Superballistic and superdiffusive scaling limits of stochastic harmonic chains with long-range interactions

Hayate Suda

Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, Kanagawa 223-8522, Japan
E-mail: hayates@keio.jp

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Abstract
We consider one-dimensional infinite chains of harmonic oscillators with random exchanges of momenta and long-range interaction potentials which have polynomial decay rate $|x|^{-\theta}, x \to \infty, \theta > 1$ where $x \in \mathbb{Z}$ is the interaction range. The dynamics conserve total momentum, total length and total energy. We prove that the systems evolve macroscopically on superballistic space-time scale $(y \varepsilon^{-1}, t \varepsilon^{-1})$ when $1 < \theta < 3$, $(y \varepsilon^{-1}, t \varepsilon^{-1} \sqrt{\log(\varepsilon^{-1})})^{-1}$ when $\theta = 3$, and hyperbolic space-time scale $(y \varepsilon^{-1}, t \varepsilon^{-1})$ when $\theta > 3$. Combining our results and the results in (Suda 2021 Ann. Inst. Henri Poincare B 57 2268–2314), we show the existence of two different space-time scales on which the systems evolve. In addition, we prove the fluctuations of the normal modes of the superballistic wave equation, which are analogues of the Riemann invariants and capture fluctuations along the characteristics. For the normal modes, the space-time scale is superdiffusive when $2 < \theta \leq 4$ and diffusive when $\theta > 4$.

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1. Introduction
In recent years, one-dimensional chains of harmonic oscillators with random exchanges of momenta have been considered as good approximations of some non-random anharmonic

*Author to whom any correspondence should be addressed.
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chains and mostly studied from the point of view of anomalous heat transport and corresponding superdiffusion of energy [4–8, 13, 15], see also the review [17, chapter 5] for related models. Such chains are broadly defined as Hamiltonian dynamics perturbed by momentum-exchange noise which acts only on momenta locally:

\[
\begin{align*}
\text{d}q_i(t) &= \partial_{p_i} H(p(t), q(t)) \, \text{d}t \\
\text{d}p_i(t) &= -\partial_{q_i} H(p(t), q(t)) \, \text{d}t + \gamma S_\alpha(p(t)),
\end{align*}
\]

where the formal Hamiltonian \( H(p, q) \) is given by

\[
H(p, q) := \sum_{x \in \mathbb{Z}} \epsilon_x, \quad \epsilon_x := \frac{1}{2} p_x^2 - \frac{1}{4} \sum_{z \in \mathbb{Z}} \alpha_{x-z} (q_x - q_z)^2.
\]

Here \( p_x, q_x \) and \( \epsilon_x \) stand for the momentum, the position, and the energy of the particle labelled by \( x \in \mathbb{Z} \), respectively. In addition, \( S_\alpha(p) \) is a stochastic noise involving the local momenta \( p_x \) around \( x \in \mathbb{Z} \), and \( \gamma \geq 0 \) is the strength of the noise. Typical assumptions of interaction potential, \( \alpha_x, x \in \mathbb{Z} \), are as follows [5, 13, 15]:

(a.1) [Non-negativity of energy]: \( \alpha_x \leq 0 \) for all \( x \in \mathbb{Z} \setminus \{0\} \), \( \alpha_x \neq 0 \) for some \( x \in \mathbb{Z} \).

(a.2) [Symmetry]: \( \alpha_x = \alpha_{-x} \) for all \( x \in \mathbb{Z} \).

(a.3) [Exponential decay rate]: there exists some positive constant \( C > 0 \) such that \( |\alpha_x| \leq C e^{-\frac{\pi k}{4}} \) for all \( x \in \mathbb{Z} \).

(a.4) [Positive definite (no pinning, acoustic)]: \( \dot{\alpha}(k) > 0 \) for all \( k \neq 0 \), \( \dot{\alpha}(0) = 0 \), \( \alpha''(0) > 0 \).

Here \( \dot{\alpha} \) is the discrete Fourier transform defined as

\[
\dot{\alpha}(k) := \sum_{x \in \mathbb{Z}} \alpha_x e^{-2\pi ik}, \quad k \in \mathbb{T},
\]

and \( \mathbb{T} = [-1/2, 1/2] \) is the one-dimensional torus. In particular, exponential decay of the interaction potential in the interaction range is assumed. The condition \( \dot{\alpha}(0) = 0 \) in (a.4) implies that there is no pinning potential, in which case total momentum is conserved. In addition, the condition \( \alpha''(0) > 0 \) ensures that the speed of sound, \( \lim_{k \to 0} \dot{\alpha}'(k)\sqrt{\alpha(k)}^{-1} \), is positive. The stochastic noise is added to destroy infinite number of local conservation law, but formally conserve total momentum \( \sum_x p_x \), total length of the system \( \sum_x \tau_x \), \( \tau_x := q_x - q_{x-1} \), and total energy \( \sum_x \epsilon_x \) [11]. Here the physical meaning of \( \tau_x \) is the inter-particle distance or tension between neighbour particles. The precise definition of the stochastic noise is given in section 3. Note that when dealing with the problems considered in this paper, there are several choices for stochastic perturbations that exchange momentum locally, see remark 4.8 for details. We call the model described above exponential decay model.

In [15], the authors pointed out an interesting feature of exponential decay models: there exists two different equilibrium states and corresponding different space-time scales of the conserved quantities due to the stochastic noise. One relates to mechanical equilibrium, which means that the macroscopic profiles of momenta and tensions are constant, and the other relates to thermal equilibrium, which means that the macroscopic profile of energy is constant. At first, the system go to mechanical equilibrium at hyperbolic scaling. Actually, they prove the weak
convergence of the scaled empirical measure of \((p_\varepsilon(t), r_\varepsilon(t), 0_\varepsilon(t))\):

\[
\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon \left[ \begin{pmatrix} p_\varepsilon \left( \frac{\varepsilon}{\varepsilon_0} \right) \\ r_\varepsilon \left( \frac{\varepsilon}{\varepsilon_0} \right) \\ 0_\varepsilon \left( \frac{\varepsilon}{\varepsilon_0} \right) \end{pmatrix} \right] \cdot \begin{pmatrix} J_1(\varepsilon x, t) \\ J_2(\varepsilon x, t) \\ J_3(\varepsilon x, t) \end{pmatrix} = \int_\mathbb{R} dy \begin{pmatrix} p(y, t) \\ r(y, t) \\ e(y, t) \end{pmatrix} \cdot \begin{pmatrix} J_1(y, t) \\ J_2(y, t) \\ J_3(y, t) \end{pmatrix}
\]

for any test function \(J_i \in C_0^0(\mathbb{R} \times [0, \infty)), i = 1, 2, 3\), where \((p(y, t), r(y, t))\) is the solution of the following linear wave equation:

\[
\begin{align*}
\partial_t r(y, t) &= \partial_y p(y, t) \\
\partial_t p(y, t) &= C_\alpha \partial_y r(y, t),
\end{align*}
\]

(1.2)

and \(C_\alpha\) is an explicit positive constant that depends on \(\alpha\). In addition, \(e(y, t) := 1/2(p^2(y, t) + C_\alpha r^2(y, t)) + T(y)\) and \(T(y)\) is the macroscopic temperature profile. After the system reaches mechanical equilibrium, then \((p, r)\) terms are macroscopically constant, but the energy will evolve in time. In the superdiffusive time scale \(t^{-3/2}\), i.e., a scale faster than the diffusive scale \(t^{-2}\), the energy distribution converges weakly,

\[
\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon \left[ \int_0^\infty dr \int_\mathbb{R} dy \left( \frac{1}{\varepsilon^{3/2}} \right) J(\varepsilon x, t) \right] = \int_\mathbb{R} dy \int_0^\infty dt T(y, t) J(y, t),
\]

for any \(J \in C_0^0(\mathbb{R} \times [0, \infty))\) and the limit point \(T(y, t)\) satisfies a \(3/4\)-fractional diffusion equation

\[
\partial_t T(y, t) = -C_{\alpha, \gamma} (-\Delta)^{3/4} T(y, t)
\]

\(T(y, 0) = T(y)\),

where \(C_{\alpha, \gamma}\) is an explicit positive constant that depends on \(\alpha, \gamma\). Note that in a longer time scale than the ballistic scaling, the empirical measures of \((p_\varepsilon(t), r_\varepsilon(t), 0_\varepsilon(t))\) converge to stationary solutions \(p(y, t) = r(y, t) = 0\) of (1.2), see [15, theorem 4.8]. In their paper, the authors show that there are two types of microscopic contributions to the convergence of their model:

- The phononic terms that can be observed in the absence of thermal energy \((T(y) = 0)\), and converge to some mechanical equilibrium.
- The thermal terms, which appear after mechanical equilibrium has been enforced.

Each term has a different initial distribution, called phononic/thermal type distribution respectively, see [15, section 2] for details. The stochastic noise is essential to observe the two different convergences described above. Actually, in the absence of the stochastic noise \((\gamma = 0)\), the macroscopic energy transport is purely ballistic [5, 10].

In this paper, we consider instead a model given by (1.1) with strong long-range interactions with interaction potential given by

\[
\alpha_\varepsilon := -|x|^{-\theta}, \quad x \in \mathbb{Z}\setminus\{0\}, \quad \alpha_0 := 2 \sum_{x \in \mathbb{N}} |x|^{-\theta} \quad \theta > 1.
\]

(1.3)

Our model has no pinning potential, and the conditions (a.3) is not satisfied. In addition, for \(\theta \leq 3, \sum_{x \in \mathbb{Z}} |x|^{2-\theta} = \infty\), thus the second derivative of \(\hat{\alpha}(k)\) cannot be defined as a continuous function on \(\mathbb{T}\) and the condition \(\hat{\alpha}''(0) > 0\) in (a.4) is not satisfied. From this polynomial decay model, we obtain the following new phenomena for stochastic harmonic chains.
1.1. Two different space-time scales without stochastic noise

For the polynomial decay model, in [21] we show that at mechanical equilibrium the thermal energy also evolves on a superdiffusive space-time scale and the time evolution law of the macroscopic thermal energy \( T(y, t) \) is given by a fractional diffusion equation and the exponent of the fractional Laplacian depends on \( \theta > 2 \):

Theorem (Theorem 1 in [21]). Under suitable initial conditions and \( \gamma > 0 \), the following weak convergence holds for any test function \( J \in C_0^\infty(\mathbb{R} \times [0, \infty)) \):

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} \int_0^\infty d\varepsilon_x \left[ \frac{t}{f_\theta(\varepsilon)} \right] J(\varepsilon x, t) = \int_\mathbb{R} \int_0^\infty dy \int_0^\infty dt T(y, t) J(y, t),
\]

where \( f_\theta(\varepsilon) = \varepsilon^{\frac{3}{2}} \) for \( \theta > 3 \) and \( f_\theta(\varepsilon) \gg \varepsilon^{\frac{3}{2}} \) for \( 2 < \theta \leq 3 \), and \( T(y, t) \) is the solution of the following fractional diffusion equation:

\[
\partial_t T(y, t) = \begin{cases} 
-C_{\theta, \gamma}(-\Delta)^{\frac{\theta}{2}} T(y, t) & 2 < \theta \leq 3, \\
-C_{\theta, \gamma}(-\Delta)^{\frac{3}{2}} T(y, t) & \theta > 3,
\end{cases}
\]

for some explicit positive constant \( C_{\theta, \gamma} \).

For the explicit formulas for \( f_\theta(\varepsilon) \) and \( C_{\theta, \gamma} \) and some discussion of \( C_{\theta, \gamma} \), see appendix C. Note that without the stochastic noise \( (\gamma = 0) \), the energy transport is also ballistic, see [21, theorem 2].

In the present paper, we consider the regime in which mechanical equilibrium is reached. We explain our main results in the latter half of Introduction. We emphasise that by combining [21, theorems 1 and 2], theorems 4.1 and 4.2 in this paper, we verify that if \( 2 < \theta \leq 3 \), then there exist two different space-time scales regardless of the existence of the stochastic noise. Especially, the phononic terms evolve on a superballistic scale while the thermal terms evolve on a hyperbolic \( (\gamma = 0) \) or a superdiffusive scale \( (\gamma > 0) \). That is, the long-range interactions decompose the energy into two terms in the sense of space-time scales. As mentioned above, in exponential decay models without stochastic noise, both phononic/thermal terms converge with hyperbolic scaling.

Before we explain our main results, i.e., the derivation of the limiting equation with superballistic space-time scaling, we will introduce new variable, which is dual variable of the momentum for the polynomial decay model.

1.2. New conserved quantity for the polynomial decay model

For our model, \( r_x = q_x - q_{x-1} \) is no longer the proper dual variable for \( p_x \), because \( r_x \) only captures the tension between neighbour particles. Actually, when \( \theta \leq 3 \), we cannot derive the limiting equations for the variables \( (p_x, r_x) \). This leads us to introduce the generalised tension, denoted by \( I_x \), \( x \in \mathbb{Z} \), which is formally defined as

\[
I_x = \mathcal{H}_\mathbb{Z}(\hat{\omega} * q)_x,
\]

where \( \hat{\omega} : \mathbb{Z} \to \mathbb{R} \) is the inverse Fourier transform of \( \omega(k) := \sqrt{\alpha(k)}, k \in \mathbb{T} \), and \( \mathcal{H}_\mathbb{Z} \) is the Hilbert transform on \( \mathbb{Z} \) defined in section 2. Since \( \hat{l}(k) = -i \text{sgn}(k) \omega(k) \hat{q}(k) \) where \( \text{sgn}(k) \),
We see that \( \sum_{x} l_x = l(0) \) is formally conserved, and if \( \theta > 3 \) then \( \tau_x, l_x \) are macroscopically equivalent up to a constant multiple, see corollary 4.2. Hence we can think that \( l_x \) is a natural generalisation of \( \tau_x \). One of our results is the weak convergence of the scaled empirical measure of \( (p_x(t), l_x(t)) \) in the time scaling \( j_{\theta}(\varepsilon) \) given by

\[
 j_{\theta}(\varepsilon) := \begin{cases} 
 \varepsilon^{2 + \frac{1}{\theta}}, & 1 < \theta < 3, \\
 \varepsilon(-\log \varepsilon)^{\frac{1}{2}}, & \theta = 3, \\
 \varepsilon, & \theta > 3. 
\end{cases}
\]

(1.4)

Note that \( j_{\theta}(\varepsilon) > j_3(\varepsilon) \) for \( \theta > 2 \), where \( j_3(\varepsilon) \) is the time scaling for thermal energy defined in the previous subsection (see figure 1). In theorem 4.1 we show the scaling limit of the scaled dynamics \( \{p_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right), l_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right) \}_{x \in \mathbb{Z}, t \geq 0} \) with initial distribution fitting a given macroscopic profile for \( p \) and \( l \) (see (4.1) below). Especially, theorem 4.1 implies the weak convergence of the scaled empirical measure of \( (p_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right), l_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right) ) \), that is,

\[
 \lim_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}} \mathbb{E}_x \left[ \left( \frac{1}{j_{\theta}(\varepsilon)} \int l_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right) \right) \cdot \left( \frac{1}{j_{\theta}(\varepsilon)} \int l_x \left( \frac{t}{j_{\theta}(\varepsilon)} \right) \right) \right] = \int_{\mathbb{R}} dy \left( p(y, t) \right) \cdot \left( l(y, t) \right),
\]

(1.5)

for any \( t \geq 0 \) and any test functions \( J_i \in C^0_c(\mathbb{R}), i = 1, 2 \), where the limit \( (p(y, t), l(y, t)) \) is the solution of the following superballistic wave equation:

\[
\begin{align*}
\frac{\partial}{\partial t} l(y, t) &= \sqrt{C_1(\theta)} \mathcal{D}_\theta p(y, t), \\
\frac{\partial}{\partial t} p(y, t) &= \sqrt{C_1(\theta)} \mathcal{D}_\theta l(y, t),
\end{align*}
\]

(1.6)

where \( C_1(\theta) \) is a positive constant defined as

\[
 C_1(\theta) := \begin{cases} 
 \int_{0}^{\infty} \frac{dy}{y^\theta} \left( \frac{2 - 2 \cos y}{y^\theta} \right) & 1 < \theta < 3, \\
 1 & \theta = 3, \\
 \sum_{x \geq 1} |x|^{-\theta} & \theta > 3. 
\end{cases}
\]

(1.7)

In addition, \( \mathcal{D}_\theta \) is an operator defined as

\[
 (\mathcal{D}_\theta f)(y) := \begin{cases} 
 \mathcal{H}_\mathbb{R} \left( \frac{(-\Delta)^{\frac{\theta - 1}{2}} f}{y} \right)(y) & 1 < \theta \leq 3, \\
 f'(y) & \theta > 3,
\end{cases}
\]

(1.8)
for any \( f \in \mathcal{S}(\mathbb{R}) \), and \( \mathcal{H}_{\mathbb{R}} \) is the Hilbert transform on \( \mathbb{R} \). The reason (1.6) is called the superballistic wave equation is that \( p(y, t) \) satisfies
\[
\partial_t^2 p(y, t) = \begin{cases} 
- C_1(\theta)(-\Delta)^{\frac{\theta}{2}} p(y, t) & 1 < \theta \leq 3, \\
C_1(\theta) \Delta p(y, t) & \theta > 3.
\end{cases}
\] (1.8)

Note that in a longer time scale than \( j_{\beta}(\varepsilon)^{-1} \), the empirical measures of \((p_x(t), l_x(t))\) converge to stationary solutions \( p(y, t) = l(y, t) = 0 \) of (1.6), see section 5.3 for details. In addition, in theorem 4.2, we prove the following scaling limit of the scaled empirical measure of the phononic energy:
\[
\lim_{\varepsilon \to 0} \mathbb{E}_{x \in \mathbb{Z}} \left[ e_x \left( \frac{t}{j_{\beta}(\varepsilon)} \right) \right] J(\varepsilon x) = \int_{\mathbb{R}} dy e(y, t) J(y),
\]
where \( e(y, t) \) is given by
\[
e(y, t) := \begin{cases} 
\frac{1}{2} p^2(y, t) + \frac{1}{4} L_{\theta}(y, t) & 1 < \theta < 3, \\
\frac{1}{2} p^2(y, t) + \frac{1}{2} \hat{f}(y, t) & \theta \geq 3,
\end{cases}
\] (1.9)
and \( L_{\theta}(y, t), 1 < \theta < 3 \) is defined as
\[
L_{\theta}(y, t) := \left(-(-\Delta)^{\frac{\theta}{2}} (D_{\theta}^{-1})^2\right)(y, t) - 2 \left(-(-\Delta)^{\frac{\theta}{2}} D_{\theta}^{-1} l\right) D_{\theta}^{-1} l(y, t).
\] (1.10)

In addition, \( D_{\theta}^{-1}, 1 < \theta < 3 \) is an operator defined as
\[
(D_{\theta}^{-1} f)(y) := \int_{\mathbb{R}} d\xi e^{2\pi i y \xi} (-i) \text{sgn}(\xi) |2\pi\xi|^{-\frac{\theta}{2}} f(\xi),
\]
for any \( f \in \mathcal{S}(\mathbb{R}) \), where \( \hat{f} \) is the Fourier transform on \( \mathbb{R} \) and \( \text{sgn}(y), y \in \mathbb{R} \) is the sign function defined as
\[
\text{sgn}(y) := \begin{cases} 
1 & y > 0, \\
0 & y = 0, \\
-1 & y < 0.
\end{cases}
\]

Note that because we consider phononic initial conditions, there is no thermal term \( T(y) \).

