QED Exponentiation for quasi-stable charged particles: the $e^- e^+ \rightarrow W^- W^+$ process†

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Abstract

All real and virtual infrared singularities in the standard analysis of the perturbative Quantum Electrodynamics (like that of Yennie-Frautschi-Suura) are associated with photon emissions from the external legs in the scattering process. External particles are stable, with the zero decay width. Such singularities are well understood at any perturbative order and are resummed. The case of production and decay of the semi-stable neutral particles like $Z$ boson or $\tau$ lepton, with the narrow decay width, $\Gamma/M \ll 1$, is also well understood at any perturbative order and soft photon resummation can be done. For the absent or loose upper cut-off on the total photon energy $\omega$ the production and decay process of the semi-stable (neutral) particles decouples approximately and can be considered quasi-independently. In particular soft photon resummation can be done separately for the production and decay process treating semi-stable (neutral) particle as stable. The QED interference contributions between production and decay are suppressed by the $\Gamma/M$ factor. In case of experimental precision $\omega$ comparable with or better than $\Gamma$, these interferences have to be included. In case of $\omega \ll \Gamma$ decoupling of production and decay does not work any more and the role of semi-stable particles is reduced to the same role as that of other internal off-shell particles. So far, consistent treatment of the soft photon resummation for semi-stable charged particles like $W^\pm$ boson is not available in the literature and the aim of this work is to present a solution for this problem. Generally, it should be feasible because the underlying physics is the same as in the case of the neutral semi-stable resonances – in the limit $\Gamma/M \ll 1$ production and decay processes for charged particles also necessarily decouple due to long lifetime of the particles. The technical problems to be solved in this work are related to the fact that semi-stable charged particle are able to emit photons. The practical importance of the presented technique to $e^+ e^- \rightarrow W^+ W^-$ process at the Future Circular Collider (FCC-ee) will be underlined.

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1 Introduction

The Electroweak (EW) Field Theory was confirmed at LEP as a correct physics theory by the precision measurements of LEP \cite{1,2}. LEP data were precise enough to test all important dynamical properties of the EW theory, such as quantum loop effects, consequences of the renormalization, multiple photon emission etc. In particular EW gauge cancellations and quantum loop effects were verified experimentally at LEP in the $e^+e^- \rightarrow W^+W^-$ process at the precision tag for the total cross section at the level of 0.3-0.5%. Mass of $W$ was also measured directly with the precision of 33MeV.

The electron-positron Future Circular Collider (FCC-ee) \cite{3,4}, considered as the future project in CERN, will be able to produce the number of $W$ pairs factor $10^3$ larger than at LEP. This will serve to determine total $WW$ cross section, mass and width of $W$ with unprecedented precision and search for any anomalous phenomena beyond the Standard Model (SM) of the EW and strong interactions. Obviously, analysing FCC-ee data will also require new SM perturbative calculations for the $e^+e^- \rightarrow W^+W^-$ process, much more precise than these available at the LEP era \cite{5,6}. The precision tag expected in FCC-ee experiments is at the level of about 0.01%, a factor 10 better than at LEP. This will require to go beyond the state of art of LEP era in the calculations of the SM predictions for the $e^+e^- \rightarrow W^+W^-$ or $e^+e^- \rightarrow 4f$ process.

For general discussion of the theoretical issues in the $W$-pair production process the reader should consult reviews of refs. \cite{7,8}. In particular the delicate question of the EW gauge invariance for the Dyson summation leading to imaginary part of the $W$ and $Z$ propagators is covered there.

Here we shall focus on the important QED part of the EW/SM perturbative corrections to $W$-pair production process. More precisely on this part of the QED corrections, which are related to soft and collinear (SC) singularities for real and virtual photon emissions on the external legs\footnote{It is tempting to call them "universal" but in fact non-soft subleading collinear perturbative corrections are process-dependent, hence non-universal, while all soft corrections are universal.}. According to the accumulated knowledge on the SC photonic contribution, it is quite clear that they factorise either at the amplitude level, or for the differential distributions and can be calculated separately to a much higher order than the remaining genuine EW corrections\footnote{The genuine EW part of the SM perturbative corrections include non-soft, non-collinear remnants of the QED origin.}. This is very convenient, because SC contributions are much bigger numerically than genuine EW corrections, simpler to calculate, and can be resummed to an infinite order. Once separation of QED and EW parts is established, resummation of higher order contributions in each of these two classes can be done independently. The important nontrivial final step is then merging/matching them in the final results.

There is little doubt that the factorisation and resummation of the QED soft/collinear corrections is the key to the success in the high precision calculations of the SM predictions for $W$-pair production process at FCC-ee.

There are four classes of QED corrections to $W$-pair production and decay process:
initial state corrections (ISR), final state corrections (FSR) in the decays of two $W^\pm$, final state Coulomb corrections (FSC) and the so-called non-factorisable interferences. between ISR and decays (IFI) and between two $W^\pm$ decays (FFI). The IFI and corrections are suppressed due to relatively long lifetime of $W$'s and FFI due to large space separation. The effects due to ISR are numerically the biggest but also easier to control, while the FSR effects can be also quite sizable for typical experimental cut-offs.

The IFI and FFI interferences are small, suppressed by the factor $\Gamma_W/M_W$ away from the threshold, strongly cut-off dependent and algebraically most complicated. At LEP they could be neglected but at FCC-ee precision they have to be handled with the great care! In reality, IFI and FFI contributions to differential cross sections do factorise – for the real photon emission at the amplitude level and for the virtual ones at the loop integrand level. Hence naming them as non-factorisable is somewhat misleading. Nevertheless, we shall stick to this traditional naming, NFI in short.

The relative narrowness of the $W$ boson resonance not only causes suppression of the QED interferences, but also provides for the expansion in terms of $\Gamma_W/M_W \sim O(\alpha)$, of the matrix element of the $e^+e^- \rightarrow 4f$ process, into the biggest physically most interesting double-resonant $e^+e^- \rightarrow W^+W^-$ part and smaller single-resonant and non-resonant background parts. In the following we shall refer to them as double-pole (DP), single-pole (SP) and non-pole (NP), as it was common in the LEP era literature.

The above pole expansion (POE) in the powers of $\Gamma_W/M_W$, disentangling DP, SP and NP components at the scattering amplitude-level is very useful because it allows for each of these three components to calculate genuine EW correction at the different perturbative order and to perform resummation of QED soft/collinear contributions at the different sophistication level. In the final stage of the calculation, the best way is to sum POE contributions coherently at the amplitudes level, before summing over spin and taking modulus squared, rather than for differential cross sections, thus avoiding proliferation of many interference terms.

At the time of LEP experiments, two solutions based on the pole expansion were worked out, in which EW $O(\alpha^1)$ corrections were complete only for the DP component of the $e^+e^- \rightarrow W^+W^-$ of the $e^+e^- \rightarrow 4f$ process. One of them nicknamed KandY \cite{9,10} was based on the combination of YFSWW \cite{11,12} Monte Carlo for the $e^+e^- \rightarrow W^+W^-$ and $W^\pm$ decays process, with another MC program KORALW \cite{17} for the remaining background. The multiphoton emission for ISR, including higher orders, was implemented using soft photon resummation inspired by the Yennie-Frautschi-Suura (YFS) work \cite{18}. The QED FSR was added in $W$ decays using PHOTOS \cite{19,20}. Another POE-based solution was that of RACOONWW \cite{21,22}, also with the complete EW $O(\alpha^1)$ corrections implemented only for the signal $e^+e^- \rightarrow W^+W^-$ process, and not for the background part.

Implementation of QED corrections in RACOONWW was very different from that in KandY. On one hand RACOONWW was using exact matrix element for the entire $e^+e^- \rightarrow 4f\gamma$ process but it was lacking sophisticated soft photon resummation of the KandY. For more detailed comparison of the two approaches see review of ref. \cite{8} or more recent of ref. \cite{23}. Both

\footnote{Including EW $O(\alpha^1)$ corrections of refs. Refs. \cite{13,16}.}
approaches were instrumental in the analysis of the LEP data for the $e^+e^- \rightarrow W^+W^-$ process \cite{2}, where gauge cancellations and quantum effects of the EW theory were tested experimentally for the first time.

Both approaches, \texttt{KandY} and \texttt{RACOONWW}, are neglecting terms of $O(\alpha\Gamma_W/M_W)$. The QED NFI interferences between $W$ production and decays were either neglected completely (\texttt{KandY}) or included in the soft approximation (\texttt{RACOONWW}) without resummation. The overall precision of there calculation was about 0.3-0.5%. FCC-ee experiments will require new calculations with the precision tag below 0.1% adding missing $\alpha\Gamma_W/M_W$ corrections, $O(\alpha^2)$ electroweak corrections to DP component, more advanced QED factorisation/resummation scheme, the inclusion of subleading QED $O(\alpha^2)$ corrections and more \cite{5,6}. In particular, the inclusion QED NFI corrections in the fully exclusive way\cite{4} including $\Gamma_W/M_W$ suppression will be necessary.

The aim of the present work is to work out a new methodology of the soft photon resummation including NFI corrections for charged unstable particles, similarly as it was done for the production and decay of the narrow neutral $Z$ boson in the process $e^+e^- \rightarrow f\bar{f} + n\gamma$ with a built-in $\Gamma_Z/M_Z$ suppression for the QED initial-final interferences (IFI) at any perturbative order \cite{24,25}. This method was already tested for $Z$ resonance in the Monte Carlo event generator KKMC \cite{24}. Its matrix element is built according to the so called coherent exclusive exponentiation (CEEX) scheme, in which factorisation of the infrared (IR) divergences is done entirely at the amplitude level (before squaring and spin-summing). The older version of the exclusive exponentiation (EEX) was done at the level of the differential distribution of ref. \cite{26,27} for the same $e^+e^- \rightarrow f\bar{f} + n\gamma$ features multiphoton resummation of ISR and FSR. Both approaches CEEX and EEX are inspired by the pioneering work of Yennie-Frautschi-Suura \cite{18}.

In the present work we shall generalize the CEEX scheme to the case of any number of the narrow charged intermediate resonances like $W$ – the scheme is however quite general and applies to any charged resonance of any spin. The new CEEX scheme provides exclusive (unintegrated) description for multiple real photons of any energy, for $E_{\gamma} \sim \Gamma_W$, $E_{\gamma} \ll \Gamma_W$ and $E_{\gamma} \sim \sqrt{s}$, with all QED interferences between production and decays properly accounted for. Multiple real and virtual photon emission from all external stable particles and the intermediate semi-stable charged resonance will be described correctly in the soft photon limit and summed up to infinite order. As in the case of CEEX of refs. \cite{24,25}, its present extension will provide for well defined methodology of incorporating non-soft contributions\cite{4} (including genuine EW corrections) calculated up to finite perturbative order into multiphoton amplitudes of the soft photon resummation scheme. In particular, sizable but easier to calculate QED non-soft collinear contributions can be also included easily up to arbitrarily high order.

The consistent resummation of the apparently IR-divergent contribution due to photon emissions from the semi-stable intermediate charged particle (narrow resonances) in the

\footnote{They depend strongly on the experimental cut-offs.}

\footnote{This will be done without introducing any parameter in the photon energy distinguishing between soft and hard photons. Minimum photon energy in the Monte Carlo implementation can be set to arbitrarily low value without any effect on the physical results.}
perturbative expansion is a non-trivial issue. Let us first consider $\Gamma \to 0$ limit. The best illustrative example is that of the $\tau^\pm$ pair production and decay in the $e^+e^-$ annihilation where time scale of the $\tau$-pair formation (production process) is shorter than $\tau$ lifetime by at least $\Gamma_\tau/m_\tau \simeq 3 \cdot 10^{-12}$ factor, hence photons emitted in these two stages get completely decoupled and QED effects in production and decay can be implemented separately \[24,28,29\].

The situation in the $W$-pair production is similar but the suppression factor $\Gamma_W/M_W \simeq 0.026$ is not that small. The QED interferences are therefore expected to be of the order of $\alpha \Gamma_W/M_W \simeq 2 \cdot 10^{-4}$. In LEP experiment this size could be neglected but at FCC-ee precision effects of this size have to be calculated and taken into account. Moreover, such interferences depend on the kinematical cut-offs – from the experience from $Z$ case we know that they may grow by factor of 2-5 even for relatively mild cut-offs on the photon energies. Also, in the case when photon energy resolution $\omega$ of the detector approaches 2GeV, which is the case for FCC-ee detectors, photon emission from FSR in the production process and from $W$ decays cannot be separated and treated in the soft photon approximation, consequently the off-shell $W$’s have to be treated the same way as other internal exchanges in the $e^+e^- \to 4f$ process.

Our aim is to construct a variant of CEEX spin amplitudes in which we profit as much as possible from the smallness of $\Gamma_W/M_W$ and the classic YFS soft limit for the entire $e^+e^- \to 4f$ process is correctly reproduced for $\omega \ll \Gamma_W$. The basic technical problem will be that if we want to treat $W$’s as stable particles in the $W$-pair production process with zero width, then amplitudes of photon emission from $W$ must be IR-singular, while for the semi-stable $W$’s they are not ($W$ width acts as IR regulator). Our aim is to reconcile these two contradictory situations in a single algebraic framework.

In the YFSWW3 program photon emission was treated in a similar way as in the above $\tau$-pair production and decay calculations, except that $W$ masses were not fixed but modeled according to the Breit-Wigner shape. QED matrix element in YFSWW3 for $e^+e^- \to W^+W^-$ with the soft photon resummation is of the EEX type, including ISR, FSR and IFI. Decays of $W^\pm$s are supplemented with additional photons using PHOTOS. However, it could be easily replaced with the multiphoton MC implementation of the EEX of the WINHAC program \[30\]. Once EEX implementation is available in the $W$-pair process for the production and decays, the new CEEX matrix element developed in the present work can be introduced using an additional multiplicative MC weight\[6\] without any change in the underlying MC program. The above would be the solution for the resummed QED corrections of the DP part of the $e^+e^- \to 4f$ process. The $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(\alpha^2)$ genuine EW corrections can be added in the on-shell approximation within the CEEX matrix element the similar way as it was done in the $e^+e^- \to 2f$ process in refs. \[24,25\]. So far only $\mathcal{O}(\alpha^1)$ EW corrections are available. In order to exploit fully FCC-ee data $\mathcal{O}(\alpha^2)$ EW corrections will have to be needed. As pointed out in ref. \[5\] clear and clean separation of the QED and genuine EW correction in any perturbative order is a useful built-in feature of the CEEX factorization/resummation scheme.

\[6\]The same way as in KKMC.
The single-pole group of diagrams of the $e^+e^- \to 4f$ process process is separated at the amplitude level in the CEEX scheme. It would be enough to include genuine EW corrections to SP part at the $\mathcal{O}(\alpha^1)$. They are in principle known, because they are part of the $\mathcal{O}(\alpha^1)$ corrections to $e^+e^- \to 4f$ process in Refs. [31,32], although it may be not simple to disentangle them from the rest of the existing calculations. For the non-pole part of the $e^+e^- \to 4f$ process it would be probably enough to take them at the tree-level concerning EW corrections and take care of the QED corrections either in CEEX or in EEX scheme.

In this work CEEX scheme will be defined only for DP part leaving easier SP and NP variants for the future development. On the other hand, we shall also discuss in a more detail the explicit algebraic relation between the CEEX scheme and the EEX scheme of the YFSWW3. This will provide better understanding of the theoretical foundation of the existing EEX scheme of the YFSWW3. The main result of this work is, however, that it provides an important building block for the future high precision calculations for the $W$ pair production process, and also for any other process with charged narrow resonances.

Close to the $WW$ threshold, where the $W$ mass is planned in the FCC-ee experiment to be measured with $\lesssim 0.5\text{MeV}$ precision (using total cross section [3,4]). The problem is that near the $WW$ threshold the pole expansion for the non-QED part of the scattering matrix element is not efficient any more. The partial suppression of the QED IFI and IFF corrections will still work close to threshold as long as resonant curve of $W$’s are not fully ”destroyed” by the threshold factor. However, as shown in works based on the effective field theory (EFT) [33,34], near the threshold one may exploit expansion in Lorentz velocity $\beta = \sqrt{s - 4M_W^2}/2M_W \ll 1$ of the $W$’s in order to reduce substantially number of diagrams, such that higher order EW and QED corrections are again within the reach of the practical evaluation. This kind of expansion should be exploited in the standard diagrammatic approach as well.

