EXPONENTIAL STABILITY OF AXIALLY MOVING KIRCHHOFF-BEAM SYSTEMS WITH NONLINEAR BOUNDARY DAMPING AND DISTURBANCE

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Abstract. This paper examines the stabilization problem of the axially moving Kirchhoff beam. Under the nonlinear damping criterion established by the slope-restricted condition, the existence and uniqueness of solutions of the closed-loop system equipped with nonlinear time-delay disturbance at the boundary is investigated via the Faedo-Galerkin approximation method. Furthermore, the solution is continuously dependent on initial conditions. Then the exponential stability of the closed-loop system is established by the direct Lyapunov method, where a novel energy function is constructed.

1. Introduction. Recently, many engineering materials of components such as textile fibers, paper sheets, magnetic tapes, conveyor belts and overhead cables, etc have been modeled as axially moving systems. The vibration of moving parts of these systems becomes the main factor affecting the work efficiency. Therefore, how to reduce the vibration of systems has become an important engineering core problem. In order to suppress the vibration of the system, many control methods have been investigated, such as the Lyapunov method [19], optimal control [4] and adaptive control [2], to name a few.

According to the flexible and geometric parameters of the system, the axially moving system can be divided into four kinds of models: string, beam, belt and plate systems. When the bending stiffness of an axially moving material is large enough, it should be regarded as a beam model. For the stability analysis of linear beam systems, we refer the reader to the axially moving Euler-Bernoulli beam [5, 16], the linear thermoelastic beam [11] and the viscoelastic Timoshenko beam

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However, the well posedness and stability analysis of nonlinear beam systems are considerable complex due to the fact that the nonlinear term leads to the failure of some common approaches dealing with these systems, such as, contraction semigroups and frequency domain methods. It is worth mentioning that the Kirchhoff-type beam proposed by Woinowsky-Krieger in [21] is as follows

\[ y_{tt}(x, t) + y_{xxxx}(x, t) - \left( a + b \int_0^L y_x^2(x, t) \, dx \right) y_{xx}(x, t) = 0, \tag{1} \]

where \( a, b \) are positive constants, and \( (\cdot)_x, (\cdot)_t \) represent \( \frac{\partial(\cdot)}{\partial x} \) and \( \frac{\partial(\cdot)}{\partial t} \), respectively. Throughout this article, we will discard the term \((x, t)\) for brevity in the following equations. The existence of classical solutions for the initial boundary value problem of equation (1) was completed by Ball [1]. Later, Guo and Guo [6] considered a more general nonlinear beam equation of Kirchhoff-type represented by the following nonlinear PDE

\[
\begin{aligned}
&y_{tt} + y_{xxxx} - F(\int_0^L y_x^2 \, dx) y_{xx} = 0, \forall x \in (0, L), t > 0, \\
y(0, t) = y_x(0, t) = y_{xx}(L, t) = 0, \forall t \geq 0, \\
y_{xx}(L, t) - F(\int_0^L y_x^2 \, dx) y_x(L, t) = U(t), \forall t \geq 0, \\
y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \forall x \in (0, L), \\
y_{out}(t) = y_t(L, t).
\end{aligned}
\tag{2}
\]

where \( F \) is an abstract nonnegative function satisfying some conditions, and the exponential stability of the system is established using the Lyapunov method under the following feedback controller

\[
\begin{aligned}
&U(t) = k(t) y_t(L, t), \\
&\dot{k}(t) = r y_t^2(L, t), \ k(0) = k_0 > 0.
\end{aligned}
\]

For this kind of Kirchhoff-type beam with boundary disturbances, we refer interested readers to [8, 13].

Time-delay caused by various reasons and different sources often occurs in engineering systems, which results in a considerable impact on the system control, therefore time-delay feedback is necessary in the design of the controller. In [15], Liang et al. proposed the Smith predictor for the Euler-Bernoulli beam equipped by delayed boundary measurement. Morguil [18] put forward a dynamic feedback controller to decline the small delays. The exponential stability of Euler-Bernoulli beams with boundary input delays was established by a kind of predictor proposed in [10]. Based on predictor and observer designs, the stabilization problems of the Euler-Bernoulli beam with delayed observation are investigated in [7, 22].

It is observed that in the above work, the time-delay was examined as a known term, and the methods dealing with stabilization depends on the determinant parameters of these time delays. When the time-delay term produced in the actual systems is an unknown disturbance, it is of significant value to study how to design a controller suppressing the vibration of these systems. However, there are only a few published works on this aspect. The feedback stabilization of the following Euler-Bernoulli beam

\[
\begin{aligned}
y_{tt} + y_{xxxx} + U(x, t) = 0, \ x \in (0, 1), t > 0, \\
y(0, t) = y_x(0, t) = y_{xx}(1, t) = 0, t > 0, \\
y_{xx}(1, t) = \beta y_t(1, t - \tau), t > 0, \\
y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), x \in (0, 1), \\
y_t(1, \theta) = \eta(\theta), \ \theta \in (-\tau, 0),
\end{aligned}
\]
where $U(x,t)$ is the control input and $\beta y_t(1,t-\tau)$ is the boundary time-delay disturbance, was studied in [14]. Shang and Xu [20] considered the Euler-Bernoulli beam

$$
\begin{align*}
\begin{cases}
y_{tt} + y_{xxxx} &= 0, \\ y_{xxx}(1,t) &= \alpha u(t) + \beta u(t-\tau), \\ y(0,t) &= y_x(0,t) = y_{xx}(1,t) = 0, \\ y(x,0) &= w_0(x), \\ u(\theta) &= f(\theta), 
\end{cases}
\end{align*}
$$

where the boundary control is $\alpha u(t) + \beta u(t-\tau)$, and analyzed the instability and uniform stability of the system with the different range of parameters $\alpha, \beta$.

In this paper, we consider the axially moving beam of Kirchhoff-type driven by the following nonlinear PDE

$$
\begin{align*}
\begin{cases}
y_{tt} - \mathcal{M}(||y_x(t)||^2)y_{xx} + 2y_{xt} + y_{xxxx} &= 0, \\ \mathcal{M}(||y_x(t)||^2)y_x(1,t) - y_{xxx}(1,t) - vy_t(1,t) &= U(t), \\ y(0,t) &= y_x(0,t) = y_{xx}(1,t) = 0, \\ y(x,0) &= h(x), \\ y_t(1,t-\tau) &= f_0(1,t-\tau), 
\end{cases}
\end{align*}
$$

(3)

for all $x \in (0,1)$, where $\mathcal{M}$ is a continuous differentiable function ($\mathcal{M} \in C^1(0,\infty)$) with $\mathcal{M}(s) \geq m_0 > 0$ ($\forall s > 0$), the real number $\tau > 0$ represents the time-delay, $h, f$ and $f_0$ are the initial displacement, the initial velocity and the given value of the system respectively. When $\mathcal{M}(||y_x(t)||^2) = a + b \int_0^1 y_x^2(x,t)dx$ ($a, b > 0$), the dynamic behavior (state response) of the system (3) is analyzed in [3]. However, to the best of our knowledge, there is no result for the boundary stabilization problem of the axially moving Kirchhoff beam (3). Nonlinear boundary control is a practical method, because the controller needs to take advantage of the nonlinear behavior of actuators and sensors when dealing with large deformation, the use of smart materials or saturation phenomenon. Motivated by this, the existence result and exponential stability of the system (3) is investigated using the Faedo-Galerkin approximation and the Lyapunov method under the feedback control given by

$$
U(t) = -F_1(y_t(1,t)) + F_2(y_t(1,t-\tau)),
$$

(4)

where the nonlinear function $F_1$ is regarded as the control input and the nonlinear function $F_2$ is considered to be a kind of time-delay disturbance.

