Before embarking on details, here is one general piece of advice. One often hears that modal (or some other) logic is pointless because it can be translated into some simpler language in a first-order way. Take no notice of such arguments.

Dana Scott (1970)
Preface

This is the first part of a threefold survey on the subject of modal $\lambda$-calculi. Why threefold? We believe that it is fair to say that work on constructive modal logic spans three distinct yet intercommunicating streams:

Curry-Howard: Work with a Curry-Howard concerns $\lambda$-calculi that arose as byproducts of the proof theory of modal logic, and their associated computational and categorical interpretations. This stream of work can be seen to date as far back as the 1950s, but witnessed rapid development during the 1990s, following the discovery of Girard’s Linear Logic. Tracing the development of this material forms the subject of the first part of our survey.

Metaprogramming: It was sooner or later realized—as a result of the work on the Curry-Howard—that modal annotations can form the logical foundation that is much needed for all sorts of metaprogramming, whether that comes in the form of binding-time analysis, staging of computations, or even partial evaluation. A flurry of developments on this stream happened in the early 2000s, and further work continues to appear in the 2010s.

Other applied calculi: A third stream of work, which constitutes a radical break from the proof-theoretic origins, is the use of modalities in all sorts of calculi that are fine-tuned to specific task. There are many possible such applications, ranging from security and dependency analysis to calculi for homomorphic encryption.

Whereas the first part of the survey is mainly the territory of the present author, the task of surveying the metaprogramming landscape—which should take up the second part of this survey—is the domain of Mario Alvarez-Picallo (mario.alvarez-picallo@cs.ox.ac.uk).

We are actively on the lookout for a third co-author, in order to cover the vast subject of applied calculi; please contact us for more information.

Finally, we also welcome any suggestions on comments on the present material. The author can be reached by e-mail at alex.kavvos@cs.ox.ac.uk.
1. Introduction: Proof theory for modal logics

The situation has given rise to various suggestions. One is that the Gentzen format, which works so well for truth-functional operators, should not be expected to work for intensional operators, which are far from truth-functions. (But then Gentzen works well for intuitionistic logic which is not truth-functional either.) Another suggestion is that the great proliferation of modal logics is an epidemic from which modal logic ought to be cured: Gentzen methods work for the important systems, and the other should be abolished. 'No wonder natural deduction does not work for unnatural systems!'

The history of modal proof theory is a long and tortuous one, and not without good reason. Attention shifted away from proof systems for modal logic quite early in its development, for it was not at all evident whether any structural proof system was appropriate. The blame was put on the ‘intensionality’ of modal logic, or the lack of appropriate symmetries. Indeed, these views are explicitly stated by Bull and Segerberg (2001) and echoed in the survey of Negri (2011). To make matters worse, attempting to combine the Kripke semantics of intuitionistic logic with those of modal logic leads to a bewildering variety of possibilities.

We shall survey a stream of work on natural deduction systems for a very specific class of normal modal logics, which has been gradually handpicked over the years—the constructive modal logics. These share several characteristics:

- The modalities □ and ◊ are introduced independently, and not by the classic law □A ≡ ¬◊¬A or its dual, as is common. This is a welcome feature of intuitionistic analysis of logical connectives.
- They possess a variety of reasonably well-behaved proof systems. In fact, these logics have been selected over the years to admit analyses as close as possible to the popular menu of proof-theoretic results: the cut-elimination theorem of Gentzen (1935a,b) for sequent calculi, and the normalisation theorem and subformula property of Prawitz (1965) for natural deduction. Whether this goal has been achieved to a satisfying extent—or whether this is possible at all—has been the subject of debate for a long time.
- As a consequence of the striking similarity of the proof-theoretic analysis of these modal logic logics with that of intuitionistic propositional logic (IPL), they admit a Curry-Howard isomorphism (Curry and Feys, 1958; Howard, 1980): each proof may be seen as a construction for a type, which is its formula. Consequently, we may read formulas as types, and proofs as programs. See, for example, the classic text of Girard et al. (1989), or the modern treatment by Sørensen and Urzyczyn (2006). This leads to computational interpretations of modalities as type formers, as well as associated modal λ-calculi.
Following the path, these logics all admit some sort of categorical semantics, as pioneered for the typed $\lambda$-calculus and cartesian closed categories by Lambek (1980). These are invariably some kind of cartesian-closed category, coupled with a monoidal endofunctor perhaps coupled with some additional categorical gadgets.

In the same way that obtaining a constructive logic required eschewing the Law of the Excluded Middle (LEM), there are some modal principles that we have to eschew in order to obtain constructive modal logic. As in the case of LEM, the principles we have to reject are regarded as self-evident in traditional Kripkean analysis. These are usually

$$\Diamond (A \lor B) \rightarrow \Diamond A \lor \Diamond B \quad \text{and} \quad \neg \Diamond \bot$$

or of a very similar form.

It is fair to say that, even though the first axiomatization of this kind is due to Wijesekera (1990), the pioneers in adapting the Curry-Howard-Lambek isomorphism to modal logic were Bierman and de Paiva (1992, 1996, 2000), and Moggi (1989, 1991).

There are also other multiple other directions situated within the intersection of intuitionism and modal logics. One of the most popular alternatives, most often referred to simply as intuitionistic modal logic, was championed by Alex Simpson in his thesis (Simpson, 1994). This approach is motivated by Kripkean model theory, and indeed most of the effort is expended in devising appropriate Kripke semantics for some modal logic whose propositional component is intuitionistic. Invariably, the proof systems considered by Simpson do not model validity, but truth at a particular world: judgments are of the form $\Gamma \vdash A@x$, which is intended to be read as “under assumptions $\Gamma$, $A$ holds at world $x$.” Propositional assumptions aside, the context $\Gamma$ may also contain assumptions of the form $xRy$, with the intended meaning that “world $y$ is accessible from world $x$.” Ranalter (2010) studies embeddings of the ‘constructive’ systems into the ‘intuitionistic’ ones.

Even though there are very many sequent systems for intuitionistic modal logic, we shall not survey sequent calculi here, despite their close relationship to natural deduction systems. There is a wide variety of those, ranging from austere Gentzen-style to far more radical proposals. The former were pioneered by Ohnishi and Matsumoto (1957, 1959), and are surveyed by Ono (1998). Wansing (2002) surveys both ordinary and ‘generalized’ sequent systems. Negri (2011) adopts an even wider viewpoint, also covering work on some ‘non-traditional’ formalizations. A broader survey of the intersection of intuitionism, and modal logic, including applications to computer science, is the editorial (de Paiva et al., 2004), which is followed up by (Stewart et al., 2015).

Returning to our present concerns, we may list the following systems that fit our aforementioned criteria:

- a constructive version of the Lewis system $S_4$, which is known $CS_4$;
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- a constructive version of the smallest normal modal logic $K$, which is known as $\text{CK}$;
- constructive versions of the logic $K4$ and the logic of provability $\text{GL}$, both entirely due to Bellin (1985); and
- propositional lax logic $\text{PLL}$, also known as computational logic, denoted CL.
- constructive linear temporal logic, $\text{CLTL}$

Of these, the logics $\text{CS4}$ and PLL/CL have been the subject of intense scrutiny. The proof theory of $\text{CS4}$ has been a popular subject, ever since the days of Prawitz (1965). Its categorical interpretation reveals that there is a latent universal property underlying it, which makes for rather well-behaved proof systems, with plenty of harmony. On the other hand, PLL/CL is the logic underlying Moggi’s monadic metalanguage, which has enjoyed considerable attention as (a) a popular way to structure denotational semantics, (b) a way to encapsulate impurities and effects in functional languages, and (c) a pattern in which many programs can be structured, in a modular way. As such, the underlying logic has generated some interest, even if it was originally discovered and discarded by Curry (1952), mainly because of its unusual properties.

The logic $\text{CK}$, a minimal modal extension of the intuitionistic propositional logic, has received very little attention. It was devised as a fragment of proof systems for $K4$ and $\text{GL}$ by Bellin (1985), whose other proof systems have not been studied since their publication.

Finally, CLTL is a very simple constructive fragment of classical Linear Temporal Logic, involving only the ‘next’ operator. Notwithstanding its simplicity, it has numerous applications in metaprogramming.

1.1. Preliminaries. All but two of our modal logics $\mathcal{L}$ shall be sets of formulae—the theorems of the logic. These formulae are generated by the following Backus-Naur form:

$$A, B ::= P | \bot | A \land B | A \lor B | A \rightarrow B | \Box A | \diamond A$$

where $P$ is drawn from a countable set of propositions. The two exceptions to this rule replace the cases of $\Box A$ and $\diamond A$ by a single modality, $\odot A$.

