

Sharp estimates for screened Vlasov-Poisson system around Penrose-stable equilibria in $\mathbb{R}^d$, $d \geq 3$

Lingjia Huang; Quoc-Hung Nguyen† and Yiran Xu‡

Abstract

In this paper, we study the asymptotic stability of Penrose-stable equilibria among solutions of the screened Vlasov-Poisson system in $\mathbb{R}^d$ with $d \geq 3$ that was first established by Bedrossian, Masmoudi, and Mouhot in [8] with smooth initial data. More precisely, we prove the sharp decay estimates for the density of the perturbed system, exactly like the free transport with only Hölder (i.e., $C^a$ for $0 < a < 1$) perturbed initial data. This improves the recent works in [21] by Han-Kwan, Nguyen, and Rousset for lower derivatives of the density and in [15] by T. Nguyen for higher derivatives with a logarithmic correction in time. Furthermore, we establish new estimates and cancellations of the kernel to the linearized problem to obtain this result. Moreover, we also prove this result for the Vlasov-Poisson system in which the electric field obeys a general nonlinear Poisson equation containing massless electrons/ions case.

1 Introduction

This paper is devoted to the study of the asymptotic stability of equilibria for the Vlasov-Poisson system of the form:

$$\begin{align*}
\partial_t f_i + v \cdot \nabla_x f_i + E \cdot \nabla_v f_i &= 0, \\
E &= -\nabla_x U_i - \Delta U + U = \rho_i - 1 + A(U), \quad \rho_i(t, x) = \int_{\mathbb{R}^d} f_i(t, x, v) dv
\end{align*}$$ (1.1)

on the whole space $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, $d \geq 3$, where $f_i = f_i(t, x, v) \geq 0$ is the probability distribution of charged particles in plasma, $\rho_i(t, x)$ is the electric charge density, and $E = E(t, x)$ is electric field and $A : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies $A(r) = o(r)$ as $r \to 0$. In particular, massless electrons/ions case if $A(r) = r + 1 - e^r$; screened case if $A(r) = 0$; and unscreened case if $A(r) = r$.

This system describes a hot, unconfined, electrostatic plasma, ions in $d$ dimensions of electrons on a uniform, static, background of ions with $d \geq 3$. This system has been extensively studied ([1–4, 10, 13, 14, 17–20, 23, 24, 28, 32, 38, 39], focusing on the global existence, regularity results and longtime behavior of solutions. The first paper on the global existence of weak solutions with $A(r) = r$ is due to Arsen’ev in [1]. Then Batt in [4] established global existence for spherically symmetric data, which was extended by Horst in [23] to global classical solvability with cylindrically symmetric data. After that in [2], Bardos and Degond obtained global existence for small data with $d \geq 3$ by using the Lagrangian approach. Next, in [35] Pfaffelmoser proved the global existence of a smooth solution with large (unrestricted size) data. Later, simpler proofs of the same were published by Schaeffer [39], Horst [24], and Lions and Perthame [32]. Moreover, the system with massless electrons (i.e $A(r) = r + 1 - e^r$) in $\mathbb{R}^d$ with $d \leq 3$ was also studied by Bouchut in [10], (see also [17, 20]). In recent years, in [26] they indicated the sharp faster decay of derivatives. Later on, in [12] they extended the results of (1.1) to $d \geq 2$ with better electric field decay in the case of screened interactions. Furthermore, the vector fields approach in [10] and the Fourier analysis approach in [28, 41] were also developed in the last few years.

In this paper, we study the stability and the long time behavior of solutions $f_i$ to (1.1) in the form

$$f_i(t, x, v) = \mu(v) + f(t, x, v),$$

where $\mu(v)$ is a stable equilibrium with $\int_{\mathbb{R}^d} \mu(v) dv = 1$ and $f_i(t = 0, x, v)$ closes to $\mu(v)$. So, $f$ solves the following perturbed system

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f &= -E \cdot \nabla_v \mu, \\
E &= -\nabla_x U_i - \Delta U + U = \rho + A(U), \quad \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv, \\
f|_{t=0} &= f_0.
\end{align*}$$ (1.2)

*E-mail address: ljhuang20@fudan.edu.cn, Fudan University, 220 Handan Road, Yangpu, Shanghai, 200433, China.
†E-mail address: qhnguyen@iamss.ac.cn, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China.
‡E-mail address: yrxn20@fudan.edu.cn, Fudan University, 220 Handan Road, Yangpu, Shanghai, 200433, China.
Here are our assumptions in this paper.

**Assumption 1** $\mu$ satisfies the Penrose stability condition:

$$\inf_{\tau \in \mathbb{R}, \xi \in \mathbb{R}^d} \left| 1 - \int_0^{+\infty} e^{-i\tau s} \frac{1}{1 + |s\xi|^2} \nabla_{v\mu}(s\xi) ds \right| \geq \hat{c},$$

(1.3)

for some constant $\hat{c} > 0$, $\nabla_{v\mu}$ is the Fourier transform of $\nabla_v \mu$ in $\mathbb{R}^d$.

**Assumption 2** $\mu \in L^1$ satisfies

$$\| \langle \cdot \rangle^N \nabla_{v\mu}(\cdot) \|_{W^{2,\infty}} + \| \langle \cdot \rangle^{d+5} \nabla \mu(\cdot) \|_{W^{2d+7,1}} + \frac{1}{N - d} \leq M^*,$$

for some $N > d$ and $M^* < \infty$.

**Assumption 3** $A : \mathbb{R} \to \mathbb{R}$ is $C^2$ and satisfies

$$\sup_{|r| \leq 1} \left( \frac{|A(r)|}{r^2} + \frac{|A'(r)|}{r} + |A''(r)| \right) \leq C_A,$$

for some constant $C_A > 0$.

Note that a particular example for the assumption 3 is $A(r) = r + 1 - e^r$ corresponding to the Vlasov–Poisson system with electrons mass, see [10].

It is well-known that the free transport equation $\partial_t f + v \nabla_x f = 0$ exhibits phase mixing which the spatial density $\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$ decays in time. It was an observation of Landau in [31] that the linearized Vlasov-Poisson equations near homogeneous Penrose stable equilibria also have the spatial density decaying in time. Under the Penrose condition (1.3), nonlinear Landau damping was proved on $\mathbb{T}^d \times \mathbb{R}^d$ in the pioneering work [29] by Mouhot and Villani for data with Gevrey regularity (see also [19] for refinements and simplifications). In [4] Bedrossian showed that the results therein do not hold in finite regularity (see also [10]). We note that related mechanisms in the fluid are the vorticity mixing by shear flows [5, 17, 30, 34]. In the whole space $\mathbb{R}^d \times \mathbb{R}^d$ with $d \geq 3$, it was established for the screened Vlasov-Poisson system in [8] by Bedrossian, Masmoudi, and Mouhot for data with finite Sobolev regularity. Their proof relies on the dispersive mechanism in Fourier space to control the plasma echo resonance. However, a decay in time of their result is far from optimal, as the dispersion in the physical space was not taken into account. In the recent work [21], Han-Kwan, Nguyen, and Rousset revisited the asymptotic stability of Penrose stable equilibria. They obtained the decay estimates for the density $\rho$ as follows:

$$\| \rho(t) \|_{L^1} + (1 + t) \| \nabla \rho(t) \|_{L^1} + (1 + t)^d \| \rho(t) \|_{L^\infty} + (1 + t)^{d+1} \| \nabla \rho(t) \|_{L^\infty} \lesssim \epsilon_0 \log(t + 2), \quad \forall t > 0.$$  

(1.4)

This is achieved by a pointwise dispersive estimate directly on the resolvent kernel for the linearized system around Penrose stable equilibrium $\mu$. Moreover, (1.3) is optimal up to a logarithmic correction. Very recently, in [20] Ionescu, Pausader, Wang, and Widmayer proved the first asymptotic stability result for the unscreened Vlasov-Poisson system in $\mathbb{R}^d$ around the Poisson equilibrium, see also in [37] for the case of a repulsive point charge. The unscreened case is open for the general equilibria. However, in [7, 22] they studied the linearized unscreened Vlasov-Poisson equation around suitably stable homogeneous equilibria in $\mathbb{R}^d \times \mathbb{R}^d$.

Our goal in this paper is to prove (1.3) for (1.2) without log ($t + 2$) in (RHS). We recall the Besov norm and Triebel-Lizorkin norm as

$$\| g \|_{\dot{B}_{p,q}^s} := \sup_{\alpha} \| \delta_{\alpha} g(x) \|_{L^p_x}, \quad \| g \|_{\dot{F}_{p,q}^s} := \| \sup_{\alpha} |\delta_{\alpha} g(x)| \|_{L^p_x},$$

and for $\kappa \in (0, 1)$ we define a norm weighted in time:

$$\| g \|_{\kappa} := \sum_{p=1,\infty} \sup_{s \in [0,\infty)} \left( (s)^{\frac{d(p-1)}{p}} \| g(s) \|_{L^p} + (s)^{\kappa + \frac{d(p-1)}{p}} \| g(s) \|_{B_{p,q}^s} \right), \quad \| g \|_{t,\kappa} := \sum_{j=0}^l \| (s)^j \nabla^j g \|_{\kappa}.$$

By Gagliardo-Nirenberg interpolation inequality, we know that

$$\| g \|_{t,\kappa} \sim \| g \|_\kappa + \| g \|_\kappa \| \nabla^j g \|_\kappa \sim \sum_{p=1,\infty} \sup_{s \in [0,\infty)} \left( (s)^{\frac{d(p-1)}{p}} \| g(s) \|_{L^p} + (s)^{l+\kappa + \frac{d(p-1)}{p}} \| \nabla^j g(s) \|_{B_{p,q}^s} \right).$$

Let us introduce some notations.
Notation 1.1 Let \( g : \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R}^m \). We define, for \((x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v \) and \( a \in (0,1) \):

\[
\mathcal{D}_1^a g(x,v) = \sup_{z \in \mathbb{R}^d} \frac{|g(x,v) - g(x-z,v)|}{|z|^a}, \quad \mathcal{D}_2^a g(x,v) = \sup_{z \in \mathbb{R}^d} \frac{|g(x,v) - g(x,v-z)|}{|z|^a},
\]

\[
\mathcal{D}^a g = \mathcal{D}_1^a g + \mathcal{D}_2^a g, \quad \mathcal{D}^a g = |g| + \mathcal{D}^a g, \quad i = 1,2.
\]

Our main result is as follows.

Theorem 1.2 Let \( a \in \left( -\frac{1}{2}, 1 \right) \). There exist \( C_0 > 0, \varepsilon \in (0,1) \) such that for \( 0 < \varepsilon \leq \varepsilon \) if

\[
\sum_{p=1}^{\infty} \sum_{i=0}^{1} \left\| \mathcal{D}^a(\mathcal{V}^1_{x,v} f_0) \right\|_{L_1 L_2 \cap L_1 L_2 \varepsilon} \leq \frac{1}{C_0 \varepsilon},
\]

then the problem (1.2) has a unique global solution \( f \) with

\[
\|\rho\|_a + \varepsilon^\frac{3}{2} \|U\|_a \leq \varepsilon \quad \text{and} \quad \|\rho\|_{1,a} + \|U\|_{1,a} \lesssim \varepsilon, a, M^{-1}.
\]

In addition, with \( \tilde{\rho}_0(t,x) = \int_{\mathbb{R}^d} f_0(x - tv,v)dv \), we have the estimate that

\[
\|\rho - \tilde{\rho}_0 - G \ast (t,v) \tilde{\rho}_0\|_a + \varepsilon^\frac{3}{2} \|U\|_a \lesssim \varepsilon^4,
\]

where the kernel \( G \) is defined in (2.5). Moreover, for \( m \in \mathbb{N}^+ \) and \( b \in (0,1) \) if

\[
\sum_{j=2}^{m+1} \sup_{|r| \leq 1} \left| A^{(j)}(r) \right| \leq C_{A,m}, \quad \sum_{j=0}^{m} \sum_{p=1}^{\infty} \left\| \mathcal{D}^b(\mathcal{V}^1_{x,v} f_0) \right\|_{L_1 L_2 \cap L_1 L_2 \varepsilon} \leq c,
\]

\[
\left\| \langle r \rangle^{d+\alpha+\varepsilon} |\nabla \mu| \right\|_{W^{2d+2\alpha,1}} \langle r \rangle \leq c, \quad (1.6)
\]

for some \( N > d \), then

\[
\|\rho\|_{m,b} + \|U\|_{m,b} \lesssim \varepsilon, a, M^{-1}.
\]

Remark 1.3 If we have initial data \( f_0 \) with higher-order derivatives, we can get more regularity of the solution \( \rho \). It is different from the condition in (3.38), which make smallness assumptions on higher derivatives of \( f_0 \).

Remark 1.4 The crucial decay estimates of \( E(t,x) \) depend on the fact that we work in dimension \( d \geq 3 \). And the decay will be stronger when \( d \) becomes higher. The decay estimates are insufficient in the case \( d \leq 2 \) which can be seen in the proof of Proposition 4.4, where we need to estimate \( \sup_{\alpha} \left\| \frac{\|\nabla v_{\gamma,\mu}\|_{L_\infty}}{|\alpha|^{\gamma,\mu}} \right\| \). That is the reason why we only consider \( d \geq 3 \) in our paper. We deal with the 2d case in (2.5).

Remark 1.5 While finishing our article, we learned that Toan Nguyen was working on this problem and had a similar result to ours. But the two works are independent.

As [21 Corollary 1.1], thanks to Theorem 1.2 we also obtain the following scattering property for the solution to (1.2). The proof is omitted as it is very analogous to [21 Proof of Corollary 1.1].

Corollary 1.6 With the same assumptions and notations as in Theorem 1.2, there is a function \( f_\infty \in W^{1,\infty} \) given by

\[
f_\infty(x,v) = f_0(x + Y_\infty(x,v),v + W_\infty(x,v)) + \mu(v + W_\infty(x,v)) - \mu(v),
\]

such that

\[
\|f(t,x+tv,v) - f_\infty(x,v)\|_{L_1 L_2} \lesssim \varepsilon, \quad t \geq 0,
\]

and \( \|Y_\infty\|_{L_1 L_2} + \|W_\infty\|_{L_1 L_2} \lesssim \varepsilon. \)

Let us discuss the main idea in this paper. First, we give the equivalence of \( (\rho,U) \) and \( (f,\rho) \) satisfying the system

\[
(p,u) = (F_1(p,u),F_2(p,u)),
\]

\[
F_1(p,u) = G \ast (I + R)(p,u) + (I + R)(p,u),
\]

\[
F_2(p,u) = (1 - \Delta)^{-1}(p + A(u)),
\]

where the kernel \( G \) is defined in (2.5) and \( I, R \) are defined in (2.3). This means, \( (\rho,U) \) is a fixed point of \( F = (F_1,F_2) \). We establish the boundedness of the mapping \( G(\cdot) \) under the norm \( \|\cdot\|_a \). To do this,
we need to estimate some pointwise decay estimates of the kernel $G(t,x)$, which gives a new proof to be different from the scaling method introduced by Han-Kwan, Nguyen, and Roussin in [21]. It should be emphasized that the cancellation of $G(t,x)$, i.e. $\int_{\mathbb{R}^d} G(t,x)dx = 0$, $\forall t \geq 0$, and the estimate of $\| |x|^a \delta_x G(t,x) \|_{L^p_{\omega}}$ play a very important role in proving the boundedness of the mapping $\mathcal{G}$. Using the decay estimates for the characteristics to prove maps $(\rho, U) \mapsto \mathcal{I}(\rho, U)$, $(\rho, U) \mapsto \mathcal{R}(\rho, U)$ be compressed mappings with a small contracting constant. Finally, we apply the contracting mapping principle to get a fixed point of $\mathcal{F}$.

The paper is organized as follows. In section 2, we establish the equivalence of $(\rho, U)$ and $f$, then we focus on the estimate of $(\rho, U)$ in the following sections. In section 3, we give the pointwise decay estimates of the kernel $G(t,x)$ and its derivatives. Then, we prove the bound: $\| \mathcal{G}(f) \|_{\gamma, \alpha} \leq M \| f \|_{\gamma, \alpha}$, for some constant $M = M(\hat{c}, d, M_1)$. In section 4, we establish the pointwise estimate of the characteristics $(X_{s,\tau}(x,v), V_{s,\tau}(x,v))$ and their derivatives. In section 5, we estimate the contribution of the initial data and explain that why we need the norm $L^1_t L^1_{\omega} \cap L^1_t L^2_{\omega} \cap L^1_t L^{\infty}_{\omega}$ of the initial data $f_0$. In Section 6, we estimate the reaction term $\mathcal{R}(\rho, U)$ and its higher derivatives. To handle it, we shall introduce a general map

$$T[F, \eta](t, x) = -\int_0^t \int_{\mathbb{R}^d} F(s, X_{s,\tau}(x,v)) \eta(V_{s,\tau}(x,v))dvds + \int_0^t \int_{\mathbb{R}^d} F(s, x - (t-s)v) \eta(v)dvds,$$

and give the estimate in Proposition [6.1]. In Section 7, we prove the main theorem.

The following are some notations in our paper.

Notation 1.7 Throughout this paper, $A \lesssim B$ means that there exists constant $C$ only depending dimension $d$ such that $A \leq CB$. $A \lesssim M B$ means that there exists constant $C = C(d, M)$ such that $A \leq CB$. The same convention is adopted for $A \gtrsim B$ and $A \sim B$.

Notation 1.8 In this paper, $\tilde{}$ is the "space-time" Fourier transform on $\mathbb{R}^{d+1}$ as follows

$$\tilde{g}(\tau, \xi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\tau \cdot x} e^{-ix \cdot \xi} g(t, x)dxdt.$$

And we define inverse Fourier transform

$$\mathcal{F}^{-1}(h)(t, x) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\tau \cdot x} e^{ix \cdot \xi} h(\tau, \xi)d\xi d\tau.$$

Notation 1.9 For multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, we denote $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$.

Notation 1.10 For a vector $x \in \mathbb{R}^d$, and $k \in \mathbb{N}$, we define $x \otimes^k = x \otimes x \otimes ... \otimes x$.

Acknowledgements: Q.H.N. is supported by the Academy of Mathematics and Systems Science, Chinese Academy of Sciences startup fund, and the National Natural Science Foundation of China (No. 1205410257 and No. 12288201) and the National Key R&D Program of China under grant 2021YFA1000800. Q.H.N. also wants to thank Benoit Pausader for his stimulating comments and suggestion to consider the Vlasov-Poisson system with massless electrons.

Data availability: Data will be made available on reasonable request.

2 Equivlance of $(\rho, U)$ and $f$

In this section, we will show that the system (1.2) is equivalent to an equation in terms of $(\rho, U)$. Let $(X_{s,\tau}(x,v), V_{s,\tau}(x,v))$ be the flow associated to the vector field $(v, E(t, x))$, i.e. $(X_{s,\tau}(x,v), V_{s,\tau}(x,v))$ satisfies the ODE system:

$$\begin{align*}
\frac{d}{dt} X_{s,\tau}(x,v) &= V_{s,\tau}(x,v), \\
\frac{d}{dt} V_{s,\tau}(x,v) &= E(s, X_{s,\tau}(x,v)),
\end{align*}$$

(2.1)

for any $0 \leq s \leq t$, $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. Then, the solution $f$ of the equation (1.2) is given by

$$f(t, x, v) = f_0(X_{0,t}(x,v), V_{0,t}(x,v)) - \int_0^t E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))ds,$$

(2.2)
We define the norm
\[ \| \cdot \| = \| \cdot \|_{L^2_t L^2_x} = \left( \int \int |\langle \cdot \rangle|^2 \right)^{1/2}. \]

where \( F \) has a fixed point i.e. \( F(p, U) = (p, U) \).

We define the operator \( F \) as:
\[ F(p, U) = \left( F_1(p, U), F_2(p, U) \right), \]
(2.7)

where
\[ F_1(p, U) = G *_{(t, x)} (I(p, U) + R(p, U) + A(U)) + I(p, U) + R(p, U), \]
\[ F_2(p, U) = (1 - \Delta)^{-1} (p + A(U)). \]

We need to prove that \( F \) has a fixed point i.e. \( F(p, U) = (p, U) \).

We define the norm
\[ \| (g_1, g_2) \|_{S^\varepsilon} := \| g_1 \|_\infty + \varepsilon^{1/2} \| g_2 \|_\infty, \quad \| (g_1, g_2) \|_{l, \infty} := \| g_1 \|_{l, \infty} + \| g_2 \|_{l, \infty}. \]

The main Theorem [12] is a consequence of the following theorems:

**Theorem 2.2** Let \( a \in (\sqrt{2} - 1, 1) \). There exist constants \( C_0 > 0 \) and \( \varepsilon > 0 \) such that for any \( \varepsilon \in (0, \bar{\varepsilon}] \) we have
\[ \| F(p, U) \|_{S^\varepsilon} \leq \varepsilon, \quad \| F(p_1, U_1) - F(p_2, U_2) \|_{S^\varepsilon} \leq \frac{1}{2} \| (p_1 - p_2, U_1 - U_2) \|_{S^\varepsilon}. \]
In this section, we prove the boundedness of the operator $G$ and boundedness of operator $f$. For some constant $M$, we have

$$\sum_{i=0,1} \sum_{p=1,\infty} \|D^i(\nabla_x, \cdot f_0)\|_{L_x^1 L_t^{\infty} L_z^{\infty}} \leq \frac{1}{C_0}.$$  \hfill (2.8)

Under condition $\mathcal{F}$, $F$ has a unique fixed point $(\rho, U)$ in $\mathcal{B}$ with $\|(\rho, U)\|_{1,a} \lesssim_{\varepsilon,M} 1$. Moreover, set $\tilde{\rho}_0(t,x) = \int_{\mathbb{R}^d} f_0(x - tv, v) dv$, we have

$$\|(\rho - \tilde{\rho}_0 - G \ast (t,x) \tilde{\rho}_0, U)\|_{\mathcal{S}_0} \lesssim_{\varepsilon,M,a} \varepsilon^\frac{1}{2}.$$  \hfill (3.3)

**Theorem 2.3** Let $\rho$ be in Theorem 2.2. Then for $b \in (0,1)$ and $m \geq 2$, we have

$$\|(\rho, U)\|_{m,b} \lesssim_{\varepsilon,a,M,e,c',C_{A,m}} 1,$$

provided the initial data $f_0$, the stationary state $u$ and function $A$ satisfying

$$\sum_{j=2}^{m+1} \sup_{|r| \leq 1} |A^{(j)}(r)| \leq C_{A,m}, \quad \sum_{j=0}^{m} \sup_{p=1,\infty} \|D^j(\nabla_x, \cdot f_0)\|_{L_x^1 L_t^{\infty} L_z^{\infty}} \leq \epsilon',$$

$$\|\langle \eta \rangle_{d+\gamma+5} \nabla \mu(\cdot)\|_{W^{2d+2\gamma+7,1}} + \langle \eta \rangle_{d+\gamma+5} \sum_{j=0}^{m} |\nabla^j \mu(\cdot)| \leq \epsilon'.$$  \hfill (2.9)

### 3 Estimate of the kernel $G$ and boundedness of operator $G$

In this section, we prove the boundedness of the operator $G : f \mapsto G \ast (t,x) f$ under the norm $\| \cdot \|_{\gamma,a}$, where $\gamma \geq 0$. The main result is

$$\|G(f)\|_{\gamma,a} \leq M\|f\|_{\gamma,a},$$

for some constant $M = M(\varepsilon, d, M_{\gamma+1})$.

To do this, we need the following Lemmas:

**Lemma 3.1** Define

$$Q_0(u, \tau, \xi) = \int_0^{+\infty} e^{-i\tau t} u(t\xi) dt.$$  \hfill (3.1)

Then,

$$|Q_0(u, \tau, \xi)| \lesssim \sup_{z} \left( |u(x)| |x|^{-1} \right).$$  \hfill (3.2)

**Proof.** Firstly, we can get the result (3.1) directly as follows,

$$|Q_0(u, \tau, \xi)| \leq \int_0^{+\infty} |u(t\xi)| dt \leq \sup_{z \in \mathbb{R}^d} \left( |u(z)| |z|^{-1} \right) \int_0^{+\infty} \frac{1}{(t\xi)^2} dt \sim \sup_{z \in \mathbb{R}^d} \left( |u(z)| |z|^{-1} \right) \frac{1}{|\xi|}.$$  \hfill (3.3)

Moreover, we integrate by parts $m$-times to obtain that for any nonnegative integer $m$,

$$Q_0(u, \tau, \xi) = - \left( \sum_{j=0}^{m-1} \frac{1}{(i\tau)^{m+j}} \xi^{\otimes j} : \nabla^j u(0) \right) + \frac{1}{(i\tau)^m} Q(\xi^{\otimes m} : \nabla^m u(\tau, \xi), \tau, \xi).$$

Combining the above equality with (3.3) to get the conclusion (3.2).

**Lemma 3.2** For $k \in \mathbb{Z}^+$, define

$$Q_k(u, \tau, \xi) = \int_0^{+\infty} t^k e^{-i\tau t} u(t\xi) dt.$$  \hfill (3.4)
Then,
\[ |Q_k(u, \tau, \xi)| \lesssim \sup_x \left\langle \frac{(u(x))}{(x)^{2+k}} \right\rangle, \]  
(3.5)
\[ |Q_k(u, \tau, \xi)| \lesssim \frac{1}{|\tau|^{k+1}} \left( |u(0)| + \sum_{l=0}^{k} \sup_x (|\nabla^{k-l+1} u(x)|)^{2+k-l} \right), \]  
(3.6)
\[ |Q_k(u, \tau, \xi)| \lesssim \frac{1}{|\tau| + |\xi|^{k+1}} \left( \sup_x (|u(x)|)^{2+k} + \sum_{l=0}^{k} \sup_x (|\nabla^{k-l+1} u(x)|)^{2+k-l} \right). \]  
(3.7)

Moreover, if \( u(0) = 0 \), we have
\[ |Q_k(u, \tau, \xi)| \lesssim \frac{|\xi|}{|\tau|^{k+2}} \sum_{l=0}^{k+1} \sup_x (|\nabla^{k-l+2} u(x)|)^{3+k-l}, \]  
(3.8)
\[ |Q_k(u, \tau, \xi)| \lesssim \frac{|\xi|}{(|\tau| + |\xi|)^{k+2}} \left( \sup_x (|u(x)|)^{2+k} + \sum_{l=0}^{k+1} \sup_x (|\nabla^{k-l+2} u(x)|)^{3+k-l} \right). \]  
(3.9)

**Proof.** First of all, \( \forall j \in \{ 1, 2, \ldots, d \} \), we obtain
\[ \xi_j^k Q_k(u, \tau, \xi) = \int_0^{+\infty} (t \xi_j)^k e^{-\tau t} u(t \xi) dt = Q_0(u_1, \tau, \xi), \]
where \( u_1(\xi) = \xi_j^k u(\xi) \). By (3.1), one has,
\[ |\xi_j^k Q_k(u, \tau, \xi)| \lesssim \sup_x \left\langle \frac{(u(x))}{(x)^{2+k}} \right\rangle, \]
which implies (3.3).

