A General Form of the Constraints in the Path Integral Formula

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Abstract

A form of the constraints, specifying a $D$-dimensional manifold embedded in $D + 1$ dimensional Euclidean space, is discussed in the path integral formula given by a time discretization. Although the mid-point prescription is privileged in the sphere $S^D$ case, it is more involved in generic cases. An interpretation on the validity of the formula is put in terms of the operator formalism. Operators from this path integral formula are also discussed.

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1. Introduction

Dynamical system, constrained on a $D$-dimensional manifold, $M^D$, which is now supposed to be given by the equation,

$$f(x) = 0,$$

(1.1)

where $x \equiv (x^1, \ldots, x^{D+1})$ is the $D+1$-dimensional Cartesian coordinate, can be described classically as follows: $f(x)$ is assumed to obey

$$(\nabla_x f(x))^2 \neq 0; \quad \forall x \in M^D,$$

(1.2)

where we have written $\nabla_x$ for the usual $\nabla$ vector. The equation of motion in a flat $D + 1$-dimensional space,

$$\ddot{x}^a = -\frac{\partial V(x)}{\partial x^a} \equiv -\partial_a V(x),$$

(1.3)

with $V(x)$ being a potential, is modified to

$$\Pi_{ab}(\nabla_x f)\dot{x}^b = -\Pi_{ab}(\nabla_x f)\partial_b V(x),$$

(1.4)

in $M^D$, where $\Pi_{ab}(X)$ is a projection operator,

$$\Pi_{ab}(X) \equiv \delta_{ab} - \frac{X^a X^b}{X^2},$$

(1.5)

onto the plane perpendicular to the vector $X$: $X^a \Pi_{ab}(X) = \Pi_{ab}(X)X^b = 0$. Here and hereafter repeated indices imply summation. The significance of (1.4) is easily grasped; since the motion is restricted on $M^D$ so that any deviation to the direction $\nabla_x f$ must be suppressed.

It is well-known that the Lagrangian,

$$L = \frac{\dot{x}^2}{2} - V(x) - \lambda f(x),$$

(1.6)

with $\lambda$ being the multiplier, leads to the equations (1.4) and (1.1). Also the canonical formalism can be developed under the guidance of Dirac[1]: regard (1.1) as the (primary) constraint

$$\phi_1(x) \equiv f(x) \quad (= 0; \quad \forall x \in M^D),$$

(1.7)
and consider the consistency condition: a Hamiltonian,

\[ H = H(p, x) + \lambda f(x) \equiv \frac{p^2}{2} + V(x) + \lambda f(x), \quad (1.8) \]
gives

\[ \dot{\phi}_1 = \{\phi_1, H\} = p \cdot \nabla x f(x), \quad (1.9) \]
thus to find

\[ \phi_2(x) \equiv p \cdot \nabla x f(x) \left( = 0; \quad \forall x \in M^D \right). \quad (1.10) \]
(Here \( \{A, B\} \) designates the Poisson bracket.) They belong to the second class:

\[ \{\phi_1(x), \phi_2(x)\} = (\nabla x f(x))^2 \neq 0, \quad (1.11) \]
on account of (1.2), which enables us to obtain the Dirac bracket,

\[ \{A, B\}_D \equiv \{A, B\} + \frac{1}{(\nabla x f)^2} \left( \{A, \phi_1(x)\}\{\phi_2(x), B\} - (A \leftrightarrow B) \right). \quad (1.12) \]
Therefore we find

\[ \{x^a, x^b\}_D = 0, \]
\[ \{x^a, p_b\}_D = \Pi_{ab}(\nabla x f) = \delta_{ab} - \frac{\partial_a f \partial_b f}{(\nabla x f)^2}, \quad (1.13) \]
\[ \{p_a, p_b\}_D = p_c \left( \partial_a \Pi_{cb} - \partial_b \Pi_{ca} \right) = p_c \frac{\partial_a \partial_c f \partial_b f - \partial_b \partial_c f \partial_a f}{(\nabla x f)^2}, \]
those which correctly reproduce the equation (1.4).

