Local scaling asymptotics for the Gutzwiller trace formula in Berezin-Töplitz quantization

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Abstract

Under certain hypothesis on the underlying classical Hamiltonian flow, we produce local scaling asymptotics in the semiclassical regime for a Berezin-Töplitz version of the Gutzwiller trace formula on a quantizable compact Kähler manifold, in the spirit of the near-diagonal scaling asymptotics of Szegö and Töplitz kernels. More precisely, we consider an analogue of the ‘Gutzwiller-Töplitz kernel’ previously introduced in this setting by Borthwick, Paul and Uribe, and study how it asymptotically concentrates along the appropriate classical loci defined by the dynamics, with an explicit description of the exponential decay along normal directions. These local scaling asymptotics probe into the concentration behavior of the eigenfunctions of the quantized Hamiltonian flow. When globally integrated, they yield the analogue of the Gutzwiller trace formula.

1 Introduction

Let \((M,J)\) be a connected and compact complex manifold, of complex dimension \(d\). Suppose that there exists on \(M\) a positive Hermitian holomorphic line bundle \((\mathcal{A},h)\), and let \(\Theta\) be the curvature form of its unique compatible covariant derivative \(\nabla\). Then \(\omega := (i/2) \Theta\) is a Kähler form on \((M,J)\), and \(dV_M = \omega^d/d!\) is a volume form on \(M\). We shall denote by \(g\) the Riemannian structure associated to \(\omega\) and \(J\).

This paper is concerned with the semiclassical aspects of the Berezin-Töplitz quantization of the Kähler manifold \((M,J,2\omega)\), a theme which has attracted great attention in recent years (see for instance, of course with
no pretense of completeness whatsoever. [B], [CGR], [BMS], [BG], [BPU1], [BPU2], [Z1], [KS], [Ch1], [MZ], [MM1], [MM2], [BMS], [S], [G]). As we point out at the end of this introduction, the present results generalize to the almost complex setting, where one considers a quantizable compact symplectic manifold \((M, \omega)\), endowed with a compatible almost complex structure \(J\).

In [BPU2], in particular, an analogue of the Gutzwiller trace formula was investigated in the Berezin-To"oplitz setting. Roughly speaking, this formula deals with the asymptotics of the trace of the distributional kernel of a smoothed spectral projector relative to a spectral band of width \(O(\hbar)\) about an energy value \(E\). The basic motivation for the present work is to revisit the general theme of [BPU2] in light of the approach to scaling asymptotics in geometric quantization more recently emerged in the series of papers [Z2], [BSZ1], [SZ], [BSZ2] and built on the microlocal theory of the Szeg"o kernel in [BS].

In order to clarify the scope of the present analysis, let us recall a Definition:

**Definition 1.1.** A function \(f \in C^\infty(M, \mathbb{R})\) will be called *compatible* (with the K"ahler structure) if the Hamiltonian flow associated to it, \(\phi^M_t : M \to M\) is a 1-parameter group of holomorphic automorphisms of \((M, J)\).

These functions are also called *quantizable* in the literature [CGR]. Compatible functions are of course quite special, but play a very important role in complex geometry and geometric quantization, as they are tightly related to Hamiltonian group actions preserving the quantization set-up.

Let then \(f\) be a compatible function on \(M\), and consider its associated Hamiltonian vector field \(\nu_f\) on \((M, 2\omega)\). It generates the flow \(\phi^M_t : M \to M\), which heuristically represents the classical evolution of the system with Hamiltonian \(f\).

On the quantum side of the picture, we replace \((M, J, \omega)\) with the spaces of global holomorphic sections \(H^0(M, A^{\otimes k})\) of higher tensor powers of \(A\). Given the volume form \(dV_M\) and the Hermitian metric \(h\), we may consider the Hilbert space \(L^2(M, A^{\otimes k})\) of square summable sections of \(A^{\otimes k}\), and \(H^0(M, A^{\otimes k})\) sits in it as a closed subspace. In geometric quantization, one views \(H^0(M, A^{\otimes k})\) as the quantum Hilbert space associated to \((M, J, 2\omega)\), at Planck’s constant \(\hbar = 1/k\), \(k = 1, 2, \ldots\). The semiclassical regime corresponds to studying the asymptotic aspects of this picture for \(k \to +\infty\).

The quantum counterpart of the observable \(f\) is often taken to be the self-adjoint Berezin-To"oplitz operator

\[
T^{(k)}_f : s \in H^0(M, A^{\otimes k}) \mapsto P_k(s) \in H^0(M, A^{\otimes k}) ,
\]
where $P_k : L^2(M, A^\otimes k) \to H^0(M, A^\otimes k)$ denotes the orthogonal projector. This is a (zeroth-order) Berezin-To"plitz operator (see Definition 1.2 below), and its eigenvalues $\eta_{kj}$ satisfy the elementary bound $a_f \leq \eta_{kj} \leq A_f$, where $a_f =: \min(f)$ and $A_f =: \max(f)$ (see e.g. Lemma 2.1 of [P2]).

When $f$ is compatible, however, there is a more natural choice for the quantization of $f$. Before explaining this, let us reformulate the picture in the frame of Hardy spaces and Szeg"o kernels, following the general approach in [BG], [BPU1], [BPU2], [Z2], [BSZ2], [SZ], [BSZ2].

Let $A^\vee \supset X \xrightarrow{\pi} M$ be the unit circle bundle in the dual line bundle (with the naturally induced Hermitian structure). The covariant derivative $\nabla$ corresponds on $X$ to a connection 1-form $\alpha$, satisfying $d\alpha = 2\pi^*(\omega)$. Thus $(X, \alpha)$ is a contact manifold, and $dV_X(x) =: (\alpha/2\pi) \wedge \pi^*(dV_M)$ is a volume form on $X$. The horizontal tangent bundle $\mathcal{H} =: \ker(\alpha) \subseteq TX$, with the pull-back to it of the complex structure $J$, is a CR structure on $X$.

The corresponding Hardy space $H(X) \subseteq L^2(X)$ then splits equivariantly and unitarily under the $S^1$-action as an orthogonal direct sum of isotypical components,

$$H(X) = \bigoplus_{k \geq 0} H(X)_k.$$  

There is natural unitary isomorphism $H(X)_k \cong H^0(M, A^\otimes k)$ for every $k$.

The orthogonal projector $\Pi : L^2(X) \to H(X)$ is called the Szeg"o projector on $X$; its distributional kernel $\Pi \in \mathcal{D}'(X \times X)$ is the Szeg"o kernel of $X$. If $(e_{kj})_j$ is any orthonormal basis of $H(X)_k$, then $\Pi = \sum_k \Pi_k$, where for each $k$

$$\Pi_k(x, y) = \sum_{j=1}^{N_k} e_{kj}(x) \cdot e_{kj}(y) \quad (x, y \in X); \quad (2)$$

$\Pi_k$ is the $C^\infty$ $k$-th equivariant Szeg"o kernel, that is, the Schwartz kernel of the orthogonal projector $\Pi_k : L^2(X) \to H(X)_k$.

Consider an Hamiltonian vector field $v_f$ on $M$, with $f$ not necessarily compatible; there is a natural lift of $v_f$ to a contact vector field $\tilde{v}_f$ on $X$, given by

$$\tilde{v}_f =: v_f^\# - f \frac{\partial}{\partial \theta}. \quad (3)$$

Here notation is as follows: $v_f^\#: x \mapsto v_f^\#(x) \in \mathcal{H}_x$ is the horizontal lift of $v_f$ to $X$ (with respect to $\alpha$), and $\partial/\partial \theta$ is the generator of the standard $S^1$-action on $X$ (in local coordinates the latter action will be expressed by translation in an angular coordinate); in addition, we have written $f$ for the pull-back

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1 We shall generally use the same notation for an operator and its Schwartz kernel.
\( \pi^*(f) \). While \( \nu_f = \nu_{f+c} \), with \( c \) a real constant, clearly \( \tilde{\nu}_f \neq \tilde{\nu}_{f+c} \) unless \( c = 0 \): the same Hamiltonian flow has many contact lifts to \( X \).

Consider the 1-parameter group of contact transformations \( \phi^X_\tau : X \to X \) generated by \( \tilde{\nu}_f \). Given that \( f \) is compatible, \( \phi^X_\tau \) preserves the CR structure and the Hardy space \( H(X) \). This corresponds to a 1-parameter group of holomorphic automorphism of \( A^{\otimes k} \) for every \( k \), preserving the Hermitian structure.

We have a 1-parameter group of unitary automorphisms of \( H(X)_k \), induced by pull-back:

\[
U(\tau)_k : H(X)_k \to H(X)_k, \quad s \mapsto (\phi^{X}_{\tau_s})^* (s) =: s \circ \phi^{X}_{-\tau}.
\]  

(4)

The latter is evidently the 1-parameter group generated by the self-adjoint operator

\[
T(f)_k =: i \tilde{\nu}_f|_{H(X)_k} : H(X)_k \to H(X)_k,
\]  

(5)

where \( \tilde{\nu}_f \) is viewed as a differential operator. In other words,

\[
U(\tau)_k = e^{i\tau T(f)_k} : H(X)_k \to H(X)_k.
\]  

(6)

Thus, if \( \lambda_{kj} \) are the eigenvalues of \( T(f)_k \), repeated according to multiplicity, the eigenvalues of \( U(\tau)_k \) are given by \( e^{i\tau \lambda_{kj}} \).

Let us recall the following definition from [BG]:

**Definition 1.2.** A Töplitz operator of degree \( k \) on \( X \) is an operator of the form \( T = \Pi \circ Q \circ \Pi \), where \( Q \) is a pseudodifferential operator of degree \( k \) on \( X \); its principal symbol \( \sigma_T \) is the restriction of the principal symbol of \( Q \) to the closed symplectic cone sprayed by the connection 1-form,

\[
\Sigma =: \{(x, r_\alpha x) : x \in X, r > 0 \} \subseteq T^*X \setminus (0).
\]

Here \( T^*X \setminus (0) \) is the complement of the zero section in the cotangent bundle of \( X \).

A Töplitz operator is commonly viewed as a (generally unbounded) linear endomorphism of the Hardy space. For instance, if \( Q = -i \partial / \partial \theta \), then \( T \) is the so-called ‘number operator’ \( \mathcal{N} =: \bigoplus_{k \geq 0} k \cdot \text{id}_{H(X)_k} \).

Therefore, the formally self-adjoint operator

\[
T_f =: \bigoplus_{k \geq 0} T(f)_k : H(X) \to H(X)
\]

is a first order Töplitz operator in the sense of [BG], with principal symbol

\[
\sigma_{T(f)}(x, r_\alpha x) = r f(\pi(x));
\]
one can see from this (see e.g. Corollary 2.1 of [P2]) that
\[ a_f k + O(1) \leq \lambda_{kj} \leq A_f k + O(1). \]

The $C^\infty$ distributional kernels of $U(\tau)_k$ and $T(f)_k$ may be described in terms of spectral data, as follows. Let $(e_{kj})_j$ be an orthonormal basis of $H(X)_k$, composed of eigenvectors of $T(f)_k$ relative to the $\lambda_{kj}$’s. Then
\begin{align*}
T(f)_k(x, y) &= \sum_{j=1}^{N_k} \lambda_{kj} e_{kj}(x) \cdot e_{kj}(y), \quad (7) \\
U(\tau)_k(x, y) &= \sum_{j=1}^{N_k} e^{i\tau \lambda_{kj}} e_{kj}(x) \cdot e_{kj}(y); \quad (8)
\end{align*}
here $N_k = h^0 (M, A^\oplus k) = \dim H(X)_k$.

Let us now consider $\chi \in C^\infty_0 (\mathbb{R})$ and a real number $E \in \mathbb{R}$. Let $G_k = G_k(\chi) : H(X)_k \to H(X)_k$ be the operator with Schwartz kernel
\begin{equation}
G_k(x, y) =: \sum_{j=1}^{N_k} \hat{\chi}(kE - \lambda_{kj}) e_{kj}(x) \cdot e_{kj}(y); \quad (9)
\end{equation}
we shall call $G_k$ a level-$k$ Gutwziller-Töplitz kernel. Although this will be mostly left implicit for notational simplicity, $G_k$ depends on $f, \chi$ and $E$ (see [BPU2]) (however, in the proof of Theorem 1.3 below we shall need to emphasize the dependence on $\chi$).

Operators of this kind in the Berezin-Töplitz context were studied in [BPU2]. Heuristically, $G_k$ is a smoothed spectral projector, associated to a spectral band of $T(f)_k$ about $kE$, of width $O(1)$. If $T(f)$ is turned into a zeroth-order Töplitz operator $T^0(f)_k$ by composing it with a parametrix for the number operator, we should speak of a spectral band about $E$, of width $O(1/k)$.

Theorem 1.1 of [BPU2] deals with an operator of type (11), in the Hardy space formulation, and describes the asymptotics as $k \to +\infty$ of $G_k(x, y)$ at fixed $x, y \in X$ ($f$ is not required to be compatible). One has $G_k(x, y) = O(k^{-\infty})$, unless $m_x =: \pi(x), m_y =: \pi(y) \in M$ satisfy the following conditions:

1. there exists $\tau \in \text{supp}(\chi)$ such that $m_x = \phi^M_\tau(m_y)$;
2. $f(m_x) = E$. 
Furthermore, let $\mathcal{R}_\chi(f, E) \subseteq M \times M$ be the locus of points satisfying these two conditions, and denote by $\tilde{\mathcal{R}}(f, E)$ its inverse image in $X \times X$. Suppose $(x, y) \in \tilde{\mathcal{R}}(f, E)$ and that $f$ is submersive at $m_x$. Then $\mathcal{G}_k(x, y)$ is described by an asymptotic expansion in descending integer powers of $k$, with leading order term of degree $d - 1/2$. The leading coefficient depends on Poincaré type data of the Hamiltonian flow, and the subprincipal symbol of an appropriate pseudodifferential operator.

The arguments in [BPU2] are based on the theory of Fourier-Hermite distributions and their symbolic calculus in terms of symplectic spinors. This ‘jumping behavior’ in the asymptotics of $\mathcal{G}_k(x, y)$ as $(m_x, m_y)$ moves away from the classically defined special locus $\mathcal{R}_\chi(f, E)$ motivates studying the asymptotic concentration of $\mathcal{G}_k$ in shrinking neighborhoods of $\tilde{\mathcal{R}}(f, E)$. In other words, one is naturally led to study the behavior of $\mathcal{G}_k$ not at fixed points, but at sequences $(x_k, y_k) \to \tilde{\mathcal{R}}(f, E)$ at appropriate rates. We shall attack this problem working in rescaled local coordinates at points $(x, y) \in \tilde{\mathcal{R}}(f, E)$; in doing so we shall build, rather than on Fourier-Hermite distributions, on the approach to the near-diagonal scaling asymptotics of level-$k$ Szegő kernels developed in the series of papers [Z2], [BSZ1], [SZ], [BSZ2].

The present analysis is restricted to compatible Hamiltonians and to first order Töplitz operators of type (5). Compatible Hamiltonians are of course quite special, but nonetheless of exceptional importance in complex geometry and harmonic analysis, given their tight relation to holomorphic Hamiltonian actions. In addition, the proofs, the geometric significance and the dynamical interpretation of the scaling asymptotics are particularly transparent in this case, and can thus serve as a guide for future developments.

In order to state our results, we first need to introduce some notation.

**Definition 1.3.** Let $\mu : G \times D \to D$, $(g, d) \mapsto \mu(g, d) = \mu_g(d)$, be an action of a group $G$ on a set $D$. For any $d \in D$, the set of periods of $d$ in $G$ is the stabilizer subgroup of $d$:

$$\text{Per}_D^G(d) := \{ g \in G : \mu(g, d) = d \}.$$ 

If $(d, d') \in D \times D$, we shall set

$$\text{Per}_D^G(d' \mapsto d) := \{ g \in G : \mu(g, d') = d \} = \{ g \in G : d' = d_g \},$$

where $d_g := \mu_g^{-1}(d)$.

**Definition 1.4.** With the hypothesis and notation of Definition 1.3, given a $G$-invariant subset $S \subseteq D$ let us set

$$\text{Per}_M^G(S) := \bigcup_{d \in S} \text{Per}_D^G(d).$$
In the present picture, we have the following built-in actions:

1. the Hamiltonian action $\phi^M: \mathbb{R} \times M \to M$;
2. the contact action $\phi^X: \mathbb{R} \times X \to X$;
3. the action of $\mathbb{R} \times S^1$ on $X$ obtained by composing $\phi^X$ with the structure action $r: S^1 \times X \to X$.

If $m \in M$, let us set $X_m =: \pi^{-1}(m) \subseteq X$. If $(m,n) \in M \times M$ and $(x,y) \in X_m \times X_n$, then there is a natural bijection

$$\text{Per}_M(n \mapsto m) \cong \text{Per}_X^S(y \mapsto x), \quad \tau \mapsto (\tau, g_\tau),$$

which for $n = m$ and $x = y$ is a group isomorphism (§2.1 below).

The scaling asymptotics of $\mathcal{G}_k$ at a point $(m,n) \in \mathcal{R}(f,E)$ with $v_f(m) \neq 0$ are controlled by a universal exponent, given by a quadratic function $Q$ on pairs of tangent vectors (see (12) below). The latter is defined in terms of a natural orthogonal decomposition of the tangent spaces $T_mM$ and $T_nM$ of $M$ at $m$ and $n$, respectively.

Let us set

$$M_E =: f^{-1}(E) \subseteq M, \quad X_E =: \pi^{-1}(M_E) \subseteq X. \quad (10)$$

Suppose that $m \in M_E$, and that $f$ is submersive at $m$. The (Riemannian) normal space to $M_E$ at $m$ is

$$N_m(M_E) = \text{span}_\mathbb{R}\{J_m(v_f(m))\}.$$ 

Let

$$R_m =: \text{span}_\mathbb{R}(v_f(m), J_m v_f(m)) = \text{span}_C(v_f(m)), \quad S_m =: R_m^\perp, \quad (11)$$

where the orthocomplement in $T_mM$ may be taken equivalently in the Riemannian or in the Hermitian sense.

