COMPETING STYLES OF STATISTICAL MECHANICS: I. Systematization and Clarification in a General Theory

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Competing styles of Statistical Mechanics have been introduced as practical succedaneous to the conventional well established Boltzmann-Gibbs statistical mechanics, when in the use of the latter the researcher is impaired in his/her capacity for satisfying the Criteria of Efficiency and/or Sufficiency in statistics [Fisher, 1922], that is, a failure in the characterization (presence of fractality, scaling, etc.) of the system related to some aspect relevant to the given physical situation. To patch this limitation on the part of the observer, in order to make predictions on the values of observables and response functions, are introduced unconventional approaches. We present a detailed description of their construction and a clarification of its scope and interpretation. Also, resorting to the use of the particular case of Renyi’s unconventional statistics is built a nonequilibrium ensemble formalism. The unconventional distribution functions of fermions and bosons are obtained, and in the follow-up article we describe applications to the study of experimental results in semiconductor physics and in electro-chemistry.
involving nanometric scales and fractal-like structures, and some additional theoretical analysis is added.
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1. INTRODUCTION

More than twenty years ago Montroll and Shlesinger wrote that in the world of the investigation of complex phenomena that requires statistical modelling and interpretation several competing styles have been emerging, each with its own champions [1]. In the intervening years up to this beginning of the 21st century, a good amount of effort – with a flood of papers – has been dispensed to the topic. What is at play consists in that in the study of certain physico-chemical systems we may face difficulties when handling situations involving fractal-like structures, correlations (spatial and temporal) with some type of scaling, turbulent or chaotic motion, small size (nanometric scale) systems with eventually a low number of degrees of freedom, etc. These difficulties consist, as a rule, in that the researcher is unable to satisfy Fisher’s Criteria of Efficiency and/or Sufficiency [2] in the conventional, well established, physically and logically sound Boltzmann-Gibbs statistics, meaning an impairment on his/her part, to include the relevant and proper characterization of the system. To mend these difficulties, and to be able to make predictions (providing an understanding, even partial, of the physics of the system but of interest in, for example, analyzing characteristics of devices technologically relevant, as illustrated in the follow up article) one may resort to alternative statistics other than the Boltzmann-Gibbs one, which are not at all extensions of the latter but, as said, introduce a patching method.

Several approaches do exist and we can mention what can be labelled as Generalized Statistical Mechanics (see for example P. T. Landsberg, in Ref. [3]), Superstatistics (see for example E. G. D. Cohen and C. Beck in Refs. [4, 5]), Nonextensive Statistics (see for example the Conference Proceedings in Ref. [6]), and some particular cases are statistical mechanics based on Renyi Statistics (see for example I. Procaccia in Ref. [7] and T. Arimitsu in Refs. [8, 9]), Kappa (sometimes called Deformational) statistics (see for example V. M. Vasyliunas in Ref. [10] and Kaniadakis in Ref. [11]). A systematization of the subject, accompanied of a description of a large number of different possibilities, are described in what we have dubbed as Unconventional Statistical Mechanics, whose general theory and its discussion is presented in this paper while in the follow up one illustrations of its application in several physico-chemical systems are presented.

We begin noticing that Statistical Mechanics of many-body systems has a long and successful history. The introduction of the concept of probability in physics originated mainly from the fundamental essay of Laplace [12], who incorporated and extended some earlier seminal ideas (see for example [13]). As well known,
Statistical Mechanics attained the status of a well established discipline at the hands of Maxwell, Boltzmann, Gibbs, and others, and went through some steps related to changes, not in its fundamental structure, but just on the substrate provided by microscopic mechanics. Beginning with classical dynamics, statistical mechanics incorporated – as they went appearing in the realm of Physics – relativistic dynamics and quantum dynamics. Its application to the case of systems in equilibrium proceeded rapidly and with exceptional success: equilibrium statistical mechanics gave – starting from the microscopic level – foundations to Thermostatics, and the possibility to build a Response Function Theory. Applications to nonequilibrium systems began, mainly, with the case of local equilibrium in the linear regime following the pioneering work of Lars Onsager [14] (see also [15]).

For systems arbitrarily deviated from equilibrium and governed by nonlinear kinetic laws, the derivation of an ensemble-like formalism proceeded at a slower pace than in the case of equilibrium, and somewhat cautiously, with a long list of distinguished scientists contributing to such development. It can be noticed that Statistical Mechanics gained in the fifties an alternative approach sustained on the basis of Information Theory [13, 16–23]: It invoked the ideas of Information Theory accompanied with ideas of scientific inference [24, 25], and a variational principle (the latter being Jaynes’ principle of maximization of informational uncertainty – also referred-to as informational-entropy – and called MaxEnt for short), compounding from such point of view a theory dubbed as Predictive Statistical Mechanics [13, 16–21, 26]. It should be noticed that this is not a new paradigm in Statistical Physics, but a quite useful and practical variational method which codifies the derivation of probability distributions, which can be obtained by either heuristic approaches or projection operator techniques [27–29]. It is particularly advantageous to build nonequilibrium statistical ensembles, as done here, when it systematizes the relevant work on the subject that renowned scientists provided along the past century. The informational-based approach is quite successful in equilibrium and near equilibrium conditions [16, 17, 22, 23], and in the last decades has been, and is being, also applied to the construction of a generalized ensemble theory for systems arbitrarily away from equilibrium [28–30]. The nonequilibrium statistical ensemble formalism (NESEF for short) provides mechanical-statistical foundations to irreversible thermodynamics (in the form of Informational Statistical Thermodynamics – IST for short [31–34]), a nonlinear quantum kinetic theory [28, 29, 35] and a response function theory [29, 36] of a large scope for dealing with many-body systems arbitrarily away from equilibrium. NESEF has been applied
with success to the study of a number of nonequilibrium situations in the physics of semiconductors (see for example the review article of Ref. [37]) and polymers [38], as well as to studies of complex behavior of boson systems in, for example, biopolymers (e.g. Ref. [39]). It can also be noticed that the NESEF-based nonlinear quantum kinetic theory provides, as particular limiting cases, far-reaching generalizations of Boltzmann [40], Mori (together with statistical foundations for Mesoscopic Irreversible Thermodynamics [41]) [42], and Navier-Stokes [43] equations and a, say, Informational Higher-Order Hydrodynamics, linear [44] and nonlinear [45].

NESEF is built within the scope of the variational method on the basis of the maximization of the informational-entropy in Boltzmann-Gibbs-Shannon-Jaynes sense, that is, the average of minus the logarithm of the time-dependent – i.e. depending on the irreversible evolution of the macroscopic state of the system – nonequilibrium statistical operator. It ought to be emphasized that informational-entropy – a concept introduced by Shannon – is in fact the quantity of uncertainty of information, and has the role of a generating functional for the derivation of probability distributions (for tackling problems in Communication Theory, Physics, Mathematical Economics, and so on). There is one and only one situation when Shannon-Jaynes informational-entropy coincides with the true physical entropy of Clausius in thermodynamics, namely, the case of strict equilibrium [8, 46–49]. For short, we shall refer to informational-entropy as infoentropy. As already noticed the variational approach produces the well established equilibrium statistical mechanics, and is providing a satisfactory formalism for describing nonequilibrium systems in a most general form. This Boltzmann-Gibbs Statistical Mechanics allows for a proper description of the physics of condensed matter, but in some kind of situations, for example, involving nanometric-scale systems with some type or other of fractal-like structures or systems with long-range space correlations, or particular long-time correlations, it becomes difficult to apply because of a deficiency in the proper knowledge of the characterization of the states of the system in the problem one is considering (at either the microscopic or/and macroscopic or mesoscopic level). This is, say, a practical difficulty (a limitation of the researcher) in an otherwise extremely successful physical theory.

In fact, in a classical and fundamental paper of 1922 [2] by R.A.Fisher, titled “On the Mathematical Foundations of Theoretical Statistics”, are presented the basic criteria that a statistics should satisfy in order to provide valuable results. In what regards present day Statistical Mechanics in Physics two of them are of major relevance, namely the Criterion of Efficiency and the Criterion of Sufficiency.
This is so because of particular constraints that impose recent developments in physical situations involving small systems (nanotechnology, nanobiophysics, quantum dots and heterostructures in semiconductor devices, one-molecule transistors, fractals-electrodes in microbatteries, and so on), where on the one hand the number of degrees of freedom entering in the statistics may be small, and on the other hand boundary conditions of a fractal-like character are present which strongly influence the properties of the system, what makes difficult to introduce sufficient information for deriving a proper Boltzmann-Gibbs probability distribution. Other cases when sufficiency is difficult to satisfy is the case of large systems of fluids whose hydrodynamic motion is beyond the domain of validity of the classical standard approach. It is then required the use of a nonlinear higher-order hydrodynamics, eventually including correlations and other variances (a typical example is the case of turbulent motion). Also we can mention other cases where long-range correlations have a relevant role (e.g. velocity distribution in clusters of galaxies at a cosmological size, or at a microscopic size the already mentioned case of one-molecule transistors where Coulomb interaction between carriers is not screened and then of long range creating strong correlations in space with problems of scaling).

Hence, we may say that the proper use of the universal Boltzmann-Gibbs statistics is simply impaired because of either a great difficulty to handle the required information relevant to the problem in hands, or incapacity on the part of the researcher to have a correct access to such information, and consequently, out of practical convenience or the force of circumstances, respectively, a way to circumvent this inconveniency in such kind of “anomalous” situations, consists to resort to the introduction of modified forms of the informational-entropy, that is, other than the quite general one of Shannon-Jaynes, the one that leads to the well established and physically and logically sound statistics of Boltzmann-Gibbs. These modified infoentropies are built in terms of the deficient characterization one does have of the system, and are dependent on parameters – called information-entropic indexes, or infoentropic indexes for short with the understanding that refer to the infoentropy.

