HYPERBOLIC ANGLES FROM HEEGNER POINTS

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Abstract. We study lattice points on hyperbolic circles centred at Heegner points of class number one. Our main result is that, on a density one subset of radii tending to infinity, the angles of such points equidistribute on the unit circle. To prove this, we establish a connection between lattice points and algebraic integers in the associated field having norm of a special form and satisfying a congruence condition. As a by-product of this, we obtain an explicit formulation of the classical hyperbolic circle problem as a shifted convolution sum for the function that counts the number of algebraic integers with given norm. Along the way, we also prove a lower bound for shifted $B$-numbers, which is done by sieve methods.

1. Introduction

The distribution of lattice points in the hyperbolic plane and hyperbolic $n$-dimensional space is a well-studied subject, with contributions dating back to Selberg [31] and spanning until recent years. The first type of problem one can investigate asks whether the number of lattice points inside a hyperbolic ball is proportional (with a prescribed constant) to the volume of the ball. Asymptotics with explicit error terms are available in the literature and usually rely on non-trivial inputs such as a “spectral gap” property of the hyperbolic Laplacian (see e.g. [20]).

A feature of hyperbolic geometry is that the volume of a ball is of the same order of magnitude as the measure of its boundary, making ineffective a geometric-type argument which would give the correct asymptotic up to miscounting elements “near” the boundary. For this reason, the behaviour of lattice points on the boundary deserves special attention.

In this paper, we work on the hyperbolic plane $\mathbb{H} = \{ x + iy, y > 0 \}$, equipped with the metric $ds^2 = y^{-2}(dx^2 + dy^2)$ and associated distance $\rho(z, w)$. We focus on hyperbolic circles (the boundary of balls) centred at Heegner points of class number one. There are exactly nine imaginary quadratic fields $K$ with class number one; if $q$ denotes their discriminant, then

$$q = 3, 4, 7, 8, 11, 19, 43, 67, 163.$$  

For each value of $q$ as above we have a Heegner point $z_q \in \mathbb{H}$, defined by

$$z_q := \mu + i\lambda,$$

where

$$\mu = \begin{cases} 0 & q = 4, 8, \\ 1 & \text{otherwise}, \end{cases} \quad \lambda = \frac{\sqrt{|q|}}{2}.$$

Note that the pair $\{1, z_q\}$ is a basis of $\mathcal{O}_K$, the ring of integers of $K$. For instance, specialising to $q = 4$ (i.e. when the underlying field is $\mathbb{Q}(i)$), we have $z_q = i$.

The modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$ acts on $\mathbb{H}$ by linear fractional transformations and we will consider (for fixed $q$) the set of lattice points $\{ \gamma z_q, \gamma \in \Gamma \}$. To a given lattice point, we
associate an “arithmetic radius” $\Re(\gamma; z_q)$ given by the relation
\[ \Re(\gamma; z_q) = 2\lambda^2 \cosh(\rho(z_q, \gamma z_q)) \]
and show that this is always an integer or half an integer (see Section 2). Therefore, circles are parametrised by the set of arithmetic radii
\[ \mathcal{N}_{z_q} := \{ \Re(\gamma; z_q), \gamma \in \Gamma \}. \]
For $n \in \mathcal{N}_{z_q}$, we also define
\[ \Gamma_{z_q, n} := \{ \gamma \in \Gamma : \Re(\gamma; z_q) = n \}. \]

Let $w \in \mathbb{H}$. We define the angle of $w$ with respect to $z_q$ as follows: first, we find the unique geodesic segment going from $z_q$ to $w$; then, we consider the tangent line to this curve in $z_q$; finally, we take the angle it forms with the horizontal axis (see Figure 1 for a visual explanation of the construction). When $w = \gamma z_q$, we will denote the angle by $\theta(\gamma)$.

Our main result is that, for a density one subset of $\mathcal{N}_{z_q}$, the lattice points $\{\gamma z_q : \gamma \in \Gamma_{z_q, n}\}$ become equidistributed, by which we mean that the angles $\theta(\gamma)$ equidistribute on the unit circle $S^1$, as $n \to \infty$. We prove this in a quantitative form, giving a bound on the discrepancy.

**Theorem 1.1.** Let $z_q$ be a Heegner point of class number one as defined in (1.1). Define $\mathcal{N}_{z_q}(x) = \{ n \in \mathcal{N}_{z_q} : n \leq x \}$. Then $\mathcal{N}_{z_q}(x) \asymp x/\log x$. Moreover, for all but $o(|\mathcal{N}_{z_q}(x)|)$ elements in $\mathcal{N}_{z_q}(x)$, we have $|\Gamma_{z_q, n}| \asymp (\log n)^{\log 2 / \log 2}$ and
\[ \sup_{\lambda \leq S^1} \left| \frac{\{ \gamma \in \Gamma_{z_q, n} : \theta(\gamma) \in I \}}{|\Gamma_{z_q, n}|} - \frac{|I|}{2\pi} \right| \asymp q \left( \frac{1}{|\Gamma_{z_q, n}|} \right)^{\log \log q}, \]
where $C = \log(\pi/2)/\log 2$.

When $q = 4$, i.e. $z_q = i$, Theorem 1.1 was proved by Chatzakos–Lester–Kurlberg–Wigman [6, Theorem 1.1]. Our paper extends their results by showing an underlying structure for imaginary quadratic fields other than $\mathbb{Q}(i)$. As far as we know, very few papers study hyperbolic circles centred at points other than $i$: in his thesis [32, §10], Steeples considered base points $(i, i)$ and $(i, 2i)$; later Malcolm [21] looked at $(2i, 2i)$. See also Chamizo [5, §3] for arithmetic applications of lattice point counting in the hyperbolic plane. In a different direction, Petridis and Risager [28] considered the classical hyperbolic circle problem on average over Heegner points of discriminant $D$, as $D \to \infty$. If one allows lattice points to lie in a full ball rather than only on its boundary, then the angular equidistribution has been established in several works [2, 25, 30] and refined statistics have been studied too [3, 4, 19, 29, 22].

One key observation in the proof of [6, Theorem 1.1] is that lattice points on a given circle of (arithmetic) radius $n$ can be mapped to integer points on a Euclidean circle of radius $\sqrt{n^2 - 4}$ satisfying a congruence condition. This leads to study angles of complex points on such a Euclidean circle, which is highly convenient due to the arithmetic nature of the coordinates. An analogous situation occurs for all the points we consider.

**Proposition 1.2.** Let $K$ be an imaginary quadratic field of class number one, with discriminant $-q$ and ring of integers $\mathcal{O}_K$. Let $z_q = \mu + i\lambda$ be as in (1.1) and let $\mathcal{N}_{z_q}, \Gamma_{z_q, n}$ be as in (1.2) and (1.3). For $n \in \mathcal{N}_{z_q}$, we have
\[ \{ \theta(\gamma) : \gamma \in \Gamma_{z_q, n} \} = \left\{ \arg(x + iy) \middle| y + ix \in \mathcal{O}_K, N(y + ix) = n^2 - 4\lambda^4, 2y \equiv 2n \pmod{q} \right\}. \]

We expect that Proposition 1.2 remains valid if one works with points of the form $z_q = ki$, $k \in \mathbb{N}$, but one may have to exclude certain values (arithmetic progressions) for $n$. For instance, if $z_q = 2i$ and $n = 10$, the equality in the proposition fails, as the left-hand side
is empty while the right-hand side is not (see Remark 2.7). This was already observed by Malcolm [21, Theorem 4.5].

In relation to the classical hyperbolic circle problem we mention that, as a by-product of the proof of Proposition 1.2, we obtain that the cardinality of $\Gamma_{z_q,n}$ can be expressed in terms of the function $r_K(n)$, the number of algebraic integers in $O_K$ with norm $n$.

**Proposition 1.3.** Let $K$ be an imaginary quadratic field of class number one and let $\Gamma_{z_q,n}$ be defined as in (1.3). Then

$$|\Gamma_{z_q,n}| = c_n r_K(n - 2\lambda^2) r_K(n + 2\lambda^2),$$

where

$$c_n = \begin{cases} 1/2 & q \text{ even and } 2|n \text{ or } q \text{ odd and } q|2n, \\ 1/4 & \text{otherwise}. \end{cases}$$

On summing over $n$, we obtain a shifted convolution sum which is familiar in the case $z_q = i$ [9, (1.14)], but has not appeared before for points off the imaginary axis. To give an example, we write the sum explicitly in the case $q = 3$ (with the standard asymptotic due to Selberg, see e.g. [15, Theorem 12.1]).

**Corollary 1.4.** Let $q = 3$, $z_q = \frac{1 + i\sqrt{3}}{2}$ and $K = \mathbb{Q}(z_q)$. Let $r_K(n)$ denote the number of algebraic integers in $O_K$ with norm $n$. Then

$$\# \{ \gamma \in \Gamma : \cosh(\rho(z_q, \gamma z_q)) \leq x \} = \sum_{n \leq x} c_n r_K(n + 2) r_K(n - 1) = 6x + O(x^{2/3}),$$

where $c_n = 1/2$ if $n \equiv 1 \pmod{3}$ and $c_n = 1/4$ otherwise.

Another ingredient in the proof of Theorem 1.1 is a lower bound for $B$-numbers for number fields other than $\mathbb{Q}(i)$. In the literature, integers that can be represented as a sum of two squares are sometimes called $B$-numbers [12, 26] and the corresponding indicator function is denoted by $b(\cdot)$, see also [8, §14.3]. By extension, we say that an integer $n$ is a $B$-number for the field $K$ if $b_K(n) = 1$, where

$$b_K(n) = \begin{cases} 1 & \text{if } n \text{ is the norm of an ideal in } O_K, \\ 0 & \text{otherwise}. \end{cases}$$

Restricting to number fields with class number one, we can (and will) simply talk about algebraic integers rather than ideals. With the above notation for $b_K$, we prove the following.

