ON AN ELEMENTARY DERIVATION OF
THE HAMILTON-JACOBI EQUATION FROM
THE SECOND LAW OF NEWTON.

Alex Granik∗

Abstract

It is shown that for a relativistic particle moving in an electromagnetic
field its equations of motion written in a form of the second law of Newton
can be reduced with the help of elementary operations to the Hamilton-
Jacobi equation. The derivation is based on a possibility of transforming
the equation of motion to a completely antisymmetric form. Moreover, by
perturbing the Hamilton-Jacobi equation we obtain the principle of least
action.

The analogous procedure is easily extended to a general relativistic
motion of a charged relativistic particle in an electromagnetic field. It
sis also shown that the special-relativistic Hamilton-Jacobi equation for
a free particle allows one to easily demonstrate the wave-particle duality
inherent to this equation and, in addition, to obtain the operators of
the four-momentum whose eigenvalues are the classical four-momentum
03.20.+i, 03.30.+p

In analytical mechanics we arrive at Newton’s second law (the experimentally
verified phenomenological equations with the observable parameters) by postu-
lating the principle of least action. In turn, the action S obeys the Hamilton-
Jacobi equation. The latter is a partial differential equation of the first order.
A transition from Newton’s second law to the Hamilton-Jacobi equation can be
achieved with the help of the algorithm for transforming a system of ordinary
differential equations into a partial differential equation. Despite the fact that
such transformation algorithm is well-known (e.g., [1]) the actual transformation
of the equations of motion of a charged relativistic particle in the electromag-
netic field into a respective PDE (the Hamilton-Jacobi equation) is not quoted
in the physical literature to the best of our knowledge. The usual approach to
the problem of derivation of the Hamilton-Jacobi equation is to heuristically
introduce classical action S and to vary it (for fixed initial and final times).

∗Department of Physics, University of the Pacific, Stockton, CA 95211; E-mail: agranik@uop.edu
Here we provide an elementary derivation of the Hamilton-Jacobi where the concept of action emerges in a natural way by considering the momentum as a function of both temporal and spatial coordinates. This can be seen by considering first a non-relativistic classical particle moving from p.A to p.F (see Fig.1). The particle can do that by taking any possible paths connecting these two points. Therefore for any fixed moment of time, say \( t = 1 \) the momentum would depend on the spatial coordinate, that is \( \vec{p} = \vec{p}(\vec{x}, t) \). In a sense we have replaced watching the particle evolution in time by watching the evolution of its velocity (momentum) in space and time. This situation is analogous to the Euler’s description of motion of a fluid (an alternative to the Lagrange description). The other way to look at that is to consider a “flow” of an “elemental” path and describe its “motion” in terms of its coordinates and velocity (determined by a slope of the path at a given point).

This allows us to represent Newton’s second law for a particle moving in a conservative field \( U(\vec{x}) \) as follows

\[
\frac{d\vec{p}}{dt} = \frac{\partial \vec{p}}{\partial t} + \frac{1}{m} (\vec{p} \cdot \vec{\nabla}) \vec{p} = -\nabla U \tag{1}
\]

We apply \( \text{curl} \) to both sides of this equation and get

\[
\text{curl} \frac{d\vec{p}}{dt} = \frac{\partial}{\partial t} \text{curl} \vec{p} + \frac{1}{m} \text{curl}(\vec{p} \cdot \vec{\nabla}) \vec{p} = 0 \tag{2}
\]

Using the vector identity

\[
(\vec{a} \cdot \vec{\nabla}) \vec{a} \equiv \frac{\vec{a}^2}{2} + \text{curl} \vec{a} \times \vec{a} \tag{3}
\]

we rewrite Eq.\((2)\)

\[
\frac{\partial}{\partial t} \text{curl} \vec{p} + \frac{1}{m} \text{curl}(\text{curl} \vec{p} \times \vec{p}) = 0 \tag{4}
\]

One obvious solution to Eq.\((4)\) is

\[
\text{curl} \vec{p} = 0
\]

similar to an \textit{irrotational} motion in Euler’s picture of a fluid motion.

