Inequalities between Partial Domination and Independent Partial Domination in Graphs

Odile Favaron and Pawaton Kaemawichanurat

1Theoretical and Computational Science Center and Department of Mathematics, Faculty of Science, King Mongkut’s University of Technology Thonburi, Bangkok, Thailand
Email: odile.favaron@lri.fr; pawaton.kae@kmutt.ac.th

Abstract

For a graph $G$, a vertex subset $S \subseteq V(G)$ is said to be $K_k$-isolating if $G - N_G[S]$ does not contain $K_k$ as a subgraph. The $K_k$-isolation number of $G$, denoted by $\iota_k(G)$, is the minimum cardinality of a $K_k$-isolating set of $G$. Analogously, $S$ is said to be independent $K_k$-isolating if $S$ is a $K_k$-isolating set of $G$ and $G[S]$ has no edge. The independent $K_k$-isolation number of $G$, denoted by $\iota'_k(G)$, is the minimum cardinality of an independent $K_k$-isolating set of $G$. A vertex subset $D \subseteq V(G)$ is said to be dominating if $V(G) \setminus N_G[D] = \emptyset$. Moreover, if $G[D]$ has no edge, then $D$ is an independent dominating set. The cardinality of a smallest dominating set is the domination number and is denoted by $\gamma(G)$, similarly, the cardinality of a smallest independent dominating set is the independent domination number and is denoted by $i(G)$. Clearly, when $k = 1$, we have $\gamma(G) = \iota_1(G)$ and $i(G) = \iota'_1(G)$. For classic results between $\gamma(G)$ and $i(G)$, in 1978, Allan and Laskar proved that $\gamma(G) = i(G)$ for all $K_1,3$-free graphs and this result was generalized to $K_1,r$-free graphs by Bollobás and Cockayne in 1979. In 2013, Rad and Volkman proved that the ratio $i(G)/\gamma(G)$ is at most $\Delta(G)/2$ when $\Delta(G) \in \{3, 4, 5\}$. Further, Furuya et. al. proved that when $\Delta(G) \geq 6$, we have $i(G)/\gamma(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2$. In this paper, for a smallest $K_k$-isolating set $S$, we prove that $\iota'_k(G) \leq -\frac{i_k(G)}{\iota_k(G)} + i_k(G)(\Delta + 2) - \ell \Delta$ where $\ell$ is the number of some specific vertices of $S$ such that the union of their closed neighborhoods in $S$ is $S$. We prove that this bound is sharp. A special case of our main theorem implies $\iota'_k(G)/\iota_k(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2$. Further, we find an inequality between $\iota'_k(G)$ and $\iota_k(G)$ when $G$ is $K_1,r$-free graph. This also generalizes the result of Bollobás and Cockayne.

Keywords: Partial-domination.
AMS subject classification: 05C69

1 Introduction and Background

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$ of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. We denote the degree of $v$ in $G$ by $\deg_G(v)$ and denote the maximum
degree of $G$ by $\Delta(G)$. A neighbor of a vertex $v$ in $G$ is a vertex $u$ which is adjacent to $v$. The open neighborhood $N_G(v)$ of a vertex $v$ in $G$ is the set of neighbors of $v$. That is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The closed neighborhood of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we use $N_S(v)$ to denote $N_G(v) \cap S$ and $\text{deg}_S(v) = |N_G(v) \cap S|$, moreover, we use $N_S[v]$ to denote $N_G[v] \cap S$. The neighborhood of a vertex subset $S$ of $G$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$. The closed neighborhood of $S$ in $G$ is the set $N_G[S] = N_G(S) \cup S$. The subgraph of $G$ induced by $S$ is denoted by $G[S]$. The subgraph obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $(u,v)$-path in $G$ and is denoted by $d_G(u,v)$. We denote the clique on $n$ vertices by $K_n$. A star $K_{1,n}$ is a graph of $n + 1$ vertices obtained by joining $n$ vertices to one vertex. A graph $G$ is $H$-free if $G$ does not contain $H$ as an induced subgraph.

A vertex subset $S$ of a graph $G$ is a dominating set of $G$ if every vertex in $V(G) \setminus S$ is adjacent to a vertex in $S$. The cardinality of a smallest dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. Moreover, $S$ is an independent dominating set of $G$ if $S$ is a dominating set of $G$ and there is no edge in $G[S]$. The cardinality of a smallest independent dominating set of $G$ is called the independent domination number of $G$ and is denoted by $i(G)$.

