Quasibound states, stability and wave functions of the test fields in 4D Einstein–Gauss–Bonnet gravity

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Abstract

In this work we examine the interaction between quantum particles and the gravitational field generated by a black hole solution that was recently obtained in the 4-dimensional Einstein–Gauss–Bonnet theory of gravity. While quasinormal modes of bosonic, electromagnetic and fermionic fields for this theory have been recently studied, there is no such study for the quasibound states of these quantum particles. Here we calculate the quasibound states of the test fields in a spherically symmetric asymptotically flat black hole spacetime in the consistent 4D Einstein–Gauss–Bonnet theory. The quasispectrum of resonant frequencies is obtained by using the polynomial condition associated to the general Heun functions. We discuss the (in)stability of the systems for values of the Gauss–Bonnet coupling constant being in the range $-16M^2 \leq a \leq 2M^2$.

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I. INTRODUCTION

The quasibound states, quasinormal modes and shadows of black hole are among the most interesting characteristic of such a astrophysical object in the observational (measurable) spectra. In addition, some physical phenomena have been observed in modern experiments involving condensate matter and optical systems. In order to interpret these data, it is used the general theory of relativity, as well as some alternative theories of gravity, since some fundamental questions cannot be solved with the Einstein’s theory, as for example, the issues related to the quantum gravity phenomenon [1–5]. Among these alternative theories, we can mention the f(R), the Lovelock and the Einstein–Gauss–Bonnet gravity, where the last two deals with higher curvature corrections [6, 7].

The Einstein–Gauss–Bonnet (EGB) theory is one of the most promising approaches developed to include higher curvature corrections to the Einstein’s theory (see [8] and references therein). It is quadratic in curvature and only leads to non-trivial corrections of the equation of motion when the Gauss–Bonnet term is coupled to a matter field, which can be, for instance, a dilaton. On the other hand, the so-called consistent theory of 4D EGB gravity was developed by Aoki et al. [9], where they used the Arnowitt–Deser–Misner (ADM) decomposition [10] to construct the Hamiltonian theory that there is not infinite coupling. In this present work, it is of fundamental matter that the black hole solution satisfies the field equations of both approaches [8] and [9]. Here, we will investigate only the test (Klein–Gordon, Maxwell and Dirac) fields, the black hole metric under consideration can be safely used, since it is considered as a consistent solution; the same cannot be said for the case of gravitational perturbations.

In this paper, we calculate the quasibound states of a bosonic, electromagnetic and fermionic perturbations by using the polynomial condition of the Heun’s differential equations and find the wave functions of an asymptotically flat 4-dimensional Einstein–Gauss–Bonnet black hole (4DEGBBH). We show that the quasibound states depend on the Gauss–Bonnet coupling. There is instability when the coupling constant has positive values $0 < a \leq 0.5$ for states with $l \geq 1$. We also show that there is no such instability for negative values of the coupling constant. In addition, we also compute the wave functions of the test fields in the 4DEGBBH.

The paper is organized as follows. In Sec. III we introduce the metric corresponding
to the 4-dimensional Einstein–Gauss–Bonnet black hole. In Sec. III we solve the master wave equations in the background under consideration. Section IV is devoted to quasibound states of the test fields. In Sec. V we provide the eigenfunctions by using some properties of the general Heun functions. Finally, in Sec. VI we summarize the obtained results. Here we adopt the natural units where $G \equiv c \equiv \hbar \equiv 1$.

II. THE 4-DIMENSIONAL EINSTEIN–GAUSS–BONNET BLACK HOLE

A crucial aspect for our studies of black hole radiation (emission, transmission and/or reflection) is such that the black hole solution obtained from the dimensional regularization approach [8] should also be a solution of the truly 4-dimensional theory [9], and from the theories with extra scalar degrees of freedom [11–14] as well.

Thus, a novel 4-dimensional Einstein–Gauss–Bonnet theory, as well as the black hole metric, was described by Konoplya and Zinhailo [15], in which all of the aforementioned approaches are taken into account. Next, we briefly review the basic ideas behind this novel approach.

