Improved readout of qubit-coupled Gottesman–Kitaev–Preskill states

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Abstract

The Gottesman–Kitaev–Preskill encoding of a qubit in a harmonic oscillator is a promising building block towards fault-tolerant quantum computation. Recently, this encoding was experimentally demonstrated for the first time in trapped-ion and superconducting circuit systems. However, these systems lack some of the Gaussian operations which are critical to efficiently manipulate the encoded qubits. In particular, homodyne detection, which is the go-to method for efficient readout of the encoded qubit in the vast majority of theoretical work, is not readily available, heavily limiting the readout fidelity. Here, we present an alternative read-out strategy designed for qubit-coupled systems. Our method can improve the readout fidelity with several orders of magnitude for such systems and, surprisingly, even surpass the fidelity of homodyne detection in the low squeezing regime.

1. Introduction

Scalable fault-tolerant quantum computation requires physical qubits which can be stored, manipulated and read-out with very high fidelity. One promising approach for realising such high quality qubits, which has received an increasing attention in recent years, is to encode each qubit into the bosonic mode of a quantum harmonic oscillator, such as the motion of a trapped particle, or a microwave or optical field mode. There are several proposals for such encodings [1–10], including the Gottesman–Kitaev–Preskill (GKP) code [1–5] which has many advantageous properties. In particular, with GKP states, a universal set of operations can be performed using solely Gaussian resources [11, 12], which are generally considered easy and efficient to implement, particularly in optics. Additionally, GKP states can be combined with continuous variable cluster states [13, 14] or the surface code [15, 16] to achieve fault-tolerance. Furthermore, the GKP code has been shown to outperform other encoding schemes in terms of its efficiency in correcting against loss [17, 18], which is the main noise factor in many bosonic systems.

These favourable features have sparked numerous new studies on applying GKP states for optical quantum computing [2, 19, 20]. Still, the generation of GKP states in the optical regime has proven extremely challenging and has so far not been demonstrated experimentally, despite several theoretical proposals [21–25]. However, recently GKP states were generated for the first time in the motional state of a trapped ion [4] and in a microwave cavity field coupled to a superconducting circuit [5]. These experiments were made possible by the strong coupling between a bosonic mode and an ancillary qubit, enabling non-Gaussian transformations of the bosonic mode, which is required to produce GKP states. Yet, these experimental platforms lack some of the crucial Gaussian operations that are required for complete manipulation, stabilization and read-out of the encoded GKP qubit [1]. Therefore, new methods specifically designed to qubit-coupled systems are required to take full advantage of the GKP encoding in these systems. For example, stabilization has already been demonstrated using the qubit-coupling [5], but the lack of homodyne detection, i.e. direct measurement of the bosonic quadrature operators, severely limits the read-out fidelity [3].
Here we propose an improved readout scheme for qubit-coupled GKP states. Our method relies on mapping the logical information of the GKP qubit onto the ancilla qubit state. This is similar to the known method based on phase-estimation [26], but by adding an additional interaction between the qubit and the oscillator, we achieve much higher read-out fidelity. In particular, for a GKP state composed of peaks of width $\Delta$, the read-out error of our scheme scales as $\Delta^6$ whereas the previously used method based on qubit-coupling scales as $\Delta^3$. Thus, as an example, for a 10 dB squeezed GKP state our method improves the read-out fidelity from 96.22% with known techniques to 99.98%.

2. Preliminaries

We consider GKP states which are encoded into a bosonic mode with quadrature operators $\hat{q}$ and $\hat{p}$ satisfying $[\hat{q}, \hat{p}] = i$. The code states of the square GKP encoding are defined in the common $+1$ eigenspace of the commuting displacement operators $D(\sqrt{2\pi})$ and $D(i\sqrt{2\pi})$, where $D(\alpha) = e^{\frac{\alpha^2}{4}}e^{i\alpha\hat{q}}$. The computational basis states are then defined as the $\pm 1$ eigenstates of $D(i\sqrt{\pi/2})$, which acts as a logical Pauli $Z$ operator. However, ideal GKP states are unphysical, as they require infinite energy. Instead, the physically relevant basis states are thus only approximate eigenstates of the logical Pauli $Z$ operator, i.e. $\langle D(i\sqrt{\pi/2}) \rangle \approx \pm 1$. There are multiple ways of expressing such approximate states. In this work we consider the commonly used expression for which the basis states consist of a superposition of multiple squeezed states of width $\Delta$, under a Gaussian envelope of width $\kappa$:

