HIGHER GENUS AFFINE LIE ALGEBRAS
OF KRICEHER – NOVIKOV TYPE

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Abstract. Classical affine Lie algebras appear e.g. as symmetries of infinite
dimensional integrable systems and are related to certain differential equations.
They are central extensions of current algebras associated to finite-dimensional
Lie algebras $\mathfrak{g}$. In geometric terms these current algebras might be described
as Lie algebra valued meromorphic functions on the Riemann sphere with two
possible poles. They carry a natural grading. In this talk the generalization
to higher genus compact Riemann surfaces and more poles is reviewed. In
case that the Lie algebra $\mathfrak{g}$ is reductive (e.g. $\mathfrak{g}$ is simple, semi-simple, abelian,
...) a complete classification of (almost-) graded central extensions is given.
In particular, for $\mathfrak{g}$ simple there exists a unique non-trivial (almost-)graded
extension class. The considered algebras are related to difference equations,
special functions and play a role in Conformal Field Theory.

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1. Introduction

Classical current algebras (also called loop algebras) and their central exten-
sions, the affine Lie algebras, are of fundamental importance in quite a number of
fields in mathematics and its applications. These algebras are examples of infinite
dimensional Lie algebras which are still tractable. They constitute the subclass of
Kac-Moody algebras of untwisted affine type.

If one rewrites the original purely algebraic definition in geometric terms the
classical current algebras correspond to Lie algebra valued meromorphic functions
on the Riemann sphere (i.e. on the unique compact Riemann surface of genus zero)
which are allowed to have poles only at two fixed points.

If this rewriting is done, a very useful generalization (e.g. needed in string theory)
is to consider the objects over a compact Riemann surface of arbitrary genus with
more than two points where poles are allowed. The main problem is to introduce
a replacement of the grading in the classical case, which is necessary to construct
highest weight and Fock space representations. This is obtained by an almost-
graded structure (see Section 3), a weaker structure but still strong enough to do
the job. Furthermore, to obtain representations of certain types one is forced to
pass over to central extensions.

Such objects (vector fields, functions, etc.) and central extensions for higher
genus with two possible poles were introduced by Krichever and Novikov
and
generalized by me to the multi-point situation \[9\]. These objects are of importance in a global operator approach to Conformal Field Theory \[10, 11\]. More generally, the current algebra resp. their central extensions, the affine algebras, correspond to symmetries of infinite dimensional systems. Their $q$-deformed version (i.e. the quantum affine algebras) are in close connection with difference equation and with special functions.

There is a very interesting direct relation to difference equation. Krichever and Novikov constructed higher genus analogues of Baker – Akhiezer difference functions which are eigen functions of suitable difference equations. Starting from these functions, representations of higher genus affine algebras associated to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ are obtained. These are related to the two-dimensional Toda lattice \[4\].

In this write-up of the talk I report on uniqueness and classification results for higher genus multi-point affine Lie algebras which I recently obtained. In particular, it turns out that for the current algebra associated to a finite-dimensional simple Lie algebra (e.g. for $\mathfrak{sl}(n, \mathbb{C})$) there exists up to equivalence and rescaling a unique non-trivial central extension which extends the almost-grading of the current algebra. The proofs can be found in \[9\]. There also further references, historical remarks and corresponding results for the Lie algebras of Lie algebra valued meromorphic differential operators can be found. The results depend also on a complete classification of central extensions of scalar functions, vector fields and differential operators of Krichever–Novikov type obtained in \[5\].

I am indebted to Paul Terwilliger who asked me to supply explicit examples of such algebras. They can be found in Section \[5\] and Section \[6\].

2. The classical situation and some algebraic background

Let us first consider the nowadays classical affine Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. A special example of fundamental importance is given by the algebra of trace-less matrices

$$\mathfrak{sl}(n, \mathbb{C}) := \{ A \in \text{Mat}(n, \mathbb{C}) \mid \text{tr}(A) = 0 \},$$

with $[A, B] := AB - BA$ the commutator as Lie product.

