2A-MAJORANA REPRESENTATIONS OF $A_{12}$

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Abstract. Majorana representations have been introduced by Ivanov in [15] in order to provide an axiomatic framework for studying the actions on the Griess algebra of the Monster and of its subgroups generated by Fischer involutions. A crucial step in this programme is to obtain an explicit description of the Majorana representations of $A_{12}$ (by [7], the largest alternating group admitting a Majorana representation) for this might eventually lead to a new and independent construction of the Monster group (see [17, Section 4, pag.115]).

In this paper we prove that $A_{12}$ has a unique Majorana representation on the set of its involutions of type $2^2$ and $2^6$ (that is the involutions that fall into the class of Fischer involutions when $A_{12}$ is embedded in the Monster) and we determine the degree and the decomposition into irreducibles of such representation. As a consequence we get that Majorana algebras affording a $2A$-representation of $A_{12}$ and of the Harada-Norton sporadic simple group satisfy the Straight Flush Conjecture (see [16] and [17]). As a by-product we also determine the degree and the decomposition into irreducibles of the Majorana representation induced on the $A_8$ subgroup of $A_{12}$. We finally state a conjecture about Majorana representations of the alternating groups $A_n$, $8 \leq n \leq 12$.

1. Introduction

Let $k$ be a field and $R$ a commutative associative ring containing $k$ as a subring. Given a finite subset $S$ of $R$, containing 1 and 0, a fusion law on $S$ is a map

$$\star: S \times S \to 2^S.$$ 

The pythagorean table of $\star$ is the matrix whose rows and columns are indexed by the elements of $S$ and, for $a$ and $b$ in $S$, the entry of row $a$ and column $b$ is $a \star b$.

Let $S$ and $\star$ be as above and let $V$ be a commutative non-associative $R$-algebra. For an element $a \in V$ define the adjoint action of $a$ on $V$ to be the map

$$\text{ad}(a): V \to V, \quad v \mapsto av$$

An idempotent element $a \in V$ is called a $\star$-axis (or simply axis) if

Ax1 $\text{ad}(a)$ is a semisimple endomorphism of $V$ with spectrum contained in $S$.

Ax2 for every $\lambda, \mu \in S$, $V_\lambda V_\mu \leq \bigoplus_{\delta \in \lambda \star \mu} V_\delta$, where, for every $\theta \in S$, $V_\theta$ is the eigenspace $\{v \in V | av = \theta v\}$ (we allow the possibility that $V_\theta = \{0\}$).

For example, if $\star$ is the fusion law with pythagorean table as in [11] and if $u$ and $v$ are, e.g., $1/32$-eigenvectors for the adjoint action of the $\star$-axis $a$, their product lies in the sum of the 1, 0, and $1/4$-eigenspaces for $\text{ad}(a)$. If the algebra $V$ is generated by $\star$-axes, we say that $V$ is an axial algebra over $R$ with spectrum $S$ and fusion law $\star$. An axial algebra $V$ is called primitive if
Ax3 for every axis \( a \), the 1-eigenspace \( V_1 \) of \( ad(a) \) has dimension 1 (or, equivalently, \( V_1 \) is the linear span of \( a \))

and **dihedral** if

Ax4 \( V \) is generated by two axes.

A **Frobenius axial algebra** is a pair \((V, \sigma)\) where \( V \) is an axial algebra and

\[
\sigma : V \times V \to k
\]

is a bilinear form on \( V \) such that

Ax5 \( \sigma \) **associates** with the algebra product, i.e.: \( \sigma(uv, w) = \sigma(u, vw) \), for every \( u, v, w \in V \)

and

Ax6 for each axis \( a \), \( \sigma(a, a) \neq 0 \).

For an element \( v \in V \), define, as usual, the (squared) **length** of \( v \) to be the value \( \sigma(v, v) \).

**Majorana algebras** are primitive real Frobenius axial algebras \((V, \sigma)\) such that \( \sigma \) is positive definite, axes have length 1, with spectrum \( \{1, 0, 1/4, 1/32\} \), and **Ising (or Monster) fusion law**:}

\[
\begin{array}{cccc}
+ & 1 & 0 & 1/4 & 1/32 \\
1 & \{1\} & \emptyset & \{1/4\} & \{1/32\} \\
0 & \emptyset & \{0\} & \{1/4\} & \{1/32\} \\
1/4 & \{1/4\} & \{1/4\} & \{1, 0\} & \{1/32\} \\
1/32 & \{1/32\} & \{1/32\} & \{1/32\} & \{1, 0, 1/4\} \\
\end{array}
\]

Moreover it is also required that \( \sigma \) satisfies the **Norton inequality**\(^1\) for every \( u, v \in V \),

\[
\sigma(uu, vv) \geq \sigma(uv, uv).
\]

As usual, when the form \( \sigma \) needs not to be specified, we write simply \( V \) for \((V, \sigma)\).

An **automorphism** of a Majorana algebra \( V \) is an isometry of \( V \) that preserves the algebra product, in particular it sends axes to axes. The set of automorphisms of a Majorana algebra \( V \) is a group and will be denoted by \( \text{aut}(V) \). The Ising fusion law implies that, for a Majorana algebra \( V \), the setting

\[
V_+ := V_1 \oplus V_0 \oplus V_{1/4} \text{ and } V_- := V_{1/32}
\]

is a \( \mathbb{Z}_2 \)-grading on \( V \). Since, by [Ax5], the scalar product is associative, it follows that, for each axis \( a \), the map \( \tau_a \) that inverts every element of its 1/32-eigenspace and fixes each element of the other eigenspaces is an involutory automorphism of \( V \). The automorphism \( \tau_a \) is called the **Miyamoto involution** associated to the axis \( a \) (the name comes after Masahiko Miyamoto who introduced these involutions in \[24\] in the context of vertex operator algebras). By a result of John Conway (see \[4\]), the (196884-dimensional) Griess algebra constructed by Robert Griess in \[12\] is a Majorana algebra (see also \[15\], Section 8.5). In particular, identifying the Monster with the automorphism group of the Griess algebra, the Miyamoto involutions of the Griess algebra are precisely the **Fischer involutions** of the Monster, i.e. the involutions in the Monster whose centraliser is the double cover of the Baby Monster.

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\(^1\)Norton inequality is satisfied by the Griess algebra and, because of that, has become part of the axiomatics of Majorana algebra. We observe, however, that within this theory it has never been used so far.
These involutions form a unique conjugacy class in the Monster that is labelled 2A in the ATLAS [5] and their name comes after Bernd Fischer, who, together with Griess, first foreshadowed the existence of the Monster. Dihedral Majorana algebras are called Norton-Sakuma algebras. The complete classification of Norton-Sakuma algebras goes under the name of Norton-Sakuma Theorem and it has been achieved by Simon P. Norton [25] for the subalgebras of the Griess algebra, by Shinya Sakuma [27] in the context of certain vertex operator algebras, and, within the Majorana algebra axiomatics, by Ivanov, Dima Pasechnik, Ákos Seress, and Sergey Shpectorov [18].

**Norton-Sakuma Theorem.** There are nine isomorphism classes of Norton-Sakuma algebras. The representatives of these classes, are described in the rows of Table 1.

In Table 1 the basis given for each representative is called the Norton basis, ρ denotes the product of the two Miyamoto involutions associated to the generating axes a₀ and a₁ and, for ε ∈ {0, 1} and i ∈ Z, aᵣ+2i is the image of aᵣ under ρ. Note that, since ρ is an algebra automorphism, aᵣ+2i is still an axis. The type of a Norton-Sakuma algebra is the name given in the ATLAS [5] for the conjugacy class in the Monster of the product of the Miyamoto involutions associated to the generating axes (when this algebra is identified with a subalgebra of the Griess algebra). The vectors aᵣ, aᵣ², aᵣ³ appearing in the algebras of type 2A, 4B, and 6A are axes whose associated Miyamoto involutions are ρ, ρ², and ρ³, respectively. Further, the vectors uᵣ, vᵣ, wᵣ, and uᵣ², appearing in algebras of type NA, for N ∈ {3, 4, 5, 6}, are called odd axes or, more specifically, N-axes, for N < 6, and 3-axis for N = 6. Odd axes do not depend on the generating pair (a₀, a₁) but (up to a sign in the case 5A) only on the cyclic subgroup (ρ) of index 2 in the dihedral subgroup of the Monster generated by the associated Miyamoto involutions. 3-, 4-, and 5-axes will also be called odd axes.

The properties of the action of Monster on the Griess algebra, together with the correspondence between axes and Miyamoto involutions are axiomatised in the definition of Majorana representation: given

- a group G,
- a G-invariant set of involutions T generating G (the Majorana set)
- a Majorana algebra V,
- an injective map ψ from T to the set of axes of V,

a Majorana representation of G on V with respect to ψ, is a group homomorphism

φ: G → aut(V)

such that

M1 Tψ generates V as an algebra,
M2 for every t ∈ T and every g ∈ G, (tψ)ψ = (tψ)gψ,
M3 for every t ∈ T, tψ is the Miyamoto involution associated to the axis tψ,
M4 for t₁ and t₂ in T, the Norton-Sakuma algebra generated by t₁ψ and t₂ψ has type 2A if and only if t₁t₂ ∈ T and in this case (t₁t₂)ψ = t₁ψ + t₂ψ - 8t₁ψ t₂ψ (cfr. Table 1 with ρ = (t₁t₂)ψ),
M5 if t₁, t₂, t₃, and t₄ are elements of T such that t₁t₂ = t₃t₄ and the subalgebras generated by t₁ψ, t₂ψ and t₃ψ, t₄ψ both have type 3A, 4A, or 5A, then u₄(t₁t₂)ψ = u₄(t₃t₄)ψ, v₄(t₁t₂)ψ = v₄(t₃t₄)ψ, or w₄(t₁t₂)ψ = w₄(t₃t₄)ψ, respectively.
| Type | Basis | Structure constants | Scalar products |
|------|-------|---------------------|-----------------|
| 1A   | $a_0$ | $a_0 \cdot a_0 = a_0$ | $(a_0, a_0) = 1$ |
| 2A   | $a_0$, $a_1$, $a_\rho$ | $a_0 \cdot a_1 = \frac{1}{2}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{2}(a_0 + a_\rho - a_1)$, $a_\rho \cdot a_\rho = a_\rho$ | $(a_0, a_1) = \frac{1}{2}$, $(a_0, a_\rho) = \frac{1}{2}$, $(a_1, a_\rho) = \frac{1}{2}$ |
| 2B   | $a_0$, $a_1$ | $a_0 \cdot a_1 = 0$ | $(a_0, a_1) = 0$ |
| 3A   | $a_{-1}$, $a_0$, $a_1$, $a_\rho$ | $a_0 \cdot a_1 = \frac{1}{2}(2a_0 + 2a_1 - a_{-1}) - \frac{3}{2} \frac{5}{2} u_\rho$, $a_0 \cdot u_\rho = \frac{1}{2}(2a_0 - a_1 - a_{-1}) + \frac{1}{2} \frac{5}{2} u_\rho$, $a_\rho \cdot u_\rho = u_\rho$ | $(a_0, a_1) = \frac{13}{2}$, $(a_0, u_\rho) = \frac{1}{2}$, $(u_\rho, u_\rho) = \frac{3}{2}$ |
| 3C   | $a_{-1}$, $a_0$, $a_1$ | $a_0 \cdot a_1 = \frac{1}{2}(a_0 + a_1 - a_{-1})$ | $(a_0, a_1) = \frac{1}{2}$ |
| 4A   | $a_{-1}$, $a_0$, $a_1$, $a_2$, $a_\rho$ | $a_0 \cdot a_1 = \frac{1}{2} (3a_1 + 3a_2 + a_{-1} + a_2 - 3\rho_\rho)$, $a_0 \cdot a_2 = 0$, $a_0 \cdot v_\rho = \frac{1}{2} (5a_0 - 2a_1 - 2a_{-1} - a_2 + 3\rho_\rho)$, $v_\rho \cdot v_\rho = v_\rho$ | $(a_0, a_1) = \frac{1}{2}$, $(a_0, a_2) = 0$, $(a_0, v_\rho) = \frac{3}{2}$, $(v_\rho, v_\rho) = 2$ |
| 4B   | $a_{-1}$, $a_0$, $a_1$, $a_2$, $a_{\rho,2}$ | $a_0 \cdot a_1 = \frac{1}{2} (a_0 + a_1 - a_{-1} - a_2 - a_{\rho,2})$, $a_0 \cdot a_2 = \frac{1}{2} (a_0 + a_2 - a_{\rho,2})$ | $(a_0, a_1) = \frac{1}{2}$, $(a_0, a_2) = \frac{1}{2}$, $(a_0, a_{\rho,2}) = \frac{1}{2}$ |
| 5A   | $a_{-2}$, $a_{-1}$, $a_0$, $a_1$, $a_2$, $a_\rho$ | $a_0 \cdot a_1 = \frac{1}{2} (3a_0 + 3a_1 - a_{-1} - a_2 - a_{-2}) + w_\rho$, $a_0 \cdot a_2 = \frac{1}{2} (3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$, $a_0 \cdot w_\rho = \frac{7}{2} (a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2} \frac{5}{2} w_\rho$, $w_\rho \cdot w_\rho = \frac{5}{2} w_\rho$ | $(a_0, a_1) = \frac{1}{2}$, $(a_0, a_2) = 0$, $(w_\rho, w_\rho) = \frac{5}{2} \frac{5}{2}$ |
| 6A   | $a_{-2}$, $a_{-1}$, $a_0$, $a_1$, $a_2$, $a_{\rho,2}$ | $a_0 \cdot a_1 = \frac{1}{2} (a_0 + a_1 - a_{-1} - a_2 - a_{-2} - a_3 + a_{\rho,3}) + \frac{5}{2} \frac{5}{2} u_\rho^2$, $a_0 \cdot a_2 = \frac{1}{2} (2a_0 + 2a_2 + a_{-2}) - \frac{5}{2} \frac{5}{2} u_\rho^2$, $a_0 \cdot a_3 = \frac{1}{2} (a_0 + a_3 - a_{\rho,3})$, $a_0 \cdot u_\rho^2 = \frac{5}{2} (2a_0 - a_2 - a_{-2}) + \frac{5}{2} \frac{5}{2} u_\rho^2$, $a_{\rho,3} \cdot u_\rho^2 = 0$ | $(a_0, a_1) = \frac{5}{2}$, $(a_0, a_2) = \frac{5}{2}$, $(a_0, a_3) = \frac{5}{2}$, $(a_{\rho,3}, u_\rho^2) = 0$ |

Table 1. The nine types of Norton-Sakuma algebras
Given a Majorana representation \( \phi \) as above, we shall also refer to it as the quintuple \((G, T, V, \phi, \psi)\).

The shape of a Majorana representation \( \phi \) is the map \( sh \) that assigns to each orbital of \( G \) on \( T \) the isomorphism class of the dihedral algebra generated by the two axes associated to one (every) pair of involutions in that orbital.

M6 the map \( sh \) respects the following embeddings of dihedral algebras:

\[
2A \rightarrow 4B, \ 2A \leftrightarrow 6A, \ 2B \leftrightarrow 4A, \ 3A \leftrightarrow 6A
\]

in the sense that, for \( t, r_1, r_2 \in T \), if \( \langle \langle t^\psi \rangle \rangle < \langle \langle t^\psi, r_1^\psi \rangle \rangle < \langle \langle t^\psi, r_2^\psi \rangle \rangle \), then

\[
(\text{sh}((t, r_1)^G), \ \text{sh}((t, r_2)^G)) \in \{(2A, 4B), (2A, 6A), (2B, 4A), (3A, 6A)\}.
\]

Axioms M4, M5, M6, and Norton-Sakuma Theorem imply that, if \( t_1^\psi \) and \( t_2^\psi \) generate a dihedral subalgebra of \( V \) of type 2A, 4B, or 6A, then \( t_1 t_2, (t_1 t_2)^2 \), or \( (t_1 t_2)^3 \) belongs to \( T \), and \( (t_1 t_2)^\psi, ((t_1 t_2)^2)^\psi \), or \( ((t_1 t_2)^3)^\psi \) coincide with \( a_{(t_1 t_2)} \), \( a_{(t_1 t_2)^2} \), or \( a_{(t_1 t_2)^3} \), respectively. The next result follows immediately from the definition of Majorana representation.

**Lemma 1.1.** If \( T_0 \) is a nonempty subset of \( T \) such that \( T_0 \) is \( \langle T_0 \rangle \)-invariant, then \( \phi|_{\langle T_0 \rangle} \) is again a Majorana representation of \( \langle T_0 \rangle \) on the subalgebra of \( V \) generated by \( T_0^\psi \) with respect to \( \psi|_{\langle T_0 \rangle} \).

The standard example of a Majorana representation is given by the usual action of the Monster on the Griess algebra, where the Majorana set is the conjugacy class of Fischer involutions (see [13] Proposition 8.6.2). More generally, let \( H \) be any finite group, \( T_H \) an \( H \)-invariant generating set of involutions of \( H \). A Majorana representation \( (H, T_H, V, \phi, \psi) \) is called 2A-Majorana representation if there is an embedding \( \epsilon \) of \( H \) in the Monster such that

\[
T_H^\epsilon = H^\epsilon \cap T_{2A}.
\]

In particular, if \( \phi \) can also be factorised as the composition of \( \epsilon \) with the Majorana representation induced on \( H^\epsilon \) by the usual action of the Monster on the Griess algebra, we say that \( \phi \) is based on the embedding \( \epsilon \) of \( H \) in the Monster.

Note that, by the Norton-Sakuma Theorem, dihedral groups \( D_n \) admit Majorana representations if and only if \( n \leq 6 \), there are 9 such representations and they all arise from an embedding in the Monster (in particular they are 2A-Majorana representations). Conversely, the ideal goal would be to prove that a 2A-Majorana representation is almost always based on an embedding in the Monster, and find the exceptions.

Majorana representations of various groups have been studied and Markus Pfeiffer and Madeleine Whybrow have produced the GAP package MajoranaAlgebras [26] for this purpose. For small groups this programme can compute the structure constants and the inner products of a Majorana representation of a group \( G \) once the Majorana set and the shape are given. On the other hand the computational complexity increases rapidly with order of the group involved and still Majorana representations of many groups are not completely known. In particular, the Majorana representations of the alternating groups \( A_n \) have been determined only for \( n \in \{5, 6, 7\} \) ([21], [19], [20]). From [7] one can deduce that \( A_{12} \) is the largest alternating group admitting a Majorana representation, but there is only partial information about the representations of \( A_n \), for \( n \in \{8, 9, 10, 11, 12\} \) (see [2], [7], [23]).
and [19]. In the spirit of [30], we point out that, in this paper, the package MajoranaAlgebras is needed only for for the proofs of Lemma 5.3 and Theorem 5, and the GAP software [11] only for long, but routine, computations of elementary linear algebra with tabloids.

The main result of the present paper is the following.

**Theorem 1.** There is, up to equivalence, a unique $2A$-Majorana representation $(A_{12}, T, V, \phi, \psi)$ of $A_{12}$. In particular

1. $\phi$ is based on the embedding of $A_{12}$ in the Monster as the centraliser of a $(2A, 3A, 5A)$-subgroup isomorphic to $A_5$.
2. $V$ has dimension 3,960 and its decomposition into irreducible submodules is given in the last column of Table 2.
3. $V$ is 2-closed and it is linearly spanned by the Majorana axes and the 3A-axes associated to permutations of cycle type 3.$^2$.

Recall that, given a partition $\lambda$ of $n$, there is a unique irreducible $\mathbb{R}[S_n]$ module $S^\lambda$ associated to $\lambda$ (the Specht module, see e.g. citeJ). When restricted to the action of $A_n$, $S^\lambda$ remains irreducible if $\lambda$ is not self conjugate, otherwise $S^\lambda$ splits into the direct sum of two non isomorphic irreducibles $\mathbb{R}[A_n]$-modules of the same dimension. In Table 2 we write $1 + 1$ when this splitting occurs, meaning that both modules appear.

As consequences of Theorem 1 we get the following two results.

**Theorem 2.** The Harada-Norton group has a unique (up to equivalence) Majorana representation, which is the one based on its embedding in the Monster.

Recall the following conjecture made by Ivanov in [17, 16].

**Straight Flush Conjecture.** Suppose $A$ is an indecomposable Majorana algebra in which, for every $i \in \{2, 3, 4, 5, 6\}$, there exists a pair of Miyamoto involutions $t_i$...
and $t_2$, such that the order of the product $t_1 t_2$ is $i$. Then $A$ embeds into the Monster algebra.

Since, for every $2A$-Majorana representation $(A_{12}, T, V, \phi, \psi)$ of $A_{12}$, the set $T$ contains involutions of cycle type $2^6$, it follows that $(A_{12}, T, V, \phi, \psi)$ is a $2A$-Majorana representation if and only if $V$ contains a subalgebra $\langle a_s, a_t \rangle$ of type $4A$ for some $s, t \in T$ (see Lemma 3.3).

