Numerical analysis of three $\theta$ methods for neutral stochastic delay differential equations

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Abstract. This article focus on the numerical analysis of stochastic $\theta$-method, split-step $\theta$ method and one-leg $\theta$ method for neutral stochastic differential equations with time lag. When some stability conditions are met, the three $\theta$ methods turned out to be mean-square stable asymptomatically. From the numerical analysis, we can see that the asymptomatically mean-square stability for both linear and nonlinear stochastic differential equations depends on the step size $h$ and parameter $\theta$, which numerical analysis are based on stochastic $\theta$-method, split-step $\theta$ method and one-leg $\theta$ method.

1. Introduction

It is well known that because of the random variables affect the process of time change and the state on its certain past history, stochastic functional differential equations named SFDEs are used to simulate many physical phenomena. The stochastic delay differential equations (SDDEs) is changed by SFDEs which time delay is replaced by a constant. In recent years, both the relative theory and the numerical analysis for SFDEs have been got rapid development (see, for example, [1, 2, 3, 5, 6]). Actually, neutral stochastic functional differential equations often called NSFDEs can be seen as a special type of SFDEs. More and more experts and scholars have shown great interest in the research of NSFDEs recently, which make many well-known theorems in SFDEs applied to NSFDEs successfully. Like SFDEs, neutral stochastic delay differential equations (NSDDEs) is a special case of NSFDEs. The form of scalar NSDDE is

$$d \left( x(t) - N(x(t - \tau)) \right) = f(t, x(t), x(t - \tau))dt + g(t, x(t), x(t - \tau))dw(t), t \in J,$$

$$x(t) = \xi(t), t \in [-\tau, 0].$$

where $\tau$ represents a positive lag term, $N(X(t-\tau))$ represents a neutral term, $J = [0,T]$, $W(t)$ represents a one-dimensional standard Wiener process, then the functions $f$ and $g : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

We see eye to eye on this issue that there are very few conclusions on stochastic multiply step methods such as the $\theta$-methods and split-step $\theta$-method for NSDDEs, while conclusions for SDDEs is few else. In [4], X. Li and W. Cao funded a set of two-step Maruyama platform, and then it show out that linear scalar NSDDEs is mean-square stable exponentially, and numerical analysis on linear and nonlinear neutral stochastic differential systems with time-lag. This article study the asymptomatically mean-square stability of the three $\theta$-methods for the linear and nonlinear scalar NSDDEs. Under mild assumptions, three $\theta$-methods are turned out to be asymptomatically mean-square stable. In next section, numerical analysis are proved the stability for NSDDEs.
2. Numerical analysis

We use the sampled average over 2000 paths of the second-order moment in all our numerical simulations, the simulation software used is Matlab.

\[ E(x_n^2) = \frac{1}{2000} \sum_{i=1}^{2000} |x_n(\omega_i)|^2 \]

We list three \( \theta \) methods with different test parameters in the table below. As can be seen these equations meet the necessary conditions and are therefore mean square stable, as we will see later.

2.1 The following linear NSDDE considered

\[
\frac{d}{dt} \left( x(t) - \frac{1}{4}x(t - \tau) \right) = \left( ax(t) + bx(t - \tau) \right)dt + \left( cx(t) + dx(t - \tau) \right)dw(t), t \geq 0,
\]

\[ x(t) = t + \tau, t \in [-1, 0]. \]

to research the asymptotically mean square stability of three \( \theta \)-methods with \( \tau = 1 \). Combining the equations to solve the stability conditions, we enumerate two sets of coefficients for the above equations which maintain exponentially mean-square stable.

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![Fig. 1. Numerical analysis with three \( \theta \)-methods for above equation with coefficients.](image1.png)

Fig. 1. Numerical analysis with three \( \theta \)-methods for above equation with coefficients \( a \) is equal to \(-0.5\), \( b \) is equal to \(0.1\), \( c \) is equal to \(0.1\), \( d \) is equal to \(0.1\), where \( \theta = 1 \), \( \Gamma = 1.2667 \), \( \Lambda = 1.0667 \). then (a) \( h = 1 \); (b) \( h = 1/2 \).

