Morphological transitions in supercritical generalized percolation and moving interfaces in media with frozen randomness

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We consider the growth of clusters in disordered media at zero temperature, as exemplified by supercritical generalized percolation and by the random field Ising model. We show that the morphology of such clusters and of their surfaces can be of different types: they can be standard compact clusters with rough or smooth surfaces, but there exist also a completely different “spongy” phase. Clusters in the spongy phase are ‘compact’ as far as the size-mass relation $M \sim R^D$ is concerned (with $D$ the space dimension), but have an outer surface (or ‘hull’) whose fractal dimension is also $D$ and which is indeed dense in the interior of the entire cluster. This behavior is found in all dimensions $D \geq 3$. Slightly supercritical clusters can be of either type in $D = 3$, while they are always spongy in $D \geq 4$. Possible consequences for the applicability of KPZ (Kardar-Parisi-Zhang) scaling to interfaces in media with frozen randomness are studied in detail.

I. INTRODUCTION

Rough surfaces and interfaces have a huge number of applications in nature and in technology. Accordingly, there is a very large number of papers devoted to them in the physics literature, as well as several monographs [1–5]. For statistical physicists, one of the major reasons of studying them is the fact that they typically show anomalous (fractal) scaling, with deep connections to phenomena like phase transitions, critical points, and the renormalization group.

One very important distinction in the theory and phenomenology of rough surfaces is between growing and pinned ones. Pinned rough surfaces can only occur in media with frozen randomness at zero temperature, while moving rough surfaces can be found both in random media and in ordered media with temporal (thermal or athermal) disorder.

The prototypical model for driven interfaces without frozen disorder is the Kardar-Parisi-Zhang (KPZ) model [1], a discrete version of which is the Eden model for cancer growth [6]. When starting from a flat interface at time $t = 0$, it never forgets the initial growth direction in any dimension, so that the interface never becomes isotropic either globally or locally (this is, e.g., different for diffusion limited aggregation, where interfaces become locally isotropic for $t \to \infty$ [7, 8]). Thus growing interfaces in non-random media are self-affine.

This is not always the case for pinned interfaces. In two dimensions, it is shown in [9, 10] that critically pinned interfaces in isotropic random media are always in the universality class of critical percolation and thus fractal and isotropic on large scales. More precisely, the interfaces are in the class of percolation hulls, i.e., of externally accessible surfaces of critical percolation clusters. In $D \geq 3$, there are two different universality classes for critically pinned interfaces, depending on control parameters (plus a tricritical region for intermediate control parameters) [11–14]. One of them is again the percolation universality class, while the other is believed to be self-affine [2–4, 12] – although there are numerical indications that this might not be strictly true, in particular in the weak disorder limit [15]. The tricritical point separating these two regimes was studied by renormalization group method in [13], and numerically in [14]. While the agreement is far from perfect in $D = 3$, at least the locations of the tricritical points are known with high precision for all $D$.

In the present paper we shall deal with the case of moving interfaces in (isotropic) media with frozen disorder. There is a wide spread believe that they should also be described by KPZ scaling [2–5], because the ‘frozenness’ of the noise should not be very relevant as long as the interface moves. It might occur occasionally that a particularly strong obstacle prevents the interface from progressing locally, in which case the interface will stop to grow locally and close again behind the obstacle. This leaves then a bubble, but these bubbles should not modify the basic scaling laws.

As we shall see, this is wrong. In some control parameter regions this scenario is completely and qualitatively overthrown. It is still true qualitatively in other regions, but whether the KPZ scaling laws hold there is not clear.

In the next section we shall define the model which we used for our numerics, but we stress that the phenomena discussed should hold much more generally. In Sec. 3 we present theoretical arguments for what we call sponge phases and numerical results supporting them. In Sec. 4 we discuss whether KPZ scaling holds in nonspongy phases, and in Sec. 5 we summarize and draw our conclusions.

