Generalized fusion frame in tensor product of Hilbert spaces

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Abstract

Generalized fusion frame and some of their properties in tensor product of Hilbert spaces are described. Also, the canonical dual g-fusion frame in tensor product of Hilbert spaces is considered. Finally, the frame operator for a pair of g-fusion Bessel sequences in tensor product of Hilbert spaces is presented.

Keywords: Frame, fusion frame, g-frame, g-fusion frame, frame operator,
Tensor product of Hilbert spaces, Tensor product of frames.

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1 Introduction

Frame for Hilbert space was first introduced by Duffin and Schaeffer [5] in 1952 to study some fundamental problems in non-harmonic Fourier series. The formal definition of frame in the abstract Hilbert spaces was given by Daubechies et al. [4] in 1986. Frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on. Several generalizations of frames namely, K-frames, g-frames, fusion frames, g-fusion frames etc. had been introduced in recent times. Sun [16] introduced g-frame and g-Riesz basis in complex Hilbert spaces and discussed several properties of them. P. Casazza and G. Kutyniok [2] were first to introduced the notion of fusion frames. P. Gavruta [7] discussed the duality of fusion frames and defined the frame operator for pair of fusion Bessel sequences in Hilbert spaces. Generalized fusion frames in Hilbert spaces were presented by Sadri et al. [15] to generalize the theory of fusion frame and g-frame. Generalized atomic subspaces for operators in Hilbert spaces were studied by P. Ghosh and T. K. Samanta [9] and they were also presented the stability of dual g-fusion frames in Hilbert spaces in [8].

The basic concepts of tensor product of Hilbert spaces were presented by S. Rabinson [14]. Frames and Bases in Tensor Product of Hilbert spaces were introduced by A. Khosravi and M. S. Asgari [12]. Reddy et al. [17] also studied the frame in tensor
product of Hilbert spaces and presented the tensor frame operator on tensor product of Hilbert spaces. The concepts of fusion frames and $g$-frames in tensor product of Hilbert spaces were introduced by Amir Khosravi and M. Mirzaee Azandaryani [13].

In this paper, generalized fusion frame or $g$-fusion frame in tensor product of Hilbert spaces is presented and some of their properties are going to be established. The canonical dual $g$-fusion frame in tensor product of Hilbert spaces is also discussed. At the end, the relation between the frame operators for the pair of $g$-fusion Bessel sequences in Hilbert spaces and their tensor product are obtained.

Throughout this paper, $H$ and $K$ are considered to be separable Hilbert spaces with associated inner products $\langle \cdot , \cdot \rangle_1$ and $\langle \cdot , \cdot \rangle_2$. $I_H$ and $I_K$ denotes the identity operators on $H$ and $K$, respectively. $B(H, K)$ is the collection of all bounded linear operators from $H$ to $K$. In particular $B(H)$ denote the space of all bounded linear operators on $H$. $P_V$ denote the orthogonal projection onto the closed subspace $V \subset H$. $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$ are the collections of closed subspaces of $H$ and $K$, where $I, J$ are subsets of integers $\mathbb{Z}$. $\{H_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$ are the collections of Hilbert spaces. $\{\Lambda_i \in B(H, H_i)\}_{i \in I}$ and $\{\Gamma_j \in B(K, K_j)\}_{j \in J}$ denotes the sequences of operators. Define the space $l^2(\{H_i\}_{i \in I}) = \{\{f_i\}_{i \in I} : f_i \in H_i, \sum_{i \in I} \|f_i\|^2 < \infty\}$ with inner product is given by $\langle \{f_i\}_{i \in I} , \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i , g_i \rangle_{H_i}$. Clearly $l^2(\{H_i\}_{i \in I})$ is a Hilbert space with respect to the above inner product [15]. Similarly, we can define the space $l^2(\{K_j\}_{j \in J})$.

2 Preliminaries

**Theorem 2.1.** [7] Let $V \subset H$ be a closed subspace and $T \in B(H)$. Then $P_V T^* = P_V T^* P_{TV}$. If $T$ is an unitary operator (i.e., $T^*T = I_H$), then $P_{TV} T = TP_V$.

**Theorem 2.2.** [10] The set $S(H)$ of all self-adjoint operators on $H$ is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in S(H)$

$$T \leq S \iff \langle Tf, f \rangle \leq \langle Sf, f \rangle \ \forall \ f \in H.$$

**Definition 2.3.** [3] A sequence $\{f_i\}_{i \in I}$ of elements in $H$ is a frame for $H$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f , f_i \rangle|^2 \leq B \|f\|^2 \ \forall \ f \in H.$$

The constants $A$ and $B$ are called frame bounds.
Definition 2.4. [2] Let \( \{ v_i \}_{i \in I} \) be a collection of positive weights i.e., \( v_i > 0 \ \forall \ i \in I \). A family of weighted closed subspaces \( \{ (V_i, v_i) : i \in I \} \) is called a fusion frame for \( H \) if there exist constants \( 0 < A \leq B < \infty \) such that

\[
A \| f \|^2 \leq \sum_{i \in I} v_i^2 \| P_{V_i}(f) \|^2 \leq B \| f \|^2 \ \forall f \in H.
\]

The constants \( A, B \) are called fusion frame bounds.

