Quantum Fidelities, Their Duals, And Convex Analysis

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Abstract
We study tree kinds of quantum fidelity. Usual Uhlmann’s fidelity, minus of \( f \)-divergence when \( f(x) = -\sqrt{x} \), and the one introduced by the author via reverse test. All of them are quantum extensions of classical fidelity, where the first one is the largest and the third one is the smallest. We characterize them in terms of convex optimization, and introduce their ‘dual’ quantity, or the polar of the minus of the fidelity. They turned out to be monotone increasing by unital completely positive maps, concave, and linked to its classical version via optimization about classical-to-quantum maps and quantum-to-classical maps.

1 Introduction
We study tree kinds of quantum fidelity. Usual Uhlmann’s fidelity, minus of \( f \)-divergence when \( f(x) = -\sqrt{x} \), and the one introduced by the author via reverse test. All of them are quantum extensions of classical fidelity, where the first one is the largest and the third one is the smallest. We characterize them in terms of convex optimization, and introduce their ‘dual’ quantity, or the polar of the minus of the fidelity. They turned out to be monotone increasing by unital completely positive maps, concave, and linked to its classical version via optimization about classical-to-quantum maps and quantum-to-classical maps.

2 Notations and conventions
In the paper, it is assumed that dimensions of Hilbert spaces are finite. The set of operators, self-adjoint operators, positive operators, and density operators over the Hilbert space \( \mathcal{H} \) will be denoted by \( \mathcal{L}(\mathcal{H}), \mathcal{L}_{sa}(\mathcal{H}), \mathcal{P}(\mathcal{H}), \) and \( \mathcal{S}(\mathcal{H}) \), respectively. When \( \mathcal{H} = \mathbb{C}^k \), they are denoted by \( \mathcal{L}_k, \mathcal{L}_{sa,k}, \mathcal{P}_k \) and \( \mathcal{S}_k \). The identity operator in \( \mathbb{C}^k \) and identity transform in \( \mathcal{L}_k \) will be denoted by \( I_k \) and
\( I_k \), respectively. \( \mathcal{L} \) denotes \( \bigcup_{k \in \mathbb{N}} \mathcal{L}_k \), and \( \mathcal{L}_{sa}, \mathcal{P}, \) and \( \mathcal{S} \) are defined similarly. We define

\[
\mathcal{L}_{sa}^{\times 2} := \bigcup_{k \in \mathbb{N}} \mathcal{L}_{sa,k}^{\times 2}, \quad \mathcal{P}^{\times 2} := \bigcup_{k \in \mathbb{N}} \mathcal{P}_k^{\times 2},
\]

e etc., where

\[
\mathcal{L}_{sa,k}^{\times 2} := \mathcal{L}_{sa,k} \times \mathcal{L}_{sa,k},
\]
\[
\mathcal{P}_k^{\times 2} := \mathcal{P}_k \times \mathcal{P}_k,
\]

e etc.

We fix a standard orthonormal basis \( \{|i\rangle\} \) of \( \mathbb{C}^k \), and denote the commutative algebra spanned by \( \{|i\rangle \langle i|\}_{i=1}^k \) by \( \mathcal{C}_k \). Also,

\[
\mathcal{CP}_k = \mathcal{C}_k \cap \mathcal{P}_k, \quad \mathcal{CS}_k = \mathcal{C}_k \cap \mathcal{S}_k,
\]
\[
\mathcal{C}_k^{\times 2} := \mathcal{C}_k \times \mathcal{C}_k, \quad \mathcal{CP}_k^{\times 2} := \mathcal{CP}_k \times \mathcal{CP}_k, \quad \mathcal{CS}_k^{\times 2} := \mathcal{CS}_k \times \mathcal{CS}_k,
\]
\[
\mathcal{C}^{\times 2} := \bigcup_{k \in \mathbb{N}} \mathcal{C}_k^{\times 2}, \quad \mathcal{CP}^{\times 2} := \bigcup_{k \in \mathbb{N}} \mathcal{CP}_k^{\times 2}, \quad \mathcal{CS}^{\times 2} := \bigcup_{k \in \mathbb{N}} \mathcal{CS}_k^{\times 2}.
\]

Any unital completely positive (CP) map from \( \mathcal{C}_n \) to operators \( \mathcal{L}_k \) is in the following form;

\[
\Phi^*_M \left( \sum_{i=1}^n l_i |i\rangle \langle i| \right) = \sum_{i=1}^n l_i M_i,
\]

where \( M = \{ M_i ; M_i \in \mathcal{P}_k, i = 1, \ldots, n \} \) is a POVM over \( \mathbb{C}^k \). Since a member of \( \mathcal{C}_n \) is represented by an array \( l = (l_1, \ldots, l_n) \), we also write

\[
\Phi^*_M (l) = \sum_{i=1}^n l_i M_i.
\]

Also, any completely positive completely positive (CPTP) map from \( \mathcal{L}_k \) to \( \mathcal{C}_n \) is in the form of

\[
\Phi_M (L) = \sum_{i=1}^n (\text{tr} \ L M_i) |i\rangle \langle i|.
\]

With \( l = (l_i)_{i=1}^k = (\text{tr} \ L M_i)_{i=1}^k \), we also write this as

\[
\Phi_M (L) = l.
\]

Any unital CP map from \( \mathcal{L}_k \) to \( \mathcal{C}_n \) is in the form of

\[
\Psi^*_\tilde{\rho} (L) = \sum_{i=1}^n (\text{tr} \ \rho_i \ L) |i\rangle \langle i|,
\]

where \( \tilde{\rho} = \{ \rho_i ; \rho_i \in \mathcal{S}_k, i = 1, \ldots, n \} \) is a set of states. With \( l = (l_i)_{i=1}^k = (\text{tr} \ \rho_i \ L)_{i=1}^k \), we also write this as

\[
\Psi^*_\tilde{\rho} (L) = l.
\]
Also, any CPTP map from $\mathcal{C}_n$ to $\mathcal{L}_k$ is in the form of
\[
\Psi_{\tilde{\rho}} (l) := \Psi_{\tilde{\rho}} \left( \sum_{i=1}^n l_i |i\rangle \langle i| \right) = \sum_{i=1}^n l_i \rho_i.
\]

We denote by $\Phi_C$ the pinching operation
\[
\Phi_C (X) = \sum_{i=1}^n \langle i| X |i\rangle |i\rangle \langle i|.
\]

When the operator $X$ is not invertible, $X^{-1}$ means Moore-Penrose generalized inverse.

3 Classical fidelity, fidelity, and minimum fidelity

For probability distributions $p = (p_x)_{x=1}^k$ and $q = (q_x)_{x=1}^k$, we define
\[
F^C (p, q) := \sum_{i=1}^k \sqrt{p_i q_i}.
\]

For $\rho, \sigma \in \mathcal{S} (\mathbb{C}^k)$, Uhlmann’s fidelity is
\[
F_{\max} (\rho, \sigma) := \text{tr} \sqrt{\sigma \rho} \sqrt{\sigma}.
\]

It is known that
\[
F_{\max} (\rho, \sigma) = \min_{M: \text{measurement}} F^C (M (\rho), M (\sigma))
\]

where $M (\rho)$ is the probability distribution of measurement $M$ applied to $\rho$. A ”dual” of $F (\rho, \sigma)$ \cite{6} is
\[
F_{\min} (\rho, \sigma) := \max_{\Phi} \{ F^C (p, q) : \Phi \text{ is a CPTP with } \Phi (p) = \rho, \Phi (q) = \sigma \}.
\]

When $\text{supp} \rho \subset \text{supp} \sigma$,
\[
F_{\min} (\rho, \sigma) = \text{tr} \sigma \sqrt{\sigma^{-1/2} \rho \sigma^{-1/2}},
\]

where $\sigma^{-1}$ is the generalized inverse. When $\text{supp} \rho \not\subset \text{supp} \sigma$,
\[
F_{\min} (\rho, \sigma) = \text{tr} \sigma \sqrt{\sigma^{-1/2} \tilde{\rho} \sigma^{-1/2}},
\]

where
\[
\begin{align*}
\tilde{\rho} &= \rho_{11} - \rho_{12} \rho_{22}^{-1} \rho_{21}, \\
\rho_{11} &= \pi_\sigma \rho \pi_\sigma, \ \rho_{12} := \pi_\sigma \rho (I_k - \pi_\sigma), \\
\rho_{21} &= \rho_{12}^\dagger, \ \rho_{22} := (I_k - \pi_\sigma) \rho (I_k - \pi_\sigma), \\
\pi_\sigma : \text{projection onto } \text{supp} \sigma.
\end{align*}
\]
Also,

\[ F_{1/2}(\rho, \sigma) := \text{tr} \rho^{1/2}\sigma^{1/2}. \]

From here, we extend \( F_{\text{max}} \), \( F_{\text{min}} \) and \( F_{1/2} \) to functionals on \( L_{sa}^{\times 2} \) in the following manner.

\[ F^Q(X,Y) := \begin{cases} \\
\sqrt{\text{tr} X \text{tr} Y} F \left( \frac{1}{\text{tr} X} X, \frac{1}{\text{tr} Y} Y \right), & (X,Y) \in \mathcal{P}^{\times 2}, X \neq 0, Y \neq 0, \\
-\infty & (X,Y) \in \mathcal{P}^{\times 2}, XY = 0, \\
& (X,Y) \not\in \mathcal{P}^{\times 2}. 
\end{cases} \]

Also \( F_{\text{cl}} \) is extended to a functional over two signed measures, in the analogous manner.

All of \( F_{\text{max}} \), \( F_{\text{min}} \) and \( F_{1/2} \) satisfy the following properties:

- (positive homogeneity)
  \[ F^Q(cX, cY) = c F^Q(X, Y), \forall c \geq 0. \]

- (concavity)
  \[ F^Q(\lambda X_1 + (1 - \lambda) X_2, \lambda Y_1 + (1 - \lambda) Y_2) \geq \lambda F^Q(X_1, Y_1) + (1 - \lambda) F^Q(X_2, Y_2), \forall \lambda \in [0,1]. \]

- (CPTP monotonicity) \( F^Q(X,Y) \leq F^Q(\Lambda(X), \Lambda(Y)) \) for any CPTP map \( \Lambda \).

- (positivity) For any \( X,Y \in \mathcal{P}_k \), \( F^Q(X,Y) \geq 0 \).

- (strong homogeneity)
  \[ F^Q(\lambda X, \mu Y) = \sqrt{\lambda\mu} F^Q(X,Y). \]

- (normalization) for any positive vectors \( x = (x_i)_{i=1}^k \) and \( y = (y_i)_{i=1}^k \), and for an orthogonal basis \( \{|i\rangle\}_{i=1}^k \)
  \[ F^Q\left( \sum_{i=1}^k x_i |i\rangle \langle i|, \sum_{i=1}^k y_i |i\rangle \langle i| \right) = F^C(x,y). \]

- (symmetry)
  \[ F^Q(X,Y) = F^Q(Y,X) \]

- (additivity)
  \[ F^Q\left( \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} \right) = F^Q(X_1, Y_1) + F^Q(X_2, Y_2) \]
By definition of $F_{\min}(X,Y)$ and (1), it is obvious that

$$F_{\min}(X,Y) \leq F_{\max}(X,Y).$$

Also, observe that joint concavity and homogeneity implies

$$F^Q(X_1 + X_2, Y_1 + Y_2) = 2F^Q\left(\frac{1}{2}(X_1 + X_2), \frac{1}{2}(Y_1 + Y_2)\right)$$

$$\geq 2 \cdot \frac{1}{2}(F^Q(X_1, Y_1) + F^Q(X_2, Y_2))$$

$$= F^Q(X_1, Y_1) + F^Q(X_2, Y_2).$$

If $F^Q$ is positive in addition,

$$F^Q(X_1 + X_2, Y_1 + Y_2) \geq F^Q(X_1, Y_1),$$

for any $(X_2, Y_2) \in \mathcal{P} \times 2$.

Strong homogeneity and joint concavity implies strong joint concavity:

$$F^Q\left(\sum_{i=1}^{m} \lambda_i X_i, \sum_{i=1}^{m} \mu_i Y_i\right) \geq \sum_{i=1}^{m} \lambda_i F^Q\left(X_i, \frac{\mu_i}{\lambda_i} Y_i\right)$$

$$= \sum_{i=1}^{m} \sqrt{\lambda_i \mu_i} F^Q(X_i, Y_i).$$

We define $\mathcal{F}_0$ as the set of all the proper closed concave functionals, which satisfies positive homogeneity, positivity, and $\text{dom } F^Q = \mathcal{P} \times 2$. $\mathcal{F}_1$ is the subset of $\mathcal{F}_0$ whose element satisfies CPTP monotonicity, normalization, strong homogeneity, symmetry, and additivity. The following lemma is almost immediate from Lemma B.4.

**Lemma 3.1** Consider a family $\{F_i\}_{i \in I}$, where $F_i \in \mathcal{F}_0$. Then, $\inf_{i \in I} F_i \in \mathcal{F}_0$. If in addition each $f_i$ is a member of $\mathcal{F}_1$, so is $\inf_{i \in I} F_i$.

Also, it is known that:

**Theorem 3.2** Suppose that a functional $F^Q$ on $\mathcal{L}_+^{\otimes 2}$ is normalized and CPTP monotone,

$$F_{\min}(X,Y) \leq F^Q(X,Y) \leq F_{\max}(X,Y).$$

**4 Convex Programming representations**

**Lemma 4.1** A functional $F^Q$ on $\mathcal{L}_+^{\otimes 2}$ is a member of $\mathcal{F}_0$ if and only if there is a closed convex subset $\mathcal{M}_{F^Q}$ of $\mathcal{P} \times 2$ such that

$$F^Q(X,Y) = \inf_{(L_0, L_1) \in \mathcal{M}_{F^Q}} \text{tr } L_0 X + \text{tr } L_1 Y,$$
and \(0^+\mathcal{M}_{FQ} = \mathcal{P}^{\times 2}\), or
\[
(L_0, L_1) \in \mathcal{M}_{FQ} \Rightarrow (L_0 + M_0, L_1 + M_1) \in \mathcal{M}_{FQ}, \quad \forall M_0, M_1 \geq 0.
\]
\(\text{(5)}\)

In addition, the correspondence between \(F^Q\) and \(\mathcal{M}_{FQ}\) is one-to-one. In fact,
\[
\mathcal{M}_{FQ} = \{ (L_0, L_1) : (L_0, L_1) \in \mathcal{L}_a^{\times 2}, \forall (X, Y) \in \mathcal{P}^{\times 2}, \text{ tr } L_0 X + \text{ tr } L_1 Y \geq F^Q (X, Y) \}
\]

**Proof.** By Lemma \(\text{B.7}\), it is obvious that \(\text{(4)}\) holds for a closed convex set \(\mathcal{M}_{FQ}\). Let \(\mathcal{M}_{FQ}\) be a closed convex set which may not satisfy \(\text{(5)}\). Then, \(\mathcal{M}_{FQ} := \{ (L_0 + M_0, L_1 + M_1) : (L_0, L_1) \in \mathcal{M}_{FQ}, M_0, M_1 \geq 0 \}\) is a closed convex set satisfying \(\text{(4)}\). Also, if \((X, Y) \in \mathcal{P}^{\times 2}\),
\[
\inf_{(L_0, L_1) \in \mathcal{M}_{FQ}} \text{ tr } L_0 X + \text{ tr } L_1 Y = \inf_{(L_0, L_1) \in \mathcal{M}_{FQ}} \text{ tr } L_0 X + \text{ tr } L_1 Y = F^Q (X, Y).
\]
If \((X, Y) \not\in \mathcal{P}^{\times 2}\),
\[
\inf_{(L_0, L_1) \in \mathcal{M}_{FQ}} \text{ tr } L_0 X + \text{ tr } L_1 Y = -\infty = F^Q (X, Y).
\]
Thus, for a given \(F^Q\), there is a closed convex set \(\mathcal{M}_{FQ}\) satisfying \(\text{(4)}\) and \(\text{(5)}\). By positivity of \(F^Q\), \(\mathcal{M}_{FQ} \subset \mathcal{P}^{\times 2}\).

That the correspondence between \(F^Q\) and \(\mathcal{M}_{FQ}\) is one-to-one is obvious by Lemma \(\text{B.7}\). ■

**Lemma 4.2** Suppose a closed convex set \(\mathcal{M}_{FQ} \subset \mathcal{P}^{\times 2}\) satisfies \(\text{(5)}\). Then, for any \(M_0, M_1 > 0\), there is a positive number \(t_0\) such that
\[
\forall t \geq t_0 \quad (tM_0, tM_1) \in \mathcal{M}_{FQ},
\]
\[
\forall t < t_0 \quad (tM_0, tM_1) \not\in \mathcal{M}_{FQ}
\]

**Proof.** To prove the statement, suppose \((L_0, L_1) \in \mathcal{M}_{FQ}\) and \(M_0, M_1 > 0\). Then there is \(t_0 \geq 0\) such that
\[
L_0 \leq t_0 M_0, \quad L_1 \leq t_0 M_1.
\]
Since \((L_0, L_1) \in \mathcal{M}_{FQ}\), \((t_0 M_0, t_0 M_1) \in \mathcal{M}_{FQ}\) by \(\text{(5)}\). Thus, for any \(t \geq t_0\), we have \((tM_0, tM_1) \in \mathcal{M}_{FQ}\). Since the set \(\{(tM_0, tM_1) : t \geq 0\}\) is closed, its intersection with \(\mathcal{M}_{FQ}\) is also closed. So the minimum
\[
t_0 = \min \{ t ; (tM_0, tM_1) \in \mathcal{M}_{FQ} \}
\]
exists. ■

**Lemma 4.3** Let \(\mathcal{M}_{FQ} \subset \mathcal{P}^{\times 2}\) be a closed convex set with \(0^+\mathcal{M}_{FQ} = \mathcal{P}^{\times 2}\). Then for any \(X > 0, Y > 0\), \((X, Y) \to \text{ tr } L_0 X + \text{ tr } L_1 Y\) has minimum in \(\mathcal{M}_{FQ}\). Also, its infimum is finite if and only if \((X, Y) \in \mathcal{P}^{\times 2}\).
**Proof.** The second statement is trivial. So we prove the only first one. Choose a which is strictly larger then the infimum, and consider a level set
\[
\{(L_0, L_1) \in \mathcal{M}_{FQ}; \text{tr} L_0 X + \text{tr} L_1 Y \leq \alpha \}
\]
which is closed. If \(X > 0\) and \(Y > 0\), the recession cone of this is empty, due to the following reasons. If it has direction of recession, it should be a member of \(\mathcal{P}^{\times 2}\), because the set is subset of \(\mathcal{P}^{\times 2}\). But, for any \((L'_0, L'_1) \in \mathcal{P}^{\times 2}\), there is \(t\) such that
\[
\text{tr} (L_0 + tL'_0) X + \text{tr} (L_1 + tL'_1) Y > \alpha.
\]
So there is no direction of recession. Therefore, the set is bounded. Therefore, \((X, Y) \rightarrow \text{tr} L_0 X + \text{tr} L_1 Y\) has minimum over the set, which coincide with the minimum over \(\mathcal{M}_{FQ}\). 

