A NOTE ON TRUNCATED DEGENERATE BELL POLYNOMIALS

TAEKYUN KIM AND DAE SAN KIM

ABSTRACT. The aim of this paper is to introduce truncated degenerate Bell polynomials and numbers and to investigate some of their properties. In more detail, we obtain explicit expressions, identities involving other special polynomials, integral representations, Dobinski-like formula and expressions of the generating function in terms of differential operators and linear incomplete gamma function. In addition, we introduce truncated degenerate modified Bell polynomials and numbers and get similar results for those polynomials. As an application of our results, we show that the truncated degenerate Bell numbers can be expressed as a finite sum involving moments of a beta random variable with certain parameters.

1. INTRODUCTION

Carlitz is the first one who initiated a study of degenerate versions of some special polynomials and numbers, namely degenerate Bernoulli polynomials and numbers and degenerate Euler polynomials and numbers. In recent years, extensive researches have been done for various degenerate versions of some special polynomials and numbers and have yielded many interesting arithmetical and combinatorial results. These include the degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, and so on.

Truncated polynomials have been shown to play an important role in various areas. However, not much are known for the properties of these polynomials. The following are some of the old results related to such polynomials. The minimum variance unbiased estimation is discussed in [4] for the zero class truncated bivariate Poisson and logarithmic series distributions, the maximum likelihood estimation of the Poisson parameter \( \lambda \) is concerned in [5] when the zero class has been truncated and a new family of Hermite polynomials is constructed in [7] by using the truncated exponential with applications to flattened beams in optics. More recently, degenerate exponential truncated polynomials and numbers are studied in [9], the degenerate zero-truncated Poisson random variables are introduced in [14], the truncated-exponential-based Apostol-type polynomials are investigated in [23] and the truncated exponential-based Mittag-Leffler polynomials are considered in [25]. Further, in [12] an umbral calculus approach is given for Bernoulli-Padé polynomials of fixed order, which include the truncated Bernoulli polynomials as a special case and whose generating function is based on the Padé approximant of the exponential function.

The aim of this paper is to introduce truncated degenerate Bell polynomials and numbers (see [15]) and to explore their various properties. In more detail, for the truncated degenerate Bell polynomials and numbers we obtain explicit expressions, identities involving other special polynomials, integral representations, Dobinski-like formula and expressions of the generating function in terms of differential operators and linear incomplete gamma function. In addition, we introduce truncated
degenerate modified Bell polynomials (see (43)) and numbers and get similar results for those polynomials. Finally, as an application of our results, we show that, if $X$ is the beta random variable with parameters $1, \rho$, then the truncated degenerate Bell numbers can be expressed as a finite sum involving moments of $X$. In the rest of this section, we recall some necessary facts that are needed throughout this paper.

For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

\begin{equation}
(1) \quad e^x_{\lambda}(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e^1_{\lambda}(t) = e^t_{\lambda}(t), \quad (\text{see [16, 17]}),
\end{equation}

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda)$, $(n \geq 1)$.

Note that $\lim_{\lambda \to 0} e^x_{\lambda}(t) = e^x_{0}(t)$. The Stirling numbers of the first kind $S_1(n,k)$ appear as the coefficients in the expansion

\begin{equation}
(2) \quad (x)_n = \sum_{k=0}^{n} S_1(n,k) x^k, \quad (n \geq 0), \quad (\text{see [6, 20]}),
\end{equation}

where $(x)_0 = 1$, $(x)_n = x(x-1) \cdots (x-n+1)$, $(n \geq 1)$.

As the inversion formula of (2), the Stirling numbers of the second kind $S_2(n,k)$ are defined as the coefficients in the expansion

\begin{equation}
(3) \quad x^n = \sum_{k=0}^{n} S_2(n,k) (x)_k, \quad (n \geq 0), \quad (\text{see [6, 20]}).
\end{equation}

The Stirling number of the second kind $S_2(n,k)$ counts the number of ways of partitioning a set of $n$ elements into $k$ non-empty subsets. The number of all partitions of a set of $n$ elements is the Bell number $\text{Bel}_n$, $(n \geq 0)$. Thus we note that

\begin{equation}
(4) \quad \text{Bel}_n = \sum_{k=0}^{n} S_2(n,k), \quad (n \geq 0), \quad (\text{see [6, 20]}).
\end{equation}

Further, the Bell polynomials are given by

\begin{equation}
(5) \quad \text{Bel}_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k,
\end{equation}

with the generating function

\begin{equation}
(6) \quad e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}, \quad (\text{see [2, 3, 8]}).
\end{equation}

Many researchers have studied Bell numbers in connection with various areas (see [1,2,3,8,15,18]). Recently, the degenerate Stirling numbers of the first kind are defined as the coefficients in the expansion

\begin{equation}
(7) \quad (x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k) (x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see [1, 11]}).
\end{equation}

As the inversion formula of (7), the degenerate Stirling numbers of the second kind appear as the coefficients in the expansion

\begin{equation}
(8) \quad (x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k) (x)_k, \quad (n \geq 0), \quad (\text{see [10]}).
\end{equation}
Let $\log_x t$ be the compositional inverse of $e_x(t)$. Then we note that the generating function of the degenerate Stirling numbers of the first kind is given by

$$\left(\log_x (1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (\text{see [10, 13]}).$$

(9)

It is well known that the degenerate Bernoulli polynomials of order $r \in \mathbb{N}$ are given by

$$e^{\lambda e_x(t)-1} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [15, 19]}).$$

(11)

When $x = 1$, $B_{n,\lambda} = B_{n,\lambda}(1)$ are called the degenerate Bell numbers. Note that

$$\lim_{\lambda \to 0} B_{n,\lambda} = B_n, \quad (n \geq 0).$$

As in [17, 18], we note that

$$B_{n,\lambda}(x) = \sum_{k=0}^{n} s_{2,\lambda}(n,k)x^k, \quad (n \geq 0).$$

(12)

As is well known, the beta function is given by the integral

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, dt, \quad (\text{Re} \alpha, \text{Re} \beta > 0), \quad (\text{see [24]}).$$

(13)

The beta function in (13) is equal to

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad (\text{see [24]}),$$

(14)

where $\Gamma(\alpha)$ is the gamma function with the property $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$.

