OPTIMAL ENERGY DECAY RATES FOR SOME WAVE EQUATIONS WITH DOUBLE DAMPING TERMS

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Abstract. We consider the Cauchy problem in $\mathbb{R}^n$ for some wave equations with double damping terms, that is, one is the frictional damping $u_t(t, x)$ and the other is very strong structural damping expressed as $(-\Delta)^\theta u_t(t, x)$ with $\theta > 1$. We will derive optimal decay rates of the total energy and the $L^2$-norm of solutions as $t \to \infty$. These results can be obtained in the case when the initial data have a sufficient high regularity in order to guarantee that the corresponding high frequency parts of such energy and $L^2$-norm of solutions are remainder terms. A strategy to get such results comes from a method recently developed by the first author [11].

1. Introduction. We are concerned with the following Cauchy problem for doubly damped wave equations in $\mathbb{R}^n$ ($n \geq 1$)

$$u_{tt}(t, x) - \Delta u(t, x) + u_t(t, x) + (-\Delta)^\theta u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (1)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \quad (2)$$

where $\theta > 1$, and the nonlocal operator $(-\Delta)^\theta$ is defined by

$$((-\Delta)^\theta v)(x) := \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} |\xi|^{2\theta} \hat{v}(\xi) d\xi$$

with the Fourier transform $\hat{v}(\xi)$ of the function $v(x)$ (see (6) below). The initial data $u_0$ and $u_1$ are chosen from the usual energy space $[u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

It is standard to check that the problem (1)-(2) has a unique weak solution (cf. [20])

$$u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)).$$

After celebrated $L^p$-$L^q$ estimates to the equation below are introduced by Ponce [26] and/or Shibata [28]:

$$u_{tt}(t, x) - \Delta u(t, x) - \Delta u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (3)$$

several mathematicians participated in studying the Cauchy and/or mixed problems for (3). For example, in 2001 Ikehata [10] has started to study a decay property of

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the total energy to the mixed problem for (3), and this problem was considered in the exterior smooth domain of a bounded obstacle. After those investigations, for a while, no any trials to derive more precise properties of solutions to the Cauchy problem for (3) seem to be done. However, since Ikehata-Todorove-Yordanov [20] investigated the asymptotic profile of solutions to (3)-(2) from the view point of abstract form, and in the framework of concrete setting Ikehata [11] also derived the leading term of solutions, and obtained the optimal decay rates of the $L^2$-norm of solutions for (3) in the higher dimensional case $n \geq 3$, many interesting research articles are rapidly published. In particular, in the low dimensional case for $n = 1, 2$ Ikehata-Onodera [15] discovers that $L^2$-norm of the solution to (3) and (2) blows up in infinite time. Furthermore, the newest paper due to Michihisa [24] studies higher order asymptotic expansions of solutions to problem (3)-(2).

On the other hand, concerning the following wave equations with a single fractional damping term with $\theta \in (0, 1)$:

$$u_{tt}(t, x) - \Delta u(t, x) + (-\Delta)^\theta u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

recently so many new results are intermittently announced. In particular, we can cite Karch [21], D’Abbicco-Ebert [4, 5, 6], D’Abbicco-Reissig [8], D’Abbicco-Ebert-Picon [7], Charaõ-da Luz-Ikehata [1], Ikehata-Natsume [14], Ikehata-Takeda [19], and all these papers have contributed to derive several decay estimates and asymptotic profiles of solutions to problem (4)-(2) with $\theta \in (0, 1]$ (for $\theta = 0$, one can cite Matsumura [23], Nishihara [25], Racke [27], Sobajima-Wakasugi [29], Taylor [30], Todorova-Yordanov [31], and the references therein). In this connection, to the best of authors’ knowledge, the paper due to Lu-Reissig [22] first presented a Cauchy problem of (4) with a more generalized time dependent structural damping $b(t)(-\Delta)^\theta u_t$ to study parabolic aspects of the equation from the viewpoint of energy estimates. Anyway, in this situation, Ikehata-Sawada [16] presented the following wave equations with double damping terms:

$$u_{tt}(t, x) - \Delta u(t, x) + u_t(t, x) - \Delta u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

and they have pointed out that the asymptotic profile as $t \to \infty$ of the solution to (5)-(2) is dominated by the frictional damping $u_t(t, x)$, that is, it is pointed out that the damping term $u_t$ is more dominant than the strong damping expressed by $-\Delta u_t$ in the equation (5). Soon after [16], D’Abbicco [3] also studied the decay estimates of solutions to problem (5)-(2) for all $n \geq 1$, and applied it to the nonlinear problems with power nonlinearities $|u|^p$, $|u_t|^p$, and $|\nabla u|^p$ to find the so-called critical exponent for $p$, which guarantees the unique global existence of solutions to the corresponding semi-linear problems. Independently from [3], Ikehata-Takeda [17, 18] also considers the semi-linear problems (5)-(2) with nonlinearity $|u|^p$ for the case $n = 1, 2, 3$, and derives the critical exponent together with asymptotic profiles of solutions as $t \to \infty$. From these results of [16], [3] and [17, 18], we can find that the equation (5) has purely a diffusive structure as $t \to \infty$, and the critical exponent is the same as the so-called Fujita exponent in the heat equation case. In addition, the asymptotic profile of solutions to problem (5)-(2) is a constant multiple of the Gauss kernel. The effect of the frictional damping term $u_t$ is extremely strong as $t \to \infty$. Concerning higher order asymptotic expansions of solutions to problem (5)-(2) one can also cite a recent paper due to Ikehata-Michihisa [13].

By the way, quite recently Ghisi-Gobbino-Haraux [9] presented the so-called GGH-model (4) with $\theta > 1$, and the unique global existence and the smoothing property of solutions are studied. $\theta > 1$ shows a kind of very strong damping effect.
Thereafter, Ikehata-Iyota [12] recently studies the equation (4) with \( \theta > 1 \), and derives its asymptotic profile and optimal decay rates of the solution. Especially, they have newly found a regularity loss structure in the high frequency estimates of solutions for (4) with \( \theta > 1 \).

So, a natural question arises that what kind of structure the equation (1) has in the case of \( \theta > 1 \). In particular, it should be emphasized that this question is investigated from the viewpoint of the energy norm in this paper.

Our results are read as follows.

**Proposition 1.** Let \( n \geq 1, \theta > 1 \) and \( \ell \geq 0 \). If \( [u_0, u_1] \in (H^{\ell+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (H^\ell(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \), then it is true that

\[
\|u(t, \cdot)\|_2^2 + \|
abla u(t, \cdot)\|_2^2 \leq C(1+t)^{-\frac{\theta}{2}} \|u_1\|_1^2 + C(1+t)^{-\frac{n+2}{2}} \|u_0\|_1^2 + C((1+t)\|u_1\|_1^2 + \|Du_0\|_2^2),
\]

where \( C > 0 \) is a constant.

**Proposition 2.** Let \( n \geq 3, \theta > 1 \) and \( \ell \geq 1 \). If \( [u_0, u_1] \in (H^\ell(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (H^{\ell-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \), then it is true that

\[
\|u(t, \cdot)\|_2^2 \leq C(1+t)^{-\frac{\theta-n}{2}} \|u_1\|_1^2 + C(1+t)^{-\frac{n}{2}} \|u_0\|_1^2 + C((1+t)\|u_1\|_1^2 + \|Du_0\|_2^2),
\]

where \( C > 0 \) is a constant.

**Remark 1.** The decay rates of solutions just obtained in Propositions 1 and 2 seem not to be optimal unfortunately (at least) in the case of high regularity \( \ell \gg 1 \). Such optimality can be discussed in section 5. The results of Propositions 1 and 2 are just intermediate ones.

**Remark 2.** Consider the case of \( n \geq 3 \). If one chooses \( \ell = 0 \) in Proposition 1, and \( \ell = 1 \) in Proposition 2, then one can obtain

\[
\|u(t, \cdot)\|_2^2 + \|
abla u(t, \cdot)\|_2^2 \leq C(1+t)^{-\frac{\theta}{2}} \|u_1\|_1^2 + C(1+t)^{-\frac{n+2}{2}} \|u_0\|_1^2 + C((\|u_1\|_1^2 + \|Du_0\|_2^2),
\]

\[
\|u(t, \cdot)\|_3^2 \leq C(1+t)^{-\frac{\theta-n}{2}} \|u_1\|_1^2 + C(1+t)^{-\frac{n}{2}} \|u_0\|_1^2 + C(1+t)^{-\frac{n}{2}} \|Du_0\|_2^2 + \|u_1\|_1^2)
\]

One can observe interesting phenomena such that the \( L^2 \)-norm of the weak solution \( u(t, x) \) necessarily decays with some rate, however the total energy still does not necessarily decay under the low regularity assumption on the initial data. This phenomenon shows a completely different aspect as compared with the usual damped wave equation

\[
uu(t, x) - \Delta u(t, x) + u(t, x) = 0.
\]

In the framework of low regularity on the initial data, the doubly damped waves (1) with \( \theta > 1 \) do not seem to have any usual diffusive aspects.

