Inequalities for convolutions of functions on commutative hypergroups

by

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Abstract

The generalized Young inequality on the Lorentz spaces for commutative hypergroups is introduced and an application of it is given to the theory of fractional integrals. The boundedness on the Lorentz space and the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups is proved.

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1 Introduction and preliminaries

It is known that a convolution of two functions on $\mathbb{R}^n$ is defined by

$$ f *_{\mathbb{R}^n} g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy. $$

Classical Young’s inequality on the $L^p(\mathbb{R}^n)$ spaces the convolution of two functions on $\mathbb{R}^n$ states that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then

$$ \| f *_{\mathbb{R}^n} g \|_r \leq C \| f \|_p \| g \|_q, $$

where $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

The generalized Young inequality give us the boundedness on the Lorentz spaces for the convolution of two functions on $\mathbb{R}^n$.

**Theorem 1.1** ([10] Theorem 2.10.1) If $f \in L^{p_1, q_1}(\mathbb{R}^n)$, $\varphi \in L^{p_2, q_2}(\mathbb{R}^n)$ and $\frac{1}{p_1} + \frac{1}{p_2} > 1$, then $(f * \varphi) \in L^{p_0, q_0}(\mathbb{R}^n)$ where $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_0}$ and $q_0 \geq 1$ is any number such that

$$ \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q_0}. $$

Moreover,

$$ \|(f * \varphi)\|_{p_0, q_0} \leq 3p_0 \| f \|_{p_1, q_1} \| \varphi \|_{p_2, q_2}. $$
An extension of the Young inequality to the convolution

\[ f *_G g(x) = \int_G f(xy)g(y^{-1})d\mu(y), \]

where \( \mu \) is the Haar measure on local compact group \( G \), was given in [6] (see Theorem 20.18 in [6]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup \((K, *_K)\) consists of a locally compact Hausdorff space \( K \) together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on \( K \) with the following properties:

1. For all \( x, y \in K \), the convolution of the point measures \( \delta_x *_K \delta_y \) is a probability measure with compact support.

2. The mapping: \( K \times K \to C(K) \), \((x, y) \mapsto \text{supp}(\delta_x *_K \delta_y)\) is continuous with respect to the Michael topology on the space \( C(K) \) of all nonvoid compact subsets of \( K \), where this topology is generated by the sets

\[ U_{V,W} = \{ L \in C(K) : L \cap V \neq \emptyset, L \subset W \} \]

with \( V, W \) open in \( K \).

3. There is an identity \( e \in K \) with \( \delta_e *_K \delta_x = \delta_x *_K \delta_e = \delta_x \) for all \( x \in K \).

4. There is a continuous involution \( \sim \) on \( K \) such that

\[ (\delta_x *_K \delta_y)^\sim = \delta_y^\sim *_K \delta_x^\sim \]

and \( e \in \text{supp}(\delta_x *_K \delta_y) \iff x = y^\sim \) for \( x, y \in K \) (see [7], [8], [2]).

A hypergroup \( K \) is called commutative if \( \delta_x *_K \delta_y = \delta_y *_K \delta_x \) for all \( x, y \in K \). It is well known that every commutative hypergroup \( K \) possesses a Haar measure which will be denoted by \( \lambda \) (see [8]). That is, for every Borel measurable function \( f \) on \( K \),

\[ \int_K f(\delta_x *_K \delta_y)d\lambda(y) = \int_K f(y)d\lambda(y) \ (x \in K). \]

Define the generalized translation operators \( T^x \), \( x \in K \), by

\[ T^x f(y) = \int_K f(\delta_x *_K \delta_y)d\lambda(y) \]

for all \( y \in K \). If \( K \) is a commutative hypergroup, then \( T^x f(y) = T^y f(x) \) and the convolution of two functions is defined by

\[ f *_K \varphi(x) = \int_K T^x f(y)\varphi(y^\sim)d\lambda(y). \]
Note that $f *_K \varphi = \varphi *_K f$.

For $1 \leq p \leq \infty$, the Lebesgue space $L^p (K, \lambda)$ is defined as

$$L^p (K, \lambda) = \{ f : f \text{ is } \lambda\text{-measurable on } K, \| f \|_{K,p} < \infty \}$$

where $\| f \|_{K,p}$ is defined by

$$\| f \|_{K,p} = \begin{cases} \left( \int_K |f(x)|^p d\lambda(x) \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{x \in K} f(x), & \text{if } p = \infty. \end{cases}$$

Let $1 \leq p \leq \infty$. If $f$ is in $L^p (K, \lambda)$ and $\varphi$ is in $L^1 (K, \lambda)$, then the function $f *_K \varphi$ belongs to $L^p (K, \lambda)$ and

$$\| f *_K \varphi \|_{K,p} \leq \| f \|_{K,p} \| \varphi \|_{K,1}$$

Let $f$ be a $\lambda$-measurable function defined on the hypergroup $K$. The distribution function $\lambda_f$ of the function $f$ is given by

$$\lambda_f (s) = \lambda \{ x : x \in K, |f(x)| > s \}, \text{ for } s \geq 0.$$  

The distribution function $\lambda_f$ is non-negative, non-increasing and continuous from the right. With the distribution function we associate the non-increasing rearrangement of $f$ on $[0, \infty)$ defined by

$$f^{*_K} (t) = \inf \{ s > 0 : \lambda_f (s) \leq t \}.$$  

Some elementary properties of $\lambda_f$ and $f^{*_K}$ are listed below. The proofs of them can be found in [1].

