ABSTRACT
Motivated by display advertising on the internet, the online stochastic matching problem is proposed by Feldman, Mehta, Mirrokni, and Muthukrishnan (FOCS 2009). Consider a stochastic bipartite graph with offline vertices on one side and with i.i.d. online vertices on the other side. The algorithm knows the offline vertices and the distribution of the online vertices in advance. Upon the arrival of each online vertex, its type is realized and the algorithm immediately and irrevocably decides how to match it. In the vertex-weighted version of the problem, each offline vertex is associated with a weight and the goal is to maximize the total weight of the matching.

In this paper, we generalize the model to allow non-identical online vertices and focus on the fractional version of the vertex-weighted stochastic matching. We design fractional algorithms that are 0.718-competitive and 0.731-competitive for non i.i.d. arrivals and i.i.d. arrivals respectively. We also prove that no fractional algorithm can achieve a competitive ratio better than 0.75 for non i.i.d. arrivals. Furthermore, we round our fractional algorithms by applying the recently developed multiway online correlated selection by Gao et al. (FOCS 2021) and achieve 0.666-competitive and 0.704-competitive integral algorithms for non i.i.d. arrivals and i.i.d. arrivals. Our results for non i.i.d. arrivals are the first algorithms beating the $1 - 1/e \approx 0.632$ barrier of the classical adversarial setting. Our 0.704-competitive integral algorithm for i.i.d. arrivals slightly improves the state-of-the-art 0.701-competitive ratio by Huang and Shu (STOC 2021).

CCS CONCEPTS
• Theory of computation → Online algorithms; Graph algorithms analysis.

KEYWORDS
online algorithms; stochastic matching

1 INTRODUCTION
Since the seminal work of Karp, Vazirani, and Vazirani [28], online bipartite matching has been extensively studied in the online algorithms literature. In the classical setting, the vertices on one side of an underlying bipartite graph are known upfront to the algorithm. We refer to this set of vertices as offline vertices. The vertices on the other side of the graph, known as online vertices, are revealed in a sequence. Upon the arrival of each vertex, the algorithm makes an irrevocable matching decision between the vertex and its offline neighbors. The goal is to maximize the size of the matching produced by the algorithm.

Karp et al. proposed the Ranking algorithm that achieves the optimal competitive ratio of $1 - \frac{1}{e}$. Later, Aggarwal et al. [1] generalized the algorithm to the vertex-weighted setting and attained the same competitive ratio.

Motivated by display advertising on the internet, Feldman et al. [13] initiated the study of online stochastic matching. In the stochastic setting, each online vertex is drawn i.i.d. from a priori known distribution and the realization of each vertex is revealed on its arrival. Feldman et al. [13] designed the first 0.67-competitive algorithm that bypasses the $1 - \frac{1}{e}$ impossibility result of the worst-case model, under an extra assumption of integral arrival rate. Later on, a line of subsequent works improved the competitive ratio to 0.729 [2, 4, 24, 30] for unweighted graphs, and extended the results to vertex-weighted graphs [24] and edge-weighted graphs [4, 16].

Without the assumption of integral arrival rate, Manshadi, Oveis Gharan, and Saberi [30] designed a 0.702-competitive algorithm, and the ratio is improved to 0.706 by Jaillet and Lu [24] and to 0.711 by Huang and Shu [19]. In the vertex-weighted online stochastic matching, the state-of-the-art competitive ratio is 0.701 by Huang and Shu [19]. Manshadi, Oveis Gharan, and Saberi [30] also proved that no algorithm can be better than 0.823-competitive.

1.1 Our Results
Fractional Online Stochastic Matching. In this paper, we study the fractional version of online stochastic matching that allows us to fractionally match between a pair of vertices, as long as the total matched portion of each vertex is at most one. Furthermore, we generalize the online stochastic matching problem to non-identical vertices. That is, the type of each vertex is drawn independently while the distributions of all vertices can be distinct.
Table 1: A summary of the state-of-the-art results and our results.

| Arrival | Previous Results | Our Results (Vertex-weighted) |
|---------|------------------|-------------------------------|
|         | Integral         | Hardness                      | Integral | Fractional | Hardness |
| I.I.D.  | Unweighted: 0.711 [19] | 0.823 [30]                        | 0.704    | 0.731      | -        |
|         | Vertex-weighted: 0.701 [19] |
| Non I.I.D. | 1 − 1/e ≈ 0.632 [1, 28]   | -                              | 0.666    | 0.718      | 0.75     |

We propose 0.731-competitive and 0.718-competitive fractional algorithms for online stochastic matching with i.i.d. arrivals and non i.i.d. arrivals, respectively. Our algorithms and analysis automatically apply to the vertex-weighted version without any adjustment. We also provide a 0.75 hardness result for the fractional online stochastic matching problem with non i.i.d. arrivals.

**Online Rounding via Online Correlated Selection.** A general and widely-applicable technique in the online algorithms literature is a two-step approach that first design a fractional online algorithm and then apply an online rounding scheme to it. As implications of our fractional algorithms, we apply the recently developed online rounding tool, namely multway online correlated selection, as a blackbox to our fractional algorithm and achieve 0.704-competitive and 0.666-competitive integral algorithms for the vertex-weighted online stochastic matching problem with i.i.d. arrivals and non i.i.d. arrivals, respectively. The online correlated selection technique was introduced in the breakthrough work of Fahrbach et al. [11] as a key subroutine for the edge-weighted online bipartite matching problem. Recently, Blanch and Charikar [3], and Gao et al. [15] extended the rounding scheme to multway online correlated selection. For the convenience of our work, we apply the multway online correlated selection of Gao et al. [15].

For the non-i.i.d. arrival setting, our fractional 0.718-competitive and integral 0.666-competitive algorithms are the first algorithms beating the $1 − 1/e$ barrier of the classical adversarial setting.\(^1\)

For the i.i.d. arrival setting, our fractional algorithm significantly improves the 0.711 ratio of Huang et al. [19], and is even slightly better than the state-of-the-art 0.729 ratio of Jaillet and Lu [24], under the extra integral arrival rate assumption. Our integral algorithm slightly improves the state-of-the-art 0.701 competitive ratio for vertex-weighted graphs.

### 1.2 Our Techniques

**Beyond Two-choice Algorithms.** Prior to our work, the arguably most successful approach [19, 24, 30] for attacking the online stochastic matching problem is a two-choice algorithm. Upon the arrival of each vertex, it makes at most two tries to match its neighbors, and if both attempts failed, the vertex is left unmatched even if there exists other unmatched neighbors. The restriction of two-choice facilitates the analysis, while at the same time, leads to obviously suboptimal behaviors of the algorithm. We import the multway online correlated selection as a rounding scheme that allows us to explore beyond two-choice algorithms.

Unbiased Estimators: Fine-Grained Offline Statistics. Similar to previous works, our algorithms heavily rely on the offline statistics of the instance and indeed, we use more fine-grained offline statistics. All previous works explicitly or implicitly use the following offline statistics which we call independent estimators:

$$x_{u,v}(t_v) = \Pr \{ (u, v) \in \text{OPT} | t_v \},$$

i.e. the probability that edge $(u, v)$ is matched in the optimal matching conditioning on the realized type $t_v$ of the online vertex $v$. A first natural fractional algorithm that we study is to fractionally match each edge $(u, v)$ by $x_{u,v}(t_v)$ on the arrival of every $v$. In Section 6, we fully characterize the performance of this algorithm in the non i.i.d. setting. Surprisingly, our analysis shows that this algorithm is the best among a much larger family of algorithms in the non-i.i.d. setting.

Furthermore, we consider more fine-grained offline statistics, e.g.

$$x_{u,v,j} (t_{v_1}, t_{v_2}, \ldots, t_{v_j}) = \Pr \{ (u, v_j) \in \text{OPT} | t_{v_1}, t_{v_2}, \ldots, t_{v_j} \} ,$$

where $v_k$ denotes the $k$-th arriving vertex. This value is the probability that $(u, v_j)$ is matched in the optimal matching conditioning on the realized types $\{t_{v_k}\}_{k < j}$ of all arrived vertices. By considering such statistics, our matching decision is inherently adaptive to the historical information of the instance. More generally, we can select an arbitrary subset of arrived vertices that includes the current vertex, and calculate the conditional probability with respect to the types of those vertices. We name such offline statistics as unbiased estimators since the expectation of such random variables $x_{u,v,j}$ equals to the offline probability of $(u, v)$ being matched in the optimal matching, for arbitrary selection of the subset.

Our fractional algorithm for the i.i.d. arrival setting in Section 5 is a mix of different unbiased estimators. To the best of our knowledge, we are the first to utilize such fine-grained statistics for the online stochastic matching problem.

Unbiased Estimator with Minimum Variance. Instead of using the size of the fractional matching or the size of the rounded matching as a natural objective function, we provide a unified framework by reducing the maximum matching problems to the design of an unbiased estimator with minimum variance. Fixing an offline vertex, the expected fraction it receives is the same for all unbiased estimators. On the other hand, the total fraction received by each vertex is a random variable and in some of the realizations, it can be larger than 1. This is wasteful since we cannot collect the part of the gain that exceeds 1. In an ideal case, if the random variable is never greater than 1, our fractional algorithm would then be optimal. Unfortunately, it is an impossible task to design an unbiased estimator with such properties, due to the online nature of the problem. Intuitively, if we can design an unbiased estimator with small variance, it should lead to a good performance. We establish

---

\(^1\)The Ranking algorithm achieves a competitive ratio of $1 − 1/e$ for adversarial arrivals and breaks this bound in the i.i.d. setting. However, it does not beat the $1 − 1/e$ barrier for non-i.i.d. arrivals when the instance is the deterministic upper-triangle graph for which Ranking is exactly $1 − 1/e$-competitive.
the connection between the variance of an unbiased estimator to competitive ratio in Section 3.

1.3 Paper Organization
We formally define the family of unbiased estimator algorithms in Section 2. In Section 3, we establish a connection between the second moment of an unbiased estimator to its competitive ratio. In Section 4, as a warm up, we provide a simple algorithm that breaks the $1 - 1/e$ barrier with non i.i.d. arrivals. In Section 5, we present our main algorithm for the i.i.d. arrival setting. In Section 6, we present our main algorithm for the non i.i.d. arrival setting and the hardness result.

1.4 Related Work
Simultaneous to our work, Huang, Shu, and Yan [20] apply the online correlated selection techniques to the i.i.d. online stochastic matching and also exploit the power of multiple-choice algorithms. They also study the edge-weighted setting with free disposal. The online correlated selection techniques have also been applied to the Adwords problem [23] and the online bipartite matching problem with reusable resources [7].