Then $\text{span}_\mathbb{R}(v_f(m)) = T_mM_E \cap \text{span}_C(v_f(m))$ and $T_mM_E$ splits as a Riemannian orthogonal direct sum

$$T_mM_E = \text{span}_\mathbb{R}(v_f(m)) \oplus S_m.$$ 

**Definition 1.5.** Let us now provide an orthocomplement direct sum decomposition of $T_mM$ which reflects the local geometry induced by the Hamiltonian flow. Given the orthogonal direct sum decomposition

$$T_mM = \text{span}_\mathbb{R}(J_m(v_f(m)) \oplus \text{span}_\mathbb{R}(v_f(m)) \oplus S_m,$$
at any $m \in M_E(\sigma)$ with $d_m f \neq 0$, we can uniquely write any $w \in T_m M$ as $w = w_t + w_v + w_h$, where $w_t \in \text{span}_\mathbb{R}(J_m(v_f(m)))$, $w_v \in \text{span}_\mathbb{R}(v_f(m))$, $w_h \in S_m$. Thus $w_t = b J_m(v_f(m))$ and $w_v = a v_f(m)$ for certain $a, b \in \mathbb{R}$. The suffix $t$ stands for ‘transverse’ (to $M_E$), $v$ stands for ‘vertical’ and $h$ for ‘horizontal’.

Verticality and horizontality heuristically refer to the quotient fibration locally induced by the orbits of the $\mathbb{R}$-action in $M_E$.

Let us now recall a definition from [SZ].

**Definition 1.6.** Let $(V,h)$ be an Hermitian vector space, and write $g = \Re(h)$ and $\omega = -\Im(h)$ (real and imaginary parts). Let $\| \cdot \|$ be the induced norm, and set

$$\psi_2(v_1, v_2) = -i \omega(v_1, v_2) - \frac{1}{2} \|v_1 - v_2\|^2 \quad (v_1, v_2 \in V).$$

This is the universal exponent appearing in near-diagonal scaling asymptotics of level-$k$ Szegö kernels when the latter are expressed in Heisenberg local coordinates.

Let us now define $Q : T_m M \times T_m M \to \mathbb{C}$ by

$$Q(w_1, w_2) =: \psi_2(w_{1h}, w_{2h}) + \left[ i \left( \omega(w_{1v}, w_{1t}) - \omega(w_{2v}, w_{2t}) \right) - \left( \|w_{1t}\|^2 + \|w_{2t}\|^2 \right) \right].$$

The scaling asymptotics of $G_k$ below will be expressed in Heisenberg local coordinates (HLC for short), which were introduced in [SZ]. These local coordinates render the universal nature of the near-diagonal scaling asymptotics of $\Pi_k$ especially transparent. We refer to [SZ] for a precise definition and discussion, and to [2.3] below a quick review.

A Heisenberg local coordinate chart for $X$ centered at $x$, $\gamma_x : (-\pi, \pi) \times B_{2d}(0, \delta) \to X \subseteq X$, is often denoted in additive notation, $\gamma_x(\theta, v) = x + (\theta, v)$; it satisfies the following properties:

1. Let $V_x =: \ker(d_x \pi)$, $H_x =: \ker(\alpha_x)$ be the vertical and horizontal tangent spaces of $X$; then the differential

$$d_{(0, 0)} \gamma_x : \mathbb{R} \oplus \mathbb{C}^d \to T_x X = V_x \oplus H_x$$

is unitary and preserves the direct sum (in particular, the isomorphism $\mathbb{C}^d \to H_x$ is $\mathbb{C}$-linear).

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2 these should not be confused with the ‘vertical’ and ‘horizontal’ distributions defined above on $M$ along $M_E$
2. The standard circle action on $X$ is expressed in HLC by translation in the angular coordinate, that is, whenever both terms are well-defined we have

$$r_{e^{i\vartheta}}(x + (\vartheta, v)) = x + (\vartheta + \vartheta, v);$$

we shall generally write $r_{\vartheta}$ for $r_{e^{i\vartheta}}$.

3. In particular, if $m =: \pi(x)$ then in additive notation

$$v \in B_{2d}(0, \delta) \mapsto m + v =: \pi(x + (\vartheta, v)) \in M' =: \pi(X') \subseteq M$$

is a well-defined coordinate chart on $M$ centered at $m$ (that is, it does not depend on $\vartheta$), and its differential at $0 \in \mathbb{R}^{2d}$ determines a unitary isomorphism $\mathbb{C}^d \to T_mM$. This coordinate system is called preferred, or adapted.

We shall generally write $x + v$ for $x + (0, v)$.

Lastly, let us give the following Definition:

**Definition 1.7.** If $m \in M$ and $\chi \in C_0^\infty(\mathbb{R})$, we shall denote by $m^\chi$ the portion of the $\mathbb{R}$-orbit of $m$ determined by the support of $\chi$:

$$m^\chi =: \{m_\tau : \tau \in \text{supp}(\chi)\}.$$

The locus $\mathcal{R}_\chi(f, E)$ (page 6) of pairs $(m, n) \in M \times M$ over which $\mathcal{G}_k$ asymptotically concentrates for $k \to +\infty$ is determined by the conditions

$$\text{dist}_M(n, m^\chi) = 0, \quad f(m) - E = 0,$$

where $\text{dist}_M$ is the distance function on $M$ associated to the Kähler metric.

We shall now formulate a quantitative estimate of the rate of concentration in terms of these two quantities.

**Theorem 1.1.** Let $f \in C^\infty(M)$ be a compatible Hamiltonian and suppose $\chi \in C_0^\infty(\mathbb{R})$. Let $\mathcal{G}_k = \mathcal{G}_k^\chi \in C^\infty(X \times X)$ be the level-$k$ Gutzwiller-Töplitz kernel associated to the first order Töplitz operator $\mathfrak{a}$. For any $\epsilon, C > 0$, we have

$$\mathcal{G}_k(x, y) = O(k^{-\infty})$$

uniformly for

$$\max\{\text{dist}_M(m_y, m^\chi_x), |f(m_x) - E|\} \geq C k^{\epsilon - \frac{1}{2}}.$$
A more explicit formulation is the following: there exist constants
\[ C_j = C_j(C, \epsilon) > 0, \quad j = 1, 2, \ldots, \]
such that for any choice of sequences \( m_k, n_k \in M \) satisfying
\[ \max \{ \text{dist}_M(n_k, m_k^x), |f(m_k) - E| \} \geq C k^{-s} \]
for \( k = 1, 2, \ldots \) we have
\[ |G_k(x_k, y_k)| \leq C_j k^{-j} \]
whenever \((x_k, y_k) \in X_{m_k} \times X_{n_k}\).

**Corollary 1.1.** In the hypothesis of Theorem 1.1, we have \( G_k(x, x) = O(k^{-\infty}) \) uniformly for \( |f(m_x) - E| \geq C_1 k^{-s/2} \).

Let us heuristically interpret \( G_k \) as a smoothed spectral projector, associated to a spectral band of the zeroth order Töplitz operator \( T^\chi(f)_k \) (page 5), of width \( O(k^{-1}) \) about \( E \). Then, in the range of Corollary 1.1, all the eigenvalues in the band stay at a distance \( O(k^{-s/2}) \) from \( f(m_x) \). Thus Corollary 1.1 tallies in spirit with the results of \[ P2 \], where it is proved that the eigenfunctions \( e_{kj} \) pertaining to eigenvalues \( \lambda_{kj} \) with \( |\lambda_{kj} - f(m_x)| \geq C_1 k^{-s/2} \) contribute negligibly for \( k \to +\infty \) to the equivariant Szegö kernel at \( x, \Pi_k(x, x) \).

Given this, we look for explicit asymptotic expansions for \( G_k \) in shrinking neighborhoods of \( \tilde{R}_\chi(f, E) \) (pages 6 and 9). We shall furthermore restrict our analysis to the open sublocus
\[ \mathcal{R}_\chi(f, E)' = \{ (m, n) \in \mathcal{R}_\chi(f, E) : v_f(m) \neq 0 \}, \]
and to its inverse image \( \tilde{\mathcal{R}}_\chi(f, E)' \subseteq X \times X \). Since \( f \) is \( \phi^M \)-invariant, if \( n = m_x \) and \( v_f(m) \neq 0 \), then also \( v_f(n) \neq 0 \).

Let us fix \((x, y) \in \tilde{\mathcal{R}}_\chi(f, E)' \), and HLC systems centered at \( x \) and \( y \), respectively, that we shall denote additively as above. Then \( \text{Per}_M^\mathbb{R}(m_y \mapsto m_x) \) is a discrete subset of \( \mathbb{R} \), either infinite or reduced to a point, and so \( \text{Per}_M^\mathbb{R}(m_y \mapsto m_x) \cap \text{supp}(\chi) \) is a finite set. We shall write
\[ \text{Per}_M^\mathbb{R}(m_y \mapsto m_x) = \{ \tau_a \}_{\mathcal{A}} \]
where the index set \( \mathcal{A} \) depends on \( x \); either \( \mathcal{A} = \{ 0 \} \), or else \( \mathcal{A} = \mathbb{Z} \). Accordingly,
\[ \text{Per}_X^{\mathbb{R} \times S^1}(y \mapsto x) = \{ (\tau_a, e^{i\vartheta_a}) \}_{\mathcal{A}}. \]
Thus, for every \( a \in \mathcal{A} \) we have (see Definition 1.3)

\[
x_{\tau_a} = \phi_{X_{\tau_a}}(x) = e^{i\theta_a} y.
\]  (15)

We shall, furthermore, denote by \( A_a (a \in \mathcal{A}) \) the unitary (that is, symplectic and orthogonal) \( 2d \times 2d \) matrix representing the differential

\[
d_{m_x} \phi_{-\tau_a}^M : T_{m_x} M \rightarrow T_{m_y} M
\]  (16)

with respect to the given HLC systems.

**Theorem 1.2.** Adopt the hypothesis and notation of Theorem 1.1, and the notation (14). Suppose \( (x,y) \in \tilde{R} \chi(f, E) \) (see (13)), and adopt HLC systems centered at \( x \) and \( y \). Fix \( C > 0 \) and \( \epsilon \in (0, 1/6) \). Then uniformly in \( v_1, v_2 \in \mathbb{R}^{2d} \) satisfying \( \|v_1\|, \|v_2\| \leq Ck^\epsilon \) the following asymptotic expansion holds for \( k \rightarrow +\infty \):

\[
G_k \left( x + \left( \frac{\theta_1}{\sqrt{k}}, \frac{v_1}{\sqrt{k}} \right), y + \left( \frac{\theta_2}{\sqrt{k}}, \frac{v_2}{\sqrt{k}} \right) \right)
\sim \frac{\sqrt{2} \chi(\tau_a) + \sum_{j=1}^{+\infty} k^{-j/2} \left( m_x, m_y; v_1, v_2 \right)}{\|v_f(m_x)\| \|v_f(m_y)\|}
\]  (17)

where for each \( a \)

\[
G_k^{(a)} \left( x + \frac{v_1}{\sqrt{k}}, y + \frac{v_2}{\sqrt{k}} \right)
\sim e^{ik(\theta_a - \tau_a E) + Q(A_v v_1 v_2)} \left( m_x, m_y; v_1, v_2 \right)
\]

here \( Q \) is as in (12) and \( P_{aj}^X \) is a polynomial in \( (v_1, v_2) \), of degree \( \leq 3j \) and parity \(-1)^j\), whose coefficients depend smoothly on \( m_x \) and \( m_y \), and which vanishes identically when \( \tau_a \notin \text{supp}(\chi) \). More precisely \( P_{aj}^X \) is, as a function of \( \chi \), the evaluation at \( \tau_a \) of a differential polynomial in \( \chi \) of degree \( \leq j \).

Notice that \( \|v_f(m_x)\| = \|v_f(m_y)\| \) and that, by the previous remarks, the sum over \( a \) in the statement of Theorem 1.2 is finite. In addition, the \( j \)-th summand in (17) satisfies

\[
|e^{ik(\theta_a - \tau_a E) + Q(A_v v_1 v_2)} k^{-j/2} P_{j}(m_x, m_y; v_1, v_2)| \leq D_a k^{-3j(1/6 - \epsilon)},
\]  (18)

so that (17) is indeed an asymptotic expansion.

In view of the previous parity statement, we recover an asymptotic expansion at fixed points akin to the one in [BPU2], by setting \( \theta_j = 0 \) and \( v_j = 0 \).
Corollary 1.2. Suppose \((x, y) \in \tilde{\mathcal{R}}_\chi(f, E)\). Then the following asymptotic expansion holds for \(k \to +\infty\):

\[
\mathcal{G}_k(x, y) \sim \frac{\sqrt{2}}{\|u_f(m_x)\|} \left( \frac{k}{\pi} \right)^{d-1/2} \sum_{a \in \mathcal{A}} \mathcal{G}^{(a)}_k(x, y),
\]

where for each \(a\)

\[
\mathcal{G}^{(a)}_k(x, y) \sim e^{ik(\tau_a - \tau_x)} \left[ \chi(\tau_a) + \sum_{j=1}^{+\infty} k^{-j} A_{aj}(m_x, m_y) \right];
\]

here \(A_{aj} = A_{aj}^\chi\) is a \(C^\infty\) function on \(\mathcal{R}_\chi(f, E)\).

Let us consider the special case \(x = y\), so that \(\mathcal{A}\) indexes the periods of \(x \in X_E\) (defined in (10)). We can heuristically interpret Corollary 1.2 as saying that the eigensections of the zeroth order Töplitz operator \(T^0(f)_k\) (defined on page 5) associated to a spectral band of width \(O(k^{-1})\) about \(E\) yield a contribution to \(\Pi_k(x, x)\) which is \(O(k^{d-1/2})\). This should be contrasted with the results in \([\text{P}2]\), which imply that the eigensections pertaining to a ‘slowly shrinking’ spectral band of \(T^0(f)_k\) of width \(O(k^{-1/2})\) about \(E\) asymptotically capture all of \(\Pi_k(x, x)\), up to a negligible contribution; recall that \(\Pi_k(x, x) = O(k^d)\) (1.2).

It is in order to show how, by integrating the local rescaled asymptotic expansion in Theorem 1.2 term by term, one obtains a asymptotic expansion for the trace of \(\mathcal{G}_k\), analogue to the one of Theorem 1.2 of \([\text{BPU}2]\); this expansion probes into the asymptotic clustering, for \(k \to +\infty\), of the eigenvalues \(\lambda_{kj}\) in a spectral band of width \(O(1)\) centered at \(kE\).

To this end, we shall make the stronger assumption that \(E\) be a regular value of \(f\), which implies that \(\mathcal{R}_\chi(f, E)' = \mathcal{R}_\chi(f, E)\) in (13); under this condition, by Proposition 2.4 below the set \(\text{Per}_M^R(M_E)\) (Definition 1.4) is a discrete subset of \(\mathbb{R}\). In fact, either \(\text{Per}_M^R(M_E) = \{0\}\) (when there are no closed orbits of \(\phi^M\) on \(M_E\)), or else it is infinite countable (and drifting to infinity).

Let us write \(\text{Per}_M^R(M_E) = \{\sigma_b\}_{b \in \mathcal{B}}\). For any \(b \in \mathcal{B}\), let \(M(\sigma_b) \subseteq M\) be the fixed point locus of \(\phi^M_{\sigma_b}\), and let \(M(\sigma_b)_l, 1 \leq l \leq n_b\), be its connected components. Each \(M(\sigma_b)_l\) is a compact and connected complex submanifold of \(M\), of complex dimension \(d_B = d - c_B\).

Given the unitarity of \(d_m\phi^M_{\sigma_b}\), the tangent and normal spaces \(T_mM(\sigma_b)\) and \(N_m(M(\sigma_b)) = T_mM(\sigma_b)^\perp\) at each \(m \in M(\sigma_b)_l\) are given by, respectively,

\[
T_mM(\sigma_b)_l = \ker (d\phi^M_{\sigma_b} - \text{id}_{T_mM})\quad \text{and} \quad N_mM(\sigma_b)_l = \text{im} (d\phi^M_{\sigma_b} - \text{id}_{T_mM}), \quad (19)
\]
where im(h) denotes the image of a map h.

The \(\mathbb{C}\)-linear map \(\text{id}_{T_mM} - d\phi^M_{-\sigma_b}\) determines a complex linear automorphism of the holomorphic normal bundle of \(M(\sigma_b)_l\), whose determinant

\[
D(b, l) =: \det \left( \text{id}_{T_mM} - d\phi^M_{-\sigma_b} \big|_{N_mM(\sigma_b)_l} : N_mM(\sigma_b)_l \to N_mM(\sigma_b)_l \right)
\]

(20) is therefore a non-zero constant on \(M(\sigma_b)_l\).

One can see that if \(E\) is a regular value of \(f\), then \(M_E\) and each \(M(\sigma_b)_l\) meet transversally (§2.2). We shall set

\[
M_E(\sigma_b) =: M_E \cap M(\sigma_b), \quad M_E(\sigma_b)_l =: M_E \cap M(\sigma_b)_l;
\]

(21) this is a submanifold of \(M\), perhaps not connected. Its (real) dimension is \(2d_b - 1\).

For each \(b \in B\) and and \(l = 1, \ldots, n_b\), let

\[
X_E(\sigma_b) =: \pi^{-1}(M_E(\sigma_b)_l), \quad X_E(\sigma_b)_l =: \pi^{-1}(M_E(\sigma_b)_l) \subseteq X.
\]

(22) Then, in view of the connectedness of \(M(\sigma_b)_l\), there is a unique \(g_{bl} = e^{i\vartheta_{bl}} \in S^1\), such that \(x_{\sigma_b} = r_{g_{bl}}(x)\) for every \(x \in X(\sigma_b)_l\).

With this notation, we can now formulate the following global spectral counterpart of Theorem 1.2 (see Theorem 1.2 of [BPU2]):

**Theorem 1.3.** Adopt the hypothesis of Theorem 1.2, and suppose in addition that \(E\) is a regular value of \(f\). Then the following asymptotic expansion holds for \(k \to +\infty\):

\[
\sum_j \hat{\chi}(kE - \lambda_{kj}) \sim \sum_{b \in B} e^{-ik\sigma_b} E \sum_{l=1}^{n_b} \left( \frac{k}{\pi} \right)^{d_b - 1} e^{ik\vartheta_{bl}} \int_{M_E(\sigma_b)_l} \frac{1}{\|v_f(m)\|} dV_{M_E(\sigma_b)_l}(m)
\]

\[
\cdot \left[ \chi(\sigma_b) + \sum_{j \geq 1} k^{-j} A_j(b, l) \right],
\]

where \(A_j(b, l) = A_j(b, l)^X \in \mathbb{C}\) are appropriate constants, which vanish for \(\sigma_b \not\in \text{supp}(\chi)\), and \(dV_{M_E(\sigma_b)_l}\) is the Riemannian density on \(M_E(\sigma_b)_l\).

