An Optimal High-Order Tensor Method for Convex Optimization

Bo JIANG  
Research Institute for Interdisciplinary Sciences, School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai 200433, China

Haoyue WANG  
Department of Mathematics, Fudan University, Shanghai, China

Shuzhong ZHANG  
Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, USA; joint appointment with Institute of Data and Decision Analytics, The Chinese University of Hong Kong, Shenzhen, China

Abstract

This paper is concerned with finding an optimal algorithm for minimizing a composite convex objective function. The basic setting is that the objective is the sum of two convex functions: the first function is smooth with up to the $d$-th order derivative information available, and the second function is possibly non-smooth, but its proximal tensor mappings can be computed approximately in an efficient manner. The problem is to find – in that setting – the best possible (optimal) iteration complexity for convex optimization. Along that line, for the smooth case (without the second non-smooth part in the objective), Nesterov (1983) proposed an optimal algorithm for the first-order methods ($d = 1$) with iteration complexity $O\left(\frac{1}{k^2}\right)$. A high-order tensor algorithm with iteration complexity of $O\left(\frac{1}{k^{d+1}}\right)$ was proposed by Baes (2009) and Nesterov (2018). In this paper, we propose a new high-order tensor algorithm for the general composite case, with the iteration complexity of $O\left(\frac{1}{k^{(3d+1)/2}}\right)$, which matches the lower bound for the $d$-th order methods as established in Nesterov (2018); Arjevani et al. (2018), and hence is optimal. Our approach is based on the Accelerated Hybrid Proximal Extragradient (A-HPE) framework proposed in Monteiro and Svaiter (2013), where a bisection procedure is installed for each A-HPE iteration. At each bisection step a proximal tensor subproblem is approximately solved, and the total number of bisection steps per A-HPE iteration is bounded by a logarithmic factor in the precision required.

Keywords: convex optimization; tensor method; acceleration; iteration complexity.

1. Introduction

In this paper, we consider the following composite unconstrained convex optimization:

$$
\min_{x \in \mathbb{R}^n} F(x) := f(x) + h(x),
$$

where $f$ is differentiable and convex, and $h$ is convex but possibly non-smooth. In this context, we assume that convex tensor (polynomial) proximal mappings regarding $h$ can be approximately computed efficiently. Given that structure, a fundamental question is to find an optimal algorithm that solves the above problem, using the available derivative information of the smooth part $f$.

1. Accepted for presentation at the Conference on Learning Theory (COLT) 2019
In case \( F(x) = f(x) \), and only the gradient information of \( f \) is available, Nesterov (1983) proposed a gradient-type algorithm, which achieves the overall iteration complexity of \( O(1/k^2) \), matching the lower bound on the iteration complexity of this class of solution methods, hence is known to be an optimal algorithm among all the first-order methods. Since Nesterov’s seminal work Nesterov (1983), especially in the recent years when the large scale machine learning applications have come under the spotlight, there has been a surge of research effort to extend Nesterov’s approach to more general settings; see e.g. Beck and Teboulle (2009); Cotter et al. (2011); Lan (2012); Drori and Teboulle (2014); Shalev-Shwartz and Zhang (2014), and/or to incorporate certain adaptive strategies to enhance the practical performances of the acceleration; see e.g. Lin and Xiao (2014); Scheinberg et al. (2014); Calatroni and Chambolle (2017). At the same time, there has also been a considerable research effort to fully understand the underpinning mechanism of the first-order acceleration phenomenon; see e.g. Bubeck et al. (2015); Su et al. (2016); Wibisono et al. (2016); Wilson et al. (2016).

When the Hessian information is available, Nesterov (2008) proposed an acceleration scheme for cubic regularized Newton’s method, and he showed that the iteration complexity bound improves from \( O(1/k^2) \) to \( O(1/k^3) \). A few years later, Monteiro and Svaiter (2013) proposed a different acceleration scheme, which they termed as Accelerated Hybrid Proximal Extragradient Method (A-HPE) framework, and they proved that if the second-order information is incorporated into the A-HPE framework then the corresponding accelerated Newton proximal extragradient method has a superior iteration complexity bound of \( O(1/k^{7/2}) \) over \( O(1/k^3) \). In 2018, Arjevani et al. (2018) showed that \( O(1/k^{7/2}) \) is actually a lower bound for the oracle complexity of the second-order methods for convex smooth optimization. This shows that the accelerated Newton proximal extragradient method is an optimal second-order method.

As evidenced by the special cases \( d = 1 \) and \( d = 2 \), there is a clear tradeoff between the level of derivative information required and the overall iteration complexity improved. Therefore, a natural and important question arises:

**What is the exact tradeoff relationship between \( d \) and the worst-case iteration complexity?**

Such question has been in fact raised and addressed in some way in recent works Birgin et al. (2017); Cartis et al. (2017, 2018); Martinez (2017) in the context of nonconvex optimization. For convex optimization, Baes (2009) extended the accelerated cubic regularized Newton method to the general high-order case with the iteration complexity of \( O(1/k^{d+1}) \), where \( d \) is the order of derivative information used in the algorithm. Such extension was recently revisited by Nesterov (2018) with a discussion on the efficient implementation of the method when \( d = 3 \). Jiang et al. (2018) extended Nesterov’s approach to accommodate the composite optimization (1) and relaxed the requirement on the knowledge of problem parameters such as the Lipschitz constants and the requirement on the exact solutions of the subproblems while maintaining the same iteration bound as in Nesterov (2018). Along the line of bounding the worst case iteration complexity using up to the \( d \)-th order derivative information, there have also been significant progresses as well. Arjevani et al. (2018) showed that the worst case iteration complexity of any algorithm in that setting cannot be better than \( O(1/k^{(3d+1)/2}) \). A simplified analysis of the bound can be found in Nesterov (2018). So, there was a gap between the achieved iteration bound \( O(1/k^{d+1}) \) and the best possible bound of \( O(1/k^{(3d+1)/2}) \). Clearly at least one of the two bounds is improvable. In this paper, we aim to settle the above theoretical quest by providing a new implementable algorithm whose iteration complexity...
complexity is precisely $O\left(1/k^{(3d+1)/2}\right)$. As a result, the tradeoff relationship discussed above is pinned down to be exactly $O\left(1/k^{(3d+1)/2}\right)$. We note the independent work Gasnikov et al. (2018) first in Russian and then translated to English in late December 2018, and two other independent works Bubeck et al. (2018) and Bullins (2018), which were posted on arxiv in December 2018 as well. They derive similar results to ours with a limitation on the smooth objective functions (or more specifically—quartics in Bullins (2018)), while we allow a composite objective.