1. If $\lambda_f$ is continuous and strictly decreasing, then $f^{*_K}$ is the inverse of $\lambda_f$, that is $f^{*_K} = (\lambda_f)^{-1}$.

2. $f^{*_K}$ is continuous from the right.

3. $$m_{f^{*_K}} (s) = \lambda_f (s), \text{ for all } s > 0,$$

   where $m_{f^{*_K}}$ is a distribution function of the function $f^{*_K}$ with respect to Lebesgue measure $m$ on $(0, \infty)$.

4. $$\int_0^t f^{*_K} (s) ds = tf^{*_K} (t) + \int_{f^{*_K} (t)}^\infty \lambda_f (s) ds \quad (1)$$

5. If $f \in L^p (K, \lambda)$, $1 \leq p < \infty$, then

$$\left( \int_K |f(x)|^p d\lambda(x) \right)^{\frac{1}{p}} = \left( p \int_0^\infty s^{p-1} \lambda_f (s) ds \right)^{\frac{1}{p}} = \left( \int_0^\infty (f^{*_K} (t))^p dt \right)^{\frac{1}{p}}.$$
Furthermore, in the case $p = \infty$,

$$\text{ess sup}_{x \in K} f(x) = \inf \{s : \lambda_f(s) = 0\} = f^{*K}(0)$$

$f^{**K}$ will denote the maximal function of $f^{*K}$ defined by

$$f^{**K}(t) = \frac{1}{t} \int_0^t f^{*K}(u)du, \text{ for } t > 0.$$  

Note the following properties of $f^{**K}$:

(1’) $f^{**K}$ is nonnegative, non-increasing and continuous on $(0, \infty)$ and $f^{*K} \leq f^{**K}$.

(2’) $(f + g)^{**K} \leq f^{**K} + g^{**K}$

(3’) If $|f_n| \uparrow |f|$ $\lambda$-a.e., then $f^{**K}_n \uparrow f^{**K}$.

For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L^{p,q}(K, \lambda)$ is defined as

$$L^{p,q}(K, \lambda) = \{f : f$ is $\lambda$-measurable on $K, \|f\|_{K,p,q} < \infty\}$$

where $\|f\|_{K,p,q}$ is defined by

$$\|f\|_{K,p,q} = \begin{cases} \left( \int_0^\infty \left( \int_0^1 \frac{1}{t} f^{**K}(t) \frac{dt}{t} \right)^\frac{q}{p} \right)^\frac{1}{q}, & 1 \leq p < \infty, 1 \leq q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**K}(t), & 1 \leq p \leq \infty, q = \infty. \end{cases}$$

Note that if for $1 < p \leq \infty$ then $L^{p,p}(K, \lambda) = L^{p}(K, \lambda)$. Moreover,

$$\|f\|_{K,p} \leq \|f\|_{K,p,p} \leq p' \|f\|_{K,p}, \quad (2)$$

where $p' = \begin{cases} \frac{p}{p-1}, & 1 < p < \infty, \\ 1, & p = \infty. \end{cases}$

For $p > 1$, the space $L^{p,\infty}(K, \lambda)$ is known as the Marcinkiewicz space or as Weak $L^{p}(K, \lambda)$. Also note that $L^{1,\infty}(K, \lambda) = L^{1}(K, \lambda)$.

If $1 < p < \infty$ and $1 < q < r < \infty$, then

$$L^{p,q}(K, \lambda) \subset L^{p,r}(K, \lambda).$$

Moreover

$$\|f\|_{K,p,r} \leq \left( \frac{q}{p} \right)^{\frac{1}{r}-\frac{1}{p}} \|f\|_{K,p,q} \quad (3)$$

The Young inequality on Lebesgue spaces for compact commutative hypergroups was given in [9]. The generalized Young inequality on the Lorentz spaces for Bessel and Dunkl convolution operators were introduced in [3] and [5] correspondingly.

