ON THE FREDHOLM PROPERTY OF BISINGULAR
PSEUDODIFFERENTIAL OPERATORS

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Abstract. For operators belonging either to a class of global bisingular pseudodifferential operators on $\mathbb{R}^m \times \mathbb{R}^n$ or to a class of bisingular pseudodifferential operators on a product $M \times N$ of two closed smooth manifolds, we show the equivalence of their ellipticity (defined by the invertibility of certain associated homogeneous principal symbols) and their Fredholm mapping property in associated scales of Sobolev spaces. We also prove the spectral invariance of these operator classes and then extend these results to the even larger classes of Toeplitz type operators.

1. Introduction

Calculi of bisingular pseudodifferential operators can be seen as a systematic approach for studying tensor products of pseudodifferential operators. Focusing on elliptic theory, a typical question would be the following: Given classical (or polyhomogeneous) pseudodifferential operators $A_j \in \mathcal{L}^\mu_{\text{cl}}(M)$ and $B_j \in \mathcal{L}^\nu_{\text{cl}}(N)$ for $j = 1, \ldots, k$, on smooth manifolds $M$ and $N$, how can we characterize the existence of a parametrix, the Fredholm property or the invertibility of the operator $A_1 \otimes B_1 + \ldots + A_k \otimes B_k$? Here, the tensor product $A \otimes B$ denotes an operator acting on functions defined on $M \times N$ with the property that

$$A \otimes B(u \otimes v) = Au \otimes Bv, \quad u \in \mathcal{C}^\infty(M), \ v \in \mathcal{C}^\infty(N),$$

where $(f \otimes g)(x, y) = f(x)g(y)$ for any two functions $f$ and $g$ on $M$ and $N$, respectively. Such tensor products, in general, do not define a classical pseudodifferential operator on $M \times N$, hence the question cannot be answered using only the standard calculus.

Questions of this kind are not only of academic interest but arose, in particular, naturally in the framework of the famous Atiyah-Singer index theorem. In fact, Atiyah and Singer in [1] were led to study systems of the form

$$A \boxtimes B = \begin{pmatrix} A \otimes 1 & -1 \otimes B^* \\ 1 \otimes B & A^* \otimes 1 \end{pmatrix},$$

where both $A$ and $B$ are zero-order classical pseudodifferential operators on $M$ and $N$, respectively. Again, $A \boxtimes B$ is not a classical pseudodifferential operator on $M \times N$. However, if both $A$ and $B$ are elliptic, then $A \boxtimes B$ is a Fredholm operator in $L^2(M \times N, \mathbb{C}^2)$ with index $\text{ind } A \boxtimes B = \text{ind } A \cdot \text{ind } B$. 
Motivated by these phenomena, Rodino in [6] introduced a pseudodifferential calculus of operators acting on sections of vector bundles over a product of smooth, closed (i.e., compact and without boundary) manifolds $M \times N$, containing such kinds of tensor product type operators. We recall the main features and ideas in Section 3. In this calculus, operators can be composed and parametrices to elliptic elements can be constructed. Ellipticity in this context refers to the invertibility of two operator-valued principal symbols associated with each operator (roughly speaking, each such principal symbol is defined on the co-tangent bundle of one of the two manifolds and takes values in the classical pseudodifferential operators of the other manifold). In Section 3.1.2 we carefully discuss these principal symbols, developing a formalism necessary for our application to so-called Toeplitz type operators presented in Section 4.

As a consequence of the existence of parametrices to elliptic operators, as shown in [6], elliptic operators act as Fredholm operators in a certain scale of naturally associated $L^2$-Sobolev spaces. The main result in the present paper is the proof of the reverse statement: If a bisingular pseudodifferential operator in the calculus of [6] is Fredholm it necessarily must be be elliptic. In other words, the ellipticity condition used in the calculus is “optimal”. Also, as a consequence, we obtain that the calculus of Rodino is spectrally invariant. The method of our proof is based on techniques introduced in Gohberg [3] and Hörmander [4].

Of course one can pose analogous questions also in case where $M$ and $N$ are not compact. It then depends very much on the sort of non-compactness which kind of operators one would consider. In the present paper, we investigate the case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ and work with bisingular operators based on pseudodifferential operators of Shubin type, cf. [9]. Such a calculus was recently considered in Battisti, Gramchev, Rodino and Pilipović [2], where a Weyl law for the spectral counting function of global bisingular operators has been obtained, and also in Nicola, Rodino [10], where the noncommutative residue is studied. Again we show, in Section 2, equivalence of ellipticity and Fredholm property as well as spectral invariance. To our best knowledge, such results are even new in the case of $m = 0$, i.e., for usual pseudodifferential operators of Shubin type in $\mathbb{R}^n$.

As a matter of fact, our results allow us to treat even more general kinds of bisingular operators, of so-called Toeplitz type, both in the context of bisingular operators on $M \times N$ and $\mathbb{R}^m \times \mathbb{R}^n$, respectively. To this end we show in Section 4 that general results of Seiler [8] on abstract pseudodifferential operators of Toeplitz type apply in the present two settings of bisingular operator classes.

### 2. Bisingular operators of Shubin type

In the present section we show the equivalence of ellipticity and Fredholm property for a certain class of global bisingular operators on $\mathbb{R}^m \times \mathbb{R}^n$, a bisingular version
of operators of Shubin type [9]. For the more technical details of this calculus we refer the reader to the recent paper [2].

2.1. Shubin type symbols with values in a Fréchet space. Let $F$ be a Fréchet space with topology given by the system of semi-norms $p_0, p_1, p_2, \ldots$. For $\nu \in \mathbb{R}$ we let $\Gamma^\nu(\mathbb{R}^n; F)$ denote the space of all smooth functions $a : \mathbb{R}^n \times \mathbb{R}^n \to F$ satisfying, for any $k \in \mathbb{N}$,

$$q_k(a) := \sup_{x, \xi \in \mathbb{R}^n} \sup_{j + |\alpha| + |\beta| \leq k} p_j \left( \left| D_\xi^\alpha D_\eta^\beta a(x, \xi) \right| (x, \xi)^{\alpha + |\beta| - \nu} \right) < +\infty.$$  

(2.1)

These semi-norms turn $\Gamma^\nu(\mathbb{R}^n; F)$ into a Fréchet space.

The subspace $\Gamma^\nu_{cl}(\mathbb{R}^n; F)$ of classical (or poly-homogeneous) symbols consists of those elements of $\Gamma^\nu(\mathbb{R}^n; F)$ for which there exist smooth functions $a^{(\nu-j)} : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{0\} \to F$, $j = 0, 1, 2, \ldots$, that are positively homogeneous of degree $\nu - j$ in $(x, \xi)$, i.e.,

$$a^{(\nu-j)}(tx, t\xi) = t^{\nu-j} a^{(\nu-j)}(x, \xi) \quad \forall t > 0 \quad \forall (x, \xi) \neq 0,$$

such that

$$r_N(a) := a - \sum_{j=0}^{N-1} \chi a^{(\nu-j)} \in \Gamma^\nu_{-N}(\mathbb{R}^n; F) \quad \forall N = 0, 1, 2, \ldots,$$

where $\chi(x, \xi)$ is a smooth zero-excision function, i.e., $\chi \equiv 1$ near the origin and $1 - \chi$ has compact support. Note that the homogeneous components $a^{(\nu-j)}$ are uniquely determined by $a$; the component $a^{(\nu)}$ is called the homogeneous principal symbol of $a$. By homogeneity, we may identify every component with a smooth $F$-valued function on the unit-sphere in $\mathbb{R}^n \times \mathbb{R}^n$. Then the maps

$$a \mapsto r_N(a) : \Gamma^\nu_{cl}(\mathbb{R}^n; F) \longrightarrow \Gamma^\nu_{-N}(\mathbb{R}^n; F),$$

$$a \mapsto a^{(\nu-j)} : \Gamma^\nu_{cl}(\mathbb{R}^n; F) \longrightarrow \mathcal{E}^\infty(S^{2n-1}; F)$$

with $j, N = 0, 1, 2, \ldots$, induce a Fréchet topology on $\Gamma^\nu_{cl}(\mathbb{R}^n; F)$.