The sets of theorems will be generated by axioms, closed under some inference rules. The set of axioms will always contain a complete axiomatization of intuitionistic propositional logic ($\text{IPL}$). The set of inference rules will consist of the two rules necessary to capture $\text{IPL}$, namely the axiom rule:

$$A \text{ is an axiom} \quad \frac{}{\vdash A}$$

and the rule of modus ponens:

$$\vdash A \rightarrow B \quad \vdash A \quad \frac{}{\vdash B}$$
As for the modal part, in all but two cases, we shall employ the necessitation rule, namely
\[ \vdash A \quad \vdash \Box A \]
Finally, any logic $\mathcal{L}$ will be closed under substitution of theorems for propositional constants.

2. Constructive S4

The founding stone of constructive modal logics has been the constructive logic underlying the Lewis system $\text{S4}$ (Lewis and Langford, 1932). A simple axiomatization of $\text{CS4}$, due to Alechina et al. (2001), comprises the axioms of IPL, and also

(K) \[ \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \]
(4) \[ \Box A \rightarrow \Box \Box A \]
(T) \[ \Box A \rightarrow A \]

To obtain the diamond fragment, we add

(\Box K) \[ \Box (A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B) \]
(\Diamond 4) \[ \Diamond \Diamond A \rightarrow \Diamond A \]
(\Diamond T) \[ A \rightarrow \Diamond A \]

Studies of the proof theory of $\text{CS4}$ abound, making it the most widely studied constructive modal logic. The fundamental reason for this success can only be grasped once one interprets its axioms categorically: take $\Box$ to be a endofunctor on a cartesian closed category. To interpret axiom K then, we need a natural transformation
\[ m_{A,B} : \Box A \times \Box B \rightarrow \Box (A \times B) \]
which would make the functor monoidal. Then, to interpret axiom T we need
\[ \epsilon : \Box \Rightarrow (-) \]
and to interpret axiom 4 we need
\[ \delta : \Box \Rightarrow \Box^2 \]
These contraptions look like they could take the place of counit and comultiplication of a monoidal comonad. As such, they give rise to an adjunction, either through the Eilenberg-Moore or Kleisli constructions (see e.g. Awodey (2010)). Thus, there is a latent universal property, which lends the proof systems of $\text{CS4}$ reasonable symmetry, from which one can mould introduction and elimination rules.

The Kripke and algebraic semantics for $\text{CS4}$ have been thoroughly investigated by Alechina et al. (2001).
2.1. Early attempts. The first attempt at modal natural deduction was the system for the \((\Box)\) fragment of \(S4\) and \(S5\) defined and studied by Prawitz (1965). A rule for diamond is also given, but was neither analyzed nor discussed. Prawitz considers the obvious solution: if all assumptions are modalized, then we are allowed to put a box in front of the conclusion. In sequent-style notation,

\[
\begin{align*}
\Box \Gamma &\vdash A \\
\Box \Gamma &\vdash \Box A
\end{align*}
\]

where

\[
\Box (A_1, \ldots, A_n) := \Box A_1, \ldots, \Box A_n
\]

A system for \(S5\) is also obtained if assumptions are either modalized, or the negation of a modalized formula.

Prawitz' systems are thoroughly criticized in (Bierman and de Paiva, 2000). Their main defect are that they are not closed under substitution: if we substitute any derivation of \(\Box A_i\) which has open non-modal assumptions at the top, then we do not necessarily get a derivation whose final step is not valid: it is no longer the case that all assumptions are modalized.

Here is a classic example of this phenomenon, taken from op. cit. Suppose that we have a derivation of the form

\[
\begin{align*}
\vdots \\
\Box A &\vdash B \\
\Box A &\vdash \Box B
\end{align*}
\]

Letting \(\Gamma = C \rightarrow \Box A, A\), we derive

\[
\begin{align*}
\Gamma &\vdash C \rightarrow \Box A \\
\Gamma &\vdash C
\end{align*}
\]

Substituing, now, this derivation of \(\Box\) for the assumption in \(\Box A \vdash \Box B\), we get that

\[
\begin{align*}
\vdots \\
\Gamma &\vdash C \rightarrow \Box A \\
\Gamma &\vdash C \\
\Box A &\vdash B \\
\Box A &\vdash \Box B
\end{align*}
\]

Eliminating the detour, we would have

\[
\begin{align*}
\vdots \\
\Gamma &\vdash B
\end{align*}
\]

which is not a valid application of the introduction rule at all: \(A\) bears no box in \(\Gamma\).

The inherent difficulties of this system were suspected by Prawitz, mainly because there are roundabout ways to derive formulas, i.e. ineliminable detours. In Prawitz (1965, §VI.2), he introduces a weaker restriction by the name of essentially...
modal formulas. However, it is noted in (Bierman and de Paiva, 2000, §8) that this system eliminates too many detours: categorically speaking, it forces $\Box A \cong \Box \Box A$.

Some more early work, none of which explicitly attributed to Prawitz, is surveyed by Satre (1972). Prawitz’ rule appears as $\Box I$ and is attributed to Ohnisi and Matsumoto (1957, 1959), but its proof-theoretic significance is not discussed.

2.2. The Experience of Linear Logic. Things took a radical turn with the discovery of Linear Logic by Girard (1987). The explicit control of assumptions seemed to pave the way for a multitude of applications in logic and computation.

One of the first goals of the community was to isolate an natural deduction system for the smallest fragment of Linear Logic, namely Intuitionistic Linear Logic (denoted ILL). This would amount to a very basic linear $\lambda$-calculus, which would occupy central ground in numerous applications. It so happened that the advancements made in this field had a significant impact on the study of natural deduction for modal logics, and so we rapidly survey them here.

An attempt to produce a system of categorical combinators by Girard and Lafont (1986) and Lafont (1988) led to the first formulation of natural deduction for ILL by Abramsky (1990, 1993). Even though it served its original purpose, Abramsky’s treatment of the “of course” (!) modality was similar to Prawitz’ for $S_4$, and hence not closed under substitution. A counterexample was discovered by Wadler (1991, 1992, 1994), who also realized that whenever substitution accidentally leads to a valid deduction, then soundness of the obvious interpretation in the categorical semantics of Seely (1989) necessitates that $! A \cong ! A \otimes ! A$.

The challenge was immediately taken up by Bierman et al. (1992) and Benton et al. (1993), whose treatment led to a rather complicated system: as their formulation of natural deduction was based on generalized rules in the style of Schroeder-Heister (1984), the $\lambda$-terms of their language contained explicit substitutions (Abadi et al., 1991). This system was very influential in subsequent developments for modal logic.

Alternative syntax for ILL was also proposed by Troelstra (1995), the system of whom is based on adapting Prawitz ‘essentially modal formulae’ restriction from $S_4$ to ILL; this solves the problem with substitution, but also forces $! A \cong ! A \otimes ! A$.

A further breakthrough occurred when the style of Girard’s Logic of Unity (Girard, 1993) was distilled by Andreoli (1992) and Wadler (1992, 1994) into a new syntax involving two contexts—one for ‘linear’ and one for ‘intuitionistic’ (“banged”) variables. An even simpler system of the same kind was rediscovered by Plotkin (1993), and extensively studied by his student Barber (1996, 1997). This logic is known as DILL (Dual Intuitionistic Linear Logic), and still comprises one of the two state-of-the-art systems for ILL.

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1It is interesting to remind the reader that controlling weakening and contraction of assumptions was an early and now-defunct goal of modal logics, through the notion of strict implication; see Girard (1987, §II.2).

2Girard’s presentation was in terms of an expertly symmetric sequent calculus for Classical Linear Logic.
Another kind of system was produced in \cite{Benton1994}, which involves two different types of sequent: intuitionistic and linear. Benton’s system offers a very convenient view of the semantics of ILL—for which see \cite{Melliess2009}—as a linear-non-linear adjunction. This is also another type of system which we may view as the state of the art, and often the initial point of further study. Even though this style of system is found in more programming applications nowadays, it does not appear to have exerted any influence at all on the study of modalities.

2.3. The Bierman & de Paiva system. Very soon after contributing to the ILL system in \cite{Bierman1992, Benton1993}, Bierman and de Paiva (1992, 1996, 2000) noticed that the rules for the “of course” (!) modality of ILL are really the same as those for $S4$. It was an easy step to remove the structural restrictions of ILL, in order to uncover a natural deduction and term assignment system for $CS4$.