Our algorithm is based on the A-HPE framework of Monteiro and Svaiter (2013), which is presented as Algorithm 1 in this paper. In fact, our algorithm specifies a way to generate an approximate solution through the use of high order derivative information by Taylor expansion. In each iteration, such approximate solution is computed by means of a bisection process. At each bisection step, a regulated convex tensor (polynomial) optimization subproblem is approximately solved. Moreover, we show that, to implement one A-HPE iteration, the number of bisection steps—each calling to solve a convex tensor subproblem—is upper bounded by a logarithmic factor in the inverse of the required precision. Our bisection procedure is similar to the one proposed in Monteiro and Svaiter (2013) for the case $d=2$; however, a key modification is applied which enables the removal of the so-called “bracketing stage” used in Monteiro and Svaiter (2013).

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries including the assumptions and the high-order oracle model used throughout this paper. Then we present our optimal tensor method and its iteration complexity analysis in Section 3. Finally, the line search subroutine being used in the main procedure of our optimal tensor method is presented in Section 4.

2. Preliminaries

2.1. Notations

We denote $\nabla^d f(x)$ to be the $d$-th order derivative tensor at point $x$ of function $f$ with the $(i_1,\ldots,i_d)$ component given as:

$$\nabla^d f(x)_{i_1,\ldots,i_d} = \frac{\partial^d f}{\partial x_{i_1}\cdots\partial x_{i_d}}(x), \forall 1 \leq i_1,\ldots,i_d \leq n.$$ 

Given a $d$-th order tensor $\mathcal{T}$ and vectors $z^1,\ldots,z^d \in \mathbb{R}^n$, we denote

$$\mathcal{T}[z^1,\ldots,z^d] := \sum_{i_1,\ldots,i_d=1}^{n} \mathcal{T}_{i_1,\ldots,i_d}z_{i_1}^1 \cdots z_{i_d}^d, \text{ and } \mathcal{T}[z]^d = \mathcal{T}[z,z,\ldots,z].$$

The operator norm associated with $\mathcal{T}$ is defined as:

$$||\mathcal{T}|| := \max_{||z^i||=1, i=1,\ldots,d} \mathcal{T}[z^1,\ldots,z^d].$$

As a matter of convention, for quantities $x$ and $y$, we use the notation $y = \Theta(x)$ to indicate the relation that there are positive constants $a$ and $b$ such that $ax \leq y \leq bx$. If $a$ is absent, then we shall indicate the relation as $y = O(x)$. 

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2.2. High-Order Oracle Model and Regularized Tensor Approximation

In this paper, we consider the following high-order oracle model and the algorithm we are going to propose belongs to such oracle model.

**d-th Order Oracle Model**

- $f$ is $d$ times Lipschitz-continuous and differentiable with Lipschitz constant $L_d$ for $d$-th order derivative tensor; i.e.
  \[
  \|\nabla^d f(x) - \nabla^d f(y)\| \leq L_d \|x - y\| \quad \forall x, y \in \mathbb{R}^n, \quad (2)
  \]
  where the left side is the $d$-th order tensor operator norm.
- Given any $x$, the oracle returns $f(x), \nabla f(x), \nabla^2 f(x), ..., \nabla^d f(x)$.
- At iteration $k$, $x_k$ is generated from a deterministic function $h$ and the oracle’s responses at any linear combination of $x_1, x_2, ..., x_{k-1}$ and $\nabla^i f(x_1), \nabla^i f(x_2), ..., \nabla^i f(x_{k-1})$, where $1 \leq i \leq d$.

Define the exact proximal minimization at point $x$ with stepsize $\lambda > 0$ as

\[
\min_{y \in \mathbb{R}^n} f(y) + h(y) + \frac{1}{2\lambda} \|y - x\|^2. \quad (3)
\]

To utilize all the derivative information, we consider the regularized tensor approximation of $f(y)$ at point $x$:

\[
f_x(y) := f(x) + \nabla f(x)^\top (y-x) + \frac{1}{2} \nabla^2 f(x)[y-x]^2 + \cdots + \frac{1}{d!} \nabla^d f(x)[y-x]^d + \frac{M}{(d+1)!} \|y-x\|^{d+1}, \quad (4)
\]

where $\nabla^d f(x)[y-x]^d = \sum_{i_1, \ldots, i_d} \nabla^d f(x)_{i_1, \ldots, i_d} [y-x]_{i_1} \cdots [y-x]_{i_d}$ and $M > 0$ is the parameter of the high-order regularization term $\|y - x\|^{d+1}$. Moreover $f_x(y)$ is convex (see Nesterov (2018) when $M \geq dL_d$. Then, by (2) and the Taylor expansion, we can bound the gap between $f_x(\cdot)$ and $f(\cdot)$ for any $x$ (see Nesterov (2018)):

**Lemma 1** For every $x, y \in \mathbb{R}^n$,

\[
\|\nabla f(y) - \nabla f_x(y)\| \leq \frac{L_d + M}{d!} \|y - x\|^d.
\]

Therefore, it is natural to consider the tensor approximation of (3):

\[
\min_{y \in \mathbb{R}^n} f_x(y) + h(y) + \frac{1}{2\lambda} \|y - x\|^2. \quad (5)
\]

Note that the unique solution $y$ of (5) is characterized by the following optimality condition:

\[
u \in (\nabla f_x + \partial h)(y), \quad \lambda u + y - x = 0. \quad (6)
\]

