 \right| \leq \max_{i,j\in D} O(p_{ij}(\zeta_D + \rho_C)) = o(1). \]

Using the uniform bounds above, we can express \( \Delta^{(2)}_{ij} \) from (57) in terms on the function \( h(x) \). This gives, uniformly over \( i, j, D \),

\[ \Delta^{(2)}_{ij} = H_{p_{ij}}(p_{ij}\zeta_D) - 2H_{\rho C_i p_{ij}}(p_{ij}\zeta_D) \]

\[ = (1 + o(1)) \left( p_{ij} h\left(\zeta_D - 1\right) - 2\rho C_i p_{ij} h\left(\frac{\zeta_D}{\rho C_i} - 1\right) \right). \]
Therefore, for \( D \in \mathcal{E}_{C_1} \), we have

\[
\frac{1}{|D|} \sum_{i<j \in D} \Delta_{ij}^{(2)} - \log \left( \frac{n|D|}{r^2} \right) = (1 + o(1)) \frac{1}{|D|} \sum_{i<j \in D} \left[ p_{ij} h \left( \zeta_D - 1 \right) - 2 \rho_{C_1} p_{ij} h \left( \frac{\zeta_D}{\rho_{C_1}} - 1 \right) \right] - \log \left( \frac{n|D|}{r^2} \right)
\]

\[
= (1 + o(1)) \left[ \frac{\mathbb{E}[r(D)]}{|D|} h \left( \zeta_D - 1 \right) - 2 \frac{\mathbb{E}_{C_1}[e(D)]}{|D|} h \left( \frac{\zeta_D}{\rho_{C_1}} - 1 \right) \right]
\]

\[
- \left( \log \left( \frac{n}{|D|} \right) - 2 \log \left( \frac{r}{|D|} \right) \right).
\]

Then, by definition of \( \zeta_D \) in Lemma \ref{lem:technical_results} and \( a_n \) in Lemma \ref{lem:asymptotic}, this becomes

\[
\max_{C_1 \subseteq V, |C_1| = r} \frac{1}{|D|} \sum_{i<j \in D} \Delta_{ij}^{(2)} - \log \left( \frac{n|D|}{r^2} \right)
\]

\[
\leq \max_{C_1 \subseteq V, |C_1| = r} 2 \left( \log \left( \frac{r}{|D|} \right) - (1 + o(1)) \frac{\mathbb{E}_{C_1}[e(D)] h \left( \frac{\zeta_D}{\rho_{C_1}} - 1 \right)}{|D|} \right)
\]

\[
\leq -2a_n - \infty.
\]

Combining the above and grouping the sets \( D \in \mathcal{E}_{C_1} \) by their size \( |D| \), together with \ref{eq:maxlog}, we obtain

\[
P_2 \leq \mathbb{P}^2 \left[ 1_{\{D \in \mathcal{E}_{C_1}\}} \exp \left( \sum_{i<j \in D} \Delta_{ij}^{(2)} \right) \right]
\]

\[
\leq \sum_{k=1}^r \exp \left( k \left( -2a_n + \log \left( \frac{nk}{r^2} \right) \right) \right) \tilde{\mathbb{P}}(|D| = k)
\]

\[
\leq \sum_{k=1}^r \exp \left( k \left( -2a_n + \log \left( \frac{nk}{r^2} \right) - \log \left( \frac{nk}{r^2} \right) + \mathcal{O}(1) \right) \right)
\]

\[
\leq \sum_{k=1}^r \exp (k(-2a_n + \mathcal{O}(1))) \rightarrow 0,
\]

where the final step follows because \( a_n \rightarrow \infty \) by Lemma \ref{lem:asymptotic}. This shows that \( P_2 = o(1) \).