The approach employed by \cite{Benton1993} for ILL, and exploited by \cite{Bierman1992} for $CS4$ is based on the same underlying principle:

\begin{center}
If the modal substitution (cut) rule is not admissible, then “bolt it” onto the introduction rule.
\end{center}

That is: if the strategy of requiring that all assumptions are modalized in order to infer a modal formula breaks substitution, then simply include a side condition of the form “…and, by the way, we can fill in some deduction for these modalized formulae already.” Hence their introduction rule:

\[
\begin{array}{c}
\Gamma \vdash M_1 : \Box A_1 \quad \ldots \quad \Gamma \vdash M_n : \Box A_n \\
\quad x_1 : \Box A_1, \ldots, x_n : \Box A_n \vdash N : B \\
\end{array}
\]

\[
\Gamma \vdash \text{box } N \text{ with } M_1, \ldots, M_n \text{ for } x_1, \ldots, x_n : \Box B
\]

Except for the fact it works, there is almost nothing good about this rule. It appears to be an introduction rule, yet the connective it introduces also appears in the conclusion of the (minor) premises, so it is also a cut rule! This is coupled with the obvious way to eliminate boxes, namely

\[
\begin{array}{c}
\Gamma \vdash M : \Box A \\
\Gamma \vdash \text{unbox } M : A
\end{array}
\]

which corresponds to the axiom $\Box A \to A$. The associated $\beta$-rule is

\[
\text{unbox } (\text{box } N \text{ with } M_1, \ldots, M_n \text{ for } x_1, \ldots, x_n) \to N[M_1/x_1, \ldots, M_n/x_n]
\]

Thus, to modalize a formula, (1) we need to be able to prove it only on modal assumptions, and (2) we must be able to fulfill all of them. Their proofs are then captured as part of the proof term, which, in the spirit of \cite{Abadi1991}, constitute an explicit substitution. The explicit substitutions are then effected en masse when the proof term is ‘unboxed’ (cf. evaluated, interpreted).

Contrary to what is stated in \cite{Bierman1992}, this introduction rule for box does not appear in \cite{Satre1972}, and is their own invention. The rest
of (Bierman and de Paiva, 2000) is dedicated to the study of the resulting natural
deduction system and $\lambda$-calculus, as well as its categorical semantics.

The original system of Bierman and de Paiva (1992, 1996) was extended by
Kobayashi (1997) with a rule for diamond, which was then incorporated into the
journal version of their article (Bierman and de Paiva, 2000). We may introduce a
diamond at any time:

$$
\Gamma \vdash M : A \\
\Gamma \vdash \text{dia } M : \Diamond A
$$

Eliminating a diamond can only be done in conjunction with a bunch of boxed
assumptions. Namely, if we use exactly some boxed variables $x_i : \Box A_i$ and a single
variable $y : B$ to produce a term of diamond type, then, if we fulfill all the boxed
assumption and provide something of diamond type for the $y : B$ variable, we can
indeed produce the term of diamond type:

$$
\Gamma \vdash M_1 : \Box A_1 \ldots \Gamma \vdash N : \Diamond B \\
\quad \quad x_1 : \Box A_1, \ldots, x_n : \Box A_n, y : B \vdash N : \Diamond C \\
\Gamma \vdash \text{let dia } y = N \text{ in } N \text{ with } M_1, \ldots, M_n \text{ for } x_1, \ldots, x_n
$$

with the reduction rule

$$
\text{let dia } y = \text{dia } P \text{ in } N \text{ with } M_1, \ldots, M_n \text{ for } x_1, \ldots, x_n
\rightarrow N[\vec{M}_i/\vec{x}_i, P/y]
$$

Putting the lack of ‘good symmetries’ aside, there are deeper flaws in the afore-
mentioned rules. Notably, a large number of commutative conversions are needed
to expose ‘hidden’ redices, which spoil the so called subformula property. As an
example of this phenomenon, quoted by Pfenning and Wong (1995), consider the
term

$$
\text{box } (\text{unbox } f \ c) \text{ with } (\text{box } (\lambda x : A. x)) \text{ for } f
$$

and notice that it has ‘hidden’ an obvious redex by ‘splitting it’ between the term
and the explicit substitution. The subformula property breaks, and this detour
cannot be eliminated—not even in conversion, let alone in reduction.

The issue of commutative conversions is known to arise from rules for positive
connectives in Natural Deduction, such as those for disjunction ($\lor I$) and existence
($\exists I$)—for a particularly perspicuous discussion, see (Girard et al., 1989, §10.1).
These rules are very similar to rules in the style of Schroeder-Heister (1984), which
in turn are very similar to those of Bierman and de Paiva’s system.

In designing their system for $\text{ILL}$, Bierman et al. (1992) and Benton et al. (1993)
include some of commutative conversions in their definition of reduction, whilst
only recovering the complete set in the categorical interpretation. However, they
do not investigate the subformula property at all. Similarly, Bierman and de Paiva
(2000) do present most commutative conversions as conversions between natural
deduction proofs, but do not push them down to the term level. As such, they prove
neither the subformula property nor completeness of the categorical semantics.

These issues were resolved by Goubault-Larrecq (1996a), who discusses the
present system and systematically remedies its shortcomings. In the process, he
provides full proofs of all common logical properties, but without considering categorical models. Goubault-Larrecq remarks that the introduction of explicit substitutions necessitates the addition of further structural rules concerning them. For example, the term

$$\text{box } \lambda u.uyz \text{ with } x, x \text{ for } y, z$$

does not correspond to a cut-free proof, but the ability to contract the two $x$’s to

$$\text{box } \lambda u.uyy \text{ with } x \text{ for } y$$

does. Similarly, the non-occurrence of variables that have been explicitly substituted for, i.e.

$$\text{box } \lambda x.x \text{ with } M \text{ for } y$$

should really be weakened to

$$\text{box } \lambda x.x \text{ with } \langle \rangle \text{ for } \langle \rangle$$

All is well with these rules, which explicitly mirror contraction and weakening on the level of explicit substitutions, and indeed they are mentioned by Bierman and de Paiva (2000) as well. Surely, we also need exchange, which would provide us with

$$\text{box } M \text{ with } P, Q \text{ for } x, y = \beta \text{ box } M \text{ with } Q, P \text{ for } y, x$$

which he also includes. He then proceeds to argue that these rules, whilst necessary, they jeopardize the computational meaning of the system. Indeed, these are rules traditionally found as structural rules in sequent calculi, but in natural deduction they are usually admissible.

To highlight why the above issues constitute an appreciable shortcoming, we need to recall that, in proof theory, it is natural deduction proofs that are thought to comprise the “real proof objects.” In contrast, sequent calculi allow for more symmetry precisely so they can facilitate proof-theoretic analysis. To quote (Girard et al., 1989, §5.4):

“The translation from sequent calculus into natural deduction is not 1-1: different proofs give the same deduction, [...] In some sense, we should think of the natural deduction rules as the true “proof” objects. The sequent calculus is only a system which enable [sic] us to work on these objects [...] In other words, the system of sequents is not primitive, and the rules of the calculus are in fact more or less complex combinations of rules of natural deduction [...]”

It follows, argues Goubault-Larrecq, that the Bierman and de Paiva calculus does not expose the computational behaviour of S4 proofs: if it did, it would need no structural rules at all.

2.4. The Goubault-Larrecq system. To mitigate the deficit introduced by the extra structural rules in Bierman and de Paiva’s system, Goubault-Larrecq (1996a,b,c).
proceeds to define his own system, namely the \( \lambda evQ \) calculus. The calculus itself is a very peculiar mixture of categorical combinators (Curien, 1993) and explicit substitutions (Abadi et al., 1991). He then proceeds to exhaustively study it, in four manuscripts of epic proportions.

One of Goubault-Larrecq’s fundamental contributions is that, in (Goubault-Larrecq, 1996a, §2) and (Goubault-Larrecq, 1996b, §2.3), he is the first to draw attention to the similarity of the computational behaviour of CS4 to the quoting features of LISP (see Bawden (1999)) and the reflective towers of Smith (1984) (see also the discussion in Goubault-Larrecq (1997a)). Even though the connection with staged metaprogramming was thrust into the limelight by Davies and Pfenning (2001)—see §2.6—this particular line of thought has not been subjected to further investigation, save from one comment in op. cit., who remark that the behaviour of their modal language “is closer to the quotations of a ‘semantically rationalized dialect’ of Lisp, called 2-Lisp,” referring to the intermediate, ‘non-reflective’ stepping stone for Smith’s reflective towers (Smith, 1984). However, it seems that the connection between constructive linear temporal logic and metaprogramming in LISP is closer—see §6.