The current algebra $\mathfrak{g}[\mathbb{C}[z, z^{-1}]]$ (sometimes also called loop algebra) is obtained by tensoring $\mathfrak{g}$ by the (associative and commutative) algebra $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials, i.e. $\mathfrak{g}[\mathbb{C}[z, z^{-1}]] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ with the Lie product

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}, \quad x, y \in \mathfrak{g}, \quad n, m \in \mathbb{Z}.$$ 

If $\mathfrak{g}$ is a matrix algebra, then $\mathfrak{g}[\mathbb{C}[z, z^{-1}]]$ can be considered as matrices with Laurent polynomials as entries, e.g. a typical element of $\mathfrak{gl}(2, \mathbb{C})[\mathbb{C}[z, z^{-1}]]$ can be written as

$$\begin{pmatrix} a(z, z^{-1}) & b(z, z^{-1}) \\ c(z, z^{-1}) & -a(z, z^{-1}) \end{pmatrix},$$

where $a, b, c$ are polynomials in $z$ and $z^{-1}$.

By setting $\text{deg}(x \otimes z^n) := n$ the Lie algebra $\mathfrak{g}[\mathbb{C}[z, z^{-1}]]$ is graded (see \[2\]). Clearly, $\mathfrak{g}[\mathbb{C}[z, z^{-1}]]$ is an infinite dimensional Lie algebra. These algebras are candidates for symmetry algebras of systems with infinitely many independent symmetries. Unfortunately, in the process of constructing representations of certain types (e.g. highest weight representations) one is forced to “regularize” certain natural actions. As a result
one obtains only so called \emph{projective representations}, which in turn define honest representations of certain central extensions of $\mathfrak{g}$.

What is a central extension $\widehat{V}$ of a Lie algebra $V$? As vector space we take $\widehat{V} = V \oplus \mathbb{C}$. We set $t := (0,1)$ and $\hat{a} := (a,0)$ and consider the following product

\begin{equation} [\hat{a}, \hat{b}] := [\hat{a}, \hat{b}] + \psi(a, b) t, \quad [\hat{a}, t] = 0, \end{equation}

with $\psi : V \times V \to V$ a bilinear map. Now $\widehat{V}$ is a Lie algebra if and only if $\psi$ is a Lie algebra two-cocycle of $V$ (with values in the trivial module $\mathbb{C}$). The cocycle conditions are

\begin{equation} \psi(a, b) = -\psi(b, a), \quad \psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0, \end{equation}

for all $a, b, c \in V$.

Two central extensions of the same algebra are called equivalent if they are the same up to some change of basis of the type $\hat{a} \mapsto \tilde{a} = (a, \phi(a))$. In more precise terms, given two extensions defined by $\psi_1$ and $\psi_2$ respectively, the two extensions are called equivalent if there exists a linear form $\phi : V \to \mathbb{C}$ such that

\begin{equation} \psi_1(a, b) - \psi_2(a, b) = \phi([a, b]). \end{equation}

In other words, the difference is a Lie algebra cohomology coboundary.

\textbf{Fact.} The set of equivalence classes of central extensions is via \eqref{eq:central-extension} in 1 to 1 correspondence to the space of Lie algebra two-cohomology classes $H^2(V, \mathbb{C}) = Z^2(V, \mathbb{C})/B^2(V, \mathbb{C})$ (cocycles modulo coboundaries).

How do we obtain central extensions for our current algebra? Let $\alpha$ be an invariant, symmetric bilinear form on $\mathfrak{g}$. Invariance means that $\alpha([a, b], c) = \alpha(a, [b, c])$ for all $a, b, c \in \mathfrak{g}$. For a simple Lie algebra the Cartan-Killing form is up to a rescaling the only such form. In particular for $\mathfrak{sl}(n, \mathbb{C})$ it is given by $\alpha(A, B) = \text{tr}(AB)$. Then a central extension $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}t$ is defined by

\begin{equation} [x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m} - \alpha(x, y) \cdot n \cdot \delta_{n+m} \cdot t. \end{equation}

To avoid cumbersome notation I dropped the $\hat{\cdot}$ in the notation. It is called the (classical) affine Lie algebra associated to $\mathfrak{g}$. By setting $\deg t := 0$ (and using $n = \deg(x \otimes z^n)$) the affine algebra is a graded algebra. If the finite dimensional Lie algebra $\mathfrak{g}$ is simple then $\widehat{\mathfrak{g}}$ defined via \eqref{eq:affine-extension} is up to equivalence of extensions and rescaling of the central element the only non-trivial central extension of $\mathfrak{g}$. In this case the algebras are exactly the Kac-Moody algebras of untwisted affine type $\widehat{\mathfrak{g}}$.