We restate the fundamental fact that these infoentropies are generating functionals for the derivation of probabilities distributions, and are not at all to be confused with the physical entropy of the system. Recently it has been considered the proposition that a particular one among the infinitely-many that can be defined –as shown as we proceed – comes to supersede the supposedly more restricted one of Boltzmann-Gibbs as the entropy of systems in Nature [50–52].
Such “entropy” has the form adapted to Physics of the structural infoentropy of Havrda-Charvat of Table II below, which is, we insist, a generating functional for deriving heterotypical distributions to patch the difficulties with the universal Boltzmann-Gibbs-Shannon-Jaynes one (or measure of Kullback-Leibler of Table I) when we face our (not of the statistics) limitations in satisfying Fisher’s criteria of efficiency and/or sufficiency or, according to Renyi [53] when dealing with incomplete information.

This alternative approach originated in the decades of the 1950’s and 1960’s at the hands of statisticians, being extensively used in different disciplines (economy, queueing theory, regional and urban planning, nonlinear spectral analysis, and so on). Some approaches were adapted for use in physics, and we present here an overall picture leading to what can be called Unconventional Statistical Mechanics (USM for short), consisting, as noticed, in a way to patch the lack of knowledge of characteristics of the physical system which are relevant for properly determining one or other property (see also P. T. Landsberg in Refs. [47] and [3]) impairing the correct use of the conventional one.

A large number of possible infoentropies can be explored, and Peter Landsberg quite properly titled an article Entropies Galore! [47]. An infinite family is the one that can be derived from Csiszer’s general measure of cross-entropy (see for example [54]); other family has been proposed by Landsberg [3]; and specific informational entropies are, among others, the ones of Skilling [55] – which have been used in mathematical economy –, and of Kaniadakis [56] who used it in the context of special relativity [11]. They, being generating functionals of probability distributions, give rise to particular forms of statistics: the one of next section which, as noticed, we have dubbed Unconventional Statistical Mechanics; we do also have the so-called Superstatistics proposed by C. Beck and E. G. D. Cohen for driven nonequilibrium systems with a stationary state and intensive parameter fluctuations [4, 5]; what can be called Deformational Statistics [11, 56], and other approaches could be possible.

We present here a derivation of USM in terms of unconventional informational-entropies. They are related to a family of so-called statistical measures in a metric space of statistical distributions, when it is provided a distance of the sought-after statistical distribution with a reference distribution: a principle of minimization of this distance (MinxEnt for short) is equivalent to the maximization of the associated infoentropy (MaxEnt) [54]. This is discussed in the next section, whereas in Section 3 we consider the formulation of a nonequilibrium-statistical ensemble formalism for far-from-equilibrium systems based on the use of one particular
unconventional infoentropy, namely the one due to Renyi [57]. In Section 4 we derive generalized distribution functions for fermions and bosons, which in Renyi statistics enter in place of the standard Fermi-Dirac and Bose-Einstein distributions. They are used in the follow up article to analyze experiments in condensed matter physics. Finally, Section 5 is devoted to the presentation of some additional general remarks and a summary of the results together with some further considerations.

2. INFORMATIONAL ENTROPY OPTIMIZATION PRINCIPLE

Use of the variational MaxEnt for building NESEF provides a powerful, practical, and soundly-based procedure of a quite broad scope, which is encompassed in what is sometimes referred-to as Informational-Entropy Optimization Principles (see for example Ref. [54]). To be more precise we should say constrained optimization, that is, restricted by the constraints consisting in the available information. Such optimization is performed through calculus of variation with Lagrange’s method for finding the constrained extremum being the preferred one.

Jaynes’ variational method of maximization of the informational-statistical entropy is connected – via information theory in Shannon-Brillouin style – to a principle of maximization of uncertainty of information. This is the consequence of resorting to a principle of scientific objectivity [24, 25], which can be stated as: Out of all probability distributions consistent with a given set of constraints, we must take the one that has maximum uncertainty.

As noticed in the Introduction, its use leads to a construction wholly equivalent to the one in Gibbs’ ensemble formalism, recovering the traditional results in equilibrium [16, 17, 22], and allowing for the extension to systems far from equilibrium [23, 28–30].

Jaynes’ MaxEnt is a major informational-entropy optimization principle requiring, as noticed, that we should use only the information which is accessible but scrupulously avoiding to use information not proven to be available. This is achieved by maximizing the uncertainty that remains after all the given information has been taken care of. However, this maximization of uncertainty can be looked at from a different approach. This is the MinxEnt principle, consisting into, first, to introduce a space of probability distributions and an associated metric defining a distance between two probability distributions and, second, a referential a priori distribution. According to the principle: Out of all proba-
bility distributions satisfying the given constraints, choose the one that is closest (minimum distance) to the given referential distribution.

Consequently, to carry this programme we must:

1. Introduce a metric considered to be appropriate for the problem in hand;
2. To have two types of information, namely,
   i. information consisting into giving the referential probability distribution, what would be based on intuition or experience related to the given problem;
   ii. information consisting of the constraints, through accessible observation and theoretical knowledge.

The MinxEnt principle can be considered to be based on common sense, as it is MaxEnt. In it the distribution that is derived is consistent with the given information, but among all that satisfy the given constraints we choose the one that is nearest to our intuition and experience. However, if we do not have a priori experience or an intuition to guide us, we must choose the uniform distribution as the referential one. This is so because we would be satisfying the principle of indifference in Logic, adjudicating to each event the same probability because doing otherwise we would be introducing information we do not have (we would be “playing with a loaded dice”). Introducing as the referential probability the uniform one, the probability distribution derived from MinxEnt, i.e., once it is defined a proper distance to the one that is minimized subjected to a set of given constraints, coincides the probability distribution which is obtained in MaxEnt, as shown below.

The distance \( d(\rho | \rho_r) \) between distribution \( \rho \) and the reference distribution \( \rho_r \) takes the usual definition of being a single-valued, nonnegative, real quantity satisfying the properties of invariance by inversion, the triangular inequality, and being a convex function of \( \rho \).

Let us consider the case when the uniform distribution is taken as the reference one, which we call \( U \), and then MinxEnt in terms of \( U \) is restated as: Out of all probability distributions satisfying given constraints, it is to be taken the one that is closest (i.e. at the minimum distance) to the uniform distribution, i.e. \( d(\rho | U) \) is minimum under the given constraints, for, of course, a given metric (a given \( d \)). In other words, for given constraints (information) the optimized – in the sense already discussed – distribution is the “nearest” to the uniform distribution corresponding to “maximal ignorance”: thus the uncertainty is maximized as it is also required by MaxEnt.
But now arises the question of which should be such distance. We begin to discuss the one which leads to recover Boltzmann-Gibbs formalism in Shannon-Jaynes approach, consisting in the so-called Kullback-Leibler metric, namely \[d_{KL}(\rho | \mathbb{U}) = Tr \{ \rho (\ln \rho - \ln W^{-1}) \} \tag{1}\]

where we have called \(W^{-1}\) the uniform probabilities corresponding to the physical states accessible to the system in the problem under consideration. Hence,

\[d_{KL}(\rho | \mathbb{U}) = \ln W + Tr \{ \rho \ln \rho \} = \ln W - S_{BG} \tag{2}\]

with

\[S_{BG} = -Tr \{ \rho \ln \rho \} \tag{3}\]

being Boltzmann-Gibbs-Shannon-Jaynes infoentropy for distribution \(\rho\).

Evidently, to minimize \(d_{KL}(\rho | \mathbb{U})\) under given constraints is equivalent to maximize \(S_{BG}\) under such constraints, once \(\ln W\) is a constant. Consequently Shannon-Jaynes MaxEnt is equivalent to use MinxEnt in terms of Kullback-Leibler metric. Moreover, we call the attention to the fact that the set of constraints may contain quantities (basic variables) related to correlations (i.e. second order, third order, etc. variances) besides additive quantities.

Jaynes’ MaxEnt aims at maximizing uncertainty when subjected to a set of constraints which depend on each particular situation (given values of observables and theoretical knowledge and some reliable guessing). But uncertainty can be a too deep and complex concept for admitting a unique measure under all conditions. We may face situations where uncertainty can be associated to different degrees of fuzziness in data and information. As already noticed, this is a consequence, in Statistical Mechanics, of a lack of a proper description of the physical situation. This corresponds to being violated the Criterion of Sufficiency in the characterization of the system (“the statistics chosen should summarize the whole of the relevant information supplied by the sample”) \cite{2}. This could occur at the level of the microscopic dynamics (e.g. lack of knowledge of the proper eigenstates, all important in the calculations), or at the level of macroscopic dynamics (e.g. when we are forced, because of deficiency of knowledge, to introduce a low-order truncation in the higher-order hydrodynamics that the situation may require): both situations are illustrated in the follow up article. Hence, in these circumstances it may arise the necessity of introducing alternative kind of measures, with the accompanying indexed (or structural) informational-entropies, (infoentropies for short) to build statistical descriptions other than the conventional, well established and logically sound of Boltzmann-Gibbs.
Let us consider some cases of particular measures: A large family of measures (distances) is the one provided by I. Csiszer [59], namely

\[ d_C (\rho | \rho_r) = Tr \{ \rho \Phi (R) \} \]

where \( R = \rho \rho_r^{-1} \), with \( \Phi (z) \) being a twice differentiable convex function of \( z \) and \( \Phi (1) = 0 \) (i.e. for \( \rho = \rho_r \)). Let us specify it for \( \rho_r = U \); then Kullback-Leibler measure follows for \( \Phi (R) = \ln R \). In Table I we present a few examples of the infinitely-many measures that are possible, all for \( \rho_r = U \), as defined by several authors, where \( W^{-1} \), we recall, is the value of the uniform probability for each state, and \( \alpha, \beta \) are numerical indexes (called infoentropic indexes).

Applying MinxEnt to any of these distances we would get the probability distribution deemed appropriate for the given problem in hands, namely, the conventional one in Kullback-Leibler metric, and others, so-called in Pearsons’ nomenclature, heterotypical probability distributions. But, as shown in the case of the Kullback-Leibler metric such minimizing principle is equivalent to Jaynes MaxEnt [cf. Eq. (2)], and similarly it follows that all the cases considered have an associated informational statistical entropy (ISE), whose maximization provides the corresponding optimal probability distributions. The structural-informational entropies corresponding to the measures of Table I, except for multiplicative and additive constants, are given in Table II: we recall that they are a quite few among the enormous number of possibilities, and which are cross-entropies for which the uniform probability distribution has been chosen as the reference one.