As for quantum mechanics, a recipe of path integral quantization had been given by Faddeev [2] and later by Senjanovic [3] (FS): the FS-formula reads formally

\[ \langle \phi | e^{-iTH} | \psi \rangle = \int D\mu \phi^*(x_f) \exp \left[ i \int_{-T/2}^{T/2} dt \left\{ p \cdot \dot{x} - H(p, x) \right\} \right] \psi(x_i), \quad (1.14) \]
with

\[ D\mu \equiv DpDx \left| \det \{\phi_1, \phi_2\}\right|^{1/2} \delta(\phi_1)\delta(\phi_2), \quad (1.15) \]
and \( x_f \equiv x(T/2), x_i \equiv x(-T/2) \). Here \( Dp \) and \( Dx \) are functional measures which must be specified somehow. The issue is then how to define the above functional measure properly.

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to confirm the well-defined form of (1.14): the most well-known and primitive approach is to discretize the time, obtaining

\[ \mathcal{D} p \mapsto \prod_j dp(j), \quad \mathcal{D} x \mapsto \prod_j dx(j). \tag{1.16} \]

In this approach it was stressed by the present author \cite{4} that the mid-point prescription is privileged in the case of \( D \)-dimensional sphere \( S^D \) given as

\[ x^2 = \rho^2. \tag{1.17} \]

We try to generalize the case in this paper.

In section 2, we review the \( S^D \) case. With this in mind, a generic case \( f(x) = 0 \) is discussed in section 3. The next section 4 deals with operators obtained from the path integral formula, then the final section 5 is devoted to discussion.

## 2. The case of \( D \)-dimensional sphere

The \( D \)-dimensional sphere \( S^D \) is given, in view of (1.17), by

\[ f(x) \equiv \frac{1}{2} \left( x^2 - \rho^2 \right) (\equiv \phi_1). \tag{2.1} \]

The secondary constraint (1.10) is read as

\[ \phi_2 \equiv p \cdot \nabla_x f(x) = p \cdot x. \tag{2.2} \]

The FS-formula (1.14) and (1.13) in a discretized form is found as

\[
\langle \phi | e^{-iTH} | \psi \rangle \equiv \lim_{N \to \infty} \prod_{j=0}^N \int d^{D+1}x(j) \, \delta(\phi_1(x(j))) \\
\times \prod_{j=1}^N \int \frac{d^{D+1}p(j)}{(2\pi)^D} \delta(\phi_2(j)) \, | \det \{ \phi_1(x(j)), \phi_2(j) \} |^{1/2} \\
\times \phi^*(x(N)) \exp \left[ i \sum_{j=1}^N \left\{ p(j) \cdot \Delta x(j) - \Delta t H(p(j), x(j)) \right\} \right] \psi(x(0)),
\]

with

\[ \Delta t \equiv \frac{T}{N}. \tag{2.4} \]
\[ \Delta x(j) \equiv x(j) - x(j - 1), \quad (2.5) \]

and
\[ \bar{x}(j) \equiv \frac{x(j) + x(j - 1)}{2}. \quad (2.6) \]

Here we have employed the mid-point prescription (2.6) to the argument of Hamiltonian, which can be interpreted as a consequence of the Weyl ordering \[5\] \[6\]. The issue is to fix the form of \( \phi_2(j) \): the correct form has been found also as the mid-point type \[4\]:
\[ \phi_2(j) = p(j) \cdot \bar{x}(j). \quad (2.7) \]

The way to (2.7) can be convinced by the following discussion.

