STRINGY POWER OPERATIONS IN TATE K-THEORY

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ABSTRACT. We study the loop spaces of the symmetric powers of an orbifold and use our results to define equivariant power operations in Tate K-theory. We prove that these power operations are elliptic and that the Witten genus is an $H_\infty$-map. As a corollary, we recover a formula by Dijkgraaf, Moore, Verlinde and Verlinde for the orbifold Witten genus of these symmetric powers. We outline some of the relationship between our power operations and notions from (generalized) Moonshine.

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1. Introduction

In recent years, a string theoretic result by Dijkgraaf, Moore, Verlinde, and Verlinde has found much attention by mathematicians. The celebrated paper [DMVV97] considers the symmetric powers $\mathcal{M}^n/\Sigma_n$ of a compact, closed, complex manifold $\mathcal{M}$ and expresses an invariant of these symmetric powers, called the \textit{orbifold elliptic genus}, merely in terms of the elliptic genus of $\mathcal{M}$ itself:

$$\sum_{n \geq 0} \phi_{\text{orb}}(\mathcal{M}^n/\Sigma_n) t^n = \exp \left( \sum_{m \geq 1} T_m(\phi(\mathcal{M})) t^m \right).$$

Here the $T_m$ are Hecke operators. In [BL03a], [BL05], Borisov and Libgober proved (1), using algebro-geometric methods, and proceeded to prove a McKay correspondence result for $\phi_{\text{orb}}$. In [Tam01], Tamanoi made a connection to homotopy theory. In [Tam03] and [Tam], he explored the geometric side of Formula (1), generalizing the result to equivariant manifolds and higher genus world-sheets and giving a beautiful mathematical account of the loop space picture underlying the geometry of (1), which is closer to the original argument in [DMVV97] and [Dij99] than the proof in [BL03b].

In [Gan06], one finds another link to homotopy theory: interpreting the elliptic genus as a natural transformation between cohomology theories (namely complex cobordism and elliptic cohomology), I proved that Formula (1) holds whenever the natural transformation preserves cohomology operations.

From this point of view it is no surprise that some of the non-equivariant picture in [DMVV97] and [Tam03] was independently discovered by Lupercio, Uribe, and Xicoténcatl, in [LUX], as part of a program to define cohomology operations in Chas-Sullivan cohomology of (the Borel construction of) orbifolds.

However striking the formal analogies between the homotopy theoretic notions introduced in [Gan06] on one hand and results from orbifold string theory on the other, in [Gan06], they remained analogies. The present paper aims to bridge between the different points of view by studying a version of equivariant elliptic cohomology introduced by Devoto, whose objective was to capture the behavior of orbifold loop spaces.

To motivate our results, let us recall the setup of [Gan06] in more detail: let Ell be an elliptic cohomology theory as defined in [AHS01], and let

$$\{\text{Ell}_G \mid G \text{ is a finite group}\}$$

be a compatible family of equivariant versions of Ell. Let $\phi: MU \to \text{Ell}$ be a map of ring spectra, and assume that we are given equivariant versions $\phi_G: MU_G \to \text{Ell}_G$ of $\phi$. The reader not familiar with elliptic cohomology should merely note that, typically, one can interpret elements $\chi \in \text{Ell}_G(pt)$ as class functions on pairs of commuting elements of $G$, in which
case we say that \( \text{Ell} \) has a Hopkins-Kuhn-Ravenel (HKR) character theory. Two important examples of such elliptic cohomology theories with HKR-theory are Borel-equivariant Lubin-Tate-Morava \( \mathbb{E}_2 \)-theory [HKR00] and Devoto’s equivariant Tate \( K \)-theory [Dev96]. The former was the original object of study in [HKR00] and [Gan06]; the latter will be the framework for the paper at hand.

**Definition 1.1** (Orbifold elliptic genera). Assume that \( \text{Ell}_G \) has a Hopkins-Kuhn-Ravenel theory, and let \( \chi \in \text{Ell}_G(\text{pt}) \). We define

\[
\varepsilon_G(\chi) := \frac{1}{|G|} \sum_{gh=hg} \chi(g,h),
\]

and

\[
\phi_{\text{orb}} := \varepsilon_G \circ \Phi_G.
\]

Often, \( \varepsilon_G \) and thus \( \phi_{\text{orb}} \) take values in the coefficient ring \( \text{Ell}_* \). Consider the following diagram:

\[
\begin{array}{ccccccc}
\text{MU}_* & \xrightarrow{\phi} & \text{Ell}_* \\
\oplus \mathbb{P}_n & \downarrow & \oplus \text{Ell}_{\Sigma_n} \cdot t^n \\
\oplus \mathbb{MU}_{\Sigma_n} \cdot t^n & \oplus \text{Ell}_{\Sigma_n} \cdot t^n & \text{Ell} [t]
\end{array}
\]

Here the left vertical arrow is the total power operation in cobordism,

\[
\mathbb{P}_n([M]) = [M^n \emptyset \Sigma_n],
\]

and \( t \) is a dummy variable. The horizontal dotted arrow exists if \( \text{Ell} \) has a Hopkins-Kuhn-Ravenel theory, and then the composite of the left vertical arrow with the two lower horizontal arrows becomes the left-hand side of the DMVV formula (1).

Assume now that \( \text{Ell} \) possesses power operations

\[
\mathbb{P}_n : \text{Ell}_G(X) \to \text{Ell}_G(X^n).
\]

For \( X \) the one point space and \( G \) the trivial group, this implies that the (right) vertical dotted arrow of the above diagram exists. The map \( \phi \) is called an \( H_{\infty} \)-map if for each \( n \), the equality

\[
\mathbb{P}_n^{\text{Ell}} \circ \phi = \phi_{\Sigma_n} \circ \mathbb{P}_n^{\text{MU}}
\]

holds. For such \( \phi \), the diagram commutes.

If \( \text{Ell} \) has a Hopkins-Kuhn-Ravenel theory, one can use the \( \mathbb{P}_n \) to define Hecke operators

\[
T_n : \text{Ell}(X) \to \text{Ell}(X)
\]

(c.f. [And92]). The \( \mathbb{P}_n \) are called elliptic if this definition of Hecke operators agrees with another one, which is in terms of isogenies [AHS04]. By [Gan06], the composite of the dotted arrows is

\[
\sum_{n \geq 0} \varepsilon_{\Sigma_n} (\mathbb{P}_n(x)) t^n = \exp \left( \sum_{m \geq 1} T_m(x) t^m \right).
\]

If \( \phi \) is an \( H_{\infty} \)-map, this implies formula (1).
1.1. Plan. In the present paper, we treat the case where $\text{Ell}$ is Devoto’s equivariant Tate $K$-theory and $\phi$ is the equivariant Witten genus. Section 3 introduces the equivariant Witten genus as a map of spectra. Motivated by the work of Lupercio, Uribe and Xicoténcatl on orbifold loop spaces, we define Thom classes in Devoto $K$-theory for $G$-equivariant $U_2$-bundles, where $G$ is a finite group. We prove that the induced map of spectra realizes the $G$-equivariant Witten genus defined in [dFLNU06]. We prove that the associated orbifold genus is Morita-invariant and takes values in

$$q^{-d/12}K_{\text{Tate}}(pt) = q^{-d/12}\mathbb{Z}[q],$$

where $d$ is the complex dimension of the manifold.

We proceed to study loop spaces of symmetric powers of orbifolds: If $M$ is a manifold, then a loop in $M^n//\Sigma_n$ is given by $n$ paths in $M$ and a permutation $\sigma$, where for each $i$, the end point of the path $\gamma_i$ is the starting point of the path $\gamma_{\sigma(i)}$. For each $k$-cycle of $\sigma$, this produces one loop of length $k$ in $M$. The fact that the loop space of $M^n//\Sigma_n$ is made up from loops in $M$ is the key argument in [DMVV97]. For the symmetric powers of a (global quotient) orbifold $M//G$, a similar picture is true, but the situation is complicated by the additional action of $G$.

In Section 4, we prove a theorem expressing the orbifold loop spaces of the symmetric powers of $M//G$ in terms of the orbifold loop space of $M//G$. This is a variation on Tamanoi’s result, and we claim no originality. However, in order to introduce the notation for Section 5, we give a complete proof. We also show that the construction is compatible with iterated symmetric powers.

In Section 5, we use the analysis of 4 to define equivariant power operations in Devoto $K$-theory. We compute a formula for the Hecke operators associated to our power operations and conclude that the $P_n$ are elliptic. Interestingly, this agrees exactly with the formula for the twisted Hecke operators in generalized Moonshine found in [Gan]. If $G$ is the trivial group, we recover the operators of [And00].

We proceed to prove that a variant of the equivariant Witten genus is an $H_\infty$-map. As a corollary, we obtain the Dijkgraaf-Moore-Verlinde-Verlinde formula 1 for our equivariant Witten genus.

The total symmetric power $\text{Sym}^{\text{string}}_t$ is defined as the composite of the two dotted arrows in Diagram (2). It is computed by Formula (3). If $\phi(M)$ is the Witten genus of $M$, then $\text{Sym}^{\text{string}}_t(\phi(M))$ becomes the right-hand side of the DMVV-formula. On the other hand, we compute $\text{Sym}^{\text{string}}_t(V)$, where $V$ is a complex vector bundle, and find that it equals Witten’s exponential characteristic class

$$\text{Sym}^{\text{string}}_t(V) = \bigotimes_{k \geq 1} \text{Sym}^{t_k}(V),$$

where $\text{Sym}^{t_k}$ is the total symmetric power in $K$-theory. This sheds some light on the fact that the roles of $t$ and $q$ become symmetric, when the right-hand side of (1) is written in its product form

$$\prod_{i,j} \left( \frac{1}{1 - q^i t^j} \right)^{c(ij)}. $$
Here the $c(i)$ are the coefficients of the Witten genus of $M$,

$$\phi(M) = \sum c(i)q^i.$$  

To conclude this introduction, I will comment in some more detail on how this paper compares to [Gan06] and [DMVV97].

In [Gan06] I worked in a purely homotopy-theoretic setup and applied my results to the $\sigma$-orientation of Morava $E_2$-theory, which was shown to be $H_\infty$ in [AHS04]. Morava $E$-theory has no known geometric definition, and the fact that I could reproduce a DMVV-type formula for the $\sigma$-orientation suggests a deep and somewhat mysterious connection between homotopy-theoretic notions such as the $K(n)$-local categories and string-theoretic notions such as orbifold genera.

The spectrum $K_{\text{Tate}}$ is related to both of these: It is an elliptic spectrum, and Devoto’s equivariant versions fit into the general framework of equivariant elliptic cohomology and level structures on elliptic curves, axiomatically formalized in [GKV]. Further, when restricted to $MU(6)$, the Witten genus becomes the $\sigma$-orientation of $K_{\text{Tate}}$ (cf. [AHS01]).

On the other hand, as we have explained in [AFG], the Witten genus is closely related to the genus considered by Borisov and Libgober. Moreover, Devoto’s definition of $K_{\text{Dev}}$ was inspired by orbifold loop spaces, and so are our definitions of Thom classes and power operations.

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1.2. Notation index.

- $\text{age}(g)$ see Page 13,
- $C_g = C_G(g)$ the centralizer of $g$ in $G$,
- $C^k_g$ the group $C_g \times (\mathbb{Z}/k|g|\mathbb{Z})/(g, -k)$, c.f. Definition 2.2,
- $C^{\mathbb{R}/k\mathbb{Z}}_g$ the group $C_g \times (\mathbb{R}/k|g|\mathbb{Z})/(g, -k)$, c.f. Definition 2.2,
- $\text{det}$ determinant line bundle (top exterior power) of a complex line bundle
- $D_n$ c.f. Definitions 4.7 and 4.10,
- $d_n$ c.f. Definition (10),
- $e^\hat{A}$ K-theory Euler class corresponding to the $\hat{A}$-genus,
- $e^{Td}$ K-theory Euler class corresponding to the Td-genus,
- $\text{eq}(f, g)$ equalizer of the maps $f$ and $g$,
- $F_{[g, \sigma]}$ defined in Theorem 4.3,
- $f_{[g, \sigma]}$ defined on page 25,
- $G$ one of the groups Spin$(2n)$ or $U(n)$,
- $(g, \sigma)$ an element $((g_1, \ldots, g_n), \sigma)$ of the wreath product $G \wr \Sigma_n$,
- $K_{G, \text{Dev}}$ Devoto equivariant $K$-theory, c.f. Definition 3.1,
- $K_{G, \text{Dev}, r}$ Devoto equivariant $K$-theory for loops of length $r$, c.f. Definition 3.1,
- $K_G$ $G$-equivariant $K$-theory,
\( k \mathcal{L}(M//G) \) orbifold loops of length \( k \) in \( M//G \), c.f. Definition 2.3,
\( k \mathcal{V} \) the loop bundle \( \mathcal{V} \) rescaled, c.f. Definition 4.6,
\( k(-) \) see Definition 5.9,
\( \Lambda_t \) total exterior power
\( \Lambda(M//G) \) the inertia orbifold \( \bigsqcup_{[g]} M^g//C_g \),
\( M//G \) global quotient orbifold of an action of \( G \) on \( M \),
\( M^g \) the \( g \)-fixed points of \( M \),
\( MU \) complex cobordism,
\( MUP \) periodic complex cobordism, c.f. Example A.3,
\( MSpinP \) periodic spin cobordism, c.f. Example A.3,
\( N^g \) normal bundle of \( M^g \) in \( M \),
\( p^n = p^k_n \) \( n \)th Atiyah power operation in \( K \)-theory,
\( p^n_{\text{string}} \) \( n \)th stringy power operation on \( K_{\text{Dev}} \), c.f. Definition 5.10,
\( p^n_{\text{top}} \) \( n \)th Atiyah power operation on \( K_G[q] \), c.f. Definition 5.2 or on \( K_{\text{Dev}} \), c.f. page 31,
\( P_t \) total power operation,
\( \pi_t \) push-forward along the map \( \pi \) (typically \( \pi \) is the unique map to the one point space), c.f. (15),
\( q \) the defining representation of the circle group \( \mathbb{R}/\mathbb{Z} \),
\( q^j \) the representation of \( \mathbb{R}/\mathbb{Z} \) obtained by scaling \( q \) with a factor \( 1/l \),
\( q^j \) the \( j \)th tensor power \( (q^j)^{\otimes c j} \),
\( S_{\text{E}}(X) \) (symmetric algebra) the graded ring \( \bigoplus_{k \geq 0} E_{G\wr \Sigma_n}(X^n)^{t^n} \), c.f. page 28,
\( \text{Sym}_t \) total symmetric power,
\( U^2 \)-bundle a vector bundle with compatible Spin- and complex structures, c.f. (17),
\( u^\wedge \) \( K \)-theory Thom class corresponding to the \( \wedge \)-genus,
\( u^\text{Td} \) \( K \)-theory Thom class corresponding to the Td-genus,
\( u^\text{string} \) stringy (equivariant) \( \wedge \)-Thom class,
\( v^\text{string} \) stringy (equivariant) Td-Thom class,
\( V_i \) the summand of the bundle \( V|_{M^g} \) fixed by \( g \),
\( (V_C)_\zeta \) the summand of the complexified bundle \( V_C|M^g \) on which \( g \) acts with eigenvalue \( \zeta \),
\( X \) an orbifold,
\( X^\mathcal{V} \) Thom space of a vector bundle \( \mathcal{V} \) over a space \( X \),
\( \zeta_t \) the primitive \( t \)th root of unity \( e^{2\pi i/t} \),
\( g \sim h \) the group elements \( g \) and \( h \) are conjugate,
\( [g] \) conjugacy class of \( g \),
\( M \rtimes G \) the translation groupoid of an action of \( G \) on \( M \),
\( \boxtimes \) external tensor product (different groups), c.f. Definition 3.3,
\( \otimes \) depending on the context, external or internal tensor product over \( G \).
2. Localization and Fourier expansion

This section introduces some basic definitions. We start with the spaces modelling loops of length $k$ in $M//G$. Our definitions are a generalization of the familiar case $k = 1$ (see for example [dFLNU06]). We need to include the case $k > 1$ for technical reasons: it will be necessary when we consider iterated symmetric powers. For the time being the reader is invited to ignore this issue and to set $k = 1$. Throughout the paper we will drop $k$ from the notation if $k = 1$. All our paths are piecewise smooth, and path-spaces carry the compact-open topology.

2.1. Orbifold loop spaces. Let $G$ be a finite group acting smoothly from the right on a manifold $M$. Let $g$ be an element of $G$, and let $l = |g|$ be the order of $g$. We write $C_g = C_G(g)$ for the centralizer of $g$ in $G$.

**Definition 2.1.** For a natural number $k$, we define

$$kP_g M := \{ \gamma: [0, k] \to M \mid \gamma(k) = \gamma(0)g \}$$

and

$$kL_g M := \text{maps}_{Z/lZ}(R/k\mathbb{Z}, M)$$

($Z/lZ$-equivariant maps), where $1 \in Z/lZ$ acts as $k$ on $R/k\mathbb{Z}$ and as $g$ on $M$.

The centralizer $C_g$ acts on both of these spaces via its action on $M$, and there is a $C_g$-equivariant homeomorphism

$$kP_g M \cong kL_g M \quad \gamma \mapsto \gamma \ast g \ast \cdots \ast g^{l-1}.$$  

*Figure 1.* A path $\gamma \in P_g M$ and its image in $L_g M$. Here $l = 4$.

