Jeans equations with account of gravitational field correlations

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In this paper, we would like to adjust this result taking into account anisotropic pressure and turbulent motion of the gravitating gas of stars on the post-Newtonian approximation. This approach was used earlier in description of a giant molecular cloud [7].

First who considered turbulence to generalize the expression for the Jeans wavelength was Chandrasekhar [8]. From his estimation [8] it follows that the gravitational constant, \( G \)

where \( G \) is the gravitational constant, \( \rho_0 \) is the equilibrium value of the density of the gas of stars, \( \sigma \) is the velocity dispersion.

However, more appropriate corrections to the Jeans theory can be found taking into account anisotropic pressure and turbulent motion of the gravitating gas of stars on the post-Newtonian approximation. In these papers, small relativistic corrections were found to low from the first order post-Newtonian theory [4, 5].

Recently, the Jeans instability was analysed within the framework of Boltzmann and Poisson equations that follow from the first order post-Newtonian theory [1, 2]. In these papers, small relativistic corrections were found to low from the first order post-Newtonian theory [4, 5].

In this paper we would like to generalize an approach to analyse the stability of the homogeneous and isotropic cloud of stars, being in stable state. Switching to the non-relativistic limit on the first-order post-Newtonian (1PN) approximation, we assume that the gravitational field to be self-consistent, so that in equilibrium it has zero first correlation moment and non-zero second one. We take into account anisotropy of pressure and turbulent-like motion of stars and investigate low-frequency acoustic oscillations with small amplitudes in such a cloud and apply the Jeans theory to analyse the stability of the system.

Further paper is organized as follows. In Section II the wave equation describing the acoustic oscillations in the cloud of stars is derived. In Section III acoustic oscillations in the cloud are studied on the 1PN approximation of the Einstein equations with account of the self-consistent gravitational field. Results are applied to observational data for star clusters. In Section IV conclusions are given.

\[ k_J = \frac{\sqrt{4\pi G \rho_0}}{\sigma}, \]

\[ \lambda_C = \sqrt{\frac{\pi (v_s^2 + \frac{1}{3} \langle v^2 \rangle_0)}{G \cdot \rho_0}}, \]
\[
\frac{\partial f}{\partial t} + v_i \partial_i f + \frac{\partial f}{\partial v_i} F_i = 0.
\]

Here and further, bold letters denote three-vectors in ordinary three-dimensional Euclidean space. Latin indexes take on values 1, 2, 3 and mark spatial coordinates, repetitive indexes are summed; \(\partial_k\) is the partial derivative with respect to the \(k\)-th coordinate; \(\partial_k = \partial/\partial x_k\); \(F_i\) are components of the force of gravitational interaction. It is convenient to express the force in terms of derivative from the momentum flux density of the gravitational field \(\pi_{ik}^g\):

\[
F_i = \partial_k \pi_{ik}^g.
\]

The momentum flux density of the gravitational field is defined as follows

\[
\pi_{ik}^g = \frac{1}{4\pi G} \frac{g_i g_k - g_k g_i}{g_{lk}},
\]

where \(g_{ik}\) are components of the gravitational field strength \(\mathbf{g}\). In the expression \((5)\), the 1PN approximation for the Landau-Lifshitz pseudotensor is used so that the gravitomagnetic field and time derivatives of the Newtonian potential are neglected (see \([10]\), Eq. (4.1b)). As it will be shown below, the gravitational field strength \(\mathbf{g}\) is not a potential vector.

The self-consistent gravitational field in the cloud is a random quantity. After averaging over a physically small spatial value, the first correlation moment of the gravitational field strength equals zero, unlike the second moment which is non-zero.

Using the definition \((4)\), we rewrite \((3)\) in the form

\[
\frac{\partial f}{\partial t} + v_i \partial_i f + \frac{\partial f}{\partial v_i} \partial_k \pi_{ik}^g = 0.
\]

The Boltzmann equation \((6)\) gives equations for velocity correlation moments. Introducing the definition for the mass density

\[
\rho = m \int f d^3v,
\]

and the mass velocity components

\[
\rho v_i = m \int v_i f d^3v,
\]

where \(m\) is the average star mass, and taking the first (continuity equation) and the second moments of \((6)\), we find the following Jeans equations \([9]\)[11]:

\[
\partial_t \rho + \partial_i (\rho v_i) = 0,
\]

\[
\partial_t (\rho v_i) + \partial_k \pi_{ik} = 0,
\]

where \(\pi_{ik}\) is the total momentum flux density of the system. Equation \((10)\) corresponds to the equation \((4-27)\) in \([11]\), but with account of the transverse gravitational field.