Note that
\[ \nabla^l u_1(0) = 0, \quad \forall l \leq k - 1, \quad |\nabla^l u_1(0)| \lesssim |\nabla^{l-k} u(0)|, \quad \forall l \geq k. \]

Then, we know from (3.2) with \( m = k + 1 \) that
\[ |\xi_j^k Q_k(u, \tau, \xi)| = Q_0(u_1, \tau, \xi) \lesssim \frac{|\xi_j^k|}{|\tau|^{k+1}} |\nabla^k u_1(0)| + \frac{|\xi_j^k|}{|\tau|^{k+1}} \sup_x (|\nabla^{k+1} u_1(x)|)^2 \]
\[ \lesssim \frac{|\xi_j^k|}{|\tau|^{k+1}} |u(0)| + \frac{|\xi_j^k|}{|\tau|^{k+1}} \sum_{l=0}^{k} \sup_x (|\nabla^{k-l+1} u(x)|)^{2+k-l}. \]

Then,
\[ |\xi_j^k |Q_k(u, \tau, \xi)| \lesssim \frac{|\xi_j^k|}{|\tau|^{k+1}} |u(0)| + \frac{|\xi_j^k|}{|\tau|^{k+1}} \sum_{l=0}^{k} \sup_x (|\nabla^{k-l+1} u(x)|)^{2+k-l}, \]
which implies (3.6).

Then (3.5) and (3.6) imply (3.7) directly.

Moreover, if \( u(0) = 0 \), thanks to (3.2) with \( m = k + 2 \), we have
\[ |\xi_j^k Q_k(u, \tau, \xi)| \lesssim \frac{|\xi_j^{k+1}|}{|\tau|^{k+2}} |\nabla^{k+1} u_1(0)| + \frac{|\xi_j^{k+1}|}{|\tau|^{k+2}} \sup_x (|\nabla^{k+2} u_1(x)|)^2 \]
\[ \lesssim \frac{|\xi_j^{k+1}|}{|\tau|^{k+2}} |\nabla u(0)| + \frac{|\xi_j^{k+1}|}{|\tau|^{k+2}} \sum_{l=0}^{k+1} \sup_x (|\nabla^{k-l+2} u(x)|)^{3+k-l} \]
\[ \lesssim \frac{|\xi_j^{k+1}|}{|\tau|^{k+2}} \sum_{l=0}^{k+1} \sup_x (|\nabla^{k-l+2} u(x)|)^{3+k-l}. \]
which implies (3.8). Finally, (3.5) and (3.8) imply (3.4).

We define
\[ \| F \|_{W^\beta_{0}} := \sum_{j=0}^{\beta} \int_{\mathbb{R}^d} (x)^{\gamma} |\nabla^j F(x)| dx, \]
and use this norm in the following lemmas.
Lemma 3.3 For integer $\alpha \geq 0$ and $\beta \geq 2$, we have following estimates:

\[
\left| \nabla_{\xi} \tilde{K}(\tau, \xi) \right| \lesssim \frac{\left| \xi \right|}{\left( \xi^2 + \left| \xi \right| \right)^{1/2}} \left\| \nabla \mu \right\|_{W_{3}^{2}},
\]
(3.10)
\[
\left| \nabla_{\xi}^{\beta} \tilde{K}(\tau, \xi) \right| \lesssim \left( \frac{1}{\left( \xi^2 + \left| \xi \right| \right)^{1/2}} + \frac{1}{\left| \xi \right|^{( \beta + 1)/2}} \right) \left\| \nabla \mu \right\|_{W_{\alpha+2}^{2\beta+1}},
\]
(3.11)
\[
\left| \partial_{\tau}^{\alpha} \tilde{K}(\tau, \xi) \right| \lesssim \frac{\left| \xi \right|^2}{\left( \xi^2 + \left| \xi \right| \right)^{1/2}} \left\| \nabla \mu \right\|_{W_{\alpha+3}^{\alpha+2}}.
\]
(3.12)

Proof. Recall that

\[
\tilde{K}(\tau, \xi) = \int_{0}^{+\infty} e^{-\imath \tau t} \frac{1}{1 + \left| \xi \right|^2} \imath \xi \cdot \nabla_{\xi} \mu(t \xi) dt.
\]
(3.13)

Then for $\alpha \geq 0$, we have

\[
\partial_{\tau}^{\alpha} \tilde{K}(\tau, \xi) = \sum_{j=1}^{d} \frac{(-\imath)^{\alpha}}{(\xi^2)^{\alpha/2}} \int_{0}^{+\infty} t^{\alpha} e^{-\imath \tau t} P_{j}(t \xi) dt,
\]

where

\[
P(\xi) = (P_{1}, P_{2}, ..., P_{d})(\xi) = \nabla_{\xi} \mu(\xi), \quad P(0) = 0.
\]

Thanks to (3.9), we have

\[
\left| \int_{0}^{+\infty} t^{\alpha} e^{-\imath \tau t} P_{j}(t \xi) dt \right| \lesssim \frac{\left| \xi \right|}{\left( \xi^2 + \left| \xi \right| \right)^{\alpha/2}} \left( \sup_{x} (P(x) \left( \left| x \right| \right)^{2+\alpha}) + \sum_{l=0}^{\alpha+1} \sup_{x} (|\nabla^{\alpha-l+1} P(x)\left( \left| x \right| \right)^{3+\alpha-l}) \right)
\]

\[
\lesssim \frac{\left| \xi \right|^2}{\left( \xi^2 + \left| \xi \right| \right)^{\alpha/2}} \left\| \nabla \mu \right\|_{W_{\alpha+3}^{\alpha+2}}.
\]

Thus we obtain that

\[
\left| \partial_{\tau}^{\alpha} \tilde{K}(\tau, \xi) \right| \lesssim \sum_{j=1}^{d} \frac{\left| \xi \right|}{(\xi^2)^{\alpha/2}} \left| \int_{0}^{+\infty} t^{\alpha} e^{-\imath \tau t} P_{j}(t \xi) dt \right| \lesssim \frac{\left| \xi \right|^2}{(\xi^2 + \left| \xi \right|)^{\alpha/2}} \left\| \nabla \mu \right\|_{W_{\alpha+3}^{\alpha+2}}.
\]

Thanks to (3.7) with $k = \beta_{1}$, one has

\[
\left| \int_{0}^{+\infty} t^{\beta_{1}} e^{-\imath \tau t} (\nabla^{\beta_{1}} P_{j})(t \xi) dt \right|
\]

\[
\lesssim \frac{1}{\left( \xi^2 + \left| \xi \right| \right)^{\beta_{1}/2}} \left( \sup_{x} (|\nabla^{\beta_{1}} P(x)| \left( \left| x \right| \right)^{2+\beta_{1}}) + \sum_{l=0}^{\beta_{1}} \sup_{x} (|\nabla^{2\beta_{1}-l+1} P(x)| \left( \left| x \right| \right)^{3+\beta_{1}-l}) \right)
\]

\[
\lesssim \frac{1}{\left( \xi^2 + \left| \xi \right| \right)^{\beta_{1}/2}} \left\| \nabla \mu \right\|_{W_{\beta_{1}+2}^{\beta_{1}+1}}.
\]

Therefore,

\[
\left| \nabla_{\xi} \tilde{K}(\tau, \xi) \right| \lesssim \sum_{j=1}^{d} \frac{\left| \xi \right|}{(\xi^2)^{\alpha/2}} \left| \int_{0}^{+\infty} t e^{-\imath \tau t} (\nabla P_{j})(t \xi) dt \right| + \sum_{j=1}^{d} \frac{1}{(\xi^2)^{\alpha/2}} \left| \int_{0}^{+\infty} e^{-\imath \tau t} P_{j}(t \xi) dt \right| \lesssim \frac{\left| \xi \right|}{(\xi^2 + \left| \xi \right|)^{1/2}} \left\| \nabla \mu \right\|_{W_{3}^{2}}.
\]

For $\beta \geq 2$, we have

\[
\left| \nabla_{\xi}^{\beta} \tilde{K}(\tau, \xi) \right| \lesssim \sum_{j=1}^{d} \frac{\beta}{\left( \xi^2 \right)^{\beta/2-1}} \left( \frac{\xi}{\xi^2} \right) \left| \int_{0}^{+\infty} t^{\beta-\beta_{1}} e^{-\imath \tau t} (\nabla^{\beta_{1}} P_{j})(t \xi) dt \right|
\]

\[
\lesssim \sum_{j=1}^{d} \sum_{\beta_{1}=0}^{\beta-1} \frac{1}{(\xi^2)^{\beta_{1}/2-\beta_{1}/2+1}} \left| \int_{0}^{+\infty} t^{\beta_{1}} e^{-\imath \tau t} (\nabla^{\beta_{1}} P_{j})(t \xi) dt \right|
\]

\[
+ \sum_{j=1}^{d} \frac{\left| \xi \right|}{(\xi^2)^{\alpha/2}} \left| \int_{0}^{+\infty} t^{\beta} e^{-\imath \tau t} (\nabla^{\beta} P_{j})(t \xi) dt \right| + \sum_{j=1}^{d} \frac{1}{(\xi^2)^{\alpha/2}} \left| \int_{0}^{+\infty} e^{-\imath \tau t} P_{j}(t \xi) dt \right|
\]

\[
\lesssim \left\| \nabla \mu \right\|_{W_{\beta_{1}+2}^{\beta_{1}+1}} \left( \frac{1}{(\xi^2)^{\beta/2}} + \frac{1}{(\xi^2)^{\beta_{1}/2}} \right).
\]

Thus, the proof is complete.

\[\blacksquare\]
Lemma 3.4  For any integer \( j \geq 1 \), there holds
\[
\left| \nabla^j \left( \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right) \right| \lesssim M_j \left( \frac{1}{\langle \xi \rangle^j (|\tau| + |\xi|)^2} + \frac{1}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^2} \right),
\]
(3.14)
\[
\left| \partial_\tau^j \left( \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right) \right| \lesssim M_j \frac{|\xi|^2}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^{j+2}},
\]
(3.15)
where
\[
M_j = \| \nabla \mu \|_{W^{j+2}_{1,2}} \left( 1 + \| \nabla \mu \|_{W^{j+2}_{1,2}} \right)^{j-1}.
\]

Proof. Thanks to the Penrose condition, one has \( |1 - \tilde{K}(\tau, \xi)| \geq \tilde{c} \). Firstly, we have
\[
\left| \nabla \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right| = \left| \nabla \left( \frac{1}{1 - K(\tau, \xi)} \right) \right| \lesssim \tilde{c} \| \nabla \tilde{K}(\tau, \xi) \| \lesssim \tilde{c} \frac{\| \nabla \mu \|_{W^{j+2}_{1,2}}}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^2} \| \nabla \mu \|_{W^{j+2}_{1,2}}.
\]
Note that we get the following lines for any \( j \geq 1 \),
\[
\left| \nabla^j \left( \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right) \right| \left| \nabla^j \left( \frac{1}{1 - K(\tau, \xi)} \right) \right| \lesssim \tilde{c} \sum_{i=1}^{j} \| \nabla^i \tilde{K}(\tau, \xi) \| \lesssim \tilde{c} \sum_{i=1}^{j} \frac{\| \nabla^i \mu \|_{W^{j+2}_{1,2}}}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^2} \| \nabla \mu \|_{W^{j+2}_{1,2}}.
\]
Moreover, one has
\[
\left| \partial_\tau^j \left( \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right) \right| \left| \partial_\tau^j \left( \frac{1}{1 - K(\tau, \xi)} \right) \right| \lesssim \tilde{c} \sum_{i=1}^{j} \| \partial_\tau^i \tilde{K}(\tau, \xi) \| \lesssim \tilde{c} \sum_{i=1}^{j} \frac{\| \partial_\tau^i \mu \|_{W^{j+2}_{1,2}}}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^2} \| \nabla \mu \|_{W^{j+2}_{1,2}}.
\]
This completes the proof.

Lemma 3.5  Define \( G_n = \mathcal{F}^{-1} \left( \tilde{G} \chi_n \right) \), where \( \chi \in [0, 1] \) is a smooth compactly supported function in the annulus \( 1/4 \leq |\xi| \leq 4 \) and \( \chi_n(\xi) = \chi(\tilde{\xi}) \). Then, we have for any integer \( \gamma \geq 0 \)
\[
|\nabla^\gamma G_n(t, x)| \lesssim \tilde{c} 2^{(d+\gamma+1)n-2n^+} M_{d+\gamma+2} \min \left\{ 1, \frac{1}{2^n(t + |x|)} \right\}^{d+\gamma+2},
\]
where \( n^+ = \max\{0, n\} \).

Proof. Firstly, we estimate \( |\nabla^\gamma G_n(t, x)| \) directly.
\[
|\nabla^\gamma G_n(t, x)| \leq \mathcal{F}^{-1} \left( \hat{\tilde{K}} \hat{\chi}_n \right) \lesssim \int_{\mathbb{R} \times |\xi| \sim 2^n} \left| \frac{\tilde{K}(\tau, \xi)}{1 - K(\tau, \xi)} \right| \xi^\gamma \chi_n(\xi) \, d\xi d\tau \lesssim \tilde{c} \int_{\mathbb{R} \times |\xi| \sim 2^n} 2^n |\tilde{K}(\tau, \xi)| \, d\tau d\xi.
\]
Using (3.12), one has
\[
|\nabla^\gamma G_n(t, x)| \lesssim \| \nabla \mu \|_{W^2} \int_{\mathbb{R} \times |\xi| \sim 2^n} 2^n \frac{|\xi|^2}{\langle \xi \rangle^{j+1} (|\tau| + |\xi|)^2} \, d\tau d\xi \lesssim \tilde{c} 2^{(d+\gamma+1)n-2n^+} \| \nabla \mu \|_{W^2}. \quad (3.16)
\]
Then we integrate by parts in $\tau$ to get that

$$\|\nabla^\gamma G_n(t, x)\| \sim \left| \int_{\mathbb{R} \times \mathbb{R}^d} e^{i \tau t} e^{i \xi \cdot x} \frac{\tilde{K}(\tau, \xi)}{1 - \tilde{K}(\tau, \xi)} \xi^{\otimes \gamma} \chi_n(\xi) d\xi d\tau \right|.$$ 

Using (3.15), we obtain that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon M_m \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^{2+\gamma}}{|(\tau| + |\xi|)^m} |\nabla^m (\tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi))| d\xi d\tau$$

Finally, we integrate by parts in $\xi$ to get that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^{2+\gamma}}{|(\tau| + |\xi|)^m} |\nabla^{m-1} (\tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi))| d\xi d\tau$$

Using (3.10) and (3.14), we obtain that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^{2+\gamma}}{|(\tau| + |\xi|)^m} |\nabla^{m-1} (\tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi))| d\xi d\tau$$

Finally, we integrate by parts in $\xi$ to get that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^{2+\gamma}}{|(\tau| + |\xi|)^m} |\nabla^{m-1} (\tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi))| d\xi d\tau$$

Using (3.10) and (3.14), we obtain that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon \|\nabla^\mu\|_{W^{1, \infty}} \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^2}{|(\tau| + |\xi|)^2} 2^{(\gamma-m_2)n} |\nabla^\xi \tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi)| d\xi d\tau$$

Finally, we integrate by parts in $\xi$ to get that

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon \frac{1}{|t|^m} \int_{|\tau| \lesssim 2} \frac{|\xi|^{2+\gamma}}{|(\tau| + |\xi|)^m} |\nabla^{m-1} (\tilde{K}(\tau, \xi) \xi^{\otimes \gamma} \chi_n(\xi))| d\xi d\tau$$

Using (3.10), (3.17) and (3.18) and taking $m_1 = m_2 = d + \gamma + 2$, we obtain

$$|\nabla^\gamma G_n(t, x)| \lesssim \varepsilon 2^{(d+\gamma+1)n-2n^+} M_{d+\gamma+2} \min \left\{ 1, \frac{1}{2n(t + |x|)} \right\}^{d+\gamma+2}.$$ 

This gives the result.

**Theorem 3.6** Let $\gamma \geq 0$, $0 \leq a_0 < 1$ and $0 < b_0 < 1$. Then,

$$\|\nabla^\gamma G(t, x)\| \lesssim \varepsilon \frac{\widetilde{M}_\gamma}{(t + |x|)^{d+\gamma+1}}.$$  

In particular,

$$\|\nabla^\gamma G(t, x)\|_{L^\infty} \lesssim \varepsilon \frac{\widetilde{M}_\gamma}{d+\gamma+1(1 + t)^2}$$

$$\|\nabla_{\alpha}^a \nabla^\gamma G(t, x)\|_{L^1} \lesssim \varepsilon \frac{\widetilde{M}_\gamma}{(1 + a_0(1 + t))^2} + \widetilde{M}_1 1_{t \leq 1} + \widetilde{M}_1 1_{\gamma = 1 + a_0} \log^+ \left( \frac{1}{t} \right),$$

$$\|\nabla_{\alpha}^a \delta_\alpha G(t, x)\|_{L^1} \lesssim \varepsilon \frac{\widetilde{M}_\gamma}{(1 + t + |\alpha|)^{2b_0}} + \widetilde{M}_1 1_{|\alpha| > 1} + \widetilde{M}_1 1_{|\alpha| \leq \sqrt{\frac{1}{1 + t}} \left( \frac{|\alpha|^{d+b_0}}{(1 + t)^2} \right)}$$

where

$$\widetilde{M}_\gamma = \overline{M}_\gamma (1 + \overline{M}_\gamma)^{d+\gamma+1}, \quad \text{and} \quad \overline{M}_\gamma = \|\langle \cdot \rangle^{d+\gamma+4} \nabla^\mu(\cdot)\|_{W^{2d+\gamma+5, 1}}.$$
Proof. Firstly, we prove (3.19). Note that
\[ \| \nabla \mu \|_{W^\alpha_0} \lesssim \| \langle \cdot \rangle^\alpha \nabla \mu \|_{W^{\gamma, 1}}, \]
thus
\[ M_{d+\gamma+2} \lesssim \overline{M}_\gamma (1 + \overline{M}_\gamma)^{d+\gamma+1} = \overline{M}_\gamma. \]
Since
\[ |\nabla \gamma G(t, x)| \leq \sum_{n \leq 0} |\nabla \gamma G_n(t, x)| + \sum_{n > 0} |\nabla \gamma G_n(t, x)|, \]
for \( n > 0 \), combining this with Lemma 3.5 we can yield that
\[
\sum_{n > 0} |\nabla \gamma G_n(t, x)| \lesssim \varepsilon \overline{M}_\gamma \sum_{n > 0} 2^{(d+\gamma-1)n} \left( 1 + \frac{1}{2^n(t + |x|)} \right)
\]
\[ \sim \varepsilon \overline{M}_\gamma \sum_{n > 0} \left( 1 + \frac{1}{2^{n(t + |x|)^{d+\gamma+2}}} \right) \]
\[ + \varepsilon \overline{M}_\gamma \sum_{n > 0} \frac{1}{2^{n(t + |x|)^{d+\gamma+2}}} \]
\[ \lesssim \varepsilon \overline{M}_\gamma \left( 1 + \frac{1}{(t + |x|)^{d+\gamma+1}} \right). \]
Similarly, we have for \( n \leq 0 \),
\[
\sum_{n \leq 0} |\nabla \gamma G_n(t, x)| \lesssim \varepsilon \overline{M}_\gamma \left( 1 + \frac{1}{(t + |x|)^{d+\gamma+1}} \right). \]
Finally, we deduce that
\[ |\nabla \gamma G(t, x)| \lesssim \varepsilon \overline{M}_\gamma \left( 1 + \frac{1}{(t + |x|)^{d+\gamma+1}} \right). \]
This implies (3.19) and (3.20).

(1) Now we prove (3.21). One has,
\[ \| |x|^{\alpha_0} \nabla \gamma G(t, x) \|_{L^1} \lesssim \varepsilon \overline{M}_\gamma \int \frac{|x|^{\alpha_0}}{(t + |x|)^{d+\gamma+1}} dx. \]
If \( t \geq 1 \), we estimate
\[ \| |x|^{\alpha_0} \nabla \gamma G(t, x) \|_{L^1} \lesssim \varepsilon \overline{M}_\gamma \int \frac{|x|^{\alpha_0}}{(t + |x|)^{d+\gamma+1}} dx \sim \varepsilon \overline{M}_\gamma \frac{|x|^{\alpha_0}}{(1 + |x|)^{d+\gamma+1}} dx \sim \varepsilon \overline{M}_\gamma. \]
If \( 0 \leq t \leq 1 \), we estimate
\[ \| |x|^{\alpha_0} \nabla \gamma G(t, x) \|_{L^1} \lesssim \varepsilon \overline{M}_\gamma \int \frac{|x|^{\alpha_0}}{(t + |x|)^{d+\gamma+1}} dx \lesssim \varepsilon \overline{M}_\gamma \left( \frac{1}{t^{\gamma+1-a_0}} + 1 + \sum_{n > 0} \frac{1}{(t + |x|)^{d+\gamma+1}} dx \right) \]
\[ \lesssim \varepsilon \overline{M}_\gamma \left( \frac{1}{t^{\gamma+1-a_0}} + 1 + \frac{1}{(1 + |x|)^{d+\gamma+1}} + \gamma=1+\gamma \log \left( \frac{1}{t} \right) + \frac{1}{t^\gamma} \right). \]
Combining these two cases to deduce that
\[
\| |x|^\alpha \nabla^\gamma G(t, x) \|_{L^2} \lesssim \epsilon \ 1_{\alpha \geq 1} \frac{\overline{M}_\gamma}{t^{\gamma+1-\alpha}} + 1_{\alpha \leq 1} \overline{M}_\gamma \left( \frac{1}{t^{\gamma-1-\alpha_0}} + 1 + 1_{\gamma = 1+\alpha_0} \log \left( \frac{1}{t} \right) \right) \\
\sim \epsilon \frac{\overline{M}_\gamma}{t^{\gamma-1-\alpha_0}(1+t)^2} + \overline{M}_\gamma 1_{\alpha \leq 1} + 1_{\gamma = 1+\alpha_0} \log^+ \left( \frac{1}{t} \right).
\]

(2) Now we prove (3.22). One has
\[
\| |x|^{b_0} \delta_0 G(t, x) \|_{L^2} = \left( \int_{|x| \leq \alpha/2} + \int_{|x| \geq \alpha/2} \right) |x|^{b_0} |\delta_0 G(t, x)| dx =: I^1_{G, b_0} + I^2_{G, b_0}.
\]
For the term \( I^1_{G, b_0} \),
\[
I^1_{G, b_0} \lesssim \epsilon |\alpha| \int_{|x| \leq \alpha/2} \int_{|x| \leq \alpha/2} |x|^{b_0} \frac{M_1}{(t + |x - \alpha|)^d} \frac{1}{(1+t + |x - \alpha|)^2} dx d\tau,
\]
\[
\sim \epsilon |\alpha| \int_{|x| \leq \alpha/2} \frac{M_1}{(t + |x|/2)^d(1+t + |x|/2)^2} dx,
\]
\[
\lesssim \epsilon |\alpha| \int_{|x| \leq \alpha/2} \frac{M_1}{(t + |x|)^2} + 1_{|\alpha| \geq 1} \int_{|x| \leq \alpha/2} \frac{M_1}{(1+|x|)^2} dx,
\]
\[
= : I^1_{G, b_0} + I^2_{G, b_0}.
\]

Then, we estimate that
\[
I^1_{G, b_0} \lesssim \epsilon |\alpha| \int_{|x| \leq \alpha/2} \frac{M_1}{(t^d(1+t)^2) dx}
\]
\[
+ 1_{|\alpha| \geq 1} \int_{|\alpha| \leq \alpha/2} \frac{M_1}{(t^d(1+t)^2) dx}
\]
\[
\lesssim \epsilon |\alpha| \left( 1_{|\alpha| \geq 1} \int_{|x| \leq \alpha/2} \frac{M_1}{(1+|x|)^2} dx + 1_{|\alpha| \geq 1} \int_{|x| \leq \alpha/2} \frac{M_1}{(1+|x|)^2} dx \right)
\]
\[
\lesssim \epsilon \overline{M}_1 \left( \frac{|\alpha|}{(1+|\alpha|)^2} + 1 \right) \sim \epsilon \overline{M}_1 (1 + |\alpha|)^{-2}.
\]

(3.24)

and

\[
I^2_{G, b_0} \lesssim 1_{|\alpha| < 1/2} \int_{|\alpha| \leq |x| \leq \alpha/2} \frac{M_1}{(1+|x|)^2} dx,
\]

\[
\sim \epsilon 1_{|\alpha| < 1/2} \overline{M}_1 \left( 1 + \frac{1}{\alpha} \right),
\]

Combining (3.23) with (3.24) to yield
\[
I^1_{G, b_0} \lesssim \epsilon \overline{M}_1 \left( 1 + \frac{|\alpha|}{(1+|\alpha|)^2} \right),
\]

(3.25)

For the term \( I^2_{G, b_0} \), one has
\[
I^2_{G, b_0} \lesssim \epsilon \overline{M}_1 \left( \frac{|\alpha|}{(1+|\alpha|)^2} + 1 \right) \sim \epsilon \overline{M}_1 \left( 1 + \frac{|\alpha|}{(1+|\alpha|)^2} \right).
\]
Combining this with \((3.25)\) and note that \(\tilde{M}_0 \leq \tilde{M}_1\), one gets

\[
\|x|^{b_0} \delta_0 G(t,x)\|_{L^1_t} \lesssim \tilde{M}_1 \frac{|\alpha|}{(1 + t + |\alpha|)^{2-b_0}} + \tilde{M}_1 \mathbf{1}_{|\alpha| > t} \frac{|\alpha|^{b_0}}{1 + t} + \tilde{M}_1 \mathbf{1}_{|\alpha| \leq t} \frac{|\alpha|^{d+b_0}}{t^{d-1}(1 + t)^2},
\]

which implies \((3.22)\).

\[\text{Lemma 3.7} \quad \text{For the kernel } G \text{ defined in } (2.6), \text{ there holds}
\]

\[
\int_{\mathbb{R}^d} G(t,x) dx = 0, \quad \forall t \geq 0.
\]

\[\text{(3.26)}\]

\[\text{Proof.} \quad \text{By } (3.12) \text{ with } \alpha = 0, \text{ one has } \tilde{K}(\tau,0) = 0. \text{ Then from } (2.6), \text{ we obtain that } \tilde{G}(\tau,0) = 0, \text{ for all } \tau \in \mathbb{R}, \text{ which implies the conclusion } (3.26). \]

Then we can prove the main result in this section.

\[\text{Theorem 3.8} \quad \text{There holds}
\]

\[
\|G *_{(t,x)} f\|_{\gamma,a} \leq M \|f\|_{\gamma,a},
\]

\[\text{with } M = M(\epsilon, \delta, M_{\epsilon+1}).\]

It is easy to check that for any \(t \in [0, 1]\)

\[
\sum_{p=1, \infty} \left(\|G *_{(t,x)} f(t)\|_{L^p} + \|\nabla^\gamma (G *_{(t,x)} f(t))\|_{B^p_{\infty}}\right) \leq \|f\|_{\gamma,a} \int_0^1 \|G(s)\|_{L^1_t} ds \lesssim \epsilon, \delta, M_{\epsilon+1} \|f\|_{\gamma,a}.
\]

To prove the theorem \(3.8\), we need to have the following Lemmas.

\[\text{Lemma 3.9} \quad \text{Let } t \geq 1, \text{ and } p = 1, \infty, \text{ we have}
\]

\[
\langle t \rangle^{\frac{d(p-1)}{p}} \|G *_{(t,x)} f(t)\|_{L^p} \lesssim \epsilon, M_0 \sup_{s \in (0, \infty)} \left(\|f(s)\|_{L^1_t} + \|s\|^\gamma \langle s \rangle^{\frac{d(p-1)}{p}} \|f(s)\|_{B^p_{\infty}}\right).
\]

\[\text{Proof.} \quad \text{We use the idea in } [11]. \text{ Since we have the cancellation shown in the } (3.26), \text{ we rewrite lines as follows when changing of variable } z = x - y \text{ in the second term,}
\]

\[
\|G *_{(t,x)} f\|_{L^p} \leq \left|\int_0^{t/2} \int_{\mathbb{R}^d} G(t-s,x-y)f(s,y)dyds\right|_{L^p_t}
\]

\[
+ \left|\int_{t/2}^{t} \int_{\mathbb{R}^d} G(t-s,z)(f(s,x) - f(s,x + z))dzds\right|_{L^p_t} := I_1 + I_2.
\]

First, we consider the term \(I_1\),

\[
I_1 \lesssim \epsilon, M_0 \int_0^{t/2} \frac{1}{\langle t-s \rangle^{1+\frac{d(p-1)}{p}} \|f(s)\|_{L^1_t} ds} \lesssim \epsilon, M_0 \tilde{M}(t)^{-\frac{d(p-1)}{p}} \sup_{s \in (0, \infty)} \|f(s)\|_{L^1_t}.
\]

\[\text{(3.27)}\]

Similarly, we have

\[
I_2 = \left|\int_{t/2}^{t} \int_{\mathbb{R}^d} G(t-s,z)|z|^a \frac{f(s,x) - f(s,x + z)}{|z|^a} dzds\right|_{L^p_t}
\]

\[
\leq \int_{t/2}^{t} \int_{\mathbb{R}^d} |G(t-s,z)| |z|^a \left\|\frac{f(s,x) - f(s,x + z)}{|z|^a}\right\|_{L^p_t} dzds
\]

\[
\leq \int_{t/2}^{t} \sup_z \frac{\|\partial_z f(s,x)\|_{L^p_t}}{|z|^a} \|G(t-s,z)|z|^a\|_{L^3_t} dzds.
\]

\[\text{(3.28)}\]
Using (3.21) in Theorem 3.6, we have,

$$I_2 \lesssim \tilde{c}_x \tilde{M} \int_{t/2}^{t} \|f(s)\|_{B_{p,\infty}^{a}} \frac{1}{(t-s)^{1-a}} ds$$

$$\lesssim \tilde{c}_x \tilde{M} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{\frac{d(p-1)}{p} + a} \|f(s)\|_{B_{p,\infty}^{a}} \right] \langle t \rangle^{-\frac{d(p-1)}{p} - a} \int_{t/2}^{t} \frac{1}{(t-s)^{1-a}} ds$$

Combining (3.21) and (3.23), we obtain that

$$\langle t \rangle^{\frac{d(p-1)}{p}} \|G \ast (t,x) f(t)\|_{L^p} \lesssim \tilde{c}_x \tilde{M} \sup_{s \in [0, \infty)} \left( \|f(s)\|_{L^1} + \langle s \rangle^{\frac{d(p-1)}{p} + a} \|\nabla \gamma f(s)\|_{B_{p,\infty}^{a}} \right).$$

This implies the result.

