ESSENTIAL FINITE GENERATION OF EXTENSIONS OF VALUATION RINGS

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Abstract. Given a generically finite local extension of valuation rings \( V \subset W \), the question of whether \( W \) is the localization of a finitely generated \( V \)-algebra is significant for approaches to the problem of local uniformization of valuations using ramification theory. Hagen Knaf proposed a characterization of when \( W \) is essentially of finite type over \( V \) in terms of classical invariants of the extension of associated valuations. Knaf’s conjecture has been verified in important special cases by Cutkosky and Novacoski using local uniformization of Abhyankar valuations and resolution of singularities of excellent surfaces in arbitrary characteristic, and by Cutkosky for valuation rings of function fields of characteristic 0 using embedded resolution of singularities. In this paper we prove Knaf’s conjecture in full generality.

1. Introduction

Let \( L/K \) be a finite field extension. Given a domain \( R \) with fraction field \( K \), results that characterize when the integral closure of \( R \) in \( L \) is a finite type \( R \)-algebra have fundamental applications in algebraic geometry, commutative algebra and number theory. For instance, finite generation of integral closures was studied extensively, among others, by Krull, Akizuki, Noether, Zariski, Grothendieck and especially Nagata, resulting in applications to fundamental topics such as rings of algebraic integers and resolution of singularities. We investigate a local valuative analogue of the finite generation of integral closures in this paper. Let \( \omega \) be a valuation of \( L \) with valuation ring \( (\mathcal{O}_\omega, \mathfrak{m}_\omega, \kappa_\omega) \) and value group \( \Gamma_\omega \). Let \( \nu \) be the restriction of \( \omega \) to \( K \) with valuation ring \( (\mathcal{O}_\nu, \mathfrak{m}_\nu, \kappa_\nu) \) and value group \( \Gamma_\nu \). Inclusion induces a local homomorphism

\[
(\mathcal{O}_\nu, \mathfrak{m}_\nu, \kappa_\nu) \hookrightarrow (\mathcal{O}_\omega, \mathfrak{m}_\omega, \kappa_\omega),
\]

and the valuation ring \( \mathcal{O}_\omega \) is a local ring of the integral closure of \( \mathcal{O}_\nu \) in \( L \) [Bou98, Chap. VI, §8.6, Prop. 6]. Thus, as a local version of the question of the finite generation of integral closures, it is natural to ask when \( \mathcal{O}_\omega \) is the localization of a finite type \( \mathcal{O}_\nu \)-algebra. Knaf proposed the following necessary and sufficient condition in terms of classical invariants of the extension \( \omega/\nu \) [CN19 Conjecture 1.2].

Conjecture 1.1. Let \( L/K \) be a finite field extension, \( \omega \) be a valuation of \( L \) and \( \nu \) be the restriction of \( \omega \) to \( K \). Let \( L^h \) (resp. \( K^h \)) denote the fraction field of the Henselization of \( \mathcal{O}_\omega \) (resp. \( \mathcal{O}_\nu \)). Then \( \mathcal{O}_\omega \) is essentially of finite type over \( \mathcal{O}_\nu \) if and only if both the following conditions are satisfied:

1. \([L^h : K^h] = [\Gamma_\omega : \Gamma_\nu][\kappa_\omega : \kappa_\nu]\).
2. \(\epsilon(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu] \epsilon(\kappa_\omega|\kappa_\nu)\), where \(\epsilon(\omega|\nu)\) is the cardinality of \(\{x \in \Gamma_{\omega, \geq 0} : x < y \text{ for all } y \in \Gamma_{\nu, > 0}\}\).

Here for a totally ordered abelian group \( \Gamma \), we define

\[
\Gamma_{\geq 0} := \{x \in \Gamma : x \geq 0\} \text{ and } \Gamma_{> 0} := \{x \in \Gamma : x > 0\}.
\]

In ramification theory, \([\Gamma_\omega : \Gamma_\nu]\) is called the ramification index, \([\kappa_\omega : \kappa_\nu]\) the inertia index and \(\epsilon(\omega|\nu)\) the initial index of the extension \(\omega/\nu\). The first equality \([L^h : K^h] = [\Gamma_\omega : \Gamma_\nu][\kappa_\omega : \kappa_\nu]\) is the assertion that the extension \(\omega/\nu\) is defectless; see Definition 1.3 for the notion of defect. The second equality \(\epsilon(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu]\) means that every element of the quotient \(\Gamma_\omega/\Gamma_\nu\) is the class of some element of \(\Gamma_{\omega, > 0}\). Indeed, if \(x_1, x_2\) are distinct elements of \(\mathcal{S} := \{x \in \Gamma_{\omega, \geq 0} : x < y \text{ for all } y \in \Gamma_{\nu, > 0}\}\), then we may assume without loss of generality that \(0 \leq x_1 < x_2\). Consequently, \(0 < x_2 - x_1 \leq x_2\), and so, \(x_2 - x_1\) cannot be an element of \(\Gamma_\nu\) because \(x_2\) is strictly smaller than every element of \(\Gamma_{\nu, > 0}\). This shows that every element of \(\mathcal{S}\) is the representative of a distinct class of \(\Gamma_\omega/\Gamma_\nu\). Thus, \(\epsilon(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu]\) would mean that the classes of the elements of \(\mathcal{S}\) constitute the whole group \(\Gamma_\omega/\Gamma_\nu\).
The question of the essential finite generation of extension of valuation rings arises naturally in approaches to the open problem of local uniformization of valuations using ramification theory. For example, an affirmative answer to this question for extensions of Abhyankar valuations is an important ingredient in Knaf and Kuhlmann’s proof of the local uniformization of Abhyankar valuations [KK05]. In addition, the essential finite generation of extensions of valuation rings also features in Knaf and Kuhlmann’s valuation-theoretic argument of the local uniformization of a valuation in a finite extension of its fraction field [KK09]. Conjecture 1.1 can be viewed as a generalization of the beautiful ramification-theoretic characterization of the module-finiteness of the integral closure of a valuation ring in a finite extension of its fraction field [Bou98, Chap. VI, §8.5, Thm. 2]. We refer the interested reader to [CN19, Cut19] for additional background on this problem.

Conjecture 1.1 is known in specific cases, often using different techniques. The necessity of conditions (1) and (2) for the essential finite generation of \( O_\omega \) over \( O_\nu \) was proved by Knaf; his argument is reproduced in [CN19, Thm. 4.1] (see also Remark 5.1 for a different approach using Zariski’s Main Theorem). The sufficiency of conditions (1) and (2) for the essential finite generation of \( O_\omega \) over \( O_\nu \) is known when

- \( L/K \) is normal using the transitive action of \( \text{Gal}(L/K) \) on the fibers of the integral closure of \( O_\nu \) in \( L \) [CN19, Cor. 2.2];
- \( \kappa_\omega/\kappa_\nu \) is separable using the theory of Henselian elements [KNT4, Thm. 1.3];
- \( \omega \) is the unique extension of \( \nu \) to \( L \) using the theory of defect [CN19, Cor. 2.2];
- \( \nu \) is centered on an excellent local two dimensional domain with fraction field \( K \) using resolution of singularities for excellent surfaces [CN19, Thm. 1.4];
- \( \nu \) is an Abhyankar valuation of a function field \( K/k \) by [CN19, Thm. 1.5] and [Cut20, Thm. 1.7] using local uniformization of Abhyankar valuations;
- \( K \) is the function field over a field of characteristic 0 using an explicit form of embedded resolution of singularities [Cut19, Thm. 1.3].