II. GENERALIZED PERCOLATION

We consider (hyper-)cubic lattices in dimensions $D = 3, 4, 5,$ and $6$. The class of models we study can be con-
sidered as a generalization of the susceptible-infected-removed (SIR) epidemic model, where we keep track of how often a (not yet infected) site had been “attacked” by an infected neighbor. Thus each lattice site can be either ‘removed’ (i.e., it had been infected, but it no longer is), infected, or in one of $N + 1$ susceptible states, where $N$ is the coordination number. Time is discrete, and updating is done in parallel: At each time steps, every infected site attacks each of its neighbor, after which it becomes removes. If a neighbor had already been attacked $k - 1$ times (either in previous or during the present time step) but is still susceptible, it becomes infected with probability $p_k$. Thus, after $k$ attacks it will have been infected with probability $q_k$ with

$$q_k = q_{k-1} + (1 - q_{k-1})p_k.$$  

We shall call this model generalized percolation. For efficiency of the code, all infected (or ‘growth’) sites are written in a list, and during each time step this list is gone through, a new list of growth sites is built, and at the end the old list is replaced by the new one. Notice that resulting configurations might depend on the the order in which these lists are gone through, for a fixed sequence of random numbers. Therefore, after each time step the list of growth sites is randomly permuted.

Special cases of this model are site and bond percolation. In the former, a site cannot be infected at all, if the first attack did not already succeed. Thus $p_1 = p$ and $p_k = 0$ for $k \geq 2$. For bond percolation, in contrast, $p_k$ is independent of $k$: $p_k = p$ for all $k$.

Consider now the random field Ising model (RFIM), with an initial state where all spins are down except for ‘seed sites’, whose spins are up. Dynamics is single flips with parallel update, and we assume that that the spin at site $i$ can only flip if (i) this reduces the energy (i.e., we are at $T = 0$), and (ii) at least one of it neighbors have flipped during the previous time step. It is easily seen \[9, 10\] that this can be mapped exactly onto the above generalized percolation model, for any distribution of local fields.

Alternatively, one could consider a porous medium that is wetted by a fluid with non-zero surface tension. Although one now cannot prove a general mapping, it is clear that generalized percolation will be a good model for a wide range of local geometries and surface tensions.

There are strong reasons \[10, 13, 14\] to believe that there are only three universality classes for critical generalized percolation. One is ordinary percolation. The second is found in cases where first attacks are very unlikely to succeed, but lead to much weakened sites. Thus $p_k$ increases strongly with $k$. In that case, clusters tend to be less fractal and their surfaces tend to be more smooth. Indeed, ‘bays’ and ‘fjords’ will become infected (because sites there have many infected neighbors), while tips and spikes will be formed less likely. This is precisely the case for wetting of porous media by a fluid with high surface

FIG. 1. (color online) The lower curves in each panel represent the critical curves. Below them, clusters starting from point seeds are finite and interfaces move only a finite distance, before they stop. Above them, interfaces would grow forever on infinite lattices, and clusters starting from point seeds would have a finite probability to grow forever. On these critical curves are indicated the tricritical points and the critical bond and site percolation points. The upper curves separate, for $D = 4$ and $5$, the sponge phase (between both curves) from the non-spongy phase. For $D = 3$, the same is true to the right of the tricritical point. On the left of the tricritical point, there is no sponge phase for $D = 3$. 
tension, and for the RFIM with very small randomness of the local fields. Indeed, it was found in [11, 12] that there is a sharp morphological transition in critically pinned interfaces in porous media and in random magnets, where the interface changes from percolation-like at small surface tensions and strong disorder to rough but non-fractal at large surface tension and weak disorder. Between these two regimes is a tricritical point [13, 14]. Essentially, we shall in the present paper deal with analogous (but very different in details) transitions for non-pinned interfaces.

In view of our claim of universality, we shall study only the simplest non-trivial model for generalized percolation, which was called the ‘minimal model’ (MM) in [10]. There, we assume \( p_1 \neq p_2 \) but \( p_k = p_2 \) for all \( k > 2 \). Thus, we distinguish between virgin sites and sites that had already been attacked by an infected neighbor, but we do not keep track of the number of previous attacks. If a site is infected from the bottom, we do not keep track of the number of previous attacks.

In all dimensions, we follow the upward evolution of an interface that was initially equal to the (hyper-)plane \( z = 0 \), on lattices that were big enough so that the upper boundary of the lattice was never reached during the simulation. Lateral boundary conditions were helical. The critical lines in the \( p_1 \) versus \( p_2 \) control parameter space for \( D = 3, 4, \) and \( 5 \) are shown in Fig. 1.