Consider \( T = 0 \) case: put \( N = 1 \) in (2.3) to obtain
\[ \langle \phi | \psi \rangle = \int d^{D+1}x \, d^{D+1}x' \, \delta \left( \frac{x^2 - \rho^2}{2} \right) \delta \left( \frac{x'^2 - \rho^2}{2} \right) \]
\[ \times \int \frac{d^{D+1}p}{(2\pi)^D} \delta \left( p \cdot x^{(a)} \right) \left| x \cdot x^{(a)} \right| \phi^*(x)e^{ip \cdot (x-x')\psi(x')}, \quad (2.8) \]

where we have written \( x, x' \), and \( p \) for \( x(1), x(0) \), and \( p(1) \) respectively and set the form of (2.2) as
\[ \phi_2(j = 1) = p \cdot x^{(a)} \equiv p \cdot \left( \frac{1}{2} - \alpha \right) x + \left( \frac{1}{2} + \alpha \right) x' \],
with \( \alpha \) being a parameter \[6\] to be determined. Decompose the \( p \)-vector such that
\[ p = p_\| + p_\perp, \quad (2.10) \]

where
\[ p_\| \equiv \frac{p \cdot x^{(a)}}{x^{(a)}_2}, \]
\[ (p_\perp)_a \equiv \Pi_{ab} \left( x^{(a)} \right) p_b, \quad (2.11) \]

are the parallel and the perpendicular components to the vector \( x^{(a)} \). Then perform the \( p \)-integration to find
\[ \int \frac{d^{D+1}p}{(2\pi)^D} \delta \left( p \cdot x^{(a)} \right) e^{ip \cdot (x-x')} = \frac{1}{x^{(a)}_2} \delta^{D}(x - x'_\perp), \quad (2.12) \]

where
\[ (x - x')_\perp^a \equiv \Pi_{ab} \left( x^{(a)} \right) (x - x')^b. \quad (2.13) \]
Therefore the $D$-dimensional $\delta$-function, in the right hand side of (2.12), implies
\[
0 = (x - x')^a = (x - x')^a - \frac{x^{(a)} \cdot (x - x')}{(x^{(a)})^2} (x^{(a)})^a, \tag{2.14}
\]
with the aid of (1.5). The solution is
\[
x = x', \quad \text{for } \alpha = 0, \tag{2.15}
\]
since the second term of (2.14) vanishes:
\[
x^{(a=0)} \cdot (x - x') = \frac{1}{2} (x^2 - x'^2) = 0, \tag{2.16}
\]
owing to the constraint (2.1). But an additional point emerges if $\alpha \neq 0$
\[
x = -x'. \tag{2.17}
\]
Thus in $\alpha \neq 0$ the $\delta$-function in (2.12) is double-valued. To avoid the situation we must take $\alpha = 0$, that is, (2.9) turns out to be (2.7).

3. A path integral formula in generic cases

In this section we wish to generalize the previous result to $M^D$, given by $f(x) = 0$. Start from (2.3) by putting
\[
\phi_2(j) \equiv p(j) \cdot \nabla f(j), \tag{3.1}
\]
and study the form of $\nabla f(j)$. The $p(j)$-integral in this case becomes
\[
\int \frac{d^{D+1}p(j)}{(2\pi)^D} \delta(p(j) \cdot \nabla f(j)) e^{ip(j) \cdot \Delta x(j)} = \frac{1}{|\nabla f(j)|} \delta^D(\Delta x_a(j)), \tag{3.2}
\]
where
\[
\Delta x_a(j) = \Pi_{ab}(\nabla f(j)) \Delta x^b(j) = \Delta x^a(j) - \frac{\Delta x(j) \cdot \nabla f(j)}{\nabla f(j)^2} (\nabla f(j))^a, \tag{3.3}
\]
which is again the consequence of the decomposition of $p$’s into the parallel and the perpendicular components with respect to a (still unknown) vector $\nabla f(j)$. 