We introduce the parameters as follows:

\[
\nu_j := \max\{\frac{n+2j}{2}(\theta-1), 1\} \quad (j = 0, 1, 2).
\]

Furthermore, one defines

\[
P_j := \int_{\mathbb{R}^n} u_j(x)dx \quad (j = 0, 1).
\]

If one assumes more regularity on the initial data, then one can get the following new results concerning the asymptotic profiles as \( t \to \infty \) of the solution to problem (1)-(2).
Theorem 1.1. Let $n \geq 1$, $\theta > 1$ and $\ell > \nu_0$. If $[u_0, u_1] \in (H^\ell(R^n) \cap L^{1,1}(R^n)) \times (H^{\ell-1}(R^n) \cap L^{1,1}(R^n))$, then the solution $u(t, x)$ to problem (1)-(2) satisfies
\[
\int_{R^n} |\hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2}|^2 d\xi 
\leq C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|D^\ell u_0\|_{L^2}^2 + \|D^{\ell-1} u_1\|_{L^2}^2)(1 + t)^{-\frac{\nu}{2\ell}}.
\]

Theorem 1.2. Let $n \geq 1$, $\theta > 1$ and $\ell > \nu_1$. If $[u_0, u_1] \in (H^\ell(R^n) \cap L^{1,1}(R^n)) \times (H^{\ell-1}(R^n) \cap L^{1,1}(R^n))$, then the solution $u(t, x)$ to problem (1)-(2) satisfies
\[
\int_{R^n} |\hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2}|^2 d\xi 
\leq C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|D^\ell u_0\|_{L^2}^2 + \|D^{\ell-1} u_1\|_{L^2}^2)(1 + t)^{-\frac{\nu}{2\ell}}.
\]

Theorem 1.3. Let $n \geq 1$, $\theta > 1$ and $\ell > \nu_2$. If $[u_0, u_1] \in (H^\ell(R^n) \cap L^{1,1}(R^n)) \times (H^{\ell-1}(R^n) \cap L^{1,1}(R^n))$, then the solution $u(t, x)$ to problem (1)-(2) satisfies
\[
\int_{R^n} |\hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2}|^2 d\xi 
\leq C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2)(1 + t)^{-\frac{n+6}{2}} + C(\|D^\ell u_0\|_{L^2}^2 + \|D^{\ell-1} u_1\|_{L^2}^2)(1 + t)^{-\frac{\nu}{2\ell}}.
\]

Remark 3. As a result above the asymptotic profile of solutions to problem (1)-(2) is a constant multiple of the Gauss kernel $G(t, x)$ as $t \to \infty$:
\[
u(t, \cdot) \sim (P_0 + P_1)G(t, \cdot) \quad (t \to \infty)
\]
in $L^2$-sense, where $G(t, x) := (\sqrt{4\pi t})^{-n}e^{-\frac{|x|^2}{4t}}$. However, this statement is only true in the case when the initial data have a high regularity such as $\ell > \nu_0$. It is still open to find a leading term of the solution as $t \to \infty$ in the case of low regularity $\ell \leq \nu_0$ on the initial data (see also Remark 2).

Our plan in this paper is as follows. In section 2, we prepare several notation which will be used later. In section 3, we shall prove Propositions 1 and 2 by the energy method in the Fourier space due to [32], and in section 4, we prove Theorems 1.1, 1.2 and 1.3 by use of the method introduced in [11]. As an application, we will discuss the optimality concerning the decay rate of the $L^2$-norm of solutions and the total energy in Section 5.

2. Notation. Throughout this paper, $\| \cdot \|_q$ stands for the usual $L^q(R^n)$-norm. For simplicity of notations, in particular, we use $\| \cdot \|$ instead of $\| \cdot \|_2$.
\[
f \in L^{1,\gamma}(R^n) \iff f \in L^1(R^n), \|f\|_{1,\gamma} := \int_{R^n} (1 + |x|)^\gamma |f(x)| dx < +\infty, \quad \gamma \geq 0.
\]
Furthermore, we denote the Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ by
\[
\hat{\phi}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ix\cdot\xi} \phi(x) dx, \quad (6)
\]
where $i := \sqrt{-1}$, and $x \cdot \xi = \sum_{i=1}^{n} x_{i} \xi_{i}$ for $x = (x_{1}, \cdots, x_{n})$ and $\xi = (\xi_{1}, \cdots, \xi_{n})$.

When we estimate several functions by applying the Fourier transform sometimes we can also use the following definition in place of (6)

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^{n}} e^{-ix \cdot \xi} \phi(x) dx$$

without loss of generality. We also use the notation

$$v_{t} = \frac{\partial v}{\partial t}, \quad v_{tt} = \frac{\partial^{2} v}{\partial t^{2}}, \quad \Delta = \sum_{i=1}^{n} \frac{\partial^{2}}{\partial x_{i}^{2}}, \quad x = (x_{1}, \cdots, x_{n}),$$

and

$$||D^{\ell} f|| := \left( \int_{\mathbb{R}^{n}} |\xi|^{2\ell} |\hat{f}(\xi)|^{2} d\xi \right)^{1/2} (f \in H^{\ell}(\mathbb{R}^{n})).$$

In order to prove Propositions 1 and 2 we shall prepare several important functions. One of them is closely related with the so-called Lyapunov function: for $\theta \in [0, \infty), \beta \in \mathbb{R}^{n}$ and $t \geq 0$, we set

$$E_{0}(t, \xi) := \frac{1}{2} |\hat{u}_{t}(t, \xi)|^{2} + \frac{1}{2} |\xi|^{2} |\hat{u}(t, \xi)|^{2},$$

$$E(t, \xi) := \frac{1}{2} |\hat{u}_{tt}(t, \xi)|^{2} + \frac{1}{2} |\xi|^{2} |\hat{u}(t, \xi)|^{2} + \beta \rho(\xi) \Re(\hat{u}_{1}(t, \xi) \overline{\hat{u}(t, \xi)})$$

$$+ \frac{1}{2} \beta \rho(\xi)(1 + |\xi|^{2\theta}) |\hat{u}(t, \xi)|^{2},$$

$$F(t, \xi) := (1 + |\xi|^{2\theta}) |\hat{u}_{t}(t, \xi)|^{2} + \beta \rho(\xi) |\xi|^{2} |\hat{u}(t, \xi)|^{2},$$

$$R(t, \xi) := \beta \rho(\xi) |\hat{u}_{1}(t, \xi)|^{2}.$$ Here, for $\theta > 1$, we have just defined a key function $\rho : \mathbb{R}^{n}_{\xi} \to \mathbb{R}$ in order to apply the energy method in the Fourier spaces due to [32]:

$$\rho(\xi) = \left\{ \begin{array}{ll} |\xi|^{2} & |\xi| \leq 1, \\ 2|\xi|^{2\theta} & 1 + |\xi|^{4\theta} \leq 1. \end{array} \right.$$ Note that $\rho(\xi)$ is continuous at $|\xi| = 1$, and $\rho(\xi)$ itself is bounded.

3. Proofs of propositions 1 and 2. In this section, let us apply the Fourier transform to both sides of (1) together with the initial data (2). Then in the Fourier space $\mathbb{R}^{n}_{\xi}$, one has the following Cauchy problem for ODE:

$$\hat{u}_{tt}(t, \xi) + |\xi|^{2} \hat{u}_{t}(t, \xi) + (1 + |\xi|^{2\theta}) \hat{u}_{1}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbb{R}^{n}_{\xi}, \quad (7)$$

$$\hat{u}(0, \xi) = \hat{u}_{0}(\xi), \quad \hat{u}_{t}(0, \xi) = \hat{u}_{1}(\xi), \quad \xi \in \mathbb{R}^{n}_{\xi}. \quad (8)$$

Multiply both sides of (7) by $\overline{\hat{u}_{t}}$, and further $\beta \rho(\xi) \overline{\hat{u}_{1}}$. Then, by taking the real part of the resulting identities one has

$$\frac{d}{dt} E_{0}(t, \xi) + (1 + |\xi|^{2\theta}) |\hat{u}_{1}|^{2} = 0, \quad (9)$$

$$\frac{d}{dt} \left( \beta \rho(\xi) \Re(\hat{u}_{1} \overline{\hat{u}_{t}}) + \frac{1}{2} \beta \rho(\xi)(1 + |\xi|^{2\theta}) |\hat{u}_{t}|^{2} \right) + \beta \rho(\xi) |\xi|^{2} |\hat{u}_{t}|^{2} = \beta \rho(\xi) |\hat{u}_{1}|^{2}. \quad (10)$$

By adding (9) and (10), one has

$$\frac{d}{dt} E(t, \xi) + F(t, \xi) = R(t, \xi). \quad (11)$$
Now we prove the following lemma.

**Lemma 3.1.** For \( \beta > 0 \), it is true that
\[
R(t, \xi) \leq 2\beta F(t, \xi), \quad \xi \in \mathbb{R}^n.
\]

**Proof.** In the case when \( |\xi| \leq 1 \), it follows from the definition of \( \rho(\xi) \) that
\[
R(t, \xi) = \beta \rho(\xi)|\hat{u}_t|^2 = \beta|\xi|^2|\hat{u}_t|^2 \leq \beta (1 + |\xi|^{2\theta})|\hat{u}_t|^2 \leq \beta F(t, \xi),
\]
and in the case of \( |\xi| \geq 1 \) one has
\[
R(t, \xi) = \beta \rho(\xi)|\hat{u}_t|^2 = \beta \frac{2|\xi|^{2\theta}}{1 + |\xi|^{4\theta - 2}}|\hat{u}_t|^2 \leq 2\beta|\xi|^{2\theta}|\hat{u}_t|^2 \leq 2\beta F(t, \xi),
\]
which imply the desired estimate. \( \square \)

It follows from (11) and Lemma 3.1 that
\[
\frac{d}{dt} E(t, \xi) + (1 - 2\beta) F(t, \xi) \leq 0,
\]
provided that the parameter \( \beta > 0 \) is small enough.