The remainder of this paper is organized as follows. The existence and uniqueness of solutions for the closed-loop system is established by the Faedo-Galerkin approximation method in Section 2. In Section 3, the global stability analysis is considered by the direct Lyapunov method. The paper ends with concluding remarks in Section 4.

2. Global existence result. In this section, the objective is to establish the existence and uniqueness of solutions for the following closed-loop system

$$
\begin{align*}
\begin{cases}
y_{tt} - \mathcal{M}(||y_x(t)||^2)y_{xx} + 2y_{xt} + y_{xxxx} &= 0, \\ \mathcal{M}(||y_x(t)||^2)y_x(1,t) - y_{xxx}(1,t) - vy_t(1,t) &= -F_1(y_t(1,t)) + F_2(y_t(1,t-\tau)), \\ y(0,t) &= y_x(0,t) = y_{xx}(1,t) = 0, \\ y(x,0) &= h(x), \\ y_t(1,t-\tau) &= f_0(1,t-\tau), 
\end{cases}
\end{align*}
$$

(5)

given by substituting (4) into (3), for any $t > 0$ and $x \in (0,1)$. Consider the following assumptions.
(H1) $F_1$ is a continuous function satisfying the slope-restricted condition
\[
F_1(0) = 0, \quad k_1 \leq \frac{F_1(s_1) - F_1(s_2)}{s_1 - s_2} \leq k_2, \quad \forall \ s_1 \neq s_2 \in R,
\]
with given constants $k_2 \geq k_1 > 0$;

(H2) $F_2$ is a continuous function such that
\[
F_2(0) = 0, \quad |F_2(s_1) - F_2(s_2)| \leq k_3|s_1 - s_2|, \quad \forall \ s_1, s_2 \in R,
\]
where $0 < k_3 < k_1$.

**Remark 1.** The slope-restricted condition (6) proposed by [9, 12] is a standard of control design in the sense of absolute stability for ODE systems, and it follows that more flexible choices of the actuators can be selected in real dynamic systems. In fact, $F_1$ covers a large number of nonlinear functions, such as
\[
F_1(x) = \begin{cases}
2x + e^2 - e + 2 + \ln(x-1), & x > 2, \\
e^x - x - e + 8, & 1 < x \leq 2, \\
5x + 2x^2, & 0 < x \leq 1, \\
5x - 2x^2, & -1 < x \leq 0, \\
- e^{-x} + x + e - 8, & -2 < x \leq -1, \\
2x - e^2 + e - 6 + \ln(-x-1), & x < -2,
\end{cases}
\]
or
\[
F_1(x) = 3x + \cos x + \ln(1 + x^2),
\]
with $k_1 = 1$ and $k_2 = 10$.

To cope with the delay feedback term, we use the technology from [14], where the new dependent variable is introduced by
\[
\lambda(\rho, t) = y(1, t - \tau \rho), \quad \rho \in (0, 1),
\]
which gives
\[
\tau \lambda(\rho, t) + \lambda(\rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0.
\]
Then system (5) can be rewritten as
\[
\begin{cases}
y_{tt} - \mathcal{M}(\|y_{x}(t)\|) y_{xx} + 2 v y_{xt} + y_{xxxx} = 0, \\
\tau \lambda(\rho, t) + \lambda(\rho, t) = 0, \quad \rho \in (0, 1), \\
\mathcal{M}(\|y_{x}(t)\|)^2 y_{x}(1, t) - y_{xxxx}(1, t) - v y_{x}(1, t) = - F_1(\lambda(0, t)) + F_2(\lambda(1, t)), \\
y(0, t) = y_{x}(0, t) = y_{xx}(1, t) = 0, \\
y(x, 0) = h(x), \quad y_{x}(x, 0) = f(x), \\
\lambda(0, t) = y_{t}(1, t), \\
\lambda(\rho, 0) = f_0(1, -\rho \tau), \quad \rho \in (0, 1),
\end{cases}
\]
for all $t > 0$ and $x \in (0, 1)$. Let
\[
E(t) = \frac{1}{2} \|y(\cdot, t)\|^2 + \frac{1}{2} \mathcal{M}(\|y_{x}(\cdot, t)\|^2) + \frac{\xi}{2} \|\lambda(\cdot, t)\|^2 + \frac{1}{2} \|y_{xx}(\cdot, t)\|^2
\]
with $\mathcal{M}(z) = \int_0^z \mathcal{M}(z) \, dz$ and $\tau k_3 < \xi < \tau (2k_1 - k_3)$ is a constant, represents the energy-like function of the axially moving Kirchhoff-beam (9).

Next, the global existence result (well-posedness) of system (9) will be proved via the Faedo-Galerkin method. First, we introduce the following spaces. Set
\[
W_1 = \{y \in H^2(0, 1) : y(0) = y_{x}(0) = 0\}, \\
W_2 = \{y \in H^4(0, 1) \cap W_1 : y_{xx}(1) = 0\},
\]
which are closed subspaces of $H^2(0,1)$ and $H^4(0,1)$ respectively.

