Is the photon paramagnetic?

H. Pérez Rojas and E. Rodriguez Querts
Instituto de Cibernética, Matematica y Física,
Calle E 309, Vedado, Ciudad Habana, Cuba.

(Dated: April 7, 2009)

A photon exhibits a tiny anomalous magnetic moment \( \mu_e \) due to its interaction with an external constant magnetic field in vacuum through the virtual electron-positron background. It is paramagnetic \( (\mu_e > 0) \) in the whole region of transparency, i.e. below the first threshold energy for pair creation and has a maximum near this threshold. The photon magnetic moment is different for eigenmodes polarized along and perpendicular to the magnetic field. Explicit expressions are given for \( \mu_e \) for the cases of photon energies smaller and closer to the first pair creation threshold. The region beyond the first threshold is briefly discussed.

PACS numbers: 12.20.-m, 12.20.Ds, 13.40.Em, 14.70.Bh.

I. INTRODUCTION

In recent years, due to the development of high power lasers and ion accelerators, the problem of pair creation in strong external electromagnetic fields has attracted the interest of several researchers both in experimental and theoretical aspects (see for instance [1]–[3] and references therein). In this connection, it is interesting to turn our attention to the study of some elementary particle properties arisen from radiative corrections in strong external electromagnetic fields. For instance, in analogy to the anomalous magnetic moment \( \mu_e = \alpha \mu_B / 2 \pi \) (being \( \mu_B = e / 2 m \) the Bohr magneton) for electrons shown by Schwinger [4] as due to their interaction with the virtual photon background through its self-energy, we want to show in the present paper that a similar effect exist for photons. Due to the magnetic properties of the photon self-energy, a photon anomalous magnetic moment \( \mu_e > 0 \) arises, which is paramagnetic in the region of transparency (which is the region of momentum space where the photon self-energy, and in consequence, its frequency, is real), and has a maximum value for \( \omega \) close to the first threshold for pair creation \( \omega = 2 m \). The photon magnetic moment vanishes only when its momentum \( \mathbf{k} \) is parallel to the magnetic field \( \mathbf{B} \).

The photon magnetic properties are due to the dependence of \( \omega \) on \( B \) expressed by the photon dispersion equation dependence on the self-energy tensor \( \Pi_{\mu\nu}(x, x'|A^{ext}) \). For an observable photon the quantity \( \mu_e = -\partial \omega / \partial B \) can be obtained analytically from the expression for \( \Pi_{\mu\nu}(x, x'|A^{ext}) \). In terms of the \( e^\pm \) Green functions \( G(x, x') \) in presence of the external field \( B \) it is \( \Pi_{\mu\nu}(x, x'|A^{ext}) = \alpha \int Tr \gamma_{\mu} G(x, z) \gamma_{\nu} G(z, x') dz^3 \) and in the one-loop approximation it was calculated by Batalin and Shabad in the Schwinger proper time representation [8]. Such expression contains the sum over all Landau numbers and spin quantum numbers of the virtual \( e^\pm \) pairs. From the dispersion equations for the eigenmodes resulting from its diagonalization, it was found in [6, 7] that the photon suffers a strong deviation from the light cone curve at frequencies near the pair creation energy thresholds, indicating that the photon dynamics in the external magnetic field is strongly influenced by the virtual \( e^\pm \) pairs, showing a behavior similar to a massive particle. We emphasize here that the photon propagating in magnetized vacuum behaves in a similar way to a polariton, which in condensed matter are quasiparticles resulting from strong coupling of a photon with a dipole-carrying excitation.

The contribution from the electron anomalous magnetic moment to the photon anomalous magnetic moment would appear in the two loop approximation, of order \( \alpha^2 \) with regard to the radiation field, and not in the present one-loop term, proportional to \( \alpha \).

We start in the first Section by recalling some basic features of the quantum relativistic electron dynamics in the magnetic field \( B \) and on the photon eigenmodes kinematics and dynamics. In the second we obtain the general expression for the photon magnetic moment in terms of the self-energy eigenvalues. In the third Section, we obtain the magnetic moment in the small frequency limit, which leads actually to expressions valid up to frequencies close to the threshold limit \( 2m \). The fourth Section is devoted to obtain the approximate expressions for the photon magnetic moment at frequencies near and below the pair creation thresholds, where it has a maximum value. In the last Section a resume is made of the results obtained in the paper, which correspond to the so-called region of transparency, and mention is made about some features characterizing the region of absorption, beyond the first threshold for pair creation.

