Brief paper

A multiple-comparison-systems method for distributed stability analysis of large-scale nonlinear systems

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A B S T R A C T
Lyapunov functions provide a tool to analyze the stability of nonlinear systems without extensively solving the dynamics. Recent advances in sum-of-squares methods have enabled the algorithmic computation of Lyapunov functions for polynomial systems. However, for general large-scale nonlinear networks it is yet very difficult, and often impossible, both computationally and analytically, to find Lyapunov functions. In such cases, a system decomposition coupled to a vector Lyapunov functions approach provides a feasible alternative by analyzing the stability of the nonlinear network through a reduced-order comparison system. However, finding such a comparison system is not trivial and often, for a nonlinear network, there does not exist a single comparison system. In this work, we propose a multiple comparison systems approach for the algorithmic stability analysis of nonlinear systems. Using sum-of-squares methods we design a scalable and distributed algorithm which enables the computation of comparison systems using only communications between the neighboring subsystems. We demonstrate the algorithm by applying it to an arbitrarily generated network of interacting Van der Pol oscillators.

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1. Introduction

A key to maintaining the successful operation of real-world engineering systems is to analyze the stability of the systems under disturbances. Lyapunov functions methods provide powerful tools to directly certify stability under disturbances, without solving the complex nonlinear dynamical equations (Haddad & Chellaboina, 2008; Lyapunov, 1892). However for a general nonlinear system, there is no universal expression for Lyapunov functions. Recent advances in sum-of-squares (SOS) methods and semi-definite programming (SDP), Papachristodoulou et al. (2013), Prajna, Papachristodoulou, Seiler, and Parrilo (2005) and Sturm (1999), have enabled the algorithmic construction of polynomial Lyapunov functions for nonlinear systems that can be expressed as a set of polynomial differential algebraic equations (Chesi, 2011; Tan, 2005). Unfortunately, such computational methods suffer from scalability issues and, in general, become intractable as the system size grows (Anderson, Chang, & Papachristodoulou, 2011). For this reason more tractable alternatives to SOS optimization have been proposed. One such approach, known as DSOS and SDSOS optimization, is significantly more scalable since it relies on linear programming and second order cone programming (Ahmadi & Majumdar, 2014). A different approach chooses Lyapunov functions with a chordal graphical structure in order to convert the semidefinite constraints into an equivalent set of smaller semidefinite constraints which can be exploited to solve the SDP programs more efficiently (Mason & Papachristodoulou, 2014). Nevertheless, the increased scalability decreases performance since both approximations are usually more conservative than SOS approaches.

Despite these computational advances, global analysis of large-scale systems remains problematic when computational and communication costs are considered. Often, a decomposition-aggregation approach offers a scalable distributed computing framework, together with a flexible analysis of structural perturbations (Šiljak, 1991) and decentralized control designs (Šiljak, 1978), as required by the locality of perturbations. Thus, for large-scale systems, it is often useful to model the system as a network of
small interacting subsystems and study the stability of the full interconnected system with the help of the Lyapunov functions of the isolated subsystems. For example, one approach is to construct a scalar Lyapunov function expressed as a weighted sum of the subsystem Lyapunov functions and use it to certify stability of the full system (Araki, 1978; Michel, 1983; Siljak, 1972; Weissenberger, 1973). However, such a method requires centralized computations and does not scale well with the size of the network. Alternatively, methods based on vector Lyapunov functions, Bailey (1966) and Bellman (1962), are computationally very attractive due to their parallel structure and scalability, and have generated considerable interest in recent times (Karafyllis & Papageorgiou, 2015; Kundu & Anghel, 2015a,c; Xu, Wang, Hong, Jiang, & Xu, 2016). However, applicability of these methods to large-scale nonlinear systems with guaranteed rate of convergence still remains to be explored. For example, in Kundu and Anghel (2015a,c) the authors consider asymptotic stability while the works in Karafyllis and Papageorgiou (2015) and Xu et al. (2016) are demonstrated on small-scale systems.

Inspired by the results on comparison systems, Beckenbach and Bellman (1961), Brauer (1961) and Conti (1956), it has been observed that the problem of stability analysis of an interconnected nonlinear system can be reduced to the stability analysis of a linear dynamical system (or, ‘single comparison system’) whose state space consists of the subsystem Lyapunov functions. Success in finding such stable linear comparison system then guarantees exponential stability of the full interconnected nonlinear system. However, for a given interconnected system, computing these comparison systems still remained a challenge. In absence of suitable computational tools, analytical insights were used to build those comparison systems, such as trigonometric inequalities in power systems networks (Jocic, Ribbens-Pavella, & Siljak, 1978). In a recent work (Kundu & Anghel, 2015b), SOS-based direct methods were used to compute the single comparison system for generic nonlinear polynomial systems, with some performance improvements over the traditional methods. However, in major challenges before such a method can be used in large-scale systems. For example, it is generally difficult to construct a single comparison system that can guarantee stability under a wide set of disturbances. Also, while Kundu and Anghel (2015b) present a decentralized analysis where the computational burden is shared between the subsystems, the scalability of the analysis is largely dependent on the cumulative size of the neighboring subsystems.