The group $\mathbb{R}/k\mathbb{Z}$ (and hence also its subgroup $\mathbb{Z}/k\mathbb{Z}$) acts on $kL_g M$ by rotation of the loops. Note that this action commutes with that of $C_g$, and that the actions of $k \in \mathbb{R}/k\mathbb{Z}$ and $g \in C_g$ agree. This motivates the following definition.

**Definition 2.2.** Let $k$ be a natural number. We write $C^k_g$ or $C^k_G(g)$ for the quotient of $C_g \times \mathbb{Z}/k\mathbb{Z}$ by the normal subgroup generated by $(g^{-1}, k)$, and we write $C^\mathbb{R}/k_g$ for the quotient of $C_g \times \mathbb{R}/k\mathbb{Z}$ by the same normal subgroup.
It follows from the discussion above that we have actions of $C_{g}^{R/k}$ and $C_{k}^{g}$ on $k\mathcal{L}_g M$. Using the homeomorphism (4), we obtain $C_{g}^{R/k}$ and $C_{k}^{g}$ actions on $kP_g M$. Throughout the paper we will use the homeomorphism (4) to identify $k\mathcal{L}_g M$ with $kP_g M$ and, by abuse of notation, we will write $k\mathcal{L}_g M$ for both.

**Definition 2.3.** Let $k\mathcal{L}(M//G)$ be defined by

$$k\mathcal{L}(M//G) := \coprod_{[g]} k\mathcal{L}_g M//\mathcal{C}_{g}^k,$$

where the disjoint union is over the conjugacy classes of $G$.

Think of $k\mathcal{L}(M//G)$ as the groupoid of $k$ open strings in $M//G$ joining together to form one long closed string, where the order of the $k$ strings does not matter (see Figure 2).

![Figure 2](image_url)

**Figure 2.** An element $\gamma = \gamma_1 \cdots \gamma_5 \in k\mathcal{L} M$ for trivial $G$.

### 2.2. Representations of $\mathbb{R}/l\mathbb{Z}$.

The goal of this section is to set the stage for our Fourier expansion principle in Section 2.3.

**Definition 2.4.** We let

$$q : \mathbb{R}/\mathbb{Z} \to \mathbb{U}(1) \quad t \mapsto e^{2\pi it}$$

denote the fundamental (complex) representation of the circle group $\mathbb{R}/\mathbb{Z}$. More generally, we write

$$q^l : \mathbb{R}/\mathbb{Z} \to \mathbb{U}(1) \quad t \mapsto e^{2\pi it/l}$$

for the representation of the long circle $\mathbb{R}/l\mathbb{Z}$ obtained by scaling the exponent $t$ by $1/l$.

Note that

$$1 \in \mathbb{Z}/l\mathbb{Z} \subseteq \mathbb{R}/l\mathbb{Z}$$

acts on $q^l$ by multiplication with a primitive $l^{th}$ root of unity, which we will denote $\zeta_l$. Recall that all non-trivial irreducible real representations of $\mathbb{R}/l\mathbb{Z}$ are of the form

$$q^l_r := \text{res} |_{\mathbb{R}}^{C_r} q^l,$$
where $\text{res}|_{\mathbb{C}}^\mathbb{C} \rho$ denotes the underlying real vector space of $\rho$. Further, $q^{\dagger} \cong_{\mathbb{R}} q^{\dagger}$ if and only if $i = \pm j$. Let $\mathcal{G}$ be one of the groups $\mathfrak{U}(n)$ or $\text{Spin}(2n)$, and let $\rho: \mathbb{R}/l\mathbb{Z} \to \mathcal{G} \to \text{SO}(2n)$ be a $\mathcal{G}$-representation of $\mathbb{R}/l\mathbb{Z}$ with underlying vector space $V$. Then $V$ decomposes into a direct sum

$$V \cong_{\mathbb{R}} V_0 \bigoplus_{j \geq 1} V_j q^{\dagger}.$$  

Here $V_j q^{\dagger}$ denotes the summand with rotation number $j$, and each of the summands is a complex/Spin representation. The notation “$V_j q^{\dagger}$” is motivated as follows: for $j \geq 1$, we can endow the underlying vector space $V_j$ of $V_j q^{\dagger}$ with a complex structure, where multiplication with $i$ is defined as the action of $\frac{1}{4j}$. Then

$$V_j q^{\dagger} = \text{res}|_{\mathbb{R}}^\mathbb{C} \left( V_j \otimes_{\mathbb{C}} (q^{\dagger})^\otimes j \right).$$

**Warning:** if the structure group $\mathcal{G}$ is $\mathfrak{U}(n)$, then $V$ decomposes as a complex representation

$$V \cong_{\mathbb{C}} \bigoplus_{j \in \mathbb{Z}} W_j q^{\dagger}.$$ 

However, (5) endowed with the complex structure defined by the rotations is

$$V \cong_{\mathbb{R}} W_0 \bigoplus_{j \geq 1} (W_{-j} \oplus W_j) q^{\dagger}.$$  

2.3. **Loop bundles and Fourier decomposition.** We are now ready to define loop vector bundles over orbifold loop spaces. The definition is somewhat technical; our main interest will be in the motivating Example 2.6.

**Definition 2.5.** Let $\mathcal{G}$ be one of the groups $\mathfrak{U}(n)$ or $\text{Spin}(2n)$. Let $G$ be a finite group acting smoothly from the right on $M$, let $g \in G$ have order $|g| = 1$. An (orbifold) $S^1$-equivariant $\mathcal{L}G$-vector bundle over $k\mathcal{L}gM//C^\circ_g$ is a (typically infinite dimensional) vector bundle $V$ over $k\mathcal{L}gM$ with structure group $k\mathcal{L}G$, which is $C^\circ_{g/k}$-equivariant as $k\mathcal{L}G$-bundle (i.e., the groups act by $k\mathcal{L}G$-bundle automorphisms), such that the action of $\mathbb{R}/k\mathbb{Z}$ on $V$ intertwines with its action on $k\mathcal{L}G$.

**Example 2.6.** Let $V \in \text{Vect}(X)$ be a $\mathcal{G}$-vector bundle on the orbifold $X$. View $V$ as an orbifold over $X$. Then the loop orbifold $\mathcal{L}V$ of $V$ is an orbifold $S^1$-equivariant $\mathcal{L}G$-vector bundle over $\mathcal{L}X$.

**Definition 2.7.** Two $\mathcal{L}G$-vector bundles $V$ and $W$ over a space $X$ are called densely isomorphic if there are dense sub-bundles $V' \subseteq V$ and $W' \subseteq W$ and an isomorphism of vector bundles with structure group $\mathcal{L}G$ between $V'$ and $W'$. If in addition $V$ and $W$ are $G$-equivariant bundles over a $G$-space $X$, then $V$ is densely isomorphic to $W$ as a loop bundle with this extra structure, if the inclusions and the vector bundle isomorphisms preserve this extra structure. We will denote dense isomorphisms by “$\cong$” and often drop the word “densely”.

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Let $\mathcal{V}$ be an orbifold $S^1$-equivariant $\mathcal{L}\mathcal{G}$ vector bundle over $k\mathcal{L}_g\mathcal{M}/G^k$. The condition that the action of $\mathbb{R}/kl\mathbb{Z}$ on $\mathcal{V}$ intertwines with that of $kl\mathcal{L}\mathcal{G}$ spells out to the following: the two actions combine to a (left-) action of $\mathbb{R}/kl\mathbb{Z} \ltimes kl\mathcal{L}\mathcal{G}$ on $\mathcal{V}$, where
\[(t, \gamma') \in \mathbb{R}/kl\mathbb{Z} \ltimes kl\mathcal{L}\mathcal{G}\]
maps $v \in V_\gamma$ to
\[(t, \gamma') \cdot v = t \cdot (\gamma' \cdot v) = (t \cdot \gamma')(t \cdot v) \in V_{t\gamma}.\]
Restricting to constant loops, we get an action of $\mathbb{C}\mathbb{R}/k\mathbb{G}$ on $V|_{M^g}$, where $\mathbb{R}/kl\mathbb{Z} \times G$ acts fiber preserving. Therefore, the restriction $V|_{M^g}$ decomposes into a direct sum of $\mathbb{R}/kl\mathbb{Z}$-representations
\[V|_{M^g} \cong \mathbb{R} V_0 \oplus \bigoplus_{j \geq 1} V_j q^{\frac{j}{l}}\]
as in (5) above, where $V_j q^{\frac{j}{l}}$ is a (finite dimensional) $\mathbb{C}\mathbb{R}/k\mathbb{G}$-equivariant $G$-vector bundle over $M^g$. The actions of $g \in C_g$ and $k \in \mathbb{R}/kl\mathbb{Z}$ on $V_j$ agree, so that $g$ acts on $V_j$ as (complex) multiplication by $\zeta_j^{l}$.

**Example 2.8.** Let $V$ be a finite dimensional real vector space, and recall (e.g. from [HBJ92, 6]) that there is a dense isomorphism of real $\mathbb{R}/l\mathbb{Z}$-representations maps $\mathbb{R}/l\mathbb{Z}, V \cong V_1 \oplus \bigoplus_{j \geq 1} (V_\mathbb{C})_{\zeta_j^l} q^{\frac{j}{l}}$ given by the Fourier expansion principle. Here $V_\mathbb{C}$ denotes the complexification of $V$. Let now $\langle g \rangle$ act on $V$, where $|g| = l$. Note that $1 \in \mathbb{R}/l\mathbb{Z}$ acts by multiplication with $\zeta_j^l$ on the $j^{th}$ summand of the right-hand side. A loop is $\mathbb{Z}/l\mathbb{Z}$-equivariant, if and only if its Fourier expansion $\sum v_j q^{\frac{j}{l}}$ satisfies
\[(\forall j \geq 0) \quad v_j g = \zeta_j^l v_j.\]
Hence we have
\[\text{maps}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}, V) \cong V_1 \oplus \bigoplus_{j \geq 1} (V_\mathbb{C})_{\zeta_j^l} q^{\frac{j}{l}}.\]
Here $(V_\mathbb{C})_{\zeta}$ denotes the $\zeta$-eigenspace of the complexification of $V$, and similarly, $V_1$ denotes the vectors of $V$ fixed by $g$.

**Example 2.9.** Let $\mathcal{V} = \mathcal{L}(\mathcal{V}/G)$ be the orbifold loop bundle of a $G$-equivariant $G$-vector bundle over $M$ as in Example 2.6. Let $l$ be the order of $g \in G$, and let $m$ be a point in $M^g$. We have
\[\mathcal{L}_g V|_m \cong \text{maps}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}, V|_m)\]
and hence
\[\mathcal{L}_g V|_{M^g} \cong V_1 \oplus \bigoplus_{j \geq 1} (V_\mathbb{C})_{\zeta_j^l} q^{\frac{j}{l}},\]
where $(V_\mathbb{C})_{\zeta}$ denotes the $\zeta$-eigenbundle of the action of $g$ on $V_\mathbb{C}|_{M^g}$. (Compare also [dFLNU06, (5.1.1)].)
3. Thom classes

3.1. Devoto’s equivariant Tate $K$-theory. Taking $g$-fixed points is a functor

$$(-)^g: G \rightarrow \text{-spaces} \rightarrow C_g \rightarrow \text{-spaces}$$

which preserves cofibre sequences, wedges and weak equivalences. Therefore

\begin{equation}
X \mapsto \bigoplus_{[g]} K_{C_g}(X^g)[\mathbb{q}^{\frac{1}{|g|}}]
\end{equation}

satisfies the axioms of a $G$-equivariant cohomology theory in [May96]. If $g$ is conjugate to $h$ in $G$ (denoted $g \sim h$) there is a canonical isomorphism

$$K_{C_g}(X^g) \cong K_{C_h}(X^h).$$

Thus (7) is, up to canonical isomorphism, independent of choices of the representatives. Example 2.9 motivates the following definition.

Definition 3.1. Consider the ring

$$\bigoplus_{[g]} K_{C_g}(X^g)[\mathbb{q}^{\frac{1}{|g|}}].$$

Note that in the $[g]$-summand, the coefficient $a^{[g]}_j$ is a virtual $C_g^r$-equivariant vector bundle. The ring $K_{Dev,G,r}(X)$ is defined as the subring of those power series where

$$1 \in \mathbb{Z}/r|g|\mathbb{Z} \subseteq C_g^r$$

acts by complex multiplication with $c^j_{r|g|}$ on the coefficient $a^{[g]}_j$. We will refer to this ring as the Devoto equivariant Tate $K$-theory ring (for strings of length $r$) of $X$. As usual, if $r = 1$, we will drop it from the notation.

Remark 3.2. For $r = 1$, the condition of the definition should be compared to Condition (a) of the generalized Moonshine axioms (c.f. [Nor87]): the Möbius transformation $\tau \mapsto \tau + 1$ (sending $\mathbb{q}^{\frac{1}{|g|}}$ to $\mathbb{q}^{\frac{1}{|g|}}c_{|g|}^j$) should correspond to the transition from $(g, h)$ to $(g, gh)$, i.e., to the action of $g$ on the summand corresponding to $[g]$.

Definition 3.3. Let $\alpha: H \rightarrow G$ be a map of finite groups, and let $x \in K_{Dev,G}(X)$. Then

$$(\text{res}_{[\alpha]} x)_{[h]} := x_{[\alpha(h)]}.$$ 

Let $x \in K_{Dev,G}(X)$ and $y \in K_{Dev,H}(Y)$. Then the external tensor product of $x$ and $y$ is the element

$$x \boxtimes y \in K_{Dev,G \times H}(X \times Y)$$

defined by

$$\ (x \boxtimes y)_{[g,h]} := x_{[g]} \boxtimes y_{[h]} \in K_{C_{C(g)} \times C_{C(h)}}(X^g \times Y^h)[\mathbb{q}^{\frac{1}{r|g|h}}],$$

where $N$ is the smallest common multiple of $|g|$ and $|h|$, $\boxtimes$ is the external tensor product of vector bundles, and $\mathbb{q}^r \boxtimes \mathbb{q}^s := \mathbb{q}^{r+s}$. If $G = H$, the $G$-equivariant external tensor product is defined as

$$x \boxtimes y := \text{res}_{[\alpha]}(x \boxtimes y),$$
where \( \delta \) denotes the diagonal inclusion of \( G \) in \( G \times G \). If in addition \( X = Y \), then the internal tensor product of \( x \) and \( y \) is defined to by the pullback along the diagonal \( d \) of \( X \times X \) of \( x \otimes y \):

\[
x \otimes y := d^* (x \otimes y).
\]

### 3.2. Stringy orbifold Thom classes.

This section serves as a morivation for Definition 3.4. Much of the discussion here is based on the circle of ideas described in the appendix. Let \( G \) be a finite group, let \( g \) be an element of \( G \), and let \( l \) be the order of \( g \). Let \( V \) be a \( G \)-equivariant Spin\((2d)\)-vector bundle over a \( G \)-space \( X \). By Example 2.9, we have

\[
(8) \quad \mathcal{L}_g V|_{X^g} \cong V_1 \oplus \bigoplus_{j \geq 1} (V_C)_j q^j.
\]

Assume that \( X^g \) is connected. The bundle \( V_1 \) is the bundle of \( g \)-fixed points of \( V|_{X^g} \), and the \( g \)-fixed points of the Thom space of \( V \) are, as a \( C_g \)-space, homeomorphic to the Thom space \( (X^g)^V_1 \) of \( V_1 \).

Assume first that \( g \) acts trivially on \( V|_{X^g} \), then \( V_1 = V|_{X^g} \), and (8) contains only the terms with integral powers of \( q \). This case was treated in [Milb], we shortly recall the discussion there: Since \( V \) is oriented, \( V_C \) is an SU-vector bundle. Hence it has a Spin\((4d)\) structure as well as a complex structure with trivial determinant bundle

\[
\det V_C = 1 \in K_{C_g}(X^g),
\]

and we have have an equality of the \( \hat{A} \)-Euler class and the Todd genus Euler class

\[
e_{\hat{A}}^C(V|_{X^g}) = e_{Td}^C(V|_{X^g})
\]

(see appendix). Formula (22) applied to the \( \mathbb{R}/l\mathbb{Z} \)-action suggests to define the \( C_g \times \mathbb{R}/l\mathbb{Z} \)-equivarint Thom class of \( \mathcal{L}_g V|_{X^g} \) as

\[
u_{\hat{A}}^g(V) \cdot \prod_{j \geq 1} q^{-dj} \cdot \Lambda_{-qj} V_C.
\]

Here \( \nu^g \) denotes the \( \hat{A} \)-Thom class, and the exterior powers are exterior powers of \( C_g \)-representations. It is customary (c.f. [Seg88]) to use a \( \zeta \)-function renormalization to replace the divergent sum \( \sum_{j > 0} j \) in the exponent of \( q \) with \( \zeta(-1) = -1/12 \), and to define

\[
u_{\text{string}}^g(V) := q^{12} \cdot \nu_{\hat{A}}^g(V) \cdot \prod_{j = 1}^{\infty} \Lambda_{-qj} V_C.
\]

Next, assume that \( g \) acts fixed point free on \( V|_{X^g} \) and that \(-1\) is not an eigenvalue of \( g \). For \( j = 1, \ldots, \lfloor \frac{1}{12} \rfloor \), let \( \theta_j = 2\pi \frac{j}{l} \), and let \( V_j \) be the subspace of \( V \) on which \( g \) acts by rotations with angle \( \theta_j \). We equip \( V \) with the unique complex structure such that \( g \) acts on \( V_j \) by multiplication with \( e^{i \theta_j} \). Then \( V_j \otimes \mathbb{C} \) is of the form

\[
V_j \otimes \mathbb{C} \cong V_j \oplus V_j^{\perp}.
\]

Setting \( V_{-j} := V_j \), we have

\[
\mathcal{L}V|_{X^g} \cong \bigoplus_{j = 1}^{\lfloor \frac{1}{2} \rfloor} V_j q^j \bigoplus \bigoplus_{k = 1}^{\infty} \bigoplus_{j = -\lfloor \frac{1}{2} \rfloor}^{\lfloor \frac{1}{2} \rfloor} V_j q^{k+j}.
\]
For fixed $k$, 

$$\det \left( \bigoplus_{j=-\lfloor \frac{l}{2} \rfloor}^{\lfloor \frac{l}{2} \rfloor} V_j q^{k+\frac{j}{l}} \right) \approx \bigotimes_{j=-\lfloor \frac{l}{2} \rfloor}^{\lfloor \frac{l}{2} \rfloor} q^{(k+\frac{j}{l})d_j} \cdot \det(V_j) \approx q^{2dk} \cdot \det(V_C).$$

Here $d_j = \dim_{\mathbb{C}} V_j$, and

$$\sum_{j=-\lfloor \frac{l}{2} \rfloor}^{\lfloor \frac{l}{2} \rfloor} d_j(k + \frac{j}{l}) = 2k \left( \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} d_j \right) = 2dk,$n while $\det V_C$ is trivial. In a similar manner, the determinant line bundle of the first $\lfloor \frac{l}{2} \rfloor$ summands becomes

$$q^{\frac{\text{age}(g)}{2}} \cdot \det(V)$$

with

$$\text{age}(g) = 2 \cdot \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor} \frac{j}{l} d_j.$$

Note that $\det(V)$ is not trivial, but has a square root.

