Maxwell’s demons with finite size and response time

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Nearly all theoretical analyses of the Maxwell’s demon focuses on its energetic and entropic costs of operation. Here, we focus on its rate of operation. In our model, a demon’s rate limitation stems from its finite response time and gate area. We determine the rate limits of mass and energy transfer, as well as entropic reduction for four such demons: Those that select particles according to (1) direction, (2) energy, (3) number and (4) entropy. In addition to rate limitations we also calculate their coefficients of performance and compare it with that of an ideal demon. Lastly, we determine the optimal gate size for a demon with finite response time.

INTRODUCTION

A Maxwell’s demon is a device that can measure the microstate of a closed system, thereby reducing its entropy, seemingly in violation with the second law \[1\]. A century of physics literature modelling the measurement protocols and internal workings of the demon \[2,3\] culminated to the conclusion that logically irreversible operations taking place within the demon \[6\] such as information erasure \[7\], account for the lost entropy

Contemporary incarnations of the demon are capable of feedback control \[8,9\] and universal computation \[10–12\]. Some demons measure and modify a tape of bits \[13,15\] or qubits \[16\] representing the state of a system, while others omit measurement altogether, instead, sorting through the microstates mechanically \[17–20\]. Non-ideal demons have also been explored: \[13,14\] accounts for the thermal equilibration of the demon with the system, and \[20\] studies the efficiency of an imperfect ratchet with finite mass. Today, we can build demons in the laboratory \[21–32\], and even make practical use of them for harvesting energy \[29,33–37\], or sorting atoms \[38\].

Experimentalists often discuss the temporal limitations of their demons, but nevertheless still operate under the simplifying assumption that the time it takes to sense, process and respond to information is negligible compared to all other times \[21,27,29–32\].

Cells are natural information engines that perform measurements and computations to process energy in a highly stochastic environment \[39,40\]. Here too, the timescale at which the cell operates relative to the timescale of its environment, impacts its efficiency of information processing \[42\].

In this paper we study how the finite size \(A\) and response time \(\tau\) restricts the transport rate and efficiency attainable by a Maxwell’s demon. In practice, \(A\) and \(\tau\) would be constrained by experimental practicalities such as the inertia and friction of the gate. Ultimately however, theoretical bounds on speed, length and mass set the true limits on how quickly a demon can transport energy or particles. For example, the gate cannot close faster than the speed of light, and must necessarily be larger than the thermal wavelength of an atom.

To this end, we study four spatiotemporally-limited demons that make decisions based on direction, number, energy or entropy measurements. For all four, we obtain heat, mass and entropy transport as a function of \(\tau, A\). Lastly, we use the Landauer-Bennett principle to obtain the efficiency of a spatiotemporally-restricted demon that operates as a heat pump.

PROBLEM SETUP

Consider a gated partition separating two equal volumes, \(V\). The left and right sides contain ideal gases of total energies \(E_l, E_r\), and numbers \(N_l, N_r\). The speed \(v\) distribution on side \(s \in \{l, r\}\) is

\[ p(v) = \Omega_d v^{d-1} e^{-\beta_s m v^2/2}/Z, \quad Z = [2\pi/(m\beta_s)]^{d/2} \]

where \(\Omega_d = 2, 2\pi, 4\pi\) is the solid angle in \(d = 1, 2, 3\) dimensions and \(m\) is the particle mass. The temperature \(1/\beta_s\) and average energy \(E_s\) are related by \(\beta_s E_s = N_s d/2 = \beta_s N_s E_s\). We assume that the demon takes measurements and decides on the state of the gate every \(\tau\).

