STABILITY AND SYMMETRY BREAKING
FOR CLOSED STRING WITH MASSIVE POINT

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Abstract
The closed relativistic string carrying a point-like mass in the space with nontrivial geometry is considered. For rotational states of this system (resulting in non-trivial Regge trajectories) the stability problem is solved. It was shown that rotations of the folded string with the massive point placed at the rotational center are stable (with respect to small disturbances) if the mass exceeds some critical value: $m > m_{cr}$. But these rotational states are unstable in the opposite case $m < m_{cr}$. We can treat this effect as the spontaneous symmetry breaking for the string state. Other classes of rotational motions of this system have appeared to be stable. These results were obtained both in numerical experiments and the analytical investigation of small disturbances for the rotational states.

1. Introduction
We consider the closed relativistic string carrying one massive point. This system moves in the space $\mathcal{M} = R^{1,3} \times T^{D-4}$, where $R^{1,3}$ is 3+1-dimensional Minkowski space and the compact manifold $T^{D-4}$ is $D-4$-dimensional torus resulting from the compactification procedure [1]. The structure of homotopic classes for the closed string in the manifold $\mathcal{M}$ is non-trivial (except for the simplest variant $\mathcal{M} = R^{1,3}, D = 4$, but it is also included as the particular case into our consideration).

The action of this system is [2]

$$S = -\gamma \int_{\Omega} \sqrt{-g} \, d\tau \, d\sigma - m \int \sqrt{\dot{x}_1^2(\tau)} \, d\tau. \quad (1)$$

Here $\gamma$ is the string tension, $m$ is the point-like mass, $g$ is the determinant of the induced metric $g_{ab} = G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu$ on the string world surface $X^\mu(\tau, \sigma)$, embedded in the manifold $\mathcal{M} = R^{1,3} \times T^{D-4}$. We denote $x^0, x^1, x^2, x^3$ the coordinates in Minkowski space $R^{1,3}$ and the torus $T^{D-4}$ has cyclic coordinates $x^k$ ($k = 4, 5 \ldots$) with periods $\ell_k$, that is, points with coordinates $x^k$ and $x^k + N_k \ell_k$, $N_k \in \mathbb{Z}$ are identified. We suppose $T^{D-4}$ to be flat manifold so the metric of $\mathcal{M}$ is equal to that of Minkowski space $R^{1,D-1}$, $G_{\mu\nu}(X) = \eta_{\mu\nu}$.

In the action (1) the speed of light $c = 1$, $\Omega = \{ \tau, \sigma: \tau_1 < \tau < \tau_2, \sigma_1(\tau) < \sigma < \sigma_2(\tau) \}$; the equations $x_i^\mu(\tau) = X^\mu(\tau, \sigma_i(\tau))$, $i = 1, 2$ describe the same trajectory of the massive point

$$x_i^\mu(\tau) = X^\mu(\tau, \sigma_1(\tau)) = X^\mu(\tau^*, \sigma_2(\tau^*)) + \sum_k N_k \ell_k \delta_k^\mu \quad (2)$$

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on the tube-like world surface [3, 4]. Here \( \delta_k^\mu = \begin{cases} 1, & \mu = k, \\ 0, & \mu \neq k. \end{cases} \) This line can be parameterized with two different parameters \( \tau \) and \( \tau^* \). They are connected by the relation \( \tau^* = \tau^*(\tau) \). This relation should be added to the closure condition (2) of the world surface.

Equations of motion for this system result from the action (1) via its variation. They may be reduced to the simplest form [2] without loss of generality under the orthonormality conditions on the world surface

\[
(\partial_\tau X \pm \partial_\sigma X)^2 = 0,
\]

and the conditions

\[
\sigma_1(\tau) = 0, \quad \sigma_2(\tau) = 2\pi.
\]

Under these conditions the string equations of motion take the mentioned simplest form

\[
\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0,
\]

\[
\frac{m d}{d\tau} \dot{X}^\mu(\tau, 0) \sqrt{X^2(\tau, 0)} + \gamma [X'^\mu(\tau^*, 2\pi) - X'^\mu(\tau, 0)] = 0.
\]

Here \( \dot{X}^\mu \equiv \partial_\tau X^\mu, X'^\mu \equiv \partial_\sigma X^\mu \); scalar square is \( a^2 = \eta_{\mu\nu} a^\mu a^\nu \). Equation (6) may be treated as the boundary condition for Eq. (5).

Thus, dynamics of the closed relativistic string with the massive point in the manifold \( \mathcal{M} \) is described by the system (2) – (6).

