Bell inequality, nonlocality and analyticity

M. Socolovsky

Departamento de Física Teórica, Universidad de Valencia, Burjassot 46100, España
and
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Circuito Exterior, Cd. Universitaria, 04510, México D.F., México

The Bell and the Clauser-Horne-Shimony-Holt inequalities are shown to hold for both the cases of complex and real analytic nonlocality in the setting parameters of Einstein-Podolsky-Rosen-Bohm experiments for spin $\frac{1}{2}$ particles and photons, in both the deterministic and stochastic cases. Therefore, the theoretical and experimental violation of the inequalities by quantum mechanics excludes all hidden variables theories with that kind of nonlocality. In particular, real analyticity leads to negative definite correlations, in contradiction with quantum mechanics.

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1. Introduction

In a recent paper, in the context of hidden variables theories, Fahmi [1] derived the deterministic Clauser-Horne-Shimony-Holt (CHSH) inequality [2], from which the Bell inequality is easily obtained [3], allowing certain conditions of nonlocality in the parameters specifying the orientation of the Stern-Gerlach (SG) apparatus or polarizers in an EPRB experiment [4,5]. Since the inequalities are violated by quantum mechanics both theoretically [2,3] and experimentally [6,7], Fahmi concluded that a broader class of deterministic hidden variables theories are excluded, namely those obeying Bell locality [3] and those which being non local, are of the type that he considered.

It arises the question as to which is the most general form of nonlocality that allows to obtain the Bell and/or the CHSH inequalities. In this paper we investigate the particular case of nonlocalities which are expressable as real valued analytic functions of the setting parameters, both in the cases that these parameters are considered complex or real valued quantities. In the complex case the proof is simple and it is based on the fact that as a consequence of the Cauchy-Riemann (CR) equations, a real valued complex analytic function with connected domain is a constant. The argument for using complex parameters is based on the fact that the set of unit vectors $a$ and $b$ specifying the directions of two SG apparatus consists of the product of two spheres, $S^2 \times S^2$, each of which, through the stereographic projection is equivalent to the Riemann sphere or complex projective line: $P^1_c = C \cup \{\infty\}$; as is well known, for spin $\frac{1}{2}$ non relativistic quantum particles, $P^1_c$ is a natural space to describe their physics [8]. Also, several authors [9-11] have pointed out the necessary complex character of quantum mechanics, which we assume here for any of its extensions. So, one is naturally guided to the use of complex valued setting parameters.

The case of real analytic nonlocality is nevertheless anaylized, and it is also shown to be violated by quantum mechanics, this time at the level of the correlations functions, since it produces for them a negative definite result.

The analysis is extended to the non deterministic stochastic case, and therefore our conclusion is that if we assume the existence of hidden variables, then all deterministic hidden variables theories as well as all stochastic hidden variables theories with complex or real analytic nonlocality in the detecting settings (SG) are excluded.

2. Complex analytic nonlocality in Bell theorem
Let 1 and 2 be the Einstein-Podolsky-Rosen-Bohm-Bell [3-5] pair of particles, that is, 1 and 2 is a system of two quantum particles which at the moment of their preparation are in the singlet state for the spin $\frac{1}{2}$ case, or in a state of opposite polarization for the photon case; and at the moment of their detection they are separated by a space-like interval. The state of the particles is assumed to be described by the wave function $\psi$ and by a set of unspecified real quantities $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, the so called hidden variables, classically distributed with a non negative normalized probability density $\rho(\lambda)$ such that $\int_{\Lambda} d\lambda \rho(\lambda) = 1$. (We recall that a hidden variables theory is an extension of ordinary quantum mechanics, namely, it does not replace $\psi$ by $\lambda$ but adds $\lambda$ to $\psi$.) The quantities which are measured, $A$ and $B$, for the particles 1 and 2 respectively, and which take values in the set $\{+1, -1\}$ (determinism), are functions of the hidden variables $\lambda$. 

In the original derivation by Bell [3], his strong locality hypothesis [12-14] says that $A$ depends on $a$ but not on $b$, and viceversa: $B$ depends on $b$ but not on $a$. The average over $\lambda$ of the product of $A$ and $B$ measures the statistical correlation between the particles, and is given by

$$P(a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda) A(a, b; \psi, \lambda) B(a, b; \psi, \lambda). \quad (1)$$

