CATALYSIS IN THE TRACE CLASS
AND WEAK TRACE CLASS IDEALS

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Abstract. Given operators $A,B$ in some ideal $\mathcal{I}$ in the algebra $\mathcal{L}(H)$ of all bounded operators on a separable Hilbert space $H$, can we give conditions guaranteeing the existence of a trace-class operator $C$ such that $B \otimes C$ is submajorized (in the sense of Hardy–Littlewood) by $A \otimes C$? In the case when $\mathcal{I} = \mathcal{L}_1$, a necessary and almost sufficient condition is that the inequalities $\text{Tr}(B^p) \leq \text{Tr}(A^p)$ hold for every $p \in [1,\infty]$. We show that the analogous statement fails for $\mathcal{I} = \mathcal{L}_{1,\infty}$ by connecting it with the study of Dixmier traces.

1. Introduction

Let $H$ be an infinite-dimensional separable Hilbert space, $\mathcal{L}(H)$ be the algebra of all bounded operators on $H$ and $\mathcal{C}_0 = C_0(\mathcal{H})$ the set of compact operators.

Given $A \in \mathcal{C}_0$, we denote by $\mu(A) := \{\mu(k, A)\}_{k \geq 0}$ the sequence of singular values of the operator $A$ (that is, eigenvalues of the operator $|A|$) arranged in decreasing order and taken with multiplicities (if any). We say that $B \in \mathcal{C}_0$ is submajorized by $A \in \mathcal{C}_0$ in the sense of Hardy–Littlewood (written $B \ll A$) if for every integer $n$

$$
\sum_{k=0}^{n} \mu(k, B) \leq \sum_{k=0}^{n} \mu(k, A).
$$

If $A, B \in \mathcal{C}_0$ are such that $B \ll A$, then $B \otimes C \ll A \otimes C$ for every $C \in \mathcal{C}_0$. The converse does not hold, even in the finite-dimensional setting: if $A, B, C$ are such that $\mu(A) = (0.5, 0.25, 0.25, 0, \ldots)$, $\mu(B) = (0.4, 0.4, 0.1, 0.1, 0, \ldots)$ and $\mu(C) = (0.6, 0.4, 0, \ldots)$, one checks easily that $B \otimes C \ll A \otimes C$ while $B$ is not submajorized by $A$. This example appears in [7] and is related to the phenomenon of catalysis in quantum information theory (the operator $C$ being called a catalyst). This corresponds to the situation where the transformation of some quantum state (in that case, $B$) into another quantum state (in that case, $A$) is only possible in

1Suppose first that $C \geq 0$ has finite rank. That is, $C = \sum_{k=0}^{n-1} \mu(k, C)p_k$, where $p_k$, $0 \leq k < n$, are pairwise orthogonal rank one projections. Set $A_k = A \otimes \mu(k, C)p_k$ and $B_k = B \otimes \mu(k, C)p_k$. It is immediate that $B_k \ll A_k$ for $0 \leq k < n$. It follows from Lemma 2.3 in [4] that $\sum_{k=0}^{n-1} B_k \ll \sum_{k=0}^{n-1} A_k$ or, equivalently, $B \otimes C \ll A \otimes C$. For an arbitrary $C$, the assertion follows by approximation.

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the presence of an extra quantum state (in that case, $C$), although the latter is not consumed in the process. It is argued in \[7\] that this phenomenon can be used to improve the efficiency of entanglement concentration procedures.

In the following we restrict ourselves to $A, B$ being positive elements in $\bigcap_{p>1} \mathcal{L}_p$ ($\mathcal{L}_p$ denoting the Schatten–von Neumann ideal) and compare the following statements:

1. There exists a nonzero $C \in \mathcal{L}_1$ such that $B \otimes C \prec \prec A \otimes C$.
2. For every $p > 1$, we have $\text{Tr}(B^p) \leq \text{Tr}(A^p)$.

One checks that 1 implies 2. This follows from the monotonicity of $A \mapsto \text{Tr}(A^p)$ with respect to submajorization and from the formula

$$\text{Tr}(S \otimes T) = \text{Tr}(S) \cdot \text{Tr}(T), \quad S, T \in \mathcal{L}_1.$$ 

There is some hope to reverse the implication 1 $\Rightarrow$ 2 if we allow closure of the set

$$\{B : \exists C \in \mathcal{L}_1 \text{ such that } B \otimes C \prec \prec A \otimes C\}$$

with respect to some topology (for the finite-dimensional case, see \[1,9,15\]).