We will give a uniform argument that settles Conjecture 1.1 in full generality. Recall that the implication that remains to be shown is that if conditions (1) and (2) of Conjecture 1.1 hold, then \( O_\omega \) is essentially of finite type over \( O_\nu \). We take as our starting point the veracity of Conjecture 1.1 for unique extensions of valuations [CN19, Cor. 2.2]. Note that if \( h \) (resp. \( h^h \)) is the valuation ring of \( L \) (resp. \( K \)) whose valuation ring is the Henselization \( O_\omega^h \) (resp. \( O_\nu^h \)), then \( h \) is the unique extension of \( h^h \) to \( L \) up to equivalence of valuations by the Henselian property (see Corollary 4.2). Moreover, Henselizations do not alter value groups, and \( L/K \) is normal using the transitive action of \( \text{Gal}(L/K) \) on the fibers of the integral closure of \( O_\omega \) in \( L \) [CN19, Cor. 2.2];

\[
\begin{align*}
(i) & \quad O_\omega \text{ is essentially of finite presentation over } O_\nu, \\
(ii) & \quad O_\omega \text{ is essentially of finite type over } O_\nu, \\
(iii) & \quad [L^h : K^h] = [\Gamma_\omega : \Gamma_\nu][\kappa_\omega : \kappa_\nu] \text{ and } \epsilon(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu], \\
(iv) & \quad O_\omega^h \text{ is essentially of finite type over } O_\nu^h, \\
(v) & \quad O_\omega^h \text{ is a module finite } O_\nu^h\text{-algebra.}
\end{align*}
\]

Instead of using known cases of local uniformization or resolution of singularities, we use Knaf’s result [CN19, Thm. 4.1] (see also Remark 5.1) to reduce the proof of Theorem 1.2 to showing that \( (v) \Rightarrow (i) \). We then prove this implication by analyzing the behavior of integral maps under base change along Henselizations. This consideration turns out to be independent of valuation theory and is carried out in Section 3. This section is the heart of our paper and does most of the heavy lifting for the proof of \( (v) \Rightarrow (i) \). The key is the approximation result of Corollary 3.7 which, combined the fact that valuation rings are maximal subrings of a field with respect to the partial order induced by domination of local rings, leads to a proof of Theorem 1.2.

The paper is structured as follows. In Section 2 we fix our conventions for the paper. Section 3 examines base change properties along Henselizations. In Section 4 we collect some basic results and definitions about
extensions of valuation rings, Henselian valuation rings and the ramification theory of extensions of valuations.

Finally, we prove Theorem 1.2 in Section 5.

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2. Conventions and basic terminology

All rings are commutative with a multiplicative identity. For a ring $A$, $\text{MaxSpec}(A)$ will denote the set of maximal ideals of $A$. Note that rings in this paper will rarely be noetherian.

The term local ring will mean a ring $(R, m, \kappa)$ with a unique maximal ideal $m$ and residue field $\kappa$. Please note that local rings are not necessarily noetherian. We will use $R^h$ to denote the Henselization of the local ring $R$ with respect to the maximal ideal $m$. Recall that $R^h$ is a faithfully flat local extension of $R$ whose maximal ideal is the expansion of the maximal ideal of $R$ and whose residue field is isomorphic to the residue field of $R$ [Sta20, Tag 07QM]. We will say that a local ring $(B, m_B)$ dominates a local ring $(A, m_A)$ if $A \subset B$ and $m_A = m_B \cap A$. Recall that a valuation ring of a field $K$ is a local subring of $K$ that is maximal in the collection of local subrings of $K$ under the partial order induced by domination of local rings. Whenever we talk about an extension of valuation rings $V \subset W$, we will always assume $W$ dominates $V$.

Valuation rings also arise from valuations, which are denoted additively in this paper. We assume the reader is familiar with valuations and valuation rings, and we will skip their definitions. A good introduction to valuation theory is [Bou98, Chap. VI].

Let $B$ be an $A$-algebra. We say $B$ is essentially of finite type over $A$ if $B$ is the localization of a finite type $A$-algebra. We say $B$ is essentially of finite presentation over $A$ if $B$ is the localization of a finitely presented $A$-algebra. While a finitely presented algebra is always of finite type, the converse is not true in a non-noetherian setting. For example, if $k$ is a field and $A = k[x_n : n \in \mathbb{N}]$ is the polynomial ring in infinitely many variables over $k$, then the canonical map $A \rightarrow k$ obtained by killing all the variables is of finite type but not of finite presentation because the ideal $(x_n : n \in \mathbb{N})$ is not finitely generated [Sta20, Tag 00R2]. However, finite generation and finite presentation often coincide in the valuative setting because of the following result and the fact that any torsion-free module over a valuation ring is free [Bou98, Chap. VI, §3.6, Lem. 1].

Lemma 2.1. [RG71, Cor. (3.4.7)] Let $A$ be a domain and $B$ be a finite type $A$-algebra. If $B$ is $A$-flat, then $B$ is a finitely presented $A$-algebra.

3. Henselization and base change

In this section we establish some base change properties of integral maps along Henselizations. The results do not use any valuation theory. The non-valuative considerations of this section will provide the main ingredients for the proof of Theorem 1.2. We will frequently use the fact that the Henselization of a local ring is flat [Sta20, Tag 07QM], that flat maps satisfy the Going-Down property [Sta20, Tag 00HS] and that the property of being an integral ring map is preserved under base change [Sta20, Tag 02JK].

We first recall a characterization of Henselian local domains that will be important for the results that follow.

Lemma 3.1. Let $(R, m)$ be a local domain. The following are equivalent:

1. $R$ is Henselian.
2. For every integral extension $R \rightarrow A$, if $A$ is a domain then $A$ is a local ring.

If the equivalent conditions hold, then any integral extension of $R$ that is also a domain is Henselian.

Indication of proof. The equivalence follows from [Nag75, Chap. VII, Thm. (43.12)]. The fact that integral extension domains of $R$ are Henselian follows by [Nag75, Chap. VII, Cor. (43.13)]. Note that in Nagata’s terminology, an integral extension of a domain is automatically a domain [Nag75, Chap. I, Pg. 30].
The next result highlights a key base change property along Henselizations.

**Lemma 3.2.** Let \((R, m)\) be a local ring, \(\varphi : R \to A\) be a ring map and \(i : R \to R^h\) be the canonical map from \(R\) to its Henselization \(R^h\). Suppose \(\mathfrak{P} \in \text{Spec}(A)\) contracts to \(m \in \text{Spec}(R)\), that is, \(\varphi^{-1}(\mathfrak{P}) = m\). Consider the induced map \(i : A \rightarrow A \otimes_R R^h\).

