Multishot Codes for Network Coding: Bounds and a Multilevel Construction

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Abstract—The subspace channel was introduced by Koetter
and Kschischang as an adequate model for the communication
channel from the source node to a sink node of a multicast
network that performs random linear network coding. So far,
attention has been given to one-shot subspace codes, that is,
codes that use the subspace channel only once. In contrast, this
paper explores the idea of using the subspace channel more than
once and investigates the so called multishot subspace codes.
We present definitions for the problem, a motivating example,
lower and upper bounds for the size of codes, and a multilevel
construction of codes based on block-coded modulation.

I. INTRODUCTION

Random linear network coding, first introduced in [1], is
an attractive proposal for networks with unknown or changing
topology, in particular for multicast communication, in which
there is only one source but many sink nodes. In this scheme,
the network operates with packets, each consisting of $m$
symbols from a finite field $\mathbb{F}_q$. A packet, then, can be interpreted
as a vector in the vector space $\mathbb{F}_q^m$. Each node in the network
transmits random linear combinations of the packets it has
received. As noted in [2], even if the random coefficients of
the linear combinations are not known, it is still possible to
carry out a multicast communication. The key idea is that the
vector subspace spanned by the packets sent by the source
node is preserved over the network and therefore information
can be encoded into subspaces.

Koetter and Kschischang defined in [2] the subspace channel,
a discrete memoryless channel with input and output alphabets
given by the projective space $\mathcal{P}(\mathbb{F}_q^m)$, which is
the collection of all possible vector subspaces of the vector
space $\mathbb{F}_q^m$. The source node selects and transmits an input
subspace from the projective space and, in the absence of
errors, the sink nodes receive that same subspace. To deal
with the problem of packet errors and erasures that may
happen during the communication, one can limit the choice of
input subspaces to a particular subcollection of the projective
space, i.e., a subspace code. Such choice is driven by a metric
known as subspace distance, which is adequate to the subspace
channel, according to [2].

We call the codes just described one-shot subspace codes,
since they use the subspace channel only once. Many bounds
and fundamental results for one-shot subspace coding, as well
as constructions of codes, have been presented in [2], [3], [4].

In contrast, codes that use the subspace channel many times
are called multishot subspace codes, in which the permissible
sequences of subspaces to be transmitted are limited to a
predetermined subset of the set of all possible sequences. The
present paper explores this direction.

One of the basic problems in the realm of one-shot sub-
space coding is to find codes with good rates and good
error correcting/detecting capabilities. To achieve both goals
simultaneously, it may be unavoidable to increase the field
size $q$ or the packet size $m$. In view of that, there are two main
reasons that motivate us to consider multishot subspace coding
as an alternative. First, the system under consideration
may be such that it is not possible to change the field and packet
size. And second, even if those parameters are under designer
control, complexity reasons may be determinant—e.g., one-
shot codes in $\mathcal{P}(\mathbb{F}_q^m)$ can be considerably more complicated
(although better) than $n$-shot codes over $\mathcal{P}(\mathbb{F}_q^m)$.

We begin in Section II by reviewing definitions for the one-
shot case and introducing new definitions for the multishot
case. In Section III we present a motivation for multishot
coding with a simple example. In Section IV we make
some pertinent remarks. Section V addresses the relationship
between one-shot and multishot codes. Section VI derives
Hamming-, Gilbert-Varshamov- and Singleton-like bounds for
multishot codes. Section VII presents a construction of mul-
tishot codes borrowing ideas from block-coded modulation.
Finally, Section VIII concludes this paper.

II. DEFINITIONS

A. Background

We start by reviewing some concepts and definitions for
one-shot subspace coding, presented in [2].