2. TRUNCATED DEGENERATE BELL NUMBERS AND POLYNOMIALS

Truncating and then normalizing the degenerate Bell polynomials in (11), we define truncated degenerate Bell polynomials $B_{n,\lambda}^{(p)}(x)$ by

$$\sum_{n=0}^{\infty} B_{n,\lambda}^{(p)}(x) \frac{t^n}{n!} = \frac{p!}{x^p(e_x(t) - 1)^p} \left( e^{\lambda e_x(t)-1} - \sum_{k=0}^{p-1} x^k \frac{(e_x(t) - 1)^k}{k!} \right),$$

(15)

where $p$ is a nonnegative integer. In addition, $B_{n,\lambda}^{(p)} = B_{n,\lambda}^{(p)}(1)$ are called truncated degenerate Bell numbers.

From (15), it is immediate to see that

$$\sum_{n=0}^{\infty} B_{n,\lambda}^{(p)}(x) \frac{t^n}{n!} = p! \sum_{k=0}^{\infty} \frac{x^k (e_x(t) - 1)^k}{(k+p)!}. $$

(16)
From (16), we observe that

\[
\sum_{n=0}^{\infty} \frac{\operatorname{Bel}_{n, \lambda}^{(p)}(x)}{n!} t^n = \sum_{k=0}^{\infty} \frac{p! k!}{(k+p)!} x^k \frac{1}{k!} (e^t - 1)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{x^k}{(k+p)} \sum_{n=k}^{\infty} S_{2, \lambda}^{(n,k)} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{S_{2, \lambda}^{(n,k)}}{\binom{k+p}{k}} x^k \right) \frac{t^n}{n!}.
\]

Therefore, by comparing the coefficients on both sides (17), we obtain the following theorem.

**Theorem 1.** For \( n, p \geq 0 \), we have

\[
\operatorname{Bel}_{n, \lambda}^{(p)}(x) = \sum_{k=0}^{n} \frac{S_{2, \lambda}^{(n,k)}}{\binom{k+p}{k}} x^k.
\]

When \( p = 0 \), we have \( \operatorname{Bel}_{n, \lambda}^{(0)}(x) = \operatorname{Bel}_{n, \lambda}(x), \) \((n \geq 0)\). Let us take \( p = 1 \). Then, by (17), we get

\[
\sum_{n=0}^{\infty} \frac{\operatorname{Bel}_{n, \lambda}^{(1)}(x)}{n!} t^n = \frac{1}{xe^t - 1} \frac{e^t - 1}{t}
\]

\[
= \frac{1}{x} \sum_{l=0}^{\infty} \beta_{l, \lambda} t^l \frac{1}{l!} \sum_{m=1}^{\infty} \frac{\operatorname{Bel}_{m, \lambda}(x)}{m!} t^m
\]

\[
= \frac{1}{x} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{m} \frac{\beta_{n-m, \lambda} \operatorname{Bel}_{m, \lambda}(x) t^n}{n!}
\]

\[
= \frac{1}{x} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m+1} \frac{n+1}{m+1} \beta_{n-m, \lambda} \operatorname{Bel}_{m+1, \lambda}(x) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{x} \sum_{m=0}^{n} \frac{1}{m+1} \frac{n}{m} \beta_{n-m, \lambda} \operatorname{Bel}_{m+1, \lambda}(x) \right) \frac{t^n}{n!},
\]

where \( \beta_{n, \lambda} = \beta_{n, \lambda}^{(1)}(0) \) are called the Carlitz’s degenerate Bernoulli numbers.

Therefore, by (18), we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), we have

\[
x \operatorname{Bel}_{n, \lambda}^{(1)}(x) = \sum_{m=0}^{n} \frac{1}{m+1} \binom{n}{m} \beta_{n-m, \lambda} \operatorname{Bel}_{m+1, \lambda}(x).
\]

In the case of \( x = 1 \), we obtain

\[
\operatorname{Bel}_{n, \lambda}^{(1)} = \sum_{m=0}^{n} \frac{1}{m+1} \binom{n}{m} \beta_{n-m, \lambda} \operatorname{Bel}_{m+1, \lambda}.
\]
We observe from (16) that

\begin{align*}
p \int_0^1 e^{x(e_\lambda(t)-1)} (1-x)^{p-1} \, dx &= p \sum_{n=0}^\infty \frac{(e_\lambda(t)-1)^n}{n!} \int_0^1 x^n (1-x)^{p-1} \, dx \\
&= p \sum_{n=0}^\infty \frac{(e_\lambda(t)-1)^n}{n!} B(n+1, p) \\
&= p! \sum_{n=0}^\infty \frac{(e_\lambda(t)-1)^n}{(n+p)!} \\
&= \sum_{n=0}^\infty \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!}.
\end{align*}

On the other hand, we also have

\begin{align*}
p \int_0^1 e^{x(e_\lambda(t)-1)} (1-x)^{p-1} \, dx &= \sum_{n=0}^\infty p \int_0^1 \text{Bel}_{n,\lambda}(x) (1-x)^{p-1} \, dx \frac{t^n}{n!}.
\end{align*}

Therefore, by (19) and (20), we obtain the following proposition.