Definition 2.5. [16] A sequence \( \{ \Lambda_i \in B(H, H_i) : i \in I \} \) is called a generalized frame or g-frame for \( H \) with respect to \( \{ H_i \}_{i \in I} \) if there exist two positive constants \( A \) and \( B \) such that

\[
A \| f \|^2 \leq \sum_{i \in I} \| \Lambda_i(f) \|^2 \leq B \| f \|^2 \ \forall f \in H.
\]

\( A \) and \( B \) are called the lower and upper bounds of g-frame, respectively.

Definition 2.6. [15] Let \( \{ v_i \}_{i \in I} \) be a collection of positive weights. Then the family \( \Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I} \) is called a generalized fusion frame or a g-fusion frame for \( H \) with respect to \( \{ H_i \}_{i \in I} \) if there exist constants \( 0 < A \leq B < \infty \) such that

\[
A \| f \|^2 \leq \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i}(f) \|^2 \leq B \| f \|^2 \ \forall f \in H.
\]

The constants \( A \) and \( B \) are called the lower and upper bounds of g-fusion frame, respectively. If \( A = B \) then \( \Lambda \) is called tight g-fusion frame and if \( A = B = 1 \) then we say \( \Lambda \) is a Parseval g-fusion frame. If \( \Lambda \) satisfies the inequality

\[
\sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i}(f) \|^2 \leq B \| f \|^2 \ \forall f \in H
\]

then it is called a g-fusion Bessel sequence with bound \( B \) in \( H \).

Definition 2.7. [15] Let \( \Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I} \) be a g-fusion Bessel sequence in \( H \) with a bound \( B \). The synthesis operator \( T_{\Lambda} \) of \( \Lambda \) is defined as

\[
T_{\Lambda} : l^2(\{ H_i \}_{i \in I}) \rightarrow H,
\]

\[
T_{\Lambda}(\{ f_i \}_{i \in I}) = \sum_{i \in I} v_i P_{V_i} \Lambda_i^* f_i \ \forall \{ f_i \}_{i \in I} \in l^2(\{ H_i \}_{i \in I})
\]

and the analysis operator is given by

\[
T_{\Lambda}^* : H \rightarrow l^2(\{ H_i \}_{i \in I}), \ T_{\Lambda}^*(f) = \{ v_i \Lambda_i P_{V_i}(f) \}_{i \in I} \ \forall f \in H.
\]

The g-fusion frame operator \( S_{\Lambda} : H \rightarrow H \) is defined as follows:

\[
S_{\Lambda}(f) = T_{\Lambda} T_{\Lambda}^*(f) = \sum_{i \in I} v_i^2 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i}(f) \ \forall f \in H.
\]
Note 2.8. [15] Let $\Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I}$ be a $g$-fusion Bessel sequence in $H$. Then it can be easily verify that for all $f \in H$

$$
\langle S_{\Lambda} (f), f \rangle = \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i}(f) \|^2 = \| T^*_\Lambda (f) \|^2.
$$

If $\Lambda$ is a $g$-fusion frame with bounds $A$ and $B$ then from (1),

$$
\langle A f, f \rangle \leq \langle S_{\Lambda}(f), f \rangle \leq \langle B f, f \rangle \quad \forall f \in H.
$$

Now, according to the Theorem (2.2), we can write,

$$
A_{IH} \leq S_{\Lambda} \leq B_{IH}.
$$

The operator $S_{\Lambda}$ is bounded, self-adjoint, positive and invertible. Also,

$$
B_{IH}^{-1} \leq S_{\Lambda}^{-1} \leq A_{IH}^{-1}.
$$

Hence, reconstruction formula for any $f \in H$, is given by

$$
f = \sum_{i \in I} v_i^2 P_{V_i} \Lambda^*_i \Lambda_i P_{V_i} (S_{\Lambda}^{-1} f) = \sum_{i \in I} v_i^2 S_{\Lambda}^{-1} P_{V_i} \Lambda^*_i \Lambda_i P_{V_i} (f).
$$

Definition 2.9. [15] Let $\Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I}$ be a $g$-fusion frame for $H$ with frame operator $S_{\Lambda}$. Then the $g$-fusion frame $\{ (S_{\Lambda}^{-1} V_i, \Lambda_i P_{V_i} S_{\Lambda}^{-1}, v_i) \}_{i \in I}$ is called the canonical dual $g$-fusion frame of $\Lambda$.

Definition 2.10. [9] Let $\Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I}$ and $\Lambda' = \{ (V'_i, \Lambda'_i, v'_i) \}_{i \in I}$ be two $g$-fusion Bessel sequences in $H$ with bounds $D_1$ and $D_2$, respectively. Then the operator $S_{\Lambda \Lambda'} : H \rightarrow H$, defined by

$$
S_{\Lambda \Lambda'} (f) = \sum_{i \in I} v_i v'_i P_{V_i} \Lambda^*_i \Lambda'_i P_{V'_i} (f) \quad \forall f \in H,
$$

is called the frame operator for the pair of $g$-fusion Bessel sequences $\Lambda$ and $\Lambda'$.

There are several ways to introduced the tensor product of Hilbert spaces. The tensor product of Hilbert spaces $H$ and $K$ is a certain linear space of operators which was represented by Folland in [6], Kadison and Ringrose in [11].