**Theorem 1.5.** Let $K$ be an imaginary quadratic field of class number one and let $h \in \mathbb{Z}$. Then

$$\sum_{n \leq x} b_K(n) b_K(n + h) \gg_{K,h} \frac{x}{\log x}.$$  

When $h = 0$, since $b_K^2 = b_K$, Theorem 1.5 follows from Bernays’ work [1, p.91–92] (see also Odoni [27]). When $h \neq 0$ and $K = \mathbb{Q}(i)$, the result was proved by Hooley [11] and independently by Indlekofer [12]. For different fields we found no reference, although Nowak [26] proved an upper bound of the correct order of magnitude (in fact he even allows several linear factors). The proof of Theorem 1.5 uses sieve methods and follows the lines of [12].

The paper is organised as follows: in Section 2 we describe the geometric settings for the problem and prove Propositions 1.2 and 1.3; in Section 3 we prove Theorem 1.5; finally, in Section 4 we combine the results of the previous sections to obtain the proof of Theorem 1.1.
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2. Geometric considerations

In this section we discuss several geometric aspects of the problem and prepare the ground for the proof of Theorem 1.1, which will be given in Section 4. Along the way, we prove Proposition 1.2 and Proposition 1.3.

The hyperbolic distance $\rho(z, w)$ between points $z, w \in \mathbb{H}$ can be expressed in terms of the Euclidean one by the identity (see e.g. [18, Theorem 1.2.6])

$$
\cosh(\rho(z, w)) = 1 + \frac{|z - w|^2}{2\Im(z)\Im(w)},
$$

where $|z - w|$ is the usual absolute value in $\mathbb{C}$. Let $z = \mu + i\lambda \in \mathbb{H}$ and define the function

$$
\Re(a, b, c, d; z) := a^2|z|^2 + b^2 + c^2|z|^4 + d^2|z|^2 + 2\mu((a - d)(b - c|z|^2) - \mu(ad + bc)).
$$

When $a, b, c, d$ are the entries of a matrix in $\text{PSL}(2, \mathbb{R})$, then a calculation using (2.1) shows that the distance $\rho(z, \gamma z)$ is closely related to $\Re(a, b, c, d; z)$.

Lemma 2.1. Let $z = \mu + i\lambda \in \mathbb{H}$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $ad - bc = 1$. Then

$$
\cosh(\rho(z, \gamma z)) = \frac{\Re(a, b, c, d; z)}{2\lambda^2}.
$$

Proof. First we have, by directly expanding and multiplying out all the terms,

$$
|\gamma z - z|^2 = \left| \frac{az + b - cz^2 - dz}{cz + d} \right|^2 = \frac{((a - d)z + b - cz^2)(a - d)z + b - cz^2}{|cz + d|^2} = \frac{\Re(a, b, c, d; z) - 2\lambda^2}{|cz + d|^2}.
$$

Since $\Im(z) = \lambda$ and the determinant condition implies $\Im(\gamma z) = \lambda|cz + d|^{-2}$, the result follows from (2.1). □

Notation. From now on, we will simply write $\Re(\gamma; z)$ instead of $\Re(a, b, c, d; z)$. In the rest of this section, $a, b, c, d$ will always denote the entries of a matrix $\gamma \in \text{PSL}(2, \mathbb{R})$.

Next, we look at angles of lattice points. As mentioned in the introduction, they are defined as angles between geodesics. However, in order to visualize and study them in a more comfortable way, it is convenient to map the hyperbolic plane to the unit disc model (see Figure 1). Thus, for any given $z \in \mathbb{H}$, we define the map

$$
f(w) = \frac{i(w - z)}{w - \bar{z}}.
$$

Clearly, $f$ maps $z$ to the origin. Also, $f$ maps the real line to the unit circle and the hyperbolic plane to its interior. Regarding the action of $\text{PSL}(2, \mathbb{R})$, let us write again $z = \mu + i\lambda$ and let $\gamma \in \text{PSL}(2, \mathbb{R})$ with $\Re(\gamma; z) = n$. In Lemma 2.2 we will show that, up to a constant, $f$ maps the point $\gamma z$ to the point $x_\gamma + iy_\gamma$, where

$$
\begin{cases}
x_\gamma = 2\lambda((\mu a + b)(\mu c + d) + \lambda^2 ac - \mu((\mu c + d)^2 + \lambda^2 c^2)), \\
y_\gamma = n - 2\lambda^2((\mu c + d)^2 + \lambda^2 c^2).
\end{cases}
$$

Since $f$ is a holomorphic diffeomorphism, hence conformal, angles of lattice points can be studied by simply looking at angles of $x_\gamma + iy_\gamma$ as $\gamma$ varies in $\text{PSL}(2, \mathbb{Z})$. 
Lemma 2.2. Let \( z = \mu + i\lambda \in \mathbb{H} \) and \( \gamma \in \text{PSL}(2, \mathbb{R}) \). Let \( f \) be the map defined in (2.2). Set \( n = \Re(\gamma; z) = 2\lambda^2 \cosh \rho(z, \gamma z) \). Then

\[
f(\gamma z) = \frac{1}{n + 2\lambda^2}(x_\gamma + iy_\gamma),
\]

where \( x_\gamma, y_\gamma \) are as in (2.3). Furthermore, we have \( x_\gamma^2 + y_\gamma^2 = n^2 - 4\lambda^2 \).

Proof. By the identities \(|w - z|^2 = |w - z|^2 + 4\Im(w)\Im(z)\) and \(\Im(\gamma z) = \lambda|cz + d|^{-2}\) we deduce

\[
|\gamma z - \gamma|^2 = \frac{2\lambda^2(\cosh \rho(z, \gamma z) + 1)}{|cz + d|^2} = \frac{(n + 2\lambda^2)}{|cz + d|^2}.
\]

Therefore,

\[
f(\gamma z) = i \frac{|cz + d|^2}{n + 2\lambda^2}(\gamma z - z)(\overline{\gamma z} - z).
\]

Expanding the product gives

\[
(n + 2\lambda^2)f(\gamma z) = 2\lambda(\mu a + b)(\mu c + d) + \lambda^2 ac - \mu((\mu c + d)^2 + \lambda^2 c^2))
+ i(|z|^2a^2 + b^2 + \mu^2|z|^2c^2 + \mu^2d^2 - \lambda^2|cz + d|^2)
+ 2\mu(a(b - |z|^2c) - d(b - \mu^2c) - \mu(ad + bc)).
\]

We recognize the real part as \(x_\gamma\). As for the imaginary part, we add and subtract \(\lambda^2|cz + d|^2\) and obtain

\[
|z|^2a^2 + b^2 + |z|^4c^2 + |z|^2d^2 + 2\mu((a - d)(b - |z|^2c) - \mu(ad + bc)) - 2\lambda|cz + d|^2.
\]

The sum of all the terms but the last one gives \(\Re(a, b, c, d; z)\) and so we obtain \(y_\gamma\). Now let us show the last part of the lemma, i.e., \(x_\gamma^2 + y_\gamma^2 = n^2 - 4\lambda^2\). Proceeding as in the beginning of the proof, we can write

\[
|f(w)|^2 = \frac{|w - z|^2}{|w - z|^2} = \frac{|w - z|^2}{2\Im(w)\Im(z)} \left( \frac{|w - z|^2}{2\Im(w)\Im(z)} + 2 \right)^{-1} = \frac{\cosh \rho(z, w) - 1}{\cosh \rho(z, w) + 1}.
\]

Setting \(w = \gamma z\) and recalling that \(n = 2\lambda^2 \cosh \rho(z, \gamma z)\) gives

\[
|f(\gamma z)|^2 = \frac{n - 2\lambda^2}{n + 2\lambda^2}.
\]

Since \(x_\gamma^2 + y_\gamma^2 = (n + 2\lambda^2)^2|f(\gamma z)|^2\), we obtain the claim. \(\square\)
2.1. Heegner points. Let us specialise to the Heegner points $z_q$ defined in (1.1). If $\gamma \in \Gamma$, then $\Re(\gamma; z_q)$ is an integer (if $q$ is even) or half an odd integer (if $q$ is odd). Indeed, when $q$ is even we have $\mu = 0$ and thus
\[
\Re(\gamma; z_q) = a^2|z_q|^2 + b^2 + c^2|z_q|^4 + d^2|z_q|^2,
\]
which is an integer since $|z_q|^2 \in \mathbb{N}$. When $q$ is odd, and so $\mu = 1/2$, recalling also the identity $ad - bc = 1$, we obtain
\[
2\Re(\gamma; z_q) = 2a^2|z_q|^2 + 2b^2 + 2c^2|z_q|^4 + 2d^2|z_q|^2
+ 4\mu(a - d)(b + c|z_q|^2) - 4\mu^2(ad + bc) \equiv ad + bc \equiv 1 \pmod{2},
\]
which shows that $\Re(\gamma; z_q)$ is half an odd integer. Combining the two cases, we can summarize by saying that the set of arithmetic radii $\mathcal{N}_{z_q}$ is a subset of $(1 - \mu)\mathbb{N}$.

For $n \in \mathcal{N}_{z_q}$, we wish to study angles attached to lattice points $\gamma z_q$ with $\gamma \in \Gamma_{z_q}$. The first step consists in using Lemma 2.2 to map hyperbolic circles to Euclidean ones centred at the origin. To do this, we use the map
\[
\phi : \gamma \mapsto x_\gamma + iy_\gamma = (n + 2\lambda^2)f(\gamma z_q),
\]
where $f, x_\gamma$ and $y_\gamma$ are defined in (2.2) and (2.3). Note that, as a map from matrices to points, the function $\phi$ is not always injective.

Lemma 2.3. Let $z_q$ be a Heegner point of class number one, as defined in (1.1), and let $\phi$ be the map defined in (2.4). Then every point in the image of $\phi$ appears a number of times equal to the cardinality of the stabilizer of $z_q$ in $\text{PSL}(2, \mathbb{Z})$. Equivalently, it appears $\frac{1}{2}|O_K^*|$ times, where $|O_K^*|$ is the number of units in $O_K$.