In this article we present a novel conceptual and computational framework which generalizes the single comparison system approach into a sequence of stable comparison systems, that collectively ascertain stability, while also offering better scalability by parallelizing the subsystem-level SOS problems. The set of multiple comparison systems are to be constructed adaptively in real-time, after a disturbance has occurred. With the help of SOS and semi-definite programming methods, we develop a fully distributed, parallel and scalable algorithm that enables computation of the comparison systems under a disturbance, with only minimal communication between the immediate neighbors. While this approach is applicable to any generic dynamical system, we choose an arbitrarily generated network of modified1 Van der Pol oscillators (Van der Pol, 1926) for illustration. Under a disturbance, the subsystems communicate with their neighbors to algorithmically construct a set of multiple comparison systems, the successful construction of which can certify stability of the network. The rest of this article is organized as follows. Following some brief background in Section 2 we describe the problem in Section 3. We present the traditional approach to single comparison systems and an SOS-based direct method of computing the comparison systems in Section 4. In Section 5, we introduce the concept of multiple comparison systems, and propose a parallel and distributed algorithmic construction of the comparison systems in real-time. We demonstrate an application of this algorithm to a network of Van der Pol oscillators in Section 6, before concluding the article in Section 7.

2. Preliminaries

Let us consider the dynamical system

\[ \dot{x}(t) = f(x(t)), \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad f(0) = 0, \]  

(1)

with an equilibrium at the origin.2 and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is locally Lipschitz. Let us use \(| \cdot |\) to denote both the Euclidean norm (for a vector) and the absolute value (for a scalar).

Definition 1. The equilibrium point at the origin is said to be asymptotically stable in a domain \( D \subseteq \mathbb{R}^n, 0 \in D \), if \( \lim_{t \to \infty}|x(t)| = 0 \) for every \( |x(0)| \in \mathcal{D} \), and it is exponentially stable if there exists \( b, c > 0 \) such that \( |x(t)| < ce^{-bt} |x(0)| \) for every \( |x(0)| \in \mathcal{D} \).

Theorem 1 (Lyapunov, 1892, Khalil, 1996, Thm. 4.1). If there exists a domain \( \mathcal{D} \subseteq \mathbb{R}^n, 0 \in \mathcal{D} \), and a continuously differentiable positive definite function \( \hat{V} : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0} \), i.e. the ‘Lyapunov function’ (LF), then the origin of (1) is asymptotically stable if \( \nabla^T \hat{V}(x) \) is negative definite in \( \mathcal{D} \), and is exponentially stable if \( \nabla^T \hat{V}(x) \leq -\alpha \hat{V} \forall x \in \mathcal{D}, \) for some \( \alpha > 0 \).

The region of attraction (ROA) of the stable equilibrium point at origin can be (conservatively) estimated as Genesio, Tartaglia, and Vicino (1985)

\[ \mathcal{D} := \{ x \in \mathcal{D} | V(x) \leq 1 \}, \quad \text{with} \quad V(x) = \hat{V}(x)/\gamma_{\text{max}}, \]  

(2a)

where \( \gamma_{\text{max}} := \max \{ \gamma \mid \{ x \in \mathbb{R}^n | \hat{V}(x) \leq \gamma \} \subseteq \mathcal{D} \} \),  

(2b)

i.e. the boundary of the ROA is estimated by the unit level-set of a suitably scaled LF \( V(x) \).

Relative recent studies have explored how sum-of-squares (SOS) based methods can be utilized to find LFs by restricting the search space to SOS polynomials (Anghel, Milano, & Papachristodoulou, 2013; Jarvis-Wloszek, 2003; Parrilo, 2000; Tan, 2006). Let us denote by \( \mathbb{R}[x] \) the ring of all polynomials in \( x \in \mathbb{R}^n \).

Definition 2. A multivariate polynomial \( p \in \mathbb{R}[x], x \in \mathbb{R}^n \), is a sum-of-squares (SOS) if there exist some polynomial functions \( h_i(x), i = 1 \ldots s \) such that \( p(x) = \sum_{i=1}^{s} h_i^2(x) \). We denote the ring of all SOS polynomials in \( x \in \mathbb{R}^n \) by \( \Sigma[x] \).

Checking if \( p \in \mathbb{R}[x] \) is an SOS is a semi-definite problem which can be solved with a MATLAB® toolbox SOSTOOLS (Papachristodoulou et al., 2013; Prajna et al., 2005) along with a semidefinite programming solver such as Sedumi (Sturm, 1999). The SOS technique can be used to search for polynomial LFs by restricting the search conditions to Theorem 1 to equivalent SOS conditions (Chesi, 2010; Jarvis-Wloszek, 2003; Papachristodoulou et al., 2013; Papachristodoulou & Prajna, 2005; Prajna et al., 2005; Włodzicki, Feeley, Tan, Sun, & Packard, 2005). An important result from algebraic geometry, called Putinar’s Positivistensatz theorem (Lasserre, 2009; Putinar, 1993), helps in translating the SOS conditions into SOS feasibility problems. Putinar’s Positivistensatz theorem states (see Lasserre, 2009, Ch. 2):

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1 Parameters are chosen to make the equilibrium point stable.