A particle with speed \(v\) will pass through an area \(A\) during \(\tau\) with probability \(p(\text{pass}|v) = c_d v \tau A/V\), where \(c_d = 1/2, 1/\pi, 1/4\) in \(d = 1, 2, 3\) dimensions, (we define \(A \equiv 1\) for \(d = 1\)). Thus, the probability that a random particle on side \(s\) passes through an open gate is

\[ p(\text{pass}) = \int_0^{\infty} p(\text{pass}|v)p(v)dv = \kappa_s/N \]

(1)

\[ \kappa_s = \rho_s \tau A \sqrt{2/\pi m \beta_s}, \quad \rho_s = \frac{E_s}{d \sqrt{m \pi}} \]

(2)

We also define \(v_s \equiv \kappa_s/\tau\), which does not depend on \(\tau\). The number density, \(\rho = N/V\), has units of \(1/\text{volume}\).

Knowing the probabilities that a particle escapes, and that a particle with a specific velocity escapes allows us to compute the probability that exactly \(n\) particles carrying total energy \(E\) pass through the gate during \(\tau\). The probability that exactly \(n\) particles carry an energy \(E\) through the gate is (see appendix B)

\[ p(E, n) = \frac{\kappa^n (\beta E)^n D e^{-\beta E - \kappa}}{\Gamma(nD)n!} \overline{E} \]

(3)
which can be marginalized over number or energy to find the probability of number and the probability of energy,
\[ p(n) = \frac{\kappa^n}{n!} e^{-\kappa} \tag{4} \]
\[ p(E) = \frac{1}{E} e^{-\beta E - \kappa} \sum_{n=1}^{\infty} \frac{\kappa^n (\beta E)^n D}{n! (nD)} \tag{5} \]
where \( D = (d + 1)/2 \). Note that \( p(E, n \to 0) = e^{-\kappa} \delta(E) \). Also, for \( d = 1 \), \( p(E) = \sqrt{3\beta} E e^{-\beta E - \kappa} I_1(2\sqrt{3\beta E}) \), which is a non-central chi-squared distribution with zero degrees of freedom.

The incomplete energy moments can be found in terms of incomplete gamma functions, \( \Gamma(\cdot, \cdot) \),
\[ \langle E^n \rangle_{E_0} = \int_{E_0}^{\infty} E^n p(E) = \frac{e^{-\kappa}}{\beta^n} \sum_{n=1}^{\infty} \frac{\kappa^n \Gamma(nD + s, \beta E_0)}{\Gamma(nD)} \]
For complete energy moments, the incomplete gamma function is replaced with a gamma function. For \( p = 1 \) we get the average \( \langle E \rangle = \kappa D/\beta \). The number distribution moments can be found similarly, \( \langle n^p \rangle = e^{-\kappa} (\kappa \partial_\kappa)^p e^{\kappa} \).

**Entropy reduction and generation by a demon**

Differentiating the Sackur-Tetrode equation, we get the entropy reduction due to heat and mass transport,
\[ \frac{\dot{S}_s}{k_B} = \left[ \beta_s \dot{E}_s - \left( \frac{d}{2} \log \frac{d}{2} - \frac{d}{2} - 1 - \log \left( \beta_s \dot{E}_s \right) \right) \right] \dot{N}_s . \]
From mass and energy conservation, \( \dot{N}_l = -\dot{N}_r, \dot{E}_l = -\dot{E}_r, \)
\[ \frac{\dot{S}_\text{tot}}{k_B} = (\beta_r - \beta_l) P_r + \left( \frac{d}{2} \log \left( \frac{\beta_r}{\beta_l} \right) - \log \left( \frac{\rho_r}{\rho_l} \right) \right) I_r , \]
where \( I_r \) and \( P_r \) are the mass and heat currents. As expected, when there is no difference in temperature and density, the change in entropy is zero. This is reasonable: if the left chamber is hotter, the demon actually increases entropy, while if the left is colder the demon decreases entropy. From continuity, the demon must not be changing entropy when the two sides are exactly equal.

