0. Introduction

The representation theory of Kac–Moody algebras, and in particular that of affine Lie algebras, has been extensively studied over the past twenty years. The representations that have had the most applications are the integrable ones, so called because they lift to the corresponding group.

The affine Lie algebra associated to a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ is the universal (one–dimensional) central extension of the Lie algebra of polynomial maps from $\mathbb{C}^* \to \mathfrak{g}$. There are essentially two kinds of interesting integrable representations of this algebra: one is where the center acts as a positive integer, or positive energy representations; and the other is where the center acts trivially, or the level zero. Both kinds of representations have interesting applications: representations of the first kind have connections with number theory through the Rogers–Ramanujan identities, the theory of vertex algebras and conformal field theory; representations of the second kind are connected with the six–vertex and XXZ–model [11] and the references therein, the Kostka polynomials and the fermionic formulas of Kirillov and Reshetikhin, [18]. The study of such level zero representations was begun in [4] and continued in [5],[6]. More recently a geometric approach to such representations was developed in [21] for the corresponding quantum algebras.

The affine Lie algebras admit an obvious generalization. Namely, we can consider central extensions of the polynomial maps $(\mathbb{C}^*)^\ell \to \mathfrak{g}$. Not surprisingly, these algebras are a lot more complicated, for instance the central extension is now infinite—dimensional. A systematic study of such algebras can be found in [2] and the representation theory has been studied in [3], [9], [10], [20]. In general, interesting theories have been found for the quotients of this algebra by a central ideal of finite–codimension.

One such quotient is the double affine algebra, this algebra is obtained from the affine algebra in the same way that the affine algebra is obtained from the finite–dimensional algebra. One can also define a corresponding quantum object, and representations of these have been studied in [21], [22], [23]. However, relatively little is still known about the integrable representations of the double affine algebras.

In this paper we study representations of the double affine algebra $\mathfrak{g}_{\text{tor}}$ when one of the centers acts trivially; this is also the situation studied in the quantum case mentioned above. The category of such representations is not semisimple and our interest is in indecomposable integrable representations of $\mathfrak{g}_{\text{tor}}$ rather than the irreducible ones. We are motivated by considerations coming from the study of quantum affine algebras [3] and modular Lie algebras. Thus, we believe that these indecomposable representations should be the limit as $q \to 1$ of the irreducible representations of the corresponding quantum algebra. In the case of quantum affine algebras this is in fact a conjecture which has been checked in many cases; for double
affine algebras, no such conjecture is possible at the moment, since the notion of $q \to 1$ is not well-defined. However, Theorem 4 can be viewed as the classical analog of the result in [22].

The paper is organized as follows. In section 1, we set up the notation to be used in the rest of the paper. In section 2, we recall several results on the representations of the affine algebra. In section 3, we define a family of indecomposable representations and identify their irreducible quotients.

Sections 4 and 5 are devoted to the study of the structure of these representations. Thus we give a sufficient condition for the indecomposable modules to be irreducible. We also show that in fact the modules are almost always reducible in Section 5. In the case of affine algebras, this was proved by passing to the quantum situation. In this paper, however, we prove it by using the notion of fusion product representations, an idea introduced recently by Feigin and Loktev in [12].

1. Preliminaries

1.1. Let $\mathfrak{g}_{\text{fin}}$ be a complex finite-dimensional simple Lie algebra of rank $n$, and let $h_{\text{fin}}$ be a Cartan subalgebra of $\mathfrak{g}_{\text{fin}}$. Fix a set $\{\alpha_i : 1 \leq i \leq n\}$ (resp. $\{\omega_i : 1 \leq i \leq n\}$) of simple roots (resp. fundamental weights) of $\mathfrak{g}_{\text{fin}}$ with respect to $h_{\text{fin}}$. Let $R_{\text{fin}}^+$ be the corresponding set of positive roots and let $\theta \in R_{\text{fin}}^+$ be the highest root. Given $\alpha \in R_{\text{fin}}^+$, let $\mathfrak{g}_{\text{fin}}^\alpha$ be the corresponding root spaces and fix non-zero elements $x_\alpha^\pm \in \mathfrak{g}_{\text{fin}}^\pm\alpha$ and $h_\alpha \in h_{\text{fin}}$ such that the elements $x_\alpha^\pm, h_\alpha$ span a subalgebra isomorphic to $sl_2$. Set $x_i^\pm = x_{\alpha_i}^\pm, h_i = h_{\alpha_i}$. Define subalgebras

$$n_\text{fin}^\pm = \oplus_{\alpha \in R_{\text{fin}}^+} \mathfrak{g}_{\text{fin}}^\pm\alpha.$$ 

We have,

$$\mathfrak{g}_{\text{fin}} = n_{\text{fin}}^- \oplus h_{\text{fin}} \oplus n_{\text{fin}}^+.$$ 

Let $Q_{\text{fin}}^+$ (resp. $P_{\text{fin}}^+$) be the non-negative root (resp. weight) lattice of $\mathfrak{g}_{\text{fin}}$.

1.2. The corresponding untwisted affine Lie algebra $\mathfrak{g}_{\text{aff}}$ is the universal central extension of the Lie algebra of Laurent polynomial maps from $\mathbb{C}^* \to \mathfrak{g}_{\text{fin}}$. Thus

$$\mathfrak{g}_{\text{aff}} = \mathfrak{g}_{\text{fin}} \otimes \mathbb{C}[t_1, t_1^{-1}] \oplus \mathbb{C}C_1,$$

with bracket given by

$$[xt_1^r, yt_1^m] = [x, y]t_1^{r+m} + r\delta_{r, -m} x, y > c_1, \quad [c_1, xt_1^r] = 0, \quad x, y \in \mathfrak{g}_{\text{fin}}, \quad r, m \in \mathbb{Z}$$

where $\langle, \rangle$ is the Killing form of $\mathfrak{g}_{\text{fin}}$. Let $d_1$ be the derivation $t_1 d/dt_1$ of $\mathbb{C}[t_1, t_1^{-1}]$; then $d_1$ acts on $\mathfrak{g}_{\text{aff}}$ in the obvious way. The extended affine algebra $\mathfrak{g}_{\text{aff}}^e$ is the semi-direct product of $Cd_1$ and $\mathfrak{g}_{\text{aff}}$, i.e. $\mathfrak{g}_{\text{aff}}^e = \mathfrak{g}_{\text{aff}} \oplus Cd_1$ with the Lie bracket between $d_1$ and $\mathfrak{g}_{\text{aff}}$ being given by

$$[d_1, c_1] = 0, \quad [d_1, xt_1^m] = mx_1^m, \quad x \in \mathfrak{g}_{\text{fin}}, \quad m \in \mathbb{Z}.$$ 

The algebra $\mathfrak{g}_{\text{aff}}^e$ admits a symmetric, invariant bilinear form, defined as follows:

$$\langle xt_1^r, yt_1^m \rangle_{\text{aff}} = \delta_{r, -m} x, y \quad x, y \in \mathfrak{g}_{\text{fin}}, \quad r, m \in \mathbb{Z}$$

and

$$\langle c_1, d_1 \rangle = 1, \quad \langle c_1, \mathfrak{g}_{\text{aff}} \rangle = \langle d_1, \mathfrak{g}_{\text{aff}} \rangle = 0.$$
Set
\[ \mathfrak{h}^e_{aff} = \mathfrak{h}_{fin} \oplus \mathbf{C}c_1 \oplus \mathbf{C}d_1, \quad \mathfrak{n}^+_{aff} = \mathfrak{g} \otimes t_1 \mathbf{C}[t_1] \oplus \mathfrak{n}^+_{fin}, \quad \mathfrak{n}^-_{aff} = \mathfrak{g} \otimes t_1^{-1} \mathbf{C}[t_1^{-1}] \oplus \mathfrak{n}^-_{fin}. \]