Eq.\((4)\) implies

\[
\vec{p} = \nabla S \tag{5}
\]

where \( S(\vec{x}, t) \) is some scalar function. Generally speaking, we can choose the negative value of \( \nabla S \). The conventional choice is connected with the fact that the corresponding value of the kinetic energy has to be positive. Upon substitution of Eq.\((5)\) into Eq.\((1)\) we obtain with the help of Eq.\((3)\) the following equation

\[
\nabla \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + U \right\} = 0 \tag{6}
\]
Figure 1: A few paths of a path set connecting the initial and the final points travelled by a particle in \( t = 3 \text{ sec} \). It is clearly seen that particle’s velocity (momentum) is a function of both coordinate \( x \) and time \( t \).

In turn Eq. (6) means that

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + U = f(t)
\]  

(7)

where \( f(t) \) is some function of time. By introducing a new function

\[ S' = S - \int f(t) dt \]

we get from (6) the Hamilton-Jacobi equation with respect to the function \( S' \) (representing the classical action):

\[
\frac{\partial S'}{\partial t} + \frac{1}{2m} (\nabla S')^2 + U = 0
\]  

(8)

If we use relation (5), \( \vec{v} = \vec{p}/m \), and drop the prime at \( S' \), the Hamilton-Jacobi equation can be rewritten as follows

\[
\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = \frac{mv^2}{2} - U
\]  

(9)

Since by definition

\[
\frac{dS}{dt} = \frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S
\]

we obtain from (9) the expression for the action \( S \) by integrating (9) from \( p.A \) to \( p.F \)

\[
S = \int_{t_A}^{t_F} \left( \frac{mv^2}{2} - U \right) dt \equiv \int_{t_A}^{t_F} L(\vec{x}, \vec{v}, t) dt
\]  

(10)
where \( L(\vec{x}, \vec{v}, t) = mv^2/2 - U \) is the lagrangian of a particle of mass \( m \).

Now we can arrive at the principle of least action (without postulating it \textit{a priori}) directly from the Hamilton-Jacobi equation. To this end we subject the action \( S \) to small perturbations \( \delta S \ll S \) and (by dropping the term \((\nabla \delta S)^2\)) get from (8) the equation with respect to \( \delta S \)

\[
\frac{\partial \delta S}{\partial t} + \frac{1}{m} (\nabla S) \cdot (\nabla \delta S) = 0 \quad (11)
\]

Since \( \nabla S/m = \vec{v} \) Eq. (11) represents the substantial derivative of \( \delta S \):

\[
\frac{d\delta S}{dt} = 0 \quad (12)
\]

This means that

\[
\delta S = \text{const} \quad (13)
\]

Thus on one hand, for a specific function \( S \) satisfying the Hamilton-Jacobi equation the respective perturbations \( \delta S = \text{const} \). On the other hand, according to Eq. (10) the action \( S \) is defined on a set of all possible paths connecting point \( A \) and point \( F \). This means that perturbations \( \delta S \) correspond to perturbations of all these path.

However, only for one of these paths \( \delta S = \text{const} \), according to (13). To determine this constant we take into account that at the fixed points \( A \) and \( F \) the paths are also fixed, that is the respective perturbations \( \delta S = 0 \) at these points. Therefore only for the specific path determined by the Hamilton-Jacobi equation (that is by the second law of Newton) \( \delta S = 0 \), thus yielding the principle of least action:

\[
\delta \int_{t_A}^{t_F} L(\vec{x}, \vec{v}, t) = 0 \quad (14)
\]

The above derivation serves as a guide for a derivation of the Hamilton-Jacobi equation for a relativistic particle of charge \( q \) and mass \( m \) moving in the electromagnetic field. Our approach is to reduce the respective equations of motion to the form which would be analogous to an irrotational motion in Euler’s picture. The very structure of the space-time metric allows one to arrive at the required result in a natural way.

Therefore we begin with the second law of Newton for a relativistic charged particle of a charge \( q \) and mass \( m \) moving in the electromagnetic field:

\[
\frac{dp^\alpha}{dt} = q[E^\alpha + \epsilon^{\alpha\beta\gamma}v^\beta B^\gamma] \quad (15)
\]

where Greek indices \( \alpha, \beta, \gamma, \ldots \) take the values 1, 2, 3, \( \epsilon^{\alpha\beta\gamma} \) is the absolutely antisymmetric tensor of the third rank, \( p^\alpha = mv^\alpha/(1 - v^\delta v^\delta)^{1/2} \) is the momentum.
of the particle, $E^\alpha$ is the electric field, $v^\alpha = \vec{v}$ is the velocity of the particle and $B^\alpha$ is the magnetic field.

