UNIQUENESS AND STABILITY OF TIME-PERIODIC PYRAMIDAL FRONTS FOR A PERIODIC COMPETITION-DIFFUSION SYSTEM

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Abstract. The existence, non-existence and qualitative properties of time periodic pyramidal traveling front solutions for the time periodic Lotka-Volterra competition-diffusion system have already been studied in $\mathbb{R}^N$ with $N \geq 3$. In this paper, we continue to study the uniqueness and asymptotic stability of such time-periodic pyramidal traveling front in the three-dimensional whole space. For any given admissible pyramid, we show that the time periodic pyramidal traveling front is uniquely determined and it is asymptotically stable under the condition that given perturbations decay at infinity. Moreover, the time periodic pyramidal traveling front is uniquely determined as a combination of two-dimensional periodic V-form waves on the edges of the pyramid.

1. Introduction. Recently, there have been great progress on the study of multi-dimensional traveling wave solution of competition or cooperative-diffusive system in high-dimensional space and many new types of nonplanar traveling waves have been observed for autonomous reaction-diffusion systems. For example, see Wang [41] and Wang et al.[44] for the existence, uniqueness and stability of two-dimensional traveling front solutions and three-dimensional pyramidal traveling front of bistable reaction-diffusion system for any admissible wave speed, also see Ni and Taniguchi [29] for the existence of pyramidal traveling fronts of competition-diffusion system with constant coefficient in $N$-dimensional space ($N \geq 3$). By utilizing the results in pyramidal traveling fronts, Wang et al.[45] has established the existence and qualitative properties of axisymmetric traveling fronts for competition-diffusion system. More recently, for cooperative system in $\mathbb{R}^3$ ($N \geq 3$), Taniguchi [39] has showed that there exist traveling wave solutions associated with a given $(N - 2)$-dimensional smooth surface with the boundaries being compact set in $\mathbb{R}^{N-1}$. Compared with autonomous systems, little attention has been paid to non-autonomous reaction-diffusion system. For time-periodic case, Bao et al. [3] recently proved the existence, uniqueness and stability of two-dimensional periodic V-form waves on the edges of the pyramid.

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traveling fronts of time periodic Lotka-Volterra system for \( N = 2 \). More recently, the existence, non-existence and qualitative properties of time-periodic pyramidal traveling fronts for time periodic Lotka-Volterra competition-diffusion system in \( N \)-dimensional space \( (N \geq 3) \) have been established, see \cite{1}. The purpose of the current paper is to show the uniqueness and asymptotic stability of time-periodic pyramidal traveling fronts for the time periodic competition diffusion system in three-dimensional whole space.

Consider the following time-periodic Lotka-Volterra competition-diffusion system,

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \Delta u_1 + u_1(x,t)(r_1(t) - a_1(t)u_1(x,t) - b_1(t)u_2(x,t)), \\
\frac{\partial u_2}{\partial t} &= d\Delta u_2 + u_2(x,t)(r_2(t) - a_2(t)u_1(x,t) - b_2(t)u_2(x,t)),
\end{align*}
\tag{1.1}
\]

where \( \Delta \) denotes \( \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \) and \( d \) is a positive constant, the parameters \( a_i(t), b_i(t) \) and \( r_i(t) \) \((i = 1, 2)\) satisfy the following basic hypothesis,

(A1) \( r_i(t), a_i(t), b_i(t) \in C^\theta(\mathbb{R}) \) \((i = 1, 2)\) are periodic functions, that is, \( r_i(t+T) = r_i(t), a_i(t+T) = a_i(t) \) and \( b_i(t+T) = b_i(t) \) for any \( t \in \mathbb{R} \), where \( \theta \in (0, 1) \). In addition, \( a_i(t) > 0 \) and \( b_i(t) > 0 \) for any \( t \in [0, T] \) and \( \tau_i = \frac{1}{T} \int_0^T r_i(t) dt > 0 \).

(A2) \( \tau_1 < \min \left( \frac{b_1(t)}{a_1(t)} \right) \tau_2, \tau_2 < \min \left( \frac{a_2(t)}{b_2(t)} \right) \tau_1 \).

(A3) \( \tau_1 + \tau_2 > \max \left( \frac{a_1(t)}{b_1(t)} \right) \tau_1, \tau_1 + \tau_2 > \max \left( \frac{b_2(t)}{a_2(t)} \right) \tau_2 \).

Under the assumptions (A1)-(A3), the corresponding kinetic system of (1.1)

\[
\begin{align*}
\frac{d}{dt} u_1(t) &= u_1(t)(r_1(t) - a_1(t)u_1(t) - b_1(t)u_2(t)), \\
\frac{d}{dt} u_2(t) &= u_2(t)(r_2(t) - a_2(t)u_1(t) - b_2(t)u_2(t))
\end{align*}
\tag{1.2}
\]

has four nonnegative period \( T \) solutions \((0,0), (p(t),0), (0,q(t))\) and a unique coexistence state \((u_1^*(t), u_2^*(t))\), see \cite{2, 48}, for the explicit expression of \( p(t) \) and \( q(t) \). In particular, the two semitrivial periodic solutions \((p(t),0)\) and \((0,q(t))\) are stable and the coexistence state \((u_1^*(t), u_2^*(t))\) is unstable and satisfies that \( 0 < u_1^*(t) < p(t) \) and \( 0 < u_2^*(t) < q(t) \) for \( t \in [0, T] \), see \cite{2, 21}. It is known from \cite{2} that if the assumptions (A1)-(A3) hold, there exists a unique function \((\Phi_1(\cdot, \cdot), \Phi_2(\cdot, \cdot), c)\) of (1.1) in one-dimensional space satisfying

\[
\begin{align*}
\frac{\partial \Phi_1}{\partial t} + c \frac{\partial \Phi_1}{\partial \xi} - \frac{\partial^2 \Phi_1}{\partial \xi^2} - \Phi_1(\xi,t)(r_1(t) - a_1(t)\Phi_1(\xi,t) - b_1(t)\Phi_2(\xi,t)) &= 0, \\
\frac{\partial \Phi_2}{\partial t} + c \frac{\partial \Phi_2}{\partial \xi} - d \frac{\partial^2 \Phi_2}{\partial \xi^2} - \Phi_2(\xi,t)(r_2(t) - a_2(t)\Phi_1(\xi,t) - b_2(t)\Phi_2(\xi,t)) &= 0, \\
\lim_{\xi \to -\infty} (\Phi_1(\xi,t), \Phi_2(\xi,t)) &= (0, q(t)), \quad \lim_{\xi \to \infty} (\Phi_1(\xi,t), \Phi_2(\xi,t)) = (p(t), 0), \\
\Phi_1(\xi, T+t) &= \Phi_1(\xi, t), \quad \Phi_2(\xi, T+t) = \Phi_2(\xi, t), \quad \forall (\xi, t) \in \mathbb{R}^2,
\end{align*}
\tag{1.3}
\]

where the function \((\Phi_1(\cdot, \cdot), \Phi_2(\cdot, \cdot)) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2\) is the wave profile and the constant \( c \in \mathbb{R} \) is the wave speed. In addition, \( \frac{\partial \Phi_1}{\partial \xi}(\xi,t) > 0 \) and \( \frac{\partial \Phi_2}{\partial \xi}(\xi,t) < 0 \) in \((\xi, t) \in \mathbb{R}^2\). We remark that when one of two solutions \((p(t),0)\) and \((0, q(t))\) is unstable and another is stable, Zhao and Ruan \cite{48} have established the existence, uniqueness and stability of time-periodic traveling wave solutions of (1.3) connecting two solutions \((p(t),0)\) and \((0, q(t))\). See \cite{4, 12, 13, 22, 28, 40, 27} and
references therein for more results on the one-dimensional traveling wave solutions of autonomous Lotka-Volterra competition-diffusion system and \cite{26, 47, 25} and references therein for the stability of traveling wave solution for reaction-diffusion equations.

We further assume that
\[(A4)\quad c > 0, \text{ where } c \text{ is the wave speed in (1.3).}
\]
Under the assumptions \((A1)-(A4)\), the authors have established the existence of time periodic pyramidal traveling fronts of (1.1) in \cite{1}, see Theorem 1.1 later. Here we further study the uniqueness and asymptotic stability of time periodic pyramidal traveling wave for (1.1).

Without loss of generality, we assume that the solutions travel towards \(-x_3\) direction. Let
\[ u_i(\mathbf{x}, t) = w_i(\mathbf{x}', x_3 + st, t), \quad \mathbf{x}' = (x_1, x_2), \quad \mathbf{x} = (\mathbf{x}', x_3), \quad i = 1, 2. \]

Then we have the following initial value problem
\[
\begin{aligned}
&\frac{\partial w_1}{\partial t} = \Delta w_1(\mathbf{x}, t) - s \frac{\partial w_1}{\partial x_3} + w_1(r_1(t) - a_1(t)w_1 - b_1(t)w_2), \\
&\frac{\partial w_2}{\partial t} = d\Delta w_2(\mathbf{x}, t) - s \frac{\partial w_2}{\partial x_3} + w_2(r_2(t) - a_2(t)w_1 - b_2(t)w_2), \\
w_1(\mathbf{x}, 0) = u_{10}(\mathbf{x}), \quad w_2(\mathbf{x}, 0) = u_{20}(\mathbf{x})
\end{aligned}
\]
where \(\mathbf{x} \in \mathbb{R}^3, t > 0\). Let \(u_0(\mathbf{x}) = (u_{10}(\mathbf{x}), u_{20}(\mathbf{x}))\). We write the solution of (1.4) with initial value \(u_0(\mathbf{x}) \in C(\mathbb{R}^3, \mathbb{R}^2)\) as \((w_1(\mathbf{x}, t; u_0), w_2(\mathbf{x}, t; u_0))\). Since the curvature often accelerates the speed, we fix \(s > c > 0\). For \(s < c\), we have showed that maybe not exist time periodic pyramidal traveling fronts of (1.1) in some special case, see \cite[Theorem 1.3]{1}.

To describe the problems studied and the results obtained in the current paper, let \(n \geq 3\) be a given integer and
\[ m_\ast = \frac{\sqrt{s^2 - c^2}}{c}. \]
Assume \(A_j = (A_j, B_j) \in \mathbb{R}^2\) satisfies \(A_j^2 + B_j^2 = 1\) for all \(j = 1, \ldots, n\) and
\[ A_jB_{j+1} - A_{j+1}B_j > 0, \quad 1 \leq j \leq n - 1; \quad A_nB_1 - A_1B_n > 0. \tag{1.5} \]
Then \((m_\ast A_j, 1)\) is the normal vector of \(\{x \in \mathbb{R}^3 \mid x_3 = m_\ast(A_jx_1 + B_jx_2)\}\). Set
\[ h_j(\mathbf{x}') := m_\ast(A_j, \mathbf{x}') \quad \text{and} \quad h(\mathbf{x}') := \max_{1 \leq j \leq n} h_j(\mathbf{x}') = m_\ast \max_{1 \leq j \leq n} (A_j, \mathbf{x}') \tag{1.6} \]
for \(\mathbf{x}' \in \mathbb{R}^2\). We call \(\{x \in \mathbb{R}^3 \mid x_3 = h(\mathbf{x}')\}\) a three-dimensional pyramid in \(\mathbb{R}^3\). For any \(j = 1, \ldots, n\), let
\[ \Omega_j = \{x' \in \mathbb{R}^2 \mid h(\mathbf{x}') = h_j(\mathbf{x}')\}, \]
then we have \(\mathbb{R}^2 = \bigcup_{j=1}^n \Omega_j\). Denote the boundary of \(\Omega_j\) by \(\partial \Omega_j\). Let \(E = \bigcup_{j=1}^n \partial \Omega_j\).