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According to the foregoing discussion, (2.14) ∼ (2.17), a sufficient condition for a single-valued $\delta$-function on $M^D$ is read from (3.3)
\[
\Delta x(j) \cdot \nabla f(j) = 0; \quad \forall x \in M^D. \tag{3.4}
\]
A simple solution therefore is
\[
\Delta x(j) \cdot \nabla f(j) = f(x(j)) - f(x(j - 1)). \tag{3.5}
\]
(This would make sense; since a naive continuum limit, defined by $x(j), p(j) \to x(t), p(t)$, $x(j - 1) \to x(t - dt)$, implies $\nabla f(j) \to \nabla_x f(x)$, yielding the classical result (1.10). ) Write
\[
x(j) = \mathcal{R}(j) + \frac{\Delta x(j)}{2},
\]
\[
x(j - 1) = \mathcal{R}(j) - \frac{\Delta x(j)}{2}, \tag{3.6}
\]
and expand the right hand side of (3.5) with respect to $\Delta x(j)$ to obtain
\[
\nabla f(j) = \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \left( \frac{\Delta x(j) \cdot \nabla \mathcal{R}}{2} \right)^{2n} \right\} \nabla_x f(\mathcal{R}(j)), \tag{3.7}
\]
where $\nabla_x$ denotes differentiation with respect to $\mathcal{R}(j)$. With this in mind a path integral formula on $M^D$ is found as
\[
\langle \phi | e^{-iTH} | \psi \rangle \equiv \lim_{N \to \infty} \prod_{j=0}^{N} \int d^{D+1}x(j) \delta(f(x(j)))
\times \prod_{j=1}^{N} \int \frac{d^{D+1}p(j)}{(2\pi)^D} \delta(p(j) \cdot \nabla f(j)) |\nabla_x f(x(j)) \cdot \nabla f(j)|
\times \phi^*(x(N)) \exp \left[ i \sum_{j=1}^{N} \{ p(j) \cdot \Delta x(j) - \Delta tH(p(j), \mathcal{R}(j)) \} \right] \psi(x(0)). \tag{3.8}
\]
Needless to say, (3.1) with (3.7) matches (2.7), the $S^D$ case, where symmetry is higher so that the mid-point prescription was valid. But as can be recognized from (3.7) there is no privilege of the mid-point prescription in general cases.

Before closing this section let us argue another aspect of the relation (3.2) with (3.7): on $M^D$, $x^a$ can be expressed by some coordinate, say, $\theta^i$ ($i = 1, 2, \cdots, D$):
\[
x^a = x^a(\theta), \quad \theta \in M^D. \tag{3.9}
\]
There should be an orthonormal as well as complete set, $Y_n(\theta)$:

$$
\int d^D\theta \sqrt{g(\theta)} Y_n^*(\theta) Y_{n'}(\theta) = \delta_{n,n'},
$$

(3.10)

$$
\sum_n Y_n(\theta) Y_n^*(\theta') = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta'),
$$

(3.11)

where $n$ represents generic labels and $g(\theta)$ is the determinant of the induced metric,

$$
g_{ij}(\theta) = \sum_{a=1}^{D+1} \frac{\partial x^a}{\partial \theta^i} \frac{\partial x^a}{\partial \theta^j}.
$$

(3.12)

Specifically, $Y_n(\theta)$ may be an eigenfunction of the Laplace-Beltrami operator:

$$
- \left[ g^{-1/2} \frac{\partial}{\partial \theta^i} \left( g^{ij} g^{1/2} \right) \frac{\partial}{\partial \theta^j} \right] Y_n(\theta) = h(n) Y_n(\theta).
$$

(3.13)

Suppose that Hamiltonian is given by

$$
\hat{H} = -g^{-1/2} \frac{\partial}{\partial \theta^i} \left( g^{ij} g^{1/2} \right) \frac{\partial}{\partial \theta^j} + V(\hat{\theta}),
$$

(3.14)

where the caret denotes operators, then the Feynman kernel,

$$
K(\theta, \theta'; T) \equiv \langle \theta | e^{-iT\hat{H}} | \theta' \rangle = \lim_{N \to \infty} \langle \theta | (I - i\Delta t \hat{H})^N | \theta' \rangle,
$$