In this paper we establish the generalized Young inequality on the Lorentz spaces for commutative hypergroups and give an application of it to the theory of fractional integrals. The boundedness on the Lorentz spaces of the fractional integrals on the commutative hypergroups is proved. We also prove the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups.
2 Lemmas

Lemma 2.1 Let $f$ and $\varphi$ be $\lambda$-measurable functions on the hypergroup $K$ where $\sup_{x \in K} |f(x)| \leq \beta$ and $f$ vanishes outside of a measurable set $E$ with $\lambda(E) = r$. Then, for $t > 0$,

$$(f * K \varphi)^* K(t) \leq \beta r \varphi^* K(r)$$

and

$$(f * K \varphi)^* K(t) \leq \beta r \varphi^* K(t).$$

Proof. Without loss of generality we can assume that the functions $f$ and $\varphi$ are nonnegative. Let $h = f * K \varphi$. For $a > 0$, define

$$\varphi_a(x) = \begin{cases} \varphi(x), & \text{if } \varphi(x) \leq a \\ a, & \text{if } \varphi(x) > a, \end{cases}$$

$$\varphi^a(x) = \varphi(x) - \varphi_a(x)$$

Also define functions $h_1$ and $h_2$ by

$$h = f * K \varphi_a + f * K \varphi^a = h_1 + h_2.$$ 

Then we have the following three estimates.

$$\sup_{x \in K} h_2(x) \leq \sup_{x \in K} f(x) \|\varphi^a\|_{K,1} \leq \beta \int_0^\infty \lambda_{\varphi^a}(s) ds = \beta \int_a^\infty \lambda_\varphi(s) ds,$$ \hfill (6)

$$\sup_{x \in K} h_1(x) \leq \|f\|_{K,1} \sup_{x \in K} \varphi_a(x) \leq \beta ra,$$ \hfill (7)

and

$$\sup_{x \in K} h_2(x) \leq \|f\|_{K,1} \|\varphi^a\|_{K,1} \leq \beta r \int_a^\infty \lambda_\varphi(s) ds.$$ \hfill (8)

Now set $a = \varphi^*_K(r)$ in (6) and (7) and obtain

$$h^{* K}(t) = \frac{1}{t} \int_0^t h^{* K}(s) ds \leq \|h\|_{K,\infty} \leq \|h_1\|_{K,\infty} + \|h_2\|_{K,\infty}$$

$$\leq \beta r \varphi^*_K(r) + \beta \int_{\varphi^*_K(r)}^\infty \lambda_\varphi(s) ds,$$

and using (1) we have the inequality (4).

Let us prove the inequality (5). For this purpose set $a = \varphi^*_K(t)$ and use (7) and (8). Then

$$th^{* K}(t) = \int_0^t h^{* K}(s) ds \leq \int_0^t h_1^{* K}(s) ds + \int_0^t h_2^{* K}(s) ds$$
\[ \leq t\|h_1\|_{K,\infty} + \int_0^t h_2^*(s) ds = t\|h_1\|_{K,\infty} + t\|h_2\|_{K,1} \]

\[ \leq t\beta r\phi^*(t) + \beta rt \int_{\phi^*(t)}^\infty \lambda\phi(s) ds \]

\[ = \beta rt \left( \phi^*(t) + \int_{\phi^*(t)}^\infty \lambda\phi(s) ds \right) = \beta rt\phi^{**}(t). \]

\[ \square \]

**Lemma 2.2** Let \( f \) and \( \phi \) be \( \lambda \)-measurable functions on hypergroup \( K \), then for all \( t > 0 \) the following inequality holds:

\[ (f *_{K} \phi)^{**}(t) \leq tf^{**}(t)\phi^{**}(t) + \int_t^\infty f^*(s)\phi^*(s) ds \quad (9) \]

**Proof.** Without loss of generality we can assume that the functions \( f \) and \( \phi \) are nonnegative. Let \( h = f *_{K} \phi \) and fix \( t > 0 \). Select a nondecreasing sequence \( \{s_n\}_{-\infty}^{+\infty} \) such that \( s_0 = f^*_K(t) \), \( \lim_{n \to +\infty} s_n = +\infty \), \( \lim_{n \to -\infty} s_n = 0 \).

Also let

\[ f(x) = \sum_{n=-\infty}^{+\infty} f_n(x) \]

where

\[ f_n(x) = \begin{cases} 
0, & \text{if } f(x) \leq s_{n-1} \\
(f(x) - s_{n-1}) & \text{if } s_{n-1} < f(x) \leq s_n \\
s_n - s_{n-1} & \text{if } s_n < f(x). 
\end{cases} \]

Since the series \( \{s_n\}_{-\infty}^{+\infty} \) converges absolutely we have

\[ h = \int_K T^x\phi(y) \left( \sum_{n=-\infty}^{+\infty} f_n(y^\sim) \right) d\lambda(y) \]

\[ = \sum_{n=-\infty}^{+\infty} \int_K T^x\phi(y)f_n(y^\sim) d\lambda(y) = \sum_{n=-\infty}^{+\infty} (f_n *_{K} \phi) \]

Define functions \( h_1 \) and \( h_2 \) by

\[ h = \sum_{n=1}^{+\infty} (f_n *_{K} \phi) + \sum_{n=-\infty}^{0} (f_n *_{K} \phi) = h_1 + h_2. \]
Estimate $h^{**}_1(t)$. For this purpose use the inequality (5) with $E = \{x : f(x) > s_{n-1}\}$ and $\beta = s_n - s_{n-1}$. We have

$$h^{**}_1(t) \leq \sum_{n=1}^{+\infty} ((f_n * K \varphi)^{**K})$$

$$\leq \sum_{n=1}^{+\infty} (s_n - s_{n-1}) \lambda_f(s_{n-1}) \varphi^{**K}(t)$$

$$= \varphi^{**K}(t) \sum_{n=1}^{+\infty} \lambda_f(s_{n-1})(s_n - s_{n-1}).$$

Hence

$$h^{**}_1(t) \leq \varphi^{**K}(t) \int_{f^*_K(t)}^{+\infty} \lambda_f(s) ds. \tag{10}$$

