On the c-entropy of L-systems with Schrödinger operator

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In loving memory of Moshe Livšic

Abstract. We study L-systems whose main operators are extensions of one-dimensional half-line Schrödinger operators with deficiency indices $(1, 1)$, the Schrödinger L-systems. Introducing new concepts of an c-entropy and dissipation coefficient for an L-system we discuss the following dual problems: describe Schrödinger L-systems (1) with a given c-entropy and minimal dissipation coefficient, and (2) with a given dissipation coefficient and maximal c-entropy. Also, we analyze in detail the dual c-entropy problems for Schrödinger L-systems with sectorial and extremal main operators.

This paper is dedicated to the memory of Moshe Livšic, a remarkable Human Being and Mathematician. His pioneering research in the theory of non-selfadjoint operators and system theory [16] has made writing this paper possible. The following lines from a Russian poet Nikolaj Gumilev (Prophets, 1905) describe the life and the light of scientific accomplishments of Livšic really well:

И ныне есть еще пророки,
Хотя упали алтари,
Их очи ясны и глубоки,
Грядущим пламенем зари.

1. Introduction

In the current paper we set focus on L-systems whose main operators are extensions of the minimal one-dimensional Schrödinger operators with deficiency indices $(1, 1)$ on the half-line. We refer to such L-systems as Schrödinger L-systems. As a main highlight of the current paper we introduce a new concept of an L-system coupling entropy (or c-entropy), which is an additive quantity with respect to the coupling of L-systems (see [7] for the concept of coupling). We relate the c-entropy of a Schrödinger L-system to the dissipation coefficient of its main operator $T_h$ and pose the following dual c-entropy problems. The first problem is to describe a Schrödinger L-system with a given c-entropy and minimal dissipation coefficient,
while the second one is to construct a Schrödinger L-system that has a given dissipation coefficient and maximal c-entropy. We solve these dual problems for several classes of Schrödinger L-systems, in particular, for the Schrödinger L-systems whose impedance functions belong to one of the generalized Donoghue classes $\mathcal{M}_\kappa$ and $\mathcal{M}_\kappa^{-1}$ introduced in [6]. As an auxiliary result, we obtain a criteria (in terms of the boundary value of the main operators) for the impedance functions to belong to one of (generalized) Donoghue classes $\mathcal{M}$, $\mathcal{M}_\kappa$, and $\mathcal{M}_\kappa^{-1}$ and then solve the dual c-entropy problems for Schrödinger L-systems with extremal and $\beta$-sectorial main operators.

The paper is organized as follows. The formal definitions of general and Schrödinger L-systems as well as corresponding function classes are presented in Sections 2, 3 and 4. We capitalize on the fact that class of Schrödinger L-systems $\Theta_{\mu,h}$ forms a two-parametric family whose members are uniquely defined by a real-valued parameter $\mu$ and a complex boundary value $h$, $(\text{Im } h > 0)$ of the main operator. Here the parameter $\mu \in \mathbb{R} \cup \{\infty\}$ uniquely defines (for a given $h$) the state-space operator $k_{\mu,h}$ of the L-system $\Theta_{\mu,h}$, thus fixing $\Theta_{\mu,h}$ in a unique way.

In Section 4 we establish a connection between the absolute value of the von Neumann parameter of the main operator $T_h$ and its boundary value parameter $h$ and put forward the definition of the dissipation coefficient. Also, we obtain a criteria for the impedance functions of L-systems with Schrödinger operator to fall into one of the (generalized) Donoghue classes $\mathcal{M}$, $\mathcal{M}_\kappa$, and $\mathcal{M}_\kappa^{-1}$ (see Theorems 8-10). It is worth mentioning that Theorem 8 describes an entire family of Schrödinger L-systems whose impedance functions belong to the Donoghue class $\mathcal{M}$, while Theorems 9-10 deal with explicitly defined Schrödinger L-systems whose impedance functions are members of the generalized Donoghue classes $\mathcal{M}_\kappa$ and $\mathcal{M}_\kappa^{-1}$, respectively.

In Sections 5 and 6 we discuss an analogue of the Phillips-Kato extension problem for a non-negative Schrödinger operator with deficiency indices $(1,1)$ in the Hilbert space $H = L^2(\ell, \infty)$. Recall that in general the Phillips-Kato extension problem concerns the existence and description of all maximal accretive and/or sectorial non-self-adjoint extensions $\tilde{A}$ of a non-negative symmetric operator $\tilde{A}$ such that $\tilde{A} \subset A \subset \tilde{A}^*$.

In Section 7 we introduce the concept of an L-system c-entropy and relate it to L-system’s dissipation coefficient introduced in Section 4. Two following dual problems associated with the c-entropy of Schrödinger L-systems are considered:

- Give a description of an L-system with the Schrödinger dissipative main operator that has a given c-entropy and the minimal dissipation coefficient.
- Describe an L-system with Schrödinger main operator that has a given dissipation coefficient and the maximal c-entropy.

In Sections 7 and 8 we respectively present the solutions to both problems posed for the Schrödinger L-systems whose impedance functions belong to one of the generalized Donoghue classes $\mathcal{M}_\kappa$ and $\mathcal{M}_\kappa^{-1}$.

In Section 9 we solve the dual c-entropy problems for the classes of Schrödinger L-systems with extremal and $\beta$-sectorial main operators. Dealing with these two cases we do not require that the impedance functions of the Schrödinger L-systems in question belong to one of the generalized Donoghue classes. As a result, we
present the solution to dual c-entropy problems for the entire one-parametric family of Schrödinger L-systems $\Theta_{\mu,h}$. We also treat a combined case of extremal and $\beta$-sectorial main operators to see when the corresponding L-system has a maximal c-entropy. It turns out that the Schrödinger L-system with accretive (either $\beta$-sectorial ($\beta \in (0, \pi/2)$ or extremal) main operator $T_h$ attains the maximum c-entropy when the main operator is extremal accretive. Moreover, in the case when both main and state-space operators are extremal, the quasi-kernel in the corresponding Schrödinger L-system coincides with the Krein-von Neumann extension of the underlying symmetric operator. In addition, we find the conditions when the impedance function of the Schrödinger L-system with the main extremal operator $\hat{\kappa}$ belongs to the generalized Donoghue classes $\mathcal{M}_{\kappa}$ or $\mathcal{M}_{\kappa}^{-1}$, respectively.

We conclude the paper with providing examples that illustrate main results.

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\hat{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f, g), f, g \in \mathcal{H}$. Any non-symmetric operator $T$ in $\mathcal{H}$ such that

$$\hat{A} \subset T \subset \hat{A}^*$$

is called a quasi-self-adjoint extension of $\hat{A}$.

Consider the rigged Hilbert space (see [2, 10]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \text{Dom}(\hat{A}^*)$ and

$$\begin{align*}
(f, g)_+ &= (f, g) + (\hat{A}^* f, \hat{A}^* g), \quad f, g \in \text{Dom}(\hat{A}^*).
\end{align*}$$

Let $\mathcal{R}$ be the Riesz-Berezansky operator $\mathcal{R}$ (see [2, 10]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f, g) = (f, \mathcal{R} g)_+ (\forall f \in \mathcal{H}_+, g \in \mathcal{H}_-)$ and $\|\mathcal{R} g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\hat{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\hat{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$. An operator $\hat{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator $\hat{A}$ if $\hat{A} = \hat{A}^*$ and $\hat{A} \supset \hat{A}$. Let $\hat{A}$ be a self-adjoint bi-extension of $\hat{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\text{Dom}(\hat{A}) = \{ f \in \mathcal{H}_+ : \hat{A} f \in \mathcal{H}_- \}, \quad \hat{A} = \hat{A}^* | \text{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a quasi-kernel of a self-adjoint bi-extension $\hat{A}$ (see [2 Section 2.1], [26]). According to the von Neumann Theorem (see [2 Theorem 1.3.1]) the domain of $\hat{A}$, a self-adjoint extension of $\hat{A}$, can be expressed as

$$\text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) \oplus (I + U)\mathcal{N}_i,$$

where von Neumann’s parameter $U$ is both a ($\cdot$)-isometric as well as ($\cdot$)-isometric operator from $\mathcal{N}_i$ into $\mathcal{N}_{-i}$ and

$$\mathcal{N}_{\pm i} = \text{Ker} (\hat{A}^* \mp iI)$$

are the deficiency subspaces of $\hat{A}$. A self-adjoint bi-extension $\hat{A}$ of a symmetric operator $\hat{A}$ is called t-self-adjoint (see [2 Definition 4.3.1]) if its quasi-kernel $\hat{A}$ is a self-adjoint operator in $\mathcal{H}$. An operator $\hat{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a quasi-self-adjoint bi-extension of an operator $T$ if $\hat{A} \supset T \supset \hat{A}$ and $\hat{A}^* \supset T^* \supset \hat{A}$.

We will be mostly interested in the following type of quasi-self-adjoint bi-extensions. Let $T$ be a quasi-self-adjoint extension of $\hat{A}$ with nonempty resolvent
set \( \rho(T) \). A quasi-self-adjoint bi-extension \( \mathcal{A} \) of an operator \( T \) is called (see [2 Definition 3.3.5]) a \((*)\)-extension of \( T \) if \( \text{Re} \mathcal{A} \) is a \( t \)-self-adjoint bi-extension of \( \dot{A} \). In what follows we assume that \( \dot{A} \) has deficiency indices \((1, 1)\). In this case it is known [2] that every quasi-self-adjoint extension \( T \) of \( \dot{A} \) admits \((*)\)-extensions. In what follows we assume that \( \dot{A} \) has deficiency indices \((1, 1)\). In this case it is known [2] that every quasi-self-adjoint extension \( T \) of \( \dot{A} \) admits \((*)\)-extensions. The description of all \((*)\)-extensions via the Riesz-Berezansky operator \( R \) can be found in [2, Section 4.3].

Recall that a linear operator \( T \) in a Hilbert space \( H \) is called accretive [15] if \( \text{Re} (Tf, f) \geq 0 \) for all \( f \in \text{Dom}(T) \). We call an accretive operator \( T \) \( \alpha \)-sectorial [15] if there exists a value of \( \alpha \in (0, \pi/2) \) such that

\[
(3) \quad (\cot \alpha)|\text{Im}(Tf, f)| \leq \text{Re}(Tf, f), \quad f \in \text{Dom}(T).
\]

We say that the angle of sectoriality \( \alpha \) is exact for an \( \alpha \)-sectorial operator \( T \) if

\[
\tan \alpha = \sup_{f \in \text{Dom}(T)} \frac{|\text{Im}(Tf, f)|}{\text{Re}(Tf, f)}.
\]

An accretive operator is called extremal accretive if it is not \( \alpha \)-sectorial for any \( \alpha \in (0, \pi/2) \). In what follows, when we say that an accretive operator is \( \alpha \)-sectorial, we mean that \( \alpha \) is its exact angle of sectoriality unless otherwise is specified.

The following definition is a “lite” version of the definition of L-system given for a scattering L-system with one-dimensional input-output space. It is tailored for the case when the symmetric operator of an L-system has deficiency indices \((1, 1)\). The general definition of an L-system can be found in [2, Definition 6.3.4].

**Definition 1.** An array

\[
(4) \quad \Theta = \begin{pmatrix} \mathcal{A} & K \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix}
\]

is called an L-system if:

1. \( T \) is a dissipative \((\text{Im}(Tf, f) \geq 0, f \in \text{Dom}(T))\) quasi-self-adjoint extension of a symmetric operator \( \dot{A} \) with deficiency indices \((1, 1)\);
2. \( \mathcal{A} \) is a \((*)\)-extension of \( T \);
3. \( \text{Im} \mathcal{A} = KK^*, \) where \( K \in [\mathbb{C}, \mathcal{H}_-] \) and \( K^* \in [\mathcal{H}_+, \mathbb{C}] \).

Operators \( T \) and \( \mathcal{A} \) are called the main and state-space operators respectively of the system \( \Theta \), and \( K \) is the channel operator. It is easy to see that the operator \( \mathcal{A} \) of the system \( \Theta \) can be chosen in such a way that \( \text{Im} \mathcal{A} = (\cdot, \chi)\chi, \chi \in \mathcal{H}_- \) and \( Kc = c \cdot \chi, c \in \mathbb{C} \). A system \( \Theta \) in [1] is called minimal if the operator \( \mathcal{A} \) is a prime operator in \( \mathcal{H} \), i.e., there exists no non-trivial reducing invariant subspace of \( \mathcal{H} \) on which it induces a self-adjoint operator. Notice that minimal L-systems of the form \( \Theta \) with one-dimensional input-output space were also considered in [6].

We associate with an L-system \( \Theta \) the function

\[
(5) \quad W_\Theta(z) = I - 2iK^*(\mathcal{A} - zI)^{-1}K, \quad z \in \rho(T),
\]

which is called the transfer function of the L-system \( \Theta \). We also consider another function related to an L-system \( \Theta \) called the impedance function and given by the formula

\[
(6) \quad V_\Theta(z) = K^*(\text{Re} \mathcal{A} - zI)^{-1}K.
\]
The transfer function $W_{\Theta}(z)$ of the L-system $\Theta$ and function $V_{\Theta}(z)$ of the form (6) are connected by the following relations valid for $\text{Im} \, z \neq 0$, $z \in \rho(T)$,

\[ V_{\Theta}(z) = i[W_{\Theta}(z) + I]^{-1}[W_{\Theta}(z) - I], \]
\[ W_{\Theta}(z) = (I + iV_{\Theta}(z))^{-1}(I - iV_{\Theta}(z)). \]

The class of all Herglotz-Nevanlinna functions, that can be realized as impedance functions of L-systems, and connections with Weyl-Titchmarsh functions can be found in [2], [6], [11], [14] and references therein. In particular it is shown there that any impedance function $V_{\Theta}(z)$ admits the integral representation

\[ V_{\Theta}(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma, \]

where $Q$ is a real number and $\sigma$ is an infinite Borel measure such that
\[ \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty. \]

3. Donoghue classes and L-systems with one-dimensional input-output

Suppose that $\hat{A}$ is a closed prime densely defined symmetric operator with deficiency indices $(1, 1)$. Assume also that $T \neq T^*$ is a maximal dissipative extension of $\hat{A}$,

\[ \text{Im}(Tf, f) \geq 0, \quad f \in \text{Dom}(T). \]

Since $\hat{A}$ is symmetric, its dissipative extension $T$ is automatically quasi-self-adjoint [2], that is,

\[ \hat{A} \subset T \subset \hat{A}^*, \]

and hence, (see [6])

\[ g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some } |\kappa| < 1, \]

where $g_+ \in \mathfrak{R}_{g_+} = \text{Ker}(\hat{A}^* \mp iI)$ and $\|g_+\| = 1$. Throughout this paper $\kappa$ will be referred to as the von Neumann parameter of the operator $T$. The next lemma contains a characterization of sectorial operators in terms of the modulus of von Neumann’s parameter.

**Lemma 2.** If $T$ is an $\alpha$-sectorial operator with $\alpha \in (0, \pi/2)$, then its von Neumann’s parameter $\kappa$ cannot equal zero.

**Proof.** Assume the contrary, let $T$ be an $\alpha$-sectorial operator in a Hilbert space $\mathcal{H}$ with $\alpha \in (0, \pi/2)$ and the von Neumann parameter $\kappa = 0$. Then (8) implies (see [2]) that there exists a non-zero vector $x \in \text{Dom}(T)$ such that $Tx = ix$, $x \neq 0$. Moreover,

\[ (Tx, x) = (ix, x) = i\|x\|^2 \quad \text{and} \quad (x, Tx) = (x, ix) = -i\|x\|^2. \]

Then

\[ \text{Re} \, (Tx, x) = \frac{(Tx, x) + (x, Tx)}{2} = 0 \]

and

\[ \text{Im} \, (Tx, x) = \frac{(Tx, x) - (x, Tx)}{2i} = \frac{i\|x\|^2 + i\|x\|^2}{2i} = \|x\|^2. \]

But $T$ is $\alpha$-sectorial and hence [3] takes place implying

\[ 0 \leq (\cot \alpha) \text{Im}(Tx, x) \leq \text{Re}(Tx, x) = 0. \]
This yields \(0 \leq (\cot \alpha)\|x\|^2 \leq 0\) or \((\cot \alpha)\|x\|^2 = 0\). Since neither \(\cot \alpha\) nor \(\|x\|\) can equal zero, we reached a contradiction. Therefore, \(\kappa \neq 0\). \(\square\)

Recall that Donoghue \([12]\) introduced a concept of the Herglotz-Nevanlinna function \(M_{(\hat{A}, A)}(z)\) associated with the pair \((\hat{A}, A)\) by
\[
M_{(\hat{A}, A)}(z) = ((Az + I)(A - zI)^{-1}g_+, g_+), \quad z \in \mathbb{C}_+,
\]
where \(\hat{A}\) is a symmetric operator with deficiency indices \((1, 1)\), and \(A\) is its self-adjoint extension. Let \(\mathcal{M}\) (see \([9]\)) be a class of all Herglotz-Nevanlinna functions \(M(z)\) that admit the representation
\[
M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma,
\]
where \(\sigma\) is an infinite Borel measure with
\[
\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty.
\]
Following our earlier developments in \([6], [9], [18], [19]\) denote by \(\mathcal{M}\) the Donoghue class of all analytic mappings \(M(z)\) from \(\mathbb{C}_+\) into itself that admits the representation \((9)\) and has a property
\[
\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = 1, \quad \text{equivalently}, \quad M(i) = i.
\]

It is known \([12], [13], [14], [17]\) that \(M(z) \in \mathcal{M}\) if and only if \(M(z)\) can be realized as the Weyl-Titchmarsh function \(M_{(\hat{A}, A)}(z)\) associated with the pair \((\hat{A}, A)\). Furthermore, we say (see \([6]\)) that a function \(M(z) \in \mathcal{M}\) belongs to the generalized Donoghue class \(\mathcal{M}_\kappa\), \((0 \leq \kappa < 1)\) if in the representation \((9)\)
\[
\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = 1 - \kappa, \quad \text{equivalently}, \quad M(i) = i \frac{1 - \kappa}{1 + \kappa}.
\]
Similarly (see \([7]\)), a function \(M(z) \in \mathcal{M}\) belongs to the generalized Donoghue class \(\mathcal{M}_\kappa^{-1}\) if in the representation \((9)\)
\[
\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = 1 + \kappa, \quad \text{equivalently}, \quad M(i) = i \frac{1 + \kappa}{1 - \kappa}.
\]
Clearly, when \(\kappa = 0\) the generalized Donoghue classes \(\mathcal{M}_\kappa\) and \(\mathcal{M}_\kappa^{-1}\) coincide with the Donoghue class \(\mathcal{M}\), that is \(\mathcal{M}_0 = \mathcal{M}_0^{-1} = \mathcal{M}\). If \(M(z)\) is an arbitrary function from \(\mathcal{M}\) with a normalization condition
\[
\int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = a,
\]
for some \(a > 0\), then it is easy to see that \(M(z) \in \mathcal{M}\) if and only if \(a = 1\). Also, if \(a < 1\), then \(M(z) \in \mathcal{M}_\kappa\) with
\[
\kappa = \frac{1 - a}{1 + a},
\]
and if \(a > 1\), then \(M(z) \in \mathcal{M}_\kappa^{-1}\) with
\[
\kappa = \frac{a - 1}{1 + a}.\]
Hypothesis 3. Suppose that \( T \neq T^* \) is a maximal dissipative extension of a symmetric operator \( \dot{A} \) with deficiency indices \((1, 1)\). Assume, in addition, that \( \dot{A} \) is a self-adjoint extension of \( \dot{A} \). Suppose, that the deficiency elements \( g_{\pm} \in \text{Ker}(\dot{A}^* \mp iI) \) are normalized, \( \|g_{\pm}\| = 1 \), and chosen in such a way that
\[
g_+ - g_- \in \text{Dom}(\dot{A}) \quad \text{and} \quad g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some} \quad |\kappa| < 1.
\]

It is known \([17]\) that if \( \kappa = 0 \), then quasi-self-adjoint extension \( T \) coincides with the restriction of the adjoint operator \( \dot{A}^* \) on \( \text{Dom}(T) = \text{Dom}(\dot{A}) + \text{Ker}(\dot{A}^* - iI) \).

