Expressing the electromagnetic interaction energy

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Abstract. The interaction energy of a \((\rho, j)\) distribution of electric charges and currents with an electromagnetic external field is expressed by the Cartesian components of the multipole tensors of the given distribution. Special attention is paid to the reduction of these tensors to the symmetric traceless ones. Although one uses the Cartesian tensor components in the explicit calculations, the final results are given in a consistent tensorial form.

1. Introduction

As is well known, a charged system given by the densities \(\rho(\mathbf{r}, t)\) and \(j(\mathbf{r}, t)\) of electric charges and currents, localized in a finite domain \(D\) may be described by an infinite system of electric and magnetic multipoles. As the electromagnetic field associated to this distribution in the exterior of \(D\) may be expressed as a multipole expansion, the interaction of this system with an external electromagnetic field may be also expanded in terms of the multipole moments. In [1,2] the reduction of multipole Cartesian tensors is studied in the cases of electrostatic and magnetostatic fields. In [3-5] there are some attempts to give consistent multipole expansions in Cartesian coordinates in the dynamic case. In the present paper we use such an approach by expressing the interaction energy of a charged system with an arbitrary electromagnetic external field. Our principal goal is to give general formulae for the interaction energy in a consistent tensorial form. Particular cases and applications and, particularly, the contributions of the toroidal moments are given in the related literature.

2. Multipole expansion of the interaction energy

Let the interaction energy

\[
W_{\text{int}} = \int_D [\rho(\mathbf{r}, t)\Phi_{\text{ext}}(\mathbf{r}, t) - j(\mathbf{r}, t) \cdot A_{\text{ext}}(\mathbf{r}, t)] \, d^3x.
\]  

(1)

Let us the origin \(O\) of the coordinates in the domain \(D\) and the Taylor series of \(\Phi_{\text{ext}}\) and \(A_{\text{ext}}\) introduced in the equation (1):

\[
W_{\text{int}} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_D \rho(\mathbf{r}, t)x_{i_1} \ldots x_{i_n} d^3x \left[ \partial_{i_1} \ldots \partial_{i_n} \Phi_{\text{ext}}(\mathbf{r}, t) \right]_{r=0}
\]
Expressing the electromagnetic interaction energy

\[ - \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{D}} j_i(r,t)x_{i_1} \cdots x_{i_n} d^3x \left[ \partial_{i_1} \cdots \partial_{i_n} A_{i(\text{ext})} (r,t) \right]_{r=0}. \]  

(2)

Denoting in the following \((\Phi, A)\) instead of \((\Phi_{\text{ext}}, A_{\text{ext}})\), and introducing the \(n\)th order electric multipole tensor

\[ P^n(t) = \int_{\mathcal{D}} r^n \rho(r,t) d^3x \]  

(3)

we may write

\[ W_{\text{int}} = \sum_{n \geq 0} \frac{1}{n!} \left[ P^n(t) \right] \Phi \]

\[ - \sum_{n \geq 0} \frac{1}{n!} \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n} j_i(r,t) d^3x \left[ \partial_{i_1} \cdots \partial_{i_n} A_i (r,t) \right]_{r=0}. \]  

(4)

Here \(a^n\) is the \(n\)-fold tensorial product \((a \otimes \cdots \otimes a)_{i_1 \cdots i_n} = a_{i_1} \cdots a_{i_n}\) and denoting by \(T^{(n)}\) an \(n\)th order tensor, \(A^{(n)}||B^{(m)}\) is an \(|n-m|\)th order tensor with the components

\[ \left( \frac{A^{(n)}||B^{(m)}}{i_1 \cdots i_{|n-m|}} \right) = \begin{cases} A_{i_1 \cdots i_{n-m} j_1 \cdots j_m} B_{j_1 \cdots j_m}, & n > m \\ A_{j_1 \cdots j_n} B_{i_1 \cdots i_n}, & n = m \\ A_{j_1 \cdots j_n} B_{i_{n+1} \cdots i_{n+m}}, & n < m \end{cases}. \]

