Two Enumerative Results on Cycles of Permutations

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Abstract

Answering a question of Bóna, it is shown that for $n \geq 2$ the probability that 1 and 2 are in the same cycle of a product of two $n$-cycles on the set \{1, 2, \ldots, n\} is $1/2$ if $n$ is odd and $\frac{1}{2} - \frac{2}{(n-1)(n+2)}$ if $n$ is even. Another result concerns the polynomial $P_\lambda(q) = \sum_w q^{\kappa((1,2,\ldots,n) \cdot w)}$, where $w$ ranges over all permutations in the symmetric group $S_n$ of cycle type $\lambda$, $(1,2,\ldots,n)$ denotes the $n$-cycle $1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1$, and $\kappa(v)$ denotes the number of cycles of the permutation $v$. A formula is obtained for $P_\lambda(q)$ from which it is deduced that all zeros of $P_\lambda(q)$ have real part 0.

1 Introduction.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $n$, denoted $\lambda \vdash n$. In general, we use notation and terminology involving partitions and symmetric functions from [12, Ch. 7]. Let $S_n$ denote the symmetric group of all permutations of $[n] = \{1, 2, \ldots, n\}$. If $w \in S_n$ then write $\rho(w) = \lambda$ if $w$ has cycle type $\lambda$, i.e., if the (nonzero) $\lambda_i$’s are the lengths of the cycles of $w$. The conjugacy classes of $S_n$ are given by $K_\lambda = \{w \in S_n : \rho(w) = \lambda\}$.

The “class multiplication problem” for $S_n$ may be stated as follows. Given $\lambda, \mu, \nu \vdash n$, how many pairs $(u, v) \in S_n \times S_n$ satisfy $u \in K_\lambda$, $v \in K_\mu$,

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uv ∈ Kν? The case when one of the partitions is (n) (i.e., one of the classes consists of the n-cycles) is particularly interesting and has received much attention. For a sample of some recent work, see [1][6][9]. In this paper we make two contributions to this subject. For the first, we solve a problem of Bóna and Flynn [4] asking what is the probability that two fixed elements of \([n]\) lie in the same cycle of the product of two random n-cycles. In particular, we prove the conjecture of Bóna that this probability is 1/2 when \(n\) is odd. Our method of proof is an ugly computation based on a formula of Boccara [2]. The technique can be generalized, and as an example we compute the probability that three fixed elements of \([n]\) lie in the same cycle of the product of two random n-cycles.

For our second result, let \(κ(w)\) denote the number of cycles of \(w ∈ S_n\), and let \((1, 2, \ldots, n)\) denote the n-cycle \(1 → 2 → \cdots → n → 1\). For \(λ \vdash n\), define the polynomial

\[
P_λ(q) = \sum_{\rho(w) = λ} q^{κ((1, 2, \ldots, n)·w)}. \tag{1}
\]

In Theorem 3.1 we obtain a formula for \(P_λ(q)\). We also prove from this formula (Corollary 3.3) that every zero of \(P_λ(q)\) has real part 0.

2 A problem of Bóna.

Let \(π_n\) denote the probability that if two n-cycles \(u, v\) are chosen uniformly at random in \(S_n\), then 1 and 2 (or any two elements \(i\) and \(j\) by symmetry) appear in the same cycle of the product \(uv\). Miklós Bóna conjectured (private communication) that \(π_n = 1/2\) if \(n\) is odd, and asked about the value when \(n\) is even. For the reason behind this conjecture, see Bóna and Flynn [4]. In this section we solve this problem. Let us note that it is easy to see (a straightforward generalization of [3 Prop. 6.18]) that the probability that \(1, 2, \ldots, k\) appear in the same cycle of a random permutation in \(S_n\) is \(1/k\) for \(k ≥ n\).

