Research Article

Chao Ma, Hao Liang, Qimiao Xie*, and Pengcheng Wang

Some inequalities on the spectral radius of nonnegative tensors

https://doi.org/10.1515/math-2020-0143
received May 3, 2019; accepted March 15, 2020

Abstract: The eigenvalues and the spectral radius of nonnegative tensors have been extensively studied in recent years. In this paper, we investigate the analytic properties of nonnegative tensors and give some inequalities on the spectral radius.

Keywords: nonnegative tensor, spectral radius, continuity, Hadamard product

MSC 2010: 15A18, 15A39, 15A69

1 Introduction

In recent years, problems related to tensors have drawn much people’s attention. As a generalization of matrix theory, fruitful research achievements have been made in topics such as tensor decomposition, tensor eigenvalues and structured tensors \[1–3\]. Tensors also have wide applications in quantum entanglement, higher order Markov chains, magnetic resonance imaging, machine learning, data analysis, polynomial optimization, nonlinear optimization, hypergraph partitioning, etc. \[4–19\].

In 2005, Qi \[20\] and Lim \[21\] independently defined the concept of eigenvalues of tensors. In 2008, Chang et al. established the Perron–Frobenius theorem for nonnegative tensors \[22\]. In 2010, Yang and Yang introduced the definition of spectral radius of tensors \[23\]. Some bounds on the spectral radius of nonnegative tensors are given in \[24–28\].

In the proof of Theorem 2.3 in \[23\], the authors considered the sequence of nonnegative tensors and gave the limit formula regarding the spectral radius. Note that the result holds when the sequence is monotonic. We want to know whether the result still holds when the sequence is not monotonic and try to investigate the continuity of the spectral radius of nonnegative tensors by some inequalities. Recently, Sun et al. generalized some inequalities on the spectral radius of the Hadamard product of nonnegative matrices to nonnegative tensors \[29\]. Their beautiful results make us interested in the further study of the Hadamard product of tensors.

In this paper, we mainly investigate the analytic properties of the spectral radius of nonnegative tensors. We discuss the continuity of the spectral radius by means of limit formulas as well as tensor inequalities involving norms. We also give some inequalities on the spectral radius of the Hadamard product of nonnegative tensors. These results can be seen as a generalization of the existing inequalities on the spectral radius of nonnegative matrices.

* Corresponding author: Qimiao Xie, College of Ocean Science and Engineering, Shanghai Maritime University, Shanghai 201306, China, e-mail: qmxie@shmtu.edu.cn
Chao Ma: School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; Department of Mathematics, Shanghai Maritime University, Shanghai 201306, China
Hao Liang: Department of Mathematics, East China Normal University, Shanghai 200241, China
Pengcheng Wang: College of Ocean Science and Engineering, Shanghai Maritime University, Shanghai 201306, China

Open Access. © 2020 Chao Ma et al., published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 Public License.
The paper is organized as follows. In Section 2, we collect some definitions, notations and helpful lemmas. In Section 3, we discuss the continuity of the spectral radius. In Section 4, we give some inequalities on the spectral radius involving the Hadamard product.

2 Preliminaries

A real $m$th order $n$-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{ij_1 \cdots j_m})$ is a multiarray of real entries $a_{ij_1 \cdots j_m}$, where $i_j \in \{1, 2, \ldots, n\}$ for $j \in \{1, 2, \ldots, m\}$. When $m = 2$, $\mathcal{A}$ is a matrix of order $n$. The set of all $m$th order $n$-dimensional real tensors is denoted as $T_{m,n}$. Throughout this paper, we assume that $m, n \geq 2$.

A tensor is said to be nonnegative (positive) if each of its entry is nonnegative (positive). Denote by $O$ the zero tensor, and by $\mathcal{F}$ the tensor with each entry equal to 1. For a tensor $\mathcal{A}$, $\mathcal{A} \geq (>) O$ implies that $\mathcal{A}$ is nonnegative (positive). For two tensors $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \geq \mathcal{B}$ or $\mathcal{B} \leq \mathcal{A}$ implies that $\mathcal{A} - \mathcal{B}$ is nonnegative. Let $|\mathcal{A}|$ be the tensor obtained from $\mathcal{A}$ by taking the absolute values of the entries. Then, $|\mathcal{A}|$ is nonnegative and $\mathcal{A} \leq |\mathcal{A}|$. $\mathcal{A} = (a_{ij_1 \cdots j_m}) \in T_{m,n}$ is said to be reducible, if there is a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{ij_1 \cdots j_m} = 0, \quad \forall \ i_1 \in I, \quad \forall \ i_2, i_3, \ldots, i_m \notin I.$$  

A tensor is said to be irreducible if it is not reducible.

