HYPEREXPANSIVE WEIGHTED COMPOSITION OPERATORS

Y. ESTAREMI

Abstract. In this note unbounded hyperexpansive weighted composition operators are investigated. As a consequence unbounded hyperexpansive multiplication and composition operators are characterized.

1. Introduction and Preliminaries

Weighted composition operators are a general class of operators and they appear naturally in the study of surjective isometries on most of the function spaces, semigroup theory, dynamical systems, Bremsnas conjecture, etc. This type of operators are a generalization of multiplication operators and composition operators. The main subject in the study of composition operators is to describe operator theoretic properties of $C_\phi$ in terms of function theoretic properties of $\phi$. The book [3] is a good reference for the theory of composition operators. Weighted composition operators had been studied extensively in past decades. The basic properties of weighted composition operators on measurable function spaces are studied by Lambert [8, 9], Singh and Manhas [11], Takagi [12], Hudzik and Krbec [7], Cui, Hudzik, Kumar and Maligranda [4], Arora [1], Piotr Budzynski, Zenon Jan Jablonksi, Il Bong Jung and Jan Stochel [2] and some other mathematicians.

In this paper we consider unbounded weighted composition operators on the Hilbert space $L^2(\Sigma)$ and study hyperexpansive weighted composition operators. As a consequence hyperexpansive multiplication and composition operators are characterized.

Let $\mathcal{H}$ be stand for a Hilbert space and $B(\mathcal{H})$ for the Banach algebra of all bounded operators on $\mathcal{H}$. By an operator on $\mathcal{H}$ we understand a linear mapping $T : \mathcal{D}(T) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ defined on a linear subspace $\mathcal{D}(T)$ of $\mathcal{H}$ which is called the domain of $T$. Set $\mathcal{D}^\infty(T) = \cap_{n=1}^\infty \mathcal{D}(T^n)$. Given an operator $T$ on $\mathcal{H}$, we define the graph norm $\|\cdot\|_T$ on $\mathcal{D}(T)$ by

$$\|f\|^2_T = \|f\|^2 + \|Tf\|^2,$$

for $f \in \mathcal{D}(T)$.

The next proposition can be easily deduced from the closed graph theorem.

Proposition 1.1. If $T$ is a closed operator on $\mathcal{H}$ such that $T(\mathcal{D}(T)) \subseteq (\mathcal{D}(T))$, then $T$ is a bounded operator on the Hilbert space $(\mathcal{D}(T), \|\cdot\|_T)$.

For an operator $T$ on $\mathcal{H}$ we set

2000 Mathematics Subject Classification. 47B47.

Key words and phrases. weighted composition operator, hyperexpansive, unbounded operator.
\[ \Theta_{T,n}(f) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \| T^i(f) \|^2, \quad f \in \mathcal{D}(T^n), \quad n \geq 1. \]

We recall that an operator \( T \) on \( \mathcal{H} \) is:

(i) \( k \)-isometry (\( k \geq 1 \)) if \( \Theta_{T,k}(f) = 0 \) for \( f \in \mathcal{D}(T^k) \),

(ii) \( k \)-expansive (\( k \geq 1 \)) if \( \Theta_{T,k}(f) \leq 0 \) for \( f \in \mathcal{D}(T^k) \),

(iii) \( k \)-hyperexpansive (\( k \geq 1 \)) if \( \Theta_{T,n}(f) \leq 0 \) for \( f \in \mathcal{D}(T^n) \) and \( n = 1, 2, \ldots, k \).

(iv) completely hyperexpansive if \( \Theta_{T,n}(f) \leq 0 \) for \( f \in \mathcal{D}(T^n) \) and \( n \geq 1 \).