2.5. The Martini & Masini, Pfenning & Wong systems. Following previous work by Masini (1992, 1993) on 2-sequent calculi and an associated system of natural deduction, Martini and Masini (1996) present and study a system of intuitionistic 2-sequents and an associated levelled \( \lambda \)-calculus that, with minor structural variations, which seemingly captures all three logics \( \mathsf{CK} \), \( \mathsf{CK4} \), and \( \mathsf{CS4} \) (this latter correspondence they do not prove). In some ways, their system is elegant and crisp; but the metatheory is difficult, and the levelling on the \( \lambda \)-calculus is complicated. No treatment of the \( \lozenge \) modality is provided.

Drawing impetus from the above presentation, Pfenning and Wong (1995) “flatten out” its two-dimensional structure into a stack of contexts. This provides them with a more workable calculus based on the same idea, and allows them to easily prove various meta-theoretic properties. For illustration, we use the improved version found in (Davies and Pfenning, 2001, §5). A sequent has the form

\[
\Psi; \Gamma \vdash M : A
\]

where \( \Psi \) is a list of contexts, the context stack. The idea is that, every \( \Gamma_i \) in \( \Psi = \Gamma_1, \ldots, \Gamma_n \) represents an ‘arbitrary world’ reachable from \( \Gamma_{i-1} \). The word ‘world’ in this context is supposed to invoke the look and feel of Kripke semantics (Kripke, 1963), but the idea is more subtle, and no actual accessibility relation is involved.

A context stack of the form

\[
\Psi; \Gamma; ·
\]

(where · is the empty context) is meant to be thought of as a ‘path’ through \( \Psi \), leading to the world \( \Gamma \), and then into some arbitrary world reachable from \( \Gamma \). So,
if something holds at the end of this path, i.e.

\[ \Psi; \Gamma; \cdot \vdash M : A \]

then it follows that the same thing holds at every world we can reach from \( \Gamma \). In the language of Kripke, \( \Box A \) holds at \( \Gamma \):

\[ \Psi; \Gamma \vdash \text{box } M : \Box A \]

This is the introduction rule. Of course, we can only use variables from the current world, i.e. the variable rule is

\[ \Psi; \Gamma, x : A, \Gamma' \vdash x : A \]

The fundamental tenet is the following:

Force terms of type \( \Box A \) to be closed with respect to any world accessible from the current one, but allow them to use whatever is available in the ‘current world’ (\( \Gamma \)), or a world visited in the path to the current one (\( \Psi \)).

The elimination rule, which allows us to ‘push’ boxed terms in some world forwards, in order to recover them in any ‘accessible’ world. For example,

\[ \cdot; x : \Box A \vdash x : \Box A \]

\[ x : \Box A; \Gamma \vdash \text{unbox} \_1 \_ x : A \]

is a way to push the variable \( x \) to any arbitrary accessible world. In full generality, if we have a term \( \Psi; \Gamma \vdash M : \Box A \) then we can push the it to whichever accessible world we desire:

\[ \Psi; \Gamma; \Gamma_1; \ldots; \Gamma_n \vdash \text{unbox} \_ n \_ M : A \]

no matter how far in the ‘future’ (\( \Gamma_1, \ldots, \Gamma_n \)) we go. The corresponding \( \beta \)-rule is

\[ \text{unbox} \_ n \_ (\text{box } M) \rightarrow \{n/1\} M \]

where \( \{n/p\} \) is a relatively involved operation on syntax which relabels occurrences of \( \text{unbox} \_ n \_ \) in a consistent way.

The notion of ‘accessible world’ is very flexible, in that restricting it allows us to limit the system to the logics \( \mathcal{C}K \) or \( \mathcal{C}K4 \)—see\[ \footnote{\text{We refer to this as \textit{transitive}.}} \] But this is a system for \( \mathcal{C}S4 \), and if—as in the Kripke semantics for \( \mathcal{S}4 \)—our accessibility relation is reflective, then the present is accessible, and we may as well ‘unbox’/evaluate anything we have:

\[ \Psi; \Gamma \vdash M : \Box A \]

\[ \Psi; \Gamma \vdash \text{unbox} \_ 0 \_ M : A \]

This corresponds to \textit{axiom T} : \( \Box A \rightarrow A \). And if our accessibility relation is transitive, we allow all constructs \( \text{unbox} \_ n \_ \), for \( n = 1, 2, \ldots \), so we may push anything as far down an accessible chain of worlds, obtaining a version of the logic \( \mathcal{C}K4 \) with \textit{axiom 4}. Even further, if we limit ourselves to the case \( n = 1 \), we get simply \( \mathcal{C}K \). This is an observation that neither \text{Pfenning and Wong (1995)} nor \text{Davies and Pfenning (2001)} capitalized on even though they were the first to make it.
2.6. The Davies & Pfenning systems. Davies and Pfenning (1996, 1999, 2001) suggested that one can intuitively read values of type □A as “code of type A,” in an interpretation that the present author calls modality-as-intension. The fundamental idea was first aired in Pfenning and Wong (1995), and is this: if we interpret box as an intensional type, then programming in that type theory really corresponds to metaprogramming: manipulating terms at modal types exactly corresponds to manipulating code, in a logical and type-safe way.

Davies and Pfenning (2001) present two systems for CS4: the leaner version of the system of Pfenning and Wong (1995) that we have just described (therein called the ‘implicit’ formulation), and a two-context calculus (the ‘explicit’ system), which is eerily similar to what one were to obtain if they were to remove linear restrictions from the system DILL of Barber (1996) and Plotkin (1993). This is, however, independent of the work of Barber and Plotkin, but shares common ancestry with it in the calculi of Girard (1993), Andreoli (1992), and Wadler (1993, 1994). We call the dual-context calculus of Davies and Pfenning by the name DCS4 (‘dual constructive S4’).

Translations are provided between the ‘implicit’ and ‘explicit’ version, along with proofs of correctness. The connection with metaprogramming comes from an embedding of two-level λ-calculus of Nielson and Nielson (1992) in the ‘implicit system.’ The two-level λ-calculus, descended from an untyped predecessor devised by Gomard and Jones (1991), is a language where expressions are annotated (and type-checked) as being available in two stages: either at compile-time, or runtime. This is done in such a way that that compile-time expressions never depend on run-time expressions and can be evaluated in advance. The contribution of Davies and Pfenning (1996, 1999, 2001) is the demonstration that the two-level calculus can be embedded in a fragment of their ‘implicit’/context-stack CS4 system, which corresponds to the logic CK. This led to a flurry of developments in calculi for metaprogramming, and is the beginning point of the second part of this survey; see also §6.

Sequents of DCS4 have the form

\[ \Delta ; \Gamma \vdash M : A \]

The variables occurring in \( \Delta \) are considered to be ‘boxed’: \( \Delta \) is a modal context. In contrast, \( \Gamma \) is exactly like the run-of-the-mill intuitionistic context we are used to.

What is code? Surely something that does not depend on normal, value variables; only code can contribute to its making:

\[ \Delta ; \vdash M : A \]

\[ \Delta ; \Gamma \vdash \text{box } M : \square M \]

This is the introduction rule. As it may be readily witnessed, it encapsulates an elegant way to enforce the constraint that code should only depend on code.

The pattern in dual-context systems it that the elimination rule is a cut rule. That is, the elimination construct is a very simple explicit substitution. In this
particular case, it is used to substitute code for a modal variable:

\[ \Delta ; \Gamma \vdash \text{let } u \leftarrow M \text{ in } N : C \]

The reduction rule is the obvious one:

\[ \text{let } u \leftarrow M \text{ in } N \rightarrow N[M/u] \]

The rationale is this: the rest of the calculus is responsible for controlling how modal assumptions are used, and the elimination rule is used to substitute for one of them. This constitutes splitting up of the introduction rule of Bierman and de Paiva (2000) into two halves, which we then use in the pattern of introduction and elimination.

What happens to the unbox construct in op. cit. then? The answer is that we hide it in a second variable rule. A modal variable is code; surely we can evaluate this code to obtain a value:

\[ \Delta, u : A, \Delta' ; \Gamma \vdash \text{let } u \leftarrow M \text{ in } N : A \]

This seemingly innocuous ‘variable rule’ is actually a defining feature of the system: it is the only way in which we can use a modal variable, and it involves unboxing it. Using this elimination rule, rule, one is able to write down a term of type \( \Box A \rightarrow A \), which corresponds to the axiom \( T \) of CS4.