To explain why some closure is needed, we give an example of a pair $A,B$ of positive operators satisfying (ii) but not (i). Consider positive operators $\mu_k$ with $\mu_k(A) \neq \mu_k(B)$ and such that $\text{Tr}(B^p) \leq \text{Tr}(A^p)$ for $p \in (1, \infty)$, while $\text{Tr}(B^p_0) = \text{Tr}(A^p_0)$ for some $p_0 \in (1,\infty)$ (such an example exists among finite rank operators). Note that the norm in $\mathcal{L}_{p_0}$ is strictly monotone with respect to submajorization (see Proposition 2.1 in \[3\]). That is, if $K \in \mathcal{L}_{p_0}(H)$ and if $L \ll K$, then either $\mu(L) = \mu(K)$ or $\|L\|_{p_0} < \|K\|_{p_0}$. Suppose that (i) holds, i.e. that $B \otimes C \prec \prec A \otimes C$ for some nonzero $C \in \mathcal{L}_1$ (that is, no closure is taken). We then have $\text{Tr}((B \otimes C)^{p_0}) = \text{Tr}((A \otimes C)^{p_0})$ and, by strict monotonicity, $\mu_k(B \otimes C) = \mu_k(A \otimes C)$ for all $k \geq 0$. Now, taking into account that the sequences $\mu_k(B \otimes C)$ and $\mu_k(A \otimes C)$ coincide with decreasing rearrangements of sequences $\mu_k(B \otimes C) \otimes \mu_k(C)$ and $\mu_k(A) \otimes \mu_k(C)$ respectively, we infer that $\mu_k(A) = \mu_k(B)$.

As we shall see, the choice of the topology plays a crucial role. Prior to stating the precise question, we recall a few definitions and relevant facts.

There is a remarkable correspondence between sequence spaces and two-sided ideals in $\mathcal{L}(H)$ due to J.W. Calkin \[2\]. Recall that a linear subspace $\mathcal{J}$ in $\mathcal{L}(H)$ is a two-sided ideal if $X \in \mathcal{J}$ and $Y \in \mathcal{L}(H)$ imply $XY, YX \in \mathcal{J}$. Every nontrivial ideal necessarily consists of compact operators. A Calkin space $\mathcal{J}$ is a subspace of $\mathcal{C}_0$ (the space of all vanishing sequences) such that $x \in \mathcal{J}$ and $\mu(x) \leq \mu(x)$ imply $y \in \mathcal{J}$, where $\mu(x)$ is the decreasing rearrangement of the sequence $|x|$. The Calkin correspondence may be explained as follows. If $\mathcal{J}$ is a Calkin space, then associate to it the subset $\mathcal{J} \subset \mathcal{L}(H)$,

$$\mathcal{J} := \{X \in \mathcal{C}_0 : \mu(X) \in \mathcal{J}\}.$$ 

Conversely, if $\mathcal{J}$ is a two-sided ideal, then associate to it the sequence space

$$\mathcal{J} := \{x \in \mathcal{C}_0 : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{J}\}.$$ 

For the proof of the following theorem we refer to Calkin’s original paper, \[2\], and to B. Simon’s book, \[13\] Theorem 2.5.

\[2\]Here is an example of such a couple. Let $p_0$ be a rank 4 projection and let $p_1$ be a rank 1 projection orthogonal to $p_0$. Set $A = 2^{-\frac{2}{p}} p_0 + 2^\frac{2}{p} p_1$ and $B = p_0$. It is immediate that $\|A\|_p^p = 2^{2-p} + 2^p$ and $\|B\|_p^p = 4$ for every $p > 0$. Thus, $\|B\|_p \leq \|A\|_p$ for every $p > 0$ and $\|B\|_2 = \|A\|_2$. 

Note that the norm in
Theorem 1 (Calkin correspondence). The correspondence \( J \leftrightarrow \mathcal{J} \) is an inclusion lattice preserving bijection between Calkin spaces and two-sided ideals in \( \mathcal{L}(H) \).

In the recent papers [8, 14] this correspondence has been specialised to quasi-normed symmetrically-normed ideals and quasi-normed symmetric sequence spaces [10]. We use the notation \( \| \cdot \|_{\infty} \) to denote the uniform norm on \( \mathcal{L}(H) \).

Definition 2. (i) An ideal \( \mathcal{E} \) in \( \mathcal{L}(H) \) is said to be symmetrically (quasi)-normed if it is equipped with a Banach (quasi)-norm \( \| \cdot \|_{\mathcal{E}} \) such that
\[
\|XY\|_{\mathcal{E}}, \|YX\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}\|Y\|_{\infty}, \quad X \in \mathcal{E}, Y \in \mathcal{L}(H).
\]

(ii) A Calkin space \( E \) is a symmetric sequence space if it is equipped with a Banach (quasi)-norm \( \| \cdot \|_{E} \) such that \( \|y\|_{E} \leq \|x\|_{E} \) for every \( x \in E \) and \( y \in c_{0} \) such that \( \mu(y) \leq \mu(x) \).

For convenience of the reader, we recall that a map \( \| \cdot \| \) from a linear space \( X \) to \( \mathbb{R} \) is a quasi-norm if for all \( x, y \in X \) and scalars \( \alpha \) the following properties hold:

(i) \( \|x\| \geq 0 \), and \( \|x\| = 0 \Leftrightarrow x = 0 \);
(ii) \( \|\alpha x\| = |\alpha|\|x\| \);
(iii) \( \|x + y\| \leq C(\|x\| + \|y\|) \) for some \( C \geq 1 \).

