ON THE MAXIMUM OF A GAUSSIAN PROCESS WITH UNIQUE MAXIMUM POINT OF ITS VARIANCE

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Abstract. Gaussian random processes whose variances reach their maximum values at unique points are considered. Exact asymptotic behavior of probabilities of large absolute maximums of their trajectories have been evaluated using the double sum method under the widest possible conditions.

1. Introduction. Preliminaries

This note is a generalization of [9]. Our aim is to show the maximum capability of Pickand’s double sum method for asymptotic behavior of the maximum tail distribution for a Gaussian stationary process (see [5], with corrections in [6]). This method has been generalized to Gaussian random fields in [7], where stationary fields with power like behavior of the correlation function at zero are considered as well as fields with a similar behavior of the correlation function at the unique maximum point of variance. However, while the power behavior of the correlation function with possible light generalization to regular variation of it [6] is quite essential for Pickand’s method, the required in [7,9] power behavior of the variance looks somewhat artificial. In the present note, we give the widest possible conditions on the variance and on the correlation function under which the double sum method still works. Note also that in the recent paper [2] it is proved that in the non-stationary case the variance behavior need not be power but can just be regularly varying.

Let $X(t), t \in [-S,S]$, be a zero mean a.s. continuous Gaussian process with covariance function $r(s,t)$, denote $\sigma^2(t) = r(t,t)$. Here we study the asymptotic behavior of the probability

$$P([-S,S]; u) = P\left(\max_{t \in [-S,S]} X(t) > u\right) \text{ as } u \to \infty. \quad (1)$$

We assume that $\sigma(t)$ reaches its absolute maximal value only at zero, since in the case of another point of the absolute maximum one can simply shift the time.

Assume the following.

A1. Suppose that $X$ has a.s. continuous sample paths.

In particular, the above assumption is satisfied under the following standard Hölder condition, namely for some positive $\Gamma$ and $\gamma$,

$$E(X(t) - X(s))^2 \leq \Gamma|t - s|^\gamma, \quad s,t \in [-S,S]. \quad (2)$$

Under this condition there exits an a.s. continuous version of $X$. Here, in contrast of [9] (see also [7,8]), we do not assume (2).

A2. $\sigma(t)$ reaches its global maximum on $[-S,S]$ only at 0 and $\sigma(0) = 1$. Moreover, there exist finite or infinite limits

$$\lim_{t \to 0, s \to 0} \frac{1 - \sigma^2(t)}{1 - r(s,t)} \in [0, \infty] \quad \text{and} \quad \lim_{t \to 0, s \to 0} \frac{1 - \sigma^2(t)}{1 - r(s,t)} \in [0, \infty]. \quad (3)$$

Furthermore, if at least one of these limits is equal to 0, then there exist $c > 0$ and $K < \infty$ such that for every $x \in [0,c]$, the number of roots of the equation $1 - \sigma^2(t) = x$ does not exceed $K$. 

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Note again that the above specification of the location and the maximal value of $\sigma(t)$ is just for convenience. Note further that A2 implies $r(s, t) \leq 1$, for all $s, t \in [−S, S]$ with equality holding only for $s = t = 0$.

Denote by $\rho(s, t) := r(s, t)/\sigma(s)\sigma(t)$, the correlation function of $X$.

A3 (local stationarity at 0). There exists a covariance function $\rho(t)$ of a stationary process such that

$$\lim_{s,t \to 0} \frac{1 - \rho(s,t)}{1 - \rho(t - s)} = 1.$$  

A4. For $\rho$ from A3, there exist a positive function $q(u)$ and a function $h(t)$ such that $h(t) > 0$ for all $t \neq 0$ and

$$\lim_{u \to \infty} u^2(1 - \rho(q(u)t)) = h(t)$$  

uniformly over $t \in [−\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

Note that $\rho(0) = 1$ and A3 imply that $\rho$ is continuous, hence $q(u) \to 0$ as $u \to \infty$, and therefore (4) is fulfilled uniformly over any compact set. Furthermore, it also follows from (4) that for any positive $s, t$,

$$\lim_{u \to \infty} \frac{1 - \rho(q(u)t)}{1 - \rho(q(u)s)} = \frac{h(t)}{h(s)},$$  

