A NOTE ON TRACE SCALING ACTIONS
AND FUNDAMENTAL GROUPS OF C*-ALGEBRAS

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(Communicated by Marius Junge)

Abstract. Using the Effros-Handelman-Shen theorem and Elliott’s classification theorem of AF algebras, we show that there exists a unital simple AF algebra $A$ with unique trace such that $A \otimes K$ admits no trace scaling action of the fundamental group of $A$.

1. Introduction

Let $M$ be a factor of type II$_1$ with a normalized trace. Murray and von Neumann introduced the fundamental group $F(M)$ of $M$ in [14]. They showed that if $M$ is hyperfinite, then $F(M) = \mathbb{R}_+^\infty$. Since then there have been many works on the computation of the fundamental groups. Voiculescu [24] showed that $F(L(F_\infty))$ of the group factor of the free group $F_\infty$ contains the positive rationals and Radulescu proved that $F(L(F_\infty)) = \mathbb{R}_+^\infty$ in [21]. Connes [3] showed that if $G$ is an ICC group with property (T), then $F(L(G))$ is a countable group. Popa showed that any countable subgroup of $\mathbb{R}_+^\infty$ can be realized as the fundamental group of some factor of type II$_1$ in [15]. Furthermore, Popa and Vaes [19] exhibited a large family $S$ of subgroups of $\mathbb{R}_+^\infty$, containing $\mathbb{R}_+^\infty$ itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension in $(0, 1)$, such that for each $G \in S$ there exist many free ergodic measure preserving actions of $F_\infty$ for which the associated II$_1$ factor $M$ has the fundamental group equal to $G$. In our previous paper [16] (see also [15]), we introduced the fundamental group $F(A)$ of a simple unital C*-algebra $A$ with a normalized trace $\tau$ based on the computation of Picard groups by Kodaka [11], [12] and [13]. The fundamental group $F(A)$ is defined as the set of the numbers $\tau \otimes \text{Tr}(p)$ for some projection $p \in M_n(A)$ such that $pM_n(A)p$ is isomorphic to $A$. We computed fundamental groups of several C*-algebras and showed that any countable subgroup of $\mathbb{R}_+^\infty$ can be realized as the fundamental group of a separable simple unital C*-algebra with unique trace [17].

The fundamental group of a II$_1$ factor $M$ is equal to the set of trace scaling constants for automorphisms of $M \otimes B(H)$. We have a similar fact; that is, the fundamental group of a C*-algebra $A$ is equal to the set of trace scaling constants for automorphisms of $A \otimes K$ [16] (see also [15]). It is of interest to know whether $A \otimes K$ admits a trace scaling action of $F(A)$. In the case where $M$ is a factor of type II$_1$, the existence of a trace scaling (continuous) $\mathbb{R}_+^\infty$-action on $M \otimes B(H)$ is equivalent...
to the existence of a type $\text{III}_1$ factor having a core isomorphic to $M \otimes B(H)$ by the continuous decomposition of type $\text{III}_1$ factors. (See \cite{23} and \cite{1}.) Hence this question is important in the theory of von Neumann algebras. Radulescu showed that $L(F_\infty) \otimes B(H)$ admits a trace scaling action of $\mathbb{R}_+^\times$ in \cite{22}. Therefore there exists a type $\text{III}_1$ factor having a core isomorphic to $L(F_\infty) \otimes B(H)$. Popa and Vaes \cite{20} showed that there exists a $\text{II}_1$ factor $M$ such that $\mathcal{F}(M) = \mathbb{R}_+^\times$ and $M \otimes B(H)$ admits no trace scaling (continuous) action of $\mathbb{R}_+^\times$.

In this paper we consider trace scaling actions on certain AF algebras. If $A$ is a UHF algebra, then $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$. Using the Effros-Handelman-Shen theorem and Elliott’s classification theorem of AF algebras, we show that there exists a unital simple AF algebra $A$ with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$. Note that there exist remarkable works of the classification of trace scaling automorphisms in \cite{1}, \cite{8} and \cite{9}. But we do not consider the classification of trace scaling actions in this paper.