As above, Formula (22) suggests to define the $\mathbb{C}g \times \mathbb{R}/l\mathbb{Z}$-equivariant Euler/Thom class of $\mathcal{L}_g V|_{X^g}$ as

$$q^{\frac{\text{age}(g)}{2}} \left( \sqrt{\det V} \right)^{-1} \cdot \prod_{j=1}^{\infty} \Lambda_{-q^\frac{j}{l}} \left( (V_C)_{\bar{c}_l} \right).$$

If we want to allow the eigenvalue $-1$, we need to assume that the eigenspace $V_{\frac{l}{2}}$ of the action of $-1$ on $V$ has a complex structure. Then we can make sense of the expression $(\sqrt{\det V})$, and the above discussion goes through.

This is for example the case if $V$ is a $G$-equivariant $\mathbb{U}^2$-bundle (i.e., if $V$ has a complex and a spin-structure that are compatible, see (17)). Note that in that case we have two complex structures on the other $V_j$ – the given complex structure and the auxiliary one we defined in formula (5) above. We choose to work with the latter. This is an issue similar to the one in the warning after Formula (5). Write $V_1$ for the subspace of $V|_{X^g}$ that is fixed by the action of $g$, write $N^g$ for its orthogonal complement, and note that

$$u^\wedge_{\mathcal{C}_g}(V_1) (\sqrt{\det N^g})^{-1} = u^{T_d}_{\mathcal{C}_g}(V_1) (\sqrt{\det V_1})^{-1} (\sqrt{\det N^g})^{-1} = u^{\wedge}_{\mathcal{C}_g}(V_1) (\sqrt{\det V})^{-1}.$$

All of the above discussion taken together motivates the following definition:

**Definition 3.4.** Let $V$ be a $G$-equivariant $\mathbb{U}^2(d)$-vector bundle over a compact $G$-space $X$. Let $g$ be an element of $G$ with order $l$. Assume that $X^g$ is connected. We define

$$u_{g}^{\text{string}}(V) := q^{\frac{d}{2} \cdot \frac{\text{age}(g)}{l}} \cdot u^{\wedge}_{\mathcal{C}_g}(V_1) \cdot (\sqrt{\det V})^{-1} \cdot \prod_{j=1}^{\infty} \Lambda_{-q^\frac{j}{l}} \left( (V_C)_{\bar{c}_l} \right)$$

and set

$$u^{\text{string}}(V) := \sum_{\{g\}} u_{g}^{\text{string}}(V).$$
This is an element of $K_{\text{Dev},G}(X^V)[q^{\pm 1/2}]$. If $X^g$ is not connected, one sums over its connected components.

**Proposition 3.5.** The $u^\text{string}_G(V)$ satisfy the axioms of Definition A.1.

**Proof:** The naturality axiom and the change of groups axiom are straight-forward. Multiplicativity follows from that of $u^\text{Td}$ and from the multiplicative properties of $\sqrt{\det(-)}$ and of the total exterior power $\Lambda_t$. Further, $\Lambda_{-t}$ is invertible with inverse $\text{Sym}_t(V)$, which implies the periodicity axiom. □

3.2.1. *Comparison to the work of Kitchloo and Morava.* Let $G$ be the trivial group, and let $V$ be a complex vector bundle of dimension $d$. Using (6) and the fact that

$$\Lambda_{-1}(Vq^{-k} \oplus Vq^{k}) \equiv \det(-Vq^{-k}) \cdot \Lambda_{-1}(V_Cq^{k}) = q^{dk}(\det V)^{-1} \cdot \Lambda_{-1}(V_Cq^{k}),$$

we see that if $\det(V)$ is trivial, a similar discussion as above suggests the definition

$$u^\text{string}_{\text{Td}}(V) := q^{-1/2}u_{\text{Td}}(V) \cdot \prod_{k \geq 1} \Lambda_{-q^k}V_C.$$

The periodicity element $\alpha = u^\text{string}_{\text{Td}}(\mathbb{C})$ (c.f. (15)) is

$$\alpha = \left(q^{-1/2} \cdot \prod_{j \geq 1} (1 - q^j)^2\right) \cdot \beta,$$

where $\beta$ is the Bott-element. Hence the renormalized Thom class in degree $2d$

$$\tau^\text{string}_{\text{Td}}(V) := u^\text{string}_{\text{Td}}(V) \cdot \alpha^{-d} = \tau_{\text{Td}}(V) \cdot \prod_{j \geq 1} \frac{\Lambda_{-q^j}V_C}{(1 - q^j)^{2d}}$$

is the Thom class defined in [AM01], [KM, 4.1]. Here $\tau_{\text{Td}}(V) = u_{\text{Td}}(V) \cdot \beta^{-d}$.

3.3. **The equivariant Witten genus.** Along with the existence of equivariant Thom classes comes an equivariant Witten genus. We are now going to calculate this equivariant Witten genus in terms of characteristic classes.

**Corollary 3.6** (of Proposition 3.5). *There is a unique map of $G$-spectra*

$$\phi_G: \text{MU}_G^2 \rightarrow K_{\text{Dev},G}$$

*taking Thom classes to Thom classes.* The induced genus

$$\hat{A}^\text{string}_G(M) := \phi_G(M)$$

*of a $G$-equivariant $\mathbb{U}^2$ manifold $M$ of dimension $2n$ is given by the formula*

$$\left(\hat{A}^\text{string}_G(M)\right)_{[g]} = q^{-\frac{d}{2} + \frac{\text{ord}(g)}{2}} \cdot \text{Td}_{C_g} \left(M^g; \sqrt{(TM)} \cdot \prod_{j \geq 1} \text{Sym}_{q^j} \left(\left((TM)^C\right)_{\ell_j}\right)\right),$$

*where $\ell$ is the order of $g$, and we sum over the connected components of $M^g$.***
The existence and uniqueness of $\phi_G$ follow from Proposition A.4. Further, assuming $M^g$ connected, we have

$$
A_{\text{string}}^G(M)_{[g]} \cong \left( \pi_t^{\text{string}}(1) \right)_{[g]} \\
\cong \pi_{Cg,0}^{Td} \left( \left( q^{\frac{d - \text{age}(g)}{2}} (\sqrt{TM})^{-1} \prod_{j > 1} \Lambda_{-q^j} (TM^C)_{\zeta_l} \right)^{-1} \right) \\
\cong q^{-\frac{d}{12} + \frac{\text{age}(g)}{2}} \cdot Td_{Cg} \left( M^g, \sqrt{TM} \prod_{j > 1} \text{Sym}_{q^j} (TM^C)_{\zeta_l} \right),
$$

where the first and last equation are (16), and the second equation follows from Definition 3.4.

It is traditional to express equivariant genera in the following more explicit and somewhat more complicated form.

**Corollary 3.7.** Let $h \in C_g$, and $k = |h|$. Assume $M^g$ to be connected. Then we have

$$
\hat{A}_{\text{G}}(M)_{[g]}(h) = \pm q^{-\frac{d}{12} + \frac{\text{age}(g)}{2}} e^{i\pi \text{age}(h)} \int_{M^g,h} e^H (M^g,h) \cdot \left( \prod_{\tilde{x}_i} \chi_k^{\tilde{x}_i} \right) \\
\cdot \left[ \prod_{s,x_i} (1 - \zeta_k^s e^{x_i}) \prod_{j \geq 1} \prod_{s,y_i} (1 - \zeta_k^s e^{y_i} q^{j}) \right]^{-1},
$$

where $e^H$ denotes the Euler class in ordinary cohomology, $s$ runs from 0 to $k - 1$, the $\tilde{x}_i$ run over the Chern roots of $TM$, the $x_i$ are the Chern roots of $TM_{g=1,h=\zeta_k}$, and the $y_i$ are the Chern roots of the simultaneous eigenspaces $(TM^C)_{g=\zeta_k, h=\zeta_k}$. (All bundles are over the double fixed points $M^{g,h}$.)

**Proof:** By (19), we have

$$
Td_{Cg} (M^g; a)(h) = \int_{M^g,h} e^H (M^g,h) \cdot ch(a(h)) \cdot \prod_{s,x_i} \frac{1}{1 - \zeta_k^s e^{x_i}},
$$

where $s$ runs from 0 to $k - 1$, and the $x_i$ are the Chern roots of $TM_{g=1,h=\zeta_k}$. We have

$$
\text{ch}(\sqrt{TM}(h)) = \pm \prod_{s, \tilde{x}_i} \zeta_k^s e^{\tilde{x}_i} = \pm e^{i\pi \text{age}(h)} \cdot \prod_{\tilde{x}_i} \chi_k^{\tilde{x}_i},
$$

where the $\tilde{x}_i$ are the Chern roots of $(TM)_{h=\zeta_k}$. The Chern character of the remaining factor, evaluated at $h$, computes to

$$
\prod_{j \geq 1} \prod_{s=0}^{k-1} \prod_{y_i} (1 - \zeta_k^s e^{y_i} q^j),
$$

where the $y_i$ are the Chern roots of the simultaneous eigenspaces $(TM^C)_{g=\zeta_k, h=\zeta_k}$. □
Corollary 3.7 should be compared to the calculations on the last page of [dFLNU06]. Basically, our discussion starts, once the authors of [dFLNU06] replace their variable \( q \) with its monodromy \( 2\pi i \). For our purposes, the double loop space picture is not necessary. It will, however, become very important when one wants to study modularity properties, i.e., the "moonshine-like" behavior discussed in [Dev96]: We already observed that replacing the pair of commuting elements \( (g, h) \) with \( (g, gh) \) corresponds to replacing \( \tau \) with \( \tau + 1 \), where \( q = e^{2\pi i \tau} \). This is half of the first Moonshine condition, the other half demands that switching to \( (h^{-1}, g) \) should correspond to \( \tau \mapsto -\frac{1}{2} \). In the picture described in the beautiful paper [dFLNU06], the elements \( g \) and \( h \) describe the monodromy of a \( G \)-principal bundle \( P \) over a torus, where \( P \) maps equivariantly to \( M \). As described in [FQ93], [Gan], the transformation behaviour of generalized Moonshine corresponds to a different choice of circles, generating the same torus. Our approach sees the first circle, but not the second one. This is the reason why we can explain the first transformation geometrically, but not the second one.

**Definition 3.8.** Let \( M \) be a \( G \)-equivariant \( U^2(d) \)-manifold. Then the orbifold Witten genus of \( M//G \) is defined by

\[
\hat{A}_{\text{orb}}^{\text{string}}(M//G) := \frac{1}{|G|} \sum_{gh=hg} \hat{A}_G^{\text{string}}(M)_{[g]}(h),
\]

where the sum is over all pairs of commuting elements of \( G \).

One of the advantages of working with our setup is that the proofs of integrality and Morita invariance become fairly straightforward.

### 3.4. Integrality.

**Proposition 3.9.** The orbifold genus \( \hat{A}_{\text{orb}}^{\text{string}} \) takes values in \( \mathbb{Z}[q][q^{\pm \frac{1}{2}}] \).

**Proof:** Let \( g \in G \) have order \( l \). Then \( \hat{A}_G^{\text{string}}(M)_{[g]} \) is an element of \( \mathbb{R}((C_g))[q^\frac{1}{l}][q^{\pm \frac{1}{2}}] \), and

\[
\hat{A}_{\text{orb}}^{\text{string}}(M//G) = \sum_{[g]} \frac{1}{|C_g|} \sum_{h \in C_g} \hat{A}_G^{\text{string}}(M)_{[g]}(h) = \sum_{[g]} \langle \hat{A}_G^{\text{string}}(M)_{[g]}, 1 \rangle_{C_g}.
\]

Here \( \langle -,- \rangle_{C_g} \) denotes the inner product in \( \mathbb{R}(C_g) \), so that

\[
\langle \sum V_j q^{\frac{j}{l}}, 1 \rangle_{C_g} := \sum \langle V_j, 1 \rangle_{C_g} q^{\frac{j}{l}} = \sum \dim_C \left( V_{\frac{j}{l}} \right) q^{\frac{j}{l}}
\]

computes the dimensions of the maximal trivial summands of the \( V_j \). Fix \( g \) for the remainder of the proof, and assume, for simplicity, that \( M^g \) is connected. The element \( g \in C_g \) acts by multiplication with \( \zeta_g^i \) on the coefficient of \( q^{\frac{i}{l}} \) in

\[
\prod_{j \geq 1} \Lambda_{-q^{\frac{j}{l}}(TM_C)} \zeta_g^i.
\]
Further, $g$ acts by multiplication with $\pm e^{-i\text{age}(g)}$ on
\[ q^{-\frac{\text{age}(g)}{2}} u_{\text{Td}}(TM^g) \cdot (\sqrt{\det TM})^{-1}. \]
To determine the sign, note that $g$ acts trivially on $u_{\text{Td}}(M^g)$ and by multiplication with $e^{2i\pi \text{age}(g)}$ on $\det TM$, and that the sign depends on the choice of an identification of $\text{Spin}(2)$ with $U(1)$. For our calculation of $\hat{\Lambda}_{\text{string}}(M)_{[g]}$, we fix this identification in such a way that the sign becomes positive. Then only the coefficients of integral powers of $q$ in
\[ \pi_{\text{Td},!} \left( q^{-\frac{\text{age}(g)}{2}} \cdot \sqrt{\det TM}^{-1} \cdot \prod_{j \geq 1} \Lambda_{-q^{-\frac{1}{2}}(TM^C)^j}^{-1} \right) \]
contribute trivial summands. If we had chosen to work with the other choice of identification, we would have to multiply the above result for $\hat{\Lambda}_{\text{string}}(M)_{[g]}$ with $-1$, which would not destroy the integrality. $\square$

3.5. The Atiyah-Segal map for orbifolds and Morita invariance of the orbifold Witten genus. A priori, it is not clear that Definition 3.8 is independent of the presentation of the orbifold as a global quotient $M//G$. For instance, why should an isomorphism of orbifolds $M//G \cong N//H$ imply
\[ \hat{\Lambda}_{\text{orb}}(M//G) = \hat{\Lambda}_{\text{orb}}(N//H)? \]
This section proves this fact for complex orbifolds with a holomorphic root of the line bundle $\det(TX)$. This condition is, for instance, satisfied by Calabi-Yau orbifolds. The main tool of the proof is the Atiyah-Segal character map for orbifolds. This map was defined in [AR03] and later in [Moe02], but we will use a definition that is somewhat more elementary and closely follows the original approach by Atiyah and Segal [Seg].

For the basic definitions of orbifolds, we refer the reader to the introductory paper [Moe02]. Let $G$ be an orbifold groupoid, and recall that the inertia groupoid $\Lambda(G)$ as objects and morphisms
\[ \Lambda(G)_0 = \text{eq}(s, t) \quad \text{and} \quad \Lambda(G)_1 = \text{eq}(s, t) \times_{G_0} G_1. \]
Here $\text{eq}(s, t)$ stands for the equalizer of the source and target of $G$, the map $G_1 \rightarrow G_0$ defining the fibred product is that target map $t$, and $(g, h) \in \Lambda(G)_1$ is an arrow from $g$ to $h^{-1}gh$.

Consider the map
\[ A: \text{eq}(s, t) \rightarrow \text{eq}(s, t) \times_{G_0} G_1 \]
\[ g \mapsto (g, g). \]
This map $A$ is simultaneously a section of $s$ and $t$, and one checks that $A$ is a natural transformation from the identity map of $\Lambda(G)$ to itself.

Let $V$ be a vector bundle over the groupoid $\Lambda(G)$. By definition, $V$ is a vector bundle over $\Lambda(G)_0$ together with an isomorphism
\[ \mu: t^*V \rightarrow s^*V \]
over $\Lambda(G)_1$ such that a few diagrams commute. The section $A$ pulls back $\mu$ to a (fiber preserving) automorphism of
\[ V = A^*t^*V = A^*s^*V \]
over $\Lambda(G)_0$, which turns out to be an automorphism of $V$ as orbifold vector bundle. All of the above is preserved by equivalences of orbifold groupoids $G \simeq H$.