Definition 2.11. [17] The tensor product of Hilbert spaces $H$ and $K$ is denoted by $H \otimes K$ and it is defined to be an inner product space associated with the inner product

$$
\langle f \otimes g, f' \otimes g' \rangle = \langle f, f' \rangle_1 \langle g, g' \rangle_2 \quad \forall f, f' \in H \text{ and } g, g' \in K.
$$

(2)

The norm on $H \otimes K$ is given by

$$
\| f \otimes g \| = \| f \|_1 \| g \|_2 \quad \forall f \in H \text{ and } g \in K.
$$

(3)

The space $H \otimes K$ is complete with respect to the above inner product. Therefore the space $H \otimes K$ is a Hilbert space.
Tensor Product of $g$-fusion frames

For $Q \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$, the tensor product of operators $Q$ and $T$ is denoted by $Q \otimes T$ and defined as

$$(Q \otimes T)A = QAT^* \quad \forall \ A \in H \otimes K.$$ 

It can be easily verified that $Q \otimes T \in \mathcal{B}(H \otimes K)$ [6].

**Theorem 2.12.** [6] Suppose $Q, Q' \in \mathcal{B}(H)$ and $T, T' \in \mathcal{B}(K)$, then

(I) $Q \otimes T \in \mathcal{B}(H \otimes K)$ and $\|Q \otimes T\| = \|Q\| \|T\|.$

(II) $(Q \otimes T)(f \otimes g) = Q(f) \otimes T(g)$ for all $f \in H, g \in K$.

(III) $(Q \otimes T)(Q' \otimes T') = (QQ') \otimes (TT').$

(IV) $Q \otimes T$ is invertible if and only if $Q$ and $T$ are invertible, in which case $(Q \otimes T)^{-1} = (Q^{-1} \otimes T^{-1}).$

(V) $(Q \otimes T)^* = (Q^* \otimes T^*).$

3 Construction of $g$-fusion frame in tensor product of Hilbert spaces

In this section, we discuss the generalized fusion frame or $g$-fusion frame and the canonical dual $g$-fusion frame in tensor product of Hilbert spaces.

**Definition 3.1.** Let $\{v_i\}_{i \in I}, \{w_j\}_{j \in J}$ be two families of positive weights i.e., $v_i > 0 \ \forall \ i \in I, \ w_j > 0 \ \forall \ j \in J$ and $\Lambda_i \otimes \Gamma_j \in \mathcal{B}(H \otimes K, H_i \otimes K_j)$ for each $i \in I$ and $j \in J$. Then the family $\Lambda \otimes \Gamma = \{ (V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j) \}_{i,j}$ is said to be a generalized fusion frame or $g$-fusion frame for $H \otimes K$ with respect to $\{H_i \otimes K_j\}_{i,j}$ if there exist constants $0 < A \leq B < \infty$ such that for all $f \otimes g \in H \otimes K$

$$A \|f \otimes g\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (f \otimes g) \|^2 \leq B \|f \otimes g\|^2 \quad (4)$$

where $P_{V_i \otimes W_j}$ is the orthogonal projection of $H \otimes K$ onto $V_i \otimes W_j$. The constants $A$ and $B$ are called the frame bounds of $\Lambda \otimes \Gamma$. If $A = B$ then it is called a tight $g$-fusion frame. If the family $\Lambda \otimes \Gamma$ satisfies the inequality

$$\sum_{i,j} v_i^2 w_j^2 \| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (f \otimes g) \|^2 \leq B \|f \otimes g\|^2 \quad \forall \ f \otimes g \in H \otimes K,$$

then it is called a $g$-fusion Bessel sequence in $H \otimes K$ with bound $B$. 
Note 3.2. For $i \in I$ and $j \in J$, define the space $l^2 \left( \{ H_i \otimes K_j \} \right)$

$$\{ f_i \otimes g_j \} : f_i \otimes g_j \in H_i \otimes K_j, \& \sum_{i,j} \| f_i \otimes g_j \|^2 < \infty \right\}$$

with the inner product $\langle \{ f_i \otimes g_j \} : \{ f_i' \otimes g_j' \} \rangle_{l^2} = \sum_{i,j} \langle f_i \otimes g_j, f_i' \otimes g_j' \rangle$

$$= \sum_{i,j} \langle f_i, f_i' \rangle_{H_i} \langle g_j, g_j' \rangle_{K_j} = \left( \sum_{i \in I} \langle f_i, f_i' \rangle_{H_i} \right) \left( \sum_{j \in J} \langle g_j, g_j' \rangle_{K_j} \right)$$

$$= \left( \{ f_i \}_{i \in I} \right)_{l^2(\{ H_i \}_{i \in I})} \left( \{ g_j \}_{j \in J} \right)_{l^2(\{ K_j \}_{j \in J})}.$$

The space $l^2 \left( \{ H_i \otimes K_j \} \right)$ is complete with the above inner product. Then it becomes a Hilbert space with respect to the above inner product.

Note 3.3. Since $\{ V_i \}_{i \in I}$, $\{ W_j \}_{j \in J}$ and $\{ V_i \otimes W_j \}_{i,j}$ are the families of closed subspaces of the Hilbert spaces $H$, $K$ and $H \otimes K$, respectively, it is easy to verify that $P_{V_i \otimes W_j} = P_{V_i} \otimes P_{W_j}$.

Theorem 3.4. The families $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ and $\Gamma = \{(W_j, \Gamma_j, w_j)\}_{j \in J}$ are g-fusion frames for $H$ and $K$ with respect to $\{ H_i \}_{i \in I}$ and $\{ K_j \}_{j \in J}$, respectively if and only if the family $\Lambda \otimes \Gamma = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i \otimes w_j)\}_{i,j}$ is a g-fusion frame for $H \otimes K$ with respect to $\{ H_i \otimes K_j \}_{i,j}$.