**Lemma 3.10** Let $t \geq 1$, and $p = 1, \infty$, we have estimates as follows,

$$\langle t \rangle^{\alpha} \langle t \rangle^{\frac{d(p-1)}{p}} \|\nabla \gamma (G \ast (t,x) f(t))\|_{B_{p,\infty}^{a}} \lesssim \tilde{c}_x \tilde{M} \sup_{s \in [0, \infty)} \left( \|f(s)\|_{L^1} + \langle s \rangle^{\frac{d(p-1)}{p} + \gamma} \|\nabla \gamma f(s)\|_{B_{p,\infty}^{a}} \right).$$

**Proof.** When referring to $\|G \ast (t,x) f(t)\|_{L^p}$ at first. Set

$$u_1(t,x) = \int_{0}^{t/2} \int_{\mathbb{R}^d} G(t-s,y)f(s,x-y)dyds,$$

$$u_2(t,x) = \int_{t/2}^{t} \int_{\mathbb{R}^d} G(t-s,y)f(s,x-y)dyds.$$

Thus,

$$\|\delta_a \nabla \gamma (G \ast (t,x) f(t))\|_{L^p} \lesssim \|\delta_a \nabla \gamma u_1(t,x)\|_{L^p} + \|\delta_a \nabla \gamma u_2(t,x)\|_{L^p}.$$

Firstly, we focus on the second term. Thanks to (3.21) and (3.24), we have for $|\alpha| > t/2 \geq \frac{1}{2}$

$$\frac{\|\delta_a \nabla \gamma u_2(t,x)\|_{L^p}}{|\alpha|^a} \lesssim \frac{2}{|\alpha|^a} \|\nabla \gamma u_2(t,x)\|_{L^p} = \frac{2}{|\alpha|^a} \left\| \int_{t/2}^{t} \int_{\mathbb{R}^d} G(t-s,y)\nabla \gamma f(s,x-y)dyds \right\|_{L^p}$$

$$\lesssim \frac{2}{|\alpha|^a} \int_{t/2}^{t} \|y|^a G(t-s,y)\|_{L^1} \sup_y \left( \|\delta_a \nabla \gamma f(s)\|_{L^p} \right) ds$$

$$\lesssim \frac{1}{|\alpha|^a} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{\frac{d(p-1)}{p} + \gamma} \|\nabla \gamma f(s)\|_{B_{p,\infty}^{a}} \right] \langle t \rangle^{-\frac{d(p-1)}{p} - a - \gamma} \int_{t/2}^{t} \frac{M}{(t-s)^{1-a}} ds$$

$$\lesssim \frac{\tilde{c}_x \tilde{M}(t)}{|\alpha|^a} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{\frac{d(p-1)}{p} + \gamma} \|\nabla \gamma f(s)\|_{B_{p,\infty}^{a}} \right] \langle t \rangle^{-\frac{d(p-1)}{p} - a - \gamma} \sup_{s \in [0, \infty)} \left( \langle s \rangle^{\frac{d(p-1)}{p} + \gamma} \|\nabla \gamma f\|_{B_{p,\infty}^{a}} \right).$$

For the case $|\alpha| \leq t/2$, we know that

$$\frac{\|\delta_a \nabla \gamma u_2(t,x)\|_{L^p}}{|\alpha|^a} \lesssim \frac{1}{|\alpha|^a} \left\| \int_{t/2}^{t} \int_{\mathbb{R}^d} 1_{|t-s| \leq |\alpha|} G(t-s,y)\nabla \delta_a f(s,x)dyds \right\|_{L^p}$$

$$+ \frac{1}{|\alpha|^a} \left\| \int_{t/2}^{t} \int_{\mathbb{R}^d} 1_{|t-s| > |\alpha|} G(t-s,y)\nabla \delta_a f(s,x)dyds \right\|_{L^p} =: \mathcal{J}_1 + \mathcal{J}_2.$$

For the term $\mathcal{J}_1$, one has

$$\mathcal{J}_1 \leq \frac{1}{|\alpha|^a} \left\| \int_{t/2}^{t} \int_{|t-s| \leq |\alpha|} G(t-s,y)\delta_a \nabla \gamma f(s,x)dyds \right\|_{L^p}$$

$$\leq \frac{1}{|\alpha|^a} \int_{t/2}^{t} \int_{|t-s| \leq |\alpha|} \|y|^a G(t-s,y)\|_{L^1} \sup_y \|\delta_a \nabla \gamma f(s)\|_{L^p} dyds$$

$$\lesssim \frac{1}{|\alpha|^a} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{\frac{d(p-1)}{p} + \gamma} \|\nabla \gamma f(s)\|_{B_{p,\infty}^{a}} \right] \langle t \rangle^{-\frac{d(p-1)}{p} - a - \gamma} \int_{0}^{t/2} \|y|^a G(s,y)\|_{L^p} 1_{|s| \leq |\alpha|} ds.$$
Since we have
\[
\int_0^{t/2} \| y |^\alpha G(s, y) \|_{L^p_y} \mathbf{1}_{s \leq t} \, ds \lesssim \frac{M}{\alpha} \int_0^{[\alpha]} \frac{1}{s^{1-\alpha}} \, ds \leq M [\alpha]^{\alpha},
\]
then
\[
\mathcal{J}_1 \lesssim \tilde{c}, M_0 \left( \frac{d(p-1)}{p} - a - \gamma \right) \sup_{s \in [0, \infty)} \left( \langle s \rangle^{d(p-1) + a + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right).
\]
(3.29)

Now we deal with the term \( \mathcal{J}_2 \),
\[
\mathcal{J}_2 \leq \frac{1}{|\alpha|^a} \int_{t/2}^t 1_{t-s < |\alpha|} \left( \int_{\mathbb{R}^d} |y|^\alpha |\delta_\alpha G(t-s, y)| \, dy \right) \sup_y \| \tilde{\delta}_y \nabla^\gamma f(s, x) \|_{L^p_y} \, ds
\]
\[
\leq \frac{1}{|\alpha|^a} (t)^{-a - \frac{d(p-1)}{p} - \gamma} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right] \int_{t/2}^t \| y |^\alpha \|_{L^p_y} \, ds
\]
\[
\lesssim \tilde{c}, M_0 \left( t \right)^{-a - \frac{d(p-1)}{p} - \gamma} \left[ \sup_{s \in [0, \infty)} \langle s \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right] \int_{t/2}^t \left( \frac{1}{1 + s} \right)^{1-a} \, ds
\]
\[
\lesssim \tilde{c}, M_0 \left( t \right)^{-a - \frac{d(p-1)}{p} - \gamma} \sup_{s \in [0, \infty)} \left( \langle s \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right).
\]
Combining this with (3.29) to conclude that
\[
\| \tilde{\delta}_\alpha u_2(t, x) \|_{L^p_y} \lesssim \tilde{c}, M_0 \left( t \right)^{-a - \frac{d(p-1)}{p} - \gamma} \sup_{s \in [0, \infty)} \left( \langle s \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right).
\]

Now we estimate \( u_1(t, x) \). One
\[
\| \tilde{\delta}_\alpha \nabla^\gamma u_1(t, x) \|_{L^p_y} = \left\| \int_0^{t/2} \int_{\mathbb{R}^d} \tilde{\delta}_\alpha \nabla^\gamma G(t-s, x-y) f(s, y) \, dy \, ds \right\|_{L^p_y}
\]
\[
\leq \int_{t/2}^t \| \tilde{\delta}_\alpha \nabla^\gamma G(s) \|_{L^p_y} \sup_{s \in [0, \infty)} \| f(s) \|_{L^1} \, ds.
\]
Note that for any \( s \geq 1 \), we have
\[
\| \tilde{\delta}_\alpha \nabla^\gamma G(s) \|_{L^p_y} \leq \min\{ \| \nabla^\gamma G(s) \|_{L^p_y}, |\alpha| \| \nabla^\gamma+1 G(s) \|_{L^p_y} \}
\]
\[
\lesssim \tilde{c}, M_{r+1} \min\left\{ \frac{1}{\langle s \rangle^{1 + \frac{d(p-1)}{p} + \gamma}}, \frac{|\alpha|}{\langle s \rangle^{2 + \frac{d(p-1)}{p} + \gamma}} \right\} \lesssim \tilde{c}, M_{r+1} \frac{|\alpha|^a}{\langle s \rangle^{1 + a + \frac{d(p-1)}{p} + \gamma}}.
\]
Then
\[
\int_{t/2}^t \| \tilde{\delta}_\alpha \nabla^\gamma G(s) \|_{L^p_y} \, ds \lesssim \tilde{c}, M_{r+1} \frac{|\alpha|^a}{\langle t \rangle^{1 + a + \frac{d(p-1)}{p} + \gamma}} \lesssim \tilde{c}, M_{r+1} \left| t \right|^{-a - \frac{d(p-1)}{p} - \gamma}.
\]
Thus we obtain
\[
\| \tilde{\delta}_\alpha \nabla^\gamma u_2(t, x) \|_{L^p_y} \lesssim \tilde{c}, M_{r+1} \left| t \right|^{-a - \frac{d(p-1)}{p} - \gamma} \sup_{s \in [0, \infty)} \| f(s) \|_{L^1}.
\]
Therefore, we deduce
\[
\| \langle t \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| G s(t, x) f(t) \|_{B_{p, \infty}} \lesssim \tilde{c}, M_{r+1} \sup_{s \in [0, \infty)} \left( \| f(s) \|_{L^1} + \langle s \rangle^{a + \frac{d(p-1)}{p} + \gamma} \| \nabla^\gamma f(s) \|_{B_{p, \infty}} \right).
\]
This gives the result.

4 Estimates for Characteristics

In this section, we study the characteristics \((X_s, t(x, v), V_s, t(x, v))\) defined in (2.1). First, we have for any \( 0 \leq s \leq t < \infty \),
\[
X_s(t, x) = x - v(t-s) + \int_s^t (\tau-s) E(\tau, X_{\tau', t}(x, v)) \, d\tau,
\]
\[
V_s(t, x) = v - \int_s^t E(\tau, X_{\tau', t}(x, v)) \, d\tau.
\]
(4.1)
Set
\[ Y_{s,t}(x,v) = \int_s^t (\tau - s)E(\tau, x + \tau v + Y_{\tau,t}(x,v))d\tau, \] (4.2)
\[ W_{s,t}(x,v) = -\int_s^t E(\tau, x + \tau v + Y_{\tau,t}(x,v))d\tau. \]

Hence,
\[ X_{s,t}(x,v) = x - (t-s)v + Y_{s,t}(x,v), \quad V_{s,t}(x,v) = v + W_{s,t}(x-vt,v). \] (4.3)

We obtain the bounds on characteristics as follows.

**Proposition 4.1** Let \( a \in (\frac{\sqrt{5}-1}{2}, 1) \), then there exists \( \tilde{\varepsilon}_0 \in (0, 1) \) such that for any \( \|(\rho, U)\|_{S_{a^0}^0} \leq \varepsilon_0 \leq \tilde{\varepsilon}_0 \) and \( 0 < \delta < a \), we have the following estimates:
\[ \sup_{0 \leq s \leq t} \langle s \rangle^{d-2+\alpha} \left( \|Y_{s,t}\|_{L^\infty_{x,v}} + \|\nabla_x Y_{s,t}\|_{L^\infty_{x,v}} + \sup_{\alpha} \frac{\|\delta_\alpha \nabla_x Y_{s,t}\|_{L^\infty_{x,v}}}{|\alpha|^\alpha} \right) \]
\[ \quad + \sup_{0 \leq s \leq t} \langle s \rangle^{d-3+\alpha} \|\nabla_t Y_{s,t}\|_{L^\infty_{x,v}} + \sup_{\alpha} \frac{\|\delta_\alpha \nabla_t Y_{s,t}\|_{L^\infty_{x,v}}}{|\alpha|^\alpha} \right) \leq_a \|(\rho, U)\|_{S_{a^0}^0}. \] (4.4)

And
\[ \sup_{0 \leq s \leq t} \langle s \rangle^{d-1+\alpha} \left( \|W_{s,t}\|_{L^\infty_{x,v}} + \|\nabla_x W_{s,t}\|_{L^\infty_{x,v}} + \sup_{\alpha} \frac{\|\delta_\alpha \nabla_x W_{s,t}\|_{L^\infty_{x,v}}}{|\alpha|^\alpha} \right) \]
\[ \quad + \sup_{0 \leq s \leq t} \langle s \rangle^{d-2+\alpha} \|\nabla_t W_{s,t}(x,v)\|_{L^\infty_{x,v}} + \sup_{\alpha} \frac{\|\delta_\alpha \nabla_t W_{s,t}(x,v)\|_{L^\infty_{x,v}}}{|\alpha|^\alpha} \right) \leq_a \|(\rho, U)\|_{S_{a^0}^0}. \] (4.5)

Here we use the notations
\[ \delta_\alpha^p Y_{s,t}(x,v) = Y_{s,t}(x,v) - Y_{s,t}(x - \alpha, v), \quad \delta_\alpha^p Y_{s,t}(x,v) = Y_{s,t}(x,v) - Y_{s,t}(x,v - \alpha). \]

Before we prove this Proposition, we state the following Lemma:

**Lemma 4.2** For \( p = 1, \infty \) and \( \kappa, \kappa_1 \in (0, 1) \), one has
\[ \sum_{j=0}^1 \|\nabla^j (1-\Delta)^{-1} \psi\|_{L^p} + \|\nabla (1-\Delta)^{-1} \psi\|_{B^{\kappa}_{p,\infty}} \leq c_0 \|\psi\|_{L^p}, \]
\[ \sum_{j=1}^2 \|\nabla^j (1-\Delta)^{-1} \psi\|_{L^p} + [2 + \kappa - (\kappa_1 + j)] \sum_{j=0,1,2 \atop \kappa_1 + j < 2 + \kappa} \|\nabla^j (1-\Delta)^{-1} \psi\|_{B^{\kappa_1}_{p,\infty} \cap F_{p,\infty}^{\kappa_1}} \leq c_0 \|\psi\|_{B^{\kappa}_{p,\infty}}, \]
\[ \|\nabla^2 (1-\Delta)^{-1} \psi\|_{B^{\kappa_1}_{p,\infty}} \leq c_0 \|\psi\|_{B^{\kappa_1}_{p,\infty}}. \]

for some \( c_0 = c_0(\kappa) \).

**Proof of Proposition 4.1** Note that \( \|(\rho, U)\|_{S_{a^0}^0} \leq \varepsilon_0 \) implies \( \|U\|_{a} \leq \varepsilon_0^\frac{\kappa}{\kappa_1} \), then by Assumption 3, we have that
\[ \|A(U)\|_{B^\kappa_{p,\infty}} \leq \max_{|\tau| \leq \varepsilon_0^\frac{\kappa}{\kappa_1}} \|A'(\tau)\|_{B^\kappa_{p,\infty}} \leq C_A \varepsilon_0^\frac{\kappa}{\kappa_1} \|U\|_{B^\kappa_{p,\infty}}. \] (4.6)
\[ \|A(U)\|_{L^p} \leq C_A \|U\|_{L^p} \leq C_A \varepsilon_0^\frac{\kappa}{\kappa_1} \|U\|_{L^p}. \]

Thus taking \( \varepsilon_0 \leq C_A^{-\kappa} \), we get
\[ \|\rho + A(U)\|_{a} \leq \|\rho\|_{a} + C_A \varepsilon_0^\frac{\kappa}{\kappa_1} \|U\|_{a} \leq \|\rho\|_{a} + \varepsilon_0^\frac{\kappa}{\kappa_1} \|U\|_{a} = \|(\rho, U)\|_{S_{a^0}^0}. \] (4.7)

Thanks to Lemma 4.2,
\[ \|E(\tau)\|_{L^\infty} + \|\nabla_x E(\tau)\|_{L^\infty} + \|\nabla_x E(\tau)\|_{B^{\kappa}_{2,\infty}} \leq c_0(\alpha) \|\rho(\tau) + A(U)(\tau)\|_{B^{\kappa}_{2,\infty}} \leq c_0(\alpha) \|(\rho, U)\|_{S_{a^0}^0}, \] (4.8)
and there exists a constant \( c_1 > 0 \) such that
\[
\int_s^t \frac{1}{(\tau - s)^{d-1+a}} \, d\tau \leq \frac{c_1}{\langle \tau \rangle^{d-1+a}}, \quad i = 0, 1, 2,
\] (4.9)

where \( c_1 \) only depends on \( d \).

(1) By (4.8) one has

\[
\|Y_s,t\|_{L^\infty_{x,v}} \leq \int_s^t (t-s) \|E(\tau, X_{\tau, \tau}(x,v))\|_{L^\infty_{x,v}} \, d\tau \leq \int_s^t \frac{c_0 \| (\rho, U) \|_{S^0}}{\langle \tau \rangle^{d-1+a}} \, d\tau \leq c_0 c_1 \frac{\| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-2+a}}.
\]

\[
\|\nabla Y_s,t\|_{L^\infty_{x,v}} \leq (1 + \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2+a} \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}) \int_s^t (t-s) \|\nabla E(\tau)\|_{L^\infty_{x,v}} \, d\tau
\]
\[
\leq c_0 \| (\rho, U) \|_{S^0} \left(1 + \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2+a} \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right) \int_s^t \frac{1}{\langle \tau \rangle^{d-1+a}} \, d\tau
\]
\[
\leq \frac{c_0 c_1 \| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-2+a}} \left(1 + \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2+a} \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right).
\]

Then, the right hand side can be absorbed by the left hand side provided that \( \| (\rho, U) \|_{S^0} \leq \frac{1}{400 c_0 c_1} \).

By the same method, we have

\[
\|\nabla Y_s,t\|_{L^\infty_{x,v}} \leq \int_s^t (t-s) \|\nabla E(\tau)\|_{L^\infty_{x,v}} \left(|\tau| + \|\nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right) \, d\tau
\]
\[
\leq c_0 c_1 \| (\rho, U) \|_{S^0} \left(\frac{1}{\langle \tau \rangle^{d-1+a}} + \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-3+a} \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right) \int_s^t \frac{1}{\langle \tau \rangle^{d-1+a}} \, d\tau
\]

Hence,

\[
\sup_{0 \leq s \leq t} \langle s \rangle^{d-2+a} \left(\|Y_s,t\|_{L^\infty_{x,v}} + \|\nabla Y_s,t\|_{L^\infty_{x,v}}\right) + \sup_{0 \leq s \leq t} \langle s \rangle^{d-3+a} \| \nabla Y_s,t\|_{L^\infty_{x,v}} \leq 2 c_0 c_1 \| (\rho, U) \|_{S^0},
\] (4.10)

provided \( \| (\rho, U) \|_{S^0} \leq \frac{1}{400 c_0 c_1} \).

(2) By (4.8), (4.10), one obtains

\[
\sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_s,t\|_{L^\infty_{x,v}}}{|\alpha|^a} \leq \sup_{\alpha} \frac{1}{|\alpha|^a} \left[ \int_s^t (t-s) \|\nabla E(\tau)\|_{L^\infty_{x,v}} \| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}} \, d\tau \right.
\]
\[
+ \int_s^t (t-s) \|\nabla E(\tau)\|_{B_1, \infty} \left(|\alpha| + \| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right)^a \left(1 + \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right) \, d\tau \right]
\]
\[
\leq \int_s^t (t-s) \|\nabla E(\tau)\|_{L^\infty_{x,v}} \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2+a} \sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}}{|\alpha|^a} \, d\tau
\]
\[
+ \int_s^t (t-s) \|\nabla E(\tau)\|_{B_1, \infty} \left(1 + \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right)^{1+a} \, d\tau
\]
\[
\leq c_0 c_1 \frac{\| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-2+a}} \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2+a} \sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}}{|\alpha|^a} + c_0 c_1 \frac{\| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-2+a}}.
\]

Similarly,

\[
\sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_s,t\|_{L^\infty_{x,v}}}{|\alpha|^{a-\delta}} \leq \sup_{\alpha} \frac{1}{|\alpha|^{a-\delta}} \left[ \int_s^t (t-s) \|\nabla E(\tau)\|_{L^\infty_{x,v}} \| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}} \, d\tau \right.
\]
\[
+ \int_s^t (t-s) \|\nabla E(\tau)\|_{B_1, \infty} \left(|\alpha|^{1-\delta} + \| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right)^{a-\delta} \left(|\tau| + \| \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}\right) \, d\tau \right)
\]
(4.11)
\[
\leq c_0 c_1 \frac{\| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-2+a-\delta}} \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-3+a-\delta} \sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_{\tau,t}\|_{L^\infty_{x,v}}}{|\alpha|^{a-\delta}} + c_0 c_1 \frac{\| (\rho, U) \|_{S^0}}{\langle s \rangle^{d-3+a-\delta}}.
\]

These imply

\[
\sup_{0 \leq s \leq t} \langle s \rangle^{d-2+a} \sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_s,t\|_{L^\infty_{x,v}}}{|\alpha|^a} + \delta \langle s \rangle^{d-3+a} \sup_{\alpha} \frac{\| \delta^\alpha \nabla Y_s,t\|_{L^\infty_{x,v}}}{|\alpha|^{a-\delta}} \lesssim_{\delta} \| (\rho, U) \|_{S^0},
\] (4.12)

provided \( \| (\rho, U) \|_{S^0} \leq \frac{1}{400 c_0 c_1} \).

Combining this with (4.10), one gets (4.3). By the same argument, we also obtain (4.5). Thus we choose \( \delta_0 = \min \{ C_A, \frac{1}{400 c_0 c_1} \} \) to finish the proof.
Remark 4.3 For $d \geq 4$, using the same method in (4.11), we obtain

$$\sup_{0 \leq s \leq t} \langle s \rangle^{d-3} \sup_{\alpha} \frac{\|\delta_\alpha^{\nu} \nabla_v Y_{s,t}\|_{L^\infty}}{|\alpha|^s} \lesssim_{\rho} \|(\rho, U)\|_{S^0_v}.$$  \hfill (4.13)

Proposition 4.4 Let $\alpha \in (1, 1/2, 1)$, $\varepsilon_0$ be as in Proposition 4.1. Assume that $\|(\rho, U)\|_{S^0_v} \leq \varepsilon_0$ and $\|(\rho, U)\|_{\gamma, b} \leq \varepsilon_0$ for any $\gamma > 0$ and $b \in (0,1)$. Then, for $j = 1, \ldots, \gamma$ and $\delta \in (0, b)$, we have

$$\sum_{j=0}^{\gamma} \sup_{0 \leq s \leq t} \langle s \rangle^{d-2-b} \|\nabla_v Y_{s,t}\|_{L^\infty} + \langle s \rangle^{d-1-b} \|\nabla_v W_{s,t}\|_{L^\infty} \lesssim_{\varepsilon_0} 1,$$

$$\sum_{j=0}^{\gamma} \sup_{0 \leq s \leq t} \langle s \rangle^{d-3-\delta} \|\nabla_v Y_{s,t}\|_{L^\infty} + \langle s \rangle^{d-2} \|\nabla_v W_{s,t}\|_{L^\infty} \lesssim_{\varepsilon_0} 1,$$

$$\sum_{j=0}^{\gamma} \sup_{0 \leq s \leq t} \langle s \rangle^{d-1-b} \|\nabla_v Y_{s,t}\|_{L^\infty} + \langle s \rangle^{d-1+b} \|\nabla_v W_{s,t}\|_{L^\infty} \lesssim_{\varepsilon_0} 1,$$

provided that

$$\sum_{j=1}^{\gamma+1} \sup_{|r| \leq 1} |A^{(j)}(r)| \leq C_{A, \gamma}.$$  \hfill (4.14)

Proof. Note that for $0 \leq j \leq \gamma$ and $\|(\rho, U)\|_{S^0_v} \leq \varepsilon_0$, we have

$$\|\nabla^j (A(U))\|_{B^0_{p, \infty}} \leq \sum_{k=1}^{j} |A^{(k)}(U)|_{L^\infty} \sum_{m_1, \ldots, m_k \geq 1} \sum_{m_1 + \ldots + m_k = j} \frac{\|\prod_{i=1}^{k} \nabla^{m_i} U\|_{B^0_{p, \infty}}}{\prod_{i=1}^{k} \nabla^{m_i} U}_{L^\infty} + \sum_{k=1}^{j} |A^{(k)}(U)|_{B^0_{p, \infty}} \sum_{m_1, \ldots, m_k \geq 1} \sum_{m_1 + \ldots + m_k = j} \frac{\|\prod_{i=1}^{k} \nabla^{m_i} U\|_{L^\infty}}{\prod_{i=1}^{k} \nabla^{m_i} U}_{B^0_{p, \infty}} \lesssim_{c, \gamma} 1.$$  \hfill (4.15)

Combining this with Lemma 4.2, there exists a constant $c_3 = c_3(d, b, b', C_{A, \gamma}, \varepsilon_0)$ such that

$$\sup_{\tau \geq 0} \sum_{j=1}^{\gamma+1} \langle \tau \rangle^{d+\min(j+1, b'+1, \gamma+b)} \|\nabla^j E(\tau)\|_{L^\infty} \lesssim_{b, b'} \rho + A(U) \|_{\gamma, b} \leq c_3, \quad \text{for any } b' \in (0, b).$$  \hfill (4.16)

(1) Let $c_1$ be in (4.10). As the proof of (4.10), we have

$$\|\nabla_v Y_{s,t}\|_{L^\infty_{s,t}} \leq \int_{s}^{t} |\tau - s| \|\nabla_v E(\tau)\|_{L^\infty} |\tau| d\tau + \int_{s}^{t} |\tau - s| \|\nabla_v E(\tau)\|_{L^\infty} \|\nabla_v Y_{\tau,t}\|_{L^\infty_{\tau,t}} |\tau| d\tau,$$

$$\lesssim_{\varepsilon_0} \int_{s}^{t} \frac{\langle \tau \rangle^2 |\tau|^2}{\langle \tau \rangle^{d+b+1}} d\tau + c_0 \frac{\langle s \rangle^{d+\min(j+1, b'+1, \gamma+b)} \|\nabla^j Y_{s,t}\|_{L^\infty_{s,t}}}{\langle \tau \rangle^{d+b+1}} \lesssim_{c, \gamma} 1.$$  \hfill (4.17)

Since $\|(\rho, U)\|_{S^0_v} \leq \frac{1}{400 \varepsilon_0 c_{\gamma} d^2}$, then we have

$$\sup_{0 \leq s \leq t} \langle s \rangle^{d-2+b} \|\nabla_v Y_{s,t}\|_{L^\infty_{s,t}} \lesssim_{c, \gamma} 1.$$  \hfill (4.18)

Similarly, we obtain

$$\sup_{0 \leq s \leq t} \langle s \rangle^{d-1+b} \|\nabla_v W_{s,t}\|_{L^\infty_{s,t}} \lesssim_{c, \gamma} 1.$$  \hfill (4.19)

(2) As (4.10), it is easy to check that

$$\sum_{j=0}^{\gamma+1} \|\nabla^j_v (Y_{s,t}, W_{s,t})\|_{L^\infty_{s,t}} + \sum_{j=0}^{\gamma} \sup_{\alpha} \frac{\|\delta_\alpha^{\nu} \nabla_v (Y_{s,t}, W_{s,t})\|_{L^\infty_{s,t}}}{|\alpha|^b} \lesssim_{c, \gamma} 1.$$  \hfill (4.20)
for any $0 \leq s \leq t < \infty$.