The proof of the following two propositions are immediate, thus omitted.

**Proposition 4.4** Suppose \(F^Q\) is a member of \(\mathcal{F}_0\). Then, \(F^Q\) is CPTP monotone if and only if \(\mathcal{M}_{FQ}\) satisfies
\[
(L_0, L_1) \in \mathcal{M}_{FQ} \Rightarrow (\Lambda^* (L_0), \Lambda^* (L_1)) \in \mathcal{M}_{FQ}
\]
for any CPTP map \(\Lambda\).

**Proposition 4.5** Suppose \(F^Q\) is a member of \(\mathcal{F}_0\). Then, \(F^Q\) is strongly homogeneous if and only if \(\mathcal{M}_{FQ}\) satisfies
\[
(L_0, L_1) \in \mathcal{M}_{FQ} \Rightarrow \left( tL_0, \frac{1}{t} L_1 \right) \in \mathcal{M}_{FQ}
\]
for any \(t > 0\).

**Proposition 4.6** Suppose \(F^Q\) is a member of \(\mathcal{F}_0\) that is CPTP monotone and normalized. Then,
\[
\mathcal{M}_{FQ} \cap C^{\times 2} = \left\{ (L_0, L_1) \in C; L_0 > 0, L_1 \geq \frac{1}{4} L_0^{-1} \right\}.
\]

**Proof.** Let \(l_0 := (l_0, i)\), \(X = \sum_{i=1}^{k} x_i |i\rangle \langle i|\), \(Y = \sum_{i=1}^{k} y_i |i\rangle \langle i|\). Then, by normalization,
\[
F^Q (X, Y) = \frac{\sum_{i=1}^{k} \sqrt{x_i y_i}}{\min_{l_0}} = \min_{l_0, i} \left( \sum_{i=1}^{k} x_i l_{0,i} + \frac{y_i}{4 l_{0,i}} \right)
\]
\[
= \min_{(l_0, l_1) \in \mathcal{P}} \left\{ \sum_{i=1}^{k} (x_i l_{0,i} + y_i l_{1,i}) ; l_{1,i} \geq \frac{1}{4 l_{0,i}} \right\}
\]
\[
= \min \left\{ (\text{tr} l_0 X + \text{tr} l_1 Y) ; (L_0, L_1) \in C; L_0 > 0, L_1 \geq \frac{1}{4} L_0^{-1} \right\}.
\]
By CPTP monotonicity, $\Phi_C(M_{FQ}) \subset M_{FQ}$. Thus,

$$\Phi_C(M_{FQ}) \subset M_{FQ} \cap C^{\times 2}.$$ 

Since each element $X$ of $C$ is unchanged by $\Phi_C$, $\Phi_C(X) = X$, the opposite inclusion is also true:

$$M_{FQ} \cap C^{\times 2} = \Phi_C(M_{FQ} \cap C^{\times 2}) \subset M_{FQ} \cap C^{\times 2}.$$ 

Therefore, we have

$$\Phi_C(M_{FQ}) = M_{FQ} \cap C^{\times 2}.$$ 

Observe

$$F^Q(X,Y) = \min_{(L_0, L_1) \in M_{FQ}} (\text{tr} L_0 X + \text{tr} L_1 Y)$$

$$= \min_{(L_0, L_1) \in M_{FQ}} (\text{tr} L_0 \Phi_C(X) + \text{tr} L_1 \Phi_C(Y))$$

$$= \min_{(L_0, L_1) \in M_{FQ}} (\text{tr} \Phi_C(L_0) X + \text{tr} \Phi_C(L_1) Y)$$

$$= \min_{(L_0, L_1) \in M_{FQ}} (\text{tr} L_0 X + \text{tr} L_1 Y)$$

$$= \min_{(L_0, L_1) \in M_{FQ} \cap C^{\times 2}} (\text{tr} L_0 X + \text{tr} L_1 Y).$$

Since this and (6) holds for any $(X,Y) \in PC^{\times 2}$, by Lemma B.3, we have the assertion. 

## 5 The minimum points of convex programs

Suppose a member $F^Q$ of $F_0$ has the derivative

$$DF^Q(X,Y)(T,S) = \text{tr} L_0 T + \text{tr} L_1 S.$$ 

Then, for any $\lambda > 0$,

$$DF^Q(\lambda X, \lambda Y)(T,S) = \frac{d}{dt} F^Q(\lambda X + tT, \lambda Y + tS) \bigg|_{t=0}$$

$$= \lambda \frac{d}{dt} F^Q\left(\lambda X + \frac{t}{\lambda} T, \lambda Y + \frac{t}{\lambda} S\right) \bigg|_{t=0}$$

$$= \lambda DF^Q(X,Y)\left(\frac{1}{\lambda} T, \frac{1}{\lambda} S\right)$$

$$= DF^Q(X,Y)(T,S).$$

Also, since $F^Q$ is concave,

$$F^Q(\lambda X + T, \lambda Y + S) - F^Q(\lambda X, \lambda Y)$$

$$= F^Q(\lambda X + T, \lambda Y + S) - \lambda F^Q(X,Y)$$

$$\leq \text{tr} L_0 T + \text{tr} L_1 S.$$
Since $F^Q$ is closed, it is upper semi continuous. Thus, taking $\lim_{\lambda \to 0}$ of both ends, for any $T \geq 0$, $S \geq 0$,

$$
\lim_{\lambda \to 0} \{ F^Q (\lambda X + T, \lambda Y + S) - \lambda F^Q (X, Y) \} = F^Q (T, S) \leq \text{tr} \ L_0 \ast T + \text{tr} \ L_1 \ast S,
$$

which means $(L_0 \ast, L_1 \ast) \in \mathcal{M}_{F^Q}$.

Also, since $F^Q$ is positively homogeneous,

$$
F^Q (0, 0) - F^Q (X, Y) = D F^Q (X, Y) (-X, -Y)
$$

holds for any $X \geq 0$, $Y \geq 0$. Thus,

$$
F^Q (X, Y) = D F^Q (X, Y) (X, Y) = \text{tr} \ L_0 \ast X + \text{tr} \ L_1 \ast Y.
$$

So $(L_0 \ast, L_1 \ast)$ achieves (1).

Define, for each $Z \in \mathcal{P}_k$, the linear transform $S_Z$ on $\mathcal{L}_{sa,k}$ by the equation

$$
X = S_Z (X) Z + Z S_Z (X).
$$

When $Z > 0$,

$$
S_Z (X) = \int_0^{\infty} e^{-tZ} X e^{-tZ} dt. \tag{7}
$$

In fact $S_Z$ is self-dual with respect to Hilbert-Schmidt inner product, $S_Z = S_Z^*$.

When $Z > 0$, this is obvious from the second expression of $S_Z$. When is positive but may have null eigenspace,

$$
\text{tr} S_Z (X) Y = \text{tr} S_Z (X) (S_Z (Y) Z + Z S_Z (Y)) = \text{tr} (S_Z (X) Z + Z S_Z (X)) S_Z (Y) = \text{tr} X S_Z (Y).
$$

So $S_Z$ is self-dual, if viewed as a linear transform on $\mathcal{L}_{sa,k}$.

The derivative of $f_1 (X) = \sqrt{X}$ is

$$
D f_1 (X) (T) = S_{\sqrt{X}} (T),
$$

since the differentiation of both sides of $X = \{ f_1 (X) \}^2$ yields

$$
T = \{ D f_1 (X) (T) \} X + X \{ D f_1 (X) (T) \}.
$$

First, consider $F_{\text{max}} (X, Y) = \text{tr} \ \sqrt{Y^{1/2}X Y^{1/2}}$. The derivative of $f_2 (X) = \sqrt{Y^{1/2}X Y^{1/2}}$.

$$
D f_2 (X) (T) = S_{Y^{1/2}X Y^{1/2}} (Y^{1/2} T Y^{1/2}).
$$
Therefore,

\[
\begin{align*}
\text{DF}_{\text{max}} (X, Y) (T, S) &= \text{tr} S \sqrt{Y^{-1/2} X Y^{-1/2}} \left( Y^{1/2} T Y^{1/2} \right) + \text{tr} S \sqrt{X^{-1/2} Y X^{-1/2}} \left( X^{1/2} S X^{1/2} \right) \\
&= \text{tr} S \sqrt{Y^{-1/2} X Y^{-1/2}} (I) Y^{1/2} T Y^{1/2} + \text{tr} S \sqrt{X^{-1/2} Y X^{-1/2}} (I) X^{1/2} S X^{1/2} \\
&= \text{tr} Y^{-1/2} S \sqrt{Y^{-1/2} X Y^{-1/2}} (I) Y^{1/2} T + \text{tr} X^{-1/2} S \sqrt{X^{-1/2} Y X^{-1/2}} (I) X^{1/2} S \\
&= \frac{1}{2} \text{tr} Y^{1/2} \left( Y^{1/2} X Y^{1/2} \right)^{-1/2} Y^{1/2} + \frac{1}{2} \text{tr} S X^{1/2} \left( X^{1/2} Y X^{1/2} \right)^{-1/2} X^{1/2},
\end{align*}
\]

and

\[
\begin{align*}
L_{0,*} &= \frac{1}{2} Y^{1/2} \left( Y^{1/2} X Y^{1/2} \right)^{-1/2} Y^{1/2}, \\
L_{1,*} &= \frac{1}{2} X^{1/2} \left( X^{1/2} Y X^{1/2} \right)^{-1/2} X^{1/2}.
\end{align*}
\]

Here, ‘\(-1\)' stands for generalized inverse. Observe

\[
\begin{align*}
Y &= 4L_{0,*} X L_{0,*}, \\
X &= 4L_{1,*} Y L_{1,*}.
\end{align*}
\]

Thus, \(2L_{0,*}\) and \(2L_{1,*}\) is non-commutative version of Radon-Nikodym derivative ‘d \(Y/dX\)’ and ‘d \(X/dY\)’, respectively. Also,

\[
(2L_{0,*}) (2L_{1,*}) = I.
\]

Indeed,

\[
F_{\text{max}} (X, Y) = \min_{L \geq 0} \text{tr} LX + \text{tr} L^{-1} Y.
\] (8)

This is verified by differentiation of the right hand side:

\[
\frac{\partial}{\partial L} \left\{ \text{tr} LX + \text{tr} L^{-1} Y \right\} = X - L^{-1} Y L^{-1}.
\]

So the minimum is achieved by a positive \(L\) with

\[
Y = L X L.
\]

Thus, \(L = 2L_{0,*}\).

Next, consider \(F_{\text{min}} (X, Y)\), supposing that \(X > 0\) and \(Y > 0\),

\[
F_{\text{min}} (X, Y) = \text{tr} Y \sqrt{Y^{-1/2} X Y^{-1/2}} = \text{tr} X \sqrt{X^{-1/2} Y X^{-1/2}}.
\]

So,

\[
\begin{align*}
\text{DF}_{\text{min}} (X, Y) (T, S) &= \text{tr} Y S \sqrt{Y^{-1/2} X Y^{-1/2}} \left( Y^{-1/2} T Y^{-1/2} \right) + \text{tr} X S \sqrt{X^{-1/2} Y X^{-1/2}} \left( X^{-1/2} S X^{-1/2} \right) \\
&= \text{tr} Y^{-1/2} S \sqrt{Y^{-1/2} X Y^{-1/2}} (Y^{-1/2} T) + \text{tr} X^{-1/2} S \sqrt{X^{-1/2} Y X^{-1/2}} (X) X^{-1/2} S.
\end{align*}
\]

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This means
\[ L_{0,*} = Y^{-1/2} S_{\sqrt{Y^{-1/2} X Y^{-1/2}}}(Y) Y^{-1/2}, \]
\[ L_{1,*} = X^{-1/2} S_{\sqrt{X^{-1/2} X Y^{-1/2}}}(X) X^{-1/2}. \]
Lastly, we consider \( F_{1/2}(X,Y) \), where \((X,Y) \in \mathcal{P}_k^2\).
\[
\begin{align*}
DF_{1/2}(X,Y)(T,S) &= \text{tr} \, S_{\sqrt{X}}(T) \sqrt{Y} + \text{tr} \, S_{\sqrt{Y}}(S) \sqrt{X} \\
&= \text{tr} \, T S_{\sqrt{X}}(\sqrt{Y}) + \text{tr} \, S S_{\sqrt{Y}}(\sqrt{X}).
\end{align*}
\]
So,
\[ L_{0,*} = S_{\sqrt{X}}(\sqrt{Y}), \quad L_{1,*} = S_{\sqrt{Y}}(\sqrt{X}). \tag{9} \]
They give another non-commutative version of Radon-Nikodym derivative \( d\sqrt{Y}/d\sqrt{X} \) and \( d\sqrt{X}/d\sqrt{Y} \).

6 \quad SDP representations

It is known \cite{2,7} that
\[
F_{\text{max}}(X,Y) = \max \left\{ \frac{1}{2} (\text{tr} \, C + \text{tr} \, C^\dagger) : \begin{bmatrix} X & C \\ C^\dagger & Y \end{bmatrix} \geq 0 \right\}, \tag{10}
\]
\[
\quad = \min \left\{ \text{tr} \, X L_0 + \text{tr} \, Y L_1 : (L_0, L_1) \in \mathcal{M}_{F_{\text{max}}} \right\}, \tag{11}
\]
where
\[
\mathcal{M}_{F_{\text{max}}} = \left\{ (L_0, L_1) : \begin{bmatrix} 2L_0 & -I_k \\ -L_1 & 2L_1 \end{bmatrix} \geq 0, \ L_0, L_1 \in \mathcal{L}_{sa,k} \right\}. \tag{12}
\]
The equality between (10) and (11) is due to duality theorem of semi definite programing. By Lemma \cite{A.1} it is easy to verify
\[
\mathcal{M}_{F_{\text{max}}} = \left\{ (L_0, L_1) : 2L_0 \geq L, 2L_1 \geq L^{-1}, \ \exists \ L \geq 0 \right\}, \tag{13}
\]
which leads to (8). Conversely, (8) leads to (13).

Also, in the case of \( \text{supp} \, X \subset \text{supp} \, Y \), it is known \cite{1} that
\[
\begin{bmatrix} X & C \\ C & Y \end{bmatrix} \geq 0, \ C = C^\dagger \tag{14}
\]
holds if and only if
\[ C \geq \sqrt{Y} \sqrt{Y^{-1/2} X Y^{-1/2} \sqrt{Y}}. \]
Therefore,

\[
F_{\text{min}} (X, Y) = \max \{ \text{tr } C ; \quad (\ref{14}), \ C \in \mathcal{L}_{sa,k} \} \tag{15} \\
= \min \{ \text{tr } X L_0 + \text{tr } Y L_1 ; \ (L_0, L_1) \in \mathcal{M}_{\text{Fmin}} \}, \tag{16}
\]

\[
\mathcal{M}_{\text{Fmin}} = \left\{ (L_0, L_1) ; \begin{bmatrix} 2L_0 & -I_k - \sqrt{-1}A \\ -I_k + \sqrt{-1}A & 2L_1 \end{bmatrix} \geq 0, \ L_0, L_1, A \in \mathcal{L}_{sa,k} \right\}, \tag{17}
\]

where the second identity is by the duality theorem of SDP.

In the case of \( \text{supp } X \not\subset \text{supp } Y \), we still have (15), as proved in the following.

