EXISTENCE OF GROUND STATE SOLUTIONS FOR CHOQUARD EQUATION INVOLVING THE GENERAL UPPER CRITICAL HARDY-LITTLEWOOD-SOBOLEV NONLINEAR TERM

Gui-Dong Li and Chun-Lei Tang*

School of Mathematics and Statistics
Southwest University, Chongqing 400715, China

(Communicated by Camil Muscalu)

Abstract. In this paper, we investigate the following a class of Choquard equation
\[-Δu + u = (I_α * F(u))f(u) \quad \text{in } \mathbb{R}^N,\]
where \(N \geq 3, \ α \in (0, N), \ I_α \) is the Riesz potential and \(F(s) = \int_0^s f(t)dt\). If \(f\) satisfies almost necessary the upper critical growth conditions in the spirit of Berestycki and Lions, we obtain the existence of positive radial ground state solution by using the Pohožaev manifold and the compactness lemma of Strauss.

1. Introduction and main result. In recent years, many authors considered the following Choquard equation
\[
\begin{aligned}
-Δu + u &= (I_α * F(u))f(u), \\
u &\in H^1(\mathbb{R}^N),
\end{aligned}
\]
where \(N \geq 3, \ I_α \) is the Riesz potential of order \(α \in (0, N)\) defined for every \(x \in \mathbb{R}^N \setminus \{0\}\) by
\[I_α = \frac{Γ(\frac{N-2}{2})}{2^απ^{\frac{N}{2}}Γ(\frac{α}{2})|x|^{N-α}},\]
\(F(s) = \int_0^s f(t)dt\) and \(f \in C(\mathbb{R}, \mathbb{R})\).

The Choquard equation (1) appeared in the contexts of various physical models. Especially, when \(N = 3, \ α = 2, \ F(u) = |u|^2\) and \(f(u) = u\) in equation (1), that is,
\[-Δu + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3,\]
which was used to describe the quantum theory of a polaron at rest by Pekar [22] in 1954. Later, equation (2) was proposed to describe as a certain approximation to Hartree-Fock theory of one-component plasma in the modeling of an electron trapped in its own hole by Choquard [12] in 1976. In the 1990s, the same equation reemerged as a model of self-gravitating matter [5, 10, 21, 23] and was known in that context as the Schrödinger-Newton equation.

2000 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Choquard equation, Pohožaev manifold, compactness lemma of Strauss, positive ground state solution, critical growth.

The research is supported by National Natural Science Foundation of China (No.11471267).

* Corresponding author.
For the case that \( F(s) = |s|^p \) with \( \frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{2} \) in problem (1), there exist lots of papers in which the authors considered the existence and qualitative properties of solutions, where the exponent \( p = \frac{N+\alpha}{N-2} \) (or \( \frac{N}{N-2} + 1 \)) is called the upper (or lower) critical exponent with respect to the Hardy-Littlewood-Sobolev inequality (see [14]), which we sketch here for the readers’ convenience. Assume that \( k \in L^r(\mathbb{R}^N) \) and \( h \in L^t(\mathbb{R}^N) \). Then one has
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{k(x)h(y)}{|x-y|^\alpha} dx dy \leq C_{s,t,\alpha} |k|_s |h|_t,
\]
in which \( 1 < s, t < +\infty, 0 < \alpha < N \) and \( \frac{1}{s} + \frac{1}{t} + \frac{\alpha}{N} = 2 \). Further results for related problems can be found in [11, 13, 16, 17, 19, 20, 25] and references therein.

While many researchers considered the autonomous Choquard equation involving general nonlinear term for recent years. This situation contrasts with the striking problems can be found in [11, 13, 16, 17, 19, 20, 25] and references therein.

In this spirit, we are interested in the existence of ground state solutions for problem (1). Moroz and Van Schaftingen [18] supposed that nonlinearity \( f \) satisfies the assumptions
\begin{align*}
(k_1) \quad & k \in C(\mathbb{R}, \mathbb{R}) \text{ is an odd function.} \\
(k_2) \quad & -\infty < \liminf_{s \to 0^+} \frac{k(s)-s}{s} \leq \limsup_{s \to 0^+} \frac{k(s)-s}{s} < 0.
\end{align*}
For the case that \( f \) is an odd function and has constant sign on \((0,\infty)\), their result reads as follows

\textbf{Theorem A.} Assume that \( N \geq 3, \alpha \in (0, N) \) and \((f_1) - (f_4)\) hold. Then problem (1) has a ground state solution. Furthermore, if \( f \) satisfies additional condition \((f_5)\), then every ground state solution of problem (1) has constant sign and is radially symmetric with respect to some point in \( \mathbb{R}^N \).

In the following, we are interested in the existence of ground state solutions for problem (1) with the upper critical nonlinear term in the present paper. For the critical case, we would like to mention [1, 6, 7, 8] and the references therein. [1] obtained the existence, multiplicity and concentration behavior of the semi-classical solutions of problem (1) with singularly perturbation. [7, 8] considered problem (1) in bounded domain. [6] considered problem (1) with periodic potential and nonlinearity. Based on the works above, we consider that \( f \) satisfies
\begin{align*}
(f_6) \quad & f \in C(\mathbb{R}, \mathbb{R}) \text{ is an odd function and has constant sign on } (0,\infty), \\
(f_7) \quad & \lim_{s \to 0} \frac{f(s)}{s^{\frac{\alpha}{N}}} = 0.
\end{align*}
Theorem 1.2. Hence theorem 1.1 is equivalent to the following Palais-Smale sequence has a converging subsequence. At last, the nonlinear term $g$ will overcome this difficulty by the compactness lemma of Strauss so that bounded $R$ we are working on infimum of functional restrained on the Pohozaev manifold (see Lemmas 2.2-2.5 in Remark 3.

(5) has a positive radial ground state solution.

Here is our main result.

Theorem 1.1. Assume that $N \geq 3$, $\alpha \in (0, N)$ and $(f_0) - (f_9)$ hold. Then problem (1) has a ground state solution, which has constant sign and is radially symmetric with respect to some point in $R^N$.

Remark 1. In this paper, we study the autonomous Choquard equation (1) with the assumptions of Theorem 1.1 which has never been investigated. There are several difficulties in our paper. The main one is the reformative condition which means nonlinear term $F$ is the upper critical growth. Contrasting with [18], this is what makes the present problem more complicated, and also means that their methods aren’t suited to our problem. To get over these difficulties, we employ a similar argument in [2, 15].

Remark 2. In fact, the continuous differentiability of functional can be ensured by (f_7) and (f_8). Moreover, it follows from (f_9) and (f_0) that $F \neq 0$ on $R^N$, which ensures that equation (1) has a nontrivial solution. At the same time, (f_9) also plays an important role in estimating the infimum of functional restrained on the Pohozaev manifold.