**Theorem 3.** The Majorana algebras affording a $2A$-Majorana representation of either $A_{12}$ or the Harada-Norton sporadic group satisfy the Straight Flush Conjecture.

Further, for $n \in \{8, 9, 10, 12\}$, we determine the $2$-closure of the subalgebra of the Majorana algebra $V$ affording a $2A$-Majorana representation of $A_{12}$ generated by the Majorana axes corresponding to involutions of cycle type $2^2$ of $A_n$ (in its natural embedding in $A_{12}$ as the stabiliser of $12 - n$ points). We denote by $X_b$ the set of bitranspositions of $A_n$.

**Theorem 4.** For $n \in \{8, \ldots, 12\}$ let $(A_n, X_b, W, \phi_b, \psi_b)$ be the Majorana representation of $A_n$ induced by the restriction to $A_n = \langle X_b \rangle$ of the $2A$-Majorana representation of $A_{12}$ and let $W^\circ$ be the $2$-closure of $W$. Then, the irreducible $\mathbb{R}[A_n]$-submodules of $W^\circ$ and their multiplicities are the ones given in Table 2.

The results of this paper enable us to determine, for $n = 8$, the subalgebra of $V$ that is generated by Majorana axes corresponding to the bitranspositions of $A_8$ and thus enlight the general setting of the Majorana representations of the groups $A_n$ for $n \in \{8, \ldots, 12\}$.

**Theorem 5.** Let $(A_8, X_b, W, \phi_b, \psi_b)$ be the Majorana representation of $A_8$ induced by the restriction of a $2A$-Majorana representation of $A_{12}$ to the stabiliser of 4 points. Then $W$ has dimension 476 and it is generated by $W^\circ$ together with the set

$$\{v_\rho \mid \rho \text{ is a permutation of cycle type } 4^2 \text{ in } A_8\}.$$  

In particular, $W$ is not $2$-closed.

We prove in Section 3 that, for each $n \in \{8, \ldots, 11\}$, the Majorana set and the shape of a Majorana representation of the group $A_n$ are unique. This fact, together with Theorems 4 and 5, leads us to formulate the following conjecture.

**Conjecture 1.** Let $n \in \{8, \ldots, 12\}$, let $(A_n, X_b, V, \phi, \psi)$ be a Majorana representation of $A_n$ and let $V^\circ$ its $2$-closure. Then, for every element $\tau \in A_n$ of cycle type $2^4$ there exists a vector $\delta_\tau \in V$, depending only on $\tau$, such that

$$V = V^\circ + \langle \delta_\tau \mid \tau \text{ is a permutation of type } 2^4 \text{ in } A_n \rangle \cong V^\circ \oplus S(4^2, n-8).$$

2. Strategy

Let $n \in \mathbb{N}$ with $8 \leq n \leq 12$, let $G := A_n$, $\hat{G} := S_n$, and let $(G, T, V, \phi, \psi)$ be a Majorana representation of $G$. Our goal is to determine the structure of the algebra $V$ which, by definition, is generated (as an algebra) by the images via $\psi$ of the elements in $T$. The first step is to determine the shape of $\phi$, which will be accomplished in Section 3. Since the Norton-Sakuma algebras are not, in general, linearly spanned by axes (see Table 1), one cannot expect $V$ to be just the linear
span of the set $T^\psi$. The next step is, therefore, to get control of the linear span of the products of two Majorana axes. For a subspace $U$ of $V$, we therefore define

$$U^\circ := \langle u \cdot v | u, v \in U \rangle.$$ 

We shall call $U^\circ$ the 2-closure of $U$ and, when $U^\circ = U$, we say that $U$ is 2-closed.

By the Norton-Sakuma Theorem, there is a subset $O_{ax}$ of the set of the odd axes appearing in the Norton-Sakuma subalgebras of $V$ generated by two elements of $T^\psi$, such that

$$V^\circ = \langle T^\psi \cup O_{ax} \rangle.$$ 

Let
- $\mathcal{X}_b$ be the set of permutations of cycle type $2^2$,
- $\mathcal{X}_s$ be the set of permutations of cycle type $2^6$ (which is nonempty only if $n = 12$),
- $\mathcal{X}_t$ be the set of subgroups generated by permutations of cycle type $3$, and
- $\mathcal{X} = \mathcal{X}_b \cup \mathcal{X}_s \cup \mathcal{X}_t \cup \mathcal{X}_t$.

In Section 3 we’ll prove that $\mathcal{X}_b \subseteq \mathcal{T} \subseteq \mathcal{X}_b \cup \mathcal{X}_s$ and, in Section 5 we’ll prove that we can choose the set $O_{ax}$ to be the set of 3-axes associated to the (generators of the) elements of $\mathcal{X}_t \cup \mathcal{X}_t$. In this case we have an obvious bijection

$$ax : \mathcal{X} \to T^\psi \cup O_{ax},$$ 

sending the elements of $\mathcal{X}$ to their associated axes. With respect to the action of $G$ on $\mathcal{X}$ by conjugation (with orbits $\mathcal{X}_b, \mathcal{X}_s, \mathcal{X}_t$, and $\mathcal{X}_t$) $ax$ is an isomorphism of $G$-sets and induces a surjective homomorphism of $\mathbb{R}[G]$-modules

$$\pi : M_X \to V^\circ,$$ 

where $M_X$ is the real permutation module of $G$ on $\mathcal{X}$. For $x \in \{b, s, r, t\}$, denote by $M_x$ the permutation $\mathbb{R}[G]$-module on the set $\mathcal{X}_x$ on which $G$ acts by conjugation. Since

$$M_X = M_b \oplus M_s \oplus M_t \oplus M_t,$$

the structure of $M_X$ follows from the results in [7], [9], [9], and [31] and we are left to compute the kernel of $\pi$. The Majorana scalar product $(,)_V$ on $V$ induces, in an obvious way, a symmetric bilinear form on the permutation module

$$f : M_X \times M_X \to \mathbb{R}$$

$$\langle u, v \rangle \mapsto \langle u^\pi, v^\pi \rangle_V$$

and an elementary and well known argument shows that the kernel of $\pi$ coincides with the radical of the form $f$. Note that, since $f$ is positive semidefinite,

$$\text{rad}(M_X) \cap M = \text{rad}(M)$$

(2)

for every submodule $M$ of $M_X$. So, in order to determine which irreducible submodules of $M_X$ are contained in $\text{rad}(M_X)$, we need to compute the restrictions of $f$ to these submodules. Due to the dimensions involved, a direct computation of these restrictions is out of question, even for machine computing, so we need to use the machinery developed in [9] (see also [10, Section 6]). Note that since the set $\mathcal{X}_x$ is invariant under the action of $\hat{G}$ by conjugation, the (permutation) $\mathbb{R}[\hat{G}]$-module $M_x$ lifts to a permutation $\mathbb{R}[\hat{G}]$-module. Since, as we shall see in Section 4, the form
$f$ is invariant under the action of $\hat{G}$, we may consider $M_x$ to be an $\mathbb{R}[\hat{G}]$-module and use the representation theory of the symmetric groups. We next focus our investigation to the module $M_x$. Let

$$M_x = \bigoplus_{i=1}^{l} \left( \bigoplus_{h=1}^{n_i} M_{x,h}^{\lambda_i} \right),$$

be a decomposition of $M_x$ into irreducible submodules, where, for $i \in \{1, \ldots, l\}$, $\lambda_i$ is a partition of $n$ and $M_{x,h}^{\lambda_i}$ is an $\mathbb{R}[\hat{G}]$-submodule isomorphic to the Specht module $S^{\lambda_i}$ (see [22]), with $\lambda_i \neq \lambda_j$ for $i \neq j$, and

$$\bigoplus_{h=1}^{n_i} M_{x,h}^{\lambda_i}$$

is the homogeneous component relative to the Specht module $S^{\lambda_i}$. Let

$$\pi_{x,h}^{\lambda_i} : M_x \to M_{x,h}^{\lambda_i}$$

be the projection of $M_x$ onto the submodule $M_{x,h}^{\lambda_i}$ with respect to the decomposition (3). Let

$$\Gamma_1, \ldots, \Gamma_{r_x}$$

be the orbitals of $\hat{G}$ on $\mathcal{X}_x$ (i.e. the orbits of $\hat{G}$ on the set $\mathcal{X}_x \times \mathcal{X}_x$ with respect to the action defined, for every $g \in \hat{G}$ and $u, v \in \mathcal{X}_x$, by $(u, v)^g = (u^g, v^g)$) and, for $j \in \{1, \ldots, r_x\}$, let

$$\Delta_j(u) := \{ v \in \mathcal{X}_x \mid (u, v) \in \Gamma_j \}.$$

Let $P(x)$ be a generalised first eigenmatrix (see [7, §2]) associated to $M_x$ with respect to the decomposition (3) and denote by

$$(P(x))_{jl}^{\lambda_i}$$

the $(i,j)$-entry of $P(x)$. Note that $P(x)^{\lambda_i}$ is a matrix of size $n_i \times n_i$ whose $(k,l)$-entry will be denoted by

$$(P(x))_{jl}^{\lambda_i}.$$

**Lemma 2.1.** [7 Equation (13) and Lemma 1(i)] With the above notation, we have

$$u_{\pi,x,h}^{\lambda_i} = \sum_{j=1}^{r_x} \frac{\dim(M_{x,h}^{\lambda_i})}{|\Gamma_j|} \left( (P(x))_{jl}^{\lambda_i} \right)_{hh} \left( \sum_{v \in \Delta_j(u)} v \right).$$

Since $(\ , \ )_v$ is $G$-invariant, the values of $f$ are constant on the orbitals of $G$ on $\mathcal{X}$. Having the shape of $\phi$ at hand, these values are given by the Norton-Sakuma Theorem for pairs of Majorana involutions (see the fourth column of Table [1]). The remaining cases are worked out in Section [4] in particular one sees that these values are constant also on the $\hat{G}$-orbitals. For $j \in \{1, \ldots, r_x\}$, denote by

$$\gamma_{x,j}$$

the value of the form $f$ on any pair $(y, z)$ belonging to $\Gamma_j$ and set

$$f_{x,h}^{\lambda} := \sum_{j=1}^{r_x} \gamma_{x,j} \cdot (P(x))_{jl}^{\lambda_i}.$$
Corollary 2.3. is the unique scalar product on $M$ multiplication, a unique nondegenerate scalar product (see [23, p.534]). In particular if
\[ \kappa: M \times M \to \mathbb{R} \]
is the unique scalar product on $M$ for which $X$ is an orthonormal basis, the relation between the form $f$ and the form $\kappa$ on the irreducible submodules of $M$ is given by the following lemma.

Lemma 2.2. [9] Lemma 4.14] Let $x \in \{b, t, r, s\}$, $i \in \{1, \ldots, I\}$, $h \in \{1, \ldots, n_i\}$, and $\lambda := \lambda_i$. Then for every $v \in M_{x,h}^\lambda$ we have
\[ f(v, v) = f_{x,h}^\lambda \kappa(v, v). \]
In particular $M_{x,h}^\lambda$ is contained in $rad(M)$ if and only if $f_{x,h}^\lambda = 0$.

Corollary 2.3. With the above notation, the values $f_{x,h}^\lambda$ are nonnegative.

Proof. This follows immediately from Lemma 2.2, since the form $\kappa$ is positive definite and the form $f$ is positive semidefinite. □

Lemma 2.2 allows us to determine $rad(M_x)$ if the action of $\hat{G}$ on $X$ is multiplicity-free, which is the case when $x = s$. For $x = b$ and $x = r$, $rad(M_x)$ has been determined in [7, Theorem 2] and [9, Theorem 1.4] respectively. Finally, for $x = t$, we’ll see in Section 7 that there exists an involutory $\mathbb{R}[\hat{G}]$-automorphism $\beta$ of $M_t$, induced by a permutation of $X$, which reduces us to computing $rad(X)$. Since $\beta$ is a $\mathbb{R}[\hat{G}]$-automorphism, for every element $u \in X$, $\beta$ induces in the obvious way a permutation of the $G_u$-orbits $\Delta_1(u), \ldots, \Delta_{r_1}(u)$, whence, for each $j \in \{1, \ldots, r_1\}$, there is a unique $j^\beta \in \{1, \ldots, r_1\}$, such that
\[ \Delta_j(u)^\beta = \Delta_{j^\beta}(u). \]
The computation of the rows of the generalised first eigenmatrix $P(t)$ needed to determine $rad([M_t, \beta])$ can be simplified by the following result.

Lemma 2.4. Let $\beta$ be as above, $u \in X$, $i \in \{1, \ldots, I\}$, $h \in \{1, \ldots, n_i\}$, and suppose that $u^{\lambda_i,h,t} \neq 0$. Then
\[ (1) \quad M^\lambda_{t,h} \leq [M_t, \beta] \iff \text{for every } j \in \{1, \ldots, r_t\}, (P(t)_j^{\lambda_i})_{hh} = -(P(t)_j^{\lambda_i})_{hh}. \]
\[ (2) \quad M^\lambda_{t,h} \leq C_{M_t}(\beta) \iff \text{for every } j \in \{1, \ldots, r_t\}, (P(t)_j^{\lambda_i})_{hh} = (P(t)_j^{\lambda_i})_{hh}. \]

Proof. Since $\beta$ acts as the identity on $C_{M_t}(\beta)$ and as the multiplication by $-1$ on $[M_t, \beta]$, by Equation [5] it follows that $u^{\lambda_i,h,t} \in [M_t, \beta]$ if and only if, for every $j \in \{1, \ldots, r_t\}$, $(P(t)_j^{\lambda_i})_{hh} = -(P(t)_j^{\lambda_i})_{hh}$, while $u^{\lambda_i,h,t} \in C_{M_t}(\beta)$ if and only if, for every $j \in \{1, \ldots, r_t\}$, $(P(t)_j^{\lambda_i})_{hh} = (P(t)_j^{\lambda_i})_{hh}$. Since $u^{\lambda_i,h,t} \neq 0$, $C_{M_t}(\beta)$, $[M_t, \beta]$, and $M^\lambda_{t,h}$ are $\mathbb{R}[\hat{G}]$-modules and $M^\lambda_{t,h}$ is irreducible, the claims follow. □

Once the quotients of $M_b$, $M_r$, $M_s$, and $[M_t, \beta]$ by their respective radicals have been determined, we need to consider the case when isomorphic irreducible submodules are contained in two or more of those quotients (see Section 6 and Section 5). To deal with this situation we use the following result.
Lemma 2.5. [9 Corollary 2.4] Let \( x, y \in \{b, r, s, t\} \), let \( \lambda \) be a partition of \( n \) and let \( M^\lambda_{x,h} \) and \( M^\lambda_{y,k} \) be irreducible submodules of \( M_x \) and \( M_y \) respectively, appearing in decomposition \([3]\), such that

1. the multiplicities of \( M^\lambda_{x,h} \) in \( M_x \) and \( M^\lambda_{y,k} \) in \( M_y \) are not 0,
2. \( \operatorname{rad}(M_x) \cap M^\lambda_{x,h} = \{0\} \),
3. \( \operatorname{rad}(M_y) \cap M^\lambda_{y,k} = \{0\} \).

Let \( H \) be a subgroup of \( \hat{G} \) such that \( \dim(C_{S^3}(H)) = 1 \) and let \( v_x \in M^\lambda_{x,h} \) and \( v_y \in M^\lambda_{y,k} \) be two \( H \)-invariant non-trivial vectors. Then

\[
\operatorname{rad}(M_x) \cap (M^\lambda_{x,h} \oplus M^\lambda_{y,k}) \cong S^\lambda
\]

if and only if

\[
\det \begin{pmatrix} f(v_x, v_x) & f(v_x, v_y) \\ f(v_y, v_x) & f(v_y, v_y) \end{pmatrix} = 0.
\]

Finally, once \( V^\circ \) has been cleared (as shown in Table [2]), in Section [10] we restrict to the case where \( n = 12 \) and prove that, in this case, \( V^\circ \) is closed under the algebra product.

For the rest of this paper we let \( (G, W, \lambda_b, \phi_b, \psi_b) \) be the Majorana representation of \( G \) induced by the restriction of a 2A-Majorana representation of \( A_{12} \) to \( G = \langle \lambda_b \rangle \). Further, set

\[
V^{(2A)} := \langle T^\psi \rangle \quad \text{and} \quad W^{(2A)} := \langle \lambda_b^\psi \rangle
\]

and for \( U \in \{V, W\} \) and \( 3 \leq N \leq 5 \), let \( U^{(N^A)} \) denote the linear span of the \( N^A \)-axes contained in all the Norton-Sakuma algebras generated by pairs of Majorana axes in \( U \). Finally, For \( 2 \leq N \leq 5 \), \( m \in \mathbb{N} \), denote by \( U^{(N^m)} \) the linear span of \( N^A \)-axes corresponding to permutations of cycle type \( N^m \). Note that \( V^{(3^2)} = W^{(3^2)} \).

NA-axes corresponding to permutations of cycle type \( N^m \) will be called simply \( N \)-axes of type \( N^m \).

3. The shape

In this section we keep the notation of Section [2] in particular \( n \in \{8, \ldots, 12\} \), \( G = A_n \), and \( (G, T, V, \phi, \psi) \) is a Majorana representation of \( G \). We determine the shape of \( \phi \) and prove that it depends only on \( G \).

Lemma 3.1. \( T \subseteq \lambda_b \cup \lambda_s \).

Proof. The cycle types of the involutions in \( G \) are \( 2^2, 2^4, \) or \( 2^6 \). Assume, by means of contradiction, that there are permutations of cycle type \( 2^4 \) in \( T \). Since these are all conjugate in \( G \) and \( T \) is invariant under conjugation, \( T \) contains all such involutions. Let \( H := \langle t_1, t_2, t_3 \rangle \), where \( t_1 := (1, 2)(3, 4)(5, 6)(7, 8) \), \( t_2 := (1, 3)(2, 4)(5, 8)(6, 7) \), and \( t_3 := (1, 8)(2, 7)(3, 5)(4, 6) \). Then \( H \) is elementary abelian of order 8 and every nontrivial element of \( H \), being of cycle type \( 2^4 \), is contained in \( T \), which is a contradiction to [32] Lemma 4.2] (note that, if \( n \geq 9 \), the result follows immediately, since the set of involutions of cycle type \( 2^4 \) in \( G \) is not a set of 6-transpositions). \( \square \)

Set \( r_1 := (1, 2)(3, 4) \), \( s_1 := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \), and denote the orbitals of \( G \) on \( \lambda_b \cup \lambda_s \) as in the first column of Table [3]. Here the pairs of orbitals of \( A_{12} \) that fuse under the natural action of \( S_{12} \) are marked with * and **.
| ORBITALS | REPRESENTATIVES | CYCLE TYPE OF $r_1 r_j$ | SHAPE |
|----------|----------------|------------------------|-------|
| $Σ_{1,b}$ | $(r_1, r_1)$ | 1 | 1A |
| $Σ_{2,b}$ | $(r_1, (1,3)(2,4))$ | 2$^2$ | 2A |
| $Σ_{3,b}$ | $(r_1, (1,5)(3,4))$ | 3 | 3A |
| $Σ_{4,b}$ | $(r_1, (1,5)(2,3))$ | 5 | 5A |
| $Σ_{5,b}$ | $(r_1, (5,6)(3,4))$ | 2$^2$ | 2A |
| $Σ_{6,b}$ | $(r_1, (5,6)(2,3))$ | 2 · 4 | 4B |
| $Σ_{7,b}$ | $(r_1, (1,5)(2,6))$ | 2 · 4 | 4B |
| $Σ_{8,b}$ | $(r_1, (1,6)(3,5))$ | 3$^2$ | 3A |
| $Σ_{9,b}$ | $(r_1, (1,5)(6,7))$ | 2$^2$, 3 | 6A |
| $Σ_{10,b}$ | $(r_1, (5,6)(7,8))$ | 2$^4$ | 2B |

| CYCLE TYPE OF $s_1 s_j$ |
|------------------------|
| $Σ_{1,s}^*$ | $(s_1, s_1)$ | 1 | 1A |
| $Σ_{2,s}$ | $(s_1, (1,2)(3,4)(5,6)(7,8)(9,11)(10,12))$ | 2$^2$ | 2A |
| $Σ_{3,s}$ | $(s_1, (1,2)(3,4)(5,7)(6,8)(9,11)(10,12))$ | 2$^4$ | 2B |
| $Σ_{4,s}$ | $(s_1, (1,2)(3,4)(5,6)(7,9)(8,11)(10,12))$ | 3$^2$ | 3A |
| $Σ_{5,s}$ | $(s_1, (1,3)(2,4)(5,7)(6,8)(9,11)(10,12))$ | 2$^6$ | 2A |
| $Σ_{6,s}$ | $(s_1, (1,3)(2,4)(5,7)(6,8)(9,12)(10,11))$ | 2$^6$ | 2A |
| $Σ_{7,s}$ | $(s_1, (1,2)(3,5)(4,6)(7,9)(8,11)(10,12))$ | 2$^2$, 3$^2$ | 6A |
| $Σ_{8,s}$ | $(s_1, (1,3)(2,5)(4,6)(7,9)(8,11)(10,12))$ | 3$^4$ | 3A |
| $Σ_{9,s}$ | $(s_1, (1,2)(3,4)(5,7)(6,9)(8,11)(10,12))$ | 4$^2$ | 4A |
| $Σ_{10,s}$ | $(s_1, (1,2)(3,5)(4,7)(6,9)(8,11)(10,12))$ | 2$^2$, 4$^2$ | 4A |
| $Σ_{11,s}$ | $(s_1, (1,3)(2,5)(4,7)(6,9)(8,11)(10,12))$ | 2$^2$, 4$^2$ | 4A |
| $Σ_{12,s}$ | $(s_1, (1,3)(2,5)(4,7)(6,9)(8,12)(10,11))$ | 5$^2$ | 5A |

| CYCLE TYPE OF $s_1 r_j$ |
|------------------------|
| $Σ_{1,ab}$ | $(s_1, r_1)$ | 2$^4$ | 2B |
| $Σ_{2,ab}$ | $(s_1, (1,3)(2,4))$ | 2$^6$ | 2A |
| $Σ_{3,ab}$ | $(s_1, (2,3)(5,6))$ | 2$^3$, 3$^3$ | 4B |
| $Σ_{4,ab}$ | $(s_1, (2,3)(4,5))$ | 2$^3$, 6 | 6A |
| $Σ_{5,ab}$ | $(s_1, (2,3)(6,7))$ | 2$^2$, 2$^2$ | 4A |

| CYCLE TYPE OF $r_j s_1$ |
|------------------------|
| $Σ_{1,bs}$ | $(r_1, s_1)$ | 2$^2$ | 2B |
| $Σ_{2,bs}$ | $((1,3)(2,4), s_1)$ | 2$^6$ | 2A |
| $Σ_{3,bs}$ | $((2,3)(5,6), s_1)$ | 2$^3$, 3$^3$ | 4B |
| $Σ_{4,bs}$ | $((2,3)(4,5), s_1)$ | 2$^3$, 6 | 6A |
| $Σ_{5,bs}$ | $((2,3)(6,7), s_1)$ | 2$^2$, 2$^2$ | 4A |

| CYCLE TYPE OF $r_1 s_j$ |
|------------------------|
| $Σ_{1,bs}$ | $(r_1, s_1)$ | 2$^2$ | 2B |
| $Σ_{2,bs}$ | $((1,3)(2,4), s_1)$ | 2$^6$ | 2A |
| $Σ_{3,bs}$ | $((2,3)(5,6), s_1)$ | 2$^3$, 3$^3$ | 4B |
| $Σ_{4,bs}$ | $((2,3)(4,5), s_1)$ | 2$^3$, 6 | 6A |
| $Σ_{5,bs}$ | $((2,3)(6,7), s_1)$ | 2$^2$, 2$^2$ | 4A |

Table 3. The $A_{12}$-orbits on $T_1 × T$ and their representatives for $r_1 := (1,2)(3,4)$, $s_1 := (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$, and $T = X_b ⊔ X_s$.