![Fig. 2. Numerical analysis with three \( \theta \)-methods for the above equation.](image2.png)

Fig. 2. Numerical analysis with three \( \theta \)-methods for the above equation with coefficients \( a \) is equal to \(-10\), \( b \) is equal to \(1\), \( c \) is equal to \(1\), \( d \) is equal to \(1\), where \( \theta = 0.9999 \), \( \Gamma = 0.0636 \), \( \Lambda = 0.0091 \). (a) \( h = 1 \); (b) \( h = 1/2 \).
Firstly, we study the coefficients of the equation satisfy $\Gamma > 1, \Lambda > 1$, which meet mean square stability of the three $\theta$ methods asymptotically. In Figure 1, it is shown that for the large step size $h=1$, the second-order moment of numerical solution tends to zero. Secondly, when $\Gamma < 1, \Lambda < 1$, the effect of the parameter $\theta$ and the step size $h$ on the exponential mean-square stability of the three $\theta$-methods is proved, in which $a$ is equal to $-10$, $b$ is equal to $1$, $c$ is equal to $1$, $d$ is equal to $1$. By Theorem, when $\theta \in [0.9999, 1]$ the three $\theta$-methods are mean-square stable asymptotically. In addition, when $\theta \in [0, 0.9999)$ with $h < h_0 = 0.0001/(1 - \theta)$, it is shown that the second-order moment of numerical solution tends to zero. In Figure 2, it is shown that the second-order moment declines, when the step size $h$ is equal to $1/2$ and $h$ is equal to $1$ for $\theta$ is equal to $0.9999$.

2.2 The following nonlinear NSDDE considered

$$d\left(x(t) - \frac{1}{2}\sin(x(t - \tau))\right) = \left(a(t) + bx(t - \tau)\sin(x(t - \tau))\right)dt + \left(c(t) + \cos(x(t - \tau))\right)dw(t), t \geq 0,$$

$$x(t) = t + \tau, t \in [-1, 0].$$

to research the asymptotically mean-square stability of three $\theta$-methods with $\tau = 1$. Combining the equations to solve the stability conditions, we enumerate two sets of coefficients for the above equations which maintain mean-square stable exponentially.

![Image](image1.png)

Fig. 3. Numerical analysis with three $\theta$-methods for the above equation with coefficients $a$ is equal to $-0.5$, $b$ is equal to $0.1$, $c$ is equal to $0.1$, where $\theta = 1$, $\Gamma = 1.3$, $\Lambda = 1.2$. (a) $h = 1$; (b) $h = 1/2$.

![Image](image2.png)

Fig. 4. Numerical analysis with three $\theta$-methods for Eq.(6.2) with coefficients $a$ is equal to $-10$, $b$ is equal to $1$, $c$ is equal to $1$, where $\theta = 0.96$, $\Gamma = 0.0727$, $\Lambda = 0.0455$. (a) $h = 1$; (b) $h = 1/2$. 


Firstly, when the coefficients of equation such as $\Gamma \geq 1, \Lambda \geq 1$, we study the mean-square stability of the three $\theta$ methods asymptotically. In Figure 3, when the large step size $h$ and $\tau$ are both equal to 1 for the above equation, it is proved that the second-order moment of numerical solution tends to zero. In figure 3, it is shown that when the step size $h$ is equal to $1/2$ and $h$ is equal to 1 for $\theta$ is equal to 0.96, the second-order moment declines. The figure illustrates that when $\theta$ ($\theta = 0.2$) and the step size $h$ is small enough, the second-order moment becomes recession.

3. Conclusion
In this article, we study the asymptotically mean-square stability of the stochastic $\theta$-method, split step $\theta$-method and one leg $\theta$-method for general NSDDEs numerically. This paper gives the conditions under which the stochastic $\theta$-method, split-step $\theta$-method and one-leg $\theta$-method maintain asymptotically mean-square stability. In mild stable assumptions, numerical simulations confirm the relationship that relative theory indicates the parameter $\theta$ and the step size $h$ for mean-square stability of the NSDDEs asymptotically. In further study work, numerical studies such as convergence are required.

Acknowledgement
This work was partially supported by the NSF of China (nos.10901036 and11271068)

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