For simplicity, we may assume that $f \geq 0$ on $(0, +\infty)$, $\mu = 1$ and $f(s) = g(s) + \frac{2s}{s^2} |s|^{\frac{2\alpha}{\alpha-2}}$. Therefore equation (1) is equivalent to the following equation

$$-\Delta u + u = \left( I_\alpha * \left( G(u) + |u|^{\frac{2\alpha}{\alpha-2}} \right) \right) \left( g(u) + \frac{2s}{s^2} |u|^{\frac{2\alpha}{\alpha-2}} \right) \quad \text{in } R^N, \quad (5)$$

where $N \geq 3$, $2_\alpha^* = \frac{2(N+\alpha)}{N-2}$, $G(s) = \int_0^s g(t)dt$ and $g$ verifies the following conditions, $(g_1)$ $g \in C(R, R)$ is an odd function and for any $s \in (0, +\infty)$,

$$g(s) + \frac{N + \alpha}{N - 2} |s|^{(2+\alpha)/(N-2)} \geq 0.$$

$(g_2)$ $\lim_{s \to 0} \frac{g(s)}{s^{\frac{2\alpha}{\alpha-2}}} = \lim_{s \to +\infty} \frac{g(s)}{s^{\frac{2\alpha}{\alpha-2}}/|s|^{(2+\alpha)/(N-2)}} = 0$.

$(g_3)$ $\lim_{s \to +\infty} \frac{G(s)}{s^{\frac{(N+\alpha)/\alpha}{N}}} = +\infty$ for $N \geq 5$,

$$\lim_{s \to +\infty} \frac{G(s)}{s^{\frac{(N+\alpha)/\alpha}{N}}} = +\infty \quad \text{for } N = 4,$$

$$\lim_{s \to +\infty} \frac{G(s)}{s^{\frac{(N+\alpha)/\alpha}{N}}} = +\infty \quad \text{for } N = 3.$$

Hence theorem 1.1 is equivalent to the following

Theorem 1.2. Assume that $N \geq 3$, $\alpha \in (0, N)$ and $(g_1) - (g_3)$ hold. Then problem (5) has a positive radial ground state solution.

Remark 3. The main argument of the proof consists rather careful estimates the infimum of functional restrained on the Pohozaev manifold (see Lemmas 2.2-2.5 in section 2 for detail) which are much more precise than the ones seen so far. Besides, we are working on $R^N$ suggests that we may have to face a lack of compactness. We will overcome this difficulty by the compactness lemma of Strauss so that bounded Palais-Smale sequence has a converging subsequence. At last, the nonlinear term $g$
in our paper need not be differentiable, then the constrained manifold need not be of class $C^1$ in our case, which implies that the minimizing sequence of the infimum isn’t Palais-Smale sequence. Motivated by [2, 15], we apply a new approach to seek Palais-Smale sequence (see section 3 for detail).

The present paper is organized as follows. In section 2 we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries. From now on, we will use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm
  $$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx\right)^{1/2}.$$

- $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
  $$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{1/2}.$$

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm
  $$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{1/p} \quad \text{and} \quad |u|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)| \quad \text{for all} \quad p \in [1, +\infty).$$

- $H_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\}$.

- $\langle \cdot, \cdot \rangle$ denotes action of dual.

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| \leq r\}$ and $B_r := B_r(0)$.

- $C, C_i (i = 0, 1, 2, \cdots)$ denote various positive constants.

From $(f_6) - (f_8)$, for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that
$$|F(s)| \leq \delta |s|^{\frac{N+\alpha}{\alpha}} + C_\delta |s|^{\frac{N+\alpha}{N-\alpha}} \quad \text{for any} \quad s \in \mathbb{R}. \quad (6)$$

By the Hardy-Littlewood-Sobolev inequality [14] and Sobolev embedding, one has
$$\int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx \leq C_{N,\alpha} \left(\int_{\mathbb{R}^N} |F(u)|^{\frac{2N}{N+\alpha}} \, dx\right)^{\frac{N+\alpha}{N}}$$
$$\leq C_{N,\alpha} \left(\int_{\mathbb{R}^N} (\delta |u|^{\frac{N+\alpha}{\alpha}} + C_\delta |u|^{\frac{N+\alpha}{N-\alpha}}) \, dx\right)^{\frac{2N}{N+\alpha}}$$
$$< +\infty. \quad (7)$$

Then the energy functional $\mathcal{I} : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (1) is well defined by
$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx.$$

This also implies that $\mathcal{I}$ is $C^1$ functional whose derivative is given by
$$\langle \mathcal{I}'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + uv \, dx - \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) v \, dx$$
for all $v \in H^1(\mathbb{R}^N)$. Formally, the critical points of $\mathcal{I}$ are solutions for problem (1). We recall the Pohožaev manifold
$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : J(u) = 0\},$$
where

\[ J(u) = \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{N + \alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx. \]

In order to complete the proof, we give the following lemmas.

**Lemma 2.1.** Assume that \( N \geq 3, \alpha \in (0, N) \) and \((f_6) - (f_9)\) hold. Then

(a) there exists \( a \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that \( u \in \mathcal{P} \);
(b) for any \( u \in H^1(\mathbb{R}^N) \) satisfying \( \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx > 0 \), there exists a unique \( t_u \) such that \( u(xe^{-t_u}) \in \mathcal{P} \) and \( \mathcal{I}(u(xe^{-t_u})) = \max_{t \in \mathbb{R}} \mathcal{I}(u(xe^{-t})) \);
(c) \( \mathcal{I} \) is bounded from below on \( \mathcal{P} \) by a positive constant.

**Proof.** (a) From \((f_9)\), there exists \( s_0 > 0 \) such that \( F(s_0) \neq 0 \). Let \( w = s_0 \chi_{B_1} \), we obtain

\[ \int_{\mathbb{R}^N} (I_\alpha * F(w)) F(w) dx = F^2(s_0) \int_{B_1} \int_{B_1} I_\alpha(x - y) dy dx > 0. \] (8)

The left hand side of (8) is continuous in \( L^2(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) from \((f_6) - (f_9)\). Since \( H^1(\mathbb{R}^N) \) is dense in \( L^2(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \), there exists \( v \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that

\[ \int_{\mathbb{R}^N} (I_\alpha * F(v)) F(v) dx > 0. \]

We will take the function \( v \) in the family of functions \( v_t \in H^1(\mathbb{R}^N) \) defined for \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^N \) by

\[ v_t(x) := v(xe^{-t}). \]

On this family, we compute for every \( t \),

\[ J(v_t) = \frac{(N - 2)e^{(N-2)t}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2} e^{Nt} \int_{\mathbb{R}^N} v^2 dx - \frac{(N + \alpha)e^{(N+\alpha)t}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(v)) F(v) dx. \]

It is easy to see that there exists a constant \( R_1 > 0 \) such that \( J(v_t) > 0 \) as \( t < -R_1 \), and \( J(v_t) < 0 \) as \( t > R_1 \), which implies that there exists \( t_v \) such that \( J(v_{t_v}) = 0 \) and \( v_{t_v} \neq 0 \), namely \( u = v_{t_v} \).