To estimate $h^{**}_2(t)$ we use the inequality (4)

$$h^{**}_2(t) \leq \sum_{n=-\infty}^{0} ((f_n * K \varphi)^{**K})$$

$$\leq \sum_{n=1}^{+\infty} (s_n - s_{n-1}) \lambda_f(s_{n-1}) \varphi^{**K}(\lambda_f(s_{n-1}))$$

$$= \sum_{n=1}^{+\infty} \lambda_f(s_{n-1}) \varphi^{**K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).$$

This implies that

$$h^{**}_2(t) \leq \int_{0}^{f^*_K(t)} \lambda_f(s) \varphi^{**K}(\lambda_f(s)) ds. \tag{11}$$

We will estimate the integral on the right-hand side of (11) by making the substitution $s = f^*_K(\xi)$ and then integrating by parts. In order to justify the change of variable in the integral, consider a Riemann sum

$$\sum_{n=1}^{+\infty} \lambda_f(s_{n-1}) \varphi^{**K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).$$

that provides a close approximation to

$$\int_{0}^{f^*_K(t)} \lambda_f(s) \varphi^{**K}(\lambda_f(s)) ds.$$

By adding more points to the Riemann sum if necessary, we may assume that the left-hand end point of each interval on which $\lambda_f$ is constant is included among the $s_n$. Then the Riemann sum is not changed if each $s_n$ that is contained in the interior of an interval on which $\lambda_f$ is constant,
is deleted. It is now an easy matter to verify that for each of the remaining $s_n$ there is precisely one element, $\xi_n$, such that $s_n = f^*K(\xi_n)$ and that $\lambda(f^*K(\xi_n)) = \xi_n$. Therefore

$$
\sum_{n=1}^{+\infty} \lambda_f(s_{n-1})\varphi^{**K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).
$$

$$
= \sum_{n=1}^{+\infty} \xi_{n-1}\varphi^{**K}(\xi_{n-1})(f^*K(\xi_n) - f^*K(\xi_{n-1}))
$$

which, by adding more points if necessary, provides a close approximation to

$$
- \int_{t}^{\infty} \xi \varphi^{**K}(\xi)df^*K(\xi).
$$

If we recall (11) we get

$$
h_{2}^{**K}(t) \leq \int_{0}^{f^*K(t)} \lambda_f(s)\varphi^{**K}(\lambda_f(s))ds = - \int_{t}^{\infty} \xi \varphi^{**K}(\xi)df^*K(\xi).
$$

(12)

Now let $\delta$ be an arbitrarily large number and choose $\xi_j$ such that $t = \xi_1 \leq \xi_2 \leq \ldots \leq \xi_{j+1} = \delta$. Then

$$
\delta\varphi^{**K}(\delta) f^*K(\delta) - t\varphi^{**K}(t) f^*K(t)
$$

$$
= \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1}) (f^*K(\xi_{n+1}) - f^*K(\xi_n))
$$

$$
+ \sum_{n=1}^{j} f^*K(\xi_n) (\varphi^{**K}(\xi_{n+1})\xi_{n+1} - \varphi^{**K}(\xi_n)\xi_n)
$$

$$
= \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1}) (f^*K(\xi_{n+1}) - f^*K(\xi_n))
$$

$$
+ \sum_{n=1}^{j} f^*K(\xi_n) \int_{\xi_n}^{\xi_{n+1}} \varphi^{*K} d\tau
$$

$$
\leq \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1}) (f^*K(\xi_{n+1}) - f^*K(\xi_n))
$$

$$
+ \sum_{n=1}^{j} f^*K(\xi_n) \varphi^{*K}(\xi_n) (\xi_{n+1} - \xi_n).
$$

This means that

$$
\delta\varphi^{**K}(\delta) f^*K(\delta) - t\varphi^{**K}(t) f^*K(t) \leq \int_{t}^{\delta} \xi \varphi^{**K}(\xi)df^*K(\xi) + \int_{t}^{\delta} f^*K(\xi) \varphi^{*K}(\xi)d\xi.
$$

(13)
Now we estimate the expression \( \delta \varphi^{**K}(\delta) f^{*K}(\delta) - t \varphi^{**K}(t) f^{*K}(t) \) below.

