Well-posedness and exponential stability of a thermoelastic-Bresse system with second sound and delay

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Abstract

In this paper, we consider a one-dimensional thermoelastic-Bresse system with a delay term, where the heat conduction is given by Cattaneo’s law effective in the shear angle displacement. We prove that the system is well-posed by using the semigroup method, and show, using the multiplier method, that the dissipation induced by the heat is strong enough to exponentially stabilize the system in the presence of a “small” delay when the stable number is zero.

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Keywords. thermoelastic-Bresse system, second sound, exponential decay, time delay

1. Introduction

In this paper, we consider the following thermoelastic-Bresse system with a constant internal delay:

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\psi + \varphi_x + l\omega)_x - lk_0(\omega_x - l\varphi) + \mu \varphi_t(x, t - \tau_0) &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma \varphi_x &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + lk(\varphi_x + \psi + l\omega) &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_3 \theta_t + q_x + \gamma \psi_{tx} &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\tau q_t + \beta q + \theta_x &= 0, \\
\varphi(x, 0) &= \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in [0, 1], \\
\psi(x, 0) &= \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad q(x, 0) = q_0(x), \quad x \in [0, 1], \\
\omega(x, 0) &= \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), \quad \varphi_t(x, -t) = f_0(x, t), \quad x \in [0, 1], \quad t \in (0, \tau), \\
\varphi(0, t) &= \varphi_x(1, t) = \psi_x(0, t) = \psi(1, t) = 0, \quad t \in [0, +\infty), \\
\omega_x(0, t) &= \omega(1, t) = \theta(0, t) = q(1, t) = 0, \quad t \in [0, +\infty). 
\end{aligned}
\] (1.1)

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This is a thermoelastic system of Bresse type ([3, 12, 13]) which governs the mechanical deformations in elastic structures of circular arch type, where the heat flux is given by Cattaneo’s law. It is composed of five functions, three of which representing the mechanical deformations: the longitudinal displacement $\omega$, the vertical displacement $\varphi$ and the shear angle displacement $\psi$; $\theta$ is the difference temperature, $q$ is the heat flux ([15, 20, 21]). The coefficients $\rho_i (i = 1, 2, 3), k, l, k_0, b, k, \gamma, \tau, \beta$ are positive constants, $\mu$ is a real number, and $\tau_0 > 0$ represents the time delay.

With respect to asymptotic behavior of solutions for thermoelastic Bresse systems, some results can be obtained. Fatori and Rivera [8] considered Bresse system with thermal dissipation effective only in one equation wrote as

$$\begin{align*}
\rho_1 \varphi_{tt} - k(\psi + \varphi_x + l\omega)_x - lk_0(\omega_x - l\varphi) &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma \theta_x &= 0, \\
\rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + lk(\varphi_x + \psi + l\omega) &= 0,
\end{align*}$$

and showed that there exist exponential stability if and only if the wave propagation is equal. They also showed that, in general, the system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. In [10], Keddi et al. studied the well-posedness and the asymptotic stability of a one-dimensional thermoelastic Bresse system, where the heat conduction is given by Cattaneo’s law effective in the shear angle displacement, wrote as

$$\begin{align*}
\rho_1 \varphi_{tt} - k(\psi + \varphi_x + l\omega)_x - lk_0(\omega_x - l\varphi) &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma \theta_x &= 0, \\
\rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + lk(\varphi_x + \psi + l\omega) &= 0, \\
\rho_3 \theta_t + q_x + \gamma \psi_{tx} &= 0, \\
\tau q_t + \beta q + \theta_x &= 0,
\end{align*}$$

They established the well-posedness of the system and proved that the system was exponentially stable depending on the stable number of the system, and showed that in general, the system was polynomially stable. If $l \equiv 0$, Bresse system reduces to the well-known Timoshenko system (see [1, 5–7, 14, 22]).

Time delays so often arise in many physical, chemical, biological, thermal and economical phenomena (see [4, 9, 16–19, 23–25, 27, 29–34]). The presence of delay may be a source of instability. In recent years, the control of partial differential equations with time delay effects has become an active area of research. For example, Kaïni et al [9] studied the Timoshenko system of thermoelasticity of type III with delay of the form

$$\begin{align*}
\rho_1 \phi_{tt} - k(\phi_x + \phi_x + \mu \phi_t(x, t) + \mu_2 \theta_t(x, t - \tau) &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\phi_x + \psi + \beta \theta_{tx}) &= 0, \\
\rho_3 \theta_t + \delta \theta_{xx} + \gamma \psi_{tx} - K \theta_{txx} &= 0, \\
\theta(\cdot, 0) &= \theta_0, \quad \theta_t(\cdot, 0) = \theta_1, \quad \psi(\cdot, 0) = \psi_0, \\
\phi(\cdot, 0) &= \phi_0, \quad \phi_t(\cdot, 0) = \phi_1,
\end{align*}$$

and proved that under suitable conditions on the initial data the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay. And they also got the result that the energy decays polynomially under different wave speeds
assumption. In [2], Apalara and Messaoudi considered the following one-dimensional linear thermoelastic system of Timoshenko type with delay, where the heat flux is given by Cattaneo’s law:

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\psi + \varphi_x)x + \mu \varphi_t(x, t - \tau_0) = 0, & \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \gamma \theta_x = 0, & \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_3 \theta_t + q_x + \gamma \psi_x = 0, & \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\tau q + \beta q + \theta = 0, & \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), & \quad x \in [0, 1], \\
\theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), \varphi_t(x, -t) = f_0(x, t), & \quad x \in [0, 1], \\
\varphi(0, t) = \varphi(1, t) = \psi_x(0, t) = \psi_x(1, t) = \theta(0, t) = \theta(1, t) = 0, & \quad t \in (0, +\infty).
\end{aligned}
\]

They proved an exponential decay result under a smallness condition on the delay and a stability number, and reproduced the polynomial decay of Santos et al. [28] using the multiplier method in the case of absence of delay.

Based on the above results, in this paper, we study the thermoclastic-Bresse system (1.1) with second sound and delay. Introducing a delay term in the internal feedback of thermoclastic-Bresse system with second sound makes our problem different from those considered so far in the literature (such as [10]). For our purpose, we use the idea of Apalara and Messaoudi in [2] to take into account the effect of the delay. We first use the semigroup method to prove the well-posedness result of the system. Then, we show, using the multiplier method, that the dissipation induced by the heat is strong enough to stabilize the system in the presence of a “small” delay when the stable number is zero.