which implies, by definition, the regular variation at zero of $1 - \rho(t)$ [1]. The index of the regular variation, say $\alpha$, is positive, and $h(t) = t^\alpha$. Indeed, if $\alpha < 0$, $\rho(t)$ is not continuous at zero, if $\alpha = 0$, $h(t) = 1$ for all $t > 0$ and $h(t) = 0$ for $t = 0$, so it is not continuous again. Further, if $\alpha > 2$ it follows from A3 and A4 that $\rho''(t) \equiv 0$ which contradicts the positive definiteness of $\rho$. Consequently, we have that $\alpha \in (0, 2]$. As well, the same is valid for $\alpha = 2$ and $t^{-2}(1 - \rho(t)) \to 0$. Thus, assumption A4 is equivalent to the corresponding assumption in [6], and therefore this condition is crucial for our method, the double sum method. Thus, we have

$$1 - \rho(t)$$  

is regularly varying at zero with index $\alpha \in (0, 2]$. (6)

Further, since $1 - \rho(t) = \ell(t)t^\alpha$, where $\ell(t)$ is slowly varying function at zero, we have

$$q(u) = (1 - \rho)^{-1}(u^{-2}),$$  

where $^{-1}$ means the generalized inverse. Now using Theorems 1.5.12 and 1.5.13 (the de Bruijn lemma) and Proposition 1.5.15 from [1], we get that

$$q(u) \sim u^{-2/\alpha}\ell^\#(u^{-2})^{1/\alpha} \quad \text{as} \quad u \to \infty.$$  

(7)

In our notation, $\sim$ stands for asymptotic equivalence, and $\ell^\#$ is the de Bruijn conjugate of $\ell$. In view of (5), we have that (4) holds for any $q'$ such that $\lim_{u \to \infty} q(u)/q'(u) = 1$. Consequently, since $q$ is regularly varying at infinity, without loss of generality, we assume hereinafter that $q$ is monotone.

Note that the slowly varying function $\ell^\#$ can be often explicitly calculated, see Theorem 2.3.3 (the Bojanic and Seneta theorem) and Corollary 2.3.4 from [1]. For example, if

$$\frac{\ell(u^{-2}\ell(u^{-2}))}{\ell(u^{-2})} \to 1 \quad \text{as} \quad u \to \infty,$$

then $\ell^\# \sim 1/\ell$.

In Sec. 2, we repeat the results from [6] in these new conditions. In Sec. 3, the main result of the paper is presented. In short Sec. 4, we present two examples to demonstrate the generality of our result.
2. Stationary Processes

In this section, we assume that \(X(t), t \in [0, S]\), is a stationary Gaussian process with mean zero, unit variance and covariance function \(\rho\) described above. We formulate here for convenience the results from [6] with some obvious further generalizations.

**Lemma 1.** If A1 and A4 hold, then, for any \(T > 0\),

\[
P((0, q(u)T) \setminus u) = (1 + \gamma(u)) H_\alpha(T) \Psi(u),
\]

with \(\gamma(u) \to 0\) as \(u \to \infty\), where \(\alpha \in (0, 2]\) is defined in (6), by the arguments below A4,

\[
\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp \left( -\frac{1}{2} x^2 \right) dx, \quad H_\alpha(T) = E \exp \left( \max_{[0,T]} \chi(t) \right),
\]

and \(\chi(t)\) is a Gaussian process with continuous trajectories, \(\chi(0) = 0\), and

\[
\var(\chi(t) - \chi(s)) = 2h(|t - s|), \quad E\chi(t) = -h(t).
\]

**Theorem 1.** Suppose that the conditions of Lemma 1 hold. Let, furthermore, \(\rho(t) < 1\) for all \(t > 0\). Then for any \(E \subset \mathbb{R}\), a bounded closure of an open set,

\[
P(E; u) = \text{mes}(E) H_\alpha \frac{\Psi(u)}{q(u)} (1 + o(1)) \quad \text{as} \quad u \to \infty,
\]

where

\[
H_\alpha = \lim_{T \to \infty} T^{-1} H_\alpha(T) \in (0, \infty).
\]

This assertion holds even if \(E = E(u)\), provided there exist segments \(E^-(u), E^+(u) \subset \mathbb{R}\) such that \(E^-(u) \subset E \subset E^+(u)\) with \(\lim_{u \to \infty} \text{mes}(E^-(u))/q(u) = \infty\), and for some \(\delta \in (0, 1/2)\), we have \(\text{mes}(E^+(u))e^{-\delta u^2} \to 0\) as \(u \to \infty\).