Proof. First we suppose that $\Lambda$ and $\Gamma$ are g-fusion frames for $H$ and $K$. Then there exist positive constants $(A, B)$ and $(C, D)$ such that

$$A \| f \|^2 \leq \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i}(f) \|^2_1 \leq B \| f \|^2_1 \forall f \in H \tag{5}$$

$$C \| g \|^2 \leq \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j}(g) \|^2_2 \leq D \| g \|^2_2 \forall g \in K. \tag{6}$$

Multiplying (5) and (6), and using the definition of norm on $H \otimes K$, we get

$$AC \| f \|^2_1 \| g \|^2_2 \leq \left( \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i}(f) \|^2_1 \right) \left( \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j}(g) \|^2_2 \right) \leq B D \| f \|^2_1 \| g \|^2_2. \Rightarrow AC \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| \Lambda_i P_{V_i}(f) \otimes \Gamma_j P_{W_j}(g) \|^2_2 \leq B D \| f \otimes g \|^2$$

$$\Rightarrow AC \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| \Lambda_i P_{V_i}(f) \otimes \Gamma_j P_{W_j}(g) \|^2 \leq B D \| f \otimes g \|^2.$$
Therefore, for all \( f \otimes g \in H \otimes K \), using the Theorem (2.12), we get
\[
AC \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| (\Lambda_i \otimes \Gamma_j) P_{V_i} \otimes P_{W_j} (f \otimes g) \|^2 \leq BD \| f \otimes g \|^2
\]
\[
\Rightarrow AC \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| (\Lambda_i \otimes \Gamma_j) P_{V_i} \otimes W_j (f \otimes g) \|^2 \leq BD \| f \otimes g \|^2.
\]

Hence, \( \Lambda \otimes \Gamma \) is a \( g \)-fusion frame for \( H \otimes K \) with respect to \( \{H_i \otimes K_j\}_{i,j} \) with bounds \( AC \) and \( BD \).

Conversely, suppose that \( \Lambda \otimes \Gamma \) is a \( g \)-fusion frame for \( H \otimes K \) with respect to \( \{H_i \otimes K_j\}_{i,j} \). Then there exist constants \( A, B > 0 \) such that for all \( f \otimes g \in H \otimes K - \{\theta \otimes \theta\} \),
\[
A \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| (\Lambda_i \otimes \Gamma_j) P_{V_i} \otimes W_j (f \otimes g) \|^2 \leq B \| f \otimes g \|^2.
\]
\[
\Rightarrow A \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| \Lambda_i P_{V_i} (f) \otimes \Gamma_j W_j (g) \|^2 \leq B \| f \otimes g \|^2.
\]

Using (2) and (3), we obtain
\[
A \| f \|_2^2 \| g \|_2^2 \leq \left( \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|_1^2 \right) \left( \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|_2^2 \right) \leq B \| f \|_1^2 \| g \|_2^2.
\]

Since \( f \otimes g \) is non-zero vector, \( f \) and \( g \) are also non-zero vectors and therefore \( \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|_1^2 \) and \( \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|_2^2 \) are non-zero.

\[
\Rightarrow \frac{A \| g \|^2 \| f \|^2}{\sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|^2} \leq \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|^2 \leq \frac{B \| g \|^2 \| f \|^2}{\sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|^2}
\]
\[
\Rightarrow A_1 \| f \|^2 \leq \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|^2 \leq B_1 \| f \|^2 \quad \forall f \in H,
\]

where \( A_1 = \frac{A \| g \|^2}{\sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|^2} \) and \( B_1 = \frac{B \| g \|^2}{\sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|^2} \). This shows that \( \Lambda \) is a \( g \)-fusion frame for \( H \) with respect to \( \{H_i\}_{i \in I} \). Similarly, it can be shown that \( \Gamma \) is \( g \)-fusion frame for \( K \) with respect to \( \{K_j\}_{j \in J} \). This completes the proof. \( \square \)

**Note 3.5.** Let \( \Lambda \otimes \Gamma = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i,j} \) be a \( g \)-fusion frame for the Hilbert space \( H \otimes K \). According to the definition (2.7), the synthesis operator \( T_{\Lambda \otimes \Gamma} : l^2 (\{H_i \otimes K_j\}) \to H \otimes K \) is given by
\[
T_{\Lambda \otimes \Gamma} (\{f_i \otimes g_j\}) = \sum_{i,j} v_i w_j P_{V_i} \otimes P_{W_j} (\Lambda_i \otimes \Gamma_j)^* (f_i \otimes g_j)
\]
This implies that $S$.

This shows that $T$.

Theorem (2.12), it follows that $T$.

If $f \otimes g \in H \otimes K$, we have

$$S_{\Lambda \otimes \Gamma} (f \otimes g) = \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (f \otimes g)$$

for all $f \otimes g \in H \otimes K$.

Theorem 3.6. If $S_{\Lambda}, S_{\Gamma}$ and $S_{\Lambda \otimes \Gamma}$ are the associated $g$-fusion frame operators and $T_{\Lambda}, T_{\Gamma}$ and $T_{\Lambda \otimes \Gamma}$ are the synthesis operators of $g$-fusion frames $\Lambda, \Gamma$ and $\Lambda \otimes \Gamma$ for $H, K$ and $H \otimes K$, respectively, then $S_{\Lambda \otimes \Gamma} = S_{\Lambda} \otimes S_{\Gamma}$, $S_{\Lambda \otimes \Gamma}^{-1} = S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1}$, and $T_{\Lambda \otimes \Gamma} = T_{\Lambda} \otimes T_{\Gamma}$, $T_{\Lambda \otimes \Gamma}^* = T_{\Lambda}^* \otimes T_{\Gamma}^*$.