**Proposition 3.** For \( p, n \geq 0 \), we have

\[ p \int_0^1 \text{Bel}_{n,\lambda}(x) (1-x)^{p-1} \, dx = \text{Bel}_{n,\lambda}^{(p)}. \]

By (12) and Proposition 3, we get

\[ \text{Bel}_{n,\lambda}^{(p)} = p \sum_{k=0}^n S_2,\lambda(n, k) \int_0^1 x^k (1-x)^{p-1} \, dx \]

\[ = p \sum_{k=0}^n S_2,\lambda(n, k) B(k+1, p) = \sum_{k=0}^n \frac{S_2,\lambda(n, k)}{(k+p)} , \]

which also follows from Theorem 1 with \( x = 1 \). From (19), we note that

\begin{align*}
\sum_{n=0}^\infty \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!} &= p \int_0^1 e^{x(e_\lambda(t)-1)} (1-x)^{p-1} \, dx \\
&= p \int_0^1 e^{e_\lambda(t)} e^{-x} (1-x)^{p-1} \, dx \\
&= p \sum_{k=0}^\infty \frac{1}{k!} e_\lambda^k(t) \int_0^1 e^{-x} x^k (1-x)^{p-1} \, dx \\
&= \sum_{n=0}^\infty \left( p \sum_{k=0}^\infty \frac{(k)_{n,\lambda}}{k!} \int_0^1 e^{-x} (1-x)^p x^k \, dx \right) \frac{t^n}{n!}.
\end{align*}
For $n \geq 0$, by (22), we get
\begin{equation}
Bel^{(p)}_{n,\lambda} = p \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} \int_{0}^{1} e^{-x}(1-x)^{p-1}x^k dx
= p \sum_{k=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_{0}^{1} (1-x)^{p-1}x^{l+k} dx
= p \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(k)_{n,\lambda}}{k!} \frac{(-1)^l}{l!} B(p, l + k + 1)
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{(k+l)!}{(k+l+p)!} \frac{(k)_{n,\lambda}}{(k+l)!}.
\end{equation}

Therefore, by (23), we obtain the following Dobinski-like formula.

**Theorem 4** (Dobinski-like formula). For $n \geq 0$, we have
\[
Bel^{(p)}_{n,\lambda} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{(k+l)!}{(k+l+p)!} \frac{(k)_{n,\lambda}}{(k+l)!}.
\]

Note that
\[
\lim_{\lambda \to 0} \frac{Bel^{(0)}_{n,\lambda}}{k!} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = Bel_n, \quad (n \geq 0).
\]

For $n \geq 0$, by Proposition 3, we get
\begin{equation}
\sum_{k=0}^{n} p \int_{0}^{1} Bel_{n,\lambda}(x)(1-x)^{p-1} dx = p \sum_{k=0}^{n} S_{2,\lambda}(n, k) \int_{0}^{1} x^k (1-x)^{p-1} dx
= p \sum_{k=0}^{n} S_{2,\lambda}(n, k) \sum_{m=0}^{p-1} (-1)^m \frac{(p-1)}{m} \int_{0}^{1} x^{k+m} dx
= \sum_{k=0}^{n} \sum_{m=0}^{p-1} \frac{(m+1)}{(m+1)} \sum_{m=0}^{\infty} (-1)^m S_{2,\lambda}(n, k)
\end{equation}

By (19), we see that
\begin{equation}
\sum_{n=0}^{\infty} p \frac{Bel^{(p)}_{n,\lambda}}{n!} = p \int_{0}^{1} e^{e_{\lambda}(t)-1}x(1-x)^{p-1} dx
= p \int_{0}^{1} e^{e_{\lambda}(t)-1}(1-x)^{p-1} dx
= p \frac{e^{e_{\lambda}(t)-1}}{(e_{\lambda}(t)-1)^{p}} \int_{0}^{e_{\lambda}(t)-1} e^{-y^{p-1}} dy
= p \frac{e^{e_{\lambda}(t)-1}}{(e_{\lambda}(t)-1)^{p}} d\left(p, e_{\lambda}(t) - 1\right),
\end{equation}

where $d(s, z)$ is the linear incomplete gamma function defined by
\begin{equation}
d(s, z) = \int_{0}^{z} e^{-t^{s-1}} dt = \int_{0}^{\infty} e^{-t^{s-1}} dt - \int_{z}^{\infty} e^{-t^{s-1}} dt.
\end{equation}

We summarize our results in (24) and (25) in the next proposition.
Proposition 5. For \( n \geq 0 \), we have the following identities:

\[
\text{Bel}^{(p)}_{n,\lambda} = \sum_{k=0}^{n} \sum_{m=0}^{p-1} \frac{(m+1)(m)}{(k+m+1)} (-1)^m S_{\lambda, \lambda}(n,k),
\]

and

\[
\sum_{n=0}^{\infty} \text{Bel}^{(p)}_{n,\lambda} \frac{t^n}{n!} = p e^{\lambda(t-1)} \frac{d(p, e^{\lambda(t-1)})}{(e^{\lambda(t-1)} - 1)^p},
\]

where \( d(p, e^{\lambda(t-1)}) \) is the linear incomplete gamma function in (26).