Summarizing, a combination of the pole expansion and of QED exclusive exponentiation has already proven to be an economical solution for precision calculations of the SM prediction for $W$ pair production process at LEP and is the best candidate for the further development in future electron-positron collider projects, especially for FCC-ee. The inclusion of the QED interferences between the $W$ production and decays, and of other missing corrections of order $\alpha \Gamma_W/M_W$ will require applying more sophisticated soft/collinear photon factorisation and resummation scheme, combined with POE. We propose here a new solution based on coherent exclusive exponentiation, CEEX, in which resummation of the infrared (IR) divergences is done entirely at the amplitude level. The interesting feature of this new scheme is that $\Gamma_W/M_W$ suppression of the QED interferences between production and decay is a built in feature valid in any order and at any photon energy scale/resolution, all over the entire multiphoton phase space. The new scheme is similar to CEEX scheme previously formulated and successfully applied, to the case of the neutral intermediate resonances ($Z$).

One should not give up on more traditional EEX schemes, however. We shall discuss briefly alternative solutions within the traditional EEX schemes (extensions of EEX of YFSWW3). We shall also examine approximations or simplification done in the transition
from CEEX to EEX schemes, and between various variants of them.

Summarizing, this work provides an important building block for the future high precision Standard Model calculations for the $W$-pair production process at the future $e^+e^-$ colliders.

2 Pole expansion for $W$-pair production

As pointed out by Stuart [35] it is always possible to decompose the matrix element into a combination of Lorentz covariant tensors and Lorentz invariant functions. If unstable particles are involved in a process, then one can then perform a Laurent expansion about complex poles corresponding to those unstable particles. However, only the Lorentz invariant functions (mathematically, analytic functions of complex variables) are subject to this expansion, while the Lorentz covariant and spinor structure of the matrix element should remain untouched. In the so-called leading-pole approximation (LPA) one retains only the leading terms in the above expansion neglecting the rest of the Laurent series. As discussed in Ref. [35] the whole procedure does not violate gauge invariance of the matrix element. This is guaranteed by the fact that all terms in the pole expansion are independent of each other, e.g. in the case of two unstable particles doubly-resonant terms are independent of the singly-resonant and non-resonant ones, therefore there cannot be gauge cancellations between those terms. In Ref. [35] the process of $Z$-pair production and decay was presented as an example.

Here, we discuss the process of $W$-pair production and decay:

$$e^- (p_1) + e^+ (p_2) \rightarrow W^- (Q_1) + W^+ (Q_2) \rightarrow f_1 (q_1) + \bar{f}_2 (q_2) + f_3 (q_3) + \bar{f}_4 (q_4), \quad (2.1)$$

where $W^-$ decays into $f_1, \bar{f}_2$ and $W^+$ into $f_3, \bar{f}_4$. At the lowest order, the minimum gauge invariant subset of Feynman diagrams needed for this process is the so-called CC11-class of graphs. It includes apart from doubly-resonant $WW$ graphs (so-called CC03) also singly-resonant $W$ graphs. Below we discuss how to apply the pole expansion this process.

Since we are interested only in LPA (a double-pole approximation in this case) we start from extracting a part of the full matrix element that can give rise to doubly-resonant contributions (the rest will drop in LPA anyway). It can be written as follows:

$$\mathcal{M} = \sum_i \left[ \bar{v}_e (p_2) T_{\mu \nu}^i u_e (p_1) \right] M_i (s, t, s_1, s_2) \times D_W^{-1} (s_2) \left[ \bar{u}_{f_3} (q_3) \gamma^\mu V_W (s_2) \omega_- v_{f_4} (q_4) \right] \times D_W^{-1} (s_1) \left[ \bar{u}_{f_1} (q_1) \gamma^\nu V_W (s_1) \omega_- v_{f_1} (q_1) \right], \quad (2.2)$$

where

$$D_W (s) = s - M_W^2 + \Pi_W (s) \quad (2.3)$$

is a Dyson-resumed $W$ propagator with $\Pi_W (s)$ being the $W$ self-energy correction. In the above we have used the following notation:

$$\omega_- = \frac{1}{2} (1 - \gamma_5),$$
\[ s_1 = Q_1^2, \ s_2 = Q_2^2, \]
\[ Q_1 = q_1 + q_2, \ Q_2 = q_3 + q_4. \]  

\( T_{\mu\nu} \) are the Lorentz covariant tensors spanning the tensor structure of the matrix element, while \( M_i, \Pi_W, V_{Wf} \) are Lorentz scalars that are analytic functions of independent Lorentz invariants of the process. These functions then undergo the Laurent expansion about the complex poles corresponding to a finite-range propagation of two \( W \)'s. Keeping only the leading terms in the above expansion we end up with the LPA matrix element \[ 10,36 \]

\[ \mathcal{M}_{\text{LPA}} = \sum_i [\bar{v}_e(p_2) T_{\mu\nu}^i u_e(p_1)] M_i(s, t, s_p, s_p) \]
\[ \times \frac{F_W(s_p)}{s_2 - s_p} [\bar{u}_{f_3}(q_3) \gamma^\mu V_{Wf}(s_p) \omega v_{f_4}(q_4)] \]
\[ \times \frac{F_W(s_p)}{s_1 - s_p} [\bar{u}_{f_1}(q_1) \gamma^\nu V_{Wf}(s_p) \omega v_{f_1}(q_1)] , \]  

where the pole position \( s_p \) is a solution to the equation

\[ s - M_W^2 + \Pi_W(s) = 0, \quad F_W(s_p) = [1 + \Pi'(s_p)]^{-1}. \]  

At the lowest order the Lorentz tensors read

\[ T_{\mu\nu}^{1,2} = \gamma^\lambda \Gamma_{\lambda\mu\nu}(Q, Q_1, Q_2), \quad (2.7) \]
\[ T_{\mu\nu}^3 = \gamma_\mu (p_2 - Q_2) \gamma_\nu, \quad (2.8) \]

and the Lorentz scalars are

\[ M_1 = e^2 \frac{1}{s}, \quad (2.9) \]
\[ M_2 = -e^2 \frac{s_W}{c_W} [v_e - a_e \gamma_5] \frac{1}{s - M_Z^2 + iM_Z \Gamma_Z}, \quad (2.10) \]
\[ M_3 = \frac{e^2}{s_W^2} \frac{1}{t}, \quad (2.11) \]
\[ V_{Wf} = \frac{eU_{ij} \sqrt{N_c}}{2s_W^2}, \quad (2.12) \]

where \( Q = p_1 + p_2, \ s = Q^2, \ t = (p_2 - Q_2)^2, \ U_{ij} \) is the CKM matrix element, \( N_c \) is the QCD colour factor, \( s_W = \sin \theta_W, \ c_W = \cos \theta_W, \ v_e \) and \( a_e \) are the vector and axial couplings of a \( Z \) boson to electrons, \( \Gamma_{\lambda\mu\nu} \) is the \( VWW \) coupling (\( V = \gamma, Z \)):  

\[ \Gamma_{\lambda\mu\nu}(Q, Q_1, Q_2) = (Q + Q_1)_{\nu} g_{\lambda\mu} + (Q_2 - Q_1)_{\lambda} g_{\mu\nu} - (Q + Q_2)_{\mu} g_{\nu\lambda}. \]  

In the scalar function \( M_2 \) we have also applied LPA to the intermediate \( Z \)-boson. It is done in a similar way as for \( W \)'s. The \( W \)-pole position, up to \( \mathcal{O}(\alpha^2) \), is given by

\[ s_p = M_W^2 - iM_W \Gamma_W + \mathcal{O}(\alpha^2), \quad (2.14) \]
where \( M_W, \Gamma_W \) are the usual on-shell scheme \( W \) mass and width, and \( F_W = 1 \). One can easily check that at the lowest order this LPA matrix element has the same form as the CC03 matrix element calculated in ’t Hooft-Feynman gauge and in the constant \( W \)-width scheme. It was noticed in Ref. [37] that when the CC03 matrix element is calculated in the axial gauge also singly-resonant terms appear. This indicates that the singly-resonant graphs are needed to guarantee gauge invariance of the matrix element, i.e. that CC03 itself is not gauge-invariant, but one has to take at least CC11 for hadronic, CC10 for semi-leptonic and CC09 for leptonic final states. In the LPA approach described above it does not matter what gauge is used in the calculations. We start from the gauge-invariant matrix element and then apply the pole expansion. In the resulting LPA matrix element all non-double-pole terms drop out.

One of the complications that arises when going to higher orders is the fact that \( W \)’s are charged and therefore radiate photons. When a real or virtual photon is emitted from the \( W \) one has more than just two \( W \) propagators in the matrix element and the question is how to apply the pole expansion in such a case. Here, however one can exploit a partial-fraction decomposition of a product of two propagators, namely:

\[
\frac{1}{Q^2 - M^2} \frac{1}{Q'^2 - M^2} \equiv \frac{1}{2kQ' + k^2} \frac{1}{Q'^2 - M^2} - \frac{1}{Q^2 - M^2} \frac{1}{2kQ - k^2},
\]

where \( M^2 = M_W^2 + i\Gamma_W M_W \), \( Q, Q' \) are the \( W \) four-momenta before and after radiation of a photon of the four-momentum \( k \), respectively. So, a product of two propagators can be replaced by a sum of single propagators multiplied by eikonal factors. This corresponds to splitting the photon radiation into the radiation in the \( W \)-production stage and the radiation in the \( W \)-decay stage. These two stages are separated by the finite-range \( W \) propagation. The above decomposition can be applied both to the real and virtual photon emissions. In the case of the real photons the radiation amplitude splits into the sum of the amplitudes corresponding to photon emission in the \( WW \)-production and two \( W \)-decays. At the level of the cross section this results in the sum of contributions corresponding to the photon radiation at each stage of the process – the factorizable corrections, and the contributions corresponding to interferences between various stages – the non-factorizable corrections. Similarly, for the virtual corrections, the contributions with photons attached to the same stage give rise to the factorizable corrections, while the ones where photons interconnect different stages of the process contribute to the non-factorizable corrections. In this way all radiative corrections can be split in a gauge-invariant way into the factorizable and non-factorizable ones.

Since the non-factorizable corrections are negligible for main LEP2 observables we may optionally drop them\(^7\) and concentrate on the factorizable ones. For factorizable corrections one can employ the existing calculation for the on-shell \( WW \)-production and the on-shell \( W \)-decay. We want to treat the QED corrections according to the YFS exclusive exponentiation procedure and also apply the LPA, described above, in order to obtain the gauge-invariant formulation. How to do this? Extraction of infra-red (IR)

\(^7\)In fact, we use an approximation for the non-factorizable corrections in terms of the so-called screened Coulomb ansatz of Ref. [38].
contributions for both real and virtual photons can be done in a gauge-invariant way according to the YFS theory for each of the stages separately. These contributions are then sum up to infinite order and result in the so-called YFS form-factor. This means that the YFS form-factors and the IR real photon \( \tilde{S} \)-factors involving \( W \)'s do not have to be taken on-pole but can be calculated like for stable particles. After having done this we can apply the pole expansion to the IR-residuals – the YFS \( \tilde{\beta} \)-functions. We proceed in the way described at the beginning of this section and retain only the leading-pole (double-pole) terms. The \( \mathcal{O}(\alpha) \) LPA matrix element for the real photon contribution reduces, similarly to the lowest order, to the form that can be obtained from doubly-resonant Feynman graphs with single photon emission in the 't Hooft-Feynman gauge. The \( \mathcal{O}(\alpha) \) virtual correction form-factors should, in principle, be evaluated on the complex pole. This would require an analytic continuation of the usual one-loop results to the second Riemann sheet (this may be a technical problem). However, for the aimed LPA accuracy, it is sufficient to use the approximation \( s_p \simeq M_W^2 \). This would correspond to neglecting terms of \( \mathcal{O}(\frac{\alpha}{\pi} \Gamma_W M_W) \). More details about implementation of the \( \mathcal{O}(\alpha) \) corrections in the \( WW \)-production process in the MC event generator YFSWW3 can be found in Ref. [11].

3 General discussion

In this section we collect discussion on various aspects of the photon radiation in the \( W \) pair production process in particular we discuss various exponentiation schemes preparing ground for defining them explicitly in the following sections. We define more precisely our aims, discuss various constraints, introduce notation and terminology.

The fact that \( W \)'s are narrow resonances and behave like almost stable particles is of great practical importance for the evaluation of the radiative corrections, because it provides an additional small parameter \( \Gamma_W/M_W \) which can be used as an additional expansion parameter, leading to reduction of the complexity of the calculation of radiative corrections. As a result, the dominant double resonant part of the process (3.3) can be very well approximated as three independent processes, production process and two decay processes. For the double resonant part it is possible to use simpler on-shell radiative corrections, while for the single-resonant part we may stay at the Born-level or use some crude leading order (LO) approximations for the radiative corrections. Of course, we have to have at our disposal a method of splitting Born amplitude and the amplitude with the radiative corrections into double- and single-resonant parts, without breaking gauge invariance and other elementary principles. The pole expansion (POE) seems to be the best method available. Once POE is used to isolate for \( W \)-pair production process the double pole (DP), single pole (SP) and non-pole (NP) parts, photon emission from the intermediate unstable \( W \)'s has to be reorganized in a consistent way. In addition, it would be desirable to sum up photon emission from \( W \)'s to infinite order (exponentiate), for instance using one of EEX or CEEX schemes.

In the following we shall characterize various methods of the known soft photon resummation and then characterize problems related to soft photon emission from the charged
3.1 Various kinds of exclusive exponentiation

Generally, there are two kinds of exclusive exponentiation schemes, older one, which we call EEX, in which isolation of IR singularities due to real photons is done for the differential distributions (probabilities), as in the classic work of Yennie-Frautschi-Suura (YFS) \[^{18}\] and the newer one, of refs. \[^{24,25,40}\], referred to as CEEX, in which the same isolation of the real photon IR singularities is done for the amplitudes themselves, that is before squaring and spin-summing them. CEEX has a number of advantages over EEX. The price to pay is that it can be more complicated in the implementation and slower in the numerical evaluation.

Since we are interested mainly in the exclusive exponentiation for the processes with the narrow resonances, it is worth to note that, within EEX and CEEX families, there are two distinct subgroups of implementations which differ rather strongly in the treatment of the narrow resonances (or of sharp \(t\)-channel peaks). The key difference is in the treatment of the shift of the energy-momentum in the propagator of the resonance due to emission of the real or virtual photons. Let us call this effect for the purpose of this work a “recoil effect” or shortly “recoil”.

Within the EEX family there is a baseline variant based on the original YFS work \[^{18}\], in which recoil is realized in an order-by-order way. Let us denote them with EEX\(_B\). Examples of EEX\(_B\) variants are: unpublished MC code YFS1 described in ref. \[^{39}\] and BHLUMI 1.x in ref. \[^{41}\]. In EEX\(_B\) the recoil is absent completely at the level of \(\mathcal{O}(\alpha^0)\) EEX. Then, it is gradually introduced in an order-by-order manner, through so called IR-finite \(\beta\)-functions. For instance in \(\mathcal{O}(\alpha^2)_{\text{EEX}}\) the exact recoil in the differential distribution is

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Table 1: List of exclusive soft photon resummation schemes and their implementations. 2nd column indicates primary reference for the formalism. Inclusion of non-factorisable interference is marked in 3rd column. Practical implementations in the MC codes are listed in 4th column. Maximum (LO) order of the complete non-soft QED corrections is indicated in the last column.

| Resummation | Formalism | NFI interf. | Implementations | Order |
|-------------|-----------|-------------|-----------------|-------|
| EEX\(_B\)   | \[^{18,39}\] | –           | YFS1            | \(\mathcal{O}(\alpha^1)\) |
| CEE\(_N\)   | None      | –           | None            | –     |
| Neutral semi-stable intermediate particles |
| EEX\(_R\)   | \[^{27}\] | No          | YFS3, KORALZ    | \(\mathcal{O}(\alpha^3)\) |
| CEE\(_R\)   | \[^{24,25}\] | Yes         | KKMC            | \(\mathcal{O}(\alpha^2)\) |
| Charged semi-stable intermediate particles |
| EEX\(_R\)   | \[^{11,12}\] | No          | YFSWW3          | \(\mathcal{O}(\alpha^3)\) |
| CEE\(_R\)   | This work | Yes         | None            | –     |
realized due to two hard real photons – if there is a third “spectator” hard photon, then its contribution to resonance propagator is simply ignored. The problem is that, from the point of view of the strong variation of the resonance propagator, photon with the energy of order of the resonance width $\Gamma$ is already hard! This is why $EEX_B$ can be disastrous for narrow resonances, where in order to realize recoil it would be mandatory to jump immediately to a very high perturbative orders, otherwise the perturbative convergence for the QED corrections would be miserable. The $EEX_B$ can be a convenient and natural choice if there are no resonances at all.