The Gaussian binomial defined by

$$\binom{m}{k}_q = \prod_{i=0}^{k-1} \frac{q^{m-i} - 1}{q^{k-i} - 1}$$

quantifies the the number of $k$-dimensional vector subspaces
of $\mathbb{F}_q^m$. Therefore, the number of elements in the projective
space $\mathcal{P}(\mathbb{F}_q^m)$ is given by

$$|\mathcal{P}(\mathbb{F}_q^m)| = \sum_{k=0}^{m} \binom{m}{k}_q.$$
The **subspace distance** between two elements $V$ and $U$ of the projective space $\mathcal{P}(\mathbb{F}_q^n)$ is defined as

$$d_S(V, U) = \dim(V + U) - \dim(V \cap U),$$

(1)

where $V \cap U$ is the intersection of subspaces $V$ and $U$ (which is clearly a subspace) and $V + U$ is the sum of subspaces $V$ and $U$, given by $V + U = \{v + u : v \in V, u \in U\}$ (which is the smallest subspace containing $V \cup U$). The function $d_S(\cdot, \cdot)$ is indeed a metric over $\mathcal{P}(\mathbb{F}_q^n)$.

In the **subspace channel**, we transmit a subspace $V \in \mathcal{P}(\mathbb{F}_q^n)$ and receive another subspace $U \in \mathcal{P}(\mathbb{F}_q^n)$. If $V \neq U$, an error has occurred. The **weight** of the error is defined as $d_S(V, U)$. We call an error of weight 1 a **single error**, an error of weight 2 a **double error**, and so on.

### B. Multishot Subspace Coding

We now introduce definitions for the multishot case by considering **block codes** of length $n$ over a projective space. In other words, we consider codes in which the subspace channel just defined is used $n$ times.

The **$n$th extension of the projective space** $\mathcal{P}(\mathbb{F}_q^n)$ is defined as $\mathcal{P}(\mathbb{F}_q^n)^n$, that is, the $n$th Cartesian power of the projective space. Thus, elements of $\mathcal{P}(\mathbb{F}_q^n)^n$ are $n$-tuples of subspaces in $\mathcal{P}(\mathbb{F}_q^n)$. Of course, the number of elements in $\mathcal{P}(\mathbb{F}_q^n)^n$ is given by

$$|\mathcal{P}(\mathbb{F}_q^n)^n| = |\mathcal{P}(\mathbb{F}_q^n)|^n.$$

The **extended subspace distance** between two elements $V = (V_1, \ldots, V_n)$ and $U = (U_1, \ldots, U_n)$ of $\mathcal{P}(\mathbb{F}_q^n)^n$ is defined as

$$d_S(V, U) = \sum_{i=0}^{n} d_S(V_i, U_i),$$

(2)

where $d_S(\cdot, \cdot)$ in the right-hand side is given by $\mathbb{1}$.

Here, we transmit an $n$-tuple of subspaces $V = (V_1, \ldots, V_n)$ and receive another $n$-tuple of subspaces $U = (U_1, \ldots, U_n)$. In the absence of errors, $V = U$. Otherwise, an error of total weight $d_S(V, U)$ has occurred. We note that, for example, two single errors occurring in different transmissions amounts to one double error occurring in some transmission, since both cases gives a total weight of 2.

A **multishot (block) subspace code of length** $n$ (also called a **$n$-shot subspace code**) over $\mathcal{P}(\mathbb{F}_q^n)^n$ is a non-empty subset of $\mathcal{P}(\mathbb{F}_q^n)^n$. The **size** of a code $C$ is given by $|C|$, and the **rate** of that code is defined as

$$R(C) = \frac{\log |C|}{n},$$

measured in information symbols per subspace channel use. Finally, the **minimum distance** of $C$ is defined as

$$d_S(C) = \min\{d_S(V, U) : U, V \in C, U \neq V\}.$$

We have $1 \leq d_S(C) \leq mn$ and $0 \leq R(C) \leq 1$, if the logarithm base is taken as $|\mathcal{P}(\mathbb{F}_q^n)|$.