II. SOME BASIC RESULTS

If the constant uniform magnetic field \( B \) is along the \( x_3 \) axis, it breaks the space symmetry so that for electrons and positrons \( (e^\pm) \) physical quantities are invariant only under rotations around \( x_3 \) or displacements along it. Angular momentum and spin components \( J_3, L_3, s_3 \) as well as linear momentum \( p_3 \) are conserved. By using units \( h = c = 1 \), the energy eigenvalues for \( e^\pm \) are...
\[ E_{n,p_3} = \sqrt{p_3^2 + m^2 + eB(2n + 1 + s_3)} \] where \( s_3 = \pm 1 \) are spin eigenvalues along \( x_3 \) and \( n = 0, 1, 2, \ldots \) are the Landau quantum numbers [8]. In other words, in presence of \( B \), the transverse squared energy \( E^2_{n,p_3} - p_3^2 \) is quantized by integer multiples of \( eB \). For the ground state \( n = 0, s = -1 \), the integer is zero. Quantum states degeneracy with regard spin is expressed by a term \( a_n = 2 - \delta_{0\mu} \), whereas degeneracy with regard to orbit’ center coordinates leads to a factor \( eB \). The quantity \( 1/eB \) characterizes the spread of the \( e^\pm \) spinor wave-functions in the plane orthogonal to \( B \). Due to the explicit symmetry breaking, the four momentum operator acting on the vacuum state does not have a vanishing four-vector eigenvalue, \( P_\mu(0, B) \neq 0 \). The \( e^\pm \) quantum vacuum energy density is given by \( \Omega_{EH} = -eB \sum_{n=0}^\infty a_n \int dp_3 E_{n,p_3} \). After removing divergences it gives the well-known Euler-Heisenberg expression \( \Omega_{EH} = \frac{aB^2}{8\pi^3} f_0^\infty e^{-Bz/|B|} \left[ \frac{\sinh k}{k} - \frac{1}{2} \right] \frac{dk}{k} \) which is an even function of \( B \) and \( B_c \), where \( B_c = m^2/e \simeq 4.4 \times 10^{13} \text{G} \) is the Schwinger critical field. The magnetized vacuum is paramagnetic \( M_V = -\partial \Omega_{EH}/\partial B > 0 \) and is an odd function of \( B \) [9]. For \( B << B_c \) it is \( M_V = \frac{2e^2}{4\pi^2} B^2 \), where \( \alpha \) is the fine structure constant.

The diagonalization of the photon self-energy tensor leads to the equations [10]

\[ \Pi_{\mu\nu}a^{(i)}_\mu = \kappa_i a^{(i)}_\mu, \tag{1} \]

having three non vanishing eigenvalues and three eigenvectors for \( i = 1, 2, 3 \), corresponding to three photon propagation modes. One additional eigenvector is the photon four momentum vector \( k_\mu \) whose eigenvalue is \( \kappa_4 = 0 \). The first three eigenvectors

\[
\begin{align*}
a_\mu^1 &= k^2 F^2_{\mu\lambda} k^\lambda - k_\mu (kF^2 k), \\
a_\mu^2 &= F^*_{\mu\lambda} k^\lambda, \\
a_\mu^3 &= F_{\mu\lambda} k^\lambda,
\end{align*}
\]

satisfy the four dimensional transversality condition \( a^{(1,2,3)}_{\mu} k_\mu = 0 \). Here \( k_\mu \) is the photon four momentum, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( F^{\mu*\nu} = \gamma^{\mu\nu} F_{\mu\nu} \) are the external electromagnetic field tensor and its dual pseudo-tensor, respectively. The vectors \( k_\perp \) and \( k_\parallel \) are the components of \( k \) across and along \( B \). In what follows \( k^2 = z_1 + z_2 \) where \( z_{1,2} \) are invariant variables defined by

\[
\begin{align*}
z_1 &= \frac{kF^2 k}{2\delta} = k_\parallel - \omega^2, \\
z_2 &= -\frac{kF^2 k}{2\delta} = k^2_\perp.
\end{align*}
\]

In reference frames which are at rest or moving parallel to \( B \) we define \( n_\perp = k_\perp/k_\parallel \) and \( n_\parallel = k_\parallel/k_\parallel \) as the transverse and parallel unit vectors respectively.

By considering \( a^{(i)}_\mu(x) \) as the electromagnetic four vector describing these eigenmodes, its electric and magnetic fields \( e^{(i)} = -\frac{\partial a^{(i)}}{\partial x^i} - \frac{1}{\omega} a^{(i)} \) and \( h^{(i)} = \nabla \times a^{(i)} \) are obtained in [10]. It is easy to see that the mode \( i = 3 \) is a transverse plane polarized wave whose electric unit vector is \( e^{(3)} = (n_\perp \times n_\parallel) \) orthogonal to the plane \((B, k)\). For \( k \perp B \), \( a^{(1)}_\mu \) is longitudinal, polarized along \( e^{(1)} = n_\perp \) and it is a non physical mode, whereas \( a^{(2)}_\mu \) is transverse, since \( e^{(2)} = n_\parallel \). Thus, in that case modes \( a^{(2,3)}_\mu \) are superposition of waves of opposite helicity. For \( k \parallel B \), the mode \( a^{(2)}_\mu \) becomes pure electric and longitudinal with \( e^{(2)} = n_\parallel \) (and also non physical), whereas \( a^{(1)}_\mu \) is transverse \( e^{(1)} = n_\perp \), as it is \( e^{(3)} \) [6, 7, 10]. In that case \( \kappa_1 = \kappa_3 \), and the circular polarization unit vectors \( (e^{(1)} \pm ie^{(3)})/\sqrt{2} \) are common eigenvectors of \( \Pi_{ij} \) and of the rotation generator matrix \( A^{(3)} \).