Proof. For each $f \otimes g \in H \otimes K$, we have

$$S_{\Lambda \otimes \Gamma} (f \otimes g) = \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (f \otimes g)$$

$$= \sum_{i,j} v_i^2 w_j^2 (P_{V_i} \otimes P_{W_j}) (\Lambda_i^* \otimes \Gamma_j^*) (\Lambda_i \otimes \Gamma_j) (P_{V_i} \otimes P_{W_j}) (f \otimes g)$$

$$= \sum_{i,j} v_i^2 w_j^2 (P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} (f) \otimes P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} (g)) \quad \text{[by Theorem (2.12)]}$$

$$= \left( \sum_{i \in I} v_i^2 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} (f) \right) \otimes \left( \sum_{j \in J} w_j^2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} (g) \right)$$

$$= S_{\Lambda} (f) \otimes S_{\Gamma} (g) = (S_{\Lambda} \otimes S_{\Gamma}) (f \otimes g).$$

This implies that $S_{\Lambda \otimes \Gamma} = S_{\Lambda} \otimes S_{\Gamma}$. Since $S_{\Lambda}$ and $S_{\Gamma}$ are invertible, by (IV) of Theorem (2.12), it follows that $S_{\Lambda \otimes \Gamma}^{-1} = S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1}$. On the other hand, for all $\{f_i \otimes g_j\} \in l^2 (\{H_i \otimes K_j\})$, we have

$$T_{\Lambda \otimes \Gamma} (\{f_i \otimes g_j\}) = \sum_{i,j} v_i w_j P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (f_i \otimes g_j)$$

$$= \left( \sum_{i \in I} v_i P_{V_i} \Lambda_i^* f_i \right) \otimes \left( \sum_{j \in J} w_j P_{W_j} \Gamma_j^* g_j \right)$$

$$= T_{\Lambda} (\{f_i\}) \otimes T_{\Gamma} (\{g_j\}) = (T_{\Lambda} \otimes T_{\Gamma}) (\{f_i \otimes g_j\}).$$

This shows that $T_{\Lambda \otimes \Gamma} = T_{\Lambda} \otimes T_{\Gamma}$. Again by (V) of Theorem (2.12), it follows that $T_{\Lambda \otimes \Gamma}^* = T_{\Lambda}^* \otimes T_{\Gamma}^*$. This completes the proof. \qed

Theorem 3.7. Let $\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}$ and $\Gamma = \{(W_j, \Gamma_j, w_j)\}_{j \in J}$ be $g$-fusion frames for $H$ and $K$ with $g$-fusion frame operators $S_{\Lambda}$ and $S_{\Gamma}$, respectively. Then $\Theta = \{S_{\Lambda \otimes \Gamma}^{-1} (V_i \otimes W_j), (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda \otimes \Gamma}^{-1}, v_i w_j \}_{i,j}$ is a $g$-fusion frame for $H \otimes K$. 
Proof. Let \((A, B)\) and \((C, D)\) be the \(g\)-fusion frame bounds of \(\Lambda\) and \(\Gamma\), respectively. Now, for each \(f \otimes g \in H \otimes K\), we have

\[
\sum_{i,j} v_i^2 w_j^2 \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda_i^{-1}} P_{\Gamma_j^{-1}} (V_i \otimes W_j) (f \otimes g) \right\|^2
\]

\[
= \sum_{i,j} v_i^2 w_j^2 \left\| (\Lambda_i \otimes \Gamma_j) \left( P_{V_i \otimes W_j} S_{\Lambda_i^{-1}} \right) \left( P_{\Gamma_j^{-1}} (V_i \otimes W_j) \right) (f \otimes g) \right\|^2
\]

\[
= \sum_{i,j} v_i^2 w_j^2 \left\| \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} V_i (f) \otimes \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} W_j (g) \right\|^2 \quad \text{[by Theorem (2.1)]}
\]

\[
= \left( \sum_{i \in I} v_i^2 \left\| \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} V_i (f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} W_j (g) \right\|_2^2 \right) \quad \text{[by Theorem (2.1)]}
\]

\[
\leq B \left\| S_{\Lambda_i^{-1}} (f) \right\|_1^2 D \left\| S_{\Gamma_j^{-1}} (g) \right\|_2^2 \quad \text{[since \(\Lambda, \Gamma\) are \(g\)-fusion frames]}
\]

\[
\leq B D \left\| S_{\Lambda_i^{-1}} \right\|_2^2 \left\| f \right\|_1^2 \left\| S_{\Gamma_j^{-1}} \right\|_2^2 \left\| g \right\|_2 \leq \frac{B D}{A^2 C^2} \left\| f \otimes g \right\|^2.
\]

[since \(B^{-1} I_H \leq S_{\Lambda_i^{-1}} \leq A^{-1} I_H\) and \(D^{-1} I_K \leq S_{\Gamma_j^{-1}} \leq C^{-1} I_K\).]