**Theorem 2.1.** Assume that $f_0 \in H^1(0,1)$, $h, f \in W_2$, $(H_1)$, $(H_2)$ and the compatibility condition

$$\mathcal{M}(\|h_x\|)h_x(1) - h_{xxx}(1) - vf(1) = -F_1(f(1)) + F_2(f_0(1))$$

(11) hold. Then, the system (9) has a unique global solution which depends continuously on initial conditions. Furthermore, for all $T > 0$,

$$y \in L^\infty([0,T),W_2), \quad y_t \in L^\infty([0,T),W_1), \quad y_{tt} \in L^\infty([0,T),L^2(0,1)).$$

**Proof.** First, we give the variational structure corresponding to (9), as follows

$$\int_0^1 y_{tt}w dx + \int_0^1 y_{xx}w_{xx} dx + \mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_{xx}w_x dx + 2v \int_0^1 y_{xt}w dx$$

$$= [v y_t(1, t) - F_1(\lambda(0, t)) + F_2(\lambda(1, t))]w(1),$$

(12)

for any $w \in W_1$. Suppose that $\{w_i\}_{i=1}^\infty$ which is an orthogonal basis for $L^2(0,1)$ is the complete basis for $W_2$. Taking account of $h, f \in W_2$, without loss of generality, assume $h, f \in \text{Span}\{w_1, w_2\}$. Let $\prod_m := \text{Span}\{w_1, \cdots, w_m\}$ for $m \in N$. Now, for fixed $m \in N$ and $m \geq 2$, set

$$\begin{align*}
  y^m(x,t) &:= \sum_{j=1}^m q_{jm}(t)w_i, \\
  \lambda^m(\rho, \tau) &= y^m_0(1, \tau - \rho),
\end{align*}$$

(13)

with the initial data $y^m(x,0) = h(x)$, $y^m_0(x,0) = f(x)$ and $\lambda^m(\rho, 0) = f_0(1, -\rho \tau)$, for any $\rho \in (0, 1)$, $x \in (0, 1)$ and $t \geq 0$. The Faedo-Galerkin approximation problem is

$$\begin{align*}
  &\int_0^1 y_{tt}^m w dx + \int_0^1 y_{xx}^m w_{xx} dx + 2v \int_0^1 y_{xt}^m w dx + \mathcal{M}(\|y_x^m(t)\|^2) \int_0^1 y_{xx}^m w_x dx \\
  &= [v y_t^m(1, t) - F_1(\lambda^m(0, t)) + F_2(\lambda^m(1, t))]w(1), \\
  y^m(x,t) &= \sum_{j=1}^m q_{jm}(t)w_j(x) \in \prod_m, \\
  \tau \lambda^m(\rho, \tau) &= \lambda^m(\rho, \tau) = 0, \quad \rho \in (0, 1),
\end{align*}$$

(14)

for all $w \in \prod_m$. It is worth noting that an ODE system driven by $q_{jm}(t)$ in (14) are known to have local solutions in the interval $[0, t_m]$, which can be extended to the whole interval $[0, T)$ for any $T > 0$ by the following two important estimates.

**Estimate 1:** $\sup_{m \in N} E_m(t) \leq E(0)$ for almost all $t \geq 0$, where

$$E_m(t) = \frac{1}{2}\|y^m(\cdot, t)\|^2 + \frac{1}{2} \mathcal{M}(\|y_x^m(\cdot, t)\|^2) + \frac{\xi}{2} \|\lambda^m(\cdot, t)\|^2 + \frac{1}{2} \|y_{xx}^m(\cdot, t)\|^2.$$  

(15)

From (15), with (8) we have

$$\frac{dE_m(t)}{dt} = \int_0^1 y_{tt}^m y^m dx + \mathcal{M}(\|y_x^m(t)\|^2) \int_0^1 y_{xx}^m y_{xt}^m dx + \int_0^1 y_{xx}^m y_{xx}^m dx$$

$$- \frac{\xi}{2\tau} (\langle \lambda^m(1, t) \rangle^2 - [\lambda^m(0, t)]^2).$$

(16)

Taking $w = y^m_t$ in (14), and using (16), Young’s inequality (with $\beta = \frac{1}{2}$), it follows that

$$\frac{dE_m(t)}{dt} = [-F_1(\lambda^m(0, t)) + F_2(\lambda^m(1, t))]y^m_t(1, t) - \frac{\xi}{2\tau} (\langle \lambda^m(1, t) \rangle^2 - [\lambda^m(0, t)]^2)$$

$$-2v \int_0^1 y_{xt}^m y_t^m dx + v |y_t^m(1, t)|^2.$$  

(17)
From the slope-restricted condition \((H_1)\) and \((H_2)\), it is easy to get that
\[
F_1(\lambda^m(0, t))\lambda^m(0, t) \geq k_1[\lambda^m(0, t)]^2,
\]
\[
|F_2(\lambda^m(1, t))| \leq k_3|\lambda^m(1, t)|,
\]
which leads to
\[
[-F_1(\lambda^m(0, t)) + F_2(\lambda^m(1, t))]|\lambda^m(0, t)
\]
\[
\leq -k_1[\lambda^m(0, t)]^2 + \frac{k_3}{2}[\lambda^m(1, t)]^2 + \frac{k_3}{2}[\lambda^m(0, t)]^2.
\]  
Notice that \(y^m(0, t) = 0\), then \(y_t^m(0, t) = 0\). This gives
\[
2v \int_0^1 y_{xt}^m y_t^m \, dx = v[y_t^m(1, t)]^2.
\]  
The substitution of (18), (19) into (17) yields
\[
\frac{dE_m(t)}{dt} \leq -k_1[\lambda^m(0, t)]^2 + \frac{k_3}{2}[\lambda^m(1, t)]^2 + \frac{k_3}{2}[\lambda^m(0, t)]^2
\]
\[
- \frac{\xi}{2\tau} ([\lambda^m(1, t)]^2 - [\lambda^m(0, t)]^2)
\]
\[
\leq -L_1[\lambda^m(0, t)]^2 - L_2[\lambda^m(1, t)]^2;
\]  
where \(L_1 = k_1 - \frac{k_3}{2} - \frac{\xi}{2\tau} > 0\) and \(L_2 = \frac{\xi}{2\tau} - \frac{k_3}{2} > 0\) due to \(\tau k_3 < \xi < \tau(2k_1 - k_3)\) and \(k_3 < k_1\). Thus, we have
\[
E_m(t) \leq \frac{1}{2} \|f\|^2 + \frac{1}{2} \hat{\mathcal{M}}(\|h_x\|^2) + \frac{1}{2} \|h_{xx}\|^2 + \frac{\xi}{2} \|f_0\|^2 = E(0),
\]  
which gives the estimate 1.