2. Examples

We recall some definitions in \cite{16}. Let $A$ be a unital simple $C^*$-algebra with a unique normalized trace $\tau$ and $\text{Tr}$ the usual unnormalized trace on $M_n(C)$. Put $\mathcal{F}(A) := \{\tau \otimes \text{Tr}(p) \in \mathbb{R}_+^\times \mid p \text{ is a projection in } M_n(A) \text{ such that } pM_n(A)p \cong A\}$. Then $\mathcal{F}(A)$ is a multiplicative subgroup of $\mathbb{R}_+^\times$ by Theorem 3.1 in \cite{16}. For an additive subgroup $E$ of $\mathbb{R}$ containing 1, we define the positive inner multiplier group $IM_+(E)$ of $E$ by

$$IM_+(E) = \{t \in \mathbb{R}_+^\times \mid t \in E, t^{-1} \in E, \text{ and } tE = E\}.$$ 

Then we have $\mathcal{F}(A) \subset IM_+(\tau_n(K_0(A)))$ by Proposition 3.7 in \cite{16}. This obstruction enables us to compute fundamental groups easily. For $x \in (A \otimes \mathbb{K})_+$, set $\hat{\tau}(x) = \sup\{\tau \otimes \text{Tr}(y) : y \in \bigcup_n M_n(A), y \leq x\}$. Define $\mathcal{M}^+_\hat{\tau} = \{x \geq 0 : \hat{\tau}(x) < \infty\}$ and $\mathcal{M}_\hat{\tau} = \text{span} \mathcal{M}^+_\hat{\tau}$. Then $\hat{\tau}$ is a densely defined (with the domain $\mathcal{M}_\hat{\tau}$) lower semicontinuous trace on $A \otimes \mathbb{K}$. Since the normalized trace on a unital $C^*$-algebra $A$ is unique, the lower semicontinuous densely defined trace on $A \otimes \mathbb{K}$ is unique up to constant multiple. It is clear that for any $\alpha \in \text{Aut}(A \otimes \mathbb{K})$, $\hat{\tau} \circ \alpha$ is a densely defined (with the domain $\alpha^{-1}(\mathcal{M}_\hat{\tau}$)) lower semicontinuous trace on $A \otimes \mathbb{K}$. Therefore there exists a positive number $\lambda$ such that $\hat{\tau} \circ \alpha = \lambda \hat{\tau}$, and hence $\alpha^{-1}(\mathcal{M}_\hat{\tau}) = \mathcal{M}_\hat{\tau}$. We define the set of trace scaling constants for automorphisms:

$$\mathcal{G}(A) := \{\lambda \in \mathbb{R}_+^\times \mid \hat{\tau} \circ \alpha = \lambda \hat{\tau} \text{ for some } \alpha \in \text{Aut}(A \otimes \mathbb{K})\}.$$ 

Then $\mathcal{F}(A) = \mathcal{G}(A)$ by Proposition 3.28 in \cite{16}. Therefore it is of interest to know whether $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$.

It is clear that if the fundamental group of $A$ is singly generated, $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$. See \cite{16} and \cite{17} for such examples. We shall show some examples of AF algebras $A$ such that $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$.