We want to remark three points: first, from the diagrammatic point of view, the non-zero photon magnetic moment is due to the non-vanishing three-leg diagram resulting from differentiating the field dependent photon self-energy with regard to \( B \). Obviously, such diagram vanishes in the limit \( B = 0 \), as it is demanded by Furry’s theorem. As a second remark, as \( \mu_\parallel \) depends through \( \Pi_{\mu\nu}(k, k' | A^{ext}) \) on the sums over infinite pairs of Landau quantum numbers and spins of the \( e^\pm \) pairs, it cannot depend on any specific eigenvalue of angular momentum, spin or orbit’s center coordinates. Our third remark is that due to the degeneracy of the orbit’s centers of the pair, a factor \( eB \) is contained in \( \Pi_{\mu\nu}(k, k' | A^{ext}) \), which is also even in \( eB \) (as it is \( \Omega_{EH} \)), and \( \mu_\parallel \) is an odd function of \( B \).

The dispersion equations, obtained as the zeros of the photon inverse Green function \( D^{-1}_{\mu\nu} = 0 \), after diagonalizing the polarization operator \( \Pi_{\mu\nu}(z_1, z_2, B) \), are

\[
k^2 = \kappa_i(z_2, z_1, B) \quad i = 1, 2, 3. \tag{4}
\]

After solving the dispersion equations for \( z_1 \) in terms of \( z_2 \) we get

\[
\omega^{(i)2} = |k|^2 + \frac{m^{2(i)}}{\omega^2} \tag{5}
\]

Let us remark at this point that the refraction index \( n^{(i)} \) can be defined as

\[
n^{(i)} = \frac{|k|}{\omega} = \left(1 + \frac{m^{2(i)}}{\omega^2}\right)^{1/2} \tag{6}
\]

being different for each eigenmode, leading to the phenomenon of birefringence. The propagation of light in magnetized vacuum is thus similar to that in an anisotropic medium. Gauge invariance implies that \( \kappa^{(i)}(0, 0, B) = 0 \) and \( \mathfrak{M}^{(i)}(0, 0, B) = 0 \). Thus, for parallel propagation \( n^{(i)} = 1 \). By differentiating \( \omega^{(i)} \) with regard to \( B \) we get the relation \( \mu^{(i)}_{\parallel} = -\frac{1}{2\delta} \frac{\partial \mathfrak{M}^{(i)}}{\partial B} \), and in consequence \( \mathfrak{M}^{(i)} = -2 \int_0^B \omega^{(i)}(B')dB' + f^{(z_2)} \). The function \( f^{(z_2)} \) is zero if the series expansion of \( \kappa^{(i)} \) in powers of \( B \) is taken as linear in \( z_1, z_2 \) (see below and Appendix). From [5], we have the approximate dispersion equation

\[
\omega = |k| - \int_0^B \mu^{(i)}(z_2, B', |k|)dB'. \tag{7}
\]

For nonparallel propagation the fact that \( n^{(i)} < 1 \) is a consequence of photon paramagnetism.
Let us point out more explicitly the observable consequences of the paramagnetic properties of the photon. As mentioned before, quantum vacuum has a magnetization $\mathcal{M}_V > 0$ at any point of space $P$. If there is a nonzero photon density $N_i^\lambda$ around $P$, an additional magnetization $\mathcal{M}_\lambda$, which in the monochromatic case it is simply $\mathcal{M}_\lambda = \sum_i N_i^\lambda \mu_i^\lambda$, is produced on $P$. This magnetization is essentially determined by the photon momentum component $k_\perp / \sqrt{\epsilon_B}$ transferred to the electron–positron virtual quanta, and the photon gets its magnetic properties from its interaction with the virtual pairs. The photon magnetization $\mathcal{M}_\lambda$ contributes to an (usually small) increase of the microscopic field $B$ to $B' = B + 4\pi \mathcal{M}_\lambda$.

A possible way to measure the photon magnetic moment would be provided by the measurement of the refraction indexes $n^{(i)}$. Through them, we obtain $\mathcal{M}^{(i)}$ and by knowing $B$ and $k$, the values of $\mu^{(i)}$ would be obtained. In the astrophysical scenario, the dependence of $n^{(i)}$ with regard $\mu^{(i)}$ play an important role in the magnetic lensing effect. See for instance [11].