### III. MULTIPLE PERCOLATING CLUSTERS AND SPONGE PHASES

Let us first consider site percolation in \( D \geq 3 \). For \( D = 3 \) the percolation threshold on the SC lattice is at \( p_c = 0.3116 \ldots \), and for \( D \geq 4 \) it is even smaller [10]. Thus, \( p_c \) is in all these dimensions less than 1/2. Take now a hypercubic lattice and color its sites randomly black and white, each with probability \( p = 1/2 \). Then both the black and the white sites will percolate, i.e. we have at least two coexisting percolating clusters. Indeed, since the density of black sites is supercritical, there will be exactly one infinite black cluster, all other black clusters will be small. Similarly, there will be precisely one infinite white cluster. These two infinite clusters will then penetrate each other. If site \( i \) is black and is on the infinite cluster, there is a finite probability that one of its neighbors is white and on the white infinite cluster. Finally, the same will be true not only for \( p = 1/2 \), but also if sites are black with probability \( p \in [p_c, 1 - p_c] \) and white with probability \( 1 - p \).

Consider now a ‘cluster’ of black sites grown – for a finite time \( t \), and for \( p \in [p_c, 1 - p_c] \) – by starting from an infected hyperplane \( z = 0 \) that can infect its upper neighbors. Otherwise said, consider the set of all black sites (in a configuration with a fraction \( p \) of black sites) which are connected to the bottom of the lattice by a path of length \( \leq t \). Roughly, they will occupy densely a layer of thickness \( \propto t \). Similarly, when starting from a point seed, the cluster will occupy densely a (hyper-) sphere of radius \( \propto t \). By ‘densely’ we mean that this layer or sphere has no big voids (near every point in the layer/sphere there is a point belonging to the cluster), but it contains many holes, and these holes are not only connected, but also dense. Indeed, the white infinite cluster will also be dense everywhere, so that the black cluster has fjords that penetrate it all the way down to the bottom \( z = 0 \).

We call the phase with two clusters penetrating each other and being dense everywhere a sponge phase. While the above arguments clearly show (even if not in a strict mathematical sense) that a sponge phase exists in site percolation for \( D > 3 \), the situation is much more subtle for bond percolation. But the existence of multiple clusters was proven rigorously at least for \( D \geq 8 \) by Bock et al. [17], and their denseness was proven in [15].

In order to understand the situation for bond percolation with all \( D \geq 3 \) and for generalized percolation, we used Monte Carlo simulations. We studied mainly the planar geometry. Thus in a first step, we started with a seed consisting of the hyperplane \( z = 0 \) and let the cluster grow until the layer \( z = L_z - 1 \) was reached for the first time. At that moment, the growth was stopped, and in a second step a cluster was grown on the not yet infected sites by starting from a (hyper-)planar seed at \( z = L_z \) and moving down. Alternatively, in order to speed up the simulation, we did not grow the entire second cluster but just followed the external hull of the first cluster (by which we mean the set of sites which are not on the first cluster but are neighboring it, and which are accessible by paths starting at \( z = L_z \) and avoiding the first cluster).

As observables we measured the penetration depth of the second cluster, its density profile, the mass of the hull, and its variance. In general, the lateral size \( L \) of the lattice was somewhat smaller than \( L_z \).

As typical results we show in Fig. 2a the average mass density of the hull and in Fig. 2b its variance, for \( D = 5 \) and for \( p_1 = p_{c, bond} \). The mass density is defined per unit area of the base surface. Both plots show very clearly a phase transition at \( p_2 = p_{2, sponge} = 0.2586(5) \). Notice that \( p_{c, bond} = 0.11817 \) for \( D = 5 \), i.e. there is a rather wide region with a sponge phase. For \( p_2 > 0.2586 \) the first cluster is so dense that the second one cannot penetrate and there is no sponge phase.

Similar plots were made for many more values of \( p_1 \) and for all dimensions between 3 and 6. Final results are plotted in Fig. 1 together with the curves for the percolation (or pinning) transitions.

Notice that it would not be easy to obtain the order of the sponge transition or any critical exponents from plots like Fig. 2. Partly this is because these plots depend strongly on \( L_z \). For \( L_z \rightarrow \infty \), not only the variance but also the average mass would diverge. But fortunately, the transition can be studied much more precisely by another type of simulation.