3. Gaussian Processes with a Unique Maximum Point of Variance

In this section, we consider a centered nonstationary Gaussian process \(X(t), t \in [-S, S]\). In view of A4, it follows from A2 that there exists the limit

\[
\lim_{u \to \infty} u^2(1 - \sigma^2(q(u)t)) = h_1(t) \in [0, \infty].\tag{8}
\]

Note that the limit relations in A2 follow from (8) as well. The limit \(h_1(t)\) can be equal to zero, it can be positive and finite, it can be equal to infinity. These assertions do not change for any other \(t\) of the same sign, i.e., from the same half-line. We say that the stationary-like case takes place if the limit equals zero for all \(t\), see discussion below. If the limit is equal to infinity, then we shall refer to the Talagrand-like case, in this case, for any set \(S\) containing zero

\[
P(S; u) \sim P(X(0) > u), \quad u \to \infty
\]

(see the proof below). Talagrand has shown this for general Gaussian processes and under the most general conditions (see [7] for references and discussions). Finally, for nonzero and non-infinity \(h_1(t)\), the third case is called the transition case. Since we do not assume that \(\sigma\) is symmetric with respect to zero, consideration of left and right limits in (8) is a combination of the three cases above. For instance, \(h_1(t) = \infty\) for any \(t \in [-S, 0]\) and \(h_1(t) \in (0, \infty)\) for any \(t \in (0, S]\). In the latter case, the arguments given for \(1 - \rho(q(u)t)\) imply that for some \(\beta \geq 0\), \(1 - \sigma^2(t), t > 0\), is regularly varying at 0 and moreover \(h_1(t) = h_1(1)t^\beta\). Since

\[
\lim_{u \to \infty} \frac{u^2(1 - \sigma^2(q(u)t))}{u^2(1 - \rho(q(u)t))} = \frac{h_1(t)}{h(t)},
\]

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we conclude that $\alpha = \beta$ and further, the regularly varying functions $1 - \sigma^2(t)$ and $1 - \rho(t)$ have to be equivalent up to a positive constant, namely we have
\[
\lim_{t \to 0} \frac{1 - \sigma^2(t)}{1 - \rho(t)} = \frac{h_1(1)}{h(1)} > 0.
\]

Now we formulate two general results for all described above types of behavior of $\sigma(t)$. The first one is a standard local lemma of the double sum method, a generalization of Lemma 1 (see [7,8]).

**Lemma 2.** Under the assumptions $A1$–$A4$, for any $T > 0$,
\[
P([0, q(u)T]; u) = P^+_{\alpha}(T)\Psi(u)(1 + o(1))
\]
and
\[
P([-q(u)T, q(u)T]; u) = P_\alpha(T)\Psi(u)(1 + o(1))
\]
as $u \to \infty$, where
\[
P^+_{\alpha}(T) = E \max_{t \in [0, T]} e^{\chi_1(t)}, \quad P_\alpha(T) = E \max_{t \in [-T, T]} e^{\chi_1(t)},
\]
with $\chi_1(t) = \chi(t) - h_1(t)$ for $h_1(t) < \infty$, and $\chi_1(t) = 0$ for $h_1(t) = \infty$.

The proof of this lemma is a simple repetition of the proof of Lemma 6.1 [7] by using the assumptions $A1$–$A4$ and the relation (8). The case $h_1(t) = \infty$ can be treated by similar arguments. Note that in the Talagrand case, the detailed consideration of the weak convergence in $C([-T, T])$ of the process
\[
\chi_u(t) = u(X(q(u)t) - u) + w
\]
given $X(0) = u - w/u$, can be restricted to $C([-T, 0])$ for $t > 0$, or to $C([0, T])$ for $t < 0$. It can be proved that in these cases given $X(0) = u - w/u$,
\[
\max_{t \in [-T, T]} \chi_u(t) \to \max_{t \in [-T, 0]} (\chi(t) - h_1(t)) \quad \text{as} \quad u \to \infty,
\]
weakly for $t > 0$, and similar convergence hold for $t < 0$. If $h_1(t) = \infty$ for all nonzero $t$, the above weak limit is equal to 0.

