Non(anti)commutative Superspace *

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ABSTRACT

We investigate the most general non(anti)commutative geometry in \( N = 1 \) four dimensional superspace, invariant under the classical (i.e. undeformed) supertranslation group. We find that a nontrivial non(anti)commutative superspace geometry compatible with supertranslations exists with non(anti)commutation parameters which may depend on the spinorial coordinates. The algebra is in general nonassociative. Imposing associativity introduces additional constraints which however allow for nontrivial commutation relations involving fermionic coordinates. We obtain explicitly the first three terms of a series expansion in the deformation parameter for a possible associative \( \star \)-product. We also consider the case of \( N = 2 \) euclidean superspace where the different conjugation relations among spinorial coordinates allow for a more general supergeometry.

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1 Introduction

During the past two years a clear connection between string theory and noncommutative geometry has emerged. In the presence of a constant magnetic Neveu-Schwarz field, the low energy dynamics of D3 brane excitations is described by noncommutative $\mathcal{N} = 4$ super Yang-Mills theory [1]. The result by Seiberg and Witten followed earlier work [2], where it was found that noncommutative geometry arises very naturally in the framework of M(atrix) theory. Apart from the string theory context, noncommutative field theories are interesting in their own right. This interest is motivated by many intriguing features of field theories on noncommutative spaces, like the UV/IR mixing [3] or the Morita equivalence [4] between Yang-Mills theories on noncommutative tori. Noncommutative field theories also play a role in solid state physics, e.g. noncommutative Chern Simons theory was recently proposed by Susskind [5] to provide a description of the fractional quantum Hall effect. Also in the physics of black holes, noncommutativity of spacetime naturally emerges from 't Hooft’s S-matrix ansatz [6] (Cf. also [7] for a recent review). The hope is that this may eventually lead to a better understanding of some of the puzzles of black hole physics, such as the information loss paradox.

Up to now, the concept of noncommutativity has been limited essentially to the bosonic coordinates, i.e. one has

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \quad (1.1)$$

where $\Theta^{\mu\nu}(x)$ is antisymmetric. In view of the fact that supersymmetry plays a fundamental role in string theory, it seems natural and compelling to ask what happens if we deform also the anticommutators between fermionic coordinates of superspace\footnote{Non-anticommutative structures in field theory and gravity have been studied in a different context in [8]. Furthermore, nonvanishing anticommutators of fermionic coordinates have been considered in [9] in the context of a possible fermionic substructure of spacetime.}, or the commutators between bosonic and fermionic coordinates. To investigate the most general deformations compatible with supersymmetry is the main purpose of this paper. First steps in this direction were undertaken in [10], where quantum deformations of the Poincaré supergroup were considered, and in [11], where it was shown that in general chiral superfields are not closed under star products that involve also deformations of fermionic coordinates. Here we will be mainly concerned with the conditions imposed on the possible deformations of superspace by requirements such as covariance under classical translations and supertranslations, Jacobi identities, associativity of the star product, and closure of chiral superfields under the star product.

The paper is organized as follows: In Section 2 we determine the most general deformation of four dimensional, $N = 1$ Minkowski superspace that is covariant
under undeformed supertranslations, and discuss the deformation of the supersymmetry algebra which follows. The result we obtain is a supersymmetric nontrivial extension of the “constant \( \Theta \)” noncommutative bosonic geometry studied in a string theory context [1]. While our geometries remain flat in the bosonic sector, they are curved along the fermionic directions. In Section 3 we study the restriction imposed on this general structure by the Jacobi identities, i.e. by the requirement to have a super Poisson structure on superspace. We will see that these additional constraints necessarily impose the spinorial coordinates to be anticommuting, but allow for possible nontrivial commutation relations among bosonic and fermionic coordinates. In this case the standard supersymmetry algebra is restored. It is then shown that the violation of the Jacobi identities, implied by the deformation of the fermionic coordinates, is equivalent to the nonvanishing of a super three-form field strength. It might be interesting to study whether deformations of fermionic variables arise in superstring theory with backgrounds that involve nonvanishing super p–form field strengths. In the following Section we obtain the first three terms in a series expansion in the deformation parameter \( \hbar \) for a possible noncommutative product of superfields. Due to the violation of the Jacobi identities, this product will be in general nonassociative. In the cases where the Jacobi identities are satisfied, we show that this product is associative up to quadratic order. Finally, in the last Section we discuss possible non(anti)commutative deformations for superspaces with euclidean signature. In the simplest \( N = 2 \) case, we find that deformations involving nontrivial anticommutation relations among spinorial variables are in this case allowed by the request of consistency with supercovariance and associativity. We conclude with some final remarks.

2 Covariant non(anti)commutative geometry

We consider a four dimensional \( N = 1 \) Minkowski superspace. The set of superspace coordinates are \( Z^A = (x^{\alpha \dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \), where \( x^{\alpha \dot{\alpha}} \) are the four real bosonic coordinates and \( \theta^\alpha, \bar{\theta}^{\dot{\alpha}} \) two–component complex Weyl fermions. The conjugation rule \( \bar{\theta}^{\dot{\alpha}} = (\theta^\alpha)^\dagger \) follows from the requirement to have a four component Majorana fermion (we use conventions of Superspace [12]).

