Parameterized combinatorial curvatures and parameterized combinatorial curvature flows for discrete conformal structures on polyhedral surfaces

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Abstract

Discrete conformal structure on polyhedral surfaces is a discrete analogue of the smooth conformal structure on surfaces that assigns discrete metrics by scalar functions defined on vertices. It unifies and generalizes tangential circle packing, Thurston’s circle packing, inversive distance circle packing and the vertex scaling described by Luo [29] and others. In this paper, we introduce combinatorial $\alpha$-curvature for discrete conformal structures on polyhedral surfaces, which is a parameterized generalization of the classical combinatorial curvature. Then we prove the local and global rigidity of combinatorial $\alpha$-curvature with respect to discrete conformal structures on polyhedral surfaces, which confirms parameterized Glickenstein rigidity conjecture in [22]. To study the Yamabe problem for combinatorial $\alpha$-curvature, we introduce combinatorial $\alpha$-Ricci flow for discrete conformal structures on polyhedral surfaces, which is a generalization of Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings [4] and Luo’s combinatorial Yamabe flow for vertex scaling [29] on polyhedral surfaces. To handle the potential singularities of the combinatorial $\alpha$-Ricci flow, we extend the flow through the singularities by extending the inner angles in triangles by constants. Under the existence of a discrete conformal structure with prescribed combinatorial curvature, the solution of extended combinatorial $\alpha$-Ricci flow is proved to exist for all time and converge exponentially fast for any initial value. This confirms a parameterized generalization of another conjecture of Glickenstein in [22] on the convergence of combinatorial Ricci flow, gives an almost equivalent characterization of the solvability of Yamabe problem for combinatorial $\alpha$-curvature in terms of combinatorial $\alpha$-Ricci flow and provides an effective algorithm for finding discrete conformal structures with prescribed combinatorial $\alpha$-curvatures.

Keywords: Combinatorial Ricci flow; Rigidity; Discrete conformal structure; Combinatorial curvature
1 Introduction

This is a continuation of [41] studying discrete conformal structures on polyhedral surfaces. Discrete conformal structure on polyhedral surfaces is a discrete analogue of the smooth conformal structure on Riemannian surfaces, which defines discrete metrics by scalar functions defined on the vertices. There are mainly four special types of discrete conformal structures on polyhedral surfaces that have been extensively studied in history, including the tangential circle packing, Thurston's circle packing, inversive distance circle packing and the vertex scaling described by Luo [29] and others. These special types of discrete conformal structures are introduced and studied individually in the past. The generic notion of discrete conformal structure on polyhedral surfaces was introduced recently independently by Glickenstein [21] and Glickenstei-Thomas [23] from Riemannian geometry perspective and by Zhang-Guo-Zeng-Luo-Yau-Gu [45] using 3-dimensional hyperbolic geometry, which unifies and generalizes the existing special types of discrete conformal structures on polyhedral surfaces. In [41], the first author studied the classical combinatorial curvature for discrete conformal structures on polyhedral surfaces. In this paper, we introduce a new combinatorial curvature (combinatorial $\alpha$-curvature) for discrete conformal structures on polyhedral surfaces, which is a parameterized generalization of the classical combinatorial curvature. Then we prove its rigidity and further study the corresponding Yamabe problem for the new combinatorial curvature using combinatorial Ricci flow and combinatorial Calabi flow.

Suppose $(M, T)$ is a triangulated connected closed surface with a triangulation $T = (V, E, F)$, where $V, E, F$ represent the sets of vertices, edges and faces respectively and $V$ is a finite subset of $M$ with $|V| = N$. Denote a vertex, an edge and a face in the triangulation $T$ by $i, \{ij\}, \{ijk\}$ respectively, where $i, j, k$ are natural numbers. If a map $l : E \to (0, +\infty)$ assigns a length to every edge in such a way that every triangle $\{ijk\} \in F$ with edge lengths $l_{ij}, l_{ik}, l_{jk}$ is embedded in 2-dimensional Euclidean space $\mathbb{E}^2$ (2-dimensional hyperbolic space $\mathbb{H}^2$ or 2-dimensional spherical space $\mathbb{S}^2$ respectively), then $(M, T, l)$ is called as a triangulated polyhedral surface with Euclidean (hyperbolic or spherical respectively) background geometry and $l : E \to (0, +\infty)$ is called a Euclidean (hyperbolic or spherical respectively) polyhedral metric. One can also take the triangulated polyhedral surface $(M, T, l)$ being obtained by gluing triangles in $\mathbb{E}^2$ ($\mathbb{H}^2$ or $\mathbb{S}^2$ respectively) isometrically along the edges in pair. For a triangulated polyhedral surface $(M, T, l)$, the classical combinatorial curvature $K : V \to (-\infty, 2\pi)$ is used to describe the conic singularities of polyhedral metrics at the vertices, which is defined to be

$$K_i = 2\pi - \sum_{\{ijk\} \in F} \theta_i^{jk}$$

(1)
with summation taken over all the triangles with \( i \) as a vertex and \( \theta_{ij}^k \) being the inner angle of the triangle \( \{ijk\} \in F \) at the vertex \( i \). The classical combinatorial curvature satisfies the following discrete Gauss-Bonnet formula ([4], Proposition 3.1)

\[
\sum_{i \in V} K_i = 2\pi \chi(M) - \lambda \text{Area}(M),
\]

where \( \lambda = -1, 0, +1 \) for hyperbolic, Euclidean and spherical background geometry respectively and \( \text{Area}(M) \) is the area of the surface \( M \).

**Definition 1.1.** ([23, 45]) Suppose \((M, T)\) is a triangulated connected closed surface with two weights \( \varepsilon : V \to \{-1, 0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying \( \eta_{ij} = \eta_{ji} \). A discrete conformal structure on \((M, T, \varepsilon, \eta)\) is a map \( f : V \to \mathbb{R} \) determining a polyhedral metric \( l : E \to \mathbb{R} \) by assigning the edge length \( l_{ij} \) for \( \{ij\} \in E \) as

\[
l_{ij} = \sqrt{\varepsilon_ie^{2f_i} + \varepsilon_je^{2f_j} + 2\eta_{ij}e^{f_i} + f_j}
\]

in the Euclidean background geometry,

\[
l_{ij} = \cosh^{-1}\left(\sqrt{(1 + \varepsilon_ie^{2f_i})(1 + \varepsilon_je^{2f_j}) + \eta_{ij}e^{f_i} + f_j}\right)
\]

in the hyperbolic background geometry and

\[
l_{ij} = \cos^{-1}\left(\sqrt{(1 - \varepsilon_ie^{2f_i})(1 - \varepsilon_je^{2f_j}) - \eta_{ij}e^{f_i} + f_j}\right)
\]

in the spherical background geometry. The weight \( \varepsilon : V \to \{-1, 0, 1\} \) is called the scheme coefficient and \( \eta : E \to \mathbb{R} \) is called the discrete conformal structure coefficient.

Two discrete conformal structures defined on the same weighted triangulated surface \((M, T, \varepsilon, \eta)\) with the same background geometry is said to be conformally equivalent. In this paper, we focus on the cases of Euclidean and hyperbolic background geometry.

**Remark 1.** The relationships of the discrete conformal structure in Definition 1.1 and the existing main special types of discrete conformal structures are summarized in Table 1. By Table 1, the discrete conformal structure in Definition 1.1 not only contains vertex scaling and different types of circle packings as special cases, but also contains other types of discrete conformal structures, for example the mixed type with \( \varepsilon_i = 0 \) for vertices \( i \in V_0 \neq \emptyset \) and \( \varepsilon_j = 1 \) for the other vertices \( j \in V \setminus V_0 \neq \emptyset \). Please refer to [23, 45] for more information on this.
The classical combinatorial curvature $K$ on polyhedral surfaces is proved to converge to the smooth Gaussian curvature on surfaces in the sense of measure by Cheeger-Müller-Schrader [3]. However, the convergence is not a purely local phenomenon. As mentioned in [3, 19], the classical combinatorial curvature $K$ defined by (1) is scaling invariant in the Euclidean background geometry and does not approximate the smooth Gaussian curvature pointwisely on smooth surfaces as the triangulation of the surface becomes finer and finer. The key issue is how to decompose such a measure over the triangulated surfaces. Motivated by this idea and the original definition of smooth Gaussian curvature via Gauss map on surfaces, Ge and the first author [19] introduced the combinatorial $\alpha$-curvature for Thurston’s Euclidean circle packing metrics on triangulated surfaces. After that, there are lots of works on new combinatorial curvatures on polyhedral surfaces and triangulated 3-manifolds. Please refer to [5, 12, 14, 16–19, 27, 28, 36, 37, 40, 42, 43] and other works.

In this paper, we introduce the following combinatorial $\alpha$-curvature for discrete conformal structures on a weighted triangulated surface $(M, T, \varepsilon, \eta)$, which is a parameterized generalization of the classical combinatorial curvature $K$.

**Definition 1.2.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$. The combinatorial $\alpha$-curvature at the vertex $i \in V$ is defined to be

$$R_{\alpha,i} = \frac{K_i}{e^{\alpha u_i}},$$

where $K_i$ is the classical combinatorial curvature at $i \in V$ defined by (1), $u_i = f_i$ for $i \in V$ in the Euclidean background geometry and

$$u_i = \begin{cases} f_i, & \varepsilon_i = 0, \\ \frac{1}{2} \ln \left| \frac{\sqrt{1 + e^{2\eta_i}} - 1}{\sqrt{1 + e^{2\eta_i}} + 1} \right|, & \varepsilon_i = 1 \end{cases}$$

for $i \in V$ in the hyperbolic background geometry.

We also call the function $u$ in Definition 1.2 as a discrete conformal structure on $(M, T, \varepsilon, \eta)$, if it causes no confusion in the context.
Remark 2. The combinatorial $\alpha$-curvature $R_\alpha$ in Definition 1.2 is motivated by the following facts. The first is that the smooth Gaussian curvature on surfaces is defined to be the limit of area distortion of the Gauss map. In the Euclidean background geometry, if we set $e^{l_i} = r_i$, which is taken to be the radius of the disk attached to the vertex $i \in V$, then $e^{2u_i} = r_i^2$ is the area of the disk with radius $r_i$ up to a constant. Similarly, in the hyperbolic background geometry, set $e^{f_i} = \sinh r_i$, then $e^{2u_i}$ is a 1-st order approximation of the area of hyperbolic disk with radius $r_i$ up to a positive constant. In this sense, the combinatorial 2-curvature could be taken as an approximation of the smooth Gaussian curvature and the combinatorial $\alpha$-curvature $R_\alpha$ is a parameterized generalization of it. The second is that the matrix $(\partial \theta / \partial u_i)$ is symmetric by Lemma 2.1 and Lemma 2.6. This provides lots of convenience in the arguments. The third is that this form of combinatorial $\alpha$-curvature unifies and generalizes the previously defined combinatorial curvatures on polyhedral surfaces and preserves similar scaling property of the smooth Gaussian curvature on surfaces. If $\alpha = 0$, the combinatorial curvature $R_\alpha$ in Definition 1.2 is reduced to the classical combinatorial curvature $K$. If $\varepsilon \equiv 1$, the combinatorial curvature $R_\alpha$ in Definition 1.2 is reduced to the parameterized combinatorial curvature for circle packings studied in [12, 14, 16–19, 36]. If $\varepsilon \equiv 0$, the combinatorial curvature $R_\alpha$ in Definition 1.2 is reduced to the parameterized combinatorial curvature for vertex scaling studied in [40, 42, 43]. The parameterized combinatorial curvature $R_\alpha$ in Definition 1.2 further covers the case of mixed type discrete conformal structures, for example, $\varepsilon_i = 0$ for vertices $i \in V_0 \neq \emptyset$ and $\varepsilon_j = 1$ for the other vertices $j \in V \setminus V_0 \neq \emptyset$.

Remark 3. In the Euclidean background geometry, if we take $g_i = e^{u_i}$ as a discrete analogue of the smooth Riemannian metric, then for any constant $\lambda > 0$, we have $R_{\alpha,i}(\lambda g_1, \ldots, \lambda g_N) = \lambda^{-\alpha} R_{\alpha,i}(g_1, \ldots, g_N)$. In the special case of $\alpha = 1$, $R_{1,i}(\lambda g_1, \ldots, \lambda g_N) = \lambda^{-1} R_{1,i}(g_1, \ldots, g_N)$, which is parallelling to the transformation of smooth Gaussian curvature $K_{\lambda g} = \lambda^{-1} K_g$ with $g$ being the Riemannian metric. Note that, in the hyperbolic background geometry, $u_i \in \mathbb{R}$ for vertex $i$ with $\varepsilon_i = 0$ and $u_i \in \mathbb{R}_{<0}$ for vertex $i$ with $\varepsilon_i = 1$, thus $u = (u_1, \ldots, u_N) \in \mathbb{R}^{N_0} \times \mathbb{R}^{N_1}_{<0}$, where $N_0$ is the number of vertices $i \in V$ with $\varepsilon_i = 0$ and $N_1 = N - N_0$.

A basic problem in discrete conformal geometry is to understand the relationships between the discrete conformal structure and its combinatorial curvature. For the combinatorial $\alpha$-curvature $R_\alpha$, we have the following rigidity with respect to discrete conformal structures on polyhedral surfaces, which confirms a parameterized generalization of Glickenstein’s rigidity conjecture in [22].