The renormalized eigenvalues of the polarization operator, calculated in one-loop approximation [7], are given by the expressions

$$\phi_i = \frac{1}{m^2} \left[ \frac{\rho_i e^\zeta}{\sinh t} \left( \frac{t}{b} - \zeta \right) + \frac{k^2 \eta^2}{2t} \right],$$

$$\varphi_i = \frac{e^\zeta}{\sinh t} \left( \frac{\partial \rho_i}{\partial z_1} - \frac{\rho_i e^\zeta}{b^2 \gamma^2} \right) + \frac{\eta^2}{2t},$$

and taking into mind that $\frac{\partial \varphi_i}{\partial B} = -2\omega_0 \frac{\varphi_i}{\pi^2}$ in (9), we obtain an expression for the photon anomalous magnetic moment

$$\mu^i = -\frac{\partial \omega}{\partial B} = \frac{m^2}{2} \frac{2\alpha}{\pi} \int_0^\infty dt \int_{-1}^1 d\eta e^{-\frac{t}{2}} \zeta_i,$$

By starting from the exact expressions (9), (10) in Figs. 1, 2 it is depicted the result of a numerical calculation for $\mu_\gamma$ for the second and third modes, in the interval $0 < B < B_c$, for frequencies such that $0 < -z_1, z_2 < m^2$. It confirms that the paramagnetic behavior is maintained throughout such interval for the modes 2, 3.

$$\mathcal{F} 1: \text{Photon magnetic moment for modes 2, 3 drawn for } -z_1, z_2 \leq m^2 \text{ in a logarithmic scale.}$$

It is easy to see that for propagation along $B$, the vacuum behaves as in the limit $B = 0$ for all eigenmodes. That is why we are mainly interested in studying the perpendicular photon propagation case $k_\parallel = 0$, for which the first mode is non physical, as was pointed out before. As different from modes 2, 3 the dispersion equation for mode 1 (which is longitudinal and unphysical for propagation orthogonal to $B$) is the light cone in the low frequency limit $\omega_0^2 - |k|^2 = 0$ (its dependence on $B$ starts to appear in the term quadratic in $z_1, z_2$; see Appendix).
the third modes, respectively, are

\[ z_1 = -2z_2 \frac{1 - 2\pi G_{22}}{1 - 2\pi G_{21}}, \]
\[ z_2 = -2z_2 \frac{1 - 2\pi G_{32}}{1 - 2\pi G_{31}}. \]

From (12), (13), after a straightforward calculation we get \( \mu_{\gamma}^2 = -\frac{\partial^2}{\partial \nu^2}. \)

\[ \mu_{\gamma}^2 = \frac{\alpha z_2}{\pi \omega B_c} \left( 1 - 2\pi G_{21} \right) \tilde{G}_{22} - \left( 1 - 2\pi G_{22} \right) \tilde{G}_{21}, \]
\[ \mu_{\gamma}^3 = \frac{\alpha z_2}{\pi \omega B_c} \left( 1 - 2\pi G_{31} \right) \tilde{G}_{32} - \left( 1 - 2\pi G_{32} \right) \tilde{G}_{31}, \]
\[ \frac{dG_{ij}}{db} = \frac{1}{b^2} \int_{0}^{\infty} dt e^{-\frac{r}{\sqrt{G}}}. \]

It is easy to see that \( \mu_{\gamma}^3 > 0 \) due to the smallness of \( \alpha < 1 \) and also because \( \tilde{G}_{21} > \tilde{G}_{32} > \tilde{G}_{22} > 0, \tilde{G}_{32} = \tilde{G}_{31} \).

As an example, we take the first two terms in the \( \kappa_i^{(0)} \) series expansion. We can write \( d^{(2)} = \frac{7\alpha}{45\pi}, \)
\[ d^{(3)} = \frac{4\alpha}{45\pi}, c^{(2)} = \frac{26}{49}, c^{(3)} = -12/7, \]
and thus the expressions for \( d^{(2,3)} \) agree with those obtained earlier in refs. 12, 13 in calculating the indexes of refraction of magnetized vacuum orthogonal and parallel to \( B \), by properly differentiating the Euler-Heisenberg Lagrangian. We have then

\[ \text{Re}^{2(i)}(z_2, B) = -d^{(i)} z_2 (b^2 + c^{(i)} b^4), \]

and the magnetic moments

\[ \mu_{\gamma}^2 = \frac{14\alpha z_2}{45\pi B_c |k|} \left( b - \frac{52b^3}{49} \right) > 0 \]

and

\[ \mu_{\gamma}^3 = \frac{8\alpha z_2}{45\pi B_c |k|} \left( b - \frac{24b^3}{7} \right) > 0. \]

In the present limit \( \mu_{\gamma}^{2,3} \) are small quantities which grow with the frequency and magnetic field intensity. For instance, for radiation of frequency \( 10^{15}\text{Hz} \) and magnetic field intensities of order \( 5 \times 10^4 \text{G} \) (of order similar to those used in some of the PVLAS experiments 17), \( \mu_{\gamma}^{2,3} \sim 10^{-16} \mu \).