In the equation \((10)\), the force of gravitational interaction is completely defined by the second correlation moment of the gravitational field strength \(\langle g_i g_k \rangle\).

Here \(\langle v_i v_k \rangle = \frac{\pi_{ik}}{\rho}\) is the velocity second correlation moment, and \(\frac{\pi_{ik}}{\rho g} = \sigma_{ik}\) is the velocity central second correlation moment, which is called the velocity dispersion. Similarly, the second correlation moment of the gravitational field strength is \(\langle g_i g_k \rangle = \frac{\pi_{ik}}{\rho g}\), and the central second correlation moment of the gravitational field strength is \(\frac{\pi_{ik}}{\rho g}\). By analogy with the theory of turbulence, the term \(-\rho \pi_{ik} v_k\) is a stress tensor that describes the anisotropic pressure and corresponds to the Reynolds stress tensor in the theory of turbulence \([12]\).

It is a common approach not to take into account the pressure anisotropy in \((10)\) and to deal with ordinary pressure that satisfies the state equation. However, one should not be so sure of the laminarity of flows in the cloud of stars. Because there is no viscosity, that would cause laminaritity, the nonstationary flows in such a system become vortical. Besides, taking into account both potential and transverse gravitational fields, the transverse anisotropic effects within the correspondent order accuracy should be included into the stress tensor in a medium.

Multiplying \((6)\) by \(v_i v_k\), we find a temporal equation for the velocity second correlation moment. In equilibrium, the homogeneous and isotropic system is described by the following parameters: \(\rho_0 = \text{const}\), \(\langle v_i v_k \rangle_0 = \text{const}\), \(\langle g_i g_k \rangle_0 = \text{const}\) and \(v_0 = 0\). We are interested in small deflections from these equilibrium values, so we find from \((6)\)

\[
\partial_t \rho \langle v_i v_k \rangle = -\partial_i \rho \langle v_i v_k v_l \rangle.
\]

The third correlation moment arises here, which, due to isotropy, can be expressed in the form \(\langle v_i v_k v_l \rangle = \langle v_i v_k v_l \rangle_0 + v_k \langle v_i v_l \rangle_0 + v_l \langle v_i v_k \rangle_0\). The equilibrium second correlation moment of the velocity in the case of isotropy has a tensor structure:

\[
\langle v_i v_m \rangle_0 = \frac{1}{3} \langle v^2 \rangle_0 \delta_{im} = \text{const}.
\]

Thus, \((12)\) with account of \((9)\) gives the temporal equation

\[
\partial_t \langle v_i v_k \rangle = -\frac{1}{3} \partial_l (v_i \delta_{kl} + v_k \delta_{il}) \langle v^2 \rangle_0.
\]
In turbulence theory, the similar to (14) equation for the Reynolds stress includes the production term that represents a transfer of energy from the mean flow to the fluctuating velocity field.

Treating the gravitational field as a random value, we shall further obtain the temporal equation for the value \( \langle g_1 g_k \rangle \) from the Einstein equations. Let us formulate the conditions of the passage to the limit of the weak field approximation. The metric composed of the Minkowski metric \( \eta_{\alpha \beta} \) with small deviations \( h_{\alpha \beta} \ll 1 \) reads

\[
g_{\alpha \beta} = \eta_{\alpha \beta} + h_{\alpha \beta}, \tag{15}
\]

where Greek indexes take on values 0, 1, 2, 3 and mark the components of 4-vectors. Zero component of the partial derivative is \( \partial_0 = \frac{1}{c} \partial_t \).

In the non-relativistic case, all velocities are much less than the speed of light, and \( \partial_t h_{\alpha \beta} \gg \partial_t \eta_{\alpha \beta} \).

On the 1PN approximation, (see e.g. [10], Eq. (2.5a)) the "electric-like" part of gravitational field is described by the strength

\[
g_i \varepsilon_i = \frac{1}{2} \partial_t g_{00} - \partial_0 g_{0i}, \tag{16}
\]

and "magnetic-like" part ([10], (2.5b)) reads

\[
H_i = \varepsilon_{ijk} \partial_j g_{0k}. \tag{17}
\]

To write the Einstein equations on the 1PN approximation it is convenient to use the metric components definition made in [15] (Eqs. (3.3)). Following then [10] (Eq. (2.6d)), it is possible to write the temporal equation for gravitational field strength in the form

\[
\partial_t g_i - \frac{c}{4} [\nabla \times \mathbf{H}]_i = 4\pi G \rho v_i, \tag{18}
\]

where \( \nabla \) is a vector operator with components \( \partial_i \).