In the second class of the EEX scenarios the recoil in the resonance propagator (or sharp $t$-channel exchange) is a built-in feature of the scheme, which is present already in $O(\alpha^0)_{CEEX}$. Let us call such a scheme $EEX_R$. It is realized for the first time in the YFS3 event generator [27] and later on included in the KORALZ [42], KKMC [24] programs and finally in the YFSWW3 program [11][12]. The analogous scheme for process dominated by the $t$-channel was implemented in the the BHLUMI [43][44] MC program. In $EEX_R$ the total energy-momentum in the resonance propagator (or $t$-channel exchange) includes contribution from all real photons emitted prior to resonance formation. That means that for each photon we have to know whether it belongs to resonance production or decay process (ISR or FSR). This is possible because in this scenario one always neglects completely and irreversibly the QED interferences between the ISR and FSR. Neglecting these interferences may be not so harmful as compared to experimental precision, because they are suppressed by the $\Gamma/M$ factor. The $EEX_R$ is obviously very well suited for narrow resonances, as long as we can afford neglecting $O(\alpha \Gamma/M)$ interference corrections, and we do not attempt to examine experimentally spectra of photons with energies $E_\gamma \simeq \Gamma$.

In the CEEX family of exponentiations there are analogous two sub-classes – either recoil is implemented in the infinite order ($CEEX_R$) or in the order-by-order manner ($CEEX_B$). One great advantage of CEEX is that, in the process with the resonant component and the non-resonant background, one may apply $CEEX_R$ to the resonant part of the amplitude and $CEEX_B$ to the background and add the two coherently afterwards.

Finally, let us comment on the relation of the above schemes to classic YFS work and the relation of $EEX_R$ to other ones. All the above exponentiation schemes are inspired by the classic YFS work [18] in one way or another. However, it is in fact only the $EEX_B$ scheme which was formulated explicitly in the original YFS work. CEEX is a non-trivial extension of the YFS exponentiation scheme, see ref. [25] for more discussion. So far there is no implementation of the $CEEX_B$ scheme, while more sophisticated $CEEX_R$ is successfully implemented in KKMC [24] program for the neutral semi-stable $Z$ boson production and decay in the electron-positron annihilation and recently in the proton-proton collision [45].

The above inventory of all schemes of the exclusive QED exponentiations and their implementations are summarised in Table 1.

Finally let us note that there is another variant of the $EEX_R$ scheme implemented in the BHWIDE program of ref. [46], featuring partial implementation of the QED NFI

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9 In the case of the low angle Bhabha process neglected are interferences between the electron and positron lines in the Feynman diagram.
interferences for semi-stable neutral boson exchanges. It will be discussed in the following whether this kind of scheme could be extended to include QED NFI interferences for charged semi-stable $W$ boson.

### 3.2 Photons from an intermediate semi-stable charged particle

Let us present an introductory discussion on the photon emission from the intermediate charged unstable $W$’s.

In order to better grasp physics of the photon emission from unstable charged particles, let us consider one more time the case of $e^+e^-\rightarrow \tau^+\tau^- + n\gamma$, $\tau^\pm \rightarrow X^\pm$ process. In this case, with $\Gamma_{\tau}/m_{\tau} = 2.27 \cdot 10^{-12}$, production and decay processes are well separated in time by this factor. For instance formation time of the $\tau$-pair at $\sqrt{s} = 100\text{GeV}$ is $\sim 10^{-24}\text{sec}$ while $\tau$ lifetime is much longer, $2.9 \cdot 10^{-13}\text{sec}$. This is why ISR photon emitted from initial beams have no chance to interfere with these of the $\tau$ decays. FSR photons emitted from ultrarelativistic outgoing $\tau$’s are quite copious, because $\ln(s/m_{\tau}^2) = 8.06$, but still, the emissions of FSR photons and photons in the decays are time-separated by the $\Gamma_{\tau}/m_{\tau} \sim 10^{-14}$. Consequently, all practical calculation for QED effects in the $\tau$-pair production and decay process from the production threshold onwards were implemented in the Monte Carlo programs independently for the production and decay parts [24,28,29]. In these calculations $\tau$ leptons in the production process are treated in the perturbative/diagrammatic QED calculations and in the phase space integration as stable particles with fixed mass and zero decay width. Photon emission from the unstable intermediate $\tau$’s is of course exponentiated. The same way in the decay parts. Can the above production-decay separation break down? Yes, in case if the energy resolution in the photon energy (cut of photon energies) would be smaller than the tau width, that is below 0.003eV, which is experimentally unfeasible.

In order to see that the problem of the photon emission form the unstable intermediate $W$’s is not a completely trivial, let us recall a well known elementary fact [18]: the emission of the photons from the stable initial beams and four final fermions can be factorized off into a product of the soft factors $\prod_i J_{6f}^\mu(k_i)$ with the total electric current for all six external particles:

$$J_{6f}^\mu(k) = \hat{J}_a^\mu(k) + \hat{J}_b^\mu(k) + \hat{J}_c^\mu(k) + \hat{J}_d^\mu(k) + \hat{J}_e^\mu(k) + \hat{J}_f^\mu(k),$$  \hspace{1cm} (3.1)

where

$$\hat{J}_x^\mu(k) = \theta_x Q_x \frac{2p_x^\mu \theta_x + k}{k^2 + 2k \cdot p_x \theta_x + i\varepsilon},$$  \hspace{1cm} (3.2)

$p_x$ and $Q_x$ are momentum and charge (in the units of positron charge) of the emitter particle $x$, and $\theta_x = +1, -1$ for initial and final state particle respectively. For the virtual photons there might be contractions among the pairs of the currents $J_{6f}^\mu(k_i)$ and $J_{6f}^\mu(k_j)$.

\hspace{10cm} 10At LEP energies $\tau$ decays are separated from the production by the giant 2 millimeters distance.
see next sections for the explicit formulation. Fig. [I] provides for the visual representation. All possible contractions (loops) for the virtual photons are not explicitly marked there.

Strictly speaking, in the orthodox YFS scheme [18] the emissions from the intermediate W’s should not be included in the IR soft factors, because W’s are internal exchanges and the corresponding emission does not contribute any IR singularity. This is true, not only because each W resonance is off-shell ($p_{W}^2 \neq M_{W}^2$), but also because photons with energy below W width, $E_\gamma \ll \Gamma_W$, emitted according to the above $J_{\mu f}$, “know nothing” about W’s [11]. The reason is that, W’s live too shortly to affect the distributions of such very soft long-living photons.

On the other hand, looking into example of $\tau$-pair production and decay, the emission of soft and hard photons out of W’s definitely makes a lot of sense. However, in the case of W-pair the time separation of production and decays is not that extremely long – this is why it is desirable to implement smooth analytical transition from the situation in which emission of photons with $E_\gamma < \Gamma_W$ is governed solely by the $J_{\mu f}$ currents to a situation in which the emission of photons with $E_\gamma > \Gamma_W$ gets well defined contribution from the intermediate W’s. The above situation is visualized in fig. [2] which describes a double-resonant process

$$e^-(p_a) + e^+(p_b) \rightarrow W^-(p_g) + W^+(p_h) + n\gamma,$$
$$W^-(p_g) \rightarrow f_c(p_e) + \bar{f}_d(p_d) + n\gamma, \quad W^+(p_h) \rightarrow f_e(p_e) + \bar{f}_f(p_f) + n\gamma,$$

where we understand again that we may also contract any pair of the photon lines into a virtual photon exchange (loop). Here and in the following we use the following shorthand notation:

$$p_{ab} = p_a + p_b, \quad p_{cd} = p_c + p_d, \quad p_{ef} = p_e + p_f.$$

11 Finite W width acts as IR regulator.
The key point is a very special way in which recoil is implemented in the resonance propagators. To understand this problem better, let us consider first the case with one real photon $n = 1$ in the two soft limit regimes: (i) semi-soft, $k^0 \ll \sqrt{s}$ and (ii) true-soft, $k^0 \ll \Gamma_W \ll \sqrt{s}$. The true-soft case (ii) is the case of standard YFS, in which we have

\[ M_1^{(0)\mu_1}(k_1) \approx \text{Const} \frac{1}{p_{cd}^2 - M^2} \frac{1}{p_{cf}^2 - M^2} \times \left\{ Q_a \frac{2p_a^\mu}{2p_a k_1} + Q_b \frac{2p_b^\mu}{2p_b k_1} - Q_c \frac{2p_c^\mu}{2p_c k_1} - Q_d \frac{2p_d^\mu}{2p_d k_1} - Q_e \frac{2p_e^\mu}{2p_e k_1} - Q_f \frac{2p_f^\mu}{2p_f k_1} \right\}. \] (3.5)

In eq. (3.5) there is no emission from any internal $W$ line and no dependence in the resonance propagators due to photon emission. In the semi-soft regime (i) we have to restore such dependence in the resonance propagators, that is recoil. This cannot be done without introducing photon emission from the intermediate charged resonance into the total electromagnetic current (unless we drop the NFI corrections altogether, as we already discussed.) In order to see this point more clearly, let us write down a naive extension of the formula of eq. (3.5) in the complete analogy with the CEEX for the neutral resonances.
like $Z$ boson:

$$\mathcal{M}_1^{(0)\mu_1}(k_1) \simeq \frac{1}{p_{cd}^2 - M^2} \frac{1}{p_{ef}^2 - M^2} \left\{ Q_a \frac{2p_a^\mu}{2p_a k_1} + Q_b \frac{2p_b^\mu}{2p_b k_1} \right\} + \frac{1}{(p_{cd} + k_1)^2 - M^2} \frac{1}{p_{ef}^2 - M^2} \left\{ -Q_c \frac{2p_c^\mu}{2p_c k_1} - Q_d \frac{2p_d^\mu}{2p_d k_1} \right\} + \frac{1}{p_{cd}^2 - M^2 \ (p_{ef} + k_1)^2 - M^2} \left\{ -Q_e \frac{2p_e^\mu}{2p_e k_1} - Q_f \frac{2p_f^\mu}{2p_f k_1} \right\} \tag{3.6}$$

The above extension is, however, useless, because it is not gauge invariant. We have  

to restore emission from the internal $W$ in order to cure the gauge invariance, while  

maintaining recoil in the resonance propagator!

We therefore restore photon emission from the internal $W$ in the soft photon approximation (starting from Feynman diagrams) and next, factorize it into the product of the  

emission factors using identity [A.2] in Appendix A. This identity also shows why it is necessary to sum up coherently over two photon assignments, either to $W$ in the production or to $W$ in the decay.

For the single real semi-soft photon under consideration, we obtain immediately the  

following gauge invariant amplitude being the sum of three parts, each of them gauge  

invariant by itself$^{12}$

$$\mathcal{M}_1^{(0)\mu}(k_1) \simeq \frac{1}{p_{cd}^2 - M^2} \frac{1}{p_{ef}^2 - M^2} \left\{ Q_a \frac{2p_a^\mu}{2p_a k_1} + Q_b \frac{2p_b^\mu}{2p_b k_1} - Q_c \frac{2p_c^\mu}{2p_c k_1} - Q_d \frac{2p_d^\mu}{2p_d k_1} \right\} + \frac{1}{(p_{cd} + k_1)^2 - M^2} \frac{1}{p_{ef}^2 - M^2} \left\{ Q_g \frac{2p_g^\mu}{2p_g k_1} - Q_e \frac{2p_e^\mu}{2p_e k_1} - Q_f \frac{2p_f^\mu}{2p_f k_1} \right\} + \frac{1}{p_{cd}^2 - M^2 \ (p_{ef} + k_1)^2 - M^2} \left\{ Q_h \frac{2p_h^\mu}{2p_h k_1} - Q_c \frac{2p_c^\mu}{2p_c k_1} - Q_f \frac{2p_f^\mu}{2p_f k_1} \right\} \tag{3.7}$$

\[= \frac{1}{p_{cd}^2 - M^2} \frac{1}{p_{ef}^2 - M^2} \left\{ j_P^\mu + \frac{p_{cd}^2 - M^2}{(p_{cd} + k_1)^2 - M^2} j_D_1^\mu + \frac{p_{ef}^2 - M^2}{(p_{ef} + k_1)^2 - M^2} j_D_2^\mu \right\} \]

\[= \sum_{\psi = (P, D_1, D_2)} \frac{1}{p_{G}^2 - M_W^2} \frac{1}{p_{H}^2 - M_W^2} \frac{1}{j_\psi^\mu}, \]

\[12 \text{Gauge invariance is manifest: } j_P^\mu k_1^\mu = j_D_1^\mu k_1^\mu = j_D_2^\mu k_1^\mu = 0. \]
where \( p_g = p_c + p_d + k_1 \) and \( p_h = p_c + p_f + k_1 \). In the last line we used
\[
\begin{align*}
  p_G &= p_c + p_d + K_D, & \quad p_H &= p_e + p_f + K_D, & \quad K_X &= \sum_{i \in X} k_i \\
  j^{\mu_1}_{p} &= \frac{2p_a^{\mu_1}}{2p_a k_i} - \frac{2p_b^{\mu_1}}{2p_b k_i} - \frac{2p_G^{\mu_1}}{2p_c k_i} - \frac{2p_H^{\mu_1}}{2p_H k_i}, \\
  j^{\mu_1}_{D_1} &= \frac{2p_G^{\mu_1}}{2p_G k_i} - \frac{2p_c^{\mu_1}}{2p_c k_i} - \frac{2p_d^{\mu_1}}{2p_d k_i}, & \quad j^{\mu_1}_{D_2} &= \frac{2p_G^{\mu_1}}{2p_H k_i} - \frac{2p_c^{\mu_1}}{2p_c k_i} - \frac{2p_f^{\mu_1}}{2p_f k_i}.
\end{align*}
\]

We keep in mind that in general \( p^2_{g,h} \neq M^2 \). The strange looking notation in the last line with the sum over partitions assigning photon to production or decays is done for the purpose of easy generalisation to the N emission case.

The single-photon amplitude of eq. (3.7) coincides precisely (up to fermion spinors) with the \( n = 1 \) case of the multiphoton \( \mathcal{O}(\alpha^0)_{\text{exp}} \) amplitude of eq. (4.10) in the next section. It features proper dependence of the resonance propagators on the photon momentum in the entire photon energy region \( k^0 \ll \sqrt{s} \), including \( k^0 \sim \Gamma_W \), and interpolates smoothly with the classic YFS formula of eq. (3.5), in the limit \( k^0 \ll \Gamma_W \). The same will be true for the amplitude of eq. (4.10) in a more general case of \( n > 1 \).

Let us close this section with the multiple photon extension of the formula (3.7), with the notation of (3.8). Details of the derivation can be found in Appendix B:
\[
M^{(0)\mu_1,\ldots,\mu_N}_{N}(k_1, \ldots, k_N) \simeq \sum_{\psi = (P,D_1,D_2)^N}^{3^N} \frac{1}{p_G^2 - M_W^2} \frac{1}{p_H^2 - M_W^2} \prod_{i=1}^{N} j^{\mu_i}_{\psi_i}. \quad (3.9)
\]

## 4 CEEX scheme for charged unstable emitters

In the following we shall implicitly assume that IR-singularities are regularized with photon mass \( m_\gamma \). The exact IR cancellations between the real photons phase space integrals \( \int_{m_\gamma} d\Phi \) and the virtual formfactor \( \alpha B(m_\gamma) \) work very schematically as follows:
\[
\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{m_\gamma} d\Phi \sum_{\text{spin}} |e^{\alpha B(m_\gamma)} M(k_1 \ldots k_n)|^2. \quad (4.1)
\]

One may, of course, introduce for all real photons the traditional IR-cut \( E_\gamma > E_{\text{min}} \), see refs. \[18, 25\] for details. This we shall not do in the following, because it would obscure notation and is in fact unnecessary (even in the MC realization we could stick to \( m_\gamma \) regulator).

In the following we shall present formalism of the CEEX for \( e^-e^+ \rightarrow W^+W^- \rightarrow X^\pm \), however, this formalism is quite general and applies also to single \( W^\pm \) production and decay (also in the hadron-hadron collision) and also to any other process with any unstable intermediate charged particles of arbitrary spin.
4.1 Non-resonant variant of $O(\alpha^1)$ CEEX for $e^- e^+ \rightarrow 4f$

Let us start defining CEEX for $e^- e^+ \rightarrow 4f$ process with the simplest possible variant of $O(\alpha^1)$ CEEX, in which the exponentiation procedure is not influenced by the presence of any narrow charged resonances in the Born matrix element $M^{(0)}$. This CEEX$_B$ scheme (according to notation of the Introduction) can be used for the non-resonant background in the $e^- e^+ \rightarrow 4f$ process. It is a kind of warm-up example in which we introduce some notation and terminology employed in the following.

