On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for the white noise dispersion

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Abstract

In this paper, we investigate a stochastic Hardy-Littlewood-Sobolev inequality. Due to the stochastic nature of the inequality, the relation between the exponents of integrability is modified. This modification can be understood as a regularization by noise phenomenon. As a direct application, we derive Strichartz estimates for the white noise dispersion which enables us to address a conjecture from [3].

Key words: Stochastic regularization; Stochastic Partial Differential Equations; Nonlinear Schrödinger equation, Hardy-Littlewood-Sobolev inequality.

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1 Introduction

Let \((\Omega, \mathbb{P})\) be the standard probability space endowed with the Wiener filtration \((\mathcal{F}_t)_{t \geq 0}\). The main objective of this paper is to address the local Cauchy problem for the following nonlinear Schrödinger equation with white noise dispersion

\[
\begin{align*}
\left\{ \begin{array}{l}
    i\psi(t, x) &= -\Delta \psi(t, x) \circ dW_t(\omega) + \lambda |\psi|^{2\sigma} \psi(t, x) dt, \\
    \psi(0, \cdot) &= \psi_0 \in L^2(\mathbb{R}^d),
\end{array} \right. \\
\forall (t, x, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega,
\end{align*}
\]

where \(\lambda \in \mathbb{R}\) and \((W_t)_{t \geq 0}\) is the Wiener process and the product \(\circ\) is understood in the Stratonovich sense.

For \(d = 1\) and \(\sigma = 1\), this equation arises in the field of nonlinear optics as a limit model for the propagation of light pulse in an optical fiber where the dispersion varies along the fiber [2, 1]. These variations in the dispersion accounts for the so-called dispersion management which aims to improve the transmission of a light signal by constructing a zero-mean dispersion fiber in order to avoid the problem of the chromatic dispersion of the light signal. When the variations are assumed to be random, a noisy dispersion can be derived (see [24, 12]) which leads, in the white noise case, to Equation (1.1).
As part of the problems concerning the propagation of waves in random media, there
is a vast literature around random Schrödinger equations. Let us mention in particular
the cases of random potentials [3, 14] and noisy potentials [2] [10] [11]. In these works,
the effects of the stochastic potential greatly affect the dynamic of the Schrödinger
equation and are, in a broader context, a motivation to introduce randomness in PDEs.
 Specifically, there is a well known effect which attracted a lot of attention: the so-called
regularization by noise phenomenon (see [17] for a survey). This phenomenon can be
summarized as an improvement, due to the presence of noise, of the well-posedness of
differential equations and has been studied in the context of SDEs [28, 27, 23, 25, 5],
transport equation [18, 16, 4], SPDEs [8] and scalar conservation laws [19]. We remark
that obtaining a regularization by noise in the context of nonlinear random PDE is a
challenging task and most of the results are obtained in a linear setting. For instance,
an open problem is to obtain a regularization by noise for the Euler or Navier-Stokes
equations.

We are not the first one to investigate the Cauchy problem of Equation (1.1). It
was first studied in [12] where the global Cauchy problem was solved for \( \sigma < d/2 \) which
is the case for the deterministic nonlinear Schrödinger equation and, thus, hints for a regularization by noise effect. In [17], the authors study the case where the
Wiener process is replaced by a fractional Wiener process and recover similar results
as in [12]. By a simple scaling argument on the space and time variables of (1.1) and
thanks to the scaling invariance of the Wiener process, it was conjectured in [3] that, in
fact, the critical nonlinearity should be \( \sigma = 4/d \), a \( L^2 \)-supercritical nonlinearity, which
is twice as large as the deterministic \( L^2 \)-critical nonlinearity. Furthermore, this fact
was supported by numerical simulations in 1D and leads to believe that the white noise
dispersion has a strong regularization effect.

In this paper, we address the global Cauchy problem (1.1) for \( \sigma < 4/d \). To be more
specific, we obtain the following result.

**Theorem 1.1.** Let \( \sigma < \frac{4}{d} \), \( \psi_0 \in L^2(\mathbb{R}^d) \) and \( a \in (2, \infty) \) such that \( \frac{d a}{4} < \frac{2(\sigma+1)}{\sigma} < 1 \). Then, for almost all \( \omega \in \Omega \), there exists a unique solution \( \psi \in L^a([0, +\infty]; L^{2\sigma+2}(\mathbb{R}^d)) \)
to (1.1).

