Estimates for Coefficients of Certain Analytic Functions

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Abstract. For $-1 \leq B \leq 1$ and $A > B$, let $S^*[A,B]$ denote the class of generalized Janowski starlike functions consisting of all normalized analytic functions $f$ defined by the subordination $zf'(z)/f(z) < (1 + A\beta)/(1 + B\beta)$ ($|z| < 1$). For $-1 \leq B \leq 1 < A$, we investigate the inverse coefficient problem for functions in the class $S^*[A,B]$ and its meromorphic counterpart. Also, for $-1 \leq B \leq 1 < A$, the sharp bounds for first five coefficients for inverse functions of generalized Janowski convex functions are determined. A simple and precise proof for inverse coefficient estimations for generalized Janowski convex functions is provided for the case $A = 2\beta - 1$ ($\beta > 1$) and $B = 1$. As an application, for $F := f^{-1}$, $A = 2\beta - 1$ ($\beta > 1$) and $B = 1$, the sharp coefficient bounds of $F/F'$ are obtained when $f$ is a generalized Janowski starlike or generalized Janowski convex function. Further, we provide the sharp coefficient estimates for inverse functions of normalized analytic functions $f$ satisfying $f'(z) < (1 + z)/(1 + Bz)$ ($|z| < 1, -1 \leq B < 1$).

1. Introduction and Preliminaries

Let $D$ denote the unit disc. Let $A$ be the class of all normalized analytic functions $f: D \rightarrow \mathbb{C}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. The subclass of $A$ consisting of univalent functions is denoted by $S$. An analytic function $f$ is said to be subordinate to an analytic function $g$, written $f \prec g$, if $f = g \circ w$ for some analytic function $w: D \rightarrow D$ with $w(0) = 0$. If $g$ is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$. Let $\varphi$ be an analytic univalent function with positive real part mapping $D$ onto domains symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Let $P(\varphi)$ denote the class of all analytic functions $p: D \rightarrow \mathbb{C}$ such that $p \prec \varphi$. For such $\varphi$, Ma and Minda \cite{22} introduced the subclasses $S^*(\varphi)$ ($K(\varphi)$) of $S$ consisting of functions $f \in S$ such that $zf'(z)/f(z) (1 + zf''(z)/f'(z)) \in P(\varphi)$. For different choices of $\varphi$, several well-known classes can be easily obtained from these classes which were earlier considered and studied one by one for their geometric and analytic properties. For instance, $S^*((1 + z)/(1 - z)) =: S^*$ and $K(((1 + z)/(1 - z))) =: K$, the usual classes of starlike and convex functions respectively; for $0 \leq \alpha < 1$, $S^*((1 + (1 - 2\alpha)z)/(1 - z)) =: S^*(\alpha)$ and $K(((1 + (1 - 2\alpha)z)/(1 - z))) =: K(\alpha)$, the well-known classes of starlike and convex functions of order $\alpha$, respectively introduced in \cite{30}; for $0 < \alpha \leq 1$, $S^*((1 + z)/(1 - z))^{\alpha} =: SS^*(\alpha)$ is the well-known class of strongly starlike functions of order $\alpha$ introduced

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the sharp coefficient bounds of \( \Sigma \) estimation for functions in the class \( \Sigma \) coefficient bounds for the inverse functions of functions in the class \( \Sigma \) coefficient problem is completely settled in \cite{1} for functions in the classes known for these classes, for details see \cite{12–14}. This leads to several works related to the inverse coefficient problem for functions in certain subclasses of \( \Sigma \).

We observe that the distortion theorem, upper bound of \( |f| \), rotation theorem, upper bound of Feketo-Szego coefficient functional \( |a_3 - \mu a_2^3| \) for \( f \in \mathcal{K}(\varphi) \) given in \cite{21} still hold for a normalized locally univalent function \( f \) satisfying \( 1 + z f''(z)/f'(z) \prec \varphi(z) \) if we drop the condition that \( \varphi \) has positive real part. Consequently, the upper bound and the lower bound of Feketo-Szego coefficient functional \( |a_3 - \mu a_2^3| \) follow for a normalized analytic function \( f \) satisfying \( z f'(z)/f(z) \prec \varphi(z) \) even if \( \varphi \) does not have positive real part. This motivates one to consider the following subclasses of \( A \), for \(-1 \leq B \leq 1, A > B\),

\[
\mathcal{K}[A, B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A, B] \right\} \quad \text{and} \quad S^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B] \right\}
\]

where \( \mathcal{P}[A, B] := \mathcal{P}((1 + A z)/(1 + B z)) \). For \(-1 \leq B < A \leq 1\), \( S^*[A, B] \) is a subclass of \( S^* \) introduced by Janowski \cite{11} and for particular values of \( A \) and \( B \), it reduces to several known subclasses of \( S^* \). Precisely, \( S^*[1 - 2\alpha, -1] := S^*(\alpha) \) (\( 0 \leq \alpha < 1 \)) \cite{20}; \( S^*[1, 1/M - 1] := S^*(M) \) (\( M > 1/2 \)) \cite{10}; \( S^*[\beta, -\beta] := S^*(\beta) \) (\( 0 < \beta \leq 1 \)) \cite{22}; \( S^*[1 - \beta, 0] := S^*_1[1 - \beta, 0] \) \cite{13}. Note that, for \(-1 < B < 1 \), the functions in the classes \( \mathcal{K}[A, B] \) and \( S^*[A, B] \) may not be univalent but must be locally univalent in \( \mathbb{D} \) and non-vanishing in \( \mathbb{D} \backslash \{0\} \), respectively.