\[
\delta \varphi^{**K}(\delta) f^{*K}(\delta) - t \varphi^{**K}(t) f^{*K}(t) = \sum_{n=1}^{j} \xi_n \varphi^{**K}(\xi_n) (f^{*K}(\xi_n + 1) - f^{*K}(\xi_n))
\]

\[
+ \sum_{n=1}^{j} f^{*K}(\xi_n + 1) (\varphi^{**K}(\xi_n + 1) \xi_n + 1 - \varphi^{**K}(\xi_n) \xi_n)
\]

\[
= \sum_{n=1}^{j} \xi_n \varphi^{**K}(\xi_n) (f^{*K}(\xi_n + 1) - f^{*K}(\xi_n))
\]

\[
+ \sum_{n=1}^{j} f^{*K}(\xi_n + 1) \int_{\xi_n}^{\xi_n + 1} \varphi^{*K}(\tau) d\tau
\]

\[
\geq \sum_{n=1}^{j} \xi_n \varphi^{**K}(\xi_n) (f^{*K}(\xi_n + 1) - f^{*K}(\xi_n))
\]

\[
+ \sum_{n=1}^{j} f^{*K}(\xi_n + 1) \varphi^{*K}(\xi_n + 1) (\xi_n + 1 - \xi_n).
\]

In other words

\[
\delta \varphi^{**K}(\delta) f^{*K}(\delta) - t \varphi^{**K}(t) f^{*K}(t) \geq \int_{t}^{\delta} \xi \varphi^{**K}(\xi) d f^{*K}(\xi) + \int_{t}^{\delta} f^{*K}(\xi) \varphi^{*K}(\xi) d\xi. \quad (14)
\]

From (13) and (14) we obtain

\[
- \int_{t}^{\delta} \xi \varphi^{**K}(\xi) d f^{*K}(\xi) = t \varphi^{**K}(t) f^{*K}(t) - \delta \varphi^{**K}(\delta) f^{*K}(\delta) + \int_{t}^{\delta} f^{*K}(\xi) \varphi^{*K}(\xi) d\xi.
\]

\[
\leq t \varphi^{**K}(t) f^{*K}(t) + \int_{t}^{\delta} f^{*K}(\xi) \varphi^{*K}(\xi) d\xi.
\]

Thus

\[
- \int_{t}^{\infty} \xi \varphi^{**K}(\xi) d f^{*K}(\xi) \leq t \varphi^{**K}(t) f^{*K}(t) + \int_{t}^{\infty} f^{*K}(\xi) \varphi^{*K}(\xi) d\xi.
\]

By using this inequality and (12) we have

\[
h^{**K}_{2}(t) \leq \int_{0}^{f^{*K}(t)} \lambda f(s) \varphi^{**K}(\lambda f(s)) ds \leq t \varphi^{**K}(t) f^{*K}(t) + \int_{t}^{\infty} f^{*K}(\xi) \varphi^{*K}(\xi) d\xi. \quad (15)
\]
Finally, from (10), (15) and (1) we get
\[
h^{**K}(t) \leq h_1^{**K}(t) + h_2^{**K}(t)
\]
\[
\leq \varphi^{**K}(t) \int_{f^{**K}(t)}^{\infty} \lambda_f(s)ds + t\varphi^{**K}(t)f^{**K}(t) + \int_t^{\infty} f^{*K}(\xi)\varphi^{*K}(\xi)d\xi
\]
\[
= f^{*K}(t)\varphi^{**K}(t) + \int_t^{\infty} f^{*K}(\xi)\varphi^{*K}(\xi)d\xi
\]
\[
= tf^{**K}(t)\varphi^{**K}(t) + \int_t^{\infty} f^{*K}(\xi)\varphi^{*K}(\xi)d\xi.
\]
\[\square\]

**Lemma 2.3** Let \( f \) and \( \varphi \) be \( \lambda \)-measurable functions on hypergroup \( K \), then for all \( t > 0 \) the following inequality holds:
\[
(f *_K \varphi)^{**K}(t) \leq \int_t^{\infty} f^{**K}(s)\varphi^{**K}(s)ds
\]

*Proof.* Assume that the integral on the right of (16) is finite. Then it is easy to see
\[
s f^{**K}(s)\varphi^{**K}(s) \to 0, \text{ as } s \to \infty.
\]

Let \( h = f *_K \varphi \). By Lemma 2.2 we have

\[
h^{**K}(t) \leq tf^{**K}(t)\varphi^{**K}(t) + \int_t^{\infty} f^{*K}(s)\varphi^{*K}(s)ds
\]
\[
\leq tf^{**K}(t)\varphi^{**K}(t) + \int_t^{\infty} f^{*K}(s)\varphi^{*K}(s)ds.
\]