Renyi approach appears to be a particularly convenient one to deal with fractal systems as discussed in Ref. [8], where it is pointed out that predictions obtained resorting to the approach of maximization in Shannon-Jaynes approach including fractality can be equivalently obtained using Renyi approach ignoring fractality (see also follow up article). Renyi ISE has been studied by Takens and Verbitski [63], and a variation of it is Hentschel-Proccia infoentropy [7] (see also the contributions of Refs. [64, 65]. For the Havrda-Charvat structural \( \alpha \)-entropy, one akin to the case \( \alpha = 2 \) has been considered by I. Prigogine in connection with practical and theoretical difficulties with Boltzmann ideas when extending them from the dilute gas to dense gases and liquids [66]. Prigogine argues that to cope with such situations one would need a statistical expression of entropy that depends explicitly on correlations, as is the case of the Havrda-Charvat structural \( \alpha \)-entropy for \( \alpha = 2 \) (also in the case of Renyi infoentropy).
TABLE I: Special cases of Csiszer’s Measure

| Measure                  | Formula                                                                 |
|--------------------------|-------------------------------------------------------------------------|
| Kullback-Leibler [58]    | \( \ln W + Tr \{ \varrho \ln \varrho \} \)                           |
| Havrda-Charvat [60]      | \( \frac{1}{\alpha-1} Tr \{ W^{\alpha-1} \varrho^{\alpha} - \varrho \} \) \( \alpha > 0 \) and \( \alpha \neq 1 \) |
| Sharma-Mittal [61]       | \( \frac{1}{\alpha-\beta} Tr \{ [W^{\alpha-1} \varrho^{\alpha} - \varrho] - [W^{\beta-1} \varrho^{\beta} - \varrho] \} \) \( \alpha > 1 \), \( \beta \leq 1 \) or \( \alpha < 1 \), \( \beta \geq 1 \) |
| Renyi [57]               | \( \ln W + \frac{1}{\alpha-1} \ln Tr \{ \varrho^{\alpha} \} \) \( \alpha > 0 \) and \( \alpha \neq 1 \) |
| Kapur [62]               | \( \ln W + \frac{1}{\alpha-\beta} [\ln Tr \{ \varrho^{\alpha} \} - \ln Tr \{ \varrho^{\beta} \}] \) \( \alpha > 0 \), \( \beta > 0 \) and \( \alpha \neq \beta \) |

TABLE II: Informational-Statistical Entropies

Conventional (Universal) ISE

| Boltzmann-Gibbs-Shannon-Jaynes ISE | \( -Tr \{ \varrho \ln \varrho \} \) |
|-------------------------------------|--------------------------------------|
| (from K{"u}lback-Leibler measure)  |                                      |

Unconventional (entropic-index-dependent) ISEs

| From Havrda-Charvat measure | \( \frac{-1}{\alpha-1} Tr \{ \varrho^{\alpha} - \varrho \} \) \( \alpha > 0 \) and \( \alpha \neq 1 \) |
|----------------------------|---------------------------------------------------------------|
| From Sharma-Mittal measure | \( \frac{-W^{\beta-1}}{\alpha-\beta} Tr \{ [W^{\alpha-\beta} \varrho^{\alpha-\beta+1} - \varrho] \varrho^{\beta-1} \} \) \( \alpha > 1 \), \( \beta \leq 1 \) or \( \alpha < 1 \), \( \beta \geq 1 \) |
| From Renyi measure          | \( \frac{-1}{\alpha-1} \ln Tr \{ \varrho^{\alpha} \} \) \( \alpha > 0 \) and \( \alpha \neq 1 \) |
| From Kapur measure          | \( \frac{-1}{\alpha-\beta} [\ln Tr \{ \varrho^{\alpha} \} - \ln Tr \{ \varrho^{\beta} \}] \) \( \alpha > 0 \), \( \beta > 0 \) and \( \alpha \neq \beta \) |
It can be noticed that taking $\beta = 1$ reduces Kapur ISE to the one of Renyi, and Sharma-Mittal ISE to the one of Havrda-Charvat. Moreover, taking also $\alpha = 1$, is obtained an ISE which is of the form of Boltzmann-Gibbs-Shannon-Jaynes ISE. What we do have in these ISE's, or in any other one of the infinitely-many which are possible, is that when the adjustment of the parameters (the infoentropic indexes) on which they depend – let it be in a calculation or as a result of the comparison with the experimental data (see follow-up article) – produces Boltzmann-Gibbs result, this gives an indication that the principle of sufficiency is being satisfied, i.e., for such particular situation the description of the system we are doing includes all the relevant characterization that properly determines the physical property that is measured in the given experiment being analyzed. The point has also recently been discussed by Nauenberg [51], and it is illustrated in the follow-up article: In the insufficient descriptions – as there described – the parameter $\alpha$ (as noticed called infoentropic index) is different from 1 and depends on each case on the system geometry, boundary conditions, mainly its thermodynamic state (in equilibrium or out of it in steady states or time-evolving conditions), the experimental protocol, and so on.

Moreover, we again stress the fundamental fact that the structural informational-entropies (quantity of uncertainty of information) are not to be confused with the Clausius-Boltzmann physical entropy: There is one and only one case when there is an equivalence, consisting of Shannon infoentropy when the system is strictly in equilibrium [8, 46–48]. Boltzmann-Gibbs-Shannon-Jaynes informational entropy and its role in NESEF is extensively discussed in Refs. [29, 34, 67].

It is quite relevant to notice that for each kind of statistical entropy it is necessary in an ad hoc manner, to introduce definitions of average values of observables with particular forms, what is required to obtain a posteriori consistent results. For the case of Kullback-Leibler measure, or Shannon-Jaynes statistical informational-entropy, we must use the usual expression, i.e. the average of quantity $\hat{A}$ is given by

$$\langle \hat{A} \rangle = Tr \left\{ \hat{A} \varrho \right\},$$

while for the case of Renyi ISE, needs be introduced an average of the form

$$\langle \hat{A} \rangle = Tr \left\{ \hat{A} D_{\alpha} \left\{ \varrho \right\} \right\},$$

that is, in terms of the so-called escort probability [68,69]

$$D_{\alpha} \left\{ \varrho \right\} = \varrho^{\alpha}/Tr \left\{ \varrho^{\alpha} \right\},$$

(7)
which is also the one to be used in the case of Havrda-Charvat statistics. Apparently, the use of the altered distribution of Eq. (7) – later called escort probability – was originally proposed by Renyi: It appears that the motivation behind is that the quantity of information using the insufficient description in the unconventional approach (incomplete probabilities in Renyi’s nomenclature) equals the quantity of information using the conventional Shannon expression but in terms of the escort probability of Eq. (7) plus the gain in information when one introduces $D_\alpha$ in place of the incomplete $\varrho$ [see Chapter IX, p. 569 et seq., in Ref. [53]). Generalization of the concept of escort distributions is given by Beck and Schlögl (see Chapter 9 in [68]), who have also shown that for the particular case of the Renyi measure of order $\alpha$ (see Tables I and II) it follows that

$$
(1 - \alpha)^2 \frac{\partial I_\alpha}{\partial \alpha} = Tr \{D_\alpha \{\varrho_\alpha\} (\ln D_\alpha \{\varrho_\alpha\} - \ln \varrho_\alpha)\} \quad ,
$$

where $I_\alpha$ is Renyi information function (the negative of Renyi $\alpha$-dependent entropy of Table II), and the right-hand side can be interpreted as the information gain when using the escort probability $D_\alpha$ built in terms of the original one $\varrho_\alpha$ (see Chapter 5 in [68]). We also call the attention to the fact that the introduction of the escort probability of a given distribution $\varrho$, said incomplete in Renyi’s sense (Chapter IX pp. 569 et seq. in Ref. [53]), adds to the normal definition of average value the presence of second and higher-order variances. In fact, and this is detailed in Appendix A, for the average value of an observable $\hat{A}$ in terms of the escort probability of order $\gamma$, if we write $\hat{S} = -\ln \varrho$ and $\gamma = 1 + \epsilon$, it follows that (see Appendix A)

$$
\langle \hat{A} \rangle = Tr \{\hat{A} D_\alpha \{\varrho\}\} = \langle \hat{A} \rangle_o + \epsilon \left\{ \langle \hat{A} \hat{S} \rangle_o - \langle \hat{A} \rangle_o \langle \hat{S} \rangle_o \right\} + \\
+ \frac{\epsilon^2}{2} \left\{ \langle \hat{A} \hat{S} \hat{S} \rangle_o - \langle \hat{A} \rangle_o \langle \hat{S} \hat{S} \rangle_o + 2 \langle \hat{A} \rangle_o \langle \hat{S} \rangle_o^2 - 2 \langle \hat{A} \hat{S} \rangle_o \langle \hat{S} \rangle_o \right\} + \mathcal{O}(\epsilon^3) \quad ,
$$

where

$$
\langle ... \rangle_o = Tr \{...\varrho\} \quad ,
$$

that is, the normal average value.
For illustration let us take for \( \hat{A} \) the Hamiltonian \( \hat{H} \) and a canonical distribution \( \varrho = Z^{-1} \exp \left\{ -\beta \hat{H} \right\} \), and then up to second order in \( \epsilon \) Eq. (9) becomes

\[
E = \langle \hat{H} \rangle = \langle \hat{H} \rangle_o + \epsilon \beta \Delta_2 E + \frac{\epsilon^2}{2} \beta^2 \Delta_3 E ,
\]

(11)

where

\[
\Delta_2 E = \left\langle \left( \hat{H} - \langle \hat{H} \rangle_o \right)^2 \right\rangle_o = \langle \hat{H}^2 \rangle_o - \langle \hat{H} \rangle_o^2 ,
\]

(12)

\[
\Delta_3 E = \left\langle \left( \hat{H} - \langle \hat{H} \rangle_o \right)^3 \right\rangle_o = \langle \hat{H}^3 \rangle_o - 3 \langle \hat{H} \rangle_o \langle \hat{H} \rangle_o^2 + 2 \langle \hat{H} \rangle_o^3 ,
\]

(13)

are the second and third order variances of the energy.