### III. A Motivating Example

Suppose we wish a multishot subspace code using the projective space $\mathcal{P}(\mathbb{F}_2^3)$ whose Hasse graph [2] is shown in Figure 1. Suppose also that our goal is to be able to detect a single error occurring in any of the $n = 3$ transmissions. So, it suffices to find a 3-shot code with minimum distance $d = 2$.

A first approach is simply to extend the best one-shot subspace code in $\mathcal{P}(\mathbb{F}_2^3)$ with minimum distance 2, which is

$$C_1 = \{S_1, S_2, S_3\}.$$

By doing so we obtain the code

$$C_1 = C_1 \times C_1 \times C_1 = \{S_1, S_1, S_1, S_1, S_2, S_1, S_2, S_3, S_3, \ldots, S_3, S_3, S_3\}$$

with $|C_1| = 27$.

Can we do better? Let us try to consider the projective space $\mathcal{P}(\mathbb{F}_2^3)$ as an alphabet of a “classical” code. Accordingly, take any bijective mapping between $\mathcal{P}(\mathbb{F}_2^3) = \{O, S_1, S_2, S_3, W\}$ and $Z_5 = \{0, 1, 2, 3, 4\}$, for example, $O \mapsto 0$, $S_1 \mapsto 1$, $S_2 \mapsto 2$, $S_3 \mapsto 3$ and $W \mapsto 4$. The best classical code of length 3 over $Z_5$ with minimum Hamming distance 2 is a parity-check code, such as

$$C_2 = \{x_1x_2x_3 \in \mathbb{Z}_5^3 : x_1 + x_2 + x_3 = 0\} = \{000, 014, 023, \ldots, 442\},$$

which is mapped back to

$$C_2 = \{OOO, OS_1W, OS_2S_3, \ldots, WWS_2\}$$

with $|C_2| = 25$, smaller than $|C_1|$.

The second approach did not succeed because it disregarded the subspace structure behind $\mathcal{P}(\mathbb{F}_2^3)$ and used only classical coding. If we want to achieve better results, we must, in fact, design codes in the metric space $\mathcal{P}(\mathbb{F}_2^3)^3$, taking into account both the subspace structure and time evolution. In Section VII following this idea, we find a code $C_3$ in $\mathcal{P}(\mathbb{F}_2^3)^3$ with minimum distance 2 and $|C_3| = 63$ by means of a multilevel construction.

1That is, we are considering an adversarial error model in which at most a single error can occur in a block of 3 transmissions.
IV. SOME REMARKS ON MULTISHOT CODES

A. Rate of a Code

In Section III we have defined the rate of a code \( C \) as
\[
R(C) = \frac{1}{n} \log |C|,
\]
measured in information symbols per subspace channel use. However, such definition may not be suitable for all situations. A good definition for rate is one which captures the notion of “cost” for the transmission of codewords. Although information is coded into subspaces, in practice we transmit vectors (packets) that form a basis for the subspace and not the subspace itself.

With this in mind, and following the work in [2], it may be interesting to redefine the rate of \( C \) either as
\[
R(C) = \frac{1}{\ell(C)} \log |C|, \quad \text{measured in information symbols per packet transmitted,}
\]
or as
\[
R(C) = \frac{1}{m \cdot \ell(C)} \log |C|, \quad \text{measured in information symbols per q-ary symbol transmitted.}
\]
In the definitions, the quantity \( \ell(C) \) can be either the average or the maximum dimension of the subspaces in code \( C \). This is specially valid for a generation-based model [5], in which “to transmit a subspace would require the transmitter to inject on average (or up to) \( \ell(C) \) packets into the network, corresponding to the transmission of \( m \cdot \ell(C) \) q-ary symbols”\(^2\), still according to [2].

