I. INTRODUCTION

Despite its long and venerable history, light propagation in disordered media continues to be a fascinating and intensely studied problem. In particular, the discovery of the coherent back scattering peak has triggered an intensive search for Anderson localization of light. Clearly, the unambiguous demonstration of Anderson localization of electromagnetic radiation in an appropriate disordered dielectric medium continues to be the unambiguous demonstration of Anderson localization of light.

In fact, early works reported anomalously low values of the diffusion constant in strongly scattering suspensions. However, it was soon realized that these low values of the diffusion constant are associated with the resonant (Mie) scattering of individual scatterers. This leads to a frequency-dependent dwell-time that has to be added to the time-of-flight between successive scattering events. As a result, the energy transport velocity acquires a corresponding renormalization, while the transport mean free path remains essentially unchanged.

Similarly, an analysis of the coherent back scattering peak together with the dependence of the transmission through disordered semiconductor powders on the sample size, suggested a scaling behavior consistent with the onset of Anderson localization. However, a reexamination of these data ignited a heated debate as to how to discriminate Anderson localization from absorption effects. Subsequent experiments on similarly strongly scattering semiconductor powders did not produce evidence of Anderson localization and the experimental data could be well explained using a recently developed effective medium theory that incorporates the resonant scattering effects alluded to above.

This unsatisfactory state of affairs has generated renewed interest in determining novel and unambiguous pathways to Anderson localization of light.

One class of highly interesting systems for multiple scattering of light are disordered dielectric media whose constituent materials exhibit one or more forms of optical anisotropies. Moreover, most optical anisotropies exhibit a certain degree of tunability through external control parameters, thus creating the possibility of a tunable disorder.

Equally intriguing is to combine the effects of multiple scattering of light with optical gain and to investigate how the two phenomena mutually influence each other. From the optical gain point of view, diffusing light will spend more time interacting with active material in a characteristic volume than ballistically propagating light. If this interaction time exceeds the (sponta-
neous) decay time of the active material, avalanche-like intensity bursts, induced by incoherent feedback, could occur. In fact, early theoretical work suggested this very possibility and has recently been observed. From a wave propagation point of view, optical gain increases the relative weight of long trajectories in the sample and, therefore, will modify a number of wave interference effects such as coherent back scattering and possibly Anderson localization. Very recent experiments point to the possibility that such long trajectories provide a feedback mechanism which leads to modes with narrow laser-like emission lines that extend across the entire sample. However, the coherence properties of the emitted light still need to be analyzed in order to establish that laser action is indeed taking place in these systems. True laser action from localized regions in disordered dielectric samples with optical gain, termed Random Lasing, has been observed earlier, where measurements of the photon statistics of the emitted light have unambiguously demonstrated a coherent laser feedback mechanism. It has been discussed in the form of varying degrees of coupling between so-called quasi-states. For a recent review of multiple scattering in amplifying media, we refer the reader to Ref. Random Lasing has potential applications ranging from micro-lasers and optical fingerprint markers to the detection of cancerous tissue. In addition, we want to note that recently, electrically pumped Random Laser action has been achieved in Nd-doped powders, thus creating potential applications of these systems in omnidirectional lighting devices and displays.

Despite these exciting developments, a convincing connection between Random Lasing and Anderson localization of light has not been demonstrated as of yet. This may be attributed to the fact, that to date the theory of random amplifying media either employs purely numerical methods in one spatial dimension or treats the multiple scattering part within certain approximation schemes that cannot account for the interference effects that lead to Anderson localization. These schemes include modeling the electromagnetic wave propagation through diffusion equations for the intensity or within the so-called ladder-approximation of the Bethe-Salpeter equation. As a result, it is unclear, whether Anderson localization of light is a necessary condition for true Random Lasing (coherent feedback) nor whether the modified coherence properties in the lasing state have, in turn, an influence on the transition from the diffusion regime to the Anderson localized regime itself. In fact, the very concept of describing the Anderson localization transition in terms of a vanishing diffusion coefficient as an order parameter, familiar from systems with energy or particle number conservation, becomes questionable in dissipative or active media, where another channel for change of the energy density exists besides diffusion.

In the present paper, we report our progress towards answering these questions. We develop a fully vectorial transport theory for multiple scattering of light in random media whose constituent materials are isotropic and exhibit linear absorption or gain. In Section II, we derive the tensorial kinetic equation for the intensity correlation function of electromagnetic radiation. The conservation of energy in media without loss or gain is incorporated in a field theoretical way by means of an exact Ward identity (WI), the effects of loss, gain as well as frequency dependence of the material parameters being represented by additional terms. The proof of this generalized WI is presented in Section III. Subsequently, we solve the kinetic equation in the hydrodynamic limit by formulating, with the help of the WI, the continuity equation in Section IV and Fick’s law in Section V, respectively. These equations relate the energy density correlation tensor and the energy current correlation tensor to each other. They allow to identify, in the hydrodynamic limit, exact expressions for quantities like the energy transport velocity, the transport mean free path and the absorption/gain length in terms of the irreducible single-photon self-energy and the two-photon irreducible vertex. We recover the well-known Mie scattering renormalization term to the transport velocity, albeit in vectorial form, along with additional, characteristic renormalizations originating from absorption or gain. Section VI features numerical results for the energy velocity in dilute systems of spherical scatterers, where all quantities can be evaluated within the independent scattering approximation. The Mie resonance dips in the energy transport velocity acquire a characteristic, absorption-induced broadening and gain-induced narrowing, which may be interpreted as due to the reduction (enhancement) of higher-order scattering contributions by absorption (gain). In contrast to previous approaches, our theory can be extended to systematically include wave interference effects (“Cooperon” contributions) and, thus, to address the Anderson localization transition by virtue of a self-consistent extension together with a replacement of the linear absorption/gain through a coupling to the master equation of the active medium.