In a remarkably lucid sequel, Pfenning and Davies (2001) attempt to justify their dual-context system using the philosophical approach of Martin-Löf (1996), and also include the possibility modality \( \Diamond \), as previously presented by Bierman and de Paiva (2000) and Kobayashi (1997). They also show how one may embed the propositional lax logic CL/PLL in the system, thus recapturing the monadic metalanguage of Moggi (1991) through their system—see §5.2.

To deal with \( \Diamond \), a special type of sequent is introduced. Let us call it a ‘possibility judgments,’ and write \( \Delta ; \Gamma \vdash M \Diamond A \) for it. Possibility judgments \( \vdash M \Diamond A \) shall be read as ‘\( M \) is a proof that \( A \) is possible.’ If \( A \) is true, then it is certainly possible; i.e. normal sequents are included in possibility judgments:

\[ \Delta ; \Gamma \vdash M : A \quad \Rightarrow \quad \Delta ; \Gamma \vdash M \Diamond A \]

This step happens ‘silently’ in op. cit. but we prefer to make it explicit for the sake of clarity. Next, we internalise the possibility judgment, using the \( \Diamond \) to stand for it. The introduction rule proceeds almost tautologically: if \( M \) is possible, then it is true that \( M \) is possible:

\[ \Delta ; \Gamma \vdash M \Diamond A \quad \Rightarrow \quad \Delta ; \Gamma \vdash M : \Diamond A \]

So far we have done nothing more than ensure that truth implies possibility. The elimination rule completes this picture. Suppose that there exists a ‘possible world’ of which we know nothing else, except that \( A \) is true. Suppose, furthermore, that assuming only this fact, that \( A \) is true, we can show \( C \) to be possible. Then, if \( A \)
is possible in this world, then \( C \) is possible in this world as well. In symbols:

\[
\Delta ; \Gamma \vdash M : \Diamond A \quad \Delta ; x : A \vdash E \vdash \Diamond C
\]

The associated \( \beta \)-rule is

\[
\text{let } \text{dia } x = M \text{ in } E \vdash \text{dia } x = M \text{ in } E
\]

and we shall now define the substitution operator \( \langle \langle \cdot / \cdot \rangle \rangle \).

The reader may think that, surely, we could avoid introducing a separate type
of sequent for possibility. Yet, there is subtlety that is easy to miss, and which we
will use to define the substitution operator. Notice that if \( \Delta ; \Gamma \vdash M \vdash \Diamond C \)
then \( M \) can be of one of two forms:

- \( M \) can be any term; or
- \( M \) can be of the form \( \text{let } \text{dia } x = M \text{ in } E \)

So, let us define a syntactic category of proof expressions, \( E \):

\[
E ::= M \mid \text{let } x = M \text{ in } E
\]

It follows that, since we only use \( \langle \langle E/x \rangle \rangle \) with proof expressions \( E \), we can define
it by induction on \( E \) rather than by induction on \( F \) in \( F(\langle \langle E/x \rangle \rangle) \). In the case of a
term, it devolves to regular substitution:

\[
F(\langle \langle M/x \rangle \rangle) := F[M/x]
\]

But in the case of a proof expression \( \text{let } x = M \text{ in } E \), we define

\[
F(\langle \langle \text{let } y = M \text{ in } E \rangle \rangle) := \text{let } y = M \text{ in } F(\langle \langle E/x \rangle \rangle)
\]

This definition unfolds proof expressions in such a way that fewer commuting
conversions are necessary. A similar trick is used in the \( \lambda \)-calculus for the lax modality
also introduced in op. cit.—see \[5.2\]

The computational interpretation of the diamond modality was further elabo-
rated by Pfenning (2001), who demonstrates that we may think of terms of type
\( \Diamond A \) as proofs of \( A \) whose structure has been hidden. That is, they are proofs of \( A \),
but their internal structure is invisible: they are proof-irrelevant. Pfenning (2001)
develops a type theory based on the juxtaposing the two extremes of this spectrum:
on one hand, terms of box type are only equal up to \( \alpha \)-equivalence; on the other
hand, terms of diamond type are all equal. Note that, in order to avoid elimina-
tion rules that look like cut rules, this type theory involves three different types
of \( \lambda \)-abstraction: one for ‘boxed’ (intensional), one for intuitionistic, and one for
‘diamond’ (proof-irrelevant) variables. Pfenning’s type theory has not yet been put
to the test of semantical (and/or categorical) analysis.

Ghani et al. (1998) also discuss DCS4, which they use to repeat the exercise of
embedding two-level languages in it, after suitably extending it with more compi-
lcated forms of explicit substitutions.
2.7. Categorical Semantics of CS4. The first to work out the categorical semantics for CS4 seem to be Bierman and de Paiva (1992, 1996, 2000): they showed that, to interpret the box fragment of CS4 soundly, one only needs a cartesian closed category with a monoidal comonad.

To interpret the diamond, one needs a □-strong monad, a notion introduced by Kobayashi (1997): that is, a monad \((\diamond, \eta, \mu)\), along with a □-strength

\[
t_{A,B} : \Box A \times \Diamond B \to \Diamond (\Box A \times B)
\]

which is a natural transformation that satisfies appropriate coherence conditions. The full definition may be found in (Kobayashi, 1997) or (Bierman and de Paiva, 2000).

However, because of the deficiencies of their term calculus, Bierman and de Paiva failed to take into account all the necessary commuting conversions, and hence did not prove a completeness result. This achievement is again due to Kobayashi (1997), and also encompasses his rules for \(\Diamond\). Kobayashi also shows that, were one were to adopt \(\eta\)-rules for box constructs, then one needs a cartesian comonad to achieve completeness, namely a strong monoidal comonad, which also happens to be product-preserving between cartesian categories. Further discussion of categorical semantics for CS4 also appears in (Alechina et al., 2001).

Concrete constructions of models are rare. Goubault-Larrecq and Goubault (1999, 2003) produce some of the few known constructions of concrete models based on (1) the standard category Cpo of complete partial orders (ubiquitously present in domain theory and most early attempts at semantics for programming languages); (2) the category CGHaus of compactly-generated Hausdorff spaces; and (3) the category of augmented simplicial sets. Of these, the last model is proven to be complete, in the sense of Friedman (1975): if two terms are equal in every simplicial augmented model, then the proof theory equates them as well; see (Goubault-Larrecq and Goubault, 2003).

A different categorical model for CS4, again on the category Cpo of complete partial orders, is presented in (Kobayashi, 1997). The construction is based on a comonad of Brookes and Geva (1991, 1992). Note, however, that this comonad also satisfies \(A \to \Box A\) naturally in \(A\), so it is perhaps not the best model to illustrate the computational behaviour of CS4.

3. Constructive K

The logic CK can be understood as the smallest normal modal logic extending intuitionistic propositional logic (IPL). That is, we only add the axiom

\[
(K) \quad \Box (A \to B) \to (\Box A \to \Box B)
\]

in order to obtain its box fragment. To also obtain the diamond fragment, we throw in

\[
(\Diamond K) \quad \Box (A \to B) \to (\Diamond A \to \Diamond B)
\]
A logic very close to $\text{CK}$ was the very first constructive modal logic to be axiomatized, and can almost be credited to Wijesekera (1990), but not entirely: Wijesekera’s logic also included $\Diamond \bot \to \bot$, which—though adequate for his application—complicates the analysis of the logic substantially. ‘Pure $\text{CK}$’ itself was introduced alongside its Curry-Howard interpretation by Bellin et al. (2001), and its Kripke semantics were worked out by Mendler and de Paiva (2005).

The natural deduction system for $\text{CK}$ introduced by Bellin et al. (2001) is a fragment of a natural deduction system for classical provability logic ($\text{GL}$), discovered and studied by one of the authors in the 1980s (Bellin, 1985). The central idea underlying it is the same as those of the Bierman and de Paiva system for $\text{CS4}$: if the substitution/cut rule is not admissible, build it in the introduction rule using explicit substitutions. The difference is that we drop the requirement that assumptions are boxed. The rule becomes

$$\Gamma \vdash M_1 : \Box A_1, \ldots, \Gamma \vdash M_n : \Box A_n, x_1 : A_1, \ldots, x_n : A_n \vdash N : B$$

$$\Gamma \vdash \text{box } N \text{ with } M_1, \ldots, M_n \text{ for } x_1, \ldots, x_n : \Box B$$

As one may see, $x_1, \ldots, x_n$ are available as ‘values’ within $N$, but we may only place a box in front of $B$ when all the assumptions going into its making are ‘boxed.’ Curiously, even though Bellin, de Paiva and Ritter were the first to seriously propose a proof-theoretic study of this formulation, it seems that—according to Satre (1972)—a rule ‘much like it’ was suggested by E. J. Lemmon during a lecture in 1966. This rule appears as $\Box I_1$ in op. cit. where it is shown that adding it to classical natural deduction comprises a system deductively equivalent to classical $K$.