Suppressing momenta and spin indices of the fermions, the $O(\alpha^0)$-exp and $O(\alpha^1)$-exp $n$-photon spin amplitudes can be written in a straightforward way

$$M^{(0)}_{\mu_1, \mu_2, \ldots, \mu_n}(k_1, k_2, \ldots k_n) = \frac{1}{n!} e^{\alpha B_6^{\text{YFS}}} \tilde{\beta}^{(0)} \prod_{i=1}^{n} j^{\mu_i}(k_i), \quad \tilde{\beta}^{(0)} = M^{(0)}$$

$$M^{(1)}_{\mu_1, \mu_2, \ldots, \mu_n}(k_1, k_2, \ldots k_n) = \frac{1}{n!} e^{\alpha B_6^{\text{YFS}}} \left\{ \tilde{\beta}^{(1)} \prod_{i=1}^{n} j^{\mu_i}(k_i) + \sum_{j=1}^{n} \tilde{\beta}^{(1)}_{\mu_j}(k_j) \prod_{i \neq j} j^{\mu_i}(k_i) \right\}$$

where the total electric current

$$j^{\mu}(k_i) = i e \sum_{X=a, b, c, d, e, f} \tilde{j}^{\mu}_{X}(k_i), \quad \tilde{j}^{\mu}_{X}(k_i) \equiv Q_{X} \theta_{X} \frac{2 p^{\mu}_{X}}{2 p_{X}k_{i}}$$

sums contributions from all six external fermions $X = a, b, \ldots, f$, see fig. 1, and $\theta_{X} = +1$ for the incoming particle $X$ (in the initial state), $\theta_{X} = -1$ for the outgoing particle $X$ (in the final state). No emission from $W$’s is seen in $j^{\mu}$. The IF-finite $\beta$-functions are defined in the usual way

$$\tilde{\beta}^{(1)} = \left[ e^{-\alpha B_6^{\text{YFS}}} M^{(0)} \right]_{O(\alpha^1)}$$

$$\tilde{\beta}^{(1)}_{\mu}(k) = M_{\mu}^{(1)}(k) - j^{\mu}(k) M^{(0)}$$

The UV-finite, IR-divergent, gauge-invariant YFS formfactor is defined in the standard way, see also Appendix B:

$$B_6^{\text{YFS}} = \int \frac{i}{(2\pi)^3} \frac{d^4 k}{k^2 - m^2 + i\varepsilon} J^{\mu}(k) \circ J^{\mu}(k),$$

$$J^{\mu}(k) = \sum_{X=a, b, c, d, e, f} \tilde{J}^{\mu}_{X}(k), \quad \tilde{J}^{\mu}_{X}(k) \equiv Q_{X} \theta_{X} \frac{2 p^{\mu}_{X} + k^{\mu}}{k^2 + 2 p_{X}k \theta_{X} + i\varepsilon},$$

where $\theta_{X}$ is defined as above, and we use the following short-hand notation:

$$S(k) = J(k) \circ J(k) = \sum_{X=a, b, c, d, e, f} J_{X}(k) \circ J_{X}(k),$$

$$J_X(k) \circ J_Y(k) \equiv J_X(k) \cdot J_Y(-k), \text{ for } X \neq Y,$$

$$J_X(k) \circ J_X(k) \equiv J_X(k) \cdot J_X(k).$$

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As we see, $B_{6}^{YFS}$ sums up the contributions from all six external fermions. IR-cancellations occur after squaring, spin-summing and integrating over the phase space, in a way which was shown using several methods in refs. [18,25]

4.2 Resonant variant of CEEX $\mathcal{O}(\alpha^{1})$ for $e^{-}e^{+} \rightarrow 4f$

In the following we shall discuss the $\mathcal{O}(\alpha^{1})$ variant of CEEX for $e^{-}e^{+} \rightarrow 4f$ in which recoil in resonance propagators is realized at any perturbative order and the $\Gamma_{W}/M_{W}$ suppression of NFI contributions is a natural, built-in feature, valid in every perturbative order $\mathcal{O}(\alpha^{r})_{exp}, r = 0, 1, 2, ...$. In order to formulate such a scheme completely, one has to re-consider the isolation of IR-singular photon emission factor to infinite order from the internal $W$ lines, going beyond the scope of the classic scheme of YFS61. The important element of the isolation of apparent IR-singularities due to emission of photons from the resonant charged particles is the reorganisation of the product for the internal propagators derived in Appendix A. The virtual exponential formfactor has also more complicated structure and is re-derived in Appendix B. Our derivation of the CEEX amplitudes is based on rearrangement of the infinite perturbative expansion in terms of Feynman diagrams, as in refs [18, 25] and the use of the pole-expansion[13]. Although our aim are the $\mathcal{O}(\alpha^{1})_{CEEX}$ amplitudes, the main features of the scheme can be already defined and discussed for the simpler $\mathcal{O}(\alpha^{0})_{CEEX}$ case, which will be discussed first. The extension of the presented technique to $\mathcal{O}(\alpha^{2})_{CEEX}$ with complete non-soft second order photonic corrections and pure electroweak corrections is straightforward.

4.2.1 Introductory double-pole $\mathcal{O}(\alpha^{0})$ CEEX

Let us assume that for the $e^{-}e^{+} \rightarrow 4f$ process depicted in fig. 1 we have at our disposal Born matrix element $M_{0}^{(0)}$ which we expand into non-pole part $M_{0}^{(0)}()$ single-pole part $M_{0}^{(0)}(Q)$ and double-pole part $M_{0}^{(0)}(Q,R)$, where $Q$ and $R$ are four momenta in the $W$ propagators

$$M_{0}^{(0)}(\mu) = M_{0}^{(0)}() + M_{0}^{(0)}(Q) + M_{0}^{(0)}(Q,R), \quad (4.7)$$

The same pole-expansion is done for the exact single-photon spin amplitudes

$$M_{1}^{(1)}(\mu) = M_{1}^{(1)}(k) + M_{1}^{(1)}(Q,k) + M_{1}^{(1)}(Q,R,k), \quad (4.8)$$

where $k$ is photon momentum and index $\mu$ is understood to be contracted with the photon polarization vector. The one-loop corrected complete $\mathcal{O}(\alpha^{1})$ spin amplitudes in the POE we denote as $M_{0}^{(1)}(), M_{0}^{(1)}(Q^{2})$ and $M_{0}^{(1)}(Q,R)$,

$$M_{0}^{(1)}(\mu)(k) = M_{0}^{(1)}(k) + M_{0}^{(1)}(Q,k) + M_{0}^{(1)}(Q,R,k), \quad (4.9)$$

Let us focus now on the double-resonant part of the amplitudes $M_{0}^{(0)}(Q,R)$ and $M_{0}^{(1)}(Q,R)$. The single-resonant part is completely analogous (we shall list the differences) and the non-resonant case was already discussed in the previous section.

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[13] We hope that the mathematical rigour of this proof will be improved in the future works.
The CEEX $O(\alpha^0)$ spin amplitudes for $n$ photons can be derived as the following gauge invariant subset of the complete perturbative series

$$M_n^{(0)\mu_1\mu_2,\ldots,\mu_n}(k_1, k_2, \ldots, k_n) = \sum_{\varphi\in\{P,D_1,D_2\}^n} e^{\alpha B_{10}(U_\varphi, V_\varphi)} \hat{\beta}^{(0)}_0(U_\varphi, V_\varphi) \prod_{i=1}^n j_{(\varphi_i)}^{\mu_i}(k_i),$$

(4.10)

$$U_\varphi = p_c + p_d + \sum_{\varphi_i=P} k_i, \quad V_\varphi = p_e + p_f + \sum_{\varphi_i=D_1} k_i.$$  

Here fermion four-momenta $p_A$ and helicities $\lambda_A, A = a,b,c,d,e,f$ are suppressed. Photons are grouped into 3 groups, production, first decay and second decay denoted as $P, D_1, D_2$. The coherent sum is taken over all $3^n$ assignments of photon to 3 stages of the process. Each assignment is represented by the vector $(\varphi_1, \ldots, \varphi_n)$ whose components are taking 3 possible values $\varphi_j = P, D_1, D_2$. The cornerstone of this construction are three gauge invariant electric currents

$$j_{(P)}^\mu(k_i) = ie \sum_{X=a,b,g,h} \hat{j}_X^\mu(k), \quad j_{D_1}^\mu(k_i) = ie \sum_{X=g,c,d} \hat{j}_X^\mu(k), \quad j_{D_2}^\mu(k_i) = ie \sum_{X=h,e,f} \hat{j}_X^\mu(k),$$

(4.11)

$$p_g = U_\varphi, \quad p_h = V_\varphi,$$

defined in terms of elementary currents $\hat{j}_X(k)$ of eq. (4.3). They include also $\hat{j}$'s for two $W$'s, see eq. (4.3). The essential steps in derivation of the CEEX formula of eq. (4.10) are given in the appendices A and B.

The dependence of the amplitude in eq. (4.10) on the four-momenta was already analysed in the case of the single real photon in the previous section. The case of many real photons is completely analogous. Let us turn now our attention to more interesting case of multiple virtual photos which contribute to the virtual formfactor $\exp(B_{10})$.

The virtual IR-singularities factorize off in eq. (4.10) into the factor $\exp(B_{10})$. Let us recall that our aim is to reproduce the $\Gamma/M$ suppression of the NFI corrections already at the $O(\alpha^0)_{\text{exp}}$ level. It would be incorrect to employ here the classic YFS formfactor $B_{10}^\text{YFS}$ of eq. (4.5). This choice would render eq. (4.10) IR-finite, however, it would fail to resum the $\alpha \ln(\Gamma/M)$ contributions and miss the $\Gamma/M$ suppression of NFI corrections, at the $O(\alpha^0)_{\text{exp}}$ level. How to see it? One may check it by explicit analytical calculation, similar to the one performed in ref. [25], or numerically. Quite generally, the reason for the above failure is that, the effective energy scale for NFI is not $\sqrt{s}$ but $\Gamma_W$. The NFI contributions for the real photon energies above $\Gamma_W$ are suppressed strongly by the resonance propagator. However, this works for the real but not for virtual photons in $B_{10}^\text{YFS}$, hence the energy scale for virtual photons is necessarily $\sqrt{s}$. The mismatch between the scale for real and virtual photon will cause the NFI contribution to blow up at the $O(\alpha^0)_{\text{exp}}$ by orders of magnitude, and even for $O(\alpha^1)_{\text{exp}}$ they may be far from the reality.

The remedy for the above problem is well known for the neutral resonances [25,47,48] and also can be deduced from the $O(\alpha^1)$ calculation (without exponentiation) of the NFI term for the charged resonance of $W$, see Refs. [38,49,50]. The modified CEEX formfactor
which should be used in eq. (4.10) is the following:

\[
B_{10}(p_{cd}, p_{ef}) = \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2 - \lambda^2 + i\varepsilon} \left\{ J_P(k) \circ J_P(k) + J_{D_1}(k) \circ J_{D_1}(k) + J_{D_2}(k) \circ J_{D_2}(k) \right. \\
+ \frac{p^2_{cd} - M^2}{(p_{cd} - k)^2 - M^2} 2J_P(k) \circ J_{D_1}(k) + \frac{p^2_{ef} - M^2}{(p_{ef} - k)^2 - M^2} 2J_P(k) \circ J_{D_2}(k) \\
+ \left. \frac{p^2_{cd} - M^2}{(p_{cd} + k)^2 - M^2} \frac{p^2_{ef} - M^2}{(p_{ef} - k)^2 - M^2} 2J_{D_1}(k) \circ J_{D_2}(k) \right\},
\]

where

\[
J^\mu_P(k) = \sum_{X=a,b,g,h} \tilde{J}^\mu_X(k), \quad J^\mu_{D_1}(k) = \sum_{X=g,c,d} \tilde{J}^\mu_X(k), \quad J^\mu_{D_2}(k) = \sum_{X=h,e,f} \tilde{J}^\mu_X(k),
\]

\[p_g = p_{cd} + K_1, \quad p_h = p_{ef} + K_2,\]

see eq. (4.6) for definition of elementary virtual current \( \tilde{J}_X \) and of its circle-products. In the eq. (4.10) four momenta \( U_\psi, V_\phi \) in \( B_{10}(U_\psi, V_\phi) \) should be identified with \( p_{cd} + K_1 \) and \( p_{ef} + K_2 \) in eq. (4.12), where \( K_1 \) and \( K_2 \) are total four momenta of all real photons in the two decay processes. Note that the above formfactor is gauge invariant and UV-finite. Moreover, each of its six components is also separately gauge invariant and UV-finite. Almost all its components are already available in the literature. We have omitted from discussion the important Coulomb effect, see ref. [34] for more details.

The index 10 in \( B_{10} \) reflects the fact that we have 10 emission currents in \( B_{10} \): 6 for fermions and 4 for \( W \)'s – that is 2 for \( W \)'s in production and 2 for \( W \)'s in decays.

Heuristic derivation of the above CEEX form-factor, directly from the Feynman diagram, is done in Appendix B using similar techniques like in sect. 3.2.2 of ref. [25]. In this derivation one may see explicitly why the first three components for the production and decays are exactly like in standard YFS, while three interferences are modified.

### 4.2.2 The \( \mathcal{O}(\alpha^1) \) CEEX for double pole component

The construction of the \( \mathcal{O}(\alpha^0)_{\text{exp}} \) for \( e^+e^- \to 4f \) process of the previous subsection was based, on one hand, on the gauge invariant POE of Born spin amplitudes into double-, single- and non-pole parts and, on the other hand, on the soft photon approximation in which real and virtual photon emission/absorption is represented as a product of the universal (spin-independent) factors, taking care of the recoil in all resonance propagators.

We intend now to extend the above scheme in such a way that the complete \( \mathcal{O}(\alpha^1) \) to \( e^+e^- \to 4f \) process are or can be included. The immediate question is to what extent the POE into double-, single- and non-pole parts can be kept at all at the \( \mathcal{O}(\alpha^1) \)?

Concerning POE at the \( \mathcal{O}(\alpha^1) \), we assume that both \( \mathcal{O}(\alpha^1) \) amplitudes \( M_{1(1)}^{(1)}(k) \) with emission of additional single photon and \( M_{0}^{(1)} \) with complete one loop corrections can be
pole-expanded into double-, single- and non-pole parts.\footnote{The ultimate proof will be provided by someone who will do it in practice.} Obviously, this can be done in many ways. Essentially it can be done (in principle) because the two propagators for the internal $W$ line due to photon emission can always be replace by a sum of “two poles” using identity (A.2). Each of these terms can be made gauge invariant by taking residue value for the entire expression multiplying the pole term, or more selectively, in its scalar line due to photon emission can always be replace by a sum of “two poles” using identity (A.2). Each of these terms can be made gauge invariant by taking residue value for the entire expression multiplying the pole term, or more selectively, in its scalar part. This can be done (in principle) for both amplitudes $M_1^{(1)}(k)$ and $M_0^{(1)}$ representing exact results of the Feynman diagrams in the $\mathcal{O}(\alpha^1)$. The soft-photon-approximated universal part is already included in the calculation due to exponentiation in the same way as in $\mathcal{O}(\alpha^0)$.

The double-pole $\mathcal{O}(\alpha^1)$ CEEX amplitude including terms of $\mathcal{O}(\frac{\alpha}{\pi M})$ term due to NFI interferences reads as follows:

$$M_{\alpha}^{(1)}(k_1, k_2, ..., k_n)_{DP} = \sum_{\varphi \in \{P, D_1, D_2\}^n} e^{\alpha B_{10}(U, V)} \beta_0^{(1)}(U, V) \prod_{i=1}^n j_{(\varphi_i)}^{(\mu_i)}(k_i)$$

$$+ \sum_{j=1}^n \sum_{\varphi \in \{P, D_1, D_2\}^{n-1}} e^{\alpha B_{10}(U, V)} \hat{\beta}_1^{(1)}(U, V, k_j) \prod_{i \neq j} j_{(\varphi_i)}^{(\mu_i)}(k_i),$$

where

$$\hat{\beta}_1^{(1)}(U, V, k) = M_1^{(1)}(U, V, k) - \sum_{\varphi = P, D_1, D_2} j_{(\varphi)}^{(\mu)}(k) M_0^{(0)}(U, V).$$

The IR-finite $\hat{\beta}_0$-functions is here defined as follows

$$\hat{\beta}_0^{(1)}(U, V) = \left[e^{-\alpha B_{10}(U, V)} M_0^{(1)}(U, V)\right]_{\mathcal{O}(\alpha^1)} = M_0^{(1)}(U, V) - B_{10}(U, V) M_0^{(0)}(U, V)$$

where $B_{10}(U, V)$ is the complete variant of eq. (4.12) and the one-loop corrections in the double-pole $M_0^{(1)}(U, V)$ have to be complete at the $\mathcal{O}(\alpha^1)$, including terms of $\mathcal{O}(\frac{\alpha}{\pi M})$. Special care should be taken in order to preserve gauge invariance. Infrared regulation using $m_\gamma$ or any other method may be employed in the intermediate steps, but the final $B_{10}(U, V)$ will be IR-finite.