Following our steps, we conclude that \( \mathbb{E}_n[L] \rightarrow 1 \) and \( \mathbb{E}_n[L^2] = P_1 + P_2 \leq 1 + o(1) \), and therefore \( \bar{R}_n(\psi_n^2) \rightarrow 1 \). Finally, the risk of any test \( \psi_n \) is bounded by the average risk of the likelihood ratio test, that is \( R_n(\psi_n) \geq \bar{R}_n(\psi_n^2) \rightarrow 1 \), completing the proof. \( \square \)

### A Proof of technical results

In this section we prove the technical results omitted from the proof of Theorem \ref{thm:main_result}.

#### A.1 Proof of Lemma \ref{lem:technical_results}

Begin by defining \( \tilde{q}_{ij} \) by

\[
\tilde{q}_{ij} p_{ij} = \frac{(\rho_{C} p_{ij})^2}{1 - \rho_{C} p_{ij}} \frac{1 - p_{ij}}{p_{ij}} (1 - \rho_{C} p_{ij})^2,
\]

which implies that \( q_{ij} (\tilde{q}_{ij} p_{ij}) = 2 \theta_{ij} (\rho_{C} p_{ij}) \). By Assumption \ref{assump:psi} we have \( p_{ij} \rightarrow 0 \) and \( \rho_{C} p_{ij} \rightarrow 0 \) for every \( i, j \in V \) and therefore it follows that \( \tilde{q}_{ij} \leq \rho_{C}^2 \).
We show below that \( h(\tilde{q}_{ij} - 1) \geq (2 + o(1))(\rho_C - 1)^2 \) for all \( i, j \in D \) when \( n \) is large enough. Using this and the fact that \( D \in \mathcal{E}_C \) gives

\[
(1 + \varepsilon) \frac{1}{|D|} \mathbb{E}_0[e(D)]h(\tilde{q}_{ij} - 1) \geq (2 + o(1))(1 + \varepsilon)(\rho_C - 1)^2 \mathbb{E}_0[e(D)] \left| \frac{|D|}{n} \right|
\]

\[
\geq (2 + o(1))(1 + \varepsilon)(1 - \varepsilon/2)\left( \log \left( \frac{n|D|}{\varepsilon^2} \right) - b_n \right)
\]

\[
\geq 2\left( \log \left( \frac{n|D|}{\varepsilon^2} \right) - b_n \right)
\]

\[
\geq 2\log\left( \frac{n}{|D|} \right),
\]

where we used \( |D| \leq r \leq n^{o(1)} \) in the last inequality, which follows from Assumption 1. Note that \( h(x - 1) \) is continuous and increasing on \( x \geq 1 \). This means that, for large enough \( n \), there is a unique solution \( \zeta_D \in (1, \min_{i,j \in D} \tilde{q}_{ij}) \) such that

\[
(1 + \varepsilon) \frac{1}{|D|} \mathbb{E}_0[e(D)]h(\zeta_D - 1) = \log\left( \frac{n}{|D|} \right).
\]

Moreover, because \( \zeta_D \in (1, \min_{i,j \in D} \tilde{q}_{ij}) \) it follows that \( \theta_{ij}(\zeta_Dp_{ij}) \leq \theta_{ij}(\tilde{q}_{ij}p_{ij}) = 2\theta_{ij}(\rho_Cp_{ij}) \) for every \( i, j \in D \).

It is left to show \( h(\tilde{q}_{ij} - 1) \geq (2 + o(1))(\rho_C - 1)^2 \), which we do by considering different cases depending on the asymptotic behavior of \( \rho_C \) (which is sufficient by Remark 1).