Recently, the classes \( S^*[2\beta - 1, 1] \) and \( \mathcal{K}[2\beta - 1, 1] \) \( (\beta > 1) \) have been studied by several authors, see \cite{23, 24, 36}. Moreover, the upper bound of the Feketo-Szego coefficient functional \( |a_3 - \mu a_2^3| \) for \( f \in \mathcal{K}[2\beta - 1, 1] \) or \( f \in S^*[2\beta - 1, 1] \); the distortion theorem, upper bound of \( |f| \), rotation theorem for \( f \in \mathcal{K}[2\beta - 1, 1] \); and the growth theorem for \( f \in S^*[2\beta - 1, 1] \) are given in \cite{1} which can actually be deduced, even for the functions in the generalized classes \( S^*[A, B] \) and \( \mathcal{K}[A, B] \) \((-1 \leq B \leq 1 < A)\), from the results in \cite{21}. Also, for \(-1 \leq B \leq 1 \) and \( A > B \), one can consider the meromorphic counterpart part of \( S^*[A, B] \), namely, the class \( \Sigma^*[A, B] \) consisting of analytic functions of the form

\[
g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots
\]

defined on \( \mathbb{C} \backslash \mathbb{D} \) such that \( zg'(z)/g(z) = p_0(\mathbb{D}) \) where \( p_0 : \mathbb{D} \to \mathbb{C} \) is defined by \( p_0(z) = (1 + A z)/(1 + B z) \). For \(-1 \leq B < A \leq 1\), the class \( \Sigma^*[A, B] \) has been considered in \cite{3} and the particular choices of \( A \) and \( B \) give the meromorphic counterpart parts of the classes corresponding to those of \( S^*[A, B] \) such as \( \Sigma^*[1 - 2\alpha, -1] := \Sigma^*(\alpha) \) (\( 0 \leq \alpha < 1 \)) \cite{20}; \( \Sigma^*[1, 1/M - 1] := \Sigma^*(M) \) (\( M > 1/2 \)) \cite{33}; \( \Sigma^*[\beta, -\beta] := \Sigma^*(\beta) \) (\( 0 < \beta \leq 1 \)) \cite{22}; \( \Sigma^*[1 - \beta, 0] := \Sigma^*_1[1 - \beta, 0] \) \cite{13}. Haltenbeck \cite{3} introduced the class \( \Sigma \) consisting of functions \( f \in \mathcal{S} \) such that \( f' \in \mathcal{P} \), where \( \mathcal{P} := \mathcal{P}((1+z)/(1-z)) \). Further, Libera and Złotkiewicz \cite{15, 17} investigated the inverse coefficient problem of functions in the class \( \Sigma \). For \(-1 \leq B < A \leq 1\), let \( \mathcal{S}[A, B] \) denote the subclass of \( \mathcal{S} \) consisting of functions \( f \in \mathcal{S} \) such that \( f' \in \mathcal{P}[A, B] \).

The problem of estimating the coefficients of inverse functions lay its origin in 1923 when Löwner \cite{20} gave the sharp coefficient estimates for inverse function of \( f \in \mathcal{S} \) along with the sharp coefficient estimation for the third coefficient of \( f \in \mathcal{S} \). Later, several authors \cite{5, 6, 7, 27, 32} gave alternate proofs for the inverse coefficient problem for functions in the class \( \mathcal{S} \) but the inverse coefficient problem is still an open problem even for the well-known classes \( \mathcal{K} \) and \( \Sigma^*(\alpha) \) \( (0 \leq \alpha < 1) \), although the sharp estimates for initial inverse coefficients are known for these classes, for details see \cite{12, 14}. This leads to several works related to the inverse coefficient problem for functions in certain subclasses of \( \mathcal{S} \), see \cite{2, 16, 18, 13, 22, 28, 34, 35}. Recently, the inverse coefficient problem is completely settled in \cite{1} for functions in the classes \( S^*[2\beta - 1, 1] \) or \( \Sigma^*[2\beta - 1, 1] \) or \( \mathcal{K}[2\beta - 1, 1] \); \( \beta > 1 \).

In this paper, we are mainly concerned about the determination of the sharp inverse coefficient bounds for functions in the classes \( \Sigma^*[A, B] \) or \( \Sigma^*[A, B] \) \((-1 \leq B \leq 1 < A)\). Also, we are giving the sharp coefficient bounds for the inverse functions of functions in the class \( \mathcal{S}[1, B] \) \((-1 \leq B < 1) \) and the sharp first five coefficient bounds for the inverse functions of functions in the class \( \mathcal{K}[A, B] \) for \(-1 \leq B \leq 1 < A \). Apart from this, we present a slightly simpler proof than the proof given in \cite{1} for the sharp inverse coefficient estimation for functions in the class \( \mathcal{K}[2\beta - 1, 1] \); \( \beta > 1 \). As an application, for \( F := f^{-1} \) and \( \beta > 1 \), the sharp coefficient bounds of \( F/F' \) are obtained when \( f \in S^*[2\beta - 1, 1] \) or \( f \in \mathcal{K}[2\beta - 1, 1] \). Further,
under some conditions, the sharp coefficient estimates are determined for functions in the class $\Sigma^*[A, B]$ ($-1 \leq B < A$).

We need the following lemmas to prove our results.

**Lemma 1.1.** [3] Theorem II, p. 547] Let $\Omega$ be the family of functions $f$ such that for $|z| < \rho$ with $\rho > 0$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ($a_1 \neq 0$). If $f \in \Omega$ and $\phi$ is the inverse function of $f$, then $\phi \in \Omega$. For any integer $t$, let $f(z)^t = \sum_{n=-\infty}^{\infty} a_n^{(t)} z^n$ and $\phi(w)^t = \sum_{n=-\infty}^{\infty} b_n^{(t)} w^n$ in some neighbourhoods of the origin, where $a_n^{(t)}$ and $b_n^{(t)}$ are zero for $n < t$. Then

$$b_n^{(t)} = \frac{t}{n} a_n^{(-n)}, \quad n \neq 0.$$

For $n = 0$, $b_0^{(t)}$ is defined by

$$\sum_{t=-\infty}^{\infty} b_0^{(t)} z^{-t-1} = \frac{f'(z)}{f(z)}.$$

**Lemma 1.2.** [32] Theorem X, p. 70] Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ ($z \in \mathbb{D}$) be such that $f \prec g$. If $g$ is univalent in $\mathbb{D}$ and $g(\mathbb{D})$ is convex, then $|a_n| \leq |b_n|$.

