On the Ground State of the Supersymmetric Five–Brane

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Abstract

We examine if there exists a zero–energy supersymmetric ground state for the fundamental five–brane. Looking for an $SO(6) \times SO(2)$–invariant ground state, we construct, in the light–cone gauge, perturbatively a Nicolai map up to third order in the inverse five–brane tension. We show that the Nicolai map equilibrates and the five–brane has a zero–energy normalizable supersymmetric vacuum state. For the other p–branes, we argue that only the three–brane has a zero–energy ground state.
1. Introduction

The existence of extended objects, p–branes, in the string spectrum promises that interesting information about the non-perturbative structure of string theory can be obtained. These p–branes configurations appear as solitons in the low–energy string field theory and they are necessary in establishing the various string dualities. There exists extensive literature dealing with their properties and their dynamics [1]–[4].

On the other hand, solitonic p–branes are quite different in many aspects from fundamental ones [1]–[7]. For example, solitonic branes have internal structure, lost in the long wave–length limit, while by definition there is no structure for fundamental p–branes. The excitations of the latter are interpreted as ordinary particles. Thus, in order for one to really think of extended objects as being fundamental, the question of the existence of massless states in the spectrum has to be answered. This issue, as far as we know, has extensively be studied only for the fundamental membrane [9]–[12]. However, the same question should also be addressed for the other extended objects which admit space–time supersymmetry, namely, for the three–, four– and five–branes. Here we will consider explicitly the latter and in particular the neutral ones since the heterotic five–brane action is not known. We will also make some comments on the other p–branes.

There exists a serious difference between the five–brane and the fundamental membrane. For the latter, there exist supersymmetric quantum mechanical models with finite degrees of freedom for which the Schrödinger equation can explicitly be studied. These models are supersymmetric $SU(N)$ Yang–Mills theories dimensionally reduced to 0 + 1 dimensions (time). The fundamental membrane is then recovered in the $N \to \infty$ limit [3]. For this class of models, the result is that the spectrum is continuous starting from zero and filling the positive real axis [3]. Moreover, there is no normalizable zero–energy state. This is consistent with the proposal that the membrane is effectively described by condensation of D0 branes of the type IIA theory [14],[15]. However, it should be mentioned that there exists also the claim that the membrane has discrete spectrum [16] as well, as a consequence of the finite size core of the D–particle [17].

For the five–brane on the other hand [18],[19], there is no quantum mechanical model and so one is forced to study a system with infinite degrees of freedom. In this case, instead of solving functional differential equations, we preferred to follow another way, namely, to find a Nicolai map [20] perturbatively in the inverse five–brane tension. We determined such a map in the case of $SO(6) \times SO(2)$–symmetric target space. Moreover, the Nicolai map we have constructed equilibrates for large times and thus there exists a zero–energy supersymmetric vacuum state. This state corresponds to the $N = 1$ ten–dimensional vector multiplet [21].
In sect. 2 we set up the formalism and we give the partition function for the five–brane in the light–cone gauge. Next we discuss the Nicolai map in sect. 3 and in sect. 4 we find the ground state wave function and we make some comments for the corresponding state of three– and four–branes. Finally, we summarize our results in sect. 5.

2. The five-brane action

Quantization in the light–cone gauge is sometimes convenient since unitarity is guaranteed. A drawback is that Lorentz invariance may be lost as a consequence of quantization and should be checked at the end. It is also possible that this gauge is the only one in which a Hamiltonian formulation of a theory can be performed as for example the string theory. Here we will study the supersymmetric five–brane in the light–cone gauge which is described by the Lagrangian [6]

\[
\mathcal{L} = \frac{T_5}{2} \left( \partial_I X^I \partial_I X^I - \det \partial_a X^I \partial_b X^J + i \bar{S} \partial \gamma S + \frac{i}{4!} \epsilon^{abcde} \partial_a X^I \partial_b X^J \partial_c X^K \partial_d X^L S \gamma_{IJKL} \partial_e S \right),
\]

(2.1)

\(T_5\) is the five–brane tension and we will assume, if it is not explicitly indicated that \(T_5 = 1\). The covariant derivative \(\mathcal{D}\) is given by

\[
\mathcal{D} = \partial / \partial t + u^a \partial / \partial \sigma^a \quad (a, b = 1, \cdots, 5),
\]

where \(\sigma^a\) are coordinates on the brane and \(u^a\) is a divergence free vector field. In the light–cone gauge only \(d - 2\) of the original \(d\) fields remain and since a supersymmetric five–brane may live only in \(d = 10\), there exists eight bosonic fields \(X^I\), \((I = 1, \cdots, 8)\). However, as we will see below, due to gauge symmetries only the transverse excitations of the brane remain which represent the physical degrees of freedom. The fermion \(S\) is a real \(SO(8)\) spinor which we will assume to be the \(8_c\) and \(\bar{S} = S^T\).

