Calculation of the Energy of Permanent Magnets by the Method of Point Magnetic Moments

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Abstract. The article proposes to use the gridless method of point magnetic moments to calculate the energy of devices with permanent magnets. The method makes it possible to exclude the operations of integration over the volume and surfaces, replacing them by summing the contributions of the moments, and significantly reducing the total number of unknowns. The method allows the use of well-known sources, namely charges. The stages of the method implementation are given and examples of its implementation are considered. The calculation results coincide with the known analytical solutions for bodies in the form of a ball and a cylinder. Numerical analysis showed that the total field of a permanent magnet, including the field of the permanent magnet itself and the field of the surrounding space, consisting of piecewise homogeneous subdomains, is equal to zero. An example of the placement of point magnetic moments in a permanent magnet in the form of a parallelepiped is shown. The calculation of its field showed the effectiveness of the method in comparison with the finite element method. The results obtained make it possible to recommend the use of the method of point magnetic moments for calculating the energy of permanent magnets.

1. Introduction
The widespread use of permanent magnets (PM) in electrical devices in recent years is due to the fact that the use of PM allows you to save energy, reduce the size and weight of equipment. The development of effective methods for their analysis and synthesis is becoming relevant to accelerate the introduction of such devices into practice.

In recent years, the gridless method of fundamental solutions (MFS) [1–4] and its variants in combination with grid methods (finite differences or finite elements) [5–9] have been widely used in field calculations. The idea of the method is to represent the approximate solution of the problem in the form of a superposition of the fields of point field sources (charges, dipoles) located outside the solution region at some distance from its boundary. However, in the field of application of the method, as well as in its theoretical foundations and development, there are still a number of limitations. Point charges are used to analyze potential fields. The use of point magnetic dipoles instead of magnetic charges made it possible to increase the accuracy of solutions, but in a number of cases there was a numerical instability [5].

This article is devoted to the application of the previously proposed by the authors of the article of gridless method of point magnetic moments (MPMM) for calculating the energy of devices with PM [10–13]. The method is distinguished by the use of vector point sources (moments) that provide a reduction in the dimension of the system of algebraic equations and the elimination of the numerical instability of solutions in unbounded domains containing subdomains with nonlinear characteristics of materials.
The force and torque of the devices can be determined by the change in energy:

\[ F = -\frac{\partial E}{\partial x}, \quad M = -\frac{\partial E}{\partial \theta} \]

where \( E \) – energy, \( x \) – variable representing the position of the object on which the force is determined, \( \theta \) – angular displacement [14].

2. Method and examples

The essence of MPMM consists in replacing sources with distributed parameters (PM with magnetization \( \vec{M} \)) by a set of point sources. Each point of \( i \)-th source is characterized by a vector quantity \( \vec{m}_i \), called the magnetic moment. The PM fields can be equalizing with high accuracy by choosing the number of values of the modules \( m_i \) and the location of point sources. In this case, the integration over volumes and surfaces is replaced by the summation of the contributions of point sources, which simplifies the calculations. The method allows the use of well-known sources, namely, charges.

The direction of the vector \( \vec{m} \) coincides with the direction of the magnetization \( \vec{M} \).

The use of MPMM consists in performing the following steps. At the first stage, the form of representation of the scalar potentials of the fields created by point sources is selected. In this case, it is assumed that the PM is uniformly magnetized with \( \vec{M} = \text{const} \), which has the magnetic permeability of vacuum \( \mu_0 \).

At the second stage, the placement of point sources is performed, a mathematical model is drawn up, and the modules of the magnetic moments \( m_i \) are calculated. At the third stage, formulas are drawn up and the fields and energy of subdomains of magnetic systems with PM are calculated.

The use of MPMM is shown in the examples below. Let it be required to determine the energy of the magnetic field in the region filled with the PM, and in the space surrounding the PM by the MPMM. The PM has the shape of a ball with \( \vec{M} = M\vec{e}_z \), the magnetic permeability of the ball \( \mu_b = \mu_0 \). The equation in a spherical coordinate system for the scalar potential of an object with axial symmetry ( \( \partial \phi/\partial \alpha = 0 \) ) has the form

\[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0 \]

with a solution in the first area occupied by the PM,

\[ \phi^{(1)}(\eta, \alpha) = \frac{M}{3} \eta \cos \theta \]

and in the second region (the space surrounding the PM)

\[ \phi^{(2)}(\eta, \alpha) = \frac{MR^3}{3\eta^2} \cos \theta \]

which can be obtained using known approaches [15].