The next result concerns the extraction of an informative parameter set depending on the level $u$, which provides the required asymptotic behavior. Consider the set
\[
B_u = \{t: 1 - \sigma^2(t) \leq u^{-2} \log^A u\}, \quad A > 1.
\]

**Lemma 3.** If $X$ is a centered Gaussian process satisfying $A1$–$A4$, then for any $E \subset [-S, S]$, which is a closure of a bounded open set containing zero, and for any $B \in (1, A)$, we have
\[
P(E; u) = P(E \cap B_u; u)\left(1 + O(e^{-Bu})\right) \quad \text{as} \quad u \to \infty.
\]

**Proof.** By A1, $X$ has bounded sample paths almost surely. Then the Borell-TIS inequality (see, for example, [7]) and the fact that $\sigma(0) = 1$ is the unique maximum of the continuous on $[-S, S]$ function $\sigma(t)$ imply that for some $a > 1/2, b > 0$, and all positive $u, \varepsilon$,
\[
P(E \setminus [-\varepsilon, \varepsilon]; u) \leq b \exp(-au^2).
\]
By assumptions A3 and A4, for the standardized process $\bar{X}(t) = X(t)/\sigma(t), t \in [-S, S]$, for any small enough $\varepsilon > 0$ (hence $\sigma(t) > 0, t \in [-\varepsilon, \varepsilon]$), and for any $s, t \in [-\varepsilon, \varepsilon]$, the following relation holds:
\[
E(\bar{X}(s) - \bar{X}(t))^2 = 2(1 - \rho(s, t)) \leq c_0|t - s|^{\gamma},
\]
where $c_0$
Theorem 2. Under the conditions of Lemma \( \sigma \) conditions imposed on \( 0 \). Hence, the claim follows for any 

Note that the behavior of these integrals as \( \alpha \) in notation of [7–9]. Denote for any 

First, we consider a simplified model for \( u \) as 

Proof. We use this 

where \( c_0, \gamma \) are some positive values. Applying [7, Theorem 8.1] to \( X \) and definition of \( B_u \), we obtain that 

\[
P \left( \sup_{t \in E \cap [\varepsilon, \varepsilon]} X(t) \sigma(t) > u \right) \leq P \left( \sup_{t \in [\varepsilon, \varepsilon]} X(t) > \frac{u}{\sqrt{1 - u^{-2}\log^4 u}} \right)
\]

\[
\leq C_1 u^{\varepsilon_1} \exp \left( -\frac{u^2}{2 - 2u^{-2}\log^4 u} \right) \leq C_2 u^{\varepsilon_1} \exp(-c_2 \log^4 u) \exp \left( -\frac{u^2}{2} \right),
\]

for some positive \( c_i, C_i, i = 1, 2 \). Since \( 0 \in E \) by assumption, \( P(E; u) \geq P(X(0) > u) = \Psi(u) \) for any \( u > 0 \). Hence, the claim follows for any \( B \in (1, A) \). \( \square \)

3.1. The Stationary-Like Case. Consider first the stationary-like case, which generalizes the case \( \beta > \alpha \) in notation of [7–9]. Denote for any \( t \in [-S, S] \),

\[
f(t) = \frac{1}{2} (1 - \sigma^2(t)),
\]

and

\[
L_+(\lambda) := \int_0^\delta e^{-\lambda f(t)} dt, \quad L_-(\lambda) := \int_{-\delta}^0 e^{-\lambda f(t)} dt, \quad \lambda > 0.
\]

Note that the behavior of these integrals as \( \lambda \to \infty \) does not depend on the choice of \( \delta > 0 \) due to the conditions imposed on \( \sigma(t) \).

