Models of Quantum Computers and Decoherence Problem

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Abstract

Mathematical models of quantum computers such as a multidimensional quantum Turing machine and quantum circuits are described and its relations with lattice spin models are discussed. One of the main open problems one has to solve if one wants to build a quantum computer is the decoherence due to the coupling with the environment. We propose a possible solution of this problem by using a control of parameters of the system. This proposal is based on the analysis of the spin-boson Hamiltonian performed in the stochastic limit approximation.

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1 Introduction

Quantum computation touches upon the foundations of mathematics, quantum physics and computer science. Various mathematical and physical aspects of quantum computations have been discussed in many works, see for example [1]-[9]. Feynman [2] has pointed out that there is no efficient way of simulating a quantum mechanical system on computer and suggested that, perhaps, a computer based on quantum principles might be able to carry out the simulation efficiently. One hopes that remarkable properties of quantum systems such as quantum parallelism and entanglement will lead to a major breakthrough in computer science.

Quantum computers violate the modern form of the Church-Turing thesis. It was shown that certain problems, in particular the recursive Fourier sampling problem, can be solved in polynomial time on a quantum Turing machine, but relative to an oracle, requires superpolynomial time on a classical probabilistic Turing machine [4]. An efficient quantum algorithm for factorizing integers was found by Shor [5].

There exist already small quantum computers working with few qubits. However it is not yet clear whether large quantum computers suitable for practical use are possible to build. One reason that quantum computer is difficult to build is decoherence of quantum superpositions due to the interaction with the environment. Decoherence can be defined as the decay of the off-diagonal matrix elements of the density operator in the computational basis. By using simple models like spin system coupling with quantum field it was found that decoherence is quite large. Different methods were discussed to reduce the decoherence in quantum computers [10]-[14], still it is considered to be as one of the main open problems in this topic.

In this work we propose an approach to the problem of reducing decoherence in quantum memory by using a control on the parameters of the system [15]. We show that one can choose the parameters of the system in such a way that decoherence is drastically reduced. This proposal is based on the previous analysis [16] of the spin-boson Hamiltonian performed in the stochastic limit approximation. Considerations of the decoherence problem in quantum computers performed in [10,11,14] used a special case of the spin-boson interaction when no spin-flip transitions (or tunneling, in another interpretation) are present. Here we consider the complete spin-boson model, including the spin-flip term, and show, follow [16], that this term plays in fact the crucial role for the reducing decoherence. Recently Viola and Lloyd [17] have proposed to use the spin-flip transitions for the dynamical suppression of decoherence. The control procedure in their scheme is implemented as a sequence of radiofrequency pulses that repetitively flip the state of the system.

Turing machines and circuits are two well known models of classical and quantum computers [3]. Models for the hardware of computers are given by circuits that compute Boolean functions. For a given sequence of Boolean functions $f_n$ one looks for circuits $S_n$ computing $f_n$ with small size and small depth. A hardware problem is a Boolean function $f_n$ and obviously each $f_n$ is computable. A software problem is a language $L$, i.e. a subset of the set of all finite 0 – 1-sequences and it is known that many languages are not computable. Circuits $S_n$ for $f_n$ are efficient (or uniform) if there is
an efficient algorithm for \( L \), the union of all \( f_n^{-1}(1) \). A circuit works only for inputs of a definite length whereas a reasonable program works for inputs of arbitrary length. Models for the software of computers are given by Turing machines. A universal TM models programmable computers. Uniform circuits can be simulated efficiently by Turing machines.

In the next section we discuss mathematical models of quantum computers such as a multidimensional Turing machine and quantum circuits and its relations with spin and gauge theories on the lattice. Then the problem of decoherence is considered.

## 2 Quantum Turing machine

A quantum Turing machine (QTM) is obtained by quantization of a classical Turing machine (TM), so let us first describe the classical TM. If \( \Sigma \) is an alphabet, i.e. a finite set of symbols, then \( \Sigma^* \) denotes the set of all strings (words) over \( \Sigma \). Let \( \Sigma \) and \( Q \) be two different alphabets.