II. BASIC THEORY OF ELECTROMAGNETIC WAVE TRANSPORT

We consider the propagation of light in a system of randomly positioned scatterers with isotropic dielectric constant $\epsilon_s$ immersed in a host medium with isotropic dielectric constant $\epsilon_h$. In addition, we allow for the possibility of having absorption or amplification in both the scatterer material and the host medium by ascribing an imaginary part to the respective dielectric constants. By virtue of the Kramers-Kronig relations between real and imaginary parts of the dielectric constant, we are required to consider materials with complex frequency dependent dielectric constant $\epsilon_s(\omega) = \epsilon_h(\omega)$. The resulting dielectric constant

$$\epsilon(\vec{r}, \omega) = \epsilon_h + (\epsilon_s - \epsilon_h) V(\vec{r}), \quad (1)$$
describes the arrangement of scatterers through the function $V(\vec{r}) = \sum_{j} S_{R}(\vec{r} - \vec{R})$ which consists of a set of localized shape functions $S_{R}(\vec{r})$ of the individual scatterers at random locations $\vec{R}$. The corresponding wave equation for a time harmonic wave with frequency $\omega$ and amplitude $E_{\omega}(\vec{r})$ reads

$$\nabla \times \nabla \times E_{\omega}(\vec{r}) - \frac{\omega^2}{c^2} \epsilon(\vec{r}, \omega) E_{\omega}(\vec{r}) = \omega J_{\omega}(\vec{r}).$$  \hspace{1cm} (2)

Here, $c$ denotes the vacuum speed of light, and $J_{\omega}(\vec{r}) = (4\pi i/c^2) J_{\omega}(\vec{r})$ represents an excitations of the wave field through an external current source $\vec{j}$. The dielectric function $\epsilon(\vec{r}, \omega)$ carries a positive or negative imaginary part for absorbing or coherently amplifying media, respectively.

For a specific realization $\epsilon(\vec{r}, \omega)$ of disorder, the formal solution to Eq. (1) is given in terms of the Green’s tensor (of 2nd rank in the spatial vector components) $G(\vec{r}, \vec{r}', \omega)$ as

$$E_{\omega}(\vec{r}) = \int d^{3}r' G(\vec{r}, \vec{r}', \omega) J_{\omega}(\vec{r}').$$  \hspace{1cm} (3)

For the analytical developments as well as for the comparison with experiments it is advantageous to calculate disorder-averaged quantities. Since these are translationally invariant, the transport theory must be formulated for correlation functions rather than single-particle properties. Denoting the average over disorder realizations by $\langle \ldots \rangle$, and changing to a Fourier representation of averaged quantities, the disorder averaged Green’s tensor $G_{\omega}(\vec{r} - \vec{r}') \equiv \langle G(\vec{r}, \vec{r}', \omega) \rangle$ and its Fourier transform $G_{\omega}^{\omega}(\vec{k})$ may be expressed in terms of the free Green’s tensor $G_{\omega}(\vec{k}, \omega)$ in the background medium,

$$G_{\omega}(\vec{k}, \omega) = \left( \epsilon_{k}(\omega)^{2} \omega^{2} / c^{2} - |\vec{k}|^2 \right)^{-1} \left( \mathbb{I} - \vec{k} \vec{k}^T \right),$$  \hspace{1cm} (4)

and the self-energy tensor $\Sigma_{\omega}^{\omega}(\vec{k})$, which represents the effects of scattering by the random perturbation or “scattering potential”, $(\omega^2/c^2)(\epsilon_{k}(\omega) - \epsilon_{k}(\omega))V(\vec{r})$, as

$$G_{\omega}^{\omega}(\vec{k}) = \left( G_{\omega}^{-1}(\vec{k}, \omega) - \Sigma_{\omega}^{\omega}(\vec{k}) \right)^{-1}.$$  \hspace{1cm} (5)

In the above expressions, we have introduced the three-dimensional unit tensor $\mathbb{I}$ and the dyadic product $\vec{k} \vec{k}^T$ of unit vectors $\vec{k}$ in the direction of $\vec{k}$.

Throughout this paper the propagators $G$ are understood as the retarded ones and complex conjugated propagators $\bar{G}$ as the advanced ones. For practical calculations, the self-energy tensor $\Sigma_{\omega}^{\omega}(\vec{k})$ has to be evaluated within consistent approximations such as the independent scatterer approximation (see section VI) or the Coherent Potential Approximation\textsuperscript{45,46}.

The disorder-averaged field correlation tensor is defined as

$$I(\vec{r}_{1}, \vec{r}_{2}; \omega_{1}, \omega_{2}) = \langle \bar{E}_{\omega_{1}}(\vec{r}_{1}) \bar{E}_{\omega_{2}}^{*}(\vec{r}_{2}) \rangle = \int d^{3}r_{3} d^{3}r_{4} \Gamma_{\omega_{1}, \omega_{2}}(\vec{r}_{1}, \vec{r}_{2}, \vec{r}_{3}, \vec{r}_{4}) \times S_{\omega_{1}, \omega_{2}}(\vec{r}_{3}, \vec{r}_{4})$$  \hspace{1cm} (6)

where the system’s (averaged) correlation tensor (of fourth rank) $\Gamma_{\omega_{1}, \omega_{2}}(\vec{r}_{1}, \vec{r}_{2}, \vec{r}_{3}, \vec{r}_{4})$ is independent of the source correlation tensor $S_{\omega_{1}, \omega_{2}}(\vec{r}_{3}, \vec{r}_{4}) = J_{\omega_{1}}(\vec{r}_{3}) J_{\omega_{2}}^{*}(\vec{r}_{4})$ and is given in terms of the disorder averaged tensor product of (unaveraged) Green’s tensors according to

$$\Gamma_{\omega_{1}, \omega_{2}}(\vec{r}_{1}, \vec{r}_{2}, \vec{r}_{3}, \vec{r}_{4}) = \langle G(\vec{r}_{1}, \vec{r}_{3}, \omega_{1}) \otimes G^{*}(\vec{r}_{2}, \vec{r}_{4}, \omega_{2}) \rangle,$$  \hspace{1cm} (7)

where $(\otimes)$ denotes the tensor product of two 2nd rank tensors operating in the space of retarded and advanced propagators, respectively. Similar to the disorder averaged Green’s tensor $G_{\omega}^{\omega}(\vec{k}, \omega)$, we introduce the spatial Fourier transform $\Gamma_{\omega}^{\omega}(\vec{Q}, \Omega)$ of the correlation tensor

$$\Gamma_{\omega}^{\omega}(\vec{Q}, \Omega) = \langle G_{\omega_{+}}(\vec{q}_{+}, \vec{q}_{+}) \otimes G_{\omega_{-}}^{*}(\vec{q}_{-}, \vec{q}_{-}) \rangle,$$  \hspace{1cm} (8)

where the transition to center of mass and relative frequencies, $\Omega$, $\omega$, and momenta, $\vec{Q}$, $\vec{q}$, respectively, with $\omega_{1,2} = \omega \pm \Omega/2 \equiv \omega_{\pm}$, $\vec{q}_{1,2} = \vec{q} \pm \vec{Q}/2 \equiv \vec{q}_{\pm}$ and $\vec{q}_{3,4} = \vec{q} \pm \vec{Q}/2 \equiv \vec{q}_{\pm}$, facilitates an investigation of the correlation tensor’s long-time ($\Omega \rightarrow 0$) and long-distance ($|\vec{q}| \rightarrow 0$) behavior. $\vec{Q}$ is associated with the time and position dependence of the electromagnetic energy density in the system, while $\omega$ represents the frequency of light.