Again, the sum over \(b\) is finite since \(B \subseteq \mathbb{R}\) is discrete and \(\chi\) has compact support.

It is in order to conclude this introduction by remarking that there is a wider scope for the previous results. Namely, consider a compact symplectic manifold \((M, \omega)\), with a compatible almost complex structure \(J\) [McDS].
The definition of compatible Hamiltonian readily generalizes to this almost Kähler setting. By way of example, given a Hamiltonian action of a compact Lie group $G$ on $(M, \omega)$, with moment map $\Phi : M \rightarrow \mathfrak{g}^\vee$, one can find a $G$-invariant compatible almost complex structure $J$ on $(M, \omega)$ [McDS]. Then any component of $\Phi$ is a compatible Hamiltonian for the almost Kähler structure $(M, \omega, J)$.

Suppose now that $(M, \omega)$ is quantizable, and let $(A, h)$ be a quantizing Hermitian line bundle. Then the theory of [BG] provides analogues $H^0_\Omega(M, A^{\otimes k})$ of the spaces of global holomorphic sections of the integrable case, lying in the range of a generalized Szegő projector, which is the first step of a resolution generalizing the $\overline{\partial}$-complex of the integrable case (see especially §A.5 of [BG]; furthermore, a detailed review of this construction is given in [SZ]). The Hilbert space direct sum of the $H^0_\Omega(M, A^{\otimes k})$'s corresponds in the usual manner to a generalized Hardy space on the unit circle bundle $X$.

The spaces $H^0_\Omega(M, A^{\otimes k})$ are not fully intrinsic, but depend on some non-canonical choices; it is then not a priori clear whether a compatible Hamiltonian induces an action on the generalized Hardy space of $(A, h)$. However, given as above a compact group action, we may assume that $G$ acts on $(A, h)$ preserving all the data involved, and so it acts unitarily on the spaces $H^0_\Omega(M, A^{\otimes k})$ ([BG], §A.5, Theorem 5.9). The same holds for the contact flow of any component of the moment map. It is thus natural to consider analogues of previous results in this wider picture.

In [SZ], the microlocal description of the Szegő kernel in [BS] is generalized to the almost complex situation, and based on this near diagonal scaling asymptotics for the equivariant components of the generalized Szegő kernel are provided. Granting the latter extension, the arguments in the present paper go over unchanged to this more general setting.

2 Preliminaries

2.1 Periods

If $m \in M$ is a critical point of $f$, then $v_f(m) = 0$ and $m$ is a fixed point of the Hamiltonian flow $\phi^M$, that is, $\text{Per}_M^R(m) = \mathbb{R}$. If on the other hand $d_m f \neq 0$ then $\text{Per}_M^R(m)$ is a discrete subgroup of $\mathbb{R}$. In fact, suppose $\tau \in \text{Per}_M^R(m)$ and choose a system of local coordinates centered at $m$. Then, in additive notation, we have for $\delta \sim 0$:

$$\phi^M_{\tau + \delta}(m) = \phi^M_{\delta}(m) = m + \delta v_f(m) + O(\delta^2),$$

which equals $m$ only for $\delta = 0$. 

If \( d_m f \neq 0 \) and \( \text{Per}_M^R(m) \) is not trivial, then it is isomorphic to \( \mathbb{Z} \). More precisely, if \( \tau_1 \in \mathbb{R} \) is the least positive element of \( \text{Per}_M^R(m) \), then \( \text{Per}_M^R(m) = \mathbb{Z} \cdot \tau_1 \). We shall write \( \tau_a = a \tau_1 \).

Given any \( x \in X_m =: \pi^{-1}(m) \), there exists a unique \( g_1 \in S^1 \) such that \( x = g_1 \cdot \phi^X_{\tau_1}(x) \), that is \( x_{\tau_1} = r_{g_1}(x) \). Thus, if \( g_a := g_1^a \) then
\[
\text{Per}_{X \times S^1}^R(x) = \{(a \tau_1, g_1^a) : a \in \mathbb{Z}\} = \{(\tau_a, g_a) : a \in \mathbb{Z}\}.
\]

Similarly, suppose \( m, n \in M \) lie in the same \( \mathbb{R} \)-orbit. Then \( \text{Per}_M^R(n \mapsto m) \) is a non-empty coset of \( \text{Per}_M^R(n) \). If \( x \in X_m \) and \( y \in Y_n \), then \( \text{Per}_{X \times S^1}^R(y \mapsto x) \) is a non-empty coset of \( \text{Per}_{X \times S^1}^R(y) \). For any \( \tau \in \text{Per}_M^R(n \mapsto m) \), there is a unique \( g_\tau \in S^1 \) such that \( x = g_\tau \cdot \phi^X_{\tau}(y) \). Thus
\[
\tau \in \text{Per}_M^R(n \mapsto m) \mapsto (\tau, g_\tau) \in \text{Per}_{X \times S^1}^R(y \mapsto x)
\]
is a natural bijection.

Let us now consider the periods on the hypersurface \( M_E \). By a standard compactness argument, \( \text{Per}_M^R(M_E) \) is a closed subset of \( \mathbb{R} \).

**Proposition 2.1.** Suppose that \( E \) is a regular value of \( f \). Then \( \text{Per}_M^R(M_E) \) is a discrete subset of \( \mathbb{R} \).

**Proof of Proposition 2.1.** Given \( \tau_0 \in \text{Per}_M^R(M_E) \), consider a sequence \( \tau_j \in \text{Per}_M^R(M_E) \), \( j = 1, 2, \ldots \), with \( \tau_j \rightarrow \tau_0 \) for \( j \rightarrow +\infty \). Thus \( \epsilon_j =: \tau_j - \tau_0 \rightarrow 0 \) for \( j \rightarrow +\infty \). We aim to prove that \( \epsilon_j = 0 \) for all \( j \gg 0 \).

For every \( j \geq 1 \) there exists by definition \( m_j \in M_E \) such that \( \phi^M_{\tau_j}(m_j) = m_j \). By compactness of \( M_E \), perhaps after passing to a subsequence we are reduced to the case where \( m_j \rightarrow m_0 \) for some \( m_0 \in M_E \). By continuity, we then have \( \phi^M_{\tau_0}(m_0) = m_0 \).

Let us fix a unitary isomorphism \( T_{m_0} M \cong \mathbb{R}^{2d} \). On an open neighborhood \( U \subseteq M \) of \( m_0 \), we have a geodesic local coordinate system centered at \( m_0 \)
\[
\Gamma : \mathbf{v} \in B_{2d}(\mathbf{0}, \delta) \subseteq \mathbb{R}^{2d} \mapsto m_0 + \mathbf{v} =: \exp_{m_0}(\mathbf{v}) \in U.
\]  
(23)

For any \( j = 1, 2, \ldots \) we have \( m_j = m_0 + \mathbf{v}_j \), for a unique \( \mathbf{v}_j \in B_{2d}(\mathbf{0}, \delta) \), and \( \mathbf{v}_j \rightarrow \mathbf{0} \) for \( j \rightarrow +\infty \).

For \( \tau \sim 0 \in \mathbb{R} \) we have
\[
\phi^M_{\tau}(m) = m + (\tau \mathbf{v}_j(m) + R_j(\tau)),
\]  
(24)
where, here and in the following, \( R_j \) is a smooth function on some open neighborhood of the origin in a Euclidean space, vanishing to \( j \)-th order at the origin; generally dependence on additional variables will be left implicit.
Since $\Gamma$ is a system of geodesic local coordinates centered at $m_0$, and $\phi^M_{\tau_0} : M \to M$ is a Riemannian isometry fixing $m_0$, we have

$$\phi^M_{\tau_0}(m_0 + v_j) = m_0 + A^{-1}_{\tau_0}v_j,$$

(25)

where $A_{\tau_0}$ is the matrix representing $d_{m_0}\phi^M_{-\tau_0}$.

We deduce from (24) and (25) that

$$m_0 + v_j = m_j = \phi^M_{\tau_0}(m_j) = \phi^M_{\tau_0+\epsilon_j}(m_0 + v_j)$$

$$= \phi^M_{\epsilon_j}(m_0 + A^{-1}_{\tau_0}v_j)$$

$$= m_0 + (A^{-1}_{\tau_0}v_j + \epsilon_j v_f(m_0) + R_2(\epsilon_j) + \epsilon_j R_1(v_j)).$$

(26)

This implies

$$v_j - A^{-1}_{\tau_0}v_j = \epsilon_j v_f(m) + R_2(\epsilon_j) + \epsilon_j R_1(v_j).$$

(27)

Since $v_f(m_0) \in \ker (I - A_{\tau_0}^{-1})$ (in local coordinates), and on the other hand $\im (I - A_{\tau_0}^{-1}) \perp \ker (I - A_{\tau_0}^{-1})$ in view of the unitarity of $A_{\tau_0}$, we deduce by taking the scalar product with $v_f(m_0)$ on both sides of (27) that

$$0 = \epsilon_j \left[ \|v_f(m)\|^2 + R_1(\epsilon_j) + R_1(v_j) \right],$$

whence that $\epsilon_j = 0$ for any $j \gg 0$.

Therefore, for any $m \in M(\sigma)$ we have

$$v_f(m), J_m v_f(m) \in T_m M(\sigma).$$

(28)

The tangent and normal spaces to $M(\sigma)$ are given (with $\sigma = \sigma_b$) by (19).

### 2.2 Transversality issues and normal bundles

As on page 12, for any $\sigma \in \mathbb{R}$ let us denote by $M(\sigma) \subseteq M$ the fixed locus of $\phi^M_{\sigma}$. Then $M(\sigma)$ is a $\phi^M_{\sigma}$-invariant closed complex submanifold of $M$. Therefore, for any $m \in M(\sigma)$ we have

$$v_f(m), J_m v_f(m) \in T_m M(\sigma).$$

(28)
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On the other hand, since $J(v_f)$ is the Riemannian gradient of $f$, at any $m \in M_E = f^{-1}(E)$ with $d_m f \neq 0$ the Riemannian normal space to $M_E$ is

$$N_m(M_E) = \text{span}(J_m v_f(m)).$$

(29)

Clearly, $\sigma \in \text{Per}^R_M(M_E)$ if and only if $M_E \cap M(\sigma) \neq \emptyset$.

**Lemma 2.1.** For any $\sigma \in \text{Per}^R_M(M_E)$, $M_E$ and $M(\sigma)$ meet transversally.

**Proof of Lemma 2.1.** Suppose $m \in M(\sigma) \cap M_E$. By (28) and (29), we have

$$N_m(M_E) \cap N_m(M(\sigma)) = \text{span}(J_m v_f(m)) \cap T_m M(\sigma) = \emptyset\text{.}$$

(30)

Here $\perp_g$ denotes the Riemannian orthocomplement.

Thus $M_E(\sigma) =: M_E \cap M(\sigma)$ is a 1-codimensional submanifold of $M(\sigma)$, and at any $m \in M_E(\sigma)$ we have $T_m M_E(\sigma) = T_m M_E \cap T_m M(\sigma)$. To be more explicit, let $R_m, S_m \subseteq T_m M$ be as in (11); then

$$T_m M_E = \text{span}_R(v_f(m)) \oplus S_m, \quad T_m M(\sigma) = \text{span}_C(v_f(m)) \oplus \tilde{T}_m M(\sigma),$$

(31)

where $\tilde{T}_m M(\sigma) =: S_m \cap T_m M(\sigma)$. Thus, with notation (21),

$$T_m M_E(\sigma) = \text{span}_R(v_f(m)) \oplus \tilde{T}_m M(\sigma).$$

(32)

For every $b \in \mathcal{B}$, as on page 12, let $M(\sigma_b), l = 1, \ldots, n_b$, be the connected components of $M(\sigma_b)$. For each $l$, let $d_{b_l}$ be the complex dimension of $M(\sigma_b)_l$, and $c_{b_l} = d - d_{b_l}$ its complex codimension in $M$. Then by transversality, $M_E(\sigma_b)_l =: M_E \cap M(\sigma_b)_l$ is a submanifold of $M$, and if non-empty it has (real) dimension $2d_{b_l} - 1$.

For any $m \in M_E(\sigma_b)_l$, in view of transversality the normal space to $M_E(\sigma_b)_l$ at $m$ is given by the orthogonal direct sum

$$N_m(M_E(\sigma_b)_l) = \text{span}_R(J_m v_f(m)) \oplus \text{im}(\text{id}_{M_E} - \text{id}_{T_m M}).$$

(33)

Now for any $m_0 \in M_E(\sigma_b)_l$ we can find an open neighborhood $U \subseteq M$ of $m$ and a smoothly varying family of geodesic local coordinates on $M$ centered at points $m \in U$, as in (23). We can also find a local $C^\infty$ unitary trivialization of the normal bundle $N_m(M_E(\sigma_b)_l) \cong \mathbb{R} \oplus \mathbb{C}^{c_{b_l}} \cong \mathbb{R}^{1+2c_{b_l}}$. In terms of these data, we obtain a local parametrization of a tubular neighborhood of $M_E(\sigma_b)_l$, and we shall write this additively in the form

$$(m, \lambda, n) : U' \times (-\delta, \delta) \times B_{2\omega}(0, \delta) \mapsto m + \left(\lambda \frac{J_m v_f(m)}{\|v_f(m)\|} + n\right),$$

(34)

where $U'$ is an open neighborhood of $m_0$ in $M_E(\sigma_b)_l$, $\delta > 0$ is suitably small, and $n \in \mathbb{R}^{2c_{b_l}}$ is identified with its image in im $\left(\text{id}_{T_m M} - d_m \phi_{M}^*\right)$.

**Remark 2.1.** In the notation of Definition 1.5 in (33) we have $n = n_0$. 
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2.3 Szegő kernel asymptotics

It is in order to make some recalls on the off-diagonal scaling asymptotics of the equivariant Szegő kernels $\Pi_k$, expressed in HLC ([BSZ1], [BSZ2], [SZ]). We refer in particular to [SZ] for a detailed discussion and definition of Heisenberg local coordinates, which we have touched upon on page 8. As is well-known, alternative approaches to the general theme of near-diagonal asymptotics of Bergman-Szegő kernels have been developed by other authors; see for instance [MZ], [MM1], [MM2], [Ch2], and references therein.

To begin with, given any fixed $C > 0$ and $\epsilon \in (0, 1/6)$, we have the following well-known rapid decrease away from slowly shrinking neighborhoods of the inverse image in $X \times X$ of the diagonal in $M \times M$:

**Proposition 2.2.** Uniformly for $\text{dist}_M (m_x, m_y) \geq C k^{\epsilon - 1/2}$, we have

$$\Pi_k(x, y) = O\left(k^{-\infty}\right).$$

This may be derived directly from the representation of $\Pi$ as in FIO in [BS], but see also the discussion in [C] for much more precise statements.

To describe the near-diagonal asymptotics, let us consider $x \in X$ and a HLC system centered at $x$; any $y$ close to the $S^1$-orbit of $x$ may be written in the form $y = x + (\theta, v)$. Let $\psi_2$ be as in Definition 1.6. Then for fixed $\epsilon \in (0, 1/6)$ and $C > 0$ the following asymptotic expansion holds for $k \to +\infty$ in the range $\|v_1\|, \|v_2\| \leq C k^{\epsilon}$:

$$\Pi_k\left(x + \left(\theta_1, \frac{v_1}{\sqrt{k}}\right), x + \left(\theta_2, \frac{v_2}{\sqrt{k}}\right)\right) \sim \left(\frac{k}{\pi}\right)^d e^{ik(\theta_1 - \theta_2) + \psi_2(v_1, v_2)} \left[1 + \sum_{j=1}^{+\infty} k^{-j/2} P_j(x; v_1, v_2)\right],$$

where $\psi_2$ is as in Definition 1.6, and $P_j(x; \cdot, \cdot)$ is a polynomial, with coefficients depending $C^\infty$-wise on $x$, of degree $\leq 3j$ and parity $j$ (see for instance the first two lines of (81) in [SZ]). If we write $P_j$ as the sum of its homogeneous components, we obtain

$$\Pi_k\left(x + \left(\theta_1, \frac{v_1}{\sqrt{k}}\right), x + \left(\theta_2, \frac{v_2}{\sqrt{k}}\right)\right) \sim \left(\frac{k}{\pi}\right)^d e^{ik(\theta_1 - \theta_2) + \psi_2(v_1, v_2)} \left[1 + \sum_{j=1}^{+\infty} k^{-j/2} \sum_{l=0}^{3j} Q_{j,l}(x; v_1, v_2)\right],$$

where now $Q_l(x, \cdot, \cdot)$ is a homogeneous polynomial of degree $l$, and vanishes unless $l - j$ is even.
Passing to unrescaled coordinates, we multiply $v_j$ by $\sqrt{k}$, and obtain

$$\Pi_k \left( x + (\theta_1, v_1), x + (\theta_2, v_2) \right)$$

$$\sim \left( \frac{k}{\pi} \right)^d e^{k[i(\theta_1 - \theta_2) + \phi_2(v_1, v_2)]} \cdot \left[ 1 + \sum_{j=1}^{+\infty} \sum_{l=0}^{3j} k^{(l-j)/2} Q_{j,l}(x; v_1, v_2) \right],$$

where only integer powers of $k$ contribute to the inner summation. The asymptotic expansion (36) holds in the range $\|v_j\| \leq Ck^{\epsilon-1/2}$.

It is convenient to rearrange the inner summands in (36) in the following manner. Let us set, for $j - l$ even, $b = (l - j)/2$; then $b$ is an integer, and $\left\lceil -j/2 \right\rceil \leq b \leq j$. Thus we have

$$\Pi_k \left( x + (\theta_1, v_1), x + (\theta_2, v_2) \right) \sim \left( \frac{k}{\pi} \right)^d e^{k[i(\theta_1 - \theta_2) + \phi_2(v_1, v_2)]} \cdot \left[ 1 + \sum_{j=1}^{+\infty} \sum_{b=\left\lceil -j/2 \right\rceil}^{j} k^b Q_{j,j+2b}(x; v_1, v_2) \right],$$

where

$$A_j(k, x; v_1, v_2) = \sum_{b=\left\lceil -j/2 \right\rceil}^{j} k^b Q_{j,j+2b}(x; v_1, v_2).$$

(38)

It follows from the previous discussion, and may readily checked directly, that (37) is an asymptotic expansion for $k \to +\infty$, in the range $\|v_j\| \leq C k^{\epsilon-1/2}$ for $\epsilon \in (0, 1/6)$. Indeed, we have $|A_j| \leq C_j k^{-\delta j}$ for $\delta = 1/6 - \epsilon$.