The couple \( (X, \| \cdot \|) \) is a quasi-normed space and the least constant \( C \) satisfying inequality (iii) above is called the modulus of concavity of the quasi-norm \( \| \cdot \| \) and denoted by \( C_{X} \). A complete quasi-normed space is called quasi-Banach.

It easily follows from Definition 2 that if \( (\mathcal{E}, \| \cdot \|_{\mathcal{E}}) \) is a quasi-Banach ideal and \( X \in \mathcal{E} \) and \( Y \in \mathcal{L}(H) \) are such that \( \mu(Y) \leq \mu(X) \), then \( Y \in \mathcal{E} \) and \( \|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}} \).

In particular, it is easy to see that if \( E \) is a Calkin space corresponding to \( \mathcal{E} \), then setting \( \|x\|_{E} := \|X\|_{\mathcal{E}} \) (where \( X \in \mathcal{E} \) is such that \( \mu(x) = \mu(X) \)) we obtain that \( (E, \| \cdot \|_{E}) \) is a quasi-Banach symmetric sequence space. The converse implication is much harder and follows from Theorem 8.11 in [8] and Theorem 4 in [14].

With these preliminaries out of the way, we are now in a position to formulate the main question.

Question 3. Let \( \mathcal{I} \) be a (quasi-)Banach ideal such that \( \mathcal{I} \subset \bigcap_{p>1} \mathcal{L}_{p} \). Let \( 0 \leq A \in \mathcal{I} \). Consider the sets
\[
\text{PM}(A, \mathcal{I}) = \left\{ 0 \leq B \in \mathcal{I} : \text{Tr}(B^{p}) \leq \text{Tr}(A^{p}) \quad \forall p > 1 \right\},
\]
\[
\text{Catal}(A, \mathcal{I}) = \left\{ 0 \leq B \in \mathcal{I} : \exists 0 \leq C \in \mathcal{L}_{1} : C \neq 0, \quad B \otimes C \prec \prec A \otimes C \right\}.
\]

Let also \( \overline{\text{Catal}}(A, \mathcal{I}) \) denote the closure of \( \text{Catal}(A, \mathcal{I}) \) with respect to the quasi-norm of \( \mathcal{I} \). Is it true that \( \text{PM}(A, \mathcal{I}) = \overline{\text{Catal}}(A, \mathcal{I}) \)?

Note that \( \text{PM}(A, \mathcal{I}) \) is a closed subset in \( \mathcal{I} \). Indeed, let \( B_{n} \in \text{PM}(A, \mathcal{I}) \) and let \( B_{n} \to B \) in \( \mathcal{I} \) as \( n \to \infty \). Observe that it follows from Definition 2 that \( \mathcal{I} \) is continuously embedded into \( \mathcal{L}(H) \), and therefore it follows from the Closed Graph Theorem that for every fixed \( p > 1 \), the identity embedding \( \mathcal{I} \subset \mathcal{L}_{p} \) is continuous;

\[\text{We have to show that } \|A\|_{\infty} \leq \text{const.} \|A\|_{\mathcal{I}} \text{ for every } A \in \mathcal{I}. \text{ Without loss of generality, } A \geq 0. \text{ Set } p = E_{A}(\|A\|_{\infty}) \text{ (the spectral projection corresponding to the one-point set } \{\|A\|_{\infty}\} \text{ is nonzero since } A \text{ is compact) and let } q \leq p \text{ be a rank one projection. Clearly, } qA = q \cdot \|A\|_{\infty}p = \|A\|_{\infty}q \text{ and, similarly, } Aq = \|A\|_{\infty}q. \text{ Thus, } A \text{ commutes with } q \text{ and } A \geq qAq = \|A\|_{\infty}q. \text{ Therefore, } \|A\|_{\infty} \geq \|A\|_{\infty}q |x| \text{. Since all rank one projections are unitarily equivalent, it follows that they have the same norm. This proves the assertion.}\]
in particular, there exists a constant $c(p,\mathcal{I})$ such that $\|C\|_p \le c(p,\mathcal{I})\|C\|_{\mathcal{I}}, C \in \mathcal{I}$. Thus,
\[ \|B_n\|_p - \|B\|_p \le \|B - B_n\|_p \le c(p,\mathcal{I})\|B - B_n\|_{\mathcal{I}} \to 0. \]
Hence,
\[ \text{Tr}(B^p) = \lim_{n \to \infty} \text{Tr}(B_n^p) \le \text{Tr}(A^p), \quad p > 1. \]
We also have that $\text{Catal}(A,\mathcal{I}) \subset \text{PM}(A,\mathcal{I})$. Indeed, if $B \otimes C \ll A \otimes C$, then
\[ \text{Tr}(B^p) = \frac{\text{Tr}((B \otimes C)^p)}{\text{Tr}(C^p)} \le \frac{\text{Tr}((A \otimes C)^p)}{\text{Tr}(C^p)} = \text{Tr}(A^p), \quad p > 1. \]
Since $\text{PM}(A,\mathcal{I})$ is closed, it follows that the inclusion $\text{Catal}(A,\mathcal{I}) \subset \text{PM}(A,\mathcal{I})$ always holds.