By using the above lemma, the following result is proved. This has been proved in [4] for the case $-1 \leq B < A \leq 1$.

**Lemma 1.3.** If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ is in $P[A, B]$ $(-1 \leq B \leq 1, A > B)$ then $|c_n| \leq A - B$. The bounds are sharp.

**Proof.** Since $p \in P[A, B]$, $p(z) < (1 + Az)/(1 + Bz)$. Let $g(z) := (1 + Az)/(1 + Bz)$. Clearly, $g$ is univalent in $\mathbb{D}$. For $-1 < B < 1$, $g(\mathbb{D})$ is the disc $|w - (1 - AB)/(1 - B^2)| < |(A - B)/(1 - B^2)|$. For $B = 1$ and $B = -1$, $g(\mathbb{D})$ is the left half plane $\text{Re}(w) < (1 + A)/2$ and the right half plane $\text{Re}(w) > (1 - A)/2$ respectively. Therefore, $g(\mathbb{D})$ is convex and hence by Lemma 1.2 $|c_n| \leq A - B$ for each $n$. Define a function $p_n : \mathbb{D} \to \mathbb{C}$ as

$$p_n(z) = \frac{1 + Az^n}{1 + Bz^n} = 1 + (A - B)z^n - B(A - B)z^{2n} + \cdots.$$

Clearly, the result is sharp for the function $p_n$. \qed

The following lemma follows easily by induction on $m$ and for $-1 \leq B < A \leq 1$, it is given in [4, Lemma 2, p. 737].

**Lemma 1.4.** Let $A > B$, $-1 \leq B \leq 1$. Then for any integer $t$ and $m \in \mathbb{N}$, we have

$$m^2 \prod_{j=0}^{m-1} \left( \frac{(A - B)(t + Bj)}{j + 1} \right)^2 = (A - B)^2 t^2 + \sum_{k=1}^{m-1} \left( ((A - B)t + Bk)^2 - k^2 \right) \prod_{j=0}^{k-1} \left( \frac{(A - B)(t + Bj)}{j + 1} \right)^2.$$

**2. Main Results**

The following theorem gives estimates for inverse coefficient of functions in the class $S^*[A, B]$ ($-1 \leq B \leq 1 < A$).

**Theorem 2.1.** Let $f \in S^*[A, B]$ $(-1 \leq B \leq 1 < A)$ and $f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. Then for each $n \geq 2$,

$$|\gamma_n| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{n(A - B) + mB}{m + 1} \right).$$

(2)

The result is sharp.
Proof. For any integer \( t > 0 \), let
\[
g(z) := \left( \frac{f(z)}{z} \right)^{-t} = 1 + \sum_{j=1}^{\infty} a_j^{(-t)} z^j \quad (|z| < 1).
\]
Then
\[
-\frac{z g'(z)}{g(z)} = \frac{zf'(z)}{f(z)} - 1.
\]
Since \( f \in \mathcal{S}^*[A, B] \), we have
\[
\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}
\]
for some analytic function \( w : \mathbb{D} \to \mathbb{D} \) with \( w(0) = 0 \). The equations (3) and (4) give
\[
\sum_{j=1}^{\infty} j a_j^{(-t)} z^j = -w(z) \left( (A - B)t + \sum_{j=1}^{\infty} (B(j - t) + A t) a_j^{(-t)} z^j \right)
\]
which can be rewritten as
\[
\sum_{j=1}^{s} j a_j^{(-t)} z^j + \sum_{j=s+1}^{\infty} b_j^{(-t)} z^j = -w(z) \left( (A - B)t + \sum_{j=1}^{s-1} (B(j - t) + A t) a_j^{(-t)} z^j \right)
\]
where
\[
\sum_{j=s+1}^{\infty} b_j^{(-t)} z^j := \sum_{j=s+1}^{\infty} j a_j^{(-t)} z^j + w(z) \left( \sum_{j=s}^{\infty} (B(j - t) + A t) a_j^{(-t)} z^j \right).
\]
Since \( |w(z)| < 1 \ (|z| < 1) \), squaring the moduli of both sides, we have
\[
\left| \sum_{j=1}^{s} j a_j^{(-t)} z^j + \sum_{j=s+1}^{\infty} b_j^{(-t)} z^j \right|^2 < \left| (A - B)t + \sum_{j=1}^{s-1} (B(j - t) + A t) a_j^{(-t)} z^j \right|^2.
\]
Integrating along \( |z| = r, 0 < r < 1 \) with respect to \( \theta \) \((0 \leq \theta \leq 2\pi)\) and applying Parseval’s identity that for an analytic function \( g : \mathbb{D} \to \mathbb{C} \) of the form \( g(z) = \sum_{n=0}^{\infty} A_n z^n \),
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |g(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |A_n|^2 r^{2n} \quad (0 < r < 1)
\]
we have
\[
\sum_{j=1}^{s} |j a_j^{(-t)}|^2 r^{2j} + \sum_{j=s+1}^{\infty} |b_j^{(-t)}|^2 r^{2j} \leq (A - B)^2 t^2 + \sum_{j=1}^{s-1} |B(j - t) + A t|^2 |a_j^{(-t)}|^2 r^{2j}.
\]
Letting \( r \to 1 \) yields
\[
\sum_{j=1}^{s} |j a_j^{(-t)}|^2 \leq (A - B)^2 t^2 + \sum_{j=1}^{s-1} |B(j - t) + A t|^2 |a_j^{(-t)}|^2
\]
and therefore,
\[
|sa_s^{(-t)}|^2 \leq (A - B)^2 t^2 + \sum_{j=1}^{s-1} ((A - B)t + B(j - t) - j^2)|a_j^{(-t)}|^2.
\]
We shall show that, for $-1 \leq B \leq 1$, $A > B$, $t \geq (s - 1)(1 - B)/(A - B)$ and $s \geq 1$,

$$|a^{(-t)}_s| \leq \prod_{m=0}^{s-1} \left( \frac{(A - B)t + mB}{m + 1} \right).$$  \hfill (6)

We proceed by induction on $s$. For $s = 1$, equation (5) gives

$$|a^{(-t)}_1| \leq (A - B)t.$$  

Since $-1 \leq B \leq 1$ and $A > B$, for fixed $j \geq 1$, $((A - B)t + Bj)^2 - j^2 = ((A - B)t - j(1 - B))(\frac{(A - B)t + t(1 - B)}{A - B} \geq 0$ if $t \geq j(1 - B)/(A - B)$. Assume that (6) holds for $s \leq q - 1$ and $t \geq (q - 1)(1 - B)/(A - B)$.