From (15), we note that

\[
\sum_{n=0}^{\infty} \frac{\text{Bel}^{(p)}_{n,\lambda}(x) t^n}{n!} = \frac{p!}{x^p e^{\lambda(t-1)}} \sum_{n=0}^{\infty} \frac{x^n (e^{\lambda(t-1)} - 1)^n}{n!} - \sum_{k=1}^{p} \frac{(p)_k}{x^k} \frac{1}{k!} \left( \frac{t}{e^{\lambda(t-1)}} \right)^k
\]

\[
= \frac{p!}{x^p e^{\lambda(t-1)}} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\beta^{(p)}_{l+m,\lambda} t^m}{m!} - \sum_{k=1}^{p} \frac{k!}{x^k} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{\beta^{(k)}_{n,\lambda} t^n}{n!}
\]

\[
= \frac{p!}{x^p e^{\lambda(t-1)}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\beta^{(p)}_{n,m,\lambda} t^m}{m!} - \sum_{k=1}^{p} \frac{k!}{x^k} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{\beta^{(k)}_{n,\lambda} t^n}{n!}
\]

Therefore, by comparing the coefficients on both sides of (27), we obtain the following theorem.

Theorem 6. For \( n, p \geq 0 \), we have

\[
x^p \text{Bel}^{(p)}_{n,\lambda}(x) = \sum_{m=0}^{n+p} \frac{(n+p)!}{(n+m)!} \beta^{(p)}_{n+p-m,\lambda} \text{Bel}^{(p)}_{m,\lambda}(x) - \sum_{k=1}^{p} \frac{(p)_k}{(n+k)!} \beta^{(k)}_{n+k,\lambda} x^{p-k}.
\]

In the case of \( x = 1 \), we obtain

\[
\text{Bel}^{(p)}_{n,\lambda} = \sum_{m=0}^{n+p} \frac{(n+p)!}{(n+m)!} \beta^{(p)}_{n+p-m,\lambda} \text{Bel}_{m,\lambda} - \sum_{k=1}^{p} \frac{(p)_k}{(n+k)!} \beta^{(k)}_{n+k,\lambda}.
\]
We observe that

\begin{equation}
(\lambda - 1)(t) \frac{d}{dt} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (1 - e^{-t})^n
\end{equation}

(28)

\begin{align*}
&= (-1)^n \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (1 - e^{-t})^{n-1}
&= (-1)^2 \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-2)!} (1 - e^{-t})^{n-2}
&= \cdots \\
&= (-1)^p \sum_{n=p}^{\infty} \frac{1}{(n+1)(n-p)!} (1 - e^{-t})^{n-p}
&= (-1)^p \sum_{n=0}^{\infty} \frac{1}{(n+p+1)n!} (1 - e^{-t})^n.
\end{align*}

From (15) and noting that \( \frac{1}{(p+n)!} = (p+1)! = \sum_{l=0}^{\infty} \frac{(\lambda)^l}{l!} \), we have

\begin{equation}
\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!} = p! \sum_{n=0}^{\infty} \frac{(e^{-t} - 1)^n}{(n+p)!} = p \sum_{n=0}^{\infty} \frac{(p-1)!n!}{(n+p)!} \left(e^{-t} - 1\right)^n
\end{equation}

(29)

\begin{align*}
&= p \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(\lambda)^l}{p+l} \right) \frac{1}{n!} \left(e^{-t} - 1\right)^n
&= p \sum_{n=0}^{\infty} \frac{1}{p+1} \left(e^{-t} - 1\right)^n \sum_{l=0}^{n} \frac{(-1)^l}{l!} \left(e^{-t} - 1\right)^l
&= pe^{e^{-t}-1} \sum_{l=0}^{\infty} \frac{1}{p+1} \frac{1}{l!} \left(1 - e^{-t}\right)^l.
\end{align*}

By (28) and (29), we get

\begin{equation}
\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!} = p! \sum_{n=0}^{\infty} \frac{(e^{-t} - 1)^n}{(n+p)!} = pe^{e^{-t}-1} \sum_{n=0}^{\infty} \frac{1}{p+n} \left(1 - e^{-t}\right)^n
\end{equation}

(30)

\begin{align*}
&= (-1)^p \frac{1}{pe^{t}} \left(e^{-t} - 1\right)^{p-1} \left(e^{-t} - 1\right)^{(p-2)} e^{t}.
\end{align*}

Therefore, by (30), we obtain the following theorem.

**Theorem 7.** For \( p \geq 1 \), we have

\begin{equation}
\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!} = (-1)^{p-1} \frac{1}{pe^{t}} \left(e^{-t} - 1\right)^{p-1} \left(e^{-t} - 1\right)^{(p-2)} e^{t}.
\end{equation}
From (29), we have

\begin{equation}
\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}^{(p)} \frac{t^n}{n!} = p e^{e_\lambda(t)} \sum_{l=0}^{\infty} \frac{1}{l!} (1 - e_\lambda(t))^l
\end{equation}

\begin{align*}
&= p e^{e_\lambda(t)} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{\infty} S_{2,\lambda}(m, l) \frac{t^m}{m!} \\
&= p \sum_{k=0}^{\infty} \text{Bel}_{k,\lambda} \frac{k!}{k!} \sum_{m=0}^{\infty} \frac{1}{p+l} S_{2,\lambda}(m, l) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left( p \sum_{m=0}^{n} \sum_{l=0}^{m} \frac{n!}{m!} \frac{(-1)^l}{p+l} S_{2,\lambda}(m, l) \text{Bel}_{n-m,\lambda} \right) \frac{t^n}{n!}
\end{align*}

Therefore, by comparing the coefficients on both sides of (31), we obtain the following theorem.

