Stochastic Evolution of Stock Market Volume-Price Distributions

Paulo Rocha¹, Frank Raischel², João P. da Cruz³,⁴, and Pedro G. Lind⁴,⁵

¹ Mathematical Department, FCUL University of Lisbon, 1749-016 Lisbon, Portugal
(e-mail: paulorocha99@hotmail.com)
² Instituto Dom Luiz, CGUL, University of Lisbon, 1749-016 Lisbon, Portugal
(e-mail: raischel@cii.fc.ul.pt)
³ Closer Consulting LTD, 4-6 University Way, London E16-2RD, United Kingdom
(e-mail: joao.cruz@closer.pt)
⁴ Centro Física Teórica e Computacional, Avenida Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
(e-mail: joao.cruz@closer.pt)
⁵ ForWind and Institute of Physics, University of Oldenburg, DE-26111 Oldenburg, Germany
(e-mail: pedro.g.lind@forwind.de)

Abstract. Using available data from the New York stock market (NYSM) we test four different biparametric models to fit the correspondent volume-price distributions at each 10-minute lag: the Gamma distribution, the inverse Gamma distribution, the Weibull distribution and the log-normal distribution. The volume-price data, which measures market capitalization, appears to follow a specific statistical pattern, other than the evolution of prices measured in similar studies. We find that the inverse Gamma model gives a superior fit to the volume-price evolution than the other models. We then focus on the inverse Gamma distribution as a model for the NYSM data and analyse the evolution of its distribution parameters as a stochastic process. Assuming that the evolution of these parameters is governed by coupled Langevin equations, we derive the corresponding drift and diffusion coefficients, which then provide insight for understanding the mechanisms underlying the evolution of the stock market.

Keywords: Stochastic Distributions, Volatility, Stock Market.

1 Scope and Motivation

In 1973 a breakthrough in financial modelling was proposed by Black and Scholes, who reinterpreted the Langevin equation for Brownian motion to predict value European options, assuming the underlying asset follows a stochastic process in the form¹²

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \]

for \( S_0 > 0 \), where \( S_t \) is the asset price, \( \mu \) is the mean rate of the asset return and \( W_t \) describes a Wiener process, with distribution \( W_t \sim N(0,t) \). The value of \( \sigma \),...
so-called volatility, measures the risk associated to the fluctuation of the asset return. Thus, by making a good estimate of its value one is able to establish a criterion for selling and buying in order to optimize the profit.

The BS, and similar stochastic approaches based on Gaussian uncorrelated noise sources, have since then received both strong criticism and improvements, such as stochastic volatility models\[3\]. It has been acknowledged that in more realistic models the statistics of extreme events, leading to heavy tails in the distributions, as well as correlations between noise sources and other components need to be taken into account.

In this paper we put this important extension in a more general context. From a purely mathematical perspective, for each stochastic variable obeying a given Langevin equation there is a probability density function (PDF) associated to it that fulfils a Fokker-Planck equation\[4\]. Probability density functions are defined by a few parameters that characterize the corresponding statistical moments. The generalization of the Black-Scholes model to incorporate stochastic volatility is a particular case of having one probability density function whose parameters are themselves stochastic variables governed by stochastic differential equations. By modelling such “stochastic” probability density functions one is able to properly describe how they evolve and, thus, evaluate how uncertain is a given prediction of the corresponding variable. We focus here on the evolution of the volume-price, i.e. on changes in capitalization, which should have more the character of a conserved quantity than the price per se. While the price and volume distribution are useful for portfolio purposes, to have access to the overall distribution of volume-prices provides information about the entire capital traded in the market.

In this paper, we show that heavy tails are present in the statistics of the capitalization, and we specifically present a stochastic evolution equation for the tail parameter. In the context of finance models, such approach can eventually enable one to improve measures of risk and to provide additional insight in risk management.

We start in Sec.\[2\] by describing the data collected from the New York stock market and in Sec.\[3\] we apply four typical models in finance to fit the empirical data. We will argue that inverse Gamma is a good model for the cumulative distributions of volume-prices and therefore, in Sec.\[4\] we concentrate in its fit parameters to mathematically describe the stochastic evolution of volume-price distributions. Conclusions close the paper in Sec.\[5\].

2 Data

We construct a database of several listed shares extracted from the New York stock market (NYSM) every ten minutes starting in March 16th, 2011 to January 1st, 2014. From the data, we compute volumes distributions for each ten minutes, in order to obtain a full description of the temporal evolution of the transactions. All the data were collected from the website \url{http://finance.yahoo.com} every 10 minutes during almost three years (907 days), yielding a total of $N_p \sim 10^5$ data points.
Fig. 1. Illustration of the volume and price evolution for one company during four days: (a) volume $V$, (b) price $p$ and (c) volume-price $pV$ time-series.

Each register refers to one specific listed company and is composed by the following fields: last trade price, volume, day’s high price, day’s low price, last trade date, 200 days-moving average, average daily volume and company name. In total, we were able to have a total of $N_e \sim 2000$ listed companies for each time-span of 10 minutes. Since we do not have access to the instantaneous trading price of each transaction for each company, we consider the last trade price as the estimate of the price change on each set of ten minutes trading volume.