For the subsequent analysis we cast Eq. (15) into the standard co- and contra-variant forms. To this end we use the metric $g^{ik} = g_{ik} = [1, -1, -1, -1]$ and use units where the speed of light is $c = 1$. In this metric $x^0 = x_0 = t$, $x^\alpha = \vec{x} = -x_\alpha$, the four- potential $A^\alpha(A^0, A^\alpha)$ whose scalar part $A^0 = \phi$ (where $\phi$ is the scalar potential) and $A^\alpha \equiv \vec{A}$ is the vector potential, and the roman indices $i, j, k, \ldots$ take the values 0, 1, 2, 3. From the Maxwell equations then follows (e.g., [2]) that the electric field $E^\alpha$ intensity and the magnetic induction $B^\alpha$ are

$$E^\alpha = -\left( \frac{\partial A^0}{\partial x^\alpha} + \frac{\partial A^\alpha}{\partial x^0} \right) \tag{16}$$

$$B^\alpha = \epsilon^{\alpha\beta\gamma} \frac{\partial A^\gamma}{\partial x^\beta} \tag{17}$$

Using (17) we express the second term on the right-hand side of Eq. (15) in terms of the vector-potential $A^\alpha \equiv \vec{A}$

$$\epsilon^{\alpha\beta\gamma} v^\beta B^\gamma = \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\lambda} \frac{\partial A^\lambda}{\partial x^\beta} = v^\beta \left( \frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta} \right) \tag{18}$$

Substitution of (17) and (18) into (15) yields

$$\frac{dp^\alpha}{dx^0} = q\left[ -\left( \frac{\partial A^0}{\partial x^\alpha} + \frac{\partial A^\alpha}{\partial x^0} \right) + \beta^\gamma \left( \frac{\partial A^\gamma}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\gamma} \right) \right] \tag{19}$$

where $\beta^\gamma = v^\gamma$.

If we use in (19) the antisymmetric tensor $F^{ik}$ (e.g., [2])

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \tag{20}$$

the relation between contra- ($A^\alpha$) and co-variant ($A_\alpha$) vectors ($A^\alpha = -A_\alpha$), introduce the space-time interval

$$ds \equiv dt \sqrt{1 - \beta^\alpha \beta^\alpha} \equiv dt \sqrt{1 - \beta^2}$$

and the four-velocity

$$u^i(u^0 = 1/\sqrt{1 - \beta^2}, u^\alpha = -u_\alpha = \beta^\alpha/\sqrt{1 - \beta^2})$$

we get

$$\frac{dp^\alpha}{ds} = q F^{\alpha k} u_k = -q F^{k\alpha} u_k \tag{21}$$

As a next step, we find the zeroth components of Eqs. (21). Using the special-relativistic identity for the momentum $p_i = m u_i$

$$p_i u^i = m$$

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we find
\[
p_0 \frac{dp^0}{ds} = -p_\alpha \frac{dp^\alpha}{ds} = p^\alpha \frac{dp^\alpha}{ds}
\] (22)

Upon insertion of (22) into (21) we obtain
\[
p^\alpha \frac{dp^\alpha}{ds} = qp^\alpha [F^{\alpha \beta} u_\beta + F^{\alpha 0} u_0]
\] (23)

On the other hand, since \(F^{ik} = -F^{ki}\) (\(F^{00} = F_{00} = 0\))
\[p^\alpha u_\beta F^{\alpha \beta} = 0
\]
Hence from (22) and (23) follows that
\[
\frac{dp}{ds} = qu^\alpha F^{\alpha 0} = qF^{\alpha 0} u_\alpha = qF^{0i} u_i
\] (24)

Adding Eqs. (24) and (21) and using the definition of \(F^{ik}\), Eq. (20), we arrive at the equation of motion in the contra-variant form:
\[
\frac{dp^i}{ds} = qF^{ik} u_k = q \left( \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \right) u_k
\] (25)

The respective co-variant form follows from raising and lowering indices in (25):
\[
\frac{dp_i}{ds} = qF_{ik} u^k = q \left( \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right) u^k
\] (26)