Similar to Hypothesis 3 it is convenient to adopt “anti-Hypothesis”.

Hypothesis 4. Suppose that \( T \neq T^* \) is a maximal dissipative extension of a symmetric operator \( \dot{A} \) with deficiency indices \((1, 1)\). Assume, in addition, that \( \dot{A} \) is a self-adjoint extension of \( \dot{A} \). Suppose, that the deficiency elements \( g_{\pm} \in \text{Ker}(\dot{A}^* \mp iI) \) are normalized, \( \|g_{\pm}\| = 1 \), and chosen in such a way that
\[
g_+ + g_- \in \text{Dom}(\dot{A}) \quad \text{and} \quad g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some} \quad |\kappa| < 1.
\]

Remark 5. Without loss of generality, in what follows we assume that \( \kappa \) is real and \( 0 \leq \kappa < 1 \): if \( \kappa = |\kappa|e^{i\theta} \), change (the basis) \( g_- \) to \( e^{i\theta}g_- \) in the deficiency subspace \( \text{Ker}(\dot{A}^* + iI) \).

This remark means the following: let
\[
(17) \quad \Theta = \begin{pmatrix} \mathbb{K} & K \\ \mathcal{H} & \mathbb{C} \end{pmatrix}
\]
be a minimal L-system with one-dimensional input-output space \( \mathbb{C} \). If the main operator \( T \) of \( \Theta \) is parameterized with a complex von Neumann’s parameter \( \kappa \) that corresponds to a chosen normalized pair of deficiency vectors \( g_+ \) and \( g_- \), then we can change the deficiency basis as explained in Remark 5 and represent \( T \) using real value \( |\kappa| \) with respect to the new deficiency basis. This procedure will change the parameter \( U \) of the quasi-kernel \( \hat{A} \) of \( \text{Re} \dot{A} \) in (2) and ultimately the way \( \hat{A} \) is described. Thus, for the remainder of this paper (unless otherwise is specified) we will consider L-systems (17) such that \( \kappa \) is real and \( 0 \leq \kappa < 1 \).

Definition 6. We say that an L-system \( \Theta \) of the form (17) satisfies Hypothesis \( \Theta \) if its main operator \( T \) and the quasi-kernel \( \hat{A} \) of \( \text{Re} \dot{A} \) satisfy the conditions of Hypothesis \( \dot{A} \) (or \( \dot{A} \)) for a fixed set of deficiency vectors of the symmetric operator \( \dot{A} \).

Let \( \Theta \) be a minimal L-system of the form (17) that satisfies Hypothesis \( \Theta \). It is shown in [6] that the impedance function \( V_\Theta(z) \) can be represented as
\[
(18) \quad V_\Theta(z) = \left( \frac{1 - \kappa}{1 + \kappa} \right) V_{\Theta_0}(z),
\]
where \( V_{\Theta_0}(z) \) is the impedance function of an L-system \( \Theta_0 \) with the same set of conditions but with \( \kappa_0 = 0 \), where \( \kappa_0 \) is the von Neumann parameter of the main operator \( T_0 \) of \( \Theta_0 \).

Let \( \Theta_1 \) and \( \Theta_2 \) be two minimal L-system of the form (17) whose components satisfy Hypothesis \( \Theta \) and Hypothesis \( \dot{A} \) respectively. Then it was proved in [7]
Lemma 5.1] that the impedance functions $V_{\Theta_1}(z)$ and $V_{\Theta_2}(z)$ admit the integral representation

$$V_{\Theta_k}(z) = \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma_k(t), \quad k = 1, 2. \quad (19)$$

Now let us consider a minimal L-system $\Theta$ of the form \((17)\) that satisfies Hypothesis \(3\). Let also

$$\Theta_\alpha = \begin{pmatrix} \mathbb{A}_\alpha & K_\alpha \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix}, \quad \alpha \in [0, \pi), \quad (20)$$

be a one parametric family of L-systems such that

$$W_{\Theta_\alpha}(z) = W_\Theta(z) \cdot (-e^{2i\alpha}), \quad \alpha \in [0, \pi). \quad (21)$$

The existence and structure of $\Theta_\alpha$ were described in details in \([2, \text{Section } 8.3]\). In particular, it was shown that the L-system $\Theta$ and $\Theta_\alpha$ share the same main operator $T$ and that

$$V_{\Theta_\alpha}(z) = \frac{\cos \alpha + (\sin \alpha) V_\Theta(z)}{\sin \alpha - (\cos \alpha) V_\Theta(z)}. \quad (22)$$

Let $\Theta$ be a minimal L-system $\Theta$ of the form \((17)\) that satisfies Hypothesis \(3\). Also let $\Theta_\alpha$ be a one parametric family of L-systems given by \((20), (21)\). It was shown in \([7, \text{Theorem } 5.2]\) that in this case the impedance function $V_{\Theta_\alpha}(z)$ has an integral representation

$$V_{\Theta_\alpha}(z) = \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma_\alpha(t)$$

if and only if $\alpha = 0$ or $\alpha = \pi/2$.

The next result describes the relationship between two L-systems of the form \((17)\) that comply with different hypotheses.

Let

$$\Theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix} \quad (23)$$

be a minimal L-system whose main operator $T$ and the quasi-kernel $\hat{A}_1$ of $\text{Re} \mathbb{A}_1$ satisfy the conditions of Hypothesis \(3\) and let

$$\Theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix} \quad (24)$$

be another minimal L-system with the same operators $\hat{A}$ and $T$ as $\Theta_1$ but with the quasi-kernel $\hat{A}_2$ of $\text{Re} \mathbb{A}_2$ that satisfies Hypothesis \(4\). It was shown in \([7, \text{Theorem } 5.3]\) that

$$W_{\Theta_1}(z) = -W_{\Theta_2}(z), \quad z \in \mathbb{C}_+ \cap \rho(T) \quad (25)$$

and

$$V_{\Theta_1}(z) = -\frac{1}{V_{\Theta_2}(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T). \quad (26)$$
4. L-systems with Schrödinger operator $T_h$ and their impedance functions

Let $\mathcal{H} = L_2(\ell, +\infty)$ and $l(y) = -y'' + q(x)y$, where $q$ is a real locally summable function. Suppose that the minimal symmetric operator

$$\tag{27} \begin{cases} \hat{A}y = -y'' + q(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases}$$

has deficiency indices (1,1). Let $D^*$ be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2(\ell, +\infty)$. Consider $\mathcal{H}_+ = \text{Dom}(\hat{A}^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_{a}^{\infty} \left(y(x)\overline{z(x)} + l(y)\overline{l(z)}\right)dx, \ y, z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider the operators (cf. [21])

$$\tag{28} \begin{cases} T_h y = l(y) = -y'' + q(x)y \\ h\gamma(\ell) - y'(\ell) = 0 \end{cases}, \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}\gamma(\ell) - y'(\ell) = 0 \end{cases}.$$

Let $\hat{A}$ be a symmetric operator of the form (27) with deficiency indices (1,1), generated by the differential expression $l(y) = -y'' + q(x)y$. Let also $\varphi_k(\ell, \lambda)(k = 1, 2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1 \\ \varphi_1(\ell, \lambda) = 0 \end{cases}, \quad \begin{cases} l(\varphi_2) = \lambda\varphi_2 \\ \varphi_2(\ell, \lambda) = -1 \end{cases}.$$

It is well known [20] that there exists a Weyl function $m_\infty(\lambda)$ such that

$$\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda)\varphi_1(x, \lambda)$$

belongs to $L_2(\ell, +\infty)$.

Now we shall construct an L-system associated with a non-self-adjoint Schrödinger operator $T_h$. It was shown in [2, 4] that the set of all $(*)$-extensions of the non-self-adjoint Schrödinger operator $T_h$ of the form (28) in $L_2(\ell, +\infty)$ can be represented as

$$\mathcal{A}_{\mu, h} y = -y'' + q(x)y - \frac{1}{\mu - h} \left[y'(\ell) - h\gamma(\ell)\right] \left[\mu\delta(x - \ell) + \delta'(x - \ell)\right],$$

$$\mathcal{A}_{\mu, h}^* y = -y'' + q(x)y - \frac{1}{\mu - h} \left[y'(\ell) - \overline{h}\gamma(\ell)\right] \left[\mu\delta(x - \ell) + \delta'(x - \ell)\right].$$

Moreover, the formulas (29) establish a one-to-one correspondence between the set of all $(*)$-extensions of the Schrödinger operator $T_h$ of the form (28) and all values $\mu \in \mathbb{R} \cup \{\infty\}$. One can easily check that the $(*)$-extension $\mathcal{A}_{\mu, h}$ in (29) of the non-self-adjoint dissipative Schrödinger operator $T_h$, $(\Im h > 0)$ of the form (28) satisfies the condition

$$\Im \mathcal{A}_{\mu, h} = \frac{\mathcal{A}_{\mu, h} - \mathcal{A}_{\mu, h}^*}{2i} = (\cdot, g)g,$$

where

$$g = \frac{(\Im h)^{1/2}}{\mu - h} \left[\mu\delta(x - \ell) + \delta'(x - \ell)\right],$$

$$\tag{30}$$
\( \delta(x - \ell) \) and \( \delta'(x - \ell) \) are the delta-function and its derivative at the point \( \ell \), respectively. Furthermore,

\[
(y, g) = \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu y(\ell) - y'(\ell)],
\]

\( y \in \mathcal{H}_+ \), \( g \in \mathcal{H}_- \), where \( \mathcal{H}_+ \subset L^2(\ell, +\infty) \subset \mathcal{H}_- \) is the triplet of Hilbert spaces introduced above.

Let \( y \in \text{Dom}(T_h) \), then \( y'(\ell) = hy(\ell) \) and

\[
\text{Im} \kappa_{\mu, h} y = \text{Im} T_h y = (y, g) = \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu y(\ell) - y'(\ell)]g
\]

\[
= \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu y(\ell) - hy(\ell)]g
\]

\[
= \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} (\mu - h) y(\ell) \cdot \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu \delta(x - \ell) + \delta'(x - \ell)]
\]

\[
= (\mu - h) \text{Im} h |\mu - h|^2 [\mu \delta(x - \ell) + \delta'(x - \ell)].
\]

Consequently,

\[
(31) \quad \text{Im}(T_h y, y) = y(\ell) \frac{(\text{Im} h)(\mu - h)}{|\mu - h|^2} \cdot (\mu - \bar{h}) y(\ell) = (\text{Im} h)|y(\ell)|^2.
\]

Having in mind (31) we will call \( \text{Im} h \) a **coefficient of dissipation** and denote it by \( \mathcal{D} = \text{Im} h \).

It was also shown in [2] that the quasi-kernel of \( \text{Re} \kappa_{\mu, h} \) is given by

\[
\frac{\mu \text{Re} h - |h|^2}{\mu - \text{Re} h},
\]

\[
(32) \quad \left\{ \begin{array}{l}
\hat{A}_\xi(y) = -y'' + q(x)y \\
y'(\ell) = \xi y(\ell)
\end{array} \right., \quad \text{where} \quad \xi = \frac{\mu \text{Re} h - |h|^2}{\mu - \text{Re} h}.
\]

Let \( E = \mathbb{C}, \ Kc = cg \ (c \in \mathbb{C}) \). It is clear that

\[
(33) \quad K^* y = (y, g), \quad y \in \mathcal{H}_+,
\]

and \( \text{Im} \kappa_{\mu, h} = KK^* \). Therefore, the array

\[
(34) \quad \Theta_{\mu, h} = \left( \begin{array}{cc}
\kappa_{\mu, h} & 1 \\
\mathcal{H}_+ \subset L^2(\ell, +\infty) & \mathcal{H}_- \subset \mathbb{C}
\end{array} \right),
\]

is an L-system with the main operator \( \kappa_{\mu, h} \) of the form \( (29) \) with the channel operator \( K \) given by \( (33) \). We will say that an L-system \( \Theta_{\mu, h} \) of the form \( (34) \) has the coefficient of dissipation \( \mathcal{D} \) if its main operator \( T_h \) has the coefficient of dissipation \( \mathcal{D} = \text{Im} h \). It was shown in [2], [4] that the transfer and impedance functions of \( \Theta_{\mu, h} \) can be evaluated as

\[
(35) \quad W_{\Theta_{\mu, h}}(z) = \frac{\mu - h}{\mu - \bar{h}} \frac{m_\infty(z) + \bar{h}}{m_\infty(z) + 1},
\]

and

\[
(36) \quad V_{\Theta_{\mu, h}}(z) = \frac{(m_\infty(z) + \mu) \text{Im} h}{(\mu - \text{Re} h) m_\infty(z) + \mu \text{Re} h - |h|^2}.
\]

Suppose that the main operator \( T_h \) of the L-system \( (34) \) has the von Neumann representation \( (5) \) with the parameter \( \kappa \) related to some normalized deficiency basis
It was shown in [6] that if the point \( z = i \) belongs to the resolvent set \( \rho(T_h) \), then in this case (by (33))

\[
|\kappa| = \frac{1}{\|W_{\Theta_{\mu,h}}(i)\|} = \left| \frac{\mu - \overline{h}}{\mu - h} \cdot \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} \right| \left| m_{\infty}(i) + h \right|
\]

Taking into account that \( \text{Im} h > 0 \) and \( \text{Im} m_{\infty}(i) < 0 \) one can easily see that \( |\kappa| < 1 \) in (37).

**Remark 7.** Now suppose \( h \neq -m_{\infty}(i) \) and hence \( \kappa \neq 0 \). If \( \Theta_{\mu,h} \) satisfies Hypothesis \( \mathbf{3} \) then

\[
\kappa = \frac{1}{W_{\Theta_{\mu,h}}(i)} = \frac{\mu - \overline{h}}{\mu - h} \cdot \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}}
\]

is real.

We can reverse the logic and try to find a value of parameter \( \mu \) for which \( \kappa \) in (38) is real and thus \( \kappa = |\kappa| \). This can be achieved if we set

\[
\frac{\mu - \overline{h}}{\mu - h} = e^{-i\alpha},
\]

where \( \alpha \in (-\pi, \pi] \) is such that

\[
\frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} = \left| \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} \right| e^{i\alpha}.
\]

Clearly, such a choice of the angle \( \alpha \) will make \( \kappa \) real and \( 0 < \kappa < 1 \). That is,

\[
\kappa = \left| \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} \right|.
\]

Solving (40) for \( \mu \) yields

\[
\mu = \mu_1 = \frac{e^{i\alpha} \overline{h} - h}{e^{i\alpha} - 1}, \quad \text{where} \quad \alpha = \text{Arg} \left( \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} \right).
\]

Therefore, we can describe an L-system

\[
\Theta_{\mu_1,h} = \left( \mathcal{K}_{\mu_1,h} \subset \mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_- \right)
\]

with main Schrödinger operator \( T_h \) that satisfies Hypothesis \( \mathbf{3} \) with \( 0 < \kappa < 1 \). The state-space operator \( \mathcal{K}_{\mu_1,h} \) of this L-system will be uniquely defined via (29) by the parameter \( h \) of \( T_h \) and parameter \( \mu_1 \) given by (41).

Similarly, we can give the description of an L-system

\[
\Theta_{\mu_2,h} = \left( \mathcal{K}_{\mu_2,h} \subset \mathcal{H}_+ \subset L_2(\ell, +\infty) \subset \mathcal{H}_- \right)
\]

with the same main operator \( T_h \) that satisfies the conditions of Hypothesis \( \mathbf{4} \) with \( 0 < \kappa < 1 \). Applying (25) yields

\[
\mu = \mu_2 = \frac{e^{i\alpha} \overline{h} + h}{e^{i\alpha} + 1}, \quad \text{where} \quad \alpha = \text{Arg} \left( \frac{m_{\infty}(i) + h}{m_{\infty}(i) + \overline{h}} \right).
\]

The state-space operator \( \mathcal{K}_{\mu_2,h} \) from Hypothesis \( \mathbf{4} \) satisfying L-system will be uniquely defined via (29) by the parameter \( h \) of \( T_h \) and parameter \( \mu_2 \) given by (44). As shown in [7],

\[
V_{\Theta_{\mu_1,h}}(z) \in \mathcal{M}_\kappa \quad \text{and} \quad V_{\Theta_{\mu_2,h}}(z) \in \mathcal{M}_\kappa^{-1}.
\]
We will rely on formula (37) to prove the following result.

**Theorem 8.** Let $\Theta_{\mu,h}$ be a Schrödinger L-system (44) with the main operator $T_h$ of the form (28). Then for any $\mu \in \mathbb{R} \cup \{\infty\}$ the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathcal{M}$ if and only if $h = -m_\infty(i)$.

**Proof.** It is well known (see [3], [7]) that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the class $\mathcal{M}$ if and only if it has the integral representation

$$V_{\Theta_{\mu,h}}(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma(\lambda), \quad \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = 1. \tag{45}$$

Suppose that the Schrödinger L-system $\Theta_{\mu,h}$ contains the main operator $T_h$ such that $h = -m_\infty(i)$ and arbitrary $\mu \in \mathbb{R} \cup \{\infty\}$. Then (36) yields

$$V_{\Theta_{\mu,h}}(i) = \frac{- (m_\infty(z) + \mu) \text{Im} m_\infty(i)}{(\mu + \text{Re} m_\infty(i)) m_\infty(i) - \mu \text{Re} m_\infty(i) - |m_\infty(i)|^2} \tag{46}$$

$$= \frac{\mu (\text{Re} m_\infty(i) + i \text{Im} m_\infty(i)) + \text{Re} m_\infty(i) \cdot m_\infty(i) - \mu \text{Re} m_\infty(i) - |m_\infty(i)|^2}{(\mu + \text{Re} m_\infty(i)) m_\infty(i)} \tag{47}$$

$$= \frac{i \mu \text{Im} m_\infty(i) + (\text{Re} m_\infty(i) - m_\infty(i)) m_\infty(i)}{\mu m_\infty(i)} \tag{48}$$

Therefore, $V_{\Theta_{\mu,h}}(z)$ admits the integral representation (45) and hence belongs to the class $\mathcal{M}$.