Devanur et al. [9] studied a similar non-identical setting for general online resource allocation problems. On the other hand, they focused on the case of small bids and their results are not directly comparable to ours. Closely related to the i.i.d. stochastic matching problem is the random arrival model. In this model, we assume that the online vertices of an underlying graph arrive in a random order. It is observed that any algorithm in the random arrival model would apply for the stochastic model. For unweighted graphs, Karande, Mehta, and Tripathi [27] and Mahdian and Yan [29] proved that the Ranking algorithm is 0.696-competitive. For vertex-weighted graphs, Huang et al. [21] and Jin and Williamson [25] achieved a 0.662 competitive ratio.

There is an extensive study of fractional online matching algorithms in the adversarial setting, including $b$-matching [26], fractional edge-weighted online bipartite matching [12], online matching with concave return [8], fully online matching [17, 18, 22], online matching with general vertex arrival [32], adwords [5, 31]. On the other hand, the state-of-the-art integral algorithms are mostly not directly related to the fractional algorithms in these models. In other words, they do not admit a good online rounding scheme. Recently, Buchbinder, Naor, and Wajc [6] revisit the classical online bipartite matching problem and study how much randomness is necessary to beat the 1/2 barrier for deterministic algorithms, for which they apply the same two-step approach of rounding a fractional algorithm as us.

2 PRELIMINARIES

Online Stochastic Matching. We study the online vertex-weighted stochastic matching problem with non i.i.d. vertex arrivals. Consider a bipartite graph with offline vertices on one side, and with stochastic online vertices on the other side. We use $L$ to denote the offline vertices and $R = \{v_j\}$ to denote the online vertices. Each offline vertex $u \in L$ is associated with a weight $w_u$. The type of each online vertex $v_j$ is drawn independently from a priori known distribution $D_j$. Here, each type specifies its edges incident to the offline vertices. The offline vertices together with their associated weights, and the distributions $D_j$’s of online vertices are known in advance. We use $T_j$ to denote the support of $D_j$. The type $t_j \in T_j$ of each $v_j$ is only realized on its arrival and the algorithm makes immediate and irrevocable matching decisions. The goal is to maximize the total weight of the matched offline vertices. We assume the arrival order of the vertices is unknown to the algorithm but is fixed in advance. We use $t$ to denote the realized types $(t_1, t_2, \ldots, t_n)$ of all vertices and $t_{\leq j}, t_{> j}$ to denote the realized types $(t_1, t_2, \ldots, t_j)$ of the first $j$ vertices and the types $(t_{j+1}, \ldots, t_n)$ of the vertices after $v_j$. For every index set $I \subseteq [n]$, we use $t_I$ to denote the types $(t_{I,i})_{i \in I}$.

We shall also study the i.i.d. setting in which all $D_j$’s are the same. We make no assumptions on the arrival rate of each type.

2.1 Fractional Online Stochastic Matching
We focus on the design and analysis of fractional algorithms, and then apply the multiway online correlated selection by Gao et al. [15] as our rounding scheme.

On the arrival of each vertex $v_j$, based on the realized type $t_j$ of $v_j$, we construct a vector $x_j \in [0,1]^L$ such that $x_{u,v} > 0$ only if $u$ is a neighbor of $v_j$ and $\sum_{u \in L} x_{u,v} \leq 1$. Then we match $v_j$ to each $u \in L$ by a fraction of $x_{u,v}$. For the ease of our presentation, we use $y_u = \sum_j x_{u,j}$ to denote the total fraction that $u$ received, and allow it to be greater than 1. Then, the performance of our algorithm equals

$$\text{ALG} = \sum_{u \in L} w_u \cdot \min(y_u, 1).$$

The vector $x_j$ is naturally decided based on the realized types of all arrived vertices. We abuse the notation $x_j$ to denote a function that maps from types of vertices to (fractional) matching decisions. We consider the following family of fractional algorithms that we call unbiased estimators.

Definition 2.1 (Unbiased Estimator). $x_j : T_1 \times T_2 \times \cdots \times T_j \rightarrow [0,1]^L$ is unbiased if:

- $\sum_{j \in \mathcal{L}} x_{u,j}(t_{\leq j}) \leq 1$ for all $t_{\leq j} \in T_1 \times T_2 \times \cdots \times T_j$;
- $x_{u,j}(t_{\leq j}) > 0$ only if the vertex $v_j$ with type $t_j$ has an edge to $u$;
- $E_{t_{\leq j}} [x_{u,j}(t_{\leq j})] = \Pr[(u,v_j) \in \text{OPT}]$ for all $u \in L$.

Further, we say an fractional algorithm is an unbiased estimator, if for every step $j \in [n]$, the constructed vector $x_j$ is unbiased.

Remark 2.1. All the estimators $x_{u,j}$ used in the work shall be the probability that edge $(u,v_j)$ is in the optimal matching, conditioning on the types of a subset of online vertices. With a sample access to each online vertex, this family of statistics can be estimated by standard Monte-Carlo algorithm within arbitrary additive accuracy (with high probability). Hence, our algorithms can be implemented efficiently with only an $\epsilon$ loss in the competitive ratio.

2.2 Rounding via Online Correlated Selection
We import the multiway online correlated selection formulation by Gao et al. [15] and present their setting and result for the ease of our application.

Multiway Online Selection Problem. [15]. Considers a set of elements $U$ and a selection process that proceeds in $n$ rounds. Each
round \(j \in [n]\) is associated with a non-negative vector \(x_j\) with \(\sum_{u \in U} x_{u,j} \leq 1\). The vectors are unknown at the beginning and are revealed to a multiway online selection algorithm at the corresponding rounds. Let \(U_j = \{u \in U : x_{u,j} > 0\}\) be the set of elements with positive masses in round \(j\). Upon observing the vector \(x_j\) for round \(j\), the algorithm selects an element from \(U_j\). Let \(y_u = \sum_{j \in [n]} x_{u,j}\) be the cumulative mass of each element \(u\).

Theorem 2.2 (Theorem 6, [15]). There exists a multiway online correlated selection such that any element with accumulated mass \(y_u\) is selected with probability at least

\[
p(y_u) \overset{\text{def}}{=} 1 - \exp\left(-y_u - \frac{1}{2} \cdot y_u^2 - \left(\frac{4 - 2\sqrt{3}}{3}\right) \cdot y_u^3\right).
\]

We remark that the original paper requires \(\sum_{u \in U} x_{u,j} \leq 1\) for each \(x_j\). In order to apply the algorithm to arbitrary vector \(\sum_{u \in U} x_{u,j} \leq 1\), we introduce a dummy vertex \(u_0\) that connects to every online vertex and for each \(j\), let \(x_{u_0,j} = 1 - \sum_{u \in U} x_{u,j}\).

We also observe the concavity of function \(p(y)\), that shall be used later in the proof of Lemma 6.1 and Lemma 6.4.

Lemma 2.1. The function \(p(y) = 1 - \exp(-y - \frac{1}{2} \cdot y^2 - \left(\frac{4 - 2\sqrt{3}}{3}\right) \cdot y^3)\) is concave.

Proof. Let \(c = \frac{4 - 2\sqrt{3}}{3}\), we calculate the second-order derivative of \(p(y)\):

\[
p'(y) = \exp(-y - \frac{1}{2} \cdot y^2 - \left(\frac{4 - 2\sqrt{3}}{3}\right) \cdot y^3) \cdot \left(-1 + 6c - 2\sqrt{3} - 9c^2y^2\right);
\]

\[
p''(y) = \left((-6c - 2)y - (1 + 6c) \cdot y^2 - 9c^2y^3\right) \cdot \exp(-y - \frac{1}{2} \cdot y^2 - \left(\frac{4 - 2\sqrt{3}}{3}\right) \cdot y^3) \cdot \left(-1 + 6c - 2\sqrt{3} - 9c^2y^2\right).
\]

\[
< 0,
\]

where the last inequality follows from the fact that \(6c - 2 \approx -0.928 < 0\).

We then apply this OCS rounding scheme as a black-box to our fractional algorithm. See the following pseudocode for a formal description of our integral algorithm.

Algorithm 1: Unbiased Estimator + Online Correlated Selection

On the arrival of vertex \(v_j\),

1. Construct a vector \(x_j \in [0, 1]^L\) according our Unbiased Estimator algorithm.
2. Select a vertex \(u \in L\) by applying the multiway OCS w.r.t. \(x_j\).
3. Match \(v_j\) to \(u\) if \(u\) is not matched yet.

2.3 Competitive Ratio

We provide a unified analysis for the fractional algorithms and their corresponding integral versions by rounding with OCS. We conclude a competitive ratio of \(\Gamma\) of our algorithms by proving a stronger statement that for every offline vertex, its matched fraction (probability of being matched) is at least \(\Gamma\) times the probability it is matched in the optimal matching. We remark that a standard competitive analysis only need to show that the total expected matching is at least \(\Gamma\) times the optimal matching.

As a reward, our analysis works for vertex-weighted graphs without any loss in the competitive ratio. In contrast, even in the i.i.d. arrival setting, prior works [4, 19, 24] need to do different analyses for the unweighted and vertex-weighted settings. Formally, we prove the following lemma, that allows us to study each offline vertex separately.