Recently, Caro and Hansberg [3] generalized the concept of domination by focusing on a vertex subset $S \subseteq V(G)$ so that $G - N_G[S]$ contains no forbidden subgraph. Let $G$ be a graph and $F$ a family of graphs. A vertex subset $S \subseteq V(G)$ is said to be $F$-isolating if $G - N_G[S]$ does not contain $H$ as a subgraph for all $H \in F$. Obviously, when $F = \{K_1\}$, $S$ is a dominating set. The $F$-isolation number of $G$, denoted by $i_F(G)$, is the minimum cardinality of an $F$-isolating set of $G$. Analogously, $S$ is said to be independent $F$-isolating if $S$ is an $F$-isolating set of $G$ and $G[S]$ has no edge. The independent $F$-isolation number of $G$, denoted by $i'_F(G)$, is the minimum cardinality of an independent $F$-isolating set of $G$. In this paper, we consider a family $F$ reduced to a clique $K_k$ for a positive integer $k$ and we denote $i(K_k)(G)$ and $i'_K(K_k)(G)$ by $i_k(G)$ and $i'_k(G)$. A smallest $(K_k)$-isolating set is called an $i_k$-set and a smallest independent $(K_k)$-isolating set is called an $i'_k$-set. We generalize to $i_k(G)$ and $i'_k(G)$ two known results related to $i_1(G) = \gamma(G)$ and $i'_1(G) = i(G)$.

For classic results between the domination number and the independent domination number, Allan and Laskar [1] proved that $\gamma(G) = i(G)$ for all $K_{1,3}$-free graphs and this result was generalized to $K_{1,r}$-free graphs by Bollobás and Cockayne [2]. That is:

**Theorem 1** [2] Let $G$ be a $K_{1,r}$-free graph where $r \geq 3$. Then $i(G) \leq (r - 2)(\gamma(G) - 1) + 1$.

In 2013, Rad and Volkmann proved that the ratio $i(G)/\gamma(G) \leq \Delta(G)/2$ when $3 \leq \Delta(G) \leq 5$. When $\Delta(G) \geq 6$, they conjectured analogously that $i(G)/\gamma(G) \leq \Delta(G)/2$. However, the conjecture was disproved by Furuya et. al. [4] with the upper bound $\Delta(G) - 2\sqrt{\Delta(G)} + 2$ sharp for every $\Delta$ equal to a square. That is:

**Theorem 2** [4] For a graph $G$, $i(G)/\gamma(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2$. 

2
2 Main results

In this section, we state our main results and prove that all these results are sharp. It is worth noting that an \(\iota_k\)-set is a \(\{K_k\}\)-isolating set. By the minimality, \(\iota_k(G) \leq \iota'_k(G)\) for any graph \(G\). In the following, when no ambiguity can occur, we write \(\iota_k, \iota'_k\) and \(\Delta\) rather than \(\iota_k(G), \iota'_k(G)\) and \(\Delta(G)\), respectively. We prove that :

**Theorem 3** Let \(G\) be a graph with maximum degree \(\Delta\) and let \(S\) be an \(\iota_k(G)\)-set for some positive integer \(k\). Let \(v_1, v_2, \ldots, v_\ell\) be a sequence of vertices of \(S\) such that \(v_1\) has minimum degree in \(S\) and recursively \(v_{i+1}\) has minimum degree in \(S \setminus N_S[v_1, v_2, \ldots, v_i]\) until \(S = \bigcup_{i=1}^\ell N_S[v_i]\). Then

\[
\iota'_k(G) \leq -\frac{\iota_k^2(G)}{\ell} + \iota_k(G)(\Delta + 2) - \ell \Delta.
\]

We will give the proof in Section 3. The following corollary of Theorem 3 generalizes Theorem 2 to all positive values of \(k\).

**Corollary 1** For a graph \(G\) and an integer \(k \geq 1\), \(\iota'_k(G)/\iota_k(G) \leq \Delta - 2\sqrt{\Delta} + 2\).