Conversely, let $V_{\Theta_{\mu,h}}(z)$ belong to the class $\mathcal{M}$ and hence has the integral representation (45). Then $V_{\Theta_{\mu,h}}(z) \in \mathcal{M}$ and consequently (see [3]) $\kappa = 0$, where $\kappa$ is the von Neumann parameter of $T_h$. In this case (37) implies that $h = -m_\infty(i)$. \[\square\]

Clearly, it follows from (37) that $\kappa = 0$ if and only if $h = -m_\infty(i)$. Therefore,

$$\begin{cases} T_h y = -y'' + q(x) y \\ y'(\ell) = h y(\ell) \end{cases} \tag{46}$$

has $\kappa = 0$ if and only if $h = -m_\infty(i)$.

The state-space operator $A_{\mu,h}$ of the L-system (44) (and its adjoint $A_{\mu,h}^*$) in this case will depend on the parameter $\mu$ only and take the form

$$A_{\mu,-m_\infty(i)} y = -y'' + q(x) y - \frac{y'(\ell) + m_\infty(i) y(\ell)}{\mu + m_\infty(i)} \left[ \mu \delta(x - \ell) + \delta'(x - \ell) \right], \tag{47}$$

$$A_{\mu,-m_\infty(i)}^* y = -y'' + q(x) y - \frac{y'(\ell) + m_\infty(-i) y(\ell)}{\mu + m_\infty(-i)} \left[ \mu \delta(x - \ell) + \delta'(x - \ell) \right].$$

It can be checked directly that in this case the impedance function $V_{\Theta_{\mu,h}}(\lambda)$ belongs to the class $\mathcal{M}$ (see also [3]).

The following theorem is similar to Theorem 8 result for the class $\mathcal{M}_\kappa$.

**Theorem 9.** Let $0 < a < 1$ and $\kappa = \frac{1-a}{1+a}$. Also let $\Theta_{\mu,h}$ be a Schrödinger L-system (44) with the main operator $T_h$ of the form (28) that has the modulus of
von Neumann’s parameter \( \kappa \). Then the impedance function \( V_{\Theta_{\mu, h}}(z) \) belongs to the class \( \mathfrak{M}_\kappa \) if and only if either

\[
(48) \quad h = -\text{Re} \ m_\infty(i) - \left( \frac{1}{a} \right) \text{Im} \ m_\infty(i),
\]

or

\[
(49) \quad h = \frac{a^2 d^2 \mu - d^2 (c + \mu) - c(c + \mu)^2}{a^2 d^2 + (c + \mu)^2} - \frac{i \mu}{a} - \frac{c d}{a^2 d^2 + (c + \mu)^2},
\]

where \( c = \text{Re} \ m_\infty(i) \) and \( d = \text{Im} \ m_\infty(i) \).

**Proof.** It is well known (see [7]) that the impedance function \( V_{\Theta_{\mu, h}}(z) \) belongs to the class \( \mathfrak{M}_\kappa \) for \( \kappa = \frac{1-a}{1+a} \), \( 0 < a < 1 \) if and only if it has the integral representation

\[
(50) \quad V_{\Theta_{\mu, h}}(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma(\lambda), \quad \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{1 + \lambda^2} = a, \quad 0 < a < 1,
\]

Suppose that the impedance function \( V_{\Theta_{\mu, h}}(z) \in \mathfrak{M}_\kappa \) and thus has the integral representation (50) with \( 0 < a < 1 \). Then, (50) and (30) imply

\[
(51) \quad V_{\Theta_{\mu, h}}(i) = \frac{(m_\infty(i) + \mu) \text{Im} \ h}{(\mu - \text{Re} \ h) m_\infty(i) + \mu \text{Re} \ h - |h|^2} = ai.
\]

Setting

\[
(52) \quad c = \text{Re} m_\infty(i), \quad d = \text{Im} m_\infty(i), \quad x = \text{Re} h, \quad y = \text{Im} h,
\]

transforms (51) into

\[
(53) \quad \frac{cy + \mu y + dyi}{(\mu - x)c + \mu x - x^2 - y^2 + (\mu d - xd)i} = ai.
\]

Cross multiplying (53) and equating the real parts yields \( cy + \mu y = axd - a\mu d \), or

\[
(54) \quad y = \frac{ad(x - \mu)}{c + \mu}.
\]

Equating the imaginary parts in (53) leads to

\[
dy = ac(\mu - x) + a\mu x - ax^2 - ay^2.
\]

Substituting (54) into the above equation results in

\[
\frac{d^2(\mu - x)}{c + \mu} + c(\mu - x) + x(\mu - x) - \frac{a^2 d^2 (\mu - x)^2}{(c + \mu)^2} = 0,
\]

or

\[
(\mu - x) \left( \frac{d^2}{c + \mu} + c + x - \frac{a^2 d^2 (\mu - x)}{(c + \mu)^2} \right) = 0.
\]

Exploring the first solution of the above equation when \( \mu = x \) and applying (54) leads us to \( x = -c \) and \( y = -d/a \), or

\[
\text{Re} h = -\text{Re} m_\infty(i) \quad \text{and} \quad \text{Im} h = -\left( \frac{i}{a} \right) \text{Im} m_\infty(i),
\]

that confirms (48).

Assuming that \( \mu \neq x \) and setting the second factor to be zero yields

\[
a^2 d^2 (\mu - x) - c(c + \mu)^2 - x(c + \mu)^2 - d^2 (c + \mu) = 0,
\]
that we solve for $x$ to obtain
\begin{equation}
    x = \frac{a^2 d^2 \mu - d^2 (c + \mu) - c(c + \mu)^2}{a^2 d^2 + (c + \mu)^2}.
\end{equation}
Substituting (55) into (54) results in
\begin{equation}
    y = \frac{-ad^3 + ad(c + \mu)^2}{a^2 d^2 + (c + \mu)^2}.
\end{equation}
Keeping in mind that $x = \text{Re} \ h$ and $y = \text{Im} \ h$ we confirm (49).

Conversely, suppose either (48) or (49) hold true for
\begin{equation}
    a > 0 \quad \text{and} \quad (55) \text{ for } a > 1.
\end{equation}
Then, as we have shown in the proof of Theorem 9, relations (36) and (50) imply
\begin{equation}
    (\Theta_{\mu,h}(i) = ai) \quad \text{for any real } \mu \neq -\text{Re} \ m_\infty(i).
\end{equation}
Combining this normalization condition of $\Theta_{\mu,h}(z)$ with its known integral representation we conclude that
\begin{equation}
    \Theta_{\mu,h}(z) \in \mathcal{M}_\kappa
\end{equation}
and is described by (50).

A very similar result takes place for the class $\mathcal{M}_\kappa^{-1}$.

**Theorem 10.** Let $a > 1$ and $\kappa = \frac{a-1}{1+a}$. Also, let $\Theta_{\mu,h}$ be a Schrödinger L-system (44) with the main operator $T_h$ of the form (28) that has the modulus of von Neumann’s parameter equal $\kappa$. Then the impedance function $\Theta_{\mu,h}(z)$ belongs to the class $\mathcal{M}_\kappa^{-1}$ with $\kappa = \frac{a-1}{1+a}$ if and only if either (48) or (49) hold true for $a > 1$.

**Proof.** It is well known (see [7]) that the impedance function $\Theta_{\mu,h}(z)$ belongs to the class $\mathcal{M}_\kappa^{-1}$ for $\kappa = \frac{a-1}{1+a}$, $a > 1$ if and only if it has the integral representation
\begin{equation}
    \Theta_{\mu,h}(z) = \int_R \left( \frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2} \right) \frac{d\sigma(\lambda)}{1+\lambda^2} = a, \quad 1 < a.
\end{equation}
The rest of the proof of Theorem 10 resembles the one of Theorem 9 taking into account that $a > 1$ this time. Suppose that the impedance function $\Theta_{\mu,h}(z) \in \mathcal{M}_\kappa^{-1}$ with $\kappa$ given by (44) and hence has the integral representation (57) with $a > 1$. Then, as we have shown in the proof of Theorem 9 relations (50) and (51) imply (51) for $a > 1$. Following the steps in the proof of Theorem 9 we use (51) to obtain (55) and (56) for $a > 1$ thus confirming either (48) or (49).

Conversely, suppose either (48) or (49) hold true for $a > 1$. Following the steps of the proof of the necessity in Theorem 9 we obtain that $\Theta_{\mu,h}(i) = ai$ for $a > 1$ and conclude that $\Theta_{\mu,h}(z) \in \mathcal{M}_\kappa^{-1}$.$\square$

5. On von Neumann’s parameter of extremal Schrödinger operator $T_h$

Recall that, the Phillips-Kato extension problem is about giving a description of all maximal accretive and/or sectorial extensions $\tilde{A}$ of a non-negative symmetric operator $A$ such that $A \subset \tilde{A} \subset \tilde{A}^\ast$. In this and the next sections we discuss the Phillips-Kato extension problem for a non-negative Schrödinger operator with deficiency indices $(1, 1)$ in $\mathcal{H} = L^2(\ell, \infty)$. For the one-dimensional Schrödinger operator $T_h$ ($\text{Im} \ h > 0$) on the semi-axis the Phillips-Kato extension problem in restricted sense was solved by one of the authors in [23] (see also [1], [3], [22]). The solution in terms of boundary values is presented in Theorem 11 below. Our goal, however, is in presenting the necessary condition to the existence of solution for the Phillips-Kato problem in terms of the modulus of von Neumann’s parameter.
Suppose that the symmetric operator $\hat{A}$ of the form (27) with deficiency indices (1,1) is nonnegative, i.e., $(\hat{A}f, f) \geq 0$ for all $f \in \text{Dom}(\hat{A})$.

**Theorem 11** ([22], [23], see also [3]). Let $\hat{A}$ be a nonnegative symmetric Schrödinger operator of the form (27) with deficiency indices (1,1) and locally summable potential in $\mathcal{H} = L^2(\ell, \infty)$. Consider operator $T_h$ of the form (28). Then

1. operator $\hat{A}$ has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension $A_F$ and the Krein-von Neumann extension $A_K$ do not coincide if and only if $m_{\infty}(-0) < \infty$;
2. operator $T_h$ coincides with the Krein-von Neumann extension if and only if $h = -m_{\infty}(-0)$;
3. operator $T_h$ is accretive if and only if

$$\text{Re} h \geq -m_{\infty}(-0);$$

4. operator $T_h$, $(h \neq \bar{h})$ is $\alpha$-sectorial if and only if $\text{Re} h > -m_{\infty}(-0)$ holds; 
5. operator $T_h$, $(h \neq \bar{h})$ is accretive extremal if and only if $\text{Re} h = -m_{\infty}(-0)$;
6. if $T_h$, $(\text{Im} h > 0)$ is $\alpha$-sectorial, then the exact angle of sectoriality $\alpha$ can be calculated as

$$\tan \alpha = \frac{\text{Im} h}{\text{Re} h + m_{\infty}(-0)}.$$ 

It follows from item (2) of Theorem 11 that if $m_{\infty}(-0) = \infty$, then the operator $T_h$ turns into a self-adjoint operator $T_\infty$ corresponding to the Dirichlet problem

$$\begin{cases} T_\infty y = -y'' + q(x)y \\ y(\ell) = 0. \end{cases}$$

For the remainder of this paper we assume that $m_{\infty}(-0) < \infty$. Then according to Theorem 11 above (see also [24], [25]) we have the existence of the operator $T_h$, $(\text{Im} h > 0)$ which is accretive and/or sectorial. The following was shown in [2]. Let $T_h$ $(\text{Im} h > 0)$ be an accretive Schrödinger operator of the form (28). Then for all real $\mu$ satisfying the inequality

$$\mu \geq \frac{(\text{Im} h)^2}{m_{\infty}(-0) + \text{Re} h} + \text{Re} h,$$

the operators (29) form the set of all accretive ($*$)-extensions $\hat{A}$ of the operator $T_h$. The accretive operator $T_h$ has a unique accretive ($*$)-extension $\hat{A}$ if and only if

$$\text{Re} h = -m_{\infty}(-0).$$

In this case this unique ($*$)-extension (and its adjoint) has the form

$$\begin{align*}
\hat{A}y &= -y'' + q(x)y + [hy(\ell) - y'(\ell)] \delta(x - \ell), \\
\hat{A}^* y &= -y'' + q(x)y + [\bar{h}y(\ell) - y'(\ell)] \delta(x - \ell).
\end{align*}$$

Now suppose that $\Theta_{\mu, h}$ of the form (34) is an L-system with the main Schrödinger operator $T_h$ defined by (28). We are going to tackle the Phillips-Kato extension problem with the help of the von Neumann parameter $\kappa$ of $T_h$. Assume also that $\mu$ is given by (41) and hence $T_h$ satisfies the Hypothesis 3 with real $\kappa$ given by (40). Set

$$m_{\infty}(i) = A - iB, \quad B > 0, \quad m_{\infty}(-0) = m, \quad C = (A - m)^2, \quad D = C + B^2 > 0.$$
Then (60) yields
\[
\kappa = \left| \frac{m_{\infty}(i) + h}{m_{\infty}(i) + h} \right| = \left| \frac{A - iB + \text{Re} h + i \text{Im} h}{A - iB + \text{Re} h - i \text{Im} h} \right| = \left| \frac{(A + \text{Re} h) + i(\text{Im} h - B)}{(A + \text{Re} h) - i(\text{Im} h + B)} \right|
\]
\[
= \sqrt{\frac{(A + \text{Re} h)^2 + (\text{Im} h - B)^2}{(A + \text{Re} h)^2 + (\text{Im} h + B)^2}}
\]
or
\[
(63) \quad \kappa^2 = \frac{(A + \text{Re} h)^2 + (\text{Im} h - B)^2}{(A + \text{Re} h)^2 + (\text{Im} h + B)^2}.
\]

Suppose that \(T_h\) is an extremal accretive operator. According to Theorem 11, we have that \(\text{Re} h = -m\). Applying this to (63) and using notations (62) gives
\[
\kappa^2 = \frac{(A - m)^2 + (\text{Im} h - B)^2}{(A - m)^2 + (\text{Im} h + B)^2} = \frac{C + (\text{Im} h)^2 - 2B \text{Im} h + B^2}{C + (\text{Im} h)^2 + 2B \text{Im} h + B^2}
\]
\[
= \frac{(\text{Im} h)^2 - 2B \text{Im} h + D}{(\text{Im} h)^2 + 2B \text{Im} h + D}.
\]

Consider \(\kappa^2\) as a function \(f\) of \(H = \text{Im} h\), that is
\[
(64) \quad f(H) = \kappa^2(H) = \frac{H^2 - 2BH + D}{H^2 + 2BH + D}.
\]

Taking the derivative of \(f(H)\) in (64) and simplifying yields
\[
f'(H) = \frac{4B(H^2 - H)}{(H^2 + BH + D)^2}.
\]

Setting \(f'(H) = 0\) and keeping in mind that \(H = \text{Im} h > 0\) and \(D > 0\) we obtain a critical number
\[
H = \sqrt{D},
\]
that can be checked to be a point of minimum of \(f(H)\) for all \(H > 0\). Also, direct substitution gives
\[
f(\sqrt{D}) = \sqrt{\frac{D - B}{D + B}}.
\]

Consequently,
\[
\frac{\sqrt{D - B}}{\sqrt{D + B}} \leq \kappa^2 < 1,
\]
and hence
\[
\sqrt{\frac{D - B}{D + B}} \leq \kappa < 1,
\]
or, after backward substitution and simplification,
\[
(65) \quad \sqrt{\frac{|m_{\infty}(i)|^2 - 2m_{\infty}(\text{Re}) m_{\infty}(i) + m_{\infty}^2(\text{Re})}{|m_{\infty}(i)|^2 - 2m_{\infty}(\text{Re}) m_{\infty}(i) + m_{\infty}^2(\text{Re}) - \text{Im} m_{\infty}(i)}} \leq \kappa < 1.
\]

Thus, \(T_h\) is extremal accretive operator if and only if (65) holds, that is, the von Neumann parameter \(\kappa\) of \(T_h\) is such that \(\kappa_0 \leq \kappa < 1\), with
\[
\kappa_0 = \sqrt{\frac{|m_{\infty}(i)|^2 - 2m_{\infty}(\text{Re}) m_{\infty}(i) + m_{\infty}^2(\text{Re}) + \text{Im} m_{\infty}(i)}{|m_{\infty}(i)|^2 - 2m_{\infty}(\text{Re}) m_{\infty}(i) + m_{\infty}^2(\text{Re}) - \text{Im} m_{\infty}(i)}}
\]

A sample graph of \(\kappa\) as a function of \(\text{Im} h\) is shown in Figure 1.
We can summarize the above reasoning in the following theorem.

**Theorem 12.** Under the condition (1) of Theorem 11 if a Schrödinger operator $T_h$ of the form (28) with the modulus of von Neumann’s parameter $\kappa$ is extremal accretive, then $\kappa_0 \leq \kappa < 1$, where $\kappa_0$ is given by (66).

The proof follows from part (5) of Theorem 11 and formulas (65) and (66).

The following lemma contains a useful property of the function $m_\infty(z)$ that makes $\kappa_0$ (given by (66)) vanish.

**Lemma 13.** Under the conditions of Theorem 12 the value of $\kappa_0$ given by (66) is zero if and only if
\begin{equation}
\text{Re } m_\infty(i) = m_\infty(-0).
\end{equation}

**Proof.** Following our notation that we set in (62), from (66) we get that
\begin{equation}
\kappa_0 = \sqrt{\frac{D - B}{D + B}}.
\end{equation}
Since $A = \text{Re } m_\infty(i)$ and $m = m_\infty(-0)$, we need to show that $\kappa_0 = 0$ if and only if $A = m$.

If $A = m$, then using (62) we obtain
\[ \sqrt{D - B} = \sqrt{C + B^2} - B = \sqrt{(A - m)^2 + B^2} - B = \sqrt{0 + B^2} - B = 0, \]
and hence the numerator of the fraction in (68) is zero making $\kappa_0 = 0$.

Conversely, assume that $\kappa_0 = 0$. Then $\sqrt{D - B} = 0$ and hence $D = B^2$. This implies $C = A - m = 0$ yielding (67).

Now we will state and prove an inverse version of Theorem 12.

**Theorem 14.** Let $\hat{A}$ be the same as in Theorems 11 and 12. If $\kappa_0 \leq \kappa < 1$, with $\kappa_0$ given by (66), then there exist two (just one if $\kappa = \kappa_0$) extremal dissipative Schrödinger operators $T_h$, ($\hat{A} \subset T_h \subset \hat{A}^*$), of the form (28) such that their modulus of von Neumann’s parameters equals $\kappa$.

In this case, the boundary value(s) of $h$ are given by
\begin{equation}
h = -m_\infty(-0) + \frac{B(1 + \kappa^2) \pm \sqrt{4B^2\kappa^2 - C(1 - \kappa^2)^2}}{1 - \kappa^2} i,
\end{equation}
where $B$ and $C$ are given by (62).