$\Gamma_{\omega}^{\omega}(\vec{Q}, \Omega)$ can be expressed in terms of the irreducible vertex tensor $\gamma_{\omega}^{\omega}(\vec{Q}, \Omega)$, the two-photon analogue of the self-energy tensor, via the Bethe-Salpeter equation

$$\Gamma_{\omega}^{\omega}(\vec{Q}, \Omega) = G_{\omega_{+}}^{\omega_{+}} \left( 2\pi \right)^{3} \delta(\vec{q} - \vec{q}') \mathbb{I} \otimes \mathbb{I} + \int \frac{d^{3}q'}{(2\pi)^{3}} \gamma_{\omega}^{\omega}(\vec{Q}, \Omega) \Gamma_{\omega}^{\omega}(\vec{Q}, \Omega).$$  \hspace{1cm} (9)

In Eq. (9), both the irreducible vertex tensor $\gamma_{\omega}^{\omega}(\vec{Q}, \Omega)$ and the correlation tensor $\Gamma_{\omega}^{\omega}(\vec{Q}, \Omega)$ are tensors of fourth rank which operate both in retarded and advanced space.
(see Eq. [5]). The notation \( \gamma_\Omega^\omega (\vec{q}, \Omega) \) in Eq. [9] implies contraction of both, retarded and advanced, intermediate indices,

\[
\begin{bmatrix}
\gamma_\Omega^\omega (\vec{q}, \Omega) \\
\gamma_\Omega^\omega (\vec{q}, \Omega)
\end{bmatrix}_{\alpha_0 \beta_0 \tau} = \begin{bmatrix}
\gamma_\Omega^\omega (\vec{q}, \Omega) \\
\gamma_\Omega^\omega (\vec{q}, \Omega)
\end{bmatrix}_{\alpha_0 \beta_0 \tau} \quad [10]
\]

where summation over repeated indices is implied, compare Fig. [4]. For tensor products of 2nd rank tensors in retarded and advanced space, \( B, C \) and \( E, F \), respectively, we have the multiplication rule

\[
(B \otimes E) \cdot (C \otimes F) = (BC) \otimes (EF), \quad [11]
\]

which allows us, after integration over the momentum \( \vec{q}' \), to rewrite the Bethe-Salpeter equation [2] as a kinetic equation (a generalized Boltzmann equation) for the system’s integrated intensity correlation tensor

\[
\Phi_\vec{q}^\omega (\vec{q}, \Omega) = \int \frac{d^3 \vec{q}'}{(2\pi)^3} \Gamma_\vec{q} (\vec{q}, \Omega). \quad [13]
\]

Explicitly, the generalized Boltzmann equation reads,

\[
\begin{bmatrix}
\Delta g_\omega 1 \otimes 1 - \Delta \tilde{L}_q + \Delta \Sigma_\vec{q}^\omega \Phi_\vec{q}^\omega \\
\Delta G_\vec{q}^\omega [1 \otimes 1 + \int \frac{d^3 \vec{q}''}{(2\pi)^3} \gamma_\vec{q} (\vec{q}, \vec{q}'' \Phi_\vec{q}^\omega)]
\end{bmatrix} = 0. \quad [14]
\]

Here, we have introduced short-hand notations for certain differences and, for later use, sums of 4th rank tensors,

\[
\Delta g_\omega (\Omega) = g(\omega_+) - g^*(\omega_-) \quad [15]
\]
\[
\Delta \tilde{L}_q (\vec{q}) = \tilde{L} (\vec{q}_+) \otimes 1 - 1 \otimes \left( \tilde{L} (\vec{q}_-) \right)^* \quad [16]
\]
\[
\Delta \Sigma_\vec{q}^\omega (\vec{q}, \Omega) = \Sigma_{\vec{q}^\omega}^+ \otimes 1 - 1 \otimes \left( \Sigma_{\vec{q}^-}^\omega \right)^* \quad [17]
\]
\[
\Delta G_\vec{q}^\omega (\vec{q}, \Omega) = G_{\vec{q}^\omega}^+ \otimes 1 - 1 \otimes \left( G_{\vec{q}^-}^\omega \right)^* \quad [18]
\]
\[
\Pi \Sigma_\vec{q}^\omega (\vec{q}, \Omega) = \Sigma_{\vec{q}^\omega}^+ \otimes 1 + 1 \otimes \left( \Sigma_{\vec{q}^-}^\omega \right)^* \quad [19]
\]
\[
\Pi G_\vec{q}^\omega (\vec{q}, \Omega) = G_{\vec{q}^\omega}^+ \otimes 1 + 1 \otimes \left( G_{\vec{q}^-}^\omega \right)^* \quad [20]
\]

which employ the definitions \( g_\omega = (\omega^2/c^2) \epsilon_\omega (\omega) \) and \( \tilde{L} (\vec{r}) = \nabla \times \nabla \times \) with corresponding Fourier representation \( \tilde{L} (\vec{q}) = -\vec{q} \times \vec{q} \times \).