The remaining part of this paper is organized as follows. In Section 2, we establish the well-posedness result of the system. In Section 3, we give the exponential decay result by modifying some classical multipliers.

2. Well-posedness

In this section, we use the semigroup techniques to prove the well-posedness of problem (1.1). In order to exhibit the dissipative nature of system (1.1), as in [23], we introduce the new variable

\[ z(x, \rho, t) = \varphi_t(x, t - \rho \tau_0) \quad x \in (0, 1), \ \rho \in (0, 1), \ t > 0. \]

A simple differentiation shows that the variable satisfies

\[ \tau_0 z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0 \quad x \in (0, 1), \ \rho \in (0, 1), \ t > 0. \]
Hence, problem (1.1) is equivalent to the following:

\[
\begin{aligned}
\rho_1 \varphi_u - k(\varphi + \psi + l\omega)_x - l\kappa_0(\omega_x - \varphi) + \mu z(x, 1, t) &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_2 \psi_u - b \psi_x + k(\varphi + \psi + l\omega) + \gamma \theta_x &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_1 \omega_u - k_0(\omega_x - \varphi)_x + l\kappa(\varphi + \psi + l\omega) &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\rho_3 \theta_t + q_x + \gamma \psi_t &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\tau q_t + \beta q + \theta_x &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\tau_0 z_t(x, \rho, t) + z_\rho(x, \rho, t) &= 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\
\varphi(x, 0) &= \varphi_0(x), \varphi_t(x, t) = f_0(x, t), \theta(x, 0) = \theta_0(x), \quad x \in [0, 1], \\
\psi(x, 0) &= \psi_0(x), \psi_t(x, 0) = \psi_1(x), q(x, 0) = q_0(x), \quad x \in [0, 1], t \in (0, \tau), \\
\omega(x, 0) &= \omega_0(x), \omega_t(x, 0) = \omega_1(x), \zeta(x, 0) = \varphi_1(x, t), \quad x \in [0, 1], t \in (0, +\infty), \\
\varphi(0, t) &= \varphi_x(1, t) = \psi_x(0, t) = \psi(1, t) = 0, \quad t \in [0, +\infty), \\
\omega(0, t) &= \omega(1, t) = \theta(0, t) = q(1, t) = 0, \quad t \in [0, +\infty). 
\end{aligned}
\tag{2.1}
\]

Now, we let

\[\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z),\]

then system (2.1) can be written as an evolutionary equation:

\[
\begin{aligned}
\Phi'(t) + (\mathcal{A} + \mathcal{B}) \Phi(t) &= 0, \quad t > 0, \\
\Phi(0) &= \Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, q_0, z_0)^T, 
\end{aligned}
\tag{2.2}
\]

where \(\mathcal{A}\) is a linear operator defined by

\[
\mathcal{A} \Phi = \begin{pmatrix}
-u \\
-\frac{k}{\rho_1}(\varphi + \psi + l\omega)_x - \frac{k_0 l}{\rho_1}(\omega_x - \varphi) + \frac{\mu}{\rho_1} u + \frac{\mu}{\rho_1} z(\cdot, 1) \\
-v \\
-\frac{b}{\rho_2} \psi_x + \frac{b}{\rho_2} (\varphi + \psi + l\omega) + \frac{\gamma}{\rho_2} \theta_x \\
-w \\
\frac{k_0}{\rho_1}(\omega_x - \varphi)_x + \frac{k l}{\rho_1}(\varphi + \psi + l\omega) \\
\frac{1}{\rho_3} q_x + \frac{\gamma}{\rho_3} \psi_x \\
\frac{\beta}{\tau} q + \frac{1}{\tau} \theta_x \\
\frac{1}{\tau_0} z_\rho
\end{pmatrix},
\]
and the operator $\mathbb{B} : D(\mathbb{B}) = \mathcal{H} \to \mathcal{H}$ is defined by

$$
\mathbb{B}\Phi = \frac{|\mu|}{\rho_1} \begin{pmatrix}
0 \\
-u \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
$$

We give the following spaces:

$$
H^1_+(0,1) = \{ f \in H^1(0,1) : f(0) = 0 \},
$$

$$
\dot{H}^1_+(0,1) = \{ f \in H^1(0,1) : f(1) = 0 \},
$$

$$
H^2_+(0,1) = H^2(0,1) \cap \dot{H}^1_+(0,1),
$$

$$
\dot{H}^2_+(0,1) = H^2(0,1) \cap \dot{H}^1_+(0,1),
$$

and the energy space:

$$
\mathcal{H} = H^1_+(0,1) \times L^2(0,1) \times \dot{H}^1_+(0,1) \times L^2(0,1) \times \dot{H}^1_+(0,1)
$$

$$
\times L^2(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1)),
$$

equipped with the inner product

$$
(\Phi, \tilde{\Phi})_{\mathcal{H}} = k \int_0^1 (\varphi_x + \psi + l\omega) \left( \tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega} \right) dx + k_0 \int_0^1 (\omega_x - l\varphi)(\tilde{\omega}_x - l\tilde{\varphi}) dx
$$

$$
+ \rho_1 \int_0^1 u\tilde{u} dx + b \int_0^1 \varphi_x \tilde{\varphi}_x dx + \rho_2 \int_0^1 \varphi \tilde{\varphi} dx
$$

$$
+ \rho_1 \int_0^1 \omega \tilde{\omega} dx + \rho_3 \int_0^1 \theta \tilde{\theta} dx + \tau \int_0^1 q \tilde{q} dx + \tau_0 |\mu| \int_0^1 \int_0^1 z \tilde{z} d\rho dx.
$$

$\mathcal{H}$ is a Hilbert space for $l$ small enough. In this case, the above inner product is equivalent to the natural inner product defined on $\mathcal{H}$. To this end, the operator $\mathcal{A}$ with its domain is

$$
D(\mathcal{A}) = \left\{ \Psi \in \mathcal{H} \mid \varphi \in H^2_+(0,1), \psi, \omega \in \dot{H}^1_+(0,1); \ u, \theta \in H^1_+(0,1), \ v, w, q \in H^2_+(0,1), \varphi_x(1) = 0, \psi_x(0) = \omega_x(0) = 0, \ z, z_\rho \in L^2((0,1), L^2(0,1)), \ z(x,0) = \varphi(x) \right\}.
$$

In what follows, we have the well-posedness result of problem (2.2).