(3.15)

can be expressed as “path integral”: by inserting the identities, (3.10) and (3.11), which are now read as

$$
\int d^D\theta \sqrt{g(\theta)} |\theta\rangle \langle \theta| = I,
$$

(3.16)

$$
\sum_n |n\rangle \langle n| = I,
$$

(3.17)

with $I$ being the identity operator,

$$
\langle \theta | \theta' \rangle = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta'),
$$

(3.18)

$$
\langle n | n' \rangle = \delta_{nn'},
$$

and $\langle \theta | n \rangle \equiv Y_n(\theta)$, (3.15) becomes

$$
K(\theta, \theta'; T) = \lim_{N \to \infty} \left( \prod_{j=1}^{N-1} \int d^D\theta(j) \sqrt{g(\theta(j))} \right) \left( \prod_{j=1}^{N} \sum_{n(j)} \right)
$$

$$
\times Y_{n(j)}(\theta(j)) Y_{n(j)}^*(\theta(j) - 1) \exp[-i\Delta t \{ h(n(j)) + V(\theta(j)) \}] \bigg|_{\theta(0)=\theta}^{\theta(n)=\theta}.
$$

(3.19)
However the expression of (3.19) is unsatisfactory as a “path integral” formula if $M^D$ is nontrivial, $g_{ij} \neq \delta_{ij}$; since some of the labels are discrete so that we are left with summation not integration. Moreover $Y_n(\theta)$ is generally far from a plane wave form: in a trivial case, $g_{ij} = \delta_{ij}$, (which is given by an $f(x)$ linear in $x$,) $Y_n(\theta)$ is read as,

$$Y_n(\theta) \equiv \frac{1}{(2\pi)^{D/2}} e^{i\mathbf{p} \cdot \mathbf{x}}.$$  

(3.20)

($n$ and $\theta$ correspond to $\mathbf{P}$ and $\mathbf{X}$ respectively.) Therefore we obtain a usual path integral formula:

$$K(\mathbf{X}, \mathbf{X}'; T) = \lim_{N \to \infty} \left( \prod_{j=1}^{N-1} \int d^D X(j) \right) \left( \prod_{j=1}^{N} \int d^D P(j) \right) \frac{1}{(2\pi)^D} \times \exp \left[ i \sum_{j=1}^{N} \left\{ \mathbf{P}(j) \cdot \Delta \mathbf{X}(j) - \Delta t \left( h(\mathbf{P}(j)) + V(\mathbf{X}(j)) \right) \right\} \right] \bigg|_{\mathbf{X}(0) = \mathbf{X}'}^{\mathbf{X}(N) = \mathbf{X}}. $$  

(3.21)

(It might be natural, however, to think that the situation is same even in the trivial case if we work with the polar coordinate; since in which there arises the spherical harmonics, being far from the plane wave except the $S^1$ case. But in these cases we can find a desired path integral formula consisting purely of an exponential form as well as integration by means of the canonical transformation [7] from the Cartesian expression (3.21).)

Now it is almost clear that the relation (3.2) with (3.7) cures the above situation for nontrivial cases: according to our discussion, the completeness condition (3.11) can be put into a plane wave type provided solely with integration:

$$\sum_n Y_n(\theta)Y_n^*(\theta') = \frac{1}{\sqrt{g(\theta)}} \delta^D(\theta - \theta') = |\nabla f| \int \frac{d^{D+1} P}{(2\pi)^D} \delta(\mathbf{p} \cdot \nabla f) e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \bigg|_M,$$  

(3.22)

where from (3.7)

$$\nabla f \equiv \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \left( \frac{\Delta \mathbf{x} \cdot \nabla \mathbf{x}}{2} \right)^{2n} \right\} \nabla \mathbf{x} f(\mathbf{x}), $$  

(3.23)

and the subscript $M$ designates that $\mathbf{x}$ and $\mathbf{x}'$ are on $M^D$. The relation (3.22) thus can be implied as the plane wave representation of the completeness condition on $M^D$. In other words the FS formula is a rigorous consequence from the operator formalism owing to this completeness condition (3.22).
4. Operators from the path integral formula

