Design of LDPC Codes for the Unequal Power Two-User Gaussian Multiple Access Channel

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Abstract—In this work, we describe an LDPC code design framework for the unequal power two-user Gaussian multiple access channel using EXIT charts. We show that the sum-rate of the LDPC codes designed using our proposed method can get close to the maximal sum-rate of the two-user Gaussian multiple access channel. Moreover, we provide numerical simulation results that demonstrate the excellent finite-length performance of the designed LDPC codes.

Index Terms—LDPC codes, EXIT charts, Gaussian multiple access channel.

I. INTRODUCTION

It is known that corner points of the two-user Gaussian multiple access channel (GMAC) can be achieved by successive decoding, while other points can be achieved by time-sharing, rate-splitting [1], or joint decoding [2]. The first work to provide an explicit low-complexity EXIT chart based method to design LDPC codes for the two-user GMAC under joint decoding was [3]. However, the authors of [3] considered only the case where the two users have equal power. The problem was recently re-examined in [4], where a Gaussian mixture (GM) model was used for the state-to-variable messages. This GM model was shown to improve the accuracy of the EXIT chart based method of [3], at the cost of increased code design complexity. The GM model was further improved in [5] and used in conjunction with EXIT charts to design LDPC codes for the multi-user GMAC where all users have equal power. Finally, the work of [6] optimized the BPSK amplitudes used to transmit groups of LDPC-coded bits under an equal average power constraint for the two-user GMAC.

Contribution: In this work, we describe an LDPC code design method for the unequal power two-user GMAC, which generalizes the method of [3] and is a low-complexity alternative to [4]. As part of the derivation of this method, we also provide a rigorous proof of the all-one/one-half codeword assumption that has been used throughout the relevant literature [3–6] without an explicit proof.

II. BACKGROUND

A. Two-User Gaussian Multiple Access Channel

Let the length-\(n\) binary codewords of the two users be \(c^{[j]} \in C_j, j = 1, 2\), where \(C_j, j = 1, 2\), are the respective codes. The BPSK modulated codewords are \(x^{[j]} = 1 - 2c^{[j]}, j = 1, 2\). The output of the GMAC channel is

\[
y = \sqrt{P_1}x^{[1]} + \sqrt{P_2}x^{[2]} + w, \quad w \sim N(0, I_n),
\]

where \(P_1\) and \(P_2\) denote the powers of the two users and, without loss of generality, the noise is assumed to have unit variance. We define the design SNR for each user as \(\text{SNR}_1 = \frac{P_1}{2}\) and \(\text{SNR}_2 = \frac{P_2}{2}\). The following symmetry property of the GMAC can be easily verified

\[
p(y_i|x_i^{[1]}, x_i^{[2]}) = p(-y_i| -x_i^{[1]}, -x_i^{[2]}).
\]

B. LDPC Codes

An ensemble of LDPC codes can be described by its edge perspective variable and check node degree distributions, \(\lambda(x)\) and \(\rho(x)\), respectively, where \([7,\ p.\ 79]\)

\[
\lambda(x) = \sum_i \lambda_i x_i^{i-1}, \quad \rho(x) = \sum_i \rho_i x_i^{i-1}.
\]

The node perspective variable node degree distribution and the design rate of the code are

\[
L(x) = \int_0^x \frac{\lambda(z)dz}{\int_0^1 \lambda(z)dz}, \quad r = 1 - \sum_i \frac{\rho_i / i}{\sum_i \lambda_i / i}.
\]

C. Belief Propagation Decoding for the GMAC

The belief propagation (BP) message-passing algorithm can be used to efficiently decode LDPC codes by exchanging log-likelihood ratio (LLR) messages over the code’s Tanner graph \([7,\ Sec.\ 4.2]\). If the Tanner graph is cycle-free, BP decoding is optimal with respect to the bit error rate. An example of a Tanner graph for joint BP decoding of two codewords transmitted over the GMAC is depicted in Fig. 1.