**Theorem 2.1.** Assume that $\Phi_0 \in \mathcal{H}$, then problem (2.2) exists a unique solution $U \in C(\mathbb{R}^+, \mathcal{H})$. Moreover, if $\Phi_0 \in D(\mathcal{A})$ then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.

**Proof.** It is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. For $\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z)^T \in D(\mathcal{A})$, a direct computation gives that

$$
(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = |\mu| \int_0^1 u^2 dx + \beta \int_0^1 q^2 dx + \mu \int_0^1 uz(\cdot,1) dx + |\mu| \int_0^1 \int_0^1 zz_\rho d\rho dx.
$$

(2.3)

By using Young’s inequality, the third term in the right hand side of (2.3) gives

$$
-\mu \int_0^1 uz(\cdot,1) dx \leq \frac{|\mu|}{2} \int_0^1 z^2(\cdot,1) dx + \frac{|\mu|}{2} \int_0^1 u^2 dx,
$$

which implies that

$$
\mu \int_0^1 uz(0,1) dx \geq -\frac{|\mu|}{2} \int_0^1 z^2(\cdot,1) dx - \frac{|\mu|}{2} \int_0^1 u^2 dx.
$$
Also, using integration by parts and the fact that \( z(x,0) = u(x) \), the last term in the right-hand side of (2.3) gives
\[
\int_0^1 \int_0^1 zz_\rho d\rho dx = \frac{1}{2} \int_0^1 z^2(\cdot,1) dx - \frac{1}{2} \int_0^1 u^2 dx.
\]
Consequently, (2.3) yields
\[
(A\Phi, \Phi)_{\mathcal{H}} \geq \beta \int_0^1 q^2 dx.
\]
Hence \( A \) is monotone. Next, we will prove that the operator \( J + A \) is surjective.

For all \( G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9)^T \in \mathcal{H} \), we solve the equation
\[
(J + A)\Phi = G.
\]
That is
\[
\begin{cases}
-u + \varphi = g_1 \in H^1_0(0,1), \\
-k(\varphi_x + \psi + l\omega)x - k_0 l(\omega_x - l\varphi) + (|\mu| + \rho_1)u + \mu z(\cdot,1) = \rho_1 g_2 \in L^2(0,1), \\
v + \psi = g_3 \in \tilde{H}^1_0(0,1), \\
b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma \theta_x + \rho_2 v = \rho_2 g_4 \in L^2(0,1), \\
w + \omega = g_5 \in \tilde{H}^1_0(0,1), \\
k_0(\omega_x - l\varphi) + k(\varphi_x + \psi + l\omega) + \rho_1 w = \rho_1 g_6 \in L^2(0,1), \\
q_x + \gamma v_x + \rho_3 \theta = \rho_3 g_7 \in L^2(0,1), \\
(\beta + \tau)q + \theta_x = \tau g_8 \in L^2(0,1), \\
z_\rho + \tau_0 z = \tau_0 g_9 \in L^2((0,1) \times (0,1)).
\end{cases}
\]
From (2.5)_8, we know that
\[
\theta = \tau \int_0^x g_8(y) dy - (\beta + \tau) \int_0^x q(y) dy,
\]
which conclude \( \theta(0,t) = 0 \). Inserting \( u = \varphi - g_1, v = \psi - g_3, w = \omega - g_5 \), the last equation in (2.5) together with the fact that \( z(x,0) = u(x) \), one has
\[
z(x,\rho) = \varphi(x)e^{-\tau \rho} - e^{-\tau \rho} g_1(x) + \tau_0 e^{-\tau \rho} \int_0^\rho s \tau \rho g_9(x,s) ds.
\]
It can be easily shown that \( \varphi, \psi, \omega \) and \( q \) satisfy
\[
\begin{cases}
-k(\varphi_x + \psi + l\omega)x - k_0 l(\omega_x - l\varphi) + (|\mu| + \rho_1 + \mu e^{-\tau}) \varphi = h_1 \in L^2(0,1), \\
b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \rho_2 \psi - \gamma (\beta + \tau) q = h_2 \in L^2(0,1), \\
k_0(\omega - \varphi) + k(\varphi_x + \psi + l\omega) + \rho_1 w = h_3 \in L^2(0,1), \\
q_x + \rho_3 (\beta + \tau) \int_0^x q(y) dy - \gamma \psi_x = h_4 \in L^2(0,1),
\end{cases}
\]
where
\[
\begin{align*}
    h_1 &= (\rho_1 + |\mu| + \mu) g_1 + \rho_1 g_2 - \mu \tau_0 e^{-\tau} \int_0^1 e^{\tau \rho} g_9(x,s) ds, \\
h_2 &= \rho_2(g_3 + g_4) - \tau \gamma g_8, \\
h_3 &= \rho_1(g_5 + g_6), \\
h_4 &= -\gamma g_3 x - \rho_3 \left( g_7 - \tau \int_0^x g_8(y) dy \right).
\end{align*}
\]
The variational formulation corresponding to (2.7) takes the form
\[
B \left( (\varphi, \psi, \omega, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}, \tilde{q}) \right) = F(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}, \tilde{q}),
\]
where
where $B : \mathcal{H}^1_1(0,1) \times \tilde{\mathcal{H}}^1_1(0,1) \times L^2(0,1) \to \mathbb{R}$ is the bilinear form defined by

$$
B \left( (\varphi, \psi, \omega, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}, \tilde{q}) \right) = k \int_0^1 (\varphi_x + \psi + l\omega) \left( \varphi_x + \tilde{\psi} + l\tilde{\omega} \right) dx + (\beta + \tau) \int_0^1 q \tilde{q} dx + b \int_0^1 \psi_x \tilde{\psi}_x dx + \rho_2 \int_0^1 \psi \tilde{\psi} dx - \gamma(\beta + \tau) \int_0^1 \psi \tilde{\psi} dx + \gamma(\beta + \tau) \int_0^1 \tilde{\psi} \psi dx + \rho_1 \int_0^1 \varphi \tilde{\varphi} dx + \rho_1 \int_0^1 \omega \tilde{\omega} dx + k_0 \int_0^1 (\omega_x - l\varphi) (\tilde{\omega}_x - l\tilde{\varphi}) dx + \rho_3(\beta + \tau)^2 \int_0^1 \left( \int_0^x (q(y)dy \right)^2 dx.
$$

and $F : \mathcal{H}^1_1(0,1) \times \tilde{\mathcal{H}}^1_1(0,1) \times \tilde{\mathcal{H}}^1_1(0,1) \times L^2(0,1) \to \mathbb{R}$ is the linear functional given by

$$
F(\varphi, \tilde{\psi}, \tilde{\omega}, \tilde{q}) = \int_0^1 h_1 \varphi dx + \int_0^1 h_2 \tilde{\psi} dx + \int_0^1 h_3 \tilde{\omega} dx + (\beta + \tau) \int_0^1 h_4 \tilde{q} dy dx.
$$