The classical approach to investigate the Cauchy problem for nonlinear Schrödinger
equations is to derive local Strichartz estimates [6]. These estimates are a direct conse-
quence of the dispersive property of the linear operator \( i\Delta \). However, as pointed out in
[13], it is much harder to obtain such estimates in the case of a white noise dispersion
because of the presence of the Wiener process. We remark that the strategy used in
[12] does rely on stochastic Strichartz estimates but these were not efficient enough to
handle \( L^2 \)-supercritical nonlinearities.

Let us now explain our approach to deduce Strichartz estimates for (1.1). We recall
from [24, 12] that the propagator associated to the linear part of (1.1) is explicitly given
by, \( \forall t, s \in (0, \infty), \forall \omega \in \Omega \) and \( \forall \varphi \in C_0^\infty(\mathbb{R}^d) \),

\[
P_{s,t}(\omega) \varphi(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i|x|^2/(2(t_1 - t_2)\omega)} e^{i\xi \cdot x} d\xi
= \frac{1}{(4\pi(W_t(\omega) - W_s(\omega)))^{d/2}} \int_{\mathbb{R}^d} e^{i(x-y)^2/(4(W_t(\omega) - W_s(\omega)))} \varphi(y) dy. \tag{1.2}
\]

Following the classical proof of Strichartz estimates (see for instance [22]), a fun-
damental tool is the Hardy-Littlewood-Sobolev inequality [20, 21, 26] which is stated
below.

**Theorem 1.2.** Let $T > 0$, $\alpha \in (0, 1)$ and $f \in L^p([0, T])$ and $g \in L^q([0, T])$ such that

$$2 - \alpha = \frac{1}{p} + \frac{1}{q}.$$  

Then, there exists a constant $C > 0$ which depends on $p$ and $q$ such that the following inequality holds

$$\left| \int_0^T \int_0^t f(s)|t - s|^{-\alpha} g(s)dsdt \right| \leq C \|f\|_{L^p([0, T])} \|g\|_{L^q([0, T])}. \quad (1.3)$$

From here, if we wish to follow the classical arguments to derive dispersive estimates, the main difficulty is to prove inequality $(1.3)$ but replacing the potential $|t - s|^{-\alpha}$ with $|W_t(\omega) - W_s(\omega)|^{-\alpha}$. This is the point of the following theorem, which is the second result of this paper.

**Theorem 1.3.** Let $T > 0$, $p, q \in (1, \infty)$ and $\alpha \in (0, 1)$ such that

$$2 - \alpha < \frac{1}{2} + \frac{1}{q}.$$  

Then, there exists a set $\mathcal{N} \subset \Omega$ of zero measure which depends on $T$ and $\alpha$ such, $\forall \omega \not\in \mathcal{N}$, $\forall f \in L^p([0, T])$, $\forall g \in L^q([0, T])$, the following inequality holds

$$\left| \int_0^T \int_0^t f(s) |W_t(\omega) - W_s(\omega)|^{-\alpha} g(s)dsdt \right| \leq \|f\|_{L^p([0, T])} \|g\|_{L^q([0, T])}. \quad (1.4)$$

We can see that our result does not give an equality between the exponents of integrability and $\alpha$ but an inequality. However, it is enough for our purpose. We also remark that the relation is very different from the one in Theorem 1.2 since $\alpha$ is divided by a factor of 2. This is due to the stochastic nature of the potential $|W_t - W_s|^{-\alpha}$ and, somehow, is a consequence of the scaling invariance of the Wiener process. This modification has a dramatic impact on the integrability assumptions of $f$ and $g$ and, as a consequence, we obtain the following stochastic Strichartz estimates for the propagator $(P_{s,t})_{s,t \geq 0}$ given by $(1.2)$.

**Definition 1.1.** For any $(q, p) \in (1, \infty)^2$ we say that $(q, p)$ is sub-admissible if

$$\frac{2}{q} > \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right).$$

**Proposition 1.1.** Let $T > 0$ and $(q, p)$ sub-admissible. Then, there exists two constants $C_1, C_2 > 0$ and a set $\mathcal{N} \subset \Omega$ of zero measure which depends on $d$, $T$, $p$ and $q$ such that, $\forall \omega \not\in \mathcal{N}$, $\forall f \in L^2(\mathbb{R}^d)$ and $\forall g \in L^p([0, T]; L^q(\mathbb{R}^d))$, the following inequalities hold

$$\|P_0(\omega)f\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_1 \|f\|_{L^2}; \quad (1.5)$$

$$\left\| \int_0^T P_{s, t}(\omega) g(s)ds \right\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_2 \|g\|_{L^p([0, T]; L^q(\mathbb{R}^d))}, \quad (1.6)$$

for any $(r, l)$ sub-admissible.