2. Hyperexpansive weighted composition operators

Let \((X, \Sigma, \mu)\) be a \( \sigma \)-finite measure space. We denote the collection of (equivalence classes modulo sets of zero measure of) \( \Sigma \)-measurable complex-valued functions on \( X \) by \( L^0(\Sigma) \) and the support of a function \( f \in L^0(\Sigma) \) is defined as \( S(f) = \{ x \in X : f(x) \neq 0 \} \). We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a \( \mu \)-null set. Denote by \( L^2(\mu) \) the Hilbert space of all square summable (with respect to \( \mu \)) \( \Sigma \)-measurable complex functions on \( X \).

For each \( \sigma \)-finite subalgebra \( \mathcal{A} \) of \( \Sigma \), the conditional expectation, \( E^\mathcal{A}(f) \), of \( f \) with respect to \( \mathcal{A} \) is defined whenever \( f \geq 0 \) almost everywhere or \( f \in L^2 \). For a sub-\( \sigma \)-finite algebra \( \mathcal{A} \subseteq \Sigma \), the conditional expectation operator associated with \( \mathcal{A} \) is the mapping \( f \to E^\mathcal{A}f \), defined for all non-negative \( f \) as well as for all \( f \in L^2(\Sigma) \), where \( E^\mathcal{A}f \), by the Radon-Nikodym theorem, is the unique \( \mathcal{A} \)-measurable function satisfying

\[ \int_A f d\mu = \int_A E^\mathcal{A}f d\mu, \quad \forall A \in \mathcal{A}. \]

As an operator on \( L^2(\Sigma) \), \( E^\mathcal{A} \) is an idempotent and \( E^\mathcal{A}(L^2(\Sigma)) = L^2(\mathcal{A}) \). If there is no possibility of confusion we write \( E(f) \) in place of \( E^\mathcal{A}(f) \) \cite{10, 13}.

For a complex \( \Sigma \)-measurable function \( u \) on \( X \). Define the measure \( \mu_u : \Sigma \to [0, \infty] \) by

\[ \mu_u(E) = \int_E |u|^2 d\mu, \quad E \in \Sigma. \]

It is clear that the measure \( \mu_u \) is also \( \sigma \)-finite. By the Radon-Nikodym theorem, if \( \mu_u \circ \phi^{-1} \ll \mu \), then there exists a unique (up to a.e. \( \mu \) equivalence) \( \Sigma \)-measurable function \( J : X \to [0, \infty] \) such that

\[ \mu_u(\phi^{-1}(E)) = \mu_u \circ \phi^{-1}(E) = \int_E J d\mu, \quad E \in \Sigma. \]

If \( \mu \circ \phi^{-1} \ll \mu \), then \( \mu_u \circ \phi^{-1} \ll \mu \). So, by definition of \( \mu_u \circ \phi^{-1} \) and applying conditional expectation with respect to \( \phi^{-1}(\Sigma) \), we get that \( J = hE(|u|^2) \circ \phi^{-1} \), where \( h \) is the Radon-Nykodim derivative \( \frac{d\mu_u \circ \phi^{-1}}{d\mu} \).
Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \(u\) be a \(\Sigma\)-measurable complex function and suppose that \(\phi\) is a mapping from \(X\) into \(X\) which is measurable (i.e. \(\phi^{-1}(\Sigma) \subseteq \Sigma\)). Define the operator \(uC_\phi : D(uC_\phi) \subseteq L^2(\mu) \to L^2(\mu)\) by

\[
D(uC_\phi) = \{ f \in L^2(\mu) : u.f \circ \phi \in L^2(\mu) \},
\]

\[
uC_\phi(f) = u.f \circ \phi.
\]

Of course such operators may not be well-defined. One can see by direct computation that if \(\mu_u \circ \phi^{-1} \ll \mu\), then \(uC_\phi\) is well-defined. And so, if \(\phi\) is a non-singular transformation, then the operator \(uC_\phi\) is well-defined. Well-defined operators of the form \(uC_\phi(f) = u.f \circ \phi\) acting in \(L^2(\mu) = L^2(X, \Sigma, \mu)\) with \(D(uC_\phi) = \{ f \in L^2(\mu) : u.f \circ \phi \in L^2(\mu) \}\) are called weighted composition operators. If \(\mu \circ \phi^{-1} \ll \mu\), then for every \(f \in D(uC_\phi)\) we have

\[
\|uC_\phi(f)\|^2 = \int_X |u|^2|f \circ \phi|^2d\mu
\]

\[
= \int_X E(|u|^2)|f|^2 \circ \phi d\mu
\]

\[
= \int_X hE(|u|^2) \circ \phi^{-1}|f|^2d\mu.
\]