For a scalar $\epsilon \geq 0$, the $\epsilon$-subdifferential of a proper closed convex function $h$ is defined as:

\[
\partial_{\epsilon} h(x) := \{u \mid h(y) \geq h(x) + \langle y - x, u \rangle - \epsilon, \forall y \in \mathbb{R}^n\}.
\]

With the above notion in mind, let us consider the following approximate solution for (6) (hence (5)).
Definition 2  Given \((\lambda, x) \in \mathbb{R}^{++} \times \mathbb{R}^n\) and \(\hat{\sigma} \geq 0\), the triplet \((y, u, \epsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+\) is called a \(\hat{\sigma}\)-approximate solution of (5) at \((\lambda, x)\) if

\[
u \in (\nabla f_x + \partial h)(y) \quad \text{and} \quad \|\lambda u + y - x\|^2 + 2\lambda \epsilon \leq \hat{\sigma}^2\|y - x\|^2.
\] (7)

Obviously, if \((y, u)\) is the solution pair of (6), then \((y, u, 0)\) is a \(\hat{\sigma}\)-approximate solution of (5) at \((\lambda, x)\) for any \(\hat{\sigma} \geq 0\). In the rest of our analysis, we assume the availability of a subroutine which, for given \((\lambda, x)\) and \(\hat{\sigma} > 0\), returns a \(\hat{\sigma}\)-approximate solution \((y, u, \epsilon)\). Let us call this subroutine \text{ATS} (Approximate Tensor Subroutine). In the case \(F(x) = f(x)\), \text{ATS} is invoked in every iteration of the algorithm proposed by Nesterov (2018). In this paper, we consider the general composite case \(F(x) = f(x) + h(x)\), and a proximal version of \text{ATS} is called in each step of the bisection search, which itself is a subroutine in the main procedure of our algorithm.

3. The Optimal Tensor Method

3.1. The tensor algorithm and its iteration complexity

Our bid to the optimal tensor algorithm is based on the so-called Accelerated Hybrid Proximal Extragradient (A-HPE) framework proposed by Monteiro and Svaiter (2013) for problem (1), whose main steps can be schematically sketched below:

Algorithm 1 A-HPE framework

\textbf{STEP 1.} Let \(x_0, y_0 \in \mathbb{R}^n\), \(0 < \sigma < 1\) and \(\theta > 0\) be given, and set \(A_0 = 0\) and \(k = 0\).

\textbf{STEP 2.} If \(0 \in \partial F(y_k)\), then \textbf{STOP}.  

\textbf{STEP 3.} Otherwise, find \(\lambda_{k+1} > 0\) and a triplet \((\tilde{y}_{k+1}, v_{k+1}, \epsilon_{k+1})\) such that

\[
v_{k+1} \in \partial \epsilon_{k+1} F(\tilde{y}_{k+1}),
\]

\[
\|\lambda_{k+1} v_{k+1} + \tilde{y}_{k+1} - \tilde{x}_k\|^2 + 2\lambda_{k+1} \epsilon_{k+1} \leq \sigma^2 \|\tilde{y}_{k+1} - \tilde{x}_k\|^2.
\] (9)

where

\[
\tilde{x}_k = \frac{A_k}{A_k + a_{k+1}} y_k + \frac{a_{k+1}}{A_k + a_{k+1}} x_k,
\]

\[
a_{k+1} = \frac{\lambda_{k+1} + \sqrt{\lambda_{k+1}^2 + 4\lambda_{k+1} A_k}}{2}.
\]

\textbf{STEP 4.} Choose \(y_{k+1}\) such that \(F(y_{k+1}) \leq F(\tilde{y}_{k+1})\) and let

\[
A_{k+1} = A_k + a_{k+1},
\]

\[
x_{k+1} = x_k - a_{k+1} v_{k+1}.
\]

\textbf{STEP 5.} Set \(k \leftarrow k + 1\), and go to \textbf{STEP 2}.
In the following, we quote some technical results derived in Monteiro and Svaiter (2013) for A-HPE. Since our proposed algorithm is within that framework, the results in Lemma 3 hold true for our method as well, and they will be used in the subsequent analysis.

**Lemma 3** Suppose the sequence \( \{x_k, y_k, \bar{x}_k, \bar{y}_k\} \) is generated from Algorithm 1. Let \( \bar{x}_0 \) be the projection of \( x_0 \) onto the set of optimal value points \( X_\star \). \( F_\star \) be the optimal value, and \( D \) be the distance from \( x_0 \) to \( X_\star \). Then for any integer \( k \geq 1 \), it holds that (Theorem 3.6 in Monteiro and Svaiter (2013)),

\[
\frac{1}{2} \| \bar{x}_0 - x_k \|^2 + A_k(F(y_k) - F_\star) + \frac{1 - \sigma^2}{2} \sum_{j=1}^{k} \frac{A_j}{\lambda_j} \| y_j - \bar{x}_{j-1} \|^2 \leq \frac{1}{2} D^2. \tag{10}
\]

Therefore,

\[
\sum_{j=1}^{k} \frac{A_j}{\lambda_j} \| y_j - \bar{x}_{j-1} \|^2 \leq \frac{D^2}{1 - \sigma^2}. \tag{11}
\]

Furthermore, \( A_k \) and \( \lambda_k \) has the following relation (Lemma 3.7 in Monteiro and Svaiter (2013)),

\[
A_k \geq \frac{1}{4} \left( \sum_{j=1}^{k} \sqrt{\lambda_j} \right)^2, \tag{12}
\]

and the distance between \( y_k \) and \( \bar{x}_0 \) can be bounded as follows (Theorem 3.10 in Monteiro and Svaiter (2013)),

\[
\| y_k - \bar{x}_0 \| \leq \left( \frac{2}{\sqrt{1 - \sigma^2}} + 1 \right) D. \tag{13}
\]

Now we are ready to propose our optimal tensor method in Algorithm 2.