**Theorem 8.** For \( n \geq 0 \) and \( p \geq 1 \), we have

\[ \text{Bel}_{n,\lambda}^{(p)} = p \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{(-1)^l}{p+l} S_{2,\lambda}(m, l) \text{Bel}_{n-m,\lambda}. \]

From (8), we can derive the following equation.

\begin{equation}
\int_{0}^{2\pi} \frac{1}{k!} (e^{\lambda (e^{i\theta})} - 1)^k \sin n \theta d\theta = \sum_{m=k}^{\infty} S_{2,\lambda}(m, k) \frac{1}{m!} \int_{0}^{2\pi} e^{m \theta} \sin n \theta d\theta
\end{equation}

\begin{align*}
&= \sum_{m=k}^{\infty} S_{2,\lambda}(m, k) \frac{1}{m!} \int_{0}^{2\pi} (\cos m \theta + i \sin m \theta) \sin n \theta d\theta \\
&= i \sum_{m=k}^{\infty} S_{2,\lambda}(m, k) \frac{1}{m!} \int_{0}^{2\pi} \sin m \theta \sin n \theta d\theta \\
&= \frac{i \pi}{n!} S_{2,\lambda}(n, k),
\end{align*}

where \( i = \sqrt{-1} \), and \( n, k \) are integers with \( n > 0 \), \( 0 \leq k \leq n \).

Thus, by (32), we obtain the following lemma.

**Lemma 9.** For \( n, k \in \mathbb{Z} \) with \( n > 0 \), \( 0 \leq k \leq n \), we have

\[ S_{2,\lambda}(n, k) = \frac{n!}{\pi} \text{Im} \int_{0}^{2\pi} \frac{1}{k!} (e^{\lambda (e^{i\theta})} - 1)^k \sin n \theta d\theta. \]

By Lemma 9, we get

\begin{equation}
\int_{0}^{2\pi} e^{e_\lambda(e^{i\theta})} - 1 \sin n \theta d\theta = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{0}^{2\pi} (e^{\lambda (e^{i\theta})} - 1)^k \sin n \theta d\theta
\end{equation}

\begin{align*}
&= \sum_{k=0}^{\infty} S_{2,\lambda}(m, k) \frac{1}{m!} \int_{0}^{2\pi} e^{m \theta} \sin n \theta d\theta \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{S_{2,\lambda}(m, k)}{m!} \int_{0}^{2\pi} (\cos m \theta + i \sin m \theta) \sin n \theta d\theta \\
&= i \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{S_{2,\lambda}(m, k)}{m!} \int_{0}^{2\pi} \sin m \theta \sin n \theta d\theta \\
&= \frac{i \pi}{n!} \sum_{k=0}^{n} S_{2,\lambda}(n, k) = \frac{i \pi}{n!} \text{Bel}_{n,\lambda},
\end{align*}
where \( n \) is a positive integer.

Therefore, by (33), we obtain the following corollary.

**Corollary 10.** For \( n > 0 \), we have

\[
\frac{n!}{\pi} \text{Im} \int_0^{2\pi} e^{i\lambda \theta} (e^{\theta} - 1)^{n-1} \sin n\theta d\theta = \text{Bel}_{n, \lambda}.
\]

From (15), we note that

\[
\begin{align*}
\text{Bel}^{(p)}_{m, \lambda} &= \frac{1}{m!} \int_0^{2\pi} e^{im\theta} \sin n\theta d\theta \\
&= \sum_{m=0}^{\infty} \text{Bel}^{(p)}_{m, \lambda} \frac{1}{m!} \int_0^{2\pi} (\cos m\theta + i \sin m\theta) \sin n\theta d\theta \\
&= i \sum_{m=0}^{\infty} \text{Bel}^{(p)}_{m, \lambda} \frac{1}{m!} \int_0^{2\pi} \sin m\theta \sin n\theta d\theta = \frac{\pi}{n!} \text{Bel}^{(p)}_{n, \lambda},
\end{align*}
\]

where \( n \) is a positive integer.

Therefore, by (34), we obtain the following theorem.

**Theorem 11.** For \( n > 0 \), we have

\[
\text{Bel}^{(p)}_{n, \lambda} = \frac{n!p!}{\pi} \text{Im} \int_0^{2\pi} \left( \frac{e^{i\lambda \theta} (e^{\theta} - 1)^{n-1}}{(e^{\theta} - 1)^p} - \sum_{l=0}^{n-1} \frac{(e^{\theta} - 1)^l}{l!} \right) \sin n\theta d\theta.
\]

From (15) and (17), we have

\[
\begin{align*}
ed^{-1}(t) \frac{d}{dt} \sum_{n=0}^{\infty} \text{Bel}^{(p)}_{n, \lambda} \frac{t^n}{n!} &= e^{\lambda-1}(t) \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(e^{\lambda}(t) - 1)^n}{(n+p)!} \\
&= p! \sum_{n=0}^{\infty} \frac{n+1}{(n+p+1)!} (e^{\lambda}(t) - 1)^n \\
&= p \sum_{n=0}^{\infty} B(n+2, p) \frac{1}{n!} (e^{\lambda}(t) - 1)^n \\
&= p \int_0^1 x \sum_{n=0}^{\infty} \frac{1}{n!} x^n (e^{\lambda}(t) - 1)^n (1-x)^{p-1} dx \\
&= p \int_0^1 x \sum_{n=0}^{\infty} \frac{1}{n!} x^n (e^{\lambda}(t) - 1)^n (1-x)^{p-1} dx.
\end{align*}
\]