By [41], one has

\[
\sup_{\alpha} \frac{\|\delta^{\alpha}_{\nu} \nabla^{k+1} Y_{s,t}\|_{L^\infty}}{|\alpha|^{b-d}} \leq \int_s^t \frac{(\tau-s)\|\nabla E(\tau)\|_{L^\infty}}{|\alpha|^{b-d}} \sup_{\alpha} \left\| \delta^{\alpha}_{\nu} \nabla^{k+1} Y_{\nu,\tau,t} \right\|_{L^\infty} d\tau + \int_s^t \frac{(\tau-s)\|\nabla E(\tau)\|_{B^{b-d}_{\infty,\infty}}}{|\tau| + \|\nabla Y_{\nu,\tau,t}\|_{L^\infty}} \left\| \nabla^{k+1} Y_{\nu,\tau,t} \right\|_{L^\infty} d\tau \\
+ C(\epsilon) \sum_{j=1}^{k} \int_s^t (\tau-s)\|\nabla^{j+1} E(\tau)\|_{B^{b-d}_{\infty,\infty}} (|\tau| + 1)^{j+1} d\tau \\
+ C(\epsilon) \sum_{j=1}^{k} \int_s^t (\tau-s)\|\nabla^{j+1} E(\tau)\|_{L^\infty} (|\tau| + 1)^{j+1} d\tau
\]

One derives that

\[
\omega(s) \sup_{\alpha} \frac{\|\delta^{\alpha}_{\nu} \nabla^{k+1} Y_{\nu,\tau,t}\|_{L^\infty}}{|\alpha|^{b-d}} \leq \frac{1}{4} B(t) + C(\epsilon, b, C_{A,\gamma}),
\]

where

\[
B(t) = \sup_{0 \leq \tau \leq t} \omega(\tau) \sup_{\alpha} \frac{\|\delta^{\alpha}_{\nu} \nabla^{k+1} Y_{\nu,\tau,t}\|_{L^\infty}}{|\alpha|^{b-d}}, \quad \omega(s) = \begin{cases} \langle s \rangle^{d-2} & \text{if } k \leq \gamma - 1, \\ \delta(s)^{d-3+\delta} & \text{if } k = \gamma. \end{cases}
\]

This implies

\[
B(t) \lesssim_{\epsilon, b, C_{A,\gamma}} 1.
\]

In particular,

\[
\sum_{k=0}^{\gamma-1} \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-2} \sup_{\alpha} \frac{\|\delta^{\alpha}_{\nu} \nabla^{k+1} Y_{\nu,\tau,t}\|_{L^\infty}}{|\alpha|^{b-d}} + \delta \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{d-3+\delta} \sup_{\alpha} \frac{\|\delta^{\alpha}_{\nu} \nabla^{\gamma+1} Y_{\nu,\tau,t}\|_{L^\infty}}{|\alpha|^{b-d}} \lesssim_{\epsilon, b, C_{A,\gamma}} 1.
\]

Similarly, one also has

\[
\sup_{\alpha} \frac{\|\delta^{\alpha}_{\alpha} \nabla^{k+1} Y_{\tau,t}\|_{L^\infty}}{|\alpha|^{b}} \leq \int_s^t \frac{(\tau-s)\|\nabla E(\tau)\|_{L^\infty}}{|\alpha|^{b}} \sup_{\alpha} \left\| \delta^{\alpha}_{\alpha} \nabla^{k+1} Y_{\tau,t}\right\|_{L^\infty} d\tau \\
+ \int_s^t \frac{(\tau-s)\|\nabla E(\tau)\|_{B^{b-d}_{\infty,\infty}}}{|\tau| + \|\nabla Y_{\tau,t}\|_{L^\infty}} \left\| \nabla^{k+1} Y_{\tau,t} \right\|_{L^\infty} d\tau \\
+ C(\epsilon) \sum_{j=1}^{k} \int_s^t (\tau-s)\|\nabla^{j+1} E(\tau)\|_{B^{b-d}_{\infty,\infty}} + \|\nabla^{j+1} E(\tau)\|_{L^\infty} d\tau \\
\lesssim \alpha_0 \int_s^t \frac{\|\langle \nu, U \rangle \|_{S^0}}{(\tau) \langle \tau \rangle^{d-1+\alpha}} \sup_{\alpha} \frac{\|\delta^{\alpha}_{\alpha} \nabla^{k+1} Y_{\tau,t}\|_{L^\infty}}{|\alpha|^{b}} + C(\epsilon, b, C_{A,\gamma}) \frac{1}{(\tau)^{d-1+\alpha}}.
\]

This gives

\[
\sup_{0 \leq s \leq t} \langle s \rangle^{d-1+b} \sup_{\alpha} \frac{\|\delta^{\alpha}_{\alpha} \nabla^{k+1} Y_{s,t}\|_{L^\infty}}{|\alpha|^{b}} \lesssim_{\epsilon, b, C_{A,\gamma}} 1.
\]

By the same argument, we obtain

\[
\sup_{0 \leq s \leq t} \langle s \rangle^{d-3} \sup_{\alpha} \frac{\|\delta^{\alpha}_{\alpha} \nabla^{\gamma+1} Y_{s,t}\|_{L^\infty}}{|\alpha|^{b}} \lesssim_{\epsilon, b, C_{A,\gamma}} 1.
\]
Proposition 4.6 Let $a \in (\sqrt{2}-1,1)$, $\varepsilon_0$ be as in Proposition $[\text{23}]$, $p_1, p_2$ be such that $\|(\rho_1, U_1)\|_{S^{n_0}}|, \|(\rho_2, U_2)\|_{S^{n_0}}| \leq \varepsilon_0 \leq \varepsilon_0'$. Let $(X_{1,t}^i, V_{1,t}^{i+1})$, $(X_{2,t}^i, V_{2,t}^{i+1})$ be solutions to (4.21) associated to $E^1(s, x) = \nabla_x (1 - \Delta)^{-1}(p_1 + A(U_1))(x)$ and $E^2(s, x) = \nabla_x (1 - \Delta)^{-1}(p_2 + A(U_2))(x)$. Let $(Y_{1,t}^i, W_{1,t}^{i+1})$, $(Y_{2,t}^i, W_{2,t}^{i+1})$ be such that

$$X_{i,t}^i(x, v) = x - (t - s)v + X_{i,t}^{i+1}(x - v, t, v), V_{i,t}^i(x, v) = v + W_{i,t}^{i+1}(x - v, t, v), \quad i = 1, 2. \tag{4.20}$$

Then for $Y_{s,t}^i = Y_{s,t}^{i+1}$, $W_{s,t}^i = W_{s,t}^{i+1}$, we have

$$\sup_{0 \leq s \leq t} (s)^{-2+\alpha} (\|Y_{s,t}^i\|_{L^\tau_{\infty}} + \sup_{\alpha} \frac{\|\delta^{\alpha}_0 Y_{s,t}^i\|_{L^\tau_{\infty}}}{|\alpha|^a}) \leq \alpha \|(\rho_1 - \rho_2, U_1 - U_2)\|_{S^{n_0}}, \tag{4.21}$$

$$\sup_{0 \leq s \leq t} (s)^{-1+\alpha} (\|W_{s,t}^i\|_{L^\tau_{\infty}} + \sup_{\alpha} \frac{\|\delta^{\alpha}_0 W_{s,t}^i\|_{L^\tau_{\infty}}}{|\alpha|^a}) \leq \alpha \|(\rho_1 - \rho_2, U_1 - U_2)\|_{S^{n_0}}. \tag{4.22}$$

**Proof.** Let $c_0, c_1$ be the constant mentioned in the proof of Theorem $[\text{13}]$. We have

$$Y_{s,t}^k(x, v) = \int_s^t (\tau - s)E^k(\tau, x + \tau v + Y_{\tau,t}^k(x, v))d\tau, \quad W_{s,t}^k(x, v) = -\int_s^t E^k(\tau, x + \tau v + Y_{\tau,t}^k(x, v))d\tau, \quad k = 1, 2.$$  

From $\|(\rho_k, U_k)\|_{S^{n_0}}| \leq \varepsilon_0$, one has $\|U_k\|_a \leq \varepsilon_0^{\frac{d}{a}}$, for $k = 1, 2$. Combining this with Assumption $[\text{23}]$, we obtain that

$$\|A(U_1) - A(U_2)\|_{\dot{B}^{a}_{\infty, \infty}} \leq \|U_1 - U_2\|_{L^\infty} \int_0^1 1 A'(\alpha U_1 + (1 - \alpha U_2))d\alpha \|_{\dot{B}^{a}_{\infty, \infty}},$$

$$\leq \|U_1 - U_2\|_{L^\infty} \int_0^1 A'(\alpha U_1 + (1 - \alpha U_2))d\alpha \|_{\dot{B}^{a}_{\infty, \infty}},$$

$$+ \|U_1 - U_2\|_{\dot{B}^{a}_{\infty, \infty}} \int_0^1 A'(\alpha U_1 + (1 - \alpha U_2))d\alpha \|_{L^\infty},$$

$$\leq \|U_1 - U_2\|_{L^\infty} \left( \max_{|r| \leq \varepsilon_0^{\frac{d}{a}}} |A'(r)| \right)^{1-a} \left( \varepsilon_0^{\frac{d}{a}} \max_{|r| \leq \varepsilon_0^{\frac{d}{a}}} |A'(r)| \right)^{a} + \|U_1 - U_2\|_{\dot{B}^{a}_{\infty, \infty}} \max_{|r| \leq \varepsilon_0^{\frac{d}{a}}} |A'(r)|,$$

$$\leq C_A \varepsilon_0^{\frac{d}{a}} \left( \|U_1 - U_2\|_{L^\infty} + \|U_1 - U_2\|_{\dot{B}^{a}_{\infty, \infty}} \right).$$

Therefore,

$$\|(E^1 - E^2)(\tau)\|_{L^\infty} + \|\nabla_x (E^1 - E^2)(\tau)\|_{L^\infty} + \|\nabla_x (E^1 - E^2)(\tau)\|_{B^{a}_{\infty, \infty}} \leq c_0(\alpha) \|(A_1) - (A_2)\|_{\dot{B}^{a}_{\infty, \infty}} + \|A_1 - A_2\|_{\dot{B}^{a}_{\infty, \infty}} \leq c_0(\alpha) \left( \|(\rho_1 - \rho_2, U_1 - U_2)\|_{\dot{B}^{a}_{\infty, \infty}} \right) \leq c_0(\alpha) \left( \|(\rho_1 - \rho_2, U_1 - U_2)\|_{S^{n_0}} \right).$$

These imply

$$Y_{s,t}(x, v) = \int_s^t (\tau - s)(E^1 - E^2)(\tau, x + \tau v + Y_{\tau,t}(x, v))d\tau$$

$$+ \int_s^t (\tau - s)(E^2(\tau, x + \tau v + Y_{\tau,t}(x, v)) - E^2(\tau, x + \tau v + Y_{\tau,t}(x, v)))d\tau.$$

Hence we know that

$$\|Y_{s,t}\|_{L^\tau_{\infty}} \leq \int_s^t (\tau - s)(E^1 - E^2)(\tau, x, x + \tau v + Y_{\tau,t}(x, v))d\tau$$

$$\leq C_0 \int_s^t \frac{\|U_1 - U_2\|_{S^{n_0}}}{(\tau)^{d+1+\alpha}}d\tau + C_0 \int_s^t \frac{\|U_1 - U_2\|_{S^{n_0}}}{(\tau)^{d-1+\alpha}}d\tau \|Y_{\tau,t}\|_{L^\tau_{\infty}}d\tau,$$

$$\leq C_0 \left( \frac{\|U_1 - U_2\|_{S^{n_0}}}{(s)^{d+1+\alpha}} + \frac{\|U_1 - U_2\|_{S^{n_0}}}{(s)^{d-1+\alpha}} \right) \sup_{0 \leq \tau \leq t} \left( \frac{(s)^{d+1+\alpha}}{\tau} \right)^{d+1+\alpha} \|Y_{\tau,t}\|_{L^\tau_{\infty}}.$$
Since \( \| (\rho_2, U_2) \|_{S^0_a} \leq \frac{1}{400\alpha_0 c_1} \),
\[
\sup_{0 \leq s \leq t} (s)^{d-2+a} \| Y_{s,t} \|_{L^\infty_{x,v}} \leq 2c_0c_1 \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^0_a} \leq \alpha \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^0_a}.
\]

For the next terms, we have
\[
\sup_{\alpha} \frac{\| \delta_\alpha Y_{s,t} \|_{L^\infty_{x,v}}}{|\alpha|^a} \leq \int_s^t (\tau - s) \| (E^1 - E^2)(\tau) \|_{B^2_{2,\infty}} d\tau + \int_s^t (\tau - s) \| \nabla E^2(\tau) \|_{B^2_{2,\infty}} \| Y_{\tau,t} \|_{L^\infty_{x,v}} d\tau
\]
\[
+ \int_s^t (\tau - s) \| \nabla E^2(\tau) \|_{L^\infty} \sup_{\alpha} \frac{\| \delta_\alpha Y_{\tau,t} \|_{L^\infty_{x,v}}}{|\alpha|^a} d\tau
\]
\[
\leq c_0c_1 \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^0_a} + 2c_0c_1 \| (\rho_2, U_2) \|_{S^0_a} \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^0_a}
\]
\[
+ c_0c_1 \| (\rho_2, U_2) \|_{S^0_a} \sup_{0 \leq \tau \leq t} (\tau)^{d-2+a} \sup_{\alpha} \frac{\| \delta_\alpha Y_{\tau,t} \|_{L^\infty_{x,v}}}{|\alpha|^a} \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^0_a}.
\]

Therefore, we get \((4.21)\) provided \( \| (\rho_2, U_2) \|_{S^0_a} \leq \frac{1}{400\alpha_0 c_1} \). Similarly, we also get \((4.22)\). The proof is complete. 

**Proposition 4.7** Let \( a \in (\frac{d}{d+1}, 1), \alpha_0 \) be as in Proposition 4.1. Let \( (\rho, U) \) be such that \( \| (\rho, U) \|_{S^0_a} \leq \varepsilon_0 \). Then, for any \( 0 \leq s \leq t < \infty \), we have a \( C^1 \) map \( (x, v) \mapsto \Psi_{s,t}(x, v) \) satisfying for all \( x, v \in \mathbb{R}^d \):
\[
X_{s,t}(x,\Psi_{s,t}(x, v)) = x - (t-s)v,
\]
and
\[
(s)^{d-1+a} \| (\Psi_{s,t}(x, v) - v) + [\nabla_x \Psi_{s,t}(x, v)] \| + (s)^{d-2+a} \| \nabla_v (\Psi_{s,t}(x, v) - v) \| \leq \alpha \| (\rho, U) \|_{S^0_a}.
\]

**Proof.** Define
\[
\Phi_{s,t}(x, v) = -\frac{1}{t-s} \int_s^t (\tau - s) E(\tau, x - (t-\tau)v + Y_{\tau,t}(x - vt, v)) d\tau.
\]
Hence, it follows from \((4.11)\) that
\[
X_{s,t}(x, v) = x - v(t-s) + Y_{s,t}(x - vt, v) = x - (t-s)(v + \Phi_{s,t}(x, v)).
\]

For the first estimate, since \( \| \nabla_x Y_{s,t} \|_{L^\infty_{x,v}} \leq 1 \), so
\[
\| \Phi_{s,t} \|_{L^\infty_{x,v}} \leq \int_s^t \| E(\tau) \|_{L^\infty_x} d\tau \leq \frac{1}{(s)^{d+a-1}} \| (\rho, U) \|_{S^0_a}, \tag{4.27}
\]
and
\[
\| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} \leq \frac{1}{t-s} \int_s^t (\tau - s) \| \nabla_x E(\tau) \|_{L^\infty} (1 + \| \nabla_x Y_{\tau,t} \|_{L^\infty_{x,v}}) d\tau \leq \frac{1}{(s)^{d+a-1}} \| (\rho, U) \|_{S^0_a}, \tag{4.28}
\]
we have
\[ \sup_{0 \leq s,t \leq T} \langle s \rangle^{d+a-1} \left( \| \Phi_{s,t} \|_{L^\infty_{x,v}} + \| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} \right) \lesssim_a \| (\rho, U) \|_{S^{\alpha}_{a}}. \] (4.29)

Now one has
\[ \| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} \leq \frac{1}{1-s} \int_s^t (\tau-s) \| \nabla_x E(\tau) \|_{L^\infty} \left( t \tau + (t-\tau) + (t-\tau) + (t-\tau) + 1 \right) d\tau \]
\[ \leq \frac{c_0 \langle (\rho, U) \rangle}{s} \| \nabla_x \Phi_{s,t} \|_{S^{\alpha}_{a}} \int_s^t \frac{1}{\langle \tau \rangle^{d+a}} d\tau, \]
\[ \leq 4c_0 \langle (\rho, U) \rangle \| \nabla_x \Phi_{s,t} \|_{S^{\alpha}_{a}}. \]

Let $c_0, c_1$ be in (4.3) and (4.9) by (4.3) and (4.8). One obtains that
\[ \| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} \leq \frac{4c_0c_1 \langle (\rho, U) \rangle}{\langle s \rangle^{d+a-1}} \leq \frac{1}{100d^2}, \]
provided $\| (\rho, U) \|_{S^{\alpha}_{a}} \leq \frac{1}{200a c_1 d^2}$, and it is clear that under the assumption $\| (\rho, U) \|_{S^{\alpha}_{a}} \leq \frac{1}{200a c_1 d^2}$, the map $(x,v) \mapsto (x,v + \Phi_{s,t}(x,v))$ is a $C^1$-diffeomorphism. Thus, there exists a map $(x,v) \mapsto \Psi_{s,t}(x,v)$ satisfying (4.21). Moreover, combining (4.21) with (4.20), we have $\Psi_{s,t}(x,v) + \Phi_{s,t}(x,v, \Psi_{s,t}(x,v)) = v$.

Now we have
\[ |\Psi_{s,t}(x,v) - v| \leq \| \Phi_{s,t} \|_{L^\infty_{x,v}} \lesssim_a \langle (\rho, U) \rangle \| \nabla_x \Phi_{s,t} \|_{S^{\alpha}_{a}} \]
\[ |\nabla_x \Psi_{s,t}(x,v) - \nabla_x \Phi_{s,t} | \leq \| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} + \| \nabla_x \Psi_{s,t} | \nabla_x \Phi_{s,t} |_{L^\infty_{x,v}} \]
\[ |\nabla_v (\Psi_{s,t}(x,v) - v) - \nabla_v (\Phi_{s,t}(x,v) - v)| \leq \| \nabla_v \Phi_{s,t} \|_{L^\infty_{x,v}} + \| \nabla_v \Psi_{s,t} | \nabla_v (\Psi_{s,t}(x,v) - v) \| (\nabla_v (\Psi_{s,t}(x,v) - v) \| (\nabla_v (\Psi_{s,t}(x,v) - v) \| 1). \]

Then, the right hand side can be absorbed by the left. Hence, we get
\[ \| \nabla_x \Psi_{s,t}(x,v) \|_{L^\infty_{x,v}} \leq 2\| \nabla_x \Phi_{s,t} \|_{L^\infty_{x,v}} \lesssim_a \| (\rho, U) \|_{S^{\alpha}_{a}} \]
\[ |\nabla_v (\Psi_{s,t}(x,v) - v)| \leq 3\| \nabla_v \Phi_{s,t} \|_{L^\infty_{x,v}} \lesssim_a \| (\rho, U) \|_{S^{\alpha}_{a}}. \]
These follow (4.25).

\begin{remark}
For $\alpha_1 \in (0,1)$, if we also have $\| (\rho, U) \|_{S^{\alpha_{1}}_{a}} \lesssim 1$,
\[ \langle s \rangle^{d-2+a_1} |\nabla_v (\Psi_{s,t}(x,v) - v)| \lesssim_a \| (\rho, U) \|_{S^{\alpha_1}_{a}}. \] (4.30)
\end{remark}

5 Contribution of the initial data

For $h = h(x,v) : \mathbb{R}^d_x \times \mathbb{R}_v^d \to \mathbb{R}$, define
\[ \mathcal{I}_h(\rho, U)(t,x,v) = \int_{\mathbb{R}^d} h(X_{0,t}(x,v), V_{0,t}(x,v))dv. \] (5.1)

\begin{proposition}
Let $\alpha \in \left( \frac{\sqrt{2} - 1}{2}, 1 \right)$, $\tilde{\varepsilon}_0$ be as in Proposition [7.1]. Let $(\rho, U)$ be such that $\| (\rho, U) \|_{S^{\alpha}_{a}} \leq \varepsilon_0 \leq \tilde{\varepsilon}_0$. Then there holds
\[ \| \mathcal{I}_h(\rho, U) \|_{S^{\alpha}_{a}} \lesssim_a \sum_{p=1}^{\infty} \| D^p(h) \|_{L^1 L^\infty_{x,v} L^1 L^\infty_{v} L^1}, \] (5.2)
for any $\sigma \in [a,1)$. Moreover, for $0 < \delta_0 < 2 - \sqrt{3}$, if $\| (\rho, U) \|_{L^{1-\delta_0}} \leq \mathbf{c}$ for some $l \geq 0$, then for any $b \in (0,1 - \delta_0)$, we have
\[ \| \mathcal{I}_h(\rho, U) \|_{L^{l+1} L^\infty_{x,v} L^1 L^\infty_{v}} \lesssim_{\mathbf{c}, a, b, C, A} \frac{1}{1 - \delta_0 - b} \sum_{i=1}^{l+1} \| D^p(\nabla_i h) \|_{L^1 L^\infty_{x,v} L^1 L^\infty_{v}}, \] (5.3)
provided that $\sum_{j=2}^{l+1} \sup_{|r| \leq 1} \| A^{(i)}(r) \| \leq C_{A,l}$. 