(b) For any \( u \in H^1(\mathbb{R}^N) \) satisfying \( \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx > 0 \), it follows from the proof of (a) that there exists \( t_u \) such that \( J(u_{t_u}) = 0 \). Next we show the maximum point is unique.

For any \( t \in \mathbb{R} \), we set

\[ \Phi(t) := \mathcal{I}(u(xe^{-t})) \]

\[ = \frac{e^{(N-2)t}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{e^{Nt}}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{e^{(N+\alpha)t}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx. \] (9)

For the sake of convenience in writing, we define

\[ a = \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad b = \frac{N}{2} \int_{\mathbb{R}^N} u^2 dx \quad \text{and} \quad c = \frac{N + \alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) dx. \]
One can easily see that
\[ J(u_t) = \Phi'(t) = e^{(N-2)t}a + e^{Nt}b - e^{(N+\alpha)t}c = e^{(N-2)t} \left[ a + e^{2t}b - e^{(2+\alpha)t}c \right]. \]

Note that, if \( a \geq 0, b \geq 0 \) and \( c > 0 \), if \( b = 0 \), the maximum point is unique obviously.

If \( b > 0 \), we define \( \tau(t) := a + e^{2t}b - e^{(2+\alpha)t}c \), then \( \tau'(t) = e^{2t}[2b-(2+\alpha)e^{\alpha t}c] \). One gets that there exists a unique \( t' \) such that \( \tau'(t') = 0 \), which means that \( \tau(t) > 0 \) on \((-\infty, t')\) and \( \tau(t) < 0 \) on \((t', +\infty)\). Hence \( \tau(t) \) is strictly increasing on \((-\infty, t')\) and strictly decreasing on \((t', +\infty)\). Combining that \( \tau(t) \geq 0 \) as \( t \to -\infty \), and \( \tau(t) < 0 \) as \( t \to +\infty \), thus there exists a unique \( t_u \) such that \( \tau(t_u) = 0 \), really, which is equivalent to that there exists a unique \( t_u \) such that \( \Phi'(t_u) = 0 \). Moreover, \( \Phi(t_u) = \max_{t \in \mathbb{R}} \Phi(t) \), namely, \( I(u(xe^{-t_u})) = \max_{t \in \mathbb{R}} I(u(xe^{-t})) \).

(c) For \( u \in \mathcal{P} \), then \( J(u) = 0 \), it is quite clear that
\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 dx = \frac{N + \alpha}{2} \int_{\mathbb{R}^N} (I_{\alpha} * F(u)) F(u) dx. \tag{10}
\]
Combining (7) with (10), one obtains
\[
\frac{N-2}{2} \|u\|^2 \leq C_{N,\alpha} \left( \int_{\mathbb{R}^N} \left( \delta \|u\|^{\frac{N+\alpha}{N}} + C_{\delta} \|u\|^{\frac{N+\alpha}{N}} \right)^\frac{2N}{N+\alpha} dx \right)^\frac{N+\alpha}{N}.
\]
\[
\leq C_{N,\delta,\alpha} \left( \|u\|^{\frac{2(N+\alpha)}{N}} + \|u\|^{\frac{2(N+\alpha)}{N}} \right). \tag{11}
\]
In fact, we only discuss one case that \( \|u\| < 1 \). Notice that \( u \neq 0 \) from the definition of \( \mathcal{P} \), combining with (11) one gets \( \frac{N-2}{2} \|u\|^2 \leq 2C_{N,\delta,\alpha} \|u\|^{\frac{2(N+\alpha)}{N}} \), which means \( 1 \leq C_{\delta}' \|u\|^{\frac{2N}{N+\alpha}} \). Thus there exists \( \rho > 0 \) such that \( \|u\| \geq \rho \) and \( \rho \) is independent of \( u \). At the same time, it follows from (10) that
\[ I(u) = I(u) - \frac{1}{N+\alpha} J(u) \]
\[ = \frac{2 + \alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} u^2 dx \]
\[ \geq \frac{\alpha}{2(N+\alpha)} \rho^2. \]
It is quite clear that for any \( u \in \mathcal{P} \) such that \( I(u) \geq \frac{\alpha}{2(N+\alpha)} \rho^2 > 0 \). \( \square \)

Use \( S_{\alpha} \) to denote the best constant defined by
\[ S_{\alpha} := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_{\alpha} * |u|^2) |u|^\frac{2N}{N} dx = 1 \right\}. \]
From [6, 7, 8, 14, 19], we know that \( S_{\alpha} \) is achieved by
\[ U_{\varepsilon} = \frac{C_{\varepsilon} \varepsilon^{-\frac{N-2}{2}}}{(\varepsilon + |x|)^{\frac{N-2}{2}}}, \]
where \( C \) is a fixed positive constant and \( \varepsilon > 0 \). Let \( \psi \in C_0^{\infty}(\mathbb{R}^N, [0, 1]) \) be a cut-off function satisfying \( \psi = 1 \) for \( x \in B_{\bar{\rho}} \) and \( \psi = 0 \) for \( x \in \mathbb{R}^N \setminus B_{2\bar{\rho}} = 0 \), where \( \bar{\rho} \) is some positive constant. Define the test function by
\[ v_{\varepsilon} = \frac{u_{\varepsilon}}{\left( \int_{\mathbb{R}^N} (I_{\alpha} * |u_{\varepsilon}|^{\frac{2N}{N}}) |u_{\varepsilon}|^{\frac{2N}{N}} dx \right)^{\frac{1}{N}}}, \]
where \( u_\varepsilon = \psi U_\varepsilon \). According to [4, 8, 26], one obtains for \( \varepsilon \) small enough,
\[
\int_{\mathbb{R}^N} \left( I_\alpha * |v_\varepsilon|^{\frac{2N}{N-\alpha}} \right) |v_\varepsilon|^{\frac{2N}{N-\alpha}} \, dx = 1, \quad \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx = S_\alpha + O(\varepsilon^{\frac{N-2}{2}}),
\]
and
\[
\gamma(\varepsilon) = \int_{\mathbb{R}^N} |v_\varepsilon|^2 \, dx = \begin{cases} O(\varepsilon), & N \geq 5; \\ O(\varepsilon^{\frac{1}{2}}), & N = 3, \end{cases}
\]
and
\[
\int_{\mathbb{R}^N} |u_\varepsilon|^2 \, dx \leq l, \quad \left( \int_{\mathbb{R}^N} \left( I_\alpha * |u_\varepsilon|^{\frac{2N}{N-\alpha}} \right) |u_\varepsilon|^{\frac{2N}{N-\alpha}} \, dx \right)^{\frac{N}{2N}} \leq l,
\]
where \( l \) is some positive constant.