The action for the five–brane is invariant under the supersymmetry transformations

\[
\delta X^I = 2i \bar{\epsilon} \gamma^I S, \\
\delta S = -2 \partial X^I \gamma_I \epsilon + \frac{2}{5!} \epsilon^{abcde} \partial_a X^I \partial_b X^J \partial_c X^K \partial_d X^L \partial_e X^M \gamma_{IJKL} \epsilon, \\
\delta u^a = -\frac{i}{3} \epsilon_{IJK} \epsilon^{abcde} \partial_b X^I \partial_c X^J \partial_d X^K \partial_e S,
\]

(2.3)
where $\gamma^I$ are $SO(8)$ $\gamma$–matrices and $\epsilon^{abcde}$ is the totally antisymmetric symbol in five dimensions. It is also invariant under reparametrizations $\sigma^a \rightarrow \sigma^a + \epsilon^a$ and the fields in (2.1) transform as

$$\delta X^I = \epsilon^a \partial_a X^I,$$

$$\delta S = \epsilon^a \partial_a S,$$

$$\delta u^a = -\frac{d\epsilon^a}{dt} + \epsilon^b \partial_b u^a - u^b \partial_b \epsilon^a.$$  \hspace{1cm} (2.4)

The vector field $\epsilon^a$ generates diffeomorphisms on the five–brane and due to the constraint eq.(2.2), $\epsilon^a$ is also divergence free

$$\partial_a \epsilon^a = 0.$$  

Hence, $\epsilon^a$ generates, in fact, volume preserving diffeomorphisms. We will gauge fix the reparametrization invariance by choosing the gauge

$$u^a = 0.$$  \hspace{1cm} (2.5)

Then ghosts $c^a$ and anti–ghosts $\bar{c}^a$ are also divergence free, i.e. they satisfy the condition

$$\partial_a c^a = \partial_a \bar{c}^a = 0,$$

which may implemented in the action by Lagrange multipliers $(\lambda, \bar{\lambda})$. The Faddeev–Popov determinant for the gauge fixing eq.(2.5) is

$$\Delta_{FP} = \det \left( \frac{d}{dt} \delta_{ab} \delta(\sigma - \sigma') \delta(t - t') \right),$$

and the ghost action is then

$$I_{gh} = \int dt d\delta \sigma \left( ic^a \frac{d\epsilon^a}{dt} + i\bar{\lambda} \partial_a c^a + \h.c. \right).$$  \hspace{1cm} (2.6)

Let us now introduce the Nambu bracket \[22\]

$$\{X^I_1, \ldots, X^I_5\} = \epsilon^{a_1 \cdots a_5} \partial_{a_1} X^I_1 \cdots \partial_{a_5} X^I_5,$$  \hspace{1cm} (2.7)

which is skew–symmetric, satisfies the Leibniz rule and the fundamental identity \[23\]

$$\{\{X^I_1, \ldots, X^I_5\}, X^I_6, \ldots, X^I_9\} + \{X^I_5, \{X^I_1, \ldots, X^I_4X^I_6\}, X^I_7, X^I_8, X^I_9\}
+ \cdots + \{X^I_5, \ldots, X^I_8, \{X^I_1, \ldots, X^I_4, X^I_9\}\} = \{X^I_1, \ldots, X^I_4, \{X^I_5, \ldots, X^I_9\}\},$$
which is a generalization of the Jacobi identity. The determinant in the Lagrangian (2.1) may be expressed in terms of the Nambu bracket as
\[ det(\partial_a X^I \partial_b X^I) = \frac{1}{5!} \{X^{I_1}, \ldots, X^{I_5}\}^2. \] (2.8)