**Case 1** \( (\rho_C \to 1) \): By definition of \( \tilde{q}_{ij} \),

\[
\tilde{q}_{ij} - 1 = (\rho_C - 1)\left( 1 + \frac{(1 - p_{ij})\rho_C^2}{1 - p_{ij}(2\rho_C - \rho_C^2)} \right) \geq 2(\rho_C - 1).
\]

Then, using the above together with \( h(x - 1) \approx (x - 1)^2/2 \) as \( x \to 1 \), we obtain

\[
h(\tilde{q}_{ij} - 1) = (\tilde{q}_{ij} - 1)^2/2 \approx 2(\rho_C - 1)^2.
\]

**Case 2** \( (\rho_C \to \alpha \in (1, \infty)) \): Using \( \tilde{q}_{ij} \approx \rho_C^2 \), we obtain

\[
\frac{h(\tilde{q}_{ij} - 1)}{(\rho_C - 1)^2} \approx \frac{h(\rho_C^2 - 1)}{(\rho_C - 1)^2} \approx \frac{\rho_C^2 \log(\rho_C^2) - \rho_C + 1}{(\rho_C - 1)^2} \approx 1 + \frac{2\rho_C(\log(\rho_C) - \rho_C + 1)}{(\rho_C - 1)^2} \geq (2 + o(1)).
\]

**Case 3** \( (\rho_C \to \infty) \): Using \( \tilde{q}_{ij} \approx \rho_C^2 \) and \( h(x - 1) \approx x \log(x) \) as \( x \to \infty \), we obtain

\[
\frac{h(\tilde{q}_{ij} - 1)}{(\rho_C - 1)^2} = (1 + o(1))\frac{\tilde{q}_{ij}^2 \log(\tilde{q}_{ij})}{\rho_C^2} \geq (2 + o(1))\log(\rho_C) \to \infty.
\]

In particular, \( h(\tilde{q}_{ij} - 1) \geq 2(\rho_C - 1)^2 \) when \( n \) is large enough. \( \square \)

**A.2 Proof of Lemma 4**

First note that [4] implies

\[
h(\rho_C - 1) \leq (1 - \varepsilon)\frac{|D|\log(n/|D|)}{\mathbb{E}_0[e(D)]},
\]

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and Lemma \[3\] implies
\[
h(\zeta_D - 1) = \frac{1}{1 + \varepsilon} \frac{|D| \log(n/|D|)}{E_0[\varepsilon(D)]}.\]
Therefore,
\[
\frac{h(\zeta_D - 1)}{h(\rho_C - 1)} \geq \frac{1}{1 - \varepsilon^2}.
\] (58)

To prove the lemma we consider three different cases depending on the asymptotic behavior of \(\rho_C\) (any other case is handled as in Remark \[1\]).

Case 1 (\(\rho_C \to 1\)): From the proof of Lemma \[3\] we have \(\zeta_D \in (1, \min_{i, j} \tilde{q}_{ij})\), where \(\tilde{q}_{ij} \approx \rho_C^2 \to 1\), and therefore \(\zeta_D \to 1\). Then using \(h(x - 1) \approx (x - 1)^2/2\) as \(x \to 1\) together with (58), we obtain
\[
\frac{(\zeta_D - 1)^2}{(\rho_C - 1)^2} \geq \frac{h(\zeta_D - 1)}{h(\rho_C - 1)} \geq \frac{1}{1 - \varepsilon^2}.
\]
Using this, we obtain
\[
\rho_C h\left(\frac{\zeta_D}{\rho_C} - 1\right) \geq \frac{(\zeta_D - \rho_C)^2}{2\rho_C}
\]
\[
= \frac{1}{2} (\zeta_D - 1)^2 \left(1 - \frac{\rho_C - 1}{\zeta_D - 1}\right)^2
\]
\[
\geq (1 + o(1)) h(\zeta_D - 1)(1 - \sqrt{1 - \varepsilon^2}) = \Omega(1) h(\zeta_D - 1).
\]
This result, together with Lemma \[3\] yields
\[
\frac{1}{|D|} \mathbb{E}_{C}[\varepsilon(D)] h\left(\frac{\zeta_D}{\rho_C} - 1\right) \geq \Omega(1) \frac{1}{|D|} \mathbb{E}_0[\varepsilon(D)] h(\zeta_D - 1) \geq \Omega(1) \log(n/|D|).
\]
Finally, because Assumption \[1\] ensures that \(r \leq n^{o(1)}\), it follows that
\[
(1 - \varepsilon) \frac{1}{|D|} \mathbb{E}_{C}[\varepsilon(D)] h\left(\frac{\zeta_D}{\rho_C} - 1\right) - \log\left(\frac{r}{|D|}\right) \geq \Omega(1) \log\left(\frac{n}{|D|}\right) - \log\left(\frac{r}{|D|}\right) \to \infty.
\]