$$\begin{align}
|\vec{0}\rangle &\propto \sum_{s\in Z} e^{-\left(\frac{\sqrt{2\pi}}{2}\right)^2 \frac{\kappa^2}{4}} \hat{D}\left(\sqrt{\frac{\pi}{2}} 2s\right) \hat{S}_\Delta |\text{vac}\rangle, \\
|\vec{1}\rangle &\propto \sum_{s\in Z} e^{-\left(\frac{\sqrt{2\pi}}{2}\right)^2 \frac{\kappa^2}{4}} \hat{D}\left(\sqrt{\frac{\pi}{2}} 2(s+1)\right) \hat{S}_\Delta |\text{vac}\rangle,
\end{align}$$

where $|\text{vac}\rangle$ is the vacuum state and $\hat{S}_\Delta = e^{-\frac{1}{4} \ln(\Delta)} (\hat{q}^2 + \hat{p}^2)$ is the squeezing operator. These approximate code states approach the ideal states for $(\Delta, \kappa^{-1}) \to 0$. The amount of squeezing is often expressed in dB as $\Delta_{\text{dB}} = -10 \log_{10}(\Delta^2)$. Figure 1 shows example Wigner functions and quadrature distributions of the GKP $|\vec{0}\rangle$ states with 9 and 15 dB squeezing respectively. For comparison, experimentally produced states have so far demonstrated measured stabilization expectation values of $\langle \hat{D}(\sqrt{2\pi}) \rangle = 0.56$ and $\langle \hat{D}(i\sqrt{2\pi}) \rangle = 0.41$ corresponding to effective squeezing levels [27] of 5.5–7.3 dB in a trapped ion [4] and $\langle \hat{D}(\sqrt{2\pi}) \rangle = 0.62$ and $\langle \hat{D}(i\sqrt{2\pi}) \rangle = 0.5$ corresponding to 6.6–8.2 dB effective squeezing in a microwave cavity [5]. Note that since the measurement of the stabilizers in these experiments are prone to imperfections during the measurement process, the actual squeezing levels could be slightly higher and care should be taken when directly comparing the different platforms. In principle, the protocols used in those experiments can produce arbitrarily large amounts of squeezing, but in practice the performance is limited by noise, such as bosonic dephasing and losses as well as errors introduced via qubit couplings [4, 5].

It is common to consider the symmetric case where $\Delta = \kappa^{-1}$, but in this paper only $\Delta$ is relevant, as we consider only read-out in the computational, i.e., Pauli $Z$ basis. If instead one wishes to read out the GKP qubit in the Pauli $X$ basis, this is done by changing $\hat{q} \to \hat{p}$ and $\hat{p} \to -\hat{q}$ in the following operations in which case $\kappa^{-1}$ becomes the relevant quantity. While we initially consider the pure states of (1) we will later turn to more realistic mixed states.

**Figure 1.** Wigner functions and quadrature distributions of the approximate GKP states given by equation (1a) for symmetric $\Delta = \kappa^{-1}$ with 9 and 15 dB squeezing respectively. The dashed lines in the quadrature distributions show the vacuum quadrature distribution (not normalized) for comparison.
We now consider the problem of how to reliably distinguish between the states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ in a physically relevant setting. In particular, we wish to minimize the measurement error probability

$$p_{err} = \frac{1}{2}(p(1|0) + p(0|1)), \quad (2)$$

where $p(x|y)$ is the probability of obtaining measurement outcome $x$ given the input state $y$. Since the approximate states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are not orthogonal, this problem is ultimately bounded by the Helstrom bound [28]:

$$p_{err} \geq p_{err, Helstrom} = \frac{1}{2} \left(1 - \sqrt{1 - |\langle \bar{0} | \bar{1} \rangle|^2}\right). \quad (3)$$