Proof. Since $f$ is bijective, we have $f(\gamma_1 z_q) = f(\gamma_2 z_q)$ if and only if $\gamma_1 z_q = \gamma_2 z_q$, that is, $\gamma_2^{-1} \gamma_1$ stabilizes $z_q$. Apart from the identity matrix, the imaginary unit $i$ is fixed by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while the point $z_q = \frac{a + \sqrt{-d}}{2}$ is fixed by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. All other points have trivial stabilizer. The last sentence follows from the fact that $|O_K^*| = 4, 6$ for $K = Q(i)$ and $Q(\frac{1 + \sqrt{-d}}{2})$ respectively, whereas $|O_K^*| = 2$ in all other cases. □

We now describe a connection between $x_\gamma + iy_\gamma$, which has norm $n^2 - 4\lambda^4$ by Lemma 2.2, and two algebraic integers in $O_K$ of norm $n \pm 2\lambda^2$, from which we can recover $\gamma$. Such a connection will be crucial for the proof of Proposition 1.2 and Proposition 1.3.

Starting from $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$ and writing $z_q = \mu + i\lambda$ as in (1.1), we define four integers $r, u, s, t$ by:
\[
\begin{aligned}
r &= a + d, \\
u &= b - |z_q|^2c - 2\mu d, \\
s &= a - d - 2\mu c, \\
t &= b + |z_q|^2c.
\end{aligned}
\]

A direct computation shows that
\[
\begin{aligned}
u^2 + 2\mu\nu r + |z_q|^2r^2 &= \Re(\gamma; z_q) + 2\lambda^2(ad - bc), \\
t^2 + 2\mu st + |z_q|^2s^2 &= \Re(\gamma; z_q) - 2\lambda^2(ad - bc).
\end{aligned}
\]

The quantities on the left are norms of algebraic integers in $O_K$, respectively of $u + rz_q$ and $t + sz_q$. Upon setting $n = 2\Re(\gamma; z_q)$ and recalling that $ad - bc = 1$, we have
\[
N(u + rz_q) = n + 2\lambda^2, \quad N(t + sz_q) = n - 2\lambda^2.
\]

The definition of $r, u, s, t$ can be written in matrix form as
\[
\begin{pmatrix}
r \\
u \\
s \\
t
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -|z_q|^2 & -2\mu \\ 1 & 0 & -2\mu & -1 \\ 0 & 1 & |z_q|^2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.
\]
The determinant of the above matrix is $4(|z_q|^2 - \mu^2) = 4\lambda^2 = q$, so the transformations can be inverted and give

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{1}{q} \begin{pmatrix} (2|z_q|^2 - 4\mu^2) & -2\mu & 2|z_q|^2 & 2\mu \\ -2\mu & 2|z_q|^2 & -2\mu|z_q|^2 & 2 \\ -2 & 2\mu & -2|z_q|^2 & -2\mu \\ 2 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} r \\ u \\ s \\ t \end{pmatrix}. \tag{2.6}$$

Denote by $M$ the above $4 \times 4$ matrix and use the shorthand $v = (r, u, s, t)^T$. Clearly, the numbers $a, b, c, d$ obtained from $v$ in this way are integers if and only if

$$Mv \equiv 0 \pmod{q}. \tag{2.7}$$

Therefore, the above construction shows that $\Gamma_{z_q,n}$ is in bijection with the set

$$\mathcal{C}_{z_q,n} = \left\{ u + rz_q, t + sz_q \in \mathcal{O}_K \mid \begin{array}{l} N(u + rz_q) = n + 2\lambda^2, \\
N(t + sz_q) = n - 2\lambda^2 \\
\text{and } (2.7) \text{ holds} \end{array} \right\}/\{\pm 1\}. \tag{2.8}$$

The quotient accounts for the fact that $(r, u, s, t)$ must be identified with $-(r, u, s, t)$ since $\gamma$ is identified with $-\gamma$ in $\text{PSL}(2, \mathbb{Z})$.

As an example, when $q = 4$ then $\mu = 0$, $\lambda = |z_q| = 1$ and (2.7) reduces to

$$2r \pm 2s \equiv 2t \pm 2u \equiv 0 \pmod{4},$$

which amounts to saying that $r$ and $s$ have the same parity, as do $u$ and $t$ (cf. [6, §2.2]). Similarly, when $q = 8$ the system simplifies to

$$r \equiv s \pmod{2} \quad u \equiv t \pmod{4}. \tag{2.9}$$

When $q$ is odd, the system (2.7) gives four congruences involving all of $r, u, s$ and $t$. However, they are in fact all equivalent. To see this, note that $4|z_q|^2 \equiv 1 \pmod{q}$ and that 2 is invertible modulo $q$. Multiplying the first row of $M$ by 2 and reducing modulo $q$ gives the third row. The same happens if we multiply the second and fourth rows respectively by $-4$ and by $-2$. Therefore, when $q$ is odd, the system (2.7) is equivalent to the single condition

$$r + 2u \equiv s + 2t \pmod{q}. \tag{2.10}$$

Furthermore, (2.8) suggests a direct relation between the cardinality $|\Gamma_{z_q,n}|$ and the product $r_K(n + 2\lambda^2)r_K(n - 2\lambda^2)$, the total number of representations of $n \pm 2\lambda^2$ as norms in $\mathcal{O}_K$.

Making this precise yields Proposition 1.3.

**Proof of Proposition 1.3.** The case $q = 4$ is discussed e.g. in [6, §2A] or [9, (1.12)–(1.14)], so we only treat the cases $q = 8$ and $q$ odd. Our goal is to compare $\mathcal{C}_{z_q,n}$ with the set

$$\{(\alpha, \beta) \in \mathcal{O}_K^2 : \ N(\alpha) = n + 2\lambda^2, \ N(\beta) = n - 2\lambda^2 \}/\{\pm 1\}. \tag{2.11}$$

**Case $q = 8$.** In this case, (2.7) reduces to the two congruences in (2.9). At the same time, the condition on the norms gives

$$u^2 + 2r^2 = n + 4, \quad t^2 + 2s^2 = n - 4. \tag{2.12}$$

We distinguish according to the value of $n$ modulo 8. Assume $n$ is odd, so $n \equiv 5, 7 \pmod{8}$, for otherwise (2.12) has no solutions. If $n \equiv 5 \pmod{8}$, then reducing (2.12) (mod 8) we see that $r$ and $s$ are even while $t$ and $u$ are odd. In particular, the first condition in (2.9) is always satisfied. On the other hand, since the pair $(\epsilon_1u + rz_q, \epsilon_2t + sz_q)$ is in (2.11) for any choice of signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, we deduce that only half the elements in (2.11) satisfy the second condition in (2.9). In other words, in this case we have

$$|\Gamma_{z_q,n}| = |\mathcal{C}_{z_q,n}| = c_n r_K(n + 2\lambda^2)r_K(n - 2\lambda^2) \quad \text{with } c_n = \frac{1}{4}. \tag{2.13}$$
When \( n \equiv 7 \text{ (mod 8)} \), again by looking at (2.12) modulo 8, we deduce that \( r, u, s, t \) are all odd. Therefore, like before, the first condition in (2.9) is always satisfied while the second one is satisfied by only half the elements in (2.11), which gives (2.13) with \( c_n = 1/4 \).

When \( n \) is even, we claim that \( c_n = 1/2 \) always. Indeed, if \( n \equiv 2 \text{ (mod 8)} \), then reducing (2.12) modulo 8 gives \( r, s \) odd and \( u \equiv t \equiv 2 \text{ (mod 4)} \), so (2.9) is automatically satisfied. Similarly, when \( n \equiv 6 \text{ (mod 8)} \) we get \( r, s \) odd and \( u \equiv t \equiv 0 \text{ (mod 4)} \). If \( n \equiv 4 \text{ (mod 8)} \), then \( r \) and \( s \) are even and \( u \equiv t \equiv 0 \text{ (mod 4)} \). Finally, \( n \equiv 0 \text{ (mod 8)} \) leads to \( r, s \) even and \( u \equiv t \equiv 2 \text{ (mod 4)} \).

Case \( q \ odd. \) In this case the system (2.7) reduces to the single condition (2.10). Let \( u + rz_q \in \mathcal{O}_K \) have norm \( n + 2\lambda^2 \) and \( t + sz_q \in \mathcal{O}_K \) have norm \( n - 2\lambda^2 \). In particular,

\[(2u + r)^2 \equiv 4n \equiv (2t + s)^2 \text{ (mod } q)\]

Therefore, \( r + 2u \equiv s + 2t \text{ (mod } q) \). If \( q \not| 2n \), then (2.10) is automatically satisfied and \( c_n = 1/2 \). When \( q | 2n \), only half the points in (2.11) will satisfy (2.10), hence \( c_n = 1/4 \) in this case.

We return now to the points \( x_{\gamma} + iy_{\gamma} \). Lemma 2.3 gives a \( \frac{1}{2}|\mathcal{O}_K^\times| \)-to-one map \( \phi : \Gamma_{x_{\gamma},n} \rightarrow L_n \), where

\[(2.14) \quad L_n = \{(x_{\gamma}, y_{\gamma}) \text{ as in (2.3), } \gamma = (a\ b\ c\ d) \in PSL(2, \mathbb{Z})\}.\]

We make one step further in order to forget about matrices: we will show that \( L_n \) is equal to the set

\[(2.15) \quad \mathcal{L}_n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = n^2 - 4\lambda^4; \ y \equiv n \text{ (mod } 2\lambda^2), \ x \equiv 0 \text{ (mod } \lambda)\}\]

To show the equality, we will use the algebraic integers \( u + rz_q \) and \( t + sz_q \) as an intermediate step to pass from \( L_n \) to \( \mathcal{L}_n \). In particular, we will obtain that elements in either set correspond to algebraic integers. For \( L_n \), this is straightforward from the following lemma.