The inner product of $\mathcal{A} = (a_{ij_1 \cdots j_m})$, $\mathcal{B} = (b_{ij_1 \cdots j_m}) \in T_{m,n}$, denoted by $\langle \mathcal{A}, \mathcal{B} \rangle$, is defined as follows:

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1, j_1, \ldots, j_m = 1}^{n} a_{ij_1 \cdots j_m} b_{ij_1 \cdots j_m}.$$  

The Frobenius norm of $\mathcal{A}$ is defined and denoted as $||\mathcal{A}||_F = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}$. Denote by $\mathbb{R}^n$ the set of real vectors of dimension $n$. $\mathbb{R}_n^+((\mathbb{R}^n_+))$ represents the cone $\{x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid x_i \geq (>) 0, \; i = 1, 2, \ldots, n\}$.

Let $\mathcal{A} = (a_{ij_1 \cdots j_m}) \in T_{m,n}$, and let $x = (x_1, x_2, \ldots, x_n)^T$ be a complex vector of dimension $n$. Then, $\mathcal{A}x^{m-1}$ is a vector of dimension $n$ with its $i$th component as

$$(\mathcal{A}x^{m-1})_i = \sum_{i_1, \ldots, i_m = 1}^{n} a_{ii_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$  

for $i = 1, 2, \ldots, n$. Denote by $0$ the zero vector. A complex number $\lambda$ is called an eigenvalue of $\mathcal{A}$ if it together with $x \neq 0$ forms a solution to the following system of homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{m-1},$$  

where $x^{m-1} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. The nonzero vector $x$ is called an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\lambda$. The spectral radius of $\mathcal{A}$ is defined and denoted as

$$\rho(\mathcal{A}) = \max||\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}||.$$  

It is well known that the Perron–Frobenius theorem is a fundamental result for nonnegative matrices [30, p. 123]. Chang, Pearson and Zhang generalized this theorem to nonnegative tensors. Yang and Yang gave some further results on the Perron–Frobenius theorem for nonnegative tensors. We summarize some of their results as follows.

Lemma 2.1. [22] Let $\mathcal{A} \in T_{m,n}$ be nonnegative. Then, there exists $\lambda_0 \geq 0$ and $x_0 \in \mathbb{R}^n_+$ such that $\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{m-1}$.
Lemma 2.2. [22] Let \( \mathcal{A} \in T_{m,n} \) be irreducible nonnegative. Then, there exists \( \lambda_0 > 0 \) and \( x_0 \in \mathbb{R}^n_+ \), such that \( \mathcal{A}x_0^{m-1} = \lambda_0 x_0^{m-1} \). Moreover, if \( \lambda \) is an eigenvalue with a nonnegative eigenvector, then \( \lambda = \lambda_0 \). If \( \lambda \) is an eigenvalue of \( \mathcal{A} \), then \( |\lambda| \leq \lambda_0 \).

Lemma 2.3. [23] Let \( \mathcal{A}, \mathcal{B} \in T_{m,n} \) be nonnegative. Then,

(i) \( \rho(\mathcal{A}) \) is an eigenvalue of \( \mathcal{A} \) with a nonnegative eigenvector corresponding to it;
(ii) If \( \lambda \) is an eigenvalue of \( \mathcal{A} \) with a positive eigenvector, then \( \lambda = \rho(\mathcal{A}) \);
(iii) If \( \mathcal{A} \leq \mathcal{B} \), then \( \rho(\mathcal{A}) \leq \rho(\mathcal{B}) \);
(iv) \( \rho(\mathcal{A}) = \max_{x \in \mathbb{R}^n_+} \min_{\lambda \neq 0} \frac{((\mathcal{A}+\mathcal{B})x^{m-1})_i}{x_i^{m-1}} \),
(v) If \( x \in \mathbb{R}^n_+ \), then \( \rho(\mathcal{A}) \leq \max_{j \leq n} \frac{(\mathcal{A}x^{m-1})_j}{x_j^{m-1}} \);
(vi) Let \( \mathcal{A}_k = \mathcal{A} + \frac{1}{k} \mathcal{J} \), \( k = 1, 2, ... \). Then, \( \lim_{k \to \infty} \rho(\mathcal{A}_k) = \rho(\mathcal{A}) \).