3. THE HIGHER GENUS CASE

Before we can extend the construction to higher genus we have to geometrize the classical situation. Recall that the associative algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ can equivalently be described as the algebra consisting of those meromorphic functions on the Riemann sphere (resp. the complex projective line $\mathbb{P}^1(\mathbb{C})$) which are holomorphic outside $z = 0$ and $z = \infty$ ($z$ the quasi-global coordinate). The current algebra $\mathfrak{g}$ can be interpreted as Lie algebra of $\mathfrak{g}$-valued meromorphic functions on the Riemann sphere with possible poles only at $z = 0$ and $z = \infty$.

The Riemann sphere is the unique compact Riemann surface of genus zero. From this point of view the next step is to take $X$ any compact Riemann surface of arbitrary genus $g$ and an arbitrary finite set $A$ of points where poles of the meromorphic
objects will be allowed. In this way we obtain the higher genus (multi-point) current algebra as the algebra of $g$–valued functions on $X$ with only possibly poles at $A$. We need gradings, central extensions etc.

For this goal we split $A$ into two non-empty disjoint subsets $I$ and $O$, $A = I \cup O$. In the interpretation of string theory, $I$ corresponds to incoming free strings and $O$ to outgoing free strings. Let $K$ be the number of points in $I$. See Figure 1 for an example given by a Riemann surface of genus two with $I = \{P_1, P_2\}$ and $O = \{Q_1\}$.

Let $\mathcal{A}$ be the associative algebra of functions meromorphic on $X$ and holomorphic outside of $A$. In some earlier work I introduced
\begin{equation}
\{ A_{n,p} \mid n \in \mathbb{Z}, \; p = 1, \ldots, K \}
\end{equation}
a certain adapted basis of $\mathcal{A}$. For the exact definition I refer to this publication. Here we only note that
\begin{equation}
\text{ord}_{P_i}(A_{n,p}) = n + 1 - \delta_{n,p}, \quad \forall P_i \in I.
\end{equation}
For genus zero and $I = \{0\}$, $O = \{\infty\}$ we get $A_{n,p} = z^n$. Let $\mathcal{A}_n := \langle A_{n,p} \mid p = 1, \ldots, K \rangle$ be the $K$-dimensional subspace of $\mathcal{A}$. We have $\mathcal{A} = \oplus_{n \in \mathbb{Z}} \mathcal{A}_n$ and there exist constants $L_1, L_2$ (independent of $n$ and $m$) such that
\begin{equation}
\mathcal{A}_n \cdot \mathcal{A}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{A}_h, \quad \forall n, m \in \mathbb{Z}.
\end{equation}
We call the elements of $\mathcal{A}_n$ homogeneous elements of degree $n$. As long as $L_1$ and $L_2$ cannot to be chosen to be 0 the algebra is not honestly graded. It is only almost-graded. In a similar way one introduces almost-gradedness for Lie algebras. This notion was introduced by Krichever and Novikov (they called it quasi-graded) and they constructed such an almost-grading in the higher genus and two point case. To find an almost-grading in the multi-point case is more difficult. This weaker grading is enough to introduce and study highest weight representations. As a remark aside: with a special choice of basis one has $L_1 = 0$ and the $L_2$ depends in a known manner on the genus $g$ and the number of points in $I$ and $O$.

The higher genus multi-point current algebra $\mathfrak{g}$ is the tensor product \(\mathfrak{g} = g \otimes \mathcal{A}\) with the Lie product
\begin{equation}
[x \otimes f, y \otimes g] = [x, y] \otimes (f \cdot g)
\end{equation}
and almost-grading
\[ \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n, \quad \mathfrak{g}_n = \mathfrak{g} \otimes A_n. \]

4. Central extensions in higher genus

The next task is to construct central extensions and to study the question of uniqueness.

Proposition 4.1. \((14)\) Let \(\alpha\) be an invariant, symmetric bilinear form of \(\mathfrak{g}\) and \(C\) a closed contour on \(X\) not meeting \(A\), then
\[ \psi_{\alpha,C}(x \otimes f, y \otimes g) := \alpha(x, y) \oint_C f dg \]
is a Lie algebra two-cocycle for the current algebra \(\mathfrak{g}\). Hence, it defines a central extension \(\hat{\mathfrak{g}}_{\alpha,C}\).