The representation of the scalar potentials of the fields created by point sources can be represented in forms that satisfy equation (1) (Fig. 1), that is, in the form

\[ \phi^{(1)} = m_1 \eta \cos \theta \]

\[ \phi^{(2)} = \left( \frac{m_2}{\eta^2} \right) \frac{\cos \theta}{\eta} \]

The force and torque of the devices can be determined by the change in energy:
where \((\vec{m}_1, \vec{n})\), \((\vec{m}_2, \vec{n}_2)\) – dot product of vectors.

The relation (5) was obtained by K. Shimoni for the electric field by the passage to the limit \(l \to 0\), where \(l\) is the distance between the charges of the dipole [16].

In the second step, point sources are placed. Obviously, in the case of axial symmetry, both point sources should be located at the origin (point 0, Fig. 1).

It is necessary to use the boundary conditions at the interface \((r = R)\) to determine \(m_1\) and \(m_2\): continuity of the scalar potential and the normal component of the magnetic induction

\[
\varphi^{(1)}(R,0) = \varphi^{(2)}(R,0); \quad \mu_0(M + H_r^{(1)}) = \mu_0H_r^{(2)}
\]

(6)

The components of the gradients are determined by the formulas

\[
H_r^{(2)} = -\frac{\partial \varphi^{(2)}}{\partial r} = \frac{2m_2}{r^3}\cos \theta; \quad H_r^{(2)} = -\frac{\partial \varphi^{(2)}}{\partial r} = \frac{2m_2}{r^3}\cos \theta;
\]

\[
H_{\alpha}^{(1)} = -\frac{1}{r} \frac{\partial \varphi^{(1)}}{\partial \alpha} = m_1 \sin \theta; \quad H_{\alpha}^{(2)} = -\frac{\partial \varphi^{(2)}}{\partial \alpha} = \frac{m_2}{r^3}\sin \theta
\]

(7)

The following system is obtained by substituting (7) in (6)

\[
m_1R = \frac{m_2}{R^3}; \quad M - m_1 = \frac{2m_2}{r^3}\;
\]

(8)

The solution to system (8) allows obtaining

\[
m_1 = \frac{M}{3}; \quad m_2 = \frac{MR^3}{3}
\]

(9)

Representations coinciding with the analytical solution of problem (2) and (3) are obtained by substituting (9) in (4) and (5). Thus, for a ball-shaped PM, the MPMM gives an exact result: the magnetic fields of the PM and point moments are identical.

The third stage is the calculation of the field energy in \(V_1\) and \(V_2\) by the MPMM.
Taking into account that the fields $\vec{H}^{(1)}$, $\vec{M}$, $\vec{B}^{(1)}$ in $V_1$ (PM), are homogeneous, the energy in $V_1$ can be calculated by the formula

$$E_{pm} = \frac{B^{(1)} H^{(1)}}{2} V_1 = \mu_0 (M + H^{(1)}) \frac{4}{3} \pi R^3 = \frac{2 \pi \mu_0}{3} R^3 (M - m_1) (-m_1)$$

Taking into account (9), is obtained

$$E_{pm} = -\frac{4}{27} \pi \mu_0 R^3 M^2$$

Formula (10) can be represented as

$$E_{pm} = E_{ppm} + E_{hpm}$$

where

$$E_{ppm} = -\frac{2}{3} \pi \mu_0 R^3 M m_1, \quad E_{hpm} = \frac{2}{3} \pi \mu_0 m_1^2.$$

The $E_{ppm}$ value is called the demagnetization energy, and $E_{hpm}$ is the field energy $\vec{H}^{(1)}$ [17]. The energy $E_{pm}$ is negative, since the magnetic induction $\vec{B}^{(1)}$ and the magnetic field strength $\vec{H}^{(1)}$ in the PM are opposite.

