On the seven-diagonals splitting for the cubic spline wavelets with six vanishing moments on an interval

B M Shumilov
Department of Applied Mathematics, Tomsk State University of Architecture and Building, Solyanaya sq. 2, Tomsk 634003, Russia
E-mail: sbm@tsuab.ru

Abstract. This study uses a zeroing property of the first six moments for constructing a splitting algorithm for the cubic spline wavelets. First, we construct a system of cubic basic spline-wavelets, realizing orthogonal conditions to all polynomials up to any degree. Then, using the homogeneous Dirichlet boundary conditions, we adapt spaces to the orthogonality to all polynomials up to the fifth degree on the closed interval. The originality of the study consists of obtaining implicit finite relations connecting the coefficients of the spline decomposition at the initial scale with the spline coefficients and wavelet coefficients at the nested scale by a tape system of linear algebraic equations with a non-degenerate matrix. After excluding the even rows of the system, the resulting transformation matrix has seven diagonals, instead of five as in the previous case with four zero moments. A modification of the system is performed, which ensures a strict diagonal dominance, and, consequently, the stability of the calculations. The comparative results of numerical experiments on approximating and calculating the derivatives of a discrete function are presented.

1. Introduction
A wavelet is a short or rapidly decaying wave function (burst), a set of contractions and displacements of whose generate the space of measurable functions on the entire number an axis [1, 2]. Due to the compressions, the wavelets reveal, with varying degrees of details, a change in the characteristics of a measured signal, and using the shifts, they can analyze the properties of the signal at different points of the studied interval. When analyzing non-stationary signals, the locality property of wavelets provides them with an advantage over the Fourier transform, which gives only the global information about the properties of the signal under study, since the basis functions used in this case (sines and cosines) have infinite support. When solving problems of numerical analysis, since wavelets transform a system of basis functions with distributed parameters to a direct sum of systems with lumped parameters, such a basis turns out to be more efficient from the point of conditioning and convergence [3].

The basis for constructing wavelets is the presence of a set of approximating spaces \( V_{L-1} \subset V_L \subset V_{L+1} \ldots \) such that each basis function in \( V_L \) can be expressed as a linear combination of basis functions in \( V_{L+1} \). In particular, splines, i.e. smooth functions glued from pieces of polynomials of degree \( m \) on an embedded sequence of grids, have this property. The essence of the wavelet transform can be formulated as follows: it allows one to decompose a given function \( V_{L+1} \) to a rough approximate representation \( V_L \) and the local specifying details \( W_L = V_{L+1} - V_L \). The main thing here is to find a suitable basis for the space \( W_L \) and to construct for it fast
one-to-one formulas for direct and inverse wavelet transforms. In the author’s paper [4], using cubic nonorthogonal wavelets with four first zero moments at the edges of the segment [5], the five-diagonals splitting method of cubic wavelets with the first six zero moments was studied and the absence of a strict diagonal dominance in the splitting system was proved. In this paper, we propose a complete solution for a seven-diagonals splitting method, including the construction of cubic nonorthogonal wavelets with the first six zero moments at the edges of the segment; the results of the numerical experiments on the approximation and calculation of the derivatives of a discretely given function are presented.

2. Constructing the cubic spline wavelets with six zero moments on a finite segment

Let the space \( V_L \) be the space of cubic splines of the smoothness \( C^2 \) on a uniform grid of the nodes \( \Delta^L : x_i = a + h \cdot i, i = 0, 1, \ldots, 2^L, h = (b - a)/2^L \), and the functions \( \varphi_3(v - i)\varphi_i \), where \( v = (x - a)/h \), are generated by contractions and shifts of a function of the form [6, p. 89]:

\[
\varphi_3(t) = \frac{1}{6} \sum_{j=0}^{4} \binom{4}{j} (-1)^j (t - j)^3_+^+, \quad 0 \leq t \leq 4,
\]

where \( t^n_+ = (\max\{t, 0\})^n \). It is well known that these functions generate a basis of the spline space on a grid infinitely extended in both directions, and satisfy the calibration relation [1, p. 154]:

\[
\varphi_3(t) = \frac{1}{8} \sum_{k=0}^{4} \binom{4}{k} \varphi_3(2t - k).
\]