**Lemma 2.2.** Suppose that \( N \geq 3, \alpha \in (0, N) \) and \((g_1) - (g_3)\) hold. Then there exists a positive constant \( C_4 \) such that, for \( \varepsilon \) small enough,
\[
-2C_4 \gamma(\varepsilon)\varepsilon^{\frac{N}{N-\alpha}} \leq \int_{\mathbb{R}^N} (I_\alpha * G(v_\varepsilon)) G(v_\varepsilon) \, dx.
\]

**Proof.** Let \( G^+(s) = \max\{G(s), 0\} \) and \( G^-(s) = \min\{G(s), 0\} \) for any \( s \in \mathbb{R} \). Indeed, from (7) and (13) we have for \( \varepsilon > 0 \) small enough,
\[
\int_{\mathbb{R}^N} |G(v_\varepsilon)|^\frac{N+\alpha}{N} \, dx \leq \int_{\mathbb{R}^N} \delta |v_\varepsilon|^2 + C_6 |v_\varepsilon|^{2^*} \, dx \leq C_1.
\]
On the other hand, it follows from \((g_1) - (g_3)\) that there exists \( C_2 \) such that
\[
G(s) \geq -C_2|s|^{\frac{N+\alpha}{N}} \quad \text{for any } s \in \mathbb{R},
\]
which also implies that \( G^-(s) \geq -C_2|s|^{\frac{N+\alpha}{N}} \) for any \( s \in \mathbb{R} \). According to (3), (12) and (14), one obtains
\[
\begin{align*}
\int_{\mathbb{R}^N} (I_\alpha * G^-(v_\varepsilon)) G^+(v_\varepsilon) \, dx & \geq -C_3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y)|v_\varepsilon(y)|^\frac{N+\alpha}{N} G^+(v_\varepsilon(x)) \, dydx \\
& \geq -C_3 \left( \int_{\mathbb{R}^N} |v_\varepsilon|^2 \, dy \right)^\frac{N+\alpha}{2N} \left( \int_{\mathbb{R}^N} |G^+(v_\varepsilon)|^\frac{2N}{N+\alpha} \, dx \right)^\frac{N+\alpha}{2N} \\
& \geq -C_4 \gamma(\varepsilon)^\frac{N}{N+\alpha}.
\end{align*}
\]
At the same time, from the Fubini theorem we have
\[
\begin{align*}
\int_{\mathbb{R}^N} (I_\alpha * G(v_\varepsilon)) G(v_\varepsilon) \, dx & = \int_{\mathbb{R}^N} (I_\alpha * G^-(v_\varepsilon)) G^-(v_\varepsilon) \, dx + \int_{\mathbb{R}^N} (I_\alpha * G^+(v_\varepsilon)) G^+(v_\varepsilon) \, dx \\
& + 2 \int_{\mathbb{R}^N} (I_\alpha * G^-(v_\varepsilon)) G^+(v_\varepsilon) \, dx \\
& \geq 2 \int_{\mathbb{R}^N} (I_\alpha * G^-(v_\varepsilon)) G^+(v_\varepsilon) \, dx \\
& \geq -2C_4 \gamma(\varepsilon)^\frac{N}{N+\alpha}.
\end{align*}
\]
This completes the proof. \( \Box \)
Lemma 2.3. Suppose that $N \geq 3$, $\alpha \in (0, N)$ and $(g_1) - (g_3)$ hold. Then
\[
\lim_{\varepsilon \to 0} \frac{\int_{|x|<\varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx}{\gamma(\varepsilon)} = +\infty,
\]
and for $\varepsilon > 0$ small enough there exists a positive constant $C_\varepsilon$ such that
\[
-C_\varepsilon \gamma(\varepsilon)^{\frac{N-\alpha}{2N}} \leq \int_{|x| \geq \varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx.
\]

Proof. Note that, for $\varepsilon$ small enough one has
\[
v_\varepsilon \geq \frac{1}{l} u_\varepsilon \geq \frac{1}{l} C \frac{C_\varepsilon^{\frac{N-\alpha}{2}}}{(\varepsilon + |x|^2)^{\frac{N-\alpha}{2}}} \geq C_\varepsilon^{-\frac{N-\alpha}{4}}.
\]
(16)

It follows from $(g_3)$ that there exists $A_R > 0$ such that for all $s \in [A_R, +\infty),$ \[G(s) \geq \begin{cases} R s^{\frac{N-\alpha}{2}}, & N \geq 5; \\ R (s^2 \ln s)^{\frac{N-\alpha}{2}}, & N = 4; \\ R s^{2(\frac{N-\alpha}{2})}, & N = 3. \end{cases} \]

Combining with (16), when $\varepsilon$ small enough one obtains for $N \geq 5,$
\[
\int_{|x|<\varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx \\
\geq C_\varepsilon R \int_{|x|<\varepsilon^\frac{1}{2}} \int_{|y|<\varepsilon^\frac{1}{2}} I_\alpha(x-y)|v_\varepsilon \frac{\varepsilon}{2} G(v_\varepsilon) dydx \\
\geq C_\varepsilon R \left[ 2\varepsilon^\frac{1}{2} \left| \alpha - N \right| \left| \varepsilon \right|^2 \right]^{2N} \varepsilon^{-\frac{(N-\alpha)(N-2)}{4N}} \\
\geq 2^{\frac{N-N}{2}} C_\varepsilon R \varepsilon^{\frac{N-\alpha}{2N}},
\]
for $N = 4,$
\[
\int_{|x|<\varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx \\
\geq C_\varepsilon R \int_{|x|<\varepsilon^\frac{1}{2}} \int_{|y|<\varepsilon^\frac{1}{2}} I_\alpha(x-y) \varepsilon^{-\frac{N-\alpha}{2}} \varepsilon^{-1} \ln \varepsilon^{-\frac{1}{2}} dydx \\
\geq C_\varepsilon R \left[ 2\varepsilon^\frac{1}{2} \left| \alpha - 4 \right| \left| \varepsilon \right|^3 \right]^{2N} \varepsilon^{-\frac{(N-\alpha)(N-2)}{8N}} \\
\geq 2^{N-4} C_\varepsilon R (\varepsilon \ln \varepsilon^{-\frac{1}{2}})^{\frac{N-\alpha}{2}} \\
\]
and for $N = 3,$
\[
\int_{|x|<\varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx \\
\geq C_\varepsilon R \int_{|x|<\varepsilon^\frac{1}{2}} \int_{|y|<\varepsilon^\frac{1}{2}} I_\alpha(x-y) \varepsilon^{-\frac{3+\alpha}{4}} \varepsilon^{-\frac{3+\alpha}{8}} dydx \\
\geq C_\varepsilon R \left[ 2\varepsilon^\frac{1}{2} \left| \alpha - 3 \right| \left| \varepsilon \right|^6 \right]^{2N} \varepsilon^{-\frac{(N-\alpha)(N-2)}{16N}} \\
\geq 2^{N-3} C_\varepsilon R \varepsilon^{\frac{3+\alpha}{4}}.
\]