1. The fiber of \(\text{Spec}(i) : \text{Spec}(A \otimes_R R^h) \rightarrow \text{Spec}(A)\) over \(\mathfrak{P}\) is a singleton.
2. If \(\mathfrak{Q}\) is the unique prime ideal of \(A \otimes_R R^h\) that contracts to \(\mathfrak{P}\), then \((A \otimes_R R^h)_{\mathfrak{Q}}^h \cong (A_{\mathfrak{P}})^h\).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A \otimes i} & A \otimes_R R^h \\
\downarrow{\varphi} & & \downarrow{\varphi \otimes \text{id}_{R^h}} \\
R & \xrightarrow{i} & R^h.
\end{array}
\] (3.2.1)

Since \(i\) is faithfully flat, so is \(\text{id}_A \otimes i\). Therefore, \(\text{Spec}(\text{id}_A \otimes i) : \text{Spec}(A \otimes_R R^h) \rightarrow \text{Spec}(A)\) is surjective.

1. Let \(\kappa(\mathfrak{P})\) denote the residue field of \(\mathfrak{P}\) (resp. \(\kappa(m)\)). Then the fiber of \(\text{Spec}(i)\) over \(\mathfrak{P}\) can be identified with \(\text{Spec}(\kappa(\mathfrak{P}) \otimes_A (A \otimes_R R^h))\). Now,

\[
\kappa(\mathfrak{P}) \otimes_A (A \otimes_R R^h) \cong \kappa(\mathfrak{P}) \otimes_R R^h \cong \kappa(\mathfrak{P}) \otimes_{\kappa(m)} (\kappa(m) \otimes_R R^h).
\]

The maximal ideal \(m^h\) of \(R^h\) is \(mR^h\) and the induced map of residue fields \(\kappa(m) \rightarrow \kappa(m^h)\) is an isomorphism \([\text{Sta}20, \text{Tag} \, 07QM]\). Therefore \(\kappa(m) \otimes_R R^h = \kappa(m^h)\) and

\[
\kappa(\mathfrak{P}) \otimes_{\kappa(m)} (\kappa(m) \otimes_R R^h) \cong \kappa(\mathfrak{P}),
\]

that is, \(\text{Spec}(\kappa(\mathfrak{P}) \otimes_A (A \otimes_R R^h))\) is the spectrum of a field. This proves (1).

2. There exists a unique prime ideal \(\mathfrak{Q}\) of \(A \otimes_R R^h\) that contracts to \(\mathfrak{P}\) by (1). By the commutativity of (3.2.1), \(\mathfrak{Q}\) contracts to \(m^h\) along \(\varphi \otimes \text{id}_{R^h} : R^h \rightarrow A \otimes_R R^h\) because \(m^h\) is the unique prime ideal of \(R^h\) that contracts \(m\) in \(R\). The rest of (2) now follows from \([\text{Sta}20, \text{Tag} \, 08HU]\). \(\square\)

We now focus on the base change properties of integral ring maps along Henselizations. When we use the term ‘integral ring map’, we do not necessarily mean an integral extension. For instance, a surjective ring map is an integral map but not an integral extension.

**Lemma 3.3.** Let \((R, m)\) be a local ring, \(\varphi : R \to A\) be an integral ring map and \(i : R \to R^h\) be the canonical map from \((R, m)\) to its Henselization \((R^h, m^h)\). Then the map

\[
\text{Spec}(i) : \text{Spec}(A \otimes_R R^h) \rightarrow \text{Spec}(A)
\]

induced by \(i : A \rightarrow A \otimes_R R^h\) has the following properties:

1. For every maximal ideal \(\mathfrak{M}\) of \(A\), the fiber of \(\text{Spec}(i)\) over \(\mathfrak{M}\) is a singleton.
2. Let \(\mathfrak{Q} \in \text{Spec}(A \otimes_R R^h)\). The following are equivalent:
   1. \(\mathfrak{Q}\) is a maximal ideal of \(A \otimes_R R^h\).
   2. \(\mathfrak{Q}\) contracts to a maximal ideal of \(\text{Spec}(A)\).
3. \(\text{Spec}(i)\) induces a bijection \(\text{MaxSpec}(A \otimes_R R^h) \leftrightarrow \text{MaxSpec}(A)\).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A \otimes i} & A \otimes_R R^h \\
\downarrow{\varphi} & & \downarrow{\varphi \otimes \text{id}_{R^h}} \\
R & \xrightarrow{i} & R^h.
\end{array}
\]

Since \(\varphi\) is integral, so is \(\varphi \otimes \text{id}_{R^h}\). Moreover, \(\text{Spec}(i) : \text{Spec}(A \otimes_R R^h) \rightarrow \text{Spec}(A)\) is surjective because \(\text{id}_A \otimes i\) is faithfully flat by base change.

1. \(\varphi^{-1}(\mathfrak{M}) = m\) since \(\varphi\) is integral and \(\mathfrak{M}\) is maximal. Then (1) follows by part (1) of Lemma 3.2.


(2) Suppose $\mathfrak{Q}$ is a maximal ideal of $A \otimes_R R^h$. Then $\mathfrak{Q}$ contracts to $m^h$ in $R^h$ along $\varphi \otimes \text{id}_{R^h}$ because this map is integral. Thus, (2a) $\Rightarrow$ (2c).

Suppose $\mathfrak{Q}$ contracts to $m^h$ in $R^h$, and hence to $m$ in $R$. Then the contraction $\mathfrak{Q}_c$ of $\mathfrak{Q}$ to $A$ must be maximal because $\mathfrak{Q}_c$ contracts to $m$ along the integral map $\varphi$, and only maximal ideals can contract to maximal ideals along integral maps. This proves (2c) $\Rightarrow$ (2a).

Finally, suppose $\mathfrak{Q}$ contracts to a maximal ideal $\mathfrak{M}$ of $A$. Since $\varphi^{-1}(\mathfrak{M}) = m$, it follows by the commutativity of the above diagram that $\mathfrak{Q}$ contracts to $m^h$ along the integral ring map $\varphi \otimes \text{id}_{R^h}$. Then $\mathfrak{Q}$ must be maximal, thereby establishing (2b) $\Rightarrow$ (2a).

(3) The equivalent statements of part (2) tell us that the inverse image of $\text{MaxSpec}(A)$ under $\text{Spec}(\text{id}_A \otimes i)$ is precisely $\text{MaxSpec}(A \otimes_R R^h)$, and part (1) shows that the induced map $\text{MaxSpec}(A \otimes_R R^h) \to \text{MaxSpec}(A)$ is both injective and surjective. □

An analogue of Lemma 3.3 exists for minimal primes. 

**Lemma 3.4.** Let $(R,m)$ be an integrally closed domain, and $\varphi : R \to A$ be an integral extension of domains. Then we have the following:

1. $R^h$ is an integrally closed domain and $\text{Frac}(R) \otimes_R R^h = \text{Frac}(R^h)$.
2. Let $\mathfrak{Q} \in \text{Spec}(A \otimes_R R^h)$. The following are equivalent:
   - (2a) $\mathfrak{Q}$ is a minimal prime of $A \otimes_R R^h$.
   - (2b) $\mathfrak{Q}$ lies over $(0)$ in $A$.
   - (2c) $\mathfrak{Q}$ lies over $(0)$ in $R^h$.
3. There is a bijection $\{\text{minimal prime of } A \otimes_R R^h\} \leftrightarrow \text{Spec}(\text{Frac}(A) \otimes_{\text{Frac}(R)} \text{Frac}(R^h))$.
4. If $\text{Frac}(A)$ is a finite extension of $\text{Frac}(R)$, then $A \otimes_R R^h$ has finitely many minimal primes.

