Random polytopes obtained by matrices with heavy tailed entries.

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Abstract

Let $\Gamma$ be an $N \times n$ random matrix with independent entries and such that in each row entries are i.i.d. Assume also that the entries are symmetric, have unit variances, and satisfy a small ball probabilistic estimate uniformly. We investigate properties of the corresponding random polytope $\Gamma^* B_1^N$ in $\mathbb{R}^n$ (the absolute convex hull of rows of $\Gamma$). In particular, we show that

$$\Gamma^* B_1^N \supset b^{-1} \left( B_\infty^n \cap \sqrt{\ln(N/n)} B_2^n \right),$$

where $b$ depends only on parameters in small ball inequality. This extends results of [18] and recent results of [17]. This inclusion is equivalent to so-called $\ell_1$-quotient property and plays an important role in compressive sensing (see [17] and references therein).

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1 Introduction

In this note, we dealt with a rectangular $N \times n$ random matrices $\Gamma = \{\xi_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}$, where $\xi_{ij}$ are independent symmetric random variables with unit variance satisfying uniform small ball probabilistic estimate. More precisely, in the main theorem we assume that there exist $u, v \in (0, 1)$ such that

$$\forall i, j \sup_{\lambda \in \mathbb{R}} \mathbb{P} \{ |\xi_{ij} - \lambda| \leq u \} \leq v. \quad (1)$$

Of course, if variables have a bounded moment $r > 2$, we will have better estimates. We are interested in geometric parameters of the random polytope generated by $\Gamma$, that is, the absolute convex hull of rows of $\Gamma$. In other words, the random polytope under consideration is $\Gamma^* B_1^N$, where $B_1^N$ is the $N$-dimensional octahedron. Such random polytopes have been extensively studied in the literature, especially in the Gaussian case and in the Bernoulli case. The Gaussian random polytopes in the case when $N$ is proportional to $n$ have many applications in the Asymptotic Geometric Analysis (see e.g., [9] and [30], and the survey
The Bernoulli case corresponds to 0/1 random polytopes. For their combinatorial properties we refer for e.g. to [7, 3] (see also the survey [32]). Their geometric parameters have been studied in [8, 18]. In the compress sensing it was shown that the so-called $\ell_1$-quotient property is responsible for robustness in certain $\ell_1$-minimizations (see [17] and references therein). More precisely, an $n \times N$ (with $N \geq n$) matrix $A$ satisfies $\ell_1$-quotient property with a constant $b$ relative to a norm $\| \cdot \|$ if for every $y \in \mathbb{R}^N$ there exists $x \in \mathbb{R}^N$ such that $Ax = y$ and $\|x\|_1 \leq b\sqrt{n/\ln(eN/n)}\|y\|$, where $\| \cdot \|_1$ denotes the $\ell_1$-norm. It is easy to see that geometrically this means

$$B_{\| \cdot \|} \subset b\sqrt{n/\ln(eN/n)}AB_1^N,$$

where $B_{\| \cdot \|}$ is the unit ball of $\| \cdot \|$. To prove robustness of noise-blind compressed sensing, the authors of [17] dealt with the norm

$$\| \cdot \| = \max\{\| \cdot \|_2, \sqrt{\ln(eN/n)} \cdot \| \cdot \|_\infty\},$$

where $\| \cdot \|_2$ is the standard Euclidean norm and $\| \cdot \|_\infty$ is the $\ell_\infty$-norm. Theorem 5 in [17] states that assuming that entries of $A$ are symmetric i.i.d. random variables with unit variances, and that they have regular (in fact, $\psi_\alpha$) behaviour of all moments till the moment of order $\ln n$, the matrix $A/\sqrt{n}$ has $\ell_1$-quotient property with high probability. Geometrically it means

$$AB_1^N \supset b^{-1}\left(B_\infty^n \cap \sqrt{\ln(N/n)}B_2^n\right). \quad (2)$$

The work [17] complements results of [18], where this inclusion was proved for random matrices with symmetric i.i.d. entries having at least third bounded moment and such that the operator norm of the matrix is bounded with high probability.

The main purpose of this note is to prove such an inclusion with much weaker assumptions on the distribution of the entries. In fact, we require only boundedness of second moments. More precisely, Theorem 4.1 below claims that if the entries $N \times n$ matrix $\Gamma$ are independent symmetric entries with unit variances satisfying condition (1) and such that in each row the entries are i.i.d. then with probability at least $1 - \exp(-cn)$ the inclusion (2) holds for $A = \Gamma^*$, where $b > 0$ depends only on $u$ and $v$ and $c$ is an absolute positive constant. In particular, it shows that “robustness” Theorem 8 in [17] holds under much weaker assumptions on the random matrix. We also investigate other properties of random polytopes $K_N = \Gamma^*B_1^N$ and $K_0^N$, such as behavior of their volumes and mean widths.

Our proof follows the scheme of [18] with a very delicate change – in [18] there was an assumption that the operator norm of $\Gamma$ is bounded by $C\sqrt{N}$ with high probability. However it is known that such a bound does not hold in general unless fourth moments are bounded ([29], see also [20] for quantitative bounds). To avoid using the norm of $\Gamma$, we use ideas appearing in [25], where the authors constructed a certain deterministic $\varepsilon$-net (in $\ell_2$-metric) $\mathcal{N}$ such that $A\mathcal{N}$ is a good net for $AB_2^N$ for most realizations of a square random matrix $A$. We extend their construction in three directions. First, we work with rectangular random matrices, not only square matrices. Second, we need a net for the image of a given convex body (not only for the image of the unit Euclidean ball).
Finally, instead of approximation in the Euclidean norm only, we use approximation in the following norm

\[ \|a\|_{k,2} = \left( \sum_{i=1}^{k} (a_i^*)^2 \right)^{1/2}, \tag{3} \]

where \( 1 \leq k \leq N \) and \( a_1^* \geq a_2^* \geq \ldots \geq a_m^* \) is the decreasing rearrangement of the sequence of numbers \(|a_1|, \ldots, |a_m|\). This norm appears naturally and plays a crucial role in our proof of inclusion (2). The generalization of the net from [25] is a new key ingredient, see Theorem 3.1. We would like to emphasize, that norms \( \| \cdot \|_{k,2} \) played an important role in proofs of many results of Asymptotic Geometric Analysis, see e.g. [11, 13, 14]. For the systematic studies of norms \( \| \cdot \|_{k,2} \) and their unit balls we refer to [12]. We believe that the new approximation in \( \| \cdot \|_{k,2} \) norms will find other applications in the theory. In the last section we present one more application of Theorem 3.1 – we show that it can be used to estimate the smallest singular value of a tall random matrix – see the discussion at the beginning of Section 5.

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## 2 Some notations

By \( \langle \cdot, \cdot \rangle \) we denote the canonical inner product on the \( m \)-dimensional real space \( \mathbb{R}^m \) and for \( 1 \leq p \leq \infty \), the \( \ell_p \)-norm is defined for any \( a \in \mathbb{R}^m \) by

\[ \|a\|_p = \left( \sum_{i=1}^{m} |a_i|^p \right)^{1/p} \quad \text{for} \quad p < \infty \quad \text{and} \quad \|a\|_\infty = \sup_{i=1, \ldots, m} |a_i|. \]

As usual, \( \ell^m_p = (\mathbb{R}^m, \| \cdot \|_p) \), and the unit ball of \( \ell^m_p \) is denoted by \( B^m_p \). The unit sphere of \( \ell^m_2 \) is denoted by \( S^{m-1} \), and the canonical basis of \( \ell^m_2 \) is denoted by \( e_1, \ldots, e_m \). Given an integer \( k \in \{1, \ldots, m\} \), we denote by \( X_{k,2} \) the normed space \( \mathbb{R}^m \) equipped with the norm \( \| \cdot \|_{k,2} \) defined by (3). The unit ball of \( X_{k,2} \) is denoted by \( B_{k,2} \). Note that \( k = N \), \( \|a\|_{k,2} = \|a\|_2 \) and that for any \( 1 \leq k \leq N \), for any vector \( a \in \mathbb{R}^N \),

\[ \|a\|_{k,2} \leq \|a\|_2 \leq \sqrt{\frac{N}{k}} \|a\|_{k,2} \quad \text{or, equivalently,} \quad B^N_2 \subseteq B_{k,2} \subseteq \sqrt{\frac{N}{k}} B^N_2. \]

Given integers \( \ell \geq k \geq 1 \), we denote \( [k] = \{1, 2, \ldots, k\} \) and \( [k, \ell] = \{k, k+1, \ldots, \ell\} \). Given a number \( a \) we denote the largest integer not exceeding \( a \) by \([a]\) and the smallest integer larger than or equal to \( a \) by \([a]\).

Given points \( x_1, \ldots, x_k \) in \( \mathbb{R}^m \) we denote their convex hull by \( \text{conv} \{x_i\}_{i \leq k} \) and their absolute convex hull by \( \text{abs} \text{conv} \{x_i\}_{i \leq k} = \text{conv} \{\pm x_i\}_{i \leq k} \). Given \( \sigma \subset [m] \) by \( P_{\sigma} \) we denote the coordinate projection onto \( \mathbb{R}^\sigma = \{x \in \mathbb{R}^m \mid x_i = 0 \text{ for } i \notin \sigma \} \).
Given a finite set \( E \) we denote its cardinality by \(|E|\). We also use \(|K|\) for the volume of a body \( K \subset \mathbb{R}^m \) (and, more generally, for the \( m \)-dimensional Lebesgue measure of a measurable subset of \( \mathbb{R}^m \)). Let \( K \) be a symmetric convex body with non empty interior. We denote its Minkowski’s functional by \( \|x\|_K \). The support function of \( K \) is \( h_K(x) = \sup_{y \in K} \langle x, y \rangle \), the polar of \( K \) is
\[
K^0 = \{ x \in \mathbb{R}^m \mid \langle x, y \rangle \leq 1 \text{ for every } y \in K \}.
\]
Note that \( h_K(\cdot) = \| \cdot \|_{K^0} \).