Multiplying (18) by \( g_k \) at the same point of the spacetime and making the symmetrization by tensor indexes, we find

\[
\partial_t g_i - \frac{c}{4} [\nabla \times \mathbf{H}]_i g_k = 4\pi G \rho v_i. \tag{19}
\]

We now make the statistical averaging of the equation (19). For this purpose, we need to find the part of the gravitomagnetic field that arises due to the motion of particles with the velocity \( \mathbf{v} \) in the Newtonian gravitational field \( \mathbf{g}_N \), where \( g_{Ni} = \frac{1}{2} \partial_t g_{00} c^2 \). As far as values \( (g_{00}, g_{0i}) \) compose a 4-vector, we switch to the medium intrinsic frame of reference so that the reference frame moves with the velocity \( \mathbf{v} \) at a given point.

On the non-relativistic approximation, the Lorentz transformations give the relation \( g_{0i} = g_{0i} - u_i g_{00}, u_i = -v_i/c \). The derivatives with respect to the spatial coordinates do not change under these transformations, thus we find for the gravitomagnetic field of the system in rest \( \mathbf{H'} = \mathbf{H} - \frac{c}{2} \mathbf{v} \times \mathbf{g}_N \). As far as in convoving reference frame \( \mathbf{H'} = 0 \), the gravitomagnetic field can be expressed in terms of the Newtonian field needed for the non-relativistic approximation \( \mathbf{H} = \frac{2}{c} \mathbf{v} \times \mathbf{g}_N \):

\[
\partial_t (g_i g_k) = \frac{1}{2} \nabla \times [\mathbf{v} \times \mathbf{g}_N], g_k \\
+ \frac{1}{2} [\nabla \times (\mathbf{v} \times \mathbf{g}_N)] g_i \\
+ 4\pi G \rho (v_i g_k + v_k g_i). \tag{20}
\]

In an analogous way, the equation for the second moment of the magnetic induction was obtained on the magnetohydrodynamic approximation in [10], and the equation for the second moment of the electric field strength was found on the electrohydrodynamic approximation in [17].

The isotropy and the negligible thermal fluctuations require the following equilibrium average values of the first and second correlation moments of the field

\[
\langle g_i \rangle_0 = 0, \quad \langle g_{N1} g_{N1} \rangle_0 = \langle g^2 \rangle_0 \frac{1}{3} \delta_{lm} = \text{const}. \tag{21}
\]

When linearising (20), the last term there has the second order of smallness by amplitude due to its non-linearity, so it can be neglected. Using (21), we obtain from (20) the following linearised equation for the second correlation moment of the gravitational field strength

\[
\partial_t \langle g_i g_k \rangle = \frac{1}{6} \partial_t \langle v_i + \partial_t v_k - 2 \partial_0 v_j \delta_{ik} \rangle \langle g^2 \rangle_0. \tag{22}
\]

linearising the Euler equation (10), we find

\[
\partial_t v_i + \frac{1}{\rho_0} \partial_k \left( \rho_0 \langle v_i v_k \rangle + \frac{\delta_{ik}}{4\pi G} \frac{\langle g^2 \rangle_0}{\rho} \right) + \frac{\langle g_i g_k \rangle - \frac{1}{2} \langle g^2 \rangle_0 \delta_{ik}}{4\pi G} = 0. \tag{23}
\]

With account of (9), (14) and (22), we differentiate (23) with respect to the time and obtain the equation for small acoustic oscillations in isotropic cloud of stars:

\[
\partial_t^2 v_i - \partial_t \partial_k \langle v_i \delta_{kl} + v_k \delta_{il} \rangle \left( \frac{1}{3} \langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{24\pi G \rho_0} \right) - \frac{1}{3} \langle v^2 \rangle_0 \partial_t \partial_k v_k = 0. \tag{24}
\]

### III. SOLUTION

Solving the equation (24), it is convenient to switch to Fourier components according to the rule

\[
\mathbf{v}(x, t) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \, d\omega \, \mathbf{v}(k, \omega) e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t}. \tag{25}
\]
Substituting (25) into (24), we obtain dispersion equations for longitudinal acoustic oscillations:

$$\omega^2 = \left( \langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{12 \pi G \rho_0} \right) k^2,$$

and two branches of transverse acoustic oscillations:

$$\omega^2_1 = \frac{1}{3} \left( \langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{8 \pi G \rho_0} \right) k^2.$$  

