Quantum pseudo-telepathy in spin systems: the magic square game under magnetic fields and the Dzyaloshinskii–Moriya interaction

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Received 13 June 2019
Accepted for publication 30 November 2019
Published 7 January 2020

Abstract

The Dzyaloshinskii–Moriya (DM) interaction has been proved to excite entanglement of spin systems, enhancing the ability to realizing various quantum tasks such as teleportation. In this work we consider DM interaction—to the best of our knowledge for the first time—in quantum game theory. We study the winning probability of the magic square game played with the thermally entangled state of spin models under external magnetic fields and DM interaction. We analytically show that although DM interaction excites the entanglement of the system as expected, it surprisingly reduces the winning probability of the game, acting like temperature or magnetic fields, and also show that the effects of DM interaction and an inhomogeneous magnetic field on the winning probability are identical. In addition, we show that the XXZ model is considerably more robust than the XX model against destructive effects. Our results open up new insights for quantum information processing with spin systems.

Keywords: magic square game, quantum game theory, Dzyaloshinskii–Moriya (DM) interaction

(Some figures may appear in colour only in the online journal)

1. Introduction

Game theory has recently attracted intense attention in quantum information science due to its importance in constructing scenarios where quantum mechanical resources, communication and operations outperform their classical counterparts. Meyer showed that a quantum player (i.e. ‘a player who can implement a quantum strategy utilizing quantum superposition’) can beat a classical player with certainty in a coin tossing game [1], and introducing quantization to the famous prisoner’s dilemma game Eisert et al. showed that the presence of entanglement gives rise to superior performance [2]. Following these pioneering works, huge effort has been devoted to elucidating the role of non-classical resources in game theory by considering various games [3–17]. One such interesting game is the so-called magic square game (MSG), which we detail in the next section. The MSG is played by two players against a referee using a pre-shared quantum system of four qubits in a specific pure entangled state; players can win the game with unit probability, while players equipped only with classical resources can achieve a winning probability at most 8/9 [18, 19]. In practical applications, however, due to inevitable interactions of the state with the environment [20–22], or even accelerating players [23], the entanglement of the state may decrease considerably, leading to a decrease of the winning probability far below even the classical limit. On the other hand, Pawela et al. have recently showed that in spite of decoherence, the winning probability can be enhanced if each party performs local operations (determined by semi-definite...
programming) to their qubits [24]. There is also increasing interest in playing games with Heisenberg spin chains [25] and thermal entanglement [26].

In spin systems, the presence of anisotropic antisymmetric exchange between neighboring magnetic spins [27–29] plays an important role in the entanglement dynamics. In antiferromagnetic materials such as α-Fe2O3 crystals, it leads to a weak ferromagnetic behavior. This exchange is historically called the Dzyaloshinskii–Moriya (DM) interaction, and is represented as an additional term \( D \cdot (\vec{r}_i \times \hat{\sigma}^{i+1}_j) \) in the total Hamiltonian of the system. Zhang showed that DM interaction excites the entanglement of a two-qubit Heisenberg chain, and therefore enhances the ability to realize quantum tasks requiring entanglement, such as teleportation [30]. Following this work, the DM interaction has been intensely studied from the perspective of entanglement dynamics of various spin systems, showing that the presence of DM interaction overwhels the destructive effects of temperature and magnetic fields by exciting the entanglement, i.e. increasing the number of entanglement measures such as negativity and concurrence [31–38]. Very recently, the role of DM interaction in quantum metrology has also been studied, showing that it enhances the precision of parameter estimation [39–41]. On the other hand, DM interaction does not always increase entanglement; it can also decrease entanglement as a decoherence factor [42]. This finding makes the problems in quantum tasks with DM interaction and entanglement at their heart, even more interesting: whether DM interaction excites or decreases the entanglement of the system concerned and, in either case, whether it provides any contribution to the performance of the task. To the best of our knowledge, however, DM interaction has not been considered in game theory.

In this paper, considering a thermalized spin system with DM interaction under homogeneous and inhomogeneous external magnetic fields, we study the influence of DM interaction and magnetic fields on the winning probability of the MSG. We show that although DM interaction excites the amount of entanglement of the system it surprisingly does not increase but decreases the winning probability. We also report that although they result in different density matrices and entanglement values, an inhomogeneous magnetic field and DM interaction of the same strength yield the same winning probabilities. This paper is organized as follows. In section 2 we briefly introduce the MSG. In section 3 we present our model for playing the game and in section 4 we present our results and discussions. Finally we conclude in section 5.