By the arbitrariness of $R$, we get
\[
\lim_{\varepsilon \to 0} \frac{\int_{|x|<\varepsilon^\frac{1}{2}} \left( I_\alpha |v_\varepsilon \frac{\varepsilon}{2} \right) G(v_\varepsilon) dx}{\gamma(\varepsilon)} = +\infty.
\]
On the other hand, combining (3), (14) and (15) one has
\[
\int_{|x| \geq \varepsilon^{1/2}} \left( I_\alpha * |v_\varepsilon|^{2^*_\alpha} \right) G(v_\varepsilon) dx \geq \int_{|x| \geq \varepsilon^{1/2}} \int_{|y| \geq \varepsilon^{1/2}} I_\alpha(x-y) |v_\varepsilon(y)|^{2^*_\alpha} G(v_\varepsilon(x)) dy dx
\]
\[
\geq -C_7 \left( \int_{\mathbb{R}^N} |v_\varepsilon|^2 dy \right)^{\frac{N+\alpha}{2\alpha}} \left( \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*_\alpha} dx \right)^{\frac{N+\alpha}{2\alpha}}
\]
\[
\geq -C_8 |\varepsilon|^{\frac{N+\alpha}{\alpha}}.
\]

This completes the proof. □

**Lemma 2.4.** Suppose that \( N \geq 3, \alpha \in (0, N) \) and \((g_1)-(g_3)\) hold. Then there exists \( t_{v_\varepsilon} \in \mathbb{R} \) such that \( v_\varepsilon(xe^{-t_{v_\varepsilon}}) \in \mathcal{P} \) and \( -\infty < T_* \leq t_{v_\varepsilon} \leq T^* < +\infty \) for \( \varepsilon \) small enough.

**Proof.** Note that, it follows from Lemmas 2.2 and 2.3 that for \( \varepsilon \) small enough,
\[
\int_{\mathbb{R}^N} (I_\alpha * F(v_\varepsilon)) F(v_\varepsilon) dx > 0. \tag{17}
\]
Then there exists an unique \( t_{v_\varepsilon} \) such that \( v_\varepsilon(xe^{-t_{v_\varepsilon}}) \in \mathcal{P} \) from Lemma 2.1.

Next, we seek that \( T_* \leq t_{v_\varepsilon} \leq T^* \) for \( \varepsilon > 0 \) small enough. By contradiction, suppose that \( t_{v_\varepsilon} \to -\infty \) as \( \varepsilon \to 0 \). Then \( \mathcal{I}(v_\varepsilon(xe^{-t_{v_\varepsilon}})) \to 0 \) from (9), which is in contradiction with Lemma 2.1. Thus there exists \( T_* \in \mathbb{R} \) such that \( t_{v_\varepsilon} \geq T_* \) for \( \varepsilon \) small enough.

Suppose that \( t_{v_\varepsilon} \to +\infty \) as \( \varepsilon \to 0 \). Hence \( \mathcal{I}(v_\varepsilon(xe^{-t_{v_\varepsilon}})) \to -\infty \) from (9), which also contradicts Lemma 2.1. Thus there exists \( T^* \in \mathbb{R} \) such that \( t_{v_\varepsilon} \leq T^* \) for \( \varepsilon \) small enough. This completes the proof. □

**Lemma 2.5.** Suppose that \( N \geq 3, \alpha \in (0, N) \) and \((g_1)-(g_3)\) hold. Then \( \inf_{u \in \mathcal{P}} \mathcal{I}(u) < S_* \), where
\[
S_* = \left( \frac{N-2}{N} \right)^{\frac{N-2}{2}} \left( \frac{\alpha + 2}{2(N + \alpha)} \right)^{\frac{N+\alpha}{\alpha}} S_0^{\frac{N+\alpha}{\alpha}}.
\]

**Proof.** From Lemma 2.1, we can easily see that there exists a positive constant \( m \) such that \( m = \inf_{u \in \mathcal{P}} \mathcal{I}(u) \). Now, we define for \( t \in \mathbb{R},
\[
\Psi(t) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon(xe^{-t})|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( I_\alpha * |v_\varepsilon(xe^{-t})|^{2^*_\alpha} \right) |v_\varepsilon(xe^{-t})|^{2^*_\alpha} dx
\]
\[
= \frac{e^{(N-2)t}}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx - \frac{e^{(N+\alpha)t}}{2}.
\]
Then one can easily see that when \( t_* = \frac{\ln \left( \frac{\alpha}{\alpha+2} \left( \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx \right)^{\frac{\alpha+2}{\alpha+2}} \right)}{\alpha+2} \),
\[
\Psi(t_*) = \sup_{t \in \mathbb{R}} \Psi(t)
\]
\[
= \left( \frac{N-2}{N+\alpha} \right)^{\frac{N-2}{2}} \left( \frac{\alpha + 2}{2(N + \alpha)} \right)^{\frac{N+\alpha}{\alpha}} \left( \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx \right)^{\frac{N+\alpha}{\alpha}}
\]
\[
= S_* + O \left( \varepsilon^{\frac{N-2}{2}} \right).
\]
At the same time, we recall that there exists $t_\varepsilon$ such that $v_\varepsilon(xe^{-t_\varepsilon}) \in P$ from (17). Combining with the Fubini theorem, we obtain

$$m \leq I(v_\varepsilon(xe^{-t_\varepsilon}))$$

$$= \frac{e^{(N-2)t_\varepsilon}}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx + \frac{e^{Nt_\varepsilon}}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx$$

$$- \frac{e^{(N+\alpha)t_\varepsilon}}{2} \int_{\mathbb{R}^N} \left( I_\alpha * (G(v_\varepsilon) + |v_\varepsilon|^{2^*_N}) \right) \left( G(v_\varepsilon) + |v_\varepsilon|^{2^*_N} \right) dx$$

$$= \Psi(t_\varepsilon) + \frac{e^{Nt_\varepsilon}}{2} \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx - \frac{e^{(N+\alpha)t_\varepsilon}}{2} \int_{\mathbb{R}^N} \left( I_\alpha * |v_\varepsilon|^{2^*_N} \right) G(v_\varepsilon) dx$$

$$\leq S^* + O(e^{-\beta^*_N}) + \frac{e^{Nt_\varepsilon}}{2} \gamma(\varepsilon) - \frac{e^{(N+\alpha)t_\varepsilon}}{2} \int_{\mathbb{R}^N} \left( I_\alpha * G(v_\varepsilon) \right) |v_\varepsilon|^{2^*_N} dx$$

$$- \frac{e^{(N+\alpha)t_\varepsilon}}{2} \int_{\mathbb{R}^N} \left( I_\alpha * G(v_\varepsilon) \right) G(v_\varepsilon) dx.$$  

According to Lemmas 2.2-2.4, we have $m \leq S_*$ for $\varepsilon$ small enough. This completes the proof.  