It is easy to see that

\[
\begin{align*}
ed^{-1}(t) \frac{d}{dt} \sum_{n=0}^{\infty} \frac{\text{Bel}^{(p)}_{m, \lambda} t^n}{n!} &= \sum_{m=0}^{\infty} \text{Bel}^{(p)}_{m+1, \lambda} \frac{t^m}{m!} \sum_{l=0}^{\infty} (\lambda - 1)_{l_m, \lambda} \frac{t^l}{l!} \\
&= \sum_{m=0}^{\infty} \left( \sum_{m=0}^{\infty} \frac{t^n}{n!} \text{Bel}^{(p)}_{m+1, \lambda}(\lambda - 1)_{n-m, \lambda} \right) \frac{t^l}{l!}.
\end{align*}
\]
On the other hand, we also have

\[
\int_{0}^{1} e^{x(e_{k}(t)-1)}x(1-x)^{p-1}dx = -p \int_{0}^{1} e^{x(e_{k}(t)-1)}(1-x)^{p}dx + p \int_{0}^{1} e^{x(e_{k}(t)-1)}(1-x)^{p-1}dx
\]

\[
= -\frac{p}{p+1}(p+1) \sum_{n=0}^{\infty} \frac{(e_{k}(t)-1)^{n}}{(n+p+1)!} + p \sum_{n=0}^{\infty} \frac{(e_{k}(t)-1)^{n}}{(n+p)!}
\]

\[
= \sum_{n=0}^{\infty} \left( -\frac{p}{p+1} \text{Bel}_{n,\lambda}^{(p+1)} + \text{Bel}_{n,\lambda}^{(p)} \right) \frac{t^{n}}{n!}
\]

Therefore, by (35), (36) and (37), we obtain the following theorem.

**Theorem 12.** For \( p \geq 0 \) and \( n \geq 2 \), we have

\[
\text{Bel}_{n+1,\lambda}^{(p)} = (n+1-n\lambda)\text{Bel}_{n,\lambda}^{(p)} - \frac{p}{p+1} \text{Bel}_{n,\lambda}^{(p)} - \sum_{m=0}^{n-2} \binom{n}{m} \text{Bel}_{m+1,\lambda}^{(p)} (\lambda - 1)_{n-m,\lambda}.
\]

In the case of \( p = 0 \), we obtain

\[
\text{Bel}_{n+1,\lambda} = (n+1-n\lambda)\text{Bel}_{n,\lambda} - \sum_{m=0}^{n-2} \binom{n}{m} \text{Bel}_{m+1,\lambda} (\lambda - 1)_{n-m,\lambda}
\]

\[
= (n+1-n\lambda)\text{Bel}_{n,\lambda} - \sum_{m=0}^{n-1} \binom{n}{m-1} \text{Bel}_{m,\lambda} (\lambda - 1)_{n-m+1,\lambda}.
\]

It is known that the degenerate Stirling polynomials of the second kind are defined by

\[
(y+x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k|x)(y)_{k}, \quad (n \geq 0), \quad (\text{see } [10,13]).
\]

Thus, we note that

\[
\sum_{n=0}^{\infty} \binom{1}{n} (e_{\lambda}(t)-1)^{n}e_{\lambda}^{x}(t) (y)_{n} = e_{\lambda}^{y+x}(t) = \sum_{k=0}^{\infty} (y+x)_{k,\lambda} \frac{t^{k}}{k!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{k} S_{2,\lambda}(k,n|x)(y)_{n} \frac{t^{k}}{k!} = \sum_{n=0}^{\infty} \left( \sum_{k=n}^{\infty} S_{2,\lambda}(k,n|x) \frac{t^{k}}{k!} \right) (y)_{n}.
\]

By comparing the coefficients on both sides of (39), we get

\[
\frac{1}{n!} (e_{\lambda}(t)-1)^{n}e_{\lambda}^{x}(t) = \sum_{k=n}^{\infty} S_{2,\lambda}(k,n|x) \frac{t^{k}}{k!}, \quad (n \geq 0).
\]

From (7), we note that

\[
\sum_{n=0}^{\infty} \frac{(x+y)_{n,\lambda}}{n!} t^{n} = e_{\lambda}^{x+y}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (x)_{i,\lambda} \frac{t^{i}}{i!} \sum_{j=0}^{\infty} (y)_{j,\lambda} \frac{t^{j}}{j!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (x)_{n-i,\lambda} (y)_{i,\lambda} \frac{t^{n}}{n!}.
\]
By (38) and (41), we get
\[
\sum_{l=0}^{n} S_{2,\lambda}(n,l|x)(y)_l = (y+x)_{n,\lambda} = \sum_{l=0}^{n} \binom{n}{l} (x)_{n-i,\lambda} (y)_i, 
\]
(42)
\[
= \sum_{l=0}^{n} \binom{n}{l} (x)_{n-i,\lambda} \sum_{l=0}^{n} S_{2,\lambda}(l,l)(y)_l 
= \sum_{l=0}^{n} \left( \sum_{l=0}^{n} \binom{n}{l} (x)_{n-i,\lambda} S_{2,\lambda}(l,l) \right)(y)_l. 
\]
By comparing the coefficients on both sides of (42), we obtain the following theorem.