One might think that the value \( \theta = 3 \) is the threshold and when \( \theta > 3 \) the macroscopic behaviours of the conserved quantities, \((p, l, \epsilon)\), are essentially the same as those of the exponential decay models. However, as will be explained in the next subsection, for \( 3 < \theta < 4 \), some effects of long-range interactions appear macroscopically.

1.3. Non-monotonic superdiffusive behaviours of normal modes

We consider the fluctuations of (1.6) by considering the normal modes, which are analogs of the Riemann invariants for (1.6). Here, the Riemann invariants are the quantities that describe the time evolution of the transport equation and are constant along the characteristics. For exponential decay models, in [15] the authors show the diffusive fluctuations of the microscopic normal modes of (1.2) along the characteristics \( y \pm \sqrt{C_0}t = \text{const} \). Note that in [20], the author
predicts similar behaviour for some anharmonic chains, e.g. the $\beta$-Fermi–Pasta–Ulam chain at zero pressure.

For the polynomial decay model, the normal modes of (1.6) are defined by $f(y, t) := p(y, t) + l(y, t)$, where $(p(y, t), l(y, t))$ is the solution of (1.6). The corresponding microscopic normal modes are defined by $f_{\pm}(x, t) := p_{\pm}(x, t) \pm l_{\pm}(x, t)$. For $\theta < 3$, (1.6) is a system of fractional differential equations, so the characteristics of (1.6) cannot be defined in usual way for hyperbolic differential equations, but by using the semigroups $S_{\theta}(t)$ and $S_{\theta}^{-1}(t)$, we can consider the fluctuations of the normal modes. In theorem 4.3, we show that

$$
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ \int_0^1 \left( \frac{t}{n_\theta(\varepsilon)} \right) \left( S_\theta^+ \left( \frac{t}{m_\theta(\varepsilon)} \right) J(\varepsilon x) \right) \right] = \int_\mathbb{R} dy F_+ \left( y, tJ(y) \right),
$$

for any $J \in C^\infty_0(\mathbb{R})$, where $(m_\theta(\varepsilon), n_\theta(\varepsilon))$ are the macroscopic and microscopic time scaling, respectively, defined as

$$
m_\theta(\varepsilon) := \begin{cases}
\varepsilon^{3-\theta} & 2 < \theta < 3, \\
(\log \varepsilon^{-1})^{-1} & \theta = 3, \\
\varepsilon^{\theta-3} & 3 < \theta \leq 4, \\
\varepsilon & \theta > 4,
\end{cases}
$$

$$
n_\theta(\varepsilon) := m_\theta(\varepsilon)j_\theta(\varepsilon),
$$

and $j_\theta(\varepsilon)$ is defined in (1.4). Note that the relation $n_\theta(\varepsilon) = m_\theta(\varepsilon)j_\theta(\varepsilon)$ implies that the ratio of the macroscopic to microscopic time scale of the normal modes is the same as that of the phononic terms. In addition, the limit $F_+^\varepsilon(y, t)$ is the solution of the following system of linear differential equations:

$$
\partial_t F_+^\varepsilon(y, t) = \begin{cases}
\pm \frac{C_2(\theta)}{C_1(\theta)} D_{6-\theta} F_+^\varepsilon & 2 < \theta < 3, \\
\pm \left( \sqrt{C_1(3)D_{3\theta}} F_+^\varepsilon + \frac{C_3(3)}{\sqrt{C_1(3)}} \partial_t F_+^\varepsilon \right) & \theta = 3, \\
\pm \frac{C_2(\theta)}{C_1(\theta)} D_{2\theta-3} F_+^\varepsilon & 3 < \theta < 4, \\
\pm \frac{C_2(4)}{\sqrt{C_1(4)}} H_{\theta}(\Delta F_+^\varepsilon) + \frac{3\gamma}{2} \Delta(F_+^\varepsilon + F_-^\varepsilon) & \theta = 4, \\
\frac{3\gamma}{2} \Delta(F_+^\varepsilon + F_-^\varepsilon) & \theta > 4,
\end{cases}
$$

![Figure 1. The graphs of log $f_\theta(\varepsilon)$ and log $j_\theta(\varepsilon)$.]
where $\mathcal{D}_{3J}$ is an operator defined as

$$
(\mathcal{D}_{3J} f)(y) := \int_{\mathbb{R}} d\xi \ e^{2\pi i \xi y} \text{sgn}(\xi) |2\pi\xi| \log\left(|2\pi\xi|^{-1}\right) \tilde{f}(\xi),
$$

for any $f \in \mathcal{S}(\mathbb{R})$. Note that for $\theta \geq 3$, $(S^\theta_\theta(t) J)(y) = J(y \mp \sqrt{C_1(\theta) t})$ and thus the above limit captures the fluctuations along the characteristics $y \mp \sqrt{C_1(\theta) t} = \text{const.}$. From (1.11), it is revealed that for $2 < \theta < 4$, the space-time scaling is superdiffusive due to the long-range interactions. We would like to emphasise that in the case $3 < \theta < 4$, the conserved quantities $(p, l, e)$ converge at ballistic scaling, but their superdiffusive fluctuations are obtained, that is, the effect of the long-range interactions appears macroscopically for $\theta < 4$. Another interesting behaviour is the non-monotonic dependence on the exponent $\theta > 2$ (see figure 2). This non-monotonic dependence is caused by the first and second order of $\hat{\alpha}(k)$ as $k \to 0$.

We follow the idea presented in [15] to prove above results. If we assume that the total energy at $t = 0$ is macroscopically finite and that the variance of $(p_x(0), l_x(0))$ around the macroscopic initial profile $(p_0(x), l_0(x))$ vanishes, then we can use a simple method based on the Fourier transform and the $L^2$ bound of $(p_x(t), l_x(t))$. In particular, we can derive the scaling limits without proving the ergodicity of the dynamics, i.e., without proving that every regular translation-invariant steady state of the polynomial decay model is of Gibbs type. Regularity in this context means that the state has finite relative entropy per unit volume with respect to a Gibbs state. Note that, since our proofs rely on the continuity of $\hat{\alpha}(k)$ and its explicit asymptotic behaviour as $k \to 0$, we do not consider the scaling limit of the phononic terms for the case $\theta \leq 1$ or the fluctuations of the normal modes with $\theta \leq 2$.

Recently, heat conduction in anharmonic chains with long-range interactions have been studied numerically [1–3, 12, 19], and anomalous heat transport has been observed. In [2], it has been observed that the thermal conductivity exhibits non-monotonic dependence on $\theta > 0$ with a maximum at $\theta = 2$. On the other hand, in [9, 14, 16, 22], some fractional-order differential equations are derived from anharmonic chains with long-range interactions as continuum limit of the equations of motion. Note that the difference between our work and previous studies on this problem is that we deal with harmonic but stochastically perturbed chains and derive (1.6) and (1.8) rigorously.

One of the important open problems is to generalise some of our results for anharmonic chains with long-range interactions. For anharmonic chains, the method in [15] cannot be used, so the relative entropy method introduced in [23] is required. A main difficulty in using the relative entropy method for anharmonic chains is to show the ergodicity of the dynamics [11, 18].
As we mentioned above, for the polynomial decay model, the variable $r_x$ is not a proper dual of $p_x$, so it is expected that a Gibbs state can be defined by using $(p_x, l_x)$ terms. However, since the definition of $l_x$ depends heavily on the specific form of the interaction potential and the Fourier transform, for anharmonic chains with long-range interactions, it seems difficult to find a dual variable for the momentum and characterise every regular translation-invariant steady state as a certain Gibbs type.

Our paper is organised as follows: in section 2 we prepare some notations. In section 3 we give the rigorous definition of the dynamics. In section 4 we state our main results, theorems 4.1–4.4. Proofs of theorems 4.1 and 4.2 are given in sections 5 and 6 respectively. In section 7 we prove theorems 4.3 and 4.4.

2. Notations

Let $\mathbb{R}$ be the real line, $\mathbb{Z}$ be the set of all integers and $\mathbb{T} = [-1/2, 1/2)$ be the one-dimensional torus. We use the notation $\mathbb{T}_a = [-a/2, a/2)$ for the torus of size $a > 0$.

Denote by $\ell^2(\mathbb{Z})$ the space of all complex valued sequences $(f_x)_{x \in \mathbb{Z}}$ equipped with the norm

$$\|f\|_{\ell^2(\mathbb{Z})} := \left( \sum_{x \in \mathbb{Z}} |f_x|^2 \right)^{1/2}.$$

Denote by $L^2(\mathbb{T})$ the space of all complex valued functions $f(k)$, $k \in \mathbb{T}$ equipped with the norm

$$\|f\|_{L^2(\mathbb{T})} := \left( \int_{\mathbb{T}} |f(k)|^2 \right)^{1/2}.$$

Denote by $S(\mathbb{R})$ the Schwartz space on $\mathbb{R}$.

For $f, g : \mathbb{Z} \to \mathbb{R}$, $h \in \ell^2(\mathbb{Z})$, we define $f * g : \mathbb{Z} \to \mathbb{R}$ and $\hat{h} \in L^2(\mathbb{T})$ as

$$(f * g)_x := \sum_{z \in \mathbb{Z}} f_{x-z}g_z,$$

$$\hat{h}(k) := \sum_{x \in \mathbb{Z}} e^{-2\pi ikx}h_x.$$

For $f \in L^2(\mathbb{T})$, we define $\hat{f} \in \ell^2(\mathbb{Z})$ as

$$\hat{f}_x := \int_{\mathbb{T}} dk e^{2\pi ikx}f(k).$$

For any integrable function $J : \mathbb{R} \to \mathbb{C}$, we define $\tilde{J} : \mathbb{R} \to \mathbb{C}$ as

$$\tilde{J}(\xi) := \int_{\mathbb{R}} dy e^{-2\pi i\xi y} J(y).$$

In section 4 and later, we will also use the notation $\tilde{J}$ for the function on $\mathbb{T}_a$, $a > 0$, defined as

$$\tilde{J}(k + na) = \tilde{J}(k) := \int_{\mathbb{R}} dy e^{-2\pi ikx}J(y), \quad k \in \mathbb{T}_a, \quad n \in \mathbb{Z}. \quad (2.1)$$
We define the Hilbert transform on $\mathbb{R}$ and $\mathbb{Z}$, denoted by $\mathcal{H}_\mathbb{R}$ and $\mathcal{H}_\mathbb{Z}$ respectively, via their Fourier transforms:
\[
\begin{align*}
\overline{(\mathcal{H}_\mathbb{R})}(\xi) &:= -i\text{sgn}(\xi)\hat{J}(\xi), \quad J \in L^2(\mathbb{R}), \\
\overline{(\mathcal{H}_\mathbb{Z})}(k) &:= -i\text{sgn}(k)\hat{f}(k), \quad f \in l^2(\mathbb{Z}).
\end{align*}
\]
Note that $\mathcal{H}_\mathbb{R}J$ and $\mathcal{H}_\mathbb{Z}f$ can be defined directly on the physical space as
\[
\begin{align*}
(\mathcal{H}_\mathbb{R}J)(y) &= \frac{1}{\pi} \text{p.v.} \int_\mathbb{R} dy' \frac{J(y')}{y - y'}, \\
(\mathcal{H}_\mathbb{Z}f)_x &= \begin{cases} 
\frac{1}{\pi} \sum_{z \text{ odd}} f(z) & x : \text{even}, \\
\frac{2}{\pi} \sum_{z \text{ even}} f(z) & x : \text{odd},
\end{cases}
\end{align*}
\]
where the integral is understood in the sense of the Cauchy principal value.

For two functions $f, g$ defined on a common domain $A$, we write $f \lesssim g$ or $g \gtrsim f$ if there exists some positive constant $C > 0$ such that $f(a) \leq Cg(a)$ for any $a \in A$. If $f \lesssim g$ and $g$ is a positive constant, then we also write $\sup_{a \in A} f(a) \lesssim 1$.

For two functions $f, g$ defined on a common open subset $A \subset \mathbb{R}$ and a number $y_0 \in \overline{A}$, where $\overline{A}$ is the closure of $A$, we write $f(y) \sim g(y)$ as $y \to y_0$ or $f(y) = O(g(y))$ as $y \to y_0$ if
\[
0 < \lim_{y \to y_0} \left| \frac{f(y)}{g(y)} \right| < \infty.
\]
In addition, we write $f(y) = o(g(y))$ as $y \to y_0$ if
\[
\lim_{y \to y_0} \left| \frac{f(y)}{g(y)} \right| = 0.
\]

3. The dynamics

In this section we define harmonic chains with noise and long-range interactions. Since we analyse the system with finite total energy, it is appropriate for us to define the dynamics through the wave functions $\{\hat{\psi}(k, t); \ k \in \mathbb{T}, \ t \geq 0\}$ as $L^2(\mathbb{T})$ solution of the stochastic differential equation (3.4). Then we can reconstruct the classical variables $\{p_x(t), q_x(t); \ x \in \mathbb{Z}, \ t \geq 0\}$ from $\{\hat{\psi}(k, t); \ k \in \mathbb{T}, \ t \geq 0\}$, and then we define the energy $\{e_x(t); \ x \in \mathbb{Z}, \ t \geq 0\}$ and the generalised tension $\{l_x(t); \ x \in \mathbb{Z}, \ t \geq 0\}$. However, it may be difficult to understand the physical meaning of the important functions from (3.4) (e.g. $\omega(k)$ and $R(k)$). To clarify the meaning of the feature values, we first give a formal description of the dynamics in terms of $\{p_x(t), q_x(t); \ x \in \mathbb{Z}, \ t \geq 0\}$ in the introduction.

3.1. Formal description of the dynamics and corresponding wave function

We consider the Hamiltonian dynamics perturbed by momentum-exchange noise which acts only on momenta locally, that is, the dynamics is governed by the following stochastic
dynamical system:
\[
\begin{align*}
\frac{dq_x(t)}{dt} &= p_x(t) dt \\
\frac{dp_x(t)}{dt} &= \left(-\alpha_x q_x(t) - \frac{\gamma}{2}(\beta * p)_x(t)\right) dt + \sqrt{\gamma} \sum_{z=-1,0,1} (Y_{x+z} p_x(t)) dw_{x+z},
\end{align*}
\]
(3.1)
where \(\alpha_x\) is defined by (1.3), \(\gamma \geq 0\) is the strength of the noise and \(Y_x, x \in \mathbb{Z}\) are operators defined as
\[
Y_x := (p_x - p_{x+1}) \partial_{p_{x-1}} + (p_{x+1} - p_{x-1}) \partial_{p_x} + (p_{x-1} - p_x) \partial_{p_{x+1}}.
\]
In addition, \(\{w_x(t); x \in \mathbb{Z}, t \geq 0\}\) are i.i.d. one-dimensional standard Brownian motions, and \(\beta_x\) is defined as
\[
\beta_x := \begin{cases} 
6, & x = 0, \\
-2, & x = \pm 1, \\
-1, & x = \pm 2, \\
0, & \text{otherwise.}
\end{cases}
\]
In other words, \(\{p_x(t), q_x(t); x \in \mathbb{Z}, t \geq 0\}\) is a Markov process generated by \(A + \gamma S\), where
\[
A := \sum_{x \in \mathbb{Z}} \left(p_x \partial_{q_x} - \sum_{z \in \mathbb{Z}} \alpha_{x-z} q_z\right), \quad S := \sum_{x \in \mathbb{Z}} (Y_x)^2.
\]
Note that \(Y_x\) conserves \(p_{x-1} + p_x + p_{x+1}\) and \(p_{x-1}^2 + p_x^2 + p_{x+1}^2\), and the terms \((\beta * p)_x, x \in \mathbb{Z}\) correspond to the first-order differential operators in \(S\).

