KANTOWSKI-SACHS COSMOLOGY WITH VLASOV MATTER

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ABSTRACT. We analyse the Kantowski-Sachs cosmologies with Vlasov matter of massive and massless particles using dynamical systems analysis. We show that generic solutions are past and future asymptotic to the non-flat locally rotationally symmetric Kasner vacuum solution. Furthermore, we establish that solutions with massive Vlasov matter behave like solutions with massless Vlasov matter towards the singularities.

1. Introduction

The Einstein-Vlasov system describes spacetimes containing ensembles of self-gravitating, collisionless particles and constitutes an excellent model for the large-scale structure of the universe where collisions of particles (modeling galaxies or galaxy clusters) are indeed rare. For the class of cosmological spacetimes certain stability results were resolved in recent years [1, 32] (cf [11, 18, 34] for recent stability results for the Einstein-Vlasov system in the asymptotically flat case).

Understanding cosmological dynamics of Einstein-matter systems is in general based on the study of dynamics of homogeneous cosmological models, whose stability properties are analyzed in a next step. The program of classifying dynamics of homogeneous spacetimes with Vlasov matter contains a number of results but for certain classes this problem is still open, among those are the Kantowski-Sachs models, which are the subject of the present paper.

1.1. The dynamical systems approach to spatially homogeneous Einstein-Vlasov cosmology. Spatially homogenous (SH) cosmologies with non-tilted perfect fluids with linear equations of state have been analysed with great success using dynamical systems methods; cf [8, 15, 35]. This is due to the fact that in this context, the Einstein equations reduce to a system of ODEs. In contrast, the Einstein-Vlasov system remains a system of PDEs even in the spatially homogenous context; cf [2, 28, 29].

Despite that, a route to analyse SH Einstein-Vlasov cosmologies with dynamical systems methods has been initiated by Rendall [26]. The core idea is to work with distribution functions which exhibit the same symmetries as the underlying spacetime in a manner which we make precise in Section 3 (Definition 2), and which then satisfy the Vlasov equation. The general functional form of such distributions has been investigated by Maartens and Maharaj in [20] in the context of spatial homogeneity, and in [19] for SH cosmologies which are also locally rotationally symmetric (LRS). The cosmologies for which they succeeded without additional assumptions, and for which the found distributions are non-trivial are:

(1) SH but not LRS: Bianchi I.
(2) SH and LRS: LRS Bianchi I, II, III, VII0, VIII, IX, Kantowski-Sachs.

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The latter are all SH cosmologies which are compatible with local rotational symmetry, except Bianchi V and VII.¹

Though restricting to these distribution functions comes with the price of losing generality it has the advantage that the Vlasov equation is solved explicitly and thus eliminated from the system. Hence what is left to solve is the Einstein part, which then again reduces to an autonomous system of ODEs. As with perfect fluids, the Vlasov matter then enters the dynamics only by the energy-momentum tensor. However, since the latter is given by integrals over the distribution function which is arbitrary up to its general functional form, Vlasov matter is still more difficult to deal with than perfect fluids, for which the energy-momentum tensor is fixed by a single parameter. The approach taken by Rendall [26] is to make the matter integrals tractable by imposing a further symmetry on the distribution functions – reflection symmetry, which we will define in Section 3 (Definition 3).

1.2. Alternatives. The strength of the dynamical systems approach to SH Einstein-Vlasov cosmology described above is that it allows for a global analysis of the state-space. The downside is that one has to impose certain symmetry assumptions in order to obtain a tractable dynamical system, thereby loosing generality. We thus refer to Nungesser et al. [16, 17, 21, 22, 23, 24] for an alternative route – a small data future stability analysis within the class of SH spacetimes. The strength of this approach is that the Vlasov distributions are of full generality. The downside is that it does not yield global results on the full state-space.

1.3. Literature. SH spacetimes can be classified by the properties of the underlying symmetry group. One distinguishes between Bianchi and Kantowski-Sachs spacetimes; cf [35, Subsec 1.2.2] and Subsection 2.1 below. The class of Bianchi spacetimes is divided further into types I–IX. Bianchi I has been analysed in the dynamical systems approach described in Subsection 1.1 in [26] and [12]. LRS Bianchi I, II and III are covered by [27, 30, 31] and LRS Bianchi VIII and IX by [6, 13]. Kantowski-Sachs has been investigated in [30], however with the restriction to Vlasov matter of massless type. This leaves the corresponding Kantowski-Sachs analysis with massive Vlasov matter as the only open problem from the list (1), (2). The present paper closes this gap.

There is also a series of papers [5, 7, 13] initiated by Calogero and Heinzle which is closely related. These are concerned with a broadly defined, generally anisotropic, matter family. The reason why this is of interest here is threefold. Firstly, as pointed out in [5, Sec 7.1] and [7, Sec 12.1], massless Vlasov matter subject to the same symmetry assumptions which we impose here naturally fits into their matter family as a special case. Secondly, in [6, 13] the formalism could be extended to incorporate massive Vlasov matter as well. Thirdly, [7] readily lists the dynamical system for Kantowski-Sachs cosmologies in their original anisotropic matter framework. We are adopt this system here.

1.4. Results and outline. In this paper we give a dynamical system analysis of Kantowski-Sachs cosmologies with massive or massless Vlasov, thereby closing the literature gap identified in subsection 1.3. The distribution functions considered are invariant under the symmetries of the spacetime (Definition 2), reflection symmetric (Definition 3) as well as split supported (Definition 4). Our main result (Theorem 1) is that generic solutions are past and future asymptotic to the non-flat LRS Kasner vacuum solution, and that there are non-generic solutions which are asymptotic to a non-isotropic Bianchi I matter solution or the flat Friedman

¹Note that Bianchi type III is the same as Bianchi type VI−1, cf [35, p 37], and that LRS Bianchi types I and VIIb can be identified with each other; cf [36, p 2579–80, p 2584], [30, p 1715] or [7, App B, p 667].
matter solution. We give the metric to all these solutions. For the case of massless particles, we thereby recover [30, Thm 5.1]. The result for massive particles is new. Our results are in accordance with results on recollapse of the spacetime, cf Subsection 2.3, we also confirm that all solutions recollapse. Finally, we establish that solutions with massive particles approach solutions with massless particles towards the singularities.

In Section 2 we give some background on Kantowski-Sachs cosmologies. Section 3 outlines the assumptions imposed on the distribution functions. The dynamical system for Kantowski-Sachs cosmologies with Vlasov matter is then formulated in Section 4, while Section 5 deals with the corresponding state-space. Finally, we present our results in Section 6. Appendix A is concerned with the analysis of the flow at infinity.

We will assume some familiarity with dynamical systems theory. For a background we refer to [25] or [35, Chap 4]. Greek indices are space-time indices, while latin indices are spatial. We use the Einstein summation convention and sum over repeated indices if not indicated otherwise.