**Proof.** The maximum of the function \(f: \mathbb{R}^+ \rightarrow \mathbb{R}\) defined by

\[
f(x) = -\frac{x^2}{\ell} + \iota_k(G)(\Delta + 2) - x\Delta
\]

is attained when \(x = \frac{\iota_k(G)}{\sqrt{\Delta}}\) and is equal to \(-\sqrt{\Delta\iota_k} + (\Delta + 2)\iota_k - \sqrt{\Delta}\iota_k\). By Theorem 3

\[
\iota'_k \leq f\left(\frac{\iota_k}{\sqrt{\Delta}}\right) = (\Delta - 2\sqrt{\Delta} + 2)\iota_k.
\]

Hence

\[
\iota'_k(G)/\iota_k(G) \leq \Delta - 2\sqrt{\Delta} + 2.
\]

\[\square\]

In our last main result, we generalize Theorem 1 by establishing the upper bound of \(\iota'_k\) in terms of \(\iota_k\) and \(r\) in \(K_{1,r}\)-free graphs. We find the same upper bound as that of Theorem 1. The proof is provided in Section 4.

**Theorem 4** Let \(G\) be a \(K_{1,r}\)-free graph where \(r \geq 3\). Then \(\iota'_k \leq (r - 2)(\iota_k - 1) + 1\).

We conclude this section by giving a construction of graphs satisfying the equality in Theorems 3, 4 and Corollary 1.
The graphs $G(t, s)$

Let $s, t, k$ be positive integers such that $s + t - 1 \geq k$. For $1 \leq i \leq t$, we let $K_{k}^{i,1}, K_{k}^{i,2}, \ldots, K_{k}^{i,s}$ be $s$ disjoint copies of a clique $K_{k}$. Let $x_{1}, x_{2}, \ldots, x_{t}$ be $t$ vertices. The graph $G(t, s)$ is obtained from $K_{k}^{i,1}, K_{k}^{i,2}, \ldots, K_{k}^{i,s}$ for $1 \leq i \leq t$ and $x_{1}, x_{2}, \ldots, x_{t}$ by joining each $x_{i}$ to a vertex of $K_{k}^{i,i'}$, $y_{i,i'}$ say, for all $1 \leq i' \leq s$ and form $x_{1}, x_{2}, \ldots, x_{t}$ a clique. Observe that $\text{deg}_{G(t,s)}(x_{i}) = \Delta(G(t,s)) = s + t - 1 \geq k$.

We see that $\{x_{1}, x_{2}, \ldots, x_{t}\}$ is a $\{K_{k}\}$-isolating set of $G(t, s)$. Thus, $\nu_{k}(G(t,s)) \leq t$. Let $S$ be an $\nu_{k}$-set of $G(t,s)$. To be adjacent to cliques $K_{k}^{i,1}$, we have that $(\{x_{i}\} \cup V(K_{k}^{i,1})) \cap S \neq \emptyset$. Thus, $\nu_{k}(G(t,s)) = |S| \geq t$ implying that $\nu_{k}(G(t,s)) = t$.

Now, we let $\nu_{k}'(G(t,s)) = t'$. We will show that $t' = s(t-1)+1$. Clearly, $\{x_{1}\} \cup \{y_{i,i'} : 2 \leq i \leq t$ and $1 \leq i' \leq s\}$ is an independent $\{K_{k}\}$-isolating set of $G(t,s)$. So, $\nu_{k}'(G(t,s)) \leq s(t-1)+1$. Let $S'$ be an $\nu_{k}'$-set of $G(t,s)$. By the independence of $S'$, $|S' \cap \{x_{1}, x_{2}, \ldots, x_{t}\}| \leq 1$. Without loss of generality, we let $x_{1}, x_{2}, \ldots, x_{t-1} \notin S'$. Hence, $S' \cap V(K_{k}^{i,i'}) \neq \emptyset$ for all $1 \leq i \leq t-1$ and $1 \leq i' \leq s$. Moreover, to be adjacent to $K_{k}^{i,1}$, we have that $S' \cap (V(K_{k}^{i,1}) \cup \{x_{i}\}) \neq \emptyset$. Hence, $\nu_{k}'(G(t,s)) = |S'| \geq s(t-1) + 1$ implying that $t' = s(t-1) + 1$.

If we let $s = t^{2} - t + 1$ with $t^{2} \geq k$, then $\Delta(G(t,s)) = t^{2}$ and $t' = s(t-1)+1 = t^{3} - 2t^{2} + 2t$.

Hence $t'/t = \Delta - 2\sqrt[3]{\Delta} + 2$ and $t' = -t^{2} + t(\Delta + 2) - \Delta$.

This shows that the bounds of Corollary 1 and of Theorem 3 in the case $\ell = 1$ are attained by arbitrarily large graphs.

