GRAPH $W^*$-PROBABILITY ON THE FREE GROUP FACTOR
$L(F_N)$

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Abstract. In this paper, we will consider the free probability on the free group factor $L(F_N)$, in terms of Graph $W^*$-probability, where $F_k$ is the free group with $k$-generators. The main result of this paper is to reformulate the moment series and the R-transform of the operator $T_0 = \sum_{j=1}^{N} (x_j)$, where $x_j$'s are free semicircular elements, $j = 1, ..., N$, generating $L(F_N)$, by using the Graph $W^*$-probability technique. To do that, we will use the graph $W^*$-probability technique to find the moments and cumulants of the identically distributed random variable $T_0$. This will be a good example of an application of Graph $W^*$-Probability Theory. We also can see how we can construct the $W^*$-subalgebra which is isomorphic to the free group factor $L(F_N)$ in the graph $W^*$-probability space $(W^*(G), E)$.

In this paper, we will reformulate the moments and cumulants of the so-called generating operator $T_0 = \sum_{j=1}^{N} x_j$, where $x_1$, ..., $x_N$ are free semicircular elements generating the free group factor $L(F_N)$, in terms of the graph $W^*$-probability theory considered in [14], [15], [16] and [17]. Voiculescu showed that the free group factor $L(F_N)$ is generated by $N$-semicircular elements which are free from each other (See [9]). The moments and cumulants of such elements are known but we will recompute them, by using the graph $W^*$-probability technique. We will construct a $D_G$-semicircular subalgebra $S_G$ which is isomorphic to the free group factor $L(F_N)$ embedded in a certain graph $W^*$-algebra $W^*(G)$. And, by constructing an operator $T$ which is identically distributed with $T_0$ in $S_G$, we will recompute the moments and cumulants of $T_0$. This would be an application of graph $W^*$-probability theory. Also, we will embed the free group factor $L(F_N)$ into an arbitrary graph $W^*$-probability space $(W^*(G), E)$, where $G$ is a countable directed graph containing at least one vertex $v_0$ having $N$-basic loops concentrated on $v_0$.

Let $G$ be a countable directed graph and $W^*(G)$, the graph $W^*$-algebra. By defining the diagonal subalgebra $D_G$ and the canonical conditional expectation $E : W^*(G) \to D_G$, we can construct the graph $W^*$-probability space $(W^*(G), E)$ over $D_G$, as a $W^*$-probability space with amalgamation over $D_G$. All elements in $(W^*(G), E)$ are called $D_G$-valued random variables. The $D_G$-freeness is observed in [14] and [15]. The generators $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ if and only if $w_1$ and $w_2$ are diagram-distinct, in the sense that they have different diagram on the graph $G$, graphically. There are plenty of interesting examples of $D_G$-valued

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random variables in this structure (See [15]), including $D_G$-semicircular elements, $D_G$-valued R-diagonal elements and $D_G$-even elements.

In this paper, we will regard the free group factor $L(F_N)$ as an embedded $W^*$-subalgebra of the graph $W^*$-algebra $W^*(G)$, where $G$ is a directed graph with

$$V(G) = \{v\} \quad \text{and} \quad E(G) = \{l_1, \ldots, l_N\},$$

where $l_j = vl_jv$ is a loop-edge, for all $j = 1, \ldots, N$. Then the generating operator $T_0$ of $L(F_N)$ is identically distributed with the operator $T$ in $W^*(G)$ such that

$$T = \sum_{j=1}^{N} \frac{1}{\sqrt{2}} \left( L_{l_j} + L_{l_j}^* \right).$$

Notice that, since $l_j$’s are diagram-distinct, in the sense that they have mutually different diagrams in the graph $G$, $L_{l_j} + L_{l_j}^*$’s are free from each other over $D_G = \mathbb{C}$ in $(W^*(G), E)$, for $j = 1, \ldots, N$. Furthermore, by [14], we know that the summands $L_{l_j} + L_{l_j}^*$’s are all $D_G = \mathbb{C}$-semicircular. So, we can get the cumulants of $T$ somewhat easily. And the moments and cumulants gotten from this are same as those of the generating operator $T_0$ in the free group factor $L(F_N)$.

Recall that studying moment series of a random variable is studying distribution of the random variable. By the moment series of the random variable, we can get the algebraic and combinatorial information about the distribution of the random variable. Also, alternatively, the R-transforms of random variables contains algebraic and combinatorial information about distributions of the random variables. So, to study moment series and R-transforms of random variables is very important to study distributions of random variables. Moreover, to study R-transform theory allows us to understand the freeness of random variables. This paper deals with the operator

$$T_0 = \sum_{j=1}^{N} x_j$$

of the free group factor $L(F_N)$, where $x_1, \ldots, x_N$ are semicircular elements, which are free from each other, generating $L(F_N)$. Notice that, by Voiculescu, we can regard the free group factor $L(F_N)$ as the von Neumann algebra $vN \left( \{x_j\}_{j=1}^{N} \right)$. We will consider this free probabilistic data about $T_0$, by using the graph $W^*$-probability technique.

In Chapter 1, we will review the graph $W^*$-probability theory. In Chapter 2, we will construct the graph $W^*$-probability space induced by the one-vertex-$N$-loop-edge graph and define a $W^*$-subalgebra of the graph $W^*$-algebra, which is isomorphic to $L(F_N)$, in the sense of [9]. In Chapter 3, we will re-compute the moments and cumulants of the generating operator $T_0$ of $L(F_N)$, by using the Graph $W^*$-probability technique. In Chapter 4, we will observe the embeddings of $L(F_N)$ into $W^*(G)$, where $G$ is an arbitrary countable directed graph containing a vertex $v_0$ having $N$-basic loops concentrate on it.
Let \( G \) be a countable directed graph and let \( \mathbb{F}^+(G) \) be the free semigroupoid of \( G \), i.e., the set \( \mathbb{F}^+(G) \) is the collection of all vertices as units and all admissible finite paths of \( G \). Let \( w \) be a finite path with its source \( s(w) = x \) and its range \( r(w) = y \), where \( x, y \in V(G) \). Then sometimes we will denote \( w \) by \( w = xwy \) to express the source and the range of \( w \).