Now we set
\[ S_j := \{x \in \mathbb{R}^3 \mid x_3 = h_j(\mathbf{x}') \text{ for } \mathbf{x}' \in \Omega_j\} \]
for each \(j = 1, \cdots, n\), and call \(\bigcup_{j=1}^n S_j \subset \mathbb{R}^3\) the lateral surfaces of a pyramid. Denote
\[ \Gamma_j := S_j \cap S_{j+1}, \quad \Gamma_n := S_n \cap S_1, \quad j = 1, \cdots, n - 1. \]
Then \(\Gamma := \bigcup_{j=1}^n \Gamma_j\) represents the set of all edges of a pyramid. For each \(\gamma > 0\), we define
\[ D(\gamma) = \{x \in \mathbb{R}^3 \mid \text{dist} (x, \bigcup_{j=1}^n \Gamma_j) > \gamma\}. \]
We note that the above setting on a pyramid comes from Taniguchi [35]. For any
\( j = 1, \ldots, n, (\Phi_1(\frac{x_3 + h_j(x')}{s}), t), \Phi_2(\frac{x_3 + h_j(x')}{s}) \) is the planar traveling
front solution of (1.1). Let
\[
\begin{align*}
  u_1^-(x, t) &:= \Phi_1 \left( \frac{c}{s} (x_3 + h(x')), t \right) = \max_{1 \leq j \leq n} \Phi_1 \left( \frac{c}{s} (x_3 + h_j(x')), t \right), \\
  u_2^-(x, t) &:= \Phi_2 \left( \frac{c}{s} (x_3 + h(x')), t \right) = \min_{1 \leq j \leq n} \Phi_2 \left( \frac{c}{s} (x_3 + h_j(x')), t \right).
\end{align*}
\]
Consequently, \((u_1^-(x, t), u_2^-(x, t))\) is a subsolution of (1.4) in the sense that
\[
\begin{align*}
  \frac{\partial u_1^-}{\partial t} &\geq \Delta u_1^- (x, t) - s \frac{\partial u_1^-}{\partial x_3} + u_1^- (r_1 (t) - a_1 (t) u_1^- - b_1 (t) u_2^-), \\
  \frac{\partial u_2^-}{\partial t} &\leq d \Delta u_2^- (x, t) - s \frac{\partial u_2^-}{\partial x_3} + u_2^- (r_2 (t) - a_2 (t) u_1^- - b_2 (t) u_2^-).
\end{align*}
\]
In particular, \(\frac{\partial u^-}{\partial x_3} (x, t) > 0, \frac{\partial u^-}{\partial x_3} (x, t) < 0\) and \(u_i^- (\cdot, \cdot + T) \equiv u_i^- (\cdot, \cdot) \) \((i = 1, 2)\) for \(x \in \mathbb{R}^s\) and \(t \in \mathbb{R}^t\).

The existence of time periodic pyramidal traveling fronts is proved in [1].

**Theorem 1.1** (See [1]). Assume that (A1)-(A4) hold. Then for each \(s > c\), there exists a solution \((u_1(x, t), u_2(x, t)) = (U_1(x', x_3 + st, t), U_2(x', x_3 + st, t))\) of (1.1) satisfying
\[
\begin{align*}
  \frac{\partial U_1}{\partial t} &= \Delta U_1 (x, t) - s \frac{\partial U_1}{\partial x_3} + U_1 (r_1 (t) - a_1 (t) U_1 - b_1 (t) U_2), \\
  \frac{\partial U_2}{\partial t} &= d \Delta U_2 (x, t) - s \frac{\partial U_2}{\partial x_3} + U_2 (r_2 (t) - a_2 (t) U_1 - b_2 (t) U_2), \\
  U_i (x, t + T) &= U_i (x, t), \quad (x, t) \in \mathbb{R}^s \times \mathbb{R}^t, \quad i = 1, 2,
\end{align*}
\]
\(u_1(x, t) > u_1^-(x, t), u_2(x, t) < u_2^-(x, t), \quad \forall x \in \mathbb{R}^s, t > 0\)

and
\[
\lim_{\gamma \to \infty} \sup_{x \in D(t), t \in [0, T]} |U_1(x, t) - u_1^-(x, t)| + |U_2(x, t) - u_2^-(x, t)| = 0. \tag{1.8}
\]

Following from (1.8), we know that the nonplanar traveling wave \((U_1, U_2)\) has pyramidal structures and is characterized as a combination of time periodic planar traveling fronts on the lateral surface. The following theorem further shows that such time periodic pyramidal traveling front \((U_1, U_2)\) is unique and asymptotically stable.

**Theorem 1.2.** In addition to the assumptions as in Theorem 1.1, if the initial value \(u_0 = (u_{10}(x), u_{20}(x)) \in C(\mathbb{R}^s, \mathbb{R}^s) with (u_{10}(x), u_{20}(x)) \in [0, p(0)] \times [0, q(0)]\) satisfies
\[
\lim_{\gamma \to \infty} \sup_{x \in D(t)} |u_{10}(x) - u_1^- (x, 0)| + |u_{20}(x) - u_2^- (x, 0)| = 0, \tag{1.9}
\]
then the solution \(u(x, t; u_0) := (u_1(x, t; u_0), u_2(x, t; u_0))\) of (1.4) satisfies
\[
\lim_{k \to +\infty} \|u(x, t + kT; u_0) - U(x', \cdot + st, t)\|_{C(\mathbb{R}^s \times [0, T])} = 0.
\]
Moreover, \((U_1(x', \cdot + st, t), U_2(x', \cdot + st, t))\) as in Theorem 1.1 is uniquely determined by (1.7) and (1.8).
We end the introduction with the following remarks. Firstly, Theorems 1.1-1.2 show that the time periodic pyramidal traveling front $(U_1, U_2)$ exists and is asymptotically stable. Since it is uniquely determined, now we call that $(U_1(x', x_3 + st, t), U_2(x', x_3 + st, t))$ is the pyramidal traveling wave associated with a pyramid $-x_3 = h(x')$. In the end of Section 4, we further characterize the time periodic pyramidal traveling fronts as a combination of two-dimensional periodic V-form traveling fronts on the edges of the pyramid, see Corollary 4.1 later.

Secondly, we remark that there also are many studies on the nonplanar traveling wave solutions of the scalar reaction-diffusion equations, for example, see [5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 23, 30, 31, 34, 32, 33, 35, 36, 37, 38, 39, 42, 43, 46].

Thirdly, in this paper we establish the uniqueness and stability of time periodic pyramidal fronts for competition-diffusion system (1.1) in $N = 3$. However, as the spatial dimension $N$ becomes higher, the uniqueness and stability of $N$-dimensional periodic pyramidal traveling fronts of (1.1) with $N \geq 4$ become more interesting and difficult and are left to be as interesting open problems.

The paper is organized as follows. In Section 2, we summarize some preliminaries. In Section 3, we prove some basic properties on the time-periodic pyramidal traveling fronts, which will be used to prove our main results. We then show that the time-periodic pyramidal traveling front is asymptotically stable and unique in Section 4.

2. Preliminaries. In order to consider the uniqueness and asymptotic stability of time-periodic pyramidal traveling fronts, we first transform (1.1) to a cooperative system via the transformation

\[
\begin{align*}
    u_1'(x, t) &= \frac{u_1(x, t)}{p(t)}, & u_2'(x, t) &= \frac{q(t) - u_2(x, t)}{q(t)}.
\end{align*}
\]  

Dropping the prime, (1.1) is then transformed into

\[
\begin{align*}
    \frac{\partial u_1}{\partial t} &= \Delta u_1(x, t) + u_1 [a_1(t)p(t)(1 - u_1) - b_1(t)q(t)(1 - u_2)], \\
    \frac{\partial u_2}{\partial t} &= d\Delta u_2(x, t) + (1 - u_2) [a_2(t)p(t)u_1 - b_2(t)q(t)u_2].
\end{align*}
\]  

Observe that (2.2) is cooperative in the region $u_1 > 0$ and $0 < u_2 < 1$ in $\mathbb{R}^3 \times \mathbb{R}^+$. Observe also that by the transformation (2.1), the solutions $(0, 0), (0, q(t)), (p(t), 0)$ and $(u_1^0(t), u_2^0(t))$ of (1.1) become solutions $(0, 1), (0, 0), (1, 1)$ and $(u_1^*(t), u_2^*(t))$ of (2.2), respectively. Moreover, there are $0 < u_i^*(t) < 1$ and $u_i^*(t + T) = u_i^*(t)$ for $i = 1, 2$ and $t \in \mathbb{R}$. For convenience, let

\[
\begin{align*}
    g(u_1, u_2, t) &= u_1[a_1(t)p(t)(1 - u_1) - b_1(t)q(t)(1 - u_2)], \\
    f(u_1, u_2, t) &= (1 - u_2)[a_2(t)p(t)u_1 - b_2(t)q(t)u_2].
\end{align*}
\]

Then we can rewrite (2.2) into

\[
\begin{align*}
    \frac{\partial u_1}{\partial t} &= \Delta u_1(x, t) + g(u_1(x, t), u_2(x, t), t), \\
    \frac{\partial u_2}{\partial t} &= d\Delta u_2(x, t) + f(u_1(x, t), u_2(x, t), t),
\end{align*}
\]  

\[
\begin{align*}
    \text{for } x \in \mathbb{R}^3, \quad t > 0.
\end{align*}
\]

Obviously, $g(0, 0, t) = h(0, 0, t) = 0$, $g(1, 1, t) = h(1, 1, t) = 0$ and $g(\cdot, \cdot, t + T) = g(\cdot, \cdot, t)$, $h(\cdot, \cdot, t + T) = h(\cdot, \cdot, t)$ in $\mathbb{R}^2 \times \mathbb{R}^+$. Moreover, it follows from (1.3) that
\((\Psi_1(\xi, t), \Psi_2(\xi, t)) = \left(\frac{\Psi_1(\xi, t)}{p(t)}, \frac{q(t) - \Psi_2(\xi, t)}{q(t)}\right)\) is the planar traveling wave solution of system (2.3) with wave speed \(c\), which satisfies
\[
\begin{aligned}
&\frac{\partial \Psi_1}{\partial t} + c \frac{\partial \Psi_1}{\partial \xi} - \frac{\partial^2 \Psi_1}{\partial \xi^2} - g(\Psi_1, \Psi_2, t) = 0, \\
&\frac{\partial \Psi_2}{\partial t} + c \frac{\partial \Psi_2}{\partial \xi} - \frac{d \partial^2 \Psi_2}{\partial \xi^2} - f(\Psi_1, \Psi_2, t) = 0 \\
&\lim_{\xi \to -\infty} (\Psi_1(\xi, t), \Psi_2(\xi, t)) = (0, 0), \\
&\lim_{\xi \to +\infty} (\Psi_1(\xi, t), \Psi_2(\xi, t)) = (1, 1), \\
&\Psi_i(\xi, T + t) = \Psi_i(\xi, t), \quad i = 1, 2,
\end{aligned}
\]
and \(\frac{\partial \Psi_i}{\partial \xi} > 0\), \(i = 1, 2\) for all \((\xi, t) \in \mathbb{R}^2 \times \mathbb{R}^+\). Consider
\[
\begin{aligned}
&\frac{\partial v_1}{\partial t} = \Delta v_1(x, t) - s \frac{\partial v_1}{\partial x} + g(v_1(x, t), v_2(x, t), t), \\
&\frac{\partial v_2}{\partial t} = d \Delta v_2(x, t) - s \frac{\partial v_2}{\partial x} + f(v_1(x, t), v_2(x, t), t),
\end{aligned}
\]
\(x \in \mathbb{R}^3, \ t > 0\). (2.5)

For \(x \in \mathbb{R}^3\), let \(v_0(x) = (v_{10}(x), v_{20}(x)) \in C(\mathbb{R}^3, [0, 1])\). We denote \((v_1(x, t; \mathbf{v}_0), v_2(x, t; \mathbf{v}_0))\) be the solution of (2.5) with \((v_1(x, 0; \mathbf{v}_0), v_2(x, 0; \mathbf{v}_0)) = v_0(x)\). Thus
\[
(v_1^-(x, t), v_2^-(x, t)) = \left(\Psi_1 \left(\frac{\xi}{s}(x_3 + h(x')), t\right), \Psi_2 \left(\frac{\xi}{s}(x_3 + h(x')), t\right)\right)
\]
is a subsolution of system (2.5). Then by Theorem 1.1 and the transformation (2.1), for each \(s > c\), there exists a solution \((V_1(x', x_3 + st, t), V_2(x', x_3 + st, t))\) of (2.5) satisfying
\[
V_i(\cdot, \cdot, \cdot + T) = V_i(\cdot, \cdot, \cdot) \quad \text{and} \quad V_i(x', x_3 + st, t) > v_i^-(x, t), \quad i = 1, 2
\]
and
\[
\lim_{\gamma \to +\infty} \sup_{x \in D(\gamma), t \in [0, T]} \sum_{t=1}^2 |V_i(x', x_3 + st, t) - v_i^-(x, t)| = 0.\]
(2.7)

To study the uniqueness and stability of \((U_1, U_2)\), it is then equivalent to study the uniqueness and stability of \((V_1, V_2)\) for (2.5).