At this point, neither Algorithm 1 nor Algorithm 2 has been shown to be implementable. In fact, STEP 3 in both algorithms presented above remain unspecified. It is even unclear why such solutions as required by STEP 3 exist at all. In Section 4, we shall establish a practical implementation of STEP 3 in Algorithm 2 via the Approximate Tensor Proximal (ATP) mappings in combination with a line-search subroutine. First, let us remark that Algorithm 2 is indeed a specialization of A-HPE. For simplicity, we let \( y_{k+1} = \bar{y}_{k+1} \) in STEP 4 of Algorithm 1. Because \((y_{k+1}, u_{k+1}, \epsilon_{k+1})\) is a \( \sigma \)-approximate solution at \((\lambda_{k+1}, \bar{x}_{k+1})\), one has that \( u_{k+1} \in (\nabla f_{\bar{x}_k} + \partial_{\epsilon_{k+1}} h)(y_{k+1}) \), and so we have

\[
v_{k+1} \in \nabla f(y_{k+1}) - \nabla f_{\bar{x}_k}(y_{k+1}) + (\nabla f_{\bar{x}_k} + \partial_{\epsilon_{k+1}} h)(y_{k+1})
= \nabla f(y_{k+1}) + \partial_{\epsilon_{k+1}} h(y_{k+1}) \subseteq \partial_{\epsilon_{k+1}} (f + h)(y_{k+1}),
\]

which satisfies (8). To establish (9), we need the following proposition.

**Proposition 4** Let \( (y, u, \epsilon) \) be a \( \sigma \)-approximate solution of (5) at \((\lambda, \bar{x})\) such that (16) holds. Define \( v := \nabla f(y) + u - \nabla f_{\bar{x}}(y) \). Then,

\[
\| \lambda v + y - \bar{x} \|^2 + 2\lambda \epsilon \leq \left( \sigma + \lambda \frac{L_d + M}{d!} \| y - \bar{x} \|^{d-1} \right)^2 \| y - \bar{x} \|^2. \tag{14}
\]

Consequently,

\[
\| \lambda v + y - \bar{x} \|^2 + 2\lambda \epsilon \leq \sigma^2 \| y - \bar{x} \|^2 \quad \text{with} \quad \sigma = \sigma_u + \hat{\sigma}. \tag{15}
\]
Combining the above inequality with (7), one has that
\[ \left\| \lambda u + y - \hat{x} \right\|^2 + 2\lambda \leq (\left\| \lambda u + y - \hat{x} \right\|^2 + \lambda \left\| u - v \right\|)^2 + 2\lambda \leq \left( \left\| \lambda u + y - \hat{x} \right\|^2 + 2\lambda \right) + 2\lambda \left\| u - v \right\| \left\| \lambda u + y - \hat{x} \right\| + \lambda^2 \left\| u - v \right\|^2 \]
\[ \leq \sigma^2 \left\| y - \hat{x} \right\|^2 + 2 \left( \lambda \frac{L_d + M}{d!} \right) \left\| y - \hat{x} \right\|^d \left( \left( \sigma + \lambda \frac{L_d + M}{d!} \right) \left\| y - \hat{x} \right\|^d - \left\| y - \hat{x} \right\|^d \right)^2 \]
\[ = \left( \sigma + \lambda \frac{L_d + M}{d!} \right) \left\| y - \hat{x} \right\|^d \left( \left\| y - \hat{x} \right\|^d - 1 \right)^2 \left\| y - \hat{x} \right\|^2, \]
proving the first inequality. Then, by the left hand side of (16), \( \lambda \frac{L_d + M}{d!} \left\| y - \hat{x} \right\|^d - \left\| y - \hat{x} \right\|^d \leq \sigma_u \), and so the second inequality follows. \( \square \)
We summarize the above discussion in the theorem below.

**Theorem 5** Algorithm 2 is a manifestation of the A-HPE framework, and thus the results of Lemma 3 hold for the sequence generated by Algorithm 2.

**Proof.** To verify that the A-HPE framework covers Algorithm 2 as a special case, it suffices to show the iterates generated by Algorithm 2 satisfy conditions (8) and (9). Note that (9) is exactly the conclusion (15) in Proposition 4 by letting \( y = y_{k+1}, \lambda = \lambda_{k+1} \) and \( \tilde{x} = \tilde{x}_k \). In addition, condition (8) is a result from Definition 2 and the updating formula for \( v_{k+1} \) in (19). \( \square \)

Before addressing the implementation of STEP 3 in Algorithm 2, let us first present the overall iteration complexity of Algorithm 2, assuming STEP 3 could be implemented. The key here is to obtain a lower bound on \( A_k \), as the following theorem stipulates.

**Theorem 6** Let \( D \) be the distance of \( x_0 \) to \( X^* \). Suppose that \( \{A_k\}^\infty_{k=1} \) is generated from Algorithm 2. Then for any integer \( k \geq 1 \), it holds that

\[
A_k \geq \left( \frac{1}{2} \right)^{d+1} \frac{d! \sigma_l}{L_d + M} \left( 1 - (\sigma_u + \hat{\sigma})^2 \right)^{\frac{d-1}{2}} \left( \frac{2}{d+1} \right)^{\frac{3d+1}{2}} k^{\frac{3d+1}{2}}. \tag{20}
\]

The next iteration complexity result readily follows from Theorem 6, whose proof can be found in the appendix.