Theorem 1.3. Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying

$$\varepsilon_i \varepsilon_j + \eta_{st} > 0, \quad \forall \{st\} \in E$$

(5)
and

\[ \varepsilon_q \eta_{st} + \eta_{qs} \eta_{qt} \geq 0, \quad \{q, s, t\} = \{i, j, k\} \]  \hspace{1cm} (6)

for any triangle \( \{ijk\} \in F \). \( \alpha \in \mathbb{R} \) is a constant and \( \overrightarrow{R} : V \rightarrow \mathbb{R} \) is a given function.

(a) In the case of Euclidean background geometry, if \( \alpha \overrightarrow{R} \equiv 0 \), there exists at most one discrete conformal structure \( f : V \rightarrow \mathbb{R} \) with combinatorial \( \alpha \)-curvature \( \overrightarrow{R} \) up to a vector \( c(1, 1, \ldots, 1) \), \( c \in \mathbb{R} \); if \( \alpha \overrightarrow{R} \leq 0 \) and \( \alpha \overrightarrow{R} \neq 0 \), there exists at most one discrete conformal structure \( f : V \rightarrow \mathbb{R} \) with combinatorial \( \alpha \)-curvature \( \overrightarrow{R} \).

(b) In the case of hyperbolic background geometry, if \( \alpha \overrightarrow{R} \leq 0 \), there exists at most one discrete conformal structure \( f : V \rightarrow \mathbb{R} \) with combinatorial \( \alpha \)-curvature \( \overrightarrow{R} \).

Remark 4. In the special case of \( \alpha = 0 \), Theorem 1.3 was proved by the first author in [41]. If \( \varepsilon_i = 1 \) for all \( i \in V \), Theorem 1.3 is reduced to the rigidity of circle packings on surfaces obtained in [4, 12, 16, 18, 19, 26, 31, 36] and others. If \( \varepsilon_i = 0 \) for all \( i \in V \), Theorem 1.3 is reduced to the rigidity of vertex scaling on polyhedral surfaces obtained in [1, 29, 40, 42] and others. Theorem 1.3 unifies these results and further covers the case of mixed type that \( \varepsilon_i = 0 \) for some vertices \( i \in V_0 \neq \emptyset \) and \( \varepsilon_i = 1 \) for the other vertices \( j \in V \setminus V_0 \neq \emptyset \).

It is interesting to consider the following combinatorial Yamabe problem for combinatorial \( \alpha \)-curvature \( R_\alpha \).

**Combinatorial Yamabe problem** Does there exist a discrete conformal structure with constant combinatorial \( \alpha \)-curvature or prescribed combinatorial \( \alpha \)-curvature on \( (M, T, \varepsilon, \eta) \)? Furthermore, how to find it?

To study the combinatorial Yamabe problem for combinatorial \( \alpha \)-curvature \( R_\alpha \), we introduce the following combinatorial \( \alpha \)-Ricci flow for discrete conformal structures on polyhedral surfaces, which is a generalization of Chow-Luo’s combinatorial Ricci flow for Thurston’s circle packings in [4], Luo’s combinatorial Yamabe flow for vertex scaling in [29] and Zhang-Guo-Zeng-Luo-Yau-Gu’s combinatorial Ricci flow for discrete conformal structures in [45].

**Definition 1.4.** Suppose \( (M, T, \varepsilon, \eta) \) is a weighted triangulated connected closed surface with weights \( \varepsilon : V \rightarrow \{0, 1\} \) and \( \eta : E \rightarrow \mathbb{R} \). \( \alpha \in \mathbb{R} \) is a constant and \( u : V \rightarrow \mathbb{R} \) is a discrete conformal structure defined by \( u_i = f_i \) in the Euclidean background geometry and by (4) in the hyperbolic background geometry. The combinatorial \( \alpha \)-Ricci flow for discrete conformal structures on polyhedral surfaces is defined to be

\[ \frac{du_i}{dt} = -R_{\alpha, i} \]  \hspace{1cm} (7)

for Euclidean and hyperbolic background geometry.
Remark 5. If $\alpha = 0$, the combinatorial $\alpha$-Ricci flow in Definition 1.4 is Zhang-Guo-Zeng-Luo-Yau-Gu’s combinatorial Ricci flow in [45], which unifies and generalizes Chow-Luo’s combinatorial Ricci flow for circle packing in [4, 11, 13] and Luo’s combinatorial Yamabe flow for vertex scaling in [29]. If $\varepsilon \equiv 1$, the combinatorial $\alpha$-Ricci flow in Definition 1.4 is reduced to the combinatorial $\alpha$-Ricci flow for circle packings studied in [12, 14, 16–19]. If $\varepsilon \equiv 0$, the combinatorial $\alpha$-Ricci flow in Definition 1.4 is reduced to the combinatorial $\alpha$-Yamabe flow for vertex scaling studied in [41, 42]. The combinatorial $\alpha$-Ricci flow in Definition 1.4 further covers the case of mixed type discrete conformal structures.

In the Euclidean background geometry, the normalized combinatorial $\alpha$-Ricci flow is defined to be

\[ \frac{du_i}{dt} = R_{\alpha,av} - R_{\alpha,i}, \]

where $R_{\alpha,av} = \frac{2\pi \chi(M)}{\sum_{i=1}^{N} e^{\alpha u_i}}$ is the average combinatorial $\alpha$-curvature. In the Euclidean and hyperbolic background geometry, we usually generalize the combinatorial $\alpha$-Ricci flow to the following modified combinatorial $\alpha$-Ricci flow

\[ \frac{du_i}{dt} = \overline{R}_i - R_{\alpha,i}, \]

where $\overline{R}$ is a given function defined on the vertices.

Along the combinatorial $\alpha$-Ricci flows and , singularities may develop, which correspond to that some triangle degenerates or the discrete conformal structure tends to infinity. If some triangle degenerates along the combinatorial curvature flow, we call the combinatorial curvature flow develops a removable singularity. If the discrete conformal structure tends to infinity, we call the the combinatorial curvature flow develops an essential singularity. To handle the potential removable singularities along the combinatorial $\alpha$-Ricci flows, we extend the combinatorial $\alpha$-curvature and then extend the combinatorial $\alpha$-Ricci flows through the removable singularities. We have the following result on the longtime existence and convergence for the solution of the combinatorial $\alpha$-Ricci flow, which confirms a parameterized generalization of another conjecture of Glickenstein in [22] on the convergence of combinatorial Ricci flow, gives an almost equivalent characterization of the solvability of combinatorial Yamabe problem for combinatorial $\alpha$-curvature in terms of extended combinatorial $\alpha$-Ricci flow and provides effective algorithms for finding discrete conformal structures with constant combinatorial $\alpha$-curvature or prescribed combinatorial $\alpha$-curvatures.

Theorem 1.5. Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions and . $\alpha \in \mathbb{R}$ is a constant and $\overline{R} : V \to \mathbb{R}$ is a given function.
(a) The solution of the combinatorial $\alpha$-Ricci flows (8) and (9) could be extended by extending the inner angles in triangles by constants. Furthermore, if $\alpha \chi(M) \leq 0$, the extended solution of the normalized combinatorial $\alpha$-Ricci flow (8) is unique for any initial Euclidean discrete conformal structure; If $\alpha R \leq 0$, the extended solution of modified combinatorial $\alpha$-Ricci flow (9) is unique for any initial Euclidean and hyperbolic discrete conformal structure.

(b) Suppose there exists a Euclidean discrete conformal structure with constant combinatorial $\alpha$-curvature and $\alpha \chi(M) \leq 0$. Then the normalized combinatorial $\alpha$-Ricci flow (8) develops no essential singularities. If the solution of (8) develops no removable singularities in finite time, then the solution of (8) exists for all time, converges exponentially fast for any initial Euclidean discrete conformal structure and does not develop removable singularities at time infinity. Furthermore, the extended solution of the normalized combinatorial $\alpha$-Ricci flow (8) in the Euclidean background geometry exists for all time and converges exponentially fast for any initial Euclidean discrete conformal structure.

(c) Suppose there exists a hyperbolic discrete conformal structure with combinatorial $\alpha$-curvature $R$ satisfying one of the following three conditions

1. $\alpha > 0$ and $R_i \leq 0$ for all $i \in V$,
2. $\alpha < 0$ and $R_i \in [0, 2\pi)$ for all $i \in V$,
3. $\alpha = 0$, $R_i \in (-\infty, 2\pi)$ for all $i \in V$ and $\sum_{i=1}^{N} R_i > 2\pi \chi(M)$.

Then the modified combinatorial $\alpha$-Ricci flow (9) in the hyperbolic background geometry develops no essential singularities. If the solution of (9) develops no removable singularities in finite time, then the solution of (9) exists for all time, converges exponentially fast for any initial hyperbolic discrete conformal structure and does not develop removable singularity at time infinity. Furthermore, the extended solution of modified combinatorial $\alpha$-Ricci flow (9) in the hyperbolic background geometry exists for all time and converges exponentially fast for any initial hyperbolic discrete conformal structure.

Remark 6. If $\alpha = 0$, Theorem 1.5 can be extended to a form that holds for any reasonable prescribed combinatorial curvature with Euclidean and hyperbolic background geometry, which was proved by the first author in [41]. The result in Theorem 1.5 (b) can also generalized to the case of prescribed combinatorial $\alpha$-curvature in the Euclidean background geometry. Please refer to Theorem 3.8 for details. The idea of extension to handle the singularities of the combinatorial $\alpha$-Ricci flow comes from Bobenko-Pinkall-Springborn [1].
and Luo [31]. There is another approach to extend the combinatorial $\alpha$-Ricci flow for vertex scaling on polyhedral surfaces introduced in [24, 25], which is called surgery by flipping under the Delaunay condition. In the approach in [24, 25], the condition on the existence of discrete conformal structure with prescribed combinatorial curvature is removed. The approach in [24, 25] was recently used to study combinatorial curvature flows for vertex scaling on polyhedral surfaces in [40, 42, 46].

Combinatorial Calabi flow is another effective combinatorial curvature flow to study combinatorial Yamabe problem, which was first introduced by Ge [6] (see also [7]) for Thurston’s Euclidean circle packing and then further studied in [8, 14, 15, 19, 20, 37, 38, 40–42, 46] and others. We introduce the following combinatorial $\alpha$-Calabi flow for discrete conformal structures on polyhedral surfaces.

**Definition 1.6.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$. $\alpha \in \mathbb{R}$ is a constant and $u : V \to \mathbb{R}$ is a discrete conformal structure defined by $u_i = f_i$ in Euclidean background geometry and by (4) in the hyperbolic background geometry. The combinatorial $\alpha$-Calabi flow for discrete conformal structures on polyhedral surfaces is defined to be

$$\frac{du_i}{dt} = \Delta_\alpha R_{\alpha,i}$$

for Euclidean and hyperbolic background geometry, where the discrete $\alpha$-Laplace operator $\Delta_\alpha$ is defined by

$$\Delta_\alpha g_i = -\frac{1}{\varepsilon_{i,m}} \sum_{j \in V} \frac{\partial K_i}{\partial u_j} g_j$$

for $g : V \to \mathbb{R}$.

**Remark 7.** If $\alpha = 0$, the combinatorial $\alpha$-Calabi flow in Definition 1.6 is the combinatorial Calabi flow introduced in [41], which unifies combinatorial Calabi flow for circle packings studied in [6, 8, 15] and combinatorial Calabi flow for vertex scaling studied in [6, 46] and generalizes them to a very general context. If $\varepsilon \equiv 1$, the combinatorial $\alpha$-Calabi flow in Definition 1.6 is reduced to the combinatorial $\alpha$-Calabi flow for circle packings studied in [14, 19]. If $\varepsilon \equiv 0$, the combinatorial $\alpha$-Calabi flow in Definition 1.6 is reduced to the combinatorial $\alpha$-Calabi flow for vertex scaling studied in [40, 42]. The combinatorial $\alpha$-Calabi flow in Definition 1.6 further covers the case of mixed type discrete conformal structures.

We have the following result on the longtime existence and convergence for the solution of combinatorial $\alpha$-Calabi flow (10).
Theorem 1.7. Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (5) and (6). $\alpha \in \mathbb{R}$ is a constant and $R : V \to \mathbb{R}$ is a given function.

(a) In the case of Euclidean background geometry, if the solution of the combinatorial $\alpha$-Calabi flow (10) converges to a nondegenerate discrete conformal structure, then there exists a discrete conformal structure on $(M, T, \varepsilon, \eta)$ with constant combinatorial $\alpha$-curvature. Furthermore, suppose that there exists a Euclidean discrete conformal structure $u$ on $(M, T, \varepsilon, \eta)$ with constant combinatorial $\alpha$-curvature and $\alpha \chi(M) \leq 0$, then there exists a constant $\delta > 0$ such that if the initial value $u(0)$ satisfies $||R_\alpha(u(0)) - R_\alpha(\overline{u})|| < \delta$ and $\sum_{i=1}^N e^{\alpha u(0)} = \sum_{i=1}^N e^{\alpha \overline{u}_i}$ in the case of $\alpha \neq 0$ or $||u(0) - \overline{u}|| < \delta$ and $\sum_{i=1}^N u(0) = \sum_{i=1}^N \overline{u}_i$ in the case of $\alpha = 0$, the solution of Euclidean combinatorial $\alpha$-Calabi flow (10) exists for all time and converges exponentially fast to $\overline{u}$.