In the standard (anti)commutative superspace the algebra of the coordinates is

\[
\begin{align*}
\{ \theta^\alpha, \theta^\beta \} &= \{ \bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}} \} = \{ \theta^\alpha, \bar{\theta}^{\dot{\alpha}} \} = 0 \\
[x^{\alpha \dot{\alpha}}, \theta^\beta] &= [x^{\alpha \dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = 0 \\
[x^{\alpha \dot{\alpha}}, x^{\beta \dot{\beta}}] &= 0
\end{align*}
\]

and it is trivially covariant under the superpoincaré group. The subgroup of the classical (super)translations (spacetime translations and supersymmetry transfor-
mations)

\[ \begin{align*}
\theta'_{\alpha} &= \theta_{\alpha} + \epsilon_{\alpha} \\
\bar{\theta}'_{\dot{\alpha}} &= \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} \\
x'_{\alpha\dot{\alpha}} &= x_{\alpha\dot{\alpha}} + a_{\alpha\dot{\alpha}} - \frac{i}{2} \left( \epsilon_{\alpha} \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} \theta_{\alpha} \right)
\end{align*} \]  

(2.2)

is generated by two complex charges \( Q_{\alpha} \) (\( \bar{Q}_{\dot{\alpha}} = Q_{\dot{\alpha}}^{\dagger} \)) and the four–momentum \( P_{\alpha\dot{\alpha}} \) subjected to

\[ \{ Q_{\alpha}, Q_{\beta} \} = \{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0 \quad , \quad \{ Q_{\alpha}, \bar{Q}_{\dot{\alpha}} \} = P_{\alpha\dot{\alpha}} \]  

(2.3)

Representations of supersymmetry are given by superfields \( V(x_{\alpha\dot{\alpha}}, \theta_{\alpha}, \bar{\theta}_{\dot{\alpha}}) \) whose components are obtained by expanding \( V \) in powers of the spinorial coordinates. The set of superfields is closed under the standard product of functions. The product of two superfields is (anti)commutative,

\[ V \cdot W = (-1)^{\dim(V) \cdot \dim(W)} W \cdot V \quad \text{and associative,} \]  

\[ (K \cdot V) \cdot W = K \cdot (V \cdot W). \]  

In order to define a non(anti)commutative superspace, we consider the most general structure of the algebra for a set of four bosonic real coordinates and a complex two–component Weyl spinor with \( (\theta_{\alpha})^\dagger = \bar{\theta}_{\dot{\alpha}} \)

\[ \begin{align*}
\{ \theta^\alpha, \theta^\beta \} &= \mathcal{A}^{\alpha\beta}(x, \theta, \bar{\theta}) \quad , \quad \{ \bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}} \} = \mathcal{A}^{\dot{\alpha}\dot{\beta}}(x, \theta, \bar{\theta}) \\
\{ \theta^\alpha, \bar{\theta}^{\dot{\alpha}} \} &= \mathcal{B}^{\alpha\dot{\alpha}}(x, \theta, \bar{\theta}) \\
[x^\alpha, \theta^\beta] &= i \mathcal{C}^{\alpha\beta}(x, \theta, \bar{\theta}) \quad , \quad [x^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = i \mathcal{C}^{\dot{\alpha}\dot{\beta}}(x, \theta, \bar{\theta}) \\
[x^\alpha, x^{\dot{\alpha}}] &= i \mathcal{D}^{ab}(x, \theta, \bar{\theta})
\end{align*} \]  

(2.4)

Here, \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) are local functions of the superspace variables and we have defined \( \mathcal{A}^{\dot{\alpha}\dot{\beta}} \equiv (\mathcal{A}^{\alpha\beta})^\dagger, \mathcal{C}^{\dot{\alpha}\dot{\beta}} \equiv (\mathcal{C}^{\alpha\beta})^\dagger \). From the conjugation rules for the coordinates it follows also \( (\mathcal{B}^{\alpha\dot{\alpha}})^\dagger = \mathcal{B}^{\dot{\alpha}\alpha} \) and \( (\mathcal{D}^{ab})^\dagger = \mathcal{D}^{ba} \).

To implement (2.4) to be the algebra of the coordinates of a non(anti)commutative \( N = 1 \) superspace we require its covariance under the group of space translations and supertranslations (2.2). The covariance under observer–Lorentz transformations is manifest in (2.4), while we do not require covariance under particle–Lorentz transformations which is in general broken in a noncommutative geometry (for a discussion of the two different Lorentz transformations see [13]). We restrict our analysis to the case of an undeformed group where the parameters \( a_{\alpha\dot{\alpha}}, \epsilon_{\alpha} \) and \( \bar{\epsilon}_{\dot{\alpha}} \) in (2.2) are kept (anti)commuting \( \parallel \).

\[ ^1\text{More general constructions of non(anti)commutative geometries in grassmannian spaces have been considered, where also the algebra of the parameters is deformed [10].} \]
Imposing covariance amounts to ask the functional dependence of the $A, B, C, D$ in (2.4) to be the same at any point of the supermanifold. To work out explicitly the constraints which follow, we perform a (super)translation (2.2) on the coordinates and compute the algebra of the new coordinates in terms of the old ones. We find that the functions appearing in (2.4) are constrained by the following set of independent equations

\[
A^{\alpha \beta}(x', \theta', \bar{\theta}') = A^{\alpha \beta}(x, \theta, \bar{\theta}) \quad \text{and} \quad B^{\alpha \dot{\alpha}}(x', \theta', \bar{\theta}') = B^{\alpha \dot{\alpha}}(x, \theta, \bar{\theta}) \quad (2.5)
\]

\[
C^{\alpha \dot{\alpha}}(x', \theta', \bar{\theta}') = C^{\alpha \dot{\alpha}}(x, \theta, \bar{\theta}) - \frac{1}{2} \epsilon^\alpha B^{\beta \dot{\beta}}(x, \theta, \bar{\theta}) - \frac{1}{2} \bar{\epsilon}^{\dot{\alpha}} A^{\alpha \beta}(x, \theta, \bar{\theta}) \quad (2.6)
\]