**Proof.** (1) That $R^h$ is an integrally closed domain is a well-known permanence property of Henselization; see [Sta20 Tag 00DI]. Since $R^h$ is a colimit of local étale extensions, $\text{Frac}(R^h)$ is an algebraic extension of $\text{Frac}(R)$. So $\text{Frac}(R) \otimes_R R^h$ is a field because it contains $\text{Frac}(R)$ and is contained in $\text{Frac}(R^h)$. Since $\text{Frac}(R) \otimes_R R^h$ is a localization of $R^h$, we get $\text{Frac}(R) \otimes_R R^h = \text{Frac}(R^h)$.

(2) Let $i : R \to R^h$ be the canonical map. Consider the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{id_A \otimes i} & A \otimes_R R^h \\
\downarrow{\varphi} \quad & & \quad \downarrow{\varphi \otimes id_{R^h}} \\
R & \xrightarrow{i} & R^h.
\end{array}
$$

Since $id_A \otimes i : A \to A \otimes_R R^h$ is flat, a minimal prime of $A \otimes_R R^h$ must contract to the unique minimal prime $(0)$ of $A$ by Going-Down. This proves (2a) $\Rightarrow$ (2b).

Suppose $\mathfrak{Q}$ contracts to $(0)$ in $A$. Then $\mathfrak{Q}$ must contract to $(0)$ in $R$ because $\varphi$ is injective. Let $p := (\varphi \otimes \text{id}_{R^h})^{-1}(\mathfrak{Q})$.

By the commutativity of the above diagram, $p$ contracts to $(0)$ in $R$. But (1) shows the generic fiber of $i$ is a singleton, consisting of the unique minimal prime $(0)$ of $R^h$. Consequently, $p = (0)$, proving (2b) $\Rightarrow$ (2c).

Assume (2c). If $\mathfrak{Q}$ is not a minimal prime of $A \otimes_R R^h$, then we can find $\mathfrak{Q}' \in \text{Spec}(A \otimes_R R^h)$ such that $\mathfrak{Q} \subseteq \mathfrak{Q}'$.

Then $(\varphi \otimes \text{id}_{R^h})^{-1}(\mathfrak{Q}') = (0)$, which is a contradiction because $\varphi \otimes \text{id}_{R^h}$ is an integral extension, and integral extensions have zero dimensional fibers [Sta20 Tag 00GT]. Thus, (2c) $\Rightarrow$ (2a).

(3) By part (2), the set of minimal primes of $A \otimes_R R^h$ is precisely the generic fiber of $id_A \otimes i : A \to A \otimes_R R^h$, which is in bijection with $\text{Spec}(\text{Frac}(A) \otimes_R (A \otimes_R R^h))$. The assertion now follows because

$$
\text{Frac}(A) \otimes_A (A \otimes_R R^h) \cong \text{Frac}(A) \otimes_{\text{Frac}(R)} (\text{Frac}(R) \otimes_R R^h) \cong \text{Frac}(A) \otimes_{\text{Frac}(R)} \text{Frac}(R^h),
$$

where the last isomorphism is a consequence of part (1).
(4) If \( \text{Frac}(A) \) is a finite extension of \( \text{Frac}(R) \), then \( \text{Frac}(A) \otimes_{\text{Frac}(R)} \text{Frac}(R^h) \) is a finite \( \text{Frac}(R^h) \)-algebra. Consequently, \( \text{Spec}(\text{Frac}(A) \otimes_{\text{Frac}(R)} \text{Frac}(R^h)) \) is a finite set. We are then done by part (3). \( \square \)

The next result is well-known. We include a proof for the reader’s convenience.

**Lemma 3.5.** Let \( A \) be a ring such that for all maximal ideals \( \mathfrak{m} \) of \( A \), \( A_{\mathfrak{m}} \) is a domain.

(1) If \( \mathfrak{p} \) and \( \mathfrak{q} \) are distinct minimal primes of \( A \), then \( \mathfrak{p} + \mathfrak{q} = A \).

(2) If \( A \) has finitely many distinct minimal primes \( \mathfrak{p}_1, \ldots, \mathfrak{p}_n \), then the canonical map \( A \to A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_n \) is an isomorphism.

**Proof.** (1) Suppose for contradiction that \( \mathfrak{p} + \mathfrak{q} \subseteq A \). Then there exists a maximal ideal \( \mathfrak{m} \) of \( A \) such that \( \mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{m} \). Now both \( \mathfrak{p} \mathfrak{m} \) and \( \mathfrak{q} \mathfrak{m} \) are distinct minimal prime ideals of the domain \( A_{\mathfrak{m}} \), which is impossible.

(2) The hypothesis implies that \( A \) is reduced, that is, \( \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = (0) \). Since \( \mathfrak{p}_i + \mathfrak{p}_j = A \) for \( i \neq j \), the result now follows by the Chinese Remainder Theorem [Sta20, Tag 00DT]. \( \square \)

**Proposition 3.6.** Let \((R, \mathfrak{m})\) be a local domain that is integrally closed in its fraction field \( K \). Let \( L \) be a finite field extension of \( K \) and let \( A \) be the integral closure of \( R \) in \( L \). We have the following:

(1) \( A \) has finitely many maximal ideals, that is, \( A \) is semi-local.

(2) \( A \otimes_R R^h \) is a semi-local ring.

(3) If \( \mathfrak{M} \) is a maximal ideal of \( A \otimes_R R^h \), then \( (A \otimes_R R^h)_{\mathfrak{M}} \) is an integrally closed domain.

(4) \( A \otimes_R R^h \) has finitely many minimal primes.

(5) Each maximal ideal \( \mathfrak{M} \) of \( A \otimes_R R^h \) contains a unique minimal prime \( \mathfrak{p} \), and conversely, \( \mathfrak{M} \) is the unique maximal ideal that contains \( \mathfrak{p} \). Moreover, the canonical map \( A \otimes_R R^h \to (A \otimes_R R^h)_{\mathfrak{M}} \) has kernel \( \mathfrak{p} \) and induces an isomorphism

\[
\frac{A \otimes_R R^h}{\mathfrak{p}} \cong (A \otimes_R R^h)_{\mathfrak{M}}.
\]

Consequently, \( (A \otimes_R R^h)_{\mathfrak{M}} \) is Henselian.

(6) If \( \mathfrak{m}_1, \ldots, \mathfrak{m}_n \) are the maximal ideals of \( A \) (assumed to be distinct), then

\[
A \otimes_R R^h \cong (A_{\mathfrak{m}_1})^h \times \cdots \times (A_{\mathfrak{m}_n})^h.
\]

Moreover, if \( \mathfrak{M}_i \) is the unique prime ideal of \( A \otimes_R R^h \) that contracts to \( \mathfrak{m}_i \), then \( \mathfrak{M}_i \) is maximal and

\[
(A \otimes_R R^h)_{\mathfrak{M}_i} \cong (A_{\mathfrak{m}_i})^h.
\]

(7) The sets \( \text{MaxSpec}(A), \text{MaxSpec}(A \otimes_R R^h), \text{Spec}(L \otimes_R \text{Frac}(R^h)) \) and \( \{\text{minimal prime of }A \otimes_R R^h\} \) have the same cardinality.