Let us

\[ w^{(n)} = \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n} j_i(r,t) \partial_{i_1 \cdots i_n} A_i \]  

(5)

with

\[ \partial_{i_1 \cdots i_n} = \partial_{i_1} \cdots \partial_{i_n}, \quad \partial_i f(r) = \partial_i f(r) |_{r=0}. \]

Considering the identity \(\nabla [x_i j(r,t)] = j_i(r,t) + x_i \nabla \cdot j(r,t)\) and the continuity equation we have

\[ j_i(r,t) = \nabla [x_i j(r,t)] + x_i \frac{\partial}{\partial t} \rho(r,t). \]  

(6)

Using this equation in the equation (5) and applying a procedure given in [6, 7] we may write

\[ w^{(n)} = - \int_{\mathcal{D}} x_i j \cdot \nabla (x_{i_1} \cdots x_{i_n}) d^3x \partial_{i_1 \cdots i_n} A_i + \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n} x_i \frac{\partial}{\partial t} d^3x \partial_{i_1 \cdots i_n} A_i \]

\[ = - n \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n-1} x_i j_n d^3x \partial_{i_1 \cdots i_n} A_i + \hat{P}_{i_1 \cdots i_n} \partial_{i_1 \cdots i_n} A_i \]

\[ = - n \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n-1} (x_i j_{i_n} - x_{i_n} j_i) d^3x \partial_{i_1 \cdots i_n} A_i - n w^{(n)} + \left[ \hat{P}^{(n+1)} || \nabla^0 \right] \cdot A \]

where some null surface terms are considered because \(j = 0\) on \(\partial \mathcal{D}\) and the super dot notation for the time derivatives is used. So, we get

\[ w^{(n)} = - \frac{n}{n+1} \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n-1} (x_i j_{i_n} - x_{i_n} j_i) d^3x \partial_{i_1 \cdots i_n} A_i + \frac{1}{n+1} \left[ \hat{P}^{(n+1)} || \nabla^0 \right] \cdot A \]

\[ = - \frac{n}{n+1} \varepsilon_{i_n k} \int_{\mathcal{D}} x_{i_1} \cdots x_{i_n-1} (r \times j)_k d^3x \partial_{i_1 \cdots i_n} A_i + \frac{1}{n+1} \left[ \hat{P}^{(n+1)} || \nabla^0 \right] \cdot A \]  

(7)
Expressing the electromagnetic interaction energy

By introducing the "vectorial product" $T^{(n)} \times a$ as the $n$th order tensor with the components

$$\left( T^{(n)} \times a \right)_{i_1 \ldots i_n} = \varepsilon_{i_1 i_2} T_{11 \ldots i_{n-1} i} a_j,$$

and observing that, particularly,

$$(\beta^n \times a)_{i_1 \ldots i_n} = \beta_{i_1} \ldots \beta_{i_{n-1}} (\beta \times a)_{i_n},$$

we may use in the equation (7) the definition of the $n$th order magnetic multipolar momentum [7]

$$M^{(n)}(t) = \frac{n}{n+1} \int \nu^n \times j(r, t) \, d^3 x$$

such that the equation (7) may be written as

$$w^{(n)} = M_{i_1 \ldots i_{n-1} k} \partial_{i_1 \ldots i_{n-1} \epsilon} \partial_{i_0} A_k + \frac{1}{n+1} \left[ \hat{p}^{(n+1)} \| \nabla^0 \| A \right.$$  

and observing that, particularly,

$$M^{(n)}(t) = \frac{n}{n+1} \int \nu^n \times j(r, t) \, d^3 x$$

Using this last result we may write

$$W_{\text{int}} = \sum_{n \geq 0} \frac{1}{n!} \left[ \hat{p}^{(n)} \| \nabla^0 \| \Phi - \sum_{n \geq 1} \frac{1}{n!} \left[ \hat{p}^{(n)} \| \nabla^0 \| \right. \right.$$  