22
Remark 5.2 By (5.13), for $d \geq 4$, we can obtain better result as follows

$$
\left\| I_h(\rho, U) \right\|_{L^{t+1} \cap L^{1} \cap L^{2} \cap L^{p}} \lesssim_{e,b,c,A,t} \sum_{i=1}^{t+1} \left\| D_{x,v}^{1-\delta_0} (\nabla_{x,v} h) \right\|_{L^{1} \cap L^{2}}.
$$

(5.4)

Remark 5.3 We can not only use $\|h\|_{L^{1} \cap L^{2} \cap L^{p}}$ to control $\|I_h(\rho, U)(t)\|_{L^{\infty}}$ for $0 \leq t \leq 1$. For example we take

$$
h(x,v) = \frac{1}{|x|^\theta (|x|^2 + |v|^2 + 1)^d}, \quad X_0(t,x,v) = x - tv, \quad V_0(t,x,v) = v, \quad U = 0,
$$
for $0 < \theta < d$. And it is easy to check that for $t \in [0,1]$, we have

$$
\|h\|_{L^{1} \cap L^{2} \cap L^{p}} < \infty, \quad \|h\|_{L^{1} \cap L^{2}} = \infty, \quad \text{and} \quad \|I_h(\rho, U)(t)\|_{L^{\infty}} \sim \frac{1}{t^q}.
$$

Proof. (1) Firstly, we deal with $\|I_h(\rho, U)(t,x)\|_{L^{p}}$. Note that $(x,v) \mapsto (X_0(t,x,v),V_0(t,x,v))$ corresponding to the transport matrix

$$
\mathfrak{A}(t,x,v) := \begin{bmatrix}
I_d + \nabla_x Y_0(t,x-vt,v) & -t I_d + \nabla_v Y_0(t,x-vt,v) \\
\nabla_x W_0(t,x-vt,v) & I_d + \nabla_v W_0(t,x-vt,v)
\end{bmatrix}.
$$

Since we have

$$
\|\nabla_x Y_0(t,x-vt,v)\|_{L^{\infty}} + \|\nabla_x W_0(t,x-vt,v)\|_{L^{\infty}} \leq 2e_0 \epsilon_1 \|(\rho, U)\|_{S^{0}} \leq 2e_0 \epsilon_1 \epsilon_0 \leq \frac{1}{400d^2},
$$
and $\|\det \mathfrak{A}^{-1}(t,x,v)\|_{L^{\infty}} \lesssim 1$. Then we directly obtain that

$$
\left\| I_h(\rho, U)(t,x) \right\|_{L^{p}} = \left\| \int_{\mathbb{R}^d} h(x-vt+Y_0(t,x-vt,v),v+W_0(t,x-vt,v))dv \right\|_{L^{p}}
$$

(5.5)

Thus

$$
\|\nabla_x X(t,x,v)\|_{L^{\infty}} + \|\nabla_x V(t,x,v)\|_{L^{\infty}} \lesssim_{a} 1,
$$

$$
\sup_{\alpha \leq t} \frac{1}{|\alpha|} \left\| \frac{\delta_{x} X(t,x,v)}{\alpha} \right\|_{L^{p \infty}} \lesssim_{a} 1,
$$

(5.6)

(5.7)

For $t \geq 1$, change of variable $w = x - vt$ to get more decay in time:

$$
I_h(\rho, U)(t,x) = \int_{\mathbb{R}^d} h \left( w + Y_0(t,x - vt), \frac{x - w}{t} \right) \left( w, \frac{x - w}{t} \right) \, dw.
$$

Thus

$$
\left\| I_h(\rho, U)(t,x) \right\|_{L^{p}} = \left\| \int_{\mathbb{R}^d} h(w, V_0(t,x,\Psi(t,x,w)) \delta_{\alpha}((\nabla_v \Psi)(t,x,\frac{x-w}{t})) \frac{dw}{|\alpha|} \right\|_{L^{p \infty}}
$$

(5.8)

Combine (5.5) with (5.6) and (5.7), we have

$$
(t) \frac{d^{1/2}}{dt^{1/2}} \left\| I_h(\rho, U)(t) \right\|_{L^{p}} \lesssim_{a} \|h\|_{L^{1} \cap L^{2} \cap L^{p}}, \quad p = 1, \infty.
$$

(5.9)

(2) Now we deal with the term $\|I_h(\rho, U)(t)\|_{L^{p \infty}}$. From Proposition 4.1 we have

$$
\|\nabla_x X(t,x,v)\|_{L^{p \infty}} \leq_1 1,
$$

$$
\sup_{\alpha \leq t} \frac{1}{|\alpha|} \left\| \frac{\delta_{x} Y_0(t,x,v)}{\alpha} \right\|_{L^{p \infty}} \leq_1 1,
$$

$$
\sup_{\alpha \leq t} \frac{1}{|\alpha|} \left\| \frac{\delta_{x} W_0(t,x,v)}{\alpha} \right\|_{L^{p \infty}} \leq_1 1.
$$
Thus, we get
\[
\frac{|\delta_t \mathcal{I}_h(p, U)(t, x)|}{|\alpha|^\mu} \lesssim_a \sum_{z = x, x - \alpha} \int_{\mathbb{R}^d} \mathcal{D}^\sigma(h)(X_{0,t}(z, v), V_{0,t}(z, v))dv,
\]
and
\[
\frac{|\delta_t \mathcal{I}_h(p, U)(t, x)|}{|\alpha|^\mu} \lesssim_a \int_{\mathbb{R}^d} \mathcal{D}^\sigma_1(h) \left( w + Y_{0,t} \left( \frac{w, x - w}{t} \right), \frac{x - w}{t} + W_{0,t} \left( \frac{w, x - w}{t} \right) \right) \frac{dw}{t^{d+\sigma}} + \int_{\mathbb{R}^d} \mathcal{D}^\sigma_2(h) \left( w + Y_{0,t} \left( \frac{w, x - \alpha - w}{t} \right), \frac{x - \alpha - w}{t} + W_{0,t} \left( \frac{w, x - \alpha - w}{t} \right) \right) \frac{dw}{t^{d+\sigma}}.
\]
Thus, we get
\[
\lesssim_a \frac{1}{t^\delta} \sum_{z = v, v - \frac{\alpha}{t}} \int_{\mathbb{R}^d} \mathcal{D}^\sigma(h)(X_{0,t}(z, v), V_{0,t}(z, v))dv.
\]
Thanks to (5.3) with \( h = \mathcal{D}^\sigma(h) \), we get
\[
\langle t \rangle^{d+\frac{d-1}{p}} \| \mathcal{I}_h(p, U)(t) \|_{B_{p,\infty}} \lesssim_a \| \mathcal{D}^\sigma(h) \|_{L^1_tL^\infty_vL^p_v}, \quad p = 1, \infty. \tag{5.9}
\]
Until now we finish the proof of (5.2).

(3) Finally, we consider the term \( \| \mathcal{I}_h(p, U)(t) \|_{t^{1+b}} \). Thanks to Proposition 4.3, one has
\[
\sum_{k = 1}^{l+1} \| \mathcal{D}^\sigma(Y, U)(t) \|_{L^\infty_v} \lesssim_{e, b, C, A, t} \sum_{j = 1}^{l} \| \mathcal{D}^\sigma_j \|_{L^1_vL^p_v}. \tag{5.10}
\]
On the other hand, changing variable as \( w = x - vt \) and using the estimation
\[
\sum_{k = 1}^{l+1} (1 - \delta_0 - b) \| \mathcal{D}^\sigma(Y, U)(t) \|_{L^\infty_v} \lesssim_{e, b, C, A, t} \sum_{j = 1}^{l} \| \mathcal{D}^\sigma_j \|_{L^1_vL^p_v},
\]
we have
\[
\lesssim_{e, b} \frac{1}{1 - \delta_0 - b} \sum_{j = 1}^{l+1} \int_{\mathbb{R}^d} \mathcal{D}^\sigma \mathcal{D}^\sigma_j(X_{0,t}(z, v), V_{0,t}(z, v)) \left( w + Y_{0,t} \left( \frac{w, x - w}{t} \right), \frac{x - w}{t} + W_{0,t} \left( \frac{w, x - w}{t} \right) \right) dw.
\]
Thus,
\[
\| \mathcal{D}^\sigma(Y, U)(t) \|_{L^\infty_v} \lesssim_{e, b, C, A, t} \sum_{j = 1}^{l} \| \mathcal{D}^\sigma_j \|_{L^1_vL^p_v}. \tag{5.11}
\]
Therefore, we obtain (5.3). 

\[\Box\]

**Proposition 5.4** Let \( a \in (\frac{\gamma - 1}{2}, 1), \quad \varepsilon_0 \) be as in Proposition 4.2. Suppose that \( \| (p_0, U_0) \|_{S^a_0}, \| (p_2, U_2) \|_{S^a_0} \leq \varepsilon_0 \leq \varepsilon_0 \). There holds,
\[
\| \mathcal{I}_h(p_1, U_1) - \mathcal{I}_h(p_2, U_2) \|_a \lesssim_a \| (p_1 - p_2, U_1 - U_2) \|_{S^a_0} \sum_{p = 1}^\infty \| \mathcal{D}^\sigma(\mathcal{D}^\sigma_j) \|_{L^1_tL^\infty_vL^p_v}. \tag{5.12}
\]
Thus Combining (5.15) with (5.16), we have after this we go back to the original variable to obtain

$$Thus Combining (5.15) with (5.16), we have after this we go back to the original variable to obtain

$$X_{s,t}(x,v, v) = (1 - \varpi)X_{s,t}(x,v) + \varpi X_{s,t}(x,v), \quad V_{s,t}(x,v, v) = (1 - \varpi)Y_{s,t}(x,v) + \varpi Y_{s,t}(x,v), \quad Y_{s,t}(x,v, v) = (1 - \varpi)Y_{s,t}(x,v) + \varpi Y_{s,t}(x,v), \quad W_{s,t}(x,v, v) = (1 - \varpi)W_{s,t}(x,v) + \varpi W_{s,t}(x,v).$$

(1) For the term $\|\mathcal{I}_h(p_1, U_1)(t) - \mathcal{I}_h(p_2, U_2)(t)\|_{L^p_v}$, we have

$$\|\mathcal{I}_h(p_1, U_1)(t) - \mathcal{I}_h(p_2, U_2)(t, v)\| \lesssim \|\rho - (\rho_1 - \rho_2, U_1 - U_2)\|_{S^{\alpha}} \sum_{p=1}^{\infty} \|\nabla_x h\|_{L^p_1 L^2_1 \cap L^p_1 L^2_1}.$$  

As the proof of (5.8), one obtains

$$\sum_{p=1}^{\infty} \langle t \rangle^{\alpha - 1} \|\mathcal{I}_h(p_1, U_1)(t) - \mathcal{I}_h(p_2, U_2)(t)\|_{L^p_v} \lesssim \|\rho - (\rho_1 - \rho_2, U_1 - U_2)\|_{S^{\alpha}} \sum_{p=1}^{\infty} \|\nabla_x h\|_{L^p_1 L^2_1 \cap L^p_1 L^2_1}.$$  

(2) Now we consider the term $\|\mathcal{I}_h(p_1, U_1)(t) - \mathcal{I}_h(p_2, U_2)(t)\|_{B^\infty_{p, p}}$. We compute directly to get

$$\|\delta_\alpha (\mathcal{I}_h(p_1, U_1) - \mathcal{I}_h(p_2, U_2))(t, x)\|
\leq \left(\sup_z \frac{\|\delta_\alpha Y_{0,t}\|_{L^\infty_{x,v}} + \|\delta_\alpha W_{0,t}\|_{L^\infty_{x,v}}}{|z|^{\alpha}}\right) \int_{\mathbb{R}^d} \int_0^1 |\nabla_x h(X_{0,t}(x,v, v), V_{0,t}(x,v, v))| \, dz \, dv
+ \left(\frac{\|Y_{0,t}\|_{L^\infty_{x,v}} + \|W_{0,t}\|_{L^\infty_{x,v}}}{|z|^{\alpha}}\right) \int_{\mathbb{R}^d} \int_0^1 \sum_{z=x \pm \delta} \nabla^\alpha |(\nabla_x h)(X_{0,t}(z, v, v), V_{0,t}(z, v, v))| \, dz \, dv.$$  

On the other hand, we first make the change of variable as $w = x - vt$, then taking $\delta_\alpha$ to get the decay of $t^{-\alpha}$, after this we go back to the original variable to obtain

$$\|\delta_\alpha (\mathcal{I}_h(p_1, U_1) - \mathcal{I}_h(p_2, U_2))(t, x)\|
\leq \frac{1}{t^\alpha} \left(\sup_z \frac{\|\delta_\alpha Y_{0,t}\|_{L^\infty_{x,v}} + \|\delta_\alpha W_{0,t}\|_{L^\infty_{x,v}}}{|z|^{\alpha}}\right) \int_{\mathbb{R}^d} \int_0^1 |\nabla_x h(X_{0,t}(x,v, v), V_{0,t}(x,v, v))| \, dz \, dv
+ \frac{1}{t^\alpha} \left(\frac{\|Y_{0,t}\|_{L^\infty_{x,v}} + \|W_{0,t}\|_{L^\infty_{x,v}}}{|z|^{\alpha}}\right) \int_{\mathbb{R}^d} \int_0^1 \sum_{z=x \pm \delta} \nabla^\alpha |(\nabla_x h)(X_{0,t}(z, v, v), V_{0,t}(z, v, v))| \, dz \, dv.$$  

Combining (5.15) with (5.16), we have

$$\|\delta_\alpha (\mathcal{I}_h(p_1, U_1) - \mathcal{I}_h(p_2, U_2))(t, x)\|
\lesssim \|\rho - (\rho_1 - \rho_2, U_1 - U_2)\|_{S^{\alpha}} \sum_{p=1}^{\infty} \|D^\alpha (\nabla_x h)(X_{0,t}(x,v, v), V_{0,t}(x,v, v))\|_{L^p_1 L^2_1}.$$  

Now the proof is complete.
6 Contribution Of The Reaction Term

In this section, we focus on estimate of the reaction term:

$$
\mathcal{R}(\rho, U)(t, x) = \int_0^t \int_{\mathbb{R}^d} E(s, x - (t - s)v) \cdot \nabla_v \mu(v) dv ds - \int_0^t \int_{\mathbb{R}^d} E(s, X_{s,t}(x, v)) \cdot \nabla_v \mu(V_{s,t}(x, v)) dv ds.
$$

(6.1)

Denote that for any given $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ and $\eta : \mathbb{R}^d \to \mathbb{R}$,

$$
\mathcal{T}[F, \eta](t, x) = -\mathcal{T}_{NL}[F, \eta](t, x) + \mathcal{T}_L[F, \eta](t, x),
$$

with

$$
\mathcal{T}_L[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, x - (t - s)v) \eta(v) dv ds,
$$

$$
\mathcal{T}_{NL}[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, X_{s,t}(x, v)) \eta(V_{s,t}(x, v)) dv ds.
$$

Hence, it is clear that

$$
\mathcal{R}(\rho) = \sum_{i=1}^d \mathcal{T}[E_i, \partial_{x_i}\mu].
$$

By changing variable, we reformulate as follows:

$$
\mathcal{T}_L[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, w + \frac{s}{t}(x - w)) \eta(\frac{x - w}{t}) \frac{dv ds}{t^d},
$$

$$
\mathcal{T}_{NL}[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, X_{s,t}(w, \frac{x - w}{t})) + \frac{s(x - w)}{t} \eta \left( W_{s,t}(w, \frac{x - w}{t}) + \frac{x - w}{t} \right) \frac{dv ds}{t^d}.
$$

Then, we have following Lemmas.

**Lemma 6.1** Let $a \in (\frac{N-1}{d}, 1)$ and $\tilde{\varepsilon}_0 > 0$ be as in Proposition 4.1. Assume that $\|(\rho, U)\|_{a, u_0^\infty} \leq \tilde{\varepsilon}_0$. Let $(X_{s,t}, V_{s,t})$ be solution to (2.1) associated to $E(s, x) = \nabla_x (1 - \Delta)^{-\gamma}(\rho(s, x))$. Let $Y_{s,t}, W_{s,t}$ be such that

$$
X_{s,t}(x, v) = x - (t - s)v + Y_{s,t}(x - v, t, v), V_{s,t}(x, v) = v + W_{s,t}(x - v, t, v).
$$

(6.2)

Suppose $\eta$ be such that

$$
\sum_{j=0}^2 \langle v \rangle^N |\nabla j \eta(v)| \leq c_1,
$$

(6.3)

for $N > d$. Then,

(1)

$$
|\mathcal{T}[F, \eta](t, x)| \lesssim a, c_1 \|(\rho, U)\|_{a, u_0^\infty} \int_0^t \int_{\mathbb{R}^d} |F(s, x - (t - s)v)| \frac{1}{(s)^{d-2+a}} \frac{dv ds}{(\langle v \rangle^N)^{2}}.
$$

(6.4)

In particular, for $p = 1, \infty$

$$
(t) \frac{d(p-1)}{p} \|\mathcal{T}[F, \eta](t)|_{L^p} \lesssim a, c_1 \|(\rho, U)\|_{a, u_0^\infty} \left( \sup_{s \in [0, t]} \langle s \rangle \frac{d(p-1)}{p} \|F(s)\|_{L^p} + \sup_{s \in [0, t]} \|F(s)\|_{L^1} \right).
$$

(6.5)

(2) For any $\sigma \in [a, 1)$

$$
\sum_{p=1, \infty} \langle t \rangle \frac{d(p-1)}{p} \|\mathcal{T}[F, \eta](t)|_{B_p^{\infty}} \lesssim a, c_1 \|(\rho, U)\|_{a, u_0^\infty} \sum_{p=1, \infty} \left( \sup_{s \in [0, t]} \langle s \rangle \frac{d(p-1)}{p} \|F(s)\|_{L^p} + \sup_{s \in [0, t]} \langle s \rangle \frac{d(p-1)}{p} \|F(s)\|_{F_p^{\infty}} \right).
$$

(6.6)

**Proof.** (1) First of all, we change of variable and obtain

$$
|\mathcal{T}[F, \eta](t, x)| = \left| -\int_0^t \int_{\mathbb{R}^d} F(s, x - (t - s)v) \eta(V_{s,t}(x, \Psi_{s,t}(x, v))) |\det(\nabla_v \Psi_{s,t}(x, v))| dv ds 
+ \int_0^t \int_{\mathbb{R}^d} F(s, x - (t - s)v) \eta(v) dv ds \right|
$$

$$
\leq \int_0^t \int_{\mathbb{R}^d} |F(s, x - (t - s)v)| |\eta(V_{s,t}(x, \Psi_{s,t}(x, v))| - |\eta(v)| dv ds 
+ \int_0^t \int_{\mathbb{R}^d} |F(s, x - (t - s)v)| |\det(\nabla_v \Psi_{s,t}(x, v))| - 1 |\eta(V_{s,t}(x, \Psi_{s,t}(x, v)))| dv ds.
$$
Hence, since we have

\[ |\eta(V_{s,t}(x, \Psi_{s,t}(x, v))) - \eta(v)| \leq \int_0^t |\nabla \eta(\mathbf{w} V_{s,t}(x, v) + (1 - \mathbf{w}) v) V_{s,t}(x, v) - v| \, d\mathbf{w} \]

\[ \lesssim c_1 \sup_{\mathbf{w} \in [0,1]} |\langle \nabla \mathbf{w} V_{s,t}(x, v) \rangle V_{s,t}(x, v) - v| \]

\[ \sim c_1 |W_{s,t}(x - vt, v)| \lesssim c_1 \left( \frac{\|\rho, U\|_{S_\infty^0}}{\langle s \rangle^{d-1+a}} \right)^N , \]

and \( |\eta(V_{s,t}(x, \Psi_{s,t}(t, x)))| \lesssim c_1 (W_{s,t}(x - \mathbf{w} t \Psi_{s,t}(t, x), \Psi_{s,t}(t, x)) + v)^{-N} \lesssim c_1 \langle v \rangle^{-N} \), where we use the fact that \( \sup_{0 \leq s \leq t} \|W_{s,t}\|_{L^\infty} \leq 1 \) in (4.5). Hence

\[ |T[F, \eta](t, x)| \lesssim c_1 \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \left( \frac{\|\rho, U\|_{S_\infty^0}}{\langle s \rangle^{d-1+a}} + \frac{\|\rho, U\|_{S_\infty^0}}{\langle s \rangle^{d-2+a}} \right) \, dv \, ds \]

\[ \lesssim c_1 \|\rho, U\|_{S_\infty^0} \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{1}{\langle s \rangle^{d-2+a}} \, dv \, ds . \]

This gives (6.4). Moreover,

\[ \|T[F, \eta](t)\|_{L^p} \lesssim c_1 \|\rho, U\|_{S_\infty^0} \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{dv}{\langle s \rangle^{N}} \, ds . \]

Then, we apply Lemma 8.1 with \( \mathcal{H}_1 = F \) and \( \mathcal{H}_2 = 0 \) to obtain

\[ \|T[F, \eta](t)\|_{L^p} \lesssim c_1 \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} + \|\rho, U\|_{S_\infty^0} \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} \]

\[ \lesssim c_1 \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} \]

\[ \lesssim c_1 \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} \]

\[ \lesssim c_1 \int_0^t \int_{\mathbb{R}^d} \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} \]

Combining this with

\[ \|T[F, \eta](t)\|_{L^p} \lesssim c_1 \|\rho, U\|_{S_\infty^0} \int_0^t \left| F(s, x - (t - s)v) \right| \frac{ds}{\langle s \rangle^{d-2+a}} \lesssim c_1 \|\rho, U\|_{S_\infty^0} \sup_{s \in [0, t]} \|F(s)\|_{L^p} , \]

we obtain (6.5).

(2) Now we estimate \( \|T[F, \eta](t)\|_{B_{p, \infty}^s} \). For \( 0 \leq t < 1 \).

We divide \( \delta_\alpha T[F, \eta](t, x) \) into three terms:

\[ \delta_\alpha T[F, \eta](t, x) = T^1_\alpha[F, \eta](t, x) + T^2_\alpha[F, \eta](t, x) + T^3_\alpha[F, \eta](t, x) , \]

where

\[ T^1_\alpha[F, \eta](t, x) = - \int_0^t \int_{\mathbb{R}^d} \delta_\alpha F(s, \cdot) (X_{s,t}(x, v)) \eta (V_{s,t}(x, v)) \, dv \, ds + \int_0^t \int_{\mathbb{R}^d} \delta_\alpha F(s, \cdot) (x - (t - s)v) \eta(v) \, dv \, ds , \]

\[ T^2_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} |F(s, X_{s,t}(x - \alpha, v)) - F(s, X_{s,t}(x, v) - \alpha)| \eta (V_{s,t}(x, v)) \, dv \, ds , \]

\[ T^3_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F(s, X_{s,t}(x - \alpha, v)) \eta (V_{s,t}(x - \alpha, v)) - \eta (V_{s,t}(x, v)) \, dv \, ds . \]
For the first term, using (6.3) and (6.4) we have

\[
\|T_\alpha[F,\eta](t)\|_{L^p} = \|T[\delta_\alpha F,\eta](t)\|_{L^p} \lesssim_{\alpha,\epsilon_1} \|\delta_\alpha F(s)\|_{L^p} + \sup_{s \in [0,t]} \|\delta_\alpha F(s)\|_{L^p}
\]

\[
\lesssim_{\alpha,\epsilon_1} |\alpha|^\sigma \|\delta_\alpha F(s)\|_{L^p} + \sup_{s \in [0,t]} \|F(s)\|_{L^p}.
\]

Since

\[
|X_{s,t}(x - \alpha, v) - (X_{s,t}(x - \alpha, v) - \alpha)| = |Y_{s,t}(x - tv, v) - Y_{s,t}(x - tv, v)| \leq |\nabla_x Y_{s,t}|_{L^\infty},
\]

we can estimate \(T_\alpha[F,\eta](t)\) as follows

\[
\|T_\alpha[F,\eta](t,x)\|_{L^p} \lesssim_{\epsilon_1} |\alpha|^\sigma \int_0^t \int_{\mathbb{R}^d} \|\delta_\alpha F(s,.)\|_{L^p} \|\nabla_x Y_{s,t}\|_{L^\infty} (\eta(w)) \frac{dw}{\rho} ds
\]

\[
\lesssim_{\alpha,\epsilon_1} |\alpha|^\sigma \|\delta_\alpha F(s,.)\|_{L^p} \|\nabla_x Y_{s,t}\|_{L^\infty} (\eta(w)) \frac{dw}{\rho} ds.
\]

where in the last inequality, we use \((d + a - 3)\sigma + \alpha \geq (a + 1)\alpha > 1\).

Note that by (1.3), one has \(\|W_{s,t}\|_{L_{x}^{\infty}} \leq 1\). Then,

\[
(\varpi V_{s,t}(x + v) + (1 - \varpi)V_{s,t}(x - \alpha, v)) = \langle v + \varpi W_{s,t}(x - tv, v) + (1 - \varpi)W_{s,t}(x - \alpha, tv, v) \rangle \sim \langle v \rangle \quad (6.7)
\]

We can estimate \(T_\alpha[F,\eta](t)\).

\[
\|T_\alpha[F,\eta](t,x)\|_{L^p} \leq \int_0^t \int_{\mathbb{R}^d} \int_0^1 \|F(s)\|_{L^p} |\nabla \eta(s)| \|\varpi V_{s,t}(x, v) + (1 - \varpi)V_{s,t}(x - \alpha, v)\| \delta_\alpha V_{s,t}(x, v) dv ds dv
\]

\[
\lesssim_{\epsilon_1} |\alpha|^\sigma \int_0^t \int_{\mathbb{R}^d} \|F(s)\|_{L^p} \frac{1}{\rho} \sup_{\alpha} \|\delta_\alpha V_{s,t}(x, v)\| dv ds
\]

\[
\lesssim_{\alpha,\epsilon_1} |\alpha|^\sigma \|\delta_\alpha F(s,.)\|_{L^p} \|\nabla_x Y_{s,t}\|_{L^\infty} \sup_{s \in [0,t]} \|F(s)\|_{L^p},
\]

where we apply (1.3).

In conclusion, for \(0 < t < 1\), we get

\[
\|T[F,\eta](t)\|_{L^p} \lesssim_{\epsilon_1} \|\rho,\eta\|_{L^\infty} \left( \sup_{s \in [0,t]} \|F(s)\|_{L^p} + \sup_{s \in [0,t]} \|F(s)\|_{L^p} \right).
\]

2.2) Estimate \(\|T[F,\eta](t)\|_{L^p} \geq 1\) for \(t \geq 1\).

Set

\[
Z_1(x) = Y_{s,t}(w, x - w + \frac{x - w}{t}), \quad Z_2(x) = W_{s,t}(w, x - w + \frac{x - w}{t}),
\]

\[
Z_3(x) = W_{s,t}(w, x - \alpha - w + \frac{x - w}{t} + \frac{w}{t}), \quad Z_4(x) = Y_{s,t}(w, x - w + \frac{s(x - \alpha - w)}{t}).
\]

We have

\[
\delta_\alpha T[F,\eta](t,x) = T_\alpha[F,\eta](t,x) + T_\alpha[F,\eta](t,x) + T_\alpha[F,\eta](t,x) + T_\alpha[F,\eta](t,x),
\]

where

\[
T_\alpha[F,\eta](t,x) = -\int_0^t \int_{\mathbb{R}^d} \delta_\alpha F(s,.) (Z_1(x)) \eta(Z_2(x)) dv ds \frac{dw}{t^d} + \int_0^t \int_{\mathbb{R}^d} \delta_\alpha F(s,.) (Z_2(x)) \eta(Z_3(x)) dv ds \frac{dw}{t^d}.
\]

\[
T_\alpha[F,\eta](t,x) = \int_0^t \int_{\mathbb{R}^d} F(s, Z_1(x)) (\delta_\alpha \eta)(Z_2(x)) dv ds - \int_0^t \int_{\mathbb{R}^d} F(s, w + \frac{s(x - w)}{t}) (\delta_\alpha \eta)(Z_2(x)) dv ds \frac{dw}{t^d}.
\]

\[
T_\alpha[F,\eta](t,x) = \int_0^t \int_{\mathbb{R}^d} F(s, Z_1(x)) - F(s, Z_4(x)) \eta(Z_2(x)) dv ds \frac{dw}{t^d}.
\]

\[
T_\alpha[F,\eta](t,x) = \int_0^t \int_{\mathbb{R}^d} F(s, Z_3(x)) (\eta(Z_2(x)) - \eta(Z_4(x))) dv ds \frac{dw}{t^d}.
\]
Applying (6.5) with \( \delta \), we reformulate as follows:

\[
T^1_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left( -\delta \frac{\partial}{\partial v} F(s, \cdot) (X_{s,t}(x,v)) \right) \eta (V_{s,t}(x,v)) + \delta \frac{\partial}{\partial v} F(s, \cdot)(x-(t-s)v) \eta (v) \, dv \, ds,
\]

\[
T^2_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left( F (s, X_{s,t}(x,v)) \right) \left( \delta \eta \right) (V_{s,t}(x,v)) - F(s, x-(t-s)v) \left( \delta \eta \right) (v) \, dv \, ds,
\]

\[
T^3_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} \left( F \left( s, X_{s,t}(x,v) - \frac{s \alpha}{t} \right) - F (s, Z_0(x)) \right) \eta (V_{s,t}(x,v)) \, dv \, ds,
\]

\[
T^4_\alpha[F, \eta](t, x) = \int_0^t \int_{\mathbb{R}^d} F (s, Z_0(x)) \left( \eta (V_{s,t}(x,v)) - \eta (Z_0(x)) \right) \, dv \, ds.
\]

Note that for \( p = 1, \infty \) and \( |\alpha| \geq t \)

\[
\left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \delta \leq 2 \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \delta \leq 4 \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \delta \leq \frac{d(p-1)}{p} \| F \|_{L^p} \left( \sup_{s \in [0,t]} \| s \|_{L^p} + \| F \|_{L^1} \right).
\]

So, it is enough to consider \( |\alpha| \leq t \).