We also consider the entropy generated by the demon, \( S_{\text{dem}} \). By gathering information about the system, the full phase space reduces into a subset, the full phase space given the measurement outcome. Since the demon operates cyclically, it must erase all the bits it has used, thereby producing entropy,
\[ \dot{S}_{\text{dem}} \cdot \tau/k_B \geq H[M] = H[\hat{X}] - H[\hat{X}|\hat{M}] . \]
where the random variables \( \hat{X} \) denotes the system state and \( Y \) a measurement of sub-state necessary to make a decision. This formula is a consequence of Bayes law for conditional entropy, \( H[M|X] = H[X|M] - H[\hat{X}] + H[M] \), and that fact that \( H[\hat{M}|\hat{X}] = 0 \) since the state of the system will completely determine the measurement \( M \) for all our demons. For details on how to use \( D \) for each of our demon models, see appendix D.

**DEMON MODELS**

We model four demons who make decisions based on direction, energy, number and entropy. By convention, all demons attempt to move mass, heat, or entropy from left to right. For all four demons, we calculate heat \( P_r \), number \( I_r \), and entropy \( J_r \) currents as a function of its finite size and response time, and compare these with Monte Carlo simulations (Fig. 1).

1. **A direction demon** opens the gate only if there are no particles moving from right to left. Since the probability that no particles approach from the right is \( e^{-\kappa r} \), the average energy approaching the gate from the left is \( D\kappa/\beta \), and the average number approaching the gate from the left is \( \kappa_l \), the average heat and mass currents are
\[ P_r^{(d)} = D\nu_l e^{-\kappa r} / \beta_l , \quad I_r^{(d)} = \nu_l e^{-\kappa r} \] (8)

Thus, the performance of the demon falls exponentially with \( \tau / (\kappa / \tau) \) does not depend on \( \tau \). For an infinitely precise demon that can process all incoming particles (\( \tau 

2. **An energy demon** opens the gate whenever the right moving particles have greater energy than left moving ones. The energy demon’s heat and mass transport rate converges to that of the direction demon as \( \kappa \to 0 \), since the probability that multiple particles approach the gate from the right and left simultaneously, vanishes. Therefore, we can write the energy demon’s heat and mass currents as the direction demon’s, plus correction terms (appendix C).
\[ P_r^{(e)} = P_r^{(d)} + \frac{\nu_l \nu_r \Gamma(2D + 1)}{\Gamma(D)^2} (-1)^D \left( \frac{f_1}{\beta_l} + \frac{f_2}{\beta_r} \right) \tau e^{-\kappa l - \kappa r} \]
\[ I_r^{(e)} = I_r^{(d)} + \frac{\Gamma(3D)\tau^2 e^{-\kappa r}}{2\Gamma(2D)\Gamma(D + 1)} \left( \nu_l^2 \nu_r f_3 - \frac{\nu_l \nu_r^2}{2} f_4 \right) \] (9)
where \( f_1 = B(-\beta_l / \beta_l, D - 2D) \) and \( f_2 = B(-\beta_l / \beta_r, D + 1, -2D) \) are Euler beta functions, and \( f_3 = F_{2,1}(D, 3D; D + 1; -\beta_l / \beta_l) \), and \( f_4 = F_{2,1}(2D, 3D; 2D + 1; \beta_l / \beta_r) \) are hypergeometric functions.
Note that if $\kappa_r f_4 > 2 \kappa_l f_3$, the number current of the energy demon will, to leading order, be less than that of the direction demon. This hints to an interesting phenomenon we observe in simulations: not only can number and energy demons perform worse than the direction demon, in some cases they can pump heat or mass in the wrong direction.

An exact solution for heat and mass transport for $d = 1$ is given in appendix A.

(3) A number demon opens the gate if right-moving particles are more than left-moving ones. We again obtain the leading order correction (appendix C),

$$P_{\tau}(n) = P_{\tau}(d) + \frac{\nu_r^2 \nu_r}{2 \beta_l} \tau^2 (d + 2) \left[ 1 - \frac{1}{2} d + 3 \beta_l \right] e^{-\kappa_l - \kappa_r}$$

$$I_{\tau}(n) = I_{\tau}(d) + \frac{\nu_r^2 \nu_r}{2} \tau^2 e^{-\kappa_l - \kappa_r}$$

As before, if $\beta_l < \frac{1 + 3}{2}$, the number demon (to leading order) will underperform the direction demon. An exact solution for $d = 1$ is given in appendix A.