In this paper, we show that the answer to Question 3 is positive when $\mathcal{I} = \mathcal{L}_1$ and negative when $\mathcal{I} = \mathcal{L}_{1,\infty}$. Recall that $\mathcal{L}_{1,\infty}$ is the principal ideal generated by the element $A_0 = \text{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \cdots \})$. Equivalently,
\[ \mathcal{L}_{1,\infty} = \{ A \in \mathcal{C}_0 : \sup_{k \ge 0} (k+1)\mu(k, A) < +\infty \}. \]
It becomes a quasi-Banach space (see e.g. [8,14]) when equipped with the quasi-norm
\[ \|A\|_{1,\infty} = \sup_{k \ge 0} (k+1)\mu(k, A), \quad A \in \mathcal{L}_{1,\infty}. \]
Here are our main results. We leave open the question of giving a complete description of the set $\text{Catal}(A,\mathcal{L}_{1,\infty})$.

**Theorem 4.** For every $0 \le A \in \mathcal{L}_1$, the sets $\text{PM}(A,\mathcal{L}_1)$ and $\text{Catal}(A,\mathcal{L}_1)$ coincide.

**Theorem 5.** There exists $0 \le A \in \mathcal{L}_{1,\infty}$ such that the set $\text{PM}(A,\mathcal{L}_{1,\infty})$ strictly contains the set $\text{Catal}(A,\mathcal{L}_{1,\infty})$.

It is actually simple to deduce Theorem 4 from the finite-dimensional considerations from [1], as we explain in Section 2. This is in sharp contrast with Theorem 5 whose proof is infinite-dimensional in its nature and uses crucially fine properties of Dixmier traces, which we introduce in Section 3. The heart of the argument behind Theorem 5 appears in Section 4 where we relegate some needed computations to Section 5.

2. The case of $\mathcal{L}_1$

We derive Theorem 4 from the following result which appears in [15] (see also Lemma 2 in [1]).

**Lemma 6.** Let $A, B$ be positive finite rank operators. Assume that for every $1 \le p \le +\infty$, we have the strict inequality $\|B\|_p < \|A\|_p$. Then there exists a nonzero finite rank operator $C$ such that $B \otimes C \ll A \otimes C$.

**Proof of Theorem 4.** Let us show the nontrivial inclusion, i.e. that every $B \in \text{PM}(A,\mathcal{L}_1)$ belongs to $\text{Catal}(A,\mathcal{L}_1)$.

Let $p_k, k \ge 0$, be a rank one eigenprojection of the operator $A$ which corresponds to the eigenvalue $\mu(k, A)$. Similarly, let $q_k, k \ge 0$, be a rank one eigenprojection of the operator $B$ which corresponds to the eigenvalue $\mu(k, B)$. We have
\[ A = \sum_{k=0}^{\infty} \mu(k, A)p_k, \quad B = \sum_{k=0}^{\infty} \mu(k, B)q_k. \]
Without loss of generality, \( \mu(0, A) = 1 \). It follows that
\[
(1 - (1 - \varepsilon)^p) \text{Tr}(A^p) \geq (1 - (1 - \varepsilon)^p)\mu(0, A)^p = 1 - (1 - \varepsilon)^p \geq \varepsilon^p.
\]
The latter readily implies
\[
\text{Tr}(A^p) - \varepsilon^p \geq (1 - \varepsilon)^p \text{Tr}(A^p), \quad p \geq 1, \quad \varepsilon \in (0, 1).
\]
Now, fix \( \varepsilon \in (0, 1) \) and select \( n \) such that
\[
\sum_{k=n}^{\infty} \mu(k, A) < \varepsilon, \quad \sum_{k=n}^{\infty} \mu(k, B) < \varepsilon.
\]
Set
\[
A_n = \sum_{k=0}^{n-1} \mu(k, A)p_k, \quad B_n = \sum_{k=0}^{n-1} \mu(k, B)q_k.
\]
It is clear that
\[
\text{Tr}(A_n^p) = \text{Tr}(A^p) - \sum_{k=n}^{\infty} \mu(k, A)^p \geq \text{Tr}(A^p) - (\sum_{k=n}^{\infty} \mu(k, A))^p
\]
\[
> \text{Tr}(A^p) - \varepsilon^p \geq (1 - \varepsilon)^p \text{Tr}(A^p), \quad p \geq 1.
\]
Therefore,
\[
(1 - \varepsilon)^p \text{Tr}(B_n^p) \leq (1 - \varepsilon)^p \text{Tr}(B^p) \leq (1 - \varepsilon)^p \text{Tr}(A^p) < \text{Tr}(A_n^p), \quad p \geq 1.
\]
Since both \( A_n \) and \( B_n \) are finite rank operators, it follows from Lemma 6 and the first footnote that there exists a finite rank operator \( C_n \) such that
\[
(1 - \varepsilon)B_n \preceq C_n \preceq A_n \otimes C_n \preceq A \otimes C_n.
\]
In particular, we have that \((1 - \varepsilon)B_n \in \text{Catal}(A, \mathcal{L}_1)\). Observing that \(\|B - B_n\|_1 \leq 1\), we further obtain
\[
\|B - (1 - \varepsilon)B_n\|_1 \leq \varepsilon\|B\|_1 + (1 - \varepsilon)\|B - B_n\|_1 \leq \varepsilon(\|B\|_1 + 1).
\]
Since \( \varepsilon \) is arbitrarily small, it follows that \(B \in \overline{\text{Catal}}(A, \mathcal{L}_1)\). \(\square\)