Now, for $V = \mathcal{H}^1_1(0,1) \times \tilde{\mathcal{H}}^1_1(0,1) \times \tilde{\mathcal{H}}^1_1(0,1) \times L^2(0,1)$ equipped with the norm

$$
\|(\varphi, \psi, \omega, q)\|_V = \|(\varphi_x + \psi + l\omega)\|^2_2 + \|\omega_x - l\varphi\|^2_2 + ||\psi_x\|^2_2 + \|q\|^2_2,
$$

and combining with

$$
\int_0^1 (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx \leq c \int_0^1 (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2 dx,
$$

for $l$ small enough, it follows that $B$ and $F$ are bounded. Furthermore, using the definition of $B$, we get

$$
B((\varphi, \psi, \omega, q), (\varphi, \psi, \omega, q)) = k \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + (\beta + \tau) \int_0^1 q^2 dx + b \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi^2 dx + \rho_1 \int_0^1 \omega^2 dx + \rho_1 \int_0^1 \omega^2 dx + k_0 \int_0^1 (\omega_x - l\varphi)^2 dx + \rho_3(\beta + \tau)^2 \int_0^1 \left( \int_0^x (q(y)dy \right)^2 dx \\
\geq c\|(\varphi, \psi, \omega, q)\|^2_V.
$$

Thus, $B$ is coercive. Consequently, Lax-Milgram Lemma provides that system (2.7) has a unique solution $\varphi \in \mathcal{H}^1_1(0,1)$, $\psi, \omega \in \tilde{\mathcal{H}}^1_1(0,1)$, $q \in L^2(0,1)$. Substituting $\varphi, \psi, \omega, q$ into (2.5)1, (2.5)3, (2.5)5 and (2.5)8 respectively, we get $u \in \mathcal{H}^1_1(0,1), v \in \tilde{\mathcal{H}}^1_1(0,1), w \in \tilde{\mathcal{H}}^1_1(0,1), \theta \in \mathcal{H}^1_1(0,1)$.

If $\left( \tilde{\psi}, \tilde{\omega}, \tilde{q} \right) \equiv (0,0,0) \in \tilde{\mathcal{H}}^1_1(0,1) \times \mathcal{H}^1_1(0,1) \times L^2(0,1)$, then (2.8) reduces to

$$
k \int_0^1 (\varphi_x + \psi + l\omega) \tilde{\varphi}_x dx - k_0 \int_0^1 (\omega_x - l\varphi) \tilde{\varphi} dx + \rho_1 \int_0^1 \varphi \tilde{\varphi} dx = \int_0^1 h_1 \tilde{\varphi} dx,
$$

for all $\tilde{\varphi}$ in $\mathcal{H}^1_1(0,1)$, which implies

$$
-k \varphi_{xx} = k \psi_x + l(k + k_0) \omega_x - (k_0 l^2 + \rho_1) \varphi + h_1 \in L^2(0,1).
$$

Consequently, by the regularity theory for the linear elliptic equations, we obtain

$$
\varphi \in \mathcal{H}^2_1(0,1).
$$

Moreover, (2.9) is also true for any $\phi \in C^1([0,1]), \phi(0) = 0$ which is in $H^1_1(0,1)$. Hence, taking any $\phi \in C^1([0,1]), \phi(0) = 0$, one has

$$
k \int_0^1 \varphi_x \phi dx - \int_0^1 \left( k \psi_x + l(k + k_0) \omega_x - (k_0 l^2 + \rho_1) \varphi + h_1 \right) \phi dx = 0.
$$
Thus, using integration by parts and taking into account (2.10), we get
\[ \varphi_x(1)\phi(1) = 0, \forall \phi \in C^1([0, 1]), \phi(0) = 0. \]

Therefore,
\[ \varphi_x(1) = 0. \]

Similarly, we get
\[
\begin{align*}
-b\psi_{xx} &= -k\varphi_x - (k + \rho - 2)\psi - lk\omega - \gamma(\beta + \tau)q + h_2 \in L^2(0, 1), \\
-k\omega_{xx} &= -l(k + k_0)\varphi_x - lkk\psi + (\rho_1 + l^2k_0)\omega + h_3 \in L^2(0, 1), \\
-q_x &= \gamma\psi_x - (\beta + \tau)\rho_3 \int_0^x q(y)dy + h_4 \in L^2(0, 1).
\end{align*}
\]

Thus, we have
\[ \psi, \omega \in H^2_s(0, 1), \quad q \in H^1_s(0, 1), \quad \omega_x(0) = \psi_x(0) = 0. \]

Hence, there exists a unique \( \Phi \in D(\mathcal{A}) \) such that (2.4) is satisfied, which conclude that

the operator \( \mathcal{A} \) is maximal. With this, it is easy to obtain that \( \mathcal{A} \) is a maximal monotone operator. On the other hand, it is obvious that operator \( \mathcal{B} \) is Lipschitz continuous. Consequently, \( \mathcal{A} + \mathcal{B} \) is the infinitesimal generator of a linear contraction \( C_0 \)-semigroup on \( \mathcal{H} \).

This completes the proof (see [26] and [11]). \( \square \)

3. Exponential stability

In this section, we state and prove our stability result for the solution of system (2.1) by using the multiplier technique. We first introduce the following energy functional:
\[
E(t) = \frac{1}{2} \int_0^1 \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 \omega_t^2 + b\psi_x^2 + \rho_3 \theta^2 + \tau q^2 + k(\varphi_x + \psi + l\omega)^2 \right] dx
+ \frac{1}{2} \int_0^1 \left[ k_0(\omega_x - l\varphi)^2 + |\mu|\tau_0 \int_0^1 z^2(x, \rho, t)d\rho \right] dx.
\]

(3.1)

Our main result of this section reads as follows.