Example 2.1. Consider a UHF algebra $M_{2^\infty,3^\infty}$. Then the fundamental group of $M_{2^\infty,3^\infty}$ is a multiplicative subgroup generated by 2 and 3. Hence $\mathcal{F}(M_{2^\infty,3^\infty})$ is isomorphic to $\mathbb{Z}^2$ as a group. Since $M_{2^\infty,3^\infty} \otimes \mathbb{K}$ is isomorphic to $M_{2^\infty} \otimes \mathbb{K} \otimes M_{3^\infty} \otimes \mathbb{K}$, there exists a trace scaling $\mathbb{Z}^2$-action on $M_{2^\infty,3^\infty} \otimes \mathbb{K}$. In general, if $A$ is a UHF algebra, then $\mathcal{F}(A)$ is a free abelian group (see \cite{16}) and $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$ (see also \cite{5}).
Example 2.2. Let $A$ be a unital simple AF algebra such that $K_0(A) = \mathbb{Z} + \mathbb{Z}\sqrt{3}$, $K_0(A)_+ = (\mathbb{Z} + \mathbb{Z}\sqrt{3}) \cap \mathbb{R}_+$ and $[1]_0 = 1$. Then $\mathcal{F}(A) = \{(2 + \sqrt{3})^n : n \in \mathbb{Z}\}$ (see Proposition 3.17 and Corollary 3.18 in [16]). Consider $B = M_5 \otimes A$. Then it is easily seen that $\tau_*(K_0(B)) = \mathbb{Z}O[\frac{1}{5}] + \mathbb{Z}O[\frac{1}{5}]\sqrt{3}$ and $\tau_*$ is an order isomorphism. We shall show that $IM_+(\tau_*(K_0(B)))$ is generated by 5 and $2 + \sqrt{3}$. Since $\tau_*(K_0(B))$ is a subring of $\mathbb{R}$, $IM_+(\tau_*(K_0(B)))$ is a group of positive invertible elements. Define a multiplicative map $N$ of $\mathbb{Z}O[\frac{1}{5}] + \mathbb{Z}O[\frac{1}{5}]\sqrt{3}$ to $\mathbb{Z}O[\frac{1}{5}]$ by $N(a + b\sqrt{3}) = a^2 - 3b^2$ for any $a, b \in \mathbb{Z}O[\frac{1}{5}]$. If $a + b\sqrt{3}$ is an invertible element in $\mathbb{Z}O[\frac{1}{5}] + \mathbb{Z}O[\frac{1}{5}]\sqrt{3}$, then there exists an integer $n$ such that $N(a + b\sqrt{3}) = \pm 5^n$. Elementary computations show that $x^2 - 3y^2 \equiv 0 \mod 25$ implies $x \equiv 0 \mod 5$ and $y \equiv 0 \mod 5$. It is easy to see that no integers $x$ and $y$ satisfy equations $x^2 - 3y^2 = 5$, $x^2 - 3y^2 = -5$ or $x^2 - 3y^2 = -1$. There exist integers $x$ and $y$ that satisfy the equation $x^2 - 3y^2 = 1$. Indeed, $x = 2, y = 1$ is such an example. (See, for example, [10] and [16].) Therefore, it can be easily checked that $IM_+(\tau_*(K_0(B)))$ is generated by 5 and $2 + \sqrt{3}$. Hence we see that $\mathcal{F}(B) = \{5^n(2 + \sqrt{3})^m : n, m \in \mathbb{Z}\}$ by Proposition 3.17 in [16] and $B \otimes K$ admits a trace scaling action.

We shall show that there exists a unital simple AF algebra $A$ with unique trace such that $A \otimes K$ admits no trace scaling action of $\mathcal{F}(A)$. Define

$$E = \left\{ \left( \frac{j + k\sqrt{3}}{5^6}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \in \mathbb{R} \times \mathbb{Z}^2 \mid i, j, k, x, y \in \mathbb{Z}, x \equiv j \mod 9, y \equiv k \mod 3 \right\},$$

$$E_+ = \left\{ \left( r, \begin{pmatrix} x \\ y \end{pmatrix} \right) \in E \mid r > 0 \right\} \cup \left\{ \left( 0, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \right\} \text{ and } [u]_0 = (1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Then there exists a simple AF algebra $A$ with a unique normalized trace $\tau$ such that $(K_0(A), K_0(A)_+, [1]_0) = (E, E_+, u)$ by the Effros-Handelman-Shen theorem [6].

Lemma 2.3. With notation as above the fundamental group of $A$ is equal to the multiplicative group generated by 5 and $2 + \sqrt{3}$.

Proof. Since $\tau_*(K_0(A))$ is equal to $\mathbb{Z}O[\frac{1}{5}] + \mathbb{Z}O[\frac{1}{5}]\sqrt{3}$, $\mathcal{F}(A)$ is a subgroup of $\{5^n(2 + \sqrt{3})^m : n, m \in \mathbb{Z}\}$ by an argument in Example 2.2. Define an additive homomorphism $\phi : E \to E$ by

$$\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = (5r, \begin{pmatrix} 5 \\ 6 \end{pmatrix}) \begin{pmatrix} 9 \\ 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).$$