**Proof.** (1) The integral closure of an integrally closed domain in a finite extension of its fraction field has finite fibers [Bou98, Chap. V, §2.3, Cor. 2]. Since \( \text{MaxSpec}(A) \) is the closed fiber of \( R \subseteq A \), it is finite.

(2) \( \text{MaxSpec}(A \otimes_R R^h) \) is in bijection with the finite set \( \text{MaxSpec}(A) \) by Lemma 3.3.

(3) \( \mathfrak{M} \) contracts to a maximal ideal \( \mathfrak{P} \) in \( A \) by Lemma 3.3 and so, \( \mathfrak{P} \) contracts to \( \mathfrak{m} \) in \( R \). Since \( A_{\mathfrak{P}} \) is integrally closed, \( (A_{\mathfrak{P}})^h \), is also an integrally closed domain [Sta20, Tag 00DT]. By Lemma 3.2

\[
(A_{\mathfrak{P}})^h \cong ((A \otimes_R R^h)_{\mathfrak{M}})^h.
\]

Since \( (A \otimes_R R^h)_{\mathfrak{M}} \to ((A \otimes_R R^h)_{\mathfrak{M}})^h \) is faithfully flat, descent of integral closedness [Sta20, Tag 033G] implies \( (A \otimes_R R^h)_{\mathfrak{M}} \) is an integrally closed domain.

(4) This follows from part (4) of Lemma 3.4 because \( L \) is the fraction field of \( A \).

(5) A maximal ideal \( \mathfrak{M} \in A \otimes_R R^h \) contains a unique minimal prime because \( (A \otimes_R R^h)_{\mathfrak{M}} \) is a domain by part (3). Let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_k \) be the minimal primes of \( A \otimes_R R^h \). Then part (3) and Lemma 3.5 imply that

\[
A \otimes_R R^h \cong (A \otimes_R R^h)/\mathfrak{p}_1 \times \cdots \times (A \otimes_R R^h)/\mathfrak{p}_k.
\]

(3.6.1)
Lemma 3.4 shows that a minimal prime of $A \otimes_R R^h$ contracts to $(0)$ in $R^h$. Thus, the composition

$$R^h \twoheadrightarrow \otimes_{i \in I} R^h \rightarrow (A \otimes_R R^h)/p_i$$

is an integral extension for all $i = 1, \ldots, k$, and so, each $(A \otimes_R R^h)/p_i$ is a Henselian local domain by Lemma 3.4. Hence each minimal prime $p_i$ is contained in a unique maximal ideal, say $\mathfrak{M}_i$. Since $(A \otimes_R R^h)_{\mathfrak{M}_i}$, is a domain by part (3), the kernel of $A \otimes_R R^h \rightarrow (A \otimes_R R^h)_{\mathfrak{M}_i}$ has to be $p_i$. Moreover, $(A \otimes_R R^h)/p_i$ is local with maximal ideal $\mathfrak{M}_i/p_i$, so the induced injection $(A \otimes_R R^h)/p_i \hookrightarrow (A \otimes_R R^h)_{\mathfrak{M}_i}$ is an isomorphism. In particular, $(A \otimes_R R^h)_{\mathfrak{M}_i}$ is a Henselian local domain.

(6) The uniqueness and maximality of $\mathfrak{M}_i$ follow from Lemma 3.3 as does the fact that MaxSpec$(A \otimes_R R^h) = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_n\}$. The decomposition \(\{\mathfrak{M}_i\}_{i \in I}\) and part (5) then show that

$$A \otimes_R R^h \cong (A \otimes_R R^h)_{\mathfrak{M}_1} \times \cdots \times (A \otimes_R R^h)_{\mathfrak{M}_n},$$

and that each $(A \otimes_R R^h)_{\mathfrak{M}_i}$ is Henselian. Thus, $((A \otimes_R R^h)_{\mathfrak{M}_i})^h \cong (A \otimes_R R^h)_{\mathfrak{M}_i}$. On the other hand, Lemma 3.2 implies that $((A \otimes_R R^h)_{\mathfrak{M}_i})^h \cong (A_{\mathfrak{M}_i})^h$. Thus, $A \otimes_R R^h \cong (A_{\mathfrak{M}_1})^h \times \cdots \times (A_{\mathfrak{M}_n})^h$.

(7) All the sets are finite because of parts (1), (4) and the bijections of Lemma 3.3 and Lemma 3.4. It remains to check that $|\text{MaxSpec}(A \otimes_R R^h)| = |\{\text{minimal prime of } A \otimes_R R^h\}|$. This follows by part (5).

**Corollary 3.7.** Let $(R, m)$ be a local domain that is integrally closed in its fraction field $K$. Let $L$ be a finite field extension of $K$ and $A$ be the integral closure of $R$ in $L$. Suppose MaxSpec$(A) = \{m_1, \ldots, m_n\}$ (the maximal ideals are assumed to be distinct).

1. Let $\Sigma$ be the collection of finite (equivalently, finitely generated) $R$-subalgebras $B$ of $A$ such that $\text{Frac}(B) = \text{Frac}(A) = L$ and $m_i \cap B \neq m_j \cap B$, for $i \neq j$. Then $\Sigma$ is filtered under inclusion and

$$A = \text{colim}_{B \in \Sigma} B.$$

2. Let $\mathfrak{M} \in \text{MaxSpec}(A \otimes_R R^h)$ and $B \in \Sigma$. If $\mathfrak{M}_B$ is the contraction of $\mathfrak{M}$ to the subring $B \otimes_R R^h$ of $A \otimes_R R^h$, then the induced map on local rings

$$(B \otimes_R R^h)_{\mathfrak{M}_B} \rightarrow (A \otimes_R R^h)_{\mathfrak{M}}$$

is injective, $(B \otimes_R R^h)_{\mathfrak{M}_B}$ is a Henselian domain, and

$$(A \otimes_R R^h)_{\mathfrak{M}} = \text{colim}_{B \in \Sigma} (B \otimes_R R^h)_{\mathfrak{M}_B}.$$

**Proof.** (1) Since every element of $\Sigma$ is integral over $R$, finitely generated is equivalent to being module finite as an $R$-algebra. Note that $\Sigma$ is non-empty. Indeed, since $\text{Frac}(A) = L$ is a finite extension of $K$, one can choose a $K$-basis of $L$ consisting of elements $b_1, \ldots, b_n \in A$. By prime avoidance, for all $i = 1, \ldots, n$, choose $a_i \in m_i$ such that $a_i$ is not contained in any of the other maximal ideals of $A$ (here we need that $A$ is semi-local). Then by construction, the $R$-subalgebra $R[a_1, \ldots, a_n, b_1, \ldots, b_n]$ of $A$ is an element of $\Sigma$.

If $B \in \Sigma$, then any finitely generated $B$-subalgebra $C$ of $A$ is also in $\Sigma$. Therefore if $B_1, B_2 \in \Sigma$, then so is $B_1[B_2]$, that is, $\Sigma$ is filtered under inclusion. Since $A$ is the filtered union of finitely generated $B$-subalgebras for any $B \in \Sigma$ and $\Sigma \neq \emptyset$, we have $A = \text{colim}_{B \in \Sigma} B$.