**Lemma 3.2.** There is a constant \( M_1 > 0 \) depending on \( \beta > 0 \) such that for all \( \xi \in \mathbb{R}^n \) with \( \xi \neq 0 \) it follows that
\[
(i) \quad \frac{\rho(\xi) + \beta \rho(\xi)^2 |\xi|^{-1}}{2(1 + |\xi|^{2\theta})} \leq M_1, \quad (ii) \quad \frac{1}{2\beta} + \frac{\rho(\xi)}{2|\xi|} + \frac{\rho(\xi)(1 + |\xi|^{2\theta})}{2|\xi|^2} \leq M_1.
\]

**Proof.** (i) For \( |\xi| \leq 1 \), one has
\[
\frac{\rho(\xi)}{2(1 + |\xi|^{2\theta})} = \frac{|\xi|^2}{2(1 + |\xi|^{2\theta})} \leq \frac{|\xi|^2}{2} \leq \frac{1}{2},
\]
\[
\frac{\beta \rho(\xi)^2 |\xi|^{-1}}{2(1 + |\xi|^{2\theta})} = \frac{\beta}{2} \cdot \frac{|\xi|^3}{1 + |\xi|^{2\theta}} \leq \frac{\beta |\xi|^3}{2} \leq \frac{\beta}{2},
\]
and for \( |\xi| \geq 1 \), one can get
\[
\frac{\rho(\xi)}{2(1 + |\xi|^{2\theta})} = \frac{1}{2} \cdot \frac{2|\xi|^{2\theta}}{1 + |\xi|^{2\theta}} \cdot \frac{1}{1 + |\xi|^{2\theta}} \leq \frac{1}{2} \cdot \frac{2}{|\xi|^{2\theta - 2}} \leq 1,
\]
\[
\frac{\beta \rho(\xi)^2 |\xi|^{-1}}{2(1 + |\xi|^{2\theta})} = \frac{\beta}{2} \cdot \frac{2|\xi|^{2\theta}}{1 + |\xi|^{2\theta}} \cdot \frac{1}{1 + |\xi|^{2\theta}} \cdot \frac{2|\xi|^{2\theta - 1}}{1 + |\xi|^{2\theta}} \leq \frac{\beta}{2} \cdot \frac{2}{|\xi|^{2\theta - 2}} \cdot \frac{1}{|\xi|^{2\theta - 1}} \leq \beta,
\]
so the statement of (i) is true.

(ii) For \( |\xi| \leq 1 \) with \( \xi \neq 0 \), one has
\[
\frac{\rho(\xi)}{2|\xi|} = \frac{|\xi|}{2} \leq \frac{1}{2},
\]
\[
\frac{\rho(\xi)(1 + |\xi|^{2\theta})}{2|\xi|^2} = \frac{1 + |\xi|^{2\theta}}{2} \leq 1,
\]
and for \( |\xi| \geq 1 \) one can get
\[
\frac{\rho(\xi)}{2|\xi|} = \frac{2|\xi|^{2\theta - 1}}{2(1 + |\xi|^{4\theta - 2})} = \frac{1}{|\xi|^{2\theta - 1} + \frac{1}{|\xi|^{4\theta - 2}}} \leq \frac{1}{2},
\]
\[
\frac{\rho(\xi)(1 + |\xi|^{2\theta})}{2|\xi|^2} = \frac{1 + |\xi|^{2\theta}}{2} \leq 1,
\]
\[ \rho(\xi)(1 + |\xi|^{2\theta}) \leq \frac{|\xi|^{2\theta} - 2(1 + |\xi|^{2\theta})}{1 + |\xi|^{4\theta - 2}} \leq \frac{1}{|\xi|^{2\theta}} + 1 \leq 2, \]

which implies the desired estimate of (ii).

\[ \square \]

**Lemma 3.3.** There is a constant \( M_2 > 0 \) such that for all \( \xi \in \mathbb{R}^n \), it follows that
\[ \rho(\xi)E(t, \xi) \leq M_2 F(t, \xi). \]

**Proof.** Since one has
\[ \mathcal{R}(\dot{u}_t \overline{u}) \leq \frac{1}{2} \left( |\dot{u}_t|^2 \right) \quad \xi \neq 0, \]

it follows that
\[ \rho(\xi)E(t, \xi) \leq \frac{\rho(\xi)}{2} |\dot{u}_t|^2 + \frac{\rho(\xi)}{2} |\xi|^2 |\dot{u}_t|^2 + \frac{\beta \rho(\xi)^2}{2} (1 + |\xi|^{2\theta}) |\dot{u}_t|^2 + \frac{\beta \rho(\xi)^2}{2} |\dot{u}_t|^2 \]
\[ + \frac{\beta \rho(\xi)^2}{2} |\xi| |\dot{u}_t|^2 \]
\[ = \left( \frac{\rho(\xi) + \beta \rho(\xi)^2 |\xi|^{-1}}{2(1 + |\xi|^{2\theta})} \right) (1 + |\xi|^{2\theta}) |\dot{u}_t|^2 + \frac{\rho(\xi)}{2|\xi|} \rho(\xi) (1 + |\xi|^{2\theta}) |\dot{u}_t|^2, \]

so that from Lemma 3.2 one has
\[ \rho(\xi)E(t, \xi) \leq M_1 (1 + |\xi|^{2\theta}) |\dot{u}_t|^2 + M_1 \beta \rho(\xi) |\xi|^2 |\dot{u}_t|^2, \]

which implies the desired estimate with \( M_2 := M_1 \) just defined in Lemma 3.2. According to the definitions of \( E(t, \xi) \) and \( F(t, \xi) \), the above inequality also holds true for \( \xi = 0 \). \( \square \)

Lemma 3.3 and (14) imply
\[ \frac{d}{dt} E(t, \xi) + M_2^{-1}(1 - \beta) \rho(\xi) E(t, \xi) \leq 0 \]
for any \( \xi \in \mathbb{R}^n \). From (18) we find
\[ E(t, \xi) \leq e^{-\alpha \rho(\xi)t} E(0, \xi), \quad \xi \in \mathbb{R}^n, \]

where \( \alpha := (1 - \beta)M_2^{-1} > 0 \) with small \( \beta \in (0, 1) \).

On the other hand, in the case of \( \xi \neq 0 \), since we have
\[ \pm \beta \rho(\xi) \mathcal{R}(\dot{u}_t \overline{u}) \leq \frac{\beta}{2} \rho(\xi) |\xi|^2 |\dot{u}_t|^2 + \frac{\beta \rho(\xi)}{2} |\dot{u}_t|^2, \]

it follows from the definition of \( E(t, \xi) \) and (20) with minus sign that
\[ E(t, \xi) \geq \left( 1 - \frac{\beta \rho(\xi)}{|\xi|^2} \right) \frac{1}{2} |\dot{u}_t|^2 + (1 - \beta \rho(\xi)) \cdot \frac{1}{2} |\xi|^2 |\dot{u}_t|^2. \]

And also we see that for \( |\xi| \leq 1 \) with \( \xi \neq 0 \),
\[ 1 - \frac{\beta \rho(\xi)}{|\xi|^2} = 1 - \left( \frac{\beta}{|\xi|^2} \cdot |\xi|^2 \right) = 1 - \beta > 0, \]
and for \( |\xi| \geq 1 \), one has
\[ 1 - \frac{\beta \rho(\xi)}{|\xi|^2} = 1 - \beta \cdot \frac{|\xi|^{2\theta - 2}}{1 + |\xi|^{4\theta - 2}} \geq 1 - \frac{2\beta}{|\xi|^{2\theta}} \geq 1 - 2\beta > 0. \]

While, for \( |\xi| \leq 1 \), it follows that
\[ 1 - \beta \rho(\xi) = 1 - \beta |\xi|^2 \geq 1 - \beta > 0, \]

and for \( |\xi| \geq 1 \), one has
\[ 1 - \beta \rho(\xi) = 1 - \beta |\xi|^2 \geq 1 - \beta > 0. \]
Lemma 3.4. Let \( \theta \in (1, \infty) \). Then, there is a constant \( C = C(\beta) > 0 \) and \( \alpha = \alpha(\beta) > 0 \) such that for all \( \xi \in \mathbb{R}_\xi^\beta \) it is true that

\[
E_0(t, \xi) \leq Ce^{-\alpha(\xi)t}E_0(0, \xi).
\]

Proof of Proposition 1.1. By Lemma 3.4 and the Plancherel theorem one has

\[
\int_{\mathbb{R}_x^\beta} (|u(x)|^2 + |\nabla u(x)|^2)dx \leq C \int_{\mathbb{R}_\xi^\beta} (|\hat{u}_t(\xi)|^2 + |\xi|^2 |\hat{u}(\xi)|^2) d\xi \leq C \int_{\mathbb{R}_\xi^\beta} E_0(t, \xi) d\xi
\]

\[
\leq C \left( \int_{|\xi| \leq 1} + \int_{|\xi| \geq 1} \right) e^{-\alpha(\xi)t} (|\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2) d\xi =: C(I_{\text{low}} + I_{\text{high}}).
\]

We first prepare the following standard formula:

\[
\int_{|\xi| \leq 1} e^{-\alpha|\xi|^2t} |\xi|^k d\xi \leq C(1 + t)^{-\frac{k+\alpha}{2}} (t \geq 0)
\]

(33)

for each \( k \in \mathbb{N} \cup \{0\} \).

Now, let us start with estimating both \( I_{\text{low}} \) and \( I_{\text{high}} \) based on the form of \( \rho(\xi) \).