A Turing machine \( M \) is a triple \( M = \{Q, \Sigma, \delta\} \). Here \( Q \) is a set of states, including the initial state \( q_0 \) and the final state \( q_1 \) and \( \Sigma \) is the tape alphabet including a blank symbol \( a_0 \). The program is given by the transition function

\[
\delta : (Q - q_1) \times \Sigma \rightarrow Q \times \Sigma \times \{R, L, N\}
\]

The tape of TM is a sequence of cells, i.e. the one-dimensional lattice \( \mathbb{Z} \). Each cell of the tape contains a letter of \( \Sigma \). At the beginning the input \( x = (x_1, ..., x_n) \) is contained in the cells \( 0, ..., n - 1 \), all other cells contain the blank symbol \( a_0 \). The central unit of the machine is at each point of time in one state \( q \in Q \). The state at the beginning of computation is defined by \( q_0 \). The machine has a head which can read one cell and which can move from the cell \( i \in \mathbb{Z} \) to the cell \( i + 1 \) or \( i - 1 \). The head starts in the state \( q_0 \). If the machine is in state \( q \) and reads \( a \) and if \( \delta(q, a) = (q', a', \sigma) \), then the machine replaces the contents of the considered cell by \( a' \), the new state of machine is \( q' \), and the head moves one step in direction \( \sigma \) (R= right, L= left, N= no move). The computation stops in the state \( q_1 \). The result of computation can be read consecutively in the cells starting at 0 until the first cell contains the letter \( a_0 \).

For input \( x \) let \( t(x) \) be the number of steps until the machine stops and let \( s(x) \) be the number of different cells which are scanned by the head of the machine. The computational time \( t \) and the space \( s \) of the Turing machine are defined by

\[
t(n) = \max\{t(x) | |x| = n\}
\]

where \( |x| \) is the length of \( x \) and

\[
s(n) = \max\{s(x) | |x| = n\}
\]

A TM decides a language \( L \) if it computes 1 if \( x \in L \) and 0 if \( x \notin L \). A configuration of TM is a vector

\[
K = (i, q, a_{j_1}, ..., a_{j_p})
\]
where \( i \in \mathbb{Z} \) is the position of the head on the tape, \( q \in Q \) is the current state and \( a_j \in \Sigma \) is the contents of the cell \( j \). Equivalently a configuration can be written as

\[
K = (i, q, F)
\]

where \( F : \mathbb{Z} \to \Sigma \) is a function with only a finite set of values different from \( a_0 \). A multidimensional tape TM has a \( d \)-dimensional tape \( \mathbb{Z}^d \). A \( d \)-dimensional tape TM can move its head in \( 2^d \) directions.

Now let us perform quantization of the classical Turing machine. Let \( \Sigma \) and \( Q \) be two languages described above. The Hilbert space \( \mathcal{H} \) of states of quantum Turing machine (QTM) is formed by complex vectors \( \psi = \{ \psi_K \} \), \( \psi_K \in \mathbb{C} \) indexed by configurations \( K \) of the classical TM with the scalar product

\[
(\psi, \phi) = \sum_K \bar{\psi}_K \phi_K
\]

In \( \mathcal{H} \) there is an orthonormal basis \( \{ e(K) \} \), \( e(K), e(P) \) = \( \delta_{KP} \) where \( e(K)_{K'} = \delta^K_{K'} \). Because of (1) one can write \( e(K) = e(i, q, F) \).