Now we reduce these equations to a form similar to the condition defining an irrotational flow in fluid mechanics. To this end we rewrite (25) and (26) in the following form
\[
\begin{align*}
u_k & \left( \frac{\partial}{\partial x_k} (mu^i + qA^i) - \frac{\partial}{\partial x_i} (qA^k) \right) = 0 \\
u^k & \left( \frac{\partial}{\partial x^k} (mu_i + qA_i) - \frac{\partial}{\partial x^i} (qA_k) \right) = 0
\end{align*}
\] (27)

and add to the third term the identity
\[
u_k \frac{\partial u^k}{\partial x_i} = u^k \frac{\partial u_k}{\partial x^i} \equiv \frac{1}{2} \frac{\partial}{\partial x_i} (u_k u^k) = 0
\]

As a result, we get
\[
\begin{align*}
u_k & \left( \frac{\partial}{\partial x_k} (mu^i + qA^i) - \frac{\partial}{\partial x_i} (mu^k + qA^k) \right) = 0
\end{align*}
\] (28)

or equivalently
\[
\begin{align*}
u^k & \frac{\partial}{\partial x^k} (mu_i + qA_i) - \frac{\partial}{\partial x^i} (mu_k + qA_k) = 0
\end{align*}
\] (29)
The expressions in square brackets represent a four-curl of the four-vector $mu_i + qA_i$ (or $mu^i + qA^i$). Both equations are identically satisfied if this four-curl is 0. Once again, this can be interpreted as the fact that the respective vector field is irrotational, that is the four-vector $\vec{m}u + q\vec{A}$ (here we use notation $\vec{a}$ for a four-vector) is the four-gradient of a scalar function, say $-S$

$$mu^i + qA^i = -\frac{\partial S}{\partial x^i} \quad (30)$$

$$mu_i + qA_i = -\frac{\partial S}{\partial x^i} \quad (31)$$

This scalar function $S$ (a "potential function") is the classical relativistic action, and our choice of the sign is dictated by the consideration that expressions (30) must become the expressions for the momentum and energy in the non-relativistic limit.

To find the explicit expression for $S$ we integrate Eq. (30) [or (31)] and obtain:

$$S = -\int_a^b (mu^i + qA^i)dx_i \equiv -\int_a^b (m + A^i u_i)ds \quad (32)$$

where $a$ and $b$ are points on the world line of the particle, $ds = (dx^idx_i)^{1/2}$, and $u_i = dx_i/ds$. Expression (32) coincides (as it should be) with the conventional definition of the action (introduced on the basis of considerations not connected to the second law of Newton). It is interesting to note that in a conventional approach to the action, the term $A^idx_i$ "cannot be fixed on the basis of general considerations alone" [2]. Here however this term is "fixed" by the very nature of the equations of motion.

Eqs. (30) and (31) yield the determining PDE for the function $S$ (the relativistic Hamilton-Jacobi equation for a charged particle in the electromagnetic field) if we eliminate $u_i$ and $u^i$ from these equations with the help of the identity $u_i u^i = 1$:

$$\left(\frac{\partial S}{\partial x^i} + qA^i\right)\left(\frac{\partial S}{\partial x^i} + qA_i\right) = m^2, \ i = 0, 1, 2, 3 \quad (33)$$

where we have to retain (in the classical region) only one sign, either plus or minus.\(^1\)

\(^1\)We would like to point out that a unified way to describe wave and particle phenomena inherent to the Hamilton-Jacobi equation (which was the main motivation of Hamilton) is conventionally demonstrated by comparing it and the eikonal equation and by showing that they are identical. On the other hand, as we show in the Appendix, there exists a simple way to do that directly from the Hamilton-Jacobi equation without resorting to the eikonal equation.
The usual way to derive the equations of motion (25) or (26) from the action, Eq.(32) is to vary it. Here we follow the well-known procedure of reducing the integration of the partial differential equation of the first order to the integration of a system of the respective ordinary differential equations [1]. In particular, given the Hamilton-Jacobi equation (33) we derive (25). To this end we subject action $S$ to small perturbations $\delta S$

$$S = S_0 + \delta S$$

and find the equation governing these perturbations. Here $S_0$ must satisfy the original unperturbed Hamilton-Jacobi equation (33), and $\delta S \ll S_0$.