The solutions of system (2) – (6) was obtained in Ref. [2] under the restrictions

\[
\frac{\gamma m}{\sqrt{X^2(\tau, 0)}} = Q = \text{const}, \quad \tau^* = \tau + \tau_0, \quad \tau_0 = \text{const}
\]

in the form of the following Fourier series:

\[
X^\mu = \sum_{n=-\infty}^\infty \chi_n^\mu(\sigma) \exp(-i \omega_n \tau).
\]

Frequencies \( \omega_n \) (in pairs with values \( \tau_0 \)) are solutions of system of the equations

\[
\cos 2\pi \omega_n - \cos \omega_n \tau_0 = \frac{\omega_n}{2Q} \sin 2\pi \omega_n, \tag{9}
\]

\[
1 + \frac{\tau_0^2}{4\pi^2} - \frac{\tau_0}{2\pi} \frac{1 - \cos \omega_n \tau_0 \cdot \cos 2\pi \omega_n}{\sin \omega_n \tau_0 \cdot \sin 2\pi \omega_n} = \left(1 - \frac{\tau_0 \sin 2\pi \omega_n}{2\pi \sin \omega_n \tau_0}\right) \frac{\gamma^2}{m^2 Q^2} \sum_{k>3} b_k^2. \tag{10}
\]

Here \( b_k = \frac{\ell_k N_k}{2\pi} \) are the factors connected with winding numbers \( N_k \) for the cyclic coordinates \( x^k, k \geq 4 \).

We shall consider below the so called one-frequency states. They are solutions of the type (8) containing only one nonzero frequency \( \omega_n \) and describing rotational motions of the system. These states are the most interesting ones, because they generate non-trivial spectrum of Regge trajectories [2] and may be applied in the hadron spectroscopy.

One-frequency states of the string system (1) may be divided into three classes. In the case \( \tau_0 \neq 0 \) they have the form [2]

\[
X^\mu = e_0^\mu a_0 \left( \tau - \frac{\tau_0}{2\pi} \sigma \right) + \sum_{k>3} e_k^\mu b_k \sigma + A_n \left[ S \cos \omega_n \sigma + (C_0 - C) \sin \omega_n \sigma \right] e^\mu(\omega_n \tau) - S_0 \sin \omega_n \sigma \cdot \dot{e}^\mu(\omega_n \tau), \tag{11}
\]
Here $e_0, e_1, e_2, \ldots e_{D-1}$ are vectors of some fixed orthonormal basis in the manifold $\mathcal{M}$ ($e_0, e_1, e_2, e_3$ is the basis in $\mathbb{R}^{1,3}$) and

\[ e^\mu(\omega_n \tau) = e_1^\mu \cos \omega_n \tau + e_2^\mu \sin \omega_n \tau, \quad \dot{e}^\mu(\omega_n \tau) = \omega_n^{-1} \frac{d}{d\tau} e^\mu(\omega_n \tau) \]  

are unit orthogonal rotating vectors. The following notations are used

\[ C = \cos 2\pi \omega_n, \quad S = \sin 2\pi \omega_n, \quad C_0 = \cos \omega_n \tau_0, \quad S_0 = \sin \omega_n \tau_0, \]  

where the values $\omega_n$ and $\tau_0$ are taken from Eqs. (9), (10).

Parameters $a_0, A_n$ are related by the conditions

\[ a_0 = \frac{mQ}{\gamma} (1 - v^2)^{-1/2}, \quad A_n = \frac{a_0 v}{\omega_n S}, \quad v^2 = \frac{\tau_0 S}{2\pi S_0}. \]  

Here $v$ is the speed of the massive point moving along a circle; $0 \leq v < 1$.

Solutions (11) describe uniform rotations of the closed string. In the case $b_k = 0$ the string has the form of a closed hypocycloid joined at non-zero angle in the massive point. This hypocycloid uniformly rotates in the $e_1, e_2$ plane. The similar solutions for the string baryon model “triangle” were obtained and studied in Ref. [3].

In the case $\tau_0 = 0$ rotational states of the system are divided into two classes, because the equation (9) (defining their frequencies $\omega_n$) has the form

\[ \sin \pi \omega_n \cdot \left( \sin \pi \omega_n + \frac{\omega_n}{2Q} \cos \pi \omega_n \right) = 0. \]  

The roots $\omega_n = -2Q \tan \pi \omega_n$ of Eq. (15) determine rotational motions in the form

\[ X^\mu = e_0^\mu a_0 \tau + \sum_{k>3} e_k^\mu b_k \sigma + A_n \cos [\omega_n (\sigma - \pi)] \cdot e^\mu(\omega_n \tau). \]  

Solutions (16) describe uniform rotations of the sinusoidal string. The massive point with mass $m$ (at $\sigma = 0$) rotates with the string. The trajectory of this point is a circle.

The roots of the equation (15) $\omega_n = n$ result in the following solutions:

\[ X^\mu = e_0^\mu a_0 \tau + \sum_{k>3} e_k^\mu b_k \sigma + A_n \sin n \sigma \cdot e^\mu(n \tau), \]  

Here $a_0 = \sqrt{n^2 A_n^2 + \sum b_k^2}$. These solutions describe rotations of the string similar to motions (16) but the massive point for states (17) is placed at the rotational center.

In the case $b_k = 0$ expressions (16) and (17) describe uniform rotations of the string folded several times. It rotates in Minkowski space $\mathbb{R}^{1,3}$ and the cyclic coordinates of $T^{D-4}$ do not “work”. Such string motions have been classified in Ref. [5].