2. Kantowski-Sachs cosmologies

In this section we give some background on the class of SH cosmologies called Kantowski-Sachs cosmologies. In Subsection 2.1 we define the definition in terms of the underlying symmetry group, and discuss the structure of the associated Lie-algebra. In Subsection 2.2 we give the metric in a symmetry adapted frame, and list the topologies on which it can be realised. Finally, in Subsection 2.3 we review some results concerning the recollapse property of this class, and comment on how our results relate to those.

2.1. Symmetry group. By a cosmology we refer to a spacetime that solves Einstein’s equations.

**Definition 1.** Kantowski-Sachs cosmologies are the class of cosmologies with a four-dimensional continuous Lie-group of isometries which acts multiply-transitively on three-dimensional spatial hypersurfaces, but which does not exhibit a three-dimensional subgroup which acts simply-transitively on them; cf [9] or [35, Sec 1.2.2].

By the former statement of Definition 1, Kantowski-Sachs cosmologies are SH and LRS. The latter statement of Definition 1 is what distinguishes Kantowski-Sachs cosmologies from SH and LRS Bianchi cosmologies, for which such a subgroup does exist.\(^2\) The Kantowski-Sachs cosmologies however do admit a three-dimensional subgroup of isometries, which acts simply transitively on two-spheres. It can hence be identified with \(SO(3, \mathbb{R})\), and we denote the associated generators, ie Killing vector fields, by \(\{\xi_1, \xi_2, \xi_3\}\). In addition to this spherical symmetry, Kantowski-Sachs cosmologies also exhibit a (radial) translation symmetry. Denoting the corresponding Killing vector field by \(\eta\), the Lie-algebra associated with the full isometry group can be represented by \([\xi_1, \xi_2] = \xi_3\) and cyclic permutations thereof, together with \([\eta, \xi_i] = 0 \forall i \in \{1, 2, 3\}\); cf [9, App B] or [7, App B.2].\(^3\)

2.2. Geometry and topology. A convenient choice of coframe for Kantowski-Sachs is given by a time independent covector \(\hat{\omega}^1\) which is invariant under the radial translation symmetry, together with an arbitrary time independent orthonormal
frame $\{\hat{\omega}^2, \hat{\omega}^3\}$ on the two-spheres; cf \cite{30}, Sec 2. The hat emphasises time independence. Choosing an appropriate time coordinate $t$, a general Kantowski-Sachs metric then takes the form

\begin{equation}
4g = -dt \otimes dt + g_{11}(t) \hat{\omega}^1 \otimes \hat{\omega}^1 + g_{22}(t) \left( \hat{\omega}^2 \otimes \hat{\omega}^2 + \hat{\omega}^3 \otimes \hat{\omega}^3 \right).
\end{equation}

This choice is canonical in the sense that spatial homogeneity is reflected by the spatial metric depending on $t$ only, and we choose the spatial frame such, that the time dependence rests solely on the spatial metric components, while the 1-forms $\hat{\omega}^i$ are time-independent. Local rotational symmetry on the other hand is reflected by the spatial metric components satisfying $g_{22}(t) = g_{33}(t)$.

The spatial part of the metric (3) can be realised on topologies of the form $\mathbb{R} \times S^2$, or topologies which can be derived from this by (i) identifying points under a translation in $\eta$-direction, ie in the $\mathbb{R}$-part of the topology, or a translation in $\eta$-direction together with a rotation or reflection, (ii) an identification of antipodal points in each $S^2$ or (iii) a combination of (i) and (ii); cf \cite[Sec 2]{9}. A prominent example is $S \times S^2$. Different realisations of these topologies do not effect the analysis we perform. In particular our results hold for all of those topologies.

2.3. Recollapse. There are several results concerning recollapse, ie the presence of both, a big-bang singularity and a big-crunch singularity. \cite[Thm 2]{9} shows recollapse by geodesic incompleteness for Kantowski-Sachs cosmologies with perfect fluids. \cite[Thm 1.2]{4} proofs recollapse by showing that the length of timelike curves in Kantowski-Sachs cosmologies is bounded from above, given that the stress tensor of the matter is positive definite, ie that it satisfies the non-negative-sum-pressures condition. This result is embedded in the more general result \cite[Thm 1.1]{4}, which shows that there is an upper bound to the length of timelike curves in spherically symmetric spacetimes that possess $S \times S^2$ Cauchy surfaces and that satisfy the non-negative-pressure and dominant energy conditions. It thereby also includes Kantowski-Sachs cosmologies with Vlasov matter. A related recollapse result concerning Einstein-Vlasov spacetimes with spherical symmetry, but in general without additional translation symmetry, is also given in \cite[Thm 3.11]{14}. However, none of these results establishes the precise behaviour towards the singularities which we obtain.

3. Spatially homogenous, locally rotationally symmetric Vlasov distributions

The purpose of this section is to define Vlasov distributions exhibiting the symmetries of the underlying spacetime, which is what we assume in our analysis. In Subsection 3.1 we briefly introduce the Einstein-Vlasov system. Subsection 3.2 discusses the way in which distribution functions inherit the symmetries of the underlying spacetime. In Subsection 3.3 we give the definition of symmetry invariant distribution functions, and discuss the resulting general functional forms of those functions int he context of SH as well as SH LRS spacetimes. Finally, in Subsection 3.4 we discuss further assumptions imposed on the distribution functions.

3.1. The Einstein-Vlasov system. Consider the mass-shell

\[ P_m := \{(x^\lambda, v^\lambda) | x^\lambda \in M, v^\lambda \in T_x M, v_{\mu}v^\mu = -m^2, v^0 > 0\} \]

associated with a spacetime $(M, 4g)$. Let there be an ensemble of collisionless particles of mass $m = 1$ or 0, described by the distribution function $f(x^\lambda, v^\lambda)$, where $x^\lambda$ and $v^\lambda$ stand for the particles positions and four-momenta with domain $P_m$. In the case $m = 0$ we understand $v^i = 0$ to be excluded from the domain. Since
the particles are collisionless, $f$ satisfies the Vlasov equation, which in coordinates $\{t, x^i\}$ and with Christoffel symbols $\Gamma^\nu_{\mu\nu}$ reads

$$v^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^k_{\mu\nu} v^\mu v^\nu \frac{\partial f}{\partial v_k} = 0. \tag{4}$$

The Einstein-Vlasov system is then given by (4) together with the Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$, with an energy-momentum tensor of the form

$$T_{\mu\nu} = \int f(x^\lambda, v^\lambda) v^\mu v^\nu \text{vol},$$

where

$$\text{vol} := \sqrt{|\det 4g|} \, \text{d}v^1 \text{d}v^2 \text{d}v^3,$$

denotes the volume form induced by $4g$ on $P_m$. Hence, in (5), $|v_0|$ is understood to be constrained by the relation

$$4g(v, v) = \begin{cases} -1, & m = 1 \\ 0, & m = 0 \end{cases}.$$  

One obtains directly from the definitions (4) and (5) that Vlasov matter satisfies the dominant and strong energy conditions, as well as the non-negative pressures condition; cf [28, Sec 1]. For more details on the Einstein-Vlasov system we refer to [2, 28, 29].