**Theorem 3.1.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1), assume that \(k = k_0\) and
\[
\xi := \left( \tau - \frac{\rho_1}{k\rho_3} \right) \left( \frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau\rho_1\gamma^2}{bk\rho_3} = 0.
\]

(3.2)

Then for \(|\mu|\) small enough, the energy functional (3.1) satisfies
\[ E(t) \leq k_1 e^{-k_2 t}, \forall t \geq 0, \]

(3.3)

where \(k_1, k_2\) are two positive constants.

We need the following lemmas to show that the associated energy non-increase in time.

**Lemma 3.2.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1), the energy functional defined by (3.1) satisfies
\[
E'(t) = -\beta \int_0^1 q^2 dx + |\mu| \int_0^1 \varphi_t^2 dx.
\]

**Proof.** (2.1)1, (2.1)2, (2.1)3, (2.1)4 and (2.1)5, by multiplying \(\varphi_t, \psi_t, \omega_t, \theta\) and \(q\) respectively, then integrating over \((0, 1)\) and summing up, using the boundary conditions, we get
\[
\frac{1}{2} \frac{d}{dt} \left\{ \rho_1 \int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \rho_1 \int_0^1 \omega_t^2 dx + b \int_0^1 \psi_x^2 dx + \rho_3 \int_0^1 \theta^2 dx + \tau \int_0^1 q^2 dx \right\}
+ \frac{1}{2} \frac{d}{dt} \left\{ \int_0^1 k_0(\omega_x - l\varphi)^2 dx + k \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + |\mu|\tau_0 \int_0^1 z^2(x, \rho, t)d\rho dx \right\}
\]
Let \( \phi(t) \) be the solution of (2.1), and using (2.1)\(_1\) and (2.1)\(_3\), we conclude that

\[
F_1(t) = - \rho_1 \int_0^1 (\varphi \phi + \omega) \, dx,
\]

satisfies

\[
F_1'(t) \leq - \rho_1 \int_0^1 \varphi^2 \, dx - \rho_1 \int_0^1 \omega^2 \, dx - \int_0^1 \phi \left( k (\varphi + \psi + l \omega) + l k_0 (\omega_x - l \varphi) - \mu z(x, 1, t) \right) \, dx
\]

for all \( \varepsilon > 0 \).

**Proof.** By differentiating \( F_1 \) and using (2.1)\(_1\) and (2.1)\(_3\), we obtain

\[
F_1'(t) = - \beta \int_0^1 \varphi^2 \, dx + |\mu| \int_0^1 \varphi^2 \, dx. \tag{3.4}
\]

Now, multiplying (2.1)\(_6\) by \( |\mu| z \) and integrating over \((0,1) \times (0,1)\), bearing in mind \( z(x, 0, t) = \varphi_t(x, t) \), we obtain

\[
\frac{|\mu| \tau_0}{2} \frac{d}{dt} \int_0^1 z^2(x, \rho, t) \, d\rho \, dx = \frac{|\mu|}{2} \int_0^1 \varphi^2 \, dx - \frac{|\mu|}{2} \int_0^1 z^2(x, 1, t) \, dx. \tag{3.5}
\]

The result follows by the combination of (3.4)-(3.5) and Young’s inequality. \( \square \)

**Lemma 3.3.** Let \( (\varphi, \psi, \omega, \theta, q, z) \) be the solution of (2.1). The functional

\[
F_1(t) = - \rho_1 \int_0^1 (\varphi \phi + \omega) \, dx,
\]

satisfies

\[
F_1'(t) \leq - \rho_1 \int_0^1 \varphi^2 \, dx - \rho_1 \int_0^1 \omega^2 \, dx + \int_0^1 \phi^2 \, dx + k_0 \int_0^1 (\omega_x - l \varphi)^2 \, dx
\]

\[
+ |\varepsilon| \int_0^1 (\varphi + \psi + l \omega)^2 \, dx + \varepsilon \int_0^1 z^2(x, 1, t) \, dx, \tag{3.6}
\]

for all \( \varepsilon > 0 \).

**Proof.** By differentiating \( F_1 \) and using (2.1)\(_1\) and (2.1)\(_3\), we conclude that

\[
F_1'(t) = - \beta \int_0^1 \varphi^2 \, dx + |\mu| \int_0^1 \varphi^2 \, dx
\]

Now, multiplying (2.1)\(_6\) by \( |\mu| z \) and integrating over \((0,1) \times (0,1)\), bearing in mind \( z(x, 0, t) = \varphi_t(x, t) \), we obtain

\[
\frac{|\mu| \tau_0}{2} \frac{d}{dt} \int_0^1 z^2(x, \rho, t) \, d\rho \, dx = \frac{|\mu|}{2} \int_0^1 \varphi^2 \, dx - \frac{|\mu|}{2} \int_0^1 z^2(x, 1, t) \, dx. \tag{3.5}
\]

The result follows by the combination of (3.4)-(3.5) and Young’s inequality. \( \square \)

**Lemma 3.4.** Let \( (\varphi, \psi, \omega, \theta, q, z) \) be the solution of (2.1). The functional

\[
F_2(t) = \rho_2 \int_0^1 \psi \, dx
\]

satisfies the estimate

\[
F_2'(t) \leq - \frac{b}{2} \int_0^1 \psi^2 \, dx + \rho_2 \int_0^1 \psi^2 \, dx + \int_0^1 (\varphi + \psi - l \omega)^2 \, dx + C \int_0^1 \theta^2 \, dx. \tag{3.7}
\]

**Proof.** Taking the derivative of \( F_2 \) with respect to \( t \) and using (2.1)\(_2\), it follows that

\[
F_2'(t) = \rho_2 \int_0^1 \psi^2 \, dx - \int_0^1 \psi (b \psi_{xx} - k (\varphi_x - \psi + l \omega) - \gamma \theta_x) \, dx.
\]

Using Young’s and Poincaré inequalities, we obtain (3.7). \( \square \)

**Lemma 3.5.** Let \( (\varphi, \psi, \omega, \theta, q, z) \) be the solution of (2.1). The functional

\[
F_3(t) = - \frac{\rho_2 \rho_3}{\gamma} \int_0^1 \psi \, dy \, dx,
\]

satisfies

\[
F_3'(t) \leq - \frac{\rho_2}{2} \int_0^1 \psi^2 \, dx + \varepsilon_3 \int_0^1 (\varphi + \psi + l \omega)^2 \, dx + \varepsilon_3 \int_0^1 \psi^2 \, dx + C(\varepsilon_3) \int_0^1 \theta^2 \, dx
\]
\[ + C \int_{0}^{1} q^2 dx. \]  

(3.8)

for all \( \varepsilon_3 > 0 \).