(b) In the case of hyperbolic background geometry, if the solution of the combinatorial $\alpha$-Calabi flow (10) converges to a nondegenerate discrete conformal structure, then there exists a discrete conformal structure on $(M, T, \varepsilon, \eta)$ with zero combinatorial $\alpha$-curvature. Furthermore, suppose that there exists a hyperbolic discrete conformal structure $u$ on $(M, T, \varepsilon, \eta)$ with zero combinatorial $\alpha$-curvature, then there exists a constant $\delta > 0$ such that if $||R_\alpha(u(0)) - R_\alpha(\overline{u})|| < \delta$, the solution of hyperbolic combinatorial $\alpha$-Calabi flow (10) exists for all time and converges exponentially fast to $\overline{u}$.

Remark 8. In the case of $\alpha = 0$, Theorem 1.7 was proved by the first author in [41]. Similar to the case in [41], for generic initial discrete conformal structure in Definition 1.1, the combinatorial $\alpha$-Calabi flow can not be extended in the way used for the combinatorial $\alpha$-Ricci flow in this paper. The global convergence of the combinatorial $\alpha$-Calabi flow (10) is not known up to now. In the special case of vertex scaling, which corresponds to $\varepsilon \equiv 0$ for the discrete conformal structure in Definition 1.1 one can do surgery by flipping under Delaunay condition on the combinatorial $\alpha$-Calabi flow to extend it and obtain the global convergence of the combinatorial $\alpha$-Calabi flow with surgery. Please refer to [40, 42, 46] for more information on this.

The paper is organized as follows. In Section 2 we study the rigidity for combinatorial $\alpha$-curvature of Euclidean and hyperbolic discrete conformal structures on polyhedral surfaces and prove a generalization of Theorem 1.3. In Section 3 we study the combinatorial Yamabe problem for combinatorial $\alpha$-curvature of discrete conformal structures using combinatorial $\alpha$-Ricci flow and combinatorial $\alpha$-Calabi flow and prove generalizations of Theorem 1.5 and Theorem 1.7. In Section 4 we discuss some open problems related to
combinatorial \(\alpha\)-curvatures and combinatorial \(\alpha\)-curvature flows.

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**2 Rigidity of combinatorial \(\alpha\)-curvatures**

**2.1 Euclidean discrete conformal structures**

By Definition 1.2, \(u_i = f_i\) for all \(i \in V\) in Euclidean background geometry. Set \(r_i = e^{u_i}\) for all \(i \in V\), we also call \(r \in \mathbb{R}^N_0\) as a Euclidean discrete conformal structure. The admissible space \(\Omega^E_{ijk}\) of Euclidean discrete conformal structures for a triangle \(\{ijk\} \in F\) in \((M,T,\varepsilon,\eta)\) is defined to be the set of \((r_i, r_j, r_k) \in \mathbb{R}^3_{>0}\) such that the triangle with edge lengths \(l_{ij}, l_{ik}, l_{jk}\) defined by (2) exists in 2-dimensional Euclidean space \(E^2\), i.e.

\[
\Omega^E_{ijk} = \{(r_i, r_j, r_k) \in \mathbb{R}^3_{>0} | l_{rs} + l_{rt} > l_{ts}, \{r, s, t\} = \{i, j, k\}\}.
\]

The admissible space of Euclidean discrete conformal structures on \((M,T,\varepsilon,\eta)\) is defined to be the vectors \(r \in \mathbb{R}^N\) such that \((r_i, r_j, r_k) \in \Omega^E_{ijk}\) for every triangle \(\{ijk\} \in F\) and we use \(\Omega^E\) to denote it. One can also define the admissible space in terms of \(f\), here we take the parameter \(r\) for simplification of notations.

**Lemma 2.1** \([11]\). Suppose \((M,T,\varepsilon,\eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0,1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions \([4]\) and \([6]\). \(\{ijk\} \in F\) is a topological triangle of the weighted triangulated surface \((M,T,\varepsilon,\eta)\).

1. The admissible space \(\Omega^E_{ijk}\) of Euclidean discrete conformal structures for \(\{ijk\} \in F\) is nonempty and simply connected with analytic boundary.

2. The matrix \(A^E_{ijk} := \frac{\partial (\theta^k_i, \theta^i_j, \theta^j_k)}{\partial (u_i, u_j, u_k)}\) is symmetric and negative semi-definite with rank 2 and kernel \(\{c(1,1,1)^T | c \in \mathbb{R}\}\) on \(\Omega^E_{ijk}\), which implies that the matrix \(A^E := \frac{\partial (K_1, \ldots, K_N)}{\partial (u_1, \ldots, u_N)}\) is symmetric and positive semi-definite with rank \(N - 1\) and kernel \(\{c(1,1,\cdots,1)^T \in \mathbb{R}^N | c \in \mathbb{R}\}\) on \(\Omega^E\).
(3) The inner angles \( \theta_{ij}^k, \theta_{jk}^i, \theta_{ki}^j \) defined for \((r_i, r_j, r_k) \in \Omega_{ijk}^E \) could be extended by constants to be continuous functions \( \tilde{\theta}_{ij}^k, \tilde{\theta}_{jk}^i, \tilde{\theta}_{ki}^j \) defined for \((r_i, r_j, r_k) \in \mathbb{R}_0^3 \).

By Lemma 2.1 we can extend the classical combinatorial curvature \( K \) defined on \( \Omega^E \) to be defined for \( r \in \mathbb{R}^N_{>0} \) by setting
\[
\tilde{K}_i = 2\pi - \sum_{(ijk) \in F} \tilde{\theta}_{ij}^k,
\]
which still satisfies the discrete Gauss-Bonnet formula \( \sum_{i=1}^{N} \tilde{K}_i = 2\pi \chi(M) \). As a result, the combinatorial \( \alpha \)-curvature \( R_{\alpha} \) could be extended by setting
\[
\tilde{R}_{\alpha,i} = \frac{\tilde{K}_i}{e^{\alpha u_i}} = \frac{\tilde{K}_i}{r_i^{\alpha}}
\]
for any \( r \in \mathbb{R}^N_{>0} \) and \( i \in V \). We call both vectors \( u = (u_1, ..., u_N) \in \mathbb{R}^N \) and \( r = (r_1, ..., r_N) = (e^{u_1}, ..., e^{u_N}) \in \mathbb{R}^N_{>0} \) as generalized Euclidean discrete conformal structures on \((M, T, \varepsilon, \eta)\), if there is no confusion.

According to Lemma 2.1 the Ricci energy function
\[
F_{ijk}(u_i, u_j, u_k) = \int_0^{(u_i, u_j, u_k)} \theta_{ij}^k du_i + \theta_{jk}^i du_j + \tilde{\theta}_{ki}^j du_k
\]
for a triangle \( \{ijk\} \in F \) is well-defined on \( \mathcal{U}_{ijk}^E = \ln \Omega_{ijk}^E \). Furthermore, \( F_{ijk}(u_i, u_j, u_k) \) is locally concave on \( \mathcal{U}_{ijk}^E \) and locally strictly concave on \( \mathcal{U}_{ijk}^E \cap \{u_i + u_j + u_k = 0\} \) with \( \nabla_{u_i} F_{ijk} = \theta_{ij}^k \) and \( F_{ijk}(u_i + t, u_j + t, u_k + t) = F_{ijk}(u_i, u_j, u_k) + t\pi \).

For further applications, we need to extend \( F_{ijk} \) to be a globally defined function. Let us recall Luo’s generalization [31] of Bobenko-Pinkall-Spingborn’s extension [1] first.

**Definition 2.2** (31, Definition 2.3). A differential 1-form \( w = \sum_{i=1}^{n} a_i(x)dx^i \) in an open subset \( U \subset \mathbb{R}^n \) is said to be continuous if each \( a_i(x) \) is continuous on \( U \). A continuous differential 1-form \( w \) is called closed if \( \int_{\partial \tau} w = 0 \) for each triangle \( \tau \subset U \).

**Theorem 2.3** (31, Corollary 2.6). Suppose \( X \subset \mathbb{R}^n \) is an open convex set and \( A \subset X \) is an open subset of \( X \) bounded by a real analytic codimension-1 submanifold in \( X \). If \( w = \sum_{i=1}^{n} a_i(x)dx_i \) is a continuous closed 1-form on \( A \) so that \( F(x) = \int_a^x w \) is locally convex on \( A \) and each \( a_i \) can be extended continuous to \( X \) by constant functions to a function \( \tilde{a}_i \) on \( X \), then \( \tilde{F}(x) = \int_a^x \sum_{i=1}^{n} \tilde{a}_i(x)dx_i \) is a \( C^1 \)-smooth convex function on \( X \) extending \( F \).

Combining Lemma 2.1 and Theorem 2.3 \( F_{ijk}(u_i, u_j, u_k) \) defined on \( \mathcal{U}_{ijk}^E \) could be extended to be
\[
\tilde{F}_{ijk}(u_i, u_j, u_k) = \int_0^{(u_i, u_j, u_k)} \tilde{\theta}_{ij}^k du_i + \tilde{\theta}_{jk}^i du_j + \tilde{\theta}_{ki}^j du_k,
\]
which is a $C^1$-smooth concave function defined on $\mathbb{R}^3$ with $\nabla_u \tilde{F}_{ijk} = \tilde{\theta}_i^{jk}$. Using the extension $\bar{F}_{ijk}$ of Ricci energy function $F_{ijk}$, we can prove the following rigidity for the extended combinatorial $\alpha$-curvature $\bar{R}_\alpha$, which is a generalization of Theorem 1.3 (a).

**Theorem 2.4.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions [1] and [1]. $\alpha \in \mathbb{R}$ is a constant and $\mathcal{R} : V \to \mathbb{R}$ is a given function. Suppose there exist $r_A \in \Omega^E$ and $r_B \in \mathbb{R}^N_{>0}$ with the same extended combinatorial $\alpha$-curvature $\bar{R}$. If $\alpha \mathcal{R} \equiv 0$, then $r_A = \lambda r_B$ for some positive constant $\lambda \in \mathbb{R}_{>0}$. If $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \neq 0$, then $r_A = r_B$. 

**Proof.** Define the following Ricci energy $F(u)$ for $\mathcal{R}$ by

$$F(u) = - \sum_{\{ijk\} \in F} F_{ijk}(u_i, u_j, u_k) + \int_{u_0}^u (2\pi - \bar{R}_i r_i^\alpha) du_i.$$ 

By direct calculations, we have

$$\nabla_u F(u) = - \sum_{\{ijk\} \in F} \theta_i^{jk} + 2\pi - \bar{R}_i r_i^\alpha = K_i - \bar{R}_i r_i^\alpha$$

and

$$\text{Hess}_u F = \Lambda^E - \alpha \left( \begin{array}{ccc} \bar{R}_1 r_1^\alpha & \cdots & \bar{R}_N r_N^\alpha \end{array} \right)$$

for $u \in \mathcal{U}^E = \ln \Omega^E$. By Lemma 2.1, if $\alpha \mathcal{R} \equiv 0$, then $\text{Hess}_u F$ is positive semi-definite with kernel $\{c(1, 1, \cdots, 1)^T | c \in \mathbb{R}\}$ and $F$ is locally convex on $\mathcal{U}^E$. If $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \neq 0$, then $\text{Hess}_u F$ is positive definite and $F$ is locally strictly convex on $\mathcal{U}^E$.

By the extension $\tilde{F}_{ijk}(u_i, u_j, u_k)$ of $F_{ijk}(u_i, u_j, u_k)$ in (11), the Ricci energy function $F(u)$ defined on $\mathcal{U}^E$ could be extended to be

$$\tilde{F}(u) = - \sum_{\{ijk\} \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{u_0}^u (2\pi - \bar{R}_i r_i^\alpha) du_i,$$

which is a $C^1$-smooth convex function defined on $\mathbb{R}^N$. Furthermore,

$$\nabla_u \tilde{F} = - \sum_{\{ijk\} \in F} \tilde{\theta}_i^{jk} + 2\pi - \bar{R}_i r_i^\alpha = \tilde{K}_i - \bar{R}_i r_i^\alpha,$$

where $\tilde{K}_i = 2\pi - \sum_{\{ijk\} \in F} \tilde{\theta}_i^{jk}$. Then $\tilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $\mathcal{U}^E \cap \{\sum_{i=1}^N u_i = 0\}$ in the case of $\alpha \mathcal{R} \equiv 0$. Similarly, $\tilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $\mathcal{U}^E$ in the case of $\alpha \mathcal{R} \leq 0$ and $\alpha \mathcal{R} \neq 0$. 