Given a set \( L \subset \mathbb{R}^m \), a convex body \( K \subset \mathbb{R}^m \), and \( \epsilon > 0 \) we say that a subset \( \mathcal{N} \subset \mathbb{R}^m \) is an \( \epsilon \)-net of \( L \) with respect to \( K \) if
\[
\mathcal{N} \subset L \subset \bigcup_{x \in \mathcal{N}} (x + \epsilon K).
\]
The cardinality of the smallest \( \epsilon \)-net of \( L \) with respect to \( K \) we denote by \( N(L, \epsilon K) \).

For a given probability space, we denote by \( P(\cdot) \) and \( E \) the probability of an event and the expectation respectively. A \( \pm 1 \) random variable taking values 1 and \(-1\) with probability \( 1/2 \) is called a Rademacher random variable.

In this paper we are interested in rectangular \( N \times n \) matrices \( \Gamma = \{ \xi_{ij} \}_{1 \leq i \leq N}^{1 \leq j \leq n} \), with \( N \geq n \), where the entries are real-valued random variables on some probability space \((\Omega, A, P)\). We will mainly consider the following model of matrix \( \Gamma \):
\[
\begin{align*}
&\forall i, j, \xi_{ij} \text{ are independent, symmetric and } E\xi_{ij}^2 = 1, \\
&\text{in each row, the entries are identically distributed.}
\end{align*}
\]
(4)

At the beginning of Section 4, we will also assume that the entries of \( \Gamma \) satisfy a uniform small ball estimate. If \( \xi_{ij} \sim \mathcal{N}(0, 1) \) are independent Gaussian random variables we say that \( \Gamma \) is a Gaussian random matrix and denote it by \( G \).

### 3 Construction of a good deterministic net.

In this section we present a key result of this paper. Let \( T \) be a subset of \( \mathbb{R}^n \), we aim at constructing a deterministic net such that for every general random operators \( \Gamma : \mathbb{R}^n \to \mathbb{R}^N \), with overwhelming probability, the image of the net by the random operator \( \Gamma \) is a good approximation of \( \Gamma T \). We show that we can quantify this approximation by almost any norm \( \| \cdot \|_{k,2} \) defined in (3). For any integers \( 1 \leq n \leq N \) and \( 0 \leq \delta \leq 1 \), let
\[
F(\delta, n, N) = \begin{cases} (32\delta N/n)^n & \text{if } \delta \geq n/(2N), \\
(en/(\delta N))^{\delta N} & \text{if } \delta \leq n/(2N). \end{cases}
\]
(5)

**Theorem 3.1.** Let \( n \in [N] \), \( 0 \leq \delta \leq 1 \), \( 0 < \epsilon \leq 1 \). Let \( k \in [N] \) such that \( k \ln(\epsilon N/k) \geq n \).

Let \( T \) be a non-empty subset of \( \mathbb{R}^n \) and denote \( M := N(T, \epsilon B_n^\infty) \). There exists a set \( \mathcal{N} \subset T \) and a collection of parallelepipeds \( \mathcal{P} \) in \( \mathbb{R}^n \) such that
\[
\max\{|\mathcal{N}|, |\mathcal{P}|\} \leq M F(\delta, n, N)e^{\delta N}.
\]
Moreover, for any random matrix $\Gamma$ satisfying assumption (4), with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$, we have

$$\forall x \in T \exists y \in \mathcal{N} \text{ such that } \| \Gamma(x - y) \|_{k,2} \leq C\varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)}$$

$$\forall x \in T \exists P \in \mathcal{P} \text{ such that } x \in P \text{ and } \Gamma P \subset \Gamma x + C\varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \mathcal{B}_{k,2}$$

where $C$ is a positive absolute constant.

**Remark 3.2.** This result extends Theorem A and Corollary A from [25], where they considered the case of square matrices, $T = S^{n-1}$ and $k = N$, which corresponds to the approximation of $\Gamma x$ in the Euclidean norm.

### 3.1 Basic facts about covering numbers and operator norms of random matrices.

We begin by recalling some classical estimates for covering numbers that will be used later. It is well known that for any two centrally symmetric bodies $K$ and $L$ in $\mathbb{R}^m$ and any $\varepsilon > 0$ there exists an $\varepsilon$-net $\mathcal{N}$ of $L$ with respect to $K$ with cardinality

$$|\mathcal{N}| \leq \left\lceil \frac{(2/\varepsilon)L + K}{|K|} \right\rceil$$

(see e.g. Lemma 4.16 in [24]). In particular, if $K = L$ is a centrally symmetric bodies in $\mathbb{R}^m$ (or if $L$ is the boundary of a centrally symmetric body $K$) then $|\mathcal{N}| \leq (1 + 2/\varepsilon)^m$.

**Lemma 3.3.** a) For every $\varepsilon \in (0, 1/\sqrt{m}]$

$$N(B_2^m, \varepsilon B_\infty^m) \leq (7/(\varepsilon \sqrt{m}))^m$$

and for every $\varepsilon \in (1/\sqrt{m}, 1]$

$$N(B_2^m, \varepsilon B_\infty^m) \leq (17\varepsilon^2 m)^{1/2}.$$

b) For $J \subset [m]$, let $S^J = \{ x \in \mathbb{R}^J \mid \|x\|_2 = 1 \}$. For every $\varepsilon \in (0, 1)$, for every integer $k \leq m$, there exists a finite set $\mathcal{N} \subset \bigcup_{|J|=k} S^J$ such that

$$|\mathcal{N}| \leq e^{k \ln(3/\varepsilon) + k \ln(em/k)} \forall J \subset [m] \text{ with } |J| = k, \forall y \in S^J, \exists z \in \mathcal{N} \cap S^J, \text{ such that } \|y - z\|_2 \leq \varepsilon \quad . (7)$$

**Proof.** a) Note that for every $m \geq 1$ one has $(1/\sqrt{m})B_\infty^m \subset B_2^m$ and $|B_2^m| \leq (2\pi e/m)^{m/2}$. Therefore, by (6), we obtain for every $\varepsilon \leq 1/\sqrt{m}$

$$N(B^m_2, \varepsilon B_\infty^m) \leq \frac{2B_2^m + B_\infty^m}{|B_\infty^m|} \leq \left( \frac{3}{\varepsilon} \right)^m \frac{|B_2^m|}{|B_\infty^m|} \leq \left( \frac{3\sqrt{\pi e}}{\varepsilon \sqrt{2m}} \right)^m.$$
This implies the first bound. For the second bound note that for every $x \in B^n_2$ the number of coordinates of $x$ larger than $\varepsilon$ is at most $1/\varepsilon^2$. Thus every $x \in B^n_2$ can be presented as $x = y + z$, where the cardinality of support of $y$ is at most $1/\varepsilon^2$, $z \in \varepsilon B^n_\infty$, and supports of $y$ and $z$ are mutually disjoint. Therefore, it is enough to cover $B^n_2$ by $\varepsilon B^n_\sigma$ for all $\sigma \subset [n]$ with $|\sigma| = m := \lceil 1/\varepsilon^2 \rceil$. Using the above bound we obtain

$$N(B_2^n, \varepsilon B^n_\infty) \leq \left( \frac{n}{m} \right) \left( \frac{3\sqrt{\pi}e}{\varepsilon \sqrt{2m}} \right)^m \leq \left( \frac{3en\sqrt{\pi}e}{\varepsilon m\sqrt{2m}} \right)^m,$$

which implies the desired result as $m \leq 1/\varepsilon^2$.

b) Fix $\varepsilon \in (0,1)$. For any fixed $J \subset [m]$ of cardinality $k$, we cover $S^J$ by an $\varepsilon$-net (of points in $S^J$) of cardinality at most $(1 + 2/\varepsilon)^k \leq (3/\varepsilon)^k$ and we take the union of these nets over all sets $J$ of cardinality $k$. We conclude using that $\binom{m}{k} \leq (em/k)^k$.

The next lemma is a classical consequence of estimates for covering numbers for evaluating operator norms of random matrices.

**Lemma 3.4.** Let $B = \{b_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}$ be a fixed $N \times n$ matrix. Let $k \in [N]$ such that $k \ln \frac{eN}{k} \geq n$. Let $\varepsilon_{ij}$ be i.i.d. Rademacher random variables. Denote $B_\varepsilon = \{\varepsilon_{ij}b_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}$. Then for every $t \geq 1$ one has

$$\mathbb{P} \left( \|B_\varepsilon : \ell^n_\infty \to X_{k,2}\| \geq 6t \sqrt{k \ln \left( \frac{eN}{k} \right)} \max_{i \leq N} \|R_i(B)\|_2 \right) \leq e^{-t^2 k \ln (eN/k)},$$

where $R_i(B)$, $i \leq N$, are the rows of $B$.