**Estimate 2:** For any \(T > 0\), there exists a constant \(C_T\) such that
\[
\sup_{m \in \mathbb{N}} \{\|y_{tt}^m(\cdot, t)\|^2 + \|y_{xxt}^m(\cdot, t)\|^2\} \leq C_T,
\]  
for \(t > 0\) a.e.. To begin with, we estimate \(\|y_{tt}^m(0)\|^2 < \infty\). In view of the variational structure regarding (13) and the compatibility condition, by setting \(t = 0\) in (12), we arrive at
\[
\int_0^1 y_{tt}^m(x, 0)w \, dx = \mathcal{M}(\|y_x^m(0)\|^2) \int_0^1 wy_{xx}^m(x, 0) \, dx
\]
\[
+ \int_0^1 wy_{xxt}^m(x, 0) \, dx + 2v \int_0^1 wy_{xt}^m(x, 0) \, dx = 0,
\]  
for any \(w \in W_1\). Taking \(w = y_{tt}^m(0)\) in (23), it follows by applying the initial value condition that
\[
\|y_{tt}^m(0)\|^2 = \mathcal{M}(\|h_x\|^2) \int_0^1 h_{xx}y_{tt}^m(x, 0) \, dx - \int_0^1 y_{tt}^m(x, 0)h_{xxx}(x) \, dx
\]
\[
-2v \int_0^1 y_{tt}^m(x, 0) f_x \, dx.
\]
Apply the Cauchy-Schwarz inequality on (24) to obtain
\[
\mathcal{M}(\|h_x\|^2) \int_0^1 h_{xxx} y_{tt}^m(x,0)dx \leq \mathcal{M}(\|h_x\|^2)\|h_{xxx}\|y_{tt}^m(0),
\]
\[
\int_0^1 y_{tt}^m(x,0) h_{xxx}(x)dx \leq \|y_{tt}^m(0)\|\|h_{xxx}\|,
\]
\[
\int_0^1 y_{tt}^m(x,0) f_x(x)dx \leq \|y_{tt}^m(0)\|\|f_x\|.
\]
The above estimates are inserted into (24) to get
\[
\|y_{tt}^m(0)\| \leq \|h_{xxx}\| + \mathcal{M}(\|h_x\|^2)\|h_{xxx}\| + 2v\|f_x\|.
\] (25)

Then, recalling the variational structure (12), we have
\[
\int_0^1 y_{tt}^m wdx + \mathcal{M}(\|y_{tt}^m(t)\|^2) \int_0^1 y_{tt}^m w_xdx + \int_0^1 y_{xx}^m w_{xx}dx + [F_1(\lambda^m(0,t) - F_2(\lambda^m(1,t)))w(1) + 2v \int_0^1 y_{tt}^m wdx - vy_t(1,t)w(1) = 0,
\] (26)
for any \(w \in W_1\). Let \(t, \alpha > 0\) satisfy \(\alpha < T - t\). Then replace \(t\) with \(t + \alpha\), and subtract (26) to obtain
\[
\int_0^1 (y_{tt}^m(t, t + \alpha) - y_{tt}^m(t, t))w(x)dx + \int_0^1 (y_{xx}^m(t, t + \alpha) - y_{xx}^m(t, t))w_{xx}(x)dx
+ \mathcal{M}(\|y_{tt}^m(t + \alpha)\|^2) \int_0^1 y_{tt}^m(t, t + \alpha)w_x(x)dx - \mathcal{M}(\|y_{tt}^m(t)\|^2) \int_0^1 y_{xx}^m(t, t)w_{xx}(x)dx
+ [F_1(\lambda^m(0, t + \alpha)) - F_2(\lambda^m(1, t + \alpha))]w(1) - [F_1(\lambda^m(0, t)) - F_2(\lambda^m(1, t))]w(1)
+ 2v \int_0^1 (y_{tt}^m(t, t + \alpha) - y_{tt}^m(t, t))w(x)dx - v[y_{tt}^m(1, t + \alpha) - y_{tt}^m(1, t)]w(1) = 0.
\] (27)
Taking \(w = y_{tt}^m(t + \alpha) - y_{tt}^m(t)\) in (27) gives
\[
\frac{1}{2} \frac{d\phi(\alpha, t)}{dt} + P_1 + P_2 + P_3 = 0
\] (28)
where
\[
\phi(\alpha, t) = \|y_{xx}^m(\cdot,t + \alpha) - y_{xx}^m(\cdot,t)\|^2 + \|y_{tt}^m(\cdot,t + \alpha) - y_{tt}^m(\cdot,t)\|^2,
\]
\[
P_1 = \mathcal{M}(\|y_{tt}^m(t + \alpha)\|^2) \int_0^1 y_{tt}^m(t, t + \alpha)[y_{xx}^m(t, t + \alpha) - y_{xx}^m(t, t)]dx
- \mathcal{M}(\|y_{tt}^m(t)\|^2) \int_0^1 y_{tt}^m(t, t)[y_{xx}^m(t, t + \alpha) - y_{xx}^m(t, t)]dx,
\]
\[
P_2 = [F_1(\lambda^m(0, t + \alpha)) - F_2(\lambda^m(1, t + \alpha))]\|\lambda^m(0, t + \alpha) - \lambda^m(0, t)\|
- [F_1(\lambda^m(0, t)) - F_2(\lambda^m(1, t))]\|\lambda^m(0, t + \alpha) - \lambda^m(0, t)\|
\]
\[
P_3 = 2v \int_0^1 (y_{tt}^m(t, t + \alpha) - y_{tt}^m(t, t))[y_{tt}^m(t, t + \alpha) - y_{tt}^m(t, t)]dx
- v[y_{tt}^m(1, t + \alpha) - y_{tt}^m(1, t)]^2.
\] (29)
Let us first estimate $P_1$. From integration by parts we have

\[
P_1 = \mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 y^m_x(x, t + \alpha) [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx
\]

\[
-\mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 y^m_x(x, t) [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx
\]

\[
+\mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 y^m_x(x, t) [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx
\]

\[
-\mathcal{M}(\|y^m_x(t)^2\|) \int_0^1 y^m_x(x, t) [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx
\]

\[
= I_1 + I_2 - I_3,
\]

(30)

where

\[
I_1 = \mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 [y^m_x(x, t + \alpha) - y^m_x(x, t)] [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx,
\]

\[
I_2 = \mathcal{M}(\|y^m_x(t + \alpha)\|^2) - \mathcal{M}(\|y^m_x(t)\|^2) y^m_x(1, t) [y^{m}_{xx}(1, t + \alpha) - y^m_x(1, t)],
\]

\[
I_3 = \mathcal{M}(\|y^m_x(t + \alpha)\|^2) - \mathcal{M}(\|y^m_x(t)\|^2) \int_0^1 y^m_x(x, t) [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx.
\]

Then, applying the estimate 1 and Young’s inequality on $I_1$, we can obtain from integration by parts that

\[
I_1 = \mathcal{M}(\|y^m_x(t + \alpha)\|^2) [y^m_x(1, t + \alpha) - y^m_x(1, t)] [\lambda^m(0, t + \alpha) - \lambda^m(0, t)]
\]

\[
-\mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 [y^m_x(x, t + \alpha) - y^m_x(x, t)] [y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)]dx
\]

\[
\leq \mathcal{M}(\|y^m_x(t + \alpha)\|^2) \int_0^1 [y^m_x(x, t + \alpha) - y^m_x(x, t)]dx [\lambda^m(0, t + \alpha) - \lambda^m(0, t)]
\]

\[
+\mathcal{M}(\|y^m_x(t + \alpha)\|^2) [y^m_x(x, t + \alpha) - y^m_x(x, t)] [\|y^{m}_{xx}(x, t + \alpha) - y^m_x(x, t)\|]
\]

\[
\leq \left(\frac{1}{4\beta} + \frac{1}{2}\right) N_1 [y^m_x(\cdot, t + \alpha) - y^m_x(\cdot, t)]^2 + N_1 \beta [\lambda^m(0, t + \alpha) - \lambda^m(0, t)]^2
\]

\[
+\frac{1}{2} N_1 [y^m_x(\cdot, t + \alpha) - y^m_x(\cdot, t)]^2,
\]