(2) Fix $B \in \Sigma$. Since $B \hookrightarrow A$ is an integral extension, each $m_i \cap B$ is a maximal ideal of $B$. Furthermore, since every maximal ideal of $B$ is contracted from a maximal ideal of $A$, we get

$$\text{MaxSpec}(B) := \{m_1 \cap B, \ldots, m_n \cap B\}.$$

Let $\mathfrak{M}_i \in \text{Spec}(A \otimes_R R^h)$ be the unique prime ideal that contracts to $m_i$. Then Lemma 3.3 shows that

$$\text{MaxSpec}(A \otimes_R R^h) = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_n\}.$$
The defining property of $\Sigma$ implies that for $i \neq j$, $m_i \cap B \neq m_j \cap B$. As $(\mathfrak{M}_i)_B$ lies over $m_i \cap B$ by the commutativity of the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & A \otimes_R R^h \\
\uparrow & & \uparrow \\
B & \longrightarrow & B \otimes_R R^h,
\end{array}
$$

we have $(\mathfrak{M}_i)_B \neq (\mathfrak{M}_j)_B$ for $i \neq j$. In other words, $B \otimes_R R^h$ consists of $n$ distinct maximal ideals $(\mathfrak{M}_i)_B$ for $i = 1, \ldots, n$, and $\mathfrak{M}_i$ is the unique prime ideal of $A \otimes_R R^h$ that contracts to $(\mathfrak{M}_i)_B$.

Since $B$ is a finite extension of $R$, $B \otimes_R R^h$ is a finite extension of $R^h$. The decomposition of finite extensions of Henselian local domains $\text{Sta20}$ giving us that

$$
B \otimes_R R^h \cong (B \otimes_R R^h)(\mathfrak{M}_1)_B \times \cdots \times (B \otimes_R R^h)(\mathfrak{M}_n)_B,
$$

(3.7.1)

and that each $(B \otimes_R R^h)(\mathfrak{M}_i)_B$ is a Henselian local ring. Moreover, $B \otimes_R R^h$ is a subring of $A \otimes_R R^h$, which is reduced because it decomposes as a finite product of domains by part (6) of Proposition 3.6. Thus, $(B \otimes_R R^h)(\mathfrak{M}_i)_B$ is reduced for all $i$.

Applying part (3) of Lemma 3.4 to the integral extension $R \hookrightarrow B$ we see that the number of minimal primes of $B \otimes_R R^h$ equals the cardinality of $\text{Spec}(L \otimes_K \text{Frac}(R^h))$. But

$$\mid \text{Spec}(L \otimes_K \text{Frac}(R^h))\mid = \mid \text{MaxSpec}(A)\mid = n$$

by part (7) of Proposition 3.6 and the fact that $\text{Frac}(A) = \text{Frac}(B) = L$. Consequently, each factor in the decomposition (3.7.1) has exactly one minimal prime. Combined with reducedness, it follows that each $(B \otimes_R R^h)(\mathfrak{M}_i)_B$ is a domain.

In particular, $(\mathfrak{M}_i)_B$ contains a unique minimal prime (which expands to the zero ideal in $(B \otimes_R R^h)(\mathfrak{M}_i)_B$). Using part (5) of Proposition 3.6 if $\mathfrak{P}_i$ is the unique minimal prime of $A \otimes_R R^h$ contained in $\mathfrak{M}_i$, then

$$(\mathfrak{P}_i)_B := \mathfrak{P}_i \cap (B \otimes_R R^h)$$

must be the unique minimal prime of $B \otimes_R R^h$ contained in $(\mathfrak{M}_i)_B$. Indeed, by part (2) of Lemma 3.4, $\mathfrak{P}_i$ contracts to $(0)$ in $R^h$. Thus $(\mathfrak{P}_i)_B$ also contracts to $(0)$ in $R^h$. Applying part (2) of Lemma 3.4 again, but this time to the integral extension $R \hookrightarrow B$, then shows that $(\mathfrak{P}_i)_B$ is a minimal prime of $B \otimes_R R^h$.

Using the commutative diagram

$$
\begin{array}{ccc}
B \otimes_R R^h & \longrightarrow & A \otimes_R R^h \\
\downarrow & & \downarrow \\
(B \otimes_R R^h)(\mathfrak{M}_i)_B & \longrightarrow & (A \otimes_R R^h)_{\mathfrak{M}_i},
\end{array}
$$

it follows that $\mathfrak{P}_i(A \otimes_R R^h)_{\mathfrak{M}_i}$ must contract to $(\mathfrak{P}_i)_B(B \otimes_R R^h)(\mathfrak{M}_i)_B$ in $(B \otimes_R R^h)(\mathfrak{M}_i)_B$. Since $(A \otimes_R R^h)_{\mathfrak{M}_i}$ and $(B \otimes_R R^h)(\mathfrak{M}_i)_B$ are both domains (the former ring is a domain by Proposition 3.6), we have $(\mathfrak{P}_i(A \otimes_R R^h)_{\mathfrak{M}_i}) = (0)$ and $(\mathfrak{P}_i)_B(B \otimes_R R^h)(\mathfrak{M}_i)_B = (0)$. Thus, the bottom horizontal arrow is injective.

A maximal ideal $\mathfrak{M}$ of $A \otimes_R R^h$ coincides with some $\mathfrak{M}_i$. Therefore the argument above shows that for any $B \in \Sigma$, $(B \otimes_R R^h)_{\mathfrak{M}_i}$ is a Henselian local domain and the induced local map

$$
(B \otimes_R R^h)_{\mathfrak{M}_i} \rightarrow (A \otimes_R R^h)_{\mathfrak{M}_i}
$$

is injective. As tensor product commutes with filtered colimits, we have $A \otimes_R R^h = \text{colim}_{B \in \Sigma} B \otimes_R R^h$ by part (1), and so, $(A \otimes_R R^h)_{\mathfrak{M}} = \text{colim}_{B \in \Sigma} (B \otimes_R R^h)_{\mathfrak{M}_i}$. In this case the filtered colimit is actually a filtered union because $(B \otimes_R R^h)_{\mathfrak{M}_i}$ is a subring of $(A \otimes_R R^h)_{\mathfrak{M}_i}$, for all $B \in \Sigma$. □

4. HENSELIAN VALUATION RINGS AND SOME RAMIFICATION THEORY

This section discusses some background from the ramification theory of extensions of valuations relevant to Conjecture 1.1. We first recall how extensions of valuation rings arise in algebraic field extensions.
Proposition 4.1. Let $L/K$ be an algebraic extension of fields. Let $V$ be a valuation ring of $K$ and $A$ be the integral closure of $V$ in $L$. Then localization $m \mapsto A_m$ induces a bijection

\[ \text{MaxSpec}(A) \leftrightarrow \{ \text{valuation rings of } L \text{ that dominate } V \}. \]

In particular, if $L/K$ is finite, then there are finitely many valuation rings of $L$ that dominate $V$.

**Indication of proof.** For the bijection see [Bou98, Chap. VI, §8.6, Prop. 6]. If $L/K$ is finite, then $A$ has finitely many maximal ideals by part (1) of Proposition 3.6. That there are finitely many valuation rings of $L$ that dominate $V$ now follows from the bijection of this Proposition. \qed

As a consequence of the Henselian property, one can now deduce:

**Corollary 4.2.** Let $V$ be a valuation ring of a field $K$. The following are equivalent:

1. $V$ is Henselian.
2. If $L$ is an algebraic extension of $K$ and $A$ is the integral closure of $V$ in $L$, then $A$ is the unique valuation ring of $L$ that dominates $V$.