2 Note that by shifting the state variables any equilibrium point of interest can be moved to the origin.
Theorem 2. Let $\mathcal{X} = \{ x \in \mathbb{R}^n | k_i(x) \geq 0, \ldots, k_m(x) \geq 0 \}$ be a compact set, where $k_i \in \mathbb{R}[x], \forall j \in [1, \ldots, m]$. Suppose there exists a $\mu \in \{ a_0 + \sum_{j=1}^m a_j k_j | a_0, a_j \in \mathbb{R}[x], \forall j \}$ such that $\{ x \in \mathbb{R}^n | \mu(x) \geq 0 \}$ is compact. Then, if $p(x) > 0 \forall x \in \mathcal{X}$, then $p \in \{ a_0 + \sum_{j=1}^m a_j k_j | a_0, a_j \in \mathbb{R}[x], \forall j \}$.

Remark 1. Using Theorem 2, we can translate the problem of checking that $p > 0$ on $\mathcal{X}$ into an SOS feasibility problem where we seek the SOS polynomials $a_0, a_j$ such that $p - \sum_{j=1}^m a_j k_j$ is SOS.

The polynomial LFs, $V_i \in \mathbb{R}[x], \forall i$, for the isolated subsystems, $\dot{x}_i = f_i(x_i), \forall i$, are computed using an SOS-based expanding interior algorithm (Anghel et al., 2013; Jarvis-Wloszek, 2003) (alternatively, the methods in Chesi, 2011, Tibken, 2000 can be used), with the isolated ROAs

$$\mathcal{A}^0 = \{ x_i \in \mathbb{R}^n | V_i(x_i) \leq 1 \}, \forall i \in [1, 2, \ldots, m].$$

The $V_i$ satisfy, for some $\eta_1, \eta_2, \eta_3 > 0$ and $\mathcal{X} \subset \mathcal{A}^0$,

$$\forall x_i \in \mathcal{A}^0, \eta_1 |x_i|^{\eta_2} \leq V_i(x_i) \leq \eta_2 |x_i|^{\eta_3},$$

and

$$\nabla^2 V_i f_i \leq -\eta_3 |x_i|^\eta_3.$$
4. Single comparison system

Let us first review the traditional approach towards stability analysis of interconnected dynamical systems using a single comparison system (or, single CS). In Araki (1978), Jocic et al. (1978), Siljak (1972) and Weissinger (1973), and related works, authors laid out a formulation of the linear CS using certain conditions on the LFs and the neighbor interactions. It was observed that if there exists a set of LFs, \( v_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \) \( i = 1, 2, \ldots, m, \) satisfying the following

(11a) \[ \forall (i,j) : \tilde{\eta}_1 \cdot |x_i| \leq v_i(x_i) \leq \tilde{\eta}_2 \cdot |x_i|, \quad \forall \xi_i \in \tilde{\xi}_1 \subset \mathbb{R}^d, \]

(11b) \[ (\nabla v_i)^T f_j \leq -\tilde{\eta}_3 \cdot |x_j|, \quad \forall \xi_j \in \tilde{\xi}_2 \subset \mathbb{R}^d, \]

and \[ |(\nabla v_i)^T g_j| \leq \tilde{\xi}_4 \cdot |x_j|, \quad \forall \xi_j \in \tilde{\xi}_2, \xi_j \in \tilde{\xi}_2, \]

(11c)

for some \( \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3 > 0 \) and \( \tilde{\xi}_4 \geq 0 \) \( \forall (i,j), \) with \( \tilde{\xi}_4 = 0 \) \( \forall j \not\in \mathcal{N}_i, \) then the corresponding vector LF \( v = [v_1, v_2, \ldots, v_m]^T \) satisfies a CS on the domain \( \mathcal{D} = \{ x \in \mathbb{R}^n | x_i \in \tilde{\xi}_1 \} \), with a comparison matrix \( \tilde{A} = [\tilde{a}_{ij}] \) given by

(12)

If \( \tilde{A} \) is Hurwitz, then any invariant domain \( \mathcal{D} \subseteq \tilde{\mathcal{D}} \) is an estimate of a region of exponential stability (Jocic et al., 1978; Weissinger, 1973).