Assume for a moment that \(\gamma = 0\), that is, the system is deterministic harmonic chain. Taking the Fourier transform of both sides of (3.1), we have
\[
\frac{d}{dt} \begin{pmatrix} \hat{q}(k,t) \\ \hat{p}(k,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\hat{\alpha}(k) & 0 \end{pmatrix} \begin{pmatrix} \hat{q}(k,t) \\ \hat{p}(k,t) \end{pmatrix}.
\]
The eigenvalues of the above matrix are \(\pm i\omega(k)\), where
\[
\omega(k) := \sqrt{\hat{\alpha}(k)},
\]
and corresponding eigenvectors \(\{\hat{\psi}(k,t), \hat{\phi}(k,t); k \in \mathbb{T}, t \geq 0\}\), called wave functions, are given by
\[
\hat{\psi}(k,t) := \omega(k) \hat{q}(k,t) + i\hat{p}(k,t).
\]
(3.3)
Actually, we can check that
\[
\frac{d}{dt} \hat{\psi}(k,t) = -i\omega(k) \hat{\psi}(k,t).
\]
For $\gamma > 0$, we also define wave functions $\{\hat{\psi}(k, t); k \in \mathbb{T}, t \geq 0\}$ as (3.3). Then from (3.1), the time evolution of $\{\hat{\psi}(k, t); k \in \mathbb{T}, t \geq 0\}$ are given by

$$
\begin{aligned}
\mathrm{d}\hat{\psi}(k, t) &= \left[-\mathbf{i}k\hat{\psi}(k, t)\mathrm{d}t - \gamma R(k) \left(\hat{\psi}(k, t) - \hat{\psi}^*(\mathbf{0}, t)\right)\right] \mathrm{d}t \\
&\quad + \mathbf{i}\sqrt{\gamma} \int r(k, k') \left(\hat{\psi}(k - k', t) - \hat{\psi}^*(k - k', t)\right) \mathrm{d}B(k', \mathrm{d}t),
\end{aligned}
$$

(3.4)

where

$$
r(k, k') := 2 \sin^2(\pi k) \sin \left(2\pi(k - k')\right) + 2 \sin(2\pi k) \sin^2 \left(\pi(k - k')\right),
$$

$$
R(k) := \frac{\beta(k)}{4} = 2 \sin^2(\pi k) + \frac{3}{2} \sin^2(2\pi k),
$$

(3.5)

and $B(\mathrm{d}k, \mathrm{d}t)$ is a cylindrical Wiener process on $\mathbb{L}^2(\mathbb{T})$ defined as

$$
B(\mathrm{d}k, \mathrm{d}t) := \sum_{x \in \mathbb{Z}} e^{2\pi i x k} \mathrm{d}k \omega_x(\mathrm{d}t).
$$

### 3.2. Rigorous definition of the dynamics and generalised tension

Now we give the rigorous definition of our dynamics. Let $(E, \mathcal{F}, \mathbb{P})$ be a probability space and assume that the cylindrical Wiener process $B(\mathrm{d}k, \mathrm{d}t)$ is defined on $(E, \mathcal{F}, \mathbb{P})$. For any $T > 0$, we introduce a Banach space $\mathcal{H}_T$ defined as

$$
\mathcal{H}_T := \left\{ f : \mathbb{T} \times [0, T] \rightarrow \mathbb{C}; \|f\|_{\mathcal{H}} := \left( \sup_{0 \leq t \leq T} \mathbb{E} \left[ \|f(t)\|^2_{\mathbb{L}^2(\mathbb{T})} \right] \right)^{1/2} < \infty \right\}.
$$

Then we define $\hat{\psi}(k, t) \in \mathcal{H}_T; k \in \mathbb{T}, 0 \leq t \leq T$ as the solution of (3.4) with initial distribution $\mu_0$, where $\mu_0$ is an arbitrary probability measure on $\mathbb{L}^2(\mathbb{T})$. The existence and uniqueness of the solution is proved by using classical fixed point theorem, see [21, appendix A]. Note that from (3.4) and Itô’s formula we have the energy conservation law

$$
\int_{\mathbb{T}} \mathrm{d}k \mathbb{E}_{\mu_0} \left[ |\hat{\psi}(k, t)|^2 \right] = \int_{\mathbb{T}} \mathrm{d}k \mathbb{E}_{\mu_0} \left[ |\hat{\psi}(k)|^2 \right],
$$

(3.6)

where $\mathbb{E}_{\mu_0}$ is the expectation with respect to the dynamics which starts from $\mu_0$ and $\mathbb{E}_{\mu_0}$ is the expectation with respect to $\mu_0$.

From $\{\hat{\psi}(k, t); k \in \mathbb{T}, t \geq 0\}$, we can reconstruct the classical variables $\{p_x(t), q_x(t); x \in \mathbb{Z}, t \geq 0\}$ as

$$
\begin{aligned}
p_x(t) &:= \int_{\mathbb{T}} \mathrm{d}k \ e^{2\pi i k x} \hat{\psi}(k, t) \\
q_x(t) &:= \int_{\mathbb{T}} \mathrm{d}k \ e^{2\pi i k x} \hat{\psi}(k, t),
\end{aligned}
$$

$$
\begin{aligned}
p(k, t) &:= \frac{1}{2i} (\hat{\psi}(k, t) - \hat{\psi}(\mathbf{0}, t)^*) \\
q(k, t) &:= \frac{1}{2\omega(k)} (\hat{\psi}(k, t) + \hat{\psi}^*(\mathbf{0}, t)),
\end{aligned}
$$

(3.7)
where \( \omega(k) \) is defined in (3.2). Note that \( \hat{\omega}(k) \) may not be well-defined as an \( L^2(\mathbb{T}) \) element, because \( \omega(\cdot)^{-1} \notin L^2(\mathbb{T}) \). However, we do not use \( q \) variable to prove our results, so we do not have to worry about the well-definedness.

Now we introduce the dual variable of \( \{p_x(t); x \in \mathbb{Z}, t \geq 0\} \), called the generalised tension \( \{l_x(t); x \in \mathbb{Z}, t \geq 0\} \) which is given by

\[
l_x(t) := \int_\mathbb{T} dk\ e^{2\pi ikx} \hat{q}(k, t),
\]

\[
\hat{l}(k, t) := \frac{i \text{sgn}(k)}{2} \left( \hat{\psi}(k, t) + \hat{\psi}(-k, t)^* \right).
\]

**Remark 3.1.** As \( r_x, x \in \mathbb{Z} \) is a function of \( q_x, x \in \mathbb{Z} \), we can formally write \( l_x, x \in \mathbb{Z} \) as a non-local function of \( q_x, x \in \mathbb{Z} \). Actually, from (3.7) we have

\[
\hat{l}(k) = l_\omega(k) \text{sgn}(k) \hat{q}(k)
\]

and thus we obtain

\[
l_x = \mathcal{H}_\omega(\hat{\omega} * q)_x.
\]

We also note that the Fourier transform of the tension \( \hat{r}(k), k \in \mathbb{T} \) is defined by using wave functions as

\[
\hat{r}(k) := \frac{1 - e^{-2\pi ik}}{2\omega(k)} \left( \hat{\psi}(k, t) + \hat{\psi}^*(-k, t) \right),
\]

and thus we have

\[
\hat{r}(k) = \frac{1 - e^{-2\pi ik}}{i \text{sgn}(k)\omega(k)} \hat{l}(k).
\]  

(3.8)

Next we define the energy \( \{e_x(t); x \in \mathbb{Z}, t \geq 0\} \) as a function of \( \{\hat{\psi}(k, t); k \in \mathbb{T}, t \geq 0\} \). For harmonic chains, \( e_x(t) \) is usually defined as

\[
e_x(t) = \frac{1}{2} p_x^2(t) - \frac{1}{4} \sum_{x,l} \alpha_{x-l}(q_x(t) - q_{x-l})^2.
\]

As mentioned above, \( \{q_x(t); x \in \mathbb{Z}, t \geq 0\} \) may not be well-defined, but we can define \( \sum_{x,l} \alpha_{x-l}(q_x(t) - q_{x-l})^2 \) directly as a function of \( \{\hat{\psi}(k, t); k \in \mathbb{T}, t \geq 0\} \) by using the following argument: suppose that \( \{q_x; x \in \mathbb{Z}\} \) is an \( \ell^2(\mathbb{Z}) \) element, then the Fourier transform of \( \sum_{x \in \mathbb{Z}} \alpha_{x-l}(q_x - q_{x-l})^2 \) is equal to

\[
\int d\bar{k}' \hat{\alpha}(k') \hat{\psi}(k' - k') \hat{\psi}^*(-k') (\hat{\psi}(k') + \hat{\psi}^*(-k')) \left( \hat{\psi}(k - k') + \hat{\psi}(k' - k)^* \right),
\]

where

\[
F(k, k') := \frac{\hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k')}{\omega(k)\omega(k')}.
\]
Therefore we can define \( \sum_{\xi \in \mathbb{Z}} \alpha_{\xi} (q_{\xi} - q_{\xi}^0)^2 \) as the Fourier coefficient of the above integration, that is,
\[
\sum_{\xi \in \mathbb{Z}} \alpha_{\xi} (q_{\xi} - q_{\xi}^0)^2 := \frac{1}{4} \int_{\mathbb{T}^2} dk \, dk' \, e^{2\pi i k x} F(k - k', k') \\
\times \left( \hat{\psi}(k') + \hat{\psi}(-k') \right) \left( \hat{\psi}(k - k') + \hat{\psi}(k - k') \right),
\]
and thus the energy of the particle \( x \in \mathbb{Z} \) is given by
\[
\varepsilon_{\xi}(t) := \frac{1}{2} \hat{p}_{\xi}^2(t) - \frac{1}{16} \int_{\mathbb{T}^2} dk \, dk' \, e^{2\pi i k x} F(k - k', k') \\
\times \left( \hat{\psi}(k', t) + \hat{\psi}(-k', t) \right) \left( \hat{\psi}(k - k', t) + \hat{\psi}(k - k', t) \right).
\]
Note that from lemmas A.1 and A.2 we see that \( F(k, k') \) is uniformly bounded, and thus \( \varepsilon_{\xi}(t) \) is well-defined for any \( t \geq 0 \).

4. Main results

4.1. Superballistic scaling limit

Let \( (\mu_t)_{0 < t < 1} \) be a family of probability measures on \( \mathbb{L}^2(\mathbb{T}) \). We define \( \{\hat{\psi}(k, t) = \hat{\psi}_t(k, t); k \in \mathbb{T}, t \in \mathbb{R}_{\geq 0} \} \) as the solution of (3.4) with initial condition \( \mu_k \). Denote by \( E_{\varepsilon} \) the expectation with respect to the dynamics which starts from \( \mu_k \). We assume that \( \{(p_\varepsilon(t), l_\varepsilon(t)) = (p_\varepsilon(\hat{\psi})(t), l_\varepsilon(\hat{\psi})(t)); x \in \mathbb{Z}, t \geq 0 \} \) satisfies the following initial condition, called phononic or mechanical type condition (cf [15, definition 2.3]): there exists some \( p_0, l_0 \in C_0^\infty(\mathbb{R}) \) such that
\[
\lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} E_{\mu_k} \left[ |p_\varepsilon(k) - p_0(k\varepsilon)|^2 + |l_\varepsilon(k\varepsilon) - l_0(k\varepsilon)|^2 \right] = 0. \tag{4.1}
\]
Note that (4.1) is equivalent to the following condition
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T} \times \mathbb{T}} dk E_{\mu_k} \left[ |\varepsilon \hat{p}(\varepsilon k) - \tilde{p}_0(\varepsilon k)|^2 + |\varepsilon \tilde{l}(\varepsilon k) - \tilde{l}_0(\varepsilon k)|^2 \right] = 0, \tag{4.2}
\]
where we regard \( \tilde{p}_0 \) and \( \tilde{l}_0 \) as functions on \( \mathbb{T}_\varepsilon \times \mathbb{T} \), as explained in (2.1). We check the equivalence in appendix B. In addition, from (3.6) and (4.1) we have
\[
\int_{\mathbb{T}} dk E_{\varepsilon} [|\hat{p}(k, t)|^2 + |\hat{l}(k, t)|^2] = \int_{\mathbb{T}} dk E_{\mu_k} [|\hat{p}_0(k)|^2 + |\hat{l}_0(k)|^2],
\]
and
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}} dk E_{\varepsilon} [|\hat{p}(k, t)|^2 + |\hat{l}(k, t)|^2] = \int_{\mathbb{R}} d\xi |\hat{p}_0(\xi)|^2 + |\hat{l}_0(\xi)|^2.
\]
Now we state one of our main results. Let us recall that the time scaling \( f_\theta(\varepsilon) \) is defined in (1.4).

**Theorem 4.1.** Suppose that \( \theta > 1, \gamma \geq 0 \) and (4.1). For any \( t \geq 0 \), we have
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T} \times \mathbb{T}} dk E_{\varepsilon} \left[ |\varepsilon \hat{p}(\varepsilon k, t) - \tilde{p}_0(\varepsilon k, t)|^2 + |\varepsilon \tilde{l}(\varepsilon k, t) - \tilde{l}_0(\varepsilon k, t)|^2 \right] = 0.
\]
and consequently we have

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_\varepsilon \left[ \left| \left( \frac{t}{j_0(\varepsilon)} \right) \right| - p(\varepsilon x, t) \right|^2 + \left| \left( \frac{t}{j_0(\varepsilon)} \right) - l(\varepsilon x, t) \right|^2 \right] = 0,
\]

where \((p(y, t), l(y, t))\) is the solution of (1.6) with initial condition \((p_0(y), l_0(y))\).

From theorem 4.1, we obtain the weak convergence of the empirical measure of \((p_x(t), l_x(t))\).

**Corollary 4.1.** Suppose that \(\theta > 1, \gamma \geq 0\) and (4.1). For any \(t \geq 0\) and \(J_i \in C_0^\infty(\mathbb{R}), i = 1, 2\) we have (1.5).

Furthermore, from (3.8) and lemma A.1, we also obtain the scaling limit of the tension \(r_x(t)\), which shows that \(r_x\) and \(l_x\) are macroscopically consistent if \(\theta > 3\).

**Corollary 4.2.** Suppose that \(\theta > 3, \gamma \geq 0\) and (4.1). For any \(t \geq 0\), we have

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}_{-1}} d\varepsilon \mathbb{E}_\varepsilon \left[ \left| \left( \frac{t}{j_0(\varepsilon)} \right) - \varepsilon \right| \right] = 0,
\]

and consequently we have

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_\varepsilon \left[ \left| \left( \frac{t}{j_0(\varepsilon)} \right) - \frac{1}{\sqrt{C_1(\theta)(\varepsilon x, t)}} \right|^2 \right] = 0.
\]

**Remark 4.1.** Note that from (3.8) and (4.1) and lemma A.1, for \(1 < \theta \leq 3\), we have

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_\varepsilon \left[ |\tau_x|^2 \right] = \lim_{\varepsilon \to 0} \int_{\mathbb{T}_{-1}} d\varepsilon \mathbb{E}_\varepsilon \left[ |\varepsilon(t)|^2 \right] \leq \lim_{\varepsilon \to 0} \int_{\mathbb{T}_{-1}} d\varepsilon \mathbb{E}_\varepsilon \left[ |\tilde{l}(k)|^2 \right] \times \begin{cases} |\varepsilon k|^{3-\theta} & 1 < \theta < 3, \\ \log (|\varepsilon k|) & \theta = 3, \end{cases}
\]

where we use the boundedness \(|1 - e^{-2\pi i k}| \leq \omega(k) \) on \(\mathbb{T}\). Therefore, under the conditions (4.1) and \(1 < \theta \leq 3\), the macroscopic profile of \(\tau_x, x \in \mathbb{Z}\) vanishes. We also note that (4.1) means that the total energy is macroscopically finite and that the variance of \(\tilde{\psi}(k)\) around the macroscopic profile vanishes.

**Remark 4.2.** The fractional Laplacians and the Hilbert transform are Fourier multipliers:

\[
(-\Delta)^a f(\xi) = -2\pi \xi^{2a} \tilde{f}(\xi) \quad 0 < a \leq 1,
\]

\[
(\mathcal{H}_a f)(\xi) = -i \text{sgn}(\xi) \tilde{f}(\xi).
\]
By using the above properties we easily see that
\[
\begin{align*}
\partial_t \tilde{p}(\xi, t) &= \sqrt{C_1(\theta)} 2\pi \xi \tilde{I}(\xi, t) \\
\partial_t \tilde{I}(\xi, t) &= \sqrt{C_1(\theta)} 2\pi \xi \tilde{p}(\xi, t)
\end{align*}
\]
and
\[
\partial^2_p \tilde{p}(\xi, t) = \begin{cases} -C_1(\theta)|2\pi \xi|^{\theta-1}\tilde{p}(\xi, t) & 1 < \theta \leq 3, \\
-C_1(\theta)|2\pi \xi|^2\tilde{p}(\xi, t) & \theta > 3. \end{cases}
\]
(4.3)
Note that from (4.3) we can derive (1.8) by using the inverse Fourier transform.

**Remark 4.3.** At a slightly longer time scale than \( j_0(\varepsilon)^{-1} \), the scaled empirical measures of \((p, t), (l, t)\) converge to stationary solutions of (1.6), \( p(y, t) = l(y, t) = 0 \), \( y \in \mathbb{R}, t \geq 0 \). In other words, the macroscopic profile of \((p, t), (l, t)\) vanishes. See section 5.3.

Next we consider the scaling limit of the energy.

**Theorem 4.2.** Suppose that \( \theta > 1, \gamma \geq 0 \) and (4.1). For any \( t \geq 0 \) and \( J \in C^\infty_0(\mathbb{R}) \), we have
\[
\lim_{\varepsilon \to 0} \int e(y, t)J(y) = \int e(y, t)J(y),
\]
where \( e(y, t) \) is defined in (1.9) and \((p(y, t), l(y, t))\) is the solution of (1.6) with initial condition \((p_0(y), l_0(y))\).