**Proof.** Let $T_h$ be a Schrödinger operator of the form (28) with the same potential as in (27). All we need is to show that there exists a value of $h$ that makes $T_h$ an extremal dissipative quasi-self-adjoint extension of $\hat{A}$ whose modulus of von Neumann’s parameter is equal to the given $\kappa \in [\kappa_0, 1)$. Note that equation (64) holds for any $T_h$ with von Neumann’s parameter $\kappa$. Setting Re $h = -m$ in (64) will guarantee (see Theorem 11) that $T_h$ is extremal and yields (63), that is,
\begin{equation}
\kappa^2 = \frac{x^2 - 2Bx + D}{x^2 + 2Bx + D},
\end{equation}
where $x = \text{Im } h$. Modifying (70) leads to the quadratic equation
\begin{equation}
x^2 - 2B \left( \frac{1 + \kappa^2}{1 - \kappa^2} \right) x + D = 0.
\end{equation}
We are going to show that equation (71) has at least one real positive solution. Setting
\[ \xi = \frac{1 + \kappa^2}{1 - \kappa^2}, \]
note that when \( \kappa \in [\kappa_0, 1) \) we have \( \xi \in [\xi_0, +\infty) \), where
\[ \xi_0 = \frac{1 + \sqrt{\mathcal{D} - B}}{1 - \sqrt{\mathcal{D} + B}} = \frac{\sqrt{\mathcal{D}}}{B}. \]

Consider the discriminant of the quadratic equation (71) as a function of \( \xi \)
\[ f(\xi) = 4B^2\xi^2 - 4D. \]
Then its derivative \( f'(\xi) = 8B^2\xi \) is always positive on \( \xi \in [\xi_0, +\infty) \), indicating that \( f(\xi) \) is an increasing function of \( \xi \in [\xi_0, +\infty) \). Moreover, the direct check reveals that
\[ f(\xi_0) = 4B^2\xi_0^2 - 4D = 0, \]
and hence \( f(\xi) \) takes positive values at \( \xi \in (\xi_0, +\infty) \). Applying the quadratic formula to equation (71) and taking into account that \( B > 0 \), \( D > 0 \) and \( \xi > 0 \), we obtain that
\[ x = \frac{2B\xi \pm \sqrt{4B^2\xi^2 - 4D}}{2} = B\xi \pm \sqrt{B^2\xi^2 - D} \]
yields two positive real solutions. Therefore, the value of \( h \) such that \( \text{Re} \, h > -m_\infty(-0) \) and \( \text{Im} \, h \) equal to one of the values of \( x \) from (72) is the one that makes \( T_h \) an extremal dissipative quasi-self-adjoint extension of \( \dot{A} \). Substituting the expression for \( \xi \) into (72) and simplifying yields (69).

Clearly, if \( \kappa \in (\kappa_0, 1) \), then (69) yields two distinct values of the boundary parameter \( h \) and hence there are two different Schrödinger operators \( T_h \) with von Neumann’s parameter \( \kappa \). In the case when \( \kappa = \kappa_0 \) the expression under the radical in (69) turns into zero and hence there is only one value of \( h \) yielding a unique extremal Schrödinger operator \( T_h \) with von Neumann’s parameter \( \kappa = \kappa_0 \). □

Figure 1 gives good visual illustration of Theorem 14. There are two extremal operators \( T_{h(-)} \) and \( T_{h(+)} \) shown that share the same (modulus of) von Neumann’s parameter \( \kappa \). Here \( h(-) \) and \( h(+) \) are the values of the parameter \( h \) in (69) that correspond to the (−) and (+) signs of the formula.

6. On von Neumann’s parameter of sectorial Schrödinger operator \( T_h \)

The main goal of this section is (for a given \( \beta \in (0, \pi/2) \)) to explicitly describe the interval for the modulus of von Neumann’s parameter \( \kappa \) of the operator \( T_h \) such that \( T_h \) is \( \beta \)-sectorial. Throughout this section we keep the assumption of the previous section that the symmetric operator \( \dot{A} \) of the form (27) with deficiency indices (1,1) is nonnegative.

Consider \( T_h \) of the form (28) and assume that \( T_h \) is \( \beta \)-sectorial. Then according to Theorem 14 we have that \( \text{Re} \, h > -m_\infty(-0) \) and (see (69))
\[ \cot \beta = \frac{\text{Re} \, h + m_\infty(-0)}{\text{Im} \, h}, \]
or equivalently
\[ \text{Re} \, h + m_\infty(-0) = \cot \beta \cdot \text{Im} \, h. \]
As in Section 5 we assume that Θ is an L-system of the form (34) with the main β-sectorial Schrödinger operator $T_h$. Suppose also that $\mu$ is given by (41) and hence $T_h$ satisfies Hypothesis 3 with real $\kappa$ given by (40). Once again, to simplify the calculation process we use our conventions described in (62). Combining (63) with (73) yields

$$\kappa^2 = \left( A + \text{Re} \, h \right)^2 + (\text{Im} \, h - B)^2$$

$$= \frac{(A - m + \cot \beta \cdot \text{Im} \, h)^2 + (\text{Im} \, h - B)^2}{(A - m + \cot \beta \cdot \text{Im} \, h)^2 + (\text{Im} \, h + B)^2}$$

$$= \frac{(A - m)^2 + 2(A - m) \cot \beta \cdot \text{Im} \, h + \cot^2 \beta (\text{Im} \, h)^2 + (\text{Im} \, h)^2 - 2B \text{Im} \, h + B^2}{(A - m)^2 + 2(A - m) \cot \beta \cdot \text{Im} \, h + \cot^2 \beta (\text{Im} \, h)^2 + (\text{Im} \, h)^2 + 2B \text{Im} \, h + B^2}$$

$$= \frac{(\cot^2 \beta + 1)(\text{Im} \, h)^2 + 2((A - m) \cot \beta - B) \text{Im} \, h + (A - m)^2 + B^2}{(\text{Im} \, h)^2 + 2 \sin^2 \beta ((A - m) \cot \beta + B) \text{Im} \, h + D \sin^2 \beta}.$$

As in Section 5 we consider $\kappa^2$ as a function $f$ of $H = \text{Im} \, h$, that is

$$f(H) = \kappa^2(H) = \frac{H^2 + 2 \sin^2 \beta ((A - m) \cot \beta - B) H + D \sin^2 \beta}{H^2 + 2 \sin^2 \beta ((A - m) \cot \beta + B) H + D \sin^2 \beta}.$$

Taking the derivative of $f(H)$ in (74) we get

$$f'(H) = \frac{4 \sin \beta \cdot B (H^2 - D \sin^2 \beta)}{(H^2 + 2 \sin^2 \beta ((A - m) \cot \beta + B) H + D \sin^2 \beta)^2}.$$

Setting $f'(H) = 0$ and keeping in mind that $H = \text{Im} \, h > 0$ and $D > 0$ we obtain a critical number

$$H = \sin \beta \cdot \sqrt{D},$$
that can be checked to be a point of minimum of \( f(H) \) for all \( H > 0 \). Also, direct substitution gives
\[
f(\sin \beta \sqrt{D}) = \frac{\sqrt{D} + (A - m) \cos \beta - B \sin \beta}{\sqrt{D} + (A - m) \cos \beta + B \sin \beta}
\]
Consequently,
\[
\frac{\sqrt{D} + (A - m) \cos \beta - B \sin \beta}{\sqrt{D} + (A - m) \cos \beta + B \sin \beta} \leq \kappa^2 < 1,
\]
and hence
\[
\sqrt{D} + (A - m) \cos \beta - B \sin \beta \leq \kappa < 1,
\]
or, after backward substitution and simplification, \( \kappa_0 \leq \kappa < 1 \), where
\[\kappa_0 = \sqrt{\frac{\sqrt{D} + (A - m) \cos \beta - B \sin \beta}{\sqrt{D} + (A - m) \cos \beta + B \sin \beta}}.\]
In the above formula
\[\sqrt{D} = \sqrt{|m_\infty(i)|^2 - 2m_\infty(-0)\text{Re} \ m_\infty(i) + m_\infty^2(-0)},\]
\[A - m = \text{Re} \ m_\infty(i) - m_\infty(-0), \quad B = - \text{Im} \ m_\infty(i).\]
It is easy to see that when \( \beta = \pi/2 \), then (75) and (76) match the previous (extremal) case in the formula (66). A sample graph of \( \kappa \) as a function of \( \text{Im} \ h \) is shown in Figure 2.

The following two theorems contain the main result of this section.

**Theorem 15.** Under the condition (1) of Theorem 11 if a Schrödinger operator \( T_h \) of the form (28) with the modulus of von Neumann’s parameter \( \kappa \) is \( \beta \)-sectorial (\( \beta \in (0, \pi/2) \)), then \( 0 < \kappa_0 \leq \kappa < 1 \), where \( \kappa_0 \) is given by (75) and (76).

**Proof.** The proof follows from part (4) of Theorem 11 and the reasoning of this section. We just note that \( 0 < \kappa_0 \) because of the fact that a \( \beta \)-sectorial (\( \beta \in (0, \pi/2) \)) operator cannot have the von Neumann parameter equal to zero (see Lemma 2). \( \square \)

Now we will state and prove an inverse version of Theorem 15 that is the analog of Theorem 13 from Section 5.

**Theorem 16.** Let \( \hat{A} \) be the same as in Theorem 15. Given \( \beta \in (0, \pi/2) \) and \( 0 < \kappa_0 \leq \kappa < 1 \), where \( \kappa_0 \) is given by (75), (76), there exist two (one if \( \kappa = \kappa_0 \)) \( \beta \)-sectorial dissipative Schrödinger operators \( T_h, (\hat{A} \subset T_h \subset \hat{A}^*) \), of the form (28) such that the modulus of their von Neumann’s parameters equals \( \kappa \). Moreover, the boundary value(s) of \( h \) are given by
\[h = -m + \sin^2 \beta (\cot \beta + i) \left( \xi B - (A - m) \cot \beta \pm \sqrt{E} \right),\]
where \( m, B \) and \( C \) are given by (62), \( \xi = \frac{1+\kappa^2}{2\kappa} \) and
\[E = [(A - m) \cot \beta - \xi B]^2 - D \csc^2 \beta.\]
Proof. We are going to use a similar to the proof of Theorem \textup{(77)} method to confirm \textup{(77)}. We notice that the rational function in formula \textup{(74)} resembles the one in \textup{(70)}. Therefore, the following formal change of coefficients

\begin{equation}
B_1 = \sin^2 \beta ((A - m) \cot \beta - B), \quad B_2 = \sin^2 \beta ((A - m) \cot \beta + B),
\end{equation}

leads to

\begin{equation}
\kappa^2 = \frac{x^2 + 2B_1 x + D \sin^2 \beta}{x^2 + 2B_2 x + D \sin^2 \beta},
\end{equation}

where \(x = \text{Im} \, h\). Modifying (80) brings us to the quadratic equation

\((\kappa^2 - 1)x^2 + 2(\kappa^2 B_2 - B_1)x + (\kappa^2 - 1)D \sin^2 \beta = 0.\)

Taking into account that \(\kappa^2 B_2 - B_1 = \sin^2 \beta [(\kappa^2 - 1)(A - m) \cot \beta + (\kappa^2 + 1)B]\) we obtain

\[x^2 + 2 \sin^2 \beta \left((A - m) \cot \beta + \frac{\kappa^2 + 1}{\kappa^2 - 1} B\right)x + D \sin^2 \beta = 0.\]

Setting

\[\xi = \frac{1 + \kappa^2}{1 - \kappa^2},\]

note that when \(\kappa \in [\kappa_0, 1]\) we have \(\xi \in [\xi_0, +\infty)\), where (see (75))

\[\xi_0 = \frac{1 + \kappa_0^2}{1 - \kappa_0^2} = \frac{1 + \sqrt{D + (A - m) \cos \beta} - B \sin \beta}{1 - \sqrt{D + (A - m) \cos \beta} + B \sin \beta} = \frac{\sqrt{D + (A - m) \cos \beta}}{B \sin \beta}.\]

It is easy to see that if \(\beta = \pi/2\), then the value of \(\xi_0\) above matches the corresponding value of \(\xi_0\) in the proof of Theorem \textup{(14)} written for the extremal case. Furthermore, the above quadratic equation transforms into (see (79))

\begin{equation}
(csc^2 \beta) x^2 + 2 [(A - m) \cot \beta - \xi B] x + D = 0.
\end{equation}

Consider the discriminant of the quadratic equation (81) as a function of \(\xi\)

\[f(\xi) = 4 [(A - m) \cot \beta - \xi B]^2 - 4D \csc^2 \beta.\]

Then its derivative \(f'(\xi) = 8B^2 \xi - 8B(A - m) \cot \beta\) gives rise to a critical point \(\xi_c = \frac{A - m}{2B} \cot \beta\) that is clearly smaller than \(\xi_0\), that is \(\xi_c < \xi_0\). Consequently, \(f'(\xi)\) is always positive on \(\xi \in [\xi_0, +\infty)\) indicating that \(f(\xi)\) is an increasing function of \(\xi \in [\xi_0, +\infty)\). Moreover, the direct check reveals that

\[f(\xi_0) = 4[(A - m) \cot \beta - \xi_0 B]^2 - 4D \csc^2 \beta = 4 \left(\frac{-\sqrt{D}}{\sin \beta}\right)^2 - 4D \csc^2 \beta = 0,\]

and hence \(f(\xi)\) takes positive values on \(\xi \in (\xi_0, +\infty)\). Applying the quadratic formula to equation (81) and taking into account that \(B > 0\), \(D > 0\) and \(\xi > 0\), we obtain that

\begin{equation}
x = \frac{2[\xi B - (A - m) \cot \beta] \pm \sqrt{4[(A - m) \cot \beta - \xi B]^2 - 4D \csc^2 \beta}}{2 \csc^2 \beta}
\end{equation}

\[= \sin^2 \beta \left[\xi B - (A - m) \cot \beta \pm \sqrt{(A - m) \cot \beta - \xi B}^2 - D \csc^2 \beta\right]
\]

\[= \sin^2 \beta \left[\xi B - (A - m) \cot \beta \pm \sqrt{E}\right].\]
yields two positive real solutions. Therefore, the value of \( h \) such that \( \text{Re} h = -m_\infty(-0) + (\cot \beta)x \), where \( x = \text{Im} h \) is given by (82), defines \( \beta \)-sectorial dissipative Schrödinger operators \( T_h \). Substituting the expression for \( \xi \) into (82) and simplifying yields (77).

It can be clearly seen from (77) that if \( \kappa \in (\kappa_0, 1) \), then there are two distinct values of the boundary parameter \( h \) and hence two different \( \beta \)-sectorial Schrödinger operators \( T_h \) with von Neumann’s parameter. In the case when \( \kappa = \kappa_0 \) there is only one \( \beta \)-sectorial Schrödinger operators \( T_h \) with this property.

\[ \text{Figure 2. Two } \beta \text{-sectorial operators } T_h \text{ with the same } \kappa \]

Figure 2 gives good visualization of Theorem 16. It depicts two \( \beta \)-sectorial operators \( T_h(-) \) and \( T_h(+) \) shown that share the same (modulus of) von Neumann’s parameter \( \kappa \). Here \( h(-) \) and \( h(+) \) are the values of the parameter \( h \) in (77) that correspond to the (−) and (+) signs of the formula.

We finalize the section with a lemma that contains a very useful property of the function \( m_\infty(z) \) related to a nonnegative symmetric operator \( \hat{A} \) of the form (27).

**Lemma 17.** Let \( \hat{A} \) be a nonnegative symmetric Schrödinger operator of the form (27) with deficiency indices \((1, 1)\) and locally summable potential in \( \mathcal{H} = L^2(\ell, \infty) \). If the corresponding Weyl function \( m_\infty(z) \) has the property \( m_\infty(-0) < \infty \), then
\[ \text{Re} m_\infty(i) \geq m_\infty(-0) \text{.} \]

**Proof.** We stick to the notation that we set in (62), that is \( m_\infty(i) = A - iB, \ B > 0, \ m_\infty(-0) = m, \ C = (A - m)^2, \) and \( D = C + B^2 > 0. \) We need to show that \( A \geq m. \) Assume the contrary, \( A < m. \) Since \( m < \infty, \) for every \( \beta \in (0, \pi/2) \) there exists a \( \beta \)-sectorial operator \( T_h \) of the form (28) (see [2, 23]). Take a \( \beta_0 \) such that
\[ \cos \beta_0 = \frac{m - A}{\sqrt{D}} \quad \text{or} \quad \beta_0 = \arccos \left( \frac{m - A}{\sqrt{D}} \right) . \]

Since under our assumption \( A < m, \) then \( \beta_0 \in (0, \pi/2) \) and hence there is a \( \beta_0 \)-sectorial operator \( T_h \) of the form (28). Moreover, according to Theorem 16 the
boundary value $h$ of this $T_h$ is determined by (77) and its von Neumann's parameter is $\kappa_0$ given by (75). Using the basic trig identity and (62) we get

\[(85) \sin \beta_0 = \sqrt{1 - \cos^2 \beta_0} = \sqrt{1 - \frac{C}{D}} = \frac{\sqrt{B^2}}{\sqrt{D}} = B.\]

Substituting $\cos \beta_0$ and $\sin \beta_0$ from (84) and (85) into (75) reveals

\[\kappa_0^2 = \frac{\sqrt{D} - \sqrt{C}}{\sqrt{D} - \sqrt{C}} \cdot \frac{\sqrt{C}}{\sqrt{B}} - B \cdot \frac{B}{\sqrt{D}} \cdot \sqrt{C} = \frac{D - C - B^2}{D + B^2 - C} = 0,
\]

or $\kappa_0 = 0$. Hence under our assumption we have the existence of a $\beta_0$-sectorial operator $T_h$ with von Neumann's parameter $\kappa_0 = 0$. This contradicts Lemma 2 and thus our assumption that $A < m$ is wrong. Consequently, $A \geq m$. □

7. c-Entropy of an L-system and minimal dissipation coefficient

We begin with introducing a concept of the c-entropy of an L-system.

**Definition 18.** Let $\Theta$ be an L-system of the form (17). The quantity

\[(86) S = -\ln(|W_{\Theta}(-i)|),\]

where $W_{\Theta}(z)$ is the transfer function of $\Theta$, is called the coupling entropy (or c-entropy) of the L-system $\Theta$.

There is an alternative operator-theoretic way to define the c-entropy. If $T$ is the main operator of the L-system $\Theta$ and $\kappa$ is von Neumann’s parameter of $T$ in some basis $g_{\pm}$, then (see [7]) $|W_{\Theta}(-i)| = |\kappa|$ and hence

\[(87) S = -\ln(|W_{\Theta}(-i)|) = -\ln(|\kappa|).\]

We emphasize that c-entropy defined by formula (87) does not depend on the choice of deficiency basis $g_{\pm}$ and moreover is an additive function with respect to the coupling of L-systems (see [7]). Note that if, in addition, the point $z = i$ belongs to $\rho(T)$, then we also have that

\[(88) S = \ln(|W_{\Theta}(i)|) = \ln(1/|\kappa|) = -\ln(|\kappa|).\]

This follows from the known (see [2]) property of the transfer functions for L-systems that states that $W_{\Theta}(z)\overline{W_{\Theta}(\bar{z})} = 1$ and the fact that $|W_{\Theta}(i)| = 1/|\kappa|$ (see [6]).

Next we pose for the following optimization problem associated with the c-entropy of Schrödinger L-systems:

- **Describe L-systems with the Schrödinger operator with a given c-entropy and minimal dissipation coefficient.**

The main goal of this section is to provide a solution of the problem above for the L-systems whose impedance functions belong to the (generalized) Donoghue classes $M_{\|}$, $M_{\kappa}$, and $M_{\kappa}^{-1}$.