(31)

where $\beta > 0$ is the Young’s parameter and $N_1 > 0$ is a constant. Since $\mathcal{M} \in C^1[0, \infty)$, then let $\hat{N} = \max_{1 \leq s \leq \infty} d\mathcal{M}(s) dx \leq \hat{N} \mathcal{M}(s)\frac{dx}{ds} + \lambda^m(0, t)\frac{dx}{dt}$, and Young’s inequality on $I_2$, there exists constants $N_2, N_3 > 0$ such that

\[
I_2 \leq \left[\mathcal{M}(\|y^m_x(t + \alpha)\|^2) - \mathcal{M}(\|y^m_x(t)\|^2)\right] y^m_x(1, t) [\lambda^m(0, t + \alpha) - \lambda^m(0, t)]
\]

\[
\leq \frac{N_3}{4\beta} [y^m_x(\cdot, t + \alpha) - y^m_x(\cdot, t)]^2 + N_4 \beta [\lambda^m(0, t + \alpha) - \lambda^m(0, t)]^2,
\]

(33)
where \( \hat{\beta} > 0 \) is the Young's parameter. By the estimate 1 and the Cauchy-Schwarz inequality, we can deduce that
\[
I_3 \leq N_5 \| y_{m}^m (\cdot, t + \alpha) - y_{m}^m (\cdot, t) \|^2, \tag{34}
\]
where \( N_5 > 0 \) is a constant. Let \( \beta = \hat{\beta} \) for the convenience of calculation. Putting (31), (33) and (34) into (30) and we have
\[
|P_1| \leq N_6 \| y_{m}^m (\cdot, t + \alpha) - y_{m}^m (\cdot, t) \|^2 \tag{35}
\]
\[
+ (N_1 + N_4) \beta |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2,
\]
where \( N_6 \) is a positive constant. According to the slope-restricted condition \((H_1)\), we have that
\[
[F_1 (\lambda^m (0, t + \alpha)) - F_1 (\lambda^m (0, t))] [\lambda^m (0, t + \alpha) - \lambda^m (0, t)] \geq k_1 |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2, \tag{36}
\]
for \( t \in [0, T] \) a.e.. Thus, for \( P_2 \) we immediately see that
\[
P_2 \geq k_1 |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2 + [F_2 (\lambda^m (1, t + \alpha)) - F_2 (\lambda^m (1, t))] \tag{37}
\]
\[
\times [\lambda^m (0, t + \alpha) - \lambda^m (0, t)],
\]
which together with assumption \((H_2)\) implies that
\[
-P_2 \leq -k_1 |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2 + \frac{k_3}{2} [\lambda^m (1, t + \alpha) - \lambda^m (1, t)]^2
\]
\[
+ \frac{k_3}{2} [\lambda^m (0, t + \alpha) - \lambda^m (0, t)]^2, \tag{38}
\]
for \( t \in [0, T] \) a.e.. Arguing as in (19), one has \( P_3 = 0 \). Substituting (35), (38) and \( P_3 = 0 \) into (28) yields
\[
\frac{1}{2} \frac{d}{dt} \left\{ \phi (\alpha, t) + \xi \| \lambda^m (\cdot, t + \alpha) - \lambda^m (\cdot, t) \|^2 \right\}
\]
\[
\leq N_6 \| y_{xx}^m (\cdot, t + \alpha) - y_{xx}^m (\cdot, t) \|^2 \tag{39}
\]
\[
+ (N_5 + \frac{N_1}{2}) \| y_{t}^m (\cdot, t + \alpha) - y_{t}^m (\cdot, t) \|^2
\]
\[
+ (N_1 + N_4) \beta |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2
\]
\[
+ \frac{k_3}{2} [\lambda^m (1, t + \alpha) - \lambda^m (1, t)]^2 + \frac{k_3}{2} [\lambda^m (0, t + \alpha) - \lambda^m (0, t)]^2
\]
\[
+ \frac{\xi}{2T} [\lambda^m (0, t + \alpha) - \lambda^m (0, t)]^2 - \frac{\xi}{2T} [\lambda^m (1, t + \alpha) - \lambda^m (1, t)]^2
\]
\[
\leq N_6 \| y_{xx}^m (\cdot, t + \alpha) - y_{xx}^m (\cdot, t) \|^2 \tag{40}
\]
\[
+ (N_5 + \frac{N_1}{2}) \| y_{t}^m (\cdot, t + \alpha) - y_{t}^m (\cdot, t) \|^2
\]
\[
+ (N_1 + N_4) \beta |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2
\]
\[
- L_2 |\lambda^m (0, t + \alpha) - \lambda^m (0, t)|^2,
\]
where \( L_1 = k_1 - \frac{k_3}{2} - \frac{\xi}{2T} > 0 \) and \( L_2 = \frac{\xi}{2T} - \frac{k_3}{2} > 0 \) by (20). Choose \( \beta = \frac{L_1}{N_1 + N_4} \)
\[
\text{since } \beta > 0 \text{ is a arbitrary number. Hence, the estimate (39) becomes}
\]
\[
\frac{d \Gamma (\alpha, t)}{dt} \leq 2 N_7 \Gamma (\alpha, t), \tag{40}
\]
where
\[
\Gamma (\alpha, t) = \| y_{xx}^m (\cdot, t + \alpha) - y_{xx}^m (\cdot, t) \|^2 + \| y_{t}^m (\cdot, t + \alpha) - y_{t}^m (\cdot, t) \|^2
\]
\[
+ \xi \| \lambda^m (\cdot, t + \alpha) - \lambda^m (\cdot, t) \|^2.
\]
which follows that \( \phi(\alpha, t) \leq \Gamma(\alpha, 0)e^{2NtT} \). Divide by \( \alpha^2 \) in the above equality and pass to the limit as \( \alpha \to 0 \) to get

\[
\|y_{tt}^m(\cdot, t)\|^2 + \|y_{xxxx}^m(\cdot, t)\|^2 \leq \|\|y_{tt}^m(0)\|^2 + \|f_{xx}\|^2 + \frac{\xi}{\tau^2}\|f_0\|^2\|e^{2NtT},
\]

which with (25) together leads to the estimate 2.