\[
D^{\alpha \dot{\alpha} \dot{\beta} \dot{\beta}}(x', \theta', \bar{\theta}') = D^{\alpha \dot{\alpha} \dot{\beta} \dot{\beta}}(x, \theta, \bar{\theta}) - \frac{i}{2} \left( \epsilon^\beta C^{\alpha \dot{\alpha}}(x, \theta, \bar{\theta}) + \bar{\epsilon}^{\dot{\beta}} C^{\alpha \dot{\alpha}}(x, \theta, \bar{\theta}) - \epsilon^\alpha \bar{C}^{\beta \dot{\beta} \dot{\alpha}}(x, \theta, \bar{\theta}) - \bar{\epsilon}^{\dot{\alpha}} C^{\beta \dot{\beta} \dot{\alpha}}(x, \theta, \bar{\theta}) \right)
\]

\[
- \frac{i}{4} \left( \epsilon^\alpha \bar{A}^{\dot{\alpha} \dot{\beta}}(x, \theta, \bar{\theta}) \epsilon^\beta + \epsilon^\alpha B^{\beta \dot{\beta}}(x, \theta, \bar{\theta}) \bar{\epsilon}^{\dot{\beta}} + \bar{\epsilon}^{\dot{\alpha}} B^{\beta \dot{\beta}}(x, \theta, \bar{\theta}) \epsilon^\beta + \bar{\epsilon}^{\dot{\alpha}} A^{\alpha \beta}(x, \theta, \bar{\theta}) \epsilon^\beta \right) \quad (2.7)
\]

Together with their hermitian conjugates.

Looking for the most general local solution brings to the following algebra for a non(anti)commutative geometry in Minkowski superspace consistent with (super)translations

\[
\begin{align*}
\{ \theta^\alpha, \theta^\beta \} &= A^{\alpha \beta}, \quad \{ \bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}} \} = \bar{A}^{\dot{\alpha} \dot{\beta}}, \quad \{ \theta^\alpha, \bar{\theta}^{\dot{\alpha}} \} = B^{\alpha \dot{\alpha}} \\
[x^{\alpha \dot{\alpha}}, \theta^\beta] &= i C^{\alpha \dot{\alpha} \beta}(\theta, \bar{\theta}) \\
[x^{\alpha \dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] &= i \bar{C}^{\alpha \dot{\alpha} \dot{\beta}}(\theta, \bar{\theta}) \\
[x^{\alpha \dot{\alpha}}, x^{\beta \dot{\beta}}] &= i D^{\alpha \dot{\alpha} \beta \dot{\beta}}(\theta, \bar{\theta}) \quad (2.8)
\end{align*}
\]

where

\[
C^{\alpha \dot{\alpha} \beta}(\theta, \bar{\theta}) = C^{\alpha \dot{\alpha} \beta} - \frac{1}{2} \theta^\alpha B^{\beta \dot{\beta}} - \frac{1}{2} \bar{\theta}^{\dot{\alpha}} A^{\alpha \beta}
\]

\[
D^{\alpha \dot{\alpha} \beta \dot{\beta}}(\theta, \bar{\theta}) = D^{\alpha \dot{\alpha} \beta \dot{\beta}} - \frac{i}{2} \left( \theta^\beta C^{\alpha \dot{\alpha} \beta} - \bar{\theta}^{\dot{\beta}} C^{\beta \dot{\beta} \dot{\alpha}} - \theta^\alpha C^{\beta \dot{\beta} \dot{\alpha}} + \bar{\theta}^{\dot{\beta}} C^{\alpha \dot{\alpha} \beta} \right) - \frac{i}{4} \left( \theta^\alpha \bar{A}^{\dot{\alpha} \dot{\beta}} \theta^\beta + \theta^\alpha B^{\beta \dot{\beta}} \bar{\theta}^{\dot{\beta}} + \bar{\theta}^{\dot{\beta}} B^{\alpha \beta} \theta^\beta + \bar{\theta}^{\dot{\beta}} A^{\alpha \beta} \bar{\theta}^{\dot{\beta}} \right) \quad (2.9)
\]

and $A, B, C$ and $D$ are constant functions.

We notice that, while covariance under spacetime translations necessarily requires the non(anti)commutation functions to be independent of the $x$ coordinates,
the covariance under supersymmetry is less restrictive and allows for a particular dependence on the spinorial coordinates.

On the algebra of smooth functions of the superspace variables we can formally define a graded bracket which reproduces the fundamental algebra (2.8) when applied to the coordinates. In the case of bosonic Minkowski spacetime, the noncommutative algebra (1.1) can be obtained by interpreting the l.h.s. of this relation as the Poisson bracket of classical commuting variables, where, for generic functions of spacetime, the Poisson bracket is defined as

\[ \{ f, g \} = i \Theta^{\mu \nu} \partial_\mu f \partial_\nu g. \]