Lemma 3.2. [31] Theorem III] With the notation of Section 3

$$M_s = M_{s,1}^{(12)} ⊕ M_{s,1}^{(10,2)} ⊕ M_{s,1}^{(8,4)} ⊕ M_{s,1}^{(6,4,2)} ⊕ M_{s,1}^{(4,4,2)} ⊕ M_{s,1}^{(4,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)} ⊕ M_{s,1}^{(6,2)}.$$
Lemma 3.3. For $i \in \{1, \ldots, 10\} \setminus \{5, 9, 11\}$, let $\Sigma_i := \Sigma_{i,s}$ and, for $i \in \{5, 9, 11\}$, let $\Sigma_i := \Sigma_{i,s}^* \cup \Sigma_{i,s}^{**}$. Then $\Sigma_1, \ldots, \Sigma_{11}$ are the orbitals of $S_{12}$ on its action on $X_s$. The first eigenmatrix $P(s)$ relative to this action is given in the last eleven columns of Table 3. The entries in the second column of Table 3 are the dimensions of the irreducible $\mathbb{R}[S_{12}]$-modules $S^\lambda$.

Proof. This can be computed using standard methods of algebraic combinatorics (see [II Chapter II, §2.1]) as in [6], or using [7] Lemma 3.

Proposition 3.4. Let $n = 12$ and assume that $X_s \subseteq T$. Then

1. $\phi$ is a 2A-Majorana representation;
2. the shape of $\phi$ is the one given in the last column of Table 3;
3. if $\Sigma^*$ and $\Sigma^{**}$ are orbitals of $G$ on $T$ that fuse under $\hat{G}$ then $sh(\Sigma^*) = sh(\Sigma^{**})$;
4. the restriction of the form $f$ on the module $M_b + M_s$ is $\hat{G}$-invariant.

Proof. Let $\Sigma$ be an orbital of $G$ on $T$, let $(u,y) \in \Sigma$, and let $|uy|$ be the cycle type of $uy$. If $|uy| \in \{1, 5, 6\}$, then, by the Norton-Sakuma Theorem, $sh(\Sigma) = |uy|A$. By the hypothesis and Axiom M4, $sh(\Sigma) = 2A$, if $|uy| = 2^6$, while $sh(\Sigma) = 2B$, if $|uy| = 2^4$. Assume $|uy| = 4$ and $|uy| \in \{4^2, 2^4\}$. Then, by Lemma (3.1), $(uy)^2 \not\in T$, whence $sh(\Sigma) = 4A$ by Axiom M6. This gives the shape of $\Sigma$ and, consequently, the scalar products of the axes associated to $u$ and $y$ for all cases such that $u,y \in X_4$, except for $|uy| \in \{2, 3\}$ and $|uy| \not\in \{2^4, 2^6\}$. In the latter case, we have $\Sigma \in \{\Sigma_{2,4}, \Sigma_{4,4}, \Sigma_{7,7}\}$ and, by the Norton-Sakuma Theorem, the possibilities for $sh(\Sigma)$ are $2A$ or $2B$, if $|uy| = 2$, and $3A$ or $3C$ if $|uy| = 3$. We shall eliminate the possibilities $2B$ and $3C$, by showing that in these cases the form $f$ would not be positive semidefinite, since some of the values $f_{s,1}^\lambda$ would be negative.

Since the orbitals $\Sigma_{2,4}$, $\Sigma_{4,4}$, and $\Sigma_{7,7}$ are invariant under the action of $\hat{G}$, it follows that the restriction of the form $f$ on the module $M_s$ is $\hat{G}$-invariant so we can use the structure of $M_s$ as an $\mathbb{R}[\hat{G}]$-module to compute the values $f_{s,1}^\lambda$. Let

$$v_{(6^2)} := (1, 15, 45, 40, 15, -120, 40, -90, 144, -120)$$

and

$$v_{(2^6)} := (1, 9, 33, -8, 27, 120, 136, -78, -114, -48, 96)$$

| $\lambda$ | $\Sigma_1$ | $\Sigma_2$ | $\Sigma_3$ | $\Sigma_4$ | $\Sigma_5$ | $\Sigma_6$ | $\Sigma_7$ | $\Sigma_8$ | $\Sigma_9$ | $\Sigma_{10}$ | $\Sigma_{11}$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| (12) | 1 | 1 | 30 | 180 | 160 | 120 | 960 | 640 | 720 | 1440 | 2304 | 3840 |
| (10, 2) | 54 | 19 | 48 | 72 | 120 | 80 | -64 | 192 | -144 | 192 | -384 |
| (8, 4) | 275 | 12 | 27 | 16 | 30 | 24 | -8 | -18 | 108 | -144 | -48 |
| (6, 4, 2) | 2673 | 4 | 3 | -8 | 2 | 0 | -24 | -18 | -4 | 32 | 16 |
| (4, 2$^2$) | 2640 | -3 | 3 | -8 | -9 | 0 | 4 | 24 | 24 | -24 | -12 |
| (4, 2$^*$) | 1485 | -8 | 3 | 12 | 6 | 20 | -16 | -6 | -36 | -24 | 48 |
| (6$^2$) | 132 | -15 | 45 | 40 | -15 | -120 | 40 | -90 | 90 | 144 | -120 |
| (2$^9$) | 132 | 9 | 33 | -8 | -27 | 120 | 136 | -78 | -114 | -48 | -24 |
| (8, 2$^*$) | 616 | 9 | -12 | 22 | -12 | 16 | 12 | -24 | -24 | 48 | 96 |
| (4$^2$) | 462 | 0 | 15 | -20 | 30 | -60 | 40 | 30 | -60 | 24 | 0 |
| (6, 2$^*$) | 1925 | 0 | -21 | 4 | 6 | 12 | 16 | -6 | 12 | 24 | -48 |
be the rows of the first eigenmatrix associated to the action of $\hat{G}$ on the set $X_*$ corresponding to the partitions $(6^2)$ and $(2^6)$ (see Table 1). For $W \in \{2A, 2B, 3A, 3C\}$, let $\gamma(W)$ be the Majorana inner product of the two generating axes of a Norton-Sakuma algebra of type $W$, so, by Table 1

$$\gamma(2A) = 1/2^8, \gamma(2B) = 0, \gamma(3A) = 13/2^8, \text{ and } \gamma(3C) = 1/2^6$$

and, for $X \in \{2A, 2B\}$ and $Y, Z \in \{3A, 3C\}$, let

$$\gamma(X, Y, Z) := (1, \gamma(X), 0, \gamma(Y), 1/2^3, 5/2^8, \gamma(Z), 1/2^5, 1/2^5, 3/2^7, 5/2^8).$$

Then, by Equation (6), for $\lambda \in \{(6^2), (2^6)\}$, $f_{s,1}^{A}$ is given by the rows by columns multiplication of $\gamma(X, Y, Z)$ by the transpose of $v_\lambda$. A straightforward computation shows that, in the case $(X, Y, Z) = (2A, 3A, 3A)$, $f_{s,1}^{(6^2)} = f_{s,1}^{(2^6)} = 0$, while in all the other cases either $f_{s,1}^{(6^2)}$ or $f_{s,1}^{(2^6)}$ is negative, against Corollary 2.3. Thus $(sh(\Sigma_{2A}), sh(\Sigma_{4A}), sh(\Sigma_{7B})) = (2A, 3A, 3A)$. In particular, $sh(\Sigma_{2A}) = 2A$, so Axiom M4 implies $X_0 \subseteq T$, that is $\phi$ is a 2A-Majorana representation. By Axiom M4, we get $sh(\Sigma_{2A}) = 2A$ and, by Axiom M6, $sh(\Sigma) = 4B$, if $[uy] \in \{2 \cdot 4, 2^3 \cdot 4\}$. Since every element of cycle type $3^2$ is the product of two elements in $X_*$, Axiom M5 implies that $sh(\Sigma_{4B}) = 3A$. Finally, if $u, y \in X_*$ and $[uy] = 3$, then the subgroup they generate fixes more than 4 points, so $sh(\Sigma) = 3A$, by Corollary 1.2. This proves (2). Assertions (3) and (4) then follow immediately.

\[\square\]

Corollary 3.5. Let $n = 12$. The following assertions are equivalent.

1. $\phi$ is a 2A-Majorana representation.
2. $X_0 \subseteq T$.
3. There are involutions $t_1, t_2 \in T$ such that $\langle (a_{t_1}, a_{t_2}) \rangle$ has type 4A.

Proof. Obviously (1) implies (2). Moreover, by Lemma 3.2 it follows that (2) implies (3). Assume (3) holds. Looking at the possible cycle types of the product $t_1t_2$, one can see that the algebra $\langle (a_{t_1}, a_{t_2}) \rangle$ has type 4A if and only if $\{t_1, t_2\} \cap X_1 \neq \emptyset$. This gives $X_s \subseteq T$ and, by Proposition 3.4 (1) holds. \[\square\]

4. The inner products

We keep the notation of Section 2. In this section we find the values of the inner product between axes and 3-axes needed to determine the radical of the restriction of the form $f$ to the $\mathbb{R}[G]$-submodule $M_{r + M_{1 + M_{r - M_{\ell}}}}$ of $M_X$. Note that the values of the inner products between Majorana axes are given by the Norton-Sakuma Theorem and can be found in Table 1. The results involving 3-axes of type 3 have been considered by Chien in [3]. However, Chien’s results partially depend on an earlier work of S. Norton, who computed the inner products between axes and 3-axes within the Griess’ algebra (see [25]). Therefore, to make Chien’s results independent from the Monster, we recalculated the inner products involving 3-axes of type 3 relying only on the Majorana axiomatrics, which we did either by hand or using the GAP package MajoranaAlgebras [26].

We begin by considering the Majorana inner products $\langle a_r, u_c \rangle_\ell$ between Majorana axes $a_r$, with $r \in X_{\ell} \cup X_s$, and 3-axes $u_c$, with $\langle c \rangle \in X_\ell$. Assume first $r \in X_s$ and $\langle c \rangle \in X_\ell$. Let, as in Section 3

$$r_1 := (1, 2)(3, 4).$$
Table 5. The scalar products between axes of type $2^6$ and 3-axes of type $3^2$

| $i$ | $c_i$ | $\{ x_i c_i \}$ | $\langle s_1, c_i \rangle$ | $(a_{s_1}, u_{c_i})_V$ |
|-----|-------|-----------------|-----------------|------------------|
| 1   | (7, 8, 9)(10, 11, 12) | $2^3 \cdot 4$ | $S_4$ | 1/36 |
| 2   | (7, 9, 11)(8, 10, 12) | $2^3 \cdot 6$ | $C_6$ | 0 |
| 3   | (7, 9, 12)(8, 11, 10) | $2^6$ | $S_3$ | 1/36 |
| 4   | (6, 7, 8)(10, 11, 12) | $2^2 \cdot 3^2$ | $2 \times A_4$ | 1/3 |
| 5   | (6, 7, 9)(10, 11, 12) | $2^2 \cdot 7$ | $2 \times L_2(7)$ | 1/14 |
| 6   | (6, 7, 9)(8, 10, 11) | $2^2 \cdot 4^2$ | $S_4$ | 1/12 |
| 7   | (6, 9, 7)(8, 10, 11) | $2^3 \cdot 6$ | $2 \times A_4$ | 1/3 |
| 8   | (3, 6, 7)(10, 11, 12) | $2 \cdot 3 \cdot 6$ | $S_3 \times A_4$ | 1/17 |
| 9   | (3, 6, 7)(8, 11, 9) | $2 \cdot 10$ | $2 \times A_5$ | 1/3 |
| 10  | (2, 3, 6)(8, 9, 11) | $6^2$ | $3 \times S_3$ | 1/3 |
| 11  | (2, 3, 6)(8, 11, 10) | $6^2$ | $3 \times S_3$ | 1/3 |

For $n = 12$, let
\[ s_1 := (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12), \]
let
\[ \{ (s_1, \langle c_i \rangle) | i \in \{1, \ldots, 11\} \} \]
be a set of representatives for the orbitals of $G$ on $X_s$, and let
\[ \{ (r_1, \langle d_i \rangle) | i \in \{1, \ldots, 13\} \} \]
be a set of representatives for the orbitals of $G$ on $X_b$.

**Lemma 4.1.** Let $n = 12$ and $\phi$ be a $2A$-Majorana representation of $G$. Then, the (relevant) Majorana inner products $(a_z, u_c)_V$ such that $z \in X_s$ and $\langle c \rangle \in X_t$ are given in Table 5.

**Proof.** By Proposition 3.4, the shape of $\phi$ is the one given in Table 5. For $i \in \{1, 6\}$, $(s_1, c_i) \cong S_4$, so
\[ (s_1, c_i)_V = 1/36 \]
by [15, Table 9 and Table 7]. For $i = 2$ we have $(s_1, c_2) \cong C_6$ and the algebra $\langle \langle a_{s_1}, u_{c_2} \rangle \rangle$ is contained in a Norton-Sakuma algebra of type $6A$. So, by the Norton-Sakuma Theorem,
\[ (a_{s_1}, u_{c_2})_V = 0. \]
Similarly, for $i = 3$, $\langle \langle a_{s_1}, u_{c_3} \rangle \rangle$ is contained into a Norton-Sakuma algebra of type $3A$, whence
\[ (a_{s_1}, u_{c_3})_V = 1/4. \]
Now assume \( i \in \{4, \ldots, 11\} \) and decompose \( c_i \) as the product of two permutations \( g \) and \( h \) of cycle type \( 2^2 \). Then, by the Norton-Sakuma Theorem,

\[
u_{c_i} = \frac{211}{27} \cdot \frac{1}{32} \left\{ 2a_g + 2a_h + a_{ghg} - a_g \cdot a_h \right\}
\]

and, by the associativity of the Majorana inner product, \( (9) \)

\[(a_{s_1}, u_{c_i})_V = \frac{211}{27} \cdot \frac{1}{32} \left\{ 2(a_{s_1}, a_g)_V + 2(a_{s_1}, a_h)_V + (a_{s_1}, a_{ghg})_V \right\} - (a_{s_1} \cdot a_h, a_g)_V.
\]

The first three scalar products in the second member of Equation \( (9) \) are scalar product of axes, so they can be detected by Proposition \( 3.4 \) and the Norton-Sakuma Theorem. The last one can be computed by choosing \( s \) so that \( a_{s_1} \cdot a_h \) is a linear combination either of axes (which is the case when the Norton-Sakuma subalgebra generated by \( a_{s_1} \) and \( a_h \) has type \( 2A \), \( 2B \), or \( 4B \)), or of axes together with a 3-axis whose inner product with \( a_g \) is already computed. Thus, for \( i = 4 \), choose \( g := (6,7)(10,11) \) and \( h := (7,8)(11,12) \).

Then, since \([s_1h] = 2^4\), by Proposition \( 3.4 \) \( a_{s_1} \) and \( a_h \) generate a subalgebra of type \( 2B \), whence \((a_{s_1}, a_h)_V = 0\). Similarly \((a_{s_1}, a_g)_V = (a_{s_1}, a_{ghg})_V = 1/32\). For the last scalar product, again by the Norton-Sakuma Theorem, we have \( a_{s_1} \cdot a_h = 0 \) giving \((a_{s_1} \cdot a_h, a_g)_V = 0\). Substituting these values in Equation \( (9) \) we get

\((a_{s_1}, u_{c_4})_V = \frac{2}{45} \).

For \( i = 7 \) the same argument with \( g = (6,7)(10,11) \) and \( h = (7,9)(8,10) \) gives the result.

In order to compute the scalar product \((a_{s_1} \cdot a_h, a_g)_V\) for \( i = 5 \), choose

\[g := (6,7)(10,11) \) and \( h := (7,9)(11,12) \).

Then \( a_{s_1} \) and \( a_h \) generate a subalgebra of type \( 4B \), so, by the Norton-Sakuma Theorem,

\[a_{s_1} \cdot a_h = \frac{1}{64} [a_{s_1} + a_h - a_{s_1}h_{s_1} - a_{h_{s_1}h} + a_{s_1}h_{s_1}h].\]

Therefore

\[(a_{s_1} \cdot a_h, a_g)_V = \frac{1}{64} [(a_{s_1}, a_g)_V + (a_h, a_g)_V - (a_{s_1}h_{s_1}, a_g)_V - (a_{h_{s_1}h}, a_g)_V + (a_{s_1}h_{s_1}h, a_g)_V].\]

If \( i = 8 \), the inner product \((a_{s_1}, u_{c_5})_V\) can be computed in a similar way, with \( g := (3,6)(10,11) \) and \( h := (6,7)(11,12) \) since, with this choice, \((a_{s_1}, a_h)\) is a Norton-Sakuma algebra of type \( 4B \).

Assume \( i = 9 \). Set

\[g := (3,6)(9,11) \) and \( h := (6,7)(8,9) \).