Thanks to the previous result, we are able to prove Theorem 1.1 by classical arguments. The rest of the paper is devoted to the proof of Theorem 1.3 in section 2 and the proofs of Proposition 1.1 and Theorem 1.3 in section 3.
2 Proof of Theorem 1.3

Let $T > 0$, $p, q \in (1, \infty)$, $\alpha \in (0, 1)$ and $\nu > 0$ such that

$$2 - \frac{\alpha}{2} - \nu = \frac{1}{p} + \frac{1}{q}.$$

Before proceeding any further, let us remark that we can, without loss of generality, assume that $f \in L^p([0, T])$ and $g \in L^q([0, T])$ are non-negative functions.

Let $N \in \mathbb{N}^+$ and $f^{(N)}, g^{(N)}$ be two simple function given by

$$f^{(N)}(t) = \sum_{j=1}^{2^N - 1} f_j^N \delta_{[t_j, t_{j+1})}(t) \quad \text{and} \quad g^{(N)}(s) = \sum_{k=1}^{2^N - 1} g_k^N \delta_{[t_k, t_{k+1})}(s),$$

where $(f_j^N)_{1 \leq j \leq 2^N - 1}, (g_k^N)_{1 \leq k \leq 2^N - 1} \subset \mathbb{R}^+$ and $(t_j)_{1 \leq j \leq 2^N - 1}$ is a uniform discretization of $[0, T]$ such that

$$t_j := jh_N, \quad \forall j \in \{0, \ldots, 2^N\},$$

with

$$h_n := \frac{T}{2^N}.$$

We remark that

$$\|f^{(N)}\|_{L^p} = \left(\sum_{j=1}^{2^N - 1} f_j^N h\right)^{1/p} \quad \text{and} \quad \|g^{(N)}\|_{L^q} = \left(\sum_{k=1}^{2^N - 1} g_k^N h\right)^{1/q}.$$

Furthermore, it follows that

$$\int_0^T \int_0^T f^{(N)}(t)|W_t(\omega) - W_s(\omega)|^{-\alpha} g^{(N)}(s)dsdt = \sum_{j,k=1}^{2^N - 1} f_j^N g_k^N \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t(\omega) - W_s(\omega)|^{-\alpha} dsdt.$$

From here, we need the following proposition.

**Proposition 2.1.** There exists a sequence of sets $\{\Omega_{\varepsilon}\}_{\varepsilon > 0}$ which depends on $T$ and $\alpha$ such that

1. $\Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$ for all $\varepsilon_1 < \varepsilon_2$,
2. $\mathbb{P}(\Omega_{\varepsilon}) \geq 1 - \varepsilon$ for all $\varepsilon > 0$,
3. there exists a $N_{\varepsilon}$ large enough such that, $\forall \nu > 0$, $\forall \omega \in \Omega_{\varepsilon}$,

$$\int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t(\omega) - W_s(\omega)|^{-\alpha} dsdt \leq h_{N_{\varepsilon}}^{-2\alpha/2 - \nu}. \quad (2.1)$$

Assume for a moment that Proposition 2.1 holds. Then, for any $N \geq N_{\varepsilon}$, we can deduce that, thanks to Jensen’s inequality and since $2 - \alpha/2 - \nu = 1/p + 1/q$,

$$\int_0^T \int_0^T f^{(N)}(t)|W_t(\omega) - W_s(\omega)|^{-\alpha} g^{(N)}(s)dsdt \leq \sum_{j,k=1}^{2^N - 1} f_j^N g_k^N h_{N_{\varepsilon}}^{\frac{1}{p} + \frac{1}{q}}.$$
Since this estimate is uniform in $\varepsilon$.