(The statistical character of the correlation is expressed in the fact that $|P(a, b; \psi)| \leq 1$, while a perfect correlation corresponds to $|P(a, b; \psi)| = 1$, as in EPRB with $b=-a$.) $a$ and $b$, being unit vectors in 3-space, are the elements of the 2-sphere, and so $(a, b) \in S^2 \times S^2$. Through the stereographic projection, $S^2 \times S^2$ is homeomorphic to $M \equiv P^2 \times P^2 = (C \cup \{\infty\}) \times (C \cup \{\infty\})$, which is a 2-dimensional compact connected complex manifold. In a natural way, $A$ and $B$ can be considered complex analytic functions on $M$ with $a$ and $b$ replaced by complex numbers $z_1$ and $z_2$, and the point at infinity. Using the Cauchy-Riemann equations [15] one can show that the fact that $A$ and $B$ take real values implies that they are constant functions of $a$ and $b$ (or $z_1$ and $z_2$), for fixed values of $\psi$ and $\lambda$, and, by continuity, have the same values on the 1-dimensional complex submanifold $\{(a, \infty), (\infty, b), (\infty, \infty)\}_{(a, b) \in S^2 \times S^2}$. Then the power series for $A$ and $B$ are respectively given by

$$A(a, b; \psi, \lambda) = \alpha_{ab}(\psi, \lambda), \quad B(a, b; \psi, \lambda) = \beta_{ab}(\psi, \lambda)$$

where for fixed $\lambda$ the r.h.s.’s are independent of $(a, b) \in S^2 \times S^2$.

Proceeding as usual for the difference of correlation functions, we have:

$$P(a, b; \psi) - P(a, b'; \psi) = \int_{\Lambda} d\lambda \rho(\lambda) \alpha_{ab}(\psi, \lambda) \beta_{ab}(\psi, \lambda) - \int_{\Lambda} d\lambda \rho(\lambda) \alpha_{ab'}(\psi, \lambda) \beta_{ab'}(\psi, \lambda)$$

$$= \int_{\Lambda} d\lambda \rho(\lambda)((\alpha_{ab}(\psi, \lambda) - \alpha_{ab'}(\psi, \lambda))(1 \pm \alpha_{a'b'}(\psi, \lambda)\beta_{a'b'}(\psi, \lambda)))$$

$$+ \alpha_{ab'}(\psi, \lambda)\beta_{ab'}(\psi, \lambda)(1 \pm \alpha_{a'b}(\psi, \lambda)\beta_{a'b}(\psi, \lambda)))$$

$$= P(a, b; \psi) - P(a, b'; \psi) \pm \int_{\Lambda} \rho(\lambda)I(\psi, \lambda)$$

with

$$I(\psi, \lambda) = \alpha_{ab}(\psi, \lambda)\beta_{ab}(\psi, \lambda)\alpha_{a'b'}(\psi, \lambda)\beta_{a'b'}(\psi, \lambda) - \alpha_{ab'}(\psi, \lambda)\beta_{ab'}(\psi, \lambda)\alpha_{a'b}(\psi, \lambda)\beta_{a'b}(\psi, \lambda).$$

Since

$$\alpha_{ab}(\psi, \lambda) = \alpha_{a'b'}(\psi, \lambda) = \alpha_{a'b}(\psi, \lambda) = \alpha_{ab'}(\psi, \lambda) = \text{const.} \in \{+1, -1\}$$

for each $(\psi, \lambda)$, and analogously for the $\beta$’s, then

$$I = 0,$$
and we can proceed straightforwardly to obtain the CHSH inequality (2):

\[ |P(a, b; \psi) - P(a, b'; \psi)| \leq \int_{\Lambda} d\lambda \rho(\lambda)(|1 \pm \alpha_{a'b'}(\psi, \lambda)\beta_{a'b'}(\psi, \lambda)| + |1 \pm \alpha_{a'b}(\psi, \lambda)\beta_{a'b}(\psi, \lambda)|) \]

\[ = 2 \pm \left( \int_{\Lambda} d\lambda \rho(\lambda)\alpha_{a'b'}(\psi, \lambda)\beta_{a'b'}(\psi, \lambda) + \int_{\Lambda} d\lambda \rho(\lambda)\alpha_{a'b}(\psi, \lambda)\beta_{a'b}(\psi, \lambda) \right) \]

\[ = 2 \pm (P(a', b'; \psi) + P(a', b; \psi)) \leq 2 \pm |P(a', b'; \psi) + P(a', b; \psi)| \]

which implies

\[ |P(a, b; \psi) - P(a, b'; \psi)| + |P(a', b'; \psi) + P(a', b; \psi)| \leq 2. \tag{2} \]

3. The stochastic case

The stochastic case, in which the values of \( A \) and \( B \) in \( \{+1, -1\} \) are replaced by average values \( \tilde{A} \) and \( \tilde{B} \) in the interval \([-1, +1]\), is easily obtained from the previous result, by first deriving a formal mathematical inequality and then by making use of the Bell factorization hypothesis [16], with nonlocality incorporated as before as complex analytic dependence of \( A \) and \( B \) on \( a \) and \( b \), for given \( \psi \) and \( \lambda \).