A quantum Turing machine \( M \) is a 4-tuple \( M = \{ Q, \Sigma, H, U \} \) where \( H \) is the Hilbert space described above and \( U \) is a unitary operator in \( \mathcal{H} \) that satisfies the following condition. There exist three functions \( u_\sigma : Q \times \Sigma \times Q \times \Sigma \to \tilde{\mathbb{C}} \), \( \sigma = 0, \pm 1 \) such that one has the following locality condition

\[
(e(i, q, F), U e(i', q', F')) = [\delta_{i'}^{i+1} u_1(z) + \delta_{i'}^{i-1} u_{-1}(z) + \delta_{i'}^i u_0(z)] \prod_{j \neq i} \delta_{F(j)}^{F'(j)}
\]

for all configurations \( (i, q, F) \) and \( (i', q', F') \). Here \( z = (q, F(i), q', F'(i)) \). The product \( \prod_{j \neq i} \delta_{F(j)}^{F'(j)} \) nonvanishes only in the case if functions \( F \) and \( F' \) are perhaps different at the point \( i \). Here \( \tilde{\mathbb{C}} \) is the field of complex numbers computable in the polynomial time. One can prove that it is enough to restrict ourself to the field of rational numbers. The fundamental role of the rational numbers is emphasized also in the \( p \)-adic approach to mathematical physics \([18]\). It would be very interesting to explore interrelations between these two approaches.

Unitary evolution operator \( U_t \) is defined as the product of \( U \), \( U_t = U \ldots U, t = 0, 1, \ldots \). Computations on QTM are performed as follows. Let \( i = 0 \) and \( q_0 \) corresponds to the input \( F_0 \) and let \( e(0, q_0, F_0) \) be the initial vector. We denote \( \psi_t = U_t e(0, q_0, F_0) \). If the vector \( \psi_t \) admits an expansion of the form

\[
\psi_t = \sum_F c(F) e(0, q_1, F)
\]

where every vector \( e(0, q_1, F) \) from the expansion includes the final state \( q_1 \), and moreover the vector \( \psi_t \) does not admit such an expansion for any \( \tau < t \), then the computation is considered the finished in time \( t \) and the vector \( \psi_t \) is the result of computation. In this case the probability to obtain the result \( F \) is \( |c(F)|^2 / N \) where \( N = \sum |c(F)|^2 \).
The classical TM can be considered as a particular case of QTM when the functions $u_\sigma$ take values 0 and 1.

We have discussed the QTM whose tape is the one-dimensional lattice $\mathbb{Z}$. The crucial property of the unitary evolution operator $U$ is the locality condition (2). A QTM provides perhaps not the best model for quantum computations. One could argue that models of quantum computations with a local Hamiltonian and not with the local evolution operator would be more realistic physically.

To extend the efficiency of quantum computations we can look for a QTM with the multidimensional tape. It is known that a classical TM with the multidimensional tape is polynomially equivalent to a TM with the one-dimensional tape. From the other side properties of one-dimensional and multidimensional quantum spin systems are drastically different. Therefore it is worth to study a multidimensional QTM. We define a QTM $M$ with the multidimensional tape by quantizing a TM with the multidimensional tape.

**Definition.** A quantum Turing machine $M$ with the multidimensional tape $\mathbb{Z}^d$ is a 4-tuple $M = \{Q, \Sigma, \mathcal{H}, U\}$ where the Hilbert space $\mathcal{H}$ has the basis $\{e(i, q, F)\}$ with $i \in \mathbb{Z}^d$, $q \in Q$ and $F : \mathbb{Z}^d \to \Sigma$. The locality condition reads

$$(e(i, q, F), U e(i', q', F')) = \prod_{\sigma: |i-i'|+\sigma| \leq 1} \delta_{i+\sigma} u_{\sigma}(z) \prod_{j \neq i} \delta_{F(i)}^{F'(j)}$$

for all configurations $(i, q, F)$ and $(i', q', F')$. Here $z = (q, F(i), q', F'(i))$.

Quantum Turing machines with two- and three-dimensional tapes have, perhaps, a better chance for a practical realization of quantum computer than the one-dimensional models.

Concerning the complexity of the Ising model it is known \[19\] that finding a ground-state in the antiferromagnetic case is $P$-hard but in the ferromagnetic case the problem is $NP$-hard for general graphs and is in $P$ for planar graphs.