There we use a lattice of size \( L^D \) with periodic bound-
ary conditions \[24\]. In a first step we determine the percolating cluster, if there is one. Since we do this deep in the supercritical region, this is practically always the case, and its determination is easy. In a second step we look for a percolating cluster with \( p_1 = p_2 = 1 \) on its complement. Notice that we cannot show in this way whether the second cluster penetrates densely the first one, but we can determine the transition point with higher precision and we can study the second cluster more carefully near the transition. A typical result is shown in Fig. 3, where we plotted the mass distributions of all clusters in the holes of the first (supercritical) cluster for bond percolation in \( D = 3 \). They have the standard Fisher exponent \( \tau = 2.189 \) and the fractal dimension \( D_f = 2.523 \) of ordinary percolation. Similar plots for dimensions \( D = 4 \) to 7 show that the problem is always in the ordinary percolation universality class. This might not be obvious in view of the fact that percolation inside holes of critical percolation clusters in \( D = 2 \) is in a different universality class \[19\]. But it actually is not very surprising, since the first cluster – being strongly supercritical – is very similar to the set of all black sites, and thus the second largest (white) cluster grows on a very weakly correlated set of randomly chosen sites. A list of sponge transitions for bond percolation is given in Table 1. These values should be compared to the asymptotic estimate

\[ p_{b,\text{sponge}} \sim \log(D)/2D \]  

(2)

for large \( D \). \[17\].

As seen from Fig. 1, there is a striking qualitative difference between \( D = 3 \) and \( D \geq 4 \): While there is a sponge phase for all values of \( p_1 \) when \( D \geq 4 \), no such

FIG. 2. (color online) Panel (a): Mass density (per unit base surface) of the external hull of the cluster growing upwards from \( z = 0 \) in dimension \( D = 5 \) and at \( p_1 = p_{c,bond} \). This density would be infinite in the sponge phase, if the cluster thickness \( L_z \) were infinite. Panel (b) shows the variance of the mass density.

FIG. 3. (color online) Mass distribution of clusters inside the holes of the giant cluster in supercritical bond percolation in \( D = 3 \), with \( p = p_{b,\text{sponge}} = 0.31958 \) chosen such that the holes just contain a incident giant cluster (notice that this second cluster is grown with \( p = 1 \), i.e. we are at the threshold where a giant cluster exists in the complement of the first cluster. As in ordinary 3-D percolation, we have \( P(m) \sim m^{-\gamma} \) with \( \tau = 2.189 \) in the scaling region and a peak position scaling \( m_{\text{max}} \sim L^{D_f} \) with \( D_f = 2.523 \).

TABLE I. Sponge transition points for bond percolation in dimensions \( D = 3 \) to 7. In the last column we give the thresholds for directed percolation on hypercubic lattices with the diagonal as preferred direction \[23\].
phase exists for $D = 3$ when $p_1 < p_{1,\text{tricrit}}$. Indeed, within the limits of accuracy, the spongy phase seems to exist precisely down to $p_1 = p_{1,\text{tricrit}}$. We have no theoretical explanation for this. It means that in $D = 3$ the pinning transition always leads to a non-spongy moving cluster when the pinned interface is non-fractal, but always leads to a spongy growing cluster when it is percolation-like. There is no such distinction for $D \geq 4$.

IV. ARE GROWING INTERFACES IN MEDIA WITH FROZEN DISORDER IN THE KPZ UNIVERSALITY CLASS?

A. General remarks and a second morphological transition related to directed percolation

When there is a sponge phase, the answer to the above question is, at least in naive sense, "no". The true interface is in this case extremely convoluted, has fractal dimension $D_f = D$, and penetrates densely the entire 'bulk' phase underneath the surface. One can presumably define then an 'outer interface' where all fjords are cut off, and one follows only what would look like the interface in a coarse grained sense. We will not follow this option. Then we still have several obvious questions:

- If we are not in a spongy phase, is the interface scaling described by a single universality class or are there different universality classes?
- Is at least one of these the KPZ universality class?
- If there are several classes, are there further morphological transitions between them?

A partial answer is suggested by the fact that, in $D \geq 3$, the KPZ equation has weak and strong coupling solutions [20]. The strong coupling solution is the one that shows the standard KPZ scaling, while interfaces in the weak coupling regime are asymptotically flat with logarithmic corrections, i.e. the interface width increases less fast than any power of the base length $L$, in the limit $t \to \infty$. A similar weak to strong coupling transition might also occur here, although it is not a priori clear what 'weak' and 'strong' should mean in the present case.