This quantity is very small and a photon density of order \( N_{\gamma} \sim 10^{24} \text{cm}^{-3} \), would be required to have the photon magnetization \( \mathcal{M}_{\gamma}^{3,3} = N_{\gamma} \mu_{\gamma}^{2,3} \sim 10^{-15} \mu \), larger than the vacuum magnetization \( \mathcal{M}_V \). But in magnetic fields and densities of the same order but using X-ray lasers 10, \( \mu_{\gamma}^{2,3} \sim 10^{-13} \mu \), and the magnetization might be of order \( 10^{-7} \mu \).

Larger values may occur for \( \gamma \) rays in strongly magnetized stars. For instance, for frequencies of order \( 10^{20}\text{Hz} \) and magnetic fields \( \sim 10^{12} \text{G} \), we have \( \mu_{\gamma}^{2,3} \sim 10^{-2} \mu \).
B. High frequencies and strong fields case

We will be interested now in the case \( m^2 \lesssim -z_1 \leq 4m^2, B \lesssim B_c \). In such case, the energy gap between successive Landau energy levels for electrons and positrons is of order close to the electron rest energy. The photon self-energy diverges for values of \(-z_1 = k_{\perp}^2\), where

\[
k_{\perp}^2 = m^2[(1 + nb)/2 + (1 + nb')/2)]^2 \tag{18}
\]

are the thresholds for pair creation \( \mathfrak{B} \) at the Landau quantum numbers \( n, n' = 0, 1, 2 \). Near and below the thresholds, the photon behaves like a massive vector particle. We are interested in the first threshold \( n = n' = 0 \) and photons of energy \( \omega \) such that \(-z_1 \leq 4m^2\). The eigenvalues of the propagation modes can be written approximately near the thresholds \( n, n' \) \( \mathfrak{B} \) as

\[
\kappa_{(i)} \approx -2\pi\phi_{n,n'}/|\Lambda| \tag{19}
\]

with

\[
|\Lambda| = ((k_{\perp}^2 - k_{\perp}^{2,0})(k_{\perp}^2 - \omega^2 + k_{\perp}^2))^{1/2} \tag{20}
\]

where

\[
k_{\perp}^{2,0} = m^2[(1 + nb)/2 - (1 + nb')/2)]^2, \tag{21}
\]

is the squared threshold energy for excitation between Landau levels \( n, n' \) for an observable electron or positron. This term does not lead to any singular behavior in the present quantum vacuum case. The functions \( \phi_{n,n'}^{(i)} \) are expressed in \( \mathfrak{B} \) in terms of Laguerre functions of the variable \( z_2/2eB \). The expression \( \mathfrak{B} \), being an approximation, is not even in \( B \) for a given pair \( n, n' \). An even expression would be obtained after summing over all \( n, n' \) values.

In the vicinity of the first threshold \( n = n' = 0 \) and by considering \( k_{\perp} \neq 0 \) and \( k_{\parallel} \neq 0 \), according to \( \mathfrak{B} \), the physical eigenwaves are described by the second and third modes, but only the second mode has a singular behavior near the threshold and the function \( \phi_{0,0}^{(2)} \) has the structure

\[
\phi_{0,0}^{(2)} \approx -\frac{2aeBm^2}{\pi}\exp \left( -\frac{z_2}{2eB} \right), \tag{22}
\]

In this case \( k_{\perp}^{2,0} = 0 \) and \( k_{\perp}^2 = 4m^2 \) is the threshold energy. The solutions of the dispersion equation \( k^2 = \kappa_{(2)}^{(0)} \) were obtained in \( \mathfrak{B} \). Below the thresholds it gives a real solution plus two complex conjugate ones, located on the second sheet of the complex \( \omega \) plane.

It is important to remark here the role of the function \( e^{-z_2/2eB} \) which is present in all \( \phi_{n,n'}^{(i)} \). It makes them significant for \( z_2 \lesssim 2eB \) and makes them and \( \mathfrak{B} \) vanishing small for \( z_2 \gg 2eB \). By substituting \( \mathfrak{B} \) in \( \mathfrak{B} \), and by differentiating the dispersion equation with regard to \( B \) one gets for the second mode the photon magnetic moment

\[
\mu_{\gamma}^{(2)} = \frac{PQ}{\omega B_c (P/3 + bQ)} (1 + \frac{z_2}{2eB}) > 0, \tag{23}
\]

\[
P = (4m^2 + z_1), \quad Q = \alpha m^3 \exp\left( -\frac{z_2}{2eB} \right).
\]

This solution is valid in the vicinity of the first threshold and agrees with the previously mentioned numerical result in its paramagnetic property, which is valid throughout the whole region of transparency (below the first pair creation threshold).