Generalizing to Minkowski superspace, the graded bracket must be constructed as a bidifferential operator with respect to the superspace variables. Using covariant derivatives

\[ D_A \equiv (D_\alpha, \bar{D}_{\dot{\alpha}}, \partial_{\alpha \dot{\alpha}}), \]

for generic functions \( \Phi \) and \( \Psi \) of the superspace coordinates we define the bidifferential operator

\[ \{ \Phi, \Psi \} = \Phi \hat{D}_A P^{AB} \bar{D}_B \Psi \quad (2.10) \]

where

\[ P^{AB} \equiv \begin{pmatrix} P^{\alpha \beta} & P^{\alpha \dot{\beta}} & P^{\alpha \dot{\beta}} \\ P^{\dot{\alpha} \beta} & P^{\dot{\alpha} \dot{\beta}} & P^{\dot{\alpha} \dot{\beta}} \\ P^{\alpha \dot{\beta}} & P^{\dot{\alpha} \beta} & P^{\dot{\alpha} \dot{\beta}} \end{pmatrix} = \begin{pmatrix} -A^{\alpha \beta} & -B^{\alpha \dot{\beta}} & iC^{\beta \dot{\gamma} \alpha} \\ -B^{\dot{\alpha} \beta} & -\bar{A}^{\dot{\alpha} \dot{\beta}} & i\bar{C}^{\dot{\gamma} \dot{\beta} \dot{\alpha}} \\ iC^{\alpha \dot{\gamma} \dot{\beta}} & i\bar{C}^{\dot{\gamma} \dot{\alpha} \dot{\beta}} & iD^{\alpha \beta} \end{pmatrix} \quad (2.11) \]

is a constant graded symplectic supermatrix satisfying \( P^{BA} = (-1)^{(a+1)(b+1)} P^{AB} \), \( a \) denoting the grading of \( A \). It is easy to verify that applying this operator to the superspace coordinates we obtain (2.8).

Alternatively, one could express the graded brackets (2.10) in terms of torsion free, noncovariant spinorial derivatives \( (\partial_\alpha, \bar{\partial}_{\dot{\alpha}}, \partial_{\alpha \dot{\alpha}}) \) so obtaining a matrix \( P^{AB} \) explicitly dependent on \( (\theta, \bar{\theta}) \).

It is important to notice that the non(anti)commutative extension given in (2.8) in general deforms the supersymmetry algebra. In the standard case, defining \( Q_A \equiv (Q_\alpha, \bar{Q}_{\dot{\alpha}}, -i\partial_{\alpha \dot{\alpha}}) \), the supersymmetry algebra can be written as

\[ [Q_A, Q_B] = iT_{AB}^C Q_C \quad , \quad [D_A, D_B] = T_{AB}^C D_C \quad , \quad [Q_A, D_B] = 0 \quad (2.12) \]

where \( T_{AB}^C \) is the torsion of the flat superspace \( T_{\alpha \beta} = T_{\dot{\alpha} \dot{\beta}} = i \delta^{\gamma}_{\alpha} \delta^{\dot{\gamma}}_{\dot{\beta}} \) are the only nonzero components) and we have introduced the notation \( [F_A, G_B] \equiv F_A G_B - (-1)^{ab} G_B F_A \). Turning on non(anti)commutation in superspace leads instead to

\[ [Q_A, Q_B] = iT_{AB}^C Q_C + R_{AB}^{CD} Q_C Q_D \]

\[ [D_A, D_B] = T_{AB}^C D_C + R_{AB}^{CD} D_C D_D \]

\[ [Q_A, D_B] = R_{AB}^{CD} Q_C D_D \quad (2.13) \]
where $T_{AB}^C$ is still the torsion of the flat superspace, while

$$R_{AB}^{CD} = - \frac{1}{8} P^{MNTM[A}^C T_{B)N}^D$$

(\text{ab}) means antisymmetrization when at least one of the indices is a vector index, symmetrization otherwise) is a curvature tensor whose presence is a direct consequence of the non(anti)commutation of the grassmannian coordinates. Its nonvanishing components are

$$R_{\alpha\beta}^\gamma = \frac{1}{8} p^{\gamma\delta\delta\delta} (\delta_{\alpha}^{\delta} \delta_{\beta}^{\delta} + \delta_{\beta}^{\delta} \delta_{\alpha}^{\delta})$$

$$R_{\dot{\alpha}\dot{\beta}}^\gamma = \frac{1}{8} (p^{\gamma\delta\delta\delta} \delta_{\dot{\alpha}}^{\delta} \delta_{\dot{\beta}}^{\delta} + p^{\delta\delta\delta\delta} \delta_{\dot{\beta}}^{\delta} \delta_{\dot{\alpha}}^{\delta})$$

From the terms proportional to the curvature in the algebra (2.13) are quadratic in the supersymmetry charges and covariant derivatives, we can define new graded brackets

$$[Q_A, Q_B]_q \equiv Q_A Q_B - (-1)^{ab} [\delta_B C^A] + (-1)^{ab} R_{AB}^{CD} Q_C Q_D$$

and analogous ones for $[D_A, D_B]_q$ and $[Q_A, D_B]_q$, which satisfy the standard algebra (2.12). The new brackets can be interpreted as a quantum deformation associated to a $q$–parameter which in this case is a rank–four tensor

$$q_{AB}^{CD} \equiv \delta_B C^A + (-1)^{ab} R_{AB}^{CD}$$

It would be interesting to investigate this issue further.

### 3 Discussing associativity

Given the bidifferential operator (2.10) associated to the noncommutative supergeometry defined in (2.8) it is easy to prove the following identities

\begin{align*}
\{ \Phi, \Psi \}_P &= (-1)^{1+\text{deg}(\Phi)+\text{deg}(\Psi)} \{ \Psi, \Phi \}_P \\
\{ c \Phi, \Psi \}_P &= c \{ \Phi, \Psi \}_P , \quad \{ \Phi, c \Psi \}_P = (-1)^{\text{deg}(c)-\text{deg}(\Phi)} c \{ \Phi, \Psi \}_P \\
\{ \Phi + \Psi, \Omega \}_P &= \{ \Phi, \Omega \}_P + \{ \Psi, \Omega \}_P
\end{align*}