3 Continuity of the spectral radius of nonnegative tensors

Suppose \( \mathcal{A} \in T_{m,n} \), and \( \{\mathcal{A}_k\}_{k=1}^{\infty} \) is a set of \( m \)th order \( n \)-dimensional tensors. \( \lim_{k \to \infty} \mathcal{A}_k = \mathcal{A} \) means \( \lim_{k \to \infty} \mathcal{A}_k(\mathcal{B}_{i_1}...\mathcal{B}_{i_m}) = \mathcal{A}(\mathcal{B}_{i_1}...\mathcal{B}_{i_m}) \) for any \( i_1, i_2, ..., i_m \).

Lemma 3.1. Let \( \mathcal{A} \in T_{m,n} \) be nonnegative, and let \( \{\mathcal{E}_k\}_{k=1}^{\infty} \) be a set of \( m \)th order \( n \)-dimensional nonnegative tensors with \( \lim_{k \to \infty} \mathcal{E}_k = 0 \). If \( \mathcal{A} \) has an eigenvalue with a positive eigenvector corresponding to it, then \( \lim_{k \to \infty} \rho(\mathcal{A} + \mathcal{E}_k) = \rho(\mathcal{A}) \).

Proof. By Lemma 2.3 (ii), there exists \( x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n_+ \) such that \( \mathcal{A}x^{m-1} = \rho(\mathcal{A})x^{m-1} \). Then, by Lemma 2.3 (iii) and (v),
\[
\rho(\mathcal{A}) \leq \rho(\mathcal{A} + \mathcal{E}_k) \leq \max_{1 \leq i \leq n} \frac{((\mathcal{A} + \mathcal{E}_k)x^{m-1})_i}{x_i^{m-1}} = \rho(\mathcal{A}) + \max_{1 \leq i \leq n} \frac{(\mathcal{E}_kx^{m-1})_i}{x_i^{m-1}}.
\]
Since \( \lim_{k \to \infty} \mathcal{E}_k = 0 \), \( \lim_{k \to \infty} \max_{1 \leq i \leq n} \frac{(\mathcal{E}_kx^{m-1})_i}{x_i^{m-1}} = 0 \). Thus, \( \lim_{k \to \infty} \rho(\mathcal{A} + \mathcal{E}_k) = \rho(\mathcal{A}) \).

Lemma 3.2. Let \( \mathcal{A} \in T_{m,n} \) be nonnegative, and let \( \{\mathcal{E}_k\}_{k=1}^{\infty} \) be a set of \( m \)th order \( n \)-dimensional nonnegative tensors such that \( \mathcal{E}_k \leq \mathcal{A} \) for \( k = 1, 2, ..., \) with \( \lim_{k \to \infty} \mathcal{E}_k = 0 \). Then, \( \lim_{k \to \infty} \rho(\mathcal{A} - \mathcal{E}_k) = \rho(\mathcal{A}) \).

Proof. By Lemma 2.3 (i), there exists nonzero \( y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n_+ \) such that \( \mathcal{A}y^{m-1} = \rho(\mathcal{A})y^{m-1} \). Then, by Lemma 2.3 (iii) and (iv),
\[
\rho(\mathcal{A}) \geq \rho(\mathcal{A} - \mathcal{E}_k) \geq \min_{y \neq 0} \frac{((\mathcal{A} - \mathcal{E}_k)y^{m-1})_i}{y_i^{m-1}} = \rho(\mathcal{A}) - \max_{y \neq 0} \frac{(\mathcal{E}_ky^{m-1})_i}{y_i^{m-1}}.
\]
Since \( \lim_{k \to \infty} \mathcal{E}_k = 0 \), \( \lim_{k \to \infty} \max_{y \neq 0} \frac{(\mathcal{E}_ky^{m-1})_i}{y_i^{m-1}} = 0 \). Thus, \( \lim_{k \to \infty} \rho(\mathcal{A} - \mathcal{E}_k) = \rho(\mathcal{A}) \).

Theorem 3.3. Let \( \mathcal{A} \in T_{m,n} \) be nonnegative, and suppose \( \mathcal{A} \) has an eigenvalue with a positive eigenvector corresponding to it. Let \( \{\mathcal{A}_k\}_{k=1}^{\infty} \) be a set of \( m \)th order \( n \)-dimensional nonnegative tensors with \( \lim_{k \to \infty} \mathcal{A}_k = \mathcal{A} \). Then, \( \lim_{k \to \infty} \rho(\mathcal{A}_k) = \rho(\mathcal{A}) \).

