Anatomy of entanglement and nonclassicality criteria

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We examine the internal structure of two-mode entanglement criteria for quadrature and number-phase squeezed states. (i) We first address entanglement criteria obtained via partial transpose of Schrödinger-Robertson inequality, where presence of an extra term makes them stronger. We show that this extra term performs an optimization within two intra-mode rotations, for the variables mixing in the entanglement criterion. We demonstrate this situation both for quadrature and number-phase squeezed states. (ii) We realize that Simon’s criterion performs this optimization automatically. (iii) Hence, we derive a Simon-like criterion for number-phase squeezed states which performs this optimization in the number-phase plane. The method can be generalized also for other states, e.g., amplitude-squeezed states. In addition to examining the intra-mode rotations, (iv) we also present an entanglement scheme in terms of product of the noises of the two modes, i.e., noise-area. We question, analytically and numerically, whether the well-known entanglement criteria are actually search mechanisms for a noise-area below unity. In this regard, we also numerically show the following. We consider an entangled state. We minimize the noise-area for the product form of the Duan-Giedke-Cirac-Zoller criterion (Mancini et al.) with respect to the intra-mode rotations. We observe that this noise-area (actually) is the input nonclassicality that a beamsplitter necessitates in order to generate exactly the amount of entanglement our particular state possesses. (vi) Additionally, we define an alternative entanglement measure for Gaussian states which can also be adapted to multimode entanglement. We further raise some intriguing questions, e.g., on the existence of an easier definition of an entanglement depth for number-phase squeezed states.

I. INTRODUCTION

In the new century, quantum optics showed fascinating progress, which quantum mechanics has performed in the past hundred years. Technologies, like quantum cryptography [1], quantum teleportation [2], measurements below standard quantum limit [3, 4] and quantum radars [5], are made possible by quantum nonclassicality, e.g., entanglement and squeezing, which became a well-experimented area.

Recent progresses in plasmonic nano-optics paved the way to the generation of nonclassical states in smaller and less-controlled systems, such as nanostructures [6–9]. The possibility for the integration of such systems into electronic and photonic devices created a potential for the miniaturization of the exceptional features achieved in quantum technologies [10, 11]. Epsilon-near-zero (ENZ) materials also demonstrated themselves as useful tools for quantum technologies, as they maintain/transfer coherence for longer distances [12, 13]. Such developments carried the detection and characterization of nonclassicality, e.g., squeezing, classicality between SMNc and multimode entanglement [25]. One another problem is that practical-to-observe nonclassicality tests move rather slow. Especially, it is a problem to obtain experimentally-accessible TME criteria can be overviewed as follows. (i) One can observe the presence of the TME. A similar conversion is demonstrated also between the SMNc and multimode entanglement [26]. One could expect similar studies to appear in near future on the conservation between SMNc and MPE in a squeezing transfer process [16]. Besides the conservation of nonclassicality, criteria for SMNc, TME and MPE can be derived from each other [26, 31]. These relations follow after one realizes that the quasi-particle excitations over an ensemble becomes single-mode nonclassical if the ensemble displays many-particle entanglement [29].

Despite the fascinating progresses in the experimental quantum optics and quantum plasmonics, developments in nonclassicality tests move rather slow. Especially, it is a problem to obtain experimentally-accessible TME measures for quantum states other than the Gaussian states. One another problem is that practical-to-observe TME criteria, each, works for a different class of entangled states.

The commonly employed methods for obtaining TME criteria can be overviewed as follows. (i) One can obtain a TME criterion by introducing an inequality which is satisfied by separable states [22, 32]. Thus, violation of the inequality witnesses the presence of the TME. (ii) One can also utilize the fact that a separable (e.g., two mode) state remain a physical quantum state under partial transpose (PT) operation [30, 34]. Then, a separable...
state has to satisfy, e.g., Heisenberg uncertainty relation (HUR) and Schrodinger-Robertson (SR) inequality, also after the PT operation. Thus, the violation of such inequalities witnesses the TME. Similarly, one can also utilize the condition that a positive-definite operator should stay so after the PT operation, if the two mode state is separable. One can further obtain new criteria by relating SMNc, TME and MPE criteria to each other. The nature of the operators, employed in the inequalities, determines the class of the states which the criteria can witness. For instance, Duan-Giedke-Cirac-Zoller (DGCZ) criterion (or its product form by Mancini et al.) and Simon-Peres-Horodecki (SPH) criterion work well for quadrature-squeezed like states, while Hillery&Zubairy (HZ) criterion works fine for number-phase squeezed like states.

There is another method employed for witnessing the nonclassicality of a single or two mode state. It relies on the positivity (or nonanalyticity) of the Glauber-Sudarshan $P(\alpha_1, \alpha_2)$ function. It utilizes the commutations appearing in a normal-ordering. One transforms an operator, $\hat{O}$, into normal-ordered form of a positive-definite function. Expectation value of a normal-ordered operator can be represented by a $P$-function, i.e., $\langle \hat{O} \rangle = \langle \hat{O}_1 \rangle + \int d^2\alpha_1 d^2\alpha_2 P(\alpha_1,\alpha_2)(\ldots)^2. \langle \hat{O}_1 \rangle$ appears in carrying out the normal-ordering of $\hat{O}$. Hence, negativity of $\langle \hat{O} - \hat{O}_1 \rangle$ witnesses the negativity of the $P(\alpha_1,\alpha_2)$ function, as also demonstrated in Sec. 11.3.4 in details. Here, $\langle \ldots \rangle^2$ is the normal-ordered form of the operator $\hat{O}$, where $\hat{a}_{1,2} \rightarrow \alpha_{1,2}$. The major problem with this method is: one cannot distinguish weather the origin of the negativity is (a) the TME between $\alpha_1$ and $\alpha_2$ modes, or (b) the SMNc present in the single-mode states $\alpha_{1,2}$, or (c) a combination of (a) and (b).

Then, developing an understanding on the internal structure of entanglement and nonclassicality criteria is important in achieving progresses in the entanglement/nonclassicality detection and quantification methods. We note in advance that, the main purpose of this paper is to explore the internal mechanism of entanglement tests, rather than deriving new criteria —though we also obtain new (motivating) witnesses. As we observe in the text, such an understanding is essential for an organized derivation of entanglement tests. To appreciate: a similar understanding on MPE-SMNc connection has enabled us to obtain MPE criteria from SMNc conditions, in particular, a measurable MPE criterion for superradiant phase transition.

In this paper, we aim to explore the internal structure of two mode entanglement criteria derived using the methods (i)-(iii). First, (i) we re-examine the entanglement generation in a beam-splitter. We present a geometric interpretation in terms of initial and final noise-areas of the two-modes. Noise-area is defined as the product of the noises in the two modes, which refers merely to the single-mode nonclassicalities in the modes, i.e., not refer to the two-mode correlations. In Refs. 21, 25, we show that quantification of single-mode nonclassicalities in terms of noise-area/volume demonstrates the swap of nonclassicality from SMNc into two-mode and multimode entanglement —exhibiting a conservation-like relation between SMNc and multimode entanglement. There are several reasons for calculating the single-mode nonclassicality in units of noise-area (converted into entanglement) as discussed in Sec. II.2 in the present paper and the section II of Ref. 25.

We show, in Fig. 1 that the noise-area increases (SMNc decreases) as the two modes are mixed more and more in a BS. We show that the increase in the noise-area is proportional to the entanglement. In order to show this relation, (2) we define a new entanglement measure in terms of nonclassical depth of a two-mode state.

We also study the intra-mode rotation, which yields further connections between the noise-area and TME as expressed in the fifth item below. Initially, we show that (3) in order to obtain a stronger TME criterion, one needs to take care of the intra-mode rotations in both modes, i.e., $\hat{a}_{\phi_{1,2}} = e^{i\phi_{1,2}} \hat{a}_{1,2}$. We demonstrate that the extra term in the SR inequality, absent in the HUR, takes care of an optimization with respect to the intra-mode rotations. This is also shown for criteria on number-phase squeezed states. (4) We discuss that SPH criterion takes care of the intra-mode rotation optimization automatically.

Next, (5) we realize the following intriguing connection between the noise-area and TME criteria. (5a) In one hand, we calculate the entanglement strength (log. neg. $E_N$) of a two-mode state. (5b) In the second hand, we also minimize the product $(\Delta \tilde{u})^2 (\Delta \tilde{v})^2$ —namely the product form of the DGCZ criterion (Mancini et al. 33) —for this state with respect to the intra-mode rotations. (5a-5b) Interestingly, comparing the two results, $(5b)$ is actually the noise-area (nonclassicality) input which a BS requires for generating $(5a)$ exactly $E_N$ amount of entanglement. This happens for an arbitrary mixing angle $\theta_{\text{BS}}$. (6) Inspired from this observation, we discuss if TME criteria somehow search for a noise-area below 1.

Then, we focus on the number-phase squeezed like states. We study the intra-mode rotations also in the number-phase plane. We observe that eigenvalues of the noise matrix, defined for number-phase operators, do not change with rotations and displacements in the number-phase $n$-$\Phi$ plane (analogous to $\phi_{1,2}$ rotations and displacement in the $x$-$p$ plane). (7) Inspired from this observation and the intra-mode rotational independence of the SPH criterion, mentioned in item (4) above, we derive a Simon-like criterion for number-phase squeezed like states.

1 We would like to kindly underline that the present manuscript is a work prior to the one in Ref. 25, see the dates in the arXiv please.

2 In Ref. 25, we utilize this new method to be able to obtain a multi-mode entanglement quantification without referring to bipartite entanglements.
states. This criterion, similar to SPH, can take care of the intra-mode rotations in the \( n_1 \Phi_1 \) and \( n_2 \Phi_2 \) planes automatically. We also examine the noise-area – TME connection in the number-phase plane. \( (8) \) We further observe that \( \theta_{\text{ns}} = -\pi/4 \) rotated version of the Hillery-Zubairy criterion is also a noise-area search below 1, but in the \( n_1,\phi_1,n_2,\phi_2 \) plane, i.e., not in the \( x_1,\phi_1,x_2,\phi_2 \) plane.

We aim to present an understanding on the internal structure, anatomy, of entanglement criteria. We believe that our observations will stimulate new studies in the field of entanglement and nonclassicality criteria. As an immediate example, the understanding can be useful in obtaining multi-mode entanglement criteria/quantifications without referring to bipartite entanglements, i.e., unlike performed in Refs. [29, 37]. We underline, in advance, that we aim to present intriguing relations and questions. We do not aim to present generalized proofs. For this reason, we prefer to use (for instance) the description ‘our observations on noise-area’ in Sec. II.5. We believe that, these relations will satisfactorily demonstrate the search mechanism such entanglement criteria employ. For a better comprehension of what purposes our findings address, one could consider the following instance. In Ref. [29], the achievement of a useful MPE criterion (by mimicking a SMNc condition) does not necessitate the presence of the proof for the MNP-SMNc connection —though we provide it in Ref. [29]. Instead, we obtained the new MPE criterion solely by anticipating on an MPE-SMNc connection.

The paper is organized as follows. Sec. I deals with Gaussian states. In Sec. II.1 we review the SMNc features of Gaussian states. We introduce the noise matrix and nonclassical depth. In Sec. II.2 we derive the two-mode noise matrix at the output of a BS. In Sec. II.3 we discuss that logarithmic negativity \( E_N \) at the beamsplitter output actually deals with the noise-area of the input state. In Sec. II.3.1 we derive the output noise-area \( \Omega^\text{(out)} \) for a BS input with a general (separable) Gaussian state. In Sec. II.3.2 we derive an alternative measure for TME, in terms of nonclassical depth \( \tau_{\text{ent}} \). We show that the change in the noise-area, after the BS, \( \Omega^\text{(out)}/\Omega^\text{(in)} \), is proportional to \( \tau_{\text{ent}} \). We demonstrate the noise-area change (increase) in Fig. \( \text{I} \). We explain why two input modes, with equal single-mode nonclassicalities, do not generate entanglement at the beam splitter output: because noise-area does not change. In Sec. II.3.3 we discuss if an entangled state can be rewound via a BS and be checked for a noise-area below unity, as a witness of entanglement? In other words, we initiate the discussion for: if the well-known TME criteria are actually a BS-search for a noise-area below 1 — a product of two SMNc conditions. In Sec. II.3.4, we demonstrate the derivation of nonclassicality conditions from the \( P \)-function. In Sec. II.4, we show that the extra term in SR inequality serves as an optimization for intra-mode rotations. In Sec. II.5, we present the intriguing observation on the relation between the noise-area and TME criteria, pronounced in the fifth item above. We also show that a nonclassicality criterion (e.g., noise-area below 1) is still a nonclassicality criterion after the coordinate transformation \( p_2 \rightarrow -p_2 \) or partial transposition of the quantum state.

Sec. III studies the number-phase squeezed like states. In Sec. III.1 we study the single-mode nonclassicality features of number-phase squeezed states: rotation and displacement (in the \( n-\Phi \) plane) invariance of noise matrix eigenvalues. In Sec. III.2 we show that the extra term in the SR inequality performs optimization with respect to the rotations in the \( n_1 \Phi_1 \) and \( n_2 \Phi_2 \) planes. In Sec. III.3 we derive a Simon-like criterion for number-phase squeezed like states. It deals with intra-mode rotation optimization automatically. In Sec. III.4 we derive an SR inequality for \( \hat{n}_{1,2} \) and \( \Phi_{1,2} \) operators and show that it is stronger than HZ criterion and the criterion by Raymer et al. [34]. In Sec. III.5 we present our observation on a possible relation between the noise-area and the HZ criterion — a criterion which works fine for number-phase squeezed states.