By Lemma A.1, \( C \) should be supported on \( \text{supp } Y \), for (14) to hold. Therefore,

\[
\begin{bmatrix} X & C \\ C & Y \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} & C & 0 \\ X_{21} & X_{22} & 0 & 0 \\ C & 0 & Y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.
\]

Because of \( X \geq 0 \) and Lemma A.1, \( X_{12} (I_{k_0} - \pi_{\text{supp } X_{22}}) = 0 \), where \( k_0 = \dim \text{supp } Y \). Therefore,

\[
\begin{bmatrix} I_{k_0} & -X_{12}X_{22}^{-1} & C & 0 \\ 0 & I_{k-k_0} & 0 & 0 \\ C & 0 & I_{k_0} & 0 \\ 0 & 0 & 0 & I_{k-k_0} \end{bmatrix} \begin{bmatrix} X & C \\ C & Y \end{bmatrix} \begin{bmatrix} I_{k_0} & 0 & C & 0 \\ -X_{22}^{-1}X_{21} & I_{k-k_0} & 0 & 0 \\ C & 0 & I_{k_0} & 0 \\ 0 & 0 & 0 & I_{k-k_0} \end{bmatrix} = \begin{bmatrix} X_{11} - X_{12}X_{22}^{-1}X_{21} & X_{12} (I_{k_0} - \pi_{\text{supp } X_{22}}) & C & 0 \\ (I_{k_0} - \pi_{\text{supp } X_{22}})X_{21} & X_{22} & 0 & 0 \\ C & 0 & Y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Therefore, \( (\ref{14}) \) is equivalent to

\[
\begin{bmatrix} \tilde{X} & C \\ C & Y \end{bmatrix} \geq 0, \ C = C^\dagger. \tag{18}
\]

Thus,

\[
\max \{ \text{tr } C ; \quad (\ref{14}), \ C \in \mathcal{L}_{sa,k} \} = \max \{ \text{tr } C ; \quad (\ref{15}), \ C \in \mathcal{L}_{sa,k} \} \tag{19} \\
= \text{tr } Y \sqrt{Y^{-1/2}XY^{-1/2}} = F_{\text{min}} (X, Y),
\]

and our assertion is proved.
Note that, if \((L_0, L_1) \in \mathcal{M}_{F_{\text{min}}}, \ L_0 > 0 \text{ and } L_1 > 0\), thus
\[
\mathcal{M}_{F_{\text{min}}} = \left\{ (L_0, L_1) : \begin{bmatrix} 2L_0 & -I_k - \sqrt{-1}A \\ -I_k + \sqrt{-1}A & 2L_1 \end{bmatrix} \geq 0, \ L_0 > 0, \ L_1 > 0, A \in \mathcal{L}_{sa} \right\}.
\] (19)

Suppose otherwise, that is, \(L_0\) has null eigenspace, and let \(|\psi\rangle\) be a member of it with unit length \(||\psi|| = 1\). Then,
\[
\begin{bmatrix} \langle \psi| \ c \langle \psi| \end{bmatrix} \begin{bmatrix} 2L_0 & -I_k - \sqrt{-1}A \\ -I_k + \sqrt{-1}A & 2L_1 \end{bmatrix} \begin{bmatrix} |\psi\rangle \\ c |\psi\rangle \end{bmatrix} = -2c + 2c^2 \langle \psi| PL_1 P |\psi\rangle
\]
is negative if \(c\) is sufficiently large positive number. So \(L_0\) should be strictly positive, and so should be \(L_1\).

A consequence of SDP representations for \(F_{\text{max}}\) and \(F_{\text{min}}\) is
\[
F_{\text{min}}(X, Y) = \min \left\{ \text{tr} \ X \ L_0 + \text{tr} \ Y \ L_1 : \left( L_0, (I_k - \sqrt{-1}A)^{-1} L_1 \left( I_k + \sqrt{-1}A \right)^{-1} \right) \in \mathcal{M}_{F_{\text{max}}}, A \in \mathcal{L}_{sa} \right\}
\]
\[
= \min \left\{ F_{\text{max}} (X, (I_k - \sqrt{-1}A) Y (I_k + \sqrt{-1}A)) : A \in \mathcal{L}_{sa} \right\}.
\] (20)

To show these, note that
\[
\begin{bmatrix} I_k & 0 \\ 0 & (I_k - \sqrt{-1}A)^{-1} \end{bmatrix} \begin{bmatrix} 2L_0 & -I_k - \sqrt{-1}A \\ -I_k + \sqrt{-1}A & 2L_1 \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & (I_k + \sqrt{-1}A)^{-1} \end{bmatrix} = \begin{bmatrix} 2L_0 & -I_k \\ -I_k & (I_k - \sqrt{-1}A)^{-1} (2L_1) \left( I_k + \sqrt{-1}A \right)^{-1} \end{bmatrix}.
\]

Here, \(I_k - \sqrt{-1}A\) is invertible because \((I_k - \sqrt{-1}A)^\dagger (I_k - \sqrt{-1}A)\) is invertible,
\[
(I_k - \sqrt{-1}A)^\dagger (I_k - \sqrt{-1}A) = I_k + A^2 \geq I_k.
\]

Therefore,
\[
(L_0, L_1) \in \mathcal{M}_{F_{\text{min}}}
\]
\[
\Leftrightarrow \exists A \in \mathcal{L}_{sa,k} \left( L_0, (I_k - \sqrt{-1}A)^{-1} L_1 \left( I_k + \sqrt{-1}A \right)^{-1} \right) \in \mathcal{M}_{F_{\text{max}}}, \quad (21)
\]

Therefore, by (11) and (16), we have the asserted identity. Similarly, we have
\[
F_{\text{min}}(X, Y) = \min_{A \in \mathcal{L}_{sa,k}} F_{\text{max}} \left( (I_k + \sqrt{-1}A) X (I_k - \sqrt{-1}A), Y \right).
\]

Finally, we present a SDP representation of \(F_{1/2}\). Define linear operators \(L_X\) and \(R_Y\) on \(L_k\) by
\[
L_X(A) = X A, \quad R_Y(A) = A Y.
\]
Then, if $X$ and $Y$ are self-adjoint, $L_X$ and $R_Y$ are self-adjoint when $L_k$ is equipped with the Hilbert-Schmidt inner product $\langle A, B \rangle_{\text{HS}} := \text{tr} A^\dagger B$. Also, $L_X$ and $R_Y$ commutes, and

$$F_{1/2}(X, Y) = \langle I, L_X^{1/2} R_Y^{1/2} I \rangle_{\text{HS}}.$$ 

Hence,

$$F_{1/2}(X, Y) = \max \left\{ \text{Re} \langle I, CI \rangle_{\text{HS}} ; C \leq L_X^{1/2} R_Y^{1/2} \right\}$$

$$= \max \left\{ \text{Re} \langle I, CI \rangle_{\text{HS}} ; \begin{bmatrix} L_X & C \\ C & R_Y \end{bmatrix} \geq 0 \right\}.$$

### 7 Polar

#### 7.1 Definition and Basic Properties

We define polar of $F^Q \in F_0$ by

$$\hat{F}^Q(L_0, L_1) := \inf \left\{ s ; (L_0, L_1) \in s (\mathcal{M}_{F^0})^c, s > 0 \right\}, \quad (L_0, L_1) \in \mathcal{P}^\times_2, \quad -\infty < (L_0, L_1) \notin \mathcal{P}^\times_2,$$

in analogy with a polar of a convex function. This is a sort of ‘dual’ of $F^Q$, and as is shown later, is CPTP monotone non-decreasing by applications of any CP unital maps. We study the property of this quantity rather meticulously.

**Proposition 7.1** Suppose $F^Q \in F_0$. If $(L_0, L_1)$ satisfies $\hat{F}^Q(L_0, L_1) > 0$, then

$$\hat{F}^Q(L_0, L_1) = \max \left\{ s ; (L_0, L_1) \in s (\mathcal{M}_{F^0}), s > 0 \right\}. \quad (22)$$

In particular, if $L_0, L_1 > 0$, we have $\hat{F}^Q(L_0, L_1) > 0$. Also,

$$\hat{F}^Q(L_0, L_1) \geq 1 \iff (L_0, L_1) \in \mathcal{M}_{F^0}. \quad (23)$$

If $F^Q \in F_0$ satisfies normalization and CPTP monotonicity and $L_0$ or $L_1$ has an eigenvalue $0$,

$$\hat{F}^Q(L_0, L_1) = 0. \quad (24)$$

**Proof.** If $(L_0, L_1)$ satisfies $\hat{F}^Q(L_0, L_1) > 0$, the set

$$\mathcal{T} := \{ t ; t \geq 0, (tL_0, tL_1) \in \mathcal{M}_{F^Q} \}$$

is not empty. Also, $\mathcal{T}$ is closed, since both $\mathcal{M}_{F^Q}$ and $\{(tL_0, tL_1) ; t \geq 0\}$ are closed. By [6], if $t \in \mathcal{T}$, any $t' \geq t$ is also an element of the set $\mathcal{T}$. Therefore, there is $t_0 > 0$ such that $t \in \mathcal{T}$ is equivalent to $t \geq t_0$. Therefore, we have [22].

In particular, if $L_0, L_1 > 0$, by Lemma[4.2] $\hat{F}^Q(L_0, L_1) > 0$.

Suppose $\hat{F}^Q(L_0, L_1) \geq 1$. Then, by definition, $(L_0, L_1) \in \mathcal{P}^\times_2$. By [22], $(L_0, L_1) \in \mathcal{M}_{F^Q}$. On the other hand, suppose $(L_0, L_1) \in \mathcal{M}_{F^Q}$. Then, by [5],

$$\frac{1}{s}(L_0, L_1) \in \mathcal{M}_{F^Q}, 0 < \forall s \leq 1,$$
which leads to \( \hat{F}^Q (L_0, L_1) \geq 1 \).

Suppose in addition \( F^Q \in \mathcal{F}_0 \) satisfies normalization and CPTP monotonicity. Then, by Theorem 3.2, \( \mathcal{M}_{F_{\max}} \subset \mathcal{M}_{F^Q} \). Therefore for any \( L_0 \) or \( L_1 \) with an eigenvalue 0,

\[
(tL_0, tL_1) \notin \mathcal{M}_{F_{\max}} \subset \mathcal{M}_{F^Q}, \forall t \geq 0,
\]

which leads to (24).

**Proposition 7.2.** For any \( (L_0, L_1), (X, Y) \in \mathcal{P}^\times 2 \), and \( F^Q \in \mathcal{F}_0 \),

\[
\hat{F}^Q (L_0, L_1) F^Q (X, Y) \leq \text{tr} L_0 X + \text{tr} L_1 Y.
\]

(25)

Also, for any \( (L_0, L_1) \in \mathcal{P}^\times 2 \),

\[
\hat{F}^Q (L_0, L_1) = \inf \left\{ \frac{1}{F^Q (X, Y)} (\text{tr} L_0 X + \text{tr} L_1 Y); (X, Y) \in \mathcal{P}^\times 2, F^Q (X, Y) > 0 \right\}.
\]

(26)

\[
= \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \mathcal{P}^\times 2, F^Q (X, Y) = 1 \right\} \quad (27)
\]

\[
= \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \mathcal{P}^\times 2, F^Q (X, Y) \geq 1 \right\}, \quad (28)
\]

and \( \hat{F}^Q \) is a member of \( \mathcal{F}_0 \). If in addition \( F^Q \) is not identically 0 on \( \mathcal{P}^\times 2 \), \( \hat{F}^Q \) also has that property.

**Proof.** First, we show (26). Observe

\[
\hat{F}^Q (L_0, L_1) = \inf \left\{ s; (L_0, L_1) \in s (\mathcal{M}_{F^Q})^c, s > 0 \right\}
\]

\[
= \inf \left\{ s; \frac{1}{s} (L_0, L_1) \in (\mathcal{M}_{F^Q})^c, s > 0 \right\}
\]

\[
= \inf \left\{ s; \exists (X, Y) \in \mathcal{P}^\times 2, \frac{1}{s} (\text{tr} L_0 X + \text{tr} L_1 Y) < F^Q (X, Y), s > 0 \right\}.
\]

Since \( \frac{1}{s} (\text{tr} L_0 X + \text{tr} L_1 Y) \geq 0 \), \( \frac{1}{s} (\text{tr} L_0 X + \text{tr} L_1 Y) < F^Q (X, Y) \) holds only if \( F^Q (X, Y) \) is positive. Therefore,

\[
\hat{F}^Q (L_0, L_1) = \inf \left\{ s; \exists (X, Y) \in \mathcal{P}^\times 2, s > \frac{1}{F^Q (X, Y)} (\text{tr} L_0 X + \text{tr} L_1 Y), F^Q (X, Y) > 0 \right\},
\]

which implies (26). (26) results from (26).

Next, (26) is equivalent to

\[
\hat{F}^Q (L_0, L_1) = \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \mathcal{P}^\times 2, F^Q (X, Y) \neq 0, -\infty \right\},
\]

which is equal to (27). Let \( \mathcal{P}_{F^Q} \) be the set of \( (X, Y) \in \mathcal{P}^\times 2 \) such that \( F^Q (X, Y) = 1 \). Then we obtain (25) as follows.

\[
\hat{F}^Q (L_0, L_1) = \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \mathcal{P}_{F^Q} \right\}
\]

\[
= \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \text{conv} \mathcal{P}_{F^Q} \right\}
\]

\[
= \inf \left\{ \text{tr} L_0 X + \text{tr} L_1 Y; (X, Y) \in \mathcal{P}^\times 2, F^Q (X, Y) \geq 1 \right\},
\]
where the last identity is due to concavity of $F^Q$.

By (28) and Lemma B.7, $\hat{F}^Q$ is closed, proper, concave, and positively homogeneous. Since $F^Q(L_0, L_1) \geq 0$ for all $(L_0, L_1) \in \mathcal{P}^2$, $\hat{F}^Q$ is a member of $F^Q$.

Next, we show $\hat{F}^Q$ is not identically 0 on $\mathcal{P}^2$. If $(L_0, L_1) \in M_{F^Q}$, by Proposition 7.1, $\hat{F}^Q(L_0, L_1) \geq 1$. Therefore, if $\hat{F}^Q$ is identically 0 on $\mathcal{P}^2$, $M_{\hat{F}^Q} = M_{F^Q} \cap \mathcal{P}^2$ is empty. By B.6, this contradicts with the assumption that $F^Q(X, Y)$ is a member of $F_0$.

**Theorem 7.3** If $F^Q$ is a member of $F_0$ and not identically 0 on $\mathcal{P}^2$, so is $\hat{F}^Q$. Also,

$$F^Q(X, Y) = \inf \left\{ \frac{1}{F^Q(L_0, L_1)} (\text{tr } L_0X + \text{tr } L_1Y) ; (L_0, L_1) \in \mathcal{P}^2, F^Q(L_0, L_1) > 0 \right\}$$

(29)

$$= \inf \left\{ \text{tr } L_0X + \text{tr } L_1Y ; (L_0, L_1) \in \mathcal{P}^2, F^Q(L_0, L_1) = 1 \right\}$$

(30)

and

$$F^Q(X, Y) = \inf \left\{ s ; (X, Y) \in s \left( M_{\hat{F}^Q} \right)^c, s > 0 \right\},$$

(31)

and

$$F^Q(X, Y) \geq 1 \iff (X, Y) \in M_{\hat{F}^Q}.$$ (32)

**Proof.** By (28) and the definition of $M_{F^Q}$,

$$F^Q(X, Y) = \inf \left\{ \text{tr } L_0X + \text{tr } L_1Y ; (L_0, L_1) \in M_{F^Q} \right\}$$

$$= \inf \left\{ \text{tr } L_0X + \text{tr } L_1Y ; \hat{F}^Q(L_0, L_1) \geq 1 \right\}$$

$$= \inf \left\{ \text{tr } L_0X + \text{tr } L_1Y ; \hat{F}^Q(L_0, L_1) = 1 \right\}.$$ (31)

The last end of this is equal to the RHS of (29), due to the almost parallel reason as (27) equals (28).

(31) is derived from (29) in almost parallel manner as the proof of (26). (32) results from (30) and concavity of $F^Q(L_0, L_1)$.

**Corollary 7.4** (26) establishes one-to-one map from $F_0$ to itself, whose inverse map is given by (29).

**Proof.** The mapping from $F^Q \in F_0$ to $\hat{F}^Q \in F_0$ is one-to-one since $F^Q$ is recovered from $\hat{F}^Q$ using (29).

**Proposition 7.5** $F^Q \in F_0$ is strongly homogeneous if and only if $\hat{F}^Q \in F_0$ is strongly homogeneous.
**Proof.** Suppose $F^Q \in \mathcal{F}_0$ is strongly homogeneous. With $t_0 > 0$, and $t_1 > 0$,
\[
\hat{F}^Q(t_0 L_0, t_1 L_1) = \inf \{ s ; (t_0 L_0, t_1 L_1) \in s (\mathcal{M}_{FQ})^c, s > 0 \}
\]
\[
= \inf \left\{ s ; \sqrt{t_0 t_1} \left( \sqrt{\frac{t_0}{t_1}} L_0, \sqrt{\frac{t_1}{t_0}} L_1 \right) \in s (\mathcal{M}_{FQ})^c, s > 0 \right\}
\]
\[
= \inf \left\{ s ; \sqrt{t_0 t_1} (L_0, L_1) \in s (\mathcal{M}_{FQ})^c, s > 0 \right\}
\]
\[
= \sqrt{t_0 t_1} \hat{F}^Q(L_0, L_1),
\]
where the third identity is due to Proposition 4.3. The opposite implication is proved in almost parallel manner.

**Proposition 7.6** $F^Q \in \mathcal{F}_0$ satisfies CPTP monotonicity if and only if $\hat{F}^Q$ is monotone increasing by application of any unital CP map $\Lambda^*$, 
\[
\hat{F}^Q (\Lambda^*(L_0), \Lambda^*(L_1)) \geq \hat{F}^Q (L_0, L_1).
\]

**Proof.** Suppose $F^Q \in \mathcal{F}_0$ satisfies CPTP monotonicity. If $\hat{F}^Q(L_0, L_1) = 0$, the assertion is trivial. Thus, we suppose $\hat{F}^Q(L_0, L_1) > 0$ and use (22). If $(L_0, L_1) \in s \mathcal{M}_{FQ}$, by Proposition 4.3, we have 
\[
(\Lambda^*(L_0), \Lambda^*(L_1)) \in \Lambda^*(s \mathcal{M}_{FQ})
\]
\[
= s \Lambda^*(\mathcal{M}_{FQ}) \subset s \mathcal{M}_{FQ},
\]
which implies the assertion. The opposite implication is proved in almost parallel manner.