Needless to say that in the above expressions, as usual in all resummation schemes, one has to provide recipe for extrapolating $\mathcal{O}(\alpha^1)$ results originally defined in the phase space with zero or one real photon, to phase space enriched with many additional ”spectator” photons\footnote{It is typically done using some kinematic manipulations on the four-momenta which are fed into $\mathcal{O}(\alpha^1)$ formulas, or using Mandelstam variables – they are less sensitive to presence of spectators.} The uncertainty due to freedom in this extrapolation is of the $\mathcal{O}(\alpha^2)$ class.

### 4.2.3 The $\mathcal{O}(\alpha^1)$ CEEX for single-pole component

The above implementation of the $\mathcal{O}(\alpha^1)$ CEEX for DP component QED $\mathcal{O}(\alpha^1)$ corrections are complete including $\mathcal{O}(\frac{\alpha}{\pi M})$ corrections due NFI interferences. However, $\mathcal{O}(\frac{\alpha}{\pi M})$ corrections arise also from the entire QED $\mathcal{O}(\alpha^1)$ correction to simple-pole component
The IR-finite \( \hat{\omega} \) assignment is reduced to sum over set of components, in particular one interference term instead of three, (ii) the sum over photon \( O \) (which by itself is of order \( \mathcal{O}(1) \)) CEEX for the SP part. In addition CEEX for the SP process is also of the vital importance for the \( q\bar{q} \to W, \ W \to f\bar{f} \) in hadron colliders like LHC.

On the other hand the non-pole (background) part, which is of order \( \mathcal{O}((\frac{1}{M})^2) \), may included without QED corrections or any kind of implementation of QED corrections, for instance using simple baseline \( \mathcal{O}(\alpha^0) \) CEEX version of subsection 4.2.1.

The CEEX \( \mathcal{O}(\alpha^1)_{\text{exp}} \) single-pole and double-pole spin amplitudes will be combined additively as follows\(^{10}\)

\[
\mathcal{M}_{n}^{(1)\mu_1\ldots\mu_n}(k_1, k_2, \ldots, k_n)_{\text{DSP}} = \mathcal{M}_{n}^{(1)\mu_1\ldots\mu_n}(k_1, k_2, \ldots, k_n)_{\text{SP}} + \mathcal{M}_{n}^{(1)\mu_1\ldots\mu_n}(k_1, k_2, \ldots, k_n)_{\text{DP}}
\]

(4.17)

The single-pole \( \mathcal{M}_{n,\text{SP}}^{(1)} \) amplitude is constructed analogously as in eq. (4.10). The differences are that: (i) the current \( j^\mu_\nu \) in the production process \( e^+e^- \to f_c + f_d + W^+ \) has five components instead of four, (ii) the function \( B_8 \) replaces \( B_{10} \), the \( B_8 \) has less components, in particular one interference term instead of three, (ii) the sum over photon assignment is reduced to sum over set of \( \{ \varphi \} = (P, D_1)^n \) of \( 2^n \) assignments.

\[
\mathcal{M}_{n}^{(1)}(k_1, k_2, \ldots, k_n)_{\text{SP}} = \sum_{\varphi \in (P, D_1)^n} e^{\alpha B_{8}(U_\varphi)} \hat{\beta}_0^{(1)}(U_\varphi) \prod_{i=1}^{n} j^\mu_{\varphi_i}(k_i)
\]

+ \[
\sum_{j=1}^{n} \sum_{\varphi \in (P, D_1)^{n-1}} e^{\alpha B_{8}(U_\varphi)} \hat{\beta}_1^{(1)\mu_j}(U_\varphi, k_j) \prod_{i \neq j} j^\mu_{\varphi_i}(k_i),
\]

(4.18)

where

\[
\hat{\beta}_1^{(1)\mu}(U, k) = M_1^{(1)\mu}(U, k) - \sum_{\varphi = P, D_1} j^\mu_{\varphi}(k) M_0^{(0)}(U_\varphi),
\]

(4.19)

The IR-finite \( \hat{\beta}_0 \)-functions is defined here as follows

\[
\hat{\beta}_0^{(1)}(U) = \left[ e^{-\alpha B_{8}(U)} M_0^{(1)}(U) \right]_{\mathcal{O}(\alpha^1)} = M_0^{(1)}(U) - B_{8}(U) M_0^{(0)}(U),
\]

(4.20)

where \( M_0^{(1)}(U) \) is single-pole part in the Born amplitude of the \( e^+e^- \to 4f \) process and the one-loop corrected single-pole \( M_0^{(1)}(U) \) amplitude is at the complete \( \mathcal{O}(\alpha^1) \). The \( B_{8}(U) \) function is the following variant of that in eq. (4.12)

\[
B_{8}(p_{\text{cd}}) = i e \int \frac{i d^4 k}{(2\pi)^3} \frac{k^2 - \lambda^2 + i \varepsilon}{k^2 - \lambda^2 + i \varepsilon}
\]

\[
\left\{ J_P(k) \circ J_P(k) + J_{D_1}(k) \circ J_{D_1}(k) + \frac{p^2_{\text{cd}} - M^2}{(p_{\text{cd}} + k)^2 - M^2} 2J_P(k) \circ J_{D_1}(k) \right\}.
\]

(4.21)

The above \( \mathcal{O}(\alpha^1) \) CEEX for single-pole part of \( e^+e^- \to 4f \) process implemented in \( \mathcal{M}_{n,\text{SP}}^{(1)} \) provides together with the double-pole CEEX amplitude of the previous section

\(^{10}\)In some four-fermion channels there is no possibility to form single resonant \( W \).
\( \mathcal{M}_{n,DP}^{(1)} \) the complete QED corrections of the order \( \mathcal{O}(\alpha) \), \( \mathcal{O}(\frac{1}{M}) \) and \( \mathcal{O}(\frac{\alpha}{M}) \) in the for \( e^+e^- \rightarrow 4f \) process. Let us keep in mind that the definition of the \( \mathcal{O}(\frac{\alpha}{M}) \) terms in \( \mathcal{M}_{n,SP}^{(1)} \) and in \( \mathcal{M}_{n,DP}^{(1)} \) depends on the exact definition of the SP and DP components in POE. Only the sum of them is uniquely defined – more precisely up to terms of \( \mathcal{O}(\frac{\alpha}{M})^2 \).

In the above formalism fermions labeled \( e \) and \( f \) do not form resonance. In case of single \( W \) production in the quark-antiquark annihilation in hadron-hadron collision the same formalism applies but particles \( e \) and \( f \) are just absent.

### 4.2.4 Approximate version of \( \mathcal{O}(\alpha) \) CEEX

Let us also consider one simpler case of CEEX matrix element, with incomplete \( \mathcal{O}(\frac{\alpha}{M}) \) corrections. It may be of some practical significance for applications with limited precision and will be described for the DP part only.

In this alternative scheme the \( \mathcal{O}(a^0) \) part is kept the same as in full version of CEEX scheme for DP part in subsection 4.2.2. The main difference is in the simplification of the non-soft \( \mathcal{O}(\alpha) \) remnants, in which non-factorisable QED interferences between production and decays are downgraded to the soft-photon approximation.

In such an approximation \( \mathcal{O}(\alpha) \) non-soft corrections are calculated separately for the production and two decay processes and they contribute separately and additively to both real \( \hat{\beta}_1^{(1)\mu} \) and virtual \( \hat{\beta}_0^{(1)\mu} \):

\[
\hat{\beta}_1^{(1)\mu}(U, V, k) = \sum_{X=P,D_1,D_2} \hat{\beta}_{1,X}^{(1)\mu}(U, V, k), \quad \hat{\beta}_0^{(1)\mu}(U, V) = \sum_{X=P,D_1,D_2} \hat{\beta}_{0,X}^{(1)\mu}(U, V),
\]

where \( U = p_{ed}, V = p_{ef} \). For instance, the non-soft contributions from penta-box diagrams in the NFI class are neglected completely in the \( \hat{\beta}_0^{(1)\mu}(U, V) \), because their soft part (including resonance effects) is already included in the \( B_{10}(U, V) \) function. The single real photon emission spin amplitudes factorize into production and decay parts

\[
\mathcal{M}_1^{(1)\mu}(k) = \mathcal{M}_{1,P}^{(1)\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{1,P}^{(1)\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{1,P}^{(1)\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} \mathcal{M}_{1,D_2}^{(1)\mu}(k)
\]

\[
= \mathcal{M}_{0,P}^{(0)} \left[ j_P^{\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{0,D_1}^{(0)} j_D^{\mu}(k) \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} j_D^{\mu}(k) \right]
\]

\[
+ \hat{\beta}^{(1)\mu}_P(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{1,P}^{(1)\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} + \mathcal{M}_{1,P}^{(1)\mu}(k) \mathcal{M}_{0,D_1}^{(0)} \mathcal{M}_{0,D_2}^{(0)} \mathcal{M}_{1,D_2}^{(1)\mu}(k)
\]

\[
= \beta_0^{(0)}(U, V) j_P^{\mu}(k) + \hat{\beta}^{(0)}_P(U + k, V) j_D^{\mu}(k) + \hat{\beta}^{(0)}_0(U + V + k) j_{D_2}^{\mu}(k)
\]

\[
+ \hat{\beta}^{(1)\mu}_P(k) + \hat{\beta}^{(1)\mu}_{1D_1}(k) + \hat{\beta}^{(1)\mu}_{1D_2}(k),
\]

where \( \hat{\beta}_X^{(1)\mu}(k), \ X = P, D_1, D_2 \) are the CEEX elements for production and decays separately and we have adopted a convention that \( W \) propagator is included in the lowest order decay amplitude \( \mathcal{M}_{0,D_i}^{(0)} \). An additional argument \( (k) \) in \( \mathcal{M}_{0,D_i}^{(0)}(k) \) marks that this \( W \) propagator includes momentum \( k \) of photon emitted in the decay.
The resulting variant of the \( \mathcal{O}(\alpha^1) \) CEEX amplitude reads as follows

\[
\mathcal{M}_{\mu_1,\mu_2,\ldots,\mu_n}^{(1)}(k_1, k_2, \ldots, k_n) = \sum_{\varphi \in \{P, D_1, D_2\}^n} e^{\alpha B_{10}(U_\varphi, V_\varphi)} \left\{ \beta_0^{(1)}(U_\varphi, V_\varphi) \prod_{i=1}^n j_{\{\mu_i\}}(k_i) + \sum_{j=1}^n \beta_1^{(1)}(U_\varphi, V_\varphi, k_j) \prod_{i \neq j} j_{\{\mu_i\}}(k_i) \right\} ,
\]

(4.24)

The important difference with respect to previous case is that due to split \( \hat{\beta}^{(1)} \) into production and decay parts, photon \( k_j \) entering \( \hat{\beta}^{(1)} \) is included into sum over photon assignments.

### 4.2.5 Higher order upgrades and inclusion of genuine electroweak corrections

The upgrade of the CEEX amplitudes from the \( \mathcal{O}(\alpha^1) \) to \( \mathcal{O}(\alpha^2) \) is straightforward, following the same path as in the analogous case of QED \( \mathcal{O}(\alpha^2) \) CEEX scheme implemented in the KKMC project [24, 25]. CEEX scheme offers great flexibility, allowing to truncate perturbative series at different order for ISR, FSR, IFI and IFF. This may be exploited in the convenient staging in the construction of the numerical Monte Carlo program. In particular, for the ISR corrections it would be good to include \( \mathcal{O}(\alpha^3) \) corrections.

From the experience of the KKMC project we know that calculations of the CEEX \( \mathcal{O}(\alpha^2) \) matrix element may be slow, due to the need of summations over the assignments of photons among production and decays. However, most of numerical contributions from these photon assignments are numerically negligible and one may invent methods of the effective forecasting which ones can be omitted from the evaluation. This would speed up significantly numerical MC calculations.

In the present work we concentrate on the QED part of the SM calculations for the \( e^+e^- \rightarrow W^+W^- \) process. Is it possible to factorize and treat separately QED part from the rest of the SM corrections, the genuine EW corrections? The answer is positive because soft photon factorisation for both real virtual photons is well established in the framework perturbative calculations [18]. The remaining genuine EW \( \mathcal{O}(\alpha^r) \) \( r = 1, 2 \) corrections are located in the IR-finite remnants \( \hat{\beta}_0^{(r)}(k), \hat{\beta}_1^{(r)}(k), \hat{\beta}_2^{(r)}(k_1, k_2) \). It is only important to remember that CEEX scheme works at the amplitude level and in the calculation of the loop corrections leading to \( \hat{\beta}_0^{(r)}(k) \) or \( \hat{\beta}_1^{(r)}(k) \), all IR divergences are removed by means of subtracting \( B_{10} \) function – adding real emissions a’la Bloch-Nordsieck in order to obtain finite results is a methodology mistake! Because of that it is much easier to manage pure EW corrections in CEEX scheme in any perturbative order than in any other scheme (especially beyond \( \mathcal{O}(\alpha^1) \)).

In the Kandy (YFSWW3) calculations of LEP era \( \mathcal{O}(\alpha^1) \) genuine EW corrections were included in the \( \hat{\beta}_0^{(1)} \) for the DP production part of the process (similarly as in RACOONWW). In order to match very high precision of the FCC-ee experiments it will be necessary to introduce \( \mathcal{O}(\alpha^2) \) corrections in \( \hat{\beta}_0^{(2)} \) and \( \hat{\beta}_1^{(1)}(k) \) of the DP component. They are not available yet. In addition it will be needed to introduce \( \mathcal{O}(\alpha^1) \) EW corrections in the \( \hat{\beta}_0^{(1)} \)

\[\text{This will be mandatory for LO } \mathcal{O}(\alpha^3) \text{ corrections.}\]
of the SP component. This subgroup of corrections can be in principle extracted from the existing EW $O(\alpha^1)$ calculations for the entire $e^+e^- \rightarrow 4f$ process of ref. [31,32].

5 Relations between CEEX and EEX schemes

Tracing exact relations between various CEEX and EEX schemes is quite important for at least two reasons. The EEX implementation of the exclusive exponentiation in YFSWW it the only existing one for $e^+e^- \rightarrow 4f$ process, so it is desirable to show that it can be embedded in the CEEX scheme as a kind of well defined approximation. It will also help to understand better the physics of the photon emission from unstable charged intermediate particles and the inherent limitations of the EEX exponentiation scheme in YFSWW, in particular clarifying the question: what is the exactly mechanism of neglecting NFI interferences in the EEX of YFSWW?

Another important reason is that it would be desirable to implement CEEX matrix element using MC correction weight on top of the same baseline MC distributions, which is implemented in the MC event generator for the EEX matrix element. This strategy was successfully exploited in the KKMC program and also in the KandY hybrid Monte Carlo. For these reasons it is interesting to establish the relation between CEEX and EEX distribution all over the entire multiphoton phase space.

5.1 From CEEX$_R$ to EEX$_R$ algebraically

As we have already indicated in the introduction, the EEX differential distributions for the process $e^-e^+ \rightarrow W^-W^+$, $W^\pm \rightarrow f \bar{f}$ can be obtained as a limiting case of the CEEX scheme for the process $e^-e^+ \rightarrow 4f$ defined in this paper. Let us do it in the following. This is analogous to the derivation of EEX of KORALZ out of CEEX amplitudes given in Section 4 of ref. [25]. The transition to EEX of YFSWW requires a few additional steps described in the next subsection.