Case 2 (\(\rho_C \to \alpha \in (1, \infty)\)): By (58) it clearly follows that \(\rho_C \leq \zeta_D\). Also, \(h(x - 1)\) is convex and has derivative \(\log(x)\). It follows that \(h(x - 1) - h(\rho_C - 1) \leq (x - \rho_C) \log(x)\) for \(x \geq \rho_C\).

Using this,
\[
\log(\zeta_D)(\zeta_D - \rho_C) \geq h(\rho_C - 1) \left(\frac{h(\zeta_D - 1)}{h(\rho_C - 1)} - 1\right) \geq h(\rho_C - 1) \left(1 - \frac{1}{1 - \varepsilon^2} - 1\right).
\]
In particular, this result implies that \(\zeta_D\) is lower bounded away from \(\rho_C\) (i.e., \(\zeta_D \geq \rho_C + \Omega(1)\)).

Now, using that \(h(x) \geq \frac{1}{2} \log(x + 1)\) and therefore
\[
\rho_C h\left(\frac{\zeta_D}{\rho_C} - 1\right) \geq \frac{\rho_C}{2} \log\left(\frac{\zeta_D}{\rho_C} - 1\right) \log\left(\frac{\zeta_D}{\rho_C}\right)
\]
\[
\geq \frac{\zeta_D - \rho_C}{2} (\log(\zeta_D) - \log(\rho_C))
\]
\[
\geq \Omega(1) \frac{\zeta_D - \rho_C}{2} \log(\zeta_D)
\]
\[
\geq \Omega(1) h(\rho_C - 1),
\]
where the last step follows from the fact that \(\zeta_D\) is lower bounded away from \(\rho_C\). To proceed similarly as in case 1, we need to relate \(h(\zeta_D - 1)\) to \(h(\rho_C - 1)\). From the proof of case 2 in

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Lemma 3 it follows that $\zeta_D \leq \tilde{q}_{ij} \simeq \tilde{p}_C^2$, and since $\rho_C$ is bounded away from 1 it follows that $h(\rho_C - 1) / h(\zeta_D - 1) \geq \Omega(1)$. Therefore we conclude that

$$\rho_C h \left( \frac{\zeta_D}{\rho_C} - 1 \right) \geq \Omega(1) h(\zeta_D - 1).$$

From this point onward the proof continues as in case 1.

Case 3 ($\rho_C \to \infty$): We have $\zeta_D \geq \rho_C \to \infty$ and $h(x - 1) \asymp x \log(x)$ as $x \to \infty$. Therefore it follows by (58) that

$$\frac{1}{1 - \varepsilon^2} \leq \frac{h(\zeta_D - 1)}{h(\rho_C - 1)} \asymp \frac{\zeta_D \log(\zeta_D)}{\rho_C \log(\rho_C)} \asymp \frac{\zeta_D}{\rho_C} \left( 1 + \frac{\log(\zeta_D/\rho_C)}{\log(\rho_C)} \right).$$

Hence, $\zeta_D / \rho_C \geq 1 + \Omega(1)$. Further, using $\zeta_D \leq \tilde{q}_{ij} \simeq \tilde{p}_C$, we obtain

$$\frac{\rho_C h(\zeta_D / \rho_C - 1)}{h(\zeta_D - 1)} \asymp \frac{\zeta_D \log(\zeta_D / \rho_C)}{\rho_C \log(\zeta_D)} \asymp \frac{\zeta_D}{\rho_C} \left( 1 + \frac{\log(\zeta_D/\rho_C)}{\log(\rho_C)} \right) + o(1) \geq \Omega(1) \frac{1}{\log(\zeta_D)} \geq \Omega(1) \frac{1}{\log(\rho_C)}.$$ 