B. Error Control Capability of a Code

Similarly to classical codes, multishot subspace codes with minimum distance \( d \) can detect every error of total weight \( d - 1 \) or less and correct every error of total weight \( (d - 1)/2 \) or less. So, is code \( C_3 \) of Section III better than code \( C_1 \)? If all we require is to detect a single error in any of the 3 transmissions, the answer is affirmative, since both can certainly detect a single error and code \( C_3 \) has a larger number of codewords. But code \( C_1 \) can detect 3 errors, as long as each of them occur in a different transmission. In view of that, the normalized distance \( d_S(C)/n \) may be a better parameter to settle when comparing two multishot codes. For example, code \( C_1 \), the one-shot counterpart of code \( C_1 \), has normalized distance 2, while code \( C_3 \) has normalized distance 2/3.

The purpose of the foregoing discussion was to emphasize the significance of the error model being adopted. Besides that, another important subject is the relation of subspace errors to packet errors and erasures. Such study is made in [2], [6] for one-shot subspace coding and could be extended to the multishot case.

V. RELATIONSHIP TO ONE-SHOT CODES

Obviously, one-shot codes are just a special case of n-shot codes—just set \( n = 1 \). In this section, we show how the converse statement can also be interpreted to be true in a sense.

The \( n \)th extension of a projective space, \( \mathcal{P}(\mathbb{F}_q^n) \), can be viewed as a “subset” of the larger projective space \( \mathcal{P}(\mathbb{F}_q^m) \). To see how, consider an injective mapping \( f : \mathcal{P}(\mathbb{F}_q^n) \rightarrow \mathcal{P}(\mathbb{F}_q^m) \) defined as follows. Let \( \mathbf{V} = (V_1, \ldots, V_n) \in \mathcal{P}(\mathbb{F}_q^n) \) and let \( b_{i,1}, \ldots, b_{i,m} \in \mathbb{F}_q^m \) be vectors such that
\[
V_i = \langle b_{i,1}, \ldots, b_{i,m} \rangle \quad \text{(i.e., the vector space spanned by}
\]
\( b_{i,1}, \ldots, b_{i,m} \), for \( i = 1, \ldots, n \). Then, \( f \) is defined as
\[
f(V) = \langle (b_{1,1}, \ldots, 0, \ldots, 0), (0, b_{2,1}, \ldots, \ldots, 0), (0, 0, b_{3,1}, \ldots, \ldots, 0), \ldots \rangle.
\]

It can be shown that \( f \) is really injective and that \( d_S(V, U) = d_S(f(V), f(U)) \) for every \( V, U \in \mathcal{P}(\mathbb{F}_q^n) \). So, every \( n \)-shot code \( C \subseteq \mathcal{P}(\mathbb{F}_q^m) \) leads to an one-shot code \( f(C) \subseteq \mathcal{P}(\mathbb{F}_q^n) \) with same minimum distance and size.

This also suggests a construction for multishot codes in \( \mathcal{P}(\mathbb{F}_q^m)^n \) based on one-shot codes in \( \mathcal{P}(\mathbb{F}_q^m) \). Indeed, if we take a code \( C \subseteq \mathcal{P}(\mathbb{F}_q^m) \) with minimum distance \( d \) and throw away the codewords that are not in \( f(\mathcal{P}(\mathbb{F}_q^m)) \), we get a code \( C' \), and \( f^{-1}(C') \subseteq \mathcal{P}(\mathbb{F}_q^m) \) is a \( n \)-shot code with minimum distance at least \( d \), but with a lower rate. Yet, it is not clear if good codes in \( \mathcal{P}(\mathbb{F}_q^m)^n \) always lead to good codes in \( \mathcal{P}(\mathbb{F}_q^n) \).

VI. BOUNDS ON CODES

Let \( A_q^n(m, d) \) denote the size of the largest code in \( \mathcal{P}(\mathbb{F}_q^m)^n \) with minimum distance \( d \), that is,
\[
A_q^n(m, d) = \max\{|C| : C \subseteq \mathcal{P}(\mathbb{F}_q^m)^n \text{ and } d_S(C) = d\}.
\]
In this section we derive upper and lower bounds on \( A_q^n(m, d) \).