The five–brane action turns out then to be
\[ I = \frac{1}{2} \int d\tau d^5\sigma \left( (DX^I)^2 - \frac{1}{5!} \{X^{I_1}, \ldots, X^{I_5}\}^2 + i \bar{S}dS + \frac{i}{4!} \tilde{S} \gamma_{IJKL} \{X^I, X^J, X^K, X^L, S\} \right), \] (2.9)

and the equations of motions as follow from (2.9) are
\[ \frac{d^2 X^I}{dt^2} - \frac{1}{4!} \{\{X^I, X^{I_1}, \ldots, X^{I_5}\}, X^{I_1}, \ldots, X^{I_4}\} - \frac{i}{3!} \{\bar{S} \gamma_{IJKL}, X^J, X^K, X^L, S\} = 0, \] (2.10)
\[ \frac{dS}{dt} + \frac{1}{4!} \gamma_{IJKL} \{X^I, X^J, X^K, X^L, S\} = 0. \] (2.11)

In order to discuss quantum aspects of the fundamental five–brane, we will consider the partition function of the theory which we write as
\[ Z = \int d\mu e^{-I_E - I_{gh}}. \] (2.12)

The measure \( d\mu \) is
\[ d\mu = [dX^I][d\bar{S}][dS][d\bar{c}][dc][d\bar{\lambda}][d\lambda] \]

and
\[ I_E = \frac{1}{2} \int d\tau d^5\sigma \left( (\frac{dX^I}{d\tau})^2 + \frac{1}{5!} \{X^{I_1}, \ldots, X^{I_5}\}^2 + \bar{S} \frac{dS}{d\tau} - \frac{i}{4!} \tilde{S} \gamma_{IJKL} \{X^I, X^J, X^K, X^L, S\} \right). \] (2.13)

is the gauge fixed Euclidean five–brane action after a Wick rotation of (2.9). Integrating out the ghosts and the Lagrange multiplier we get
\[ Z = \int [dX^I][d\bar{S}][dS] e^{-I_E} det \left( \frac{d}{d\tau} \delta(\sigma - \sigma') \delta(\tau - \tau') \right)^4 det(\partial_\alpha \partial^\alpha), \] (2.14)

and thus, as advertised, only four of the \( X^I \) represent the physical excitations of the brane.

We may also integrate out the fermions which appear quadratically in (2.13). Thus, finally, the partition function may be expressed as an integral over bosonic fields only as
\[ Z = \int [dX^I] e^{-I_E[X^I]} det \left( \frac{d}{d\tau} \delta(\sigma - \sigma') \delta(\tau - \tau') \right)^4 det_F, \] (2.15)

where
\[ det_F = det \left[ \left( \frac{d}{d\tau} \delta_{\alpha\beta} + \frac{i}{4!} (\gamma_{IJKL})_{\alpha\beta} \{X^I, X^J, X^K, X^L, \} \right) \delta(\sigma - \sigma') \delta(\tau - \tau') \right]^{1/2} \] (2.16)
is the fermionic determinant and \((\alpha, \beta = 1, \cdots, 8)\) are spinor indices. It should be noted here that we assume periodic boundary conditions for both bosons and fermions in order supersymmetry to be respected.

Let us suppose now that it is possible to find a transformation \(X^I \rightarrow \xi^I(X)\) such that i) the Jacobian of this transformation cancels exactly the product of determinants in eq.(2.15) and ii) \(I_E\) is proportional to the length \(|\xi|^2\). Such transformation, known as Nicolai map, reduces the partition function into a Gaussian integration. It is in general a non–local and non–polynomial transformation which, however, can be constructed order by order in perturbation theory [20],[24]. Exact expressions may be obtained for topological field theories [25]. In the next section we will see that such a \(\xi^I(X)\) can be found approximately up to third order in \(1/T_s^2\) in a similar way as in the four–dimensional supersymmetric Yang–Mills theory [20].

3. The Nicolai map

The previous considerations were quite general. Here we will study a particular case, namely, we will split \(X^I\) as \(X^I = (X^i, X^7, X^8)\), \((i = 1 \cdots, 6)\) and we will assume that \((X^7, X^8)\) are constants. This breaks the original \(SO(8)\) symmetry into \(SO(6) \times SO(2)\). The spinor \(S\) is split accordingly into \(4 + \bar{4}\) and one may form the two real spinors \(\theta_1 \sim 4 + \bar{4}\) and \(\theta_2 \sim i(4 - \bar{4})\). Similarly, we will assume that \(\theta_1\) is also constant.