Note that time-periodic V-form traveling front of Lotka-Volterra competition-diffusion system (1.1) has been studied in [3]. Let \((\bar{w}_1(\xi, \eta, t; \mathbf{w}_0), \bar{w}_2(\xi, \eta, t; \mathbf{w}_0))\) be the solution of
\[
\begin{aligned}
&\frac{\partial \bar{w}_1}{\partial t} - \frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \frac{\partial^2 \bar{w}_1}{\partial \eta^2} + s \frac{\partial \bar{w}_1}{\partial \eta} - g(\bar{w}_1, \bar{w}_2, t) = 0, \\
&\frac{\partial \bar{w}_2}{\partial t} - \frac{d \partial^2 \bar{w}_2}{\partial \xi^2} - \frac{d \partial^2 \bar{w}_2}{\partial \eta^2} + s \frac{\partial \bar{w}_2}{\partial \eta} - f(\bar{w}_1, \bar{w}_2, t) = 0, \\
&\bar{w}_1(\xi, \eta, 0) = \bar{w}_{10}(\xi, \eta), \quad \bar{w}_2(\xi, \eta, 0) = \bar{w}_{20}(\xi, \eta), \quad i = 1, 2
\end{aligned}
\]
with the initial value \(\bar{w}_0 = (\bar{w}_{10}(\xi, \eta), \bar{w}_{20}(\xi, \eta)) \in C(\mathbb{R}^2, \mathbb{R}^2)\).

**Theorem 2.1.** (see [3, Theorems 1.1-1.2]) Assume that (A1)-(A4) hold. For each \(\hat{s} > c\), let \(\hat{m}_s = \sqrt{\hat{s} - c^2} \). Then there exists a unique \((\hat{V}_1(\xi, \eta, t; \hat{s}), \hat{V}_2(\xi, \eta, t; \hat{s}))\) satisfying (2.8) and
\[
\hat{V}_i(\xi, \eta, t + T; \hat{s}) = \hat{V}_i(\xi, \eta, t; \hat{s}) \quad \text{for} \quad (\xi, \eta) \in \mathbb{R}^2, \ t \in \mathbb{R}, \ i = 1, 2.
\]
In addition,
\[
\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2, t \in [0, T]} \left| \nabla \bar{w}(\xi, \eta, t) - \nabla \bar{v}(\xi, \eta, t) \right| = 0, \quad (2.9)
\]

\[
\Psi_i \left( \frac{c}{s} (\eta + \hat{m}_s(\xi)) \right) < \bar{v}_1(\xi, \eta, t) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, t \in \mathbb{R}^+, \ i = 1, 2. \quad (2.10)
\]

Furthermore, for any initial function \( \bar{w}_0(\xi, \eta) \in C(\mathbb{R}^2, \mathbb{R}^2) \) with \( \bar{w}_0 \in [0, 1] \) and
\[
\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2} \left| \bar{w}(\xi, \eta, t) - \bar{v}(\xi, \eta, t) \right|_{C(\mathbb{R}^2)} = 0,
\]

or equivalently,
\[
\lim_{k \to \infty} \left\| \bar{w}(\xi, \eta, t + kT; \bar{w}_0) - \bar{v}(\xi, \eta, t; \hat{s}) \right\|_{C(\mathbb{R}^2 \times [0, T])} = 0.
\]

For any subset \( D \subset \mathbb{R}^3 \), we denote the characteristic function of \( D \) by \( \chi_D \), namely, \( \chi_D(x) = 1 \) for \( x \in D \) and \( \chi_D(x) = 0 \) for \( x \notin D \). For \( j = 1, 2 \), define
\[
M_{ij} := \sup_{(u_1, u_2) \in [-1, 2], t \in \mathbb{R}} g_{u_j}(u_1, u_2, t) \quad \text{and} \quad M_{2j} := \sup_{(u_1, u_2) \in [-1, 2], t \in \mathbb{R}} f_{u_j}(u_1, u_2, t).
\]

Let \( h_{ij}(x, t) \in C(\mathbb{R}^3 \times \mathbb{R}^+) \) \( (i, j = 1, 2) \) be the given continuous function with
\[
0 \leq h_{ij}(x, t) \leq M_{ij}, \quad i \neq j, \quad \sup_{x \in \mathbb{R}^3, t \in \mathbb{R}} |h_{ij}(x, t)| \leq M_{ij}, \quad i = j.
\]

For any given initial value \( w_0(x) \in C(\mathbb{R}^3, \mathbb{R}^2) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^2) \), we consider the following linear system
\[
\begin{cases}
\frac{\partial w_1}{\partial t} - \sum_{k=1}^{3} \frac{\partial^2 w_1}{\partial x_k^2} + s \frac{\partial w_1}{\partial x_3} - h_{11}(x, t)w_1 - h_{12}(x, t)w_2 = 0, \\
\frac{\partial w_2}{\partial t} - d \sum_{k=1}^{3} \frac{\partial^2 w_2}{\partial x_k^2} + s \frac{\partial w_2}{\partial x_3} - h_{21}(x, t)w_1 - h_{22}(x, t)w_2 = 0,
\end{cases}
\]

(2.11)

for \( x \in \mathbb{R}^3 \) and \( t \in \mathbb{R}^+ \).

The following Lemma follows from [44, Lemma 4.2].

**Lemma 2.1.** Let \( w(x, t) = (w_1(x, t), w_2(x, t)) \) is the solution of (2.11). Then there exist positive constants \( \bar{A}, \bar{B} \) and \( m > 0 \) such that
\[
\max_{1 \leq i \leq 2} \sup_{x \in \mathbb{R}^3} w_i(x, t) \leq e^{mt} \max_{1 \leq i \leq 2} \sup_{x \in \mathbb{R}^3} w_{i0}(x), \quad \forall t > 0,
\]
\[
e^{mt} \min_{1 \leq i \leq 2} \inf_{x \in \mathbb{R}^3} w_{i0}(x) \leq \min_{1 \leq i \leq 2} \inf_{x \in \mathbb{R}^3} w_i(x, t), \quad \forall t > 0,
\]
\[
\max_{1 \leq i \leq 2} \sup_{x \in \mathbb{R}^3} |w_i(x, t)| \leq e^{mt} \max_{1 \leq i \leq 2} \sup_{x \in \mathbb{R}^3} \|w_{i0}(x)\|_{L^\infty(\mathbb{R}^3)}, \quad \forall t > 0,
\]

and for any \( \gamma > 0 \),
\[
\sup_{1 \leq t \leq 2} \sup_{x \in D(2\gamma)} |w_i(x, t)| \leq e^{mt} 3\pi \frac{\bar{A}}{\bar{B}} \int_{\frac{\bar{B}}{\bar{A}}}^{+\infty} \exp \left( -Br^2 \right) dr \max_{1 \leq i \leq 2} \sup_{x \in D(\gamma)} |w_{i0}(x)|
\]
Lemmas 2.1-2.2, we then have system for (2.5) as that in [3]. Let \( C \) from above by \( 1 \) and \( C \) from below by \( 2 \). Note that the supersolutions and subsolutions constructed later cannot be bounded also to use the super- and subsolution technique coupled with comparison principle. We note that the following results come from Bao et al. [3]. Define 
\[
\lambda = \frac{3\sqrt{\pi A}}{B \sqrt{3}} \sup_{1 \leq i \leq 2} \sup_{x \in D(\gamma)} |w_{i0}(x)|, \quad \forall t > 0,
\]
(2.12)

where \( D(\gamma) = \{ x \in \mathbb{R}^3 \mid x \notin D(\gamma) \} \). If \( w_{i0}(x) = 0 \) for any \( x \in B(x_0, \sqrt{3}R) := \{ x \in \mathbb{R}^3 \mid |x - x_0| < \sqrt{3}R \} \) and \( i = 1, 2 \), then for any given \( t > 0 \), we have
\[
\sup_{1 \leq i \leq 2} |w_i(x, t)| \leq e^{\frac{3\sqrt{\pi A}}{B} \int_{t}^{+\infty} \exp \left( -Bt^2 \right) dr} \max_{1 \leq i \leq 2} \sup_{x \in \mathbb{R}^3} |w_{i0}(x)|
\]
for any \( x_0 \in \mathbb{R}^3 \) and \( R > 0 \).

In the current paper, we will extend the arguments of Wang et al.[44], Taniguchi [36] and Bao et al. [3] to study the uniqueness and stability of time periodic pyramidal fronts for (1.1) in \( \mathbb{R}^3 \) under the assumptions (A1)-(A4). The main method is also to use the super- and subsolution technique coupled with comparison principle. Note that the supersolutions and subsolutions constructed later cannot be bounded from above by \( 1 \) and from below by \( 0 \) and the comparison principal on \( [0, 1] \) is invalid for such super- and subsolutions. To overcome this, we introduce an auxiliary system for (2.5) as that in [3]. Let \( C^+_1 \) and \( C^-_2 \) be positive constants defined in [1, Lemmas 2.1-2.2], we then have
\[
g_{u_1}(u_1, u_2, t) + C^+_1 \{ u_1 \}^{-} \geq 0 \quad \text{and} \quad h_{u_1}(u_1, u_2, t) + C^-_2 \{ 1 - u_2 \}^{-} \geq 0
\]
for all \((u_1, u_2) \in [-1, 2]\), where \(-1 = (-1, -1), 2 = (2, 2)\) and
\[
\{ a \}^{-} = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}
\]

Consider the following initial value problem:
\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \Delta u_1(x, t) - s \frac{\partial u_1}{\partial x_3} + G(u_1, u_2, t), \\
\frac{\partial u_2}{\partial t} &= d \Delta u_2(x, t) - s \frac{\partial u_2}{\partial x_3} + F(u_1, u_2, t), \\
(\mathbf{x}_0(0), & u_2(0, 0)) = (v_{i0}(\mathbf{x}), v_{20}(\mathbf{x})) \in Y,
\end{aligned}
\]
(2.14)

where \( Y = \{(v_{i0}, v_{20}) \in C(\mathbb{R}^3, \mathbb{R}^2) : -1 \leq v_{i0}(\mathbf{x}), v_{20}(\mathbf{x}) \leq 2 \} \) and
\[
G(u_1, u_2, t) = g(u_1, u_2, t) + C^+_1 \{ u_1 \}^{-} u_2,
\]
\[
F(u_1, u_2, t) = f(u_1, u_2, t) + C^-_2 \{ 1 - u_2 \}^{-} (u_1 - 1).
\]

Note that \( G(u_1, u_2, t) = g(u_1, u_2, t) \) and \( F(u_1, u_2, t) = f(u_1, u_2, t) \) for \((u_1, u_2) \in [0, 1] \). Following from [1, Theorem 2.5] and [1, Corollary 2.6], the comparison principle holds for (2.14) for \((u_1, u_2) \in [-1, 2]\). Thus we can construct some useful supersolutions and subsolutions of (2.14), which will be used to establish the stability of time periodic pyramidal traveling front \((V_1, V_2)\) in Section 4.

We note that the following results come from Bao et al. [3]. Define \( \lambda_0 \) and \( \lambda_1 \) be the eigenvalues of the following linearized periodic system
\[
\begin{cases}
\frac{d\phi_0}{dt} - [a_1(t)p(t) - b_1(t)q(t)] \phi_0 = \lambda_0 \phi_0, \\
\frac{d\psi_0}{dt} - [a_2(t)p(t) + b_2(t)q(t)] \psi_0 = \lambda_0 \psi_0, \\
\phi_0(t + T) = \phi_0(t), \quad \psi_0(t + T) = \psi_0(t)
\end{cases}
\]