### 2.4 An integral formula for Gutzwiller-Töplitz kernels

Equation (5) is a representation of $U(\tau)$ in terms of spectral data; on the other hand, by definition $U(\tau)$ is a pull-back operator restricted to the Hardy space (recall (14)), and therefore its Schwartz kernel is also given by

$$U(\tau)_k(x, y) = \sum_j e_{k,j}(x_\tau) \cdot \overline{e_{k,j}(y)} = \Pi_k(x_\tau, y) \quad (x, y \in X).$$

(39)

Similarly, equation (9) is a representation of $G_k$ in terms of spectral data. We shall now combine (14) and (39) to obtain an integral representation of
\( G_k \) in terms of \( \Pi_k \). By definition of Fourier transform, we have

\[
G_k(x, y) = \int_{-\infty}^{+\infty} e^{-i\tau k E} \chi(\tau) \left[ \sum_{j=1}^{N_k} e^{i\lambda_j k} e_{kj}(x) \cdot e_{kj}(y) \right] d\tau
\]

\[
= \int_{-\infty}^{+\infty} e^{-i\tau k E} \chi(\tau) U(\tau_k(x, y)) d\tau
\]

\[
= \int_{-\infty}^{+\infty} e^{-i\tau k E} \chi(\tau) \Pi_k(x_{\tau}, y) d\tau
\]

(40)

### 2.5 A handy distance estimate

Let \( B_r(0, \delta) \subseteq \mathbb{R}^r \) be the open ball of radius \( \delta > 0 \) centered at the origin. Also, let \( \langle \cdot, \cdot \rangle_{st} : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R} \) be the standard Euclidean product, and let \( \| \cdot \| \) be the corresponding norm.

**Lemma 2.2.** Let \((M, g)\) be an \( r \)-dimensional Riemannian manifold, \( m \in M \), and let

\[
\eta : B_r(0, \delta) \to U = \eta(B_r(0, \delta)) \subseteq M
\]

be a local coordinate chart, satisfying \( \eta(0) = m \) and \( \eta^*(g)_0 = \langle \cdot, \cdot \rangle_{st} \). Then, perhaps after passing to a smaller \( \delta \) of \( m \) we have

\[
2\|w - w'\| \geq \text{dist}_M(\eta(w), \eta(w')) \geq \frac{1}{2}\|w - w'\|
\]

for any \( w, w' \in B_r(0, \delta) \).

Thus, \( d_0\eta : (\mathbb{R}^r, \langle \cdot, \cdot \rangle_{st}) \to (T_m M, g_m) \) is required to be an isometry.

**Proof of Lemma 2.2.** Let us denote by \( \| \cdot \|^*_p \) the norm at a point \( p \in B_r(0, \delta) \) associated to the pulled-back Riemannian metric \( \eta^*(g) \). Perhaps after passing to a smaller \( \delta \) we may assume, by continuity, that \( 2\| \cdot \| \geq \| \cdot \|^*_p \geq (1/2)\| \cdot \| \) for any \( p \in B_r(0, \delta) \).

If \( \gamma : [a, b] \to B_r(0, \delta) \) is any piecewise smooth curve, let us denote by \( \ell^*(\gamma) \) its length with respect to \( \eta^*(g) \), which is the same as the length \( \ell(\gamma) \) of \( \gamma' = \eta \circ \gamma : [a, b] \to M \) with respect to \( g \). If \( w, w' \in B_r(0, \delta) \) and \( \gamma(a) = w, \gamma(b) = w' \), then

\[
\ell^*(\gamma) = \int_a^b \| \dot{\gamma}(\tau) \|^*_\gamma(\tau) d\tau \geq \frac{1}{2} \int_a^b \| \dot{\gamma}(\tau) \| d\tau
\]

\[
\geq \frac{1}{2} \| \int_a^b \dot{\gamma}(\tau) d\tau \| = \frac{1}{2}\|w - w'\|.
\]

(41)
Suppose now $w, w' \in B_r(0, \delta/3)$. If $\gamma' : [a, b] \to M$ is any piecewise smooth curve joining $\gamma(w)$ and $\gamma(w')$, we can distinguish two cases.

If $\gamma'([a, b]) \subseteq U$, then applying the previous argument applied to $\gamma =: \eta^{-1} \circ \gamma'$ implies that $\ell(\gamma') \geq (1/2) \|w - w'\|$, where $\ell$ is the length with respect to $g$.

If $\gamma'([a, b]) \not\subseteq U$, then applying the same argument to the portion of the curve preceding the first boundary point we see that $\ell(\gamma') \geq (1/2) \left( \frac{\delta}{3} - \frac{\delta}{3} \right) = \delta/3 \geq (1/2) \|w - w'\|$.

Thus at any rate, passing to inf over $\gamma$, we conclude that the right inequality in the statement holds for any $w, w' \in B_r(0, \delta/3)$.

To prove the left inequality, consider $w, w' \in B_r(0, \delta)$ and the curve $\gamma : [0, 1] \to B_r(0, \delta)$ given by $\gamma(t) = (1 - t) w + t w'$; define $\gamma' =: \eta \circ \gamma$. Then
\[
\|w - w'\| = \int_0^1 \|\dot{\gamma}(\tau)\| \, d\tau \geq \frac{1}{2} \int_0^1 \|\dot{\gamma}(\tau)\|_{\gamma(\tau)} \, d\tau = \frac{1}{2} \ell(\gamma') \geq \frac{1}{2} \text{dist}_M(\eta(w), \eta(w')).
\]

\[
3 \text{ Proof of Theorem 1.1}
\]

**Proof of Theorem 1.1.** By Proposition 2.2 there exist constants $C_j > 0, j = 1, 2, \ldots$, such that for any choice of a sequence $(x_k, y_k) \in X \times X$ with
\[
\text{dist}_M(\pi(y_k), \pi(x_k) \chi) \geq C k^{-1/2}
\]
we have
\[
|\Pi_k(x_\tau, y)| < C_j k^{-j}
\]
for $k \geq 1$, for any $\tau \in \text{supp}(\chi)$. Since $\chi$ is compactly supported, we then deduce from (40) and (42) that $G_k(x_k, y_k) = O\left(k^{-\infty}\right)$ for $k \to +\infty$.

We are then reduced to assuming that
\[
\text{dist}_M(n_k, m_k \chi) \leq C k^{\epsilon - \frac{1}{2}},
\]
where we have set $m_k =: \pi(x_k), n_k =: \pi(y_k)$.

To begin with, let us fix some small $\epsilon_1 > 0$ and consider a bump function $\varrho_1 \in C_0^\infty(\mathbb{R})$ such that $\varrho_1 \equiv 1$ on $(-\epsilon_1, \epsilon_1)$ and $\varrho_1 \equiv 0$ on $\mathbb{R} \setminus (-2\epsilon_1, 2\epsilon_1)$. Then, by the previous considerations,
\[
G_k(x, y) \sim \int_{-\infty}^{+\infty} e^{-\imath r k E} \chi(\tau) \varrho_1\left(\text{dist}_M(m_{k\tau}, n_k)\right) \Pi_k(x_{k\tau}, y_k) \, d\tau,
\]
where $\sim$ means ‘has the same asymptotics as’.
Since our analysis is local in the neighborhood of \( y_k \), we can find a (smoothly varying) HLC system centered at \( y_k \), and in terms of the latter we shall write

\[
x_{k\tau} = y_k + \left( \theta_k(\tau), \mathbf{v}_k(\tau) \right), \quad m_{k\tau} = n_k + \mathbf{v}_k(\tau),
\]

where the second equality should be interpreted in terms of the associated adapted local coordinates on \( M \) (see the discussion on page 8). Since HLC are isometric at the origin, we have for sufficiently small \( \epsilon_1 \) that

\[
\frac{1}{2} \| \mathbf{v}_k(\tau) \| \leq \text{dist}_M(m_{k\tau}, n_k) \leq 2 \| \mathbf{v}_k(\tau) \|. \tag{44}
\]

In particular, we have

\[
\| \mathbf{v}_k(\tau) \| \leq 2C_k \epsilon_1^{-1/2}. \tag{45}
\]

Let us then choose another bump function \( \varrho \in C^\infty_0(\mathbb{R}^{2d}) \), such that \( \varrho \equiv 1 \) on \( B_{2d}(0, 2C) \) and \( \varrho \equiv 0 \) on \( \mathbb{R} \setminus B_{2d}(0, 3C) \). If \( \varrho \left( k^{1/2-\epsilon} \mathbf{v}_k(\tau) \right) \neq 1 \), then

\[
\text{dist}_M(m_{k\tau}, n_k) \geq C k^{\epsilon - 1/2}. \tag{46}
\]

Again, we conclude therefore that \( \Pi_k(x_{k\tau}, y_k) = O(k^{-\infty}) \) in the range where \( \varrho \left( k^{1/2-\epsilon} \mathbf{v}_k(\tau) \right) \neq 1 \). This implies

\[
\mathcal{G}_k(x, y) \sim \int_{-\infty}^{+\infty} e^{-i\tau k E} \chi(\tau) \varrho_1(\text{dist}_M(m_{k\tau}, n_k)) \varrho \left( (k^{1/2-\epsilon} \mathbf{v}_k(\tau)) \right) \Pi_k(x_{k\tau}, y_k) d\tau
\]

\[
= \int_{-\infty}^{+\infty} e^{-i\tau k E} \chi(\tau) \varrho \left( (k^{1/2-\epsilon} \mathbf{v}_k(\tau)) \right) \Pi_k \left( y_k + \left( \theta_k(\tau), \mathbf{v}_k(\tau) \right), y_k \right) d\tau.
\]

The latter equality follows for \( k \gg 0 \) because \( \varrho \left( (k^{1/2-\epsilon} \mathbf{v}_k(\tau)) \right) \neq 0 \) implies, in view of (44), that

\[
\text{dist}_M(m_{k\tau}, n_k) \leq 2 \| \mathbf{v}_k(\tau) \| \leq 6 C k^{\epsilon - 1/2} \ll \epsilon_1 \implies \varrho_1(\text{dist}_M(m_{k\tau}, n_k)) = 1.
\]

Now we have, in view of (37),

\[
\Pi_k \left( y_k + \left( \theta_k(\tau), \mathbf{v}_k(\tau) \right), y_k \right) \sim \left( \frac{k}{\pi} \right)^d e^{i k \theta_k(\tau) + \frac{i}{2} \| \mathbf{v}_k(\tau) \|^2} \cdot \left[ 1 + \sum_{j=1}^{+\infty} A_j(k, x; \mathbf{v}_k(\tau), 0) \right]. \tag{48}
\]
Inserting (48) in (47), we obtain
\[
G_k(x_k, y_k) \sim \left( \frac{k}{\pi} \right)^d \int_{-\infty}^{+\infty} e^{ik\Gamma(x_k, y_k, \tau)} \chi(\tau) \varrho \left( k^{1/2-\epsilon} v_k(\tau) \right)
\]
\[
\cdot \left[ 1 + \sum_{j=1}^{+\infty} A_j(k, x; v_k(\tau), 0) \right] d\tau,
\]
where
\[
\Gamma(x_k, y_k, \tau) = \theta_k(\tau) + i\frac{2}{\pi} \|v_k(\tau)\|^2 - \tau E,
\]
and the asymptotic expansion can be integrated term by term.

To proceed further, we need to make \( \Gamma \) more explicit. Let us fix \( \tau_0 \) in the support of the integrand, and make the change of variables \( \tau \to \tau_0 + \tau \) in a small neighborhood of \( \tau_0 \), so that now \( \tau \sim 0 \). Then by Corollary 2.2 of [P5] we have
\[
y_k + \left( \theta_k(\tau_0 + \tau), v_k(\tau_0 + \tau) \right) = \phi_{\tau-\tau_0}(x_k)
\]
\[
= \phi_{\tau}^x \left( \phi_{-\tau_0}^x(x_k) \right) = \phi_{-\tau}^x \left( y_k + \left( \theta_k(\tau_0), v_k(\tau_0) \right) \right)
\]
\[
= y_k + \left( \theta_k(\tau_0) + \tau f(n_k) + \tau \omega_n \left( v_f(n_k), v_k(\tau_0) \right) + R_3(\tau; v_k(\tau_0)), \right)
\]
\[
+ \left( v_k(\tau_0) - \tau v_f(n_k) + R_2(\tau; v_k(\tau_0)) \right).
\]

Therefore, we obtain
\[
\Gamma(x_k, y_k, \tau_0 + \tau) = \theta_k(\tau_0) + \tau \left[ f(n_k) + \omega_n \left( v_f(n_k), v_k(\tau_0) \right) \right] - E + R_3(\tau; v_k(\tau_0))
\]
\[
+ \frac{i}{2} \left\| v_k(\tau_0) - \tau v_f(n_k) + R_2(\tau; v_k(\tau_0)) \right\|^2
\]
\[
\left[ f(n_k) + \omega_n \left( v_f(n_k), v_k(\tau_0) \right) \right] - E - i \left( v_k(\tau_0) v_f(n_k) \right)
\]
\[
+ \frac{i}{2} \tau^2 \left\| v_f(n_k) \right\|^2 + R_3(\tau; v_k(\tau_0)).
\]

Since the \( R_3 \) term on the first line is real, the first equality in (51) implies \( \Re \Gamma \geq 0 \).

In view of (48), applied with \( \tau_0 + \tau \) in place of \( \tau \), the \( j \)-th summand in the asymptotic expansion for the amplitude in the integrand in (49) is given
by
\[ \mathcal{A}_j(x_k, y_k, \tau_0 + \tau) =: \left( \frac{k}{\pi} \right)^d \chi(\tau_0 + \tau) \theta \left( k^{1/2-\epsilon} v_k(\tau_0 + \tau) \right) A_j(k; x; v_k(\tau_0 + \tau), 0) \]
\[ = \left( \frac{k}{\pi} \right)^d \chi(\tau_0 + \tau) \theta \left( k^{1/2-\epsilon} \left[ v_k(\tau_0) - \tau u_j(n_k) + R_2(\tau, v_k(\tau_0)) \right] \right) \]
\[ \cdot \sum_{b = [-j/2]}^{j} k^b Q_{j+2b} \left( y_k; v_k(\tau_0) - \tau u_j(n_k) + R_2(\tau, v_k(\tau_0)), 0 \right). \]

This is defined for \( j \geq 1 \), but for convenience we shall extend to \( j = 0 \) by summing over \( j \geq 0 \) with \( A_0 = 1 \) in (49).

In view of the factor \( k^{1/2-\epsilon} \) appearing in (52), and the considerations following (38), we have for \( a \geq 0 \)
\[ \left| \frac{\partial^a \mathcal{A}_j}{\partial \tau^a}(x_k, y_k, \tau) \right|_{\tau = \tau_0} = \left| \frac{\partial^a \mathcal{A}_j}{\partial \tau^a}(x_k, y_k, \tau_0 + \tau) \right|_{\tau = 0} = O \left( k^{d-3j(1/6-\epsilon)+a(1/2-\epsilon)} \right). \]  

(53)

On the other hand, given (51) we have
\[ \left. \frac{\partial}{\partial \tau} \Gamma(x_k, y_k, \tau) \right|_{\tau = \tau_0} = \left. \frac{\partial}{\partial \tau} \Gamma(x_k, y_k, \tau_0 + \tau) \right|_{\tau = 0} = f(n_k) - E + R_1(v_k(\tau_0)). \]

(54)

Furthermore, in view of (45), for \( k \gg 0 \) the \( R_1 \) term on the last line of (54) is bounded by \( C_1 C k^{\epsilon-1/2} \), where \( C_1 \) only depends on \( f \) and \( E \).

If then \( |f(n_k) - E| \geq 2C_1 C k^{\epsilon-1/2} \), we conclude from (51) that for any \( \tau \in \text{supp}(\chi) \) and \( k \gg 0 \) one has
\[ \left. \frac{\partial}{\partial \tau} \Gamma(x_k, y_k, \tau) \right| \geq C_1 C k^{\epsilon-1/2}. \]

(55)

Let us consider the \( k \)-dependent differential operator on \( \mathbb{R} \)
\[ L_k =: \left( \frac{1}{f(n_k) - E + R_1(v_k(\tau))} \frac{\partial}{\partial \tau} \right), \]
where of course \( \partial_{\tau} \Gamma = \partial \Gamma / \partial \tau \).
3 Proof of Theorem 1.1

Then
\[ e^{ik\Gamma(x_k,y_k,\tau)} = \frac{1}{ik} L_k \left( e^{ik\Gamma(x_k,y_k,\tau)} \right). \] (56)

Iterating \( r \) times integration by parts, we obtain for the \( j \)-th summand in (49) (with \( A_0 = 1 \))
\[
\int_{-\infty}^{+\infty} e^{ik\Gamma(x_k,y_k,\tau)} A_j(x_k,y_k,\tau) \, d\tau \quad (57)
\]
where
\[
\tilde{L}_k(h) = \frac{\partial}{\partial \tau} \left( \frac{1}{\partial_{\tau} \Gamma(x_k,y_k,\tau)} \cdot h \right) \quad (h \in C^\infty(\mathbb{R})).
\]

In general, given \( \gamma \in C^\infty(\mathbb{R}) \) not vanishing identically, let us denote by \( U_\gamma \subseteq \mathbb{R} \) the open set where \( \gamma \neq 0 \), and define
\[
L_\gamma(h) = \frac{\partial}{\partial \tau} \left( \frac{1}{\gamma} \cdot h \right) \quad (h \in C^\infty(U_\gamma)),
\]
viewed as an operator \( C^\infty(U_\gamma) \to C^\infty(U_\gamma) \).

**Lemma 3.1.** For any integer \( r \geq 0 \),
\[
\tilde{L}_k^r(h) = \sum_{a+b \leq 2r} P_{a,b}(\gamma) \frac{\partial_{\tau}^a h}{\gamma^b} \quad (h \in C^\infty(U_\gamma)), \] (58)
where \( P_{a,b}(\gamma) \) is a differential operator of degree \( d_{ab} \) satisfying \( d_{ab} + a \leq r \).