The operator $\{ \cdot, \cdot \}_P$ will then be promoted to a graded Poisson structure on superspace if and only if the Jacobi identities hold

\begin{align*}
\{ \Phi, \{ \Psi, \Omega \}_P \}_P + (-1)^{\text{deg}(\Phi)-\text{deg}(\Psi)+\text{deg}(\Omega)} \{ \Psi, \{ \Phi, \Omega \}_P \}_P \\
+ (-1)^{\text{deg}(\Omega)-\text{deg}(\Phi)+\text{deg}(\Psi)} \{ \Omega, \{ \Phi, \Psi \}_P \}_P &= 0
\end{align*}
for any triplet of functions of the superspace variables. This property is not in general satisfied as a consequence of possible lack of associativity in the fundamental algebra (2.8). Indeed, imposing (3.2) yields the nontrivial conditions

\[ P^A R^B S^T C^A (-1)^{c+b(c+a+r)} + P^B R^C S^T A^B (-1)^{a+c(a+b+r)} + P^C R^A S^T B^C (-1)^{b+a(b+c+r)} = 0 \]  

(3.3)

\[ (-1)^{b m} P^A M^B N^C R^M N^C = 0 \]  

(3.4)

where the torsion \( T_{AB}^C \) and the curvature \( R_{AB}^{CD} \) have been introduced in (2.12) and (2.13).

If \( P^{AB} \) is invertible (\( P^{AB} P^{BC} = \delta^C_A \)), the equation (3.3) is equivalent to the vanishing of the contorsion tensor \( H_{ABC} \) defined by

\[ H_{ABC} = T_{AB}^D P_{DC} (-1)^{ac} + T_{CA}^D P_{DB} (-1)^{cb} + T_{BC}^D P_{DA} (-1)^{ba} \]  

(3.5)

Its only nonvanishing components are

\[ H_{\alpha\bar{\alpha}\beta} = -i [P_{\alpha\beta} + P_{\bar{\alpha}\bar{\beta}}] \]

\[ H_{\alpha\bar{\beta}\beta} = -i [P_{\alpha\beta} - P_{\bar{\alpha}\bar{\beta}}] \]

\[ H_{\alpha\beta\bar{\beta}} = i P_{\alpha\beta\bar{\beta}} \]  

(3.6)

We notice that its bosonic components \( H_{abc} \) vanish due to the \( x \)-independence of the noncommutation functions in (2.8). The nonvanishing of \( H \) comes entirely from the \( \theta \)-dependence of the functions in (2.9).

In a string theory context, knowing the geometric objects of the (super)space like the curvature \( R \) and the field strength \( H \) would allow to identify the class of supergravity backgrounds in which noncommutative geometries might naturally emerge. In [19] the deformation of D-brane world-volumes in the presence of NS-NS curved backgrounds was investigated in a nonsupersymmetric context. It was shown that, if both the brane and the background are curved, i.e. \( H \equiv d(B + F) \neq 0 \), then the deformation of the world-volume is a Kontsevich deformation which defines a nonassociative, noncommutative product. Noncommutativity is governed by the usual NS-NS \( B \)-field, whereas nonassociativity arises from the NS-NS field strength \( H \). This suggests that our supergeometries might naturally appear, if an analysis similar to [19] were to be performed in a manifestly supersymmetric formalism (i.e. working with a Green-Schwarz [20] or Berkovits [21] string) for backgrounds with nonvanishing super p–form field strengths. First steps in this direction were taken in [14], where open Green-Schwarz superstrings ending on a D-brane in the presence of a constant NS-NS \( B \) field in flat spacetime were considered in a manifestly supersymmetric approach. In this simple case it was found that the anticommutation relations for the fermionic variables of superspace remain unmodified. It would be
interesting to extend these calculations to more general backgrounds to see whether also a deformation of superspace can arise. Since the noncommutative geometries we have constructed are characterized by a 3–form field strength with the only nonvanishing components being fermionic, we expect to find connections even with string backgrounds with $B_{ab} = 0$.

We now search for the most general solutions of the conditions (3.3, 3.4). Writing them in terms of the $P^{AB}$ components (2.11) we obtain

$$B^{a\dot{a}} A_{a\dot{a}} + A^{a\dot{a}} B_{a\dot{a}} = 0$$
$$B^{a\dot{b}} B^{b\dot{c}} + A^{a\dot{b}} A^{b\dot{c}} = 0$$
$$\left( \tilde{C}^{a\dot{b}} A^{a\dot{b}} A^{a\dot{b}} - C^{a\dot{b}} A^{a\dot{b}} - C^{a\dot{b}} B^{a\dot{b}} \right) = 0$$
$$\Im \left( \tilde{C}^{a\dot{b}} C^{a\dot{b}} + \tilde{C}^{a\dot{b}} C^{a\dot{b}} + \tilde{C}^{a\dot{b}} C^{a\dot{b}} \right) = 0.$$ (3.7)

The first two conditions necessarily imply the vanishing of the constants $A$ and $B$. Inserting this result in the third constraint we immediately realize that it is automatically satisfied and the only nontrivial condition which survives is the last one. This equation has nontrivial solutions. For example, the matrix

$$C^{a\dot{a}, \beta} = \begin{pmatrix} \psi_{a\dot{a}} & \bar{\psi}_{a\dot{a}} \\ \bar{\psi}_{a\dot{a}} & \psi_{a\dot{a}} \end{pmatrix}$$

for any spinor $\psi_{a\dot{a}}$, is a solution. It would correspond to assume the same commutations rules among any bosonic coordinate and the spinorial variables.