If explicit substitutions for $\text{CS4}$ were problematic, then the situation for $\text{CK}$ is hopeless: the duality between introduction and elimination ceases to exist altogether, for there is no elimination rule at all! To match this perfectly, the only plausible ‘reduction rule’ is very similar to one of the ‘secondary’ commuting conversions suggested by Goubault-Larrecq (1996a) in the context of $\text{CS4}$. Its function is to unbox any ‘canonical’ terms in the explicit substitutions; e.g

$$\text{box } yx \text{ with } y, (\text{box } M \text{ with } z \text{ for } z) \text{ for } y, x$$

$$\to \text{box } y(\text{box } M) \text{ with } y, z \text{ for } y, z$$

in an appropriate context for $y$ and $z$.

In the original presentation of Bellin et al. (2001) there were multiple technical deficiencies which were remedied by Kakutani (2007). Kakutani’s paper also contains many other gems: a call-by-value version; a CPS transformation from the call-by-value to the call-by-name version; and soundness and completeness of categorical semantics.
In the final part of his paper, Kakutani (2007) extends the calculus to CK4 and CS4 by including type-indexed constants,

\[
\varepsilon_A : \Box A \rightarrow A
\]

\[
\delta_A : \Box A \rightarrow \Box \Box A
\]

along with some rather complicated equations that govern their behaviour. Finally, he proves that, were we to extend the system with the extra structural rules of Goubault-Larrecq (1996a) on explicit substitutions, as in the case for CS4, one can translate the DCS4 calculus of Pfenning and Davies (2001) to it.

The state of the art for the box fragment of CK is thus represented by a thoroughly-studied system of explicit substitutions.

3.1. Other systems. There is also another system CK, a dual-context formulation discussed by Bellin et al. (2001) and de Paiva and Ritter (2011). We do not dwell much on it, as it is seriously pathological (e.g. weakening is not admissible). de Paiva and Ritter (2011) also broadly survey ‘non-standard’ systems for CK, including those in the style of Fitch, and reiterate the need for a better system.

3.2. Categorical Semantics of CK. The categorical semantics of CK were understood by Bellin et al. (2001) to be a monoidal endofunctor, which is the sole ingredient required to interpret the (K) axiom. The proof of soundness and completeness for the system of Bellin et al. (2001), in its amended formulation, was provided by Kakutani (2007).

If one wishes to interpret the diamond as well, Bellin et al. (2001) and de Paiva and Ritter (2011) remark that we also need a natural transformation,

\[
st_{A,B} : \Box A \times \Diamond B \rightarrow \Diamond (A \times B)
\]

This is not as strong as the $\Box$-strong transformation of Kobayashi (1997) for CS4, nor as strong as the strong monads of Moggi (1991). It lies somewhere in between, and there is no discussion of any coherence conditions it may need to obey. There is no known proof of soundness or completeness for this fragment: following the amendments to the calculus presented by Kakutani (2007) in order to verify the proof, we consider the statement in (Bellin et al., 2001) wholly unreliable.

4. Constructive K4 & GL

The only attempt at a natural deduction system for logics such as K4 and GL (Provability Logic) can be found in an obscure paper of Bellin (1985). Therein, a system for classical GL is presented, with systems for K and K4 appearing as fragments. We remind the reader that the axiom 4, namely $\Box A \rightarrow \Box \Box A$, is derivable in provability logic; see, for example, Boolos (1994) or Artemov and Beklemishev (2002).

Historically speaking, Bellin’s system is the first one with explicit substitutions. One can thus expect that it is not particularly well-behaved. The situation is
worsened by the inclusion of rules for classical logic, which as is well known cause problems with normalization. Nevertheless, the paper is very much concerned with proof theory, and soon the focus of his paper shifts to the ‘intuitionistic’ fragment of his system. Unfortunately, the solution he provides is rather complicated, and we cannot reproduce it in its entirety here.

We can, however, present his rules. The key idea that yields K4 amounts to standing halfway between K and S4. In term assignment, the appropriate rule would look like so:

\[ \Gamma \vdash P_1 : \Box A_1 \ldots \Gamma \vdash P_n : \Box A_n \]
\[ x_1 : \Box A_1, \ldots, x_n : \Box A_n, y_1 : B_1, \ldots, y_m : B_m \vdash N : C \]

In the Bierman and de Paiva system for CS4, the types of all the free variables in \( N \) had to be ‘boxed’—i.e. they had to be of modal type. In the Bellin, de Paiva and Ritter system for CK, no types of variables at all had to be boxed—they were available within the (major) premise without a modality. The case of K4 sends us straight to the halfway house, for we need both. But, once more, we have to fulfill all these assumptions, modal or not, with terms of modal/‘boxed’ type.

Why might this be? Intuitively, CS4 also had a rule for ‘unboxing’ things, corresponding to the axiom T. Therefore, we did not lose anything by requiring that all assumptions are ‘boxed,’ for we could ‘unbox’ them whenever we needed that:

\[ x_1 : \Box A_1, \ldots, x_n : \Box A_n \vdash (\ldots \text{unbox } x_i \ldots) : B \]

In K4 this is not available, so we have to make provisions to accept both ‘code’ and ‘values.’

How can one adapt this to work for GL, then? The answer is to make the above rule even more complicated, by adding a ‘boxed’ diagonal reference:

\[ \Gamma \vdash P_1 : \Box A_1 \ldots \Gamma \vdash P_n : \Box A_n \]
\[ x_1 : \Box A_1, \ldots, x_n : \Box A_n, y_1 : B_1, \ldots, y_m : B_m, z : \Box C \vdash N : C \]

As in the case of its CK-fragment, this system has no elimination rule. One \( \beta \)-rule associated with the above takes box constructs in the explicit substitutions, and substitutes them in the term. Another \( \beta \)-rule defines a strange recursive procedure, where the entire term is somehow substituted for occurrences of \( z \) in itself. This has to be done very carefully in order to achieve strong normalization; in particular, some “doubly diagonal” assumptions need to be avoided.

5. **Propositional Lax Logic (PLL), or Computational Logic (CL)**

There is a peculiar constructive modality which has arisen in a multitude of contexts, and thereby repeatedly reinvented. Curry (1952) discovered its rules and
briefly considered it in the process of trying to prove a deduction theorem for modal logics. Almost twenty years later, Goldblatt (1981) discovered a similar modality on topos logic, with the meaning that “it is locally the case that.”

The true link with constructivity was made with the investigations of Moggi (1991) on the monadic metalanguage; this included a modality-like type constructor, which turns out to be logically equivalent to Curry’s modality, as eventually shown by Benton et al. (1998), who name it Computational Logic (CL). And then, only a few years later, Mendler (1993) developed a logic for the verification of hardware constraints, whose modal propositional fragment turned out to be coincide with Curry and Moggi. The resulting Propositional Lax Logic (PLL) was studied in detail by Fairtlough and Mendler (1995, 1997).

Fairtlough and Mendler (1995) characterise the logic by the following axioms:

\[(\Box R)\] \[A \rightarrow \Box A\]
\[(\Box M)\] \[\Box \Box A \rightarrow \Box A\]
\[(\Box F)\] \[(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\]

Equivalently, Fairtlough and Mendler (1997) replace axiom \((\Box F)\) with \(\Box A \land \Box \rightarrow \Box (A \land B)\), which they refer to as \((\Box S)\). One may wisely ask what modal rules we need. It seems that we do not need any! Of course, necessitation is improper in this setting. Fairtlough and Mendler (1997) suggest a rule which directly follows from \((\Box F)\) and propositional logic. Benton et al. (1998) suggest that we need no inference rules related to the modality at all.

Almost immediately, one notices that \((\Box F)\)—or equivalently, \((\Box S)\)—is characteristic of necessity modalities, usually denoted by a box (\(\Box\)), whereas \((\Box R)\) and \((\Box M)\) of a possibility modality, commonly denoted as a diamond (\(\Diamond\)). What kind of modality is this, then? Fairtlough and Mendler (1997) note that, in a classical context, \(\neg \Box A\) is true, whether \(\Box = \Box\) or \(\Box = \Diamond\). But then, we have that

\(\Box A\) implies \(\Box (A \land \bot)\) implies \(\Box A \land \Box \bot\) implies \(\Box \bot\) implies \(\bot\) implies \(A\)

and hence

\[A \leftrightarrow \Box A\]

That is, \(\Box\) cannot be either a classical diamond, or a classical box.