A similar consideration as in (2.8) leads us to the observation that an expectation value of some operator $O(\hat{p}, \hat{x})$ can be given, with the aid of the formula (3.8) with (3.7), by

$$\langle O(\hat{p}, \hat{x}) \rangle \equiv \langle \varphi | O(\hat{p}, \hat{x}) | \psi \rangle \equiv \int d^{D+1}x \int d^{D+1}x' \delta(f(x)) \delta(f(x'))$$

$$\times |\nabla f(x) \cdot \nabla f| \varphi^*(x) \psi(x') \int \frac{d^{D+1}p}{(2\pi)^D} \delta(p \cdot \nabla f) O(p, \overline{x}) e^{ip \cdot \Delta x},$$

(4.1)

where $\nabla f$ is given by (3.23),

$$\Delta x \equiv x - x',$$

(4.2)

and

$$\overline{x} \equiv \frac{x + x'}{2}.$$  

(4.3)

By noting

$$\delta(X)\delta(Y) = \delta\left(\frac{X + Y}{2}\right) \delta(X - Y),$$

(4.4)

then using (3.8), (4.1) becomes

$$\langle O(\hat{p}, \hat{x}) \rangle = \int d^{D+1}x \int d^{D+1}x' \delta(\overline{f}) \delta(\Delta x \cdot \nabla f) |\nabla f(x) \cdot \nabla f| \varphi^*(x) \psi(x')$$

$$\times \int \frac{d^{D+1}p}{(2\pi)^D} \delta(p \cdot \nabla f) O(p, \overline{x}) e^{ip \cdot \Delta x} \int d^{D+1}x \int d^{D+1}x'$$

$$\times \delta(\overline{f}) \left[ \frac{|\nabla f(x) \cdot \nabla f|}{(\nabla f)^2} \varphi^*(x) \psi(x') \mathcal{O}\left(-i \frac{\partial}{\partial \Delta x}, \overline{x}\right) \right] \delta^{D+1}(\Delta x),$$

(4.5)

where we have introduced the notation,

$$\overline{f} \equiv \frac{f(x) + f(x')}{2},$$

(4.6)

and integrated with respect to $p$'s in a similar manner as before, to find $\delta^{D}(\Delta x_{\perp})$ which is combined with $\delta(\Delta x \cdot \nabla f) \sim \delta(\Delta x_{\parallel})$ yielding $\delta^{D+1}(\Delta x)$ finally. Now changing variables $(x, x')$ to $(\overline{x}, \Delta x)$ and performing integration by parts, we find

$$\langle O(\hat{p}, \hat{x}) \rangle = \int d^{D+1}\overline{x} \mathcal{O}\left(-i \frac{\partial}{\partial \Delta x_{\perp}}, \overline{x}\right)$$

$$\times \left[ \delta(\overline{f}) \left[ \frac{|\nabla f(x) \cdot \nabla f|}{(\nabla f)^2} \varphi^*(x + \frac{\Delta x}{2}) \psi\left(x - \frac{\Delta x}{2}\right) \right] \right]_{\Delta x=0}.$$  

(4.7)
where
\[
\nabla_x f(x) = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\Delta x \cdot \nabla_{\mathbf{x}}}{2} \right)^n \right\} \nabla_{\mathbf{x}} f(\mathbf{x}),
\]
and
\[
\mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( \frac{\Delta x \cdot \nabla_{\mathbf{x}}}{2} \right)^{2n} f(\mathbf{x}),
\]
in view of (4.6). (The subscript \( \Delta x = 0 \) designates that \( \Delta x \to 0 \) must be put after all calculations have been done.) Also note that
\[
\frac{\nabla_x f(x) \cdot \nabla f}{(\nabla f)^2} = 1 + \frac{1}{(\nabla_x f(\mathbf{x}))^2} \nabla_{\mathbf{x}} f(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \left( \frac{\Delta x \cdot \nabla_{\mathbf{x}}}{2} \right) f(\mathbf{x})
\]
\[
+ \frac{1}{3 (\nabla_x f(\mathbf{x}))^2} \nabla_x f(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \left( \frac{\Delta x \cdot \nabla_{\mathbf{x}}}{2} \right)^2 f(\mathbf{x}) + O(\Delta x^3). \tag{4.10}
\]