Lemma 2.2. For any unbiased estimator, its competitive ratio for the (non-i.i.d. vertex-weighted) fractional online stochastic matching problem is at least \(\min_{u \in L} \frac{E_t[y_u(t)]}{E_t[\Delta y_u(t) \frac{\min(y_u(t), 1)}{E_t[y_u(t)]}]},\) where \(y_u(t) = \sum_{j \in [n]} x_{u,j}(t \in j).\)

Moreover, if we round the unbiased estimator with OCS, its competitive ratio is at least \(\min_{u \in L} \frac{E_t[p(y_u(t))] - \max_{u \in L} E_t[y_u(t)]}{E_t[y_u(t)]} \frac{\min(y_u(t), 1)}{E_t[y_u(t)]} ,\)

Proof. Directly write \(x_{u,j}, y_u\) and omit the inputs \(t \in j\) for notation simplicity. According to the definition of unbiased estimator, we have

\[
E[\text{OPT}] = \sum_{u \in L} w_u \cdot \text{Pr}[u \text{ is matched in OPT}]
\]

\[
= \sum_{u \in L} w_u \sum_{j \in [n]} \text{Pr}[u, v_j] \in \text{OPT}
\]

\[
= \sum_{u \in L} w_u \sum_{j \in [n]} E[x_{u,j}] = \sum_{u \in L} w_u \cdot E[y_u] .
\]

For our fractional algorithm, \(E_t[y_u(t)] = \sum_{u \in L} w_u \cdot E_t[\min(y_u, 1)]\). Therefore,

\[
\frac{E[\text{FRAC}]}{E[\text{OPT}]} = \frac{\sum_{u \in L} w_u \cdot E_t[\min(y_u, 1)]}{\sum_{u \in L} w_u \cdot E_t[y_u]} \geq \min_{u \in L} \frac{E_t[\min(y_u, 1)]}{E_t[y_u]} .
\]

Next, we consider the integral version by rounding with OCS. Recall the second condition of the definition of unbiased estimator and the property of OCS, the candidate vertex \(v_j\) from the second step of our algorithm must be a neighbor of \(v_j\). In other words, our attempt for matching \((v_j, u)\) is valid. To evaluate the performance of our algorithm, we observe the two types of randomness involved. The first type of randomness comes from the nature of the stochastic environment, i.e. the types \(t\) are drawn from the product distribution \(D_1 \times D_2 \times \cdots \times D_n\). The second type of randomness comes from the random selection of multiway OCS. Fix any type vector \(t\), the vectors \(x_j\)’s constructed in the first step of our algorithm are fixed according to the first condition above. Applying Theorem 2.2, we know that each vertex \(u \in L\) is matched with probability at least \(p(y_u)\). Consequently, \(E[\text{ALG}(t)] \geq \sum_{u \in L} w_u \cdot p(y_u)\). By taking the randomness over \(t\), we have

\[
E[\text{ALG}] \geq \sum_{u \in L} w_u \cdot E[p(y_u)] .
\]

We conclude the lemma by noticing that

\[
\frac{E[\text{ALG}]}{E[\text{OPT}]} \geq \frac{\sum_{u \in L} w_u \cdot E[p(y_u)]}{\sum_{u \in L} w_u \cdot E[y_u]} \geq \min_{u \in L} \frac{E[p(y_u)]}{E[y_u]} .
\]
3 UNBIASED ESTIMATOR WITH MINIMUM VARIANCE

Equipped with Lemma 2.2, it suffice to design an unbiased estimator such that the values $E[f(y)]/E[y]$ are large simultaneously for all offline vertices $u \in L$, where

$$f(y) = \begin{cases} 
\min(y, 1) & \text{for the fractional version} \\
\rho(y) & \text{for the integral version}
\end{cases}$$

Fix an arbitrary offline vertex $u$, recall that $E[y] = \Pr[u \text{ is matched in OPT}]$ for any unbiased estimator. Hence, we are aiming for an unbiased estimator with the largest possible $E[f(y)]$. There is a rich space of unbiased estimators and in principle, different functions of $f(y)$ might result in different optimal unbiased estimators.

On the other hand, observe that the two functions we care about, $\min(y, 1)$ and $\rho(y)$, are both concave. This suggests to design an unbiased estimator with minimum variance. Intuitively speaking, for two random variables $y_1, y_2$ with the same mean, $E[f(y_1)]$ is more likely to be larger than $E[f(y_2)]$, if the variance $\text{Var}[y_1]$ of $y_1$ is smaller than the variance $\text{Var}[y_2]$ of $y_2$.

This intuitive claim holds for arbitrary random variables. But for the purpose of our algorithm design, it is sufficient to use $\text{Var}[y]$ or $E[y^2]$ as a proxy for the objective function $E[f(y)]$.

The next lemma formalize this idea and reduces our algorithm design problem to unbiased estimator design with minimum variance. We approximate the functions $f(y) = \min(y, 1)$ and $f(y) = \rho(y)$ by quadratic functions and prove that $E[f(y)]$ must be large, provided $E[y^2]$ is small.

**Lemma 3.1.** Suppose a non-negative random variable $y$ satisfies that $E[y] = \mu$ and $E[y^2] \leq \gamma$.

(a) If $\mu \in [0, 1]$ and $\gamma = \mu + \frac{1}{2} \mu^2$, then $E[\min(y, 1)] \geq 0.646 \cdot E[y]$;
(b) If $\mu \in [0, 1]$ and $\gamma = \mu + \frac{1}{2} \mu^2$, then $E[\rho(y)] \geq 0.634 \cdot E[y]$;
(c) If $\mu \in [0, 1]$ and $\gamma = 1.0577 \mu + 0.231 \mu^2$, then $E[\min(y, 1)] \geq 0.731 \cdot E[y]$;
(d) If $\mu \in [0, 1]$ and $\gamma = 1.0577 \mu + 0.231 \mu^2$, then $E[\rho(y)] \geq 0.704 \cdot E[y]$.

The proof of the Lemma can be found in the full version of this paper. In Section 4 (respectively Section 5), we construct an unbiased estimator such that the condition $\gamma = \mu + \frac{1}{2} \mu^2$ holds (respectively the condition $\gamma = 1.0577 \mu + 0.231 \mu^2$) for all $y_0$’s simultaneously, in the non i.i.d. arrival setting (respectively in the i.i.d. arrival setting).

4 WARM UP: BREAKING $1 - \frac{1}{e}$ FOR NON I.I.D. ARRIVALS

In this section, we study the non-identical setting and propose an algorithm with competitive ratio strictly better than $(1 - 1/e)$. We first introduce the following two unbiased estimators.

**Independent Estimator.** On the arrival of each vertex $v_j$, fix its type $t_j$ and calculate the probability that $(u, v_j)$ is matched in the optimal matching when the types of all other vertices are resampled. I.e.,

$$x_{u,j}(t_j) = \Pr_{t_j}[(u, v_j) \in \text{OPT}(t_j, \hat{L}_j)] .$$

**Fully-Correlated Estimator.** On the arrival of each $v_j$, fix the types $t_{\leq j}$ of all previously arrived vertices and calculate the probability that $(u, v_j)$ is matched in the optimal matching when the types of the remaining vertices are resampled. I.e.,

$$x_{u,j}(t_{\leq j}) = \Pr_{t_{\leq j}}[(u, v_j) \in \text{OPT}(t_{\leq j}, \hat{L}_{\leq j})] .$$

The independent estimator ignores the realized types of previous vertices and calculates the probability $(u, v_j)$ is matched in OPT conditioning on the type of $v_j$ being $t_j$. This is considerably the most natural unbiased estimator and the idea of resampling all remaining vertices is standard in the prophet inequality literature and the online contention resolution scheme literature, e.g. the resampling process as in [10]. The fully-correlated estimator makes use of the full history and calculates the probability $(u, v_j)$ is matched in OPT conditioning on the types $t_{\leq j}$ of all arrived vertices. It is straightforward to check both estimators satisfy the conditions of unbiased estimator.

**Even Mix of IE/FE.** Consider an even mix between the independent and fully-correlated estimators, i.e.

$$x_{u,j}(t_{\leq j}) = \frac{1}{2} \cdot \Pr_{t_j}[(u, v_j) \in \text{OPT}(t_j, \hat{L}_j)] + \frac{1}{2} \cdot \Pr_{t_{\leq j}}[(u, v_j) \in \text{OPT}(t_{\leq j}, \hat{L}_{\leq j})] .$$

**Theorem 4.1.** An even mix between independent estimator and fully-correlated estimator is $0.646$-competitive for the non i.i.d. vertex-weighted fractional online stochastic matching. Furthermore, by rounding it with OCS, the algorithm is $0.634$-competitive.

**Proof.** Fix an arbitrary vertex $u \in L$. Next, we bound the second moment of $y_0$ of our algorithm by studying the independent estimator and fully-correlated estimator separately. For notation simplicity, we omit the subscript $u$ in all $x_{u,j}$s and $y_0$’s. We use $\hat{x}_j, \hat{y}$ (resp. $\hat{x}_j, \hat{y}$) to denote $x_{u,j}, y_0$ by applying independent estimation (resp. by applying fully-correlated estimation). Then, our algorithm uses $y_j = \frac{1}{2}(x_j + \hat{x}_j)$ and $y_0 = \frac{1}{2}(\hat{y} + \hat{y})$.

Observe that $\hat{x}_j$’s are independent, while $\hat{x}_j$’s are not. However, since $\hat{x}_j$ and $\hat{x}_k$ share the same types $t_{\min(j,k)}$, they are naturally negatively correlated. On the other hand, the variance of each $\hat{x}_k$ is larger than each $\hat{x}_k$ as more randomness is realized in $\hat{x}_k$. We formalize these two properties in the following lemma.

**Lemma 4.1.** Suppose $E[\hat{y}] = E[y] = \Pr[u \text{ is matched in OPT}] = \mu$. We have

(1) $E[\hat{y}^2] \leq \mu^2 + \sum_{j \in [n]} E[x_j^2]$

(2) $E[\hat{y}^2] \leq 2\mu - \sum_{j \in [n]} E[x_j^2]$

**Proof.** We study independent estimator first. By the independence among $\hat{x}_j$’s. We have

$E[\hat{y}^2] = E[y^2] + \text{Var}[\hat{y}] = \mu^2 + \sum_{j \in [n]} \text{Var}[\hat{x}_j] \leq \mu^2 + \sum_{j \in [n]} E[x_j^2]$

Next, we study the fully-correlated estimator. We first prove the following negative correlation between $\hat{x}_j$ and $\sum_{k > j} \hat{x}_k$. Fixing $t_{\leq j}$, we have
Taking the randomness over condition of Lemma 3.1, we have thatObserve that the random variable inequality is by Lemma 4.1; and the third inequality is by Lemma 4.2. Here, the first inequality follows by AM-GM inequality; the second is by convexity of the function .