We may consider the fields \(X^i(\sigma)\) as a map \(X^i : \Sigma \rightarrow M\) from the five–brane worldvolume \(\Sigma\) which we take to be \(S^6\) to the six–dimensional target space \(M\) parametrized by \(X^i\). In this case we may define the winding number (“degree”) of this map as

\[
q = \frac{1}{128\pi^6} \int d\tau d^5\sigma \frac{dX^j}{d\tau} \partial_{a_1} X^{i_1} \cdots \partial_{a_5} X^{i_5} \varepsilon^{a_1 \cdots a_5} \epsilon_{i_1 \cdots i_5}, \tag{3.1}
\]

which may also be written as

\[
q = \frac{1}{128\pi^6} \int d\tau d^5\sigma \frac{dX^j}{d\tau} \{X^{i_1}, \cdots, X^{i_5}\} \epsilon_{j_1 \cdots j_5}. \tag{3.2}
\]

We introduce now new bosonic fields \(\xi^i\) defined by

\[
\xi^i = \begin{cases}
\frac{dX^i}{d\tau} + \frac{1}{5!}\{X^{i_1}, \cdots, X^{i_5}\} \epsilon_{i_1 \cdots i_5} & \text{if } q < 0 \\
\frac{dX^i}{d\tau} - \frac{1}{5!}\{X^{i_1}, \cdots, X^{i_5}\} \epsilon_{i_1 \cdots i_5} & \text{if } q \geq 0
\end{cases}, \tag{3.3}
\]
where $\epsilon_{i_1\cdots i_6}$ is the six-dimensional antisymmetric symbol. We will see below that the transformation $X^i \rightarrow \xi^i(X)$ is a Nicolai map. One may easily verify that

$$\xi^i \xi^i = \left(\frac{dX^i}{d\tau}\right)^2 + \text{det}(\partial_a X^i \partial_b X^i) \pm \frac{2}{5!} \frac{dX^i}{d\tau} \{X^{i_1}, \cdots, X^{i_5}\} \epsilon_{i_1\cdots i_5},$$

for $q < 0, q \geq 0$. Then, the bosonic part of the action (2.13) is written as a quadratic form in $\xi^i$,

$$I_{E}^{\text{bos}} = \frac{1}{2} \int \xi^i \xi^i + |Q|,$$

where $Q = 128\pi^6 q$. As a result, the partition function (2.13) turns out to be

$$Z = \int [d\xi^i] e^{-\frac{1}{2} \int \xi^i \xi^i + |Q| \text{det}(\delta\xi^i / \delta X^i)^{-1} \text{det} \left(\frac{d}{d\tau} \delta(\sigma - \sigma') \delta(\tau - \tau')\right)^4 \text{det}_F},$$

where $|\text{det}(\delta\xi^i / \delta X^i)|$ is the Jacobian of the map $X^i \rightarrow \xi^i(X)$. We will show below that

$$\text{det}(\delta\xi^i / \delta X^i) = \text{det} \left(\frac{d}{d\tau} \delta(\sigma - \sigma') \delta(\tau - \tau')\right)^4 \text{det}_F,$$

up to third order in $T_5^{-2}$ so that

$$\mathcal{M} = 1 + \mathcal{O}(T_5^{-8}).$$

Thus, the partition function will be transformed into a Gaussian integration over $\xi^i$'s, as required from a Nicolai map.

The Jacobian of $X^i \rightarrow \xi^i(X)$ is

$$\text{det}(\delta\xi^i / \delta X^i) = \text{det} \left(\frac{d}{d\tau} \delta_{ij} \pm \frac{1}{4!} \epsilon_{ijklmn} \{X^k, X^l, X^m, X^n, \cdots\} \delta(\sigma - \sigma') \delta(\tau - \tau')\right),$$

which we may write as

$$\text{det}(\delta\xi^i / \delta X^i) = \text{det} \left(\frac{d}{d\tau} \delta(\sigma - \sigma') \delta(\tau - \tau')\right)^6 \text{det}(1 \pm A),$$