For specific illustrations see in the follow up paper the case of the ideal gas in a finite box, and in next Section the case of ideal quantum gases. Hence, complementing what was said previously, the use of the escort probability adds “information” through the inclusion of second and higher order particular variances.

We call the attention to the fact that USM is to be based on the use of both definitions, namely, the heterotypical probability distribution and the escort probability (notice that for probability distributions other than Renyi and Havrdá-Charvat other definitions of escort probabilities should be introduced). The role of the escort probability accompanying the heterotypical-probability distribution is that both complement each other in order to redefine, in the sense of weighting, the values of the probabilities associated to the physical states of the system; on the microscopic level and on the macroscopic level the question is illustrated in the follow-up article.

Of course other possibilities are open, that is, other statistical entropies or statistical measures. One attempt is due to W. Ebeling [70, 71] who has addressed the question of the statistical treatment of a class of systems that are in some sense “anomalous”. They contain those in nature and society which are determined by its total history. Usually the given examples are the evolution of the Universe and of our planet, phenomena at the biological, ecological, climatic, social levels, etc. The approach consists into introducing conditional probabilities in the context of Boltzmann-Gibbs formalism in Shannon-Jaynes approach, leading to a generalized statistical entropy appropriate for describing the thermodynamics of complex processes with long-ranging memory and including correlations [70–72]; it can be referred-to as Ebeling statistics.
We consider next the formulation of a nonequilibrium ensemble based on the particular case of Renyi informational-entropy.

3. NONEQUILIBRIUM $\alpha$-DEPENDENT RENYI ENSEMBLE

For systems away from equilibrium several important points need be carefully taken into account in each case under consideration [27, 29]:

1. **The choice of the basic variables** (a wholly different choice than in equilibrium when suffices to take a set of those which are constants of motion), which is to be based on an analysis of what sort of macroscopic measurements and processes are actually possible, and, moreover, one is to focus attention not only on what can be observed but also on the character and expectatives concerning the equations of evolution for these variables (e.g. Refs. [73, 74]). We also notice that even though at the very initial stages we would need to introduce all the observables of the system, as time elapses more and more contracted descriptions can be used as enters into play Bogoliubov’s principle of correlation weakening and the accompanying hierarchy of relaxation times [75].

2. **It needs be introduced historicity**, that is, the idea that it must be incorporated all the past dynamics of the system (or historicity effects), all along the time interval going from a starting description of the macrostate of the sample in the given experiment, say at $t_o$, up to the time $t$ when the measurement is performed. This is a quite important point in the case of dissipative systems as emphasized among others by John Kirkwood and Hazime Mori: It implies in that the history of the system is not merely the series of events in which the system has been involved, but it is the series of transformations along time by which the system progressively comes into being at time $t$ (when a measurement is performed), through the evolution governed by the laws of mechanics [76, 77].

3. **The question of irreversibility** (or Eddington’s arrow of time) on what Rudolf Peierles stated that: “In any theoretical treatment of transport problems, it is important to realize at what point the irreversibility has been incorporated. If it has not been incorporated, the treatment is wrong. A description of the situation which preserves the reversibility in time is bound to give the answer zero or infinity for any conductivity. If we do not see clearly where the irreversibility is introduced, we do not clearly understand what we are doing” [78].

Points (1) to (3) above are discussed in Ref. [29], where it is presented a complete description of the construction of ensembles for nonequilibrium systems,
within the general theory provided by the use of Boltzmann-Gibbs formalism in Shannon-Jaynes approach.

We present next the construction of an unconventional nonequilibrium statistical ensemble formalism. First we call the attention to the situation where it is applied, namely, the experiment in condensed matter. Consider the most general experiment one can think of, namely a sample (the open system of interest composed of very-many degrees of freedom) subjected to given experimental conditions, as it is diagrammatically described in Fig. 1.

In Fig. 1, the sample is composed of a number of subsystems, $\sigma_j$, (or better to say subdegrees of freedom, for example, in solid state matter those associated to electrons, lattice vibrations, excitons, impurity states, collective excitations as plasmons, magnons, etc., hybrid excitations as polarons, polaritons, plasmaritons and so on). They interact among themselves via interaction potentials producing exchange at certain rates, $\tau_{ij}$, of energy and momentum. Pumping sources act on the different subsystems of the sample – via particular types of fields, electric, magnetic, electromagnetic, etc. – which should of course be very well characterized on setting up the experiment, and there follows relaxation of the energy in excess of equilibrium to the external reservoirs, $\tau_{jR}$. Finally, the experiment is performed coupling an external probing source, characterized in the figure by $P(t)$, with one or more subsystems of the sample, and some kind of response, say $R(t)$, is detected by a measuring apparatus (e.g. ammeter, spectrometer, etc.) Here the pumping sources exert their influence on the given open system through the fields they generate, say, magnetic, electric, electromagnetic as produced for example from a laser machine, and so on, eventually, in scattering experiments is the interaction potential with the particles of an incoming beam.

Furthermore, for simplicity, in order to avoid a cumbersome description which would obscure the presentation of the matter, we restrict the situation to the case when it is assumed that the probed subsystem $\sigma_1$ is driven out of equilibrium, while remaining in contact (interaction) with the other subsystems which are taken as an ideal thermal bath (their macroscopic states remaining constantly in equilibrium with the external reservoirs). According to theory the nonequilibrium statistical operator is a superoperator of an auxiliary one dubbed “quasi-equilibrium instantaneous frozen” statistical operator, say, $\mathcal{R}(t, 0)$ [29, 30]. In the conditions stated above it is composed of the product of the one of the subsystem under consideration, $\tilde{\rho}(t, 0)$, times the constant one of the thermal bath and reservoirs (the coupling between the subsystems and with the reservoirs is introduced in the construction of the nonequilibrium statistical operator shown below).
We concentrate the attention on the statistical operator of the subsystem of interest — from now on simply called the system —, and then once the auxiliary operator $\bar{\rho}(t, 0)$ is given, we can build the nonequilibrium statistical operator, say $\rho_\varepsilon(t)$, which can be given in either of two equivalent forms, one being ( [77, 79])

$$
\rho_\varepsilon(t) = \epsilon \int_{-\infty}^{t} dt' e^{\varepsilon(t' - t)} \bar{\rho}(t', t' - t)
$$

where

$$
\bar{\rho}(t', t' - t) = \exp \left\{ -\frac{1}{i\hbar} (t' - t) \hat{H} \right\} \bar{\rho}(t', 0) \exp \left\{ \frac{1}{i\hbar} (t' - t) \hat{H} \right\}
$$

and

$$
\hat{S}(t, 0) = \phi(t) + \sum_{j=1}^{n} \int d^3r \ F_j(r, t) \hat{P}_j(r)
$$

is the so-called informational-statistical-entropy operator which is extensively discussed in Ref. [80]. In these expressions, $\hat{H}$ is the system Hamiltonian and $\left\{ \hat{P}_j(r) \right\}$, $j = 1, 2, \ldots$, constitutes the set of basic dynamical variables describing the nonequilibrium macroscopic state of the system, with the average values of them — in terms of the distribution of Eq. (14) — constituting the set $\left\{ Q_j(r, t) \right\}$ of basic macrovariables in the nonequilibrium thermodynamic state of the system [34]. In Eq. (17), $\left\{ F_j(r, t) \right\}$, $j = 1, 2, \ldots$, is the set of Lagrange multipliers (intensive nonequilibrium thermodynamic variables [34, 81] that the variational procedure introduces), and $\phi(t)$ ensures the normalization of the distribution and can be considered as being the logarithm of a nonequilibrium partition function, i.e. $\phi(t) \equiv \ln \bar{Z}(t)$. Finally, $\epsilon \exp \left\{ \epsilon (t' - t) \right\}$ is Abel’s kernel (in the theory of convergence of integral transforms), with $\epsilon$ being a positive infinitesimal which goes to zero after the calculation of averages have been performed. This introduces the concept of Bogoliubov’s quasiaverages [82], and leads to irreversible evolution from an initial condition, what it does by selecting the retarded solutions of the Liouville equation that $\bar{\rho}$ satisfies, i.e. the advanced solutions are discarded in a quite similar way as done by Gell-Mann and Goldberger in the case of Schrödinger equation in scattering theory [83].
Equation (14) can be rewritten, after integration by parts in time, as

$$\dot{\varrho}_e(t) = \bar{\varrho}(t,0) + \dot{\varrho}_e(t),$$  \hspace{1cm} (18)

where $\bar{\varrho}(t,0)$ is given in Eq. (16) and

$$\dot{\varrho}_e(t) = - \int_{-\infty}^{t} dt' e^{(t'-t)} \frac{d}{dt'} \bar{\varrho}(t',t'-t).$$  \hspace{1cm} (19)

According to Eq. (18), the proper statistical operator $\varrho_e$ is composed of two contributions, namely $\bar{\varrho}$ which is the so-called “instantaneously frozen” contribution of Eq. (16) and $\dot{\varrho}_e$ which is responsible for the description of the irreversible evolution of the system, and it is the contribution that introduces *historicity* in the theory. Some confusion sometimes occurs when some authors use $\bar{\varrho}$ as the proper statistical operator: This auxiliary distribution, (i) *does not* satisfy Liouville equation, (ii) *does not* describe the dissipative processes that develop in the system, (iii) *does not* provide the correct kinetic theory for the description of the dissipative processes that are unfolding in the medium, (iv) *does not* give the correct values of observables, other than those corresponding to the basic variables; this also applies to the case of steady states. We also call the attention to the fact that care must be exercised on the question of separating the state of the system from the one of the reservoirs [29]. Finally, we recall the important result that for the basic variables, and *only* for the basic variables, there follows that [28–30]

$$Q_j(r,t) = Tr \left\{ \hat{P}_j(r) \varrho_e(t) \right\} = Tr \left\{ \hat{P}_j(r) \bar{\varrho}(t,0) \right\}. \hspace{1cm} (20)$$