**Proof of Lemma 3.1.** The claim is obvious for \( r = 0 \), and for \( r = 1 \) we have
\[
L_\gamma(h) = \frac{\partial}{\partial \tau} \left( \frac{1}{\gamma} \cdot h \right) = \frac{\gamma \partial_{\tau} h - h \partial_{\tau} \gamma}{\gamma^2} = \frac{\partial_{\tau} h}{\gamma} - \partial_{\tau} \gamma \frac{h}{\gamma^2}.
\]

Let us assume that the claim is true for \( r \geq 1 \). Then
\[
\tilde{L}_k^{r+1}(h) = \frac{\partial}{\partial r} \left( \sum_{a+b \leq 2r} P_{a,b}(\gamma) \frac{\partial_{\tau}^a h}{\gamma^b+1} \right)
\]
\[
= \sum_{a+b \leq 2r} \left\{ \frac{1}{\gamma^{b+1}} \left[ (\partial_{\tau} P_{a,b}(\gamma)) \cdot \partial_{\tau}^a h + P_{a,b}(\gamma) \partial_{\tau}^{a+1} h \right] - \frac{(b+1)}{\gamma^{b+2}} \partial_{\tau} \gamma P_{a,b}(\gamma) \partial_{\tau}^b h \right\},
\]
which clearly implies the statement for \( r + 1 \). □
Let us apply Lemma 3.1 with $h = A_j$ and $\gamma = \partial \Gamma_k$. In view of (53) and (55) with obtain for each of the summands in (58) the estimate

$$\left| P_{a,b}(\partial \Gamma_k) \frac{\partial^r \Gamma_k}{(\partial \Gamma_k)}^b \right| \leq C_{ab} k^{d-3j(1/6-\epsilon)+(a+b)(1/2-\epsilon)} \leq C_{ab} k^{d-3j(1/6-\epsilon)+r-2r\epsilon}. \quad (59)$$

Inserting this estimate in (57), we deduce that for every $r$ and $j$ there exist constants $C_{rj} > 0$ such that

$$\left| \int_{-\infty}^{+\infty} e^{ik(x_k,y_k,\tau)} A_j(x_k, y_k, \tau) d\tau \right| \leq C_{rj} k^{d-3j(1/6-\epsilon)-2r\epsilon}.$$

We have thus proved the following. Given any $C > 0$ and $\epsilon \in (0, 1/6)$, we have $G_k(x_k, y_k) = O(k^{-\infty})$ uniformly for all sequences $(x_k, y_k) \in X \times X$ such that either

A): $\text{dist}_M (n_k, m_k^\chi) \geq C k^{\epsilon-1/6}$, or

B): $|f(n_k) - E| \geq DC k^{\epsilon-1/6}$, for an appropriate constant $D > 0$, depending only on $f$ and $E$.

Now suppose that:

$$\max \{ \text{dist}_M (n_k, m_k^\chi), |f(n_k) - E| \} \geq C k^{\epsilon-\frac{1}{2}}.$$

In order to prove that $G_k(x_k, y_k) = O(k^{-\infty})$, after passing to subsequences we are reduced to assuming that either

1. $\text{dist}_M (n_k, m_k^\chi) \geq (C/D) k^{\epsilon-1/6}$, or else

2. $\text{dist}_M (n_k, m_k^\chi) \leq (C/D) k^{\epsilon-1/6}$ and $|f(m_k) - E| \geq C k^{\epsilon-1/6}$.

The statement then follows by replacing $C$ by $C/D$ in A) and B) above.

Finally, let us assume that a sequence $(x_k, y_k) \in X \times X$ satisfies

$$\max \{ \text{dist}_M (n_k, m_k^\chi), |f(n_k) - E|, |f(m_k) - E| \} \geq C k^{\epsilon-\frac{1}{2}}. \quad (60)$$

There is a constant $D' \geq 1$ such that $|f(m) - f(n)| \leq D' \text{dist}_M (m, n)$ for all $m, n \in M$. Since $f$ is $\phi^M$ invariant, we deduce that in fact

$$|f(m) - f(n)| \leq D' \text{dist}_M (m, m^\chi), \quad (m, n \in M). \quad (61)$$

Fix $R > 0$ a large positive constant. After passing to subsequences, we fall under one of the following the three cases:
4 Proof of Theorem 1.2

Proof of Theorem 1.2. We shall first work in unrescaled coordinates and set \( x + v_1 = x + (0, v_1), y + v_2 = y + (0, v_2) \), where \( \|v_1\|, \|v_2\| \leq C k^{e^{-1/2}} \). We get

\[
G_k(x + \theta_1, v_1, y + \theta_2, v_2) = e^{i k (\theta_1 - \theta_2)} \int_{-\infty}^{+\infty} e^{-i \tau k E} \chi(\tau) \Pi_k((x + v_1) \tau, y + v_2) d\tau.
\]

Let us set \( m = m_x, n = m_y \).

**Proposition 4.1.** There is a constant \( \beta = \beta(x, y) > 0 \) such that the contribution of the locus in \( \mathbb{R} \) where

\[
\text{dist}_M(\tau, \text{Per}^R_M(n \mapsto m)) \geq \beta C k^{e^{-1/2}}
\]

to the asymptotics for \( k \to +\infty \) of (63) is rapidly decreasing. If \( E \) is a regular value of \( f \), then \( \beta \) can be chosen uniformly on \( \mathcal{R}(f, E) \).
Given any $\delta > 0$ for $k \gg 0$ we have (in adapted local coordinates, see page 8)

$$
(1 - \delta) \|v_1\| \leq \text{dist}_M(m_x, m_x + v_1) \leq (1 + \delta) \|v_1\|,
$$

and similarly for $m_y$ and $v_2$. Therefore, using that $\phi^M$ is a Riemannian isometry, we get

$$
dist_M(m_y, n) \leq \text{dist}_M(m_y, (m + v_1)_\tau) + \text{dist}_M((m + v_1)_\tau, n + v_2) + \text{dist}_M(n + v_2, n)
= \text{dist}_M(m, m + v_1) + \text{dist}_M((m + v_1)_\tau, n + v_2) + \text{dist}_M(n + v_2, n)
\leq C (1 + \delta) \|v_1\| + \text{dist}_M((m + v_1)_\tau, n + v_2) + C (1 + \delta) \|v_2\|
\leq 2C (1 + \delta) k^{\varepsilon - 1/2} + \text{dist}_M((m + v_1)_\tau, n + v_2).
$$

On the other hand, if (63) holds then $\tau = \sigma + \tau'$, where $\sigma \in \text{Per}_M(n \mapsto m)$ and $|\tau'| \geq \beta C k^{\varepsilon - 1/2}$. Thus $m_{\sigma} = \phi^M_{\tau'}(n)$, whence

$$
dist_M(m_{\tau}, n) \geq |\tau'| (1 - \delta) \|v_f(m)\| \geq \beta C (1 - \delta) \|v_f(m)\| k^{\varepsilon - 1/2}.\tag{66}
$$

From (65) and (66) we obtain that if $\text{dist}_R(\tau, \text{Per}_M(n \mapsto m)) \geq \beta C k^{\varepsilon - 1/2}$ then

$$
dist_M((m + v_1)_\tau, n + v_2) \geq \text{dist}_M(m_{\tau}, n) - 2C (1 + \delta) k^{\varepsilon - 1/2}
\geq \beta C (1 - \delta) \|v_f(m)\| k^{\varepsilon - 1/2} - 2C (1 + \delta) k^{\varepsilon - 1/2}
= C \left[\beta (1 - \delta) \|v_f(m)\| - 2 (1 + \delta)\right] k^{\varepsilon - 1/2}.
$$

Thus it suffices to choose $\beta > 3/\|v_f(m)\|$, say, to conclude that in (62) we have $\Pi_k((x + v_1)_\tau, y + v_2) = O(k^{-\infty})$, for $k \to +\infty$, uniformly where (63) is satisfied.

Proposition 4.1 means the following. First of all, let $\varrho_0 \in C^\infty(\mathbb{R})$ be $\geq 0$, $\equiv 1$ on $(-1, 1)$ and $\equiv 0$ on $\mathbb{R} \setminus (-2, 2)$. Given some very small $\varepsilon_1 > 0$, let us set $\varrho(\cdot) =: \sum_a \varrho_0((\cdot - a)/\varepsilon_1)$ (recall (14)). Then $\varrho$ is supported in a small neighborhood of $\text{Per}_M(n \mapsto m)$, namely

$$
supp(\varrho) \subseteq \bigcup_{a \in A} (a - 2\varepsilon_1, a + 2\varepsilon_1),
$$

and

$$
\varrho \equiv 1 \text{ on } \bigcup_{a \in A} (a - \varepsilon_1, a + \varepsilon_1),
$$

(69)
Thus first of all we have:
\[
\mathcal{G}_k(x + (\theta_1, \mathbf{v}_1), y + (\theta_2, \mathbf{v}_2)) = e^{ik(\theta_1 - \theta_2)} \int_{-\infty}^{+\infty} e^{-i\tau kE} \varrho(\tau) \chi(\tau) \Pi_k((x + \mathbf{v}_1)_\tau, y + \mathbf{v}_2) \, d\tau
\]
\[
\sim e^{ik(\theta_1 - \theta_2)} \sum_a \int_{-\infty}^{+\infty} e^{-i\tau kE} \varrho_0((\tau - \tau_a)/\varepsilon_1) \chi(\tau) \Pi_k((x + \mathbf{v}_1)_\tau, y + \mathbf{v}_2) \, d\tau.
\]

By the Proposition, only a rapidly decreasing contribution to the asymptotics is lost, if the integrand is further multiplied by \( \varrho_0(k^{1/2-\varepsilon}(\tau - \tau_a)/\beta C) \); in addition, for \( k \gg 0 \) this factor is supported where \( \varrho_0((\tau - \tau_a)/\varepsilon_1) = 1 \). Therefore, we conclude
\[
\mathcal{G}_k(x + (\theta_1, \mathbf{v}_1), y + (\theta_2, \mathbf{v}_2)) \sim e^{ik(\theta_1 - \theta_2)} \sum_a \int_{-\infty}^{+\infty} e^{-i\tau kE} \varrho_0(k^{1/2-\varepsilon}(\tau - \tau_a)/\beta C) \chi(\tau) \Pi_k((x + \mathbf{v}_1)_\tau, y + \mathbf{v}_2) \, d\tau
\]
\[
= e^{ik(\theta_1 - \theta_2)} \sum_a \mathcal{G}^{(a)}_k(x + \mathbf{v}_1, y + \mathbf{v}_2),
\]

where we have set, with \( \nu(\cdot) =: \varrho_0(\cdot/\beta C) \),
\[
\mathcal{G}^{(a)}_k(x + \mathbf{v}_1, y + \mathbf{v}_2) = \int_{-\infty}^{+\infty} e^{-i\tau kE} \nu(k^{1/2-\varepsilon}(\tau - \tau_a)) \chi(\tau) \Pi_k((x + \mathbf{v}_1)_\tau, y + \mathbf{v}_2) \, d\tau
\]
\[
= e^{-i\tau_a kE} \int_{-\infty}^{+\infty} e^{-i\tau kE} \nu(k^{1/2-\varepsilon}\tau) \chi(\tau + \tau_a) \Pi_k((x + \mathbf{v}_1)_{\tau + \tau_a}, y + \mathbf{v}_2) \, d\tau.
\]

In the latter integral, the integrand is supported where \( |\tau| \leq 2\beta C k^{\varepsilon-1/2} \); in addition, \( \nu(k^{1/2-\varepsilon}\tau) \equiv 1 \) for \( |\tau| \leq \beta C k^{\varepsilon-1/2} \). The constant \( \beta \) can be chosen arbitrarily large.

In order to proceed further, we need to give an explicit expression for \((x + \mathbf{v}_1)_{\tau + \tau_a}\). Given (15), a slight modification of Lemma 3.2 of [P4] shows that, with \( A_a \) as on page 11,
\[
\phi_{x+\tau_a}(x + \mathbf{v}_1) = y + (\vartheta_a + R_3(\mathbf{v}_1), A_a \mathbf{v}_1 + R_2(\mathbf{v}_1)).
\]
Therefore, applying Corollary 2.2 of [P5] we have:

\begin{align*}
(x + v_1)_{\tau + \tau_a} &= \phi^X_{-(\tau + \tau_a)}(x + v_1) = \phi^X_{-\tau}(\phi^X_{-\tau_a}(x + v_1)) \\
&= \phi^X_{-\tau}(y + (\theta_a + R_3(v_1), A_a v_1 + R_2(v_1))) \\
&= y + (\theta_a + \tau \left[ f(n) + \omega_n(v_f(n), A_a v_1) \right] + R_3(\tau, v_1), A_a v_1 - \tau f(n) + R_2(\tau, v_1)) \\
&= y + (\theta_a + \theta_a(\tau), v_a(\tau)),
\end{align*}

where $R_j$ is as usual a $C^\infty$ function vanishing to $j$-th order at the origin, possibly depending on $a$, and $\theta_a(\tau), v_a(\tau)$ are defined by the latter equality.

Thus in view of (37) we get

\begin{align*}
\Pi_k((x + v_1)_{\tau + \tau_a}, y + v_2) &= \Pi_k(y + (\theta_a + \theta_a(\tau), v_a(\tau)), y + v_2) \\
&\sim \left( \frac{k}{\pi} \right)^d e^{ik[\theta_a + \theta_a(\tau) + \psi_2(v_a(\tau), v_2)]} \cdot \left[ 1 + \sum_{j=1}^{+\infty} A_j(k, y; v_a(\tau), v_2) \right].
\end{align*}

(75)

We deduce from (72) and (75) that

\begin{align*}
G^{(a)}_k(x + v_1, y + v_2) &= e^{ik(\theta_a - \tau_a)E} \left( \frac{k}{\pi} \right)^d \int_{-\infty}^{+\infty} e^{ik[\theta_a(\tau) - \tau E - i\psi_2(v_a(\tau), v_2)]} A_k(y, v_a(\tau), v_2) \, d\tau,
\end{align*}

where

\begin{align*}
A_k(\tau, v_a(\tau), v_2) &= \nu \left( k^{1/2 - \epsilon} \tau \right) \chi(\tau + \tau_a) \cdot \left[ 1 + \sum_{j=1}^{+\infty} \sum_{b=-[j/2]}^{j} k^b Q_{j+2b}(y; v_a(\tau), v_2) \right],
\end{align*}

(77)

and the asymptotic expansion can be integrated term by term.
We can expand the exponent in (76) as follows:

\[
\begin{align*}
\theta_n(\tau) & \quad - \tau E - i \psi_2(v_a(\tau), v_2) \\
& \quad = \tau \left[ f(n) - E + \omega_n(v_f(n), A_a v_1) \right] + R_3(\tau, v_1) \\
& \quad - \omega_n(A_a v_1 - \tau v_f(n) + R_2(\tau, v_1), v_2) + \frac{i}{2} \| A_a v_1 - v_2 - \tau v_f(n) + R_2(\tau, v_1) \|^2 \\
& \quad = -\omega_n(A_a v_1, v_2) + \frac{i}{2} \| A_a v_a - v_2 \|^2 + R_3(\tau, v_1, v_2) \\
& \quad \tau \left[ f(n) - E + \omega_n(v_f(n), A_a v_1 + v_2) \right] \\
& \quad + \frac{i}{2} [\tau^2 \| v_f(n) \|^2 - 2 \tau g_n(v_f(n), A_a v_1 - v_2)] \\
& \quad = -i \psi_2(A_a v_1, v_2) + \Upsilon_a(y, v_1, v_2; \tau) + R_3(\tau, v_1, v_2),
\end{align*}
\]

where we have set, using now that \( f(n) = E \),

\[
\Upsilon_a(y, v_1, v_2; \tau) = \tau \omega_n(v_f(n), A_a v_1 + v_2) + \frac{i}{2} \tau^2 \| v_f(n) \|^2 - 2 \tau g_n(v_f(n), A_a v_1 - v_2).
\]

We can then rewrite (76) in the following manner:

\[
\mathcal{G}_k^{(n)}(x + v_1, y + v_2) = e^{ik(\theta_n - \tau n E)} \left( \frac{k}{\pi} \right) d \int_{-\infty}^{+\infty} e^{ik\Upsilon_a(y, v_1, v_2; \tau)} B_k(y, v_1(\tau), v_2) d\tau,
\]

where

\[
\begin{align*}
\Upsilon_a(y, v_1, v_2; \tau) &= -i \psi_2(A_a v_1, v_2) + \Upsilon_a(y, v_1, v_2; \tau), \\
B_k(y, v_1(\tau), v_2) &= e^{ikR_3(\tau, v_1, v_2)} A_k(y, v_1(\tau), v_2)
\end{align*}
\]

We shall view (81) as an oscillatory integral in \( d\tau \), with phase \( \Upsilon_a \) and amplitude \( B_k(y, v_1(\tau), v_2) \). The imaginary part of \( \Upsilon_a \) satisfies:

\[
\Im \Upsilon_a(y, v_1, v_2; \tau) = \frac{1}{2} \left[ \| A_a v_a - v_2 \|^2 + \tau^2 \| v_f(n) \|^2 - 2 \tau g_n(v_f(n), A_a v_1 - v_2) \right]
\]

\[
= \frac{1}{2} \| A_a v_a - v_2 - \tau v_f(n) \|^2 \geq 0.
\]

Regarding the phase, momentarily viewing \( k \) as a continuous parameter we have:
Lemma 4.1. \( B(y, v_a(\cdot), v_2) \in S_{1/2-\epsilon}^0(\mathbb{R} \times \mathbb{R}_+), \) as a function of \((\tau, k) \in \mathbb{R} \times \mathbb{R}_+, \) uniformly in \( y, v_1 \) and \( v_2 \) with \( \|v_j\| \leq C k^{\epsilon-1/2}. \)

Proof of Lemma 4.1. In view of (77) and (82), we have

\[
B_k(y, v_a(\tau), v_2) = \sum_{j=0}^{+\infty} \sum_{b=\lceil-j/2\rceil}^j F_{j,b}(\tau, k),
\]

where

\[
F_{j,b}(\tau, k) := \nu \left( k^{1/2-\epsilon} \chi(\tau + \tau_a) \cdot e^{ikR_3(\tau, v_1, v_2)} k^b Q_{j+2b}(y; v_a(\tau), v_2),
\]

Since the asymptotic expansion for \( \Pi_k \) can be differentiated any number of times, it suffices to prove that every \( F_{j,b} \in S_{-3j(1/6-\epsilon)}^{1/2-\epsilon}(\mathbb{R} \times \mathbb{R}_+). \)

Thus we need to show that for any \( j \geq 0 \) and \( \lceil-j/2\rceil \leq b \leq j \), and every \( l \in \mathbb{N} \) we have

\[
F_{j,b}^{(l)}(\tau, k) =: \frac{\partial^l F_{j,b}(\tau, k)}{\partial \tau^l} = O \left( k^{l(1/2-\epsilon)} \right)
\]

as \( k \to +\infty. \)

On the support of \( F_{j,b} \), we have

\[
\max \{|\tau|, \|v_1\|, \|v_2\|\} \leq D k^{\epsilon-1/2}
\]

for some \( D \gg 0. \) Therefore, the exponent in (82) satisfies

\[
|k R_3(\tau, v_1, v_2)| = O \left( k^{3(1/6-\epsilon)} \right).
\]

(85)

Similarly, on the same domain we have

\[
|k^b Q_{j+2b}(y; v_a(\tau), v_2)| \leq D_1 k^{b+(j+2b)(-1/2)} \leq D_1 k^{-3j(1/6-\epsilon)}.
\]

Thus the statement is clear when \( l = 0. \)

Let us make the inductive hypothesis that for \( l \leq l_0 \) \( F_{j,b}^{(l)} \) is a linear combination of terms of the form

\[
k^{b'+a'(1/2-\epsilon)} \nu \left( k^{1/2-\epsilon} \right) R_{c'}(\tau, v_1, v_2) \cdot e^{ikR_3(\tau, v_1, v_2)};
\]

(86)

where

\[
b' + (a' - c') \left( \frac{1}{2} - \epsilon \right) \leq -3j \left( \frac{1}{6} - \epsilon \right) + l \left( \frac{1}{2} - \epsilon \right).
\]

(87)

If we differentiate \( F_{j,b}^{(l_0)} \) with respect to \( \tau \), thus passing from \( l_0 \) to \( l_0 + 1 \), we obtain by the Leibnitz rule the sum of three terms, as follows. The first has the form
the second has the form
\[ k^{b'} \left( k^{1/2-\epsilon} \tau \right) R_c(\tau, v_1, v_2) \cdot e^{ikR_3(\tau, v_1, v_2)}; \] (88)

since \( \partial_\tau R_c = R_c - 1 \); finally the third has the form
\[ k^{b'+a'(1/2-\epsilon)} \nu \left( k^{1/2-\epsilon} \tau \right) R_c(\tau, v_1, v_2) kR_2(\tau, v_1, v_2) \cdot e^{ikR_3(\tau, v_1, v_2)}, \] (90)

since \( \partial_\tau [\tau R_3] = k R_2 \).