Consequently, there exist central extensions for \(\mathfrak{g}\). But contrary to the classical situation, even if \(\mathfrak{g}\) is a simple Lie algebra, there will not be a unique nontrivial cocycle class. If we choose essentially different contours \(C\) the \(\psi_{\alpha,C}\) define essentially different central extensions \(\hat{\mathfrak{g}}_{\alpha,C}\).

But in the classical situation we were able to extend our grading of \(\mathfrak{g}\) to \(\hat{\mathfrak{g}}\) by assigning a degree to the central element \(t\). This will not necessarily be true for all cocycles of the form \((13)\).

Definition 4.1. A 2-cocycle is called \textit{local} if there exist \(T_1\) and \(T_2\) such that
\[ \psi(\mathfrak{g}_n, \mathfrak{g}_m) \neq 0 \implies T_2 \leq n + m \leq T_1. \]

Given a local cocycle \(\psi\) defining a central extension, then by setting \(\text{deg}(t) = 0\) (or any other number) the almost-grading of \(\mathfrak{g}\) extends to the central extension. Vice versa, if such an extension of the almost-grading exists, the defining cocycle will be local.

We use \(H^2_{\text{loc}}(\mathfrak{g}, \mathbb{C})\) to denote the subspace of 2-cocycle classes containing a representative which is local. In general, the cocycles \(\psi_{\alpha,C}\) are not local. But if we choose as integration contour a smooth contour \(C_S\) separating the points in \(I\) from the points in \(O\) and which is of winding number 1, then it can be shown that \(\psi_{\alpha,C_S}\) is a local cocycle. In fact its values can be calculated as
\[ \psi_{\alpha,C_S}(x \otimes f, y \otimes g) = \alpha(x, y) \oint_{C_S} f dg = \sum_{i=1}^{K} \text{res}_{P_i}(f dg). \]

We call any such \(C_S\) a separating cycle.

Theorem 4.1. \((16)\) Let \(\mathfrak{g}\) be a finite-dimensional simple Lie algebra, \(\mathfrak{g} = \mathfrak{g} \otimes A\) its (higher genus) current algebra, then
\[ \dim H^2_{\text{loc}}(\mathfrak{g}, \mathbb{C}) = 1, \]
and a basis is given by the class of \((15)\), where \(C_S\) is a separating cycle and \(\alpha\) is a multiple of the Cartan–Killing form. In particular, there exists up to equivalence and rescaling a unique almost-graded non-trivial central extension of \(\mathfrak{g}\).

As a side-result we obtain that every local cocycle is cohomologous to a geometric cocycle of the form \((15)\).
Remark 4.1. The cocycles coming from Fock space representations and other type of representations are local. Hence we obtain that there exists a unique equivalence class (up to rescaling of the central element) of central extensions coming from these representations.

In [9] the more general situation of reductive Lie algebras is considered. Recall that a finite-dimensional Lie algebra \( \mathfrak{g} \) is reductive if and only if it is the direct sum (as Lie algebra)
\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_M, \quad \mathfrak{g}_0 \text{ abelian}, \mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_M \text{ simple}.
\]

If the abelian summand \( \mathfrak{g}_0 \) is missing then \( \mathfrak{g} \) is semi-simple. In the semi-simple case it is shown that every local 2-cocycle is cohomologous to a cocycle of the type (15) where \( \alpha \) is an arbitrary linear combination of the individual Cartan–Killing forms of the summands (trivially extended to the rest). Vice versa, such cocycles are local. In particular, we get \( \dim H^2_{\text{loc}}(\mathfrak{g}, \mathbb{C}) = M \).

In the reductive case we have to add another condition. We denote by \( \mathcal{L} \) the Lie algebra of meromorphic vector fields on \( X \) which are holomorphic outside of \( A \). A 2-cocycle is called \( \mathcal{L} \)-invariant if
\[
\psi(x \otimes (e \cdot f), y \otimes g) + \psi(x \otimes f, y \otimes (e \cdot g)) = 0, \quad \forall f, g \in \mathcal{A}, \quad \forall e \in \mathcal{L}.
\]
Cocycles of the form (13) are \( \mathcal{L} \)-invariant. In [9] it is shown that every \( \mathcal{L} \)-invariant local cocycle is cohomologous to (15). In particular \( \dim H^2_{\mathcal{L}, \text{loc}}(\mathfrak{g}, \mathbb{C}) = M + \frac{m(m+1)}{2} \).