**Theorem 7** Let \( D \) be the distance of \( x_0 \) to \( X^* \). Then, for any integer \( k \geq 1 \), the iterate \( y_k \) generated by Algorithm 2 satisfies:

\[
F(y_k) - F_* \leq \left( \frac{d + 1}{2} \right)^{\frac{3d+1}{2}} \frac{2^d}{(1 - (\hat{\sigma} + \sigma_u)^2)^{\frac{d+1}{2}} d! \sigma_l} D^{d+1}(L_d + M) k^{-\frac{3d+1}{2}}.
\]

**Proof.** Combining (10) and (20) yields that

\[
F(y_k) - F_* \leq \frac{1}{2 A_k} D^2 \leq \left( \frac{d + 1}{2} \right)^{\frac{3d+1}{2}} \frac{2^d}{(1 - (\hat{\sigma} + \sigma_u)^2)^{\frac{d+1}{2}} d! \sigma_l} D^{d+1}(L_d + M) k^{-\frac{3d+1}{2}}.
\] \( \square \)

The above theorem establishes the \( O(1/k^{\frac{3d+1}{2}}) \) iteration complexity for Algorithm 2. Since Algorithm 2 falls into the category of the High-Order Oracle Model, whose iteration complexity has a lower bound of \( O(1/k^{\frac{3d+1}{2}}) \); see Arjevani et al. (2018) and Nesterov (2018). The worst-case iteration complexity of Algorithm 2 matches this lower bound and it is therefore an optimal method.

### 4. A Line Search Subroutine and Its Iteration Complexity

Now, it remains to find a way to implement STEP 3 of the algorithm. We start with a special case, which is easier to illustrate.
4.1. The Non-Composite Case

Let us first consider a special case for Algorithm 2 where $F(x) = f(x)$ in the objective function and $y_{k+1}$ is the exact solution of the following convex tensor proximal point problem:

$$\min_y f_{\tilde{x}_k}(y) + \frac{1}{2\lambda_{k+1}}\|y - \tilde{x}_k\|^2.$$  

We shall discuss how to find $\lambda_{k+1}$ to satisfy the alternative condition in STEP 3 of Algorithm 2.

Note that for fixed $x_k$ and $y_k$, $\tilde{x}_k$ and $y_{k+1}$ are uniquely determined by $\lambda_{k+1}$. Therefore the functions $\tilde{x}_k(\lambda)$ and $y_{k+1}(\lambda)$ are continuous with respect to $\lambda$ (where we denote $\lambda_{k+1}$ to be $\lambda$). Next, we show that:

(i) $\lambda\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^{d-1} \to 0$, as $\lambda \to 0$;

(ii) Either there exists an increasing sub-sequence $\lambda_j \uparrow \infty$, such that $\lambda_j\|y_{k+1}(\lambda_j) - \tilde{x}_k(\lambda_j)\|^{d-1} \to \infty$ as $j \to \infty$, or there exists $\lambda$ such that $\|\nabla f(y_{k+1}(\lambda))\| \leq \tilde{\rho}$ for any $\lambda \geq \bar{\lambda}$.

Observe that

$$f_{\tilde{x}_k}(\lambda_k)(y_{k+1}(\lambda)) + \frac{1}{2\lambda}\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^2 \leq f_{\tilde{x}_k}(\lambda_k)(\tilde{x}_k(\lambda)) = f(\tilde{x}_k(\lambda)) < \infty, \forall \lambda > 0$$

where $f(\tilde{x}_k(\lambda))$ is bounded, since $\tilde{x}_k(\lambda)$ is a convex combination of $x_k$ and $y_k$. Letting $\lambda \to 0$ in the above inequality leads to $\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^2 \to 0$, which implies $\lambda\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^{d-1} \to 0$ as $\lambda \to 0$, proving (i).

To prove (ii), it suffices to show that if the “either” part does not hold, then the “or” part must hold. In this case, there must exist $C_1 > 0$ such that when $\lambda \to \infty$, $\lambda\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^{d-1} \leq C_1$, and thus $\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\| \to 0$. Moreover, for any $\lambda > 0$ the optimality condition is

$$\nabla f_{\tilde{x}_k}(\lambda_k)(y_{k+1}(\lambda)) + \frac{1}{\lambda}(y_{k+1}(\lambda) - \tilde{x}_k(\lambda)) = 0.$$  

Letting $\lambda \to \infty$ in the above identity yields that $\nabla f_{\tilde{x}_k}(\lambda_k)(y_{k+1}(\lambda)) \to 0$. Recall that in this case we have $\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\| \to 0$, thus $\nabla f(y_{k+1}(\lambda)) \to 0$ proving the “or” part.

To summarize, either we have $\lambda\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^{d-1} \to 0$ as $\lambda \to 0$ and $\lambda_j\|y_{k+1}(\lambda_j) - \tilde{x}_k(\lambda_j)\|^{d-1} \to \infty$ as $j \to \infty$, which guarantees the existence of $\lambda$ to satisfy (16) due to the continuity of $\lambda\|y_{k+1}(\lambda) - \tilde{x}_k(\lambda)\|^{d-1}$ on $\lambda$. Or we have a $\lambda_{k+1}$ such that $\|\nabla f(y_{k+1}(\lambda))\| \leq \tilde{\rho}$. In this case, since $h(x)$ is not present, $u_{k+1} = \nabla f_{\tilde{x}_k}(y_{k+1})$ and $\|\nabla f(y_{k+1}) + u_{k+1} - \nabla f_{\tilde{x}_k}(y_{k+1})\| = \|\nabla f(y_{k+1})\| \leq \tilde{\rho}$. Therefore, we have shown that the alternative condition in STEP 3 is actually satisfied.
4.2. The Composite Case

To present the algorithm that computes \( \lambda \) satisfying the conditions in STEP 3, we first construct \( \beta_{k+1} = \frac{a_{k+1}}{\beta_k + a_{k+1}} \). From (18), we can see that \( \lambda_{k+1} = \frac{\sigma_k}{A_k + a_{k+1}} \). Therefore, we are able to represent \( \lambda_{k+1} \) and \( x_k \) by means of \( \beta_{k+1} \):

\[
\begin{align*}
\lambda_{k+1} &= A_k \frac{\beta_{k+1}^2}{1 - \beta_{k+1}}, \\
x_k &= \beta_{k+1} x_k + (1 - \beta_{k+1}) y_k.
\end{align*}
\]

In the \( k \)-th iteration, we denote

\[
\lambda(\beta) = A_k \frac{\beta^2}{1 - \beta}, \quad \beta \in (0, 1).
\]  

We shall perform bisection on \( \beta \) instead of \( \lambda \) in STEP 3 of Algorithm 2 to search for \( \lambda_{k+1} \). In that way, the initial interval for the bisection is \([0, 1]\), which allows us to skip what they called the bracketing stage in Monteiro and Svaiter (2013)).