**2.2.1)** Estimate \( T^1_\alpha[F, \eta](t, x) \).

Applying (6.5) with \( \delta \), we have

\[
\| T^1_\alpha[F, \eta](t) \|_{L^p} \leq \| \delta \|_{L^p} \sup_{s \in [0,t]} \| F(s) \|_{L^1} + \| \delta \|_{L^p} \sup_{s \in [0,t]} \| F(s) \|_{L^1} \leq \frac{d(p-1)}{p} \| \delta \|_{L^p} \| F \|_{L^p} \left( \sup_{s \in [0,t]} \| s \|_{L^p} + \| F \|_{L^1} \right),
\]

where we apply \( \sigma < 2\alpha \). Thus, we can yield

\[
\left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| T^1_\alpha[F, \eta](t) \|_{L^p} \leq \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| F(s) \|_{B^1_{\infty}} + \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| F(s) \|_{B^1_{\infty}},
\]

**2.2.2)** Estimate \( T^2_\alpha[F, \eta](t, x) \).

Note that

\[
|\delta \eta(v)| + |\nabla_v (\delta \eta(v))| \leq \min \left\{ \frac{|\alpha|}{t}, 1 \right\} \left( \frac{1}{\langle v \rangle^N} + \frac{1}{\langle v - \frac{\alpha}{t} \rangle^N} \right) \leq \frac{|\alpha|}{t} \frac{1}{\langle v \rangle^N} \frac{1}{\langle v \rangle^N},
\]

since \( |\alpha| \leq t \). Thus, we have

\[
\| T^2_\alpha[F, \eta](t) \|_{L^p} = \frac{|\alpha|}{t^\sigma} \left\| T \left[ F, \frac{t^\sigma}{|\alpha|^p} \delta \eta(v) \right] \right\|_{L^p} \leq \frac{|\alpha|}{t^\sigma} \left\| \| F(s) \|_{L^1} + \| F(s) \|_{L^1} \right\|_{L^p} \leq \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| F(s) \|_{L^p} + \| F(s) \|_{L^p},
\]

which implies

\[
\left( \frac{d(p-1)}{p} + \| F(s) \|_{L^p} + \| F(s) \|_{L^p} \right) \leq \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| F(s) \|_{L^p} + \| F(s) \|_{L^1} \leq \left( \frac{d(p-1)}{p} + \frac{\alpha}{|\alpha|^p} \right) \sup_{s \in [0,t]} \| F(s) \|_{L^p} + \| F(s) \|_{L^1} \right).
\[2.2.3\] Estimate \(|T^\alpha_a[F,\eta](t,x)|\)

\[
\lesssim_{c_1} \int_0^t \int_{\mathbb{R}^d} \left| F_s(x, x_s(x,v) - \frac{s\alpha}{t}) - F_s(x, Z_5(x)) \right| \, dv \, ds < \infty
\]

Thus applying Lemma \[\text{[3.1]}\] with \(H = \sup_x |z|^{-\sigma}\|\delta_z F(s, \cdot)|\) and \(\phi(x,v) = Y_s(t - x,t,v) - \frac{\alpha}{t} - \frac{a}{t}\) to obtain

\[\|T^\alpha_a[F,\eta](t)\|_{L^p} \lesssim_{c_1} \int_0^t \left\| (\rho, U) \right\|_{S^0_a}^\sigma \int_0^{t/2} \left\| F_s(x, v) \right\|_{L^p} \, ds \, dv < \infty\]

where in the last line, we use the fact that \(a^2 + a > 1\). Thus,

\[t^{\frac{d(p-1)}{p} + \sigma} \sup_\alpha \left\| T^\alpha_a[F,\eta](t) \right\|_{L^p} \lesssim_{c_1} \left( \sup_{s \in [0,t]} \left\| F(s) \right\|_{H^\infty} + \left( \sup_{s \in [0,t]} \left\| F_s(x,v) \right\|_{L^p} \right) \right)\]

\[2.2.4\] Estimate \(|T^\alpha_a[F,\eta](t,x)|\)

One has

\[
\|T^\alpha_a[F,\eta](t,x)\| \lesssim_{c_1} \int_0^t \int_{\mathbb{R}^d} \left| W_s(t,x-t,v) - W_s(t,x,v) - \frac{\alpha}{t} \right| \, dv \, ds < \infty
\]

Then, applying Lemma \[\text{[3.1]}\] with \(H = F\) and \(\phi(x,v) = Y_s(t - x, t, v) - \frac{\alpha}{t} - \frac{a}{t}\), one gets

\[
\|T^\alpha_a[F,\eta](t)\|_{L^p} \lesssim_{c_1} \left( \sup_{s \in [0,t]} \left\| F(s) \right\|_{H^\infty} + \left( \sup_{s \in [0,t]} \left\| F_s(x,v) \right\|_{L^p} \right) \right)
\]

Hence, we obtain

\[t^{\frac{d(p-1)}{p} + \sigma} \sup_\alpha \left\| T^\alpha_a[F,\eta](t) \right\|_{L^p} \lesssim_{c_1} \left( \sup_{s \in [0,t]} \left\| F(s) \right\|_{L^p} + \left( \sup_{s \in [0,t]} \left\| F_s(x,v) \right\|_{L^p} \right) \right)\]

Summing up, we conclude that for \(t \geq 1\),

\[
\sum_{p=1,\infty} t^{\frac{d(p-1)}{p} + \sigma} \|T[F,\eta](t)\|_{L^p} \lesssim_{c_1} \left( \sup_{s \in [0,t]} \left\| F(s) \right\|_{L^p} + \left( \sup_{s \in [0,t]} \left\| F_s(x,v) \right\|_{L^p} \right) \right)
\]

Hence, we proved the Lemma with \[\text{[3.8]}\] and \[\text{[6.13]}\].
Remark 6.2 For $\alpha_1 \in (0, 1)$, if we also have $\| (\rho, U) \|_{S_{\alpha_1}^0} \lesssim 1$,
\begin{equation}
|T^i_0[F, \eta](t, x)| \lesssim_{\alpha_1, \epsilon_1} \| (\rho, U) \|_{S_{\alpha_1}^0} \int_0^t \int_{\mathbb{R}^d} |F(s, x - (t - s)v)| \frac{1}{\langle s \rangle^{d-2+\alpha_1}} \langle v \rangle^N ds dv, \quad i = 1, 2.
\end{equation}

Proposition 6.3 Let $a$, $\sigma$ and $\rho$ satisfy the same conditions in Lemma 6.7. Then we have
\begin{equation}
\| \mathcal{R}(\rho, U) \|_{\sigma} \lesssim_{a, M^*} \| (\rho, U) \|_{S_{\alpha}^0}^{1+\sigma}.
\end{equation}

Proof. Since $\langle v \rangle^N \sum_{j=0}^2 |\nabla_j (\nabla \mu(v))| \leq M^*$, thus, by Lemma 6.1 we have
\[ \| \mathcal{R}(\rho, U) \|_{\sigma} \lesssim_{a, M^*} \| (\rho, U) \|_{S_{\alpha}^0} \sum_{p=1, \infty} \left( \| \mathcal{R}(\rho, U) \|_{P}^{d+\beta} \| \mathcal{R}(\rho, U) \|_{L_p}^{\alpha} \right) \lesssim_{a, M^*} \| (\rho, U) \|_{S_{\alpha}^0}^{1+\sigma}. \]
This implies the result.

Proposition 6.4 Let $a = (\sqrt{\pi} \frac{1}{2}, 1)$ and $\varepsilon_0 > 0$ be as in Proposition 4.11. Assume $\| (\rho_1, U_1) \|_{S_{\alpha}^0}, \| (\rho_2, U_2) \|_{S_{\alpha}^0} \leq \varepsilon_0 < \varepsilon_0$. We have
\begin{equation}
\| \mathcal{R}(\rho_1, U_1) - \mathcal{R}(\rho_2, U_2) \|_{S_{\alpha}^0} \lesssim_{a, M^*} (\| (\rho_1, U_1) \|_{S_{\alpha}^0}^{\alpha} + \| (\rho_2, U_2) \|_{S_{\alpha}^0}^{\alpha}) \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S_{\alpha}^0}^{\alpha}.
\end{equation}

Proof. In this proof, we recall the notations $X_{s,t}^1, Z_{s,t}^1, Y_{s,t}^1, V_{s,t}^1, V_{s,t}^2, Y_{s,t}^2, W_{s,t}, W_{s,t}$ in Proposition 4.9 and $S_{s,t}(x, v, w)$, $V_{s,t}(x, v, w)$, $W_{s,t}(x, v, w)$ in Proposition 4.11. By using Lemma 6.1, and $\rho = (\rho_1, U_1)$, $\sigma = (\rho_1, U_1)$, we get
\begin{equation}
\sum_{i=1}^d \| \mathcal{T}[\mathcal{T}(E^1 - E^2)]_i, \|_{S_{\alpha}} \lesssim_{a, M^*} (\| (\rho_1, U_1) \|_{S_{\alpha}^0}^{\alpha} \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S_{\alpha}^0}^{\alpha}.
\end{equation}

(1) Estimate on $\mathcal{R}^1(\rho_1, \rho_2, U_1, U_2)(t, x)$, one has
\begin{equation}
\mathcal{R}^1(\rho_1, \rho_2, U_1, U_2)(t, x) = - \sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d} \mathcal{Y}_{s,t}(x - v, t)v\nabla E^1(s, X_{s,t}(x, v, w)) \partial_i \mu(V_{s,t}(x, v)) \nu dv ds.
\end{equation}

Hence,
\begin{equation}
\| \mathcal{R}^1(\rho_1, \rho_2, U_1, U_2)(t, x) \| \lesssim_{a, M^*} \| (\rho_1, U_1) \|_{S_{\alpha}^0}^{\alpha} \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S_{\alpha}^0}^{\alpha}.
\end{equation}

Note that $X_{s,t}(x, v, w) = x - (t - s)v + Y_{s,t}(x - tv, t, w)$, then we apply Lemma 4.21 with $\mathcal{H} = \nabla E^2$ and $\varphi(x, v) = Y_{s,t}(x - tv, v, w)$ to obtain that for $\tau > 1$,
\begin{equation}
\| \mathcal{R}^1(\rho_1, \rho_2, U_1, U_2)(t, x) \| \lesssim_{a, M^*} \| (\rho_1, \rho_2, U_1 - U_2) \|_{S_{\alpha}^0}^{\alpha} \int_0^t \int_{\mathbb{R}^d} \left( \| \nabla E^2(s, X_{s,t}(x, v, w)) \|_{L^\infty} \right) \frac{1}{\langle v \rangle^N} \nu dv ds dv.
\end{equation}

(6.18)
Meanwhile, for $0 \leq t \leq 1$

$$\|R^1(\rho_1, \rho_2, U_1, U_2)(t,x)\|_{L^p} \lesssim_{a,M^*} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \langle s \rangle^{-d+2-a} \|\nabla E^2(s)\|_{L^p} ds$$

\[ \lesssim_{a,M^*} \|\rho(1 - \rho_2, U_1 - U_2)\|_{S^a_0}. \]  

(6.19)

Then, with (6.19) and (6.18), we can obtain

$$\sum_{p=1}^{\infty} \langle t \rangle^{-\frac{d-1}{2}} \|R^1(\rho_1, \rho_2)\|_{L^p} \lesssim_{a,M^*} \|(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0}. \]  

(6.20)

Now we deal with $\frac{\partial_s R^1(\rho_1, \rho_2)(t,x)}{\|\rho\|^a}$, for the case $t \leq 1$ we have

$$\frac{\partial_s R^1(\rho_1, \rho_2)(t,x)}{\|\rho\|^a} \lesssim_{a,M^*} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} |\nabla E^2(s, X_{s,t}(x,v,\varpi))| \frac{dv}{\langle v \rangle^N} ds \|dw\| \|d^t\|$$

$$+ \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} \|\nabla E^2(s, X_{s,t}(x,v,\varpi))\| \frac{dv}{\langle v \rangle^N} ds \|dw\|$$

\[ + \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} \|\nabla E^2(s, X_{s,t}(x-v,\varpi))\| \frac{dv}{\langle v \rangle^N} ds \|dw\|. \]

Then thanks to (6.22), we deduce that

$$\frac{\partial_s R^1(\rho_1, \rho_2)(t,x)}{\|\rho\|^a} \lesssim_{a,M^*} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \int_0^t \left( \langle s \rangle^{-d+2-a} \|\nabla E^2(s)\|_{L^p} + \langle s \rangle^{-d+2} \|\nabla E^2(s)\|_{L^p} \right) ds$$

\[ \lesssim_{a,M^*} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \|\rho_2, U_2\|_{S^a_0}. \]

For the case $t > 1$, $\frac{\partial_s R^1(\rho_1, \rho_2, U_1, U_2)(t,x)}{\|\rho\|^a}$ can be recast as

$$\frac{\partial_s R^1(\rho_1, \rho_2, U_1, U_2)(t,x)}{\|\rho\|^a} \lesssim_{a,M^*} \frac{1}{t^a} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \int_0^t \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} \|\nabla E^2(s, X_{s,t}(x,v,\varpi))\| \frac{dv}{\langle v \rangle^N} ds \|dw\|$$

$$+ \frac{1}{t^a} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} \|\nabla E^2(s, X_{s,t}(x,v,\varpi))\| \frac{dv}{\langle v \rangle^N} ds \|dw\|$$

$$+ \frac{1}{t^a} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0} \int_0^t \int_0^t \int_0^t \int_{\mathbb{R}^d} \langle s \rangle^{-d+2-a} \|\nabla E^2(s, X_{s,t}(x-v,\varpi))\| \frac{dv}{\langle v \rangle^N} ds \|dw\|.$$ 

Thus, it holds that for $t > 1$,

$$\frac{\partial_s R^1(\rho_1, \rho_2, U_1, U_2)}{\|\rho\|^a} \lesssim_{a,M^*} \frac{1}{t^a} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \int_0^t \langle s \rangle^{-d+2-a} \|\nabla E^2(s)\|_{L^1} + \langle s \rangle^{-d+2} \|\nabla E^2(s)\|_{L^p} ds$$

$$+ \frac{1}{t^a} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \int_0^t \langle s \rangle^{-d+2-a} \|\nabla E^2(s)\|_{L^p} + \langle s \rangle^{-d+2} \|\nabla E^2(s)\|_{L^p} ds$$

\[ \lesssim_{a,M^*} \frac{1}{t^a} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \|\rho_2, U_2\|_{S^a_0}. \]

Then,

$$\sum_{p=1}^{\infty} \langle t \rangle^{\frac{d-1}{2}} \frac{\partial_s R^1(\rho_1, \rho_2, U_1, U_2)(t,x)}{\|\rho\|^a} \lesssim_{a,M^*} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0} \|\rho_2, U_2\|_{S^a_0} \|\rho(\rho_1 - \rho_2, U_1 - U_2)\|_{S^a_0}. \]  

(6.21)

Combining this with (6.20), one gets

$$\|R^1(\rho_1, \rho_2, U_1, U_2)\|_{S^a_0} \lesssim_{a,M^*} \|\rho_2, U_2\|_{S^a_0} \|\rho_1 - \rho_2, U_1 - U_2\|_{S^a_0}. \]  

(6.22)
(2) Estimate on $\mathbb{R}^2(p_1, p_2, U_1, U_2)(t, x)$. We infer that
\[
\mathbb{R}^2(p_1, p_2, U_1, U_2)(t, x) = -\frac{d}{\gamma} \int_0^1 \int_0^t \int_{\mathbb{R}^d} W_{s,t}(x-vt,v) E_i^2 (s, X_{s,t}^2(x,v))(\nabla \partial_\mu)(V_{s,t}(x,v,\omega)) dv ds d\omega.
\]
Analogously, we obtain
\[
\|\mathbb{R}^2(p_1, p_2, U_1, U_2)\|_{S^\alpha_{0,0}} \lesssim \|\rho(U_2)\|_{S^\alpha_{0,0}} (|\rho_1 - 2\rho_2| + |U_1 - U_2|)_{S^\alpha_{0,0}},
\]
which complete the proof.

\textbf{Lemma 6.5} Let $H_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ and $A_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\lambda \in \mathbb{R}$. Then, for any multi-index $\alpha \neq 0$,
\[
|\partial_\alpha^2 [H_0(\lambda x + A_0(x))] - (\partial_\alpha^2 H_0)(\lambda x + A_0(x))\lambda^{\gamma} | \lesssim |(\nabla^{\gamma} H_0)(\lambda x + A_0(x))| |\nabla A_0(x)| (|\lambda| + |\nabla A_0(x)|)_{\gamma}^{\gamma-1} \\
+ \sum_{k=1}^{\gamma-1} \sum_{i=2}^{\gamma-1} \sum_{j=0}^{k-1} |(\nabla^{k} H_0)(\lambda x + A_0(x))| (|\lambda| + |\nabla A_0(x)|)^j |\nabla^i A_0(x)|^{\frac{|\gamma-i|}{\gamma-i}}.
\]
\textbf{Proof.} Assume $\partial_\alpha^2 = \partial_\alpha^{\gamma_1} \cdot \partial_\alpha^{\gamma_2}$. We know that, for $\tilde{A}_0(x) = \lambda x + A_0(x)$, we have
\[
|\partial_\alpha^2 [H_0(\lambda x + A_0(x))] - Q | \lesssim \sum_{k=1}^{\gamma-1} |(\nabla^{k} H_0)(\tilde{A}_0(x))| \sum_{m_1, \ldots, m_k \geq 1} |\nabla^m \tilde{A}_0(x)|,
\]
with
\[
Q = (\nabla^{\gamma_1} H_0)(\lambda x + A_0(x)) : \left[ (\partial_{\gamma_1} \tilde{A}_0(x))^{\otimes \gamma_1} \otimes (\partial_{\gamma_2} \tilde{A}_0(x))^{\otimes \gamma_2} \otimes \cdots \otimes (\partial_{\gamma_d} \tilde{A}_0(x))^{\otimes \gamma_d} \right].
\]
It is easy to check that
\[
\sum_{k=1}^{\gamma-1} \sum_{m_1, \ldots, m_k \geq 1} |(\nabla^{k} H_0)(\tilde{A}_0(x))| \sum_{j=1}^{k} |\nabla^m \tilde{A}_0(x)|
\]
\[
\lesssim \sum_{k=1}^{\gamma-1} \sum_{i=2}^{\gamma-1} \sum_{j=0}^{k-1} |(\nabla^{k} H_0)(\lambda x + A_0(x))| (|\lambda| + |\nabla A_0(x)|)^j |\nabla^i A_0(x)|^{\frac{|\gamma-i|}{\gamma-i}},
\]
and
\[
|Q - (\partial_\alpha^2 H_0)(\lambda x + A_0(x))\lambda^{\gamma} | \lesssim |(\nabla^{\gamma} H_0)(\lambda x + A_0(x))| |\nabla A_0(x)| (|\lambda| + |\nabla A_0(x)|)_{\gamma}^{\gamma-1}.
\]
Combining (6.22) with (6.23), we finish the proof.

\textbf{Proposition 6.6} Let $a \in (\frac{2l}{2l+1}, 1)$ and assume $\|\rho(U)\|_{S^\alpha_{0,0}} \leq \varepsilon_0 \leq \varepsilon_0$ with $\varepsilon_0$ mentioned in Proposition 4.1. Then we have for $b \in (\frac{2s}{2s+1}, 1)$
\[
\|\mathbb{R}(\rho, U)\|_{L^{t+1} \cdot L^b} \lesssim_{a, b, \delta_0, c, c', c_{A,\delta}, 1} 1,
\]
provided $\|\rho(U)\|_{L^{t+1} \cdot L^b} \leq c$, $\sum_{j=2}^{l+1} \sup_{|r| \leq 1} |A^{(j)}(r)| \leq C_{A, l}$ and $\mu$ satisfies
\[
\sum_{i=1}^{l+3} |\gamma|^N \nabla^i \mu(v) | \leq c',
\]
where $l \geq 0$, $0 < \delta_0 < \frac{1}{2\sqrt{l+1}}$ and $N > d$.

\textbf{Proof.} In this proof, we denote $C = C(a, b, \delta_0, c, c', C_{A, l})$.

(1) We firstly consider $t \geq 1$. Set
\[
F_{1,1}(x) = \delta_0 \partial_\alpha E_i(s, ) (Z_1(x)), \quad \eta_{1,1}(x) = \partial_\alpha (Z_2(x)),
\]
\[
F_{1,2}(x) = \delta_0 \partial_\alpha E_i(w + \frac{s}{t}(x - w)), \quad \eta_{1,2}(x) = \partial_\alpha \mu(x - w),
\]
\[
F_{2,1}(x) = E_i(s, Z_1(x)), \quad \eta_{2,1}(x) = \delta_\partial \partial_\mu (Z_2(x)),
\]
\[
F_{2,2}(x) = E_i(s, w + \frac{s}{t}(x - w)), \quad \eta_{2,2}(x) = \delta_\partial \partial_\mu \mu(x - w),
\]
\[
F_3(x) = E_i(s, Z_1(x) - \alpha) - E_i(s, Z_4(x)), \quad \eta_3(x) = \partial_\mu (Z_3(x) - \partial_\mu (Z_2(x)),
\]

33
where $Z_1, Z_2, Z_3, Z_4$ are defined in (6.9) and $w = x - vt$, then we have,

$$
\delta_\alpha T[E_i, \partial_\mu](t, x) = \sum_{j=1}^4 T^j_\alpha[E_i, \partial_\mu](t, x),
$$

where

$$
T^1_\alpha[E_i, \partial_\mu](t, x) = - \int_0^t \int_{\mathbb{R}^d} (F_{1,1}(x)\eta_{1,1}(x) - F_{1,2}(x)\eta_{1,2}(x)) \frac{duds}{t^d},
$$

$$
T^2_\alpha[E_i, \partial_\mu](t, x) = \int_0^t \int_{\mathbb{R}^d} (F_{2,1}(x)\eta_{2,1}(x) - F_{2,2}(x)\eta_{2,2}(x)) \frac{duds}{t^d},
$$

$$
T^3_\alpha[E_i, \partial_\mu](t, x) = \int_0^t \int_{\mathbb{R}^d} F_3(x)\eta_{1,1}(x) \frac{duds}{t^d},
$$

$$
T^4_\alpha[E_i, \partial_\mu](t, x) = \int_0^t \int_{\mathbb{R}^d} F_{2,1}(x - \alpha)\eta_3(x) \frac{duds}{t^d}.
$$

Hence, we obtain

$$
|\nabla_x^{t+1}\delta_\alpha T^t[E_i, \partial_\mu](t, x)| \lesssim \sum_{|\beta_1|+|\beta_2|=t+1} |T^j_\alpha[E_i, \partial_\mu, \beta_1, \beta_2](t, x)|,
$$

where

$$
T^1_\alpha[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = - \int_0^t \int_{\mathbb{R}^d} (\partial_\beta^1 F_{1,1}(x)\partial_\beta^2 \eta_{1,1}(x) - \partial_\beta^3 F_{1,2}(x)\partial_\beta^2 \eta_{1,2}(x)) \frac{duds}{t^d},
$$

$$
T^2_\alpha[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = \int_0^t \int_{\mathbb{R}^d} (\partial_\beta^1 F_{2,1}(x)\partial_\beta^2 \eta_{2,1}(x) - \partial_\beta^3 F_{2,2}(x)\partial_\beta^2 \eta_{2,2}(x)) \frac{duds}{t^d},
$$

$$
T^3_\alpha[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = \int_0^t \int_{\mathbb{R}^d} \partial_\beta^1 F_3(x)\partial_\beta^2 \eta_{1,1}(x) \frac{duds}{t^d},
$$

$$
T^4_\alpha[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = \int_0^t \int_{\mathbb{R}^d} \partial_\beta^1 F_{2,1}(x - \alpha)\partial_\beta^2 \eta_3(x) \frac{duds}{t^d}.
$$

1.1) Estimate of terms in $\nabla_x^{t+1} T^t_\alpha[E_i, \partial_\mu]$.

Note that

$$
\partial_{\beta}^1 F_{1,1}(x)\partial_{\beta}^2 \eta_{1,1}(x) - \partial_{\beta}^3 F_{1,2}(x)\partial_{\beta}^2 \eta_{1,2}(x)
$$

$$
= \left( \partial_{\beta}^1 F_{1,1}(x) - \left( \frac{S}{t} \right)^{|\beta_1|} \partial_{\beta}^1 \delta_\beta^1 E_1(s, \cdot)(Z_1) \right) \left( \partial_{\beta}^2 \eta_{1,1}(x) - \left( \frac{1}{t} \right)^{|\beta_2|} \partial_{\beta}^2 \partial_\mu(Z_2) \right)
$$

$$
+ \left( \partial_{\beta}^3 F_{1,1}(x) - \left( \frac{S}{t} \right)^{|\beta_1|} \partial_{\beta}^3 \delta_{\beta}^1 E_1(s, \cdot)(Z_1) \right) \left( 1 \right)^{|\beta_2|} \left( \partial_{\beta}^2 \partial_\mu(Z_2) \right)
$$

$$
+ \left( \frac{S}{t} \right)^{|\beta_1|} \partial_{\beta}^1 \delta_{\beta}^1 E_1(s, \cdot)(Z_1) \left( \partial_{\beta}^2 \eta_{1,1}(x) - \left( \frac{1}{t} \right)^{|\beta_2|} \partial_{\beta}^2 \partial_\mu(Z_2) \right)
$$

$$
+ \left( \frac{S}{t} \right)^{|\beta_1|} \partial_{\beta}^3 \delta_{\beta}^1 E_1(s, \cdot)(Z_1) \left( 1 \right)^{|\beta_2|} \left( \partial_{\beta}^2 \partial_\mu(Z_2) - \partial_{\beta}^3 F_{1,2}(x)\partial_{\beta}^2 \eta_{1,2}(x) \right)
$$

$$
:= \sum_{k=1}^4 T_{\alpha, k}^{1, \beta_1, \beta_2}(t, x, \frac{x - w}{t}).
$$

We first deal with the fourth term directly

$$
\sum_{|\beta_1|+|\beta_2|=l+1} \left\| \int_0^t \int_{\mathbb{R}^d} T_{\alpha, l}^{1, \beta_1, \beta_2}(t, x, \frac{x - w}{t}) \frac{duds}{t^d} \right\|_{L^p_w} = \frac{1}{p^{l+1}} \sum_{|\beta_1|+|\beta_2|=l+1} \left\| T_{\alpha, l}^{1, \beta_1, \beta_2}(t, x, \frac{x - w}{t}) \right\|_{L^p_w}
$$

$$
\lesssim \delta_0 e^{\frac{|\alpha|}{t^{l+1+b}+d}} \sum_{|\beta_1|=0}^{l+1} \left( \int_0^{t/2} |s|^{|\beta_1|+b} \left\| \nabla_\beta^1 E_1(s) \right\|_{F_\infty} \| \delta_{\beta_1} \|_{L^1} ds \right) \left( s^{d-1-\delta_0} \right)
$$

$$
+ \int_{t/2}^t \sup_{t/2 \leq s \leq t} |s|^{|\beta_1|+b+d} \left\| \nabla_\beta^1 E_1(s) \right\|_{F_\infty} \| \delta_{\beta_1} \|_{L^1} ds \left( s^{d-1-\delta_0} \right)
$$

$$
\lesssim \delta_0, \delta_0 e^{c\lambda_\alpha}, \frac{|\alpha|}{t^{l+1+b}} \int_0^t \left( \frac{s}{t} \right)^{d-1-\delta_0} ds \lesssim \frac{|\alpha|}{t^{l+1+b}}
$$

(6.28)
since \(b < 1 - 2\delta_0\).