Another partial answer is also suggested by the fact that yet another morphological transition for interface growth with bounded speed of spreading is provided by the threshold of directed percolation (DP) [20,22]. Consider, more precisely, bond percolation from an initial wetted surface in $D \geq 3$ with Miller indices $(1,\ldots,1)$, such that growth is along the space diagonal. Denote by $p_{c,\text{bond}}$ the bond percolation threshold in $D$, and by $p_{b,\text{dirperc}}$ the threshold for directed bond percolation on a $D$-dimensional hypercubic lattice with growth in the $(1,\ldots,1)$ direction. As long as $p_{c,\text{bond}} < p < p_{b,\text{dirperc}}$, a typical shortest path to a wetted site with $z \gg 1$ will have many back-turns, simply because long paths without back-turns would not exist for $p < p_{b,\text{dirperc}}$. This is, however, no longer true for $p > p_{b,\text{dirperc}}$. In this regime, many such paths exist, and there is a non-zero probability that a site is wetted by such a path. Thus there is a finite probability that sites with $z = t$ get wetted at time $t$, and the interface is perfectly flat with fjords and holes penetrating it nearly everywhere. If we would cut off these fjords by some ad hoc rule, we would obtain a flat interface.

This last argument rests on the assumption that we have a regular lattice with frozen percolative disorder. It would no longer hold, if we had in addition annealed disorder (e.g. with random times needed to jump over a lattice bond), or if the frozen disorder were of a different type. In that case, there would still be a directed percolation transition at which back-bending paths become irrelevant, but paths without back-bending would not arrive at the same height at any given time. Thus the interface would still be rough, and the standard arguments for KPZ scaling in cases with annealed disorder (in particular the relationship with directed polymers [20]) would suggest that such interfaces are in the KPZ universality class.

Thus we should not expect the DP-related morphology transition to be universal, while we do expect the sponge transition to be universal. Nevertheless, it is of interest to look at the numerics. Critical vales for DP on hypercubic lattices with spreading direction along the diagonal [23] are given in Table 1. We see that $p_{b,\text{dirperc}} > p_{b,\text{sponge}}$ for $D = 3$, but the opposite is true for $D > 3$. Thus, we have no regime where we can expect true KPZ scaling of supercritical bond percolation interfaces in dimensions $D \geq 4$, there is a possible window $p \in [0.3196, 0.3822]$ for $D = 3$.

In the following we shall study only the case $D = 3$. We shall present simulation data both in this window and for small values of $p_1$, where there is no sponge phase at the critical transition line. Although our results are affected by very large finite size corrections, we shall argue that KPZ scaling is only found in the former case, while slightly supercritical interfaces are asymptotically flat when the near-by critical interfaces are self-affine.

B. Supercritical interfaces in $D = 3$

In this subsection we show simulations for the minimal model in $D = 3$. We use plane seeds oriented in the $(1,1,1)$ direction, i.e. with the normal to the plane parallel to the space diagonal of the lattice. With such an orientation, interfaces can grow even if $p_1 = 0$, i.e. even if two neighbors are needed to wet a site. Lateral b.c. are helical, but for most efficient use of the memory we used tilted coordinates $i$ for the horizontal and $z$ for the
vertical position, such that the neighbors of site \((i, z)\) are \((i, z + 1), (i, z - 1), (i + 1, z + 1), (i - 1, z - 1), (i + L, z + 1), (i - L, z - 1)\). If we had used the original tilted coordinates, every second memory location would be unused.

In the \(z\) direction, lattices were de facto infinitely large, because we used a recycling trick: First, we determined in test runs the roughness in worst cases. Assume that we estimated that the width – for the given lateral size \(L\) and for the given values of \(p_1\) and \(p_2\) – is less than \(W_{\text{max}}\) with high probability. Then, if the highest site in the interface had reached a height \(z_{\text{max}} > W_{\text{max}}\), we erased the lowest \(z_{\text{max}} = W_{\text{max}}\) planes and used them to store the next higher parts of the interface. During these simulations we checked that \(W\) indeed remained bounded by \(W_{\text{max}}\). If this was violated, we discarded the entire simulation and repeated it with a higher value of \(W_{\text{max}}\).