For rotational motions (11), (16) and (17) in the case $b_k = 0$ the string has points moving at the speed of light. There are singularities of the metrics $\dot{X}^2 = X'^2 = 0$ at the world lines of these points on the world surface. At the same time, for $b_k \neq 0$ the mentioned solutions have no such singularities.

Possible applications of the solutions (11), (16) and (17) in hadron spectroscopy essentially depend on stability or instability of these states with respect to small disturbances. In the following section the numerical experiments in this direction are described. They showed
rather unexpected results and in Sect. 3 the stability problem for the mentioned rotational states is studied analytically.

2. Numerical description and visualization of arbitrary motions of the system

To solve the stability problem for rotational motions (11), (16) and (17) we study evolution of their small disturbances numerically.

Any motion of the system with the action (1) can be determined unambiguously, if the following initial conditions of the system are given: an initial position of the system in Minkowski space and initial velocities of string points [6]. The solution of this initial-boundary value problem for various string models meson and baryons was suggested earlier in Refs. [6, 7].

We apply this approach to the closed string with the action (1) using the general solution of the equation of motion (5)

\[ X^\mu(\tau, \sigma) = \frac{1}{2}[\Psi_+^\mu(\tau + \sigma) + \Psi_-^\mu(\tau - \sigma)]. \] (18)

Here

\[ \Psi_+^2 = \Psi_-^2 = 0 \] (19)

by virtue of the orthogonality conditions (3).

It is convenient to use the following notation for the unit vector of velocity of the massive point

\[ U^\mu = \frac{\dot{X}^\mu(\tau, 0)}{\sqrt{X^2(\tau, 0)}} = \frac{\dot{X}^\mu(\tau^*, 2\pi)}{\sqrt{X^2(\tau^*, 2\pi)}}. \]

It takes the form

\[ U^\mu = \frac{\Psi_+^\mu(\tau) + \Psi_-^\mu(\tau)}{\sqrt{2\langle \Psi_+^\mu(\tau), \Psi_-^\mu(\tau) \rangle}} = \frac{\Psi_+^\mu(\tau^* + 2\pi) + \Psi_-^\mu(\tau^* - 2\pi)}{\sqrt{2\langle \Psi_+^\mu(\tau^* + 2\pi), \Psi_-^\mu(\tau^* - 2\pi) \rangle}}. \] (20)

after substitution of the expression (18). Here and below the square product is denoted \( \langle a, b \rangle = \eta_{\mu\nu} a^\mu b^\nu \).

We reduce the equations of motion (2) - (6) to the system of differential equations with respect to functions \( \Psi_\pm^\mu, U^\mu \). For this purpose we multiply the expression (20) by vectors \( \Psi_\pm^\mu(\tau), \Psi_\pm^\mu(\tau^* \pm 2\pi) \) and take into account the conditions (19).

We obtain the following relation:

\[ \langle U, \Psi_\pm^\mu(\tau) \rangle = \sqrt{\frac{\langle \Psi_\pm^\mu(\tau), \Psi_\pm^\mu(\tau) \rangle}{2}}, \quad \langle U, \Psi_\pm^\mu(\tau^* \pm 2\pi) \rangle = \sqrt{\frac{\langle \Psi_\pm^\mu(\tau^* + 2\pi), \Psi_\pm^\mu(\tau^* - 2\pi) \rangle}{2}}. \]

This equality allows us to rewrite expressions (20) as

\[ \Psi_+^\mu(\tau) + \Psi_-^\mu(\tau) = 2\langle U, \Psi_\pm^\mu(\tau) \rangle U^\mu, \] (21)

\[ \Psi_+^\mu(\tau^* + 2\pi) + \Psi_-^\mu(\tau^* - 2\pi) = 2\langle U, \Psi_\pm^\mu(\tau^* \pm 2\pi) \rangle U^\mu. \] (22)

Using these relations, we write down the boundary condition (6) as follows:

\[ \frac{dU^\mu}{d\tau} = -\frac{\gamma}{m} \left[ U^\mu(\tau) U_\mu(\tau) - \delta^\mu_\nu \right] \left[ \Psi_+^\mu(\tau) + \Psi_-^\mu(\tau^* - 2\pi) \right]. \] (23)
Differentiating the closure condition (2) on $\tau$

$$\dot{X}^\mu(\tau^*, 2\pi) \cdot \frac{d\tau^*}{d\tau} = \dot{X}^\mu(\tau, 0)$$

and squaring this equality with using Eqs. (18), (19) we obtain the relation between $\tau^*$ and $\tau$:

$$\frac{d\tau^*}{d\tau} = \frac{\langle U, \Psi^\prime_\pm(\tau) \rangle}{\langle U, \Psi^\prime_\pm(\tau \pm 2\pi) \rangle}. \quad (24)$$

Note that the function $\tau^*(\tau)$ in Eq. (2) can not be taken in an arbitrary form. It should be calculated from the dynamical equations of this system, in particular, from Eq. (24).

Equations (21) – (24) allow us to describe numerically any motion of the closed relativistic string with one point-like mass.

An initial position of the string can be given as the parametric curve

$$x^\mu = \rho^\mu(\lambda), \quad \lambda \in [\lambda_1, \lambda_2]$$

and initial velocities of string points as the function $v^\mu(\lambda), \quad \lambda \in [\lambda_1, \lambda_2]$. Here $\rho'(\lambda)$ is the space-like vector, $v^\mu$ is the time-like vector.

Consider in more detail the numerical procedure for solving the described initial-boundary value problem. This procedure includes three stages.

At the first stage it is necessary to calculate the functions $\Psi^\prime_\pm$ in the segments of influence of the initial data. We can do it without loss of a generality if we use the freedom in choice of two functions $\tilde{\tau}(\lambda)$ and $\tilde{\sigma}(\lambda)$ in parametrization of the initial curve (initial position of the string) [6, 7]:

$$X^\mu(\tilde{\tau}(\lambda), \tilde{\sigma}(\lambda)) = \rho^\mu(\lambda).$$

For the considered system it is convenient to choose them as

$$\tilde{\tau}'(\lambda) = \langle v, \rho' \rangle / v^2, \quad \tilde{\sigma}'(\lambda) = \Delta / v^2;$$

Here $\Delta(\lambda) = \sqrt{\langle v, \rho' \rangle^2 - v^2 \rho^2}$.

Then we integrate numerically $\tilde{\tau}'(\lambda)$ and $\tilde{\sigma}'(\lambda)$ using the Simpson’s method and obtain the functions $\tilde{\tau}(\lambda)$ and $\tilde{\sigma}(\lambda)$, under the assumption $\tilde{\tau}(\lambda_1) = \tilde{\sigma}(\lambda_1) = 0$.

With the help of them it possible to find the functions $\Psi^\prime_\pm$ via formulas [6]

$$\Psi^\prime_\pm(\tau(\lambda) \pm \tilde{\sigma}(\lambda)) = \frac{(\Delta \pm P)\rho^\mu \mp \rho'^2 v^\mu}{\Delta[\tilde{\tau}'(\lambda) \pm \tilde{\sigma}'(\lambda)]}$$

on initial segments

$$\Psi^\prime_+(\xi_+), \quad \xi_+ \in [0; \tilde{\tau}(\lambda_2) + \tilde{\sigma}(\lambda_2)]; \quad \Psi^\prime_-(\xi_-), \quad \xi_- \in [\tilde{\tau}(\lambda_2) - \tilde{\sigma}(\lambda_2); 0]. \quad (25)$$

They are the mentioned segments of influence of the initial data.

For the further numerical calculation it is convenient to recalculate the values of functions $\Psi^\prime_\pm$ in equidistant points of their arguments with step $h$ using linear approximation.

The second stage includes extension of the functions $\Psi^\prime_\pm(\tau)$ beyond the initial segments (25) up to given maximal value of their argument $\tau_{max}$. For this purpose we integrate the equation (23) and simultaneously calculate current values of functions $\Psi^\prime_-(\tau), \Psi^\prime_+(\tau + 2\pi)$ and $\tau^*(\tau)$ using the equations (21), (22), (24).
To increase accuracy of the difference scheme we substitute \( \frac{[U^\mu(\tau + h) - U^\mu(\tau)]}{h} \) into the left part of Eq. (23) and calculate the value of the right part at \( \tau + h/2 \). For this reason functions \( \Psi_{\pm}^\mu(\tau) \) are taken at the points \( \tau + h/2 \). Values of \( U^\mu \) at these points are replaced by \( \frac{1}{2}[U^\mu(\tau) + U^\mu(\tau + h)] \).

Eq. (23) will be transformed into the following difference scheme:

\[
U^\mu(\tau + h)
\left(1 + \frac{\gamma h}{2m} U, \Pi\right)
= U^\mu(\tau) + \frac{\gamma h}{m} \left( \Pi^\mu - \frac{U^\mu(\tau)}{2} (U, \Pi) \right).
\]

Here \( \Pi^\mu = \Psi_+^\mu(\tau + h/2) + \Psi_-^\mu(\tau^* + h/2) - 2\pi \).

Using the obtained values \( U^\mu(\tau + h) \) we calculate \( \Psi_+^\mu(\tau + h/2) \) from Eq. (21), \( \Psi_-^\mu(\tau^* + 2\pi + h/2) \) from Eq. (22), and \( \tau^*(\tau) \) from Eq. (24).

When the values of \( \Psi_{pm}^\mu, \tau^*(\tau) \) are calculated from \( \tau = 0 \) up to \( \tau = \tau_{\text{max}} \), we integrate the vector functions \( \Psi_{pm}^\mu \) with the help of the Simpson’s method (approximating the integrated function by square trinomials). If values of the function are known in half-integer points this method is reduced to the formula

\[
\int_0^h f(x) \, dx = \frac{h}{24} \left[ f\left( -\frac{h}{2} \right) + 22f\left( \frac{h}{2} \right) + f\left( \frac{3h}{2} \right) \right].
\]

At the third stage of this procedure we calculate the function \( X^\mu(\tau, \sigma) \) from Eq. (18) (the world surface) and build projections of lines \( t = X^0(\tau, \sigma) = \text{const} \) onto the \( xy \) plane where the axes \( x \) and \( y \) are directed accordingly along vectors \( e_1 \) and \( e_2 \).

Visualization of string motions was represented in various ways, in particular, as animated cartoons. In this case the second and third stages of the calculations should be made almost simultaneously (with certain delay).

![Figure 1: Quasirotational motion of the system (11).](image)

Another way of visualization is shown below in Figs. 1 and 2. In these figures projections of sequential positions of the system on the plane \( xy \) (projection of sections \( t = \text{const} \) of the
world surface) are made at regular intervals of time $\Delta t = 0.3$. Numbers of these positions in ascending order of $t$ are specified. Position of the massive point is designated by the small circle.

Fig. 1 demonstrates the example of motion of the system close to the exact solution (11) with parameters $m/\gamma = 1; \sum b_k^2 = 0.25; a_0 = 1.55; \omega = 1.285; \tau_0 = 2.006$. These curves slightly differ from hypocycloids because of $b_k \neq 0$. The initial position (marked by number 1 in Fig. 1a) corresponds to the section $t = 0$ of the world surface (11). Initial velocity contains small disturbance in the form $\delta v^\mu = e_2^\mu \cdot \sin^2 \sigma$.

Numerical experiments show, that amplitudes of small disturbances of states (11) and also (16) do not grow during enough large time intervals $t$. For example, in Fig. 1b the position 101 corresponds to $t = 30$, the curvilinear triangle after 3 rotations keeps its form in general. We may conclude that the rotational states (11) and also (16) are stable with respect to small disturbances on the level of numerical simulation.

Numerical investigations of the rotational states (17) with the massive point at the center of rotation showed interesting and unexpected results. We obtained that stability of this state depends on values of mass $m$ of the material point, or, more precisely, it depends on values of parameter $m/\gamma$. If this value is large enough, the motion is stable, but if the values of $m/\gamma$ is less than some critical value the disturbed motions (17) are unstable. Amplitudes of small disturbances of the exact solution (17) grow with growing time $t$.

The example of such a motion is shown in Fig. 2. Here $m/\gamma = 1; n = 1; b_k^\mu = 0; a_0 = A_1 = 1$ and initial conditions have the form

$$
\rho^\mu (\lambda) = e_1^\mu \sin \lambda, \quad v^\mu (\lambda) = e_0^\mu + e_2^\mu (\sin \lambda + 0.001 \cos^2 \lambda), \quad \lambda \in [0, 2\pi].
$$

We see that at the initial stage (Fig. 2a) the folded string uniformly rotates, keeping the rectilinear form, and the massive point is at rest at the center. After some time interval (depending on the amplitude of initial disturbance) the massive point leaves the center of rotation and the string shape changes. This stage is shown in Fig. 2b. The further evolution results in the essential change of the string shape (Fig. 2c), the string takes the form of rotating curvilinear triangle and the massive point periodically moves from one its corner to another. This character of quasiperiodical motion is the final stage of any disturbed state (17), it does not depend on the form of initial disturbance. But the typical time of breaking an initial state

![Figure 2: Evolution of the instability for the motion (17).](image-url)
depends on amplitude of this disturbance. If the initial data corresponding to solutions (17), are not disturbed, experiments demonstrate instability of the motion because of disturbances from errors of numerical calculations.

However, at large values of mass $m$ or parameter $m/\gamma$ (for the given initial conditions (17) this means $m/\gamma > 6$) numerical experiments shows, that the massive point does not leave the center of rotation, and motions (17) are stable.

This effect is similar to the spontaneous symmetry breaking. The state (17) remains symmetric and stable if the mass of the material point $m$ exceeds some critical value: $m > m_{cr}$. But the symmetry of this state is breaking spontaneously because of instability in the case

$$m < m_{cr}.$$ (27)

These results of numerical experiments are unexpected and an analytical verification of them is required. The analytical study of small disturbances of rotational states is presented in the following section.

### 3. Spectrum of small disturbances

We denote $\tilde{\Psi}^\mu_{\pm}(\tau \pm \sigma)$ the functions in the expression (18) for the considered rotational motions (11), (16) and (17). For example, for the world surfaces (17) (with the mass at the center) derivatives of these vector functions are

$$\tilde{\Psi}^\mu_{\pm}(\tau) = e^\mu_0 a_0 \pm \sum_{k>3} e^\mu_k b_k \pm nA_n \cdot e^\mu(n\tau)$$ (28)

and for the hypocycloidal states (11) they are

$$\tilde{\Psi}^\mu_{\pm}(\tau) = e^\mu_0 (1 \mp \theta) a_0 \pm \sum_{k>3} e^\mu_k b_k + \omega_n A_n \left[ \pm (C_0 - C) e^\mu(\omega_n \tau) + (S_0 \mp S) e^\mu(\omega_n \tau) \right].$$ (29)

Here

$$\theta = \frac{\tau_0}{2\pi},$$ (30)

the values $\tau_0$ and $2\pi$ are roots of Eqs. (9), (10), and notations (12), (13) are used.

To describe any small disturbances of the rotational motion of the system, that is motions close to states (11), (16) or (17) we consider vector functions $\Psi^\mu_{\pm}$ close to $\tilde{\Psi}^\mu_{\pm}$ in the form

$$\Psi^\mu_{\pm}(\tau) = \tilde{\Psi}^\mu_{\pm}(\tau) + \varphi^\mu_{\pm}(\tau).$$ (31)

The disturbance $\varphi^\mu_{\pm}(\tau)$ is supposed to be small, so we omit squares of $\varphi_{\pm}$ when we substitute the expression (31) into dynamical equations (2) and (6). In other words, we work in the first linear vicinity of the states (11), (16) or (17). Both functions $\Psi^\mu_{\pm}$ and $\tilde{\Psi}^\mu_{\pm}$ in expression (31) must satisfy the condition (19), hence in the first order approximation on $\varphi_{\pm}$ the following scalar product equals zero:

$$\langle \tilde{\Psi}^\mu_{\pm}, \varphi^\mu_{\pm} \rangle = 0.$$ (32)

For the disturbed motions the equality (7) $\tau^* = \tau + \tau_0$, generally speaking, is not carried out and should be replaced with the equality

$$\tau^* = \tau + \tau_0 + \zeta(\tau),$$ (33)

where $\zeta(\tau)$ is a small disturbance.
Expression (31) together with Eq. (18) is the solution of the equations of string motion (5). Therefore we can obtain the equations of evolution of small disturbances $\varphi^\mu_\pm(\tau)$, substituting expressions (31) and (33) with Eqs. (28), (29) in two other equations of motion (2) and (6). We take into account the nonlinear factor $[\dot{X}^2(\tau,0)]^{-1/2}$ and contributions from the disturbed argument $\tau^*(33)$:
\[
\ddot{\Psi}^\mu_\pm(\tau^* \pm 2\pi) \simeq \ddot{\Psi}^\mu_\pm(\pm) + \ddot{\Psi}^{\mu\prime}_\pm(\pm) \zeta(\tau).
\]
Here and below $(\pm) \equiv (\tau + \tau_0 \pm 2\pi)$.

This substitution results in the following linearized system of equations in linear (with respect to $\varphi^\mu_\pm$ and $\zeta$) approximation:
\[
\frac{d}{d\tau}\left\{\varphi^\mu(+)_\mp + \varphi^\mu(-)_\mp + 2a_0[e^0_\mu + v\dot{e}^\mu(\omega_n\tau)]\zeta(\tau) - 2a_0v\omega_n\dot{e}^\mu(\omega_n\tau)\zeta(\tau) = \varphi^\mu_+(\tau) + \varphi^\mu_-(\tau),
\right.
\]
\[
\left. + Q[\varphi^\mu(-)_\mp - \varphi^\mu_+(\tau) + \varphi^\mu_+(\tau) + 2\omega_n^2 A_0 n^0_\mu(\tau) \cdot \zeta(\tau)] = 0.\right\}
\]
Here $f^\mu(\tau) = (C - C_0)\dot{e}^\mu(\omega_n\tau) + S_0\dot{e}^\mu(\omega_n\tau)$ for the states (11) and (16), $v$ is the speed of massive point satisfying Eqs. (14), and we use the following notations for the scalar products:
\[
\varphi^0_\pm \equiv \langle e_0, \varphi \pm \rangle, \quad \varphi^k_\pm \equiv \langle e_k, \varphi \pm \rangle, \quad \varphi_\pm \equiv \langle e, \varphi \pm \rangle, \quad \dot{\varphi}_\pm \equiv \langle \dot{e}, \varphi \pm \rangle.
\]

For the rotational state (17) with the mass at the center we are to put $\omega_n = n$, the speed $v = 0$ and $f^\mu(\tau) = \dot{e}^\mu(n\tau)$ in Eqs. (34).

If we take scalar products of equations (34) onto the basic vectors $e_0$, $e_k$, $e(\tau)$, $\dot{e}(\tau)$, and add the equations (32) after substituting expressions (29)
\[
(1 \mp \theta) \varphi^0_\pm(\tau) \mp a_0^{-1} \sum_{k>3} b_k \varphi^k_\pm(\tau) \mp \frac{\omega_n v}{2Q} \varphi^0_\pm(\tau) \pm (v \mp \theta/v) \dot{\varphi}^0_\pm(\tau),
\]
we obtain the linear system of differential equations with respect to projections (35) $\varphi^0_\pm(\tau)$, $\varphi^k_\pm(\tau)$, $\varphi_\pm(\tau)$, $\dot{\varphi}_\pm(\tau)$, and the function $\zeta(\tau)$. This system has constant coefficients but it also has deviating arguments $(\pm)$ together with $(\tau)$.

We search solutions of this system in the form of harmonics
\[
\varphi^0_\pm = B^0_\pm e^{-i\omega_\tau}, \quad \varphi^k_\pm = B^k_\pm e^{-i\omega_\tau}, \quad \varphi_\pm = B_\pm e^{-i\omega_\tau}, \quad \dot{\varphi}_\pm = \dot{B}_\pm e^{-i\omega_\tau}, \quad 2a_0\zeta = \Delta e^{-i\omega_\tau}.
\]

This substitution results in the linear homogeneous system of algebraic equations with respect to the amplitudes of harmonics (37). For the rotational state (17) with the mass at the center with the function $\ddot{\Psi}^\mu_\pm$ in the form (28) this linear system is
\[
B^k_+ E_+ + B^k_- E_- = 0, \quad B^k_+(Q E_+ - i\omega) = B^k_+(Q E_- + i\omega);
\]
\[
B^0_+ E_+ + B^0_- E_- = i\omega\Delta, \quad B^0_+ E_+ = B^0_- E_-,
\]
\[
B_+ E_+ + B_- E_- = 0, \quad \dot{B}_+ E_+ + \dot{B}_- E_- = 0,
\]
\[
B_+(Q E_+ - i\omega) - B_-(Q E_- + i\omega) - n\dot{B}_+ - n\dot{B}_- = 0,
\]
\[
\dot{B}_+(Q E_+ - i\omega) - \dot{B}_-(Q E_- + i\omega) + nB_+ + nB_- Q n\omega \Delta = 0;
\]
\[
B^0_\pm + \nu B^0_\pm \pm a_0^{-1} \sum_{k>3} b_k^k B^k_\pm = 0.
\]
Here $E_\pm = \exp(\mp 2\pi i \bar{\omega}) - 1$, 

$$\nu = \frac{nA_n}{a_0} = \left(1 - a_0^{-2} \sum_{k>3} b_k^2 \right)^{1/2}.$$  \hspace{1cm} (39)

The last equations in system (38) are the consequence of Eqs. (36).

If $b_k = 0$, the first two equations of system (38) form the closed subsystem. It has non-trivial solutions if and only if $\bar{\omega}$ is a root of the equation

$$\sin \pi \bar{\omega} \cdot \left( \sin \pi \bar{\omega} + \frac{\bar{\omega}}{2Q} \cos \pi \bar{\omega} \right) = 0.$$  \hspace{1cm} (39a)

It coincides with Eq. (15). Hence, the spectrum of transversal (with respect to the $xy$ plane) small fluctuations of the string for the state (17) contains the same frequencies, as $\omega_n$, from solutions (16) and (17). All these frequencies are real numbers, therefore amplitudes of such fluctuations do not grow with growth of time $t$.

We in the greater degree are interested in disturbances in the $xy$ plane, shown in Fig. 2. Assuming for such fluctuations $B_k^\pm = 0$, we study the condition of existence of non-trivial solutions for the last 8 equations of the homogeneous system (38). This condition (vanishing the corresponding determinant) is reduced to the following equation:

$$4Q^2 \tan^2 \pi \bar{\omega} + 4Q \left( \bar{\omega} + \frac{n^2 \nu^2}{2\bar{\omega}} \right) \tan \pi \bar{\omega} + \bar{\omega}^2 - n^2 = 0.$$  \hspace{1cm} (40)

This transcendental equation contains the denumerable set of real roots (frequencies). They correspond to different modes of small oscillations of the string in the rotational state (17).

This state will be unstable, if there are complex frequencies $\bar{\omega} = \omega + i\xi$ in the spectrum, generated by Eq. (40). If its imaginary part $\xi$ will be positive, the modes of disturbances $\varphi^\mu$ (corresponding to the root $\omega + i\xi$) get the multiplier $\exp(\xi \tau)$, that is they grow exponentially.

The search of complex roots of equation (40) shows that such roots can exist only on the imaginary axis of the complex plane $\bar{\omega}$. The typical behavior of these roots is shown in Figs. 3a and 3b. Here $b_k = 0$, hence $\nu = 1$. Level lines of the real part of l.h.s. of Eq. (40) $\text{Re}F(\omega + i\xi) = 0$ are marked with black lines and zeros of the imaginary part $\text{Im}F(\omega + i\xi) = 0$ — with red lines. Roots of Eq. (40) correspond to cross points of these lines.

On the imaginary axis of $\bar{\omega}$ (in the case $\bar{\omega} = i\xi$) the equation (40) is transformed into the form

$$4Q^2 \tan^2 \pi \xi + n^2 = 4Q \left( \frac{n^2 \nu^2}{2\xi} - \xi \right) \tan \pi \xi.$$  \hspace{1cm} (41)

The left hand side of this equation grows with growth $\xi$ (for $\xi > 0$), and the right hand side decreases. It is obvious, that the root $\xi > 0$ of Eq. (41), that is the imaginary root $\bar{\omega} = i\xi$ of Eq. (40) exists, if and only if

$$2\pi Q\nu^2 > 1.$$  \hspace{1cm} (42)

If we use the expression (14) in the form $Q = \gamma a_0/m$ (remind that $\nu = 0$ in the case under consideration) and Eq. (39) we reduce the criterion (42) to the following form:

$$m < m_{cr} \equiv 2\pi \gamma a_0 \left(1 - a_0^{-2} \sum_{k>3} b_k^2 \right).$$  \hspace{1cm} (43)

Thus, we obtain the analytical proof of the threshold effect (27) observed in numerical experiments.
Figure 3: Zero level lines for real and imaginary part (red) of Eq. (40); a) $m = 6.5\gamma a_0$, b) $m = \gamma a_0$

In the particular case $b_k = 0$ (that is cyclic coordinates do not “work” or the manifold $\mathcal{M} = R^{1,3}$ is the simplest one) shown in Figs. 2 and 3, the threshold value in criterion (43) takes the form $m_{cr} = 2\pi\gamma a_0$. In Fig. 3a the value $m = 6.5\gamma a_0$ is a bit greater than $m_{cr}$, hence all roots are real ones. But in Fig. 3b the inequality (43) takes place, so the imaginary root $\tilde{\omega} = i\xi$ appears and the corresponding amplitude of disturbances grows exponentially: $\varphi = B e^{\xi \tau}$.

This threshold effect or the spontaneous symmetry breaking for the string state (17) was observed in details in numerical experiments in Sect. 2. Note that our analysis in Sect. 3 is suitable only for initial stage of an unstable motion when disturbances are small. The further evolution of the disturbed motion (shown in Figs. 2c) can be described only in the numerical procedure.

The similar investigation of small disturbances in the form (31) for the rotational states (11) and (16) includes substitution of harmonics (37) into the linearized system of differential equations for projections (35) of $\varphi_{\pm}^n u$. In this case the first two equations of system (38) keep the same form

$$B_+^k E_+ + B_-^k E_- = 0, \quad B_+^k (Q E_+ - i\tilde{\omega}) = B_-^k (Q E_- + i\tilde{\omega});$$

but here $E_{\pm} = \exp\left[-i(\tau_0 \pm 2\pi)\tilde{\omega}\right] - 1$. This subsystem has non-trivial solutions if and only if $\tilde{\omega}$ is a root of the equation

$$\cos 2\pi\tilde{\omega} - \cos \tau_0 \tilde{\omega} = \frac{\tilde{\omega}}{2Q} \sin 2\pi\tilde{\omega},$$
coinciding with Eq. (9). Hence, in this case of small transversal fluctuations the spectrum of roots $\tilde{\omega}$ also looks like the spectrum of frequencies $\omega_n$.

For the last 8 equations of the homogeneous system, generalizing (38), the condition of existence of non-trivial solutions (oscillations in $xy$ plane) for the states (16) may be reduced to the equation, decomposing into product of the following two equations:

$$2Q\tilde{\omega}\left[v^2 - C + (1 - v^2)\cos 2\pi\tilde{\omega}\right] + \omega_n^2 v^2 \sin 2\pi\tilde{\omega} = 0,$$

$$\left[\omega_n^2 + 4Q^2(1 - v^2)\right](1 - \cos 2\pi\tilde{\omega}) + 4Q(1 - v^2)\tilde{\omega}\sin 2\pi\tilde{\omega} + \omega_n^2\left[C - v^2 + (1 - v^2)\cos 2\pi\tilde{\omega}\right] = 0,$$

The analysis of their roots shows that all roots of Eq. (44) and Eq. (45) are real numbers, if all values satisfy natural physical restrictions, for example, $v < 1$, $m > 0$. The typical picture of these roots is presented in Fig. 4a for Eq. (44) and in Fig. 4b for Eq. (45).

![Figure 4: Zero level lines for real (black) and imaginary part (red) for a) Eq. (44); b) Eq. (45)](image)

Here the values of the parameters for the rotational state (16) are: $\omega_n = 0.9$, $Q \simeq 1.385$, $v^2 \simeq 0.875$, $(\gamma/m)^2 \sum b_k^2 = 0.5$. We may conclude that in linear approximation this rotational state is stable with respect to small disturbances.

**Conclusion**

The analysis of the stability problem for rotational states (11), (16) and (17) of the closed string with a point-like mass was made with two different approaches in Sects. 2 and 3. This analysis showed that the states of the class (17) (with the massive point at the rotational center) are unstable if the mass is less than the threshold $m_{cr}$ (43). This effect of the spontaneous symmetry breaking for these string states restrains applicability of these states in hadron spectroscopy.
But our investigation of rotational states (11), (16) showed, that they are stable in the mentioned sense. These rotational states generate non-trivial spectrum of Regge trajectories [2]. So we can apply them for describing excited hadrons with exotic properties, in particular, glueballs, hybrids, pentaquarks in accordance with applications of various string hadron models [8], [9] in meson and baryon spectroscopy.

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