Using the notations from [1,3-5], the separation of the symmetric part $M_{\text{sym}}^{(n)}$ of the $n$th order magnetig multipole tensor $M^{(n)}$ is given by the formula

$$M_{i_1 \ldots i_n} = M_{\text{sym}|i_1 \ldots i_n|} + \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1 i_2 \ldots i_{n-1} q} N_{i_1 \ldots i_{n-1} q}^{(\lambda)}$$

where

$$M_{\text{sym}|i_1 \ldots i_n|} = \frac{1}{n} \left[ M_{i_1 \ldots i_n} + M_{i_2 \ldots i_{n-1} i_1} + \ldots + M_{i_n \ldots i_1 i_{n-1}} \right]$$

and

$$N_{i_1 \ldots i_{n-1} q} = \varepsilon_{i_{n-1} q} M_{i_1 \ldots i_{n-2} q}$$

with the notation

$$f_{i_1 \ldots i_n}^{(\lambda)} = f_{i_1 \ldots i_{n-1} \lambda+1 \ldots i_n}.$$

The symmetric tensor $M_{\text{sym}}$ is reduced to the symmetric traceless tensor $M^{(n)}$ by the \textit{detracer theorem} [8] which gives, with our notations,

$$M_{\text{sym}|i_1 \ldots i_n|} = M^{(n)} - \sum_{m=1}^{[n/2]} \frac{(-1)^m (2n - 1 - 2m)!!}{(2n - 1)!!} \sum_{D(i)} \delta_{i_1 i_2} \ldots \delta_{i_{2m-1} i_{2m}} M^{(n;m)}_{\text{sym}|i_1 \ldots i_{2m+1} \ldots i_n|}$$

where $[n/2]$ denotes the integer part of $n/2$, $M^{(n;m)}_{\text{sym}|i_{2m+1} \ldots i_n|}$ are the components of the $(n - 2m)$th-order tensor obtained from $M_{\text{sym}}$ by the contractions of $m$ pairs of symbols
Expressing the electromagnetic interaction energy

\[ M_{i_1 \ldots i_n}(t) = \frac{(-1)^n}{(n+1)(2n-1)!} \sum_{\lambda=1}^{n} \frac{1}{D} \int r^{2n+1} [j(r,t) \times \nabla]\delta_{i_1 \ldots i_n} \frac{1}{r} d^3 x. \]  

The equation (14) may be written as

\[ M_{(\text{sym})i_1 \ldots i_n} = M_{i_1 \ldots i_n} + \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} \]  

where \( \Lambda^{(n-2)} \) is a symmetric tensor. Using the equation (14), we may express \( \Lambda^{(n-2)} \) by the formula

\[ \Lambda_{i_3 \ldots i_n} = \frac{1}{2n-1} M_{(\text{sym})qi_3 \ldots i_n} 
+ \frac{[n/2]}{(2n-1)!} \sum_{m=2} \delta_{i_3 i_4} \ldots \delta_{i_{2m-1} i_{2m}} M_{(\text{sym})i_{2m+1} \ldots i_n}. \]  

The reduction of the symmetric tensor \( P^{(n)} \) is achieved by the relation

\[ P_{i_1 \ldots i_n} = P_{i_1 \ldots i_n} + \sum_{D(i)} \delta_{i_1 i_2} \Pi_{i_3 \ldots i_n} \]  

where the symmetric tensor \( \Pi^{(n-2)} \) is defined in terms of the traces of the tensor \( P^{(n)} \) by a relation similar to equation (17). The symmetric traceless tensor \( P^{(n)} \) is given by the formula

\[ P_{i_1 \ldots i_n} = \frac{(-1)^n}{(2n-1)!} \int \rho(r,t) r^{2n+1} \nabla^n \frac{1}{r} d^3 x. \]  

Denoting by \( W_{\text{int}} \) the expression obtained from \( W_{\text{int}} \) given by the equation (11) by the substitutions \( P^{(n)} \rightarrow P^{(n)} \), \( M^{(n)} \rightarrow M^{(n)} \) for all \( n \), we write

\[ W_{\text{int}} = W_{\text{int}} - \sum_{n \geq 1} \frac{1}{n!} [n]_{i_1 \ldots i_{n-1}} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1 k q} N^{(\lambda)}_{1 \ldots i_{n-1} q} (\nabla_0 \times A)_k 
- \sum_{n \geq 1} \frac{1}{n!} \left( \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} \right) (\nabla_0 \times A)_{i_n} 
+ \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{D(i)} \delta_{i_1 i_2} \Pi_{i_3 \ldots i_n} \right) \partial_{i_1 \ldots i_{n-1}} A_{i_n}. \]  