Of course, every lower bound for \( |\mathcal{P}(\mathbb{F}_q^m)| \)-ary classical codes is a lower bound on \( A_q^n(m, d) \), a fact following from the discussion in Section III. Likewise, every upper bound for one-shot codes in \( \mathcal{P}(\mathbb{F}_q^m) \) is an upper bound on \( A_q^n(m, d) \), according to Section IV. Hence,
\[
A_q(\mathcal{P}(\mathbb{F}_q^m))^n(n, d) \leq A_q^n(m, d) \leq A_q(mn, d),
\]
where \( A_q(n, d) \) is the size of the best classical code of length \( n \) over \( \mathbb{F}_q \) with Hamming distance \( d \) and \( A_q(m', d) = A_q^n(m', d) \) is the size of the best one-shot code in \( \mathcal{P}(\mathbb{F}_q^m) \) with minimum subspace distance \( d \).

A. Sphere-Packing and Sphere-Covering Bounds

For the next two bounds we will need the notion of spheres lying in the metric space \( \mathcal{P}(\mathbb{F}_q^n) \). The sphere centered in \( \mathbf{V} = (V_1, \ldots, V_n) \) with radius \( r \) in \( \mathcal{P}(\mathbb{F}_q^n) \) is given by
\[
B_{(q,m)}(\mathbf{V}, r) = \{ U \in \mathcal{P}(\mathbb{F}_q^n) : d_S(U, \mathbf{V}) \leq r \},
\]
and the volume of that sphere is defined as
\[
\text{Vol}_{(q,m)}(\mathbf{V}, r) = |B_{(q,m)}(\mathbf{V}, r)|.
\]
It can be shown that
\[
\text{Vol}_{(q,m)}(\mathbf{V}, r) = \sum_{\substack{j \in \{0, \ldots, m\}^n: \sum_{i=1}^n j_i \leq r}} \prod_{i=1}^n \text{Vol}_{(q,m)}(V_i, j_i),
\]
where
\[V_i = \langle b_{i,1}, \ldots, b_{i,m} \rangle\] (i.e., the vector space spanned by \( b_{i,1}, \ldots, b_{i,m} \)), for \( i = 1, \ldots, n \).
average volume of a sphere of radius $r$ in $V$ with $\dim V = k$ in the projective space $\mathcal{P}(\mathbb{F}_q^n)$, as given in [2], [3]. The volume of a shell centered in $V$ depends only on $k = (\dim V_1, \ldots, \dim V_n)$, so we also adopt the notation $\text{Vol}_{(q,m)}(V, r)$ instead of $\text{Vol}_{(q,m)}^\text{shell}(V, r)$.

Given a tuple $k = (k_1, \ldots, k_n)$, there are a total of $\prod_{i=1}^n (m-k_i)$ points $V$ such that $k = (\dim V_1, \ldots, \dim V_n)$. Therefore, the average volume of a sphere of radius $r$ in $\mathcal{P}(\mathbb{F}_q^n)$ is

$$\text{Vol}^\text{avg}(r) = \frac{1}{|\mathcal{P}(\mathbb{F}_q^n)|} \sum_{V \in \mathcal{P}(\mathbb{F}_q^n)} \text{Vol}(V, r)$$

$$= \frac{1}{|\mathcal{P}(\mathbb{F}_q^n)|} \sum_{k \in \{1, \ldots, m\}^n} \text{Freq}(k) \text{Vol}(k, r).$$