**Algorithm 3** Bisection on \( \beta \) based on \( L_d \)

**INPUT:** \( M \geq L_d \), \( \hat{\sigma} \geq 0 \), \( 0 < \sigma_l < \sigma_u < 1 \) such that \( \sigma := \hat{\sigma} + \sigma_u < 1 \) and \( \sigma_l (1 + \hat{\sigma})^{d-1} < \sigma_u (1 - \hat{\sigma})^{d-1} \), \( 1 > \hat{\epsilon}, \hat{\rho} > 0 \).

**STEP 1.** Let \( \alpha_+ = \frac{d\sigma_u}{L_d + M} \) and \( \alpha_- = \frac{d\sigma_l}{L_d + M} \).

**STEP 2.** (Bisection Setup) Set \( \beta_- = 0, \beta_+ = 1, \lambda_+ = \lambda(\beta_+) = +\infty, \lambda_- = \lambda(\beta_-) \).

2.a. Let \( \beta = \frac{\beta_- + \beta_+}{2} \) and let

\[
\lambda_\beta = \lambda(\beta), \quad x_\beta = (1 - \beta)y_k + \beta x_k,
\]

and compute \((y_\beta, u_\beta, \epsilon_\beta)\) as a \( \hat{\sigma} \)-approximate solution at \((\lambda_\beta, x_\beta)\), and \( v_\beta = \nabla f(\lambda_\beta) - \nabla f_{x_\beta}(y_\beta) - u_\beta \).

2.b. If \( \|v_\beta\| \leq \hat{\rho} \) and \( \epsilon_\beta \leq \hat{\epsilon} \) then output \((\lambda_\beta, x_\beta, y_\beta, u_\beta, \epsilon_\beta)\) and STOP.

else if \( \lambda_\beta \| y_\beta - x_\beta \|^{d-1} \in [\alpha_-, \alpha_+] \) then

set \((\beta_{k+1}, \tilde{x}_{k+1}, y_{k+1}, v_{k+1}) = (\beta, x_\beta, y_\beta, v_\beta)\) and STOP.

else if \( \lambda_\beta \| y_\beta - x_\beta \|^{d-1} > \alpha_+ \) then

set \( \beta_+ \leftarrow \beta \), and go to STEP 2.a.

else if \( \lambda_\beta \| y_\beta - x_\beta \|^{d-1} < \alpha_- \) then

set \( \beta_- \leftarrow \beta \), and go to STEP 2.a.

end if

An upper bound for the overall number of iterations required by Algorithm 3 is presented in the following theorem.

**Theorem 8** Algorithm 3 needs to perform no more than

\[
\Theta \left( \max\{\log_2(\hat{\epsilon}^{-1}), \log_2(\hat{\rho}^{-1})\} \right)
\]  

(23)
bisection steps before reaching \( \lambda_{k+1} > 0 \) and a \( \hat{\sigma} \)-approximate solution \((y_{k+1}, u_{k+1}, \epsilon_{k+1})\) at \((\lambda_{k+1}, \bar{x}_k(\lambda_{k+1}))\) satisfying
\[
\alpha_+ \leq \lambda_{k+1} \| \bar{x}_k(\lambda_{k+1}) - y_{k+1} \|^{d-1} \leq \alpha_+,
\]
or to return \(v_{k+1}\) and \(\epsilon_{k+1}\) such that \(\|v_{k+1}\| \leq \bar{\rho}\) and \(\|\epsilon_{k+1}\| \leq \bar{\epsilon} \).

Due to the page limitation, we will not provide the full proof of Theorem 8 but summarize some main steps of the proof as follows. Suppose that Algorithm 3 has performed \(j\) bisection steps before triggering the stopping criteria. We aim to show that \(j \leq \Theta \left( \max\{\log_2(\bar{\sigma}^{-1}), \log_2(\bar{\rho}^{-1})\} \right)\). At that iteration let us denote \(x_+ = x_{\beta_+}, x_- = x_{\beta_-}, y_+ = y_{\beta_+}\) and \(y_- = y_{\beta_-}\), and we also have \(\beta_+ - \beta_- = \frac{1}{2j}\). Denote
\[
\bar{\lambda} = \max \left\{ \alpha_+^{1/d} \left[ \frac{1}{\rho} (1 + \hat{\sigma} + \frac{L_d + M}{d!} \alpha_-) \right]^{1 - \frac{2}{d}} \right\}.
\]
If \(\bar{\beta} \leq \frac{1}{2} \) then \(\frac{1}{1 - \beta} \leq 2\); if \(\bar{\beta} > \frac{1}{2}\), then (21) gives
\[
\frac{1}{1 - \beta} = \frac{\bar{\lambda}}{A_k \bar{\beta}^2} \leq \frac{4\bar{\lambda}}{A_k} \leq \max \left\{ \Theta \left( \bar{\rho}^{-\frac{d+1}{d-1}} \right), \Theta \left( \bar{\epsilon}^{-\frac{d+1}{d-1}} \right) \right\}.
\]
Therefore, without loss of generality we may assume \(j \geq \log_2(2/(1 - \bar{\beta}))\), for otherwise \(j < \log_2(2/(1 - \bar{\beta})) \leq \Theta \left( \max\{\log_2(\bar{\rho}^{-1}), \log_2(\bar{\epsilon}^{-1})\} \right)\) already holds. Since Algorithm 3 did not stop before iteration \(j\), it holds that
\[
\lambda_+ \| y_{\beta_+} - x_{\beta_+} \|^{d-1} > \alpha_+, \quad \lambda_- \| y_{\beta_-} - x_{\beta_-} \|^{d-1} < \alpha_-.
\]
Next, we consider the following quantitative measure of optimality of (1) to accommodate the higher-order information:
\[
\psi(\lambda; x) := \lambda \left\| (I + \lambda(\nabla f_x + \partial h))^{-1} (x) - x \right\|^{d-1}.
\]
Then we can show that
\[
\psi_+ := \psi(\lambda_+; x_+) \geq \lambda_+ (1 - \hat{\sigma})^{d-1} \| y_+ - x_+ \|^{d-1} > (1 - \hat{\sigma})^{d-1} \alpha_+,
\]
\[
\psi_- := \psi(\lambda_-; x_-) \leq \lambda_- (1 + \hat{\sigma})^{d-1} \| y_- - x_- \|^{d-1} < (1 + \hat{\sigma})^{d-1} \alpha_-.
\]
Consequently,
\[
\psi_+ - \psi_- > (1 - \hat{\sigma})^{d-1} \alpha_+ - (1 + \hat{\sigma})^{d-1} \alpha_-.
\]
Moreover, the left hand side of the above inequality can be bounded as follows:
\[
|\psi_+ - \psi_-| \leq \max \left\{ \Theta \left( \bar{\epsilon}^{-2d+1} \right), \Theta \left( \bar{\rho}^{-\frac{(2d-1)(d+1)}{d}} \right) \right\} (\beta_+ - \beta_-).
\]
Combining the above two inequalities yields that
\[
(1 - \hat{\sigma})^{d-1} \alpha_+ - (1 + \hat{\sigma})^{d-1} \alpha_- \leq \max \left\{ \Theta \left( \bar{\epsilon}^{-2d+1} \right), \Theta \left( \bar{\rho}^{-\frac{(2d-1)(d+1)}{d}} \right) \right\} \frac{1}{2j},
\]
which further implies that \(j \leq \Theta \left( \max\{\log_2(\bar{\epsilon}^{-1}), \log_2(\bar{\rho}^{-1})\} \right)\) by observing that the left hand side of the above inequality is a positive constant.