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If there exist \( r_A \in \Omega^E \) and \( r_B \in \mathbb{R}^N_0 \) with the same extended combinatorial \( \alpha \)-curvature \( \mathcal{R} \), set \( h(t) = \tilde{F}((1-t)u_A + tu_B), \ t \in [0,1] \). Then \( h(t) \) is a \( C^1 \) convex function with

\[
h'(t) = \sum_{i=1}^N \nabla_u \tilde{F}(1-t)u_A + tu_B \cdot (u_{B,i} - u_{A,i}) = \sum_{i=1}^N (\tilde{K}_i - \mathcal{R}^T_i)_{(1-t)u_A + tu_B} \cdot (u_{B,i} - u_{A,i}).
\]

By the assumption that \( \tilde{R}_\alpha(u_A) = \tilde{R}_\alpha(u_B) = \mathcal{R} \), we have \( h'(0) = h'(1) = 0 \), which implies \( h'(t) \equiv 0 \) by the convexity of \( h(t) \). Note that \( u_A \) is in the open subset \( \mathcal{U}^E \subseteq \mathbb{R}^N \), there exists \( 0 < \epsilon < 1 \) such that \( (1-t)u_A + tu_B \in \mathcal{U}^E \) and \( h(t) \) is smooth for \( t \in [0,\epsilon) \).

In the case of \( \alpha \mathcal{R} \leq 0 \) and \( \alpha \mathcal{R} \neq 0 \), \( \tilde{F}(u) \) is locally strictly convex on \( \mathcal{U}^E \), which implies \( h(t) \) is locally strictly convex on \([0,\epsilon]\) and \( h'(\epsilon) > 0 \) by \( h'(0) = 0 \). This contradicts \( h'(t) \equiv 0 \) on \([0,1]\). So \( u_A = u_B \), which implies \( r_A = r_B \).

In the case of \( \alpha \mathcal{R} \equiv 0 \), we have \( h(t) \) is \( C^1 \) convex on \([0,1]\) and smooth on \([0,\epsilon)\). \( h'(t) \equiv 0 \) for \( t \in [0,1] \) implies that \( h''(t) \equiv 0 \) for \( t \in [0,\epsilon) \). Note that

\[
h''(t) = (u_B - u_A)^T \cdot \text{Hess} F|_{(1-t)u_A + tu_B} \cdot (u_B - u_A)
\]

for \( t \in [0,\epsilon) \) and \( \text{Hess} F \) is positive semi-definite with kernel \( \{c(1,1,\cdots,1)^T \in \mathbb{R}^N | c \in \mathbb{R} \} \), we have \( u_B - u_A = \mu(1,1,\cdots,1)^T \) for some constant \( \mu \in \mathbb{R} \), which implies \( r_B = \lambda r_A \) with \( \lambda = e^\epsilon \).

As a corollary of Theorem 2.4, we have the following result on the global rigidity for discrete conformal structures with constant combinatorial \( \alpha \)-curvature.

**Corollary 2.5.** Suppose \((M,T,\varepsilon,\eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0,1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \([3]\) and \([6]\). \( \alpha \in \mathbb{R} \) is a constant. If \( \alpha \chi(M) = 0 \), there exists at most one Euclidean discrete conformal structure \( f : V \to \mathbb{R} \) with constant combinatorial \( \alpha \)-curvature up to a vector \( c(1,1,\cdots,1)^T, \ c \in \mathbb{R} \). If \( \alpha \chi(M) < 0 \), there exists at most one Euclidean discrete conformal structure \( f : V \to \mathbb{R} \) with constant combinatorial \( \alpha \)-curvature.

### 2.2 Hyperbolic discrete conformal structures

The admissible space \( \Omega^H_{ijk} \) of hyperbolic discrete conformal structures for a triangle \( \{ijk\} \in F \) is defined to be the set of \((f_i, f_j, f_k) \in \mathbb{R}^3 \) such that the triangle with edge lengths \( l_{ij}, l_{ik}, l_{jk} \) defined by \([3]\) exists in 2-dimensional hyperbolic space \( \mathbb{H}^2 \), i.e.

\[
\Omega^H_{ijk} = \{(f_i, f_j, f_k) \in \mathbb{R}^3 | l_{rs} + l_{rt} > l_{ts}, \{r,s,t\} = \{i,j,k\}\}.
\]

The admissible space of hyperbolic discrete conformal structures on \((M,T,\varepsilon,\eta)\) is defined to be the vectors \( f \in \mathbb{R}^N \) such that \((f_i, f_j, f_k) \in \Omega^H_{ijk} \) for every triangle \( \{ijk\} \in F \) and we use \( \Omega^H \) to denote it.
In the hyperbolic background geometry, for the map \( u \) defined by (4), we have \( u = (u_1, \ldots, u_N) \in \mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1} \) by Remark 3 where \( N_0 \) is the number of the vertices \( i \in V \) with \( \varepsilon_i = 0 \) and \( N_1 = N - N_0 \). For simplicity, we also call \( u \in \mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1} \) as hyperbolic discrete conformal structure.

**Lemma 2.6**. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \rightarrow \{0, 1\} \) and \( \eta : E \rightarrow \mathbb{R} \) satisfying the structure conditions \( \mathcal{T} \) and \( \mathcal{E} \). \( \{ijk\} \in F \) is a topological triangle of the weighted triangulated surface \((M, \mathcal{T}, \varepsilon, \eta)\).

1. The admissible space \( \Omega^H_{ijk} \) of hyperbolic discrete conformal structures for \( \{ijk\} \in F \) is nonempty and simply connected with analytic boundary.

2. The matrix \( \Lambda^H_{ijk} := \frac{\partial(\theta^i, \theta^j, \theta^k)}{\partial(u_i, u_j, u_k)} \) is symmetric and negative definite on \( \Omega^H_{ijk} \), which implies the matrix \( \Lambda^H := \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} \) is symmetric and positive definite on \( \Omega^H \).

3. The inner angles \( \theta^i, \theta^j, \theta^k \) defined for \( (f_i, f_j, f_k) \in \Omega^H_{ijk} \) could be extended by constants to be continuous functions \( \tilde{\theta}^i, \tilde{\theta}^j, \tilde{\theta}^k \) defined for \( (f_i, f_j, f_k) \in \mathbb{R}^3 \).

By Lemma 2.6, we can extend the classical combinatorial curvature \( K \) defined on \( \Omega^H \) to be defined on \( \mathbb{R}^N \) by setting \( \tilde{K}_i = 2\pi - \sum_{\{ijk\} \in F} \tilde{\theta}^i_{ijk} \). As a result, the combinatorial \( \alpha \)-curvature \( \tilde{R}_\alpha \) defined for \( f \in \Omega^H \) could be extended to be defined for \( f \in \mathbb{R}^N \) by setting \( \tilde{R}_{\alpha,i} = \frac{\tilde{K}_i}{\alpha^i} \). We call both vectors \( f = (f_1, \ldots, f_N) \in \mathbb{R}^N \) and \( u = (u_1, \ldots, u_N) \in \mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1} \) as generalized hyperbolic discrete conformal structures on \((M, \mathcal{T}, \varepsilon, \eta)\), if there is no confusion.

By Lemma 2.6, the Ricci energy function

\[
F_{ijk}(u_i, u_j, u_k) = \int_0^{(u_i, u_j, u_k)} \theta^i_{ijk} du_i + \theta^j_{ijk} du_j + \theta^k_{ijk} du_k
\]

for a triangle \( \{ijk\} \in F \) is a well-defined locally strictly concave function defined on \( \mathcal{U}^H_{ijk} \) with \( \nabla_u F_{ijk} = \theta^i_{ijk} \), where \( \mathcal{U}^H_{ijk} \) is the image of \( \Omega^H_{ijk} \) under the map (4). Combining Lemma 2.6 and Theorem 2.3, \( F_{ijk}(u_i, u_j, u_k) \) defined on \( \mathcal{U}^H_{ijk} \) could be extended to be

\[
\tilde{F}_{ijk}(u_i, u_j, u_k) = \int_0^{(u_i, u_j, u_k)} \tilde{\theta}^i_{ijk} du_i + \tilde{\theta}^j_{ijk} du_j + \tilde{\theta}^k_{ijk} du_k,
\]

which is a \( C^1 \)-smooth concave function defined on \( \mathbb{R}^{n_0} \times \mathbb{R}_{<0}^{n_1} \) with \( \nabla_u \tilde{F}_{ijk} = \tilde{\theta}_i \), where \( n_0 \) is the number of the vertices in \( \{i,j,k\} \) with \( \varepsilon = 0 \) and \( n_1 = 3 - n_0 \).

Paralleling to Theorem 2.4 for generalized Euclidean discrete conformal structures, we prove the following rigidity of the extended combinatorial \( \alpha \)-curvature \( \tilde{R}_\alpha \) for generalized hyperbolic discrete conformal structures, which is a generalization of Theorem 1.3 (b).
Theorem 2.7. Suppose \( (M, T, \varepsilon, \eta) \) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \( \text{(5)} \) and \( \text{(6)} \). \( \alpha \in \mathbb{R} \) is a constant and \( \overline{R} : V \to \mathbb{R} \) is a given function with \( \alpha \overline{R} \leq 0 \). If there exist \( f_A \in \Omega^H \) and \( f_B \in \mathbb{R}^N \) with the same extended combinatorial \( \alpha \)-curvature \( \overline{R} \), then \( f_A = f_B \).

Proof. Define the following Ricci energy \( F(u) \) for \( \overline{R} \) by

\[
F(u) = - \sum_{\{ijk\} \in F} F_{ijk}(u_i, u_j, u_k) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{R}_i e^{\alpha u_i}) du_i.
\]

Then

\[
\text{Hess}_u F(u) = \Lambda^H - \alpha \begin{pmatrix} \overline{R}_1 e^{\alpha u_1} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \overline{R}_N e^{\alpha u_N} \end{pmatrix}
\]

on \( U^H = u(\Omega^H) \). Combining Lemma 2.6 with the condition \( \alpha \overline{R} \leq 0 \), \( \text{Hess}_u F \) is positive definite and \( F \) is locally strictly convex on \( U^H \).

Paralleling to the Euclidean case, the Ricci energy function \( F(u) \) defined on \( U^H \) could be extended to be

\[
\tilde{F}(u) = - \sum_{\{ijk\} \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{R}_i e^{\alpha u_i}) du_i,
\]

which is a \( C^1 \)-smooth convex function defined on \( \mathbb{R}^N_0 \times \mathbb{R}_<^N \) and locally strictly convex on \( U^H \subset \mathbb{R}^N_0 \times \mathbb{R}_<^N \) with

\[
\nabla_{u_i} \tilde{F} = - \sum_{\{ijk\} \in F} \tilde{\theta}_i + 2\pi - \overline{R}_i e^{\alpha u_i} = \tilde{K}_i - \overline{R}_i e^{\alpha u_i}.
\]

If there exist \( f_A \in \Omega^H \) and \( f_B \in \mathbb{R}^N \) with the same extended combinatorial \( \alpha \)-curvature \( \overline{R} \), set \( h(t) = \tilde{F}((1-t)u_A + tu_B), \ t \in [0, 1], \) where \( u_A = u(f_A), u_B = u(f_B) \) with \( u \) given by \( \text{(1)} \). Then \( h(t) \) is a \( C^1 \) convex function for \( t \in [0, 1] \). The proof in the following is paralleling to that for Theorem 2.4, we omit the details here.

3 Deformation of discrete conformal structures

In this section, we use combinatorial curvature flows to study the combinatorial Yamabe problem for combinatorial \( \alpha \)-curvature of discrete conformal structures on polyhedral surfaces. In the Euclidean background geometry, we study constant combinatorial
α-curvature problem and prescribed combinatorial α-curvature problem using the combinatorial α-Ricci flow (8), (9) and the combinatorial α-Calabi flow (10). In hyperbolic background geometry, we take the constant combinatorial α-curvature problem as a special case of prescribed combinatorial α-curvature problem and use the modified combinatorial α-Ricci flow (9) and the following modified combinatorial α-Calabi flow

$$\frac{du_i}{dt} = \Delta_\alpha(R_\alpha - R)_i$$

(14)

to study the prescribed combinatorial α-curvature problem.

3.1 Local convergence of combinatorial α-curvature flows

We have the following properties of the normalized combinatorial α-Ricci flow (8) and the combinatorial α-Calabi flow (10) in Euclidean background geometry.

**Lemma 3.1.** In Euclidean background geometry, if \(\alpha = 0\), then \(\sum_{i=1}^{N} u_i\) is invariant along the normalized combinatorial α-Ricci flow (8) and the combinatorial α-Calabi flow (10). If \(\alpha \neq 0\), \(||r||^\alpha = \sum_{i=1}^{N} r_i^\alpha = \sum_{i=1}^{N} e^{\alpha u_i}\) is invariant along the normalized combinatorial α-Ricci flow (8) and the combinatorial α-Calabi flow (10).

**Proof.** The case of \(\alpha = 0\) has been proved in [41], we only prove the case of \(\alpha \neq 0\) here. If \(\alpha \neq 0\), along the equation (8), we have

$$\frac{d(\sum_{i=1}^{N} e^{\alpha u_i})}{dt} = \sum_{i=1}^{N} \alpha e^{\alpha u_i} \frac{du_i}{dt} = \alpha \left(2\pi\chi(M) - \sum_{i=1}^{N} K_i\right) = 0,$$

which implies \(\sum_{i=1}^{N} e^{\alpha u_i}\) is invariant along the normalized combinatorial α-Ricci flow (8). Similarly, along the equation (10), we have

$$\frac{d(\sum_{i=1}^{N} e^{\alpha u_i})}{dt} = -\alpha \sum_{j=1}^{N} \sum_{i=1}^{N} (A^E)_{ij} R_{\alpha,j} = 0$$

by Lemma 2.1 which implies \(\sum_{i=1}^{N} e^{\alpha u_i}\) is invariant along the combinatorial α-Calabi flow (10).