By induction we get that for every \(n \geq 1\)

\[
\|(uC_\phi)^n(f)\|^2 = \int_X |u_{\phi,n}|^2|f \circ \phi^n|^2d\mu
\]

\[
= \int_X J_n|f|^2d\mu,
\]

for all \(f \in D((uC_\phi)^n)\). Where \(u_{\phi,n} = u.u \circ \phi.u \circ \phi^2...u \circ \phi^n^{-1}\), \(J_n = hE(J_{n-1}|u|^2) \circ \phi^{-1}\), \(h\) is the Radon-Nykodim derivative \(\frac{d\mu \circ \phi^{-1}}{\mu}\), \(E\) is conditional expectation with respect to \(\phi^{-1}(\Sigma)\) and \(J_0 = 1\).

**Lemma 2.1.** Let \(w = 1 + J\) and \(d\nu = wd\mu\). Then we have

(a) \(S(w) = X\) and \(L^2(\nu) = D(uC_\phi)\),

(b) And also, the followings are equivalent;

(i) \(uC_\phi\) is densely defined.

(ii) \(J < \infty \ a.e. \ \mu\).

**Proof.** (a) Let \(f\) be a measurable function on \(X\). We have

\[
\|f\|_{\mu}^2 + \|uf \circ \phi\|^2 = \int_X |f|^2d\mu + \int_X |uf \circ \phi|^2d\mu
\]

\[
= \int_X (1 + J)|f|^2d\mu = \|f\|^2_{\nu}.
\]
This means that, \( f \in \mathcal{D}(uC_{\phi}) \) if and only if \( f \in L^2(\nu) \). So \( L^2(\nu) = \mathcal{D}(uC_{\phi}) \).

(b) \( (i) \to (ii) \) Set \( F = \{ J = \infty \} \). By (a), \( f \mid_{F} = 0 \) a.e. \( \mu \) for every \( f \in \mathcal{D}(uC_{\phi}) \). This and (i) implies that \( f \mid_{\sigma} = 0 \) a.e. \( \mu \) for every \( f \in L^2(\mu) \). So we have \( \chi_{\sigma \setminus F} = 0 \) a.e. \( \mu \) for all \( A \in \Sigma \) with \( \mu(A) < \infty \). By the \( \sigma \)-finiteness of \( \mu \) we have \( \chi_{F} = 0 \) a.e. i.e \( \mu(E) = 0 \).

\( (ii) \to (i) \) Here we prove that \( L^2(\nu) \) is dense in \( L^2(\mu) \). Suppose that \( f \in L^2(\mu) \) such that \( \langle f, g \rangle = \int_{X} f \cdot \bar{g} d\mu = 0 \) for all \( g \in L^2(\nu) \). For \( A \in \Sigma \) we set \( A_{n} = \{ x \in A : w(x) \leq n \} \). It is clear that \( A_{n} \subseteq A_{n+1} \) and \( X = \cup_{n=1}^{\infty} A_{n} \). Since \( (X, \Sigma, \mu) \) is \( \sigma \)-finite, hence \( X = \cup_{n=1}^{\infty} X_{n} \) with \( \mu(X_{n}) < \infty \). If we set \( B_{n} = A_{n} \cap X_{n} \), then \( B_{n} \not\supset A \) and so \( f \cdot \chi_{B_{n}} \not\Rightarrow f \chi_{A} \) a.e. \( \mu \). Since \( \nu(B_{n}) \leq (n + 1) \mu(B_{n}) < \infty \), we have \( \chi_{B_{n}} \in L^2(\nu) \) and by our assumption \( \int_{B_{n}} f d\mu = 0 \). Therefore by Fatou’s lemma we get that \( \int_{A} f d\mu = 0 \). Thus for all \( A \in \Sigma \) we have \( \int_{A} f d\mu = 0 \). This means that \( f = 0 \) a.e. \( \mu \) and so \( L^2(\nu) \) is dense in \( L^2(\mu) \).