As a last trick to use memory most efficiently, we used multispin coding. The system sizes that we could handle in this way depend on the interface roughness and thus also on the distance from the critical percolation line. For small roughnesses, i.e. far above the percolation threshold, we could simulate systems with base surfaces up to \(L \times L = 8192 \times 8192\) (although we shall present here results only for \(L \leq 4096\)), and followed them typically for \(10^6\) to \(10^7\) time steps. This is to be compared to the largest previous simulation of interfaces with overhangs (in the RFIM), where \(t \approx L \leq 250\) [25].

For small times, KPZ scaling is masked by the local roughness of the interfaces, and is visible only for inaccessibly large systems. Therefore, all data shown in Figs. 4 to 7 are results for large times, when the interface moves with constant average speed and its statistical properties are stationary. Since our interfaces had overhangs and the bulk phase below had holes, the definition of an interface height is not unambiguous. For simplicity we show results where the height at horizontal position \(i\) is given by the height of the highest wetted site in the column \(\{i, z; z \geq 0\}\). We verified however that similar results were also obtained by other definitions, e.g. if average height and variance of the interface is defined by average height and variance of the set of all growth (i.e., newly wetted) sites.

For KPZ in 3 dimensions, the width of a moving interface in the stationary state scales as

\[
W(L) \sim L^\alpha
\]

with \(\alpha = 0.3869 \pm 4\) [26]. In Fig. 4 we compare this (short straight line) to data for \(p_1 = 0.04\). Each curve here corresponds to one value of \(p_2\). The uppermost curves essentially show the scaling of nearly pinned interfaces. The upper envelope gives \(W \sim L^{0.84}\) for just marginally unpinned interfaces. All curves are convex. If this convexity prevails also for larger \(L\), each curve gives an upper bound for \(\alpha\). Indeed, the slopes of all curves seem to become the same, giving the bound \(\alpha \leq 0.10(1)\), in striking disagreement with the KPZ prediction.

FIG. 4. (color online) Interface widths in the stationary state for large times versus base size \(L\). The short red straight line represents the KPZ scaling \(W \sim L^\alpha\) with \(\alpha = 0.3869\). Each of the other curves is for one value of \(p_2\), with \(p_1 = 0.04\) being common to all of them. The lowest curve is the one for largest \(p_2\), while the uppermost curve is for \(p_2\) just slightly above the percolation threshold, i.e. for interfaces that are nearly pinned. More precisely, the values of \(p_2\) are (from top to bottom) 0.52432, 0.52436, 0.5244, 0.52452, 0.5248, 0.5252, 0.526, 0.528, 0.532, 0.54, 0.548, 0.56, 0.576, and 0.596.

FIG. 5. (color online) Interface widths in the stationary state for large times at \(p_1 = 0\) versus base size \(p_2 - p_{\text{2,crit}}\). Each curve corresponds to one fixed value of \(L\). Anticipating that \(W(L) \sim L^\alpha\) with \(\alpha = 0.01\) for large \(L\), we actually plotted \(W(L)/L^{0.1}\).
FIG. 6. (color online) Interface widths in the stationary state for large times at five pairs \( (p_1, p_2) \) versus base size. The curve for bond percolation \( (p_1 = p_2) \) is shifted up to avoid crowding of the curves. The short straight line indicates again KPZ scaling.

In order to see whether this depends on the particular value of \( p_1 \), we show in Fig. 5 analogous results for \( p_1 = 0 \). We plot there the data differently, to indicate also the scaling for small \( p_2 - p_2^{crit} \), where \( p_2^{crit} = 0.559188 \) is the pinning threshold. More precisely, we plotted in Fig. 5 the ratios \( W(L)/L^{0.1} \), anticipating that \( W(L) \) scales also for \( p_1 = 0 \) with the same exponent \( \alpha \) as for \( p_1 = 0.04 \). This seems to be indeed the case, and again our results are in striking disagreement with the KPZ prediction.

These results might suggest that there is indeed another non-trivial power law, in addition to the KPZ scaling and to the trivial scaling \( W = const \) of the weak-noise solution of the KPZ equation. We consider this, however, as extremely unlikely and interpret the data in Figs. 4 and 5 as logarithmic growth of \( W(L) \), as is indeed expected for the weak coupling solution [20].