**Proof.** (1) $\Rightarrow$ (2) By Lemma 3.1, $A$ must be a local ring. By the bijection of Proposition 4.1 it follows that $A$ must be the unique valuation ring of $L$ that dominates $V$.

Conversely, assume (2). Let $B$ be a domain that is an integral extension of $V$. By Lemma 3.1 again, it suffices to show that $B$ is local. Let $L = \text{Frac}(B)$. Then $L/K$ is algebraic. If $A$ is the integral closure of $V$ in $L$, then $B \subset A$ is integral. Since $A$ is local by (2), $B$ must also be local. Indeed, the unique maximal ideal of $A$ must contract to the unique maximal ideal of $B$ because $B \subset A$ is an integral extension [AM69, Cor. 5.8, Thm. 5.10]. \qed

**Remark 4.3.** In terms of valuations, Corollary 4.2 can be reinterpreted as saying that if $\nu$ is a valuation of a field $K$, then the valuation ring of $\nu$ is Henselian if and only if for every algebraic extension $L/K$, there exists a unique valuation $\omega$ of $L$ (up to equivalence) that extends $\nu$.

The Henselization of a valuation ring admits a purely valuation theoretic description. However, for the purposes of this paper, it is more helpful to think of Henselizations as filtered colimits of local étale extensions that induce isomorphisms on residue fields. One then has the following result.

**Lemma 4.4.** Let $\nu$ be a valuation of a field $K$ with valuation ring $O_\nu$ and value group $\Gamma_\nu$. Then the Henselization $O^h_\nu$ of $O_\nu$ is a valuation ring whose associated valuation $\nu^h$ also has value group $\Gamma_\nu$.

**Proof.** See [Sta20, Tag 0ASK]. The main points are that local étale extensions of valuation rings are valuation rings and a filtered colimit of valuation rings is a valuation ring. \qed

**Notation 4.5.** The fraction field of $O^h_\nu$ will be denoted by $K^h$. Thus, $\nu^h$ is a valuation of $K^h$ whose valuation ring is $O^h_\nu$.

We record a descent result that we will need in the proof of Theorem 1.2.

**Lemma 4.6.** Let $\varphi : V \to W$ be a ring map and $W$ be a valuation ring. The following are equivalent:

1. $V$ is a valuation ring and $\varphi$ is an injective local map.
2. $\varphi$ is faithfully flat.
3. $\varphi$ is cyclically pure, that is, for all ideals $I$ of $V$, the induced map $V/I \to W/IW$ is injective.

**Proof.** Assume (1). If $\varphi$ is injective, then $W$, being a domain, is a torsion-free $V$-module, hence flat [Bou98, Chap. VI, §3.6, Lem. 1]. Since $\varphi$ is local, $\varphi$ is faithfully flat. Thus, (1) $\Rightarrow$ (2). Furthermore, (2) $\Rightarrow$ (3) is a property of faithfully flat maps; see [Bou98, Chap. I, §3.5, Prop. 9].

Assume (3). Taking $I = (0)$, we see that $\varphi$ is injective. Thus, $V$ is a domain because $W$ is. To show that $V$ is a valuation ring, it is enough to show that for all $x, y \in V$, $xV \subseteq yV$ or $yV \subseteq xV$. Since $W$ is a valuation ring, we must have $xW \subseteq yW$ or $yW \subseteq xW$. Cyclic purity of $\varphi$ implies that $\varphi^{-1}(IW) = I$, for any ideal $I$ of
Thus, if $xW \subseteq yW$, then $xV = \varphi^{-1}(xW) \subseteq \varphi^{-1}(yW) = yV$. Similarly, $yV \subseteq xV$ if $yW \subseteq xW$. Finally, $\varphi$ is local because if $m_V$ is the maximal ideal of the valuation ring $V$, then injectivity of $V/m_V \rightarrow W/m_V W$ shows $m_V W \neq W$.

Conjecture 1.1 relates essential finite generation of extensions of valuation rings to fundamental invariants from the ramification theory of extensions of valuations. We now briefly introduce these invariants. Let $\omega$ be a field extension. Note that $\Gamma$ of $\nu$ of $L$ is the unique extension of $\omega$ to $\nu$.

Definition 4.7. Suppose $L/K$ is a finite extension and consider the extension of valuations $\omega/\nu$.

(a) The ramification index of $\omega/\nu$, denoted $e(\omega|\nu)$, is $[\Gamma_\omega : \Gamma_\nu]$.
(b) The inertia index of $\omega/\nu$, denoted $f(\omega|\nu)$, is $[\kappa_\omega : \kappa_\nu]$.
(c) The initial index of $\omega/\nu$, denoted $\epsilon(\omega|\nu)$, is the cardinality of the set $\{x \in \Gamma_{\omega, \geq 0} : x < \Gamma_{\nu, > 0}\}$.

The finiteness of the initial index follows from the inequality

$$\epsilon(\omega|\nu) \leq e(\omega|\nu),$$

which holds because if $x, y \in \Gamma_{\omega, \geq 0}$ are distinct elements such that $x, y < \Gamma_{\nu, > 0}$, then $x + \Gamma_\nu \neq y + \Gamma_\nu$ in $\Gamma_\omega/\Gamma_\nu$. Indeed, assume without loss of generality that $0 \leq x < y$. Then $y - x \in \Gamma_{\omega, > 0}$ and $y - x \leq y < \Gamma_{\nu, > 0}$, that is, $y - x \notin \Gamma_\nu$.

By Lemma 4.3 if $\omega/\nu$ is an extension of valuations, then for the extension of Henselizations $\omega^h/\nu^h$, we have $e(\omega|\nu) = e(\omega^h|\nu^h), f(\omega|\nu) = f(\omega^h|\nu^h)$ and $\epsilon(\omega|\nu) = \epsilon(\omega^h|\nu^h)$ because Henselizations do not alter value groups and residue fields. In addition, one can use the isomorphism of part (6) of Proposition 3.6 and Proposition 1.1 to conclude that $L \otimes_K K^h$ is a finite product of fields, one of which coincides with $L^h$, the fraction field of $\mathcal{O}_{\omega}^h$. Thus,

$$[L^h : K^h] \leq [L : K] < \infty.$$  (4.7.1)

Using these observations we recall the notion of the defect of $\omega/\nu$ [Kuh11], which measures to what extent equality fails in (4.6.1), at least when $\omega$ is the unique extension of $\nu$ to $L$.

Definition 4.8. Let $L/K$ be a finite field extension and $\nu$ be a valuation of $K$. If $\omega$ is the unique extension of $\nu$ to $L$, then the defect of $\omega/\nu$, denoted $d(\omega|\nu)$, is defined to be

$$d(\omega|\nu) = \frac{[L : K]}{e(\omega|\nu)f(\omega|\nu)}.$$  (4.7.2)

If the extension of valuations $\omega/\nu$ is not necessarily unique, the defect of $\omega/\nu$ is defined to be the defect of the extension of Henselizations $\omega^h/\nu^h$, that is,

$$d(\omega|\nu) = \frac{[L^h : K^h]}{e(\omega|\nu)f(\omega|\nu)}.$$  (4.7.3)

We say $\omega/\nu$ is defectless if $d(\omega|\nu) = 1$, that is, if $[L^h : K^h] = e(\omega|\nu)f(\omega|\nu)$.