Since \( f^{*K} \) and \( g^{**K} \) are non-increasing,
\[
\frac{df^{**K}(s)}{ds} = -\frac{1}{s^2} \int_0^s f^{*K}(\tau)d\tau + \frac{1}{s} f^{*K}(s) = \frac{1}{s} (f^{*K}(s) - f^{**K}(s)),
\]
\[
\frac{d(s\varphi^{**K}(s))}{ds} = \varphi^{**K}(s) + s \left( \frac{1}{s} (\varphi^{*K}(s) - \varphi^{**K}(s)) \right) = \varphi^{*K}(s)
\]
for \( m \)-almost all \( s \). Since \( f^{**K} \) and \( g^{**K} \) are absolutely continuous, we may use the integration by parts for \( \int_t^{\infty} f^{*K}(s)d(s\varphi^{**K}(s)) \). Using (19), (20) and (17) we obtain
\[
\int_t^{\infty} f^{*K}(s)\varphi^{*K}(s)ds = \int_t^{\infty} f^{*K}(s)d(s\varphi^{**K}(s))
\]
\[ f^{**}(s) \varphi^{**}(s) - \int_{t}^{\infty} s \varphi^{**}(s) df^{**}(s) = -tf^{**}(t) \varphi^{**}(t) + \int_{t}^{\infty} \varphi^{**}(s)(f^{**}(s) - f^{*}(s)) ds \]
\[ \leq -tf^{**}(t) \varphi^{**}(t) + \int_{t}^{\infty} \varphi^{**}(s) f^{**}(s) ds \] (21)

By (18) and (21) we have
\[ h^{**}(t) \leq \int_{t}^{\infty} f^{**}(s) \varphi^{**}(s) ds. \]

The next lemma is a classical estimate, known as Hardy’s inequality.

**Lemma 2.4** ([1], [10]) If \( 1 \leq p < \infty \), \( q > 0 \) and \( f \) be a nonnegative \( m \)-measurable function on \( (0, \infty) \), then
\[ \int_{0}^{\infty} \left( \frac{1}{s} \int_{0}^{s} f(\tau) d\tau \right)^{p} s^{p-q-1} ds \leq \left( \frac{p}{q} \right) q \int_{0}^{\infty} f(t)^{p} t^{p-q-1} dt. \] (22)

### 3 Generalization of Young’s inequality

**Theorem 3.1** If \( f \in L_{p_1,q_1}(K, \lambda) \), \( \varphi \in L_{p_2,q_2}(K, \lambda) \) and \( \frac{1}{p_1} + \frac{1}{p_2} > 1 \), then \( (f * K \varphi) \in L_{p_0,q_0}(K, \lambda) \) where \( \frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_0} \) and \( q_0 \geq 1 \) is any number such that \( \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q_0} \). Moreover,
\[ \|(f * K \varphi)\|_{K, p_0, q_0} \leq 3p_0 \|f\|_{K, p_1, q_1} \|\varphi\|_{K, p_2, q_2}. \] (23)

**Proof.** Let \( h = f * K \varphi \).

Suppose that \( q_1, q_2, q_0 \) are all different from \( \infty \). Then, by (16), we have
\[
\left( \left\| h \right\|_{K, p_0, q_0} \right)^{q_0} = \int_{0}^{\infty} \left( \frac{1}{s^{q_0}} h^{**}(s) \right)^{q} ds \frac{1}{s} \leq \int_{0}^{\infty} \left( \frac{1}{s^{p_0}} \int_{s}^{\infty} f^{**}(\tau) \varphi^{**}(\tau) d\tau \right)^{q} ds \frac{1}{s} = \int_{0}^{\infty} \left( \frac{1}{t^{p_0}} \int_{0}^{t} f^{**}(\eta) \varphi^{**}(\frac{1}{\eta}) \frac{d\eta}{\eta^2} \right)^{q} dt. \]
The last equality was obtained by the change of variables \( s = \frac{1}{t} \) and \( \tau = \frac{1}{\eta} \). Using (22) we get

\[
\int_0^\infty \left( \frac{1}{t^{\frac{1}{p_0}}} \int_0^t f^{**K} \left( \frac{1}{\eta} \right) \varphi^{**K} \left( \frac{1}{\eta} \right) \frac{d\eta}{\eta^2} \right)^q \frac{dt}{t} \leq p_0^{q_0} \int_0^\infty \left( t^{1-\frac{1}{p_0}} \frac{f^{**K} \left( \frac{1}{t} \right) \varphi^{**K} \left( \frac{1}{t} \right)}{t^2} \right)^{q_0} \frac{dt}{t} \leq p_0^{q_0} \int_0^\infty \left( s^{1+\frac{1}{p_0}} f^{**K} (s) \varphi^{**K} (s) \right)^{q_0} \frac{ds}{s}
\]

The last equality was obtained by the change of the variable \( t = \frac{1}{s} \). Since \( \frac{q_0}{q_1} + \frac{q_0}{q_2} \geq 1 \), one can find positive numbers \( n_1 \) and \( n_2 \) such that

\[
\frac{1}{n_1} + \frac{1}{n_2} = 1 \quad \text{and} \quad \frac{1}{n_1} \leq \frac{q_0}{q_1}, \quad \frac{1}{n_2} \leq \frac{q_0}{q_2}.
\]