Theorem 2. Under the conditions of Lemma 3 together with the equality \( h_1(t) = 0, t \in [-S, S] \), we have

\[
P([0, S], u) = H_\alpha L_{f_+} \left( u^2 q^{-1}(u) \Psi(u)(1 + o(1)) \right)
\]

and

\[
P([-S, S], u) = H_\alpha \left( L_{f_+} \left( u^2 + L_{f_-} \left( u^2 \right) \right) q^{-1}(u) \Psi(u)(1 + o(1)) \right)
\]

as \( u \to \infty \).

Proof. First, we consider a simplified model for \( X \) and then use the Slepian inequality to derive the result for general \( X \), this is a standard approach (see [7]). Let \( X_0(t), t \in [-S, S] \), be a centered stationary Gaussian process satisfying conditions of Theorem 1. Suppose for a while that

\[
X(t) = X_0(t) \sigma(t), \quad t \in [-S, S],
\]

so that \( X(t) \) satisfies the assumptions A1–A4. Recall that we consider the case

\[
\lim_{t \to 0} \frac{1 - \sigma^2(t)}{1 - \rho(t)} = 0,
\]

regardless of the sign of \( t \). Take small \( \varepsilon_+ \) and \( \varepsilon_- \) with \( \varepsilon_+ > \varepsilon_- > 0 \). Then for a sufficiently large \( u_0 \) and all \( u \geq u_0 \) there exists \( \varepsilon = \varepsilon(u) \in (\varepsilon_-, \varepsilon_+) \) with \( (\log^4 u)/\varepsilon \in \mathbb{N} \). We use this \( \varepsilon \) below on. Denote

\[
B^+_k(u) = \{ t \geq 0 : k \varepsilon \leq u^2(1 - \sigma^2(t)) < (k + 1) \varepsilon \},
\]

\[
B^-_k(u) = \{ t < 0 : k \varepsilon \leq u^2(1 - \sigma^2(t)) < (k + 1) \varepsilon \}.
\]

We consider only the sets \( B_k(u) = B^+_k(u) \) in the sequel; the considerations for \( B^-_k(u) \) are similar. Since \( q(u) \) regularly varies with index \(-2/\alpha\), from (12) and (13) we have

\[
\zeta(u) := \frac{|B_1(u)|}{q(u)} \to \infty \quad \text{as} \quad u \to \infty,
\]

where \(| \cdot |\) denotes the length. Observe that the sets of indices \( K = \{ k : B_k(u) \cap B_u \neq \emptyset \} \) and \( K^- = \{ k : B_k(u) \subset B_u \} \) coincide for \( \varepsilon \) chosen above. Denote

\[
\sigma_k = \sqrt{1 - u^{-2} \varepsilon}, \quad u_k = \frac{u}{\sigma_k}.
\]

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For all $k \in \mathcal{K}$, define the sets

$$A_k(u) = \left\{ \max_{t \in B_k(u)} X_0(t) > u_k \right\}, \quad A'_k(u) = \left\{ \max_{t \in B_k(u)} X_0(t) > u_{k+1} \right\}.$$  

Take $a \in (0, 1)$ and denote $v(u) = q(u)\zeta^a(u)$. Note that $v(u)/q(u) \to \infty$ as $u \to \infty$. Now denote two sets of indices

$$D(u) = \{ k : |B_k(u)| > v(u) \}, \quad D'(u) = \{ k : |B_k(u)| \leq v(u) \}$$

that correspond to “long” and “short” time intervals, respectively. Similarly to [10], it is easy to show that the sum of probabilities of $A_k(u)$ taken over $D'(u)$ is negligibly smaller than the sum over $D(u)$. Recall that

$$u \leq u_k \leq u + u^{-1}\log^A u, \quad k \in \mathcal{K}.$$  

It follows from the condition E2 that for all sufficiently large $u$ every $B_k(u)$, $k \geq 0$, consists of finite (not more than $K$) number of intervals, so that there exist subintervals $B'_k(u)$ of $B_k(u)$, $k \in D(u)$, such that $|B'_k(u)|/q(u) \to \infty$ as $u \to \infty$. Hence, for all $k \in D(u)$ the sets $B_k(u)$ and Gaussian processes $X_0(t)$, $t \in B_k(u)$, satisfy the conditions of Theorem 1. By this theorem, we have that