On the other hand, again using Theorem (2.1)

\[
\left\| f \otimes g \right\|^4 = \left| \langle f \otimes g, f \otimes g \rangle \right|^2 = \left| \langle f, f \rangle \right|^2 \left| \langle g, g \rangle \right|^2
\]

\[
= \left| \left\langle \sum_{i \in I} v_i^2 P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} (f), f \right\rangle \right|^2 \left| \left\langle \sum_{j \in J} w_j^2 P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} (g), g \right\rangle \right|^2
\]

\[
= \left| \sum_{i \in I} v_i^2 \left\langle \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} (f), \Lambda_i P_{V_i} (f) \right\rangle \right|^2 \left| \sum_{j \in J} w_j^2 \left\langle \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} (g), \Gamma_j P_{W_j} (g) \right\rangle \right|^2
\]

\[
\leq \left( \sum_{i \in I} v_i^2 \left\| \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} (f) \right\|_1^2 \right) \left( \sum_{i \in I} v_i^2 \left\| \Lambda_i P_{V_i} (f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} (g) \right\|_2^2 \right) \times
\]

\[
\left( \sum_{j \in J} w_j^2 \left\| \Gamma_j P_{W_j} (g) \right\|_2^2 \right) \quad \text{[by C-S inequality]}
\]

\[
\leq B D \left\| f \right\|_1^2 \left\| g \right\|_2 \left( \sum_{i \in I} v_i^2 \left\| \Lambda_i P_{V_i} S_{\Lambda_i^{-1}} P_{\Lambda_i^{-1}} V_i (f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| \Gamma_j P_{W_j} S_{\Gamma_j^{-1}} P_{\Gamma_j^{-1}} W_j (g) \right\|_2^2 \right)
\]
This implies that for all $f \otimes g \in H \otimes K$, we get
\[
\frac{1}{BD} \|f \otimes g\|^2 \leq \sum_{i,j} v_i^2 w_j^2 \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda \otimes \Gamma}^{-1} P_S S_{V_i \otimes W_j}^{-1} (f \otimes g) \right\|^2.
\]

Hence, $\Theta$ is a $g$-fusion frame for $H \otimes K$ with bounds $\frac{1}{BD}$ and $\frac{BD}{A^2C^2}$. $\square$

Result 3.8. For the $g$-fusion frame $\Theta$, frame operator is $S_{\Lambda \otimes \Gamma}^{-1}$.

Proof. Take $\Delta = (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda \otimes \Gamma}^{-1}$. Then
\[
\Delta^* \Delta = ((\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda \otimes \Gamma}^{-1})^* (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} S_{\Lambda \otimes \Gamma}^{-1} = (S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1}) (P_{V_i \otimes W_j} (\Lambda_i^* \otimes \Gamma_j^*) (\Lambda_i \otimes \Gamma_j)) (P_{V_i \otimes W_j} (S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1}))
\]

$= S_{\Lambda}^{-1} P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1} P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1}$ [by Theorem (2.12)].

Now, $P_{S_{\Lambda \otimes \Gamma}^{-1}} (V_i \otimes W_j)$
\[
= \left( P_{S_{\Lambda}^{-1} V_i} \otimes P_{S_{\Gamma}^{-1} W_j} \right) \Delta^* \Delta \left( P_{S_{\Lambda}^{-1} V_i} \otimes P_{S_{\Gamma}^{-1} W_j} \right)
\]

$= P_{S_{\Lambda}^{-1} V_i} S_{\Lambda}^{-1} P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} P_{S_{\Lambda}^{-1} V_i} \otimes P_{S_{\Gamma}^{-1} W_j} S_{\Gamma}^{-1} P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1} P_{S_{\Gamma}^{-1} W_j}$

$= (P_{V_i} S_{\Lambda}^{-1})^* \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} \otimes (P_{W_j} S_{\Gamma}^{-1})^* \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1}$ [by Theorem (2.11)]

$= S_{\Lambda}^{-1} P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1} P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1}$.

Therefore, for each $f \otimes g \in H \otimes K$, we have
\[
\sum_{i,j} v_i^2 w_j^2 P_{S_{\Lambda \otimes \Gamma}^{-1}} (V_i \otimes W_j) \Delta^* \Delta P_{S_{\Lambda \otimes \Gamma}^{-1}} (V_i \otimes W_j) (f \otimes g)
\]

$= \sum_{i,j} v_i^2 w_j^2 (S_{\Lambda}^{-1} P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1} P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1}) (f \otimes g)$

$= \left( \sum_{i \in I} v_i^2 S_{\Lambda}^{-1} P_{V_i} \Lambda_i^* \Lambda_i P_{V_i} S_{\Lambda}^{-1} (f) \right) \otimes \left( \sum_{j \in J} w_j^2 S_{\Gamma}^{-1} P_{W_j} \Gamma_j^* \Gamma_j P_{W_j} S_{\Gamma}^{-1} (g) \right)$

$= S_{\Lambda}^{-1} S_{\Lambda} (S_{\Lambda}^{-1}(f)) \otimes S_{\Gamma}^{-1} S_{\Gamma} (S_{\Gamma}^{-1}(g))$

$= S_{\Lambda}^{-1}(f) \otimes S_{\Gamma}^{-1}(g) = (S_{\Lambda}^{-1} \otimes S_{\Gamma}^{-1}) (f \otimes g) = S_{\Lambda \otimes \Gamma}^{-1} (f \otimes g)$.

This shows that the corresponding $g$-fusion frame operator for $\Theta$ is $S_{\Lambda \otimes \Gamma}^{-1}$. $\square$
Remark 3.9. According to the definition [27], the g-fusion frame \( \Theta = \{ S_{\Lambda}^{-1} \otimes \Gamma (V_i \otimes W_j), (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j}, S_{\Lambda}^{-1} \otimes \Gamma, v_i w_j \}_{i,j} \) is called the canonical dual g-fusion frame of \( \Lambda \otimes \Gamma \).