Let \(v_{vk}^{[1]}\) and \(v_{vk}^{[2]}\) denote the check-to-variable and variable-to-check messages for check and variable node \(k\) of user \(j\), respectively. These messages follow standard single-user BP rules \([7,\ Sec.\ 3.3]\). Let \(u_{vk}^{[j]}\) denote the variable-to-state message from user \(j\) towards state node \(k\) and let \(u_{vk}^{[j]}\) denote the state-to-variable message from state node \(k\) towards user \(j\). Messages \(u_{vk}^{[j]}\) follow single-user BP rules \([7,\ p.\ 59]\). Using standard function node message-passing rules \([7,\ p.\ 56]\), we derive the following update rules for \(u_{vk}^{[1]}\) and \(u_{vk}^{[2]}\)

\[
\begin{align*}
u_{vk}^{[1]} &= \log \frac{e^{-i(\sqrt{P_1} - \sqrt{P_2})^2} e^{w_{vk}^{[2]}} + e^{-i(\sqrt{P_1} + \sqrt{P_2})^2}}{e^{-i(\sqrt{P_1} + \sqrt{P_2})^2} e^{w_{vk}^{[2]}} + e^{-i(\sqrt{P_1} - \sqrt{P_2})^2}}, \\
u_{vk}^{[2]} &= \log \frac{e^{-i(\sqrt{P_1} - \sqrt{P_2})^2} e^{w_{vk}^{[1]}} + e^{-i(\sqrt{P_1} + \sqrt{P_2})^2}}{e^{-i(\sqrt{P_1} + \sqrt{P_2})^2} e^{w_{vk}^{[1]}} + e^{-i(\sqrt{P_1} - \sqrt{P_2})^2}}.
\end{align*}
\]

where we have dropped the index \(k\) for simplicity, since the update rules are identical for all state nodes. The above update rules are generalizations of the equal power update rules derived in [3]. As in [3], we assume a parallel decoding schedule, i.e., we first perform a round of BP for each user and...
III. EXIT CHARTS FOR THE UNEQUAL POWER GMAC

In the limit of infinite blocklength, the Tanner graph of the two-user GMAC is cycle-free [7, p. 310]. Thus, we can use tools such as density evolution (DE) [8] and EXIT charts [9] to design LDPC codes for this channel. Density evolution tracks the densities of the messages exchanged during BP decoding and can be used for the derivation of conditions that guarantee a vanishingly small probability of error for large code lengths. EXIT charts are a simpler analysis tool than DE that reduces the infinite-dimensional problem of tracking densities to a single-dimensional problem of tracking the mutual information between the messages in the decoder and the codeword bits.

A. Restriction to the All-One/One-Half Codewords

A crucial observation is that, for symmetric channels, the bit error probability of BP decoding is independent of the transmitted codeword [7, Lemma 4.90]. This means that the decoder analysis can be restricted to the all-one BPSK codeword, thus making the complexity of both DE and EXIT charts tractable.

Unfortunately, in general, the GMAC is not symmetric with respect to each user. However, if we separately examine the cases where $x_1^j = x_2^j$ (resp. $x_1^j = -x_2^j$), then the GMAC channel is equivalent to a BI-AWGN channel with inputs $\sqrt{T_1^j} \pm \sqrt{T_2^j}$ (resp. $\sqrt{T_1^j} - \sqrt{T_2^j}$), which is symmetric in the single user sense. Let $a_1(z)$ (resp. $a_2(z)$) denote the density of log $P(y_i | x_1^j = +1)$ (resp. log $P(y_i | x_1^j = -1)$) conditioned on $X_{i,1} = +1$ (resp. $X_{i,2} = +1$), where $x_{i,1} = x_1^j = x_2^j$ (resp. $x_{i,2} = x_2^j = -x_1^j$). We can model each of the two BI-AWGN channels multiplicatively as [7, p. 215]

$$Y_{1,i} = x_{1,i}Z_{1,i} \quad \text{and} \quad Y_{2,i} = x_{2,i}Z_{2,i},$$

where $Z_{1,i}$ is distributed according to $a_1(z)$, $Z_{2,i}$ is distributed according to $a_2(z)$, and $x_{1,i}, x_{2,i} \in \{-1, +1\}$. For a typical LDPC codeword, it holds that $p(x_1^j = 0) = p(x_2^j = 1) = 0.5$, $j = 1, 2$ [7, p. 296]. Since $x_1^1$ and $x_2^2$ are independent LDPC codewords, the events $x_2^2 = x_1^1$ and $x_1^1 = -x_2^2$ have probability 0.5. Thus, in the typical case, half the state nodes will be $x_1^1 = x_2^2$ nodes and half the state nodes will be $x_1^1 = -x_2^2$ nodes.

Proposition 1. The DE analysis of LDPC code ensembles for transmission over the two-user GMAC can be restricted to the case where $x_1^1$ is the all-one BPSK codeword and $x_2^2$ is a typical codeword of type one-half.