After some straightforward calculations one obtains

\[ W_{\text{int}} = W_{\text{int}} + \sum_{n \geq 2} \frac{n-1}{2n!} \left( n \left[ \Pi^{(n-2)} \right] \nabla_0^{n-2} \right) \Delta_0 \Phi 
- (n-2) \left[ \Pi^{(n-2)} \right] \cdot \Delta_0 A 
- \frac{2}{n} \left[ \nabla_0^{n-2} \right] \cdot \nabla_0 A 
- \frac{2}{n} \left[ \nabla_0^{n-2} \right] \cdot \nabla_0 B \right). \]  

\[ n \]
Expressing the electromagnetic interaction energy

Using the expressions of the type (17) for \( \Lambda^{(n)} \) and \( \Pi^{(n)} \) or using directly in the equation (10) the detracer theorem from [8], i.e., the relationships of the form (14), one obtains a detailed expression of \( W_{\text{int}} \) in terms of the multipole tensors:

\[
W_{\text{int}} = W_{\text{int}} + \sum_{n \geq 2}^{1} \left\{ \frac{1}{n} \left[ (n+1)! \frac{(n-1)!}{2^{n-1}m!} \right] \right\} \sum_{m=1}^{[n/2]} (-1)^m (2n - 1 - 2m)!! \left[ -n \left( \mathcal{P}^{(n:m)} \right) \sum_{n=2}^{n} \Delta_0^m \Phi \right] 
+ (n - 2m) \left( \hat{P}^{(n:m)} \right) \cdot \Delta_0^m \mathbf{A} + 2m \left( \hat{P}^{(n:m)} \right) \cdot \Delta_0^m \left( \nabla_0 \cdot \mathbf{A} \right) 
+ (n - 2m) \left( \mathcal{M}^{(n:m)} \right) \cdot \Delta_0^m \mathbf{B} - \frac{n - 1}{n!} \left( \nabla_0 \cdot \mathbf{N}^{(n-1)} \right) \cdot \left( \nabla_0 \times \mathbf{B} \right) \right] 
\]

Because \( \Delta \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \mathbf{B} \), in the equations (21) and (22) we may consider the relationships

\[
\left[ \hat{\Pi}^{(n-2)} \right] \cdot \Delta_0 \mathbf{A} = \left[ \hat{\Pi}^{(n-2)} \right] \nabla_0 \mathbf{A} - \left[ \hat{\Pi}^{(n-2)} \right] \cdot \left( \nabla_0 \times \mathbf{B} \right) 
\]

and

\[
\left[ \hat{P}^{(n:m)} \right] \cdot \Delta_0^m \mathbf{A} = \left[ \hat{P}^{(n:m)} \right] \cdot \nabla_0^m \mathbf{A} - \left[ \hat{P}^{(n:m)} \right] \cdot \Delta_0^m \mathbf{A} 
\]

Using the equations (23) and (24) and considering also the equation \( \Delta \mathbf{B} = \nabla \times (\nabla \times \mathbf{B}) \) in the equations (21) and (22) we may write

\[
W_{\text{int}} = W_{\text{int}} + \sum_{n \geq 2}^{1} \left\{ \frac{1}{n} \left[ (n+1)! \frac{(n-1)!}{2^{n-1}m!} \right] \right\} \sum_{m=1}^{[n/2]} (-1)^m (2n - 1 - 2m)!! \left[ -n \left( \mathcal{P}^{(n:m)} \right) \sum_{n=2}^{n} \Delta_0^m \Phi \right] 
+ n \left( \hat{P}^{(n:m)} \right) \cdot \Delta_0^m \nabla_0 \mathbf{A} - (n - 2m) \left( \hat{P}^{(n:m)} \right) \cdot \Delta_0^m \left( \nabla_0 \times \mathbf{B} \right) 
- (n - 2m) \left( \mathcal{M}^{(n:m)} \right) \cdot \Delta_0^m \mathbf{B} - \frac{n - 1}{n!} \left( \nabla_0 \cdot \mathbf{N}^{(n-1)} \right) \cdot \left( \nabla_0 \times \mathbf{B} \right) \right] 
\]