Although the traditional approach provides nice analytical insights into the construction of the comparison matrix, it suffers from practical limitations. One has to first compute the bounds in (11) which, interestingly, are not satisfied by any polynomial LFs. However, one can compute the bounds in (6) satisfied by the polynomial LFs, and then construct a set of non-polynomial LFs as Jocic et al. (1978) and Weissinger (1973)

(13a) \[ \forall (i,j) : v_i = \sqrt[\gamma]{V_i} \text{satisfies} (11) \]

(13b) \[ \tilde{\eta}_1 = \sqrt[\gamma]{\tilde{\eta}_1}, \quad \tilde{\eta}_2 = \sqrt[\gamma]{\tilde{\eta}_2}, \quad \tilde{\eta}_3 = \sqrt[\gamma]{\tilde{\eta}_3}, \quad \tilde{\xi}_4 = \sqrt[\gamma]{\tilde{\xi}_4}. \]

Further, the bounds in (11), while convenient for analytical insights, need not be optimal for constructing a Hurwitz comparison matrix. For example, in (11c), \( |(\nabla v_i)^T g_j| \) is a function of both \( x_i \) and \( x_j \) but is bounded using only \( |x_i| \).

4.1. SOS-based direct computation

SOS-based techniques can be used to resolve some of the issues that arise with the traditional approach (Kundu & Anghel, 2015b). The idea is to compute the single CS in a decentralized way, using directly the LFs \( V_i \in \mathbb{R}[x_i] \) (Anghel et al., 2013; Jarvis-Wloszek, 2003), which however do not satisfy the conditions in (11). For convenience, let us introduce, for all \( i \in \{1, \ldots, m\} \), the following notations,

\[ 0 \leq a_2 < a_1 \leq 1 : \mathcal{G}_1[a_1] := \left\{ x \in \mathbb{R}^n | V_i(x_i) \leq a_1 \right\}, \]

\[ \mathcal{G}_2[a_1] := \left\{ x \in \mathbb{R}^n | V_i(x_i) = a_1 \right\}, \]

(14b) and \( \mathcal{G}_1[a_2, a_1] := \left\{ x \in \mathbb{R}^n | a_2 < V_i(x_i) \leq a_1 \right\}. \)

Given a set of \( \gamma^0 \in (0, 1] \), we want to construct the single CS in a distributed way by calculating each row of the comparison matrix \( A \in \mathbb{R}^{m \times m} \) (with non-negative off-diagonal entries) locally at each subsystem level, such that,

\[ \forall i \in \{1, \ldots, m\} : \hat{V}_i \leq \sum_{j \in \mathcal{N}_i} a_{ij} V_j \text{ on } \mathcal{G}_1[\gamma^0]. \]

(15)

**Proposition 1.** The domain \( \bigcap_{i=1}^m \mathcal{G}_1[\gamma^0] \) is an estimate of the ROA of the interconnected system in (3) if for each \( i \in \{1, \ldots, m\} \), \( \sum_{j \in \mathcal{N}_i} a_{ij} \gamma^0 \leq 0 \) and \( \sum_{j \in \mathcal{N}_i} a_{ij} \gamma^0 < 0 \).

**Proof.** Because of the non-negative off-diagonal entries and \( \sum_{j \in \mathcal{N}_i} a_{ij} < 0 \forall i, \) the application of Gershgorin’s Circle theorem (Bell, 1965; Gershgorin, 1931) states that the comparison matrix \( A = [a_{ij}] \) is Hurwitz. Further, note that whenever \( V_i(x_i(t)) = \gamma_i^0 \), for some \( i \), and \( V_i(x_i(t)) \leq \gamma_i \forall t \neq i, \) for some \( \gamma > 0, \) we have \( V_i(x_i(t)) \leq 0 \forall t < 0 \), i.e. the (piecewise continuous) trajectories can never cross the boundaries defined by \( \bigcap_{i=1}^m \mathcal{G}_1[\gamma^0] \).

**Remark 2.** Henceforth, we will loosely refer to the conditions of the form \( \sum_{j \in \mathcal{N}_i} a_{ij} \gamma^0 < 0 \) as the ‘Hurwitz conditions’, while the conditions of the form \( \sum_{j \in \mathcal{N}_i} a_{ij} \gamma^0 < 0 \) will be referred to as the ‘invariance conditions’.

**Proposition 2.** If for some LFs \( v_i, i \in \{1, \ldots, m\} \), there exists a comparison matrix \( A = [a_{ij}], \) with \( \tilde{a}_i + \sum_{j \in \mathcal{N}_i} \tilde{a}_j \tilde{c}_{ij} < 0 \forall i \in \{1, \ldots, m\} \), for some \( \tilde{c}_{ij} > 0 \forall i \not\in \mathcal{N}_j \), then, for any LFs \( V_i = (v_i)^d \forall i, d \geq 1, \) the existence of a comparison matrix \( A = [a_{ij}] \) is guaranteed, with \( \tilde{a}_i + \sum_{j \in \mathcal{N}_i} a_j (c_{ij})^d < 0 \forall i. \)

**Proof.** Note from Proposition 1, that choosing \( \tilde{c}_{ij} = 1, \gamma^0 / \gamma_i^0, \) or \( \max(\gamma^0 / \gamma_i^0, 1) \) we may retrieve the Hurwitz condition, the invariance condition, or simultaneously both, respectively. The proof follows directly after we show that \( V_i \in \{1, \ldots, m\} \).