Also, the maximum and minimum volumes are

$$\text{Vol}^\text{min}(r) = \text{Vol}((\lfloor m/2 \rfloor, \ldots, \lfloor m/2 \rfloor), r),$$

$$\text{Vol}^\text{max}(r) = \text{Vol}((0, \ldots, 0), r).$$

If we consider the packing of spheres of radius $r = \lfloor (d-1)/2 \rfloor$ centered at the codewords of a code $C$ in $\mathcal{P}(\mathbb{F}_q^n)$, we get

$$|\mathcal{P}(\mathbb{F}_q^n)| \geq \sum_{V \in \mathcal{C}} \text{Vol}(V, r) \geq \sum_{V \in \mathcal{C}} \text{Vol}^\text{min}(r) = |\mathcal{C}| \text{Vol}^\text{min}(r),$$

and so we have the Hamming-like upper bound

$$A_q^n(m, d) \leq \frac{|\mathcal{P}(\mathbb{F}_q^n)|}{\text{Vol}^\text{min}(\lfloor (d-1)/2 \rfloor)},$$

where $\text{Vol}^\text{min}(\cdot)$ is given by (4).

The same approach used in [3] for the one-shot case can be used here to get the Gilbert-Varshamov-like lower bound

$$A_q^n(m, d) \geq \frac{|\mathcal{P}(\mathbb{F}_q^n)|}{\text{Vol}^\text{avg}(d-1)},$$

where $\text{Vol}^\text{avg}(\cdot)$ is given by (3).

B. Singleton Bound

We now consider a puncturing operation of a codeword

$$(\cdot)^\uparrow : \mathcal{P}(\mathbb{F}_q^n) \rightarrow \mathcal{P}(\mathbb{F}_q)^{n-1}$$

which consists in removing any coordinate of tuple $V$. The punctured code is defined as $C^\uparrow = \{V^\uparrow : V \in C\}$. One can prove that if $d_S(C) > m$ then $|C^\uparrow| = |C|$ and $d_S(C^\uparrow) \geq d_S(C) - m$.

Let $C \subseteq \mathcal{P}(\mathbb{F}_q^n)$ be a code with $d_S(C) = d$. By puncturing the code $\left\lfloor \frac{d_s}{m} \right\rfloor$ times we get a code $C' = C^{\uparrow^\cdot} \subseteq \mathcal{P}(\mathbb{F}_q^n)^{n-\left\lfloor \frac{d_s}{m} \right\rfloor}$ with $|C'| = |C|$ and $d_S(C') \geq 1$. Therefore the Singleton-like upper bound becomes

$$A_q^n(m, d) \leq |\mathcal{P}(\mathbb{F}_q^n)|^{n-\left\lfloor \frac{d_s}{m} \right\rfloor}.$$
intrasubset distances. Say we find that $L'$ is the minimum level satisfying $d_S^{(l)} \geq d$ for all $l \geq L'$. We have to make sure that all partitions up to level $L'$ are nested partitions, throwing out subspaces if necessary. Then, we must find classical block codes (called component codes) $C_l \subseteq \mathbb{F}_q^n$, $1 \leq l \leq L'$, with maximal rates and minimum Hamming distance $d_H^{(l)}$ such that $\min \{d_S^{(l-1)} \cdot d_H^{(l)} : 1 \leq l \leq L' \} \geq d$.

Also, it is guaranteed that the minimum distance of $C_l$ gives a $3$-shot subspace code with minimum distance $2$ and $62$ (resp., $63$) codewords.

VIII. CONCLUSION

The aim of this paper was to suggest multishot subspace coding as a potential alternative to one-shot subspace coding, specially when the field size $q$ or packet size $m$ cannot be changed. Multishot subspace coding introduces a new degree of freedom: the number of channel uses $n$.

Future directions of research may include the following.

1) The use of convolutional coding instead of block coding by considering ideas similar to Ungerboeck’s trellis-coded modulation [9].

2) The determination of the subspace channel capacity under a probabilistic error model and an information-theoretical point of view. The works [10], [11] deal with the so called “one-shot capacity” and find asymptotical expressions when either the symbol size or packet size (or both) increases.

3) Finally, the development of bounds and constructions for constant-dimension multishot subspace codes. For the one-shot case, refer to [2], [3], [4] and [12], [13], the last two based on a related metric called the rank-metric.

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