Then the Norton-Sakuma algebra generated by \( a_{s_1} \) and \( a_h \) has type \( 6A \) and \( (s_1h)^2 = (5,9,8)(6,7,10) \). Since \((a_{s_1}, u_{(s_1h)^2})\) is \( A_{12} \)-conjugate to \((a_{s_1}, u_{c_5})\), by the previous
Table 6. The scalar products between axes of type 2^2 and 3-axes of type 3^2

| i   | d_i | [r_1 d_i] | (r_1, d_i) | (a_{r_1}, u_{d_i})_V |
|-----|-----|-----------|------------|-----------------------|
| 1   | (6, 7, 8)(9, 10, 11) | 2^2 \cdot 3^2 | 3 \times 3 | 0                     |
| 2   | (4, 5, 6)(9, 10, 11) | 2 \cdot 3 \cdot 4 | 3 \times S_4 | \frac{1}{18}         |
| 3   | (3, 4, 5)(9, 10, 11) | 2^2 \cdot 3 | 3 \times S_3 | \frac{1}{18}         |
| 4   | (3, 5, 7)(4, 6, 8) | 2 \cdot 6 | 2 \times A_4 | \frac{1}{18}         |
| 5   | (2, 3, 4)(9, 10, 11) | 3^2 | A_4 | \frac{1}{3} |
| 6   | (2, 3, 5)(9, 10, 11) | 3 \cdot 5 | 3 \times A_5 | \frac{11}{360}       |
| 7   | (2, 3, 5)(4, 6, 7) | 7 | L_3(2) | \frac{1}{27}       |
| 8   | (2, 5, 7)(4, 6, 8) | 4^2 | S_4 | \frac{13}{180}      |
| 9   | (2, 5, 7)(3, 4, 6) | 2 \cdot 4 | S_4 | \frac{1}{27}       |
| 10  | (1, 2, 3)(4, 5, 6) | 5 | A_5 | \frac{1}{18}         |
| 11  | (1, 2, 5)(3, 4, 6) | 2^2 | S_3 | \frac{1}{9}          |
| 12  | (1, 3, 5)(2, 4, 6) | 3^2 | A_4 | \frac{1}{9}         |
| 13  | (1, 3, 6)(2, 5, 4) | 2 \cdot 4 | S_4 | \frac{1}{18}       |

case we have \((a_{s_1}, u_{(s_1 h)^2})_V = 13/180\). It follows that
\[
(a_{s_1}, a_g \cdot a_h)_V = (a_g, a_{s_1} \cdot a_h)_V
= \frac{1}{64} \left[ (a_g, a_{s_1})_V + (a_g, a_h)_V - (a_g, a_{s_1} h_{s_1})_V - (a_g, a_{h_{s_1}} h_{s_1})_V \\
- (a_g, a_{s_1} h_{s_1} h_{s_1})_V - (a_g, a_{h_{s_1}} h_{s_1} h_{s_1})_V \right] + \frac{45}{211} (a_g, u_{(s_1 h)^2})_V = \frac{13}{211}
\]
and \((a_{s_1}, u_{c_0})_V = 1/30\).

Finally, the inner products \((a_{s_1}, u_{c_i})_V\) for \(i \in \{10, 11\}\) have been computed, within the Majorana axiomatics, by Chien [3, p.86].

Lemma 4.2. Let \(\phi\) be a 2A-Majorana representation of \(A_n\). Then, the (relevant) Majorana inner products \((a_z, u_c)_V\) such that \(z \in X_b\) and \((c) \in X_t\) are given in Table 6.

Proof. By Proposition 3.3, the shape of \(\phi\) is the one given in Table 6. When \(i = 1\), \(r_1\) and \(d_1\) are disjoint and hence their inner product is 0 (see case \(i = 2\) in
Lemma 4.1. The inner products $(a_{r_i}, u_{d_i})_V$ for $i \in \{5, 7, 9, 10, 12, 13\}$ have been computed in [19], since in those cases $r_1$ and $d_i$ are contained in a stabiliser of $n-5$ points. When $i = 11$, $\langle r_1, d_{11} \rangle \cong S_3$ and the result follows from Norton-Sakuma theorem. In the remaining five cases one proceeds as in the proof of Lemma 4.1 and get the required values.

The inner products between two 3-axes have been computed by Chien in [3]. Unfortunately in Chien’s thesis it is not clear which ones have been computed relying only on the Majorana axiomatics and which ones depend on the embedding of $A_{12}$ in the Monster, so we have checked all the products needed for this paper. The group $A_{12}$ has 32 orbits on the set $\mathcal{X}_t \times \mathcal{X}_t$, which we denote by $\Omega_i$, $i \in \{1, \ldots, 32\}$. All but $\Omega_{30}$ and $\Omega_{32}$ are also $S_{12}$-orbits, while $\Omega_{30}$ and $\Omega_{32}$ merge into a unique $S_{12}$-orbit. A set of representatives $(\langle e_i \rangle, \langle e_i \rangle)$ for the first 31 orbits is

| $t$ | $e_t$ | $e_{1e_t}$ | $e_{1e_t}$ | $(u_{e_t}, u_{e_t})_V$ |
|-----|-------|------------|------------|------------------|
| 1   | 1, 2, 3(5, 6, 7) | $3A$ | $4A$ | $C_3 \times C_3$ |
| 2   | 1, 3, 2(5, 6, 7) | $3A$ | $4A$ | $C_3 \times C_3$ |
| 3   | 1, 2, 4(5, 6, 7) | $3A$ | $4A$ | $A_4$ |
| 4   | 1, 4, 2(5, 6, 7) | $3A$ | $4A$ | $A_4$ |
| 5   | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 6   | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 7   | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 8   | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 9   | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 10  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 11  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 12  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 13  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 14  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 15  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 16  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 17  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 18  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 19  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 20  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 21  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 22  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 23  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 24  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 25  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 26  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 27  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 28  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 29  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 30  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |
| 31  | 2, 3, 4(5, 6, 7) | $6$ | $5A$ | $A_4$ |

Table 7. The representatives for the orbitals of $\hat{G}$ on $\mathcal{X}_t$ and the Majorana form, for $7 \leq n \leq 12$
given in the second column of Table [7]. Double horizontal lines detect the change of the parameter $n$: $\{(e_i, x) | i \in \{1, \ldots, 10\}\}$ is a set of representatives for the $S_7$-orbits, $\{(e_i, x) | i \in \{1, \ldots, 20\}\}$ is a set of representatives for the $S_8$-orbits, and so on. A representative for $\Omega_{32}$ is $(e_1, (6, 7, 8)(9, 11, 10))$.

**Lemma 4.3.** Let $\phi$ be a $2A$-Majorana representation of $A_{12}$. Then the (relevant) inner products between two 3-axes of type $3^2$ are those given in the last column of Table [7]. Moreover the value of the inner product on $\Omega_{32}$ is the same with that on $\Omega_{30}$.

**Proof.** Inner products $(u_{e_1}, u_{e_1})$ for $i \in \{1, \ldots, 10\}$ have been computed in [19]. For $i \in \{11, 12, 17\}$, these products can be computed using [3] Lemma 4.6. For $i \in \{13, 19, 24\}$, these products can be computed using [3] Lemma 4.7. For $i \in \{14, 15, 16, 20, 23\}$, these products can be computed using [3] Lemma 4.8. Assume $i = 21$. Let

$$f_1 := (1, 2)(5, 6), \quad f_2 := (1, 3)(4, 5), \quad g_1 := (3, 4)(8, 9), \quad \text{and} \quad g_2 := (3, 7)(6, 8),$$

so that

$$e_1 = f_1 f_2 \quad \text{and} \quad e_{21} = g_1 g_2.$$  

By the Norton-Sakuma Theorem and Lemma 4.2, the inner product $(u_{e_1}, u_{e_1})$ can be computed once the value of

$$(a_{f_1} \cdot a_{f_2}, a_{g_1} \cdot a_{g_2})V$$

is known. By the associativity of the scalar product,

$$(a_{f_1} \cdot a_{f_2}, a_{g_1} \cdot a_{g_2})V = (a_{f_1} \cdot (a_{f_2} \cdot a_{g_2}), a_{g_1})V$$

and, since $f_1$ and $g_1$ commute, by the Norton-Sakuma Theorem $a_{f_1}$ and $a_{g_1}$ generate a subalgebra of type $2B$, whence $a_{g_1}$ is a 0-eigenvector for the adjoint action of $a_{f_1}$. By [18] Lemma 1.10 and the associativity of the scalar product we have

$$(a_{f_1} \cdot (a_{f_2} \cdot a_{g_1}), a_{g_2})V = (a_{f_1} \cdot (a_{f_2} \cdot a_{g_1}), a_{g_2})V = (a_{f_2} \cdot a_{g_1}, a_{f_1} \cdot a_{g_2})V.$$ 

Since the subalgebra generated by $a_{f_2}$ and $a_{g_1}$ is of type $4B$, by Norton-Sakuma, $a_{f_2} \cdot a_{g_1}$ can be written as a linear combination of Majorana axes. On the other hand, the subalgebra generated by $a_{f_1}$ and $a_{g_2}$ has type $6A$, with $u_{(f_1 g_2)^2}$ is of type 3. By [2] Corollary 3.2 the product $a_{f_1} \cdot a_{g_2}$ can be written as a linear combination of Majorana axes. Thus eventually we are reduced to computing scalar products of Majorana axes, which are given by the Norton-Sakuma Theorem. Similar arguments give the scalar products for $i \in \{25, 26, 27, 29, 30, 31\}$. Finally, by the symmetry of the scalar product, the products $(u_{e_1}, u_{e_1})V$, $(u_{e_1}, u_{e_2})V$, and $(u_{e_1}, u_{e_2})V$ coincide, respectively with $(u_{e_1}, u_{e_1})V$, $(u_{e_1}, u_{e_2})V$, and $(u_{e_1}, u_{e_2})V$. \(\square\)

5. A spanning set for the 2-closures of $V$ and $W$

We stick to the notation of the Section [2] and, for the rest of this paper, we assume that $\phi$ is a $2A$-Majorana representation of $G$. Let

$$g := (a_1, b_1)(a_2, b_2)(a_3, b_3)(a_4, b_4)$$

be a permutation of cycle type $2^4$ and let $S_a$ be the subgroup of $G$ with support contained in $\{a_1, a_2, a_3, a_4\}$. Define

$$H_{(g)} := [S_a, g].$$
Clearly $H(g)$ is a diagonal subgroup of the direct product $S_a \times S_a^3$, in particular $H(g)$ is isomorphic to $S_4$. For distinct $t_1, t_2 \in T \cap H(g)$, set

$$(10) \quad \sigma_{t_1, t_2} := a_{t_1} \cdot a_{t_2} - \frac{1}{32}(a_{t_1} + a_{t_2}).$$

By [18, Lemma 2.3], $\sigma_{t_1, t_2}$ is invariant under the actions of the subgroup of $H(g)$ generated by $t_1$ and $t_2$ and isomorphic to $S_3$ and can be indexed by the product $t_1 t_2$. Moreover, since the Norton Sakuma subalgebra generated by $a_{t_1}$ and $a_{t_2}$ has type $3A$, by Table [1] $\sigma_{t_1, t_2}$ is expressible as a linear combination of $a_{t_1}, a_{t_2}, a_{t_1 t_2 t_1}$ and $a_{t_1 t_2}$.

Let

$$\xi := (a_1, a_2)(a_3, a_4)(b_1, b_2)(b_3, b_4)$$

and define

$$(11) \quad \delta_\xi := \frac{1}{2}\sigma(a_2, a_3, a_4)(b_2, b_3, b_4) \cdot a_{(1,2,3)}(b_1, b_2)$$

$$- \frac{1}{25} \sigma(a_2, a_3, a_4)(b_2, b_3, b_4) + \frac{1}{210} a_{(1,2,3)}(b_1, b_2).$$

**Lemma 5.1.** The vector $\delta_\xi$ depends only on the involution $\xi$.

**Proof.** First note that $\xi \in H(g)$. Since $H(g)$ is isomorphic to $S_4$ and it is generated by the set $T \cap H(g)$, i.e. the set of the permutations of type $2^2$ in $H(g)$, by Proposition [3.4] it follows that every $2A$-Majorana representation of $G$ induces a Majorana representation of $H(g)$ and the product of any two elements of $T \cap H(g)$ has either cycle type $2^4$ or $3^2$. By Table [3] with the notation of [18, Proposition 4.3], the representation induced on $H(g)$ has type $(2B, 3A)$ and the claim then follows from [18, Lemma 4.6]. □

**Lemma 5.2.** In the algebra $V$ the following relation holds

$$v_{1,3,2,4,5,7,6,8} = \frac{1}{32}(a_{(1,2,3)}(5,6) + a_{(3,4)}(7,8))$$

$$+ \frac{7}{2 \cdot 3^2} a_{(1,3)}(5,7) + a_{(1,4)}(5,8) + a_{(2,3)}(6,7) + a_{(2,4)}(6,8)$$

$$+ \frac{25 \cdot 5}{3^3} \sigma(a_{(1,2,3,4)}(5,6,7,8) + \sigma(a_{(1,3,4)}(5,7,8) + \sigma(a_{(1,2,4)}(5,6,8) + \sigma(a_{(2,3,4)}(6,7,8))$$

$$- \frac{211}{3^3} \delta_{(1,2)}(3,4)(5,6,7,8) + \frac{12}{3^3} \delta_{(1,3)}(2,4)(5,7,6,8) + \delta_{(1,4)}(2,3)(5,8,6,7).$$

**Proof.** Set $g = (1,5)(2,6)(3,7)(4,8), \xi = (1,2)(3,4)(5,6)(7,8)$ and $H := H(g) \times \langle (1,5)(2,6)(3,7)(4,8)(9,10)(11,12) \rangle$, so the $H$ is a subgroup of $A_{12}$ isomorphic to $S_4 \times 2$ and generated by $T \cap H$. As in the proof of Lemma [5.1] the $2A$-Majorana representation $\phi$ of $A_{12}$ induces a $2A$-Majorana representation $\phi|_H$ of $H$. In particular, the shape of $\phi|_H$ is determined by the shape of $\phi$, which is given in Table [3] by Proposition [3.4]. Thus the relation can be checked within the smaller subalgebra (of dimension 23) generated by $(T \cap H)^{\phi}$ either by hand (not recommended) or using the GAP package MajoranaAlgebras [20]. □

The formula of Lemma [5.2] is the same as that given in [18, p.2462], which has been obtained under the assumption that a certain parameter $m$ is equal to $-3$. The latter fact seems to have already been established by Norton computing inside
Lemma 5.3. Let $O$ be the subgroup of $\hat{G}$ generated by the permutations

$$(1, 2)(3, 4)(5, 6)(7, 8), \ (1, 6)(2, 5)(3, 7)(4, 8), \ and \ (1, 3)(2, 4)(5, 8)(6, 7),$$

let $N := N_{A_8}(O)$, $S := \langle (9, 10)(11, 12) \rangle \times N$, and let $V_S$ be the subalgebra of $V$ generated by the Majorana axes $a_z$ such that $z \in T \cap S$. Then

1. $V_S$ has dimension 204,
2. $V_S$ is 2-closed,
3. a linearly independent set of generators for $V_S$ is given by

$$\mathcal{B} := \{ a_z \mid z \in T \cap S \} \cup \{ u_\rho \mid \rho \in (2, 8, 6)(4, 5, 7)^S \}.$$"
Lemma 6.1. We have
\[ W \subseteq \langle V^{(2A)}, V^{(2^2)} \rangle \]
and, by Lemma 5.3, we have
\[ V^{(2^2)} \subseteq \langle V^{(2A)}, V^{(3^2)} \rangle, \]
so the claim about \( V^o \) follows. Finally, by [7, Lemma 8], \( W^{(4A)} = \{0\} \) and the result for \( W^{o} \) also follows.

6. The submodule \( V^{(2A)} \)

In this section we determine the decomposition into irreducible submodules of \( V^{(2A)} \). Recall, from Section 2, that \( V^{(2A)} = (M_b + M_s)^{n} \).

Lemma 6.1. We have
\[ M_b = M_{b,1}^{(12)} \oplus M_{b,1}^{(11,1)} \oplus M_{b,1}^{(9,3)} \oplus M_{b,1}^{(9,2,1)} \oplus M_{b,1}^{(8,2^2)} \oplus M_{b,1}^{(10,2)} \oplus M_{b,2}^{(10,2)}, \]
(1) a generalised first eigenmatrix \( P(b) \), relative to the action of \( \hat{G} \) on \( X_b \) with respect to the decomposition of \( M_b \) in Equation (3), is given in Table 8 and
(2) \( \text{rad}(M_b) = M_{b,1}^{(9,2,1)} \oplus M_{b,2}^{(10,2)}, \) where \( M_{b,2}^{(10,2)} \) is a diagonal submodule of \( M_{b,1}^{(10,2)} \)

Proof. The first assertion follows from [7, Lemma 6], the generalised first eigenmatrix \( P(b) \) has been computed in [7, Tables 8 and 9], and the last assertion follows from [7, Theorem 2].

Lemma 6.2. We have
\[ \text{rad}(M_s) = M_{s,1}^{(6,4,2)} \oplus M_{s,1}^{(4^2,2^2)} \oplus M_{s,1}^{(4,2^4)} \oplus M_{s,1}^{(6^2)} \oplus M_{s,1}^{(2^6)}, \] and
(1) \( V^{(2^3)} \cong S^{(12)} \oplus S^{(10,2)} \oplus S^{(8,4)} \oplus S^{(8,2^2)} \oplus S^{(4^2)} \oplus S^{(6,2^3)}. \)
Proof. The decomposition of $M_\lambda$ as a direct sum of irreducible submodules is given in Lemma 3.2. By Lemma 2.2, a submodule $M_{s,1}^\lambda$ of $M_s$ is contained in $\text{rad}(M_s)$ if and only if $f_{s,1}^\lambda = 0$. The first assertion then follows from the formula for $f_{s,1}^\lambda$ given in Equation (3), using the values of the Majorana inner products given by the Norton-Sakuma Theorem (Table 1), and Table 4. Since $V(\varphi^2) \cong M_\lambda/\text{rad}(M_\lambda)$ (see Section 2), the second assertion also follows.

In the remainder of this paper we fix the following notation for vectors in $M_\lambda$, $x \in \{b, s, t\}$: suppose $H$ is a subgroup of $\hat{G}$ and $H_{1}^x, \ldots, H_{n_x}^x$ are the $H$-orbits on $X_x$, so that

\begin{equation}
H_x := \left( \sum_{v \in H_1^x} v, \ldots, \sum_{v \in H_{n_x}^x} v \right)
\end{equation}

is a basis for $C_{M_x}(H)$. For every $w \in C_{M_x}(H)$ let

$\bar{w} := (w_1^x, \ldots, w_{n_x}^x)^{H_x}$

be the $n_x$-tuple of the coefficients of $w$ written as a linear combination of the elements of the basis $H_x$.

Lemma 6.3. Let $x, y \in \{b, s, t\}$, let $H$ be a subgroup of $\hat{G}$, $H_1^x, \ldots, H_{n_x}^x$ be the $H$-orbits on $X_x$, $H_1^y, \ldots, H_{n_y}^y$ be the $H$-orbits on $X_y$, and $z_i \in H_i^x$ for every $i \in \{1, \ldots, n_x\}$. Let $u_x \in C_{M_x}(H)$ and $u_y \in C_{M_y}(H)$. Then, with the notation above,

$f(u_x, u_y) = \bar{u}_x D_x F(x, y) \bar{u}_y,$

where $D$ is the diagonal $n_x \times n_x$ matrix with $D_{ii} = |H_i^x|$ and $F(x, y)$ is the $n_x \times n_y$ matrix whose ij-entry is

$F(x, y)_{ij} = \sum_{w \in H_j^y} f(z_i, w).$

Proof. By the bilinearity and the $H$-invariance of $f$ and the definition of $F$, we have

\[
f(u_x, u_y) = f\left( \sum_{i=1}^{n_x} (u_i^x \sum_{v \in H_i^x} v), \sum_{j=1}^{n_y} (u_j^y \sum_{w \in H_j^y} w) \right) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} u_i^x u_j^y \sum_{v \in H_i^x} \sum_{w \in H_j^y} f(v, w) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} u_i^x u_j^y |H_i^x| \sum_{w \in H_j^y} f(z_i, w) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} u_i^x u_j^y |H_i^x| F(x, y) = \bar{u}_x D_x F(x, y) \bar{u}_y.
\]

Lemma 6.4. With respect to the decomposition in Equation (5), for $\lambda \in \{(12), (8, 4), (8, 2^2), (10, 2)\}$, we have