Theorem 2.1. For \(n ≥ 2\) we have

\[
π_n = \begin{cases} 
\frac{1}{2}, & n \text{ odd} \\
\frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even}.
\end{cases}
\]
Proof. First note that if \( w \in \mathfrak{S}_n \) has cycle type \( \lambda \), then the probability that 1 and 2 are in the same cycle of \( w \) is

\[
q_\lambda = \frac{\sum \binom{\lambda_i}{2}}{\binom{n}{2}} = \frac{\sum \lambda_i(\lambda_i - 1)}{n(n - 1)}.
\]

Let \( a_\lambda \) be the number of pairs \((u, v)\) of \( n\)-cycles in \( \mathfrak{S}_n \) for which \( uv \) has type \( \lambda \). Then

\[
\pi_n = \frac{1}{(n - 1)!^2} \sum_{\lambda \vdash n} a_\lambda q_\lambda.
\]

By Boccara [2] the number of ways to write a fixed permutation \( w \in \mathfrak{S}_n \) of type \( \lambda \) as a product of two \( n\)-cycles is

\[
(n - 1)! \int_0^1 \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \, dx.
\]

Let \( n!/z_\lambda \) denote the number of permutations \( w \in \mathfrak{S}_n \) of type \( \lambda \). We get

\[
\pi_n = \frac{1}{(n - 1)!^2} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \left( \sum \frac{\lambda_i(\lambda_i - 1)}{n(n - 1)} \right) \cdot (n - 1)! \int_0^1 \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \, dx
\]

\[
= \frac{1}{n - 1} \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \left( \sum \lambda_i(\lambda_i - 1) \right) \int_0^1 \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \, dx.
\]

Now let \( p_\lambda(a, b) \) denote the power sum symmetric function \( p_\lambda \) in the two variables \( a, b \), and let \( \ell(\lambda) \) denote the length (number of parts) of \( \lambda \). It is easy to check that

\[
2^{-\ell(\lambda)} \left( \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) p_\lambda(a, b)|_{a=b=1} = \sum \lambda_i(\lambda_i - 1).
\]

By the exponential formula (permutation version) [12, Cor. 5.1.9] or by [12, Prop. 7.7.4],

\[
\sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{1}{z_\lambda} 2^{-\ell(\lambda)} p_\lambda(a, b) \left( \prod_i (x^{\lambda_i} - (x - 1)^{\lambda_i}) \right) t^n
\]

3
\[
\exp \sum_{k \geq 1} \frac{1}{k} \left( \frac{a^k + b^k}{2} \right) (x^k - (x-1)^k)t^k.
\]

It follows that \((n-1)\pi_n\) is the coefficient of \(t^n\) in

\[
F(t) := \int_0^1 \left( \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b} \right) \exp \left[ \sum_{k \geq 1} \frac{1}{k} \left( \frac{a^k + b^k}{2} \right) (x^k - (x-1)^k)t^k \right] \bigg|_{a=b=1} dt.
\]

We can easily perform this computation with Maple, giving

\[
F(t) = \int_0^1 \frac{t^2(1 - 2x - 2tx^2)}{(1 - t(x-1))(1 - tx)^3} dx
= \frac{1}{t^2} \log(1 - t^2) + \frac{3}{2} + \frac{-\frac{1}{2} + t}{(1 - t^2)^2}.
\]

Extract the coefficient of \(t^n\) and divide by \(n-1\) to obtain \(\pi_n\) as claimed. \(\square\)

It is clear that the argument used to prove Theorem 2 can be generalized. For instance, using the fact that

\[
3^{-\ell(\lambda)-1} \left( \frac{\partial^3}{\partial a^3} - 3 \frac{\partial^3}{\partial a^2 \partial b} + 2 \frac{\partial^3}{\partial a \partial b \partial c} \right) p_\lambda(a, b, c)|_{a=b=c=1}
= \sum \lambda_i(\lambda_i - 1)(\lambda_i - 2),
\]

we can obtain the following result.

**Theorem 2.2.** Let \(\pi_n^{(3)}\) denote the probability that if two \(n\)-cycles \(u, v\) are chosen uniformly at random in \(S_n\), then \(1, 2,\) and \(3\) appear in the same cycle of the product \(uv\). Then for \(n \geq 3\) we have

\[
\pi_n^{(3)} = \begin{cases} 
\frac{1}{3} + \frac{1}{(n-2)(n+3)}, & n \text{ odd} \\
\frac{1}{3} - \frac{3}{(n-1)(n+2)}, & n \text{ even}.
\end{cases}
\]

Are there simpler proofs of Theorems 2.1 and 2.2, especially Theorem 2.1, when \(n\) is odd?
3  A polynomial with purely imaginary zeros

Given \( \lambda \vdash n \), let \( P_\lambda(q) \) be defined by equation (1). Let \( (a)_n \) denote the falling factorial \( a(a-1) \cdots (a-n+1) \). Let \( E \) be the backward shift operator on polynomials in \( q \), i.e., \( Ef(q) = f(q-1) \).