Sec. IV contains our summary.

II. GAUSSIAN STATES

In this section, we focus on the nonclassical properties of Gaussian states.

II.1. Properties of nonclassical states

Nonclassicality features of a Gaussian state can be represented solely by a covariance (noise) matrix \( V^{(c)} = \langle \hat{u}_i \hat{u}_j \rangle / 2 - \langle \hat{u}_i \rangle \langle \hat{u}_j \rangle \) \[38\] where the vector \( u = [x_1, p_1, x_2, p_2] \) and \( u = [x_1, p_1] \) for a two-mode (TM) and single-mode (SM) state, respectively. For a Gaussian state, the Wigner function \( (x, p\text{-representation}) \) can be written as \( W(u) = \exp(-u^T V^{(c)} u) / 2 \).

One can transform the real noise matrix into the complex form, \( u^{(c)} = [\alpha_1, \alpha_1^\dagger, \alpha_2, \alpha_2^\dagger] \) using the transformation matrices \[49\] \( \mathcal{T}_1 = [1, i; 1, -i] / \sqrt{2} \) and \( \mathcal{T}_2 = \mathcal{T}_1 \otimes \mathcal{T}_1 \) for SM and TM states, respectively, i.e., \( V^{(c)} = 3 V^{(c)} \mathcal{T}_1 \). Here, \( \alpha_{1,2} = (x_{1,2} + ip_{1,2}) / \sqrt{2} \). \( u^{(c)} \) for a SM state can be expressed similarly.

Noise matrix for a single-mode (SM) Gaussian state can be written as

\[
V^{(c)} = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix},
\]

(2.1)

where \( a = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + 1/2 \) and \( b = \langle \hat{a}_2 \rangle \). A SM Gaussian state is nonclassical if \( |\langle \hat{a}_2 \rangle| > |\langle \hat{a}_1 \rangle| \), i.e., for \( |b| > a - 1/2 \). Eigenvalues of \( V^{(c)} \), \( \Lambda_{\text{SM}} = a \pm |b| \), determine the largest (lg) and smallest (sm) uncertainty possible in this state. The noise reduces below the standard quantum limit (SQL) if \( \Lambda_{\text{SM}} < 1/2 \). For enabling an easier comparison, we define \( \lambda = 2A \), where \( \lambda < 1 \) implies the quadrature-squeezing or the SMNc of the state.
One can rotate the operators in the $x_1$-$p_1$ plane via $\hat{a}_n = e^{-i\phi_1} \hat{a}_1$ and align the $x_n$, in the largest or smallest uncertainty direction. For instance, examining the noise of $x_{\phi_1}$, one can obtain

$$\langle (\Delta \hat{x})^2 \rangle_{\phi_1} = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \frac{1}{2} + \frac{1}{2} (be^{-i2\phi_1} + b^* e^{i2\phi_1}).$$

(2.2)

In order to obtain the minimum noise, one chooses $\phi_1$ such that the last term in Eq. (2.2) assigns the minimum possible value, which is $-|b|$, by choosing $e^{-i2\phi_1} \langle \hat{a}_1^2 \rangle$ to have a (negative) phase $e^{i\pi}$. Largest uncertainty is obtained by setting the last term to $+|b|$. Fortunately, eigenvalues of matrix (2.1) performs this automatically. In a quadrature-squeezed state $\langle (\Delta x_{\phi_1})^2 \rangle = e^{-2\tau}/2 = a - |b|$ and $\langle (\Delta p_{\phi_1})^2 \rangle = e^{2\tau}/2 = a + |b|$. We note that, for brevity, we assume that a unitary transformation is performed \[49\] in the operators, i.e., we set their expectations to zero, which does not change the noise properties of the matrix determining the SMNc features.

Nonclassicality of a (general) state can as well be quantified with a nonclassical depth $\tau$ \[50\]. $\tau$ is defined as the minimum of the value in the filtering process

$$R(\alpha, \tau) = \frac{1}{\pi \tau} \int d^2 \alpha' \exp(-|\alpha - \alpha'|^2/\tau) P(\alpha'),$$

(2.3)

which makes the Glauber-Sudarshan function $P(\alpha)$ an analytical and positive-definite probability distribution. That is, it transforms the analytical and positive-definite probability distribution $P = \frac{1}{\pi} \sum_{n=0}^{\infty} N(n)$ to have a (negative) phase $e^{i\pi}$. The Glauber function in the integrand is a nonclassicality filter function \[51\]. $d^2 \alpha = d\alpha_1 d\alpha_2$ is a two dimensional integral over the real and imaginary components of $\alpha$.

For Gaussian states, entanglement depth can be simply determined \[50\] as $\tau = \max\{0, (1 - \lambda_{sm})/2\}$. Nonclassicality becomes nonzero for a noise $\lambda_{sm}$ below unity.

### II.2. After the beam splitter

When we mix the $V^{(c)}$ state, $V^{(c)} = V^{(c)}$ in Eq. (2.4), in a beam-splitter (BS), with a Gaussian state of noise matrix

$$V_2^{(c)} = \begin{bmatrix} a_2 & b_2 \\ b_2^* & a_2 \end{bmatrix},$$

(2.4)

we obtain the two-mode noise matrix

$$V_{BS}^{(c)} = \begin{bmatrix} A & C \\ C^\dagger & B \end{bmatrix},$$

(2.5)

with

$$A = \begin{bmatrix} a_1 \cos^2 \theta_{ns} + a_2 \sin^2 \theta_{ns} & a_1 \cos \theta_{ns} \sin \theta_{ns} + b_2 \sin \theta_{ns} \cos \theta_{ns} \\ b_1^* \cos \theta_{ns} \sin \theta_{ns} + b_2^* \sin \theta_{ns} \cos \theta_{ns} & a_1 \cos^2 \theta_{ns} + a_2 \sin^2 \theta_{ns} \end{bmatrix},$$

(2.6)

$$B = \begin{bmatrix} a_1 \sin^2 \theta_{ns} + a_2 \cos^2 \theta_{ns} & a_1 \sin \theta_{ns} \cos \theta_{ns} + b_2 \cos \theta_{ns} \sin \theta_{ns} \\ b_1^* \sin \theta_{ns} \cos \theta_{ns} + b_2^* \cos \theta_{ns} \sin \theta_{ns} & a_1 \sin^2 \theta_{ns} + a_2 \cos^2 \theta_{ns} \end{bmatrix},$$

(2.7)

$$C = \sin \theta_{ns} \cos \theta_{ns} \begin{bmatrix} (a_1 - a_2) & (b_1 - b_2) \\ (b_1 - b_2)^* & (a_1 - a_2) \end{bmatrix}. $$

(2.8)

Here, $\theta_{ns}$ is the beam splitter mixing angle with $R = r^2 = \sin^2 \theta_{ns}$ is the reflection coefficient.

### II.3. Noise Area

Two-mode entanglement (TME) generated at the output of the BS can be quantified by logarithmic negativity \[62\], $E_N$. The $E_N$ follows from similar considerations used in Simon-Peres-Horodecki (SPH) criterion \[45\] and is an entanglement monotone \[54\]. Ref. \[55\] calculates $E_N$ at the output of the beam splitter, for mixing of two nonclassical states and it finds

$$E_N = \max\{0, -\frac{1}{2} \log_2[(1 - 2\tau_1)(1 - 2\tau_2)]\},$$

(2.9)

where $\tau_{1,2}$ is the nonclassical depth of the first/second input mode. Similarly, Ref. \[15\] mixes a nonclassical state with a thermal (noisy) state, in order to test when the entanglement at the BS output will be destroyed. Ref. \[15\] obtains the form

$$E_N = \max\{0, -\frac{1}{2} \log_2[(1 - 2\tau_1)(1 + 2\tilde{n})]\},$$

(2.10)

where $\tilde{n}$ is the mean photon number in the thermal state. $(1 + 2\tilde{n})$ is the noise ($\lambda_{sm}$) of the thermal light, to be compared with $1$.

Eqs. (2.9) and (2.10) can be expressed in a common form

$$E_N = \max\{0, -\frac{1}{2} \log_2[\lambda_{sm}^{(1)} \lambda_{sm}^{(2)}]\}.$$ 

(2.11)

Hence, for Gaussian states, entanglement is generated at the BS output if the input “noise-area” is smaller than unity, i.e. $\Omega^{(in)} = \lambda_{sm}^{(1)} \lambda_{sm}^{(2)} < 1$.

It becomes more interesting when one notes that

$$S_N = \log_2 \frac{\lambda_{1,sm}^{(out)} \lambda_{2,sm}^{(out)}}{\lambda_{1,sm}^{(in)} \lambda_{2,sm}^{(in)}}.$$ 

(2.12)

Eq. (11) in Ref. \[21\], behaves a quantification for the entanglement equivalent to the logarithmic negativity $E_N$ at the output of a BS \[21\], \[23\]. Eq. (2.12) indicates that the entanglement generated at the output of a BS increases with the output noise-area divided by the input noise-area; that is, entanglement increases with the ratio of the output/input noise-area. $\lambda_{1,2,sm}^{(out)}$ refers to the SMNc $\tau_{1,2} = (1 - \lambda_{1,2,sm})/2$ remaining in the mode 1,2 at the BS output. BS cannot convert all of the initial SMNc into TME \[21\] and some SMNc $\lambda_{1,2,sm}^{(out)}$ remains in both modes. $\lambda_{1,2,sm}^{(out)}$ are calculated simply by wiping out the TME at the BS output, by setting $C = [0, 0; 0, 0]$ in
Eqs. (2.5) and (2.8). We also observe a similar behavior for multimode entanglement [25]. Noise-area (volume) increases as the initial SMNc is converted into multimode entanglement.

These observations make one raise the question: does the noise-area \( \Omega \) play a key role in the generation of two-mode entanglement? This question leads us, in this work, to perform a further investigation for the presence of a connection between the TME criteria [32–35] and the noise-area. For such a reason, we reconsider the BS mixing in terms of noise-area.

In the following, in Sec. II.3.1 we first calculate the noise-area \( \Omega^{(\text{out})} = \lambda_{1,\text{sm}}^{(\text{out})} \lambda_{2,\text{sm}}^{(\text{out})} \) associated with the (nonconverted) SMNcs remaining at the two output modes of a BS. We compare \( \Omega^{(\text{out})} \) with the initial noise-area \( \Omega^{(\text{in})} = \lambda_{1,\text{sm}}^{(\text{in})} \lambda_{2,\text{sm}}^{(\text{in})} \) when the two input modes are separable. We present a scheme for the geometries of the two areas in Fig. 1. Next, in Sec. II.3.2 we calculate the entanglement, in difference to Ref. [21], in terms of nonclassical depth \( \tau \) and we compare it with the two noise-areas \( \Omega^{(\text{in})} \) and \( \Omega^{(\text{out})} \).

II.3.1. Beam splitter output noise-area

One can obtain the (unused/unconverted) SMNcs remaining at the BS output modes [21] as follows. When one wipes out the entanglement (correlations) in the noise matrix (2.5), by setting \( 2 \times 2 \) matrix \( C = [0, 0; 0, 0] \) in (2.5), the remaining nonclassicality in the modes are only the SMNcs. One obtains the remaining SMNcs as

\[
\lambda_{1,\text{min}}^{(\text{out})} = (a_1 - |b_1|) \cos^2 \theta_{\text{as}} + (a_2 - |b_2|) \sin^2 \theta_{\text{as}}, \quad (2.13)
\]

\[
\lambda_{2,\text{min}}^{(\text{out})} = (a_2 - |b_2|) \cos^2 \theta_{\text{as}} + (a_1 - |b_1|) \sin^2 \theta_{\text{as}}, \quad (2.14)
\]

which result in an increased noise-area

\[
\Omega^{(\text{out})} = (a_1 - |b_1|) ((a_2 - |b_2|)) + \sin^2 \theta_{\text{as}} \cos^2 \theta_{\text{as}} [(a_1 - |b_1|) - (a_2 - |b_2|)]^2, \quad (2.15)
\]

\[
\Omega^{(\text{out})} = (a_1 - |b_1|) (a_2 - |b_2|), \quad (2.17)
\]

compared to the initial noise-area

\[
\Omega^{(\text{in})} = (a_1 - |b_1|) (a_2 - |b_2|), \quad (2.18)
\]

where the two input modes are not entangled.

Fig. 1 demonstrates the coordinate transformation (rotation, \( \theta \equiv \theta_{\text{as}} \)) of the two modes after the BS. The green and orange lines in Fig. 1 stand for the terms in Eq. (2.13). The terms in Eq. (2.14) can be extracted similarly. We kindly note that the area of the outer rectangle in Fig. 1 does not even present the new noise-area \( \Omega^{(\text{out})} \). Eqs. (2.13) and (2.14) sum the squares of the transformed coordinates, as it should be, while the outer rectangle in Fig. 1 sums the two directly. Still, Fig. 1 presents the concept for the noise-area increase fairly well when the SMNc is converted into TME in a BS.