Here it was crucial to use the fact that $\zeta_D / \rho_C$ is lower bounded away from 1. Finally, we get

$$(1 - \varepsilon) \frac{1}{|D|} E_C[e(D)] h \left( \frac{\zeta_D}{\rho_C} - 1 \right) - \log \left( \frac{r}{|D|} \right) \geq \Omega(1) \frac{1}{|D|} E_C[e(D)] \frac{h(\zeta_D - 1)}{\log(\rho_C)} - \log \left( \frac{r}{|D|} \right) \geq \Omega(1) \log(n / |D|) - \log \left( \frac{r}{|D|} \right) \geq \Omega(1) \log(n) \left( \frac{1}{\log(\rho_C)} - \frac{\log(r)}{\log(n)} \right) \to \infty,$$

where the final step follows because $\rho_C p_{ij} \leq 1$ for every $C \subseteq V$ and $i, j \in C$, and therefore $\log(r) / \log(n) \leq o(1 / \log(1 / \rho_{ij})) \leq o(1 / \log(\rho_C))$ by Assumption 1.

### A.3 Proof of Lemma 5

Define the function

$$f_p(q) := H_p(q) - ph \left( \frac{q}{p} - 1 \right) = (q - p) + (1 - q) \log \left( \frac{1 - q}{1 - p} \right).$$

Then the derivatives of $f_p(q)$ are given by

$$\frac{\partial f_p(q)}{\partial q} = \log \left( \frac{1 - p}{1 - q} \right), \quad \frac{\partial^2 f_p(q)}{\partial q^2} = \frac{1}{1 - q}, \quad \frac{\partial^3 f_p(q)}{\partial q^3} = \frac{1}{(1 - q)^2}.$$ 

Therefore, for $0 < p < q$, a Taylor expansion of $f_p(q)$ around $p$ shows that there exists $\xi \in [p, q]$ such that

$$f_p(q) = \frac{1}{2(1 - p)} (q - p)^2 + \frac{1}{6(1 - \xi)^2} (q - p)^3.$$

Now, we continue by considering two cases depending of the value of $q/p$.

**Case 1 ($q/p \leq 5$):** Here we use that $h(x - 1) \geq (x - 1)^2 / 4$ for all $1 < x \leq 5$. Therefore,

$$\frac{f_p(q)}{ph \left( \frac{q}{p} - 1 \right)} \leq \frac{4p f_p(q)}{(q - p)^2} = \frac{4p}{(q - p)^2} \left[ \frac{(q - p)^2}{2(1 - p)} + \frac{(q - p)^3}{6(1 - x)^2} \right]$$

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\[ \frac{2p}{1-p} + \frac{2}{3} \frac{q-p}{(1-\xi)^2} \leq O(p) + O(q), \]

for some \( \xi \in (p, q) \).

**Case 2** \((q/p > 5)\): Here we use that \( h(x-1) \geq (x-1) \) for all \( x > 5 \). Therefore,

\[ \frac{f_p(q)}{ph\left(\frac{q}{p} - 1\right)} \leq \frac{f_p(q)}{q-p} = \frac{q-p}{2(1-p)} + \frac{(q-p)^2}{6(1-\xi)^2} \leq O(p) + O(q), \]

for some \( \xi \in (p, q) \).

To complete the proof, note that \( f_p(q) \geq 0 \) and \( ph\left(\frac{q}{p} - 1\right) \geq 0 \) for all \( 0 < p < q < 1 \). Therefore, it follows that

\[ 0 \leq \frac{f_p(q)}{ph\left(\frac{q}{p} - 1\right)} \leq O(p + q). \]

This completes the proof. \( \square \)

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