We conclude that the most general associative and non(anti)commuting algebra in Minkowski superspace has the form

$$\{ \theta^a, \theta^\beta \} = \{ \bar{\theta}^\alpha, \bar{\theta}^\dot{\beta} \} = \{ \theta^a, \bar{\theta}^\dot{\beta} \} = 0$$
$$[x^{a\dot{a}}, \theta^\beta] = iC^{a\dot{a}, \beta}$$
$$[x^{a\dot{a}}, \bar{\theta}^\dot{\beta}] = i\bar{C}^{a\dot{a}, \dot{\beta}}$$
$$[x^{a\dot{a}}, x^{b\dot{b}}] = iD^{a\dot{a}, b\dot{b}} + \frac{1}{2} \left( \tilde{C}^{a\dot{b}} \theta^a - C^{a\dot{b}} \theta^a + C^{a\dot{b}} \bar{\theta}^\dot{a} - C^{a\dot{b}} \bar{\theta}^\dot{a} \right),$$ (3.9)

where $C$ is subject to the last constraint in (3.7). Setting $C^{a\dot{a}, \beta} = 0$ we recover the usual noncommuting superspace considered so far in literature [11, 14, 15].

Under conditions (3.7) the graded brackets (2.10) satisfy the Jacobi identities (3.2), as can be easily proved by expanding the functions in power series. In this case we have a well–defined super Poisson structure on superspace.

We notice that a non(anti)commutative but associative geometry always maintains the standard algebra (2.12) for the covariant derivatives. In fact, in this case, from (2.15) it follows $R_{AB}^{CD} = 0$. 

8
Towards the construction of a noncommutative product

In this section we describe the first few steps towards the construction of a star product defined on the class of general superfields. By definition, this product must be associative, i.e. it has to satisfy the Jacobi identities (3.2) when the fundamental algebra is associative.

In the nonsupersymmetric case the lack of associativity of the fundamental algebra is signaled by the presence of a nonvanishing 3–form $H$. A product has been constructed [16] so that the terms violating the Jacobi identities are proportional to $H$. The product is then automatically associative when the fundamental algebra is.

In the present case we have shown that the lack of associativity in superspace is related to a nonvanishing super 3–form. This suggests the possibility to construct a super star product by suitably generalizing the Kontsevich construction [16] to superspace.

We begin by considering the Moyal–deformed product defined in the usual way

$$\Phi \ast \Psi \equiv \Phi \exp(\hbar \hat{D}_AP^{AB} \hat{D}_B)\Psi,$$

where $\Phi$ and $\Psi$ are arbitrary superfields, and $\hbar$ denotes a deformation parameter. In general, due to the lack of (anti)commutativity among covariant derivatives (see eq. (2.13)), it is easy to prove that the $\ast$–product is not associative even when the Poisson brackets are. However, inspired by the Kontsevich procedure [16], we perturbatively define a modified product $\star$ with the property to be associative up to second order in $\hbar$ when the Jacobi identities are satisfied. Precisely, we find an explicit form for the product by imposing the Jacobi identities (3.2) to be violated at this order only by terms proportional to $H$. To this end we define

$$\Phi \star \Psi \equiv \Phi \Psi + \hbar \Phi \hat{D}_AP^{AB} \hat{D}_B \Psi + \frac{\hbar^2}{2} \Phi(\hat{D}_AP^{AB} \hat{D}_B)(\hat{D}_C P^{CD} \hat{D}_D) \Psi$$

$$-\frac{\hbar^2}{3} (\hat{D}_A \Phi \mathcal{M}^{ABC} \hat{D}_B \hat{D}_C \Psi - (-1)^c \hat{D}_C \hat{D}_A \Phi \mathcal{M}^{ABC} \hat{D}_B \Psi)$$

$$+ O(\hbar^3),$$

where

$$\mathcal{M}^{ABC} = P^{AD} T_{DE}^C P^{EB} (-1)^{ce} + \frac{1}{2} D^{BD} T_{DE}^A P^{EC} (-1)^{ae+a+b+ab+bc}$$

$$+ \frac{1}{2} P^{CD} T_{DE}^B P^{EA} (-1)^{bc+a+c+ac+ab}.$$
Since it is straightforward to show that

\[(\Phi \star \Psi) \star \Omega - \Phi \star (\Psi \star \Omega) = \]
\[-\frac{2}{3} \hbar^2 (-1)^{(c+b)(e+1)} + c + e + f \vec{D}_A \Phi \, P^{AE} \, P^{BF} \, P^{CG} \, H_{GFE} \, \vec{D}_C \Omega \, \vec{D}_B \Psi \]  
(4.4)

up to second order in \(\hbar\) the product is associative if and only if \(H = 0\), i.e. the fundamental algebra is associative. We note that at this order only the contorsion enters the breaking of associativity, being the curvature tensor \(R\) of order \(\hbar\).

It would be interesting to pursue the construction of the star product to all orders in \(\hbar\). We believe that to this respect there are no objections of principle in generalizing to superspace the Kontsevich procedure to all orders.

We now discuss the closure of the class of chiral superfields under the deformed products we have introduced. For a generic choice of the supermatrix \(P^{AB}\) the star product of two chiral superfields (satisfying \(\bar{D}^\dot{\alpha} \Phi = 0\)) is not a chiral superfield, both for associative and nonassociative products. However, in the particular case where the only nonvanishing components of the symplectic supermatrix \(P^{AB}\) are \(P^{\alpha \beta}\) and \(P^{ab}\), chiral superfields are closed both under the deformed product defined in (4.1) and under the Kontsevich star product (4.2) (for the latter up to terms of order \(\mathcal{O}(\hbar^3)\)). Clearly for \(P^{\alpha \beta} \neq 0\) the above star products are no more associative. For chiral superfields however, they become commutative**. This commutativity implies that there is no ambiguity in putting the parenthesis e.g. in the cubic interaction term of a deformed Wess–Zumino model, with action

\[S = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \Phi \star \bar{\Phi} + \int d^4x \left[ \int d^2\theta \left( \frac{m}{2} \Phi \star \Phi + \frac{g}{3} \Phi \star \Phi \star \Phi \right) + \text{c. c.} \right]. \]  
(4.5)

Notice that in this case the \(\star\)-product in the kinetic action cannot be simply substituted with the standard product as it happens in superspace geometries where grassmannian coordinates anticommute [11, 15].