Fixing the lemma.

\[ \sum_{j \in [n]} \hat{x}_j(t_{\xi_j}) \leq \sum_{j \in [n]} E[\hat{x}_j(t_{\xi_j})] + 2 \sum_{j \in [n]} \hat{x}_j(t_{\xi_j}) \cdot \sum_{k=j} \hat{x}_k(t_{\xi_k}) \leq \sum_{j \in [n]} E[\hat{x}_j(t_{\xi_j})] + 2 \sum_{j \in [n]} E[\hat{x}_j(t_{\xi_j})(1 - \hat{x}_j(t_{\xi_j}))] = 2\mu - \sum_{j=1}^n E[\hat{x}_j^2(t_{\xi_j})], \]

where the inequality is by Equation (4.1). We finish the proof of the lemma. \( \square \)

**Lemma 4.2.** For all \( j \in [n] \), \( E[\hat{x}_j^2] \leq E[\hat{x}_j^2] \).

**Proof.** By definition, \( \hat{x}_j \) only depends on type \( t_j \).

\[ \hat{x}_j(t_j) = \Pr[(u, v_j) \in \text{OPT}(t_j, \hat{\hat{t}}_j)], \]

\[ \hat{x}_j(t_{\xi_j}) = \Pr[(u, v_j) \in \text{OPT}(t_{\xi_j}, \hat{\hat{t}}_{\xi_j})]. \]

Fixing \( t_j \), \( \hat{x}_j \) is a constant and \( E_{t_j} [\hat{x}_j \mid t_j] = \hat{x}_j(t_j) \). Together with the convexity of the function \( x^2 \), this implies

\[ \hat{x}_j^2(t_{\xi_j}) = \hat{x}_j(t_{\xi_j})^2 \leq E[\hat{x}_j^2(t_{\xi_j})] \mid t_j]. \]

Taking the randomness over \( t_j \) concludes the proof of the lemma. \( \square \)

Finally, we have

\[ E[y^2] = E \left( \left( \frac{\hat{y} + \hat{y}}{2} \right)^2 \right) \leq \frac{1}{2} \cdot \left( E[\hat{y}^2] + E[\hat{y}^2] \right) \leq \frac{1}{2} \cdot 2\mu^2 + \mu + \frac{1}{2} \sum_{j \in [n]} \left( E[\hat{x}_j^2] - E[\hat{x}_j^2] \right) \leq \frac{1}{2} \cdot 2\mu^2 + \mu. \]

Here, the first inequality follows by AM-GM inequality; the second inequality is by Lemma 4.1; and the third inequality is by Lemma 4.2. Observe that the random variable \( y \) satisfies the first and second condition of Lemma 3.1, we have that

\[ E[\min(y, 1)] \geq 0.646 \cdot E[y] \quad \text{and} \quad E[p(y)] \geq 0.634 \cdot E[y]. \]

This holds for every vertex \( u \in L \). We conclude the proof of the theorem by Lemma 2.2. \( \square \)

5 I.I.D. ARRIVALS

In this section, we prove our main theorem for the i.i.d. arrival setting.

**Theorem 5.1.** There exists a 0.731-competitive unbiased estimator for the i.i.d. vertex-weighted fractional online stochastic matching. Furthermore, by rounding it with OCS, the algorithm is 0.704-competitive.

In Section 5.1, we first describe the space of all unbiased estimators we study and the intuitions we collect from the warm-up analysis. In Section 5.2, we provide our algorithm. The full analysis of our algorithm is provided in the full version of this paper.

5.1 Space of Unbiased Estimators

We study the following family of estimators and their convex combinations.

**Subset-Resampling Estimators.** Fix an arbitrary index set \( I_j \subseteq [j] \) that contains \( j \) itself (i.e. \( j \in I_j \)). Consider the following estimator:

\[ x_{u,j}(t_j) = \Pr_{t_{\xi_j} \in I_{\xi_j}} (u, v_j) \in \text{OPT} \mid t_j, \]

where \( t_j \) denotes the types \( t_i \) of each \( i \in I_j \). It is straightforward to check this is an unbiased estimator. Observe that the independent estimator corresponds to \( I_j = [j] \) and the fully-correlated estimator corresponds to \( I_j = [j] \).

Recall that we are aiming for an unbiased estimator with minimum variance. Fix an arbitrary vertex \( u \in L \), the objective is to minimize

\[ E \left[ y_{u,i}^2 \right] = \sum_{j=1}^n E[t_{\xi_j}] \cdot x_{u,j}^2 \leq 2 \sum_{j=1}^n \sum_{k=j+1}^n E[t_k] \cdot x_{u,j} \cdot x_{u,k}. \]

Thus, we are aiming for two goals at the same time: 1) to minimize the variance of each \( x_{u,j} \), and 2) to negatively correlate \( x_{u,j} \) and \( x_{u,k} \).

For the first goal, we have the following intuition that generalizes Lemma 4.2.

**For all \( I_j \subseteq I \), \( E \left[ x_{u,j}^2 \right] \leq E \left[ (x_{u,j}^2)^2 \right] \). This is because \( x_{u,j}^2(t_j) = E_{t_{\xi_j}} (x_{u,j}^2) \cdot t_j, \forall t_j \). An immediate implication of this observation is that the independent estimator gives the smallest \( (x_{u,j})^2 \) among all choices of \( I \). Moreover, we should consider using index sets with smaller sizes in order to minimize the first part.**

For the second goal, we have the following intuition from Lemma 4.1. Recall that in the analysis of fully-correlated estimator, we proved that \( \sum_{k=j} x_{u,k}^2 \leq 1 \) in Equation (4.1). We interpret this property as a negative correlation between \( x_{u,j} \) and \( x_{u,k} \). Indeed, since the two random variables share the same types \( t_{\xi_j} \), the more likely \( (u, v_j) \) is matched in the optimal matching, the less likely \( (u, v_k) \) is matched in the optimal matching. More generally, we observe this negative correlation is actually caused by \( t_j \) alone. Indeed, we have that

**For any \( I_j \subseteq [j], I_k \subseteq [k], j < k \), \( E \left[ x_{u,j} \cdot x_{u,k} \right] \leq E \left[ x_{u,j} \cdot x_{u,k} \right] \).**
This observation suggests a different design of the unbiased estimator by adding \( j \) into \( I_k \). In the extreme case when we have \( j \in I_k \) for all \( j < k \), the only choice is to apply the fully-correlated estimator.

Observe that the two goals are contradictory to each other. Therefore, we must quantify the trade-off between the two intuitions and design our unbiased estimator in a careful way. In the warm-up section, we simply apply an even mix between the two extreme estimators, the independent estimator and the fully-correlated estimator. Next, we provide an improved estimator for the i.i.d. arrival setting.

5.2 Our Algorithm

We consider a restricted subfamily of subset-resampling estimators. 

**Windowed Estimators.** We use \( x_{u,j}^{(r)} \) to denote \( x_{u,j}^{[r-1+1,j]} \), which we call windowed estimators, as it fix the types \( t^{[r-1+1,j]} \) of the last \( r \) arrived vertices and resample the remaining types. I.e.,

\[
x_{u,j}^{(r)}(t_{j-r+1,j}) = \Pr_{t^{[n][j-r+1,j]}}[(u, v_j) \in \text{OPT} | t^{[j-r+1,j]}].
\]

**Mix of WE.** We apply the correlated estimation \( x_{u,i}(t) \) as a linear combination of windowed estimators: for each \( j \in [n] \),

\[
x_{u,i}(t_{j}) = \frac{\beta}{n} \sum_{r=1}^{j-1} x_{u,i}^{(r)}(t_{j}) + \left( 1 - \frac{j-1}{n} \beta \right) x_{u,i}^{(j)}(t_{j}),
\]

where \( \beta = 0.79 \) is a optimized constant used for all \( j \)'s.

The following lemma is the most technical of our analysis for i.i.d. arrivals. The proof of this lemma is given in the full version. Finally, applying the lemma together with Lemma 2.2 and 3.1, we conclude the proof of Theorem 5.1.

**Lemma 5.1.** The above mix of windowed estimator satisfies that 

\[
E[y_u^2] \leq 1.05771 \cdot E[y_u] + 0.231 \cdot (E[y_u])^2, \forall u \in L.
\]

6 NON I.I.D. ARRIVALS

In this section, we focus on the independent estimator for non i.i.d. vertex arrivals and characterize the worst-case instance for it.

**Theorem 6.1.** The independent estimator is 0.718-competitive for the non i.i.d. vertex-weighted fractional online stochastic matching problem. By rounding it with OCS, the algorithm is 0.666-competitive. Moreover, the independent estimator achieves the best competitive ratio among all subset-resampling estimators.

The full proof of Theorem 6.1 is deferred to Section 6.4. Moreover, we complement it with the following hardness result.

**Theorem 6.2.** No algorithm achieves a competitive ratio better than 0.75 for the unweighted fractional online stochastic matching problem.

We first introduce some notations and definitions in Section 6.1 and then provide a proof sketch of Theorem 6.1 in Section 6.2. After that, we present the proof of Theorem 6.2 in Section 6.3. Finally, we show the full proof of Theorem 6.1 in Section 6.4 and present the details of our experiments in Section 6.5.

Previously, we have defined independent estimator with respect to OPT. Now we generalize this definition to any general selection rule.

**Independent Estimator for Selection Rule.** On the arrival of each vertex \( v_j \), fix its type \( t_j \) and calculate the probability that \( j \) is selected by \( r \) when the types of all other vertices are resampled. I.e.,

\[
x_{u,j}^{(r)}(t_j) = \mathbb{E}_{t_{j-1}}[r_j(t_j, \hat{t}_{j-1})].
\]

Note \( x_{u,j}^{(r)}(t_j) \) is exactly the independent estimator defined in Section 4. Accordingly, we denote the cumulative mass of each element \( u \) as \( y_u^r(t) = \sum_{j \in [n]} x_{u,j}^{(r)}(t_j) \).

6.1 Selection Rules and Independent Estimator

Fix an offline vertex \( u \). We are only interested in the matching status of \( u \) in the optimal matching and we think of OPT as a selection rule. That is, given the type vector \( t = (t_1, t_2, \ldots, t_n) \) of online vertices, OPT selects at most one of \( j \in [n] \) and matches \( u \) to \( v_j \). I.e.

\[
\text{OPT}(t) = \begin{cases} 
\perp & \text{if } u \text{ is unmatched in the optimal solution.} \\
 j & \text{if } u \text{ is matched to } v_j \text{ in optimal solution.}
\end{cases}
\]

**Selection Rule.** Formally, we define a (possibly randomized) selection rule \( r \) to be a function that maps a type vector to a distribution over \([n] \cup \{ \perp \}\).

\[
r : T_1 \times T_2 \times \cdots \times T_n \mapsto \Delta ([n] \cup \{ \perp \})
\]

where \( T_i \) is the support of the distribution \( D_i \) for every \( i \in [n] \). \( \Delta ([n] \cup \{ \perp \}) \) is the family of all distributions supported on \([n] \cup \{ \perp \}\). For \( j \in [n] \cup \{ \perp \} \), we use \( r_j(t) \) to denote the probability mass of \( j \). Specifically, \( r_j(t) \) is the probability of selecting nothing.

We will crucially use the following special class of selection rules, called **permutation rules**. Each permutation rule specifies a total order \( \pi \) over a subset \( Y \) of identity-type pairs \((j, t_j) \in [n] \times T_j\), and then selects the vertex whose identity together with its realized type appears first according to \( \pi \). Note that when none of the \((j, t_j)\)'s belongs to \( Y \), we select nothing and return \( \perp \). See the following for a more formal definition.