In a 4-dimensional spacetime, the Einstein’s theory of general relativity is described by the Einstein–Hilbert action, which is given by

$$S_{EH} = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R \right),$$

(1)

where $R$ is the Ricci scalar, and $M_P$ is the Planck mass characterizing the gravitational coupling strength. On the other hand, in higher than four dimensions ($D > 4$), the Gauss–Bonnet action satisfying the Lovelock’s theorem [16] is given by

$$S_{GB} = \int d^Dx \sqrt{-g} \ a \ G,$$

(2)

where $a$ is the (dimensionless) Gauss–Bonnet coupling constant. The Gauss–Bonnet invariant $G$ is defined as $G = R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} - 4 R_{\mu} R^{\mu} + R^2$. Then, by following the dimensional regularization approach [8], we rescale the coupling constant as $a \rightarrow \tilde{a}/(D - 4)$ and hence we take the limit when $D \rightarrow 4$. Therefore, the exact solution describing a 4-dimensional Einstein–Gauss–Bonnet black hole has the following form

$$ds^2 = -f(r) \ dt^2 + \frac{1}{f(r)} \ dr^2 + r^2 \ d\theta^2 + r^2 \sin^2 \theta \ d\phi^2,$$

(3)
where the metric function \( f(r) \) is given by
\[
f(r) = 1 + \frac{r^2}{a} \left( 1 \pm \sqrt{1 + \frac{4aM}{r^3}} \right). \tag{4}
\]
Note that we have redefined the coupling constant as \( a = 32\pi\tilde{a} \), as well as the Newton’s constant is \( G = 1/(8\pi M^2) \rightarrow 1 \). The parameter \( M \) is the total mass centered at the origin of the system of coordinates. The signs \( \pm \) are related to two branches for the solution of the field equations: the minus sign leads to the asymptotically flat solution, while the plus sign leads to an effective asymptotically de Sitter solution. Here we will analyze the minus case.

The exterior (+) and interior (−) event horizons are the solutions of the surface equation \( f(r) = 0 = (r - r_+)(r - r_-) \) and given by
\[
r_{\pm} = M \left( 1 \pm \sqrt{1 + \frac{a}{2M^2}} \right). \tag{5}
\]
Here, there are two branch of event horizon solutions if \( a > 0 \), otherwise there is only one (positive) event horizon if \( a < 0 \). Nevertheless, since the 4DEGBBH metric is well-behaved outside the event horizon, we will obtain some results on quasibound states that are also valid for negative values of \( a \). It is worth noticing that the exterior black hole solution exists for values of the Gauss–Bonnet coupling constant \( a \) being in the range \(-16M^2 \leq a \leq 2M^2 \). Finally, from now on, for simplicity, we define \( M = 1/2 \) in our calculations.

In what follows, we will consider the equations of motion, quasibound states and wave functions of the above black holes.

### III. MASTER WAVE EQUATIONS

The general covariant equations for the test bosonic \( \Phi \), electromagnetic \( F_{\mu\nu} \) and fermionic \( \Upsilon \) fields have the following forms
\[
\frac{1}{\sqrt{-g}} \partial_{\mu}(g^{\mu\nu} \sqrt{-g} \partial_{\nu} \Phi) = 0, \tag{6}
\]
\[
\frac{1}{\sqrt{-g}} \partial_{\mu}(F^{\mu\nu} \sqrt{-g}) = 0, \tag{7}
\]
\[
\gamma^\alpha \left( \frac{\partial}{\partial x^\alpha} - \Gamma_\alpha \right) \Upsilon = 0, \tag{8}
\]
where \( F^{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the electromagnetic field tensor, \( A_\mu \) is the 4-vector potential, \( \gamma^\alpha \) are the noncommutative gamma matrices, and \( \Gamma_\alpha \) are the spin connections in the tetrad formalism.
Next, we need to separate the angular and radial parts of these master equations. To do this, we follow the approaches described by Konoplya and Zinhailo [15], Churilova [17], and Aragón et al. [18], which have been used to deal with the Maxwell, Klein–Gordon and Dirac equations, respectively. Summarizing, after some algebra, the radial parts of the master equations are given as follows.