where $A$ is the antisymmetric matrix

$$A_{ij} = \frac{1}{4!} \epsilon_{ijklmn} \delta^{-1} \{X^k, X^l, X^m, X^n, \cdots\} \delta(\sigma - \sigma') \delta(\tau - \tau').$$
and $1_{ij} = \delta_{ij}$. Expanding the determinant $\det(1 \pm A)$ in the right hand side of eq.(3.11) we get
\[
\det(\frac{\delta \xi^i}{\delta X^j}) = \det \left( \frac{d}{d\tau} \delta (\sigma - \sigma') \delta (\tau - \tau') \right)^6 \left(1 - \frac{1}{2} Tr(A^2) + \frac{1}{4!} \left[ Tr(A^4 - (TrA^2)^2) \right] + \cdots \right). \tag{3.13}
\]
Similarly, the fermionic determinant is written as
\[
\det_F = \det \left( \frac{d}{d\tau} \delta (\sigma - \sigma') \delta (\tau - \tau') \right)^2 \det(I + \Gamma)^{1/2}, \tag{3.14}
\]
where $I_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$, ($\alpha\bar{\beta} = 1, \cdots, 4$) and
\[
\Gamma = \frac{i}{4!} \bar{\gamma}_{ijkl} \partial_\tau \{ X^i, X^j, X^k, X^l, \} \delta (\sigma - \sigma') \delta (\tau - \tau'), \tag{3.15}
\]
with $\bar{\gamma}^i$ the $SO(6)$ $\gamma$–matrices. Expanding the determinant $\det(I + \Gamma)$ in eq.(3.14) we get
\[
\det_F = \det \left( \frac{d}{d\tau} \delta (\sigma - \sigma') \delta (\tau - \tau') \right)^2 \left(1 - \frac{1}{4} Tr(\Gamma^2) + \frac{1}{48} \left[ Tr\Gamma^4 - \frac{1}{2} (Tr\Gamma^2)^2 \right] + \cdots \right). \tag{3.16}
\]
It is a straightforward matter to verify that
\[
Tr(\Gamma^2) = 2TrA^2, \quad Tr(\Gamma^4) = -\frac{3}{4} \left( TrA^4 - 2(TrA^2)^2 \right). \tag{3.17}
\]
Comparing then eqs.(3.13, 3.16) using eq.(3.17), one may verify eq.(3.7) up to third order and thus, indeed the Jacobian of the transformation $X^i \to \xi^i(X)$ cancels the fermionic and the Faddeev–Popov determinants. Since the expansion of the determinants was actually an expansion in $1/T^2$, eq.(3.8) follows trivially.

The transformation $X^i \to \xi^i(X)$ turns thus the partition function into the Gaussian integration over $\xi^i$’s
\[
Z \sim \int [d\xi^i] e^{-\frac{1}{2} \int \xi^i \xi^i - |Q| \text{sign} \det \left( \frac{d\xi^i}{dX^i} \right) }, \tag{3.18}
\]
up to cubic order. The factor sing det above is due to the fact that it is the modulus of $\det(\delta \xi / \delta X)$ rather than the determinant itself which appears in eq.(3.6). If we define the operator $\Delta_{ij} = \frac{\delta \xi^i}{\delta X^j}$, we have that
\[
\Delta_{ij} = \frac{d}{d\tau} \delta_{ij} \pm \frac{1}{4!} \epsilon_{ijklmn} \{ X^k, X^l, X^m, X^n, \} \tag{3.19}
\]
One may easily verify that
\[
sing \det \Delta_{ij} = \exp \left( i\frac{\pi}{2} [\xi(0) - \eta(0)] \right), \tag{3.20}
\]
where \( \zeta_\Delta(s) = \sum_n |\lambda_n|^{-s} \) and \( \eta_\Delta(s) = \sum_n \text{sign}(\lambda_n)|\lambda_n|^{-s} \) are the \( \zeta \)–function and the \( \eta \)–invariant of \( \Delta_{ij} \). Thus, finally, up to third order
\[
Z \sim \int [d\xi^i] \exp \left( -\frac{1}{2} \int \xi^i \xi^i - |Q| + i \frac{\pi}{2} [\zeta_\Delta(0) - \eta_\Delta(0)] \right). \tag{3.21}
\]