Theorem 3.10. Let \( \Lambda = \{ (V_i, \Lambda_i, v_i) \}_{i \in I}, \Lambda' = \{ (V_i', \Lambda_i', v_i') \}_{i \in I} \) be g-fusion Bessel sequences with bounds \( B, D \), respectively in \( H \) and \( \Gamma = \{ (W_j, \Gamma_j, w_j) \}_{j \in J}, \Gamma' = \{ (W_j', \Gamma_j', w_j') \}_{j \in J} \) be g-fusion Bessel sequences with bounds \( E, F \), respectively in \( K \). Suppose \( (T_\Lambda, T_\Lambda^*) \) and \( (T_\Gamma, T_\Gamma^*) \) are their synthesis operators such that \( T_\Lambda^* T_\Lambda = I_H \) and \( T_\Gamma^* T_\Gamma = I_K \). Then \( \Lambda \otimes \Gamma = \{ (V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j) \}_{i,j} \) and \( \Lambda' \otimes \Gamma' = \{ (V_i' \otimes W_j', \Lambda_i' \otimes \Gamma_j', v_i' w_j') \}_{i,j} \) are g-fusion frames for \( H \otimes K \).

Proof. Since \( (\Lambda, \Lambda') \) and \( (\Gamma, \Gamma') \) are g-fusion Bessel sequences in \( H \) and \( K \), respectively, by Theorem [3.4], \( \Lambda \otimes \Gamma \) and \( \Lambda' \otimes \Gamma' \) are g-fusion Bessel sequences with bounds \( B E \) and \( D F \), respectively in \( H \otimes K \). Now, for each \( f \otimes g \in H \otimes K \),

\[
\| f \otimes g \| = \| (f, f) \| \leq \| (f, f) \|_1 = \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|_1 \times \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|_2 \leq D F \| f \|_1 \| g \|_2 \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|_1 \times \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|_2
\]

[ since \( \Lambda', \Gamma' \) are g-fusion Bessel sequences ]

\[
= D F \| f \|_1 \| g \|_2 \left( \sum_{i \in I} v_i^2 \| \Lambda_i P_{V_i} (f) \|_1 \right) \left( \sum_{j \in J} w_j^2 \| \Gamma_j P_{W_j} (g) \|_2 \right)
\]

Thus, \( \Lambda \otimes \Gamma \) is a g-fusion frame for \( H \otimes K \). Similarly, it can be shown that \( \Lambda' \otimes \Gamma' \) is also a g-fusion frame for \( H \otimes K \). \( \square \)
Frame operator for a pair of $g$-fusion Bessel sequences in tensor product of Hilbert spaces

In this section, the frame operator for a pair of $g$-fusion Bessel sequences in $H \otimes K$ is presented.

**Definition 4.1.** Let $\Lambda \otimes \Gamma = \{(V_i \otimes W_j, \Lambda_i \otimes \Gamma_j, v_i w_j)\}_{i,j}$ and $\Lambda' \otimes \Gamma'$ be two $g$-fusion Bessel sequences in $H \otimes K$. Then the operator $S : H \otimes K \rightarrow H \otimes K$ defined by for all $f \otimes g \in H \otimes K$,

$$S(f \otimes g) = \sum_{i,j} v_i w_j v_i' w_j' P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda'_i \otimes \Gamma'_j) P_{V_i' \otimes W_j'} (f \otimes g)$$

is called the frame operator for the pair of $g$-fusion Bessel sequences $\Lambda \otimes \Gamma$ and $\Lambda' \otimes \Gamma'$.

**Theorem 4.2.** Let $S_{\Lambda \Lambda'}$ and $S_{\Gamma \Gamma'}$ be the frame operators for the pair of $g$-fusion Bessel sequences $(\Lambda = \{(V_i, \Lambda_i, v_i)\}_{i \in I}, \Lambda' = \{(V_i', \Lambda_i', v_i')\}_{i \in I})$ and $(\Gamma = \{(W_j, \Gamma_j, w_j)\}_{j \in J}, \Gamma' = \{(W_j', \Gamma_j', w_j')\}_{j \in J})$ in $H$ and $K$, respectively. Then $S = S_{\Lambda \Lambda'} \otimes S_{\Gamma \Gamma'}$.

**Proof.** Since $S$ is the associated frame operator for the pair of $g$-fusion Bessel sequences $\Lambda \otimes \Gamma$ and $\Lambda' \otimes \Gamma'$, for all $f \otimes g \in H \otimes K$, we have

$$S(f \otimes g) = \sum_{i,j} v_i w_j v_i' w_j' P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda'_i \otimes \Gamma'_j) P_{V_i' \otimes W_j'} (f \otimes g)$$

$$= \sum_{i,j} v_i w_j v_i' w_j' (P_{V_i} \otimes P_{W_j}) (\Lambda_i^* \otimes \Gamma_j^*) (\Lambda'_i \otimes \Gamma'_j) (P_{V_i'} \otimes P_{W_j'}) (f \otimes g)$$

$$= \left(\sum_{i \in I} v_i v_i' P_{V_i} \Lambda_i^* \Lambda'_i P_{V_i'} (f)\right) \otimes \left(\sum_{j \in J} w_j w_j' P_{W_j} \Gamma_j^* \Gamma'_j P_{W_j'} (g)\right)$$

$$= S_{\Lambda \Lambda'} (f) \otimes S_{\Gamma \Gamma'} (g) = (S_{\Lambda \Lambda'} \otimes S_{\Gamma \Gamma'}) (f \otimes g).$$

This shows that $S = S_{\Lambda \Lambda'} \otimes S_{\Gamma \Gamma'}$.