### 2. The magic square game

The game is played on a 3 × 3 matrix of binary entries by Alice and Bob against a referee, who gives the number of the row to Alice and the number of the column to Bob. In order to win the game, the sum of the entries of the row given by Alice must be even, the sum the entries of the column given Bob must be odd and the intersection bit must agree. Before starting the game, Alice and Bob can agree on any strategy and can share any physical resources, but once the game starts, i.e. they are spatially separated and the referee gives them the numbers of the row and the column, they cannot communicate or share new resources. Having shared classical resources, the highest winning probability via the best strategy they can achieve is 8/9. However, by sharing quantum mechanical resources, in particular a specific entangled state of four qubits, they can achieve a unit winning probability. This specific state is

\[
|\Psi\rangle = \frac{1}{2}(|0011\rangle - |1100\rangle - |0110\rangle - |1001\rangle),
\]

where the first and second qubits belong to Alice and the third and fourth belong to Bob. When the game starts and the referee gives the number of the row, \( m \), to Alice and the number of the column, \( n \), to Bob, each of them applies one of the unitary operators \( A_m \) and \( B_n \) below to the two qubits they possess:

\[
A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}, \quad A_2 = \frac{1}{2} \begin{bmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{bmatrix}, \quad A_3 = \frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix},
\]

\[
B_1 = \frac{1}{2} \begin{bmatrix} i & -i & 1 & 1 \\ -1 & i & -1 & 1 \\ 1 & -i & 1 & i \\ -i & -i & i & 1 \end{bmatrix}, \quad B_2 = \frac{1}{2} \begin{bmatrix} -1 & i & 1 & i \\ 1 & 1 & -1 & i \\ 1 & -1 & i & -i \\ -1 & -1 & -i & 1 \end{bmatrix}, \quad B_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},
\]

with \( m \) and \( n \) running from 1 to 3. After applying the corresponding operations, each determines the two bits and the third bits are found according to parity conditions. Note that the four-qubit state \( |\Psi\rangle \) is actually the product state of two Bell pairs, i.e. \( \frac{1}{2}(|01\rangle - |10\rangle) \otimes (|01\rangle - |10\rangle) \), such that the first and third qubits belong to Alice and the second and fourth qubits belong to Bob.

For a given arbitrary system described by the density matrix \( \sigma \), the winning probabilities of each row \( m \) and column \( n \) can be calculated as \( P_{m,n}(\sigma) = \text{tr}(\sigma \Sigma_{m,n}|\phi_{m,n}\rangle\langle \phi_{m,n}|) \) where \( \sigma = (A_m \otimes B_n)\sigma(A_m^\dagger \otimes B_n^\dagger) \). Here the states \( |\phi_{m,n}\rangle \) leading to \( P_w = 1 \) for reference are called the success states and they are obtained by Alice and Bob upon implying \( A_m \) and \( B_n \). The winning probability of the given system can be obtained by averaging the winning probabilities over rows and columns, i.e. \( P_w(\sigma) = \frac{1}{2} \Sigma_{m,n} P_{m,n} [20] \).
\[ H = \frac{1}{2} [\sigma_1 \sigma_2^z + \sigma_2 \sigma_1^z + J_2 \sigma_1 \sigma_2^z + D (\sigma_1^z \sigma_2^z - \sigma_1^z \sigma_2^z)] + (B + b) \sigma_1^z + (B - b) \sigma_2^z. \] (3)

where \( B \) and \( b \) are the strengths of the homogeneous and inhomogeneous magnetic fields, respectively, and \( D \) is the strength of the DM interaction which we will choose in the \( z \) direction for simplicity, resulting in the Hamiltonian

\[ H = \frac{1}{2} [\sigma_1 \sigma_2^z + \sigma_2 \sigma_1^z + J_2 \sigma_1 \sigma_2^z + D (\sigma_1^z \sigma_2^z - \sigma_1^z \sigma_2^z)] + (B + b) \sigma_1^z + (B - b) \sigma_2^z. \] (4)

The effective Hamiltonian of four spins is then described as

\[ H_{\text{eff}} = H \otimes I_4 + I_4 \otimes H, \] (5)

where \( I_4 \) is the \( 4 \times 4 \) identity matrix. The normalized density matrix of a thermally entangled system described by \( H_{\text{eff}} \) in the thermal equilibrium can be found as

\[ \rho_{\text{eff}} = e^{-\beta H_{\text{eff}}} / \text{tr}(e^{-\beta H_{\text{eff}}}), \] (6)

where \( \text{tr}(\cdot) \) is the trace function and \( \beta = 1/kT \) with \( k \) the Boltzmann constant (we take \( k = 1 \) for simplicity) and \( T \) is the temperature. Note that the second and the third qubits of the thermally entangled system are swapped to obtain the target system for the MSG, i.e.

\[ \rho_{\text{eff}} \rightarrow \text{SWAP}_{2,3}(\rho_{\text{eff}}). \] (7)