In fact,

\[
I_{\text{low}} = \int_{|\xi| \leq 1} e^{-\alpha|\xi|^2t} |\hat{u}_1(\xi)|^2 d\xi + \int_{|\xi| \leq 1} e^{-\alpha|\xi|^2t} |\xi|^2 |\hat{u}_0(\xi)|^2 d\xi
\]

\[
\leq C\|u_1\|_1^2 (1 + t)^{-\frac{\alpha}{2}} + C\|u_0\|_1^2 (1 + t)^{-\frac{\alpha+2}{2}}.
\]

(34)
Next, we shall estimate the high frequency part. This part is crucial in the case of \( \theta > 1 \). First, one has

\[
|\xi| \geq 1 \Rightarrow \frac{|\xi|^{2\theta}}{1 + |\xi|^{4\theta-2}} \geq \frac{1}{2|\xi|^{2\theta-2}}.
\]

Hence, it follows that

\[
I_{\text{high}} = \int_{|\xi| \geq 1} e^{-\frac{2|\xi|^{2\theta-2}}{1+|\xi|^{4\theta-2}}} (|\dot{u}_1|^2 + |\xi|^2 |\dot{u}_0|^2) d\xi \leq \int_{|\xi| \geq 1} e^{-\frac{|\xi|^{2\theta-2}}{2}} (|\dot{u}_1|^2 + |\xi|^2 |\dot{u}_0|^2) d\xi
\]

\[
\leq \left( \sup_{|\xi| \geq 1} \frac{e^{-\alpha / |\xi|^{2\theta-2}}}{|\xi|^{2\theta}} \right) \int_{|\xi| \geq 1} (|\dot{u}_1|^2 + |\xi|^{2\ell+2} |\dot{u}_0|^2) d\xi
\]

\[
\leq C(1 + t)^{-\frac{\alpha}{\sigma}} (\|D^\ell u_1\|^2 + \|D^{\ell+1} u_0\|^2),
\]

(35)

where we have just used the following fact to get the desired decay rate:

\[
\sup_{|\xi| \geq 1} \left( \frac{e^{-\alpha / |\xi|^{2\theta-2}}}{|\xi|^{2\theta}} \right) = \sup_{|\xi| \geq 1} \left( \frac{e^{-\alpha (1+t)/|\xi|^{2\theta-2}}}{|\xi|^{2\theta}} \right) e^\alpha = e^\alpha \sup_{r \geq 1} \left( \frac{e^{-\alpha (1+t)/r^{2\theta-2}}}{r^{2\theta}} \right)
\]

\[
eq e^\alpha e^{-\frac{\alpha}{\sigma} (1 + t)^{-\frac{\alpha}{\sigma}}} \sup_{\sigma \geq 0} \left( \frac{\sigma^{2\ell/(\theta-1)} e^{-\alpha \sigma} }{e^{\alpha \sigma^2}} \right)
\]

\[
\leq e^\alpha e^{-\frac{\alpha}{\sigma} (1 + t)^{-\frac{\alpha}{\sigma}}} \sup_{\sigma \geq 0} \left( \frac{\sigma^{2\ell/(\theta-1)} }{e^{\alpha \sigma^2}} \right)
\]

\[
\leq C(1 + t)^{-\frac{\alpha}{\sigma}},
\]

(36)

where \( C = C(\ell, \theta, \alpha) > 0 \) is a constant, and we have just changed the variable from \( r \) to \( \sigma = \frac{\sqrt{\alpha (1+t)}}{r^{\theta-1}} \) (\( r \geq 1 \)). This completes the proof of Proposition 1.

\[\square\]

**Proof of Proposition 2.** To begin with, from Lemma 3.4, if \( \xi \neq 0 \), then it is true that

\[
|\dot{u}(t, \xi)|^2 \leq C e^{-\alpha \rho(\xi) t} \left( \frac{|\dot{u}_1(\xi)|^2}{|\xi|^2} + |\dot{u}_0(\xi)|^2 \right).
\]

(37)

Furthermore, if \( |\xi| \leq 1 \), then we see that

\[
\rho(\xi) = |\xi|^2.
\]

By integrating (37) over \( \{\delta \leq |\xi| \leq 1\} \) with small \( \delta > 0 \), one gets

\[
\int_{\delta \leq |\xi| \leq 1} |\dot{u}(t, \xi)|^2 \leq C \int_{\delta \leq |\xi| \leq 1} e^{-\alpha t |\xi|^2} \frac{|\dot{u}_1(\xi)|^2}{|\xi|^2} d\xi + C \int_{\delta \leq |\xi| \leq 1} e^{-\alpha t |\xi|^2} |\dot{u}_0(\xi)|^2 d\xi
\]

\[
\leq \|u_1\|^2 \int_{\delta \leq |\xi| \leq 1} e^{-\alpha t |\xi|^2} |\xi|^{-2} d\xi + C \|u_0\|^2 \int_{\delta \leq |\xi| \leq 1} e^{-\alpha t |\xi|^2} d\xi
\]

\[
\leq C(1 + t)^{-\frac{\alpha}{2} \frac{\alpha}{2}} \|u_1\|^2 + C(1 + t)^{-\frac{\alpha}{2}} \|u_0\|^2,
\]

(38)

where \( C > 0 \) is a constant independent of any \( \delta > 0 \). Here, we just have used the following general formula in the case when \( n - 1 \geq k \):

\[
\int_{\delta \leq |\xi| \leq 1} e^{-\alpha t |\xi|^2} |\xi|^{-k} d\xi = \int_{\delta \leq |\xi| \leq 1} e^{-\alpha (1+t) |\xi|^2} |\xi|^{-k} e^{\alpha |\xi|^2} d\xi
\]
for $t \geq 0$, where $C > 0$ does not depend on any $\delta > 0$ and $t > 0$. By letting $\delta \downarrow 0$ in (39) one has the low frequency estimate:

$$\int_{|\xi| \leq 1} |\hat{u}(t, \xi)|^2 d\xi \leq C(1 + t)^{-\frac{2\theta}{1 + \theta}} \|u_1\|_1^2 + C(1 + t)^{-\frac{2}{1 + \theta}} \|u_0\|_1^2 \quad (n \geq 3).$$

(40)

On the other hand, if $|\xi| \geq 1$, since we have once more

$$\frac{|\xi|^{2\theta}}{1 + |\xi|^{1 + \theta}} \geq \frac{1}{2|\xi|^{1 + \theta}},$$

it follows from (37) that

$$\int_{|\xi| \geq 1} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi| \geq 1} e^{-\alpha t/|\xi|^{1 + \theta}} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^2} d\xi + C \int_{|\xi| \geq 1} e^{-\alpha t/|\xi|^{1 + \theta}} |\hat{u}_0(\xi)|^2 d\xi$$

$$\leq C \left( \sup_{|\xi| \geq 1} e^{-\alpha t/|\xi|^{1 + \theta}} \right) \int_{|\xi| \geq 1} (|\xi|^{2H-2} |\hat{u}_1(\xi)|^2 + |\xi|^{1+\theta} |\hat{u}_0(\xi)|^2) d\xi$$

$$\leq C(1 + t)^{-\frac{\alpha t}{1 + \theta}} \left( \|D^{1+\theta} u_1\|^2 + \|D^\theta u_0\|^2 \right),$$

(41)

where we have just used (36) again. Inequalities (40) and (41) imply the desired estimate. \(\square\)

4. Proof of Theorems. In this section, we prove Theorems 1.1, 1.2 and 1.3 by employing the method due to [11].

At the first stage we will find a leading term of the asymptotic expansion of solutions in the low frequency region $|\xi| \ll 1$, which is essential task to prove Theorem 1.1.

**Lemma 4.1.** Let $n \geq 1$ and $\theta > 1$. Then there exist constants $C > 0$ and $\delta_0 > 0$ such that for $t \gg 1$, it holds that

$$\int_{|\xi| \leq \delta_0} \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi$$

$$\leq C(1 + t)^{-\frac{2\theta}{1 + \theta}} \left( \|u_0\|_{L^1}^2 + \|u_1\|_{L^2}^2 \right) + C(1 + t)^{-\frac{2\theta}{1 + \theta}} \left( \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 \right).$$

(42)

In order to prove Lemma 4.1, we apply the Fourier transform with respect to the space variable $x$ of both sides of (1)-(2). Then in the Fourier space $\mathbb{R}_\xi^n$, one has

$$\hat{u}_{tt}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) + \hat{u}_t(t, \xi) + |\xi|^{2\theta} \hat{u}_t(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbb{R}_\xi^n,$$

$$\hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}_\xi^n.$$

(43)

Let us solve ODE (43) with the parameter $\xi$ directly under the condition $0 < |\xi| \leq \delta_0 \ll 1$. By solving (43), we get

$$\hat{u}(t, \xi) = \frac{\hat{u}_1(\xi) - \sigma_2 \hat{u}_0(\xi)}{\sigma_1 - \sigma_2} e^{\sigma_1 t} + \frac{\sigma_1 \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\sigma_1 - \sigma_2} e^{\sigma_2 t}$$

$$= \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \hat{u}_1(\xi) + \frac{\sigma_1 e^{\sigma_2 t} - \sigma_2 e^{\sigma_1 t}}{\sigma_1 - \sigma_2} \hat{u}_0(\xi),$$

(44)
where \( \sigma_j \in \mathbb{R} \) \((j = 1, 2)\) are expressed as

\[
\begin{align*}
\sigma_1 &= \frac{-(1 + |\xi|^{2\theta}) + \sqrt{(1 + |\xi|^{2\theta})^2 - 4|\xi|^2}}{2}, \\
\sigma_2 &= \frac{-(1 + |\xi|^{2\theta}) - \sqrt{(1 + |\xi|^{2\theta})^2 - 4|\xi|^2}}{2}.
\end{align*}
\] (45)

In this case, the smallness \( \delta_0 > 0 \) of \( |\xi| \) is assumed to guarantee \( (1+|\xi|^{2\theta})^2 - 4|\xi|^2 > 0 \), which implies \( \sigma_j \in (-\infty, 0) \) \((j = 1, 2)\). Here, we notice that

\[
\sigma_j = -\sigma_j^2 - |\xi|^{2\theta} \sigma_j - |\xi|^2 \quad (j = 1, 2).
\] (46)