Finally, note that

$$\Gamma^{-\infty}(\mathbb{R}^n; F) := \bigcap_{\nu \in \mathbb{R}} \Gamma^\nu(\mathbb{R}^n; F) = \bigcap_{\nu \in \mathbb{R}} \Gamma^\nu_{cl}(\mathbb{R}^n; F)$$

coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^n, F)$ of rapidly decreasing, $F$-valued functions.

1Actually, in [2] the authors work with a class of symbols slightly larger than the one employed here. They only require the existence of the homogeneous principal symbols while we ask the existence of complete asymptotic expansions in homogeneous components. However, our approach carries over without modification to this larger calculus and our results, i.e., Theorems 2.5, 2.10 and Corollary 2.11, remain valid. In fact, our calculus coincides with the one of [10], where it is presented with a slightly different formalism.
2.1.1. Operator-valued symbols. Of particular importance is the case $F = \mathcal{L}(E_1, E_2)$, the Banach space of all bounded, linear operators $E_1 \to E_2$ between two Hilbert spaces. In this case we associate with $a \in \Gamma^\nu(\mathbb{R}^n, \mathcal{L}(E_1, E_2))$ the pseudodifferential operator $A = \text{op}(a) : \mathcal{S}(\mathbb{R}^n, E_1) \to \mathcal{S}(\mathbb{R}^n, E_2)$ defined by

$$(Au)(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi, \quad \mathcal{S}(\mathbb{R}^n, E_1).$$

For $E_1 = E_2 = \mathbb{C}$ these are the standard pseudodifferential symbols (respectively operators) from the Shubin class as introduced in [9]. The generalisation to Hilbert spaces is straightforward and we do not comment on it. In particular, for a symbol $a$ of order $\nu$, the associated operator extends continuously to $A : Q^s(\mathbb{R}^n, E_1) \to Q^{s-\nu}(\mathbb{R}^n, E_2)$, where $Q^s(\mathbb{R}^n, E) = \Lambda^{-s}(L^2(\mathbb{R}^n, E))$ with an invertible operator $\Lambda^{-s} = \text{op}(\lambda^{-s})$, $\lambda^{-s} \in \Gamma^{-s}(\mathbb{R}^n, \mathcal{L}(E))$ is the Shubin-type Sobolev space of order $s$.

Also note that operators associated with symbols of order $-\infty$ are integral operators with an integral kernels that are Schwartz functions in both variables.

2.2. Bisingular symbols and their calculus. Let us denote by

$$\Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; C^k, C^\ell), \quad \mu, \nu \in \mathbb{R} \cup \{-\infty\}, \quad k, l \in \mathbb{N},$$

the space of all smooth functions $a : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to C^\ell \times k$ (taking values in the complex $\ell \times k$-matrices, identified with $\mathcal{L}(C^k, C^\ell)$ by using the standard basis of $C^k$ and $C^\ell$, respectively) such that

$$(x, \xi) \mapsto a_1(x, \xi) := \left((y, \eta) \mapsto a(x, \xi, y, \eta)\right)$$

defines a Fréchet space valued symbol

$$(2.2) \quad a_1 \in \Gamma^{\mu}(\mathbb{R}^m; \Gamma^{\nu}(\mathbb{R}^n; C^\ell \times k)).$$

In this case,

$$(y, \eta) \mapsto a_2(y, \eta) := \left((x, \xi) \mapsto a(x, \xi, y, \eta)\right)$$

defines a symbol

$$(2.3) \quad a_2 \in \Gamma^{\nu}(\mathbb{R}^n; \Gamma^{\mu}(\mathbb{R}^m; C^\ell \times k)).$$

**Remark 2.1.** A function $a$ belongs to $\Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; C^k, C^\ell)$ if, and only if, it satisfies the uniform estimates

$$\|D^\alpha_x D^\beta_{\xi} D^\gamma_y a(x, \xi, y, \eta)\|_{C^\ell \times k} \leq C_{\alpha\beta\gamma} \langle x, \xi \rangle^{\mu - |\alpha| - |\beta|} \langle y, \eta \rangle^{\nu - |\gamma| - |\delta|}$$

for every order of derivatives.

The spaces of classical symbols $\Gamma^{\mu,\nu}_cl(\mathbb{R}^m \times \mathbb{R}^n; C^k, C^\ell)$ are defined as above, replacing $\Gamma^{\mu}$ and $\Gamma^{\nu}$ by $\Gamma^{\mu}_cl$ and $\Gamma^{\nu}_cl$, respectively.
2.2.1. Operators and Sobolev spaces. With $a \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell)$ we associate, as usual, its pseudodifferential operator

$$A = \text{op}(a) : \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \to \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^\ell).$$

The analogous statements are true for classical symbols. We shall denote by $\Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell)$. Operators of order $(-\infty, -\infty)$ we shall refer to as regularizing or smoothing operators.

**Remark 2.2.** With $A = \text{op}(a) \in \Gamma^{\mu}(\mathbb{R}^n)$ and $B = \text{op}(b) \in \Gamma^{\nu}(\mathbb{R}^m)$, let $a \otimes b \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ be defined by $a \otimes b(x, \xi, y, \eta) = a(x, \xi)b(y, \eta)$. The associated operator we shall denote by $A \otimes B = \text{op}(a \otimes b)$. If $u(x, y) = v(x)w(y)$ with rapidly decreasing functions $v$ and $w$, then

$$[(A \otimes B)u](x, y) = (Av)(x)(Bw)(y).$$

Such tensor-products, respectively finite linear combinations, are the most simple examples of bisingular operators. Using the nuclearity of $\Gamma^\mu_\ell(\mathbb{R}^n)$ indeed it can be shown that

$$\Gamma^{\mu,\nu}_\ell(\mathbb{R}^m \times \mathbb{R}^n) = \Gamma^{\mu}_\ell(\mathbb{R}^m) \hat{\otimes}_\pi \Gamma^{\nu}_\ell(\mathbb{R}^n),$$

where $E \hat{\otimes}_\pi F$ denotes the completed, projective tensor-product of two Fréchet spaces $E$ and $F$, cf. [11]. Note that an equality as in (2.5) does not hold for the spaces of non-classical symbols.

The operator from (2.4) extends continuously to

$$A : Q^{s,\ell}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \to Q^{s-\mu,\ell-\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^\ell), \quad s, \ell \in \mathbb{R},$$

where $Q^{s,\ell}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^j)$ is the $j$-fold sum of $Q^{s,\ell}(\mathbb{R}^m \times \mathbb{R}^n)$, the latter being the closure of $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ with respect to the norm $\|u\|_{s,\ell} = \|\Lambda^{s,\ell}u\|_{L^2(\mathbb{R}^m \times \mathbb{R}^n)}$, where $\Lambda^{s,\ell}$ is an invertible operator in $\Gamma^{s,\ell}(\mathbb{R}^m \times \mathbb{R}^n)$.\(^2\)

Bisingular symbols behave well under composition and taking the formal adjoint, in the sense that

1. Composition of operators, $(A_2, A_1) \mapsto A_2A_1$, induces maps

$$\Gamma^{\mu_2,\nu_2}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^j, \mathbb{C}^\ell) \times \Gamma^{\mu_1,\nu_1}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell) \to \Gamma^{\mu_1+\mu_2,\nu_1+\nu_2}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell).$$

2. Taking the formal $L^2$-adjoint, $A \mapsto A^*$, induces maps

$$\Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell) \to \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^\ell).$$

The analogous statements are true for classical symbols.