**Proof.** By differentiating \( F_3 \) and using (2.1)\_2 and (2.1)\_4, we get

\[
F'_3(t) = \frac{\rho_2}{\gamma} \int_{0}^{1} (q_x + \gamma \psi_t) \int_{0}^{x} \psi_t(y) dy dx - \frac{\rho_3}{\gamma} \int_{0}^{1} \theta \int_{0}^{x} (b \psi_{xx} - k(\varphi_x + \psi + \omega)) dy dx
\]

\[
= - \rho_2 \int_{0}^{1} \psi_x^2 dx - \frac{\rho_2}{\gamma} \int_{0}^{1} q \psi_t dx + \rho_3 \int_{0}^{1} \theta^2 dx - \frac{b \rho_3}{\gamma} \int_{0}^{1} \theta \psi_{xx} dx + \frac{b \rho_3}{\gamma} \int_{0}^{1} \theta \psi_x dx
\]

\[
+ \frac{k \rho_3}{\gamma} \int_{0}^{1} (\varphi_x + \psi + \omega) dx. \]

The result thanks to Young’s inequality. \( \square \)

**Lemma 3.6.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1). The functional

\[
F_4(t) = \tau \rho_3 \int_{0}^{1} \theta \int_{0}^{x} q(y) dy dx,
\]

satisfies

\[
F'_4(t) \leq - \frac{\rho_3}{2} \int_{0}^{1} \theta^2 dx + \varepsilon_4 \int_{0}^{1} \psi_t^2 dx + C(\varepsilon_4) \int_{0}^{1} q^2 dx,
\]

(3.9)

for all \( \varepsilon_4 > 0 \).

**Proof.** Differentiating \( F_4 \) with respect to \( t \), using (2.1)\_4 and (2.1)\_5, one has

\[
F'_4(t) = \tau \int_{0}^{1} \int_{0}^{x} (-q_x - \gamma \psi_t) q(y) dy dx + \rho_3 \int_{0}^{1} \theta \int_{0}^{x} (-\beta q - \theta_x) dy dx
\]

\[
= - \rho_3 \int_{0}^{1} \theta^2 dx + \tau \int_{0}^{1} q^2 dx + \tau \gamma \int_{0}^{1} \theta \psi_t dx - \beta \rho_3 \int_{0}^{1} \theta \int_{0}^{x} q dy dx.
\]

Then, we use Cauchy-Schwarz and Young’s inequalities with \( \varepsilon_4 > 0 \) to obtain (3.9). \( \square \)

**Lemma 3.7.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1). The functional

\[
F_5(t) = - \rho_1 \int_{0}^{1} \varphi_t(\omega_x - l \varphi) dx - \rho_1 \int_{0}^{1} \omega_t(\varphi_x + \psi + \omega) dx,
\]

satisfies

\[
F'_5(t) \leq - \frac{l k_0}{2} \int_{0}^{1} (\omega_x - l \varphi)^2 dx - \frac{l \rho_1}{2} \int_{0}^{1} \omega_x^2 dx + l \rho_1 \int_{0}^{1} \varphi_t^2 dx + k \int_{0}^{1} (\varphi_x + \psi + \omega)^2 dx
\]

\[+ \frac{\rho_1}{2l} \int_{0}^{1} \psi_x^2 dx + 2 l k_0 \int_{0}^{1} \mu^2 x^2 (x, 1, t) dx. \]

(3.10)

for all \( \varepsilon_5 > 0 \).

**Proof.** Differentiating \( F_5 \) with respect to \( t \), using (2.1)\_1 and (2.1)\_3, it follows that

\[
F'_5(t) = - \int_{0}^{1} \omega_t(\varphi_x + \psi + \omega) dx + \int_{0}^{1} (k(\varphi_x + \psi + \omega) - \mu z(x, 1, t))(\omega_x - l \varphi) dx
\]

\[+ \int_{0}^{1} (k_0(\omega_x - l \varphi) x - lk(\varphi_x + \psi + \omega))(\varphi_x + \psi + \omega) dx
\]

\[+ \rho_1 \int_{0}^{1} \varphi_t(\omega_x - l \varphi) dx - \rho_1 \int_{0}^{1} \omega_t(\varphi_x + \psi + \omega) dx
\]

\[+ \rho_1 \int_{0}^{1} (\varphi_x + \psi + \omega) dx + \int_{0}^{1} (\omega_x - l \varphi)^2 dx + \int_{0}^{1} \omega_t^2 dx + \int_{0}^{1} \omega_t dx + \int_{0}^{1} \omega_x^2 dx
\]

\[+ \rho_1 \int_{0}^{1} \psi_x dx + \int_{0}^{1} \mu z(x, 1, t)(\omega_x - l \varphi) dx.
\]
(3.10) follows Young’s inequality with the fact that \(k = k_0\). \(\blacksquare\)

**Lemma 3.8.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1). The functional

\[
F_b(t) = \tau_0 \int_0^1 \int_0^1 e^{-70\rho} z(x, \rho, t) d\rho dx
\]

satisfies

\[
F_b(t) \leq -\frac{k}{2} \int_0^1 (\varphi_0 + \psi + \omega)^2 dx + \frac{2b2^2}{k} \int_0^1 \psi_0^2 dx + C(\varepsilon) \int_0^1 \psi_0^2 dx + \varepsilon_8 \int_0^1 \omega_0^2 dx
\]

where \(m = \min\{e^{-70}, e^{-70\rho}\}\).

**Proof.** Similarly computation, using (2.1)\(_6\), we have

\[
F_b(t) = \frac{d}{dt} \int_0^1 \int_0^1 e^{-70\rho} z^2(x, \rho, t) d\rho dx - \tau_0 \int_0^1 \int_0^1 e^{-70\rho} z^2(x, \rho, t) d\rho dx
\]

It is obvious that result (3.11).