**Remark 4.4.** From theorem 4.2 we see that when \( \theta \geq 3 \) the quantities \( e_{t}(t) \) and \(|\psi_{s}(t)|^2\) coincide macroscopically, that is, for any \( t \geq 0 \) and \( J \in C^\infty_0(\mathbb{R}) \) we have
\[
\lim_{\varepsilon \to 0} \int e_{t}(t)J(y) = \int e_{t}(t)J(y) = 0.
\]
(4.5)
However, if \( 1 < \theta < 3 \), then (4.5) does not hold. On the other hand, in [21, proposition 4.1], we show that under the thermal type condition, an analogue of (4.5) holds for \( \theta > 2 \). This difference is caused by initial conditions and from the above we see that \(|\psi_{s}(t)|^2\) may not coincide the phononic energy macroscopically.

**4.2. Fluctuations of normal modes**

In this subsection we consider the normal modes of the system (1.6), \( f^\pm(y, t) \), an analogue of the Riemann invariants, which are defined as
\[
f^\pm(y, t) := p(y, t) \pm l(y, t).
\]
(4.6)
The time evolution of the normal modes is given by
\[
\partial_t f^\pm(y, t) = \pm \sqrt{C_1(\theta)}(D_{0} f^\pm)(y, t),
\]
where \( D_{0} f^\pm \) is the Riemann invariants of the normal modes.
that is, \( f^\pm(y, t) \) are the eigenvectors and \( \pm \sqrt{C_1(\theta)}D_\theta \) are the eigenvalues of the system \((1.6)\). For the sake of clarity, at first we consider the case \( \theta > 4 \). Then the characteristics of \((1.6)\) are given by straight lines \( y = \pm \sqrt{C_1(\theta)}t \) and thus we have

\[
f^\pm(y, t) = f^\pm_0(y = \pm \sqrt{C_1(\theta)}t),
\]

where \( f^\pm_0 := f^\pm(y, 0) \). This observation motivates us to study fluctuations of microscopic normal modes around macroscopic characteristics. Considering the above, we introduce the microscopic normal modes \( f^\pm_\theta(t) \), \( x \in \mathbb{Z} \), \( t \geq 0 \), defined as

\[
f^\pm_\theta(t) := p_\theta(t) = \pm \sqrt{C_1(\theta)} J(\pm \sqrt{C_1(\theta)} x)
\]

From theorem 4.1 we see that

\[
\lim_{\theta \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ f^\pm_\theta \left( \frac{t}{\varepsilon^2} \right) \right] J(\varepsilon x) = \int_\mathbb{R} dy f^\pm_\theta(y, t) J(y),
\]

for any \( \theta > 1 \), \( t \geq 0 \) and \( J \in C_0^\infty(\mathbb{R}) \). To obtain the fluctuations around the characteristics at the diffusive time scaling, we need to recenter the dynamics by shifting the origin to \( \pm \sqrt{C_1(\theta)} t/\varepsilon \) because the space is scaled by \( \varepsilon \). Then we obtain the following scaling limit, which is the special case of theorem 4.3 stated in the end of this subsection.

**Corollary 4.3.** Assume that \( \theta > 4 \), \( \gamma > 0 \) and \((4.1)\). Then for any \( t \geq 0 \) and \( J \in C_0^\infty \) we have

\[
\lim_{\theta \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ f^\pm_\theta \left( \frac{t}{\varepsilon^2} \right) \right] J \left( \varepsilon x + \sqrt{C_1(\theta)} \left( \frac{t}{\varepsilon^2} \right) \right)
\]

\[
= \int_\mathbb{R} dy \int d\xi e^{2\pi i k y} e^{-\frac{2\pi i}{\varepsilon} t} \left( J_0^+ + J_0^- \right)(\xi) J(y),
\]

where \( J_0^\pm(y) := p_\theta(y) = \pm \sqrt{C_1(\theta)} J(\pm \sqrt{C_1(\theta)} x) \).

**Remark 4.5.** The above result is essentially the same as that of \cite[theorem 3.4]{15}. Note that the ballistic transport of the phononic terms with diffusive fluctuations and 3/4-superdiffusion of thermal energy agree with the theoretical prediction of \cite{20}. We also note that for the polynomial decay model 3/4-superdiffusion of thermal energy is proved in \cite{21} when \( \theta > 3 \). Now we generalise the above argument to the case \( 2 < \theta \leq 4 \). The macroscopic and microscopic normal modes are given by \((4.6)\) and \((4.7)\). Let \( \{S^\pm_\theta(t) : t \in \mathbb{R}, \theta > 1 \} \) be the semigroups corresponding to \( \pm \sqrt{C_1(\theta)} D_\theta \) defined via its Fourier transform:

\[
(S^\pm_\theta(t)g)(\xi) := \begin{cases} 
\varepsilon^{\frac{1}{2}} e^{\frac{\pm i \sqrt{C_1(\theta) 2\pi |\xi|0} t}{\varepsilon}} g(\xi) & 1 < \theta \leq 3, \\
\varepsilon^{\frac{1}{2}} e^{\frac{\pm i \sqrt{C_1(\theta) 2\pi |\xi|0} t}{\varepsilon}} g(\xi) & \theta > 3,
\end{cases}
\]

for any \( g \in S(\mathbb{R}) \). Note that the right-hand side of \((4.8)\) can be rewritten as

\[
\varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ f^\pm_\theta \left( \frac{t}{\varepsilon^2} \right) \right] \left( S^\pm_\theta \left( \frac{t}{\varepsilon^2} \right) J(\varepsilon x) \right).
\]

Therefore by using these semigroups we can recenter the dynamics for any \( \theta > 1 \), but the correct time scaling is not diffusive if \( \theta < 4 \).
Recall that \((m_\theta(\varepsilon), n_\theta(\varepsilon))\) are defined in (1.11).

**Theorem 4.3.** Assume that \(\theta > 2, \gamma > 0\) and (4.1). Then for any \(t \geq 0\) and \(J \in C_0^\infty(\mathbb{R})\) we have
\[
\lim_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}} \mathbb{E}_\varepsilon \left[ \int_0^1 \left( \frac{t}{m_\theta(\varepsilon)} \right) \left( S^+ \left( \frac{t}{m_\theta(\varepsilon)} \right) J \right) (\varepsilon x) \right] = \int_{\mathbb{R}} dy F^\pm(y, t) J(y),
\]
where \(F^\pm(y, t)\) is the solution of (1.12) with initial condition \(F^\pm(y, 0) := p_0(y) \pm l_0(y)\).

**Remark 4.6.** We see that if \(2 < \theta < 4\) then the right-hand side of (1.12) does not depend on the strength of the noise \(\gamma\), and thus the fluctuations of normal modes are determined by the interaction potential \(\alpha\). On the other hand, if \(\theta > 4\) then the fluctuations do not depend on \(\alpha\).

In addition, when \(2 < \theta < 4\) we have the following stronger result, which means the law of large numbers for the empirical measures of the normal modes along the characteristics.

**Theorem 4.4.** Assume that \(2 < \theta < 4, \gamma \geq 0\) and (4.1). Then for any \(t \geq 0\) and \(J \in C_0^\infty(\mathbb{R})\) we have
\[
\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon \left[ \int_{\mathbb{R}} \left( \frac{t}{m_\theta(\varepsilon)} \right) \left( S^+ \left( \frac{t}{m_\theta(\varepsilon)} \right) J \right) (\varepsilon x) - \int_{\mathbb{R}} dy F^\pm(y, t) J(y) \right] = 0,
\]
where \(F^\pm(y, t)\) is the solution of (1.12) with initial condition \(F^\pm(y, 0) := p_0(y) \pm l_0(y)\).

**Remark 4.7.** The specific choice of the interaction potential (1.3) is not so important to derive the scaling limits. Actually, for a given interaction potential, if one can explicitly calculate the first and second order terms of its Fourier transform as \(k \to 0\), then one can derive the scaling limits in the same way as in this paper, see the proofs and appendix A. Note that the exponents appearing in the limiting equations vary depending on the first and second order of the Fourier transform of the interaction potential being treated.

**Remark 4.8.** There are several options for stochastic noise with appropriate properties. For example, we can choose the jump-type exchange noise defined as follows: define an operator \(S'\) as
\[
(S'f)(p) := \sum_{x \in \mathbb{Z}} \left( f(p_{x+1}) - f(p) \right)
\]
for any bounded local function \(f\), i.e., bounded function that depend on the configurations only through a finite set of coordinates. Here \(p_{x+1}^{\pm, x+1}\) is defined as
\[
p_{x+1}^{\pm, x+1} := \begin{cases} \ p_{x+1} & \varepsilon = x, \\ \ p_x & \varepsilon = x + 1, \\ \ p_z & \text{otherwise}. \end{cases}
\]
Then, we can consider the Markov process \(\{p_x(t), q_x(t); t \geq 0\}\) generated by \(A + \gamma S'\). Note that, for example, [11] and the review [17, chapter 5] employ the jump-type noise described above. As we will see in the proofs, the asymptotic behaviours of \(\omega(k)\) and \(R_\delta(k) := R(k)\) as \(k \to 0\) determine the macroscopic behaviour of the dynamics [21, introduction], where \(R(k)\) is defined in (3.5) and \(R(k) \sim k^2\) as \(k \to 0\). For the polynomial decay model with the jump-type
exchange noise, the time evolution law of the wave functions is similar to (3.4), and we have \( R_{\beta}(k) \sim k^2 \) as \( k \to 0 \). See also [15, remark] at the end of [15, section 3.3] for exponential-decay models with jump-type noises. More generally, if stochastic noise \( \mathcal{S} \) is given and the asymptotic behaviour of the function \( R_{\beta}(k) \) is the same as that of \( R(k) \), then the same result as in this paper can be obtained.

5. Proof of theorem 4.1

In this section we prove theorem 4.1 by following the strategy presented in [15, section 4]. Since if \( 1 < \theta \leq 3 \) then the asymptotic behaviour of \( \hat{\alpha}(k) \), \( k \to 0 \) is different from that of exponential-decay models, we need some modification. The asymptotic behaviour of \( \hat{\alpha}(k) \), \( k \to 0 \) is computed in appendix A.

5.1. Mean dynamics

First we define the scaled mean dynamics \( \{ p_{x,t}(t), l_{x,t}(t); x \in \mathbb{Z}, t \geq 0 \} \) as

\[
p_{x,t}(t) := \mathbb{E}_c \left[ p_x \left( \frac{t}{j_0(\xi)} \right) \right], \quad l_{x,t}(t) := \mathbb{E}_c \left[ l_x \left( \frac{t}{j_0(\xi)} \right) \right].
\]

We also introduce the Fourier transform of the mean dynamics \( \{ \hat{p}_c(k,t), \hat{l}_c(k,t); k \in \mathbb{T}_c, t \geq 0 \} \), that is,

\[
\hat{p}_c(k,t) := \varepsilon \mathbb{E}_c \left[ \hat{p} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) \right], \quad \hat{l}_c(k,t) := \varepsilon \mathbb{E}_c \left[ \hat{l} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) \right].
\]

The following proposition states the scaling limit for the mean dynamics.

**Proposition 5.1.** Suppose that \( \theta > 1, \gamma \geq 0 \) and (4.1). For any \( t \geq 0 \), we have

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}_c} dk \left| \hat{p}_c(k,t) - \hat{p}(k,t) \right|^2 + \left| \hat{l}(k,t) - \hat{l}(k,t) \right|^2 = 0,
\]

and consequently we have

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{x \in \mathbb{Z}} \left( |p_{x,t}(t) - p(x,t)|^2 + |l_{x,t}(t) - l(x,t)|^2 \right) = 0.
\]

In the rest of this subsection we show that proposition 5.1 implies theorem 4.1. In section 5.2 we prove proposition 5.1.

5.1.1. Proof of theorem 4.1. First we define \( \hat{P}_c(k,t), \hat{L}_c(k,t), k \in \mathbb{T}, t \geq 0 \) as

\[
\hat{P}_c(k,t) := \varepsilon \hat{p} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) - \hat{p}_c(k,t) = \varepsilon \hat{p} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) - \mathbb{E}_c \left[ p_x \left( \frac{t}{j_0(\xi)} \right) \right],
\]

\[
\hat{L}_c(k,t) := \varepsilon \hat{l} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) - \hat{l}_c(k,t) = \varepsilon \hat{l} \left( \varepsilon k, \frac{t}{j_0(\xi)} \right) - \mathbb{E}_c \left[ l_x \left( \frac{t}{j_0(\xi)} \right) \right].
\]
From (3.6), for any $t \geq 0$ we have
\[
\int_{\mathbb{T}_e^{-1}} \, dkEe \left[ \left| \hat{e}(\varepsilon k, 0) \right|^2 + \left| \hat{e}(\varepsilon k, 0) \right|^2 \right] = \int_{\mathbb{T}_e^{-1}} \, dkEe \left[ \left| \hat{e}(\varepsilon k, t) \right|^2 + \left| \hat{e}(\varepsilon k, t) \right|^2 \right] = \int_{\mathbb{T}_e^{-1}} \, dk \left| \hat{P}_e(k, t) \right|^2 + \left| \hat{L}_e(k, t) \right|^2 + \int_{\mathbb{T}_e^{-1}} \, dk \left| \hat{P}_e(k, t) \right|^2 + \left| \hat{L}_e(k, t) \right|^2.
\]

By using (4.1) and proposition 5.1, we get
\[
\int_{\mathbb{R}} d\xi |\hat{P}(\xi, t)|^2 + \left| \hat{l}_0(\xi) \right|^2 = \lim_{\varepsilon \to 0} \int_{\mathbb{T}_e^{-1}} \, dkEe \left[ \left| \hat{P}_e(k, t) \right|^2 + \left| \hat{L}_e(k, t) \right|^2 \right] = \lim_{\varepsilon \to 0} \int_{\mathbb{T}_e^{-1}} \, dk \left| \hat{P}_e(k, t) \right|^2 + \left| \hat{L}_e(k, t) \right|^2.
\]

Hence from (3.6) we obtain
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}_e^{-1}} \, dkEe \left[ \left| \hat{P}_e(k, t) \right|^2 + \left| \hat{L}_e(k, t) \right|^2 \right] = 0. \tag{5.2}
\]

Combining (5.1) with (5.2), we have
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}_e^{-1}} \, dkEe \left[ \varepsilon \hat{p} \left( \varepsilon k, \frac{t}{\varepsilon} \right) \right] \left( \hat{l}_0(\xi) - \hat{l}(\xi, t) \right)^2 + \left| \hat{l}(\xi, t) \right|^2 = 0,
\]

and thus we established theorem 4.1.

5.2. Proof of proposition 5.1

From (3.4), the time evolution law of \( \left( \hat{p}_e(k, t), \hat{l}_e(k, t) \right) \) is given by
\[
\frac{d}{dt} \left( \begin{array}{c} \hat{p}_e(k, t) \\ \hat{l}_e(k, t) \end{array} \right) = A_e(k) \left( \begin{array}{c} \hat{p}_e(k, t) \\ \hat{l}_e(k, t) \end{array} \right), \tag{5.3}
\]

where
\[
A_e(k) := \frac{1}{\varepsilon} \left( \begin{array}{cc} -2\gamma R(\varepsilon k) & i \text{sgn}(k) \omega(\varepsilon k) \\ i \text{sgn}(k) \omega(\varepsilon k) & 0 \end{array} \right).
\]

Then we decompose \( A_e(k) \), \( k \in \mathbb{T}_e^{-1} \) into two parts as follows:
\[
A_e(k) = A_0(k) + B_{\varepsilon,0}(k), \quad A_0 = (A_0^{(i,j)})_{i,j=1,2}, \quad B_{\varepsilon,0} = (B_{\varepsilon,0}^{(i,j)})_{i,j=1,2},
\]

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where

\[
A^{(1,1)}(k) = A^{(2,2)}(k) = 0,
\]

\[
A^{(1,2)}(k) = A^{(2,1)}(k) := \begin{cases} 
\sqrt{C_1(\theta)i \operatorname{sgn}(k)|2\pi k|^{\frac{\theta-1}{2}}} & 1 < \theta \leq 3, \\
\sqrt{C_1(\theta)i(2\pi k)} & \theta > 3,
\end{cases}
\]

\[
B^{(i,j)}(k) := A^{(i,j)}(k) - A^{(i,j)}(k) \quad i, j = 1, 2.
\]

To prove proposition 5.1, we need the followinglemmas 5.1 and 5.2.

**Lemma 5.1.**

\[
C_* := \sup_{0 < \varepsilon < 1} \sup_{(k,t) \in \mathbb{R} \times \mathbb{R}} \|\exp(A_{\varepsilon}(k)t)\| < \infty,
\]

\[
D_* := \sup_{(k,t) \in \mathbb{R} \times \mathbb{R}} \|\exp(A_{\theta}(k)t)\| < \infty,
\]

where \(\| \cdot \|\) is the matrix norm defined as

\[
\|A\| := \sup_{x \in \mathbb{R}^2} \left| \frac{Ax}{x} \right|, \quad A \in M_2(\mathbb{R}).
\]

In addition, for any \(K > 0\) we have

\[
\sup_{|k| \leq K} \|B_{\varepsilon,\theta}(k)\| \leq c_{K,\theta} b_{\theta}(\varepsilon),
\]

where

\[
b_{\theta}(\varepsilon) := \begin{cases} 
\varepsilon & 1 < \theta < 2, \\
\varepsilon \log \varepsilon^{-1} & \theta = 2, \\
\varepsilon^{3-\theta} & 2 < \theta < 3 \\
(\log \varepsilon^{-1})^{-1} & \theta = 3, \\
\varepsilon^{\theta-3} & 3 < \theta < 4 \\
\varepsilon & \theta \geq 4,
\end{cases}
\]

and \(c_{K,\theta}\) is a positive constant which depends on \(K > 0, \theta > 1\).