Proof. Let \( \mathcal{A} - \mathcal{A}_k = \mathcal{E}_k - \mathcal{E}_k, k = 1, 2, ... \), where
\begin{equation*}
(E_k)_{i_1 \ldots i_n} = \begin{cases}
(A - A_k)_{i_1 \ldots i_n}, & \text{if } (A - A_k)_{i_1 \ldots i_n} > 0; \\
0, & \text{otherwise.}
\end{cases}
\end{equation*}

Then, $E_k \geq 0$, $E_k \geq 0$, $A - E_k \geq 0$, $\lim_{k \to \infty} E_k = 0$, $\lim_{k \to \infty} A_k = 0$. Since $A_k = A + E_k$, $\lim_{k \to \infty} A_k = A$, by Lemma 3.1, $\rho(A_k) \leq \rho(A + E_k)$. Then, by Lemma 3.1, $\lim_{k \to \infty} \rho(A_k) = \lim_{k \to \infty} \rho(A + E_k) = \rho(A)$. Since $A_k \geq A_k - E_k = A - E_k$, by Lemma 2.3 (iii), $\rho(A_k) \geq \rho(A - E_k)$. Then, by Lemma 3.2, $\lim_{k \to \infty} \rho(A_k) = \lim_{k \to \infty} \rho(A - E_k) = \rho(A)$. This completes the proof. \hfill \Box

Let $A = (a_{i_1 \ldots i_n}) \in T_{m,n}$. If the entries $a_{i_1 \ldots i_n}$ are invariant under any permutation of their indices, then $A$ is called a symmetric tensor. The set of all $m$th order $n$-dimensional real symmetric tensors is denoted as $S_{m,n}$. Let $x = (x_1, x_2, \ldots, x_n)^T$. Define

$$
A x^m = \sum_{i_1, i_2, \ldots, i_m=1}^n a_{i_1 i_2 \ldots i_m} x_1 x_2 \cdots x_m.
$$

**Lemma 3.4.** Let $A \in T_{m,n}$ be positive, and let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$, $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}_+^n$, $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$. Then,

$$
\frac{v^T(A u)^{m-1}}{A x^m} \leq \prod_{i_1, i_2, \ldots, i_m=1}^n \frac{v_{i_1} u_{i_1} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}.
$$

**Proof.** Since the function $f(t) = \log t$ is concave on $(0, +\infty)$, we have

$$
\log \left( \sum_{i_1, i_2, \ldots, i_m=1}^n a_{i_1 i_2 \ldots i_m} x_1 x_2 \cdots x_m \right) \geq \sum_{i_1, i_2, \ldots, i_m=1}^n a_{i_1 i_2 \ldots i_m} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{A x^m} \log \frac{v_{i_1} u_{i_1} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}.
$$

Then,

$$
\log \frac{v^T(A u)^{m-1}}{A x^m} \geq \log \prod_{i_1, i_2, \ldots, i_m=1}^n \frac{v_{i_1} u_{i_1} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}},
$$

which implies

$$
\frac{v^T(A u)^{m-1}}{A x^m} \geq \prod_{i_1, i_2, \ldots, i_m=1}^n \frac{v_{i_1} u_{i_1} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}.
$$

This completes the proof. \hfill \Box

**Lemma 3.5.** Let $A \in S_{m,n}$ be nonnegative, and let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$. Then, $A x^m \leq \rho(A) \sum_{i=1}^n x_i^m$.

**Proof.** By Lemma 2.3 (vi), it suffices to consider the case when $A$ is positive. First suppose $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$. Note that Lemma 2.2 implies that $\rho(A)$ is an eigenvalue of $A$ with a positive eigenvector corresponding to it. Let $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}_+^n$ be the eigenvector corresponding to $\rho(A)$, and let $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$, with $v_i = \frac{x_i^m}{\|x\|^m}$, $i = 1, 2, \ldots, n$. Since $A$ is symmetric,

$$
\frac{n}{i=1} \left( v_i u_i \cdots u_i \right)^{\frac{\|x\|^m}{x_i^m}} = \frac{n}{i=1} \left( v_i u_i^{m-1} \right)^{\frac{\|x\|^m}{x_i^m}}.
$$