In case (88), \((a', b', c')\) get replaced by \((a'+1, b', c')\), in case (89) by \((a', b', c'-1)\), and in case (90) by \((a', b'+1, c'+2)\). At any rate, the inductive step is complete.

We can in fact rewrite the expansion for \( B_k \) as we did for \( A_k \).

**Lemma 4.2.** We have the following asymptotic expansion:
\[ B_k(\tau, v_a(\tau), v_2) \sim \nu \left( k^{1/2-\epsilon} \tau \right) \chi(\tau + \tau_a) \cdot \left[ 1 + \sum_{j=1}^{+\infty} \sum_{b=\lceil -j/2 \rceil} k^b P_j^{(j,b)}(y; v_a(\tau), v_2) \right], \]

where \( P_j^{(j,b)} \) is a homogeneous polynomial of degree \( l \).

**Proof of Lemma 4.2.** First we notice that for every \( N \geq 0 \) we have:
\[ e^{ikR_3(\tau, v_1, v_2)} = \sum_{l=0}^{N} \frac{k^l}{l!} R_3(\tau, v_1, v_2) + O \left( k^{N+1} R_3(N+1)(\tau, v_1, v_2) \right). \] (91)

The remainder term satisfies
\[ \left| k^{N+1} R_3(N+1)(\tau, v_1, v_2) \right| = O \left( k^{N+1+3(N+1)(\epsilon-1/2)} \right) = O \left( k^{-1/2+3\epsilon-3N(1/6-\epsilon)} \right). \] (92)

We can thus multiply the asymptotic expansions for \( A_k \) and for the exponential term. Now the general term in the asymptotic expansion for \( A_k \) is a multiple of
\[ k^b \tau^{r_0} v_1^{r_1} v_2^{r_2}, \]
where \( r_0 \in \mathbb{N}_0, \) \( r_1 \) and \( r_2 \) are multi-indexes with \( r_0 + |r_1| + |r_2| \geq j + 2b \), and \([-j/2]\) \( \leq b \leq j \). On the other hand, the general term in the expansion for the exponential is a multiple of

\[
k^s \tau^r v_1^r v_2^r,
\]

where now

\[
r'_0 + |r'| + |r'_2| \geq 3s,
\]

with \( s \geq 0 \). If we then multiply the two expansions, we get that

\[
e^{ikR_\alpha(x,v_1,v_2)} \cdot A_k(\tau, v_1, v_2)
\]

is a linear combination of terms of the form

\[
k^{b+s} \tau^{r_0+r'_0} v_1^{r_1+r'_1} v_2^{r_2+r'_2}.
\]

Let us set \( \rho_0 = r_0 + r'_0, \rho_j = r_j + r'_j \) for \( j = 1, 2, b' = b + s, j' = j + s \). We can then rewrite (93) in the form

\[
k^{b'} \tau^{\rho_0} v_1^{\rho_1} v_2^{\rho_2}
\]

with

\[
\rho_0 + |\rho_1| + |\rho_2| = (r_0 + |r_1| + |r_2|) + (r'_0 + |r'_1| + |r'_2|) \geq j + 2b + 3s = (j + s) + 2(b + s) = j' + 2b'.
\]

On the other hand, we have

\[
[-j'/2] = [-(j + s)/2] \leq [-j/2] \leq b' = b + s \leq j + s = j'.
\]

Summing up, we have the following situation:

\[
G_k(x + (\theta_1, v_1), y + (\theta_2, v_2)) \sim e^{ik(\theta_1 - \theta_2)}
\]

\[
\cdot \sum_a e^{ik(a - \tau a E)} \left( -\frac{d}{k} \right)^d \int_{-\infty}^{+\infty} e^{ikY_a(y,v_1,v_2;\tau)} B_k(\tau, v_a(\tau), v_2) d\tau,
\]

where, in view of Lemma 4.2 and (74) we have

\[
B_k(\tau, v_a(\tau), v_2)
\]

\[
\sim \nu \left( k^{1/2-\epsilon} \chi(\tau + \tau_a) \cdot \left[ 1 + \sum_{j=1}^{+\infty} \sum_{b=|-j/2|}^{j} k^b \sum_{|\rho| \geq j+2b} \gamma_{jb\rho} \tau^{\rho_0} v_1^{\rho_1} v_2^{\rho_2} \right] \right)
\]

\[
\sim \nu \left( k^{1/2-\epsilon} \chi(\tau + \tau_a) \cdot \sum_{j=0}^{+\infty} \sum_{b=|-j/2|}^{j} k^b \sum_{|\rho| \geq j+2b} \gamma_{jb\rho} \tau^{\rho_0} v_1^{\rho_1} v_2^{\rho_2} \right).
\]
where \( \rho = (\rho_0, \rho_1, \rho_2) \in \mathbb{N}_0 \times \mathbb{N}_0^{2d} \times \mathbb{N}_0^{2d} \) is a multi-index, and \(|\rho| := \rho_0 + |\rho_1| + |\rho_2| \) denotes its length; here \( \gamma_{000} = 1 \).

Since the expansion may be integrated term by term, we are reduced to considering oscillatory integrals of the form

\[
I_\rho^{(a)}(y, v_1, v_2) = e^{ik(\rho_0 - \tau_n E)} \left( \frac{k}{\pi} \right)^{d/2} k^b \int_{-\infty}^{+\infty} e^{ik\tilde{\Upsilon}_a(y, v_1, v_2; \tau)} \cdot \nu \left( k^{1/2 - \epsilon} \tau \right) \chi(\tau + \tau_n) \cdot \tau^{\rho_1} \nu_1^\rho_2 \, d\tau, \tag{98}
\]

where \(|\rho| \geq j + 2b\), with \([-j/2] \leq b \leq j \) for some \( j \geq 0 \). By the previous considerations, furthermore, we are in a position to apply the stationary phase Lemma for complex valued phase functions \([\text{MS}], [\text{H}]\).

In view of (81), the study of the critical points of \( \tilde{\Upsilon}_a \) is clearly reduced to the study of critical points of \( \Upsilon_a \) (recall (79) and (81)). The following Lemma is proved by a straightforward computation, that we shall leave to the reader:

**Lemma 4.3.** The phase \( \Upsilon_a(y, v_1, v_2; \cdot) \) has a unique critical point \( \tau_c = \tau_c(n, v_1, v_2) \), given by

\[
\tau_c = \frac{1}{\|v_f(n)\|^2} \left[ g_n(v_f(n), A_n v_1 - v_2) + i \omega_n(v_f(n), A_n v_1 + v_2) \right].
\]

Furthermore,

\[
\partial^2_{\tau} \Upsilon_a(y, v_1, v_2; \tau) = i \|v_f(n)\|^2,
\]

so that the critical point is non-degenerate. In addition, at the critical point we have

\[
\Upsilon_a(y, v_1, v_2; \tau_c) = \frac{i}{2 \|v_f(n)\|^2} \left[ g_n(v_f(n), A_n v_1 - v_2)^2 - \omega_n(v_f(n), A_n v_1 + v_2)^2 
+ 2i g_n(v_f(n), A_n v_1 - v_2) \cdot \omega_n(v_f(n), A_n v_1 + v_2) \right]. \tag{99}
\]

Let us clarify (99) in light of the decomposition in Definition 1.5. To this end let us write:

\[
v_j = v_{j1} + v_{jv} + v_{jh} \quad (j = 1, 2). \tag{100}
\]

Since \( A \) is unitary and leaves \( v_f(n) \) invariant, it preserves the previous decomposition, meaning that \((Av_j)_1 = Av_{j1}\), and so forth. Hence we have

\[
\psi_2(A_n v_1, v_2) = \psi_2(A_n v_{11}, v_{22}) + \psi_2(A_n v_{11} + A_n v_{1v}, v_{2t} + v_{2v})
= \psi_2(A_n v_{11}, v_{22}) - i \omega_n(A_n v_{11}, v_{22}) - i \omega_n(A_n v_{11}, v_{2t})
- \frac{1}{2} \|A_n v_{11} - v_{2t}\|^2 - \frac{1}{2} \|A_n v_{1v} - v_{2v}\|^2. \tag{101}
\]
Furthermore, it is evident that
\[ g_n(v_f(n), A_n v_1 - v_2)^2 = \|v_f(n)\|^2 \cdot \|A_n v_1 - v_2\|^2 \]  
(102)
and that
\[ \omega_n(v_f(n), A_n v_1 + v_2)^2 = \|v_f(n)\|^2 \cdot \|A_n v_{1t} + v_{2t}\|^2. \]  
(103)

Therefore, we can write
\[ i\tilde{\Upsilon}_a(y, v_1, v_2; \tau_c) = \psi_2(A_n v_1, v_2) + i\Upsilon_a(y, v_1, v_2; \tau) \]
\[ = \psi_2(A_n v_{1h}, v_{2h}) - \|A_n v_{1t}\|^2 - \|v_{2t}\|^2 + i\mathcal{H}, \]  
(104)
where
\[ \mathcal{H} =: -\omega_n(A_n v_{1t}, v_{2v}) - \omega_n(A_n v_{1v}, v_{2t}) \]
\[ + \frac{1}{\|v_f(n)\|^2} g_n(v_f(n), A_n v_1 - v_2) \cdot \omega_n(v_f(n), A_n v_1 + v_2). \]
(105)

To compute \( \mathcal{H} \), let us write
\[ A_n v_1 = \alpha_1 v_f(n) + \beta_1 J_n v_f(n) + A_n v_{1h}, \]
\[ v_2 = \alpha_2 v_f(n) + \beta_2 J_n v_f(n) + v_{2h}, \]
for appropriate \( \alpha_j, \beta_j \in \mathbb{R} \). Then clearly
\[ \omega_n(A_n v_{1t}, v_{2v}) = -\alpha_2 \beta_1 \|v_f(n)\|^2, \quad \omega_n(A_n v_{1v}, v_{2t}) = \alpha_1 \beta_2 \|v_f(n)\|^2; \]
furthermore,
\[ g_n(v_f(n), A_n v_1 - v_2) = (\alpha_1 - \alpha_2) \|v_f(n)\|^2, \]
and
\[ \omega_n(v_f(n), A_n v_1 + v_2) = (\beta_1 + \beta_2) \|v_f(n)\|^2. \]
Hence we may rewrite (105) as follows:
\[ \mathcal{H} = \|v_f(n)\|^2 (\alpha_2 \beta_1 - \alpha_1 \beta_2 + (\alpha_1 - \alpha_2) (\beta_1 + \beta_2)) \]
\[ = \|v_f(n)\|^2 (\alpha_1 \beta_1 - \alpha_2 \beta_2) = \omega_n(A_n v_{1v}, A_n v_{1t}) - \omega_n(A_n v_{2v}, A_n v_{2t}). \]  
(106)

Recalling (12), we conclude from (104) and (106) that
\[ i\tilde{\Upsilon}_a(y, v_1, v_2; \tau_c) = \psi_2(A_n v_{1h}, v_{2h}) - \|A_n v_{1t}\|^2 - \|v_{2t}\|^2 \]
\[ + i \left[ \omega_n(A_n v_{1v}, A_n v_{1t}) - \omega_n(A_n v_{2v}, A_n v_{2t}) \right] \]
\[ = \mathcal{Q}(A_n v_1, v_2). \]  
(107)
Since \( \tilde{Y}_a \) is a polynomial of degree two in \( \tau \), there is no third order remainder to take care of in the stationary phase expansion for (108). In addition, upon choosing the constant \( \beta \) in Proposition 4.1 sufficiently large, we may assume that \( \rho (k^{1/2-\epsilon} \tau) \equiv 1 \) near the critical point.

Thus the asymptotic expansion coming from each summand (108) is

\[
\mathcal{I}_\rho (y, v_1, v_2) \sim e^{k \frac{i}{\tau_a - \tau} + \frac{Q(A, a, v_1, v_2)}{2}} \left( \frac{k}{\pi} \right)^d k^{b-1/2} \frac{\sqrt{2\pi}}{\| v_f (n) \|} \cdot \sum_{r=0}^{\infty} \frac{1}{r!} (-2k \| v_f (n) \|^2)^{-r} \frac{\partial^{2r}}{\partial \tau^{2r}} \chi (\tau + \tau_a) \cdot \tau^{\rho_0 - l} \bigg|_{\tau = \tau_c} v_1^\rho_1 v_2^\rho_2 \cdot (108)
\]

Since \( \tau_c \) is a complex number, in the previous expression \( \chi \) should be really replaced by an almost analytic extension, that we still denote by \( \chi \) for notational simplicity. On the other hand, with the same abuse of notation, we have

\[
\frac{\partial^{2r}}{\partial \tau^{2r}} \chi (\tau + \tau_a) \cdot \tau^{\rho_0} \bigg|_{\tau = \tau_c} = \sum_{l=0}^{\min \{2r, \rho_0 \}} C_{rl} \chi^{(2r-l)} (\tau_c + \tau_a) \tau_c^{\rho_0 - l} \cdot (109)
\]

If we Taylor expand \( \chi^{(2r-l)} (\tau_c + \tau_a) \) at \( \tau = 0 \) to estimate \( \chi^{(2r-l)} (\tau_c + \tau_a) \) asymptotically, we end up with an asymptotic expansion of the form

\[
\chi^{(2r-l)} (\tau_c + \tau_a) \sim \sum_{s \geq 0} \frac{1}{s!} \chi^{(2r-l+s)} (\tau_a) \tau_c^s \cdot (110)
\]

Therefore,

\[
\frac{\partial^{2r}}{\partial \tau^{2r}} \chi (\tau + \tau_a) \cdot \tau^{\rho_0} \bigg|_{\tau = \tau_c} \sim \sum_{l=0}^{\min \{2r, \rho_0 \}} \sum_{s \geq 0} \frac{C_{rl}}{s!} \chi^{(2r-l+s)} (\tau_a) \tau_c^{\rho_0 + s - l} \cdot (111)
\]

Inserting (111) in (108), we obtain

\[
\mathcal{I}_\rho (y, v_1, v_2) \sim e^{k \frac{i}{\tau_a - \tau} + \frac{Q(A, a, v_1, v_2)}{2}} \left( \frac{k}{\pi} \right)^d k^{b-1/2} \frac{\sqrt{2\pi}}{\| v_f (n) \|} \cdot \sum_{r=0}^{\infty} k^{-r} \sum_{l=0}^{\min \{2r, \rho_0 \}} \sum_{s \geq 0} C_{rl}^{(a)} \chi^{(2r-l+s)} (\tau_a) \tau_c^{\rho_0 + s - l} v_1^\rho_1 v_2^\rho_2 \cdot (112)
\]

Now by Lemma 4.3 \( \tau_c \) is linear in \( v_1 \) and \( v_2 \); therefore, we can rewrite (112) as

\[
\mathcal{I}_\rho (y, v_1, v_2) \sim e^{k \frac{i}{\tau_a - \tau} + \frac{Q(A, a, v_1, v_2)}{2}} \left( \frac{k}{\pi} \right)^d k^{b-1/2} \cdot \sum_{r=0}^{\infty} k^{-r} \sum_{l=0}^{\min \{2r, \rho_0 \}} \sum_{s \geq 0} \chi^{(2r-l+s)} (\tau_a) P_{l+s-l}^{(a)} (n; v_1, v_2) \cdot (113)
\]
where $P_{\ell}^{(a)}(n; \cdot, \cdot, \cdot)$ is a homogeneous polynomial of degree $\ell$.

If we insert (113) in (96), with the amplitude $B_k$, given by the asymptotic expansion (97), we obtain the following asymptotic expansion:

\[
G_k(x + (\theta_1, v_1), y + (\theta_2, v_2)) \sim e^{ik(\theta_1 - \theta_2)} \left( \frac{k}{\pi} \right)^{\frac{d}{2}} \left( \frac{1}{v_f(n)} \right) \sum_{a} e^{i(n - \tau_{0}E) + O(A_{0}, v_{1}, v_{2})} \sum_{j=0}^{+\infty} \sum_{b = -[j/2]}^{j} k^{b - 1/2} \sum_{|\rho| \geq j + 2b}^{\infty} \sum_{r=0}^{\infty} k^{-r} \sum_{l=0}^{\min\{2r, \rho_{0}\}} \sum_{s \geq 0}^{\infty} \chi^{(2r - l + s)}(\tau_{a}) P_{|\rho| + s - l}^{(a_{j}b_{l}s)}(n; v_{1}, v_{2}),
\]

where $P_{\ell}^{(a_{j}b_{l}s)}(n; \cdot, \cdot, \cdot)$ is homogeneous of degree $\ell$.

