I review recent developments in lattice calculations of $B$ decay matrix elements and other related quantities.

1. Introduction

$B$ decay phenomenology is a rapidly growing area of particle physics, as we expect that the precise measurements of various $B$ meson decays provide rich information of the Standard Model and the physics beyond it. Following the successful CLEO, LEP and CDF experiments, the next generation $B$ factories (BaBar and Belle) have just started operation, and other experimental projects will follow.

Lattice QCD may contribute to this program by calculating the $B$ meson decay matrix elements starting from the first principles. Its application is extending from the decay constant $f_B$ and bag parameter $B_B$ to various semileptonic decay form factors. Results are being checked by several groups using different lattice formulations of heavy quark, and systematic errors are carefully estimated. In addition, studies of new applications have been started, such as the zero recoil form factor of $B \rightarrow D^{(*)}\ell\nu$, the width difference of $B_s$ meson, and the lifetime ratio of $b$ hadrons. In this talk I review recent developments in such efforts.

This talk is organized as follows. Before discussing the matrix elements calculation, I will discuss new applications of lattice QCD, which include the calculation of $B_s$ width difference (Section 1.3) and $b$-hadron lifetime ratios (Section 1.4). Those calculations involve similar matrix elements as in $B_B$. A summary of the status as of 1998 was given by T. Draper at the last lattice conference [1].

2. The $b$ quark mass

Although the $b$ quark mass is not a matrix element related to any exclusive decay of $B$ meson, it is necessary in the calculation of many inclusive decay rates. It has become clear, however, that there is an ambiguity in the definition of pole mass of heavy quark itself (renormalon ambiguity), and its determination cannot be better than of order $\Lambda_{QCD}$ [2]. In order to avoid the ambiguity one has to use a short distance mass definition, such as $\overline{MS}$ quark mass $\overline{m}_b(\overline{m}_b)$.

In the lattice QCD calculation, important progress was made by Martinelli and Sachrajda [3], when they obtained the two-loop coefficient in the perturbative matching between continuum QCD and lattice HQET (static approximation)

$$\overline{m}_b(\overline{m}_b) = \Delta \times \left[ 1 + \frac{1}{\Delta \alpha} \left\{ 2.1173 \alpha_s(\overline{m}_b) ight. ight. 
\left. \left. + (3.707 \ln(\overline{m}_b a) - 1.306) \alpha_s^2(\overline{m}_b) \right\} 
\times \left[ 1 - 4 \frac{\alpha_s(\overline{m}_b)}{3 \pi} - 11.66 \left( \frac{\alpha_s(\overline{m}_b)}{\pi} \right)^2 \right] \right], \quad (1)$$

where $\Delta = M_B - \mathcal{E}$ is obtained with the “binding energy” $\mathcal{E}$ measured in lattice simulations. In this expression the perturbative expansion in the first parenthesis matches the energy shift in the...
lattice HQET to the heavy quark pole mass. A power divergence exists in the loop integral that cancels with power divergent behavior of nonperturbatively calculated $\Delta$. The last line connects the pole mass to the $\overline{\text{MS}}$ mass. The renormalon ambiguity exists in both of these two expansions, while it cancels between them. With the perturbative expansion to two-loop, these cancellations are expected to become more precise, making the result more stable.

Numerical results with the two-loop matching are very stable among three $\beta$ values (6.0, 6.2 and 6.4) as shown in Figure 1, where corresponding one-loop results are also plotted with various definitions of the coupling constant. This result suggests that the cancellation of the power divergence and the renormalon ambiguity is working fairly well at two-loop level.

A dynamical ($N_F = 2$) result along this line has been reported at this conference by the APE collaboration (V. Giménez [4]), which is shown by a square in Figure 1. Their result does not show significant shift from the quenched results.

The NRQCD collaboration presented a $b$ quark mass calculation using NRQCD action at Lattice 98 [5]. They use a one-loop matching between the kinetic mass in NRQCD and the $\overline{\text{MS}}$ mass, where no power divergence appears in the perturbative calculation and the one-loop coefficient is not large. Their results with (plus) and without (stars) two-flavour dynamical fermions are consistent with the HQET calculations.

The best estimate from the lattice calculations $\overline{m_b}(m_b) = 4.26\pm0.11$ GeV, which I take from the two dynamical results, is in good agreement with recent continuum calculations $4.25(8)$ [6], $4.20(6)$ [7], $4.20(10)$ [8] on the $\Upsilon$ resonances, and with DELPHI’s measurement 3.91(67) GeV at $Z$ pole [9].

3. Semileptonic decays

The lattice calculation of $B$ meson semileptonic decay form factors may be used to determine $|V_{cb}|$ and $|V_{ub}|$ with corresponding experimental results of exclusive decays $B \to D^{(*)}l\nu$ and $B \to \pi(\rho)l\nu$.