The physical interpretation of the generalized Boltzmann equation, Eq. [14], starts with the long-time limit, i.e., small \( \Omega \) expansion of the first term on the left-hand side (l.h.s.), \( \Delta g_\omega (\Omega) = i g_\omega (0) + g_\omega (1) \Omega + ..., \) where

\[
g_\omega (0) = 2 \frac{\omega^2}{c^2} \text{Im} [\epsilon_L (\omega)] \quad [21]
\]
\[
g_\omega (1) = \text{Re} \left[ \partial_\omega \frac{\omega^2}{c^2} \epsilon_L (\omega) \right]. \quad [22]
\]

It is seen that the term linear in \( \Omega \), \( g_\omega (1) \) corresponds to a rate of change of the correlation tensor \( \Phi_\vec{q}^\omega (\vec{q}, \Omega) \) (term of \( O(\Omega) \)), while the term of \( O(\Omega^0) \) describes absorption or emission by the host medium. The latter is non-vanishing only if \( \text{Im}_L (\omega) \neq 0 \). Similarly, the second term on the l.h.s. represents a drift term (first order in \( \vec{q} \)) for \( \Phi_\vec{q}^\omega (\vec{q}, \Omega) \). The third term on the l.h.s. of Eq. [14], \( \Delta \Sigma_\vec{q}^\omega (\vec{q}, \Omega) \), embodies single-particle scattering from the external random perturbation, while the terms on the right-hand side (r.h.s.) represent an effective two-particle collision integral induced by disorder averaging.

The physical interpretation of the generalized Boltzmann equation suggests the subsequent strategy for obtaining the solution in the hydrodynamic limit (\( \Omega \to 0 \), \( \vec{q} \to 0 \)), where the collective mode dynamics are governed by the conservation laws of the system. The electromagnetic wave equation [2] being 2nd order in time implies (in a loss- and gainless medium) the local conservation of the energy density rather than the intensity \( \Phi \).

Consequently, we will seek solutions for the energy density correlation tensor \( P \) and the energy current density correlation tensor \( J \), and recast the kinetic equation in terms of these quantities. This is possible because, as seen in section IV, \( P \) and \( J \) are the leading coefficients of the correlator \( \Phi_\vec{q}^\omega (\vec{q}, \Omega) \) in a small \( \vec{q}, \Omega \) expansion. Before we develop this solution in Sections IV and V, we will derive in the next section an exact \( WI \) for vector waves, which relates the photon self-energy \( \Sigma \) and the irreducible two-photon vertex \( g_{\gamma\gamma} \) to each other. It embodies local energy conservation as well as dissipation or gain induced deviations in a field theoretical way.

### III. WARD IDENTITY

The derivation of the \( WI \) for disordered electronic systems has been demonstrated for the first time by Vollhard and Wolfle using a diagrammatic approach. A corresponding \( WI \) for scalar classical waves has been derived, using an algebraic approach, by Barabanenkov et al. and, using a diagrammatic technique, by Kroha et al. Barabanenkov et al. have generalized their derivation to electromagnetic waves. Subsequently, the correct form of this \( WI \) has been the subject of heated debate, where a consensus has been reached in Ref. 54. However, to date, frequency dependent and/or complex dielectric functions have not been included in the derivation of \( WI \) for classical waves. Neither have all the
implications of the WI on the renormalization of transport quantities been discussed.

Therefore, the aim of this section is to derive a WI for electromagnetic waves in the presence of frequency dependent, complex dielectric functions. In the following proof, we choose to follow the approach by Barabanenkov et al. A detailed discussion as to how the WI affects the various transport properties is postponed to sections IV and V.

We start from the Green’s tensor before impurity averaging whose equation of motion (see Eqs. 2 and 3) we write as

\[
\left( \bar{L}(\bar{r}_1) + h(\omega_1) \bar{V}(\bar{r}_1) \right) \times \bar{G}(\bar{r}_1, \bar{r}_2, \omega_1) = \delta(\bar{r}_1 - \bar{r}_2) 1. 
\]

Here, we have introduced the, in general, complex quantity

\[
h(\omega) = - (\epsilon_s(\omega) - \epsilon_h(\omega)) (\omega^2/c^2) .
\]

Multiplying Eq. (24) with \(h(\omega_2)^* \bar{G}^*(\bar{r}_3, \bar{r}_4, \omega_2)\) within the appropriate tensor subspace yields

\[
\left[ h^*(\omega_2) \bar{L}(\bar{r}_1) \otimes 1 + g(\omega_2) h^*(\omega_2) 1 \otimes 1 \right. \\
\left. - h(\omega_1) h^*(\omega_2) \bar{V}(\bar{r}_1) 1 \otimes 1 \right] \\
\times \bar{G}(\bar{r}_1, \bar{r}_2; \omega_1) \otimes \bar{G}^*(\bar{r}_3, \bar{r}_4; \omega_2) = h^*(\omega_2) \delta(\bar{r}_1 - \bar{r}_2) 1 \otimes \bar{G}^*(\bar{r}_3, \bar{r}_4; \omega_2).
\]

In the time reversed case, i.e. starting with the complex conjugated equation, an analogous procedure leads to

\[
\left[ h(\omega_1) 1 \otimes \bar{L}^*(\bar{r}_3) + g(\omega_2) h(\omega_1) 1 \otimes 1 \\
- h(\omega_1) h^*(\omega_2) \bar{V}(\bar{r}_1) 1 \otimes 1 \right] \\
\times \bar{G}(\bar{r}_1, \bar{r}_2; \omega_1) \otimes \bar{G}^*(\bar{r}_3, \bar{r}_4; \omega_2) = h(\omega_1) \delta(\bar{r}_1 - \bar{r}_2) 1 \otimes \bar{G}(\bar{r}_1, \bar{r}_2; \omega_1) 1 \otimes 1.
\]

Upon substracting Eqs. (24) and Eq. (25), followed by averging over disorder and taking the limit of \(\bar{r}_1 \rightarrow \bar{r}_3\), we combine retarded and advanced quantities to what Barabanenkov et al. refer to as the pre-Ward identity.

\[
0 = \lim_{\bar{r}_1 \rightarrow \bar{r}_3} \left\{ \left[ h^*(\omega_-) \bar{L}(\bar{r}_1) \otimes 1 - h(\omega_+)(1 \otimes \bar{L}^*(\bar{r}_3)) \right. \\
+ [g(\omega_+) h^*(\omega_-) - g(\omega_-) h(\omega_+)] 1 \otimes 1 \}
\\
- (h^*(\omega_-) \delta(\bar{r}_1 - \bar{r}_2) 1 \otimes \bar{G}^*(\bar{r}_3, \bar{r}_4; \omega_-)) \\
- (h^*(\omega_-) \delta(\bar{r}_1 - \bar{r}_2) 1 \otimes \bar{G}^*(\bar{r}_3, \bar{r}_4; \omega_-)) \\
\left. - h(\omega_+) 1 \otimes \bar{G}(\bar{r}_1, \bar{r}_2; \omega_+)(1 \otimes 1) \right\}.
\]

The final step in obtaining the WI is to transfer the pre-Ward identity to momentum space and to cancel or simplify several terms with the help of both the Beth-Salpeter equation and the generalized Boltzmann equation (for details we refer to the work of Barabanenkov et al.).