By rewriting (44) with the help of (46), one has

\[
\hat{u}(t, \xi) = e^{-t|\xi|^2}(K_1(t, \xi) + K_2(t, \xi)),
\] (47)

where

\[
K_1(t, \xi) = \frac{\sigma_2 \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\sigma_2 - \sigma_1} e^{-\sigma_2 t - |\xi|^{2\theta} \sigma_1 t}, \\
K_2(t, \xi) = \frac{-\sigma_1 \hat{u}_0(\xi) + \hat{u}_1(\xi)}{\sigma_2 - \sigma_1} e^{-\sigma_1 t - |\xi|^{2\theta} \sigma_2 t}.
\] (48)

Then, \( K_1(t, \xi) \) can be decomposed into the following style when one relies on an idea coming from Chill-Haraux [2]:

\[
K_1(t, \xi) = \hat{u}_0(\xi) + \hat{u}_1(\xi) + \frac{\sigma_1 \hat{u}_0(\xi)}{\sigma_2 - \sigma_1} \left(1 - e^{-\sigma_2 t - |\xi|^{2\theta} t}\right) + \frac{\hat{u}_1(\xi)}{\sigma_2 - \sigma_1} \left(e^{-\sigma_2 t - |\xi|^{2\theta} t} - (\sigma_1 - \sigma_2)\right).
\] (49)

So, one has arrived at the following important equality

\[
\hat{u}(t, \xi) = e^{-t|\xi|^2} \left\{ \hat{u}_0(\xi) + \hat{u}_1(\xi) + \frac{\sigma_1 \hat{u}_0(\xi)}{\sigma_2 - \sigma_1} \left(1 - e^{-\sigma_2 t - |\xi|^{2\theta} t}\right) + \frac{\hat{u}_1(\xi)}{\sigma_2 - \sigma_1} \left(e^{-\sigma_2 t - |\xi|^{2\theta} t} - (\sigma_1 - \sigma_2)\right) \right\}
+ e^{-t|\xi|^2} K_2(t, \xi).
\] (50)

We are in a position to use an idea of [11] in order to find a leading term of \( \hat{u}(t, \xi) \) as \( t \to \infty \). Since we have

\[
\hat{u}_j(\xi) = A_j(\xi) - iB_j(\xi) + P_j \quad (j = 0, 1),
\] (51)

where

\[
A_j(\xi) := \int_{\mathbb{R}^n} (\cos(x \cdot \xi) - 1) u_j(x) dx, \quad B_j(\xi) := \int_{\mathbb{R}^n} \sin(x \cdot \xi) u_j(x) dx, \quad (j = 0, 1),
\]

from (50) and (51) we see that

\[
\hat{u}(t, \xi) - (P_0 + P_1) e^{-t|\xi|^2} = (A_0(\xi) - iB_0(\xi) + A_1(\xi) - iB_1(\xi)) e^{-t|\xi|^2}
+ e^{-t|\xi|^2} \left\{ \frac{\sigma_1 \hat{u}_0(\xi)}{\sigma_2 - \sigma_1} + \frac{\sigma_2 \hat{u}_0(\xi)}{\sigma_1 - \sigma_2} \left(1 - e^{-\sigma_2 t - |\xi|^{2\theta} t}\right) + \frac{\hat{u}_1(\xi)}{\sigma_2 - \sigma_1} \left(e^{-\sigma_2 t - |\xi|^{2\theta} t} - (\sigma_1 - \sigma_2)\right) \right\}
+ e^{-t|\xi|^2} K_2(t, \xi)
\] (52)

for all \( \xi \) satisfying \( 0 < |\xi| \leq \delta_0 \).

In order to estimate four quantities \( A_j(\xi) \) and \( B_j(\xi) \) \((j = 0, 1)\), we prepare the following important lemma which has its origin in [11, Lemma 3.1].
Lemma 4.2. Let \( n \geq 1 \). Then it holds that for all \( \xi \in \mathbb{R}^n \),
\[
|A_j(\xi)| \leq L|\xi||u_j|_{1,1} \quad (j = 0, 1),
|B_j(\xi)| \leq M|\xi||u_j|_{1,1} \quad (j = 0, 1),
\]
where
\[
L := \sup_{\theta \neq 0} \frac{|1 - \cos \theta|}{|\theta|} < +\infty, \quad M := \sup_{\theta \neq 0} \frac{|\sin \theta|}{|\theta|} = 1.
\]

Let us derive some decay estimates of 5 terms of the right hand side (RHS for short) of (52). For this, let us prepare several inequalities which are used to derive decay estimates of the RHS of (52).

Set
\[
D := (1 + |\xi|^{2\theta})^2 - 4|\xi|^2 > 0
\]
for small \( |\xi| \ll 1 \). At first one has the inequality
\[
0 < -\sigma_1 = \frac{1}{2} \frac{4|\xi|^2}{(1 + |\xi|^{2\theta}) + \sqrt{D}} \leq 2|\xi|^2 \tag{53}
\]
and, since
\[
\sigma_1 - \sigma_2 = \sqrt{D} \to 1 \quad (|\xi| \downarrow 0),
\]
one can assume that for small \( |\xi| \ll 1 \),
\[
\sigma_1 - \sigma_2 > \frac{1}{2} \tag{54}
\]
Therefore, because of (53) and (54), the second term of the RHS of (52) can be estimated as follows in terms of the \( L^2 \)-norm:
\[
J_0(t) := \int_{|\xi| < 1} e^{-2t|\xi|^2} \left| \frac{\sigma_1 b_0(\xi)}{\sigma_2 - \sigma_1} \right|^2 d\xi
\leq C||\hat{u}_0||^2_\infty \int_{|\xi| < 1} e^{-2t|\xi|^2} |\xi|^4 d\xi \leq Ct^{-\frac{3}{2}}||u_0||^2_1. \tag{55}
\]
Furthermore, because of the mean value theorem we get
\[
|1 - e^{-(\sigma_1^2 + \sigma_1|\xi|^{2\theta})t}| \leq t |\sigma_1^2 + \sigma_1|\xi|^{2\theta}|, \tag{56}
\]
so that from (54) the third term of the RHS of (52) can be evaluated as follows:
\[
J_1(t) := \int_{|\xi| < 1} \frac{\sigma_2^2(1 - e^{-(\sigma_1^2 + \sigma_1|\xi|^{2\theta})t})^2}{|\sigma_1 - \sigma_2|^2} |\hat{u}_0|^2 e^{-2t|\xi|^2} d\xi
\leq Ct^2||u_0||^2_1 \int_{|\xi| < 1} \sigma_2^2 |\sigma_1^2 + \sigma_1|\xi|^{2\theta}|^2 e^{-2t|\xi|^2} d\xi. \tag{57}
\]
Here, we notice that \( \sigma_1^2 \leq 16|\xi|^8 \) from (53), and
\[
0 \leq -\sigma_2 = \frac{(1 + |\xi|^{2\theta}) + \sqrt{(1 + |\xi|^{2\theta})^2 - 4|\xi|^2}}{2} \leq \frac{2(1 + |\xi|^{2\theta})}{2} = 1 + |\xi|^{2\theta} \leq 2
\]
for small \( |\xi| \ll 1 \), so that from (57), it follows that
\[
\int_{|\xi| < 1} \sigma_2^2(\sigma_1^4 + \sigma_1|\xi|^{4\theta}) e^{-2t|\xi|^2} d\xi \leq C \int_{|\xi| < 1} (|\xi|^8 + |\xi|^{4\theta+4}) e^{-2t|\xi|^2} d\xi
\leq C \left( (1 + t)^{-\frac{n+4\theta}{2}} + (1 + t)^{-\frac{n+4\theta+4}{2}} \right).
Furthermore, due to (45) and (46), one can see that

\[ \text{This implies} \]

\[ J_1(t) \leq C \left( (1 + t)^{-\frac{n+1}{2}} + t^{-\frac{n+4\theta}{2}} \right) \| u_0 \|^2_{L^2}. \]  

(58)

On the other hand, since

\[ \left| e^{-\left(\sigma_1^2 + \sigma_1 |\xi|^2\right) t} - (\sigma_1 - \sigma_2) \right| \]

\[ = \left| \left( e^{-\sigma_1^2 |\xi|^2 t - 1} + 1 - (\sigma_1 - \sigma_2) \right) \right| \]

\[ \leq e^{-\sigma_1^2 |\xi|^2 t - 1} + \left| 1 - \sqrt{D} \right| \]

\[ = \left| e^{-\sigma_1^2 |\xi|^2 t - 1} + 1 \right| + \left| 1 - \sqrt{D} \right| \]

\[ \leq \left| e^{-\sigma_1^2 |\xi|^2 t - 1} + 1 \right| + 4|\xi|^2. \]  

(59)

because of (53), (56) and the assumption \( \theta > 1 \), we see that for small \( |\xi| \ll 1 \)

\[ \left| e^{-\left(\sigma_1^2 + \sigma_1 |\xi|^2\right) t} - (\sigma_1 - \sigma_2) \right| \leq t \left| \sigma_1^2 + \sigma_1 |\xi|^2 \right| + 4|\xi|^2 \leq 6t|\xi|^4 + 4|\xi|^2. \]  

(60)

Thus, from (54) and (60), the 4th term of the RHS of (52) can be estimated as follows:

\[ J_2(t) := \int_{|\xi| \ll 1} e^{-2t|\xi|^2} |\hat{u}_1(\xi)|^2 \left| e^{-\left(\sigma_1^2 + \sigma_1 |\xi|^2\right) t} - (\sigma_1 - \sigma_2) \right|^2 d\xi \]

\[ \leq C\| u_1 \|^2 \int_{|\xi| \ll 1} e^{-2t|\xi|^2} (t^2|\xi|^8 + t|\xi|^6 + |\xi|^4) \leq C\| u_1 \|^2 (1 + t)^{-\frac{n+4}{2}}. \]  

(61)