(26)

(27)

Taking into account correlations of the gravitational field and the turbulence of flows in stable cloud in a stationary state, the equation (26) gives the following relation for the modified velocity of the longitudinal acoustic wave:

$$u_\parallel = \sqrt{\langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{12 \pi G \rho_0}},$$

and (27) gives the velocity of the transverse acoustic wave:

$$u_\perp = \sqrt{\frac{1}{3} \left( \langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{8 \pi G \rho_0} \right)}.$$  

(28)

(29)

If the frequency of perturbation $\omega$ in (26) or in (27) becomes imaginary, the correspondent mode will increase. Expressing (28) through (29) as follows

$$u_\parallel = \sqrt{\frac{1}{3} \langle v^2 \rangle_0 + 2 u_\perp^2},$$

one can see that $u_\parallel$ always exists when $u_\perp$ exists. Hence, the very transverse perturbations (27) determine the stability of the system. The stability condition thus follows from (29)

$$\langle v^2 \rangle_0 - \frac{\langle g^2 \rangle_0}{8 \pi G \rho_0} \geq 0.$$  

(30)

(31)

The energy density of the gravitational field is given by (18)

$$W = -\frac{g^2}{8 \pi G},$$

hence the cloud is stable when the following condition is satisfied

$$\langle v^2 \rangle_0 \rho_0 = 2T \geq -W,$$

where $T = \frac{1}{2} \langle v^2 \rangle_0 \rho_0$ is the average kinetic energy density.

Thus, the cloud of stars stability limit $u_\perp = 0$ coincides with the virial theorem $2T = -W$ (see e.g. (11), Eq.(4-81), [4], Eq.(9.11)). Transverse waves do not exist in this limit, and the speed of longitudinal sound is

$$u_\parallel = \sqrt{\frac{1}{3} \langle v^2 \rangle_0}.$$  

If the condition (31) is not satisfied, transverse modes may increase the kinetic energy until the onset of the equality $u_\perp = 0$.

Using (31), it is also possible to estimate the size of the stable spherical cloud of stars. The gravitational energy of the spherical cloud with the radius $R$ and the mass $M = \frac{4}{3} \pi \rho_0 R^3$ reads

$$U = -\int_0^R \frac{G}{r} M dM = -\frac{16}{15} G \rho_0^2 \pi^2 R^5.$$  

(32)

(33)

Its density can be estimated as

$$W = -\frac{4}{5} G \rho_0^2 \pi R^2.$$  

(34)

(35)

Then the condition (31) gives the following restriction on the radius of the gravitationally bound spherical cloud:

$$\langle v^2 \rangle_0 \frac{W}{\rho_0} = \langle v^2 \rangle_0 - \frac{4}{5} G \rho_0 \pi R^2 \geq 0,$$

from where one has

$$R \leq \sqrt{\frac{5}{2} \langle v^2 \rangle_0}{2 \sqrt{\pi G \rho_0}},$$

that is analogous to the Chandrasekhar condition for with turbulence included (see [8], Eq. (24)).

Taking $\lambda_C$ (2) as a limiting diameter of the stable spherical cloud of stars, we obtain the Chandrasekhar radius

$$R_C = \frac{\lambda_C}{2} \leq \sqrt{\frac{5}{2} \langle v^2 \rangle_0}{2 \sqrt{3 \pi G \rho_0}},$$

(36)

(37)

The correspondent expression for the energy of the gravitational interaction

$$W_C = G \rho_0^2 \pi \frac{\langle v^2 \rangle_0}{15 G \rho_0} = \rho_0 \frac{\pi^2}{15} \langle v^2 \rangle_0$$

does not satisfy the virial theorem

$$2T = \langle v^2 \rangle_0 \rho_0 \neq W_C = \rho_0 \langle v^2 \rangle_0 \frac{\pi^2}{15}.$$  

(38)

(39)

The relation (37) allows to adjust the Jeans mass $M_J$ arising from the Chandrasekhar condition (38) (see [11], Eq. (5-24)).

$$\frac{M_J}{M_C} = \frac{R^3}{R_C^3} = \left( \frac{\sqrt{15}}{\pi} \right)^3 \approx 1.87.$$  

(40)

(41)

This increases the mass of the stable gravitating system and is much greater than the weak relativistic corrections found in [4].