We can define the graph Hilbert space \( H_G \) by the Hilbert space \( l^2(\mathbb{F}^+(G)) \) generated by the elements in the free semigroupoid \( \mathbb{F}^+(G) \). i.e., this Hilbert space has its Hilbert basis \( B = \{ \xi_w : w \in \mathbb{F}^+(G) \} \). Suppose that \( w = e_1 ... e_k \in FP(G) \) is a finite path with \( e_1, ..., e_k \in E(G) \). Then we can regard \( \xi_w \) as \( \xi_{e_1} \otimes ... \otimes \xi_{e_k} \). So, in [10], Kribs and Power called this graph Hilbert space the generalized Fock space. Throughout this paper, we will call \( H_G \) the graph Hilbert space to emphasize that this Hilbert space is induced by the graph.

Define the creation operator \( L_w \), for \( w \in \mathbb{F}^+(G) \), by the multiplication operator \( \xi_w \) on \( H_G \). Then the creation operator \( L \) on \( H_G \) satisfies that

(i) \( L_w = L_{xwy} = L_x L_w L_y \), for \( w = xwy \) with \( x, y \in V(G) \).

(ii) \( L_{w_1} L_{w_2} = \begin{cases} L_{w_1 w_2} & \text{if } w_1 w_2 \in \mathbb{F}^+(G) \\ 0 & \text{if } w_1 w_2 \notin \mathbb{F}^+(G), \end{cases} \)

for all \( w_1, w_2 \in \mathbb{F}^+(G) \).

Now, define the annihilation operator \( L_w^* \), for \( w \in \mathbb{F}^+(G) \) by

\[
L_{w}^* \xi_{w'} \overset{def}{=} \begin{cases} \xi_h & \text{if } w' = wh \in \mathbb{F}^+(G) \\ 0 & \text{otherwise}. \end{cases}
\]

The above definition is gotten by the following observation:

\[
< L_w \xi_h, \xi_{w} > = < \xi_{wh}, \xi_{wh} >
= 1 = < \xi_h, \xi_h >
= < \xi_h, L_w \xi_{wh} >.
\]

where \( <,> \) is the inner product on the graph Hilbert space \( H_G \). Of course, in the above formula we need the admissibility of \( w \) and \( h \) in \( \mathbb{F}^+(G) \). However, even though \( w \) and \( h \) are not admissible (i.e., \( wh \notin \mathbb{F}^+(G) \)), by the definition of \( L_w^* \), we have that

\[
< L_w \xi_h, \xi_h > = < 0, \xi_h >
= 0 = < \xi_h, 0 >
= < \xi_h, L_w^* \xi_h >.
\]
Notice that the creation operator $L$ and the annihilation operator $L^*$ satisfy that

\[(1.1) \quad L^* w L = L_y \quad \text{and} \quad L w L^* w = L_x, \quad \text{for all } w = xwy \in F^+(G), \]

where $x, y \in V(G)$. Remark that if we consider the von Neumann algebra $W^*\{L_w\}$ generated by $L_w$ and $L^*_w$ in $B(H_G)$, then the projections $L_y$ and $L_x$ are Murray-von Neumann equivalent, because there exists a partial isometry $L_w$ satisfying the relation (1.1). Indeed, if $w = xwy$ in $F^+(G)$, with $x, y \in V(G)$, then under the weak topology we have that

\[(1.2) \quad L_w L^* w L_w = L_w \quad \text{and} \quad L^*_w L w L^*_w = L^*_w. \]

So, the creation operator $L_w$ is a partial isometry in $W^*\{L_w\}$ in $B(H_G)$. Assume now that $v \in V(G)$. Then we can regard $v$ as $v = vvv$. So,

\[(1.3) \quad L^*_v L_v = L_v = L_v L^*_v = L^*_v. \]

This relation shows that $L_v$ is a projection in $B(H_G)$ for all $v \in V(G)$.

Define the graph $W^*$-algebra $W^*(G)$ by

\[ W^*(G) \overset{\text{def}}{=} \mathbb{C}[\{L_w, L^*_w : w \in F^+(G)\}]^w. \]

Then all generators are either partial isometries or projections, by (1.2) and (1.3). So, this graph $W^*$-algebra contains a rich structure, as a von Neumann algebra. (This construction can be the generalization of that of group von Neumann algebra.) Naturally, we can define a von Neumann subalgebra $D_G \subset W^*(G)$ generated by all projections $L_v$, $v \in V(G)$. i.e.

\[ D_G \overset{\text{def}}{=} W^*(\{L_v : v \in V(G)\}). \]

We call this subalgebra the diagonal subalgebra of $W^*(G)$. Notice that $D_G = \Delta_{|G|} \subset M_{|G|}(\mathbb{C})$, where $\Delta_{|G|}$ is the subalgebra of $M_{|G|}(\mathbb{C})$ generated by all diagonal matrices. Also, notice that $1_{D_G} = \sum_{v \in V(G)} L_v = 1_{W^*(G)}$.