**Example 3.10.** Let $G = M \rtimes G$ be the translation groupoid of the action of a finite group $G$ on a manifold $M$, and let $V$ be a $G$-equivariant vector bundle on $M$. Then $\Lambda(G)$ is equivalent to the disjoint union over the conjugacy classes of $G$ of the translation groupoids of the actions of the centralizers on the fixed point loci:

$$\Lambda(G) \simeq \bigsqcup_{[g]} M^g \rtimes C_g.$$  

Under this equivalence, the map $A$ corresponds to the maps

$$M^g \to M^g \times C_g$$

$$m \mapsto (m, g),$$

and $A^*|_{M^g}$ becomes multiplication with $g$ on $V|_{M^g}$.

**Definition 3.11.** Let $X$ be an orbifold, and let $\Lambda(X)$ be its inertia orbifold. We define

$$\chi': K_{orb}(\Lambda(X)) \to K_{orb}(\Lambda(X)) \otimes \mathbb{C}$$

$$V \mapsto \sum_{\zeta} V_\zeta \otimes \zeta,$$

where the sum is over all eigenvalues of the action of $\Lambda^*(\mu)$ on $V$. Write $i$ for the canonical map

$$i: \Lambda(X) \to X$$

$$s = t: eq(s, t) \to G_0$$

$$pr_2: eq(s, t) \times_{G_0} G_1 \to G_1.$$  

We define the **Atiyah-Segal character map** as the composite of $i^*$ with $\chi'$.

**Example 3.12.** In the situation of Example 3.10, the map $i$ becomes the inclusion of the fixed point locus in $M$, and

$$\chi: K_0(M) \to \bigoplus_{[g]} K_{C_g}(M) \otimes \mathbb{C}$$

is the classical Atiyah-Segal character map.

Let now $X$ be a compact complex orbifold and $TX$ its tangent bundle. It follows from Example 3.10 and the analogous statement for the equivariant case that locally (and hence globally) we have $(TX)_1 \simeq T\Lambda X$. Let $L$ be the smallest common multiple of the orders of all the elements of the stabilizers of $X$, and recall that the connected components of an orbifold are well-defined (they correspond to the connected components of its quotient space). Further, it is still true for orbifolds that the dimension of a vector bundle is constant over each connected component. Hence, over each connected component of $X$,

$$q_\pi \prod_{j \geq 1} \text{Sym}^{d_{\frac{\text{vol}(\Lambda^*(\mu))}{2}}} (TX)_1^{c_i}$$

is a well defined element of $K_{orb}(\Lambda X)[q^{\pm 1}][q^{\pm 1/2}]$. Here $TX^C$ is restricted to $(\Lambda X)$ along $i$. 

Let \( \mathcal{A} \) be an abelian sheaf over the orbifold \( X \). Recall that the orbifold cohomology of \( X \) with coefficients in \( \mathcal{A} \), denoted \( H^*(X, \mathcal{A}) \), is defined as the cohomology of the complex \( \Gamma_{\text{inv}}(I^\bullet) \), where
\[
0 \to \mathcal{A} \to I^\bullet
\]
is an injective resolution of \( \mathcal{A} \) in the category of abelian sheaves over \( X \) and \( \Gamma_{\text{inv}} \) denotes the invariant sections.

**Example 3.13.** Let \( G \) be a finite group, \( M \) a complex \( G \)-manifold, \( V \) a holomorphic \( G \)-vector bundle over \( M \). We will write \( \mathcal{O}(V) \) for the sheaf of germs of holomorphic sections of \( V \). Let \( X = M//G \). Since taking \( G \)-invariants is an exact functor on complex vector spaces (denoted \( (\cdot)^G \)), we have
\[
H^*(X, \mathcal{O}(V)) \cong H^*(M, \mathcal{O}(V))^G.
\]

**Definition 3.14.** Let \( X \) be a complex orbifold, and let \( V \) be a holomorphic vector bundle on \( X \). We define the topological Todd genus (also known as topological complex Euler characteristic) of \( X \) with coefficients in \( V \) by
\[
\text{Td}_{\text{top}}(X, V) := \text{sdim} H^*(X, \mathcal{O}(V)) = \sum_p (-1)^p H^p(X, \mathcal{O}(V)).
\]

If \( W \) is a holomorphic vector bundle over \( \Lambda X \), then the orbifold Todd genus of \( X \) with coefficients in \( W \) is defined by
\[
\text{Td}_{\text{orb}}(X, W) := \text{sdim} H^*(\Lambda X, \mathcal{O}(W)).
\]

**Example 3.15.** In the situation of Example 3.13, one has [AS68, pp.543-545]
\[
\text{Td}_{\text{top}}(M//G; V) = \text{sdim} H^*(M; \mathcal{O}(V))^G
\]
\[
= \frac{1}{|G|} \sum_{g \in G} \sum_p (-1)^p \text{Trace}(g|H^p(M; \mathcal{O}(V)))
\]
\[
= \frac{1}{|G|} \sum_{g \in G} \text{Td}(M^g, \frac{V_{\mid M^g}}{\text{det}(\Lambda^1((N^g)^r)(g)))}.
\]

**Definition 3.16.** Let \( X \) be a complex orbifold. We set
\[
\text{Td}_{\text{top}} \left( X; \sum_j V_j q^{l_j} \right) := \sum_j \text{Td}_{\text{top}}(X; V_j) q^{l_j}.
\]
(here the \( V_j \) are assumed holomorphic). Assume that \( \Lambda X \) is connected and that there exists a holomorphic line bundle \( \sqrt{\det(TX)} \) on \( X \) whose square is the determinant line bundle of the holomorphic tangent bundle \( TX \). Then we set
\[
\Lambda_{\text{orb}}^{\text{string}}(X) := q^{-\frac{d}{2}} \cdot \text{Td}_{\text{orb}} \left( X; \sqrt{\det TX} \cdot \prod_{j \geq 1} \text{Sym}_{q^{l_j}} (TX^C) \right).
\]

If \( \Lambda X \) is not connected, one sums over its connected components.

In the case of a global quotient orbifold by a finite group, this definition specializes to Definition 3.4.
Example 3.17. An important class of orbifolds for which the conditions of the definition are satisfied are Calabi-Yau orbifolds: for these, the bundle $\det(TX)$ is trivial.

The author recently learned of the work of Dong, Liu and Ma, who defined Morita-invariant versions of elliptic genera in [DLM02]. It seems likely that their definitions are closely related to ours. A comparison to their work would be very interesting, because they are working with the index theorem on orbifolds. The authors of [dFLNU06] also started from an index theoretic discussion but do not formulate their results in a way that makes the Morita invariance of their arguments obvious.

4. Loop spaces of symmetric powers

In this section, we will define symmetric powers of orbifolds and study their loop spaces. In the case of a global quotient orbifold $M/G$, the $n$th symmetric power is again a global quotient orbifold, namely $M^n/(G \wr \Sigma_n)$. We start by recalling some well known facts about wreath products.

4.1. Actions of wreath products. Let $G$ be a finite group. Recall that the wreath product $G \wr \Sigma_n$ has elements

$$(g, \sigma) \in G^n \times \Sigma_n$$

which compose as follows:

$$(g_1, \ldots, g_n, \sigma) \cdot (h_1, \ldots, h_n, \tau) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}, \sigma \tau).$$

Let $l$ be the order of $g_n \cdots g_1$ in $G$, and let $\sigma$ be the $n$-cycle $(1 \ldots n)$. Then the order of $(g, \sigma)$ in $G \wr \Sigma_n$ equals $ln$. An element $(h, \tau)$ is in the centralizer of $(g, \sigma)$ if and only if

$$\tau \sigma = \sigma \tau \quad \text{and} \quad \forall i : g_{\sigma(\tau(i))} h_{\tau(i)} = h_{\tau(\sigma(i))} g_{\sigma(i)}.$$

Let $(h, \tau)$ be in the centralizer of $(g, \sigma)$, let $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ be $k$-cycles in $\sigma$, and assume that $\tau(i_r) = j_{r+m}$. Then

$$\forall r \in \mathbb{Z}/k\mathbb{Z} : g_{j_r} h_{j_{r-1}} = h_{j_r} g_{j_{r-m}}.$$ 

In particular,

$$h_{j_k} g_{i_{k-1}} \cdots g_{i_{1-m}} = g_{j_k} \cdots g_{i_1} h_{j_k},$$

and the other $h_{j_r}$ are determined by $h_{j_k}$ and $g$. Note that the element

$$h_{j_k} g_{i_{k-1}} \cdots g_{i_{1-m}} = g_{j_k}^{-1} \cdots g_{j_{m-1}}^{-1} h_{j_m}$$

conjugates $g_{j_k} \cdots g_{j_1}$ into $g_{j_k} \cdots g_{j_1}$.

Let $M$ be a right $G$-manifold. Then $G \wr \Sigma_n$ acts on $\chi \in M^n$ via

$$\chi \cdot (g, \sigma) = (\chi_{\sigma(1)} g_{\sigma(1)}, \ldots, \chi_{\sigma(n)} g_{\sigma(n)}).$$

More generally, let $G := (G_0, G_1, s, t, u, i)$ be an orbifold groupoid.

Definition 4.1. We define the $n$th symmetric power of $G$, denoted $G \wr \Sigma_n$, to be the groupoid $G^n \times \Sigma_n$. Explicitly, $G \wr \Sigma_n$ has objects $G_0^n$, and morphisms $G_1^n \times \Sigma_n$. Its source sends the morphism $(g, \sigma)$ to $(s(g_{\sigma(1)}), \ldots, s(g_{\sigma(n)}))$, while its target sends it to $(t(g_1), \ldots, t(g_n))$. The unit map sends the object $x$ to $(u(x_1), \ldots, u(x_n), 1)$. The multiplication of $(g, \sigma)$ and $(h, \tau)$, where $g_{\sigma(i)} = h_i$, is given by $(g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)});$ and the inverse of $(g, \sigma)$ is given by $(i(g_{\sigma(1)}), \ldots, i(g_{\sigma(n)}), \sigma^{-1})$. 

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One checks that this definition is Morita-invariant. We will be interested in the orbifold loop spaces of symmetric powers. We start by considering the “untwisted sector”.

**Example 4.2 (Symmetric powers of loop spaces).** We have

\[
(L(M//G)) \wr \Sigma_n = \left( \prod_{g \in G^n} L_{g_1} M \times \cdots \times L_{g_n} M \right) \mod G \wr \Sigma_n,
\]

where \((h, \tau) \in G \wr \Sigma_n\) acts on the right-hand side by

\[
(\gamma_1, \ldots, \gamma_n) \mapsto (\gamma_{\sigma(1)} h_{\sigma(1)}, \ldots, \gamma_{\sigma(n)} h_{\sigma(n)}).
\]

Assume now that \(V\) is a loop bundle over \(L(M//G)\) with Fourier decomposition

\[
V|_X \cong \sum_{j \in \mathbb{Z}} (V_j \otimes G) q^{\frac{j}{N}} =: x
\]

Assume further that \(V\) (and hence each \(V_j\) is a right \(G\)-bundle. Note that this is not the same setup as in Definition 2.5, here we are only considering the “untwisted sector”

\[
(LM//G) = L_1 M//C_1.
\]

The \(n\)th symmetric power \(V^n//\Sigma_n\) is a \(G \wr \Sigma_n\)-equivariant bundle over \((LM)^n\), on which \(S^1\) acts (diagonally). The Fourier decomposition of \((V \wr \Sigma_n)|_{\mathcal{M}^n}\) is

\[
d_n(x) := \sum_{j \in \mathbb{Z}} (V_j \times 0 \times \cdots \times 0) q^{\frac{j}{N}} \oplus \cdots \oplus (0 \times \cdots \times 0 \times V_{j_n}) q^{\frac{j_n}{N}}
\]

\[
\cong \sum_{j \in \mathbb{Z}} \sum_{k=0}^{n} W_{j,k} q^{\frac{j}{N}},
\]

where \(W_{j,k}\) consists of the \(n\) \(k\) summands in which \(V_j\) turns up exactly \(k\) times and the other summands are zero (in particular, \(W_{j,0} = 0\)). The summands of (10) are \(G^n\)-representations in the obvious way, and the symmetric group acts on \(W_{j,k}\) by permutation of these summands.

### 4.2. Loop spaces of symmetric powers.

We are now going to describe the loop space of \((M//G) \wr \Sigma_n\),

\[
\mathcal{L}((M//G) \wr \Sigma_n) = \prod_{[g, \sigma]} \mathcal{L}_{[g, \sigma]}(M^n) \mod C([g, \sigma]).
\]

Let \([\sigma]\) correspond to the partition \(n = \sum_k kN_k\), and assume that for each cycle of \(\sigma\), we have fixed a “first” element \(i_1\), and thus a representation as

\[
(i_1, \ldots, i_k).
\]

**Theorem 4.3.** The component of \(\mathcal{L}((M//G) \wr \Sigma_n)\) corresponding to \([g, \sigma]\) is homeomorphic to

\[
\mathcal{L}_{[g, \sigma]} M^n \cong \prod_k \prod_{i_1, \ldots, i_k} k \mathcal{L}_{g_{i_k} \cdots g_{i_1}} M,
\]
where the second product runs over all $k$-cycles in $\sigma$. Let $(h, \tau)$ be an element of $C(g, \sigma)$ with $\tau(i) = j_{1+m}$ as in Section 4.1, and define its action on $\gamma \in kL_{g_{k_1} \cdots g_{k_1}} M$ by

$$(\gamma \cdot (h, \tau))(t) := \gamma(m + t)h_{i_k}g_{i_{1-m}} \cdots g_{i_1}^{-1}.$$ 

Then $\gamma \cdot (h, \tau)$ is a path in $kL_{g_{k_1} \cdots g_{k_1}} M$, and the homeomorphism is $C(g, \sigma)$-equivariant with respect to this action on the right-hand side. Moreover, it preserves the $S^1$-action by reparametrization of paths. There is a canonical map from the target of (11) with this action of $C(g, \sigma)$ to

$$\prod_k (kL(M//G)) \triangleleft \Sigma_n.$$

Denote its composite with (11) by $F_{(g, \sigma)}$. As a map of orbifolds, $F_{(g, \sigma)}$ is independent of the choices made.

**Figure 3.** An element of $L_{(g_{1}, \ldots, g_{5})(135)(24)} M^5$ and its image in $L_{g_{1}g_{2}g_{3}} M \times L_{g_{4}g_{5}} M$.

**Proof:** The path

$$\gamma = (\gamma_1, \ldots, \gamma_n)$$

in $M^n$ is in $L_{(g, \sigma)} M^n$ if and only if

$$\forall i: \gamma_i(1) = \gamma_{\sigma(i)}(0)g_{\sigma(i)}.$$ 

I.e., if $\gamma(0) = (m_1, \ldots, m_n)$, then for $1 \leq i \leq n$, $\gamma_i$ is a path from $m_i$ to $m_{\sigma(i)}g_{\sigma(i)}$. Let $(i_1, \ldots, i_k)$ be a cycle in $\sigma$. The paths $\gamma_{i_k}, \gamma_{i_1}g_{i_1}$ to $\gamma_{i_{k-1}}g_{i_{k-1}} \cdots g_{i_1}$ compose to a path

$$\gamma := \gamma_{i_k} \ast \gamma_{i_1}g_{i_1} \ast \cdots \ast \gamma_{i_{k-1}}g_{i_{k-1}} \cdots g_{i_1}$$

in $kL_{g_{i_1} \cdots g_{i_1}} M$. This defines the homeomorphism of the proposition (see Figure 3). Its inverse sends a path $\gamma \in kL_{g_{i_1} \cdots g_{i_1}} M$ to $(\gamma_{i_r})_{r=1}^k$ with

$$\gamma_{i_r} = \gamma|_{[0,1]}$$

and

$$\gamma_{i_r}(t) := \gamma(t + g_{i_1}^{-1} \cdots g_{i_r}^{-1},$$

for $1 \leq r < k$. Let $(h, \tau)$ be as in the proposition. Then the $i_r^{th}$ path of $(\gamma_i)_{i=1}^n \cdot (h, \tau)$ is $\gamma_{j_{r+m} \ast h_{j_{r+m}}}$, and

$$\gamma_{j_{r+m} \ast h_{j_{r+m}} \ast g_{i_r} \cdots g_{i_1}} = \gamma_{j_{r+m} \ast g_{j_{r+m}} \cdots g_{i_1}}(g_{j_1}^{-1} \cdots g_{j_m}^{-1}h_{j_m}).$$

It follows that the centralizer acts as claimed. Recall that

$$kL(M//G) \cong \prod_{[g]} kL_g M//C_g^k.$$
and that if $h$ is conjugate to $g$ the groupoids $\mathcal{L}_gM//C^k_g$ and $\mathcal{L}_hM//C^k_h$ are related by an isomorphism which is canonical up to a natural transformation (i.e., it becomes canonical in the category of orbifolds). To be specific, fix representatives of the conjugacy classes of $G$, fix $k$, and fix an ordering $i_1, \ldots, i_{N_k}$ of the $k$-cycles of $\sigma$. For $1 \leq \alpha \leq N_k$, set $g_\alpha := g_{i_\alpha,k} \cdots g_{i_\alpha,1}$, and let $r_\alpha$ be the representative of $g_\alpha$. Then on the $\alpha$th factor, the canonical map of the proposition is represented by multiplication with a group element $s_\alpha$ which conjugates $g_\alpha$ into $r_\alpha$ (and hence maps $\mathcal{L}_{g_\alpha}M$ to $\mathcal{L}_{r_\alpha}M$). Let now $(h, \tau) \in C_G(g, \sigma)$ be as in the proposition. Then $\tau$ defines a permutation $\rho \in \Sigma_{N_k}$ of the set of $k$-cycles of $\sigma$. Let $r$ be the representative of $[g_{i_1} \cdots g_{i_1}]$. Since $h_k \cdot g_{i_1}^{-1} \cdots g_{i_k}^{-1}$ conjugates $g_{i_1} \cdots g_{i_k}$ into $g_{i_k} \cdots g_{i_1}$, $\tau$ is also the representative of the latter, and under our canonical map, right multiplication with $h_k \cdot g_{i_1}^{-1} \cdots g_{i_k}^{-1}$ translates into multiplication with an element of $C_\tau$ combined with the permutation $\rho$ of factors of the target $\prod_{\alpha} \mathcal{L}_{r_\alpha}M$.