**Proposition 7.7** $F^Q \in \mathcal{F}_0$ satisfies CPTP monotonicity and additivity if and only if $\hat{F}^Q \in \mathcal{F}_0$ is monotone increasing by any unital CP map and satisfies
\[
\hat{F}^Q \left( L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)} \right) = \min \left\{ \hat{F}^Q \left( L_0^{(1)}, L_1^{(1)} \right), \hat{F}^Q \left( L_0^{(2)}, L_1^{(2)} \right) \right\}.
\]

**Proof.** Suppose $F^Q \in \mathcal{F}_0$ is CPTP monotone and additive. Then by Proposition 7.6, $F^Q$ is monotone increasing by any unital CP map. $\mathcal{M}_{FQ}$ is invariant by any unital CP map. Therefore, for any $(X, Y) \in \mathcal{M}_{FQ}$ with 
\[
X = \begin{bmatrix} X_1 & * \\ * & X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 & * \\ * & Y_2 \end{bmatrix},
\]
we have $(X_1 \oplus X_2, Y_1 \oplus Y_2) \in \mathcal{M}_{FQ}$ (Consider the pinching operation that maps $(X, Y)$ to $(X_1 \oplus X_2, Y_1 \oplus Y_2)$). Therefore,
\[
\hat{F}^Q \left( L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)} \right)
\]
\[
= \inf_{(X,Y)\in\mathcal{M}_{FQ}} \text{tr} \left( L_0^{(1)} \oplus L_0^{(2)} \right) X + \text{tr} \left( L_1^{(1)} \oplus L_1^{(2)} \right) Y
\]
\[
= \inf_{(X,Y)\in\mathcal{M}_{FQ}} \text{tr} L_0^{(1)} X_1 + \text{tr} L_0^{(2)} X_2 + \text{tr} L_1^{(1)} Y_1 + \text{tr} L_1^{(2)} Y_2.
\]

\[
\text{tr} L_0^{(1)} X_1 + \text{tr} L_0^{(2)} X_2 + \text{tr} L_1^{(1)} Y_1 + \text{tr} L_1^{(2)} Y_2
\]

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Also, by (23),
\[(X_1 \oplus X_2, Y_1 \oplus Y_2) \in M_{FQ} \]
\[\iff F^Q(X_1 \oplus X_2, Y_1 \oplus Y_2) = F^Q(X_1, Y_1) + F^Q(X_2, Y_2) \geq 1,\]
\[\iff F^Q(X_1, Y_1) \geq \lambda, F^Q(X_2, Y_2) \geq 1 - \lambda, 0 \leq \lambda \leq 1.\]
\[\implies (X_1, Y_1) = \lambda \left(\tilde{X}_1, \tilde{Y}_1\right), (X_2, Y_2) = (1 - \lambda) \left(\tilde{X}_2, \tilde{Y}_2\right), \]
\[\exists \left(\tilde{X}_1, \tilde{Y}_1\right), \left(\tilde{X}_2, \tilde{Y}_2\right) \in M_{\tilde{F}Q}, 0 \leq \exists \lambda \leq 1.\]

Therefore,
\[\hat{F}^Q \left(L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)}\right) \]
\[= \inf_{0 \leq \lambda \leq 1} \inf_{(\tilde{X}_1, \tilde{Y}_1) \in M_{\tilde{F}Q}} \inf_{(\tilde{X}_2, \tilde{Y}_2) \in M_{\tilde{F}Q}} \lambda \left(\text{tr} \left(L_0^{(1)} \tilde{X}_1 + \text{tr} L_1^{(1)} \tilde{Y}_1\right)\right) + (1 - \lambda) \left(\text{tr} L_0^{(2)} \tilde{X}_2 + \text{tr} L_1^{(2)} \tilde{Y}_2\right) \]
\[= \inf_{0 \leq \lambda \leq 1} \left\{ \lambda \hat{F}^Q \left(L_0^{(1)}, L_1^{(1)}\right) + (1 - \lambda) \hat{F}^Q \left(L_0^{(2)}, L_1^{(2)}\right) \right\} \]
\[= \min \left\{ \hat{F}^Q \left(L_0^{(1)}, L_1^{(1)}\right), \hat{F}^Q \left(L_0^{(2)}, L_1^{(2)}\right) \right\}.\]

Conversely, suppose \(\hat{F}^Q \in \mathcal{F}_0\) is monotone increasing by any unital CP map and satisfies (33). Then by Proposition 7.6, \(F^Q \in \mathcal{F}_0\) is CPTP monotone. Therefore, \(\left(L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)}\right)\) is a member of \(M_{F^Q}\) if and only if there exists \((L_0, L_1) \in M_{\hat{F}Q}\) such that
\[L_\theta = \begin{bmatrix} L_0^{(1)} & \ast \\ \ast & L_0^{(2)} \end{bmatrix}, \theta \in \{0, 1\}.\]

Also, by (23),
\[\left(L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)}\right) \in M_{F^Q} \]
\[\iff \hat{F}^Q \left(L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)}\right) \geq \min \left\{ \hat{F}^Q \left(L_0^{(1)}, L_1^{(1)}\right), \hat{F}^Q \left(L_0^{(2)}, L_1^{(2)}\right) \right\} \geq 1 \]
\[\iff \hat{F}^Q \left(L_0^{(1)}, L_1^{(1)}\right) \geq 1 \text{ and } \hat{F}^Q \left(L_0^{(2)}, L_1^{(2)}\right) \geq 1 \]
\[\iff \left(L_0^{(1)}, L_1^{(1)}\right), \left(L_0^{(2)}, L_1^{(2)}\right) \in M_{F^Q}.\]
Therefore,

\[
F^Q (X_1 \oplus X_2, Y_1 \oplus Y_2) = \inf_{(L_0, L_1) \in \mathcal{M}_{F^Q}} \text{tr} L_0 (X_1 \oplus X_2) + \text{tr} L_1 (Y_1 \oplus Y_2) = \inf_{(L_0^{(1)} \oplus L_0^{(2)}, L_1^{(1)} \oplus L_1^{(2)}) \in \mathcal{M}_{F^Q}} \left( \text{tr} L_0^{(1)} X_1 + \text{tr} L_0^{(2)} X_2 + \text{tr} L_1^{(1)} Y_1 + \text{tr} L_1^{(2)} Y_2 \right)
\]

\[
= \inf_{(L_0^{(1)}, L_1^{(1)}), (L_0^{(2)}, L_1^{(2)}) \in \mathcal{M}_{F^Q}} \left( \text{tr} L_0^{(1)} X_1 + \text{tr} L_1^{(1)} Y_1 \right) + \left( \text{tr} L_0^{(2)} X_2 + \text{tr} L_1^{(2)} Y_2 \right) = F^Q (X_1, Y_1) + F^Q (X_2, Y_2).
\]

7.2 Classical version and \( \hat{F}_{\min}, \hat{F}_{\max} \)

For real vectors \( l_0 := (l_{0,i})_{i=1}^k \) and \( l_1 := (l_{1,i})_{i=1}^k \) with positive components we can define \( \hat{F}^C (l_0, l_1) \) in analogy with \( \hat{F}^Q \); First, let

\[
\mathcal{M}_C := \bigcup_{k=1}^{\infty} \left\{ (l_0, l_1) ; F^C (x, y) \leq \sum_{i=1}^{k} l_{0,i} x_i + \sum_{i=1}^{k} l_{1,i} y_i \right\}.
\]

Then

\[
F^C (x, y) = \inf \left\{ \sum_{i=1}^{k} l_{0,i} x_i + \sum_{i=1}^{k} l_{1,i} y_i ; (l_0, l_1) \in \mathcal{M}_C \right\}.
\]

Thus we define

\[
\hat{F}^C (l_0, l_1) := \inf \left\{ s ; (l_0, l_1) \in s (\mathcal{M}_C)^c , s > 0 \right\}.
\]

\( \hat{F}^C \) is concave, positively homogeneous, proper, and monotone increasing by transpose of stochastic map. Also,

\[
\hat{F}^C (l_0, l_1) = \inf \left\{ \frac{1}{F^C (x, y)} \left( \sum_{i=1}^{k} l_{0,i} x_i + \sum_{i=1}^{k} l_{1,i} y_i \right) ; x, y \in \mathbb{R}^k_{\geq 0}, F^C (x, y) > 0 \right\}.
\]

(35)

In addition, it is additive in the sense that

\[
\hat{F}^C (l_0, l_1) = \hat{F}^C \left( l_0^{(1)} , l_1^{(2)} \right) + \hat{F}^C \left( l_0^{(2)} , l_1^{(1)} \right),
\]

(36)

\[
l_0^{(1)} := (l_{0,1}, l_{0,2}, \cdots, l_{0,k}),
l_0^{(2)} := (l_{0,k+1}, l_{0,k+2}, \cdots, l_{0,k}),
l_1^{(1)} := (l_{1,1}, l_{1,2}, \cdots, l_{1,k}),
l_1^{(2)} := (l_{1,k+1}, l_{1,k+2}, \cdots, l_{1,k}), \quad \theta = 0, 1.
\]

The proof of these properties is almost parallel as the proof of analogous properties of \( F^Q \), thus omitted.
By additivity (36),

\[ \hat{F}^C (l_0, l_1) = \min_i \hat{F}^C (l_{0,i}, l_{1,i}) \]
\[ = \min_i \inf \left\{ \frac{1}{\sqrt{xy}} (l_{0,i}x + l_{1,i}y); x, y \in \mathbb{R}_{>0} \right\} \]
\[ = \min_i \inf \left\{ l_{0,i} \sqrt{\frac{x}{y}} + l_{1,i} \sqrt{\frac{y}{x}}; x, y \in \mathbb{R}_{>0} \right\} \]
\[ = \min_i 2 \sqrt{l_{0,i}l_{1,i}}. \] (37)

We say a functional over \( \mathcal{P} \times 2 \) is \textit{polarly normalized} if, for any \((l_0, l_1) \in \mathbb{R}_{\geq 0}^k \times \mathbb{R}_{\geq 0}^k \) and for an orthonormal basis \( \{|i\}; i = 1, \cdots, k \} \),

\[ \hat{F}^Q (L_{0,c}, L_{1,c}) = \hat{F}^C (l_0, l_1), \]

where

\[ L_{0,c} := \sum_{i=1}^k l_{0,i} |i\rangle \langle i|, \quad L_{1,c} := \sum_{i=1}^k l_{1,i} |i\rangle \langle i|. \] (38)

Below, \( \Gamma_1 \) is the pinching by the basis \( \{|i\}; i = 1, \cdots, k \} \).

**Proposition 7.8** Suppose that \( F^Q \) is a CPTP monotone and normalized member of \( \mathcal{F}_0 \). Then, \( \hat{F}^Q \) is polarly normalized.

**Proof.** Observe \( L_{0,c} \) and \( L_{1,c} \) as of (35) are unchanged by \( \Gamma_1 \). Hence, by (35), and by CPTP monotonicity and normalization of \( F^Q \),

\[ \hat{F}^Q (L_{0,c}, L_{1,c}) \]
\[ = \inf \left\{ \frac{1}{F^Q (X, Y)} \left( \text{tr} \Gamma_1 (L_{0,c}) X + \text{tr} \Gamma_1 (L_{1,c}) Y \right); (X, Y) \in \mathcal{P} \times 2, F^Q (X, Y) > 0 \right\} \]
\[ = \inf \left\{ \frac{1}{F^Q (X, Y)} \left( \text{tr} L_{0,c} \Gamma_1 (X) + \text{tr} L_{1,c} \Gamma_1 (Y) \right); (X, Y) \in \mathcal{P} \times 2, F^Q (X, Y) > 0 \right\} \]
\[ \geq \inf \left\{ \frac{1}{F^Q (\Gamma_1 (X), \Gamma_1 (Y))} \left( \text{tr} L_{0,c} \Gamma_1 (X) + \text{tr} L_{1,c} \Gamma_1 (Y) \right); (X, Y) \in \mathcal{P} \times 2, F^Q (\Gamma_1 (X), \Gamma_1 (Y)) > 0 \right\} \]
\[ = \inf \left\{ \frac{1}{F^C (x, y)} \left( \sum_{i=1}^k l_{0,i} x_i + \sum_{i=1}^k l_{1,i} y_i \right); (x, y) \in \mathbb{R}_{\geq 0}^k, F^C (x, y) > 0 \right\} \]
\[ = \hat{F}^C (l_0, l_1). \]

Also, by

\[ \hat{F}^Q (L_{0,c}, L_{1,c}) \]
\[ \leq \inf \left\{ \frac{1}{F^Q (X, Y)} \left( \text{tr} L_{0,c} X + \text{tr} L_{1,c} Y \right); (X, Y) \in \mathcal{P} \times 2, F^Q (X, Y) > 0, X, Y: \text{diagonal} \right\} \]
and by normalization of $F^Q$,

$$
\hat{F}^Q (L_{0,c}, L_{1,c}) 
\leq \inf \left\{ \frac{1}{F^C (x, y)} \left( \sum_{i=1}^k l_{0,i} x_i + \sum_{i=1}^k l_{1,i} y_i \right) ; (x, y) \in \mathbb{R}^k_{\geq 0}, F^C (x, y) > 0 \right\}
= \hat{F}^C (l_0, l_1).
$$

After all, we have polar normalization. ■

Below, we show

$$\hat{F}_{\max} (L_0, L_1) = 2 \sqrt{\min_{\| \psi \|=1} \langle \psi | L_1^{1/2} L_0 L_1^{1/2} | \psi \rangle}$$

$$= 2 \sqrt{\min_{\| \varphi \|=1} \langle \varphi | L_0^{1/2} L_1^{1/2} | \varphi \rangle}$$

$$= 2 \min_{\| \psi \|=1, \| \varphi \|=1} \left| \langle \psi | L_0^{1/2} L_1^{1/2} | \varphi \rangle \right|$$

$$= 2 \left\| L_0^{-1/2} L_1^{-1/2} \right\|^{-1}.$$  \quad (39)

If $L_0, L_1 > 0$, by (13) and (22), $s \leq \hat{F}_{\max} (L_0, L_1)$ holds if and only if there is $L > 0$ such that

$$2L_0 \geq sL, 2L_1 \geq sL^{-1}$$

$$\Leftrightarrow 2L_0 \geq sL, L \geq \frac{s^2 L_1^{-1}}{4} \Leftrightarrow L_0 \geq \frac{s^2}{4} L_1^{-1}$$

$$\Leftrightarrow \sqrt{L_1 L_0} \sqrt{L_1} \geq \frac{s^2}{4} I,$$

which leads to the asserted result. In particular, (39) and Proposition 7.8 leads to (37).

By (39) and (21), we have

$$\hat{F}_{\min} (L_0, L_1) = \inf \{ s ; (L_0, L_1) \notin s (\mathcal{M}_{F_{\min}}), s > 0 \}$$

$$= \inf \left\{ s ; \exists A \in \mathcal{L}_{sa,k}, (L_0, (I_k + \sqrt{-1}A)^{-1} L_1 (I_k - \sqrt{-1}A)^{-1}) \notin s (\mathcal{M}_{F_{\min}}), s > 0 \right\}$$

$$= \sup_{A \in \mathcal{L}_{sa,k}} \hat{F}_{\max} \left( L_0, (I_k + \sqrt{-1}A)^{-1} L_1 (I_k - \sqrt{-1}A)^{-1} \right).$$

**Theorem 7.9** Suppose that $\hat{F}^Q \in \mathcal{F}_0$ is monotone increasing by any unital CP map, and polarly normalized. Then,

$$\hat{F}_{\max} (L_0, L_1) \leq \hat{F}^Q (L_0, L_1) \leq \hat{F}_{\min} (L_0, L_1).$$
**Proof.** For each \( \hat{F}^Q \in \mathcal{F}_0 \) which is monotone increasing by any unital CP map, and polarly normalized, there is a CPTP monotone and normalized member \( F^Q \) of \( \mathcal{F}_0 \) with \( \{26\} \), due to by Corollary \( \{7.4\} \). Since \( F^Q \) is sandwiched by \( F_{\min} \) and \( F_{\max} \) by Theorem \( \{3.2\} \) implies

\[
\hat{F}^Q (L_0, L_1) \geq \inf \left\{ \frac{1}{F_{\max}^Q(X,Y)} (\text{tr} \ L_0 X + \text{tr} \ L_1 Y) ; (X,Y) \in \mathcal{P}^2, F^Q(X,Y) > 0 \right\},
\]

\[
= \hat{F}_{\max} (L_0, L_1),
\]

\[
\hat{F}^Q (L_0, L_1) \leq \inf \left\{ \frac{1}{F_{\min}^Q(X,Y)} (\text{tr} \ L_0 X + \text{tr} \ L_1 Y) ; (X,Y) \in \mathcal{P}^2, F^Q(X,Y) > 0 \right\}
\]

\[
= \hat{F}_{\min} (L_0, L_1).
\]

Thus we have the assertion. ■

Define

\[
\hat{F}_{\max}' (L_0, L_1) := \sup \left\{ \hat{F}^C (l_0, l_1) ; L_0 = \Phi^* (l_0), \theta = 0, 1, \Phi^* \text{: CP unital map from } \mathcal{C}_k \text{ to } \mathcal{L}_k \right\}
\]

\[
= \sup \left\{ \hat{F}^C (l_0, l_1) ; L_0 = \Phi^* (l_0), \theta = 0, 1, M \text{: POVM } \right\}, \quad (40)
\]

\[
\hat{F}_{\min}' (L_0, L_1) := \inf \left\{ \hat{F}^C (l_0, l_1) ; l_0 = \Psi^* (L_0), \theta = 0, 1, \Psi^* \text{: CP unital map from } \mathcal{L}_k \text{ to } \mathcal{C}_k \right\}
\]

\[
= \inf \left\{ \hat{F}^C (l_0, l_1) ; l_0 = \Psi^* (L_0), \theta = 0, 1, \rho \text{: array of density operators } \right\}
\]

\[
= \min \left\{ 2\sqrt{\text{tr} \rho L_0 \text{tr} \rho L_1} ; \rho \geq 0, \text{tr} \rho = 1 \right\}. \quad (41)
\]

**Lemma 7.10** \( \hat{F}_{\min}' \) is a member of \( \mathcal{F}_0 \), monotone increasing by any unital CP map, and is polarly normalized. Also, \( \hat{F}_{\min}' \) is continuous. Namely, if \((L_{0, \infty}, L_{1, \infty})\) is at the (relative) boundary of \( \mathcal{P}^2 \) and \( \lim_{i \to \infty} L_{\theta, i} = L_{\theta, \infty}, \) \( \theta = 0, 1, \)

\[
\lim_{i \to \infty} \hat{F}_{\min}' (L_{0, i}, L_{1, i}) = \hat{F}_{\min}' (L_{0, \infty}, L_{1, \infty}) = 0. \quad (42)
\]

**Proof.** Observe \( \hat{F}_{\min}' \) is infimum of the map

\[
(L_0, L_1) \to \hat{F}^C (\Psi^* (L_0), \Psi^* (L_1))
\]

Since this map is a member of \( \mathcal{F}_0 \) for each \( \rho \), so is \( \hat{F}_{\min}' \) by Lemma \( \{3.1\} \). Also, if \( \Lambda^* \) is a CP unital map,

\[
\hat{F}_{\min}' (\Lambda^* (L_0), \Lambda^* (L_1)) = \min \left\{ 2\sqrt{\text{tr} \rho \Lambda^* (L_0) \text{tr} \rho \Lambda^* (L_1)} ; \rho \geq 0, \text{tr} \rho = 1 \right\}
\]

\[
= \min \left\{ 2\sqrt{\text{tr} \Lambda (\rho) L_0 \text{tr} \Lambda (\rho) L_1} ; \rho \geq 0, \text{tr} \rho = 1 \right\}
\]

\[
\geq \min \left\{ 2\sqrt{\text{tr} \rho L_0 \text{tr} \rho L_1} ; \rho \geq 0, \text{tr} \rho = 1 \right\}
\]

\[
= \hat{F}_{\min}' (L_0, L_1).
\]
Thus $\hat{F}'_{\min}$ is monotone increasing by any unital CP map.