If $a \in W^*(G)$ is an operator, then it has the following decomposition which is called the Fourier expansion of $a$:

\[ a = \sum_{w \in F^+(G) : a} p^{(u_w)}_{w} L_{w}, \]

where $p^{(u_w)}_{w} \in \mathbb{C}$, $u_w \in \{1, *\}$, and $F^+(G : a)$ is the support of $a$ defined by

\[ F^+(G : a) = \{ w \in F^+(G) : p^{(u_w)}_{w} \neq 0 \}. \]
Remark that the free semigroupoid $\mathbb{F}^+(G)$ has its partition \( \{ V(G), FP(G) \} \), as a set. i.e.,

\[
\mathbb{F}^+(G) = V(G) \cup FP(G) \quad \text{and} \quad V(G) \cap FP(G) = \emptyset.
\]

So, the support of \( a \) is also partitioned by

\[
\mathbb{F}^+(G : a) = V(G : a) \cup FP(G : a),
\]

where

\[
V(G : a) \overset{\text{def}}{=} V(G) \cap \mathbb{F}^+(G : a)
\]

and

\[
FP(G : a) \overset{\text{def}}{=} FP(G) \cap \mathbb{F}^+(G : a).
\]

So, the above Fourier expansion (1.4) of the random variable \( a \) can be re-expressed by

\[
(1.5) \quad a = \sum_{v \in V(G : a)} p_v L_v + \sum_{w \in FP(G : a), \, u, w \in \{1, \ast\}} p_w^{(u \ast)} L_{uw}.
\]

We can easily see that if \( V(G : a) \neq \emptyset \), then \( \sum_{v \in V(G : a)} p_v L_v \) is contained in the diagonal subalgebra \( D_G \). Also, if \( V(G : a) = \emptyset \), then \( \sum_{v \in V(G : a)} p_v L_v = 0_{D_G} \). So, we can define the following canonical conditional expectation \( E : W^*(G) \rightarrow D_G \) by

\[
(1.6) \quad E(a) \overset{\text{def}}{=} \sum_{v \in V(G : a)} p_v L_v,
\]

for all \( a \in W^*(G) \) having its Fourier expansion (1.5). Indeed, \( E \) is a well-determined conditional expectation. Moreover it is faithful, in the sense that if \( E(a^* a) = 0_{D_G} \), then \( a = 0_{D_G} \), for \( a \in W^*(G) \).

**Definition 1.1.** We say that the algebraic pair \( (W^*(G), E) \) is the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \).

We will define the following free probability data of \( D_G \)-valued random variables in \( (W^*(G), E) \).

**Definition 1.2.** Let \( W^*(G) \) be the graph \( W^* \)-algebra induced by \( G \) and let \( a \in W^*(G) \). Define the \( n \)-th (\( D_G \)-valued) moment of \( a \) by

\[
E( d_1 a d_2 a \ldots d_n a), \quad \text{for all} \ n \in \mathbb{N},
\]

where \( d_1, \ldots, d_n \in D_G \). Also, define the \( n \)-th (\( D_G \)-valued) cumulant of \( a \) by
Theorem 1.1. (See [14]) Let $w$-valued random variables, where $c$ certain complex number and the mixed $n$-th trivial cumulant of $a$ in the sense of Speicher. We define the $E$, $a$-th cumulant multiplicative bimodule map induced by the conditional expectation $E$, in the sense of Speicher. We define the $n$-th trivial moment of $a$ and the $n$-th trivial cumulant of $a$ by

$$E(a^n) \quad \text{and} \quad k_n\left(a, a, \ldots, a_{\text{n-times}}\right) = C^{(n)} \left(a \otimes a \otimes \ldots \otimes a\right),$$

respectively, for all $n \in \mathbb{N}$.

In [14], we showed that

**Theorem 1.1.** (See [14]) Let $n \in \mathbb{N}$ and let $L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \in \left(W^*(G), E\right)$ be $D_G$-valued random variables, where $w_1, \ldots, w_n \in FP(G)$ and $u_j \in \{1, *\}$, $j = 1, \ldots, n$. Then

$$k_n\left(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}\right) = \mu^{u_1, \ldots, u_n}_{w_1, \ldots, w_n} \cdot E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}),$$

where $\mu^{u_1, \ldots, u_n}_{w_1, \ldots, w_n} = \sum_{\pi \in C_{w_1}^{u_1, \ldots, u_n}} \mu(\pi, 1_n)$. Here, $C_{w_1}^{u_1, \ldots, u_n}$ is a subset of $NC(n)$ consisting of all partitions $\pi$ in $NC(n)$, satisfying that

$$E_{\pi}\left(L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n}\right) = E(L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}) \neq 0_{DG}.$$

The above theorem show us that, different from the general case, the mixed $n$-th $D_G$-cumulants of operator-valued random variables in $(W^*(G), E)$ is the product of a certain complex number and the mixed $n$-th $D_G$-moments of the operators. Now, we consider the $D_G$-valued freeness of given two random variables in $(W^*(G), E)$. We will characterize the $D_G$-freeness of $D_G$-valued random variables $L_{w_1}$ and $L_{w_2}$, where $w_1 \neq w_2 \in FP(G)$. And then we will observe the $D_G$-freeness of arbitrary two $D_G$-valued random variables $a_1$ and $a_2$ in terms of their supports.

**Definition 1.3.** Let $w_1$ and $w_2$ be elements in the free semigroupoid $F^+(G)$. We say that they are diagram-distinct if they have the different diagrams on $G$. Also, we will say that the subsets $X_1$ and $X_2$ of $F^+(G)$ are diagram-distinct if $w_1$ and $w_2$ are diagram-distinct, for all pair $(w_1, w_2)$ in $X_1 \times X_2$.