Figure 1a and 1b show the evolution of the trading volume $V$ and the last trade price $p$ respectively for one single company during approximately 5 working days. We define the volume-price $s = pV$ as the product of both these properties (see Fig. 1c) and will concentrate henceforth in analysing its joint evolution. This image gives us an idea of how our volume-price $s$ and the separated components, volume $V$ and price $p$, change along one day in one particular company and, consequently, it reflects the change in capitalization of a given company.

In Fig. 1 we also indicate that the period of six and half hours during which the price change, corresponds exactly to the period at which the NYSE is open, generally from 9:30 am to 4:00 pm (east time). After the market closes, there is still a 4-hour window during which trading occurs, so-called after-hours trading, typically from 4:00 to 8:00 pm. We maintain these largely inactive periods for future studies on the statistics of the after-hours trade. In the context of this study, the changes in capitalization during these periods can be neglected.
For each 10-minute interval we compute the cumulative density distribution (CDF) of all $N_e$ volume-prices and record its respective average $\langle s \rangle$ over the listed companies, and standard deviation $\sigma$. For convenience, we take the volume-price normalized to its average $\langle s \rangle$ when computing the CDF. In Fig. 2a, we show the CDF for a particular 10-minute span and in Fig. 2b and 2c one plots the typical evolution of the average and standard deviation respectively.

The choice of the normalized volume-price is the best for assessing the underlying “geometry” of the market as a complex network [5], and therefore we consider henceforth the normalized volume-price $s/\langle s \rangle$. Volume-price repre-
Fig. 3. Time series of the two parameters characterizing the evolution of the cumulative density function (CDF) of the volume-price $s$: (a) $\Gamma$-distribution (b) inverse $\Gamma$-distribution, (c) log-normal distribution and (d) Weibull distribution. Each point in these time series correspond to 10-minute intervals. Periods with no activity correspond to the period where market is closed, and therefore will not be considered in our approach. (e-f) Probability density function of the resulting relative error correspondent to the fitting parameters $\phi$ and $\theta$ for each distribution. In all plots, different colors correspond to different distribution models.

sent the amount of capital of a particular listed company that is exchanged in the market. The normalized distribution of volume-price represents the distribution of links between investors and companies.

|                      | Param. err. $\Delta \phi / \phi$ | Param. err. $\Delta \theta / \theta$ |
|----------------------|-----------------------------------|-------------------------------------|
|                      | Average Std Dev. | Average Std Dev.                   |
| $\Gamma$-distribution| 2.21e-2   8.54e-3 | 2.82e-2   1.16e-2                   |
| **Inverse $\Gamma$-distribution**| **1.43e-2  6.46e-3** | **3.43e-2  5.49e-2**                |
| Weibull              | 3.13e-2   5.29e-2 | 4.89e-2   9.77e-2                   |
| Log-normal           | 3.78e-2   7.53e-2 | 5.60e-2   9.28e-2                   |

Table 1. The average and standard deviations of the value distributions for each parameter error, $\Delta \phi / \phi$ and $\Delta \theta / \theta$, in Fig. 3e-f. The best fit is indeed obtained for the inverse Gamma distribution.
3 Four models for volume-price distributions

In order to find a good fit to the empirical CDF we will consider four well-known bi-parametric distributions, namely the Gamma distribution, inverse Gamma distribution, log-normal distribution and the Weibull distribution. We fit the empirical CDF data (bullets in Fig. 2a) with these four different models, which are often used for finance data analysis[6].

The Gamma probability density function (PDF) is given by

\[ F_G(s) = \frac{s^{\phi - 1}}{\theta^\phi \Gamma(\phi)} \exp \left[ -\frac{s}{\theta} \right], \]  

(2)

the inverse Gamma PDF by

\[ F_{1/G}(s) = \frac{\theta^{\phi_{1/G}}}{\Gamma[\phi_{1/G}]} s^{-\phi_{1/G} - 1} \exp \left[ -\frac{\theta_{1/G}}{s} \right], \]  

(3)

the log-normal PDF by

\[ F_{\ln}(s) = \frac{1}{s\theta_{\ln}\sqrt{2\pi}} \exp \left[ -\frac{(\log s - \phi_{\ln})^2}{2\theta_{\ln}^2} \right] \]  

(4)

and the Weibull PDF by

\[ F_W(s) = \frac{\phi_W}{\theta_W^\phi} s^{\phi_W - 1} \exp \left[ -\left( \frac{s}{\theta_W} \right)^{\phi_W} \right]. \]  

(5)

In Fig. 2a, we plot the corresponding fit of each of these models for the empirical CDF. In Fig. 3(a–d) we show a short time-interval of the series of each pair of parameter.

For each model above, we take into account the relative error of each parameter value, \( \Delta\phi/\phi \) and \( \Delta\theta/\theta \), computed using a least square scheme when making the fit. Figure 3e and 3f show the distributions of the observed relative errors of \( \phi \) and \( \theta \) respectively. From these two plots it seems that each distribution fits quite well the empirical CDF data, since relative errors are mostly under five percent. From the inspection of Fig. 3e and 3f as well as Tab. 1, one sees that the best fit seems to be for the inverse Gamma distribution and therefore we will consider henceforth only this distribution.