\(^2\)Take, for example, $\Lambda^{s,\ell} = \Lambda^s \otimes \Lambda^\ell$ with invertible operators $\Lambda^s$ in the Shubin class $\Gamma^s(\mathbb{R}^j)$. 
2.2.2. Classical symbols and ellipticity. With a classical operator \( A = \text{op}(a) \in \Gamma^{\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^l) \) we associate two principal symbols
\[
\sigma^\mu_1(A) = a^{(1)}_1 \in \mathcal{C}^\infty(S^{2m-1}, \Gamma^{\mu}_c(\mathbb{R}^n; \mathbb{C}^{k \times k})), \\
\sigma^\nu_2(A) = a^{(2)}_2 \in \mathcal{C}^\infty(S^{2n-1}, \Gamma^{\nu}_c(\mathbb{R}^m; \mathbb{C}^{l \times l})),
\]
the homogeneous principal symbol of \( a_1 \) and \( a_2 \) as defined in (2.2) and (2.3), respectively, restricted to the corresponding unit-sphere. Note that
\[
\sigma^\mu_1(A) \in \mathcal{C}^\infty(S^{2m-1}, \mathcal{L}(Q^s(\mathbb{R}^m, \mathbb{C}^k), Q^{s-\mu}(\mathbb{R}^m, \mathbb{C}^l))), \quad s \in \mathbb{R},
\]
and similarly for \( \sigma^\nu_2(A) \). For composition and adjoints of operators we have, using notation from (1) and (2) above,
\[
\sigma^{\mu_1+\mu_2}_1(A_2A_1) = \sigma^{\mu_2}_1(A_2)\sigma^{\mu_1}_1(A_1), \quad \sigma^{\mu}_1(A^*) = \sigma^{\mu}_1(A)^*,
\]
where the \(^*\) on the right-hand side is the formal \( L^2\)-adjoint \( \Gamma^{\mu}(\mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^l) \rightarrow \Gamma^{\ast\mu}(\mathbb{R}^n; \mathbb{C}^{l \times l}, \mathbb{C}^{k \times k}) \). Analogous equations hold for the other principal symbol \( \sigma_2 \).

**Definition 2.3.** \( A \in \Gamma^{\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) is called elliptic if both \( \sigma^\mu_1(A) \) and \( \sigma^\nu_2(A) \) take values in the invertible operators.

In the previous definition invertibility of \( \sigma^\mu_1(A)(x, \xi) \) refers either to invertibility in \( \Gamma^\mu(\mathbb{R}^n; \mathbb{C}^{k \times k}) \) or to invertibility in \( \mathcal{L}(Q^s(\mathbb{R}^m, \mathbb{C}^k), Q^{s-\mu}(\mathbb{R}^m, \mathbb{C}^l)) \) for some \( s \in \mathbb{R} \). Due to the spectral invariance of the standard Shubin class (which is a particular case of the spectral invariance of bisingular operators that we shall prove in this paper) both possibilities are equivalent.

The following theorem is one of the main results for elliptic operators:

**Theorem 2.4.** An operator \( A \in \Gamma^{\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) is elliptic if, and only if, there exists an operator \( B \in \Gamma^{-\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) such that
\[
1 - AB, 1 - BA \in \Gamma^{-\infty,-\infty}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k).
\]

Any such \( B \) is called a parametrix of \( A \).

Note that parametrices of elliptic operators are uniquely determined modulo smoothing operators. Recall once more that smoothing operators are precisely those integral operators with an integral kernel which is rapidly decreasing in all variables.

2.3. Ellipticity and Fredholm property. Let \( A \in \Gamma^{\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \). If \( A \) is elliptic one can construct a parametrix \( B \in \Gamma^{-\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \), i.e., both \( 1 - AB \) and \( 1 - BA \) are smoothing operators. Since smoothing operators induce compact operators in the Sobolev spaces of any order, the implication a) \( \Rightarrow \) b) of the following theorem is evident:

**Theorem 2.5.** For \( A \in \Gamma^{\mu,\nu}_c(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) the following properties are equivalent:

a) \( A \) is elliptic.

b) For every \((s, t) \in \mathbb{R}^2\), \( A \) induces Fredholm operators
\[
Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k) \rightarrow Q^{s-\mu, t-\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k).
\]
c) There exists a tuple \((s, t) \in \mathbb{R}^2\) such that \(A\) induces a Fredholm operator

\[
Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k) \to Q^{s-\mu, t-\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k).
\]

The implication b) \(\Rightarrow\) c) is trivial. In the sequel we shall prove the implication c) \(\Rightarrow\) a). To shorten notation let us now assume \(k = 1\). By using order reductions we also may assume without loss of generality that \(\mu = \nu = s = t = 0\), i.e., we may assume that \(A \in \Gamma^{0,0}(\mathbb{R}^m \times \mathbb{R}^n)\) is a Fredholm operator in \(L^2(\mathbb{R}^m \times \mathbb{R}^n)\). The method of proof is inspired by that of Theorem 1 in Section 2.3.4.1 of [5] and by that of Theorem 1.6 in [7].

2.3.1. A family of isometries. Let \(E\) be a Hilbert space. For fixed \((x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n\) with \(|(x_0, \xi_0)| = 1\) and \(0 < \tau < 1/2\) define \(S_\lambda \in \mathcal{L}(L^2(\mathbb{R}^n, E))\), \(\lambda \geq 1\), by

\[
(S_\lambda u)(x) = \lambda^{n\tau/2} e^{i\lambda x \xi_0} u(\lambda^\tau (x - \lambda x_0)).
\]

It is straightforward to verify that any \(S_\lambda\) is an isometric isomorphism with inverse given by

\[
(S_\lambda^{-1} v)(x) = \lambda^{-n\tau/2} e^{-i\lambda(x_0 + \lambda^{-\tau} x) \xi_0} v(\lambda^\tau (\lambda x_0 + \lambda^{-\tau} x)).
\]

Moreover,

\[
\limsup_{\lambda \to +\infty} w\text{-lim} S_\lambda u = 0 \quad \forall u \in L^2(\mathbb{R}^n, E),
\]

where \(w\text{-lim}\) denotes the limit with respect to the weak topology of \(L^2(\mathbb{R}^n, E)\). In fact, this property follows from the fact that all \(S_\lambda\) are isometries and that

\[
|\langle S_\lambda u, v \rangle_{L^2(\mathbb{R}^n, E)}| = \left| \int (S_\lambda u(x), v(x))_E \, dx \right|
\]

\[
\leq \int \lambda^{n\tau/2} \|u(\lambda^\tau (x - \lambda x_0))\|_E \|v(x)\|_E \, dx
\]

\[
\leq \lambda^{-n\tau/2} \|u\|_{L^1(\mathbb{R}^n, E)} \|v\|_{L^\infty(\mathbb{R}^n, E)} \xrightarrow{\lambda \to +\infty} 0
\]

for every \(u\) and \(v\) belonging to the dense subspace \(\mathcal{S}(\mathbb{R}^n, E)\) of \(L^2(\mathbb{R}^n, E)\).