3. Dixmier traces

The crucial ingredient in the proof is the notion of a Dixmier trace on \( \mathcal{L}_{1,\infty} \). Let \( \ell_{\infty} \) stand for the Banach space of all bounded sequences \( x = (x_n)_{n \geq 0} \) equipped with the usual norm \( \|x\|_{\infty} := \sup_{n \geq 0} |x_n| \). A generalized limit is any positive linear functional on \( \ell_{\infty} \) which equals the ordinary limit on the subspace \( c \) of all convergent sequences.

**Remark 7.** Given a sequence \((x_n)_{n \geq 0} \in \ell_{\infty}\), there is a generalized limit \( \omega \) such that \( \omega((x_n)) = \limsup_{n \to \infty} x_n \).

**Proof.** Fix \( x = (x_n) \in \ell_{\infty} \) and let the sequence \((n_k)_{k \geq 0} \) be such that \( \lim_{k \to \infty} x_{n_k} = \limsup_{n \to \infty} x_n \). Consider the set of functionals \((\varphi_{n_k})_{k \geq 0} \) on \( \ell_{\infty} \) defined by \( \varphi_{n_k}(y) := y_{n_k}, y = (y_n) \in \ell_{\infty}, k \geq 0 \). The set \((\varphi_{n_k})_{k \geq 0} \) belongs to the unit ball \( B \) of the Banach dual \( \ell_{1,\infty} \). The set \( B \) is compact in the weak* topology \( \sigma(\ell_{1,\infty}, \ell_{\infty}) \), and therefore the set \((\varphi_{n_k})_{k \geq 0} \) possesses a cluster point \( \omega \in \ell_{1,\infty} \) in that topology. The fact that \( \omega \) is a generalized limit on \( \ell_{\infty} \) such that \( \omega((x_n)) = \limsup_{n \to \infty} x_n \) follows immediately from the definition of the weak* topology. \(\square\)

The Dixmier traces are defined as follows.
Theorem 8. Let \( \omega \) be a generalized limit. The mapping \( \text{Tr}_\omega : \mathcal{L}^+_{1,\infty} \to \mathbb{R}^+ \) defined for \( 0 \leq A \in \mathcal{L}_{1,\infty} \) by setting
\[
\text{Tr}_\omega(A) := \omega\left(\frac{1}{\log(N + 2)} \sum_{k=0}^{N} \mu(k, A)\right)_{N=0}^{\infty}
\]
is additive and, therefore, extends to a positive unitarily invariant linear functional on \( \mathcal{L}_{1,\infty} \) called a Dixmier trace.

Note that the positivity of generalized limits implies that
\[
|\text{Tr}_\omega(A)| \leq \|A\|_{1,\infty}
\]
for every Dixmier trace \( \text{Tr}_\omega \) and \( A \in \mathcal{L}_{1,\infty} \).

Let us comment on how additivity is proved in Theorem 8. This is usually achieved under the extra assumption that \( \omega \) is scale invariant (see Theorem 1.3.1 in [11]), i.e. that \( \omega \circ \sigma_k = \omega \) for all positive integers \( k \), where \( \sigma_k : \ell_\infty \to \ell_\infty \) is defined as
\[
\sigma_k(x_1, x_2, \ldots, x_n, \ldots) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n, \ldots)_{k \text{ times}}.
\]
Under this extra assumption the map \( \text{Tr}_\omega \) is actually additive on the larger ideal \( \mathcal{M}_{1,\infty} \) (we refer to [11, Example 1.2.9] for the definition of the latter ideal and to [11, Section 6.8] for historical background). In the form presented here, Theorem 8 follows from Theorem 17 in [12]. For the reader’s convenience we reproduce the argument here.