(4) Entropy Demon. An entropy demon opens the gate if doing so reduces the total entropy, i.e. if

$$E_r - E_l > \left[ \log \frac{\rho_r}{\rho_l} - \frac{d}{2} \log \left( \frac{\beta_l}{\beta_r} \right) \right] \frac{n_l - n_r}{\beta_l - \beta_r} \equiv \chi \cdot (n_l - n_r)$$

If $\beta_l = \beta_r$, the entropy demon opens the gate whenever $n_l > n_r$, acting as a number demon, and if $\chi = 0$, it acts as an energy demon. The average heat and mass flow is

$$J = \sum_{n_l, n_r = 0}^{\infty} \int_0^\infty dE_l dE_r p_{\nu_l}(E_l) p_{\nu_r}(E_r) \times \ldots \Theta (E_r - E_l - \chi (n_l - n_r)) \Delta (n_l, n_r, E_l, E_r)$$

with a step function enforcing the inequality above. Here $\Delta = E_l - E_r$ for $J = P_r$ and $\Delta = n_l - n_r$ for $J = I_r$.

The entropy demon behaves different than the number and energy demon, in that it does not act as a direction demon as $\tau \to 0$.

### SIMULATIONS

We distribute particles uniformly in space, assign them Boltzmann-distributed velocities, and obtain the time they approach the gate and the energy they carry. We compare our analytical results in the early time limit, before the demon can modify $E$ and $\rho$ substantially. For $d = 2, 3$, the probability of atoms hitting the gate decreases exponentially with the gate area. Thus, for economical reasons, we run simulations only for $d = 1$. In all plots the units of temperature is such that $k_B = 1$.

In Fig. 1 we compare the energy, mass and entropy currents generated by the demons to our analytical formulas. In the left and middle panels, the right chamber has a temperature four times greater than the left, and the number densities are the same. In the right panel, the temperatures of the right chamber is half of that of the left, but the number density of the right chamber is four times that of the left.

Fig. 1 illustrates an interesting phenomenon, that at large $\tau$ the demons will “fail”: for sufficiently large $\tau$, the number demon will move heat in the wrong direction (left, blue), and the energy demon can move particles in the wrong direction (right black). This is because the number demon is willing to let a few very energetic molecules move from right to left as long as a larger num-
FIG. 2. Critical response time. The critical \( \tau \) for energy and number demons, along with estimates of the critical values below which no \( \tau \) exists. Above \( \tau_c \), the demon will transfer heat or mass in the wrong direction. In both plots, the left subsystem has \( T_l = 1, \rho_l = 100 \), only the parameters of the right subsystem are varied. Top: The energy demon’s \( \tau_c \) for different right subsystem temperatures, varying right subsystem number density. Bottom: The number demon’s \( \tau_c \), for different right subsystem number densities, varying the right subsystem’s temperature.

ber of less energetic molecules move from left to right. Similarly, the energy demon will open the gate to allow a few fast particles to pass from left to right even if it lets a larger number of slow particles to pass from right to left. We define \( \tau = \tau_c \), the response time for which demons start pumping particles or energy the wrong direction.

FIG. 3. The effect of spatiotemporal restrictions on coefficients of performance. We plot the ratio of heat transferred to heat generated (left) and mass transferred to heat generated (right) for several sets of system parameters, for our energy demon (solid) and an ideal heat / mass pumps (dashed). The spatiotemporally restricted demons generate entropy, whereas the ideal pumps do not. Length and time restrictions lead to two orders of magnitude of reduction in the coefficient of performance for the indicated density and temperatures. The left subsystem has \( \rho_l = 100, \beta_l = 1 \).

To measure the efficiency of the power demon, we look at the ratio of power or number current to heat generated by the demon, and compare these values to their theoretical minimum values (when \( \dot{S}_{\text{dem}} = \dot{S}_r + \dot{S}_l \)) (Fig. 3).