If the mesh \( \Delta^{L-1}, L \geq 3 \), is obtained from \( \Delta^L \) by removing every second node, then the corresponding space \( V_{L-1} \) with the basis functions \( \varphi_3(v/2 - i)\varphi_i \), whose supports are twice as wide and the centers at even mesh nodes \( \Delta^L \), are nested in \( V_L \). The difference between the spaces \( V_L \) and \( V_{L-1} \) is the wavelet space \( W_{L-1} = V_L - V_{L-1} \). As basis functions in the space \( W_{L-1} \), we can use cubic wavelets orthogonal to all polynomials of degree \( n \) [7],

\[
w_n(t) = \frac{1}{8} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \varphi_3(2t - k),
\]

having the following support

\( \text{supp } w_n = [0, n + 1] \),

and, accordingly, \( n + 1 \) zero moments

\[
\int_0^{n+1} x^k w_n(x)dx = 0,
\]

for \( k = 0, 1, \ldots, n \).

The above functions introduced completely solve the problem of constructing the bases for spline and wavelet spaces in the case of an infinite mesh. Unfortunately, in order to construct bases on a finite segment \([a, b]\), the functions \( \varphi_3(v - i), w_n(v - i) \) cannot be simply truncated when going beyond the ends of the segment. One of the ways to eliminate this circumstance is to use composite nodes at the points \( a, b \). Additionally, in order to make the bases more symmetric, we will impose the homogeneous Dirichlet conditions \( f(a) = f(b) = 0 \) on the functions. Then the left boundary basis spline functions have the form [6]

\[
\varphi_{b1}(t) = \frac{7}{4} t^3_+^+ - \frac{9}{2} t^2_+ + 3t^1_+ - 2(t - 1)^3_+, \quad 0 \leq t \leq 2,
\]
\[ \varphi_{b2}(t) = \frac{3}{2}t^2 + \frac{11}{12}t^3 + \frac{3}{2}(t-1)^3 + \frac{3}{4}(t-2)^3, \quad 0 \leq t \leq 3, \]

and the calibration relations are satisfied

\[ \begin{align*}
\varphi_{b1}(t) &= \frac{1}{2}\varphi_{b1}(2t) + \frac{3}{4}\varphi_{b2}(2t) + \frac{3}{16}\varphi_3(2t), \\
\varphi_{b2}(t) &= \frac{1}{4}\varphi_{b2}(2t) + \frac{11}{16}\varphi_3(2t) + \frac{1}{2}\varphi_3(2t-1) + \frac{1}{8}\varphi_3(2t-2).
\end{align*} \tag{3} \tag{4} \]

At the right end of the segment, the basis functions mirror the functions \( \varphi_{b1}(t), \varphi_{b2}(t) \). As a result, for any mesh \( \Delta^L, L \geq 2 \), the third degree spline with zero boundary conditions can be represented as

\[ s^L(v) = C^L_{-2}\varphi_{b1}(v) + C^L_{-1}\varphi_{b2}(v) + \sum_{i=0}^{2^L-4} C^L_i\varphi_3(v-i) + C^L_{2^L-3}\varphi_{b2}(2^L-v) + \\
+ C^L_{2^L-2}\varphi_{b1}(2^L-v), \quad 0 \leq v \leq 2^L, \quad \text{Dim}(V_L) = 2^L + 1, \tag{5} \]

where the coefficients \( C^L_\gamma \), are the solution, for example, of the interpolation problem:

\[ s^L(i) = f(x_i), \quad i = 1, 2, \ldots, 2^L - 1, \]

\[ \left( s^L \right)'(i) = h \cdot f'(x_i), \quad i = 1, 2^L - 1. \]

Let the centers of the supports of \( w_5(v-i) \) be located at the odd nodes \( j = 5, 7, \ldots, 2^L - 5 \), then to fulfill the condition of complementing the dimensions of the considered spaces,

\[ \text{Dim}(V_L) = \text{Dim}(V_{L-1}) + \text{Dim}(W_{L-1}), \]

it is required to find four more edge wavelets, so as to end up with \( \text{Dim}(W_{L-1}) = 2^{L-1} \).