**Theorem 13.** For \(n,l \geq 0\), we have
\[
S_{2,\lambda}(n,l|x) = \sum_{l=0}^{n} \binom{n}{l} (x)_{n-i,\lambda} S_{2,\lambda}(l,l). 
\]

For \(p \geq 0\), we define truncated degenerate modified Bell polynomials \(B_{n,\lambda}^{(p)}(x)\) by
\[
\frac{p!}{(e_{\lambda}(t) - 1)^p} \left( e_{\lambda}(t) - 1 \right) e_{\lambda}(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(p)}(x) \frac{t^n}{n!}.
\]
(43)
For \(x = 1\), \(B_{n,\lambda}^{(p)} = B_{n,\lambda}^{(p)}(1)\) are called truncated degenerate modified Bell numbers. From (43), we note that
\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(p)}(x) \frac{t^n}{n!} = p! \sum_{n=0}^{\infty} \frac{(e_{\lambda}(t) - 1)^n}{(n+p)!} e_{\lambda}(t) 
= \sum_{m=0}^{\infty} \text{Bel}_{n,\lambda}^{(p)}(x) \sum_{l=0}^{\infty} \binom{n}{l} \frac{t^n}{l!} 
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{n,\lambda}^{(p)}(x)_{n-m}\lambda \right) \frac{t^n}{n!}.
\]
Thus we obtain
\[
B_{n,\lambda}^{(p)}(x) = \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{n,\lambda}^{(p)}(x)_{n-m}\lambda.
\]
Again, from (43) we observe that
\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(p)}(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{p!}{(k+p)!} \left( e_{\lambda}(t) - 1 \right)^k e_{\lambda}(t) 
= \sum_{k=0}^{\infty} \frac{1}{(k+p)!} \sum_{n=k}^{\infty} S_{2,\lambda}(n,k|x) \frac{t^n}{n!} 
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{S_{2,\lambda}(n,k|x)}{(k+p)} \right) \frac{t^n}{n!}.
\]
(45)
Therefore, by (45), we obtain the following theorem.

**Theorem 14.** For \(n,p \geq 0\), we have
\[
B_{n,\lambda}^{(p)}(x) = \sum_{k=0}^{n} \frac{1}{(k+p)} S_{2,\lambda}(n,k|x).
\]
From (43), we have
\[
 p \int_0^1 (1 - y)^{p-1} e^\lambda_n(t) e^{\psi(t)} dy = \sum_{n=0}^{\infty} \frac{(e^\lambda_n(t) - 1)^n}{n!} \int_0^1 (1 - y)^{p-1} y^n dy
\]
\[
= \sum_{n=0}^{\infty} \frac{(e^\lambda_n(t) - 1)^n}{n!} B(n+1, p)
\]
\[
= \frac{p!}{n!} \sum_{n=0}^{\infty} \frac{(e^\lambda_n(t) - 1)^n}{(n+p)!} e^\lambda_n(t)
\]
\[
= \sum_{n=0}^{\infty} B_{n, \lambda}^{(p)}(x) \frac{t^n}{n!}.
\]

Thus, we get
\[
P \int_0^1 (1 - y)^{p-1} e^\lambda_n(t) e^{\psi(t)} dy = \sum_{n=0}^{\infty} B_{n, \lambda}^{(p)}(x) \frac{t^n}{n!}.
\]

From (47), we can easily derive the following equation
\[
B_{n, \lambda}^{(p)}(x) = \frac{p!}{n!} \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m, \lambda} \int_0^1 (1 - y)^{p-1} B_{m, \lambda}(y) dy,
\]
\[\text{ (n \geq 0),} \]

By (11), we easily get
\[
B_{n, \lambda}(x) = e^{-x} \sum_{k=0}^{\infty} \frac{(k)_{m, \lambda}}{k!} x^k.
\]

Thus, by (48) and (49), we get
\[
B_{n, \lambda}^{(p)}(x) = \frac{p!}{n!} \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m, \lambda} \int_0^1 (1 - y)^{p-1} B_{m, \lambda}(y) dy
\]
\[
= \frac{p^m}{n!} \sum_{k=0}^{\infty} \frac{(x+k)_{n, \lambda}}{k!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^1 y^{m+k}(1 - y)^{p-1} dy
\]
\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(x+k)_{n, \lambda}}{k!} \frac{(-1)^m}{m!(m+k+p)}.
\]

On the other hand,
\[
P \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m, \lambda} \int_0^1 (1 - y)^{p-1} B_{m, \lambda}(y) dy
\]
\[
= \frac{p!}{n!} \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m, \lambda} \sum_{k=0}^{\infty} S_{2, \lambda}(m, k) \int_0^1 (1 - y)^{(p-1)k} dy
\]
\[
= \sum_{m=0}^{n} \sum_{k=0}^{\infty} \binom{n}{m} (x)_{n-m, \lambda} S_{2, \lambda}(m, k) \frac{(p+k)^{-1}}{k}.
\]

Therefore, by (50) and (51), we obtain the following theorem.

**Theorem 15.** For \( n, p \geq 0 \), we have
\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(x+k)_{n, \lambda}}{k!} \frac{(-1)^m}{m!(m+k+p)} = \sum_{m=0}^{n} \sum_{k=0}^{\infty} \binom{n}{m} (p+k)^{-1} S_{2, \lambda}(m, k)(x)_{n-m, \lambda}.
\]
From (50), we note that

\[ B_{n+1,\lambda}^{(p)}(x) = p \sum_{k=0}^{\infty} \frac{(x+k)n\lambda}{k!} \int_0^1 e^{-\gamma y^k(1-y)^{p-1}} dy, \quad (n \geq 0). \]