By Hölder’s inequality we obtain

\[
\left( \left\| h \right\|_{K_{p_0,q_0}}^q \right)^{q_0} \leq p_0^{q_0} \int_0^\infty \left( \frac{1}{s^{\frac{1}{p_1}}} f^{**K} (s) \right)^{q_0} \left( \frac{1}{s^{\frac{1}{p_2}}} \varphi^{**K} (s) \right)^{q_0} \frac{ds}{s} \leq p_0^{q_0} \left[ \int_0^\infty \left( \frac{1}{s^{\frac{1}{p_1}}} f^{**K} (s) \right)^{q_0 n_1} \frac{ds}{s} \right]^{\frac{1}{n_1}} \left[ \int_0^\infty \left( \frac{1}{s^{\frac{1}{p_2}}} \varphi^{**K} (s) \right)^{q_0 n_2} \frac{ds}{s} \right]^{\frac{1}{n_2}} = p_0^{q_0} \left( \left\| f \right\|_{K_{p_1,q_0 n_1}}^{q_0} \left( \left\| \varphi \right\|_{K_{p_2,q_0 n_2}}^{q_0} \right) \right).
\]

Finally, by (3) we have

\[
\left\| h \right\|_{K_{p_0,q_0}} \leq p_0 \left\| f \right\|_{K_{p_1,q_0 n_1}} \left\| \varphi \right\|_{K_{p_2,q_0 n_2}} \leq p_0 e^{\frac{1}{p_1} e^{\frac{1}{p_2}}} \left\| f \right\|_{K_{p_1,q_1}} \left\| \varphi \right\|_{K_{p_2,q_2}} \leq 3p_0 \left\| f \right\|_{K_{p_1,q_1}} \left\| \varphi \right\|_{K_{p_2,q_2}}.
\]

Similar reasoning leads to the desired result in case one or more of \( q_1, q_2, q_0 \) are \( \infty \). \( \square \)

### 4 Applications to the theory of fractional integrals

Consider the following particular case of Theorem [3,1]. If we take \( p_1 = \frac{N}{N - \alpha} \), with \( o < \alpha < N, q_1 = \infty \) in Theorem [3,1] then the condition \( \frac{1}{p_1} + \frac{1}{p_2} > 1 \) is equivalent to \( \alpha < \frac{N}{p_2} \), and the condition \( \frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_0} \) is equivalent to \( \frac{1}{p_0} = \frac{1}{p_2} - \frac{\alpha}{N} \). Thus we have the following result.
Theorem 4.1 Let \((K, \ast_K)\) be a commutative hypergroup, with Haar measure \(\lambda\). If \(f \in L^{\frac{N}{N-\alpha}, \infty}(K, \lambda)\), \(\varphi \in L^{p,q}(K, \lambda)\), where \(0 < \alpha < \frac{N}{p}\), \(1 \leq q \leq \infty\) then \((f \ast_K \varphi) \in L^{r,q}(K, \lambda)\) and
\[
\| (f \ast_K \varphi) \|_{K, r, q} \leq 3r \| f \|_{K, \frac{N}{N-\alpha}, \infty} \| \varphi \|_{K, p, q}.
\]
where \(\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}\).

Let \(K\) be a set. A function \(\rho : K \times K \to [0, \infty)\) is called quasi-metric if:
\begin{enumerate}
  \item \(\rho(x, y) = 0 \iff x = y\);
  \item \(\rho(x, y) = \rho(y, x)\);
  \item there is a constant \(c \geq 1\) such that for every \(x, y, z \in X\)
  \[
  \rho(x, y) \leq c(\rho(x, z) + \rho(z, y)).
  \]
\end{enumerate}

Define the fractional integral (or Riesz potential)
\[
I_{\alpha}f(x) = \int_{K} T^{x} \rho(e, y)^{\alpha-N} f(y) d\lambda(y), \ 0 < \alpha < N
\]
on commutative hypergroup \((K, \ast_K)\) equipped with the pseudo-metric \(\rho\).
Also define a ball \(B(e, r) = \{y \in K : \rho(e, y) < r\}\) with a center \(e\) and a radius \(r\).

Theorem 4.2 Let \((K, \ast_K)\) be a commutative hypergroup, with quasi-metric \(\rho\) and Haar measure \(\lambda\) satisfying \(\lambda B(e, r) = Ar^{N}\), where \(A\) is a positive constant. Assume that \(1 \leq q \leq \infty, 1 \leq p < \infty, 0 < \alpha < \frac{N}{p}\). If \(f \in L^{p,q}(K, \lambda)\), then \(I_{\alpha}f \in L^{r,q}(K, \lambda)\) and
\[
\| I_{\alpha}f \|_{K, r, q} \leq C \| f \|_{K, p, q},
\]
where \(\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}\) and \(C = \frac{3rN}{\alpha} A^{-\frac{\alpha}{N}}\).