In Table I, the compiled data from Messier catalogue [19], SIMBAD astronomical database [20] and HyperLEDA database for globular clusters [21] is used to find estimative values of mean square velocities of stars $\sqrt{\langle v^2 \rangle_{\text{est}}}$ with respect to the condition (37). Observed values $\sqrt{\langle v^2 \rangle_{\text{obs}}}$ were taken from [22] and Holger Baumgardt globular cluster database [26].
Table I. Structure parameters of some globular clusters. M is the cluster mass in units of masses of the Sun; \( \tau \) is the estimative age of the cluster; R is the average radius of the cluster; N is estimative number of stars in the cluster; \( \sqrt{\langle v^2 \rangle_{\text{obs}}} \) and \( \sqrt{\langle v^2 \rangle_{\text{est}}} \) are observed and estimated by (37) mean square velocities of stars in the cluster in [km/s].

| Name      | M \([10^5 M_\odot]\) | \( \tau \) [Gyr] | R [pc] | N \([10^5]\) | \( \sqrt{\langle v^2 \rangle_{\text{obs}}} \) [km/s] | \( \sqrt{\langle v^2 \rangle_{\text{est}}} \) [km/s] |
|-----------|---------------------|-----------------|--------|--------------|------------------|------------------|
| NGC 6121  | 8.71                | 12.2            | 11.55  | 0.3          | 2.0              | 2.0              |
| NGC 6864  | 3.70                | >13             | 20.54  | 0.4          | 5.0              | 5.0              |
| NGC 6218  | 1.10                | 13.8            | 11.34  | 0.7          | 5.0              | 4.0              |
| NGC 6779  | 2.30                | 13.7            | 12.88  | 0.8          | 4.5              | 4.0              |
| NGC 7078  | 6.33                | 12.5            | 26.98  | 1.0          | 9.0              | 3.8              |
| NGC 7089  | 6.20                | >13             | 26.67  | 1.5          | 8.2              | 3.8              |
| NGC 6266  | 12.2                | 11.8            | 15.02  | 1.5          | 13.7             | 5.8              |
| NGC 1851  | 3.18                | 9.20            | 19.93  | 2.0          | 7.9              | 5.1              |
| NGC 6093  | 3.38                | 12.5            | 14.72  | 2.0          | 9.0              | 5.9              |
| NGC 7006  | 3.03                | >10             | 21.55  | 2.5          | 3.0              | 5.5              |
| NGC 5272  | 4.10                | 11.4            | 27.29  | 5.0          | 4.8              | 6.9              |
| NGC 5024  | 8.26                | 12.7            | 33.74  | 5.0          | 3.8              | 6.2              |
| NGC 0104  | 7.00                | 13.6            | 18.40  | 5.0          | 10.5             | 8.4              |
| NGC 5904  | 8.57                | >13             | 24.53  | 5.0          | 5.7              | 7.3              |
| NGC 2808  | 14.2                | 12.5            | 19.62  | 10.0         | 14.2             | 11.5             |
| NGC 6715  | 17.8                | >13             | 46.91  | 10.0         | 14.0             | 7.4              |

Figure 1. Cluster mass as a function of cluster radius for the data from HyperLeda and Harris [25] catalogues (circles). Diamonds indicate massive distant clusters listed in Table I. The dashed line shows the best fit empiric law \( M = 0.02 R^{1.72} \) for the whole data. The solid line is for diamond only, plotted according to the fitting law \( M = \frac{3}{4} \pi \langle \rho \rangle R^3 \). The average cluster density \( \langle \rho \rangle \) obtained from the best fit is \( \langle \rho \rangle = 70M_\odot\text{pc}^{-3} \).

IV. CONCLUSIONS

To summarise, a hydrodynamic model describing small oscillations in the cloud of stars is built on the basis of Jeans equations. On the non-relativistic limit of 1PN approximation of Einstein equations with account of transverse part of the gravitational field strength, expressions are found for one longitudinal and two transverse branches of acoustic oscillations in the cloud.

When the velocity of transverse oscillations turns to zero, the stability condition arises for the cloud of stars, which requires the kinetic energy density to be equal to half of the gravitational interaction energy density, that exactly corresponds to the virial theorem. The radius of the stable spherical cloud of stars is estimated (37). Obtained results were applied to compiled observational data for globular clusters. The estimated mean square velocities of stars found in Table I do not contradict the observational data. The average cluster density \( \langle \rho \rangle \) for distant massive clusters is estimated to be about \( 70M_\odot\text{pc}^{-3} \).

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