DISCUSSION

Most literature on the Maxwell’s demon focuses on its thermodynamic cost of operation. Here we point out that even if all thermodynamic books are balanced, a demon will still be limited in its rate of operation. Here, we determined rate bounds for four kinds of demons. We have derived the optimal area of the gate, and how the demons’ response time and gate size determine heat, mass and entropy currents, as well as their energy efficiency.

For a square gate with \( A = 1 \mu m^2 \) that moves at the speed of light to sort air molecules at 300K, we get \( \kappa \sim 9.5 \). For a simple demon, the energy and number transfer for a demon with \( \tau > 0 \) is \( e^{-\kappa \tau} \) times less than a demon operating with \( \tau = 0 \), meaning that its currents would be \( \sim 7.5 \times 10^{-5} \) times less than an infinitely fast demon.

We speculate that the exponential inefficiencies caused by the finite size and response time obtained here is a generic property of any physically embedded information engine. This dependence will be increasingly important to understand as nano-molecular pumps and refrigerators become a reality.
Appendix A - One dimensional demon solutions

As seen in [9], etc., the expressions for the power and number rate of the demons are very complicated in general, even in the small \( \tau \) limit. For the one dimensional case, we have solved for the power and number transfer rates for the power and number demons. The solution is straightforward, but tedious, to derive.

For the energy demon in one dimension,

\[
P^{(p)}(\tau) = \frac{k_t e^{-\kappa_l}}{\beta_l \tau^n} \left( 1 + \frac{\kappa_r e^{-\kappa_r}}{\kappa_l} \sum_{j,k \geq 0} \left( \frac{k + j}{k} \right) \frac{(\gamma_l \kappa_l)^j (\gamma_r \kappa_r)^k}{(k + 1)!} \left[ \mathcal{E}_{1+j}(\kappa_l) - \gamma_l \kappa_l (k + j + 1) \mathcal{E}_{1+j+2}(\kappa_l) \right] \right) \tag{11}
\]

\[
I^{(c)}_\tau = \frac{k_t e^{-\kappa_r}}{\beta_l \tau^n} \left( 1 + \kappa_r \gamma_r e^{-\kappa_l} \sum_{j,k \geq 0} \sum_{l=0}^j \left( \frac{l + k + 1}{k} \right) (j - k) \frac{\kappa_l^j \kappa_r^k}{(j + 1)! (k + 1)!} \right). \tag{12}
\]

For the number demon in one dimension,

\[
P^{(n)}(\tau) = \frac{k_t e^{-\kappa_l}}{\beta_l \tau^n} \left( 1 + \frac{\kappa_r e^{-\kappa_l}}{\kappa_l} \sum_{j,k \geq 0} \frac{\kappa_l^j \kappa_r^k}{(j + k + 2)! (k + 1)!} \left[ j + k + 2 - (k + 1) \frac{\beta_l}{\beta_r} \right] \right). \tag{14}
\]

Here, \( \mathcal{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)} \) denotes a Mittag-Leffler function, \( \mathcal{E}' \) its derivative, and the \( I_k \) a modified Bessel function. The dimensionless gamma constants are \( \gamma_s = \beta_s / (\beta_l + \beta_r) \).

Appendix B - Integrals and identities

One useful identity used in deriving (1) is

\[
c_d \Omega_d \Gamma \left( \frac{d + 1}{2} \right) = \pi^{(d+1)/2}
\]

where \( \Omega_d = 2, 2\pi, 4\pi \) and \( c_d = 1/2, 1/\pi, 1/4 \) in \( d = 1, 2, 3 \) dimensions.