Let us now consider the case of Renyi informational entropy, i.e.

$$S_\alpha(t) = - \frac{1}{\alpha - 1} \ln Tr \left\{ [\bar{\varrho}_\alpha(t,0)]^\alpha \right\}; \hspace{1cm} (21)$$

we notice that a recent application of Renyi’s statistics for dealing with (multi)fractal systems is presented by Jizba and Arimitsu [8]: There it is addressed the question on how Renyi’s approach appears as a quite convenient one in such cases. Further considerations on Renyi’s approach can be consulted in the articles by Hentschel and Procaccia [7] and Takens and Verbitski [63]. We first proceed to find the “instantaneously frozen” auxiliary distribution, by maximizing $S_\alpha$ subjected to the conditions of normalization

$$Tr \left\{ \bar{\varrho}_\alpha(t,0) \right\} = 1, \hspace{1cm} (22)$$
and the constraints consisting of the average values, as defined by Eq. (6), of the basic dynamical variables, namely
\[
Q_j (r, t) = Tr \left\{ \hat{P}_j (r) \, \hat{D}_\alpha \{ \hat{\varrho} (t, 0) \} \right\} ,
\]
where
\[
\hat{D}_\alpha \{ \hat{\varrho} (t, 0) \} = [\hat{\varrho}_\alpha (t, 0)]^\alpha / Tr \{ [\hat{\varrho}_\alpha (t, 0)]^\alpha \}
\]
is the corresponding escort probability [68, 69] (cf. discussion after Eq. (7) above).

It follows that (see Appendix B)
\[
\bar{\varrho}_\alpha (t, 0) = \frac{1}{\bar{\eta}_\alpha (t)} \left[ 1 + (\alpha - 1) \sum_j \int d^3r \, F_{j\alpha} (r, t) \, \Delta \hat{P}_j (r, t) \right]^{-\frac{1}{\alpha - 1}} ,
\]
where
\[
\Delta \hat{P}_j (r, t) = \hat{P}_j (r) - Q_j (r, t) ,
\]
with \(Q_j (r, t)\) given in Eq. (23),
\[
\bar{\eta}_\alpha (t) = Tr \left\{ \left[ 1 + (\alpha - 1) \sum_j \int d^3r \, F_{j\alpha} (r, t) \, \Delta \hat{P}_j (r, t) \right]^{-\frac{1}{\alpha - 1}} \right\} ,
\]
ensures the normalization condition, and \(F_{j\alpha}\) are the Lagrange multipliers that the variational method introduces, which are related to the basic variables through Eq. (23).

In terms of the auxiliary \(\bar{\varrho}_\alpha\), the statistical distribution is given by [cf. Eqs. (14) and (16)]
\[
\varrho_{\alpha\epsilon} (t) = \epsilon \int_{-\infty}^t dt' e^{\epsilon (t-t')} \bar{\varrho}_\alpha (t', t'-t) ,
\]
where, we recall,
\[
\bar{\varrho}_\alpha (t', t'-t) = \exp \left\{ -\frac{1}{i\hbar} (t' - t) \, \hat{H} \right\} \bar{\varrho}_\alpha (t', 0) \exp \left\{ \frac{1}{i\hbar} (t' - t) \, \hat{H} \right\} ,
\]
The statistical distribution of Eq. (28) satisfies the Liouville equation
\[
\frac{\partial}{\partial t} \varrho_{\alpha\epsilon} (t) + \frac{1}{i\hbar} [\varrho_{\alpha\epsilon} (t), \hat{H}] = -\epsilon [\varrho_{\alpha\epsilon} (t) - \varrho_\alpha (t, 0)] ,
\]
with the presence of the infinitesimal source introducing Bogoliubov’s symmetry breaking procedure (quasiaverages), in the present case the one of time reversal (as already noticed in that way are discarded the advanced solutions of the full Liouville equation). Thus, the retarded solutions have been selected, and, \textit{a posteriori}, this is transmitted to the kinetic equations producing a \textit{fading memory} and irreversible behavior (cf. Refs. [29, 35]).

We also call the attention to the fact that for average values, as given by Eq. (6), we then have

\[
\langle \hat{A} \rangle = Tr \left\{ \hat{A} D_{\alpha\epsilon} \{ \rho_{\alpha\epsilon} (t) \} \right\} ,
\]

(31)

where

\[
D_{\alpha\epsilon} \{ \rho_{\alpha\epsilon} (t) \} = \rho_{\alpha\epsilon} (t) / Tr \{ \rho_{\alpha\epsilon} (t) \}
\]

(32)

and it is implicit the limit \( \epsilon \to 0 \) after the calculation of traces has been performed.

Because of the boundary condition \( \rho_{\alpha\epsilon} (t_o) = [\bar{\rho}_\alpha (t_o, 0)]^\alpha (t_o \to -\infty) \), we have that

\[
D_{\alpha\epsilon} \{ \rho_{\alpha\epsilon} (t_o) \} = \bar{D}_\alpha \{ \bar{\rho}_\alpha (t_o, 0) \},
\]

where \( \bar{D}_\alpha \) is given by Eq. (24). For \( \epsilon \to 0 \), \( \rho_{\alpha\epsilon} \) satisfies a true Liouville equation [cf. Eq. (30)], and so does \( D_{\alpha\epsilon} \), and we recall that the infinitesimal source on the right-hand side of Eq. (30) is selecting the retarded solutions of the true Liouville equation (via, then, Bogoliubov’s method of quasiaverages, as previously noticed). Hence, for the given initial condition and the imposition of discarding the advanced solutions, \( D_{\alpha\epsilon} \{ \rho_{\alpha\epsilon} (t) \} \) also satisfies a modified Liouville equation, and we can write

\[
D_{\alpha\epsilon} \{ \rho_{\alpha\epsilon} (t) \} = \bar{D}_\alpha \{ \bar{\rho}_\alpha (t, 0) \} + D'_{\alpha\epsilon} (t)
\]

(33)

where \( \bar{D}_\alpha \{ \bar{\rho}_\alpha (t, 0) \} \) is given by Eq. (24), and

\[
D'_{\alpha\epsilon} (t) = - \int_{-\infty}^{t} dt' e^{\epsilon (t-t')} \frac{d}{dt'} \bar{D}_\alpha \{ \bar{\rho}_\alpha (t', t'-t) \}
\]

(34)

Introducing Eq. (33) into Eq. (31), we can see that the averages are composed of an “instantaneously frozen” (at time \( t \)) contribution, plus a contribution associated to the irreversible processes and including historicity. For the basic dynamical quantities, and \textit{only} for them [cf. Eq. (20)], it follows that

\[
Q_j (r, t) = Tr \left\{ \hat{P}_j \bar{D}_\alpha \{ \bar{\rho}_\alpha (t) \} \right\} = Tr \left\{ \hat{P}_j \bar{D}_\alpha \{ \bar{\rho}_\alpha (t, 0) \} \right\}
\]

(35)
with, as already noticed, being implicit the limit of $\epsilon$ going to $+0$ to be taken after the calculation of the trace operation has been performed.

After the nonequilibrium distribution using an heterotypical index-dependent information-entropy has been derived, next step – like done in the conventional case [29, 34–36, 67, 81] – should consists in deriving for arbitrarily far-from-equilibrium systems, a nonlinear quantum kinetic theory, a response function theory, and, of course, a systematic study of experimental results, that is, a full collection of measurements of diverse properties of the system, amenable to be studied in terms of structural (infoentropic-index dependent) informational-entropies, what is fundamental for the validation of the theory (some examples are presented in the follow-up article).

In the next section we derive the corresponding unconventional distributions for free fermions and bosons in far-from-equilibrium conditions, which are always present in the calculations of physical properties and response functions (see follow up article).

Closing this Section we recall that the previous analysis was done on the basis of considering a subsystem of the sample as out of equilibrium, but keeping the rest (so-called thermal bath) in constant equilibrium (or near equilibrium) with the reservoirs. For an unconventional nonequilibrium statistical mechanics, say in Renyi’s approach, without the restriction we would need to write the auxiliary “quasi-equilibrium statistical frozen” operator as a product involving those of each and all the $n$ subsystems, namely $\mathcal{R}(t) = \tilde{\varrho}_{\alpha_1}(t) \otimes \cdots \otimes \tilde{\varrho}_{\alpha_n}(t) \otimes \varrho_{\text{Reservoirs}}$, adjudicating an infoentropic index $\alpha_j, j = 1, 2, \ldots, n$ to each subsystem.

4. UNCONVENTIONAL DISTRIBUTIONS OF INDIVIDUAL FERMIONS AND BOSONS

Let us consider the auxiliary “instantaneously frozen” nonequilibrium statistical operator of Eq. (25). After some straightforward mathematical manipulations it follows that it can be rewritten in a more convenient form for performing calculations, namely, for the homogeneous case (i.e. neglecting dependence on the space variables)

$$
\tilde{\varrho}_\alpha(t, 0) = \frac{1}{\tilde{\eta}_\alpha(t)} \left[ 1 + (\alpha - 1) \sum_j \tilde{F}_{\alpha j}(t) \tilde{P}_j \right]^{-\frac{1}{\alpha - 1}},
$$

(36)
where
\[
\tilde{\eta}_\alpha (t) = Tr \left\{ \left[ 1 + (\alpha - 1) \sum_j \tilde{F}_{ja} (t) \hat{P}_j \right]^{-\frac{1}{\alpha - 1}} \right\}, \tag{37}
\]
\[
\tilde{F}_{ja} (t) = F_{ja} (t) \left[ 1 - (\alpha - 1) \sum_m F_{ma} (t) Q_m (t) \right]^{-1}. \tag{38}
\]
Equation (37) stands for a modified form of the quantity that ensures the normal-
ization condition, and Eq. (38) for redefined Lagrange multipliers.