Let \( P(\mu, \nu; a, b; \psi, \lambda) \in [0, 1] \) be the probability that the measurements of the spins (or polarizations) on particles 1 and 2, respectively along the directions \( a \) and \( b \), give the results \( \mu \) for 1 and \( \nu \) for 2, for given \( \psi \) and \( \lambda \), with \( \mu, \nu \in \{+1, -1\} \). The crucial hypothesis is that of factorization [16] of \( P \):

\[ P(\mu, \nu; a, b; \psi, \lambda) = P_1(\mu; a, b; \psi, \lambda)P_2(\nu; a, b; \psi, \lambda). \tag{3} \]

In the Bell’s local version, \( P_1 \) depends on \( a \) but not on \( b \), and vice versa for \( P_2 \). Clearly, \( 0 \leq P_1, P_2 \leq 1 \). The average of the above probability over the fluctuations of \( \lambda \) is given by

\[ P(\mu, \nu; a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)P(\mu, \nu; a, b; \psi, \lambda) = \int_{\Lambda} d\lambda \rho(\lambda)P_1(\mu; a, b; \psi, \lambda)P_2(\nu; a, b; \psi, \lambda). \]

As in section 2, the correlation function is defined as the average of the product of the two spins (or polarizations):

\[ P(a, b; \psi) = P(+, +; a, b; \psi) + P(-, -; a, b; \psi) - P(+, -; a, b; \psi) - P(-, +; a, b; \psi) \]

with

\[ P(+, +; a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)P(+; a, b; \psi, \lambda)P_2(+; a, b; \psi, \lambda), \]

\[ P(-, -; a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)P(-; a, b; \psi, \lambda)P_2(-; a, b; \psi, \lambda), \]

\[ P(+, -; a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)P(+; a, b; \psi, \lambda)P_2(-; a, b; \psi, \lambda), \]

\[ P(-, +; a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)P(-; a, b; \psi, \lambda)P_2(+; a, b; \psi, \lambda); \]

then

\[ P(a, b; \psi) = \int_{\Lambda} d\lambda \rho(\lambda)(P_1(+; a, b; \psi, \lambda)P_2(+; a, b; \psi, \lambda) + P_1(-; a, b; \psi, \lambda)P_2(-; a, b; \psi, \lambda)) \]
\[ -P_1(+; \mathbf{a}, \mathbf{b}; \psi, \lambda)P_2(-; \mathbf{a}, \mathbf{b}; \psi, \lambda) - P_1(-; \mathbf{a}, \mathbf{b}; \psi, \lambda)P_2(+; \mathbf{a}, \mathbf{b}; \psi, \lambda) \]

\[ = \int_\Lambda d\lambda \rho(\lambda) [P_1(+; \mathbf{a}, \mathbf{b}; \psi, \lambda) - P_1(-; \mathbf{a}, \mathbf{b}; \psi, \lambda)] [P_2(+; \mathbf{a}, \mathbf{b}; \psi, \lambda) - P_2(-; \mathbf{a}, \mathbf{b}; \psi, \lambda)] \]

\[ := \int_\Lambda d\lambda \rho(\lambda) < A(\mathbf{a}, \mathbf{b}; \psi, \lambda) > < B(\mathbf{a}, \mathbf{b}; \psi, \lambda) > \]

where \(< A(\mathbf{a}, \mathbf{b}; \psi, \lambda) >\) and \(< B(\mathbf{a}, \mathbf{b}; \psi, \lambda) >\), both in the interval \([-1,+1]\), are the average values of the spins (or polarizations) of particles 1 and 2, respectively, along the directions \(\mathbf{a}\) and \(\mathbf{b}\). Assuming the complex analytic dependence of these quantities on \(\mathbf{a}\) and \(\mathbf{b}\), and using the formal mathematical result at the beginning of this section, we obtain the CHSH inequality for the stochastic case, which is again given by eq. (2) but with the \(P^i\)’s given now by eq. (4).