Example 4.1. The Lie algebra of trace-less matrices \( \mathfrak{sl}(n, \mathbb{C}) \) is simple. Hence, the unique non-trivial almost-graded central extension is given (up to equivalence and rescaling) by the cocycle
\[
\psi_1(A \otimes f, B \otimes g) = \text{tr}(A \cdot B) \oint_{C_S} f dg.
\]

Example 4.2. The Lie algebra of all matrices \( \mathfrak{gl}(n, \mathbb{C}) \) is the direct sum \( \mathfrak{gl}(n, \mathbb{C}) = \mathfrak{s}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \), where \( \mathfrak{s}(n, \mathbb{C}) \) is the abelian summand of scalar matrices. In particular \( \mathfrak{gl}(n, \mathbb{C}) \) is a reductive Lie algebra. Following the general results \( \dim H^2_{\mathcal{L}, \text{loc}}(\mathfrak{gl}(n, \mathbb{C}), \mathbb{C}) = 2 \). A basis is given by the elements \( \psi_1 \) and \( \psi_2 \)
\[
\psi_2(A \otimes f, B \otimes g) = \text{tr}(A) \cdot \text{tr}(B) \oint_{C_S} f dg.
\]

5. AN EXAMPLE: THE THREE-POINT GENUS ZERO CASE

Let us consider the Riemann sphere \( S^2 = \mathbb{P}^1(\mathbb{C}) \) and the set \( A \) consisting of 3 points. Given any triple of 3 points there exists always an analytic automorphism of \( S^2 \) mapping this triple to \( \{a, -a, \infty\} \), with \( a \neq 0 \). In fact \( a = 1 \) would suffice. Without restriction we can take
\[
I := \{a, -a\}, \quad O := \{\infty\}.
\]
Due to the symmetry of the situation it is more convenient to take a symmetrized basis of $A$:

\begin{equation}
A_{2k} := (z-a)^k(z+a)^k, \quad A_{2k+1} := z(z-a)^k(z+a)^k, \quad k \in \mathbb{Z}.
\end{equation}

It is shown in [1] that it a basis. By more or less direct calculations one can show the structure equation for the current algebra $\mathfrak{g}^*$

\begin{equation}
[x \otimes A_n, y \otimes A_m] = \begin{cases}
[x, y] \otimes A_{n+m}, & n \text{ or } m \text{ even,} \\
[x, y] \otimes A_{n+m} + a^2[x, y] \otimes A_{n+m-2}, & n \text{ and } m \text{ odd,}
\end{cases}
\end{equation}

Again $a = 1$ could be set. The reason to keep $a$ is that it can be seen that if we vary $a$ over the affine line we obtain for $a = 0$ the classical current algebra. In particular, this family gives a deformation. In [1] it was shown that this deformation is a geometrically non-trivial deformation despite the fact that for $\mathfrak{g}$ simple, $\mathfrak{g}^*$ is formally rigid, i.e. there does not exists a non-trivial formal deformation. This effect is peculiar to infinite dimensional Lie algebras and is discussed in detail there.

For the central extension $\mathfrak{g}_{a,S}$ of Section 4 we obtain the defining cocycles (see [1], A.13 and A.14)

\begin{equation}
\gamma(x \otimes A_n, y \otimes A_m) = \alpha(x, y) \cdot \frac{1}{2\pi i} \int_{C_S} A_n dA_m,
\end{equation}

with

\begin{equation}
\frac{1}{2\pi i} \int_{C_S} A_n dA_m = \begin{cases}
-n\delta_m^{-n}, & n \text{ even,} \\
0, & n \text{ odd,}
\end{cases}
\end{equation}