The estimates 1, 2 show

\[
\begin{align*}
\{y^m\}_{m \geq 1} &\text{ is bounded in } L^\infty([0, T); W_1), \\
\{y^m_t\}_{m \geq 1} &\text{ is bounded in } L^\infty([0, T); W_1), \\
\{y^m_{tt}\}_{m \geq 1} &\text{ is bounded in } L^\infty([0, T); L^2(0,1)).
\end{align*}
\]

In view of the Lions’ Lemma, we therefore obtain a subsequence from \( \{y^m\}_{m \geq 1} \in L^\infty([0, T); W_1) \), still denoted by \( \{y^m\}_{m \geq 1} \), and \( y \in L^\infty([0, T); W_1) \) satisfying

\[
\begin{align*}
y^m &\rightharpoonup y \text{ in } L^\infty([0, T); W_1) \text{ weak}^*, \\
y^m_t &\rightharpoonup y_t \text{ in } L^\infty([0, T); W_1) \text{ weak}^*, \\
y^m_{tt} &\rightharpoonup y_{tt} \text{ in } L^\infty([0, T); L^2(0,1)) \text{ weak}^*.
\end{align*}
\]

Applying the estimate 2 reveals that \( \{y_{xx}^m(\cdot, t)\}_{m \geq 1} \) is bounded in \( L^2(0, 1) \), which leads to that \( \{y_{xx}^m(\cdot, t)\}_{m = 1} \) is compact in \( L^2(0,1) \). This, together with \( y_1^m(1, t) = \int_0^1 y_1^m dx \), means that there is a subsequence of \( y_1^m(1, t) \) still denoted by itself, such that \( y_1^m(1, t) \rightharpoonup y_1(t, t) \) for almost every \( t \in [0, T) \). An application of the Lebesgue dominant convergence theorem with the continuity of \( F_i \) (i=1,2), gives \( F_1(y_1^m(1, t)) \rightharpoonup F_1(y_1(t, t)) \) and \( F_2(y_1^m(1, t - \tau)) \rightharpoonup F_2(y_1(1, t - \tau)) \). Based on the Aubin-Lions theorem, it follows that

\[
\begin{align*}
\int_0^1 y_{tt} wdx &+ \int_0^1 y_{xx} w_{xx} dx + 2v \int_0^1 y_{xx} wdx + \mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_{x} w_x dx \\
&= [v y_1(t) + F_1(\lambda(0, t)) + F_2(\lambda(1, t))]w(1), \quad \forall w \in W_1, \quad \text{and} \quad t \in [0, T) \text{ a.e.}
\end{align*}
\]

by passing to the limit as \( m \to \infty \) to (14). Let \( w \in W_0 := \{w \in W_1 : w(1) = 0\} \), then from (42) we have

\[
\begin{align*}
\int_0^1 y_{tt} wdx &+ \int_0^1 y_{xx} w_{xx} dx + 2v \int_0^1 y_{xx} wdx = -\mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_{x} w_x dx
\end{align*}
\]

for a.e. \( t \in [0, T) \), which yields the existence of generalized derivatives \( y_{xxxx} \), i.e. \( y \in W_2 \), and

\[
y_{tt} + y_{xxxx} + 2vy_{xx} = \mathcal{M}(\|y_x(t)\|^2)y_{xx} \in L^2(0, 1).
\]

An integration by parts on (42) yields

\[
\begin{align*}
\int_0^1 y_{tt} wdx &+ \int_0^1 y_{xxxx} w_{xx} dx + 2v \int_0^1 y_{xx} wdx - \mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_{x} w_x dx \\
&= [v y_1(t) - F_1(\lambda(0, t)) + F_2(\lambda(1, t))]w(1),
\end{align*}
\]

which follows by applying (44) that

\[
\mathcal{M}(\|y_x(t)\|^2)y_1(t) - y_{xxx}(1, t) = v y_1(t) - F_1(\lambda(0, t)) + F_2(\lambda(1, t)).
\]

Consequently, we can conclude the existence of the global solution to the closed-loop system (9) in \([0, T)\), for all \( T > 0 \).
Next, we prove the uniqueness of the solution. Let \( y, \tilde{y} \) be two solutions of the closed-loop system (9) with the same initial value. Using \( \tilde{y} \) instead of \( y \) in (12), and then subtract (12) to get
\[
\int_{0}^{1} (\tilde{y}_{tt}(x,t) - y_{tt}(x,t))w(x)dx + \int_{0}^{1} (\tilde{y}_{xx}(x,t) - y_{xx}(x,t))w_{xx}(x)dx
\]
\[
+ \mathcal{M}(\|\tilde{y}_{x}(t)\|^{2}) \int_{0}^{1} \tilde{y}_{x}(x,t)w_{x}(x)dx - \mathcal{M}(\|y_{x}(t)\|^{2}) \int_{0}^{1} y_{x}(x,t)w_{x}(x)dx
\]
\[
+ [F_{1}(\tilde{\lambda}(0,t)) - F_{2}(\tilde{\lambda}(1,t))]w(1) - [F_{1}(\lambda(0,t)) - F_{2}(\lambda(1,t))]w(1)
\]
\[
+ 2v \int_{0}^{1} (\tilde{y}_{x}(x,t) - y_{x}(x,t))w(x)dx - v[\tilde{y}_{t}(1,t) - y_{t}(1,t)]w(1) = 0. \tag{47}
\]
Taking \( w = \tilde{y} - y \) and arguing as in (28), we arrive at
\[
\frac{d\Upsilon(t)}{dt} \leq 2N_{T}\Upsilon(t), \tag{48}
\]
where \( N_{T} \) is the constant stated in (40) and
\[
\Upsilon(t) = \|\tilde{y}_{xx}(\cdot,t) - y_{xx}(\cdot,t)\|^{2} + \|\tilde{y}_{t}(\cdot,t) - y_{t}(\cdot,t)\|^{2} + \xi\|\tilde{\lambda}(\cdot,t) - \lambda(\cdot,t)\|^{2},
\]
which yields with \( \Upsilon(0) = 0 \) the uniqueness of the solution.

Finally, the continuous dependence of solutions regarding initial functions needs to be proved. For this, let \( y' \) be the solutions of the closed-loop system (9) for the initial value \( (h', f', f_{0}') \in W_{2} \times W_{2} \times H^{1}(0,1) \) satisfying \( h' \rightarrow h, f' \rightarrow f \) in \( W_{2} \) and \( f_{0}' \rightarrow f_{0} \) in \( H^{1}(0,1) \). Then in a similar fashion, from (48) it is easy to show that
\[
\mathcal{R}(t) \leq \mathcal{R}(0)e^{2N_{T}t}, \tag{49}
\]
where
\[
\mathcal{R}(t) = \|y''_{xx}(\cdot,t) - y_{xx}(\cdot,t)\|^{2} + \|y''_{t}(\cdot,t) - y_{t}(\cdot,t)\|^{2} + \xi\|\lambda'(\cdot,t) - \lambda(\cdot,t)\|^{2},
\]
and
\[
\mathcal{R}(0) = \|h''_{xx} - h_{xx}\|^{2} + \|f'' - f\|^{2} + \xi\|f_{0}' - f_{0}\|^{2},
\]
which implies that \( y' \rightarrow y \) in \( H^{2}(0,1) \) for any \( t > 0 \) as \( \epsilon \rightarrow \infty \). This completes the proof of Theorem 2.1. \( \square \)

3. Stability analysis. This section considers the exponential stability of the closed-loop system (9) by the direct Lyapunov method. For this a novel Lyapunov function needs to be constructed to reject the nonlinear delay term. Now we state a lemma which shows the non-increasing of the energy function \( E(t) \) from the estimate 1 of Theorem 2.1.