Lemma 2.4. Let \( \gamma \in \Gamma \), let \( x_{\gamma} + iy_{\gamma} \) be as in (2.3) and \( r, u, s, t \) be as in (2.5). Then

\[y_{\gamma} + ix_{\gamma} = (u + rz_q)(t + sz_q).\]

In particular,

\[|L_n| = \frac{2|\mathcal{L}_{x_{\gamma},n}|}{|\mathcal{O}_K^\times|} = \frac{2c_n}{|\mathcal{O}_K^\times|}r_K(n + 2\lambda^2)(n - 2\lambda^2),\]

with \( c_n \) as in Proposition 1.3.

Proof. First expand the product on the right, which gives

\[(2.16) \quad y_{\gamma} + ix_{\gamma} = rs|z_q|^2 + ut + \mu(rt + us) - \lambda i(us - rt).\]

Now (2.3) and (2.6) give

\[y_{\gamma} = n - \frac{8\lambda^4}{q^2}
\left(N(u + rz_q) + N(t + sz_q) - 2(rs|z_q|^2 + ut + \mu rt + \mu us)\right).\]

Since \( 4\lambda^2 = q \), \( N(u + rz_q) = n + 2\lambda^2 \) and \( N(t + sz_q) = n - 2\lambda^2 \), the right-hand side simplifies, giving the real part in (2.16). As for \( x_{\gamma} \), we have

\[\frac{q^2x_{\gamma}}{2\lambda} = 4\lambda^2 \left(\mu N(u + rz_q) + \mu N(t + sz_q) - 2\mu rs|z_q|^2 - 2\mu ut + 2\mu t(\lambda^2 - \mu^2) - 2us|z_q|^2\right)
- 4\lambda^2 \mu \left(N(u + rz_q) + N(t + sz_q) - 2(rs|z_q|^2 + ut + \mu rt + \mu us)\right).\]

Simplifying and using \(|z_q|^2| = \mu^2 + \lambda^2\) and again \(4\lambda^2 = q\), we obtain the imaginary part in (2.16).

Elements in \( \mathcal{L}_n \) correspond to algebraic integers too.
Lemma 2.5. Let \((x, y)\) be a point in \(L_n\). Then \(y + ix\) is an element of \(O_K\).

Proof. When \(q = 4, 8\) this is immediate, since \(y\) is then an integer and \(x\) is an integer multiple of \(z_q\). Assume \(q\) is odd and write \(x = \lambda h\) and \(y = n + 2\lambda^2k\) for some \(h, k \in \mathbb{Z}\). Then
\[
n^2 - 4\lambda^4 = x^2 + y^2 = \lambda^2h^2 + n^2 + 4\lambda^2k + 4\lambda^4.
\]
Recalling \(4\lambda^2 = q\), this implies
\[
h^2 + qk^2 + 4kn + q = 0.
\]
Reducing modulo two, we deduce that \(h\) and \(k\) have different parity. When \(h\) is even and \(k\) is odd, it follows that \(y\) is an integer and \(x\) is an integer multiple of \(\sqrt{q}\), say \(x = x_0\sqrt{q}\), and therefore \(y + ix = y - x_0 + 2x_0z_q \in O_K\). When \(h\) is odd and \(k\) is even, then \(y\) is half an odd integer and \(x\) is an odd multiple of \(\lambda\), so in this case \(y + ix = y - x + xz_q \in O_K\).

We can now prove the equality between the sets \(L_n\) and \(L_n\).

Proposition 2.6. Let \(z_q\) be as in (1.1), \(n \in \mathbb{N}_{z_q}\) and \(L_n, L_n\) be the sets defined in (2.14) and (2.15). Then \(L_n = L_n\). In particular, if \(c_n\) denotes the constant appearing in Proposition 1.3, we have
\[
|L_n| = 2c_n r_K(n^2 - 4\lambda^4).
\]

Proof. Regarding the inclusion \(L_n \subseteq L_n\), first of all we have \(x_\gamma^2 + y_\gamma^2 = n^2 - 4\lambda^4\) by Lemma 2.2. Moreover, by the definition of \(x_\gamma, y_\gamma\) we see that
\[
y_\gamma = n - 2\lambda^2(|z_q|^2c^2 + 2\mu cd + d^2) \equiv n \pmod{2\lambda^2},
\]
since the quantity in parenthesis is an integer (it is the norm in \(O_K\) of \(d + cz_q\)). Similarly, we have
\[
x_\gamma\lambda = 2ac|z_q|^2 + 2bd + 2\mu(ad + bc) - 2\mu(|z_q|^2c^2 + 2\mu cd + d^2),
\]
which is an integer since \(2\mu\) is either zero or one and \(a, b, c, d \in \mathbb{Z}\). Therefore \(L_n \subseteq L_n\).

Now for the reverse inclusion, let \((x, y)\) be a point in \(L_n\). By Lemma 2.5 we know that \(y + ix \in O_K\). It follows in particular that \(n^2 - 4\lambda^4\) is a norm in \(O_K\). From standard algebraic number theory, this is equivalent with saying that, in the factorisation of \(n^2 - 4\lambda^4\), all the primes \(p\) such that \(\chi_q(p) = -1\) (with \(\chi_q(\cdot) = (\cdot \mid \mathcal{O}_q)\) appear with even exponent. Note that \(\gcd(n + 2\lambda^2, n - 2\lambda^2)\) divides \(q\). Therefore, the property of the factorisation and of being a norm descends to \(n \pm 2\lambda^2\). In other words, we can find algebraic integers \(z_1, z_2 \in O_K\) such that
\[
y + ix = z_1\sqrt{2}, \quad N(z_1) = n + 2\lambda^2, \quad N(z_2) = n - 2\lambda^2.
\]
Write \(z_1 = u + rz_q\) and \(z_2 = t + sz_q\), for some integers \(r, u, s, t\). We claim that the pair \((u + rz_q, t + sz_q)\) belongs to the set \(\mathcal{C}_{z_q,n}\) defined in (2.8), to prove which we need to show that (2.7) holds.

We distinguish on whether \(q\) is even or odd. The case \(q = 4\) is treated in [6], so we do not discuss it here.

Case \(q = 8\). In this case (2.7) reduces (see (2.9)) to the two congruences
\[
r \equiv s \pmod{2} \quad u \equiv t \pmod{4}.
\]
If \(n\) is even, Proposition 1.3 (see also its proof) implies that the condition (2.18) is automatically satisfied once we have the condition on the norms, so there is nothing to prove. Assume \(n \equiv 1 \pmod{4}\). Then from the norms we get
\[
u^2 + 2r^2 \equiv 1 \pmod{4}, \quad t^2 + 2s^2 \equiv 1 \pmod{4},
\]
and so $ r \equiv s \equiv 0 \pmod{2} $ and $ u \equiv t \equiv 1 \pmod{2} $. In order to show that in fact $ u \equiv t \pmod{4} $ we note that by (2.17) we also have
\[
2rs + ut = y \equiv n \equiv 1 \pmod{4}.
\]
Since $ 2rs \equiv 0 \pmod{4} $, this leads to $ ut \equiv 1 \pmod{4} $, therefore $ u \equiv t \pmod{4} $ and (2.18) holds. Assume now $ n \equiv 3 \pmod{4} $. From the conditions on the norms we get
\[
u^2 + 2r^2 \equiv 3 \pmod{4}, \quad 2s^2 + t^2 \equiv 3 \pmod{4},
\]
and so $ r, u, s, t $ are all odd. To show $ u \equiv t \pmod{4} $ we argue as before and observe that
\[
2rs + ut = y \equiv n \equiv 3 \pmod{4}.
\]
Since $ 2rs \equiv 2 \pmod{4} $, it follows again $ ut \equiv 1 \pmod{4} $ and thus $ u \equiv t \pmod{4} $, so (2.18) holds in this case too.

Case $ q $ odd. In this case (2.7) simplifies (see (2.10)) to
\[
(2.19) \quad r + 2u \equiv s + 2t \pmod{q}.
\]
The condition on the norms tells us that
\[
(r + 2u)^2 \equiv 4n \equiv (s + 2t)^2 \pmod{q},
\]
so $ r + 2u \equiv \pm s + 2t \pmod{q} $. If $ q \not| 2n $ then both sides are zero mod $ q $ and so (2.19) automatically holds. If $ q \mid 2n $, assume $ r + 2u \equiv -(s + 2t) \pmod{q} $. By looking at the real part in (2.17) and recalling that $ y \equiv n \pmod{2}\lambda^2 $, we have (cf. (2.16))
\[
4n \equiv rs + 4ut + 2(rt + us) \equiv (r + 2u)(s + 2t) \equiv -(r + 2u)^2 \equiv -4n \pmod{q},
\]
a contradiction. Hence (2.19) holds.

Finally, the identity for $ |L_n| $ follows from Lemma 2.4 and the fact that $ \frac{\tau(n)}{|O_K|} $ is multiplicative (see §4.1).

**Remark 2.7.** Proposition 2.6 may fail in general. For instance, if we consider $ z_q = 2i $ and $ n = 10 $, we have $ L_n = \emptyset $ whereas $ L_n = \{ (0, -6) \} $, so the equality $ L_n = L_n $ is not true.

By combining Lemma 2.3 and Proposition 2.6, we deduce we have a $ \frac{1}{2}|O_K| $-to-one conformal map from $ \{ \gamma z_q : \gamma \in \Gamma_{z_q, n} \} $ to $ L_n $. In particular, the angles $ \{ \theta(\gamma) : \gamma \in \Gamma_{z_q, n} \} $ can be equivalently described as angles of $ x + iy $, for $ (x, y) \in L_n $ (each one repeated $ \frac{1}{2}|O_K| $ times), which proves Proposition 1.2.

We conclude this section by rewriting the discrepancy appearing in Theorem 1.1 in terms of the angles $ \theta(x + iy) $ rather than $ \theta(\gamma) $. Such a new formulation will be the starting point in the proof of Theorem 1.1 in Section 4.