Let us estimate the first term of the RHS of (52) in terms of \( L^2 \)-norm. In fact, we first get

\[ J_3(t) := \int_{|\xi| \ll 1} |A_0(\xi) - iB_0(\xi) + A_1(\xi) - iB_1(\xi)|^2 e^{-2t|\xi|^2} d\xi \]

\[ \leq C \int_{|\xi| \ll 1} \left( |A_0(\xi)| + |B_0(\xi)| + |A_1(\xi)| + |B_1(\xi)| \right)^2 e^{-2t|\xi|^2} d\xi. \]  

(62)

By using Lemma 4.2, one can deduce the following decay estimates:

\[ J_3(t) \leq C(\| u_0 \|^2_{L^2,1} + \| u_1 \|^2_{L^2,1}) \int_{|\xi| \ll 1} (L + M)^2 |\xi|^2 e^{-2t|\xi|^2} d\xi \]

\[ \leq C(L + M)^2 (\| u_0 \|^2_{L^2,1} + \| u_1 \|^2_{L^2,1})(1 + t)^{-\frac{n+2}{2}}. \]

(63)

Let us estimate the final term of the RHS of (52). Indeed, one has

\[ J_4(t) := \int_{|\xi| \ll 1} e^{-2t|\xi|^2} |K_2(t, \xi)|^2 d\xi \]

\[ = \int_{|\xi| \ll 1} e^{-2t|\xi|^2} \left| \frac{\hat{u}_0(\xi)\sigma_1 - \hat{u}_1(\xi)}{\sigma_1 - \sigma_2} \right|^2 e^{-2(\sigma_1^2 + \sigma_2 |\xi|^2) t} d\xi. \]  

(64)

Because of (53) and (54), one can get

\[ J_4(t) \leq C \int_{|\xi| \ll 1} e^{-2t|\xi|^2} \left( |\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2 \right) e^{-2(\sigma_1^2 + \sigma_2 |\xi|^2) t} d\xi. \]

Furthermore, due to (45) and (46), one can see that

\[ \sigma_1^2 + \sigma_2 |\xi|^2 \rightarrow 1 \quad (|\xi| \rightarrow +0), \]
Because of the Plancherel theorem one has
\[ e^{-2(\sigma_2^2 + \sigma_2|\xi|^{2\theta})t} \leq e^{-t}. \] (65)

Because of the Plancherel theorem one has
\[ J_4(t) \leq C e^{-t} \int_{|\xi| \leq 1} e^{-2|\xi|^2 (|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2)} d\xi \]
\[ \leq C e^{-t} \int_{|\xi| \leq 1} (|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) d\xi \leq C e^{-t} (\|u_0\|^2 + \|u_1\|^2). \] (66)

By summarizing decay estimates obtained in (55), (58), (61), (63) and (66) combined with (52) one has arrived at the desired estimate for Lemma 4.1:
\[ \int_{|\xi| \leq \delta_0} \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi \leq C(\|u_0\|_{L^1} + \|u_1\|_{L^1})(1 + t)^{-\frac{3}{2}} + C(\|u_0\|_2^2 + \|u_1\|_2^2)(1 + t)^{-\frac{3}{2}} + C\|u_0\|_2^2(1 + t)^{-\frac{3}{2}} + C\|u_0\|^2 + \|u_1\|^2)e^{-t} \] (67)
with some constants \( \delta_0 > 0 \) and \( C > 0 \).

Now, let us prove Theorem 1.1 by relying on Lemmas 3.4 and 4.1. In this connection, we use Lemma 3.4 restricted only to the high frequency region \( |\xi| \geq \gamma \). In the course of proof we encounter the so-called regularity loss type estimates.

**Proof of Theorem 1.1.** We first divide the integrand into two parts:
\[ \int_{\mathbb{R}^n} \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi = \left( \int_{|\xi| \leq \delta_0} + \int_{|\xi| \geq \delta_0} \right) \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi =: I_l(t) + I_h(t). \] (68)

Lemma 4.1 directly implies that
\[ I_l(t) \leq C(\|u_0\|_{L^1} + \|u_1\|_{L^1})(1 + t)^{-\frac{3}{2}} + C(\|u_0\|^2 + \|u_1\|^2)e^{-t}. \] (69)

On the other hand, it follows from Lemma 3.4 that
\[ |\hat{u}_1(t, \xi)|^2 + |\xi|^2 |\hat{u}(t, \xi)|^2 \leq C e^{-\alpha|\xi|^2 t} (|\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2), \quad \xi \in \mathbb{R}^n. \] (70)

In this case, we see that if \( \delta_0 \leq |\xi| \leq 1 \), then
\[ \rho(\xi) = |\xi|^2 \geq \delta_0^2, \]
and if \( |\xi| \geq 1 \), then
\[ \frac{|\xi|^{2\theta}}{1 + |\xi|^{4\theta-2}} \geq \frac{1}{2|\xi|^{2\theta-2}}. \]

So, by (70) and a similar argument to (35), one can estimate as follows:
\[ \int_{|\xi| \geq \delta_0} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi| \geq \delta_0} e^{-\alpha\rho(\xi)t} \left( \frac{|\hat{u}_1(\xi)|^2}{|\xi|^2} + |\hat{u}_0(\xi)|^2 \right) d\xi \]
\[ = \int_{|\xi| \geq \delta_0} e^{-\alpha|\xi|^2 t} \left( \frac{|\hat{u}_1(\xi)|^2}{|\xi|^2} + |\hat{u}_0(\xi)|^2 \right) d\xi \]
with some constant \(C > 0\) depending on \(\delta_0\). On the other hand,

\[
\int_{|\xi| \geq \delta_0} |P_0 + P_1|^2 e^{-t|\xi|^2} d\xi \leq C |P_0 + P_1|^2 \int_{|\xi| \geq \delta_0} e^{-2t|\xi|^2} d\xi 
\]

\[
\leq C |P_0 + P_1|^2 e^{-t\delta_0^2} \int_{|\xi| \geq \delta_0} e^{-t|\xi|^2} d\xi 
\]

\[
\leq C |P_0 + P_1|^2 e^{-t\delta_0^2} \lesssim t^{-\frac{n}{2}} \leq C |P_0 + P_1|^2 e^{-\alpha t} \quad (t \gg 1),
\]

(72)

with some \(C > 0\) and \(\alpha > 0\). Therefore, by evaluating \(I_h(t)\) based on (70) and (71), and combining it with (69) one obtains the desired estimates for \(\ell \geq 0\)

\[
\int_{\mathbb{R}^n} \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-2t|\xi|^2} \right|^2 d\xi 
\]

\[
\leq C(1 + t)^{-\frac{n+2}{4}} \left( \|u_0\|_{1,1}^2 + \|u_1\|_{1,1}^2 \right) + C e^{-\alpha t} \left( \|P_0 + P_1|^2 + \|u_1\|^2 + \|u_0\|^2 \right) 
\]

\[
+ C(1 + t)^{-\frac{n+4}{4}} \left( \|D^\ell u_0\|^2 + \|D^{\ell-1} u_1\|^2 \right) \quad (t \gg 1)
\]

with some generous constant \(\alpha > 0\). Additional regularity assumptions on the initial data assumed in Theorem 1.1 guarantees that the leading term of the solution to problem (1)-(2) is the multiple of the Gauss kernel \((P_0 + P_1)G(t, x)\) in the \(L^2\)-sense.

At the second stage we prove Theorem 1.2. For this purpose, similarly to the proof of Lemma 4.1, one can prove the following lemma based on (52). We shall omit its proof.

**Lemma 4.3.** Let \(n \geq 1\), and \(\theta > 1\). Then, there exist constants \(C > 0\) and small \(\delta_0 > 0\) such that for \(t \gg 1\), it holds that

\[
\int_{|\xi| \leq \delta_0} |\xi|^2 \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi 
\]

\[
\leq C(1 + t)^{-\frac{n+2}{4}} \left( \|u_0\|_{1,1}^2 + \|u_1\|_{1,1}^2 \right) (1 + t)^{-\frac{n+4}{4}} + C e^{-t} \left( \|u_0\|_{1,1}^2 + \|u_1\|_{1,1}^2 \right) \quad (t \gg 1)
\]

Furthermore, the next lemma can be derived easily by the same computation as those for \(I_h(t)\) in (68).

**Lemma 4.4.** Let \(n \geq 1\), and \(\theta > 1\), and \(\ell \geq 0\). Then it is true that

\[
\int_{|\xi| \geq \delta_0} |\xi|^2 \left| \hat{u}(t, \xi) - (P_0 + P_1)e^{-t|\xi|^2} \right|^2 d\xi 
\]

\[
\leq C(1 + t)^{-\frac{n+2}{4}} \left( \|u_0\|^2 + \|u_1\|^2 \right) e^{-\eta t} + C \left( \|D_\ell u_0\|^2 + \|D_\ell u_1\|^2 \right) (1 + t)^{-\frac{n+4}{4}} \quad (t > 0)
\]

with some constant \(\eta > 0\).
Once one obtains Lemmas 4.3 and 4.4, Theorem 1.2 can be proved similarly to that of Theorem 1.1. This is an easy exercise for the reader.

At the third stage we shall prove the following lemma concerning the behavior of the time derivative of the solution in asymptotic sense to problem (1)-(2). The first one is about the decay property in the low frequency region. This is also needed to prove Theorem 1.3.