5 Non(anti)commutative  \(N = 2\) Euclidean superspace

The main difference in the description of euclidean superspace with respect to Minkowski relies on the reality conditions satisfied by the spinorial variables. As it is well known [18], in euclidean signature a reality condition on spinors is applicable

**Generalized star products that are commutative but nonassociative have been considered in a different context in [17].
only in the presence of extended supersymmetry. We concentrate on the simplest case, the $N = 2$ Euclidean superspace even if our analysis can be easily extended to more general cases. In a chiral description the two-component Weyl spinors satisfy a symplectic Majorana condition

$$
(\theta^\alpha_i)^* = \theta^{\alpha}_i \equiv C^{ij} \theta^\beta_j C_{\beta\alpha}, \quad (\bar{\theta}^{\dot{\alpha},i})^* = \bar{\theta}_{\dot{\alpha},i} \equiv \bar{\theta}^{\dot{\beta},j} C_{\beta\dot{\alpha}} C_{ji}
$$

(5.1)

where $C^{12} = -C_{12} = i$. This implies that the most general non(anti)commutative algebra can be written as an obvious generalization of (2.4) with the functions on the rhs now being in suitable representations of the R–symmetry group. When imposing covariance under (super)translations we obtain that the most general non(anti)commutative geometry in Euclidean superspace is

$$
\begin{align*}
\{\theta^\alpha_i, \theta^\beta_j\} &= A_1^{\alpha\beta,ij}, \quad \{\bar{\theta}^{\dot{\alpha},i}, \bar{\theta}^{\dot{\beta},j}\} = A_2^{\dot{\alpha}\dot{\beta},ij}, \quad \{\theta^\alpha_i, \bar{\theta}^{\dot{\alpha},j}\} = B^{\alpha\dot{\alpha},ij} \\
[x^{\alpha\dot{\alpha}}, \theta^i] &= i C_1^{\alpha\beta,i}(\theta, \bar{\theta}) \\
[x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta},j}] &= i C_2^{\alpha\beta,i}(\theta, \bar{\theta}) \\
[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] &= i D^{\alpha\beta\dot{\alpha}\dot{\beta}}(\theta, \bar{\theta})
\end{align*}
$$

(5.2)

where

$$
\begin{align*}
C_1^{\alpha\beta, i}(\theta, \bar{\theta}) &\equiv C_1^{\alpha\beta, i} + \frac{i}{2} \theta^\alpha_j B^{\beta\dot{\alpha}, j,i} + \frac{i}{2} \bar{\theta}^{\dot{\alpha},j} A_1^{\alpha\beta, ji} \\
C_2^{\alpha\beta,i}(\theta, \bar{\theta}) &\equiv C_2^{\alpha\beta, i} + \frac{i}{2} \theta^\alpha_j A_2^{\dot{\alpha}\dot{\beta}, ji} + \frac{i}{2} \bar{\theta}^{\dot{\alpha},j} B^{\beta\dot{\alpha}, j,i} \\
D^{\alpha\beta\dot{\alpha}\dot{\beta}}(\theta, \bar{\theta}) &\equiv D^{\alpha\beta\dot{\alpha}\dot{\beta}} \\
&\quad + \frac{1}{2} \left( \theta^\alpha_i C_2^{\beta\dot{\alpha}, i,j} - \theta^\beta_i C_2^{\alpha\dot{\alpha}, i,j} + \bar{\theta}^{\dot{\alpha},i} C_1^{\beta\dot{\alpha}, i,j} - \bar{\theta}^{\dot{\beta},i} C_1^{\alpha\dot{\alpha}, i,j} \right) \\
&\quad + \frac{i}{4} \left( \theta^\alpha_i A_2^{\dot{\alpha}\dot{\beta}, ij} \theta^\beta_j + \theta^\alpha_i A_2^{\dot{\alpha}\dot{\beta}, ij} \bar{\theta}^{\dot{\beta}, j} + \theta^\beta_i A_2^{\dot{\alpha}\dot{\beta}, ij} \bar{\theta}^{\dot{\alpha}, j} \right)
\end{align*}
$$

(5.3)

with $A_1, A_2, B, C_1, C_2$ and $D$ constant.

Following the same steps as in the Minkowski case, we can look for the most general associative algebra. The results we obtain for associative non(anti)commuting geometries in Euclidean superspace are

$$
\begin{align*}
\{\theta^\alpha_i, \theta^\beta_j\} &= A_1^{\alpha\beta, ij}, \quad \{\bar{\theta}^{\dot{\alpha},i}, \bar{\theta}^{\dot{\beta},j}\} = 0 , \quad \{\theta^\alpha_i, \bar{\theta}^{\dot{\alpha},j}\} = 0 \\
[x^{\alpha\dot{\alpha}}, \theta^i] &= i C_1^{\alpha\beta,i} - \frac{1}{2} \bar{\theta}^{\dot{\alpha},j} A_1^{\alpha\beta, ji} \\
[x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta},i}] &= 0 \\
[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] &= i D^{\alpha\beta\dot{\alpha}\dot{\beta}} + \frac{i}{2} \left( \bar{\theta}^{\dot{\alpha},i} C_1^{\beta\dot{\alpha}, i,j} - \bar{\theta}^{\dot{\beta},i} C_1^{\alpha\dot{\alpha}, i,j} \right) - \frac{1}{4} \bar{\theta}^{\dot{\alpha},i} A_1^{\alpha\beta, ij} \bar{\theta}^{\dot{\beta}, j}
\end{align*}
$$