The ratio of the noise-area change is

\[
\frac{\Omega^{(\text{out})}}{\Omega^{(\text{in})}} = 1 + \frac{(\lambda_{1,\text{sm}} - \lambda_{2,\text{sm}})^2}{\lambda_{1,\text{sm}} \lambda_{2,\text{sm}}}, \quad (2.19)
\]

whose inverse is demonstrated [21] to be proportional to the logarithmic negativity \( E_N \) of the output entanglement.

One can quickly realize that increase in the noise-area (NA), or the NA itself, becomes maximum for \( \theta_{\text{as}} = \pi/4 \), when the two SMNcs noises \( \lambda_{1,\text{sm}}^{(\text{out})} \) and \( \lambda_{2,\text{sm}}^{(\text{out})} \) are equal to each other. \( \theta_{\text{as}} = \pi/4 \) is also recognized as the optimum mixing angle for lossless BSs, the case we study here. One another thing to be noted is: the noise-area does not change if the two input beams have equally squeezed noise. In such a case, we already know that no entanglement is generated in an ideal BS which can be equivalently observed via the calculation of \( E_N \), or \( \tau_{\text{ent}} \) which we introduce in the next subsection, see Eq. (2.27).

II.3.2. An alternative definition for entanglement

In Ref. [21] and in Sec. II.3.1 we focus on the two unconverted SMNcs remaining in the two output modes of a beam splitter. So, we wipe out the entanglement (correlations) by setting \( C = 0 \) in the noise matrix (2.5). We calculate the nonclassical depths \( \tau_1 \) and \( \tau_2 \) associated with the single mode matrices \( A \) and \( B \), Eqs. (2.6) and (2.7), respectively.

It would be illuminating if one could quantify nonclassicality and entanglement in the same units. Thus, we consider just the opposite of the procedure we carried out in Ref. [21]. That is, when we remove the SM nonclassicalities in the noise matrix (2.5) by setting \( A = B = I/2 \), as

\[
V^{(c)} = \begin{bmatrix} I/2 & C \\ C^T & I/2 \end{bmatrix}, \quad (2.19)
\]

then the remaining nonclassicality, \( \tau_{\text{ent}} \), is solely due to entanglement. Here, \( I \) is a \( 2 \times 2 \) identity matrix, whose nonclassical depth corresponds to the one of a vacuum or a coherent state.

The nonclassicality associated with the noise matrix \( V^{(c)} \) can be determined by the condition \( \text{eig}(V^{(c)} + \tau) > 0 \) [35]. One obtains the minimum value of \( \tau_1 \) which makes the noise-matrix correspond to a positive (classical) Glauber-Sudarshan \( P \) function, i.e., in the filtered

One can also define a nonclassical depth for two-mode states —this is also pointed out by Mark Hillery in a work we cannot re-find— \( \tau_{\text{rms}} \), via a generalization of (2.3). One can numerically test that (we actually did) \( \tau_{\text{rms}} \) can be visualized as some kind of a total nonclassicality, does not change under BS rotations.
function (2.3). Here, $\tau = \text{diag}[\tau_1, \tau_1, \tau_1, \tau_1]$ is a diagonal matrix where the noise-inputs (filters) for both modes are constrained to be equal $\tau_1$. Imposing such a constraint on the filter function in Eq. (2.3), however, can enforce the injection of an unnecessary amount of noise. Thus, in Ref. [25], also here, we use the more general filter function $\tau = \text{diag}[\tau_1, \tau_1, \tau_2, \tau_2]$. The eigenvalue inequality can be written as

$$\text{eig} \begin{bmatrix} \frac{1}{2} + \tau_1 & 0 & \tilde{a} & \tilde{b} \\ 0 & \frac{1}{2} + \tau_1 & \tilde{b}^* & \tilde{a} \\ \tilde{a} & \tilde{b} & \frac{1}{2} + \tau_2 & 0 \\ \tilde{b}^* & \tilde{a} & 0 & \frac{1}{2} + \tau_2 \end{bmatrix} = 0,$$  

(2.20)

where $\tilde{a} = \sin \theta_{ns} \cos \theta_{ns}(a_1 - a_2)$ and $\tilde{b} = \sin \theta_{ns} \cos \theta_{ns}(b_1 - b_2)$. The equation for the eigenvalues $\beta_{1-4}$ can be obtained as

$$\tilde{a}^2 \phi^2 - (\phi^2 + |\tilde{b}|^2) \tilde{\tau}_1 \tilde{\tau}_2 + (\phi^2 - |\tilde{b}|^2)^2 = 0,$$  

(2.21)

where $\tilde{\tau}_{1,2} = 1/2 + \tau_{1,2} - \beta$. If we define $\phi = \tilde{\tau}_1 \tilde{\tau}_2$, the solutions are

$$x^{(\pm)} = \tilde{\tau}_1 \tilde{\tau}_2 = (|\tilde{a}| \pm |\tilde{b}|)^2,$$  

(2.22)

where positivity condition for the eigenvalues $\beta_i \geq 0$ necessitate

$$(1/2 + \tau_1)(1/2 + \tau_2) > (|\tilde{a}| \pm |\tilde{b}|)^2.$$

(2.23)

For all eigenvalues to be positive, $\tau_{1,2}$ need to satisfy

$$(1/2 + \tau_1)(1/2 + \tau_2) > (|\tilde{a}| + |\tilde{b}|)^2.$$

(2.24)

In this particular problem, the minimum amount of noise, injected to remove the nonclassicality, is achieved at $\tau_1 = \tau_2$. One obtains $\tau_1 = (\tilde{a} + |\tilde{b}| - 1/2)$, or more precisely $\tau_1 = \max[0, (\tilde{a} + |\tilde{b}| - 1/2)]$. The noise-area associated with the strength of the entanglement in $V_{ent}^{(c)}$ is

$$\Omega_{ent} = (1 - 2\tau_1)(1 - 2\tau_2).$$  

(2.25)

(See Ref. [25] for a generalization.) We note that a smaller $\Omega_{ent}$ value corresponds to a larger noise injected to the system in order to remove the nonclassicality in $V_{ent}^{(c)}$, whose origin is solely the TME. Thus, smaller $\Omega_{ent}$ values stand for stronger entanglement.

Therefore, entanglement in the BS state $V_{BS}^{(c)}$, Eq. (2.5), can be expressed in units of noise-area

$$\Omega_{ent} = \left[1 - \sin \theta_{ns} \cos \theta_{ns}(|a_1 - a_2| + |b_1 - b_2|)^2 \right].$$  

(2.26)

We note that in Eq. (2.26) we become expressed the entanglement in the same units with the SMNe, e.g., with the unconverted SMNe $\Omega^{(out)}$ in Eq. (2.16).

One can also associate the nonclassicality corresponding to the entanglement

$$\tau_{ent} = |\sin \theta_{ns} \cos \theta_{ns}|(|a_1 - a_2| + |b_1 - b_2|)$$  

(2.27)

$\tau$’s, a positive-definite $R(a_1, a_2)$ function can be achieved with a smaller noise product. That is, $\tau^2 > \tau_1 \tau_2$ or $(1 - 2\tau)^2 < (1 - 2\tau_1)(1 - 2\tau_2)$. 

---

4 In Ref. [25], noise injected to both modes are constrained to be equal, $\tau$. However, we note that such a definition may result the injection of an unnecessary amount of excess noise for removing the nonclassicality [25]. When the Gaussian filter function, i.e., generalized form of Eq. (2.3), is chosen to allow two different
in units of nonclassical depth \[50\]. Yet, we need to remark that \(\tau_{\text{ent}}\) amount of noise is injected into “both” modes. Hence, quantitatively, nonclassicality associated with the entanglement is (more appropriately) to be presented as \((1 - 2\tau_{\text{ent}})^2\). While in this problem, BS mixing, the minimum injected noise is obtained for \(\tau_1 = \tau_2\), in general, e.g., in the multimode case \[25\], one filters out the nonclassicality for unequal injection of the noises to the modes.

One can observe that, noise-area associated with the entanglement, \(\Omega_{\text{ent}}\) in Eq. (2.25), decreases (i.e. entanglement increases) as the noise-area associated with the two remaining single-mode nonclassicalities, \(\Omega^{(\text{out})}/\Omega^{(\text{in})}\) in Eq. (2.18), increases (i.e. remaining SMNcs in the output mode decreases). Although the product of the two, i.e. \(\Omega^{(\text{out})}/\Omega^{(\text{in})} \times \Omega_{\text{ent}}\) is not a constant, e.g. as a conserved quantity; in this simple formalism, one can observe the noise-area swap between the remaining SMNcs and the generated entanglement. This is an alternative demonstration of our earlier results \[21\], where \(S_N\) in Eq. (2.12), or \(\Omega^{(\text{out})}/\Omega^{(\text{in})}\) in Eq. (2.18), has been shown to work as a quantification of entanglement equivalent to logarithmic negativity.

We already derive a conservation-like relation in Ref. \[21\]. Here, in difference, comparison of \(\tau_{\text{ent}}\) and \((\Omega^{(\text{out})}/\Omega^{(\text{in})})\), in the same units, is still an illuminating result. We need to underline that \(\tau_{\text{ent}}\) amount of noise is injected into both modes. Hence, the nonclassicality associated with the generated entanglement appears as the reduction of the noise as \((1 - 2\tau_{\text{ent}})^2\) in the noise-area \[2.18\].

It is also worth mentioning that: a two-mode squeezing hamiltonian \(\exp(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2)\) generates pure entanglement from initial coherent states. That is, the SMNcs in the generated entangled state —calculated by setting \(C = 0\) in Eq. (2.5)— do vanish.

**II.3.3. Search for a noise-area below unity**

The observation we make in Fig. 1 and in the text above, is not useful merely for the demonstration of a conservation-type relation between entanglement and remaining SMNcs. Considering the reverse process, i.e. Fig. 2(b) \(\rightarrow\) Fig. 1(a), and examining Eq. (2.11), one naturally raises the following question. “Are some of the entanglement criteria actually search mechanisms for an initial noise-area (Fig. 1(a)) below unity?” In other words, does an entanglement criterion rotate the entangled (original) state by \(-\theta_m\) and seek for a noise-area (at any of \(\theta_m\)) with \((\langle \Delta \hat{x}_{\phi_1} \rangle^2) / (\langle \Delta \hat{x}_{\phi_2} \rangle^2) < 1/4\) or \(\lambda_{\phi_1} \lambda_{\phi_2} < 1\)?

We are aware that not any particular two-mode entangled state can be generated via a BS. However, any two-mode entangled state can be rewound (rotated back) by \(-\theta_m\). So, the rewound state can be checked if the input noise-area, product of SMNcs in the two modes, is sufficiently small. We remark that measurements of the noises of the modes, \((\langle \Delta \hat{x}_{\phi_1} \rangle^2)\), we calculate, do not refer to the correlations among the two modes. One can also show, using the negativity of the \(P\)-function, that noise-area below unity is also a nonclassicality condition which is a necessary condition for generating entanglement at the BS output \[14\].

We remark that \(\hat{x}_{\phi_{1,2}}\), e.g. in Fig. 1(a), is along the direction at which minimum noise for each mode is achieved. An arbitrary choice of \(\hat{x}_{\phi_{1,2}}\), i.e., not along the minimum noise, would yield a higher noise-area in Fig. 1(a). Repeating ourselves, fortunately, \(\lambda_{\phi m}\) performs this choice automatically. A similar (optimum) choice is performed also by SPH \[45\] criterion and in the calculation of \(E_N\) since it is invariant under intra-mode rotations, i.e. \(a_{1,2}(\phi_{1,2}) = e^{i\phi_{1,2}} a_{1,2}\). DGCZ criterion \[32\], or its product form \[33\], \[34\], see the Appendix, is not invariant under intra-mode rotations. So, one needs to take care of optimal directions in DGCZ mixing.

Our aim, in this work, is not to prove generalized relations between “TME criteria” and “BS search for a noise-area below 1”, which is a nonclassicality condition. Instead, we aim to raise this incidence to the attention of the community.

**II.3.4. Nonclassicality and \(P\)-function**

In this section, we show that a noise-area below unity is a nonclassicality condition which makes Glauber-Sudarshan \(P\)-function negative, or nonanalytic, in some region. Noise-area conditions, we demonstrate below, do not refer to the two-mode correlations. They refer only to the SMNcs (noise) of the each mode separately \[7\].

First, we present a sample derivation, i.e., Eq. (2.29). Then we introduce the noise-area like conditions, i.e., Eqs. (2.31) and (2.34).