Until now, only the cubic wavelets orthogonal to all third-degree polynomials have been known in the published form [6],

\[ \begin{align*}
w_{b1}(t) &= 6\varphi_{b1}(2t) - \frac{57}{5}\varphi_{b2}(2t) + \frac{919}{100}\varphi_3(2t) - \frac{116}{25}\varphi_3(2t-1) + \\
&+ \varphi_3(2t-2), \\
w_{b2}(t) &= \frac{7}{3}\varphi_{b2}(2t) - \frac{319}{60}\varphi_3(2t) + \frac{101}{15}\varphi_3(2t-1) - \\
&- \frac{25}{6}\varphi_3(2t-2) + \varphi_3(2t-3).
\end{align*} \]

These cubic wavelets have the following supports

\[ \text{supp } w_{b1} = [0, 3], \quad \text{supp } w_{b2} = [0, 3.5], \]

and, accordingly, four zero moments

\[ \int_0^3 x^k w_{b1}(x) dx = \int_0^{3.5} x^k w_{b2}(x) dx = 0, \]

for \( k = 0, 1, 2, 3 \), and

\[ \int_0^3 x^4 w_{b1}(x) dx = \frac{27}{56}, \quad \int_0^{3.5} x^4 w_{b2}(x) dx = \frac{21}{32}, \]
whereas

\[ \int_0^4 x^4 w_3(x) dx = \frac{3}{4}, \quad \int_0^5 x^5 w_4(x) dx = \frac{-15}{8}. \]

Hence we find that the wavelets

\[ w_{b1}^1(t) = w_{b1}(t) - \frac{9}{14}w_3(t), \quad w_{b2}^1(t) = w_{b2}(t) - \frac{7}{8}w_3(t) \]

have five zero moments, while

\[ \int_0^4 x^5 w_{b1}^1(x) dx = -\frac{225}{112}, \quad \int_0^4 x^5 w_{b2}^1(x) dx = -\frac{35}{24}, \]

whence it immediately follows that the wavelets

\[
\begin{align*}
  w_{b1}^2(t) &= w_{b1}^1(t) - \frac{15}{14}w_4(t) = \\
  &= 6\varphi_{b1}(2t) - \frac{57}{5}\varphi_{b2}(2t) + \frac{5233}{700}\varphi_3(2t) + \frac{1151}{350}\varphi_3(2t - 1) - \\
  &\quad - \frac{95}{7}\varphi_3(2t - 2) + \frac{93}{7}\varphi_3(2t - 3) - 6\varphi_3(2t - 4) + \frac{15}{14}\varphi_3(2t - 5), \\
  w_{b2}^2(t) &= w_{b2}^1(t) - \frac{7}{9}w_4(t) = \\
  &= \frac{7}{3}\varphi_{b2}(2t) - \frac{2509}{360}\varphi_3(2t) + \frac{1271}{90}\varphi_3(2t - 1) - \frac{619}{36}\varphi_3(2t - 2) + \\
  &\quad + \frac{221}{18}\varphi_3(2t - 3) - \frac{343}{72}\varphi_3(2t - 4) + \frac{7}{9}\varphi_3(2t - 5)
\end{align*}
\]

have the desired six zero moments.

### 2.1. The construction of a defining system of equations

We write down basic spline functions as a one-row matrix,

\[ \varphi^L(\cdot) = \left[ \varphi_{b1}(\cdot), \varphi_{b2}(\cdot), \varphi_3(\cdot), \varphi_3(\cdot - 1), \ldots, \varphi_3(\cdot - 2^L + 4), \varphi_{b2}(2^L - \cdot), \varphi_{b1}(2^L - \cdot) \right]. \]

Introducing the notation for the vector consisting of the coefficients of the spline

\[ c^L = \left[ C^L_2, C^L_1, C^L_0, \ldots, C^L_{2^L-3}, C^L_{2^L-2} \right]^T, \]

we write down formula (5) in the vector form

\[ s^L(\cdot) = \varphi^L(\cdot)c^L. \]