2.3.2. Recovering the principal symbol. If \(a \in \Gamma^\mu(\mathbb{R}^n, \mathcal{L}(E))\) is an operator-valued symbol in the sense of Remark Section 2.1.1 and the \(S_\lambda\), \(\lambda \geq 1\), are as introduced in the previous Section, a direct calculation shows

\[
S_\lambda^{-1} \text{op}(a) S_\lambda = \text{op}(a_\lambda), \quad a_\lambda(x, \xi) = a(\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^\tau \xi)
\]

Note that \(a_\lambda \in \Gamma^\mu(\mathbb{R}^n, \mathcal{L}(E))\) for every \(\lambda\). The following estimate will be crucial later on:

**Lemma 2.6.** Let \(a \in \Gamma^\mu(\mathbb{R}^n, \mathcal{L}(E))\) with \(\mu \leq 0\) and \(\rho = \frac{\mu}{1-\tau}\) (note that \(0 < \rho < 1\)). Then, for any order of derivatives,

\[
\left\| D_\xi^\alpha D_\xi^\beta a_\lambda(x, \xi) \right\|_{\mathcal{L}(E)} \leq C_{\alpha\beta} \lambda^{(1-\tau)\rho |\beta|} \langle x, \xi \rangle^{|\rho|\alpha|\beta| - \mu}
\]

uniformly in \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\) and \(\lambda \geq 1\).
Proof. By chain rule and using the standard symbol estimates for \(a\), we have
\[
\left\| D_x^2 D_\xi^2 a_\lambda(x, \xi) \right\|_{L^2(E)} \leq C \lambda^{\alpha |\tau|+\|\beta|} (\lambda x_0 + \lambda \tau \xi, \lambda \xi_0 + \lambda \tau \xi)^{\mu-\rho |\alpha|}
\]
with a constant \(C\) independent of \((x, \xi)\) and \(\lambda\). Since \(\langle v + w \rangle^{-1} \leq C \langle w \rangle / |v|\) by Peetre’s inequality and \(\langle \sigma w \rangle \leq |\sigma w|\) for \(\sigma \geq 1\), we can estimate
\[
\langle \lambda x_0 + \lambda \tau \xi, \lambda \xi_0 + \lambda \tau \xi \rangle^{\mu-\rho |\alpha|} \leq C \langle \lambda^{-\tau} x, \lambda \xi \rangle^{\rho |\alpha| - \mu} |\lambda x_0, \lambda \xi_0|^{\mu-\rho |\alpha|}
\]
resulting in
\[
\left\| D_x^2 D_\xi^2 a_\lambda(x, \xi) \right\|_{L^2(E)} \leq C \lambda^{(1-\tau)(\mu-\tau)|\beta|+(\tau-\rho+\tau \rho)|\alpha|} \langle x, \xi \rangle^{\rho |\alpha| - \mu}.
\]
It remains to observe that \(\tau - \rho + \tau \rho = 0\), due to the choice of \(\rho\). \(\square\)

Lemma 2.7. Let \(\{a_\lambda \mid \lambda \geq 1\}\) be a subset of \(\Gamma^0(\mathbb{R}^n, \mathcal{L}(E))\), \(\sigma \in \mathbb{C}\) a constant, and \(u \in \mathcal{S}(\mathbb{R}^n, E)\). Assume that
\begin{enumerate}
\item[(1a)] \(a_\lambda(x, \xi) \xrightarrow{\lambda \to +\infty} \sigma\) for all \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\),
\item[(1b)] for every \(x \in \mathbb{R}^n\) there exist constants \(c_x, m_x \geq 0\) such that
\[
\left\| a_\lambda(x, \xi) \right\| \leq c_x \langle \xi \rangle^{m_x} \quad \forall \xi \in \mathbb{R}^n \quad \forall \lambda \geq 1,
\]
\item[(2)] there exists a \(g \in L^1(\mathbb{R}^n)\) such that
\[
\left\| \text{op}(a_\lambda)u(x) \right\|_E^2 \leq g(x) \quad \forall x \in \mathbb{R}^n \quad \forall \lambda \geq 1.
\]
\end{enumerate}
Then \(\text{op}(a_\lambda)u \xrightarrow{\lambda \to +\infty} \sigma u \) in \(L^2(\mathbb{R}^n, E)\).

Proof. The result follows directly from Lebesgue’s dominated convergence theorem, provided we can show that \(\text{op}(a_\lambda)u\) converges pointwise on \(\mathbb{R}^n\) to \(\sigma u\) as \(\lambda\) tends to infinity. However, with \(x \in \mathbb{R}^n\) fixed,
\[
\left| \text{op}(a_\lambda)u(x) \right| = \int e^{i\xi \cdot x} a_\lambda(x, \xi) \hat{u}(\xi) \ d\xi.
\]
By assumption (1a), the integrand converges pointwise on \(\mathbb{R}^n_\xi\) to \(\sigma e^{i\xi \cdot \hat{u}}(\xi)\). By (1b) the integrand is majorized in norm by \(h(\xi) := c_x \langle \xi \rangle^{m_x} \hat{u}(\xi) \in L^1(\mathbb{R}^n_\xi)\). Thus, by dominated convergence,
\[
\left| \text{op}(a_\lambda)u(x) \right| \xrightarrow{\lambda \to +\infty} \sigma \int e^{i\xi \cdot \hat{u}}(\xi) \ d\xi = \sigma u(x).
\]
This completes the proof. \(\square\)

The following proposition gives a method for recovering the principal symbol from the operator:

Proposition 2.8. Let \(A = \text{op}(a) \in \Gamma^0_{cl}(\mathbb{R}^n, \mathcal{L}(E))\), \(a_\lambda\) as in (2.9), and \(u \in \mathcal{S}(\mathbb{R}^n, E)\). Then
\[
\text{op}(a_\lambda)u \xrightarrow{\lambda \to +\infty} a^{(0)}(x_0, \xi_0)u \quad \text{in} \ L^2(\mathbb{R}^n, E),
\]
where \(a^{(0)} \in \mathcal{C}^\infty(\mathbb{S}^{2n-1}, \mathcal{L}(E))\) denotes the homogeneous principal symbol of \(a\).
Proof. By Lemma 2.6 with $|\alpha| = |\beta| = \mu = 0$, condition (1b) of Lemma 2.7 is obviously satisfied (with $m_x = 0$). Now let $\chi(x, \xi)$ be a zero-excision function and write $a = a^0 + r$, where

$$a^0(x, \xi) = \chi(x, \xi)a^{(0)}(x, \xi), \quad r \in \Gamma^{-1}(\mathbb{R}^n, \mathcal{L}(E)).$$

Then $a_\lambda = a^0_\lambda + r_\lambda$. By Lemma 2.6 with $|\alpha| = |\beta| = 0$ and $\mu = -1$, it is clear that $r_\lambda(x, \xi) \to 0$ for all $x$ and $\xi$. Moreover, by homogeneity of $a^{(0)}$,

$$a^0_\lambda(x, \xi) = \chi(\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^\tau \xi) a^{(0)}(x_0 + \lambda^{-1-\tau} x, \xi_0 + \lambda^{-1+\tau} \xi)$$

and thus $a_\lambda(x, \xi) \to a^{(0)}(x_0, \xi_0)$ for all $x$ and $\xi$. Therefore assumption (1a) of Lemma 2.7 with $\sigma = a^{(0)}(x_0, \xi_0)$ is satisfied.

It remains to verify assumption (2). To this end let $M \in \mathbb{N}$ and write, using integration by parts,

$$\langle x \rangle^{2M} [\text{op}(a_\lambda)u](x) = \int e^{ix\xi}(1 + \Delta \xi)^M (a_\lambda(x, \xi) \hat{u}(\xi)) d\xi.$$ 

By product rule and Lemma 2.6 there exist functions $u_\alpha \in \mathcal{S}(\mathbb{R}^n, E)$ such that

$$\langle x \rangle^{2M} \|[\text{op}(a_\lambda)u](x)\|_E \leq \sum_{|\alpha| \leq 2M} \int \langle x, \xi \rangle^{\rho|\alpha|} |\hat{u}_\alpha(\xi)| d\xi.$$ 

Hence

$$\|[\text{op}(a_\lambda)u](x)\|_E^2 \leq C \langle x \rangle^{4M(\rho-1)} =: g(x)$$

with a suitable constant independent of $x$ and $\lambda$. Since $\rho - 1 < 0$ we can choose $M$ so large that $g \in L^1(\mathbb{R}^n)$.