This finally yields the Ward identity for electromagnetic waves with complex frequency dependent dielectric functions,

\[
\Delta \Sigma_{\bar{q}}^{\omega}(\bar{Q}; \Omega) - \int \frac{d^3 q}{(2\pi)^3} \Delta \chi^{\omega}_{\bar{q}}(\bar{Q}; \Omega) \gamma^{\omega}_{\bar{q}}(\bar{Q}, \Omega) \\
= \frac{h(\omega_+)}{h(\omega_-)} \left[ \frac{h(\omega_+)}{h(\omega_-)} \int \frac{d^3 q}{(2\pi)^3} \Pi \chi^{\omega}_{\bar{q}}(\bar{Q}; \Omega) \gamma^{\omega}_{\bar{q}}(\bar{Q}, \Omega) \right].
\]

As compared to the case of electronic wave propagation in a disordered solid, the non-zero r.h.s. of Eq. (28) constitutes a novel term which originates from the dependence of the “scattering potential” \((\omega^2/c^2)/(\epsilon_s(\omega) - \epsilon_h(\omega))V(\bar{r})\) on the light frequency as well as its possible imaginary part, and will lead below to a renormalization of the energy transport velocity. In fact, in the case of frequency independent and real dielectric constant, we have that the prefactor of this term, \((h(\omega_+)(h^*(\omega_-))/h(\omega_-) = 4\omega \Omega/(4\omega^2 + \Omega^2)),\) takes on a form that has been discovered earlier. Since in this case the r.h.s. of Eq. (28) is \(\infty \Omega\) it leads to a renormalization of the energy transport velocity. However, in the present case of absorptive or amplifying media, the r.h.s. contributes also in zero-th order in \(\Omega\), signaling that absorption and gain induce more severe effects than renormalizing the energy transport velocity, namely a mass term for the diffusion modes, see below.

**IV. CONTINUITY EQUATION**

We now proceed with solving the kinetic equation (14). From the 2nd order wave equation (2) it follows that for a homogeneous system \((\epsilon(\bar{r}) = \epsilon_h = const.)\) the quantities

\[
\rho_E = i \frac{\sqrt{\epsilon_h}}{c} \left( \dot{\bar{E}}^\dagger \cdot \frac{\partial \dot{\bar{E}}^\dagger}{\partial \bar{t}} - \frac{\partial \bar{E}^\dagger}{\partial \bar{t}} \cdot \dot{\bar{E}} \right)
\]

\[
\bar{J}_E = i \frac{\epsilon}{\sqrt{\epsilon_h}} \left( \bar{E}^\dagger \cdot (\nabla \bar{E}) - (\nabla \bar{E}^\dagger) \cdot \dot{\bar{E}} \right)
\]

obey the continuity equation \(\partial \rho_E/\partial t + \nabla \cdot \bar{J}_E = 0\), and may be interpreted as energy density and energy current density, respectively, where \(\bar{c} = c/\sqrt{\text{Re} \epsilon_h(\omega)}\) is the phase velocity of the homogeneous medium. In order to obtain a similar relation for correlation functions in a random medium we, therefore, define the energy density–energy density correlation tensor \(P_{EE}\) and the longitudinal energy current–energy density correlation tensor \(J_{EE}\).
in Fourier representation as
\[
P^\omega_E(\vec{q}, \Omega) = \omega^2 e^2 \int \frac{d^3 q}{(2\pi)^3} \Phi^\omega_\vec{q}(\vec{q}, \Omega),
\]
(31)
\[
J^\omega_E(\vec{q}, \Omega) = \omega \mathbf{v}_E(\omega)e^{-1} \int \frac{d^3 q}{(2\pi)^3} (q \cdot \vec{q}) \Phi^\omega_\vec{q}(\vec{q}, \Omega).
\]
(32)
where
\[
e_\nu = e^2 \left[ 1 - \text{Re}(\Sigma^\omega_{k=\omega f/})/\omega^2 c^2 \right]^{-1}
\]
(33)
defines the averaged phase velocity tensor in the random medium, and the energy transport velocity tensor \(\mathbf{v}_E(\omega)\) is to be identified below. We stress that in Eqs. (31) and (32), the products on the r.h.s. are contractions. For instance, in full index notation Eq. (31) reads
\[
\left[P^\omega_E(\vec{q}, \Omega)\right]_{\alpha\beta} = \omega^2 \left[ e^{-1} \right]_{ij} \int \frac{d^3 q}{(2\pi)^3} \Phi^\omega_{\vec{q}}(\vec{q}, \Omega) \right]_{ij,\alpha\beta},
\]
(34)
where summation over repeated indices is implied. A corresponding expression can be written down for the current density tensor, Eq. (32).

The expressions Eqs. (31), (32) are the leading terms of a small \(\Omega, \vec{q}\) expansion of \(\Phi^{\omega}_{\vec{q}}\), analogous to the so-called \(P_1\)-approximation to the standard Boltzmann equation,6

\[
\Phi^\omega_\vec{q}(\vec{q}, \Omega) \approx \Delta G^\omega_\vec{q}(0, 0) \left[ \hat{\Phi}^\omega_E(\vec{q}, \Omega) A_0^{-1} + (\vec{q} \cdot \vec{q}) J^\omega_E(\vec{q}, \Omega) A_1^{-1} \right] + O(\Omega, \vec{q}^2),(35)
\]
\[
A_0 = \int d^3 q \Delta G^\omega_\vec{q}(0, 0)
\]
(36)
\[
A_1 = \int d^3 q (\hat{\vec{q}} \cdot \vec{q})^2 \Delta G^\omega_\vec{q}(0, 0),
\]
(37)
The tensor coefficients \(A_0\) and \(A_1\) above can be computed by projecting \(\Phi^\omega_\vec{q}(\vec{q}, \Omega)\) onto its 0th and 1st moments with respect to the longitudinal current vertex \((\vec{q} \cdot \vec{q})\), i.e., onto \(P^\omega_E(\vec{q}, \Omega)\) and \(J^\omega_E(\vec{q}, \Omega)\) respectively.