Let $\Theta_{\mu,h}$ be a Schrödinger L-system one of (34) based upon the symmetric operator $\hat{A}$ of the form (27) with corresponding Weyl function $m_{\infty}(z)$. Also, let $\kappa$ denote the modulus of the von Neumann parameter of the main operator $T_h$ of the L-system $\Theta_{\mu,h}$. Suppose that the impedance function $V_{\Theta_{\mu,h}}(z)$ given by (36), belongs to a generalized Donoghue class. Then

\[(89) V_{\Theta_{\mu,h}}(i) = ai,\]
for some $a > 0$. Representation (89) has been justified in the proof of Theorem 9 for $0 < a < 1$ (see (51)) and can be verified for $a > 1$ (cf. Theorem 10). Depending on the value of $a$ in (89) we (see (13) and (14)) have

$$
\kappa = \begin{cases}
\frac{a}{a+1}, & 0 < a < 1; \\
0, & a = 1; \\
\frac{a-1}{a+1}, & a > 1.
\end{cases}
$$

In order to address the problem above we point out that in accordance with (87) the knowledge of c-entropy fixes the value of $\kappa = |\kappa|$ and hence of $a$ as well. Moreover, by Theorem 9 for the coefficient of dissipation $\text{Im} \, h$ we have the explicit representation

$$
\text{Im} \, h(\mu) = -\frac{ad^3 + ad(c + \mu)^2}{a^2d^2 + (c + \mu)^2},
$$

where $c = \text{Re} \, m_\infty(i)$, $d = \text{Im} \, m_\infty(i)$, and $\mu$ is the parameter that determines the L-system $\Theta_{\mu,h}$. Thus, to minimize $\text{Im} \, h$ for a given $a > 0$ we seek the corresponding critical value of $\mu$. Taking the derivative of $\text{Im} \, h$ as a function of $\mu$, simplifying, and setting it equal to zero yields

$$
\frac{d}{d\mu} (\text{Im} \, h) = \frac{2ad^3(c + \mu)(1 - a^2)}{(a^2d^2 + (c + \mu)^2)^2} = 0.
$$

Therefore,

$$
\mu = -c = -\text{Re} \, m_\infty(i)
$$

is the only critical point of $\text{Im} \, h(\mu)$. It is easy to check that this critical value of $\mu$ gives the minimum for $\text{Im} \, h$ whenever $a > 1$ and the maximum whenever $0 < a < 1$. Also, clearly $\text{Im} \, h \equiv -d = -\text{Im} \, m_\infty(i)$ if $a = 1$. Moreover, as we have shown in the proof of Theorem 9 in this case

$$
\text{Im} \, h = -(1/a) \text{Im} \, m_\infty(i).
$$

This reasoning leads to the following result.

**Theorem 19.** Assume that an L-system $\Theta_{\mu,h}$ of the form (34) with a main operator $T_h$ has c-entropy $S$. Then, under the constraint that the impedance function $V_{\Theta_{\mu,h}}(z)$ belongs to the generalized Donoghue class $M^{-1}_\kappa$ with $\kappa = e^{-S}$, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation

$$
D_{\text{min}} = \text{Im} \, h = -\left(\frac{e^S - 1}{e^S + 1}\right) \text{Im} \, m_\infty(i) = -\tanh\left(\frac{S}{2}\right) \text{Im} \, m_\infty(i).
$$

In this case $\Theta_{\mu,h}$ is uniquely determined by the parameters

$$
h = -\text{Re} \, m_\infty(i) - i \tanh\left(\frac{S}{2}\right) \text{Im} \, m_\infty(i) \quad \text{and} \quad \mu = -\text{Re} \, m_\infty(i).
$$

**Proof.** Since out L-system $\Theta_{\mu,h}$ has a fixed given c-entropy $S$, then the modulus of von Neumann’s parameter $\kappa$ of its main operator $T_h$ is given by $\kappa = e^{-S}$. Moreover, since $V_{\Theta_{\mu,h}}(z) \in M^{-1}_\kappa$, then $V_{\Theta_{\mu,h}}(i) = a i$ for some $a > 1$. Using this value of $\kappa$ and formula (90) we obtain

$$
a = \frac{1 + \kappa}{1 - \kappa} > 1.
$$
We have
\[ \frac{1}{a^*} = \frac{1 - e^{-S}}{1 + e^{-S}} = \frac{e^S - 1}{e^S + 1} = \tanh \left( \frac{S}{2} \right). \]

The rest follows from the fact that \( \mu = -c = -\text{Re} m_\infty(i) \) is the point of minimum of \( \text{Im} h \) if \( a > 1 \), formulas (48) (in the context of Theorem 18) and (92). \( \square \)

The L-system \( \Theta_{\mu, h} \) with the minimum coefficient of dissipation in Theorem 19 can be written explicitly. In particular its state space operator \( A_{\mu, h} \) can be described by formulas (29) with the values of \( h \) and \( \mu \) given by (91). That is,
\begin{align*}
\kappa_{\mu, h} y &= -y'' + q(x)y + \frac{y'(x) - hy(x)}{\text{Re} m_\infty(i) + \hbar} \left[ \delta'(x) - \text{Re} m_\infty(i) \delta(x) \right], \\
A_{\mu, h} y &= -y'' + q(x)y + \frac{y'(x) - iy(x)}{\text{Re} m_\infty(i) + \hbar} \left[ \delta'(x) - \text{Re} m_\infty(i) \delta(x) \right],
\end{align*}
where
\[ h = -\text{Re} m_\infty(i) - i \tanh \left( \frac{S}{2} \right) \text{Im} m_\infty(i). \]

An analogous result similar to Theorem 19 takes place whenever \( 0 < a < 1 \).

**Theorem 20.** Assume that an L-system \( \Theta_{\mu, h} \) of the form (44) with a main operator \( T_h \) has c-entropy \( S \). Then, under the constraint that the impedance function \( V_{\Theta_{\mu, h}}(z) \) belongs to the generalized Donoghue class \( M_\kappa \) with \( \kappa = e^{-S} \), the L-system \( \Theta_{\mu, h} \) has the minimum coefficient of dissipation
\[ D_{\text{min}} = -\tanh \left( \frac{S}{2} \right) \text{Im} m_\infty(i). \]

In this case, \( \Theta_{\mu, h} \) is uniquely determined by the parameters
\[ h = -\text{Re} m_\infty(i) - i \tanh \left( \frac{S}{2} \right) \text{Im} m_\infty(i) \quad \text{and} \quad \mu = \infty. \]

**Proof.** Since \( V_{\Theta_{\mu, h}}(z) \in M_\kappa \), we have \( V_{\Theta_{\mu, h}}(i) = ai \) for some \( 0 < a < 1 \). Hence by (90), since \( 0 < a < 1 \), we have \( \kappa = e^{-S} = \frac{1}{1 + a} \). It was shown in 24 that there exists a unique Schrödinger L-system \( \Theta_{\tilde{\mu}, h} \) of the form (44) with the same main operator \( T_h \) as in \( \Theta_{\mu, h} \) such that \( V_{\Theta_{\mu, h}}(z) = -1/V_{\Theta_{\tilde{\mu}, h}}(z) \). Indeed, this L-system \( \Theta_{\tilde{\mu}, h} \) is determined via (29) by the parameters \( h \) and
\[ \tilde{\mu} = \frac{\mu \text{Re} h - |h|^2}{\mu - \text{Re} h}. \]

Furthermore, \( V_{\Theta_{\mu, h}}(i) = \tilde{\mu}i = -1/V_{\Theta_{\tilde{\mu}, h}}(i) = i/a \) and hence \( \tilde{\mu} = 1/a > 1 \). Both L-systems \( \Theta_{\mu, h} \) and \( \Theta_{\tilde{\mu}, h} \) share the same main operator \( T_h \) and thus the same c-entropy \( S \). Repeating the argument in the proof of Theorem 19 we obtain
\[ \tilde{\mu} = \frac{1 + \kappa}{1 - \kappa} > 1 \quad \text{and} \quad \frac{1}{\tilde{\mu}} = \frac{1}{1/a} = \frac{e^S - 1}{e^S + 1} = \tanh \left( \frac{S}{2} \right). \]

Applying (92) yields
\[ \text{Im} h = D_{\text{min}} = -\frac{1}{a} \text{Im} m_\infty(i) = -\left( \frac{e^S - 1}{e^S + 1} \right) \text{Im} m_\infty(i) = -\tanh \left( \frac{S}{2} \right) \text{Im} m_\infty(i). \]
To find \( \text{Re} h \) in this case we substitute \( \tilde{\mu} = -c = -\text{Re} m_{\infty}(i) \) in (29). In order to obtain the value of \( \mu \) that defines \( \Theta_{\mu,h} \) we set \( \tilde{\mu} = -c \) in (97) and solve for \( \mu \). We have

\[
- c = \frac{\mu \text{Re} h - |h|^2}{\mu - \text{Re} h},
\]

yielding

\[
\mu = \frac{c \text{Re} h + |h|^2}{c + \text{Re} h} = \infty.
\]

Thus (96) takes place and the proof is complete. \( \Box \)

We note that if \( a = 1 \) in the above consideration, then the \( c \)-entropy is infinite, \( S = +\infty \), and \( \kappa = 0 \) implying

\[
\text{Im} h = -\text{Im} m_{\infty}(i)
\]

(see Theorem 8).

As before, the L-system \( \Theta_{\mu,h} \) with the given \( c \)-entropy and minimum coefficient of dissipation in the statement of Theorem 20 can be written explicitly. In particular, its state space operator \( \mathcal{A} \) is described by formulas (29) with the values of \( h \) given by (96), and \( \mu = \infty \) as follows:

\[
\begin{align*}
\mathcal{A}_{\infty,h} y &= -y'' + q(x)y - |y'(\ell) - hy(\ell)| \delta(x - \ell), \\
\mathcal{A}^*_{\infty,h} y &= -y'' + q(x)y - |y'(\ell) - \overline{h} y(\ell)| \delta(x - \ell),
\end{align*}
\]

where

\[
h = -\text{Re} m_{\infty}(i) - i \tanh \left( \frac{S}{2} \right) \text{Im} m_{\infty}(i).
\]

8. L-systems with Schrödinger’s operator and maximal \( c \)-entropy

In this section we set focus on the second part of the dual \( c \)-entropy problem that was announced in Introduction, that is:

- Describe an L-system with Schrödinger main operator with a given dissipation coefficient and the maximal \( c \)-entropy.

Again we are going to consider L-systems with Schrödinger operator whose impedance functions belong to the generalized Donoghue classes \( \mathcal{M}, \mathcal{M}_{\kappa}, \) and \( \mathcal{M}_{-1} \). Within this class of L-systems we will describe the ones that have a given dissipation coefficient and the maximal finite \( c \)-entropy.

Let \( \Theta_{\mu,h} \) be a Schrödinger L-system of the form (34). Let also \( \kappa \) be the von Neumann parameter of the main operator \( T_h \) in \( \Theta_{\mu,h} \). Suppose \( V_{\Theta_{\mu,h}}(z) \) is the impedance function of (36) that belongs to one of the generalized Donoghue classes. Then, as we have shown in the proof of Theorem 9 the relation \( V_{\Theta_{\mu,h}}(i) = ai \) is given by (31) for some \( a > 0 \). Depending on the value of \( a \), the parameter \( \kappa \) is related to \( a \) via (90).

To address the problem of maximizing the \( c \)-entropy using a given dissipation coefficient we start at relation (31). Recall the notations we set in (32), that is \( c = \text{Re} m_{\infty}(i), \ d = \text{Im} m_{\infty}(i), \ x = \text{Re} h, \ y = \text{Im} h \). As we have shown in the proof of Theorem 9 relation (31) leads to (53) which (after equating real and imaginary parts on both sides of (53)) yields

\[
(c + \mu)y = ad(x - \mu),
\]

\[
dy = ac(\mu - x) + a\mu x - ax^2 - ay^2.
\]
Solving the first equation in (99) gives us
\[ x - \mu = \frac{(c + \mu)y}{ad} \quad \text{and} \quad \mu = \frac{adx - cy}{ad + y}. \tag{100} \]
Substituting the expressions from (100) into the second equation (99) yields
\[ dy = -ac \cdot \frac{(c + \mu)y}{ad} - ax \cdot \frac{(c + \mu)y}{ad} - ay^2, \]
or (after cancelation)
\[ d^2 = -c(c + \mu) - x(c + \mu) - ady. \]
Solving for \( a \) results in
\[ a = -\frac{(c + \mu)(c + x) + d^2}{dy}. \]
Plugging the expression for \( \mu \) from (100) in the above and simplifying leads us to
the quadratic equation for \( a \)
\[ dy a^2 + \left[ (c + x)^2 + y^2 + d^2 \right] a + dy = 0. \tag{101} \]
Notice the discriminant \( D \) of the quadratic equation (101) is non-negative and
equals zero if only if \( c = -x \) and \( y = -d \). Indeed,
\[ D = [(c + x)^2 + y^2 + d^2] - 4d^2y^2 = ((c + x)^2 + y^2 + d^2 - 2dy)((c + x)^2 + y^2 + d^2 + 2dy) = ((c + x)^2 + (y - d)^2)((c + x)^2 + (y + d)^2) \geq 0. \]
Therefore, if \( c \neq -x \) or \( y \neq -d \) equation (101) has two distinct real roots
\[ a_{1,2} = \frac{-(c + x)^2 - y^2 - d^2 \pm \sqrt{[(c + x)^2 + y^2 + d^2] - 4d^2y^2}}{2dy}. \tag{102} \]
Set
\[ \nu = (c + x)^2 + y^2 + d^2. \tag{103} \]
Our goal is to show that
\[ a_1 = \frac{-\nu + \sqrt{\nu^2 - 4d^2y^2}}{2dy} < 1. \]
Since \( d < 0 \) and \( y > 0 \), the above inequality is equivalent to
\[ -\nu + \sqrt{\nu^2 - 4d^2y^2} > 2dy \quad \text{or} \quad \nu^2 - 4d^2y^2 > (\nu + 2dy)^2, \]
which transforms to
\[ (\nu + 2dy)^2 - (\nu + 2dy)(\nu - 2dy) < 0 \quad \text{or} \quad 4dy(\nu + 2dy) < 0. \]
Taking into account that \( 4dy < 0 \), we get
\[ \nu + 2dy = (c + x)^2 + y^2 + d^2 = (c + x)^2 + (y + 2)^2 > 0, \]
for \( c \neq -x \) or \( y \neq -d \). Reversing the argument and following the chain of equivalent inequalities backward we obtain
\[ a_1 = \frac{-(c + x)^2 - y^2 - d^2 + \sqrt{[(c + x)^2 + y^2 + d^2] - 4d^2y^2}}{2dy} < 1, \tag{104} \]
for \( c \neq -x \) or \( y \neq -d \).
Our next goal is to show that
\[
a_2 = \frac{-\nu - \sqrt{\nu^2 - 4d^2y^2}}{2dy} > 1,
\]
for \(c \neq -x\) or \(y \neq -d\). The above inequality is equivalent to
\[
-\nu - \sqrt{\nu^2 - 4d^2y^2} < 2dy,
\]
which transforms to
\[
\sqrt{\nu^2 - 4d^2y^2} > -\nu - 2dy = -(c + x)^2 - y^2 - d^2 - 2dy = -(c + x)^2 - (y + d)^2.
\]
The last inequality is always true for any \(c \neq -x\) or \(y \neq -d\) (we are comparing a positive number on the left to a negative number on the right). As before, we pull the chain of equivalent inequalities backward to obtain
\[
a_2 = \frac{-(c + x)^2 - y^2 - d^2 - \sqrt{[(c + x)^2 + y^2 + d^2]^2 - 4d^2y^2}}{2dy} > 1,
\]
for \(c \neq -x\) or \(y \neq -d\).

Finally we observe that substituting \(c = -x\) and \(y = -d\) into (102) results in
\[
a = a_1 = a_2 = 1, \quad c = -x, \quad y = -d.
\]

Now we are ready to state the following.

**Theorem 21.** Assume that an L-system \(\Theta_{\mu, h}\) of the form (34) with a main operator \(T_h\) and a coefficient of dissipation \(D = \text{Im} \, h \neq -\text{Im} \, m_\infty(i)\). Then, under the constraint that the impedance function \(V_{\Theta_{\mu, h}}(z) \in \mathcal{M}_\kappa\) for some \(0 < \kappa < 1\), the L-system \(\Theta_{\mu, h}\) has the maximum finite \(c\)-entropy \(S_{\text{max}}\) if and only if it is determined by parameters

\[
(107) \quad h = -\text{Re} \, m_\infty(i) + iD \quad \text{and} \quad \mu = \begin{cases} 
-\text{Re} \, m_\infty(i), & \text{if } D > -\text{Im} \, m_\infty(i); \\
\infty, & \text{if } D < -\text{Im} \, m_\infty(i). 
\end{cases}
\]

In this case, \(S_{\text{max}}\) is determined by the formula

\[
(108) \quad S_{\text{max}} = \ln |D - \text{Im} \, m_\infty(i)| - \ln |D| + \text{Im} \, m_\infty(i)|.
\]

**Proof.** Since \(V_{\Theta_{\mu, h}}(z) \in \mathcal{M}_\kappa\), then it satisfies a normalization condition

\[
V_{\Theta_{\mu, h}}(i) = ai \quad \text{for} \quad a = \frac{1 - \kappa}{1 + \kappa}.
\]

Moreover, since \(0 < a < 1\) in the above, then formula (104) takes place and hence

\[
(109) \quad a(x) = \frac{-(c + x)^2 - D^2 - d^2 + \sqrt{[(c + x)^2 + D^2 + d^2]^2 - 4d^2D^2}}{2dD}.
\]

Taking the derivative of \(a(x)\) in (109) and setting it equal to zero yields

\[
a'(x) = \frac{c + x}{dD} \left( \frac{(c + x)^2 + D^2 + d^2}{\sqrt{[(c + x)^2 + D^2 + d^2]^2 - 4d^2D^2} - 1} \right) = \frac{c + x}{dD} \cdot \frac{\nu - \sqrt{\nu^2 - 4d^2D^2}}{\sqrt{\nu^2 - 4d^2D^2}}
\]

\[
= (c + x) \cdot \frac{\nu - \sqrt{\nu^2 - 4d^2D^2}}{dD \sqrt{\nu^2 - 4d^2D^2}} = 0,
\]

where \(\nu\) is defined in (103). By inspection,

\[
\frac{\nu - \sqrt{\nu^2 - 4d^2D^2}}{dD \sqrt{\nu^2 - 4d^2D^2}} < 0
\]
and that makes the only critical number \( x = -c \) to be a point of maximum. Moreover, this maximum value of \( a \) is

\[
a(-c) = \begin{cases} 
-d/D, & \text{if } D > -d; \\
-D/d, & \text{if } D < -d.
\end{cases}
\]

This follows from the direct substitution of \( x = -c \) into (109) giving

\[
a(-c) = \frac{-D^2 - d^2 + \sqrt{(D^2 + d^2)^2 - 4d^2D^2}}{2dD} = \frac{-D^2 - d^2 + \sqrt{(D^2 - d^2)^2}}{2dD}.
\]

Suppose that our L-system \( \Theta_{\mu,h} \) with a given coefficient of dissipation \( D = \text{Im } h \neq \text{Im } m_\infty(i) \) has the maximum finite c-entropy \( S_{\text{max}} \). It follows from the definition of c-entropy and formula (87) that the maximum finite c-entropy is achieved when the modulus of von Neumann parameter \( \kappa \) is at its minimum. Since \( 0 < a < 1 \), the corresponding \( \kappa \) is related to \( a \) via (13). Consider \( \kappa \) as a function of a single variable \( x \). Then

\[
\kappa(x) = \frac{1 - a(x)}{1 + a(x)}
\]

is a decreasing function of \( a \) and hence \( \kappa(x) \) attains its minimum whenever \( a(x) \) assumes its maximum. But as we have just shown above \( a(x) \) has its absolute maximum at \( x = -c \). Therefore, the L-system \( \Theta_{\mu,h} \) has the maximum c-entropy \( S_{\text{max}} \) when \( x = -c = -\text{Re } m_\infty(i) \). The corresponding values of system parameters \( h \) and \( \mu \) are found once we recall that \( x = \text{Re } h \) and \( D = \text{Im } h \) and then substitute \( x = -c = -\text{Re } m_\infty(i) \) in (100) considering both cases for \( a \) in (110). This proves the necessity part of the statement.