2.3.3. The proof of Theorem 2.5. First we shall proof the following result on pseudodifferential operators with operator-valued symbols:

**Proposition 2.9.** Consider $A = \text{op}(a) \in \Gamma^0(\mathbb{R}^n, \mathcal{L}(E))$ as a bounded operator in $L^2(\mathbb{R}^n, E)$ and let $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ be a unit-vector.

a) If $A$ is upper semi-fredholm, $a^{(0)}(x_0, \xi_0)$ is injective.

b) If $A$ is lower semi-fredholm, $a^{(0)}(x_0, \xi_0)$ is surjective.

**Proof.** Assume that $A = \text{op}(a) \in \Gamma^0(\mathbb{R}^n, \mathcal{L}(E))$ induces an upper semi-fredholm operator $A \in \mathcal{L}(L^2(\mathbb{R}^n, E))$. Since $E$ is a Hilbert space, there exists a $B \in \mathcal{L}(L^2(\mathbb{R}^n, E))$ such that $K := 1 - BA$ is a compact operator in $L^2(\mathbb{R}^n, E)$.

Let $u \in \mathcal{S}(\mathbb{R}^n)$ with $\|u\|_{L^2(\mathbb{R}^n)} = 1$ and define $u_e \in \mathcal{S}(\mathbb{R}^n, E), e \in E$, by $u_e(x) = u(x)e$. Then, with notations from the previous subsection,

$$\|e\|_E = \|u_e\|_{L^2(\mathbb{R}^n, E)} = \|(BA + K)S_A u_e\|_{L^2(\mathbb{R}^n, E)} \leq \|B\|_{\mathcal{L}(L^2(\mathbb{R}^n, E))}\|S_A u_e\|_{L^2(\mathbb{R}^n, E)} + \|KS_A u_e\|_{L^2(\mathbb{R}^n, E)} \lambda \to \infty \|B\|_{\mathcal{L}(L^2(\mathbb{R}^n, E))}\|a^{(0)}(x_0, \xi_0)e\|_E.$$
For the convergence we have used that $KS_\lambda u_\epsilon \to 0$, since $S_\lambda u_\epsilon \to 0$ weakly by (2.8) and $K$ is compact, and that $S_\lambda^{-1} AS_\lambda u_\epsilon = \text{op}(a_\lambda) u_\epsilon \to u_\epsilon$ in $L^2(\mathbb{R}^n, E)$ due to Proposition 2.8. Therefore,

$$\|a^{(0)}(x_0, \xi_0)e\|_E \geq \frac{1}{\|B\|_{\mathcal{L}(L^2(\mathbb{R}^n, E))}} \|e\|_E \quad \forall e \in E.$$  

This implies a). If $A$ is lower semi-Fredholm, its adjoint is an upper semi-Fredholm operator. By a), the principal symbol of $A^*$ evaluated in $(x_0, \xi_0)$, i.e., $a^{(0)}(x_0, \xi_0)^*$, is injective. Hence $a^{(0)}(x_0, \xi_0)$ is surjective. \hfill \Box

The proof of c) $\Rightarrow$ a) of Theorem 2.5 now easily follows from the previous proposition: Consider $A \in \Gamma^0,0(\mathbb{R}^m \times \mathbb{R}^n)$ as an element of $\Gamma^0(\mathbb{R}^n, \mathcal{L}(E))$ with $E = L^2(\mathbb{R}^m)$. Since, by assumption, $A$ is Fredholm, its (operator-valued) principal symbol evaluated in an arbitrary unit-vector $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ is invertible. However, this principal symbol just coincides with $\sigma_0^2(A) \in \mathcal{C}^{\infty}(\mathbb{S}^{2n-1}, \mathcal{L}(L^2(\mathbb{R}^m)))$ as introduced in Section 2.2.2. Analogously, $\sigma_0^1(A)$ evaluated in an arbitrary unit-vector of $\mathbb{R}^m \times \mathbb{R}^m$ is invertible as an operator in $L^2(\mathbb{R}^n)$.

2.4. Spectral invariance. A consequence of Theorem 2.5 is the following result, the so-called spectral-invariance of bisingular pseudodifferential operators:

**Theorem 2.10.** Let $A \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k, \mathcal{C}^k)$. Assume that $A$ induces an isomorphism $Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k) \to Q^{s-\mu,t-\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k)$ for some tuple $(s, t) \in \mathbb{R}^2$. Then there exists a $B \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k, \mathcal{C}^k)$ such that $AB = BA = 1$. In particular, $A$ induces an isomorphism $Q_{s,t}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k) \to Q^{s-\mu,t-\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathcal{C}^k)$ for every tuple $(s, t) \in \mathbb{R}^2$.

In other words, invertibility as a bounded operator between Sobolev spaces implies the invertibility within the class of bisingular pseudodifferential operators.

**Proof.** To shorten notation let us assume $k = 1$. The isomorphism is, in particular, a Fredholm operator. Due to Theorem 2.5, $A$ is elliptic. Therefore it has a parametrix $B_0 \in \Gamma^{-\mu,-\nu}(\mathbb{R}^m \times \mathbb{R}^n)$. Thus $K_R := 1 - B_0 B_0$ and $K_L := 1 - B_0 A$ are smoothing operators. Passing to the action in Sobolev spaces, and resolving both equations for $A^{-1}$ we derive

$$A^{-1} = B_0 + B_0 K_R + K_L A^{-1} K_R.$$  

Obviously, both $B_0$ and $B_0 K_R$ belong to $\Gamma^{-\mu,-\nu}(\mathbb{R}^m \times \mathbb{R}^n)$. Now let $R := K_L A^{-1} K_R$. We shall argue below that $R$ is smoothing and therefore $B = B_0 + B_0 K_R + R \in \Gamma^{-\mu,-\nu}(\mathbb{R}^m \times \mathbb{R}^n)$ is the desired operator.

Since $K_L$ and $K_R$ are smoothing it is obvious that both $R$ and $R^*$ map $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$. However, this is known to be equivalent to $R$ being an integral operator with an integral kernel that is rapidly decreasing in all variables; for convenience of the reader we sketch the argument: First of all one sees that $R$ has a kernel $k(x, y) = k(x_1, x_2, y_1, y_2) \in L^2(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ such that

$$k \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n, L^2(\mathbb{R}^m \times \mathbb{R}^m)) \cap \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n, L^2(\mathbb{R}^m \times \mathbb{R}^m)) \cap \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m, L^2(\mathbb{R}^m \times \mathbb{R}^m)) \cap \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m, L^2(\mathbb{R}^m \times \mathbb{R}^m)).$$
Thus the claim follows if we can show that
\[ \mathcal{S}(\mathbb{R}^k, L^2(\mathbb{R}^\ell)) \cap \mathcal{S}(\mathbb{R}^\ell, L^2(\mathbb{R}^k)) = \mathcal{S}(\mathbb{R}^{k+\ell}). \]

Let \( g \) be a function from the space on the left-hand side and denote by \( \| \cdot \| \) the norm of \( L^2(\mathbb{R}^{k+\ell}) \). Then, by Parseval's identity,
\[ \| g \| = (2\pi)^{-(k+\ell)/2} \| \mathcal{F}g \| = (2\pi)^{-k/2} \| \mathcal{F}_{u \to \xi} g \| = (2\pi)^{-\ell/2} \| \mathcal{F}_{v \to \eta} g \|. \]

Combining this repeatedly with the estimate \( ab \leq a^2 + b^2 \), one obtains that
\[ \| \langle u \rangle^i \langle v \rangle^j (D_u)^i (D_v)^j g \| \leq C \left( \| \langle u \rangle^i \langle v \rangle^j (D_u)^i g \| + \| (D_u)^{2i} g \| + \| (D_v)^{2j} g \| + \| (D_v)^{4j} g \| \right) < +\infty \]
for any choice of non negative integers \( i, i', j, j' \). This yields \( g \in \mathcal{S}(\mathbb{R}^{k+\ell}) \). \( \square \)

**Corollary 2.11.** Let \( A \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) be elliptic and \( \mu, \nu \geq 0 \). Then the unbounded operator
\[ A_{s,t} : \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \subset Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \rightarrow Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \]
has one, and only one, closed extension, given by the action of \( A \) on the domain \( Q^{s+t,\mu+t,\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \). The spectrum of the closure of \( A_{s,t} \) does not depend on both \( s \) and \( t \).