**Theorem 3.1.** Suppose that \( \lambda \) has length \( \ell \). Define the polynomial

\[
g_\lambda(t) = \frac{1}{1-t} \prod_{j=1}^{\ell} (1- t^\lambda_j).
\]

Then

\[
P_\lambda(q) = z^{-1}_\lambda g_\lambda(E)(q + n - 1)_n.
\]

**Proof.** Let \( x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots), \) and \( z = (z_1, z_2, \ldots) \) be three disjoint sets of variables. Let \( H_\mu \) denote the product of the hook lengths of the partition \( \mu \) (defined e.g. in [12, p. 373]). Write \( s_\lambda \) and \( p_\lambda \) for the Schur function and power sum symmetric function indexed by \( \lambda \). The following identity is the case \( k = 3 \) of [5, Prop. 2.2] and [12, Exer. 7.70]:

\[
\sum_{\mu \vdash n} H_\mu s_\mu(x)s_\mu(y)s_\mu(z) = \frac{1}{n!} \sum_{\nu \vdash n \in S_n} p_{\rho(\nu)}(x)p_{\rho(\nu)}(y)p_{\rho(\nu)}(z).
\]

For a symmetric function \( f(x) \) let \( f(1^q) = f(1, 1, \ldots, 1, 0, 0, \ldots) \) \( (q \ 1's) \). Thus \( p_{\rho(\nu)}(1^q) = q^n(\nu) \). Let \( \chi^\lambda(\mu) \) denote the irreducible character of \( S_n \) indexed by \( \lambda \) evaluated at a permutation of cycle type \( \mu \) [12, §7.18]. Recall [12 Cor. 7.17.5 and Thm. 7.18.5] that

\[
s_\mu = \sum_{\nu \vdash n} z_\nu^{-1} \chi^\mu(\nu)p_\nu,
\]

where \( \#K_\nu = n!/z_\nu \) as above. Take the coefficient of \( p_n(x)p_\lambda(y) \) in equation (3) and set \( z = 1^q \). Since there are \( (n-1)! \) \( n \)-cycles \( u \), the right-hand side becomes \( \frac{1}{n} P_\lambda(q) \). Hence

\[
P_\lambda(q) = n \sum_{\mu \vdash n} H_\mu z_\mu^{-1} \chi^\mu(n) z_\lambda^{-1} \chi^\mu(\lambda)s_\mu(1^q).
\]

Write \( \sigma(i) = (n - i, 1^i) \), the “hook” with one part equal to \( n - i \) and \( i \) parts equal to 1, for \( 0 \leq i \leq n - 1 \). Now \( z_n = n \), and e.g. by [12 Exer. 7.67(a)] we
have
\[ \chi^n(n) = \begin{cases} (-1)^i, & \text{if } \mu = \sigma(i), \ 0 \leq i \leq n - 1 \\ 0, & \text{otherwise}. \end{cases} \]

Moreover, \( s_{\sigma(i)}(1^n) = (q + n - i - 1)_n H_{\sigma(i)}^{-1} \) by the hook-content formula \cite{12}. Cor. 7.21.4. Therefore we get from equation (4) that
\[ P(\lambda)(q) = 
\begin{align*}
\lambda^{-1} \sum_{i=0}^{n-1} (-1)^i \chi^{\sigma(i)}(\lambda)(q + n - i - 1)_n.
\end{align*}
\] (5)

The following identity is a simple consequence of Pieri’s rule \cite{12} Thm. 7.15.7 and appears in \cite{7} I.3, Ex. 14:
\[ \prod_i \frac{1 + tx_i}{1 - ux_i} = 1 + (t + u) \sum_{i=0}^{n-1} s_{\sigma(i)} t^n u^{n-i-1}. \]