and
\[
\begin{aligned}
&\begin{cases}
\frac{d\phi_1}{dt} + a_1(t)p(t)\phi_1 - b_1(t)q(t)\psi_1 = \lambda_1\phi_1, \\
\frac{d\psi_1}{dt} - [b_2(t)q(t) - a_2(t)p(t)]\psi_1 = \lambda_1\psi_1,
\end{cases}
\end{aligned}
\]
respectively. Let \((\phi_0(t), \psi_0(t))\) and \((\phi_1(t), \psi_1(t))\) be the eigenfunctions correspond to \(\lambda_0\) and \(\lambda_1\), respectively. Under the hypotheses (A2) and (A3), we can calculate that
\[
\lambda_0 = -\frac{1}{T} \int_0^T (a_1(t)p(t) - b_1(t)q(t))dt > 0
\]
and
\[
\lambda_1 = -\frac{1}{T} \int_0^T (b_2(t)q(t) - a_2(t)p(t))dt > 0.
\]
In particular, the eigenfunctions \((\phi_0(t), \psi_0(t))\) associated with the eigenvalue \(\lambda_0\) and \((\phi_1(t), \psi_1(t))\) associated with the eigenvalue \(\lambda_1\) are both time periodic and positive on \(t \in \mathbb{R}\), see [2]. Since the boundedness and positivity of the eigenfunctions \((\phi_0(t), \psi_0(t))\) and \((\phi_1(t), \psi_1(t))\) on \(t \in \mathbb{R}\), take \(\eta^\pm > 0\) small enough such that \((\eta^-\phi_0(t), \eta^-\psi_0(t)) \ll (\phi_1(t), \psi_1(t))\) and \((\eta^+\phi_1(t), \eta^+\psi_1(t)) \ll (\phi_0(t), \psi_0(t))\) for all \(t \in \mathbb{R}\). Let
\[
\begin{aligned}
P^-(t) &= \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}, \\
P^+(t) &= \begin{pmatrix} p_1^+(t) \\ p_2^+(t) \end{pmatrix}, \\
Q^-(t) &= \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}, \\
Q^+(t) &= \begin{pmatrix} q_1^+(t) \\ q_2^+(t) \end{pmatrix}
\end{aligned}
\]
Define \(\omega(\zeta)\) be a smooth function such that \(\omega(\zeta) = 0\) for \(\zeta \leq -2\) and \(\omega(\zeta) = 1\) for \(\zeta \geq 2\). Moreover for \(\zeta \in \mathbb{R}, 0 \leq \omega(\zeta) \leq 1, |\omega'(\zeta)| \leq 1\). Let the positive vector functions \(P(\zeta, t) = (p_1(\zeta, t), p_2(\zeta, t))\) and \(Q(\zeta, t) = (q_1(\zeta, t), q_2(\zeta, t))\) be defined by
\[
\begin{aligned}
p_i(\zeta, t) &= \omega(\zeta)p_i^+ + (1 - \omega(\zeta))p_i^-(t) \\
q_i(\zeta, t) &= \omega(\zeta)q_i^+ + (1 - \omega(\zeta))q_i^-(t)
\end{aligned}
\]
respectively, where \(i = 1, 2\). It is easy to see that \(p_i(\zeta, t) \in [p_i^-, p_i^+(t)]\) and \(q_i(\zeta, t) \in [q_i^-, q_i^+(t)]\) for \(\zeta \in \mathbb{R}\) and \(t \in \mathbb{R}\). In particular, \(\frac{\partial p_i}{\partial \zeta}(\zeta, t) < 0\) and \(\frac{\partial q_i}{\partial \zeta}(\zeta, t) < 0\) on \(\mathbb{R} \times [0, T]\).

Recall that \(h_j(x')\) and \(h(x')\) are as in (1.6). Choose \(\rho : \mathbb{R}^2 \to \mathbb{R}\) which belongs to \(C^\infty(\mathbb{R}^2)\) and satisfies \(\int_{\mathbb{R}^2} \rho(x')dx' = 1\) and \((\rho \ast h_j)(x') = h_j(x')\) for \(x' \in \mathbb{R}^2\) and \(j = 1, ..., n\). Define \(\varphi(x') = (\rho \ast h)(x')\) for \(x' \in \mathbb{R}^2\). Then \(\varphi(x') \in C^\infty(\mathbb{R}^2)\) and let
\[
\hat{\mu}(x) = \frac{x_3 + \frac{1}{\alpha} \varphi(\alpha x')}{\sqrt{1 + |\nabla \varphi(\alpha x')|^2}} \text{ and } \sigma(x') = \varepsilon \left( \frac{s}{\sqrt{1 + |\nabla \varphi(\alpha x')|^2}} - c \right).
\]
For \(\alpha > 0, \varepsilon > 0\) and \(\varphi \in C^\infty(\mathbb{R}^2)\), we put
\[
u_0^+(x; t; \varepsilon, \alpha) = \Psi_1(\hat{\mu}(x), t) + p_i(\hat{\mu}(x), t)\sigma(x'), \quad i = 1, 2.
\]
Then by [1, Lemma 4.1], there exist a positive constant \(\varepsilon_0^+\) and a positive function \(\alpha_0^+(\varepsilon)\) such that \((\nu_1^+(x; t; \varepsilon, \alpha), u_0^+(x; t; \varepsilon, \alpha))\) is a supersolution of (2.14) on \(x \in \mathbb{R}^3\) and \(t \in \mathbb{R}^+\) for \(0 < \varepsilon < \varepsilon_0^+\) and \(0 < \alpha < \alpha_0^+(\varepsilon)\).
Lemma 2.2. Assume that (H1)-(H4) hold. For some \( \varepsilon > 0 \) and \( \alpha > 0 \), let
\[
u_i^-(x, t; \varepsilon, \alpha) = \Psi_i(\eta(x), t) - \varepsilon q_i(\eta(x), t) \sech(\beta_2 \alpha x_3), \quad i = 1, 2,
\]
where
\[
\eta(x) = \frac{h(x') + \psi(\alpha x_3)}{m_x + \frac{\alpha}{\alpha}} \sqrt{1 + \psi'(\alpha x_3)^2} \quad \text{and} \quad \psi(\vartheta) := -\frac{1}{m_x \beta_2} \ln(1 + \exp(-\beta_2 \vartheta)).
\]
Then there exist \( \varepsilon > 0 \) and \( \alpha^{-}(\varepsilon) > 0 \) such that for \( 0 < \varepsilon < \varepsilon^{-} \) and \( 0 < \alpha < \alpha^{-}(\varepsilon) \),
\((u_1^-(x, t; \varepsilon, \alpha), u_2^-(x, t; \varepsilon, \alpha)) \) is a subsolution of (2.14) for \( t > 0 \).

Proof. From [3, Lemma 4.3], we know that there exists \( \varepsilon^{-} > 0 \) and \( \alpha^{-}(\varepsilon) > 0 \) such that for \( 0 < \varepsilon < \varepsilon^{-} \) and \( 0 < \alpha < \alpha^{-}(\varepsilon) \), \((u_1^{-}(x, t; \varepsilon, \alpha), u_2^{-}(x, t; \varepsilon, \alpha)) \) is a subsolution of (2.14) for \( x \in \mathbb{R}^3 \) and \( t > 0 \), where for \( i = 1, 2, \)
\[
u_i^{-}(x, t; \varepsilon, \alpha) = \Psi_i \left( \frac{h_i(x') + \psi(\alpha x_3)}{m_x + \frac{\alpha}{\alpha}} \sqrt{1 + \psi'(\alpha x_3)^2}, t \right) - \varepsilon q_i \left( \frac{h_i(x') + \psi(\alpha x_3)}{m_x + \frac{\alpha}{\alpha}} \sqrt{1 + \psi'(\alpha x_3)^2}, t \right) \sech(\beta_2 \alpha x_3).
\]
Thus
\[
(u_1^-(x, t; \varepsilon, \alpha), u_2^-(x, t; \varepsilon, \alpha)) = \left( \max_{1 \leq i \leq n} u_1^{-}, \max_{1 \leq j \leq n} u_2^{-} \right)
\]
is a subsolution of (2.14) for \( t > 0 \). This completes the proof. \( \square \)

The following lemmas provide much more subsolutions and supersolutions of (2.14), which will be used to prove uniqueness and stability of time periodic pyramidal fronts in Section 4.

Lemma 2.3. There exist positive constants \( \rho \) sufficiently large, \( \beta \) small enough and \( \delta_1 \) such that for any \( 0 < \delta < \delta_1 \), \( w^+ = (w_1^+, w_2^+) \) defined by
\[
w_i^+(x, t; \delta) = V_1(x + 3 \rho \delta e^{-\beta t}, t) + \delta \rho_i \left( s^+, t \right) e^{-\beta t}, \quad i = 1, 2
\]
is a supersolution of (2.14), and \( w^- = (w_1^-, w_2^-) \) defined by
\[
w_i^-(x, t; \delta) = V_i(x - 3 \rho \delta e^{-\beta t}, t) - \delta q_i \left( s^-, t \right) e^{-\beta t}, \quad i = 1, 2
\]
is a subsolution of (2.14), where
\[
s^\pm = \frac{x_3 \pm \rho \delta(1 - e^{-\beta t}) + \varphi(x')}{\sqrt{1 + |\nabla \varphi(x')|^2}}.
\]

Proof. By Lemma 2 in [35], for any \( x' \in \mathbb{R}^2 \), there is a positive constant \( m_0 \) such that
\[
h(x') \leq \varphi(x') \leq m_0 + h(x').
\]
It is easy to verify that there exist constants \( N_i^+ > 0 \) such that
\[
\left| \frac{\partial^2 p_i}{\partial \mu^2} (\mu_1^2 + \mu_2^2) + \frac{\partial p_i}{\partial \mu} (\mu_1 x_1 + \mu_2 x_2) + \frac{1}{1 + |\nabla \varphi(x')|^2} \frac{\partial^2 p_i}{\partial \mu^2} \right| \leq N_i^+
\]
and
\[
\left| \frac{\partial^2 q_i}{\partial \mu^2} (\mu_1^2 + \mu_2^2) + \frac{\partial q_i}{\partial \mu} (\mu_1 x_1 + \mu_2 x_2) + \frac{1}{1 + |\nabla \varphi(x')|^2} \frac{\partial^2 q_i}{\partial \mu^2} \right| \leq N_i^-
\]
for any \( \alpha \in (0, 1) \) and \( (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+ \). We omit the rest of the proof, which is similar to that of [3, Lemma 4.5]. This completes the proof. \( \square \)
Lemma 2.4. There exist positive constants \( \rho \) sufficiently large, \( \beta \) small enough and \( \delta_2 \) such that, for any \( 0 < \delta < \delta_2 \), \( W^+ \) defined by
\[
W^+_i(x, t; \delta) = u^+_i(x', x_3 + \rho \delta(1 - e^{-\beta t}), t; \varepsilon, \alpha) + \delta p_i(t)e^{-\beta t}
\]
is a supersolution of (2.14) for any \( \delta > 0 \), where \( i = 1, 2 \) and
\[
\tau = \frac{x_3 + \rho \delta(1 - e^{-\beta t}) + \varphi(\alpha x')/\alpha}{\sqrt{1 + \varphi'^2(\alpha x')}}.
\]
The proof of the lemma is similar to that of Lemma 2.3. Following form [3, Lemma 4.7], we obtain the following lemma.

Lemma 2.5. There exist positive constants \( \rho \) sufficiently large, \( \beta \) small enough and \( \delta_3 \) such that, for any \( 0 < \delta < \delta_3 \), \( \hat{W}^j \) and \( \hat{W} \) are also subsolutions of (2.14), where \( j = 1, \ldots, n \) and
\[
\hat{\vartheta} = \frac{h_j(x')/m_* - \rho \delta(1 - e^{-\beta t}) + \psi(\alpha x_3)/\alpha}{\sqrt{1 + \psi'^2(\alpha x_3)}}.
\]

3. Basic properties. In this section, we will prove some basic properties and show that the time-periodic pyramidal traveling front \((V_1, V_2)\) converges to two-dimensional V-form traveling fronts on edges of the pyramid. To do this, we first study the exact formulation of the two-dimensional V-form front on each edge.

For each \( j (1 \leq j \leq n) \) we consider a plane perpendicular to an edge \( \Gamma_j = S_j \cap S_{j+1} \). Then the cross section of \(-x_3 = \max\{h_j(x'), h_{j+1}(x')\}\) in this plane has a time periodic V-form front. Let \( E^j = (E_1^j, E_2^j) \) be the two-dimensional periodic V-form front as in Theorem 2.1 corresponding to the cross section \(-x_3 = \max\{h_j(x'), h_{j+1}(x')\}\).