Proof. Let us show that \(\rho(e, \cdot)^{\alpha-N} \in L^{\frac{N}{N-\alpha}, \infty}(K, \lambda)\). For the distribution of \(\rho(e, \cdot)^{\alpha-N}\) we can write
\[
\lambda_{\rho(e, \cdot)^{\alpha-N}}(t) = \lambda \{x : x \in K, \rho(e, x)^{\alpha-N} > t\}
= \lambda \{x : x \in K, \rho(e, x) < t^{-\frac{1}{\alpha+N}}\} = At^{-\frac{1}{\alpha+N}}.
\]
Since \(\rho(e, \cdot)^{\alpha-N}\) is continuous and strictly decreasing we have \((\rho(e, \cdot)^{\alpha-N})^{*K}\) is the inverse of the distribution function. That is \((\rho(e, \cdot)^{\alpha-N})^{*K}(t) = \left(\frac{t}{A}\right)^{-\frac{\alpha}{N}}\). Then
\[
(\rho(e, \cdot)^{\alpha-N})^{*K}(t) = \frac{1}{t} \int_{0}^{t} \left(\frac{s}{A}\right)^{-\frac{\alpha}{N}} ds = \frac{N}{\alpha} \left(\frac{t}{A}\right)^{-\frac{\alpha}{N}}.
\]
Therefore \(\rho(e, \cdot)^{\alpha-N} \in L^{\frac{N}{N-\alpha}, \infty}(K, \lambda)\) and
\[
\| \rho(e, \cdot)^{\alpha-N} \|_{K, \frac{N}{N-\alpha}, \infty} = \frac{N}{\alpha} A^{-\frac{\alpha}{N}}.
\]
Thus, from Theorem 4.1 and we have the required result. \(\square\)
Theorem 4.3 Let $(K, *_K)$ be a commutative hypergroup, with Haar measure $\lambda$. If $f \in L^{\frac{N}{N-\alpha}, \infty}(K, \lambda)$, $\varphi \in L^p(K, \lambda)$, where $1 < p < \infty$, $0 < \alpha < \frac{N}{p}$, then $(f *_K \varphi) \in L^r(K, \lambda)$ and

$$\| (f *_K \varphi) \|_{K,r} \leq 3r \frac{p}{p-1} \left( \frac{p}{r} \right)^{\frac{1}{p} - \frac{1}{r}} \| f \|_{K, \frac{N}{N-\alpha}, \infty} \| \varphi \|_{K,p}. \tag{27}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}$.

Proof. From (2), (3) and (23) we have

$$\| (f *_K \varphi) \|_{K,r} \leq \| (f *_K \varphi) \|_{K,r,r} \leq \left( \frac{p}{r} \right)^{\frac{1}{p} - \frac{1}{r}} \| (f *_K \varphi) \|_{K,r,p} \leq 3r \frac{p}{p-1} \left( \frac{p}{r} \right)^{\frac{1}{p} - \frac{1}{r}} \| f \|_{K, \frac{N}{N-\alpha}, \infty} \| \varphi \|_{K,p}. \tag{28}$$

The following result give us the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups.

Theorem 4.4 Let $(K, *_K)$ be a commutative hypergroup, with quasi-metric $\rho$ and Haar measure $\lambda$ satisfying $\lambda B(e, r) = Ar^N$, where $A$ is a positive constant. Assume that $1 < p < \infty$, $0 < \alpha < \frac{N}{p}$. If $f \in L^p(K, \lambda)$, then $I_\alpha f \in L^r(K, \lambda)$ and

$$\| I_\alpha f \|_{K,r} \leq C \| f \|_{K,p}, \tag{29}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}$ and $C = \frac{3pr}{p-1} \left( \frac{p}{r} \right)^{\frac{1}{p} - \frac{1}{r}} \frac{N}{\alpha} A^{\frac{N-\alpha}{N}}$.

Proof. This follows immediately from Theorem 4.3 and (4).

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References

[1] C. Bennett and R. Sharpley, Interpolation of operators, *Pure and Applied Math.*, vol.129, Academic Press, Orlando, Florida, 1988

[2] W. R. Bloom and H. Heyer, Harmonic analysis of probability measures on hypergroups, *de Gruyter Stud. Math.*, vol. 20, Walter de Gruyter & Co., Berlin, 1995.

[3] A. D. Gadjiev, M. G. Hajibayov, Inequalities for $B$-convolution operators, *TWMS J. Pure Appl. Math.*, 1(1) (2010), 41-52.

[4] G. Gigante, Transference for hypergroups, *Collect. Math.*, 52(2) 2001, 127-155.
[5] M.G.Hajibayov, Boundedness of the Dunkl convolution operators, *An. Univ. Vest Timiş.*, *Ser. Mat.-Inform.* **49 (1)** (2011), 49-67.

[6] E. Hewitt and K. A. Ross, Abstract harmonic analysis , vol I, *Springer-Verlag*, (1979).

[7] R. L. Jewett, Spaces with an abstract convolution of measures. *Adv. in Math.*, **18(1)** (1975), 1-101.

[8] R. Spector, Measures invariantes sur les hypergroupes(French), *Trans. Amer. Math. Soc.*, **239** (1978), 147-165.

[9] R. C. Vrem, $L^p$-improving measures on hypergroups, Probability measures on groups, IX (Oberwolfach, 1988), 389-397, Lecture Notes in Math., 1379, Springer, Berlin, 1989

[10] W. P. Ziemer, Weakly differentiable functions, *Graduate Texts in Mathematics*, **120**, *Springer-Verlag, New York*, 1989

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