Conversely, let the L-system \( \Theta_{\mu,h} \) has parameters \( h \) and \( \mu \) defined by (107). Then as it follows from the first part of (107) we have \( x = -c = -\text{Re } m_\infty(i) \). As we have shown in the first part of the proof, this value of \( x = -c \) maximizes \( a(x) \), minimizes \( \kappa(x) \), and hence maximizes the c-entropy of our L-system. Thus, \( \Theta_{\mu,h} \) has the maximum finite c-entropy \( S_{\text{max}} \) if (107) holds.

In order to prove (108) we observe that

\[
\kappa(-c) = \begin{cases} 
\frac{D + d}{D - d}, & \text{if } D > -d; \\
\frac{D + d}{D - d}, & \text{if } D < -d,
\end{cases}
\]

is the absolute minimum of \( \kappa \) for all real \( x \). It follows from the definition of c-entropy and formula (87) that the minimum of \( \kappa \) corresponds to the maximum of c-entropy and we have

\[
S(-c) = -\ln |\kappa(-c)| = -\ln \left| \frac{D + d}{D - d} \right| = \ln |D - d| - \ln |D + d|,
\]

confirming (108).

A similar result takes place when \( a > 1 \).

**Theorem 22.** Assume that an L-system \( \Theta_{\mu,h} \) of the form (54) with a main operator \( T_\mu \) and a coefficient of dissipation \( D = \text{Im } h \neq -\text{Im } m_\infty(i) \). Then, under constraint that the impedance function \( V_{\Theta_{\mu,h}}(z) \in \mathcal{M}_\kappa^{-1} \) for some \( 0 < \kappa < 1 \), the L-system \( \Theta_{\mu,h} \) has the maximum finite c-entropy \( S_{\text{max}} \) if and only if it is determined
by parameters

\begin{align}
  h &= -\text{Re} \, m_\infty(i) + iD \quad \text{and} \quad \mu = \begin{cases} 
    \infty, & \text{if } D > -\text{Im} \, m_\infty(i); \\
    -\text{Re} \, m_\infty(i), & \text{if } D < -\text{Im} \, m_\infty(i).
  \end{cases}
\end{align}

In this case, \( S_{\text{max}} \) is determined by \( \text{(108)} \).

**PROOF.** Since \( V_{\Theta_{\mu,h}}(z) \in \mathcal{M}_\kappa^{-1} \), it satisfies a normalization condition

\[ V_{\Theta_{\mu,h}}(i) = ai \quad \text{for} \quad a = \frac{1+\kappa}{1-\kappa}. \]

Also, since \( a > 1 \) in the above, then formula \( \text{(105)} \) takes place and hence

\begin{align}
  a(x) &= \frac{-(c+x)^2 - D^2 - d^2 - \sqrt{[(c+x)^2 + D^2 + d^2]^2 - 4d^2D^2}}{2D}.
\end{align}

Taking the derivative of \( a(x) \) in \( \text{(113)} \) and setting it equal to zero yields

\[
  a'(x) = -\frac{c + x}{dD} \left( \frac{(c + x)^2 + D^2 + d^2}{\sqrt{[(c+x)^2 + D^2 + d^2]^2 - 4d^2D^2}} + 1 \right)
  = -\frac{c + x}{dD} \cdot \frac{\nu + \sqrt{\nu^2 - 4d^2D^2}}{\sqrt{\nu^2 - 4d^2D^2}} = -(c + x) \cdot \frac{\nu + \sqrt{\nu^2 - 4d^2D^2}}{dD\sqrt{\nu^2 - 4d^2D^2}} = 0,
\]

where \( \nu \) is defined in \( \text{(103)} \). Clearly, we have that

\[
  \frac{\nu - \sqrt{\nu^2 - 4d^2D^2}}{dD\sqrt{\nu^2 - 4d^2D^2}} > 0
\]

and \( a(x) \) has the only critical number \( x = -c \) a point of minimum. Moreover, the minimum value of \( a \) is

\begin{align}
  a(-c) &= \begin{cases} 
    -D/d, & \text{if } D > -d; \\
    -d/D, & \text{if } D < -d.
  \end{cases}
\end{align}

This follows from the direct substitution of \( x = -c \) into \( \text{(109)} \) giving

\[
  a(-c) = \frac{-D^2 - d^2 - \sqrt{[D^2 + d^2]^2 - 4d^2D^2}}{2D} = \frac{-D^2 - d^2 - \sqrt{[D^2 - d^2]^2}}{2D}.
\]

Suppose that the L-system \( \Theta_{\mu,h} \) with a given coefficient of dissipation \( D = \text{Im} \, h \neq \text{Im} \, m_\infty(i) \) has the maximum finite c-entropy \( S_{\text{max}} \). As we already mentioned in the proof of Theorem \( \text{(21)} \), the maximum finite c-entropy is achieved when the modulus of the von Neumann parameter \( \kappa \) is at its minimum. Since \( a > 1 \), the corresponding \( \kappa \) is related to \( a \) via \( \text{(14)} \). Consider \( \kappa \) as a function of a single variable \( x \). Then

\[
  \kappa(x) = \frac{a(x) - 1}{a(x)}
\]

is increasing function of \( a \) and hence \( \kappa(x) \) takes its minimum whenever \( a(x) \) assumes its minimum. But as we have just shown above \( a(x) \) has its absolute minimum at \( x = -c \). Therefore, the L-system \( \Theta_{\mu,h} \) has the maximum c-entropy \( S_{\text{max}} \) when \( x = -c = -\text{Re} \, m_\infty(i) \). The corresponding values of system parameters \( h \) and \( \mu \) are found once we recall that \( x = \text{Re} \, h \) and \( D = \text{Im} \, h \) and then substitute \( x = -c = -\text{Re} \, m_\infty(i) \) in \( \text{(100)} \) considering both possible cases for \( a \) in \( \text{(114)} \). This proves necessity.
Conversely, let the L-system $\Theta_{\mu,h}$ has the parameters $h$ and $\mu$ defined by (112). Then as it follows from the first part of (112) we have $\text{Re} \ h = x = -\text{Re} \ m_\infty(i) = -c$. As we have shown in the first part of the proof, this value of $x = -c$ minimizes $a(x)$, minimizes $\kappa(x)$, and hence maximizes c-entropy of our L-system. Thus, $\Theta_{\mu,h}$ has the maximum finite c-entropy $S_{\text{max}}$ if (112) holds.

In order to prove (108) we observe that (111) is still true for this case when one substitutes the values of $a(-c)$ from (114) into (14). Hence (111) again determines the absolute minimum of $\kappa$ for all real $x$. It follows from (87) that the minimum of $\kappa$ corresponds to the maximum of c-entropy which confirms (108). \hfill $\square$

We point out that both Theorems 21 and 22 describe uniquely the L-systems with maximum c-entropy for the corresponding classes of impedance functions. Also, an attentive reader notices that in Theorems 21 and 22 the hypothesis that $D = \text{Im} \ h \neq -\text{Im} \ m_\infty(i)$ holds and might wonder what happens if $h = -m_\infty(i)$ for a given L-system. The following remark addresses that question.

**Remark 23.** An L-system $\Theta_{\mu,h}$ of the form (34) with a Schrödinger operator $T_h$ and a parameter $\mu \in \mathbb{R} \cup \{\infty\}$ has infinite c-entropy, $S = \infty$ if and only if $h = -m_\infty(i)$. Indeed, one might also notice that if $h = -m_\infty(i)$, then $D = \text{Im} \ h = -\text{Im} \ m_\infty(i)$ and hence formulas (110) and (114) imply that $a(-c) = 1$ and $\kappa(-c) = 0$. The assertion then is a corollary of Theorem 8.

9. **c-Entropy in sectorial cases and the Krein-von Neumann extension**

Throughout this section we deal with L-systems whose main operators are either extremal accretive or $\beta$-sectorial.

9.1. **Dual c-entropy problems in the extremal case.** The following theorem gives the solution to the first dual c-entropy problem posed in Section 7.

**Theorem 24.** Assume that an L-system $\Theta_{\mu,h}$ of the form (34) with a main operator $T_h$ has c-entropy $S$. Then, under the constraint that $T_h$ is extremal, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation

\begin{equation}
D_{\text{min}} = -\text{Im} \ m_\infty(i) \coth S - \sqrt{B^2 \csc^2 S - (A - m)^2},
\end{equation}

where $A = \text{Re} \ m_\infty(i)$, $B = -\text{Im} \ m_\infty(i)$, and $m = m_\infty(-0)$.

In this case, the L-system $\Theta_{\mu,h}$ is determined by the parameters

\begin{equation}
h = -m_\infty(-0) + iD_{\text{min}} \quad \text{and arbitrary} \quad \mu \in \mathbb{R} \cup \{\infty\}.
\end{equation}

**Proof.** Suppose that the Schrödinger L-system $\Theta_{\mu,h}$ has a given fixed c-entropy $S$. Then, as we have shown in Section 5, the modulus of von Neumann parameter $\kappa$ is related to the coefficient of dissipation $D = \text{Im} \ h$ via (64)

\begin{equation}
\kappa^2 = \frac{D^2 - 2BD + D}{D^2 + 2BD + D},
\end{equation}

where $B = -\text{Im} \ m_\infty(i)$, $D = C + B^2$, and $C = (\text{Re} \ m_\infty(i) - m_\infty(-0))^2$. By Theorem 12 $\kappa_0 \leq \kappa < 1$, where $\kappa_0$ is given by (60). Since $\kappa = e^{-S}$, by (117), we have

\begin{equation}
e^{-2S} = \frac{D^2 - 2BD + D}{D^2 + 2BD + D}
\end{equation}

and hence

\begin{equation}(1 - e^{-2S})D^2 - 2B(1 + e^{-2S})D + (1 - e^{-2S})D = 0,
\end{equation}
or

\[(118) \quad \mathcal{D}^2 - 2B(\coth S)\mathcal{D} + D = 0.\]

This equation (see the proof of Theorem 14) has either two or one real roots (also see Figure 11 for convenience) given by

\[(119) \quad \mathcal{D} = B \coth S \pm \sqrt{B^2 \coth^2 S - D},\]

where \(B\) and \(D\) are defined above. The case of a unique solution occurs when \(e^{-S} = \kappa = \kappa_0\). As we have shown in Section 5, in this case

\[D = \sqrt{D} = \sqrt{(\text{Im} m_\infty(i) - m_\infty(-0))^2 + (\text{Im} m_\infty(i))^2}.\]

If \(\kappa_0 < \kappa < 1\), then the minimal coefficient of dissipation is the smallest of two roots of equation (119). It is easy to see that

\[D_{\text{min}} = B \coth S - \sqrt{B^2 \coth^2 S - D}\]

in this case. Substituting the values for \(B\) and \(D\) into the above formula and taking into account that \(\text{csch}^2 S = \coth^2 S - 1\) we obtain (115).

Now we address the second dual c-entropy problem.

**Theorem 25.** Assume that an L-system \(\Theta_{\mu,h}\) is of the form (34) with a main operator \(T_h\). Assume in addition that \(\text{Re} m_\infty(i) \neq m_\infty(-0)\). Then, under the constraint that \(T_h\) is extremal, the L-system \(\Theta_{\mu,h}\) has the maximum finite c-entropy \(S_{\text{max}}\) if and only if it is determined by the parameters

\[\begin{align*}
(120) \quad h &= -m + i\sqrt{(A - m)^2 + B^2} \quad \text{and} \quad (\text{arbitrary}) \quad \mu \in \mathbb{R} \cup \{\infty\},
\end{align*}\]

where \(A = \text{Re} m_\infty(i)\), \(B = -\text{Im} m_\infty(i)\), and \(m = m_\infty(-0)\).

In this case,

\[\begin{align*}
(121) \quad S_{\text{max}} &= -\frac{1}{2} \ln \left(\frac{\sqrt{A^2 + B^2 - 2mA + m^2} - B}{\sqrt{A^2 + B^2 - 2mA + m^2} + B}\right).
\end{align*}\]

**Proof.** It follows from the development in Section 5 that there are many L-system of the form (34) with an extremal main operators \(T_h\) only different by the coefficient of dissipation (see also Figure 14). Naturally, such an L-system has the maximum c-entropy \(S_{\text{max}}\) if the modulus of von Neumann parameter \(\kappa\) of \(T_h\) is at minimum, i.e., \(\kappa = \kappa_0\) (see formula (16)). Note that \(\kappa_0 \neq 0\) or otherwise \(A = m\) and our L-system \(\Theta_{\mu,h}\) has infinite entropy according to Theorem 8. We know that \(\kappa = \kappa_0\) if and only if

\[\begin{align*}
(122) \quad \text{Im} h &= \sqrt{\text{Re} m_\infty(i) - m_\infty(-0))^2 + (\text{Im} m_\infty(i))^2} = \sqrt{(A - m)^2 + B^2}.
\end{align*}\]

In this case (see (87)) \(S_{\text{max}} = -\ln(\kappa_0)\) proving (121).

**Remark 26.** If \(\text{Re} m_\infty(i) = m_\infty(-0)\), then \(h = -m_\infty(i)\) and our L-system \(\Theta_{\mu,h}\) has infinite entropy according to Theorem 8.
9.2. Dual c-entropy problems in the sectorial case. We use similar analysis to treat the sectorial case. We remind the reader that by saying that an accretive operator is $\beta$-sectorial, we mean that $\beta \in (0, \pi/2)$ is its exact angle of sectoriality unless otherwise is specified.

The following theorem is analogous to Theorem 24 for the sectorial case.

Theorem 27. Assume that an L-system $\Theta_{\mu,h}$ of the form (51) with a main operator $T_h$ has c-entropy $\mathcal{S}$. Then, under the constraint that $T_h$ is $\beta$-sectorial, the L-system $\Theta_{\mu,h}$ has the minimum coefficient of dissipation

$$D_{\min} = \sin^2 \beta \left[ B \coth \mathcal{S} - E \cot \beta - \sqrt{(E \cot \beta - B \coth \mathcal{S})^2 - D} \right],$$

where $A = \Re m_{\infty}(i)$, $B = -\Im m_{\infty}(i)$, $D = E^2 + B^2$, $E = A - m$, and $m = m_{\infty}(-0)$.

In this case, the L-system $\Theta_{\mu,h}$ is determined by the parameters

$$h = (\cot \beta)D_{\min} - m + iD_{\min} \quad \text{and} \quad \mu \in \mathbb{R} \cup \{\infty\}.$$

Proof. Let $\Theta_{\mu,h}$ have a $\beta$-sectorial main operator $T_h$ whose von Neumann’s parameter is $\kappa$. Then $\kappa$ is related to the coefficient of dissipation $\mathcal{D} = \Im h$ via (34), that is

$$\kappa^2 = \frac{D^2 + 2 \sin^2 \beta((A - m) \cot \beta - B)D + D \sin^2 \beta}{D^2 + 2 \sin^2 \beta((A - m) \cot \beta + B)D + D \sin^2 \beta},$$

According to Theorem 15, the parameter $\kappa$ in this case belongs to the interval $0 < \kappa_0 \leq \kappa < 1$, where $\kappa_0$ is given by (75), (76). Note that $\kappa_0 \neq 0$ since otherwise this will contradict Lemma 2.

In order to find the minimal coefficient of dissipation we recall that $\kappa = e^{-S}$, use (125), and solve

$$e^{-2S} = \frac{D^2 + 2 \sin^2 \beta((A - m) \cot \beta - B)D + D \sin^2 \beta}{D^2 + 2 \sin^2 \beta((A - m) \cot \beta + B)D + D \sin^2 \beta},$$

for $\mathcal{D}$ to pick the minimal root. The above equation simplifies to the quadratic one of the form

$$\frac{(1 - e^{-2S})D^2 + 2B \sin^2 \beta \left[(1 - e^{-2S})(A - m) - (1 + e^{-2S})B \right]D}{D^2 + 2 \sin^2 \beta((A - m) \cot \beta - B)D + D \sin^2 \beta} = 0,$$

or, with $E = A - m$,

$$D^2 + 2B \sin^2 \beta \left[E \cot \beta - (\coth \mathcal{S})B \right]D + D \sin^2 \beta = 0.$$  

Equation (126) has either two or one real solutions (see also Figure 2) given by

$$D = (\sin \beta) \sqrt{D} = (\sin \beta) \sqrt{|m_{\infty}(i)|^2 - 2m_{\infty}(-0)\Re m_{\infty}(i) + m_{\infty}^2(-0)}.$$  

The case of a unique solution occurs when $e^{-S} = \kappa = \kappa_0$, where $\kappa_0$ is given by (75), (76). As we have shown in Section 6, in this case

$$D = (\sin \beta) \sqrt{D} = (\sin \beta) \sqrt{|m_{\infty}(i)|^2 - 2m_{\infty}(-0)\Re m_{\infty}(i) + m_{\infty}^2(-0)}.$$  

If $\kappa_0 < \kappa < 1$, then the minimal coefficient of dissipation is the smallest of the two roots of (127). It is easy to see that in this case

$$D_{\min} = (\sin \beta)^2 \left[B \coth \mathcal{S} - E \cot \beta - \sqrt{(E \cot \beta - B \coth \mathcal{S})^2 - D} \right].$$
Formula (124) follows from the relation (73) connecting the real and imaginary part of the parameter $h$ that describes a $\beta$-sectorial operator $T_h$. □

Now we address the second dual problem of maximizing $c$-entropy for the class of Schrödinger $L$-systems with sectorial main operators.

The following theorem is similar to Theorem 25.

**Theorem 28.** Assume that an $L$-system $\Theta_{\mu,h}$ is of the form (34) with a main operator $T_h$. Then, under the constraint that $T_h$ is $\beta$-sectorial, the $L$-system $\Theta_{\mu,h}$ has the maximum finite $c$-entropy $S_{\text{max}}$ if and only if it is determined by the parameters

$$h = (\cos \beta) \sqrt{D} - m + i(\sin \beta) \sqrt{D} \quad \text{and} \quad (\text{arbitrary}) \quad \mu \in \mathbb{R} \cup \{\infty\},$$

where $A = \text{Re} m_\infty(i)$, $B = -\text{Im} m_\infty(i)$, $D = (A - m)^2 + B^2$, and $m = m_\infty(-0)$.