**Proof.** Observe that for any $a \in \mathbb{R}^N$, we have

$$\|a\|_{k,2} = \sup_{J \subset [N]} \sup_{b \in S^J} \sum_{i=1}^N a_i b_i.$$

Given $x \in \{\pm 1\}^n$, $y \in S^{N-1}$, consider the following random variable,

$$\xi_{x,y} = \sum_{j=1}^n \sum_{i=1}^N \varepsilon_{ij} b_{ij} x_j y_i.$$

Since $e^x + e^{-x} \leq 2 \exp(x^2/2)$ for every real $x$, we observe for $\lambda > 0$,

$$\mathbb{E} \exp \left( \lambda \sum_{j=1}^n \sum_{i=1}^N \varepsilon_{ij} b_{ij} x_j y_i \right) = \prod_{j=1}^n \prod_{i=1}^N \mathbb{E} \exp (\lambda \varepsilon_{ij} b_{ij} x_j y_i) \leq \exp \left( \frac{\lambda^2}{2} \sum_{i=1}^N y_i^2 \|R_i(B)\|_2^2 \right) \leq \exp \left( \frac{\lambda^2}{2} \max \|R_i(B)\|_2^2 \right).$$

Therefore, using the Laplace transform of $\xi_{x,y}$, we deduce that for any $u > 0$,

$$\mathbb{P} (\xi_{x,y} > u \max \|R_i(B)\|_2) \leq e^{-u^2/2}.$$
Note that
\[ \|B_{\varepsilon} : \ell_{\infty}^n \to X_{k,2}\| = \sup_{x \in \{\pm 1\}^n} \sup_{J \subseteq [N]} \sup_{|J|=k} \sup_{y \in S^J} \xi_{x,y}. \tag{8} \]

Now we apply the classical net argument. Let \( \mathcal{N} \) be the net defined by (7) with \( \varepsilon = 1/2 \). Then
\[
\mathbb{P} \left( \sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z} \geq u \max \|R_i(B)\|_2 \right) \leq 2^n |\mathcal{N}| e^{-u^2/2} \\
\leq 2^n \exp \left( -\frac{u^2}{2} + k \ln(6) + k \ln(eN/k) \right).
\]

Taking \( u = 3t \sqrt{k \ln(eN/k)} \) and using \( k \ln(eN/k) \geq n \), we get for every \( t \geq 1 \),
\[
\mathbb{P} \left( \sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z} \geq 3t \sqrt{k \ln(eN/k)} \max \|R_i(B)\|_2 \right) \leq e^{-t^2 k \ln(eN/k)}.
\]

By definition of \( \mathcal{N} \), for any \( J \subseteq [N] \) of cardinality \( k \) and \( y \in S^J \), there exists \( z \in \mathcal{N} \cap S^J \) such that \( \|z - y\|_2 \leq 1/2 \), hence, by the triangle inequality,
\[
\sup_{x \in \{\pm 1\}^n} \sup_{J \subseteq [N]} \sup_{y \in S^J} \xi_{x,y} \leq 2 \sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z}.
\]

This completes the proof of the lemma. \( \square \)

### 3.2 Auxiliary statements

By \( D_n \) we denote the set of all \( n \times n \) diagonal matrices whose diagonal entries belong to the set \( \{1\} \cup \{2^{-k}\}_{k \geq 0} \). The following theorem was proved in [25] in the square case. However the proof works as well in the rectangular case. One just needs to repeat the proof of Proposition 2.7 there for \( N \times n \) matrices, to combine it with Remark 2.8 following the proposition, and to substitute the upper bound on the expectation with a probability bound using Markov’s inequality.

**Theorem 3.5.** Let \( \Gamma = \{\xi_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n} \) be an \( N \times n \) random matrix on a probability space \( \Omega \). Assume that entries of \( \Gamma \) are independent centered random variables with unit variances and that in each row the entries are identically distributed. Let \( \delta \in (0, 1] \). Then there exists a random matrix \( D_\Gamma \) on \( \Omega \) taking values in \( D_n \) such that

(i) for every \( \omega \in \Omega \), \( D_\Gamma(\omega) \) depends only on the realization \( \{|\xi_{ij}(\omega)|\}_{1 \leq i \leq N, 1 \leq j \leq n} \),

(ii) for every \( \omega \in \Omega \) one has
\[ \|R_i(\Gamma(\omega)D_\Gamma(\omega))\|_2 \leq C \sqrt{n/\delta}, \]

(iii) \[ \mathbb{P} \left( \det D_\Gamma \leq e^{-4\delta N} \right) \leq e^{-\delta N}, \]

where \( C \) is an absolute positive constant.
As in [25], Theorem 3.5 has important consequences. It allows to construct, with high probability, a diagonal matrix \(D\) such that the volume of \(DB_n^\infty\) remains big enough and such that, according to Lemma 3.4, we have a good control of the operator norm of \(\Gamma D\) from \(\ell_n^\infty\) to \(X_{k,2}\). Comparing to [25], Lemma 3.4 simplifies significantly the proof and allows to extend Theorem 3.1 from [25] to the case of rectangular matrices and to approximations with respect to \(\|\cdot\|_{k,2}\) norms.

**Theorem 3.6.** Let \(1 \leq n \leq N\) be integers, \(\delta \in (0, 1]\). Let \(k \in [N]\) such that \(k \ln \frac{eN}{k} \geq n\). Let \(\Gamma\) be an \(N \times n\) random matrix satisfying the hypothesis (4). Then

\[
\mathbb{P} \left( \exists D \in D_n \mid \det D \geq e^{-\delta N} \text{ and } \|\Gamma D : \ell_n^\infty \to X_{k,2}\| \leq C \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \right) \geq 1 - e^{-\delta N/4} - e^{-k\ln(eN/k)},
\]

where \(C\) is a positive absolute constant.

**Proof.** Let \(D_\Gamma\) be the matrix given by Theorem 3.5. By property (iii) of \(D_\Gamma\) it is enough to prove that

\[
\mathbb{P} \left( \|\Gamma D : \ell_n^\infty \to X_{k,2}\| \leq C \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \right) \geq 1 - e^{-k\ln(eN/k)},
\]

Consider two probability spaces – the original \((\Omega, \mathbb{P}_\omega)\), where the matrix \(\Gamma\) is defined and the auxiliary space \((E, \mathbb{P}_\varepsilon)\), where \(E := \{-1, 1\}^{N \times n}\) and \(\mathbb{P}_\varepsilon\) is the uniform probability on \(E\). Given a matrix \(A = \{a_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}\) and \(\varepsilon \in E\), denote \(A_\varepsilon = \{\varepsilon_{ij}a_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}\). Since entries of \(\Gamma\) are symmetric, for every fixed \(\varepsilon \in E\) the matrix \(\Gamma_\varepsilon\) has the same distribution on \(\Omega\) as \(\Gamma\). By property (i) of \(D_\Gamma\), we have \(D_\Gamma = D_{\Gamma_\varepsilon}\) for every fixed \(\varepsilon \in E\). Therefore, since \(D_\Gamma\) is diagonal, we have for every \(\varepsilon \in E\)

\[(\Gamma D_\Gamma)_\varepsilon = \Gamma_\varepsilon D_\Gamma = \Gamma_\varepsilon D_{\Gamma_\varepsilon}.
\]

Then, by property (ii) of \(D_\Gamma\) from Theorem 3.5, there exists an absolute positive constant \(C_1\) such that for every \(i \leq N\) and every \((\omega, \varepsilon) \in \Omega \times E\),

\[
\|R_i ((\Gamma(\omega)D_\Gamma(\omega))_\varepsilon)\|_2 \leq C_1 \sqrt{n/\delta}
\]

Fixing \(\omega \in \Omega\) and applying Lemma 3.4 to the matrix \(B = \Gamma(\omega)D_\Gamma(\omega)\), we obtain that for every fixed \(\omega \in \Omega\) one has

\[
\mathbb{P}_\varepsilon \left( \|\Gamma_\varepsilon(\omega)D_\Gamma(\omega) : \ell_n^\infty \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \right) \leq e^{-k\ln(eN/k)}.
\]

Using that \(\Gamma_\varepsilon\) has the same distribution as \(\Gamma\) and Fubini’s theorem, we obtain

\[
\mathbb{P}_\omega \left( \|\Gamma D_\Gamma : \ell_n^\infty \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \right) = \mathbb{P}_\varepsilon \mathbb{P}_\omega \left( \|\Gamma_\varepsilon(\omega)D_\Gamma(\omega) : \ell_n^\infty \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} \right) \leq e^{-k\ln(eN/k)}.
\]
As in Lemma 3.11 from [25], we need to compute the cardinality of diagonal matrices in $\mathcal{D}_n$ such that their determinant are lower bounded.

**Lemma 3.7.** Let $n, N \geq 1$ be integers, $\delta \in (0, 1]$ and 

$$Q := \{D \in \mathcal{D}_n \mid \det D \geq \exp(-\delta N)\}$$

Then

$$|Q| \leq F(\delta, n, N) := \begin{cases} (32\delta N/n)^n & \text{if } \delta \geq n/(2N), \\ \left(\frac{en}{\delta N}\right)^{4\delta N} & \text{if } \delta \leq n/(2N), \end{cases}$$

**Proof.** Note that if $D \in \mathcal{D}_n$ and $d_1, \ldots, d_n$ its diagonal elements then for every $k \geq 0$ the set

$$Q_D(k) = \{i \leq n \mid d_i = 2^{-k}\}$$

has cardinality at most $m_k := \min\{n, |2^{-k}2\delta N|\}$. Thus there are at most

$$\sum_{\ell=0}^{m_k} \binom{n}{\ell} \leq \left(\frac{en}{m_k}\right)^{m_k}.$$

choices of $\sigma_k \subset [n]$, where matrices from $\mathcal{D}_n$ may have such coordinates. Note also that the trivial bound for the number of subsets is $2^n$. Denote $a := 4\delta N/n$. Note that $m_k \leq n/2$ if and only if $2^k \geq a$.