As a starting point we take an approximate variant of CEEX of eq. (4.24), which is obtained from the exact one of eq. (4.14) by means of neglecting some non-IR interference NFI terms:

$$\sigma = \frac{1}{f_{\text{c}}\varepsilon} \sum_{n=0}^{\infty} \frac{1}{n!} \int d\text{Lips}_{4+n}(p_a + p_b; p_c, p_d, p_e, p_f, k_1...k_n)$$

$$\times \sum_{\varphi \in \{P,D_1,D_2\}^n} e^{\alpha B_0(U_{\varphi},V_{\varphi})} \left\{ \hat{\beta}_0^{(1)}(U_{\varphi},V_{\varphi}) \prod_{i=1}^{n} j_{\{\psi_i\}}^{\mu_i}(k_i) + \sum_{j=1}^{n} \hat{\beta}_1^{(1)}(U_{\varphi},V_{\varphi},k_j) \prod_{i \neq j} j_{\{\psi_i\}}^{\mu_i}(k_i) \right\}$$

$$\times \sum_{\varphi' \in \{P,D_1,D_2\}^n} e^{\alpha B_0(U_{\varphi'},V_{\varphi'})} \left\{ \hat{\beta}_0^{(1)}(U_{\varphi'},V_{\varphi'}) \prod_{i=1}^{n} j_{\{\psi'_i\}}^{\mu_i}(k_i) + \sum_{j=1}^{n} \hat{\beta}_1^{(1)}(U_{\varphi'},V_{\varphi'},k_j) \prod_{i \neq j} j_{\{\psi'_i\}}^{\mu_i}(k_i) \right\}^*,$$

(5.1)

The analogy is however incomplete, because here we take into account photon emission from the intermediate charged $W$ boson, while in ref. [25] neutral resonance $Z$ was considered.
where $U_{\nu} = p_{cd} + \sum_{i=1}^{D_1} k_i$ and $V_{\nu} = p_{ef} + \sum_{i=1}^{D_2} k_i$.

The consistent method of omitting all of the remaining QED NFI interferences between the production and two decays requires omitting from the double sum over photon assignments all non-diagonal terms $\nu \neq \nu'$, and the interference terms in $B_{10}$. After doing that the above omission the sum over photons can be reorganized into product of separate three sums, one for production and two for two decays. In this way we get the following EEX expression:

$$\sigma = \frac{1}{flux} \sum_{n=0}^{\infty} \frac{1}{n!} \int dLip_{4+n}(p_a + p_b, p_c, p_d, p_e, p_f, k_1 \ldots k_n)$$

$$\times \sum_{\nu \in \{P,D_1,D_2\}^n} e^{2\alpha R_{PDD}(U_{\nu},V_{\nu})} \prod_{i=1}^{n} |j^{\mu_i}_{(\nu_i)}(k_i)|^2 \left\{ \left| \tilde{\beta}_0^{(1)}(U_{\nu},V_{\nu}) \right|^2 + \sum_{j=1}^{n} \left| \tilde{\beta}_1^{(1)}(U_{\nu},V_{\nu},k_j) \cdot j_{(\nu_j)}(k_j)^* \right|^2 \right\} \left| j_{(\nu_j)}(k_j) \right|^{-2},$$

(5.2)

In the above expression the YFS formfactor $e^{2\alpha R_{10}}$ factorizes into product of independent formfactors for production and two decay processes $e^{R_{PDD}} = e^{2\alpha R_P} e^{2\alpha R_D_1} e^{2\alpha R_D_2}$.

Eq. (5.2) can be rewritten in a more traditional EEX notation as follows:

$$\sigma = \frac{1}{flux} \sum_{n=0}^{\infty} \frac{1}{n!} \int dLip_{4+n}(p_a + p_b, p_c, p_d, p_e, p_f, k_1 \ldots k_n) \sum_{\nu \in \{P,D_1,D_2\}^n} e^{2\alpha R_{PDD}(U_{\nu},V_{\nu})} \prod_{i=1}^{n} \tilde{S}_{(\nu_i)}(k_i) \left\{ \left| \tilde{\beta}_0^{(1)}(U_{\nu},V_{\nu}) + \sum_{j=1}^{n} \tilde{\beta}_1^{(1)}(U_{\nu},V_{\nu},k_j)(\tilde{S}_{(\nu_j)}(k_j))^{-1} \right| \right\},$$

(5.3)

where

$$\tilde{S}_X(k) = |j^{\mu}_{X}(k)|^2, \ X = P, D_1, D_2.$$  

(5.4)

Note that in the above expression for each photon assignment we perfectly know the four momentum in each $W$ propagator – simply because each photon is associated with production or one of the decays.

In fact eq. (5.2) looks like three separate EEX exponentiation schemes for the three subprocesses. They talk to each other only through total energy conservation and spin correlations.\(^{19}\) This can be seen manifestly even more clearly when for the purpose of the MC implementation eq. (5.3) is transformed into the following form in which $\tilde{S}$-factors for production and decays are factorised. For $n$ photons in the overall sum over $3^n$ assignments of the photos $\{P,D_1,D_2\}^n$ there are groups (partitions) of $\frac{n!}{n_0!n_1!n_2!}$ choices with $n_0$ photons in the production $n_1$ photons in the first decay and $n_2$ photons in the second decay, $n_0 + n_1 + n_2 = n$. The assignments in each partition are related by the

\(^{19}\)Connecting production and decays through spin density matrix formalism is the logical solution in the EEX case, as for $\tau$ pair production and decay in KORALZ.
permutation of the photons within the partition. We may replace in eq. (5.3) the whole such partition just by one permutation member getting the following expression:

$$\sigma = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \int dLips d_{4+n_0+n_1+n_2} \left( p_a + p_b; p_c, p_d, p_e, p_f, k_1...k_{n_2} \right)$$

$$\times \frac{1}{n_0!} \prod_{i_1=0}^{n_0} \tilde{S}_{P}(k_{i}) \frac{1}{n_1!} \prod_{i_1=1}^{n_1} \tilde{S}_{D_{1}}(k_{i_1}) \frac{1}{n_2!} \prod_{i_2=1}^{n_2} \tilde{S}_{D_{2}}(k_{i_2})$$

$$\times e^{2\alpha R B_{10}(U_{1}, V_{2})} \left\{ \overline{\beta}_{0}^{(1)} (U_{1}, V_{2}) + \sum_{j=1}^{n_0} \overline{\beta}_{1(P)}^{(1)} (U_{1}, V_{2}, k_{j}) \tilde{S}_{P}(k_{j})^{-1} \right.$$  

$$+ \sum_{j=1}^{n_1} \overline{\beta}_{1(D_{1})}^{(1)} (U_{1}, V_{2}, k_{j}) \tilde{S}_{D_{1}}(k_{j})^{-1} + \sum_{j=1}^{n_2} \overline{\beta}_{1(D_{2})}^{(1)} (U_{1}, V_{2}, k_{j}) \tilde{S}_{D_{2}}(k_{j})^{-1} \right\} \right.$$  

(5.5)

where $U_1 = p_{cd} + \sum_{i=0}^{n_1} k_{i_1}$ and $V_2 = p_{ef} + \sum_{i_2=0}^{n_2} k_{i_2}$. One can always come back to configuration of eq. (5.3) by means of symmetrization over photons. In the MC the sum over photons is “randomized” in a natural way and only one partition member is generated at the time, (using effectively eq. (5.5)) so the fact that the basic distribution for EEX$_R$ is that of eq. (5.3) can be easily overlooked, see also discussion in [24].

From the above algebra we see in a detail how the EEX$_R$ can be embedded in a natural way in the full CEEX$_R$ defined in the previous Section.

5.2 Last step towards EEX$_R$ of YFSWW3

The EEX of eq. (5.5) is not exactly that of EEX of YFSWW3 and KandY as described in refs. [9][12]. Let us discuss the remaining differences. The most important difference is that QED matrix element in YFSWW3 is implemented using PHOTOS program which has matrix element is not in the EEX scheme, although very close to it. At the precision of LEP experiments this was acceptable and economic solution. There would be no problems with replacing PHOTOS with the true EEX implementation for the W decays because such an implementation is already available in the WINHAC program developed for single W production at hadron colliders [30].

The implementation EEX matrix elements for the production process in YFSWW3 is described in fine detail in refs [12]. It is based on the YFS3 event generator [27] for the $e^+e^- \rightarrow 2f$ process replacing final massive fermions with W’s. The YFS3 program does not include QED initial-final state interferences (IFI) between initial $e^\pm$ and final particles. Such interferences (present in EEX of eq. (5.5) were also added in YFSWW3 using reweighting technique of the BHWIDE program of ref. [46].

5.3 From EEX$_R$ to CEEX$_R$ in the MC implementation

The upgrade from EEX of eq. (5.5) to CEEX in the MC implementation is feasible and well defined. In the Monte Carlo program implementing EEX usually one generates MC
events according to some baseline distribution\[^{20}\] and final correcting weight introduces fine details of the EEX matrix element. The CEEX matrix elements can be implemented by reweighting events generated according to the same baseline distributions as in EEX case, just by replacing EEX final MC correcting weight with that of CEEX, without any changes in the baseline MC. This kind of flexible and economic solution was already applied in the KKMC program \[^{24}\]. Similarly as in KKMC the MC weight correcting from EEX to CEEX will be not bound from the above. There are several solutions for this purely technical problem.

### 5.4 Photon distributions around $E_\gamma \sim \Gamma$

Let us finally comments on two apparent deficiencies of the EEX\(_R\) scheme:

- Lack of transmutation of photon distributions around $E_\gamma \sim \Gamma$ and
- Excess of photons multiplicity for very soft photons $E_\gamma \leq \Gamma$.

The phenomenon of “transmutation of photon distributions” occurs when photon energy changes from the “semisoft region” $\Gamma < E_\gamma \ll E_{\text{beam}}$ down to “true soft region” $E_\gamma < \Gamma$. In the true-soft region photon distributions do not reflect the existence of the single charged object, the resonance, they reflect, instead, momenta and charges of all its decay products. For these long range photons the resonance itself is just living too short to be “felt”. On the other hand, the semi-soft photons with shorter wavelength can see the resonance as a distinct object – its presence is imprinted in the distributions of photon energy and angles. In fact, it is the interference between the production-current $j_P^\mu$ and the decay-current $j_{D1,D2}^\mu$ which enforces the transition in the photon distributions. This effect can be also seen explicitly in the instrumental identity of eq. (A.2), or in the explicit one-photon emission amplitude of eq. (3.7). The absence of this interference in EEX, where all NFI interferences are neglected, causes that in the EEX (of YFSWW3) the above beautiful transmutation phenomenon cannot be present\[^{21}\].

The lack of the above interferences causes also certain unphysical effect for very soft photons. As we know, in the real world (and in CEEX) there is no IR singularity (neither real nor virtual) for emission from the internal $W$ line, see eq. (3.7), while in the EEX there is such (real and virtual), as seen explicitly in eq. (4.14). How to explain this paradox? Is this something dangerous? The artificial IR divergence in EEX is not dangerous, as we are at the $O(\frac{\alpha}{\pi M})$ precision level for the distributions which are inclusive enough, such that we do not examine multiplicities and angular spectra of the photons with $E_\gamma < \Gamma$. Extra unphysical photons in this energy range do not contribute to integrated cross section, because their contribution is countered immediately by the virtual formfactor. They will however affect multiplicity of such very soft photons.

\[^{20}\]The baseline distribution has to include all soft and collinear singularities of the EEX distributions.

\[^{21}\]The transition between these two situations is modeled in our new CEEX in completely realistic way. It is continuous in the photon energy.
Good agreement of the soft photon spectra between YFSWW3 and RACOONWW confirms that the effect is not sizable. The numerical estimates of ref. [49] also suggest that this effect is small, negligible for LEP2. On the other hand, in the future high statistics experiments it is worth to examine of the above effects for the photons with $E_\gamma \sim \Gamma_W$. It was proposed in ref. [49] that it may even provide an independent relatively precise measurement of $\Gamma_W$.

Summarizing, the presence of the extra unphysical soft photons $E_\gamma < \Gamma$ in EEX (and its version implemented in YFSWW3) due to setting to zero all QED interference effects between production and decay processes is not harmful at the precision scale of $O(\alpha \pi \Gamma_M)$. For the higher precision requirements, like that in FCC-ee, one should go back to CEEXR, from which EEX$_R$ is derived, and get back for $E_\gamma \sim \Gamma$ fully exclusive realistic photon distributions.

6 Summary and outlook

In the present paper we have solved that long-standing problem of the systematic treatment of the soft and hard photon emission from the unstable charged particles and the interferences between production and decay parts of the process, at any perturbative order. This is of practical importance for high precision measurements of the $W$ pairs in the $e^\pm$ annihilation and for single $W$ production in the hadron colliders, an in many other processes with production and decay of charged unstable particles of any spin. So far there is no practical implementation of the full scale calculation in the proposed scheme. It was outlined how to do in the framework of the Monte Carlo event generator.

We have not discussed the issues related to the definition and resummation of mass and width of the resonance nor the UV renormalisation. Our approach exploited the similarity between virtual and real form factors guaranteed by the IR cancellations.

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Appendices

A Factoring photon emission from $W$

The following considerations are valid for charged unstable particle of any spin, eg. $W^\pm$, $\tau^\pm$ or $t$-quark. Let us start with a simple identity for two propagators related to single photon emission from an internal charged particle line

$$\frac{1}{(Q_0^2 - M^2)(Q_1^2 - M^2)} = \frac{1}{(Q_0^2 - Q_1^2)(Q_1^2 - M^2)} - \frac{1}{(Q_0^2 - Q_1^2)(Q_0^2 - M^2)}$$

(A.1)

where $M^2 = M_W^2 + i M_W \Gamma_W$.

The kinematics is depicted in fig 3. Noticing that $Q_0^2 - Q_1^2 = 2k_1 Q_0 - k_1^2 = 2k_1 Q_1 + k_1^2$

we may rewrite the above as follows

$$\frac{1}{(Q_0^2 - M^2)(Q_1^2 - M^2)} = \frac{1}{(2k_1 Q_1 + k_1^2)(Q_1^2 - M^2)} + \frac{1}{(-2k_1 Q_0 + k_1^2)(Q_0^2 - M^2)}.$$  \hspace{0.5cm} (A.2)

The reader will recognize the first term as representing photon (eikonal) emission factor in the production part of the process times resonance propagator (with the reduced four momentum $Q_1 = Q_0 - k_1$) and the second term as the analogous emission factor in the decay process times resonance propagator (with the four momentum $Q_0 = Q_1 + k_1$). Each of the two terms look IR-divergent, however the two IR divergences cancel – the difference is finite. In the original expression it was resonance width $\Gamma_W$ which was providing infrared regulator for photon with the momentum $k_1 = Q_1 - Q_2$.

Let us now consider the general case of the $n$-photon emission from the internal charged particle line depicted in fig. 4 in the soft photon approximation. The reorganization of the product of the propagators starts with the following identity:

$$\frac{1}{(Q_0^2 - M^2)(Q_1^2 - M^2) \ldots (Q_n^2 - M^2)} = \sum_{j=0}^{n} \frac{1}{\prod_{i=0}^{j-1}(Q_i^2 - Q_j^2) \prod_{i=j}^{n-j}(Q_j^2 - Q_i^2)}. \hspace{0.5cm} (A.3)$$

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It can be proven by the mathematical induction. Assuming the identity is true for \( n \) let us prove it for \( n + 1 \). Using shorthand notation \( y_i = Q_i^2 - M^2 \) one obtains\(^\text{22}\)

\[
\prod_{i=0}^{n+1} \frac{1}{y_i} = \frac{1}{y_{n+1}} \sum_{j=0}^{n} \frac{1}{y_j} \prod_{i \neq j} \frac{1}{(y_i - y_j)} = \sum_{j=0}^{n} \frac{1}{(y_{n+1} - y_j)} \prod_{i \neq j} \frac{1}{(y_i - y_j)}
\]

\[
= \sum_{j=0}^{n} \frac{1}{y_j} \prod_{i \neq j} \frac{1}{(y_i - y_j)} - \frac{1}{y_{n+1}} \sum_{j=0}^{n} \prod_{i \neq j} \frac{1}{(y_i - y_j)} - \frac{1}{y_{n+1}} \sum_{j=0}^{n} \prod_{i \neq j} \frac{1}{(y_i - y_j)}
\]

\[
= \sum_{j=0}^{n+1} \frac{n+1}{y_j} \prod_{i \neq j} \frac{1}{(y_i - y_j)} - \frac{1}{y_{n+1}} \sum_{j=0}^{n} \prod_{i \neq j} \frac{1}{(y_i - y_j)} - \frac{1}{y_{n+1}} \sum_{j=0}^{n} \prod_{i \neq j} \frac{1}{(y_i - y_j)}
\]

(\text{A.4)}

Alternatively one can prove it with the help of partial fractioning with respect to \( M^2 \):

\[
\frac{1}{(Q_0^2 - M^2)(Q_1^2 - M^2) \ldots (Q_n^2 - M^2)} = \sum_{j=0}^{n} \frac{A_j}{Q_j^2 - M^2}.
\]

(A.5)

Multiplying eq. (\text{A.5)} in a standard way by \( Q_j^2 - M^2 \) and substituting \( Q_j^2 = M^2 \) we obtain

\[
A_j = \frac{1}{\prod_{i=0}^{j-1}(Q_i^2 - Q_j^2) \prod_{i=j+1}^{n}(Q_j^2 - Q_i^2)}.
\]

(A.6)

Let us now examine soft photon limit in eq. (\text{A.3)} Taking \( j \)-th term we may identify

\[
Q_0^2 - Q_j^2 \simeq (2k_j Q_j + k_j^2) + \cdots + (2k_2 Q_j + k_2^2) + (2k_1 Q_j + k_1^2)
\]

\[
Q_1^2 - Q_j^2 \simeq (2k_j Q_j + k_j^2) + \cdots + (2k_2 Q_j + k_2^2)
\]

(A.7)

\[
Q_{j-1}^2 - Q_j^2 = (2k_j Q_j + k_j^2)
\]

and

\[
Q_{j+1}^2 - Q_j^2 = (-2k_{j+1} Q_j + k_{j+1}^2)
\]

\[
Q_{n-1}^2 - Q_j^2 \simeq (-2k_{j+1} Q_j + k_{j+1}^2) \cdots + (-2k_{n-1} Q_j + k_{n-1}^2)
\]

(A.8)

\[
Q_n^2 - Q_j^2 \simeq (-2k_{j+1} Q_j + k_{j+1}^2) \cdots + (-2k_{n-1} Q_j + k_{n-1}^2) + (-2k_n Q_j + k_n^2)
\]

In the above equations we have neglected subleading products \( k_i k_j \). This is allowed in the soft photon approximation. On the other hand, terms \( k_i^2 \) could be also omitted in the soft photon approximation, but they are kept because they render virtual photon integrals UV-finite.