Also, for any state $\rho$,
\[
\sqrt{\text{tr } \rho L_{0,c} \text{tr } \rho L_{1,c}} = \sqrt{\text{tr } \Gamma_1 (\rho) L_{0,c} \text{tr } \Gamma_1 (\rho) L_{1,c}},
\]
where $L_{0,c}, \theta = 0, 1$ are as of (38) and $\Gamma_1$ is the pinching with respect to the basis $\{|i\rangle; i = 1, \cdots, k\}$. Thus, in the minimum of (11), $\rho$ can be restricted to those which commute with $L_0$ and $L_1$. Therefore,
\[
\hat{F}'_{\max}(L_{0,c}, L_{1,c}) = \min \left\{ 2 \sqrt{\sum_i p_i l_{0,i} \sum_i p_i l_{1,i}}, \sum_i p_i = 1, p_i \geq 0 \right\} = \hat{F}^C(l_0, l_1),
\]
where the second equality holds because of the concavity of $(a, b) \rightarrow \sqrt{ab}$. Thus, $\hat{F}'_{\min}$ is also polarly normalized.

By Lemma 7.11 $\hat{F}'_{\min}$ is continuous on $\text{ri } \mathcal{P}^\times$. Also, for any sequence $\{(L_{0,i}, L_{1,i})\}$ which converges to $(L_{0,\infty}, L_{1,\infty})$,
\[
\lim_{i \to \infty} \hat{F}'_{\min}(L_{0,i}, L_{1,i}) \leq \lim_{i \to \infty} 2 \sqrt{\langle \psi | L_{0,i} \psi \rangle \langle \psi | L_{1,i} \psi \rangle} = 0,
\]
\[
\hat{F}'_{\min}(L_{0,\infty}, L_{1,\infty}) \leq 2 \sqrt{\langle \psi | L_{0,\infty} \psi \rangle \langle \psi | L_{1,\infty} \psi \rangle} = 0,
\]
where $|\psi\rangle$ satisfies $L_{0,\infty} |\psi\rangle = 0$ or $L_{1,\infty} |\psi\rangle = 0$. Since $\hat{F}'_{\min}$ is non-negative on $\mathcal{P}^\times$, we have
\[
\lim_{i \to \infty} \hat{F}'_{\min}(L_{0,i}, L_{1,i}) = 0 = \hat{F}'_{\min}(L_{0,\infty}, L_{1,\infty}).
\]

Therefore, $\hat{F}'_{\min}$ is continuous in $\mathcal{P}^\times$. ■

**Lemma 7.11** $\hat{F}'_{\max}$ is a member of $\hat{F}_0$, monotone increasing by any unital CP map, and is polarly normalized. Also, $\hat{F}'_{\min}$ is continuous.

**Proof.** $\hat{F}'_{\max}$ is obviously positively homogeneous by definition. Observe that $L_0 = \sum_{i=1}^{n} l_{\theta,i} M_i$ and $L'_{0} = \sum_{i=1}^{n'} l'_{\theta,i} M'_i$ imply
\[
L_0 + L'_{0} = \sum_{j=1}^{n+n'} l_{\theta,j} M^\lambda_j = \Phi^*_M \left( \{ l_{\theta,j}^\lambda \} \right),
\]
where
\[
l_{\theta,j}^\lambda = \left\{ \begin{array}{ll}
\frac{1}{\lambda} l_{\theta,j}, & j = 1, \cdots, n \\
\frac{1}{1-\lambda} l'_{\theta,j-n}, & j = n+1, \cdots, n+n' \end{array} \right.,
\]
\[
M^\lambda_j = \left\{ \begin{array}{ll}
\lambda M_j, & j = 1, \cdots, n \\
(1-\lambda) M'_j, & j = n+1, \cdots, n+n' \end{array} \right..
\]

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Therefore,
\[
\hat{F}'_{\max}(L_0 + L_0', L_1 + L_1') \\
\geq \sup_{l_0, l_0', M, M', \lambda \in [0, 1]} \hat{F}^C(l_0, l_1) \\
= \sup_{l_0, l_0', M, M', \lambda \in [0, 1]} \max \min \left\{ \frac{1}{\lambda} \hat{F}^C(l_0, l_1), \frac{1}{1 - \lambda} \hat{F}^C(l_0', l_1') \right\} \\
= \sup_{l_0, l_0', M, M'} \left\{ \hat{F}^C(l_0, l_1) + \hat{F}^C(l_0', l_1') \right\} \\
= \hat{F}'_{\max}(L_0, L_1) + \hat{F}'_{\max}(L_0', L_1').
\]

Therefore, combined with positive homogeneity, we have concavity of \( \hat{F}'_{\max} \).

Also,
\[
\hat{F}'_{\max} \leq \hat{F}'_{\min}
\]  \hspace{1cm} (43)
as is shown below. For any CP unital map \( \Phi^* \) from \( \mathcal{C}_k \) to \( \mathcal{L}_k \) and \( \Psi^* \) from \( \mathcal{L}_k \) to \( \mathcal{C}_k \), \( \Psi^* \circ \Phi^* \) is transpose of a stochastic map. So,
\[
\hat{F}^C(l_0, l_1) \leq \hat{F}^C(\Psi^* \circ \Phi^* (l_0), \Psi^* \circ \Phi^* (l_1)).
\]

Hence, if \( L_\theta = \Phi^* (l_0) \), \( \theta = 0, 1 \),
\[
\hat{F}^C(l_0, l_1) \leq \hat{F}^C(\Psi^* (L_0), \Psi^* (L_1)).
\]

Taking supremum of the LHS and infimum of the RHS, we obtain \( \hat{F}'_{\max} \leq \hat{F}'_{\min} \).

Therefore, \( \hat{F}'_{\max} \) nowhere takes the value \( \infty \). Also, that \( \hat{F}'_{\max} \) does not take the value \(-\infty\) in \( \mathcal{P}^{\times 2} \) is obvious by definition.

By Lemma B.2, \( \hat{F}'_{\max} \) is continuous on \( \text{ri} \mathcal{P}^{\times 2} \). Also, for any sequence \( \{ (L_{0,i}, L_{1,i}) \} \) which converges to \( (L_{0,\infty}, L_{1,\infty}) \), by (43) and (42)
\[
\lim_{i \to \infty} \hat{F}'_{\max}(L_{0,i}, L_{1,i}) \leq \lim_{i \to \infty} \hat{F}'_{\min}(L_{0,i}, L_{1,i}) = 0,
\]
\[
\hat{F}'_{\max}(L_{0,\infty}, L_{1,\infty}) \leq \hat{F}'_{\min}(L_{0,\infty}, L_{1,\infty}) = 0.
\]

Since \( \hat{F}'_{\max} \) is non-negative on \( \mathcal{P}^{\times 2} \), we have
\[
\lim_{i \to \infty} \hat{F}'_{\max}(L_{0,i}, L_{1,i}) = 0 = \hat{F}'_{\max}(L_{0,\infty}, L_{1,\infty}).
\]

Therefore, \( \hat{F}'_{\max} \) is continuous, and thus it is closed. After all, \( \hat{F}'_{\max} \) is a member of \( \mathcal{F}_0 \).

\( \hat{F}'_{\max} \) is monotone increasing by any unital CP map \( \Lambda^* \), proved as follows. If a POVM \( M = \{ M_i \} \) satisfies \( L_\theta = \sum_i l_{\theta,i} M_i \), the set \( \{ \Lambda^* (M_i) \} \) of operators
is POVM and satisfies $\Lambda^\ast (L_\theta) = \sum i l_{\theta,i} \Lambda^\ast (M_i)$. Therefore,

$$\hat{F}_{\max}' (\Lambda^\ast (L_0), \Lambda^\ast (L_1))$$

$$= \sup \left\{ \hat{F}_C (l_0, l_1); \Lambda^\ast (L_\theta) = \sum i l_{\theta,i} M_i, \theta = 0, 1, M: \text{POVM} \right\}$$

$$\geq \sup \left\{ \hat{F}_C (l_0, l_1); \Lambda^\ast (L_\theta) = \sum i l_{\theta,i} \Lambda^\ast (M_i), \theta = 0, 1, M: \text{POVM} \right\}$$

$$\geq \sup \left\{ \hat{F}_C (l_0, l_1); L_\theta = \sum i l_{\theta,i} M_i, \theta = 0, 1, M: \text{POVM} \right\}$$

$$= \hat{F}_{\max}' (L_0, L_1).$$

Suppose that the triple $l_0$, $l_1$, $M = \{M_i\}$ satisfies the constrain given in the RHS of (44) with $(L_0, L_1) = (L_{0,c}, L_{1,c})$, where $(L_{0,c}, L_{1,c})$ is as of (38). Then the triple $l_0$, $l_1$, $\{\Gamma_1 (M_i)\}$ also satisfies the constrain. Therefore, without changing the maximum, we may restrict the range of POVM to the ones which are diagonalized in the basis $\{|i\rangle \}_{i=1}^k$. Such a POVM corresponds to a transpose of a stochastic matrix. Therefore,

$$\hat{F}_{\max}' (L_{0,c}, L_{1,c}) = \sup \left\{ \hat{F}_C (\lambda_0, \lambda_1); l_\theta = T^T \lambda_\theta, \theta = 0, 1, T: \text{column stochastic matrix} \right\}$$

Since $\hat{F}_C$ is monotone increasing by the application of a transpose of a stochastic matrix, $\hat{F}_C (\lambda_0, \lambda_1)$ cannot exceed $\hat{F}_C (l_0, l_1)$ if $l_\theta = T^T \lambda_\theta$ (0, 1). Therefore, $\hat{F}_{\max}' (L_{0,c}, L_{1,c}) = \hat{F}_C (l_0, l_1)$, and $\hat{F}_{\max}'$ is polarly normalized. \(\blacksquare\)

The following theorem gives 'operational' meaning of $\hat{F}_{\max}'$ and $\hat{F}_{\min}'$.

**Theorem 7.12**

$$\hat{F}_{\max} = \hat{F}_{\max}' ; \hat{F}_{\min} = \hat{F}_{\min}' .$$

**Proof.** First, we show

$$\hat{F}_{\max}' (L_0, L_1) \leq \hat{F}_{\max} (L_0, L_1) \leq \hat{F}_{\min} (L_0, L_1) \leq \hat{F}_{\min}' (L_0, L_1).$$

Suppose $\hat{F}_Q$ is polarly normalized and monotone increasing by any unital CP maps. Then,

$$\hat{F}_Q (L_0, L_1) \geq \sup_{l_\theta, M} \left\{ \hat{F}_Q (L_{0,c}, L_{1,c}); L_\theta = \Phi_M^\ast (L_{\theta,c}) \right\}$$

$$= \sup_{l_\theta, M} \left\{ \hat{F}_C (l_0, l_1); L_\theta = \Phi_M^\ast (l_\theta) \right\} = \hat{F}_{\max}' (L_0, L_1),$$

$$\hat{F}_Q (L_0, L_1) \leq \inf_{l_\theta, \rho} \left\{ \hat{F}_Q (L_{0,c}, L_{1,c}); L_{\theta,c} = \Psi_\rho^\ast (L_{\theta}) \right\}$$

$$= \inf_{l_\theta, \rho} \left\{ \hat{F}_C (l_0, l_1); l_\theta = \Psi_\rho^\ast (L_{\theta}) \right\} = \hat{F}_{\min}' (L_0, L_1).$$

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Since $\hat{F}_{\text{max}}$ and $\hat{F}_{\text{min}}$ are examples of such $\hat{F}^Q$, we have inequalities (45).

By Lemmas 7.11-7.10, $\hat{F}_{\text{max}}$ and $\hat{F}_{\text{min}}$ are members of $F_0$ which is polarly normalized and monotone increasing by any unital CP map. Therefore, by Theorem 7.9 we have $\hat{F}'_{\text{max}} \leq \hat{F}_{\text{max}}$ and $\hat{F}'_{\text{min}} \leq \hat{F}_{\text{min}}$, which, combined with (45), lead to the assertion.

7.3 $\hat{F}_{1/2}$

Below, we give an expression of $\hat{F}_{1/2}$. By (9), the optimal $(L_{0,*}, L_{1,*})$ is given by the simultaneous linear equations

$$
\sqrt{Y} = \sqrt{X}L_{0,*} + L_{0,*}\sqrt{X}, \\
\sqrt{X} = \sqrt{Y}L_{1,*} + L_{1,*}\sqrt{Y}.
$$

Therefore,

$$
\sqrt{X} = S_{L_{0,*}}\left(\sqrt{Y}\right), \quad \sqrt{Y} = S_{L_{1,*}}\left(\sqrt{X}\right).
$$

So

$$
\partial M_{F_{1/2}} = \{(L_{0,*}, L_{1,*}); \exists X, Y \geq 0 \text{ with (46)}\}.
$$

Suppose there is $\sqrt{X}$ such that

$$
\sqrt{X} = S_{L_{0,*}} \circ S_{L_{1,*}}\left(\sqrt{X}\right).
$$

Observe $S_{L_{1,*}}$ is positive. Then defining $Y \geq 0$ by $\sqrt{Y} = S_{L_{1,*}}\left(\sqrt{X}\right), X, Y \geq 0$ satisfies (46). Therefore,

$$
\partial M_{F_{1/2}} = \{(L_{0,*}, L_{1,*}); \exists X \geq 0 \text{ with (47)}\}.
$$

For any $L_0 > 0$ and $L_1 > 0$, the map $S_{L_0} \circ S_{L_1}$ is strictly positive. Also, a map

$$
A \rightarrow \frac{1}{\text{tr}S_{L_0} \circ S_{L_1}(A)}S_{L_0} \circ S_{L_1}(A)
$$

defined on the compact convex set

$$
P_k^\times \cap \{A; \text{tr } A = 1\},
$$

is continuous. Thus, by Tychonoff’s fixed point theorem, there is $A_* \geq 0$ fixed by the map (48), or equivalently,

$$
S_{L_0} \circ S_{L_1}(A_*) = \alpha_* A_*, \quad \alpha_* > 0,
$$

or

$$
S_{\sqrt{\pi L_0}} \circ S_{\sqrt{\pi L_1}}(A_*) = A_*.
$$
Therefore, \( (\hat{F} (L_0, L_1))^{-2} \) is an eigenvalue of \( S_{L_0} \circ S_{L_1} \) corresponding to the eigenvector \( A_\lambda \geq 0 \).

But there can be two or more eigenvalues of \( S_{L_0} \circ S_{L_1} \), which corresponding eigenvectors are positive. Also, in this way one has to compute eigenvectors in addition to eigenvalues of \( S_{L_0} \circ S_{L_1} \). So we further investigate the nature of \( S_{L_0} \circ S_{L_1} \).

**Proposition 7.13** Suppose \( L_0 > 0 \) and \( L_1 > 0 \). Then, \( S_{L_0} \circ S_{L_1} \) is diagonalizable and all the eigenvalues are positive.

**Proof.** First, \( S_{L_0} \) and \( S_{L_1} \) is self-adjoint, and all the eigenvalues of them are positive. In fact, let \( |\varphi_j\rangle \) be the eigenvector of \( L_1 \) with corresponding eigenvalue \( \lambda_j (> 0, \text{by the assumption } L_1 > 0) \). Then,

\[
S_{L_1} (|\varphi_i\rangle \langle \varphi_j|) = \frac{1}{\lambda_i + \lambda_j} |\varphi_i\rangle \langle \varphi_j|.
\]

Since \( \{|\varphi_i\rangle \langle \varphi_j|\}_{i,j} \) forms a complete basis of \( L_k \), they are the only eigenvectors. Thus, all the eigenvalues of \( S_{L_1} \) are positive. Second, \( S_{L_0} \circ S_{L_1} \) has the same Jordan standard form as \( S_{L_1}^{1/2} \circ S_{L_0} \circ S_{L_1}^{1/2} = S_{L_1}^{1/2} \circ S_{L_0} \circ S_{L_1} \). Thus, \( S_{L_0} \circ S_{L_1} \) is diagonalizable, and all of its eigenvalues are positive.

**Proposition 7.14** Suppose \( L_0 > 0 \) and \( L_1 > 0 \). Let \( A \) and \( A' \) be an eigenvector of \( S_{L_0} \circ S_{L_1} \), with corresponding eigenvalue \( \alpha \) and \( \alpha' \), respectively. Further, suppose \( A > 0 \). Then \( \alpha \geq \alpha' \). Especially, if \( A' \) is also strictly positive, \( \alpha = \alpha' \).

**Proof.** Without loss of generality, we suppose \( A' \) is not a member of \( -P \). (Otherwise, we name \( -A' \) as \( A' \).) Since \( A > 0 \), there is a positive \( \lambda > 0 \) such that \( A - \lambda A' \) is on \( \partial P_k \). Suppose \( \alpha' > 0 \). Then since

\[
S_{L_0} \circ S_{L_1} (A - \lambda A') = \alpha \left(A - \frac{\alpha'}{\alpha} \lambda A'\right)
\]

is positive and \( \alpha > 0 \) by Proposition 7.13, \( \alpha' \) cannot exceed \( \alpha \). The second statement is proved by interchanging \( A \) and \( A' \).