Thus,
\[
\prod_{i_1,i_2,\ldots,i_m=1}^{n} (x_{i_1}x_{i_2}\cdots x_{i_m})^{\frac{a_{i_1}a_{i_2}\cdots a_{i_m}}{\rho(n)}} = \prod_{i=1}^{n} (x_i^{m})^{\frac{a_i}{\rho(n)}}.
\]

By Lemma 3.4, \(\mathcal{A}x^m \leq v^T(\mathcal{A}u^{m-1}) = v^T\rho(\mathcal{A})u^{m-1} = \rho(\mathcal{A}) \sum_{i=1}^{n} v_i u_i^{m-1} = \rho(\mathcal{A}) \sum_{i=1}^{n} x_i^m \).

Next suppose \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+\). For \(k = 1, 2, \ldots\), let \(x^{(k)} = x + \frac{1}{k}e\), where \(e\) is the vector with each component equal to 1. Then, \(x^{(k)} \in \mathbb{R}_n^+\) and \(\lim_{k \to \infty} x^{(k)} = x\). By what we have proved above, \(\mathcal{A}(x^{(k)})^m \leq \rho(\mathcal{A}) \sum_{i=1}^{n} \left(\frac{x_1 + \frac{1}{k}}{k}\right)^m\) for \(k = 1, 2, \ldots\). The conclusion holds when \(k \to \infty\). \(\square\)

**Theorem 3.6.** Let \(\mathcal{A}, \mathcal{B} \in S_{m,n}\) be nonnegative. Then, \(\rho(\mathcal{A} + \mathcal{B}) \leq \rho(\mathcal{A}) + \rho(\mathcal{B})\).

**Proof.** By Lemma 2.3 (i), let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+\) be the eigenvector of \(\mathcal{A} + \mathcal{B}\) corresponding to \(\rho(\mathcal{A} + \mathcal{B})\) with \(\sum_{i=1}^{n} x_i^m = 1\). Then, by Lemma 3.5, \(\rho(\mathcal{A} + \mathcal{B}) = \rho(\mathcal{A} + \mathcal{B}) \leq \rho(\mathcal{A}) \sum_{i=1}^{n} x_i^m + \rho(\mathcal{B}) \sum_{i=1}^{n} x_i^m = \rho(\mathcal{A}) + \rho(\mathcal{B})\).

Let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+\). Denote by \(x^{\otimes m}\) the tensor in \(T_{m,n}\) with its \((i_1, i_2, \ldots, i_m)\) entry as \(x_{i_1}x_{i_2}\cdots x_{i_m}\).

**Lemma 3.7.** Let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+\) with \(\sum_{i=1}^{n} x_i^m = 1\). Then, \(|\|x^{\otimes m}\|_F| \leq n^{\frac{m}{2}}\).

**Proof.** Since \(\sum_{i=1}^{n} x_i^m = 1\) with \(m \geq 2\) by the power mean inequality [31, p. 203], we have \(\left(\frac{x_1^m + x_2^m + \cdots + x_n^m}{n}\right)^{\frac{1}{m}} \leq \left(\frac{x_1^m + x_2^m + \cdots + x_n^m}{n}\right)^{\frac{1}{m}} \leq \left(\frac{1}{n}\right)^{\frac{1}{m}}\). Then, \(|\|x^{\otimes m}\|_F| = \|x_1^m x_2^m \cdots x_n^m = (x_1^m + x_2^m + \cdots + x_n^m)^{\frac{m}{2}} \leq n^{\frac{m}{2}}\). This completes the proof. \(\square\)

**Lemma 3.8.** Let \(\mathcal{A} \in T_{m,n}\) be nonnegative. Then, \(\rho(\mathcal{A}) \leq n^{\frac{m}{2}}||\mathcal{A}||_F\).

**Proof.** Let \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+\) be the eigenvector of \(\mathcal{A}\) corresponding to \(\rho(\mathcal{A})\) with \(\sum_{i=1}^{n} x_i^m = 1\). By Lemma 3.7 and the Cauchy–Schwarz inequality, \(\rho(\mathcal{A}) = \rho(\mathcal{A}) \sum_{i=1}^{n} x_i^m = \sum_{i=1}^{n} x_i^m \rho(\mathcal{A} x^{m-1}) = \sum_{i, i_2, \ldots, i_m} a_{i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} = \langle \mathcal{A}, x^{\otimes m} \rangle \leq ||\mathcal{A}||_F ||x^{\otimes m}||_F \leq n^{\frac{m}{2}} ||\mathcal{A}||_F\). \(\square\)

**Theorem 3.9.** Let \(\mathcal{A}, \mathcal{B} \in S_{m,n}\) be nonnegative. Then, \(|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leq n^{\frac{m}{2}} ||\mathcal{A} - \mathcal{B}||_F\).