If all functions \( J_{i} = hE(J_{i-1}|u|^{2}) \circ \phi^{-1}, i = 1, \ldots, n \), are finite valued, where \( h_{i} \) is the Radon-Nykodim derivative \( \frac{d\mu_{\phi^{-i}}}{d\mu} \), then we set

\[
\triangle_{J_{n}}(x) = \sum_{0 \leq i \leq n} (-1)^{i} \begin{pmatrix} n \\ i \end{pmatrix} J_{i}(x).
\]

**Proposition 2.2.** If \( \mathcal{D}(uC_{\phi}) \) is dense in \( L^2(\Sigma) \), then the following conditions are equivalent:

(i) \( uC_{\phi}(\mathcal{D}(uC_{\phi})) \subseteq \mathcal{D}(uC_{\phi}) \).

(ii) There exists \( c > 0 \) such that \( J_{2} \leq c(1 + J_{1}) \) a.e. \( \mu \).

**Proof.** \( (i) \to (ii) \). Since \( uC_{\phi} \) is closed, densely defined and \( uC_{\phi}(\mathcal{D}(uC_{\phi})) \subseteq \mathcal{D}(uC_{\phi}) \), then by closed graph theorem \( uC_{\phi} \) is a bounded operator on \( (\mathcal{D}(uC_{\phi}), \| \cdot \|_{uC_{\phi}}) \). Hence there exists \( c > 0 \) such that \( \| uC_{\phi}(f) \|_{uC_{\phi}}^{2} \leq c \| f \|_{uC_{\phi}}^{2} \) for \( f \in \mathcal{D}(uC_{\phi}) \). By replacing \( f \) with \( uC_{\phi}(f) \) we have

\[
\| uC_{\phi}(f) \|^{2} \leq \| uC_{\phi}(f) \|^{2} + \| uC_{\phi}(f) \|^{2} \leq c(\| f \|^{2} + \| uC_{\phi}(f) \|^{2})
\]

i.e,

\[
\int_{X} J_{2} |f|^{2} d\mu \leq c\left( \int_{X} |f|^{2} d\mu + \int_{X} J_{1} |f|^{2} d\mu \right) = \int_{X} c(1 + J_{1}) |f|^{2} d\mu.
\]

This implies that for all \( f \in \mathcal{D}(uC_{\phi}) \) and also for all \( f \in \overline{\mathcal{D}(uC_{\phi})} = L^2(\mu) \) we have

\[
\int_{X} (c(1 + J_{1}) - J_{2}) |f|^{2} d\mu \geq 0
\]
and so \( J_2 \leq c(1 + J_1) \) a.e. \( \mu \).

(ii) \(\rightarrow\) (i). Let \( f \in \mathcal{D}(uC\phi) \). Then by assumption \( J_2 \leq c(1 + J_1) \) a.e. \( \mu \), we have

\[
\int_X |(uC\phi)^2(f)|^2 d\mu = \int_X J_2|f|^2 d\mu \\
\leq c \left( \int_X |f|^2 d\mu + \int_X J_1|f|^2 d\mu \right) \\
= c(\|f\|^2 + \|uC\phi(f)\|^2) < \infty.
\]

Therefore \( uC\phi(f) \in \mathcal{D}(uC\phi) \).