**Proof.** By density of the rapidly decreasing functions in any Sobolev space, it is clear that \( Q^{s+t,\mu+t,\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \) is contained in the domain of the closure of \( A_{s,t} \). Moreover, if both \( u \) and \( Au \) belong to \( Q^{s+t,\mu+t,\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \) then \( u \in Q^{s+t,\mu+t,\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \) by elliptic regularity. Therefore, the domain of any closed extension is a subset of, and hence equal to, \( Q^{s+t,\mu+t,\nu}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}^k) \).

The statement on the spectrum follows directly from Theorem 2.10 and the fact that \( \lambda - A \in \Gamma^{\mu,\nu}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^k) \) for any \( \lambda \in \mathbb{C} \). \( \square \)

### 3. Bisingular operators on closed manifolds

In [6] bisingular operators acting on sections into vector bundles over products of closed manifolds are considered. We shall use the notation \( L^{\mu,\nu}_{\mathcal{V}}(M \times N; E, F) \) for such operators and \( Q^{s,t}_{\mathcal{V}}(M \times N; G) \) for the associated Sobolev spaces, where \( M \) and \( N \) are closed Riemannian manifolds and \( E, F \) and \( G \) are finite-dimensional hermitian vector-bundles over \( M \times N \).\(^3\)

#### 3.1. Description of the calculus

As usual, bisingular operators on a manifold are defined as those that in any local trivialisation of the bundles and any local coordinates correspond to bisingular operators in a product of two Euclidean spaces, with symbols taking values in \( \mathbb{C}^{\dim F \times \dim E} \). We shall not go too much into the details, but only describe how the classes \( \Gamma^{\mu,\nu} \) introduced above have to be modified to recover the situation of [6].

\(^3\)A comment analogous to the one of footnote 1 applies also here.
3.1.1. The calculus on $\mathbb{R}^m \times \mathbb{R}^n$. For a Fréchet space $F$ define the space $L^\nu(\mathbb{R}^n, F)$ as in the beginning of Section 2.1, replacing in (2.1) the term $\langle x, \xi \rangle^{[\alpha]+[\beta]-\nu}$ by $\langle \xi \rangle^{[\alpha]-\nu}$.

For defining the classical symbols $L^\nu_{cl}(\mathbb{R}^n, F)$, in the subsequent part one considers homogeneous components $a^{(\nu-j)} : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to F$ which are homogeneous in the sense of

$$a^{(\nu-j)}(x, t\xi) = t^{\nu-j} a^{(\nu-j)}(x, \xi) \quad \forall \ t > 0 \ \forall \ x \ \forall \ \xi \neq 0.$$ 

The excision function $\chi(x, \xi)$ needs to be replaced by an excision function $\chi(\xi)$.

Starting out with these symbol classes, one then introduces, as before, the bisingular symbols $L^{\mu,\nu}_{cl}(\mathbb{R}^m \times \mathbb{R}^n; C^k, C^l)$. The corresponding Sobolev spaces $Q^{s,t}(\mathbb{R}^m \times \mathbb{R}^n)$ are defined as the closure of $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ with respect to the norm $\|u\|_{s,t} = \|\Lambda^{s,t} u\|_{L^2(\mathbb{R}^m \times \mathbb{R}^n)}$, where $\Lambda^{s,t}$ is the operator with symbol $\lambda^{s,t}(\xi, \eta) = \langle \xi \rangle^s \langle \eta \rangle^t$.

The two principal symbols associated with $A = \text{op}(a) \in L^\mu_{cl}(\mathbb{R}^m \times \mathbb{R}^n; C^k, C^l)$ are then

$$\sigma_1^\mu(A) = a^{(\mu)}_1 \in \mathcal{C}^\infty(\mathbb{R}^m ; \mathcal{S}^{-m-1}_{\mathcal{E}}(\mathbb{R}^m \times \mathbb{R}^n)), \quad \sigma_2^\mu(A) = a^{(\mu)}_2 \in \mathcal{C}^\infty(\mathbb{R}^n ; \mathcal{S}^{-n-1}_{\mathcal{E}}(\mathbb{R}^m \times \mathbb{R}^n)),$$ 

and ellipticity asks the pointwise invertibility of both these symbols.

The analogue of Theorem 2.4 holds true, while Theorem 2.5 fails to be true, since smoothing operators do not induce compact operators in the Sobolev spaces. However, the analogue of Theorem 2.5 for operators on a product of compact manifolds is valid, as we shall see below.

3.1.2. The principal symbols. For an operator $A \in L^\mu_{cl}(M \times N; E, F)$ the existence of local principal symbols leads to two globally defined (on the unit co-sphere bundles $S^*M$ and $S^*N$, respectively) objects, again denoted by $\sigma_1^\mu(A)$ and $\sigma_2^\mu(A)$.

If $v = (x, \xi) \in S^*M$ then $\sigma_1^\mu(A)(v)$ is an operator in $L^\mu_{cl}(N; E(x), F(x))$, where $L^\mu_{cl}$ refers to the usual space of classical pseudodifferential operators on a closed manifold and

$$E(x) := E|_{\{x\} \times N}, \quad F(x) := F|_{\{x\} \times N} \quad x \in M,$$

considered as vector bundles over $N \cong \{x\} \times N$.

If we denote by $\pi_M : S^*M \to M$ the canonical projection and define the (infinite-dimensional) Hilbert space bundle $Q^s(N, E)$ over $M$ by taking as fibre in $m \in M$ the Sobolev space $Q^s(N, E(m))$ of sections in $E(m)$ (see Section 5 for details), then we can consider $\sigma_1^\mu(A)$ as a bundle homomorphism

$$\sigma_1^\mu(A) : \pi_M^s Q^s(N, E) \to \pi_M^s Q^{s-\mu}(N, F), \quad s \in \mathbb{R}.$$ 

Similarly,

$$\sigma_2^\mu(A) : \pi_N^s Q^s(M, E) \to \pi_N^s Q^{s-\mu}(M, F), \quad s \in \mathbb{R}.$$
Theorem 3.1. A ∈ $L^{\mu,\nu}_{cl}(M \times N; E, F)$ is called elliptic if both homomorphisms (3.2) and (3.3) are isomorphisms\(^4\). Then, the following are equivalent:

a) $A \in L^{\mu,\nu}_{cl}(M \times N; E, F)$ is elliptic.

b) There exists a $B \in L^{-\mu,-\nu}_{cl}(M \times N; F, E)$ such that both $1 - BA$ and $1 - BA$ are smoothing operators.

3.2. Ellipticity and Fredholm property. We are now going to explain that the analogue of Theorem 2.5 holds for operators $A \in L^{\mu,\nu}_{cl}(M \times N; E, F)$. Again we may assume $\mu = \nu = 0$ and consider the operator

$$A : L^2(M \times N, E) \rightarrow L^2(M \times N, F).$$

Assume that this operator is Fredholm and thus has an inverse $B$ modulo compact operators. Let $K := 1 - BA$ and $v_0 \in S^*_m M$ be a given, fixed unit co-vector. We shall verify the invertibility of $\sigma_0^0(A)(v_0) \in L^0_{cl}(N; E(m_0), F(m_0))$.