$$P(A_k) = (1 + \gamma(u_k))H_\alpha|B_k(u)|q^{-1}(u_k)\Psi(u_k)$$

and

$$P(A'_k) = (1 + \gamma(u_{k+1}))H_\alpha|B_k(u)|q^{-1}(u_{k+1})\Psi(u_{k+1}),$$

where $\gamma(u) \to 0$ as $u \to \infty$. Next, it follows from Lemma 1 that for $k \in D'(u)$,

$$P(A_k) \leq C|B_k(u)|q^{-1}(u_k)\Psi(u_k),$$

where $C$ is a positive constant independent of $k$. On the other hand, it follows from Bonferroni inequality that

$$P(B_u; u) \leq \sum_{k \in \mathcal{K}} P(A_k(u)) \quad (15)$$

and

$$P(B_u; u) \geq \sum_{k \in \mathcal{K}} P(A'_k(u)) - \sum_{k,l \in \mathcal{K}, k \neq l} P(A_k(u)A_l(u)). \quad (16)$$

The asymptotic equivalence of the sums $\sum_{k \in \mathcal{K}} P(A_k(u))$ and $\sum_{k \in \mathcal{K}} P(A'_k(u))$ as $u \to \infty$ and the relation

$$\sum_{k \in \mathcal{K}} P(A_k(u)) = H_\alpha L_\alpha(u^2)q^{-1}(u)\Psi(u)(1 + o(1))$$

are proved similarly to corresponding steps in [10]. The estimation of the double sum is quite similar to that in [7, 8], as well.

Hence in view of already mentioned standard passage from the particular $X(t) = X_0(t)\sigma(t)$ to the general Gaussian process (by applying the Slepian inequality), the proof follows easily.

3.1.1. Remarks on representations and properties of $L_{f_\pm}(\lambda)$. Introduce the monotone rearrangements $f_+(t)$ and $f_-(t)$ for $f(t)$, $t \in [0, S]$ and $f(t)$, $t \in [-S, 0]$, respectively, which are defined as the generalized inverses

$$f_\pm = F_\pm^{-1},$$

where

$$F_+(x) = \max\{ x : f(t) \leq x, \quad t \in [0, S] \}, \quad x \in [0, 1],$$

and

$$F_-(x) = \max\{ x : f(t) \leq x, \quad t \in [-S, 0] \}, \quad x \in [0, 1],$$

are the distribution functions for the corresponding occupation measures (see, for example, [4]).
An important property of monotone rearrangements is that for any monotone function $\phi$ we have
\[
\int_0^S \phi(f(t)) \, dt = \int_0^S \phi(f_+(t)) \, dt,
\]
and similar equality holds for $f_-$. 

**Remark 1.** If $\sigma(t)$ is locally monotone at zero from both sides, then for some $\varepsilon > 0$, $f_-(t) = f(t)$, $t \in [-\varepsilon, 0]$, and $f_+(t) = f(t)$, $t \in [0, \varepsilon]$, i.e., for $x \in [0, 1]$, $F_\pm(x) = f^\pm(x)$.

Lemma 3 implies that the distribution functions $F_+(x)$ and $F_-(x)$, $x \in \mathbb{R}_+$, may be defined outside $[0, u^{-2} \log^4 u]$ (see (9)) in an arbitrary way, and the asymptotic behavior of $P([-S, S]; u)$ will remain the same.

Observe that
\[
L_+(\lambda) = (1 + o(1)) \int_0^1 e^{-\lambda t} dF_+(x), \quad L_-(\lambda) = (1 + o(1)) \int_0^1 e^{-\lambda u} dF_-(x), \quad \lambda \to \infty.
\]

First, consider an important case of regularly varying $F_\pm(x)$ at zero, i.e., for some slowly varying at zero functions $\ell_\pm(t)$,
\[
F_\pm(x) \sim \ell_\pm(x) x^{a_\pm} \quad \text{as} \quad x \to 0,
\]
where $a_\pm \geq 0$. By [3, Theorems XIII.2 and XIII.3], (18) is equivalent to
\[
L_\pm(\lambda) \sim \Gamma(1 + a_\pm) \lambda^{-a_\pm} \ell_\pm \left( \frac{1}{\lambda} \right) \quad \text{as} \quad \lambda \to \infty.
\]

Using (7), we obtain the following statement.