The case of a scalar field is simpler, so that we can use the expansion in terms of the spherical harmonic functions, namely, \( \Phi(r) = \frac{R_B(r)}{r} Y_{lm}(\theta, \phi)e^{-i\omega t} \), where \( \omega \) is the frequency (energy) of the (scalar) particle, \( m = -l, \ldots, +l \) is the azimuthal quantum number, and \( l = 0, 1, 2, \ldots \) is the angular quantum number. Thus, the bosonic radial function \( R_B(r) \) satisfies

\[
\frac{d^2 R_B(r)}{dr^2} + \frac{1}{f(r)} \frac{df(r)}{dr} \frac{dR_B(r)}{dr} + \frac{1}{[f(r)]^2} \left\{ \omega^2 - f(r) \left[ \frac{l(l+1)}{r^2} + \frac{1}{r} \frac{df(r)}{dr} \right] \right\} R_B(r) = 0. \tag{9}
\]

In the case of an electromagnetic field, we expand \( A_\mu(r) \) in terms of the vector spherical harmonic functions, as well as of the functions \( f^{lm}_j(t, r) \), with \( j = 1, 2, 3, 4, 5 \). The time dependence is assumed as \( e^{-i\omega t} \). Thus, the electromagnetic radial function \( R_E(r) \) satisfies

\[
\frac{d^2 R_E(r)}{dr^2} + \frac{1}{f(r)} \frac{df(r)}{dr} \frac{dR_E(r)}{dr} + \frac{1}{[f(r)]^2} \left\{ \omega^2 - f(r) \left[ \frac{l(l+1)}{r^2} \right] \right\} R_E(r) = 0. \tag{10}
\]

Finally, for the case of a fermionic field, we can expand the wave function as \( \Upsilon = \frac{e^{-i\omega t}}{r[f(r)]^{1/4}} R_F^\pm(r) \otimes \varsigma(\theta, \phi) \), where \( \pm \) labels the spins (a 2x1 matrix), and \( \varsigma(\theta, \phi) \) is a 2-components fermion. Thus, the fermionic radial function \( R_F^\pm(r) \) satisfies

\[
\frac{d^2 R_F^\pm(r)}{dr^2} + \left[ \frac{1}{2f(r)} \frac{df(r)}{dr} + \frac{1}{r} \right] \frac{dR_F^\pm(r)}{dr} + \frac{1}{[f(r)]^2} \left\{ \omega^2 - f(r) \left[ \frac{(l+1)^2}{r^2} + \frac{i\omega}{r} \pm \frac{i\omega}{2f(r)} \frac{df(r)}{dr} \right] \right\} R_F^\pm(r) = 0. \tag{11}
\]

Now, in order to solve these radial equations, let us write all of them as a single radial equation, namely,

\[
\frac{d^2 R_s(r)}{dr^2} + \left[ \left( \frac{1}{2} \right)^{4s(1-s)} \frac{1}{f(r)} \frac{df(r)}{dr} + \frac{4s(1-s)}{r} \right] \frac{dR_s(r)}{dr} + \frac{1}{[f(r)]^2} \left[ \omega^2_s - V_s(r) \right] R_s = 0, \tag{12}
\]
with the effective potential \( V_s(r) \) given by

\[
V_s(r) = f \left\{ \frac{\lambda_l}{r^2} - 4s(1-s) \frac{i \omega_s}{r} + \left[ \frac{2(s-1)(s-\frac{1}{2})}{r} + 4s(1-s) \frac{i \omega_s}{2f(r)} \right] df(r) \right\},
\]

(13)

where \( \lambda_l = 4s(1-s) + l[l + 2^{4s(1-s)}] \) and \( s = (0, 1, \frac{1}{2}) = (B, E, F) \). Note that we have chosen the + spin for our calculation of quasibound states. In fact, the potentials \( V_{+\frac{1}{2}} \) and \( V_{-\frac{1}{2}} \) can be transformed one into another by using the Darboux transformation, which means that the both potentials give the same quasispectrum. Then, we can provide an analytic solution for the radial equation \([12]\) in terms of the general Heun functions \([19]\), by following the approach described in our recent work \([20]\). It is given by

\[
R_{sl}(x) = x^{\frac{2s - 2^{4s(1-s)} - 2}{2}} (x - 1)^{\frac{2s - 2^s - 2}{2}} (x - b)^{-\frac{2s - 2^s(1-s)}{2}}
\]

\[
\times [C_1 \ \text{HeunG}(b, q_{sl}; \alpha_{sl}, \beta_{sl}, \gamma_s, \delta_s; x)
\]

\[
+ C_2 \ x^{1 - \gamma_s} \ \text{HeunG}(b, q_{2sl}; \alpha_{2sl}, \beta_{2sl}, \gamma_{2s}, \delta_s; x)],
\]