Motivated by Propositions 1–2, we propose an SOS-based direct computation of the comparison matrix \( A = [a_{ij}] \) in (15), in which each subsystem calculates the corresponding row entries of the matrix \( A \), by solving the following SOS feasibility problem (using Theorem 2):

\[-\nabla V_i^T (f_i + g_i) + \sum_{j \in \mathcal{N}_i} (a_{ij} V_j - \sigma_j (\gamma_j^0 - V_j)) \in \Sigma[\hat{x}_i], \]

(16a)

\[ -\sum_{j \in \mathcal{N}_i} a_{ij} \in \Sigma[0], \]

(16b)

and \[ -\sum_{j \in \mathcal{N}_i} a_{ij} \gamma_j^0 \in \Sigma[0]. \]

(16c)

where, \( \sigma_j \in \Sigma[\hat{x}] \forall j \in \mathcal{N}_i, \) \( a_{ij} \in \mathbb{R} \),

\[ a_{ij} \in \Sigma[0] \forall j \in \mathcal{N}_i \].

(16d)

Here \( \mathbb{R}[0] \) denotes scalars, \( \Sigma[0] \) denotes non-negative scalars and \( \hat{x}_i \) were defined in (4). If (16) is feasible for each \( i \in \{1, \ldots, m\} \), then the origin is exponential stable and the domain \( \bigcap_{i=1}^m \mathcal{G}_1[\gamma^0] \) is an estimated ROA.

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5. \( a^1/b^{1/q} \leq a/p + b/q \) for \( a, b > 0, p > 1 \) and \( 1/p + 1/q = 1. \)
4.2 Limitations

The SOS-based single CS approach, proposed in Kundu and Anghel (2015b), has certain conceptual and computational limitations. Note that larger ‘self-decay rates’, introduced in (8), favor the CS approach. However, the function $\alpha_j(\cdot)$ is generally non-monotonic, i.e. smaller level-sets need not yield larger self-decay rates (see Fig. 2(c) in Section 6). Consequently, finding a single set of scalars $a_j \forall i, j \in A_i$, that satisfy the comparison inequalities, including (16b)–(16c), in a large domain $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$ could be difficult. Instead, we propose a better approach involving multiple comparison systems (or, multiple CSs). Secondly, the proposed direct approach requires solving subsystem-level SOS problems that involve all the state variables associated with the neighborhoods. Consequently, presence of a large neighborhood can severely affect the overall computational speed, and scalability, of the analysis. A pairwise-interactions based approach can further reduce the computational burden at the subsystems by reducing the size of the SOS problem. Finally, it is not clear how to find the scalars $\gamma_i^0 \forall i$ that define $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$, the domain of definition of the CS. It is logical that the set of values for $\gamma_i^0 \forall i$ should be found adaptively, given a disturbance, so that the domain $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$ takes a shape that resembles the particular disturbance. In Section 5 we propose a novel framework to circumvent these limitations.

5. Multiple comparison systems

In this section, we formulate a generalized CSs approach in which we use a sequence of CSs to collectively certify stability under given disturbances. We also propose a framework to parallelize the subsystem-level SOS problems using the pairwise interactions. Fig. 1 illustrates the basic idea behind our proposed formulation. Given any invariant domain $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$, $k = 0, 1, 2, \ldots$, the idea is to find the next invariant domain $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^{k+1}]$, with $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^{k+1}] \subseteq \bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$ such that any trajectory starting from $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$ converges exponentially on $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^{k+1}]$. This is done in a distributed way in which each subsystem computes its next invariant level-set and communicates that value to its neighbors, until all the sequences of level-sets converge (to zero, for asymptotic stability). The idea is that, with $\sum_{j \in A_i} a_{ij}^k \leq 0 \forall k, i$ and $\sum_{j \in A_i} a_{ij}^k (\gamma_j^{k+1} - \gamma_j) < 0 \forall i$, then the system trajectories converge exponentially to $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^*]$ where $\gamma_i^* \forall i$ is the limit of the monotonically decreasing sequence of non-negative scalars $\{\gamma_i^k\}, k \in \{0, 1, 2, \ldots\}$.