3.2. Inheritance of symmetries by the distribution function. The dynamical systems approach to spatially homogenous Einstein-Vlasov cosmology, which we will use in the following, relies on symmetries obeyed by the distribution function $f$, which, by virtue of these symmetries, then solves the Vlasov equation (4) and effectively eliminates the latter from the system of equations. In the proceeding discussion, we mainly follow the work by Maartens and Maharaj [19, 20].

Let the spacetime $(M, 4g, f)$ be subject to symmetries, described by a set of Killing vector fields $X$, then

$$\mathcal{L}_\xi 4g = 0 \quad \forall \xi \in X, \tag{6}$$

where $\mathcal{L}$ denotes the Lie-derivative. In the case of Kantowski-Sachs spacetimes there are four such fields – three corresponding to spatial homogeneity, and one to local rotational symmetry; cf [35, Sec 1.2.2 (b)]. Through the Einstein equations, the symmetries (6) are inherited by the energy-momentum tensor, such that

$$\mathcal{L}_\xi T_{\mu\nu} = 0 \quad \forall \xi \in X. \tag{7}$$

Consequently, from (7) with (5), the distribution function $f$ inherits the symmetry by

$$\mathcal{L}_\xi T_{\mu\nu} = \int \tilde{\xi} (f) v^\mu v^\nu \text{vol} = 0 \quad \forall \xi \in X, \tag{8}$$

where

$$\tilde{\xi} = \xi^i \frac{\partial}{\partial x^i} + \xi^i j v^j \frac{\partial}{\partial v^i}$$

denotes the complete lift of the Killing vector $\xi$ onto the tangent bundle; cf [19, Sec II.C], [3, Sec III.E], references therein and [33]. Hence, a Vlasov matter distribution in a space-time with symmetries $X$ has to satisfy (8).

\textsuperscript{4}This relation also holds for homothetic vector fields, for which $\mathcal{L}_\xi 4g = \kappa$, with $\kappa = \text{const}$, since the Riemann tensor is invariant with respect to a constant scaling; cf [35, Sec 1.2.4].
3.3. Symmetry invariant distribution functions.

**Definition 2.** A Vlasov distribution functions \( f \) is invariant under the symmetries of the spacetime described by a set of Killing vector fields \( X \) if and only if
\[
\tilde{\xi}(f) = 0 \quad \forall \xi \in X.
\]
(10)

An \( f \) which is invariant under spatial homogeneity (and local rotational symmetry) is called a spatially homogenous (and locally rotationally symmetric) distribution function.

Clearly, (8) follows from (10), but the reverse statement does not hold in general.\(^5\) Hence, considering such distribution functions usually mean a loss of generality. In [19, 20], Maartens and Maharaj sought to find the general functional forms of SH as well as SH and LRS distribution functions which satisfy the Vlasov equation (4). The cosmologies for which they succeeded without additional assumptions, and for which the found distributions are non-trivial are (1), (2). Those distribution functions are:

\[
\begin{align*}
(11) & \quad f = f_0(v_1, v_2, v_3), & \text{Bianchi I;} \\
(12) & \quad f = f_0(v_1, v_2^2 + v_3^2), & \text{SH LRS cosmologies listed in (2).}
\end{align*}
\]

Here \( v_i \) denote the Killing vector constants of motion corresponding to spatial homogeneity, i.e., the conserved momenta of the particles. Spatial homogeneity is reflected in (11), (12) by the fact that \( f_0 \) is independent of the spatial coordinates. In addition, local rotational symmetry is reflected in (12) by the fact that \( f_0 \) is independent of the direction of the particle momenta in the 2,3-plane. Note also that the \( f_0 \) are time independent.

3.4. Further symmetry assumptions. The route initiated by Rendall [26] is based on choosing Vlasov distribution functions of the form (11), (12) and to make the matter integrals tractable by imposing a further symmetry on the distribution functions – reflection symmetry. We will restrict here to the SH LRS case, and refer to [26] for the analogous discussion in the context of Bianchi I without additional LRS symmetry.

**Definition 3.** A non-trivial spatially homogenous and locally rotationally symmetric Vlasov distribution function \( f_0(v_1, v_2^2 + v_3^2) \) is reflection symmetric if and only if it is even in \( v_1 \).

From (5) it then follows that the corresponding energy-momentum tensor is diagonal, and so is the metric for SH LRS spacetimes; cf [19]. Approaches using reflection symmetry are thus also referred to as diagonal models. We refer to [30, 31] for more details.

One further restriction sometimes imposed, and in this paper as well, is split support; cf [31].

**Definition 4.** A spatially homogenous and locally rotationally symmetric distribution function \( f_0(v_1, v_2^2 + v_3^2) \) has split support if and only if its support does not intersect the coordinate planes \( v^i = 0 \).

The motivation for this is to ensure that certain matter functions are smooth, as we will see in Subsection 4.2.

\(^5\)A counter example is presented in [10] and also quoted in [19, Sec C]: a \( k = 0 \) Friedman spacetime on which there lives an anisotropic \( f \).
4. The dynamical system for Kantowski-Sachs cosmologies with Vlasov matter

In this section we prepare the equations for our analysis. The dynamical system for Kantowski-Sachs cosmologies with Vlasov matter is given in Subsection 4.1. The Vlasov matter parameters which enter the system are discussed in Subsection 4.2.

4.1. The dynamical system. Using their anisotropic matter family, in [7, Sec 9] Calogero and Heinzle present the corresponding three-dimensional dynamical systems for LRS Bianchi VIII, IX and III as well as for Kantowski-Sachs in a unified form. As stated in Subsection 1.3, this matter family also encompasses the case of massless Vlasov matter, subject to the symmetry conditions which we impose here – Definitions 2, 3 and 4. In the case of LRS Bianchi IX, the extension of the system to also include the case of massive Vlasov matter could be achieved by dropping the linearity of the equation of state of the matter on the one hand, cf Subsection 4.2, and introducing an additional variable and evolution equation to the system on the other hand; cf [6, Sec 3]. Due to the unified framework, the same generalisation applies to the case of LRS Bianchi VIII, cf [13], and it is straightforward to see that this is also the case for Kantowski-Sachs.

Hence, from [7, Sec 9] and [6, Sec 3], we can readily write down the four-dimensional Einstein-Vlasov dynamical system for Kantowski-Sachs cosmologies:

It consists of the evolution equations

\begin{align}
H_D' &= -(1 - H_D^2)(q - H_D \Sigma_+) , \\
\Sigma_+ &= -(2 - q)H_D \Sigma_+ - (1 - H_D^2)\left(1 - \Sigma_+^2\right) + \Omega(w_2(l, s) - w_1(l, s)) , \\
M_1' &= M_1(qH_D - 4\Sigma_+ + (1 - H_D^2)\Sigma_+), \\
l' &= 2H_D(l - l),
\end{align}

together with the Hamiltonian constraint

\begin{equation}
\Omega = 1 - \Sigma_+^2.
\end{equation}

The four dynamical quantities are defined by

\begin{align}
H_D &:= - \frac{1}{3D}(k_1^1 + 2k_2^2), \\
\Sigma_+ &:= \frac{1}{3D}(k_1^1 - k_2^2), \\
M_1 &:= \frac{1}{D} \sqrt{\frac{g_{11}}{g_{22}}}, \\
l &:= \frac{(\det g)^{1/3}}{1 + (\det g)^{2/3}},
\end{align}

where $k^i_j$ are the components of the extrinsic curvature. $H := DH_D$ is the Hubble scalar, i.e the expansion scalar; cf [35, Sec 1.1.3]. The dominant variable $D$ is given by

\begin{equation}
D := \sqrt{H^2 + \frac{1}{3g_{22}}} > 0;
\end{equation}

cf also [35, Sec 8.5.2]. $D\Sigma_+$ is the only independent degree of freedom of the components of the shear tensor; cf [35, Sec 1.1.3]. $DM_1$ gives the ratio between the spatial metric components belonging to the plane of local rotational symmetry, and the (square-root of the) spatial metric component in the orthogonal direction. $l$ is a measure of a length scale via the third root of the spatial volume element $\det g$, however compactified to the range $(0, 1)$. All four dynamical quantities are dimensionless. The prime denotes derivation with respect to $D$-rescaled time

\begin{equation}
\tau := \int_{t_0}^t D(t) \, dt \quad \text{with} \quad \tau(t_0) = 0.
\end{equation}
Next we specify the quantities $\Omega, q$ and $s$. $\Omega$ is a dimensionless measure of the energy density of the matter. More precisely, if $\rho := T_{tt}$ denotes the energy density, where $T_{ij}$ are the components of the energy-momentum tensor, then
\[
\Omega := \frac{\rho}{3D^2}, \quad q := 2\Sigma_+^2 + \frac{1}{2}(1 + 3w(l, s))\Omega
\]
denotes the deceleration parameter, and it is a function of $\Sigma_+$ and the matter parameter $w(l, s)$, which we will specified in Subsection 4.2 below. Finally, $s$ is a measure of the anisotropy of the inverse metric components $g^{ii}$, which can be expressed as function of $H_D$ and $M_1$:
\[
s := \frac{g^{22}}{\sum_i g^{ii}} = \left(2 + \frac{3(1 - H_D^2)}{M_1^2}\right)^{-1} \in (0, 1/2).
\]
In Section 5 we will see that we can also think of $s$ as a rotational angle around the $H_D = \pm 1, M_1 = 0$ axes in the state-space; cf Figure 1.

The Vlasov matter functions $w_1(l, s), w_2(l, s)$ and $w(l, s)$ are functions of $l$ and $s$, which we give in Subsection 4.2 below. We have now specified all quantities appearing in equations (13)–(17). Hence, these represent the reduced, four-dimensional, constrained and closed dynamical system of Kantowski-Sachs cosmologies with Vlasov matter.

Finally we note, that the dynamical system (13)–(17) is invariant under the discrete transformation
\[
(\tau, H_D, \Sigma_+) \rightarrow -(\tau, H_D, \Sigma_+).
\]

4.2. The Vlasov matter functions. We work with Vlasov distributions $f_0$ which are invariant under the Kantowski-Sachs symmetries in the sense of Definition 2, and hence have the general functional form (12). In addition we require $f_0$ to be reflection symmetric; cf Definition 3. From (5) we then have a diagonal energy-momentum tensor with
\[
\rho = T_{tt} = (\det g)^{-\frac{1}{2}} \int f_0 \left(m + g^{11}v_1^2 + g^{22}(v_2^2 + v_3^2)\right)^{\frac{1}{2}} dv_1 dv_2 dv_3,
\]
\[
p_i := T_{ii} = (\det g)^{-\frac{1}{2}} \int f_0 g^{ii} v_i^2 \left(m + g^{11}v_1^2 + g^{22}(v_2^2 + v_3^2)\right)^{-\frac{1}{2}} dv_1 dv_2 dv_3,
\]
without summing over $i$; cf [31, Sec 2] or [6, p 1248]. Note that these expressions are functions of the inverse metric components, and that $p_2 = p_3$. We now restrict to $m = 1$ and elaborate below how massless Vlasov matter still fits into the presented framework. Following [6] we define dimensionless principal pressures $w_i := p_i/\rho$ and a dimensionless isotropic pressure $w := \sum_i w_i/3$. Note that $w_2 = w_3$. Instead of by the inverse metric components, we express these as functions of $l$ and $s$ which yields
\[
\begin{align*}
\rho_1(l, s) &= (1 - l)(1 - 2s)F_1(l, s) \quad \text{and} \quad w_2(l, s) = (1 - l)s F_2(l, s),
\end{align*}
\]
with $F_i(l, s)$ given by
\[
\tilde{F}_i(l, s) := \frac{\int f_0 v_i^2 \left(l(s^2(2s - 1) + (1 - l)((1 - 2s)v_1^2 + s(v_2^2 + v_3^2))\right)^{-1/2} dv_1 dv_2 dv_3}{\int f_0 \left(l(s^2(2s - 1) + (1 - l)((1 - 2s)v_1^2 + s(v_2^2 + v_3^2))\right)^{1/2} dv_1 dv_2 dv_3}.
\]
To assure that these are smooth functions, we have to assume that $f_0$ has split support (Definition 4), such that the denominator of $F_i$ is strictly positive. It is the functions (26) through which the Vlasov matter enters the dynamical system (13)–(17).

Though we formulated the $w_i$ for $m = 1$, it is straightforward to show that we recover the expressions corresponding to $m = 0$ if we formally set $l = 0$ in (26);
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\( l = 0 \) thus marks the region which corresponds to solutions with massless Vlasov matter. In Section 5 we show that this is a boundary of the state-space. From the perspective of physics this is expected since in (20), \( l \) is defined via the spatial volume element \( \det g \). Thus towards a singularity, ie for \( l \to 0 \leftrightarrow \det g \to 0 \), one would expect massive particles to pick up momentum, such that the particle four-momenta become lightlike in the limit; cf [30, Sec 1]. For later use we give the values

\[
\begin{align*}
 w_1(0, 1/2) &= 0 & w_2(0, 1/2) &= 1/2 & w(0, s) &= 1/3
\end{align*}
\]

which follow directly from (26).

In analogy to the preceding argument, solutions approaching a state of infinite expansion, ie for \( l \to 1 \leftrightarrow \det g \to \infty \), one would expect particles to loose momentum, and to asymptotically resemble dust, ie a perfect fluid with equation of state \( p = 0 \); cf [31]. Indeed from (26) we find \( w_1(1, s) = w_2(1, s) = w(1, s) = 0 \), ie an isotropic state with pressures 0. \( l = 1 \) thus marks the region which corresponds to dust solutions. As we will see in Section 5 this is also a boundary of the state-space.

5. The state-space

This section discusses the Kantowski-Sachs state-space and its features. In Subsection 5.1 we give the state-space corresponding to the dynamical system discussed in Section 4. Subsection 5.2 discusses two drawbacks: it reaches out to infinity in one coordinate and the dynamical system does not yield a smooth extension onto one of its boundary subsets. We point out how we deal with these issues, and define alternative coordinates in Subsection 5.3.

5.1. The state-space \( \mathcal{X} \). To determine the state-space, we have to take into account the constraints which are imposed onto the dynamical quantities. Firstly, the range of \( H_D \) follows directly from the definition of \( D \), cf (20). Secondly, together with the requirement that the energy density should be positive, ie \( \Omega > 0 \), the Hamiltonian constraint (17) determines the range of the shear parameter \( \Sigma_+ \). Finally, the ranges of \( M_1 \) and \( l \) follow directly from their definitions (19), and in the case of \( M_1 \) also using the positivity of \( D \), cf (20).

Let \( x := (H_D, \Sigma_+, M_1, l) \). The Kantowski-Sachs state-space is given by

\[
\mathcal{X} := \{ x \in \mathbb{R}^4 | H_D \in (-1, 1), \Sigma_+ \in (-1, 1), M_1 > 0, l \in (0, 1) \}
\]

and we denote its relevant boundary subsets as follows:

\[
\begin{align*}
 \mathcal{X}_i &:= \{ x \in \partial \mathcal{X} | l = i \}, & i &= 0, 1 \\
 \mathcal{X}^\pm &:= \{ x \in \partial \mathcal{X} | H_D = \pm 1 \} \\
 \mathcal{X}_i^\pm &:= \{ x \in \partial \mathcal{X} | H_D = \pm 1, l = i \}, & i &= 0, 1 \\
 \mathcal{I}^\pm &:= \{ x \in \partial \mathcal{X} | H_D = \pm 1, M_1 = 0 \} \\
 \mathcal{I}_0^\pm &:= \{ x \in \partial \mathcal{X} | H_D = \pm 1, M_1 = 0, l = 0 \}
\end{align*}
\]

In this notation we follow the scheme that a subscript 0 or 1 refers to a restriction to \( l = 0 \) or 1, and a superscript \( \pm \) to a restriction to \( H_D = \pm 1 \). \( \mathcal{X} \) corresponds to solutions with massive Vlasov matter. In Subsection 4.2 we established that \( \mathcal{X}_0 \) corresponds to massless Vlasov matter solutions and \( \mathcal{X}_1 \) to dust solutions. \( \mathcal{X}_0 \) is depicted in Figure 1. From the Hamiltonian constraint (17) we see that the \( \Sigma_+ = \pm 1 \) boundaries correspond to vacuum solutions. Solutions in \( \mathcal{I}^\pm \) are of LRS Bianchi type I; cf [6, 7].
5.2. Main challenges. There are two main challenges with the system (13)–(17) on the state-space $\mathcal{X}$. Firstly, $\mathcal{X}$ is unbounded in $M_1$-direction. Thus we need to perform a careful analysis of the flow at infinity; cf [25, Sec 3.10]. In analogy to the approach taken in [13], we achieve this by a formal compactification via a projection of the flow onto the ‘Poincar cylinder’; cf Section 6, together with Appendix A. Secondly, the dynamical system (13)–(17) does not have a smooth extension onto $\mathcal{I}^\pm$. This is because the limit of $s$ for $H_D \to \pm 1$ and $M_1 \to 0$ simultaneously does not exist; cf (22). Note however that each of the limits exists separately, so there is a smooth extension onto $\mathcal{I}^\pm$ on the boundary $\mathcal{X}^\pm$, and on the $M_1 = 0$ boundary. In the same way as in [6, 7, 13] we deal with this issue by introducing polar coordinates centered at $H_D = \pm 1$, $M_1 = 0$. In the new coordinates a smooth extension of the system onto the corresponding boundary then does exist; cf Subsection 5.3.
5.3. The state-space $\mathcal{Y}$. We introduce new coordinates $y := (r, \theta, \Sigma_+, l)$ in order to analyse the solutions in a neighbourhood of $I^\pm$. The coordinate transformation is given by

$$
\frac{1 - H_D^2}{2} = r \cos \theta \quad \text{and} \quad \frac{M_1^2}{3} = r \sin \theta,
$$

while $\Sigma_+$ and $l$ stay unchanged, and we choose $H_D > 0$. These are polar coordinates centered at $I^+$ which cover the $H_D > 0$ region of $\mathcal{X}$; cf Figure 1. The solutions in a neighbourhood of $I^-$ then follow from the discrete symmetry (23). With (30), (22) defines a bijection $s(\theta)$ which maps $(0, \pi)$ onto $(0, 1/2)$. Hence we can identify the two.

We denote the state-space in the new coordinates by

$$\mathcal{Y} := \{ y \in \mathbb{R}^4 | r \in (0, \frac{1}{2 \cos \theta}), \theta \in (0, \pi), \Sigma_+ \in (-1, 1), l \in (0, 1) \}$$

and denote its relevant boundary subsets as follows:

$$\mathcal{Y}_0 := \{ y \in \partial \mathcal{Y} | l = 0 \}$$
$$S := \{ y \in \partial \mathcal{Y} | \theta = \pi/2 \}$$
$$S_0 := \{ y \in \partial \mathcal{Y} | l = 0, \theta = \pi/2 \}$$
$$I^Y := \{ y \in \partial \mathcal{Y} | r = 0 \}$$
$$I^Y_0 := \{ y \in \partial \mathcal{Y} | l = 0, r = 0 \}$$

From (30) we have the following identifications ($\sim$) between subsets of $\mathcal{X}$ and $\overline{\mathcal{Y}}$:

$$\mathcal{Y} \sim \mathcal{X} \text{ for } H_D > 0, \quad \overline{S \cup I^Y} \sim \overline{\mathcal{X}^+}, \quad \overline{I^Y} \sim \overline{I^+},$$

and equivalently for the respective subsets with subscript 0. Note that the Bianchi I boundary $I^Y$ is of one dimension higher than its counterpart in the $x$ coordinates. Because of this degeneracy this correspondence is not a diffeomorphism. However, (30) defines a diffeomorphism ($\cong$) between $\overline{\mathcal{Y} \setminus I^Y}$ and $\overline{\mathcal{X} \setminus I^+}$ for $H_D > 0$. Consequently the flows in these sets are topologically equivalent. In particular

$$S_0 \cong \mathcal{X}_0^+, \quad \mathcal{Y}_0 \cong \mathcal{X}_0^+,$$

so we can choose to work in either coordinate system to analyse the flow in these subsets.

While the $y$ coordinates allow us to analyse the flow in a neighbourhood of the Bianchi I boundary, they have two drawbacks. Firstly, they only cover the $H_D > 0$ region of the full state-space. Though by the discrete symmetry (23) this then also covers the $H_D < 0$ region, it does not cover the $H_D = 0$ region. Secondly, in the $y$ coordinates the region where $M_1 \to \infty$ is of one dimension lower and thus degenerate. Hence, the $y$ coordinates are not suited as a basis to analyse the flow at infinity. In our analysis we thus only use the $y$ coordinates to analyse the flow in $I^Y_0$.

6. Results

Subsection 6.1 we formulate our main theorem, which gives the past and future asymptotic solutions, and we formulate four lemmas by which we proof it. The proofs of the lemmas are given in Subsections 6.2–6.5. Subsections 6.2 and 6.3 also contain some physical interpretation concerning recollapse and the approach of massless Vlasov solutions by massive ones towards the singularities. In this section we use standard terminology of dynamical systems theory. We refer to [35, Chap 4] and [25] for a background.
Figure 2. The three-dimensional $l = 0$ boundary $\mathcal{Y}_0$ of the four-dimensional state-space $\mathcal{Y}$. It corresponds to the $H_D > 0$ region in Figure 1. The $H_D = 0$ surface is in this sense not a boundary of the Kantowski-Sachs state-space, but rather marks the end of the $y$ coordinate patch.

6.1. Main theorem.

Theorem 1. Consider Kantowski-Sachs cosmologies (Definition 1) with Vlasov matter of massive or massless particles, with distribution functions satisfying the following assumptions,

(i) invariance under the Kantowski-Sachs symmetries (Definition 2),
(ii) reflection symmetry (Definition 3),
(iii) split support (Definition 4).

The following statements hold:
(a) Generic solutions are past and future asymptotic to the non-flat LRS Kasner vacuum solution given by the metric (3) with
\[ g_{11}(t) = at^{-2/3} \quad \text{and} \quad g_{22}(t) = bt^{4/3}, \]
where \( a, b \) are positive constants, and where the time is shifted such that the big-bang (big-crunch) occurs at \( t = 0 \).

(b) There are non-generic solutions which are past (future) asymptotic to a non-isotropic Bianchi I matter solution.

(c) There are non-generic solutions which are past (future) asymptotic to the flat Friedman matter solution.

The metric to these solutions is given by (3) with components listed in Table 1.

Proof. In the preceding sections we established that we can describe the dynamics in this scenario for massive particles by the dynamical system (13)–(17) on the state-space \( X \), (28), and for massless particles by the restriction of this system to the boundary \( X_0 \), (29). We can thus proof our statement in the framework of dynamical systems theory, and do so through Lemmas 1–4 below.

Lemma 1. All past asymptotic solutions satisfy \( H_D = 1 \) and all future asymptotic solutions satisfy \( H_D = -1 \).

Proof. Cf Subsection 6.2.

Lemma 2. All past and future asymptotic solutions satisfy \( l = 0 \).

Proof. Cf Subsection 6.3.

Lemma 3. The only past (future) attractor in \( X \cup X_0 \) is the equilibrium point \( Q_\infty \) at infinity. \( R_\infty \) repels orbits from a three-dimensional (two-dimensional) set in \( X \) \( (X_0) \). \( \bar{R}_\infty \) attracts accordingly. \( F \) repels orbits from a two-dimensional (one-dimensional) set in \( X \) \( (X_0) \). The image of \( F \) under (23) attracts accordingly.

Proof. Cf Subsection 6.4.

Lemma 4. The fixed points correspond to the exact solutions listed in Table 1.

Proof. Cf Subsection 6.5

6.2. Proof of Lemma 1. Since Vlasov matter satisfies a non-negative pressures condition, cf [28, Sec 1], we have \( w(l,s) \geq 0 \) in \( \bar{X} \). With this we find from (13) that \( H_D < 0 \forall x \in \bar{X} \setminus \bar{X}^\pm \). In other words, \( H_D \) is strictly monotonically decreasing along orbits in \( \bar{X} \setminus \bar{X}^\pm \). We therefore know that

\[ \alpha(\Gamma) \subseteq \bar{X}^+ \quad \text{and} \quad \omega(\Gamma) \subseteq \bar{X}^- \quad \forall \Gamma \in \bar{X} \setminus \bar{X}^\pm, \]

where \( \Gamma \) denotes an arbitrary orbit, and \( \alpha(\Gamma) \) (\( \omega(\Gamma) \)) its \( \alpha \) (\( \omega \)) limit set, ie its past (future) asymptotic sets; cf [35, Def 4.12].

Note that (33) does not exclude the possibility that the limit sets could be empty. In other words, the solutions may satisfy \( M_1 \rightarrow \infty \) asymptotically. We thus have to analyse the flow at infinity. Since the only equation of the system (13)–(17) which depends on the coordinate \( M_1 \) is the evolution equation of \( M_1 \) itself, naively we would await \( H_D \) to also be strictly monotonically decreasing at infinity. A careful check via a formal compactification by projecting the flow onto the ‘Poincare cylinder’ entails that this intuition is indeed correct; cf Appendix A. Hence, we know that if the limit sets (33) are empty, ie if the solutions satisfy \( M_1 \rightarrow \infty \) for \( \tau \rightarrow \pm \infty \), then also \( H_D \rightarrow \pm 1 \). This completes the proof of Lemma 1.
Interpretation of Lemma 1. We have shown that all Kantowski-Sachs cosmologies with Vlasov matter satisfy
\[(34) \quad H_D \to +1 \text{ for } \tau \to -\infty, \quad H_D \to 0 \text{ for } \tau = \tau^*, \quad H_D \to -1 \text{ for } \tau \to +\infty,\]
where \(\tau^*\) is some finite time, depending on the initial data. From the physical meaning of \(H_D\) given in Section 4.1 as the \(D\)-normalised Hubble scalar, we can interpret this result as follows: Kantowski-Sachs solutions with Vlasov matter undergo an expanding phase for \(\tau < \tau^*\), at the end of which they reach a state of maximal expansion for \(\tau = \tau^*\), after which they undergo a contracting phase for \(\tau > \tau^*\).

6.3. Proof of Lemma 2. For massless particles, i.e., for solutions in \(\mathcal{X}_0\), Lemma 1 is trivial; cf Subsection 4.2. From (16) we see that for solutions in \(\mathcal{X} \setminus (\mathcal{X}_0 \cup \mathcal{X}_1)\) the sign of \(l'\) is dictated by the sign of \(H_D\). From (34) we thus know that Kantowski-Sachs solutions with massive Vlasov matter satisfy
\[(35) \quad l' > 0 \text{ for } \tau \in (-\infty, \tau^*), \quad l' = 0 \text{ for } \tau = \tau^*, \quad l' < 0 \text{ for } \tau \in (\tau^*, +\infty),\]
and together with (33) we can thus infer that
\[(36) \quad \alpha(\Gamma) \subseteq \mathcal{X}_0 \quad \text{and} \quad \omega(\Gamma) \subseteq \mathcal{X}_0 \quad \forall \Gamma \in \mathcal{X} \setminus (\mathcal{X}_0 \cup \mathcal{X}_1).\]

(36) does not exclude the possibility that the limit sets could be empty. Thus, as in the proof of Lemma 1, we have to analyse the flow at infinity. Following the same arguments, we would expect the monotonicities (35) to also hold at infinity, and in complete analogy, a projection of the flow onto the ‘Poincaré cylinder’ shows that this is indeed the case; cf Appendix A. Hence, we know that if the limit sets (36) are empty, i.e., if the solutions satisfy \(M_1 \to \infty\) for \(\tau \to \pm \infty\), then also \(l \to 0\). This completes the proof of Lemma 2.

Interpretation of Lemma 2. From the physical interpretation of \(l\) as a length scale associated with the spatial volume element \(\det g\), cf Subsection 4.1, Lemma 2 says the following: Kantowski-Sachs solutions with massive Vlasov matter exhibit a big-bang singularity towards the past and a big-crunch singularity towards the future; in other words, they recollapse. This is in compliance with more general recollapse results; cf Subsection 2.3.

Furthermore, as discussed in Subsection 4.2, \(\mathcal{X}_0\), i.e., the \(l = 0\) boundary of \(\mathcal{X}\), can be interpreted as the state-space for the case of massless Vlasov particles, which is three-dimensional. In this sense we can conclude that Kantowski-Sachs cosmologies with massive Vlasov particles, behave like Kantowski-Sachs cosmologies with massless particles towards both, the big-bang and the big-crunch singularities.

6.4. Proof of Lemma 3. So far we know (36), and that if these sets are empty, then the solutions must asymptote towards those regions of \(M_1 \to \infty\), for which \(H_D = \pm 1\) and \(l = 0\). What is left to do is to identify precisely where exactly solutions asymptote to in these two-dimensional regions. We restrict our analysis to the past asymptotics, from which we immediately get the future asymptotics as well via the discrete symmetry (23).

From the second relation of (31) we see that in terms of the \(y\) coordinates \(\alpha(\Gamma) \subseteq \mathcal{X}_0 \cup \mathcal{I}_0\). In \(\mathcal{I}_0\), we have \(M_1 = 0\), and for this the dynamical system (13)–(17) coincides with that of LRS Bianchi IX; cf [7, Sec 9]. Thus we can use the result of the analysis in [6, Sec 5]. The respective qualitative flow in \(\mathcal{I}_0\) is depicted in their Figure 3, and we reproduce it here in our Figure 3.\(^6\)

\(^6\)We note that the flow on \(\mathcal{I}_0\) is also equivalent with [7, Fig 15(g)] and the respective parts of [13, Fig 7 and 8]; the latter in the context of LRS Bianchi type VIII.
Because of (32) the flow in $S_0$ is qualitatively equivalent to the one in $X_0^+$. We choose to analyse the latter, since the $x$ coordinates are better suited as a basis for the analysis of the flow at infinity than the $y$ coordinates; cf Subsection 5.3. To restrict the dynamical system (13)–(17) to this boundary, we have to set $H_D = 1$ and $l = 0$. From (22) we then have $s = 1/2$, and from this and (26) we get (27).

With this the dynamical system on $X_0^+$ reduces to

\begin{equation}
\Sigma^+ = -\frac{1}{2}(1 - \Sigma^2)(2\Sigma^+ - 1), \quad M'_1 = M_1(\Sigma^2 - 4\Sigma^+ + 1).
\end{equation}

This system exhibits three fixed points in $X_0^+$:

\begin{align}
T &= (-1, 0) & Q &= (1, 0) & R &= (1/2, 0)
\end{align}

Their local stability properties follow from the eigenvalues and eigenvectors of the linearisation of (37), evaluated at (38); cf [25, Sec 2.6] or [35, Sec 4.3.2]. Below we follow the notation scheme $P : [\lambda_i], [u_1 u_2]$ where $\lambda_i$ denotes the eigenvalue corresponding to the eigenvector $u_i$ of the linearisation at $P$. We find

\begin{align*}
T : \begin{bmatrix} 3/6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \rightarrow \text{local source}, \\
Q : \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} & \rightarrow \text{local saddle repelling in $\Sigma^+$-direction}, \\
R : \begin{bmatrix} -3/4 & 1 & 0 \\ -3/4 & 0 & 1 \end{bmatrix} & \rightarrow \text{local sink}.
\end{align*}

Next we investigate the monotonicity of the flow. From (37)

\begin{align*}
\Sigma^+ > 0 & \text{ for } \Sigma^+ \in (-1, 1/2), \\
\Sigma^+ = 0 & \text{ for } \Sigma^+ \in \{\pm 1, 1/2\}, \\
\Sigma^+ < 0 & \text{ for } \Sigma^+ \in (1/2, 1).
\end{align*}

Therefore, the line $\Sigma^+ = 1/2$ represents a separatrix of the flow in $X_0^+$; cf [25, Sec 3.11]. For higher (lower) values of $\Sigma^+$, $\Sigma^+$ is strictly monotonically decreasing (increasing) along the flow in $X_0^+$.

What is left to do is to investigate the flow at infinity. Since the first equation of (37) is independent of $M_1$, naively one would expect that the flow at infinity resembles that of the flow at $M_1 = 0$. With a projection of the flow onto the ‘Poincare cylinder’ one can convince oneself that this is indeed the case; cf Appendix A. This means that there are three fixed points at infinity,

\begin{align*}
T_\infty & := (-1, \infty), & Q_\infty & := (1, \infty), & R_\infty & := (1/2, \infty),
\end{align*}

and the flow between these indeed resembles the flow on the line $M_1 = 0$.

We have now gathered all the information required to draw the qualitative flow diagram in $X_0^+ \cong S_0$. It is shown in Figure 3 together with the flow in $T_0^0$; cf also Figures 1 and 2. From this, together with the monotonicities of $H_D$ and $l$ obtained in Subsections 6.2 and 6.3, we obtain Lemma 3.