**Lemma 5.2.** Suppose that \((\widehat{p}_c^{(0)}(k,t), \widehat{l}_c^{(0)}(k,t))\) is the solution of the following equation:

\[
\frac{d}{dt} \begin{pmatrix} \widehat{p}_c^{(0)}(k,t) \\ \widehat{l}_c^{(0)}(k,t) \end{pmatrix} = A_{\theta}(k) \begin{pmatrix} \widehat{p}_c^{(0)}(k,t) \\ \widehat{l}_c^{(0)}(k,t) \end{pmatrix},
\]

\[
\begin{pmatrix} \widehat{p}_c^{(0)}(k,0) \\ \widehat{l}_c^{(0)}(k,0) \end{pmatrix} = \begin{pmatrix} \hat{p}_c(k,0) \\ \hat{l}_c(k,0) \end{pmatrix}.
\]
Then for any $K > 0$, we have
\[
\lim_{\varepsilon \to 0} \int_{|k| \leq K} |k| \left| \hat{p}_\varepsilon(k, t) - \hat{p}_0(k, t) \right|^2 + \left| \hat{q}_\varepsilon(k, t) - \hat{q}_0(k, t) \right|^2 = 0.
\]

From now on we prove lemmas 5.1 and 5.2, and then we verify proposition 5.1 by using these lemmas.

5.2.1. Proof of lemma 5.1.

**Proof.** Since the eigenvalues of $A_\theta(k)$ are imaginary for every $k \in \mathbb{T}$, we have $D_\varepsilon < \infty$. The eigenvalues of $A_\varepsilon(k)$ are
\[
-\frac{1}{j_\theta(\varepsilon)} \left( \gamma R(\varepsilon k) \pm \sqrt{\gamma^2 R^2(\varepsilon k) - \hat{\alpha}(\varepsilon k)} \right).
\]

Since $R(\cdot)$ and $\hat{\alpha}(\cdot)$ are positive, the real parts of the eigenvalues of $A_\varepsilon(k)$ are negative. Therefore we have $C_\varepsilon < \infty$.

Next we consider the order of $\sup_{|k| \leq K} \| B_{\varepsilon, \theta}(k) \|$. From lemma A.1 we obtain
\[
|B_{\varepsilon}^{(1,2)}(k)| = \begin{cases} 
\frac{C_1(\theta)2\pi k^{\theta-1} - \hat{\alpha}(\varepsilon k)j_\theta(\varepsilon)}{i \, \text{sgn}(k)\omega(\varepsilon k)j_\theta(\varepsilon) - \sqrt{C_1(\theta)}} & 1 < \theta < 3, \\
\frac{i \, \text{sgn}(k)\omega(\varepsilon k)j_\theta(\varepsilon) - \sqrt{C_1(\theta)}}{C_1(\theta)2\pi k^{\theta-1} - \hat{\alpha}(\varepsilon k)} & \theta > 3,
\end{cases}
\]

and
\[
|B_{\varepsilon}^{(1,1)}(k)| \lesssim \varepsilon^2 j_\theta(\varepsilon) \lesssim \varepsilon
\]
on $|k| \leq K$. Hence we complete the proof of this lemma. \qed
5.2.2. Proof of lemma 5.2.

**Proof.** Fix a positive constant \( K > 0 \) and \( 0 < \varepsilon < 1 \) such that \( K < \frac{1}{27} \). From (5.3) and the decomposition of \( A_l(k) \), we have

\[
\frac{d}{dt} \left( \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right) = A_d(k) \left( \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right) + B_{c, d}(k) \left( \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right).
\]

By using Duhamel’s formula, we have

\[
\left( \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right) \left( \hat{l}_c(k, t) - \hat{l}_c^{(0)}(k, t) \right) = \int_0^t ds \exp ((t - s)A_d(k)) B_{c, d}(k) \left( \hat{p}_c(k, s) - \hat{p}_c^{(0)}(k, s) \right).
\]

From (3.6) and lemma 5.1, we have

\[
\int_{|k| \leq K} dk \left| \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right|^2 + \left| \hat{l}_c(k, t) - \hat{l}_c^{(0)}(k, t) \right|^2 \\
\leq t \int_{|k| \leq K} dk \int_0^t ds \left| \exp ((t - s)A_d(k)) B_{c, d}(k) \left( \hat{p}_c(k, s) - \hat{p}_c^{(0)}(k, s) \right) \right|^2 \\
\leq t(b_0(\varepsilon))^2 \int_0^t ds \int_{\mathbb{R}} dk \int_{\mathbb{R}^3} \left| \hat{\psi} \left( k, \frac{s}{b_0(\varepsilon)} \right) \right|^2 \\
\leq t^2(b_0(\varepsilon))^2.
\]

We established this lemma. \( \square \)

5.2.3. Proof of proposition 5.1.

**Proof.** By using Schwarz’s inequality we have

\[
\int_{T_{-1}} dk \left| \hat{p}_c(k, t) - \hat{p}_c(0, t) \right|^2 + \left| \hat{l}_c(k, t) - \hat{l}_c(0, t) \right|^2 \\
\leq 2 \int_{T_{-1}} dk \left| \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right|^2 + \left| \hat{l}_c(k, t) - \hat{l}_c^{(0)}(k, t) \right|^2 \\
+ 2 \int_{T_{-1}} dk \left| \hat{p}_c^{(0)}(k, t) - \hat{p}_c(0, t) \right|^2 + \left| \hat{l}_c^{(0)}(k, t) - \hat{l}_c(0, t) \right|^2 \\
\leq 2 \int_{T_{-1}} dk \left| \hat{p}_c(k, t) - \hat{p}_c^{(0)}(k, t) \right|^2 + \left| \hat{l}_c(k, t) - \hat{l}_c^{(0)}(k, t) \right|^2 \\
+ 2 \int_{T_{-1}} dk \left\| \exp (A_\theta(k)) \right\|^2 \left( \hat{p}_c(k, 0) - \hat{p}_c(0, 0) \right) \left( \hat{l}_c(k, 0) - \hat{l}_c(0, 0) \right)^2.
\]

(5.5)
From (4.1) and lemma 5.1, we see that the second term of (5.5) vanishes. Now we estimate the first term. From lemma 5.2, it suffices to show that
\[ \lim_{k \to \infty} \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{\varepsilon}} dk \left| \tilde{p}_e(k, t) - \tilde{p}_e^{(0)}(k, t) \right|^2 + \left| \tilde{l}(k, t) - \tilde{l}^{(0)}(k, t) \right|^2 = 0. \]  
(5.6)

From (5.3), we can write
\[ \begin{pmatrix} \tilde{p}_e(k, t) \\ \tilde{l}(k, t) \end{pmatrix} = \exp(A_\theta(k)) \begin{pmatrix} \tilde{p}_e(0) \\ \tilde{l}(0) \end{pmatrix}. \]

In addition, \((\tilde{p}(\xi, t), \tilde{l}(\xi, t)), \xi \in \mathbb{R}, t \geq 0\) satisfies
\[ \partial_t \begin{pmatrix} \tilde{p}(\xi, t) \\ \tilde{l}(\xi, t) \end{pmatrix} = A_\theta(\xi) \begin{pmatrix} \tilde{p}(\xi, t) \\ \tilde{l}(\xi, t) \end{pmatrix}, \]
where \(A_\theta(\xi), \xi \in \mathbb{R}\) is defined as
\[ A^{(1,1)}_\theta(\xi) = A^{(2,2)}_\theta(\xi) = 0, \]
\[ A^{(1,2)}_\theta(\xi) = A^{(2,1)}_\theta(\xi) = \begin{cases} \sqrt{C_1(\theta)} \pi |\xi|^{\frac{1}{2}} & 1 < \theta \leq 3, \\ \sqrt{C_1(\theta)} \pi & \theta > 3, \end{cases} \]
and hence we obtain
\[ \begin{pmatrix} \tilde{p}(\xi, t) \\ \tilde{l}(\xi, t) \end{pmatrix} = \exp(A_\theta(\xi)) \begin{pmatrix} \tilde{p}_0(\xi) \\ \tilde{l}_0(\xi) \end{pmatrix}. \]

By using (4.1), lemma 5.1 and Schwarz’s inequality we obtain
\[ \begin{aligned} & \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{\varepsilon}} dk \left| \tilde{p}_e(k, t) - \tilde{p}_e^{(0)}(k, t) \right|^2 + \left| \tilde{l}(k, t) - \tilde{l}^{(0)}(k, t) \right|^2 \\ & = \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{\varepsilon}} dk \left| \exp(A_\theta(k)) \begin{pmatrix} \tilde{p}_e(0) \\ \tilde{l}(0) \end{pmatrix} \right|^2 \\ & \quad - \exp(A_\theta(k)) \begin{pmatrix} \tilde{p}_e(0) \\ \tilde{l}(0) \end{pmatrix} \right|^2 \\ & \leq 2 \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{\varepsilon}} dk \left| \exp(A_\theta(\xi)) \begin{pmatrix} \tilde{p}_e(0) \\ \tilde{l}(0) \end{pmatrix} \right|^2 \\ & \quad + \left| \exp(A_\theta(k)) \begin{pmatrix} \tilde{p}_e(0) \\ \tilde{l}(0) \end{pmatrix} \right|^2 \\ & \leq 2(C_* + D_*) \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{\varepsilon}} dk \left| \tilde{p}_e(k, 0) \right|^2 + \left| \tilde{l}(k, 0) \right|^2 \\ & \leq 2(C_* + D_*) \int_{|\xi| > K} d\xi \left| \tilde{p}_0(\xi) \right|^2 + \left| \tilde{l}_0(\xi) \right|^2, \end{aligned} \]
and thus we have (5.6).

\[ \square \]
5.3. Convergence on longer time scales

In this subsection we consider the scaling limit on longer time scales than \( j_\theta(\varepsilon)^{-1} \). Recall that \( b_\theta(\varepsilon) \) is defined in (5.4).

**Corollary 5.1.** Suppose that \( \theta > 1, \gamma \geq 0 \) and (4.1). Assume that \( v_\theta(\varepsilon) > 0 \) satisfies

\[
\lim_{\varepsilon \to 0} v_\theta(\varepsilon) = 0, \quad \lim_{\varepsilon \to 0} b_\theta(\varepsilon) = 0.
\]

Then for any \( t > 0 \) and \( J_i \in C_0^\infty(\mathbb{R}), i = 1, 2 \), we have

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ \begin{pmatrix} p_x \left( \frac{t}{j_\theta(\varepsilon)v_\theta(\varepsilon)} \right) \\ l_x \left( \frac{t}{j_\theta(\varepsilon)v_\theta(\varepsilon)} \right) \end{pmatrix} \right] \cdot \begin{pmatrix} J_1(\varepsilon x) \\ J_2(\varepsilon x) \end{pmatrix} = 0. \tag{5.7}
\]

**Proof.** The strategy of the proof of corollary 5.1 is essentially the same as the proof of theorem 4.1. By using lemma 5.1 and performing the same calculations as those used in the proof of lemma 5.2, we obtain

\[
\int_{|k| \leq K} \left| \tilde{\hat{p}}_{\varepsilon} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) - \hat{p}(0) \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \right|^2 + \left| \tilde{l}_{\varepsilon}(k, t) - \tilde{l}(0)(k, t) \right|^2 \leq \varepsilon^2 \left( \frac{b_\theta(\varepsilon)}{v_\theta(\varepsilon)} \right)^2,
\]

for any \( K > 0 \). Thus by performing the same calculations as those used in the proof of proposition 5.1, we get

\[
\lim_{\varepsilon \to 0} \int_{T \to 1} dk \left| \tilde{\hat{p}}_{\varepsilon} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) - \hat{p} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \right|^2 \\
+ \left| \tilde{l}_{\varepsilon} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) - \tilde{l} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \right|^2 = 0.
\]

On the other hand, for any \( s \geq 0 \), we have

\[
\int_{T \to 1} dk \left| \tilde{\hat{p}}(k, s) \right|^2 + \left| \tilde{l}(k, s) \right|^2 = \int_{T \to 1} dk \left| \exp(A_\theta(k)s) \left( \tilde{p}_0(k) \tilde{l}_0(k) \right) \right|^2 \\
= \int_{T \to 1} dk \left| \tilde{\hat{p}}_\theta(k) \right|^2 + \left| \tilde{l}_\theta(k) \right|^2.
\]
Combining the above computations, we obtain

$$
\lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} \text{dk} \mathbb{E}_\varepsilon \left[ \left| p_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \right|^2 + \left| L_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \right|^2 \right] = 0. \quad (5.8)
$$

Next we observe that

$$
\lim_{\varepsilon \to 0} \sum_{x \in \mathbb{Z}} \mathbb{E}_\varepsilon \left[ \begin{pmatrix} p_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \\ l_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \end{pmatrix} \right] \cdot \begin{pmatrix} J_1(\varepsilon x) \\ J_2(\varepsilon x) \end{pmatrix}
= \lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} \text{dk} \tilde{p}_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \tilde{J}_1(-k) + \tilde{L} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \tilde{J}_2(-k)
= \lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} \text{dk} \tilde{p}_\varepsilon \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \tilde{J}_1(-k) + \tilde{I} \left( k, \frac{t}{v_\theta(\varepsilon)} \right) \tilde{J}_2(-k)
= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{T_{\varepsilon}} \text{dk} \tilde{p}_0(k) \left[ \left( \tilde{S}^+_{\theta} J_1 \right)_1 + \left( \tilde{S}^-_{\theta} J_1 \right)_1 + \left( \tilde{S}^+_{\theta} J_2 \right)_1 + \left( \tilde{S}^-_{\theta} J_2 \right)_1 \right](-k)
+ \tilde{I}_0(k) \left[ \left( \tilde{S}^+_{\theta} J_1 \right)_1 + \left( \tilde{S}^-_{\theta} J_1 \right)_1 + \left( \tilde{S}^+_{\theta} J_2 \right)_1 + \left( \tilde{S}^-_{\theta} J_2 \right)_1 \right](-k),
$$

where in the second equation we use (5.8) and $S^\pm(t)$ is defined in (4.9). In addition, in the third equation, we omit the time variable of $S^\pm(t/v_\theta(\varepsilon))$ for simplicity. Therefore, to show (5.7), it is sufficient to show that

$$
\lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} \text{dk} \tilde{H}(k) \left( \tilde{S}^+_{\theta} \left( \frac{t}{v_\theta(\varepsilon)} \right) J \right)(k) = 0, \quad (5.9)
$$

for any $H, J \in C^\infty_0(\mathbb{R})$. From Parseval’s identity we have

$$
\lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} \text{dk} \tilde{H}(k) \left( \tilde{S}^+_{\theta} \left( \frac{t}{v_\theta(\varepsilon)} \right) J \right)(k)
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \text{d} \xi \tilde{H}(\xi) \left( \tilde{S}^+_{\theta} \left( \frac{t}{v_\theta(\varepsilon)} \right) J \right)(\xi)
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \text{d} y \tilde{H}(-y) \left( \tilde{S}^+_{\theta} \left( \frac{t}{v_\theta(\varepsilon)} \right) J \right)(y).
$$

If $\theta \geq 3$, then we obtain

$$
\left( \tilde{S}^+_{\theta} \left( \frac{t}{v_\theta(\varepsilon)} \right) J \right)(y) = J \left( y - \sqrt{C_1(\theta)} \right)
$$
and hence by using the dominated convergence theorem, we get (5.9). If $1 < \theta < 3$, then we have
\[
\left( S^j_{\epsilon} \left( \frac{t}{\nu_\theta(\epsilon)} \right) J \right)(y) = \int_\mathbb{R} d\xi \, e^{2\pi i \xi y} \exp \left( \frac{\sqrt{|C_1(\theta)|} \sgn(\xi) i \frac{|\xi|^\theta - 1}{\nu_\theta(\epsilon)}}{2\pi} \right) J(\xi) = \int_\mathbb{R} d\xi' \exp \left( \frac{\sqrt{|C_1(\theta)|} \xi'^\theta}{\nu_\theta(\epsilon)} \right) 
\times \left[ \sgn(\xi') \xi'^\theta \frac{2\pi i \sgn(\xi') \xi'^\theta}{2\pi} \right] \exp \left( \frac{\sqrt{|C_1(\theta)|} \xi'^\theta}{2\pi} \right) J \left( \frac{\sgn(\xi') \xi'^\theta}{2\pi} \right),
\]
and thus by using the Riemann–Lebesgue lemma and the dominated convergence theorem, we have (5.9).