We now separate the proof in two parts: the first part where $j > k$, and the second part when $j = k$. Hence, there exists a set $\Omega_\varepsilon$ such that, $\forall \alpha \in \Omega_\varepsilon$, $\forall f \in L^p([0,T])$, $\forall g \in L^q([0,T])$, the following inequality holds:

$$\int_0^T \int_0^T f(t)|W_t(\omega) - W_s(\omega)|^{-\alpha}g(s)dsdt \leq \|f\|_{L^p([0,T])}\|g\|_{L^q([0,T])}. \quad (2.2)$$

Since this estimate is uniform in $\varepsilon$, we can pass to the limit $\varepsilon \to 0$. Set $\varepsilon = 1/n$, we have

$$\mathbb{P}\left[\cap_{n=1}^\infty \Omega_{1/n}\right] = \lim_{n \to +\infty} \mathbb{P}[\Omega_{1/n}] \geq 1.$$

Hence, there exists a set $\mathcal{N}$ of zero measure, which depends on $\alpha$ and $T$, such that, $\forall \omega \notin \mathcal{N}$, $\forall f \in L^p([0,T])$ and $\forall g \in L^q([0,T])$, the estimate (2.2) holds. Thus, to conclude the proof of Theorem 1.3, it remains to prove Proposition 2.1. Before proceeding further, we need some technical results.

Let us begin with the following estimate.

**Lemma 2.1.** Let $p \geq 1$ and $j, k \in \mathbb{N}$. There exists a constant $C > 0$ such that the following estimate holds,

$$\mathbb{E}\left[\left(\int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha}dsdt\right)^p\right] \leq C^{p}p(p!)h_n^{2(2-\alpha)/2}.$$

**Proof.** We first remark that, since $t_j = jh$ and by the scaling property of the Brownian motion,

$$\mathbb{E}\left[\left(\int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha}dsdt\right)^p\right] = h^{p(2-\alpha)/2} \mathbb{E}\left[\left(\int_{j}^{j+1} \int_{k}^{k+1} |W_t - W_s|^{-\alpha}dsdt\right)^p\right].$$

We now separate the proof in two parts: the first part where $j > k$, which, by symmetry, also gives the case $j > k$, and the second part when $j = k$.

**Step 1: the case $j > k$.** By denoting $t = (t_1, t_2, \cdots, t_{p}) \in \mathbb{R}^p$ and $s = (s_1, s_2, \cdots, s_{p}) \in \mathbb{R}^p$, we remark that

$$\mathbb{E}\left[\left(\int_{j}^{j+1} \int_{k}^{k+1} |W_t - W_s|^{-\alpha}dsdt\right)^p\right] = \int_{[j,j+1]^p} \mathbb{E}\left[\prod_{n=1}^{p} \int_{k}^{k+1} |W_{t_n} - W_s|^{-\alpha}ds\right] dt$$

$$= (p!) \int_{\Delta_p([j,j+1])} \mathbb{E}\left[\prod_{n=1}^{p} \int_{k}^{k+1} |W_{t_n} - W_s|^{-\alpha}ds\right] dt$$

$$= (p!) \int_{\Delta_p([j,j+1])} \int_{[k,k+1]^p} \mathbb{E}\left[\prod_{n=1}^{p} |W_{t_n} - W_{s_n}|^{-\alpha}\right] dsdt.$$
where we used the fact that, for all \((s_1, s_2, \cdots, s_p) \in \mathbb{R}^p,\)
\[
\sum_{\sigma \in \mathfrak{S}_p} 1_{s_1(1) \leq s_2(2) \leq \cdots \leq s_p(p)} = 1,
\]
with \(\mathfrak{S}_p\) is the set of permutations of length \(p\) and \(\Delta_p([\alpha, \beta]) = \{t \in \mathbb{R}^p; \alpha < t_1 < t_2 < \cdots < t_p < \beta\}.\)

Since \(k + 1 \leq j\), we have \(s_p < t_1\) and, hence,
\[
s_1 < s_2 < \cdots < s_p < t_1 < t_2 < \cdots < t_p.
\]