Let us calculate some examples:

- (i) \( \mathcal{O}(\hat{p}, \hat{x}) \equiv F(\hat{x}) \):
  \[
  \langle F(\hat{x}) \rangle = \int d^{D+1} x \, \delta(f(x)) \varphi^*(x) \, F(x) \, \psi(x),
  \tag{4.11}
\]
  where we have written \( x \) for \( \mathbf{x} \). This shows
  \[
  F(\hat{x}) = F(x). \tag{4.12}
\]

- (ii) \( \mathcal{O}(\hat{p}, \hat{x}) \equiv \hat{p}_a \):
  \[
  \langle \hat{p}_a \rangle = \int d^{D+1} \mathbf{x} \, \Pi_{ab} (\nabla_{\mathbf{x}} f) \left( -i \frac{\partial}{\partial \Delta x^b} \right) \delta(\mathcal{F}) \frac{\nabla_x f(\mathbf{x}) \cdot \nabla f}{(\nabla f)^2}
  \]
  \[
  \times \varphi^* \left[ \mathbf{x} + \frac{\Delta x}{2} \right] \psi \left[ \mathbf{x} - \frac{\Delta x}{2} \right] \bigg|_{\Delta x = 0} = \int d^{D+1} x \, \delta(f(x))
  \]
  \[
  \times \Pi_{ab} (\nabla_x f) \frac{i}{2} \left\{ \partial_b \varphi^*(x) \psi(x) - \varphi^*(x) \partial_b \psi(x) + \partial_b \partial_c f \partial_c f (\nabla_x f)^2 \partial_b \varphi^*(x) \partial_b \psi(x) \right\}, \tag{4.13}
\]
  where again we have put \( \mathbf{x} \to x \). The third term in the final expression comes from the differentiation to (4.10). (There remains no effect from differentiating the \( \delta \)-function, in view of (4.9).) Finally integrating by parts with respect to the first term, while paying attention to the property of the projection operator, \( \Pi_{ab} \partial_b \delta(f(x)) = 0 \), we obtain
  \[
  \langle \hat{p}_a \rangle = \int d^{D+1} x \, \delta(f(x)) \, \varphi^*(x) \left[ -i \Pi_{ab} (\nabla_x f) \partial_b 
  \right.
  \]
  \[
  - \frac{i}{2} \partial_b \Pi_{ab} (\nabla_x f) - \frac{i}{2} \Pi_{ab} (\nabla_x f) \partial_c \Pi_{bc} (\nabla_x f) \bigg] \psi(x). \tag{4.14}
  \]
Therefore
\[
\hat{p}_a = -i\Pi_{ab}(\nabla_x f) \partial_b - \frac{i}{2} \partial_b \Pi_{ab}(\nabla_x f) - \frac{i}{2} \Pi_{ab}(\nabla_x f) \partial_c \Pi_{bc}(\nabla_x f)
\]
\[
= -i\Pi_{ab}(\nabla_x f) \partial_b + \frac{i}{2} \frac{2\partial_a \partial_b f \partial_b f + \partial_a f \nabla_x^2 f}{(\nabla_x f)^2} - \frac{3i}{2} \frac{\partial_b \partial_c f \partial_a f \partial_b f \partial_c f}{(\nabla_x f)^4},
\]
(4.15)
is the momentum operator. It can be shown by an explicit calculation that (4.15) satisfies the quantum version of (1.13):
\[
[\hat{x}^a, \hat{x}^b] = 0,
\]
\[
[\hat{p}_a, \hat{p}_b] = i\Pi_{ab}(\nabla_x \hat{f}) = i \left( \delta_{ab} - \frac{\partial_a \hat{f} \partial_b \hat{f}}{(\nabla_x \hat{f})^2} \right),
\]
(4.16)
where \{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}.