Upon substitution of (34) into (33) we get with accuracy to the first order in $\delta S$

$$\left( \frac{\partial S_0}{\partial x_i} + qA^i \right) \frac{\partial}{\partial x^i} \delta S + \left( \frac{\partial S}{\partial x^i} + qA_i \right) \frac{\partial}{\partial x_i} \delta S = 0$$

or equivalently

$$\left( \frac{\partial S}{\partial x^i} + qA_i \right) \frac{\partial}{\partial x_i} \delta S = 0$$

Equation (36) is a quasi-linear first-order PDE whose characteristics are given by the following equations

$$\frac{dx_0}{\partial S_0/\partial x^\alpha + qA_\alpha} = \frac{dx^\alpha}{\partial S_0/\partial x_\alpha + qA^\alpha}$$

Here the repeated indices do not represent summation, and $\alpha = 1, 2, 3$. It is immediately seen that the characteristics of linearized Hamilton-Jacobi equation (37) are the four-velocity $u^i$:

$$u^i = \frac{1}{m} \left( \frac{\partial S_0}{\partial x_i} + qA^i \right)$$

Inversely, these characteristics are the solutions of the equations of motion written in a form of the second law of Newton. To demonstrate that we divide both sides of (38) by $ds$, use Eqs. (30), (31) and the fact that $d/ds = u_k \partial/\partial x_k$ and obtain

$$mc \frac{du^i}{ds} = \frac{1}{m} \left( \frac{\partial S_0}{\partial x^i} + qA^i \right) \frac{\partial}{\partial x_k} \left( \frac{\partial S_0}{\partial x_i} + qA^i \right) \equiv \frac{1}{m} \left( \frac{\partial S_0}{\partial x^i} + qA^i \right) \left[ \frac{\partial}{\partial x_k} \left( \frac{\partial S_0}{\partial x_i} + qA^i \right) + q \frac{\partial A_k^i}{\partial x_i} - q \frac{\partial A^i_k}{\partial x_i} \right] \equiv \frac{1}{m} \left( \frac{\partial S_0}{\partial x^i} + qA^i \right) \left[ \frac{\partial}{\partial x_i} \left( \frac{\partial S_0}{\partial x_k} + qA^k \right) - q \frac{\partial A_k^i}{\partial x_i} - q \frac{\partial A^i_k}{\partial x_k} \right] = \frac{1}{2m} \frac{\partial}{\partial x_i} \left( u_k u^k \right) + \frac{1}{m} q u_k \left( \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i_k}{\partial x_k} \right) = qu_k F_{ik}$$

(39)
that is the second law of Newton, Eq.(25)

Now we return to the linearized equation (36) which we rewrite in the identical form

\[ m u_i \frac{\partial}{\partial x_i} \delta S \equiv \frac{d}{ds} \delta S = 0 \]  

(40)

This means that \( \delta S = \text{const} \) along a certain world line, singled out of a continuous set of possible world lines according to this condition. Without any loss of generality we can take the above \( \text{const} = 0 \).

Thus on one hand, for a specific function \( S \) satisfying the Hamilton-Jacobi equation the respective perturbations \( \delta S = \text{const} \). On the other hand, according to Eq. (32) the action \( S \) is defined on a set of all possible world lines connecting world points \( a \) and \( b \). This means that perturbations \( \delta S \) correspond to perturbations of all these world lines. However, only for one of these world lines \( \delta S = \text{const} \), according to (40). To determine this constant we take into account that at the fixed world points \( a \) and \( b \) the world lines are also fixed, that is the respective perturbations \( \delta S = 0 \) at these points. If we apply condition Eq. (40) to the action \( S \), Eq. (30), the former would “choose” out of all possible world lines the only one satisfying that condition, that is we arrive at the classical principle of least action.

\[ \delta \int_a^b (mu^l + qA^l)dx_i = 0 \] 

(41)

Finally, we demonstrate in an elementary fashion how the same technique of transforming the equations of motion in the Newtonian form to the Hamilton-Jacobi equation can be applied to a motion of a charged particle in general relativity. The equations of motion of a charged particle in gravitational and electromagnetic field are [3].

\[ M(u^l \frac{\partial u^l}{\partial x^l} + \Gamma_{kl}^i u^k u^l) = qg^{im} F_{mk} u^k \] 