Then by using induction hypothesis and the equation (5) for $F$, it can be easily seen that

$$|q a^{(-t)}_q|^2 \leq (A - B)^2 t^2 + \sum_{j=1}^{q-1} \left( ((A - B)t + Bj)^2 - j^2 \right) \prod_{m=0}^{j-1} \left( \frac{(A - B)t + mB}{m + 1} \right)^2$$

which by using Lemma [3] gives

$$|a^{(-t)}_q| \leq \prod_{m=0}^{q-1} \left( \frac{(A - B)t + mB}{m + 1} \right).$$

Thus, (6) holds for $s = q$ and hence by induction (6) holds for all $s \geq 1$. By applying Cauchy’s integral formula for $F$, it can be easily seen that

$$\gamma_n = \frac{1}{n} a^{(-n)}_{n-1} \quad (n \geq 2).$$  \hfill (7)

Since $A > 1$, therefore $(n - 2)(1 - B)/(A - B) \leq n - 2$ $(n \geq 2)$. So, for $t = n$ and $s = n - 1$, the equation (6) gives

$$|\gamma_n| = \frac{1}{n} |a^{(-n)}_{n-1}| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{(A - B)n + mB}{m + 1} \right).$$

Define a function $f_1 : \mathbb{D} \to \mathbb{C}$ by

$$f_1(z) = \begin{cases} z(1 + Bz)^{(A - B)/B}, & B \neq 0 \\ ze^{Az}, & B = 0. \end{cases}$$  \hfill (8)

The result is sharp for the function $f_1$.

For $A = 2\beta - 1$, $B = 1$ $(\beta > 1)$, the above theorem reduces to [1, Theorem 4.3, p. 14].

**Corollary 2.2.** Let $f \in S^*[2\beta - 1, 1]$ $(\beta > 1)$ and $f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. If $F(w)/F^*(w) = w + \sum_{n=2}^{\infty} \delta_n w^n$, then $|\delta_2| \leq 2(\beta - 1)$ and for $n > 2$,

$$|\delta_n| \leq 2(\beta - 1) \prod_{j=2}^{n-1} \frac{2(n - 1)(\beta - 1) + j}{j}.$$  

The result is sharp.

**Proof.** Since $f \in S^*[2\beta - 1, 1]$ $(\beta > 1)$, $zf'(z)/f(z) \in P[2\beta - 1, 1]$. This gives

$$\frac{zf'(z)}{f(z)} = p(z)$$
where \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}[2\beta - 1, 1] \). In terms of \( F := f^{-1} \), the above equation becomes

\[
\frac{F(w)}{F'(w)} = wp(F(w)).
\]

Using power series expansions of \( F/F' \), \( p \) and \( F \), we obtain

\[
\sum_{n=2}^{\infty} \delta_n w^n = \sum_{n=2}^{\infty} \left( \sum_{j=1}^{n-1} c_j \gamma_{n-1,j} \right) w^n
\]

where \( \gamma_{n-1,j} \) denotes the coefficient of \( w^{n-1} \) in the expansion of \( F(w)^2 \). In fact, \( \gamma_{n-1,j} = S_j(\gamma_2, \gamma_3, \ldots, \gamma_{n-2}) \) is a polynomial in \( \gamma_2, \gamma_3, \ldots, \gamma_{n-2} \) with non-negative coefficients and \( \gamma_{n-1,n-1} = 1 \). On comparing the coefficients of \( w^n \), we have

\[
\delta_n = \sum_{j=1}^{n-1} c_j \gamma_{n-1,j}.
\]

An application of Lemma 1.3 gives

\[
|\delta_n| \leq 2(\beta - 1) \sum_{j=1}^{n-1} S_j(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_{n-2}|).
\]

Define \( g_1(z) := e^{-iz} f_1(e^{iz} z) \) where \( f_1 \) is given by (8) for \( A = 2\beta - 1 \) and \( B = 1 \). Clearly, \( g_1 \in \mathcal{S}^*[2\beta - 1, 1] \). Then \( G_1(w) := g_1^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \) and \( G_1(w)/G_1'(w) = w - \sum_{n=2}^{\infty} B_n w^n \) where \( w \) lies in some neighbourhood of the origin,

\[
B_2 := 2(\beta - 1), \quad B_n := 2(\beta - 1) \prod_{j=2}^{n-1} \left( \frac{2(n-1)(\beta - 1) + j}{j} \right) \quad (n > 2)
\]

and

\[
A_n := \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{2n(\beta - 1) + m}{m + 1} \right) \quad (n \geq 2).
\]

Proceeding as in (9) for \( g_1 \) and then comparing the coefficients of \( w^n \) give

\[
B_n = 2(\beta - 1) \sum_{j=1}^{n-1} S_j(A_2, A_3, \ldots, A_{n-2}) \quad (n \geq 2).
\]

Since \( f \in \mathcal{S}^*[2\beta - 1, 1] \), applying Theorem 2.1 in (10) and using (11) give \( |\delta_n| \leq B_n \). Clearly, the sharpness follows for the function \( g_1 \). 

**Corollary 2.3.** Let \( g \), given by (1), be in \( \Sigma^*[A, B] \) \((-1 \leq B \leq 1 < A) \) and \( n(1-B) - (A-B) \leq 0 \). Then for each \( n \geq 0 \),

\[
|b_n| \leq \prod_{m=0}^{n} \left( \frac{(A-B)+mB}{m+1} \right).
\]

The result is sharp.