Proof of Theorem 8. Given \( A \in \mathcal{L}_{1,\infty} \), consider the sequence \( (x_N(A))_{N=0}^{\infty} \) defined by
\[
x_N(A) = \frac{1}{\log(N + 2)} \sum_{j=0}^{N} \mu(j, A).
\]
It is not hard to check that for all positive integers \( k \),
\[
\lim_{N \to \infty} x_N(A) - x_{kN}(A) = 0.
\]
Let \( E \subset \ell_\infty \) be the subspace
\[
E = \text{span} \left\{ \sigma_k(\{x_N(A)\}) : k \geq 1, A \in \mathcal{L}_{1,\infty} \right\}.
\]
It follows from (2) that the equation \( \omega \circ \sigma_k(x) = \omega(x) \) is satisfied for \( x \in E \). By a version of the Hahn–Banach theorem (see [6, Theorem 3.3.1]), the linear functional \( \omega|_E \) can be extended to a generalized limit \( \omega' : \ell_\infty \to \mathbb{R} \) which is scale-invariant.

The usual argument ([11], Theorem 1.3.1) implies that \( \text{Tr}_{\omega'} \) (which coincides with \( \text{Tr}_\omega \) on \( \mathcal{L}_{1,\infty} \)) is additive on \( \mathcal{L}_{1,\infty} \). \( \square \)

We also need a version of Fubini’s theorem for Dixmier traces.

Theorem 9. For every \( A \in \mathcal{L}_{1,\infty} \) and for every \( C \in \mathcal{L}_1 \), we have \( A \otimes C \in \mathcal{L}_{1,\infty} \) and
\[
\|A \otimes C\|_{1,\infty} \leq \|A\|_{1,\infty} \|C\|_1.
\]
Moreover, for every Dixmier trace \( \text{Tr}_\omega \) on \( \mathcal{L}_{1,\infty} \), we have
\[
\text{Tr}_\omega(A \otimes C) = \text{Tr}_\omega(A) \text{Tr}(C).
\]
Proof. We may assume $\|A\|_{1,\infty} = 1$. Recall that $A_0 = \text{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \cdots\})$. We have for all $k \geq 0$,

$$
\mu(k, A \otimes C) \leq \frac{1}{k+1} \sum_{j=0}^{k} \mu(j, C) \leq \frac{\|C\|_1}{k+1},
$$

where the second inequality follows from Proposition 3.14 in [5]. This proves (3).

Observe that both sides of (4) depend linearly on $A$ and $C$ (thanks to Theorem 8). Thus, we can assume without loss of generality that $A, C \geq 0$. When $C$ is a rank one projection, (4) follows from Theorem 8 since in that case $\mu(k, A \otimes C) = \mu(k, A)$ for all $k \geq 0$. Again appealing to linearity of Dixmier traces, we infer the result for the finite rank operator $C$ and when $A \in L_{1,\infty}$ is arbitrary. Now consider a general $C \in L_1$ and let $(C_n)$ be a sequence of finite rank operators such that $\|C - C_n\|_1 \to 0$. We have

$$
|\text{Tr}_\omega(A \otimes C_n) - \text{Tr}_\omega(A \otimes C)| \leq \|A \otimes (C - C_n)\|_{1,\infty} \leq \|A\|_{1,\infty}\|C - C_n\|_1,
$$

and this quantity tends to 0 as $n$ goes to infinity. Consequently,

$$
\text{Tr}_\omega(A \otimes C) = \lim_{n \to \infty} \text{Tr}_\omega(A \otimes C_n) = \lim_{n \to \infty} \text{Tr}_\omega(A)\text{Tr}(C_n) = \text{Tr}_\omega(A)\text{Tr}(C). \quad \square
$$

As a corollary, we obtain that Dixmier traces give necessary conditions for catalysis.

Corollary 10. Let $0 \leq A \in L_{1,\infty}$ and $0 \leq B \in \text{Catal}(A, L_{1,\infty})$. Then for every Dixmier trace $\text{Tr}_\omega$, one has

$$
(5) \quad \text{Tr}_\omega(B) \leq \text{Tr}_\omega(A).
$$

Proof. We know from (11) that Dixmier traces are continuous on $L_{1,\infty}$, and therefore we may assume that $B \in \text{Catal}(A, L_{1,\infty})$. By definition of the latter set (see Question 8), there exists a nonzero positive $C$ in $L_1$ with the property that $B \otimes C \ll A \otimes C$. Combining the definition of Hardy-Littlewood submajorization $\ll$ and the positivity from the definition of a Dixmier trace $\text{Tr}_\omega$ (see Theorem 8), we infer that the inequality $\text{Tr}_\omega(B \otimes C) \leq \text{Tr}_\omega(A \otimes C)$ holds for every Dixmier trace $\text{Tr}_\omega$. Inequality (5) now follows from (11) and from the fact that $\text{Tr}(C) > 0$. \quad \square

4. The Case of $L_{1,\infty}$: The Main Argument

Here is the main technical result used in the proof of Theorem 5. In the lemma below, we tacitly identify a sequence in the space $\ell_\infty$ with the corresponding diagonal operator. For $I \subset \mathbb{N}$, we note by $\chi_I$ the sequence defined by $\chi_I(n) = 1$ if $n \in I$ and $\chi_I(n) = 0$ otherwise.