### 5.1. Moggi’s calculus

The natural deduction system for PLL/CL was discovered by Moggi (1989, 1991), and it is—at least compared to the other logics we survey—largely uncontroversial. The introduction rule simply takes values of type \(A\) to computations of type \(\Box A\):

\[
\Gamma \vdash M : A \\
\Gamma \vdash \text{val} \ M : \Box A
\]

Under no circumstances should we be allowed to coerce such a computation back to its previous state as a value. This is because there are more things of type \(\Diamond A\) than just values of \(A\). Moggi’s motivation is to use the \(\Diamond\) types to isolate a part
of the language which may have side effects. Thus, we may also consider impure elements of type $\diamond A$.

But recall that, in some abstract sense, we know how to sequentially compose side effects. The elimination rule reflects this intuition: if we can use a value of type $A$ to produce a computation of type $\diamond B$, we may as well plug in a computation of type $\diamond A$ for that value. The resulting term first performs the side-effects that go into $\diamond A$, and then the ones embedded in the term of type $\diamond B$:

$$
\Gamma \vdash M : \diamond A \quad \Gamma, x : A \vdash N : \diamond B
$$

$$
\Gamma \vdash \text{let val } x = M \text{ in } N : \diamond B
$$

The $\beta$-rule is predictable:

$$
\text{let val } x = \text{val } M \text{ in } N \rightarrow N[M/x]
$$

The above natural deduction system, as well as equivalent sequent calculus and Hilbert-style formulations are studied by Benton et al. (1998). A sequent calculus is also discussed by Fairtlough and Mendler (1997), but their main focus is on the Kripke models for PLL/CL. In Fairtlough and Mendler (2002), the inhabitation problem for PLL/CL is discussed, as well as its interpretation as a logic of ‘constraint contexts.’ Goubault-Larrecq et al. (2008) discuss a notion of logical relation appropriate for Moggi’s monadic metalanguage.

5.2. Pfenning & Davies’ lax $\lambda$-calculus. After presenting their dual-context calculus DCS4, for which see §2.6 Pfenning and Davies (2001) spend the second half of their paper discussing PLL/CL. Using, once more, the quasi-philosophical approach of Martin-Löf (1996), they ‘judgmentally reconstruct’ lax logic, yielding a $\lambda$-calculus that is almost identical to Moggi’s, yet with a subtle distinction in the types of sequents, some are regular sequents, notated as usual, whereas some are lax, written $\Gamma \vdash E \sim A$. Regular sequents imply lax sequents:

$$
\Gamma \vdash M : A
$$

$$
\Gamma \vdash M \sim A
$$

We are only allowed to introduce the modality only if we have such a lax sequent:

$$
\Gamma \vdash E \sim A
$$

$$
\Gamma \vdash E : \diamond A
$$

Finally, the elimination rule substitutes a term of type $\diamond A$ into a lax sequent:

$$
\Gamma \vdash M : \diamond A \quad \Gamma, x : A \vdash N \sim B
$$

$$
\Gamma \vdash \text{let val } x = M \text{ in } N \sim B
$$

This may seem like a silly game to play, but there is a subtlety that is easy to miss: well-typed terms $M$ which appear in a lax sequent $\Gamma \vdash M \sim A$ are also elements of the syntactic category $E$ of proof expressions, defined by

$$
E ::= M \mid \text{let val } x = M \text{ in } E
$$
The importance of this becomes obvious once we glance at the $\beta$-rule:

$$\text{let val } x = \text{val } E \text{ in } N \rightarrow N\langle E/x \rangle$$

As $E$ is a proof expression, we can define the substitution operator $N\langle E/x \rangle$ by induction on $E$, instead of the more common pattern of induction on $N$. There are two clauses, one for regular terms, and one for let val constructs:

$$N\langle M/x \rangle \overset{\text{def}}{=} N[M/x]$$

$$N\langle \text{let val } y = M \text{ in } E/x \rangle \overset{\text{def}}{=} \text{let val } y = M \text{ in } N\langle E/x \rangle$$

This adjustment to Moggi’s system actually removes the need for its one commuting conversion.

5.3. Categorical semantics of PLL/CL. The semantics of PLL/CL is simple and natural: it consist of a cartesian closed category, with the modality $\Box$ being the functor part of a monad, say $(\Box, \eta, \mu)$. However, Moggi (1989) remarks this is not quite enough by itself: it is required that this be a strong monad, in the sense of Koch (1972). A strength, in this context, is a natural transformation

$$\text{st}_{A,B} : A \times \Box B \rightarrow \Box (A \times B)$$

satisfying some appropriate coherence conditions.

Computationally speaking, the strength corresponds to the ability to ‘impurify’ any $A$, in order to include it as part of an impure computation $\Box B$, finally yielding $\Box (A \times B)$. Following this setup, Moggi (1989, 1991) obtains soundness and completeness.

5.4. The subtle relationship to CS4. Suppose we have full CS4, diamond modality included. Then, let us throw in axioms $A \rightarrow \Box A$. Combined with the $\top$ axiom, this trivialises the $\Box$ modality, in the sense that $A \leftrightarrow \Box A$. However, if we set

$$\Box \overset{\text{def}}{=} \Diamond$$

then it is not very hard to see that the resulting logic satisfies the axioms of PLL/CL.

This is not a mere coincidence. Alechina et al. (2001) showed that, if one takes Kripke models for CS4, and requires that they satisfy the semantic counterpart of $A \rightarrow \Box A$ (namely, the accessibility relation is hereditary), then one obtains sound and complete Kripke semantics for PLL/CL. It also seems like this semantics coincides with the original Kripke semantics given in Fairtlough and Mendler (1997).

More importantly, this does not appear to be an artifact of the Kripke semantics: the rabbit hole goes deeper, for the correspondence is mirrored on the categorical level. To quote Alechina et al. (with the emphasis being our doing):

In the logic, PLL arises as a special case of CS4 when we assume the derivability of $A \rightarrow \Box A$. A similar statement holds in category theory. We have an inclusion functor from the category of PLL-categories into the category of CS4-categories: each PLL-category
is a \textit{CS4}-category where the comonad is the identity functor. Conversely, each \textit{CS4}-category such that $\square A$ is isomorphic to $A$ is a [sic] \textit{CS4}-category.

It follows, the authors argue, that $\Diamond$ really is a possibility modality.

Further evidence of this deep relationship is provided by Pfenning and Davies (2001), who embed their lax $\lambda$-calculus into DCS4 (see §2.6). The key rests in the following translation of types, where we write $\Rightarrow$ for implication in PLL:

\[
A \Rightarrow B \overset{\text{def}}{=} \square A \rightarrow B \\
\Diamond A \overset{\text{def}}{=} \Diamond \Diamond A
\]

The rules of PLL then become \textit{derivable} rules of \textit{CS4}.

6. Constructive Linear Temporal Logic (CLTL)

Whilst developing calculi for \textit{CS4}, Davies and Pfenning (1996, 1999, 2001) noticed that their type systems enforced an important restriction: their calculi could not manipulate code that is not closed, i.e. code that has free ‘value’/dynamic variables, see Davies and Pfenning (2001, §8).

That is, if we have a term of shape $\Box M$ at hand, then all of its free variables are of modal type. This is a very useful property to know when practicing metaprogramming: we know that this term only depends on code, i.e. \textit{only depends on static data}, and we are thus ready to generate code for it. This generation process may involve some static analysis or local optimizations applied to $M$, but the calculus shall not control those: they will happen \textit{once we compile to some other language}—see e.g. the survey by Wickline et al. (1998). The calculus is there only to help us ascertain that things labelled static depend only on other static data. As a result, \textit{no unbound dynamic variables will be encountered upon evaluation}.

However, there is a lot of interest in a much more simple and direct kind of metaprogramming, namely that of \textit{symbolic execution}. Consider a term $M$ with one a free \textit{dynamic} variable, $x$. We would like to be able to reduce $M$ whilst carrying $x$ around as an unevaluated ‘symbol’ until $M$ is as simple as we would like it to be. A classic example is that of the exponential function: suppose we have a term $\text{expt}$ such that $\text{expt } x \, n$ evaluates to $x^n$. We would like some ‘logical mechanism’ by which we can rewrite $\text{expt}$ to some term $\text{expt'}$, and then ‘partially evaluate’ $\text{expt'} \, x \, 3$, reducing it to $x \times x \times x$. Furthermore, we would like to do this not as a static analysis on compiled code—as languaged based on \textit{CS4} allow us to do—but in a controlled manner \textit{within the modal calculus}. This is metaprogramming \textit{with open variables}.