**Proof.** It is easy to observe that for any \( g \in \Sigma^*[A, B] \), there exists \( f \in \mathcal{S}^*[A, B] \) such that for \( z \in \mathbb{C} \setminus \mathbb{D} \), \( g(z) = 1/f(1/z) \). Also, we note that the expansions of \( f(z)^{-1} \) about the origin and \( f(1/z)^{-1} \) about the infinity have same coefficients. Thus, if \( z/f(z) = 1 + \sum_{n=1}^{\infty} a_n^{(-1)} z^n \) \((z \in \mathbb{D})\), then for \( z \in \mathbb{C} \setminus \mathbb{D} \), we have

\[
g(z) \frac{z}{z} = \frac{1}{z f(1/z)} = 1 + \sum_{n=1}^{\infty} a_n^{(-1)} z^{-n}.
\]
On comparing the coefficients, we obtain
\[ b_n = a_{n+1}^* \quad (n \geq 0). \] (12)

An application of (8) for \( t = 1 \) and \( s = n + 1 \) in the equation (12) gives the desired estimate. Define a function \( g_1 : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \) by
\[ g_1(z) = \frac{1}{f_1(1/z)} \] (13)
where \( f_1 \) is given by (8). The result is sharp for the function \( g_1 \) given by (13). \( \Box \)

For \( A = 2\beta - 1, B = 1 (\beta > 1) \), the above result is mentioned in [1, Theorem 4.5, p. 17]. Next, we prove the meromorphic counterpart of the Theorem 2.3.

**Theorem 2.4.** Let the function \( g \in \Sigma^*[A, B] \) \((-1 \leq B \leq 1 < A)\) and \( g^{-1}(w) = w + \sum_{n=0}^{\infty} \gamma_n w^{-n} \) in some neighbourhood of the infinity. Then \( |\gamma_0| \leq A - B \) and
\[ |\gamma_n| \leq \frac{1}{n} \prod_{m=0}^{n} \left( \frac{(A - B)n + mB}{m + 1} \right) \quad (n \geq 1). \]
The result is sharp.

**Proof.** Since \( g \in \Sigma^*[A, B] \), there exists \( f \in S^*[A, B] \) such that for \( z \in \mathbb{C} \setminus \overline{\mathbb{D}}, g(z) = 1/f(1/z) \) and \( g^{-1}(w) = 1/f^{-1}(1/w) \), see [27, Theorem 2.4, p. 459]. Therefore, for each \( n \geq 0 \),
\[ |\gamma_n| = |\gamma_n^{(-1)}| \] (14)
where \( \gamma_n^{(-1)} \) is the coefficient of \( w^{-(n+1)} \) in \( 1/(wf^{-1}(1/w)) = 1 + \sum_{n=1}^{\infty} \gamma_n^{(-1)} w^{-n} \).

Since \( f \in S^*[A, B] \), we have \( zf'(z)/f(z) = q(z) \in \mathcal{P}[A, B] \). If \( q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \), then by applying Lemma 1.1 we have
\[ \sum_{p=-\infty}^{\infty} \gamma_1^{(p)} z^{-p+1} = \frac{f'(z)}{f(z)} = q(z) = \frac{1}{z} \left( 1 + \sum_{n=1}^{\infty} q_n z^n \right). \]

Therefore, in view of (14) and Lemma 1.3 \( |\gamma_0| = |\gamma_1^{(-1)}| = |q_1| \leq A - B \). For \( n \geq 1 \), an application of Lemma 1.1 and the inequality (10) for \( t = n, s = n + 1 \) in (14) gives
\[ |\gamma_n| = |\gamma_n^{(-1)}| = \frac{1}{n} |a_n^{(-1)}| \leq \frac{1}{n} \prod_{m=0}^{n} \left( \frac{(A - B)n + mB}{m + 1} \right). \]
The sharpness follows for the function \( g_1 \) given by (13). \( \Box \)

For \( A = 2\beta - 1, B = 1 (\beta > 1) \), the above theorem reduces to [1, Theorem 4.8, p. 18]. Recall that for \(-1 \leq B < A \leq 1\),
\[ \mathcal{I}[A, B] := \left\{ f \in \mathcal{S} : f'(z) \sim \frac{1 + Az}{1 + Bz} \right\}. \]
The following theorem gives the sharp inverse coefficient estimates for functions in the class \( \mathcal{I}[1, B] \) and its proof is based on the fact that if \( p \in \mathcal{P}[A, B] \) \((-1 \leq B < A \leq 1)\), then \( 1/p \in \mathcal{P}[-B, -A] \) \((-1 \leq -A < -B \leq 1)\).

**Theorem 2.5.** For \(-1 \leq B < 1, let f \in \mathcal{I}[1, B] \) and \( g(z) = \int_0^1 (1-t)/(1-Bt) dt \) \(|z| < 1\). If \( f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n \) and \( g^{-1}(w) = G(w) = w + \sum_{n=2}^{\infty} A_n w^n \) where \( w \) lies in some neighbourhood of the origin, then for each \( n \geq 2 \), \( |\gamma_n| \leq A_n \). The result is sharp.
\textbf{Proof.} Since \( f' \in \mathcal{P}[1, B] \), \( f'(z) = p(z) \) for some \( p \in \mathcal{P}[1, B] \). Let \( w = f(z) \) then \( f'(z)f'(w) = 1 \) and so we have
\[ F'(w) = P(F(w)) \]
where \( P(z) := 1/p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}[-B, -1] \). This gives
\[ 1 + \sum_{n=1}^{\infty} (n+1)\gamma_{n+1} w^n = 1 + \sum_{n=1}^{\infty} c_n F(w)^n. \]