(42)

where

\[ \Gamma_{kl}^i = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) \]

is the Ricci tensor. The expression \( \Gamma_{kl}^i u^k u^l \) is significantly simplified according to the following identities:

\[ \Gamma_{kl}^i u^k u^l \equiv u^l \frac{1}{2} [u^k (\frac{\partial g^{im} g_{mk}}{\partial x^l} - g_{ml} \frac{\partial g^{im}}{\partial x^k}) - u^k g_{im} \frac{\partial g^{im}}{\partial x^k} - u^k g^{im} \frac{\partial g_{kl}}{\partial x^m}] \equiv \]

\[ -\frac{1}{2} [u^l u_m \frac{\partial g^{im}}{\partial x^l} + u^l u^k (g_{im} \frac{\partial g_{kl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^l})] \equiv -\frac{1}{2} [2u^l u_m \frac{\partial g^{im}}{\partial x^l} + u^l u^k \frac{\partial g_{kl}}{\partial x^l})] \equiv \]
If we substitute this result into (42) and use the expression (20) for $F_{ik}$, we obtain

$$g^{ik}u_l\left[\frac{\partial}{\partial x^l}(Mu_k + qA_k) - \frac{\partial}{\partial x^i}(Mu_i + qA_i)\right] = 0 \quad (43)$$

Equation (43) is identically satisfied if we set

$$Mu_k + qA_k = -\frac{\partial S}{\partial x^k} \quad (44)$$

where $S$ is the action and we use the negative sign, representing a conventional choice of positive energies in classical mechanics. Raising and lowering the indices in (44), expressing the respective 4-velocities $u_k$ and $u^k$ in terms of $\partial S/\partial x^k$, and using the identity $g^{ik}u_iu_k = 1$, we arrive at the Hamilton-Jacobi equation:

$$g^{ik}(\frac{\partial S}{\partial x^i} + A_i)(\frac{\partial S}{\partial x^k} + A_k) = m^2 \quad (45)$$

We have shown that with the help of elementary operations one can arrive at the Hamilton-Jacobi equation from the phenomenological second law of Newton, without using of a priori defined action $S$. The latter arises in a natural way as a consequence of the existence of the ”irrotational” solutions to the second law of Newton. The procedure follows from the fact that for forces determined by the potential energy $U$, the second law of Newton has a symmetry which allows us to reduce it to an antisymmetric form analogous to the form observed in the potential flow of an ideal fluid. This form lends itself to the introduction of a certain potential function (action), whose gradient represent a generalized momentum. Upon introduction of the action back into the second law of Newton we arrive at the Hamilton-Jacobi equation. We also show in an elementary fashion the wave-particle duality inherent in the Hamilton-Jacobi equation. As a result of this derivation we obtain the energy-momentum operators of quantum mechanics.
1 Appendix

Let us consider a motion of a free relativistic particle of a mass $m$, whose Hamilton-Jacobi equation

$$\frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x_i} = m^2$$

has a solution:

$$S = -p_i x^i$$

(46)

corresponding to the energy-momentum relation $p_i p^i = m^2$ of the special relativity.

On the other hand, if we introduce the function

$$2S = h \ln \Psi$$

(where $h$ is some constant having the dimension of the action $S$) the Hamilton-Jacobi equation yields:

$$\frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x_i} = m^2 \Psi^2$$

(48)

This equation admits the wave solution

$$\Psi = e^{-ik_j x^j}$$

(49)

where $k^i(\omega, \vec{k})$ is the wave four-vector. The respective dispersion relation is

$$k_j k^j = m^2$$

(50)

Thus, on one hand the Hamilton-Jacobi equation describes a free particle with the momentum-energy four vector $p^i(E, \vec{p})$ and, on the other hand, the same equation describes a monochromatic wave with the wave four-vector $k^i$

Returning back to function $S$ (Eq.48) we obtain from (49)

$$S = -hik_j x^j$$

(51)

Comparing Eqs. (49) and (51) we obtain the well-known relation between the wave four-vector and the four-momentum vector:

$$k_j = \frac{p_j}{ih}$$

(52)

Moreover, relation (52) allows us to reinterpret the four-momentum vector in terms of the eigenvalues of the certain differential operators. In fact, from (49) and (52) follows

$$hi \frac{\partial \Psi}{\partial x^j} = p_j \Psi$$

(53)

which was done for the first time by E. Schrödinger in his historical paper [4] on non-relativistic quantum mechanics.
which means that we arrive at the relations describing quantum-mechanical operators of momentum-energy.

\[ p_j \rightarrow \hbar i \frac{\partial}{\partial x_j} \]  

(54)

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