Next, by the same way in [44, 36], we will give the precise definition of \( E^j \). Let \( A_{n+1} := A_1 \) and \( B_{n+1} := B_1 \). Define
\[
p_j := A_{j+1}B_j - A_jB_{j+1} > 0 \quad \text{and} \quad q_j := \sqrt{(A_{j+1} - A_j)^2 + (B_{j+1} - B_j)^2} > 0
\]
for \( 1 \leq j \leq n \). Take \( \nu_j = \frac{1}{\sqrt{1 + m_j^2}}(m_*A_j, m_*B_j, 1) \) for \( j = 1, \ldots, n + 1 \). Let \((x_1, x_2, x_3)^T\) is the transposed vector of \((x_1, x_2, x_3)\). The direction of \( \Gamma_j \) is given by
\[
\nu_{j+1} \times \nu_j = \frac{1}{\sqrt{m_j^2p_j^2 + q_j^2}}(B_{j+1} - B_j, A_j - A_{j+1}, m_*(A_{j+1}B_j - A_jB_{j+1}))^T
\]
and the traveling direction of a two-dimensional V-form wave \( E^j \) is given by
\[
(\nu_{j+1} \times \nu_j) \times \frac{\nu_{j+1} - \nu_j}{|\nu_{j+1} - \nu_j|} = \frac{(m_*(B_{j+1} - B_j)p_j, m_*(A_j - A_{j+1})p_j, q_j^2)^T}{q_j \sqrt{m_j^2p_j^2 + q_j^2}}.
\]
Let \( s_j \) be the speed of \( E^j \) and \( 2\theta_j \) be the angle between \( S_j \) and \( S_{j+1} \). Then
\[
s_j \sin \theta_j = c, \quad \sin \theta_j = \sqrt{m_j^2p_j^2 + q_j^2}/q_j \quad \text{and} \quad s_j = \frac{s q_j}{\sqrt{m_j^2p_j^2 + q_j^2}}.
\]
The speed of $E^j$ toward the $x_3$-axis equals $s_j\sqrt{m_{j}^2p_j^2 + q_j^2}/q_j = c\sqrt{1 + m_{j}^2} = s_j$, which coincides with the speed of $V = (V_1, V_2)$. Let $(x_1, x_2, x_3)^T = R_j(\xi, \eta, \zeta)^T$ and $(\xi, \eta, \zeta)^T = R_j^T(x_1, x_2, x_3)^T$, where $R_j^T$ is the transposed matrix of $R_j$. Here we take

$$R_j = \begin{pmatrix}
\frac{A_{j+1} - A_j}{q_0} & \frac{m_0(B_{j+1} - B_j)q_1}{q_0} & \frac{B_{j+1} - B_j}{\sqrt{m_{j}^2p_j^2 + q_j^2}} \\
\frac{B_{j+1} - B_j}{q_0} & \frac{m_0(A_j - A_{j+1})q_1}{q_0} & \frac{A_j - A_{j+1}}{\sqrt{m_{j}^2p_j^2 + q_j^2}} \\
0 & \frac{m_0(A_j - A_{j+1})q_1}{q_0} & \frac{m_0(B_{j+1} - B_j)q_1}{q_0} 
\end{pmatrix}.$$ 

Define $E^j = (E^j_1, E^j_2)$, where $E^j_i$ $(i = 1, 2)$ be defined as

$$E^j_i(x, t) := \hat{V}_i \left(\frac{(A_j - A_{j+1})x_1 + (B_j - B_{j+1})x_2}{q_j}, \frac{m_0(B_j - B_{j+1})p_jx_1 + m_0(A_{j+1} - A_j)p_jx_2 + q_j^2x_3}{q_j\sqrt{m_{j}^2p_j^2 + q_j^2}}, t, \frac{sq_j}{\sqrt{m_{j}^2p_j^2 + q_j^2}}\right).$$

Direct calculations show that

$$\begin{align*}
\frac{\partial \hat{V}_1}{\partial t} - \frac{\partial^2 \hat{V}_1}{\partial \xi^2} - \frac{\partial^2 \hat{V}_1}{\partial \eta^2} + s_j \frac{\partial \hat{V}_1}{\partial \eta} - f(\hat{V}_1, \hat{V}_2, t) &= 0,
\frac{\partial \hat{V}_2}{\partial t} - d \frac{\partial^2 \hat{V}_2}{\partial \xi^2} - \frac{\partial^2 \hat{V}_2}{\partial \eta^2} + s_j \frac{\partial \hat{V}_2}{\partial \eta} - g(\hat{V}_1, \hat{V}_2, t) &= 0
\end{align*}$$

for all $(\xi, \eta) \in \mathbb{R}^2$, $t \in [0, T]$. Hence for each $j$ $(1 \leq j \leq n)$, $E^j(x, t)$ satisfies (2.5). We call $E^j$ a planar time periodic V-form traveling front corresponding to an edge $\Gamma_j$.

Let $Q_j := \{x \in \mathbb{R}^3 | \text{dist}(x, \Gamma) = \text{dist}(x, \Gamma_j)\}$ for $1 \leq j \leq n$. Then $\mathbb{R}^3 = \bigcup_{j=1}^{n} Q_j$. Define

$$\hat{E}(x, t) := \left(\hat{E}_1(x, t), \hat{E}_2(x, t)\right) = \left(\max_{1 \leq j \leq n} E^j_1(x, t), \max_{1 \leq j \leq n} E^j_2(x, t)\right).$$

Since $E^j(x, t)$ is strictly monotone increasing in $x_3$ for each $j$, we have $\hat{E}(x, t)$ is strictly monotone increasing in $x_3$. In addition, $\hat{E}(x, t)$ also has the following properties.

**Lemma 3.1.** The function $\hat{E}(x, t)$ satisfies

$$v^-(x, t) < \hat{E}(x, t) < V(x, t), \quad x \in \mathbb{R}^3, t \in [0, T]$$

and

$$\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0, T]} \left| \hat{E}(x, t) - v^-(x, t) \right| = 0. \quad (3.1)$$

**Proof.** By Theorem 2.1, we have

$$\max \left\{ \Psi_i \left(\frac{c}{s}(x_3 + h_j(x')), t\right), \Psi_i \left(\frac{c}{s}(x_3 + h_{j+1}(x')), t\right) \right\} < E^j_i(x, t)$$

for any $x \in \mathbb{R}^3$, $t \in [0, T]$ and $i = 1, 2$. It then follows that

$$v^i_-(x, t) = \Psi_i \left(\frac{c}{s}(x_3 + h(x')), t\right) < \hat{E}_i(x, t) \quad \text{for } x \in \mathbb{R}^3, t \in [0, T], i = 1, 2.$$
Let $\Psi\left(\frac{x_3 + h_j(x')}{\gamma}\right), 0)$ and $E^j(x, 0)$ be the initial values of (2.5), respectively. Namely, we consider the solutions $v(x, t + kT; \Psi\left(\frac{x_3 + h_j(x')}{\gamma}\right), 0)$ and $v(x, t + kT; E^j(x, 0))$ of (2.5). Let $k \to \infty$, we have $E^j(x, t) < V(x, t)$ for all $x \in \mathbb{R}^3, t \in [0, T]$ and $j = 1, ..., n$.

Consequently, the comparison principle implies $\hat{E}(x, t) < V(x, t)$ for all $x \in \mathbb{R}^3$ and $t \in [0, T]$. Finally, (3.1) directly follows from (2.7). This completes the proof.

Suppose that $v_0 \in [0, 1]$ satisfies
\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} |v_0(x) - v^-(x, 0)| = 0. \tag{3.2}
\]
Let $v(x, t; v_0) = (v_1(x, t; v_0), v_2(x, t; v_0))$ be the solution of (2.5) with the initial value $v_0$. For any given $t \in [0, T]$ and $k \in \mathbb{N}$, by Lemma 2.1, there exist constants $\hat{A} > 0$ and $B > 0$ such that
\[
\max_{1 \leq i \leq 2} \sup_{x \in (x, t; v_0) - V_i(x, t)} \leq \exp\left(-\frac{\hat{B}}{B}\right) \sup_{1 \leq i \leq 2} \sup_{x \in D(\gamma)} |v_0(x) - V_i(x, 0)|
\]
\[
+ \frac{\pi \sqrt{\hat{A}}}{\hat{B}} e^{\sqrt{\gamma} |Bt|} \sup_{i \leq 2} \sup_{x \in (x, t; v_0) - V_i(x, 0)} \left|v_0(x) - V_i(x, 0)\right| \tag{3.3}
\]
for any $\gamma > 0$ and $t > 0$. Then we obtain
\[
\lim_{\gamma \to \infty} \max_{i = 1, 2} \sup_{x \in (x, t; v_0) - V_i(x, t)} = 0 \tag{3.4}
\]
for any fixed $k \in \mathbb{N}$, which implies
\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0, T]} |v(x, t + kT; v_0) - v^-(x, t)| = 0, \tag{3.5}
\]
\[
\lim_{\gamma \to \infty} \max_{1 \leq j \leq n} \sup_{x \in D(\gamma), x \in Q_j, t \in [0, T]} |v(x, t + kT; v_0) - E^j(x, t)| = 0 \tag{3.6}
\]
and
\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0, T]} |v(x, t + kT; v_0) - \hat{E}(x, t)| = 0 \tag{3.7}
\]
for any fixed $k \in \mathbb{N}$.

**Lemma 3.2.** Assume that $v_0 \in [0, 1]$ satisfies (3.2). For any given $\varepsilon_1 > 0$, there exists $k_1 \in \mathbb{N}$ large enough such that, for any fixed $k \geq k_1$,
\[
\lim_{R \to \infty} \max_{1 \leq j \leq n} \sup_{|x| \geq R, t \in [0, T]} |v(x, t + kT; v_0) - E^j(x, t)| < \varepsilon_1. \tag{3.8}
\]

Lemma 3.2 will play a key role in the following estimates and can be proved by the strategy in [36, Proposition 1] and [44, Proposition 4.5]. We provide the proof of Lemma 3.2 in Appendix for interested readers.

**Lemma 3.3.** Assume that $v_0 \in [0, 1]$ satisfies (3.1). For any given $\varepsilon_1 > 0$, there is $k_\ast \in \mathbb{N}$ large enough such that
\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0, T]} |v(x, t + kT; v_0) - V(x, t)| < \varepsilon_1 \tag{3.9}
\]
for any \( k \gg k_* \). In particular,

\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0, T]} \left| V(x, t) - E(x, t) \right| = 0. \tag{3.10}
\]

**Proof.** Take \( v_0 = V(x, 0) \) in Proposition 3.2, then for any \( \varepsilon_1 > 0 \),

\[
\lim_{R \to \infty} \max_{1 \leq j \leq n} \sup_{|x| \geq R, x \in Q_j, t \in [0, T]} \left| V(x, t) - E^j(x, t) \right| < \varepsilon_1.
\]

By the arbitrariness of \( \varepsilon_1 > 0 \), we obtain (3.10) and

\[
\lim_{R \to \infty} \max_{1 \leq j \leq n} \sup_{|x| \geq R, x \in Q_j, t \in [0, T]} \left| V(x, t) - E^j(x, t) \right| = 0.
\]

Further, by Proposition 3.2, we have (3.9). This completes the proof. \( \square \)

The equality (3.10) shows that time periodic pyramidal traveling front \((V_1, V_2)\) converges to two-dimensional time periodic V-form fronts \((\tilde{E}_1, \tilde{E}_2)\) near the edges.

**Lemma 3.4.** One has

\[
\lim_{R \to \infty} \sup_{|x| \geq R, t \in [0, T]} \left| \frac{\partial}{\partial x_3} V_1(x, t) \right| + \left| \frac{\partial}{\partial x_3} V_2(x, t) \right| = 0.
\]

In addition, for any fixed \( \delta \in (0, 1) \) small enough,

\[
\min_{1 \leq i \leq 2} \left\{ \delta \leq E^i_2(x, t) \leq 1 - \delta, t \in [0, T] \right\} \inf_{1 \leq i \leq 2} \frac{\partial}{\partial x_3} E_1^i(x, t) > 0, \quad \forall j = 1, \ldots, n
\]

and

\[
\min_{1 \leq i \leq 2} \left\{ \delta \leq V_i(x, t) \leq 1 - \delta, t \in [0, T] \right\} \inf_{1 \leq i \leq 2} \frac{\partial}{\partial x_3} V_1(x, t) > 0. \tag{3.11}
\]

**Proof.** Since

\[
\frac{\partial}{\partial x_3} E_1^i(x, t) = \frac{q_j}{\sqrt{m_2^2 p_j^2 + q_j^2}} \frac{\partial}{\partial \eta} \tilde{V}_i(\xi, \eta, t; s_j),
\]

where

\[
\xi = (A_{j+1} - A_j)x_1 + (B_{j+1} - B_j)x_2
\]

and

\[
\eta = m_*(B_{j+1} - B_j)p_jx_1 + m_*(A_j - A_{j+1})p_jx_2 + s_j^2 x_3.
\]

It then follows from [3, Lemma 4.3] that

\[
\min_{1 \leq i \leq 2} \left\{ \delta \leq E^i_2(x, t) \leq 1 - \delta, t \in [0, T] \right\} \inf_{1 \leq i \leq 2} \frac{\partial}{\partial x_3} E^i_1(x, t) > 0 \quad \text{for } j = 1, 2, \ldots, n.
\]

Next, we prove (3.11). Since \( \frac{\partial}{\partial x_3} V_i > 0 \) in \( \mathbb{R}^3 \), \( \frac{\partial}{\partial x_3} V_i \) has a positive minimum on any compact subset of \( \mathbb{R}^3 \). Thus we need only to study \( \frac{\partial}{\partial x_3} V_i \) as \( |x| \to \infty \). Fix \( i = 1, 2 \). Let

\[
\Omega_i = \{ x \in \mathbb{R}^3 | |x| \geq R, x \in \tilde{\Omega}_i, t \in [0, T] \}.
\]