The continuity equation for \(P^\omega_E\) and \(J^\omega_E\), can be derived by integrating the generalized Boltzmann equation, Eq. (11), over the momentum \(\vec{q}\) and subsequent application of the WI, Eq. (28). Upon considering the long-time, long-distance limit, we arrive at
\[
-\frac{1}{2} g_{\omega}^{(1)} \left[ I \otimes I + \Delta(\omega) \right] \Omega P^\omega_E + \omega e^2 \mathbf{v}_E(\omega) \right]^{-1} QJ^\omega_E = \omega^2 e^2 \int d^3 q \Delta G^\omega_\vec{q}(\vec{q}, \Omega) + \frac{1}{2} g_{\omega}^{(0)} \left[ I \otimes I + \Lambda(\omega) \right] P^\omega_E.
\]
(38)
The renormalization terms \(\Delta(\omega)\) and \(\Lambda(\omega)\) originate from the nonzero l.h.s. of the WI, Eq. (28), and are defined as
\[
\Delta(\omega) = \Delta^{(1)}(\omega) + \Delta^{(2)}(\omega)
\]
(39)
\[
\Lambda(\omega) = \frac{a_{\omega}^{(0)} g_{\omega}^{(1)}}{a_{\omega}^{(1)}} \Delta^{(1)}(\omega),
\]
(40) with the explicit forms of \(\Delta^{(1)}(\omega)\) and \(\Delta^{(2)}(\omega)\) given by
\[
\Delta^{(1)}(\omega) = -\frac{a_{\omega}^{(1)}}{g_{\omega}^{(1)}} \int d^3 q/ (2\pi)^3 \left\{ \Pi \Sigma^\omega (0, 0) \Delta G^\omega_\vec{q}(0, 0) \right\} A_0^{-1}
\]
(41)
\[
\Delta^{(2)}(\omega) = \int d^3 q/ (2\pi)^3 \left[ (\partial_1 \Sigma^\omega (0, 0)) \Delta G^\omega_\vec{q}(0, 0) \right] A_0^{-1}
\]
(42)
In Eqs. (43) we have expanded the prefactor
\[
a_{\omega}(\Omega) = \frac{h(\omega_+ + h^*(\omega_+ + h^*(\omega_-))}{h(\omega_+ + h^*(\omega_-))} = ia_{\omega}^{(0)} + a_{\omega}^{(1)} + \cdots, \quad (43)
\]
to leading order in \(\Omega\), where we introduced
\[
a_{\omega}^{(0)} = \frac{\text{Im}[h(\omega)]}{\text{Re}[h(\omega)]}, \quad a_{\omega}^{(1)} = \frac{\text{Re}[h(\omega)]}{\text{Re}[h(\omega)]} \text{Im}[\partial_1 h(\omega)] + \text{Im}[h(\omega)] \text{Im}[\partial_2 h(\omega)].
\]
(44)
Eq. (38) takes the form of a continuity equation by identifying the energy transport velocity \(\mathbf{v}_E(\omega)\) such that the prefactors of the two terms on the l.h.s. are equal, and by multiplying with the inverse of that prefactor. In fact, this procedure defines the energy transport velocity,
\[
\mathbf{v}_E(\omega) = \frac{\omega}{g_{\omega}^{(1)}} [e^2]^{-1} \left[ I \otimes I + \Delta(\omega) \right]^{-1},
\]
(46) where the r.h.s. implies a contraction as explained in Eq. (31). The continuity equation then reads,
\[
\Omega P^\omega_E + QJ^\omega_E = \frac{1}{2} g_{\omega}^{(1)} \left[ I \otimes I + \Delta(\omega) \right]^{-1} \omega^2 e^2 \int d^3 q \Delta G^\omega_\vec{q}(\vec{q}, \Omega) + \frac{1}{2} g_{\omega}^{(0)} \left[ I \otimes I + \Lambda(\omega) \right] P^\omega_E.
\]
(47)
We want to emphasize that in the long-time, long-distance limit, the continuity equation, Eq. (38) is exact. Also note that in the disordered medium transport occurs, with different velocities, both perpendicular and parallel to the polarization directions of the in- as well as of the outgoing wave field, since each scattering process alters the polarization. Accordingly, the averaged transport velocity \(\mathbf{v}_E(\omega)\) is a 4th rank tensor, as evidenced by Eq. (40). Similarly, in a transport theory of a vector field the diffusion coefficient \(\mathbf{D}(\omega)\), the transport mean free path \(\mathbf{f}_T(\omega)\), the absorption/gain length \(\mathbf{x}_a(\omega)\), given
below by Eqs. (55, 50), and (57), respectively, and other transport quantities are tensors as well.

The microscopic derivation of the energy transport velocity, Eq. (10), and the continuity equation, Eq. (17), exhibit a number of important physical aspects:

1) The frequency dependent renormalization $\Delta(\omega)$ of the energy transport velocity tensor consists of the “dwell time” renormalization $\Delta^{(1)}(\omega)$ alluded to in the introduction which in the case of independent scatterers may be traced to their internal (Mie) resonances.

2) However, in the presence of a mismatch in absorption or amplification (“impedance mismatch”) between the scatterer and the host medium (non-vanishing $a_0^{(0)}$), Eq. (10) features additional renormalization of the transport velocity. To the best of our knowledge, this renormalization has not been discussed before. Numerical results for the transport velocity in absorptive or active media will be presented in Section VI.

3) The r.h.s. of the continuity equation, Eq. (17) features as the first term the source contribution expected for correlation functions. We find that in the presence of either absorption or gain the second term on the r.h.s. does not vanish. Moreover, a “dwell time” renormalization $\Delta(\omega) \propto \Delta^{(1)}(\omega)$ occurs in the case of unequal absorption or gain characteristics of host medium and scatterer (non-vanishing $a_0^{(0)}$). While such a renormalization of either absorption or gain could be expected on physical grounds, our calculation constitutes the first quantitative analysis of this effect. Both absorption/gain related renormalization terms may have profound influence on lasing action in random dielectric media and further explorations are necessary.