**Remark 2.3.** If \( uC\phi(D(C\phi)) \subseteq D(uC\phi) \) and \( d\nu = (1 + J_1) d\mu \), then \((X, \Sigma, \nu)\) is a \( \sigma \)-finite measure space, \( \nu \circ \phi^{-1} \) is absolutely continuous with respect to \( \nu \), \( L^2(\nu) = \mathcal{D}(uC\phi) \), \( \|\cdot\|_{L^2(\nu)} \) is the graph norm of \( uC\phi \) (considered as an operator in \( L^2(\mu) \)), and \( uC\phi \) is a bounded weighted composition operator acting on \( L^2(\nu) \). Furthermore, if \( uC\phi \) is \( k \)-isometric (resp. \( k \)-expansive, \( k \)-hyperexpansive), then so is \( uC\phi \) as an operator on \( L^2(\nu) \).

If all functions \( u^{2i} \) and \( h_i \) for \( i = 1, \ldots, n \) are finite valued, then we set

\[
\triangle u,n(x) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} u^{2i}(x), \\
\triangle h,n(x) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} h_i(x).
\]

**Corollary 2.4.** If \( \mathcal{D}(C\phi) \) is dense in \( L^2(\Sigma) \), then the following conditions are equivalent:

(i) \( C\phi(D(C\phi)) \subseteq D(C\phi) \).

(ii) There exists \( c > 0 \) such that \( h_2 \leq c(1 + h_1) \) a.e. \( \mu \).

**Corollary 2.5.** If \( \mathcal{D}(M_u) \) is dense in \( L^2(\Sigma) \), then the following conditions are equivalent:

(i) \( M_u(D(M_u)) \subseteq D(M_u) \).

(ii) There exists \( c > 0 \) such that \( u^{4} \leq c(1 + u^2) \) a.e. \( \mu \).

**Proposition 2.6.** If \( \mathcal{D}((uC\phi)^n) \) is dense in \( L^2(\mu) \) for a fixed \( n \geq 1 \), then:

(i) \( uC\phi \) is \( k \)-expansive if and only if \( \triangle J,n(x) \leq 0 \) a.e. \( \mu \).
(ii) $uC_\phi$ is $k$-isometry if and only $\triangle_{J,n}(x) = 0$ a.e. $\mu$.

**Proof.** (i). Since $\| (uC_\phi)^i (f) \|^2 = \int_X J_i |f|^2 d\mu$ for all $f \in \mathcal{D}((uC_\phi)^i)$, we have

$$\sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \| (uC_\phi)^i (f) \|^2 = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \int_X J_i |f|^2 d\mu$$

$$= \int_X \left( \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} J_i \right) |f|^2 d\mu$$

$$= \int_X \triangle_{J,n}(x) |f|^2 d\mu,$$

for all $f \in \mathcal{D}((uC_\phi)^n)$. Since $(uC_\phi)^n$ is densely defined, then we get that $uC_\phi$ is $k$-expansive if and only if $\triangle_{J,n}(x) \leq 0$ a.e. $\mu$.

(ii) Likewise we have $uC_\phi$ is $k$-isometry if and only $\triangle_{J,n}(x) = 0$ a.e. $\mu$.

**Corollary 2.7** If $\mathcal{D}((C_\phi)^n)$ is dense in $L^2(\mu)$ for a fixed $n \geq 1$, then:

(i) $C_\phi$ is $k$-expansive if and only if $\triangle_{h,n}(x) \leq 0$ a.e. $\mu$.

(ii) $C_\phi$ is $k$-isometry if and only $\triangle_{h,n}(x) = 0$ a.e. $\mu$.

**Corollary 2.8.** If $\mathcal{D}((M_u)^n)$ is dense in $L^2(\mu)$ for a fixed $n \geq 1$, then:

(i) $M_u$ is $k$-expansive if and only if $\triangle_{u,n}(x) \leq 0$ a.e. $\mu$.

(ii) $M_u$ is $k$-isometry if and only $\triangle_{u,n}(x) = 0$ a.e. $\mu$.