By the previous theorem, we can get the following theorem which shows that the diagram-distinctness characterize the $D_G$-freeness of generators of $W^*(G)$;
Theorem 1.2. (See [14]) Let $w_1, w_2 \in FP(G)$ be finite paths. The $D_G$-valued random variables $L_{w_1}$ and $L_{w_2}$ in $(W^*(G), E)$ are free over $D_G$ if and only if $w_1$ and $w_2$ are diagram-distinct. □

Corollary 1.3. (See [14]) Let $a, b \in (W^*(G), E)$ be $D_G$-valued random variables with their supports $FP(G : a)$ and $FP(G : b)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^*(G), E)$ if $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct. □

In [15], we observed certain kind of $D_G$-valued random variables in $(W^*(G), E)$. One of the most interesting elements are $D_G$-semicircular elements.

Proposition 1.4. (See [15]) Let $l \in \text{loop}(G)$ be a loop. Then the $D_G$-valued random variable $L_l + L_l^*$ is $D_G$-semicircular. □

2. One-Vertex Graph $W^*$-Probability Spaces

Throughout this chapter, fix $N \in \mathbb{N}$. Suppose that $G$ be a finite directed graph with only one vertex. Let

$$V(G) = \{v\} \quad \text{and} \quad E(G) = \{l_1, \ldots, l_N\};$$

where $l_j = vl_jv$ is a loop, for all $j = 1, \ldots, N$. Notice that, in this case, the diagonal subalgebra $D_G$ is isomorphic to $C$, i.e,

$$D_G = \overline{C[L_v]}^{aw} = C.$$ 

Thus the canonical conditional expectation $E : W^*(G) \to D_G$ is a linear map and hence the graph $W^*$-probability space $(W^*(G), E)$ over its diagonal subalgebra $D_G$ is just a (scalar-valued) $W^*$-probability space. We will denote such linear map $E$ by $tr$ and, by $(W^*(G), tr)$, we will denote the corresponding graph $W^*$-probability space $(W^*(G), E)$. Remark that the linear functional $tr = E$ is a faithful trace on $W^*(G)$. Indeed, assume that $tr(xx^*) = 0_{D_G} = 0$. Then $x = 0$. Also, indeed, $tr$ is a trace. Thus, our graph $W^*$-probability space $(W^*(G), tr)$ is a tracial $W^*$-probability space.

In this setting, the projection $L_v$ is the identity $1_{W^*(G)}$ of $W^*(G)$ and the partial isometries $L_{l_j}$, $j = 1, \ldots, N$, are unitaries in this graph $W^*$-algebra, $W^*(G)$, since

$$L_{l_j}^*L_{l_j} = L_v = 1_{W^*(G)} = L_{l_j}L_{l_j}^*,$$
for all \( j = 1, \ldots, N \), and hence
\[
L^*_{i_j} = L^{-1}_{i_j}, \quad \text{for all } j = 1, \ldots, N.
\]

Therefore, the graph \( W^* \)-algebra \( W^*(G) \) can be understood as a \( W^* \)-algebra generated by \( N \)-unitaries. By [15], we have \( D_G \)-semicircular elements \( L_{i_j} + L^*_{i_j}, \quad j = 1, \ldots, N \). Since \( D_G = \mathbb{C} \), in our case, they are indeed (scalar-valued) semicircular elements in \( (W^*(G), \text{tr}) \). So, we can consider the \( W^* \)-subalgebra \( L(G) \) of \( W^*(G) \) generated by semicircular elements \( \frac{1}{\sqrt{2}} \left( L_{i_j} + L^*_{i_j} \right) \), \( j = 1, \ldots, N \).

**Definition 2.1.** Let \( G \) be the given one-vertex-\( N \)-loop-edges directed graph. Define a \( W^* \)-subalgebra \( L(G) \) of the graph \( W^* \)-algebra \( W^*(G) \) by
\[
L(G) \overset{\text{def}}{=} \mathbb{C}[\{ \frac{1}{\sqrt{2}} \left( L_{i_j} + L^*_{i_j} \right) : j = 1, \ldots, N \}].
\]

Let \( (W^*(G), \text{tr}) \) be the graph \( W^* \)-probability space (over \( D_G = \mathbb{C} \)), with its faithful trace \( \text{tr} = \text{tr}|_{L(G)} \). We will call the \( W^* \)-probability space \( (L(G), \text{tr}) \), the semicircular algebra.

By [9], we can see that \( (L(G), \text{tr}) \) is isomorphic to \( (L(F_N), \tau) \). Recall that \( L(F_N) \) is the free group factor induced by the free group \( F_N \) with \( N \)-generators. i.e, \( F_N = \langle g_1, \ldots, g_N \rangle \) and \( L(F_N) = \mathbb{C}[F_N]^{\text{op}} \). So, if \( x \in L(F_N) \), then \( x \) can be expressed as \( x = \sum_{g \in F_N} \alpha_g g \). We can define the trace \( \tau \) on the free group factor \( L(F_N) \) by
\[
\tau : L(F_N) \rightarrow \mathbb{C}, \quad \tau \left( \sum_{g \in F_N} \alpha_g g \right) = \alpha_e,
\]
where \( e \) is the group identity of \( F_N \). Voiculescu showed that there exists a semicircular system \( \{ x_j : j = 1, \ldots, N \} \), in the sense of the set consisting of mutually free semicircular elements with covariance 1, such that
\[
L(F_N) = vN(\{ x_j : j = 1, \ldots, N \}),
\]
where \( vN(S) \) means the von Neumann algebra generated by the set \( S \). Define an operator
\[
T_0 = \sum_{j=1}^{N} x_j,
\]
where \( x_j \)'s are semicircular elements generating the free group factor \( L(F_N) \). We will call this operator \( T_0 \) the generating operator of \( L(F_N) \).
We can show that the semicircular algebra \((L(G), tr)\) has the same free probabilistic structure with \((L(F_N), \tau)\), i.e., there exists a \(W^*-\)algebra isomorphism between \(L(G)\) and \(L(F_N)\), which preserves the moments of all generators (See [9]). It is easy to do that by defining the generator-preserving linear map between \(L(G)\) and \(L(F_N)\), by regarding \(L(F_N)\) as the von Neumann algebra \(vN\) of \(\{x_j : j = 1, ..., N\}\).