Similarly, we can write down basic wavelet functions as a row matrix

\[ \psi^{L-1}(\cdot) = \left[ w_{b1}^2(\cdot), w_{b2}^2(\cdot), w_5(\cdot - 1), w_5(\cdot - 3), \ldots, w_5(\cdot - 2^L + 5), w_{b2}^2(2^L - \cdot), w_{b1}^2(2^L - \cdot) \right]. \]

The corresponding wavelet approximation coefficients at the level \( L \) will be denoted by

\[ D^{L-1}_i, \quad i = -1, 0, \ldots, 2^{L-1} - 2, \]

and we introduce the vector-column

\[ d^{L-1} = \left[ D^{L-1}_{-1}, D^{L-1}_0, \ldots, D^{L-1}_{2^{L-1}-2} \right]^T. \]
Since the spaces $V_{L-1}$ and $W_{L-1}$ are subspaces of $V_L$ by definition, the functions $\varphi^{L-1}()$ and $\psi^{L-1}()$ are linear combinations of the functions $\varphi^L()$: 

$$\varphi^{L-1}() = \varphi^L()P^L \text{ and } \psi^{L-1}() = \varphi^L()Q^L,$$

where the columns of the matrix $P^L$ are composed of the coefficients of relations (1) and (3), (4), since each wide basis function over the interval of the approximation can be constructed from five narrow basis functions, and each wide basis function at the ends of the interval can be constructed from three or four narrow basis functions; the entries of the columns of the matrix $Q^L$ are composed of the coefficients of relations (2) and (6), (7).

Therefore, there is a chain of equalities:

$$\varphi^L()c^L = \varphi^{L-1}()c^{L-1} + \psi^{L-1}()d^{L-1} = \varphi^L()P^Lc^{L-1} + \varphi^L()Q^Ld^{L-1}. \quad (8)$$

Let the coefficients $c^{L-1}$ and $d^{L-1}$ be known. Then the coefficients $c^L$ can be obtained from $c^{L-1}$ and $d^{L-1}$ as follows

$$c^L = P^Lc^{L-1} + Q^Ld^{L-1}$$

or, using the notation for block matrices,

$$c^L = \begin{bmatrix} P^L & Q^L \end{bmatrix} \begin{bmatrix} c^{L-1} \\ d^{L-1} \end{bmatrix}. \quad (9)$$

The inverse process of decomposing the coefficients $c^L$ to a rougher version $c^{L-1}$ and the refining coefficients $d^{L-1}$ consists in solving a system of linear equations (9). The solvability of the resulting system is ensured by the linear independence of the basis functions. To facilitate the numerical solution to the system of linear equations (9), following [2], the matrix $[P^L \mid Q^L]$ can be made banded by changing the order of the unknowns so that the columns of the matrices $P^L$ and $Q^L$ were interleaved. However, as is easy to see, the resulting system of equations does not have a diagonal dominance, which can complicate the wavelet analysis of large data.

Earlier in [6], with the use of some modification of edge wavelets (2), (3), the fulfillment of the Riesz basis condition was proved and thus was justified the stability of calculations with third-degree wavelets orthogonal to all third-degree polynomials. For the wavelets considered, orthogonal to all polynomials of the fifth degree, similar results are not known. Therefore, we restrict ourselves to reducing the system of equations to a matrix of half the dimension and having a smaller number of diagonals, namely, seven, provided that the system is split into even and odd rows.