In other words, denoting the system led by \( H \) as \( \rho \), the target system can be obtained as \( \rho_{\text{eff}} = \text{SWAP}_{2,3}(\rho \otimes \rho) \). The non-zero elements of the unnormalized \( \rho \) are found as

\[ \rho_{11} = \gamma^{-1} [\cosh(B T) - \sinh(B T)], \]
\[ \rho_{22} = \gamma [\cosh(\nu T) - b \nu \sinh(\nu T)], \]
\[ \rho_{23} = -\gamma \frac{1 + iD}{\nu} \sinh(\nu T), \]
\[ \rho_{24} = -\gamma \frac{1 - iD}{\nu} \sinh(\nu T), \]
\[ \rho_{33} = \gamma [\cosh(\nu T) + b \nu \sinh(\nu T)], \]
\[ \rho_{44} = \gamma^{-1} [\cosh(B T) + \sinh(B T)]. \] (8)

where \( \gamma = \text{Exp}(\frac{B T}{\beta}), \nu = \sqrt{1 + b^2 + D^2} \) and the trace function yields

\[ \text{tr}(\rho_{\text{eff}}) = 2 [\gamma^{-1} \cosh(\frac{B T}{\beta}) + \gamma \cosh(\frac{\nu T}{\beta})]. \] (9)

It is easy to see that for \( D = B = b = 0 \) and \( T \) approaches zero, \( H \) leads to \( |\{01\} - |10\} \) and \( H_{\text{eff}} \) leads to \( \frac{1}{2} ((|0011\} + |1100\} - |0110\} - |1001\}) \), i.e. the success state (after the swap operation). Now we are ready to analyze the effects of thermalization, DM interaction and magnetic fields, as well as the coupling constant in the \( z \) direction, on the probability of success in the MSG. Note that throughout the paper we take \( k = 1 \) and make the calculations and plot the figures in units of the Boltzmann constant \( k \).

4. Results and discussions

Following the method explained in section 2, we analytically calculate the winning probability \( P_w \) for \( \rho_{\text{eff}} \). For simplicity,
Figure 2. Probability of winning the magic square game, $P_w$, with respect to Dzyaloshinskii–Moriya (DM) interaction, $D$, at various temperatures from $T = 0.01$ to $T = 1$, in units of the Boltzmann constant $k$. DM interaction counter-intuitively does not increase but decreases the performance of the quantum task. However, since $P_w = 8/9$ is the classical limit, for low temperatures, quantum resources outperform classical resources even with a DM interaction of strength $D \leq 0.75$. Since the effects of $D$ and external magnetic field, $b$, on $P_w$ are identical, the same plot applies to $b$ as well.

Figure 3. Entanglement of the thermally entangled state of our system in terms of logarithmic negativity, with respect to Dzyaloshinskii–Moriya (DM) interaction $D$ and external inhomogeneous magnetic field $b$ at various temperatures. Although the amount of entanglement greatly depends on the temperature for no or low $D$ or $b$, it becomes independent of the temperature as $D$ or $b$ increases.

Figure 4. Probability of winning the magic square game, $P_w$, with respect to temperature $T$ (blue), Dzyaloshinskii–Moriya (DM) interaction $D$ (black) and an external homogeneous magnetic field $B$ (red) with $J_z = 0$ (solid), $J_z = 0$ (dashed) and $J_z = 0$ (dotted), in units of the Boltzmann constant $k$. Although $P_w$ is not affected by $J_z$ for low temperature and zero $B$, it is easy to check that for high temperature or non-zero $B$, larger $J_z$ makes $P_w$ more robust against $D$ as well.
we first choose the XX model, i.e. $J_z = 0$. In order to analyze the effect of a homogeneous magnetic field $B$ and temperature $T$ on $P_w$ we do not consider $D$ and $b$, find

$$P_w(\rho_{D=b=0}) = \frac{1}{\cosh(\frac{\nu}{T}) + \cosh(\frac{\nu}{T})^2} \left\{ 0.0694 \cosh(\frac{4B}{T}) + \cosh(\frac{3B}{T}) \left[ 0.5 \cosh(\frac{1}{T}) - 0.0555 \sinh(\frac{1}{T}) \right] \right\}$$

$$+ \cosh(\frac{2B}{T}) \left[ -0.0555 \sinh(\frac{2}{T}) + 0.7777 \cosh(\frac{2}{T}) + 0.9444 \right] + \cosh(\frac{B}{T}) \left[ -0.1111 \sinh(\frac{1}{T}) + 0.0555 \sinh(\frac{3}{T}) \right]$$

$$+ 2.8888 \cosh(\frac{1}{T}) + 0.6111 \cosh(\frac{3}{T})] + 0.0277 \sinh(\frac{4}{T}) + 1.0555 \cosh(\frac{2}{T}) + 0.0972 \cosh(\frac{4}{T}) + 1.05556 \right\}$$