• (iii) \( \mathcal{O}(\hat{p}, \hat{x}) \equiv \hat{p}^2 \): with a similar manner as above, we find
\[
\langle \hat{p}^2 \rangle = \int d^{D+1}\mathbf{x} \Pi_{ab}(\nabla_x f) \left( -\frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b} \right) \left[ \delta(f) \left| \frac{\nabla_x f(x) \cdot \nabla f}{(\nabla f)^2} \right| \right. \\
\times \varphi^*(\mathbf{x} + \frac{\Delta \mathbf{x}}{2}) \psi(\mathbf{x} - \frac{\Delta \mathbf{x}}{2}) \right] \bigg|_{\Delta \mathbf{x} = 0}
\]
\[
= \int d^{D+1}\mathbf{x} \delta(f(\mathbf{x})) \varphi^*(\mathbf{x}) \left[ -\Pi_{ab}(\nabla_x f) \frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b} \\
+ \left( \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_c f \partial_b f}{(\nabla_x f)^2} - \partial_a \Pi_{ab}(\nabla_x f) \right) \frac{\partial}{\partial x^b} + \frac{1}{2} \frac{\partial_b \left( \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_c f \partial_b f}{(\nabla_x f)^2} \right)}{(\nabla_x f)^2} \\
- \frac{1}{4} \frac{\partial_a \partial_b \Pi_{ab}(\nabla_x f) - \frac{1}{6} \Pi_{ab}(\nabla_x f) \frac{\partial_a \partial_b \partial_c f \partial_c f}{(\nabla_x f)^2}}{(\nabla_x f)^2} \right] \psi(\mathbf{x}).
\]

(4.17)
From this we obtain

\[\hat{p}^2 = -\Pi_{ab}(\nabla_x f) \frac{\partial^2}{\partial \Delta x^a \partial \Delta x^b}\]

\[+ \left\{ \frac{2 \partial_a \partial_b f \partial_a f + \nabla_x^2 f \partial_b f}{(\nabla_x f)^2} - \frac{3 \partial_a \partial_c f \partial_a f \partial_b f \partial_c f}{(\nabla_x f)^4} \right\} \frac{\partial}{\partial x^b}\]

\[+ \frac{1}{(\nabla_x f)^2} \left\{ \frac{5}{6} \partial_a \nabla_x^2 f \partial_a f + \frac{1}{4} \left( \nabla_x^2 f \right)^2 + \frac{3}{4} \partial_a \partial_b f \partial_a f \partial_b f \right\}\]

\[+ \frac{1}{(\nabla_x f)^4} \left\{ \frac{3}{2} \nabla_x^2 f \partial_a \partial_b f \partial_a f \partial_b f + \frac{7}{2} \partial_a \partial_b f \partial_b c f \partial_a f \partial_c f \right\}\]

\[+ \frac{5}{6} \partial_a \partial_b \partial_c f \partial_a f \partial_b f \partial_c f\].

(4.18)

It should be noted that \(\hat{p}^2 \neq \hat{p}_a \hat{p}_a\) unless \(f(x)\) is linear in \(x\).

5. Discussion

In this paper we have established a form of constraints in the path integral formula given by the time discretization. The main interest is how to incorporate the classical constraint \(p \cdot \nabla_x f = 0\) into the quantum one: the correct form can be found by requiring that the delta function be single-valued.

The conclusion is unchanged even if we take a nonstandard form of Hamiltonian instead of (1.8) such as

\[H(p, x) \rightarrow h(p^2) + V(x),\]

provided \(h'(p^2) \neq 0\).

Therefore we have successfully described a ‘local’ form of the path integral formula; where the word ‘local’ must be attached since if manifold is nontrivial and composed of \(G/H\) there emerge induced gauge fields according to recent studies [8][9]. Our formula apparently lacks these informations. There has been a trial [10] but we are still on the way to the final goal.
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