**Definition 6.3 (Permutation Rule \( r^\pi \)).** Suppose \( \pi \) is a permutation over \( Y \subset \{(j, t_j) \mid t \in T_j\}\). Based on \( \pi \), \( r^\pi \) denotes the following deterministic selection rule.

**Algorithm 2: Permutation Rule \( r^\pi(t) : T_1 \times T_2 \times \cdots \times T_n \mapsto [n] \cup \{ \perp \} \)**

1. for \( i \leftarrow 1 \) to \( |\pi| \) do
2. Suppose \( \pi_i = (j, t) \).
3. if \( t_j = t \) then
4. Select \( j \) and return \( r^\pi(t) = j \).
5. end
6. end
7. Select nothing and return \( r^\pi(t) = \perp \).

6.2 Selection Rule and Independent Estimator

We now give a formal definition of the independent estimator by taking \( r \) to be a permutation rule. Based on \( \pi \) and \( r^\pi \), we define the independent estimator as

\[
x_{u,i}(t_{j}) = \mathbb{E}_{t_{j-1}}[r_j(t_j, \hat{t}_{j-1})].
\]

Note \( x_{u,j}^{\text{OPT}}(t_j) \) is exactly the independent estimator defined in Section 4. Accordingly, we denote the cumulative mass of each element \( u \) as \( y_u^r(t) = \sum_{j \in [n]} x_{u,j}^{(r)}(t_j) \).
6.2 Proof Sketch

By Lemma 2.2, our algorithm is $\Gamma$-competitive if for all $u \in L$, $E[f(y^\text{OPT}_u)] \geq \Gamma \cdot E[y^\text{OPT}_u]$ (where $f(y) = \min(y, 1)$ or $p(y)$).

As discussed above, we think of OPT as a selection rule and strengthening the above statement: for an arbitrary selection rule $r$ and $u \in L$, $E[f(y^r_u)] \geq \Gamma \cdot E[y^r_u]$ holds. Therefore, the competitive ratio $\Gamma$ is lower bounded by the following optimization problem:

$$\inf_{D_1, D_2, \ldots, D_n} \inf_{\pi} \frac{E[f(y^r_u)]}{E[y^r_u]}$$

(6.1)

Our proof contains the following two steps:

- We first solve the inner optimization. We prove that for any fixed mean $\mu = E[y^r_u]$, for any concave function $f$, the value of $E[f(y^r_u)]$ is minimized by permutation rules.\footnote{We remark that the statement does not hold for all fixed distributions $D_1, D_2, \ldots, D_n$. On the other hand, we show that by optimizing the distributions simultaneously, we can without loss of generality restrict ourselves to permutation rules. See Lemma 6.1 for the formal statement.}

- Next, we solve the outer optimization. For any fixed mean $\mu = E[y^r_u]$, we characterize the worst case arrival distributions that minimize $E[f(y^r_u)]$ under permutation rules.

Putting them together, the performance of the worst-case selection rule on its worst-case arrival for independent estimator gives the optimal constant $\Gamma$ for the above optimization problem. Furthermore, we show that under the worst-case distribution for independent estimator, all subset-resampling estimators have the same behavior as independent estimator, that implies the optimality of independent estimators among all subset-resampling estimators. Finally, we use computer assistance to calculate the numerical value of $\Gamma$.

Below, we elaborate the two steps for solving the optimization problem in more detail.

**Selection Rule.** We formulate the inner optimization over $r$ by an optimization problem $CO$ with concave objective and (matroid) polytope constraints. By concavity, its optimum must be achieved at extreme points of the polytope. Next, we adopt the approach in [14] to prove that every extreme point of the polytope corresponds to a permutation $\pi$ over $Y \subseteq \{(j, i) \mid i \in S_j\}$.

**Arrival Distributions.** Next, we fix an arbitrary offline vertex $u \in L$ and a permutation rule $r$, and optimize over all possible arrival distributions. We prove that a careful subdivision of an offline vertex could decreases the value of $E[f(y^r_u)]$ while preserving the mean $\mu = E[y^r_u]$. Informally, the subdivision increases the variance of $y^r_u$ and by the concavity of $f$, the value of $E[f(y^r_u)]$ decreases. By applying the subdivision process iteratively, the worst case is achieved in the limit when each online vertex is infinitesimal.

6.3 Hardness Result for Non I.I.D. Arrival

In this subsection, we give an upper bound for the fractional online matching problem with non i.i.d. arrivals.

**Proof of Theorem 6.2:** Consider the following instance with 2 offline vertices and 2 online vertices. Suppose offline vertices are $\{u_1, u_2\}$, and online vertices are $\{v_1, v_2\}$. The vertices $v_1, v_2$ arrive in order, and $v_1$ deterministically connects to $u_1$ and $u_2$. For $v_2$, with probability 0.5, it will have an edge to $u_1$, otherwise it will connect to $u_2$. The vertex weight for each offline vertex is 1 (i.e., the unweighted case).

Figure 6.1: The hard instance for fractional general arrival matching

Notice that this instance will always have a perfect matching. Hence, the offline optimum equals 2. However, any algorithm that matches $x_1$ fraction to $u_1$ and $x_2$ fraction to $u_2$ for the first online vertex $v_1$ will receive an expected total weight of at most 1.5 after the realization of $v_2$, since

$$E[\text{ALG}] \leq \frac{1}{2} (x_1 + 1) + \frac{1}{2} (x_2 + 1) \leq \frac{3}{2},$$

where the first inequality follows from the fact that any vertex can match to at most a fraction of 1, and the second inequality uses the fact that $x_1 + x_2 \leq 1$. Therefore, no algorithm can achieve a competitive ratio better than $1.5/2 = 0.75$, that conclude the proof of the theorem. $\blacksquare$

6.4 Proof of Theorem 6.1

6.4.1 Permutation Rules. We prove that permutation rules are the worst among all selection rules for the inner optimization of (6.1) introduced in Section 6.2. Formally, we prove Lemma 6.1 in this subsection.

**Definition 6.4.** For a distribution $D$ with support $T = \{a_1, a_2, \ldots, a_n\}$, we say the distribution $D'$ with support $T' = S_1 \cup S_2 \cup \cdots \cup S_n$\footnote{We use $\sqcup$ to denote a disjoint union of sets.} is a refinement of $D$ if and only if for all $i, \sum_{a \in S_i} p^{D'}(a) = p^D(a)$, where $p^{D}, p^{D'}$ are the probability mass functions of distributions $D, D'$ respectively.

Moreover, for a sequence of distributions $D = (D_1, D_2, \ldots, D_n)$, we say $D' = (D'_1, D'_2, \ldots, D'_n)$ is its refinement if and only if each $D'_i$ is a refinement of $D_i$.\footnote{We remark that the statement does not hold for all fixed distributions $D_1, D_2, \ldots, D_n$. On the other hand, we show that by optimizing the distributions simultaneously, we can without loss of generality restrict ourselves to permutation rules. See Lemma 6.1 for the formal statement.}
where the inequality follows from the simple observation that for each type vector \( t \), the selection \( r \) selects at most one of the \( j \in [n] \). The last equality follows from the definition of function \( g \). \( \square \)

Now, consider the optimization problem \( \text{CO} \) below, where \( g(S_1, S_2, \ldots, S_n) \) is defined as in Lemma 6.2. Let \( \mathcal{P}(t) \) \( \triangleq \mathcal{P}_{t_1}(t_1) \cdots \mathcal{P}_{t_n}(t_n) \) be the probability that types \( t \) are realized.

\[
\begin{align*}
\text{minimize:} & \quad \sum_{t} \mathcal{P}(t) \cdot f \left( \sum_{j \in [n]} z_j(t) / \mathcal{P}(t_j) \right) \\
\text{subject to:} & \quad \sum_{j \in [n]} \sum_{t \in T_j} z_j(t) = \mu \\
& \quad \sum_{j \in [n]} \sum_{t \in T_j} z_j(t) \leq g(S_1, S_2, \ldots, S_n) \\
& \quad \forall (S_1, S_2, \ldots, S_n) : S_i \subseteq T_i \\
& \quad z_j(t) \geq 0 \quad \forall j \in [n], t \in T_j
\end{align*}
\]

By Lemma 6.2, we have that for every selection rule \( r \) and every offline vertex \( u \in L \), \( \mathcal{E}_D[f(y_u^r)] \geq \text{OPT}_D \), where \( \text{OPT}_D \) is the optimal solution to the above optimization problem. Next, we adapt the following lemma from [14] to show that the optimal solution of CO corresponds to a permutation \( \pi \). We define the following notations. For a set sequences \( A = (A_1, A_2, \ldots, A_n) \), we interchangeably view it as a set of index-type pairs \( A = \{(i, t) \mid t \in A_i\} \). Consequently, we use 1) \( A \subseteq B \) to denote that \( i \in [n], \ A_i \subseteq B_i \) 2) \( A \setminus B \) to denote the set \( \{(i, t) \mid t \in A_i \setminus B_i\} \), and 3) \( A \cup B \) (and \( A \cap B \)) to denote the set sequence \( (A_1 \cup B_1, A_2 \cup B_2, \ldots, A_n \cup B_n) \) (and \( (A_1 \setminus B_1, \ldots, A_n \setminus B_n) \)).

**Lemma 6.3 (Lemma 12 of [14], Adapted).** For any concave function \( f(y) \), let the optimal solution of CO be \( z^*_j(t) \) and \( Y = \{(j, t) \mid z^*_j(t) > 0\} \). Then, there exists a sequence of set sequences \( \tilde{S}_1 \subseteq \tilde{S}_2 \subseteq \cdots \subseteq \tilde{S}_\lceil |Y| \rceil \subseteq T = (T_1, T_2, \ldots, T_n) \) such that:

- For \( 1 \leq k \leq |Y| - 1 \), the constraint (6.3) corresponding to \( \tilde{S}_k \) is tight. I.e., suppose \( \tilde{S}_k = (S_{k,1}, S_{k,2}, \ldots, S_{k,n}) \) where \( S_{k,i} \subseteq T_i \), then
  \[
  \sum_{j \in [n]} \sum_{t \in S_{k,j}} z^*_j(t) = g(S_{k,1}, S_{k,2}, \ldots, S_{k,n})
  \]

- For \( k = |Y| \), we have
  \[
  \sum_{j \in [n]} \sum_{t \in S_{k,j}} z^*_j(t) = \min \left( \mu, g(S_{|Y|,1}, S_{|Y|,2}, \ldots, S_{|Y|,n}) \right)
  \]

Moreover, \( |\tilde{S}_k \setminus \tilde{S}_{k-1}| = 1 \) for every \( 1 \leq k \leq |Y| \), where \( \tilde{S}_0 = \emptyset \); and \( \tilde{S}_{|Y|} = Y \) corresponds to the set of non-zero variables.