By the Markov property of the Brownian motion and denoting the conditional expectation, \(\forall t \geq 0, \forall X \in L^1(\Omega),\)
\[
E_t[X] := E[X|\mathcal{F}_t],
\]
we obtain that
\[
E_{t_{p-1}}\left[\prod_{n=1}^{p} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha}\right] = \prod_{n=1}^{p-1} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha} E_{t_{p-1}}\left[|W_{t_p} - W_{s_{\sigma(p)}}|^{-\alpha}\right]
\]
\[
= \prod_{n=1}^{p-1} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha} \int_{\mathbb{R}} |x_p + W_{t_{p-1}} - W_{s_{\sigma(p)}}|^{-\alpha} G_{t_{p-1}}(x) dx,
\]
where \((G_t)_{t \geq 0}\) is the Gaussian kernel. We remark that the following estimate holds
\[
\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |x - y|^{-\alpha} G_1(x) dx \leq C, \quad (2.3)
\]
for a certain constant \(C > 0\). It then follows from the scaling property of the Gaussian kernel, by denoting \(\delta t_j = t_j - t_{j-1}\) and the estimate \((2.3)\) that
\[
E_{t_{p-1}}\left[\prod_{n=1}^{p} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha}\right] = (\delta t_p)^{-\alpha/2} \prod_{n=1}^{p-1} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha} \int_{\mathbb{R}} |x + (\delta t_p)^{-1} (W_{t_{p-1}} - W_{s_{\sigma(p)}})|^{-\alpha} G_1(x) dx
\]
\[
\leq C(\delta t_p)^{-\alpha/2} \prod_{n=1}^{p-1} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha}.
\]
By the tower property of the conditional expectations and an induction argument, we deduce that
\[
E\left[\prod_{n=1}^{p} |W_{t_n} - W_{s_{\sigma(n)}}|^{-\alpha}\right] \leq C^{p-1} \prod_{n=2}^{p} (\delta t_n)^{-\alpha/2} E\left[|W_{t_1} - W_{s_{\sigma(1)}}|^{-\alpha}\right]
\]
\[
\leq C^p (t_1 - s_{\sigma(1)})^{-\alpha/2} \prod_{n=2}^{p} (\delta t_n)^{-\alpha/2}.
\]
Thus, we have, since

$$\mathbb{E} \left[ \left( \int_j^{j+1} \int_k^{k+1} |W_t - W_s|^{-\alpha} ds dt \right)^p \right] \leq C^p(p!) \sum_{\sigma \in \mathcal{G}_p} \int_{\Delta_p([j,j+1])} \int_{\Delta_p([k,k+1])} (t_1 - s_{\sigma(1)})^{-\alpha/2} \prod_{n=2}^{p} (\delta t_n)^{-\alpha/2} ds dt.$$

We remark that, if \( j = k + 1 \) and since \( t_1 \in [k+1, k+2] \),

$$\int_{\Delta_p([k,k+1])} (t_1 - s_{\sigma(1)})^{-\alpha/2} ds \leq \int_{\Delta_{p-1}([k,k+1])} ds \int_k^{k+1} (t_1 - s)^{-\alpha/2} ds \leq \frac{1}{(p-1)!} \sup_{t \in [k+1, k+2]} (t_1 - k)^{1-\alpha/2} \leq \frac{1}{(p-1)!}.$$

Else, if \( j \geq k + 2 \), we have

$$\int_{\Delta_p([k,k+1])} (t_1 - s_{\sigma(1)})^{-\alpha/2} ds \leq \sup_{t \in [j,j+1], s \in [k,k+1]} (t - s)^{-\alpha/2} \int_{\Delta_p([k,k+1])} ds \leq \frac{1}{p!}.$$

In order to estimate the term

$$\int_{\Delta_p([j,j+1])} \prod_{n=2}^{p} (\delta t_n)^{-\alpha/2} dt,$$

we proceed by induction thanks to the following estimate, for all \( 2 \leq n \leq p \),

$$\int_{k}^{t_n} (\delta t_n)^{-\alpha/2} dt_{n-1} = \int_{k}^{t_n} (t_n - t_{n-1})^{-\alpha/2} dt_{n-1} = \frac{1}{1 - \alpha/2} (t_n - k)^{1-\alpha/2} \leq 1.$$

This leads us to the following bound

$$\int_{\Delta_p([j,j+1])} \prod_{n=2}^{p} (\delta t_n)^{-\alpha/2} dt \leq 1.$$

Finally, since \( \text{Card}(\mathcal{G}_p) = p! \), we obtain the desired estimate

$$\mathbb{E} \left[ \left( \int_j^{j+1} \int_k^{k+1} |W_t - W_s|^{-\alpha} ds dt \right)^p \right] \leq \sum_{\sigma \in \mathcal{G}_p} C^p \max(1, p) \leq C^p p!(p!).$$