Rescaling a path by $t \mapsto t + m$ is the action of $m \in C_{r_{\alpha}}^k$.

Assume now that we have chosen a different first element, say $i_r$, of the cycle $(i_1, \ldots, i_k)$. This leads to a different groupoid map $F'_{[g, \sigma]}$. The factor of the target of $F_{[g, \sigma]}$ corresponding to this cycle is $\mathcal{L}_{g_{i_r} \cdots g_{i_1}}M$, while that of $F'_{[g, \sigma]}$ is $\mathcal{L}_{g_{i_1} \cdots g_{i_r}}M$. There is a natural isomorphism $I$ between the two maps, sending $\gamma \in \mathcal{L}_{g_{i_1} \cdots g_{i_r}}M$ to

$$I(\gamma)(t) = \gamma(t + r)g_{i_1}^{-1} \cdots g_{i_{r-1}}^{-1}.$$  

Hence $F$ and $F'$ define the same map of orbifolds. Similarly, changing the order of the $k$-cycles of $\sigma$ translates into a permutation in $\Sigma_{N_k}$.

The orbifold loop space $\mathcal{L}(\{M//G\} \wr \Sigma_n)$ can be viewed as the space of $n$ strings moving in $M//G$, their order does not matter, they are either closed (in $M//G$) or joining together to form longer closed strings.

**Example 4.4.** Let $(h, \tau)$ be equal to $(g, \sigma) \in C_{[g, \sigma]}$. Since $\sigma(i_1) = i_2$, and $g_{i_k} \cdot g_{i_k}^{-1} = 1$, its action on $\mathcal{L}_{g_{i_1} \cdots g_{i_1}}M$ rotates the path $\gamma(t)$ to $\gamma(t + 1)$. Hence $(g, \sigma)$ maps to $1 \in C_{\tau}$ in the $k\mathcal{L}_{\tau}M$-factor of $k\mathcal{L}(M//G)$ (see Figure 4).

In the case that $G$ is the trivial group the discussion in [DMVV97] yields a formula summarizing the above description of $\mathcal{L}(M \wr \Sigma_n)$ for all $n$: write

$$\text{Sym}^\star(tM) := \prod_{n \geq 0} M^n//\Sigma_n t^n$$

for the “total symmetric power” of $M$. Here $t$ is a dummy variable.

**Corollary 4.5.** We have

$$\mathcal{L}(\text{Sym}^\star(tM)) \cong \prod_{k \geq 1} \text{Sym}^\star((k\mathcal{L}(M) \rtimes (\mathbb{Z}/k))t^k),$$

and the inertia orbifolds are

$$\Lambda(\text{Sym}^\star(tM)) := \prod_{n \geq 0} \Lambda(M \wr \Sigma_n)t^n \cong \prod_{k \geq 1} \text{Sym}^\star(t^k M \rtimes (\mathbb{Z}/k)).$$
where the action of the groups $\mathbb{Z}/k\mathbb{Z}$ on $M$ is trivial.

For non-trivial $G$, these product formulas are proved in [Tam].

4.3. Fourier decompositions of loop bundles of symmetric products. In this section we will apply the construction of the previous section to loop bundles and calculate its effect on Fourier decompositions. The following two, slightly technical, definitions are motivated by Example 4.8 below.

**Definition 4.6.** Let $V$ be an orbifold $S^1$-equivariant vector bundle over $\mathcal{LM}/G$. We write $kV$ for the orbifold $S^1$-equivariant vector bundle over $k\mathcal{LM}/G$ obtained by rescaling the $\mathbb{R}/kl\mathbb{Z}$ action on $V_g$ with $\frac{1}{k}$ to obtain an action of $\mathbb{R}/kl\mathbb{Z}$.

Let $V$ be as in the previous definition, and assume that its decomposition over the constant $g$-loops is

$$V|_{M^g} \cong V_0 \bigoplus_{j \geq 1} V_j q^{\frac{1}{j}}.$$  

Then the decomposition of $kV$ over $M^g$ is

$$kV|_{M^g} \cong V_0 \bigoplus_{j \geq 1} V_j q^{\frac{1}{j-k}}.$$  

with the same $V_j$ now viewed as $C_{g/kr}$ representations, where $C_g$ acts as before, and $\mathbb{R}/kl\mathbb{Z}$ acts with rotation number $j$. Hence the action of $g$ on $V_j$ equals that of $kr \in \mathbb{R}/kl\mathbb{Z}$, namely complex multiplication with $\zeta_j^k$.  

---

**Figure 4.** The images in $\mathcal{L}_{g_1g_2g_3} M$ of an element of $\mathcal{L}_{(g_1,g_2,g_3;\{123\})} M^3$ and its counterpart under the action of $(g_1,g_2,g_3;\{123\})$ differ by rotation by 1.
**Definition 4.7.** Let $\mathcal{V}$ be an orbifold $S^1$-equivariant vector bundle over $\mathcal{L}(M//G)$. We define the $S^1$-equivariant vector bundle $D_n(\mathcal{V})$ over $\mathcal{L}((M//G) \wr \Sigma_n)$ by

$$
(D_n(\mathcal{V}))_{(g,\sigma)} := F^*_{(g,\sigma)} \left( \prod_k (k\mathcal{V}) \wr \Sigma_{N_k} \right),
$$

where $F_{(g,\sigma)}$ is the homoemorphism of Theorem 4.3.

**Example 4.8.** In the case that $\mathcal{V} = \mathcal{L}(V)$ is the loop space of a bundle $V$ over $M//G$, we have

$$D_n(\mathcal{V}) \cong \mathcal{L}(V//G) \wr \Sigma_n.$$

Let $(g,\sigma)$ be as above, and let $f_{(g,\sigma)} = F^1_{(g,\sigma)}$ be the restriction of $F_{(g,\sigma)}$ to the constant loops.

**Proposition 4.9.** Let the orbifold $S^1$-equivariant vector bundle $\mathcal{V}$ over $\mathcal{L}(M//G)$ have Fourier decomposition

$$x := \mathcal{V}|_{\Lambda(M)} \cong \bigoplus_{j \in \mathbb{Z}} (V_j \triangleleft G)q^{1/|G|}$$

over the $G$-space $\coprod_g M^g$. Then the restriction of $D_n(\mathcal{V})$ to $(M^n)^{(g,\sigma)}$ has decomposition

$$f^*_{(g,\sigma)} (d_{N_1}(1(x)) \times d_{N_2}(z(x)) \times \cdots \times d_{N_k}(k(x)) \cdots)$$

over the $C_G \rtimes \Sigma_n$-space $(M^n)^{(g,\sigma)}$. Here $d_N$ is as in (10), and $k(x)$ is as in Definition 4.6.

**Proof:** By the definition of $f_{(g,\sigma)}$, the diagram

$$
\begin{array}{ccc}
(M^n)^{(g,\sigma)} & \xrightarrow{f} & \prod \left( \Lambda(M//G) \wr \Sigma_{N_k} \right) \\
\downarrow & & \downarrow \prod \iota_k \\
\mathcal{L}_{(g,\sigma)} M^n & \xrightarrow{F} & \prod \left( k\mathcal{L}(M//G) \wr \Sigma_{N_k} \right)
\end{array}
$$

commutes, where the products are over $k \in \mathbb{N}$, and the maps $\iota_k$ and $\iota$ are the fixed point inclusions. Thus

$$
(D_n \mathcal{V})|_{(M^n)^{(g,\sigma)}} = f^*_{(g,\sigma)} \left( \prod_k u_k^* \left( (k\mathcal{V}) \wr \Sigma_{N_k} \right) \right)
$$

$$
= f^*_{(g,\sigma)} \left( \prod_k d_{N_k} \left( (k\mathcal{V})|_{\Lambda(M//G)} \right) \right)
$$

$$
= f^*_{(g,\sigma)} \left( \prod_k d_{N_k} (kx) \right).
$$

Here $\prod$ stands for an external product, and $\Lambda(M//G)$ is the inertia orbifold as in Example 3.10. □

Proposition 4.9 motivates the following definition.
Definition 4.10. We define the map $D_n$ in $K_{Dev}^\Sigma_n$ by
\[ D_n: K_{Dev,G}(M) \to K_{Dev,G\wr\Sigma_n}(M^n) \]
\[ x \mapsto f^*_{(g,\sigma)}(d_{N_1}(x) \times d_{N_2}(x) \times \cdots \times d_{N_k}(x)) \]

4.4. Iterated symmetric powers. We will need to understand how our maps behave under iterated symmetric powers. Let
\[(\tau, \sigma) = (\tau_1, \ldots, \tau_n, \sigma) \in \Sigma_m \wr \Sigma_n.\]
For $1 \leq i \leq n$ and $1 \leq j \leq m$, we set
\[(\tau, \sigma)(i,j) := (\sigma(i), \tau_{\sigma(i)}(j)).\]
The bijection
\[\{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq m\} \to \{1, \ldots, nm\}\]
\[ (i,j) \mapsto i + (j-1)n \]
duces an inclusion
\[\iota: \Sigma_m \wr \Sigma_n \to \Sigma_{nm}.\]
If $(i_1, j_r)$ is a cycle of $(\tau, \sigma)$, then $i_r$ form a cycle of $\sigma$. Let $\sigma$ be the cycle $(1 \ldots n)$, and set
\[\tau = \tau_n \cdots \tau_1.\]
Then $(j_1 \ldots j_k)$ is a $k$-cycle of $\tau$ if and only if
\[\{(1, \tau_1(j_k))(2, \tau_2 \tau_1(j_k)) \ldots (n, j_1)(1, \tau_1(j_1)) \ldots (n, j_2)(1, \tau_1(j_2)) \ldots (n, j_k)\}\]
is an $nk$-cycle of $(\tau, \sigma)$.

Proposition 4.11. Let
\[(g_{i,j}, \tau_i, \sigma)_{i,j} \in (G \wr \Sigma_m) \wr \Sigma_n.\]
Then the following diagram commutes:
\[ \mathcal{L}(g_{i,j}, \tau_i, \sigma)_{i,j}(M^n)^m \to \prod_{(i_1, \ldots, i_l)} \prod_{(j_1, \ldots, j_k)} \mathcal{L}(g_{i,j}, \tau_i)M^m \]
\[ \downarrow \]
\[ \mathcal{L}(g_{i,j}, \iota(\tau_i), \sigma)_{i,j}(M^n)^m \to \prod_{(i_1, \ldots, i_l)} \prod_{(j_1, \ldots, j_k)} \mathcal{L}(g_{i,j}, \tau_i)M^m \]
Here $(i_1, \ldots, i_l)$ runs over all $l$-cycles of $\sigma$,
\[(g_{j_1, \tau_j} := (g_{i_1,j_1}, \tau_{i_1}) \cdots (g_{i_l,j_l}, \tau_{i_l}),\]
and the $(j_1, \ldots, j_k)$ run over all $k$ cycles of $\tau$ (hence the product in the lower right entry runs over all cycles of $(\tau, \sigma)$). The upper map is $F_{(g,\tau),\sigma}$, the right vertical map is $\prod_{k \times l} F_{(g,\tau_i)}$, the left is $\mathcal{L}\iota$ and the bottom is $F_{(g,\iota(\tau,\sigma))}$. (These are the maps defined in Theorem 4.3.)
Figure 5. An example with \( \sigma = (12) \), \( \tau_1 = (12)(3) \), and \( \tau_2 = (123) \). Hence \( \tau = (13)(2) \) and \( g_1 = g_{21}g_{13} \) and \( g_2 = g_{22}g_{11} \) and \( g_3 = g_{23}g_{12} \).

**Proof:** Without loss of generality, we may assume \( \sigma = (1 \ldots n) \). Then \( F_{(g, \tau, \sigma)} \) sends \( (\gamma_{ij})_{i,j} \) to \( (\gamma_j)_{j=1}^m \) with

\[
\gamma_j = \gamma_{n,j} * \gamma_{1, \tau_1(j)} g_{1, \tau_1(j)} * \ldots * \gamma_{n-1, \tau_{n-1}(j)} g_{n-1, \tau_{n-1}(j)} \tau_1(j) \ldots \gamma_1 \tau_1(j)
\]

(see Figure 5). Let

\[
(g_1, \ldots, g_m, \tau) := (g_{n,1}, \ldots, g_{n,m}, \tau_n) \cdots (g_{1,1}, \ldots, g_{1,m}, \tau_1).
\]

Then \( \tau = \tau_n \cdots \tau_1 \), and

\[
g_j = g_{n,j} g_{n-1, \tau_n^{-1}(j)} \cdots g_{1, \tau_2^{-1}(j)} g_1 \tau_1(j),
\]

and \( \gamma_j \) is a path from \( x_{n,j} \) to \( x_{n, \tau(j)} g_{\tau(j)} \), where the \( x_{i,j} \) are the starting points of the paths \( \gamma_{i,j} \). Hence \( (\gamma_j)_{j=1}^m \) is an element of

\[
nL_{(g_1, \ldots, g_m, \tau)} M^m.
\]

Let now \( (j_1, \ldots, j_k) \) be a cycle of \( \tau \). Then the component of \( f_{m,G} \) corresponding to this cycle sends \( (\gamma_j) \) to

\[
\gamma = \gamma_{j_k} * \gamma_{j_1} g_{j_1} * \ldots * \gamma_{j_{k-1}} g_{j_{k-1}} \cdots g_{j_1}.
\]

For \( r \in \mathbb{Z}/k\mathbb{Z} \), we have

\[
\gamma|_{[r n+i, r n+i+1]} = \gamma_{j_r} g_{j_r} \cdots g_{j_{i+1}}.
\]

Further,

\[
\gamma_{j_r}|_{[i+1, i+1]} = \gamma_{i, \tau_{i-1}(j_r)} g_{i, \tau_{i-1}(j_r)} g_{i-1, \tau_{i-1}(j_r)} \cdots g_{\tau_1(j_r)}
\]

\[
= \gamma_{(\tau, \sigma)^{r n+i}(n_{j_k})} g_{(\tau, \sigma)^{r n+i}(n_{j_k})} \cdots g_{(\tau, \sigma)^{r n+i}(n_{j_k})},
\]

and

\[
g_{j_r} = g_{(\tau, \sigma)^{r n}(n_{j_k})} \cdots g_{(\tau, \sigma)^{(r+1)n_{j_k}}},
\]

\( \square \)

**Corollary 4.12.** Let \( \iota \) be the inclusion of \( \Sigma_m \) in \( \Sigma_n \) in \( \Sigma_m \) described above. Then we have

\[
\text{res}_{\iota} \circ D_{nm} \cong D_n \circ D_m.
\]

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5. Power operations

The goal of this section is to define power operations on Devoto’s equivariant Tate $K$-theory. We want to define them in such a way that the equivariant Witten genus becomes an $H_\infty$ map. In other words, we need our power operations to take Thom classes $u_{\text{string}}^*\langle V \rangle$ to the Thom classes $u_{\text{string}}^*\langle V \wr \Sigma_n \rangle$. It turns out that the discussion of the previous section dictates our definitions. In our treatment of power operations, we will once again proceed in two steps. First, we define “Atiyah power operations” on the untwisted sector $K_G(X)[[q]]$. Then we use the Atiyah power operations to define “stringy power operations” on all of $K_{\text{Dev},G}(X)$. We briefly recall the general setup of equivariant power operations from [Gan06, Def.4.3]. Let $\{E_G | G \text{ finite}\}$ be a compatible family of equivariant cohomology theories in the sense of [LMSM86, II.8.5], and write $E_G(X)$ for $E_G^0(X)$. We also ask that our family has unitary, commutative and associative external products $\boxtimes$ that are natural in (stable) maps of $X$ and $Y$.

**Definition 5.1.** An $H_\infty$-structure on $E$ is given by a collection of natural maps

$$P_n : E_G(X) \to E_{G|\Sigma_n}(X^n)$$

called power operations satisfying the following conditions:

(a) $P_1 = \text{id}$ and $P_0(x) = 1$,

(b) the (external) product of two power operations is

$$P_j(x) \boxtimes P_k(x) = \text{res}_{\Sigma_j \times \Sigma_k} (P_{j+k}(x)),$$

(c) the composition of two power operations is

$$P_j(P_k(x)) = \text{res}_{\Sigma_{jk} / \Sigma_j} (P_{jk}(x)),$$

(d) and the $P_j$’s preserve (external) products:

$$P_j(x \boxtimes y) = \text{res}_{\Sigma_j} (P_j(x) \boxtimes P_j(y)),$$

where the restriction is along the map

$$[(X \boxtimes G)^2 \wr \Sigma_j] \to [(X \boxtimes G) \wr (\Sigma_j \times \Sigma_j)] \cong [(X \boxtimes G) \wr \Sigma_j]^2.$$

We also recall the graded ring

$$S_E(X) = \bigoplus_{n \geq 0} E_{G|\Sigma_n}(X^n)t^n.$$

Here $t$ is a dummy variable keeping track of the grading, and the multiplication of two elements of degree $m$ and $n$ respectively, is given by the external product, followed by the transfer

$$\text{ind}_{\Sigma_m \times \Sigma_n}.$$

The ring $S_E(X)$ is the target of the total power operation

$$P = \sum_{n \geq 0} P_n t^n.$$
5.1. Atiyah power operations. In this section we work over the untwisted sector, i.e., with the compatible family 
\[ E_G(X) = K_G(X)[q] \].