Clearly we can regard \( \mathfrak{g}_{fin} \) as the algebra of constant maps in \( \mathfrak{h}^e_{aff} \). Given any element \( \lambda \in (\mathfrak{h}^e_{fin})^* \) we regard it as an element of \( (\mathfrak{h}^e_{aff})^* \) by setting \( \lambda(c_1) = \lambda(d_1) = 0 \). Define \( \delta_1 \in (\mathfrak{h}^e_{aff})^* \) by
\[ \delta_1(d_1) = 1, \quad (\delta_1)|_{\mathfrak{h}_{fin} \oplus \mathbf{C}c_1} = 0. \]

Set \( \alpha_0 = \delta_1 - \theta \). The elements \( \{\alpha_i : 0 \leq i \leq n\} \) are called the affine simple roots. The corresponding set of positive affine roots is
\[ R^+_{aff} = \{\pm \alpha + r\delta_1 : \alpha \in R^+_fin, r \in \mathbf{Z}, r > 0\} \cup R^+_{fin} \cup \{r\delta_1 : r \in \mathbf{Z}, r > 0\}. \]

The root space corresponding to \( \pm \alpha + r\delta_1 \) is \( \mathfrak{g}^\pm \mathfrak{h}_{fin} \otimes \mathbf{C}t_1^r \), and that corresponding to \( r\delta_1 \) is \( \mathfrak{h}_{fin} \otimes \mathbf{C}t_1^r \). Fix non–zero elements \( x^+_0 \in \mathfrak{g}^+_0 \mathfrak{h}_{aff} \) and \( h_0 \in \mathfrak{h}^e_{aff} \) so that the subalgebra spanned by \( x^+_0, h_0 \) is isomorphic to \( sl_2 \). Then,
\[ c_1 = h_0 + h_0. \]

Let \( \omega_0 \in (\mathfrak{h}^e_{aff})^* \) be defined by
\[ \omega_0(h) = 0, \quad \omega_0(c_1) = 1, \quad \omega_0(d_1) = 0, \quad (h \in \mathfrak{h}_{fin}). \]

The affine root lattice \( Q^+_{aff} \) and the affine weight lattice \( P^+_{aff} \) in \( (\mathfrak{h}^e_{aff})^* \) are now defined in the obvious way.

1.3. The double affine algebra \( \mathfrak{g}_{tor} \) and the extended algebra \( \mathfrak{g}^e_{tor} \) are obtained from \( \mathfrak{g}_{aff} \) in the same way that \( \mathfrak{g}_{aff} \) and \( \mathfrak{g}^e_{aff} \) were obtained from \( \mathfrak{g}_{fin} \). Thus,
\[ \mathfrak{g}^e_{tor} = \mathfrak{g}_{aff} \otimes \mathbf{C}[t_2, t_2^{-1}] \oplus \mathbf{C}c_2 \oplus \mathbf{C}d_2, \quad \mathfrak{g}_{tor} = \mathfrak{g}_{aff} \otimes \mathbf{C}[t_2, t_2^{-1}] \oplus \mathbf{C}c_2 \oplus \mathbf{C}d_1, \]
with the Lie bracket given as follows. The element \( c_2 \) is central, and
\[ [d_2, xt_2^r] = rxt_2^r, \quad [xt_2^r, yt_2^m] = [x, y]t_2^{r+m} + \delta_{r, -m} x, y >_{aff} c_2, \quad (x, y \in \mathfrak{g}_{aff}, r, m \in \mathbf{Z}). \]

Clearly \( \mathfrak{g}_{aff} \) is a subalgebra of \( \mathfrak{g}_{tor} \). Set
\[ \mathfrak{h}^e_{tor} = \mathfrak{h}^e_{aff} \oplus \mathbf{C}c_2 \oplus \mathbf{C}d_2. \]

As before, we regard an element \( \lambda \in (\mathfrak{h}^e_{aff})^* \) as an element of \( (\mathfrak{h}^e_{tor})^* \) by setting it to be zero on \( \mathbf{C}c_2 \) and \( \mathbf{C}d_2 \). The set of roots of the double affine algebra are then,
\[ R_{tor} = \{\pm \alpha + n_2\delta_2 : \alpha \in R^+_{aff}, n_2 \in \mathbf{Z}\} \cup \{n_2\delta_2 : n_2 \in \mathbf{Z}\}. \]

Unlike the affine case, there is no natural choice of simple roots. However, we will work with the following partition of \( R_{tor} \) into mutually disjoint sets,
\[ R_{tor}(>) = \{\alpha + n_2\delta_2 : \alpha \in R^+_{aff}, n_2 \in \mathbf{Z}\}, \]
\[ R_{tor}(0) = \{n_2\delta_2 : n_2 \in \mathbf{Z}\}, \]
\[ R_{tor}(<) = \{-\alpha + n_2\delta_2 : \alpha \in R^+_{aff}, n_2 \in \mathbf{Z}\}. \]

The subalgebras \( \mathfrak{g}_{tor}(>) \), \( \mathfrak{g}_{tor}(0) \) and \( \mathfrak{g}_{tor}(<) \) are defined in the obvious way.
Lemma 1.1. Let $\lambda \in \mathfrak{g}_{\text{fin}}$ (resp. $\mathfrak{g}_{\text{aff}}$, $\mathfrak{g}_{\text{tor}}$) denote the enveloping algebra of $\mathfrak{g}_{\text{fin}}$ (resp. $\mathfrak{g}_{\text{aff}}$, $\mathfrak{g}_{\text{tor}}$). By the Poincare–Birkhoff–Witt theorem, we have

\[
U_{\text{fin}} = U(\mathfrak{n}^-_{\text{fin}})U(\mathfrak{h}_{\text{fin}})U(\mathfrak{n}^+_{\text{fin}}), \\
U_{\text{aff}} = U(\mathfrak{n}^-_{\text{aff}})U(\mathfrak{h}_{\text{aff}}^e)U(\mathfrak{n}^+_{\text{aff}}), \\
U_{\text{tor}} = U(\mathfrak{g}_{\text{tor}}(<))U(\mathfrak{h}_{\text{tor}})U(\mathfrak{g}_{\text{tor}}(0))U(\mathfrak{g}_{\text{tor}}(>).
\]

For $h \in \mathfrak{h}_{\text{aff}}$, let $\Lambda^\pm(h, u)$ be the power series in an indeterminate $u$ with coefficients in $U_{\text{tor}}$, defined by

\[
\Lambda^\pm(h, u) = \exp \left( \sum_{r=1}^{\infty} \frac{ht_{\pm r}^r u^r}{r} \right).
\]

Let $\Lambda^\pm(h, r)$ be the coefficient of $u^r$ in $\Lambda^\pm(h, u)$. It is not hard to see that the elements \{\(\Lambda^\pm(h, r) : h \in \mathfrak{h}_{\text{aff}}, r \in \mathbb{Z}, r > 0\}\) generate the subalgebra $U(\mathfrak{g}_{\text{tor}}(0))$.

For $\alpha \in R_{\text{fin}}^+$ and $r \in \mathbb{Z}$, the the elements \(x_{\beta}^{\pm r}t_2^{\pm m} : s \in \mathbb{Z}\) generate a subalgebra of $\mathfrak{g}_{\text{tor}}$ which is isomorphic to the affine algebra associated to $sl_2$. The following lemma was proved in [13].

Lemma 1.1. Let $\beta \in R_{\text{aff}}^+$ be of the form $\alpha + r_1 \delta_1$ for some $r_1 \in \mathbb{Z}$. For all $s \geq 1$, we have

\[
(x_{\beta}^{\pm r}t_2^{s+1}).(x_{\beta}^s)^{s+1} = \sum_{m=0}^{s} (x_{\beta}^{-s}t_2^{s-m})\Lambda^\pm(h_\beta, s - m) + X,
\]

\[
(x_{\beta}^{\pm r}t_2^{s+1}).(x_{\beta}^s)^{s+1} = \Lambda^\pm(h_\beta, s + 1) + Y,
\]

where $X$ and $Y$ are in the left ideal of $U_{\text{tor}}$ generated by the subalgebra $\mathfrak{g}_{\text{tor}}(>)$.

\[
\square
\]

2. Representation theory of $\mathfrak{g}_{\text{fin}}$ and $\mathfrak{g}_{\text{aff}}$

In this section, we discuss the representation theory of $\mathfrak{g}_{\text{fin}}$ and $\mathfrak{g}_{\text{aff}}$. We shall be interested in the finite–dimensional representations of $\mathfrak{g}_{\text{fin}}$ and their analogues, the integrable representations of $\mathfrak{g}_{\text{aff}}$.