Proof: Similarly to [7] Lemma 4.90, we will show that the probability of error when the received values at $x_1^1 = x_2^2$ nodes are $Y_1 = x_1Z_1$ and the received values at $x_1^1 = -x_2^2$ nodes are $Y_2 = x_2Z_2$ is equal to the probability of error for the case where the received values at $x_1^1 = x_2^2$ nodes are $Y_1 = Z_1$ and the received values at $x_1^1 = -x_2^2$ nodes are $Y_2 = Z_2$. This means that, for the analysis, $x = [x_1, x_2]$ can be restricted to the all-one BPSK codeword, which implies, by construction of $x$, that $x_1^1$ can be the all-one BPSK codeword but $x_2^2$ has to be a typical codeword of type one-half.

Let $i_k$ be a variable node of user $k$ and let $j_k$ be one of its neighboring check nodes. Let $v_{c_k}^{(0)}(y_i)$ denote the message sent from $i_k$ to $j_k$ in iteration $\ell$ assuming that the received value is $y_i$, let $v_{c_k}^{(\ell)}(y_i)$ denote the corresponding message sent from $j_k$ to $i_k$, and let $v_{s_k}^{(\ell)}(y_i)$ denote the message sent from variable node $i_k$ to its corresponding state node. Finally, let $s^{(\ell)}(y_i, v_{s_k}^{(\ell)}(y_i))$ denote the message from the state node connected to variable node $i_1$ of user 1 (resp. variable node $i_2$ of user 2) towards the corresponding variable node of user 2, i.e. $i_2$ (resp. $i_1$).

Due to (7) and (8), the initial messages from the state nodes to the variable nodes of user $k$ are

$$v_{i_k}^{(0)}(y_i, 0) = v_{i_k}^{(0)}(x_1z_i, 0) = x_1v_{i_k}^{(0)}(z_i, 0).$$

Due to the variable node update rule symmetry, we have

$$v_{i_k}^{(0)}(y_i) = v_{i_k}^{(0)}(x_1z_i) = x_1v_{i_k}^{(0)}(z_i).$$

Using the check and variable node symmetries [7, Sec. 4.2], we get

$$v_{s_k}^{(\ell+1)}(y_i) = x_1v_{s_k}^{(\ell+1)}(z_i).$$

Due to the state node update rule symmetry in (9), for the message from the state node connected to variable node $i_1$ towards the corresponding variable node of user 2, we have

$$s^{(\ell+1)}(x_1z_i, x_1v_{s_k}^{(\ell+1)}) = x_1s^{(\ell+1)}(z_i, v_{s_k}^{(\ell+1)}).$$

An analogous statement holds for user 2. By invoking the variable node symmetry again, we have

$$v_{i_k}^{(\ell+1)}(y_i) = x_1v_{i_k}^{(\ell+1)}(z_i).$$

As in [7] Lemma 4.90, since we can factor out the message signs at each iteration, we can conclude that $x$ can be the all-one BPSK codeword.

Remark: If we set $x_1 = -x_2^1 = x_2^2$ in (8) when $x_1^1 = -x_2^2$ (instead of $x_1 = x_2^1 = -x_2^2$), then we will reach the conclusion that user 2 can be restricted to the all-one codeword and user 1 has to transmit a typical codeword of type one-half.
\[ F_+^{[1]}(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \log \left( \frac{1 + e^{\mu + 4\mu z^2 + \mu + 2P_2}}{1 + e^{\mu + 4\mu z^2 - \mu - 2P_2 - 4\sqrt{\mu} z^2}} \right) \; dz - \mu + 2(P_1 - P_2) \]  
\[ F_-^{[1]}(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \log \left( \frac{1 + e^{-\mu + 4\mu z^2 + \mu - 2P_2}}{1 + e^{-\mu + 4\mu z^2 - \mu + 2P_2 - 4\sqrt{\mu} z^2}} \right) \; dz + \mu + 2(P_2 - P_1) \]  
\[ F_+^{[2]}(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \log \left( \frac{1 + e^{4\mu + 4\mu z^2 + \mu + 2P_1}}{1 + e^{4\mu + 4\mu z^2 - \mu + 2P_2 - 4\sqrt{\mu} z^2}} \right) \; dz - \mu + 2(P_2 - P_1) \]  
\[ F_-^{[2]}(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \log \left( \frac{1 + e^{-4\mu + 4\mu z^2 + \mu - 2P_1}}{1 + e^{-4\mu + 4\mu z^2 - \mu + 2P_2 - 4\sqrt{\mu} z^2}} \right) \; dz - \mu - 2(P_1 - P_2) \]
Additionally, (19) becomes $\lambda^{[j]}_2 < \exp \left( P_j / 2 \right) / (k_j - 1)$, $j = 1, 2$. In general, each $I_{EVC}^{[j]}$ is a non-linear function of the coefficients of $\lambda^{[1]}(x)$ and $\lambda^{[2]}(x)$, so the joint code design cannot be expressed as a linear program (LP). To overcome this problem, [3] uses the assumptions that $\lambda^{[1]}(x) = \lambda^{[2]}(x)$ and that the state nodes always connect a degree $i$ variable node of user 1 with a degree $i$ variable node of user 2. These assumptions are reasonable for the equal power case, but in the unequal power case the degree distributions for each user have to be different in general to enable communication at different rates. In [4], the authors use differential evolution to optimize the degree distributions for the unequal power GMAC, which gives good results but it is less elegant and has a much higher computational complexity than an LP approach.