Proof. Without loss of generality, we assume that \( 0 < \mathcal{P}(t) < 1 \) for all types \( t \). Then the function \( g(S_1, S_2, \ldots, S_n) \) is strictly submodular (c.f. Lemma 11 of [14]).

Note that the objective function of CO is a composition of the concave function \( f(y) \) and a linear function of the variables \( \{z_j(t_j)\}_{j \in [n], t_j \in T_j} \). Thus, the objective is concave and the optimum value must be attained at one of the extreme points of the polytope defined by the constraints. Thus, it suffices to show that all extreme points of the polytope satisfy our claim.

For extreme point \( (z_j(t_j)) \) of the polytope, let \( Y = \{(j, t) \mid z_j(t_j) > 0\} \) be the set of non-zero variables. Since it is an extreme point, at least \( |Y| \) constraints of (6.2) and (6.3) are tight. Let \( A = (A_1, A_2, \ldots, A_n) \)
where the last inequality follows from constraint (6.3). However, if \( A \not\subseteq B \) and \( B \not\subseteq A \), inequality (6.4) will contradict the strict submodularity of function \( g \). Thus, for any \( A, B \) whose corresponding constraints of (6.3) are tight, either \( A \subseteq B \) or \( B \subseteq A \).

Consequently, if \(|Y|\) constraints (6.3) are tight, then there must be \(|Y|\) corresponding sets \( S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{|Y|} \) and for all \( 1 \leq i \leq |Y| \), \( S_i \setminus S_{i-1} = 1 \).

We are left with the case when \(|Y| - 1\) constraints in (6.3) are tight and (6.2) is tight. Let \( S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{|Y|-1} \) be \(|Y| - 1\) set sequences whose corresponding constraints of (6.3) are tight. In this case, we let \( \tilde{S}_{|Y|} = Y \) and verify that \( \tilde{S}_{|Y|-1} \subseteq \tilde{S}_{|Y|} \). For the sake of contradiction, suppose \( \tilde{S}_{|Y|-1} \neq \tilde{S}_{|Y|} \). Then we must have \( g(\tilde{S}_{|Y|-1}) = \mu \) because \( \tilde{S}_{|Y|-1} \) contains all nonzero variables and the corresponding constraint of (6.3) is tight. The constraint (6.2) is therefore dominated by \( \sum_{j \in [n] \setminus \tilde{S}_{|Y|-1}} z_j(t) \leq g(\tilde{S}_{|Y|-1}) \) in (6.3). This implies that (6.2) can be removed, and there must be \(|Y| - 1\) tight constraints in (6.3) contradicting our assumption of having only \(|Y| - 1\) tight constraints in (6.3). So we must have that \( S_{|Y|-1} \) is a strict subset of \( S_{|Y|} \).

Consequently, we also have \( S_i \setminus S_{i-1} = 1 \) for all \( 1 \leq i \leq |Y| \). This finishes our proof. \( \square \)

Now, we are ready to complete the proof of our main lemma.

**Proof of Lemma 6.1:** Given distributions \( D \) and \( u \in L \), we solve the corresponding optimization problem and let \( \{z_j(t)\} \) be the optimal solution. Applying Lemma 6.3, we get \(|Y|\) set sequences \( S_1, S_2, \ldots, S_{|Y|} \) that satisfy the stated properties. Consider the two cases below.

**Case 1.** If \( g(S_{|Y|}) \leq \mu \), we simply define \( D' = D \) and \( \pi_t \) be the unique element in \( \tilde{S}_t \setminus \tilde{S}_{t-1} \). We verify that \( \text{OPT}_G = E_{D'} \{ f(y_u^{\tau_*}) \} \). For each \( i \), suppose \( \pi_t = (j, t) \). We have

\[
x_i^\tau(t)P_j(t) = E_{D'} [r_i^\tau(t) \cdot 1_{\{t_j = t\}}] = g(\tilde{S}_t) - g(\tilde{S}_{t-1})
\]

Consequently,

\[
E_{D'} \{ f(y_u^{\tau_*}) \} = \sum_{t} P(t) \cdot f \left( \sum_{j \in [n]} r_i^\tau(t) \right) = \text{OPT}_G \leq E_{D'} \{ f(y_u) \}.
\]

**Case 2.** If \( g(S_{|Y|}) > \mu \), consider the following refinement. Let the unique element in \( \tilde{S}_{|Y|} \setminus \tilde{S}_{|Y|-1} \) be \((j^*, t^*)\). We split the type \( t^* \) into two types \( t_a \) and \( t_b \) and modify the distribution to \( D_{j^*} \) with \( P_{D_{j^*}}(t_a) + P_{D_{j^*}}(t_b) = P_{D_{j^*}}(t^*) \). We explain below how the values of \( P_{D_{j^*}}(t_a), P_{D_{j^*}}(t_b) \) are chosen.

For each \( 1 \leq i \leq |Y| - 1 \), let \( \tilde{S}_i \subseteq S_i \). Suppose \( \tilde{S}_{|Y|} = (S_{|Y|}, S_{|Y|-1}, \ldots, S_{|Y|-1}, \ldots, S_{|Y|-1}, \ldots, S_{|Y|}) \). Let \( \tilde{S}_{|Y|} = (S_{|Y|}, S_{|Y|-1}, \ldots, (S_{|Y|} \setminus \{t^*\}) \cup \{t_a\}, \ldots, S_{|Y|}) \). Note that \( g(S_{|Y|}) \) is monotone in \( P_{D_{j^*}}(t_a) \), we therefore can set the value of \( P_{D_{j^*}}(t_a) \) so that \( g(\tilde{S}_{|Y|}) = \mu \).

For each \( j \neq j^* \), let \( D_j = D_j \). Define \( \pi_t \) to be the unique element in \( \tilde{S}_t \setminus \tilde{S}_{t-1} \) for \( 1 \leq i \leq |Y| \). We verify that \( E_{D_j} \{ f(y_u^{\tau_*}) \} \leq \text{OPT}_G \). Similar to the previous case, we first calculate the value of \( x_i^\tau(t) \cdot P_{D_j} \) for each \( \pi_t = (j, t) \). We consider the two cases depending on the type \( t \):

- If \( t \neq t_a \), we have that
  \[
x_i^\tau(t)P_{D_j}(t) = E_{D_j} [r_i^\tau(t) \cdot 1_{\{t_j = t\}}] = g(\tilde{S}_t) - g(\tilde{S}_t) = \sum_{j \in [n]} \sum_{t' \in S_{t+1}} z_j(t') - \sum_{j \in [n]} \sum_{t' \in S_{t+1}} z_j(t') = z_j(t).
  \]

- If \( t = t_a \), we have that
  \[
x_i^\tau(t_a)P_{D_j}(t_a) = E_{D_j} [r_i^\tau(t_a) \cdot 1_{\{t_j = t\}}] = g(\tilde{S}_t) - g(\tilde{S}_{t-1}) = \sum_{j \in [n]} \sum_{t' \in S_{t+1}} z_j(t') - \sum_{j \in [n]} \sum_{t' \in S_{t+1}} z_j(t') = z_j(t^*).
  \]

Therefore, we have

\[
E_{D_j} \{ f(y_u^{\tau_*}) \} = \sum_{t} P(t) \cdot f \left( \sum_{j \in [n]} x_i^\tau(t) \right) = \sum_{t \neq t_a} P(t) \cdot f \left( \sum_{j \in [n]} z_j(t) \right) + \sum_{t \neq t_a} P(t) \cdot f \left( \sum_{j \in [n]} z_j(t_a) \cdot P_{D_j}(t_a) \right) = \text{OPT}_G.
\]
This concludes the proof of $\P_{t-D}[f(y_i^*)] \leq \OPT_{CO} \leq \P_{t-D}[f(y_i^*)]$.  

\[\textbf{6.4.2 Arrival Distributions.}\] By Lemma 6.1, for any distribution $D$ and selection rule $r$, there exists a refinement $D'$ and a permutation rule $r''$ which gives worse ratio for the optimization problem (6.1). Hence, it is without loss of generality restricting ourselves to permutation rules. We explicitly find the worst-case arrivals under permutation rules in this subsection. Specifically, we construct a splitting operation of online vertices, and prove that the value $\E[f(y_i)]$ always decreases after each split, given $f$ is concave.\footnote{In the proof of Lemma 6.1, we did a similar operation which split the type $t' \in T_j$, of an online vertex $v_j$. It worths pointing out the difference that here the number of online vertex is increased after splitting, while in Lemma 6.1, the number of vertices is fixed.} Before formally state our main lemma, we set up necessary notations and define the operation.

\textbf{Notations.} We shall fix an offline vertex $u \in L$ in the analysis. For notation simplicity, we drop the subscript of $u$. We focus on instances $I = (\pi, R, D)$ that are parameterized by a permutation $\pi$, a set of all online vertices $R$ and corresponding distributions $D = (D_1, D_2, ..., D_{|R|})$. We use $T_j, P_j$ to denote the support and the probability mass function of distribution $D_j$ for each $j$.

\textbf{Splitting an Online Vertex.} Given an instance $I = (\pi, R, D)$, a vertex $v_j \in R$, and an arbitrary $\epsilon > 0$. Consider the following splitting operation.

Let $T_j = \{a_1, a_2, ..., a_k, \emptyset\}$ be the support of vertex $v_j$, where we use $\emptyset$ to represent the types that are not involved in the support of $\pi$. I.e., those types would not be selected by the permutation rule $r''$ and we can safely merge them to a single type that we denote by $\emptyset$. Suppose the types $\{a_1, a_2, ..., a_k\}$ are in ascending order with respect to $\pi$, i.e. $(j, a_1) < (j, a_2) < \cdots < (j, a_k)$. We split the vertex $v_j$ into two vertices $v_{j'}$ and $v_{j''}$ with the following distributions:

\[
 t_j' = \begin{cases} 
 a_1 & \text{w.p. } p_j(a_1) \epsilon, \\
 a_k & \text{w.p. } p_j(a_k) (1-\epsilon), \\
 \emptyset & \text{w.p. } 1-\epsilon. 
\end{cases}
\]

We denote the two distributions by $D_j', D_j''$ and they are well-defined for $\epsilon \in (0, P_j(a_1)]$.