$\text{rad}(M_b \oplus M_s) \cap (M_{b,1}^\lambda \oplus M_{s,1}^\lambda) \cong S^\lambda.$
Proof. Keeping the notation of Section 4 let
\[ s_1 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) \] and let \( H := C_{S_{12}}(s_1) \).
For every \( \lambda \in \{(8,4), (8,2^2), (10,2)\} \), the module \( S^\lambda \) has multiplicity 1 in \( M_s \), hence, by the Frobenius Reciprocity Theorem,
\[ \dim(C_{S^\lambda}(H)) = 1. \]
Denote the orbits of \( H \) on the set \( X_s \) as follows:
\[ R_1 := ((9,10)(11,12))^H, \quad R_2 := ((9,11)(10,12))^H, \quad R_3 := ((8,9)(11,12))^H, \quad R_4 := ((8,9)(10,11))^H, \quad R_5 := ((6,7)(10,11))^H \]
and let \( R \) be the corresponding basis for \( C_{M_s}(H) \) as in Equation (12). Set
\[ u := \sum_{v \in R} v \]
Since \( u \) is \( H \)-invariant \( u^{\pi_{b,1}} \in C_{M_{b,1}}(H) \). By Equation (5) and Table 8 we get
\[
\begin{align*}
\overline{u^{(8,2^2)}} & = \begin{pmatrix} \frac{8}{15} & -\frac{4}{15} & -\frac{1}{15} & \frac{1}{30} & 0 \end{pmatrix}^R, \\
\overline{u^{(8,4)}} & = \begin{pmatrix} 16 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}^R, \\
\overline{u^{(10,2)}} & = \begin{pmatrix} 11 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 11 \end{pmatrix}^R.
\end{align*}
\]
Since \( s_1 \) is also \( H \)-invariant \( \pi_{b,1}^{s_1} \in C_{M_{b,1}}(H) \) for every \( \lambda \). Let
\[ S_i := \{ z \in X_s \mid (s_1, z) \in \Sigma_{i,s} \}, \]
so that \( S_1, \ldots, S_{11} \) are the orbits of \( H \) on the set \( X_s \) (see Table 3 and Lemma 3.3 for the definition of \( \Sigma_{i,s} \)). Let \( S \) be the corresponding basis of \( C_{M_s}(H) \) defined as in Equation (12). By Equation (5) and Table 4 we get
\[
\begin{align*}
\overline{\pi_{(8,2^2)}^{s_1}} & = \begin{pmatrix} 8 & 4 & 8 & 11 & -4 \\ 135 & 225 & 2025 & 1350 & 675 \\ 270 & 675 & 2025 & -2025 & 810 \\ 675 & 2025 & 2025 & 2 & 1 \end{pmatrix}^S, \\
\overline{\pi_{(8,4)}^{s_1}} & = \begin{pmatrix} 5 & 2 & 1 & 5 \ 189 & 252 & 378 & 756 & 1512 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}^S, \\
\overline{\pi_{(10,2)}^{s_1}} & = \begin{pmatrix} 2 & 9 & 8 & 1 \ 385 & 5775 & 5775 & 3850 & 1925 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}^S.
\end{align*}
\]
By Lemma 2.2 and Lemma 3.3 we get
\[
\begin{align*}
\det \begin{pmatrix} f(u^{(8,2^2)}_{b,1}, u^{(8,2^2)}_{b,1}) & f(u^{(8,2^2)}_{b,1}, s_1^{(8,2^2)}) \\ f(u^{(8,2^2)}_{b,1}, s_1^{(8,2^2)}) & f(s_1^{(8,2^2)}, s_1^{(8,2^2)}) \end{pmatrix} &= \\
\det \begin{pmatrix} 135/16 & -15/16 \\ -15/16 & 5/48 \end{pmatrix} &= 0.
\end{align*}
\]
\[
\begin{align*}
\det \begin{pmatrix} f(u^{(8,4)}_{b,1}, u^{(8,4)}_{b,1}) & f(u^{(8,4)}_{b,1}, s_1^{(8,4)}) \\ f(u^{(8,4)}_{b,1}, s_1^{(8,4)}) & f(s_1^{(8,4)}, s_1^{(8,4)}) \end{pmatrix} &= \\
\det \begin{pmatrix} f(u^{(8,4)}_{b,1}, u^{(8,4)}_{b,1}) & f(u^{(8,4)}_{b,1}, s_1^{(8,4)}) \\ f(u^{(8,4)}_{b,1}, s_1^{(8,4)}) & f(s_1^{(8,4)}, s_1^{(8,4)}) \end{pmatrix} &= 0.
\end{align*}
\]
Proposition 6.5. We have
\[ \det \begin{pmatrix} f(u^{(10,2)}_{k,1}, u^{(10,2)}_{k,1}) & f(u^{(10,2)}_{k,1}, s_{1}^{(10,2)}) \\ f(u^{(10,2)}_{k,1}, s_{1}^{(10,2)}) & f(s_{1}^{(10,2)}, s_{1}^{(10,2)}) \end{pmatrix} = \det \begin{pmatrix} 75/14 & 25/28 \\ 25/28 & 25/168 \end{pmatrix} = 0 \]
and again Lemma 2.5 yields the claim. □

Then, Lemma 6.1 yields the claim when \( \lambda \in \{(8, 4), (8, 2, 2), (10, 2)\} \). Finally, when \( \lambda = (12) \), the modules \( M_{b,1}^{(12)} \) and \( M_{s,1}^{(12)} \) have dimension 1 and they are generated by the vectors \( v_{1} \) and \( v_{2} \), where
\[ \bar{v}_{1} := (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^{R} \quad \text{and} \quad \bar{v}_{2} := (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^{S} \]
respectively. By Lemma 6.3 we get
\[ \det \begin{pmatrix} f(v_{1}, v_{1}) & f(v_{1}, v_{2}) \\ f(v_{2}, v_{1}) & f(v_{2}, v_{2}) \end{pmatrix} = \det \begin{pmatrix} 467775/8 & 3274425/8 \\ 3274425/8 & 22920975/8 \end{pmatrix} = 0, \]
and again Lemma 2.5 yields the claim. □

**Proposition 6.5.** We have
\[ V^{(2A)} \cong S^{(12)} \oplus S^{(10,2)} \oplus S^{(8,4)} \oplus S^{(8,2)} \oplus S^{(4)} \oplus S^{(6,2)} \oplus S^{(11,1)} \oplus S^{(9,3)}. \]

**Proof.** This follows from Lemma 6.1, Lemma 6.2, and Lemma 6.4. □

7. The module \( W^{(3^2)} \)

In this section, \( a, b \) will always denote a pair of disjoint 3-cycles in \( \hat{G} \). Let \( \bar{\beta} \) be the map
\[ \bar{\beta} : X_{t} \to X_{t}, \quad \langle ab \rangle \mapsto \langle ab^{-1} \rangle. \]
Since \( \bar{\beta} \) is an isomorphism of \( \hat{G} \)-sets, \( \beta \) induces an involutory \( \mathbb{R} \hat{G} \)-automorphism \( \beta \) on the module \( M_{t} \), which decomposes as
\[ M_{t} = C_{M_{t}}(\beta) \oplus [M_{t}, \beta] \]
and this decomposition projects into the decomposition
\[ W^{(3^2)} = W^{(3^2)^{+}} + W^{(3^2)^{-}}, \]
where
\[ W^{(3^2)^{+}} := \langle u_{ab} + u_{ab^{-1}} | a, b \text{ disjoint 3-cycles in } \hat{G} \rangle \]
and
\[ W^{(3^2)^{-}} := \langle u_{ab} - u_{ab^{-1}} | a, b \text{ disjoint 3-cycles in } \hat{G} \rangle. \]

By Pasechnik’s relation [19], Lemma 3.4, and [2], Lemma 3.1, we have
\[ W^{(3^2)^{+}} \leq \langle W^{(2^2)}, W^{(3)} \rangle \leq V^{(2A)}. \]
Since our ultimate goal is to determine \( V^{\circ} \) and \( W^{\circ} \) and the modules \( \langle W^{(2^2)}, W^{(3)} \rangle \) and \( V^{(2A)} \) have been determined in [9], Theorem 1.2, and in Proposition 6.5 respectively, by Proposition 5.3 it follows that we only need to consider the contribution of the module \( W^{(3^2)^{-}} \), which in turn is isomorphic to \( [M_{t}, \beta]/\text{rad}(M_{t}, \beta) \).

Let, for \( 8 \leq n \leq 12 \),
- \( \Omega_n^1, \ldots, \Omega_n^n \) be the orbitals of \( \hat{G} \) on \( X_t \), where \( r_n = 20, 25, 29, 30, 31 \) for \( n = 8, 9, 10, 11, 12 \) respectively,
- \( k_t(n) \) be the valencies of the action, that is \( k_t(n) := |\Omega_t^n|/|X_t| \). For \( n = 8, 10, 11, 12 \), they are displayed in the last three columns of Table 3,
- \( \varepsilon_i, i \in \{1, \ldots, 31\} \), as in Table 7 so that for each \( i, (\varepsilon_1, \varepsilon_i) \in \Omega_n^1 \), and
- \( \sigma_i, i \in \{1, \ldots, 31\} \), as in Table 9 so that for each \( i, \varepsilon_i^\sigma = \varepsilon_i \).

The module \( C_{M_t}(\beta) \) is isomorphic to the permutation \( \mathbb{R}[\hat{G}] \)-module associated to the action of \( \hat{G} \) on the set of conjugates of \((1, 2, 3, (4, 5, 6))\), and its decomposition into irreducible submodules can be easily computed with GAP 11. Comparing this decomposition with that of \( M_t \), given in 8 Theorem 1.1], we get that we may choose \( M_{t, 1}^{(n-4,2^2)} \) and \( M_{t, 2}^{(n-4,2^2)} \) such that

\[
[M_t, \beta] = M_{t, 1}^{(n-4,2^2)} \oplus M_{t, 1}^{(n-5,2^2,1)} \oplus M_{t, 1}^{(n-6,3^2)} \oplus M_{t, 1}^{(n-4,1^4)} \\
\oplus M_{t, 1}^{(n-5,2,1^3)} \oplus M_{t, 1}^{(n-5,1^4)} \oplus M_{t, 1}^{(n-6,2,1^4)}.
\]

We now have to determine which irreducible submodules of the decomposition in Equation (17) are contained into \( \text{rad}([M_t, \beta]) \). As in Section 6 we do this by computing the restrictions of the form \( f \) to the irreducible submodules of \([M_t, \beta]\) using Lemma 22.

For a partition \( \lambda \) of \( n \), denote by \( M^\lambda \) the permutation \( \mathbb{R}[\hat{G}] \)-module associated to the action of \( \hat{G} \) on the set of \( \lambda \)-tableaux, let \( \mathcal{B}_\lambda \) be the basis of \( M^\lambda \) consisting of all \( \lambda \)-tableaux and let \( \kappa_\lambda : M^\lambda \times M^\lambda \rightarrow \mathbb{R} \) be the \( \hat{G} \)-invariant non-degenerate symmetric bilinear form on \( M^\lambda \) such that \( \mathcal{B}_\lambda \) is an orthonormal basis with respect to \( \kappa_\lambda \). We shall use in the sequel the following well known fact from the representation theory of the symmetric groups (see e.g. 22 Theorem 8.15).

**Lemma 7.1.** For every partition \( \lambda \) of \( n \), let \( \lambda' \) denote the partition conjugate to \( \lambda \) and let \( A \) be a real vector space of dimension 1 generated by a vector \( a \) on which even permutations act trivially and odd permutations act as the multiplication by \(-1\). Then the \( \mathbb{R}[S_n] \)-modules \( S^\lambda \) and \( S^{\lambda'} \otimes A \) are isomorphic.

**Lemma 7.2.** The (relevant) values \((P(t)^\lambda)_{11}\) are listed in Table 10.

**Proof.** Let \( n \in \{8, \ldots, 12\} \). For an orbital \( \Omega_n^i \), let

\[
(\Omega_n^i)^\beta := \{(X^\beta, Y^\beta)|(X, Y) \in \Omega_n^i\}.
\]

Then, for \( i \in \{1, 3, 5, 7, 9, 11, 17, 19\}, \ (\Omega_n^i)^\beta = \Omega_n^{i+1} \) and, for \( i \in \{13, \ldots, 16\} \cup \{21, \ldots, 31\}, \ (\Omega_n^i)^\beta = \Omega_n^i \). By Lemma 24 it follows that every row

\[
(P(t)^\lambda)_{11}, \ldots, (P(t)^\lambda)_{11}
\]

of \( P(t) \) corresponding to the irreducible submodules of \([M_t, \beta]\) is of the form

\[
(1, -1, x_1, -x_1, x_2, -x_2, x_3, -x_3, x_4, -x_4, x_5, -x_5, 0, 0, 0, x_6, -x_6, x_7, -x_7, 0, \ldots, 0),
\]

whence, for each row, only the seven parameters \( x_1, \ldots, x_7 \) need to be computed.

Assume first that \( \lambda \neq (n-4,2^2) \). By 8 Theorem 1.1], the module \( S^\lambda \) has multiplicity 1 in \( M_t \), so, by 7 Lemma 3,

\[
(P(t)^\lambda)_{11} = \frac{\nu_\lambda(w, w^{\sigma_1})}{\kappa_\lambda(w, w)} k_t(n),
\]
where \( w \) is any non zero vector in \( C_{S^n}(N_G(⟨e_1⟩)) \) and \( e_1 = (1, 2, 3)(4, 5, 6) \), as in Table 7. Clearly the vector \( w \) can be chosen as a non zero sum of a \( N_G(⟨e_1⟩) \)-orbit of a polytabloid in the Specht module \( S^λ \) (resp. \( S^{λ'} \otimes A \)). On turn this polytabloid is obtained by a suitable \( λ \)-tableau (resp. a \( λ' \)-tableau) \( T \). So e.g. assume \( λ = (n - 4, 1^4) \). Set

\[
T := \begin{array}{c}
1
2
3
4
5
\end{array},
\]

let \( v \) be the polytabloid associated to the tableau \( T \), and

\[
w := \sum_{\sigma \in N_G(⟨e_1⟩)} v^\sigma.
\]

| \( \sigma_i \) | \( k_i(8) \) | \( k_i(10) \) | \( k_i(11) \) | \( k_i(12) \) |
|---|---|---|---|---|
| 1 | ( ) | 1 | 1 | 1 | 1 |
| 2 | (2, 3) | 1 | 1 | 1 | 1 |
| 3 | (3, 4) | 9 | 9 | 9 | 9 |
| 4 | (2, 4, 3) | 9 | 9 | 9 | 9 |
| 5 | (1, 4, 7) | 36 | 72 | 90 | 108 |
| 6 | (1, 4, 7)(2, 3) | 36 | 72 | 90 | 108 |
| 7 | (1, 2, 3, 7) | 12 | 24 | 30 | 36 |
| 8 | (1, 2, 7) | 12 | 24 | 30 | 36 |
| 9 | (3, 4)(1, 6, 7) | 72 | 144 | 180 | 216 |
| 10 | (1, 5, 3, 4, 7) | 72 | 144 | 180 | 216 |
| 11 | (1, 7)(4, 8) | 18 | 108 | 180 | 270 |
| 12 | (1, 7)(2, 3)(4, 8) | 18 | 108 | 180 | 270 |
| 13 | (1, 4, 7)(2, 5, 8) | 36 | 216 | 360 | 540 |
| 14 | (1, 4, 8)(2, 7) | 72 | 432 | 720 | 1080 |
| 15 | (1, 7)(2, 8) | 12 | 72 | 120 | 180 |
| 16 | (1, 4, 7)(5, 8) | 72 | 432 | 720 | 1080 |
| 17 | (1, 7)(3, 4)(6, 8) | 18 | 108 | 180 | 270 |
| 18 | (1, 7)(3, 4, 8, 6) | 18 | 108 | 180 | 270 |
| 19 | (1, 7)(3, 4, 8, 5) | 18 | 108 | 180 | 270 |
| 20 | (1, 7)(3, 4)(5, 8) | 18 | 108 | 180 | 270 |
| 21 | (1, 4, 8)(2, 7)(5, 9) | 864 | 2160 | 4320 | 6480 |
| 22 | (2, 8)(3, 9)(1, 6, 7) | 144 | 360 | 1080 | 2160 |
| 23 | (1, 7)(2, 9)(4, 8) | 432 | 1080 | 2160 | 4320 |
| 24 | (1, 7)(2, 8)(3, 9, 6) | 144 | 360 | 1080 | 2160 |
| 25 | (1, 7)(2, 8)(3, 9) | 16 | 40 | 80 | 160 |
| 26 | (1, 7)(2, 9)(4, 8)(5, 10) | 540 | 1080 | 1620 |
| 27 | (1, 4, 8)(2, 7)(5, 9)(6, 10) | 360 | 1080 | 1680 |
| 28 | (1, 5, 10)(2, 7)(3, 9)(4, 8) | 360 | 1080 |
| 29 | (1, 8)(2, 9)(3, 10)(4, 7) | 240 | 720 |
| 30 | (1, 9)(2, 10)(3, 11)(4, 7)(5, 8) | 120 | 720 |
| 31 | (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12) | 360 | 1080 |

Table 9. Set \( \Sigma \) and values of the valencies \( k_i(n) \).
Note that, since the numbers 6, ..., n do not appear in the lower rows of the summands of \( v \) and \( N_G(\langle e_1 \rangle) \) is the direct product of a subgroup fixing the set \( \{1, \ldots, 6\} \) with a subgroup fixing the set \( \{7, \ldots, n\} \), the numbers 7, ..., n do not appear in \( w \) as well. Thus \( w \) and the quotients

\[
\frac{\kappa_{(n-4,1^4)}(w, w^u)}{\kappa_{(n-4,1^4)}(w, w)}
\]

are constant for \( 7 \leq n \leq 12 \). We can, therefore, reduce computations assuming \( n = 8 \).

Similarly, assume \( \lambda = (2^2, 1^4) \). By Lemma 13.1 instead of working inside the module \( S^{(2^2, 1^4)} \), we can work inside the isomorphic module \( S^{(6, 2)} \otimes A \), which simplifies the computations. Consider the \( \lambda \)-tableau

\[
\begin{array}{c|cccccccccc}
\lambda & (2^2, 1^4) & (3, 2, 1^3) & (2^4) & (3, 1^5) & (6, 2^2, 1) & (5, 2^3) & (6, 2, 1^3) & (6, 1^5) & (5, 2, 1^4) \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
3 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & -3 & -3 \\
4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
5 & 0 & -8 & -6 & 12 & 0 & -6 & -16 & 24 & 6 \\
6 & -8 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
29 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Table 10. The values \( (P(t)_{\lambda})_{11} \) for the relevant partitions (columns correspond to the rows of a first eigenmatrix)

\[\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]
As in [7], to simplify the notation, here and in the sequel we shall write tabloids by substituting their first row by a bar. The polytabloid obtained from it is

\[ v := \overline{86 + 12} = 16 - \overline{82} \]

and we choose

\[ w := \sum_{\sigma \in NS_8((e_1))} (v \otimes a)^\sigma = (86 + \overline{85} + 84 + \overline{73} + \overline{72} + \overline{71} - \overline{76} - \overline{74} - \overline{83} - \overline{82} - \overline{81}) \otimes a. \]

For all partitions but \((7, 2^2)\) and \((8, 2^2)\) we proceed in the same way, choosing \(T\) as in the following table

| \(\lambda\) | \((n - 5, 2^2, 1)\) | \((n - 6, 2^3)\) | \((3, 2, 1^3)\) | \((n - 5, 2, 1^3)\) | \((n - 4, 1^4)\) | \((n - 5, 1^5)\) | \((n - 6, 2, 1^4)\) | \((n - 4, 2^2)\) |
|---|---|---|---|---|---|---|---|---|
| \(T\) | 1 4 8 \ldots n | 1 6 8 \ldots n | 3 6 7 | 2 5 9 4 6 7 1 5 2 | 2 7 6 5 4 3 2 5 | 2 7 6 5 4 3 4 5 | 2 7 6 5 4 3 4 5 7 | 2 7 6 5 4 3 7 n |

Once the tableau \(T\) (hence the vector \(w\)) has been chosen, Equation (13) and a straightforward computation give the corresponding columns of Table 10.

Assume now \(\lambda \in \{ (7, 2^2), (8, 2^2) \}\). In this case, we can’t use the above method since \(S^\lambda\) has multiplicity 2 in \(M_t\), so we need to proceed in a different way. The rows orthogonality relations (see [13 (3.8)] and [7 Lemma 1] between the row of the first eigenmatrix \(P(t)\) corresponding to \(M_{t,1}^\lambda\) and the rows corresponding to the submodules \(M_{t,1}^\mu\) for the partitions \(\mu\) in the following set

\{(n-5, 2^2, 1), (n-6, 2^3), (n-4, 1^4), (n-5, 2, 1^3), (n-5, 1^5), (n-6, 2, 1^4), (n-4, 2^2)\}

give, respectively for \(n = 11, 12\), the following quadratic systems of equations.

\[
\begin{pmatrix}
\frac{2}{5} & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} & -\frac{2}{3} & -\frac{1}{9} & \frac{1}{9} \\
\frac{2}{3} & -\frac{2}{15} & -\frac{2}{15} & -\frac{2}{15} & \frac{2}{27} & \frac{1}{15} & -\frac{1}{15} \\
-\frac{2}{3} & -\frac{2}{15} & \frac{2}{15} & \frac{2}{15} & -\frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\
-\frac{2}{3} & -\frac{2}{15} & -\frac{2}{15} & \frac{2}{15} & \frac{2}{27} & \frac{1}{15} & -\frac{1}{15} \\
-\frac{2}{3} & \frac{2}{15} & -\frac{2}{3} & \frac{2}{15} & \frac{2}{15} & -\frac{1}{15} & \frac{1}{15} \\
\frac{2}{5} x_1 & \frac{1}{15} x_2 & \frac{1}{15} x_3 & \frac{1}{90} x_4 & \frac{1}{90} x_5 & \frac{1}{90} x_6 & \frac{1}{90} x_7 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
\end{pmatrix}
= \begin{pmatrix}
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
\end{pmatrix}
\]
Let $R$ of Table 10. $M$ (resp. $S$) and, for $n \leq 30$ CLARA FRANCHI, ALEXANDER A. IVANOV, MARIO MAINARDIS

For $\lambda \in \{1, 11\}$, we obtain, for $i \in \{1, \ldots, 30\}$ (resp. $i \in \{1, \ldots, 31\}$) the values $(P(t_i^{(7,2^2)})_{11}$ (resp. $(P(t_i^{(8,2^2)})_{11}$) of Table 10.