**Step 2: the case \( j = k \)** By denoting \( t = (t_1, t_2, \cdots, t_{2p}) \in \mathbb{R}^{2p} \), we remark that

$$\mathbb{E} \left[ \left( \int_j^{j+1} \int_j^{j+1} |W_t - W_s|^{-\alpha} ds dt \right)^p \right] = \mathbb{E} \left[ \int_{[j,j+1]^{2p}} \prod_{n=1}^{p} |W_{t_{n+p}} - W_{t_n}|^{-\alpha} dt \right]$$

$$= \sum_{\sigma \in \mathcal{G}_{2p}} \int_{\Delta_{2p}([j,j+1])} \mathbb{E} \left[ \prod_{n=1}^{p} |W_{t_{\sigma(n+p)}} - W_{t_{\sigma(n)}}|^{-\alpha} \right] dt.$$
Let \( k = \text{argmax}_{1 \leq \ell \leq 2p} \sigma(\ell) \). Then, we have

\[
\mathbb{E}_{t_{2p-1}} \left[ \prod_{n=1}^{p} |W_{t_{\sigma(n)}} - W_{t_{\sigma(n)+p}}|^{-\alpha} \right] = \prod_{n=1 \atop n \neq k}^{p} |W_{t_{\sigma(n)}} - W_{t_{\sigma(n)+p}}|^{-\alpha} \mathbb{E}_{t_{2p-1}} \left[ |W_{t_{2p}} - W_{t_{\sigma(k)+p}}|^{-\alpha} \right]
\]

We have, thanks to scaling property of the Gaussian kernel and estimate (2.3),

\[
\mathbb{E}_{t_{2p-1}} \left[ |W_{t_{2p}} - W_{t_{\sigma(k)+p}}|^{-\alpha} \right] = \int_{\mathbb{R}} \left| x + W_{t_{2p-1}} - W_{t_{\sigma(k)+p}} \right|^{-\alpha} G_{t_{2p}-t_{2p-1}}(x)dx
\]

\[= (\delta t_{2p})^{-\alpha/2} \int_{\mathbb{R}} \left| x + (\delta t_{2p})^{-1}(W_{t_{2p-1}} - W_{t_{\sigma(k)+p}}) \right|^{-\alpha} G_{1}(x)dx \leq C(\delta t_{2p})^{-\alpha/2}.
\]

By repeating this procedure, we obtain a \( p \)-tuple \( k = (k_1, k_2, \ldots, k_p) \in \{1, \ldots, 2p\}^p \) such that, by integrating out the \( p \) singularities in time,

\[
\int_{\Delta_{2p}([j,j+1])} \mathbb{E} \left[ \prod_{n=1}^{p} |W_{t_{\sigma(n)+p}} - W_{t_{\sigma(n)}}|^{-\alpha} \right] dt \leq C^p \int_{\Delta_{2p}([j,j+1])} \prod_{n \in k} (\delta t_n)^{-\alpha/2} dt \leq \frac{C^p}{p!}
\]

We deduce that

\[
\mathbb{E} \left[ \left( \int_{j}^{j+1} \int_{j}^{j+1} |W_t - W_s|^{-\alpha} ds dt \right)^p \right] \leq \sum_{\sigma \in \mathcal{S}_{2p}} \frac{C^p}{p!} \leq C^p \frac{(2p)!}{p!}
\]

It follows from Stirling approximation that

\[
\frac{(2p)!}{p!} \leq \frac{(2p)^{2p+1/2}e^{-2p}}{p^{p+1/2}e^{-p}} = 2^{1/2}4^p p^p e^{-p} \leq \frac{4^p}{\sqrt{p}} p! \leq p4^p p!,
\]

which gives the desired estimate. \(\square\)

The previous result enables us to deduce the following Lemma.

**Corollary 2.1.** We have the following limit

\[
\lim_{\kappa \to +\infty} \mathbb{P} \left\{ \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt > \kappa n^{2-\alpha/2}, 1 \leq j, k \leq 2N - 1, \forall N \geq 1 \right\} = 0.
\]

**Proof.** Let \( \kappa > 4C \log(2) \) where \( C \) is the constant from Lemma 2.1. Denote \( \theta = n^{\alpha-2}/(2C) \), we have, thanks to Chebyshev’s inequality, for all \( 1 \leq j, k \leq 2N - 1, \)

\[
\mathbb{P} \left[ \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt > \kappa Nn^{2-\alpha/2} \right] \leq \mathbb{E} \left[ e^{\left( \frac{t_{j+1}}{t_j} \int_{t_j}^{t_{j+1}} |W_t - W_s|^{-\alpha} ds dt \right)} \right]
\]

\[
\leq e^{-N} \mathbb{E} \left[ e^{\left( \frac{t_{k+1}}{t_k} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt \right)} \right].
\]