2.1. For $\lambda = \sum_{i=1}^{n} \lambda(h_i) \omega_i \in P_{\text{fin}}^+$, (resp. $\lambda = \sum_{i=0}^{n} \lambda(h_i) \omega_i \in P_{\text{aff}}^+$), let $V_{\text{fin}}(\lambda)$ (resp. $V_{\text{aff}}((\lambda))$) be the unique irreducible $\mathfrak{g}_{\text{fin}}$–module (resp. $\mathfrak{g}_{\text{aff}}$–module) with highest weight $\lambda$ and highest weight vector $v_\lambda$. Thus, $V_{\text{fin}}(\lambda) = U_{\text{fin}}.v_\lambda$ (resp. $V_{\text{aff}}(\lambda) = U_{\text{aff}}.v_\lambda$) is generated by the element $v_\lambda$, subject to the following relations:

\[
\mathfrak{n}_{\text{fin}}^+.v_\lambda = 0, \quad h.v_\lambda = \lambda(h).v_\lambda, \quad (x_i^{\lambda(h_i)+1}).v_\lambda = 0, \quad i = 1, \ldots, n, \quad h \in \mathfrak{h}_{\text{fin}},
\]

( resp. $\mathfrak{n}_{\text{aff}}^+.v_\lambda = 0, \quad h.v_\lambda = \lambda(h).v_\lambda, \quad (x_i^{-\lambda(h_i)+1}).v_\lambda = 0, \quad i = 0, \ldots, n, \quad h \in \mathfrak{h}_{\text{aff}}$.)

It is well–known that the set \(\{V_{\text{fin}}(\lambda) : \lambda \in P_{\text{fin}}^+\}\) is in one–one correspondence with the set of isomorphism classes of irreducible finite–dimensional representations of $\mathfrak{g}_{\text{fin}}$. Further, any finite–dimensional $\mathfrak{g}_{\text{fin}}$–module is isomorphic to a direct sum of irreducible $\mathfrak{g}_{\text{fin}}$–modules.
2.2. The $\mathfrak{g}_{aff}^e$–modules $V_{aff}(\lambda)$ are not finite–dimensional, but are integrable \[17\] in the following sense.

**Definition 2.1.** A $\mathfrak{g}_{aff}^e$–module $V$ is said to be integrable if

$$V = \bigoplus_{\mu \in (\mathfrak{h}_{aff}^e)^*} V_\mu,$$

where $V_\mu = \{v \in V : hv = \mu(h)v \quad \forall h \in \mathfrak{h}_{aff}^e\}$, and if for all $\alpha \in R_{fin}^+, r \in \mathbb{Z}$ the elements $x_\alpha^\pm t_r^r$ act locally nilpotently on $V$.

To describe the isomorphism classes of irreducible integrable representations of $\mathfrak{g}_{aff}^e$, we need to introduce two more families of modules. The first one is obtained by just taking the restricted dual $V_{aff}^e(\lambda)$ of $V_{aff}(\lambda)$, i.e., the $\mathfrak{g}_{aff}^e$–module generated by an element $v_\lambda^*$ subject to the relations

$$n_{aff}^- v_\lambda = 0, \quad h.v_\lambda = -\lambda(h).v_\lambda, \quad (x_i^+)^{\lambda_i+1}.v_\lambda = 0, \quad i = 0, \ldots, n, \quad h \in \mathfrak{h}_{aff}^e.$$

Notice that the center $c_1$ acts as a positive integer on $V_{aff}(\lambda)$ (if $\lambda \neq 0$) and as a negative integer on $V_{aff}^e(\lambda)$.

The second family that we need are the loop modules which were introduced in \[8\]. For $k \geq 1, \lambda_1, \ldots, \lambda_k \in P_{fin}^+, a_1, \ldots, a_k \in \mathbb{C}^*, b \in \mathbb{C}$, define a $\mathfrak{g}_{aff}^e$–module structure on

$$L(V_{fin}(\lambda_1) \otimes \cdots \otimes V_{fin}(\lambda_k)) = V_{fin}(\lambda_1) \otimes \cdots \otimes V_{fin}(\lambda_k) \otimes \mathbb{C}[t, t^{-1}],$$

as follows:

\[
\begin{align*}
c_1(v_1 \otimes \cdots \otimes v_k) \otimes t^s &= 0, \\
d_1(v_1 \otimes \cdots \otimes v_k) \otimes t^s &= (s + b)(v_1 \otimes \cdots \otimes v_k) \otimes t^s, \\
x_i^r(v_1 \otimes \cdots \otimes v_k) \otimes t^s &= \left( \sum_{i=1}^{k} a_i^r v_1 \otimes \cdots \otimes v_{i-1} \otimes xv_i \otimes v_{i+1} \otimes \cdots \otimes v_k \right) \otimes t^{r+s},
\end{align*}
\]

where $\{v_i \in V_{fin}(\lambda_i) : 1 \leq i \leq k\}$, $s, r \in \mathbb{Z}$, $x \in \mathfrak{g}_{fin}$. Denote this module by $V_{aff}(\lambda, a, b)$. The following result was proved in \[8\]

**Proposition 2.1.** Let $k \geq 1, b \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_k \in P_{fin}^+$ and assume that $a_1, \ldots, a_k$ are distinct non–zero complex numbers.

(i) The $\mathfrak{g}_{aff}^e$–module $V_{aff}(\lambda, a, b)$ is irreducible iff for every $r \in \mathbb{Z}$ there exists an integer $m_r \in \mathbb{Z}$ with $m_r \neq 0$ mod $r$ such that

$$\sum_{i=1}^{k} a_i^{m_r} \lambda_i \neq 0.$$

(ii) Suppose that there exists $r > 0$ such that $\sum_{i=1}^{k} a_i^{m} \lambda_i = 0$ for all $m \neq 0$ mod $r$. Then the $\mathfrak{g}_{aff}^e$–module $L(V_{fin}(\lambda_1) \otimes \cdots \otimes V_{fin}(\lambda_k))$ is completely reducible. The irreducible submodules are generated by the elements $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k} \otimes t^\ell, 0 \leq \ell < r$. \[\square\]
Remark. The characters of these modules have been studied in [14]. In the case of $sl_n$ a crystal model was obtained for these modules in [13]. The next result was proved in [4], [5].

Theorem 1. Let $V$ be an irreducible integrable $\mathfrak{g}_{aff}^e$-module. Assume also that $\dim(V_\mu) < \infty$ for all $\mu \in \mathfrak{h}_{aff}^e$. Let $\ell \in \mathbb{Z}$ be such that $c_1 v = \ell v$ for all $v \in V$. Then:

(i) If $\ell > 0$ (resp. $\ell < 0$) there exists $\lambda \in P_{aff}^+$ such that $V \cong V_{aff}(\lambda)$ (resp. $V \cong V_{aff}^*(\lambda)$).

(ii) If $\ell = 0$, then there exists $k \geq 1$, $\lambda_1, \ldots, \lambda_k \in P_{fin}^+$, $b \in \mathbb{C}$ and distinct non–zero complex numbers $a_1, \ldots, a_k$ such that $V$ is isomorphic to either $V_{aff}(\lambda, a, b)$ or to one of its irreducible submodules as described in Proposition 2.1.

Remark. In [16] a closely related family of bounded admissible modules for $\mathfrak{g}_{aff}$ were studied and a similar classification theorem was obtained.

The category of integrable representations is in general far from semisimple. However, one has the following result, [17].