**Remark.** Assume $M$ is an irreducible $\hat{G}$-submodule of $M_t$. Then, by Equation (2), either $M \leq \text{rad}(M_t)$ or $M \cap \text{rad}(M_t) = \{0\}$. We’ll use this fact together with the Branching Theorem (see [22, Theorem 9.2]) to reduce computations to $S_m$-subgroups of $\hat{G}$ for $8 < m < n$. So, e.g., by the Branching Theorem, the unique $\mathbb{R}[\hat{G}]$-submodule of $[M_t, \beta]$ that contains an $\mathbb{R}[S_8]$-submodule isomorphic to $S^{(2^7,1^4)}$ is $M_t^{(n-6,2,1^4)}$. Thus, $M_t^{(n-6,2,1^4)} \leq \text{rad}(M_t)$ if and only if $M_t^{(2^7,1^4)} \leq \text{rad}(M_t)$.

**Lemma 7.3.** For $n \in \{8, \ldots, 11\}$, we have
\[
\text{rad}(M_t, \beta) = M_t^{(n-5,2,1^3)} \oplus M_t^{(n-6,2,1^4)}
\]
and, for $n = 12$,
\[
\text{rad}(M_t, \beta) = M_t^{(7,2^2,1^3)} \oplus M_t^{(6,2^4,1^4)} \oplus M_t^{(8,1^4)} \oplus M_t^{(7,2^2,1)}.
\]

**Proof.** By Lemma 2.2, $M_t^{\lambda_1} \leq \text{rad}(M_t, \beta)$ if and only if $f_t^{\lambda_1} = 0$. Thus, Equation (9), Table 10 and Table 11 imply the following facts
(i) for $n \in \{11, 12\}$, the decomposition of $\text{rad}(M_t, \beta)$ is the one of the claim,
(ii) $M_t^{(n-4,1^4)} \leq \text{rad}(M_t, \beta)$ if and only if $n = 12$,
and, for $n = 8$,
(iii) $M_t^{(2^2,1^4)} \leq \text{rad}(M_t, \beta)$,
(iv) $M_t^{(2^4)} \nleq \text{rad}(M_t, \beta)$,
(v) $M_t^{(3,1^3)} \leq \text{rad}(M_t, \beta)$, and
(vi) $M_t^{(3^2,1^2)} \nleq \text{rad}(M_t, \beta)$.

Let $n \in \{9, 10\}$. By the Branching Theorem (see [22, Theorem 9.2]), the unique $\mathbb{R}[\hat{G}]$-submodule of $[M_t, \beta]$ that contains an $\mathbb{R}[S_8]$-submodule isomorphic to $S^{(2^7,1^4)}$ (resp. $S^{(2^7)}$) is $M_t^{(n-6,2,1^3)}$ (resp. $M_t^{(n-6,2^2)}$). By (iii), (iv), and the above remark we have
(vii) $M_t^{(n-6,2,1^3)} \leq \text{rad}(M_t, \beta)$ for every $n$, and
(viii) for \( n \in \{9, 10\} \), \( M_{t, 1}^{(n-6,2^2)} \not\subseteq \text{rad}([M_t, \beta]) \).

Similarly, the \( \mathbb{R}[S_6] \)-submodules of \([M_t, \beta]\) that contain an \( \mathbb{R}[S_8] \)-submodule isomorphic to \( S^{(3,2,1^3)} \) are \( M_{t, 1}^{(n-5,2,1^3)} \) and \( M_{t, 1}^{(n-6,2,1^4)} \). Then, (vi) and (vii) imply that \( M_{t, 1}^{(3,2,1^3)} \leq M_{t, 1}^{(4,2,1^3)} \) and so \( M_{t, 1}^{(4,2,1^3)} \not\subseteq \text{rad}([M_t, \beta]) \). Repeating the same argument also for \( n = 10 \), we get

(ix) for \( n \in \{9, 10\} \), \( M_{t, 1}^{(n-5,2,1^3)} \not\subseteq \text{rad}([M_t, \beta]) \).

Let \( n \in \{8, 9, 10\} \). By the Branching Theorem the submodule \( M_{t, 1}^{(5,2^2,1)} \) is contained either in \( M_{t, 1}^{(6,2^2,1)} \) or in \( M_{t, 1}^{(5,2^2,1)} \). Since, by (i) and (vii), the latter submodules are not contained in \( \text{rad}([M_t, \beta]) \), by the above remark, \( M_{t, 1}^{(5,2^2,1)} \not\subseteq \text{rad}([M_t, \beta]) \) either. The same argument for \( n = 9 \) and \( n = 8 \) gives

(x) for \( n \in \{8, \ldots, 11\} \), \( M_{t, 1}^{(n-5,2,1^3)} \not\subseteq \text{rad}([M_t, \beta]) \).

Finally, let \( n = 10 \). By (i), for \( n = 11 \), \( \text{rad}([M_t, \beta]) = M_{t, 1}^{(6,2,1^3)} \oplus M_{t, 1}^{(5,2,1^4)} \). By the Branching Theorem, none of its irreducible submodules contains an \( \mathbb{R}[S_{10}] \)-module isomorphic \( S^{(6,2,2)} \), whence \( M_{t, 1}^{(6,2^2)} \not\subseteq \text{rad}([M_t, \beta]) \). By repeating this argument when \( n \in \{8, 9\} \) we finally get

(xi) for \( n \in \{8, \ldots, 10\} \), \( M_{t, 1}^{(n-4,2^2)} \not\subseteq \text{rad}([M_t, \beta]) \).

Proposition 7.4. For \( 8 \leq n \leq 11 \),

\[
W^{(3^2)^-} \cong S^{(n-4,2^2)} \oplus S^{(n-5,2^2,1)} \oplus S^{(n-6,2^2)} \oplus S^{(n-4,1^4)} \oplus S^{(n-5,1^5)},
\]

while, for \( n = 12 \),

\[
W^{(3^2)^-} \cong S^{(6,2^3)} \oplus S^{(6,2^3)} \oplus S^{(7,1^5)}.
\]

Proof. The claim follows from Equation (17) and Lemma 7.3.\( \Box \)

In the sequel we’ll denote by \( V_1, V_2, \) and \( V_3 \) the \( \mathbb{R}[\mathcal{G}] \)-submodules \((M_{t, 1}^{(n-4,2^2)})^\pi, (M_{t, 1}^{(n-6,2^2)})^\pi, \) and \((M_{t, 1}^{(n-5,1^5)})^\pi\) of \( W^{(3^2)^-}\), which are isomorphic to \( S^{(n-4,2^2)}, S^{(6,2^3)}, \) and \( S^{(7,1^5)} \), respectively.

8. INTERSECTIONS \( W^{(2A)} \cap W^{(3^2)} \) AND \( V^{(2A)} \cap W^{(3^2)} \)

In this section we determine the intersections \( W^{(2A)} \cap W^{(3^2)} \) and \( V^{(2A)} \cap W^{(3^2)} \). We keep the notation of Sections 6 and 7. By the remark at the beginning of Section 7 we need only to determine the intersections

\[
W^{(2A)} \cap W^{(3^2)^-} \text{ and } V^{(2A)} \cap W^{(3^2)^-}.
\]

Comparing the decompositions in Proposition 7.4, Proposition 6.5, and Theorem 2 gives, for \( n = 12 \),

\[
V^{(2A)} \cap W^{(3^2)^-} \leq V_1 \oplus V_2
\]

and, for \( 8 \leq n \leq 12 \),

\[
W^{(2A)} \cap W^{(3^2)^-} \leq U,
\]

where \( U \) is the irreducible submodule of \( W^{(3^2)^-} \) isomorphic to \( S^{(n-4,2^2)} \).

Consider first the intersection \( V^{(2A)} \cap W^{(3^2)^-} \).
Lemma 8.1. For \( n = 12 \), we have \( V_1 \oplus V_2 \leq V^{(2A)} \).

Proof. Keeping the notation of Section 4 let
\[
s_1 = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)\)
and let \( H := C_S(s_1) \).

For every \( \lambda \in \{(8, 2^2), (6, 2^3)\} \), the module \( S^\lambda \) has multiplicity 1 in \( M_s \), hence, by the Frobenius Reciprocity Theorem,
\[
\dim(C_{S^\lambda}(H)) = 1.
\]

For \( i \in \{1, \ldots, 11\} \), let \( c_i \) be as in Table 3 denote by \( P_i \) the \( H \)-orbit \( \{(c_i)^h | h \in H\} \), and let \( P \) be the corresponding basis of \( C_{M_s}(H) \) as in Equation (12). Set
\[
u := \sum_{v \in P} v.
\]

Since \( u \) is \( H \)-invariant, we have \( u_{\pi^1} \in C_{M_s}(H) \). By Equation (5) and Table 10 we get
\[
u_{\pi^1} = \begin{pmatrix} 0, 1, -\frac{1}{6}, 1, 0, -\frac{1}{18}, 0, 0, 0 \end{pmatrix}^T,
\]
\[
u_{\pi^2} = \begin{pmatrix} 0, 16, -\frac{16}{5}, 0, 0, -\frac{8}{15}, 0, 0, 0 \end{pmatrix}^T.
\]

Since \( s_1 \) is also \( H \)-invariant \( s_1^{\pi^1} \in C_{M_s}(H) \) for every \( \lambda \). Let \( S_1, \ldots, S_{11} \) be the orbits \( H \) on the set \( X_s \) as in Equation (13) and let \( S \) be the corresponding basis of \( C_{M_s}(H) \) defined as in Equation (12). By Equation (5) and Table 3 we get
\[
s_1^{\pi^1} = \begin{pmatrix} 8, 4, 11, -4, 1, 2, -2, 1 \end{pmatrix},
\]
\[
s_1^{\pi^2} = \begin{pmatrix} 27, 0, 7, 1, 1, 1, 5, 1 \end{pmatrix}.
\]

By Lemma 2.2, Lemma 6.3, and Lemma 1.1 we get
\[
\det\left(\begin{array}{c} f(s_1^{\pi^1}, s_1^{\pi^1}) & f(s_1^{\pi^1}, s_1^{\pi^2}) \\ f(s_1^{\pi^1}, s_1^{\pi^2}) & f(s_1^{\pi^2}, s_1^{\pi^2}) \end{array}\right) = \begin{pmatrix} 320 & -10 \\ -10 & 5 \end{pmatrix} = 0
\]
and
\[
\det\left(\begin{array}{c} f(s_1^{\pi^1}, s_1^{\pi^1}) & f(s_1^{\pi^1}, s_1^{\pi^2}) \\ f(s_1^{\pi^1}, s_1^{\pi^2}) & f(s_1^{\pi^2}, s_1^{\pi^2}) \end{array}\right) = \begin{pmatrix} 25 & -256 \\ -256 & 1048576 \end{pmatrix} = 0
\]
whence the claim, by Lemma 2.5.

□

Proposition 8.2. \( V^{(2A)} + W^{(3^2)} = V^{(2A)} \oplus V_3 \), where \( V_3 \cong S^{(7, 1^5)} \).

Proof. Since by [1] Lemma 4.13, \( W^{(3^2)} \) is not contained in \( V^{(2A)} \), the claim follows from Equation (10), Proposition 7.4 and Lemma 8.1.
We now turn to the intersection $W^{(2A)} \cap W^{(3^2)}$, for $8 \leq n \leq 12$. As in Section 4 let $r_1 = (1, 2) (3, 4)$ and set $K := \mathbb{C}_{S_4}(r_1)$. The module $S^{(n-4,2^2)}$ has multiplicity 1 in $M_b$, hence, by the Frobenius Reciprocity Theorem, $\dim(C_{S^{(n-4,2^2)}}(K)) = 1$. For $i \in \{1, \ldots, 10\}$, let

$Q_i := \{ z \in \mathcal{X}_b \mid (r_1, z) \in \Sigma_{i,b} \}$,

so that $Q_1, \ldots, Q_{10}$ are the orbits of $K$ on the set $\mathcal{X}_b$. For $i \in \{1, \ldots, 13\}$, let $d_i$ be as in Table 8, $N_i := \{(d_i)^k \mid k \in K\}$, and let $Q$ and $N$ be the corresponding bases for $C_{M_b}(K)$ and $C_{M_i}(K)$ as in Equation (12), respectively.

**Proposition 8.3.** For $n \in \{8, \ldots, 10\}$, we have

$W^{(2A)} \cap W^{(3^2)} = 0$

and, for $n \in \{11, 12\}$,

$W^{(2A)} \cap W^{(3^2)} \cong S^{(n-4,2^2)}$.

**Proof.** As already observed, $W^{(2A)} \cap W^{(3^2)} \leq U$, where $U$ is the submodule of $W^{(3^2)}$ isomorphic to $S^{(n-4,2^2)}$, so we only need to check for which $n$ the module $U$ is contained in $W^{(2^2)}$, or, equivalently, for which $n$

$$\text{rad}(M_{b,1}^{(n-4,2^2)} + M_{t,1}^{(n-4,2^2)}) \neq 0.$$ 

If $n = 12$, then by Proposition 6.5 and Proposition 5.2, $S^{(8,2^2)}$ has multiplicity 1 in $V^{(2A)} + W^{(3^2)}$. Since $W^{(2A)} + W^{(3^2)} \leq V^{(2A)} + W^{(3^2)}$ we get the claim.

Assume $n = 11$. Set

$u_{11} := \sum_{v \in N_{11}} v$.

Then we have

$\frac{\pi^{(7,2^2)}_{b,1}}{u_{11}^{\pi^{(7,2^2)}_{b,1}}} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, -12)^{N} \in C_{M_{t,1}}^{(7,2^2)}(K)$.

From Table 8], we get

$\frac{r^{(7,2^2)}_{1,1}}{\pi^{(7,2^2)}_{b,1}} = \left(1, \frac{1}{2}, \frac{1}{7}, \frac{1}{4}, \frac{1}{14}, \frac{1}{21}, \frac{1}{42}, \frac{1}{42}, \frac{1}{84}, 0, 0\right)^Q \in C_{M_{k,1}}^{(7,2^2)}(K)$.

By Lemma 2.2 Lemma 8 in [7], Lemma 4.2 and Lemma 6.3 we get

$\det \begin{pmatrix} f(r_{1,1}^{(7,2^2)}, r_{1,1}^{(7,2^2)}) & f(r_{1,1}^{(7,2^2)}, t_{1,1}^{(7,2^2)}) \\ f(r_{1,1}^{(7,2^2)}, r_{1,1}^{(7,2^2)}) & f(r_{1,1}^{(7,2^2)}, u_{11}^{(7,2^2)}) \end{pmatrix} = \det \begin{pmatrix} 1215 & 216 \\ 216 & \frac{80016}{5} \end{pmatrix} = 0$.

Hence, by Lemma 2.3 $\text{rad}(M_{b,1}^{(7,2^2)} + M_{t,1}^{(7,2^2)})$ contains a submodule isomorphic to $S^{(7,2^2)}$. When $n = 10$ we proceed as above setting

$u_{10} := \sum_{v \in N_{10}} v$.

We have

$\frac{\pi^{(6,2^2)}_{b,1}}{u_{10}^{\pi^{(6,2^2)}_{b,1}}} = \left(0, 0, 0, 0, 0, 0, 0, 0, 28, -28, \frac{28}{3}, -\frac{28}{3}\right)^N \in C_{M_{t,1}}^{(6,2^2)}(K)$. 

From [7] Table 8 we get
\[
\bar{r}_{1,1}^{(6,2^2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 12 & 15 & 15 & 15 & 15 & 15 & 15 \\
30 & 30 & 30 & 30 & 30 & 30 & 30 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} ^{\mathbb{Q}} \in C_{M_{1,1}^{(6,2^2)}}(K).
\]
By Lemma 2.2, Lemma 8 in [7], Lemma 4.2, and Lemma 6.3 we get
\[
\det \left( \begin{pmatrix} f(r_{1,1}^{(6,2^2)}, r_{1,1}^{(6,2^2)}) \\
(f(r_{1,1}^{(6,2^2)}, r_{1,1}^{(6,2^2)})) \\
(f(r_{1,1}^{(6,2^2)}, u_{10}^{(6,2^2)})) \\
(f(r_{1,1}^{(6,2^2)}, u_{10}^{(6,2^2)})) \\
\end{pmatrix} \right) = \det \left( \begin{pmatrix} 189 \\
392 \\
64 \\
392 \\
200704 \\
\end{pmatrix} \right) = 43904.
\]
Therefore, \(\text{rad}(M_{1,1}^{(6,2^2)} + M_{1,1}^{(6,2^2)}) = 0\). The remaining cases follow using the Branching Theorem as in the proof of Lemma 7.3.

\[
\text{Proof of Theorem 4} \quad \text{The claim follows from Proposition 5.4, Equation 16, Proposition 5.8, and Lemma 1.2.}
\]

9. PROJECTION ON THE IRREDUCIBLE SUBMODULE \( V_3 \)

Let \( V_3 \) be the irreducible \( \mathbb{R}[\tilde{G}] \)-submodule of \( V \) isomorphic to \( S^{(7,1^5)} \), as defined after Proposition 7.3. In this section we study the behaviour of the \( 3 \)-axes of type \( 2 \) under the projection \( V^{(2A)} \oplus V_3 \rightarrow V_3 \). Our aim is to take advantage of the smaller dimension of \( V_3 \) and find certain dependence relations between these axes that will be needed in Section 10. To do that, we need to express the projections of the \( 3 \)-axes in terms of the basis of \( V_3 \) given by the images of polytabloids via an isomorphism \( S^{(n-5,1^5)} \rightarrow V_3 \). We use the following elementary result.

**Lemma 9.1.** Assume \( M \) and \( N \) are isomorphic irreducible \( \mathbb{R}[\tilde{G}] \)-modules. Let \( R \) be a subgroup of \( \tilde{G} \) such that \( \dim \mathbb{C}_M(R) = 1 \). Then, for every \( m \in \mathbb{C}_M(R) \setminus \{0\} \) and every \( n \in \mathbb{C}_N(R) \setminus \{0\} \) there is a unique isomorphism of \( \mathbb{R}[\tilde{G}] \)-modules between \( M \) and \( N \) sending \( m \) to \( n \).

We keep the notation of the previous sections. For \( n \in \{8, \ldots, 12\} \) and \( x = t \), we choose the decomposition in Equation (3) in such a way that Equation (17) is satisfied. By Equation (14) it follows that
\[
C_{M_t}^{(n-5,1^5)}(\beta) \leq \ker(\pi_{t,1}^{(n-5,1^5)}),
\]
where \( \pi_{t,1}^{(n-5,1^5)} : M_t \rightarrow M_{t,1}^{(n-5,1^5)} \) is the canonical projection defined in Equation (4).

Let \( e_1 = (1, 2, 3)(4, 5, 6) \), as defined in Section 4, and let \( R := N_{\tilde{G}}(e_1) \). By the Frobenius Reciprocity Theorem, \( \dim(C_{S^{(n-5,1^5)}}(R)) = 1 \). In order to find a non-zero element \( w_n \) in \( C_{S^{(n-5,1^5)}}(R) \), we proceed in a similar way as in the proof of Lemma 7.2. Here it is convenient to use the isomorphism \( S^{(n-5,1^5)} \cong S^{(6,1^{n-6})} \otimes A \), defined in Lemma 7.1, and work inside the latter module. Set
\[
\begin{aligned}
T_n := & \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 7 \\
6 \\
\vdots \\
n \\
\end{bmatrix} \\
\end{aligned}
\]
and let $e_{T_n}$ be the polytabloid associated to $T_n$. Set

$$w_7 := \sum_{\sigma \in N_{S_2}(\langle e_1 \rangle)} (e_{T_n} \otimes a)^\sigma$$

and, for $n \in \{8, \ldots, 12\}$,

$$w_n := \sum_{i=7}^n w_i^{(7,i)}.$$

Then $w_n \in C_{S(6,1^{n-6})} \otimes A(R)$. Since $M_{1,1}^{(n-5,1^5)}$ and $S^{(6,1^{n-6})} \otimes A$ are isomorphic irreducible $\mathbb{R}[\hat{G}]$-modules and $((e_1))^{(n-5,1^5)}_{1,1} \in C_{M_{1,1}^{(n-5,1^5)}}(R)$, by Lemma [9.1] there is a unique isomorphism of $\mathbb{R}[\hat{G}]$-modules

$$(21) \quad \zeta: M_{1,1}^{(n-5,1^5)} \rightarrow S^{(6,1^{n-6})} \otimes A$$

sending $((e_1))^{(n-5,1^5)}_{1,1}$ to $w_n$.