The energy in the space $V_2$ surrounding the PM (Fig. 2) is determined by formula

$$E_{air} = \iiint_{V_2} \frac{B^{(2)} H^{(2)}}{2} dV = \frac{\mu_0}{2} \iiint_{V_2} \left( H^{(2)} \right)^2 dV$$

where

$$dV = drd\theta d\phi d\alpha$$

$$\left( H^{(2)} \right)^2 = \left( H_r^{(2)} \right)^2 + \left( H_\theta^{(2)} \right)^2 = \frac{4m_2^2}{r^6} \cos^2 \theta + \frac{m_2^2}{r^6} \sin^2 \theta$$

**Figure 2.** For calculation $E_{air}$.
Substituting (12) and (13) in (11), is obtained
\[
E_{air} = \frac{\mu_0}{2} \int_0^{2\pi} \int_0^\infty \int_0^R \left(4\cos^2 \theta + \sin^2 \theta\right)^2 \sin \theta \sin \phi \, d\theta \, d\phi \, dr \, d\alpha = \frac{4}{3} \pi \mu_0 m^2 \frac{m^2}{R^3}
\]

Taking into account (9), is obtained
\[
E_{air} = \frac{4}{27} \pi \mu_0 R^3 M^2
\]

The obtained values of \(E_{pm}\) and \(E_{air}\) satisfy condition [17]
\[
E_{pm} + E_{air} = 0
\]

As a second example, the energy in a magnetic system with a PM in the form of a uniformly magnetized cylinder with four subregions is calculated: 1 – PM \((\vec{M} = \text{const}, \mu_0)\); 2 – air \((\mu_0)\); 3 – steel \((\mu_{st} = \mu_t \mu_0)\); 4 – air \((\mu_0)\) (Fig. 3).

It is necessary to make sure in the fulfillment of condition: The total energy of the magnetic system with PM is zero, that is
\[
\sum_{k=1}^{4} E_k = 0
\]

where \(k\) – the number of the subdomain in Fig. 3.

![Figure 3. Magnetic system with PM in the form of a cylinder.](image)

Let's move on to the first stage: the construction of a representation of the scalar potentials of the fields of subdomains created by point sources, namely, magnetic moments.

The equation of a plane-parallel magnetic field in a cylindrical coordinate system, taking into account equality \(\frac{\partial \phi}{\partial z} = 0\), has the form [15]
with a solution in general form \[11\]
\[
\phi(r,\alpha) = c_1 r \cos \alpha + \frac{c_2}{r} \cos \alpha
\]
(15)

Representations for scalar potentials of fields created by point magnetic moments are chosen based on (15), taking into account that the scalar potential must have finite values everywhere:

– in the subdomain 1
\[
\phi^{(1)} = m_1 r \cos \alpha
\]
(16)

– in the subdomain 2
\[
\phi^{(2)} = -m_2 r \cos \alpha + \frac{m_3}{r} \cos \alpha
\]
(17)

– in the subdomain 3
\[
\phi^{(3)} = -m_4 r \cos \alpha + \frac{m_5}{r} \cos \alpha
\]
(18)

– in the subdomain 4
\[
\phi^{(4)} = \frac{m_6}{r} \cos \alpha
\]
(19)

The components of the potential gradients required when calculating the energies are determined by the formulas:

\[
H_r^{(1)} = -m_1 \cos \alpha; \quad H_\alpha^{(1)} = m_1 \sin \alpha;
\]

\[
H_r^{(2)} = \left( m_2 + \frac{m_3}{r^2} \right) \cos \alpha; \quad H_\alpha^{(2)} = \left( -m_2 + \frac{m_3}{r^2} \right) \sin \alpha;
\]

\[
H_r^{(3)} = \left( m_4 + \frac{m_5}{r^2} \right) \cos \alpha; \quad H_\alpha^{(3)} = \left( -m_4 + \frac{m_5}{r^2} \right) \sin \alpha;
\]

\[
H_r^{(4)} = \frac{m_6}{r^2} \cos \alpha; \quad H_\alpha^{(4)} = \frac{m_6}{r^2} \sin \alpha
\]

At the second stage, point sources \( \tilde{m}_1 - \tilde{m}_6 \) are placed at point 0 due to the axial symmetry of the problem.

The boundary conditions (6) for each collocation point \( N_1, N_2, N_3 \) are used to determine the modules of magnetic moments

The result is a system of six equations with six unknowns
\begin{align}
    m_1 R_1 &= -m_2 R_1 + \frac{m_3}{R_1};
    M - m_1 &= m_2 + \frac{m_3}{R_2};
    -m_2 R_2 + \frac{m_3}{R_2} &= -m_4 R_2 + \frac{m_5}{R_2};
    m_2 + \frac{m_3}{R_2} &= \mu_r \left( m_4 + \frac{m_5}{R_2} \right); \\
    -m_4 R_3 + \frac{m_5}{R_3} &= \frac{m_6}{R_3};
    \mu_r \left( m_4 + \frac{m_5}{R_3} \right) &= \frac{m_6}{R_3}.
\end{align}