(14)

with the new radial coordinate \( x \) and the singularity parameter \( b \) defined as

\[
x = \frac{r - r_+}{r_- - r_+},
\]

(15)

\[
b = \frac{r_+}{r_- - r_+},
\]

(16)

where \( C_1 \) and \( C_2 \) are constants to be determined. These are two linearly independent solutions of the general Heun equation since \( \gamma_s \) is not a negative (positive) integer in the first (second) solution, which corresponds to the exponents 0 and \( 1 - \gamma_s \) at \( x = 0 \). The parameters \( \gamma_s, \delta_s, \epsilon_{sl} \) and \( q_{sl} \), which depend on the spin \( s \) and angular quantum number \( l \), are given by

\[
\gamma_s = 1 - \sqrt{[2^{4s(1-s)} - 1]^2 - \frac{4 \omega_s [2is(s-1)(r_+ - r_-) + \omega_s]}{(r_+ - r_-)^2},}
\]

(17)

\[
\delta_s = 1 - \frac{(-1)^{4s(1-s)}}{\sqrt{[2^{4s(1-s)} - 1]^2 - \frac{4 \omega_s [2is(s-1)(r_+ - r_-) + \omega_s]}{(r_+ - r_-)^2}},}
\]

(18)

\[
\epsilon_{sl} = 1 - \sqrt{(1 - 2s)^4 + \frac{4s(1-s) + l[l + 4^{2s(1-s)}]}{r_+ r_-},}
\]

(19)

\[
q_{sl} = \frac{1}{2} \left\{ \gamma_s \epsilon_{sl} + 2^{4s(1-s)+2}(s-1)s
\right.
\]

\[
+ b \left[ \gamma_s \delta_{s} - \frac{2l[l + 2^{4s(1-s)-s}] + 8(1-s)s}{r_+^4} + \frac{2(s-1)(2r_- s - r_- - 4is \omega_s)}{r_+}
\right.
\]

\[
- \frac{4 \omega_s^2}{(r_+ - r_-)^2} - 4s^2 - 2^{4s(1-s)} + 6s - 2
\left\}ight.
\]

(20)
The general expressions of the parameters $\alpha_{sl}$ and $\beta_{sl}$ are quite long that no insight is gained by writing them out, instead we will write their expressions for each value of the spin $s$. They are given by

\[
\alpha_{Bl} = \frac{1}{2} \left[ 5 - \sqrt{1 + \frac{4l(l+1)}{r_+r_-} - \frac{4i\omega_B}{r_+ - r_-}} \right], \\
\alpha_{El} = \frac{1}{2} \left[ 3 - \sqrt{1 + \frac{4l(l+1)}{r_+r_-} - \frac{4i\omega_E}{r_+ - r_-}} \right], \\
\alpha_{Fl} = \frac{1}{4} \left[ 6 - \frac{4(l+1)}{\sqrt{r_+r_-}} - \frac{8i\omega_F}{r_+ - r_-} \right], \quad (21)
\]

and

\[
\beta_{Bl} = -\frac{1}{2} \left[ 1 + \sqrt{1 + \frac{4l(l+1)}{r_+r_-} + \frac{4i\omega_B}{r_+ - r_-}} \right], \\
\beta_{El} = \frac{1}{2} \left[ 1 - \sqrt{1 + \frac{4l(l+1)}{r_+r_-} - \frac{4i\omega_E}{r_+ - r_-}} \right], \\
\beta_{Fl} = \frac{1}{4} \left[ 2 - \frac{4(l+1)}{\sqrt{r_+r_-}} - \frac{8i\omega_F}{r_+ - r_-} \right]. \quad (22)
\]

Finally, for the second solution, the parameters $\alpha_{2,sl}$, $\beta_{2,sl}$, $\gamma_2$, and $q_{2,sl}$ are given by

\[
\alpha_{2,sl} = \alpha_{sl} + 1 - \gamma_s, \quad (23) \\
\beta_{2,sl} = \beta_{sl} + 1 - \gamma_s, \quad (24) \\
\gamma_2 = 2 - \gamma_s, \quad (25) \\
q_{2,sl} = q_s + (\alpha_{sl}\delta_s + \epsilon_{sl})(1 - \gamma_s). \quad (26)
\]

In what follows, we will use this analytical solution of the radial equation in the 4DEG-BBH spacetime, and the properties of the general Heun functions as well, to discuss the quasibound state frequencies.