\footnote{An inference rule is \textit{derivable} just if, from a proof of the assumptions, a finite sequence of steps consisting of instances of rules can build a proof of the conclusion. In contrast, an inference rule is \textit{admissible} just if, whenever there exists a proof of the assumptions, then there exists a proof of the conclusion.}
To achieve this kind of manipulation, we need to replace the □ modality with something which has a more *temporal* flavour, and which we write □⃝. This innovation belongs to Davies (1995, 1996), who devised a different system, λ⃝. The system itself is a more radical departure than anything we have discussed up to this point. The main departure is that both assumptions and sequents are *timed*.

Sequents are of the form

\[ \Gamma \vdash^n M : A \]


The above sequent can be read as follows: at time \( n \), term \( M \) has type \( A \), under the assumptions \( \Gamma \). But assumptions in \( \Gamma \) are also timed: the assignment \( x : A^n \) means that \( x \) is a variable of type \( A \) at time \( n \). One can only use variables which are available at the ‘current’ time:

\[
(x : A^n) \in \Gamma \\
\Gamma \vdash^n x : A
\]

Accordingly, we may only λ-abstract a variable that is currently available. Moving a time step backwards into the past or a step forwards into the future is effected by the rules for the modality □⃝. If we can infer \( A \) in the next moment, then we can infer that □⃝\( A \) (‘in the next step \( A \)’):

\[
\Gamma \vdash^{n+1} M : A \\
\Gamma \vdash^n \text{next } M : \Box A
\]

Similarly, if we can infer □⃝\( A \) at some point, then we can infer \( A \) at the preceding step:

\[
\Gamma \vdash^n M : \Box A \\
\Gamma \vdash^{n+1} \text{prev } M : A
\]

These two constructs give rise to *temporal β-redices*, namely

\[
\text{prev (next } M) \rightarrow M
\]

and *temporal η-redices*, namely

\[
\text{next (prev } M) \rightarrow M
\]

The natural numbers that decorate sequents carry a very strong flavour of *time*, and can also be understood as delimiting the *stage of a computation*. This intuition was formalized by Davies (1996), who proved a *time-ordered normalization theorem*: suppose \( \Gamma \vdash^n M : A \) has no temporal redices, and no subterm whose derivation is labelled with time less than \( n' \). Then, reducing a β-redex labelled with time \( n' \) followed by reducing all resultant temporal redices does not create β-redices with time less than \( n' \). In other words: reducing a β-redex of minimal time and then cancelling prev’s with next’s will not produce a redex of lesser time.

### 6.1. Logical Aspects

Davies (1995, 1996) claims to have drawn inspiration for the design of λ⃝ through the Curry-Howard isomorphism:
“Putting this all together naturally suggests that constructive linear-time temporal logic with $\circ$ and a type system for multi-level binding-time analysis should be images of each other under the Curry-Howard isomorphism.”

Indeed, in (Davies, 1996) it is shown that there exists a term $M$ such that $\vdash^0 M : A$ if and only if there is a proof of $A$ in a very small fragment of classical Linear Temporal Logic (LTL). This fragment is known as $L^\circ$ and is due to Stirling (Stirling, 1992, p. 516). Axiomatically, $L^\circ$ consists of the normality axiom $\circ(A \to B) \to (\circ A \to \circ B)$, and the equivalence of $\circ \neg A$ and $\neg \circ A$, along with modus ponens and necessitation.

However, $L^\circ$ is a classical logic, and hence not an exact match to the type system of $\lambda^\circ$. The exact logic of $\lambda^\circ$ is intuitionistic and was isolated by Kojima and Igarashi (2011), who call it Constructive Linear-Time Temporal Logic (CLTL). In op. cit. the authors study $\lambda^\circ$ as a natural deduction system, provide an associated sequent calculus for which they prove cut elimination, and discuss appropriate Kripke semantics. They also provide an axiomatization of CLTL, which consists of all instances of intuitionistic propositional tautologies, as well as the following axioms:

\[
\begin{align*}
(\circ K) & \quad \circ(A \to B) \to (\circ A \to \circ B) \\
(\circ K^{-1}) & \quad (\circ A \to \circ B) \to \circ(A \to B)
\end{align*}
\]

under the rules of modus ponens and necessitation.

6.2. Metaprogramming Variants. Attempts to understand, implement and practice metaprogramming span multiple decades. Early efforts include quasiquote in LISP-derived languages, and which date from the mid 1970s (Bawdlen, 1999); Ershov’s idea of mixed computation (Ershov, 1982); and the Futamura projections, discovered in 1971 (Futamura, 1999). The Futamura projections were later elaborated into the theory and practice of partial evaluation; see the survey by Jones (1996) and the book by Jones et al. (1993). These developments began to acquire a logical foundation in the 1990s, mainly through the use of constructive modalities.

$\lambda^\circ$ was a seminal language in providing this logical foundation, and it spearheaded many developments. This is because the constructs next and prev essentially act like the quote and unquote operators of LISP, as explained in Bawdlen (1999).

$\lambda^\circ$ forms the essence of MetaML, a metaprogramming language developed by Taha and Sheard (1997, 2000). Only two fundamental differences distinguish MetaML to $\lambda^\circ$:

- MetaML implements cross-stage persistence: a variable can be used at a later stage than its annotation, as the variable rule essentially becomes

\[
\Gamma \vdash^m x : A \quad \frac{(x : A) \in \Gamma \quad m \geq n}{\Gamma \vdash^n x : A}
\]
• A run construct is forcibly added to the system, with reduction rule

\[
\text{run } (\text{next } e) \rightarrow e
\]

However, this construct is ‘unsafe’—in that one can encounter an bound runtime variable upon evaluation—and it also comes at a cost in ‘expressivity’; see (Taha and Sheard, 2000, §12).

MetaML was further simplified in a series of papers by Moggi, Taha, Sheard and Benaissa. First, (Moggi et al., 1999) introduced a simplified version called AIM, short for An Idealized MetaML. Their work was driven by their attempt to find a categorical semantics for MetaML, a sketch of which may be found in an unpublished manuscript (Benaissa et al., 1998). An even more simplified language, \(\lambda^{BN}\), was introduced by the same authors a year later (Benaissa et al., 1999), and their discussion also included a more refined categorical semantics.

A few years later, (Taha and Nielsen, 2003) introduced environment classifiers, a system that, amongst other improvements, constitutes an expansion of \(\lambda^\bigcirc\) from linear time to branching time. A logical foundation was then developed for this system by (Tsukada and Igarashi, 2010).

Following from the work of Moggi et al., (Yuse and Igarashi, 2006) introduced a combination of \(\lambda^\bigcirc\) and Davies and Pfenning’s DCS4, called \(\lambda^{\bigcirc\square}\), which combines the \(\bigcirc\) modality of CLTL and the CS4 modality \(\square\). The interpretation is straightforward, and \(\bigcirc\) is supposed to be read as ‘in the next step’ (or stage), whereas \(\square\) is to be read as ‘always’ (or, in all stages). The resulting calculus is somehow similar to Moggi et al’s AIM and \(\lambda^{BN}\), but arguably closer to a more Curry-Howard based interpretation. Nevertheless, not enough effort is put into determining which fragment of LTL their calculus corresponds to, but they do note that it lacks the ‘induction axiom’ \(\square(A \rightarrow \bigcirc A) \rightarrow (A \rightarrow \square A)\).

As the reader may witness, metaprogramming with modalities has grown into a vast field of its own that is still expanding. As such, it will be the main subject of the second part of the present survey, which Mario Alvarez-Picallo is currently preparing.

6.3. Categorical Semantics of CLTL. (Benaissa et al., 1998) made a start in analyzing the categorical semantics of various modalities for metaprogramming, and this included an analysis of \(\lambda^\bigcirc\). A seemingly ‘conclusive’ analysis is given by (Benaissa et al., 1999): a semantics of CLTL consists of a cartesian-closed category \(\mathcal{C}\) and a full and faithful CCC-functor,

\[
F : \mathcal{C} \hookrightarrow \mathcal{C}
\]

A CCC-functor is a finite product preserving (i.e. strong monoidal) functor for which the canonical arrow \(F(B^A) \rightarrow FB^FA\) is an isomorphism.
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