Let us now pass to rescaled coordinates, thus replacing $v_j$ by $v_j/\sqrt{k}$, we may express the previous expansion in the following manner:

\[
G_{k}\left(x + \left(\theta_1, \frac{v_1}{\sqrt{k}}\right), y + \left(\theta_2, \frac{v_2}{\sqrt{2}}\right)\right) \sim e^{ik(\theta_1 - \theta_2)} \left( \frac{k}{\pi} \right)^{d - 1/2} \left( \frac{\sqrt{2}}{v_f(n)} \right) \sum_{a} e^{i(n - \tau_{0}E) + O(A_{0}, v_{1}, v_{2})} \sum_{j=0}^{+\infty} \sum_{b = -[j/2]}^{j} k^{-\frac{1}{2}(2r - 2b + |\rho| + s - l)} \sum_{l=0}^{\min\{2r, \rho_{0}\}} \sum_{s \geq 0}^{\infty} \chi^{(2r - l + s)}(\tau_{a}) P_{|\rho| + s - l}^{(a_{j}b_{l}s)}(n; v_{1}, v_{2}),
\]

Let us remark that, since $|\rho| \geq j + 2b$,

\[
2r - 2b + |\rho| + s - l = (2r - l) + (|\rho| - 2b) + s \geq (2r - l) + j + s \geq 0;
\]

therefore, only integer or half-integer powers of $k$ of the form $k^{d - 1/2 - \ell/2}$ with $\ell \geq 0$ contribute to the asymptotic expansion.

It is also clear from (115) and (116) that the coefficient of $k^{d - 1/2 - \ell/2}$ is the evaluation at $\tau_{0}$ of a differential polynomial in $\chi$ of degree $\leq \ell$.

Let us first consider the contribution with $\ell = 0$.

**Lemma 4.4.** $2r - 2b + |\rho| + s - l = 0$ if and only if $r = l = j = b = s = 0$ and $\rho = 0$.

Thus for each $a$ there is only one summand in (115) contributing to the power $k^{d - 1/2}$, and it is readily seen from (110) that its coefficient is $\chi(\tau_{0})$.  

Proof of Lemma 4.4. By (116), if $2r - 2b + |\rho| + s - l = 0$ then $2r = l$, $j = s = 0$ and $|\rho| = 2b$; given that $[-j/2] \leq b \leq j$, then also $b = 0$ and so $|\rho| = 0$. On the other hand $l \leq \min\{2r, \rho_0\}$, and so $l = 0$ and thus $r = 0$.

Lemma 4.5. For any $n_0 \in \mathbb{N}$ there are only finitely many summands in (115) with $2r - 2b + |\rho| + s - l = n_0$.

Thus the sum of all polynomials $P^{(a_j b \rho l s)}_{|\rho| + s - l}(n; v_1, v_2)$ with $2r - 2b + |\rho| + s - l = n_0$ is itself a polynomial.

Proof of Lemma 4.5. If $2r - 2b + |\rho| + s - l = n_0$, then $0 \leq j \leq (2r - l) + s + j \leq (2r - l) + s + (|\rho| - 2b) = n_0$, and so $[-n_0/2] \leq b \leq n_0$. Consequently, $|\rho| - 2n_0 \leq |\rho| - 2b \leq (2r - l) + s + (|\rho| - 2b) = n_0$. Thus, $|\rho| \leq 3n_0$, and so also $l \leq \rho_0 \leq |\rho| \leq 3n_0$. Therefore, we also have $2r \leq l + n_0 \leq 4n_0$. Similarly, $0 \leq s \leq (2r - l) + s + (|\rho| - 2b) = n_0$.

Lemma 4.6. For any summand in (115), we have $|\rho| + s - l \leq 3(2r - 2b + |\rho| + s - l)$.

Thus, the polynomials in $(v_1, v_2)$ contributing to a given power $k^{d-1/2-l/2}$ in (115) all have degree $\leq 3\ell$.

Proof of Lemma 4.6. The claimed inequality is equivalent to $6b + 2l \leq 6r + 2|\rho| + 2s$. On the other hand, given that $l \leq 2r$ and $|\rho| \geq j + 2b \geq 3b$,

$$6b + 2l \leq 6b + 4r \leq 6b + 6r \leq 2|\rho| + 6r \leq 6r + 2|\rho| + 2s.$$ 

It is furthermore also evident that $|\rho| + s - l$ and $2r - 2b + |\rho| + s - l$ have the same parity. Thus each polynomial in $(v_1, v_2)$ contributing to a given power $k^{d-1/2-l/2}$ in (115) has parity $(-1)^\ell$.

The proof of Theorem 1.2 is complete.

5 Proof of Theorem 1.3

Proof of Theorem 1.3. Let us define

$$\mathfrak{G}_k(m) := G_k(x, x) \text{ if } x \in X_m.$$ 

The definition is well-posed, and $\mathfrak{G}_k \in C^\infty(M)$.

The following is an immediate consequence of Theorem 1.1.
Corollary 5.1. Suppose that $E$ is a regular value of $f$. Then, for any $C > 0$ and $\epsilon > 0$, we have $\mathcal{G}_k(m) = O(k^{-\infty})$, uniformly for

$$\max \{\text{dist}_M(m, M_E), \text{dist}_M(m, m^x)\} \geq C k^{\epsilon - 1/2}.$$ 

Proof of Corollary 5.1. When $E$ is a regular value of $f$, any $m$ in a small tubular neighborhood of $M_E$ may be written $m = \exp_{m_0}(\lambda J_{m_0}(\nu_f(m_0)))$, for unique $m_0 \in M_E$, $\lambda \in \mathbb{R}$. Here $\exp$ is of course the exponential map for the Riemannian manifold $(M, g)$. Clearly, for sufficiently small $\lambda$ one has $a |\lambda| \leq \text{dist}_M(m, M_E) \leq A |\lambda|$ and $a |\lambda| \leq |f(m) - E| \leq A |\lambda|$, for suitable constants $0 < a < A$; we may take, for example, $a = \|\nu_f(m_0)\|^2/2, A = 2 \|\nu_f(m_0)\|^2$. It follows that $|f(m) - E| > (b/A) \text{dist}_M(m, M_E)$, and this implies the statement.

We want to estimate

$$\text{trace}(\mathcal{G}_k) = \int_X \mathcal{G}_k(x, x) dV_X(x)$$

$$= \int_M \mathcal{G}_k(m) dV_M(m).$$

Integration in $dV_M(m)$, by Corollary 5.1, asymptotically localizes to a neighborhood of $U_E$ of $M_E$ in $M$. Let us normalize the previous parametrization by the map:

$$\eta : (m, \lambda) \in M_E \times (-\delta, \delta) \mapsto \exp \left(\frac{\lambda}{\|\nu_f(m)\|} J_m(\nu_f(m))\right) \in M. \quad (119)$$

Let us define $\mathcal{V} \in C^\infty(M_E \times (-\delta, \delta))$ by setting

$$\mathcal{V}(m, \lambda) d\lambda dV_{M_E} = \eta^*(dV_M),$$

where $dV_{M_E}$ is the volume form of the oriented manifold $M_E$ (for our practical purposes, we may as well deal with Riemannian densities); in particular, $\mathcal{V}(m, 0) \equiv 1$ identically. Thus

$$\text{trace}(\mathcal{G}_k) \sim \int_{M_E} \int_{-\delta}^{\delta} \mathcal{G}_k(\eta(m, \lambda)) \mathcal{V}(m, \lambda) d\lambda dV_{M_E}(m). \quad (120)$$

Furthermore, if $\varrho_0$ is as on page 28 then, by Corollary 5.1 and its proof, we can insert the cut-off $\varrho_0(k^{1/2-\epsilon} \lambda)$ without affecting the asymptotics:

$$\text{trace}(\mathcal{G}_k) \sim \int_{M_E} \int_{-\delta}^{\delta} \mathcal{G}_k(\eta(m, \lambda)) \mathcal{V}(m, \lambda) \varrho_0(k^{1/2-\epsilon} \lambda) d\lambda dV_{M_E}(m), \quad (121)$$
so that integration in $d\lambda$ is now over the interval $(-2k^{-1/2}, 2k^{-1/2})$.

Here $G_k = G_k^{(x)}$. Let us define $\mathcal{P}_k : \mathbb{R} \times M \to \mathbb{C}$ by setting

$$
\mathcal{P}_k(T, m) =: \Pi_k(x, x)
$$

for an arbitrary choice of $x \in X_m$. Then by (100)

$$
\mathcal{G}_k(\eta(m, \lambda)) = \int_\mathbb{R} e^{-ikT} \chi(\tau) \mathcal{P}_k(\tau, \eta(m, \lambda)) d\tau,
$$

so that given (120) we have

$$
\text{trace}(G_k) \sim \int_{M\mathcal{E}} \int_{-\delta}^{-\delta} \int_\mathbb{R} e^{-ikT} \chi(\tau) \mathcal{P}_k(\tau, \eta(m, \lambda)) \mathcal{V}(m, \lambda) \varrho_0 (k^{1/2-\epsilon} \lambda) \cdot d\tau d\lambda dV_{M\mathcal{E}}(m),
$$

In view of Proposition 2.1, $\text{Per}_M(M\mathcal{E}) \cap \text{supp}(\chi)$ is a finite set $\{\sigma_b\}_{b \in B_0}$ ($B_0$ is a finite subset of $\mathcal{B}$ in the discussion on page 16). Let us choose $\epsilon_1$ such that

$$
\min\{|\sigma_l - \sigma_h| : l, h \in B_0, h \neq l\}/5 > \epsilon_1 > 0,
$$

and set

$$
\tilde{\varrho}_0(\tau) =: \sum_{b \in B_0} \varrho_0((\tau - \sigma_b)/\epsilon_1) \quad (\tau \in \mathbb{R}).
$$

Then $\tilde{\varrho}_0 \in \mathcal{C}_\infty^0(\mathbb{R})$ is (compactly) supported in $\bigcup_{b \in B_0} (\sigma_b - 2\epsilon_1, \sigma_b + 2\epsilon_1)$, and it is $\equiv 1$ in $\bigcup_{b \in B_0} (\sigma_b - \epsilon_1, \sigma_b + \epsilon_1)$.

**Lemma 5.1.** Only a rapidly decreasing contribution to the asymptotics of trace($G_k$) is lost, if the integrand in (123) is multiplied by $\tilde{\varrho}_0(\tau)$.

**Proof of Lemma 5.1.** It suffices to show, in view of (123), (122) and Proposition 2.2 that there exists $\epsilon_2 > 0$ such that $\text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)) \geq \epsilon_2$ whenever $m \in M\mathcal{E}$, $|\lambda| \leq 2k^{-1/2}$, $\tau \in \text{supp}(\chi)$, and $|\tau - \sigma_b| > \epsilon_1$ for each $b \in B_0$.

Let us first show that there is $\epsilon_3 > 0$ such that $\text{dist}_M(m_x, m) \geq \epsilon_3$ whenever $m \in M\mathcal{E}$, $\tau \in \text{supp}(\chi)$, and $|\tau - \sigma_b| > \epsilon_1$ for each $b \in B_0$.

Indeed, if not we can find pairs $(m_h, \tau_h) \in M\mathcal{E} \times \text{supp}(\chi)$ for $h = 1, 2, \ldots$, such that $|\tau_h - \sigma_b| > \epsilon_1$ for each $b \in B_0$ and $\text{dist}_M(m_h, m_h, m_h) < 1/h$. By compactness of $M\mathcal{E}$ and $\text{supp}(\chi)$, perhaps after passing to a subsequence we may assume that $m_h \to m_\infty \in M\mathcal{E}$ and $\tau_h \to \tau_\infty \in \text{supp}(\chi)$.

By continuity, $m_\infty \tau_\infty = m_\infty$, whence $\tau_\infty \in \text{Per}_M(M\mathcal{E}) \cap \text{supp}(\chi)$. Thus $\tau_\infty = \sigma_b$ for some $b \in B_0$; however this is clearly absurd since, again by continuity, $|\tau_\infty - \sigma_b| \geq \epsilon_1$ for every $b \in B_0$. 

Given this, we have whenever \( m \in M_E, |\lambda| \leq 2 k^\epsilon - 1/2, \tau \in \text{supp}(\chi) \), and \( |\tau - \sigma_b| \geq \epsilon_1 \) for each \( b \in B_0^\epsilon \):

\[
\epsilon_3 \leq \text{dist}_M(m, m_\tau) \leq \text{dist}_M(m, \eta(m, \lambda)) + \text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)_\tau) + \text{dist}_M(\eta(m, \lambda)_\tau, m_\tau),
\]

\[
= 2 \text{dist}_M(m, \eta(m, \lambda)) + \text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)_\tau)
\]

\[
\leq 5 k^\epsilon - 1/2 + \text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)_\tau).
\]

Therefore, for \( k \gg 0 \) we have, say, \( \text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)_\tau) \geq \epsilon_3/2 \).

Let us set \( \chi_b(\tau) =: \varrho_0((\tau - \sigma_b)/\epsilon_1) \cdot \chi(\tau) \). Then by Lemma 5.1 we may rewrite (123) as follows:

\[
\sum_{j=1}^{N_k} \hat{\lambda}(kE - \lambda_{k_j}) \sim \sum_{b \in B_0^\epsilon} \int_{-\delta}^{+\delta} \left[ \int_{-\infty}^{+\infty} e^{-irkE} \chi_b(\tau) \Psi_k(\tau, \eta(m, \lambda)) \, d\tau \right] \cdot \mathcal{V}(m, \lambda) \varrho_0(k^{1/2 - \epsilon} \lambda) \, d\lambda \, dV_{M_E}(m)
\]

\[
= \sum_{b \in B_0^\epsilon} \int_{-\delta}^{+\delta} \mathcal{G}^{(\chi_b)}(\eta(m, \lambda)) \cdot \mathcal{V}(m, \lambda) \varrho_0(k^{1/2 - \epsilon} \lambda) \, d\lambda \, dV_{M_E}(m),
\]

\[
\sim \sum_{b \in B_0^\epsilon} \text{trace} \left( \mathcal{G}^{(\chi_b)} \right) (126)
\]

where \( \mathcal{G}^{(\chi_b)} \) is the Gutzwiller-Töplitz operator with \((f, E, \chi)\) replaced by \((f, E, \chi_b)\).

An argument similar to those used in the proof of Lemma 5.1 proves the following further reduction. Let \( M_E(\sigma_b) = \bigcup M_E(\sigma_b) \) be as in (21).

**Lemma 5.2.** Only a rapidly decreasing contribution to the asymptotics is lost in (123), if in the \( b \)-th summand integration in \( dV_{M_E}(m) \) is restricted to an arbitrarily small tubular neighborhood \( U_E(\sigma_b) \subseteq M_E \) of \( M_E(\sigma_b) \).

**Proof of Lemma 5.2.** It suffices to prove that given \( \delta_1 > 0 \) there exists \( \delta_2 > 0 \) such that uniformly for \((m, \tau, \lambda) \in M_E \times \text{supp}(\chi) \times \mathbb{R} \) satisfying:

\[
|\tau - \sigma_b| \leq 2 \epsilon_1, \text{ dist}_M(m, M_E(\sigma_b)) \geq \delta_1, |\lambda| \leq 2 k^\epsilon - 1/2 \quad (127)
\]

we have \( \text{dist}_M(\eta(m, \lambda), \eta(m, \lambda)_\tau) \geq \delta_2 \).
Assume, to the contrary, that for \( k = 1, 2, \ldots \) we can find 
\[
(m_k, \tau_k, \lambda_k) \in M_E \times \text{supp}(\chi) \times (-2k^{t-1/2}, 2k^{t-1/2})
\]
satisfying (127), and such that \( \text{dist}_M(\eta(m_k, \lambda_k), \eta(m_k, \lambda_k)_{\tau_k}) \to 0 \) when \( k \to 0 \). Again by compactness, we may assume, perhaps after passing to a subsequence \( k_j \), that \( m_{k_j} \to m_\infty \in M_E \) and \( \tau_{k_j} \to \tau_\infty \in [\sigma_b - 2\epsilon_1, \sigma_b + 2\epsilon_1] \) when \( j \to +\infty \).

It follows clearly that \( \eta(m_{k_j}, \lambda_{k_j}) \to m_\infty \) and that \( m_\infty \tau_\infty = m_\infty \). Therefore, \( \tau_\infty \in \text{Per}_M^R(M_E) \cap \text{supp}(\chi) \). Necessarily then we have \( \tau_\infty = \sigma_b \), since by continuity \( |\tau_k - \sigma_b| \geq 3\epsilon_1 \) for any \( b' \in B_b \setminus \{b\} \) (recall (124)). It thus follows that \( m_\infty \in M_E(\sigma_b) \). But by continuity we also have \( \text{dist}_M(m_\infty, M_E(\sigma_b)) \geq \delta_1 \), and this is absurd. \(\square\)

We may assume that \( U_E(\sigma_b) \) is the disjoint union of tubular neighborhoods \( U_E(\sigma_b)_l \subseteq M_E \) of each \( M_E(\sigma_b)_l \). Thus, we can rewrite (125) as follows:

\[
\sum_{j=1}^{N_b} \tilde{\chi}(kE - \lambda_{kj}) \sim \sum_{b \in B_0} \sum_{l=1}^{n_b} \int_{U_E(\sigma_b)_l} \int_{-\delta}^{+\delta} \Phi_k^{(\lambda)}(\eta(m, \lambda)) \cdot \mathcal{V}(m, \lambda) \varphi_0(k^{1/2-\epsilon}\lambda) \, d\lambda \, dV_{M_E}(m).
\] (128)

As discussed in (222) \( M_E(\sigma_b)_l \) is a \( \phi^M \)-invariant submanifold of \( M \), of dimension \( 2d_M - 1 \). In particular, \( \nu_j(m) \in T_m M_E(\sigma_b)_l \) for each \( m \in M_E(\sigma_b)_l \). Furthermore, the normal bundle \( N(M_E(\sigma_b)_l/M_E) \) of \( M_E(\sigma_b)_l \) in \( M_E \) is the restriction of the normal bundle of \( M(\sigma_b)_l \) in \( M \). In view of (119), at each \( m \in M(\sigma_b)_l \) the normal space of \( M_E(\sigma_b)_l \) in \( M_E \) at \( m \) is thus given by

\[
N_m(M_E(\sigma_b)_l/M_E) = \text{im} \left( d\phi^M_{\sigma_b} - \text{id}_{T_m M} \right).
\] (129)

Hence we can find a finite open cover \( \mathcal{U} = \{ U_{blj} \} \) of \( M_E(\sigma_b)_l \) such that for each \( j \) there are unitary trivializations \( N(M_E(\sigma_b)_l/M_E) \cong U_{blj} \times \mathbb{C}^{c_\mathcal{U}} \).