**IV. QUASIBOUND STATES**

The quasibound states, also known as quasistationary levels or resonance spectra, are localized in the black hole potential well and hence obey to two boundary conditions: the radial solution should describe an ingoing wave at the exterior event horizon and tend to zero far from the black hole at asymptotic infinity.
Thus, the flux of particles crosses into the black hole, so that the spectrum of bound states has complex frequencies (energies), which can be expressed as \( \omega = \omega_R + i \omega_I \), where \( \omega_R = \text{Re}[\omega] \) and \( \omega_I = \text{Im}[\omega] \) are the real and imaginary parts, respectively. In fact, the real part is the oscillation frequency, while the imaginary part determines the (in)stability of the system. The wave solution is said to be stable when the imaginary part of the resonant frequencies is negative (\( \text{Im}[\omega] < 0 \)), which means a decay rate with the time. Otherwise, the wave solution is unstable when the imaginary part of the resonant frequencies is positive (\( \text{Im}[\omega] > 0 \)), which means a growth rate with the time.

In order to derive the characteristic resonance equation, many authors have been using a method that consists of solving the radial equation in two different asymptotic regions and then matching these two radial solutions in their common overlap region. In this paper, we will use the Vieira–Bezerra–Kokkotas method \([20, 21]\) to find the resonant frequencies related to quasibound states. This method consists in to obtain the ingoing wave solution at the exterior event horizon and then to impose the polynomial condition related, in this case, to the general Heun functions.

In the limit when \( r \to r_+ \), which implies that \( x \to 0 \), the radial solution (14) behaves as

\[
\lim_{r \to r_+} \mathcal{R}_{sl}(r) \sim C_1 \left( r - r_+ \right)^{\frac{\gamma_s - 24s(s-1)}{2}} + C_2 \left( r - r_+ \right)^{\frac{-24s(s-1)}{2}},
\]

where \( C_1 \) and \( C_2 \) include all remaining constants. On the other hand, from Eq. (20), we get

\[
\frac{\gamma_s - 24s(s-1)}{2} = -\frac{i\omega_s}{2\kappa_+},
\]

where the gravitational acceleration on the exterior event horizon \( \kappa_+ (> 0 \forall s) \) is given by

\[
\kappa_+ \equiv \frac{1}{2} \left. \frac{df(r)}{dr} \right|_{r=r_+} = \frac{r_+ - r_-}{2}.
\]

Thus, we can rewrite Eq. (27) as

\[
\lim_{r \to r_+} \mathcal{R}_{sl}(r) \sim C_1 \mathcal{R}_{sl}^{\text{in}} + C_2 \mathcal{R}_{sl}^{\text{out}},
\]

where the ingoing and outgoing radial solutions are given by

\[
\mathcal{R}_{sl}^{\text{in}} = \mathcal{R}_{sl}^{\text{in}}(r > r_+) = (r - r_+)^{-\frac{i\omega_s}{2\kappa_+}},
\]

\[
\mathcal{R}_{sl}^{\text{out}} = \mathcal{R}_{sl}^{\text{out}}(r > r_+) = (r - r_+)^{\frac{i\omega_s}{2\kappa_+}}.
\]
Therefore, in order to fully satisfy the first boundary condition, we must impose that $C_2 = 0$ in Eq. (30), and in (14) as well. Thus, we get

$$\lim_{r \to r_+} R_{sl}(r) \sim C_1 R_{sl}^{in}.$$  \hspace{1cm} (33)