Consider a new instance with online vertices $R' = R \setminus \{v_j\} \cup \{v_{j'}, v_{j''}\}$, distributions $D' = (D_j', D_j', D_j'')$, and the permutation $\pi'$ defined below. Suppose

\[
 \pi' = (j, a_1), (j, a_2), \ldots, (j, a_k).
\]

Then $\pi'$ is defined as

\[
 \pi' = (j', a_1'), (j'', a_2''), (j', a_2'), (j', a_k'), (j', a_k), (j, a_1), \ldots, (j, a_k).
\]

That is, we change $(j, a_i)$ to $(j', a_i)$ for every $i \geq 2$ and substitute $(j', a_1'), (j'', a_2'')$ for $(j, a_i)$. The order of other types remains unchanged.

Now we are ready to state the main lemma of this subsection. It states that the instance $I'$ after an arbitrary split is always worse than the original instance $I$. Therefore, the worst competitive ratio is achieved in the limit case when we keep splitting the instance. The implication of the lemma shall be explained in the next subsection.

\textbf{Lemma 6.4.} For any instance $I = (\pi, R, D)$, let $I' = (\pi', R', D')$ be the instance after splitting $v_j \in R$. For any concave function $f$, we have

\[
\frac{\P_{t-D}[f(y_i^*)]}{\P_{t-D'}[f(y_i^*)]} \geq \frac{\P_{t-D}[f(y_i^*)]}{\P_{t-D'}[f(y_i^*)]}.
\]

\textbf{Proof.} For notation simplicity, within the proof of the lemma, we use $x_i(t), x'_i(t)$ to denote $x_i^*(t), x'_i^*(t)$, and $y(t), y'(t)$ to denote $y^*(t), y'^*(t)$.

We first observe that the mean of $y, y'$ are equal:

\[
\E_{t-D}[y] = 1 - \prod_{i \in R} p_{D_i}(\emptyset) = 1 - \prod_{i \notin j, a_1 \in R} p_{D_j'}(\emptyset) \cdot p_{D_{j'}}(\emptyset) = \E_{t-D'}[y'],
\]

where the third equality follows from the definition of $D_j'$ and $D_{j''}$.

Hence, it suffices to prove that $\P_{t-D}[f(y)] \geq \P_{t-D'}[f(y')]$. Note we have

\[
y(t) = \sum_{v_j \in R} x_i(t), \quad y'(t) = \sum_{v_j \in R} x'_i(t) + x'_i(t) + x'_i(t).
\]

It is straightforward to verify that for every $t \neq j$, the value of $x_i(t) = x_i'(t)$. This follows by the definition of the permutation rule and we omit the tedious proof.

Moreover, our splitting operation guarantees that for arbitrary constant of $c$:

\[
\E_{t_j}[f(c + x_j(t))] \geq \E_{t_j',t''}[f(c + x'_j(t) + x''_j(t))]
\]

whose proof is deferred to the end of the subsection. Combining this inequality with the fact that $x_i(t) = x_i'(t)$ for $t \neq j$, we have that

\[
\E_{t-D}[f(y(t))] \geq \E_{t-D'}\left[\sum_{v_j \in R, t \notin j} x_i(t_j) + x_i(t_j)\right] \geq \E_{t-D'}\left[\sum_{v_j \in R, t \notin j} x'_i(t_j) + x'_i(t_j)\right] = \E_{t-D'}[f(y'(t))],
\]

that concludes the proof of the lemma. \hfill $\square$

\textbf{Proof of (6.7):} We define $g(x) = f(c + x)$ which is also a concave function. We first expand the left hand side of (6.7) by definition. Suppose $T_j = \{a_1, a_2, ..., a_k, \emptyset\}$. We have

\[
\E_{t_j, D_j}[g(x_j(t_j))] = \sum_{i=1}^{k} p_{D_i}(a_i) \cdot g(x_j(a_i)) + \left(1 - \sum_{i=1}^{k} p_{D_i}(a_i)\right) \cdot g(0).
\]

(6.8)
We also expand the right hand side. Suppose $T_P = \{a'_1, a_2, \ldots, a_k, \emptyset\}$. For simplicity, we define $\tilde{a}_1 = a'_1$, and $\tilde{a}_i = a_i$ for $i > 1$.

For $t_P \neq \emptyset$, we get (6.10).

\[
\begin{align*}
g(x'_P(t_P) + x''_P(t_P)) &= \frac{g(x'_P(\tilde{a}_i) + x''_P(\tilde{a}_i))}{\omega \cdot \mathcal{P}D'j(\tilde{a}_i)} \
g(x'_P(\tilde{a}_i)) &= \frac{g(x'_P(\tilde{a}_i))}{\omega \cdot \mathcal{P}D'j(\tilde{a}i)} \
g(x''_P(\tilde{a}_i)) &= \frac{g(x''_P(\tilde{a}_i))}{\omega \cdot \mathcal{P}D'j(\tilde{a}i)} \
g(0) &= \frac{g(0)}{\omega \cdot \mathcal{P}D'j(\tilde{a}i)}.
\end{align*}
\]

And we get the following equation:

\[
\begin{align*}
\mathbb{E}_{t_P \neq \emptyset | D'j} & \left[ g \left( x'_P(t_P) + x''_P(t_P) \right) \right] \\
&= \sum_{i=1}^{k} \left[ \mathcal{P}D'j(\tilde{a}_i) \cdot g \left( x'_P(\tilde{a}_i) + x''_P(\tilde{a}_i) \right) + (1 - \varepsilon) \mathcal{P}D'j(\tilde{a}_i) \cdot g \left( x'_P(\tilde{a}_i) \right) \right] \\
&\quad + \left[ 1 - \sum_{i=1}^{k} \mathcal{P}D'j(\tilde{a}_i) \right] \cdot \varepsilon \cdot g \left( x''_P(\tilde{a}_i) \right) \\
&\quad + \left[ 1 - \sum_{i=1}^{k} \mathcal{P}D'j(\tilde{a}_i) \right] \cdot (1 - \varepsilon) \cdot g(0)
\end{align*}
\]

(6.9)

We now handle the terms of (6.9) with $i > 1$.

Claim 6.1. For any concave function $g$, suppose $i > 1$ (which means $\tilde{a}_i = a_i$),

\[
\varepsilon \mathcal{P}D'j(a_i) \cdot g \left( x'_P(a_i) + x''_P(a_i) \right) + (1 - \varepsilon) \mathcal{P}D'j(a_i) \cdot g \left( x'_P(a_i) \right) \leq \mathcal{P}Dj(a_i) \cdot g \left( x_j(a_i) \right) + \varepsilon \mathcal{P}D'j(a_i) \cdot g \left( x''_P(a_i) \right)
\]

(6.10)

Proof. Suppose $d > c > b > a$, and $c > a$, from the concavity of $g$, it follows that

\[
g(d) - g(c) \leq \frac{d - c}{b - a} \left( g(b) - g(a) \right).
\]

Notice that by our definition of splitting, $x'_P(a_i) = (1 - \varepsilon) \cdot x_j(a_i)$ for $i > 1$. We take $d = x'_P(a_i) + x''_P(a_i)$, $c = x''_P(a_i)$, $b = x_j(a_i)$, $a = x'_P(a_i) = x_j(a_i)$, and get

\[
g(x'_P(a_i) + x''_P(a_i)) - g(x'_P(a_i)) \geq \frac{1 - \varepsilon}{\varepsilon} \left( g(x_j(a_i)) - g(x'_P(a_i)) \right)
\]

Rearranging the formula above, we find that (6.14) is equivalent to (6.7) by (6.8) and (6.9).

6.4.3 The Worst-case Distribution for Independent Estimator. Recall that the competitive ratio $\Gamma$ is lower bounded by the following optimization problem:

\[
\inf_{D_1, D_2, \ldots, D_n} \inf_{r} \frac{\mathbb{E}[f(y^D_r)]}{\mathbb{E}[y^D_0]}.
\]

(6.15)

In Section 6.4.1, we show that permutation rules are the worst among all possible selection rules. In Section 6.4.2, we show that splitting the instance while preserving the mean $\mu = \mathbb{E}[y^D_0]$ will only decreases the ratio. Given an instance $I = (\pi, R, D)$, by applying the splitting operation multiple times to every online vertex $v_j \in R$, we first convert the distributions $D$ into “Bernoulli” distributions, i.e., each online vertex $v_j$ either has type $t_j$, or has type $\emptyset$. This can be achieved by setting $\varepsilon = \mathcal{P}j(a_i)$ in the splitting operation. Next, we fix an sufficiently small $\varepsilon$ and continue splitting the instance until the probability of each vertex being realized (i.e., its type is not $\emptyset$) is at most $\varepsilon$. According to Lemma 6.4, the ratio monotonically decreases during the above procedure. Finally, we can remove all vertices with realized probability smaller than $\varepsilon$. The impact of such change can be made vanishingly small as $\varepsilon \rightarrow 0$. At the end, we get a regularized instance defined as the following.

Definition 6.5 (Worst-case distribution for fixed $\mu$). Fixing $\mu = \mathbb{E}[y^D_0]$, $\mathcal{W}_n$ is an instance with $n$ online vertices and a permutation selection rule $r^\pi$. For the $i$-th arrived vertex $v_j$, it has probability $\varepsilon$ to be type $t_i$, and otherwise it has no edges. Here $\varepsilon$ satisfies that $1 - (1 - \varepsilon)^n = \mu$. The permutation selection rule $r^\pi$ always selects the type $t_k$ with the maximal subscripts.

Proof of Theorem 6.1: We first give the competitive ratio for both fractional and integral non-i.i.d. matching and then show that independent estimator reaches the best possible ratio among subset-resampling estimators.

According to the discussion above, the infimum of the optimization problem is achieved in the limit case of $\mathcal{W}_n$ when $n \rightarrow \infty$ for arbitrary fixed number of $\mu$. We use computer assistance to get a
numerical result. We leave the details of our implementations in Section 6.5, and plot the numerical results in the following figure:

![Competitive ratio for different μ](image)

**Figure 6.2: Competitive ratio for different μ**

We conclude that the competitive ratios for fractional and integral matching are respectively 0.718 and 0.666. Finally, notice that in the definition of $\mathcal{W}_n$, we specify the permutation $\pi$ to be the reverse order of the arrival of vertices. Observe that for such instances, all subset-resampling estimators described in Section 5 are equivalent to the independent estimator, since the presence of types before vertex $\alpha_i$ shall not influence the selection of type $\nu_i$ under such permutation rule. In other words, for this restricted family of algorithms, they all achieve the same competitive ratio as the independent estimator for such instances. As we have proved this ratio is the worst case ratio for the independent estimator, we conclude that the independent estimator is optimal among all subset-resampling algorithms. □

### 6.5 Experiments in Non i.i.d. Arrivals

In this section we will present the details of our experiments in non-i.i.d. arrivals. The codes which generate all the numerical results presented in this section are open-sourced at this [link](https://example.com).