Evolution operators with certain locality conditions on the plane lattice are studied in the theory of quantum integrable models \[20\].

One has the following example of the QTM. Let $\mathcal{H}$ be the Hilbert space with the orthonormal basis $e(i, \alpha)$ where $i \in \mathbb{Z}$ and $\alpha = 1, 2, ..., n$. Let $U$ be an operator in $\mathcal{H}$ satisfying the conditions

$$(e(i, \alpha), U e(i', \alpha')) = \delta_{i+\alpha} A_{\alpha \alpha'} + \delta_{i-\alpha} B_{\alpha \alpha'}$$

for all $i, i', \alpha, \alpha'$. Here $A = (A_{\alpha \alpha'})$ and $B = (B_{\alpha \alpha'})$ are some $n \times n$ matrices. Then the operator $U$ is unitary if and only if the matricies $A$ and $B$ satisfy the following conditions:

$$AA^* + BB^* = I, \quad A^* A + B^* B = I, \quad AB^* = 0$$

A quantum computation is a sequence of unitary transformations. Any unitary matrix can be approximated by means of the product of unitary matrices of a simple form. Such a representation is the quantum analogue of the representation of a recursive function in terms of primitive functions. It is known that the set $\{e^{2\pi i n \theta} | n \in \mathbb{Z}\}$, where
\( \theta \) is a fixed irrational number, is dense on the unite circle. This can be interpreted as saying that the \( 1 \times 1 \) matrix \( e^{2\pi i \theta} \) is universal for the set of all unitary \( 1 \times 1 \) matrices.

A unitary \( d \times d \)-matrix \( U \) is of a simple form, if it has (after possible reordering) a block-diagonal form such that every block is a \( 2 \times 2 \)-matrix of rotations

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
or it is a number \( e^{i\theta} \) with some \( \theta \).

One denotes \( QMT_\theta \) the subset of QTM with the evolution operator of the described form.

There exists a classical algorithm [4] (i.e. a classical TM) that for a given \( d \times d \)-matrix \( U \) and \( \epsilon > 0 \) it computes in the time polynomial in \( d \) and \( \log 1/\epsilon \) unitary matrices of the simple form \( U_1, \ldots, U_n \), where \( n \) depends polynomially on \( d \), such that

\[
||U - U_1 \cdots U_n|| < \epsilon
\]

Therefore to perform an arbitrary quantum computation one has to build a QTM performing the simple unitary transformations. There exists a universal QTM \( M \) that performs an approximate computation of simple unitary transformations with an arbitrary small error. Moreover the unitary evolution operator of QTM \( M \) includes only the block-matrices of rotations on the fixed angle. The Hilbert space \( C^2 \) is called qubit. Any \( d \)-dimensional unitary matrix may be written as a product of \( 2d^2 - d \) unitary matrices (quantum gates), each of which acts only within a two-qubit subspace.

Let be given two QTM \( M_1 \) and \( M_2 \) with the same tape alphabet \( \Sigma \). One says that QTM \( M_2 \) approximates QTM \( M_1 \) if the following condition is satisfied. Let \( M_1 \) for an input \( e(0, q_0, F) \) computes the output \( \psi_1 \) in time \( t \). Then \( M_2 \) for the same input computes the output \( \psi_2 \) in time polynomial in \( t \) and moreover

\[
||\psi_1 - \psi_2|| < \epsilon.
\]

A universal QTM should approximate any QTM. To this end one has to enumerate and encode all QTM in such a way that any QTM can be considered as an input for the universal QTM. Every QTM can be represented as a finite set \( \{Q, \Sigma, u_{\sigma}(q, a, q', a')\} \). All such sets can be enumerated by using the tape alphabet \( \{0, 1, a_0\} \).

There exists a QTM \( M \) (universal QTM) [4] that approximates any QTM. Moreover there exists a universal QTM \( M \) belonging to the class \( QTM_\theta \), where the number \( \theta, 0 < \theta < \pi/2 \) is such that \( \cos \theta \) and \( \sin \theta \) are rational numbers; in particular one can take \( \cos \theta = 3/5 \), \( \sin \theta = 4/5 \).