We can construct a graph $\tilde{G}$ satisfying the equality for the bound in Theorem 3 for any positive value of $\ell$ by letting $\tilde{G}$ be the disjoint union of $G_{1}, \ldots, G_{\ell}$ where each $G_{i}$ is a copy of $G(t^{2} - t + 1)$ as defined in the above paragraph. Similarly, we have $\Delta(\tilde{G}) = t^{2}$, $\nu_{k}(\tilde{G}) = t\ell$ and $\nu_{k}'(\tilde{G}) = (t^{3} - 2t^{2} + 2t)\ell$. Hence,

$$\nu_{k}'(\tilde{G}) = \frac{(t^{3} - 2t^{2} + 2t)\ell}{\ell}$$

$$= -\frac{t^{2}\ell^{2}}{\ell} + t\ell(t^{2} + 2) - \ell t^{2}$$

$$= -\frac{(\nu_{k}(\tilde{G}))^{2}}{\ell} + \nu_{k}(\tilde{G})(\Delta(\tilde{G}) + 2) - \ell \Delta(\tilde{G}).$$

The graph $\tilde{G}$ is not connected. We can make it connected when $k \geq 3$ by joining with a path of length at least four one vertex of $K_{k}^{i,s}$ of $G_{j}$ to one vertex of $K_{k+1}^{i,1}$ of $G_{j+1}$ for $1 \leq j \leq \ell - 1$. For the resulting graph $\dot{G}$, $\Delta(\dot{G}) = t^{2}$, $\nu_{k}(\dot{G}) = t\ell$ and $\nu_{k}'(\dot{G}) = (t^{3} - 2t^{2} + 2t)\ell$. Hence, $\nu_{k}'(\dot{G})$ satisfies the bound in Theorem 3.

Finally, if we let $s = r - 2$, the graph $G(t,s)$ is $K_{r}, r$-free and $t' = s(t-1)+1 = (r-2)(t-1)+1$. This shows that the bound of Theorem 4 is sharp.
3 Proof of Theorem 3

First, we restate Theorem 3.

Theorem 3 Let $G$ be a graph with maximum degree $\Delta$ and let $S$ be an $i_k(G)$-set for some positive integer $k$. Let $v_1, v_2, \cdots, v_\ell$ be a sequence of vertices of $S$ such that $v_1$ has minimum degree in $S$ and recursively $v_{i+1}$ has minimum degree in $S \setminus N_S[\{v_1, v_2, \cdots, v_i\}]$ until $S = \bigcup_{i=1}^{\ell} N_S[v_i]$. Then

$$i_1'(G) \leq -\frac{i_2^2(G)}{\ell} + i_k(\Delta + 2) - \ell \Delta.$$  

Proof. Let $\{v_1, v_2, \cdots, v_\ell\}$ be a sequence of vertices of $S$ as defined in the theorem. Initially, we let $S_0 = S$. Then, we let for $1 \leq i \leq \ell$,

$$S_i = S_{i-1} \setminus N_{S_{i-1}}[v_i] \text{ and } N_{S_{i-1}}(v_i) = \{v_i^1, v_i^2, \ldots, v_i^{j_i}\}$$

where $j_i = deg_{S_{i-1}}(v_i)$. It is worth noting that

$$\emptyset = S_\ell \subset S_{\ell-1} \subset S_{\ell-2} \subset \cdots \subset S_1 \subset S_0.$$  

Claim 1 : $\{v_1, v_2, \ldots, v_\ell\}$ is an independent set.

Proof. This is a consequence of the construction of the sequence $v_1, \cdots, v_\ell$ since for $2 \leq j \leq \ell$, $v_j \notin \bigcup_{i=1}^{j-1} N_{S_{i-1}}[v_i]$. \(\Box\)

Claim 2 : $\Sigma_{i=1}^{\ell}(deg_{S_{i-1}}(v_i) + 1) = |S|$, in particular, $\Sigma_{i=1}^{\ell} deg_{S_{i-1}}(v_i) = |S| - \ell$.

Proof. Clearly $\bigcup_{i=1}^{\ell} N_{S_{i-1}}[v_i] = S$ and $N_{S_{i-1}}[v_i] \cap N_{S_{i-1}}[v_j] = \emptyset$. Thus $\Sigma_{i=1}^{\ell} |N_{S_{i-1}}[v_i]| = |S|$. Because $|N_{S_{i-1}}[v_i]| = deg_{S_{i-1}}(v_i) + 1$, it follows that $\Sigma_{i=1}^{\ell}(deg_{S_{i-1}}(v_i) + 1) = |S|$. Hence, $\Sigma_{i=1}^{\ell} deg_{S_{i-1}}(v_i) = |S| - \ell$. This completes the proof. \(\Box\)

For a clique $K_k$ and a vertex $v \in V(G)$, we say that $K_k$ is adjacent to $v$ (or vice versa) if $v$ is adjacent to a vertex of $K_k$ or is a vertex of $K_k$.

Let $A = N_{G \setminus S}(S) - N_{G \setminus S}(v_1, v_2, \cdots, v_\ell)$.

Claim 3 : $|A| \leq \Sigma_{i=1}^{\ell} deg_{S_{i-1}}(v_i)(\Delta - deg_{S_{i-1}}(v_i))$.

Proof. Clearly $deg_{S_{i-1}}(v_i^1) + deg_{G \setminus S_{i-1}}(v_i^1) = deg_G(v_i^1) \leq \Delta$. Thus, from the choice of $v_i$,

$$deg_{G \setminus S_{i-1}}(v_i^1) \leq \Delta - deg_{S_{i-1}}(v_i^1) \leq \Delta - deg_{S_{i-1}}(v_i)$$
for $1 \leq j \leq j_i$. Therefore $v_i^j$ has at most $\Delta - \text{deg}_{S_{i-1}}(v_i)$ neighbors in $G \setminus S$. Hence, and since $j_i = \text{deg}_{S_{i-1}}(v_i)$,

$$|A| \leq \sum_{i=1}^{\ell} \sum_{j=1}^{j_i} \text{deg}_G(v_i^j)$$

$$\leq \sum_{i=1}^{\ell} \sum_{j=1}^{j_i} (\Delta - \text{deg}_{S_{i-1}}(v_i))$$

$$= \sum_{i=1}^{\ell} \text{deg}_{S_{i-1}}(v_i)(\Delta - \text{deg}_{S_{i-1}}(v_i)).$$

(\textit{i})

Now, we let $\mathcal{K}$ be the set of all cliques $K_k$ of $G$. Moreover, we let $\mathcal{K}_1$ be the set of all cliques $K_k$ of $G$ such that $V(K_k) \cap S \neq \emptyset$ and $\mathcal{K}_2$ be the set $\mathcal{K} \setminus \mathcal{K}_1$. Since $S$ is an $i_k$-set of $G$, every $K_k \in \mathcal{K}_2$ is adjacent to a vertex in $S$. We also let $\mathcal{K}_3$ be the subset of $\mathcal{K}_2$ such that all cliques $K_k$ of $\mathcal{K}_3$ are not adjacent to any vertex in $\{v_1, v_2, \ldots, v_{\ell}\}$. Since $S$ is a $\{K_k\}$-isolating set of $G$, every clique in $\mathcal{K}_3$ is adjacent to a vertex of $S \setminus \{v_1, v_2, \ldots, v_{\ell}\}$ and thus contains a vertex of $A$. Hence every clique in $\mathcal{K}_3$ is adjacent to $B$ where $B$ is an independent dominating set of $G[A]$. Therefore $\{v_1, v_2, \ldots, v_{\ell}\} \cup B$ is an independent $\{K_k\}$-isolating set of $G$ and $i'_k(G) \leq \ell + |B| \leq \ell + |A|$. By Claim 3,

$$i'_k(G) \leq \ell + \sum_{i=1}^{\ell} \text{deg}_{S_{i-1}}(v_i)(\Delta - \text{deg}_{S_{i-1}}(v_i)). \quad (1)$$

Let $\text{deg}_{S_{i-1}}(v_i) = x_i$ and define two functions $f$ and $g : (\mathcal{R}^+ \cup \{0\})^\ell \rightarrow \mathcal{R}$ by

$$f(x_1, x_2, \ldots, x_{\ell}) = \ell + \sum_{i=1}^{\ell} x_i (\Delta - x_i)$$

and

$$g(x_1, x_2, \ldots, x_{\ell}) = x_1 + x_2 + \cdots + x_{\ell} - |S| + \ell.$$

To find an upper bound on $i'_k(G)$, we look for the maximum of $f(x_1, x_2, \ldots, x_{\ell})$ under the condition, due to Claim 2, $g(x_1, x_2, \ldots, x_{\ell}) = 0$. Let

$$F(x_1, x_2, \ldots, x_{\ell}, \lambda) = f(x_1, x_2, \ldots, x_{\ell}) - \lambda g(x_1, x_2, \ldots, x_{\ell}) \quad (2)$$

with $\lambda \in \mathcal{R}$. From the Lagrange's multipliers method, we get an extremum for $f$ by letting

$$\frac{\partial F(x_1, \ldots, x_{\ell}, \lambda)}{\partial x_i} = \Delta - 2x_i - \lambda = 0 \quad \text{for all} \quad 1 \leq i \leq \ell$$

and

$$\frac{\partial F(x_1, \ldots, x_{\ell}, \lambda)}{\partial \lambda} = x_1 + x_2 + \cdots + x_{\ell} - |S| + \ell = 0.$$

For this extremum all the $x_i$’s are equal to $\frac{|S|}{\ell} - 1$ and the extremum is equal to

$$M = \ell + (\frac{|S|}{\ell} - 1)(\ell \Delta - |S| + \ell). \quad (3)$$
For the particular values \( \{x_1, x_2, \cdots, x_\ell\} = \{|S| - \ell, 0, \cdots, 0\} \) corresponding to the case \( N_G(v_i) = N_G(v_1) \) for \( 2 \leq i \leq \ell \), \( f(x_1, \cdots, x_\ell) = \ell + (|S| - \ell)(\Delta - |S| + \ell) \leq M \). Therefore the extremum \( M \) of \( f \) is a maximum and

\[
i'_k(G) \leq \ell + \left(\frac{\ell}{\ell - 1}\right)(\ell \Delta - \ell_k + \ell) = \frac{i^2_k(G)}{\ell} + \ell_k(G)(\Delta + 2) - \ell \Delta.
\]

This completes the proof. \( \square \)

\section*{4 Proof of Theorem 4}

We restate Theorem 4.

\textbf{Theorem 4} Let \( G \) be a \( K_{1,r} \)-free graph where \( r \geq 3 \). Then \( \iota'_k \leq (r - 2)(\ell_k - 1) + 1 \).

\textbf{Proof.} Let \( S \) be an \( \iota_k \)-set of \( G \). Clearly, \( |S| = \iota_k \). Let \( I \) be a maximum independent set of \( G[S] \). If \( I = S \), then \( S \) is an independent \( \{K_k\}\)-isolating set of \( G \) implying that \( \iota'_k \leq |S| = \iota_k \). This completes the proof because \( r \geq 3 \). Hence, we may assume that \( S \setminus I \neq \emptyset \). Let \( A = N_G(S) \setminus N_G[I] \) and let \( B \) be an independent dominating set of \( G[A] \). Consider a partition \( B = \bigcup_{1 \leq i \leq |S \setminus I|} B_i \) where \( B_i \subseteq N_A(v_i) \) for each \( v_i \in S \setminus I \) and \( B_i \cap B_j = \emptyset \) for all \( i \) and \( j \). Note that some \( B_i \) may be empty. For each nonempty \( B_i \), \( |B_i| \leq r - 2 \) since \( v_i \) has at least one neighbor in \( I \) and \( G \) is \( K_{1,r} \)-free. Therefore

\[
|B| \leq \sum_{1 \leq i \leq |S \setminus I|} |B_i| \leq |S \setminus I|(r - 2).
\]

Since \( S \) is a \( \{K_k\}\)-isolating set of \( G \), every clique \( K_k \) of \( G \) non-adjacent to a vertex of \( I \) has a vertex in \( A \) and is thus adjacent to a vertex of \( B \). Therefore \( I \cup B \) is an independent \( \{K_k\}\)-isolating set of \( G \) and

\[
i'_k(G) \leq |I| + |B| \leq |I| + (|S| - |I|)(r - 2) = |S|(r - 2) - |I|(r - 3)
\]

which is maximized when \( |I| = 1 \). Hence,

\[
i'_k \leq (r - 2)\iota_k - (r - 3) = (r - 2)(\iota_k - 1) + 1
\]

which completes the proof. \( \square \)

\textbf{References}

[1] R. B. Allan and R. Laskar, On domination and some related topics in graph theory. Proceeding of the 9th Southeast Conference on Graph Theory, Combinatorics and Computing, Boca Raton, February, Utilitas Mathematica (1978), 43–56.
[2] B. Bollobás and E. J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance. *Journal of Graph Theory* 3 (1979), 241–249.

[3] Y. Caro and A. Hansberg, Partial domination-the isolation number of a graph. *FiloMath* 31:12 (2017), 3925–3944.

[4] M. Furuya, K. Ozeki and A. Sasaki, On the ratio of the domination number and the independent domination number in graphs, *Discrete Applied Mathematics* 178 (2014), 157–159.

[5] N. J. Rad, L. Volkmann, A note on the independent domination number in graphs, *Discrete Applied Mathematics* 161 (2013), 3087–3089.