**Corollary 2.8.** Let $ n \in \mathcal{N}_{z_q} $ and $ L_n $ as in (2.15). Then
\[
\sup_{I \subseteq S^1} \left| \frac{1}{|\Gamma_{z_q, n}|} \sum_{\gamma \in \Gamma_{z_q, n}} 1_{\{ \theta(\gamma) \in I \}} \right| = \frac{|I|}{2\pi} \quad \sup_{I \subseteq S^1} \left| \frac{1}{|L_n|} \sum_{(x, y) \in L_n} 1_{\{ \theta(x + iy) \in I \}} \right| = \frac{|I|}{2\pi}.
\]

3. Shifted B-numbers: Proof of Theorem 1.5

In this section we prove Theorem 1.5, which is done by sieve theory methods, following the line of argument adopted in [12].

The function $ b_K(n) $ indicates whether or not $ n $ is the norm of an element in $ O_K $. This is characterised starting from the behaviour on primes. For fixed $ K $, with discriminant $ -q $, we define two sets $ \mathcal{D}_1 $ and $ \mathcal{D}_{-1} $ by
\[
\mathcal{D}_\pm := \{ n \in \mathbb{N} : p|n \Rightarrow \chi_q(p) = \pm 1 \}.
\]
Here and in the rest of the paper \( \chi_q \) will denote the real character \( \chi_q(\cdot) = \overline{\chi_2} \). An integer \( n \) is then a norm if and only if the primes from \( \mathcal{D}_{-1} \) in the factorisation of \( n \) appear with even exponent.

The first step of the proof consists in estimating from below the shifted convolution sum by a similar one where we restrict to a suitable arithmetic progression and consider only terms from the set \( \mathcal{D}_1 \). More precisely, we start with the sum

\[
B(x, h) := \sum_{n \leq x} b_K(n) b_K(n + h).
\]

Note that we can assume that \( h \) is coprime with \( q \). Indeed, if this is not the case and \( h = q^\alpha h' \) with \((h', q) = 1\) (or \( h = 2^\alpha h' \) with \( h' \) odd in the case \( q = 4, 8 \)), then by restricting to \( n = q^\alpha n' \) and observing that \( b_K(q^\alpha m) = b_K(m) \) for any \( m \in \mathbb{N} \) gives

\[
B(x, h) \geq \sum_{n' \leq x'} b_K(n') b_K(n' + h'),
\]

where \( x' = xq^{-\alpha} \), which shows it suffices to lower bound the sum on the right, where the shift \( h' \) is now coprime to \( q \). Replacing \( q^\alpha \) by \( 2^\alpha \) leads to a similar inequality in the case \( q = 4, 8 \).

Next, we assume for a moment that \( q \) is odd and restrict the summation in (3.1) to a suitable arithmetic progression. Up to replacing \( h \) by \(-h\) and interchanging the roles of \( n \) and \( n + h \), we can assume that \( h \) is a quadratic residue modulo \( q \). Once we have this, we choose \( n \equiv q (\text{mod } q^2 h) \). If \( h \) is odd, we also impose \( n \equiv 4 (\text{mod } 8) \). These choices imply \( (n, n + h) = (q, h) = 1 \) and in particular

\[
b_K(n)b_K(n + h) = b_K(4^{-\sigma} q^{-1} n(n + h)),
\]

where \( \sigma = 0, 1 \) according to whether \( h \) is even or odd, respectively, so that the argument on the right is always odd. Similarly, when \( q = 4, 8 \), we can assume that \( h \equiv 1 (\text{mod } 4) \) or \( h \equiv 1, 3 (\text{mod } 8) \). Then we take \( n \equiv 4q (\text{mod } 4q^2 h) \) so we can write again \( b_K(n)b_K(n + h) = b_K(4^{-\sigma} q^{-1} n(n + h)) \).

Write \( n = n_1 j + n_0 \) for some integers \( n_1, n_0 \in \mathbb{N} \) and \( j \leq y := n_1^{-1}(x - n_0) \). By further restricting to elements in \( \mathcal{D}_1 \), we deduce that

\[
B(x, h) \geq \sum_{j \leq y} b_K(4^{-\sigma} q^{-1} (n_1 j + n_0)(n_1 j + n_0 + h)) \geq B^\ast(y, h),
\]

where

\[
B^\ast(y, h) := |\{ j \leq y : 4^{-\sigma} q^{-1} (n_1 j + n_0)(n_1 j + n_0 + h) \in \mathcal{D}_1 \}|.
\]

Since \( y \approx x \), Theorem 1.5 will follow from a lower bound for \( B^\ast(y, h) \) of the right order of magnitude, as stated in the proposition below.

**Proposition 3.1.** Let \( B^\ast(y, h) \) be the cardinality defined in (3.2). Then

\[
B^\ast(y, h) \asymp K, h \frac{y}{\log y}.
\]

Let \( z > 2 \) and consider the product of primes

\[
P_q(z) := \prod_{p \leq z} p.
\]

Define also the set

\[
M := \{ 4^{-\sigma} q^{-1} (n_1 j + n_0)(n_1 j + n_0 + h) : j \leq y \}
\]

and the function

\[
A(M, z) := |\{ m \in M : (m, P_q(z)) = 1 \}|.
\]
The quantity $A(M, z)$ is of a shape suitable to be attacked with sieve theory and should be viewed as an approximation to $B^*(y, h)$. When $z$ is large enough (say $z \gg y^{1/2}$), we expect $B^*(y, h) \asymp A(M, z)$, so that Proposition 3.1 would follow if we could produce a lower bound for $A(M, z)$. To make this rigorous, we will compare $B^*(y, h)$ and $A(M, z)$ for $z$ chosen as a power of $y$ slightly smaller than $1/2$ and will control explicitly what happens with the bad primes $p \in \mathcal{D}_{-1}$ in the range $z \leq p \ll y^{1/2}$.

Before diving into the proof of Proposition 3.1, we discuss three auxiliary results. The first one (Lemma 3.2) is the linear sieve, which we present in a simplified form, specialized to our set $M$ above. The second result (Lemma 3.3) is an upper bound for a sieve of polynomials at primes, which can be derived from Selberg’s sieve. We use such an upper bound to prove our third result, Lemma 3.4.

**Lemma 3.2.** Let $y \geq z$ be positive real numbers and let $s = \frac{\log y}{\log z}$. Assume that $2 \leq s \leq 4$. Let $M$ and $A(M, z)$ be as in (3.3) and (3.4). Then

$$A(M, z) \geq y \prod_{2 < p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{2}{p}\right) \left\{\frac{2\gamma}{s}\log(s-1) + O\left(\frac{1}{\log y}\right)\right\},$$

where $\gamma$ is Euler’s constant.

**Proof.** The result is a consequence of [8, Theorem 12.14], let us explain briefly why (for comparison, see also [10, Theorem 8.4] and [12, Lemma 1], where the term $(\log y)^{-1}$ appears with a worse exponent). For squarefree odd integers $d$ coprime with $qh$, the equation

$$4^{-\sigma}q^{-1}(n_1j + n_0)(n_1j + n_0 + h) \equiv 0 \pmod{d}$$

has $2^{\omega(d)}$ solutions, where $\omega(d)$ is the number of prime factors of $d$. Thus, the congruence sums satisfy

$$(3.6) \quad A_d = |\{m \in M : m \equiv 0 \pmod{d}\}| = \frac{y}{d}2^{\omega(d)} + |\theta|2^{\omega(d)}, \quad |\theta| \leq 1.$$

On the other hand, if $d$ is even or if there is a prime in $\mathcal{D}_{-1}$ dividing both $d$ and $h$, there are no solutions (since the quantity on the left in (3.5) is always odd and $h$ divides $n_1$ by construction). Our sifting range $P_s(z)$ contains only primes from $\mathcal{D}_{-1}$, so the density function is defined on primes by $g(2) = 0$ and, for $p \neq 2$,

$$g(p) = \begin{cases} 
2/p & p \in \mathcal{D}_{-1}, p \not| h, \\
0 & \text{otherwise}.
\end{cases}$$

We see by (3.6) that [8, (12.74)] is satisfied by the function $g$, while by the prime number theorem in arithmetic progressions we also have that $g$ satisfies [8, (12.48)]. Therefore, the hypotheses of [8, Theorem 12.14] are satisfied and we obtain

$$A(M, z) \geq y \prod_{2 < p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{2}{p}\right) \left\{f(s) + O\left(\frac{H(s)}{\log z}\right)\right\}.$$

In particular, the inequality is valid when $s \in [2, 4]$; $f(s)$ is equal to $\frac{2\gamma}{s}\log(s-1)$ in this interval (see e.g. [8, (12.2)]; $H(s)$ is uniformly bounded for $s \in [2, 4]$ (as it is a continuous function [8, (12.39)] on a compact set). Upon extending the product to the odd primes dividing $h$ and noting that $\log z \gg \log y$, we obtain the lemma.

**Lemma 3.3 ([10, Theorem 4.2]).** Let $F$ be a polynomial of degree $g \geq 1$ with integer coefficients. For each prime $p$, denote by $\rho(p)$ the number of solutions of $F(n) \equiv 0 \pmod{p}$.
Then, for any set of primes $\mathcal{P}$, we have
\[
|\{p : p \leq x, \ (F(p), \mathcal{P}) = 1\}| \ll \prod_{p < x \atop p \in \mathcal{P}} \left(1 - \frac{\rho(p)}{p}\right) \prod_{p < x \atop p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} \frac{x}{\log x},
\]
where the implied constant depends only on $g$.

Using the above lemma we can prove the following result which is tailored to our later applications.