The Helstrom bound drops very rapidly with decreasing $\Delta$, but is generally not achievable in a realistic setting. Instead, homodyne detection is often considered as a practical and efficient read-out method. With this method, the state is measured in the bosonic $\hat{q}$-basis, and the results closer to even multiples of $\sqrt{\pi}$ are considered a 0 while results closer to an odd multiple of $\sqrt{\pi}$ are considered a 1. The measurement error probability for homodyne detection, assuming a negligible overlap between neighbouring squeezed states of the basis states, i.e. $|\langle \text{vac} | \hat{S}_\Delta \hat{D}(\sqrt{2\pi}) \hat{S}_\Delta | \text{vac} \rangle| \approx 0$, is given by:

$$p_{err, \text{homodyne}} = \text{erfc} \left(\frac{\sqrt{\pi}}{2\Delta}\right) \approx \frac{2}{\pi} \Delta e^{-\frac{\pi}{4\Delta^2}}, \quad (4)$$

where the second approximation follows from a series expansion of the complementary error function. The exponential term in equation (4) causes the measurement error probability to drop rapidly with decreasing $\Delta$, i.e. homodyne detection is very efficient for highly squeezed states.

However, while homodyne detection can be efficiently implemented in free-space optics, it is less practical for microwave cavities or trapped ions. Instead, these systems can couple to an ancilla qubit, e.g. a superconducting transmon qubit for the microwave platform or an internal spin state for the trapped ions, and the state of the ancilla qubit can subsequently be measured. In particular, it is possible to realise a Rabi-type interaction Hamiltonian, $\hat{q}\sigma_x$, where $\sigma_x$ is the Pauli X operator of the qubit [5, 29]. The action of this Hamiltonian is sometimes referred to as a conditional displacement, as the bosonic mode gets displaced in a direction depending on the state of the qubit, entangling the qubit and the oscillator. Such interaction can be used to read-out a GKP-qubit using the following simple circuit [3–5]:

$$|\psi\rangle_{\text{GKP}} \quad \text{bosonic mode} \quad |0\rangle \quad U_x \left(i\frac{\sqrt{\pi}}{2}\right) \quad \ldots \quad (5)$$

where

$$U_k(\beta) = \exp \left[i(\text{Re}[\beta]\hat{p} + \text{Im}[\beta]\hat{q})\right] \quad (6)$$

for $k \in \{x,y,z\}$. The expected measurement outcome of the qubit is $\text{Prob}(|1\rangle) = \frac{1}{2} \left(1 - \text{Re} \langle \hat{D} \left(i\frac{\sqrt{\pi}}{2}\right)\rangle\right)$. For ideal GKP basis states for which $\langle \hat{D} \left(i\frac{\sqrt{\pi}}{2}\right)\rangle = \pm 1$ we achieve a perfect read-out. For the approximate states $|\bar{0}\rangle$ and $|\bar{1}\rangle$, with negligible overlap between neighbouring squeezed states, we have $\langle \hat{D} \left(i\frac{\sqrt{\pi}}{2}\right)\rangle = e^{-\frac{\pi}{4}\Delta^2}$, so the measurement error probability is:

$$p_{err, \text{simple}} = \frac{1}{2} \left(1 - e^{-\frac{\pi}{8}\Delta^2}\right) \approx \frac{\pi}{8} \Delta^2. \quad (7)$$

This scaling is significantly worse than the homodyne strategy of equation (4). The scaling can be improved by running the circuit multiple times and considering a majority vote, but because of the measurement back-action this strategy has diminishing returns. Additionally, multiple runs of the circuit results in an increased total measurement time during which the state accumulates noise.

### 3. Protocol

In this work we propose to modify the circuit in (5), adding an additional Rabi-type interaction of the type $\hat{p}\sigma_y$ with interaction strength $\lambda$:

$$|\psi\rangle_{\text{GKP}} \quad \text{bosonic mode} \quad |0\rangle \quad U_y (-\lambda) \quad U_x \left(i\frac{\sqrt{\pi}}{2}\right) \quad \ldots \quad (8)$$
For $|\lambda| \ll 1$, the measurement error probability of this circuit is given by (derivation given in appendix A):

$$P_{err, improved} = \frac{1}{2} \left( 1 - e^{-\frac{2}{\pi} \lambda^2} \left( e^{-\frac{2}{\pi} \lambda^2} + \sin(\sqrt{\pi}\lambda) \right) \right),$$

(9)

which reduces to that of equation (7) for $\lambda = 0$ as expected. However, for $\lambda \neq 0$ it is possible to achieve a better scaling. The minimum is achieved for $\lambda$ satisfying $\frac{2}{\pi} \lambda^2 = \sqrt{\pi} \cos(\sqrt{\pi}\lambda)$, which for small $\Delta$ is approximately at $\lambda = \sqrt{\pi} \Delta^2 / 2$. Inserting this into equation (9) and expanding to lowest order in $\Delta$ we get:

$$P_{err, improved} \approx \frac{5\pi^3}{384} \Delta^6,$$

(10)

i.e. a significantly better scaling than (7). The measurement error probabilities of the different methods are compared in figure 2(a). The blue curve shows the result of circuit (8), with the optimum $\lambda$ chosen for each point. We see a clear improvement over the simple circuit in (5), i.e. for $\lambda = 0$, even when using multiple runs of the simple circuit. For a squeezing of less than 9 dB the modified circuit even outperforms homodyne detection. We found that using circuit (8) we could not further improve the performance using multiple rounds and majority voting. This is because the measurement back-action upon getting the wrong measurement heavily modifies the input state, making subsequent measurement rounds useless (see appendix B for details). One important thing to note is, that the optimum interaction parameter, $\lambda$, depends on the quality, or $\Delta$, of the input GKP state. This is different from the homodyne measurement strategy or the simple circuit, both of which are constructed independently on the quality of the input state. Therefore, it is important to calibrate the modified measurement circuit, i.e. tuning $\lambda$, according to the squeezing of the input state. Figure 2(b) shows the performance when fixing $\lambda$ at different values. For large amounts of squeezing we see that the circuit performs optimally only for input states in a narrow region. In a practical setting it might be difficult to consistently fix the squeezing level of the state to be measured, as it could depend on previous operations of the state. Therefore, the average measurement error probability will likely be higher than what is predicted by equation (9). However, from figure 2(b) we see that the results are generally improved compared to the simple circuit for a wide range of $\Delta$.

So far we have considered only the states of equation (1). However, these states might not necessarily be physically realistic as, for example, they are pure. Instead, we can construct more general mixed GKP states by applying a Gaussian displacement channel of strength $\sigma$ to the pure states of equation (1):

$$\rho_\mu = \frac{1}{\pi\sigma} \int d^2 \alpha e^{-\frac{1}{\pi^2} \tilde{D}(\alpha) [\mu] [\mu] \tilde{D}(\alpha)},$$

(11)

where $\rho_\mu$ is the density matrix of the output state and $\mu \in \{0, 1\}$. The performance of the simple circuit (5) does not depend on the exact form of the input state but only on the expectation value $\langle \tilde{D}(i\sqrt{\frac{\pi}{2}}) \rangle$. In fact, one can use a similar expectation value to define an effective squeezing parameter $\Delta_{eff}$ as [27]:

$$\Delta_{eff} = \sqrt{\frac{1}{2\pi} \ln\left(\frac{1}{\langle |\tilde{D}(i\sqrt{\frac{\pi}{2}})|^2\rangle}\right)},$$

(12)

allowing us to describe the amount of squeezing in an arbitrary state. For the states of equation (1) we simply have $\Delta_{eff} = \Delta$. For the mixed state of equation (11) we have $\Delta_{eff} = \sqrt{\Delta^2 + 2\sigma^2}$. By tuning $\Delta$ and $\sigma$ we can thus now construct GKP states of arbitrary purity, $P = \text{Tr}(\rho^2)$, and effective squeezing. Figure 2(c) shows the performance of the circuit for states of different purity. We see that the performance degrades for mixed states, although we still obtain superior behaviour compared to the simple circuit. This performance degradation can be understood by the fact that the circuit (8) is not robust against shifts of the GKP state by an integer multiples of the GKP lattice spacing along the $p$-axis, i.e. shifts of the type $\tilde{D}(i\sqrt{\frac{\pi}{2n}})$ for integer $n$. Usually, such shifts leaves the logical information of the GKP state intact, and indeed they do not affect the measurement result using homodyne detection or the simple circuit in (5). However, for the circuit in (8) we have $U_\lambda(-\lambda)\tilde{D}(i\sqrt{\frac{\pi}{2n}})U_\lambda(-\lambda) = e^{2i\pi\lambda^2/\sqrt{\pi}} D(\sqrt{\pi}\lambda)U_\lambda(-\lambda)$. Thus a displacement of one or more lattice spacings along the $p$-axis before the measurement circuit induces an unwanted qubit rotation around the $y$-axis. The mixed state can to some degree be interpreted as a pure approximate GKP state which has experienced an unknown integer lattice displacement. This therefore results in an unknown rotation of the qubit which ultimately increases the error rate of our circuit. In the literature, GKP states are commonly only quantified in terms of their squeezing level, with the purity being less relevant as it plays no role for e.g. homodyne detection. It is therefore unclear what levels can be expected in experimental setting, which will also likely vary between platforms. Note that the mixed states were constructed in one particular way in
Figure 2. (a) Measurement error probability, $p_{\text{err}}$, for various measurement strategies. The red $\lambda = 0$ lines correspond to circuit (5), while the blue line corresponds to circuit (8) with the interaction parameter $\lambda$ chosen to minimize $p_{\text{err}}$. (b) Performance for fixed $\lambda$ as a function of the input squeezing. For large amounts of squeezing the optimal performance is only achieved in a narrow range, requiring good knowledge of the input state. (c) Performance for mixed states generated by applying the Gaussian displacement channel, equation (11), to the pure input states of equation (1). For such states, the purity, $P$, heavily impacts the performance of the protocol, although the performance is always improved compared to the simple circuit.

4. Conclusion

We have presented a protocol for efficient read-out of a GKP state in a qubit-coupled oscillator. Our protocol reduces the measurement error rate from a $\Delta^2$-scaling with previously known methods to a $\Delta^6$-scaling, enabling low error rates in the absence of homodyne detection. Our protocol is sensitive to the exact form of the input state, with a reduced performance for mixed states. However, our results demonstrate that homodyne detection might not be crucial to efficiently utilize the GKP encoding, e.g. in microwave cavities or trapped ions.

Note added: after preparing this paper, we became aware of a parallel work by de Neeve et al [30] which experimentally implements the measurement protocol presented here on a trapped-ion platform. Furthermore, they provide a physical interpretation of the improvement obtained by the added interaction. Additionally, we became aware of work by Royer et al [31], which considers similar alternating Rabi-type interactions to autonomously generate and stabilize approximate GKP states. Those works thus complement the analysis presented in this paper.

This paper, i.e. by combining equations (1) and (11). The purity alone might therefore not accurately describe performance of the protocol for other states. Still, the result of figure 2(c) indicates that high quality states with features beyond just the squeezing are required to take full advantage of the improved measurement scheme.
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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix A. Improved measurement error probability**

Here we derive equation (9) of the main text. To do so we calculate the probability of measuring the qubit in state 0 after the application of circuit (8) with input GKP state \(|0\rangle\) or \(|1\rangle\). This probability is given by the norm of the (not normalized) post-measurement state, \(|\langle \psi_0 | \psi \rangle|\), where \(|\psi_0\rangle = |0\rangle U_x \left( i \frac{\lambda}{\sqrt{2}} \right) U_y (-\lambda) |0\rangle\) with input GKP state \(|\psi\rangle\) given by equation (1). First we compute the operation \(|0\rangle U_x \left( i \frac{\lambda}{\sqrt{2}} \right) U_y (-\lambda) |0\rangle\). For this it convenient to write the \(U\) gates as:

\[
U_x \left( i \frac{\lambda}{\sqrt{2}} \right) = D_1 \otimes |+\rangle \langle +| + D_1^\dagger \otimes |-\rangle \langle -|,
\]

where \(D_1 = D(-\lambda/\sqrt{2})\) and \(D_2 = D(i \sqrt{\lambda/2})\), and \(|\pm\rangle = (|0\rangle \pm i |1\rangle)/\sqrt{2}\) are the Pauli-\(y\) and Pauli-\(x\) eigenstates respectively. Thus \(|0\rangle = (|i\rangle - |\bar{i}\rangle)/\sqrt{2}\).

From the first gate we get:

\[
U_x(-\lambda) |0\rangle = \frac{1}{\sqrt{2}} \left( D_1 |\bar{i}\rangle + D_1^\dagger |-\rangle \right).
\]

(15)

Transforming the qubit states into the \(x\)-basis through \(|\bar{i}\rangle = e^{i\pi/4}(|+\rangle - |i\rangle)/\sqrt{2}\) and \(|-\rangle = e^{i\pi/4}(-|+\rangle - |\bar{i}\rangle)/\sqrt{2}\), we get:

\[
e^{i\pi/4} \left[ \left(D_1 - i D_1^\dagger\right) |+\rangle + \left(D_1^\dagger - iD_1\right) |\bar{i}\rangle \right].
\]

(16)

Applying the second gate:

\[
U_y \left( i \frac{\lambda}{\sqrt{2}} \right) U_x(-\lambda) |0\rangle = \frac{e^{i\pi/4}}{2\sqrt{2}} \left[ D_2 \left(D_1 - i D_1^\dagger\right) |+\rangle + D_2^\dagger \left(D_1^\dagger - iD_1\right) |\bar{i}\rangle \right].
\]

(17)

After measuring the qubit in state \(|0\rangle\):

\[
\langle 0 | U_x \left( \frac{\lambda}{\sqrt{2}} \right) U_y (-\lambda) |0\rangle = \frac{e^{i\pi/4}}{2\sqrt{2}} \left[ D_2 D_1 - i D_2 D_1^\dagger - i D_2^\dagger D_1 + D_2^\dagger D_1^\dagger \right],
\]

(18)

thus

\[
|\psi_0\rangle = \frac{e^{i\pi/4}}{2\sqrt{2}} \left[ D_2 D_1 - i D_2 D_1^\dagger - i D_2^\dagger D_1 + D_2^\dagger D_1^\dagger \right]|\psi\rangle.
\]

(19)

We now calculate \(p(0|0)\) by inserting the expression for the approximate GKP 0 state, \(|\tilde{0}\rangle\) (equation (1)), and calculating the norm. Here we assume that the circuit do not cause an overlap between distinct peaks of the GKP state, which is valid for small \(\lambda\). Fixing \(s\) in equation (1), the post-measurement norm of a single peak is then:

\[
\frac{1}{8} |\langle \text{vac} | \tilde{S}_\Delta^\dagger \tilde{D} \left( \sqrt{\frac{\pi}{2}} \right)^s \left[ D_2 D_1 - i D_2 D_1^\dagger - i D_2^\dagger D_1 + D_2^\dagger D_1^\dagger \right] \dagger | \tilde{D} \left( \sqrt{\frac{\pi}{2}} \right)^s \tilde{S}_\Delta | \text{vac} \rangle|.
\]

(20)

This expression consists of 16 terms, which can be evaluated using the relations \(\langle \text{vac} | \tilde{D}(\alpha) | \text{vac} \rangle = e^{-|\alpha|^2/2}\), \(\tilde{D}(\beta) \tilde{D}(\alpha) = e^{|\alpha\beta|} \tilde{D}(\alpha + \beta)\) and \(\tilde{S}_\Delta \tilde{D}(\alpha) \tilde{S}_\Delta = \tilde{D}(\Re(\alpha)/\Delta + i \Im(\alpha)/\Delta)\). The 16 terms are:
\[
\langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = 1
\]

(21)

\[
\langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = e^{-\frac{\Delta^2}{2}}
\]

(22)

\[
\langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = e^{-\frac{\Delta^2}{2}} e^{-i\sqrt{\pi s} e^{-\frac{\Delta^2}{2}}}
\]

(23)

\[
\langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = \langle \text{vac} | \hat{S}_\Delta^\dagger \hat{D} \left( \sqrt{\frac{\pi}{2} s} \right)^\dagger \hat{D}_1 \hat{D}_2 \hat{D}_3 \hat{D}_4 \left( \sqrt{\frac{\pi}{2} s} \right) \hat{S}_\Delta | \text{vac} \rangle = e^{-i\sqrt{\pi s} e^{-\frac{\Delta^2}{2}}}
\]

(24)

Inserting these into equation (20) we find that each peak contributes with:

\[
\begin{align*}
\frac{1}{2} \left[ 1 + e^{-\frac{\Delta^2}{2}} \cos(2\pi s) \left( e^{-\frac{\Delta^2}{2}} + \sin(\sqrt{\pi s}) \right) \right].
\end{align*}
\]

(26)

Since \( s \) is an integer, i.e. \( \cos(2\pi s) = 1 \), all peaks contribute equally (apart from their initial weighting from the broad Gaussian envelope), and the measurement error probability for the GKP 0 state is thus

\[
p(1|0) = 1 - p(0|0) = \frac{1}{2} \left[ 1 - e^{-\frac{\Delta^2}{2}} \left( e^{-\frac{\Delta^2}{2}} + \sin(\sqrt{\pi s}) \right) \right].
\]

(27)

Similarly, we can calculate the probability \( p(0|1) \) by considering the input state \( |\bar{1}\rangle \) for which the result can be obtained directly from equation (26) by changing \( s \to s + \frac{1}{2} \). In this case \( \cos(2\pi s) = -1 \) and we get

\[
p(0|1) = p(1|0).
\]

(28)
Thus in total the measurement error probability is
\[
p_{\text{err}} = \frac{1}{2} (p(0|1) + p(1|0)) = p(0|1) = \frac{1}{2} \left[ 1 - e^{\frac{2}{\sqrt{\pi}} \Delta^2} \left( e^{\frac{\Delta^2}{\epsilon^2}} + \sin(\sqrt{\pi} \lambda) \right) \right]. \tag{29}
\]

**Appendix B. Post-measurement state**

Since we do not directly measure the bosonic mode, one could try to repeat the measurement circuit to gain more accurate information on the input state. However, the qubit measurement also projects the bosonic mode into a post-measurement state, which is not necessarily suitable for following operations or measurements. In figure 3 we show the Wigner functions and quadrature distributions of the post-measurement states for a 10 dB input GKP \( |0\rangle \) state. Figure 3(b) shows the post-measurement states for the qubit results \(|0\rangle\) and \(|1\rangle\) using the simple circuit in (5), while figure 3(c) shows the post-measurement states when using the improved circuit in (8). In both cases observation of the wrong qubit measurement outcome, i.e. observing \(|1\rangle\) with input state \(|0\rangle\), heavily distorts the input GKP state. For this reason, subsequent measurements yield less reliable results, which limits the effect of multiple measurements on the same state. For the improved circuit the post-measurement state is even more distorted compared to the simple circuit. Thus we found that multiple runs of the improved circuit does not further reduce the measurement error.

The repeated measurement of the simple circuit can also be considered as an instance of the non-adaptive phase estimation protocol to prepare approximate GKP states presented in [26]. In that work it is shown that some unlikely measurement outcomes can lead to erroneous states, just like the distorted
state in figure 3(b). Furthermore, it was shown that the probability of obtaining such poor states could be lowered by applying an adaptive measurement sequence. It might therefore be possible to further lower the error rate by applying an adaptive measurement strategy through repeated use of the improved circuit, but with settings based on previous qubit measurements. We leave this an interesting open direction for further research.

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