(5.4)
or
\[
\{\theta^\alpha_i, \theta^\beta_j\} = 0, \quad \{\bar{\theta}^{\dot{\alpha},i}, \bar{\theta}^{\dot{\beta},j}\} = A_2^{\dot{\alpha}\dot{\beta},ij}, \quad \{\theta^\alpha_i, \bar{\theta}^{\dot{\beta},j}\} = 0
\]
\[
[x^{\alpha\dot{\alpha}}, \theta^\beta_i] = 0
\]
\[
[x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta},i}] = i C_2^{\alpha\dot{\alpha}\dot{\beta},i} - \frac{1}{2} \theta_j^\alpha A_2^{\dot{\alpha}\dot{\beta},ji}
\]
\[
[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = i D^{\alpha\dot{\alpha}\beta\dot{\beta}} + \frac{i}{2} \left( \theta_i^\alpha C_2^{\beta\dot{\beta}\dot{\alpha},i} - \theta_i^\beta C_2^{\alpha\dot{\alpha}\dot{\beta},i} \right) - \frac{1}{4} \theta_i^\alpha A_2^{\dot{\alpha}\dot{\beta},ij} \bar{\theta}^{\dot{\beta}}_j
\]  

We notice that in this case associativity imposes less restrictive constraints because of the absence of conjugation relations between $A_1$ and $A_2$. As a consequence, nontrivial anticommutation relations among $\theta$'s (or $\bar{\theta}$'s) are allowed. Moreover the $R$–symmetry group of the $N = 2$ euclidean superalgebra is broken only by the constant terms $C_1$ and $C_2$. Setting these terms equal to zero leads to nontrivial (anti)commutation relations preserving $R$–symmetry.

Again, explicit expressions for the corresponding graded brackets can be obtained as an obvious generalization of (2.10–2.11). In this case they define a super Poisson structure on the euclidean superspace. A simple example of a super Poisson structure is
\[
\{\Phi, \Psi\}_P = - \Phi \bar{D}^j_i \theta A_1^{\alpha\beta, ij} \bar{D}_\beta^j \Psi
\]

We notice that this extension is allowed only in euclidean superspace, where it is consistent with the reality conditions on the spinorial variables.

6 Final remarks

In this paper we have studied the most general non(anti)commutative geometry in $N = 1$ four dimensional Minkowski superspace that is compatible with classical (super)translations. We have shown that nonanticommutation relations among spinorial variables are allowed if the commutation relations of bosonic coordinates with the spinorial ones and bosonic coordinates among themselves acquire a particular dependence on the $\theta$–variables. In a geometric framework we can interpret the supermatrix $P^{AB}$ defining the non(anti)commutative algebra (see eq. (2.11)) as a nontrivial metric in superspace. The geometry is in general nonassociative and deforms the algebra of the superspace derivatives through a curvature term (which, however, does not affect the algebra of the coordinates). This deformation can be interpreted as a quantum deformation associated to a four–rank tensor $q$. The geometric properties of the $q$–deformed superspace haven’t been considered in this paper but certainly deserve a deeper investigation.

We have further showed that imposing associativity, i.e. the validity of the
Jacobi identities, implies additional conditions on the (anti)commutators of superspace coordinates which nevertheless allow for nontrivial deformations involving also fermionic variables. In particular, the spinorial variables are required to anticommute, but they can have nonzero commutation relations with the bosonic coordinates. As a consequence, in the associative case the algebra of the (super)derivatives is not $q$–deformed. Inspired by the Kontsevich procedure, we obtained the first three terms in a series expansion in the deformation parameter $\hbar$ for a possible noncommutative product that is associative up to second order, if the Jacobi identities are satisfied. For the general case, the deviation from associativity has been shown to be proportional to a super three–form field strength which, in a complete covariant formalism, has the interpretation of the contorsion of the non(anti)commutative superspace. For the bosonic case, it was shown in [19] that the deformation of D–brane world-volumes in curved backgrounds, with $H = d(B + F) \neq 0$, is described by a nonassociative Kontsevich star–product. It would be interesting to see if an analogous procedure performed in a manifestly supersymmetric formalism (Green–Schwarz or Berkovits) leads to superspace deformations involving fermionic coordinates.

We have extended our analysis to the case of $N = 2$ euclidean superspace. Due to the different hermiticity conditions on the spinorial variables, the non(anti)commutative euclidean superspace manifests quite different features from Minkowski. In fact, in this case nonzero anticommutation relations among spinorial variables are allowed by covariance and associativity. This implies that, even in the case of non(anti)commutative but associative geometry, the supersymmetry algebra is $q$–deformed. The results obtained for the $N = 2$ case can be easily extended to generic $N > 1$. It is however important to stress that in general the R–symmetry group is broken by the noncommutativity among fermionic and bosonic coordinates.

An interesting continuation of our work would be to generalize the results of [14], in order to see whether deformations of fermionic variables can arise for open superstrings ending on a D–brane in the presence of a more general super field strength. It would be also interesting to study field theories defined on a non(anti)commutative superspace, such as the extended Wess–Zumino model proposed in (4.5). As already noticed, when the spinorial variables satisfy a nontrivial algebra, also the kinetic action contains interaction terms.
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