3.1. $B \to D^{(*)}l\nu$

3.1.1. Zero recoil form factor

One of the most promising methods to determine $|V_{cb}|$ is to measure $|V_{cb}|^2 F_{B\to D^{(*)}}(w)$ from the differential decay rate of $B \to D^{(*)}l\nu$, and extrapolate it to the zero recoil limit of daughter meson $w \to 1$, at which the form factor $F_{B\to D^{(*)}}(1)$ is normalized to unity in the heavy quark mass limit. Theoretical calculations of the power correction to the heavy quark limit have been made using QCD sum rule, but their uncertainty is still quite large ($\sim 9\%$ for $F_{B\to D^{(*)}}(1)$) [10]. Lattice calculation could be an important alternative, if it provides determination better than $O(5\%)$.

Recently, the Fermilab group has proposed a method to calculate the deviation from the heavy quark limit by actually measuring the heavy quark mass dependence of the form factors on the lattice [11],[12]. Their key observation is that most of statistical and systematic errors cancel in
a ratio of matrix elements \[1\]

\[
|h_{+}^{B\to D}(1)|^{2} = \frac{\langle D|V_{cb}^{*}|B\rangle\langle B|V_{bc}^{*}|D\rangle}{\langle D|V_{0}^{bc}|D\rangle\langle B|V_{0}^{cb}|B\rangle},
\]

where \(h_{+}^{B\to D}(1)\) denotes a zero recoil form factor of \(B \to Dl\nu\) decay through a vector current \(V_{cb}^{*}\). The statistical error is less than one per cent for typical values of \(m_{b}\) and \(m_{c}\), and thus it is possible to study its mass dependence. The renormalization factor to match the heavy-heavy current on the lattice to its continuum counterpart is perturbatively calculated, depending on the heavy quark mass, and found to be small for the above ratio \[13\].

They fit their data for \(h_{+}(1)\) as well as for \(h_{1}(1)\) and \(h_{A_{1}}(1)\), which are form factors of \(B^{*} \to D^{*}\) and \(B \to D^{*}\) modes obtained through similar ratios, with the form of \(1/m_{Q}^{2}\) expansion predicted by the heavy quark symmetry \[14\]:

\[
\begin{align*}
    h_{+}(1) &= 1 - l_{p}\left(\frac{1}{2m_{c}} - \frac{1}{2m_{b}}\right)^{2} + O(1/m_{Q}^{3}), \\
    h_{1}(1) &= 1 - l_{V}\left(\frac{1}{2m_{c}} - \frac{1}{2m_{b}}\right)^{2} + O(1/m_{Q}^{3}), \\
    h_{A_{1}}(1) &= 1 - \left(\frac{1}{2m_{c}} - \frac{1}{2m_{b}}\right)\left(\frac{l_{V}}{2m_{c}} - \frac{l_{p}}{2m_{b}}\right) + \frac{\Delta}{4m_{c}m_{b}} + O(1/m_{Q}^{3}).
\end{align*}
\]

The parameters \(l_{p}\), \(l_{V}\) and \(\Delta\) of \(O(1/m_{Q}^{3})\) terms are determined with the fit. Their preliminary result for \(B \to D^{*}\ell\nu\) form factor obtained at \(\beta=5.7\) is \(\mathcal{F}_{B\to D^{*}}(1) = 0.935(22)(^{+13}_{-8})(8)(20)\), where errors represent statistical, mass determination, perturbative and unknown \(O(1/m_{Q}^{3})\) in the given order \[16\]. The systematic error associated with the Fermilab formalism of heavy quark \[16\] is a subtle issue, especially because the \(1/m_{Q}^{2}\) terms in the effective Hamiltonian are not correctly tuned with the use of the clover action, while the correction to be measured is of \(O(1/m_{Q}^{3})\). Nevertheless, using the \(1/m_{Q}^{2}\) expansion, it is generally shown that only \(O(1/m_{Q})\) terms in the action and currents contribute to the ratios calculated above, and thus the systematic error is well under control \[17\]. The result is consistent with the recent QCD sum rule calculation 0.89±0.08 \[11\], and the estimated error is already slightly smaller, demonstrating a possibility to improve the determination of \(|V_{cb}|\) using the lattice calculation in near future.

### 3.1.2. Shape of the form factor

In recent high statistics experiments, the extrapolation to the zero recoil limit without theoretical constraints seems good enough \[18\]. It is, however, informative to compare the shape of form factors with theoretical predictions in order to check the reliability of the extrapolation, and also to check the theoretical methods. Two new calculations of heavy-to-heavy form factors (Isgur-Wise function) have been presented by UKQCD \[14\] and by Hein et al. \[20\] at this conference.

UKQCD used the non-perturbatively \(O(a)\)-improved action at \(\beta=6.0\) and 6.2, and calculated \(D \to D\) form factor \(h_{+}\) \[19\]. They found no significant dependence on the heavy quark mass, and their preliminary result for the slope parameter is \(\rho_{u,d} = 1.10(^{+12}_{-13})(^{+5}_{-6})\) and \(1.12(^{+29}_{-15})(^{+6}_{-4})\) at \(\beta=6.0\) and 6.2, respectively. The precision is the best among previous results \[21\], and the scaling with two lattice spacings is remarkable.

Hein et al. presented a very preliminary study with a NRQCD action at \(\beta=5.7\) \[20\]. Form factor \(h_{+}\) is obtained for \(B \to D\) and also for \(B \to D^{*}\) mode, where \(D^{*}\) denotes a radially excited state of \(D\) meson. Perturbative matching factors recently calculated by Boyle \[22\] are incorporated for heavy-heavy (axial-)vector currents.