Now, in the limit when $r \to \infty$, which implies that $x \to \infty$, the radial solution (14) behaves as

$$\lim_{r \to \infty} R_{sl}(r) \sim C_1 r^{\sigma_{sl}},$$  \hspace{1cm} (34)

with $\sigma_{sl} = D_{sl} - \alpha_{sl}$, where the coefficients $D_{sl}$ are given by

$$D_{Bl} = \frac{1}{2} \left[ 1 - \sqrt{1 + \frac{4l(l + 1)}{r_1 r_2}} \right] - \frac{2i\omega_B}{r_1 - r_2} - 1,$$

$$D_{El} = \frac{1}{2} \left[ 1 - \sqrt{1 + \frac{4l(l + 1)}{r_1 r_2}} \right] - \frac{2i\omega_E}{r_1 - r_2} - 1,$$

$$D_{Fl} = \frac{1}{2} \left[ 1 - 2(l + 1) \sqrt{\frac{1}{r_1 r_2}} \right] - \frac{2i\omega_F}{r_1 - r_2} - \frac{3}{2}.$$  \hspace{1cm} (35)

The sign of the real part of $\sigma_{sl}$ determines the behavior of the wave function far from the black hole at asymptotic infinity, that is, in the limit when $r \to \infty$. The radial solution tends to zero if $\text{Re}[\sigma_{sl}] < 0$, which describes the quasibound states; whereas if $\text{Re}[\sigma_{sl}] > 0$, the radial solution diverges. Therefore, the final behavior of the radial solution will be determined when we know the values of the frequencies $\omega_s$.

The resonant frequencies, which describe quasibound states of the test fields in a 4DEG-BBH spacetime, can be found by imposing the polynomial (alpha-)condition related to the general Heun functions, namely,

$$\alpha = \alpha_{sln} = -n,$$  \hspace{1cm} (36)

where $n = 0, 1, 2, \ldots$ is the principal quantum number (or the overtone number). Thus, by imposing the polynomial (alpha-)condition (36) to the parameters $\alpha$ (21), we obtain the following expressions for the massless resonant frequencies

$$\omega_B = \omega_{Bln} = i \sqrt{\frac{1 - 2a}{16a}} \left[ \sqrt{a + 8l(l + 1)} - \sqrt{a(5 + 2n)} \right],$$

$$\omega_E = \omega_{El} = i \sqrt{\frac{1 - 2a}{16a}} \left[ \sqrt{a + 8l(l + 1)} - \sqrt{a(3 + 2n)} \right],$$

$$\omega_F = \omega_{Fl} = i \sqrt{\frac{1 - 2a}{32a}} \left[ 4(l + 1) - \sqrt{2a(3 + 2n)} \right].$$  \hspace{1cm} (37)
TABLE I. Values of the resonant frequencies $\omega_{sln}$, and the real part of the corresponding coefficients $\sigma_{sln}$. We focus on the fundamental mode $l = n = 0$.

| $a$         | $\omega_{B00}$ | Re[$\sigma_{B00}$] | $\omega_{E00}$ | Re[$\sigma_{E00}$] | $\omega_{F00}$ | Re[$\sigma_{F00}$] |
|-------------|----------------|--------------------|----------------|--------------------|----------------|--------------------|
| $-0.30$     | $-1.26491i$    | $-3.00$            | $-0.63246i$    | $2.00$             | $1.63299 - 0.94868i$ | $-2.50$            |
| $-0.15$     | $-1.14018i$    | $-3.00$            | $-0.57009i$    | $2.00$             | $2.08167 - 0.85513i$ | $-2.50$            |
| $-0.01$     | $-1.00995i$    | $-3.00$            | $-0.50497i$    | $2.00$             | $7.14143 - 0.75746i$ | $-2.50$            |
| $\lim_{a \to 0}$ | $-1.00000i$    | $-3.00$            | $-0.50000i$    | $2.00$             | Indeterminate         | $-2.50$            |
| $+0.01$     | $-0.98995i$    | $-3.00$            | $-0.49497i$    | $2.00$             | $6.25754i$             | $-2.50$            |
| $+0.15$     | $-0.83666i$    | $-3.00$            | $-0.41833i$    | $2.00$             | $0.90003i$              | $-2.50$            |
| $+0.30$     | $-0.63246i$    | $-3.00$            | $-0.31623i$    | $2.00$             | $0.34215i$              | $-2.50$            |

Note that this polynomial (alpha-)condition implies $\sigma_{sln} = D_{sln} + n$.