3. An algorithm using splitting

To solve the system of linear equations (9), we will use an artificial trick, reminiscent of multiplying the system by a preconditioning matrix [8, p. 7]. We transform the system (9) to the following form

$$c^L = \begin{bmatrix} P^L & Q^L \end{bmatrix} R^L(R^L)^{-1} \begin{bmatrix} c^{L-1} \\ d^{L-1} \end{bmatrix}$$

and introduce the notation

$$G^L = \begin{bmatrix} P^L & Q^L \end{bmatrix} R^L.$$

Usually, dismissing at once the extreme cases $G^L = E$ (an identity matrix), when the transformation results in just the original system, and $G^L = \begin{bmatrix} P^L & Q^L \end{bmatrix}$, when the complexity
of constructing the iterative matrix $(G^L)^{-1}$ does not concede to the complexity of solving the system itself, one tries to reduce the number of iterations while maintaining the easiness of calculating the inverse matrix, for example, choosing the matrix $G^L$, having a diagonal dominance. In our case, taking into account the peculiarities of the system, we can achieve more - so that both $G^L$ and $R^L$ are sparse banded matrices. As a result, an efficient sweep method can be applied to solving the resulting system [9, 10].

Let, for the levels of the expansion $L \geq 4$, the matrices $G^L$ of size $(2^L + 1) \times (2^L + 1)$ have the form

$$G^L = \begin{bmatrix} G^L_{\text{left}} & G^L_{\text{center}} & G^L_{\text{right}} \end{bmatrix},$$

where

$$G^L_{\text{left}} = \begin{bmatrix} 12032 & 8 & 84492 & 4 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35904 & 3 & 274224 & -15 & 0 & 5 & 50 & 0 \\ 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 \\ 12096 & 0 & 137376 & -3 & 1436200 & 51 & 585 & 5 \\ 0 & 0 & 10800 & 0 & 743600 & 51 & 1080 & 51 \\ 0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 \\ 0 & 0 & 53325 & 0 & 53800 & 5 & 222 & 51 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -130 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix},$$

and

$$G^L_{\text{center}} = \begin{bmatrix} 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ -15 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ -130 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ 207 & 5 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ 900 & 51 & \cdots & \cdots \\ 0 & -16 & \cdots & \cdots \\ 207 & 51 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ -130 & 5 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ -15 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix}.$$
whereas the matrix $R^L$ composed of six blocks according to $2^{L-1} + 1$ basic spline functions from $V_{L-1}$ and $2^{L-1}$ basic wavelets from $W_{L-1}$:

$$R^L = \begin{bmatrix} A^L_{\text{left}} & A^L_{\text{center}} & A^L_{\text{right}} \\ B^L_{\text{left}} & B^L_{\text{center}} & B^L_{\text{right}} \end{bmatrix},$$

where

$$A^L_{\text{left}} = \begin{bmatrix} 11464 & 16 & 81784 & 8 & 71400 & 0 & 0 & 0 \\ 24828 & 0 & 184308 & -24 & -337260 & 0 & 0 & 0 \\ 2799 & 0 & 53325 & 0 & 100821 & 24 & 240 & 0 \\ -144 & 0 & 0 & 0 & 273224 & 24 & 728 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ : & : & : & : & : & : & : & : \\ \end{bmatrix},$$

$$B^L_{\text{left}} = \begin{bmatrix} 1050 & 0 & 7350 & 0 & -5950 & 0 & 0 & 0 \\ -1215 & 0 & -10125 & 0 & -15885 & 0 & 0 & 0 \\ 144 & 0 & 0 & 0 & 157176 & 16 & 160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 112 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & -48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ : & : & : & : & : & : & : & : \\ \end{bmatrix}.$$
Here, the last six columns of the matrices $G^L$ and $R^L$ mirror the first six columns, the diagonal points mean that the previous two columns repeat the corresponding number of times, each time moving two positions down; empty positions of matrix rows being equal to zero. According to the approach [2, p. 112], placing as many zeros as possible in the upper and lower parts of each column of the matrix $R^L$, we ensure the compactness of the computational scheme, and
additional zeroing the entries at odd positions of columns of the matrix $G^L$, we provide the possibility of subsequent splitting of the system to even and odd rows. Similar to [11], the following result has been proved.

**Theorem 1** The determinant of the matrix $\det G^L \neq 0$.

The Gauss method, as a way to solve a system of linear equations (9) by decomposing the matrix $[P^L \mid Q^L]$ to the product of the lower and upper triangular matrices, is not, of course, the only way to factorize. What we propose to do can be called the inverse factorization method. Namely, relying on the existence of the inverse matrix $(G^L)^{-1}$, it is proposed to perform calculations based on the even-odd splitting of the wavelet decomposition of the form (8). Similar to [11], the following result has been proved.