### 3.2. \(B \to \pi\ell\nu\)

Exclusive semileptonic decays \(B \to \pi(\rho)\ell\nu\) could be used for the determination of \(|V_{ub}|\), provided that corresponding form factors are theoretically calculated. Unfortunately, in the lattice simulations, it is difficult to put large recoil momentum on the daughter meson, so that momentum transfer squared \(q^{2}\) is restricted in the large \(q^{2}\) region (small recoil momentum). The lattice calculations can be, however, still useful, if statistics in the experiment is precise enough to measure the partial decay rate for the large \(q^{2}\) region, and such work has already been done by CLEO for \(B \to \rho\ell\nu\) \[23\].
There are two form factors $f^+(q^2)$ and $f^0(q^2)$ involved in $B \to \pi \nu$. $f^0(q^2)$ has a negligible contribution to the physical decay rate, as it is proportional to the lepton mass. It is, however, interesting to study $f^0(q^2)$, since the soft pion theorem relates $f^0(q_{\text{max}}^2)$ to $f_B$, which may be used for a test of the lattice calculations. On the other hand, the calculation of $f^+(q^2)$ is of practical importance, as it is directly related to the physical decay rate and may be used for a precise determination of $|V_{ub}|$ of $O(10\%)$.

3.2.1. $f^0(q^2)$

Based on the LSZ reduction formula and current algebra, the soft pion theorem predicts

$$f^0(q_{\text{max}}^2) = \frac{f_B}{f_\pi}$$

in the limit $m_\pi \to 0$ and $p_\pi \to 0$, thus $q_{\text{max}}^2 = m_B^2$.

It was pointed out by Onogi at Lattice 97 that the lattice results for $f^0(q_{\text{max}}^2)$ and $f_B$ do not seem to satisfy this relation [24]. The inconsistency has become even clearer with new data as shown in Figure 3, in which I plot currently available results for $f^0(q_{\text{max}}^2)/\sqrt{M}$ (with renormalization group correction), comparing them with representative data for $(f_B/f_\pi)/\sqrt{M}$. The heavy quark scaling law predicts both quantities should be constant, up to $1/M$ corrections.

Possible cures for this problem have been proposed at this conference: nonperturbative renormalization and a method of chiral and $q^2$ extrapolation, which I discuss in the following.

The JLQCD collaboration has calculated a ratio of renormalization factor $(Z_A/Z_V)^{\text{HL}}$ nonperturbatively, using the chiral Ward identity [31]

$$Z_A Z_V^{\text{HL}} \int d^4y (\partial_\mu A_\mu - 2m_q P)(y) V_0^{\text{HL}}(x) O = -Z_A^{\text{HL}} \langle A_0^{\text{HL}}(x) O \rangle,$$

where $A_\mu$ and $P$ denote the light-light axial-current and pseudoscalar density defined on the lattice, and $Z_A$ represents the renormalization factor for $A_0$ available nonperturbatively [14]. $V_0^{\text{HL}}$ and $A_0^{\text{HL}}$ are temporal component of heavy-light vector and axial-vector currents, which appear in the definition of $f^0(q_{\text{max}}^2)$ and of $f_B$ respectively. Their simulation method follows that of Maiani and Martinelli [35], while they use $O(a)$-improved action and current for light quark and static heavy quark action. A preliminary result for the static-light currents at $\beta=6.0$, $\langle Z_A/Z_V \rangle^{\text{HL}}=0.72(1)$, is significantly smaller than the one-loop value 0.86(4). This result indicates that the one-loop calculation of the renormalization factor of the static-light current may contain a large systematic error ($\sim 20\%$), and suggests larger $f^0(q_{\text{max}}^2)$ and/or smaller $f_B$ in the static limit, which gives right direction to satisfy the soft pion relation, but magnitude is not enough.

An ongoing study to calculate $Z_A^{\text{HL}}$ nonperturbatively using the Schrödinger functional method has also been presented at this conference by Kurth and Sommer [34].
UKQCD has pointed out that a term proportional to $m_\pi$ is necessary in the chiral extrapolation as well as the usual $m_\pi^2$ term \cite{47}. That dependence appears solely from the $m_\pi$ dependence of $q_{\text{max}}^2$: $q_{\text{max}}^2 = (m_B - m_\pi)^2 \simeq m_B^2 - 2m_B m_\pi$. As a result the chiral extrapolation could be subtle, since $m_\pi$ and $m_\pi^2$ behave similarly in the region where simulations are made. In order to remove $m_\pi$ term they interpolate $f^0(q^2)$ to several fixed $q^2$ values for each of the active and spectator light quark masses. Then, the chiral extrapolation can be done at each fixed $q^2$ with the $m_\pi^2$ term only. The range of available $q^2$ for all light quark mass is substantially lower than the physical $q_{\text{max}}^2$, so they employ a pole dominance model \cite{39} to obtain the physical $f^0(q_{\text{max}}^2)$ form factor from data points. This extrapolation to the physical $q_{\text{max}}^2$ makes $f^0(q_{\text{max}}^2)$ significantly high $1.3(\pm 3)(^{+3}_{-2})$, which is consistent with $f_B/f_\pi = 1.35(\pm 0.04)(^{+0.12}_{-0.10})$.