Computations show that $\phi$ is a well-defined order isomorphism of $E$ with $\phi(u) = (5, \begin{pmatrix} 5 \\ 6 \end{pmatrix})$. There exist a natural number $n$ and a projection $p$ in $M_n(A)$ such that $[p]_0 = (5, \begin{pmatrix} 5 \\ 6 \end{pmatrix})$ and $\tau \otimes \text{Tr}(p) = 5$. Since we have $(K_0(pM_n(A)p), K_0(pM_n(A)p)_+, [p]_0) = (E, E_+, 5, \begin{pmatrix} 5 \\ 6 \end{pmatrix}))$, there exists an isomorphism $f : A \to pM_n(A)p$ with $f_* = \phi$ by Elliott’s classification theorem of AF algebras [7]. Therefore $5 \in \mathcal{F}(A)$. Define an additive homomorphism $\psi : E \to E$ by

$$\psi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = ((2 + \sqrt{3})r, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).$$

Then we see that $2 + \sqrt{3} \in \mathcal{F}(A)$. Consequently $\mathcal{F}(A)$ is the multiplicative group generated by 5 and $2 + \sqrt{3}$. $\square$
We shall consider the order automorphisms of \((E, E_+)\).

**Lemma 2.4.** Let \(\phi\) be an order automorphism of \((E, E_+)\). Then there exist integers \(a, b, c, d\) and a positive invertible element \(\lambda\) in \(\mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}\) such that \(ad - bc = \pm 1\) and
\[
\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})) = (\lambda r, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).
\]
Moreover if \(\lambda = 5\), then
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 6 & 2 \end{pmatrix} \mod 9,
\]
and if \(\lambda = 2 + \sqrt{3}\), then
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix} \mod 9.
\]

**Proof.** We denote by \((\phi_1((r, \begin{pmatrix} x \\ y \end{pmatrix})), \phi_2((r, \begin{pmatrix} x \\ y \end{pmatrix})))) the element \(\phi((r, \begin{pmatrix} x \\ y \end{pmatrix})))\) for any \((r, \begin{pmatrix} x \\ y \end{pmatrix}) \in E\). Consider a subgroup \(F\) generated by \((0, \begin{pmatrix} 9 \\ 0 \end{pmatrix})\) and \((0, \begin{pmatrix} 0 \\ 3 \end{pmatrix})\). Then \(F\) is an \(\phi\)-invariant subgroup because \(\phi\) is an order isomorphism. Hence there exist integers \(m_1, m_2, m_3, m_4\) such that \(m_1m_4 - m_2m_3 = \pm 1\) and
\[
\phi_2((0, \begin{pmatrix} x \\ y \end{pmatrix})) = \left( \frac{m_1}{m_3}, \frac{3m_2}{m_4} \right) \begin{pmatrix} x \\ y \end{pmatrix} \text{ for any } (0, \begin{pmatrix} x \\ y \end{pmatrix}) \in F.
\]
Furthermore we see that there exists a positive invertible element \(\lambda\) in \(\mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}\) such that
\[
\phi_1((r, \begin{pmatrix} x \\ y \end{pmatrix}))) = \lambda r. \text{ Since } 5^6\phi\left((\frac{9}{5^{17}}, \begin{pmatrix} 0 \\ 0 \end{pmatrix})\right) = \phi\left((9, \begin{pmatrix} 0 \\ 0 \end{pmatrix})\right)\text{ for any } i \in \mathbb{Z},\text{ we see that } \phi_2((9, \begin{pmatrix} 0 \\ 0 \end{pmatrix}))) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\]
This observation and easy computations show that
\[
\phi((1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}))) = (\lambda, \left( \frac{m_1}{m_3}, \frac{3m_2}{m_4} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix})\text{ and } \frac{m_1}{m_3} \in \mathbb{Z}.\]
In a similar way, we see that
\[
\phi((\sqrt{3}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}))) = (\lambda\sqrt{3}, \left( \frac{m_1}{m_3}, \frac{3m_2}{m_4} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}).\]
It is easily seen that \(\phi\) is determined by the values of \(\phi((1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}))), \phi((\sqrt{3}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}))), \phi((0, \begin{pmatrix} 9 \\ 0 \end{pmatrix})))\) and \(\phi((0, \begin{pmatrix} 0 \\ 3 \end{pmatrix})))\). Therefore there exist integers \(a, b, c, d\) and a positive invertible element \(\lambda\) in \(\mathbb{Z}[\frac{1}{5}] + \mathbb{Z}[\frac{1}{5}]\sqrt{3}\) such that \(ad - bc = \pm 1\) and
\[
\phi((r, \begin{pmatrix} x \\ y \end{pmatrix}))) = (\lambda r, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).
\]
Let \(\lambda = 5\); then \(a \equiv 5 \mod 9\), \(b \equiv 0 \mod 9\), \(c \equiv 0 \mod 3\) and \(d \equiv 5 \mod 3\) by the definition of \(E\). If \(ad - bc = 1\), then \(d \equiv 5^5 \mod 9\), \(-b \equiv 0 \mod 9\), \(-c \equiv 0 \mod 3\) and \(a \equiv 5^5 \mod 3\) because \(\phi\) is an isomorphism. Therefore computations show that
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 6 & 2 \end{pmatrix} \mod 9.
\]
If \(ad - bc = -1\), then \(-d \equiv 5^5 \mod 9\), \(b \equiv 0 \mod 9\), \(c \equiv 0 \mod 3\) and \(-a \equiv 5^5 \mod 3\). There does not exist an integer \(a\) such that \(a \equiv 5 \mod 9\) and \(-a \equiv 5^5 \mod 3\).
Therefore we reach a conclusion in the case $\lambda = 5$. In the case $\lambda = 2 + \sqrt{3}$, a similar argument as above proves the lemma.