Theorem 2. Let $V$ be an integrable $\mathfrak{g}_{aff}^e$–module such that $\dim V_\mu < \infty$ for all $\mu \in (\mathfrak{h}_{aff}^e)^*$. Assume that there exists $\mu_1, \ldots, \mu_s \in P_{aff}^+$ such that

$$V_\mu \neq 0 \implies \mu \in \mu_i - Q_{aff}^+,$$

$$(\text{resp.} V_\mu \neq 0 \implies \mu \in -\mu_i - Q_{aff}^-),$$

for some $1 \leq i \leq s$. Then, $V$ is isomorphic to a direct sum of representations of the form $V_{aff}(\lambda)$ (resp. $V_{aff}^*(\lambda)$) where $\lambda \in P_{aff}^+$. In particular, if $\lambda, \mu \in P_{aff}^+$, the tensor product $V_{aff}(\lambda) \otimes V_{aff}(\mu)$ is a reducible but completely reducible $\mathfrak{g}_{aff}^e$–module.

2.3. We give an easy example of an integrable representation $V = \bigoplus_{\mu \in (\mathfrak{h}_{aff}^e)^*} V_\mu$ (with $\dim V_\mu < \infty$ for all $\mu \in (\mathfrak{h}_{aff}^e)^*$) of $\mathfrak{g}_{aff}^e$ which is indecomposable and reducible. Such representations are studied in greater detail in [3]. Assume that $\mathfrak{g}_{fin} = sl_2$ and that $h, x, y$ is the standard basis of $sl_2$. Let $V$ be the free module over $\mathbb{C}[t_1, t_1^{-1}]$ of rank 4, with basis $v_0, v_1, v_2, w_0$. It is not hard to check (using the Chevalley presentation [7]) that the following formulae define an action of $\mathfrak{g}_{aff}^e$ on $V$. The center $c_1$ acts trivially, and for any $v \in V$, $n \in \mathbb{Z}$, $d_1 v t^n = n v t^n$. In addition,

$$h(v_1 t_1^i) = (2 - 2i)(v_1 t^r), \quad x(v_1 t^r) = (3 - i)(v_{i-1} t^r), \quad y(v_1 t^r) = (i + 1)(v_{i+1} t^r),$$

$$(x t^{-1})(v_0 t^r) = 0, \quad (x t^{-1})(v_1 t^r) = 2(v_0 t^{r-1}), \quad (xt^{-1})(v_2 t^r) = (v_1 + w_0) t^{r-1},$$

$$g_{aff} w_0 t^r = 0, \quad (yt)(v_0 t^r) = (v_1 + w_0) t^{r+1}, \quad (yt)v_1 t^r = 2v_2 t^{r+1}, \quad (yt)v_2 = 0.$$

It is clear that this module is generated by the element $v_0$ and that for all $r \in \mathbb{Z}$, the elements $w_0 t^r$ generate a proper submodule of it.
2.4. We shall need the following result, \[6\] in our study of representations of the double affine algebra.

**Theorem 3.** Let \( \lambda \in P_+^{\text{aff}} \) and let \( \mu_1, \ldots, \mu_k \in P_+^{\text{fin}} \). Assume also that \( a_1, \ldots, a_k \) are distinct non-zero complex numbers. The \( \mathfrak{g}_\text{aff} \)-module \( V_\text{aff}(\lambda) \otimes V_\text{aff}(\mu, a) \) is irreducible if

(i) \( \sum_{i=1}^{k} a_i \mu_i \neq 0 \),
(ii) there exists \( \alpha \in R_+^{\text{fin}} \) such that either

\[
(k + 1)\lambda(c_1) < (\mu + \lambda)(h_\alpha),
\]

or

\[
k\lambda(c) < (\mu^* - \lambda)(h_\alpha),
\]

where \( \mu = \sum_{i=1}^{k} \mu_i \) and \( \mu^* \) is the highest weight of the \( \mathfrak{g}_\text{fin} \)-module that is dual to \( V_\text{fin}(\mu) \).

Notice that, in particular, the theorem implies the existence of irreducible integrable representations with infinite-dimensional weight spaces. A partial converse to the theorem was proved in \[6\]. More recently, a converse has been proved in \[7\] in the case of \( \mathfrak{sl}_2 \).

3. Representations of \( \mathfrak{g}_\text{tor} \)

The representation theory of \( \mathfrak{g}_\text{tor} \) is substantially more complicated than the affine case, and a classification theorem of the kind in Theorem \[4\] seems much more difficult; however, see \[8\], \[9\]. The study of \( \mathfrak{g}_\text{tor} \)-modules is closely related to that of \( \mathfrak{g}_\text{tor} \)-modules and for simplicity we restrict our attention to the representation theory of \( \mathfrak{g}_\text{tor} \). We shall be interested in the category of integrable representations of \( \mathfrak{g}_\text{tor} \) on which \( c_2 \) acts trivially, but \( c_1 \) acts non-trivially. Since this category is not semisimple, we shall discuss both irreducible and indecomposable representations of \( \mathfrak{g}_\text{tor} \).

3.1. We begin with the definition of integrable \( \mathfrak{g}_\text{tor} \)-modules.

**Definition 3.1.** A representation \( V \) of \( \mathfrak{g}_\text{tor} \) is called integrable if

\[
V = \bigoplus_{\lambda \in (\mathfrak{h}_\text{tor})^*} V_\lambda
\]

where

\[
V_\lambda = \{ v \in V : h.v = \lambda(h)v \forall h \in \mathfrak{h}_\text{tor} \},
\]

and if the elements \( x_1^{m_1} t_2^{m_2} \) act locally nilpotently on \( V \) for all \( \alpha \in R_+^{\text{fin}}, m_1, m_2 \in \mathbb{Z} \). We say that \( V \) is admissible if \( \dim V_\lambda < \infty \) for all \( \lambda \in (\mathfrak{h}_\text{tor})^* \). \( \square \)

It is clear that if \( V \) is integrable, and \( 0 \neq v \in V_\lambda \), then \( U_\text{fin}.v \) is a finite-dimensional \( U_\text{fin} \)-module and that \( U_\text{aff}.v \) is an integrable \( \mathfrak{g}_\text{aff} \)-module.
3.2. Given \( \lambda_1, \cdots, \lambda_k \in (h_{aff}^\circ)^*, a_1, \cdots, a_k \in C^* \), define an action of \( g_{tor}^e \) on \( V_{aff}(\lambda_1) \otimes \cdots \otimes V_{aff}(\lambda_k) \) as follows: \( c_2 \) acts as zero and for \( x \in g_{aff}, m \in Z \),

\[
x t_2^m (v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} a_j^m (v_1 \otimes \cdots \otimes v_{j-1} \otimes xv_j \otimes v_{j+1} \otimes \cdots \otimes v_k), \quad v_i \in V_{aff}(\lambda_i), 1 \leq i \leq k.
\]

Denote this module by \( V_{tor}(\lambda, a) \). Notice that

\[
V_{tor}(\lambda, a) \cong V_{tor}(\lambda', a')
\]

if and only if there exists a permutation \( \sigma \) of \( \{1, \cdots, k\} \) such that

\[
\lambda_i = \lambda'_{\sigma(i)}, \quad a_i = a'_{\sigma(i)}.
\]

Lemma 3.1. For all \( k \geq 1 \), \( \lambda = (\lambda_1, \cdots, \lambda_k) \in (P_{aff}^+)^k \), \( a = (a_1, \cdots, a_k) \in (C^*)^k \), the module \( V_{tor}(\lambda, a) \) is an integrable admissible \( g_{tor}^e \)-module which is irreducible iff \( a_r \neq a_s \) for all \( 1 \leq r \neq s \leq k \). Let \( v_{\lambda_1} \) be the highest weight vector in \( V_{aff}(\lambda_r) \). We have

\[
(3.1) \quad U_{tor}(>).v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k} = 0,
\]

\[
(3.2) \quad \Lambda^\pm(h_i, u)(v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}) = \prod_{j=1}^{k} (1 - a_j^\pm u)^{\lambda_j(h_i)}(v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}), \quad \forall 0 \leq i \leq n.
\]