Although it is a nice observation, the method is, to some extent, model dependent. Therefore, a cross check seems necessary with the direct extrapolation method including $m_\pi$ and $m_\pi^2$ terms, which requires high statistics data at several (active and spectator) light quark masses.

MILC collaboration have started such a study using fatlink clover quark action for both heavy and light quarks on their dynamical quark configurations \cite{48}. They presented a very preliminary result that soft pion theorem is satisfied when extrapolated including $m_\pi$ term albeit with a large error.

### 3.2.2. $f^+(q^2)$

Model independent calculation of $f^+(q^2)$ has great phenomenological importance, since the differential decay rate $d\Gamma(B \rightarrow \pi\nu)/dq^2$ is proportional to $|V_{ub}|^2 f^+(q^2)$ and its measurement may be used for the determination of $|V_{ub}|$.

Near $q_{\text{max}}^2 = (m_B - m_\pi)^2$, where lattice calculation is most powerful, the pole dominance picture should give a good approximation, as the corresponding pole of $B^*$ meson is very close. Using the $B^*B\pi$ coupling $g_{B^*B\pi}$ and $B^*$ meson decay

\footnote{Strictly speaking they combine $f^+$ and $f^0$ when they fit with dipole-pole ansatz \cite{26} or with Becirevic-Kaidalov model \cite{27}. The functional form of $f^0(q^2)$ are pole-type in both models.}

\begin{align*}
\langle B(p)\pi(k)|B^*(p')\rangle &= gb\cdot B\pi(k\cdot \epsilon), \\
\langle 0|V_\mu|B^*\rangle &= if_{B^*}m_{B^*}\epsilon,
\end{align*}

the form factor $f^+(q^2)$ is given by

\begin{equation}
\sum\frac{g^{B^*B\pi}f_{B^*}m_{B^*}}{m_{B^*}^2 - q^2}
\end{equation}

up to the contribution of higher excited states, which is relatively small for large $q^2$.

Many previous lattice calculations supported or assumed the shape of the pole model \cite{25,40,26,27}. At this conference JLQCD has further tested the heavy quark scaling of the pole model using the NRQCD action \cite{31}. They fitted observed the measured form factor with \cite{37} and extracted the numerator (pole residue $\text{Res}f^+$). The heavy quark scaling law predicts $g_{B^*B\pi} \sim M$ and $f_{B^*} \sim M^{-1/2}$, so that the pole residue should behave as $M^{3/2}$ \cite{3}. They confirmed this behavior as shown in Figure 3 and obtained a very preliminary result $g = 0.33(4)$ for the $B^*B\pi$ coupling, where $g$ is defined through $g_{B^*B\pi}(2m_B/f_\pi)/g$.

A direct lattice calculation of the $B^*B\pi$ coupling has recently been made by UKQCD\footnote{This scaling law is compatible with the scaling predicted for $f^+(q_{\text{max}}^2)$ itself ($\sim M^{1/2}$), if the daughter pion mass is kept finite. The denominator of \cite{37} behaves as $(m_{B^*}^2 - m_\pi^2) + 2m_Bm_\pi \sim 2Mm_\pi$.}. The functional form of $f^0(q^2)$ are pole-type in both models.\\
collaboration [11]. They use the reduction formula to relate the matrix element \( \langle B(p)\pi(k)|B^+(p') \rangle \) to a ‘semi-leptonic transition’ amplitude \( \langle B(p)|A_\mu|B^+(p+k) \rangle \), where \( A_\mu \) is a light-light axial current. Then they use the stochastic propagator to evaluate the latter matrix element. Their result \( g = 0.42(4)(8) \) is consistent with a phenomenological determination through \( D^* \to D\pi \) \[22\] and other phenomenological model calculations \[23\].

In the other limit \( q^2 = 0 \), where the pion recoil momentum is large, the light cone scaling law \( f^+(0) \sim M^{-3/2} \) holds, which is not compatible with the pole dominance model that predicts \( \sim M^{-1/2} \). Becirevic and Kaidalov proposed a model to parametrize \( f^+(q^2) \), introducing a term representing the effects of higher excited state contributions \[33\]. By choosing a parameter of the new term \( \alpha \) it is possible to produce an additional suppression of \( \sim M^{-1} \), and the model becomes consistent with the light cone scaling law. Using this and other models, such as the pole-dipole model, one may extrapolate the lattice data to \( q^2 = 0 \). APE \[30\] and UKQCD \[37\] presented \( f^+(0) = f^0(0) \), which are consistent with each other and with previous calculations \[21\].

As I discussed before, the comparison with experiment and extraction of \( |V_{ub}| \) is possible with partial decay rate without introducing model dependence. Fermilab group proposed to compare the differential decay rate \( \frac{d\Gamma}{d|p_\pi|} \) in the region \( 400 \text{ MeV} \leq |p_\pi| \leq 850 \text{ MeV} \), where systematic error is minimized \[20\]. An update was reported at this conference \[14\], where they found good scaling between \( \beta = 5.7 \) and 5.9.