**Lemma 9.2.** The following assertions hold:

(1) for $n = 8$, $S^{(6,1^7)} \otimes A$ is linearly generated by the set

$$G_8 := \{w^g_8 | g \in S_8 \text{ and } ((e_1), (e_1^g)) \in \Omega^8_1, i \in \{1, 3, 5, 7, 10\}\},$$

in particular the elements $w(8)(1,7)(4,8)$ and $w(8)(1,4,7)(2,5,8)$ of $S^{(6,1^7)} \otimes A$ can be written as linear combinations of the elements of $G_8$ as follows

$$w(8)(1,7)(4,8) = -w_8 + w(8)(1,4,7) + w(8)(1,4,8) - w(8)(1,2,3,7) + w(8)(4,5,6,7)$$

and

$$w(8)(1,4,7)(2,5,8) = w(8)(3,4) - w(8)(3,6) - w(8)(1,4) + w(8)(1,4,7) + w(8)(1,5,7) - w(8)(1,6,7)$$

$$-w(8)(2,4,7) - w(8)(2,5,7) + w(8)(1,2,3,7) - w(8)(1,7,2,3) + w(8)(1,5,3,4,8);$$

(2) for $n = 9$, $S^{(6,1^8)} \otimes A$ is linearly generated by the set

$$G_9 := \{w^g_9 | g \in S_9 \text{ and } ((e_1), (e_1^g)) \in \Omega^9_1, i \in \{1, 3, 5, 7, 11, 19\}\};$$

(3) for $n = 10$, $S^{(6,1^9)} \otimes A$ is linearly generated by the set

$$G_{10} := \{w^g_{10} | g \in S_{10} \text{ and } ((e_1), (e_1^g)) \in \Omega^{10}_1, i \in \{1, 5, 7, 11, 22, 23, 24, 25, 26, 27, 28\}\}.$$

**Proof.** (1) $S^{(6,1^7)} \otimes A$ has dimension 21 and

$$w_8 = [6 \quad 8 \quad 8 \quad 7 \quad 7 \quad 8 \quad 1 \quad 7 \quad 6 \quad 6 \quad 7 \quad 8 \quad 8 \quad 1 \quad 7 \quad 6 \quad 6 \quad 7 \quad 8 \quad 1 \quad 7 \quad 6 \quad 6 \quad 7 \quad 8 \quad 1\] \otimes a.$$

If we set

$$S_8 := \{(1,3,4,3,5,3,6),(1,4),(1,4,7),(1,5,7),(1,6,7),(1,4,8),(1,5,8),(1,6,8),(2,4,7),$$

$$2,5,7),(2,6,7),(2,4,8),(2,6,8),(3,4,7),(3,5,7),(3,6,7),(3,4,8),(3,5,8),(3,6,8),(1,2,3,7),$$

$$1,2,3,8),(1,7,2,3),(1,8,2,3),(1,2,7,3),(1,2,8,3),(1,2,8,3),(1,2,8,3),(1,2,8,3),(1,2,8,3),(1,2,8,3),(1,2,8,3),$$

$$1,4,5,6,7),(4,5,6,8),(4,7,5,6),$$

then, for every $g \in S_8$, $((e_1), (e_1^g)) \in \Omega^8_1$ with $i \in \{1, 3, 5, 7, 10\}$ and the tuple

$$(w^g_8)_{g \in S_8}$$

is a basis for the module $S^{(6,1^7)} \otimes A$. The expressions for $w(8)(1,7)(4,8)$ and $w(8)(1,4,7)(2,5,8)$ are easily found.
(2) \(S^{(6,13)} \otimes A\) has dimension 56. As in the previous case, one can easily write explicitly the vector \(w_9\) and check that we get a basis for the module \(S^{(6,13)} \otimes A\) by taking the tuple \((w_{9}^{g})_{g \in S_9}\), where

\[S_{9} := \{(1), (3, 4, 5), (3, 6), (1, 4), (1, 4, 7), (1, 5, 7), (1, 6, 7), (1, 4, 5), (1, 5, 8), (1, 6, 8), (1, 4, 9), (2, 4, 7), (2, 5, 7), (2, 6, 7), (2, 4, 8), (2, 5, 8), (2, 6, 8), (2, 4, 9), (2, 5, 9), (2, 6, 9), (3, 4, 7), (3, 5, 7), (3, 6, 7), (3, 4, 8), (3, 5, 8), (3, 6, 8), (3, 4, 9), (3, 5, 9), (3, 6, 9), (1, 2, 3, 7), (1, 2, 3, 8), (1, 7, 2, 3), (1, 8, 2, 3), (1, 9, 2, 3), (1, 2, 7, 3), (1, 2, 8, 3), (1, 2, 9, 3), (4, 5, 6, 7), (4, 5, 6, 8), (4, 5, 6, 9), (4, 7, 5, 6), (4, 8, 5, 6), (4, 5, 7, 6), (1, 7)(5, 9), (1, 8)(4, 7), (1, 9)(5, 8), (2, 7)(4, 8), (2, 7)(5, 9), (2, 8)(4, 7), (2, 8)(5, 9), (3, 7)(5, 9), (1, 8, 2, 6)(4, 9), (2, 9)(1, 4, 8, 5), (3, 9)(2, 5, 8, 6).

Moreover, for every \(g \in S_{9}\), \((e_{1}^{g}), (e_{2}^{g})\) \(\in \Omega_{1}^{10}\) for some \(i \in \{1, 3, 5, 7, 11, 19\}\).

(3) \(S^{(6,14)} \otimes A\) has dimension 126. Using GAP [13], one can check that the tuple \((w_{9}^{g})_{g \in S_{10}}\) is a basis for the module \(S^{(6,14)} \otimes A\), where

\[S_{10} := \{(1), (3, 4, 5, 6, 7), (4, 5, 6, 9, 8, 7), (4, 5, 6, 10, 9, 8, 7), (4, 5, 7, 6), (4, 5, 8, 7, 6), (4, 5, 9, 8, 7, 6), (4, 5, 10, 9, 8, 7, 6), (4, 7)(5, 8)(6, 9), (4, 7)(5, 8)(6, 10, 9), (4, 7)(5, 6, 9, 10, 8), (4, 8, 5, 9, 6, 10, 7), (4, 9, 8, 7, 5, 6), (4, 10, 9, 8, 7, 5, 6), (3, 4), (3, 7, 4)(5, 8)(6, 10, 9), (3, 7, 4)(5, 9, 6, 10, 8), (3, 7, 4)(5, 8, 9), (3, 7)(5, 9), (3, 7)(5, 9, 8, 5, 4)(6, 9), (3, 8, 5, 9, 6, 10, 7, 4), (3, 8, 9, 5, 7), (3, 8, 7)(5, 9), (3, 9, 6, 8, 5, 7, 4), (3, 9, 6, 10, 8, 5, 7, 4), (3, 9, 6, 10, 7, 4)(5, 8), (3, 9, 5, 7), (3, 9, 5, 8, 7), (3, 10, 9, 8, 5, 7, 4), (2, 4, 7), (2, 4, 8, 7), (2, 4, 9, 8, 7), (2, 4, 10, 9, 8, 7), (2, 5, 7), (2, 5, 8, 7), (2, 5, 9, 8, 7), (2, 6, 7), (2, 6, 8, 7), (2, 6, 9, 8, 7), (2, 6, 9)(4, 7)(5, 8), (2, 6, 10, 8, 9, 7)(5, 8), (2, 6, 10, 8, 5, 9)(4, 7), (2, 6, 10, 7, 4, 8, 5, 9), (2, 7, 4)(5, 8)(6, 9), (2, 7, 4)(5, 8)(6, 10, 9), (2, 7, 4)(5, 8, 6, 10, 9), (2, 7)(5, 8, 9), (2, 7)(5, 9), (2, 7)(4, 8), (2, 7)(4, 9, 8), (2, 7)(3, 8, 4, 6, 9), (2, 7)(3, 8, 6, 10, 9), (2, 7)(3, 8, 5, 9, 4)(6, 9), (2, 7)(3, 8, 4, 9, 7), (2, 7)(3, 8, 4, 10, 9), (2, 7)(3, 9)(6, 8), (2, 7)(3, 9)(6, 10, 8), (2, 7)(3, 9)(5, 8), (2, 7)(3, 9)(5, 10, 8), (2, 7)(3, 9)(4, 8), (2, 7)(3, 9)(4, 10, 8), (2, 7)(3, 10, 9)(6, 8), (2, 7)(3, 10, 9)(5, 8), (2, 7)(3, 10, 9)(4, 8), (2, 8, 5, 9, 6, 10, 7, 4), (2, 8, 9, 5, 7), (2, 8, 7)(5, 9), (2, 8, 4, 9, 7), (2, 8, 6, 7)(3, 9), (2, 8, 6, 10, 7)(3, 9), (2, 8, 5, 7)(3, 9), (2, 8, 5, 10, 7)(3, 9), (2, 8, 4, 7)(3, 9), (2, 8, 4, 10, 7)(3, 9), (2, 8, 6, 7)(3, 10, 9), (2, 9, 5, 7), (2, 9, 5, 8, 7), (2, 9, 3, 10, 8, 6, 7), (1, 4, 7), (1, 4, 8, 7), (1, 4, 9, 8, 7), (1, 4, 8)(2, 7)(5, 9), (1, 4, 8)(2, 7)(5, 10, 9), (1, 4, 9, 5, 10, 8)(2, 7), (1, 4, 9, 3, 8, 2, 7), (1, 4, 10, 9, 3, 8, 2, 7), (1, 4, 10, 8, 2, 7)(3, 9), (1, 4, 7, 2, 8)(5, 9, 4)(5, 10, 9), (1, 4, 9, 5, 10, 7, 2, 8), (1, 4, 10, 7)(2, 8)(3, 9), (1, 4, 7, 2, 9, 5, 8, 5, 8), (1, 4, 7, 2, 9, 5, 10, 8), (1, 4, 8)(2, 9, 5, 10, 7), (1, 4, 7, 2, 10, 9, 5, 8), (1, 5, 9, 3, 8, 2, 7), (1, 6, 9)(4, 7)(5, 8), (1, 6, 10, 9)(4, 7)(5, 8), (1, 6, 10, 8, 5, 9)(4, 7), (1, 7)(5, 8, 9), (1, 7)(5, 9), (1, 7, 5, 9, 2, 8, 4), (1, 7)(2, 8, 6, 9), (1, 7)(2, 8, 4, 9), (1, 9)(2, 7)(5, 8)(6, 10), (2, 5, 8)(3, 7)(4, 10)(6, 9).\)

Moreover, for every \(g \in S_{10}\), \((e_{1}^{g}), (e_{2}^{g})\) \(\in \Omega_{1}^{10}\) for some \(i \in \{1, 5, 7, 11, 22, 23, 24, 25, 26, 27, 28\}\).

For a finite set \(\Phi\) of natural numbers we denote by \(A_{\Phi}\) the alternating group on \(\Phi\).

**Proposition 9.3.** In the algebra \(V\) the following assertions hold:
Proof. \[\pi\] Thus, the map \(W\) Composing this map with the isomorphism \(u\) mapping \(\nu\) The claims then follow from Lemma 9.2 when we consider the map \(W\)tion 8.2, \(i\)

\[(1) \text{ For every } i \in \{11, \ldots, 20\}, \text{ the vector } u_{e_i} \text{ is a linear combination of Majo-
ranas and 3-axes } u_{\rho} \text{ of type } 3^2 \text{ such that } \rho \text{ is contained in } A_{\{1, \ldots, 6, 7\}} \cup \ A_{\{1, \ldots, 6, 7\}}. \text{ In particular}
\]

\[u_{e_{11}} \in -u_{\rho_1} + u_{\rho_2} + u_{\rho_5} - u_{\rho_4} + u_{\rho_6} + V(2A),\]

and
\[u_{e_{13}} \in u_{\rho_2} + u_{\rho_4} - u_{\rho_7} - u_{\rho_8} + u_{\rho_9} - u_{\rho_{11}} - u_{\rho_{12}} - u_{\rho_{13}} + u_{\rho_{14}} + V(2A).
\]

where

\[\rho_1 = e_1, \rho_2 = e_5, \rho_3 = e_3, \rho_4 = e_7, \rho_5 = (2, 3, 4)(5, 6, 8), \rho_6 = (1, 2, 3)(5, 6, 7), \rho_7 = (1, 2, 6)(3, 4, 5), \rho_8 = (1, 5, 6)(2, 3, 4), \rho_9 = (2, 3, 5)(4, 7, 6), \rho_{10} = (2, 3, 6)(4, 5, 7), \rho_{11} = (1, 4, 3)(5, 6, 7), \rho_{12} = (1, 5, 3)(4, 7, 6), \rho_{13} = (1, 7, 3)(4, 5, 6), \rho_{14} = (2, 4, 5)(3, 6, 8).
\]

\[(2) \text{ For every } i \in \{21, \ldots, 25\}, \text{ the vector } u_{e_i} \text{ is a linear combination of Majo-
ranas and 3-axes } u_{\rho} \text{ of type } 3^2 \text{ such that } \rho \text{ is contained in}
\]

\[\bigcup_{k,l \in \{7, 8, 9\}} A_{\{1, \ldots, 6, k, l\}}.\]

\[(3) \text{ For every } i \in \{26, \ldots, 29\}, \text{ the vector } u_{e_i} \text{ is a linear combination of Majo-
ranas and 3-axes } u_{\rho} \text{ of type } 3^2 \text{ such that } \rho \text{ is contained in}
\]

\[\bigcup_{k,l,m \in \{7, 8, 9, 10\}} A_{\{1, \ldots, 6, k, l, m\}}.\]

Proof. By Equation (11), Equation (20), and Lemma 3.3 for every \(n \in \{8, \ldots, 12\}, \)

\[\text{rad}(M_t) \leq \ker \pi^{(n-5, 1^5)}_{t, 1}.
\]

Thus, the map \(\pi^{(n-5, 1^5)}_{t, 1}\) induces a well defined homomorphism

\[M_t/\text{rad}(M_t) \rightarrow M^{(n-5, 1^5)}_{t, 1}.
\]

Composing this map with the isomorphism \(W(3^2) \cong M_t/\text{rad}(M_t), \) we get a well defined homomorphism

\[\nu_n: \quad W(3^2) \rightarrow M^{(n-5, 1^5)}_{t, 1}.
\]

mapping \(u_{\rho}\) to \(\rho^{(n-5, 1^5)}_{t, 1}, \) for every permutation \(\rho\) of cycle type \(3^2. \) By Proposition 8.2 \(W(3^2) \cap V(2A)\) contains no submodule isomorphic to \(M^{(n-5, 1^5)}_{t, 1}, \) so

\[W(3^2) \cap V(2A) \leq \ker \nu_n.
\]

The claims then follow from Lemma 0.2 when we consider the map \(\nu_n \circ \zeta. \) \(\square\)
10. Closure of $V^\circ$

In this section we prove that $V^\circ$ is closed under the algebra product.

**Lemma 10.1.** Let $z$ be an element in $X_b \cup X_6$, let $\sigma$ be a permutation of type $3^2$, and $a, b$ disjoint 3-cycles in $A_{12}$. Then, the following assertions hold:

1. $a_z \cdot u_{ab} \in V^\circ$ if and only if $a_z \cdot u_{ab-1} \in V^\circ$.
2. $u_\sigma \cdot u_{ab} \in V^\circ$ if and only if $u_\sigma \cdot u_{ab-1} \in V^\circ$.

*Proof.* By Equation (10),

$$W(3^2)^+ \subseteq V(2A).$$

Since

$$a_z \cdot u_{ab} = a_z \cdot (u_{ab} + u_{ab-1}) - a_z \cdot u_{ab-1} \in -a_z \cdot u_{ab-1} + V^\circ$$

and similarly

$$u_\sigma \cdot u_{ab} = u_\sigma \cdot (u_{ab} + u_{ab-1}) - u_\sigma \cdot u_{ab-1} \in -u_\sigma \cdot u_{ab-1} + V^\circ$$

the claims follow. \hfill \Box

**Lemma 10.2.** $V(3^2) \cdot V^\circ \subseteq V^\circ$ and, for every $z \in X_b$ and every pair of disjoint 3-cycles $a, b \in A_{12}$, the product $a_z \cdot u_{ab}$ is uniquely determined by the factors.

*Proof.* By Proposition 5.4 we need to prove that, for every $z \in X_b$ and every pair of disjoint 3-cycles $a, b \in A_{12}$, the product $a_z \cdot u_{ab}$ is uniquely determined by the factors and is contained in $V^\circ$. Possibly substituting $t$ with one of its conjugates in $A_{12}$, we may assume that $z = (1, 2)(3, 4)$. Let $l$ be the order of the intersection of the supports of $z$ and $ab$. Suppose first $l > 2$; possibly substituting $ab$ with one of its conjugates under the stabiliser of $z$ in $A_{12}$, we may assume that the supports of $z$ and $ab$ are contained in $\{1, \ldots, 7\}$ and the result follows by 19. Suppose $l = 2$, then, as above, we may assume that the supports of $z$ and $ab$ are contained in $\{1, \ldots, 8\}$, but in none of its proper subsets. Then, with the notation of Table 7, $\langle (e_1), (ab) \rangle \in \Omega^1_2$ for some $i \in \{11, \ldots, 20\}$. By Proposition 9.3(1), $u_{ab}$ is a linear combination of Majorana axes and 3-axes $u_\rho$ of type $3^2$ such that $\rho$ belongs to the point stabiliser $A$ of $\alpha_8$ of either 7 or 8. Since $A \cong A_7$, the result follows by the previous case. Suppose $l = 1$, again we may assume that the supports of $z$ and $ab$ are contained in $\{1, \ldots, 9\}$, but in none of its proper subsets, whence $\langle (e_1), (ab) \rangle \in \Omega^1_1$ for some $i \in \{21, \ldots, 25\}$. By Proposition 9.3(2), $u_{ab}$ is a linear combination of Majorana axes and 3-axes $u_{\rho_j}$ of type $3^2$ such that the $\rho_j$'s belong to some subgroup $A_{1, \ldots, 6, k_j, l_j}$, $\{k_j, l_j\} \subseteq \{7, 8, 9\}$. Then the intersection between the supports of $z$ and $\rho_j$ is at least 2, so, by the previous case, the product $a_z \cdot u_{\rho_j}$ is uniquely determined by the factors and is contained in $V^\circ$ for every $j$, whence the claim for $a_z \cdot u_{ab} \in V^\circ$. Finally, assume that the supports of $z$ and $ab$ are disjoint. Then there are $g, h \in X_b$ disjoint from $z$ such that $ab = gh$. It follows that the Norton-Sakuma subalgebras generated by $a_z$ and $a_\rho$ and by $a_z$ and $a_h$ are of type $2B$. In particular, by the Norton-Sakuma Theorem, $a_q$ and $a_\rho$ lie in the 0-eigenspace for the adjoint action of $a_z$ (see Table 1). Thus the fusion law implies that $a_z \cdot u_{ab} = 0$. \hfill \Box

**Lemma 10.3.** $V(2^6) \cdot V^\circ \subseteq V^\circ$ and, for every $z \in X_6$ and every pair of disjoint 3-cycles $a, b \in A_{12}$, the product $a_z \cdot u_{ab}$ is uniquely determined by its factors.
Proof. As in Lemma 10.2 by Proposition 5.3, we only have to prove that, for every \( z \in X_s \) and every every pair of disjoint 3-cycles \( a, b \in A_{12} \), the product \( a_s \cdot u_{ab} \) is determined by its factors and is contained in \( V^\circ \). The orbits of \( A_{12} \) on the set \( X_s \times X_s \) correspond to the 11 orbits \( P_1, \ldots, P_{11} \) of the centraliser \( H \) of the involution \( s_1 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) \) on the set \( X_s \), considered in the proof of Lemma 5.1 and Table 3. As in Lemma 10.2, by Proposition 5.4, we only have to prove that, for every \( i \in \{1, \ldots, 11\} \). For the convenience of the reader, denote by \( a_i \) (resp. \( b_i \)) the first (resp. second) factor appearing in the decomposition of \( e_i \) in the second column of Table 5 so, e.g. \( a_1 = (7,8,9) \) and \( b_1 = (10,11,12) \).

The representatives of the orbits \( P_2 \) and \( P_3 \) are \( \langle a_2b_2 \rangle \) and \( \langle a_3b_3 \rangle \), respectively. We have

\[
a_2b_2^{-1} = s_1r_2 \text{ with } r_2 := (1,2)(3,4)(5,6)(7,9)(10,11)(7,12)
\]

and

\[
a_3b_3 = s_1r_3 \text{ with } r_3 := (1,2)(3,4)(5,6)(7,11)(8,9)(10,12),
\]

so that, by Axiom (M5) and Table 3 \( u_{a_2b_2^{-1}} = u_{s_1r_2} \) and \( u_{a_3b_3} = u_{s_1r_3} \). Thus, for \( i \in \{2,3\} \), \( \langle (a_{s_i}, r_i) \rangle \) is a Norton-Sakuma algebra of type 3A containing the 3-axis \( u_{a_{s_i}} \) and the product \( a_{s_i} \cdot u_{s_i} \). In this case, then, the claim follows by the Norton-Sakuma Theorem.