The normalized combinatorial α-Ricci flow (8) and the combinatorial α-Calabi flow (10) are ODE systems with smooth coefficients. Therefore, the solutions always exist locally around the initial time \(t = 0\). We further have the following result on the longtime behaviors for the solutions of normalized combinatorial α-Ricci flow (8) and combinatorial α-Calabi flow (10) in Euclidean background geometry, which is a generalization of Theorem 1.7 (a).
Theorem 3.2. Suppose \((M, \mathcal{T}, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions \([5]\) and \([6]\). \(\alpha \in \mathbb{R}\) is a constant.

(1) If the solution of normalized combinatorial \(\alpha\)-Ricci flow \([5]\) or the solution of combinatorial \(\alpha\)-Calabi flow \([10]\) converges in \(\mathcal{U}_E\), there exists a discrete conformal structure with constant combinatorial \(\alpha\)-curvature.

(2) Suppose that there exists a discrete conformal structure \(\pi \in \mathcal{U}_E\) with constant combinatorial \(\alpha\)-curvature and \(\alpha \chi(M) \leq 0\), then there exists a constant \(\delta > 0\) such that if the initial value \(u(0)\) satisfies \(\|R_\alpha(u(0)) - R_\alpha(\pi)\| < \delta\) and \(\sum_{i=1}^{N} \epsilon_i^{\alpha}u_i(0) = \sum_{i=1}^{N} \epsilon_i^{\alpha}\pi_i\) in the case of \(\alpha \neq 0\) or \(\|u(0) - \pi\| < \delta\) and \(\sum_{i=1}^{N} u_i(0) = \sum_{i=1}^{N} \pi_i\) in the case of \(\alpha = 0\), the solutions of normalized combinatorial \(\alpha\)-Ricci flow \([5]\) and the combinatorial \(\alpha\)-Calabi flow \([10]\) exist for all time and converge exponentially fast to \(\pi\) respectively.

Proof. The case of \(\alpha = 0\) has been proved in \([41]\), we only prove the case of \(\alpha \neq 0\) here. Suppose \(u(t)\) is a solution of the normalized combinatorial \(\alpha\)-Ricci flow \([5]\). If \(\bar{\pi} := u(+) = \lim_{t \to +\infty} u(t)\) exists in \(\mathcal{U}_E\), then \(R_\alpha(\bar{\pi}) = \lim_{t \to +\infty} R_\alpha(u(t))\) exists. Furthermore, there exists a sequence \(\xi_n \in (n, n+1)\) such that

\[
\begin{align*}
\lim_{n \to \infty} u_i(n + 1) - u_i(n) = u_i'(\xi_n) &= R_{\alpha, av} - R_{\alpha, i}(u(\xi_n)) \to 0
\end{align*}
\]

which implies \(R_\alpha(\bar{\pi}) = R_{\alpha, av}\) and \(\bar{\pi}\) is a discrete conformal structure with constant combinatorial \(\alpha\)-curvature \(R_{\alpha, av}\). Similarly, if the combinatorial \(\alpha\)-Calabi flow \([10]\) converges, there exists a discrete conformal structure with constant combinatorial \(\alpha\)-curvature.

Suppose there exists a discrete conformal structure \(\pi \in \mathcal{U}_E\) with constant combinatorial \(\alpha\)-curvature \(R_{\alpha, av}\). For the normalized combinatorial \(\alpha\)-Ricci flow \([5]\), set \(\Gamma_i(u) = R_{\alpha, av} - R_{\alpha, i}\). By direct calculations, we have

\[
\begin{align*}
D\Gamma|_{u=\pi} = \alpha R_{\alpha, av} I - \sum_{\epsilon_i} \left( \Lambda^E + \alpha R_{\alpha, av} \frac{r^\alpha \cdot (r^\alpha)^T}{\|r\|^2_{\alpha}} \right) \\
&= -\sum_{\epsilon_i} \left( L - \alpha R_{\alpha, av} \left[ I - \frac{r^\alpha \cdot (r^\alpha)^T}{\|r\|^2_{\alpha}} \right] \right) \Sigma \varphi,
\end{align*}
\]

where \(\Sigma = \text{diag}\{r_1, r_2, ..., r_N\}\), \(L = \sum_{\epsilon_i} \Lambda^E \Sigma^{-\frac{\varphi}{2}}\). Note that the matrix \(I - \frac{r^\alpha \cdot (r^\alpha)^T}{\|r\|^2_{\alpha}}\) has eigenvalues 1 \((N-1)\) times and 0 \((1)\) time and its kernel is \(\{cr^\varphi | c \in \mathbb{R}\}\). Further note that the matrix \(L\) is positive semi-definite with 1-dimensional kernel \(\{cr^\varphi | c \in \mathbb{R}\}\) by Lemma \([2,1]\). Therefore, by Lemma \([3,1]\) and the condition \(\alpha \chi(M) \leq 0\), \(D\Gamma|_{u=\pi}\) has \(N - 1\) negative eigenvalues and a zero eigenvalue with 1-dimensional kernel \(\{c(1, 1, ..., 1)\}\). Let \(\tilde{u}_t = \)
\[ e^{\frac{\partial}{\partial t}} u_i = r_i^2 u_i \text{ and } \Sigma = \Sigma(u) = \text{diag}\{r_1, \ldots, r_N\}, \text{ then } \tilde{u} = \Sigma^2 u. \] The combinatorial \( \alpha \)-Ricci flow with respect to \( \tilde{u} \) is \( \frac{du_i}{dt} = r_i^2 du_i - r_i^2 \Gamma_i (u) \). Let \( \tilde{\Gamma}(u) = \Sigma^2 \Gamma(\Sigma^{-\frac{2}{3}} \tilde{u}) \), then the matrix
\[
\left( \frac{\partial \tilde{\Gamma}}{\partial u} \right)_{\tilde{u} = \Sigma^2 \tilde{u}} = \Sigma^{-\frac{2}{3}} \cdot \left( \frac{\partial \Gamma}{\partial u} \right)_{u = \Sigma^2 \tilde{u}} \cdot \Sigma^{-\frac{2}{3}}.
\]
is symmetric and negative semi-definite with 1-dimensional kernel \( \{c\Sigma^2 | c \in \mathbb{R} \} \) by the property of \( D\Gamma|_{u = \Sigma^2} \). The hypersurface \( \{ r \in \mathbb{R}^N | \sum_{i=1}^N r_i^a = \sum_{i=1}^N e^{\alpha u_i} = \sum_{i=1}^N r_i^a(0) \} \) is equivalent to \( \{ \tilde{u}_i \in \mathbb{R} | \sum_{i=1}^N e^{\alpha v_i} \tilde{u}_i = \sum_{i=1}^N r_i^a(0) \} \), the normal of which at \( \tilde{u}_i \) is \( n_i = \alpha e^{-\frac{2}{3} \Sigma^2} e^{\alpha - \frac{2}{3} \Sigma^2} \tilde{u}_i = e^{\alpha - \frac{2}{3} \Sigma^2} e^{\alpha \Sigma^2} = \alpha e^{\frac{2}{3} \Sigma^2}. \) Thus the matrix \( \left( \frac{\partial \tilde{\Gamma}}{\partial u} \right)_{\tilde{u} = \Sigma^2 \tilde{u}} \) restricted to the hypersurface \( \{ r \in \mathbb{R}^N | \sum_{i=1}^N r_i^a = \sum_{i=1}^N r_i^a(0) \} \) is negative definite, which implies that \( D\Gamma|_{u = \Sigma^2} \) restricted to the hypersurface \( \{ r \in \mathbb{R}^N | \sum_{i=1}^N r_i^a = \sum_{i=1}^N r_i^a(0) \} \) is negative definite. Therefore, \( \Sigma \) is a local attractor of the normalized combinatorial \( \alpha \)-Ricci flow \( (8) \). Then the conclusion follows from Lyapunov Stability Theorem \( (33), \text{ Chapter 5} ) \).

Similarly, for the combinatorial \( \alpha \)-Calabi flow \( (14) \), set \( \Gamma_i(u) = \Delta_\alpha R_{\alpha, i}. \) By direct calculations, we have
\[
D\Gamma|_{u = \Sigma^2} = -\Sigma^{-\alpha} \Lambda^E \Sigma^{-\alpha} \Lambda^E + \alpha R_{\alpha, av} \Sigma^{-\alpha} \Lambda^E
= -\Sigma^{-\frac{2}{3}} \left( \Sigma^{-\frac{2}{3}} \Lambda^E \Sigma^{-\alpha} \Lambda^E \Sigma^{-\frac{2}{3}} - \alpha R_{\alpha, av} \Sigma^{-\frac{2}{3}} \Lambda^E \Sigma^{-\frac{2}{3}} \right) \Sigma^2.
\]
By Lemma \( 2.16 \) and the condition \( \alpha \chi(M) \leq 0, D\Gamma|_{u = \Sigma^2} \) has \( N - 1 \) negative eigenvalues and a zero eigenvalue with 1-dimensional kernel \( \{c(1, 1, \ldots, 1) \} \). Using the above trick again, \( D\Gamma|_{u = \Sigma^2} \) restricted to the hypersurface \( \{ r \in \mathbb{R}^N | \sum_{i=1}^N r_i^a = \sum_{i=1}^N r_i^a(0) \} \) is negative definite. By Lemma \( 3.1 \) \( \Sigma^2 \) is a local attractor of the combinatorial \( \alpha \)-Calabi flow \( (10) \). Then the conclusion follows from Lyapunov Stability Theorem \( (33), \text{ Chapter 5} ) \).

Similar to Theorem \( 3.2 \) for Euclidean background geometry, we have the following result on the longtime existence and convergence for the solutions of modified combinatorial \( \alpha \)-Ricci flow \( (9) \) and modified combinatorial \( \alpha \)-Calabi flow \( (14) \) in the hyperbolic background geometry, which is a generalization of Theorem \( 1.7 \) \( (b) \).

**Theorem 3.3.** Suppose \( (M, \mathcal{T}, \varepsilon, \eta) \) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \( (5) \) and \( (6) \), \( \alpha \in \mathbb{R} \) is a constant and \( \overline{R} : V \to \mathbb{R} \) is a given function.

1. If the solution of the modified combinatorial \( \alpha \)-Ricci flow \( (9) \) or the solution of modified combinatorial \( \alpha \)-Calabi flow \( (14) \) converges in \( \mathcal{U}^H \), there exists a hyperbolic discrete conformal structure in \( \mathcal{U}^H \) with combinatorial \( \alpha \)-curvature \( \overline{R} \).

2. Suppose that there exists a discrete conformal structure \( \overline{\Sigma} \in \mathcal{U}^H \) with combinatorial \( \alpha \)-curvature \( \overline{R} \) and \( \alpha \overline{R} \leq 0 \), then there exists a constant \( \delta > 0 \) such that if \( || R_{\alpha}(u(0)) || \)
\[ R_\alpha(\overline{\pi}) \| < \delta, \] the solutions of modified combinatorial \( \alpha \)-Ricci flow (9) and modified combinatorial \( \alpha \)-Calabi flow (14) exist for all time and converge exponentially fast to \( \overline{\pi} \) respectively.

**Proof.** The proof of Theorem 3.3 is similar to that of Theorem 3.2, so we only give some necessary steps of the proof for the second part here. For the modified combinatorial \( \alpha \)-Ricci flow (9), set \( \Gamma_i(u) = R_i - R_{\alpha,i} \). By direct calculations, we have

\[ D\Gamma|_{u=\overline{\pi}} = -\Sigma^{-\frac{3}{2}} \Lambda^H + \alpha L' = -\Sigma^{-\frac{3}{2}} (\Sigma^{-\frac{3}{2}} \Lambda^H \Sigma^{-\frac{3}{2}} - \alpha L') \Sigma^{-\frac{3}{2}}, \]

where \( \Sigma = \text{diag}\{e^{u_1}, e^{u_2}, ..., e^{u_N}\} \), \( L' = \text{diag}\{R_1, R_2, ..., R_N\} \). By Lemma 2.6 and the condition \( \alpha R \leq 0 \), \( D\Gamma|_{u=\overline{\pi}} \) has \( N \) negative eigenvalues, which implies \( \overline{\pi} \) is a local attractor of the modified combinatorial \( \alpha \)-Ricci flow (9). Then the conclusion follows from Lyapunov Stability Theorem ([33], Chapter 5).

Similarly, for the modified combinatorial \( \alpha \)-Calabi flow (14), set \( \Gamma_i(u) = \Delta (R_\alpha - \overline{R})_i \). By direct calculations, we have

\[ D\Gamma|_{u=\overline{\pi}} = -\Sigma^{-\frac{3}{2}} \Lambda^H + \alpha L' = -\Sigma^{-\frac{3}{2}} (\Sigma^{-\frac{3}{2}} \Lambda^H \Sigma^{-\frac{3}{2}} - \alpha \Sigma^{-\frac{3}{2}} \Lambda^H \Sigma^{-\frac{3}{2}} L') \Sigma^{-\frac{3}{2}}, \]

where \( Q = \Sigma^{-\frac{3}{2}} \Lambda^H \Sigma^{-\frac{3}{2}} \) is a symmetric and positive definite matrix by Lemma 2.6. By \( \alpha \overline{R} \leq 0 \), \( D\Gamma|_{u=\overline{\pi}} \) has \( N \) negative eigenvalues, which implies \( \overline{\pi} \) is a local attractor of the modified combinatorial \( \alpha \)-Calabi flow (14). Then the conclusion follows from Lyapunov Stability Theorem ([33], Chapter 5).