4. \( B\bar{B} \) mixing

The mass difference of two neutral \( B_d \) mesons is measured quite precisely \( \Delta M_d = 0.481 \pm 0.017 \text{ ps}^{-1} \), and a bound is known for \( B_s \) meson \( \Delta M_s > 14.3 \text{ ps}^{-1} \) \[15\]. To extract the CKM matrix elements \( |V_{ud}| \) and \( |V_{ts}/V_{td}| \), hadronic parameters \( f_B^2 B_B \) and \( f_{B_s}^2 B_{B_s}/f_B^2 B_B \) must be obtained theoretically, for which the lattice calculation has been proven to be the best tool. Here I describe updates on the calculation of these quantities.

![Figure 4. Recent quenched lattice calculations of \( f_B \) using \( O(a) \)-improved actions. Results with the Fermilab formalism of heavy quarks \[16\] are given by filled symbols: JLQCD \[32\], Fermilab \[32\]. Shaded symbols represent calculation involving an extrapolation in heavy quark mass: APE \[17\], UKQCD \[18\]. NRQCD results are given by open symbols: GLOK \[19\], JLQCD \[33\], CP-PACS \[50\].](image)
increase of $f_B$ with unquenching.

At this conference MILC updated their calculation with three new dynamical lattices at $\beta=5.6$ (three different sea quark masses), which is consistent with their previous dynamical result \cite{55}. A problem in their result is a large $a$ dependence seen in the quenched data due to the use of unimproved Wilson quark. A linear extrapolation to the continuum limit gives a substantially lower value compared to the data at finite $a$. On the other hand, their dynamical results do not show a similar $a$ dependence and the continuum limit remains high. For this reason, although their result suggests $f_B^{N_F=2} > f_B^{N_F=0}$, the conclusion is not solid enough. Therefore, they started a new calculation using the fatlink clover action for heavy quark, with which scaling behavior is expected to be improved. A preliminary result favors lower value of $f_B$, albeit with large statistical error.

The CP-PACS collaboration presented two new calculations of $f_B$ on their dynamical lattices ($N_F=2$) generated with an RG improved gauge action \cite{56}:

Ali Khan discussed a NRQCD calculation at $\beta=1.95 (1/a \sim 1 \text{ GeV})$ with two sea quark masses \cite{57}. A correction of order $\alpha_s/M$ of the heavy-light current induced by the operator mixing \cite{58} is included. They compared the dynamical result with their quenched result obtained at a similar lattice spacing, and found clear increase with unquenching ($\sim 15\% - 20\%$), while no difference was found between two sea quark masses.

Shanahan presented another systematic study of unquenching at three $\beta$ values ($1/a=0.7\sim 1.7 \text{ GeV}$) with four sea quark masses (for each $\beta$) \cite{61}. They used the $O(a)$-improved relativistic action for both heavy and light quarks and applied the Fermilab reinterpretation for heavy quark \cite{16}. The chiral extrapolation was performed with $m_{\text{sea}} = m_{\text{valence}}$, so the real unquenching was made for the first time.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Dynamical lattice calculations of $f_B$. Results are from MILC \cite{51,55,57}, Collins et al. \cite{52} and CP-PACS \cite{57,61}. Quenched results as shown in Figure 4 are also plotted with small open symbols.}
\end{figure}

All dynamical results for $f_B$ are plotted in Figure 5 together with the recent quenched data. We observe clear upward shift of $f_B$ with the inclusion of dynamical quarks. Although it is a difficult task to combine the results from different groups, we can crudely say that all available data is consistent with the following estimates:

\begin{align*}
N_F=2 & \quad N_F=0 \\
\begin{array}{lc}
 f_B (\text{MeV}) & 210 \pm 30 \quad 170 \pm 20 \\
 f_{B_s} (\text{MeV}) & 245 \pm 30 \quad 195 \pm 20 \\
f_{B_s}/f_B & 1.16 \pm 4 \quad 1.15 \pm 4 
\end{array}
\end{align*}

where I also list the results for $f_{B_s}$ and $f_{B_s}/f_B$. I do not attempt to extrapolate these results to the physical $N_F = 3$ limit. To do so, it seems necessary to understand the systematic errors coming from the use of different actions and lattice spacings. The sea quark mass dependence should also be clarified.

\subsection{4.2. $B_B$}

In contrast to the achievement for $f_B$, the lattice calculation of $B_B$ is still premature.

In the static approximation, the $O(a)$-improved results by Giménez and Martinelli \cite{62} and by UKQCD \cite{63} have been reanalyzed in a recent

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$N_F$ & \$f_B$ (MeV) \\
\hline
2 & 210 \pm 30 \\
0 & 170 \pm 20 \\
\hline
\end{tabular}
\caption{Dynamical lattice calculations of $f_B$.}
\end{table}
paper by Giménez and Reyes [14], using corrected one-loop matching coefficient, and a disagreement, which existed between the static-clover and static-Wilson results [66], has been greatly reduced.

The Hiroshima group performed a calculation using the NRQCD action [67,68], and found significant decrease of $B_B(m_b)$ as one includes $1/M$ corrections. The matching, however, was done with the coefficient in the static limit, and thus large $O(\alpha/(aM))$ systematic error is expected for the slope in $1/M$.