**Lemma 4.5.** Let $n \geq 1$, and $\theta > 1$. Then it is true that

\[
\left. \int_{|\xi| \leq \delta_0} \left| \hat{u}_t(t, \xi) + (P_0 + P_1)|\xi|^2 e^{-t|\xi|^2} \right|^2 d\xi \leq C(\|u_0\|_1^2 + \|u_1\|_1^2)(1 + t)^{-\frac{n+6}{2}} \right.
\]

\[
+ \left( \|u_0\|_{1,1}^2 + \|u_1\|_{1,1}^2 \right)(1 + t)^{-\frac{n+6}{2}} + Ce^{-t} \left( \|u_0\|^2 + \|u_1\|^2 \right).
\]

**Proof.** By differentiating both sides of (52) with respect to the time variable $t$ one has

\[
\hat{u}_t(t, \xi) + (P_0 + P_1)|\xi|^2 e^{-t|\xi|^2} = -|\xi|^2 e^{-t|\xi|^2} \left( A_0 - iB_0 + A_1 - iB_1 \right)
\]

\[
- |\xi|^2 e^{-t|\xi|^2} \left\{ \frac{\sigma_1 \hat{u}_0(\xi)}{\sigma_2 - \sigma_1} + \frac{\sigma_2 \hat{u}_0(\xi)(1 - e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t})}{\sigma_1 - \sigma_2} \right\}
\]

\[
+ \frac{\hat{u}_1(\xi) \left( e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t} - \sigma_1 \right)}{\sigma_1 - \sigma_2}
\]

\[
+ e^{-t|\xi|^2} \left \{ \frac{\sigma_2 \hat{u}_0(\xi)}{\sigma_1 - \sigma_2} \left( \sigma_1^2 + \sigma_1 |\xi|^{2\theta} \right) e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t} - \frac{\hat{u}_1(\xi)e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t}(\sigma_1^2 + \sigma_1 |\xi|^{2\theta})}{\sigma_1 - \sigma_2} \right \}
\]

\[
- e^{-t|\xi|^2} \left( \sigma_2^2 + \sigma_2 |\xi|^{2\theta} \right) K_2(t, \xi).
\]

Now, let us estimate 8 quantities above in order to check that $L^2(|\xi| \leq \delta_0)$-norms of each terms in the RHS of (73) are remainder ones as $t \to \infty$. Indeed, one has a series of inequalities below to show the statement of Lemma 4.5.

First, because of (53) and (54), one has

\[
I_1 := \int_{|\xi| \leq 1} |\xi|^4 e^{-2t|\xi|^2} \frac{\sigma_1 \hat{u}_0(\xi)}{\sigma_2 - \sigma_1} d\xi \leq C\|u_0\|_1^2 \int_{|\xi| \leq 1} |\xi|^8 e^{-2t|\xi|^2} d\xi
\]

\[
\leq C\|u_0\|_1^2 (1 + t)^{-\frac{n+6}{2}}.
\]

Next, it follows from (53), (54), (56) and the fact $0 \leq -\sigma_2 \leq 2$ (see around (58)) that

\[
I_2 := \int_{|\xi| \leq 1} |\xi|^4 e^{-2t|\xi|^2} \left| \frac{\sigma_2 \hat{u}_0(\xi)(1 - e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t})}{\sigma_1 - \sigma_2} \right|^2 d\xi
\]

\[
\leq C\|u_0\|_1^2 \int_{|\xi| \leq 1} |\xi|^4 e^{-2t|\xi|^2} \left( 1 - e^{-\sigma_1^2(1 - \sigma_1|\xi|)^2 t} \right)^2 d\xi
\]

\[
\leq C\|u_0\|_1^2 \int_{|\xi| \leq 1} |\xi|^4 e^{-2t|\xi|^2} \left( \sigma_1^4 + \sigma_1^2 |\xi|^{4\theta} \right) d\xi
\]

\[
\leq C\|u_0\|_1^2 \int_{|\xi| \leq 1} |\xi|^4 e^{-2t|\xi|^2} \left( |\xi|^8 + |\xi|^{4\theta+4} \right) d\xi
\]

\[
\leq C\|u_0\|_1^2 \{ (1 + t)^{-\frac{n+12}{2}} + (1 + t)^{-\frac{n+6+\theta}{2}} \}.
\]

(74)
Because of (54) and (60) one has
\[ I_3 := \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left| \frac{\hat{u}_1(\xi)(e^{-\sigma_1^2 t - \sigma_1 |\xi|^2 t} - (\sigma_1 - \sigma_2))}{\sigma_1 - \sigma_2} \right|^2 d\xi \]
\[ \leq C \|u_1\|_1^2 \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left| t|\xi|^4 + |\xi|^2 \right|^2 d\xi \]
\[ \leq C \|u_1\|_1^2 \int_{|\xi| < 1} e^{-2|\xi|^2} \left( t^2|\xi|^4 + t|\xi|^{10} + |\xi|^8 \right) d\xi \]
\[ \leq C \|u_1\|_1^2 \left\{ t^2(1 + t)^{-\frac{n+12}{2}} + t(1 + t)^{-\frac{n+10}{2}} + (1 + t)^{-\frac{n+8}{2}} \right\}. \]
From (53), (54) and (65) one has
\[ I_4 := \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left| K_2(t, \xi) \right|^2 d\xi \]
\[ = \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left| \hat{u}_0(\xi) - \hat{u}(\xi) \right|^2 \left| \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_2} \right|^2 e^{-2(\sigma_1^2 t + |\xi|^2 \sigma_2 t)} d\xi \]
\[ \leq C e^{-t} \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left( |\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2 \right) d\xi \]
\[ \leq C e^{-t} \left( \|u_0\|^2 + \|u_1\|^2 \right). \quad (75) \]
In order to estimate \( I_5 \) below one uses Lemma 4.2 again to obtain
\[ I_5 := \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left| A_0(\xi) - iB_0(\xi) + A_1(\xi) - iB_1(\xi) \right|^2 d\xi \]
\[ \leq C \int_{|\xi| < 1} |\xi|^4 e^{-2|\xi|^2} \left( |A_0(\xi)| + |B_0(\xi)| + |A_1(\xi)| + |B_1(\xi)| \right) \left( |A_0(\xi)| + |B_0(\xi)| + |A_1(\xi)| + |B_1(\xi)| \right) d\xi \]
\[ \leq C (L + M)^2 (\|u_0\|^2_{1,1} + \|u_1\|^2_{1,1}) (1 + t)^{-\frac{n+4}{2}}. \quad (76) \]
Because of \( 0 \leq -\sigma_2 \leq 2, (53) \) and (54), one can estimate
\[ I_6 := \int_{|\xi| < 1} e^{-2|\xi|^2} \left| \hat{u}_0(\xi) \right|^2 \left| \frac{\sigma_2}{\sigma_1 - \sigma_2} \left( \sigma_1^2 + \sigma_1 |\xi|^2 \right) e^{-\sigma_1^2 t - \sigma_1 |\xi|^2 t} \right|^2 d\xi \]
\[ \leq C \|u_0\|_1^2 \int_{|\xi| < 1} e^{-2|\xi|^2} \left( \sigma_1^2 + \sigma_1 |\xi|^2 \right)^2 e^{-2(\sigma_1^2 + \sigma_1 |\xi|^2) t} d\xi \]
\[ \leq C \|u_0\|_1^2 \int_{|\xi| < 1} e^{-2|\xi|^2} \left( \sigma_1^4 + \sigma_1^2 |\xi|^4 \right) d\xi \]
\[ \leq C \|u_0\|_1^2 \int_{|\xi| < 1} e^{-2|\xi|^2} \left( |\xi|^8 + |\xi|^{10} + 4 \right) d\xi \]
\[ \leq C \|u_0\|_1^2 \left\{ (1 + t)^{-\frac{n+8}{2}} + (1 + t)^{-\frac{n+10}{4}} \right\}. \quad (77) \]
Similarly, from (53) and (54), one has
\[ I_7 := \int_{|\xi| < 1} e^{-2|\xi|^2} \left| \hat{u}_1(\xi) \right|^2 \left| \frac{\sigma_2}{\sigma_1 - \sigma_2} \left( \sigma_1^2 + \sigma_1 |\xi|^2 \right) e^{-\sigma_1^2 t - \sigma_1 |\xi|^2 t} \right|^2 d\xi \]
\[ \leq C \|u_1\|_1^2 \int_{|\xi| < 1} e^{-2|\xi|^2} \left( |\xi|^4 + |\xi|^{10} + 4 \right) d\xi \]
\[ \leq C \|u_1\|_1^2 \left\{ (1 + t)^{-\frac{n+6}{2}} + (1 + t)^{-\frac{n+10}{4}} \right\}. \quad (78) \]
Finally, it follows from (65) that
\[ I_8 := \int_{|\xi| < 1} e^{-2|\xi|^2} \left| \sigma_2^2 + \sigma_2 |\xi|^{2\theta} \right|^2 |K_2(t, \xi)|^2 d\xi \]
\[ = \int_{|\xi| < 1} e^{-2|\xi|^2} \left| \sigma_2^2 + \sigma_2 |\xi|^{2\theta} \right|^2 \left| -\sigma_1 \hat{u}_0(\xi) + \hat{u}_1(\xi) \right|^2 e^{-2(\sigma_2^2 + \sigma_2 |\xi|^{2\theta})t} d\xi \]
\[ \leq C e^{-t} \int_{|\xi| < 1} \left| -\sigma_1 \hat{u}_0(\xi) + \hat{u}_1(\xi) \right|^2 d\xi \leq C e^{-t} \int_{|\xi| < 1} (|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) d\xi \]
\[ \leq C e^{-t} (\|u_0\|^2 + \|u_1\|^2). \] (79)

In the computations of (77), (78) and (79) above we have just used the positivity of the quantities \(\sigma_2^2 + \sigma_2 |\xi|^{2\theta}\) \((j = 1, 2)\) in the case of \(|\xi| \ll 1\). Here, we just check the case of \(j = 1\). Indeed, from (46) it follows that
\[ \sigma_1^2 + \sigma_1 |\xi|^{2\theta} = -\sigma_1 - |\xi|^2 \]
\[ = \frac{1}{2} \left( 1 + |\xi|^{2\theta} - \sqrt{(1 + |\xi|^{2\theta})^2 - 4|\xi|^2} \right) - |\xi|^2 \]
\[ = \frac{1}{2} \left( 1 + |\xi|^{2\theta} - 2|\xi|^2 - \sqrt{(1 + |\xi|^{2\theta})^2 - 4|\xi|^2} \right) \]
\[ = \frac{1}{2} \left( 1 + |\xi|^{2\theta} - 2|\xi|^2 + \sqrt{1 + 2|\xi|^{2\theta} + |\xi|^{4\theta} - 4|\xi|^2} \right) \]
\[ = \frac{1}{2} \left( 1 + |\xi|^{2\theta} - 2|\xi|^2 + \sqrt{1 + 2|\xi|^{2\theta} + |\xi|^{4\theta} - 4|\xi|^2} \right), \]
where we have used \(\theta > 1\). This implies \(\sigma_1^2 + \sigma_1 |\xi|^{2\theta} > 0\) for \(|\xi| \ll 1\). The desired estimates of Lemma 4.5 now follows from a series of inequalities \(I_j\) \((j = 1, 2, 3, 4, 5, 6, 7, 8)\).