**Lemma 3.4.** Let $h \in \mathbb{Z}$, $h \neq 0$, and $a = 4^\sigma mp_0$, with $m \in \mathcal{D}_1$, $p_0 \in \mathcal{D}_{-1}$, $p_0$ prime and $\sigma \in \{0,1\}$. Set
\[
M^*: = \{n = ap + h : p \leq x/a, \ p \text{ prime}\}.
\]
Then for $2 \leq \sqrt{x/a} \leq z \leq x/a$ we have
\[
|\{n \in M^* : (n, P_q(z)) = 1\}| \ll_h \frac{x}{a} \left(\log \frac{x}{a}\right)^{-3/2}.
\]
The implied constant depends on $h$ but is independent of $a, x$ and $z$. Furthermore, the result still holds when $2 \in \mathcal{D}_{-1}$ and $P_q(z)$ is replaced by $2^{-1}P_q(z)$.

**Proof.** The quantity we want to bound is
\[
A(M^*, z) := |\{p \leq x/a : (ap + h, P_q(z)) = 1\}|.
\]
When $p_0 < z$, we may assume that $p_0 \nmid h$ for otherwise this cardinality is zero. We apply Lemma 3.3 with the linear polynomial $F(n) = an + h$, the set of primes $\mathcal{P} = \{p \in \mathcal{D}_{-1} : p < z\}$ and with $x/a$ in place of $x$, obtaining
\[
A(M^*, z) \ll \prod_{p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{\rho(p)}{p}\right) \prod_{p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{1}{p}\right)^{-1} \frac{x}{a \log(x/a)}.
\]
By extending the second product to all the divisors of $h$ we obtain, with an implicit constant that depends only on $h$,
\[
A(M^*, z) \ll_h \prod_{p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{\rho(p)}{p}\right) \frac{x}{a \log(x/a)}.
\]
(3.7)

As for the value of $\rho(p)$, we observe that $F(n) \equiv 0 \pmod{p}$ has one solution for all $p < z$, $p \in \mathcal{D}_{-1}$, unless $p = p_0$, in which case it has no solutions, or $p = 2$, $\sigma = 1$ and $h$ is even, in which case again there are no solutions. Therefore, by the prime number theorem in arithmetic progressions and using that $z \geq \sqrt{x/a}$, we have
\[
\sum_{p < z \atop p \in \mathcal{D}_{-1}} \frac{\rho(p)}{p} = \sum_{p < z \atop p \in \mathcal{D}_{-1}} \frac{1}{p} + O(1) \geq \frac{1}{2} \log \log \frac{x}{a} + O(1).
\]
As a consequence, we deduce
\[
\prod_{p < z \atop p \in \mathcal{D}_{-1}} \left(1 - \frac{\rho(p)}{p}\right) \ll \exp\left(-\frac{1}{2} \log \log \frac{x}{a}\right) \ll \left(\log \frac{x}{a}\right)^{-1/2}.
\]
Inserting this in (3.7) gives the desired estimate. The final part of the lemma follows by the same proof with minor modifications. $\square$
Proof of Proposition 3.1. In order to show that \( B^*(y, h) \) is large, we wish to show that there are many elements of \( \mathcal{D}_1 \) in the set \( M \) defined in \((3.3)\). To approach this, we consider sets that approximate \( \mathcal{D}_1 \): for \( z > 2 \) and any \( l \geq 1 \), define
\[
\mathcal{D}_1(z) := \{ m \in \mathbb{N} : p|m, \; p \text{ prime}, p < z \Rightarrow p \in \mathcal{D}_1 \},
\]
\[
\mathcal{D}_1(z) := \{ n = m \prod_{i=1}^{l} p_i : m \in \mathcal{D}_1, p_i > z, p_i \text{ prime}, p_i \in \mathcal{D}_1 \; \forall \; i = 1, \ldots, l \}.
\]
We then have the inclusions, for every \( l \geq 1 \) and \( z_1 < z_2 \),
\begin{equation}
\label{inclusion}
\mathcal{D}_1(z) \subseteq \mathcal{D}_1(z), \quad \mathcal{D}_1 \subseteq \mathcal{D}_1(z_1) \subseteq \mathcal{D}_1(z_2).
\end{equation}
Pick \( z = y^{1/s} \) with \( 2 < s < 5/2 \). Let \( A(M, z) \) be the quantity defined in \((3.4)\). By looking at the size of possible harmful primes, we see that
\begin{equation}
\label{A_M_z}
A(M, z) = B^*(y, h) + \sum_{\substack{m \in M, \; m \in \mathcal{P}_1(y^{1/s})}} 1 + \sum_{\substack{m \in M, \; m \in \mathcal{P}_1(y^{1/s})}} 1.
\end{equation}
Now, \( m \) is the product of two integers, \( 4^{-s}q^{-1}(n_1j + n_0) =: m_1 \) and \( n_1j + n_0 + h =: m_2 \). Say, for our choice of arithmetic progression, we have \( \chi_q(m_1) = \chi_q(m_2) = 1 \). Therefore, primes from \( \mathcal{P}_1 \) will divide each of \( m_1 \) and \( m_2 \) in pairs. In turn, this means that the sums above decompose further as
\[
\sum_{4^{-s}q^{-1}(n_1j + n_0) \in \mathcal{P}_1(y^{1/s})} 1 + \sum_{4^{-s}q^{-1}(n_1j + n_0) \in \mathcal{P}_1(y^{1/s})} 1 + \sum_{4^{-s}q^{-1}(n_1j + n_0) \in \mathcal{P}_1(y^{1/s})} 1.
\]
Denote the three sums by \( S_1, S_2, S_3 \), respectively. Treating first \( S_1 \) and \( S_3 \), we can bound, recalling the inclusions in \((3.8)\)
\[
S_1 + S_3 \ll \sum_{4^{-s}q^{-1}(n_1j + n_0) \in \mathcal{P}_1(y^{1/s})} 1 \ll \sum_{m \in \mathcal{P}_1} \sum_{n_1y + n_0 < R < n_1y + n_0 + h} \frac{1}{mr},
\]
By applying Lemma 3.4 to the last sum with \( x = n_1y + n_0, \; a = 4^s \), \( z = y^{1/s} \) (so that for \( 2 < s < 5/2 \) and \( y \) sufficiently large we have \( \sqrt{x/a} \leq z \leq x/a \)), we can bound the above by
\[
\ll \frac{y}{(\log y)^{3/2}} \sum_{m < n_1y + n_0} \frac{1}{mr}.
\]
The sum over \( r \) can be easily bounded by using the prime number theorem and gives
\[
\ll \frac{y}{(\log y)^{3/2}} (\log(s-1) + o(1)) \sum_{m < n_1y + n_0} \frac{1}{m}.
\]
Finally, since
\[
\sum_{m \leq \sqrt{s}} \frac{1}{m} \leq \prod_{p \leq \sqrt{s}} \left( 1 - \frac{1}{p} \right)^{-1} \ll (\log x)^{1/2},
\]
we arrive at the estimate
\begin{equation}
\label{S_1_S_3}
S_1 + S_3 \ll \frac{y}{(\log y)^{3/2}} (\sqrt{x-2\log(s-1)} + o(1)).
\end{equation}
The sum $S_2$ is bounded similarly after swapping the roles of $n_1j+n_0$ and $n_1j+n_0+h$, since we have the estimate

$$S_2 \ll \sum_{n_1j+n_0+h \in \mathcal{D}_2(y^{1/s})} 1,$$

so that when $2 \in \mathcal{D}_-1$ the last part of Lemma 3.4, with the negative shift $-h$, is to be applied. Going back to (3.9), we can apply Lemma 3.2 to obtain the lower bound

$$A(M, y^{1/s}) \gg \frac{y}{\log y} (\log(s) + o(1)).$$

Combining (3.9) and (3.10)–(3.11), we arrive at the inequality

$$B^*(y, h) \geq \frac{y}{\log y} (c_1 - c_2 \sqrt{s} - 2 + o(1)) \log(s - 1),$$

where $c_1, c_2 > 0$ depend on $K$ and $h$. Picking $s$ sufficiently close to 2 gives a positive constant on the right and proves the proposition. \(\square\)

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In §4.1 we collect a few preliminary results and introduce a function $r_K$ defined as an exponential sum over elements in $\mathcal{O}_K$, proving it is multiplicative. Then in §4.2 we give the proof of Theorem 1.1, using the results of §4.1 as well as those of Sections 2 and 3.

4.1. Preliminary results. Let $M \geq 1$ and let $r_K(M)$ denote the number of algebraic integers in $\mathcal{O}_K$ with norm $M$. By [14, (11.9)] we have

$$r_K(M) = \sum_{d|M} \chi_q(d),$$

where $|\mathcal{O}_K^\times|$ is the number of units in $\mathcal{O}_K$. In particular, the above is a multiplicative function, so if we define

$$\omega_K(M) = \sum_{p|M, \chi_q(p)=1} 1 \quad \text{and} \quad \Omega_K(M) = \sum_{p^a||M, \chi_q(p)=1} a,$$

then we have

$$2^{\omega_K(M)} \leq r_K(M) \leq 2^{\Omega_K(M)}.$$

Assume now that $M$ is a norm. When $q \neq 8$, this implies that $M$ is a quadratic residue modulo $q$ and so we can write $M \equiv m^2 \pmod{q}$ for some integer $m$. When $q = 8$, we set $m = 1$ or $m = 0, 2$ according to whether $M$ is odd or $M \equiv 0, 2 \pmod{8}, M \equiv 4, 6 \pmod{8}$ respectively. With this notation, we define the function

$$r^*_K(M) := \sum_{y+ix \in \mathcal{O}_K} 1.$$

Lemma 4.1. Let $M \geq 1$ be a norm in $\mathcal{O}_K$. The function $r^*_K$ given in (4.4) is well defined and we have

$$r^*_K(M) = \begin{cases} r_K(M) & \text{if } (M, q) > 1, \\ \frac{1}{2}r_K(M) & \text{if } (M, q) = 1. \end{cases}$$
below we show that, for the generic $\omega$ not congruent to $1$ modulo $4$, for the other units, so the value of $r_K^*(M)$ is unchanged and the function is well defined.