In order to prove the positive of solutions, the following propositions obtained by Moroz and Van Schaftingen [18] is of great significance, which relies on a nonlocal version of the Brézis-Kato estimate [3]. We give a sketch here for the reader’s convenience and omit their proof.

**Proposition 1.** If $H, K \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) + L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $u \in H^1(\mathbb{R}^N)$ solves

$$-\Delta u + u = (I_\alpha * Hu)K,$$  

then $u \in L^p(\mathbb{R}^N)$ for every $p \in \left(2, \frac{2N}{N-2}\right)$. Moreover, there exists a constant $C_p$ independent of $u$ such that

$$\left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}} \leq C_p \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$  

**Proposition 2.** Assume that $N \geq 3$, $\alpha \in (0, N)$ and $(f_6) - (f_9)$ hold. If $u \in H^1(\mathbb{R}^N)$ solves problem (1), then $J(u) = 0$.

3. **Proof of Theorem 1.2.** In order to complete the proof of Theorem 1.2, we are inspired by [4, 15] to introduce the following equation

$$-\Delta u + u = (I_\alpha * F_q(u)) F_q(u),$$  

where $F_q(t) = \int_0^t f_q(s)ds$ and $f_q(s) = g(s) + q|s|^{q-2}s + p\varphi(q)|s|^{p-2}s$ with $p, q \in (\frac{N+\alpha}{N}, \frac{2N}{N+\alpha})$, $\varphi(q) \geq 0$. Then the associated energy functional is

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F_q(u)) F_q(u) dx,$$  

and the corresponding Pohožaev manifold is

$$P_q = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : J_q(u) = 0 \right\},$$
where

\[ J_q(u) = \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{N + \alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F_q(u)) F_q(u) \, dx. \]

We define

\[ m_q = \inf_{u \in \mathcal{P}_q} \mathcal{I}_q(u). \]

It follows from \((g_1)-(g_2)\) that there exists \(\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)\) such that \(f_q \geq 0\) on \((0, +\infty)\) and \(\varphi(q) \to 0\) as \(q \to 2^*_\alpha/2\). According to Theorem A, it is quite clear that there exists a positive radial ground state solution \(u_q \in H^1(\mathbb{R}^N)\) of problem (18).

**Lemma 3.1.** Suppose that \(N \geq 3\), \(\alpha \in (0, N)\) and \((g_1)-(g_3)\) hold. Then

\[ \limsup_{q \to 2^*_\alpha/2} m_q \leq m. \]

**Proof.** For any \(\varepsilon \in (0, \frac{1}{2})\), there exists \(u \in \mathcal{P}\) such that \(\mathcal{I}(u) < m + \varepsilon\). Taking the function \(u\) in the family of functions \(u_t \in H^1(\mathbb{R}^N)\) defined for \(t \in \mathbb{R}\) and \(x \in \mathbb{R}^N\) by

\[ u_t(x) := u(xe^{-t}). \]

From lemma 2.1, there exists \(T \in \mathbb{R}\) such that

\[ \mathcal{I}(u_T) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_T|^2 + u_T^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_T)) F(u_T) \, dx \leq -1. \]

Since the continuity of \(\int_{\mathbb{R}^N} (I_\alpha * F_q(u_t)) F_q(u_t) \, dx\) on \((t, q) \in \mathbb{R} \times \left[\frac{N+\alpha}{N}, \frac{2^*_\alpha}{2}\right]\), there exists \(\theta > 0\) such that for all \(\frac{2^*_\alpha}{2} - \theta < q < \frac{2^*_\alpha}{2}\) and \(t \leq T\),

\[ |\mathcal{I}_q(u_t) - \mathcal{I}(u_t)| = \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F_q(u_t)) F_q(u_t) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_t)) F(u_t) \, dx \leq \varepsilon, \]

which states that \(\mathcal{I}_q(u_T) \leq -\frac{1}{2}\) for all \(\frac{2^*_\alpha}{2} - \theta < q < \frac{2^*_\alpha}{2}\). By Lemma 2.1, it is quite clear that there exists \(T_1 > |T|\) such that \(\mathcal{I}_q(u_t) > 0\) for \(t < -T_1\). Therefore there exists \(t_q^* \in (-T_1, T)\) such that \(\frac{d}{dt}\mathcal{I}_q(u_t)|_{t=t_q^*} = 0\) and then \(u_{t_q^*} \in \mathcal{P}_q\). From Lemma 2.1, one has \(\mathcal{I}(u_{t_q^*}) \leq \mathcal{I}(u)\). Thus for all \(\frac{2^*_\alpha}{2} - \theta < q < \frac{2^*_\alpha}{2}\),

\[ m_q \leq \mathcal{I}_q(u_{t_q^*}) \leq \mathcal{I}(u_{t_q^*}) + \varepsilon \leq \mathcal{I}(u) + \varepsilon < m + 2\varepsilon \]

which implies that \(\limsup_{q \to 2^*_\alpha/2} m_q \leq m\). \(\square\)

Note that, for sequence \(\{q_n\} \subset \mathbb{R}\), it follows from Theorem A that there exists a positive and radial symmetric sequence \(\{u_n\} \subset H^1(\mathbb{R}^N)\) such that

\[ \mathcal{I}_{q_n}'(u_n) = 0 \quad \text{and} \quad \mathcal{I}_{q_n}(u_n) = m_{q_n}, \]

where \(q_n \in \left(\frac{N+\alpha}{N}, \frac{2^*_\alpha}{2}\right)\) and \(q_n \to \frac{2^*_\alpha}{2}\) as \(n \to +\infty\).

**Lemma 3.2.** Suppose that \(\{u_n\} \subset H^1(\mathbb{R}^N)\) satisfies (19). Then \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^N)\) and \(\liminf_{n \to +\infty} m_{q_n} > 0\).
Proof. As a matter of fact, from (10) and Lemma 3.1, it is easy to see that

\[ m + 1 > m_{q_n} = \mathcal{I}_{q_n}(u_n) = \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} u_n^2 dx \geq \frac{\alpha}{2(N + \alpha)} \|u_n\|^2, \]

which implies that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). Indeed, from lemma 2.1 there exists \( \rho > 0 \) such that \( \|u_n\| \geq \rho \), we obtain

\[ m_{q_n} \geq \frac{\alpha}{2(N + \alpha)} \|u_n\|^2 \geq \frac{\alpha}{2(N + \alpha)} \rho^2. \]

Hence \( \liminf_{n \to +\infty} m_{q_n} > 0 \). This completes the proof. \( \square \)

To prove Theorem 1.2, we have to overcome the lack of compactness. Then the following two lemmas are central to our proof, which we sketch here for the readers’ convenience.