**Proof.** Since \(\mathcal{A} - \mathcal{B} \leq |\mathcal{A} - \mathcal{B}|, \mathcal{A} \leq |\mathcal{A} - \mathcal{B}| + \mathcal{B}\). By Theorem 3.6, \(\rho(\mathcal{A}) \leq \rho(|\mathcal{A} - \mathcal{B}| + \mathcal{B}) \leq \rho(|\mathcal{A} - \mathcal{B}|) + \rho(\mathcal{B})\). Thus, \(\rho(\mathcal{A}) - \rho(\mathcal{B}) \leq \rho(|\mathcal{A} - \mathcal{B}|)\). Similarly, \(\rho(\mathcal{B}) - \rho(\mathcal{A}) \leq \rho(|\mathcal{A} - \mathcal{B}|)\). Then, by Lemma 3.8, \(|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leq n^{\frac{m}{2}} ||\mathcal{A} - \mathcal{B}||_F\). \(\square\)

### 4 Some inequalities involving the Hadamard product

Let \(\mathcal{A} = (a_{i,j})_{m,n}, \mathcal{B} = (b_{i,j})_{m,n} \in T_{m,n}\). The Hadamard product of \(\mathcal{A}\) and \(\mathcal{B}\) is defined and denoted as \(\mathcal{A} \circ \mathcal{B} = (a_{i,j}b_{i,j})_{m,n}\). Let \(\mathcal{A} = (a_{i,j})_{m,n}\) be nonnegative, and let \(\alpha\) be a positive real number.
The $\alpha$th Hadamard power of $\mathcal{A}$ is the tensor $\mathcal{A}^{[\alpha]} = (a_{\ell_1 \cdots \ell_n}^{\alpha})$. Similarly, for two vectors $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$, let $x \odot y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)^T$; for $x \in \mathbb{R}_+^n$ and $\alpha > 0$ let $x^{[\alpha]} = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)$.

**Theorem 4.1.** Let $\mathcal{A} \in T_{m,n}$ be nonnegative. Then, $(\rho(\mathcal{A}^{[\alpha]}))^\frac{n}{\alpha} \leq (\rho(\mathcal{A}^{[\beta]}))^\frac{n}{\beta}$, with $r \geq s > 0$.

**Proof.** By Lemma 2.3 (vi), we may suppose $\mathcal{A}$ is positive. Let $\mathcal{A}^{[\alpha]} = \mathcal{B} = (b_{\ell_1 \cdots \ell_n}) > 0$, and let $t = \frac{\beta}{\alpha} \geq 1$. It suffices to prove that $\rho(\mathcal{B}^{[\alpha]}) \leq (\rho(\mathcal{B}))^\beta$. By Lemma 2.2, there exists $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$, such that $\mathcal{B} x^{[\alpha]} = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)^T$. Then, for $i = 1, 2, \ldots, n$,

$$(\mathcal{B}^{[\alpha]})^{[\alpha]}(x_i) = \sum_{\ell_1, \ldots, \ell_n=1}^n b_{\ell_1 \cdots \ell_n} x_1^{\alpha_{\ell_1}} \cdots x_n^{\alpha_{\ell_n}} \leq \left( \sum_{\ell_1, \ldots, \ell_n=1}^n b_{\ell_1 \cdots \ell_n} x_1 \cdots x_n \right)^\frac{n}{\alpha}.$$

By Lemma 2.3 (v),

$$\rho(\mathcal{B}^{[\alpha]}) \leq \max_{1 \leq i \leq n} \left( (\mathcal{B}^{[\alpha]})^{[\alpha]}(x_i) \right)^{\frac{1}{\alpha}} \leq (\rho(\mathcal{B}))^\beta.$$

This completes the proof $\square$.

**Theorem 4.2.** Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \in T_{m,n}$ be nonnegative. Then, $\rho(\mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]}) \leq (\rho(\mathcal{A}_1))^{a_1} (\rho(\mathcal{A}_2))^{a_2} \cdots (\rho(\mathcal{A}_k))^{a_k}$ with $a_i > 0$ and $\sum_{i=1}^k a_i = 1$.