**Proposition 1.** Suppose that the above assumptions hold and $h_1(t) \equiv 0$. If (18) holds with $\ell_\pm$, $a_\pm$ as above, then $a_\pm \leq 1/\alpha$ and
\[
P([0, S], u) = H_\alpha \Gamma(1 + a_\pm) u^{2/\alpha - 2a_\pm} \ell_\pm(u^2) \left( \ell^\#(u^{-2}) \right)^{1/\alpha} \Psi(u) \left( 1 + o(1) \right),
\]
\[
P([-S, S], u) = H_\alpha \left( \Gamma(1 + a_\pm) u^{2/\alpha - 2a_\pm} \ell_\pm(u^2) + \Gamma(1 + a_-) u^{2/\alpha - 2a_-} \ell_-(u^2) \right)
\times \left( \ell^\#(u^{-2}) \right)^{1/\alpha} \Psi(u) \left( 1 + o(1) \right)
\]

as $u \to \infty$.

**Remark 2.**

1. If $f_\pm(x) = x^{\beta_\pm} \ell_\pm(x)$ are regularly varying at zero with positive indices $\beta_\pm$ ($\ell_\pm(x)$ are slowly varying), then by [1, Theorem 1.5.3] there exist asymptotically monotone equivalents to $f_\pm$, say, $f_{*,\pm}(t)$. Hence, in view of the above argument, one can take $F_\pm(x) = f_{*,\pm}^\pm(x)$. Further, by the same argument as before in (7), we have
\[
F_\pm(x) \sim x^{1/\beta_\pm} \ell^\#_\pm(x), \quad x \to 0.
\]

2. The case where $f$ is regularly varying at 0 has been recently investigated in [2], where the authors established (19). In fact, $1 - \sigma$ is assumed to be symmetric around 0 therein, the nonsymmetric case can be established with no additional efforts. Note further that the case $\ell_\pm(x) = 1$, $x \in [-S, S]$, was considered in [9]; in this case $L_\pm(\lambda) \sim \Gamma(1 + 1/\beta) \lambda^{-1/\beta}$.

Now consider some other representations of $L_\pm(\lambda)$. Integration by parts and choice of sufficiently large $A$ imply that in the stationary-like case for any $\lambda > 0$,
\[
L_\pm(\lambda) = e^{-2 \log^4 \lambda^2} \lambda + \lambda^{2 \lambda^{-1} \log^4 \lambda/2} \int_0^1 e^{-\lambda x} F_\pm(x) \, dx = e^{-2 \log^4 \lambda^2} \lambda + J_\pm(\lambda) \sim J_\pm(\lambda) \quad \text{as} \quad \lambda \to \infty.
\]
Moreover, if $\sigma(t)$ is locally monotone at 0,
\[
J_\pm(\lambda) \sim \int_{\exp(-2\log A/2\lambda)}^1 f^-(\lambda^{-1} \log v) \, dv, \quad \lambda \to \infty. \tag{21}
\]

3.2. The Talagrand Case. It immediately follows from Lemmas 2 and 3 that if $h_1(t) = \infty$ for all $t \neq 0$, then
\[
P([0, S], u) = P([-S, S], u) = \Psi(u)(1 + o(1)) \quad \text{as} \quad u \to \infty. \tag{22}
\]

3.3. The Transition Case. We already know that in this case $1 - \sigma^2(t) = Ct^\alpha \ell_1(t)$ with $\ell(t)/\ell_1(t) \to 1$ as $t \to 0$, where $1 - r(s, t) \sim |t - s|^{\alpha} \ell(t - s)$, $s, t \to 0$. In this case, the exceeding probability asymptotic evaluation is very similar to the corresponding evaluations in [7–9]. The only difference is that the case $\ell(t) = \ell_1(t) = 1$ is considered in these papers. As shown in [2], the slowly varying function does not play any role in the asymptotics. Consequently, in this case we obtain
\[
P((0, S], u) = (1 + o(1)) P^+_{\alpha} \Psi(u), \tag{23}
\]
\[
P([-S, S], u) = (1 + o(1)) P_{\alpha} \Psi(u) \tag{24}
\]
as $u \to \infty$, where
\[
P^+_{\alpha} = \lim_{T \to \infty} P^+_{\alpha}(T) \in (0, \infty), \quad P_{\alpha} = \lim_{T \to \infty} P_{\alpha}(T) \in (0, \infty).
\]

Note that in contrast with the stationary like case, in the Talagrand and transition cases the double side probabilities are not asymptotically equal to the sum of one side ones.