**Proof.** Note that the CSs can be written compactly as, $\forall k : \dot{V}_i \leq A^k (V_i - \gamma_i^{k+1})$, where $A^k = [a_{ij}^k]$ and $\gamma_i^{k+1} = \{\gamma_1^{k+1}, \gamma_2^{k+1}, \ldots, \gamma_m^{k+1}\}$. The conditions $\sum_{j \in A_i} a_{ij}^k \leq 0 \forall k, i$ and $\sum_{j \in A_i} a_{ij}^k (\gamma_j^{k+1} - \gamma_j) < 0 \forall i$ imply that $A^k$ is Hurwitz and $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^{k+1}]$ is invariant. Hence, for each $k$, the system trajectories starting inside $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$ converge exponentially to $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^{k+1}]$, while always staying within $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^0]$. Note that now the new state variables in the comparison system are $(V_i - \gamma_j^{k+1})$. As the subsystems cross the level sets $\gamma_i^{k+1}$, the comparison system changes. Indeed, let us say, the subsystems cross into $\gamma_j^{k+1}$ in the order $\{1, 2, 3, \ldots\}$. After subsystem 1 crosses, the new comparison system is

$$\dot{V}_{2:m} \leq A_{2:m} \tilde{V}_{2:m}$$

where $\tilde{V}_{2:m} = \begin{bmatrix} (V_2 - \gamma_2^{k+1}) \\ \vdots \\ (V_m - \gamma_m^{k+1}) \end{bmatrix}$, $A_{2:m} = \begin{bmatrix} a_{22}^k & \ldots & a_{2m}^k \\ \vdots & \vdots & \vdots \\ a_{m2}^k & \ldots & a_{mm}^k \end{bmatrix}$.

Note that each matrix in the sequence $A_{2:m}, \ldots, A_{m:m}$ remains Hurwitz. Moreover, regardless of the order in which the subsystems cross the level sets $\gamma_i^{k+1}$, the sequence of $A$ matrices remains Hurwitz, thus proving finite time convergence to the new level sets.

**Corollary 1.** $\gamma_i^* = 0 \forall i$ implies exponential stability.

Such a formulation, however, is difficult to implement in a distributed algorithm, since the computation of the i-th row of the comparison matrices at the iteration $k$ requires that subsystem $i$ has knowledge of the $\gamma_j^{k+1}$ of $j \in A_i \setminus \{i\}$ of its neighbors. A possible approach could be, for each iteration-$k$, compute the $k$th CS iteratively, i.e. using iterations within iterations. But in this work, we restrict ourselves to a simpler formulation, by seeking only diagonal comparison matrices.

**Lemma 3.** If the subsystem LF s of the system in (3) satisfy $\forall (k, i) : \dot{V}_i \leq a_i^k (V_i - \gamma_i^{k+1})$ on $\bigcap_{j \in A_i \setminus \{i\}} \mathcal{A}[\gamma_j^{k+1}]$ with $a_i^k < 0 \forall k, i$, then the system trajectories converge exponentially to $\bigcap_{i=1}^{m} \mathcal{A}[\gamma_i^*]$ where $\gamma_i^* \forall i$ is the limit of the monotonically decreasing sequence of non-negative scalars $\{\gamma_i^k\}, k \in \{0, 1, 2, \ldots\}$.

**Proof.** If the subsystem $i$ satisfies $\gamma_i^{k+1} < \gamma_i^k \forall k, i$ which immediately proves necessity. The sufficiency follows from the continuity of the polynomial functions.

**Corollary 2.** Exponential stability is guaranteed if $\alpha_i(\gamma_i^k) > \max_{e \in \mathcal{E}[\gamma_i^k]} \nabla V_i^g \forall (k, i)$. 

![Fig. 1. Distributed coordinated sequential stability certification.](image-url)
The computation of the multiple CSs in Lemma 3 involves two phases. In Phase 1, we search for the level-sets \( \gamma_i^0 \in [0, 1) \forall i \) such that the system trajectories starting from any initial point converge to these level-sets. We do this by finding the invariant envelope \( \gamma_i^0 \) for each subsystem, which is the smallest \( \gamma_i^0 \) such that \( \forall i \) the system trajectories starting from some initial level-sets will always converge to \( \gamma_i^0 \). In Phase 2, we determine the diagonal comparison matrices \( \hat{D} \) based on the pairwise interaction terms. Notably, the final comparison matrices \( \hat{D} \) are guaranteed to be stable if the system trajectories starting from some initial level-sets converge to \( \hat{D} \) at an exponential rate.

5.1. Distributed construction

5.1.1. Phase 1: find the invariant envelope \( \gamma_i^0 \)

We search for the smallest \( \gamma_i^0 \) \( \in [0, 1) \) that satisfies the following inequality:

\[
\forall i : \hat{V}_i \leq 0 \text{ on } \gamma_i^0 \text{ and } \gamma_i^0 = \bigcap_{k \in \mathbb{N}(i)} \gamma_i^0.
\]

This requires knowledge of the neighbors' expanded level-sets, and hence can only be solved iteratively. The inequality is solved for each subsystem-\( i \), finding the smallest \( \gamma_i^0 \) that satisfies the following inequality:

\[
\forall i : \hat{V}_i \leq 0 \text{ on } \gamma_i^0 \text{ and } \gamma_i^0 = \bigcap_{k \in \mathbb{N}(i)} \gamma_i^0.
\]