To this end, let $U$ be a coordinate system of $M$ near $m_0$ such that $v_0$ corresponds to $(x_0, \xi_0)$ and that $E_{|U \times N} \cong U \times E(m_0)$, $F_{|U \times N} \cong U \times F(m_0)$ in the sense of Proposition 5.1. Moreover, let $\chi_1, \chi_2, \chi_3 \in C_0^\infty(U_0)$ such that $\chi_{i+1} \equiv 1$ on the support of $\chi_i$ for $i = 1, 2$. Consider the $\chi_i$ as functions on $M \times N$, not depending on the variable of $N$. Multiplying the identity $K = 1 - BA$ from the left with $\chi_1$, from the right with $\chi_3$, and rearranging terms yields

$$\chi_1 B\chi_2 \chi_3 A\chi_3 = \chi_1 - \chi_1 K\chi_3 - \chi_1 B(1 - \chi_2)A\chi_3.$$

Note that $(1 - \chi_2)A\chi_3 \in L^{-\infty,0}_{cl}(M \times N; E, F)$ due to the disjoint supports of $(1 - \chi_2)$ and $\chi_3$, and that all involved operators $B' := \chi_1 B\chi_2$, $A' := \chi_3 A\chi_3$, $K_1' := \chi_1 K\chi_3$ and $K_2' := \chi_1 B(1 - \chi_2)A\chi_3$ are localized in $U \times N$. In particular, they can be considered – by passing to local coordinates – as operators from $L^2(\mathbb{R}^n, L^2(N, E(m_0)))$ into $L^2(\mathbb{R}^n, L^2(N, F(m_0)))$. Note that $K_1'$ is not a compact operator, but extends to a continuous map $L^1(\mathbb{R}^n, L^2(N, E(m_0)))$ into $L^2(\mathbb{R}^n, L^2(N, F(m_0)))$. Moreover, $A'$ is a pseudodifferential operator with symbol

$$a \in L^0_0(\mathbb{R}^n, \mathcal{L}(L^2(N, E(m_0)), L^2(\mathbb{R}^n, L^2(N, F(m_0))))).$$

Since $a^{(0)}(x_0, \xi_0)$ is just the local expression of $\sigma_0^0(A)(v_0)$, the injectivity of $\sigma_0^0(A)(v_0)$ follows from the following proposition (the surjectivity, hence invertibility, then follows by considering the adjoint of $A$):

Proposition 3.2. Let $E$ be a Hilbert space, $A = \text{op}(a) \in L^0_{cl}(\mathbb{R}^n, \mathcal{L}(E))$, $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi_0| = 1$, and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near $x_0$. Moreover assume that there exists a $B \in \mathcal{L}(L^2(\mathbb{R}^n, E))$ such that

$$BA = \chi - K_1 - K_2$$

where $K_1$ is a compact operator in $L^2(\mathbb{R}^n, E)$ and $K_2$ induces a continuous operator $L^1(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, E)$. Then $a^{(0)}(x_0, \xi_0)$ is injective.

\(^4\)Due to spectral invariance, this condition is independent of $s$. 
Proof. The proof is very similar to the one of Proposition 2.9. Instead of the operator-family \( S_\lambda \), defined in (2.7), we shall now use \( S_\lambda \in \mathcal{L}(L^2(\mathbb{R}^n, E)), \lambda \geq 1 \), defined by
\[
(S_\lambda u)(x) = \lambda^{n/4} e^{i\lambda x_0} u(\lambda^{1/2}(x - x_0)).
\]
Similarly to Section 2.3.1 we can verify that these \( S_\lambda \) are isometric isomorphisms and, for every \( u \in \mathcal{S}(\mathbb{R}^n, E) \),
\[
\begin{align*}
&i) \quad S_\lambda^{-1} \text{op}(a) S_\lambda u \xrightarrow{\lambda \to +\infty} a^{(0)}(x_0, \xi_0) u \text{ in } L^2(\mathbb{R}^n, E), \\
&ii) \quad S_\lambda u \xrightarrow{\lambda \to +\infty} 0 \text{ weakly in } L^2(\mathbb{R}^n, E), \\
&iii) \quad S_\lambda u \xrightarrow{\lambda \to +\infty} 0 \text{ in } L^1(\mathbb{R}^n, E).
\end{align*}
\]
Now let us choose \( u \in C_0^\infty(\mathbb{R}^n) \) such that \( \|u\|_{L^2(\mathbb{R}^n)} = 1 \) and \( \chi \equiv 1 \) on the support of \( u(\cdot - x_0) \). Note that then also \( \chi \equiv 1 \) on the support of \( S_\lambda u \). With \( u_e \) defined by \( u_e(x) = u(x)e \) with \( e \in E \), we obtain
\[
\|e\|_E = \|\chi S_\lambda u_e\|_{L^2(\mathbb{R}^n, E)} = \|(BA + K_1 + K_2)S_\lambda u_e\|_{L^2(\mathbb{R}^n, E)} \\
\leq \|B\|_{\mathcal{L}(L^2(\mathbb{R}^n, E))} \|S_\lambda^{-1} AS_\lambda u_e\|_{L^2(\mathbb{R}^n, E)} + \|K_1 S_\lambda u_e\|_{L^2(\mathbb{R}^n, E)} + \|K_2 S_\lambda u_e\|_{L^2(\mathbb{R}^n, E)}.
\]
By passing to the limit as \( \lambda \to +\infty \), using i)–iii) from above, we derive the estimate
\[
\|a^{(0)}(x_0, \xi_0)e\|_E \geq \frac{1}{\|B\|_{\mathcal{L}(L^2(\mathbb{R}^n, E))}} \|e\|_E \quad \forall e \in E.
\]
This implies the injectivity. \( \square \)

Also the results of Section 2.4 on the spectral invariance extend to the present setting. Let us state this explicitly:

**Theorem 3.3.** Theorems 2.5, 2.10 and Corollary 2.11 remain valid, with obvious adaptations, in the framework of bisingular pseudodifferential operators from \( L^p_{\partial u}(M \times N; E, F) \).

## 4. Operators of Toeplitz Type

Assume we consider a class of operators that act in an associated scale of Sobolev spaces and that in this class we can characterize the Fredholm property of an operator by its ellipticity which, by definition, means the invertibility of certain principal symbols associated with the operator. It is natural to pose the following problem: Take an operator \( A \) and two projections \( P_0, P_1 \) in that class of operators (where projection means that \( P_j^2 = P_j \)), such that the composition \( \tilde{A} = P_1 AP_0 \) makes sense. The range spaces of the projections determine closed subspaces of the Sobolev spaces. How can we characterize the Fredholm property of \( \tilde{A} \), considered as an operator acting between these closed subspaces?

This question has been answered in [8], in a quite general context of “abstract” pseudodifferential operators. We shall apply these results here to the case of bisingular pseudodifferential operators. We focus on the case of operators defined on a product \( M \times N \) of compact manifolds, as described in the preceding Section 3; an
Proof. First of all let us justify that we may assume without loss of generality that
\[ \mu = \nu = s = t = 0. \]
Let \( E_0 \) and \( E_1 \) be two vector bundles over \( M \times N \) and \( P_j \in L^{0,0}(M \times N; E_j, E_j) \),
\( j = 0, 1 \) be two projections. The range spaces
\[ Q^{s,t}(M \times N, E_j; P_j) := P_j(Q^{s,t}(M \times N, E_j)), \quad s \in \mathbb{R}, \]
are closed subspaces of \( Q^s(M \times N, E_j) \). The principal symbols \( \sigma^s_t(P_j) \) and \( \sigma^0_t(P_j) \),
see (3.2) and (3.3), are projections when acting as bundle homomorphisms in
\( \pi^*_M Q^s(N, E_j) \) and \( \pi^*_N Q^s(M, E_j) \), respectively. Thus they determine subbundles
which we shall denote by
\[ Q^s(N, E_j; P_j) \subset \pi^*_M Q^s(N, E_j), \quad Q^s(M, E_j; P_j) \subset \pi^*_N Q^s(M, E_j). \]
Note that these are bundles on \( S^* M \) and \( S^* N \), respectively, that generally do not arise as liftings from bundles over \( M \) and \( N \), respectively.