**Proof.** By continuity of the spectral radius, we may suppose $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are all positive. Let $\alpha = \sum_{i=1}^k a_i$, and let $\mathcal{A}_i = (a_{i_{\ell_1 \cdots \ell_n}}^{[a_i]})$, $i = 1, 2, \ldots, k$. By Theorem 4.1,

$$\rho(\mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]}) = \rho \left( \mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]} \right)^{\frac{n}{\alpha}} \leq \left( \rho \left( \mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]} \right) \right)^{\frac{n}{\alpha}}.$$

(1)

By Lemma 2.2, for $i = 1, 2, \ldots, k$, there exist $x_i^{[\alpha]} = (x_i^{[\alpha]}(1), x_i^{[\alpha]}(2), \ldots, x_i^{[\alpha]}(n))^T \in \mathbb{R}_+^n$, such that $\mathcal{A}_i x_i^{[\alpha]} = (x_i^{[\alpha]}(1), x_i^{[\alpha]}(2), \ldots, x_i^{[\alpha]}(n))^T$. Let $\mathcal{A} = \mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]}$, and let $x = (x_1^{[\alpha]}(1), x_2^{[\alpha]}(1), \ldots, x_k^{[\alpha]}(1))^T = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$. For $i = 1, 2, \ldots, n$,

$$(\mathcal{A} x^{[\alpha]})_i = \sum_{\ell_1, \ldots, \ell_n=1}^n \left( (a_{i_{\ell_1 \cdots \ell_n}}^{[a_i]}) x_1^{[\alpha]} \cdots x_n^{[\alpha]} \right) \leq \left( \sum_{\ell_1, \ldots, \ell_n=1}^n a_{i_{\ell_1 \cdots \ell_n}}^{[a_i]} x_1^{[\alpha]} \cdots x_n^{[\alpha]} \right)^{\frac{n}{\alpha}}.$$

Then, by Lemma 2.3 (v), $\rho(\mathcal{A}) \leq \max_{1 \leq i \leq n} (\mathcal{A} x^{[\alpha]})_i^{\frac{1}{\alpha}} \leq (\rho(\mathcal{A}))^{\frac{1}{\alpha}} (\rho(\mathcal{A}))^{\frac{1}{\alpha}}$. Thus, by (1),

$$\rho(\mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]}) \leq \left( \rho \left( \mathcal{A}_1^{[a_1]} \odot \mathcal{A}_2^{[a_2]} \odot \cdots \odot \mathcal{A}_k^{[a_k]} \right) \right)^{\alpha} \leq (\rho(\mathcal{A}))^{a_1} \cdots (\rho(\mathcal{A}))^{a_k}.$$

$\square$
5 Conclusion

In this paper, we focus on the analytic properties of the spectral radius of nonnegative tensors. First, we discuss the continuity of the spectral radius. Then, we give some inequalities on the spectral radius involving the Hadamard product. These results generalize some existing results on the spectral properties of nonnegative matrices to nonnegative tensors.

Acknowledgement: The authors would like to express their sincere thanks to referees and editor for their enthusiastic guidance and help. This research was supported by the National Natural Science Foundation of China (Grant No. 11601322, 71503166 and 61573240).

References

[1] T. G. Kolda and B. W. Bader, *Tensor decompositions and applications*, SIAM Rev. 51 (2009), 455–500, DOI: 10.1137/07070111X.

[2] L. Qi, H. Chen, and Y. Chen, *Tensor Eigenvalues and Their Applications*, Springer, New York, 2018.

[3] L. Qi and Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, Philadelphia, 2017.

[4] F. Bohnet-Waldraff, D. Braun, and O. Giraud, *Tensor eigenvalues and entanglement of symmetric states*, Phys. Rev. A 94 (2016), 042324, DOI: 10.1103/PhysRevA.94.042324.

[5] S. Hu, L. Qi, and G. Zhang, *Computing the geometric measure of entanglement of multipartite pure states by means of non-negative tensors*, Phys. Rev. A 93 (2016), 012304, DOI: 10.1103/PhysRevA.93.012304.

[6] C.-K. Li and S. Zhang, *Stationary probability vectors of higher-order Markov chains*, Linear Algebra Appl. 473 (2015), 114–125, DOI: 10.1016/j.laa.2014.03.043.