**Case 1.** $a \geq 2$. Set $m := \lfloor \log_2 a \rfloor \geq 1$. By above we have

$$|Q| \leq \prod_{k<m} 2^n \prod_{k\geq m} \left(\frac{en}{m_k}\right)^{m_k} \leq 2^n \prod_{k\geq m} \left(\frac{e}{2^\delta N}\right)^{2\delta N/2k} \prod_{k\geq m} 2^{2k\delta N/2k} \leq a^n \left(\frac{2e}{a}\right)^{4\delta N/a} 2^{2\delta N(2m+1)/2m} \leq (2e)^n a^{4\delta N/2m} 2^{4\delta N/2m} \leq (8a)^n.$$

**Case 2.** $a \leq 2$. Similarly we have

$$|Q| \leq \prod_{k\geq 0} \left(\frac{en}{m_k}\right)^{m_k} \leq \prod_{k\geq 0} \left(\frac{en}{2^\delta N}\right)^{2\delta N/2k} \prod_{k\geq 0} 2^{2k\delta N/2k} \leq \left(\frac{en}{2^\delta N}\right)^{4\delta N} 2^{3\delta N},$$

which implies the desired result. \hfill \Box

### 3.3 Proof of Theorem 3.1

Let $Q$ be as in Lemma 3.7. Note that every $D \in Q$ is diagonal with reciprocal of integers on the diagonal. Therefore, we can define $\mathcal{N}_D \subset T$ of cardinality

$$|\mathcal{N}_D| \leq N(T, \varepsilon DB^\infty_n) \leq N(T, \varepsilon B^\infty_n) N(B^\infty_n, DB^\infty_n) \leq M \det D^{-1} \leq M e^{\delta N}$$
which satisfies that for any \( x \in T \), there exists \( y \in \mathcal{N}_D \) such that \( x - y \in \varepsilon DB^n_{\infty} \). Let
\[
P = \{ y + \varepsilon DB^n_{\infty} \mid D \in Q, y \in \mathcal{N}_D \}.
\]
Thus, by Lemma 3.7, \(|P| \leq M e^{\delta N} F(\delta, n, N)\) and for any \( x \in T \) and for any \( D \in Q \) there exists \( P = y_{x,D} + \varepsilon DB^n_{\infty} \in P \) such that \( x \in P \).

Theorem 3.6 implies that with probability at least \( 1 - e^{-k \ln(eN/k)} - e^{-\delta N/4} \) there exists \( D \in Q \) such that
\[
\Gamma(\varepsilon DB^n_{\infty}) \subset C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2}.
\]
Therefore, for such \( D \),
\[
\Gamma(x - y_{x,D}) \subset \Gamma(\varepsilon DB^n_{\infty}) \subset C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2},
\]
hence,
\[
\Gamma(P) \subset \Gamma x + \Gamma(y_{x,D} - x) + \Gamma(\varepsilon DB^n_{\infty}) \subset \Gamma x + 2C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2}.
\]
This proves the existence of a "good" collection \( P \).

Finally, let \( P' \) be the set of all \( P \in P \) such that \( P \cap T \neq \emptyset \). For every \( P \in P' \) choose an arbitrary \( z_P \in P \cap T \) and let \( \mathcal{N} = \{ z_P \}_{P \in P'} \). By above, for every \( x \in T \) there exists \( D \in Q \) and \( P = y_{x,D} + \varepsilon DB^n_{\infty} \in P \) such that \( x \in P \), in particular \( P \in P' \), and
\[
\Gamma(P) \subset \Gamma x + 2C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2}.
\]
Thus, \( \Gamma z_P \in \Gamma x + 2C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2} \). This implies the desired result.

\[\square\]

**Remark 3.8.** We apply Theorem 3.1 for \( T \subset tB^{n}_2 \), \( t \geq 1 \), \( \varepsilon \leq 1/\sqrt{n} \), and \( \delta \geq n/(2N) \) so that, \( F(\delta, n, N) = (32\delta N/n)^n \). Then Theorem 3.1 combined with Lemma 3.3 implies that there exists \( \mathcal{N} \subset T \) with cardinality at most
\[
\left( \frac{224\delta t N}{\varepsilon n^{3/2}} \right)^n e^{\delta N}
\]
such that with probability at least \( 1 - e^{-k \ln(eN/k)} - e^{-\delta N/4} \) one has
\[
\forall x \in T \; \exists y \in \mathcal{N} \; \text{such that} \; \Gamma(x - y) \in C \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)} B_{k,2}.
\]
4 Geometry of Random Polytopes

In this section, we study some classical geometric parameters associated to random polytopes of the form $K_N := \Gamma^* B_1^N$, where $\Gamma = \{\xi_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}$ is an $N \times n$ random matrix. In other words, $K_N$ is the absolute convex hull of the rows of $\Gamma$. We provide estimates on the asymptotic behavior of the volume and the mean widths of $K_N$ and its polar. In this section, the random operator $\Gamma$ satisfies the hypothesis (4): the random variables $\xi_{ij}$ are independent symmetric with unit variances such that in each row of $\Gamma$ the entries are identically distributed. Moreover we assume that the random variables $\xi_{ij}$ satisfy a uniform small ball probability condition which means that we can fix $u, v \in (0, 1)$ such that

$$\forall i, j \sup_{\lambda \in \mathbb{R}} \mathbb{P}\{|\xi_{ij} - \lambda| \leq u\} \leq v.$$ 

4.1 Inclusion Theorem

We begin by showing that for a random $N \times n$ random matrix $\Gamma$ satisfying conditions described above, the body $K_N = \Gamma^* B_1^N$ contains a large “regular” body with high probability.

**Theorem 4.1.** Let $\beta \in (0, 1)$. There are two positive constants $M = M(u, v, \beta)$ and $C(u, v, \beta)$ which depend only on $u, v, \beta$, such that the following holds. For every positive integers $n, N$ satisfying $N \geq Mn$ one has

$$\mathbb{P}\left( K_N \supset C(u, v, \beta) \left( B_\infty^n \cap \sqrt{\ln(N/n)} B_2^n \right) \right) \geq 1 - 4 \exp\left(-cn^\beta N^{1-\beta}\right),$$

where $c$ is an absolute positive constant.

**Remark 4.2.** It is known that for a Gaussian random matrix one has

$$\mathbb{P}\left( K_N \supset C' \sqrt{\beta \ln(N/n)} B_2^n \right) \geq 1 - 3 \exp\left(-cn^\beta N^{1-\beta}\right),$$

where $C, c$ are absolute positive constants ([10]). Moreover, the probability estimate cannot be improved. Indeed, for a Gaussian random matrix and $\beta \in (0, c'')$ one has

$$\mathbb{P}\left( K_N \supset C'' \sqrt{\beta \ln(N/n)} B_2^n \right) \leq 1 - \exp\left(-c'n^\beta N^{1-\beta}\right),$$

where $C', c' > 0$ and $0 < c'' \leq 1$ are absolute constants.

Since $B_\infty^n \subset \sqrt{n} B_2^n$, Theorem 4.1 has the following consequence.

**Corollary 4.3.** Under the assumptions and notations of Theorem 4.1, for $Mn < N \leq e^n$ one has

$$\mathbb{P}\left( K_N \supset C(u, v, \beta) \sqrt{\ln(N/n)} B_\infty^n \right) \geq 1 - 4 \exp\left(-cn^\beta N^{1-\beta}\right).$$
In fact, our proof of Theorem 4.1 gives that if

$$N \geq n \max \left\{ \exp(4C_v/\beta), \left( \frac{C \ln(e/(1-\beta))}{c_{uv} (1-\beta)} \right)^{1/(1-\beta)} \right\},$$

where $C > 1$ is an absolute positive constant, $c_{uv}$ is a constant from Lemma 4.4, and $C_v = 5 \ln(2/(1-v))$, then

$$P \left( K_N \supset \frac{c_{uv}}{2\sqrt{2}} (B_\infty^n \cap RB_2^n) \right) \geq 1 - 4 \exp \left( - \frac{n^\beta N^{1-\beta}}{40} \right) \tag{9}$$

with $R = \sqrt{\beta \ln(N/n)/C_v}$. Note that $K_N = \text{abs conv}\{x_j\}_{j \leq N}$ where $x_j = \Gamma^* e_j$ are the columns of $\Gamma^*$. Hence for every $z \in \mathbb{R}^n$,

$$h_{K_N}(z) = \sup_{j \leq N} |\langle z, x_j \rangle| = \|\Gamma z\|_\infty.$$

Let $L = c_{uv}(B_\infty^n \cap RB_2^n)$, then, to prove (9), our goal is to show that

$$P \left( \exists z \in \partial L^0 \mid \|\Gamma z\|_\infty < \frac{1}{4} \right) \leq 4 \exp \left( - \frac{n^\beta N^{1-\beta}}{40} \right). \tag{10}$$

The proof of this statement will be divided into two steps. First we will show an individual estimate for any fixed $z \in \partial L^0$. In a second step, we will use the net introduced in Theorem 3.5 to get a global estimate for any point of this net, using that the net is a subset of $\partial L^0$. A crucial point is that this net is a good covering of $\Gamma(\partial L^0)$.

4.1.1 Basic facts about small ball probabilities.

Recall that for a (real) random variable $\xi$ its Lévy concentration function $Q(\xi, \cdot)$ is defined on $(0, \infty)$ as

$$Q(\xi, t) := \sup_{\lambda \in \mathbb{R}} \mathbb{P}\{ |X - \lambda| \leq t \}.$$

For any centered random variable with unit variance, there exist $u, v \in (0, 1)$ such that

$$Q(\xi, u) \leq v. \tag{11}$$

The following lemma is a consequence of a theorem of Rogozin [26] that was used for example in [25] (see Lemma 4.7 there).