Therefore for the practical realisation of quantum computers it is sufficient to implement rotations on two qubits subsystems of a quantum system.

## 3 Quantum circuits

Quantum circuits are quantum analogues of the classical circuits computing Boolean functions. A classical circuit can be represented as a directed acyclic graph. Similarly
a quantum circuit is a sequence of unitary matrices of the special form associated with a (hyper)graph.

Let us consider a quantum system with the Hilbert space $\mathcal{H}$ from $l$ qubits, i.e. $\mathcal{H} = H^\otimes l$, where qubit $H = \mathbb{C}^2$. It is convenient to treat $\mathcal{H}$ as the set of functions on the set $L = \{1, 2, ..., l\}$ with values in $H$, i.e. $\mathcal{H} = H^\otimes l = H^L$. If $g$ is a subset of $L$, $g \subset L$, then one has $H^L = H^g \otimes H^{L \setminus g}$. Let $U$ be a unitary operator in $H^{\otimes r}$, $r \leq l$ and the cardinality of the set $g \subset L$ is $r = |g|$. Let us denote $U^{(g)}$ the corresponding unitary operator in $H^g$ and let $\Lambda_g(U)$ be its extension to a unitary operator in $H^L$ obtained by tensoring to the identity, i.e. $\Lambda_g(U) = U^{(g)} \otimes \text{id}(H^{L \setminus g})$. Let $U = \{V, W, \ldots\}$ be a finite set of unitary operators (basis of quantum gates) each of which acts in one of the spaces $H^{r_\alpha}$, $r_\alpha \leq l$, $\alpha = 1, \ldots, q$.

A pair $(L, \Gamma)$ where $L$ is a set and $\Gamma$ is a family of its subsets is called a hypergraph. If one of the members of the family is marked then such a hypergraph is called a network or circuit. We will deal with the hypergraph $(L, \Gamma)$, where $L = \{1, \ldots, l\}$ and $\Gamma = \{L_1, \ldots, L_T\}$ is a family of subsets from $L$, $L_i \subset L$.

**Definition.** A quantum circuit $S_l$ is a 5-tuple $S_l = \{(L, \Gamma), H^L, U, U_\Gamma\}$

Here $(L, \Gamma)$, $H^L$ and $U$ are the described above hypergraph, Hilbert space and the basis of gates, and $U_\Gamma$ is a unitary operator in $H^L$ of the form

$$U_\Gamma = \Lambda_{L_T}(U_T)\ldots\Lambda_{L_1}(U_1),$$

where $U_i$ are unitary operators from the basis $U$. It is assumed that the hypergraph and the basis are consistent in the sense that $U_i$ acts in $H^{r_{\alpha(i)}}$ and $|L_i| = r_{\alpha(i)}$, $i = 1, \ldots, T$. Here $\alpha$ is a function from $\{1, \ldots, T\}$ to $\{1, \ldots, q\}$.

If $\{e_a\}$, $a = 0, 1$ is a basis in qubit $\mathbb{C}^2$ then the unitary operation in $\mathbb{C}^4$ of the form $e_a \otimes e_b \rightarrow e_a \otimes e_{a \oplus b}$ is called the quantum controlled NOT gate. Here $\oplus$ denotes addition modulo 2. One can show that the quantum controlled NOT gate, together with one-qubit gates, is sufficient for any arbitrary quantum computation.

By using the Dirac notations one writes $|a_1, \ldots, a_t> := e_{a_1} \otimes \ldots \otimes e_{a_t}$. The basis $\{|a> = |a_1, \ldots, a_t>\}$ in $H^l$ is called the computational basis. Here $a = \sum_{i=0}^{l-1} 2^i a_i$ is the binary decomposition on the number $a$.

More details about mathematical models of quantum computers one can find for example in [21].