**Lemma 3.9.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1). The functional

\[
F_7(t) = -\rho_1 \int_0^1 (\varphi_x - l\varphi) \int_0^x \omega_t(y) dy dx - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_0 + \psi + \omega)(y) dy dx
\]

satisfies

\[
F_7(t) \leq \frac{\rho_1}{2} \int_0^1 \varphi^2 dx - \int_0^1 (\varphi_x - l\varphi) \int_0^x (k_0(\varphi_x - \omega_x) - lk(\varphi_0 + \omega)) dy dx
\]

\[
- \int_0^1 (k(\varphi_x + \psi + \omega)_x + l k_0(\varphi_x - \varphi) - \mu z(0, 1, t)) \int_0^x (\varphi_x + \psi + \omega)(y) dy dx
\]

\[
= \rho_1 \int_0^1 \omega_t^2 dx + \int_0^1 \mu z(x, 1, t) \int_0^x (\varphi_0 + \psi + \omega)(y) dy dx
\]

The result follows Young’s and Cauchy-Schwarz inequalities with the fact that \(k = k_0\). \(\blacksquare\)

**Lemma 3.10.** Let \((\varphi, \psi, \omega, \theta, q, z)\) be the solution of (2.1). The functional

\[
F_8(t) = \rho_2 \int_0^1 \psi_t(\varphi_0 + \psi + \omega) dx + \frac{b \rho_3}{k} \int_0^1 \varphi_t \psi dx + \frac{b \rho_3}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \psi dx
\]

satisfies

\[
F_8(t) \leq \frac{k}{2} \int_0^1 (\varphi_0 + \psi + \omega)^2 dx + \frac{2b^2 l^2}{k} \int_0^1 \psi_0^2 dx + C(\varepsilon) \int_0^1 \psi_0^2 dx + \varepsilon_8 \int_0^1 \omega_0^2 dx
\]
A differentiation of above functional gives

\[ C(\varepsilon_8) \int_0^1 q^2 dx + C(\varepsilon_8) \int_0^1 \theta^2 dx + \varepsilon_8 \int_0^1 (\omega_x - l \varphi)^2 dx + C \int_0^1 \mu z^2(x, 1, t) dx, \]

for all \( \varepsilon_8 > 0 \).

Proof. A differentiation of above functional gives

\[
F_3'(t) = \int_0^1 (b\psi_{xx} - k(\varphi_x + \psi + l\omega) - \gamma \theta_x)(\varphi_x + \psi + l\omega) dx
\]

\[ + \rho_2 \int_0^1 \psi_t(\varphi_x + \psi + l\omega) dx + \frac{b}{k} \int_0^1 (k(\varphi_x + \psi + l\omega)_x + l k_0(\omega_x - l \varphi) - \mu z(x, 1, t)) \psi_x dx \]

\[ + \frac{b \rho_1}{k \gamma} \int_0^1 \varphi_x (-\rho_3 \theta_t - q_x) dx + \frac{b}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 (-q_x - \gamma \varphi_t) \varphi_t dx \]

\[ + \frac{b \rho_3}{\gamma \rho_1} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta_t(k(\varphi_x + \psi + l\omega)_x + l k_0(\omega_x - l \varphi) - \mu z(x, 1, t)) dx \]

\[ - \frac{b}{\gamma \tau} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 (-\beta q - \theta_x)(\varphi_x + \psi + l\omega) dx - \frac{b}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q(\varphi_x + \psi + l\omega)_t dx \]

\[ - \frac{b l^2}{k_0} \int_0^1 (b \psi_{xx} - k(\varphi_x + \psi + l\omega) - \gamma \theta_x) \psi dx - \frac{b l^2}{k_0} \int_0^1 \psi_t dx \]

\[ + \frac{b l}{k_0} \int_0^1 (k_0(\omega_x - l \varphi)_x -lk(\varphi_x + \psi + l\omega)) \psi dx + \frac{b \rho_1 l}{k_0} \int_0^1 \omega_t \psi dx \]

\[ = -k \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \left( \rho_2 - \frac{bl^2}{k_0} \right) \int_0^1 \psi_t^2 dx + \left( \frac{l \rho_2}{k_0} + \frac{bl \rho_1}{k_0} \right) \int_0^1 \psi^2 dx \]

Noting that \( k = k_0 \) and \( \xi = 0 \), the above equation turns into

\[
F_3'(t) = -k \int_0^1 (\varphi_x + \psi + l\omega)^2 dx + \left( \rho_2 - \frac{bl^2}{k_0} \right) \int_0^1 \psi_t^2 dx + \left( \frac{l \rho_2}{k_0} + \frac{bl \rho_1}{k_0} \right) \int_0^1 \psi^2 dx
\]

\[ + \frac{b}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q \psi_t dx - \frac{bl}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q_\omega dx + \frac{b k_0 \rho_3}{\gamma \rho_1} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta(\omega_x - l \varphi)_t dx \]

\[ - \frac{\gamma b l^2}{k_0} \int_0^1 \theta \psi_t dx + \frac{b \beta}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q(\varphi_x + \psi + l\omega)_t dx + \frac{b^2 l^2}{k_0} \int_0^1 \psi^2 dx \]

\[ - \frac{b}{k} \int_0^1 \mu z(x, 1, t) \psi dx - \frac{b \rho_3}{\gamma \rho_1} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \mu z(x, 1, t)_t dx. \]

Using Young’s inequality, we get (3.13). \[\square\]

Now, we are ready to prove an exponential decay result under a smallness condition on the delay.
Proof of Theorem 3.1

We define a Lyapunov functional

\[ \mathcal{L}(t) := NE(t) + \sum_{i=1}^{8} N_i F_i(t), \]  