As we have shown in Section 6.4.3, the worst case distribution $\mathcal{W}_n$ is as following: Given the parameter $\mu$ which is the probability that the selection function $f^\pi$ will select one online vertex, there are $n$ online vertices arrive in order. For the $i$-th arrived vertex, it has a probability $p$ to have type $t_i$, and otherwise it has an empty type that will never be selected by the selection function $f^\pi$. The selection function $f^\pi$ will always try to select a type with the maximal index, and so for the $j$-th arrived online vertex, the independent estimator $x_j^\pi$ is the probability that no vertex with an index larger than $j$ has a non-empty set, which is $(1 - p)^{n-j}$. Therefore, we could easily draw a sample of

$$y = \sum_{j=1}^{n} x_j^\pi (t_i)$$

from the worst case distribution given $n$ and $\mu$ by the following algorithm:

**Algorithm 3:** Draw one sample from the worst case distribution

**Parameters:** $\mu$: The probability that selection function selects one vertex.

$n$: Number of offline vertices.

**Result:** $y \leftarrow \sum_{j=1}^{n} x_j^\pi (t_i)$: A sample of sum of independent estimators drawn from worst case distribution.

1. $p \leftarrow 1 - (1 - \mu)^n$; // $p$ is the number satisfying $1 - (1 - p)^n = \mu$
2. $y \leftarrow 0$;
3. for $i \leftarrow 1$ to $n$ do
   4. $x_i \leftarrow (1 - p)^{n-i}$, with probability $p$, and $x_i \leftarrow 0$ otherwise
5. $y \leftarrow y + x_i$;
6. end

Notice that the ratio of fractional and integral matching are the minimum of the following optimization problem when $f(x)$ respectively equals to $\min(x, 1)$ and $f(x) = 1 - \exp(-x + \frac{x^2}{2} - \frac{x^3}{3})$:

$$\inf_{D_1, D_2, \ldots, D_n} \inf_{\epsilon} \frac{E[f(D_1)]}{E[y]}$$

We discretize $\mu$ to multiples of 0.01, and set $n$ to be 1000 for our numerical experiment. Due to the concentration bound for summation of independent random variable, our experiment leads to a reasonably accurate estimation of the real performance of our algorithm.

### REFERENCES

[1] Gagan Aggarwal, Gagan Goel, Chinmay Karande, and Aranyak Mehta. 2011. Online Vertex-Weighted Bipartite Matching and Single-Bid Budgeted Allocations. In SODA. 1253–1264. https://doi.org/10.1137/1.9781611973082.95

[2] Bahman Bahmani and Michael Kapralov. 2010. Improved Bounds for Online Stochastic Matching. In ESA (1) (Lecture Notes in Computer Science, Vol. 6346). Springer, 170–181. https://doi.org/10.1007/978-3-642-15775-2_15

[3] Guy Blanc and Moses Charikar. 2022. Multiway online correlated selection. In 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS). IEEE, 1277–1284. https://doi.org/10.1109/FOCS52979.2021.00124

[4] Brian Brubach, Karthik Abinav Srinivasan, and Pan Xu. 2020. Online Stochastic Matching: New Algorithms and Bounds. Algorithmica 82, 10 (2023), 2737–2783. https://doi.org/10.1007/s00453-023-00347-2

[5] Niv Buchbinder, Kamal Jain, and Joseph Naor. 2007. Online Primal-Dual Algorithms for Maximizing Ad-Auctions Revenue. In ESA (Lecture Notes in Computer Science, Vol. 4699). Springer, 253–264. https://doi.org/10.1007/978-3-540-73529-2_24

[6] Niv Buchbinder, Joseph Naor, and David Wajc. 2021. A Randomness Threshold for Online Bipartite Matching, via Locally Online Rounding. CoRR abs/2106.04863 (2021). https://doi.org/10.48550/arXiv.2106.04863

[7] Steven Delong, Alireza Farhadi, Rad Niazadeh, and Balasubramanian Sivan. 2021. Online Bipartite Matching with Reusable Resources. CoRR abs/2110.07084 (2021). https://doi.org/10.48550/arXiv.2110.07084

[8] Nikhil R. Devanur and Kamal Jain. 2012. Online matching with concave returns. In STOC ’22, June 20–24, 2022, Rome, Italy. ACM, 137–144. https://doi.org/10.1145/2213977.2213992

[9] Nikhil R. Devanur, Kamal Jain, Balasubramanian Sivan, and Christopher A. Wülcken. 2019. Near Optimal Online Algorithms and Fast Approximation Algorithms for Resource Allocation Problems. J. ACM 66, 1 (2019), 7:1–7:41. https://doi.org/10.1145/3284177

[10] Tomer Ezra, Michal Feldman, Nick Gravin, and Zhaihao Gavin Tang. 2020. Online Stochastic Max-Weight Matching: Prophet Inequality for Vertex and Edge Arrival Models. In EC. ACM, 769–787. https://doi.org/10.1145/3391403.3399513

[11] Matthew Fahrbach, Zhiyi Huang, Runzhou Tao, and Mortaza Zadimoghaddam. 2020. Edge-Weighted Online Bipartite Matching. In FOCS IEEE, 412–423. https://doi.org/10.1109/FOCS546700.2020.00046
[12] Jon Feldman, Nitish Korula, Vahab S. Mirrokni, S. Muthukrishnan, and Martin Pál. 2009. Online Ad Assignment with Free Disposal. In WINE 2009. 374–385. https://doi.org/10.1007/978-3-642-10841-9_34

[13] Jon Feldman, Aranyak Mehta, Vahab S. Mirrokni, and S. Muthukrishnan. 2009. Online Stochastic Matching: Beat 1-1/e. In FOCS (IEEE Computer Society), 117–126. https://doi.org/10.1109/FOCS.2009.72

[14] Buddhima Gamlath, Sagar Kale, and Ola Svensson. 2019. Beating greedy for stochastic bipartite matching. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, 2841–2854. https://doi.org/10.1137/19M1265629

[15] Ruiquan Gao, Zhongtian He, Zhiyi Huang, Zipei Nie, Bijun Yuan, and Yan Zhong. 2022. Improved Online Correlated Selection. In Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms. SIAM, 1265–1276. https://doi.org/10.1137/1.9781611976345.102

[16] Bernhard Haeupler, Vahab S. Mirrokni, and Morteza Zadimoghaddam. 2011. Online Stochastic Weighted Matching: Improved Approximation Algorithms. In WINE (Lecture Notes in Computer Science, Vol. 6992). Springer, 170–181. https://doi.org/10.1007/978-3-642-25510-6_15

[17] Zhiyi Huang, Ning Kang, Zhihao Gavin Tang, Xiaowei Wu, Yuhao Zhang, and Xue Zhu. 2020. Fully Online Matching. J. ACM 67, 3 (2020), 17:1–17:25. https://doi.org/10.1145/3398890

[18] Zhiyi Huang, Binghui Peng, Zhihao Gavin Tang, Runzhou Tao, Xiaowei Wu, and Yuhao Zhang. 2019. Online Vertex-Weighted Bipartite Matching: Beating 1-1/e with Random Arrivals. ACM Transactions on Algorithms 15, 3 (2019), 1–15. https://doi.org/10.1145/3326169

[19] Zhiyi Huang, Zhihao Gavin Tang, Xiaowei Wu, and Yuhao Zhang. 2020. Fully Online Matching II: Beating Ranking and Water-filling. In FOCS (IEEE), 1380–1391. https://doi.org/10.1109/FOCS46700.2020.00130

[20] Zhiyi Huang, Qiankun Zhang, and Yuhao Zhang. 2020. AdWords in a Panorama. In FOCS (IEEE), 1416–1426. https://doi.org/10.1109/FOCS46700.2020.00133

[21] Zhiyi Huang, Zhihao Gavin Tang, Xiaowei Wu, and Yuhao Zhang. 2022. The Power of Multiple Choices in Online Stochastic Matching. CoRR abs/2203.02883 (2022).

[22] Zhiyi Huang, Zhihao Gavin Tang, Xiaowei Wu, and Yuhao Zhang. 2020. Fully Online Matching II: Beating Ranking and Water-filling. In FOCS (IEEE), 1380–1391. https://doi.org/10.1109/FOCS46700.2020.00130

[23] Zhiyi Huang, Qiankun Zhang, and Yuhao Zhang. 2020. AdWords in a Panorama. In FOCS (IEEE), 1416–1426. https://doi.org/10.1109/FOCS46700.2020.00133

[24] Patrick Jaillet and Xin Lu. 2014. Online Stochastic Matching: New Algorithms with Better Bounds. Math. Oper. Res. 39, 3 (2014), 624–646. https://doi.org/10.1287/moor.2013.0621

[25] Billy Jin and David P Williamson. 2021. Improved Analysis of RANKING for Online Vertex-Weighted Bipartite Matching in the Random Order Model. In International Conference on Web and Internet Economics. Springer, 207–225. https://doi.org/10.1007/978-3-030-94676-9_12

[26] Bala Kalyanasundaram and Kirk Pruhs. 2000. An optimal deterministic algorithm for online b-matching. Theoretical Computer Science 233, 1-2 (2000), 319–325. https://doi.org/10.1016/S0304-3975(99)00140-1

[27] Chinmay Karande, Aranyak Mehta, and Pushkar Tripathi. 2011. Online bipartite matching with unknown distributions. In STOC 2011. 587–596. https://doi.org/10.1145/1993636.1993715

[28] Richard M. Karp, Umesh V. Vazirani, and Vijay Vazirani. 1990. An Optimal Algorithm for On-line Bipartite Matching. In STOC 1990. 352–358. https://doi.org/10.1145/100216.100262

[29] Mohammad Mahdian and Qiqi Yan. 2011. Online bipartite matching with random arrivals: an approach based on strongly factor-revealing LPs. In STOC 2011. 597–606. https://doi.org/10.1145/1993636.1993716

[30] Vahideh H. Manshadi, Shayan Oveis Gharan, and Amin Saberi. 2012. Online Stochastic Matching: Online Actions Based on Offline Statistics. Math. Oper. Res. 37, 4 (2012), 559–573. https://doi.org/10.1287/moor.1120.0551

[31] Aranyak Mehta, Amin Saberi, Umesh Vazirani, and Vijay Vazirani. 2005. Adwords and generalized on-line matching. In FOCS 2004. 264–273. https://doi.org/10.1145/1084320.1084321

[32] Yajun Wang and Sam Chiu-wai Wong. 2015. Two-sided Online Bipartite Matching and Vertex Cover: Beating the Greedy Algorithm. In IARCS. 1070–1081. https://doi.org/10.1007/978-3-662-47672-7_87