Proof. Since \( x^\pm_1 t_1^m \) acts locally nilpotently on \( V_{aff}(\lambda_i) \) for all \( \alpha \in R_{fin}^+ \), it follows from the definition that \( V_{tor}(\lambda, a) \) is integrable. The module is admissible because the weight spaces of \( V_{aff}(\lambda_i) \) are finite-dimensional. If \( a_r \neq a_s \) for all \( r \neq s \), notice that for all \( x \in g_{aff} \), the element \( x \prod_{s \neq r} (t_2 - a_s) \) acts only on the \( r \)th component of the tensor product. The irreducibility of \( V_{tor}(\lambda, a) \) now follows from the irreducibility of the \( g_{aff}^e \)-module \( V_{aff}(\lambda_i) \). Suppose now that \( a_r = a_s \) for some pair \( (r, s) \); we can assume without loss of generality (by applying a suitable permutation) that \( r = 1, s = 2 \). Recall from [17] that \( V_{aff}(\lambda_1) \otimes V_{aff}(\lambda_2) \) is a reducible \( g_{aff}^e \)-module; let \( W \) be a proper submodule. It is clear that \( W \otimes V_{aff}(\lambda_3) \otimes \cdots \otimes V_{aff}(\lambda_k) \) is a proper non-zero submodule of \( V_{tor}(\lambda, a) \).

Finally, note that (3.1) and (3.2) follow from the definition of the \( g_{tor}^e \)-action on \( V_{tor}(\lambda, a) \) and (1.1). \( \square \)

3.3. We now construct a family of integrable indecomposable \( g_{tor}^e \)-modules. This family will be maximal in suitable sense. We begin with the following definition.

Given a collection \( p = \{ p_r^\pm \in (h_{aff})^* : r \neq 0 \} \) and \( \lambda \in (h_{tor})^* \), let \( M_{tor}(\lambda, p) \) be the left \( U_{tor} \)-module defined as follows. Let \( I(\lambda, p) \) be the left ideal in \( U_{tor} \) generated by

\[
U_{tor}(>), \quad \Lambda^\pm(h_i, r) - p_r^\pm(h_i), \quad h' - \lambda(h'), \quad (h' \in h_{tor}, 0 \leq i \leq n).
\]

Set

\[
M_{tor}(\lambda, p) = U_{tor}/I(\lambda, p).
\]

Clearly \( M_{tor}(\lambda, p) \) is a left \( U_{tor} \)-module. Let \( v_{\lambda, p} \) be the image of 1 in \( M_{tor}(\lambda, p) \). Standard arguments prove that \( M_{tor}(\lambda, p) \) is a free \( U_{tor}(<) \) module generated by \( v_{\lambda, p} \) and hence that \( M_{tor}(\lambda, p) \) has a unique irreducible quotient, which we denote by \( V_{tor}(\lambda, p) \).

The following result determines a necessary and sufficient condition for \( M_{tor}(\lambda, p) \) to have an integrable quotient and identifies the irreducible quotient in this case.
Lemma 3.2. The quotient of \( M_r \) and we set it is not hard to see that there exists a map of in the proof of Proposition (1.1)(iv),(v) in [8], we omit the details. Since the elements \( \lambda \) Proposition 3.1. The module \( M_\text{tor}(\lambda, p) \) has an integrable quotient iff there exists an \((n+1)\)-tuple of polynomials (with constant term one) \( \pi = (\pi_0, \cdots, \pi_n) \) in an indeterminate \( u \), such that the following conditions hold for all \( i = 0, \cdots, n \):

\[
(\text{i}) \quad \lambda(h_i) = \deg \pi_i,
\]

\[
(\text{ii}) \quad \sum_{r \geq 0} p_r^\pm(h_i)u^r = \pi^\pm_i(u), \text{ where } \pi^+_i(u) = \pi_i \text{ and } \pi^-_i(u) = \frac{u^{\deg \pi_i} \pi^+(u^{-1})}{(u^{\deg \pi_i} \pi^+(u^{-1}))(0)}.
\]

Proof. Suppose that \( M_\text{tor}(\lambda, p) \) has an integrable \( g_\text{tor} \)-quotient \( W \) and let \( w_p \) be the image of \( v_{\lambda,p} \) in \( W \). By the representation theory of \( sl_2 \) it follows that \( \lambda(h_i) \) is a non-negative integer for all \( 0 \leq i \leq n \). Further \( \lambda(h_i) = r_i \) is the smallest non-negative integer such that

\[
(x_i^\pm)^{r_i+s}.w_p = 0, \quad 0 \leq i \leq n, \ \forall s > 0.
\]

Applying \((x_i^\pm t_2)^{r_i+s} \) to the preceding equation and using Lemma 3.1, we see that

\[
\Lambda^+(h_i, m).w_p = \pi_i^m(h_i)(0) = 0 \text{ if } m \geq r_i + 1.
\]

To see that \( p_i^\pm(h_i) \neq 0 \), note that for all \( 0 \leq i \leq n \) we have

\[
(h_i t_2^{-1})(x_i^{-} t_2^{-1})(x_i^{-})^{r_i}.w_p = 0.
\]

Since the elements \( x_i^\pm t_2^{-1} \) generate a subalgebra isomorphic to \( sl_2 \) it follows from the representation theory of \( sl_2 \) that,

\[
(x_i^\pm t_2)^{r_i}.w_p \neq 0.
\]

Using Lemma 3.1 we see that this means that \( p_i^\pm(h_i) \neq 0 \) for all \( 0 \leq i \leq n \). Hence \( \pi^+_i(u) = \sum_{s} p_s^+(h_i)u^s \) is a polynomial of degree \( r_i \). Similarly one can prove that \( \pi^-_i(u) = \sum_{s} p_s^-(h_i)u^s \) is a polynomial of degree \( r_i \). To see that \( \pi^\pm_i \) are related as in the proposition, one proceeds as in the proof of Proposition (1.1)(iv),(v) in [8], we omit the details.

Conversely, given \( \pi = (\pi_0, \cdots, \pi_n) \), consider the set \( \{a_1, a_2, \cdots, a_r\} \) of distinct roots of \( \pi = \prod_{j=0}^r \pi_j \). Let \( m_{ij} \) be the multiplicity with which \( a_i \) occurs as a root in \( \pi_j \). Set

\[
\mu_j = \sum_{i=0}^n m_{ij} \omega_i, \quad 1 \leq j \leq r.
\]

It is not hard to see that there exists a map of \( g_\text{tor} \)-modules \( M_\text{tor}(\lambda, p) \to V_\text{tor}(\mu, a) \). Since \( V_\text{tor}(\mu, a) \) is integrable, the theorem follows.

3.4. We shall only be interested in the modules \( M_\text{tor}(\lambda, p) \) which are given by an \((n+1)\)-tuple of polynomials as in the preceding proposition. Set \( \lambda_{\pi} = \sum_{i=0}^n (\deg \pi_i) \omega_i \). Then \( \lambda_{\pi} = \lambda \), and we set \( M_\text{tor}(\pi) = M_\text{tor}(\lambda, p) \) and denote by \( V_\text{tor}(\pi) \) the unique irreducible quotient of \( M_\text{tor}(\pi) \). Let \( v_\pi = v_{\lambda,p} \). Clearly \( V_\text{tor}(\pi) \cong V_\text{tor}(\mu, a) \) for a suitable choice of \( \mu, a \).

We now define the maximal integrable quotient \( W_\text{tor}(\pi) \) of \( M_\text{tor}(\pi) \). Thus, let \( W_\text{tor}(\pi) \) be the quotient of \( M_\text{tor}(\pi) \) by the submodule generated by the elements

\[
(x_i^-)^{r_i+1}.v_\pi, \quad r_i = \deg \pi_i, \quad 0 \leq i \leq n.
\]

Let \( w_\pi \) be the image of \( v_\pi \) in \( W_\text{tor}(\pi) \). The following lemma is immediate.