In the calculation with relativistic actions, UKQCD presented the first calculation with the $O(\alpha)$-improved action at $\beta=6.0$ and 6.2 at Lattice 98 [69], which has recently been updated [48]. To obtain the result at the $B$ meson mass, an extrapolation from charm mass regime is necessary, and they found a clear negative slope in $1/M$.

At this conference, APE group [70] has presented the first result obtained using nonperturbative renormalization [71]. They found a similar dependence of $B_B$ on $1/M$, but their final numerical results are not yet available at the time I wrote this contribution.

Figure 4 presents a compilation of lattice data for $\Phi_B(B)(\mu_b) \equiv (\alpha_s(M_P)/\alpha_s(M_B))^{2/\beta_0}B_B(\mu_b)$ with $\mu_b = 5$ GeV as a function of $1/M_P$. The renormalization factor is introduced to cancel the $\ln(M/\mu_b)$ dependence appearing in the matching factor [48]. It is encouraging that all relativistic results including the early works [72,73] show a reasonable agreement with each other, and that the recent UKQCD data show a nice scaling between $\beta=6.0$ and 6.2. The extrapolation to the static limit ($\sim 0.92$), however, seems considerably higher than the $O(\alpha)$-improved results in that limit. It suggests that there is unknown sources of systematic error in either or both of static (NRQCD) and relativistic calculations. Higher order perturbative corrections (for both) and $O((aM)^2)$ uncertainty in the relativistic calculations are their potential candidates. For this reason, my summary of the current available data includes a large systematic uncertainty: $B_B(m_b) = 0.80(15)$.

The allowed region on the $(\rho, \eta)$ plane of the CKM matrix is shown in Figure 5. The two flavour result $f_B = 210 \pm 30$ MeV and the conservative estimate of $B_B$ are used to draw the constraint from $\Delta M_d$. Due to the upward shift of $f_B$ from the previous quenched results, the allowed region favors $\rho > 0$.

### 4.3. $B_s$ width difference

The width difference in the $B_s - \bar{B}_s$ mixing is given as

$$\Delta \Gamma_s \propto \imath m \frac{1}{2 M_{B_s}} \langle B_s | i \int d^4 x T \mathcal{H}_{\text{eff}}^{\text{eff}}(x) \mathcal{H}_{\text{eff}}^{\text{eff}}(0) | B_s \rangle$$

where $\mathcal{H}_{\text{eff}}^{\text{eff}}$ represents the $\Delta B=1$ effective Hamiltonian [74]. Only the final states into which both of $B_s$ and $\bar{B}_s$ can decay contribute. The $1/m_b$ expansion induces two four-quark $\Delta B=2$ operators,
whose matrix elements with $B_s$ and $\bar{B}_s$ states are $B_B$ and $B_S$, where $B_S$ is defined through

$$\langle \bar{B}_s | O_S(\mu) | B_s \rangle = \frac{-5}{3} f_B^2 M_B^2 \frac{M_B^2}{m_b + m_s} B_S(\mu), \quad (10)$$

and $O_S = \bar{b}(1 - \gamma_5)s\bar{b}(1 - \gamma_5)s$.

At this conference, the Hiroshima group presented a calculation of $B_S$ using the NRQCD action [18]. Their calculation method is the same as that of $B_B$ and they obtain $B_S(m_b) = 1.19(2)(20)$. Using a next-to-leading order formula of Beneke et al. [2], the width difference is obtained as $(\Delta \Gamma/\Gamma)_{B_s} = 0.16(3)(4)$, where errors are from $f_{B_s}$ and $B_S$ respectively. The two-flavour result for $f_{B_s}$, discussed in Section 4.1, is used. The latest experimental bound from DELPHI is $(\Delta \Gamma/\Gamma)_{B_s} < 0.42$ [78].

4.4. Lifetime ratios

The ratios of lifetime of $b$ hadrons, such as $\tau(B^-)/\tau(B^0)$ and $\tau(\Lambda_b)/\tau(B^0)$, provide an important test of the theoretical method to calculate the inclusive hadronic decay rates [77]. In the $1/m_b$ expansion, the leading contribution to the decay rate comes from a diagram in which the $b$-quark decay proceeds without touching the spectator quark, so that it does not contribute to the lifetime ratios. The $O(1/m_b^2)$ correction to the ratios is also small for the same reason, and the first correction involving the spectator quark effect is of $O(1/m_b^4)$, which is parametrized by the ‘$B$ parameters’ of $\Delta B=0$ four-quark operators. UKQCD computed these matrix elements for the first time and obtained $\tau(B^-)/\tau(B^0) = 1.03(2)(3)$ [2], which is consistent with the recent experimental result 1.07(2) [80].

It is a known problem that the lifetime of $\Lambda_b$ is surprisingly shorter than that of $B$ mesons $\tau(\Lambda_b)/\tau(B^0) = 0.79(5)$ [51]. It is, therefore, interesting to see whether it is explained with the theoretical calculation, in which the similar matrix elements of four quark operators for $\Lambda_b$ are required. The UKQCD group has studied these matrix elements at $\beta = 5.7$ with 20 gauge configurations [51], and found that the spectator effect is large $\sim -6\%$. Although their result $\tau(\Lambda_b)/\tau(B^0) = 0.91(1)-0.93(1)$, depending on the light quark mass, is much higher than the experimental value, higher statistics calculations at higher $\beta$ values seem necessary to draw a definite conclusion.