The values of the modules of the magnetic moments are determined by solving of the system (20):

\begin{align}
    m_3 &= \frac{MR_1^2}{2}; \\
    m_4 &= \frac{MR_1^2 (1 - \mu_r)}{2}; \\
    m_5 &= \frac{MR_1^2 R_2^2 (1 + \mu_r)}{2}; \\
    m_6 &= \frac{2 \mu_r MR_1^2 R_2^2}{a_1}; \\
    m_2 &= \frac{m_2}{R_2^2} + m_4 - \frac{m_5}{R_2^2}; \\
    m_1 &= \frac{m_3}{R_1^2} - m_2.
\end{align}

where \( a_1 = R_3^2 (1 + \mu_r)^2 - R_2^2 (1 + \mu_r) \).

It is easy to verify that representations (16) – (19) with relations (21) coincide with the analytical solution of the problem [15].

At the third stage, the energy in the subdomains of the magnetic system with PM is calculated (Fig. 3).

In the first subdomain (filled with PM), the energy is determined by the formula

\begin{align}
    E_1 &= E_{pm} = \frac{B_1^{(i)} H_1^{(i)}}{2} V_1 = \frac{\pi \mu_0 L R_1^2 (M + H_2)}{2} = \frac{\pi \mu_0 L R_1^2 (M - m_1) (-m_1)}{2}
\end{align}

where \( m_1 \) is taken from the ratios (21), \( L \) – length of cylinders on the axis \( 0z \).

In the second subdomain, the energy is determined by the formula

\begin{align}
    E_2 &= E_{air} = \int \int V_2 \frac{B_2^{(i)} H_2^{(i)}}{2} dV = \frac{\mu_0 L \int \int (H_2^{(i)})^2 dD}{2} = \frac{\mu_0 L \int \int (H_2^{(i)})^2 dD}{2}
\end{align}
where \( dD = drd\alpha \), \( (H_r^{(2)})^2 = (H_r^{(2)})^2 + (H_\alpha^{(2)})^2 = \left(m_3 + \frac{m_5}{r^2}\right)^2 \cos^2 \alpha + \left(-m_2 + \frac{m_3}{r^2}\right)^2 \sin^2 \alpha \), where \( m_2 \) and \( m_3 \) is taken from (21).

After integration is obtained

\[
E_2 = \frac{\pi \mu_0}{2} L \left(R_2^2 - R_1^2\right) \left(m_3^2 + \frac{m_5^2}{R_1^2 R_2^2}\right)
\]

(23)

In the third subdomain, the energy is determined by formula

\[
E_3 = E_{Fe} = \left[\left[H_r^{(3)}\right]^2 + \left(H_\alpha^{(3)}\right)^2\right] = \left(m_4 + \frac{m_5}{r^2}\right)^2 \cos^2 \alpha + \left(-m_4 + \frac{m_5}{r^2}\right)^2 \sin^2 \alpha
\]

After integration is obtained

\[
E_3 = \frac{\pi \mu_0 \mu_r}{2} L \left(R_3^2 - R_2^2\right) \left(m_4^2 + \frac{m_5^2}{R_2^2 R_3^2}\right)
\]

(24)

In the fourth subdomain, the energy is determined by formula

\[
E_4 = E_{air2} = \left[\left[H_r^{(4)}\right]^2 + \left(H_\alpha^{(4)}\right)^2\right] = \frac{m_6^2}{r^4} \cos^2 \alpha + \frac{m_6^2}{r^4} \sin^2 \alpha = \frac{m_6^2}{r^4}
\]

After integration is obtained

\[
E_4 = \frac{\pi \mu_0}{2} L \frac{m_6^2}{R_3^2}
\]

(25)

In order to make sure that condition (14) is fulfilled, the energy of the subdomains (Fig. 3) is calculated by formulas (22) – (25) for the following initial data: \( R_1 = 50 \cdot 10^{-3} \) m; \( R_2 = 60 \cdot 10^{-3} \) m; \( R_3 = 80 \cdot 10^{-3} \) m; \( M = 10^6 \) A/m; \( \mu_r = 10^{-3} \); \( L = 1 \) m. Calculation results: \( E_1 = -647.156 \) J; \( E_2 = 635.043 \) J; \( E_3 = 12.073 \) J; \( E_4 = 0.040 \) J.

Substituting the obtained energy values into (14), is obtained

\[
\sum_{k=1}^{4} E_k = -647.156 + 635.043 + 12.073 + 0.040 = -9.849 \cdot 10^{-13} \approx 0
\]

Condition (14) is satisfied.