### 4 Reducing of decoherence in quantum computers

The maintenance of quantum coherence is a crucial requirement of the ability of quantum computers to be more efficient in certain problems than classical computers. One can simulate the environment as a classical or quantum white noise [22]. In a simple model of spin coupling with the massless quantum field it was found [10] that for quantum computations not only the coupling with the environment must be small, but the
time taken in the quantum calculation must be less than the thermal time scale $1/T$ where $T$ is the temperature of the initial state of the environment.

The tape (memory) cells are taken to be two-level systems (qubits), with each of the levels having the same energy and the two states are taken to be the eigenstates of the spin operator $\sigma_z$.

Any (pure or mixed) state of a quantum circuit $S_l$ can be described by a density operator of the form

$$\rho_S(t) = \sum_{a,b=0}^{2^l-1} \rho_{ab}(t) |a><b|$$  \hspace{1cm} (3)

where $\{|a>\}$ is the computational basis in $H^l$. The degree of the quantum coherence is described by the off-diagonal elements $\rho_{ab}, a \neq b$ of the density operator. The decoherence is the decay of the off-diagonal elements of the density operator in the computational basis.

In the simple model of the quantum computer interacting with the environment (reservoir) represented by a quantum field (a family of harmonic oscillators) one assumes that the total Hilbert space is $H^l \otimes F$ where $H^l$ is the $l$-qubit space while $F$ is the bosonic Fock space. If $\rho_{Tot}(t)$ is the density operator of the total system then to get the density operator $\rho_S(t)$ of the quantum computer one has to take the partial trace over the reservoir space

$$\rho_S(t) = Tr_F(\rho_{Tot}(t))$$

The Hamiltonian describing the coupling of qubit with the environment (the spin–boson Hamiltonian) has the form

$$H_\lambda = -\frac{1}{2}\Delta \sigma_x + \frac{1}{2}\epsilon \sigma_z + \int dk \omega(k) a^+(k)a(k) + \lambda \sigma_z(A(g^*) + A^+(g))$$  \hspace{1cm} (4)

where $\sigma_x$ and $\sigma_z$ are Pauli matrices, $\epsilon$ and $\Delta$ are real parameters interpreted respectively as the energy of the spin and the spin-flip parameter. Here

$$A^+(g) = \int a^+(k)g(k)dk, \quad A(g^*) = \int a(k)g^*(k)dk$$

where $a(k)$, and $a^+(k)$ are bosonic annihilation and creation operators

$$[a(k), a^+(k')] = \delta(k - k')$$

which describe the environment.

The one–particle energy of the environment is denoted $\omega(k)$ and we assume $\omega(k) \geq 0$. The function $g(k)$ is a form factor describing the interaction of the system with the environment, $\lambda$ is the coupling constant. It is well known that, in times of order $t/\lambda^2$, the interaction produces effects of order $t$. Thus $\lambda$ provides a natural time scale for the observable effects of the interaction system–environment.

Leggett et al. [23] have found a very rich behavior of the dynamics of the Hamiltonian (4) ranging from undamped oscillations, to exponential relaxation, to power–law
types of behavior and to total localization. Main qualitative features of the system
dynamics can be described in terms of the temperature (i.e. the initial state of the
environment) and of the behavior, for low frequencies $\omega$, of the spectral function

$$J(\omega) = \int dk |g(k)|^2 \delta(\omega(k) - \omega)$$  

(5)

The dynamics of the Hamiltonian (4) in the so called stochastic approximation have
been considered in [16]. It was found that the pure oscillating regime when no damping
and no decoherence is present is described by the simple equation

$$J(\nu \Delta) = 0$$  

(6)

where

$$\nu = \sqrt{1 + \left(\frac{\epsilon}{\Delta}\right)^2}$$

The basic idea of the stochastic approximation (see [24]) is the following. If one
has a Hamiltonian of the form

$$H_\lambda = H_0 + \lambda V$$  

(7)

then the stochastic limit of the evolution operator

$$U^{(\lambda)}(t) = e^{itH_0}e^{-itH_{\lambda}}$$

is the following limit (when it exists in the sense of the convergence of matrix elements):

$$U(t) = \lim_{\lambda \to \infty} U^{(\lambda)} \left( \frac{t}{\lambda^2} \right)$$

The limiting evolution operator $U(t)$ describes the behavior of the model in the time
scale $t/\lambda^2$.