Lemma 11. Let $I$ be the subset of $\mathbb{N}$ defined as

$$
I = \bigcup_{n \geq 0} [2^{2n}, 2^{2n+1}).
$$

Consider the operator

$$
B = \bigoplus_{m \in I} 2^{-m} \chi_{(0, 2^m)}.
$$

\footnote{In the subsequent formulas, the symbol $\oplus$ stands for the direct sum of operators.}
Then $B \in \mathcal{L}_{1,\infty}$. Moreover,

$$\limsup_{s \to 0^+} s \text{Tr}(B^{1+s}) \leq \frac{5}{9 \log 2} < \frac{2}{3 \log 2} \leq \limsup_{N \to \infty} \frac{1}{\log N} \sum_{k=0}^{N} \mu(k, B).$$

Let us postpone the proof of Lemma 11 and show how it implies the result stated in Theorem 5. Consider $B$ as in Lemma 11 and fix a number $\alpha$ such that $\frac{5}{9 \log 2} < \alpha < \frac{2}{3 \log 2}$. Recall that $A_0 = \text{diag}(\{1, \frac{1}{2}, \frac{1}{3}, \cdots\})$. Since

$$\lim_{s \to 0^+} s \text{Tr}((\alpha A_0)^{1+s}) = \lim_{s \to 0^+} s \zeta(s) \alpha^{1+s} = \alpha,$$

it follows from (6) that there exists $\delta > 0$ such that the inequality

$$\text{Tr}(B^{1+s}) \leq \text{Tr}((\alpha A_0)^{1+s})$$

holds whenever $0 < s \leq \delta$. Define the operator $A = \alpha A_0 \oplus \|B\|_{1+\delta}^p$, where $p$ is a rank one projection. We claim that $B \in \text{PM}(A, \mathcal{L}_{1,\infty})$: indeed, for $s > \delta$ we may write

$$\text{Tr}(B^{1+s}) = \text{Tr}((B^{1+\delta})^{\frac{s}{1+\delta}}) \leq (\text{Tr}(B^{1+\delta}))^{\frac{s}{1+\delta}} = \|B\|_{1+\delta}^s \leq \text{Tr}(A^{1+s}),$$

while for $0 < s \leq \delta$ the inequality $\text{Tr}(B^{1+s}) \leq \text{Tr}(A^{1+s})$ follows immediately from (7).

We now assume by contradiction that $B$ belongs to the set $\text{Catal}(A, \mathcal{L}_{1,\infty})$. We know from Corollary 10 that $\text{Tr}_\omega(B) \leq \text{Tr}_\omega(A)$ for every Dixmier trace $\text{Tr}_\omega$. Observing that any such trace vanishes on finite rank operators, we see that the value $\text{Tr}_\omega(A)$ coincides with $\text{Tr}_\omega(\alpha A_0)$ and hence is equal to $\alpha$ for every Dixmier trace $\text{Tr}_\omega$ (see the definition given in Theorem 8). On the other hand, we may choose a generalized limit $\omega$ such that

$$\text{Tr}_\omega(B) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{k=0}^{N} \mu(k, B)$$

and obtain from (6) that $\frac{2}{3 \log 2} \leq \alpha$, a contradiction.

We note that the Dixmier trace considered in the proof does not behave in a monotone way with respect to trace of powers: we have $\text{Tr}(B^p) \leq \text{Tr}(A^p)$ for every $p > 1$, but $\text{Tr}_\omega(B) > \text{Tr}_\omega(A)$.

5. PROOF OF LEMMA 11

Let $I$ and $B$ be as defined in Lemma 11 and denote by $E_B$ the spectral measure of $B$. First, note that for every integer $m$,

$$\text{Tr}(E_B(2^{-m}, \infty)) \leq \sum_{l<m} 2^l \leq 2^m.$$

Hence, for every positive integer $n$, writing $2^m \leq n < 2^{m+1}$, we infer

$$\text{Tr}(E_B(\frac{1}{n}, \infty)) \leq \text{Tr}(E_B(2^{-m-1}, \infty)) \leq 2^{m+1} \leq 2n.$$

Recall also (e.g., see [11, Chapter 2, Section 2.3]) that $\mu(k, B), k \geq 0$, can be computed via the formula

$$\mu(k, B) = \inf \{s \geq 0 : \text{Tr}(E_B(s, \infty)) \leq k\}.$$
Hence, it follows from (9) that \( \mu(k, B) \leq 2^{k+1} \) for every \( k \geq 0 \) and, in particular, \( B \in \mathcal{L}_{1,\infty} \). We now prove the right inequality in (9). For a given \( n \), let \( N = \text{Tr}(E_B(2^{-2^{2n+1}}, \infty)) \). We know from (8) that \( N \leq 2^{2n+1} \cdot 2^{2n+1} - \frac{1}{3} \).