As a third example, the energy in a magnetic system with a PM in the form of a parallelepiped is calculated. An example of the placement of four point sources (points \( D_1 - D_4 \) on the middle section \( S_{air} \), perpendicular to the 0z PM axis, and collocation points \( N_1 - N_4 \) on the upper side \( S_6 \) of the PM) is shown in Fig. 4. A similar problem was solved by us earlier [12].
Figure 4. PM in the form of a parallelepiped.

In this case, the boundary conditions at the collocation points take the form

$$ \varphi_i(N_i) = \varphi_2(N_i) ; \quad \mu_i H_z^{(1)}(N_i) = \mu_2 H_z^{(2)}(N_i) , \quad i = 0,1, \ldots , k - 1 $$

where

$$ \varphi_2(N_i) = \sum_{i=0}^{k-1} \frac{m_i \cos \theta_i}{r_{D_h N_i}} ; \quad H_z^{(2)}(N_i) = \sum_{i=0}^{k-1} \frac{m_i \left(3 \cos^2 \theta_i - 1\right)}{r_{D_h N_i}} ; \quad k \text{ – number of sources.} $$

MPMM is meshless and has a number of significant advantages over mesh methods. The calculation of the PM field was carried out with an error of less than 2% with a reduction in the total number of unknowns by almost 60 times as compared with the finite element method with 25 point sources (7 unknown moments due to symmetry). A more detailed calculation of the energy of a PM in the form of a parallelepiped is given in [17].

3. Conclusion

A gridless method for calculating magnetic fields and energy of magnetic systems with PM is proposed, which is distinguished by the use of point magnetic moments (except for the known point sources, charges). This makes it possible to exclude the operations of integration over the volume and surfaces, replacing them by summing the contributions of the moments, and significantly reduce the total number of unknowns. It is shown analytically that the external magnetic field of the PM in the form of a ball and the field of one point magnetic moment located in the center of the ball are identical. The total field of the PM, including the field of the PM itself and the field of the surrounding space, consisting of piecewise homogeneous subdomains (for example, air-steel-air, etc.), is equal to zero, which is shown by numerical analysis. An example of the placement of point magnetic moments in a PM in the form of a parallelepiped is given. The dimension of the problem, the total number of unknowns is reduced by 60 times in comparison with the finite element method. In this case, the error in calculating the PM field was less than 2 %. The results obtained show the effectiveness of the method of point magnetic moments in calculating the energy of permanent magnets.
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References
[1] Kupradze V D and Alexidze M A 1964 USSR Comput. Math. Phys. 4 82
[2] Mathon R and Johnston R L 1977 SIAM Journal on Numerical Analysis 14 4 638
[3] Chen C S, Karageorghis A and Smyrlis Y S 2008 The Method of Fundamental Solutions – A Meshless Method (Atlanta: Dynamic) p 165
[4] Knyazev S Yu, Sherbakova E Ye and Sherbakov A A 2012 Computer modeling of potential fields using point sources (Rostov-on-Don: DGTU Publ.) p 156
[5] Golberg M A and Chen C S 1998 The method of fundamental solutions for potential, Helmholtz and diffusion problems Boundary Integral Methods (Boston: WIT Press) p 103-176
[6] Chen C S, Reutksiy S Y and Rozov V Y 2009 Comput. Assist. Mech. Eng. Sci. 16 21
[7] Wang H and Qin Q 2011 Comput Mech 48 515
[8] Zhou J, Wang K, Li P and Miao X 2018 Engineering Analysis with Boundary Elements 91 82
[9] Cao C, Qin Q H 2015 Advances in Mathematical Physics 916029
[10] Balaban A L, Bakhvalov Y A and Grechikhin V V 2019 AIP Conf. Proc. 2188 050015
[11] Bakhvalov Y A and Grechikhin V V 2019 Journal of Physics: Conference Series 1415 012004
[12] Balaban A L, Bakhvalov Y A and Grechikhin V V 2019 Russian Electromechanics 4 48
[13] Bakhvalov Y A and Grechikhin V V 2019 Russian Electromechanics 4 48
[14] Lovatt H C and Watterson P A 1999 IEEE Trans. Magn. 35 505
[15] Stafl M. 1966 Electrodynamik problems in electrical machines (Moscow: Energy) p 200
[16] Shimoni K 1964 Theoretical electrical engineering (Moscow: Mir) p 774
[17] Yonnet J P, Chillet C, Allag H and Chouikhi L 2016 Journal of Modern Physics 7 2281