In this case,

$$S_{\text{max}} = -\frac{1}{2} \ln \left( \frac{\sqrt{D} + (A - m) \cos \beta - B \sin \beta}{\sqrt{D} + (A - m) \cos \beta + B \sin \beta} \right).$$

**Proof.** It follows from the development in Section 6 and formulas (73) and (76) that an $L$-system of the form (34) with a $\beta$-sectorial main operator $T_h$ has the maximum $c$-entropy $S_{\text{max}}$ if the modulus of von Neumann parameter $\kappa$ of $T_h$ is at minimum (see also Figure 2). That happens if and only if $\kappa = \kappa_0$ (see (75) and (76)) and hence

$$\text{Im} \ h = \sin \beta \cdot \sqrt{D} = \sin \beta \cdot \sqrt{\text{Re} m_\infty(i) - m_\infty(-0))^2 + (\text{Im} m_\infty(i))^2}.$$

Note that $\kappa_0 \neq 0$ or otherwise this will violate Lemma 2. Moreover, in this case (see (74)) $S_{\text{max}} = -\ln(\kappa_0)$ proving (130). The expression in the right hand side of (130) is well defined as the fraction inside the logarithm represents $\kappa_0^2$, the positive quantity. □

**9.3. Accretive $T_h$ case.** In this subsection we are going to look at the situation when the main operator $T_h$ of the Schrödinger $L$-system under consideration is just accretive. Recall (see Section 2 for the definition) that the operator $T_h$ is accretive if $\text{Re} (T_h y, y) \geq 0$ for all $y \in \text{Dom}(T_h)$. Clearly, the set of all such accretive operators consists of the class of $\beta$-sectorial (for some $\beta \in (0, \pi/2)$) operators plus the class of extremal operators.

The first question we address is which $L$-systems $\Theta_{\mu,h}$ with accretive main operators $T_h$ have the maximal finite $c$-entropy.

**Theorem 29.** Assume that an $L$-system $\Theta_{\mu,h}$ is of the form (34) with a main operator $T_h$. Then, under the constraint that $T_h$ is accretive, the $L$-system $\Theta_{\mu,h}$ with extremal accretive main operator achieves the maximum finite $c$-entropy $S_{\text{max}}$.

In this case, $S_{\text{max}}$ is given by (121) and $\Theta_{\mu,h}$ is determined by the parameters

$$h = -m + i\sqrt{(A - m)^2 + B^2} \quad \text{and} \quad (\text{arbitrary}) \quad \mu \in \mathbb{R} \cup \{\infty\},$$

where $A = \text{Re} m_\infty(i)$, $B = -\text{Im} m_\infty(i)$, and $m = m_\infty(-0)$.

**Proof.** We have already mentioned above that since our main operator $T_h$ is accretive, then it is either $\beta$-sectorial (for some $\beta \in (0, \pi/2)$) or extremal accretive. In the first case the $L$-system $\Theta_{\mu,h}$ with $\beta$-sectorial main operator has the maximum
of the reader to the parameters $f$ and $\kappa$. Note that under our current assumption (130) that the minimal von Neumann’s parameter $\kappa$ is such that $\kappa = 0$ and hence the maximum c-entropy of L-system with such extremal $T_h$ is infinite, i.e., $S_{\text{max}}^{\text{ext}} = \infty$. In this theorem we are only interested in the case of finite maximum c-entropy and thus the case when $A = m$ should not be considered. Thus, we can assume without loss of generality that $A > m$.

Consider the function

$$f(\beta) = \frac{\sqrt{D} + (A - m)\cos \beta - B\sin \beta}{\sqrt{D} + (A - m)\cos \beta + B\sin \beta} \quad \text{on} \quad \beta \in (0, \pi/2).$$

Since $A > m$, we have that the derivative

$$f'(\beta) = -\frac{2B(A - m + \sqrt{D}\cdot \cos \beta)}{(\sqrt{D} + (A - m)\cos \beta + B\sin \beta)^2} < 0$$

for any $\beta \in (0, \pi/2)$. Hence the function $f(\beta)$ in (134) decreases on $\beta \in (0, \pi/2)$ and

$$f\left(\frac{\pi}{2}\right) = \frac{\sqrt{D} - B}{\sqrt{D} + B}.$$ 

Note that under our current assumption $f(\pi/2)$ is not finite or otherwise (133) holds for $A > m$. Consequently, a Schrödinger L-system $\Theta_{\mu, h}$ with an accretive main operator $T_h$ achieves the maximum finite c-entropy whenever $T_h$ is extremal. In this case, according to Theorem 25, $S_{\text{max}}$ is determined by (121) while $h$ is given by (122) and $\mu$ is arbitrary in $\mathbb{R} \cup \{\infty\}$. □

**Remark 30.** Note that if $\Theta_{\mu, h}$ is the Schrödinger L-system referred to in Theorem 29, then the modulus of von Neumann’s parameter of the corresponding main operator $T_h$ equals $\kappa_0$ given by (106). Moreover, if under the assumptions of Theorem 29 $\Theta_{\mu_1, h}$ and $\Theta_{\mu_2, h}$ are such L-systems that the corresponding impedance functions $V_{\Theta_{\mu_1, h}}(z)$ and $V_{\Theta_{\mu_2, h}}(z)$ belong to the generalized Donoghue classes $M_{\kappa_0}$ or $M_{\kappa_0^{-1}}$ respectively, then

$$\mu_1 = \frac{(A - mBF)(BF - 1) + (m^2F + DF - B - mAF)(A - m)F}{(BF - 1)^2 + (A - m)^2F^2}.$$
Thus, the Schrödinger L-systems $\Theta_{\mu_1, h}$ and $\Theta_{\mu_2, h}$ with the desired properties are uniquely constructed using the values of $h$ from (132), $\mu_1$ from (136), and $\mu_2$ from (137).
9.4. Maximal c-entropy in accretive case. In this subsection we consider Schrödinger L-systems with maximal finite c-entropy and accretive state-space operator. The following theorem describes an L-system with \( \beta \)-sectorial main and state-space operators and maximal c-entropy.

**Theorem 31.** An L-system \( \Theta_{\mu,h} \) of the form (34) with a \( \beta \)-sectorial (\( \beta \in (0, \pi/2) \)) Schrödinger main operator \( T_h \) and the maximum finite c-entropy \( S_{\text{max}} \) determined by the formula (130) has a \( \beta \)-sectorial (with the same angle of sectoriality) state-space operator \( \mathcal{A}_{\mu,h} \) if and only if \( \mu = \infty \) in (29).

**Proof.** It was shown in [2, Theorem 10.6.4] (see also [5, 8]) that if \( \mathcal{A}_{\mu,h} \) is a \((\ast)\)-extension of an \( \beta \)-sectorial operator \( T_h \) with the exact angle of sectoriality \( \beta \in (0, \pi/2) \), then \( \mathcal{A}_{\mu,h} \) is an \( \beta \)-sectorial \((\ast)\)-extension of \( T_h \) (with the same angle of sectoriality) if and only if \( \mu = \infty \). The rest follows from Theorem 28. \( \square \)

**Theorem 32.** An L-system \( \Theta_{\mu,h} \) of the form (34) with a \( \beta \)-sectorial (\( \beta \in (0, \pi/2) \)) Schrödinger main operator \( T_h \) and the maximum finite c-entropy \( S_{\text{max}} \) determined by the formula (130) has an extremal state-space operator \( \mathcal{A}_{\mu,h} \) if and only if

\[
\mu = (2 \csc 2\beta) \sqrt{(\text{Re} \, m_{\infty}(i) - m_{\infty}(-0))^2 + (\text{Im} \, m_{\infty}(i))^2 - m_{\infty}(-0)).
\]

**Proof.** It follows from [2, Theorem 10.6.5] (see also [5, 8]) that if \( \mathcal{A}_{\mu,h} \) is a \((\ast)\)-extension of an \( \beta \)-sectorial operator \( T_h \) with the exact angle of sectoriality \( \beta \in (0, \pi/2) \), then \( \mathcal{A}_{\mu,h} \) is accretive but not \( \beta \)-sectorial for any \( \beta \in (0, \pi/2) \) \((\ast)\)-extension of \( T_h \) if and only if in (29)

\[
\mu = \frac{(\text{Im} \, h)^2}{m_{\infty}(-0) + \text{Re} \, h} + \text{Re} \, h.
\]

Taking into account that in our case \( \text{Im} \, h \) and \( \text{Re} \, h \) are related by (73), we substitute their values into (143) to get

\[
\mu = \frac{(\text{Im} \, h)^2}{m + \text{Re} \, h} + \text{Re} \, h = \frac{(\text{Im} \, h)^2}{m + (\cot \beta) \text{Im} \, h - m} + (\cot \beta) \text{Im} \, h - m
\]

\[
= (\tan \beta) \text{Im} \, h + (\cot \beta) \text{Im} \, h - m = (\tan \beta + \cot \beta) \text{Im} \, h - m
\]

\[
= (2 \csc 2\beta) \text{Im} \, h - m,
\]

where \( m = m_{\infty}(-0) \). We know from Theorem 28 that if the L-system \( \Theta_{\mu,h} \) has the maximum c-entropy, then \( \text{Im} \, h \) is given by (131). Substituting the value of \( \text{Im} \, h \) into the above we obtain (142). \( \square \)

Now we focus on the case when the main operator of a Schrödinger L-system is extremal.

**Theorem 33.** An L-system \( \Theta_{\mu,h} \) of the form (34) with an extremal Schrödinger main operator \( T_h \) and the maximum finite c-entropy \( S_{\text{max}} \) determined by the formula (121) has an accretive state-space operator \( \mathcal{A}_{\mu,h} \) if and only if \( \mu = \infty \).

**Proof.** As it was shown in [2, Theorem 10.6.4] (see also [5, 8]), a state-space operator \( \mathcal{A}_{\mu,h} \) preserves the exact angle of sectoriality of \( T_h \) if and only if \( \mu = \infty \). Here our operator \( T_h \) is extremal and thus can only admit one extremal \((\ast)\)-extension \( \mathcal{A}_{\mu,h} \) with \( \mu = \infty \). \( \square \)
At this point again we are going to look at the combined class of \( \beta \)-sectorial and extremal accretive operators to see what case provides us with an L-system with maximal finite \( c \)-entropy.

**Theorem 34.** An L-system \( \Theta_{\mu,h} \) of the form (34) with an accretive Schrödinger main operator \( T_h \) and accretive state-space operator \( A_{\mu,h} \) achieves the maximum finite \( c \)-entropy \( S_{\text{max}} \) determined by (121) when \( T_h \) is extremal.

In this case \( h \) is given by (132), \( \mu = \infty \), and the quasi-kernel of \( \Re A_{\mu,h} \) is the Krein-von Neumann extension defined by the formula

\[
\begin{align*}
A_K y &= -y'' + q(x)y, \\
y'(\ell) + m_\infty (-0)y(\ell) &= 0.
\end{align*}
\]

**Proof.** We know that according to Theorem 29 among all the L-system \( \Theta_{\mu,h} \) of the form (34) with an accretive Schrödinger main operator \( T_h \) and any \( \mu \in \mathbb{R} \cup \{\infty\} \), the L-system that has the maximum finite \( c \)-entropy is the one with the extremal main operator \( T_h \). On the other hand, since we are given an extra condition that the state-space operator \( A_{\mu,h} \) is also accretive, then Theorem 33 provides us with \( \mu = \infty \) for this case. Consequently, \( A_{\mu,h} \) is given by (98) that is

\[
\begin{align*}
A_{\mu,h} &= A_{\infty,h} y = -y'' + q(x)y - [y'(\ell) - \bar{h} y(\ell)] \delta(x - \ell), \\
A_{\mu,h}^* &= A_{\infty,h}^* y = -y'' + q(x)y - [y'(\ell) - \bar{h} y(\ell)] \delta(x - \ell).
\end{align*}
\]

Using (145) one easily gets

\[
\Re A_{\infty,h} = -y'' + q(x)y + [y'(\ell) + m_\infty (-0)y(\ell)] \delta(x - \ell).
\]

Clearly, (146) implies that \( \Re A_{\infty,h} \supset A_K \) and \( A_K \) given by (144) is the quasi-kernel for \( \Re A_{\infty,h} \).

**Remark 35.** Note that if an L-system \( \Theta_{\mu,h} \) of the form (34) has an accretive state-space operator, then its impedance function \( V_{\Theta_{\mu,h}}(z) \) cannot belong to any of the generalized Donoghue classes.

Indeed, we know from [2, Theorem 9.8.2] that if an L-system has an accretive state-space operator, then its impedance function is a Stieltjes function and thus has the integral representation

\[
V_{\Theta_{\mu,h}}(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t - z},
\]

where \( \gamma \geq 0 \) and \( \sigma(t) \) is a non-decreasing on \([0, +\infty)\) function such that \( \int_0^\infty \frac{dG(t)}{1+t} < \infty \). On the other hand, if we assume that a Stieltjes function \( V_{\Theta_{\mu,h}}(z) \) belonged to a generalized Donoghue class, then it would have admited integral representation of the form (14), that is

\[
V_{\Theta_{\mu,h}}(z) = \int_0^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t).
\]

However, calculating the real part of \( V_{\Theta_{\mu,h}}(i) \) using both (147) and (148) and then setting the results equal leads to

\[
\Re V_{\Theta_{\mu,h}}(i) = 0 = \gamma + \int_0^\infty \frac{t d\sigma(t)}{t^2 + 1},
\]
which is a contradiction since $\gamma \geq 0$ and $\int_0^\infty \frac{td\sigma(t)}{t+1} > 0$ (see [2]). Therefore we have shown that a Stieltjes function $V_{\Theta_{\mu,h}}(z)$ can be written as

$$V_{\Theta_{\mu,h}}(z) = Q + \int_0^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t),$$

with

$$Q = \gamma + \int_0^{+\infty} \frac{td\sigma(t)}{t^2 + 1} > 0.$$  

Consequently, none of the Stieltjes impedance functions $V_{\Theta_{\mu,h}}(z)$ can belong to any of the generalized Donoghue classes.

10. Table of dual c-entropy problems solutions

In this subsection we summarize the results on dual c-entropy optimization problems and present them in the tabular form (see Table 1).

| L-system property | First dual problem | Second dual problem | L-system uniqueness |
|-------------------|--------------------|---------------------|--------------------|
| $V_{\Theta}(z) \in M_b$ | Theorem 20 | Theorem 21 | Unique |
| $V_{\Theta}(z) \in M_b^{-1}$ | Theorem 19 | Theorem 22 | Unique |
| $T_h$ is extremal | Theorem 24 | Theorem 25 | Not unique |
| $T_h$ is $\beta$-sectorial | Theorem 27 | Theorem 28 | Not unique |

Table 1. Solutions to dual c-entropy problems

In the very left column of the table we list the properties of the Schrödinger L-systems under consideration. The second and the third columns of the table indicate the theorems that contain solutions to the first and the second dual c-entropy problems, respectively. The last column of Table 1 addresses the uniqueness of the Schrödinger L-system solving either of the dual c-entropy problems. We have already noted in Sections 7 and 8 that the Schrödinger L-systems $\Theta_{\mu,h}$ referred to in Theorems 19–22 are unique with the exact values of the defining parameters $\mu$ and $h$ provided. On the other hand, when the dual c-entropy problems are being solved for the classes of Schrödinger L-systems with extremal or $\beta$-sectorial main operators $T_h$ the answer is not unique. Namely, Theorems 24–28 describe the entire family of solutions. These solutions $\Theta_{\mu,h}$ are parameterized by a particular value of $h$ and an arbitrary parameter $\mu \in \mathbb{R} \cup \{\infty\}$. Note that the case when the impedance
function $V_0(z)$ belongs to the Donoghue class $\mathcal{M}$ deals with infinite $c$-entropy (see Theorem 3) and therefore is not presented in the table.

In the next section we will illustrate our findings by two examples.

11. Examples

We conclude this paper with a couple of simple illustrations. Consider the differential expression

$$l_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad x \in [1, \infty)$$

of order $\nu \geq \frac{1}{2}$ in the Hilbert space $\mathcal{H} = L^2(1, \infty)$. The minimal symmetric operator

(150)

$$\begin{cases}
\hat{A} y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\
y(1) = y'(1) = 0
\end{cases}$$

generated by this expression and boundary conditions has deficiency indices $(1, 1)$ and is obviously nonnegative. Consider also the operator

(151)

$$\begin{cases}
T_h y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\
y'(1) = hy(1)
\end{cases}$$

Example 1. Let $\nu = 1/2$. It is known [2] that in this case

$$m_{\infty, \frac{1}{2}}(z) = -i\sqrt{z} \quad \text{and hence} \quad m_{\infty, \frac{1}{2}}(i) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$  

The minimal symmetric operator then becomes

$$\begin{cases}
\hat{A} y = -y'' \\
y(1) = y'(1) = 0.
\end{cases}$$

Note that in this case

$$\text{Re} \ m_{\infty, \frac{1}{2}}(i) = \frac{1}{\sqrt{2}} > m_{\infty, \frac{1}{2}}(0) = 0,$$

which taking into account that the symmetric operator $\hat{A}$ is nonnegative, illustrates the result of Lemma [17].

To construct a family of L-systems with von Neumann’s parameter $\kappa = 0$ for this case we set

$$h = -m_{\infty, \frac{1}{2}}(i) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$  

Then the main operator is given by

(152)

$$\begin{cases}
T_{h_0} y = -y'' \\
y'(1) = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)y(1)
\end{cases}$$

and it will be shared by all the family of L-systems with $\kappa = 0$ and

(153)

$$h_0 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$
Clearly, any L-system $\Theta_{\mu,h_0}$ has infinite c-entropy for all $\mu \in \mathbb{R} \cup \{\infty\}$. The quasi-kernel of the real part of the state-space operator of this family of L-systems is determined by (32) as follows

$$\begin{cases}
\dot{A}_\mu y = -y'' \\
y'(1) = -\frac{\mu + \sqrt{\mu + 1}}{\sqrt{2\mu + 1}}y(1).
\end{cases}$$

(154)

To construct an L-system with an accretive state-space operator, we take $\mu = \infty$ to have

$$\begin{cases}
\dot{A}_\infty y = -y'' \\
y'(1) = -\frac{\sqrt{2}}{\sqrt{2}}y(1).
\end{cases}$$

(155)

The state-space operator of the L-system $\Theta_{\infty,h_0}$ with $\kappa = 0$, $h_0$ defined by (156), and $\mu = \infty$ is (see (29))

$$A_{\infty,h_0} y = -y'' - \frac{1}{\sqrt{2}}\sqrt{2}y'(1) + (1 - i)y(1) + \delta(x - 1),$$

(156)

$$A_{\infty,h_0}^* y = -y'' - \frac{1}{\sqrt{2}}\sqrt{2}y'(1) + (1 + i)y(1) + \delta(x - 1).$$

(157)

Also the channel operator $K_{\infty,h_0} e = cg_{\infty,h_0}$, $(c \in \mathbb{C})$, where (see (30))

$$g_{\infty,h_0} = 2^{-i}x(x - 1).$$

Then $\Theta_{\infty,h_0}$ has the form

$$\Theta_{\infty,h_0} \begin{pmatrix} A_{\infty,h_0} & K_{\infty,h_0} \end{pmatrix} = \begin{pmatrix} A_{\infty,h_0} & K_{\infty,h_0} \end{pmatrix},$$

(156)

where all the components are described above. Using formulas (155) and (156) we obtain

$$W_{\Theta_{\infty,h_0}}(z) = \frac{m_{\infty,h}(z) + \overline{h}}{m_{\infty,h}(z) + h} = \frac{i\sqrt{2}z + 1 + i}{i\sqrt{2}z + 1 - i}$$

(157)

and

$$V_{\Theta_{\infty,h_0}}(z) = \frac{\text{Im} h}{m_{\infty,h}(z) + \text{Re} h} = -\frac{1}{i\sqrt{2}z + 1}.$$

(158)

Direct substitution confirms that $V_{\Theta_{\infty,h_0}}(i) = i$. Clearly, $V_{\Theta_{\infty,h_0}}(z) \in \mathfrak{M}$ and our L-system $\Theta_{\infty,0}$ has infinite c-entropy according to Remark (23).