In order to apply the stochastic approximation to the Hamiltonian (4), we write
(4) in the form (7) where

$$H_0 = H_S + H_R$$

The system Hamiltonian $H_S$ is

$$H_S = -\frac{1}{2} \Delta \sigma_x + \frac{1}{2} \epsilon \sigma_z$$

and the reservoir Hamiltonian $H_R$ is

$$H_R = \int dk \omega(k)a^+(k)a(k)$$

The evolution operator $U^{(\lambda)}(t)$ satisfies equation

$$\frac{dU^{(\lambda)}(t)}{dt} = -i\lambda V(t)U^{(\lambda)}(t)$$
where $V(t) = e^{itH_0}V e^{-itH_0}$ has the form

$$V(t) = \sigma_z(t)(A(e^{-it\omega}g^*) + A^+(e^{it\omega}g))$$

and

$$\sigma_z(t) = e^{itH_S}\sigma_z e^{-itH_S}$$

Let us compute (8). The eigenvalues of the Hamiltonian $H_S$ are

$$H_S|e_\pm > = \lambda_\pm |e_\pm >$$

where

$$\lambda_\pm = \pm \frac{1}{2} \Delta \nu , \quad |e_\pm > = \frac{1}{\sqrt{1 + \mu_\pm^2}} \left( \begin{array}{c} 1 \\ \mu_\pm \end{array} \right)$$

and

$$\mu_\pm = \frac{\epsilon}{\Delta} \pm \nu , \quad \nu = \sqrt{1 + \left( \frac{\epsilon}{\Delta} \right)^2}$$

Notice that:

$$\langle e_\pm | \sigma_z | e_\pm \rangle = \frac{1 - \mu_\pm^2}{1 + \mu_\pm^2} ; \quad \langle e_+ | \sigma_z | e_- \rangle = \langle e_- | \sigma_z | e_+ \rangle = 1/\nu$$

Therefore

$$\sigma_z(t) = \frac{1 - \mu_-^2}{1 + \mu_-^2} DD^* + \frac{1 - \mu_+^2}{1 + \mu_+^2} D^*D + \nu^{-1} e^{it\Delta} D + \nu^{-1} e^{-it\Delta} D^*$$

where

$$D = |e_+ > < e_-|$$

The interaction Hamiltonian can now be written in the form:

$$V(t) = \sum_{\alpha=1}^3 (D^*_\alpha \otimes A(e^{-it\omega_\alpha}g^*) + h.c.)$$

where the three spectral frequencies correspond respectively to the down, zero, and up transitions of the 2–level system, i.e.

$$\omega_1(k) = \omega(k) - \nu \Delta , \quad \omega_2(k) = \omega(k) , \quad \omega_3(k) = \omega(k) + \nu \Delta$$

$$D_1 = \nu^{-1} D^+ \quad , \quad D_2 = \frac{1 - \mu_-^2}{1 + \mu_-^2} DD^* + \frac{1 - \mu_+^2}{1 + \mu_+^2} D^*D \quad , \quad D_3 = \nu^{-1} D^*$$