Substitute \(-t\) for \(t\), set \(u = 1\) and take the scalar product with \(p_\lambda\). Since \( \langle s_\mu, p_\lambda \rangle = \chi^n(\lambda) \) the right-hand side becomes \( (1 - t) \sum_{i=0}^{n-1} (-1)^i \chi^{\sigma(i)}(\lambda)t^i \). On the other hand, the left-hand side is given by
\[ \left\langle \exp \left( \sum_{n \geq 1} \frac{p_n}{n} \right) \cdot \exp \left( - \sum_{n \geq 1} \frac{p_n t^n}{n} \right), p_\lambda \right\rangle = \left\langle \exp \left( \sum_{n \geq 1} \frac{p_n}{n} (1 - t^n) \right), p_\lambda \right\rangle \]
\[ \quad \quad = \prod_{i=1}^{\ell} (1 - t^{\lambda_i}), \]
by standard properties of power sum symmetric functions \cite{12} §7.7. Hence
\[ \sum_{i=0}^{n-1} (-1)^i \chi^{\sigma(i)}(\lambda)t^i = g_\lambda(t). \]
Comparing with equation (5) completes the proof. \hfill \Box

Note. Since \((1 - E)(q + n)_{n+1} = (n + 1)(q + n - 1)_n\), equation (2) can be rewritten as
\[ P(\lambda)(q) = \frac{1}{(n + 1)z_\lambda} g'_\lambda(E)(q + n)_{n+1}, \] (6)
where \(g'_\lambda(t) = \prod_{i=1}^{\ell} (1 - t^{\lambda_i})\).

The zeros of the polynomial \(P(\lambda)(q)\) have an interesting property that will follow from the following result.
Theorem 3.2. Let $g(t)$ be a complex polynomial of degree exactly $d$, such that every zero of $g(t)$ lies on the circle $|z| = 1$. Suppose that the multiplicity of 1 as a root of $g(t)$ is $m \geq 0$. Let $P(q) = g(E)(q + n - 1)_{n-1}$.

(a) If $d \leq n - 1$, then

$$P(q) = (q + n - d - 1)_{n-d}Q(q),$$

where $Q(q)$ is a polynomial of degree $d - m$ for which every zero has real part $(d - n + 1)/2$.

(b) If $d \geq n - 1$, then $P(q)$ is a polynomial of degree $n - m$ for which every zero has real part $(d - n + 1)/2$.

Proof. First, the statements about the degrees of $Q(q)$ and $P(q)$ are clear; for we can write $g(t) = c \prod(t - u)$ and apply the factors $t - u$ consecutively. If $h(q)$ is any polynomial and $u \neq 1$ then $\deg(E - u)h(q) = \deg h(q)$, while $\deg(E - 1)h(q) = \deg h(q) - 1$.

The remainder of the proof is by induction on $d$. The base case $d = 0$ is clear. Assume the statement for $d < n - 1$. Thus for $g(E)(q + n - 1)_{n-1}$ we have

$$g(E)(q + n - 1)_{n-1} = (q + n - d - 1)_{n-d}Q(q),$$

for certain real numbers $\delta_j$. Now

$$(E - u)g(E)(q + n - 1)_{n-1}$$

$$= (q + n - d - 1)_{n-d}Q(q) - u(q + n - d - 2)_{n-d}Q(q - 1)$$

$$= (q + n - d - 2)_{n-d-1}[(q + n - d - 1)Q(q) - u(q - 1)Q(q - 1)]$$

$$= (q + n - d - 2)_{n-d-1}Q'(q),$$

say. The proof now follows from a standard argument (e.g., [3, Lemma 9.13]), which we give for the sake of completeness. Let $Q'(\alpha + \beta i) = 0$, where $\alpha, \beta \in \mathbb{R}$. Thus

$$(\alpha + \beta i + n - d - 1) \prod_j \left(\alpha + \beta i - \frac{d - n + 1}{2} - \delta_j i\right)$$

say.
\[ u(\alpha + \beta i - 1) \prod_j \left( \alpha - 1 + \beta i - \frac{d - n + 1}{2} - \delta_j \right). \]