8
It then follows from Lemma 2.1 that

\[
\mathbb{E} \left[ e^{\theta \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt} \right] = \sum_{p=1}^{+\infty} \frac{\theta^p}{p!} \mathbb{E} \left[ \left( \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt \right)^p \right] \\
\lesssim \sum_{p=0}^{+\infty} p\theta^p C^p h_N^{p(2-\alpha/2)} = \sum_{p=0}^{+\infty} p2^{-p} \lesssim 1.
\]

Thus, we have

\[
P \left[ \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt > \kappa N h_N^{2-\alpha/2} \right] \lesssim e^{-N \frac{\kappa}{2\delta}}
\]

which leads to

\[
P \left\{ \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt > \kappa N h_N^{2-\alpha/2}, \, 1 \leq j, k \leq 2^N - 1, \, \forall N \geq 1 \right\} \\
\lesssim \sum_{N=1}^{+\infty} \sum_{j,k=1}^{2^N-1} e^{-N \frac{\kappa}{2\delta}} = \sum_{N=1}^{+\infty} 2^{2^N-2} e^{-N \frac{\kappa}{2\delta}} \lesssim \sum_{N=1}^{+\infty} e^{-N(\frac{\kappa}{2\delta}-2\log(2))} =: I(\kappa).
\]

Hence, since \( \kappa > 4C \log(2) \), we have that \( I(\kappa) < +\infty \). Furthermore, by theorem of dominated convergence, we deduce

\[
I(\kappa) \xrightarrow{\kappa \to +\infty} 0.
\]

We can now proceed to prove Proposition 2.1. By denoting

\[
\Omega_\varepsilon := \left\{ \int_{t_j}^{t_{j+1}} \int_{t_k}^{t_{k+1}} |W_t - W_s|^{-\alpha} ds dt \leq \kappa_\varepsilon N h_N^{2-\alpha/2}, \, 1 \leq j, k \leq 2^N - 1, \, \forall N \geq N_0 \right\},
\]

we deduce from Corollary 2.1 that, \( \forall \varepsilon > 0 \), there exist \( \kappa_\varepsilon > 0 \) such that

\[
P(\Omega_\varepsilon) \geq 1 - \varepsilon.
\]

Furthermore, we see that \( \kappa_\varepsilon \) is non-increasing with respect to \( \varepsilon \) and, thus, we deduce that the sequence \( \{\Omega_\varepsilon\}_{\varepsilon > 0} \) is increasing, i.e. \( \Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2} \), for \( \varepsilon_1 < \varepsilon_2 \). We finally remark that \( \Omega_\varepsilon \) depends on \( \alpha \) and \( T \).

3 Proof of Proposition 1.1 and Theorem 1.1

We can now proceed to prove the Strichartz estimates by the \( TT^* \) strategy (see [22]) and use them in a fixed-point argument (see [6]) to prove the global well-posedness of Equation (1.1).
3.1 Proof of Proposition 1.1

We deduce, since \( P_{s,t}(\omega) \) is an isometry from \( L^2 \) to itself, thanks to the Hausdorff-Young inequality and an interpolation argument, that, \( \forall p \in [2, \infty], \forall \varphi \in L^{p'}(\mathbb{R}^d) \),

\[
\| P_{s,t}(\omega) \varphi \|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|W_t(\omega) - W_s(\omega)|^{d(1/2 - 1/p)}} \| \varphi \|_{L^{p'}(\mathbb{R}^d)}, \tag{3.1}
\]

where \( p' \) is the Hölder conjugate of \( p \).

Let \( T > 0, (q,p) \) sub-admissible and \( \omega \notin \mathcal{N} \) where \( \mathcal{N} \) is given by Theorem 1.3 with \( \alpha = d(1/2 - 1/p) \). We denote \( (P_{s,t}^*) \) the adjoint of the propagator of the white noise dispersion, that is

\[
P_{s,t}^*(x) := \mathcal{F}^{-1} \left( e^{i|q|^2(W_t - W_s)} \hat{\varphi}(\xi) \right) = P_{t,s} \varphi(x).
\]

This yields, on one hand, that

\[
P_{s,t}^* = P_{t,s}, \quad P_{0,s}^* P_{0,t} = P_{s,t} \quad \text{and} \quad P_{s,t} P_{r,t}^* = P_{s,r}, \quad \forall r \in [s,t].
\]