**Lemma 4.4.** Let $\xi_1, \ldots, \xi_m$ be independent random variables satisfying (11) with the same $u, v \in (0, 1)$. Then for every $x \in S^{m-1}$ one has

$$Q \left( \sum_{i=1}^m x_i \xi_i, c_{uv} \right) \leq v,$$

where $c_{uv} = cu^2 \sqrt{1-v}$ and $c \in (0, 1]$ is an absolute constant.
Remark 4.5. If we have a bounded moment of order larger than 2, then we could use a consequence of the Paley-Zygmund inequality, which also provides a lower bound on the small ball probability of a random sum. The following statement was proved in [19, Lemma 3.1] following the lines of [18, Lemma 3.6] with appropriate modifications to deal with centered random variables (rather than symmetric):

Let \( 2 < r \leq 3 \) and \( \mu \geq 1 \). Suppose \( \xi_1, \ldots, \xi_m \) are independent centered random variables such that \( \mathbb{E}|\xi|^2 \geq 1 \) and \( \mathbb{E}|\xi|^r \leq \mu^r \) for every \( i \leq m \). Let \( x = (x_i) \in \ell_2 \) be such that \( |x| = 1 \). Then for every \( \lambda \geq 0 \)

\[
P\left( \left| \sum_{i=1}^m \xi_i x_i \right| > \lambda \right) \geq \left( \frac{1 - \lambda^2}{8 \mu^2} \right)^{r/(r-2)}.
\]

(12)

Proof of Lemma 4.4. Fix \( x \in S^{m-1} \). We clearly have \( Q(x_i \xi_i, |x_i| u) \leq v \) for every \( x_i \neq 0 \). Applying Theorem 1 of [26] to random variables \( x_i \xi_i, i \leq m \), we observe there exists and absolute constant \( C \geq 1 \) such that for every \( w \geq \max_i |x_i| u / 2 \),

\[
Q\left( \sum_{i=1}^m x_i \xi_i, w \right) \leq \frac{Cw}{\sqrt{\sum_{i=1}^m |x_i|^2 u^2 (1 - Q(x_i \xi_i, |x_i| u))}} \leq \frac{Cw}{u \sqrt{1 - v}}.
\]

Take \( w = u^2 \sqrt{1 - v} / C \). If \( \|x\|_\infty \leq 2u \sqrt{1 - v} / C \) then we have that \( w \geq \max_i |x_i| u / 2 \). Therefore for such \( x \) we have

\[
Q\left( \sum_{i=1}^m x_i \xi_i, w \right) \leq v.
\]

If there exists \( \ell \leq m \) such that \( |x_\ell| > 2u \sqrt{1 - v} / C \), then we have

\[
Q\left( \sum_{i=1}^m x_i \xi_i, w \right) \leq Q(x_\ell \xi_\ell, w) = Q(\xi_\ell, w/|x_\ell|) \leq Q(\xi_\ell, u) \leq v,
\]

which completes the proof. \( \square \)

4.1.2 The individual small ball estimate.

To prove Theorem 4.1 we need to extend a result by Montgomery-Smith ([23]), which originally was proved for Rademacher random variables. Note that this lemma does not require any conditions on the moments of random variables.

Lemma 4.6. Let \( \xi_i, i \leq n \), be independent symmetric random variables satisfying condition (11). Let \( \alpha \geq 1 \) and \( L = c_{uv}(B_\infty^n \cap \alpha B_2^n) \), where \( c_{uv} \) is a constant from Lemma 4.4. Then for every non-zero \( z \in \mathbb{R}^n \) one has

\[
P\left( \sum_{i=1}^n \xi_i z_i > h_L(z) \right) > \left( (1 - v)/2 \right)^{5 \alpha^2}.
\]
We postpone the proof of this lemma to the end of this section. Note that if our variables satisfy $1 \leq \mathbb{E} \xi_i^2 \leq \mathbb{E} |\xi_i|^r \leq \mu^r$ for some $r > 2$ then using (12) and repeating the proof of Lemma 4.3 from [18] we could consider $L = (1 - \delta)(B_{\infty}^{\alpha} \cap \alpha B_2^{\beta})$ and estimate the corresponding probability from below by $\exp (-C_{\mu,\delta,\alpha} \alpha^2)$, where $C_{\mu,\delta,\alpha}$ depends only on $\mu, \delta, \alpha$.

Lemma 4.6 has the following consequence.

**Lemma 4.7.** Under assumptions of Lemma 4.6 for every $z \in \mathbb{R}^n$ and every $\sigma \subset [N]$ one has
\[
\mathbb{P} \left( \|P_\sigma \Gamma z\|_\infty < h_L(z) \right) < \exp \left( -|\sigma| \exp(-C_v \alpha^2) \right),
\]
where $P_\sigma : \mathbb{R}^N \rightarrow \mathbb{R}^\sigma$ is the coordinate projection and $C_v = 5 \ln(2/(1 - v))$.

**Proof.** Applying Lemma 4.6 to the $|\sigma| \times n$ random matrix $P_\sigma \Gamma = (\xi_{ij})_{i \in \sigma, j \leq n}$ we have for every $z = \{z_j\}_{j=1}^n \in \mathbb{R}^n$ and every $i \in \sigma$
\[
\mathbb{P} \left( \sum_{j=1}^n z_j \xi_{ij} < h_L(z) \right) \leq 1 - \exp(-C_v \alpha^2) \leq \exp \left( -|\sigma| \exp(-C_v \alpha^2) \right).
\]
Thus
\[
\mathbb{P} \left( \|P_\sigma \Gamma z\|_\infty < h_L(z) \right) = \mathbb{P} \left( \sup_{i \in \sigma} \left| \sum_{j=1}^n z_j \xi_{ij} \right| < h_L(z) \right) = \prod_{i \in \sigma} \mathbb{P} \left( \left| \sum_{j=1}^n z_j \xi_{ij} \right| < h_L(z) \right) < \exp \left( -|\sigma| \exp(-C_v \alpha^2) \right).
\]

We can now state the main individual small ball estimate.

**Lemma 4.8.** Let $\beta \in (0, 1)$ and define $m = 8[(N/n)^\beta]$ (if the latter number is greater than $N/4$ we take $m = N$) and $k = \lfloor N/m \rfloor$. Let $L = c_{uv}(B_{\infty}^n \cap RB_2^n)$, where $R = \sqrt[\beta]{\ln(N/n)/C_v}$. Then for any $z \in \partial L^\sigma$ one has
\[
\mathbb{P} \left( \frac{1}{\sqrt{k}} \|\Gamma z\|_{k,2} < \frac{1}{2} \right) \leq \exp(-0.3 \ n^\beta N^{1-\beta}).
\]

**Proof.** Below we assume $m < N/4$ (then $k \geq 4$, hence $km > 4N/5$); the proof in the case $m = N, k = 1$ repeats the same lines with simpler calculations. Let $\sigma_1, \ldots, \sigma_k$ be a partition of $[N]$ such that $m \leq |\sigma_i|$ for every $i \leq k$. Then, for any $a \in \mathbb{R}^N$
\[
\frac{1}{\sqrt{k}} \|a\|_{k,2} \geq \frac{1}{\sqrt{k}} \left( \sum_{i=1}^k \|P_i a\|_\infty^2 \right)^{1/2} \geq \frac{1}{k} \sum_{i=1}^k \|P_i z\|_\infty,
\]
where $P_i = P_{\sigma_i} : \mathbb{R}^N \rightarrow \mathbb{R}^{\sigma_i}$ is the coordinate projection. Define $\|\| \cdot \||$ on $\mathbb{R}^N$ by
\[
\|\|z\|| = \frac{1}{k} \sum_{i=1}^k \|P_i z\|_\infty
\]

14
for every $z \in \mathbb{R}^N$. Note that if for some $z \in \mathbb{R}^n$ we have $|||z||| < h_L(z)/2$ then there exists $I \subset [k]$ of cardinality at least $k/2$ such that for every $i \in I$ one has $||P_1z||_\infty < h_L(z)$. Applying Lemma 4.7 with $\alpha = R$ (note that $\alpha \geq 2$, by the condition on $n$ and $N$), we obtain for every $z = \{z_i\}_{i=1}^n \in \mathbb{R}^n$

$$
\mathbb{P}(||z|| < h_L(z)/2) \leq \sum_{|I|=(k+1)/2} \mathbb{P}(||P_1z||_\infty < h_L(z) \text{ for every } i \in I)
$$

$$
\leq \sum_{|I|=(k+1)/2} \prod_{i \in I} \mathbb{P}(||P_1z||_\infty < h_L(z))
$$

$$
\leq \sum_{|I|=(k+1)/2} \prod_{i \in I} \exp(-|\sigma_i| \exp(-C_v\alpha^2))
$$

$$
\leq \left(\frac{k}{[k/2]}\right) \exp(- (km/2) \exp(-C_v\alpha^2))
$$

$$
\leq \exp(k \ln 2 - (km/2) \exp(-C_v\alpha^2)),
$$

where $C_v = 5 \ln(2/(1-v))$. By our choice of $k$ and $m$ we have $km > 4N/5$, therefore $(km/2) \exp(-C_v\alpha^2) \geq 2N^{1-\beta}n^3/5$. We also have $k \leq N^{1-\beta}n^3/8$. Thus

$$
\mathbb{P}(||z|| < h_L(z)/2) \leq \exp(-0.3 N^{1-\beta}n^3).
$$

This completes the proof. \hfill \Box

Finally we prove Lemma 4.6. For a positive integer $m$, define $||| \cdot |||_m$ on $\mathbb{R}^n$ by

$$
|||z|||_m = \sup \left( \sum_{i=1}^m \left( \sum_{k \in B_i} |z_k|^2 \right)^{1/2} \right),
$$

where the supremum is taken over all partitions $B_1, \ldots, B_m$ of $[n]$. We will need the following lemma, which was essentially proved in [23] (see Lemma 2 there).