In the next step we perform the usual sum over permutation over all photons. This will lead to a "poissonian" emission formula, separately for resonance production and decay

\(^{22}\) The identity \( \sum_{j=0}^{n} \prod_{i \neq j} \frac{1}{x_i - x_j} = 0 \) is used in the last step.
stages of the entire process, with the explicit sum over the assignments of photons to production denoted by index \( P \) and decay by index \( D \). We start from eq. (A.3) switching to a more compact notation

\[
R(Q_i^2) = Q_i^2 - M^2, \\
N_i^P = 2k_iQ_j + k_i^2 = (Q_j + k_i)^2 - Q_j^2, \\
N_i^D = -2k_iQ_j + k_i^2 = (Q_j - k_i)^2 - Q_j^2.
\]

(A.9)

Inserting relations of eqs. (A.7) and (A.8) into eq. (A.3) and summing over permutations we obtain:

\[
\sum_{\text{permut.}} \frac{1}{R(Q_0^2)R(Q_1^2)\ldots R(Q_n^2)}
\]

\[
= \sum_{\text{permut.}} \sum_{j=0}^{n} \left[ \frac{1}{N_1^P + N_2^P + N_3^P + \ldots N_j^P} \frac{1}{N_2^P + N_3^P + \ldots N_j^P} \ldots \frac{1}{N_j^P} \right]
\]

\[
\times \frac{1}{R(Q_j^2)} \times \left[ \frac{1}{N_{j+1}^D} \frac{1}{N_{j+1}^D + N_{j+2}^D} \ldots \frac{1}{N_{j+2}^D + N_{j+1}^D + N_{j+2}^D + \ldots N_n^D} \right]
\]

(A.10)

where for \( j = 0 \) and \( j = n \) respectively term in the first/second square bracket pair should read as 1. Next, for each \( j \)-th term we split the sum over all permutations of \((1, 2, 3, \ldots, n)\) into two separate sums, one over permutations over \((1, 2, 3, \ldots, j)\) and another over permutations of \((j + 1, j + 2, \ldots, n)\). These two sums are performed\(^{23}\). The sum over \( \binom{n}{j} \) assignments of photons to production and decay remains. Alternatively, the entire remaining sum can be represented as a sum over \( \sum_j \binom{n}{j} = (1 + 1)^n = 2^n \) terms (photon assignments) as follows

\[
\sum_{\text{permut.}} \frac{1}{R(Q_0^2)R(Q_1^2)\ldots R(Q_n^2)}
\]

\[
= \sum_{\phi=(P,D)^n} \prod_{\phi_i=P} \frac{1}{(Q_0 - k_i)^2 - Q_i^2} \times \prod_{\phi_i=D} \frac{1}{(Q_0 - k_i)^2 - Q_i^2},
\]

(A.11)

where

\[
Q_\phi = Q_0 - \sum_{\phi_i=P} k_i = Q_n + \sum_{\phi_i=D} k_i.
\]

(A.12)

The vectors \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \) of the photon assignments whose components have values equal to \( P \) or \( D \), while the sum \( \sum_{\phi_i=P} (\prod_{\phi_i=P}) \) denotes sum over (product of) all \( i \) for which \( \phi_i = P \), i.e., all photon which belong to the production stage of the process.

Main features of eq. (A.11), the principal result of this Appendix, are the following:

\(^{23}\) Here we use twice the well known identity \( \sum_{\text{perm.}} \frac{1}{a_1(a_1+a_2)(a_1+a_2+a_3)\ldots(a_1+a_2+\ldots+a_n)} = \frac{1}{a_1a_2\ldots a_n} \), where the sum is over all permutations of \((1, 2, 3, \ldots, n)\).
• Its left hand side represents “raw” Feynman diagrams for multiple photon emission from the charged particle internal line.

• Its right hand side includes two photon emission factors, one for emissions for the production part of the process (resonance formation) and the second one for the decay part of the process (resonance decay).

• It includes single resonance propagator of the standard form, with complex mass $M$ and the four-momentum $Q$, which comprises momenta of all photons assigned to resonance decay.

• It is rather striking that all photon emission factors look as if photons were emitted by the charged particle of the mass $Q$. This is, of course, intuitively well justified and quite appealing.

• The fact that a coherent sum is performed over all assignments of photons to decay and productions reflects QED gauge invariance and Bose statistics.

• It holds both for virtual and real photons (this is why we have kept $k^2$).

B Resummation of the real emission

In this Appendix we show how to do the resummation of the amplitude of the multiple-real-photon emission. We expect that because of IR cancellations the basic algebraic structure of our derivation holds for the integrands of multiloop corrections.

Let us begin with a short summary of the YFS method performed in a combinatorial way. The process under consideration is $e(p_a)\bar{\nu}_e(p_b) \rightarrow W \rightarrow \mu(p_c)\bar{\nu}_\mu(p_d)$. At first we consider the standard YFS scheme without radiation nor recoil from $W$. As proven by YFS, the IR radiation comes entirely from the charged external legs ($e$ and $\mu$) and has a form of soft currents. The sum of graphs with $N$ real emissions is the following:

$$M^{(0)i_1,\ldots,i_N}_{N}(k_1, \ldots, k_N) \simeq \sum_{l=0}^{N-1} \sum_{\pi} \frac{2p_{i_l}\pi_1}{2p_\mu k_{\pi_1}} \frac{2p_{i_2}\pi_2}{2p_\mu k_{\pi_2}} \frac{2p_{i_l}^{\mu_l}}{2p_\mu k_{\pi_l}} \left( \frac{-2p_c^{\mu_{l+1}}}{2p_c k_{\pi_{l+1}}} \frac{-2p_c^{\mu_{l+2}}}{2p_c k_{\pi_{l+2}}} \cdots \frac{-2p_c^{\mu_N}}{2p_c k_{\pi_N}} \right) \frac{1}{p_{ab}^2 - M^2}. \quad (B.1)$$

We execute now the sum over permutations of photons within the $a$ and $c$ sub-groups according to the formula of footnote $23$. This turns complicated sums into simple products:

$$M^{(0)i_1,\ldots,i_N}_{N}(k_1, \ldots, k_N) \simeq \frac{1}{p_{ab}^2 - M^2} \sum_{l=0}^{N-1} \sum_{\pi/\pi_{N-l}} \left( \prod_{i=1}^{l} \frac{2p_{i_l}\pi_1}{2p_\mu k_{\pi_1}} \right) \left( \prod_{i=1}^{N-l} \frac{-2p_c^{\mu_{l+1}}}{2p_c k_{\pi_{l+1}}} \right). \quad (B.2)$$
It takes now a few moments to realize that combinatorial sum over permutations can be replaced by the sum over partitions (cf. eqs. (A.10) and (A.11))

\[
\sum_{l=0}^{N} \sum_{\pi/\pi_a/\pi_c}^{N!/(N-l)!} = \sum_{\varphi=(a,c)^N}^{2^N}.
\] (B.4)

Consequently we get

\[
\mathcal{M}_N^{(0)\mu_1,\ldots,\mu_N}(k_1, \ldots, k_N) \simeq \frac{1}{p_{ab}^2 - M^2} \sum_{\varphi=(a,c)^N}^{2^N} \left( \prod_{i=1}^{N} \frac{2\theta_{\phi_i}p_{\mu_i}}{2p_{\phi_i}k_i} \right).
\] (B.5)

where \(\theta\) equals +1 for initial state and −1 for final state. Finally we notice that sum over partitions in eq. (B.5) can be rewritten in a compact form as

\[
\mathcal{M}_N^{(0)\mu_1,\ldots,\mu_N}(k_1, \ldots, k_N) \simeq \frac{1}{p_{ab}^2 - M^2} \prod_{i=1}^{N} \left( \frac{2p_{\mu_i}^a}{2p_{a}k_i} - \frac{2p_{\mu_i}^c}{2p_{c}k_i} \right).
\] (B.6)

Let us now allow for the radiation from \(W\). We begin by analysing numerator of the multiple emission graph of Fig. 4, i.e. of LHS of eq. (A.3). The numerator of the single photon emission with two accompanying \(W\) propagators looks like (in the small photon momentum limit)

\[
(-g^{\lambda\nu} + p^\nu p^\lambda /M_W^2)V(p, k, p - k)\lambda_{\nu\rho\sigma}(-g^{\sigma',\sigma} + (p - k)^{\sigma'}(p - k)^{\sigma}/M_W^2)
\]

\[
k \rightarrow 0 (-g_{\lambda\sigma} + p_\lambda p_\sigma /M_W^2)(-2p_\rho) + g_{\lambda\rho}p_\sigma(p^2 - M_W^2) + g_{\sigma\rho}p_\lambda(p^2 - M_W^2)
\] (B.7)

where \(V(p, k, p - k)\lambda_{\nu\rho\sigma}\) is the \(W\gamma W\) vertex. Dropping also the terms proportional to \(p^2 - M_W^2\) (i.e. putting \(p\) on-shell) we obtain self-repeating structure and the numerator of the whole line becomes

\[
(-g_{\lambda\sigma} + Q_j\lambda Q_j/\sigma /M_W^2) \prod_{i=1}^{n} (-2p_{\mu_i}).
\] (B.8)

\textsuperscript{24} Note that identity (B.4) generalizes to more than two particles. For example:

\[
\sum_{i_a, i_c, i_e=0}^{N} \sum_{\pi/\pi_a/\pi_c}^{N!/(i_a!i_c!i_e!)} = \sum_{\varphi=(a,c,e)^N}^{3^N}.
\] (B.3)

\textsuperscript{25} For instance for \(N=2\) we have four partitions

\[
\frac{2p_{\mu_1}^a}{2p_{a}k_1} \frac{2p_{\mu_2}^a}{2p_{a}k_2} - \frac{2p_{\mu_1}^a}{2p_{a}k_1} \frac{2p_{\mu_2}^c}{2p_{c}k_2} - \frac{2p_{\mu_1}^c}{2p_{c}k_1} \frac{2p_{\mu_2}^a}{2p_{a}k_2} + \frac{2p_{\mu_1}^c}{2p_{c}k_1} \frac{2p_{\mu_2}^c}{2p_{c}k_2} = \left( \frac{2p_{\mu_1}^a}{2p_{a}k_1} - \frac{2p_{\mu_1}^c}{2p_{c}k_1} \right) \left( \frac{2p_{\mu_2}^a}{2p_{a}k_2} - \frac{2p_{\mu_2}^c}{2p_{c}k_2} \right).
\]
Now inclusion of the radiation from $W$ into eq. (B.1) amounts to the following modification:

$$
M^{(0)\mu_1,\ldots,\mu_N}_{N}(k_1, \ldots, k_N) \simeq \sum_{l_a,l_c,n=0}^{N} \sum_{l_a+l_c+n=N}^{N} \left[ \left( \frac{2p_a^{\mu_1}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \left( \frac{2p_a^{\mu_2}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \cdots \left( \frac{2p_a^{\mu_l}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \right] 
$$

$$
- \frac{g^\lambda g^\sigma + Q^\lambda Q^\sigma / M_W^2}{Q^2_{\pi_0} - M_W^2} \prod_{i=1}^{n} \left( -2Q^{\mu_{l_a+i}}_{\pi_0} \right) 
$$

$$
\left( \left( \frac{-2p_c^{\mu_{l_a+n+1}}}{2p_c k_{\pi_{l_a+n+1}} + 2p_c k_{\pi_{l_a+n+2}} + \cdots + 2p_c k_{\pi_{l_a+n+2}}} \right) \left( \frac{-2p_c^{\mu_{l_a+n+2}}}{\cdots} \right) \cdots \left( \frac{-2p_c^{\mu_{l_a+n+2}}}{\cdots} \right) \right),
$$

where we temporarily chose $Q_0$ as momentum in the numerators. The unmatched indices $\lambda \sigma$ in the $W$ propagator are to be treated "symbolically" as we do not write a complete expression for $M$. At this moment we plug in the formula (A.10)

$$
M^{(0)\mu_1,\ldots,\mu_N}_{N}(k_1, \ldots, k_N) \simeq \sum_{l_a,l_c,n=0}^{N} \sum_{l_a+l_c+n=N}^{N} \left[ \left( \frac{2p_a^{\mu_1}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \left( \frac{2p_a^{\mu_2}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \cdots \left( \frac{2p_a^{\mu_l}}{2p_a k_{\pi_1} + 2p_a k_{\pi_2} + \cdots + 2p_a k_{\pi_l}} \right) \right] 
$$

$$
- \frac{g^\lambda g^\sigma + Q^\lambda Q^\sigma / M_W^2}{Q^2_{\pi_0} - M_W^2} \prod_{i=1}^{n} \left( -2Q^{\mu_{l_a+i}}_{\pi_0} \right) 
$$

$$
\left( \left( \frac{-2p_c^{\mu_{l_a+n+1}}}{2p_c k_{\pi_{l_a+n+1}} + 2p_c k_{\pi_{l_a+n+2}} + \cdots + 2p_c k_{\pi_{l_a+n+2}}} \right) \left( \frac{-2p_c^{\mu_{l_a+n+2}}}{\cdots} \right) \cdots \left( \frac{-2p_c^{\mu_{l_a+n+2}}}{\cdots} \right) \right),
$$

The double sum can be converted into a single one

$$
\sum_{l_a,l_c,n=0}^{N} \sum_{l_a+l_c+n=N}^{N} = \sum_{l_a,l_c,l_g,l_h=0}^{N} \sum_{l_a+l_c+l_g+l_h=N}^{N}
$$

(B.11)
and the four groups of permutations can be executed as in eq. (B.2)

\[
\mathcal{M}_N^{(\mu_1, \ldots, \mu_N)}(k_1, \ldots, k_N) \simeq \sum_{l_a, l_b, l_c, l_d, j_a, j_b = 0}^{N!} \sum_{\pi_1, \pi_2, \pi_3, \pi_4} \frac{-g\lambda^\sigma + Q_{\pi_1g}^\lambda Q_{\pi_2j_g}^\sigma / M_W^2}{Q_{\pi_1g}^2 - M_W^2} 
\]

\[
\left(\prod_{i=1}^{l_a} \frac{2p_{\mu_i}}{2p_{\mu_i}}\right) \left(\prod_{i=1}^{l_b} \frac{-2Q_{\mu_i}^0}{2Q_{\pi_1g} k_{\pi_1a+i}}\right) \left(\prod_{i=1}^{l_c} \frac{2Q_{\mu_i}^0}{2Q_{\pi_1g} k_{\pi_1a+i}}\right) \left(\prod_{i=1}^{l_d} \frac{-2p_{\mu_i}^c k_{\pi_1a+i} l_{g+i}}{2p_{\mu_i}^c k_{\pi_1a+i} l_{g+i}}\right).
\]

The first two terms in curly brackets describe the emission form the production part (from lines \(a\) and \(g\)). The \(Q_g\) is defined there as \(Q_g = p_{ab} - K_P, \ K_P = K_a - K_g\) and is the same for all terms. Therefore these two products can be combined into one as in eqs. (B.5) (B.6).