**Proposition 7.15** Let \( A \) be an eigenvector of \( S_{L_0} \circ S_{L_1} \), \( L_0 > 0 \), \( L_1 > 0 \). If \( A \) is positive but may not be strictly positive, then the subspace \( \text{supp } A \) is invariant by \( L_0 \) and \( L_1 \).

**Proof.** Since \( S_{L_1} \) and \( S_{L_0} \) are completely positive map,

\[
\text{supp } A \subset \text{supp } S_{L_1} (A) \subset \text{supp } S_{L_0} \circ S_{L_1} (A) = \text{supp } A.
\]

Therefore,

\[
\text{supp } S_{L_1} (A) = \text{supp } A.
\]
Recall
\[ S_{L_1}(A) = \int_0^\infty e^{-tL_1} Ae^{-tL_1} \, dt. \]

So, as is proved in the following,
\[ \text{supp} e^{-tL_1} A e^{-tL_1} \subset \text{supp} A, \ \forall t \geq 0. \]

Suppose otherwise, or there is \( t_0 \geq 0 \) such that
\[ \text{supp} e^{-tL_1} A e^{-tL_1} \not\subset \text{supp} A. \]

Then, by continuity of \( t \to e^{-tL_1} A e^{-tL_1}, \ \varepsilon > 0 \) such that
\[ \text{supp} e^{-tL_1} A e^{-tL_1} \not\subset \text{supp} A, \ \forall t \in [t_0, t_0 + \varepsilon]. \]

Thus
\[ \text{supp} S_{L_1}(A) \subset \text{supp} \int_{t_0}^{t_0 + \varepsilon} e^{-tL_1} A e^{-tL_1} \, dt \not\subset \text{supp} A, \]

which leads to contradiction.

Since
\[ \dim \text{supp} e^{-tL_1} A e^{-tL_1} = \dim \text{supp} A, \]

we should have
\[ \text{supp} e^{-tL_1} A e^{-tL_1} = \text{supp} A, \ \forall t \geq 0. \]

Therefore,
\[ \text{supp} A = \text{supp} \frac{e^{-tL_1} A e^{-tL_1} - A}{-t}, \ \forall t > 0, \]

which means
\[ \text{supp} L_1 A + AL_1 = \text{supp} A. \]

Since
\[ L_1 A + AL_1 = \begin{bmatrix} L_{1,11} & L_{1,12} \\ L_{1,21} & L_{1,22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_{1,11} & L_{1,12} \\ L_{1,21} & L_{1,22} \end{bmatrix}, \]

we have \( L_{1,21} A_{11} = 0 \). Since \( A_{11} \) is strictly positive, \( L_{1,21} = 0 \). Therefore, \( \text{supp} A \) is invariant by \( L_1 \).

Replacing \( A \) by \( S_{L_1}(A) \) and \( L_1 \) by \( L_0 \) in the above argument, we can conclude that \( \text{supp} S_{L_1}(A) = \text{supp} A \) is invariant also by \( L_0 \).

Using these propositions, \( \tilde{F}_{1/2}(L_0, L_1) \) \( (L_0 > 0, \ L_1 > 0) \) is computed as follows.

**Theorem 7.16**

\[
\tilde{F}_{1/2}(L_0, L_1) = 2 \left\| S_{L_0}^{1/2} \circ S_{L_1} \circ S_{L_0}^{1/2} \right\|^{-1/2}.
\]
In practice, it is easier to compute the square root of the smallest eigenvalue of $S^{-1}_{L_0} \circ S^{-1}_{L_1}$, which is the linear map sending $X$ to $\{L_1, \{L_0, X\}\}$.

**Proof.** Decompose $L_\theta (\theta = 0, 1)$ into

$$L_\theta := L^{(1)}_\theta \oplus L^{(2)}_\theta \oplus L^{(3)}_\theta \oplus \cdots (\theta = 0, 1)$$

so that $L^{(i)}_0$ and $L^{(i)}_1$ are acting on the same subspace $\mathcal{H}^{(i)}$, and do not have any smaller common invariant subspace. Then by (53), the problem reduces to the computation of each $\hat{F}_{1/2} \left( L^{(i)}_0, L^{(i)}_1 \right)$. Since $L^{(i)}_1$ and $L^{(i)}_0$ has no smaller common invariant subspace, Proposition 7.15 implies the following: Let us view $S_{L^{(i)}_0}$ as a linear transform on $\mathcal{L} \left( \mathcal{H}^{(i)} \right)$. If the eigenvector $A^{(i)} \in \mathcal{L} \left( \mathcal{H}^{(i)} \right)$ of $S_{L^{(i)}_0} \circ S_{L^{(i)}_1}$ is a positive operator, it is strictly positive. Therefore, by Proposition 7.14 the largest eigenvalue $\alpha^{(i)}$ of $S_{L^{(i)}_0} \circ S_{L^{(i)}_1}$ is the only eigenvalue whose corresponding eigenvector positive definite operator. Thus, we have

$$\hat{F}_{1/2} \left( L^{(i)}_0, L^{(i)}_1 \right) = 2 \left( \alpha^{(i)} \right)^{-1/2},$$

$$\hat{F}_{1/2} (L_0, L_1) = 2 \min_i \left( \alpha^{(i)} \right)^{-1/2} = 2 \left( \max_i \alpha^{(i)} \right)^{-1/2}.$$

Denote by $[X^{(i,j)}]$ the matrix whose $(i, j)$ block is a linear map $X^{(i,j)}$ from $\mathcal{H}^{(j)}$ to $\mathcal{H}^{(i)}$. Then, by (7),

$$S_{L_0} \circ S_{L_1} (X) = \int_0^\infty \int_0^\infty \left[ e^{-sL_0^{(i)}} e^{-tL_1^{(i)}} X^{(i,j)} e^{-tL_1^{(j)}} e^{-sL_0^{(j)}} \right] \, dt \, ds.$$

Therefore, the $(i, i)$ block of $S_{L_0} \circ S_{L_1} (X)$ is $S_{L_0^{(i)}} \circ S_{L_1^{(i)}} (X^{(i,i)})$. Therefore, if $X$ is an eigenvector of $S_{L_0} \circ S_{L_1} (X)$ corresponding to the eigenvalue $\alpha$, $X^{(i,i)}$ is an eigenvector of $S_{L_0^{(i)}} \circ S_{L_1^{(i)}}$ corresponding to the eigenvalue $\alpha$. Therefore, an eigenvalue of $S_{L_0} \circ S_{L_1}$ cannot exceed $\max_i \alpha^{(i)}$. On the other hand,

$$0 \oplus \cdots \oplus A^{(i_*)} \oplus 0 \cdots \oplus 0$$

is an eigenvector of $S_{L_0} \circ S_{L_1}$ corresponding to the eigenvalue $\max_i \alpha^{(i)}$, where $i_* := \arg\max_i \alpha^{(i)}$. Therefore,

$$\max_i \alpha^{(i)} = \text{the largest eigenvalue of } S_{L_0} \circ S_{L_1},$$

$$= \left\| S_{L_0}^{1/2} \circ S_{L_1} \circ S_{L_0}^{1/2} \right\|,$$

and we have the asserted result. \( \blacksquare \)

### 8 Extreme points and boundary

The set of all extreme points $\text{ext } \mathcal{M}_{P^2}$ of $\mathcal{M}_{P^2}$, by Lemma 11, satisfies

$$\mathcal{M}_{P^2} = \text{conv ext } \mathcal{M}_{P^2} + \mathcal{P} \times 2. \quad (51)$$

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Thus, \( \text{ext } \mathcal{M}_{FQ} \) is the key part in considering minimization of \( \text{tr } L_0 X + \text{tr } L_1 Y \).

By (8),

\[
\text{ext } \mathcal{M}_{F_{\text{min}}} = \left\{ (L_0, L_1) ; L_0 > 0, L_1 = (4L_0)^{-1} \right\}.
\]

To see geometry of \( \text{ext } \mathcal{M}_{FQ} \) and \( \mathcal{M}_{FQ} \), for each \( L_0 \), define

\[
\mathcal{M}_{FQ} (L_0) := \left\{ L_1 ; (L_0, L_1) \in \mathcal{M}_{FQ} \right\}.
\]

By Lemma 8.1, \( \mathcal{M}_{FQ} (L_0) = \text{conv } \text{ext } \mathcal{M}_{FQ} (L_0) \cap \mathcal{P} \).

(52)

\( \mathcal{M}_{FQ} \) is specified if we specify \( \mathcal{M}_{FQ} (L_0) \), because of the following.

**Proposition 8.1** Suppose \( F^Q \in \mathcal{F}_0 \) is CPTP monotone and normalized. Then,

\[
\left\{ L_0 ; (L_0, L_1) \in \mathcal{M}_{FQ} \right\} = \left\{ L_0 ; L_0 > 0 \right\}
\]

**Proof.** By Theorem 3.2, \( \mathcal{M}_{F_{\text{max}}} \subset \mathcal{M}_{FQ} \subset \mathcal{M}_{F_{\text{min}}} \). Therefore, (13) implies

\[
\left\{ L_0 ; L_0 > 0 \right\} \subset \left\{ L_0 ; (L_0, L_1) \in \mathcal{M}_{FQ} \right\},
\]

and (19) implies opposite inclusion. Thus we have the assertion. \( \blacksquare \)

Due to this Proposition, we have

\[
\mathcal{M}_{FQ} (L_0) = \left\{ (L_0, L_1) ; L_0 > 0, L_1 \in \mathcal{M}_{FQ} (L_0) \right\}
\]

\[
\text{ext } \mathcal{M}_{FQ} = \left\{ (L_0, L_1) ; L_0 > 0, L_1 \in \text{ext } \mathcal{M}_{FQ} (L_0) \right\}.
\]

Thus, \( \mathcal{M}_{FQ} (L_0) \) and \( \text{ext } \mathcal{M}_{FQ} (L_0) \) determines \( \mathcal{M}_{FQ} \) and \( \text{ext } \mathcal{M}_{FQ} \), respectively. By (8),

\[
\mathcal{M}_{F_{\text{max}}} (L_0) = \left\{ L_1 ; L_1 \geq (4L_0)^{-1} \right\},
\]

\[
\text{ext } \mathcal{M}_{F_{\text{max}}} (L_0) = \left\{ (4L_0)^{-1} \right\},
\]

and the latter is consists of only a single point.

As is shown below, if \( L_0 \) and \( L_1 \) are strictly positive and have no common non-trivial invariant subspace,

\[
L_{1,\#} \in \text{conv } \mathcal{M}_{F_{1/2}} (L_0),
\]

where

\[
L_{1,\#} := \frac{1}{\left\{ \mathcal{F}^{1/2} (L_0, L_1) \right\}} L_1.
\]

This means the dimension of \( \text{conv } \mathcal{M}_{F_{1/2}} (L_0) \cap \mathcal{L}_k \) is full, i.e., \( k^2 \).

First, by the definition of \( \mathcal{F}^{1/2} (L_0, L_1) \), Proposition 4.5 and Proposition 7.5, \( (L_0, L_{1,\#}) \) is a member of \( \partial \mathcal{M}_{F_{1/2}} \), where Suppose \( (M_0, M_1) \in \mathcal{M}_{F_{1/2}} \) that is
In this section, we determine the 9-qubit case. So while convolutional ext $M_0^{\text{ext}}$ contradicts with the assumption. Therefore, there is no $(X, Y) \in P_{\mathcal{L}}$ such that

$$\min_{(L'_0, L'_1) \in M_{F_{1/2}}} (\text{tr} X L'_0 + \text{tr} Y L'_1) = \text{tr} X L_0 + \text{tr} Y L_{1,\#}$$

$$\geq \text{tr} X M_0 + \text{tr} Y M_1$$

$$\geq \min_{(L'_0, L'_1) \in M_{F_{1/2}}} (\text{tr} X L'_0 + \text{tr} Y L'_1).$$

Therefore,

$$\text{tr} X L_0 + \text{tr} Y L_{1,\#} = \text{tr} X M_0 + \text{tr} Y M_1.$$ 

This can hold true only if $X$ and $Y$ has null eigenspace. By Proposition 7.15 in turn this means $L_0$, and $L_1$ has a common non-trivial invariant subspace, contradicting with the assumption. Therefore, there is no $(M_0, M_1)$ with $M_0 \leq L_0$, $M_1 \leq L_{1,\#}$. So if $(L_0, L_{1,\#})$ is not a member of $\text{conv ext } M_{F_{1/2}}$, it contradicts with (51). Therefore, we have (53).

On the other hand, as is shown in detail in the next section,

$$\dim \text{conv ext } M_{F_{\min}} (L_0) \cap \mathcal{L}_2 = 3.$$ 

So while $\text{conv ext } M_{F_{\max}} (L_0)$ and $\text{conv ext } M_{F_{\max}} (L_0)$ are confined to lower dimensional subspace, $\text{conv ext } M_{F_{1/2}} (L_0)$, which lies between them, is extends to the full space.

9 Qubit case

9.1 $M_{F_{\min}} (L_0)$

In this section, we determine $M_{F_{\min}} (L_0)$, when $L_0$ is living in qubit space, $L_0 \in \mathcal{L}_2$. In what follows, $\sigma_x$, $\sigma_y$, and $\sigma_z$ are Pauli matrices.

By (17),

$$M_{F_{\min}} (L_0) \cap \mathcal{L}_2 = \left\{ L_1; L_1 \geq \frac{1}{4} \left( I_2 - \sqrt{-1} A \right) L_0^{-1} I_2 + \sqrt{-1} I_2, A \in \mathcal{L}_{sa,2} \right\}.$$

Define

$$M_0 (M) := \left\{ L; L \geq \left( M + \sqrt{-1} B \right) \left( M - \sqrt{-1} B \right) - M^2, B \in \mathcal{L}_{sa,2} \right\}.$$

and suppose, without loss of generality, $L_0^{-1} = l\sigma_z + mI_2$. Then by Lemma 6.15,

$$\text{ext, } M_{F_{\min}} (L_0) \cap \mathcal{L}_2$$

$$= \frac{1}{4} \left( L_0^{-1} + \sqrt{L_0^{\text{ext}, M_0} (L_0^{-1})} \sqrt{L_0} \right)$$

$$= \left\{ \frac{1}{4} L_0^{-1} + \frac{l^2}{4} \sqrt{L_0} \left( s \cos \alpha \sigma_x + \sin \alpha \sigma_y \right) + \frac{s^2}{4} I_2 \right\} \sqrt{L_0}; \alpha \in \mathbb{R}, s \in [-2, 2]$$

$$= \left\{ \frac{1}{4} L_0^{-1} + \frac{l^2}{4} \left( \frac{s}{\sqrt{l^2 - m^2}} \cos \alpha \sigma_x + \sin \alpha \sigma_y \right) + \frac{s^2}{4} L_0 \right\}; \alpha \in \mathbb{R}, s \in [-2, 2]$$.
Thus the dimension of the smallest affine plane spanned by $\text{ext} \mathcal{M}_{F_{\min}} (L_0)$ is 3. By Lemma C.4,

$$\mathcal{M}_{F_{\min}} (L_0) \cap \mathcal{L}_2 = \frac{1}{4} \left( L_0^{-1} + \sqrt{L_0 \mathcal{M}_0 (L_0^{-1})} \sqrt{L_0} \right) = \left\{ \frac{1}{4} (I_2 - \sqrt{-1} A) L_0^{-1} (I_2 + \sqrt{-1} A) ; A \in \mathcal{L}_{sa,2} \right\}.$$

### 9.2 $\hat{F}_{\max}$ and $\hat{F}_{\min}$

In this subsection, we deal with $\hat{F}_{\max}$, $\hat{F}_{\min}$ and $\hat{F}_{1/2}$.

First, we compute $\hat{F}_{\max}$ by (39). Since $\text{tr} L_0^{1/2} L_1^{1/2} = \text{tr} L_0 L_1$ and $\det L_0^{1/2} L_1^{1/2} = \det L_0 \det L_1$,

$$\hat{F}_{\max} (L_0, L_1) = \frac{\text{tr} L_0 L_1 - \sqrt{\left( \text{tr} L_0 L_1 \right)^2 - 4 \det L_0 \det L_1}}{2 \det L_0 \det L_1 \text{tr} L_0 L_1 + \sqrt{\left( \text{tr} L_0 L_1 \right)^2 - 4 \det L_0 \det L_1}}.$$

Suppose

$$\text{tr} L_0 = \text{tr} L_1 = 2, \quad \frac{1}{2} \text{tr} L_0^2 - 1 = \frac{1}{2} \text{tr} L_1^2 - 1 = r^2. \quad (54)$$

Then

$$\det L_0 = \det L_1 = 1 - r^2.$$

Thus

$$\hat{F}_{\max} (L_0, L_1) = \frac{2 (1 - r^2)^2}{\text{tr} L_0 L_1 + \sqrt{\left( \text{tr} L_0 L_1 \right)^2 - 4 (1 - r^2)^2}}.$$

The general explicit formula for $\hat{F}_{\min}$ is awfully complicated even for qubit case. But when (54) and (55) hold, it takes very simple form. Let

$$L_0 = I_2 + a \sigma_x + b \sigma_z, L_1 = I_2 + a \sigma_x - b \sigma_z,$$
where \( a^2 + b^2 = r^2 \leq 1 \). Then by (44) and Theorem 7.12

\[
\hat{F}_{\text{min}} (L_0, L_1) = 2 \sqrt{\min_{x,z: x^2 + z^2 = 1} (1 + ax + bz)(1 + ax - bz)}
\]

\[
= 2 \sqrt{\min_{x,z: x^2 + z^2 = 1} (r^2x^2 + 2ax + 1 - b^2)}
\]

\[
= 2 \min_{x \in [-1,1]} (r^2x^2 + 2ax + 1 - b^2)
\]

\[
= \begin{cases}
2\sqrt{(1 - r^2)} \left(1 - \frac{a^2}{2r^2}\right), & (r^2 \geq |a|), \\
2 (1 - |a|), & (r^2 < |a|),
\end{cases}
\]

where the first equality is due to \( x^2 + z^2 = 1 \).