Then we start to estimate \(T_{\alpha,k}^{1,\beta_1,\beta_2}\), where \(k = 1, 2, 3\). Applying Lemma 5.3 and Proposition 4.4, we get

\[
\left| \frac{\partial_{\beta_1}^3 E_t}{\partial_{\beta_1}^3} (x, \frac{x}{t}) \right| \leq \frac{1}{\delta_0} \sum_{|\beta|=1} \sum_{m=1}^{\frac{|\beta|}{2}} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}})
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

Thus, we have

\[
\sum_{|\beta_1| + |\beta_2| = 1} \sum_{|\beta_1| = 1} \frac{|T_{\alpha,1}^{1,\beta_1,\beta_2} (t, x, \frac{x}{t})|}{\rho t} \lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

Hence, we get

\[
\sum_{k=1}^{3} \sum_{|\beta_1| + |\beta_2| = 1} \frac{|T_{\alpha,k}^{1,\beta_1,\beta_2} (t, x, \frac{x}{t})|}{\rho t} \lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

Note that we have (S.1) and (S.2) in the Appendix with \(\nabla^m \phi E_t \) and \(\phi = Y_{s,t}(x, v)\), we make the change of variables \(v = \frac{x-w}{t} \), then we apply (4.15) to obtain

\[
\left\| \int_0^t \int_{\mathbb{R}^d} \sum_{|\beta_1| + |\beta_2| = 1} \frac{|T_{\alpha,k}^{1,\beta_1,\beta_2} (t, x, \frac{x}{t})|}{\rho t} dv \right\|_{L^p_x} \lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

\[
\lesssim C 1_{|\beta_1| > 1} \left( \frac{(\rho, U)}{\rho} \sum_{|\beta_2|} |\nabla^m \phi E_t(x, \frac{x}{t})| (|s| + |\nabla E_t(x, \frac{x}{t})|^{\frac{1}{2}}) \right)
\]

(6.30)
Thanks to (6.28) and (6.30), we have
\[
L^{\frac{d}{4}p-1} t^{1+b} \sup_{\alpha} \left\| \nabla^{l+1}_x \mathcal{T}^{a}_{\alpha} \left[ E_i, \partial_i \mu \right](t) \right\|_{L^p_x} \lesssim_{a,b,\delta_0,\epsilon, C_{A,t}} 1.
\]

1.2) Estimate of terms in \( \nabla^{l+1}_x \mathcal{T}^{2}_{\alpha}[E_i, \partial_i \mu] \). It is clear that
\[
\partial^\beta_2 F_{2,1}(x) \partial^\beta_2 \eta_2(x) - \partial^\beta_2 F_{2,2}(x) \partial^\beta_2 \eta_2(x)
\]
\[
= \left( \partial^\beta_2 F_{2,1}(x) - \left( \frac{8}{7} \right) \partial^\beta_2 E_i(s, \cdot)(Z_1) \right) \left( \partial^\beta_2 \eta_2(x) - \left( \frac{1}{7} \right) \partial^\beta_2 \left( \delta^\beta_2 \partial_\mu \right)(Z_2) \right)
\]
\[
+ \left( \partial^\beta_2 F_{2,1}(x) - \left( \frac{8}{7} \right) \partial^\beta_2 E_i(s, \cdot)(Z_1) \right) \left( \partial^\beta_2 \eta_2(x) - \left( \frac{1}{7} \right) \partial^\beta_2 \left( \delta^\beta_2 \partial_\mu \right)(Z_2) \right)
\]
\[
+ \left( \frac{8}{7} \right) \partial^\beta_2 E_i(s, \cdot)(Z_1) \left( \partial^\beta_2 \eta_2(x) - \left( \frac{1}{7} \right) \partial^\beta_2 \left( \delta^\beta_2 \partial_\mu \right)(Z_2) \right)
\]
\[
+ \left( \frac{8}{7} \right) \partial^\beta_2 E_i(s, \cdot)(Z_1) \left( \partial^\beta_2 \eta_2(x) - \left( \frac{1}{7} \right) \partial^\beta_2 \left( \delta^\beta_2 \partial_\mu \right)(Z_2) \right)
\]
\[
\lesssim \sum_{k=1}^{4} \mathcal{T}^{2,\beta_1,\beta_2}_{a,k}(t, x, \frac{x-w}{t}).
\]

Then, we also compute the last term first as follows
\[
\sum_{|\beta_1|+|\beta_2|=l+1} \left\| \int_0^t \int \mathcal{T}^{2,\beta_1,\beta_2}_{a,k}(t, x, \frac{x-w}{t}) \frac{dwds}{t^{d+1}} \right\|_{L^p_x} = \frac{1}{t^{l+1+\delta_0}} \sum_{|\beta_1|+|\beta_2|=l+1} \left\| \mathcal{T}^{2,|\beta_1|\partial^\beta_1 E_i, \partial^\beta_2 \partial_\mu}(t, x, \frac{x-w}{t}) \right\|_{L^p_x}
\]

\[
\lesssim_{\delta_0, \epsilon, C_{A,t}} |\alpha|^b \sum_{d+l+\delta_0} \sum_{|\beta_1|=0}^{l+1} \left( \int_0^{t/2} \left| \nabla^{|\beta_1|} E_i(s, \cdot)(Z_1) \right| \left( s \right)^{d-1-\delta_0} \right) \left( s \right)^{d-1-\delta_0} \left( s \right)^{d-1-\delta_0} \left( s \right)^{d-1-\delta_0}
\]

\[
\lesssim_{\delta_0, \epsilon, C_{A,t}} |\alpha|^b \sum_{d+l+\delta_0} \sum_{|\beta_1|=0}^{l+1} \left( \int_0^{t/2} \left| \nabla^{|\beta_1|} E_i(s, \cdot)(Z_1) \right| \left( s \right)^{d-1-\delta_0} \right) \left( s \right)^{d-1-\delta_0} \left( s \right)^{d-1-\delta_0} \left( s \right)^{d-1-\delta_0}
\]
Define
\[ \Xi(x, w, \alpha, t) = \frac{1}{(\frac{t}{\vartheta_1})^N} + \frac{1}{(\frac{t}{\vartheta_2})^N}. \]

Thus, we have
\[
\sum_{|\beta_1| + |\beta_2| = t + 1} |T^2_{\alpha, \beta_1, \beta_2}(t, x, \frac{x - w}{t})| \lesssim_C |\alpha|^b |t|^{l + 1 + b} \sum_{|\beta_1| = 1} \sum_{m=1}^{l + 1} |\nabla^{m} E_i(s, \cdot)(Z_1)| (|s|)^{m - 2d + 2} \Xi(x, w, \alpha, t),
\]
\[
\sum_{|\beta_1| + |\beta_2| = t + 1} |T^2_{\alpha, \beta_1, \beta_2}(t, x, \frac{x - w}{t})| \lesssim_C |\alpha|^b |t|^{l + 1 + b} \sum_{|\beta_1| = 1} \sum_{m=1}^{l} |\nabla^{m} E_i(s, \cdot)(Z_1)| (|s|)^{m - d + 1} \Xi(x, w, \alpha, t),
\]
\[
\sum_{|\beta_1| + |\beta_2| = t + 1} |T^2_{\alpha, \beta_1, \beta_2}(t, x, \frac{x - w}{t})| \lesssim_C |\alpha|^b |t|^{l + 1 + b} \sum_{|\beta_1| = 0} \sum_{l=1}^{l} |\partial^2_{x} E_i(s, \cdot)(Z_1)| (|s|)^{|\beta_1| - 1} \Xi(x, w, \alpha, t).
\]

We observe to get
\[
\sum_{k=1}^{3} \sum_{|\beta_1| + |\beta_2| = t + 1} |T^2_{\alpha, \beta_1, \beta_2}(t, x, \frac{x - w}{t})| \lesssim_C |\alpha|^b |t|^{l + 1 + b} \sum_{|\beta_1| = 1} \sum_{m=1}^{l + 1} |\nabla^{m} E_i(s, \cdot)(X_{s,t}(x, v))| \left( \frac{1}{(\frac{t}{\vartheta})^N} + \frac{1}{(\frac{t}{2 - \vartheta})^N} \right) dv |ds| (6.33)
\]

As the proof of (6.30), we have Remark [8.2] in the Appendix with \( H = \nabla^m E_1 \) and \( \varphi = Y_{s,t}(x, v) \), we make the change of variable \( v = \frac{d}{dX} \), then we apply (1.15) to obtain
\[
\left\| \int_{0}^{t} \int_{\mathbb{R}^d} \sum_{k=1}^{3} \sum_{|\beta_1| + |\beta_2| = t + 1} |T^1_{\alpha, \beta_1, \beta_2}(t, x, \frac{x - w}{t})| dv ds \right\|_{L^p_x} \lesssim_C |\alpha|^b |t|^{l + 1 + b} \sum_{|\beta_1| = 1} \sum_{m=1}^{l + 1} \int_{0}^{t} (|s|)^{m - d + 1} \left( \int_{\mathbb{R}^d} |\nabla^{m} E_i(s, \cdot)(X_{s,t}(x, v))| \left( \frac{1}{(\frac{t}{\vartheta})^N} + \frac{1}{(\frac{t}{2 - \vartheta})^N} \right) dv \right) ds \lesssim_C |\alpha|^b |t|^{l + 1 + b}. \quad (6.34)
\]

Thanks to (6.32) and (6.34), we have
\[
\left( \int_{t}^{t + \frac{1}{p - 1} \cdot \sup_{\alpha} \| T^1_{x, \beta} E_i(\partial_\alpha \mu_\alpha) \|_{L^p_x} \right) \lesssim_C 1. \quad (6.35)
\]

**1.3) Estimate of terms in \( T^3_{\alpha} E_i, \partial_\alpha \mu_\alpha \).**

Firstly, we have
\[
|\partial^2_{x} F_3(x)| \lesssim \sum_{1 \leq |\beta_1|, |\beta_2|, |\delta_1|, |\delta_2| > 0} |\prod_{j=1}^{2} (\partial^j_{x} Z_4(x)) (\partial^j_{x} E_i)(s, Z_4(x)) - \prod_{j=1}^{2} (\partial^j_{x} Z_4(x - \alpha)) (\partial^j_{x} E_i)(s, Z_1(x - \alpha))| \leq \sum_{1 \leq |\beta_1|, |\beta_2|, |\delta_1|, |\delta_2| > 0} \left| \prod_{j=1}^{2} (\partial^j_{x} Z_4(x)) - \prod_{j=1}^{2} (\partial^j_{x} Z_4(x - \alpha)) \right| \left| \nabla^{[\beta_1]} E_i(s, Z_4(x)) \right| + \prod_{j=1}^{2} |\partial^j_{x} Z_4(x - \alpha)| \left| \nabla^{[\beta_1]} E_i(s, Z_4(x)) - \nabla^{[\beta_1]} E_i(s, Z_1(x - \alpha)) \right|. \]
Then, by Proposition 3.11 and 3.13 for $|\beta_1| \geq 1$,

$$|\partial^2_x F_3(x)| \lesssim \delta \rho \leq \frac{|\alpha|}{t^{1/2}} \sum_{1 \leq |\beta_1| \leq |\beta_1|} \left( \left( |s| |\beta_1|-d+1 \right)_{(|\beta_1|,|\beta_1|)} (1, t+1) + (s)^{-d+2+\delta_0+b} 1_{(|\beta_1|,|\beta_1|)=(1, t+1)} \right) \|\nabla |\beta_1| E_i(s, Z_4)\|.$$ 

And if $|\beta_1| = 0$, we can obtain

$$|F_3(x)| \leq \frac{|\alpha|}{t^{1/2}} \|\nabla x Y_{\alpha, t}\|_L^\infty \lesssim \delta \rho.$$ 

Now we estimate the $L^1$ and $L^\infty$ norm of $\partial^2_x F_3(x)$ as follows,

$$\|\partial^2_x F_3(x)\|_{L^1} \lesssim \frac{|\alpha|}{t^{1/2}} \sum_{1 \leq |\beta_1| \leq |\beta_1|} \left( \left( |s| |\beta_1|-d+1 \right)_{(|\beta_1|,|\beta_1|)} (1, t+1) + (s)^{-d+2+\delta_0+b} 1_{(|\beta_1|,|\beta_1|)=(1, t+1)} \right) \|\nabla |\beta_1| E_i(s)\|_L^p,$$

$$\|\partial^2_x F_3(x)\|_{L^\infty} \leq \frac{|\alpha|}{t^{1/2}} \left( \sum_{1 \leq |\beta_1| \leq |\beta_1|} (s)^{-d_0-(d-1-\delta_0) b} \right) \lesssim C \frac{|\alpha|}{t^{1/2}} \left( \sum_{1 \leq |\beta_1| \leq |\beta_1|} (s)^{-d_0-(d-1-\delta_0) b} \right) \|\partial^2_x F_3(x)\|_{L^p}.$$ 

(6.36)

On the other hand, for $|\beta_2| \geq 0$, we have

$$|\partial^2_x \eta_1(x)\| \lesssim \frac{|\alpha|}{t^{1/2}} \sum_{1 \leq |\beta_1| \leq |\beta_1|} \left( \left( |s| |\beta_1|-d+1 \right)_{(|\beta_1|,|\beta_1|)} (1, t+1) + (s)^{-d+2+\delta_0+b} 1_{(|\beta_1|,|\beta_1|)=(1, t+1)} \right) \|\nabla |\beta_1| E_i(s)\|_L^p \|\partial^2_x \mu(Z_2(x))\| \lesssim C \frac{|\alpha|}{t^{1/2}} \left( \sum_{1 \leq |\beta_1| \leq |\beta_1|} \left( |s| |\beta_1|-d+1 \right)_{(|\beta_1|,|\beta_1|)} (1, t+1) + (s)^{-d+2+\delta_0+b} 1_{(|\beta_1|,|\beta_1|)=(1, t+1)} \right) \|\nabla |\beta_1| E_i(s)\|_L^p \|\partial^2_x \mu(Z_2(x))\|.$$ 

(6.37)

1.4) Estimate of terms in $T^3_\alpha[E_i, \partial \mu].$

Firstly, we have for $|\beta_1| \geq 1$,

$$|\partial^2 x F_{21}(x) - \alpha| \lesssim \sum_{1 \leq |\beta_1| \leq |\beta_1|} \left( \left( |s| |\beta_1|-d+1 \right)_{(|\beta_1|,|\beta_1|)} (1, t+1) + (s)^{-d+2+\delta_0+b} 1_{(|\beta_1|,|\beta_1|)=(1, t+1)} \right) \|\nabla |\beta_1| E_i(s)\|_L^p \|\partial^2 x \mu(Z_2(x) - \alpha)\|.$$ 

(6.38)

and for $|\beta_1| = 0$, we obtain

$$|F_{21}(x) - \alpha| = |E_i(s, Z_1(x) - \alpha)|.$$ 

38
Then, we estimate the $L^1$ and $L^\infty$ norm of $\partial_x^3 F_{2,1}$,
\[
\|\partial_x^3 F_{2,1}\|_{L^p} \lesssim \|\partial_x^3 E_t(s, Z_1(x - \alpha))\|_{L^p} + \|\partial_x^3 E_t(s, Z_1(x - \alpha))\|_{L^p} 
\lesssim C \|\partial_x^3 \eta (s)\|_{L^p} + \frac{1}{\|\partial_x^3 \eta (s)\|_{L^p}}.
\]
(6.38)

Meanwhile, for $|\beta_2| \geq 1$, we have
\[
|\partial_x^{\beta_2} \eta_3(x)| \lesssim \sum_{j=1}^{|\beta_2|} \prod_{j=1}^{|\beta_2|} (\partial_x^{\beta_2} E_t(s, Z_2(x))) (\partial_x^{\beta_2} \partial_\mu(s, Z_3(x))) - \prod_{j=1}^{|\beta_2|} (\partial_x^{\beta_2} E_t(s, Z_3(x))) (\partial_x^{\beta_2} \partial_\mu(s, Z_3(x))) 
\lesssim C \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}}.
\]

For $|\beta_2| = 0$, we have
\[
|\eta_3(x)| = |(\partial_\mu(Z_2(x)) - (\partial_\mu)(Z_3(x)))| = \frac{|\alpha|^b (\partial_\mu(Z_2(x)) - (\partial_\mu)(Z_3(x)))}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \lesssim C \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}}.
\]

Thus, with (6.38), we have
\[
\|T_4^f E_t, \partial_\mu(t, x)\|_{L^p} \lesssim C \sum_{|\beta_1| + l + t = l + 1} \sum_{|\beta_2| = l} \int_0^{t/2} \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} 
\]
\[
+ \int_0^{t/2} \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} 
\]
\[
+ \sum_{|\beta_1| + l + t = l + 1} \sum_{|\beta_2| = l} \int_0^{t/2} \int_{R^d} \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} 
\]
\[
\lesssim C \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} 
\]
\[
\lesssim C \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} 
\]

This implies
\[
\left(\sum_{\alpha} \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} \right) \lesssim C 1.
\]
(6.39)

Hence, combining (6.31), (6.35), (6.37) with (6.39) we obtain
\[
\left(\sum_{\alpha} \frac{|\alpha|^b \|\mu, U\|_{L^p} \alpha_{\delta - \delta_0} - \delta^b_{\delta - \delta_0}}{\|\partial_x^{\beta_2} \eta (s)\|_{L^p}} \|\partial_x^{\beta_2} F_{2,1}(x)\|_{L^p} \frac{ds}{(t - s)^{\frac{d + 1}{2}}} \right) \lesssim C 1.
\]
(6.40)

(2) We now consider $t \leq 1$. Denote
\[
T_{1,1} (x) = \delta_1 E_t(s, x(t - x), v) \quad \text{and} \quad T_{1,1} (x) = \partial_\mu (v_t(x, v)),
\]
\[
T_{1,2} (x) = \delta_1 E_t(s, x(t - x), v) \quad \text{and} \quad T_{1,2} (x) = \partial_\mu (v_t(x, v)),
\]
\[
T_{2,1} (x) = E_t(s, x(t - x), v) \quad \text{and} \quad T_{2,1} (x) = \partial_\mu (v_t(x, v)),
\]
\[
T_{2,1} (x) = E_t(s, x(t - x), v) \quad \text{and} \quad T_{2,1} (x) = \partial_\mu (v_t(x, v)).
\]
Thus, we have

$$\eta T[E_i, \partial_\mu](t, x) = \sum_{j=1}^{3} \eta T^j_h[E_i, \partial_\mu](t, x),$$

with

$$\eta T^1_h[E_i, \partial_\mu](t, x) = - \int_0^t \int_{\mathbb{R}^d} (F_{1,1}(x)\eta_{1,1}(x) - F_{1,2}(x)\eta_{1,2}(x)) \, dvds,$$

$$\eta T^2_h[E_i, \partial_\mu](t, x) = \int_0^t \int_{\mathbb{R}^d} F_3(x)\eta_{1,1}(x) \, dvds,$$

$$\eta T^3_h[E_i, \partial_\mu](t, x) = \int_0^t \int_{\mathbb{R}^d} F_{2,1}(x - \alpha)\eta_3(x) \, dvds.$$

Hence, we obtain

$$|\nabla_x^{t+1}\eta T^j_h[E_i, \partial_\mu](t, x)| \lesssim \sum_{|\beta_1|+|\beta_2|=t+1} |\eta T^j_h[E_i, \partial_\mu, \beta_1, \beta_2](t, x)|,$$

where

$$\eta T^1_h[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = - \int_0^t \int_{\mathbb{R}^d} (\partial_x^{\beta_1} F_{1,1}(x)\partial_x^{\beta_2} \eta_{1,1}(x) - \partial_x^{\beta_1} F_{1,2}(x)\partial_x^{\beta_2} \eta_{1,2}(x)) \, dvds,$$

$$\eta T^2_h[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = \int_0^t \int_{\mathbb{R}^d} \partial_x^{\beta_1} F_3(x)\partial_x^{\beta_2} \eta_{1,1}(x) \, dvds,$$

$$\eta T^3_h[E_i, \partial_\mu, \beta_1, \beta_2](t, x) = \int_0^t \int_{\mathbb{R}^d} \partial_x^{\beta_1} F_{2,1}(x - \alpha)\partial_x^{\beta_2} \eta_3(x) \, dvds.$$

2.1) Estimate of \( \eta T^j_h[E_i, \partial_\mu, \beta_1, \beta_2] \)

We can write

$$|\partial_x^{\beta_1} F_{1,1}(x)\partial_x^{\beta_2} \eta_{1,1}(x) - \partial_x^{\beta_1} F_{1,2}(x)\partial_x^{\beta_2} \eta_{1,2}(x)|$$

$$\leq \left| \partial_x^{\beta_1} F_{1,1}(x) - \partial_x^{\beta_1} E_i(s,)(X_{s,t}(x,v)) \right| \left| \partial_x^{\beta_2} \eta_{1,1}(x) - (\partial_x^{\beta_2} \partial_\mu)(V_{s,t}(x,v)) \right|$$

$$+ \left| \partial_x^{\beta_1} F_{1,1}(x) - \partial_x^{\beta_1} \delta_0 E_i(s,)(X_{s,t}(x,v)) \right| \left| \partial_x^{\beta_2} \eta_{1,1}(x) - (\partial_x^{\beta_2} \partial_\mu)(V_{s,t}(x,v)) \right|$$

$$+ \left| \partial_x^{\beta_1} \delta_0 E_i(s,)(X_{s,t}(x,v)) \right| \left| \partial_x^{\beta_2} \eta_{1,1}(x) - (\partial_x^{\beta_2} \partial_\mu)(V_{s,t}(x,v)) \right|$$

$$+ \left| \partial_x^{\beta_1} \delta_0 E_i(s,)(X_{s,t}(x,v)) \right| (\partial_x^{\beta_2} \partial_\mu)(V_{s,t}(x,v)) - F_{1,2}(x) \eta_{1,2}(x) \right|.$$

Thanks to Lemma 6.5 we have

$$|\partial_x^{\beta_1} F_{1,1}(x) - \partial_x^{\beta_1} E_i(s,)(X_{s,t}(x,v))|$$

$$\lesssim 1_{|\beta_1| \geq 1} |\nabla_x^{[\beta_1]} E_i(s,)(X_{s,t}(x,v))|(|\nabla_x Y_{s,t}|) \left( 1 + |\nabla_x Y_{s,t}| \right)^{|\beta_1|}$$

$$+ 1_{|\beta_1| \geq 1} \sum_{m=1}^{[\beta_1]} \sum_{j=0}^{[\beta_1] - j} \sum_{i=2}^{|\beta_1|} |\nabla_x^{m-j} E_i(s,)(X_{s,t}(x,v))| (1 + |\nabla_x Y_{s,t}|) \left( 1 + |\nabla_x Y_{s,t}| \right)^{|\beta_1| - j}$$

$$\lesssim C 1_{|\beta_1| \geq 1} \sum_{m=1}^{[\beta_1]} |\nabla_x^{m} E_i(s,)(X_{s,t}(x,v))| (s)^{-d+\delta_0}.$$

Then, for \(|\beta_2| \geq 1,

$$|\partial_x^{\beta_2} \eta_{1,1}(x) - (\partial_x^{\beta_2} \partial_\mu)(V_{s,t}(x,v))|$$

$$\lesssim 1_{|\beta_2| \geq 1} \left( |\nabla_x^{[\beta_2]} \partial_\mu(V_{s,t}(x,v))| |\nabla_x W_{s,t}| \right)^{[\beta_2]}$$

$$\lesssim C 1_{|\beta_2| \geq 1} \sum_{m=1}^{[\beta_2]} |\nabla_x^{m} \partial_\mu(V_{s,t}(x,v))| (s)^{-d+\delta_0} \|\rho(U)\|_{S_{0}^\infty} \lesssim C 1_{|\beta_2| \geq 1} (\frac{|\alpha|}{|v|})^N (s)^{-1-d+\delta_0}.$$
Then, we have for $p = 1, \infty$,
\[
\left\| \mathcal{T}_\alpha[E_i, \partial_\mu] \right\|_{L^p} \lesssim \sum_{|\beta_1| + 1 \leq m = 0}^{t+1} \int_{\mathbb{R}^d} \left\| \nabla^m \delta_\alpha E_i(s) \right\|_{L^p} \left( s^{-d-\delta_0} \right) \frac{ds}{v^N} + \sum_{|\beta_1| + |\beta_2| = t+1} \left\| \mathcal{T}[\partial^2_\beta \delta_\alpha E_i, \partial^2_\beta \partial_\mu] \right\|_{L^p} \lesssim_C |\alpha|^b.
\]
(6.41)

2.2) Estimate of $\mathcal{T}_\alpha[E_i, \partial_\mu]$.
Firstly, we consider another term. For $|\beta_1| \geq 1$, we get
\[
|\partial^2_\beta \mathcal{F}_3(x) | \lesssim \sum_{1 \leq |\beta_1| \leq |\beta|, |\beta_1| > 0} \left| \prod_{\beta_1} (\partial^2_\beta X_{s,t}(x,v))(\partial^2_\beta E_i)(s, X_{s,t}(x,v) - \alpha) \right|
\leq \sum_{1 \leq |\beta_1| \leq |\beta|, |\beta_1| > 0} \left( \prod_{\beta_1} |(\partial^2_\beta X_{s,t}(x,v)) - \prod_{\beta_1} |(\partial^2_\beta X_{s,t}(x,v) - \alpha)| \right)
+ \prod_{\beta_1} |(\partial^2_\beta X_{s,t}(x,v)) \left( (\nabla|\beta_1| E_i)(s, X_{s,t}(x,v) - \alpha) - (\nabla|\beta_1| E_i)(s, X_{s,t}(x,v)) \right) \right)
\lesssim_C |\alpha|^b \sum_{1 \leq |\beta_1| \leq |\beta|} \left( s^{-d-\delta_0} |\nabla|\beta_1| E_i|(s, X_{s,t}(x,v) - \alpha) + s^{-(d-\delta_0)} b \sup_b \frac{|\beta_1|^{\beta_1}}{|\beta_1|^b \left. \right. |\nabla|\beta_1| E_i|}(s, X_{s,t}(x,v)) \right).
\]
If $|\beta_1| = 0$, we can yield
\[
|\mathcal{F}_3(x) | \leq \sup_{s, t} \frac{|\mathcal{F}_3(x)|}{\mathcal{F}_3(x,v)} \| \nabla_x Y_{s,t}(x,v) \|_{L^\infty} \lesssim_{b, \delta_0, b} |\alpha|^b \sup_b \frac{|\beta_1|^{\beta_1}}{|\beta_1|^b \left. \right. |\nabla|\beta_1| E_i|}(s, X_{s,t}(x,v)) \right).
\]

Then, we consider another term. For $|\beta_2| \geq 0$,
\[
|\partial^2_\beta \mathcal{F}_3(x) | \lesssim \sum_{1 \leq |\beta_1| \leq |\beta|, |\beta_1| > 0} \left( s^{-d-\delta_0} |\nabla|\beta_1| E_i|(s, X_{s,t}(x,v) - \alpha) + s^{-(d-\delta_0)} b \sup_b \frac{|\beta_1|^{\beta_1}}{|\beta_1|^b \left. \right. |\nabla|\beta_1| E_i|}(s, X_{s,t}(x,v)) \right).
\]
Hence, we obtain
\[
\left\| \mathcal{T}_\alpha[E_i, \partial_\mu](t) \right\|_{L^p} \lesssim_C |\alpha|^b \int_0^t \int_{\mathbb{R}^d} \left( s^{-d-\delta_0} |\nabla^{|\beta_1|} E_i|(s, X_{s,t}(x,v)) \right) \left( s^{-(d-\delta_0)} b \sup_b \frac{|\beta_1|^{\beta_1}}{|\beta_1|^b \left. \right. |\nabla|\beta_1| E_i|}(s, X_{s,t}(x,v)) \right) \frac{ds}{v^N} \lesssim_C \frac{1}{(v)^N}.
\]
(6.42)

2.3) Estimate of $\mathcal{T}_\alpha[E_i, \partial_\mu]$.
Firstly, we estimate $\partial^2_\beta \mathcal{F}_{2,1}(x - \alpha)$. If $|\beta_1| \geq 1$,
\[
|\partial^2_\beta \mathcal{F}_{2,1}(x - \alpha) | \lesssim \sum_{1 \leq |\beta_1| \leq |\beta|} \left( \prod_{\beta_1} (\partial^2_\beta X_{s,t}(x,v))(\partial^2_\beta E_i)(s, X_{s,t}(x,v)) \right) \lesssim_C |\alpha|^b.
\]
Combining above with the definition of $\partial^2_x F_{2,1}(x - \alpha)$, we get for any $|\beta_1| \geq 0$,

$$|\partial^2_x F_{2,1}(x - \alpha)| \lesssim_C \sum_{0 \leq |\beta_3| \leq |\beta_1|} |(\partial^{\beta_3} E_i(s, X_{s,t}(x - \alpha, v))|.$$

Then, we compute the other term. For $|\beta_2| \geq 1$,

$$|\partial^2_x \eta_3(x)| \lesssim \sum_{1 \leq |\beta_2| \leq |\beta_1|, |\beta_3| > 0} \left| \prod_{j=1}^{\beta_3} \partial^2_x W_{\xi,t}(x - vt, v)(\partial^{\beta_3} \partial_{\xi}(V_{s,t}(x,v))) \right| \lesssim \sum_{1 \leq |\beta_2| \leq |\beta_1|, |\beta_3| > 0} \left| \prod_{j=1}^{\beta_3} \partial^2_x W_{\xi,t}(x - vt, v)(\partial^{\beta_3} \partial_{\xi}(V_{s,t}(x,v))) \right| \lesssim |C| |\alpha|^{N} \frac{1}{(v)^N} + \frac{1}{(v - \alpha)^N} \langle s \rangle^{-d+\delta_0-1} \lesssim_C |\alpha|^{N} \frac{1}{(v)^N} + \frac{1}{(v - \alpha)^N} \langle s \rangle^{-d+\delta_0-1}.$$  

If $|\beta_2| = 0$, we have

$$|\eta_3(x)| \lesssim_C |\alpha|^{N} \frac{1}{(v)^N} \langle s \rangle^{-d+\delta_0-1}.$$  

Hence, we deduce that

$$\left\| \alpha^{E_i, \partial_{\xi}}(t, x) \right\|_{L^p} \lesssim_b \delta_0, c, e \frac{1}{(v)^N} \frac{1}{(v - \alpha)^N} \langle s \rangle^{-d+\delta_0-1} ds.$$  

Thus, combining (6.41), (6.42) with (6.43), we have for $0 \leq t \leq 1$,

$$\sup_{\alpha} \left\| \alpha^{E_i, \partial_{\xi}}(t, x) \right\|_{L^\infty} \lesssim\, 1. \quad (6.44)$$

In conclusion, thanks to (6.40) and (6.44), we finish the proof.