**Lemma 4.9.** Let $\alpha \geq 1$ and $m \geq 1 + 4\alpha^2$ be an integer. For all $x \in \mathbb{R}^n$ one has

$$
h_{B^n_\infty \cap \alpha B^2_2}(x) \leq |||x|||_m.
$$

**Proof.** Fix $x \in \mathbb{R}^n$ and choose $y \in B^n_\infty \cap \alpha B^2_2$ so that $h(x) = \sum_i x_i y_i$. For every $k$ with $y_k^2 \geq 1/2$ choose $B_{1,k} = \{k\}$. Since $|y| \leq \alpha$ there are at most $2\alpha^2$ such sets. Denote $B := \cup_k B_{1,k}$. Now let $z_i$ denote $y_i$ if $|y_i| \leq 1/\sqrt{2}$ and $z_i = 0$ otherwise. Let $n_0 = 0$ and define $n_0 < n_1 < n_2 < \ldots$ by

$$
n_{k+1} = 1 + \sup \left\{ \ell \in [n_k + 1, n - 1] \mid \sum_{i=n_{k+1}}^\ell z_i^2 \leq 1/2 \right\}
$$

(if $n_k = n$ we stop the procedure). Denote $B_{2,k} := [n_{k+1}, n_k] \setminus B$. Since $|y| \leq \alpha$ we have at most $2\alpha^2 + 1$ such sets. Moreover, we have

$$
\sum_{i \in B_{2,k}} z_i^2 = \sum_{i \in B_{2,k}} y_i^2 \leq 1.
$$
Since \( y \in B_n^\infty \) and \( m \geq 4\alpha^2 + 1 \), we obtain

\[
h(x) = \sum_{i=1}^{n} x_i y_i \leq \sum_{j=1}^{2} \sum_{k} \left( \sum_{i \in B_{j,k}} x_i^2 \right)^{1/2} \left( \sum_{i \in B_{j,k}} y_i^2 \right)^{1/2} \leq \sum_{j=2,k} \left( \sum_{i \in B_{j,k}} x_i^2 \right)^{1/2} \leq ||x||_m.
\]

\( \blacksquare \)

**Proof of Lemma 4.6:** We follow the lines of Montgomery-Smith’s proof. Let \( m = \lceil 1 + 4\alpha^2 \rceil \). Given \( z \in \mathbb{R}^n \), let \( m' \leq m \) and \( B_1, \ldots, B_{m'} \) be a partition of \([n]\) such that and

\[
\forall i \leq m' \sum_{k \in B_i} |z_k|^2 \neq 0 \quad \text{and} \quad ||z||_m = \sum_{i=1}^{m'} \left( \sum_{k \in B_i} |z_k|^2 \right)^{1/2}.
\]

Then, using Lemma 4.9, we have

\[
p := P \left( \sum_{i=1}^{n} \xi_i z_i > h_L(z) \right) \geq P \left( \sum_{i=1}^{n} \xi_i z_i > c_{uv} \cdot ||z||_m \right)
\]

\[
= P \left( \sum_{i=1}^{m'} \sum_{k \in B_i} \xi_k z_k > c_{uv} \cdot \sum_{i=1}^{m'} \left( \sum_{k \in B_i} |z_k|^2 \right)^{1/2} \right)
\]

\[
\geq P \left( \bigcap_{i \leq m'} \left( \sum_{k \in B_i} \xi_k z_k \geq c_{uv} \left( \sum_{k \in B_i} |z_k|^2 \right)^{1/2} \right) \right).
\]

Since \( \xi_i \)'s are independent we obtain

\[
p \geq \prod_{i=1}^{m'} P \left( \sum_{k \in B_i} \xi_k z_k > c_{uv} \left( \sum_{k \in B_i} |z_k|^2 \right)^{1/2} \right).
\]

For \( i \leq m' \) set

\[
f_i = \left( \sum_{k \in B_i} \xi_k z_k \right) \cdot \left( \sum_{k \in B_i} |z_k|^2 \right)^{-1/2}.
\]

Using that \( \xi_i \)'s are symmetric and applying Lemma 4.4 we get

\[
P (f_i > c_{uv}) = \frac{1}{2} P (|f_i| > c_{uv}) \geq \frac{1 - v}{2}.
\]

Therefore,

\[
p \geq \left((1 - v)/2\right)^{m'} \geq \left((1 - v)/2\right)^{m} \geq \left((1 - v)/2\right)^{5\alpha^2},
\]

which implies the desired result.
4.1.3 The global small ball estimate.

In this section, we prove Theorem 4.1. As we mentioned after its statement, our goal is to prove (10), when \( N \geq Mn \) where \( M \) depends only on \( \beta, u \) and \( v \).

Let \( \beta \in (0, 1) \) and as in Lemma 4.8, define \( m = 8\lceil (N/n)^{\beta} \rceil \) and \( k = \lfloor N/m \rfloor \) so that \( N^{1-\beta}n^\beta/10 \leq k \leq N^{1-\beta}n^\beta/8 \). By the choice of \( M \), we obviously have \( k \ln(eN/k) \geq n \).

Let \( \delta = 0.1(n/N)\beta \) and \( \varepsilon = \frac{1}{c_{uv} \sqrt{n} \exp((N/n)^{1-\beta/20})} \).

Let \( T = \partial L_0 \) and set \( \delta = 0 \) and \( \varepsilon = 1 \).

Since \( T \subset L^0 = c_{uv}^{-1}(\text{conv } B^n_1 \cup (B^n_2/R)) \subset c_{uv}^{-1}B^n_2 \), we use Theorem 3.1 (see Remark 3.8) to define a set \( N \subset T \) of cardinality at most

\[
\left( \frac{224\delta N}{\varepsilon c_{uv} n^{3/2}} \right)^n e^{\delta N}
\]

such that with probability at least \( 1 - e^{-k\ln(eN/k)} - e^{-\delta N/4} \) one has

\[
\forall x \in T \exists z \in N \text{ such that } \|\Gamma(x - z)\|_{k,2} \leq C_1 \varepsilon \sqrt{\frac{kn}{\delta} \ln \left( \frac{eN}{k} \right)}
\]

where \( C_1 > 0 \) is an absolute constants. Since

\[
\exp \left( n \ln(224\delta N/n) + n \ln(1/(\varepsilon c_{uv} n^{1/2})) + \delta N - 0.3 \ N^{1-\beta}n^\beta \right) \leq \exp \left( -0.1 \ N^{1-\beta}n^\beta \right),
\]

provided that \((N/n)^{1-\beta}\) is large enough, and \( N \subset T \), we deduce from Lemma 4.8 that

\[
\mathbb{P} \left( \exists z \in N : \frac{1}{\sqrt{k}} \|\Gamma z\|_{k,2} < 1/2 \right) \leq \sum_{z \in N} \mathbb{P} \left( \frac{1}{\sqrt{k}} \|\Gamma z\|_{k,2} < 1/2 \right) \leq \exp \left( -0.1 \ N^{1-\beta}n^\beta \right).
\]

Let \( \overline{\Omega} \) be subset of \( \Omega \), where (13) holds. We obtain that on \( \overline{\Omega} \), for every \( x \in T \) there exists \( z \in N \) such that

\[
\frac{1}{\sqrt{k}} \|\Gamma z\|_{k,2} \leq \frac{1}{\sqrt{k}} \|\Gamma x\|_{k,2} + \frac{1}{\sqrt{k}} \|\Gamma(z - x)\|_{k,2} \leq \frac{1}{\sqrt{k}} \|\Gamma x\|_{k,2} + C_1 \varepsilon \sqrt{\frac{n}{\delta} \ln \left( \frac{eN}{k} \right)}.
\]

\[
\leq \frac{1}{\sqrt{k}} \|\Gamma x\|_{k,2} + \frac{C_2 \sqrt{\left( \frac{N}{n} \right)^{\beta} \ln \left( 10e \left( \frac{N}{n} \right)^{\beta} \right)}}{c_{uv} \exp((N/n)^{1-\beta/20})},
\]

where \( C_2 \) is an absolute positive constant. Since \( N \geq Mn \) (for large enough \( M \) depending only on \( u, v \) and \( \beta \)), we observe

\[
c_{uv}^2 \exp((N/n)^{1-\beta}/10) > 16 C_2^2 \left( \frac{N}{n} \right)^{\beta} \ln \left( 10e \left( \frac{N}{n} \right)^{\beta} \right).
\]
Therefore,

\[
\mathbb{P} \left( \left\{ \omega \in \Omega \mid \exists x \in \partial L^0 : \frac{1}{\sqrt{k}} \|\Gamma x\|_2 < \frac{1}{4} \right\} \right) \\
\leq \mathbb{P} \left( \left\{ \omega \in \Omega \mid \exists z \in \mathcal{N} : \frac{1}{\sqrt{k}} \|\Gamma z\|_2 < \frac{1}{2} \right\} \right) \\
\leq \exp \left( -0.1 N^{1-\beta} n^\beta \right).
\]

The desired result follows since \( h_{K_N}(x) = \|\Gamma x\|_\infty \geq \frac{1}{\sqrt{k}} \|\Gamma x\|_2 \) for every \( x \in \mathbb{R}^n \) and since

\[
\mathbb{P}(\Omega) \geq 1 - e^{-k \ln(eN/k)} - e^{-N/4} \geq 1 - 2 \exp(N^{1-\beta} n^\beta /40).
\]

\( \square \)

### 4.2 Volumes and mean widths of \( K_N \) and \( K_N^0 \)

In this section we apply the results of the previous subsection to obtain asymptotically sharp estimates for the volumes and the mean widths of \( K_N, K_N^0 \). We refer to [24] for general knowledge about these parameters. We recall that by Santaló inequality and Bourgain-Milman [4] inverse Santaló inequality there exists an absolute positive constant \( c \) such that for every convex symmetric body \( K \) one has

\[
c^n \|B_2^n\|^2 \leq |K||K^0| \leq \|B_2^n\|^2.
\]

Below we fix constants \( M = M(u,v,\beta) \) and \( C(u,v,\beta) \) from Theorem 4.1.

We start estimating the volumes of \( K_N \) and \( K_N^0 \). For convenience we separate upper and lower estimates (some bounds require an additional condition on the matrix \( \Gamma \)). Corollary 4.3 and (14) imply the following volume estimates for \( K_N \) and \( K_N^0 \).

**Theorem 4.10.** Let \( Mn < N \leq e^n, \beta \in (0,1) \). There exists an absolute positive constant \( C \) such that

\[
|K_N|^{1/n} \geq 2C(u,v,\beta) \sqrt{\frac{\ln(N/n)}{n}} \quad \text{and} \quad |K_N^0|^{1/n} \leq \frac{C}{C(u,v,\beta) \sqrt{n \ln(N/n)}},
\]

with probability at least \( 1 - \exp \left( -cn^\beta N^{1-\beta} \right) \), where \( c \) is an absolute positive constant.

To prove the remaining bounds on volumes of \( K_N \) and \( K_N^0 \) we introduce one more condition on the matrix \( \Gamma \), namely we require that

\[
\mathbb{P} \left( \max_{i \leq N} |\Gamma^* e_i| > \lambda \sqrt{n} \right) \leq p_0
\]

for some \( 0 < p_0 < 1 \) and \( \lambda \geq 1 \). Such condition holds for example when entries of \( \Gamma \) are i.i.d. centered random variables with finite \( p \)-th moment for some \( p > 4 \), provided that \( N \leq C_p n^{p/4} \) (this can be proved using Rosenthal’s inequality, see Corollary 6.4 in [15]).

The lower bound on \( |K_N| \) (and the upper bound on \( |K_N^0| \)) follows from (14) and a well known estimate on the volume of the convex hull of \( k \) points ([2], [6], [10]):
Let \( 2n \leq k \leq e^n \) and \( z_1, \ldots, z_k \in S^{n-1} \), then
\[
|\text{abs conv}\{z_i\}_{i \leq k}|^{1/n} \leq c \sqrt{\ln(k/n)/n},
\]
where \( c > 0 \) is an absolute constant.

**Theorem 4.11.** Let \( M_n < N \leq e^n \) and \( \beta \in (0, 1) \). Assume that the matrix \( \Gamma \) satisfies (15). There exist absolute positive constants \( c \) and \( C \) such that one has
\[
|K_N|^{1/n} \leq C \sqrt{\frac{\ln(N/n)}{n}} \quad \text{and} \quad |K_N^0|^{1/n} \geq c/(\lambda \sqrt{n \ln(N/n)})
\]
with probability larger than or equal to \( 1 - p_0 \).

An important geometric parameter associated to a convex body is the (half of) mean width of \( K \) defined by
\[
M_K = M(K) = \int_{S^{n-1}} \|x\|_K \, d\nu,
\]
where \( \nu \) is the normalized Lebesgue measure on \( S^{n-1} \). It is well known that there exists a constant \( c_n > 1 \) such that
\[
M_K = \frac{c_n}{\sqrt{n}} \mathbb{E} \| \sum_{i=1}^{n} e_i g_i \|_K,
\]
for every \( K \subset \mathbb{R}^n \). Also, \( c_n \to 1 \) as \( n \to \infty \). The (half of) mean width of \( K \), \( M(K^0) \), we denote by \( M_K^* = M^*(K) \). Observe that
\[
M^*(K) = \frac{c_n}{\sqrt{n}} \mathbb{E} \| \sum_{i=1}^{n} e_i g_i \|_{K^0} = \frac{c_n}{\sqrt{n}} \mathbb{E} \sup_{t\in K} \sum_{i=1}^{n} t_i g_i = \frac{c_n}{\sqrt{n}} \ell^*(K),
\]
where \( \ell^*(K) = \mathbb{E} \sup_{t\in K} \sum_{i=1}^{n} t_i g_i \) is a Gaussian complexity measure of the convex body \( K \). We recall the following inequality, which holds for every convex body \( K \) (see e.g. [24])
\[
M_K^* \geq (|K|/|B^n|)^{1/n} \geq 1/M_K.
\]
Now we calculate the mean widths \( M(K_N) \) and \( M(K_N^0) \).

**Theorem 4.12.** Let \( M_n < N \leq e^n \) and \( \beta \in (0, 1) \). Then
\[
M(K_N) \leq C C^{-1}(u, v, \beta) \left( \sqrt{(\ln(2n))/n} + 1/\sqrt{\ln(N/n)} \right)
\]
with probability at least \( 1 - \exp \left( -cn^\beta N^{1-\beta} \right) \), where \( C \) and \( c \) are absolute positive constants. Moreover, if the matrix \( \Gamma \) satisfies (15), then there exists an absolute positive constant \( c_1 \) such that
\[
M(K_N) \geq c_1/(\lambda \sqrt{\ln(N/n)})
\]
with probability at least \( 1 - p_0 \).
Proof. By Theorem 4.1 we have
\[
M(K_N) \leq M \left( C(u, v, \beta) \left( B_\infty^n \cap \sqrt{\ln(N/n)} B_2^n \right) \right) \\
\leq (1/C(u, v, \beta)) \left( M(B_\infty^n) + M \left( \sqrt{\ln(N/n)} B_2^n \right) \right),
\]
which proves the upper bound.

By (16) and Theorem 4.11 there exists an absolute positive constant \( c_1 \) such that
\[
M(K_N) \geq \left( \frac{|B_2^n|}{|K_N|} \right)^{1/n} \geq \frac{c_1}{(\lambda \sqrt{\ln(2N/n)})},
\]
with probability larger than or equal to \( 1 - p_0 \). This proves the lower bound.

\( \square \)

Remark 4.13. Note that by Theorem 4.12, for \( N \leq \exp(n/\ln n) \) we have
\[
M(K_N) \approx \frac{1}{\sqrt{\ln(N/n)}}.
\]
If \( N \geq \exp(n/\ln(2n)) \) there is a gap between lower and upper estimates. Both estimates could be asymptotically sharp as was shown in [18].

Theorem 4.14. There exist positive absolute constants \( c, c_0 \), and \( C \) such that the following holds. Let \( Mn < N \leq e^n \). Then
\[
M(K_0^N) \geq c_0 \sqrt{\ln(N/n)}
\]
with probability at least \( 1 - \exp(-cn^\beta N^{1-\beta}) \). Moreover, assuming that the matrix \( \Gamma \) satisfies (15), one has
\[
M(K_0^N) \leq C \lambda \sqrt{\ln N}
\]
with probability at least \( 1 - p_0 \).

Proof. By (16) we have
\[
M(K_0^N) \geq \left( \frac{|B_2^n|}{|K_0^N|} \right)^{1/n}.
\]
Therefore, the lower bound follows by Theorem 4.10.

Let \( G = \sum_{i=1}^n g_i e_i \). Recall that \( K_N \) is the absolute convex hull of \( N \) vertices \( \Gamma^* e_i \). Thus we have
\[
M(K_0^N) \leq \frac{c_1}{\sqrt{n}} \mathbb{E}\|G\|_{K_0^N} = \frac{c_1}{\sqrt{n}} \mathbb{E} \max_{i \leq N} \langle G, \Gamma^* e_i \rangle,
\]
where \( c_1 \) is an absolute constant. Since with probability at least \( 1 - p_0 \) we have \( |\Gamma^* e_i| \leq \lambda \sqrt{n} \) for every \( i \leq N \), using standard estimate for the expectation of maximum of Gaussian random variables (see, e.g., [24]), we obtain that there is an absolute constant \( c_2 \) such that
\[
M(K_0^N) \leq c_2 \lambda \sqrt{\ln N},
\]
with probability larger than or equal to \( 1 - e^{-n} \).  

\( \square \)
Finally we note that the bounds of Theorem 4.14 are sharp, whenever \( \ln N \) and \( \ln(N/n) \)
are comparable, for example if \( N > n^2 \). However, when \( N \) is close to \( n \) we have a gap
between upper and lower bounds. Below we provide a better lower bound for \( M(K_N^0) \) in
the case \( N \leq n^2 \), which closes this gap. We will need two more conditions on the matrix
\( \Gamma \), namely
\[
\mathbb{P} \left( \| \Gamma \|_{HS} < \sqrt{Nn}/2 \right) \leq p_1, \tag{17}
\]
for some \( p_1 \in (0, 1) \) and where \( \| \Gamma \|_{HS} \) denotes the Hilbert–Schmidt norm of \( \Gamma \); and
\[
\mathbb{P} \left( \| \Gamma \| > \mu \sqrt{N} \right) \leq p_2, \tag{18}
\]
for some \( p_2 \in (0, 1) \), \( \mu \geq 1 \) and where \( \| \Gamma \| \) denotes the operator (spectral) norm of \( \Gamma \).
Both conditions are satisfied for example when entries of \( \Gamma \) are i.i.d. centered random
variables with finite \( p \)-th moment for some \( p > 4 \). Indeed, Rosenthal’s inequality (see proof
of Corollary 6.4 in \([15]\)) implies (17) with \( p_1 \leq (C_p \mathbb{E}[|\xi|^p]/(Nn)^{p/4}) \); while Theorem 2.1
combined with Corollary 6.4 in \([15]\) implies (18) with \( \mu = C_p' \) and
\[
p_2 \leq 1/N^{\gamma_p} + (C_p \mathbb{E}[|\xi|^p]/Nn^{p/4})
\]
(to make \( p_2 < 1 \) we have to ask \( C_p \mathbb{E}[|\xi|^p]N \leq n^{p/4} \)). We would like also to note that the
proof below will works also for \( N \leq n^\alpha \) for some \( \alpha \in (1, 2] \) if we substitute the condition (18) with
\[
\mathbb{P} \left( \| \Gamma \| > \mu(Nn)^\gamma \right) \leq p_2,
\]
for some \( \gamma \in (0, 1/2) \), which could be the case in the absence of 4-th moment (see for
example Corollary 2 in \([1]\) and Remark 2 in \([20]\)). Note also that the condition (18) implies
(15), since \( \| \Gamma \| \geq \max_{i \leq N} |\Gamma^* e_i| \).