So if (54) and (55) hold, \( \hat{F}_{\text{max}} (L_0, L_1) \) and \( \hat{F}_{\text{min}} (L_0, L_1) \) are decreasing in the overlap \( \text{tr}L_0L_1 \). While fidelity is increasing in the overlap between the states, its dual is decreasing in the overlap of observables. One may wonder whether such quantity can be of any use. But, since \( \hat{F}_{\text{max}} \) and \( \hat{F}_{\text{min}} \) are CPTP monotone increasing by CP unital map \( \Lambda^* \), \( \hat{F}_{\text{max}} (L_0, L_1) \leq \hat{F}_{\text{max}} (L'_0, L'_1) \) or \( \hat{F}_{\text{min}} (L_0, L_1) \leq \hat{F}_{\text{min}} (L'_0, L'_1) \) or is a necessary condition for

\[
\Lambda^* (L_0) = L'_0, \theta \in \{0, 1\}
\]

(56)
to hold for some CP unital map \( \Lambda^* \). In fact, using these conditions, one can prove the following assertion.

**Proposition 9.1** Consider a qubit system, and suppose \((L_0, L_1)\) and \((L'_0, L'_1)\) satisfy (54) and

\[
\text{tr} L'_0 = \text{tr} L'_1 = \text{tr} (L'_0)^2 = \text{tr} (L'_1)^2.
\]

Then, there is a CP unital map \( \Lambda^* \) with (56) if and only if

\[
\text{tr} L_0L_1 = \text{tr} L'_0L'_1.
\]

In other words, \((L_0, L_1)\) and \((L'_0, L'_1)\) are unitary equivalent.

**Proof.** Without loss of generality, we put

\[
L_0 = I_2 + a\sigma_x + b\sigma_z, \quad L_1 = I_2 + a\sigma_x - b\sigma_z.
\]

Then, under the condition of the present proposition,

\[
\|L_0 - L_1\| = \|2b\sigma_z\| = 2|b| = \sqrt{\text{tr} L'_0 - \text{tr} L_0L_1}
\]

is monotone decreasing in \( \text{tr} L_0L_1 \). Since \( \|L_0 - L_1\| \) is monotone decreasing by application of any CP unital map, \( \text{tr} L_0L_1 \) is monotone increasing by any CP unital map.

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\( \hat{F}_{\text{max}}(L_0, L_1) \) and \( \hat{F}_{\text{min}}(L_0, L_1) \) are decreasing in \( \text{tr} L_0 L_1 \) and increasing by application of any CP unital map. Therefore, \( \text{tr} L_0 L_1 \) is monotone decreasing by any CP unital map. Combining the above argument, \( \text{tr} L_0 L_1 \) is invariant by any CP unital map under the condition of the present proposition. Thus we have the assertion. ■

**Proposition 9.2** Consider a qubit system, and suppose \( \text{rank} L_0 = 1 \) or \( \text{rank} L_1 = 1 \). There is a CP unital map \( \Lambda^* \) with (56) only if \( \text{rank} L'_0 = 1 \) or \( \text{rank} L'_1 = 1 \).

**Proof.** Since \( \hat{F}_{\text{max}}(L_0, L_1) = \hat{F}_{\text{min}}(L_0, L_1) = 0 \) and \( \hat{F}^Q \) is monotone decreasing by CP unital map, \( \hat{F}_{\text{max}}(L'_0, L'_1) = \hat{F}_{\text{min}}(L'_0, L'_1) = 0 \). Therefore, either \( L'_0 \) or \( L'_1 \) is not full-rank. ■

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**A Matrix**

**Lemma A.1** Let \( X, Y \) be a positive definite matrices. Then,

\[
\begin{bmatrix}
X & C \\
C^\dagger & Y
\end{bmatrix} \geq 0
\] (57)

if and only if

\[
(I - \pi_X) C = 0, \ C (I - \pi_Y) = 0.
\] (58)

and

\[
X \geq CY^{-1} C^\dagger
\] (59)
Proof. Suppose \((57)\) holds. To prove \((I - \pi_X)C = 0\), suppose \((I - \pi_X)C \neq 0\). Then, there is a unit vector \(|\varphi\rangle\) in the support of \(I - \pi_X\) such that \(\langle \varphi | C \neq 0\). Therefore, for a sufficiently large \(c > 0\),

\[
\begin{bmatrix}
-c \langle \varphi | C \\
\langle C^\dagger | \varphi \rangle
\end{bmatrix}
\begin{bmatrix}
X \\
C^\dagger Y
\end{bmatrix}
\begin{bmatrix}
-c |\varphi\rangle \\
C^\dagger |\varphi\rangle
\end{bmatrix}
= 0
-2c \langle \varphi | C C^\dagger | \varphi \rangle
+ \langle \varphi | C^\dagger Y C | \varphi \rangle < 0.
\]

This contradicts with \((57)\). Therefore, we have \((I - \pi_X)C = 0\). The proof of \(C (I - \pi_Y) = 0\) is almost parallel.

If \((57)\) holds,

\[
\begin{bmatrix}
I & -CY^{-1} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
X & C \\
C^\dagger Y
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-Y^{-1}C^\dagger & I
\end{bmatrix}
= \begin{bmatrix}
X - CY^{-1}C^\dagger & C - C\pi_Y \\
C^\dagger - \pi_Y C^\dagger & Y
\end{bmatrix}
= \begin{bmatrix}
X - CY^{-1}C^\dagger & 0 \\
0 & Y
\end{bmatrix}
geq 0,
\]

which implies \((59)\).

Suppose, on the other hand, \((58)\) and \((59)\) holds. Tracking back the chain of identities in \((60)\), we have

\[
\begin{bmatrix}
I & -CY^{-1} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
X & C \\
C^\dagger Y
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-Y^{-1}C^\dagger & I
\end{bmatrix}
geq 0.
\]

Since

\[
\det \begin{bmatrix}
I & -CY^{-1} \\
0 & I
\end{bmatrix} = 1 \neq 0,
\]

this matrix is invertible. Therefore, we have \((57)\). □

B Convex analysis

Below, unless otherwise mentioned, a function \(f\) is defined on \(\mathbb{R}^n\) and takes values in \(\mathbb{R} \cup \{\infty, -\infty\}\). The epigraph \(\text{epi} f\) of a function \(f\) defined on \(\mathbb{R}^n\) is

\[
\text{epi} f = \{(x, y) : x \in \mathbb{R}^n, y \geq f(x)\}.
\]

\(f\) is said to be \textit{convex} if \(\text{epi} f\) is convex, and \textit{concave} if \(-f\) is convex.

The effective domain \(\text{dom} f\) of a convex (concave, resp.) function is

\[
\text{dom} f := \{x : x \in \mathbb{R}^n, f(x) < \infty\} \subset \mathbb{R}^n,
\]

(\(\text{dom} f := \{x : x \in \mathbb{R}^n, f(x) > -\infty\} \subset \mathbb{R}^n\), resp. ). A convex (concave, resp.) function \(f\) is said to be \textit{proper} if \(f(x) \neq -\infty \) (\(f(x) \neq \infty\), resp.) for any \(x\) and \(f(x) \neq \infty \) (\(f(x) \neq -\infty\), resp.) for some \(x\). A sublinear function \(f\) is a function which is convex and homogeneous.
A function \( f \) is said to be lower semi continuous (upper semi continuous, resp.) if
\[
 f(x) = \lim_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} (\inf \{ f(y) : \|y - x\| < \varepsilon \})
\]
(resp.). The lower semicontinuous hull (upper semicontinuous hull, resp.) of \( f \) is the greatest lower semicontinuous (the smallest upper semicontinuous) function which is not larger than (not smaller than, resp.) \( f \).

**Lemma B.1** For any family of functions \( \{f_i; i \in I\} \),
\[
 \lim_{y \to x} \sup_{i \in I} f_i(y) = \sup_{i \in I} \lim_{y \to x} f_i(y),
\]
\[
 \lim_{y \to x} \inf_{i \in I} f_i(y) = \inf_{i \in I} \lim_{y \to x} f_i(y).
\]

Therefore, if each \( f_i \) is lower semicontinuous (upper semicontinuous, resp.), so is \( \sup_{i \in I} f_i \) (\( \inf_{i \in I} f_i \)).

**Proof.** Observe
\[
 \lim_{y \to x} \sup_{i \in I} f_i(y) = \lim_{\varepsilon \downarrow 0} \sup_{i \in I} \left\{ f_i(y) : \|y - x\| < \varepsilon \right\}
\]
\[
 = \sup_{\varepsilon > 0} \left\{ \sup_{i \in I} f_i(y) : \|y - x\| < \varepsilon \right\}
\]
\[
 \geq \sup_{i \in I} \sup_{\varepsilon > 0} \left\{ f_i(y) : \|y - x\| < \varepsilon \right\}
\]
\[
 = \sup_{i \in I} \lim_{y \to x} f_i(y).
\]

Since \( \lim_{y \to x} f(y) \leq f(x) \) by definition, this means
\[
 \lim_{y \to x} \sup_{i \in I} f_i(y) = \sup_{i \in I} \lim_{y \to x} f_i(y).
\]
The second identity is shown in almost parallel manner. 

The closure \( \text{cl} f \) of a convex (concave, resp.) function \( f \) is defined as follows. If \( f \) nowhere has the value \(-\infty \) (\( \infty \), resp.), \( \text{cl} f \) is the lower semicontinuous hull (upper semicontinuous hull, resp.) of \( f \). If \( f(x) = -\infty \) (\( = \infty \), resp.) for some \( x \), \( \text{cl} f \) is the constant function \(-\infty \) (\( \infty \), resp.). A convex or concave function \( f \) is said to be closed if \( \text{cl} f = f \). If \( f \) nowhere has the value \(-\infty \) and \( f \) is convex, \( f \) is closed if and only if \( \text{epi} f \) is closed.

The affine hull \( \text{aff} C \) of a set \( C \) is the smallest affine set which includes \( C \). The relative interior \( \text{ri} C \) of a convex set \( C \) is
\[
 \text{ri} C = \{ x \in \text{aff} C : \exists \varepsilon > 0, (x + B_\varepsilon) \cap \text{aff} C \subset C \},
\]
where \( B_\varepsilon \) is \( \varepsilon \)-ball centered at 0. The relative boundary of \( C \) is \( \text{cl} C \setminus \text{ri} C \).
Lemma B.2 (Theorem 10.1 and Theorem 7.4, [3]) A convex function \( f \) on \( \mathbb{R}^n \) is continuous on \( \text{ri}(\text{dom } f) \). Let \( f \) be a proper convex function on \( \mathbb{R}^n \). Then \( \text{cl } f \) agrees with \( f \) except perhaps at relative boundary points of \( \text{dom } f \).

Lemma B.3 (Theorem 7.4, [3]) If \( f \) is proper and convex, so is \( \text{cl } f \).

Lemma B.4 If \( f_i \) is convex, closed, and nowhere has the value \(-\infty\) for each \( i \in I \), so is \( \sup_{i \in I} f_i \). Also, if \( f_i \) is concave, closed and has nowhere has the value \( \infty \) for each \( i \in I \), so is \( \inf_{i \in I} f_i \).

Proof. We only have to show the first statement, since the second one follows by considering \(-f_i\). Observe

\[
\text{epi } \sup_{i \in I} f_i = \bigcap_{i \in I} \text{epi } f_i.
\]

Therefore, if each \( \text{epi } f_i \) is convex and closed, so is \( \text{epi } \sup_{i \in I} f_i \).

The dual \( f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty, -\infty\} \) of \( f \) is

\[
f^* (x^*) := \sup_{x \in \mathbb{R}^n} \langle x^*, x \rangle - f(x).
\]

Lemma B.5 (Theorem 12.2 and Corollary 12.2.1, [3]) Let \( f \) be a convex function. The conjugate function \( f^* \) is then a closed convex function, proper if and only if \( f \) is proper. Moreover, \( (\text{cl } f)^* = f^* \) and \( f^{**} = \text{cl } f \). Thus, The conjugacy operation \( f \to f^* \) induces a symmetric one-to-one correspondence in the class of all closed proper convex functions on \( \mathbb{R}^n \).

The indicator function \( \delta (x|C) \) and the support function \( \delta^* (x^*|C) \) of a convex set \( C \) is

\[
\delta (x|C) := \begin{cases} 
0, & (x \in C), \\
\infty, & (x \notin C), 
\end{cases}
\]

and

\[
\delta^* (x^*|C) := \sup_{x \in C} \langle x^*, x \rangle \\
= \sup_{x \in \mathbb{R}^n} \langle x^*, x \rangle - \delta (x|C) \\
= (\delta (\cdot|C))^* (x^*).
\]

For any convex set \( C \) (p. 112 of [3])

\[
\delta^* (x^*|C) = \delta^* (x^*|\text{cl } C).
\] (61)

If \( C \) is a closed convex set, \( \delta (\cdot|C) \) is closed, since it is lower semicontinuous.

Lemma B.6 (Theorem 13.2, [3]) The indicator function and the support function of a closed convex set are conjugate to each other. The support function of a non-empty convex set is closed, proper, convex, and positively homogeneous. Also, any closed, proper, convex, and positively homogeneous function is the support function of a non-empty convex set.
Lemma B.7 Suppose \( f^* \) is a closed proper convex functions which are positively homogeneous. Then, there is a non-empty closed convex set \( C_{f^*} \) such that

\[
f^* (x^*) = \delta^* (x^*|C_{f^*}),
\]

and the correspondence between \( f^* \) and \( C_{f^*} \) is one-to-one.

Proof. By Lemma B.6 there is a non-empty convex set with (62). By (61), we can suppose that \( C_{f^*} \) is closed, and thus, that its indicator function \( \delta (\cdot|C_{f^*}) \) is closed. Therefore, by Lemma B.5,

\[
(f^*)^* = (\delta^* (\cdot|C_{f^*}))^* = (\delta (\cdot|C_{f^*}))^{**} = \text{cl} \delta (\cdot|C_{f^*}) = \delta (\cdot|C_{f^*}).
\]

Thus, for each given \( f^* \), \( \delta (\cdot|C_{f^*}) \) is uniquely decided, and we have the assertion.

The recession cone \( 0^+C \) of the convex set \( C \) is

\[
0^+C := \{ y; x + \lambda y \in C, \forall x \in C, \forall \lambda \geq 0 \},
\]

and the recession function \( f \) \( 0^+ \) of the convex function \( f \) is the function such that

\[
\text{epi } f0^+ = 0^+\text{epi } f.
\]

Let \( C_1 \) and \( C_2 \) be non-empty sets in \( \mathbb{R}^n \). A hyperplane \( H \) is said to separate \( C_1 \) and \( C_2 \) if \( C_1 \) is contained in one of the closed half-spaces associated with \( H \) and \( C_2 \) lies in the opposite closed half-space. It is said to separate \( C_1 \) and \( C_2 \) properly if \( C_1 \) and \( C_2 \) are not both actually contained in \( H \) itself.

Lemma B.8 (Theorem 11.1 of [3]) Let \( C_1 \) and \( C_2 \) be non-empty sets in \( \mathbb{R}^n \). There exists a hyperplane separating \( C_1 \) and \( C_2 \) properly if and only if there exists a vector \( b \) such that

\[
\inf \{ \langle x, b \rangle; x \in C_1 \} \geq \sup \{ \langle x, b \rangle; x \in C_2 \},
\]

\[
\sup \{ \langle x, b \rangle; x \in C_1 \} > \inf \{ \langle x, b \rangle; x \in C_2 \}.
\]

A face of a convex set \( C \) is a convex subset \( C' \) of \( C \) such that every (closed) line segment in \( C \) with a relative interior point in \( C' \) has both endpoints in \( C' \). A face consists of a single point is called an extreme point. \( x \in C \) is an extreme point if and only if it cannot be expressed as a convex combination of points of \( C \) other than \( x \). The set of all extreme points of \( C \) is expressed as \( \text{ext } C \). If \( C' \) is a half-line face of a convex set \( C \), we shall call the direction of \( C' \) an extreme direction of \( C \) (extreme point of \( C \) at infinity). Obviously, an extreme direction of \( C \) is, viewed as a point in \( \mathbb{R}^n \setminus \{0\} \), is a member of recession cone \( 0^+C \).

Lemma B.9 (Theorem 18.5 of [3]) Let \( C \) be a closed convex set containing no lines. Then, any point \( x \in C \) can be written as

\[
x = \sum \lambda_i y_i + \sum \mu_i z_i,
\]
where \( \lambda_i \geq 0, \sum_i \lambda_i = 1, \mu_i \geq 0, y_i \in \text{ext } C, \) and \( z_i \) is an extreme direction of \( C, \) for each \( i.\)

**Lemma B.10** (Corollary 18.3.1 of [3]) Let \( C \) be a closed convex set. Let \( S_1 \) be a subset of \( C, \) and \( S_2 \) be a set of directions such that

\[
x = \sum_i \lambda_i y_i + \sum_i \mu_i z_i
\]

stands for some \( \lambda_i \geq 0, \sum_i \lambda_i = 1, \mu_i \geq 0, y_i \in S_1, \) and \( z_i \in S_2. \) Then, \( \text{ext } C \) is a subset of \( S_1.\)

From these, the following lemma is immediate.

**Lemma B.11** Let \( C \) be a closed convex set containing no lines. Then,

\[
C = \text{conv } \text{ext } C + 0^+ C.
\]

Also, if a subset \( S \) of \( C \) satisfies

\[
C = \text{conv } S + 0^+ C,
\]

\( S \) contains \( \text{ext } C.\)

If a certain linear function \( h \) achieves maximum over \( C \) at \( x \in C \) and not achieved at any other point \( x' \in C, \) \( x \) is called an exposed point of \( C. \) Any exposed point is an extreme point, but not vice versa.