What is still missing is the range of integration of the \( \xi \)–fields which can be found by determine how many times the space of \( X^i \)'s covers the \( \xi \)–space. This may be specified by counting the number of times the \( \xi \)–fields pass through zero and in which direction. The zeroes of the \( \xi^i(X) \) are given by the instanton and anti–instanton configurations
\[
\frac{dX^i}{d\tau} = \frac{1}{4!} \{X^{i_1}, \ldots, X^{i_5}\} \epsilon^{i_1 \ldots i_5}_{i} , \tag{3.22}
\]
\[
\frac{dX^i}{d\tau} = -\frac{1}{4!} \{X^{i_1}, \ldots, X^{i_5}\} \epsilon^{i_1 \ldots i_5}_{i} . \tag{3.23}
\]
One may easily check that these configurations satisfy the field equations eq.(2.11) and that they are absolute minima of the action (2.9). Fields obeying eq.(3.22) have \( Q \geq 0 \) (instantons) while fields obeying eq.(3.23) have \( Q < 0 \) (anti–instantons). We expect that solutions to eqs.(3.22,3.23) will exist for all \( Q \) and thus the winding number of the Nicolai map is infinity.

4. The five–brane ground state

Now, we define the “superpotential” \( W \) through the equation
\[
\xi^i = \frac{dX^i}{d\tau} \pm \frac{\delta W}{\delta X^i} , \tag{4.1}
\]
which is the most famous of the stochastic equations, the Langevin equation. It reflects the relation of the Nicolai map to the stochastic process of the classical Euclidean vacuum [24]–[26]. Using the Fokker–Planck equation for eq.(4.1) one can show that if \( W(X) \to \pm \infty \) as \( |X^i| \to \infty \) then there exists a large–time limit corresponding to thermal equilibrium. In this case, the probability distributions \( P_{\pm}[X^i, \tau] \), which obey the appropriately regularized Fokker–Planck equation
\[
\frac{\partial P_{\pm}}{\partial \tau} = \int d^5\sigma \frac{\delta}{\delta X^i} \left( \pm \frac{\delta W}{\delta X^i} + \frac{\delta}{\delta X^i} \right) P_{\pm} , \tag{4.2}
\]
satisfy
\[
\lim_{\tau \to \infty} P_{\pm}[X^i, \tau] = |\Psi_0^\pm(X)|^2 , \tag{4.3}
\]
where

\[ \Psi_0^\pm(X) = C_\pm e^{\pm W(X)}, \quad (4.4) \]

\[ |C_\pm|^2 = \int [dX^i] e^{\pm W(X)}, \]

is the zero–energy supersymmetric ground state, provided that it is normalizable \[24\]. In our case, by solving

\[ \frac{\partial W}{\partial X^j} = \frac{1}{5!} \{X^{i_1}, \ldots, X^{i_5}\} \epsilon_{j i_1 \cdots i_5}, \quad (4.5) \]

one may easily verify that the superpotential \( W \) is

\[ W = \frac{1}{6!} \int d^5 \sigma X^j \{X^{i_1}, \ldots, X^{i_5}\} \epsilon_{j i_1 \cdots i_5}. \quad (4.6) \]

Thus, for the five–brane there exist the \( SO(6) \times SO(2) \)–invariant vacuum states

\[ \Psi_0^\pm(X) \sim \exp \left[ \mp \frac{1}{6!} \int d^5 \sigma X^j \{X^{i_1}, \ldots, X^{i_5}\} \epsilon_{j i_1 \cdots i_5} \right]. \quad (4.7) \]

These states correspond to “forward” and “backward” stochastic processes. One of the states \( (4.7) \) is in addition normalizable and thus, there exists zero–energy ground state. It has zero fermion charge and corresponds to the \( N = 1 \) vector multiplet in ten dimensions \[21\].

We may also generalize the above discussion for the other \( p \)-branes of the brane scan. Let us consider a \( p \)-brane in \( D \) dimensions. In the light–cone gauge there exist \( D - p - 1 \) degrees of freedom describing transverse excitations. In this case, there exists the Nicolai map

\[ \xi^i = \frac{dX^i}{d\tau} \pm \frac{1}{p!} \{X^{i_1}, \ldots, X^{i_p}\} \epsilon_{i_1 \cdots i_p}, \quad (i = 1, \ldots, p + 1) \quad (4.8) \]

analogous to eq.\( (3.5) \), where \( \{X^{i_1}, \ldots, X^{i_p}\} \) is the Nambu “p-bracket”

\[ \{X^{I_1}, \ldots, X^{I_p}\} = \varepsilon^{a_1 \cdots a_p} \partial_{a_1} X^{I_1} \cdots \partial_{a_p} X^{I_p}. \quad (4.9) \]