6. Proof of theorem 4.2

First we observe that from theorem 4.1 it is sufficient to show that
\[
\lim_{\epsilon \to 0} \sum_{x \in \mathbb{Z}} \mathbb{E}_x \left[ \epsilon \left( \frac{t}{\nu_\theta(\epsilon)} \right) - \frac{1}{2} p_x \left( \frac{t}{\nu_\theta(\epsilon)} \right) \right] J(\epsilon x)
\]
\[
= \begin{cases} 
\frac{1}{2} \int_{\mathbb{R}} dy \, L(y, t) J(y) & 1 < \theta < 3, \\
\frac{1}{2} \int_{\mathbb{R}} dy \, \mathcal{L}^2(y, t) J(y) & \theta \geq 3,
\end{cases}
\]
\[
= \begin{cases} 
\int_{\mathbb{R}^2} d\xi \, dk \, \sgn(k) \sgn(-k - \epsilon \xi) \frac{|\xi|^{\theta - 1} - |k|^{\theta - 1} - \epsilon \xi + |\xi|^{\theta - 1}}{4k |k + \xi|^\theta} \mathcal{J}(k, t) \tilde{J}(\tilde{t}) & 1 < \theta < 3, \\
\frac{1}{2} \int_{\mathbb{R}^2} d\xi \, dk \tilde{J}(k, t) \tilde{J}(\tilde{t}) & \theta \geq 3,
\end{cases}
\]
where $L(y, t)$ is defined in (1.10). To simplify the notation, we may omit the variable $t \geq 0$. By using the Poisson summation formula, we obtain
\[
\sum_{x} \mathbb{E}_x \left[ \epsilon \left( \frac{t}{\nu_\theta(\epsilon)} \right) - \frac{1}{2} p_x \left( \frac{t}{\nu_\theta(\epsilon)} \right) \right] J(\epsilon x)
\]
\[
= \frac{\epsilon}{4} \sum_{x \in \mathbb{Z}} \int_{\mathbb{R}^2} dk' \, e^{2\pi i k' y} \sgn(k') \sgn(k) F(k, k') \mathbb{E}_x \left[ \tilde{J}(k') \tilde{J}(k) \right] \int_{\mathbb{R}} d\xi \, e^{2\pi i \epsilon \xi x} \tilde{J}(\xi)
\]
\[
= \frac{1}{4} \int_{\mathbb{R} \times \mathbb{T}^1} d\xi \, dk \, \sgn(k) \sgn(-k - \epsilon \xi) F(k, -k - \epsilon \xi) \mathbb{E}_x \left[ \tilde{J}(k) \tilde{J}(-k - \epsilon \xi) \right] \tilde{J}(\xi)
\]
\[
= \frac{1}{4} \int_{\mathbb{R} \times \mathbb{T}^1} d\xi \, dk \, \sgn(k) \sgn(-k - \xi) F(\epsilon k, -\epsilon k - \epsilon \xi) \mathbb{E}_x \left[ \tilde{J}(\epsilon k) \tilde{J}(-\epsilon k - \epsilon \xi) \right] \tilde{J}(\xi).
\]
From theorem 4.1 and the boundedness of $F(k, k')$, we have

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \, d\xi \, dk \, \text{sgn}(k) \, \text{sgn}(-k - \xi) \, F(\varepsilon k, -\varepsilon k - \varepsilon \xi) \\
\times \varepsilon^2 \text{Re}_q \left[ \hat{i} \left( \frac{t}{f_{0}(\xi)} \right) \hat{i} \left( -\varepsilon k - \varepsilon \xi, \frac{t}{f_{0}(\xi)} \right) \right] \tilde{J}(\xi) \\
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \, d\xi \, dk \, \text{sgn}(k) \, \text{sgn}(-k - \xi) \, F(\varepsilon k, -\varepsilon k - \varepsilon \xi) \tilde{i}(k, t) \tilde{i}(-k - \xi, t) \tilde{J}(\xi).
$$

Since $1_{\{k \in \mathcal{T}_{\varepsilon}^{-} \}} F(\varepsilon k, -\varepsilon k - \varepsilon \xi)$ is uniformly bounded and

$$
\lim_{\varepsilon \to 0} 1_{\{k \in \mathcal{T}_{\varepsilon}^{-} \}} \text{sgn}(k) \, \text{sgn}(-k - \xi) \, F(\varepsilon k, -\varepsilon k - \varepsilon \xi)
$$

almost every $(\xi, k) \in \mathbb{R}^2$, we obtain

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \, d\xi \, dk \, \text{sgn}(k) \, \text{sgn}(-k - \xi) \, F(\varepsilon k, -\varepsilon k - \varepsilon \xi) \tilde{i}(k, t) \tilde{i}(-k - \xi, t) \tilde{J}(\xi)
$$

and thus we get (4.4).

7. Proof of theorems 4.3 and 4.4

In this section we show theorems 4.3 and 4.4 by using the strategy which is similar to that used in section 5.

First we note that the solution of (1.12) with initial condition $f_{0}^{+}(y) := p_{0}(y) \pm l_{0}(y)$ is given by

$$
\left( \begin{array}{c}
F_{0}^{+} \\
F_{0}^{-}
\end{array} \right)(y, t) = \int_{\mathbb{R}} d\xi \, e^{2\pi iky} \exp\{M_{0}(\xi) t\} \left( \begin{array}{c}
f_{0}^{+} \\
\bar{f}_{0}^{-}
\end{array} \right)(\xi),
$$

...
where $M_\theta(\xi) = (M_\theta^{(i,j)}(\xi))_{i,j=1,2}$ is a $2 \times 2$ matrix defined as

\[
M_\theta^{(1,1)}(\xi) := \begin{cases} 
0 & 2 < \theta < 3, \\
\frac{\sqrt{C(3)} \gamma}{\sqrt{C(3)} \gamma} \log |2\pi \xi|^{-1} + \frac{\sqrt{C(3)} \gamma}{\sqrt{C(3)} \gamma} |\ln \xi| & \theta = 3, \\
-\frac{3\gamma}{2} (2\pi \xi)^2 & \theta > 4,
\end{cases}
\]

\[
M_\theta^{(2,1)}(\xi) := \begin{cases} 
\frac{\sqrt{C(3)} \gamma}{\sqrt{C(3)} \gamma} \log |2\pi \xi|^{-1} + \frac{\sqrt{C(3)} \gamma}{\sqrt{C(3)} \gamma} |\ln \xi| & \theta = 3, \\
-\frac{3\gamma}{2} (2\pi \xi)^2 & \theta > 4,
\end{cases}
\]

Next we observe that

\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} E_x \left[ \tilde{f}_\varepsilon^+ \left( \frac{t}{m_\varepsilon(\varepsilon)} \right) \left( S_\theta^\varepsilon \left( \frac{t}{m_\varepsilon(\varepsilon)} \right) J \right) (\varepsilon x) \right] = \lim_{\varepsilon \to 0} \int_{\mathbb{T}_{-1}} dk \hat{\Delta}_\varepsilon^+(k, t) \hat{J}(-k),
\]

where $\hat{\Delta}_\varepsilon^+(k, t), k \in \mathbb{T}_{-1}, t \geq 0$ is defined as

\[
\hat{\Delta}_\varepsilon^+(k, t) := \begin{cases} 
\exp \left( \frac{\pm \sqrt{C(3)} \gamma \sgn(k) 2\pi k t}{m_\varepsilon(\varepsilon)} \right) \hat{f}_\varepsilon^+(k, t) & 2 < \theta \leq 3, \\
\exp \left( \frac{\pm \sqrt{C(3)} \gamma \sgn(k) 2\pi k t}{m_\varepsilon(\varepsilon)} \right) \hat{f}_\varepsilon^+(k, t) & \theta > 3,
\end{cases}
\]

and

\[
\hat{f}_\varepsilon^+(k, t) := \varepsilon \mathbb{E}_x \left[ \hat{f}_\varepsilon^+ \left( \varepsilon k, \frac{t}{m_\varepsilon(\varepsilon)} \right) \right].
\]

The main subject of this section is to show the following proposition, which implies theorem 4.3.

**Proposition 7.1.** Suppose that $\theta > 2, \gamma > 0$ and (4.1). For any $t \geq 0$, we have

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{T}_{-1}} dk \left| \frac{\hat{\Delta}_\varepsilon^+(k, t)}{\hat{\Delta}_\varepsilon^-(k, t)} - \frac{\hat{F}^+(k, t)}{\hat{F}^-(k, t)} \right|^2 = 0.
\]

We prove proposition 7.1 in section 7.1. In section 7.2 we show theorem 4.4 by using proposition 7.1. In the rest of this subsection, we consider the time evolution law of $\{\hat{\Delta}_\varepsilon^+(k, t); k \in \mathbb{T}_{-1}, t \geq 0\}$ and prepare two lemmas to show proposition 7.1.
From (3.4) we have

\[
\frac{d}{dt} \left( \begin{array}{c} \hat{v}_+^T \\ \hat{v}_-^T \end{array} \right) = \frac{1}{n_\theta(\varepsilon)} \begin{pmatrix} i \text{ sgn}(k)\omega(\varepsilon k) - \gamma R(\varepsilon k) & -\gamma R(\varepsilon k) \\ -\gamma R(\varepsilon k) & -i \text{ sgn}(k)\omega(\varepsilon k) - \gamma R(\varepsilon k) \end{pmatrix} \times \left( \begin{array}{c} \hat{v}_+^T \\ \hat{v}_-^T \end{array} \right).
\]

Thus the time evolution law of \( \{ \hat{\delta}_+^\varepsilon(k,t); k \in \mathbb{T}_{\varepsilon-1}, t \geq 0 \} \) is given by

\[
\frac{d}{dt} \left( \begin{array}{c} \hat{\delta}_+^\varepsilon(k,t) \\ \hat{\delta}_-^\varepsilon(k,t) \end{array} \right) = M_{\varepsilon,\theta}(k) \left( \begin{array}{c} \hat{\delta}_+^\varepsilon(k,t) \\ \hat{\delta}_-^\varepsilon(k,t) \end{array} \right), \quad M_{\varepsilon,\theta}(k) = (M_{\varepsilon,\theta}^{(i,j)}(k))_{i,j=1,2},
\]

\[M_{\varepsilon,\theta}^{(1,1)}(k) := \frac{1}{n_\theta(\varepsilon)} \text{ sgn}(k)\omega(\varepsilon k) - \sqrt{C_1(\theta)\varepsilon} \text{ sgn}(k)\frac{2\pi k}{n_\theta(\varepsilon)} - \frac{\gamma R(\varepsilon k)}{n_\theta(\varepsilon)} \]

\[M_{\varepsilon,\theta}^{(1,2)}(k) = M_{\varepsilon,\theta}^{(2,1)}(k) : = -\frac{\gamma R(\varepsilon k)}{n_\theta(\varepsilon)}, \quad M_{\varepsilon,\theta}^{(2,2)}(k) := (M_{\varepsilon,\theta}^{(1,1)}(k))^*(k).
\]

Then we have the following decomposition of \( M_{\varepsilon,\theta}(k) \):

\[M_{\varepsilon,\theta}(k) = M_\theta(k) + \text{Rem}_{\varepsilon,\theta}(k), \quad M_\theta = (M_\theta^{(i,j)})_{i,j=1,2},
\]

\[\text{Rem}_{\varepsilon,\theta} = (\text{Rem}_{\varepsilon,\theta}^{(i,j)})_{i,j=1,2},
\]

\[\text{Rem}_{\varepsilon,\theta}^{(i,j)}(k) := M_{\varepsilon,\theta}^{(i,j)}(k) - M_\theta^{(i,j)}(k) \quad i,j = 1,2,
\]

where the matrix \( M_\theta \) is defined right before theorem 4.3. We can obtain the following estimates of the matrix norm of \( M_\theta, M_{\varepsilon,\theta} \) and \( \text{Rem}_{\varepsilon,\theta} \).

**Lemma 7.1.**

\[C_1' := \sup_{0 < \varepsilon < 1} \sup_{(k,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}} \| \exp (M_\theta(k)t) \| < \infty,
\]

\[D_1' := \sup_{(k,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}} \| \exp (M_\theta(k)t) \| < \infty.
\]

**In addition, for any** \( K > 0 \), **we have**

\[\sup_{|k| \leq K} \| \text{Rem}_{\varepsilon,\theta}(k) \| \leq C_{K,\theta}(\varepsilon),
\]
where

\[
  r_\theta(\varepsilon) := \begin{cases} 
    \varepsilon^\theta - 2 & 2 < \theta < \frac{5}{2} \\
    \varepsilon^{3-\theta} & \frac{5}{2} < \theta < 3, \\
    (\log(\varepsilon^{-1}))^{-1} & \theta = 3, \\
    \varepsilon^{4-\theta} & \frac{7}{2} < \theta < 4, \\
    \varepsilon \log(\varepsilon^{-1}) & \theta = 4, \\
    \varepsilon^{\theta-4} & 4 < \theta < 5, \\
    \varepsilon \log(\varepsilon^{-1}) & \theta = 5, \\
    \varepsilon & \theta > 5.
  \end{cases}
\]

**Proof.** $M_\varepsilon$ and $M$ have the form

\[
  \begin{pmatrix}
    \omega k + b & b \\
    b & -\omega a + b
  \end{pmatrix} \quad a, b \in \mathbb{R},
\]

and the eigenvalues of the matrix are $b \pm \sqrt{b^2 - a^2}$. Since $R(k)$ is non-negative, we see that the eigenvalues of $M_\varepsilon$ and $M$ are non-positive and thus we obtain $C_* D'_* < \infty$.

Next we consider the order of $\sup_{|k| \leq K} |\text{Rem}_{\varepsilon, b}(k)|$, $K > 0$. Since

\[ |\text{Rem}^{(1,2)}_{\varepsilon, b}(k)| \lesssim |\text{Rem}^{(1,1)}_{\varepsilon, b}(k)|, \]

it is sufficient to estimate $|\text{Rem}^{(1,1)}_{\varepsilon, b}(k)|$. From Lemma A.1, we have

\[
  \frac{\omega(\varepsilon k)}{n_\varepsilon(\varepsilon k)} = \frac{\theta_\varepsilon(\varepsilon k) n_\varepsilon(\varepsilon k) - C_1(3) [2\pi k]^2 \log(2\pi k) + C_2(3) [2\pi k]^2}{\omega(\varepsilon k) \theta_\varepsilon(\varepsilon k)^{-1} + \sqrt{C_1(3) [2\pi k]^3}} \quad \text{as } \theta_\varepsilon(\varepsilon k) \rightarrow 0
\]

\[
  = \begin{cases} 
    \frac{C_2(\theta) [2\pi k]^2}{\omega(\varepsilon k) j_\varepsilon(\varepsilon k) - C_1(3) [2\pi k]^2 \log(2\pi k) + C_2(3) [2\pi k]^2 + O(\varepsilon k)^2} + O(\varepsilon |k|^\theta) & 2 < \theta < 3, \\
    \frac{C_2(\theta) [2\pi k]^3}{\omega(\varepsilon k) j_\varepsilon(\varepsilon k) - C_1(3) [2\pi k]^2 \log(2\pi k) + C_2(3) [2\pi k]^2 + O(\varepsilon k)^2} + O(\varepsilon |k|^\theta) & 3 < \theta < 4, \\
    \frac{C_2(\theta) [2\pi k]^4}{\omega(\varepsilon k) j_\varepsilon(\varepsilon k) - C_1(3) [2\pi k]^2 \log(2\pi k) + C_2(3) [2\pi k]^2 + O(\varepsilon k)^2} + O(\varepsilon |k|^\theta) & \theta = 4.
  \end{cases}
\]
and

\[
\frac{\omega(\varepsilon k)}{n_0(\varepsilon)} - \sqrt{C_1(\theta)} \frac{2\pi k}{n_0(\varepsilon)} = \begin{cases}
O(\varepsilon^{\theta-4}|k|^\theta - 2 + \varepsilon|k|^3) & 4 < \theta < 5, \\
O(\varepsilon \log(\varepsilon^{-1})|k|^3) & \theta = 5, \\
O(\varepsilon|k|^3 + \varepsilon^{\theta-4}|k|^\theta - 2) & 5 < \theta < 7, \\
O(\varepsilon|k|^3 + \varepsilon^3 \log(\varepsilon^{-1})|k|^3) & \theta = 7, \\
O(\varepsilon|k|^3 + \varepsilon^3|k|^2) & \theta > 7.
\end{cases}
\]

In addition, we have

\[
\frac{R(\varepsilon k)}{n_0(\varepsilon)} = \begin{cases}
O(\varepsilon^2 n_0(\varepsilon)^{-1}|k|^2 + \varepsilon n_0(\varepsilon)^{-1}|k|^4) & 2 < \theta < 4, \\
\frac{3}{2}(2\pi k)^2 + O(\varepsilon^2|k|^4) & \theta \geq 4.
\end{cases}
\]

Thus we obtain

\[
\text{Rem}_{\varepsilon,\theta}^{(1,1)}(k) = \begin{cases}
O\left(\varepsilon^{3-\theta}|k|^{\frac{11-3\theta}{2}} + \varepsilon^{\theta-2}|k|^2 + \varepsilon^\theta k^2\right) & 2 < \theta < 3, \\
O\left((\log(\varepsilon^{-1}))^{-1}|k| + \varepsilon|k|^2 + \varepsilon(\sqrt{\log(\varepsilon^{-1})})k^2\right) & \theta = 3, \\
O(\varepsilon^{\theta-3}|k|^{2\theta-5} + \varepsilon|k|^{\theta-1} + \varepsilon^{4-\theta}k^2) & 3 < \theta < 4, \\
O(\varepsilon \log(\varepsilon^{-1})|k|^3 + \varepsilon^2|k|^4) & \theta = 4, \\
O(\varepsilon^{\theta-4}|k|^\theta - 2 + \varepsilon|k|^3 + \varepsilon^2|k|^4) & 4 < \theta < 5, \\
O(\varepsilon \log(\varepsilon^{-1})|k|^3 + \varepsilon^2|k|^4) & \theta = 5, \\
O(\varepsilon|k|^3 + \varepsilon^{\theta-2}k^2 + \varepsilon^2|k|^4) & 5 < \theta < 7, \\
O(\varepsilon|k|^3 + \varepsilon^3 \log(\varepsilon^{-1})|k|^4 + \varepsilon^2|k|^4) & \theta = 7, \\
O(\varepsilon|k|^3 + \varepsilon^3|k|^5 + \varepsilon^2|k|^4) & \theta > 7.
\end{cases}
\]
Hence on $|k| \leq K$ we have

$$\begin{cases}
\varepsilon^{\theta-2} & 2 < \theta \leq \frac{5}{2} \\
\varepsilon^{3-\theta} & \frac{5}{2} < \theta < 3, \\
(\log(\varepsilon^{-1}))^{-1} & \theta = 3, \\
\varepsilon^{\theta-3} & 3 < \theta \leq \frac{7}{2}, \\
\varepsilon^{4-\theta} & \frac{7}{2} < \theta < 4, \\
\varepsilon \log(\varepsilon^{-1}) & \theta = 4, \\
\varepsilon^{\theta-4} & 4 < \theta < 5, \\
\varepsilon \log(\varepsilon^{-1}) & \theta = 5, \\
\varepsilon & \theta > 5.
\end{cases}$$