The representatives of the orbits \( P_i \) and \( P_4 \) are \( \langle a_1b_1 \rangle \) and \( \langle a_2b_1 \rangle \), respectively, and, for \( i \in \{1,4\} \), we may write \( a_i b_1 = g s_i \), where \( g := (7,8)(9,10) \) and \( s_i \in X_s \). Since \( g s_i = s_i g \), by Axiom (M4) and Table 3 it follows that \( a_g \) is a 0-eigenvector for the adjoint action of \( a_{s_i} \) and, by [18] Lemma 1.10, we have

\[
a_{s_i} \cdot (a_{s_i} \cdot a_g) = (a_{s_i} \cdot a_{s_i}) \cdot a_g.
\]

Since, by definition, \( a_{s_i} \cdot a_{s_i} \in V^\circ \), by Lemma 10.2 \( (a_{s_i} \cdot a_{s_i}) \cdot a_g \in V^\circ \). On the other hand, by the Norton-Sakuma Theorem, we have

\[
a_{s_i} \cdot a_g = \frac{27 \cdot 5}{211} u_{a_i b_1} + w_i \text{ with } w_i \in V^{(2A)},
\]

hence

\[
a_{s_i} \cdot u_{a_i b_1} = \frac{211}{27 \cdot 5} \left[ a_{s_i} \cdot (a_{s_i} \cdot a_g) - a_{s_i} \cdot w_i \right] \in V^\circ.
\]

Let \( i \in \{5,6\} \). Let \( g_5 := (2,3,6)(5,7)(9,10), \ c_5 := (5,6,7)(2,8,4) \) and \( c_6 := (4,7,5)(2,8,6) \), and let \( S \) be the group defined in Lemma 5.3. Then \( \{s_1, c_1\} \subseteq S^g \) and \( c_i \in P_i \). By Lemma 5.3(2), we have

\[
a_{s_i} \cdot u_{c_i} \in V^\circ.
\]

Let \( i = 7 \). Then \( \langle a_5b_5 \rangle \in P_6 \) and \( \langle a_5b_5^{-1} \rangle \in P_7 \), so the claim follows by the previous case and Lemma 10.1.

Let \( i = 8 \) and \( g_8 := (1,4,5,11,2,3,6,12)(8,10,9) \). Then \( \langle e_{81} \rangle \in P_8 \). By Proposition 9.3(1) we have

\[
u_{e_{81}} \in -u_{(3,6,4)(5,11,12)} + u_{(3,6,5)(7,11,12)} + u_{(3,6,5)(10,11,12)} -u_{(3,6,7)(5,11,12)} + u_{(3,6,4)(7,11,12)} + V^{(2A)}.
\]

For every \( u_e \) appearing on the right hand side of Equation (22), \( e \) belongs to some of the orbits \( P_1, \ldots, P_7 \). Hence the claim follows by the previous cases.
Let $i = 9$ and $g_9 := (1, 4, 6, 8, 9, 12)(2, 5, 7, 11)$. Then $\langle e_{13}^{g_9} \rangle \in \mathcal{P}_9$ and, by Proposition 9.3(1), we have

$$u_{e_{13}^{g_9}} \in u_{(3, 7, 8)(4, 5, 6)} - u_{(3, 6, 7)(4, 5, 8)} - u_{(3, 6, 5)(4, 7, 8)} + u_{(3, 6, 5)(7, 8, 11)} + u_{(3, 7, 5)(6, 11, 8)} - u_{(3, 8, 5)(6, 7, 11)} - u_{(3, 4, 6)(7, 8, 11)} - u_{(3, 4, 7)(6, 11, 8)} + u_{(3, 11, 5)(6, 7, 8)} - u_{(3, 4, 11)(6, 7, 8)} + u_{(3, 8, 9)(5, 6, 7)} + V^{(2A)}.$$ 

For every $u_c$ appearing on the right hand side of Equation (23), $(c)$ belongs to some of the orbits $\mathcal{P}_1, \ldots, \mathcal{P}_8$, so the claim follows as before.

For $i \in \{10, 11\}$ choose $g_{10} := (5, 8, 11)(6, 9, 7)$ and $g_{11} := (5, 8, 10)(6, 11, 7)$, then $\langle e_{11}^{g_i} \rangle \in \mathcal{P}_i$ and again the result follows as in the previous cases. □

We consider now the products between two 3-axes of type $3^2$.

**Proposition 10.4.** $V = V^\circ$ and for every pair of permutations $e$ and $c$ of cycle type $3^2$, the product $u_e \cdot u_c$ is uniquely determined by the factors.

**Proof.** Clearly $V = V^\circ$ if and only if $V^\circ$ is a subalgebra of $V$. By Proposition 5.4 Lemma 10.2 and Lemma 10.3, $V^\circ$ is a subalgebra of $V$ if and only if, for every pair of permutations $e$ and $c$ of cycle type $3^2$,

$$u_e \cdot u_c \in V^\circ.$$ 

Since $V^\circ$ is $\hat{G}$-invariant and $\hat{G}$ acts on $V$ as a group of algebra automorphisms, we may assume that $e = e_1$ and $c = e_i$ for $i \in \{1, \ldots, 31\}$. Let $I_0 := \{1, \ldots, 10\}$, $I_1 := \{1, \ldots, 20\}$, $I_2 := \{1, \ldots, 25\}$, and $I_3 := \{1, \ldots, 29\}$. We first prove recursively that Equation (24) holds for every $j \in \{0, 1, 2, 3\}$, every $i \in I_j$, and every pair of permutations $e$ and $c$ of type $3^2$ such that $((e), (c)) \in \Omega^1$. As above, we may assume that $e = e_1$ and $c = e_i$, for every $i \in I_j$. Assume first $i \in I_0$, then $e_i \in A_7$. Then Equation (24) follows by [19, Proposition 2.7(iii)]. Let $j \in \{1, 2, 3\}$. By Proposition 9.3(1), for every $i \in I_j$, $u_{e_i}$ is a linear combination of Majorana axes and 3-axes $u_p$ of type $3^2$ such that each $p$ is contained in a subgroup $A_{Y_p}$ of $A_{12}$, for a suitable set $Y_p$ of cardinality $7 + (j - 1)$ with $\{1, \ldots, 6\} \subseteq Y_p \subseteq \{1, \ldots, 7 + j\}$. The claim then follows by Lemma 10.2, Lemma 10.3, and a recursive argument. Finally, if $i \in \{30, 31\}$ set

$$t_1 := (1, 2)(4, 5), \quad t_2 := (1, 3)(4, 6), \quad s_1 := (7, 8)(10, 11), \quad s_2 := (6, 7)(9, 10), \quad \text{and} \quad s_3 := (7, 9)(10, 12),$$

so that $e_1 = t_1t_2$, $e_{30} = s_1s_2$, and $e_{31} = s_1s_3$. By the Norton-Sakuma Theorem (Table 1), there exist vectors $w_1, w_2 \in V^{(2A)}$ such that

$$u_{e_1} = \frac{a_{11}}{2^7 \cdot 9}(a_{t_1} \cdot a_{t_2} + w_1) \quad \text{and} \quad u_{e_{30}} = \frac{a_{11}}{2^7 \cdot 9}(a_{s_1} \cdot a_{s_2} + w_2).$$

Since $t_1$ commutes with all the $s_i$’s, by Axiom M4 and Table 1, the Norton-Sakuma subalgebra generated by $a_{t_1}$ and $a_{s_i}$ is of type $2B$. Thus $a_{s_i}$ is a 0-eigenvector for the adjoint action of $a_{t_1}$ and the fusion law for eigenvectors implies that $a_{s_1} \cdot a_{s_2}$ and $a_{s_1} \cdot a_{s_3}$ are 0-eigenvalues for the quotient action of $a_{t_1}$. Hence, by [18, Lemma 1.10],
we have
\[ u_{e_1} \cdot u_{e_{30}} = \frac{2^{11}}{27 \cdot 5} \left[ (a_1 \cdot a_{t_2} + w_1) \cdot (a_{s_1} \cdot a_{s_2} + w_2) \right] \]
\[ = \frac{2^{11}}{27 \cdot 5} \left[ (a_1 \cdot a_{t_2} \cdot (a_{s_1} \cdot a_{s_2}) + (a_{t_1} \cdot a_{t_2}) \cdot w_2 \right. \]
\[ + w_1 \cdot (a_{s_1} \cdot a_{s_2}) + w_1 \cdot w_2 \right] \]
\[ = \frac{2^{11}}{27 \cdot 5} \left[ (a_{t_1} \cdot (a_{t_2} \cdot (a_{s_1} \cdot a_{s_2})) + (a_{t_1} \cdot a_{t_2}) \cdot w_2 \right. \]
\[ + w_1 \cdot (a_{s_1} \cdot a_{s_2}) + w_1 \cdot w_2 \right] \]

By Lemma 10.2 and Lemma 10.3, the last expression lies in \( V^\circ \), proving that \( u_{e_1} \cdot u_{e_{30}} \in V^\circ \). Similarly, we get \( u_{e_1} \cdot u_{e_{31}} \in V^\circ \). Clearly, in all cases, the product \( u_c \cdot u_c \) is uniquely determined by the factors.

Corollary 10.5.
\[ \frac{V}{(V^{(2^2)}, V^{(3A)})} \cong S^{(4,4,4)}. \]

Proof. This follows by comparing the decompositions into irreducible submodules of \( V = V^\circ \) and \( (V^{(2^2)}, V^{(3A)}) \).

Proof of Theorem 1. Let \( \{A_{12}, \mathcal{T}, V, \phi, \psi\} \) be a 2A-Majorana representation of \( A_{12} \). By Proposition 3.4, the shape of \( \phi \) is the one given in Table 3. It follows that the scalar products between axes and 3-axes are the ones given by the Norton-Sakuma Theorem (Table 1, Table 5, Table 6, and Table 7). By Proposition 5.4 and Proposition 10.4, \( V \) is 2-closed and generated by the Majorana axes and the 3-axes of type \( 3^2 \), proving the third assertion of Theorem 1. By Lemma 10.2, Lemma 10.3, and Proposition 10.4, it follows that the algebra is uniquely determined and this implies the first assertion. Finally, the second assertion follows by Proposition 6.5 and Proposition 8.2.

Proof of Theorem 2. Let \( F_5 \) be the Harada-Norton group and let \( \{F_5, \mathcal{T}_F, V_{F_5}, \phi_{F_5}, \psi_{F_5}\} \) be a 2A-Majorana representation of \( F_5 \). Then \( F_5 \) contains a subgroup \( H \) isomorphic to \( A_{12} \) and \( \phi_{F_5} \) induces, by restriction, a 2A-Majorana representation \( \{H, \mathcal{T}_H, V_H, \phi_H, \psi_H\} \) of \( H \). By [6, Lemma 3.1], every 3-axis (resp. 4- and 5-axis) of \( V_{F_5} \) is conjugate by an element of \( F_5 \) to an element of \( V_H \). Since, again by [6, Lemma 3.1], the permutations of cycle type \( 3^2 \) and those of cycle type \( 3 \) in \( H \) are fused in \( F_5 \), by Theorem 1(3) and [2, Corollary 3.2], \( V_{F_5} \) is generated by Majorana axes and so the algebra is uniquely determined by [6, Theorem 1.1] and the Norton-Sakuma Theorem.

Proof of Theorem 3. This is immediate.

Proof of Theorem 4. We identify \( A_8 \) with the subgroup of \( A_{12} \) that fixes points \( 9, \ldots, 12 \). Using the GAP program "MajoranaAlgebras" [26] we computed the Majorana representation of the group \( L := A_8 \times ((9, 10)(11, 12)) \) with generating set of involutions \( T \cap L \) and shape induced by the unique shape of a 2A-Majorana representation of \( A_{12} \). This representation has dimension 554 and it is 2-closed. The subalgebra \( W \) generated by Majorana involutions \( a_t \), with \( t \in A_5 \cap A_8 \) has dimension 476. On the other hand, Theorem 4 implies that \( W^\circ \) has dimension 462,
whence it follows that $W$ is not 2-closed. We checked that the subset $Z$ of 4-axes of $V$ defined by

$$Z := \left\{ v_p \mid \rho = (a, b, c, d)(e, f, g, h) \in A_8 \text{ and } a b c d e f g h \text{ is a semistandard } (4^2)\text{-tableau} \right\}$$

is linearly independent in $W$ and $W = W' \oplus (Z)$. □

11. Appendix

A formula for expressing a 4-axis $u_p$ of type $4^2$ as a linear combination of Majorana axes and 3-axes of type $3^2$ has been obtained working within the subgroup $S$ of $A_{12}$ defined in Lemma [53]. We give it here explicitly for $\rho = (1, 3, 2, 4)(5, 7, 6, 8)$:

\[
\begin{align*}
\nu(1,3,2,4)(5,7,6,8) &= \frac{1}{3}u_{(1,2)}(3,4)(3,5)(1,2)(7,8)(1,12) \\
&+ \frac{1}{3}u_{(2,6)}(3,8) + q_{(2,6)}(4,7) + q_{(2,5)}(3,7) + q_{(2,5)}(4,8) + q_{(1,6)}(3,7) + q_{(1,6)}(4,8) + \\
&- q_{(1,5)}(3,8) + q_{(1,5)}(4,7) \\
&+ \frac{1}{3}u_{(3,7)}(4,8) + q_{(3,8)}(4,7) + q_{(1,6)}(2,5) + q_{(1,5)}(2,6) \\
&+ \frac{1}{3}u_{(3,4)}(7,8) + q_{(1,2)}(5,6) + q_{(1,8)}(2,7)(3,5)(4,6)(9,10)(11,12) + \\
&q_{(1,4)}(2,3)(5,7)(6,8)(9,10)(11,12) + q_{(1,7)}(2,8)(3,6)(4,5)(9,10)(11,12) + \\
&q_{(1,3)}(2,4)(5,8)(6,7)(9,10)(11,12) \\
&+ \frac{1}{3}u_{(1,8)}(2,7)(3,6)(4,5)(9,10)(11,12) + q_{(1,4)}(2,3)(5,8)(6,7)(9,10)(11,12) + \\
&q_{(1,7)}(2,8)(3,5)(4,6)(9,10)(11,12) + q_{(1,3)}(2,4)(5,7)(6,8)(9,10)(11,12) + \\
&q_{(1,6)}(2,5)(3,8)(4,7)(9,10)(11,12) + q_{(1,6)}(2,5)(3,7)(4,8)(9,10)(11,12) + \\
&q_{(1,5)}(2,6)(3,7)(4,8)(9,10)(11,12) + q_{(1,5)}(2,6)(3,8)(4,7)(9,10)(11,12) \\
&- \frac{1}{3}u_{(5,6)}(7,8) + q_{(1,2)}(3,4) \\
&- \frac{1}{3}u_{(3,4)}(5,6) + q_{(1,2)}(7,8) \\
&- \frac{1}{3}u_{(1,8)}(2,3)(4,6)(5,7)(9,10)(11,12) + q_{(1,8)}(2,4)(3,5)(6,7)(9,10)(11,12) + \\
&q_{(1,2)}(3,7)(4,8)(5,6)(9,10)(11,12) + q_{(1,2)}(3,8)(4,7)(5,6)(9,10)(11,12) + \\
&q_{(1,4)}(2,7)(3,5)(6,8)(9,10)(11,12) + q_{(1,4)}(2,8)(3,6)(5,7)(9,10)(11,12) + \\
&q_{(1,7)}(2,4)(3,6)(5,8)(9,10)(11,12) + q_{(1,7)}(2,3)(4,5)(6,8)(9,10)(11,12) + \\
&q_{(1,3)}(2,8)(4,5)(6,7)(9,10)(11,12) + q_{(1,3)}(2,7)(4,6)(5,8)(9,10)(11,12) + \\
&q_{(1,6)}(2,5)(3,4)(7,8)(9,10)(11,12) + q_{(1,5)}(2,6)(3,4)(7,8)(9,10)(11,12) \\
&+ \frac{1}{3}u_{(1,3,4)}(5,6,8) + u_{(1,2,3)}(5,8,7) + u_{(1,7,8)}(3,6,5) + u_{(1,8,2)}(3,5,4) + u_{(1,2,4)}(5,7,8) + \\
&u_{(1,4,2)}(6,7,8) + u_{(1,3,2)}(6,8,7) + u_{(1,4,3)}(5,6,7) + u_{(2,3,4)}(5,8,6) + u_{(2,4,3)}(5,7,6) + \\
&u_{(1,7,8)}(4,5,6) + u_{(1,2,7)}(3,5,4) + u_{(2,8,7)}(3,6,5) + u_{(1,2,8)}(3,6,4) + u_{(1,7,2)}(3,6,4) + \\
&u_{(2,7,8)}(4,6,5) \\
&+ \frac{1}{3}u_{(1,3,2)}(5,8,6) + u_{(1,2,3)}(5,6,7) + u_{(1,2,4)}(4,6,5) + u_{(1,7,2)}(4,6,5) + u_{(1,4,2)}(5,6,8) + \\
&u_{(1,3,2)}(5,6,7) + u_{(1,5,2)}(3,8,4) + u_{(1,4,3)}(5,7,8) + u_{(1,6,2)}(3,7,4) + u_{(1,2,6)}(3,8,4) + \\
&u_{(1,3,4)}(6,7,8) + u_{(2,3,4)}(5,7,8) + u_{(2,6,5)}(3,8,7) + u_{(1,5,2)}(3,4,7) + u_{(1,2,8)}(4,6,5) + \\
&u_{(1,2,7)}(3,6,5) + u_{(2,3,4)}(6,8,7) + u_{(2,5,6)}(4,8,7) + u_{(1,6,5)}(4,8,7) + u_{(1,5,6)}(3,8,7) + \\
&u_{(1,7,8)}(3,5,4) + u_{(2,7,8)}(3,6,4) + u_{(2,7,8)}(3,4,5) + u_{(1,8,7)}(3,6,4)
\end{align*}
\]
\[ \frac{1}{n!} \left[ u(1,3,5)(2,4,6) + u(1,7,3)(4,8,5) + u(1,6,3)(2,8,5) + u(1,3,6)(2,4,7) + u(1,4,7)(2,3,8) + \\
+ u(1,5,8)(2,6,7) + u(1,5,3)(2,7,6) + u(1,8,6)(2,5,3) + u(1,4,8)(3,5,7) + u(1,7,5)(2,8,6) + \\
+ u(4,6,8)(4,5,7) + u(2,3,7)(4,6,8) + u(3,5,8)(4,6,7) + u(1,5,4)(2,6,3) + u(1,4,6)(2,3,5) + \\
+ u(1,6,8)(2,5,7) + u(1,8,4)(2,7,3) + u(3,7,5)(4,8,6) + u(1,3,6)(2,4,5) + u(1,5,8)(2,4,6) + \\
+ u(1,8,5)(4,7,6) + u(2,7,4)(3,8,5) + u(1,6,4)(2,7,5) + u(1,7,6)(2,5,4) + u(2,8,4)(3,6,7) + \\
+ u(1,7,5)(2,6,3) + u(1,4,5)(2,6,8) + u(2,8,3)(4,7,5) + u(1,7,3)(2,8,4) + u(1,4,7)(3,8,6) + \\
+ u(3,6,7)(4,5,8) + u(1,6,7)(2,5,8) \right] \]

\[ \frac{1}{n!} \left[ u(1,3,8)(2,6,4) + u(1,5,7)(3,8,6) + u(1,3,5)(2,7,4) + u(1,4,5)(3,8,6) + u(1,7,4)(2,6,8) + \\
+ u(1,3,7)(2,8,5) + u(1,6,3)(4,5,8) + u(1,8,5)(4,6,7) + u(1,5,4)(2,3,8) + u(1,6,4)(2,3,7) + \\
+ u(2,6,8)(4,7,5) + u(2,6,7)(3,8,5) + u(2,5,7)(4,8,6) + u(1,3,5)(4,7,6) + u(1,5,8)(2,7,4) + \\
+ u(4,6,5)(3,8,5) + u(1,8,3)(2,6,7) + u(1,4,7)(2,6,3) + u(2,3,6)(4,7,5) + u(1,5,7)(2,8,3) + \\
+ u(1,6,8)(3,7,5) + u(1,8,6)(2,3,7) + u(1,6,3)(2,4,8) + u(2,5,3)(4,6,8) + u(2,8,5)(3,6,7) + \\
+ u(1,7,5)(2,4,5) + u(1,6,7)(4,8,5) + u(1,6,4)(3,5,7) + u(1,4,8)(2,7,5) + u(2,4,5)(3,7,6) + \\
+ u(1,7,6)(2,4,8) + u(1,8,4)(2,3,5) \right] \]

\[ \frac{1}{n!} \left[ u(1,5,6)(3,4,8) + u(2,5,6)(3,4,7) + u(1,2,5)(3,8,7) + u(1,2,6)(4,8,7) + u(1,6,2)(3,8,7) + \\
+ u(2,6,5)(3,4,8) + u(1,5,6)(3,7,4) + u(1,2,5)(4,7,8) \right] \]

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