Similarly we can consider L-systems for which $h \neq -m_{\infty,\frac{1}{2}}(i)$ and hence $\kappa \neq 0$. Then according to (10) we have

$$\kappa = \left| \frac{m_{\infty,\frac{1}{2}}(i) + h}{m_{\infty,\frac{1}{2}}(i) + \overline{h}} \right| = \left| \frac{\sqrt{2} + \sqrt{2}i + h}{\sqrt{2} - \sqrt{2}i + \overline{h}} \right| = \frac{1 - i + \sqrt{2}\text{ Re} h + \sqrt{2}\text{ Im} h i}{1 - i + \sqrt{2}\text{ Re} h - \sqrt{2}\text{ Im} h i}.$$

(159)

$$= \frac{(1 + \sqrt{2}\text{ Re} h)^2 + (\sqrt{2}\text{ Im} h - 1)^2}{(1 + \sqrt{2}\text{ Re} h)^2 + (\sqrt{2}\text{ Im} h + 1)^2}.$$

If we want our operator $T_h$ to be extremal, we (according to Theorem (11) set $\text{Re} h = -m_{\infty,\frac{1}{2}}(0) = 0$. Then by the above calculations

$$\kappa_{\text{ext}} = \sqrt{\frac{1 + (\sqrt{2}\text{ Im} h - 1)^2}{1 + (\sqrt{2}\text{ Im} h + 1)^2}} = \sqrt{\frac{1 + (\text{Im} h)^2 - \sqrt{2}\text{ Im} h}{1 + (\text{Im} h)^2 + \sqrt{2}\text{ Im} h}},$$

(160)
where $\kappa_{ext}$ is the von Neumann parameter of the operator $T_h$. As we have established in Section 5, if the main operator $T_h$ is extremal, then $\kappa_{ext}$ satisfies $\kappa_0 \leq \kappa_{ext} < 1$, where $\kappa_0$ is given by (66). In our case $m_{\infty, \frac{1}{2}}(0) = 0$ and $|m_{\infty, \frac{1}{2}}(i)| = 1$, and hence

\[(159) \quad \kappa_0 = \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} = \sqrt{2} - 1.\]

Let us describe L-systems having the main extremal operator with von Neumann’s parameter $\kappa_0$ as in (159). The corresponding to $\kappa_0$ value of $h$ is (see (122))

\[h = -m_{\infty, \frac{1}{2}}(0) + i\sqrt{(\text{Re} m_{\infty}(i) - m_{\infty, \frac{1}{2}}(0))^2 + (\text{Im} m_{\infty, \frac{1}{2}}(i))^2} = i|m_{\infty}(i)| = i.\]

We have

\[(160) \quad \begin{cases} T_i y = -y'' \\
y'(1) = iy(1). \end{cases}\]

The quasi-kernel of the real part of the state-space operator of our family of L-systems is determined by (32) as follows

\[(161) \quad \begin{cases} \hat{A}_{\mu} y = -y'' \\
y'(1) = -\frac{1}{\mu}y(1). \end{cases}\]

Now we construct two L-systems having $T_i$ as their main operator and attaining maximal $c$-entropy. We note that in the case in question the coefficient of dissipation $D = 1$ and $\text{Im} m_{\infty, \frac{1}{2}}(i) = -1/\sqrt{2}$. First we use (136) to obtain $\mu_1 = -1$. We get

\[(162) \quad \begin{cases} \hat{A}_{-1} y = -y'' \\
y'(1) = y(1). \end{cases}\]

The state-space operator of the L-system $\Theta_{-1,i}$ with $\kappa_0$ from (159) and $\mu = -1$ is (see (24))

\[(163) \quad \Theta_{-1,i} = \begin{pmatrix} \hat{A}_{-1,i} & K_{-1,i} \\ \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- & 1 \end{pmatrix}, \]

where all the entries are described above. Using formulas (35) and (36) we obtain

\[(164) \quad W_{\Theta_{-1,i}}(z) = \frac{1}{-1 + i} \frac{m_{\infty, \frac{1}{2}}(z) + \bar{h}}{m_{\infty, \frac{1}{2}}(z) + h} = i\sqrt{z} + 1 \]

and

\[(165) \quad V_{\Theta_{-1,i}}(z) = -\frac{i\sqrt{z} - 1}{i\sqrt{z} + 1} = -\frac{\sqrt{z} - i}{\sqrt{z} + i}.\]
Direct substitution yields
\[ V_{\Theta_{-1,i}}(i) = -\frac{\sqrt{i} - i}{\sqrt{i} + i} = -\frac{\sqrt{i} + \frac{\sqrt{i}}{2} - i}{\frac{\sqrt{i}}{2} + \frac{\sqrt{i}}{2} + i} = (\sqrt{2} - 1)i. \]

Clearly, \( V_{\Theta_{-1,i}}(z) \in \mathfrak{M}_{\kappa_0} \), where \( \kappa_0 \) is given by \( (159) \). The L-system \( \Theta_{-1,i} \) in \( (163) \) exemplifies the first L-system described in Remark 30.

In order to construct the second L-system described in Remark 30 we apply \( (137) \) and obtain \( \mu = 1 \) first. Then we get
\[
\begin{align*}
\hat{A}_1 y &= -y'' \\
y'(1) &= -y(1).
\end{align*}
\]

The state-space operator of the L-system \( \Theta_{1,i} \) with \( \kappa_0 \) from \( (159) \) and \( \mu = 1 \) is (see \( (29) \))
\[
\begin{align*}
A_{1,i} y &= -y'' - \frac{1}{1-i} |y'(1) - iy(1)| [\delta(x-1) + \delta'(x-1)], \\
A_{1,i}^* y &= -y'' - \frac{1}{1+i} |y'(1) + iy(1)| [\delta(x-1) + \delta'(x-1)].
\end{align*}
\]

Also the channel operator is given by \( K_{1,i} c = cg_{1,i}, (c \in \mathbb{C}) \), where (see \( (30) \))
\[ g_{1,i} = \sqrt{2}\delta(x-1) + \delta'(x-1). \]

Then \( \Theta_{1,i} \) has the form
\[
\Theta_{1,i} = \begin{pmatrix} A_{1,i} & K_{1,i} \\ \mathcal{H}_+ \subset L_2(1, +\infty) \subset \mathcal{H}_- & 1 \end{pmatrix},
\]
where all the components are described above. Using formulas \( (35) \) and \( (56) \) we obtain
\[
W_{\Theta_{1,i}}(z) = \frac{1-i}{1+i} \frac{m_{\infty,1}(z) + \overline{h}}{m_{\infty,1}(z) + h} = -i \frac{\sqrt{2} + 1}{\sqrt{2} - 1}
\]
and
\[
V_{\Theta_{1,i}}(z) = \frac{\sqrt{2} + i}{\sqrt{2} - i}.
\]

Hence
\[ V_{\Theta_{1,i}}(i) = \frac{\sqrt{2} + i}{\sqrt{2} - i} = \frac{\sqrt{2} + \frac{\sqrt{2}}{2} + i}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i} = (\sqrt{2} + 1)i. \]

Clearly, \( V_{\Theta_{1,i}}(z) \in \mathfrak{M}_{\kappa_0}^{-1} \), where \( \kappa_0 \) is given by \( (159) \). The L-system \( \Theta_{1,i} \) in \( (167) \) exemplifies the second L-system described in Remark 30.

**Example 2.** Let \( \nu = 3/2 \). It is known \( [2] \) that in this case
\[ m_{\infty,1}(z) = -\frac{iz - \frac{3}{2}\sqrt{2} - \frac{3}{2}i}{\sqrt{2} + i} - \frac{1}{2} = \frac{\sqrt{2} - iz + i}{\sqrt{2} + i} = 1 - \frac{iz}{\sqrt{2} + i} \]
and
\[ m_{\infty,1}(-0) = 1, \quad m_{\infty,1}(i) = 1 + \frac{1}{\sqrt{2}} - \frac{i}{2} = \frac{2 + \sqrt{2} - i}{2}. \]

The minimal symmetric operator then becomes
\[
\begin{align*}
\hat{A} y &= -y'' + \frac{2}{\sqrt{2}} y \\
y'(1) &= y(1) = 0.
\end{align*}
\]
Note that in this case
\[
\text{Re} m_{\infty, \frac{1}{2}}(i) = 1 + \frac{1}{\sqrt{2}} > m_{\infty, \frac{1}{2}}(0) = 1,
\]
which taking into account that the symmetric operator ˙\(A\) is nonnegative, illustrates the result of Lemma 17.

We are going to construct an L-system with an extremal main operator and maximum c-entropy. In order to do that we take ˙\(h\) defined by (132). Then
\[
(170) \quad h = -1 + i \sqrt{1 + \frac{1}{\sqrt{2}} - 1} + \frac{1}{4} = -1 + \frac{\sqrt{3}}{2}i.
\]
Then the corresponding main operator ˙\(T_h\) is
\[
(171) \quad \begin{cases}
  T_h y = -y'' + \frac{2}{x^2} y \\
  y'(1) = \left(-1 + \frac{\sqrt{3}}{2}i\right) y(1).
\end{cases}
\]
This value of ˙\(h\) in (170) corresponds to the value of \(\kappa_0\) given by (66) that is in this case
\[
(172) \quad \kappa_0 = \frac{\sqrt{2}}{\sqrt{3} + 1}.
\]

Having in mind the result of Theorem 33, we would like to have an L-system with maximum c-entropy and accretive extremal state-space operator. To do so we take \(\mu = \infty\) and construct the corresponding L-system \(\Theta_{h, \infty}\). According to (32) the quasi-kernel in this case is
\[
\begin{cases}
  \hat{A}_{\infty} y = -y'' + \frac{2}{x^2} y \\
  y'(1) = -y(1),
\end{cases}
\]
which is the Krein-von Neumann extension of ˙\(A\). The state-space operator of this L-system \(\Theta_{h, \infty}\) with \(\kappa = \kappa_0\) and \(\mu = \infty\) is (see (29))
\[
\begin{align*}
\tilde{A}_{\infty, h} y &= -y'' + \frac{2}{x^2} y - \left[y'(1) + \left(1 - \frac{\sqrt{3}}{2}i\right) y(1)\right] \delta(x - 1), \\
\tilde{A}^*_{\infty, h} y &= -y'' + \frac{2}{x^2} y - \left[y'(1) + \left(1 + \frac{\sqrt{3}}{2}i\right) y(1)\right] \delta(x - 1).
\end{align*}
\]
Its impedance function \(V_{\Theta_{\infty, h}}(z)\) is such that
\[
V_{\Theta_{\infty, h}}(i) = \frac{\text{Im} h}{m_{\infty, \frac{1}{2}}(i) + \text{Re} h} = \frac{\sqrt{3}}{2 + \sqrt{2}i} - 1 = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3} + 1} = \sqrt{3} - \sqrt{2}.
\]
Now we are going to construct two L-systems having ˙\(T_h\) from (171) as their main operator and such that their corresponding impedance functions are from the classes \(\mathcal{M}_{\kappa_0}\) and \(\mathcal{M}_{-\frac{1}{\kappa_0}}\) (as discussed in Remark 39), where \(\kappa_0\) is given by (172).

First find the normalizing parameter \(a\) that corresponds to \(\kappa_0\),
\[
a = \frac{1 - \kappa_0}{1 + \kappa_0} = \frac{1 - \frac{\sqrt{3}}{2 + \sqrt{2}i}}{1 + \frac{\sqrt{3}}{2 + \sqrt{2}i}} = \frac{\sqrt{3} - \sqrt{2} + 1}{\sqrt{3} + \sqrt{2} + 1} = \sqrt{3} - \sqrt{2}.
\]
Then we use \( V_{\Theta_{\mu_1,h}}(i) = ai \) and (55) for above value of \( a \) to obtain

\[
V_{\Theta_{\mu_1,h}}(i) = \frac{(\frac{2i+\sqrt{3}i}{2} + \mu_1) \sqrt{3}}{\mu_1 + 1} = (\sqrt{3} - \sqrt{2})i.
\]

Solving the above equation for \( \mu_1 \) we get \( \mu_1 = -\frac{2 + \sqrt{3}}{2} \). This yields the quasi-kernel (see (52))

\[
\begin{cases}
A_{\mu_1} y = -y'' + \frac{2}{\kappa^2} y \\
y'(1) = (\sqrt{3} - 1)y(1).
\end{cases}
\]

The state-space operator of the L-system \( \Theta_{\mu_1,h} \) with \( \kappa_0 \) from (159) and \( \mu_1 = -\frac{2 + \sqrt{3}}{2} \) is (see (29))

\[
A_{\mu_1,h} y = -y'' + \frac{2}{\kappa^2} y - \frac{2y'(1) + (1 - \sqrt{3})y(1)}{2\sqrt{3}(1 + i)} \left[ (2 + \sqrt{3})\delta(x - 1) - 2\delta'(x - 1) \right],
\]

\[
A_{\mu_1,h}^* y = -y'' + \frac{2}{\kappa^2} y - \frac{2y'(1) + (1 + \sqrt{3})y(1)}{2\sqrt{3}(1 - i)} \left[ (2 + \sqrt{3})\delta(x - 1) - 2\delta'(x - 1) \right].
\]

Also the channel operator is given by \( K_{\mu_1,h} c = cg_{\mu_1,h}, (c \in \mathbb{C}) \), where (see (30))

\[
g_{\mu_1,h} = \frac{1}{2 \cdot 3^{1/4}} \left[ 2\delta'(x - 1) - (2 + \sqrt{3})\delta(x - 1) \right].
\]

Then \( \Theta_{\mu_1,h} \) has the form

\[
\Theta_{\mu_1,h} = \begin{pmatrix} A_{\mu_1,h} & K_{\mu_1,h} \end{pmatrix},
\]

where all the components are described above. Using formulas (35) and (36) we obtain

\[
W_{\Theta_{\mu_1,h}}(z) = \frac{-1-i}{-1+i} \cdot \frac{m_{\infty,\frac{1}{2}}(z) + h}{m_{\infty,\frac{1}{2}}(z) + h} = (-i) \frac{2z + \sqrt{3}z + \sqrt{3}i}{2z - \sqrt{3}z - \sqrt{3}i}
\]

and

\[
V_{\Theta_{\mu_1,h}}(z) = \frac{\left( m_{\infty,\frac{1}{2}}(z) + \mu_1 \right) \text{Im} h}{\mu_1 - \text{Re} h} \cdot \frac{m_{\infty,\frac{1}{2}}(z) + \mu_1 \text{Re} h - |h|^2}{2iz - \sqrt{3}z - i\sqrt{3}} = \frac{2iz - \sqrt{3}z - i\sqrt{3}}{2iz + \sqrt{3}z + i\sqrt{3}}.
\]

Direct substitution confirms that

\[
V_{\Theta_{\mu_1,h}}(i) = \frac{2i - i - \sqrt{3}i - \sqrt{3}i}{2i + \sqrt{3}i + i\sqrt{3}} = (\sqrt{3} - \sqrt{2})i.
\]

Clearly,

\[
V_{\Theta_{\mu_1,h}}(z) \in \mathfrak{H}_{\kappa_0},
\]

where \( \kappa_0 \) is given by (172).

Now we apply (137) and obtain

\[
\mu_2 = \frac{\mu_1 \text{Re } h - |h|^2}{\mu_1 - \text{Re } h} = \frac{2 + \sqrt{2}}{2} \cdot \frac{\frac{1}{2} - \frac{7}{4}}{1 - \frac{2 + \sqrt{2}}{2}} = \frac{\sqrt{3} - 2}{2}.
\]

This yields

\[
\begin{cases}
A_{\mu_2} y = -y'' + \frac{2}{\kappa^2} y \\
y'(1) = (\sqrt{3} - 1)y(1).
\end{cases}
\]
The state-space operator of the L-system $\Theta_{\mu_2,h}$ with $\kappa_0$ from (159) and $\mu_2 = \frac{\sqrt{3} - i}{2}$ is (see (29))

$$A_{\mu_2,h} y = -y'' - \frac{2}{x^2} y + \frac{2y'(1) + (1 - \sqrt{3} i)y(1)}{2\sqrt{3}(1 - i)} \left[ (\sqrt{3} - 2)\delta(x - 1) - 2\delta'(x - 1) \right],$$

$$A^*_{\mu_2,h} y = -y'' - \frac{2}{x^2} y + \frac{2y'(1) + (1 + \sqrt{3} i)y(1)}{2\sqrt{3}(1 + i)} \left[ (\sqrt{3} - 2)\delta(x - 1) - 2\delta'(x - 1) \right].$$

As the channel operator is given by $K_{\mu_2,h} c = cg_{\mu_2,h}$, ($c \in \mathbb{C}$), where (see (31))

$$g_{\mu_2,h} = \frac{1}{2 \cdot 3^{1/4}} \left[ 2\delta'(x - 1) + (\sqrt{3} - 2)\delta(x - 1) \right].$$

Then the L-system $\Theta_{\mu_2,h}$ has the form

$$(175) \quad \Theta_{\mu_2,h} = \begin{pmatrix} A_{\mu_2,h} & K_{\mu_2,h} \\ \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- & \mathbb{C} \end{pmatrix}.$$ 

Using formulas (35) we obtain

$$(176) \quad W_{\Theta_{\mu_2,h}}(z) = \frac{1 - i}{1 + i} \cdot \frac{m_{\infty,2} \bar{z} + \bar{h}}{m_{\infty,2} \bar{z} + h} = (i) \frac{2z + \sqrt{3}z + \sqrt{3}i}{2z - \sqrt{3}z - \sqrt{3}i},$$

and

$$(177) \quad V_{\Theta_{\mu_2,h}}(z) = \frac{1}{V_{\Theta_{\mu_1,h}}(z)} = -\frac{2iz + \sqrt{3}z + i\sqrt{3}}{2iz - \sqrt{3}z - i\sqrt{3}}.$$ 

So that

$$V_{\Theta_{\mu_2,h}}(i) = \frac{1}{\sqrt{3} - \sqrt{2}} i = (\sqrt{3} + \sqrt{2})i,$$

and therefore

$$V_{\Theta_{\mu_2,h}}(z) \in \mathfrak{M}_{\kappa_0}^{-1},$$

where $\kappa_0$ is given by (152).

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