Finally, combining Theorem 7 and Theorem 8, we obtain the total number of ATS to be solved in Algorithm 2.
Theorem 9 Suppose the STEP 3 of Algorithm 2 is implemented via the bisection strategy in Algorithm 3. For given $\epsilon > 0$, set $\bar{\epsilon} = \bar{\rho} \leq \epsilon \left(\frac{2}{\sqrt{1 - \sigma^2}}\right)D + 1^{-1}$. Then the total number of calls of ATS to find $y_k$ such that $F(y_k) - F_* \leq \epsilon$ is bounded above by

$$\Theta \left( \left\lceil \frac{D^{d+1}(L_d + M)}{\epsilon} \right\rceil \frac{2^d + 1}{\log_2 \left( \frac{1}{\epsilon} \right)} \right),$$

where we have ignored the logarithmic factors in $D$ and $L_d$.

Proof. If the Algorithm 2 does not terminate at Step 2, then the bisection stage stops at the else if part of 2.b in every iteration of Algorithm 2 and the conclusion follows by multiplying the two bounds in Theorem 7 and Theorem 8. Otherwise, we have an early stop and the number of calls of ATS is less than $\Theta \left( \left\lceil \frac{D^{d+1}(L_d + M)}{\epsilon} \right\rceil \frac{2^d + 1}{\log_2 \left( \frac{1}{\epsilon} \right)} \right)$. In this case, from Theorem 8 we get a point $y_{k+1}$ such that $v_{k+1} \in \partial_{k+1} F(y_{k+1})$, $\|v_{k+1}\| \leq \bar{\rho}$ and $\|\epsilon_{k+1}\| \leq \bar{\epsilon}$. Recall that we denote $\hat{x}_0$ as the projection of $x_0$ onto the set of optimal value points $X_*$, and the bound in (13), we have

$$F(y_{k+1}) - F_* \leq v_{k+1}^\top (y_{k+1} - \hat{x}_0) + \epsilon_{k+1}$$
$$\leq \|v_{k+1}\| \|y_{k+1} - \hat{x}_0\| + \epsilon_{k+1}$$
$$\leq \|v_{k+1}\| \left( \frac{2}{\sqrt{1 - \sigma^2}} + 1 \right)D + \epsilon_{k+1}$$
$$\leq \epsilon.$$ 

\[\square\]

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Appendix A. Proof of Theorem 6

To establish the lower bound of \( \{ A_k \}_{k=1}^{\infty} \) in Theorem 6, we first provide a recursive bound as an intermediate step.

**Proposition 10** Let \( D \) be the distance of \( x_0 \) to \( X_+ \). Suppose \( \{ A_k \}_{k=1}^{\infty} \) is generated from Algorithm 2, then

\[
A_k \geq \frac{1}{4} C^{-\frac{2p}{q}} \left( \sum_{j=1}^{k} A_j^\frac{1}{q} \right)^{2p}
\]

where \( q = \frac{3d+1}{d-1} \), \( p = \frac{3d+1}{2d+2} \) and \( C = \frac{D^2}{(1-(\hat{\sigma}+\sigma_u)^2)} \left( \sqrt{d\sigma_l} \right)^{-\frac{2}{q-1}} \).

**Proof.** Suppose \( \{ x_k, y_k, \hat{x}_k \} \) is the sequence generated by Algorithm 2. Then, according to (11) and Proposition 4, it holds that

\[
\sum_{j=1}^{k} \frac{A_j}{\lambda_j^\frac{2}{q+1}} ||y_j - \hat{x}_{j-1}||^2 \leq \frac{D^2}{1-(\hat{\sigma}+\sigma_u)^2},
\]

which together with the left hand side of (16) implies

\[
\sum_{j=1}^{k} \frac{A_j}{\lambda_j^\frac{2}{q+1}} = \sum_{j=1}^{k} \frac{A_j}{\lambda_j} \cdot \frac{1}{\lambda_j^{\frac{2}{q+1}}} ||y_j - \hat{x}_{j-1}||^2 \leq \frac{D^2}{(1-(\hat{\sigma}+\sigma_u)^2)} \left( \sqrt{d\sigma_l} \right)^{-\frac{2}{q-1}} = C. \tag{26}
\]