We proceed next to derive the distribution functions for fermions and for
bosons using USM in terms of Renyi structural statistical approach. We choose
as basic dynamical variables, i.e. the \( \hat{P}_j \), the set of occupation number operators
\[
\{ \hat{n}_k \} = \{ c_k^\dagger c_k \}, \tag{39}
\]
where \( c (c^\dagger) \) are the usual annihilation (creation) operators in states \( |k\rangle \), satisfying the corresponding commutation and anticommutation rules of, respectively,
obsons and fermions (the spin index is ignored). Their average values are the
infoentropic-index \( \alpha \)-dependent distribution functions
\[
f_k (t) = Tr \left\{ c_k^\dagger c_k D_{\alpha \epsilon} \{ \bar{\rho}_{\alpha \epsilon} (t) \} \right\} = Tr \left\{ c_k^\dagger c_k \bar{D}_{\alpha} \{ \bar{\rho}_{\alpha} (t, 0) \} \right\}, \tag{40}
\]
where we have used Eq. (35) valid for the basic variables. The auxiliary statistical
operator is then [cf. Eq. (36)]
\[
\bar{\rho}_{\alpha} (t, 0) = \frac{1}{\tilde{\eta}_\alpha (t)} \left[ 1 + (\alpha - 1) \sum_k \tilde{F}_{ka} (t) c_k^\dagger c_k \right]^{-\frac{1}{\alpha - 1}}, \tag{41}
\]
with [cf. Eq. (38)]
\[
\tilde{F}_{ka} (t) = F_{ka} (t) \left[ 1 - (\alpha - 1) \sum_{k'} F_{k'a} (t) f_{k'} (t) \right]^{-1}. \tag{42}
\]

The populations of Eq. (40), according to the calculation described in Appen-
dix C, take the form
\[
f_k (t) = \tilde{f}_k (t) + \mathcal{C}_k (t), \tag{43}
\]
\[ f_k(t) = \frac{1}{1 + (\alpha - 1) \tilde{F}_{k\alpha}(t)} \pm 1, \quad (44) \]

where upper plus sign stands for fermions, and the lower minus sign for bosons, and

\[ c_k(t) = \alpha (1 - \alpha) (1 - f_k(t)) \sum_{k'} \tilde{F}_{k\alpha}(t) \tilde{F}_{k'\alpha}(t) Tr \left\{ c_k^{\dagger} c_k^{\dagger} c_{k'}^{\dagger} D_{\alpha} \{ \tilde{\rho}(t, 0) \} \right\} + \ldots, \quad (45) \]

Involving two, three, etc. particle correlations, which in general are minor corrections to the first, and main, contribution, the one given by Eq. (44).

In the limit of \( \alpha \) going to 1, which applies when the criteria of efficiency and/or sufficiency is satisfied, Renyi statistical entropy acquires the form of Boltzmann-Gibbs-Shannon-Jaynes one, \( C \) becomes null, \( \tilde{F}_{k\alpha}(t) \) becomes \( F_{k\alpha}(t) \), and then

\[ f_k(t) = \frac{1}{e^{F_{k\alpha}(t)} \pm 1}. \quad (46) \]

(In equilibrium \( F_{k\alpha}(t) \to (\epsilon_k - \mu)/k_B T \) and there follows the traditional Fermi-Dirac and Bose-Einstein distributions).

We can see that the distribution of Eq. (43) is composed of a term \( \tilde{f} \) corresponding to the individual particle in state \( |k\rangle \), plus the contribution \( C \) containing correlations (of order two, three, etc.) among the individual particles. This type of calculation but for systems in equilibrium, and not using the average value defined in Eq. (6), in terms of the escort probability, was reported in Ref. [84].

Let us now give some attention to the Lagrange multipliers \( F_{k\alpha}(t) \). The most general statistical operator for nonequilibrium systems can be expressed in the form of a generalized nonequilibrium grand-canonical statistical operator for a system of individual quasiparticles, where the basic variables are independent linear combinations of the single-quasiparticle occupation number operators [cf. Eq. (39)], consisting of the energy and particle densities and their fluxes of all order [29, 42, 85, 86]. In this description we have that (see also Section 4 and Appendix B in the follow up article)

\[ F_{k\alpha}(t) = \tilde{\beta}_\alpha(t) [\epsilon_k - \tilde{\mu}_\alpha(t)] - \tilde{\nu}_{h\alpha}(t) \cdot \epsilon_k u(k) - \tilde{\nu}_{n\alpha}(t) \cdot u(k) - \sum_{r \geq 2} \left[ \tilde{F}_{hr\alpha}(t) \otimes \epsilon_k u^{[r]}(k) + \tilde{F}_{nr\alpha}(t) \otimes u^{[r]}(k) \right], \quad (47) \]
where has been introduced the quantities \( \tilde{\beta}(t) = \frac{1}{k_B T^* (t)} \), playing the role of a reciprocal of a quasitemperature [87, 88], \( \tilde{\mu}_\alpha(t) \) is a quasi-chemical potential, \( \tilde{\nu}_{h\alpha}(t) \) and \( \tilde{\nu}_{n\alpha}(t) \) are vectors, and \( \tilde{F}^{[r]}_{h\alpha} \) and \( \tilde{F}^{[r]}_{n\alpha} \) \( r \)-th rank tensors. Moreover,

\[
u^{[r]}(k) = [u(k) \ldots (r \text{-times}) \ldots u(k)]
\]

is the tensorial product of \( r \)-times the characteristic velocity \( u(k) = h^{-1} \nabla_k \epsilon_k \), where \( \epsilon_k \) is the energy dispersion relation of the single-particle, and then \( u(k) \) is the group velocity in state \( |k\rangle \). Dot stands as usual for scalar product of vectors, and \( \otimes \) for fully contracted product of tensors.

To better illustrate the matter, we introduce a simplified description, or better to say a quite truncated description, proceeding to neglect in Eq. (47) all the contributions arising out of the fluxes, i.e. we put \( \nu = 0 \) and \( F^{[r]} = 0 \), retaining only the first term on the right-hand side. Therefore, we do have that

\[
\tilde{f}_k(t) = \frac{1}{1 + (\alpha - 1) \tilde{\beta}_\alpha(t) [\epsilon_k - \mu_\alpha(t)]} \pm 1
\]

where

\[
\tilde{\beta}_\alpha(t) = \beta_\alpha(t) / \{ 1 - (\alpha - 1) \beta_\alpha(t) [E(t) - \mu_\alpha(t) N(t)] \}
\]

In this Eq. (50) \( E(t) \) is the energy

\[
E(t) \simeq \sum_k \epsilon_k \tilde{f}_k(t)
\]

and \( N \) the number of particles

\[
N(t) \simeq \sum_k \tilde{f}_k(t)
\]

where the correlations \( C \) in Eq. (43) have been neglected. Moreover, in many cases we can use an approximate expression for the populations, that is, in the one of Eq. (44) we admit that \( \pm 1 \) can be neglected in comparison with the other term. This is considered as taking a statistical nondegenerate limit, once, if we put \( \alpha \) going to 1 (what, we again stress, strictly corresponds to the situation when the principle of sufficiency is satisfied), the population takes the form of a
Maxwell-Boltzmann distribution with quasitemperature $T^*(t)$ at time $t$. In this condition the expression for the population can be written as

$$\bar{f}_k(t) = A_\alpha(t) [1 + (\alpha - 1) B_\alpha(t) \epsilon_k]^{-\frac{\alpha}{\alpha-1}}, \quad (53)$$

where

$$A_\alpha(t) = \left[ 1 - (\alpha - 1) \tilde{\beta}_\alpha(t) \mu_\alpha(t) \right]^{-\frac{1}{\alpha-1}}, \quad (54)$$

and

$$B_\alpha(t) = \tilde{\beta}_\alpha(t) \left[ 1 - (\alpha - 1) \tilde{\beta}_\alpha(t) \mu_\alpha(t) \right]. \quad (55)$$

Consider a parabolic dispersion relation, that is, $\epsilon_k = \hbar k^2/2m^*$. Using Eq. (53) in Eqs. (51) and (52), we arrive at the result that

$$n(t) = \frac{N(t)}{V} = A_{3/2}^\alpha(t) \frac{\lambda_\alpha^{-3}(t)}{4\pi^2} I_{1/2}(\alpha), \quad (56)$$

$$e(t) = \frac{E(t)}{V} = n(t) \frac{I_{3/2}(\alpha)}{I_{1/2}(\alpha)} k_B T_\alpha(t), \quad (57)$$

with the integrals $I_\nu(\alpha)$ shown in Appendix D, and we have introduced the definition

$$B_{\alpha}^{-1}(t) = k_B T_\alpha(t), \quad (58)$$

where $T$ plays the role of a pseudotemperature and where $\lambda_\alpha$ in Eq. (56) is a characteristic length given by $\lambda_\alpha^2(t) = \hbar^2/m^* k_B T_\alpha(t)$ (that is, de Broglie wave length for a particle of mass $m^*$ and energy $k_B T_\alpha(t)$).