### 3.2 Uniqueness for the solution of extended combinatorial \( \alpha \)-Ricci flow

Theorem 3.2 and Theorem 3.3 give the longtime existence and convergence for the solutions of the combinatorial \( \alpha \)-curvature flows for initial values with small energy. But for general initial value, the combinatorial \( \alpha \)-curvature flows may develop singularities, including the removable singularities and the essential singularities. For the combinatorial \( \alpha \)-Ricci flow, one can extend it through the removable singularities by extending the combinatorial \( \alpha \)-curvature.

**Definition 3.4.** Suppose \( (M, \mathcal{T}, \varepsilon, \eta) \) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0,1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions (4).
and (6). For Euclidean and hyperbolic background geometry, the extended combinatorial \(\alpha\)-Ricci flow for discrete conformal structures on polyhedral surfaces is defined to be

\[
\frac{du_i}{dt} = -\tilde{R}_{\alpha,i}
\]

and the extended modified combinatorial \(\alpha\)-Ricci flow is defined to be

\[
\frac{du_i}{dt} = \overline{R}_i - \tilde{R}_{\alpha,i},
\]  

(15)

where \(\overline{R} : V \to \mathbb{R}\) is a function and \(\tilde{R}_{\alpha,i} = \frac{\tilde{R}_i}{e^{u_i}} = \frac{1}{e^{u_i}}(2\pi - \sum_{\{ijk\} \in F} \tilde{\theta}_{ijk}^i).

Note that for the modified combinatorial \(\alpha\)-Ricci flow (9), \(\overline{R}_i - R_{\alpha,i}\) is smooth and locally Lipschitz as a function of \(u \in U\) (or \(U_E\) or \(U_H\)). By Picard’s uniqueness for the solution of ODE, the modified combinatorial \(\alpha\)-Ricci flow (9) has a unique solution \(u(t), t \in [0, T]\) for some \(T > 0\). As noted by the first author in [41], the extension \(\tilde{K}\) of the classical combinatorial curvature \(K\) is not a locally Lipschitz function of the generalized discrete conformal structures. This implies that the extended combinatorial \(\alpha\)-curvature \(\tilde{R}_\alpha\) is only a continuous function of the generalized discrete conformal structures \(f \in \mathbb{R}^N\) and not a locally Lipschitz function. Therefore, there may exist more than one solution for the extended combinatorial \(\alpha\)-Ricci flow by the standard ODE theory. However, we can prove the following uniqueness for the solutions of the extended modified combinatorial \(\alpha\)-Ricci flow (15) under the structure conditions (5) and (6), which is a generalization of Theorem 1.5 (a).

**Theorem 3.5.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \(\varepsilon : V \to \{0, 1\}\) and \(\eta : E \to \mathbb{R}\) satisfying the structure conditions (5) and (6). \(\alpha \in \mathbb{R}\) is a constant and \(\overline{R} : V \to \mathbb{R}\) is a given function with \(\alpha \overline{R} \leq 0\). For any initial generalized discrete conformal structure, the solution of the extended modified combinatorial \(\alpha\)-Ricci flow (15) is unique.

**Proof.** In the case of \(\alpha = 0\), Theorem 3.5 was proved by the first author in [41]. Here we only prove the case of \(\alpha \neq 0\), the idea of which comes from [41]. Define the following map in Euclidean background geometry

\[
w^E(u) : \mathbb{R}^N \to W^E \subseteq \mathbb{R}^N
\]

\(u \mapsto w^E(u) := (w^E_1(u_1), ..., w^E_N(u_N))\),

where \(w^E_i(u_i) = \int_{u_i}^{u_i + 1} e^{\alpha x} dx\) and \(W^E = w^E(\mathbb{R}^N)\). Note that \(w^E_i(u_i)\) is strictly increasing in \(u_i\), the map \(w^E(u) : \mathbb{R}^N \to W^E\) is a diffeomorphism. The map \(w^H(u)\) in hyperbolic background geometry can be defined similarly. Except for different domains and images, the map \(w^E_i\) and \(w^H_i\) are similar, we use \(w_i\) to represent \(w^E_i\) and \(w^H_i\) for simplicity, if there
is no confusion. Denote the inverse map of $w(u)$ by $u(w)$, we define a new function on $W$ ($W^E$ or $W^H$) by

$$
\hat{F}(w) := \hat{F}(u(w)),
$$

where $\hat{F}(u)$ is defined by (12) for Euclidean background geometry and by (13) for hyperbolic background geometry. Note that the extended modified combinatorial $\alpha$-Ricci flow (13) can be written as

$$
u'_i(t) = -\frac{1}{\eps(w_i)} \nabla_{w_i} \hat{F}.
$$

Therefore, the equation (16) is equivalent to the following equation in the $w$-coordinate

$$
u'_i(t) = -\nabla_{w_i} \hat{F},
$$

which means that the extended modified combinatorial $\alpha$-Ricci flow (15) is equivalent to a negative gradient flow of the function $\hat{F}$ in the $w$-coordinate.

Suppose $u_A(t)$ and $u_B(t)$ are two solutions of the extended modified combinatorial $\alpha$-Ricci flow (15) with $u_A(0) = u_B(0)$. Then $w_A(0) = w_B(0)$ as the map $w(u)$ is a diffeomorphism. There exists $T \in (0, +\infty)$ such that $\{u_A(t)|t \in [0, T]\}$ and $\{u_B(t)|t \in [0, T]\}$ lie in a compact set and hence $\{w_A(t)|t \in [0, T]\}$ and $\{w_B(t)|t \in [0, T]\}$ lie in a compact convex subset $W'$ of $W$. We claim that $\hat{F}$ is $(-\lambda)$-convex on $W'$, i.e. there exists a positive constant $\lambda$ such that for any $w_A, w_B \in W'$,

$$(\nabla_{w} \hat{F}(w_A) - \nabla_{w} \hat{F}(w_B)) \cdot (w_A - w_B) + \lambda|w_A - w_B|^2 \geq 0.$$ 

Set $h(t) = |w_A(t) - w_B(t)|^2$, then

$$
h'(t) = 2(w_A(t) - w_B(t)) \cdot (w'_A(t) - w'_B(t)) \\
= -2(w_A(t) - w_B(t)) \cdot (\nabla_{w} \hat{F}(w_A(t)) - \nabla_{w} \hat{F}(w_B(t))) \\
\leq 2\lambda |w_A(t) - w_B(t)|^2 = 2\lambda h(t)
$$

by the claim, which implies $h(t) \leq h(0)e^{2\lambda t}$. Note that $h(0) = 0$ and $h(t) \geq 0$, then we have $h(t) \equiv 0$. Therefore, $w_A(t) = w_B(t)$, which implies $u_A(t) = u_B(t)$.

We use Ge-Hua’s trick in (8) to prove the claim. Note that $\hat{F}$ is a $C^1$-smooth convex function defined on $\mathbb{R}^N$ by the proof of Theorem 2.3 for Euclidean background geometry and by the proof of Theorem 2.7 for hyperbolic background geometry. By mollifying $\hat{F}$ using the standard mollifier $\varphi_{\epsilon}(u) = \frac{1}{\epsilon^N} \varphi\left(\frac{u}{\epsilon}\right)$ with

$$\varphi(u) = \begin{cases} 
C e^{1-|u|^2} & |u| < 1, \\
0 & |u| \geq 1,
\end{cases}$$

and
where the positive constant $C$ is chosen such that $\int_{\mathbb{R}^N} \varphi(u) du = 1$, we have $\tilde{F}_\epsilon = \tilde{F} * \varphi_\epsilon$ is a smooth convex function of $u$ and $\tilde{F}_\epsilon \to \tilde{F}$ in $C^1_{loc}$ as $\epsilon \to 0$. Moreover, $\nabla_{u_i} \tilde{F}_\epsilon = \nabla_{u_i} \tilde{F} * \varphi_\epsilon = (\tilde{K}_i - \tilde{R}_ie^{\alpha u_i}) * \varphi_\epsilon$. Set $\tilde{F}_\epsilon(w) = \tilde{F}_\epsilon(u(w))$, then by the chain rules, we have $\nabla_{w} \tilde{F}_\epsilon = \nabla_{u} \tilde{F}_\epsilon \frac{\partial u}{\partial w}$ and

$$\nabla^2_{w_i, w_j} \tilde{F}_\epsilon = \frac{\partial^2 \tilde{F}_\epsilon}{\partial u_i \partial u_j} \frac{\partial u_i}{\partial w_i} \frac{\partial u_j}{\partial w_j} + \frac{\partial \tilde{F}_\epsilon}{\partial u_i} \frac{\partial^2 u_i}{\partial w_i \partial w_j} - \frac{1}{2} (\tilde{K}_i - \tilde{R}_ie^{\alpha u_i}) * \varphi_\epsilon e^{-\alpha u_i} \delta_{ij}.$$

Note that on the compact subset $W'$, there exists a positive constant $\lambda$ such that

$$\frac{1}{2} |(\tilde{K}_i - \tilde{R}_ie^{\alpha u_i}) * \varphi_\epsilon e^{-\alpha u_i}| \leq C |\alpha| e^{-\alpha u_i} \leq \lambda.$$

By the convexity of $\tilde{F}_\epsilon$, we have

$$\nabla^2_{w} \tilde{F}_\epsilon \geq \frac{\partial u}{\partial w} \nabla^2_{u} \tilde{F}_\epsilon \frac{\partial u}{\partial w} - \lambda I \geq -\lambda I,$$

which implies that $\tilde{F}_\epsilon$ is $(-\lambda)$-convex on $W'$, i.e. for any $w_A, w_B \in W'$,

$$(\nabla_{w} \tilde{F}_\epsilon(w_A) - \nabla_{w} \tilde{F}_\epsilon(w_B)) \cdot (w_A - w_B) + \lambda |w_A - w_B|^2 \geq 0.$$

The claim follows by letting $\epsilon \to 0$. $\square$

**Remark 9.** By Definition 3.4, $\tilde{R}_\alpha|_U = R_\alpha$ and the solutions of (9) and (15) agree on $U$ by Picard’s uniqueness for the solution of ODE. Therefore, any solution of the extended modified combinatorial $\alpha$-Ricci flow (15) extends the solution of the modified combinatorial $\alpha$-Ricci flow (9).

**Remark 10.** By the proof of Theorem 3.5, one can further prove the uniqueness of the solution of extended modified combinatorial $\alpha$-Ricci flow (15) without the assumption $\alpha \tilde{R} \leq 0$.

### 3.3 Longtime existence and global convergence of the combinatorial $\alpha$-Ricci flow in the Euclidean background geometry

According to Definition 3.4 the extended normalized combinatorial $\alpha$-Ricci flow for discrete conformal structures in the Euclidean background geometry is defined to be

$$\frac{du_i}{dt^*} = R_{\alpha,av} - \tilde{R}_{\alpha,i},$$

(17)
where $R_{\alpha, av} = \frac{2\pi \chi(M)}{\sum_{i=1}^{N} e_{\alpha u_i}}$ is the average combinatorial $\alpha$-curvature. We have the following result on the longtime existence and convergence for the solution of the normalized combinatorial $\alpha$-Ricci flow under the existence of the Euclidean discrete conformal structure with constant combinatorial $\alpha$-curvature, which is a slight generalization of Theorem 1.5(b).

**Theorem 3.6.** Suppose $(M, T, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (5) and (6). $\alpha \in \mathbb{R}$ is a constant and $\alpha \chi(M) \leq 0$. Suppose there exists a Euclidean discrete conformal structure $u \in U$ with constant combinatorial $\alpha$-curvature. Then the normalized combinatorial $\alpha$-Ricci flow (8) develops no essential singularities. If the solution of (8) develops no removable singularities in finite time, then the solution of (8) exists for all time, converges exponentially fast to $u$ for any initial Euclidean discrete conformal structure $u(0) \in U^E$ with $\sum_{i=1}^{N} e_{\alpha u(0)} = \sum_{i=1}^{N} e_{\alpha u}$ for $\alpha \neq 0$ and $\sum_{i=1}^{N} u(0) = \sum_{i=1}^{N} u_i$ for $\alpha = 0$.

**Proof.** If $\alpha = 0$, Theorem 3.6 has been proved by the first author in [41]. Here we only prove the case of $\alpha \neq 0$. Similar to the proof of Lemma 3.1, $\sum_{i=1}^{N} r_{\alpha}^i$ is invariant along the extended normalized combinatorial $\alpha$-Ricci flow (17). Without loss of generality, assume $\sum_{i=1}^{N} r_{\alpha}^i(0) = N$, then the solution of the extended normalized combinatorial $\alpha$-Ricci flow (17) stays in the hypersurface $\Pi := \{u \in \mathbb{R}^N | \sum_{i=1}^{N} r_{\alpha}^i(0) = N\}$.