Lemma 3.2. The \( U_{aff} \)-submodule of \( W_\text{tor}(\pi) \) generated by \( w_\pi \) is isomorphic to \( V_{aff}(\lambda_{\pi}) \), where \( \lambda_{\pi} = \sum_{i=0}^n (\deg \pi_i) \omega_i \in P_{aff}^+ \).
Proposition 3.2. The $\mathfrak{g}_{tor}$-module $W_{tor}(\pi)$ is integrable and admissible. Further, any integrable quotient of $M_{tor}(\pi)$ is a quotient of $W_{tor}(\pi)$.

Proof. To see that $W_{tor}(\pi)$ is integrable, first observe that $W_{tor}(\pi) = U_{tor}(\cdot, \pi)$. Since the elements $x_{\alpha}^+ t_{i_1}^{m_1} t_{i_2}^{m_2}$, $\alpha \in R_{fin}^+$, $m_1, m_2 \in \mathbb{Z}$, act locally nilpotently (via the adjoint action) on $U_{tor}$ it is enough to prove that they act locally nilpotently on $w_\pi$. If $m_2 = 0$, the result follows since $V_{aff}(h)$ is an integrable $\mathfrak{g}_{aff}$-module, and so we have

$$(x_{\alpha}^- t_{i_1}^{m_1})^N \cdot w_\pi = 0,$$

for some $N \geq 0$ depending on $\alpha$ and $m_1$. Applying $h_{\alpha} t_{i_2}^r$ to the preceding equation, we get

$$(x_{\alpha}^- t_{i_1}^{m_1})^{N-1} (x_{\alpha}^- t_{i_1}^{m_1} t_{i_2}^r) \cdot w_\pi = 0.$$ Repeating, we find that for any $r_1, \ldots, r_N \in \mathbb{Z}$

$$(x_{\alpha}^- t_{i_1}^{m_1})^{N-1} (x_{\alpha}^- t_{i_1}^{m_1} t_{i_2}^{r_1}) (x_{\alpha}^- t_{i_1}^{m_1} t_{i_2}^{r_2}) \cdots (x_{\alpha}^- t_{i_1}^{m_1} t_{i_2}^{r_N}) \cdot w_\pi = 0.$$ This proves that the elements $(x_{\alpha}^- t_{i_1}^{m_1} t_{i_2}^{m_2})$ act nilpotently on $w_\pi$ and hence that $W_{tor}(\pi)$ is integrable.

To see that $W$ is admissible, fix a total order on the set $R_{aff}^+$. By the Poincare–Birkhoff–Witt theorem it follows that the weight space $W_{\lambda-\eta}$ for $\eta \in Q_{aff}^+$ is spanned by elements of the form

$$x_{\beta_1}^- t_{i_2}^{k_1} \cdots x_{\beta_r}^- t_{i_2}^{k_r} \cdot w_\pi,$$

where $r \geq 1$, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$ are elements of $R_{aff}^+$ such that $\sum \beta_s = \eta$ and $k_1, \ldots, k_r \in \mathbb{Z}$. Since the number of roots in $R_{aff}^+$ that add up to a fixed $\eta \in Q_{aff}^+$ is finite, to see that $W_{\lambda-\eta}$ is finite-dimensional it suffices to prove that the number of choices for $k_1, \ldots, k_r$ is finite. In fact, it suffices to prove that for every $\beta \in R_{aff}^+$ there exists an integer $N_\beta$ such that $x_{\beta}^- t_{i_2}^s \cdot w_\pi$ is in the span of elements of the form $x_{\beta}^- t_{i_2}^s \cdot w_\pi$, with $-N_\beta \leq s \leq N_\beta$. For then an obvious induction on $r$ proves that, for every $\eta \in Q_{aff}^+$, there exists an integer $N_\eta$ such that $W_{\lambda-\eta}$ is spanned by elements of the form

$$x_{\beta_1}^- t_{i_2}^{k_1} \cdots x_{\beta_r}^- t_{i_2}^{k_r} \cdot w_\pi,$$

$-N_\eta \leq k_i \leq N_\eta$, $\sum \beta_s = \eta$.

If $\beta \in R_{aff}^+ \setminus \{m \delta_1 : m \in \mathbb{Z}, m > 0\}$, it follows from Lemma 1.1 and the fact that $(x_{\beta}^-)^{\lambda(h_\beta)+1} = 0$ that $x_{\beta}^- t_{i_2}^s \cdot w_\pi$ is in the span of $\{x_{\beta}^- t_{i_2}^s \cdot w_\pi : -\lambda(h_\beta) \leq r \leq \lambda(h_\beta)\}$. If $\beta = m \delta_1, m > 0$, the corresponding negative root vectors in $\mathfrak{g}_{aff}^+$ are $\{h_i t_{i_1}^{-m} : 1 \leq i \leq n\}$. Take $\beta = \alpha_i + m \delta_1$ in the first equation in Lemma 1.1 and apply $x_i^+$ to it. This gives

$$\sum_{r=0}^{s} h_i t_{i_1}^{-m} t_{i_2}^{s_2} \lambda^+(h_i, s-r) \cdot w_\pi = 0.$$ Again, it follows that the element $h_i t_{i_1}^{-m} t_{i_2}^{m_2} \cdot w_\pi$ is in the span of the elements $\{h_i t_{i_1}^{-m} t_{i_2}^{s_2} \cdot w_\pi : -\lambda(h_i) \leq s_2 \leq \lambda(h_i)\}$. This completes the proof of the proposition. 

□

Corollary 3.1. We have $W_{tor}(\pi) = U(\mathfrak{g}_{aff}^+ \otimes \mathbb{C}[t_2]) \cdot w_\pi$. 

Proof. Since \( W_{\text{tor}}(\pi) \) is an integrable module, it follows from Lemma [3] that if \( \beta \in R^\text{aff}_+ \setminus \{r_1 \delta : r_1 \in \mathbb{Z} \} \), then
\[
\sum_{m=0}^{N_{\beta}} (x_{\beta}^{-m})^t \Lambda^\pm (h_{\beta}, N_{\beta} - m) \cdot w_\pi = 0,
\]
where \( N_{\beta} = \lambda_{\pi}(h_{\beta}) \). Applying \( h_{\beta}t_2^{m_2} \), \( m_2 < 0 \), to both sides of the equation gives
\[
\sum_{m=0}^{N_{\beta}} (x_{\beta}^{-m})^t \Lambda^+(h_{\beta}, N_{\beta} - m) \cdot w_\pi = 0.
\]
This proves that \( (x_{\beta}^{-m})^t \cdot w_\pi \) is in the span of elements \( \{ (x_{\beta}^{-m})^t \cdot w_\pi : m > m_2 \} \) and hence by a simple induction in the span of elements of the form \( \{ (x_{\beta}^{-m})^t \cdot w_\pi : m \geq 0 \} \). A similar statement then follows for roots of the form \( \{ r_1 \delta_1 : r_1 \in \mathbb{Z} \} \), exactly as in the proof of the proposition. The corollary follows.

We conclude this section with some result on the structure of the modules \( W_{\text{tor}}(\pi) \) that we shall need in the last section of the paper.

Lemma 3.3. Regarded as a module for \( \mathfrak{g}_{\text{aff}} \), we have
\[
W_{\text{tor}}(\pi) \cong \bigoplus_{\mu \in P^+_\text{aff}} m(\mu)V_{\text{aff}}(\mu),
\]
for some \( m(\mu) \in \mathbb{Z}, m(\mu) \geq 0 \). Further, \( m(\lambda_{\pi}) = 1 \) and
\[
m(\mu) \neq 0 \implies \mu = \lambda_{\pi} - \mathcal{Q}^+_\text{aff}.
\]
Proof. To prove (i), notice that
\[
(W_{\text{tor}}(\pi))_\mu \neq 0 \implies \mu \in \lambda_{\pi} - \mathcal{Q}^+_\text{aff},
\]
and \( \dim (W_{\text{tor}}(\pi))_\mu \) is finite. The Lemma now follows from Proposition [3].