We can see that the above Eqs. (56) and (57) define the Lagrange multipliers, $\tilde{\beta}_\alpha(t)$ and $\mu_\alpha(t)$, present in $A_\alpha(t)$, $\lambda_\alpha(t)$, and $B_\alpha(t)$, in terms of the basic variables energy and number of particles. Moreover, using Eq. (56) we can obtain an expression for the quasi-chemical potential in terms of quasitemperature and density, namely

$$1 - (\alpha - 1) \tilde{\beta}_\alpha(t) \mu_\alpha(t) = \left[ 4\pi^2 \lambda_\alpha^3(t) / I_{1/2}(\alpha) \right]^{\frac{2(\alpha-1)}{\alpha-3}} \left[ n(t) \right]^{\frac{2(\alpha-1)}{\alpha-3}}. \quad (59)$$

Also, it can be noticed that for $\alpha=1$ (provided that the condition of sufficiency is satisfied) one recovers the equivalent of the results of conventional nonequilibrium statistical mechanics [29, 34], which are

$$e(t) = \frac{3}{2} n(t) k_B T^*(t), \quad (60)$$
where we have introduced the so-called quasitemperature \([29, 34, 87]\), defined by 
\[ k_B T^* (t) = B_{\alpha=1}^{-1} (t), \]
this equation standing for a kind of equipartition of energy at time \(t\), and
\[
\mu (t) = -\mu_\alpha k_B T^* (t) \ln \left[ T^* (t) / \theta_{tr} (t) \right],
\]
where \(\theta_{tr} (t) = \hbar^2 n^{2/3} (t) / 2m^*\) is the characteristic temperature (here in nonequilibrium conditions and at time \(t\)) for translational motion. This suggests us to define a so-called “kinetic temperature” \(\Theta_K (t) [89]\) by equating \(e (t)\) to \((3/2) n (t) k_B \Theta_K (t)\), given, after Eq. (57) is used, by
\[
\Theta_K (t) = \mathcal{T}_\alpha (t) / (5 - 3\alpha),
\]
where we can see that \(\alpha\) must be smaller than \(5/3\), as shown in the follow up article where connection of theory with experiment is presented, together with other illustrations and discussions.

How does the \(\alpha\)-dependent distribution of Eq. (49) compares with the usual Fermi-Dirac and Bose-Einstein distributions? For illustration we consider the nondegenerate limit of Eq. (53), common to both, where parameter \(B\) is related to the kinetic temperature \(\Theta_K\) by Eqs. (58) and (62). Taking for \(T_\alpha\) of Eq. (58) the unique value of 300 K, we do find in Figures 2 and 3 a comparison of the population of Eq. (53) corresponding to several values of the infoentropic index \(\alpha\). It can be noticed the characteristic of a different weighting of the values of the standard distribution \((\alpha \simeq 1)\), such that: (1) for \(\alpha < 1\) the population of the modes at low energies are increased at the expense of those of higher energies \((\varepsilon > 7 \times 10^{-3} eV)\), while (2) for \(\alpha > 1\) we can see the opposite behavior.

5. COMMENTS AND CONCLUDING REMARKS

Summarizing, we first notice the relevant point that in the construction of a statistical mechanics, the derivation of an appropriate (for the problem in hands) probability distribution – associated to a set of constraints imposed on the system – can be obtained in a compact and practical way by means of optimization-variational principles in a context related to information theory. These are methods of maximization of the so-called informational-entropies (better called quantities of uncertainty of information) or minimization of distances in a space of probability distributions (MaxEnt and MinxEnt respectively).

In the original formulation of Shannon and Jaynes use was made of Boltzmann-Gibbs statistical-entropy, which in MaxEnt provides the canonical-like (exponential) distributions of classical, relativistic, and quantum statistical mechanics. In
Ref. [29] it is described its use for the case of many-body systems arbitrarily far removed from equilibrium, and the discussion of the dissipative phenomena that unfold in such conditions (mainly ultrafast relaxation processes; see Ref. [37]). These statistical distributions also follow from MinxEnt once we use Kullback-Leibler measure with the uniform probability as the referential one in the definition of the corresponding distance.

This approach has been exceedingly successful in conditions of equilibrium, and is a very promising one for nonequilibrium conditions. To have a reliable statistical theory in these situations is highly desirable since in very many situations – as for example are the case of electronic and optoelectronic devices, chemical reactors, fluid motion, and so on – the system is working in far-from-equilibrium conditions.

However the enormous success and large application of Shannon-Jaynes method to Laplace-Maxwell-Boltzmann-Gibbs statistical foundations of physics, as it has been noticed, some cases look as difficult to be properly handled within the Boltzmann-Gibbs formulation, as a result of existing some kind of fuzziness in data or information, that is, the presence of a condition of insufficiency in the characterization of the (microscopic and/or macroscopic or mesoscopic) state of the system. Such, say, difficulty with the proper characterization of the system in the problem in hands, (which is a practical one and, we stress, not intrinsic to the most general and complete Boltzmann-Gibbs formalism) can be, as shown, patched with the introduction of peculiar parameter-dependent alternative structural informational-entropies (see Table II).

Particularly, to deal with systems with some kind of fractal-like structure the use of Boltzmann-Gibbs-Shannon-Jaynes infoentropy would require to introduce as information the highly correlated conditions that are in that case present. Two examples in condensed matter physics (described in the follow up article) are “anomalous” diffusion [90] and “anomalous” optical spectroscopy [91], when fractality enters via the non-smooth topography of the boundary surfaces which have large influence on phenomena occurring in constrained geometries (nanometer scales in the active region of the sample). In the conventional and more general approach, the spatial correlations that the granular boundary conditions introduce need be given as information (to satisfy the criterion of sufficiency, since they are quite relevant for determining the behavior of the system in the nanometric scales involved), but to handle them is generally a nonfeasible task. For example, in the second case above mentioned one has no easy access to the determination of the detailed topography of the surfaces which limit the active region of the sample (the
nanometric quantum wells in semiconductor heterostructures), what can be done in the first case using atomic-force microscopy and the determination of the fractal dimension involved is possible. Hence the most general and complete Boltzmann-Gibbs formalism in Shannon-Jaynes approach becomes hampered out and is difficult to handle, and then, as shown, use of other types of informational-entropies (better called generating functionals for deriving probability distributions) may help to circumvent such inconvenience by introducing alternative algorithms (dependent on the so-called informational-entropic indexes), that is, the derivation of heterotypical probability distributions on the basis of the constrained maximization of unconventional informational-statistical entropies (quantity of uncertainty of information), to be accompanied, as noticed in the main text, with the use of the so-called escort probabilities.

Summarizing, Unconventional Statistical Mechanics consists of two steps: 1. The choice of a deemed appropriate structural informational-entropy for generating the heterotypical statistical operator, and 2. The use of a escort probability in terms of the heterotypical distribution of item 1.

As shown in the main text, and illustrated in the follow-up article, the escort probability introduces corrections to the insufficient description by including correlations and higher-order variances of the observables involved. On the other hand, the heterotypical distribution introduces corrections to the insufficient description (or incomplete probabilities in Renyi’s nomenclature) by modifying the statistical weight of the dynamical states of the conventional approach involved in the situation under consideration. Moreover, we have considered a particular case, namely the statistics as derived from the use of Renyi informational entropy (also used in the analysis of the experiments described in the follow-up article). We centered the attention on the derivation of an Unconventional Statistical Mechanics appropriate for dealing with far-removed-from-equilibrium systems. Moreover, we have reported the calculation, in such conditions, of the distribution functions of single fermions and bosons, the counterpart in these unconventional statistics of the usual Fermi-Dirac and Bose-Einstein distributions, which are used and the results compared with experimental data in the follow-up article. These distributions are illustrated in Figs. 2 and 3.

In conclusion, we may say that USM appears as a valuable approach, in which the introduction of informational-entropic-indexes-dependent informational-entropies leads to a particularly convenient and sophisticated tool for fitting theory to experimental data for certain classes of physical systems, for which the criterion of sufficiency in its characterization cannot be properly satisfied. Among them we
can pinpoint fractal-like structured nanometric-scale systems, which, otherwise, would be difficult to deal with within the framework of the conventional Statistical Mechanics. While in the latter case one would need to have a detailed description of the spatial characteristics of the structure of the system, the unconventional one needs to pay the price of having an open adjustable index to be fixed by best fitting with experimental results. It is relevant to notice the fact that the infoentropic index(es) is(are) dependent on the dynamics involved, the system’s geometry and dimensions, boundary conditions, its macroscopic-thermodynamic state (in equilibrium, or out of it when becomes a function of time), and the experimental protocol.

Finally, we call the attention to the fact that we have presented several alternatives of cross-entropies (see Table II), for which, as stated in the main text, the uniform probability distribution is taken as the reference one, and such generating functionals provide a corresponding family of heterotypical probability distributions. However, other choices of the reference probability can be made and then we have at our disposal very-many possibilities: It is tempting to look for the construction of a theory using for the probability of reference, instead of the uniform distribution, Shannon-Jaynes informational-entropy in its incomplete formalism, that is, when suffering from the deficiency that the researcher cannot satisfy Fisher’s criteria of efficiency and/or sufficiency.

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Appendix A. The escort probability

Let us consider the probability distribution $\rho$ and construct the associated escort probability of order $\gamma$

$$D_\gamma \{\rho\} = \rho^\gamma / Tr \{\rho^\gamma\} \quad .$$  \hspace{1cm} (A.1)

We write $\gamma = 1 + \epsilon$ and proceed with a series expansion of $D_\gamma$ around the value $\gamma = 1$, to obtain, on the one hand

$$\rho^\gamma = \rho \left[ 1 + \epsilon \hat{S} + \frac{\epsilon^2}{2} \hat{S} \hat{S} + ... \right] \quad ,$$  \hspace{1cm} (A.2)

and

$$Tr \{\rho^\gamma\} = 1 + \epsilon Tr \{\rho \hat{S}\} + \frac{\epsilon^2}{2} Tr \{\rho \hat{S} \hat{S}\} + ... \quad ,$$  \hspace{1cm} (A.3)

where we have introduced the nomenclature

$$\hat{S} = - \ln \rho \quad .$$  \hspace{1cm} (A.4)

Using these results, given any observable $\hat{A}$ its average value in terms of the escort probability is given by

$$\langle \hat{A} \rangle = Tr \{\hat{A} \hat{D}_\gamma \{\rho\}\} =$$

$$= Tr \{\hat{A} \rho\} + \epsilon \left[ Tr \{\hat{A} \hat{S} \rho\} - Tr \{\hat{A} \rho\} Tr \{\hat{S} \rho\} \right] +$$

$$+ \frac{\epsilon^2}{2} \left[ Tr \{\hat{A} \hat{S} \hat{S} \rho\} - Tr \{\hat{A} \rho\} Tr \{\hat{S} \hat{S} \rho\} + Tr \{\hat{A} \rho\} \left[ Tr \{\hat{S} \rho\} \right]^2 -$$

$$- Tr \{\hat{A} \hat{S} \rho\} Tr \{\hat{S} \rho\} \right] .$$  \hspace{1cm} (A.5)

For illustration, let $\rho$ be the auxiliary nonequilibrium statistical operator of Eq. (16), that is

$$\rho(t,0) = \exp \left\{ -\phi(t) - \sum_{j=1}^{n} F_j(t) \hat{P}_j \right\} \quad ,$$  \hspace{1cm} (A.6)

and the average of any of the basic observables, say $\hat{P}_m$ [cf. Eq. (20)] in terms of the associated escort probability, given by

$$Q_m(t) = \langle \hat{P}_m \mid t \rangle_{ep} = Tr \{\hat{P}_m \tilde{\rho}(t,0)\} +$$
\[ + \epsilon \sum_{j=1}^{n} F_j(t) \left[ Tr \left\{ \hat{P}_m \hat{P}_j \rho(t, 0) \right\} - Tr \left\{ \hat{P}_m \bar{\rho}(t, 0) \right\} Tr \left\{ \hat{P}_j \rho(t, 0) \right\} \right] = \ldots . \]  \hfill (A.7)

In terms of Renyi probability distribution, when the order of the escort probability is to be chosen as equal to the infoertronic index, that is \( \tilde{D}_\alpha \{ \bar{\rho}_\alpha(t, 0) \} \) we do have the result of Eq. (A.7) but where \( \bar{\rho}_\alpha(t, 0) \) enters as the probability \( \bar{\rho}(t, 0) \).