Lemma 3.3 (Compactness lemma of Strauss, see [2, 24]). Let \( P, Q : \mathbb{R} \to \mathbb{R} \) be two continuous functions satisfying

\[ \frac{P(s)}{Q(s)} \to 0 \text{ as } |s| \to +\infty. \]

Let \( \{u_n\} \) be a sequence of measurable functions: \( \mathbb{R}^N \to \mathbb{R} \) such that

\[ \sup_n \int_{\mathbb{R}^N} |Q(u_n)|dx < +\infty \]

and \( P(u_n(x)) \to v(x) \) a.e. in \( \mathbb{R}^N \), as \( n \to +\infty \). Then for any bounded Borel set \( B \) one has

\[ \int_B |P(u_n(x)) - v(x)|dx \to 0 \text{ as } n \to +\infty. \]

If one further assumes that

\[ \frac{P(s)}{Q(s)} \to 0 \text{ as } |s| \to 0 \]

and \( u_n(x) \to 0 \) as \( |x| \to +\infty \), uniformly with respect to \( n \), then \( P(u_n(x)) \) converges to \( v \) in \( L^1(\mathbb{R}^N) \) as \( n \to +\infty \).

Lemma 3.4 (Strauss inequality, see [26]). If \( N \geq 2 \), there exists \( C_N > 0 \) such that

\[ |u(x)| \leq C_N |u|^\frac{1}{2} \left| \nabla u \right|^\frac{1}{2} |x|^\frac{1-N}{2} \text{ a.e. in } \mathbb{R}^N \]

for every \( u(x) = u(|x|) \in H^1(\mathbb{R}^N) \).

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Recall that \( \{u_n\} \subset H^1(\mathbb{R}^N) \) in (19). Lemma 3.2 implies that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). Then there exists a subsequence, still denoted \( \{u_n\} \), such that

- \( u_n \to u \in H^1(\mathbb{R}^N) \);
- \( u_n \to u \in L^p(\mathbb{R}^N) \) for any \( p \in (2, 2^*) \);
- \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \),
where $u \in H^1(\mathbb{R}^N)$ is a nonnegative and radially symmetric function obviously. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, one has

$$0 = \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \phi + u_n \phi dx - \int_{\mathbb{R}^N} (I_\alpha * F_q(u_n)) f_q(u_n) \phi dx + o(1)$$

$$= \int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi + u \phi dx - \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) \phi dx,$$

which means that $u$ is a solution of problem (5). We claim that $u \neq 0$. By contradiction, suppose that $u = 0$.

Set $P(s) = |G(s)|^{\frac{2N}{N + \alpha}}$ and $Q(s) = |s|^2 + |s|^2$. It follows from $(g_2)$ and $(g_3)$ that

$$\lim_{|s| \to 0} P(s) = 0 \quad \text{and} \quad \lim_{|s| \to +\infty} P(s) = 0.$$

Obviously, $P(u(x)) \to 0$ a.e. in $H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} Q(u) dx \leq C$. Moreover, Lemma 3.4 states that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to $n$. Then Lemma 3.3 implies that $\int_{\mathbb{R}^N} P(u_n) dx \to 0$ as $n \to +\infty$. By the Hardy-Littlewood-Sobolev inequality and Sobolev embedding, we have as $n \to +\infty$,

$$\int_{\mathbb{R}^N} (I_\alpha \ast G(u_n)) |u_n |^q dx = \left( \int_{\mathbb{R}^N} |G(u_n)|^{\frac{2N}{N + \alpha}} dx \right) \left( \int_{\mathbb{R}^N} |u_n|^{\frac{2Nq}{N + \alpha}} dx \right)^{\frac{N + \alpha}{2N}}$$

$$= o(1).$$

As the above process, set $P_1(s) = |g(s)s|^{\frac{2N}{2N + \alpha}}$, we also get that $\int_{\mathbb{R}^N} P_1(u_n) dx \to 0$ as $n \to +\infty$, and then

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u_n |^q g(u_n)) u_n dx = o(1).$$

Indeed, it is easy to see that as $n \to +\infty$,

$$\int_{\mathbb{R}^N} (I_\alpha \ast G(u_n)) g(u_n) u_n dx = o(1).$$

By the Young inequality, one gets

$$|u_n |^q \leq \frac{\frac{2N}{2N + \alpha} \left( \frac{2N}{2N + \alpha} - q_n \right)}{\frac{2N}{2N + \alpha} - \frac{2N}{2N + \alpha}} + \frac{\frac{q_n - \frac{2N}{2N + \alpha}}{2N - \frac{2N}{2N + \alpha}}}{\frac{q_n - \frac{2N}{2N + \alpha}}{2N - \frac{2N}{2N + \alpha}}} |u_n |^q.$$

Hence, for $q_n \to \frac{2\alpha}{N}$ as $n \to +\infty$, one sees

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u_n |^q) |u_n |^{q_n} dx \leq \int_{\mathbb{R}^N} \left( I_\alpha \ast |u_n |^{\frac{2\alpha}{N}} \right) |u_n |^{\frac{2\alpha}{N}} dx + o(1)$$

$$\leq S_\alpha^{\frac{2\alpha}{N}} \left( \int_{\mathbb{R}^N} |\nabla u_n |^2 dx \right)^{\frac{2\alpha}{N}} + o(1).$$
Note that, \((\mathcal{I}_n'(u_n), u_n) = 0\) and \(\varphi(q_n) \to 0\) as \(n \to +\infty\). So we obtain
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 + u_n^2 dx
\]
\[
= \int_{\mathbb{R}^N} (\mathcal{I}_n * (G(u_n) + |u_n|^q + \varphi(q_n)|u_n|^p)) (g(u_n)u_n + q_n|u_n|^q + p\varphi(q_n)|u_n|^p) dx
\]
\[
\leq \frac{2s}{2} \int_{\mathbb{R}^N} \left( I_n * |u_n|^\frac{2s}{s-2} \right) |u_n|^\frac{2s}{s-2} dx + o(1)
\]
\[
\leq \frac{2a}{2} \frac{\mathcal{S}^\frac{2s}{s-2}}{\mathcal{S}^\frac{2s}{s-2}} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^\frac{2s}{s-2} + o(1).
\]
This deduces that
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq \frac{2a}{2} \mathcal{S}^\frac{2s}{s-2} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^\frac{2s}{s-2} + o(1).
\]
Therefore we have either
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 0
\]
or
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \left( \frac{N-2}{N+\alpha} \right)^\frac{N-2}{s-2} \mathcal{S}^\frac{N+\alpha}{s+\alpha}.
\]
If \(\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 0\), from (20) we have \(\lim_{n \to +\infty} \|u_n\| = 0\). Combining (10) and (19), one has
\[
\lim_{n \to +\infty} m_{q_n} = \lim_{n \to +\infty} \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} u_n^2 dx = 0,
\]
which is in contradiction with Lemma 3.2. If \(\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \left( \frac{N-2}{N+\alpha} \right)^\frac{N-2}{s-2} \mathcal{S}^\frac{N+\alpha}{s+\alpha}\). From Lemma 3.1, one can easily see that
\[
m \geq \limsup_{n \to +\infty} m_{q_n}
\]
\[
\geq \lim_{n \to +\infty} \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} u_n^2 dx
\]
\[
\geq S^*.
\]
which contradicts Lemma 2.5. In either case, we arrive at a contradiction. So \(u \neq 0\).