with

\[
\nabla_0 \times (\nabla_0 \times \mathbf{B}) = \mu_0 \nabla_0 \times \mathbf{j}_{\text{ext}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}.
\]
4. Concluding remarks

Considering firstly the cases of the static external fields, we have the separate electric and magnetic terms:

\[ W^{(e)}_{\text{int}} = W^{(e)}_{\text{int}} - \frac{1}{\varepsilon_0} \sum_{n \geq 2} \frac{n(n-1)}{n!} \left[ \Pi^{(n-2)} || \nabla_0^{n-2} \right] \rho_{\text{ext}} \]  \hspace{1cm} (27)

and

\[ W^{(\mu)}_{\text{int}} = W^{(\mu)}_{\text{int}} - \mu_0 \sum_{n \geq 2} \frac{1}{n!} \left[ \frac{n-1}{n} \left( N^{(n-1)} || \nabla^{n-2} \right) \cdot j_{\text{ext}} \right. \]
\[ \left. - \frac{(n-1)(n-2)}{2} \left( \Lambda^{(n-2)} || \nabla^{n-3} \right) \cdot \left( \nabla_0 \times j_{\text{ext}} \right) \right]. \]  \hspace{1cm} (28)

If the supports of the external sources do not intersect the supports of the given \((\rho, j)\) distribution, then the interaction energies are invariant in respect to the substitutions of multipole tensors by the symmetric traceless ones. Differences appear when the intersection is not empty.

Denoting \( W'_{\text{int}} = W_{\text{int}} - W'_{\text{int}} \), it is easy to see that the gauge invariance of the theory is satisfied separately by \( W_{\text{int}} \) and \( W'_{\text{int}} \).

Let the external field potential satisfying the Lorenz constraint

\[ \nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0. \]  \hspace{1cm} (29)

Because in this case

\[ \Delta A = -\mu_0 j_{\text{ext}} + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2}, \quad \Delta \Phi = -\frac{1}{\varepsilon_0} \rho_{\text{ext}} + \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}, \]  \hspace{1cm} (30)

we may write

\[ W'_{\text{int}} = \sum_{n \geq 2} \frac{n-1}{2n!} \left\{ -n \left[ \Pi^{n-2} || \nabla_0^{n-2} \right] \frac{1}{\varepsilon_0} \rho_{\text{ext}} \right. \]
\[ \left. + (n-2) \left[ \Pi^{n-2} || \nabla^{n-3} \right] \cdot (\nabla_0 \times B) - (n-2) \left[ \Lambda^{(n-2)} || \nabla_0^{n-3} \right] \cdot \Delta_0 B \right. \]
\[ \left. - \frac{2}{n} \left[ \nabla^{n-2} || N^{(n-1)} \right] \left( \nabla_0 \times B \right) \right\} - \frac{\partial}{\partial t} \left\{ \sum_{n \geq 2} \frac{1}{2(n-2)!} \left[ \Pi^{(n-2)} || \nabla_0^{n-3} \right] \frac{\partial \Phi}{\partial t} \right\}. \]  \hspace{1cm} (31)

We see that in the Lorenz gauge the potentials do not contribute actually to the part \( W'_{\text{int}} \) of the interaction energy. Moreover, in the case of a free external field with the radiative gauge, \( \nabla_0 \cdot A = 0, \Phi = 0 \), the potential \( A \) contribute actually only in the part \( W_{\text{int}} \) of the interaction energy.

From the equation (25) we see also that in the Coulomb gauge for the external field the potentials do not contribute to \( W'_{\text{int}} \).

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Expressing the electromagnetic interaction energy

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