Suppose there exists a Euclidean discrete conformal structure $\bar{u} \in U^E \cap \Pi$ with constant combinatorial $\alpha$-curvature $R_{\alpha, av}$. Define the following Ricci energy function

$$F(u) = - \sum_{\{ijk\} \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{\bar{u}} \sum_{i=1}^{N} (2\pi - R_{\alpha, av} r_{\alpha}^i) du_i.$$ 

By direct calculations, we have

$$\nabla_{u_i} \tilde{F}(u) = - \sum_{\{ijk\} \in F} \tilde{g}_{ijk} + 2\pi - R_{\alpha, av} r_{\alpha}^i = \tilde{K}_i - R_{\alpha, av} r_{\alpha}^i$$

on $\mathbb{R}^N$ and

$$\text{Hess}_{u} \tilde{F}(u) = \Lambda^E - \alpha R_{\alpha, av} \sum_{\alpha} \left( I - \frac{\tilde{r}_{\alpha} \cdot (\tilde{r}_{\alpha})^T}{||\tilde{r}_{\alpha}||^2} \right) \sum_{\alpha}$$

$$= \sum_{\alpha} \left( L - \alpha R_{\alpha, av} \frac{\tilde{r}_{\alpha} \cdot (\tilde{r}_{\alpha})^T}{||\tilde{r}_{\alpha}||^2} \right) \sum_{\alpha}$$

on $\mathbb{R}^N$.
on $\mathcal{U}^E$, where $\Sigma = \text{diag}\{r_1, r_2, \ldots, r_N\}$, $L = \Sigma^{-\frac{2}{5}}\Lambda^E\Sigma^{-\frac{2}{5}}$. Note that $\tilde{F}(u)$ is a $C^1$ smooth convex function defined on $\mathbb{R}^N$ and $\nabla_u \tilde{F}(\pi) = \tilde{K}_i(\pi) - R_{\alpha,av}e^{\alpha\pi} = \tilde{R}_{\alpha,i}(\pi)e^{\alpha\pi} - R_{\alpha,av}e^{\alpha\pi} = 0$ by assumption, we have $\tilde{F}(u) \geq \tilde{F}(\pi) = 0$ and $\lim_{u \to \infty} \tilde{F}(u)|_\Pi = +\infty$ by the following property of convex functions, a proof of which could be found in [18] (Lemma 4.6).

**Lemma 3.7.** Suppose $f(x)$ is a $C^1$ smooth convex function on $\mathbb{R}^N$ with $\nabla f(x) = 0$ for some $x_0 \in \mathbb{R}^N$, $f(x)$ is $C^2$ smooth and strictly convex in a neighborhood of $x_0$, then $\lim_{x \to \infty} f(x) = +\infty$.

Note that

$$\frac{d}{dt} \tilde{F}(u(t)) = \sum_{i=1}^N \nabla_{u_i} \tilde{F} \cdot \frac{du_i}{dt} = \sum_{i=1}^N (\tilde{K}_i - R_{\alpha,av}r_i^\alpha)(R_{\alpha,av} - \tilde{R}_{\alpha,i}) = -\sum_{i=1}^N (R_{\alpha,av} - \tilde{R}_{\alpha,i})^2 r_i^\alpha \leq 0$$

along the extended normalized combinatorial $\alpha$-Ricci flow (17), we have $\tilde{F}(u(t))$ is bounded along (17), which implies the solution $u(t)$ of the extended normalized combinatorial $\alpha$-Ricci flow (17) stays in a compact subset of $\Pi$ by $\lim_{u \to \infty} \tilde{F}(u)|_\Pi = +\infty$. Therefore, the normalized combinatorial $\alpha$-Ricci flow (8) develops no essential singularities and the solution of extended normalized combinatorial $\alpha$-Ricci flow (17) exists for all time. Furthermore, if the solution of (8) develops no removable singularities in finite time, $\lim_{t \to +\infty} \tilde{F}(u(t))$ exists and there exists $\xi_n \in (n, n+1)$ such that

$$\tilde{F}(u(n+1)) - \tilde{F}(u(n)) = \frac{d}{dt} \tilde{F}(u(t))|_{t=\xi_n} = -\sum_{i=1}^N (R_{\alpha,av} - \tilde{R}_{\alpha,i})^2 r_i^\alpha|_{t=\xi_n} \to 0, \quad n \to \infty. \quad (18)$$

As $u(\xi_n)$ is bounded, there exists a convergent subsequence of $u(\xi_n)$, still denoted by $u(\xi_n)$ for simplicity, such that $u(\xi_n) \to u^* \in \mathbb{R}^N$ and then $\tilde{R}_\alpha(u(\xi_n)) \to \tilde{R}_\alpha(u^*)$. The equation (18) shows that $\tilde{R}_\alpha(u^*) = R_{\alpha,av} = R_{\alpha}(\pi)$. Therefore, $u^* = \pi$ by Theorem 2.4.

Similar to Theorem 3.2 set $\Gamma_i(u) = R_{\alpha,av} - \tilde{R}_{\alpha,i}$, then $D\Gamma|_{u=\pi}$ restricted to the hypersurface $\Pi$ is negative definite, which implies that $\pi$ is a local attractor of (17). Then the conclusion follows from Lyapunov Stability Theorem (33, Chapter 5).

For the prescribed combinatorial $\alpha$-curvature problem in the Euclidean background geometry, we have following result paralleling to Theorem 3.6.

**Theorem 3.8.** Suppose $(M, \mathcal{T}, \varepsilon, \eta)$ is a weighted triangulated connected closed surface with the weights $\varepsilon : V \to \{0, 1\}$ and $\eta : E \to \mathbb{R}$ satisfying the structure conditions (11) and (10). $\alpha \in \mathbb{R}$ is a constant and $\mathcal{F} : V \to \mathbb{R}$ is a given function.

1. If the solution of modified combinatorial $\alpha$-Ricci flow (9) in the Euclidean background geometry converges, then there exists a Euclidean discrete conformal structure $\pi \in \mathcal{U}^E$ with combinatorial $\alpha$-curvature $\tilde{R}$.
(2) Suppose there exists a Euclidean discrete conformal structure $u \in U^E$ with combinatorial $\alpha$-curvature $R$ satisfying $\alpha R \leq 0$ and $\alpha R \neq 0$. Then the modified combinatorial $\alpha$-Ricci flow \([9]\) in the Euclidean background geometry develops no essential singularities along the flow. If the solution of \([9]\) develops no removable singularities in finite time, then the solution of \([9]\) exists for all time, converges exponentially fast to $\tilde{u}$ for any initial Euclidean discrete conformal structure $u(0) \in U^E$ and does not develop removable singularity at time infinity. Furthermore, the unique solution of the extended modified combinatorial $\alpha$-Ricci flow \([15]\) exists for all time and converges exponentially fast to $\tilde{u}$ for any initial generalized Euclidean discrete conformal structure $u(0) \in \mathbb{R}^N$.

Proof. The proof of (1) in Theorem 3.8 is similar to that of (1) in Theorem 3.2, so we omit the details of the proof here. By Theorem 2.4, the function

$$\tilde{F}(u) = -\sum_{\{ijk\} \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{\pi}^{u} \sum_{i=1}^{N} (2\pi - \tilde{R}_i r_i^\alpha) du_i$$

is a $C^1$-smooth convex function defined on $\mathbb{R}^N$ and locally strictly convex on $U^E$ in the case of $\alpha R \leq 0$ and $\alpha R \neq 0$. By direct calculations, we have

$$\nabla u_i \tilde{F}(u) = -\sum_{\{ijk\} \in F} \tilde{\theta}_{ij}^k + 2\pi - \tilde{R}_i r_i^\alpha = \tilde{K}_i - \tilde{R}_i r_i^\alpha.$$ 

By assumption, $\nabla u_i \tilde{F}(\pi) = \tilde{K}_i(\pi) - \tilde{R}_i e^\alpha = \tilde{R}_{\alpha,i}(\pi) e^\alpha - \tilde{R}_i e^\alpha = 0$. By Lemma 3.7, since $\tilde{F}(u)$ is strictly convex in a neighborhood of $\pi \in U^E$, we have $\lim_{u \to \infty} \tilde{F}(u) = +\infty$. Note that

$$\frac{d}{dt} \tilde{F}(u(t)) = \sum_{i=1}^{N} \nabla u_i \tilde{F} \cdot \frac{du_i}{dt} = \sum_{i=1}^{N} (\tilde{K}_i - \tilde{R}_i r_i^\alpha)(\tilde{R}_i - \tilde{R}_{\alpha,i}) = -\sum_{i=1}^{N} (\tilde{R}_i - \tilde{R}_{\alpha,i})^2 r_i^\alpha \leq 0$$

along the extended modified combinatorial $\alpha$-Ricci flow \([15]\), we have $\tilde{F}(u(t))$ is bounded, which implies that the solution $u(t)$ of the extended modified combinatorial $\alpha$-Ricci flow \([15]\) stays in a compact subset of $\mathbb{R}^N$ and $\lim_{t \to \infty} \tilde{F}(u(t))$ exists. Therefore, the solution of the extended modified combinatorial $\alpha$-Ricci flow \([15]\) exists for all time and the modified combinatorial $\alpha$-Ricci flow \([9]\) in the Euclidean background geometry develops no essential singularities. By the mean value theorem, there exists $\xi_n \in (n, n+1)$ such that

$$\tilde{F}(u(n+1)) - \tilde{F}(u(n)) = \frac{d}{dt} \tilde{F}(u(t))|_{t=\xi_n} = -\sum_{i=1}^{N} (\tilde{R}_i - \tilde{R}_{\alpha,i})^2 r_i^\alpha|_{t=\xi_n} \to 0, \quad n \to \infty. \quad (19)$$
As \( \{u(\xi_n)\} \) is bounded, there exists a convergent subsequence of \( \{u(\xi_n)\} \), still denoted by \( \{u(\xi_n)\} \), for convenience, such that \( u(\xi_n) \to u^* \in \mathbb{R}^N \) and then \( \tilde{R}_\alpha(u(\xi_n)) \to \tilde{R}_\alpha(u^*) \). The equation (19) shows that \( \tilde{R}_\alpha(u^*) = \tilde{R} = R_\alpha(\pi) \). Therefore, \( u^* = \pi \) by Theorem 2.4.

Set \( \Gamma_i(u) = \tilde{R}_i - \tilde{R}_{\alpha,i} \), then

\[
D\Gamma_i|_\pi = -\Sigma^{-\alpha} \Lambda^E + \alpha L' = -\Sigma^{-\frac{\alpha}{2}} (\Sigma^{-\frac{\alpha}{2}} \Lambda^E \Sigma^{-\frac{\alpha}{2}} - \alpha L') \Sigma^\frac{\alpha}{2},
\]

where \( \Sigma = \text{diag}\{\varepsilon^{u_1}, \varepsilon^{u_2}, ..., \varepsilon^{u_N}\} \), \( L' = \text{diag}(\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_N) \). Therefore, \( D\Gamma_i|_\pi \) is negative definite by \( \alpha \tilde{R} \leq 0 \) and \( \alpha \tilde{R} \neq 0 \), which implies that \( \pi \) is a local attractor of (15). Then the conclusion follows from Lyapunov Stability Theorem ([33], Chapter 5). \( \square \)

### 3.4 Longtime existence and global convergence of the combinatorial \( \alpha \)-Ricci flow in the hyperbolic background geometry

In the hyperbolic background geometry, we take the constant combinatorial \( \alpha \)-curvature problem as a special case of the prescribed combinatorial \( \alpha \)-curvature problem. To study the prescribed combinatorial \( \alpha \)-curvature problem in the hyperbolic background geometry using the extended modified combinatorial \( \alpha \)-Ricci flow (15), we need the following lemma, which was proved by the first author in [41] (Corollary 4.9).

**Lemma 3.9.** ([41]) Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \([3] \) and \([4] \). \( \{ijk\} \in F \) is a triangle with \( \varepsilon_i = 1 \). Then for any \( \varepsilon > 0 \), there exists a positive number \( L = L(\varepsilon, \eta, \varepsilon) \) such that if \( f_i > L \), the extended inner \( \tilde{\theta}_{ijk} \) at the vertex \( i \in V \) in the generalized hyperbolic triangle \( \{ijk\} \in F \) with edge lengths given by \([3] \) is smaller than \( \varepsilon \).

We have the following result on the longtime existence and convergence of the solution of extended modified combinatorial \( \alpha \)-Ricci flow (15) in hyperbolic background geometry, which is a generalization of Theorem 1.5 \((\varepsilon)\).

**Theorem 3.10.** Suppose \((M, T, \varepsilon, \eta)\) is a weighted triangulated connected closed surface with the weights \( \varepsilon : V \to \{0, 1\} \) and \( \eta : E \to \mathbb{R} \) satisfying the structure conditions \([3] \) and \([4] \). \( \alpha \in \mathbb{R} \) is a constant and \( \tilde{R} : V \to \mathbb{R} \) is a given function. Suppose there exists a hyperbolic discrete conformal structure \( \pi \in \mathcal{U}^H \) with combinatorial \( \alpha \)-curvature \( \tilde{R} \) such that one of the following three conditions is satisfied

1. \( \alpha > 0 \) and \( \tilde{R}_i \leq 0 \) for all \( i \in V \),
2. \( \alpha < 0 \) and \( \tilde{R}_i \in [0, 2\pi) \) for all \( i \in V \),
3. \( \alpha = 0 \) and \( \tilde{R}_i \in (-\infty, 2\pi) \) for all \( i \in V \) and \( \sum_{i=1}^{N} \tilde{R}_i > 2\pi \chi(M) \).