On \( K_G(X) \) we have Atiyah’s power operations
\[ P_n: K_G(X) \to K_G(\Sigma_n(X^n)) \]
\[ [V] \mapsto [V^{\otimes n}] \].

Here \( \Sigma_n \) acts with a sign on \( V^{\otimes n} \).

**Definition 5.2.** We define the total Atiyah power operation on \( K_{\text{Tate}} \) by
\[ P_{\text{top}}: K_G(X)[q] \to \bigoplus_{n \geq 0} K_G(\Sigma_n(X^n))[q]t^n \]
\[ x_1 q^{i_1} \mapsto \sum_{n \geq 0} x_1^{\otimes n} q^{i_1} t^n, \text{ and} \]
\[ \sum_{i \geq 0} x_i q^i \mapsto \prod_{j \geq 0} P_{\text{top}}(x_j q^j). \]

The coefficient of \( t^n \) is called the \( n \)th Atiyah power operation, it is given by
\[ P_n^{\text{top}}(x) = \sum_{i} P_{\hat{i}}(x), \]
where \( \hat{i} = (i_j)_{j \geq 0} \) runs over all sequences of natural numbers whose elements add up to \( n \), and
\[ P_{\hat{i}}(x) = \prod_{j} P_{i_j}(x_j) q^{i_j}. \]

Note that for each \( \hat{i} \), this product has only finitely many non-trivial factors. It is taken in the graded ring \( S_{K_G}(X) \) and hence involves the induced representations
\[ \text{ind}_{\hat{i}}^n := \text{ind}_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_j} \times \cdots} \]
The transfer \( \text{ind}_{\hat{i}}^n \) adds all possible shuffles of the partition \( \hat{i} \). More precisely, we have
\[ (12) \]
\[ P_n^{\text{top}}(x) = \sum_{\hat{i} \in \Sigma^n} x_{i_1} q^{i_1} \otimes \cdots \otimes x_{i_n} q^{i_n}, \]
and \( \Sigma_n \) acts (with a sign) by permutation of the summands and the factors therein. The infinite sum \( P_n^{\text{top}} \) is a well defined formal power series in \( q \).

**Proposition 5.3.** The \( P_n^{\text{top}} \) satisfy the axioms of Definition 5.1.

**Proof:** Part (a) is clear. Part (b) and (c) follow from Equation (12). Let \( x \in K_G(X)[q] \) and \( y \in K_H(Y)[q] \). Then the external product of \( x \) and \( y \) is
\[ x \otimes y = (\sum_{i} x_i q^i) \otimes (\sum_{j} y_j q^j) = \sum_{i,j} (x_i \otimes y_j) q^{i+j}, \]
and
\[ P_n(x \otimes y) = \sum_{\hat{i} \in \Sigma^n} (x_{i_1} \otimes y_{j_1}) q^{i_1+j_1} \otimes \cdots \otimes (x_{i_n} \otimes y_{j_n}) q^{i_n+j_n}, \]

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Under the isomorphism
\[(X \times Y)^n \cong X^n \times Y^n,\]
this is identified with
\[\text{res}_{(G \times H) \wr \Sigma_n} P_n(x) \boxtimes P_n(y).\]

### 5.2. The topological Witten genus of orbifolds and its product formula.

**Definition 5.4.**
Let \(V\) be a \(G\)-equivariant Spin (respectively complex) vector bundle, and set
\[u^\text{top}_G(V) := u^A_G(V) \cdot \prod_{j=1}^{\infty} \Lambda_{-q_j^t} V_c.\]
We write
\[\phi^\text{top}_G : MS\text{Spin}P_G \to K_G[[q]],\]
(respectively with \(MS\text{Pin}P_G\) replaced by \(MUP_G\)) for the corresponding genus (c.f. Proposition A.4).

**Proposition 5.5.** The map \(\phi^\text{top}_G\) is \(H^4_{\infty}\) (\(H^2_{\infty}\) in the complex case).

**Proof:** By [Gan06, 4.6], [tD68, (A4)], it suffices to show that
\[u^\text{top}_G \wr \Sigma_n(V^n) = P_n(u^\text{top}_G(V))\]
for any Spin(4k)-bundle \(V\). This follows from Axiom (d), \(\dim V^n = 4nk\), the known analogue for \(u^A\) and \(u^{Td}\):
\[u^A_G \wr \Sigma_n(V^n) = P_n(u^A_G(V))\]
[BMMS86], and the fact that
\[\Lambda_t(V^n) \cong (\Lambda_t V)^{\Sigma n},\]
where \(\Sigma_n\) acts by permuting the factors inside on the left-hand side, and by permuting the factors outside (with a sign) on the right-hand side.

**Definition 5.6.** We define the topological Witten genus of a global quotient orbifold \(M//G\) by
\[\phi^\text{top}(M//G) := \frac{1}{|G|} \sum_{g \in G} \text{Trace}(g|\phi^\text{top}_G(M)),\]
where \(\phi\) stands for \(\hat{A}\) or \(Td\).

**Definition 5.7.** We define the (topological) total symmetric power operation \(\text{Sym}^\text{top}_t\) on \(K_{\text{Tate}}\) to be the composite
\[\text{Sym}^\text{top}_t : K_G(X)[[q]] \to \bigoplus_{n \geq 0} K_G \wr \Sigma_n(X^n)[[q]] t^n \to \bigoplus_{n \geq 0} K_{G \times \Sigma_n}(X)[[q]] t^n \to K_G(X)[[q]][t],\]
where the first map is \(p^\text{top}\), and the second map is restriction along the diagonals of \(X^n\) and \(G^n\). The last map \(\varepsilon\) is (in degree \(n\)) given by
\[\varepsilon_n = \text{id} \otimes (-1)^n : K_G(X) \otimes R(\Sigma_n) \to K_G(X).\]
Note that the total power operation takes sums to products, and that $\varepsilon$ is a ring map. Hence $S_t$ is exponential. In the special case that $X$ is a point and $G$ is the trivial group, we obtain the following corollary of Proposition 5.5:

**Corollary 5.8** (product formula for the topological Witten genus). If $M$ is a Spin$(4k)$ or complex manifold with Witten genus $\phi(M) = \sum_j c(j)q^j$, then

$$\sum_{n \geq 0} \phi_{\text{top}}^n(M^n/S_n)t^n = \prod_j \left( \frac{1}{1 - tq^j} \right)^{c(j)}.$$ 

**Proof**: Since $\phi_{\text{top}}$ preserves power operations, we have

$$\sum_{n \geq 0} \phi_{\Sigma_n}^n(M^n)t^n = \Phi_{\text{top}}^n(\phi(M)).$$

Applying $\varepsilon$ to both sides, we get

$$\sum_{n \geq 0} \phi_{\Sigma_n}^n(M^n/S_n)t^n = \text{Sym}_{\text{top}}^n(\phi(M)).$$

Since $\text{Sym}_{\text{top}}^n$ is exponential, it suffices to observe that $\text{Sym}_{\text{top}}^n(q^j) = 1/(1-tq^j)$ to complete the proof. $\square$

### 5.3. Stringy power operations.

The following definitions should be compared to those of Section 4.3.

**Definition 5.9.** Let $\kappa(-)$ be the ring map

$$\kappa(-) : K_{\text{Dev}, G, r} \to K_{\text{Dev}, G, rk}$$

$$\sum V_j q^{\tau^g_j} \mapsto \sum V_j q^{\tau^g_{rk|g|}}.$$ 

Here the action of $C_g$ on $V_j$ remains unchanged, while $1 \in \mathbb{Z}/rk|g|\mathbb{Z}$ acts as $\zeta_{rk|g|}^j$ on the $j^{th}$ coefficient of the $[g]^{th}$ summand on the right-hand side.

In the following, for $x \in K_{\text{Dev}, G, k}(X)$, by $P_{n, \text{top}}^n(x)$ we will mean the image of $x$ under the composite

$$K_{\text{Dev}, G, k}(X) \to K_{G \times \mathbb{Z}/kL}(\prod_g X^g)[q^{\frac{1}{\tau^g}}] \to K_{(G \times \mathbb{Z}/kL)/\Sigma_n}(\prod_g (X^g)^n)[q^{\frac{1}{\tau^g}}].$$

Here $L$ is the order of $G$, and the second map is $P_{n, \text{top}}^n$.

**Definition 5.10.** Let $x \in K_{\text{Dev}, G}(X)$. We define

$$P_{n, \text{string}}^n(x) \in K_{\text{Dev}, G/\Sigma_n}(X^n)$$

by

$$(P_{n, \text{string}}^n(x))_{[g, \sigma]} := f_{[g, \sigma]}^* (P_{n_1}^n(1(x)) \boxtimes \cdots \boxtimes P_{n_k}^n(1(x))) \boxtimes \cdots.$$
We need to verify that $\mathbf{p}^\text{string}_n$ indeed takes values in $K_{Dev,G}(X^n)$. First, we have

$$P_{N_k}^\text{top}(k(x)) \in K_G(X_n)(\bigotimes_{h \in G} X^h)^{N_k}[[q^\frac{1}{h}]]$$

$$\cong \bigoplus_{|h|} K_{\text{Stab}[h]}(\bigotimes_{i=1}^{N_k} X^{h_i})[[q^\frac{1}{h}]],$$

where the direct sum is over the orbits of the action of $(G \times \mathbf{Z}/kL) \wr \Sigma_{N_k}$ on $G^{N_k}$. In fact, the $h$th summand of $P_{N_k}^\text{top}(k(x))$ is a power series in $q^{\frac{1}{|h|}}$, since it is obtained by multiplying power series in $q^{\frac{1}{|h|}}$ for $1 \leq i \leq N_k$. Further, recall from Section 4.2 that

$$(X^n)^{\text{string}} \cong \prod_{k \geq 1} \prod_{\{i_1, \ldots, i_k\}} X^{g_{i_1} \cdots g_{i_k}},$$

where the second product runs over the $k$-cycles of $\sigma$. Hence, for fixed $k$, the target of the $k$th factor of $f_{(g,\sigma)}$ is $\prod_{i=1}^{N_k} X^{h_i}$, where $h_i$ runs over the $k$-cycles of $\sigma$, and $h_i = g_{i_1} \cdots g_{i_k}$. By Example 4.4, $(g,\sigma)$ acts by multiplication with $\zeta_{|h_i|}^j$ on the coefficient of $q^{\frac{1}{|h_i|}}$ of $X^{h_i}$. Since $|(g,\sigma)|$ is the smallest common multiple of all $k \cdot |g_{i_1} \cdots g_{i_k}|$ such that $k \in \mathbb{N}$ and $(i_1 \ldots i_k)$ is a $k$-cycle of $\sigma$, it follows that $(\mathbf{p}^\text{string}_n(\mathbf{x}))^{(g,\sigma)}$ is a power series in $q^{\frac{1}{|g,\sigma|}}$, where $(g,\sigma)$ acts by multiplication with $\zeta_{|g,\sigma|}^j$ on the coefficient of $q^{\frac{1}{|g,\sigma|}}$.

**Theorem 5.11.** The $\mathbf{p}^\text{string}_n$ satisfy the axioms of Definition 5.1.

**Proof:** Axiom (a) is clear. For Axiom (b), let $(\sigma,\tau) \in \Sigma_m \times \Sigma_n$. Then the set of $k$-cycles of the element $(\sigma,\tau) \in \Sigma_{m+n}$ is identified with the disjoint union of the sets of $k$-cycles of $\sigma$ and $\tau$, and thus

$$\left(\mathbf{p}^\text{string}_n(\mathbf{x})\right)^{(g,\sigma,\tau)} = \left(\mathbf{p}^\text{string}_m(\mathbf{g},g_{m+1},\ldots,g_{m+n},\mathbf{\tau})\right)^{(g_{m+1},\ldots,g_{m+n},\mathbf{\tau})} = \left(\mathbf{p}^\text{string}_m \boxtimes \mathbf{p}^\text{string}_n\right)^{(g,\sigma,\tau)},$$

Axiom (c) follows from Proposition 4.11: With the notation of that Proposition, and omitting the equivariant information, we have

$$(\mathbf{p}^\text{string}_n(\mathbf{p}^\text{string}_m(\mathbf{x})))^{(g_{i_1},\tau_{i_1},\mathbf{\sigma})} = \prod_{i_1(i_1,\ldots,i_k)} \prod_{k_{i_1}(j_1,\ldots,j_k)} X^{g_{i_1} \cdots g_{i_k}}$$

$$= \mathbf{p}^\text{string}_n(\mathbf{x})^{(g_{i_1},\tau_{i_1},\mathbf{\sigma})},$$

Here all the products denote the external tensor product $\boxtimes$. For the non-equivariant information it makes no difference whether $\boxtimes$ is inside or outside the product, and when restricting along the inclusion of centralizers

$$C_{G;\Sigma_m} \subseteq C_{G;\Sigma_{m+n}}$$

we restrict that part along the inclusion $\mathbb{Z}/k \mathbb{Z} \subseteq (\mathbb{Z}/k \mathbb{Z})^{N_k}$. To prove Axiom (d), let $x \in K_{Dev,G}(X)$ and $y \in K_{Dev,H}(Y)$. Then $x \boxtimes y$ is the element of $K_{Dev,G \times H}$ with

$$(x \boxtimes y)_{(g,h)}^{(g,h)} = x_{[g]} \boxtimes y_{[h]},$$
and the statement follows from the analogous statement for the $p^{\text{top}}_{N_k}$. \hfill \square

In this section, we will work with the following variant of the stringy equivariant Witten genus:

**Definition 5.12.** We define $T_d^{\text{string}}: \text{MU}_G \to \mathcal{K}_{\text{Dev}}$ to be the map of spectra corresponding to the Thom class

$$u^{\text{string}}_{T_d,G}(V) := u^{T_d}_{C_G} (V^g) \cdot \prod_{j \geq 1} \Lambda_{-1} \left( (V_C)_{\mathbb{C} [q]} \right).$$

The computation of $T_{d[g]}^{\text{string}}(M)(h)$ is the same as that for $\hat{A}^{\text{string}}$ with the terms $q^{-\frac{d}{2} + \text{age}(g)}$ and $e^{i \pi \text{age}(h)}$ missing. For SU-bundles, we have $\sum \tilde{x}_i = 0$, so that the missing factors are just a renormalization constant.

**Theorem 5.13.** The map $T_d^{\text{string}}$ is $H^2_{\infty}$.

**Proof:** We write $u^{\text{string}}_G$ for $u^{\text{string}}_{T_d,G}$ and $u$ for $u^{T_d}$. We need to prove

$$u^{\text{string}}_{G \wr \Sigma_n} (V^n) = P_{n}^{\text{string}} (u^{\text{string}}_G (V)).$$

Let $(g, \sigma) \in G \wr \Sigma_n$. Then

$$V^n |_{X^n [g, \sigma]} \cong \prod_k \prod_i V^{|g_{i_k} \cdots g_{i_1}|}_{X^i_g_{i_k} \cdots g_{i_1}},$$

where the product signs denote external direct sums and the second one runs over the $k$-cycles of $\sigma$. On the $i$th summand of this, $(g, \sigma)$ acts by

$$(v_1, \ldots, v_k) \mapsto (v_2 g_{i_2}, \ldots, v_k g_{i_k}, v_1 g_{i_1}).$$

Hence $\zeta$ is an eigenvalue of this action with eigenvector $(v_1, \ldots, v_k)$ if and only if $\zeta^k$ is an eigenvalue of $g_{i_k} \cdots g_{i_1}$ with eigenvector $v_k$, and $v_j = v_{j+1} g_{i_{j+1}} g_{i_{j+1}}$ for $1 \leq j \leq k$. Thus

$$(V^{|g_{i_k} \cdots g_{i_1}|})_{\mathbb{C} [q]} \cong (V^{|g_{i_k} \cdots g_{i_1}|})_{\mathbb{C} [q]},$$

and

$$(V^n |_{X^n [g, \sigma]})_{\mathbb{C} [q]} \cong \prod_k \prod_i (V^{|g_{i_k} \cdots g_{i_1}|})_{\mathbb{C} [q]}.$$

Now Definition 5.12 gives

$$u^{\text{string}}_{(g, \sigma), g} (V^n) = u^{\text{string}}_{G, [g, \sigma]} (V^n) \cdot \prod_{j=1}^{\infty} \Lambda_{-q [g, \sigma]} \left( (V_C)_{\mathbb{C} [g, \sigma]} \right)$$

$$= \prod_{k, i} \left( u^{\text{string}}_{g_i} (V_i) \cdot \prod_{j=1}^{\infty} \Lambda_{-q g_i} \left( (V_C)_{\mathbb{C} [g_i]} \right) \right),$$

where $g_i = g_{i_k} \cdots g_{i_1}$, in the first line, $V$ and $V_C$ get restricted to $(X^n) [g, \sigma]$ and in the second line, they get restricted to $X^{g_{i_k} \cdots g_{i_1}}$. \hfill \square
5.4. **Hecke operators.** In this section we prove that the power operations \( p_n^{\text{string}} \) are elliptic in the sense explained in the introduction. We recall from [Gan06] that any cohomology theory with power operations and a level 2 Hopkins-Kuhn-Ravenel theory has Hecke operators, acting as (internal) cohomology operations:

**Definition 5.14.** The \( n \)-th Hecke operator \( T_n \) is defined by

\[
T_n(x) := \frac{1}{n} \sum_{\alpha \in A} \psi_\alpha(x).
\]

Here \( A \) is a system of representatives of those conjugacy classes of pairs of commuting elements\(^1\) \((\sigma, \tau)\) of \( \Sigma_n \), with the property that the subgroup of \( \Sigma_n \) generated by \( \sigma \) and \( \tau \) acts transitively on \( \{1, \ldots, n\} \). For such a representative \( \alpha \in A \), the operation \( \psi_\alpha \) is defined by

\[
\psi_\alpha(x) := \text{eval}_\alpha \left( \text{res}_\delta |P_n(x)\right),
\]

where \( \delta \) denotes the inclusion of the groupoid \( X \rtimes (G \times \Sigma_n) \) into \( (X \rtimes G) \Sigma_n \), which is induced by the diagonal inclusion of \( X \rtimes G \) into \( (X \rtimes G)^n \). (On the source of \( \delta \), \( \Sigma_n \) acts trivially on the \( G \)-space \( X \).)

The rest of this section is dedicated to the computation of \( T_n \). We start by choosing a system of representatives \( A \) as in (13). In order for \((\sigma, \tau)\) to act transitively, any two cycles of \( \sigma \) must have equal length. Let \( c_1 \) be a cycle of \( \sigma \), w.l.o.g. say \( c_1 = (1, \ldots, k) \). Let \( c_2 \) be the \( k \)-cycle of \( \sigma \) starting with \( \tau(1) \). Noting that \( c_2 \) cannot equal \( c_1 \), because of the transitivity of the action, we may assume \( c_2 = (k+1, \ldots, 2k) \). Repeating this argument, we arrive at \( n \) \( k \)-cycles \( c_1, \ldots, c_n \) of \( \sigma \), where \( n = nk \), and for \( 1 \leq j < n \), \( \tau \) maps the first element of \( c_j \) to the first element of \( c_{j+1} \). For definiteness, we let \( c_j = ((j-1)k+1, \ldots, jk) \). Now \( \tau \) has to map the first element of \( c_N \) to an element of \( c_1 \), and each of the elements of \( c_1 \) is possible. The conjugacy class of \((\sigma, \tau)\) is uniquely determined by \( k \) and \( 1+m := \tau((N-1)k+1) \in \{1, \ldots, k\} \).

Fix \( N \) and \( k \) such that \( Nk = n \), and let \( \sigma \) be the element of \( \Sigma_n \) with \( N \) \( k \)-cycles as above. Let \( g \in G \) and \( \overline{g} := \delta(g) = (g, \ldots, g) \in G^n \). Then

\[
p_{[g, \sigma]}^{\text{string}}(x) = \tau^*_g |P_N^{\text{top}}((kX^g)_{X^g}^k).\]

Consider \((h, \tau) \in C_G(g) \times C_{\Sigma_n}(\sigma)\), with \( \tau \) as above. In order to determine its action on \((X^g_k)^N\), we note that we can simplify the discussion in the proof of Theorem 4.3 by choosing all the representatives \( r_\alpha \) to equal \( g^k \). Then \((h, \tau)\) acts as

\[
(h, \ldots, h, (h^g - m, m), C) \in C_g^k \Sigma_N\]
on \((X^g_k)^N\), where \( C = (1, \ldots, N) \).

Assume first that \( m = 0 \). Then \( P_N^{\text{top}} \) composed with \( \text{Trace}(-, \tau) \) is the \( N \)-th Adams operator \( \psi_N \). The set of \( \tau \)-fixed points of \((X^g_k)^N\) is the image of the diagonal map

\[
\delta': X^g_k \rightarrow (X^g_k)^N.
\]

Hence the \([g]-\)component of \( \psi_{\sigma, \tau}(x) \) is

\[
(\psi_{\sigma, \tau}(x))[g] = i^*(\delta')^*(\psi_N((kX^g)_{X^g}^k)),
\]

where

\[
i: X^g \odot C_g \rightarrow X^g_k \odot C_g^k
\]

\(^1\)In this context, “conjugacy class” refers to simultaneous conjugation.
is the inclusion. On coefficients, we get the following formula for characters:

\[ (P_n(x)) (g, \sigma, h, \tau) = \left( \sum_{j \geq 0} V_j q^{nj} (g^k, h^N) \right), \]

where \( l \) is the order of \( g \), and \( x = \sum_{j \geq 0} V_j q^j \). Let now \( m \) be arbitrary. Then \((h, m) \in C_k \) acts as \( h \cdot \zeta_{kl}^m \) on \( V_j \). Non-equivariantly, (14) is still true, however, now the action of \( h \) on \( \psi_N(kx|Xgk) \) is twisted with \( g^{-m} \zeta_{kl}^m \). On coefficients, we get

\[ (P_n(x)) (g, \sigma, h, \tau) = \left( \sum_{j \geq 0} V_j q^{nj} \zeta_{kl}^m (g^k, g^{-m}h^N) \right), \]

where \( x \) and \( l \) are as above.

\[ \text{Remark 5.15.} \] Viewing \( x(g, h) \) as the \( q \) expansion of a function \( x(g, h; z) \), where \( z \) is in the upper half-plane, and \( q = e^{2\pi iz} \), we get (on coefficients)

\[ (T_n(x)) (g, h; z) = \frac{1}{n!} \sum_{\alpha} x \left( g^k, g^{-m}h^N, \frac{Nz + m}{k} \right). \]

This is exactly the formula for the twisted Hecke operators of generalized Moonshine in [Gan]. It follows from the way the twisted Hecke operators are defined in [ibid.] (namely using isogenies of elliptic curves) that our \( P_n \) are elliptic. Let \( G \) be the trivial group. Using the theory of isogenies on the Tate-curve, Matthew Ando has defined power operations on \( K_{Tate} \) in [And00]. In this case, our definition specializes to his. For \( G = 1 \) and \( X \) a point, the Hecke operators are the usual ones acting on the \( q \)-expansions of modular forms.

As we noted above, our geometric picture only sees one of the two circles of the elliptic curve. The case where \( k = n \) and \( N = 1 \) corresponds to replacing this circle by one of length \( n \) and pulling back along an \( n \)-fold covering map from this long circle to the short one. The analogous power operation for the circle we cannot see \( (k = 1 \text{ and } N = n) \), is the \( n \text{th} \) Adams operation.

6. The DMVV-formula, Borcherds products and replicability

We recall from [Gan06] that in a cohomology theory with power operations and a level 2 Hopkins-Kuhn-Ravenel theory, the \( n \text{th} \) (stringy) symmetric power is defined by

\[ \text{sym}_n^\text{string}(x) = \frac{1}{n!} \sum_{\alpha} \psi_\alpha(x), \]

where \( \alpha \) runs over all pairs of commuting elements of \( \Sigma_n \). A similar argument to that of Section 3.4 shows that the symmetric powers take values in \( K_{Dev,G}(X) \). The total (stringy) symmetric power is defined as

\[ \text{Sym}_t^\text{string}(x) := \sum \text{sym}_n^\text{string}(x)t^n. \]

With these definitions, the generating function argument in [Gan06, 9.2] goes through and yields:
Proposition 6.1. On the level of cohomology operations, we have

\[ \text{Sym}_{1}^{\text{string}}(x) = \exp \left( \sum_{m \geq 1} T_m(x)t^m \right). \]

In terms of elliptic curves, this generating function can be interpreted as follows: the Hecke operators average over the pullback along a system of representatives for isogenies onto the elliptic curve. The \( n \)-th symmetric power averages over all possible \( n \)-fold coverings of the elliptic curve. Each covering is made up from its connected components, and those are isogenies. For a detailed discussion of this picture, we refer the reader to [Gan].

Together with Theorem 5.13, and applied to the case where \( X \) is a point, Proposition 6.1 becomes the Dijkgraaf-Moore-Verlinde-Verlinde formula:

Corollary 6.2. Let \( M \) be a compact closed complex manifold. Then

\[ \sum_{n \geq 0} T_{d_{\text{orb}}^n}(M^n)t^n = \exp \left( \sum_{m \geq 1} T_m(T_{d_{\text{string}}}(M))t^m \right). \]

The right-hand side of the DMVV-formula can be rewritten as a Borcherds product:

\[ \prod_{i,j} \left( \frac{1}{1 - q^it^j} \right)^{c(ij)}, \]

where \( \phi(M) = \sum c(j)q^j \). The best known example of a product formula is the case where the \( c(i) \) are the coefficients of the \( q \)-expansion of the modular function \( j - 744 \). In this case, the product is equal to the inverse of

\[ t(j(t) - j(q)). \]

In other words, all the mixed terms in \( t \) and \( q \) in the product \( t^{-1} \prod (1 - q^it^j)^{c(ij)} \) are zero. This property encodes the replicability of the function \( j - 744 \). Similar identities hold for the other Moonshine functions.

Note that \( j(q) - 744 \) equals the Witten genus of the Witten manifold \( M \) constructed in [MH02].

One rather peculiar property of these kind of product formulas is the change of the role of \( t \) from a dummy variable to a variable which plays the same role as \( q \). In our context, both the variables \( t \) and \( q \) are counting winding numbers, which makes it less surprising that they should play a similar role. Indeed, the symmetry of their roles is clarified by the following proposition:

Proposition 6.3. Let \( G \) be the trivial group, and view \( x \in K(X) \) as an element of \( K_{\text{Tate}}(X) \). Then the total symmetric power \( \text{Sym}_{q}^{\text{string}}(x) \) is Witten’s exponential characteristic class

\[ \text{Sym}_{q}^{\text{string}}(x) = \bigotimes_{k \geq 1} \text{Sym}_{q^k}(x), \]

where \( \text{Sym}_{q} \) stands for the total symmetric power in K-theory.

Proof: Note that \( kx = \text{res}_{\mathbb{Z}/k}^1 x \). Hence, for every natural number \( N \), we have

\[ \langle P_{N}^{\text{top}}(kx), 1 \rangle_{\mathbb{Z}/k\Sigma N} = \langle P_{N}^{\text{top}}(x), 1 \rangle_{\Sigma N} = \text{Sym}^N(x). \]
The coefficient of $q^n$ in $\text{Sym}^n_q(x)$ is
\[
\text{sym}^n_{\text{string}}(x) = \frac{1}{|\Sigma_n|} \sum_{\sigma \tau = \tau \sigma} p_n^\text{string}(x)|_{\delta_n}(\sigma, \tau) \\
= \sum_{[\sigma]} \frac{1}{C_{\sigma}} \sum_{\tau \in C_{\sigma}} \left( \bigotimes_{k \geq 1} p_{N_k}^{\text{top}}(kx)|_{\delta_{N_k}}(\tau) \right),
\]
where $[\sigma]$ corresponds to the partition $n = \sum kN_k$, so that $C_{\sigma} \cong \prod_{k \geq 1} (\mathbb{Z}/k) \wr \Sigma_{N_k}$. (Since we are pulling back along the diagonal, the tensor product is an internal tensor product.)

The expression becomes
\[
\sum_{n=\sum N_k} \bigotimes_{k \geq 1} \left( \frac{1}{|k^N_k N_k|} \sum_{\tau_k} \left( p_{N_k}(kx)|_{\delta_{N_k}}(\tau_k) \right) \right) = \sum_{n=\sum N_k} \bigotimes_{k \geq 1} \text{Sym}^N_k(x).
\]
(Here $\tau_k \in \mathbb{Z}/k \wr \Sigma_{N_k}$.) This is the coefficient of $q^n$ in $\bigotimes_{k \geq 1} \text{Sym}^k_q(x)$.

\[\square\]

6.1. **Replicability.** Let $F(q)$ be a Laurent series with coefficients in $\mathbb{R}(G)$ which is of the form
\[
F(q) = q^{-1} + a_1 q + a_2 q^2 + \ldots.
\]
From the Moonshine literature, such $F$ are known as McKay-Thompson series. We recall\(^2\) that the Faber polynomials $\Phi_{n,F}$ of $F$ are defined by
\[
-\sum_{n=1}^{\infty} \Phi_{n,F}(w)t^n = \log \left( t(F(t) - w) \right).
\]
Hence $\Phi_{n,F}$ is a polynomial in $w$ of degree $n$, which depends on the first $n$ coefficients of $F$ and is uniquely characterized by the fact that it is of the form
\[
\Phi_{n,F}(F(q)) = q^{-n} + b_1 q + b_2 q^2 + \ldots.
\]
Viewing $F(q)$ as an element in (the $[1]$-component of) $q^{-1}K_{\text{Dev},G}(pt)$, we arrive at the following definition:

**Definition 6.4.** Let $F$ be a McKay-Thompson series. We write $F^{(a)}$ for the $a$\textsuperscript{th} Adams operator applied to $F$. We call $F$ *replicable*, if for every natural number $n$, we have
\[
\Phi_{n,F}(F(q)) = \sum_{\substack{\alpha d - n \\ \delta \in b < d}} F^{(a)} \left( \frac{\alpha \tau + b}{d} \right),
\]
Here $q = e^{2\pi i \tau}$.

This appears to be the right notion of replicability of McKay-Thompson series, it is the one that turns up in [Bor92]. Note that the right-hand side of the equation in the definition equals $n \cdot T_n(F(q))$, where $T_n$ is the Hecke operator computed in Section 5.4. It follows

\(^2\)Compare e.g. [Teo03, (2.1)] with $b = 1$, $t = 1/z$ and $F(q) = g(z)$.
immediately from the definitions and from Corollary 6.1 that a McKay-Thompson series \( F \) is replicable if and only if it satisfies the following identity:

\[
F(t) - F(q) = t^{-1} \cdot \text{Sym}^\text{string}_t(-F(q)) = t^{-1} \cdot \Lambda^\text{string}_t(F(q)),
\]

where \( \Lambda^\text{string}_t(x) \) is defined as the multiplicative inverse of \( \text{Sym}^\text{string}_t(x) \). For \( F(q) \) as above, \( \Lambda^\text{string}_t(F(q)) \) can be written as

\[
\Lambda^\text{string}_t(F(q)) = \Lambda^{-1}_t \left( \sum a_{m,n} q^m t^n \right).
\]

This is the form in which it appears in [Bor92, p.410].

There is a lot more to be said about the connection between power operations in elliptic cohomology and the notion of replicability in (generalized) Moonshine. We will come back to these topics at a different occasion.

### Appendix A. Thom classes and the fixed point formula

#### A.1. Thom classes in equivariant cohomology theories.

**Definition A.1.** We say that a compatible family of equivariant cohomology theories \( \{E_G\} \) has natural Thom classes for complex vector bundles, if for every complex \( G \)-vector bundle \( V \) over a pointed \( G \)-space \( X \) there exists a class \( u^E_G(V) \in \tilde{E}^0(X^v) \) with the following properties:

1. **Naturality:** If \( f: X \to Y \) is a pointed \( G \)-map, then

\[
u^E_G(f^* V) = f^* u^E_G(V).
\]

2. **Multiplicativity:** The family \( \{E_G\} \) has external products, and if \( V \) is a complex \( G \)-vector bundle over \( X \) and \( W \) is a complex \( H \)-vector bundle over \( Y \), then

\[
u^E_G(V \oplus W) = u^E_G(V) \boxtimes u^E_H(W).
\]

3. **Periodicity:** If \( V \) is a complex \( G \)-representation, then

\[
u^E_G(V) \in \tilde{E}^0(S^V)
\]

is a unit in \( E^* \).

4. **Change of groups:** If \( \alpha: H \to G \) is a map of groups, then

\[
\text{res}|_a u^E_G(V) = u^E_H(V).
\]

If \( G \) is the trivial group we omit it from the notation. We might also omit \( E \) from the notation if it is clear from the context which cohomology theory is meant.

Fix a group \( G \).

**Definition A.2.** We say that \( E_G \) has natural Thom classes for complex vector bundles if it satisfies axioms (1) and (3) of the previous definition and

\[
(2') \quad u_G(V \oplus W) = u_G(V) \otimes u_G(W).
\]
Note that (2) and (4) together imply (2'). In the situation of Definition A.2, let
\[ \alpha := u(C) \in \tilde{E}^{-2}(pt) \]
be the periodicity element, and
\[ \tau_G(V) := u_G(V) \cdot \alpha^{-d} \in \tilde{E}^{2d}(X^V), \]
where \( d = \dim_C V \). Then the \( \tau_G \) satisfy tom Dieck’s axioms for equivariant Thom classes (c.f. [May96, p.335]). On the other hand, if \( E_G \) satisfies the axioms of [May96, p.335] and \( E^2 \) contains a unit, then the axioms of Definition A.2 follow.

**Example A.3.** Complex equivariant K-theory and Borel equivariant \( E \)-theory for even periodic \( E \) satisfy the axioms of Definition A.1. For any \( E \) satisfying the axioms of [May96, p.335],
\[ EP_G := \bigvee_{n \in \mathbb{Z}} \Sigma^{2n}E_G \]
satisfies the axioms of Definition A.2.

Recall (c.f. [Oko82], [May96]) that \( MU_G \) is universal among the \( G \)-equivariant cohomology theories with Thom classes in the sense of tom Dieck.

**Proposition A.4.** The conditions of Definition A.2 are equivalent to the existence of a map of \( G \)-ring spectra
\[ \phi_G: MUP_G \to E_G. \]
It is the unique map of ring spectra taking Thom classes to Thom classes.