3.4. Main Result. Now we combine all the obtained results concerning the asymptotic behavior in the non-stationary case and formulate our main result. We say that we have

S-S case, the stationary-like case (considered in Proposition 2);
S-T case, when $h_1(t) = 0$, $t \leq 0$, and $h_1(t) = \infty$, $t > 0$;
P-S case, when $h_1(t) \in (0, \infty)$, $t \leq 0$, and $h_1(t) = 0$, $t > 0$;
and so on, similarly for the remaining six cases.

**Theorem 3.** If $X(t)$, $t \in [-S, S]$ is a Gaussian zero mean process satisfying conditions A1–A4, then

- in S-S case, (11) is valid;
- in other four cases concerning S, the asymptotic behavior is equal to the right-hand side of (10);
- in T-T case, (22) is valid;
- in P-P case, (24) is valid;
- in T-P, P-T cases, the asymptotic behavior is equal to the right-hand side of (23).

**Remark 3.**

(i) If a set $E \subset [-S, S]$ is a closure of an open bounded set containing a unique point of the variance maximum, then $P(E, u) \sim P([-S, S], u)$ as $u \to \infty$. In particular, all the asymptotic results above do not depend on $S > 0$.
(ii) In the case where the maximum point of $\sigma^2$ is a boundary point, the corresponding one side relations hold.

4. Examples

Below we present two illustrating examples of S-S case. Exotic cases when $\sigma(t)$ is not locally monotone can be dealt similarly by calculating first the monotone rearrangement of $f(t) = 1 - \sigma^2(t)$. 511
Example 1. The case of a very gentle sharp maximum. Let for some \( \varepsilon, \beta > 0 \) and positive \( t \),

\[
f(t) = 1 - \sigma^2(t) = e^{-t^{\beta}}, \quad t \in (0, \varepsilon].
\]

We have that \( \sigma(t) \) is symmetric and locally monotone on both sides; so we write simply \( F_{\pm} = F \). For \( x > 0 \),

\[
F(x) = f^{-}(x) = \log^{-1/\beta} \left( \frac{1}{x} \right),
\]

i.e., \( a_{\pm} = 0 \), \( \ell(x) = \log^{-1/\beta}(1/x) \). Hence by Proposition 1,

\[
P([-S, S], u) = 2^{1-1/\beta} H_\alpha \log^{-1/\beta}(u) \Psi(u) \left( 1 + o(1) \right) \quad \text{as} \quad u \to \infty.
\]

The asymptotic behavior of \( P([-S, S], u) \) for this and similar “gentle” cases has not been known in the literature so far.

Example 2. Consider the case where \( 1 - \sigma^2(t) \) is close to \( 1 - r(s, t) \), \( s, t \to 0 \). Take \( 1 - \sigma^2(t) = t^\alpha \log(1/t) \) with, of course, \( \log(1/t)/\ell(t) \to 0 \) as \( t \to 0 \), where, combining A3 and A4, \( 1 - r(s, t) \sim |t - s|^{\alpha} \ell(t - s), \ s, t \to 0 \). By [1, Corollary 2.3.4], see also Remark 2,

\[
\left( \log \left( \frac{1}{t} \right) \right)^\# \sim \frac{1}{\log(1/t)},
\]

so that

\[
F_{\pm}(x) \sim \frac{x^{1/\alpha}}{\log(1/x)}, \quad x \to 0,
\]

and

\[
L_{\pm}(u^2) \sim \frac{\Gamma(1 + 1/\alpha)u^{-2/\alpha}}{2 \log u} \quad \text{as} \quad u \to \infty.
\]

Thus,

\[
P([-S, S], u) = H_\alpha \frac{\ell^\#(u^{-2})^{1/\alpha}}{\log u} \Psi(u) \left( 1 + o(1) \right) \quad \text{as} \quad u \to \infty
\]

(see (7)).

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