The expression \( \mathfrak{B} \) has a maximum near the threshold, \( z_2 \simeq k_{\perp}^2 \). If we consider \( \omega \) close to \( 2m \), the function \( \mu_{\gamma}^{(2)} = f(X) \), where \( X = \sqrt{4m^2 + z_1} \) has a maximum for \( X = (2\pi\phi_{00}^{(2)}/m)^{1/3} \), which is very close to the threshold. By calling

\[
m_{\gamma} = \omega(k_{\perp}^2) = \sqrt{4m^2 - m^2[2ab\exp(-\frac{2}{b})]^{2/3}} \tag{24}
\]

that maximum is \( \mu_{\gamma}^{(2)} = \frac{e(1+2b)}{3m^2} \sqrt{2ab\exp\left( -\frac{2}{b} \right)} \). Numerically for \( b \sim 1 \), \( \mu_{\gamma}^{(2)} \approx 3\mu'(\frac{1}{\alpha})^{1/3} \approx 12.85\mu' \).

In Fig.1 it is depicted the dependence of \( \mu_{\gamma}^{(2)} \) with regard to \( k_{\perp}^2 \). It shows that \( \mu_{\gamma}^{(2)} > 0 \) and has an absolute maximum for \( B \lesssim B_c \).

In \( \mathfrak{B} \) we introduced \( m_{\gamma} \), which has meaning near the thresholds, and we name the photon dynamical mass in presence of a strong magnetic field. It is a consequence of the enhanced coexistence of the massless photon with the massive virtual pair (as a polariton). For energies near the thresholds, it has a behavior similar to a neutral massive vector particle moving parallel to \( B \). However, it does not violate gauge invariance since the condition \( \Pi_{\mu
u}(0,0,B) = 0 \) is preserved. The idea of a photon mass has been introduced previously, for instance in \( \mathfrak{B} \), in a regime different from ours, in which \( k_{\parallel} \gg 4m^2 \).

Beyond the first threshold \( n = n' = 0 \), for frequencies such that \(-z_1 > 4m^2 \) starts the so-called region of absorption, i.e., \( 4m^2 + z_1 < 0, \kappa_2 \) becomes complex, its imaginary part leading to complex frequencies \( \omega + i\Gamma \) after solving the dispersion equations (the thresholds for absorption would be slightly lower if the photon decays in a pair in a bound state, forming positronium. The expression \( \mathfrak{B} \) has been introduced previously, for instance in \( \mathfrak{B} \), in a regime different from ours, in which \( k_{\parallel} \gg 4m^2 \).

In \( \mathfrak{B} \) we have seen that a photon moving in magnetized vacuum shows a paramagnetic behavior, its magnetic moments, which are different for each polarization mode, being increasing functions of \( B \). This behavior is shown in the low frequency, low magnetic field limit \(-z_1 \ll m^2, B \ll B_c \), as well as in the large frequency large magnetic field limit, \( m^2 \ll -z_1 \ll 4m^2, B \approx B_c \). Both limits correspond to the region of transparency. The measurement
of $\mu_{q,3}^2$ in the low frequency limit could be accessible to
next future laboratory conditions. The large frequency
limit is interesting in connection to astroparticle physics
(strongly magnetized stars).

The absorptive region is the continuation of the large
frequency, large magnetic field limit to the region $-z_1 \geq
4m^2$ and fields $B \gtrsim B_c$. That region is also interesting
in astrophysics, and in cosmology (stars with fields
$B \gtrsim B_c$ and early universe). A study on the photon
magnetic moment in this new scenario is in progress
by the present authors. It is interesting, however, to
remark some of the new features. For instance, al-
though larger values are expected for the photon mag-
netic moment than in the region of transparency, a neg-
ative peak is found for the first threshold of the third
mode. This has no absolute meaning since in that region
photons coexist with electron-positron pairs (the pho-
ton has some nonzero probability of decaying in pairs
or either in positronium) and the magnetic moment of
the created electron-positron pairs (or positronium) must
be added to that of photons. Also, the magnetized vacuum
background is no longer satisfactorily described
by the Euler-Heisenberg expression $\Omega_{EH}$, and radiative
corrections containing the photon self-energy must be
taken into account. These corrections can be written as
$\Omega_{EH} = \sum_j \int_0^\infty \int d^3kd\omega \kappa(i)/\kappa(q) -$ (or either, a
similar expression in terms of the electron Green function
and self energy) [21] (appropriate counterterms must be
subtracted to make the integrals convergent). For some
ranges of $k$ and $\omega$, $\kappa(i)$ become complex, and this suggests
that quantum vacuum modes at these frequencies might
become unstable and decay for fields $B \gtrsim B_c$. At present
QED in unstable vacuum (see for instance [22], [23] and
its references) has growing interest, and the possibility of
observing vacuum decay in critical electric fields in ter-
restrial laboratories (see [24]) is becoming realistic thanks
to the development of high power pulse lasers technology.