By the definition of \( p \) and \( q \), we have \( \frac{1}{p} + \frac{1}{q} = 1 \). Using Hölder’s inequality, together with (26), we have

\[
\left( \sum_{j=1}^{k} \sqrt{\lambda_j} \right)^{\frac{q}{p}} C^\frac{1}{q} \geq \left( \sum_{j=1}^{k} \sqrt{\lambda_j} \right)^{\frac{q}{p}} \left( \sum_{j=1}^{k} A_j \lambda_j\right)^{\frac{1}{q}} \geq \sum_{j=1}^{k} \lambda_j^{\frac{1}{q}} A_j^{\frac{1}{q} \cdot \frac{d+4}{4(d-1)}} = \sum_{j=1}^{k} A_j^{\frac{1}{q} \cdot \frac{d+4}{4(d-1)}}.
\]

Finally, by (12) we obtain

\[
A_k \geq \frac{1}{4} \left( \sum_{j=1}^{k} \sqrt{\lambda_j} \right)^{2} \geq \frac{1}{4} C^{-\frac{2p}{q}} \left[ \sum_{j=1}^{k} A_j^{\frac{1}{q} \cdot \frac{d+4}{4(d-1)}} \right]^{2p}.
\]

\[
\square
\]

**Proof of Theorem 6.** Let \( p, q \) and \( C \) be defined as in Proposition 10. Construct \( \{ B_k \} \) such that \( B_1 = A_1 \) and \( B_i = T^{\frac{1}{2} - \frac{2p}{q} \cdot i - 1} (A_i)^{\frac{2p}{q} - i - 1} \) for \( i \geq 2 \), where \( T := \left( \frac{1}{4} \right) \left( \frac{1}{2} \right)^{\frac{2p}{q} - 1} (2^{2p}) \). Next, we shall apply induction to show that for any \( k \geq 1 \),

\[
A_k \geq B_k k^{\frac{2p}{q}}, \quad \forall k \geq 1,
\]

(27)
where \( r_i = \frac{3d+1}{2} [1 - (2p/q)^{i-1}] \). When \( i = 1 \), this is obvious because \( A_k \geq A_1 = B_1 k^{r_1} \). Now suppose that for any \( k \geq 1 \), \( A_k \geq B_i k^{r_i} \) for some \( i \). Then, by the induction hypothesis and (25) it holds that

\[
A_k \geq \frac{1}{4} C^{-\frac{2p}{q}} \left( \sum_{j=1}^{k} A_{j}^{\frac{1}{q}} \right) ^{2p} \geq \frac{1}{4} C^{-\frac{2p}{q}} \left( \sum_{j=1}^{k} (B_{j}^{r_j})^{\frac{1}{q}} \right) ^{2p} = \frac{1}{4} \left( \frac{B_i}{C} \right) ^{\frac{2p}{q}} \left( \frac{1}{1 + r_i/q} \right) ^{2p} \left( \frac{2}{d+1} \right)^{2p} k^{2p (r_i/q + 1)}.
\]

where the last inequality follows from

\[
\frac{q}{q + r_i} = \frac{\frac{3d+1}{d-1} + \frac{3d+1}{2} [1 - (2p/q)^{i-1}]}{1 + \frac{d-1}{2} [1 - (2p/q)^{i-1}]} \geq \frac{1}{1 + \frac{d-1}{2}} = \frac{2}{d+1}.
\]

Let us further simplify the expression. First of all, from the definition of \( T \) and \( B_i \), one observes that

\[
\frac{1}{4} \left( \frac{B_i}{C} \right)^{\frac{2p}{q}} \left( \frac{2}{d+1} \right)^{2p} = B_i^{\frac{2p}{q}} T = \left[ T^{-\frac{1-(2p/q)^{i-1}}{1-2p/q} A_{i}^{(2p/q)^{i-1}}} \right]^{\frac{2p}{q}} T = T^{-\frac{2p}{1-2p/q} + 1} A_{i}^{(2p/q)^{i}} = T^{-\frac{1-(2p/q)^{i}}{1-2p/q}} A_{i}^{(2p/q)^{i}} = B_{i+1}.
\]

Then, the construction of \( q \) and \( r_i \) implies that

\[
2p \left( \frac{r_i}{q} + 1 \right) = \frac{3d+1}{d+1} \left( 1 + \frac{3d+1}{2} (1 - (2p/q)^{i-1}) \right) = \frac{3d+1}{d+1} \left( 1 + \frac{d-1}{2} (1 - (2p/q)^{i-1}) \right) = \frac{3d+1}{d+1} \left( \frac{d+1}{2} - \frac{d-1}{2} (2p/q)^{i-1} \right) = \frac{3d+1}{2} (1 - (2p/q)^{i}) = r_{i+1},
\]

where the second last equality holds true due to the fact that \( 2p/q = (d-1)/(d+1) \). Now the desired inequality (27) follows by combining (28), (29) and (30). To further prove (20), we observe that \( 2p/q = (d-1)/(d+1) < 1 \), and so \( \lim_{i \to \infty} B_i = T^{-\frac{1-(2p/q)^{i}}{1-2p/q}} = T^{\frac{3d+1}{4}} \) and \( \lim_{i \to \infty} r_i = \frac{3d+1}{2} \). Finally, by letting \( i \to \infty \) in (27) and using the definition of \( C \) in (26), we have
\[ A_k \geq T^{\frac{d+1}{2}} k^{\frac{3d+1}{2}} = \left[ \frac{1}{4} \left( \frac{1}{C} \right)^{\frac{4}{d+1}} \left( \frac{2}{d+1} \right)^{\frac{2d+1}{d+1}} \right]^{\frac{d+1}{2}} k^{\frac{3d+1}{2}} \]

\[ = \left( \frac{1}{2} \right)^{d+1} \frac{d! \sigma_l}{L_d + M} \left( \frac{1 - \sigma^2}{D^2} \right)^{\frac{d-1}{2}} \left( \frac{2}{d+1} \right)^{\frac{3d+1}{2}} k^{\frac{3d+1}{2}}. \]