We introduce the dynamics $\{\hat{\delta}_{\varepsilon}(k, t); k \in \mathbb{T}_{\varepsilon^{-1}}, t \geq 0\}$, which is generated by $M(k)$ with the same initial condition as $\{\hat{\delta}_{\varepsilon}(k, t); k \in \mathbb{T}_{\varepsilon^{-1}}, t \geq 0\}$, that is,

$$\frac{d}{dt} \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-, (0)}(k, t)
\end{pmatrix} (k, t) = M_0(k) \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-, (0)}(k, t)
\end{pmatrix},$$

$$\begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+(0)}(k, 0) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, 0)
\end{pmatrix} = \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+(0)}(k, 0) \\
\hat{\delta}_{\varepsilon}^{-(0)}(k, 0)
\end{pmatrix}.$$

**Lemma 72.** For any $K > 0$ and $T > 0$, we have

$$\lim_{\varepsilon \to 0} \int_{|k| \leq K} \left\| \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix} - \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+(0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-(0)}(k, t)
\end{pmatrix} \right\|^2 = 0.$$

**Proof.** Since

$$\frac{d}{dt} \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) - \hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t) - \hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix} = M_0(k) \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) - \hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t) - \hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix}$$

$$+ \text{Rem}_{+, \beta}(k) \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix},$$

by using Duhamel’s formula, we obtain

$$\begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) - \hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t) - \hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix} = \int_0^t ds \exp(M(k)(t - s)) \text{Rem}_{+, \beta}(k) \begin{pmatrix}
\hat{\delta}_{\varepsilon}^{+, (0)}(k, t) \\
\hat{\delta}_{\varepsilon}^{-,(0)}(k, t)
\end{pmatrix}.$$
From (3.6) and (4.1) and lemma 7.1, we have

\[
\int_{|k| \leq K} \frac{d k}{\left\| \frac{\tilde{v}_{\varepsilon}^{+0}(k, t)}{\tilde{v}_{\varepsilon}^{+0}(k, t)} - \left( \frac{\tilde{v}_{\varepsilon}^{+}}{\tilde{v}_{\varepsilon}^{-}}(k, s) \right) \right\|^2} \leq t \int_{|k| \leq K} \frac{d k}{\int_0^t ds \left\| \exp(M\vartheta(k)(t - s)) \text{ Rem}_{\varepsilon, \vartheta}(k) \left( \frac{\tilde{v}_{\varepsilon}^{+}}{\tilde{v}_{\varepsilon}^{-}}(k, s) \right) \right\|^2} \leq r^2_0(\varepsilon) \int_{|k| \leq K} \frac{d k}{\int_0^t ds \left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, s)}{\tilde{v}_{\varepsilon}^{-}(k, s)} \right\|^2} = r^2_0(\varepsilon) \int_{|k| \leq K} \frac{d k}{\int_0^t ds \left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, s)}{\tilde{v}_{\varepsilon}^{-}(k, s)} \right\|^2} = 2r^2_0(\varepsilon) \int_0^t ds \left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, s)}{\tilde{v}_{\varepsilon}^{-}(k, s)} \right\|^2 \leq r^2_0(\varepsilon). \]

7.1. Proof of proposition 7.1

Proof. Thanks to Schwarz’s inequality, we get

\[
\int_{T_{\varepsilon^{-1}}} \frac{d k}{\left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, t)}{\tilde{v}_{\varepsilon}^{-}(k, t)} - \left( \tilde{F}^+(k, t) \right) \right\|^2} \leq 2 \int_{T_{\varepsilon^{-1}}} \frac{d k}{\left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, t)}{\tilde{v}_{\varepsilon}^{-}(k, t)} - \left( \frac{\tilde{v}_{\varepsilon}^{+0}(k, t)}{\tilde{v}_{\varepsilon}^{-0}(k, t)} \right) \right\|^2} + 2 \int_{T_{\varepsilon^{-1}}} \frac{d k}{\left\| \frac{\tilde{v}_{\varepsilon}^{+}(k, t)}{\tilde{v}_{\varepsilon}^{-}(k, t)} - \left( \tilde{F}^-(k, t) \right) \right\|^2} = 2 \int_{T_{\varepsilon^{-1}}} \frac{d k}{\left\| \exp(M_{\varepsilon, \vartheta}(k)t) - \exp(M\vartheta(k)t) \right\|^2} \left( \frac{\tilde{v}_{\varepsilon}^{+}(k, 0)}{\tilde{v}_{\varepsilon}^{-}(k, 0)} \right)^2 + 2 \int_{T_{\varepsilon^{-1}}} \frac{d k}{\left\| \exp(M\vartheta(k)t) \right\|^2} \left( \frac{\tilde{v}_{\varepsilon}^{+}(k, 0) - \tilde{F}^+(k, 0)}{\tilde{v}_{\varepsilon}^{-}(k, 0) - \tilde{F}^-(k, 0)} \right)^2. \]  

From (4.1) and lemma 7.1, we see that the second term of (7.1) vanishes as \( \varepsilon \to 0 \). Now we estimate the first term of (7.1). From lemma 7.2, it is sufficient to show that

\[
\lim_{K \to \infty} \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{4\varepsilon}} \frac{d k}{\left\| \exp(M_{\varepsilon, \vartheta}(k)t) - \exp(M\vartheta(k)t) \right\|^2} \left( \frac{\tilde{v}_{\varepsilon}^{+}(k, 0) - \tilde{F}^+(k, 0)}{\tilde{v}_{\varepsilon}^{-}(k, 0) - \tilde{F}^-(k, 0)} \right)^2 = 0.
\]
By using (4.1) and lemma 7.1 we have

\[ \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{2\varepsilon}} |\exp (M_{\varepsilon, \theta}(k) t) - \exp (M_\theta(k) t)| \left( \hat{\theta}^+ (k, 0) \right)^2 \]

\[ \lesssim \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{2\varepsilon}} \left| \left( \hat{\theta}^+ (k, 0) \right)^2 \right| \]

\[ = \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{2\varepsilon}} \left| \left( \tilde{\theta}^+ (k, 0) \right)^2 \right| \]

\[ = 2 \lim_{\varepsilon \to 0} \int_{|k| < \frac{1}{2\varepsilon}} |\hat{\theta}^+ (k, 0)|^2 + |\tilde{\theta}^+ (k, 0)|^2 \]

\[ = 2 \int_{|\xi| > \frac{1}{2\varepsilon}} d\xi |\tilde{\theta}^0 (\xi)|^2 + |\tilde{\theta}^0 (\xi)|^2 \]

Hence we complete the proof of this lemma. \( \square \)

7.2. Proof of theorem 4.4

Define

\[ \hat{\mathcal{F}}^\pm_\varepsilon (k, t) := \begin{cases} 
\exp \left( \mp \sqrt{C_1 (\theta) \text{sgn}(k)} |2\pi k| \frac{|t|}{m_\theta (\varepsilon)} \right) \hat{\xi}^\pm \left( \varepsilon, \frac{t}{n_\theta (\varepsilon)} \right) \hat{\theta}^\pm (k, t) & 2 < \theta \leq 3, \\
\exp \left( \mp \sqrt{C_1 (\theta) |2\pi k| t} \right) \frac{\hat{\xi}^\pm \left( \varepsilon, \frac{t}{n_\theta (\varepsilon)} \right)}{m_\theta (\varepsilon)} \hat{\theta}^\pm (k, t) & \theta > 3. 
\end{cases} \]

From (3.6) we have

\[ \int_{\mathcal{T}_\varepsilon} d\xi E_\varepsilon \left[ |\hat{\xi}^\pm \left( \varepsilon, \frac{t}{n_\theta (\varepsilon)} \right)|^2 + |\hat{\xi}^\pm \left( \varepsilon, \frac{t}{n_\theta (\varepsilon)} \right)|^2 \right] \]

\[ = 2 \int_{\mathcal{T}_\varepsilon} d\xi E_\varepsilon \left[ |\tilde{\xi} \hat{\theta}^0 (\xi, 0)|^2 + |\tilde{\xi} \hat{\theta}^0 (\xi, 0)|^2 \right] \]

\[ = 2 \int_{\mathcal{T}_\varepsilon} d\xi E_\varepsilon \left[ |\tilde{\xi} \hat{\theta}^0 (\xi, 0)|^2 + |\tilde{\xi} \hat{\theta}^0 (\xi, 0)|^2 \right] \]

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for any \( t \geq 0 \). Thus we obtain

\[
2 \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} \left[ |\epsilon \hat{p}(\epsilon k, 0)|^2 + |\epsilon \hat{i}(\epsilon k, 0)|^2 \right] = \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} \left[ |\mathcal{F}^+_{\epsilon}(k, t)|^2 + |\mathcal{F}^-_{\epsilon}(k, t)|^2 \right]
\]

\[
+ \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} |\tilde{\delta}^+_{\epsilon}(k, t)|^2 + |\tilde{\delta}^-_{\epsilon}(k, t)|^2.
\]

By using (4.1) and proposition 7.1 we get

\[
\lim_{\epsilon \to 0} \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} \left[ |\hat{F}^+_{\epsilon}(k, t)|^2 + |\hat{F}^-_{\epsilon}(k, t)|^2 \right]
\]

\[
= 2 \int_{R} \mathrm{d}\xi |\hat{p}_0(\xi)|^2 + |\hat{i}_0(\xi)|^2 - \int_{R} \mathrm{d}\xi |\hat{F}^+(\xi, t)|^2 + |\hat{F}^-(\xi, t)|^2.
\]

If \( 2 < \theta < 4 \), then for any \( t \geq 0 \), we have

\[
|\hat{F}^\pm(\xi, t)|^2 = |\hat{F}^\pm(\xi, 0)|^2 = |\hat{p}_0(\xi) \pm \hat{i}_0(\xi)|^2
\]

and thus we obtain

\[
\lim_{\epsilon \to 0} \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} \left[ |\mathcal{F}^+_{\epsilon}(k, t)|^2 + |\mathcal{F}^-_{\epsilon}(k, t)|^2 \right] = 0. \quad (7.2)
\]

Proposition 7.1 and (7.2) imply theorem 4.4 because

\[
\lim_{\epsilon \to 0} \mathbb{E}_\epsilon \left[ \left| \sum_{x \in \mathbb{Z}} \hat{F}^\pm(\xi, t) \left( \frac{t}{n_\theta(\epsilon)} \right) \left( \frac{t}{m_\theta(\epsilon)} \right) J \right| (\epsilon x) \right] \leq \|J\|_{L^2(R)} \lim_{\epsilon \to 0} \left( \int_{T_{\epsilon^{-1}}} \frac{\mathrm{d}k \mathbb{E}_\epsilon}{\epsilon} \left[ |\mathcal{F}^+_{\epsilon}(k, t)|^2 + |\mathcal{F}^-_{\epsilon}(k, t)|^2 \right] \right)^{\frac{1}{2}} = 0.
\]
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Appendix A. Asymptotic behaviour of $\hat{\alpha}(k)$, $k \to 0$

Lemma A.1.

\[
\begin{align*}
C_1(\theta)(2\pi)^{\theta-1}|k|^{\theta-1} + O(|k|^\theta) & \quad 1 < \theta < 2, \\
C_1(2)2\pi|k| + O(k^2 \log |k|^{-1}) & \quad \theta = 2, \\
C_1(\theta)(2\pi)^{\theta-1}|k|^{\theta-1} + C_2(\theta)(2\pi)^2|k|^{3} + O(|k|^\theta) & \quad 2 < \theta < 3, \\
C_1(3)(2\pi)^2|k|^2 \log |2\pi k|^{-1} + C_2(3)(2\pi)^3|k|^3 + O(|k|^\theta) & \quad \theta = 3, \\
C_1(\theta)(2\pi)^2|k|^2 + C_2(\theta)(2\pi)^{\theta-1}|k|^{\theta-1} + O(|k|^\theta) & \quad 3 < \theta < 4, \\
\hat{\alpha}(k) = C_1(\theta)(2\pi)^2|k|^2 + C_2(\theta)(2\pi)^{\theta-1}|k|^{\theta-1} + O(|k|^4 \log |k|^{-1}) & \quad \theta = 4, \\
C_1(\theta)(2\pi)^2|k|^2 + C_2(\theta)(2\pi)^{\theta-1}|k|^{\theta-1} + O(|k|^4) & \quad 4 < \theta < 5, \\
C_1(5)(2\pi)^2|k|^2 + C_2(5)(2\pi)^4|k|^4 \log |k|^{-1} + O(|k|^4) & \quad \theta = 5, \\
C_1(\theta)(2\pi)^2|k|^2 + C_2(\theta)(2\pi)^4|k|^4 + O(|k|^\theta) & \quad 5 < \theta < 7, \\
C_1(\theta)(2\pi)^2|k|^2 + C_2(\theta)(2\pi)^4|k|^4 + O(|k|^\theta) & \quad \theta > 7,
\end{align*}
\]

as $k \to 0$, where $C_1(\theta)$ is defined in (1.7), and $C_2(\theta)$ is defined as

\[
C_2(\theta) := \begin{cases} 
-\int_0^1 \frac{1}{y^{\theta-2}} + \int_1^\infty dy (y^{-\theta-2} - y^{\theta-2}) & \quad 2 < \theta < 3, \\
\frac{3}{2} & \quad \theta = 3, \\
\int_0^\infty dy \frac{2 - 2 \cos y - y^2}{y^\theta} & \quad 3 < \theta < 5, \\
-\frac{1}{12} & \quad \theta = 5, \\
-\frac{1}{12} \sum_{x \geq 1} \frac{1}{x^{\theta-4}} & \quad \theta > 5.
\end{cases}
\]

Moreover, $[y], y \in \mathbb{R}$ is the greatest integer less than or equal to $y$. 

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Proof. First we observe that \( \hat{\alpha} \) can be divided into three parts:

\[
\hat{\alpha}(k) = \sum_{\chi \in \mathbb{Z}} \alpha_k e^{-2\pi\chi k} = 2 \int_1^\infty \frac{1 - \cos 2\pi k\gamma}{y^\theta} = \alpha_1(k) + \alpha_2(k) + \alpha_3(k),
\]

where

\[
\alpha_1(k) := \begin{cases} 
(2\pi)^{\theta-1}|k|^{\theta-1} \int_0^\infty dy \frac{2 - 2 \cos y}{y^\theta} & 1 < \theta < 3, \\
(2\pi)^2 |k|^2 \log |2\pi k|^{-1} & \theta = 3, \\
(2\pi)^2 k^2 \sum_{x > 1} \frac{1}{|x|^{\theta-2}} & \theta > 3,
\end{cases}
\]

\[
\alpha_2(k) := \begin{cases} 
0 & 1 < \theta \leq 2, \\
-(2\pi)^{\theta-1}|k|^{\theta-1} \int_0^{2\pi/k} dy \frac{1}{y^\theta} + (2\pi)^2 |k|^2 \int_1^\infty dy \frac{1}{|y|^{\theta-2}} - \frac{1}{y^\theta} & 2 < \theta < 3, \\
(2\pi)^2 |k|^2 \left\{ \int_1^\infty dy \frac{1}{|y|^\theta} - \frac{1 + 2 - 2 \cos y}{y^\theta} + \int_1^\infty dy \frac{2 - 2 \cos y - y^2}{y^\theta} \right\} & \theta = 3, \\
(2\pi)^{\theta-1}|k|^{\theta-1} \int_0^\infty dy \frac{2 - 2 \cos^2 y - y^2}{y^\theta} & 3 < \theta < 5, \\
\frac{(2\pi)^4 |k|^4 \log |k|^{-1}}{12} & \theta = 5, \\
-\frac{(2\pi)^4 |k|^4}{12} \sum_{x > 1} \frac{1}{x^{\theta-4}} & \theta > 5,
\end{cases}
\]

and

\[
\alpha_3(k) := \begin{cases} 
\int_1^{2\pi/k} dy \frac{2 - 2 \cos 2\pi k\gamma}{y^\theta} - \frac{2 - 2 \cos 2\pi k\gamma}{y^\theta} & 1 < \theta \leq 2, \\
-(2\pi)^{\theta-1}|k|^{\theta-1} \int_0^{2\pi/k} dy \frac{2 - 2 \cos y}{y^\theta} - \frac{2 - 2 \cos 2\pi k\gamma - (2\pi)^2 k^2 y^2}{y^\theta} & 2 < \theta < 5, \\
\int_1^\infty dy \frac{2 - 2 \cos 2\pi k\gamma - (2\pi)^2 k^2 |y|^2}{y^\theta} - \frac{2 - 2 \cos 2\pi k\gamma - (2\pi)^2 k^2 y^2}{y^\theta} & \theta = 5, \\
\frac{(2\pi)^4 |k|^4}{12} \int_1^\infty dy \frac{2 - 2 \cos y - y^2}{y^\theta} & \theta > 5.
\end{cases}
\]

Each \( \alpha_i \) corresponds to \( i \)-th order term of \( \hat{\alpha} \) and we can easily see the asymptotic behaviour of \( \alpha_1, \alpha_2 \). In the following subsections we compute the order of the remainder term \( \alpha_3 \). Since
in the case \( \theta > 5 \) we can repeat almost the same argument, we only compute the cases \( 1 < \theta \leq 2, 2 < \theta < 5, \theta = 5 \). Before showing the asymptotic behaviour of \( \alpha_3 \), we note that if \( \theta = 3 \) then \( \alpha_2(k) = \frac{k}{4} \) because

\[
\int_{1}^{\infty} \frac{1}{y} - \frac{1}{y} = \gamma_E,
\]

\[
\int_{1}^{\infty} \frac{2 - 2 \cos y}{y^3} = 1 - \cos 1 + \int_{1}^{\infty} \frac{\sin y}{y^2} \]
\[= 1 - \cos 1 + \sin 1 - \text{Ci}(1),
\]

\[
\int_{0}^{1} \frac{2 - 2 \cos y - y^2}{y^3} = -\frac{1}{2} + \cos 1 + \int_{0}^{1} \frac{\sin y - y}{y^2} \]
\[= -\frac{1}{2} + \cos 1 - \sin 1 + \int_{0}^{1} \frac{\cos y - 1}{y} \]
\[= -\frac{1}{2} + \cos 1 - \sin 1 - \gamma_E + \text{Ci}(1),
\]

where \( \gamma_E \) is the Euler’s constant and \( \text{Ci} \) is the cosine integral. At the last line we use the following equation

\[
\text{Ci}(x) := -\int_{x}^{\infty} \frac{\cos y}{y} = \gamma_E + \log x + \int_{0}^{x} \frac{\cos y - 1}{y},
\]