4. Real analyticity

It can be easily shown that if we assume real analytic expansions for the functions \(A\) and \(B\) of section 2, then the correlation functions (1) become negative definite, in clear contradiction with the quantum formula (7) of section 5. In fact, let \(A\) and \(B\) be given by the convergent series

\[ A(\mathbf{a}, \mathbf{b}; \psi, \lambda) = \sum_{i,j=1}^{\infty} \sum_{r,s=1}^{3} \alpha_{ij}^r a_i^r b_s^j \]

and

\[ B(\mathbf{a}, \mathbf{b}; \psi, \lambda) = \sum_{k,l=1}^{\infty} \sum_{i,s=1}^{3} \beta_{kl}^t b_t^k a_s^l \]

with \(\alpha_{ij}^r\) and \(\beta_{kl}^t\) real coefficients (functions of \(\psi\) and \(\lambda\)). (In the real case the CR equations do not hold and the power series expansions do not reduce to constant terms.) For \(\mathbf{b}=\mathbf{a}\), (1) gives the perfect correlation

\[ -1 = P(\mathbf{a}, \mathbf{a}; \psi) = \int_\Lambda d\lambda \rho(\lambda) A(\mathbf{a}, \mathbf{a}; \psi, \lambda) B(\mathbf{a}, \mathbf{a}; \psi, \lambda) \]

which implies

\[ \int_\Lambda d\lambda \rho(\lambda) (A(\mathbf{a}, \mathbf{a}; \psi, \lambda) B(\mathbf{a}, \mathbf{a}; \psi, \lambda) + 1) = 0 \]

and then

\[ B(\mathbf{a}, \mathbf{a}; \psi, \lambda) = -A(\mathbf{a}, \mathbf{a}; \psi, \lambda) \]

for all \(\lambda\), except for a possible set of measure zero in \(\Lambda\). So,

\[ A(\mathbf{a}, \mathbf{a}; \psi, \lambda) = \sum_{i,j=1}^{\infty} \sum_{r,s=1}^{3} \alpha_{ij}^r a_i^r a_s^j = \sum_{i,j=1}^{\infty} \sum_{r,s=1}^{3} (-\beta_{ij}^r) a_i^r a_s^j \]

i.e.

\[ \sum_{i,j}^{\infty} \sum_{r,s=1}^{3} (\alpha_{ij}^r + \beta_{ij}^r) a_i^r a_s^j = 0 \]

which holds for all unit vectors \(\mathbf{a} \in \mathbb{R}^3\). Then

\[ \beta_{ij}^r = -\alpha_{ij}^r \]

(5)

and therefore

\[ P(\mathbf{a}, \mathbf{b}; \psi) = -\int_\Lambda d\lambda \rho(\lambda) (A(\mathbf{a}, \mathbf{b}; \psi, \lambda))^2 = -1, \]

(6)
which is a negative constant value for the correlation function.

5. Quantum correlations

For completeness, and to emphasize the difference between standard quantum mechanics and its extension with hidden variables, we consider the quantum correlation between two spin \( \frac{1}{2} \) particles in the singlet state, given by the well known formula \[10,17\]

\[
P_Q(a, b; \psi) = -a \cdot b. \tag{7}
\]

\(P_Q\) is a real valued real analytic function of \(a\) and \(b\) considered as real vectors in \(R^3\); however, since we are now in the framework of pure quantum mechanics, this does not contradict the main conclusion in the Introduction since hidden variables are absent.

On the other hand, considered as a real valued complex function on the connected space \(P^1_c \times P^1_c\), it can be shown to be non analytic (and therefore not necessarily a constant): in fact, using the stereographic projection

\[
\Phi : C \cup \{\infty\} \to S^2, \quad \Phi(z) = \frac{(z + \bar{z}), i(z - \bar{z}), 1 - |z|^2}{1 + |z|^2}, \quad \Phi(\infty) = (0, 0, -1),
\]

and identifying \(a\) with \(z\) or \(\infty\) and \(b\) with \(w\) or \(\infty\), we obtain

\[
P_Q(z, \bar{z}; w, \bar{w}; \psi) = -\frac{(z + \bar{z})(w + \bar{w}) - (\bar{z} - z)(\bar{w} - w) + (1 - |z|^2)(1 - |w|^2)}{(1 + |z|^2)(1 + |w|^2)},
\]

\[
P_Q(z, \bar{z}; \infty; \psi) = P_Q(\infty; z, \bar{z}) = \frac{1 - |z|^2}{1 + |z|^2}, \quad \text{and} \quad P_Q(\infty, \infty; \psi) = -1.
\]

The dependence of \(P_Q\) on both \(z\) and \(\bar{z}\) and on \(w\) and \(\bar{w}\) shows that \(P_Q\) is non analytic. In this form, \(P_Q\) has the same non analytic character of the correlation function in the case of quantum mechanics extended with hidden variables; clearly, the agreement is not necessary since we are dealing with different theories, however, as mentioned above, it supports the view of quantum mechanics as a necessarily complex theory \[9,11\].

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