Remark 4.9.

1. If $L/K$ is a finite extension, then $\omega^h$ is the unique extension of $\nu^h$ to $L^h$ by Corollary 1.2 and (4.7.1). Thus, the definition of the defect of an extension of valuations that is not necessarily unique in terms of the defect of the extension of henselizations makes sense.
Example 4.10. Let $\nu$ be a valuation of $K/k$. Then for all $B \in \Sigma$,

\[ 2\mathfrak{m} := \text{the unique prime (equivalently, maximal) ideal of } A \otimes_{O_{\nu}} O_{\nu}^h \text{ that contracts to } \mathfrak{m}. \]
For all \( B \in \Sigma \), let \( \mathfrak{M}_B \) denote the contraction of \( \mathfrak{M} \) to the \( O^h_\nu \)-subalgebra \( B \otimes_{O_\nu} O^h_\nu \) of \( A \otimes_{O_\nu} O^h_\nu \) (it is a subalgebra by flatness of \( O^h_\nu \)). Then by the commutativity of the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & A \otimes_{O_\nu} O^h_\nu \\
\uparrow & & \uparrow \\
B & \longrightarrow & B \otimes_{O_\nu} O^h_\nu
\end{array}
\]

and Lemma 3.3 again, \( \mathfrak{M}_B \) is the unique prime (equivalently, maximal) ideal of \( B \otimes_{O_\nu} O^h_\nu \) that contracts to the maximal ideal \( m_B \) of \( B \).

By Corollary 3.7 for all \( B \in \Sigma \),

\[(B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B}\]

is a \( O^h_\nu \)-subalgebra of

\[(A \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}},\]

and

\[(A \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}} = \text{colim}_{B \in \Sigma} (B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B}.
\]

Note that the filtered colimit is a filtered union. By part (6) of Proposition 3.6 we have that

\[(A \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}} \cong (A_m)^h = O^h_\omega.
\]

Since \( O^h_\omega \) is a module-finite \( O^h_\nu \)-algebra by the hypothesis of (v), we can find \( B \in \Sigma \) such that

\[(A \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}} = (B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B}.
\]

Therefore \( (B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B} \) is a Henselian valuation ring, and by part (2) of Lemma 3.2 we conclude

\[(B_{m_B})^h \cong ((B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B})^h = (B \otimes_{O_\nu} O^h_\nu)_{\mathfrak{M}_B}.
\]

In other words, \((B_{m_B})^h\) is a valuation ring, so by descent (Lemma 1.6), \( B_{m_B} \) is a valuation ring as well. By the definition of the collection \( \Sigma \), we have

\[\text{Frac}(B_{m_B}) = \text{Frac}(B) = L, \]

that is, \( B_{m_B} \) is a valuation ring of \( L \). Since \( O_\omega = A_m \) is also a valuation ring of \( L \) that dominates \( B_{m_B} \), we must have

\[O_\omega = B_{m_B}\]

because valuation rings are maximal with respect to domination of local rings. Thus, \( O_\omega \) is the localization of the finite \( O_\nu \)-algebra \( B \). But \( B \) is \( O_\nu \)-flat since it is a torsion-free \( O_\nu \)-module [Kou8, Chap. VI, §3.6, Lem. 1]. Therefore, \( B \) is a finitely presented \( O_\nu \)-algebra by Lemma 2.1. This completes the proof of (v) \( \Rightarrow \) (i), hence also of the Theorem.

The proof of Theorem 1.2 establishes the stronger result that if conditions (1) and (2) of Conjecture 1.1 hold, then \( O_\omega \) is the localization of a finite \( O_\nu \)-algebra \( B \) contained in the integral closure of \( O_\nu \) in \( L \). Since a finitely generated torsion-free module over a valuation ring is free, \( B \) is a free \( O_\nu \)-module of finite rank.

**Remark 5.1.** One can prove (i) \( \Rightarrow \) (v) (or, (ii) \( \Rightarrow \) (v)) using Zariski’s Main Theorem. Suppose \( O_\omega \) is the localization of a finite type \( O_\nu \)-algebra \( B \) at a prime ideal \( \mathfrak{p} \). Then

\[[\kappa(\mathfrak{p}) : \kappa_\omega] = [\kappa_\omega : \kappa_\nu] \leq [L : K] < \infty,
\]

where the first inequality follows from (4.6.1). Moreover,

\[\dim(B_{\mathfrak{p}}/m_{\mathfrak{p}}B_{\mathfrak{p}}) = \dim(O_{\omega}/m_{\mathfrak{p}}O_{\omega}) = 0\]

because \( m_{\omega} \) is the only prime ideal of \( O_{\omega} \) that contracts to \( m_{\nu} \) (if not, a non-maximal prime of \( O_{\omega} \) that contracts to \( m_{\nu} \) will give a non-maximal prime of the integral closure \( A \) of \( O_\nu \) in \( L \) that contracts to the maximal ideal \( m_{\omega} \)). Thus, \( B \) is quasi-finite at \( \mathfrak{p} \) by part (6) of [Sta20 Tag 00PK]. Then Zariski’s Main Theorem [Sta20 Tag 00QBB] implies that there exists a finite \( O_\nu \)-subalgebra \( B' \) of \( B \) such that \( O_{\omega} \) is a localization of \( B' \) at a maximal ideal \( \mathfrak{q} \) of \( B' \) (\( \mathfrak{q} \) is maximal because it contracts to \( m_{\nu} \)). By Lemma 3.2, \( O^h_\nu \) is the Henselization of \( (B' \otimes_{O_\nu} O^h_\nu)_{\mathfrak{q}} \), where \( \Omega \) is the unique prime ideal of \( B' \otimes_{O_\nu} O^h_\nu \) that contracts to \( \mathfrak{q} \). But \( \mathfrak{q} \) is maximal, so \( \Omega \) is a maximal ideal as well by Lemma 3.3. Since \( B' \otimes_{O_\nu} O^h_\nu \) is a finite \( O^h_\nu \)-algebra, it decomposes as a finite product of finite Henselian local rings [Sta20 Tag 04GG]. Then \((B' \otimes_{O_\nu} O^h_\nu)_{\mathfrak{q}}\)
must coincide with one these local factors, that is, \((B' \otimes \mathcal{O}_\nu \mathcal{O}^h_{\nu})_\Omega\) is Henselian and finite. Consequently, \(\mathcal{O}^h_\nu \cong ((B' \otimes \mathcal{O}_\nu \mathcal{O}^h_{\nu})_\Omega)^h = (B' \otimes \mathcal{O}_\nu \mathcal{O}^h_{\nu})_\Omega\) is a finite \(\mathcal{O}^h_{\nu}\)-algebra, proving (v). Once we know that \(\mathcal{O}^h_\nu\) is a module-finite \(\mathcal{O}^h_{\nu}\) algebra, part (iii) of Theorem 1.2 now follows by [Bou98, Chap. VI, §8.5, Thm. 2]. This gives a different proof of [CN19, Thm. 4.1].

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