(10)

As shown in figure 1, we find that as temperature decreases, $P_w$ becomes more robust against an increasing homogeneous magnetic field $B$ and at the same time it exhibits a sudden change at $B = 1$. For clearer observation of this sudden change we focus on the region around $B = 1$ in the inset of the figure. Note that this critical point shifts to $B = 1.5$ and to $B = 2$ for $J_z = 0.5$ and for $J_z = 1$, respectively, as shown in figure 4 (red curves).

For a fixed $B$, although $D$ and $b$ lead to different Hamiltonians, density matrices and entanglement values, they yield the same game-winning probability, i.e.

$$P_w(\rho_{D=b=0}) = \frac{\text{sech}^2(\frac{\nu}{T})}{\nu^3/2[\cosh(\frac{\nu}{T})] + 1} \left\{ -0.0833 \nu^2 \sinh(\frac{\nu}{T}) \right.$$\n
$$+ 0.0277 \nu^2 \sinh(\frac{3\nu}{T}) + (0.0694 \nu^2 + 0.0278) \nu \cosh(\frac{3\nu}{T}) \right\}$$

$$+ (0.9305 \nu^2 - 0.0278) \nu \cosh(\frac{\nu}{T}) + (0.3611 \nu^2 + 0.0555) \nu \cosh(\frac{2\nu}{T}) + (0.6388 \nu^2 - 0.055) \nu \right\}$$

(11)

with $\nu = \sqrt{1 + b^2 + D^2}$. Therefore for a fixed material, i.e. a fixed DM interaction strength, the desired effect due to DM interaction can be achieved by applying an external inhomogeneous magnetic field of the same strength (or, vice versa, to realize a task with a desired strength of external inhomogeneous magnetic field, it may be possible to choose a material with the same strength of DM interaction). We plot the dependence of $P_w$ on $D$ (or the same, $b$) for various temperatures in figure 2.

Here, although DM interaction excites entanglement of the system it decreases the overall success of the task, in contrast to the usual cases where DM interaction excites both the amount of entanglement of the system and the performance of the task [30]. The physical explanation for this result is that the performance of realizing a quantum task depends not only on the amount of entanglement but also on the type of entanglement. In [30] and the following works on the influences of DM interaction, systems of two particles were usually studied and just the amount of entanglement is sufficient to determine the performance of quantum teleportation or violation of Bell’s inequality, for instance, leaving room to overlook controversial cases such as the one presented herein. However, when a system of four particles is considered with a more complex procedure, some of the specific elements of the density matrix of the system become more significant in determining the performance. That is why an increase in the amount of entanglement of the system alone does not imply an increase in performance of the task.

In order to clarify that $D$ and $b$ affect the entanglement of the system in opposite ways we plot the logarithmic negativity [43] of the system with respect to $D$ and $b$ in figure 3. At each temperature value, $D$ increases but $b$ decreases the entanglement of the system, and as the strength of $D$ or $b$ increases dependence of $P_w$ on $T$ decreases.

Finally, we analyze the effect of $J_z$ on the winning probability for three instances, $J_z = 0$, $J_z = 0.5$ and $J_z = 1$ with respect to $T$, $D$ and $B$, setting other effects to zero for simplicity. We find that for larger $J_z$, $P_w$ becomes considerably more robust against $B$ and more robust against $T$ but is the same against $D$, as shown in figure 4. However, it is straightforward to see that in the case of non-zero $B$ or $T$, a larger $J_z$ makes $P_w$ more robust against $D$ as well.

5. Conclusion

In conclusion, we have studied the influence of DM interaction in a quantum game for the first time, to the best of our knowledge. We constructed a model of four spins including external homogeneous and inhomogeneous magnetic fields and DM interaction to serve as the quantum resource for playing the MSG. Analytically obtaining $P_w$, the probability of winning the game with the thermally entangled state of the spin system, we show how it decreases with respect to increasing temperature and external magnetic field. Exciting the entanglement of the system, although DM interaction excites the performance of realizing tasks in general, here we showed that DM interaction decreases the probability of winning the game. We also showed that the effects of DM interaction and inhomogeneous magnetic fields on the winning probability are identical. In addition, we showed that an XXZ model is more robust than an XX model in playing the MSG. We believe that our work could be useful in quantum information processing with spin systems.
Acknowledgments

This work has been funded by Isik University Scientific Research Funding Agency under grant no. BAP 15B103. The author thanks I Karakurt and A A Altintas for fruitful discussions.

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