It suffices to prove

\[
\lim_{R \to \infty} \liminf_{|x| > R, x \in \Omega_i, t \in [0, T]} \frac{\partial V_i}{\partial x_3}(x, t) > 0.
\]
By (3.10) we have
\[
\lim_{R \to \infty} \sup_{x \in B(Q_j, 2), |x| > R, t \in [0, T]} |V(x, t) - E_j(x, t)| = 0,
\]
where \( B(Q_j, 2) := \{x \in \mathbb{R}^3 \mid \text{dist}(x, Q_j) \leq 2\} \) and \( j \in \{1, \ldots, n\} \). Then there exists \( \tilde{R}_j > 0 \) such that
\[
\sup_{x \in B(Q_j, 2), |x| \geq \tilde{R}_j, t \in [0, T]} |V(x, t) - E_j(x, t)| < \frac{\delta}{2}.
\]
Consequently, we have \( \frac{\delta}{2} < E_j(x, t) < 1 - \frac{\delta}{2} \) for \( x \in B(Q_j, 2) \cap \overline{\Omega}_i \) with \( |x| \geq \tilde{R}_j \) and \( t \in [0, T] \). For any \( x_0 \in Q_j \), we have
\[
\lim_{R \to \infty} \sup_{x_0 \in Q_j, |x_0| \geq R} \left| g(V_1, V_2, t) - g(E_1^j, E_2^j, t) \right|_{L^p(B(x_0, 2) \times [0, T])} = 0
\]
and
\[
\lim_{R \to \infty} \sup_{x_0 \in Q_j, |x_0| \geq R} \left| f(V_1, V_2, t) - f(E_1^j, E_2^j, t) \right|_{L^p(B(x_0, 2) \times [0, T])} = 0
\]
where \( p > 3 \), \( B(x_0, r) := \{x \in \mathbb{R}^3 \mid |x - x_0| < r\} \). Applying the interior \( L^p \) estimate for second order parabolic equations in [24, Theorem 7.22] to
\[
\begin{align*}
\frac{\partial}{\partial t}(V_1 - E_1^j) - \Delta(V_1 - E_1^j) + s \frac{\partial}{\partial x_3}(V_1 - E_1^j) &= g(V_1, V_2, t) - g(E_1^j, E_2^j, t), \\
\frac{\partial}{\partial t}(V_2 - E_2^j) - d\Delta(V_2 - E_2^j) + s \frac{\partial}{\partial x_3}(V_2 - E_2^j) &= f(V_1, V_2, t) - f(E_1^j, E_2^j, t)
\end{align*}
\]
in \( B(x_0, 2) \times \mathbb{R} \) with \( x_0 \in Q_j \), we obtain
\[
\lim_{R \to \infty} \sup_{x_0 \in Q_j, |x_0| \geq R} \left\| V(\cdot, \cdot) - E_j(\cdot, \cdot) \right\|_{W^{2,1}(B(x_0, 1) \times [0, T])} = 0.
\]
Then we have
\[
\lim_{R \to \infty} \sup_{x \in Q_j, |x| \geq R, t \in [0, T]} \left| \frac{\partial}{\partial x_3} V_i(x, t) - \frac{\partial}{\partial x_3} E_j^i(x, t) \right| = 0.
\]
Thus, by virtue of the estimate on \( E_j^i \) there exist \( \tilde{R}_j > R_j \) such that
\[
\min_{x \in \Omega_i \cap Q_j, |x| \geq \tilde{R}_j, t \in [0, T]} \frac{\partial}{\partial x_3} V_i(x, t) > 0.
\]
Apply the above argument to all \( j = 1, \ldots, n \) and \( i = 1, 2 \), we can obtain (3.11).

The assumption \( |x_3 + h(x')| \to \infty \) implies \( \text{dist}(x, \Gamma) \to \infty \). It follows that
\[
\lim_{R \to +\infty} \sup_{x_3 + h(x') \geq R, t \in [0, T]} |V(x, t) - 1| = 0
\]
and
\[
\lim_{R \to +\infty} \sup_{x_3 + h(x') \leq -R, t \in [0, T]} |V(x, t)| = 0,
\]
which yields
\[
\lim_{R \to +\infty} \sup_{|x_3 + h(x')| \geq R, t \in [0, T]} |g(V_1, V_2, t)| + |f(V_1, V_2, t)| = 0.
\]
By the interior \( L^p \) estimate [24, Theorem 7.22] again, we have
\[
\lim_{R \to \infty} \sup_{1 \leq i \leq 2} \left\{ |V_i|_{L^p(B(x, 1) \times [0, T])} \mid x \in \mathbb{R}^3, |x_3 + h(x')| \geq R \right\} = 0
\]
for $p > 3$. Therefore, we have

$$
\lim_{R \to +\infty} \sup_{|x_3 + h(x')| \geq R, t \in [0, T]} \left| \frac{\partial V_i}{\partial x_3}(x, t) \right| = 0, \quad i = 1, 2.
$$

This completes the proof. □

**Lemma 3.5.** Fix $\delta \in (0, 1)$ small. For any $x \in \mathbb{R}^3$ with $\delta \leq \hat{E}_i(x, t) = \max_{1 \leq j \leq n} E_i^j(x, t) \leq 1 - \delta$, we have

$$
\inf_{0 < \rho < \rho_0} \frac{\hat{E}_i(x', x_3 + \rho, 0) - \hat{E}_i(x, 0)}{\rho} \geq \min_{1 \leq i \leq 2} \left\{ \min_{1 \leq j \leq n} \inf_{z \in E_i^j(x, t) \leq 1 - \frac{\delta}{2}, t \in [0, T]} \frac{\partial E_i^j(x, t)}{\partial x_3} \right\} > 0,
$$

where $\rho_0$ is a positive constant depending on $\delta$ and is independent of $x$.

**Proof.** Fix $i = 1, 2$. By the continuity of $\hat{E}_i(x, t)$, there is $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$,

$$
\frac{\delta}{2} \leq \hat{E}_i(x', x_3 + \rho, t) \leq 1 - \frac{\delta}{2},
$$

where $x \in \mathbb{R}^3$ and $t \in [0, T]$ with $\delta \leq \hat{E}_i(x, t) \leq 1 - \delta$. Obviously, for any $x_0 = (x_{10}, x_{20}, x_{30})$ with $\delta \leq \hat{E}_i(x_0, t) \leq 1 - \delta$, there is $j_0 = \{1, 2, ..., n\}$ such that

$$
\hat{E}_i(x_0, t) = E_i^{j_0}(x_0, t).$$

It then follows that

$$
\hat{E}_i(x_{10}, x_{20}, x_{30} + \rho, t) - \hat{E}_i(x_0, t) = \hat{E}_i(x_{10}, x_{20}, x_{30} + \rho, t) - E_i^{j_0}(x_0, t)
$$

$$
\geq \rho \min_{1 \leq i \leq 2} \left\{ \min_{1 \leq j \leq n} \inf_{z \in E_i^j(x, t) \leq 1 - \frac{\delta}{2}, t \in [0, T]} \frac{\partial E_i^j(x, t)}{\partial x_3} \right\}.
$$

By the arbitrariness of $\rho$ and $x_0$, we can obtain the results. This completes the proof. □

4. **Asymptotic stability and uniqueness.** In this section, we develop the arguments in [36] and [44] to establish the stability and uniqueness of time periodic pyramidal fronts $(V_1(x, t), V_2(x, t))$. We divide into two cases: $v_0(x) \geq v^-(x, 0)$ and $v_0(x) \leq v^-(x, 0)$ to prove Theorem 1.2.

First of all, we prove the asymptotic stability of $(V_1(x, t), V_2(x, t))$ for (2.5) with the initial value $v_0(x)$ satisfies

$$
v_0(x) \geq v^-(x, 0) \quad \forall x \in \mathbb{R}^3.
$$

Take $0 < \varepsilon < \varepsilon_0^+$ and $0 < \alpha < \alpha^+(\varepsilon)$. Let $(u_1^+(x; t; \varepsilon, \alpha), u_2^+(x; t; \varepsilon, \alpha))$ be the supersolution of (2.5). Define

$$
(V_1^+(x, t), V_2^+(x, t)) := \lim_{k \to \infty} (u_1(x, t + kT; u_1^+, 0), u_2(x, t + kT; u_2^+, 0)) \quad (4.1)
$$

for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, where $u_1^+, u_2^+$ is a supersolution of (2.5) and satisfies $u_i^+ (\cdot, t + \varepsilon, \alpha)$ for $i = 1, 2$. Consequently, we have $(u_1(x, t + kT; u_1^+, 0), u_2(x, t + kT; u_2^+, 0))$ is non-increasing for any $x \in \mathbb{R}^3$, $t \in [0, T]$ and $k \in \mathbb{N}$. Then $(u_1(x, t + kT; u_1^+, 0), u_2(x, t + kT; u_2^+, 0))$ converges to $(V_1^+(x, t), V_2^+(x, t))$ under the norm $\| \cdot \|_{C^{\beta,1}(\mathbb{R}^3 \times [0, T], \mathbb{R}^2)}$ as $k \to \infty$. Obviously, $(V_1^+(x, t), V_2^+(x, t)) = (V_1^+(x, t + T), V_2^+(x, t + T))$ and

$$
(V_1(x, t), V_2(x, t)) \leq (V_1^+(x, t), V_2^+(x, t)) \quad \text{for all } x \in \mathbb{R}^3, t \in [0, T].
$$
It then follows from the similar arguments of [44, Lemma 4.12] that $V_i(x, t) = V^*_i(x, t)$ for any $x \in \mathbb{R}^3$ and $t \in [0, T]$.

**Theorem 4.1.** Assume that (H1)-(H4) hold. Suppose that $v_0(x) \in C(\mathbb{R}^3, [0, 1])$ satisfies $v_0(x) \geq v^-(x, 0)$ for any $x \in \mathbb{R}^3$ and

$$\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} |v_0(x) - v^-(x, 0)| = 0,$$

then the solution $(v_1(x, t; v_0), v_2(x, t; v_0))$ of (2.5) satisfies

$$\lim_{k \to \infty} \|v(\cdot, kT + \cdot; v_0) - V(\cdot, \cdot)\|_{C([\mathbb{R}^3 \times [0, T]])} = 0.$$

**Proof.** We develop the arguments of [44, Theorem 4.13]. Note that $u^+(x, t; \varepsilon, \alpha)$ is the supersolution of (2.5) for $0 < \varepsilon < \varepsilon_0^*$ and $0 < \alpha < \alpha^+(\varepsilon)$. Let $p_* = \min_{t \in [0, T]} \{p^+_1(t), p^+_2(t)\}$. Choose $\delta > 0$ small enough such that $v_i(x, t + kT; v_0) \leq u^+_i(x, t; \varepsilon, \alpha) + \delta p_*$ for all $x \in \mathbb{R}^3$ and $t \in [0, T]$. By using arguments similar to that in [36] and [34], we have

$$\lim_{k \to \infty} \|v(x, t + kT; v^-) - V(x, t)\|_{L^\infty(\mathbb{R}^3 \times \{0, T\})} = 0$$

and

$$\lim_{k \to \infty} \|v(x, t + kT; \hat{u}^{+,0}_i) - V(x, t)\|_{L^\infty(\mathbb{R}^3 \times \{0, T\})} = 0.$$

Take $\hat{k} > 0$ large enough such that

$$v_i(x, t + kT; v^-) \leq v_i(x, t + kT; \hat{u}^{+,0}_i) < V_i(x, t) + \delta p_*$$

for $x \in \mathbb{R}^3$, $k > \hat{k}$ and $t \in [0, T]$. Let $\rho$ and $\beta$ be as in Lemma 2.4 and note that $\rho$ and $\beta$ are independent of $\delta$. We have that $W^+(x, t; \delta)$ is a supersolution of (2.14). Then there exist $\hat{k} > 0$ large enough such that

$$v_i(x, t + (k + k_1)T; v_0) \leq u^+_i(x', x_3 + \rho \delta, t; \varepsilon, \alpha) + \delta e^{-\beta \hat{k} T} p_*$$

for $x \in \mathbb{R}^3$, $t \in [0, T]$ and $k \geq \hat{k}$. Let

$$\hat{u}^{+,\delta}_i(x) = \left(\min \{u^+_i(x', x_3 + \rho \delta, 0, 1), \min \{u^+_2(x', x_3 + \rho \delta, 0, 1)\}\right).$$