For the case $q = 4$ we refer to \cite[(3.2)]{cite}, so assume $q = 8$. If $M$ is even then by reducing modulo $8$ the identity $y^2 + 2x^2 = M$ we deduce that $y \equiv m \pmod{q}$ always, so $r_K^*(M) = r_K(M)$ in this case. If $M$ is odd, then the condition on the norm only gives $y \equiv \pm 1 \pmod{4}$ and therefore again half of the elements in $\mathcal{O}_K$ with norm $M$ satisfy the congruence in (4.4), so $r_K^*(M) = \frac{1}{2}r_K(M)$.

Next, for $X \geq 1$ we define an index set

\begin{equation}
\mathcal{I}_q(X) := \begin{cases}
\{n \in \mathbb{N}, n \leq X\} & \text{if } q \text{ is even}, \\
\{n = \frac{2k+1}{2} \leq X, k \in \mathbb{N}\} & \text{if } q \text{ is odd}.
\end{cases}
\end{equation}

In this notation, the set $\mathcal{N}_{\alpha}(X)$ of non-empty arithmetic radii defined in (1.2) is a subset of $\mathcal{I}_q(\infty)$ by Lemma 2.4 and Proposition 2.6; up to height $X$, we have

\begin{equation}
\mathcal{N}_{\alpha}(X) = \{n \in \mathcal{N}_{\alpha} : n \leq X\} = \{n \in \mathcal{I}_q(X) : b_K(n^2 - 4\lambda^4) = 1\}.
\end{equation}

In Lemma 4.2 below we show that, for the generic $n \in \mathcal{I}_q(X)$ such that $b_K(n^2 - 4\lambda^4) = 1$, both $\omega_K(n^2 - 4\lambda^4)$ and $\Omega_K(n^2 - 4\lambda^4)$ are close to $\log \log X$. The lemma is proved with the aid of a result by Nair and Tenenbaum on shifted convolution sums of multiplicative functions \cite[(7)]{cite} (see also \cite[Theorem 15.6]{cite}), which states that for any non-negative multiplicative functions $f$ and $g$ such that $f(n), g(n) \leq \tau(n)$ for some divisor function $\tau$, and for any integer $h$, we have

\begin{equation}
\sum_{n \leq X} f(n)g(n + h) \ll h X \prod_{p \leq X} \left(1 + \frac{f(p) - 1}{p}\right) \left(1 + \frac{g(p) - 1}{p}\right).
\end{equation}

\textbf{Lemma 4.2.} Fix $\epsilon \in (0, 1/2)$ and let $\omega_K, \Omega_K$ be as in (4.2). Then, as $X \to \infty$, we have

\begin{equation}
\sum_{n \in \mathcal{I}_q(X)} b_K(n^2 - 4\lambda^4) + \sum_{n \in \mathcal{I}_q(X)} b_K(n^2 - 4\lambda^4) \ll_{K, \epsilon} X \frac{X}{\log X^{1+\epsilon^2}}.
\end{equation}

Therefore,

\begin{equation}
r_K^*(n^2 - 4\lambda^4) \ll (\log n)^{2+o(1)}
\end{equation}

for $n \in \mathcal{I}_q(X)$ outside of an exceptional set of size at most $X(\log X)^{-1-\frac{1}{2}\epsilon^2}$.

\textbf{Proof.} Let $\alpha \in (0, 1)$. By Chernoff’s bound, we have

\begin{equation}
\sum_{n \in \mathcal{I}_q(X)} b_K(n^2 - 4\lambda^4) \leq (\log X)^{\alpha(1-\epsilon)} \sum_{n \in \mathcal{I}_q(X)} b_K(n^2 - 4\lambda^4) e^{-\alpha \omega_K(n^2 - 4\lambda^4)}.
\end{equation}

Note that the function $b_K(\cdot)e^{-\alpha \omega_K(\cdot)}$ is multiplicative. Since we can write $n^2 - 4\lambda^4 = (n + 2\lambda^2)(n - 2\lambda^2)$ and the two factors share at most a divisor of $q$, the above can be bounded by

\begin{equation}
\leq e^{\alpha(\log X)^{\alpha(1-\epsilon)} \sum_{n \in \mathcal{I}_q(X)} b_K(n + 2\lambda^2)e^{-\alpha \omega_K(n + 2\lambda^2)} b_K(n - 2\lambda^2)e^{-\alpha \omega_K(n - 2\lambda^2)}}.
\end{equation}
We then apply (4.7) and Mertens’ theorem to bound
\[ \ll_K X (\log X)^{\alpha(1-\epsilon)} \prod_{p \leq X} \left( 1 + \frac{b_K(p)e^{-\alpha} - 1}{p} \right)^2 \]
\[ \ll_K \frac{X}{\log X} (\log X)^{\alpha(1-\epsilon)-1+e^{-\alpha}}. \]

Picking \( \alpha = \epsilon \) and using that \( e^{-\epsilon} \leq 1 - \epsilon + \epsilon^2/2 \) we obtain the result for \( \omega_K \). The argument for \( \Omega_K \) is similar, except that in the end the exponent gives \( -\epsilon(1 + \epsilon) - 1 + e^\epsilon \leq -\epsilon^2/3. \)

Finally, (4.9) follows from (4.8), Lemma 4.1 and the bounds in (4.3). \( \square \)

We conclude this part by introducing a new function closely related to \( r_K^* \). Let \( M \geq 1 \) be a norm in \( \mathcal{O}_K \) and let \( m \) denote the same integer used to define \( r_K^*(M) \), see before (4.4). For any \( k \in \mathbb{Z} \) we define
\[ v_k(M) := \frac{1}{r_K^*(M)} \left| \sum_{y+ix \in \mathcal{O}_K} e^{ik\theta(y+ix)} \right|, \]
(4.10)
when \( r_K^*(M) = 0 \), we define \( v_k(M) = 0 \). Note that selecting \(-m\) instead of \( m \) leads to a new sum which corresponds to the original one multiplied by an element of absolute value one. Therefore \( v_k \) is well defined. Note also that \( v_k \) is real since \( y+ix \) appears in the sum if and only if \( y-ix \) does. Hence, we have \( v_k(M) = v_{-k}(M) \).

**Lemma 4.3.** Let \( v_k \) be the function defined in (4.10).

(i) \( v_k \) is multiplicative.

(ii) if \( q = 3 \), then \( v_k = 0 \) when \( 3 \nmid k \).

(iii) if \( q = 4 \), then \( v_k = 0 \) when \( k \) is odd.

(iv) if \( q \) is odd (resp. even), then \( v_k(q^a) = v_k(q^b) \) (resp. \( v_k(2^a) = v_k(2^b) \)), for all \( a, b \geq 0 \).

**Proof.** (i) Assume \( q \neq 3,4 \).

Let \( M_1, M_2 \geq 1 \) with \( (M_1, M_2) = 1 \) and \( r_K^*(M_1), r_K^*(M_2) \neq 0 \). For \( j = 1,2 \), write \( M_j \equiv m_j^2 \) (mod \( q \)). Then
\[ v_k(M_1)v_k(M_2) = \frac{1}{r_K^*(M_1)r_K^*(M_2)} \left| \sum_{N(y+ix) = M_1, \ N(y_2+ix_2) = M_2} \sum_{2y \equiv 2m_1(q)} e^{ik\theta_1}e^{ik\theta_2} \right|, \]
(4.11)
where for brevity we wrote \( \theta_j \) in place of \( \theta(y_j + ix_j) \), \( j = 1,2 \). Consider the product \( y+ix := (y_1+ix_1)(y_2+ix_2) \). Since the norm is multiplicative, \( N(y+ix) = M_1M_2 \). Moreover, \( \theta(y+ix) \equiv \theta_1 + \theta_2 \) modulo \( 2\pi \). Regarding the congruence modulo \( q \), we distinguish two cases depending on whether \( q \) is odd or \( q = 8 \).

When \( q \) is odd, recalling that \( 2x_1 \) and \( 2x_2 \) are integer multiples of \( \sqrt{7} \), we can write
\[ 4y = 4y_1y_2 - 4x_1x_2 \equiv 4y_1y_2 \equiv 4m_1m_2 \] (mod \( q \).

Note that \( 2 \) is invertible modulo \( q \), so this is equivalent to \( 2y \equiv 2m_1m_2 \) (mod \( q \)). In other words, \( y+ix \) satisfies the congruence in the definition of \( v_k(M_1M_2) \). Finally, since \( r_K^* \) is multiplicative when \( q \neq 3,4 \) (by Lemma 4.1 and (4.1)), we deduce that (4.11) equals \( v_k(M_1M_2) \).

When \( q = 8 \), the argument is the same except when \( M_1 \equiv M_2 \equiv 3 \) (mod \( 8 \)). In this case, the above calculations lead to \( y = y_1y_2 - x_1x_2 \equiv -1 \) (mod \( 4 \)), whereas in the definition of \( v_k \) we had \( y \equiv 1 \) (mod \( 4 \)). However, we saw that changing the residue class to its negative amounts to multiplying the sum by an element of modulus 1, so after taking absolute value we obtain again \( v_k(M_1M_2) \).
(ii) The sum defining \( v_k \) contains the elements \( y + ix, (y + ix)e^{2\pi i/3} \) and \( (y + ix)e^{-2\pi i/3} \). Therefore, when \( 3 \nmid k \), the contribution of any such three points vanishes, giving \( v_k = 0 \). Assume \( 3|k \) and \( 3 \nmid M \) (if \( 3|M \) the congruence in the definition of \( v_k \) is always satisfied and the argument simplifies). Then \( v_k \) can be written as a sum of primary elements (see e.g. [13, Ch.9 §3], although they choose the residue class 2 modulo 3), namely we have

\[
v_k(M) = \frac{3}{r_K(M)} \left| \sum_{N(y+ix)=M, y+ix \equiv 1 \pmod{3}} e^{ik\theta(y+ix)} \right|,
\]

which can be showed to be multiplicative by a similar argument as the one used to prove (i).