(3.14)

which it is equivalent to the energy functional \( E \). Now, gathering the estimates in Lemmas 3.3-3.10, we obtain

\[
\mathcal{L}'(t) \leq - \left( N_1 \rho_1 + \frac{\rho_1}{2} N \right) \int \varphi_t^2 \, dx \\
- \left( N_1 \rho_1 + \frac{\rho_1}{2} N - \varepsilon_8 N_8 \right) \int_0^1 \omega_t^2 \, dx \\
- \left( \frac{1}{2} N_2 - C N_1 - \varepsilon_3 N_3 - \frac{2b^2 l^2}{k} N_7 \right) \int \psi_t^2 \, dx \\
- \left( \frac{\rho_2}{2} N_4 - C N_2 - C(\varepsilon_3) N_3 - C(\varepsilon_8) N_8 \right) \int_0^1 \theta^2 \, dx \\
- \left( \frac{\rho_2}{2} N_3 - \rho_2 N_2 - \varepsilon_4 N_4 - \frac{\rho_1}{2} N_5 - \frac{\rho_1}{2} N_7 - C(\varepsilon_8) N_8 \right) \int \psi_t^2 \, dx \\
- \left( l k_0 N_5 - k_0 N_1 + k_0 N_7 - \varepsilon_8 N_8 \right) \int_0^1 (\omega_x - l \varphi)^2 \, dx \\
- (N \beta - C N_3 - C(\varepsilon_4) N_4 - C(\varepsilon_8) N_8) \int_0^1 q^2 \, dx \\
- \left( k N_8 - C(\varepsilon_1) N_1 - \frac{k^2}{b} N_2 - \varepsilon_3 N_3 - l k N_5 - \left( k + \frac{1}{2} \right) N_7 \right) \int \int_0^1 (\varphi_x + \psi + l \omega)^2 \, dx \\
- \left( m N_6 - N_1 \varepsilon_1 - 2l k_0 \mu^2 N_5 - \frac{\mu^2}{2} N_7 - C \mu N_8 \right) \int_0^1 z^2(x, 1, t) \, dx \\
- N_6 \mu \int_0^1 \int_0^1 z^2(x, \rho, t) \, d \rho \, dx.
\]

Then, we let

\[ N_6 = 1, \ N_1 = N_7 = l N_5 = \frac{3}{\rho_1}, \]

the choices yield

\[
\mathcal{L}'(t) \leq - \left( \frac{1}{2} - N \mu \right) \int \int_0^1 \varphi_t^2 \, dx - \left( \frac{3}{2} - N \varepsilon_8 \right) \int \int_0^1 \omega_t^2 \, dx \\
- \left( \frac{1}{2} N_2 - \left( C + \frac{2b^2 l^2}{k} \right) \frac{3}{\rho_1} - \varepsilon_3 N_3 \right) \int \int_0^1 \psi_t^2 \, dx \\
- \left( \frac{\rho_2}{2} N_3 - \rho_2 N_2 - \varepsilon_4 N_4 - 3 - C(\varepsilon_8) N_8 \right) \int \int_0^1 \psi_t^2 \, dx - \left( \frac{3k_0}{2\rho_1} - \varepsilon_8 N_8 \right) \int \int_0^1 (\omega_x - l \varphi)^2 \, dx \\
- \left( \frac{\rho_2}{2} N_4 - C N_2 - C(\varepsilon_3) N_3 - C(\varepsilon_8) N_8 \right) \int \int \theta^2 \, dx - m \tau_0 \int_0^1 \int_0^1 z^2(x, \rho, t) \, d \rho \, dx \\
- \left( \frac{k}{2} N_8 - C(\varepsilon_1) + \left( k + \frac{1}{2} \right) + k \right) \frac{3}{\rho_1} - \frac{k^2}{b} N_2 - \varepsilon_3 N_3 \right) \int_0^1 (\varphi_x + \psi + l \omega)^2 \, dx \\
- (N \beta - C N_3 - C(\varepsilon_4) N_4 - C(\varepsilon_8) N_8) \int_0^1 q^2 \, dx \\
- \left( m - \left( \varepsilon_1 + 2k_0 \mu^2 + \frac{1}{2} \mu^2 \right) \frac{3}{\rho_1} - C \mu N_8 \right) \int_0^1 z^2(x, 1, t) \, dx.
\]
As follows, we need to choose our constants carefully. we let \( \varepsilon_1 = \frac{\rho_1}{6m} \) and choose \( N_2 \) large, such that

\[
\frac{b}{2}N_2 - \left( C + \frac{2b^2\gamma^2}{k} \right) \frac{3}{\rho_1} \geq \alpha_1 > 0.
\]

At this point, we take \( N_8 \) large enough so that

\[
\frac{k}{2}N_8 - \left( C(\varepsilon_1) + \left( k + \frac{1}{2} \right) + k \right) \frac{3}{\rho_1} - \frac{k^2}{b}N_2 \geq \alpha_2 > 0,
\]

and then select \( \varepsilon_8 \) such that \( \varepsilon_8 \leq \min \left\{ \frac{3}{2N_8}, \frac{3k_0}{2\rho_1N_8} \right\} \). We choose \( N_3 \) large enough so that

\[
\frac{\rho_2}{2}N_3 - \rho_2N_2 - \varepsilon_4 N_4 - 3 - C(\varepsilon_8)N_8 \leq \alpha_3 > 0,
\]

and choose \( \varepsilon_3 \) such that \( \alpha_1 - \varepsilon_3 N_3 > 0 \) and \( \alpha_2 - \varepsilon_3 N_3 > 0 \). We then choose \( N_4 \) large enough so that

\[
\frac{\rho_3}{2}N_4 - CN_2 - C(\varepsilon_3)N_3 - C(\varepsilon_8)N_8 \geq 0,
\]

and choose \( \varepsilon_4 \) such that \( \alpha_3 - \varepsilon_4 N_4 > 0 \). We set \( N \) so large to satisfies

\[
N\beta - CN_3 - C(\varepsilon_4)N_4 - C(\varepsilon_8)N_8 > 0.
\]

Finally, by taking \( |\mu| \) so small that

\[
\frac{m}{2} - \left( 2k_0\mu^2 + \frac{1}{2}\mu^2 \right) \frac{3}{\rho_1} - C\mu N_8 > 0.
\]

Utilizing the definition of \( E(t) \), we have

\[
\mathcal{L}'(t) \leq -c_1 E(t).
\]

On the other hand, exploiting (3.14), we get

\[
(N - c_2)E(t) \leq \mathcal{L}(t) \leq (N + c_2)E(t),
\]

which deduces that

\[
\mathcal{L}'(t) \leq -k_2 \mathcal{L}(t), \quad \forall t > 0.
\]

A simple integration over \((0, 1)\) leads to

\[
\mathcal{L}(t) \leq \mathcal{L}(0)e^{-k_2t}.
\]

It gives the desired result in Theorem 3.1 when combined with the equivalence of \( \mathcal{L}(t) \) and \( E(t) \).

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