Then

$$v_i(x, (\hat{k} + k_1)T; v_0) \leq \hat{u}^{+,\delta}_i(x) + \delta e^{-\beta \hat{k} T} p_*$$

for $x \in \mathbb{R}^3$. Lemma 2.1 implies

$$v_i(x, (\hat{k} + k_1 + k)T; v_0) \leq v_i(x, (\hat{k} + k)T; \hat{u}^{+,\delta}_i) + \delta p_*$$

for $x \in \mathbb{R}^3$. By (4.3), we have

$$v_i(x, (k_1 + \hat{k} + k)T; v_0) \leq V_i(x', x_3 + \rho \delta, 0) + 2\delta p_*, \quad i = 1, 2$$

for $x \in \mathbb{R}^3$. By Lemma 2.4, it follows that

$$v_i(x, t + (k + k_1 + \hat{k} + k)T; v_0) \leq W^+_i(x', x_3 + \rho \delta, t + kT; 2\delta), \quad i = 1, 2$$

for $k \geq 0$ and $t \in [0, T]$. Therefore, we have

$$V_i(x, t) \leq v_i(x, t + kT; v_0) \leq V_i(x', x_3 + \rho \delta + 2\rho \delta, t) + 2\delta p_* \leq V_i(x, t) + M^* \delta,$$

for $k > k_1 + \hat{k} + \hat{k}$ and $t \in [0, T]$, where $M^* > 0$ is a constant and is independent of $\delta$. Due to the arbitrariness of $\delta$, we obtain the conclusion of this theorem. This completes the proof. $\square$
Next, we consider the case that the initial value $v_0(x) = (v_{10}(x), v_{20}(x))$ satisfies $v_0(x) \leq v^-(x, 0)$ in $\mathbb{R}^3$. Take $0 < \varepsilon < \min \varepsilon_0$ small. Define $\overline{\Psi}(x, t) = (\overline{\Psi}_1(x, t), \overline{\Psi}_2(x, t))$ and $\overline{\Psi}_1(x, t) = (\overline{\Psi}_1(x, t), \overline{\Psi}_2(x, t))$ as follows:

$$\overline{\Psi}_i(x, t) := V_i(x', x_3 - M', t) \quad \text{and} \quad \overline{\Psi}_i(x, t) := \max \left\{ u_i^-(x; t, \varepsilon, \alpha), \overline{\Psi}_i(x, t) \right\},$$

where $i = 1, 2$ and $M'$ is a positive constant that will be defined later. Recall that $\overline{w}$ is defined in Lemma 2.5 and $w^-$ is defined in Lemma 2.3, we also define

$$\overline{w}(x, t; \delta) = w^-(x', x_3 - M', t), \quad (\overline{w}_1(x, t; \delta), \overline{w}_2(x, t; \delta)) = \max \left\{ (\overline{w}_1(x, t; \delta), \overline{w}_2(x, t; \delta)) \right\}.$$

**Lemma 4.1.** For any small constant $\delta$ and any initial value $v_0(x) \in C(\mathbb{R}^3, [0, 1])$ with

$$\lim_{\gamma \to \infty, x \in D(\gamma)} \sup_{\gamma \to \infty} |v_0(x) - v^-(x, 0)| = 0,$$

there exist positive constants $\alpha < \alpha^-(\varepsilon)$, $\varepsilon < \varepsilon_0$, $M'$ and an integer $k_1'$ such that

$$\overline{\Psi}(x, t) - \delta q^+(t) \leq v(x, k_1'T + t; v_0)$$

for $x \in \mathbb{R}^3$ and $t \in [0, T]$.

The proof is similar to that of [44, Lemma 4.14] and is omitted. Take $0 < \delta < 1$ small. For $x \in \mathbb{R}^3$, $t \in [0, T]$, define $v^\delta_i(x, t) = (\overline{\Psi}_i^1(x, t), \overline{\Psi}_i^2(x, t))$ as follows

$$v^\delta_i(x, t) := \max \left\{ \psi^\delta_i(x', t, x_3 - m_* \rho \delta, t) \right\}, \quad i = 1, 2,$$

where

$$v^\delta_i(x, t) = \overline{\Psi}(\bar{\psi}(x), t) - \varepsilon Q(\bar{\psi}(x), t) \sech (\alpha x_3), \quad \bar{\psi}(x) = \frac{h(x')/m_* - \rho \delta + \psi(\alpha x_3)/\alpha}{\sqrt{1 + \psi'(\alpha x_3)^2}}.$$

Since $\overline{w}(x, t; \delta) \leq v(x, t + k'T; v_0)$, let $t \to \infty$ we have

$$\overline{v}^\delta_i(x, t) \leq \liminf_{t \to \infty} v(x, t; v_0). \quad (4.4)$$

**Lemma 4.2.** One has

$$\lim_{R \to \infty, \sup_{|x| \leq R, t \in [0, T]} |v^\delta_i(x, t) - v^-(x', x_3 - m_* \rho \delta, t)| \geq 0. \quad (4.5)$$

**Proof.** It is obvious that $\lim_{x_3 \to +\infty} |v^\delta_i(x, t) - v(x', x_3 - m_* \rho \delta, t)| = 0$ uniformly for $x' \in \mathbb{R}^2$ and $t \in [0, T]$. Moreover, we also can prove that

$$\lim_{R \to \infty, \sup_{|x_3 + h(x')| \leq R, t \in [0, T]} |v^\delta_i(x, t) - v(x', x_3 - m_* \rho \delta, t)| = 0.$$

Next we consider $|x_3 + h(x')| \leq X_2$ for some $X_2 > 0$ sufficiently large and $x_3 < X_1$. To ensure that $|x| \to +\infty$, there must be $x_3 \to -\infty$. By the definition of $v^\delta_i(x, t), \overline{w}(x, t; \delta), \overline{w}(x, t; \delta)$, we obtain (4.5). $\square$

**Theorem 4.2.** Assume that (H1)-(H4) hold. If initial value $v_0(x) \in C(\mathbb{R}^3, [0, 1])$ satisfies $v_0(x) \leq v^-(x, 0)$ for any $x \in \mathbb{R}^3$ and

$$\lim_{\gamma \to \infty, x \in D(\gamma)} \sup_{\gamma \to \infty} |v_0(x) - v^-(x, 0)| = 0,$$

then the solution $v(x, t; v_0)$ of (2.5) satisfies

$$\lim_{k \to \infty} \|v(x, kT + t; v_0) - V_2(x, t)\|_{C(\mathbb{R}^3 \times [0, T])} = 0.$$
Proof. Fix $\delta > 0$ small enough. By $v_0(x) \leq v^-(x, 0)$ we have $v(x, t; v_0) \leq V(x, t)$ for any $x \in \mathbb{R}^3$ and $t > 0$. Let $\nabla^{\delta}_{0}(x) = \nabla^{\delta}(x, 0)$. Note that the limit of $v(x, kT + t; \nabla^{\delta}_{0})$ as $k \to \infty$ exists. Let

$$V^{\delta}(x, t) := \lim_{k \to \infty} v(x, kT + t; \nabla^{\delta}_{0})$$

and then $V^{\delta}(x, t)$ satisfies (2.5). Moreover, for $x \in \mathbb{R}^3$ and $t \in [0, T]$, $V(x', x_3 - m_\star \rho \delta, t) \leq V^{\delta}(x, t) \leq V(x, t)$ and $V^{\delta}(\cdot, \cdot) = V^{\delta}_{1}(\cdot, \cdot + T)$. It then follows that there exists $\hat{k} > 0$ such that $v_i(x, t + kT; V^{\delta}_{0}) \geq V_i(x', x_3 - m_\star \rho \delta, t) - \delta q^*$ for $k \geq \hat{k}$, where $q^* = \max_{t \in [0, T]} \{q^+_1(t), q^+_2(t)\}$. It then follows from Lemma 4.1 that

$$v(x, t + kT; v_0) \geq \nabla(x, t) - \delta q^+(t).$$

By (4.4), there exist $k' > 0$ so that

$$v_i(x, t + kT; v_0) \geq \nabla^{\delta}_{i}(x, t) - \delta q^* e^{-\beta kT}, \quad i = 1, 2$$

for $k \geq k'$ and $t \in [0, T]$. Then by Lemma 2.1, we have

$$v_i(x, (k' + \hat{k} + k')T; v_0) > v(x, \hat{k}T; V^{\delta}_{0}) - \delta q^*.$$

Therefore, we have

$$v_i(x, (k' + \hat{k} + k')T; v_0) > V_i(x', x_3 - m_\star \rho \delta, 0) - 2\delta q^*, \quad i = 1, 2$$

for $x \in \mathbb{R}^3$. By Lemma 2.5, we have

$$v_i(x, t + kT + (k' + \hat{k} + k')T; v_0) > W_i^-(x', x_3 - m_\star \rho \delta, t + kT; 2\delta) \quad i = 1, 2$$

for $k \geq 0$ and $t \in [0, T]$. Then

$$V_i(x, t) \geq v_i(x, t + (k + k' + \hat{k} + k')T; v_0)$$

$$\geq V_i(x', x_3 - m_\star \rho \delta - 2\rho \delta, t) - 2\delta q^* e^{-\beta (t + kT)}$$

for $k \geq 0$, $t \in [0, T]$ and $i = 1, 2$. It then follows that for $k > k_{\delta} := k' + \hat{k} + k'$,

$$v_i(x, t + kT; v_0) \geq V_i(x, t) - 2\delta q^* - 2M' \rho \delta - M'm_\star \rho \delta,$$

where $M' = \max_{x \in \mathbb{R}^3, t \in [0, T]} \left| \frac{\partial}{\partial x} V_i(x, t) \right|$. By the arbitrariness of $\delta > 0$, we have that $v(\cdot, t; v_0)$ converges to $V(\cdot, \cdot)$ as $t \to \infty$ in $\| \cdot \|_{C(\mathbb{R}^3 \times [0, T] ; \mathbb{R}^3)}$. This completes the proof. \hfill \Box

Proof of Theorem 1.2. Let the initial value $v_0(x) \in C(\mathbb{R}^3, [0, 1])$ satisfies

$$\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} |v_0(x) - v^-(x, 0)| = 0.$$
Corollary 4.1. Let \((V_1, V_2)\) be the three-dimensional periodic traveling front associated with the pyramidal \(-x_3 = h(x^i)\). If (2.5) has a solution \((V_1, V_2)\) with

\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), t \in [0, T]} |V_1'(x, t) - \hat{E}_1(x, t)| = 0,
\]

then we have \(V_1'(x, t) \equiv V_i(x, t)\) for any \(i = 1, 2\).

5. Appendix. This appendix is devoted to the proof of Lemma 3.2.

Proof of Lemma 3.2. Let

\[
I_j := \Omega_j \cap \Omega_{j+1} = \left\{ r \left( \frac{A_j + A_{j+1}}{B_j + B_{j+1}} \right) \left| 1 \leq j \leq n - 1, \right. \right\},
\]

\[
I_n := \Omega_n \cap \Omega_1 = \left\{ r \left( \frac{A_n + A_1}{B_n + B_1} \right) \left| 1 \leq j \leq n - 1, \right. \right\}.
\]

Then \(I_j\) is the projection of \(\Gamma_j\) onto the \(x_1 - x_2\) plane and \(\bigcup_{j=1}^{n-1} I_j\) is the projection of \(\Gamma\) onto the \(x_1 - x_2\) plane.

Fix \(j \in \{1, \ldots, n\}\). Without loss of generality, we assume \(x \in Q_j\) for some \(j\) as \(|x| \to \infty\). Since \((\partial/\partial x_1)^2 + (\partial/\partial x_2)^2\) is invariant under rotations on the \(x_1 - x_2\) plane, we assume \(\Omega_j \cap \Omega_{j+1} = \{(0, x_2, 0) | x_2 \geq 0\}\), \((A_j, B_j) = (A, B)\) and \((A_{j+1}, B_{j+1}) = (-A, B)\), where \(A > 0, B > 0\) and \(A^2 + B^2 = 1\). Two planes \(S_{j+1}\) and \(S_j\) are \(-x_3 = m_s(-Ax_1 + Bx_2)\) and \(-x_3 = m_s(Ax_1 + Bx_2)\), respectively. The common line \(\Gamma_j\) of them is \(x_1 = 0, -x_3 = m_sBx_2\). The projection of \(Q_j\) onto \(x_1 - x_2\) plane is given by \(\{x_2 \geq a|x_1|, x_1 \geq 0\} \cup \{x_2 \geq b|x_1|, x_1 \leq 0\}\) for some constants \(a > 0, b > 0\).

In fact, (3.2) implies

\[
\lim_{\gamma \to \infty} \sup_{x \in D(\gamma), x \in Q_j} \left| v_0(x) - \Psi \left( \frac{c}{\delta}(x_3 + m_sBx_2 + m_sA|x_1|, 0) \right) \right| = 0.
\]

The unit normal vector of the common line \(\Gamma_j\) directing downwards and lying on \(\{x_1 = 0\}\) is given by \(\frac{1}{\sqrt{1 + m_sB^2}}(0, m_sB, -1)^T\).