**Proposition 2.1.** Let \(G\) be the given one-vertex-\(N\)-loop-edges directed graph. The semicircular algebra \((L(G), tr)\), generated by the semicircular system

\[
\left\{ \frac{1}{\sqrt{2}} \left( L_{ij} + L_{ij}^* \right) : j = 1, ..., N \right\}
\]

is isomorphic to \((L(F_N), \tau)\), in the sense of [9]. □

Now, we will consider the generating operator contained in \((L(G), tr)\).

**Definition 2.2.** Let \(T \in (L(G), tr)\) be a random variable defined by

\[
T = \sum_{j=1}^{N} \left( \frac{1}{\sqrt{2}} \left( L_{ij} + L_{ij}^* \right) \right),
\]

where \(\left\{ \frac{1}{\sqrt{2}} \left( L_{ij} + L_{ij}^* \right) : j = 1, ..., N \right\}\) is the generator set of \(L(G)\). We will call \(T\) the generating operator of \(L(G)\).

Recall the generating operator \(T_0\) of the free group factor \(L(F_N)\), \(T_0 = \sum_{j=1}^{N} x_j\). By [9] and by the previous proposition, we have the following result;

**Proposition 2.2.** Let \(T_0 = \sum_{j=1}^{N} x_j\) be the generating operator of \((L(F_N), \tau)\) and let \(T = \sum_{j=1}^{N} \left( L_{ij} + L_{ij}^* \right)\) be the generating operator of \((L(G), tr)\), where \(G\) is the given one-vertex-\(N\)-loop-edge directed graph. Then

\[
\tau(T_0^n) = tr(T^n)
\]

and hence

\[
k_\tau^n(T_0, ..., T_0) = k_n(T, .., T),
\]

for all \(n \in \mathbb{N}\), where \(k_\tau^n(\ldots)\) and \(k_n(\ldots)\) are cumulants with respect to the traces \(\tau\) and \(tr\), respectively. In other words, the operators \(T_0\) and \(T\) are identically distributed. □

Now, let \((A, \varphi)\) be a \(W^*-\)probability space and let \(a \in (A, \varphi)\) be a random variable. Then we can define the \(n\)-th moments and the \(n\)-th cumulants of \(a\) by

\[
\varphi(a^n) \quad \text{and} \quad k_n^\varphi(a, ..., a),
\]
for all $n \in \mathbb{N}$, where $k_n^\varphi(\ldots)$ is the cumulant function induced by $\varphi$. Defin $\Theta_1$ as a set of all formal power series in the indeterminate $z$, without the constant terms, in $\mathbb{C}[[z]]$, where $\mathbb{C}[[z]]$ is the set of all formal power series. For the given random variable $a \in (A, \varphi)$, we can define the following two elements in $\Theta_1$:

$$M_a(z) = \sum_{n=1}^\infty \varphi(a^n) z^n$$
and

$$R_a(z) = \sum_{n=1}^\infty k_n^\varphi(a, \ldots, a) z^n,$$

called the moment series of $a$ and the R-transform of $a$, respectively. By the previous proposition, we can get that:

**Corollary 2.3.** Let $T_0$ and $T$ be given as before. Then

$$M_{T_0}(z) = M_T(z) \quad \text{and} \quad R_{T_0}(z) = R_T(z),$$
in $\Theta_1$. \(\Box\)

Again, let $(A_i, \varphi_i)$ be a $W^*$-probability space, for $i = 1, 2$, and let $a_i \in (A_i, \varphi_i)$ be random variables, for $i = 1, 2$. We say that the random variables $a_1$ and $a_2$ are identically distributed if their R-transforms are same in $\Theta_1$, i.e, the random variables $a_1$ and $a_2$ are identically distributed if

$$R_{a_1}(z) = R_{a_2}(z) \quad \text{in} \quad \Theta_1.$$

Notice that, by the Möbius inversion, if $a_1$ and $a_2$ are identically distributed, then

$$M_{a_1}(z) = M_{a_2}(z) \quad \text{in} \quad \Theta_1.$$

The above corollary says that, as random variables, the generating operators $T_0 \in (L(F_N), \tau)$ and $T \in (L(G), tr)$ are identically distributed. Therefore, by computing the moment series or R-transform of $T_0$, we can get those of $T$. So, by using the graph $W^*$-probability technique, we can get the moment series and the R-transform of $T_0 \in (L(F_N), \tau)$.

### 3. Moment and Cumulants of $T_0$

Throughout this chapter, fix $N \in \mathbb{N}$ and let $G$ be a one-vertex-$N$-loop-edge directed graph with
\[ V(G) = \{ v \} \quad \text{and} \quad E(G) = \{ l_j = vl_jv : j = 1, \ldots, N \}. \]

Recall that \( L_v = 1_{W^*(G)} = 1_{L(G)} \) and \( L_{l_j}'s \) are unitaries in \( W^*(G) \), for all \( j = 1, \ldots, N \). Also, let \((W^*(G), tr)\) be the graph \( W^* \)-probability space (over its diagonal subalgebra \( D_G = \mathbb{C} \)), with its faithful trace on \( W^*(G) \). For the random variables \( L_{l_1} + L_{l_1}^*, \ldots, L_{l_N} + L_{l_N}^* \), we can form the semicircular system and then we can construct the semicircular algebra \((L(G), tr)\), defined by

\[
L(G) \overset{\text{def}}{=} \mathbb{C}[\{ \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) : j = 1, \ldots, N \}] \quad \text{and} \quad tr = tr \mid_{L(G)}.
\]

Again, remark that \((L(G), tr) = (L(F_N), \tau)\), where \((L(F_N), \tau)\) is the free group factor induced by the free group \( F_N \), with \( N \)-generators. Notice that, by regarding \( L(F_N) \) as the von Neumann algebra \( v\mathbb{C}[\{ x_j : j = 1, \ldots, N \}] \), generated by the semicircular system \( \{ x_1, \ldots, x_N \} \), we can get the above equality, by Voiculescu. In this chapter, we will compute the moments and cumulants of the generating operator \( T = \sum_{j=1}^{N} \left( L_{l_j} + L_{l_j}^* \right) \) of \( L(G) \).