**Theorem 2** For any level of decomposition of $L \geq 4$, the matrix of the wavelet decomposition of cubic splines satisfies, that the equality holds:

$$[P^L \mid Q^L] R^L = G^L.$$

After that, the solution to the system of equations (9) can be written down in the matrix form as

$$\begin{bmatrix} c^{L-1} \\ d^{L-1} \end{bmatrix} = [P^L \mid Q^L]^{-1} c^L = R^L \left( G^L \right)^{-1} c^L.$$

Thus, instead of direct solution to a system of the form of (9), we can solve the system

$$G^L \xi^L = c^L \quad (10)$$

relative to some values of $\xi^L$ and then just calculate the values of $c^{L-1}$ and $d^{L-1}$ using the linear transformations $c^{L-1} = A^L \xi^L$, $d^{L-1} = B^L \xi^L$.

Nevertheless, for large values of $L$, we still need to divide the system (10) into even and odd rows to reduce the algorithm to a seven-diagonals sweep over odd rows. Then the remaining coefficients are calculated from averaging relations, thus realizing the task of filtering. We will achieve this goal using the odd-even reduction method studied in [10, Ch. IV, §3], and to ensure stability objective, at the first stage we will solve a slightly truncated system, postponing to the second stage the search for the unknowns $\xi_{-2}$, $\xi_{2L-2}$. Let us note that from the point of view of stability, performing the Gaussian elimination by rows is equivalent to performing the Gaussian elimination on a transposed matrix by columns [9, pp. 70–72].

4. **An example**

Let $L = 4$ and the discrete signal be represented as values of the analytic function $f(x) = (x^2 - 16)^2$, given at the points $\Delta^4 : x = -4, -3.5, \ldots, 4$. Since at the extreme points of $x = \pm 4$, the homogeneous boundary conditions necessary for constructing the wavelet decomposition are satisfied, there is everything to investigate the application of the above wavelets to the problem of the wavelet analysis. After solving the corresponding interpolation problems and sequentially executing the wavelet analysis procedure, at the last stage $L = 2$, we reject all the wavelet expansion coefficients obtained.

In this case, let the mean square error of approximation be estimated by the expression

$$\text{MSE} = \left( \frac{1}{15} \sum_{i=1}^{15} \left( f(x_i) - S^2(x_i/2 + 2) \right)^2 \right)^{1/2}.$$
Table 1 presents five calculation options: 1) from the solution of the interpolation problem; 2) according to the scheme presented (6 moments); 3) using wavelets orthogonal to the third degree polynomials [5, 11] (4 moments); 4) using wavelets orthogonal to polynomials of the first degree [12, 13] (2 moments); 5) using interpolation cubic wavelets [14, 15] (ICW).

| Calculation | Interpolation | 6 moments | 4 moments | 2 moments | ICW |
|-------------|---------------|-----------|-----------|-----------|-----|
| (S^2)''(-4) | 127.5         | 117.75    | 118.1     | 121.7     | 120 |
| (S^2)''(0)  | -64.498       | -72.29    | -72       | -70.3     | -72 |
| MSE         | 0             | 0.323     | 0.332     | 0.551     | 0.66|

Thus, in this example, the approximation scheme based on the use of wavelets with two zero moments gives the values closest to the values of the second derivative. The use of more complicated approximation schemes with four and six zero moments leads to an increase in the approximation accuracy, but also to a distance from the true values of the approximated derivative. The algorithm built based on the interpolation of cubic spline wavelets with the property of the best mean-square approximation of the second derivative of the function being approximated, is in general, inferior in all indicators to algorithms that have zero moments.

5. Conclusion
This paper considers the further development of the author’s procedure for an even-odd partition of a defining system of the Hermite wavelet expansion to the practically important case that does not require specifying the values of the derivatives of approximated functions, based on the B-splines of the third degree. Extension of the approach proposed to splines of a higher degree and a larger number of zero moments can provide new opportunities for the development of algorithms for constructing and applying spline wavelets.

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