A state (density matrix \(\hat{\rho}\)) is classical if it can be represented as \[40\]

\[
\hat{\rho}_{12} = \int d^2 \alpha_1 d^2 \alpha_2 P(\alpha_1, \alpha_2) |\alpha_1\rangle \langle \alpha_1| \otimes |\alpha_2\rangle \langle \alpha_2|, \tag{2.28}
\]

with \(P(\alpha_1, \alpha_2)\) are positive-definite probabilities in the coherent state representation. A normal-ordered operator, e.g. \(\hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger}\), can be replaced by their coherent (c-number) correspondence, e.g. \(\hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger} \hat{a}_{1,2}^{\dagger}\), in the \(P\)-representation. One can develop a nonclassicality condition while putting an operator into normal-ordered form of a positive-definite function, e.g. \[56\]

\[
\langle (\hat{n}_1 - \hat{n}_2)^2 \rangle = \langle \hat{n}_1 + \hat{n}_2 \rangle + 4\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + 4\hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 - 2\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1, \tag{2.29}
\]

\[5\] We recall that a \(P(\alpha_1, \alpha_2) < 0\) input is a necessary and sufficient condition for TME at the BS output \[14\].
where the last term is in normal-order and can be $P$-represented as
\[
\langle (\hat{n}_1 - \hat{n}_2)^2 \rangle = \langle \hat{n}_1 + \hat{n}_2 \rangle 
+ \int d^2 \alpha_1 d^2 \alpha_2 \, P(\alpha_1, \alpha_2) \langle |\alpha_1|^2 - |\alpha_2|^2 \rangle^2. \tag{2.30}
\]
If $\langle (\hat{n}_1 - \hat{n}_2)^2 \rangle < \langle \hat{n}_1 + \hat{n}_2 \rangle$, this can originate only from the negative values of $P(\alpha_1, \alpha_2)$ in some finite region, which shows that $\hat{\rho}$ is a nonclassical state. Here, we note that $\langle \hat{n}_1 + \hat{n}_2 \rangle$ shows up because of the commutations appearing in the normal-ordering. This realization is important in the discussions of item (vi) at Sec. II.5.

The following nonclassicality conditions, which do not refer to correlations but refer merely to two single-mode uncertainties, can be obtained similarly. For quadrature variables one obtains
\[
\langle (\Delta \hat{x}_1)^2 \rangle \langle (\Delta \hat{x}_2)^2 \rangle < 1/4 \quad \text{or,} \quad \lambda_{1,sm} \lambda_{2,sm} < 1, \tag{2.31}
\]
if $\hat{x}_{1,2}$ are chosen along the minimum noise direction. A weaker form of Eq. (2.31)
\[
\langle (\Delta \hat{x}_1)^2 \rangle + \langle (\Delta \hat{x}_2)^2 \rangle < 1 \tag{2.33}
\]
\[
\text{can also be obtained. Comparison of conditions (2.33) and (2.31) resembles the DGCZ criterion} \tag{32} \text{and its stronger (product) form by Mancini et al.} \tag{33}, \text{respectively.}
\]
For number-squeezed like states, one can also obtain
\[
\Omega_n = \frac{\langle (\Delta \hat{n}_1)^2 \rangle}{\langle \hat{n}_1 \rangle} \frac{\langle (\Delta \hat{n}_2)^2 \rangle}{\langle \hat{n}_2 \rangle} < 1 \tag{2.34}
\]
and its weaker form
\[
\langle (\Delta \hat{n}_1)^2 \rangle + \langle (\Delta \hat{n}_2)^2 \rangle < \langle \hat{n}_1 + \hat{n}_2 \rangle. \tag{2.35}
\]
As it is well-known, $\langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle < 1$ is squeezing in the number fluctuations below the SQL. So, we also define LHS of Eq. (2.34) as a noise-area, $\Omega_n$, proper for number-squeezed like states.

Stating one more time, these nonclassicality conditions do not refer to two-mode correlations.

\subsection*{II.4. Role of the extra term in Schrödinger-Roberson inequality—inefficient use of SMNC}

In this section, we briefly describe how one can derive stronger TME criteria from the Schrödinger-Roberson (SR) inequality \cite{10,11}. We show that the extra term in the SR inequality, Eq. (2.36), compensates (partially) for the non-optimum choice of $\hat{x}_{1,2}$, i.e. not along the minimum noise direction $\hat{x}_{\phi_{1,2}}$. We note that, SPH criterion (and $E_N$) performs this choice automatically in Fig. I and in Sec. III.1. In Sec. III.2, we observe that the extra term in the TME criterion for number-phase squeezed like states, Eq. (11) in Ref. \cite{40}, accounts a similar mixing optimization in the $n$-$\Phi$ plane.

Priorly, DGCZ \cite{32} and Mancini \textit{et al.} \cite{33} have obtained the entanglement criteria based on the following principle. Sum, $(\Delta \hat{u})^2 + (\Delta \hat{v})^2$, or product, $(\Delta \hat{u})^2 (\Delta \hat{v})^2$, of the noises of two observables $\hat{u}, \hat{v}$ must be larger than a critical value for a separable state $\hat{\rho}_{1,2} = \sum_k \rho_1^{(k)} \otimes \rho_2^{(k)}$. Violation of this inequality witnesses the presence of TME. Ref. \cite{38} shows that DGCZ criterion and the one by Mancini \textit{et al.} can be derived also using the partial transpose (PT) condition in a Heisenberg uncertainty relation (HUR). That is, PT of a separable state should also satisfy the HUR inequality. Following studies \cite{39,41} show that stronger forms of these criteria can be obtained using the PT condition in the Schrödinger-Roberson (SR) inequality
\[
(\Delta \hat{u})^2 (\Delta \hat{v})^2 \geq \frac{1}{4} \langle [\hat{u}, \hat{v}]^2 \rangle + (\Delta \hat{u} \Delta \hat{v})^2, \tag{2.36}
\]
where $(\Delta \hat{u} \Delta \hat{v})^2 = \frac{1}{4} \langle [\hat{u}, \hat{v}]^2 \rangle + (\Delta \hat{u} \Delta \hat{v})^2$ is the extra term compared to HUR. Stronger inseparability criteria are obtained \cite{40} by considering the PT of the state in Eq. (2.36). If the two modes are separable, PT yields a physical density matrix. So, Eq. (2.36) must be satisfied for partial transposed separable states. Then, violation of inequality (2.36) witnesses the presence of two-mode entanglement. HUR version of Eq. (2.36) yields the product form of DGCZ \cite{32}, the one by Mancini \textit{et al.} \cite{33}, under the PT condition for the operators
\[
\hat{u} = \cos \theta \hat{x}_1 + \sin \theta \hat{x}_2, \tag{2.37}
\]
\[
\hat{v} = \cos \theta \hat{p}_1 - \sin \theta \hat{p}_2. \tag{2.38}
\]
Inclusion of the $(\Delta \hat{u} \Delta \hat{v})^2$ term results in stronger TME criteria.

\textbf{The extra term.}—Here, we show that the extra term $(\Delta \hat{u} \Delta \hat{v})^2$ \cite{10} accounts the inefficiently used SMNC, for the mixing of the operators $\hat{u}$ and $\hat{v}$, in the weaker entanglement criterion $(\Delta \hat{u})^2 (\Delta \hat{v})^2$. \cite{33}. When $\hat{x}_{1,2}$ (in $\hat{u}, \hat{v}$) are chosen along the minimum noise direction $\hat{x}_{\phi_{1,2}}$, the extra term is zero. However, for a non-optimal choice of $\hat{x}_{1,2}$ directions, the extra term grows with the unused SMNCs in the $\hat{u}$-$\hat{v}$-mixing \cite{32,33}. Actually, in the close vicinity of separability, one can show that the extra term $(\Delta \hat{u} \Delta \hat{v})^2$ becomes zero for the choice of the optimum direction for $\hat{x}_{1,2}$, in $\hat{u}, \hat{v}$. But it grows as $\hat{x}_{1,2}$ deviates more from the minimum noise direction.

We note that the scheme in Fig. I presents the minimum uncertainty area possible, when both mixing coordinates $\hat{x}_1$ and $\hat{x}_2$ are chosen along the minimum uncertainty direction $x_{\phi_1}$ and $x_{\phi_2}$. This choice yields the minimum noise-area, $\Omega_n^{(in)} = \lambda_{sm}^{(1)} \lambda_{sm}^{(2)}$ in the BS (or $\hat{u}, \hat{v}$)-mixing. However, the TME criterion obtained by DGCZ, or by Mancini \textit{et al.}, perform the mixing given by Eqs. (2.37, 2.38), where $\hat{x}_{1,2}$ and $\hat{p}_{1,2}$ are not the minimum noise directions $\hat{x}_{\phi_1}$ and $\hat{x}_{\phi_2}$.

We are aware that the connection between the BS mixing and $\hat{u}$-$\hat{v}$ mixing is not apparent at this stage. Here,
we pronounce merely the finger prints of such a connection in advance. For instance, let us consider a two-mode state $|\psi\rangle$, or $\rho_{1,2}$ of entanglement strength $E_N$. In Sec. II.5 we observe that the optimized DGCZ noise-area $(|\Delta u|)(|\Delta v|)_{\phi_{1,2}}$ with $u$ and $v$ given in Eqs. (2.37) and (2.38) actually gives the initial noise-area for a BS in order to generate the $E_N$ amount of entanglement at the BS output.

In Fig. 2 we numerically demonstrate that the extra term in SR inequality (2.36), $(\Delta u)(\Delta v)_{\phi_{1,2}}$, accounts for the non-optimal $\hat{u} \hat{v}$ mixing. We fix $\hat{x}_1$ in the min noise direction and rotate the $\hat{x}_1$ by $\phi_1$ from the min noise direction. When the mixing is performed along the minimum uncertainty quadratures $x_{\phi_1}$, $x_{\phi_2}$, i.e. $\phi_1 = 0$, the extra term is “zero”. However, when two coordinates are not mixed efficiently, i.e., they are different than the scheme in Fig. 1 $(\Delta u)(\Delta v)_{\phi_1}$, in SR inequality accounts for the unused SMNcs in the $\hat{u} \hat{v}$ mixing.

Simon’s criterion should be stronger.— It is not a common belief, but also analytically demonstrated [57] that Simon-Peres-Horodecki (SPH) criterion is stronger than the other ones used for Gaussian states, e.g. DGCZ or the one by Mancini et al. Logarithmic negativity $E_N$, closely related with the SPH criterion, is also used as an entanglement measure [52,54] for Gaussian states.

After realizing the deficiency in the DGCZ mixing [Eqs. (2.37) and (2.38)] via Fig. 2 one can appreciate the invariance feature of the SPH criterion (or $E_N$) under the intra-mode rotations $\hat{a}_{\phi_1,\phi_2} \rightarrow e^{i\phi_{1,2}}$. SPH criterion chooses the optimum mixing automatically, similar to $\lambda_{1,2,sm}$ in SMNcs. Definition of a measure [52,54] becomes possible because an SPH-like approach does not fluctuate with intra-mode rotations. Actually, SPH criterion is invariant under any local canonical transformation. Second, Simon already uses the SR inequality in the derivation of the SPH criterion [45,49]. One another property of Simon’s treatment is SPH criterion is also invariant under mirror reflection [49]. That is, it treats both signs in the sets $\{u = \alpha x_1 + \beta x_2, \hat{v} = \beta \hat{p}_1 \pm \alpha \hat{p}_2\}$ and $\{\hat{u} = \alpha \hat{x}_1 + \beta \hat{x}_2, \hat{v} = \alpha \hat{p}_1 \pm \beta \hat{p}_2\}$.

II.5. Noise-area and Entanglement

In Secs. II.3.1, II.3.2, II.3.3 and via Fig. 1(a) $\rightarrow$ Fig. 1(b), we demonstrate that entanglement is associated with the increase in the initial noise-area (decrease in the product of SMNcs). Here, we consider the reverse process, Fig. 1(b) $\rightarrow$ Fig. 1(a). We raise the question: Does an entangled state yield a noise-area, $\Omega = \mu_{1,sm}\lambda_{2,sm}$, below 1 if it is rewound in a BS? That is, can $\Omega < 1$ be done by converting the entanglement (if exists) into a smaller noise-area, product of the two SMNcs? Noise-area accounts only the SMNcs in the two modes, it does not refer to correlations. Noise-area below unity is also a nonclassicality condition, necessary for a BS to generate TME.

We kindly indicate that our aim is not to provide formal proofs, but to present our observations about: the relations between noise-area and entanglement. We present similar discussions also for number-phase squeezed like states in Sec. II.5.

Numerical calculations, we conduct here, show very interesting and illuminating results/observations. First, (a) we recall Eqs. (2.9)-(2.11). The maximum amount of entanglement that can be generated at the output of a BS, $E_N = -\frac{1}{2} \log_2 \Omega_{sm}$, is determined by the initial noise-area, $\Omega_{sm} = \lambda_{1,sm}\lambda_{2,sm}$, of the two input beams. (b) We also remind that DGCZ criterion (or its stronger product form by Mancini et al.) does not take into account the optimization of the SM noises by intra-mode rotations $\hat{x}_{\phi_{1,2}}$. It is apparent that, in Eqs. (2.37) and (2.38), such an optimization would yield smaller fluctuations in the product form of the DGCZ criterion, in $(\Delta u)^2(\Delta v)^2$.

6 Ref. [57] shows that optimized form of DGCZ criterion [32] results the Simon’s criterion [15].

7 We are aware that not any particular two-mode entangled state can be generated via a BS. However, any two-mode entangled state can be rewound (rotated back) by $-\theta_{sm}$. So, the rewound state can be checked if the input noise-area, product of SMNcs in the two modes, is sufficiently small.