**Probability derivation:** The probability that exactly \( n \) carrying total energy \( E \) pass through the gate during \( \tau \) can be derived by first calculating the probability that exactly \( n \) particles, with velocities \( v_1, v_2, \ldots, v_n \), pass through the gate

\[
p_n(\text{pass}, v_1, \ldots, v_n) = \binom{N}{n} p(\text{pass}, v_1) \cdots p(\text{pass}, v_n) p^n(\text{pass})^{N-n}
\]

\[
= \binom{N}{n} \left( \frac{c_d \tau A \Omega_d}{VZ_\beta} \right)^n v_1^d \cdots v_n^d e^{-\kappa} \exp \left( -\frac{\beta}{2} m(v_1^2 + \cdots + v_n^2) \right).
\]

Note that \( p(\text{pass}, v) = p(\text{pass} | v) p(v) \), and \( p^n(\text{pass}) = 1 - p(\text{pass}) \). From this, we can integrate over the \( v_k \) using a delta function to ensure that the kinetic energy is \( E \),

\[
p(E, n) = \int_0^\infty dv_1 \cdots dv_n \delta \left( \frac{1}{2} m(v_1^2 + \cdots + v_n^2) - E \right) p_n(\text{pass}, v_1, \ldots, v_n)
\]

This can be done using an integral representation of the delta function, and using the formula

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikE} (\beta + ik)^n \, dk = \frac{E^{n-1} e^{-\beta E}}{\Gamma(n)}.
\]

In the thermodynamic limit, \( p^n(\text{pass})^N = (1 - \frac{E}{N})^N \rightarrow e^{-\kappa}, \) \( p^n(\text{pass})^{N-n} \rightarrow 1 \), and the product \( \binom{N}{n} V^{n} = \frac{N-N+1}{n!} \frac{N-1}{1} \rightarrow \rho^n/n! \). The remainder of the constants to the \( n \)-th power, and the \( E^{n-1} = E^n / E \) become \( \kappa^n / E \). This leaves us with the expression for \( p(E, n) \), [3], as desired.
Incorporating chemical potentials: If there are chemical potentials, some changes must be made to the probabilities. Let us assume that $\mu_r = \mu > 0$, $\mu_l = 0$ (only the difference in the $\mu$s will matter). Since all particles that hit the gate from the right will be able to pass through the door, the previously derived formula for $p(n, E)$ is still valid. For the left side though, particles can only pass through the gate if they have $v \geq v_{\text{min}} \equiv \sqrt{2\mu/m}$. The probabilities of passing, $p(\text{pass})$ and $p(\text{pass}|v)$, as derived before, are actually just the probability of hitting the door area. To describe whether the particle actually passes through the door, they must be amended

$$p'(\text{pass}) = \int_{v_{\text{min}}}^{\infty} p(\text{pass}|v)p(v)dv = \kappa(v_{\text{min}})/N$$

$$\kappa(v_{\text{min}}) = \frac{\rho \tau A}{\sqrt{2\pi \beta m}} \left( \frac{m \beta}{2} \right) \frac{v_{\text{min}}^{d+1} E_{\beta v_{\text{min}}/2}}{\frac{D}{2} \Gamma(D)}.$$

The $E_n(z)$ is the exponential integral function. Since $v_{\text{min}}^{d+1} E_{\beta v_{\text{min}}/2} \rightarrow \Gamma(D)(m\beta/2)^{(d+1)/2}$ as $v_{\text{min}} \rightarrow 0$, this equation reproduces (2) when $\mu = 0$.

The expression for $p_n$ for particles on the left side must be replaced with

$$p'_n(\text{pass}, v_1, \ldots, v_n) = \binom{N}{n} p(\text{pass}, v_1) \cdots p(\text{pass}, v_n) \left( 1 - p'(\text{pass}) \right)^{N-n} \Theta(v_1 - v_{\text{min}}) \cdots \Theta(v_n - v_{\text{min}}).$$

Integrating this over all velocities with the delta function constraining the energy will result in the new expression $p'(E, n)$, analogous to (3). This in turn can be marginalized and results in analogues to (5) and (4).

A final change that must be made: whenever particles move from right to left, they gain kinetic energy $\mu$. Because of this, the energy demon must not make sure that $E_l > E_r$, but instead it must open the gate only when $E_l > E_r + n\mu$.