**Proof:** If \( \phi_G \) exists, we use it to push-forward the Thom classes of \( MUP_G \) to \( E_G \). In the situation of Definition A.2, the \( \tau_G \)'s give rise to a map \( \psi_G: MU_G \to E_G \), and we set
\[ \phi_G|_{\Sigma^{2n}MU_G} := (\Sigma^{2n}\psi_G) \cdot \alpha^n. \]
Then \( \phi_G \) is a map of \( G \)-ring spectra, and the two constructions are inverse to each other. \( \square \)

Traditionally, topologists like to work with graded rings and define the genus corresponding to \( \psi_G \) as the composite
\[ \mathcal{N}^*_U \to MU^*_G(pt) \to E^*_G(pt) \\
[M] \mapsto \psi_G(M), \]
where the first map is the Pontrjagin-Thom map from the complex equivariant cobordism ring to the graded coefficient ring of \( MU_G \). Alternatively, we can consider the composite of ungraded maps
\[ \mathcal{N}^*_U \to MUP^0_G(pt) \to E^0_G(pt) \]
to define \( \phi_G(M) \). Hence if \( [M] \in \mathcal{N}^{2d}_{U,G} \), then
\[ \psi_G(M) = \phi_G(M) \alpha^{-d}. \]

**Example A.5.** For the Atiyah-Bott-Shapiro K-theory Thom classes for complex vector bundles, \( u_G^{Td} \), the periodicity element \( \alpha \) is the Bott element \( \beta \), and we have \( \phi_G(M) = Td(M) \) and \( \psi_G(M) = Td_G(M) \cdot \beta^{-d} \).
We write $\theta_E$ for the Thom isomorphism

$$\theta_E(x) = x \cdot u_E^C.$$ 

Further, $z: X_+ \to X^V$ denotes the zero section, and

$$e^E_G(V) = z^* (u^E_G(V)) \in E^G_0(X)$$

denotes the Euler class of $V$. Using the fact that Thom classes of trivial bundles are given by units, one extends the definitions of the $u^E_G$ and $\theta_E$ to virtual bundles in the usual way. Recall further that the push-forward along the map $\pi: M \to \text{pt}$, where $M$ is a stably almost complex oriented $G$-manifold, is defined as the composite

$$\pi_!: E^G_0(M) \to \tilde{E}^G_0(M - TM) \to \tilde{E}^G_0(S^0),$$

where $TM$ is the tangent bundle of $M$, the first map is the Thom isomorphism, and the second map is the Pontrjagin-Thom collapse.

The genus $\phi_G$ is then computed as the push-forward of 1:

$$\phi_G(M) = \pi_G!(1) \in \tilde{E}^G_0(S^0).$$

Similarly, the renormalized genus $\psi_G(M)$ is the graded push-forward of 1 which is obtained by using the graded Thom isomorphism associated to $\tau$:

$$\psi_G(M) = \pi_G^\tau!(1) \in \tilde{E}^{-2d}(S^0)$$

(c.f. [Oko82], [CF66]).

A.1.1. The $\hat{A}$-genus and the Todd genus Thom classes. For the remainder of this appendix, we will closely follow the exposition in [Mila]. Consider the inclusion

$$i: \text{Spin}(n) \to \text{Spin}^C(n) \cong (\text{Spin}(n) \times U(1))/\mathbb{Z}/2,$$

and let $V$ be a real, $8k$-dimensional vector bundle over $X$ which has a (chosen) Spin$(8k)$-structure. Let

$$u_{\text{ABS}}(V) \in \text{KO}(X^V)$$

be the Atiyah-Bott-Shapiro Thom class of $V$. Forgetting the Spin-structure and viewing $V$ as a Spin$^C(8k)$-bundle via the forgetful map $i$, we get a $K$-theory Thom class

$$u_{\text{ABS}}^C(V) \in K(X^V),$$

and a close look at the definition of these Thom classes on p.31 of [ABS64] shows:

$$u_{\text{ABS}}^C(V) = u_{\text{ABS}}(V) \otimes \mathbb{C}.$$

We further recall the map

$$\tilde{i}: U(k) \to \text{Spin}^C(2k),$$

which allows us to view any complex vector bundle $V$ as a Spin$^C$ bundle. Atiyah, Bott and Shapiro prove [ABS64, Thm.11.6]

$$u_{\text{Td}}(V) = u_{\text{ABS}}^C(V).$$
These statements are summarized in the following commuting diagram of maps of ring spectra:

\[
\begin{array}{ccc}
\text{MSU} & \xrightarrow{\hat{\phi}} & \text{KO} \\
\downarrow & & \downarrow \\
\text{MU} & \xrightarrow{\phi} & \text{K}
\end{array}
\]

Let \( \mathbb{U}^2(n) \) be the pull-back in the diagram

\[
\begin{array}{ccc}
\mathbb{U}^2(n) & \xrightarrow{j} & \text{Spin}(2n) \\
\downarrow & & \downarrow \\
\mathbb{U}(n) & \xrightarrow{\pi} & \text{SO}(2n)
\end{array}
\]

In particular, \( \mathbb{U}^2(1) \) is the non-trivial double cover of \( \mathbb{U}(1) \) and hence isomorphic to it. Then \( \mathbb{U}^2(n) \) is also the pull-back in the square

\[
\begin{array}{ccc}
\mathbb{U}^2(n) & \xrightarrow{j} & \text{Spin}(2n) \\
\downarrow & & \downarrow \\
\mathbb{U}(n) & \xrightarrow{\pi} & \text{SO}(2n)
\end{array}
\]

(c.f. [Milat]). Vector bundles with a \( \mathbb{U}^2 \)-structure are bundles which have at the same time a complex structure and a Spin structure. The second pull-back square implies that these are exactly those complex bundles whose determinant bundle possess a square root, together with a choice of this square root.

Note that (17) gives rise to two different factorizations of the map

\[
\mathbb{U}^2(n) \longrightarrow \text{SO}(2n)
\]

through \( \text{Spin}^C(2n) \): The composition of \( j \) with the inclusion \( i \) becomes

\[
(j, 1): \mathbb{U}^2(n) \longrightarrow (\text{Spin}(2n) \times \mathbb{U}(1))/(\mathbb{Z}/2).
\]

while composing \( \pi \) with \( \tilde{\iota} \) yields the map

\[
(j, \tilde{\det}): \mathbb{U}^2(n) \longrightarrow (\text{Spin}(2n) \times \mathbb{U}(1))/(\mathbb{Z}/2)
\]

Let \( \mathcal{M} \) be a complex Clifford module. Then the two above maps make \( \mathcal{M} \) into \( \mathbb{U}^2 \) representations, and if we denote the representation obtained from \( (j, 1) \) by \( \rho_{\mathcal{M}} \) then the other one becomes \( \rho_{\mathcal{M}} \otimes_C \tilde{\det} \).

Let \( \mathcal{P} \) be a principal \( \mathbb{U}^2(n) \)-bundle, and let

\[
\mathcal{V} = \mathcal{P} \times_{\mathbb{U}^2(n)} \mathbb{R}^{2n}
\]

be its associated vector bundle. Viewing \( \mathcal{V} \) as \( \text{Spin}^C(2n) \) bundle via the map \( (j, 1) \) (i.e. by first viewing it as a \( \text{Spin}(2n) \) bundle and then forgetting some of the structure), gives the complex Atiyah-Bott-Shapiro Thom class

\[
u_{\mathcal{A}}^C(\mathcal{V}) := \nu^C_{\text{ABS}}(\mathcal{V}) = \chi^C_{\mathcal{V}}(\mathcal{P} \times_{\mathbb{U}^2(n)} \rho_{\mathcal{M}} \otimes_C \tilde{\det}).
\]
Here $\chi_C$ and $\mu_C$ are as defined in [ABS64]. Note that this is really an abuse of notation, since $u^C_{\text{ABS}}(V)$ depends on the Spin$^C$ structure of $V$, not just on $V$. Viewing $V$ as Spin$^C$-bundle via the other map $(j, \tilde{\det})$, we obtain a different class for $u^C_{\text{ABS}}(V)$, namely

$$u_{TD}(V) = u^C_{\text{ABS}}(V) = \chi_C \left( P \times u^C (n) \left( \rho_{\mu_C} \otimes C \tilde{\det} \right) \right) = u_A(V) \otimes C \sqrt{\det(V)}.$$  

(The associated fibers of these two bundles are the same as complex Clifford modules, but are viewed as $U^2(n)$-representations in different ways. Hence the associated bundles are not necessarily isomorphic.) The class $u_{TD}(V)$ is the Todd genus Thom class of $V$ viewed as a complex vector bundle. If $n = 4k$, we have

$$u^C_A(V) = u_{\text{ABS}}(V) \otimes \mathbb{C},$$

where $u_{\text{ABS}}(V)$ is the KO-Thom class of $V$ viewed as Spin$(8k)$-bundle.

### A.2. Atiyah and Segal’s Lefschetz fixed point formula

Let $E$ be an even periodic cohomology theory. Fix a group $G$ and an element $g \in G$. Then $X \mapsto E^*(X^g) := F_G(X)$ is a $G$-equivariant cohomology theory. Let $X$ be a $G$-pace, and let $V$ be a $G$-equivariant complex vector bundle over $X$. Then the $g$-fixed points of $V$ form a vector bundle over $X^g$, namely $V_{g=1}$, the eigenbundle of the eigenvalue 1 of the action of $g$ on $V|_{X^g}$.

Assume that we have chosen a complex orientation of $E$. Then $F_G$ inherits natural Thom classes

$$u^G_E(V) := u^E(V_{g=1}) \in \tilde{E}^0((X^g)^g).$$

Let now $E_G$ be an equivariant version of $E$ with natural Thom classes $u^G_E$ continuing those of the complex orientation of $E$, and assume that $E^0_{(g)}$ is flat over $E^0$. Consider the natural transformation

$$r: E_G(X) \to E_{(g)}(X^g)$$

defined by

$$r := i^* \circ \text{res}|^G_{(g)},$$

where $i: X^g \to X$ is the inclusion of the fixed points.

**Definition A.6.** For a complex $G$-vector bundle $V$ over $X$, we write $V_{g=\zeta}$ for the $\zeta$-eigenbundle of the action of $g$ on $V|_{X^g}$. Write $V_{g \neq 1}$ for the orthogonal complement of $V_1$,

$$V_{g \neq 1} = \bigoplus_{\zeta \neq 1} V_\zeta.$$

**Example A.7.** Let $M$ be a smooth complex $G$-manifold, and let $T$ its tangent bundle. Then $T_{g=1}$ is the tangent bundle $TM^g$ of $M^g$, and $T_{g \neq 1}$ is the normal bundle $N^g$ of $i$.

**Proposition A.8.** For a complex $G$-vector bundle $V$ over $X$, we have

$$r \left( \theta_E(x) \right) = \theta_E(r(x) \cdot e^E_G(V_{g \neq 1})).$$

Its proof relies on the following lemma about relative zero sections:
Lemma A.9 ([Rud98]). Let $V$ and $W$ be (equivariant) vector bundles over $X$, let 
\[ s: W \to V \oplus W \]
be the inclusion, and write $X^s$ for the induced map of Thom spaces 
\[ X^s: X^W \to X^{V \oplus W}. \]
Then $X^s$ pulls back $u_G(V \oplus W)$ to 
\[ e_G(V) \cdot u_G(W) \in \tilde{E}_G^0(X^W). \]

Proof of Proposition A.8: The fixed points inclusion 
\[ i: (X^V)_g \to X^V \]
factors into the composition of the two maps 
\[ i_1 := (X^g)^*: (X^g)^{V_g-1} \to (X^g)^{V_g \neq 1} \]
and 
\[ i_2: (X^g)^{V_g \neq 1} \to X^V, \]
where $i_2$ is the map of Thom spaces obtained from the fixed points inclusion $i_g: X^g \to X$ and the corresponding bundle map 
\[ \tilde{i}: i^*(V) \cong V_{g \neq 1} \oplus V_{g=1} \to V. \]
We have 
\[ r(\theta_E(x)) = r(x \cdot u_E^G(V)) = r(x) \cdot r(u_G^E(V)) = r(x) \cdot i_1^* \circ i_2^* \res^G_{(g)} u_E^G(V) = r(x) \cdot i_1^* \circ i_2^* u_E^G(V) = r(x) \cdot e_E^G(V_{g \neq 1}) \cdot u_E^G(V_{g=1}) = \theta_f(r(x) \cdot e_E^G(V_{g \neq 1})), \]
where the second to last equality is Lemma A.9. Let $\alpha$ be the unique map from $\langle g \rangle$ to the trivial group. Since the action of $\langle g \rangle$ on $V_{g=1}$ is trivial, we have 
\[ u_E^G(V_{g=1}) = \res_{\alpha} u_E^G(V_{g=1}) = u_G^F(V). \]

The correction factor $e_{\langle g \rangle}(V_{g \neq 1})$ is an exponential characteristic class. If we assume the Euler classes of $\langle g \rangle$-representations to be invertible and set 
\[ e_{\langle g \rangle}(-V) := e_{\langle g \rangle}(V)^{-1} \in \tilde{E}_{\langle g \rangle}^0(S^V), \]
it follows that Proposition A.8 holds for virtual bundles, too. In the case $E_G = K_G$, and $u_E^G = u_G^{Td}$, the Riemann-Roch formula in [Dye69] 
\[ \pi_k^E(x) = \pi_k^E(r(x) \cdot e_{\langle g \rangle}^{Td}(-N^g)) \]
becomes exactly the Atiyah-Segal result [AS68, 2.10]

\[(\text{ind}_{\mathcal{G}}^{\mathcal{H}}(x))_g = (\text{ind}_{\mathcal{I}}^{\mathcal{H}} \otimes \text{id}_{R(g)})_g \left( \frac{r(x)}{\lambda_{-1}(N^g)} \right). \]

Here \(\lambda_{-1}(V)\) denotes the alternating sum of the exterior powers of the \((g)\)-representation \(V\). Let now \(u^A\) be the complex Atiyah-Bott-Shapiro Thom class for Spin\((2d)\)-vector bundles. For simplicity, we assume that \(\mathcal{M}\) is a \(U^2\)-manifold. We combine the classical Riemann-Roch formula for the Chern character

\[\pi^\mathcal{A}_!(a) = \pi^\mathcal{H} \left( \frac{\text{ch}(e^\mathcal{A}(-TM))}{e^\mathcal{H}(-TM)} \cdot \text{ch}(a) \right)\]

\[= \int_{\mathcal{M}} \mathcal{A}(TM) \cdot \text{ch}(a)\]

with the Lefschetz formula (19) to obtain

\[\left(M\pi^\mathcal{A}_!(a)\right)(g) = \int_{\mathcal{M}} \frac{e^\mathcal{H}(TM^g)}{\text{ch}(e^\mathcal{A}(TM)|_{M^g})(g)} \cdot \text{ch}([a]|_{M^g})(g)\]

\[= \int_{\mathcal{M}} \frac{e^\mathcal{H}(TM^g) \cdot \text{ch} \left( \sqrt{\det TM}|_{M^g} \right)}{\text{ch}(e^{\mathcal{T}d}(TM)|_{M^g})(g)} \cdot \text{ch}([a]|_{M^g})(g)\]

\[= \varepsilon \left( \prod_{x_i} x_i \right) \left( \prod_{r=0}^{k-1} \prod_{y_j} \frac{e^{\frac{y_j + ni}{2}}}{1 - e^{y_j + 2ni}} \right) \cdot \text{ch}([a]|_{M^g})(g) \right) [M^g].\]

Here the \(x_i\) are the Chern roots of \(TM^g\), for fixed \(r\), the \(y_j\) run over the Chern roots of the \(e^{2\pi i \xi}\) eigenbundle \((TM)|_{M^g}\), and \(\varepsilon = \pm 1\) depends on the way that the action of \(g\) on \(\det TM\) lifts to an action on \(\sqrt{\det TM}\) and on our choice of identification of \(U^2(1)\) with \(U(1)\) above.

**Remark A.10.** The above discussion remains valid if we replace \((g)\) by a topologically cyclic group e.g. \(S^1\) or \(\mathbb{R}/\mathbb{Z}\).

Let now \(P\) be a principal Spin\((2n)\) bundle over \(X\), and \(V = P \times_{\text{Spin}(2n)} \mathbb{R}^{2n}\) its associated vector bundle. We assume that \(V\) has an even \(S^1\)-action, i.e., one that is induced by an \(S^1\)-action on \(P\). We write

\[u^A_{S^1}(V) := u^A_{S^1}C(V)\]

for its equivariant Atiyah-Bott-Shapiro Thom class in complex K-theory. Note that \(V_{g=1}\) and its orthogonal complement are still even dimensional Spin bundles. Hence the above discussion goes through with \(U(n)\) replaced by Spin\((2n)\), and if we let \(\pi^\mathcal{A}\) denote the (equivariant) push-forward in K-theory defined by \(u^A\), we get

\[\pi^\mathcal{A}_{S^1}(x) = \pi^\mathcal{A}_{S^1} \left( r(x) \cdot e^\mathcal{A}_{S^1}(-N) \right),\]

where \(N\) is the normal bundle of the fixed point inclusion. In order to compute the equivariant Euler class of \(N\) (or more generally of \(V_{g\neq1}\)), recall that, since \(S^1\) acts fibre preserving and fixed-point free on \(N\), we have by (5)

\[N \cong \bigoplus_{j \geq 1} V_j q^j,\]

(21)
and $N$ can be equipped (uniquely) with a complex structure making this a complex isomorphism. Then (18) gives

$$e_{S^1}^A(N) = e_{S^1}^{Td}(N) \cdot \left( \sqrt{\det(N)} \otimes S^1 \right)^{-1} \cdot \prod_{j \geq 1} \Lambda_{-q^j} V_j,$$

(22)

where $d = \sum_j j \dim_R V_j$.

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