8 We note that we do not restrict to an entangled state generated in a BS. For instance, it can be generated via two-mode squeezing hamiltonian.

9 SMNcs increases as smaller the noise is.
We do the following. We consider an entangled Gaussian state. We numerically minimize the product
\[ \Omega_{\text{DGCZ}} = \langle (\Delta \hat{u})^2 (\Delta \hat{\nu})^2 \rangle, \]  
(2.39)
a kind of noise-area, in the entanglement criterion by Mancini et al., or Raymer et al. [31] (see the Appendix) under the intra-mode rotations \( \hat{a}_{\phi_1,2} = e^{i \phi_{1,2}} \hat{a}_{1,2} \). \( \hat{u} \) and \( \hat{v} \) are defined in Eqs. (2.37) and (2.38). For the same state, we also calculate the logarithmic negativity \( E_N \).

(i) We observe that, at each instance, the area \( \Omega_{\text{neg}} \) obtained from the inversion of \( E_N = -\frac{1}{2} \log_2(\Omega_{\text{neg}}) \) is equal to the area in Eq. (2.39), minimized under \( \phi_1, \phi_2 \) rotations, i.e. \( \Omega_{\text{neg}} = \Omega_{\text{DGCZ}}^{(\text{min})} \).

This observation can alternatively be stated as follows. The amount of entanglement \( E_N \) in this particular state is the maximum amount of entanglement which can be generated at a BS output, when the BS is input by an initial noise-area of \( \Omega_{\text{neg}} = \Omega_{\text{DGCZ}}^{(\text{min})} = (\Delta \hat{u})^2 (\Delta \hat{\nu})^2 \) minimized under \( \phi_1, \phi_2 \) rotations, i.e. \( \Omega_{\text{neg}} = \Omega_{\text{DGCZ}}^{(\text{min})} \).

We kindly remark that the two quantities \( \Omega_{\text{neg}} = \Omega_{\text{DGCZ}}^{(\text{min})} = (\Delta \hat{u})^2 (\Delta \hat{\nu})^2 \) are quite different. Evaluation of \( \Omega_{\text{neg}} \) refers to the complete two-mode state, whereas, \( \Omega_{\text{DGCZ}}^{(\text{min})} \) refers only to the two single-mode noises of the same state.

This can be interpreted as follows. (i.a) \( \Omega_{\text{DGCZ}}^{(\text{min})} \) behaves as if it searches a noise-area below unity, to yield a positive logarithmic negativity \( E_N = -\frac{1}{2} \log_2(\Omega_{\text{neg}}) \) in Eq. (2.11). We also have the outcome (i.b) when minimized with respect to intra-mode rotations, the product form of DGCZ becomes equivalent to the SPH criterion.

(ii) We observe that, retaining the \( \phi_{1,2} \) minimization, \( \Omega_{\text{DGCZ}}^{(\text{min})} \) becomes minimum for \( \theta = -\pi/4 \). (Also for \( \theta = +\pi/4 \) See Eqs. (2.37) and (2.38). It is interesting that, \( \Omega_{\text{DGCZ}}^{(\text{min})} \) is still minimum at \( \theta = -\pi/4 \), even if the entangled state is generated in a BS at \( \theta_{\text{BS}} \neq \pi/4 \) using an initially separable state. This shows that, thereby, \( \Omega_{\text{DGCZ}}^{(\text{min})} \) becomes somehow associated with the entanglement only, i.e., not with the SMNC known to remain in the modes [21]. Otherwise, \( \Omega_{\text{DGCZ}}^{(\text{min})} \) would be max at \( \theta = \theta_{\text{BS}} \).

(iii) When \( \theta = \pi/4 \) in Eqs. (2.37) (2.38) and (2.39), they can be shown to rewind the entangled (the original) state back in a BS by \( \theta_{\text{BS}} \) and check if
\[ (\Delta \hat{u})^2 (\Delta \hat{\nu})^2 < 1/4 \]  
(2.40)
or
\[ (\Delta \hat{u})^2 (\Delta \hat{\nu})^2 < 1 \]  
(2.41)
That is, they check if the noise-area is below unity. We note that inequality (2.41) is a weaker form of (2.40), similar to DGCZ and Mancini et al., respectively.

(iv) General form of DGCZ criterion, for \( \theta \neq \pi/4 \) in Eqs. (2.37) and (2.38), at first, seems not possible to obtain via a BS rotation. This is because, a BS can generate the form
\[ u' = \cos \theta \hat{v}_1 + \sin \theta \hat{v}_2 \]  
(2.42)
\[ v' = \cos \theta \hat{v}_1 + \sin \theta \hat{v}_2 \]  
(2.43)
where \( v' \) is different than \( \hat{v} \) in (2.38) used for DGCZ criterion. Even worse, \( (\Delta \hat{u})^2 + (\Delta \hat{\nu})^2 < 1 \), which is also a nonclassicality condition, cannot be satisfied by Eqs. (2.42) and (2.43), at all. Because the HUR sets the constraint \( (\Delta \hat{u})^2 + (\Delta \hat{\nu})^2 \geq \cos^2 \theta + \sin^2 \theta = 1 \).

Accordingly, the picture becomes reversed after realizing, in Fig. 1, that noise-area does not change under the coordinate transformation \( \hat{p}_2 \rightarrow -\hat{p}_2 \). This can be interpreted as follows. (i.a) \( \Omega_{\text{HUR}} \) or \( \Omega_{\text{DGCZ}} \) in (2.37) and (2.38), at all. Because the HUR sets the constraint \( (\Delta \hat{u})^2 + (\Delta \hat{\nu})^2 \geq \cos^2 \theta + \sin^2 \theta = 1 \).

Stating one more time, we used the following feature of the noise-area. It remains unchanged under the coordinate transformation \( \hat{p}_2 \rightarrow -\hat{p}_2 \). We did not partial transpose the state. As illuminated in the next item, this feature for the “noise-area below unity” (a nonclassicality condition) works for any nonclassicality condition obtained following the method introduced in Eq. (2.30).

(v) We finally note the following observation, in relation with the item (iv). Nonclassicality conditions are derived by performing normal-ordering in an operator, see Sec. 4.3.4. Normal-ordered operators can be expressed as \( \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \rightarrow \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \) in the \( \hat{P} \)-representation. This way, an operator can be rewritten in the form of a positive-definite function, e.g., as in Eqs. (2.29) and (2.30). Normal-ordered results in commutations, e.g. \( \hat{a}_2 \hat{a}_2^\dagger = \hat{a}_2^\dagger \hat{a}_2 + 1 \) or \( \hat{a}_2 \hat{a}_2^\dagger = \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2 \hat{a}_2^\dagger \).

One feature of the coordinate transformation \( \hat{p}_2 \rightarrow -\hat{p}_2 \) is that, \( \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \rightarrow \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \), it leaves the operators normal-ordered. So, \( \Omega_{\text{neg}} \) in the positive-definite function needs to be complex conjugated. This leaves the positive-definite function still a nonclassicality condition. Hence, \( (\Delta \hat{u})^2 + (\Delta \hat{\nu})^2 < 1 \) is also a nonclassicality condition.

It is well-known to the community that \( \hat{p}_2 \rightarrow -\hat{p}_2 \) transformation is conducted also for evaluating the PT of a two mode state. As already discussed in Sec. 4.3, it is used to obtain separability conditions by utilizing the phenomenon: a separable state remains physical under the PT operation [35] [40]. One exploits the fact that, partial transpose (PT) of a two mode state can alternatively be calculated via \( \hat{p}_2 \rightarrow -\hat{p}_2 \) transformation in the expectations.

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10 This becomes puzzling if we recall that the main difficulty for \( P(\alpha_1, \alpha_2) < 0 \) witnessing the inseparability is it may refer also to SMNCs in the modes.
In item (iv), however, \( p_2 \rightarrow -p_2 \) appears in a different concept: noise-area below 1 (a nonclassicality condition) is still a nonclassicality condition when the coordinates are transformed as \( p_2 \rightarrow -p_2 \). This makes the noise-area< 1 search by BS transformation (2.42)-(2.43) equivalent with the noise-area search via the DGCZ (or Mancini et al.) criterion in Eqs. (2.37)-(2.38). In other words, here, in this item, concerning the nonclassicality condition; \( p_2 \rightarrow -p_2 \) transformation appears a providing a new nonclassicality condition via unaltering the normal-ordering.

Commutators, e.g. \( \langle \hat{n}_1 + \hat{n}_2 \rangle \) in Eq. (2.30), appear in similarity with the HUR version of Eq. (2.36), where PT (\( p_2 \rightarrow -p_2 \)) is utilized for obtaining TME criteria. The physics in item (iv), however, is completely different.

II.6. Section Summary

In summary, in this section we (I) introduced the concept of noise-area as the product of the two quadrature noises \( \langle (\Delta x_1)^2 \rangle \langle (\Delta x_2)^2 \rangle \) in the two coupled modes. (II) We showed that the noise-area increases in the generation of entanglement (\( E_N \)) is a beam splitter (BS). (III) We defined an alternative entanglement measure \( \tau_{ent} \) by wiping out the SMNCs in the two modes. (IV) We presented defined an alternative entanglement measure \( E \) of entanglement \( (\min) \) in Eq. (2.30), appear in similarity with the HUR version of Eq. (2.36), where PT (\( p_2 \rightarrow -p_2 \)) is utilized for obtaining TME criteria. The physics in item (iv), however, is completely different.

(V) We showed that widely used TME criteria, in particular DGCZ [32] and Mancini et al. [33], are actually search mechanisms for achieving a noise-area below unity, i.e., \( \langle (\Delta x_1)^2 \rangle \langle (\Delta x_2)^2 \rangle < 1 \). (VI) Let a two-mode Gaussian state has an \( E_N \) amount of entanglement. Numerical calculations showed that minimum noise-area \( \Omega_{DGCZ}^{(\min)} \) of \( \hat{n}_1, \hat{n}_2 \) is achievable via \( \phi_{1,2} \)-optimization, actually, corresponds exactly to the amount of input noise-area in \( E_N = \frac{1}{2} \log_2 \Omega_{\text{input}} \) for making \( E_N = E_N \). Restating, \( E_N(\Omega_{DGCZ}^{(\min)}) = \frac{1}{2} \log_2 \Omega_{DGCZ}^{(\min)} = E_N \).

(VII) At first attempt, the DGCZ type of mixing, Eqs. (2.37) and (2.38), appears as unachievable via a backward BS rotation. Nevertheless, after realizing that noise-area (a nonclassicality condition) does not change in the coordinate transformation \( p_2 \rightarrow -p_2 \); one can observe that DGCZ and Mancini et al. criteria are actually BS-like search mechanisms.

(VIII) We understood that retention “noise-area below unity” as a nonclassicality condition in the coordinate transformation \( p_2 \rightarrow -p_2 \) is not a coincidence. Because, \( p_2 \rightarrow -p_2 \) transformation keeps a normal-ordered expression again in the normal-ordered form. This makes a nonclassicality condition, obtained via method in Eq. (2.30), still a nonclassicality condition after the coordinate transformation. As a final remark, \( p_2 \rightarrow -p_2 \) transformation (we employ to nonclassicality conditions) is not to be confused with the one performed for evaluating partial transposed states exploited in obtaining TME criteria [40, 41].

III. NUMBER-PHASE SQUEEZED LIKE STATES

In this section, we focus on the nonclassical properties of number-phase squeezed like states.

III.1. Single-mode nonclassicality

In Sec. II.1 we demonstrated that: smallest (squeezed) and largest noise of a single-mode state can be obtained from the eigenvalues of the noise matrix \( V^{(c)} \), in Eq. (2.1). \( V^{(c)} \) is represented in \( [\alpha_1, \alpha_1^*] \) with \( \alpha_1 = (x_1 + i p_1)/\sqrt{2} \). The smallest eigenvalue corresponds to the noise minimized with respect to intra-mode rotations \( \hat{a}_\phi = e^{i\phi} \).

So, one does not need to find the optimum direction \( \hat{a}_\phi \) manually in Fig. [1].

In this subsection, we perform a similar treatment for number-phase squeezed states. We define the annihilation and creation operators as

\[
\hat{A}_n = \hat{n} + i\gamma \hat{\Phi} \quad \text{and} \quad \hat{A}_n^\dagger = \hat{n} - i\gamma \hat{\Phi},
\]

where \( \hat{\Phi} \) is the phase operator and \( \gamma = 2\langle \hat{n} \rangle \) [58, 59].

Eigenstates of the operator

\[
\hat{E}_n = \hat{n} + i\gamma \hat{\Phi},
\]

with a generalized coefficient \( \gamma' = r\langle \hat{n} \rangle = r\gamma \), are intelligent states [58]. Intelligent states satisfy the minimum uncertainty relations. For \( r = 1 \), \( \gamma' = \gamma \) or \( \hat{E}_n = \hat{A}_n \), uncertainties \( \langle (\Delta \hat{n})^2 \rangle = \langle \hat{n} \rangle \) and \( \langle (\Delta \hat{\Phi})^2 \rangle = 1/4\langle \hat{n} \rangle \) determine the threshold for the standard quantum limit (SQL) for coherent states when \( \langle \hat{n} \rangle \gg 1 \) [59]. Eigenstates of \( \hat{E}_n \) possess squeezing in the number uncertainty \( \langle (\Delta \hat{n})^2 \rangle \) below the standard quantum limit (SQL) \( \langle (\Delta \hat{n})^2 \rangle < \langle \hat{n} \rangle \) for \( r > 1 \). For \( r > 1 \), eigenstates possess squeezing in the phase uncertainty \( \langle (\Delta \hat{\Phi})^2 \rangle \) below the SQL 1/4\langle \hat{n} \rangle. In both cases, the product \( \langle (\Delta \hat{n})^2 \rangle \langle (\Delta \hat{\Phi})^2 \rangle = 1/4 \langle \hat{n} \rangle \) achieves the minimum uncertainty.