Although, we have so far been interested only in the representations of \( \mathfrak{g}_\text{tor} \), we shall also need the corresponding results for the algebra \( \mathfrak{g}_{\text{tor}}^+ = \mathfrak{g}_{\text{aff}}[t_2] \) of polynomial maps \( \mathbb{C} \rightarrow \mathfrak{g}_{\text{aff}} \). One can define, in the obvious way, a family of \( \mathfrak{g}_{\text{tor}}^+-\text{modules} \) \( M^+_{\text{tor}}(\lambda, \mathfrak{p}^+) \), where \( \lambda \in (\mathfrak{h}_{\text{aff}}^*)^+, \mathfrak{p}^+ = \{ p_r^+ : r \in \mathbb{Z}, r > 0 \} \). It is not hard to see that there exists an injective map of \( \mathfrak{g}_{\text{tor}}^+-\text{modules} \)
\[
M^+_{\text{tor}}(\lambda, \mathfrak{p}^+) \rightarrow M(\lambda, \mathfrak{p}),
\]
where \( \mathfrak{p} = \{ p_r : r \in \mathbb{Z}, r \neq 0 \} \) is such that \( p_r = p_r^+ \) for all \( r > 0 \). Further, one can show as in Proposition [3] that \( M^+_{\text{tor}}(\lambda, \mathfrak{p}^+) \) has an integrable quotient if and only if \( \mathfrak{p}^+ \) is given by an \( (n+1) \)-tuples of polynomials. Let \( W^+_{\text{tor}}(\pi) \) be the corresponding maximal integrable quotient, i.e. the quotient of \( M^+_{\text{tor}}(\pi) \) by the submodule generated by elements of the form \( (x_i^-)^{\deg \pi_i+1} \cdot v_\pi \).

Proposition 3.3. Assume that \( \pi \) is an \( (n+1) \)-tuple of polynomials with constant term one. Then,
\[
W^+_{\text{tor}}(\pi) \cong W_{\text{tor}}(\pi)
\]
as \( \mathfrak{g}_{\text{tor}}^+-\text{modules} \).
Proof. It is clear that there exists a map of $g_{\text{tor}^+}$–modules $\iota : M_{\text{tor}^+}(\pi) \to W_{\text{tor}}(\pi)$, which by Corollary 3.1 is surjective. Suppose that $v = g\pi \in M_{\text{tor}^+}(\pi)$ maps to zero. This means that $\iota(g\pi) = g\iota(v) \in g_{\text{tor}^+}$–modules $M_{\text{tor}^+}(\pi)$ generated by the elements $(x_i^{-\deg \pi})^{\deg \pi_i+1}$. Since

$$
\sum_{i=0}^{n} U(g_{\text{tor}}(<))(x_i^{-\deg \pi_i})^{\deg \pi_i+1} \iota(v) = \sum_{i=0}^{n} U(n_{aff}^{-\deg \pi_i} C[t_2]) U(n_{aff}^{-\deg \pi_i} C[t_2]) (x_i^{-\deg \pi_i})^{\deg \pi_i+1} \iota(v),
$$

and $M_{\text{tor}}(\pi)$ is a free $g_{\text{tor}^-}$–module, it follows from the PBW–theorem that $g \in \sum_{i=0}^{n} U(n_{aff}^{-\deg \pi_i} C[t_2]) (x_i^{-\deg \pi_i})^{\deg \pi_i+1} \iota(v)$.

Hence, we find that the induced map $W_{\text{tor}^+}(\pi) \to W_{\text{tor}}(\pi)$ is injective, and the proposition is proved.

The final result of this section is an analog for the modules $W_{\text{tor}}(\pi)$ of the factorization that holds for the irreducible modules $V_{\text{tor}}(\mu, a)$. The proof of this Proposition is a modification of the proof of Proposition 3.1 in \cite{8}, the details of the proof can be found in \cite{19}.

**Proposition 3.4.** Let $a_1, \cdots, a_k$ be the distinct roots of $\prod_{i=0}^{n} \pi_i$. For $1 \leq s \leq k$, and $0 \leq i \leq n$, assume that $a_s$ occurs as a root of $\pi_i$ with multiplicity $m_{i,s}$. For $1 \leq j \leq k$, let $\pi_j = (\pi_{j,0}, \cdots, \pi_{j,n})$ be defined by,

$$
\pi_{j,i} = (1 - a_j^{-1} u)^{m_{i,j}}.
$$

Then we have an isomorphism

$$
W(\pi) \cong W(\pi_1) \otimes \cdots \otimes W(\pi_k),
$$

of $g_{\text{tor}^-}$–modules.

**4. An irreducibility criterion for $W_{\text{tor}}(\pi)$**

It follows from Proposition 3.2 that the unique irreducible quotient of $W_{\text{tor}}(\pi)$ is $V_{\text{tor}}(\pi)$. In this section we give a condition for the quotient map to be an isomorphism.

Recall that $\theta$ is the highest root of $R_{\text{fin}}^+$ and write $h_\theta = \sum_{i=1}^{n} m_i h_i$. For $0 \leq i \leq n$, and $a \in \mathbb{C}^\times$, define an $(n+1)$–tuple of polynomials $\pi_{i,a}$ by

$$
\pi_j = 1, \quad j \neq i, \quad \pi_i = (1 - au).
$$

**Theorem 4.** Assume that either $i = 0$ or that $1 \leq i \leq n$ is such that $m_i = 1$. Then,

$$
W_{\text{tor}}(\pi_{i,a}) \cong V_{\text{tor}}(\pi_{i,a})
$$

as $U_{aff}$–modules.

Proof. Set $\pi = \pi_{i,a}$. Consider the map

$$
\phi_r : (g_{aff} t_2^\mathbb{Z}) \otimes W_{\text{tor}}(\pi) \to W_{\text{tor}}(\pi)
$$

Proof.
given by \( \phi_r(xt^r_2, w) = (xt^r_2)w \). This is clearly a map of \( g_{aff} \)-modules, where \( g_{aff} \) acts on the first factor through the adjoint representation. Let \( p_\mu \) denote the projection of
\[
p_\mu : W_{tor}(\pi) = \bigoplus_{\nu \in P^+_{aff}} V_{aff}(\nu) \to W(\mu)
\]
where \( W(\mu) \) is the \( \mu \)th isotypical component of \( W_{tor}(\pi) \). Then \( p_\mu \) is a map of \( g_{aff} \)-modules, and hence for every \( r \in \mathbb{Z} \) we get a map of \( g_{aff} \)-modules
\[
\phi_{r,\mu} = p_\mu \phi_r : (g_{aff} t^r_2) \otimes W_{tor}(\pi) \to W(\mu).
\]
We show that for all \( r \in \mathbb{Z} \) the restriction of \( \phi_{r,\mu} \) to \( (g_{aff} t^r_2) \otimes V_{aff}(\pi) \) is zero if \( \mu \neq \lambda_{\pi} \).