**Lemma 3.1.** The energy functional \( E(t) \) defined in (10) satisfies along solutions of (9)
\[
\frac{dE(t)}{dt} \leq -L_{1}\lambda^{2}(0,t) - L_{2}\lambda^{2}(1,t), \tag{50}
\]
where \( L_{1} = k_{1} - \frac{k_{2}}{2} - \frac{k_{3}}{2T} > 0 \) and \( L_{2} = \frac{k_{3}}{2T} - \frac{k_{4}}{2} > 0 \) are given in (20).

**Remark 2.** Integrate over \([0,t] \) on (50) to obtain
\[
L_{1} \int_{0}^{t} \lambda^{2}(0,s)ds + L_{2} \int_{0}^{t} \lambda^{2}(1,s)ds \leq E(0) \text{ for any } t > 0.
\]
Combining this with assumptions \((H_{1})\) and \((H_{2})\) guarantees that the control input \( F_{1}(\lambda(0,\cdot)) \) and the delay disturbance \( F_{2}(\lambda(1,\cdot)) \) are uniformly bounded, i.e. \( F_{1}(\lambda(0,\cdot)), F_{2}(\lambda(1,\cdot)) \in L^{2}(0, +\infty) \).
The modified energy function \( E(t) \) along the solution of (9) is defined by
\[
E(t) = E(t) + \mu(\varphi_1(t) + \varphi_2(t)),
\]
where \( \mu \) is a positive constant and
\[
\varphi_1(t) = \int_0^1 x(y_x y_t + vy_x^2)dx,
\]
\[
\varphi_2(t) = \xi \int_0^1 e^{-2\sigma \chi^2/\chi^2} \lambda^2(\rho, t)d\rho,
\]
with \( \tau k_3 < \xi < \tau(2k_1 - k_3) \) for any \( t \geq 0 \). The equivalence relationship between \( E(t) \) and \( E(t) \) is established in the following result.

**Lemma 3.2.** Suppose that the constant \( \mu \) satisfies
\[
0 < \mu < \frac{1}{\max\{2, \frac{L_2}{m_0}\}}.
\]
Then
\[
\chi_1 E(t) \leq E(t) \leq \chi_2 E(t), \quad \forall t \geq 0,
\]
where \( \chi_1 = 1 - \mu \max\{2, \frac{1+2v}{m_0}\} \) and \( \chi_2 = 1 + \mu \max\{2, \frac{1+2v}{m_0}\} \).

**Proof.** From (51), we can deduce that
\[
\varphi_1(t) + \varphi_2(t) = \int_0^1 x(y_x y_t + vy_x^2)dx + \xi \int_0^1 e^{-2\sigma \chi^2} \lambda^2(\rho, t)d\rho
\leq \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1+2v}{2} \int_0^1 y_x^2 dx + \xi \int_0^1 \lambda^2(\rho, t)d\rho
\leq \frac{1}{2} \|y_x(\cdot, t)\|^2 + \frac{1+2v}{2m_0} \mathcal{M}(\|y_x(\cdot, t)\|^2) + \xi \|\lambda(\cdot, t)\|^2
\leq \max\{2, \frac{1+2v}{m_0}\} E(t)
\]
for all \( t \geq 0 \). Analogously, it is easy to get
\[
\varphi_1(t) + \varphi_2(t) \geq -\max\{2, \frac{1+2v}{m_0}\} E(t),
\]
which together with (51) gives the estimate (53). \( \square \)

Now, we state the exponential stability of the energy function \( E(t) \) defined in (10) along the solution of (9).

**Theorem 3.3.** Let all assumptions of Theorem 2.1 be satisfied and the real number \( \mu \) satisfies
\[
\mu < \min\left\{ \frac{1}{\max\{2, \frac{1+2v}{m_0}\}}, \frac{L_1}{K_1 + \xi/\tau}, \frac{L_2}{K_2} \right\},
\]
where \( K_1 = \frac{4(\varepsilon^2 + k_2^2) + m_0}{2m_0} \) and \( K_2 = \frac{\varepsilon^2}{m_0} \), and \( L_1, L_2 \) are the constants stated in (20).
Then there exist constants \( K > 0 \) and \( \theta > 0 \) such that \( E(t) \leq Ke^{-\theta t} \) for all \( t \geq 0 \).

**Proof.** It follows from (51) that
\[
\frac{dE(t)}{dt} = \mu \frac{d\varphi_1(t)}{dt} + \mu \frac{d\varphi_2(t)}{dt},
\]
where
\[
\frac{d\varphi_1(t)}{dt} = \int_0^1 xy_x y_tdx + \int_0^1 xyy_x tdx + 2v \int_0^1 xy_x y_tdx,
\]
(58)
and
\[
\frac{d\varphi_2(t)}{dt} = 2\xi \int_0^1 e^{-2\rho \tau} \lambda(\rho, t) \lambda_t(\rho, t) \, d\rho
\]
\[
= -\frac{\xi}{\tau} \int_0^1 e^{-2\rho \tau} \frac{\partial}{\partial \rho} \lambda^2(\rho, t) \, d\rho
\]
\[
= -\frac{\xi}{\tau} (e^{-2\tau} \lambda^2(1, t) - \lambda^2(0, t)) - 2\xi \int_0^1 e^{-2\rho \tau} \lambda^2(\rho, t) \, d\rho
\]  
(59)

using (8) and integration by parts. To deal with (58), we first analyse the term
\[
\int_0^1 x_y y_t \, dx.
\]
In light of the system equation (9), integration by parts implies
\[
\int_0^1 x_y y_t \, dx = M(\|y_x(t)\|^2) \int_0^1 x_y y_{xx} \, dx - \int_0^1 x_y y_{xxx} \, dx - 2v \int_0^1 x_y y_{xt} \, dx
\]
\[
= \frac{1}{2} M(\|y_x(t)\|^2) y_x^2(1, t) - \frac{1}{2} M(\|y_x(t)\|^2) \int_0^1 y_t^2 \, dx
\]
\[
y_x(1, t) y_{xxx}(1, t) - \frac{3}{2} \int_0^1 y_{xx}^2 \, dx - 2v \int_0^1 x_y y_{xt} \, dx,
\]  
(60)