\[
\mathcal{M}_N^{(\mu_1, \ldots, \mu_N)}(k_1, \ldots, k_N) \simeq \sum_{l_a, l_b, l_c, l_d, j_a, j_b = 0}^{N!} \sum_{\pi_1, \pi_2, \pi_3, \pi_4} \frac{-g\lambda^\sigma + Q_{\pi_1g}^\lambda Q_{\pi_2j_g}^\sigma / M_W^2}{Q_{\pi_1g}^2 - M_W^2} 
\]

\[
\prod_{i=1}^{l_a} \left(\frac{2p_{\mu_i}^g}{2p_{\mu_i}^g} - \frac{2Q_{\mu_i}^0}{2Q_{\pi_1g} k_{\pi_1a+i}}\right) \prod_{i=1}^{l_b} \left(\frac{2Q_{\mu_i}^0}{2Q_{\pi_1g} k_{\pi_1a+i}} - \frac{2p_{\mu_i}^c}{2p_{\mu_i}^c k_{\pi_1a+i}}\right)
\]

where \(l_a + l_b = l_P\) and \(l_h + l_c = l_D\). The sum over permutations can be once more replaced by the sum over partitions (cf. eq. (B.4))

\[
\mathcal{M}_N^{(\mu_1, \ldots, \mu_N)}(k_1, \ldots, k_N) \simeq \sum_{\varphi = (P,D)}^{2N} \frac{-g\lambda^\sigma + Q_g^\lambda Q_g^\sigma / M_W^2}{Q_g^2 - M_W^2} \prod_{i=1}^{N} \left(\frac{2\theta_{\varphi i} p_{\mu_i}^g}{2\theta_{\varphi i} k_i} - \frac{2\theta_{\varphi i} Q_{\mu_i}^0}{2Q_g k_i}\right)
\]

\[
\sum_{\varphi = (P,D)}^{2N} \frac{-g\lambda^\sigma + Q_g^\lambda Q_g^\sigma / M_W^2}{Q_g^2 - M_W^2} \prod_{i=1}^{N} \left(\frac{2\theta_{\varphi i} p_{\mu_i}^g}{2\theta_{\varphi i} k_i} - \frac{2\theta_{\varphi i} Q_{\mu_i}^0}{2Q_g k_i}\right)
\]

We used in eq. (B.14) a freedom of defining \(Q_{\mu_i}^0\) to replace it with \(Q_{\varphi i} \equiv Q_g\). Note that, contrary to \(p_X\), the vectors \(Q_X\) depend on the choice of partitions, i.e. vary from partition to partition. This prevents us from collapsing the remaining sum over partitions, quite analogously as in case of neutral resonance.

### C. Details of the virtual formfactor

In the following we are going to generalize the YFS \[18\] virtual formfactor function \(\alpha B\) to general case with charged intermediate resonances. In order to introduce notation let us first write down explicitly the emission factor for single real photon

\[
j^\mu(k) = ie \sum_{X=a,b,c,d,e,f} Q_X \theta_X \frac{2p_X^\mu}{2p_X k}, \quad (C.1)
\]
Figure 5: Example of one real and one virtual photon emissions. Electric current is a sum of contribution from all external particles. This is why it is attached to dashed line which crosses all relevant external lines. The rest of Feynman diagram is visualized as the internal dark box.

where $\theta_X = +1, -1$ for particles in the initial and final state, the $Q_X$ is the charge of the particle (in the units of positron charge $e$) and the single virtual photon current reads:

$$J^\mu(k) = \sum_{X=a,b,c,d,e,f} \tilde{J}^\mu_X(k), \quad \tilde{J}^\mu_X(k) \equiv Q_X \theta_X \frac{2p_X^\mu \theta_X + k^\mu}{k^2 + 2p_X k \theta_X + i\varepsilon},$$

see fig. 5. For the virtual corrections we always have an even number of the $J^\mu(k)$ currents paired in the so called virtual $S$-factor

$$S(k) = J(k) \circ J(k) = \sum_{X=a,b,c,d,e,f} \sum_{Y=a,b,c,d,e,f} J_X(k) \circ J_Y(k),$$

where

$$J_X(k) \circ J_Y(k) = J_X(k) \cdot J_Y(-k), \text{ for } X \neq Y, \quad J_X(k) \circ J_X(k) = J_X(k) \cdot J_X(k).$$

In fig. 5 we illustrate all that in a visual way. The contribution $J_X(k) \cdot J_X(k)$ looks diagrammatically like self-energy, but in fact it comes from the charge renormalization, see discussion in refs. [18,51].

In the derivation of $\exp(\alpha B)$ of ref. [18], (taking as an example four fermion production process) we arrive at certain stage in which contributions from all real and virtual photons are factorized. The corresponding scattering amplitude with $m$ real and any number of
virtual photons taken in soft photon approximation is visualized in fig. 6 and it reads

\[ M_{\mu_1\mu_2...\mu_m}(k_1, k_2, ..., k_m) = \mathcal{M} \prod_{l=1}^{m} j^\mu(k_l) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \alpha \int \frac{i}{(2\pi)^3} \frac{d^4 k_i}{k_i^2 - \lambda^2 + i\varepsilon} J^\mu(k_i) \circ J_{\mu}(k_i) \]  

(C.5)

Sum over virtual photons is done trivially resulting in exponential formfactor:

\[ M_{\mu_1\mu_2...\mu_m}(k_1, k_2, ..., k_m) = \mathcal{M} \prod_{l=1}^{m} j^\mu(k_l) \, e^{\alpha B_6}, \]  

(C.6)

Note that in the residual function \( \mathcal{M} \) there is no “recoil” dependence on photon momenta, we are therefore limited to very soft photons \( (E_\gamma \ll \Gamma_W) \) in the process of our interest.

Let us now take into account the double-resonant character of the process, see fig. 7. After factorizing all real and virtual soft photons, and introducing new source of emission from resonant intermediate \( W \)'s, we will use the identity (A.11) of appendix A, to arrive
Figure 7: CEEX amplitude for $WW$ production and two decays in soft photon approximation. Visualized are all classes of virtual and real photon emissions.

at the amplitude depicted in fig. 7 which can be written explicitly as follows

$$M_{n_1 n_2 n_3}^{\mu_1 \cdots \mu_{n_3}} (k_1, \ldots, k_{3n_3}) = M_0 \prod_{i_1=1}^{n_1} j_{P_i} (k_{i_1}) \prod_{i_2=1}^{n_2} j_{D_1} (k_{i_2}) \prod_{i_3=1}^{n_3} j_{D_2} (k_{i_3})$$

$$\sum_{n_4=0}^{\infty} \frac{1}{n_4!} \prod_{i_4=1}^{n_4} \frac{1}{\alpha} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_4}}{k_{i_4}^2 - m_7^2 + i\varepsilon} J_P (k_{i_4}) \circ J_P (k_{i_4})$$

$$\sum_{n_5=0}^{\infty} \frac{1}{n_5!} \prod_{i_5=1}^{n_5} \frac{1}{\alpha} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_5}}{k_{i_5}^2 - m_2^2 + i\varepsilon} J_{D_1} (k_{i_5}) \circ J_{D_1} (k_{i_5})$$

$$\sum_{n_6=0}^{\infty} \frac{1}{n_6!} \prod_{i_6=1}^{n_6} \frac{1}{\alpha} \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_6}}{k_{i_6}^2 - m_7^2 + i\varepsilon} J_{D_2} (k_{i_6}) \circ J_{D_2} (k_{i_6})$$

$$\sum_{n_7=0}^{\infty} \frac{1}{n_7!} \prod_{i_7=1}^{n_7} 2\alpha \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_7}}{k_{i_7}^2 - m_7^2 + i\varepsilon} J_P (k_{i_7}) \circ J_{D_1} (k_{i_7})$$

$$\sum_{n_8=0}^{\infty} \frac{1}{n_8!} \prod_{i_8=1}^{n_8} 2\alpha \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_8}}{k_{i_8}^2 - m_7^2 + i\varepsilon} J_P (k_{i_8}) \circ J_{D_2} (k_{i_8})$$

$$\sum_{n_9=0}^{\infty} \frac{1}{n_9!} \prod_{i_9=1}^{n_9} 2\alpha \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_9}}{k_{i_9}^2 - m_7^2 + i\varepsilon} J_{D_1} (k_{i_9}) \circ J_{D_2} (k_{i_9})$$

$$\sum_{n_{10}=0}^{\infty} \frac{1}{n_{10}!} \prod_{i_{10}=1}^{n_{10}} 1 \int \frac{i}{(2\pi)^3} \frac{d^4 k_{i_{10}}}{k_{i_{10}}^2 - m_7^2 + i\varepsilon} J_{D_2} (k_{i_{10}})$$

$$\frac{1}{(p_{cd} + K_2 - K_7 + K_9)^2 - M^2} \frac{1}{(p_{ef} + K_3 - K_8 - K_9) - M^2}$$
where $K_l = \sum_{i=1}^{n_l} k_{li}$. We have defined nine groups of photons (labeled by $l$). The $l = 1, 2, 3$ corresponds respectively to real emission from $W$-pair production and their decays, virtual photons corresponding to the same sources are denoted as groups (4,5,6), the (7,8) corresponds to virtual photons attached to production and decay. Finally virtual photons denoted as group (9) connect decay of the first and second $W$ boson.

The most interesting part in the above expression is that the product of two resonance propagators includes all relevant recoil dependence on the real and virtual photon momenta. This dependence can be read easily from fig. 7. We are now not limited by $E_\gamma \ll \Gamma_W$, but rather by $E_\gamma \ll \sqrt{s}$. One important feature is that propagators do not depend on $K_4, K_5$ and $K_6$ – this is why the sums over relevant photons can be immediately folded into three standard YFS formfactor $e^{\alpha B}$, for production and decay processes. This, however, cannot be done for the three virtual interference contributions because propagators do depend on $K_7, K_8$ and $K_9$. The dependence on the real photons $K_2$ and $K_3$ is not harmful for our task of summing up virtual contributions to infinite order.

\[
M_{\mu_1 \ldots \mu_3} (k_1, \ldots, k_{3n_3}) = M_0 \prod_{i=1}^{n_1} j_P^{\mu_1} (k_{i1}) \prod_{i=1}^{n_2} j_D^{\mu_2} (k_{i2}) \prod_{i=1}^{n_3} j_D^{\mu_3} (k_{i3})
\]

\[
e^{\alpha B_P} e^{\alpha D_1} e^{\alpha D_2} \sum_{n_7=0}^{\infty} \frac{1}{n_7!} \int \frac{i}{(2\pi)^3} \frac{d^4k_{i_7}}{k_{i_7}^2 - m_7^2} J_P (k_{i_7}) \circ J_D (k_{i_7})
\]

\[
\sum_{n_8=0}^{\infty} \frac{1}{n_8!} \prod_{i_8=1}^{n_8} 2\alpha \int \frac{i}{(2\pi)^3} \frac{d^4k_{i_8}}{k_{i_8}^2 - m_7^2} J_P (k_{i_8}) \circ J_D (k_{i_8}) \quad (C.8)
\]

\[
\sum_{n_9=0}^{\infty} \frac{1}{n_9!} \prod_{i_9=1}^{n_9} 2\alpha \int \frac{i}{(2\pi)^3} \frac{d^4k_{i_9}}{k_{i_9}^2 - m_7^2} J_D (k_{i_9}) \circ J_D (k_{i_9})
\]

\[
(U_2 - K_7 + K_9)^2 - M^2 \quad (V_3 - K_8 - K_9)^2 - M^2
\]

where $U_2 = p_{cd} + K_2$ and $V_3 = p_{ef} + K_3$ and

\[
\alpha B_X = \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2 - m_7^2 + i\varepsilon} J_X (k) \circ J_X (k), \quad X = P, D_1, D_2. \quad (C.9)
\]

At this point we use the following approximations (valid in soft photon limit)

\[
\frac{1}{(U_2 - K_7 + K_9)^2 - M^2} \approx \frac{1}{U_2^2 - M^2 - 2U_2 K_7 + 2U_2 K_9}
\]

\[
= \frac{1}{U_2^2 - M^2} \frac{1}{1 - \sum_{i_7} \frac{2U_2 k_{i_7}}{U_2^2 - M^2} + \sum_{i_9} \frac{2U_2 k_{i_9}}{U_2^2 - M^2}} \quad (C.10)
\]

\[
\approx \frac{1}{U_2^2 - M^2} \prod_{i_7} \frac{U_2^2 - M^2}{1 + \frac{2U_2 k_{i_7}}{U_2^2 - M^2}} \prod_{i_9} \frac{U_2^2 - M^2}{1 + \frac{2U_2 k_{i_9}}{U_2^2 - M^2}}
\]

\[
\prod_{i_7} \frac{U_2^2 - M^2}{(U_2 - k_{i_7})^2 - M^2} \prod_{i_9} \frac{U_2^2 - M^2}{(U_2 + k_{i_9})^2 - M^2}
\]
which, together with renormalization issues, are to be addressed in a future work.

Technical details related to the definition of the width and mass of $W$-bosons (see Appendix B) and the cancellation between real and virtual charged resonance, or more than two charged resonances, can be treated in the same way.

Presented derivation of the virtual formfactor is based to a large extent on the analogy with the real emission part (see Appendix B) and the cancellation between real and virtual emissions. Technical details related to the definition of the width and mass of $W$-bosons along with renormalization issues are to be addressed in a future work.

\[ M_{\mu_1 \cdots \mu_n} (k_1, \ldots, k_{3n}) = M_0 \prod_{i=1}^{n_1} j_{\mu_1}^{i} (k_i) \prod_{i=2}^{n_2} j_{\mu_2}^{i} (k_{i_2}) \prod_{i=3}^{n_3} j_{\mu_3}^{i} (k_{i_3}) \]

where

\[
\begin{align*}
\alpha_{B_{10}}^{\text{CEEX}}(U, V) &= \alpha B_P + \alpha B_{D_1} + \alpha B_{D_2} + 2\alpha B_{P \otimes D_1}(U) + 2\alpha B_{P \otimes D_1}(U) + 2\alpha B_{D_1 \otimes D_2}(U, V), \\
\alpha_{B_{P \otimes D_1}}(U) &= \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2 - m_r^2 + i\varepsilon} J_P(k) \circ J_{D_1}(k) \frac{U^2 - M^2}{(U - k)^2 - M^2}, \\
\alpha_{B_{P \otimes D_2}}(V) &= \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2 - m_r^2 + i\varepsilon} J_P(k) \circ J_{D_2}(k) \frac{V^2 - M^2}{(V - k)^2 - M^2}, \\
\alpha_{B_{D_1 \otimes D_2}}(U, V) &= \int \frac{i}{(2\pi)^3} \frac{d^4k}{k^2 - m_r^2 + i\varepsilon} J_{D_1}(k) \circ J_{D_2}(k) \frac{U^2 - M^2}{(U + k)^2 - M^2} \frac{V^2 - M^2}{(V - k)^2 - M^2}.
\end{align*}
\]

Let us note that in the no-recoil limit $U - k \to U$, $V - k \to V$, i.e. $k \ll \Gamma_W$ the $B_{10}^{\text{CEEX}}(U, V)$ reduces to $B_0^{\text{CEEX}}$ in an analogous way to eq. 3.7.

In eq. (C.11) and in all previous steps the contributions of real photon were taken as just one term (in which we know to which subprocess every real photon belongs) from the grand sum (as defined e.g. in formula 4.14), over all $3^n$ photon assignments $(P, D_1, D_2)^n$, in which we know to which subprocess every real photon belongs. Let us restore this coherent sum over all photon assignments in the following compact final expression:

\[
M_{\mu_1 \cdots \mu_n} (k_1, k_2, \ldots, k_n) = \sum_{\varphi \in (P, D_1, D_2)^n} M_0 \prod_{i=1}^{n} j_{\mu_i}^{\varphi_i} (k_i) e^{\alpha_{B_{10}}^{\text{CEEX}}(U_\varphi, V_\varphi)} \frac{1}{U_\varphi^2 - M^2} \frac{1}{V_\varphi^2 - M^2},
\]

where $U_\varphi = p_{cd} + \sum_{\varphi_i = D_1} k_i$ and $V_\varphi = p_{ef} + \sum_{\varphi_i = D_2} k_i$.

Eq. (C.13) is the principal result of this Appendix. The CEEX formfactor of eq. (C.13) is valid for production of a pair of any charged resonances of any spin. The case of single charged resonance, or more than two charged resonances, can be treated in the same way.

The final form of the result took shape thanks to the use, at the earlier step, the identity A.10 of the Appendix A.

\[\text{The final form of the result took shape thanks to the use, at the earlier step, the identity A.10 of the Appendix A.}\]
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