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Then the modified combinatorial $\alpha$-Ricci flow (15) in the hyperbolic background geometry develops no essential singularities. If the solution of (15) develops no removable singularities in finite time, then the solution of (15) exists for all time, converges exponentially fast to $\overline{\nu}$ for any initial hyperbolic discrete conformal structure $u(0) \in \mathcal{U}^H$ and does not develop removable singularities at time infinity. Furthermore, the unique solution of the extended modified combinatorial $\alpha$-Ricci flow (15) exists for all time and converges exponentially fast to $\overline{\nu}$ for any initial generalized hyperbolic discrete conformal structure $u(0)$.

Proof. The case of $\alpha = 0$ has been proved by the first author in [41]. As the proofs for the cases of $\alpha > 0$ and $\alpha < 0$ are all the same, we only prove the case of $\alpha > 0$.

Suppose there exists a hyperbolic discrete conformal structure $\overline{\nu} \in \mathcal{U}^H$ with combinatorial $\alpha$-curvature $\overline{R}$, then the following Ricci energy function

$$\tilde{F}(u) = -\sum_{\{ijk\} \in F} \tilde{F}_{ijk}(u_i, u_j, u_k) + \int_{\overline{\nu}} \sum_{i=1}^{N}(2\pi - \overline{R}_i e^{\alpha u_i}) du_i$$

is a $C^1$-smooth convex function defined on $\mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1}$ by the proof of Theorem 2.7, where $N_0$ is the number of the vertices $i \in V$ with $\varepsilon_i = 0$ and $N_1 = N - N_0$. Moreover, $\nabla_{u_i} \tilde{F}(\overline{\nu}) = \tilde{K}_i(\overline{\nu}) - \overline{R}_i e^{\alpha u_i} = \tilde{R}_{\alpha,i}(\overline{\nu}) e^{\alpha u_i} - \overline{R}_i e^{\alpha u_i} = 0$ by assumption, which implies $\tilde{F}(u) \geq \tilde{F}(\overline{\nu}) = 0$ and $\lim_{u \to -\infty} \tilde{F}(u) = +\infty$ by Lemma 3.7. Furthermore,

$$\frac{d}{dt} \tilde{F}(u(t)) = \sum_{i=1}^{N} \nabla_{u_i} \tilde{F} \cdot \frac{du_i}{dt} = \sum_{i=1}^{N} (\tilde{K}_i - \overline{R}_i e^{\alpha u_i})(\overline{R}_i - \tilde{R}_{\alpha,i}) = -\sum_{i=1}^{N} (\overline{R}_i - \tilde{R}_{\alpha,i})^2 e^{\alpha u_i} \leq 0$$

along the extended modified combinatorial $\alpha$-Ricci flow (15), which implies the solution $u(t)$ of the extended modified combinatorial $\alpha$-Ricci flow (15) is bounded in $\mathbb{R}^N$. By Remark 3, $u = (u_1, ..., u_N) \in \mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1}$. We claim that $u_i(t)$ is uniformly bounded from above in $\mathbb{R}_{<0}$ for vertices $i \in V$ with $\varepsilon_i = 1$. We shall prove the theorem assuming the claim and then prove the claim.

By the claim, we have the solution $u(t)$ of the extended modified combinatorial $\alpha$-Ricci flow (15) lies in a compact subset of $\mathbb{R}^{N_0} \times \mathbb{R}_{<0}^{N_1}$, which implies that the modified combinatorial $\alpha$-Ricci flow (9) in the hyperbolic background geometry develops no essential singularities and the solution of the extended modified combinatorial $\alpha$-Ricci flow (15) exists for all time. As $u(t)$ is bounded, $\tilde{F}(u(t))$ is bounded along the extended modified combinatorial $\alpha$-Ricci flow (15) and $\lim_{t \to +\infty} \tilde{F}(u(t))$ exists. By the mean value theorem, there exists $\xi_n \in (n, n+1)$ such that

$$\tilde{F}(u(n + 1)) - \tilde{F}(u(n)) = \frac{d}{dt} \tilde{F}(u(t))|_{t=\xi_n} = -\sum_{i=1}^{N} (\overline{R}_i - \tilde{R}_{\alpha,i})^2 e^{\alpha u_i}|_{t=\xi_n} \to 0, \ n \to \infty. \ (20)$$
As \( u(\xi_n) \) is bounded, then there exists a subsequence of \( u(\xi_n) \), still denoted by \( u(\xi_n) \) for simplicity, such that \( u(\xi_n) \to u^* \in \mathbb{R}^{N_0} \times \mathbb{R}^{N_1}_{<0} \) and then \( \tilde{R}_\alpha(u(\xi_n)) \to \tilde{R}_\alpha(u^*) \). The equation (20) shows that \( \tilde{R}_\alpha(u^*) = R = R_{\alpha}(\varpi) \). Therefore, \( u^* = \varpi \) by Theorem 2.7. Similar to Theorem 3.3, set \( \Gamma_i(u) = \tilde{R}_\alpha - \tilde{R}_{\alpha,i} \), then \( D\Gamma|_{\varpi=\varpi} \) is negative definite, which implies that \( \varpi \) is a local attractor of (15). Then the conclusion follows from Lyapunov Stability Theorem (33), Chapter 5).

We use Ge-Xu’s trick in (17) to prove the claim. Suppose that there exists at least one vertex \( i \in V \) such that \( \lim_{t \to T} u_i(t) = 0 \) for \( T \in (0, +\infty) \), which is equivalent to \( \lim_{t \to T} f_i(t) = +\infty \) by the map (4). For the vertex \( i \), by Lemma 3.9 there exists \( c \in (-\infty, 0) \) such that whenever \( u_i(t) > c \), the extended inner angle \( \tilde{\theta}_i \) is smaller than \( \epsilon = \frac{2\pi}{d_i} > 0 \), where \( d_i \) is the degree at the vertex \( i \). Thus \( \tilde{K}_i = \left[ 2\pi - \sum_{\{ijk\} \in F} \tilde{\theta}_i \right] > 0 \geq \tilde{R}_i \epsilon^{\alpha u_i} \) and \( \tilde{R}_{\alpha,i} = \frac{\tilde{K}_i}{\epsilon^{\alpha u_i}} \tilde{K}_i > \tilde{R}_i \). Choose a time \( t_0 \in (0, T) \) such that \( u_i(t_0) > c \), this can be done because \( \lim_{t \to T} u_i(t) = 0 \). Set \( a = \inf\{t < t_0 | u_i(s) > c, \forall s \in (t, t_0)\} \), then \( u_i(a) = c \). Note that \( \frac{du_i}{dt} = \tilde{R}_i - \tilde{R}_{\alpha,i} < 0 \) on \( (a, t_0] \), we have \( u_i(t_0) < u_i(a) = c \), which contradicts \( u_i(t_0) > c \). Therefore, \( u_i(t) \) is uniformly bounded from above in \( \mathbb{R}_{<0} \) for vertex \( i \) with \( \epsilon_i = 1 \).

**Remark 11.** If \( \epsilon_i = 0 \) for all \( i \in V \), the condition (1)(2)(3) in Theorem 3.10 can be generalized to be the condition that \( \alpha \in \mathbb{R} \) is a constant and \( \tilde{R} : V \to \mathbb{R} \) is a given function defined on \( V \) with \( \alpha \tilde{R} \leq 0 \), which was proved in (42).

**Remark 12.** By the proof of Theorem 3.10 for vertices \( i \in V \) with \( \epsilon_i = 1 \), the fact that \( u_i(t) \) is upper bounded in \( \mathbb{R}_{<0} \) is independent of the existence of hyperbolic discrete conformal structure with combinatorial \( \alpha \)-curvature \( \tilde{R} \).

## 4 Open problems

### 4.1 Rigidity of combinatorial \( \alpha \)-curvature

The rigidity of combinatorial curvature for discrete conformal structures on polyhedral surfaces are usually studied under the structure conditions (5) and (6) for the weight \( \varpi : V \to \{0, 1\} \), which corresponds to the Euclidean and hyperbolic background geometry. However, the rigidity of discrete conformal structures in the spherical background geometry is seldom studied, which corresponds to \( \varpi = -1 \). Please refer to (2), (33), (34) and others for some results. It would be interesting to know a general version of the rigidity of discrete conformal structures on surfaces in the spherical background geometry.

Another problem on rigidity is that the rigidity of combinatorial \( \alpha \)-curvature for different types of discrete conformal structures on polyhedral surfaces are usually studied under the condition \( \alpha \tilde{R} \leq 0 \), where \( \tilde{R} \) is the prescribed combinatorial \( \alpha \)-curvature. Please
refer to \([17, 19, 40, 42]\) and others. By the proof of the rigidity, one can also generalize this condition to be a condition on the lower bound of the eigenvalue of the \(\alpha\)-Laplace operator \(\Delta_\alpha\). Please refer to \([19]\) (Section 5) or the proof of Theorem 2.4 in this paper. A natural question on the rigidity of combinatorial \(\alpha\)-curvature is the following.

**Question 4.1.** What is the sharp lower bound of the eigenvalues of the \(\alpha\)-Laplace operator \(\Delta_\alpha\) that ensures the rigidity of the combinatorial \(\alpha\)-curvature? Does the sharp lower bound contain any geometric or topological information on the triangulated surfaces?

### 4.2 Prescribed combinatorial \(\alpha\)-curvature problem

Prescribed Gaussian curvature on smooth surfaces is a fundamental problem in Riemannian geometry. Prescribed combinatorial curvature on polyhedral surfaces also has lots of applications in geometry, topology and applications, which asks the following question.

**Prescribed combinatorial curvature problem** Suppose \(M\) is a connected closed surface and \(V\) is a finite nonempty subset of \(M\). Which functions defined on \(V\) are combinatorial curvatures of polyhedral metrics on \((M, V)\)?

There are lots of important works on this problem for the classical combinatorial curvature on polyhedral surfaces, which corresponds to \(\alpha = 0\) in this paper. Thurston \([34]\) gave a full characterization of the image of the classical curvature map for Thurston’s circle packing metrics on surfaces, which corresponds to \(\varepsilon \equiv 1\) and \(\eta : E \to [0, 1]\) in this paper. Gu-Luo-Sun-Wu \([22]\) and Gu-Guo-Luo-Sun-Wu \([24]\) proved the fundamental discrete uniformization theorem for Luo’s vertex scaling by introducing a new definition of discrete conformality, which corresponds to \(\varepsilon \equiv 0\) in this paper. In the case of \(\alpha \neq 0\), the authors \([40, 42, 43]\) obtained some Kazdan-Warner type existence results for prescribed combinatorial \(\alpha\)-curvature problem in the setting of vertex scaling \((\varepsilon \equiv 0)\) using the discrete conformal theory established in \([24, 22]\). Some equivalent conditions for the solution of prescribed combinatorial \(\alpha\)-curvature problems on triangulated polyhedral surfaces are given using the convergence of combinatorial \(\alpha\)-curvature flows in \([10, 13, 17, 14]\) and others.

As the combinatorial \(\alpha\)-curvature can be taken as an approximation of the smooth Gaussian curvature on surfaces, it would be interesting to get some existence results for the prescribed combinatorial \(\alpha\)-curvature problem for generic \(\alpha, \varepsilon\) and \(\eta\).
4.3 Combinatorial $\alpha$-curvature flows with surgery

In Theorem 1.5, we prove the convergence of combinatorial $\alpha$-Ricci flow by extending the flow through removable singularities along the flow under the assumption that there exists a discrete conformal structure with the prescribed combinatorial $\alpha$-curvature. However, as noted in Remark 8, the combinatorial $\alpha$-Calabi flow cannot be extended in this way. On the other hand, in the case of vertex scaling, Gu-Luo-Sun-Wu [23] and Gu-Guo-Luo-Sun-Wu [24] introduced a different extension of Luo’s combinatorial Yamabe flow [29] via doing surgery along the flow by edge flipping under the Delaunay condition. Using the surgery by edge flipping, Gu-Luo-Sun-Wu [23] and Gu-Guo-Luo-Sun-Wu [24] proved the convergence of the combinatorial Yamabe flow without the assumption of the existence of conformal factors with prescribed combinatorial curvature. This method is then applied by Zhu-Xu [46] to prove the convergence of the combinatorial Calabi flow with surgery. The first author [40] and the authors [43] further applied this surgery to combinatorial $\alpha$-curvature flows and obtained some Kazdan-Warner type existence results for prescribed combinatorial $\alpha$-curvature problem in the setting of vertex scaling. Motivated by these facts, it is natural to introduce surgery by edge flipping under the weighted Delaunay condition along the combinatorial $\alpha$-Ricci flow and $\alpha$-Calabi flow for the discrete conformal structures in Definition 1.1. We have the following conjecture on the combinatorial $\alpha$-curvature flows with surgery.

**Conjecture 4.2.** Under the structure conditions (5) and (6) with $\varepsilon : V \to \{0, 1\}$, the solution of modified combinatorial $\alpha$-Ricci flow and $\alpha$-Calabi flow with surgery for discrete conformal structures in Definition 1.1 exists for all time and converges to the target combinatorial $\alpha$-curvature given by the prescribed combinatorial $\alpha$-curvature problem.

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