5.1.2. Phase 2: find the diagonal comparison matrices

With the invariant envelope \( \gamma_i^0 \) already found, we can compute the multiple CSs in Lemma 3. At each iteration \( k \), each subsystem-\( i \) computes \( a_i \) and the smallest \( \gamma_i^{k+1} \) using a bisection-search on \( \gamma_i^{k+1} \) over \( \gamma_i^0, \gamma_i^1 \), which satisfies:

\[
\forall (k, i) : \hat{V}_i \leq a_i(V_i - \gamma_i^{k+1}) \text{ with } a_i^k < 0,
\]

where the decision variables are defined as:

\[
\text{s.t. } \forall i \in \mathbb{N}(i), \gamma_i^{k+1} \in \gamma_i^0 \text{ and } \gamma_i^{k+1} = \bigcap_{k \in \mathbb{N}(i)} \gamma_i^0.
\]

5.2. Distributed parallel construction

The computational complexities in the SOS problems in Section 5.1 are largely dominated by the dimension of the state-space of the associated neighborhood. To circumvent this limitation, we propose a parallel formulation of the SOS problems based on the pairwise interactions. Note that the problem can be solved in parallel on each subsystem,\( i \), with each subsystem solving the SOS problem in parallel.

Lemma 4. If the subsystem LF's of the system in (3) satisfy

\[
\forall (k, i, j) : \hat{V}_i(V_i + \gamma_i^{k+1}) \leq a_i^+, a_i^+ \geq 0,
\]

on \( \gamma_i^0 \) and \( \gamma_i^{k+1} \) with \( \sum_{k \in \mathbb{N}(i)} a_i^+ < 1 \) and \( a_i^+ < 0 \forall (k, i) \), then the system trajectories converge exponentially to \( \gamma_i^{k+1} = 0 \forall i \).
Appendix for details), the subsystem dynamics in the presence of the neighbor interactions are given by
\[
f_i(x_i) = \left[ x_{i,2} + \mu_i x_{i,2} (c_i^{(1)} - c_i^{(2)} x_{i,1} + c_i^{(3)} x_{i,1}^2) \right]^T
\]
where, \( c_i^{(1)} = 1 - (0.5 c_i^{(2)})^2 \), \( c_i^{(3)} = 1 - \sum_{j \neq i} \beta_{ij} (0.5 \beta_{ij} c_i^{(2)} - \beta_{ij}^{(1)}) \), \( \mu_i \) and \( \beta_{ij}^{(1)} \) and \( \beta_{ij}^{(2)} \) are chosen randomly and \( c_i^{(2)} \) are related to the equilibrium point before shifting. Polynomial LFs for the isolated (no interaction) subsystems are computed using the expanding interior algorithm (Section 3). Fig. 2(a) shows that a quartic LF estimates the ‘true’ ROA of the isolated subsystems (obtained via time-reversal simulation) better than a quadratic LF. However, a better estimate of the isolated ROAs does not necessarily translate into better stability certificates for the interconnected system, as illustrated later. The ‘self-decay rates’ are plotted in Fig. 2(a)–(c) for a range of level-sets from 0 to 1. For each subsystem, as \( \gamma_i \) approaches 1, \( \alpha_i \gamma_i^2 \) approaches 0. Thus it is impossible to obtain a Hurwitz comparison matrix when the initial conditions lie close to the boundary of the estimated ROAs. Moreover, note that the evolution of the self-decay rates is generally non-monotonic. An attempt to find a single CS valid all the way to the origin is generally difficult, since the row-sum values of the single comparison matrix will be limited by the lowest self-decay rate. A multiple CS approach, however, can still guarantee exponential convergence to some level-sets close to the origin.

In Fig. 3, we compare the traditional and the direct approaches of computing a CS (using the quadratic LFs). Choosing \( \gamma_1 = \gamma_2 = \cdots = \gamma_9 = 0 \), and varying their values, we compute the comparison matrices in (15) using SOS-based direct approach in (16), by replacing the constraint (16b) by an objective of minimizing \( \sum_i \omega_i \gamma_i \). Also, we find the comparison matrices using the traditional approach, where the maximum of the real parts of the eigenvalues (denoted by \( \text{Re}(\lambda) \)) and the maximum row-sum of the comparison matrices are plotted, for varying level-sets. Clearly, the SOS-based direct method yields improved stability certificates. Further note that the maximal (uniform) level-set for which both the maximum row-sum value and the maximum of \( \text{Re}(\lambda) \) are negative gives an estimate of the ROA of the full interconnected system. Thus \( \bigcap_{i=1}^9 \gamma_i^{0.683} \) is an estimate of the ROA. Next we compare the performances of the single CS approach and the multiple CS approach (with and without the parallel computation), in Fig. 4. For each subsystem, we plot the maximal (uniform) level-set for which either the row-sum is negative (single CS), or a strict convergence is achieved at iteration-0, i.e. \( \gamma_i^* < \gamma_i^{0.1} \) (multiple CS). When not considering the parallel implementation, the multiple CS approach outperforms the single CS in estimating the invariance (since we focus only at iteration-0). However, the parallel implementation, while achieving computational tractability for larger systems, yields more conservative certificates.