The superpotential \( W_p \) turns out then to be

\[ W_p = \frac{1}{(p + 1)!} \int d^5 \sigma X^j \{X^{i_1}, \ldots, X^{i_p}\} \epsilon_{j i_1 \cdots i_p}. \quad (4.10) \]

Then, the \( SO(p + 1) \times SO(D - p - 3) \)–invariant ground state is given by eq.\( (4.4) \) and it is normalizable for \( p = 3, 5 \). The wave–function for \( p = 2 \), in particular, is

\[ \Psi_0^\pm \sim \exp \left( \mp \frac{1}{6} \int d^2 \sigma X^i \{X^j, X^k\} \epsilon_{ijk} \right), \quad (4.11) \]
which is non-normalizable and has been given by de Wit [12]. Thus, only the three- and five-brane seem to have a zero-energy ground state.

It should be noted that, in particular for the membrane, following this method one may also construct a $G_2$-invariant vacuum wave function. A Nicolai map for this case may be chosen to be

$$\xi^i = \frac{dX^i}{d\tau} \pm \frac{1}{2} c^i_{jk}\{X^j, X^k\}, \quad (i = 1, \cdots, 7), \quad (4.12)$$

where $c_{ijk}$ are the octonionic structure constants [13]. The “forward” ground state wave function is then

$$\Psi^+_0 \sim \exp \left( -\frac{1}{6} \int d^2\sigma X^i\{X^j, X^k\} c_{ijk} \right). \quad (4.13)$$

It is non-normalizable and coincides with what was reported in [10]. Details will be given elsewhere.

5. Conclusions

The purpose of this work has been to report some results concerning the ground state of supersymmetric five-branes. It was initiated by the fact that although much is known about the ground state of membranes, similar results for the other branes are lacking. Based on the “p-brane democracy” [28] as follows from U-duality arguments [29], the spectrum of all branes are equally important. However, to determine the spectrum of extended objects other than strings is a notoriously treacherous subject and one may only deal with their ground states at the moment. For the membrane, it seems that there are no massless particles since there is no zero-energy ground state at least in the quantum mechanical models considered so far. Claims about the opposite have also been made.

To find the ground state of the five-brane one may follow the standard way of solving the corresponding Schrödinger equation as in the case of the membrane. Here, however, we chose another, indirect way, in order to find the vacuum state. Namely, we tried to form a Nicolai map for the theory and then to read off the vacuum wave function in the equilibrium limit. The existence of the latter is equivalent to the existence of normalizable zero-energy ground state and thus after constructing the Nicolai map, one may check if such a state exists. We showed this explicitly for the five-brane and we argued that only the three-brane besides the five-brane has a normalizable zero-energy ground state. The wave function we found here is valid up to third order in inverse brane tension and there will be higher order corrections which however do not spoil its normalizability.
As a final comment, let us note that we studied here the neutral five–brane since the action of the heterotic one is not known. Although the former is anomalous \[30\], we expect our results for the ground state to carried over the heterotic five–brane.

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Appendix

The \(SO(8)\) \(\gamma\)–matrices we use here are

\[
\Gamma_I = \begin{pmatrix} 0 & \gamma_I \\ \gamma_I^\dagger & 0 \end{pmatrix}, \quad I = 1, \cdots, 8, \tag{A.1}
\]

where \(\gamma_I\) are \(8\times8\) real matrices and \(\gamma_I\gamma_J^\dagger + \gamma_J^\dagger\gamma_I = 2\delta_{IJ}\). One of the \(\gamma_I\) can be chosen to be Hermitian while the rest are anti-Hermitian. In particular

\[
\begin{align*}
\gamma_i &= i\bar{\gamma}_i, \quad i = 1, \cdots, 6 \\
\gamma_7 &= i\bar{\gamma}_7, \\
\gamma_8 &= 1, \tag{A.2}
\end{align*}
\]

where \(\bar{\gamma}_i\) are \(SO(6)\) \(\gamma\)–matrices and \(\bar{\gamma}_7 = i\bar{\gamma}_1 \cdots \bar{\gamma}_6\) is the corresponding chiral matrix.

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