Exponential integrals: The following integral identity is extremely useful both in calculating the partial moments of energy, and with calculating the leading order power and number rates for the smart demons,

$$\int_{E_0}^{\infty} dE E^p e^{-\beta E} = \frac{(p+1, \beta E_0)}{\beta^{p+1}} = \frac{p!}{\beta^{p+1}} e^{-\beta E_0} \sum_{j=0}^{p} (\beta E_0)^j / j!.$$

For full moments ($E_0 = 0$), this reduces to

$$\int_{0}^{\infty} E^p e^{-\beta E} = \frac{p!}{\beta^{p+1}} = \frac{(p+1)}{\beta^{p+1}} : p \geq 0.$$

Some properties of the incomplete gamma function: The incomplete gamma function is defined to be

$$\Gamma(s, z) = \int_{z}^{\infty} t^{s-1} e^{-t} dt.$$

As a consequence, we have the recursive formula $\Gamma(s+1, z) = s \Gamma(s, z) + z^s e^{-z}$, and the special cases $\Gamma(s, 0) = \Gamma(s)$ and $\Gamma(1, z) = e^{-z}$. For integer values of $s$, this recursion can be expanded to give us

$$\Gamma(s+1, z) = e^{-z} \sum_{k=0}^{s} \frac{s!}{(s-k)!} z^{s-k}.$$

One integrals that occurs that involves the incomplete gamma function is

$$\int_{0}^{\infty} dE e^{-\beta_1 E} E^k \Gamma(n, \beta_2 E) = \frac{\Gamma(n+k+1)}{(k+1) \beta_2^{k+1}} F_{2,1} \left( k+1, n+k+1; k+2; -\frac{\beta_1}{\beta_2} \right)$$

which is useful in deriving (9).

Appendix C - Demon energy and number currents for $d > 1$

When $\tau$ is very small (strictly speaking, when $\kappa_l$ and $\kappa_r$ are very small), both the energy and number demons act like the direction demon, since they always let particles pass from left to right if no particles are passing from right
to left, and the probability that particles pass from right to left and from left to right during a time window becomes very small.

Because of this, we can expand the energy and number currents of the energy and number demons as the direction demon current, plus terms of increasing order in \( \kappa_l \) or \( \kappa_r \) that correspond to events where the demon opens the door even though some particles are passing from right to left. We will treat \( \kappa_l \) and \( \kappa_r \) as having the same order.

For the energy demon, the next order term for \( P_{1r}^{(e)} \) after the direction demon term should be have a factor of \( \kappa^2 \). The only relevant event is where one particle approaches from each side, and \( E_l > E_r \). The event where two particles approach from the left, but none from the right is already included in the direction demon term, and the event where two particles approach from the left, but none approach from the right always has \( E_r > E_l = 0 \), so the door will not open and allow this event.

We integrate the probabilities from (3) over allowed energies, with \( n_l = 2, n_r = 1 \), to find the leading correction for \( P_{1r}^{(e)} \)

\[
\frac{1}{\tau} \int_0^\infty \int_0^\infty dE_r \int_{E_r}^\infty dE_l \frac{\kappa_l(\beta_l E_l)^D e^{-\beta_l E_l - \kappa_l}}{\Gamma(D) E_l} \frac{\kappa_r(\beta_r E_r)^D e^{-\beta_r E_r - \kappa_r}}{\Gamma(D) E_r} \times (E_l - E_r)
\]

with the result being the correction term of \( P_{1r}^{(e)} \) from (9).

To evaluate the number current for the energy demon, we first notice that there are no relevant terms of order \( \kappa^2 \). The event that was the most significant for the power does not count towards the number current since \( n_l = n_r = 0 \). The next relevant term is from the events \( n_l = 2, n_r = 1 \), and \( n_l = 1, n_r = 2 \). We must still integrate this over the space of energies with \( E_l > E_r \).