**Proposition 2.9.** If $\mathcal{D}((uC_\phi)^2)$ is dense in $L^2(\mu)$ and $uC_\phi$ is $2$-expansive, then:

(i) $uC_\phi$ leaves its domain invariant:

(ii) $J_k \geq J_{k-1}$ a.e. $\mu$ for all $k \geq 1$.

**Proof.** (i). By the Proposition 2.3 we get that $J_2 \leq 2J_1 - 1$. Hence for every $f \in \mathcal{D}(uC_\phi)$ we have

$$\| (uC_\phi)^2 (f) \|^2 = \int_X J_2 |f|^2 d\mu$$

$$\leq 2 \int_X J_1 |f|^2 d\mu - \int_X |f|^2 d\mu < \infty,$$

so $uC_\phi(f) \in \mathcal{D}(uC_\phi)$. 


(ii) Since $uC_\phi$ leaves its domain invariant, then $\mathcal{D}(uC_\phi) \subseteq \mathcal{D}^\infty(uC_\phi)$. So by lemma 3.2 (iii) of [6] we get that $\|(uC_\phi)^k(f)\|^2 \geq \|(uC_\phi)^{k-1}(f)\|^2$ for all $f \in \mathcal{D}(uC_\phi)$ and $k \geq 1$ we have

$$\int_X (J_k - J_{k-1})|f|^2d\mu \geq 0, \quad f \in \mathcal{D}(uC_\phi),$$

so this leads to $J_k \geq J_{k-1}$ a.e. $\mu$.

**Corollary 2.10.** If $\mathcal{D}((C_\phi)^2)$ is dense in $L^2(\mu)$ and $C_\phi$ is 2-expansive, then:

(i) $C_\phi$ leaves its domain invariant:

(ii) $h_k \geq h_{k-1}$ a.e. $\mu$ for all $k \geq 1$.

**Corollary 2.11.** If $\mathcal{D}((M_u)^2)$ is dense in $L^2(\mu)$ and $M_u$ is 2-expansive, then:

(i) $M_u$ leaves its domain invariant:

(ii) $u^{2k} \geq u^{2(k-1)}$ a.e. $\mu$ for all $k \geq 1$.

Recall that a real-valued map $\varphi$ on $\mathbb{N}$ is said to be completely alternating if

$$\sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \varphi(m + i) \leq 0$$

for all $m \geq 0$ and $n \geq 1$. The next theorem is a direct consequence of proposition 2.3 and 2.4.

**Theorem 2.12.** If $\mathcal{D}((uC_\phi)^2)$ is dense in $L^2(\mu)$ and $k \geq 1$ is fixed, then:

(i) $uC_\phi$ is $k$-hyperexpansive if and only if $\triangle_{J,i}(x) \leq 0$ a.e. $\mu$ for $i = 1, \ldots, k$.

(ii) $uC_\phi$ is completely hyperexpansive if and only if $\{J_i\}_{i=0}^\infty$ is a completely alternating sequence for almost every $x \in X$.

**Corollary 2.13.** If $\mathcal{D}((C_\phi)^2)$ is dense in $L^2(\mu)$ and $k \geq 1$ is fixed, then:

(i) $C_\phi$ is $k$-hyperexpansive if and only if $\triangle_{h,i}(x) \leq 0$ a.e. $\mu$ for $i = 1, \ldots, k$.

(ii) $C_\phi$ is completely hyperexpansive if and only if $\{h_i\}_{i=0}^\infty$ is a completely alternating sequence for almost every $x \in X$.

**Corollary 2.14.** If $\mathcal{D}((M_u)^2)$ is dense in $L^2(\mu)$ and $k \geq 1$ is fixed, then:

(i) $M_u$ is $k$-hyperexpansive if and only if $\triangle_{u,i}(x) \leq 0$ a.e. $\mu$ for $i = 1, \ldots, k$. 
(ii) $M_u$ is completely hyperexpansive if and only if $\{u^{2i}\}_{i=0}^{\infty}$ is a completely alternating sequence for almost every $x \in X$.