V. ACKNOWLEDGMENTS

The authors thank A.E. Shabad, for several comments
and important remarks and to S. Villalba Chavez for sev-
eral discussions. They thank also OEA-ICTP for support
under Net-35. H.P.R. thanks G. Altarelli and J. Ellis for
comments, and to CERN for hospitality at early stages
of this research.

VI. APPENDIX

We start from the renormalized eigenvalues of the
polarization operator in presence of a constant ho-
monogeneous magnetic field in the one-loop approxima-
tion, given by Shabad [8]. We are interested in a wide range
of frequencies characterized by the condition $z_1, z_2 << m^2$.
We can express $\kappa_i$ as

$$\kappa_i = \sum_{l=0}^{\infty} \kappa_i^{(l)}, \quad (25)$$

$$\kappa_i^{(0)} = \frac{2\alpha}{\pi} \int_0^\infty dt \int_{-1}^{1} d\eta e^{-\frac{t}{\eta}} \left( \frac{\rho_i}{\sinh t} + (z_1 + z_2)^2 \eta^2 \right),$$

$$\kappa_i^{(l)} = \frac{2\alpha}{\pi} \frac{1}{l!} \int_0^\infty dt \int_{-1}^{1} d\eta e^{-\frac{t}{\eta}} \frac{\rho_i}{\sinh t} \eta^l,$$

and consider the first few terms of $\kappa_i^{(l)}$ in the series expansion [25]. Here we explicitly compute $\kappa_i^{(0)}$ and $\kappa_i^{(1)}$.

A. The linear in $z_1, z_2$ term $\kappa_i^{(0)}$

After integrating on $\eta$ we get for the term $\kappa_i^{(0)}$ ($i = 1, 2, 3$), which is linear in $z_1, z_2$

$$\kappa_i^{(0)} = \frac{2\alpha}{\pi} \int_0^\infty dt e^{\frac{t}{\eta}} \sum_{j=1,2} g_{ij} z_j, \quad (26)$$

$$g_{11} = -\frac{1}{12} t^2 - \frac{\coth t}{4t} + \frac{\coth^2 t}{4t} > 0, \quad (27)$$

$$g_{21} = \frac{1}{6t} \frac{\cosh t}{6 \sinh t} < 0, \quad (28)$$

$$g_{32} = \frac{1}{2} + \frac{2t}{6 \sinh^2 t} - \frac{\cosh t}{2 \sinh^2 t} > 0,$$

and

$$g_{12} = g_{22} = g_{31} = g_{11}. \quad (29)$$

For fields $eB << m^2$ (actually it is enough that
$\frac{e^2 B}{m^2} \lesssim 10^{-1}$) the functions inside the integrals in $\kappa_i^{(0)}$
are significantly different from zero only for $t << 1$ and we
can expand the following expressions around $t = 0$ and
retain the first four terms:

$$g_{11} \approx \frac{t}{45} + \frac{t^3}{315} + \frac{2t^5}{4725} - \frac{t^7}{1871},$$

$$g_{21} \approx \frac{1}{6} \left( \frac{t}{3} + \frac{t^3}{45} + \frac{2t^5}{945} + \frac{t^7}{4725} \right),$$

$$g_{32} \approx \frac{t}{15} + \frac{t^3}{63} + \frac{2t^5}{675} - \frac{t^7}{2079}.$$  

Finally, by using (29) we easily get approximate ex-
pressions for the first term $\kappa_i^{(0)}$ of the series expansion
of the eigenvalues [25]. We have

$$\kappa_i^{(0)} = \frac{2\alpha}{\pi} \sum_{j=1,2} G_{ij} z_j, \quad (30)$$
are even functions of $b$ (from (28) $G_{11} = G_{12} = G_{22} = G_{31}$)

$$G_{11} = \frac{1}{45} b^2 - \frac{2}{15} b^4 + \frac{16}{315} b^6 - \frac{80}{297} b^8,$$

$$G_{21} = \frac{1}{6} \left[ - \frac{1}{3} b^2 + \frac{2}{15} t^4 - \frac{16}{63} b^6 + \frac{16}{15} b^8 \right],$$

$$G_{32} = \frac{1}{15} b^2 - \frac{2}{21} b^4 + \frac{16}{45} b^6 - \frac{80}{33} b^8,$$