Some characteristic values of the massless resonant frequencies $\omega_{sln}$, as well as the corresponding coefficients $\sigma_{sln}$, are shown in Table I as functions of the Gauss–Bonnet coupling $a$.

From Table I we conclude that all solution for the massless resonant frequencies, $\omega_{sln}$, are physically admissible in the fundamental mode, which represent the quasibound states of massless test particles in the 4DEGBBH spacetime. In this case, the radial solution (14) tends to zero far from the 4DEGBBHs at asymptotic infinity, since Re[$\sigma_{sln}$] < 0, as required by the conditions for quasibound states. In addition, we see that the motion of both bosonic and electromagnetic fields are over-damped (with purely imaginary frequencies) in the fundamental mode. On the other hand, the case of fermionic fields may describe an unstable system, since there is a change in the sign of their imaginary part when $a > 0$.

The behavior of the massless resonant frequencies $\omega_{sln}$ is shown in Fig. 1, as functions of the Gauss–Bonnet coupling $a$.

In Fig. 1 we see that the imaginary part of the massless resonant frequencies $\omega_{sln}$ increases (in modulus) as the Gauss–Bonnet coupling $a$ approaches to zero (for all the test fields in the first excited modes), but it reaches a limiting value when $a \to 0$, as displayed in Table I. In these modes, we may conclude that the 4DEGBBHs are unstable, since the imaginary part of the massless resonant frequencies changes its sign for $a > 0$.

The time dependence $\Psi_{sln}(t) = e^{-i\omega_{sln}t}$ is shown in Figs. 2 and 3 for the Gauss–Bonnet
FIG. 1. Massless resonant frequencies in the 4DEGBBH spacetime. The right plots show the decay (or growth) rate $\text{Im} [\omega_{sln}]$, while the left plots show the oscillation frequency $\text{Re} [\omega_{sln}]$. We focus on the first excited modes.
FIG. 2. Time dependence of the test fields in the 4DEGBBH spacetime for $a = -0.15$. We focus on the first excited mode $l = n = 1$.

coupling $a = \pm 0.15$.

From Figs. 2 and 3 we can realize the “final flight” of the test particles crossing the 4DEGBBH exterior event horizon. Note that the motion (in the first excited mode) is over-damped for $a = +0.15$.

In addition, we can calculate the transition frequency, $\Delta \omega$, between two highly damped ($n \to \infty$) neighboring states [22]. It is given by

$$\Delta \omega \approx \text{Im}[\omega_{s(n-1)}] - \text{Im}[\omega_{sn}] = \kappa_+ = \frac{1}{2} \sqrt{1 - 2a}. \quad (38)$$

On the other hand, the natural adiabatic invariant quantity, $I_{ad}$, for a test field-black hole system with total energy $E$, is given by

$$I_{ad} = \int \frac{dE}{\Delta \omega} = \int T_H dS_{\text{BH}} = \frac{\hbar S_{\text{BH}}}{2\pi}, \quad (39)$$

where $T_H$ is the Hawking temperature and $S_{\text{BH}} (= A_+/4\hbar)$ is the Bekenstein–Hawking entropy, with $A_+$ being the surface area of the exterior event horizon. In this limit, the Bohr–Sommerfeld quantization condition applies and hence $I_{ad}$ is a quantized quantity, namely, $I_{ad} = nh$. Thus, from Eq. (39) we get

$$S_{\text{BH},n} = 2\pi n. \quad (40)$$
FIG. 3. Time dependence of the test fields in the 4DEGBBH spacetime for $a = +0.15$. We focus on the first excited mode $l = n = 1$.

Therefore, the area spectrum is given by

$$A_{+n} = 8\pi n\hbar,$$  \hspace{1cm} (41)

so that its minimum change becomes

$$\Delta A_{+}^{\text{min}} = 8\pi \hbar.$$  \hspace{1cm} (42)

From Eqs. (40) and (41), we can conclude that both entropy and area spectrum are equally spaced. Furthermore, they do not depend on the black hole parameters. Finally, Eq. (42) shows that the 4DEGBBH exterior event horizon is made by patches with equal area.

It is worth pointing out that these massless resonant frequencies were obtained directly from the general Heun functions by using a polynomial condition, and, to our knowledge, there is no similar result in the literature for the background under consideration.