Observe that \( \lambda_{\pi} = \omega_i \) where \( i \) is such that \( m_i = 1 \) and hence \( \omega_i(c_1) = 1 \). This implies that
\[
(\lambda_{\pi} + \theta)(h_\theta) > 2, \quad \text{if} \ i \neq 0
\]
and
\[
(\theta - \lambda_{\pi})(h_\theta) > 1, \quad \text{if} \ i = 0.
\]
Further, since \( c_1 t^r_2.w_{\pi} \) is a scalar multiple of \( w_{\pi} \), it follows that \( \phi_{r,\mu}(c_1 t^r_2, v) = 0 \) for all \( v \in V_{aff}(\lambda_{\pi}) \) if \( \mu \neq \lambda_{\pi} \). This implies that \( \phi_{r,\mu} \) factors through to a map of \( g_{aff} \)-modules
\[
g_{fin}(C[t_1, t_1^{-1}])t^r_2 \otimes V_{aff}(\lambda_{\pi}) \to W(\mu).
\]
Since
\[
g_{fin}(C[t_1, t_1^{-1}])t^r_2 \cong L(V_{fin}(\theta))
\]
as \( g_{aff} \)-modules, it follows from Theorem 3 that \( (g_{aff} t^r_2) \otimes V_{aff}(\lambda_{\pi}) \) is irreducible. On the other hand since this module has infinite-dimensional weight spaces, it follows from Schur’s lemma that \( \phi_{r,\mu} = 0 \). This means that as a \( g_{aff}^e \)-module we have
\[
W_{tor}(\pi_{i,a}) \cong V_{aff}(\lambda_i).
\]
On the other hand the irreducible quotient of \( W_{tor}(\pi_{i,a}) \) is the module \( V_{aff}(\lambda_i, a) \) and this by definition is isomorphic to \( V_{aff}(\lambda_i) \) as a \( g_{aff}^e \)-module. The theorem now follows.

\[ \square \]

5. Fusion product and reducibility of \( W_{tor}(\pi) \)

To understand the \( g_{aff} \)-module structure of \( W_{tor}(\pi) \) and in particular to prove that it is not isomorphic to \( V_{tor}(\pi) \), we need to introduce the notion of the fusion product of representations of \( g_{aff} \otimes C[t_2] \). This was introduced by Feigin and Loktev in [14].

Thus, let \( a \) be any Lie algebra and let \( a[t] \) be the algebra of polynomial maps \( C \to a \). For \( k \geq 1 \), let \( V_1, \ldots, V_k \) be representations of \( a \) and let \( a_1, \ldots, a_k \in C \) be arbitrary complex numbers. As in the previous sections, one sees that the tensor product \( V_1 \otimes \cdots \otimes V_k \) admits a structure of \( a[t] \)-module given by
\[
x t^r (v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^k a_j^r v_1 \otimes \cdots \otimes v_{j-1} x v_j \otimes \otimes v_{j+1} \cdots \otimes v_k,
\]
for all \( x \in a, \ r \geq 0, \ v_i \in V_i, \ 1 \leq i \leq k \). Denote this module by \( V_1(a_1) \otimes \cdots \otimes V_k(a_k) \). The following lemma is easily proved.
Lemma 5.1. Assume that there exist elements \( v_j \in V_j \) such that \( V_j \) is generated as an \( \mathfrak{a}[t] \)-module by \( v_1 \otimes \cdots \otimes v_k \).

The Lie algebra \( \mathfrak{a}[t] \) and its enveloping algebra \( U(\mathfrak{a}[t]) \) are obviously graded by powers of \( t \). Let \( U(\mathfrak{a}[t])_r \) be the \( r \)-th graded piece. If \( V_1, \ldots, V_k \) are generated by elements \( v_1, \ldots, v_k \) and \( a_1, \ldots, a_k \) are distinct complex numbers, the \( \mathfrak{a}[t] \)-module \( V = V_1(a_1) \otimes \cdots \otimes V_k(a_k) \) admits an \( \mathfrak{a} \) (but not an \( \mathfrak{a}[t] \)) equivariant filtration. The \( r \)-th filtered piece is given by

\[
V_r = \bigoplus_{0 \leq s \leq r} (U(\mathfrak{a}[t]))_s.(v_1 \otimes \cdots \otimes v_k).
\]

The associated graded vector space \( \oplus_{r \geq 0} V_r / V_{r-1} \) is obviously a \( \mathfrak{a} \)-module. Since \( \mathfrak{a} t^s. V_r \subset V_{r+s}^t \), one can define an \( \mathfrak{a}[t] \)-module structure on \( \oplus_{r \geq 0} V_r / V_{r-1} \). This module, denoted by \( V_1(a_1) \ast V_2(a_2) \ast \cdots \ast V_k(a_k) \), is called the fusion product of the modules \( V_1, \ldots, V_k \) with respect to \( a_1, \ldots, a_k \). Let \( v_1 \ast \cdots \ast v_k \) be the image of \( v_1 \otimes \cdots \otimes v_k \) in \( V_1(a_1) \ast V_2(a_2) \ast \cdots \ast V_k(a_k) \).

Remark As an \( \mathfrak{a} \)-module, it is clear that

\[
(5.1) \quad V_1(a_1) \ast V_2(a_2) \ast \cdots \ast V_k(a_k) \cong V_1 \otimes \cdots \otimes V_k.
\]

Assume now that \( \mathfrak{a} = \mathfrak{g}_{aff} \) and that \( V_j = V_{aff}(\lambda_j) \), where \( \lambda_j \in P_{aff}^+, 1 \leq j \leq k \). The next lemma is not hard to check.

Lemma 5.2. Let \( k \geq 1 \), and let \( \lambda_1, \ldots, \lambda_k \in P_{aff}^+ \). Assume that \( a_1, \ldots, a_k \) are distinct scalars. The fusion product \( V_{aff}(\lambda_1)(a_1) \ast \cdots \ast V_{aff}(\lambda_k)(a_k) \) is generated as an \( \mathfrak{a}[t] \)-module by the element \( v = v_1 \otimes \cdots \otimes v_k \). Further,

\[
n_{aff}^r[t].v = 0, \quad (h_{aff}) t \mathfrak{C}[t].v = 0, \quad h.v = \sum_j \lambda_j(h).v, \quad h \in \mathfrak{h}_{aff}.
\]

\[ \square \]

For any \( \lambda = \sum_{i=0}^n \lambda(h_i) \omega_i, \lambda(h_i) \geq 0 \), fix distinct complex numbers \( \{c_{ij} : 0 \leq i \leq n, 1 \leq j \leq \lambda(h_i)\} \). Consider the fusion product

\[
W(\lambda) = V_{aff}(\omega_0)(c_{01}) \ast \cdots \ast V_{aff}(\omega_0)(c_{0\lambda_0}) \ast \cdots \ast V_{aff}(\omega_n)(c_{n1}) \ast \cdots \ast V_{aff}(\omega_n)(c_{n\lambda_n}).
\]

For any complex number \( a \in \mathbb{C} \), let \( W(\lambda, a) \) be the \( \mathfrak{g}_{tor}[t] \)-module obtained by pulling back \( W(\lambda) \) through the Lie algebra homomorphism \( \mathfrak{g}_{aff}[t] \to \mathfrak{g}_{aff}[t] \) given by sending \( xt^r \to x(t - a)^r \).

We turn now to the \( \mathfrak{g}_{tor} \)-module \( W_{tor}(\pi) \). We restrict our attention to the case where there exists \( a \in \mathbb{C}^\ast \) and \( n_j \in \mathbb{Z}, n_j \geq 0 \), such that if \( \pi = (\pi_1, \cdots, \pi_n) \) then

\[
\pi_j(u) = (1 - au)^{n_j}, \quad 1 \leq j \leq n.
\]

Theorem 5. There exist a surjective map of \( \mathfrak{g}_{aff}[t_2] \)-modules \( W_{tor}(\pi) \to W(\lambda, \pi, a) \).

Proof. By Proposition 5.3 it is enough to prove that the element \( (v_{\omega_0})^{\otimes \lambda_0} \otimes \cdots \otimes (v_{\omega_n})^{\otimes \lambda_n} \) satisfies the defining relations of the element \( w_{\pi} \). But this is clear from Lemma 5.2. \( \square \)
It is now not hard to see that $W_{\text{tor}}(\pi)$ is generally reducible. This is because we know from the results of Section 3 that the irreducible quotient of $W_{\text{tor}}(\pi)$ in this case is $V_{\text{tor}}(\lambda\pi, a)$, and $V_{\text{tor}}(\lambda\pi, a) \cong V_{\text{aff}}(\lambda\pi)$ as $\mathfrak{g}_a^{\text{aff}}$–modules. On the other hand the module $W(\lambda\pi, a)$ is isomorphic as a $\mathfrak{g}_a^{\text{aff}}$–module to a tensor product of integrable irreducible $\mathfrak{g}_a^{\text{aff}}$–modules, and hence must be reducible as a $\mathfrak{g}_a^{\text{aff}}$–module.

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