Since the generating operator \( T_0 = \sum_{j=1}^{N} x_j \) of \((L(F_N), \tau)\) and the generating operator \( T \) of \((L(G), tr)\) are identically distributed, the computations for \( T \) will be the reformulation of the moments and cumulants of \( T_0 \). In fact, the moments and cumulants of such element \( T_0 \) is solved in various articles. However, in this section, we will provides the graph \( W^* \)-probability approach.

**Theorem 3.1.** Let \( G \) be the given one-vertex-\( N \)-loop-edge directed graph and let \((L(G), tr)\) be the semicircular algebra generated by semicircular elements \( \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) \), \( j = 1, \ldots, N \). If \( T = \sum_{j=1}^{N} \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) \) is the generating operator of \( L(G) \), then it has all vanishing odd moments and cumulants and

\[
\begin{align*}
(1) \quad & tr \left( T^n \right) = c_{\frac{n}{2}} \cdot N^{\frac{n}{2}}, \\
(2) \quad & k_n^{tr} (T, \ldots, T) = \begin{cases} \\
N & \text{if } n = 2 \\
0 & \text{otherwise}, \\
\end{cases}
\end{align*}
\]

for all \( n \in 2\mathbb{N} \), where \( c_k = \frac{1}{k+1} \left( \begin{array}{c} 2k \ \\
2 \end{array} \right) \) is the \( k \)-th Catalan number.
Proof. Fix \( n \in \mathbb{N} \). If \( n \) is odd, then we have the vanishing moments of \( T \), because of the \(*\)-axis-property. Thus all odd cumulants of \( T \) also vanish. We will prove (2), first.

(2) Notice that \( L_{l_1} + L_{l_1}^* \), ..., \( L_{l_N} + L_{l_N}^* \) are free from each other in \((L(G), tr)\), by the diagram-distinctness of \( l_1, \ldots, l_N \). So, we have that

\[
k_n^{tr} (T, \ldots, T) = k_n^{tr} \left( \sum_{j=1}^{N} \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) \right)
= \sum_{j=1}^{N} k_n^{tr} \left( \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) \right)
\]

by the mutually freeness of \( L_{l_1} + L_{l_1}^* \), ..., \( L_{l_N} + L_{l_N}^* \)

\[
= \sum_{j=1}^{N} \sum_{(u_1, \ldots, u_n) \in \{1, \ast\}^n} k_n^{tr} \left( \frac{1}{\sqrt{2}} L_{l_j}^{u_1}, \ldots, \frac{1}{\sqrt{2}} L_{l_j}^{u_n} \right)
\]

by the semicircularity of \( L_{l_j} + L_{l_j}^* \), for all \( j = 1, \ldots, N \)

\[
= \begin{cases} 
\sum_{j=1}^{N} \frac{1}{2} (2L_v) & \text{if } n = 2 \\
0 & \text{otherwise}
\end{cases}
\]

by Section 2.5

\[
= \begin{cases} 
\sum_{j=1}^{N} 1 = N & \text{if } n = 2 \\
0 & \text{otherwise},
\end{cases}
\]

since \( L_v = 1_{L(G)} = 1 \in \mathbb{C} \).

(1) Now, remark that the generating operator \( T \) is semicircular, by (2). Fix \( n \in 2\mathbb{N} \). Then we have that

\[
tr (T^n) = \sum_{\pi \in NC(n)} k_\pi (T, \ldots, T)
\]

by the Möbius inversion

\[
= \sum_{\pi \in NC_2(n)} k_\pi (T, \ldots, T)
\]
by the semicircularity of \( T \), where

\[
NC_2(n) = \{ \pi \in NC(n) : V \in \pi \Rightarrow |V| = 2 \},
\]

and then

\[
\begin{align*}
\sum_{\pi \in NC_2(n)} \left( \prod_{V \in \pi} k_V(T, \ldots, T) \right) &= \sum_{\pi \in NC_2(n)} (k_2(T, T))^{||\pi||} = \sum_{\pi \in NC_2(n)} N^{||\pi||}
\end{align*}
\]

since \( k_2(T, T) = N \), by (2)

\[
\sum_{\pi \in NC_2(n)} N^{\frac{\pi}{2}} = |NC_2(n)| \cdot N^{\frac{\pi}{2}} = c_n \cdot N^{\frac{\pi}{2}},
\]

since \( |NC_2(n)| = |NC(\frac{\pi}{2})| = c_n \), where \( c_k \) is the \( k \)-th Catalan number. \( \square \)

By the previous theorem we can get that:

**Corollary 3.2.** Let \( T \) be the generating operator of the semicircular algebra \((L(G), tr)\), where \( G \) is the given one-vertex-N-loop-edge directed graph. Then the moment series \( MT(z) \) of \( T \) and the R-transform \( RT(z) \) of \( T \) are

\[
MT(z) = \sum_{n=1}^{\infty} \left( c_n \cdot N^{\frac{\pi}{2}} \right) z^n
\]

and

\[
RT(z) = N \cdot z^2,
\]

in \( \Theta_1 \). \( \square \)

The above corollary shows that the generating operator \( T_0 \) of the free group factor \( L(F_N) \) satisfies that

\[
MT_0(z) = \sum_{n=1}^{\infty} \left( c_n \cdot N^{\frac{\pi}{2}} \right) z^n
\]

and

\[
RT_0(z) = N \cdot z^2,
\]

in \( \mathbb{C}[[z]] \), too.