Recall that \(2\theta_j\) is the angle between \(S_j\) and \(S_{j+1}\) \((0 < \theta_j < \frac{\pi}{2})\). Then we have \(\sin \theta_j = \sqrt{1 + m_sB^2}/\sqrt{1 + m_s^2}\). The change of variable is as follows

\[
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & \frac{m_s}{\sqrt{1 + m_sB^2}} & \frac{0}{\sqrt{1 + m_sB^2}} \\
0 & \frac{0}{\sqrt{1 + m_sB^2}} & \frac{1}{\sqrt{1 + m_sB^2}} \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]
For this change of variables, we have

$$
\Psi_i \left( \frac{c}{s_j} \left( \eta + \frac{s_j^2 - c^2}{c} |x_i| \right), t \right) = \Psi_i \left( \frac{c}{s} (x_3 + m_s B x_2 + m_s A |x_1|), t \right), \forall i = 1, 2,
$$

where \( s_j = \frac{c}{\sqrt{1 + m_s^2 B^2}} \). Observe that

$$
E^j(x, t) = \tilde{V}(\eta, \eta, t; s_j) = \tilde{V} \left( -x_1, \frac{x_3 + m_s B x_2}{\sqrt{1 + m_s^2 B^2}}, t; s_j \right)
$$

is the solution of (2.5). Let \( \tilde{W}(\xi, \eta, t) := (\tilde{W}_1(\xi, \eta, t; \tilde{W}_0), \tilde{W}_2(\xi, \eta, t; \tilde{W}_0)) \) be the solution of

$$
\begin{cases}
\frac{\partial \tilde{W}_1}{\partial t} - \frac{\partial^2 \tilde{W}_1}{\partial \eta^2} - \frac{\partial^2 \tilde{W}_1}{\partial \xi^2} + s_j \frac{\partial \tilde{W}_1}{\partial \gamma} - g(\tilde{W}_1, \tilde{W}_2, t) = 0, \\
\frac{\partial \tilde{W}_2}{\partial t} - d\frac{\partial^2 \tilde{W}_2}{\partial \eta^2} - d\frac{\partial^2 \tilde{W}_2}{\partial \xi^2} + s_j \frac{\partial \tilde{W}_2}{\partial \gamma} - f(\tilde{W}_1, \tilde{W}_2, t) = 0,
\end{cases}
$$

(5.1)

Take

$$
W_0(x) = \tilde{W}_0 \left( -x_1, \frac{x_3 + m_s B x_2}{\sqrt{1 + m_s^2 B^2}} \right),
$$

we have

$$
(W_1(x, t; W_0), W_2(x, t; W_0)) = (\tilde{W}_1(\xi, \eta, t; \tilde{W}_0), \tilde{W}_2(\xi, \eta, t; \tilde{W}_0))
$$
satisfying

$$
\begin{cases}
\frac{\partial W_1}{\partial t} - \Delta W_1 + s \frac{\partial W_1}{\partial x_3} - g(W_1, W_2, t) = 0, \\
\frac{\partial W_2}{\partial t} - d\Delta W_2 + s \frac{\partial W_2}{\partial x_3} - f(W_1, W_2, t) = 0, \\
W_i(x, 0) = W_{i0}(x), \quad (\xi, \eta) \in \mathbb{R}^2, i = 1, 2.
\end{cases}
$$

(5.2)

By (3.1) and (3.2), \( \lim_{\gamma \to \infty} \sup_{x \in D(\gamma) \cap Q_j, |v_0(x) - E^j(x, 0)| > 0} \) choose a function \( g_i(\cdot) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}) \) (i = 1, 2) satisfying

$$
g_i(\gamma) = \sup_{x \in D(\gamma) \cap Q_j} \left| v_{i0}(x) - E^j_i(x, 0) \right|, \quad \forall \gamma \geq 1,
$$

$$
\sup_{x \in D(\gamma) \cap Q_j} \left| v_{i0}(x) - E^j_i(x, 0) \right| \leq g_i(\gamma) \leq 2 + \| V_{0i} \|_{L^\infty(\mathbb{R})}, \quad 0 < \gamma < 1,
$$

$$
g'_i(\gamma) \leq 0, \quad 0 < \gamma < 1,
$$

$$
g_i(\gamma) = g_i(-\gamma), \quad \forall \gamma \in \mathbb{R}.
$$

Obviously, \( g_i(\gamma) \) is monotone non-increasing in \( \gamma > 0 \) and \( \lim_{\gamma \to \infty} g(\gamma) = 0 \). Since

$$
dist(x, \Gamma) = dist(x, \Gamma_j) = \frac{\sqrt{(1 + m_s^2 B^2)x_1^2 + (x_3 + m_s B x_2)^2}}{\sqrt{1 + m_s^2 B^2}}
$$

for all \( x \in Q_j, \) we have

$$
\left| v_{i0}(x) - E^j_i(x, 0) \right| \leq g_i(dist(x, \Gamma)) = g_i \left( \frac{\sqrt{(1 + m_s^2 B^2)x_1^2 + (x_3 + m_s B x_2)^2}}{\sqrt{1 + m_s^2 B^2}} \right)
$$

(5.3)
for all $x \in Q_j$. Let

$$\tilde{W}_i^0(\xi, \eta) := \min \left\{ \tilde{V}_i(\xi, \eta, 0; \sigma) + g_i(\sqrt{x^2 + \eta^2}), 1 \right\}, \quad i = 1, 2,$$

$$\tilde{W}_i^2(\xi, \eta) := \max \left\{ \tilde{V}_i(\xi, \eta, 0; \sigma) - g_i(\sqrt{x^2 + \eta^2}), 0 \right\}, \quad i = 1, 2.$$  

We study (5.1) with the initial value $(\tilde{W}_i^\pm(\xi, \eta), \tilde{W}_2^\pm(\xi, \eta))$ is equivalent to study (5.2) for $W_i^\pm(x) = (W_i^\pm(x), V_i^\pm(x))$, where

$$W_i^0(x) = \min \left\{ E_i^j(x, 0) + g_i \left( \sqrt{x_1^2 + \frac{1}{1 + m_2^2 B^2}(x_3 + m_s B x_2)^2} \right), 1 \right\},$$

$$W_i^2(x) = \max \left\{ E_i^j(x, 0) - g_i \left( \sqrt{x_1^2 + \frac{1}{1 + m_2^2 B^2}(x_3 + m_s B x_2)^2} \right), 0 \right\}.$$

Then we have

$$\lim_{R \to \infty} \sup_{\xi^2 + \eta^2 > R^2} \left| \tilde{W}_i^\pm(\xi, \eta) - \tilde{V}_i(\xi, \eta, 0; s_j) \right| = 0, \quad i = 1, 2.$$  

For $s_j = \frac{c}{\sqrt{1 + m_2^2 B^2}}$, applying Theorem 2.1 we have

$$\lim_{k \to \infty} \left\| \tilde{W}(\xi, \eta, t + kT; \tilde{W}_0^\pm) - \tilde{V}(\xi, \eta, t; s_j) \right\|_{C(R^2 \times [0, T])} = 0,$$

which implies that

$$\lim_{k \to \infty} \left\| W(x, t + kT; W_0^\pm) - E^j(x, t) \right\|_{C(R^3 \times [0, T])} = 0.$$  

Take $k_j' \in \mathbb{N}$ large enough such that

$$\sup_{k \geq k_j'} \left\| W(\cdot, t + kT; W_0^\pm) - E^j(\cdot, t) \right\|_{C(R^3 \times [0, T])} < \frac{\varepsilon_1}{2}. \quad (5.4)$$  

Let $v^+(x, t) = v(x, t; v_0) - W(x, t; W_0^\pm)$. Then $(v_1^+, v_2^+)$ satisfies

$$\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + s \frac{\partial}{\partial x_1} \right) v_1^+(x, t) + \sum_{k=1}^{2} a_k v_k^+(x, t) = 0,$$

$$\left( \frac{\partial}{\partial t} - d \frac{\partial^2}{\partial x_1^2} - d \frac{\partial^2}{\partial x_2^2} + s \frac{\partial}{\partial x_1} \right) v_2^+(x, t) + \sum_{k=1}^{2} b_k v_k^+(x, t) = 0$$

with $v_i^+(x, 0) = v_{i0}(x) - W_{i0}^+(x)$, $x \in \mathbb{R}^3$ for $i = 1, 2$, where

$$a_k := \int_{0}^{1} \frac{\partial}{\partial u_k} g(\theta v_1(x, t) + (1 - \theta) W_1, \theta v_2(x, t) + (1 - \theta) W_2, t) d\theta,$$

$$b_k := \int_{0}^{1} \frac{\partial}{\partial u_k} f(\theta v_1(x, t) + (1 - \theta) W, \theta v_2(x, t) + (1 - \theta) W, t) d\theta.$$  

In particular, from (5.3) we have $v^+(x, 0) \leq 0$ and $v^-(x, 0) \geq 0$ for all $x \in Q_j$. Let $\tilde{v}^\pm(x, t) = \left( \tilde{v}_1^\pm(x, t), \tilde{v}_1^\pm(x, t) \right)$ be defined by

$$\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + s \frac{\partial}{\partial x_1} \right) \tilde{v}_1^+(x, t) + \sum_{k=1}^{2} a_k \tilde{v}_k^+(x, t) = 0, \\
\left( \frac{\partial}{\partial t} - d \frac{\partial^2}{\partial x_1^2} - d \frac{\partial^2}{\partial x_2^2} + s \frac{\partial}{\partial x_1} \right) \tilde{v}_2^+(x, t) + \sum_{k=1}^{2} b_k \tilde{v}_k^+(x, t) = 0,
\end{array} \right. \quad x \in \mathbb{R}^3, t > 0.$$
with 
\[ \hat{v}_i^+(x,0) = \max \{ v_{i_1}^+(x,0), 0 \}, \quad \hat{v}_i^-(x,0) = \max \{ -v_{i_1}^-(x,0), 0 \}, \quad i = 1, 2. \]

It is easy to see \( \hat{v}_i^+(x,0) \geq v_i^+(x,0) \) and \( -\hat{v}_i^-(x,0) \leq v_i^-(x,0) \) for \( x \in \mathbb{R}^3 \) and \( i = 1, 2. \) By the comparison principle, we obtain
\[
\hat{v}_i^+(x,t) \geq v_i^+(x,t), \quad -\hat{v}_i^-(x,t) \leq v_i^-(x,t) \quad \text{for} \ (x,t) \in \mathbb{R}^3 \times [0,T], i = 1, 2. \tag{5.5}
\]

Note that \( |\hat{v}_i^+(x,0)| \leq 3 \) for \( x \in \mathbb{R}^3 \) and \( \hat{v}_i^+(x,0) = 0 \) for \( x \in Q_j \), where \( i = 1, 2. \)

Applying Lemma 2.1 to \( \hat{v}^\pm(x,t) \), for any \( t > 0 \) there exists \( \bar{A} > 0 \) and \( \bar{B} > 0 \) such that
\[
0 \leq \hat{v}_i^+(x,t) \leq 3e^{mt} \frac{3\pi \bar{A}}{B} \int_0^{t+\epsilon} \exp(-\bar{B}r^2) dr, \quad i = 1, 2
\]

for \( x \in Q_j \) and \( \sqrt{3}R < \text{dist}(x, \partial Q_j) \). For any fixed \( k \in \mathbb{N}^+ \),
\[
\lim_{R \to \infty} \sup_{x \in Q_j, \text{dist}(x, \partial Q_j) \geq R, t \in [0,T]} \hat{v}_i^+(x,t+kT) = 0, \quad i = 1, 2. \tag{5.6}
\]

Applying (5.3), (5.5) and (5.6) to
\[
v(x,t+kT; v_0) = v^\pm(x,t+kT) + W(x,t+kT; W_0),
\]

for given \( k \geq k_j' \) we can take a constant \( R_j > 0 \) large enough such that
\[
\sup_{x \in Q_j, \text{dist}(x, \partial Q_j) \geq R_j, t \in [0,T]} |v(x,t+kT; v_0) - E_j'(x,t)| < \epsilon_1. \tag{5.7}
\]

Thus we have obtain the estimates on \( Q_j \) for any given \( j \).

Set
\[
k_* = \max\{k_1', ..., k_n'\} \quad \text{and} \quad \hat{R} = \max\{R_1, ..., R_n\}.
\]

From the definition of \( \Gamma \) and \( Q_j \), we get
\[
\lim_{R \to \infty} \inf_{|x| \geq R, \text{dist}(x, \partial Q_j) \leq \hat{R}} \text{dist}(x, \Gamma) = \infty, \quad \forall \ 1 \leq j \leq n.
\]

Applying the argument stated above for each \( j \) \((1 \leq j \leq n) \), we have
\[
\max_{1 \leq j \leq n} \sup_{|x| \geq R, x \in Q_j, \text{dist}(x, \partial Q_j) \leq \hat{R}, t \in [0,T]} |v(x,t+kT; v_0) - E_j'(x,t)| < \epsilon_1
\]

for any \( k \geq k_* \). This together with (5.7), we obtain (3.8). This completes the proof. \( \square \)

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