In order to express the code design as an LP, we propose to fix the variable node degree distribution of one user and optimize the variable node degree distribution of the other user by alternately solving the LP, for $j = 1, 2$,

$$\text{maximize} \sum_i \lambda^{[j]}_i / i \quad (27)$$

subject to

$$I_{EVC}^{[j]} \leq \sum_i \lambda^{[j]}_i I_{EVC}^{[j]} \quad (28)$$

$$\sum_i \lambda^{[j]}_i = 1, \quad \lambda^{[j]}_i \geq 0, \quad i = 2, 3, \ldots, v_{\text{max}}, \quad (29)$$

$$\lambda^{[j]}_2 < \exp \left( P_j / 2 \right) / (k_j - 1). \quad (30)$$

The same procedure is repeated for several $(k_1, k_2)$ pairs and the best pair, in terms of sum-rate, is kept.

In order to test our method, we set $P_1 = 1.5$, $P_2 = 1$, and $v_{\text{max}} = 200$. After 4 iterations of the alternating LP procedure, the best resulting variable node degree distributions are

$$\lambda^{[1]}(x) = 0.2428 x + 0.3653 x^2 + 0.0152 x^{18} + 0.1839 x^{19} + 0.1787 x^{20} + 0.0141 x^{31},$$

$$\lambda^{[2]}(x) = 0.1855 x + 0.2775 x^2 + 0.0057 x^{11} + 0.1138 x^{12} + 0.3209 x^{175} + 0.0960 x^{176},$$

with $\rho^{[1]} = \rho^{[2]} = x^7$ and rates $r^{[1]} = 0.5104$ and $r^{[2]} = 0.3646$, respectively. Additional iterations do not provide any significant rate improvement. The maximal sum-rate for this power allocation is 0.8860 bits/ch. use, so we only have 0.0101 bits/ch. use away from this limit. We note that, when we set $P_1 = P_2$ and $\rho^{[1]}(x) = \rho^{[2]}(x)$, we experimentally observe that our optimization results satisfy $\lambda^{[1]}(x) = \lambda^{[2]}(x)$, which corroborates the validity of the assumptions made in [3].

The above-designed codes should provide asymptotically error-free communication for each user at their corresponding design SNRs. However, the EXIT chart analysis is approximate and, in practice, we have to use finite-length codes. Thus, in order to assess the accuracy of our analysis and the finite-length performance loss, we create random codes of length $n = 5 \cdot 10^4$ according to the optimized $\lambda^{[j]}(x)$ and $\rho^{[j]}(x)$, $j = 1, 2$. The maximum number of BP decoding iterations is set to 200. In Fig. 2 we see that the BER is less than $10^{-6}$ at an SNR only approximately 0.5 dB away from the design SNR for both users.

V. CONCLUSION

We presented an EXIT-chart based method for the design of LDPC codes for the unequal power two-user GMAC. We showed that, under certain assumptions, the optimization problem can be expressed as an alternating sequence of LPs, which can be solved efficiently. Numerical results demonstrate that, for $P_1 = 1$ and $P_2 = 1.5$, the resulting codes are only 0.0101 bits/ch. use away from the maximal sum-rate and only 0.5 dB away from their design SNR when using an LDPC code of length $n = 5 \cdot 10^4$.

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