Results for larger values of \( p_1 \) are shown in Fig. 6. They are all for values of \( p_2 \) which are slightly \( \approx 5\% \) higher than the critical ones, except for those for bond percolation. There, \( p \) had to be substantially larger than \( p_c \) because it had to be outside the sponge phase. To avoid crowding of curves, we thus shifted the curve for bond percolation by multiplying \( W \) by a factor 3.5. We see that curves for \( p_1 < 0.1 \) bend down for all \( L \), while curves for \( p_1 > 0.1 \) veer up for large \( L \). Although none of the curves reaches the KPZ scaling at large \( L \) (for this we would need much larger system sizes), we interpret this as an indication that the transition between weak and strong coupling happens near \( p_1 \approx 0.1 \). This also supported by Fig. 7, where we show again the \( p_1 \) versus \( p_2 \) plane, and indicate by bullets (crosses) points in the weak (strong) coupling regime. All this suggests a weak/strong transition near the short straight line in Fig. 7. It is very tempting to suggest that this line meets the critical line precisely at the tricritical point. This fascinating conjecture would mean that four different transition lines meet at this point: The critical curve for self-affine pinned surfaces, the critical percolation curve, the sponge transition, and the weak/strong KPZ transition. It would also explain why the very careful RG study of Janssen et al. [13] disagreed so dramatically with the numerics of [14].

V. DISCUSSION AND CONCLUSIONS

The main result of the present paper is the establishment of a sponge phase. Similar phases with two tightly intermingled microscopic phases were previously known only for systems with at least three microscopic phases, e.g., in emulsions with two liquids and one membrane [27]. In the present case, there are no explicit membranes between the two microphases, which are made up of interpenetrating supercritical percolation clusters. The striking property which distinguishes a sponge phase from other multiphase systems is that both microphases are connected and everywhere dense in the mathematical sense, i.e., in the scaling limit every point in space is infinitely close to points in both microphases.

For \( D > 3 \) we found that the transition from pinned to moving interfaces in media with frozen randomness al-
ways leads first to a sponge phase, while non-spongy bulk phases appear only in a second morphological transition. This is at least what we found in what was called the ‘minimal model’ in [10], but we conjecture that the same is true also for the RFIM, with any distribution of local random fields.

For $D=2$ no sponge phases can exist (for topological reasons), and for $D=3$ we found that the de-pinning transition can lead either to a sponge or to a non-sponge phase. The first seems to happen when the de-pinning transition is in the percolation universality class, while the latter happens when the critically pinned interface is self-affine. Again, this claim is only based on the minimal model. Simulations with other models in the generalized percolation family (which includes the zero-T RFIM) would be extremely welcome to verify it, as would be analytic arguments. So far, these claims are entirely based on simulations.

At the sponge / non-sponge transition, the incipient infinite clusters in the voids of the the main phase are in the universality class of critical ordinary percolation. Again we claim that this is true in general, although we have only simulation data to support it.

In the last part of the paper, we asked whether the interfaces in the parameter region above the sponge phase satisfy KPZ scaling, as is often assumed – in spite of the fact that theoretical arguments for KPZ scaling only exist in cases without frozen randomness. Here we studied only the case $D=3$. We found huge finite size corrections, but our simulations were nevertheless able to support the following scenario: For all control parameters, the interfaces are asymptotically either in the usual (strong-coupling) KPZ universality class, or in the weak-coupling class. In the latter, the interface width increases, for large times, less fast that any power of the base length $L$, so that it becomes asymptotically flat.

Moreover, marginally supercritical (i.e., nearly pinned) interfaces seem to be rough only when the critically pinned interfaces are percolation hulls in the standard percolation class, while de-pinning via self-affine critical interfaces leads to asymptotically flat interfaces. This might not be the most important finding of the present paper, but it certainly is the most surprising.

Since the present work is mainly numerical, and moreover based only on one particular model, the most obvious open problems are to support our claims with mathematical arguments, and to do simulations for more general model classes. Apart from that, there are several minor open problems. For instance, at the sponge transition for bond percolation a supercritical cluster with $p > p_c$ coexists with a critical cluster with $p = 1$. A more general problem would be to find the region(s) in control parameter space where one giant cluster with $p = p_1$ coexists with another one with $p = p_2$.

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