**Theorem 4.3.** The frame operator for the pair of $g$-fusion Bessel sequences in $H \otimes K$ is bounded.

**Proof.** Let $S$ be the corresponding frame operator for the pair of $g$-fusion Bessel sequences $\Lambda \otimes \Gamma$ and $\Lambda' \otimes \Gamma'$ with bounds $B_1$ and $B_2$ in $H \otimes K$. Now, for $f \otimes g, f_1 \otimes g_1 \in H \otimes K$, we have

$$\langle S(f \otimes g), f_1 \otimes g_1 \rangle$$

$$= \left\langle \sum_{i,j} v_i w_j v_i' w_j' P_{V_i \otimes W_j} (\Lambda_i \otimes \Gamma_j)^* (\Lambda'_i \otimes \Gamma'_j) P_{V_i' \otimes W_j'} (f \otimes g), f_1 \otimes g_1 \right\rangle$$
\[
\sum_{i,j} v_i w_j v_i' w_j' \left( P_{V_i} \Lambda_i^* \Lambda_i' P_{V_i'} (f) \otimes P_{W_j} \Gamma_j^* \Gamma_j' P_{W_j'} (g) , f_1 \otimes g_1 \right)
\]

\[
= \left( \sum_{i \in I} (v_i')^2 \left\| \Lambda_i' P_{V_i'} (f) \right\|^2 \right)^{1/2} \left( \sum_{i \in I} v_i^2 \left\| \Lambda_i P_{V_i} (f_1) \right\|^2 \right)^{1/2} \times \left( \sum_{j \in J} (w_j')^2 \left\| \Gamma_j' P_{W_j'} (g) \right\|^2 \right)^{1/2} \left( \sum_{j \in J} w_j^2 \left\| \Gamma_j P_{W_j} (g_1) \right\|^2 \right)^{1/2}
\]

\[
= \left( \sum_{i,j} (v_i')^2 (w_j')^2 \left\| \Lambda_i' P_{V_i'} (f) \right\|^2 \left\| \Gamma_j' P_{W_j'} (g) \right\|^2 \right)^{1/2} \times \left( \sum_{i,j} v_i^2 w_j^2 \left\| \Lambda_i P_{V_i} (f_1) \right\|^2 \left\| \Gamma_j P_{W_j} (g_1) \right\|^2 \right)^{1/2}
\]

\[
= \left( \sum_{i,j} (v_i')^2 (w_j')^2 \left\| (\Lambda_i' \otimes \Gamma_j') P_{V_i' \otimes W_j'} (f \otimes g) \right\|^2 \right)^{1/2} \times \left( \sum_{i,j} v_i^2 w_j^2 \left\| (\Lambda_i \otimes \Gamma_j) P_{V_i \otimes W_j} (f_1 \otimes g_1) \right\|^2 \right)^{1/2}
\]

\[
\leq \sqrt{B_1 B_2} \left\| f \otimes g \right\| \left\| f_1 \otimes g_1 \right\|.
\]

Let \( S_{\Lambda \Lambda'} \) and \( S_{\Gamma \Gamma'} \) be the frame operators for the pair of \( g \)-fusion Bessel sequences \((\Lambda, \Lambda')\) and \((\Gamma, \Gamma')\), respectively. Then by above calculation

\[
\left\| \langle (S_{\Lambda \Lambda'} \otimes S_{\Gamma \Gamma'}) (f \otimes g) , f_1 \otimes g_1 \rangle \right\| \leq \sqrt{B_1 B_2} \left\| f \otimes g \right\| \left\| f_1 \otimes g_1 \right\|
\]

\[
\Rightarrow \left\| \langle S_{\Lambda \Lambda'} (f) , f_1 \rangle \right\| \left\| \langle S_{\Gamma \Gamma'} (g) , g_1 \rangle \right\| \leq \sqrt{B_1 B_2} \left\| f \right\| \left\| f_1 \right\| \left\| g \right\| \left\| g_1 \right\|
\]

\[
\Rightarrow \sup_{\left\| f_1 \right\| = 1} \left\| \langle S_{\Lambda \Lambda'} (f) , f_1 \rangle \right\| \sup_{\left\| g_1 \right\| = 1} \left\| \langle S_{\Gamma \Gamma'} (g) , g_1 \rangle \right\| \leq \sqrt{B_1 B_2} \left\| f \right\| \left\| g \right\|
\]
\[ \| S_{\Lambda \Lambda'}(f) \|_1 \| S_{\Gamma \Gamma'}(g) \|_2 \leq \sqrt{B_1 B_2} \| f \|_1 \| g \|_2 \]
\[ \| S_{\Lambda \Lambda'}(f) \|_1 \| S_{\Gamma \Gamma'}(g) \|_2 \leq \sqrt{B_1 B_2}. \]
Again taking supremum on both sides with respect to \( \| f \|_1 = 1 \) and \( \| g \|_2 = 1 \),
\[ \| S \| = \| S_{\Lambda \Lambda'} \otimes S_{\Gamma \Gamma'} \| = \| S_{\Lambda \Lambda'} \| \| S_{\Gamma \Gamma'} \| \leq \sqrt{B_1 B_2}. \]
This completes the proof. \( \square \)

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