Since \(I'(u) = 0\), by the weaker lower semi-continuity of the norm, we obtain
\[
m \leq I(u)
\]
\[
= \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} u^2 dx
\]
\[
\leq \liminf_{n \to +\infty} \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} u_n^2 dx
\]
\[
= \liminf_{n \to +\infty} m_{q_n}
\]
\[
\leq m,
\]
which implies that \( I(u) = m \). Moreover, \( u \in H^1(\mathbb{R}^N) \) is a nonnegative and radially symmetric function obviously. Then, to complete the proof of Theorem 1.2, we should seek that \( u \) is positive. Define \( H : \mathbb{R}^N \to \mathbb{R} \) and \( K : \mathbb{R}^N \to \mathbb{R} \) for \( x \in \mathbb{R}^N \) by \( H(x) = \frac{F(u(x))}{u(x)} \) and \( K(x) = f(u(x)) \). Observe that for every \( x \in \mathbb{R}^N \), we have
\[
|K(x)| \leq C \left( |u(x)|^\frac{p}{N} + |u(x)|^{\frac{N+2}{N}} \right)
\]
and
\[
|H(x)| \leq C \left( |u(x)|^\frac{p}{N} + |u(x)|^{\frac{N+2}{N}} \right).
\]
By Proposition 1, we obtain that \( u \in L^p(\mathbb{R}^N) \) for every \( p \in [2, \frac{2N}{N+2}] \). In view of \((g_1) - (g_2)\), one gets \( F(u) \in L^p \) for every \( p \in [\frac{2N}{N+2}, \frac{N}{N+2}] \) (see [19, 18]). It follows from the Hardy-Littlewood-Sobolev inequality that
\[
\int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{N}{N+\alpha}} \, dx \leq \left( \int_{\mathbb{R}^N} |F(u)|^s \, dx \right)^{\frac{N}{N-s}}.
\]
Since \( \frac{2N}{N+\alpha} < \frac{N}{\alpha} < \frac{2N}{N+\alpha} \), we have
\[
I_\alpha * F(u) \in L^{\infty}(\mathbb{R}^N),
\]
and thus
\[
| - \Delta u + u | \leq C \left( |u|^{\frac{p}{N}} + |u|^{\frac{N+2}{N}} \right).
\]
By the classical bootstrap method for subcritical local problems in bounded domains, we deduce that \( u \in L^{\infty}_{loc}(\mathbb{R}^N) \). Combining with the strong maximum principle [9, 19, 18], we know that \( u \) is positive. According to Proposition 2, we may conclude that \( u \) is a positive and radial ground state solution of problem (4). This completes the proof. \( \square \)

Acknowledgments. The authors would like to express sincere thanks to the referees and the handling editor whose careful reading of the manuscript and valuable comments greatly improve the original manuscript.

REFERENCES

[1] C. O. Alves, F. Gao, M. Squassina and M. Yang, Singularly perturbed critical Choquard equations, J. Differential Equations, 263 (2017), 3943–3988.
[2] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal., 82 (1983), 313–345.
[3] H. Brézis and T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. (9), 58 (1979), 137–151.
[4] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math., 36 (1983), 437–477.
[5] J. Chabrowski, On the existence of G-symmetric entire solutions for semilinear elliptic equations, Rend. Circ. Mat. Palermo (2), 41 (1992), 433–440.
[6] F. Gao and M. Yang, A strongly indefinite Choquard equation with critical exponent due to Hardy-Littlewood-Sobolev inequality, Commun. Contemp. Math.
[7] F. Gao and M. Yang, On nonlocal Choquard equations with Hardy-Littlewood-Sobolev critical exponents, J. Math. Anal. Appl., 448 (2017), 1006–1041.
[8] F. Gao and M. Yang, On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation, Sci China Math.
[9] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition. Classics in Mathematics. Springer–Verlag, Berlin, 2001.
[10] K. R. W. Jones, Newtonian quantum gravity, Australian J. Phys., 48 (1995), 1055–1081.
[11] T. Küpper, Z. Zhang and H. Xia, Multiple positive solutions and bifurcation for an equation related to Choquard’s equation, Proc. Edinb. Math. Soc. (2), 46 (2003), 597–607.
[12] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Studies in Appl. Math.*, 57 (1976/77), 93–105.
[13] E. H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, *Ann. of Math. (2)*, 118 (1983), 349–374.
[14] E. H. Lieb and M. Loss, *Analysis*, 2nd edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
[15] J. Liu, J. F. Liao and C. L. Tang, Ground state solution for a class of Schrödinger equations involving general critical growth term, *Nonlinearity*, 30 (2017), 899–911.
[16] P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.*, 4 (1980), 1063–1072.
[17] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, 195 (2010), 455–467.
[18] V. Moroz and J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.*, 367 (2015), 6557–6579.
[19] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, 265 (2013), 153–184.
[20] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, *Commun. Contemp. Math.*, 17 (2015), 12 pp.
[21] I. M. Moroz, R. Penrose and P. Tod, Spherically–symmetric solutions of the Schrödinger–Newton equations, *Classical Quantum Gravity*, 15 (1998), 2733–2742.
[22] S. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin. 1954.
[23] R. Penrose, On gravity’s role in quantum state reduction, *Gen. Relativity Gravitation*, 28 (1996), 581–600.
[24] W. A. Struwe, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, 55 (1977), 149–162.
[25] P. Tod and I. M. Moroz, An analytical approach to the Schrödinger-Newton equations, *Nonlinearity*, 12 (1999), 201–216.
[26] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

Received November 2017; revised November 2017.

E-mail address: 190799542@qq.com
E-mail address: tangcl@swu.edu.cn