V. FICK’S LAW AND DIFFUSION POLE

In order to close the set of equations for $P^\omega_E$ and $J^\omega_E$, another relation besides the continuity equation has to be derived from the kinetic equation, Eq. (13), which relates the current correlator $J^\omega_E$ to the gradient of the density correlator $P^\omega_E$, i.e. a version of Fick’s law for the correlation tensors. This can be achieved by first multiplying Eq. (13) with $\hat{q} \cdot \hat{Q}$, subsequent integration over the momenta $\hat{q}$ and employing the WI, Eq. (28). The resulting relation should be evaluated to first order in momentum $\hat{Q}$ at zero external frequency, $\Omega = 0$ (see also the book by Case and Zweifel for a detailed discussion of this so-called P1-approximation). This procedure leads to Fick’s law,

$$ Q [A + \kappa(\omega)] P^{\omega}_E = g^{(1)}_\omega \ell^{-1} [\mathbb{1} \otimes \mathbb{1} + \Delta(\omega)] J^{\omega}_E, $$

where the renormalization terms $A$ and $\kappa$ are given by

$$ A = \int \frac{d^3 q}{(2\pi)^3} (\hat{q} \cdot \hat{Q})^2 \Delta G^{\omega}_{\hat{q}}(0,0) A_0^{-1} $$

$$ \kappa(\omega) = i g^{(0)}_\omega \int \frac{d^3 q}{(2\pi)^3} (\hat{q} \cdot \hat{Q}) \partial_{\hat{q}} [\Pi \Sigma^{\omega}_{\hat{q}}(0,0)] + \int \frac{d^3 q'}{(2\pi)^3} \Pi G^{\omega}_{\hat{q}}(0,0) \gamma_{\hat{q}'} \Delta G^{\omega}_{\hat{q}}(0,0) a_0^{-1}. $$

The tensor quantity $\ell^{-1}(\omega)$ (of dimension inverse length)

$$ \ell^{-1}(\omega) = i g^{(0)}_\omega \mathbb{1} \otimes \mathbb{1} + \Delta^{(3)} + \ell_0^{-1}(\omega) $$

consists of a “standard” term $\ell_0^{-1}(\omega)$ that would also be there in electron transport theory,

$$ \ell_0^{-1}(\omega) = - \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} (\hat{q} \cdot \hat{Q})^2 \Delta G^{\omega}_{\hat{q}}(0,0) \times \gamma_{\hat{q}'} G^{\omega}_{\hat{q}}(0,0) a_0^{-1} $$

$$ + \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} (\hat{q} \cdot \hat{Q}) (\hat{q}' \cdot \hat{Q}) \Delta G^{\omega}_{\hat{q}}(0,0) \times \gamma_{\hat{q}'} G^{\omega}_{\hat{q}}(0,0) a_0^{-1}, $$

and two additional contributions $g^{(0)}_\omega \mathbb{1} \otimes \mathbb{1}$ and $\Delta^{(3)}$ that are non-vanishing only in the presence of either absorption or gain,

$$ \Delta^{(3)} = - i A^{(0)}_\omega \int \frac{d^3 q}{(2\pi)^3} (\hat{q} \cdot \hat{Q})^2 [\Pi \Sigma^{\omega}_{\hat{q}}(0,0)] $$

$$ + \int \frac{d^3 q'}{(2\pi)^3} \Pi G^{\omega}_{\hat{q}}(0,0) \gamma_{\hat{q}'} \Delta G^{\omega}_{\hat{q}}(0,0) A_1^{-1}. $$

Similar to the case of the energy transport velocity, any discrepancy between the absorption/gain characteristics of scatterers and host medium (“impedance mismatch” given by a non-vanishing $a_0^{(0)}$) leads to a renormalization of $\ell^{-1}$. Although the renormalizations $\Delta^{(3)}(\omega)$ and $\kappa(\omega)$ originate from isotropic and anisotropic scattering effects, respectively, they contribute only in the presence of absorption/gain mismatch between scatterer and host medium. This is in contrast to earlier results by Livdan and Lisiansky for media with frequency independent absorbing spheres, due to the fact that these authors inconsistently employed the WI without absorption/gain when deriving the current-density relation Eq. (18).

Combining the continuity equation, Eq. (38), with the microscopic version of Fick’s law, Eq. (18), we are finally in a position to solve for the energy density tensor in the long-time, long-distance limit

$$ P^{\omega}_E = \left( \frac{i}{2} g^{(1)}_\omega (1 + \Delta(\omega)) \left( -i \Omega + D \left( Q^2 + x_a^2 \right) \right)^{-1} \right. $$

$$ \times \omega^2 c_p^{-2} \int d^3 q \Delta G^{\omega}_{\hat{q}}(0,0) $$

which exhibits a familiar diffusion pole structure. In addition, the presence of absorption or gain leads to the
appearance of a mass term $D(\omega)/\omega^2(\omega)$. This term is absent in the work of Livdan and Lisyansky, although they considered absorbing spheres, albeit within a scalar model. From Eq. (16), the diffusion tensor $D(\omega)$ is explicitly given by
\[ D(\omega) = \frac{2i}{g^2(\omega)} (1 \otimes 1 + \Delta(\omega))^{-1} \ell (A + \kappa(\omega)) \] (55)
which establishes the transport mean free path tensor $\ell_T(\omega)$ as
\[ \ell_T(\omega) = \frac{2i}{\omega} c_\ell (A + \kappa(\omega)) \] (56)
Finally, the absorption/gain length tensor $x_a(\omega)$ explicitly reads
\[ x_a^2 = \frac{1}{3} \ell_A \ell_T, \] (57)
where the absorption mean free path tensor $\ell_A$ is given by
\[ \ell_A = \frac{\omega}{g^2(\omega)} (1 \otimes 1 + \Lambda(\omega))^{-1}. \] (58)
Eqs. (16), (55), (56), and (57) represent the central results of our investigation and make explicit how the various renormalizations discussed above affect the transport quantities of a disordered dielectric medium in the presence of linear absorption or gain. In the following section, we provide illustrations of these results for a system of dilute scatterers.