**Lemma B.12** (Straszewicz’s Theorem, Theorem 18.6 of [3]) For any closed convex set \( C, \) the set of exposed points of \( C \) is a dense subset of the set of extreme points of \( C.\)

## C Determination of certain convex set in \( L_{sa,2} \)

In this section, we determine

\[
\mathcal{M}_0 (M) := \{ L; L \geq (M + \sqrt{-1} B) (M - \sqrt{-1} B) - M^2, B \in L_{sa,2} \}
\]

\[
= \{ L; L \geq \sqrt{-1} [B, M] + B^2, B \in L_{sa,2} \}.
\]

By Lemma B.11, \( \mathcal{M}_0 (M) \) is determined by \( \text{ext } \mathcal{M}_0 (M),\)

\[
\mathcal{M}_0 (M) = \text{conv } \mathcal{M}_0 (M) + P_2. \tag{63}
\]

So first we determine \( \text{ext } \mathcal{M}_0 (M). \) Below, \( \sigma_x, \sigma_y, \sigma_z \) are Pauli matrices.

**Lemma C.1** If \( M = l \sigma_z + m I_2, \)

\[
\text{ext } \mathcal{M}_0 (M)
\]

\[
= \left\{ t^2 U \begin{pmatrix} \sigma_x + \frac{s^2}{4} I_2 \\ s \end{pmatrix} U^{-1}; s \in [-2, 2], [U, L_0^{-1}] = 0, U \in SU(2) \right\} \tag{64}
\]

\[
= \left\{ t^2 \left( s (\cos \alpha \sigma_x + \sin \alpha \sigma_y) + \frac{s^2}{4} I_2 \right); \alpha \in \mathbb{R}, s \in [-2, 2] \right\} \tag{65}
\]

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Proof. Lemma B.12, we have to determine the set of all exposed points of \( \mathcal{M}_0(M) \). Thus, we investigate
\[
\min_{L \in \mathcal{M}_0(M)} \text{tr} Y L.
\]
If \( Y \) is not positive, the target function is unbounded from below, thus the minimum is never attained. Also, if \( Y \) has eigenvalue 0, the minimum is achieved by any member of a certain convex set containing more than single point. This means the corresponding minimum points are not exposed. Therefore, we suppose
\[
Y > 0.
\]
In this case, the above minimum equals the minimum of
\[
f_Y(B) := \min_{A \in \mathcal{L}_{sa}} \text{tr} Y \left( \sqrt{-1} |B, M| + B^2 \right),
\]
Observe \( f_Y(B) \) is a proper, convex, and differentiable function. Hence, at the minimum point \( B \), the derivative \( Df_Y(B) \left( \dot{B} \right) \) must vanish for any \( \dot{B} \).
Thus,
\[
\sqrt{-1} (MY - YM) + BY + YB = 0,
\]
(66)
Observe that if \( Y \) satisfies (66), so does real multiple of \( Y \). Therefore, without loss of generality, we suppose
\[
Y = a\sigma_x + b\sigma_y + c\sigma_z + I_2,
\]
where \( a^2 + b^2 + c^2 < 1 \). Observe also if the pair \((Y, B)\) satisfies (66), so does \((UYU^\dagger, UBU^\dagger)\), where \( U \) is any unitary commutative with \( M \). Therefore, we first solve (66) fixing \( b = 0 \), and then rotate the result by unitaries commutative with \( M \).
Then, if \( b = 0 \), after some calculations, one can easily verify that \( B = la\sigma_y \) satisfies (66). Since \( B \) satisfying (66) is unique for each \( M \) and \( Y > 0 \), this is the only solution to (66). Applying above ”gauge transform”,
\[
B = laU\sigma_y U^\dagger,
\]
where \( U \in SU(2) \) commute with \( M \), are the solutions to (66). Also, \( a \in (0, 1) \), so that \( Y > 0 \). Therefore, the set of all the exposed points of \( \mathcal{M}_0(M) \) is
\[
\left\{ t^2U \left( -2a\sigma_x + a^2I_2 \right) U^\dagger; a \in (-1, 1), U \in SU(2), [U, M] = 0 \right\}.
\]
The closure of this, by Lemma B.12, is ext \( \mathcal{M}_0(M) \). Thus the asserted result is obtained. 

Below we use the following functions to describe the results.
\[
f_1(x) := \begin{cases} |x| - 1, & (|x| \geq 2) \\ \frac{1}{4} x^2, & (|x| \leq 2) \end{cases},
\]
\[
f_2(x, z) := \begin{cases} \sqrt{x^2 + z^2} - 1, & (x^2 + z^2 \geq 4) \\ \frac{1}{4} \{x^2 + z^2\}, & (x^2 + z^2 \leq 4) \end{cases},
\]

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and

\[
D(x, z, w) := 16w^4 + (-8x^2 + 8z^2 + 32)w^3 + (x^4 + 2x^2z^2 - 32x^2 + z^4 - 8z^2 + 16)w^2
\]

\[
+ (10x^4 + 2x^2z^2 - 8x^2 - 8z^4 - 32z^2)w + (x^4 - 3x^4z^2 - x^6 - 3x^2z^4 + 20x^2z^2 - z^6 - 8z^4 - 16z^2).
\]

**Lemma C.2** For each \((x, z) \in \mathbb{R}^2\) with \(z \neq 0\), \(w\) satisfying

\[
D(x, z, w) = 0
\]

and \(w \geq f_2(x, z)\) is unique.

**Proof.** We view (68) as a equation for \(w\) and prove it has a unique solution in the region defined by \(w \geq f_2(x, z)\).

First we study the case of \(x \neq 0\) and \(x^2 + z^2 - 4 > 0\). Let \(D_2\) be the discriminant of (68) viewed as a equation for \(w\), then

\[
D_2 = c \times -x^2z^2 \left(84x^2z^2 + 3x^4z^2 + 3x^2z^4 + 48x^2 - 12x^4 + x^6 + 48z^2 - 12z^4 + z^6 - 64\right)^3
\]

\[
= c \times -x^2z^2 \left\{(x^2 + z^2)^3 + 15(x^2 + z^2)^2 + 48(x^2 + z^2) - 27(x^2 - z^2)^2 - 64\right\}^3
\]

\[
\leq c \times -x^2z^2 \left\{(x^2 + z^2)^3 + 15(x^2 + z^2)^2 + 48(x^2 + z^2) - 27(x^2 + z^2)^2 - 64\right\}^3
\]

\[
= c \times -x^2z^2 \left\{(x^2 + z^2)^3 - 12(x^2 + z^2)^2 + 48(x^2 + z^2) - 64\right\}^3
\]

\[
= c \times -x^2z^2 (x^2 + z^2 - 4)^9 < 0.
\]

where \(c\) is a positive constant. Therefore, (68) has two distinct real roots and two (non-real) complex roots. Also,

\[
D(x, z, \sqrt{x^2 + z^2} - 1) = z^2 \left\{ -2(x^2 + z^2 + 12) \sqrt{x^2 + z^2} + 12x^2 + 16 - 15z^2 \right\}
\]

\[
\leq z^2 \left\{ -2(x^2 + z^2 + 12) \sqrt{x^2 + z^2} + 12(x^2 + z^2) + 16 \right\}
\]

\[
= -2z^2 \left( \sqrt{x^2 + z^2} - 2 \right)^3 < 0.
\]

Since \(\lim_{w \to \infty} D(x, z, w) = \lim_{w \to -\infty} D(x, z, w) = \infty\), by intermediate value theorem, one of two real solutions of (68) is smaller than \(\sqrt{x^2 + z^2} - 1\), and the other is larger. This means there is only one real solution of (68) satisfying \(w \geq f_2(x, z)\).

Second, we consider \(x = 0\)-case, where

\[
D(x, z, w) = (w - z)(w + z)(z^2 + 4w + 4)^2.
\]

Obviously, in this case, only positive solution of (68) is \(w = z\), which satisfies \(w \geq f_2(x, z)\).
Third, we study the case of \( x \neq 0 \) and \( x^2 + z^2 \leq 4 \). Observe that the third derivative is positive of the function \( w \rightarrow D(x, z, w) \) in the region \( \left[ \frac{x^2 + z^2}{4}, \infty \right) \),

\[
\frac{\partial^3}{\partial w^3} D(x, z, w) = 384 \left\{ w - \frac{1}{8} (x^2 - z^2 - 4) \right\}.
\]

and

\[
\frac{\partial^2}{\partial w^2} D\left(x, z, \frac{x^2 + z^2}{4}\right) = 2 \left( (x^2 - 4)^2 + 14x^2z^2 + 13z^4 + 16z^2 \right) > 0.
\]

So the second derivative is positive in the region \( \left[ \frac{x^2 + z^2}{4}, \infty \right) \). Since

\[
D\left(x, z, \frac{x^2 + z^2}{4}\right) = \frac{1}{4} z^2 \left\{ (x^2 + z^2 - 4)^3 - 108z^2 \right\} < 0,
\]

the function \( w \rightarrow D(x, z, w) \) is increasing at the smallest solution \( w_0 \) of (68) in the region \( \left[ \frac{x^2 + z^2}{4}, \infty \right) \). Therefore, there cannot be any larger solution than \( w_0 \).

This proves that (68) has only one solution in the region \( \left[ \frac{x^2 + z^2}{4}, \infty \right) \).

**Lemma C.3** Suppose \( M = l \sigma_z + mI_2 \). Then

\[
l^2 (x \sigma_x + y \sigma_y + z \sigma_z + wI_2) \in \mathcal{M}_0(M)
\]

if and only if

\[
z = 0, \ w \geq f_1(x') \quad (69)
\]

or \( z \neq 0, \ w \geq f_2(x', z), \ D(x', z, w) \geq 0 \quad (70)
\]

where we have defined \( x' := \sqrt{x^2 + y^2} \).

**Proof.** By (65) and (67),

\[
l^2 (x \sigma_x + y \sigma_y + z \sigma_z + wI_2) + M \in \mathcal{M}(M)
\]

\[
\Leftrightarrow \exists s \in [-1, 1], \ \exists \alpha \in \mathbb{R}, \ w \geq \frac{s^2}{4} + \sqrt{(x - s \cos \alpha)^2 + (y - s \sin \alpha)^2 + z^2}, \quad (71)
\]

\[
\Leftrightarrow \exists s \in [-1, 1], \ w \geq w_2(s) := \frac{s^2}{4} + \sqrt{(x' - s)^2 + z^2}, \quad (72)
\]

\[
\Leftrightarrow \exists s \in \mathbb{R}, \ w \geq w_2(s) := \frac{s^2}{4} + \sqrt{(x' - s)^2 + z^2}, \quad (73)
\]

where the second "\( \Leftrightarrow \)" is due to the fact that

\[
s \sigma_x + \frac{s^2}{4} I_2 \geq 2 \sigma_x + \frac{2}{4} I_2, \text{ if } s \geq 2,
\]

\[
s \sigma_x + \frac{s^2}{4} I_2 \geq -2 \sigma_x + \frac{(-2)^2}{4} I_2, \text{ if } s \leq -2.
\]
For each given \((x', z)\), \(w_2(s)\) goes to \(+\infty\) as \(s \to \infty\). So we are interested in the minimum \(w_2\) of \(w_2(s)\) over \(\mathbb{R}\). First, suppose \(z = 0\). Then,
\[
\begin{align*}
\overline{w_2} & = \min_{s \in \mathbb{R}} w_2(s) = \min_{s \in \mathbb{R}} \left( \frac{s^2}{4} + |x' - s| \right) \\
& = f_1(x'),
\end{align*}
\]
verifying (69).

Next, suppose \(z \neq 0\). Since \(w_2(s)\) is differentiable, bounded below, and defined on the open set \(\mathbb{R}\), its minimum \(w_2\) should satisfy \(dw_2/ds = 0\). The definition of \(w_2\) is equivalent to
\[
\begin{align*}
f_M(s) := & \left( w_2 - \frac{s^2}{4} \right)^2 - (x' - s)^2 - z^2 \\
& = \frac{1}{16} s^4 - \left( \frac{1}{2} w_2 + 1 \right) s^2 + 2x's + \left( w_2^2 - x'^2 - z^2 \right) = 0, \tag{75}
\end{align*}
\]
and
\[
s^2 < w_2. \tag{76}
\]
Note here \(w_2 = \frac{z^2}{4}\) cannot happen because of (73) and \(z^2 > 0\). Differentiating both ends of (75) by \(s\),
\[
f_M'(s) + 2 \frac{dw_2}{ds} \left( w_2 - \frac{s^2}{4} \right) = 0,
\]
where
\[
f_M'(s) = \frac{3}{4} w_2 - (w_2 + 2)s + 2x'.
\]
is the derivative of \(f_M(s)\) by \(s\) considering \(w_2\) as a constant. Thus, because of the restriction (76), \(dw_2/ds = 0\) is equivalent to
\[
f_M'(s) = 0. \tag{77}
\]
So if \(w_2 = \overline{w_2}\) (75) and (77), viewed as algebraic equations for \(s\), has a real common root \(s\) satisfying (76). Therefore, the discriminant of \(f_M(s)\) has to be zero. After some computation, the discriminant coincide with \(D(x', z, \overline{w_2})\) up to constant factor. Therefore, we should have
\[
D(x', z, \overline{w_2}) = 0.
\]
\((x', z, \overline{w_2})\) should also have to satisfy :
\[
\begin{align*}
\overline{w_2} & \geq \min_{(s, t) \in \mathbb{R}^2} \frac{s^2 + t^2}{4} + \sqrt{(x' - s)^2 + (z - t)^2} \\
& = \min_{(s, t) \in \mathbb{R}^2} \frac{s^2 + t^2}{4} + \sqrt{(\sqrt{x'^2 + z^2} - s)^2 + t^2} \\
& = \min_{s \in \mathbb{R}} \frac{s^2}{4} + \sqrt{x'^2 + z^2 - s} \\
& = f_2(x', z).
\end{align*}
\]
Therefore, $w_2$ satisfies

$$D (x', z, w_2) = 0, \quad w_2 \geq f_2 (x', z).$$

By Lemma C.2, the above condition specifies $w_2$ uniquely.

Since $\lim_{w \to \infty} D (x', z, w) = \infty$, $w \geq w_2$ is equivalent to $D (x', z, w) \geq 0$ and $w \geq f_2 (x', z)$. Thus we have (70). □

**Lemma C.4** If the dimension of the Hilbert space is 2,

$$\mathcal{M}_0 (M) = \{ \sqrt{-1} [B, M] + B^2; B \in \mathcal{L}_{sa,2} \}.$$

**Proof.** Without loss of generality, let $M = l\sigma + mI_2$. Let us parameterize $B \in \mathcal{L}_{sa,2}$ as follows,

$$B = lU (\beta\sigma + \gamma\sigma + \delta I) U^\dagger,$$

where $U$ is a unitary commuting with $M$. Then,

$$\frac{1}{l^2} U^\dagger \left\{ \sqrt{-1} [B, L_0^{-1}] + (B)^2 \right\} U$$

$$= x\sigma_x + y\sigma_y + z\sigma_z + wI,$$

where

$$x = -2\beta, y = 2\beta\delta,$$
$$z = 2\gamma\delta, w = \beta^2 + \gamma^2 + \delta^2.$$

Therefore, erasing $\beta, \gamma$ and replacing $t := \delta^2$,

$$w (t) := w = \frac{x'^2}{4 (1 + t)} + \frac{z^2}{4t} + t, \quad (78)$$

where $x' = \sqrt{x^2 + y^2}$. Observe $t = \delta^2$ can take any non-negative value. Observe also $\lim_{t \to \infty} w (t) = \infty$, for any $x', z$. Hence, $w$ can take any value larger than or equal to the minimum $w$ of $w (t)$ over $t \in [0, \infty)$. Below, we determine relation satisfied by $w$, $x'$, and $z$, and shows that $w \geq w$ is equivalent to (69) and (70). Then, since $x' = \sqrt{x^2 + y^2}$ is invariant by the conjugation of unitary $U$ commuting with $M$, Lemma C.3 implies our assertion.

If $z = 0$, (78) is very simple and easy to minimize.

$$w = \min_{t \in [0, \infty)} w (t) = \min_{t \in [0, \infty)} \frac{x'^2}{4 (1 + t)} + t = f_1 (x'). \quad (79)$$

Hence, in this case, $w \geq w$ is equivalent to (69).

Next, suppose $z \neq 0$. Then $t$ cannot be 0, $t \in (0, \infty)$. Since $w (t)$ is differentiable, defined on the open interval, and bounded below, it has minimum,
and at the minimum, the derivative of \(w(t)\) should vanish. Rearranging the terms of (78), \((x', z, w)\) satisfies
\[
f_N(t) := 4t^3 + 4(1 - w)t^2 + (x'^2 + z^2 - 4w)t + z^2 = 0. \tag{80}
\]
Differentiating both sides by \(t\),
\[
f'_N(t) - 4(t^2 + t) \frac{dw}{dt} = 0,
\]
where
\[
f'_N(t) = 12t^2 + 8(1 - w)t + x'^2 + z^2 - 4w = 0
\]
is the derivative \(f_N(t)\) by \(t\) viewing \(w\) as a constant. Since \(t > 0\), \(dw/dt = 0\) is equivalent to
\[
f'_N(t) = 0. \tag{81}
\]
(80) has a multiple root if and only if its discriminant is zero. After some tedious calculations (in fact done by computer algebra system), this is equivalent to
\[
D(x', z, w) = 0, \tag{82}
\]
In addition to this, \((x', z, w)\) has to obey other constrains. Since
\[
w(t) \geq \frac{x'^2 + z^2}{4(1 + t)} + t,
\]
we should have
\[
w \geq f_2(x', z). \tag{83}
\]
By Lemma C.2 (82) and (83) uniquely determines \(w\). Since \(\lim_{w \to \infty} D(x', z, w) = \infty\), Therefore, \(w \geq \underline{w}\) is equivalent to (70).
After all, \(w \geq \underline{w}\) is equivalent to (69) and (70), and we have the assertion.