[7] C. Cartis, J. M. Fowkes, and N. I. M. Gould, *Convergence analysis of a block improvement method for polynomial optimization problems*, Math. Oper. Res. 39 (2014), 1084–1108, DOI: 10.1287/moor.2014.0644.

[8] K. Hou and A. M. C. So, *Hardness and approximation results for $L_p$-ball constrained homogeneous polynomial optimization problems*, Math. Oper. Res. 39 (2014), 1084–1108, DOI: 10.1287/moor.2014.0644.

[9] B. Jiang, Z. Li, and S. Zhang, *On cones of nonnegative quartic forms*, Found. Comput. Math. 17 (2017), 161–197, DOI: 10.1007/s10208-015-9286-4.

[10] Y. Wang, L. Caccetta, and G. Zhou, *Convergence analysis of a block improvement method for polynomial optimization over unit spheres*, Numer. Linear Algebra Appl. 22 (2015), 1059–1076, DOI: 10.1002/nla.1996.

[11] Y. Yang, Q. Yang, and L. Qi, *Approximation bounds for trilinear and biquadratic optimization problems over nonconvex constraints*, J. Optim. Theory Appl. 163 (2014), 841–858, DOI: 10.1007/s10957-014-0538-2.

[12] C. Cartis, J. M. Fowkes, and N. I. M. Gould, *Branching and bounding improvements for global optimization algorithms with Lipschitz continuity properties*, J. Global Optim. 61 (2015), 429–457, DOI: 10.1007/s10898-014-0199-6.

[13] G. Li, B. S. Mordukhovich, and T. S. Pham, *New fractional error bounds for polynomial systems with applications to Holderian stability in optimization and spectral theory of tensors*, Math. Program. 153 (2015), 333–362, DOI: 10.1007/s10107-014-0806-9.

[14] Y. Yang, Y. Feng, X. Huang, and J. A. K. Suykens, *Rank-1 tensor properties with applications to a class of tensor optimization problems*, J. Optim. Theory Appl. 163 (2014), 841–858, DOI: 10.1007/s10957-014-0538-2.

[15] Y. Chen, L. Qi, and X. Zhang, *The Fiedler vector of a Laplacian tensor for hypergraph partitioning*, SIAM J. Sci. Comput. 39 (2017), A2508–A2537, DOI: 10.1137/16M1094828.

[16] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symb. Comput. 40 (2005), 1302–1324, DOI: 10.1016/j.jsc.2005.05.007.

[17] L.-H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, in: CAMSAP’05: Proceeding of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, pp. 129–132, 2005.
Some inequalities on the spectral radius of nonnegative tensors

[22] K. C. Chang, K. Pearson, and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2008), 507–520, DOI: 10.4310/cms.2008.v6.n2.a12.

[23] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31 (2010), 2517–2530, DOI: 10.1137/090778766.

[24] J. Zhao and C. Sang, Two new eigenvalue localization sets for tensors and their applications, Open Math. 15 (2017), 1267–1276, DOI: 10.1515/math-2017-0106.

[25] C. Bu, X. Jin, H. Li, and C. Deng, Brauer-type eigenvalue inclusion sets and the spectral radius of tensors, Linear Algebra Appl. 512 (2017), 236–248, DOI: 10.1016/j.laa.2016.09.041.

[26] W. Li and M. Ng, Some bounds for the spectral radius of nonnegative tensors, Numer. Math. 130 (2015), 315–335, DOI: 10.1007/s00211-014-0666-5.

[27] S. Li, C. Li, and Y. Li, A new bound for the spectral radius of nonnegative tensors, J. Inequal. Appl. 2017 (2017), 88, DOI: 10.1186/s13660-017-1362-7.

[28] J. Cui, G. Peng, Q. Lu, and Z. Huang, A new estimate for the spectral radius of nonnegative tensors, Filomat 32 (2018), 3409–3418, DOI: 10.2298/FIL1810409C.

[29] L. Sun, B. Zheng, J. Zhou, and H. Yan, Some inequalities for the Hadamard product of tensors, Linear Multilinear Algebra 66 (2018), 1199–1214, DOI: 10.1080/03081087.2017.1346060.

[30] X. Zhan, Matrix Theory, Grad. Stud. Math., vol. 147, Amer. Math. Soc., Providence, 2013.

[31] P. S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Press, Dordrecht, 2003.