**Appendix B. Derivation in MaxEnt of Eq. (18)**

Given the constraints of Eqs. (22) and (23), with \( \tilde{D}_\alpha(t, 0) \) defined in Eq. (24), and the statistical \( \alpha \)-entropy of Eq. (21), according to Lagrange method we look for a maximum of the functional

\[
I(\bar{\rho}) = -\frac{1}{\alpha - 1} \ln Tr \left\{ [\bar{\rho}_\alpha(t, 0)]^\alpha \right\} + \phi(t) Tr \left\{ \tilde{D}_\alpha(t, 0) \right\} - \sum_j \int dr^3 F_{j\alpha}(r, t) Tr \left\{ \hat{P}_j(r) \tilde{D}_\alpha(t, 0) \right\}, \quad (B.1)
\]

where \( \phi \) and \( F_{j\alpha} \) are the corresponding Lagrange multipliers. The variational differential of \( I \) for a variation \( \delta \bar{\rho}_\alpha \) is given by

\[
\frac{\delta I(\bar{\rho})}{\delta \bar{\rho}_\alpha} = -\frac{\alpha}{\alpha - 1} \frac{[\bar{\rho}_\alpha(t, 0)]^{\alpha-1}}{Tr \left\{ [\bar{\rho}_\alpha(t, 0)]^\alpha \right\}} + \phi(t) - \sum_j \frac{1}{Tr \left\{ [\bar{\rho}_\alpha(t, 0)]^\alpha \right\}} \int dr^3 F_{j\alpha}(r, t) \Delta \hat{P}_j(r, t) \left[ \tilde{D}_\alpha(t, 0) \right]^{\alpha-1}, \quad (B.2)
\]

where

\[
\Delta \hat{P}_j(r, t) = \hat{P}_j(r) - Tr \left\{ \hat{P}_j(r) \tilde{D}_\alpha(t, 0) \right\} = \hat{P}_j(r) - Q_j(r, t) \quad , \quad (B.3)
\]

Making null Eq. (B.2) it follows that

\[
[\bar{\rho}_\alpha(t, 0)]^{\alpha-1} = \frac{(\alpha - 1) \phi(t) Tr \left\{ [\bar{\rho}_\alpha(t, 0)]^\alpha \right\} / \alpha}{1 + (\alpha - 1) \sum_j \int dr^3 F_{j\alpha}(r, t) \Delta \hat{P}_j(r, t)}, \quad (B.4)
\]
which can be written in the form

\[
\bar{\rho}_\alpha(t,0) = \frac{1}{\bar{\eta}_\alpha(t)} \left[ 1 + (\alpha - 1) \sum_j \int d^3r \ F_j(t) \ \Delta \hat{P}_j(t) \right]^{-\frac{1}{\alpha-1}}, \quad (B.5)
\]

where

\[
\bar{\eta}_\alpha(t) = \int d\Gamma \left[ 1 + (\alpha - 1) \sum_j \int d^3r \ F_j(t) \ \Delta \hat{P}_j(t) \right]^{-\frac{1}{\alpha-1}}, \quad (B.6)
\]

ensures the normalization of \(\bar{\rho}_\alpha\) and we have the expressions of Eqs. (25) and (27). We recall, and stress, that \(\bar{\rho}_\alpha(t,0)\) of Eq. (B.5) is an auxiliary operator, with the proper statistical operator resulting as a functional of this one once historicity is introduced, as indicated in Eq. (28).

**Appendix C. Calculation of Distribution Functions**

To proceed with the calculation of \(f_k(t)\) of Eq. (40) we first write

\[
Tr \left\{ c_k^\dagger c_k \ \bar{\rho}_\alpha \right\} = Tr \left\{ \bar{\rho}_\alpha \ c_k^\dagger c_k \right\} = Tr \left\{ \bar{\rho}_\alpha \ c_k^\dagger c_k \right\} = Tr \left\{ \bar{\rho}_\alpha \ c_k^\dagger c_k \right\}
\]

where \(\bar{\rho}_\alpha\) is given by Eq. (28). We define

\[
\hat{A} = (\alpha - 1) \sum_k \tilde{F}_k c_k^\dagger c_k \quad , \quad (C.2)
\]

\[
\hat{B} = c_k^\dagger \quad ; \quad \nu = \alpha / (1 - \alpha) \quad , \quad (C.3)
\]

and use that [92]

\[
\left( 1 + \hat{A} \right)^\nu = 1 + \sum_n a_{n\nu} \hat{A}^n \quad , \quad (C.4)
\]

where

\[
a_{n\nu} = \frac{1}{n!} \nu (\nu - 1) \ldots (\nu - n + 1) \quad , \quad (C.5)
\]

considering the eigenvalues of \(\hat{A}\) as being smaller than 1 to ensure the convergence. Then, after some lengthy but straightforward calculations we find that

\[
\bar{\rho}_\alpha \ c_k^\dagger \ [\bar{\rho}_\alpha]^{-\alpha} = \left( 1 + \hat{A} \right)^\nu \hat{B} \left( 1 + \hat{A} \right)^{-\nu} =
\]
\[ \hat{B} + a_{1\nu} \left[ \hat{A}, \hat{B} \right] + a_{2\nu} \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] + ... + \left( a_{2\nu} - a_{2,-\nu} \right) \left[ \hat{A}, \hat{B} \right] \hat{A} + ... \] , (C.6)

which, on account that,

\[ \left[ \hat{A}, \hat{B} \right] = \lambda \hat{B} \quad ; \quad \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] = \lambda^2 \hat{B} \quad ; \quad ... \] , (C.7)

where \( \lambda = -(1 - \alpha) \tilde{F}_k \), can be rewritten as

\[ \left[ \bar{\rho}_\alpha \right]^\alpha c_k \dagger \left[ \bar{\rho}_\alpha \right]^{-\alpha} = \left[ 1 + (\alpha - 1) \tilde{F}_k \right]^{-\frac{1}{\alpha - 1}} c_k \dagger - \hat{N}_k \] , (C.8)

with

\[ \hat{N}_k = \alpha (\alpha - 1) \sum_k \tilde{F}_k \tilde{F}_k c_k \dagger c_k \dagger c_k + ... \] (C.9)

being a series composed of terms involving three, four, etc., single-particle creation annihilation operators. Using Eq. (C.8) in Eq. (40) there follows Eq. (44), after taking into account that \( c_k c_k \dagger = 1 \mp c_k \dagger c_k \); \((-\) for fermions and \((+\) for bosons respectively.

**Appendix D. The Beta Functions of Eqs. (49) and (50)**

The functions of the parameter \( \alpha \) of Eqs. (56) and (57)

\[ I_\nu (\alpha) = \int_0^\infty dx x^\nu \left[ 1 + (\alpha - 1) \right]^{\frac{\alpha}{\alpha - 1}} \] (D.1)

are of the family of the so-called Beta functions, which are [92]

\[ B(x, y) = \int_0^\infty dt \frac{t^{x-1}}{(t + 1)^{x+y}} = \int_0^1 dt t^{x-1} (1 - t)^{y-1} = \Gamma (x) \Gamma (y) / \Gamma (x + y) \] . (D.2)

Using Eq. (D.2), after some handling, we find for \( I_{1/2} (\alpha) \) and \( I_{3/2} (\alpha) \) that for \( \alpha > 1 \)

\[ I_{1/2} (\alpha) = \frac{1}{(\alpha - 1)^{3/2}} \frac{\Gamma (3/2) \Gamma \left( \frac{\alpha}{\alpha - 1} - \frac{3}{2} \right)}{\Gamma \left( \frac{\alpha}{\alpha - 1} \right)} \] , (D.3)
with the restriction $1 \leq \alpha < 3$, 
\[ I_{3/2}(\alpha) = \frac{1}{(\alpha - 1)^{5/2}} \frac{\Gamma(5/2) \Gamma \left( \frac{\alpha}{\alpha - 1} - \frac{5}{2} \right)}{\Gamma \left( \frac{\alpha}{\alpha - 1} \right)} \] 
with the restriction $1 \leq \alpha < 5/3$.

Using the property that $\Gamma(z + 1) = z \Gamma(z)$ it follows Eq. (57).
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**Figure 1:** Diagramatic description of a typical pump-probe experiment in an open dissipative system.
Figure 2: The distribution of Eq. (53) for a kinetic temperature of $300K$ and values of Renyi’s infoentropic-index $\alpha$ smaller than 1.
Figure 3: The distribution of Eq. (53) for a kinetic temperature of 300K and values of Renyi’s infoentropic-index $\alpha$ larger than 1.