VI. NUMERICAL RESULTS
As alluded to above, any numerical evaluation of Eqs. (16), (55), (56), and (57) requires the computation of consistent values for the self-energy tensor $\Sigma_{\omega_T}^T$ and the irreducible vertex tensor $\gamma_{\omega_T}^\times(Q, \Omega)$. For dense systems and in the diffusive regime, a Coherent Potential Approximation has to be employed for both, the self-energy tensor and the irreducible vertex tensor. To date, such an approach has been carried out for scalar waves and in the presence of point scatterers only. Moreover, near the Anderson localization transition, the irreducible vertex tensor is expected to exhibit critical behavior and more sophisticated schemes such as a self-consistent theory of localization have to be employed. Again, to date such a program has been realized for scalar waves and point scatterers only.

However, in realistic strongly scattering electromagnetic systems, the scatterers cannot be approximated by point scatterers. Instead, they exhibit internal (Mie) resonances which - besides the vectorial nature of the electromagnetic radiation itself - greatly complicate the approach discussed above. Therefore, we have chosen to illustrate our results in a simpler system consisting of a dilute collection of identical absorbing or amplifying spherical scatterers in a host medium without absorption or gain. In this case, the independent scatterer approximation applies and the self-energy as well as the irreducible vertex tensor may be expressed through the density $n$ of scatterers and the full $t$-matrix of such a scatterer according to
\[ \Sigma_{\omega_T}^{\omega_T} = n t_{\omega_T}(\omega_T) \] (59)
\[ \gamma_{\omega_T}^\times(Q, \Omega) = n t_{\omega_T}(\omega_T) \otimes t_{\omega_T}^\times(\omega_T). \] (60)
Explicit expressions for the t-matrix of a spherical scatterers are fairly involved and can be found in a recent work by K. Arya et al. Consequently, even within this approximation, all transport quantities have to be evaluated numerically.

Note that the integrations with respect to momentum of, e.g., certain convolutions of the irreducible Vertex $\gamma$, which occur in the evaluation of the transport quantities, are in general ultraviolet divergent. These divergencies are remedied by applying the regularization scheme familiar from quantum electrodynamics, i.e. by subtracting appropriate mass terms from the integrands such that after subtraction, these terms, and sufficiently many of their derivatives, do not diverge, thus making the integrals convergent. Note that the corresponding regularization masses in general have to be determined numerically, since the prefactors of the ultraviolet divergencies are known numerically only.

In the following we restrict ourselves to a discussion of the numerical results for the energy transport velocity tensor. We would like to point out that for isotropic...
dielectric materials, the energy transport velocity tensor becomes an isotropic tensor, so that we can further restrict our discussion to a single valued, frequency dependent energy velocity.

In Fig. 2, we display the frequency dependence of the energy transport velocity for a typical concentration of 6% by volume of absorption-free scatterers with different (real) dielectric constants in an air background. Clearly visible are the pronounced dips in the energy transport velocity near frequencies that correspond to the single scatterer Mie resonances. As the dielectric constant increases, the Mie scattering becomes stronger and leads to correspondingly stronger renormalization of the energy transport velocity through the “dwell time” effect mentioned above. These data are, therefore, consistent with earlier results for scalar waves\textsuperscript{5,7} and vector waves\textsuperscript{14,15}. This behavior of the energy transport velocity has to be contrasted with the corresponding frequency dependence of the phase velocity, Eq. (33) in Fig. 3. Near the Mie resonances, this phase velocity may exhibit values that significantly exceed the vacuum speed of light. Since the phase velocity in a random medium does not correspond to a physical quantity, this behavior is not in conflict with the laws of relativity.

Adding linear gain to the system, dramatically modifies the situation. In Fig. 4, we show the frequency dependence of the energy transport velocity for the weakest scattering system of Fig. 2, linear gain is added to the scatterer. Although the concentration is only $n = 6\%$ by volume, adding a relatively small negative imaginary part to the dielectric constant results in a significant narrowing of the energy transport velocity resonances. This gain narrowing may be understood by noting that the Mie resonance dips in the transport velocity arise because of multiple reflection and interference of light within a single scatterer (“dwell time effect”). In the presence of gain, the relative importance of long wave paths for the interference processes is enhanced, leading to a narrowing of the resonance lines, analogous to the narrowing of the transmission lines in a Fabry-Perot filter with increasing number of reflections. Increasing the gain to rather unrealistic values shows further narrowing of the resonances together with a slight increase in the energy transport velocity. It should be noted that the energy transport velocity retains physical values for all values of the gain that we have considered.

The opposite behavior occurs when we add linear absorption to the weakest scattering spheres of Fig. 2. This is illustrated in Fig. 5, where we display the frequency dependence of the energy transport velocity for increasing values of linear absorption in this system. In this case, the dips in the energy transport velocity that are associated with the Mie resonances of the individual scatterers become washed out and disappear altogether for sufficiently large values of the linear absorption.

VII. CONCLUSION

In conclusion, we have presented a microscopic approach to the calculation of transport quantities for electromagnetic waves propagating in disordered dielectric
media which exhibit linear absorption or gain. The effects of energy conservation and of the violation of energy conservation in the presence of absorption or gain have been incorporated in the theory by means of an exact Ward identity. The kinetic equation for light in random media with absorption or gain has been derived and solved for the energy density correlation tensor, taking fully into account the vectorial nature of electromagnetic radiation. In this way we have, to the best of our knowledge, for the first time derived explicit expressions for the energy transport velocity tensor, the diffusion tensor, the absorption length tensor, in absorbing/emitting media. All these quantities exhibit renormalization due to scattering or impedance effects, analogous to the Fabry-Perrot effect. The opposite effect occurs when absorption is added to scatterers and the resonances in the energy transport velocity are washed out and ultimately disappear altogether. In all cases, the energy transport velocity retains physical values, in contrast to the phase velocity.

As a systematic transport theory, expressing all physical quantities ultimately in terms of the single-photon selfenergy and the two-photon irreducible scattering vertex, our approach may be generalized to include interference effects like weak or strong localization, e.g., in the spirit of the self-consistent theory of Anderson localization that is well-known for electronic systems. Together with a replacement of the linear gain by a direct coupling of transport equations to the semiclassical laser rate equations this may provide a microscopic theory for the interplay of optical gain and Anderson localization of light. Similarly, the theory may be extended to include optically anisotropic materials such as scatterers immersed in a liquid crystal or Faraday-active scatterers.

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