5. Conclusions

Lattice calculations provide model independent predictions for many important $B$ decay matrix elements. Progress made for the zero recoil $B \rightarrow D^{(*)}l\nu$ from factors is essential for precise determination of $|V_{cb}|$, and the shape of the form factors is also being studied with improved techniques (NP improved action, NRQCD, etc.). More study is necessary to achieve a complete understanding of the $B \rightarrow \pi l\nu$ form factors: the violation of soft pion theorem and the shape of $f^+(q^2)$. The determination of $|V_{ub}|$ with 10% precision will become possible, once we understand these questions.

The dynamical quark simulation has become practical by several groups, and its effect on $f_B$ has been identified. Further systematic study like that of MILC and CP-PACS is necessary to understand systematic errors and eventually to obtain physical prediction at $N_F = 3$. An unquenched study of other quantities should also be important.

Several new applications have also been stud-
ied, such as the width difference of $B_s$ meson, and the lifetime ratios of $b$ hadrons. Those will become useful theoretical calculations, provided that statistical and systematic errors are improved.

Acknowledgements

I thank A. Ali Khan, S. Aoki, D. Becirevic, C. Bernard, S. Collins, C. DeTur, G. Douglas, V. Giménez, L. Giusti, J. Hein, K-I. Ishikawa, A. Kronfeld, L. Lellouch, C.-J.D. Lin, C. Maynard, T. Onogi, S. Ryan, H. Shanahan, J. Simone, and N. Yamada for communications and useful discussions. I also thank M. Okawa and A. Ukawa for comments on the manuscript. S.H. is supported in part by the Ministry of Education under Grant No. 11740162.