Hence, for \( N \) as above, we have

\[
\frac{1}{\log(N)} \sum_{k=0}^{N-1} \mu(k, B) \geq \frac{1}{\log(2^{2n+1})} \cdot \left( \frac{2}{3} \cdot 2^{2n+1} - \frac{1}{3} \right) = \frac{2}{3} \cdot 2^{2n+1} + o(1),
\]

as needed.

We now focus on the left inequality in (6) and use the following summation formula, whose proof we postpone. For a given sequence \( (x_n) \in \ell_\infty \) and for a given \( s > 0 \), we have that

\[
(10) \quad \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \sum_{l=0}^{k} x_l \right) 2^{-ms} = (1 - 2^{-s})^{-2} \sum_{l=0}^{\infty} x_l 2^{-ls}.
\]

Note that \( \text{Tr}(B^{1+s}) = \sum_{m \in I} 2^{-ms} = \sum_{m \geq 0} \chi_I(m) 2^{-ms} \) (here, \( \chi_I(0) = 0 \)). Applying (10) to \( x = \chi_I \), we obtain, for every \( M > 0 \),

\[
\limsup_{s \to 0^+} s \sum_{l \geq 0} \chi_I(l) 2^{-ls} = \limsup_{s \to 0^+} s (1 - 2^{-s})^{-2} \sum_{m \geq 0} \left( \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l) \right) 2^{-ms}
\]

\[
= \limsup_{s \to 0^+} s (1 - 2^{-s})^{-2} \sum_{m \geq M} \left( \frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l) \right) \cdot (m+1)^2 2^{-ms}
\]

\[
\leq \left( \sup_{m \geq M} \frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l) \right) \cdot \left( \limsup_{s \to 0^+} s (1 - 2^{-s})^{-2} \sum_{m \geq M} (m+1)^2 2^{-ms} \right).
\]

Passing \( M \to \infty \), we infer that

\[
\limsup_{s \to 0^+} s \sum_{l \geq 0} \chi_I(l) 2^{-ls} \leq C \limsup_{s \to 0^+} s (1 - 2^{-s})^{-2} \sum_{m \geq 0} (m+1)^2 2^{-ms},
\]

where

\[
C := \limsup_{m \to \infty} \frac{1}{(m+1)^2} \sum_{k=0}^{m} \sum_{l=0}^{k} \chi_I(l).
\]

An elementary computation gives

\[
\sum_{m=0}^{\infty} (m+1)^2 2^{-ms} = \frac{1 + 2^{-s}}{(1 - 2^{-s})^3}.
\]

It follows that

\[
\limsup_{s \to 0^+} s \text{Tr}(B^{1+s}) \leq \frac{2C}{\log 2}.
\]
It remains to show that $C \leq 5/18$ (we actually show $C = 5/18$). To that end, we think of $\chi_I$ as an element of $L_\infty(0, \infty)$ and define $z \in L_\infty(0, \infty)$ by setting $z = \chi_{\bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1})}$. Observe that $\chi_I \leq z$. Therefore,

$$C \leq \limsup_{t \to \infty} \frac{1}{t^2} \int_0^t \int_0^s z(u) \, du \, ds.$$  

Since $z(4t) = z(t)$ for every $t > 0$, applying Fubini’s theorem we have

$$C \leq \sup_{t \in (1, 4)} \frac{1}{t^2} \int_0^t z(u)(t-u) \, du.$$  

However,

$$\frac{1}{t^2} \int_0^t z(u)(t-u) \, du = \begin{cases} \frac{1}{2} - \frac{2}{3t} + \frac{2}{5t^2}, & 1 \leq t \leq 2, \\ \frac{4}{3t} - \frac{8}{5t^2}, & 2 \leq t \leq 4. \end{cases}$$

Hence, the latter supremum is, in fact, a maximum which is attained at $t = \frac{12}{5}$ and equal to $\frac{5}{18}$.

**Proof of (10).** Write

$$\sum_{m \geq 0} \left( \sum_{k \geq 0} \sum_{l=0}^k x_l \right) 2^{-ms} = \sum_{m \geq k \geq 0} \sum_{l=0}^k x_l \sum_{m\geq k \geq 0} \sum_{l=0}^k 2^{-ms} = \sum_{k=0}^\infty \sum_{l=0}^k x_l \sum_{m\geq k \geq 0} \sum_{l=0}^k 2^{-ms} = (1 - 2^{-s})^{-1} \left( \sum_{k=0}^\infty \sum_{l=0}^k x_l \right) 2^{-ks} = (1 - 2^{-s})^{-1} \sum_{k \geq 0} x_l 2^{-ks} = (1 - 2^{-s})^{-2} \sum_{l=0}^\infty x_l 2^{-ls}. \quad \Box$$

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