### A.1. When \( 1 < \theta < 2 \)

Define \( f_1(y) := \frac{1 - \cos 2\pi ky}{y^\theta}, y > 0 \). Since

\[
f_1([y]) = f_1(y) + f_1'(y)([y] - y) + \frac{1}{2}f_1''(y')([y] - y)^2
\]
for some \([y] \leq y' \leq y\), and

\[
|f_1'(y)| \lesssim |k| \frac{\sin 2\pi ky}{y^\theta}, \quad |f_1''(y')| \lesssim |k|^2 \frac{1}{|y|^\theta},
\]

we have

\[
\left| \int_{1}^{\infty} df_1([y]) - f_1(y) \right| \lesssim |k| \int_{1}^{\infty} \frac{\sin 2\pi ky}{y^\theta} + |k|^2 \int_{1}^{\infty} \frac{1}{[y]^\theta} \]
\[\lesssim |k|^\theta \left\{ \int_{[k]}^{1} \frac{\sin 2\pi y}{y^\theta} + \int_{1}^{\infty} \frac{\sin 2\pi y}{y^\theta} \right\} + |k|^2
\]
\[\lesssim \begin{cases}
|k|^\theta & 1 < \theta < 2, \\
|k|^2 \log |k|^{-1} & \theta = 2.
\end{cases}
\]

In addition, we get

\[
|k|^\theta - 1 \left| \int_{0}^{2\pi |k|} \frac{1 - \cos y}{y^\theta} \right| \lesssim |k|^\theta - 1 \int_{0}^{2\pi |k|} \frac{1}{y^{\theta - 2}} \lesssim |k|^2.
\]
A.2. When $2 < \theta < 5$

Define $f_2(y) := \frac{2 - 2 \cos 2\pi ky - (2\pi ky)^2}{y^\theta}$, $y > 0$. Since

$$f_2(\|y\|) = f_2(y) + f_2'(y)(\|y\| - y) + \frac{1}{2}f_2''(y')(\|y\| - y)^2$$

for some $[y] \leq y' \leq y$, we have

$$|f_2'(y)| \lesssim |k| \frac{|\sin 2\pi ky - 2\pi ky|}{y^\theta},$$

$$|f_2''(y')| \lesssim |k|^2 \left\{ \frac{1 - \cos 2\pi ky'}{(y')^\theta} + \frac{2\pi |k|y' - \sin 2\pi |k|y'}{(y')^\theta + 1} + \frac{\pi^2 k^2(y')^2 - \sin^2 \pi ky'}{(y')^{\theta + 2}} \right\}$$

\begin{align*}
\lesssim \begin{cases} 
|k|[y]^{-\theta} & 2 < \theta \leq 3, \\
|k|^4[y]^{2-\theta} & \theta > 3,
\end{cases}
\end{align*}

and

$$\int_1^\infty dy |f_2'(y)| \lesssim \begin{cases} 
|k|^\theta \left\{ \int_1^{2|k|} dy \frac{y - \sin y}{y^\theta} + \int_1^\infty dy \frac{y - \sin y}{y^\theta} \right\} & 2 < \theta \leq 4, \\
|k|^4 \int_1^\infty dy \frac{1}{y^{\theta - 3}} & 4 < \theta < 5,
\end{cases}$$

\begin{align*}
\lesssim \begin{cases} 
|k|^\theta & 2 < \theta < 4, \\
|k|^4 \log |k|^{-1} & \theta = 4, \\
|k|^4 & \theta > 4.
\end{cases}
\end{align*}

In addition, we get

$$|k|^{\theta - 1} \left| \int_0^{2\pi |k|} dy \frac{2 - 2 \cos y - y^2}{y^\theta} \right| \lesssim |k|^{\theta - 1} \int_0^{2\pi |k|} dy \frac{1}{y^{\theta - 4}} \lesssim |k|^4.$$  

A.3. When $\theta = 5$

Since

$$|k|^4 \left| \int_{2\pi |k|}^1 dy \frac{24 - 24 \cos y - 12y^2 + y^4}{y^5} \right| \lesssim |k|^6,$$

we can see that $\alpha_3(k) = O(|k|^4)$. $\square$
Lemma A.2.

\[ \hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k') \]

\[
= \begin{cases} 
(2\pi)^{\theta-1}C_1(\theta) \left( (k + k')^{\theta-1} - |k|^{\theta-1} - |k'|^{\theta-1} \right) + \mathcal{O}(|k|^{\theta-1}|k'|^{\theta-1}) & 1 < \theta < 2, \\
2\pi C_1(2) \left( (k + k') - |k| - |k'| \right) + \mathcal{O}(|k||k'| \log |k|^{-1} + |k||k'| \log |k'|^{-1}) & \theta = 2, \\
(2\pi)^{\theta-1}C_1(\theta) \left( (k + k')^{\theta-1} - |k|^{\theta-1} - |k'|^{\theta-1} \right) + \mathcal{O}(|k||k'|) & 2 < \theta < 3, \\
(2\pi)^2 \left( |k + k'|^2 \log (|k + k'|^{-1}) - |k|^2 \log (|k|^{-1}) - |k'|^2 \log (|k'|^{-1}) \right) + \mathcal{O}(|k||k'|) & \theta = 3, \\
2(2\pi)^2 C_1(\theta)kk' + \mathcal{O}(kk') & \theta > 3,
\end{cases}
\]

for any \( k, k' \in \mathbb{T} \). In addition, if \( \theta = 3 \) then we obtain

\[
|\hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k')| 
\leq 2(2\pi)^2 |kk'| \sqrt{\log(|k|^{-1}) \log(|k'|^{-1})} + O \left( |kk'| \sqrt{\log(|k|^{-1})} \right)
\]

\[
+ |kk'| \sqrt{\log(|k'|^{-1})}.
\]

Proof. First we observe that

\[
\hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k')
= 2 \sum_{\gamma > 1} \frac{\cos(2\pi kx) + \cos(2\pi k'x) - \cos \left( 2\pi (k + k')x \right) - 1}{|x|^\theta}
= 2 \sum_{\gamma > 1} \frac{\sin(2\pi kx) \sin(2\pi k'x) - (1 - \cos(2\pi kx)) \left( 1 - \cos(2\pi k'x) \right)}{|x|^\theta}.
\]

A.4. When \( 1 < \theta < 3 \)

Define

\[ f_3(y) := 2 \frac{\sin(2\pi kx) \sin(2\pi k'x) - (1 - \cos(2\pi kx)) \left( 1 - \cos(2\pi k'x) \right)}{|y|^\theta}. \]

Then we have

\[ \hat{\alpha}(k) + \hat{\alpha}(k') - \hat{\alpha}(k + k') = \int_1^\infty dy f_3(y) + \int_1^\infty dy f_3(|y|) - f_3(y). \]

For any \( y > 1 \), there exists some \( y' \geq 1 \), \( |y| \leq y' \leq y \) such that

\[ f_3(|y|) = f_3(y) + f_3(y')|y| - y, \]

and we have

\[ |f_3(y')| \lesssim |k| \frac{|\sin 2\pi k'y|}{|y'|^\theta} + |k'| \frac{|\sin 2\pi ky|}{|y|^\theta}. \]
where we use an inequality \(|y|^{-1} \leq 2y^{-1}\) on \(\{y \geq 1\}\). Since

\[
\int_1^\infty \frac{\sin 2\pi ky}{|y|^\theta} \, dy \lesssim \begin{cases} |k|^{\theta-1} & 1 < \theta < 2, \\ |k| \log |k|^{-1} & \theta = 2, \\ |k| & \theta > 2, \end{cases}
\]

and

\[
\int_1^\infty dy f_3(y) = -\int_1^\infty dy \frac{2 - 2 \cos(2\pi k'y)}{|y|^\theta} - \int_1^\infty dy \frac{2 - 2 \cos(2\pi k'x)}{|y|^\theta} = (2\pi)^{\theta-1}C_1(\theta) \left(|k + k'|^{\theta-1} - |k|^{\theta-1} - |k'|^{\theta-1}\right) - 2 \int_0^1 dy \frac{\sin(2\pi ky) \sin(2\pi k'y) - (1 - \cos(2\pi ky))(1 - \cos(2\pi k'y))}{|y|^\theta}
\]

we have

\[
\hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k') = (2\pi)^{\theta-1}C_1(\theta) \left(|k + k'|^{\theta-1} - |k|^{\theta-1} - |k'|^{\theta-1}\right) + \begin{cases} O(|k|^{\theta-1}|k'| + |k| |k'|^{\theta-1}) & 1 < \theta < 2, \\ O(|kk'| \log |k|^{-1} + |kk'| \log |k'|^{-1}) & \theta = 2, \\ O(|kk'|) & \theta > 2. \end{cases}
\]

A.5. When \(\theta = 3\)

In this case, from the following decomposition,

\[
\int_1^\infty dy f_3(y) = (2\pi)^2|k + k'|^2 \log \left([2\pi(k + k')]^{-1}\right) - (2\pi)^2|k|^2 \log \left([2\pi k]^{-1}\right) - (2\pi)^2|k'|^2 \log \left([2\pi k']^{-1}\right) + (2\pi)^2 \left(|k + k'|^2 - |k|^2 - |k'|^2\right) \int_1^\infty dy \frac{2 - 2 \cos(y)}{|y|^3} + (2\pi)^2 \left(|k + k'|^2 - |k|^2 - |k'|^2\right) \int_0^1 dy \frac{2 - 2 \cos(y) - y^2}{|y|^3}
\]

\[
- 2 \int_0^1 dy \frac{\sin(2\pi ky) \sin(2\pi k'y) - 4\pi^2 k'k'y^2}{|y|^3} + 2 \int_0^1 dy \frac{(1 - \cos(2\pi ky))(1 - \cos(2\pi k'y))}{|y|^3},
\]

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we obtain

\[ \hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k') \\
= (2\pi)^2 \left( |k + k'|^2 \log (|k + k'|^{-1}) - |k|^2 \log (|k|^{-1}) \\
- |k'|^2 \log (|k'|^{-1}) \right) + O(|kk'|). \]

Note that by using Schwarz’s inequality we obtain the following boundedness:

\[ \left| \int_{\mathbb{R}} dy f_3(y) \right| \leq 2 \left( \int_{\mathbb{R}} \frac{\sin(2\pi ky) \sin(2\pi k'y)}{|y|^3} \right)^{1/2} \left( \int_{\mathbb{R}} \frac{\sin(2\pi k'y)}{|y|^3} \right)^{1/2} + O(|kk'|) \]

\[ = 2 (2\pi)^2 |k|^2 \log(|k|^{-1}) + O(|k'^2|) \]

\[ + O(|kk'|) \]

\[ = 2 (2\pi)^2 |kk'| \sqrt{\log(|k|^{-1}) \log(|k'|^{-1})} + O \left( |kk'| \sqrt{\log(|k|^{-1}) \log(|k'|^{-1})} \right). \]

\[ \Box \]

Appendix B. On the equivalence of (4.1) and (4.2)

First we observe that

\[ \sum_{x \geq 1} \sin(2\pi kx) \sin(2\pi k'x) \left( 1 - \cos(2\pi kx) \right) \left( 1 - \cos(2\pi k'x) \right) \]

\[ = 2(2\pi)^2 C_1(\theta) |kk'| + 2 \sum_{s \geq 1} \frac{\sin(2\pi kx) \sin(2\pi k'x) - (2\pi)^2 k'k x^2}{|x|^p} \]

\[ - 2 \sum_{s \geq 1} \frac{(1 - \cos(2\pi kx)) \left( 1 - \cos(2\pi k'x) \right)}{|x|^p} \]

\[ = 2(2\pi)^2 C_1(\theta) |kk'| + o(|kk'|). \]
\[
\varepsilon \sum_{x \in \mathbb{Z}} |p_x - p_0(\varepsilon x)|^2 \\
\leq \varepsilon \sum_{x \in \mathbb{Z}} |p_x - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k)|^2 \\
+ \varepsilon \sum_{x \in \mathbb{Z}} |p_0(\varepsilon x) - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k)|^2 \\
= \int_{T_{r-1}} dk |\varepsilon \tilde{p}(k) - \tilde{p}_0(k)|^2 + \varepsilon \sum_{x \in \mathbb{Z}} |p_0(\varepsilon x) - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k)|^2
\]
and
\[
\int_{T_{r-1}} dk |\varepsilon \tilde{p}(k) - \tilde{p}_0(k)|^2 = \varepsilon \sum_{x \in \mathbb{Z}} |p_x - \frac{1}{\varepsilon} \int_T dk e^{2\pi ik \frac{k}{\varepsilon}} p_0 \left( \frac{k}{\varepsilon} \right)|^2 \\
\leq \varepsilon \sum_{x \in \mathbb{Z}} |p_x - p_0(\varepsilon x)|^2 + \varepsilon \sum_{x \in \mathbb{Z}} |p_0(\varepsilon x) - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k)|^2,
\]
where we use Parseval’s identity. Therefore to check the equivalence of (4.1) and (4.2), it is sufficient to show that
\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{x \in \mathbb{Z}} |p_0(\varepsilon x) - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k)|^2 = 0. \quad (B.1)
\]
Since \( p_0 \in C_0^\infty (\mathbb{R}) \), we have
\[
\left| p_0(\varepsilon x) - \int_{T_{r-1}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k) \right| = \left| \int_{|k| > \frac{1}{\varepsilon}} dk e^{2\pi ik \varepsilon} \tilde{p}_0(k) \right| \\
= \left| \frac{1}{2\pi i \varepsilon} \left( e^{-\pi ik} \tilde{p}_0 (\frac{1}{2\varepsilon}) - e^{\pi ik} \tilde{p}_0 (\frac{1}{2\varepsilon}) \right) \\
- \int_{|k| > \frac{1}{\varepsilon}} dk e^{2\pi ik \varepsilon} \partial_k \tilde{p}_0(k) \right| \\
\lesssim \frac{\varepsilon}{|x|},
\]
and thus we obtain (B.1).

**Appendix C. Explicit formulas for some constants**

We restate the result of the scaling limit for thermal energy of the polynomial decay model mentioned in the introduction.
**Theorem (Theorem 1 in [21]).** Under suitable initial conditions and $\gamma > 0$, the following weak convergence holds for any test function $J \in C_0^\infty(\mathbb{R} \times [0, \infty))$:

$$
\lim_{\varepsilon \to 0} \varepsilon \sum_{\lambda \in \mathbb{Z}} \int_0^\infty dt \mathbb{E}_x \left[ \epsilon_x \left( \frac{t}{f_0(\varepsilon)} \right) \right] J(\varepsilon x, t) = \int dy \int_0^\infty \delta T(y, t) J(y, t),
$$

where $f_0(\varepsilon)$ is the time scaling defined as

$$
f_0(\varepsilon) := \begin{cases} 
\varepsilon^{\frac{6}{\theta}} & 1 < \theta < 3, \\
|\theta(\varepsilon)|^3 & \theta = 3, \\
\varepsilon^{\frac{7}{\theta}} & \theta > 3,
\end{cases}
$$

and $h(\cdot)$ is the inverse function of $y \mapsto \left(\frac{y}{\log(\gamma)}\right)^{\frac{1}{\theta}}$ on $[0, 1)$. In addition, $T(y, t)$ is the solution of the following fractional diffusion equation:

$$
\partial_t T(y, t) = \begin{cases} 
-C_{\theta, 3}(\Delta)^{\frac{3}{2}} T(y, t) & 2 < \theta \leq 3, \\
-C_{\theta, 3}(\Delta)^{\frac{3}{2}} T(y, t) & \theta > 3,
\end{cases}
$$

where the positive constant $C_{\theta, 3}$ is given by

$$
C_{\theta, 3} := \begin{cases} 
\frac{24}{\gamma} & \theta < 3, \\
\frac{\sqrt{6}}{12} & \theta > 3,
\end{cases}
$$

where $\csc(y) = (\sin(y))^{-1}$ is the cosecant function and $C_1(\theta)$ is defined in (1.7).

Note that $\lim_{\theta \to 3} C_{\theta, 3} = \infty$, which corresponds to the discontinuity of $f_0(\varepsilon)$ at $\theta = 3$. Next, we introduce the consistency between the constants appearing in the fractional diffusion equation obtained from the exponential and polynomial decay models when $\theta > 3$. In the case $\theta > 3$, $\theta \in C^2(\mathbb{T})$ and $C_{\theta, 3}$ can be written as

$$
C_{\theta, 3} = (2\pi)^{\frac{1}{2}} \frac{\left(\frac{\partial^{(\theta-1)}}{\partial \theta^{(\theta-1)}} (0)\right)^{\frac{1}{2}}}{2\pi^{(3\gamma)}},
$$

and the expression of $C_{\theta, 3}$ is equal to the constant $\delta$ defined in [13, (5.22)] multiplied by $(2\pi)^{-\frac{1}{2}}$. On the other hand, since $\hat{W}$ defined in [13, (5.21)] is the Fourier transform of the thermal energy, by taking the inverse Fourier transform, we can see that the thermal energy $T(y, t)$ for the exponential decay model evolves according to the following $3/4$-superdiffusion equation

$$
\partial_t T(y, t) = - (2\pi)^{-\frac{1}{2}} \hat{c}(\Delta)^{\frac{3}{2}} T(y, t).
$$

From the above, it can be seen that the constants appearing in the fractional diffusion equations are determined in the same way for both models.
ORCID iDs

Hayate Suda https://orcid.org/0000-0003-2150-2870

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