We also check that the leading term of the time derivative of the solution \(\hat{u}(t, \xi)\) in the high frequency region \(|\xi| \geq \delta_0\) is \((P_0 + P_1) \frac{\partial}{\partial t} e^{-t|\xi|^2}\). Our argument relies on Lemma 3.4 again. We state the following lemma.

**Lemma 4.6.** Let \(n \geq 1\), and \(\theta > 1\), and \(\ell \geq 0\). Then, it holds that
\[ \int_{|\xi| \geq \delta_0} |\hat{u}_t(t, \xi) + (P_0 + P_1)|\xi|^2 e^{-t|\xi|^2} d\xi \]
\[ \leq C e^{-\eta t} (\|u_0\|^2 + \|u_1\|^2) + C (\|D^{\ell+1}u_0\|^2 + \|D^{\ell}u_1\|^2) (1 + t)^{-\frac{\ell - \theta}{2}} \] \((t > 0)\)
with some constant \(\eta > 0\).

**Proof.** We first obtain:
\[ \int_{|\xi| \geq \delta_0} |\hat{u}_t(t, \xi) + (P_0 + P_1)|\xi|^2 e^{-t|\xi|^2} d\xi \]
\[ \leq C \left( \int_{|\xi| \geq \delta_0} |\hat{u}_t(t, \xi)|^2 d\xi + \int_{|\xi| \geq \delta_0} |(P_0 + P_1)|\xi|^2 e^{-t|\xi|^2} |^2 d\xi \right). \] (80)

It follows from Lemma 3.4 that
\[ |\hat{u}_t(t, \xi)|^2 \leq C e^{-\alpha t}(\|\hat{u}_1(\xi)|^2 + |\xi|^2 \|\hat{u}_0(\xi)|^2). \] (81)
Therefore, the first term of (80) can be estimated similarly to (35). Indeed, one can compute as follows:

\[
\int_{|\xi| \geq \delta_0} |\hat{u}(t, \xi)|^2 d\xi = \int_{|\xi| \geq \delta_0} |\hat{u}(t, \xi)|^2 d\xi + \int_{|\xi| \geq 1} |\hat{u}(t, \xi)|^2 d\xi \\
\leq C \int_{|\xi| \geq \delta_0} e^{-\alpha|\xi|^2 t} (|\hat{u}(\xi)|^2 + |\xi|^2 |\hat{u}(\xi)|^2) d\xi \\
+ C \int_{|\xi| \geq 1} e^{-\alpha \frac{|\xi|^2 + |\xi|^4}{\sqrt{\log t}} t} (|\hat{u}(\xi)|^2 + |\xi|^2 |\hat{u}(\xi)|^2) d\xi \\
\leq Ce^{-\alpha \frac{\delta_0^2 t}{2}} (\|u_1\|^2 + \|u_0\|^2) + C \int_{|\xi| \geq 1} e^{-\alpha \frac{|\xi|^2 + |\xi|^4}{\sqrt{\log t}} t} (|\hat{u}(\xi)|^2 + |\xi|^2 |\hat{u}(\xi)|^2) d\xi \\
\leq Ce^{-\alpha \frac{\delta_0^2 t}{2}} (\|u_1\|^2 + \|u_0\|^2) + C(\|D^{\ell+1} u_0\|^2 + \|D^{\ell} u_1\|^2)(1 + t)^{-\frac{n}{4\ell}}. 
\]  

(82)

For the second term of (80), one can estimate as follows:

\[
\int_{|\xi| \geq \delta_0} |P_0 + P_1|^2 |\xi|^4 e^{-2|\xi|^2 t} d\xi \\
= |P_0 + P_1|^2 \int_{|\xi| \geq \delta_0} |\xi|^4 e^{-2|\xi|^2 t} d\xi \\
\leq |P_0 + P_1|^2 e^{-\delta_0^2 t} \int_{|\xi| \geq \delta_0} e^{-\delta_0^2|\xi|^2} d\xi \\
\leq C|P_0 + P_1|^2 e^{-\delta_0^2 t} t^{-\frac{n+2}{4\ell}} \leq C|P_0 + P_1|^2 e^{-\eta t} \quad (t \gg 1)
\]

for some $\eta > 0$. Thus one can get the desired estimate. \hfill \Box

By combining Lemmas 4.5 and 4.6 one can prove Theorem 1.3 similarly to the proof of Theorem 1.1, so we shall omit its detail.

5. Optimality of the energy decay rate. In this section, we shall study the optimality of decay rates of solutions to problem (1)-(2) in the framework of $L^2$-norm. Since we are interested in decay rates of the total energy as $t \to \infty$ defined by

\[
E_u(t) := \frac{1}{2} \left( \|u(t, \cdot)\|^2 + \|\nabla u(t, \cdot)\|^2 \right),
\]

we will study decay rates of each norm of solutions computed by $\|u(t, \cdot)\|^2$, $\|\nabla u(t, \cdot)\|^2$, and $\|u(t, \cdot)\|^2$, respectively. For this, we set

\[
I_0^2 := \\
(\|u_0\|^2 + \|u_1\|^2 + \|u_0\|^2_{1,1} + \|u_1\|^2_{1,1} + \|u_0\|^2 + \|u_1\|^2 + \|D^\ell u_0\|^2 + \|D^{\ell-1} u_1\|^2).
\]

Our results read as follows.

**Proposition 3.** Under the same assumptions as in Theorem 1.2, there exist constants $C_j > 0$ $(j = 1, 2)$ such that for sufficiently large $t > 0$ it is true that

\[
C_1|P_0 + P_1| t^{-\frac{n+2}{4\ell}} \leq \|\nabla u(t, \cdot)\| \leq C_2(I_0 + |P_0 + P_1|) t^{-\frac{n+2}{4\ell}}.
\]

**Proposition 4.** Under the same assumptions as in Theorem 1.3, there exist constants $C_j > 0$ $(j = 1, 2)$ such that for sufficiently large $t > 0$ it is true that

\[
C_1|P_0 + P_1| t^{-\frac{n+4}{4\ell}} \leq \|u(t, \cdot)\| \leq C_2(I_0 + |P_0 + P_1|) t^{-\frac{n+4}{4\ell}}.
\]
Proposition 5. Under the same assumptions as in Theorem 1.1, there exist constants $C_j > 0$ ($j = 1, 2$) such that for sufficiently large $t > 0$ it is true that

$$C_1|P_0 + P_1|t^{-\frac{n}{2}} \leq \|u(t, \cdot)\| \leq C_2(I_0 + |P_0 + P_1|)t^{-\frac{n}{2}}.$$  

Now, let us prove Propositions above based on results obtained in Theorems 1.1, 1.2 and 1.3.

Proof of Proposition 3. At first, because of the Plancherel theorem it follows that

$$\|\nabla u(t, \cdot)\| = \|\xi \hat{u}(t, \xi)\|$$

$$= \left\| \xi \hat{u}(t, \xi) - (P_0 + P_1)|\xi|e^{-\xi t}||^2 + (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\|$$

$$\leq \left\| \xi \hat{u}(t, \xi) - (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\| + \left\| (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\|$$

$$\leq CI_{\text{ho}}(t^{-\frac{n+2}{2}}) + C |P_0 + P_1|t^{-\frac{n+2}{2}} \quad (t \gg 1).$$

Here one has just used Theorem 1.2 with $\ell > \frac{n+2}{2} (\theta - 1)$, and the fact that for $k \geq 0$ (cf. (33))

$$\int_{\mathbb{R}^n} |\xi|^{k}e^{-2\xi t}d\xi = \omega_n \left( \int_{0}^{\infty} \sigma^{k+n-1} e^{-\sigma^2}d\sigma \right)t^{-\frac{n+k}{2}} = Ct^{-\frac{n+k}{2}} \quad (t > 0), \quad (83)$$

where

$$\omega_n := \int_{|\omega| = 1} d\omega.$$ 

Next, we show the estimate from below in Proposition 3. Indeed, it follows from the Plancherel theorem and Theorem 1.2 that

$$\|\nabla u(t, \cdot)\| = \|\xi \hat{u}(t, \xi)\|$$

$$= \left\| \xi \hat{u}(t, \xi) - (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\| + \left\| (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\|$$

$$\geq \left\| (P_0 + P_1)|\xi|e^{-\xi t}||^2 \right\| - \left\| (P_0 + P_1)|\xi|e^{-\xi t}||^2 - |\xi|\hat{u}(t, \xi)\right\|$$

$$\geq C |P_0 + P_1|t^{-\frac{n+2}{2}} - o(t^{-\frac{n+2}{2}}) \quad (t \gg 1),$$

which implies the desired estimate from below. \qed

The results of Propositions 4 and 5 can be obtained similarly to the proof of Proposition 5.1 by relying on Theorems 1.3 and 1.1, the Plancherel theorem, and (83), so it is left to the reader’s exercise.

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