REFERENCES

1. T. Draper, Nucl. Phys. B (Proc. Suppl.) 73 (1999) 43.
2. For a recent review, see, M. Beneke, Phys. Rept. 317 (1999) 1.
3. G. Martinelli and C.T. Sachrajda, hep-lat/9812001.
4. V. Giménez et al. (APE collaboration), these proceedings, hep-lat/9909138.
5. K. Hornbostel et al. (NRQCD collaboration), Nucl. Phys. B (Proc. Suppl.) 73 (1999) 339.
6. M. Beneke and A. Signer, hep-ph/9906475.
7. A.H. Hoang, hep-ph/9905550.
8. K. Melnikov and A. Yelkhovsky, Phys. Rev. D59, 114009 (1999).
9. DELPHI Collaboration (P. Abreu et al.), Phys. Lett. B418, 430 (1998).
10. For a recent summary, see I.I. Bigi, hep-ph/9907270.
11. S. Hashimoto, A.X. El-Khadra, A.S. Kronfeld, P.B. Mackenzie, S.M. Ryan, J.N. Simone, hep-ph/9906376.
12. J.N. Simone et al., these proceedings.
13. A.S. Kronfeld and S. Hashimoto, Nucl. Phys. B (Proc. Suppl.) 73 (1999) 387.
14. A.F. Falk, M. Neubert, Phys. Rev. D47, 2965 (1993).
15. T. Mannel, Phys. Rev. D50, 428 (1994).
16. A.X. El-Khadra, A.S. Kronfeld and P.B. Mackenzie, Phys. Rev. D55, 3933 (1997).
17. A.S. Kronfeld, these proceedings, hep-lat/9909083.
18. See, for example, DELPHI collaboration (M. Margoni et al.), DELPHI 99-107 CONF 294, reported at EPS-HEP99.
19. UKQCD collaboration (G. Douglas et al.), these proceedings, hep-lat/9909120.
20. J. Hein et al., these proceedings, hep-lat/9908058.
21. For a review, see J.M. Flynn and C.T. Sachrajda, in Heavy Flavours (2nd ed.), ed. by A.J. Buras and M. Linder (World Scientific, Singapore), hep-lat/9710057.
22. P. Boyle, these proceedings.
23. CLEO collaboration, (B.H. Behrens et al.), hep-ex/9905056.
24. T. Onogi, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 59.
25. A. Abada et al., Nucl. Phys. B416 (1994) 675.
26. UKQCD Collaboration (D.R. Burford et al.), Nucl. Phys. B447 (1995) 425.
27. S. Hashimoto, K-I. Ishikawa, H. Matsufuru, T. Onogi, N. Yamada, Phys. Rev. D58, 014502 (1998).
28. JLQCD collaboration (S. Aoki et al.), Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 380.
29. S. Ryan et al., Nucl. Phys. B (Proc. Suppl.) 73 (1999) 390.
30. APE collaboration (D. Becirevic et al.), these proceedings.
31. JLQCD collaboration (T. Onogi et al.), these proceedings.
32. JLQCD collaboration (S. Aoki et al.), Phys. Rev. Lett. 80, 5711 (1998).
33. JLQCD Collaboration (K-I. Ishikawa et al.), hep-lat/9905030.
34. M. Lüscher, S. Sint, R. Sommer and H. Wittig, Nucl. Phys. B491 (1997) 344.
35. L. Maiani and G. Martinelli, Phys. Lett. 178B (1986) 265.
36. M. Kurth and R. Sommer, these proceedings, hep-lat/9908039.
37. UKQCD collaboration (C.M. Maynard et al.), these proceedings, hep-lat/9909100.
38. D. Becirevic and A.B. Kaidalov, hep-ph/9904490.
39. MILC collaboration (C. DeTar et al.), these proceedings, hep-lat/9909076.
40. APE Collaboration (C.R. Allton et al.), Phys. Lett. B345 (1995) 513.
41. UKQCD Collaboration (G.M. de Divitiis et al.), JHEP 10 (1998) 010.
42. I.W. Stewart, Nucl. Phys. B529 (1998) 62.
43. For a review see, R. Casalbuoni et al., Phys. Lett. 281 (1997) 145.
44. S. Ryan et al., these proceedings, hep-lat/9910010.
45. LEP B Oscillation Working Group, see their web page for the latest summary, http://www.cern.ch/LEPBOSC/.
46. A.X. El-Khadra et al., Phys. Rev. D58, 014506 (1998).
47. D. Becirevic, P. Boucaud, J.P. Leroy, V. Lubicz, G. Martinelli, F. Mescia, F. Rapuano, Phys. Rev. D60, 074501 (1999); see also [30] for an update.
48. UKQCD collaboration (L. Lellouch and C.J.D. Lin), to appear in the proceedings of Heavy Flavours 8, Southampton, July 1999.
49. A. Ali Khan et al., Phys. Lett. B427 (1998) 132.
50. CP-PACS collaboration (A. Ali Khan et al.), these proceedings.
51. C. Bernard et al., Phys. Rev. Lett. 81, 4812 (1998).
52. S. Collins et al., Phys. Rev. D60, 074504 (1999).
53. M.J. Booth, Phys. Rev. D51, 2338 (1995); S.R. Sharpe, Y. Zhang, Phys. Rev. D53, 5125 (1996).
54. G.M. de Divitiis, R. Frezzotti, M. Masetti and R. Petronzio, Phys. Lett. B382 (1996) 398.
55. MILC collaboration (S. Gottlieb et al.), these proceedings, hep-lat/9909121.
56. R. Burkhalter (CP-PACS collaboration), Nucl. Phys. B (Proc. Suppl.) 73 (1999) 3.
57. CP-PACS collaboration (A. Ali Khan et al.), these proceedings.
58. C.J. Morningstar and J. Shigemitsu, Phys. Rev. D57 (1998) 6741; see also, J. Shigemitsu, Nucl. Phys. B (Proc. Suppl.) 60A (1998) 134.
59. C. Bernard and T. DeGrand, these proceedings, hep-lat/9909083.
60. K-I. Ishikawa et al., these proceedings, hep-lat/9909159.
61. CP-PACS collaboration (H. Shanahan et al.), these proceedings, hep-lat/9909052.
62. V. Giménez and G. Martinelli, Phys. Lett. B398 (1997) 135.
63. UKQCD collaboration, (A.K. Ewing et al.), Phys. Rev. D54 (1996) 3526.
64. V. Giménez and J. Reyes, Nucl. Phys. B545 (1999) 576.
65. UKQCD collaboration (M. Di Pierro and C.T. Sachrajda), Nucl. Phys. B534 (1998) 373.
66. J. Christensen, T. Draper and C. McNeile, Phys. Rev. D56, 6993 (1997).
67. S. Hashimoto, K-I. Ishikawa, H. Matsufuru, T. Onogi, S. Tominaga, N. Yamada, Phys. Rev. D60, 094503 (1999).
68. N. Yamada et al., these proceedings, hep-lat/9910006.
69. UKQCD collaboration (L. Lellouch and C.J.D. Lin), Nucl. Phys. B (Proc. Suppl.) 73 (1999) 357.
70. A. Donini, V. Giménez, L. Giusti and G. Martinelli, these proceedings, hep-lat/9909041.
71. G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa, A. Vladikas, Nucl. Phys. B445 (1995) 81.
72. C. Bernard, T. Draper, G. Hockney and A. Soni, Phys. Rev. D45, 3540 (1988).
73. A. Abada et al., Nucl. Phys. B376 (1992) 172.
74. A. Soni, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 43.
75. R. Gupta, T. Bhattacharya and S. Sharpe, Phys. Rev. D55, 4036 (1997).
76. M. Beneke, G. Buchalla, I. Dunietz, Phys. Rev. D54, 4419 (1996).
77. M. Beneke, G. Buchalla, C. Greub, A. Lenz and U. Nierste, Phys. Lett. B459 (1999) 631.
78. DELPHI collaboration (A. Borgland et al.), DELPHI 99-109 CONF 296, reported at EPS-HEP99.
79. M. Neubert and C.T. Sachrajda, Nucl. Phys. B483 (1997) 339.
80. LEP B Lifetime Working Group, see
their web page for the latest summary. [http://home.cern.ch/~claires/lepblife.html](http://home.cern.ch/~claires/lepblife.html)

81. UKQCD collaboration (M. Di Pierro, C.T. Sachrajda and C. Michael), [hep-lat/9906031](http://arxiv.org/abs/hep-lat/9906031).