B. The term $\kappa_i^{(1)}$, quadratic in $z_1, z_2$

The quadratic terms in $z_1, z_2$ appear in the term $\kappa_i^{(1)}$, which can be calculated by following the same procedure we used previously to get $\kappa_i^{(0)}$. We obtain, after integrating in $\eta$,

$$\kappa_i^{(1)} = \frac{\alpha}{\pi c B} \int_0^\infty \frac{d\tau}{\tau^{3/2}} \sum_{j,n=1,2} f_{ij} z_j z_n,$$

where

$$f_{ij} \approx \frac{1}{15} t^2 + \frac{23}{70} t^4 + \frac{19}{9450} t^6 + \frac{1}{212837625} t^8,$$

$$f_{211} \approx \frac{1}{15} t^2 + \frac{2}{70} \frac{t^4}{105} + \frac{19}{51975} t^6 + \frac{7213}{141891750} t^8,$$

$$f_{311} \approx \frac{1}{2} \left[ 2 \frac{t^2}{15} + \frac{13}{1575} t^4 - \frac{6237}{70945875} t^6 + \frac{10664398}{425675250} t^8 \right],$$

$$f_{312} \approx \frac{1}{2} \left[ \frac{2}{15} t^2 + 13 t^4 + \frac{253}{28350} t^6 - \frac{1}{10664398} t^8 \right],$$

$$f_{322} \approx \frac{1}{2} \left[ \frac{2}{15} t^2 + \frac{4}{25} t^4 + \frac{4}{13828} t^6 \right],$$

We finally integrate once more in $t$ and express $\kappa_i^{(1)}$ as

$$\kappa_i^{(1)} = \frac{\alpha}{\pi m^2} \sum_{j,n=1,2} F_{ij} z_j z_n,$$

where

$$F_{ij} = \frac{1}{b} \int_0^\infty \frac{d\tau}{\tau^{3/2}} f_{ij} z_j z_n.$$
[1] C. Muller, A.B. Voitkov, N. Grun, Phys. Rev. Lett. 91, 223601 (2003).
[2] A.I. Milstein, C. Muller, K.Z. Hatsagortsyan, U.D. Jentschura, C.H. Keitel, Phys. Rev. A 73, 062106 (2006).
[3] Carsten Muller, Carlus Deneke and Christoph H. Keitel, Phys. Rev. Lett. 101, 060402 (2008).
[4] J. Schwinger, Phys. Rev. 82, 664 (1951).
[5] I. A. Batalin and A. E. Shabad, JETP 33, 483 (1971).
[6] A. E. Shabad, Lettere al Nuovo Cimento 2, 457 (1972).
[7] A. E. Shabad, Ann. Phys. 90, 166 (1975).
[8] M.H. Johnson, B.A. Lippmann, Phys. Rev. 76, 828 (1949)
[9] H. Pérez Rojas and E. Rodríguez Querts, International Jour. of Mod. Phys. A 21, 3761, 2006.
[10] A. E. Shabad, Astrophys. Space Sci. 102, 327 (1984).
[11] A. Dupays, C. Robilliard, C. Rizzo, G. F. Bignami, Phys. Rev. Lett. 94, 161101 (2005)
[12] S. Adler, Ann. Phys. 67, 599 (1971).
[13] W. Dittrich, H. Gies Phys. Rev. D58, 025004 (1998).
[14] H. Pérez Rojas and A. E. Shabad, Ann. Phys. 138, 1 (1982).
[15] A. V. Kuznetsov, N. V. Mikheev, and M. V. Osipov Mod. Phys. Lett. A 17, 231 (2002).
[16] G A. Mourou, T. Tajima and S. V. Bulanov, Rev. Mod. Phys. 78, 309 (2006).
[17] E. Zavattini et al., Nucl. Phys. Proc. Suppl. 164, 264 (2007).
[18] H. Pérez Rojas and A. E. Shabad, Ann. Phys. 121, 432 (1979).
[19] A. E. Shabad, JETP 98, 186 (2004).
[20] V. V. Usov, A. E. Shabad Sov. Phys. -JETP Letters 42, 19 (1985).
[21] E.S. Fradkin, in *Quantum Field Theory and Hydrodynamics*, Proc. of the Lebedev Phys. Inst. No. 29, Consultants Bureau, New York, 1967
[22] A.A. Grib, S.G. Mamayev, V.M. Mostepanenko *Vacuum quantum effects in strong fields*, Friedmann Lab. Publ., St.Petersburg, (1994)
[23] E.S. Fradkin, D.M. Gitmann, S.M. Shvartsman *Quantum Electrodynamics with unstable vacuum*, Springer, Berlin (1991)
[24] Lance Labun and Johann Rafelski, Phys.Rev. D 79: 057901 (2009).