The \( \hat{A}_n \) operator, given in Eq. (3.1), can be expressed also in a scaled form

\[
\hat{a}_n = \frac{1}{\sqrt{2}} \left( \hat{n} + i\gamma \hat{\Phi} \right) / \sqrt{\gamma}.
\]

Analogous to Sec. II.1 noise matrix can be composed as

\[
V^{(i)}_n = \begin{bmatrix}
\langle (\Delta \hat{n}')^2 \rangle & \langle (\Delta \hat{n}') \hat{\Phi} + \hat{\Phi} \hat{n}' \rangle /2 - \langle \hat{n}' \rangle \langle \hat{\Phi} \rangle \\
\langle (\Delta \hat{n}') \hat{\Phi} + \hat{\Phi} \hat{n}' \rangle /2 - \langle \hat{n}' \rangle \langle \hat{\Phi} \rangle & \langle (\Delta \hat{\Phi})^2 \rangle
\end{bmatrix}
\]

in the real representation \( \xi = [\hat{n}', \hat{\Phi}'] \), where \( \hat{n}' = \hat{n}/\gamma \) and \( \Phi' = \sqrt{\gamma} \) are scaled coordinates. The noise-matrix
takes the form

$$V_n^{(c)} = \left[ (\hat{a}_n^\dagger \hat{a}_n) - |\hat{a}_n|^2 + 1/2 \right]$$

(3.5)

in the complex representation $\xi = [\alpha_n, \alpha_n^\ast]$, $\alpha_n = (n + i\gamma \Phi)/\sqrt{2}$ or, equivalently, $\alpha_n = (n' + i\Phi)/\sqrt{2}$. The two noise matrices can be transformed as $V_n^{(r)} = TV_n^{(i)}$, the same way described in Sec. III.1. $T$ is also the same with the one defined in Sec. III.1, since $\hat{a}^\dagger \hat{x}$ and $\hat{a}_n \hat{n}$ relations are defined similarly. One can note that when $\hat{a}_n \rightarrow \hat{n}$ in Eq. (3.5), it already yields $V_n^{(c)}$ given in Eq. (2.1).

Eigenvalues of $V_n^{(r)}$ and $V_n^{(c)}$ are the same and determine the smallest (e.g., $\Lambda_{n,sm} \leq 1/2$) and largest (e.g., $\Lambda_{n,lg} \geq 1/2$) noise available in the system for the scaled operators $\hat{n}' = \hat{n}/\sqrt{\tau}$ and $\hat{\Phi}' = \sqrt{\tau} \hat{\Phi}$. Similar to Sec. III.1, we define $\Lambda_{n,sm} = 2\Lambda_{n,sm}$ and $\Lambda_{n,lg} = 2\Lambda_{n,lg}$, where $\lambda$ to be compared with 1, the SQL for the presence of squeezing.

In a general number-phase squeezed state, minimum squeezing does not need to be along either $\hat{n}' = \hat{n}/\sqrt{\tau}$ or $\hat{\Phi}' = \sqrt{\tau} \hat{\Phi}$. Minimum squeezing can be along a rotated $\hat{a}_{\alpha,\phi} = e^{i\phi} \hat{a}_n$ coordinate, similar to $\hat{x}_\phi$ in the quadrature squeezed states. Here, we check if a rotated state $|\psi_\phi\rangle = \exp(i\hat{a}_{\alpha,\phi}^\dagger \hat{a}_n \phi) |\psi\rangle$ results the same eigenvalues in $V_n^{(c)}$ or $V_n^{(r)}$ with the original state $|\psi\rangle$. We observe that, similar to Sec. III.1, $\Lambda_{n,sm}$ and $\Lambda_{n,lg}$ do not change with such an intra-mode rotation. This saves us from seeking the minimum noise direction for the mixing depicted in Fig. 1 lets say for $\hat{n}_{\phi_1}$ and $\hat{n}_{\phi_2}$.

It is worth noting that $R_n(\phi) = \exp(i\hat{a}_{\phi}^\dagger \hat{a}_n \phi)$ performs a completely different rotation compared to the one in the x-p plane, i.e., $\hat{a}_n \rightarrow \hat{a}_n e^{i\phi}$. The $(\langle \Delta \hat{n}' \rangle^2)$ and $(\langle \Delta \hat{\Phi}' \rangle^2)$ noises become unaltered under the x-p plane rotation. Whereas, $R_n(\phi) = \exp(i\hat{a}_{\phi}^\dagger \hat{a}_n \phi)$ performs a $\phi$-angle rotation in the new $\hat{n}'-\hat{\Phi}'$ plane. We observe that the phase of $\langle \hat{a}_n \rangle$ rotates with the angle $\phi$ under the rotation $R_n(\phi)$ where $\langle |\hat{a}_n| \rangle$ keeps unchanged.

We also observe that the generalized displacement operator $D = \exp(\beta \hat{a}_{\phi} - \beta^\ast \hat{a}_n)$ shifts the state in the $\hat{n}'-\hat{\Phi}'$ plane. $D$ does not change the noise properties of number-phase squeezed state. That is, it keeps the quantities $(\langle \Delta \hat{n}' \rangle^2)$ and $(\langle \Delta \hat{\Phi}' \rangle^2)$ unchanged.

It is important to remark that $\hat{D}$ operator is defined unproblematically [33] only for $\langle \hat{n} \rangle \gg 1$. In general, one can still use the uncertainty relation

$$\langle (\Delta \hat{n}')^2 \rangle \langle (\Delta \hat{\Phi}')^2 \rangle \geq \langle \hat{C} \rangle^2 / 4$$

(3.6)

following from the commutation $[\hat{n}, \hat{S}] = i\hat{C}$, where $\hat{S}$ and $\hat{C}$ are defined [33] as

$$\hat{S} = \frac{1}{2\lambda} (\hat{e}_- - \hat{e}_+) \quad \text{and} \quad \hat{C} = \frac{1}{2} (\hat{e}_- + \hat{e}_+),$$

(3.7)

with $\hat{e}_- = (\hat{n} + 1)^{-1/2} \hat{a}_-$ and $\hat{e}_+ = \hat{e}_+^\dagger$. In this case, we evaluate the noise matrices $V_n^{(r)}$ and $V_n^{(c)}$ in Eqs. (3.4) and (3.5) using the the scaled operator

$$\hat{n} = \frac{1}{\sqrt{\gamma}} \left( \hat{n} + i\gamma \frac{\hat{S}}{\langle \hat{C} \rangle} \right) / \sqrt{\gamma},$$

(3.8)

where $\hat{\Phi} \rightarrow \hat{\Phi} / (\hat{C})$ [33] in the definition of $\hat{a}_n$ in Eq. (3.3). $\gamma = 2 \langle \hat{n} \rangle$ is the same with Eq. (3.3). The features specified above work also when $\hat{a}_n$ is defined as in Eq. (3.8).

Some questions—Here, we raise some important questions, whose answers are left to the curiosity of the audience. Nonclassicality depth $\tau_{n,sm}$ is defined via [30]

$$R(\alpha) = \frac{1}{\pi \tau} \int d^2 \alpha \exp(-|\alpha - \alpha'|^2 / \tau) P(\alpha'),$$

(3.9)

where $P(\alpha)$ is the Glauber-Sudarshan $P$-function whose negativity at some region is the necessary and sufficient condition for SMnc. The Gaussian function in Eq. (3.9) is actually a filter function [31] introduced to make $R(\alpha)$ positive at all regions. $P(\alpha)$ in Eq. (3.9) can be re-expressed in terms of Wigner function $W(x,p)$, which establishes a connection between $W$-function and the $R$-function.

The questions are as follows. (i) Can we introduce states analogous to the Gaussian states, generalized to number-phase squeezed like states, if we define a Gaussian characteristic function

$$\chi(n_1', n_2') = \exp \left( -\frac{1}{2} u_n^\dagger V_{n_1' n_2'} u_n \right),$$

(3.10)

where $u_n = [n_1', \Phi_1', n_2', \Phi_2']$. Characteristic function can also be expressed in terms of complex variables

$$\chi(n_1', n_2') = \exp \left( -\frac{1}{2} y_{n_1' n_2'} V_{n_1' n_2'} y_n \right),$$

(3.11)

with $y = [\alpha_n^1, \alpha_n^2, \alpha_n^1, \alpha_n^2]$. (ii) If we express the Gaussian states (defined for number-phase squeezed states) as in Eq. (3.9), can we define an easier $\tau$ for such states? (iii) One another question is: Can we obtain the remaining SMncs and TME via setting the generalized correlation matrix $C = 0$, as in Ref. [21] and using Eq. (2.27), respectively?

III.2. Role of the extra term in Schrodinger-Robertson inequality—inefficient use of nonclassicality

In Sec. II.4, we showed that the extra term in the Schrodinger-Robertson (SR) inequality (2.30) takes care of the non-optimal mixing of the quadratures, i.e., $\hat{u} \neq \cos \theta \hat{x}_{\phi_1} + \sin \theta \hat{\phi}_{\phi_2}$ with $\hat{x}_{\phi_1}, \hat{\phi}_{\phi_2}$ are chosen along the minimum noise direction. Regarding the number squeezed like states, Hillery-Zubairy (HZ) criterion [35] is a more successful witness [39]. HZ criterion works fine for the superpositions of number (Fock) states. Ref. [40] shows that a TME criterion, stronger than the HZ criterion, can be obtained

$$((\Delta \hat{n}')^2 + 1) \langle (\Delta \hat{\Phi}')^2 + 1 \rangle < (\langle \hat{n}_1 + \hat{n}_2 \rangle + (\Delta \hat{n} \Delta \hat{\Phi})^2)$$

(3.12)

utilizing the Schrodinger-Robertson (SR), Eq. (2.30), for the variables

$$\hat{u} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2,$$

(3.13)

$$\hat{v} = i(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_1).$$

(3.14)
Here, \(\langle \Delta \hat{u} \Delta \hat{v} \rangle^2\) is the extra term providing a stronger TME criterion. TME criterion (3.12) is obtained from the partial transpose of observables \(\hat{u} = \hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_2 \hat{a}_2\) and \(\hat{u} = i(\hat{a}_1^\dagger \hat{a}_1^\dagger - \hat{a}_1 \hat{a}_2)\). Criterion (3.12) is stronger than the HZ criterion \(^{39}\) and the one in Ref. 80.

An eigenstate of the operator \(\hat{E}_n = \hat{n} + i\gamma' \hat{\Phi}\) in Eq. (3.12), or \(\hat{E}_n = \hat{n} + i\gamma' \delta'(C)\), yields number-squeezing along the \(\hat{n}\) operator for \(\gamma = \gamma' < 1\) and phase-squeezing along the \(\hat{\Phi}\) operator for \(\gamma > 1\). Here, \(\gamma = 2(\hat{n})\). That is, minimum squeezing in the state [smallest eigenvalue of the noise matrix (3.5) \(\lambda_{n,sm}\)] is equal to \(Q_n = \langle (\Delta \hat{n})^2 \rangle / \langle \hat{n} \rangle\) for \(\gamma < 1\) and is equal to \(Q_n = \langle (\Delta \hat{\Phi})^2 \rangle / \langle \hat{n} \rangle\) for \(\gamma > 1\). When such a state is mixed with a coherent state in a BS in order to obtain TME, the extra term becomes \(\langle \Delta \hat{n} \Delta \hat{\Phi} \rangle = 0\). In other words, HUR version of Eq. (3.12), which omits the last term, becomes equally strong with the Eq. (3.12).

This is not the case, however, if the eigenstate of \(\hat{E}_n\) in Eq. (3.12) is rotated in the \(n' - \Phi'\) plane. As already discussed in Sec. 11.2 SMNC in the rotated state does not change, i.e. \(\lambda_{n,sm}\) of \(V_n^{(c)}\) and \(V_n^{(c)}\), remain the same. Nevertheless, both \(Q_n\) and \(Q_{\Phi}\) become larger than \(\lambda_{n,sm}\). That is, minimum squeezing in the rotated state, now, is not along the \(\hat{n}\) or \(\hat{\Phi}\) operators. Since the \(\hat{n}\) (or \(\hat{\Phi}\)) is not in the optimal direction \(\theta_{n,\phi}\), the forms of \(\hat{n} \hat{\Phi}\) in Eqs. (3.13) and (3.14), are not optimal. This is analogous to the case in Sec. 11.3 where the optimum direction \(x_{1,2}\) is not chosen in the mixing (2.37) and (2.38).