On comparing the coefficients of \( w^n \), we have
\[ (n+1)\gamma_{n+1} = \sum_{i=1}^{n} c_i \gamma_{n,i} \quad (n \geq 1) \quad (15) \]
where \( \gamma_{n,i} \) denotes the coefficient of \( w^n \) in the expansion of \( F(w)^i \) and \( \gamma_{n,n} = 1 \). Since \( g'(z) = (1-z)/(1-Bz) \in \mathcal{P}[1, B] \), proceeding as above, we have
\[ G'(w) = \frac{1-BG(w)}{1-G(w)} \quad (16) \]
which gives
\[ \sum_{n=1}^{\infty} (n+1)A_{n+1} w^n = \sum_{n=1}^{\infty} (1-B)G(w)^n. \]

Comparing the coefficients of \( w^{n-1} \), we get
\[ nA_n = (1-B) \sum_{i=1}^{n-1} A_{n-1,i} \quad (n \geq 2) \quad (17) \]
where \( A_{n-1,i} \) denotes the coefficient of \( w^{n-1} \) in the expansion of \( G(w)^i \) and \( A_{n-1,n-1} = 1 \). We first show that \( A_n > 0 \) for all \( n \geq 2 \). By using the power series expansion of \( G \) in \( (16) \) and on comparing the coefficients of both sides, we obtain
\[ 2A_2 = 1 - B, \quad 3A_3 = (1 - B + 2)A_2, \quad \text{and} \]
\[ (n+1)A_{n+1} = (1-B+n)A_n + \sum_{k=1}^{n-2} (k+1)A_{k+1}A_{n-k} \quad (n > 2). \]

Since \(-1 \leq B < 1 \), \( A_2 = (1-B)/2 > 0 \). By using induction on \( n \), it can be easily seen from the above relations that \( A_n > 0 \) for all \( n \geq 2 \).

Next, we shall show that for all \( n \geq 2 \), \( |\gamma_n| \leq A_n \). We proceed by induction on \( n \). Since \( P \in \mathcal{P}[-B, -1] \), by using Lemma \[ (13) \] \( |c_i| \leq 1 - B \) for each \( i \geq 1 \). Clearly, the result holds for \( n = 2 \). Assume that \( |\gamma_i| \leq A_i \) for \( i \leq n - 1 \). It is easy to observe that \( \gamma_{n,i} = S_i(\gamma_2, \gamma_3, \ldots, \gamma_{n-1}) \) is a polynomial in \( \gamma_2, \gamma_3, \ldots, \gamma_{n-1} \) with non-negative coefficients and thus \( |\gamma_{n,i}| \leq S_i(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_{n-1}|) \leq S_i(A_2, A_3, \ldots, A_{n-1}) \). Therefore, in view of \( (15) \) and \( (17) \), we have
\[ n|\gamma_n| \leq \sum_{i=1}^{n-1} |c_i||\gamma_{n-1,i}| \leq (1-B) \sum_{i=1}^{n-1} S_i(A_2, A_3, \ldots, A_{n-2}) = (1-B) \sum_{i=1}^{n-1} A_{n-1,i} = nA_n \]
where \( A_{n-1,i} = S_i(A_2, A_3, \ldots, A_{n-2}) \) is the coefficient of \( w^{n-1} \) in the expansion of \( G(w)^i \).

For \( B = -1 \), the above theorem reduces to the theorem given in \[ (17) \].

The following theorem has been proved in \[ (1) \, \text{Theorem 4.4, p. 14} \] by using the coefficient bounds of the functions in the class \( \mathcal{P} \) but we are providing a slightly different proof by making use of the coefficient bounds of the functions in the class \( \mathcal{P}[2\beta - 1, 1] \) \( (\beta > 1) \) which shortens the computations involved in the proof to some extent.
Let \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in K[2\beta - 1, 1] (\beta > 1) \) and \( f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n \) in some neighbourhood of the origin. Then for \( n \geq 2 \),

\[
|\gamma_n| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{2(\beta - 1) + m(2\beta - 1)}{m + 1} \right).
\]

The result is sharp.

**Proof.** Since \( f \in K[2\beta - 1, 1] \), we have \( 1 + zf''(z)/f'(z) = p(z) \) where \( p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \in P[2\beta - 1, 1] \) and \( \beta > 1 \). This gives

\[
\frac{d}{dw} \left( \frac{F(w)}{F'(w)} \right) = 1 - \frac{F(w)F''(w)}{(F'(w))^2} = p(F(w))
\]

where \( w = f(z) \) lies in some disk around the origin. Integrate the equation along the line segment \([0, w]\) and using the power series expansions of \( F \) and \( p \), we have

\[
\sum_{n=1}^{\infty} \gamma_n w^n = \sum_{n=1}^{\infty} n\gamma_n w^n + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} k\gamma_k \sum_{j=1}^{n-k} c_j \frac{\gamma_{n-k,j}}{n - k + 1} \right) w^n
\]

where \( \gamma_1 = 1 \) and \( \gamma_{n-k,j} \) denotes the coefficient of \( w^{n-k} \) in the expansion of \( F(w)^2 \) with \( \gamma_{n-k,2} = 1 \).

On comparing the coefficients of \( w^n \), we have

\[
-(n-1)\gamma_n = \sum_{k=1}^{n-1} k\gamma_k \sum_{j=1}^{n-k} c_j \gamma_{n-k,j} \quad (n \geq 2).
\]

Define a function \( f_1 : \mathbb{D} \to \mathbb{C} \) such that

\[
f_1'(z) = (1 - z)^{2(\beta - 1)}.
\]

Then \( F_1(w) := f_1^{-1}(w) = w + A_2 z^2 + A_3 z^3 + \cdots \) where for \( n \geq 2 \),

\[
A_n := \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{2(\beta - 1) + m(2\beta - 1)}{m + 1} \right).
\]

We shall show that for all \( n \geq 2 \), \( |\gamma_n| \leq A_n \). We proceed by induction on \( n \). Since \( p \in P[2\beta - 1, 1] (\beta > 1) \), an application of Lemma 1.3 gives \( |c_j| \leq 2(\beta - 1) \) for each \( j \geq 1 \). Therefore, the desired estimate holds for \( n = 2 \). Assume that the theorem is true for \( j \leq n - 1 \) and thus we have \( |\gamma_j| \leq A_j \) for \( j \leq n - 1 \). Since \( \gamma_{n,j} = S_j(\gamma_2, \gamma_3, \ldots, \gamma_{n-1}) \) is a polynomial in \( \gamma_2, \gamma_3, \ldots, \gamma_{n-1} \) with non-negative coefficients, we have \( |\gamma_{n,j}| \leq S_j(\gamma_2, |\gamma_3|, \ldots, |\gamma_{n-1}|) \leq S_j(A_2, A_3, \ldots, A_{n-1}) \). An application of induction hypothesis and bounds of \( c_j \) in 20 gives

\[
(n-1)|\gamma_n| \leq 2(\beta - 1) \sum_{k=1}^{n-1} k |\gamma_k| \sum_{j=1}^{n-k} |\gamma_{n-k,j}|
\leq 2(\beta - 1) \sum_{k=1}^{n-1} k A_k \sum_{j=1}^{n-k} S_j(A_2, A_3, \ldots, A_{n-k-1})
= 2(\beta - 1) \sum_{k=1}^{n-1} k A_k \sum_{j=1}^{n-k} A_{n-k,j}
\]

(23)
where $A_1 = 1$ and $A_{n-k,j}$ denotes the coefficient of $w^{n-k}$ in the expansion of $F_1(w)^j$ with $A_{n-k,n-k} = 1$.

We now show that for each $n \geq 2$,

$$2(\beta - 1) \sum_{k=1}^{n-1} \frac{kA_k}{n-k+1} \sum_{j=1}^{n-k} A_{n-k,j} = (n-1)A_n. $$

(24)

For $f_1$, given by (21), we have

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \frac{1 - (2\beta - 1)z}{1 - z}. $$

This proves (24) and hence, in view of (23), we have

$$|\gamma_n| = A_n. $$

The sharpness follows for the function $f_1$, given in (21).

**Corollary 2.7.** Let $f \in \mathcal{K}_1 \geq 2\beta - 1, 1 \text{ } (\beta > 1)$ and $f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. If $F(w)/F'(w) = w + \sum_{n=2}^{\infty} \delta_n w^n$, then $|\delta_2| \leq \beta - 1$ and for $n > 2$,

$$|\delta_n| \leq \frac{2\beta - 1}{n(n-1)} \prod_{m=0}^{n-3} \left( \frac{2\beta + m(2\beta - 1)}{m + 1} \right). $$

The result is sharp.

**Proof.** On integrating the equation (18) along the line segment $[0, w]$ and using the power series expansions of $F/F'$, $F$ and $p$, we have

$$w + \sum_{n=2}^{\infty} \delta_n w^n = w + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} c_j \gamma_{n-1,j} \frac{w^n}{n} w^n $$

(25)

where $\gamma_{n-1,j}$ denotes the coefficient of $w^{n-1}$ in the expansion of $F(w)^j$ with $\gamma_{n-1,n-1} = 1$. Note that $\gamma_{n-1,j} = S_j(\gamma_2, \gamma_3, \ldots, \gamma_{n-2})$ is a polynomial in $\gamma_2, \gamma_3, \ldots, \gamma_{n-2}$ with non-negative coefficients. On comparing the coefficients of $w^n$ in (25) and using Lemma 1.3 and Theorem 2.6 we have

$$|\delta_n| \leq \frac{2(\beta - 1)}{n} \sum_{j=1}^{n-1} S_j(A_2, A_3, \ldots, A_{n-2}) $$

$$= \frac{2(\beta - 1)}{n} \sum_{j=1}^{n-1} A_{n-1,j} $$

(26)
where $A_{n-1,j} = S_j(A_2, A_3, \ldots, A_n - 2)$ denotes the coefficient of $w^{n-1}$ in the expansion of $F_1(w)^j$ with $A_{n-1,n-1} = 1$ and $F_1$ is given by (22). Corresponding to $F_1$, $F_1(w)/F_1'(w) = w - \sum_{n=2}^{\infty} B_n w^n$ where

$$B_2 := (\beta - 1) \quad \text{and} \quad B_n := \frac{2(\beta - 1)}{n(n - 1)} \prod_{m=0}^{n-3} \left( \frac{2\beta + m(2\beta - 1)}{m + 1} \right) \quad (n > 2).$$

For $f_1$, given by (21), by proceeding as in (20), we have

$$w - \sum_{n=2}^{\infty} B_n w^n = w - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} 2(\beta - 1) A_{n-1,j} w^n.$$

On comparing the coefficients of $w^n$, we obtain

$$B_n = \frac{2(\beta - 1)}{n} \sum_{j=1}^{n-1} A_{n-1,j}. \quad (27)$$

In view of (26) and (27), the desired estimates follow. \qed

In the generalized class $K[A, B] (-1 \leq B \leq 1 < A)$, the technique used in the Theorem 2.4 does not hold true. However, we are able to give the sharp estimation for the initial inverse coefficients for functions in $K[A, B]$.

**Theorem 2.8.** Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in K[A, B] (-1 \leq B \leq 1 < A)$ and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. Then for $n = 2, \ldots, 6$,

$$|\gamma_n| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left( \frac{(A - B) + mA}{m + 1} \right).$$

The result is sharp.

**Proof.** Since $f \in K[A, B], 1 + z f''(z)/f'(z) \prec (1 + Az)/(1 + Bz)$ which is equivalent to $1 + z f''(z)/f'(z) \prec (1 - Az)/(1 - Bz)$. Let $g(z) := z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$ and $p(z) := z g'(z)/g(z) = 1 + b_1 z + b_2 z^2 + \cdots$.

Then $p(z) \prec (1 - Az)/(1 - Bz)$ and for $n > 1$, we have

$$(n - 1) n a_n = \sum_{k=1}^{n-1} (n - k) b_k a_{n-k}. \quad (28)$$

It is easy to observe that if $p \prec \varphi$, then

$$p(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right), \quad p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}. \quad (29)$$

Using (28) and (29) for $\varphi = (1 - Az)/(1 - Bz)$, the coefficients $a_i$ can be expressed in terms of $c_i, A$ and $B$, see (24). In particular, we have

$$a_2 = \frac{1}{4} (A - B) c_1,$$

$$a_3 = \frac{1}{24} (A - B) \left( (A - 2B + 1)c_1^2 - 2c_2 \right),$$

$$a_4 = \frac{1}{192} (A - B) \left( (A - 2B + 1)(A - 3B + 2)c_1^3 - 2(3A - 7B + 4)c_1c_2 + 8c_3 \right),$$

$$a_5 = \frac{1}{1920} (A - B) \left( -4(3A^2 - 17AB + 11A + 23B^2 - 29B + 9)c_1^4 + (A - 2B + 1)(A - 3B + 2)(A - 4B + 3)c_1^4 + 16(2A - 5B + 3)c_1c_3 + 12(A - 3B + 2)c_2^2 - 48c_4 \right).$$
Substituting the expressions of $a_i$ in terms of $c_i$ in the above expressions of $\gamma_i$, we have

$$a_6 = \frac{1}{23040}(A - B)(- (A - 5B + 4)(A - 4B + 3)(A - 3B + 2)(A - 2B + 1)c_1^5 + 4(5A^2 - 50A^2B + 35A^2 + 160AB^2 - 220AB + 75A - 163B^3 + 329B^2 - 219B + 48)c_1^5c_2 - 16(5A^2 - 30AB + 20A + 43B^2 - 56B + 18)c_1^5c_3 + 32(5A - 17B + 12)c_2c_3 - 4(15A^2 - 100AB + 70A + 157B^2 - 214B + 72)c_1c_2^2 + 48(5A - 13B + 8)c_1c_4 - 384c_5).$$

Using power series expansions of $f$ and $f^{-1}$ in the relation $f(f^{-1}(w)) = w$, or

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + \cdots,$$

we obtain

$$\gamma_2 = -a_2,$$
$$\gamma_3 = 2a_2^2 - a_3,$$
$$\gamma_4 = -5a_2^3 + 5a_2a_3 - a_4,$$
$$\gamma_5 = 14a_2^4 - 21a_2^3a_3 + 6a_2a_4 + 3a_3^2 - a_5$$

and

$$\gamma_6 = 7(- 6a_2^5 + 12a_2^3a_3 - 4a_2^2a_4 + a_2(a_5 - 4a_2^3) + a_3a_4) - a_6.$$

Substituting the expressions of $a_i$ in terms of $c_i$ in the above expressions of $\gamma_i$, we have

$$\gamma_2 = \frac{1}{4}(A - B)c_1,$$
$$\gamma_3 = \frac{1}{24}(A - B)((2A - B - 1)c_1^2 + 2c_2),$$
$$\gamma_4 = \frac{1}{192}(A - B)((2A - B - 1)(3A - B - 2)c_1^3 + 2(7A - 3B - 4)c_1c_2 + 8c_3),$$
$$\gamma_5 = \frac{1}{1920}(A - B)(p(A, B) c_1^5c_2 + (2A - B - 1)(3A - B - 2)(4A - B - 3)c_1^4 + 8(11A - 5B - 6)c_1c_3 + 4(7A - B - 6)c_2^2 + 48c_4)$$

and

$$\gamma_6 = \frac{1}{23040}(A - B)(q(A, B) c_1^5c_3 + r(A, B) c_1c_2^2 + 384(2A - B - 1)c_1c_4 + s(A, B) c_1^2c_3 + (2A - B - 1)(3A - B - 2)(4A - B - 3)(5A - B - 4)c_1^2 + 16(25A - B - 24)c_2c_3 + 384c_5)$$

where

$$p(A, B) := 4(23A^2 - 17AB - 29A + 3B^2 + 11B + 9),$$
$$q(A, B) := 8(101A^2 - 81AB - 121A + 16B^2 + 49B + 36),$$
$$r(A, B) := 4(127A^2 - 58AB - 196A + 3B^2 + 52B + 72)$$

and

$$s(A, B) := 4(163A^3 - 160A^2B - 329A^2 + 50AB^2 + 220AB + 219A - 5B^3 - 35B^2 - 75B - 48).$$
Since $-1 \leq B \leq 1 < A$, we can easily see that

$$\frac{\partial p(A,B)}{\partial A} = 4(29(A-1) + 17(A-B)) > 0,$$

$$\frac{\partial q(A,B)}{\partial A} = 8(121(A-1) + 81(A-B)) > 0$$

and

$$\frac{\partial r(A,B)}{\partial A} = 4(196(A-1) + 58(A-B)) > 0.$$

Therefore, $p(A,B) > p(1,B) = 12(1-B)^2 \geq 0$; $q(A,B) > q(1,B) = 128(1-B)^2 \geq 0$ and $r(A,B) > r(1,B) = 12(1-B)^2 \geq 0$. Clearly,

$$\frac{\partial s(A,B)}{\partial A} = 4(489A^2 - 658A - 320AB + 219 + 220B + 50B^2)$$

and

$$\frac{\partial^2 s(A,B)}{\partial A^2} = 4(658(A-1) + 320(A-B)) > 0.$$

Therefore, $\partial s(A,B)/\partial A$ is a strictly increasing function of $A$ and hence $\partial s(A,B)/\partial A > 200(1-B)^2 \geq 0$. Consequently, $s(A,B) > s(1,B) = 20(1-B)^3 \geq 0$. Thus, for $n = 2, \ldots, 6$, $\gamma_n$ are polynomials in $c_i$ ($i = 1, 2, \ldots, 5$) with non-negative coefficients. Since $p_1 \in P$, $|c_i| \leq 2$ ($i = 1, 2, \ldots$) and therefore, the maximum of $|\gamma_n|$ would correspond to $|c_i| = 2$. On simplification, we get the desired estimates. Define a function $f_0 : \mathbb{D} \to \mathbb{C}$ such that

$$f'_0(z) = \begin{cases} (1-Bz)(A-B)/B, & B \neq 0 \\ e^{-A^2}, & B = 0. \end{cases}$$

The result is sharp for the function $f_0$. \qed

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