(iii) When \( q = 4 \) the sum defining \( v_k \) contains both \( y + ix \) and \( -y - ix \), so if \( k \) is odd the corresponding exponentials cancel out and \( v_k = 0 \). When \( k \) is even, the sum is expressed in terms of primary elements in the Gaussian integers, i.e. of the form \( y + ix \equiv 1 \pmod{2(1+i)} \), and is again multiplicative. We refer to [6, §3.1] for more details.

(iv) Assume \( q \neq 3, 4, 8 \). Since \( q \) ramifies in \( O_K \), the elements of norm \( q \) are \( \pm \alpha \), where \( \alpha = i\sqrt{q} \in O_K \). Similarly, the elements of norm \( q^a \) are \( \pm \alpha^a \). In particular, \( r_K(q^a) = 2 \) and

\[
2v_k(q^a) = |1|^a + (-1)^k|1|^a = |1 + (-1)^k|,
\]

which is independent of \( a \). Therefore \( v_k(q^a) \) attains the same value for all \( a \geq 0 \). When \( q = 8 \), the argument is the same except that one works with \( \alpha = i\sqrt{2} \). For the case \( q = 4 \) we refer to [6, §3.2]. Finally, when \( q = 3 \) we have six units of the form \( \omega^j \), where \( \omega = \frac{1+i\sqrt{3}}{2} \) and \( j = 0, \ldots, 5 \). In particular, \( r_K(3^a) = 6 \). Recall that we can assume \( 3|k \), say \( k = 3h \), for otherwise \( v_k \) is identically zero. If we take \( \alpha = \frac{1+i\sqrt{3}}{2} \) (which has norm 3 and angle \( \frac{\pi}{3} \)), we deduce

\[
6v_k(3^a) = \left| \sum_{j=0}^{5} e^{3\pi i(j\frac{\pi}{3} + \frac{2}{3})} \right| = \left| \sum_{j=0}^{5} (-1)^h j \right|,
\]

which again is independent of \( a \). \( \square \)

If we evaluate \( v_k \) on primes, we obtain simply a cosine. More precisely, \( v_k(p) = |\cos(k\theta_p)| \), where \( \theta_p \) is the angle of a prime element in \( O_K \) above \( p \) (a primary element if \( q = 3, 4 \)). Such angles equidistribute on the unit circle (as a consequence of the prime number theorem over number fields, see [16, Theorem 5.36] and [17]). Following an argument of Erdős and Hall [7, (24)–(25)], one can derive the following asymptotic for the absolute values of the cosines. Let \( k \neq 0 \) be given. Then, uniformly for \( |k| \ll \log x \), we have

\[
\sum_{p \leq x, \chi_q(p) = 1} \left| \frac{\cos(k\theta_p)}{p} \right| = \frac{1}{\pi} \log \log x + O(1).
\]

4.2. Proof of Theorem 1.1. In Corollary 2.8 we saw that the discrepancy appearing in Theorem 1.1 can be calculated by looking at angles of \( x + iy \), where \( (x, y) \in \mathcal{L}_n \), the set \( \mathcal{L}_n \) is defined in (2.15) and \( n \in \mathcal{N}_z \), as in (1.2). Furthermore, we note that the discrepancy is unchanged if we replace \( x + iy \) by \( y + ix \), which is more convenient to have, since in Lemma 2.5 we showed that \( y + ix \in O_K \) and so we can use the results from §4.1. In other words, to prove Theorem 1.1 it suffices to estimate

\[
\mathcal{D}_n := \sup_{I \subseteq S^1} \left| \frac{1}{|\mathcal{L}_n|} \sum_{y+ix \in \mathcal{L}_n} 1_{\{\theta(y+ix) \in I\}} - \frac{|I|}{2\pi} \right|.
\]

Here and below, with a slight abuse of notation, we write \( y + ix \in \mathcal{L}_n \) rather than \( (x, y) \in \mathcal{L}_n \).
By Proposition 2.6, $|L_n| = r^n_K(n^2 - 4\lambda^4)$, where $r^n_K$ is as in (4.4). Even more precisely, the elements $y + ix \in L_n$ are exactly those appearing in the definition of $r^n_K$ and $v_k$ in (4.10), with $M = n^2 - 4\lambda^4$. Therefore, applying the Erdős–Turán inequality [23, Corollary 1.1], we can bound

$$D_n \ll \frac{1}{\log X} + \sum_{k=1}^{\log X} \frac{v_k(n^2 - 4\lambda^4)}{k}.$$ (4.13)

Let $\epsilon \in (0, 1/2)$. We now restrict $n$ to a density one subset of $N_{n_0}$, namely to

$$B_K(X; \epsilon) := \{n \in N_{n_0} : n \leq X \text{ and } \omega_K(n^2 - 4\lambda^4) \geq (1 - \epsilon) \log \log X\}.$$

Using the notation $I_q(X)$ from (4.5) and recalling (4.6), the set $B_K(X; \epsilon)$ can be written as

$$B_K(X; \epsilon) = \{n \in I_q(X) : b_K(n^2 - 4\lambda^4) = 1, \omega_K(n^2 - 4\lambda^4) \geq (1 - \epsilon) \log \log X\}.$$ Note that

$$\#\{n \in I_q(X) : b_K(n^2 - 4\lambda^4) = 1\} \geq \sum_{n \in I_q(X)} b_K(n + 2\lambda^2)b_K(n - 2\lambda^2) \gg_K \frac{X}{\log X},$$

the last inequality being true by Theorem 1.5. On the other hand, Lemma 4.2 gives

$$\#\{n \in I_q(X) : b_K(n^2 - 4\lambda^4) = 1 \text{ and } \omega_K(n^2 - 4\lambda^4) \leq (1 - \epsilon) \log \log X\} \ll \frac{X}{(\log X)^{1 + \frac{\epsilon}{1+\epsilon}}},$$

which shows that $B_K(X; \epsilon)$ is a density one subset of $\{n \in I_q(X) : b_K(n^2 - 4\lambda^4) = 1\}$. Furthermore, we distinguish between elements in $B_K(X; \epsilon)$ “coprime” to $q$ or not. Set

$$B^\circ_K(X; \epsilon) := \{n \in B_K(X; \epsilon) : (n, q) = 1\} \quad \text{odd}, \quad B^K(X; \epsilon) := B_K(X; \epsilon) \setminus B^\circ_K(X; \epsilon).$$

Define also $I_q^\circ(X)$ and $I_q^K(X)$ in a similar way. We will carry out the proof first for $B^\circ_K(X; \epsilon)$ and show how to reduce back to this case when dealing with $B^K(X; \epsilon)$.

By Chebyshev’s inequality, we can write

$$\#\left\{n \in B^\circ_K(X; \epsilon) : \sum_{1 \leq k \leq \log X} \frac{v_k(n^2 - 4\lambda^4)}{k} \geq (\log X)^{-\log \frac{\epsilon}{1+\epsilon}}\right\} \leq (\log X)^{\log \frac{\epsilon}{1+\epsilon}} \sum_{1 \leq k \leq \log X} \frac{1}{k} \sum_{n \in I_q^\circ(X)} v_k(n^2 - 4\lambda^4).$$ (4.14)

We focus on the inner sum over $n$, which we denote by $S_k$. Let $\alpha \in (0, 1)$. By Chernoff’s bound, we can estimate

$$S_k \leq (\log X)^{-\alpha(1-\epsilon)} \sum_{n \in I_q^\circ(X)} v_k(n^2 - 4\lambda^4)e^{\alpha\omega_K(n^2 - 4\lambda^4)}.$$ Moreover, $(n + 2\lambda^2, n - 2\lambda^2) = 1$ since the gcd divides $q$ but $n \in I_q^\circ(X)$. By Lemma (4.3), the function $v_k$ is multiplicative. Therefore, the right-hand side above equals

$$(\log X)^{-\alpha(1-\epsilon)} \sum_{n \in I_q^\circ(X)} v_k(n + 2\lambda^2)e^{\alpha\omega_K(n + 2\lambda^2)}v_k(n - 2\lambda^2)e^{\alpha\omega_K(n - 2\lambda^2)}.$$ Applying (4.7), (4.12) and picking $\alpha = \log \pi/2$, we obtain

$$S_k \ll \frac{X}{(\log X)^{\alpha(1-\epsilon)}} \prod_{p \leq X} \left(1 + \frac{e^\epsilon v_k(p)b_K(p) - 1}{p}\right)^2 \ll \frac{X}{(\log X)^{1 + (1-\epsilon)\log \frac{\pi}{2}}}. $$
Inserting this bound for \( S_k \) in (4.14) and going back to \( \mathcal{D}_n \) and \( r^*_K(n^2 - 4\lambda^4) \) by means of (4.13) and (4.9), we deduce that
\[
\mathcal{D}_n \ll (\log X)^{-\log \frac{5}{2} + \epsilon} \ll r^*_K(n^2 - 4\lambda^4)^{-C + \epsilon}, \quad C = \frac{\log(\pi/2)}{\log 2},
\]
for all arithmetic radii \( n \in \mathcal{N}_q \cap \mathcal{D}_n(X) \) outside of a set of cardinality
\[
\ll \frac{X}{(\log X)^{1 + \frac{\epsilon}{2}}} + \frac{X(\log \log X)}{(\log X)^{1 - \epsilon + \frac{\epsilon}{2}}},
\]
which proves Theorem 1.1 in the coprime case.

When \( n \in \mathcal{B}_q \), assuming \( q \) is odd, then \( (n + 2\lambda^2, n - 2\lambda^2) = q \). Therefore, by Lemma 4.3 (iv), we can write
\[
(4.15) \quad v_k(n^2 - 4\lambda^4) = v_k \left( \frac{n + 2\lambda^2}{q} \right) v_k \left( \frac{n - 2\lambda^2}{q} \right),
\]
where the two arguments are now coprime and one can apply the same proof as above. The case \( q \) even is analogous, except that in (4.15) one divides by an appropriate power of 2.

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