Now, when the minimum squeezing is not along \(n'\) or \(\Phi'\) directions, the HUR version of Eq. (3.12) gets help from the last term \(\langle \Delta \hat{n} \Delta \hat{\Phi} \rangle\). In other words, the last term takes care of the inefficient mixing in \(\hat{n}, \hat{\Phi}\), similar to the \(x-p\) case in Eqs. (2.37) and (2.38).

In order to check the phenomenon, we mix an eigenstate of \(\hat{E}_n\) operator (\(r = 5/7\)), \(|\psi_r\rangle\), in a BS with a coherent state. We then rotate the eigenstate as \(|\psi_r(\phi)\rangle = \hat{R}_n(\phi)|\psi_r\rangle\) and mix \(|\psi_r(\phi)\rangle\) with a coherent state in the BS. In Fig. 3 we plot the extra term \(\langle \Delta \hat{n} \Delta \hat{\Phi} \rangle\) with respect to the angle of rotation in the \(n' - \Phi'\) plane. For \(\phi = 0\), the minimum noise of the eigenstate \(|\psi_r\rangle\) is along the \(\hat{n}\) direction. As it can be observed in Fig. 3, for \(\phi = 0\), extra term is \(\langle \Delta \hat{n} \Delta \hat{\Phi} \rangle = 0\). When the \(|\psi_r\rangle\) state is rotated, the extra term increases up to \(\phi = \pi/4\). At \(\phi = \pi/4\), the minimum squeezing direction is along the middle of \(n'\) and \(\Phi'\) directions. After \(\phi > \pi/4\) the extra term starts to decrease again, since the minimum direction now approaches along the \(\hat{\Phi}'\) operator. At \(\phi = \pi/2\) min squeezing direction points along the \(\hat{\Phi}'\), while it was along \(\hat{n}\) for \(\phi = 0\). So, the extra term becomes zero again.

This shows that \(\langle \Delta \hat{n} \Delta \hat{\Phi} \rangle^2\) term in Eq. (3.12) compensates partially the inefficiency in being unable to choose the \(\bar{n}_1, \bar{\Phi}_1\) and \(\bar{n}_2, \bar{\Phi}_2\) in the minimum squeezing direction, i.e., \(\bar{a}_{n,\phi_1}\) and \(\bar{a}_{n,\phi_2}\). \(^{\ast}\)

**III.3. Simon-like criterion for number-phase squeezed states**

In Sec. 11.4 we mentioned that Simon’s (SPH) criterion \(^{15}\) takes an automatic choice of the optimal choice for the mixing direction \(x_{1,2} - P_{1,2}\) and \(x_{1,2} - P_{1,2}\) in Eqs. (3.57) and (3.59). This is because, SPH uses quantities invariant under \(\phi_1, \phi_2\) rotations, similar to \(V_r^{(c)}\) in Eq. (2.1) and \(V_n^{(c)}\) in Eq. (3.8). Hence, it is natural to raise the question can Simon’s treatment in

**FIG. 3.** Role of the extra (the last) term in the entanglement criterion (3.12) for number-phase squeezed like states. Criterion is obtained from SR inequality (2.36), \(\hat{n}_2, \Phi_2\) already aligned in minimum noise direction in the \(n_2 - \Phi_2\) plane. When \(\hat{n}_1\) or \(\Phi_1\) is aligned with the minimum noise directions \(\bar{a}_{n,\phi_1}\), \(\phi_1 = 0\), the extra term becomes zero. When \(\hat{n}_1\) deviates from the minimum noise direction, the extra term increases to compensate the inefficient use of SMNC in mode 1.

Ref. \(^{15}\) perform a similar optimization automatically also for the intra-mode rotations in the \(n_1^\prime - \Phi_1^\prime\) and \(n_2^\prime - \Phi_2^\prime\) planes. The answer is yes. Because, as \(\bar{a}_n\) has similar properties with the standard \(\hat{a}\) operator, one can obtain a similar criterion.

When we define

\[
Q_1 = (\bar{a}_{n,1} + \bar{a}_{n,1})/\sqrt{2}, \quad P_1 = i(\bar{a}_{n,1} - \bar{a}_{n,1})/\sqrt{2},
\]

\[
Q_2 = (\bar{a}_{n,2} + \bar{a}_{n,2})/\sqrt{2}, \quad P_2 = i(\bar{a}_{n,2} - \bar{a}_{n,2})/\sqrt{2},
\]

where \(\hat{a} \rightarrow \bar{a}_n\), with \(\bar{a}_n\) defined in Eq. (3.3) or (3.8), it is possible to obtain a similar criterion similar to SPH \(^{15}\). Following the same arguments in Ref. \(^{15}\), defining \(\xi = [\bar{Q}_1, \bar{P}_1, \bar{Q}_2, \bar{P}_2]\), the noise matrix \(V_n^{(c)} = \langle (\Delta \xi_1, \Delta \xi_1) \rangle / 2\)

\[
V_n^{(c)} = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}
\]

must satisfy

\[
\mu = \det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \text{tr}(AJCJBJ^TC^T) \geq 0
\]

for a separable state. \(J = [0, 1; -1, 0]\) and \(A, B, C\) are 2×2 matrices. We kindly note that TME criteria (3.18) and (3.14) are entirely regarding the forms of mixings (3.15)-(3.16) and (3.13)-(3.14), respectively.

Unlike SPH \(^{15}\) criterion, this one, Eq. (3.18), works good for number-phase squeezed like states. \(\bar{P}_2 \rightarrow -\bar{P}_2\) under partial transposition, similar to \(\bar{P}_2 \rightarrow -\bar{P}_2\) in quadrature variables. In Fig. 4 we plot \(\mu\), in Eq. (3.18), for an initial number-squeezed state mixed with a coherent state in a BS with mixing angle \(\theta_{ns}\).

If one defines the Gaussian states in terms of \(\bar{a}_n\), in place of \(\hat{a}\), it is possible to extend the logarithmic negativity \(^{52-53}\), which is an entanglement monotone \(^{54}\), to number-phase squeezed states.

Finally, we note that the method we discuss in this section can be generalized for obtaining TME criteria working well
for other type of states. For instance, regarding amplitude-squeezed like states, one can define the annihilation operator (3.1) as
\[ \hat{A}_{\text{amp}} = \hat{Y}_1 + i\gamma \hat{Y}_2, \]
with \( \hat{Y}_1 = \hat{a}^2 + \hat{a}^2 \) and \( \hat{Y}_2 = i(\hat{a}^2 - \hat{a}^2) \). So, our treatment can be generalized to entangled states by an appropriate choice of the observables according to which operators nonclassicality is associated with. For all such choices, \( \mu \) takes care of the \( \phi_1, \phi_2 \) optimization automatically.

III.4. Schrödinger-Robertson inequality for \( \hat{n} \) and \( \hat{\phi} \)

Here, we also present a TME criterion for \( \hat{n} \) and \( \hat{\phi} \) variables by utilizing the SR inequality. We employ a mixing analogous to DGCZ criterion, given in Eqs. (3.18) and (3.19), by simply replacing \( \hat{x}_{1,2} \rightarrow \hat{n}_{1,2} \) and \( \hat{p}_{1,2} \rightarrow \gamma_{1,2} \hat{P}_{1,2} \).

The choice \( \hat{u} = \hat{a}_x^1 \hat{a}_y^1 + \hat{a}_x^2 \hat{a}_y^2 \) and \( \hat{u} = i(\hat{a}_x^1 \hat{a}_y^1 - \hat{a}_x^2 \hat{a}_y^2) \) in the Schrödinger-Robertson (SR) inequality in Eq. (3.26) yields the condition (3.12) for separability under the partial transpose (PT) condition. Alternatively, one can use the operators analogous to \( \hat{u}, \hat{v} \) in Eqs. (3.37) and (3.38).

\[ \hat{u} = \cos \theta \hat{n}_1 + \cos \theta \hat{n}_2 \] (3.20)
\[ \hat{v} = \sin \theta \gamma_1 \Phi_1 + \sin \theta \gamma_2 \Phi_2 \] (3.21)
and obtain
\[ \langle (\Delta \hat{u})^2 \rangle + \langle (\Delta \hat{v})^2 \rangle \geq \left( \cos^2 \theta \langle \hat{n}_1 \rangle + \sin^2 \theta \langle \hat{n}_2 \rangle \right)^2 + \langle \Delta \hat{u} \Delta \hat{v} \rangle, \]
(3.22)
where \( \hat{v} = \cos \theta \gamma_1 \Phi_1 - \sin \theta \gamma_2 \Phi_2 \) is the PT of Eq. (3.21). For the sake of a comparison, one can obtain a criterion using the product (stronger) form of Raymer et al. [34], see Eq. (A10) in the Appendix,
\[ \langle (\Delta \hat{u})^2 \rangle + \langle (\Delta \hat{v})^2 \rangle \geq 4 \cos^2 \theta \sin^2 \theta \langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle, \]
(3.23)
which is even weaker than the first term on the RHS of Eq. (3.22), since \( (a - b)^2 = a^2 + b^2 - 2ab > 0 \).

III.5. Noise-area in the \( \hat{n}' - \hat{\phi}' \) plane and Entanglement

In Sec. III.5 we showed that the DGCZ criterion [32, 33] is actually a search mechanism for a noise-area below unity [a nonclassicality (Nc) condition] ins a beam-splitter (BS). Here, in analogy, we show that Hillery-Zubairy (HZ) criterion [35] is also a weaker form of the noise-area search for \( \Omega_n = \frac{\langle (\Delta \hat{a}_1)^2 \rangle}{\langle \hat{n}_1 \rangle} \frac{\langle (\Delta \hat{a}_2)^2 \rangle}{\langle \hat{n}_2 \rangle} < 1 \).

More explicitly, in this subsection, we show that \( \theta_{bs} = -\pi/4 \)-rotated version of the HZ criterion [35] looks like a noise-area search, similar to the DGCZ criterion discussed in Sec. III.5.

Hillery and Zubairy (HZ) criterion [35], \( \langle \hat{a}_x^1 \hat{a}_x^1 \hat{a}_x^2 \hat{a}_x^2 \rangle < |\langle \hat{a}_x^1 \hat{a}_x^1 \rangle|^2 \Rightarrow \text{inseparable, can be derived by examining the uncertainty } \langle (\Delta \hat{S}_x)^2 \rangle + \langle (\Delta \hat{S}_y)^2 \rangle \]
(3.24)
where pseudo-spin operators are defined as \( \hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2, \hat{S}_y = -i(\hat{S}_+ - \hat{S}_-) \).

For separable states, \( \hat{S}_{1,2} = \sum_k \hat{P}_k (\hat{P}_k^{(1)} \otimes \hat{P}_k^{(2)}) \), uncertainty satisfies the inequality
\[ \mu_{\text{HZ}} \rightarrow \langle (\Delta \hat{S}_x)^2 \rangle + \langle (\Delta \hat{S}_y)^2 \rangle \geq \langle \hat{N}_+ \rangle / 2, \]
(3.24)
whose violation witnesses the presence of the TME. HZ criterion works fine for superpositions of number-squeezed like states [36]. A product form of the HZ criterion [34] can as well be obtained using the partial transpose (PT) of a HUR type of inequality [11].

Noise-area an entanglement.—We now check if this is somehow related to the noise-area \( \Omega_n \) in a BS. As \( \theta_{bs} = -\pi/4 \) is known to generate the strongest TME state in at a BS output, we rotate Eq. (3.24) back by \( \theta = -\pi/4 \) in a BS, then, we investigate if it is related to \( \Omega_n < 1 \). The rotations transform the spins as \( \hat{S}_x \rightarrow \hat{S}_x \), where \( \hat{S}_x \) and \( \hat{N}_+ \) remain invariant. Hence, the \( \theta = -\pi/4 \) rotated (back) version of the HZ criterion is
\[ \mu_{\text{HZ}}(\theta_{bs} = -\pi/4) \rightarrow \langle (\Delta \hat{S}_x)^2 \rangle + \langle (\Delta \hat{S}_y)^2 \rangle \geq \langle \hat{N}_+ \rangle / 2. \]
(3.25)

Now, we show that \( \mu_{\text{HZ}}(\theta_{bs} = -\pi/4) \) is a nonclassicality condition and it is reminiscent of the DGCZ (sum) version of the noise-area \( \Omega_n \). The inequality (3.25) can be put into the form
\[ \mu_{\text{HZ}}(\theta_{bs} = -\pi/4) \rightarrow \langle (\Delta \hat{S}_x)^2 \rangle \geq \langle \hat{N}_+ \rangle / 4 - \int d^2 \alpha d^2 \alpha_2 P(\alpha_1, \alpha_2) \left[ \frac{i}{2} (\alpha_2 \alpha_1 - \alpha_1 \alpha_2) - \langle \hat{S}_y \rangle \right]^2 \]
(3.26)
using the normal-ordered form of \( \langle (\Delta \hat{S}_y)^2 \rangle \)
\[ \langle (\Delta \hat{S}_y)^2 \rangle = \langle \hat{N}_+ \rangle / 4 \]
\[ + \int d^2 \alpha d^2 \alpha_2 P(\alpha_1, \alpha_2) \left[ \frac{i}{2} (\alpha_2 \alpha_1 - \alpha_1 \alpha_2) - \langle \hat{S}_y \rangle \right]^2. \]
(3.27)

Therefore, rotated form of the violation of \( \mu_{\text{HZ}} \) leads to
\[ \mu_{\text{HZ}}(\theta_{bs} = -\pi/4) \rightarrow \langle (\Delta \hat{n}_1)^2 \rangle + \langle (\Delta \hat{n}_2)^2 \rangle < \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle. \]
(3.28)

\[ ^{11} \text{HZ criterion is a subset of conditions by Shchukin and Vogel [32].} \]
This resembles the DGCZ (weaker, $\langle (\Delta \hat{n})^2 \rangle + \langle (\Delta \hat{\phi})^2 \rangle < 1$) version of the noise-area (product) form by Mancini et al.\cite{33} ($\langle (\Delta \hat{n})^2 \rangle < 1$ studied in Sec. 11). Similar to DGCZ being a subset of the product form, Eq. (3.28) is a subset of

$$\Omega_n = \frac{\langle (\Delta \hat{n}_1)^2 \rangle}{\langle \hat{n}_1 \rangle} \frac{\langle (\Delta \hat{n}_2)^2 \rangle}{\langle \hat{n}_2 \rangle} < 1 \quad (3.29)$$

which searches a noise-area below unity.