4. Embedding \( L(F_N) \) into \( W^*(G) \)
In this chapter, we will consider the embedding of the free group factor $L(F_N)$ in the graph $W^*$-probability space $(W^*(G), E)$, where $G$ is an arbitrary countable directed graph having at least one vertex with $N$-diagram-distinct loops. This is already observed in [14]. Throughout this chapter, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Also, we will assume that there exists a vertex $v_0 \in V(G)$ such that there is a nonempty set consisting of basic loops, $\text{Loop}_{v_0}(G) = \{ l \in \text{Loop}(G) : l = v_0lv_0 \}$ and $|\text{Loop}_{v_0}(G)| = N$.

Without loss of generality, we can write $\text{Loop}_{v_0}(G) = \{ l_1, ..., l_N \}$. Notice that $l_1, ..., l_N$ are distinct basic loops. So, we can construct the $D_G$-semicircular system, $S \overset{\text{def}}{=} \{ \frac{1}{\sqrt{2}} (L_l + L_l^*) : l \in \text{Loop}_{v_0}(G) \}$.

i.e, the set $S$ is consist of mutually $D_G$-free $D_G$-semicircular elements in $(W^*(G), E)$. Now, define the (scalar-valued) semicircular subalgebra $L(S)$ by

$$L(S) = \mathbb{C}[S]^{aw}.$$ 

Notice that this subalgebra $L(S)$ is slightly different from those of [16]. In [16], we defined the $D_G$-semicircular subalgebra $L_{D_G}(S)$ by

$$L_{D_G}(S) = \overline{D_G[S]}^{aw}.$$ 

We can see that

$$(L_{D_G}(S), E) = (L(S), E|_{L(S)}) \otimes (D_G, 1),$$

where $1$ is the identity map on $D_G$. More generally, we have that;

**Theorem 4.1.** Let

$$\mathcal{L}_N = \left\{ \frac{1}{\sqrt{2}} (L_{l_j} + L_{l_j}^*) : \begin{array}{c} l_j = v_{j}l_jv_{j}, j = 1, ..., N \\ l_j's \ are \ mutually \ diagram-distinct \end{array} \right\}$$

be a $D_G$-valued semicircular system in $W^*(G)$. Then

$$(v_N(\mathcal{L}_N, D_G), E) = (v_N(\mathcal{L}_N, D_N), E) \otimes (D_G, 1),$$
where $vN(S)$ is the von Neumann algebra generated by the set $S$ and $1$ is the identity map on $D_G$, and

$$D_N = \overline{\mathbb{C}[\{L_{v_j} : l_j = v_j l_j v_j, \quad j = 1, \ldots, N\}]}^\omega.$$  

**Proof.** Let $L_N$ be the collection of $N$-$D_G$-semicircular elements which are mutually free over $D_G$, as follows;

$$L_N = \left\{ \frac{1}{\sqrt{2}} \left( L_{l_j} + L_{l_j}^* \right) : \quad l_j = v_j l_j v_j, j = 1, \ldots, N \text{ \ where } l_j \text{ are mutually diagram-distinct } \right\}.$$  

The $L_N$ is a $D_G$-semicircular system. As $W^*$-algebras,

$$vN(L_N, D_G) \simeq vN(L_N, D_N) \otimes D_G,$$

where $D_N = \overline{\mathbb{C}[\{L_{v_j} : l_j = v_j l_j v_j\}]}^\omega$. Indeed, without loss of generality, take $a \in vN(L_N, D_G)$ by

$$a = d_1 a_{i_1}^{k_1} d_2 a_{i_2}^{k_2} \cdots d_n a_{i_n}^{k_n} \text{ and } a_{l_j} = \frac{1}{\sqrt{2}} \left( L_{l_j} + L_{l_j}^* \right)$$

where $d_1, \ldots, d_n \in D_G, k_1, \ldots, k_n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$, $n \in \mathbb{N}$. Observe that, for any $j \in \{1, \ldots, N\}$, we have that

$$a_{l_j}^k = \left( L_{l_j} + L_{l_j}^* \right)^k = L_{l_j}^k + L_{l_j}^{*k} + Q \left( L_{l_j}, L_{l_j}^* \right),$$

where $Q \in \mathbb{C}[z_1, z_2]$. Also, observe that $L_{l_j}^{k_1} L_{l_j}^{k_2}$, for any $k_1, k_2 \in \mathbb{N}$, satisfies that

$$L_{l_j}^{k_1} L_{l_j}^{k_2} = L_{l_j}^{k_1} L_{l_j}^{*k_2} = \left\{ \begin{array}{ll} 0 \quad \text{if } k_1 > k_2 \\
L_{l_j}^{k_1} L_{l_j}^{k_2} \quad \text{if } k_1 < k_2 \end{array} \right. \quad \text{if } k_1 = k_2,$$

and similarly,

$$L_{l_j}^{k_1} L_{l_j}^{k_2} = L_{l_j}^{*k_1} L_{l_j}^{k_2} = \left\{ \begin{array}{ll} L_{l_j}^{k_1} L_{l_j}^{k_2} \quad \text{if } k_1 > k_2 \\
L_{l_j}^{k_1} L_{l_j}^{k_2} \quad \text{if } k_1 < k_2 \\
L_{v_j} \quad \text{if } k_1 = k_2. \end{array} \right. \quad \text{if } k_1 = k_2.$$