**Theorem 2.5.** There exists a unital simple AF algebra $A$ with unique trace such that $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$.

**Proof.** Let

$$E = \left\{ \left( \frac{j + k \sqrt{3}}{5^i}, \left( \begin{array}{c} x \\ y \end{array} \right) \right) \in \mathbb{R} \times \mathbb{Z}^2 \mid i, j, k, x, y \in \mathbb{Z}, x \equiv j \mod 9, y \equiv k \mod 3 \right\},$$

$$E_+ = \left\{ \left( r, \left( \begin{array}{c} x \\ y \end{array} \right) \right) \in E \mid r > 0 \right\} \cup \left\{ \left( 0, \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right) \right\} \text{ and } [u]_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

Then there exists a simple AF algebra $A$ with a unique normalized trace $\tau$ such that $(K_0(A), K_0(A)_+, [1_A]_0) = (E, E_+, u)$ by the Effros-Handelman-Shen theorem [6]. By Lemma 2.3, $\mathcal{F}(A) = \left\{ 5^n(2 + \sqrt{3})^m : n, m \in \mathbb{Z} \right\}$. Let $\alpha$ be an automorphism of $A \otimes \mathbb{K}$ such that $\tilde{\tau} \circ \alpha = 5\tilde{\tau}$ and $\beta$ an automorphism of $A \otimes \mathbb{K}$ such that $\tilde{\tau} \circ \beta = (2 + \sqrt{3})\tilde{\tau}$. Then $\alpha_*$ and $\beta_*$ are order isomorphisms of $(K_0(A), K_0(A)_+)$. Lemma 2.4 and computations show that $\alpha_* \circ \beta_* \neq \beta_* \circ \alpha_*$. Therefore $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$.

**Remark 2.6.** Let $A$ be a unital simple $C^*$-algebra with a unique normalized trace $\tau$. We denote by $\text{Pic}(A)$ the Picard group of $A$ (see [2]). Assume that the normalized trace on $A$ separates equivalence classes of projections. Then we have the following exact sequence [16] (see also [11]):

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1.$$

If $A \otimes \mathbb{K}$ admits a trace scaling action of $\mathcal{F}(A)$, then $\text{Pic}(A)$ is isomorphic to a semidirect product of $\text{Out}(A)$ with $\mathcal{F}(A)$. Example 2.1 and Example 2.2 are such examples. We do not know whether there exists a simple $C^*$-algebra $A$ with a unique normalized trace $\tau$ such that the normalized trace on $A$ separates equivalence classes of projections and $A \otimes \mathbb{K}$ admits no trace scaling action of $\mathcal{F}(A)$.

**Remark 2.7.** If $A$ is a $C^*$-algebra in the proof of Theorem 2.5 then it can be checked that $\text{Out}(A)$ is not a normal subgroup of $\text{Pic}(A)$ by Lemma 2.4 in this paper, Proposition 1.5 in [11] and Elliott’s classification theorem of AF algebras.

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