7 Proof of the Theorem 2.2 and Theorem 2.3

Recall the operator $\mathcal{F}$ as:

$$\mathcal{F}(\rho, U) = \left( \mathcal{F}_1(\rho, U), \mathcal{F}_2(\rho, U) \right),$$

where $\mathcal{F}_1(\rho, U) = G*(t, \varepsilon)(\mathcal{I}(\rho, U) + \mathcal{R}(\rho, U) + A(U)) + \mathcal{I}(\rho, U) + \mathcal{R}(\rho, U)$ and $\mathcal{F}_2(\rho, U) = (1 - \Delta)^{-1}(\rho + A(U))$. Now we start to prove Theorem 2.2 and 2.3

Proof of Theorem 2.2 Let $\varepsilon_0$ be in Proposition 4.1 and take $\varepsilon \leq \varepsilon_0$. Assume

$$\|\rho(t)\|_{S^2_0}, \|\rho_1, U_1\|_{S^2_0}, \|\rho_2, U_2\|_{S^2_0} \leq \varepsilon.$$
Applying Theorem 5.8 with \( f = \mathcal{I}_f(\rho, U) + \mathcal{R}(\rho, U) \) and \( f = \mathcal{I}_f(\rho, U_1) - \mathcal{I}_f(\rho, U_2) + \mathcal{R}(\rho, U_1) - \mathcal{R}(\rho, U_2) \), there exists a constant \( M = M(\varepsilon, \delta, M^*) \) such that

\[
\| F_1(\rho, U) \|_a \leq M \left( \| (\mathcal{I}_f + \mathcal{R})(\rho, U) \|_a + \| A(U) \|_a \right),
\]

\[
\| F_1(\rho, U_1) - F_1(\rho, U_2) \|_a \leq M \left( \| (\mathcal{I}_f + \mathcal{R})(\rho, U_1) - (\mathcal{I}_f + \mathcal{R})(\rho, U_2) \|_a + \| A(U_1) - A(U_2) \|_a \right). \tag{7.1}
\]

Thanks to Lemma 4.2, Proposition 5.1, Proposition 6.3 and 4.6, we get that there exists a constant \( C_{3,1} = C_{3,1}(a, \varepsilon, d, M^*) > 0 \) such that

\[
\| F_1(\rho, U) \|_a \leq C_{3,1} \left( \sum_{p=1}^{\infty} \| D^p(f_0) \|_{L^1(\mathbb{R}^n)} + \| (\rho, U) \|_{S^1_{\rho}} + C_1 \epsilon^p x \| (\rho, U) \|_{S^1_{\rho}} \right)
\]

\[
\leq C_{3,1} \left( \frac{\varepsilon}{C_2} + \varepsilon^{1+a} + \varepsilon^p x \right) \leq \left( \frac{C_{3,1}}{C_2} + C_{3,1}(1 + C_A) \varepsilon^p x \right) \varepsilon, \tag{7.3}
\]

and

\[
\| F_2(\rho, U) \|_a \leq C_{3,2} \| \rho + A(U) \|_a \leq C_{3,2} \| (\rho, U) \|_{S^1_{\rho}} \leq C_{3,2} \varepsilon, \tag{7.4}
\]

\[
\| F_2(\rho, U_1) - F_2(\rho, U_2) \|_a \leq C_{3,2} \| \rho_1 - \rho_2 + A(U_1) - A(U_2) \|_a \leq C_{3,2} \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^1_{\rho}}. \tag{7.5}
\]

Moreover, with Proposition 5.3 and Proposition 6.3 there exists a constant \( C_4 = C_4(a, \varepsilon, d, M^*) > 0 \) such that

\[
\| F_1(\rho, U_1) - F_1(\rho, U_2) \|_a \leq C_4 \| A(U_1) - A(U_2) \|_a
\]

\[
+ C_4 \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^1_{\rho}} \leq \left( \sum_{p=1}^{\infty} \| D^p(\nabla x, f_0) \|_{L^1(\mathbb{R}^n)} + \| (\rho, U_1) \|_{S^1_{\rho}} + \| (\rho_2, U_2) \|_{S^1_{\rho}} \right)
\]

\[
\leq \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^1_{\rho}} \left( \frac{C_4}{C_2} + 2C_4 \right) \varepsilon^p + C_4 C_A \varepsilon^p \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^1_{\rho}}
\]

\[
\leq \| (\rho_1 - \rho_2, U_1 - U_2) \|_{S^1_{\rho}} \left( \frac{C_4}{C_2} + (2 + C_A)C_4 \right) \varepsilon^p. \tag{7.6}
\]

Combining (7.3), (7.4), (7.5) with (7.3), we obtain that

\[
\| F(\rho, U) \|_{S^1_{\rho}} \leq \| F_1(\rho, U) \|_a + \varepsilon^p \| F_2(\rho, U) \|_a \leq \left( \frac{C_{3,1}}{C_2} + C_{3,1}(1 + C_A) + C_{3,2} \varepsilon^p \right) \varepsilon,
\]

\[
\| F(\rho, U_1) - F(\rho, U_2) \|_{S^1_{\rho}} \leq \| F_1(\rho, U_1) - F_1(\rho, U_2) \|_a + \varepsilon^p \| F_2(\rho, U_1) - F_2(\rho, U_2) \|_a
\]

\[
\leq \| (\rho_1 - \rho_2, U_1 - U_2) \|_a \left( \frac{C_4}{C_2} + (2 + C_A)C_4 \right) \varepsilon^p
\]

\[
:= C_5 \varepsilon^p \| (\rho_1 - \rho_2, U_1 - U_2) \|_a. \tag{7.7}
\]

Hence, we choose \( C_2 = \max\{1, 8C_{3,1}, 8C_4\} \), and \( \hat{\varepsilon}_1 = \min \left\{ \left( \frac{1}{4(3,1)(1+C_A)+C_{3,2}} \right)^3, \left( \frac{1}{100+4(2+C_A)C_4+C_{3,2}} \right)^3 \right\}, \) we can yield that

\[
\| F(\rho, U) \|_a \leq \varepsilon, \quad \| (\rho, U_1) - F_1(\rho, U_2) \|_a \leq \frac{1}{2} \| (\rho_1 - \rho_2, U_1 - U_2) \|_a, \quad \forall \varepsilon \leq \hat{\varepsilon}_1. \tag{7.8}
\]

Therefore, we conclude that \( F \) is a compressed mapping in \( \mathcal{B}(\varepsilon) \) for \( \varepsilon \leq \hat{\varepsilon}_1 \). Now we consider the iterative sequence \( (\rho_n, U_n) = F(\rho_{n-1}, U_{n-1}) \) with \( \rho_0(t, x) = U_0(t, x) = 0 \). So, \( \rho_1(t, x) = F_2(0, 0)(t, x) = G *_{t,x} \tilde{g}_0(t, x) \) and \( U_1 = 0 \) with \( \tilde{g}_0(t, x) = \int_{\mathbb{R}^n} f_0(x - tv, v) dv \). From (7.3) we obtain that for \( n \in \mathbb{N} \),

\[
\| F(\rho_{n+1}, U_{n+1}) - F(\rho_n, U_n) \|_a \leq C_5 \varepsilon^p \| (\rho_n - \rho_{n-1}, U_n - U_{n-1}) \|_a = C_5 \varepsilon^p \| F(\rho_n, U_n) - F(\rho_{n-1}, U_{n-1}) \|_a. \tag{7.9}
\]

Thus \( \{(\rho_n, U_n)\}_{n=1}^{\infty} \) is a Cauchy sequence. We denote

\[
(\rho, U) = \left( \sum_{n=2}^{\infty} (\rho_n - \rho_{n-1}) + \rho_1, \sum_{n=2}^{\infty} (U_n - U_{n-1}) + U_1 \right),
\]

it is clear that \( (F_1(\rho, U), F_2(\rho, U)) = (\rho, U) \in \mathcal{B}(\varepsilon) \). Moreover by (7.9), we have

\[
\| (\rho - \rho_1, U - U_1) \|_a \leq \sum_{n=2}^{\infty} \| (\rho_n - \rho_{n-1}, U_n - U_{n-1}) \|_a
\]

\[
= \sum_{n=2}^{\infty} \| F(\rho_{n-1}, U_{n-1}) - F(\rho_{n-2}, U_{n-2}) \|_a \leq \varepsilon \sum_{n=2}^{\infty} (C_5 \varepsilon^p)^{n-1} \leq 2C_5 \varepsilon^4.
\]

43
Recall the definition of $\mathcal{F}(\rho)$ in (2.4), we have that the fixed point is exactly the solution of (2.5). Now we consider $\|\langle \rho, U \rangle\|_{1,a}$, note that

$$\|\rho\|_{1,a} = \|\mathcal{F}(\rho)\|_{1,a} = \|G^{\ast}_{(U)}(\mathcal{I}_0(\rho, U) + \mathcal{R}(\rho, U) + \mathcal{A}(U))\|_{1,a} \leq M (\|\mathcal{I}_0(\rho, U)\|_{1,a} + \|\mathcal{R}(\rho, U)\|_{1,a} + \|\mathcal{A}(U)\|_{1,a})$$

$$\|\mathcal{R}(\rho, U)\|_{1,a} + \|\mathcal{A}(U)\|_{1,a} + \varepsilon \|\mathcal{R}(\rho, U)\|_{1,a} + \varepsilon \|\mathcal{A}(U)\|_{1,a}.$$ 

Therefore we have

$$\|\rho\|_{1,a} + \frac{1}{4} \|U\|_{1,a} \leq M (\|\mathcal{I}_0(\rho, U)\|_{1,a} + \|\mathcal{R}(\rho, U)\|_{1,a}) + M \varepsilon \|U\|_{1,a} + \frac{1}{4} \|\rho\|_{1,a} + \frac{1}{4} \varepsilon \|U\|_{1,a},$$

which implies that

$$\|\rho\|_{1,a} + \|U\|_{1,a} \leq 8M (\|\mathcal{I}_0(\rho, U)\|_{1,a} + \|\mathcal{R}(\rho, U)\|_{1,a}).$$

Thanks to (5.2) and (6.15), we get

$$\|\mathcal{I}_0(\rho, U)\|_{1,a} + \|\mathcal{R}(\rho, U)\|_{1,a} \lesssim_{\varepsilon,a,M^*} 1,$$

where $\delta_1 = \min\{\frac{4 - \sqrt{11}}{6}, \frac{1 - \sqrt{3}}{3}\}$. Then we use Proposition 6.26 and (5.3) to obtain

$$\|\mathcal{I}_0(\rho, U)\|_{1,a} + \|\mathcal{R}(\rho, U)\|_{1,a} \lesssim_{\varepsilon,a,M^*} 1.$$

Consequently, it gives that

$$\|\langle \rho, U \rangle\|_{1,a} \lesssim_{\varepsilon,a,M^*} 1.$$

The proof is complete. 

**Proof of Theorem 2.3.** From Theorem 2.2, we know that $\|\rho, U\|_{S_\ell} \leq \tilde{\varepsilon}_0$ and $\|\rho, U\|_{1,a} \lesssim_{\varepsilon,a,M^*} 1$. Applying the Gagliardo-Nirenberg interpolation inequality, we get $\|\rho\|_{1-\delta_0} \lesssim_{\varepsilon,a,M^*} 1$, where $\delta_0 \in (0, \min\{\frac{4 - \sqrt{11}}{6}, \frac{1 - \sqrt{3}}{3}\})$. Combining Theorem 4.3 and Proposition 5.12 with Proposition 6.3, one has

$$\|\rho\|_{1,1-3\delta_0} = \|\mathcal{F}_0(\rho, U)\|_{1,1-3\delta_0} \leq M (\|\mathcal{I}_0(\rho, U)\|_{1,1-3\delta_0} + \|\mathcal{R}(\rho, U)\|_{1,1-3\delta_0} + \|\mathcal{A}(U)\|_{1,1-3\delta_0}),$$

$$\|\mathcal{I}_0(\rho, U)\|_{1,1-3\delta_0} + \|\mathcal{R}(\rho, U)\|_{1,1-3\delta_0} \lesssim_{\varepsilon,a,M^*, c,e} 1.$$

Using the same method in (7.10), we have $\|\rho\|_{1,1-3\delta_0} = \|\langle \mathcal{F}_0(\rho, U), \mathcal{F}_0(\rho, U)\rangle\|_{1,1-3\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e'} 1$. Note that

$$\|\langle \rho \rangle\|_{2,1-9\delta_0} = \|\mathcal{F}_0(\rho, U)\|_{2,1-9\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} \|\mathcal{I}_0(\rho, U)\|_{2,1-9\delta_0} + \|\mathcal{R}(\rho, U)\|_{2,1-9\delta_0} + \|\langle \rho \rangle\|_{2,1-9\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} 1.$$

We do one more step to obtain

$$\|\rho\|_{2,1-9\delta_0} = \|\mathcal{F}_0(\rho, U)\|_{2,1-9\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} \|\mathcal{I}_0(\rho, U)\|_{2,1-9\delta_0} + \|\mathcal{R}(\rho, U)\|_{2,1-9\delta_0} + \|\langle \rho \rangle\|_{2,1-9\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} 1.$$

Then, with $m$ times iteration, we have

$$\|\rho\|_{m,1-3^m\delta_0} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} 1.$$

Note that $1 - 3^m\delta_0 > b$, thus by the Gagliardo-Nirenberg interpolation inequality again to get

$$\|\rho\|_{m,b} \lesssim_{\varepsilon,a,M^*, e,c,e',C_{A,m}} 1,$$

which completes the proof.
8 Appendix

Lemma 8.1 Assume $\mathcal{H} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\varphi(x,v)$ satisfy $\|\nabla_{x,v}\varphi(x,v)\|_{L^\infty_{x,v}} \leq \frac{1}{t}$, then for $p = 1, \infty$, $0 \leq s \leq t$ and $t \geq 0$ we have

$$\left\| \int_{\mathbb{R}^d} \mathcal{H}(\varphi(x,v) + x - (t-s)v) \frac{dv}{\langle v \rangle^N} \right\|_{L^p_s} \lesssim \|\mathcal{H}\|_{L^p}. \tag{8.1}$$

Moreover, for $0 \leq s \leq \frac{t}{2}$ and $t \geq 1$, we also get

$$\left\| \int_{\mathbb{R}^d} \mathcal{H}(\varphi(x,v) + x - (t-s)v) \frac{dv}{\langle v \rangle^N} \right\|_{L^p_s} \lesssim \frac{1}{t^{\frac{d-1}{p}}} \|\mathcal{H}\|_{L^1}. \tag{8.2}$$

Proof. We can directly prove the first inequality as follows

$$\left\| \int_{\mathbb{R}^d} \mathcal{H}(\varphi(x,v) + x - (t-s)v) \frac{dv}{\langle v \rangle^N} \right\|_{L^p_s} \leq \|\mathcal{H}\|_{L^p} \left(1 + \|\nabla_{x,v}\varphi(x,v)\|_{L^\infty_{x,v}}\right) \lesssim \|\mathcal{H}\|_{L^p}.$$

Furthermore, we change variable $\zeta = x - (t-s)v$ and obtain for $0 \leq s \leq \frac{t}{2}$,

$$\left\| \int_{\mathbb{R}^d} \mathcal{H}(\varphi(x,v) + x - (t-s)v) \frac{dv}{\langle v \rangle^N} \right\|_{L^p_s} = \frac{1}{(t-s)^d} \left\| \int_{\mathbb{R}^d} \mathcal{H} \left(\varphi(x, \frac{x-\zeta}{t-s}) + \zeta\right) \frac{dv}{\langle \frac{x-\zeta}{t-s} \rangle^N} \right\|_{L^\infty_x}$$

$$\lesssim \frac{1}{t^d} \sup_{x \in \mathbb{R}^d} \left(\|\nabla_{x,v}\varphi(x, \frac{x-\zeta}{t-s})\| (t-s)^{-d} + 1\right)^{-1} \|\mathcal{H}\|_{L^1} \lesssim \frac{1}{t^{d-1}} \|\mathcal{H}\|_{L^1}.$$

\[ \blacksquare \]

Remark 8.2 Similarly, under the same assumptions of $\mathcal{H}$ and $\varphi$, one has

$$\left\| \int_{\mathbb{R}^d} |\mathcal{H}(\varphi(x,v) + x - (t-s)v)| \left(\frac{1}{\langle v \rangle^N} + \frac{1}{\langle v - \frac{2\zeta}{t} \rangle^N}\right) dv \right\|_{L^p_s} \lesssim \|\mathcal{H}\|_{L^p}, \tag{8.3}$$

and for $0 \leq s \leq \frac{t}{2}$, $t \geq 1$,

$$\left\| \int_{\mathbb{R}^d} |\mathcal{H}(\varphi(x,v) + x - (t-s)v)| \left(\frac{1}{\langle v \rangle^N} + \frac{1}{\langle v - \frac{2\zeta}{t} \rangle^N}\right) dv \right\|_{L^p_s} \lesssim \frac{1}{t^{\frac{d-1}{p}}} \|\mathcal{H}\|_{L^1}. \tag{8.4}$$

References

[1] A. Arsenév. Global Existence of a Weak Solution of Vlasov system of equations, U.S.S.R. Comp. Math. Math. Phys., 15 (1975), pp. 131-143.

[2] C. Bardos and P. Degond. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. Annales de l’Institut Henri Poincaré C, Analyse non linéaire, 2(2):101-118, 1985.

[3] C. Bardos, F. Golse, T. Nguyen, and R. Sentis. The Maxwell-Boltzmann approximation for ion kinetic modeling. Physica. D, 376/377:94–107, 2018.

[4] J. Batt. Global Symmetric Solutions of the Initial Value Problem of Stellar Dynamics, Journal of Differential Equations, 25 (1977), pp. 342-364.

[5] J. Bedrossian and N. Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publ. Math. Inst. Hautes Études Sci., 122:195–300, 2015.

[6] J. Bedrossian. Nonlinear echoes and Landau damping with insufficient regularity. Tunis. J. of Math., Vol. 3 (2021), No. 1, 121-205, DOI: 10.2140/tunis.2021.3.121.

[7] J. Bedrossian, N. Masmoudi and C. Mouhot. Linearized wave damping structure of Vlasov-Poisson in $\mathbb{R}^3$. arXiv:2007.08580, 2020.

[8] J. Bedrossian, N. Masmoudi and C. Mouhot. Landau damping in finite regularity for unconfined systems with screened interactions. Communications on Pure and Applied Mathematics, 71(3): 537-576, 2018.

[9] J. Bedrossian, N. Masmoudi and C. Mouhot. Landau damping: paraproducts and Gevrey regularity. Annals of PDE, 2(1): 1-71, 2016.
[10] F. Bouchut. Global weak solution of the Vlasov-Poisson system for small electrons mass. *Comm. Partial Differential Equations*, 16(8-9):1337-1365, 1991.

[11] K. Chen and Q.H. Nguyen. The Peskin Problem with $B_{L^2}^{1,\infty}$ initial data, submitted, arXiv:2107.13854.

[12] S.-H. Choi, S.-Y. Ha and H. Lee. Dispersion estimates for the two-dimensional Vlasov–Yukawa system with small data. *Journal of Differential Equations*, 250(1):515–550, 2011.

[13] P. Flynn, Z. Ouyang, B. Pausader, and K. Widmayer. Scattering Map for the Vlasov–Poisson System. *Peking Math J* (2021). https://doi.org/10.1007/s42543-021-00041-x

[14] R. T. Glassey. The Cauchy problem in kinetic theory. *Society for Industrial and Applied Mathematics, Philadelphia, PA*, 1996.

[15] E. Grenier, T. T. Nguyen and I. Rodnianski. Landau damping for analytic and Gevrey data. arXiv:2004.05979, 2020.

[16] E. Grenier, T. T. Nguyen, and I. Rodnianski. Plasma echoes near stable Penrose data. *SIAM J. Math. Anal.* (to appear) preprint arXiv:2004.05984, 2020.

[17] M. Griffin-Pickering and M. Iacobelli. Global well-posedness in 3-dimensions for the Vlasov–Poisson system with massless electrons. arXiv:1810.06928.

[18] Y. Guo and B. Pausader. Global Smooth Ion Dynamics in the Euler-Poisson System. *Commun. Math. Phys.* 303, 89–125 (2011). https://doi.org/10.1007/s00220-011-1193-1.

[19] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Communications in Partial Differential Equations*, 36(8):1385-1425, 2011.

[20] D. Han-Kwan and M. Iacobelli. The quasineutral limit of the Vlasov-Poisson equation in Wasserstein metric. *Commun. Math. Sci.*, 15(2):481–509, 2017.

[21] D. Han-Kwan, T. T. Nguyen and F. Rousset. Asymptotic stability of equilibria for screened Vlasov–Poisson systems via pointwise dispersive estimates. *Annals of PDE*, 7(2): 1-37, 2021.

[22] D. Han-Kwan, T. Nguyen and F. Rousset. On the linearized Vlasov–Poisson system on the whole space around stable homogeneous equilibria. *Communications in Mathematical Physics*, 387(3): 1405-1440, 2021.

[23] E. Horst. On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation (Parts I and II), *Math. Meth. Appl. Sci.* 3 (1981), pp. 229-248 and 4 (1982), pp. 19-32.

[24] E. Horst. On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Mathematical Methods in the Applied Sciences*, 16(2):75–86, 1993.

[25] L. Huang, Q-H. Nguyen and Y. Xu. Nonlinear Landau damping for the 2d Vlasov-Poisson system with massless electrons around Penrose-stable equilibria, arXiv:2206.11744.

[26] H.-J. Hwang, A. Rendall, and J.-L. Velázquez. Optimal gradient estimates and asymptotic behaviour for the Vlasov-Poisson system with small initial data. *Archive for Rational Mechanics and Analysis*, 200(1):313–360, 2011.

[27] A. D. Ionescu and H. Jia. Inviscid damping near the Couette flow in a channel. *Communications in Mathematical Physics*, 374(3):2015–2096, 2020.

[28] A. Ionescu, B. Pausader, X. Wang and K. Widmayer. On the Asymptotic Behavior of Solutions to the Vlasov–Poisson System. *International Mathematics Research Notices*, 2021, rnab155, https://doi.org/10.1093/imrn/rnab155

[29] A. Ionescu, B. Pausader, X. Wang and K. Widmayer. Nonlinear Landau damping for the Vlasov-Poisson system in $\mathbb{R}^3$: the Poisson equilibrium. arXiv:2205.04340, 2022.

[30] A. D. Ionescu and H. Jia. Nonlinear inviscid damping near monotonic shear flows. *Acta Mathematica* (to appear), arXiv:2001.03087, 2020.

[31] L. Landau. On the vibration of the electronic plasma. *J. Phys. USSR*, 10(25), 1946.

[32] P. L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Inventiones mathematicae*, 105(1): 415-430, 1991.
[33] C. Mouhot and C. Villani. On Landau damping. *Acta Mathematica*, 207(1):29–201, 2011.

[34] N. Masmoudi, W. Zhao. Nonlinear inviscid damping for a class of monotone shear flows in finite channel, *arXiv:2001.08564*.

[35] Q. H. Nguyen. Quantitative estimates for regular Lagrangian flows with BV vector fields. *Communications on Pure and Applied Mathematics*, 74(6): 1129-1192, 2021.

[36] T. T. Nguyen. Derivative estimates for screened Vlasov-Poisson system around Penrose-stable equilibria. *Kinetic and Related Models*, 2020, 13(6): 1193-1218. doi: 10.3934/krm.2020043.

[37] B. Pausader and K. Widmayer. Stability of a point charge for the Vlasov-Poisson system: the radial case. *Communications in Mathematical Physics*, 385(3):1741–1769, 2021.

[38] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *Journal of Differential Equations*, 95(2):281–303, 1992.

[39] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Communications in Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[40] J. Smulevici. Small data solutions of the Vlasov-Poisson system and the vector field method. *Annals of PDE*, 2(2): 11-55, 2016.

[41] X. Wang. Decay estimates for the 3D relativistic and non-relativistic Vlasov-Poisson systems. *arXiv:1805.10837*, 2018.