So,

$$Q(L_{l_j}, L_{l_j}^*) = L_{v_j} \left( Q(L_{l_j}, L_{l_j}^*) \right) L_{v_j},$$

for all $j = 1, \ldots, N$. Thus

$$a_{l_j}^k = L_{v_j} a_{l_j}^k L_{v_j}, \text{ for all } j = 1, \ldots, N.$$  

(1.1)
Now, consider that 

\[ d_j = d_j^N + d_j', \quad \forall j = 1, \ldots, N. \]

where \( d_j^N = \sum_{j=1}^{N} L_{v_j}d_jL_{v_j} \) and \( d_j' = d_j - d_j^N \) in \( D_G \). So, we can rewrite that

\[
a = (d_1^N + d_1') a_{i_1}^{k_1} (d_2^N + d_2') a_{i_2}^{k_2} \ldots (d_n^N + d_n') a_{i_n}^{k_n}
\]

\[
= d_1^N a_{i_1}^{k_1} d_2^N a_{i_2}^{k_2} \ldots d_n^N a_{i_n}^{k_n} + d_1' a_{i_1}^{k_1} d_2' a_{i_2}^{k_2} \ldots d_n' a_{i_n}^{k_n}
\]

\[
= d_1^N a_{i_1}^{k_1} d_2^N a_{i_2}^{k_2} \ldots d_n^N a_{i_n}^{k_n},
\]

by (1.1). This shows that \( a = a \otimes 1 \in vN(L_N, D_N) \otimes 1 \) and

\[
E(a) = E_{D_N}^{D_G} \circ E(a) = E_N(a) = E_N \otimes 1(a \otimes 1).
\]

Trivially, if \( a \in D_G \subset L_{DG}(S) \), then \( a = 1 \otimes a \in 1 \otimes D_G \). Furthermore, if \( a \in D_G \), then

\[
E(a) = a = a \otimes 1 = E_N \otimes 1(1 \otimes a).
\]

By the previous theorem, as a corollary, we can get that:

**Corollary 4.2.** \( (L_{DG}(S), E) = (L(S), E_{|L(S)}) \otimes (D_G, 1) \). \( \square \)

Therefore, we have that:

**Corollary 4.3.** Let \( v_0, S \) and \( L(S) \) be given as above. Then \( (L(S), E_{|L(S)}) = (L(F_N), \tau) \).

**Proof.** Denote \( \frac{1}{\sqrt{2}} \left( L_{i_j} + L^1_{i_j} \right) \) by \( x_j \), for all \( j = 1, \ldots, N \). Then, for any projections \( L_v \in W^*(G) \), \( v \in V(G) \), we have that

\[
L_v x_j = x_j L_v, \quad \text{for all } j = 1, \ldots, N.
\]

Suppose that \( v \neq v_0 \) in \( V(G) \). Then

\[
L_v x_j = 0_{DG} = x_j L_v, \quad \text{for all } j = 1, \ldots, N
\]

Now, assume that \( v = v_0 \) in \( V(G) \). Then
The above corollary shows how to embed the free group factor $L(F_N)$ into $W^*(G)$. Vice versa, if a graph $G$ contains a vertex having $N$-loops based on it, then we can construct a $W^*$-subalgebra $L(S)$ isomorphic to the free group factor $L(F_N)$.

References

[1] A. Nica, R-transform in Free Probability, IHP course note.
[2] A. Nica, R-transforms of Free Joint Distributions and Non-crossing Partitions, J. of Func. Anal, 135 (1996), 271-296.
[3] A. Nica, D. Shlyakhtenko, R. Speicher, R-Cyclic Families of Matrices in Free Probability, J. of Funct. Anal, 188 (2002), 227-271.
[4] A. Nica, D. Shlyakhtenko, R. Speicher, R-Diagonal Elements and Freeness with Amalgamation, Canad. J. Math, 53, # 2, (2001), 335-381.
[5] A. Nica, R. Speicher, R-diagonal Pair-A Common Approach to Haar Unitaries and Circular Elements, (1995), Preprint.
[6] D. Shlyakhtenko, Some Applications of Freeness with Amalgamation, J. Reine Angew. Math, 500 (1998), 191-212.
[7] D. Shlyakhtenko, A-Valued Semicircular Systems, J. of Funct Anal, 166 (1999), 1-47.
[8] D. Voiculescu, Operations on Certain Non-commuting Operator-Valued Random Variables, Astérisque, 232 (1995), 243-275.
[9] D. Voiculescu, K. Dykemma and A. Nica, Free Random Variables, CRM Monograph Series Vol 1 (1992).
[10] F. Radulescu, Singularity of the Radial Subalgebra of $L(F_N)$ and the Pukánszky Invariant, Pacific J. of Math, vol. 151, No 2 (1991), 297-306.
[11] I. Cho, Amalgamated Boxed Convolution and Amalgamated R-transform Theory (2002), Preprint.
[12] I. Cho, An Example of Moment Series under the Compatibility (2003), Preprint.
[13] I. Cho, The Moment Series and R-transform of the Generating Operator of $L(F_N)$ (2003), Preprint.
[14] I. Cho, Graph $W^*$-Probability Theory (2004), Preprint.
[15] I. Cho, Random Variables in Graph $W^*$-Probability Spaces (2004), Ph.D Thesis, Univ. of Iowa.
[16] I. Cho, Amalgamated Semicircular Systems in Graph $W^*$-Probability Spaces (2004), Preprint.
[17] I. Cho, Free Product Structure of Graph $W^*$-Probability Spaces (2004), Preprint.
[18] R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, AMS Mem, Vol 132, Num 627, (1998).
[19] R. Speicher, Combinatorics of Free Probability Theory IHP course note.

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