where we use the following formulas
\[
M(\|y_x(t)\|^2) \int_0^1 x_y y_{xx} \, dx = \frac{1}{2} M(\|y_x(t)\|^2) \int_0^1 (x y_x^2)_x \, dx
\]
\[
= \frac{1}{2} M(\|y_x(t)\|^2) \int_0^1 y_x^2 \, dx
\]
and
\[
\int_0^1 x_y y_{xxx} \, dx = y_x(1, t) y_{xxx}(1, t) - \int_0^1 y_{xxx}(y_x + x y_{xx}) \, dx
\]
\[
= y_x(1, t) y_{xxx}(1, t) + \int_0^1 y_{xx}(2 y_{xx} + x y_{xxx}) \, dx
\]
\[
= y_x(1, t) y_{xxx}(1, t) + 2 \int_0^1 y_{xx}^2 \, dx
\]
\[
- \frac{1}{2} \int_0^1 y_{xx}^2 \, dx + \frac{1}{2} \int_0^1 (x y_{xxx})_x \, dx
\]
\[
= y_x(1, t) y_{xxx}(1, t) + \frac{3}{2} \int_0^1 y_{xx}^2 \, dx.
\]
Likewise, we have
\[
\int_0^1 x_y y_{xt} \, dx = \frac{1}{2} \int_0^1 (x y_t^2)_x \, dx - \frac{1}{2} \int_0^1 y_{xt}^2 \, dx
\]
\[
= \frac{1}{2} y_t^2(1, t) - \frac{1}{2} \int_0^1 y_{xt}^2 \, dx.
\]  
(61)
The substitution of (60) and (61) into (58) shows that
\[
\frac{d}{dt} \varphi_1(t) = \frac{1}{2} \lambda^2(0, t) - \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) y_x^2(1, t)
\]
\[
- y_x(1, t)y_{xxx}(1, t) - \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_x^2 dx - \frac{3}{2} \int_0^1 y_{xx}^2 dx.
\]
Considering the boundary value condition (9), we obtain from Young’s inequality that
\[
\mathcal{M}(\|y_x(t)\|^2) y_x^2(1, t) - y_x(1, t)y_{xxx}(1, t)
\]
\[
= y_x(1, t)[v y_x(t) - F_1(\lambda(0, t)) + F_2(\lambda(1, t))]
\]
\[
= y_x(1, t)[v y_x(t) - F_1(\lambda(0, t)) + F_2(\lambda(1, t))]
\]
\[
\leq 2\eta y_x^2(1, t) + \frac{2v^2 + 2k_0^2}{4\eta} - \frac{1}{2} \lambda^2(0, t) + \frac{k_0^2}{4\eta} \lambda^2(1, t).
\]
Since \(\mathcal{M} \in C^1[0, \infty)\) with \(\mathcal{M}(s) \geq m_0\) and \(D_0 = \max\{\mathcal{M}(s) | 0 \leq s \leq \max_{t \geq 0} \|y_x(t)\|^2\}\) < \(\infty\), then \(\frac{\mathcal{M}(\|y_x(t)\|^2)}{\|y_x(t)\|^2} \leq \mathcal{M}(\|y_x(t)\|^2) \int_0^1 y_x^2 dx\). Insert this with (63) into (62) to obtain
\[
\frac{d}{dt} \varphi_1(t) \leq - \frac{m_0}{D_0} \left[ \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) \right] + \frac{1}{2} \int_0^1 y_{xx}^2 dx
\]
\[
+ 2\eta y_x^2(1, t) + \left[ \frac{2v^2 + 2k_0^2}{4\eta} + \frac{1}{2} \right] \lambda^2(0, t) + \frac{k_0^2}{4\eta} \lambda^2(1, t)
\]
\[
- \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) y_x^2(1, t).
\]
Due to the arbitrariness of Young’s parameters \(\eta\), let \(\eta = \frac{m_0}{D_0}\), then (64) becomes
\[
\frac{d}{dt} \varphi_1(t) \leq - \frac{m_0}{D_0} \left[ \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) \right] + \frac{1}{2} \int_0^1 y_{xx}^2 dx
\]
\[
+ \hat{K}_1 \lambda^2(0, t) + \hat{K}_2 \lambda^2(1, t),
\]
where \(\hat{K}_1 = \frac{4(v^2 + k_0^2) + m_0}{2m_0}\) and \(\hat{K}_2 = \frac{k_0^2}{m_0}\). Putting (50), (59) and (65) into (57) shows that
\[
\frac{d\mathcal{E}(t)}{dt} \leq - \frac{m_0 \mu}{D_0} \left[ \frac{1}{2} \int_0^1 y_x^2 dx + \frac{1}{2} \mathcal{M}(\|y_x(t)\|^2) \right]
\]
\[
+ \frac{1}{2} \int_0^1 y_{xx}^2 dx + 2\xi \int_0^1 e^{-2\rho^2} \lambda^2(\rho, t) d\rho
\]
\[
\leq - \frac{m_0 \mu}{D_0} \min\left\{ \frac{L_1}{K_1 + \xi}, \frac{L_2}{K_2} \right\} \mathcal{E}(t).
\]
Notice that \(\mu < \min\left\{ \frac{L_1}{K_1 + \xi}, \frac{L_2}{K_2} \right\}\) by (56), we can find from (53) that
\[
\frac{d\mathcal{E}(t)}{dt} \leq - \frac{m_0 \mu \min\{4e^{-2\tau}, 1\}}{\chi_2 D_0} \mathcal{E}(t),
\]
\[
(67)
\]
which yields that
\[ \mathcal{E}(t) \leq \mathcal{E}(0)e^{-m_0\min\{4e^{-2\tau_1}, 1\} t}, \quad \forall t > 0, \] (68)
by an integration of (67) on (0, t). Using (53), we obtain the existence of \( \theta, K > 0 \) such that \( \mathcal{E}(t) \leq Ke^{-\theta t} \), which ends the proof. \( \square \)

**Remark 3.** Under the assumption of Theorem 3.3, applying the Sobolev inequality and \( m_0\|y_x\|^2 \leq \hat{M}(\|y_x\|^2) \) implies \( |y(x,t)| \leq \|y_x\| \leq \sqrt{2E(t)}/m_0 \) for any \( x \in (0, 1) \) and \( t > 0 \). Following Theorem 3.3, we can conclude that the state response of the axially moving Kirchhoff beam decays exponentially. In addition, if the disturbance has no delay \( (\tau = 0) \) in (5), it is easy to get that all results obtained in the paper still hold along this line. It’s worth noting that the existence result and exponential stability of the non-moving Kirchhoff beam (i.e. \( v = 0 \) in (5)) follows easily by means of the method adopted here.

4. **Conclusion.** In this paper, the stabilization problem of the axially moving Kirchhoff beam is considered. First the nonlinear damping criterion based on the the slope-restricted condition is proposed, then in this framework when the nonlinear delay disturbance is produced at the boundary, the existence result and exponential stability of the closed-loop system are proved using the Faedo-Galerkin approximation method and the direct Lyapunov method respectively. If the disturbance is the uncertainty not depending on the boundary velocity, the stability analysis of the axially moving Kirchhoff beam is still an open problem, which will be the focus of future work.

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