Can we anticipate forms for TME criteria? One may also check if the $\theta_{BS} = \pi/4$-rotated form of the nonclassicality condition (3.29) is a TME, as follows. That is, one can simply place $\hat{p}_{12} = \sum_k p_k \hat{p}_k^{(h)} \hat{p}_k^{(v)}$ into $\Omega_n(\theta_{BS} = \pi/4)$, in Eq. (3.29), and try to obtain a TME criterion using the method worked out in Refs.\cite{29} 32\cite{33} 35. The $\theta_{BS} = \pi/4$-rotated form of $\Omega_n$ is

$$\langle (\Delta (\hat{N}_+ + \hat{S}_z))^2 \rangle \langle (\Delta (\hat{N}_+ - \hat{S}_z))^2 \rangle \geq \langle (\hat{N}_+)^2 \rangle - \langle \hat{S}_z \rangle^2. \quad (3.30)$$

We could not achieve analytically to show that Eq. (3.30) is a TME criterion, i.e., by placing $\hat{p}_{12} = \sum_k p_k \hat{p}_k^{(h)} \hat{p}_k^{(v)}$ and using Cauchy-Schwartz inequalities. We, however, are not very skillful in conducting such analytical calculations. Thus, we believe that one can still show the Eq. (3.30) or a variant of it works as a TME criterion. Nevertheless, we numerically checked that Eq. (3.30) works as a TME criterion. That is, for all (thousands) of the random inseparable states Eq. (3.30) is violated.

Actually, one would not need Eq. (3.30) for entanglement detection purposes. Because, the Simon like criterion for number-phase squeezed like states, introduced in Sec. 11, would anyway be stronger than Eq. (3.30). Demonstration of Eq. (3.30) would still be beneficial for developing understanding on the structure of TME criteria.

IV. SUMMARY AND DISCUSSIONS

One can obtain observable nonclassicality (Nc) conditions using the positivist of the Glauber-Sudarshan $P(a_1, a_2)$ function\cite{16} for classical states. The use of such two-mode nonclassicality conditions either as either detecting the presence of entanglement or single-mode nonclassicality (SMNc) is quite limited. Because the nonclassicality may originate either from TME or SMNc, even from both. So, developing understanding on such conditions has vital importance for witnessing entanglement in quantum technologies.

In this study, we make a survey through the internal structure (anatomy) of two-mode entanglement (TME) criteria and two-mode nonclassicality (TMNc) conditions. We present examples demonstrating that the well-known TME criteria behave as searching for a noise-area below unity via a beam-splitter (BS) rotation. A noise-area is defined as the product of the minimum noises of the two modes, e.g., $\Omega = \langle (\Delta \hat{x}_1)^2 \rangle / \langle \hat{x}_1 \rangle^2 + \langle (\Delta \hat{x}_2)^2 \rangle / \langle \hat{x}_2 \rangle^2$.

There are several reasons for us to focus on noise-area. First, a BS creates the amount of entanglement (for Gaussian states) $E_N = -\frac{1}{2} \log_2 \Omega_{\text{input}}$ via a BS input noise-area $\Omega_{\text{input}}$. Second, noise-area demonstrates conservation like behavior with entanglement. Third, noise-area (becomes noise-volume) also displays quantification for the remaining multimode nonclassicality after the generation of multimode entanglement. Here, we further geometrically demonstrate (see Fig. 1) that increase in the noise-area (of remaining SMNc) is a swap of the nonclassicality into entanglement. We demonstrate this, in addition, by quantifying the entanglement also in terms of noise-area, see $\tau_{\text{ent}}$, i.e., SMNc and TME are represented in the same units.

The fourth reason is "noise-area below unity" is a TMNc condition. Fifth, the most important one, noise-area refers merely to the noise features of the individual modes, i.e., it does not refer to correlation between the two modes.

In the paper, we show the close relationship between the DGCZ\cite{32} (or it product form\cite{33}) entanglement criterion and noise-area search via backward rotations in a BS. It is such that, for an entangled state the optimized (with respect to intra-mode rotations $\theta_{1,2}$) form of DGCZ mixing, see Eqs. (2.37)-(2.38), $\Omega^{\text{DGCZ}} = \langle [\Delta \hat{a}]^2 (\Delta \hat{v})^2 \rangle_{\phi_1, \phi_2}$ gives the exact amount of the entanglement the state possesses, i.e., $E_N = E_N(\Omega^{\text{DGCZ}})$.

At the first attempt, backward BS rotation of the noise-area, alone, cannot mimic the DGCZ like criterion mixing given in Eqs. (2.37)-(2.38) for an arbitrary $\theta_{BS}$. However, we realize that a TMNc condition, e.g., $\Omega < 1$, is still a TMNc condition after a coordinate transformation $p_2 \rightarrow -p_2$. (Not to be mixed with partial transposition.) Because, $p_2 \rightarrow -p_2$ keeps a a normal-ordered expression still in normal-order which one utilizes for obtaining TMNc condition, see Eq. (2.30). Thus, an $\Omega < 1$ search behaves the same with the DGCZ and Mancini et al. type criterion. A similar method $p_2 \rightarrow -p_2$ is utilized for obtaining DGCZ like criteria from HUR or SR inequalities.\cite{33} 40\cite{41} for separable states. Here, however, $p_2 \rightarrow -p_2$ appears in a different concept: a TMNc criterion is still a TMNc criterion (but different one) via a preserved normal-ordering under $p_2 \rightarrow -p_2$ transformation. We further show that also the Hillery-Zubairy (HZ) criterion\cite{35}, which works good for number squeezed like states, search for a noise-area below unity, i.e., $\Omega_n < 1$.

The TMNc criteria obtained from Schrodinger-Robertson (SR) inequality via partial transposition are stronger than DGCZ\cite{32} 33 or HZ\cite{35} criteria. This is owing to an extra term $\langle (\Delta \hat{a} \Delta \hat{\phi})_S \rangle$ present in the SR inequality, see Eq. (2.36). Here, we show that this extra term takes partial care of the inefficiency in the $\nu$-$v$ mixing, Eqs. (2.37)-(2.38). Inefficiency originates form the choice of the coordinates, e.g., $\hat{x}_1, \hat{p}_1$, in the minimum squeezing directions, i.e., $\hat{x}_0, \hat{p}_0$. When the minimum squeezing directions are mixed in Eqs. (2.37)-(2.38), the extra term becomes zero. This is also true for the TME criteria obtained for dealing with number squeezed like states, see Eq. (3.14). In this case, optimization is carried in the $n_{1,2} - \Phi_{1,2}$ planes, the number of photons and phases, respectively.

Simon’s criteria\cite{35}, invariant under intra-mode rotations and $p_2 \rightarrow -p_2$ transformation, performs the pronounced optimization automatically. Thus, we utilize Simon’s criterion for obtaining a stronger TME criterion for number-phase squeezed like states by merely replacing $\hat{a} \rightarrow \hat{a}_n = (\hat{n} + \hat{\Phi})/\sqrt{2}$. The method can be applied also to other group of states, e.g., amplitude squeezed states\cite{53}.

Our findings illuminate the internal mechanism of entanglement criteria and their connection with the two-mode nonclassicality conditions where the latter appears as noise-area inequalities. We also raise some intriguing questions throughout the text, whose answers have fundamental importance in the development of new TME criteria which serve a basis for the quantum technology applications.
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Appendix A: Derivation of inequality \([3,23]\)

One can derive entanglement criteria with the help from the method by Mancini et al. \([33]\) and Raymer et al. \([34]\).

We consider two linearly mixed operators

\[
\begin{align}
\hat{u} &= \alpha \hat{A}_1 + \beta \hat{A}_2, \\
\hat{v} &= \alpha \hat{B}_1 + \beta \hat{B}_2,
\end{align}
\]

where \(\hat{A}_1, \hat{B}_1\) and \(\hat{A}_2, \hat{B}_2\) are observables randomly chosen from the subsystem 1 and subsystem 2, respectively. \(\alpha\) and \(\beta\) are any real numbers.

One can also derive a lower bound on the variance of operators \((A_{1a}, A_{1b})\) following the approach introduced by Mancini et al. \([33]\). One inserts the form

\[
\hat{\rho} = \sum_i P_i \hat{\rho}_i^{(1)} \otimes \hat{\rho}_i^{(2)}
\]

for the density matrix for separable states to obtain

\[
\begin{align}
\langle \hat{u}^2 \rangle &= \alpha^2 \sum_i P_i \langle \hat{A}_1^2 \rangle_i + \beta^2 \sum_i P_i \langle \hat{A}_2^2 \rangle_i + 2\alpha\beta \sum_i P_i \langle \hat{A}_1 \hat{A}_2 \rangle_i, \\
\sum_i P_i \langle \hat{v}^2 \rangle_i &= \sum_i P_i \left( \alpha^2 \langle \hat{A}_1^2 \rangle_i + \beta^2 \langle \hat{A}_2^2 \rangle_i + 2\alpha\beta \langle \hat{A}_1 \hat{A}_2 \rangle_i \right) \\
\langle \hat{u} \hat{v} \rangle &= \langle \hat{v} \hat{u} \rangle
\end{align}
\]

where \(\langle i \rangle\) implies the evaluation of the expectation with the quantum state, which has a classical statistical probability \(P_i\) to emerge. Similar expressions can be obtained for \(\langle \hat{v}^2 \rangle\) and \(\sum_i P_i \langle \hat{v} i \rangle^2\).

We subtract (add) the RHS (LHS) of Eq. \((A4)\) from (to) the RHS Eq. \((A3)\) and obtain the variance

\[
\langle (\Delta \hat{u})^2 \rangle = \alpha^2 \sum_i P_i \langle (\Delta \hat{A}_1)^2 \rangle_i + \beta^2 \sum_i P_i \langle (\Delta \hat{A}_2)^2 \rangle_i + \sum_i P_i \langle \hat{u}^2 \rangle_i - \langle \hat{u} \rangle^2.
\]

The second line of this equation is always greater than zero, due to Cauchy-Schwarz inequality

\[
\left( \sum_i P_i \right) \left( \sum_i P_i \langle \hat{u}^2 \rangle_i \right) \geq \left( \sum_i P_i \langle \hat{v} \rangle_{i} \right)^2.
\]

This implies the inequality

\[
\langle (\Delta \hat{u})^2 \rangle \geq \alpha^2 \sum_i P_i \langle (\Delta \hat{A}_1)^2 \rangle_i + \beta^2 \sum_i P_i \langle (\Delta \hat{A}_2)^2 \rangle_i.
\]

One can obtain the same inequality, Eq. \((A7)\), for \(\langle (\Delta \hat{v})^2 \rangle\), where the operators \(\hat{A}_{1,2}\) are replaced with \(\hat{B}_{1,2}\).

We use the identity \(a^2 + b^2 \geq 2|a||b|\) for the real numbers in the RHS of Eq. \((A7)\) to obtain

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \geq 4 \prod_i P_i |\alpha\beta| \sqrt{\langle (\Delta \hat{A}_1)^2 \rangle_i \langle (\Delta \hat{A}_2)^2 \rangle_i} \\
\times \sum_i P_i |\alpha\beta| \sqrt{\langle (\Delta \hat{B}_1)^2 \rangle_i \langle (\Delta \hat{B}_2)^2 \rangle_i}
\]

which can be rewritten as

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \geq 4\alpha^2 \beta^2 \left[ \sum_i P_i \left( \langle (\Delta \hat{A}_1)^2 \rangle_i \langle (\Delta \hat{A}_2)^2 \rangle_i \langle (\Delta \hat{B}_1)^2 \rangle_i \langle (\Delta \hat{B}_2)^2 \rangle_i \right) \right]^2
\]

using the Cauchy-Schwarz inequality. Last, we use the uncertainty relation for the products of operators belonging to the same subsystems. Finally, one obtains the inequality

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \geq \alpha^2 \beta^2 C_1 C_2
\]

where \(C_i = |\langle [\hat{A}_i, \hat{B}_i] \rangle|\). We note that this inequality is stronger than its sum form in Ref. \([33]\).

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