Notice that in the same way we can characterize alternatingly hyperexpansive weighted composition operators.

We say that the $\sigma$-algebra $\phi^{-1}(\Sigma)$ is essentially all of $\Sigma$ with respect to $\mu_u$ if and only of given $A \in \Sigma$ there is $B \in \Sigma$ with the symmetric difference $\phi^{-1}(B) \triangle A = (\phi^{-1}(B) \setminus A) \cup (A \setminus \phi^{-1}(B))$ having $\mu_u(\phi^{-1}(B) \triangle A) = 0$.

The following proposition characterizes 2-expansive weighted composition operators on the measure space $(X, \Sigma, \mu)$ such that $\mu_u(X) < \infty$.

**Theorem 2.15.** Let $uC_\phi$ be 2-expansive operator.

(i) Let $(X, \Sigma, \mu)$ is an infinite measure space such that $\mu_u(X) < \infty$ and $D((uC_\phi)^2)$ is dense in $L^2(\mu)$.

(ii) Let $(X, \Sigma, \mu)$ is a measure space such that $\mu_u(X) < \infty$, $u \leq 1$ a.e. $\mu$ and $D((uC_\phi)^2)$ is dense in $L^2(\mu)$.

If the conditions (i) or (ii) holds, then $uC_\phi$ is an isometry.

(iii) If $uC_\phi$ is densely defined, $u \neq 0$ a.e. $\mu$ and the sigma algebra $\phi^{-1}(\Sigma)$ is essentially all of $\Sigma$, with respect to $\mu$, then $uC_\phi$ is a unitary operator.

**Proof.** (i) It follows from Proposition 2.4 that $uC_\phi$ leaves its domain invariant and $J_1 \geq 1$ a.e. $\mu$. Suppose that (a) holds and contrary to our claim, there exists $B \subseteq X$ such that $\mu(B) > 0$ and $J_1 \geq \epsilon + 1$ on $B$ for some $\epsilon > 0$. Then we have

$$\infty > \mu_u(X) = \mu_u(\phi^{-1}(X \setminus B)) + \mu_u(\phi^{-1}(B)) \geq (1 + \epsilon)\mu(B) + \mu(X \setminus B) > \mu(X),$$

which is a contradiction. Thus $J_1 = 1$ a.e. $\mu$.

If (ii) holds, then by the same method we conclude that $\mu(X) > \mu(X)$ which is a contradiction. These imply that $uC_\phi$ is an isometry.

(iii) Let $u \neq 0$ a.e. $\mu$ and the sigma algebra $\phi^{-1}(\Sigma)$ be essentially all of $\Sigma$, with respect to $\mu$. This implies that $uC_\phi$ is dense range. Then by [K], proposition 3.5] we get that $uC_\phi$ is unitary.

**Corollary 2.16.** Let $C_\phi$ be 2-expansive operator.

(i) If $(X, \Sigma, \mu)$ is a finite measure space and $D((C_\phi)^2)$ is dense in $L^2(\mu)$, then $C_\phi$ is an isometry.
(ii) If \( C_\phi \) is densely defined and the sigma algebra \( \phi^{-1}(\Sigma) \) is essentially all of \( \Sigma \), then \( C_\phi \) is a unitary operator.

**Corollary 2.17.** Let \( M_u \) be 2-expansive operator.

(i) Let \((X, \Sigma, \mu)\) is an infinite measure space such that \( \mu_u(X) < \infty \) and \( \mathcal{D}((M_u)^2) \) is dense in \( L^2(\mu) \).

(ii) Let \((X, \Sigma, \mu)\) is a measure space such that \( \mu_u(X) < \infty \), \( u \leq 1 \) a.e. \( \mu \) and \( \mathcal{D}((M_u)^2) \) is dense in \( L^2(\mu) \).

If the conditions (i) or (ii) holds, then \( M_u \) is an isometry.

