WHEN POWERS OF A MATRIX COINCIDE WITH ITS HADAMARD POWERS

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Abstract. We characterize matrices whose powers coincide with their Hadamard powers.

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Let $M_n(F)$ be the algebra of all $n \times n$ matrices over the field $F$. The Hadamard product of matrices $A = [a_{ij}]_{i,j=1}^n \in M_n(F)$ and $B = [b_{ij}]_{i,j=1}^n \in M_n(F)$ is the matrix $A \circ B = [a_{ij}b_{ij}]_{i,j=1}^n$. The usual product of $A$ and $B$ is denoted by $AB$. Given a positive integer $r$, the $r$-th Hadamard power of a matrix $A = [a_{ij}]_{i,j=1}^n \in M_n(F)$ is the matrix $A^{(r)} = [a_{ij}^r]_{i,j=1}^n$, while the usual $r$-th power of $A$ is denoted by $A^r$.

Let $p(\lambda) = c_m\lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda$ be a polynomial with given coefficients $c_m, c_{m-1}, \ldots, c_1 \in F$ and without constant term. For any $A \in M_n(F)$, we can first define the usual matrix function by

$$p(A) = c_mA^m + c_{m-1}A^{m-1} + \cdots + c_1A,$$

and then also the Hadamard matrix function by

$$p^H(A) = c_mA^{(m)} + c_{m-1}A^{(m-1)} + \cdots + c_1A.$$

The Hadamard product and Hadamard matrix functions arise naturally in a variety of ways (see e.g. [3, Section 6.3]). So, it is perhaps useful to know for which matrices $A$ we have $p(A) = p^H(A)$ for all such polynomials $p$, or equivalently, $A^r = A^{(r)}$ for every $r \in \mathbb{N}$. The latter question has been recently posed in [5] for the case of real matrices, and two characterizations of such matrices have been given in [2] and [4]. In this note we give another description of such matrices.

**Theorem 1.** Let $A \in M_n(F)$ be a nonzero matrix. Then the following assertions are equivalent:

(a) $A^r = A^{(r)}$ for every positive integer $r$;

(b) $A^r = A^{(r)}$ for every integer $r \in \{2, 3, \ldots, n + 1\}$.
(c) There exist \( k \in \mathbb{N} \), distinct non-zero elements \( \lambda_1, \ldots, \lambda_k \in \mathbb{F} \), and idempotent \((0,1)\)-matrices \( E_1, \ldots, E_k \) such that

\[
A = \sum_{i=1}^{k} \lambda_i E_i \quad \text{and} \quad E_i \circ E_j = E_i E_j = 0 \quad \text{for all} \ i \neq j.
\]

**Proof.** The implication \((a) \Rightarrow (b)\) is trivial. We begin the proof of the implication \((b) \Rightarrow (c)\) by letting \( p(\lambda) = c_m \lambda^m + c_{m-1} \lambda^{m-1} + \cdots + c_1 \lambda \) be a polynomial of degree \( m \leq n + 1 \). If \( A = (a_{ij})_{i,j=1}^n \), then our assumptions give that

\[
p(A) = c_m A^m + c_{m-1} A^{m-1} + \cdots + c_1 A = [p(a_{ij})]_{i,j=1}^n.
\]

This implies that

\[
p(A) = 0 \iff p(a_{ij}) = 0 \quad \text{for all} \ i, j.
\]

Let \( m(\lambda) \) be the minimal polynomial of \( A \). If \( A \) is invertible, put \( p(\lambda) = \lambda m(\lambda) \), otherwise let \( p(\lambda) = m(\lambda) \). Then the degree of \( p(\lambda) \) is at most \( n + 1 \) and \( p(0) = 0 \), so that the equivalence (2) implies that \( p(a_{ij}) = 0 \) for all \( i, j \). Let \( q(\lambda) \) be the minimal polynomial annihilating the element 0 and all entries of \( A \). Then the polynomial \( q(\lambda) \) divides the polynomial \( p(\lambda) \), so that its degree is at most \( n + 1 \). Therefore, the equivalence (2) gives that \( q(A) = 0 \), and so \( m(\lambda) \) divides \( q(\lambda) \), as \( m(\lambda) \) is the minimal polynomial of \( A \). This means that \( m(\lambda) \) factors into distinct linear factors, so that the matrix \( A \) is diagonalizable over \( \mathbb{F} \) and the set \( \{\lambda_1, \ldots, \lambda_k\} \) of all non-zero eigenvalues of \( A \) coincides with the set of all non-zero entries of \( A \).

Now, for each \( i = 1, \ldots, k \), let \( p_i(\lambda) \) be the Lagrange interpolation polynomial such that \( p_i(\lambda_i) = 1 \), \( p_i(0) = 0 \), and \( p_i(\lambda_j) = 0 \) for all \( j \neq i \), that is,

\[
p_i(\lambda) = \frac{\lambda(\lambda - \lambda_1) \cdots (\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_k)}{\lambda_i(\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdots (\lambda_i - \lambda_k)}.
\]

Then \( E_i = p_i(A) \) is an idempotent. Furthermore, \( E_i E_j = 0 \) for all \( i \neq j \), as \( p_i(\lambda) p_j(\lambda) = 0 \) on the spectrum of the diagonalizable matrix \( A \). Since each entry of \( A \) belongs to the set \( \{0, \lambda_1, \ldots, \lambda_k\} \), it follows from (1) that \( E_1, \ldots, E_k \) are \((0,1)\)-matrices satisfying \( E_i \circ E_j = 0 \) for all \( i \neq j \). Finally, since \( \lambda = \sum_{i=1}^{k} \lambda_i p_i(\lambda) \) on the spectrum of the diagonalizable matrix \( A \), we conclude that \( A = \sum_{i=1}^{k} \lambda_i \sum_{i=1}^{k} \lambda_i E_i \). This proves the implication \((b) \Rightarrow (c)\).
For the proof of the implication \((c) \Rightarrow (a)\), we just compute the powers:

\[
A^r = \sum_{i=1}^{k} \lambda_i^r E_i = A^{(r)}.
\]

\[\square\]

We now give the canonical form of an idempotent \((0, 1)\)-matrix. When the field \(\mathbb{F}\) is the field \(\mathbb{R}\) of all real numbers, this can be obtained easily from the canonical form of a nonnegative idempotent matrix (see e.g. \[\square\] Theorem 3.1 on page 65).

**Theorem 2.** Let \(E \in M_n(\mathbb{F})\) be an idempotent \((0, 1)\)-matrix of rank \(m \in \mathbb{N}\). Suppose that the characteristic of the field \(\mathbb{F}\) is either zero or larger than \(n\). Then there exists a permutation matrix \(P\) such that

\[
P EP^T = \begin{bmatrix} I & U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ V & VU & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ V \\ 0 \end{bmatrix} \cdot \begin{bmatrix} I & U & 0 & 0 \end{bmatrix},
\]

where \(I\) is the identity matrix of size \(m\), and \(U, V\) are \((0, 1)\)-matrices such that \(U\) has no zero columns, \(V\) has no zero rows, and \(VU\) is also a \((0, 1)\)-matrix. (It is possible that \(U\) or \(V\) act on zero-dimensional spaces.)

**Proof.** Suppose first that \(E\) has no zero rows and no zero columns. We must show that \(m = n\) and \(E = I\). Assume on the contrary that \(m < n\). Since \(\text{tr} (E) = m\) and \(E\) is a \((0, 1)\)-matrix, there exists a permutation matrix \(P\) such that

\[
P EP^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

where the diagonal entries of \(A \in M_m(\mathbb{F})\) are equal to 1, while the diagonal entries of \(D \in M_{n-m}(\mathbb{F})\) are equal to 0. Since \(E\) is an idempotent, we have \(A^2 + BC = A\), so that, in view of our characteristic assumption, \(A^2\) is also a \((0, 1)\)-matrix. It follows that \(A\) must be the identity matrix. As \(P EP^T\) is an idempotent, we obtain that \(BC = 0, BD = 0, DC = 0\) and \(CB + D^2 = D\). Since \(E\) has no zero rows, the equalities \(BC = 0\) and \(BD = 0\) imply that \(B = 0\). Since \(D^2 = D\) and \(\text{tr} (D) = 0\), we conclude that \(D = 0\). This contradicts the fact that \(E\) has no zero columns. So, we must have that \(m = n\) and \(E = I\).

To prove the general case, let us group the indices \(i = 1, 2, \ldots, n\) into four sets according to whether the \(i\)-th row and the \(i\)-th column of \(E\) are both non-zero, or the \(i\)-th row is
zero but the $i$-th column is not, and so on. So, there exists a permutation matrix $P$ such that

$$PEP^T = \begin{bmatrix} T & U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ V & W & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $T, U, V, W$ are $(0, 1)$-matrices such that $T$ and $U$ have no zero rows in common, and $T$ and $V$ have no zero columns in common. Since $E^2 = E$, we have $T^2 = T$, $TU = U$, $VT = V$ and $VU = W$. It follows from $W = VU$ that $U$ has no zero columns and $V$ has no zero rows. Indeed, if $U$ had a zero column, then the whole column in $PEP^T$ would be zero, contradicting the definition of the second group of indices. As $T$ and $U = TU$ have no zero rows in common, $T$ has no zero rows. Similarly, $T$ cannot have a zero column. By the first part of the proof, we obtain that $T = I$ which gives the desired form. $\square$

In Theorem 2 we cannot omit the assumption on the characteristic of the field $\mathbb{F}$. Namely, if the field $\mathbb{F}$ has prime characteristic $p < n$, then, for example, take the $(p+1) \times (p+1)$ matrix of all ones and enlarge it by adding zeros to get an idempotent $(0, 1)$-matrix in $M_n(\mathbb{F})$ which is not of the above form.

If we apply Theorem 2 for idempotent $(0, 1)$-matrices in the assertion (c) of Theorem 1, we obtain the following description of a matrix whose powers coincide with its Hadamard powers.

**Theorem 3.** Let $A \in M_n(\mathbb{F})$ be a non-zero matrix of rank $m$, where the characteristic of the field $\mathbb{F}$ is either zero or larger than $n$. Then the assertions (a), (b) and (c) of Theorem 1 are further equivalent to the following:

(d) There exist a permutation matrix $P$, non-zero elements $\mu_1, \ldots, \mu_m \in \mathbb{F}$, and $(0, 1)$-matrices $U, V$ such that $U$ has no zero columns, $V$ has no zero rows, $VU$ is also a $(0, 1)$-matrix, and

$$PAP^T = \sum_{i=1}^{m} \mu_i v_i u_i^T,$$

where $u_1^T, \ldots, u_m^T$ are the rows of the $m \times n$ matrix $[I \ U \ 0 \ 0]$, and $v_1, \ldots, v_m$ are the columns of the $n \times m$ matrix $[I \ 0 \ V \ 0]^T$. 
Proof. We must explain only how to obtain the assertion (d) from the assertion (c) of Theorem 1. We first observe that the matrix $E = E_1 + \cdots + E_k$ is an idempotent $(0,1)$-matrix of rank $m$. By Theorem 2, there is a permutation matrix $P$ such that

$$PEP^T = \begin{bmatrix} I & U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ V & VU & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ V \\ 0 \end{bmatrix} \cdot \begin{bmatrix} I & U & 0 & 0 \end{bmatrix},$$

where $I$ is the identity matrix of size $m$, and $U$, $V$ are $(0,1)$-matrices such that $U$ has no zero columns, $V$ has no zero rows, and $VU$ is also a $(0,1)$-matrix. Let $u_1^T, \ldots, u_m^T$ be the rows of the matrix $[I \ U \ 0 \ 0]^T$, and let $v_1, \ldots, v_m$ be the columns of the matrix $[I \ 0 \ V \ 0]^T$. Then $u_i^T v_i = 1$ for all $i$, $u_i^T v_j = 0$ for all $i \neq j$, and $PEP^T = \sum_{i=1}^m v_i u_i^T$. Since $E_1, \ldots, E_k$ and $E = E_1 + \cdots + E_k$ are $(0,1)$-matrices, all the ones of a matrix $E_j$ $(j = 1, \ldots, k)$ are at positions where also $E$ has ones. Thus, we have

$$PE_j P^T = \begin{bmatrix} I_j & U_j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ V_j & (VU)_j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $U_j$ is a matrix obtained from $U$ by replacing some ones with zeros, and likewise for $I_j$, $V_j$, and $(VU)_j$. Now, it follows from $E_j = E_j E = EE_j$ that $U_j = I_j U$ and $V_j = V I_j$, so that each of the first $m$ rows (resp. columns) of $PE_j P^T$ is either equal to 0 or to a corresponding row (resp. column) of $PEP^T$. Thus, the matrix $PE_j P^T$ is a sum of some of rank-one matrices $v_i u_i^T$; for these indices $i$, put $\mu_i = \lambda_j$. Then we have

$$PAP^T = \sum_{j=1}^k \lambda_j PE_j P^T = \sum_{i=1}^m \mu_i v_i u_i^T.$$

It is worth mentioning that we can eliminate the permutation matrix $P$ in the assertion (d) of Theorem 3 by setting

$$\tilde{u}_i = P^T u_i , \quad \tilde{v}_i = P^T v_i \quad \text{for} \quad i = 1, 2, \ldots, m,$$

that gives the form

$$A = \sum_{i=1}^m \mu_i \tilde{u}_i \tilde{v}_i^T.$$
by $\mu E$, where $E$ is an idempotent $(0,1)$-matrix of the form $vu^T$ and $u$, $v$ are $(0,1)$-vectors associated with the indices in $I$, $J$, respectively.

Finally, we give a simple example showing that the assertion (c) of Theorem 1 does not imply that, up to a permutation similarity, $A$ has a block diagonal form with $k$ blocks. Given any non-zero real numbers $\alpha$ and $\beta$, define the matrix $A \in M_4(\mathbb{R})$ by
\[
A = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & \beta \\ \alpha & \beta & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}.
\]

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