**Theorem 4.15.** Let \( \mu \geq 1, n \geq 16\mu^2 \), and \( 2n < N \leq n^2 \) and assume that the matrix \( \Gamma \)
satisfies conditions (17) and (18) for some \( p_1, p_2 \in (0, 1) \). Then
\[
M(K_N^0) \geq c\sqrt{\ln(n/(8\mu^2))}
\]
with probability at least \( 1 - p_1 - p_2 \).

**Proof.** We apply Vershynin’s extension \([31]\) of Bourgain-Tzafriri theorem \([5]\). Denote
\( A = \|\Gamma^*\|_{HS}, B = \|\Gamma^*\| \). Vershynin’s theorem implies that there exists \( \sigma \subset \{1, \ldots, N\} \)
of cardinality at least \( A^2/(2B^2) \) such that for all \( i \in \sigma \) one has \( |\Gamma^* e_i| \geq c_3 A/\sqrt{N} \), where \( c_3 \)
is an absolute positive constant, and vectors \( \Gamma^* e_i, i \in \sigma \), are almost orthogonal (up to an
absolute positive constant). Since \( \Gamma \) satisfies conditions (17) and (18), with probability
at least \( 1 - p_1 - p_2 \) we have \( A \geq \sqrt{Nn}/2 \) and \( B \leq \mu \sqrt{N} \). Therefore, there exists
\( \sigma \subset \{1, \ldots, n\} \) of cardinality at least \( n/(8\mu^2) \) such that \( |\Gamma^* e_i| \geq c_3 \sqrt{n}/2 \) for \( i \in \sigma \) and
\( \{\Gamma^* e_i\}_{i \in \sigma} \) are almost orthogonal. Then,
\[
M(K_N^0) \geq \frac{1}{\sqrt{n}} \mathbb{E}\|G\|_{K_N^0} = \frac{1}{\sqrt{n}} \mathbb{E} \max_{i \leq N} \langle G, \Gamma^* e_i \rangle \geq \frac{1}{\sqrt{n}} \mathbb{E} \max_{i \in \sigma} \langle G, \Gamma^* e_i \rangle.
\]
Since \( \{\Gamma^* e_i\}_{i \in \sigma} \) are almost orthogonal, by Sudakov inequality (see, e.g., \([24]\)), the last
expectation is greater than \( c_4 \sqrt{\ln(n/(8\mu^2))} \), where \( c_4 \) is an absolute constant. This completes the proof. \( \square \)
5 Smallest singular value

In this section we provide a simple short proof of a weaker inclusion, namely, we obtain a lower bound on the radius of the largest ball inscribed into $K_N$. It is based on a lower bound for the smallest singular value for tall matrices. Although such bounds are known with possibly better constants (see the last remark in [16] or the main theorem of [28]), we would like to emphasize a simple short proof, based on our Theorem 3.1. In fact our proof is close to the corresponding proofs in [18] and [19], however it is somewhat cleaner and it uses Theorem 3.1 instead of a standard net argument via the norm of an operator. We would also like to mention that very recently G. Livshyts has extended such results to rectangular random matrices with arbitrary small aspect ratio [21].

Recall that for an $N \times n$ matrix $\Gamma$ with $N \geq n$, its smallest singular value $s_n(\Gamma)$ is defined by

$$s_n(\Gamma) = \inf_{x \in S^{n-1}} \| \Gamma x \|_2.$$ 

In this section we assume that the random matrix $\Gamma$ satisfies conditions described at the beginning of Section 4.

**Theorem 5.1.** Let $\beta \in (0,1)$. There exist a constant $C_0 = C_0(u,v,\beta) > 1$, which depends only on $u,v,\beta$, such that for $N \geq C_0n$ one has

$$\mathbb{P} \left( s_n(\Gamma) \leq c'_{uv} \sqrt{N} \right) \leq \exp \left( -n^\beta N^{1-\beta}/10 \right) ,$$

where $c'_{uv} = c_{uv} \sqrt{\ln(2/(1+v))}/\ln 4$.

Since $h_{\Gamma^*B_1^N}(x) = \| \Gamma^* x \|_\infty$ and $K_N = \Gamma^*B_1^N \supset \frac{1}{\sqrt{N}} \Gamma^*B_2^N$, this theorem immediately implies the following inclusion.

**Corollary 5.2.** Let $\beta \in (0,1)$ and $N \geq C_0n$. Then

$$\mathbb{P} \left( K_N \supset c'_{uv}B_2^n \right) \geq 1 - \exp \left( -n^\beta N^{1-\beta}/10 \right) .$$

To prove Theorem 5.1 we first provide the individual bounds.

**Proposition 5.3.** Let $1 \leq n < N$. Then for every $x \in S^{n-1}$ one has

$$\mathbb{P} \left( \| \Gamma x \|_2 \leq c'_{uv} \sqrt{N} \right) \leq \left( \frac{1+v}{2} \right)^{N/2} ,$$

where $c'_{uv} = c_{uv} \sqrt{\ln(2/(1+v))}/\ln 4$.
Proof. Fix $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $\|x\|_2 = 1$. Denote $f_j := |\sum_{i=1}^n \xi_{ij}x_i|$, so that

$$\|\Gamma x\|_2^2 = \sum_{j=1}^N f_j^2.$$  

Clearly $f_1, \ldots, f_N$ are independent. Therefore, for any $t, \tau > 0$ one has

$$\mathbb{P}\left( \|\Gamma x\|_2^2 \leq t^2 N \right) = \mathbb{P}\left( \sum_{j=1}^N f_j^2 \leq t^2 N \right) = \mathbb{P}\left( \tau N - \frac{\tau}{t^2} \sum_{j=1}^N f_j^2 \geq 0 \right) \leq \mathbb{E}\exp\left( \tau N - \frac{\tau}{t^2} \sum_{j=1}^N f_j^2 \right) = e^{\tau N} \prod_{j=1}^N \mathbb{E}\exp\left( -\frac{\tau f_j^2}{t^2} \right).$$

Lemma 4.4 implies that $\mathbb{P}(f_j < c_{uw}) \leq \nu$ for every $j \leq N$. Therefore, taking $\eta = \ln 2$, and $\tau = t^2 \eta/c_{uw}^2$ we obtain for every $j \leq N$,

$$\mathbb{E}\exp\left( -\frac{\tau f_j^2}{t^2} \right) = \int_0^1 \mathbb{P}\left( \exp\left( -\frac{\eta f_j^2}{c_{uw}^2} \right) > s \right) ds = \int_0^{e^{-\eta}} \mathbb{P}\left( \exp\left( \frac{\eta f_j^2}{c_{uw}^2} \right) < \frac{1}{s} \right) ds + \int_{e^{-\eta}}^1 \mathbb{P}\left( \exp\left( \frac{\eta f_j^2}{c_{uw}^2} \right) < \frac{1}{s} \right) ds \leq e^{-\eta} + \mathbb{P}(f_j < c_{uw})(1 - e^{-\eta}) \leq (1 + \nu)/2.$$

This and the upper bound on $t$ imply

$$\mathbb{P}(\|\Gamma x\|_2^2 \leq t^2 N) \leq e^{\tau N} \left( \frac{1 + \nu}{2} \right)^N \leq \left( \frac{1 + \nu}{2} \right)^{N/2},$$

provided that $t \leq c_{uw} \sqrt{\ln(2/(1 + \nu))/\ln 4}$. This completes the proof. \hfill \Box

Proof of Theorem 5.1. Let $c_{uv}'$ be as in Proposition 5.3,

$$\varepsilon := (c_{uv}'/(2C)) \sqrt{\delta/n} < \sqrt{1/n},$$

and $\delta \in [n/(2N), 1)$ will be specified later. By Theorem 3.1 (see Remark 3.8), applied with $T = S^{n-1}$ and $k = N$, there exists $\mathcal{N} \subset B_2^n$ with cardinality at most

$$\left( \frac{224\delta N}{\varepsilon n^{3/2}} \right)^n \left( \frac{448C\sqrt{\delta N}}{c_{uv}' n} \right)^n \leq \left( \frac{448C\sqrt{\delta N}}{c_{uv}' n} \right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-\delta N/4} - e^{-N}$ one has

$$\forall x \in B_2^n \ \exists y_x \in \mathcal{N} \ \text{such that} \quad \Gamma(x - y_x) \in C\varepsilon \sqrt{Nn/\delta} B_2^n = (c_{uv}'/2) \sqrt{N} B_2^n,$$

where $C > 0$ is an absolute constants. Condition on the corresponding event, denoted below by $\Omega_0$. Assume that $x \in S^{n-1}$ satisfies $\|\Gamma x\|_2 \leq (c_{uv}'/2) \sqrt{N}$. Then for the corresponding $y_x \in \mathcal{N}$ we have

$$\|\Gamma y_x\|_2 \leq \|\Gamma x\|_2 + \|\Gamma(y_x - x)\|_2 \leq c_{uv}' \sqrt{N}.$$
This implies
\[
q_0 := \mathbb{P} \left( \exists x \in S^{n-1} \mid \|\Gamma x\|_2 \leq (c^\prime_{uv}/2)\sqrt{N} \right) \leq \mathbb{P} (\Omega^c_0) + \mathbb{P} \left( \exists y \in \mathcal{N} \mid \|\Gamma y\|_2 \leq c^\prime_{uv}\sqrt{N} \right).
\]

Applying Proposition 5.3,
\[
q_0 \leq 2e^{-\delta N/4} + \left( \frac{448C\sqrt{\delta N}}{c^\prime_{uv}n} \right)^n e^{\delta N} \left( \frac{1 + v}{2} \right)^{N/2}
\]

The choice \( \delta = (n/(2N))^{\beta} \) completes the proof, provided that \( N \geq C_0 n \) for large enough \( C_0 \), where \( C_0 \) depends only on \( u, v, \beta \).

\begin{flushright}
\square
\end{flushright}

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