Thus, we get

\[ B_{n+1,\lambda}^{(p)}(x) = p \sum_{k=0}^{\infty} \frac{(x+k)n\lambda}{k!} \int_0^1 e^{-\gamma y^k(1-y)^{p-1}} dy \]

\[ + p \sum_{k=1}^{\infty} \frac{(x+k)n\lambda}{(k-1)!} \int_0^1 e^{-\gamma y^k(1-y)^{p-1}} dy \]

\[ = (x-n\lambda)B_{n,\lambda}(x) + p \sum_{k=0}^{\infty} \frac{(x+k+1)n\lambda}{k!} \int_0^1 y^{k-1}e^{-\gamma y(1-y)^{p-1}} dy \]

\[ + \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (1-n-j\lambda) \sum_{k=0}^{\infty} \frac{(x+k)j\lambda}{k!} \int_0^1 (1-y)^{p-1} e^{-\gamma y^k} dy \]

\[ = (x-n\lambda)B_{n,\lambda}(x) + \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (1-n-j\lambda) \left( \frac{p}{p+1}B_{j,\lambda}^{(p+1)}(x) - B_{j,\lambda}^{(p)}(x) \right). \]

Therefore, by (53), we obtain the following theorem.

**Theorem 16.** For \( n, p \geq 0 \), we have

\[ B_{n+1,\lambda}^{(p)}(x) = (x-n\lambda)B_{n,\lambda}(x) - \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (1-n-j\lambda) \left( \frac{p}{p+1}B_{j,\lambda}^{(p+1)}(x) - B_{j,\lambda}^{(p)}(x) \right). \]

3. **Further Remark**

It is well known that \( X \) is the beta random variable with parameters \( \alpha > 0 \) and \( \beta > 0 \) if the probability density function of \( X \) is given by

\[ f(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise}. \end{cases} \]

The beta random variable \( X \) with parameters \( \alpha > 0 \) and \( \beta > 0 \) is denoted by \( X \sim \text{Beta}(\alpha, \beta) \), (see [22]).

Let \( g(x) \) be a real valued function, and let \( f(x) \) be the probability density function of the continuous random variable \( X \). Then the expectation of \( g(X) \) is defined by

\[ E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx, \quad \text{(see [21])}. \]
For $X \sim \text{Beta}(\alpha, \beta)$, we have

$$E[e^{X}] = \int_{-\infty}^{\infty} f(x) e^{x} dx = \frac{1}{B(\alpha, \beta)} \int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1} e^{x} dx$$

$$= \frac{1}{B(\alpha, \beta)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{1} x^{n+\alpha-1}(1-x)^{\beta-1} dx \frac{t^{n}}{n!}$$

Thus, we get

$$E[X^{n}] = \frac{B(n+\alpha, \beta)}{B(\alpha, \beta)} = \frac{(n+\alpha-1)}{(n+\alpha+\beta-1)}, \quad (n \geq 0).$$

For $X \sim \text{Beta}(1, p)$, we have

$$E[e^{X(e_{X}(t)-1)}] = p \int_{0}^{1} (1-x)^{p-1} e^{(e_{X}(t)-1)} dx$$

$$= \sum_{n=0}^{\infty} \frac{(e_{X}(t)-1)^{n}}{n!} p \int_{0}^{1} (1-x)^{p-1} x^{n} dx = p! \sum_{n=0}^{\infty} \frac{(e_{X}(t)-1)^{n}}{(n+p)!}$$

$$(54)$$

On the other hand,

$$E[e^{X(e_{X}(t)-1)}] = \sum_{k=0}^{\infty} E[X^{k}] \frac{1}{k!} (e_{X}(t)-1)^{k}$$

$$= \sum_{k=0}^{\infty} E[X^{k}] \sum_{n=k}^{\infty} \frac{S_{2,n}(n,k)}{n!} t^{n}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_{2,n}(n,k) E[X^{k}] \right) \frac{t^{n}}{n!}.$$  

From (54) and (55), we have

$$\sum_{k=0}^{n} E[X^{k}] S_{2,n}(n,k) = \text{Be}_{n}^{(p)}, \quad (n \geq 0),$$

where $X \sim \text{Beta}(1, p)$.

4. CONCLUSION

Truncated polynomials have been shown to play an important role in various areas. In this paper, we introduced truncated degenerate Bell polynomials and numbers and investigated their various properties. In more detail, for the truncated degenerate Bell polynomials and numbers we obtained explicit expressions, identities involving other special polynomials, integral representations, Dobinski-like formula and expressions of the generating function in terms of differential operators and linear incomplete gamma function. In addition, we introduced truncated degenerate modified Bell polynomials and numbers and got similar results for those polynomials. As an application of our results, we showed that the truncated degenerate Bell numbers can be expressed as a finite sum involving moments of the beta random variable with some parameters.

We would like to conclude this section by displaying the different expressions for the truncated degenerate Bell numbers that we have obtained in this paper.
\[ \text{Bel}_{n, \lambda}^{(p)} = p \int_{0}^{1} \text{Bel}_{n, \lambda}(x)(1-x)^{p-1}dx \]
\[ = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{(k+l)^{p}}{(k+l+p)} \frac{(k)_{n, \lambda}}{(k+l)!} \]
\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{n} \frac{n^{p-1}}{m} \frac{(-1)^l}{m+l} S_{2, \lambda} (n, k) \]
\[ = \sum_{m=0}^{n} \frac{n!}{m!} \frac{S_{2, \lambda} (m, l) \text{Bel}_{n-m, \lambda}}{p+l} \]
\[ = \sum_{k=0}^{n} E[X^k] S_{2, \lambda} (n, k), \quad (n \geq 0), \]

where \( X \sim \text{Beta}(1, p) \).

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DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: tkim@kw.ac.kr

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
Email address: dskim@sogang.ac.kr