Of course, given a simple Lie algebra $\mathfrak{g}$ with generators and structure equations the relations can be written in these terms. For $\mathfrak{sl}(2, C)$ with the standard generators

\begin{equation}
h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{equation}

we set $e_n := e \otimes A_n, n \in \mathbb{Z}$ and in the same way $f_n$ and $h_n$. Recall that $\alpha(x, y) = \text{tr}(x \cdot y)$. We calculate

\begin{equation}
[e_n, f_m] = \begin{cases}
h_{n+m}, & n \text{ or } m \text{ even} \\
h_{n+m} + a^2 h_{n+m-2}, & n \text{ and } m \text{ odd}
\end{cases}
\end{equation}

\begin{equation}
[h_n, e_m] = \begin{cases}
2e_{n+m}, & n \text{ or } m \text{ even} \\
2e_{n+m} + 2a^2 e_{n+m-2}, & n \text{ and } m \text{ odd}
\end{cases}
\end{equation}

\begin{equation}
[h_n, f_m] = \begin{cases}
-2f_{n+m}, & n \text{ or } m \text{ even} \\
-2f_{n+m} - 2a^2 f_{n+m-2}, & n \text{ and } m \text{ odd}
\end{cases}
\end{equation}

For the central extension we get

\begin{equation}
[e_n, f_m] = \begin{cases}
h_{n+m} - n\delta_m^{-n}, & n \text{ or } m \text{ even} \\
h_{n+m} + a^2 h_{n+m-2} - n\delta_m^{-n} - a^2(n-1)\delta_m^{-n+2}, & n \text{ and } m \text{ odd}
\end{cases}
\end{equation}

The other commutators stay the same.
6. An example: The torus case

Let \( T = \mathbb{C}/L \) be a complex one-dimensional torus, i.e. a Riemann surface of genus 1. Here \( L \) denotes the lattice \( L = (1, \tau) \mathbb{Z} \) with \( \text{Im} \, \tau > 0 \). The field of meromorphic functions on \( T \) is generated by the doubly-periodic Weierstraß \( \wp \) function and its derivative \( \wp' \) fulfilling the differential equation

\[
(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3.
\]

with the \( e_i \) pairwise distinct and given by

\[
\wp\left(\frac{1}{2}\right) = e_1, \quad \wp\left(\frac{\tau}{2}\right) = e_2, \quad \wp\left(\frac{\tau + 1}{2}\right) = e_3, \quad e_1 + e_2 + e_3 = 0.
\]

The function \( \wp \) is an even meromorphic function with poles of order two at the points of the lattice and holomorphic elsewhere. The function \( \wp' \) is an odd meromorphic function with poles of order three at the points of the lattice and holomorphic elsewhere. It has zeros of order one at the points \( 1/2, \tau/2 \) and \( (1 + \tau)/2 \) and all its translates under the lattice.

We consider the subalgebra of functions which are holomorphic outside of \( \bar{z} = \bar{0} \) and \( \bar{z} = 1/2 \). As shown in [7] a basis is given by

\[
A_{2k} = (\wp - e_1)^k, \quad A_{2k+1} = \frac{1}{2}\wp' \cdot (\wp - e_1)^{k-1} \quad k \in \mathbb{Z}.
\]

See also [5] for a similar result in the vector field algebra case. The following is shown in [11]

\[
[x \otimes A_n, y \otimes A_m] = \begin{cases} 
[x, y] \otimes A_{n+m}, & n \text{ or } m \text{ even}, \\
[x, y] \otimes A_{n+m} + 3e_1[x, y] \otimes A_{n+m-2} + (e_1 - e_2)(2e_1 + e_2)[x, y] \otimes A_{n+m-4}, & n \text{ and } m \text{ odd}.
\end{cases}
\]

If we let \( e_1 \) and \( e_2 \) (and hence also \( e_3 \)) go to zero we obtain the classical current algebra as degeneration.

The cocycle defining the central extension is given by (11, Thm. 4.6)

\[
\gamma(x \otimes A_n, y \otimes A_m) = \alpha(x, y) \cdot \frac{1}{2\pi i} \int_{C_S} A_n dA_m
\]

with

\[
\frac{1}{2\pi i} \int_{C_S} A_n dA_m = \begin{cases} 
-n\delta_m^{-n}, & n, m \text{ even}, \\
0, & n, m \text{ diff. parity}, \\
-n\delta_m^{-n} + 3e_1(-n + 1)\delta_m^{-n+2} + (e_1 - e_2)(2e_1 + e_2)(-n + 2)\delta_m^{-n+4}, & n, m \text{ odd}.
\end{cases}
\]

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