4 The stochastic evolution of inverse Gamma tails

To explore the inverse Gamma distribution model, we first consider the meaning of its two parameters. A closer look at Eq. (3) leads to the conclusion that while \( \theta \) characterizes the shape of the distribution for the lowest range of volume-prices, the parameter \( \phi \) characterizes the power law tail \( \sim s^{-\phi - 1} \). Since it is this tail that incorporates the large fluctuations of volume-prices, in this section
we focus on the evolution of the parameter $\phi$ solely. Label $1/\Gamma$ is dropped for simplicity.

Taking the time series of the parameter $\phi$ we derive the stochastic evolution equation as thoroughly described in Ref. [7]. This approach retrieves two functions, called the drift and diffusion coefficients $D_1(\phi)$ and $D_2(\phi)$, governing the stochastic evolution of $\phi$:

$$d\phi = D_1(\phi) dt + \sqrt{D_2(\phi)} dW_t.$$  \hspace{1cm} (6)

Where $W_t$ represents the typical Wiener process, with $\langle W_t \rangle = 0$ and $\langle W_t W_{t'} \rangle = 2\delta(t - t')$. Typically the drift term governs the deterministic contributions for the overall evolution of $\phi$, while the diffusion term governs the corresponding (stochastic) fluctuations.

Functions $D_1(\phi)$ and $D_2(\phi)$ can be computed directly from the data [7] computing the first and second conditional moments respectively ($n = 1, 2$):

$$D_n(\phi_i) = \lim_{\tau \to 0} \frac{1}{n! \tau} M_n(\phi_i, \tau),$$  \hspace{1cm} (7)

where $\phi_i$ represents one specific bin-point in the range of observable values and the conditional moment is given by

$$M_n(\phi_i, \tau) = \langle (\phi(t + \tau) - \phi(t))^n \rangle_{\phi(t) = \phi_i}.$$  \hspace{1cm} (8)

Figure 4a and 4b show the first and second conditional moments respectively, as a function of $\tau$, for a given bin value $\phi_i$. For the lowest range of $\tau$ values one sees a linear dependence of the conditional moments, which enables to directly extract the corresponding value of the drift and diffusion in Eq. (7). Further, there is a clear offset in both moments, which indicates the presence of an additional stochastic process superimposed on the intrinsic stochastic
dynamics, called measurement noise \[8\], whose amplitude can be estimated as 
\[ \sigma = \sqrt{M_2(\langle \phi \rangle, 0)/2} \]. See Fig. 4b.

By computing the slopes of \( M_1 \) and \( M_2 \) for each bin in variable \( \phi \) yields a complete definition of both drift \( D_1 \) and diffusion \( D_2 \) coefficients for the full range of observed \( \phi \) values. Figures 5a and 5b show the drift and diffusion respectively. While the diffusion term has an almost constant amplitude, \( \sqrt{D_2} \sim 10^{-3} \), the drift is linear on \( \phi \) with a negative sloped and a fixed point close to one, \( \phi_f \sim 0.93 \).

This last observation is interesting from the point of view of the inverse Gamma PDF: the volume-price tails fluctuate around an inverse square law \( \sim s^{-2} \) driven by a restoring force which can be modelled through Hooke’s law. Furthermore, the fluctuations around the inverse square law are quantified by the diffusion amplitude \( \sqrt{D_2} \) of the tail parameter, which can be interpreted as a sort of “parameter volatility”.

5 Discussion and Conclusions

In this paper we analyse New York stock market volume-price distributions during the last two years sampled every ten minutes. We tested four models commonly applied to finance data and presented evidence that the inverse Gamma distribution is the model yielding the least error.

Further, we considered the parameter controlling the tail of the inverse Gamma distribution and extracted a Langevin equation governing its stochastic evolution directly from the parameter’s time series. While the deterministic contribution (drift) depends linearly on the parameter, with a restoring force around unity approximately, the stochastic contribution (diffusion) is almost constant. Considering both contributions together, our findings show that the tail of the volume-price distributions tend to evolve stochastically around an inverse square law with a constant parameter volatility.

This parameter volatility can be proposed as a risk measure for the expected tail of New York assets. The analysis propose here can be extended to other
markets or even in other contexts where non-stationary processes are observed. If the inverse Gamma distribution is commonly the best model for volume-price distributions is up to our knowledge an open question. The confidence of each model can be further tested using other methods such as the Kolmogorov-Smirnov test [10].

It must be noticed that the above approach is only valid for Markovian processes, which seems to be the case of the parameter here considered, which was tested comparing two-point and three-point conditional probabilities. Moreover, the Langevin analysis here proposed can also be extended to both parameters characterizing the inverse Gamma model. Further research will be necessary to access the reliability of the stochastic reconstruction of the volume-price evolution, and a comparison to theoretical agent models. These and other issues will be addressed elsewhere.

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