The spectral frequencies $\omega_2(k)$ and $\omega_3(k)$ are positive. Therefore in the stochastic limit one gets only one white noise field. Still a remnant of the interaction remains because, after the limit, the system evolves with a new hamiltonian, equal to the old one plus a shift term depending on the interaction and on the initial state of the field. This was called in [16] a Cheshire Cat effect.
The limiting evolution equation can then be written:

$$\frac{dU(t)}{dt} = Db^+(t)U(t) - D^+U(t)b(t) - (\gamma + i\sigma)D^+DU(t) - i\varphi U(t) \tag{9}$$

where

$$\gamma = \nu^{-2}\pi J(\nu\Delta),$$

$$\sigma = \nu^{-2}(I(-\nu\Delta) - I(\nu\Delta)) + \left(\frac{1 - \mu^2_+}{1 + \mu^2} - \frac{1 - \mu^2_-}{1 + \mu^2}\right)^2 I(0),$$

$$\varphi = \nu^{-2}I(-\nu\Delta) + \left(\frac{1 - \mu^2_-}{1 + \mu^2}\right)^2 I(0)$$

and we denote

$$J(\omega) = \int dk|g(k)|^2\delta(\omega(k) - \omega) \quad ; \quad I(\omega) = P.P.\int_0^\infty \frac{d\omega' J(\omega')}{\omega' - \omega}$$

where \(P.P.\) means the principal part of the integral. The operators \(b(t), b^+(t)\) satisfy the quantum white noise relations

$$[b(t), b^+(t')] = \gamma \delta(t - t')$$

In the notations of quantum stochastic equations the equation (9) reads

$$dU(t) = (DdB^+_t - D^+dB_t - (\gamma + i\sigma)D^+D - i\varphi)U(t)$$

Notice that all parameters \(\gamma, \sigma\) and \(\varphi\) in the evolution equation (11) are expressed in terms of the spectral density \(J(\omega)\) and parameters \(\Delta\) and \(\epsilon\) of the original Hamiltonian.

For zero temperature the stochastic approximation to the vacuum expectation value of the Heisenberg evolution of \(\sigma_z\) is given by

$$P(t) = \langle U^*(t)\sigma_z(t)U(t) \rangle$$

From equation (9) one gets the Langevin equation for \(P(t)\) which solution is

$$P(t) = \nu^{-1}e^{-\gamma t}(D^+e^{i(\sigma-\nu\Delta)t} + De^{-i(\sigma-\nu\Delta)t}) +$$

$$+ D^+D \left(\frac{1 - \mu^2_+}{1 + \mu^2} - \frac{1 - \mu^2_-}{1 + \mu^2}\right) e^{-2\gamma t} + \frac{1 - \mu^2_-}{1 + \mu^2} \tag{10}$$

We obtain the pure oscillating behaviour and no decoherence if

$$\gamma = \nu^{-2}\pi J(\nu\Delta) = 0 \tag{11}$$

For a non-zero temperature we get a stochastic evolution equation of the same form as before only with new constants \(\gamma, \sigma\) and \(\varphi\). More precisely:

$$\gamma = \nu^{-2}\pi J(\nu\Delta) \coth \frac{\beta \nu \Delta}{2},$$
\[ \sigma = \left[ \left( \frac{1 - \mu_+^2}{1 + \mu_+^2} \right)^2 - \left( \frac{1 - \mu_-^2}{1 + \mu_-^2} \right)^2 \right] (I_+(0) + I_-(0)) + \\
+ \nu^{-2}(I_+(-\nu \Delta) - I_+(-\nu \Delta) + I_-((-\nu \Delta) - I_-((-\nu \Delta))) \]

where spectral densities are

\[ J_+^\omega = \frac{J^\omega}{1 - e^{-\beta \omega}} \quad ; \quad J_-^\omega = \frac{J^\omega e^{-\beta \omega}}{1 - e^{-\beta \omega}} \]

Here \( J^\omega \) is the spectral density (5) and \( \beta \) is the inverse temperature. The functions \( I^\pm_\omega \) are defined by

\[ I^\pm_\omega = P.P. \int d\omega ' J^\pm_\omega \frac{\omega ' - \omega}{\omega ' - \omega} \]

One has the same as for the zero-temperature expression (11) for \( P(t) \) but now with new constants \( \gamma \) and \( \sigma \) depending on temperature. The condition for the reducing of coherence is still the same (11). It seems this condition is a rather weak requirement on the parameters of the interaction between quantum computer and the environment, so one can hope to use it to reduce decoherence.

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