Letting \(|u| = 1\) and taking the square modulus gives

\[
\frac{(\alpha + n - d - 1)^2 + \beta^2}{(\alpha - 1)^2 + \beta^2} \prod_j \frac{(\alpha - \frac{d - n + 1}{2})^2 + (\beta - \delta_j)^2}{(\alpha - 1 - \frac{d - n + 1}{2})^2 + (\beta - \delta_j)^2} = 1.
\]

If \(\alpha < (d - n + 2)/2\) then

\[
(\alpha + n - d - 1)^2 - (\alpha - 1)^2 < 0
\]

and

\[
\left( \alpha - \frac{d - n + 1}{2} \right)^2 < \left( \alpha - 1 - \frac{d - n + 1}{2} \right)^2.
\]

The inequalities are reversed if \(\alpha > (d - n + 2)/2\). Hence \(\alpha = (d - n + 2)/2\), so the theorem is true for \(d \leq n - 1\).

For \(d \geq n - 1\) we continue the induction, the base case now being \(d = n - 1\) which was proved above. The induction step is completely analogous to the case \(d \leq n - 1\) above, so the proof is complete. \(\square\)

**Corollary 3.3.** The polynomial \(P_\lambda(q)\) has degree \(n - \ell(\lambda) + 1\), and every zero of \(P_\lambda(q)\) has real part 0.

**Proof.** The proof is immediate from Theorem 3.1 and the special case \(g(t) = g_\lambda(t)\) (as defined in Theorem 3.1) and \(d = n - 1\) of Theorem 3.2. \(\square\)

It is easy to see from Corollary 3.3 (or from considerations of parity) that \(P_\lambda(q) = (-1)^nP_\lambda(-q)\). Thus we can write

\[
P_\lambda(q) = \begin{cases} 
R_\lambda(q^2), & \text{n even} \\
qR_\lambda(q^2), & \text{n odd}, 
\end{cases}
\]

for some polynomial \(R_\lambda(q)\). It follows from Corollary 3.3 that \(R_\lambda(q)\) has (nonpositive) real zeros. In particular (e.g., [11, Thm. 2]) the coefficients of \(R_\lambda(q)\) are log-concave with no external zeros, and hence unimodal.

The case \(\lambda = (n)\) is especially interesting. Write \(P_n(q)\) for \(P_{(n)}(q)\). From equation (6) we have

\[
P_n(q) = \frac{1}{n(n + 1)}((q + n)_{n+1} - (q)_{n+1}).
\]
Now
\[(q)_{n+1} = (-1)^{n+1}(-q + n)_{n+1}\]
and
\[(q + n)_{n+1} = \sum_{k=1}^{n+1} c(n + 1, k)q^k,
\]
where \(c(n + 1, k)\) is the signless Stirling number of the first kind (the number of permutations \(w \in S_{n+1}\) with \(k\) cycles) \([10, \text{Prop. 1.3.4}]\). Hence
\[
\frac{1}{n(n+1)}((q + n)_{n+1} - (q)_{n+1}) = \frac{1}{\binom{n+1}{2}} \sum_{k \equiv n \pmod{2}} c(n + 1, k)x^k.
\]

We therefore obtain the following result.

**Corollary 3.4.** The number of \(n\)-cycles \(w \in S_n\) for which \(w \cdot (1, 2, \ldots, n)\) has exactly \(k\) cycles is 0 if \(n-k\) is odd, and is otherwise equal to \(c(n+1, k)/(n+1)^2\).

Is there a simple bijective proof of Corollary 3.4?

Let \(\lambda, \mu \vdash n\). A natural generalization of \(P_\lambda(q)\) is the polynomial
\[
P_{\lambda, \mu}(q) = \sum_{\rho(u) = \lambda} q^{\kappa(w_{\mu} \cdot w)},
\]
where \(w_{\mu}\) is a fixed permutation in the conjugacy class \(K_{\mu}\). Let us point out that it is false in general that every zero of \(P_{\lambda, \mu}(q)\) has real part 0. For instance,
\[
P_{332,332}(q) = q^8 + 35q^6 + 424q^4 + 660q^2,
\]
four of whose zeros are approximately \(\pm 1.11366 \pm 4.22292i\).

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