**Theorem 4.1.** Let \( A \in L^{\mu,\nu}(M \times N; E_0, E_1) \) and \( P_j \) projections as described above. For \( \tilde{A} := P_j A P_0 \) the following assertions are equivalent:

(a) \( \tilde{A} : Q^{s,t}(M \times N, E_0; P_0) \rightarrow Q^{s+\mu,t-\nu}(M \times N, E_1; P_1) \) is a Fredholm operator for some \( s \in \mathbb{R} \).

(b) The following bundle homomorphisms are isomorphisms:
\[ \sigma^s_t(\tilde{A}) : Q^s(N, E_0; P_0) \rightarrow Q^{s-\nu}(N, E_1; P_1), \]
\[ \sigma^0_t(\tilde{A}) : Q^s(M, E_0; P_0) \rightarrow Q^{s-\mu}(M, E_1; P_1). \]

Moreover, the following two assertions are equivalent:

(i) \( \tilde{A} : Q^{s,t}(M \times N, E_0; P_0) \rightarrow Q^{s-\mu,t-\nu}(M \times N, E_1; P_1) \) is invertible for some \( s, t \in \mathbb{R} \).

(ii) There exists a \( B \in L^{\mu,\nu}(M \times N; E_1, E_0) \) such that \( \tilde{A}B = B\tilde{A} = 1 \) for \( B := P_j B P_0 \).

**Proof.** First of all let us justify that we may assume without loss of generality that
\( \mu = \nu = s = t = 0 \). In fact, let \( \Lambda_j^{\sigma,\rho} \in L^{\sigma,\rho}(M \times N; E_j, E_j) \), \( \sigma, \rho \in \mathbb{R} \), be invertible
with \( (\Lambda_j^{\sigma,\rho})^{-1} = \Lambda_j^{-\sigma,-\rho} \). Then the Fredholm property (respectively invertibility)
of \( \tilde{A} \) is equivalent to that of
\[ \tilde{A} := P_j A P_0 : Q^{0,0}(M \times N, E_0; P_0) \rightarrow Q^{0,0}(M \times N, E_1; P_1), \]
where \( A' := \Lambda_1^{s-\mu,t-\nu} A \Lambda_0^{-s,-t} \), \( P'_0 = \Lambda_0^{s,t} P_0 \Lambda_0^{-s,-t} \) and \( P'_1 = \Lambda_1^{s-\mu,t-\nu} P_1 \Lambda_1^{-s,-t} \).

Following [8], let \( G := \{(M \times N; E) \mid E \text{ vector bundle over } M \times N \} \), called the set
of admissible weights, and
\[ L^{\mu}(g) := L^{\mu}(M \times N; E_0, E_1), \quad g = (M \times N; E_0), (M \times N; E_1) \in G \times G \]
\[ H^s(g) := Q^s(M \times N, E), \quad g = (M \times N; E) \in G. \]

\[ \Gamma_j^{\sigma,\rho} = (\Gamma_j^{\sigma,\rho})^{-1} + 1. \]

For \( \sigma, \rho \geq 0 \) let \( \Gamma_j^{\sigma,\rho} \in L^{\sigma/2,\rho/2}(M \times N; E_j, E_j) \) be an arbitrary elliptic operator. Then define
\( \Lambda_j^{\sigma,\rho} = (\Gamma_j^{\sigma,\rho})^{-1} \Gamma_j^{\sigma,\rho} \). For \( \sigma, \rho \geq 0 \) then choose \( \Gamma_j^{\sigma,\rho} = (\Gamma_j^{\sigma,\rho})^{-1} \) and recall the spectral invariance
of bisingular operators, cf. Theorem 3.3.
Then the equivalence of a) and b) is just Theorem 3.12 of [8] (the assumptions are satisfied due to the equivalence of ellipticity and Fredholm property, cf. Theorem 3.3 and Section 3.2), while the equivalence of i) and ii) is Theorem 3.9 of [8]. □

5. Appendix: A remark on vector bundles over product spaces

Let $E$ be a vector bundle over $M \times N$, the product of two smooth compact manifolds. For every $m \in M$ we define an embedding of $N$ into $M \times N$ by

$$\iota_m : N \to M \times N, \quad n \mapsto (m, n)$$

and we denote by $E(m) := \iota_m^* E$ be the corresponding pull-back of $E$ to $N$.

**Proposition 5.1.** For every $m \in M$ exists an open neighborhood $U \subset M$ such that $E|_{U \times N} \cong U \times E(m)$ (diffeomorphism between smooth manifolds).

**Proof.** By Swan’s theorem we may assume that $E$ is a subbundle of $M \times N \times \mathbb{C}^N$ for some $N \in \mathbb{N}$. Hence there exists a function $p \in \mathcal{C}^\infty(M \times N, \mathcal{L}(\mathbb{C}^N))$ taking values in the projections of $\mathbb{C}^N$ and such that

$$E_{(m, n)} = \{(m, n, p(m, n)v) \mid v \in \mathbb{C}^N\}, \quad E(m)_n = \{(n, p(m, n)v) \mid v \in \mathbb{C}^N\}$$

are the fibres of $E$ over $(m, n)$ and of $E(m)$ over $n$, respectively. Now let $m_0 \in M$ be fixed. Define $\varphi \in \mathcal{C}^\infty(M \times N, \mathcal{L}(\mathbb{C}^N))$ by

$$\varphi(m, n) = p(m_0, n) + (1 - p)(m, n).$$

Since $\varphi(m_0, n) = 1$ for every $n$ and since $N$ is compact, we find an open neighborhood $U_0$ of $m_0$ such that $\varphi(m, n) \in \mathcal{L}(\mathbb{C}^N)$ is an isomorphism for every $(m, n) \in U_0 \times N$. In particular, $\varphi$ induces a bundle isomorphism $\Phi$ in $U_0 \times N \times \mathbb{C}^N$. Moreover,

$$\Phi(E_{(m, n)}) = \{m\} \times E(m_0)_n, \quad (m, n) \in U_0 \times N.$$

In fact, since both sides have the same dimension, this follows if the left-hand side is a subset of the right-hand side. However, this is true, since $\varphi(m, n)p(m, n)v = p(m_0, n)p(m, n)v \in \text{im} p(m_0, n)$ for every $v \in \mathbb{C}^N$. In other terms, we have verified that $\Phi : E|_{U_0 \times N} \to U_0 \times E(m_0)$ diffeomorphically. □

**Corollary 5.2.** Let $M$ be connected and $m_0 \in M$ be fixed. Then:

a) $E(m)$ is isomorphic to $E(m_0)$ for every $m \in M$.

b) $E$ is a fibre bundle over $M$ with typical fibre $E(m_0)$.

**Proof.** For a) denote by $V$ the set of all $m \in M$ such that $E(m) \cong E(m_0)$. By Proposition 5.1 both $V$ and $M \setminus V$ are open subsets of $M$. Since $m_0 \in M$ and $M$ is connected, $M \setminus V$ must be empty, hence $V = M$. Clearly, b) follows from a) and Proposition 5.1. □

In the following let $Q^s(N, F)$ denote the standard $L^2$-Sobolev space of order $s$ of sections in the vector bundle $F$ over $N$. This is a separable, infinite dimensional Hilbert space.
Corollary 5.3. Let $m_0 \in M$ be fixed (and $M$ not necessarily connected). Then

$$Q^s(N, E) := \bigcup_{m \in M} \{m\} \times Q^s(N, E(m))$$

is a Hilbert space bundle over $M$ with typical fibre $Q^s(N, E(m_0))$.

Proof. Let $M_0, \ldots, M_k$ be the connected components of $M$ and fix points $m_i \in M_i$. Corollary 5.2 implies that $Q^s(N, E)|_{M_i}$ is a bundle over $M_i$ with typical fibre $Q^s(N, E(m_i))$. It remains to observe that any $Q^s(N, E(m_i))$ is isomorphic to $Q^s(N, E(m_0))$, since all these spaces are isomorphic to $\ell^2(N)$, for example. $\square$

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