Next we use an example to illustrate a couple of key observations. Fig. 5 shows the stability analysis results on an arbitrarily generated initial condition (or disturbance) using both quadratic and quartic LFs, and multiple CS. Note in Fig. 5(b) that the initial level-set lies outside the estimated ROA obtained in Fig. 3. By allowing the analysis to be dependent on the particular initial condition, we are able to find a suitably shaped stability region. Further note that, while the quadratic LFs-based multiple CS analysis certifies exponential stability (in Fig. 5(b)), the quartic LFs can only guarantee exponential convergence to a domain \( \bigcap_{i=1}^9 \gamma_i^{0.016} \) very close to the origin, with \( \gamma_i^* = 0.023 \) and \( \gamma_i^* = 0.016 \) (in Fig. 5(c)). In fact, this domain of convergence is characteristic of the system and the (quartic) LFs used, and is independent of the initial condition. Referring to Fig. 2(c), the low self-decay rates of the quartic LFs for subsystems 2 and 3 explain the convergence away from the origin (Corollary 2). Thus, while the quartic LFs...
may yield better estimates of the isolated ROAs (Fig. 2(a)) compared to the quadratic LFs, and hence are able to analyze a larger number of initial conditions, they fail to certify exponential stability. This suggests that it may be useful to switch from quartic LFs to the quadratic LFs as the $\gamma^k V(k, i)$ decrease. We also noted that, in the quartic LF-based analysis subsystems-6 and 7 underwent an expansion (Phase 1) from $(\gamma_0^6, \gamma_0^7) = (0.015, 0.002)$ to $(\gamma_0^6, \gamma_0^7) = (0.151, 0.013)$, while none of the subsystems underwent expansion using quadratic LF-based analysis.

**Remark 4.** Note that the algorithm estimates the exponential convergence rates rather conservatively. While this issue may be resolved by constraining the row-sum values to be less than some chosen negative number, it will likely delay the convergence of the stability algorithm.

### 7. Conclusion

We have used vector Lyapunov functions to design an iterative, distributed and parallel algorithm to certify exponential stability of a nonlinear network, under initial disturbances. The algorithm requires one-time computation of Lyapunov functions of the isolated subsystems, and minimal real-time communications between the neighboring subsystems. It is shown that the proposed SOS-optimization based direct approach towards computation of the single comparison system leads to less conservative certificates than the traditional methods. Further, a generalization has been proposed via multiple comparison systems, which has been found to yield improved results compared to the single comparison system approach. Using the pairwise interactions, a parallel implementation is also proposed, which enables the algorithm to scale up smoothly with the size of the largest neighborhood. The distributed stability analysis algorithm has been tested on an arbitrary network of nine Van der Pol systems, using vectors of quadratic and quartic Lyapunov functions. It is easy to visualize a multi-agent distributed coordinated control framework where each subsystem (‘agent’) will coordinate with its neighbors to design local control policies to stabilize the system under disturbances. Finally, it would be interesting to explore the applicability of the proposed algorithm on real-world problems, such as the transient stability analysis of large-scale structure-preserving power system networks.

**Appendix. Model description**

Subsystem dynamics in the original state variables (i.e., before shifting), $\hat{x}_i = (\hat{x}_{i1} \hat{x}_{i2})^T$, are given by $\dot{\hat{x}}_i = f(\hat{x}_i) + \sum_{j \in N_i \setminus \{i\}} g_{ij}(\hat{x}_i, \hat{x}_j) \forall i$, where $f(\hat{x}_i) = [\hat{x}_{i2}, \mu_i \hat{x}_{i2} (1 - \hat{x}_{i1}) - \hat{x}_{i1}]^T$ and $g_{ij}(\hat{x}_i, \hat{x}_j) = c_{ij} + \tilde{p}_{ij}^{(1)} (\hat{x}_{i1} - \hat{x}_{j1}) + \tilde{p}_{ij}^{(2)} \hat{x}_{i1} \hat{x}_{j1} \mu_i \in [-3, -1]$, $c_{ij} \in [-0.2, 0.2]$, $\tilde{p}_{ij}^{(1)} \in [-0.1, 0.1]$ and $\tilde{p}_{ij}^{(2)} \in [-0.1, 0.1]$ are chosen randomly. By shifting the equilibrium point $\hat{x}_i^* = \sum \gamma_k^i (1 - \sum \gamma_k^j) + 0 \forall i$, we obtain (23), where $c_i^{(2)} = 2 \sum \gamma_k^j (1 - \sum \gamma_k^j)$, and $\beta_{ij}^{(1)} = 0.5 \beta_{ij}^{(2)} c_i^{(2)} - \tilde{\gamma}_i^{(1)}$.

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**Fig. 5.** Analysis of (a) a given stable initial condition using (b) quadratic and (c) quartic LFs, and multiple CSs.
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