In what follows, we will use this polynomial condition, and the frequency eigenvalues as well, to discuss the wave functions.
V. EIGENFUNCTIONS

In this section we will use the Vieira–Bezerra–Kokkotas method \[20, 21\] to derive the wave functions describing massless test particles that propagate in the 4DEGBBH spacetime. This approach consists of using some properties of the general Heun functions, as well as their polynomial (alpha-)condition, to write the general Heun polynomials and then obtains the wave eigenfunctions.

The general Heun polynomials are denoted as \( H_{p,n,m}(x) \). Here, \( m (= 0, 1, \ldots, n) \) is related to the properly choose of the accessory parameter \( q = q_{n,m} \), which are the solutions that cut (in a certain order) the power series describing the general Heun functions. In our case, general Heun polynomials depend on the spin \( s \), as well as on the angular quantum number \( l \), of the test fields and should be denoted as \( H_{p_{s,l,n,m}}(x) \) and \( q_{s,l,n,m} \). Thus, the general Heun polynomials for the fundamental and first excited modes are given, respectively, by \[20\]

\[
H_{p_{s,l,0,0}}(x) = 1,
\]

\[
H_{p_{s,l,1,0}}(x) = 1 + \frac{\sqrt{\Delta_{sl} - x}}{2b\gamma_s},
\]

\[
H_{p_{s,l,1,1}}(x) = 1 + \frac{\sqrt{\Delta_{sl} + x}}{2b\gamma_s} \tag{43}
\]

where \( \Delta_{sl} = [\gamma_s(1 + b\delta_s + \epsilon_{sl})]^2 + 4b\alpha_{sl}\beta_{sl}\gamma_s \). Then, the radial wave eigenfunctions, for massless test particles propagating in the 4DEGBBH spacetime, can be written as

\[
U_{s,l,n,m}(x) = \frac{R_{s,l,n,m}(x)}{u_{s}(x)} = C_{s,l,n,m} x^{\frac{s-2s(1-s)}{2}}(x - 1)^{\frac{s}{2} - \frac{s(1-s)}{4}} (x - b)^{\frac{s}{2} - \frac{s(s-1)}{4}} \frac{H_{p_{s,l,n,m}}(x)}{u_{s}(x)}, \tag{44}
\]

where \( C_{s,l,n,m} \) is a constant to be determined and \( x \) is given by Eq. (15). The functions \( u_{s}(r) \) are given by

\[
u_B(r) = r, \]

\[
u_E(r) = r, \]

\[
u_F(r) = r[f(r)]^{1/4}. \tag{45}\]

The first three squared radial wave eigenfunctions are presented in Figs. 4 and 5.

From Figs. 4 and 5 we conclude that the massless resonant frequencies \( \omega_{s,l,n} \) are quasi-bound states, since the radial solution tends to zero at infinity and diverges at the exterior event horizon.
FIG. 4. The first three squared eigenfunctions for $a = -0.15$. The units are in multiples of $C_{sln,m}$. We focus on the modes for $l = 1$. 
FIG. 5. The first three squared eigenfunctions for $a = +0.15$. The units are in multiples of $C_{sln,m}$. We focus on the modes for $l = 1$. 
VI. CONCLUSIONS

In this work, we obtained the exact analytical solutions of the master wave equations for the test fields in a 4-dimensional Einstein–Gauss–Bonnet black hole spacetime. The radial parts of these solutions are given in terms of the general Heun functions.

We imposed two boundary conditions on the radial solutions in order to study their asymptotic behaviors, which led to the quasibound state phenomena. Near the exterior event horizon, the radial solutions describe test fields crossing into the black hole, while far from the black hole at the asymptotic infinity, these solutions tend to zero, that is, the probability of finding some particles is null there.

The spectra of quasistationary levels for the test fields was obtained by using the polynomial (alpha-)condition of the general Heun functions.

Finally, we have discussed the (in)stability of the system. The systems constituted by bosonic or electromagnetic fields are stable and present an over-damped motion. On the other hand, a system with fermionic fields is unstable, since the imaginary part of the resonant frequencies has positive values for $a > 0$.

We hope that our results, which describe an unquestionably phenomenon associated with purely quantum effects in gravity, may be used to fit some astrophysical data in the near future and hence shed some light on the physics of black holes.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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