(iii) If \( M_u \) is densely defined, \( u \neq 0 \) a.e. \( \mu \), then \( M_u \) is a unitary operator.

Now, let \( m = \{m_n\}_{n=1}^\infty \) be a sequence of positive real numbers. Consider the space \( \ell^2(m) = L^2(\mathbb{N}, 2^\mathbb{N}, \mu) \), where \( 2^\mathbb{N} \) is the power set of natural numbers and \( \mu \) is a measure on \( 2^\mathbb{N} \) defined by \( \mu(\{n\}) = m_n \). Let \( u = \{u_n\}_{n=1}^\infty \) be a sequence of complex numbers. Let \( \varphi : \mathbb{N} \to \mathbb{N} \) be a non-singular measurable transformation; i.e. \( \mu \circ \varphi^{-1} \ll \mu \). Direct computation shows that

\[
h(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j, \quad E_\varphi(f)(k) = \sum_{j \in \varphi^{-1}(\varphi(k))} f_j m_j, \quad \sum_{j \in \varphi^{-1}(\varphi(k))} m_j,
\]

for all non-negative sequence \( f = \{f_n\}_{n=1}^\infty \) and \( k \in \mathbb{N} \). So

\[
J(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} |u_j|^2 m_j.
\]

This observations lets us to consider the weighted composition operators on discrete measure space \((\mathbb{N}, \mu, \Sigma)\). If \( uC_\phi \) is a weighted composition operator on \( \ell^2(m) \), then

\[
\mathcal{D}(uC_\phi) = \{f = \{f_n\} \in \ell^2(m) : \sum_{n=0}^\infty \left( \sum_{j \in \varphi^{-1}(n)} |u_j|^2 m_j \right) |f_n|^2 < \infty \}
\]

\[
||uf \circ \phi||^2 = \int_X |uf \circ \phi|^2 d\mu
\]

\[
= \sum_{n=0}^\infty J(n)m_n |f_n|^2
\]

\[
= \sum_{n=0}^\infty \left( \sum_{j \in \varphi^{-1}(n)} |u_j|^2 m_j \right) |f_n|^2.
\]

**References**

[1] S. C. Arora, Gopal Datt and Satish Verma, Composition operators on Lorentz spaces, Bull. Austral. Math. Soc. 76 (2007), 205-214.

[2] Piotr Budzynski, Zenon Jan Jablonski, Il Bong Jung, Jan Stochel, Unbounded Weighted Composition Operators in L2-Spaces, [arXiv:1310.3542](https://arxiv.org/abs/1310.3542)
[3] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, (1995).
[4] Y. Cui, H. Hudzik, R. Kumar and L. Maligranda, Composition operators in Orlicz spaces, J. Aust. Math. Soc. 76 (2004), 189-206.
[5] J. Jablonski, Hyperexpansive composition operators, Math. Proc. Camb. Phil. Soc. 135 (2003), 513-526.
[6] J. Jablonski and J. Stochel, Unbounded 2-hyperexpansive operators, Proc. Edinburgh Math. Soc. 44 (2001), 613-629.
[7] H. Hudzik and M. Krbec, On non-effective weights in Orlicz spaces. Indag. Math. (N.S.) 18 (2007), 215-231.
[8] A. Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy. Soc. Edinburgh Sect. A 118 (1991), 111-118.
[9] A. Lambert, Operator algebras related to measure preserving transformations of finite order, Rocky Mountain J. Math. 14 (1984), 341-349.
[10] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.
[11] R. K. Singh and J. S. Manhas, Composition operators on function spaces, North Holland Math. Studies 179, Amsterdam 1993.
[12] H. Takagi, Compact weighted composition operators on $L^p$, Proc. Amer. Math. Soc. 116 (1992), 505-511.
[13] A. C. Zaanen, Integration, 2nd ed., North-Holland, Amsterdam, 1967.

Y. ESTAREMI

E-mail address: estaremi@gmail.com

Department of Mathematics, Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran.