Quantum moduli space of the cascading
$Sp(p+M) \times Sp(p)$ gauge theory

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Abstract

We extend the detailed analysis of the quantum moduli space of the cascading $SU(p+M) \times SU(p)$ gauge theory in the recent paper of Dymarsky, Klebanov, and Seiberg for the $Sp(p+M) \times Sp(p)$ cascadig gauge theory, which lives on the world volume of $p$ D3-branes and $M$ fractional D3-branes at the tip of the orientifolded conifold. As in their paper, we also find in this case that the ratio of the deformation parameters of the quantum constraint on the different branches in the gauge theory can be reproduced by the ratio of the deformation parameters of the conifold with different numbers of mobile D3-branes.
1 Introduction

In the recent paper [1], Dymarsky, Klebanov, and Seiberg gave a detailed analysis of the moduli space of the cascading $SU(p + M) \times SU(p)$ gauge theory, which lives on the world volume of $p$ D3-branes and $M$ fractional D3-branes at the tip of the conifold [2]. They found new mesonic branches, which is characterized by the number of performing the cascading duality transformation and the phase factor of the deformation parameter. The latter can also be seen by the $Z_2$ $R$-symmetry breaking by gaugino condensation to $Z_2$. They discussed that the branches correspond to the warped deformed conifold with different numbers of mobile D3-branes. In particular, they showed that the ratio of the deformation parameters of the quantum constraints on the different branches in the field theory can be reproduced by the ratio of the deformation parameters of the conifold with different numbers of mobile D3-branes.

In this note, we will extend their analysis for the cascading $Sp(p + M) \times Sp(p)$ gauge theory [3], which is dual to the string background given by $p$ D3-branes and $M$ fractional D3-branes at the tip of the orientifolded conifold with four D7-branes on top of an orientifold plane (O7-plane) to cancel the RR charge of the O7-plane. As discussed in [3, 4], the supergravity solution is given by the Klebanov-Strassler solution [2] only projected by the $Z_2$ operation. Therefore, it is obvious that the ratio of the deformation parameters of the orientifolded conifold is identical to the one of the conifold in [1]. In this note, we demonstrate that the cascading $Sp(p + M) \times Sp(p)$ gauge theory gives the similar result on the ratio of the deformation parameters to the one of the $SU(p + M) \times SU(p)$ gauge theory in [1]. This note is organized as follows: in the next section, we will give a brief review on the result of [1], in particular on the ratio of the deformation parameters. In the section 3, we will study the moduli space of the cascading $Sp(p + M) \times Sp(p)$ gauge theory and demonstrate that the ratio of the deformation parameters gives the correct ratio given on the string side in section 2. The quantum moduli space for the case $p = 1$ has been explored in the paper [3]. Since no new mesonic branches are however available in the case, we need to consider other cases with $p > 1$. The section 4 will be devoted to summary and discussion. In the appendix, the quantum moduli space is discussed for $p = M - 1, M$, which could happen at the last step of the cascading flow.

2 The Klebanov-Strassler Solution with Mobile D3-Branes

It has been discussed in the paper [3, 4] that the Klebanov-Strassler (KS) solution [2] projected by the $Z_2$ projection is dual to $Sp(p + M) \times Sp(p)$ gauge theory. The solution is

\[^1\text{In our notation, } Sp(1) \simeq SU(2).\]
the warped deformed conifold background

\[ ds^2_{10} = h^{-1/2}(r)dx^\mu dx_\mu + h^{1/2}(r)ds^2_6, \]

where \( h(r) \) is a warp factor and \( ds^2_6 \) is given by the metric of the deformed conifold \[5\]. For large \( r \), the metric can be approximated by the singular conifold with the metric

\[ ds^2_6 \sim dr^2 + r^2 ds^2_{T,1}. \] (2.1)

Here, \( T^{1,1} \) can be described as a \( U(1) \) bundle over \( S^2 \times S^2 \) \[5\] \[6\]. In this note, we follow the notation in \[7\], where the two two-spheres are parametrized by \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \), and the \( U(1) \) fiber by \( \psi \in [0, 4\pi] \).

The deformed conifold can also be described by the polynomial

\[ xy - zw = \varepsilon^2 \] (2.2)

of the complex variables \( x, y, z, w \) in \( \mathbb{C}^4 \) with the deformation parameter \( \varepsilon \).

In order to obtain the dual description of the cascading \( Sp(p+M) \times Sp(p) \) gauge theory, we need to orientifold the warped deformed conifold by the \( \mathbb{Z}_2 \) projection

\[ z \leftrightarrow w, \] (2.3)

which is equivalent to

\[ (\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2). \] (2.4)

Since the \( \mathbb{Z}_2 \) projection has the set of fixed points

\[ xy - z^2 = \varepsilon^2 \] (2.5)

and gives rise to an orientifold plane (O7-plane), we need to introduce four D7-branes to cancel the RR-charge of the O7-plane, otherwise not only the D3-brane charge, but also the D5-brane charge would change, as we go down to the tip of the conifold \[8\]. The four D7-branes will have the effect on the gauge theory side that additional fundamental matters are introduced in the gauge theory.

On the gauge theory side, we begin with \( SU(2M + 2p) \times SU(2p) \) gauge theory which includes two bifundamentals \( A_{1,2} \) in \( \square \square \) and two conjugates \( B_{1,2} \) in \( \square \square \) and use the identification

\[ \text{tr}[A_1B_2] \sim z, \quad \text{tr}[A_2B_1] \sim w \] (2.6)

to perform the corresponding \( \mathbb{Z}_2 \) projection in the field theory to give the \( Sp(p+M) \times Sp(p) \) gauge theory \[8\]. Therefore, it is natural to begin with twice as many of the D3-brane and D5-brane charge as in the \( SU(p + M) \times SU(p) \) theory;

\[ \frac{1}{16\pi^2} \int_{T^{1,1}} F_5 = 2p, \quad \frac{1}{4\pi^2} \int_A F_3 = 2M, \] (2.7)
where $A$ is a copy of $S^3$ given by $(\theta_2, \phi_2) = (0, 0)$. Here, the D3-brane charge $2p$ is defined on the $T^{1,1}$ at the cut-off radius $r = r_c$.

In order to obtain the ratio of the deformation parameters with the different numbers of mobile D3-branes, we only need to know the large radius $r$ behavior of the warped deformed conifold. In particular, we are interested in the self-dual five-form $F_5$

$$F_5 = \widetilde{F}_5 + \ast \widetilde{F}_5, \quad \widetilde{F}_5 = 27\pi \mathcal{N}(r) \text{Vol}(T^{1,1}),$$

(2.8)

of the KS solution [2]. Here $\mathcal{N}(r)$ stands for the effective number of the 5-form flux through the deformed $T^{1,1}$ at $r$, and its radial dependence reflects the gravitational manifestation of the duality cascade. We can read off its large $r$ limit

$$\mathcal{N}(r) \rightarrow \frac{g_s(2M)^2}{2\pi} \ln \left(\frac{r^3}{\varepsilon^2}\right) + \text{const.},$$

(2.9)

where const. denotes a constant which is independent of $r$ and $\varepsilon$.

Now, we are ready to find the ratio of the deformation parameters, one of which is given by the KS solution with $(p - lM)$ mobile D3-branes and their mirrors at $r = r_{D3}$, where $r_{D3}$ satisfies $\varepsilon^{2/3} \ll r_{D3} < r_c$, and the other of which given by the one with no mobile D3-branes. The condition for $r_{D3}$ means that $(p - lM)$ mobile D3-branes and their mirrors stay far from the tip of the conifold. Furthermore, for simplicity, we also suppose that all $(p - lM)$ mobile D3-branes and their mirrors lie at the same surface $r = r_{D3}$. We will see that our following analysis is independent of the choice of $r_{D3}$. In fact, one can verify that the result does not change, even if we place the D3-branes separately. We describe the geometries in the regions $r < r_{D3}$ and $r > r_{D3}$ by the KS solutions with the deformation parameters $\varepsilon_c$ and $\varepsilon_l$, respectively. As a matching condition, due to the presence of $(p - lM)$ D3-branes and their mirrors at $r_{D3}$, we can see that the number of the five-form flux jumps by $2(p - lM)$ at $r_{D3}$.

We suppose that in the region of $r_{D3} < r \leq r_c$ the effective number of the 5-form flux is given by (2.9) with the deformation parameter $\varepsilon = \varepsilon_c$, which we denotes $\mathcal{N}_+(r)$. The boundary condition at the surface $r = r_c$ (2.7) becomes

$$2p = \frac{g_s(2M)^2}{2\pi} \ln \left(\frac{r_c^3}{\varepsilon_c^2}\right) + \text{const.}$$

(2.10)

This equation determines the $\varepsilon_c$. After going down to the surface at $r = r_{D3}$, the number of 5-form flux becomes

$$\mathcal{N}_+(r_{D3}) = \frac{g_s(2M)^2}{2\pi} \ln \left(\frac{r_{D3}^3}{\varepsilon_c^2}\right) + \text{const.}$$

(2.11)

Below the surface $r = r_{D3}$, we use (2.9) with the deformation parameter $\varepsilon = \varepsilon_c$, which we denote $\mathcal{N}_-(r)$. We must take into account the jump of the five-form flux at $r = r_{D3}$, and the matching condition is given by

$$\mathcal{N}_+(r_{D3}) = \mathcal{N}_-(r_{D3}) + 2(p - lM),$$

(2.12)
Thus, the deformation parameter for the branch $l$ is

$$\varepsilon_{l}^{2} = \varepsilon_{c}^{2} e^{\frac{2\pi l}{g_s(2M)}} \exp \left( -\frac{2\pi l}{g_s(2M)} \right).$$

(2.13)

We note that the deformation parameter is independent of $r_{D3}$, as mentioned above.

Finally, we obtain the ratio between the deformation parameter $\varepsilon_{l}$ with $(p - lM)$ pairs of mobile D3-brane and its mirror and $\varepsilon_{l+1}$ with $(p - M - lM)$ ones as

$$\left( \frac{\varepsilon_{l+1}}{\varepsilon_{l}} \right)^{\frac{2}{3}} = \exp \left( -\frac{2\pi}{3g_s(2M)} \right).$$

(2.14)

In the following sections, we will follow the proposal in [1] that the meson branch after performing the cascade duality transformation $l$ times corresponds to the KS solution with $(p - Ml)$ mobile D3-branes and their mirrors.

And using the relation explained later between the string coupling constant $g_s$ on the string side and the dynamical scale in the gauge theory, we will find that the ratio of the deformation parameters on the different meson branches is in agreement with the above result on the ratio of the deformation parameters also in the $Sp(p + M) \times Sp(p)$ gauge theory, not only in the $SU(p + M) \times SU(p)$ gauge theory.

3 The Moduli Space of the $Sp(p + M) \times Sp(p)$ Gauge Theory

We consider the four-dimensional $N = 1$ supersymmetric $Sp(p + M) \times Sp(p)$ gauge theory with two chiral fields $A^{a\alpha}$ ($\alpha = 1, 2$) in the bifundamental representation, four $Q^{aA}$ ($A = 1, \ldots, 4$) in the fundamental representation of $Sp(p + M)$ gauge group and four $q^{iI}$ ($I = 1, \ldots, 4$) in the fundamental representation of $Sp(p)$ gauge group. Here, $a = 1, \ldots, 2(p + M)$ denotes the index of the $Sp(p + M)$ gauge group and $i = 1, \ldots, 2p$ denotes the one of the $Sp(p)$ gauge group. This gauge theory lives on the world volume of $p$ D3-branes and $M$ fractional D3-branes at the tip of the orientifolded conifold with four D7-branes on top of an orientifold plane. The tree level superpotential of the model is given by

$$W_{tree} = -h \left( \frac{1}{2} J_{ab} J_{cd} J_{jk} J_{il} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} A^{a\alpha} A^{b\beta} A^{c\gamma} A^{d\delta} + J_{ab} J_{cd} J_{ij} \varepsilon_{\alpha\beta} Q^{aA} A^{b\alpha} A^{c\beta} Q^{dA} - J_{ab} J_{ij} J_{kl} \varepsilon_{\alpha\beta} q^{iI} A^{a\alpha} A^{b\beta} q^{jI} - \frac{1}{2} J_{ab} J_{cd} Q^{AB} Q^{BB} Q^{cB} Q^{dA} + \frac{1}{2} J_{ij} J_{kl} q^{iI} q^{jI} q^{kI} q^{lI} \right),$$

(3.1)
where $J_{ab}$ and $J_{ij}$ are the invariant antisymmetric tensors of the $Sp(p + M)$ and the $Sp(p)$
gauge groups, respectively. In this note, for the matrix $J$, we will use the convention

$$J = i\sigma_2 \otimes 1 = \begin{pmatrix} 1 & \cdots & -1 \\ -1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 1 \end{pmatrix}. \tag{3.2}$$

We list the symmetries of the gauge theory in the following table:

| $A^a\alpha$ | $Sp(p + M)$ | $Sp(p)$ | $SU(2)$ | $SO(4)_1$ | $SO(4)_2$ |
|-------------|-------------|---------|---------|-----------|-----------|
| $a,b,\ldots$ | $i,j,\ldots$ | $\alpha,\beta,\ldots$ | $A,B,\ldots$ | $I,J,\ldots$ |
| $Q^{aA}$    | $\square$   | $\square$ | $1$     | $1$       | $1$       |
| $q^I$       | $\square$   | $1$     | $1$     | $\square$ | $1$       |

3.1 The Classical Moduli Space

In order to compare the deformation parameter of the supergravity background with the
one of the quantum constraint on the corresponding branches of the gauge theory, we study
the branches which can be regarded as the motion of D3-branes probing the orientifolded
conifold. We will see that the corresponding branch in the classical moduli space is specified
by

$$A^{\alpha-1} = \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_p \\ b_p \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad A^{\alpha-2} = \begin{pmatrix} c_1 \\ d_1 \\ \vdots \\ c_p \\ d_p \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad q = 0,$$ \tag{3.4}
where $|a_i|^2 + |c_i|^2 = |b_i|^2 + |d_i|^2$, $(i = 1, ..., p)$ and $a^2 = b^2 + c^2$. We can verify that the vacuum expectation values \((3.4)\) satisfy the $F$-term conditions and the $D$-term conditions.

We note that at a generic point on the moduli space, the $Sp(p + M) \times Sp(p)$ gauge group are broken to $Sp(M - 1) \times U(1)^p$.

The classical moduli space \((3.4)\) can also be given in terms of the chiral fields

$$N^{ij\alpha\beta} = J_{ab} A^{a\alpha} A^{b\beta}, \quad v^{iaA} = J_{ab} A^{a\alpha} Q^{bA}, \quad M^{AB} = J_{ab} Q^{aA} Q^{bB},$$

which are left invariant only by the $Sp(p + M)$ gauge transformation. For the purpose of our discussion, it is convenient to introduce the matrix notation for the meson $N^{ij\alpha\beta}$ as $(\tilde{N}^{ij})^k = J_{kj} N^{ij\alpha\beta}$. Now the moduli space can be rewritten as

$$\sqrt{2h} \tilde{N}^{11} = \text{diag} \left( x_1, x_1, \cdots, x_p, x_p \right), \quad \sqrt{2h} \tilde{N}^{12} = \text{diag} \left( w_1, z_1, \cdots, w_p, z_p \right),$$

$$\sqrt{2h} \tilde{N}^{21} = \text{diag} \left( z_1, w_1, \cdots, z_p, w_p \right), \quad \sqrt{2h} \tilde{N}^{22} = \text{diag} \left( y_1, y_1, \cdots, y_p, y_p \right),$$

$$\sqrt{2h} M = \begin{pmatrix} 0 & X & Y & Z \\ -X & 0 & Z & -Y \\ -Y & -Z & 0 & X \\ -Z & Y & -X & 0 \end{pmatrix}, \quad v = 0. \tag{3.6}$$

The components of matrices are related to the vacuum expectation values \((3.4)\) as

$$x_i = \sqrt{2h} a_i b_i, \quad y_i = \sqrt{2h} c_i d_i, \quad w_i = \sqrt{2h} b_i c_i, \quad z_i = \sqrt{2h} a_i d_i,$$

$$X = \sqrt{2h} a^2 = \sqrt{2h} (b^2 + c^2), \quad Y = \sqrt{2h} ac, \quad Z = \sqrt{2h} ab, \tag{3.7}$$

which satisfy the constraints

$$x_i y_i - w_i z_i = 0, \quad X^2 + Y^2 + Z^2 = 0. \tag{3.8}$$

We introduced the factor $\sqrt{2h}$ in order to absorb the coupling constant $h$ in the superpotential \((3.1)\). In addition to the first equation of \((3.8)\) giving the singular conifold, there exists the $Sp(p)$ gauge transformation

$$(x_i, y_i, z_i, w_i) \leftrightarrow (x_i, y_i, w_i, z_i). \tag{3.9}$$

This gauge transformation leads to the $\mathbb{Z}_2$ identification on the conifold. Thus, the classical moduli space \((3.6)\) describes $p$ D3-branes probing the orientifolded conifold \([3]\). At least at a generic point far away from the origin of the classical moduli space, the vacuum expectation value \((3.4)\) may give a good description to the quantum moduli space. In particular, the gauge group $Sp(p + M) \times Sp(p)$ are broken to $Sp(M - 1) \times U(1)^p$ at the point, where the effective theory is given by the $Sp(M - 1)$ pure Yang-Mills theory. The classical $U(1)_R$ symmetry of the this pure Yang-Mills theory is broken by the anomaly to $\mathbb{Z}_{2M}$ symmetry, which rotates the gaugino as

$$\lambda \rightarrow \lambda e^{i \frac{2\pi}{2M}}. \tag{3.10}$$

Furthermore, the $\mathbb{Z}_{2M}$ symmetry is broken by gaugino condensation to $\mathbb{Z}_2$, and we end up with $M$ inequivalent vacua in the infrared.
3.2 The Quantum Moduli Space

In this section, we will see that the classical moduli space is deformed by the quantum effect. In the case for $p \geq M + 1$, we can perform the Seiberg duality transformation \cite{10} at least once, and in general several times. After performing the transformations say, $l$ times and integrating out the singlets under the $Sp(p - (l - 1)M)$ gauge group, we can find a branch where the mesons of the dual quarks develop the vacuum expectation values. On the branch, we will see that the classical constraints of the mesons, describing the conifold, are quantum-mechanically deformed to the one describing the deformed conifold with the deformation parameter given by the dynamical scale, as in the case $p = 1$ \cite{3}. As found in \cite{1}, we will find the deformation parameters on the meson branches depend on the numbers $l$ of performing the duality transformations. We will demonstrate the ratio of two different deformation parameters matches the results on the gravity side in the previous section.

Taking into account the 1-loop $\beta$-functions of the $Sp(p + M)$ and $Sp(p)$ gauge groups, the gauge coupling constant of $Sp(p + M)$ becomes much larger than the one of $Sp(p)$ after a finite amount of RG flow. Therefore, we will effectively regard the theory as $Sp(p + M)$ gauge theory with $N_f = 2p + 2$ flavors. We denotes the dynamical scale of the $Sp(p + M)$ gauge theory as $\Lambda_1$ and the one of $SU(p)$ as $\Lambda_2$. Since at the energy scale lower than $\Lambda_1$ the magnetic theory gives a good description, we perform the Seiberg duality transformation \cite{10} to go to the dual $Sp(p - M) \times Sp(p)$ gauge theory. The matter fields in the dual theory are listed in the table:

$$
\begin{array}{c|c|c|c|c|c}
 & Sp(p + M) & Sp(p) & SU(2) & SO(4)_1 & SO(4)_2 \\
\hline
A^\alpha_{i\alpha} & 1 & 1 & 1 & 1 \\
\hat{Q}^A & \Box & \Box & \Box & \Box \\
q^I & 1 & 1 & 1 & 1 \\
\hat{N}^{[ij][\alpha\beta]} & 1 & \Box & \Box & 1 \\
n^{[\alpha\beta]} & 1 & 1 & \Box & 1 \\
N^{(ij)[\alpha\beta]} & 1 & 1 & 1 & 1 \\
v^{\alpha A} & 1 & \Box & \Box & \Box \\
M^{AB} & 1 & 1 & 1 & \Box \\
\end{array} 
$$

(3.11)

Here, we decomposed the $Sp(p - M)$ singlet meson $N^{ij\alpha\beta}$ into the irreducible representations as

$$N^{ij\alpha\beta} = N^{(ij)[\alpha\beta]} + \hat{N}^{[ij][\alpha\beta]} + n^{(\alpha\beta)} j^{ij}. 
$$

(3.12)
The superpotential of the magnetic theory is given by

\[
W_{\text{tree}} = -h \left( \frac{1}{2} J^a_{ij} \varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta} N^{ij \alpha \beta} N^{kl \gamma \delta} - J^A_{ij} \varepsilon_{\alpha \beta} v^{iA} \varepsilon^{jA} \right) 
- J^A_{ijk} \varepsilon_{\alpha \beta} q^{iI} q^{jI} - \frac{1}{2} M^{AB} M^{BA} + \frac{1}{2} J^A_{ij} \varepsilon_{\alpha \beta} v^{iA} \varepsilon^{jA} \right) 
+ \frac{1}{\mu} \left( J^A_{a} N^{i i \alpha \beta} A^a_{i} A^b_{j} + 2 J^A_{a} v^{i A} A^a_{i} Q^b_{j} + J^A_{a} M^{AB} Q^a_{A} Q^b_{B} \right),
\]

where \( \mu \) is the duality scale.

We can see that the superpotential (3.13) gives the mesons \( N, v, \) and \( M \) masses of the order \( h \Lambda_1^2 \). On the other hand, when the fields \( N \) and \( M \) develop vacuum expectation values, \( \tilde{A} \) and \( \tilde{Q} \) acquire the masses \( \langle N \rangle / \mu \) and \( \langle M \rangle / \mu \) via the superpotential (3.13). Therefore, in order to obtain the correct low energy effective theory by integrating out massive degrees of freedom, we need to consider the case \( |\langle N \rangle / \mu|, |\langle M \rangle / \mu| \gg h \Lambda_1^2 \) and the case \( |\langle N \rangle / \mu|, |\langle M \rangle / \mu| \ll h \Lambda_1^2 \), separately.

We begin with the former case \( |\langle N \rangle / \mu|, |\langle M \rangle / \mu| \gg h \Lambda_1^2 \), where the mesons \( N \) and \( M \) obtain vacuum expectation values of full rank and its eigenvalues are all much larger than \( h \mu \Lambda_1^2 \). Since the dual quarks \( \tilde{A}, \tilde{Q} \) acquire masses much larger than the mesons \( N, v, \) and \( M \), we integrate them out to obtain the pure \( Sp(p - M) \) gauge theory. The gaugino condensation generates the superpotential

\[
W_{\text{dyn}} = (p - M + 1) \Lambda_2^3 \Lambda_L^3
\]

where \( \Lambda_L \) is the dynamical scale of the pure super Yang-Mills theory.

In order to obtain the low energy effective potential for the mesons \( N \) and \( M \) from the superpotential (3.14), it is convenient to write the mesons as a matrix

\[
V = (V^{zw}) = \begin{pmatrix}
N^{ij \alpha \beta} & v^{jB} \\
-v^{iA} & M^{AB}
\end{pmatrix},
\]

where \( z, w \) run over both of the paired indices \( (i, \alpha) \) of the \( Sp(p) \) gauge group and the global \( SU(2) \) group and the index \( A \) of the global \( SO(4) \) group. We have a matching condition of the dynamical scales

\[
\Lambda_L^{3(p-M+1)} = \tilde{\Lambda}_2^{p-3M+1} \frac{\text{Pf}(V)}{\mu^{2p+2}},
\]

where \( \tilde{\Lambda}_2 \) is the dynamical scale of the magnetic \( Sp(p - M) \) theory with the dual quarks \( \tilde{A} \) and \( \tilde{Q} \). We also have a relation between the dynamical scales \( \Lambda_1 \) and \( \tilde{\Lambda}_2 \)

\[
\Lambda_1^{p+3M+1} \tilde{\Lambda}_2^{p-3M+1} = (-1)^{p-M+1} \mu^{2p+2}.
\]
Using the relations \((3.16)\) and \((3.17)\), the superpotential \((3.14)\) can be written as

\[
W_{\text{dyn}} = -(p - M + 1) \left( \frac{\text{Pf}(V)}{\Lambda_1^{p+3M+1}} \right)^{\frac{1}{p-3M+1}}.
\]

Thus, we find the total superpotential is given by

\[
W_{\text{eff}} = -\hbar \left( \frac{1}{2} J_{jk} J_{i\ell} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} N^{ij \alpha \beta} N^{k\ell \gamma \delta} - J_{ij} \varepsilon_{\alpha \beta} v^{iA} v^{jB} A - J_{jk} J_{i\ell} \varepsilon_{\alpha \beta} N^{ij \alpha \beta} q^{kI} q^{I}\right) - \frac{1}{2} M^{AB} M^{RA} + \frac{1}{2} J_{ij} J_{kl} q^{iI} q^{jJ} q^{kI} q^{lI} \right) -(p - M + 1) \left( \frac{\text{Pf}(V)}{\Lambda_1^{p+3M+1}} \right)^{\frac{1}{p-3M+1}}.
\]

We substitute the vacuum expectation values \((3.6)\) into the superpotential \((3.19)\) and solve the \(F\)-term conditions. We find that the classical constraints \((3.8)\) are deformed into

\[
x_i y_i - z_i w_i = \varepsilon_{l=0,r}^2, \quad (i = 1, \ldots, p), \quad X^2 + Y^2 + Z^2 = \varepsilon_{l=0,r}^2,
\]

where the deformation parameter \(\varepsilon_{l=0,r}\) is given by

\[
\varepsilon_{l=0,r}^2 = \left[ (2\hbar)^{p+1} \Lambda_1^{p+3M+1} \right]^{\frac{1}{M}} e^{\frac{2\pi ir}{M}} \quad (r = 0, \ldots, M - 1).
\]

We refer this branch as the \(l = 0\) branch. The deformed constraint corresponds to the motion of \(p\) D3 branes probing the deformed conifold. The phase factor \(e^{(2\pi/M)ir}\) comes from the branch cut in the deformation parameter \(\varepsilon_{l=0,r}\) and thus gives additional \(M\) distinct branches. These \(M\) branches are generated by the symmetry breaking from \(\mathbb{Z}_{2M}\) to \(\mathbb{Z}_2\) through gaugino condensation, as discussed.

Now, let us turn to the other case \(|\langle N \rangle/\mu|, |\langle M \rangle/\mu| \ll |\hbar\Lambda_1^2|\). In this case, the mesons \(N\) and \(M\) are much heavier than the dual quarks \(\tilde{A}\) and \(\tilde{Q}\), and we integrate out the mesons. We obtain the \(Sp(p) \times Sp(p-M)\) theory with matters listed as follows:

\[
\begin{array}{c|c|c|c|c|c}
 & Sp(p) & Sp(p-M) & SU(2) & SO(4)_2 & SO(4)_1 \\
 A^{i\alpha} & \square & \square & \square & 1 & 1 \\
 \tilde{q}^{jI} & \square & 1 & 1 & \square & 1 \\
 \tilde{Q}^{\alpha A} & 1 & \square & 1 & 1 & \square \\
\end{array}
\]

The superpotential is given by

\[
\tilde{W}_{\text{tree}} = -\frac{1}{h\mu^2} \left( \frac{1}{2} J_{jk} J_{i\ell} J_{\dot{a}\dot{b}} J_{\dot{c}\dot{d}} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} \tilde{A}^{i\alpha} \tilde{A}^{j\beta} \tilde{A}^{k\gamma} \tilde{A}^{l\delta} \right.
\]

\[
+ h\mu J_{ij} J_{kl} \tilde{a}\tilde{b} \varepsilon_{\alpha\beta} q^{iI} q^{jI} - J_{ij} J_{kl} \tilde{a}\tilde{b} \varepsilon_{\alpha\beta} \tilde{Q}^{\alpha A} \tilde{Q}^{\beta A} \tilde{Q}^{\gamma B} \tilde{Q}^{\delta A} \\
- \left( \frac{(h\mu)^2}{2} \right) J_{ij} J_{kl} \tilde{a}\tilde{b} q^{iI} q^{jI} + \frac{1}{2} J_{\dot{a}\dot{b}} J_{\dot{c}\dot{d}} \tilde{Q}^{\alpha A} \tilde{Q}^{\beta B} \tilde{Q}^{\gamma C} \tilde{Q}^{\delta A} \right).
\]
The resulting theory is similar to the original electric theory, if we make the replacement $p \to p - M$, $A \to \tilde{A}$, $Q \to q$ and $q \to \tilde{q}$ in the original theory. We obtain a new quantum constraints in terms of the mesons $\tilde{N}$, $\tilde{M}$, $\tilde{v}$ of the dual quarks $\tilde{A}, \tilde{Q}$ and the electric quark $q$ in the form (3.6). We refer this branch as the $l = 1$ branch.

Now, let us calculate the deformation parameter of the $l = 1$ branch. We can borrow the result of the $l = 0$ branch. In order to this, we must introduce the scale factor $\sqrt{2/(h\mu^2)}$ for the meson $\tilde{N}, \tilde{v}$ and $2h$ for $\tilde{M}$ in the similar way as (3.6) and use the dynamical scale $\hat{\Lambda}_1$ of the $Sp(p)$ gauge theory. We find the deformation parameter $\varepsilon_{l=1,r}$ is

$$
\varepsilon_{l=1,r}^2 = \left[2^{p-M+1} h^{-p+M+1} \mu^{2(p+M)} \hat{\Lambda}_1 p+2M+1 \right]^{\frac{1}{M}} e^{\frac{2\pi i r}{M}} \quad (r = 0, \ldots, M - 1).
$$

The phase factor $e^{\frac{2\pi i r}{M}}$ has the same origin as the one for the $l = 0$ branch, and there exist $M$ branches.

Now let us recall that we can perform the duality transformations successively. When we perform the duality transformations $l_0$ times, we find the $"l = 0"$ branch in the $Sp(p-(l_0-1)M) \times Sp(p-l_0M)$ gauge theory, which we refer as the $l = l_0$ branch of the $Sp(p+M) \times Sp(p)$ gauge theory. Similarly to our discussion in the previous section, the branches of $l = 0, 1, \ldots, k$ may be regarded as $p-lM$ D3-brane probes on the orientifolded deformed conifold.

The maximum number of times that we can perform the duality transformations is $k = \lfloor p/M \rfloor$. At the end of the duality cascade we have $Sp(p'+M) \times Sp(p')$ gauge theory, where $p' = p - kM$. We can effectively regard $Sp(p'+M)$ gauge theory with $N_f = 2p' + 2$ flavors. We find $N_f \leq N_c + 1$ at the end of the duality cascade. In the case for $p' \leq M - 2$ ($N_f \leq N_c - 1$), the dynamics of the $Sp(p'+M)$ gauge theory generates the non-perturbative superpotential [13]

$$
(-p' + M - 1) \left( \frac{\Lambda_1 p'+3M+1}{\text{Pr}V} \right)^{\frac{1}{p'+M-1}},
$$

on the branch in the form (3.6) to quantum-mechanically modify the classical constraints (3.8) into the one for the deformed conifold. The case of $p' = M - 1$, $M$ ($N_f = N_c, N_c + 1$) leads to the similar result and is discussed in the appendix.

Now let us evaluate the ratio of the deformation parameters $\varepsilon_{l=1}$ and $\varepsilon_{l=0}$. For this purpose, we would like to write the deformation parameter $\varepsilon_{l=1}$ in terms of the dynamical scales of the original $Sp(p+M) \times Sp(p)$ gauge theory. We have the anomalous symmetries which rotate the fields, the coupling constant and the instanton factors:

| $A, Q, q$ | $U(1)_A$ | $U(1)_R$ |
| --- | --- | --- |
| $h$ | $-4$ | $-2$ |
| $\Lambda_1^{p+3M+1}$ | $4p + 4$ | $2(p + M) + 2$ |
| $\Lambda_2^{p-2M+1}$ | $4(p + M) + 4$ | $2p + 2$ |

(3.26)
The ratio of the two deformation parameters should be the function of the quantity which is invariant under these symmetries as well as dimensionless. Such a quantity turns out to be
\[ h^{2p+M+2} \Lambda_1^{p+3M+1} \Lambda_2^{p-2M+1}. \] (3.27)

We may assume that the scale \( \mu \) appearing in the magnetic superpotential (3.13) provides the cut-off below which the magnetic theory gives a good description for the low energy dynamics of the electric theory. At energies close to the scale \( \mu \), it is conceivable that the coupling constant \( \Lambda_2 \) of the \( Sp(p) \) gauge dynamics also matches the one \( \tilde{\Lambda}_1 \) of the \( Sp(p) \) after the duality transformation as
\[
\left| \frac{\tilde{\Lambda}_1}{\mu} \right|^{-3p+2M-1} \sim \left| \frac{\Lambda_2}{\mu} \right|^{p-2M+1}.
\] (3.28)

The mesons \( N, v \) and \( M \) are integrated out to obtain the \( l = 1 \) branch, and the matching condition at the energy scale of their mass \( h |\Lambda_1|^2 \) gives
\[
\hat{\Lambda}_1^{p+2M+1} \sim \tilde{\Lambda}_1^{-3p+2M-1} (h |\Lambda_1|^2)^4 p+2.
\] (3.29)

As in [1], we follow the convention
\[ |\mu| \sim |\Lambda_1|. \] (3.30)

Using these relations, together with the relation between \( \Lambda_1 \) and \( \tilde{\Lambda}_2 \) in (3.17), we find the deformation parameter \( \varepsilon_{l=1} \) is
\[
|\varepsilon_{l=1}^2| \sim \left| h^{2p+M+2} \Lambda_1^{2p+6M+2} \Lambda_2^{p-2M+1} \right|^{1/4} \sim \left| \varepsilon_{l=0}^2 \right| \left| h^{2p+M+2} \Lambda_1^{p+3M+1} \Lambda_2^{p-2M+1} \right|^{1/4}.
\] (3.31)

Since the ratio should be given in terms of the combination (3.27), we find that the ratio is given by
\[
\frac{\varepsilon_{l=1}}{\varepsilon_{l=0}} \sim \left| h^{2p+M+2} \Lambda_1^{p+3M+1} \Lambda_2^{p-2M+1} \right|^{1/4}.
\] (3.32)

The relation of the string coupling with the gauge coupling constant \( g_1^2 \) of the \( SU(p+M) \) gauge group and the coupling \( g_2^2 \) of the \( SU(p) \) is known as [11, 12]
\[
\frac{11}{g_1^2} + \frac{11}{g_2^2} \sim \frac{2\pi}{g_s}.
\] (3.33)

Since the \( Sp(N) \) gauge coupling constant \( 1/g_3^{Sp} \) on the worldvolume theory is half as large as the \( SU(N) \) gauge coupling constant \( 1/g_{SU}^2 \), as in [1], it leads us to the relation
\[
\exp \left( -\frac{2\pi}{2g_s} \right) \sim h^{2p+M+2} \Lambda_1^{p+3M+1} \Lambda_2^{p-2M+1}.
\] (3.34)

Using this relation, we find that the result (3.32) exactly matches with the result (2.14) on the gravity side discussed in the previous section.
4 Summary and Discussion

We have studied the quantum moduli space of the cascading \( Sp(p + M) \times Sp(p) \) gauge theory. We found that the gauge theory includes many branches labeled by \( r = 0, \ldots, M - 1 \) and \( l = 0, \ldots, [p/M] \). The label \( r \) distinguishes \( M \) vacua generated by the symmetry breaking through gaugino condensation, while the label \( l \) is the number of times we take the Seiberg duality transformations and specifies the dimension of the flat directions. The branch \( l \) is proposed in [1] to correspond to the KS solution with mobile \( (p - lM) \) D3-branes for the \( SU(p + M) \times SU(p) \) gauge theory. In this note, we applied their proposal to the \( Sp(p + M) \times Sp(p) \) gauge theory and have seen that the ratio of the deformation parameters of different two branches is in exact agreement with the result on the string dual.

It would be an interesting problem concerned with the orientifolded conifolds to study the world volume theory on the \( p \) D3-branes and \( M \) fractional D3-branes at the orientifolded conifold without D7-branes. The gauge theory living on the D3-branes has only chiral bifundamental fields and no fundamental fields. The beta function and the symmetries of the gauge theory are different from the ones we discussed. In particular, as found in [8] for the \( SU(p + M) \times SU(p) \) cascade gauge theory, at each step of the cascade duality transformation, the difference between the ranks of the one factor group and the one of the other of the gauge groups change. On the other hand, on the supergravity side, the axion-dilaton field, which couples to the O7-plane, is no longer constant, since we have no D7-branes to cancel the RR-charge of the O7-plane. The axion-dilaton field thus gets a monodromy around the orientifold fixed point. Furthermore, it turns out that the number of RR 3-form flux as well as RR 5-form flux has the radial dependence. Therefore, it would be interesting to construct the concrete supergravity solution and analyze the gauge theory via the holographic correspondence.

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Appendix

A Quantum moduli space for \( p = M - 1, M \)

In this appendix, we discuss that the classical moduli space for \( p = M - 1, M \) are also modified by the quantum effect to give rise to the constraint for the deformed conifold, in a similar way to the case for \( p < M - 1 \).
In the case $p = M - 1$, the non-perturbative superpotential

$$W_{\text{dyn}} = X \left( \text{Pf}(V) - \Lambda_1^{p+3M+1} \right),$$  \hspace{1cm} (A.1)

where $X$ is a Lagrange multiplier, is generated \cite{13} to give the low energy effective superpotential

$$W_{\text{eff}} = h \left( J_{jk}J_{li} \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} N^{ij\alpha\beta} N^{kl\gamma\delta} + J_{ij}J_{kl}q^j^l q^i^l + J_{ij}J_{kl}q^i^j q^k^l q^j^l \right) + X \left( \text{Pf}(V) - \Lambda_1^{p+3M+1} \right).$$  \hspace{1cm} (A.2)

On the branch in the form (3.6), solving the $F$-term conditions from the effective superpotential, we obtain the quantum constraint (3.20), giving the deformed conifold.

Also for $p = M$, the non-perturbative dynamics generates the superpotential \cite{13}

$$W_{\text{dyn}} = - \frac{\text{Pf}(V)}{\Lambda_1^{p+3M+1}}.$$

In order to find the quantum moduli space on the branch (3.6), since the superpotential can be obtained formally by substituting $p = M$ into (3.18), we only have to solve the same $F$-term condition as there but with $p = M$ and find the quantum deformed constraint (3.20).

References

[1] A. Dymarsky, I. R. Klebanov and N. Seiberg, “On the moduli space of the cascading $SU(M + p) \times SU(p)$ gauge theory,” JHEP 0601 (2006) 155 arXiv:hep-th/0511254.

[2] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and $\chi$SB-resolution of naked singularities,” JHEP 0008 (2000) 052 arXiv:hep-th/0007191.

[3] S. G. Naculich, H. J. Schnitzer and N. Wyllard, “A cascading $N = 1 \ Sp(2N + 2M) \times \ Sp(2N)$ gauge theory,” Nucl. Phys. B 638, 41 (2002) arXiv:hep-th/0204023.

[4] H. J. Schnitzer and N. Wyllard, “An orientifold of $AdS_5 \times T^{11}$ with D7-branes, the associated $\alpha'$ corrections and their role in the dual $N = 1 \ Sp(2N + 2M) \times \ Sp(2N)$ gauge theory,” JHEP 0208, 012 (2002) arXiv:hep-th/0206071.

[5] P. Candelas and X. C. de la Ossa, “Comments on Conifolds,” Nucl. Phys. B 342, 246 (1990).

[6] D. N. Page and C. N. Pope, “Which Compactifications Of $D = 11$ Supergravity Are Stable?,” Phys. Lett. B 144, 346 (1984).
[7] C. P. Herzog, I. R. Klebanov and P. Ouyang, “D-branes on the conifold and $N = 1$ gauge / gravity dualities,” arXiv:hep-th/0205100.

[8] P. Ouyang, “Holomorphic D7-branes and flavored $N = 1$ gauge theories,” Nucl. Phys. B 699, 207 (2004) arXiv:hep-th/0311084.

[9] I. R. Klebanov, P. Ouyang and E. Witten, “A gravity dual of the chiral anomaly,” Phys. Rev. D 65, 105007 (2002) arXiv:hep-th/0202056.

[10] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) arXiv:hep-th/9411149.

[11] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) arXiv:hep-th/9807080.

[12] D. R. Morrison and M. R. Plesser, “Non-spherical horizons. I,” Adv. Theor. Math. Phys. 3, 1 (1999) arXiv:hep-th/9810201.

[13] K. A. Intriligator and P. Pouliot, “Exact superpotentials, quantum vacua and duality in supersymmetric $SP(N_c)$ gauge theories,” Phys. Lett. B 353, 471 (1995) arXiv:hep-th/9505006.