Writing \( \exp^M_{mE} \) for the exponential map of \( M_E \) at a point \( m \in M_E(\sigma_b)_l \), we obtain a local parametrization of \( U_E(\sigma_b)_l \subseteq M_E \) along \( U_{blj} \) by setting

\[
\beta_{blj} : (m, n) \in U_{blj} \times B_{2c_\mathcal{U}}(0, \delta) \mapsto \exp^M_{mE}(n) \in U_E(\sigma_b)_l,
\]
where \( n \in B_{2c_\mathcal{U}}(0, \delta) \subseteq \mathbb{R}^{2c_\mathcal{U}} \cong \mathbb{C}^{c_\mathcal{U}} \) is identified with a normal vector through the previous unitary isomorphism.

Let \( \{ \gamma_{blj} \} \) be a partition of unity on \( M_E(\sigma_b)_l \) subordinate to the open cover \( \mathcal{U} \). We can then write \( m = \exp^M_{mE}(n) \) in (128), and express (128) in
of Theorem 1.3

\[ \sum_{j=1}^{N_k} \hat{\chi}(kE - \lambda_{kj}) \]  

(130)

\[ \sim \sum_{b \in B_0} \sum_{l=1}^{n_b} \sum_{j} \int_{U_{blj}} \gamma_{blj}(m) \int_{B_{2\varepsilon l}(0,\delta)}^{+\delta} \mathcal{G}_k^{(\chi_b)} (\eta (\exp^M_m(n), \lambda) \cdot \mathcal{V}_1(m, n, \lambda) \varrho \left( k^{1/2-\varepsilon} \lambda \right) d\lambda d\mu \)  

where \( \mathcal{V}_1(\cdot, 0, 0) \equiv 1 \), and \( d\mu \) denotes the Lebesgue measure on \( \mathbb{R}^2c_b \). In view of (119), we have

\[ \eta (\exp^M_m(n), \lambda) = \exp^{M_{\exp m}}_n (\lambda/\| v_f(m) \|) J_m(v_f(m)) \]  

(131)

in (130). This is a parametrization

\[ \hat{\beta}_{blj} : U_{blj} \times B_{2\varepsilon l}(0, \delta) \times (-\delta, \delta) \rightarrow U_E(\sigma_b)_l; \]

we obtain another parametrization (perhaps changing \( \delta \) and restricting \( U_E(\sigma_b)_l \)),

\[ \tilde{\beta}_{blj} : U_{blj} \times B_{2\varepsilon l}(0, \delta) \times (-\delta, \delta) \rightarrow U_E(\sigma_b)_l, \]

by using the unitary trivialization of the normal bundle \( N(M_E(\sigma_b)_l) \) to \( M_E(\sigma_b)_l \) on \( U_{blj} \), as in (32), and setting instead:

\[ \hat{\beta}_{blj} : (m, n, \lambda) \mapsto \exp^M_m \left( \frac{\lambda}{\| v_f(m) \|} J_m(v_f(m)) + n \right). \]  

(132)

We have \( \hat{\beta}_{blj}^*(d\mu) = \hat{V}(m, n, \lambda) d\mu \) \( d\lambda \) with \( \mathcal{V}(\cdot, 0, 0) \equiv 1 \). Also, \( \hat{\beta}_{blj} \) the same \( \hat{\beta}_{blj} \) induce the same differential for \( n = 0 \) and \( \lambda = 0 \). Thus, if we apply the change of integration variables \( \hat{\beta}_{blj}^{-1} \circ \hat{\beta}_{blj} \) we can reformulate (130) in the following form:

\[ \sum_{j=1}^{N_k} \hat{\chi}(kE - \lambda_{kj}) \]  

(133)

\[ \sim \sum_{b \in B_0} \sum_{l=1}^{n_b} \sum_{j} \int_{U_{blj}} \gamma_{blj}(m) \int_{B_{2\varepsilon l}(0,\delta)}^{+\delta} \mathcal{G}_k^{(\chi_b)} (\hat{\beta}_{blj}(m, n, \lambda)) \cdot \mathcal{V}_2(m, n, \lambda) \varrho \left( k^{1/2-\varepsilon} \lambda \right) d\lambda d\mu \)  

where again \( \mathcal{V}_2(\cdot, 0, 0) \equiv 1 \) (\( \mathcal{V}_2 \) also depends on \( b, l, j \)).
Perhaps after passing to a finer open cover, we may assume that the restriction to $M_E(\sigma_b)_I$ of the tangent bundle $TM$ is unitarily trivialized on each $U_{blj}$, and that this trivialization is consistent with the orthogonal direct sum decomposition in Definition 1.5. Paired with this trivialization, $\exp_m$ provides for each $m \in U_{blj}$ a system of preferred local coordinates on $M$ centered at $m$ (see (133)), for which $(\lambda / \|v_f(m)\|) J_m(v_f(m))$ is a transverse tangent vector, of norm $|\lambda|$, and $n$ is horizontal (see the discussion on page 17, especially Remark 2.1). It is however more convenient to use a slightly different choice of preferred local coordinates centered at $m$, as follows.

Let us unitarily identify $\text{span}(v_f(m))$ and $\text{span}(J_m v_f(m))$ with $\mathbb{R}$ by $\tau \mapsto (\tau / \|v_f(m)\|) v_f(m)$, and $\lambda \mapsto (\lambda / \|v_f(m)\|) J_m v_f(m)$. We have also a smoothly varying unitary isomorphism $S_m \cong \mathbb{R}^{2d-2}$, where $S_m$ is as in (11); recall that $\text{im}(d\phi^M_\sigma - \text{id}_{T_m M}) \subseteq S_m$, hence it corresponds to a copy of $\mathbb{R}^{2c m}$ in $\mathbb{R}^{2d-2}$.

Under these identifications, we shall denote by $s \in \mathbb{R}^{2d-2}$ the general element of $S_m$, and by $n \in \mathbb{R}^{2c m} \subseteq \mathbb{R}^{2d-2}$ the general element of $\text{im}(d\phi^M_\sigma - \text{id}_{T_m M})$.

We shall always adopt additive notation, but set

$$m + (\tau, \lambda, s) =: \phi^M_{-\tau / \|v_f(m)\|} \left( \exp^M_m \left( \frac{\lambda}{\|v_f(m)\|} J_m(v_f(m)) + s \right) \right).$$

(134)

with $(\tau, \lambda, s) \in (-\delta, \delta)^2 \times B_{2d-2}(0, \delta) \subseteq \mathbb{R} \times \mathbb{R}^{2d-2}$.

Now let $\varphi_1 \in C^\infty(\mathbb{R}^{2c m})$ satisfy $\varphi_1(n) = 0$ if $\|n\| \geq 2$ and $\varphi_1(n) = 1$ if $\|n\| \leq 1$. Recall that $\chi_b$ in (133) is supported in $(1 - 2\epsilon_1, \sigma_b + 2\epsilon_1)$.

**Lemma 5.3.** Given that $\epsilon_1 > 0$ is sufficiently small, only a negligible contribution to the asymptotics is lost in (133), if the integrand is multiplied by $\varphi_1(k^{1/2-\epsilon} n)$.

**Proof of Lemma 5.3.** Consider $m \in U_{blj} \subseteq M_E(\sigma_b)_I$, and adopt notation (134). Since $\phi^M_{\sigma_b}$ is a holomorphic Riemannian isometry, it preserves geodesics. Furthermore, it leaves $v_f$ invariant. Given that $\phi^M_{\sigma_b}(m) = m$, we then have

$$\phi^M_{-\sigma_b}(m + (0, \lambda, n)) = m + (0, \lambda, A_b n),$$

(135)

where $A_b$ is the unitary (orthogonal and symplectic) $(2c n) \times (2c n)$ matrix representing the restriction to $N_m(M(\sigma_b)) \cong \mathbb{R}^{2c m}$ of $d_m \phi^M_{-\sigma_b} : T_m M \to T_m M$.

Given (134), we then have that for $\tau \sim 0$

$$\phi^M_{\tau + \sigma_b}(m + (0, \lambda, n)) = \phi^M_{-\sigma_b}(m + (0, \lambda, A_b n)) = m + (\tau \|v_f(m)\|, \lambda, A_b n).$$

(136)
Since the local chart (134) is isometric at the origin, by Lemma 2.2 we have, perhaps after passing to a smaller $\delta$,
\[
\text{dist}_M (m + (\tau, \lambda, s), m + (\tau', \lambda', s')) \geq \frac{1}{2} \| (\tau - \tau', \lambda - \lambda', s - s') \| \quad (137)
\]
for all $(\tau, \lambda, s), (\tau', \lambda', s') \in (-\delta, \delta)^2 \times B_{2d-2}(0, \delta)$.

Let us apply this with $s = n, s' = A_b n \in \mathbb{R}^{2d} \subseteq \mathbb{R}^{2d-2}$. Recalling that $A_b - I_{2d}$ is invertible, we have that for $\tau \sim 0$
\[
\text{dist}_M (\phi_{-(\tau + \sigma_b)} (m + (0, \lambda, n)), m + (0, \lambda, n)) = \text{dist}_M (m + (\|v_f(m)\|, \lambda, A_b n), m + (0, \lambda, n)) \geq C_1 \|n\|, \quad (138)
\]
where $C_1 > 0$ depends only on $E, b$ and $l$.

Hence if $\epsilon_1 > 0$ is sufficiently small, we deduce that
\[
\text{dist}_M (m + (0, \lambda, n), (m + (0, \lambda, n))^{(x_b)}) \geq C_1 \|n\| \quad (139)
\]
for some constant $C_1 > 0$, and the statement follows from Theorem 1.1. \hfill $\square$

With the rescaling $\lambda \mapsto \lambda/\sqrt{k}$ and $n \mapsto n/\sqrt{k}$, we can then rewrite (133) as
\[
\sum_{j=1}^{N_k} \hat{\chi}(kE - \lambda_{kj}) \quad (140)
\]
\[
\approx \sum_{b \in B_0} \sum_{l=1}^{n_b} k^{-c_b l - \frac{1}{2}} \sum_{j} \int_{U_{blj}} \gamma_{blj}(m) \int_{\mathbb{R}^{2d}} \int_{-\infty}^{+\infty} \mathcal{G}_k^{(x_b)} \left( m + \left( 0, \frac{\lambda}{\sqrt{k}}, \frac{n}{\sqrt{k}} \right) \right) 
\cdot \mathcal{V}_2 \left( m, \frac{n}{\sqrt{k}}, \lambda, \frac{\lambda}{\sqrt{k}} \right) g_0 \left( k^{-\epsilon} \lambda \right) g_1 \left( k^{-\epsilon} n \right) d\lambda \ dn \ dV_{M_E(x_b)l}(m)
\]
\[
= \sum_{b \in B_0} \sum_{l=1}^{n_b} \int_{M_E(x_b)l} \sum_{j} \gamma_{blj}(m) I_{blj}(m, k) \ dV_{M_E(x_b)l}(m),
\]
where we have set
\[
I_{blj}(m, k) =: k^{-c_b l - \frac{1}{2}} \int_{\mathbb{R}^{2d}} \int_{-\infty}^{+\infty} \mathcal{G}_k^{(x_b)} \left( m + \left( 0, \frac{\lambda}{\sqrt{k}}, \frac{n}{\sqrt{k}} \right) \right) 
\cdot \mathcal{V}_2 \left( m, \frac{n}{\sqrt{k}}, \lambda, \frac{\lambda}{\sqrt{k}} \right) g_0 \left( k^{-\epsilon} \lambda \right) g_1 \left( k^{-\epsilon} n \right) d\lambda \ dn; \quad (141)
\]
integration in $d\lambda \, dn$ in (141) is over an open ball centered at the origin and of radius $O(k^\epsilon)$ in $\mathbb{R}^{1+2cbl}$. The dependence on $j$ of the right hand side of (141) is implicit through the above choices of trivializations and HLC systems.

To find an asymptotic expansion for (141), let us notice that for an arbitrary choice of $x \in X_m$ and of a preferred local frame of $A$ centered at $x$, we obtain a system of HLC centered at $x$. Let $w \in T_mM$ be defined by $w_t = (\lambda/\|v_f(m)\|) J_m v_f(m)$, $w_n = n$, and $w_v = 0$. Recalling (117), we have

$$G_k^{(\lambda)}(m + 0, \frac{\lambda}{\sqrt{k}}, \frac{n}{\sqrt{k}}) \sim G_k^{(\lambda)}\left(x + \frac{w}{\sqrt{k}}, x + \frac{w}{\sqrt{k}}\right). \quad (142)$$

Theorem 1.2 provides an asymptotic expansion for the right hand side of (142), when we take $y = x$, $\theta_1 = \theta_2 = 0$, $v_1 = v_2 = w$, $\tau_a = \sigma_b$, $\vartheta_a = \vartheta_{bl}$ ($\sigma_b$ is the only period in the support of $\chi_b$).

By construction, we have $\chi_b(\sigma_b) = \chi(b)$ and

$$Q(A_a w, w) = \psi_2(A_b n, n) - 2 |\lambda|^2. \quad (143)$$

Given this, the asymptotic expansion of Theorem 1.2 yields

$$G_k^{(\lambda)}(x + \frac{w}{\sqrt{k}}, x + \frac{w}{\sqrt{k}}) \sim \frac{\sqrt{2}}{\|v_f(m)\|} \left(\frac{k}{\pi}\right)^{d-1/2} \psi_2(A_b n, n) - 2 |\lambda|^2 \left[\chi(\sigma_b) + \sum_{s=1}^{+\infty} k^{-s/2} P_{bs}(m; \lambda, n)\right], \quad (144)$$

where $P_{as}(m, \cdot)$ is a polynomial of degree $\leq 3s$ and parity $s$.

Furthermore, if we Taylor expand $V_2(m, \cdot, \cdot)$ at the origin, and recall that $V_1(m, 0, 0) = 1$, we obtain an asymptotic expansion

$$V_2\left(m, \frac{n}{\sqrt{k}}, \frac{\lambda}{\sqrt{k}}\right) \sim 1 + \sum_{r\geq 1} k^{-r/2} F_r(m; \lambda, n), \quad (145)$$

where $F_r$ is a homogeneous polynomial of degree $r$. The step-$N$ remainder is bounded by $C_{N+1} k^{-(N+1)(1/2-\epsilon)}$.

Multiplying (144) and (145), we obtain for the product of the first two factors in the integrand of (141) an asymptotic expansion as in (144), but with the $P_{bs}$’s ($j \geq 1$) replaced with polynomials $R_{bs}$’s with the same properties. Since integration is over a ball of radius $O(k^\epsilon)$, the expansion can be integrated term by term.

On the other hand, the exponent in (143) satisfies

$$\Re \left(\psi_2(A_b n, n) - 2 |\lambda|^2\right) \leq -c \left(||n||^2 + |\lambda|^2\right). \quad (146)$$
for some constant $c > 0$; therefore, only a rapidly decreasing contribution to the asymptotics is lost, if the cut-off functions are omitted. Thus we obtain for $I_{tbj}(m, k)$ an asymptotic expansion of the form

\[
I_{tbj}(m, k) \sim \frac{\sqrt{2}}{\|v_f(m)\|} \left( \frac{k}{\pi} \right)^{d-1/2} k^{-c_{\text{W}} - c/2} e^{ik(\theta_{l, n} - \sigma_b E)}
\cdot \sum_{s=0}^{+\infty} k^{-s/2} I_{tbj}(m, s),
\]

(147)

where we have set

\[
I_{tbj}(m, s) =: \int_{\mathbb{R}^{2+d/2}} \int_{-\infty}^{+\infty} e^{i\psi_2(A_{b, \lambda, n}) - 2|\lambda|^2} R_{tbj}(m; \lambda, n) \, d\lambda \, dn.
\]

(148)

with $R_{tbj}(m; \lambda, n) =: \chi(\sigma_b)$. Since $\psi_2(A_{b, \lambda, n}) - 2|\lambda|^2$ is an even function of $(n, \lambda)$ and $R_{tbj}(m; \lambda, n)$ has the same parity as $s$, $I_{tbj}(m, s) = 0$ if $s$ is odd.

Let us insert this in (140). We obtain

\[
\sum_{j=1}^{N_k} \hat{\chi}(kE - \lambda_{kj}) \sim \sum_{b \in \mathcal{B}_0} \sum_{l=1}^{n_b} \mathcal{I}(b, l, k)
\]

(149)

where each $\mathcal{I}(b, l, k)$ is an asymptotic expansion

\[
\mathcal{I}(b, l, k) \sim \sqrt{2} \pi^{1/2-d} k^{d_{ab}-1} e^{ik(\theta_{l, n} - \sigma_b E)}
\cdot \sum_{s=0}^{+\infty} k^{-s} \int_{M_{E}(\sigma_b)_{b, l}} \frac{1}{\|v_f(m)\|} \sum_{j} \gamma_{tbj}(m) I_{tbj}(m, 2s) \, dV_{M_{E}(\sigma_b)_{b, l}}(m).
\]

(150)

Let us compute the leading order term in (150). We have in view of (148)

\[
I_{tbj}(m, 0) = \chi(\sigma_b) \left( \int_{\mathbb{R}^{2+d/2}} e^{i\psi_2(A_{b, \lambda, n})} \, dn \right) \cdot \left( \int_{-\infty}^{+\infty} e^{-2|\lambda|^2} \, d\lambda \right)
\]

(151)

\[
= \frac{\chi(\sigma_b)}{\sqrt{2}} \cdot \frac{\pi^{d_{ab}+\frac{1}{2}}}{D(b, l)},
\]

(152)

where $D(b, l)$ was defined on page 13. We have made use of the computation on page 235 of [P1].

Thus the leading order term in (150) is

\[
\frac{\chi(\sigma_b)}{D(b, l)} \left( \frac{k}{\pi} \right)^{d_{ab}-1} e^{ik(\theta_{l, n} - \sigma_b E)} \cdot \int_{M_{E}(\sigma_b)_{b, l}} \frac{1}{\|v_f(m)\|} \, dV_{M_{E}(\sigma_b)_{b, l}}(m).
\]

(153)

This completes the proof of Theorem 1.3.
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