\[
\frac{1}{\tau} \int_0^\infty \int_{E_r}^\infty dE_r \int_{E_r}^\infty dE_l \left( \frac{\kappa_l^2(\beta_l E_l)^{2D} e^{-\beta_l E_l - \kappa_l}}{\Gamma(2D) 2! E_l} \frac{\kappa_r(\beta_r E_r)^D e^{-\beta_r E_r - \kappa_r}}{\Gamma(D) E_r} - \frac{\kappa_l^2(\beta_l E_l)^{2D} e^{-\beta_l E_l - \kappa_l}}{\Gamma(2D) 2! E_l} \frac{\kappa_r(\beta_r E_r)^D e^{-\beta_r E_r - \kappa_r}}{\Gamma(D) E_r} \right) \times (E_l - E_r)
\]

The first term is from \( n_l = 2, n_r = 1 \), the second is from \( n_l = 1, n_r = 2 \). The result is the correction term of \( N_{1r}^{(e)} \) from (9).

For the number demon, the leading order event is \( n_l = 2, n_r = 1 \) since the events \( n_l = 0, n_r = 2 \), and \( n_l = 1, n_r = 1 \), and \( n_l = 0, n_r = 2 \) do not satisfy \( n_l > n_r \). Using (4), it is easy to see that

\[
\frac{1}{\tau} \frac{\kappa_l^2}{2!} \kappa_r e^{-\kappa_l - \kappa_r} \]

is just the correction term for \( I_{1r}^{(n)} \) in (10). To find the correction term for the number demon’s power, we integrate (3) over all energies,

\[
\frac{1}{\tau} \int_0^\infty \int_0^\infty dE_r \int_0^\infty dE_l \frac{\kappa_l^2(\beta_l E_l)^{2D} e^{-\beta_l E_l - \kappa_l}}{\Gamma(2D) 2! E_l} \frac{\kappa_r(\beta_r E_r)^D e^{-\beta_r E_r - \kappa_r}}{\Gamma(D) E_r} \times (E_l - E_r),
\]

obtaining the correction term of \( P_{1r}^{(n)} \) in (10).

**Appendix D - Demon entropy production**

Here we detail how to use (7) to calculate the entropy production of the demons. For convenience, we reproduce (7) here,

\[
S_{\text{dem}}/k_B \geq (H[\hat{X}] - H[\hat{X}|\hat{Y}])/\tau
\]

where \( \hat{X} \) is the state of the system (a random variable), and \( \hat{Y} \) is the demon measurement (also a random variable) that will vary from demon to demon. We will use the energy demon as our example, the other demons are analogous.

The total entropy of the system is just the entropy of the particle and number distribution,

\[
H[\hat{X}] = S[p^{(l)}_p p^{(r)}] = - \sum_{n_l, n_r \geq 0} \int_0^\infty dE_l dE_r p^{(l)}_{n_l}(E_l) p^{(r)}_{n_r}(E_r) \log \left[ p^{(l)}_{n_l}(E_l) p^{(r)}_{n_r}(E_r) \right].
\]
The measurement, $\hat{Y}$, for the energy demon is the function $\hat{Y}(\omega) = 0$ if $E_l < E_r$ for outcome $\omega$, and $\hat{Y}(\omega) = 1$ if $E_l > E_r$. Consequently, let

$$P_0(n_l, n_r, E_l, E_r) = p_{n_l}(E_l)p_{n_r}(E_r)\Theta(E_r - E_l)/p_0 \quad P_1(n_l, n_r, E_l, E_r) = p_{n_l}(E_l)p_{n_r}(E_r)\Theta(E_l - E_r)/p_1$$

$$p_0 = \sum_{n_l, n_r \geq 0} \int_0^\infty dE_l dE_r p_{n_l}(E_l)p_{n_r}(E_r)\Theta(E_r - E_l) \quad p_1 = 1 - p_0.$$

That is, $P_0$ and $P_1$ are just the probability distribution given that $E_l < E_r$ or vice-versa, and $p_0$, $p_1$ are the probabilities that $E_l < E_r$ or vice-versa.

The conditional entropy is just

$$H[\hat{X}|\hat{Y}] = p_0 \cdot S(P_0) + p_1 \cdot S(P_1).$$

For other demons, $H[\hat{X}]$ has the same value, the only different part is calculating the conditional probabilities $P_0$, $P_1$.

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