**Remark.** Note that concerning massless Vlasov matter we recover [30, Thm 5.1].

6.5. **Proof of Lemma 4.** We can identify the fixed points with exact solutions. [7, App A] gives a calculation of the components of the metric (3) and the energy-momentum tensor (24)–(25) in terms of the fixed point coordinates. The result for the metric components is

\begin{align*}
g_{11}(t) &= at^{2\gamma_1} & \text{and} & \quad g_{22}(t) &= bt^{2\gamma_2}
\end{align*}
Figure 3. The qualitative flow on $\mathcal{I}_0^Y \cup \mathcal{S}_0 \sim \mathcal{X}_0^\dagger$. As a consequence of Lemmas 1 and 2, all fixed points are repelling orbits in the two respective orthogonal directions, which cannot be seen from the graph: $F$, $T$ and $Q$ are repelling in $r$ and $l$-direction, where for $T$ and $Q$, the $r$-direction also coincides with the $H_D$-direction. $T_\infty$ and $Q_\infty$ are repelling in $H_D$ and $l$-direction. $Q_\infty$ is the past attractor.

Table 1. The components of the metric (3) and the energy-momentum tensor (24)–(25) of the fixed point solutions. $a$ and $b$ are positive constants. The fixed points listed here satisfy $l = 0$ and are depicted in Figures 1 and 2. However these have a copy at the $l = 1$ boundary. Including the latter this table is complete.

| fixed point | $g_{11}(t)$ | $g_{22}(t)$ | $\rho(t)$ | $p_1(t)$ | $p_2(t)$ | solution |
|-------------|-------------|-------------|------------|-----------|-----------|----------|
| $Q, Q_2, Q_\infty$ | $at^{-2/3}$ | $bt^{2/3}$ | 0 | 0 | 0 | non-flat LRS Kasner |
| $T, T_2, T_\infty$ | $at^2$ | $b$ | 0 | 0 | 0 | Taub (flat LRS Kasner) |
| $R, R_2, R_\infty$ | $a$ | $bt^{4/3}$ | $\frac{4}{3}t^{-2}$ | 0 | $\frac{2}{3}t^{-2}$ | no-name Bianchi I |
| $F$ | $at$ | $bt$ | $\frac{4}{3}t^{-2}$ | $\frac{1}{3}t^{-2}$ | $\frac{2}{3}t^{-2}$ | flat Friedman |

with positive and generally constrained constants $a, b$,

$$\gamma_1 = \frac{H_D - 2\Sigma_+}{H_D(1 + q) + \Sigma_+(1 - H_D^2)}$$

and

$$\gamma_2 = \frac{H_D + \Sigma_+}{H_D(1 + q) + \Sigma_+(1 - H_D^2)}.$$}

The result for the components (24)–(25) of the energy-momentum tensor is

$$\rho(t) = \frac{3(1 - \Sigma_+^2)}{(H_D(1 + q) + \Sigma_+(1 - H_D^2))^2}t^{-2}$$

and

$$p_i(t) = w_i(l, s)\rho(t).$$

Note that these expressions are only valid at the fixed points. Entering the coordinates for our fixed points and fixed points at infinity we obtain Table 1, which concludes the proofs of Lemma 4 and Theorem 1.

Appendix A. The Flow at Infinity

In dynamical systems theory, the flow at infinity is usually analysed using projections of the flow onto the Poincar sphere; cf [25, Sec 3.10]. The sphere is the natural
surface to project onto if one deals with a state-space which extends to infinity in all directions. However both, the state-space of LRS Bianchi VIII in [13] as well as that of Kantowski-Sachs in the present paper, extend to infinity in one coordinate direction only; the former in $H_D$-direction and the latter in $M_1$-direction. Hence, in these cases we seek to analyse the flow at infinity in the respective directions. However, those regions project to points on the Poincar sphere, which is why this approach is not well suited for an analysis of the flow at infinity in these cases. In [13, Sec 6] the second author adopted this technique with the slight modification that the compactification is only done in one direction; ie projected is onto the ‘Poincar cylinder’ rather than onto the Poincar sphere. In the appendix of that paper this has been presented in two and three dimensions. Though the generalisation to higher dimensions is trivial, for completeness, and to make the paper more self-contained, in the following we will present this technique in four dimensions as well.

Consider the dynamical system

$$
\begin{bmatrix}
H_D \\
\Sigma_+ \\
M_1 \\
l
\end{bmatrix}' = 
\begin{bmatrix}
P(H_D, \Sigma_+, M_1, l) \\
Q(H_D, \Sigma_+, M_1, l) \\
R(H_D, \Sigma_+, M_1, l) \\
S(H_D, \Sigma_+, M_1, l)
\end{bmatrix},
$$

where $P, Q, R$ and $S$ are polynomials of degree $p, q, r$ and $s$ in $M_1$, satisfying

$$p, q, s + 1 \geq r.
$$

The system (39) can also be written as

1. $$P(H_D, \Sigma_+, M_1, l) dM_1 - R(H_D, \Sigma_+, M_1, l) dH_D = 0,$$
2. $$Q(H_D, \Sigma_+, M_1, l) dM_1 - R(H_D, \Sigma_+, M_1, l) d\Sigma_+ = 0,$$
3. $$S(H_D, \Sigma_+, M_1, l) dM_1 - R(H_D, \Sigma_+, M_1, l) dl = 0,$$

where however the information of the flow direction is lost. Supported by the illustration in Figure 4, we now project onto the Poincar cylinder

$$\{(H_D, \Sigma_+, X, l, Z) \in \mathbb{R}^5 \mid X^2 + Z^2 = 1\}
$$

via the transformation

$$
(H_D, \Sigma_+, M_1, l) \rightarrow (H_D, \Sigma_+, X, l, Z): X^2 + Z^2 = 1, \\
M_1 = X/Z, \\
dM_1 = \frac{1}{Z} dX - \frac{X}{Z^2} dZ.
$$

Applying this to (41) and multiplying by $Z^{p+2}$, we get

$$Z^{p+1} P dX - Z^p XP dZ - Z^{p+2} R dH_D = 0.$$
three-dimensional dynamical system

\[
\begin{bmatrix}
H_D \\
\Sigma_+ \\
l
\end{bmatrix}' = \begin{bmatrix}
Z^p P(H_D, \Sigma_+, X/Z, l) \\
Z^q Q(H_D, \Sigma_+, X/Z, l) \\
Z^s S(H_D, \Sigma_+, X/Z, l)
\end{bmatrix}_{X=1, Z=0}.
\]

In the case of the Kantowski-Sachs evolution equations (13) to (16) we have

\[
p, q, s = 0
\]

and \( r = 1 \), hence (40) is satisfied. Further, because of (46) it follows that the system (45) in this case simply corresponds to the original system with the evolution equation for \( M_1 \) being dropped, ie

\[
\begin{bmatrix}
H_D \\
\Sigma_+ \\
l
\end{bmatrix}' = \begin{bmatrix}
P(H_D, \Sigma_+, l) \\
Q(H_D, \Sigma_+, l) \\
S(H_D, \Sigma_+, l)
\end{bmatrix},
\]

with \( P, Q \) and \( S \) given by the right hand sides of (13), (14) and (16).

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