We consider the integral, \( \forall f, g \in C([0,T], \mathcal{C}^0(\mathbb{R}^d)) \),

\[
I(f, g) = \left| \int_0^T \int_0^T (P_{0,s} f(s), P_{0,s} g(t))_{L^2} ds \right| \leq \left| \int_0^T \int_0^T (P_{0,s} f(s), P_{0,t} g(t))_{L^2} ds \right|
\]

It follows by Hölder’s inequality, (3.1) and Theorem 1.3 that \( \forall p \in (1, \infty) \),

\[
I(f, g) \lesssim \left( \int_0^T |W_t(\omega) - W_s(\omega)|^{d(1/2 - 1/p)} \| f(t) \|_{L^{p'}(\mathbb{R}^d)} \| g(t) \|_{L^{p'}(\mathbb{R}^d)} ds \right)^{1/q_2}
\]

where \( q_1, q_2 \in (1, \infty) \) verify

\[
2 - \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right) > \frac{1}{q_1} + \frac{1}{q_2}.
\]

Setting \( q_1 = q_2 = q' \), the previous inequality becomes

\[
2 - \frac{d}{2} \left( \frac{1}{2} - \frac{1}{p} \right) > \frac{1}{q}.
\]

This yields, on one hand, that

\[
\left\| \int_0^T P_{0,s}^*(\omega) f(s) ds \right\|_{L^2(\mathbb{R}^d)}^2 = I(f, f) \lesssim \| f \|_{L^{p'}([0,T], L^{p'}(\mathbb{R}^d))}^2, \tag{3.2}
\]

and, on another hand, by a duality argument,

\[
\left\| \int_0^T P_{s,t}^*(\omega) f(s) ds \right\|_{L^q([0,T], L^p(\mathbb{R}^d))} \lesssim \| f \|_{L^{p'}([0,T], L^{p'}(\mathbb{R}^d))} \tag{3.3}
\]
We are now in position to prove (1.5) and (1.6). It follows from (3.2) that, \( \forall f \in L^2(\mathbb{R}^d) \) and \( \forall g \in L^q([0, T]; L^p(\mathbb{R}^d)) \),

\[
\int_0^T \langle P_{0,t}(\omega) f, g(t) \rangle_{L^2} dt = \left\langle f, \int_0^T P_{0,t}^* (\omega) g(t) \right\rangle_{L^2} \leq \left\| f \right\|_{L^2(\mathbb{R}^d)} \left\| \int_0^T P_{0,t}^* (\omega) g(t) ds \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim \left\| f \right\|_{L^2(\mathbb{R}^d)} \left\| g \right\|_{L^q((0, T], L^p'(\mathbb{R}^d))},
\]

which leads to (1.5) by a duality argument. We now turn to (1.6). We have, by (3.2),

\[
\left\| \int_0^T P_{s,t}(\omega) f(s) ds \right\|_{L^q((0, T], L^p(\mathbb{R}^d))} \leq \int_0^T \left\| P_{s,t} f(s) \right\|_{L^q([0, T]; L^p(\mathbb{R}^d))} ds \leq \int_0^T \left\| f(s) \right\|_{L^2(\mathbb{R}^d)} ds = \left\| f \right\|_{L^1([0, T]; L^2(\mathbb{R}^d))}.
\]

Thanks to this estimate and an interpolation argument with (3.3), we deduce (1.6). This concludes the proof of Proposition 1.1.

### 3.2 Proof of Theorem 1.1

We can now apply the previous result to solve the global Cauchy problem of (1.1). First, we rewrite the equation in its mild formulation, \( \forall t \in [0, T] \) and \( \forall x \in \mathbb{R}^d \),

\[
\psi(t, x) = P_{0,t} \psi_0(x) - i \lambda \int_0^t P_{s,t} |\psi|^{2\sigma} \psi(s, x) ds,
\]

and assume that \( \sigma < \frac{4}{d} \). Let \( T > 0 \) and \( (q, p) \) be sub-admissible that we will fix later and we consider equation (3.4) for all \( \omega \notin \mathcal{N} \), where \( \mathcal{N} \) is given by Proposition 1.1. We consider the mapping \( \Gamma \) from \( L^q([0, T]; L^p(\mathbb{R}^d)) \) to itself given by

\[
\Gamma(\psi)(t, x) = P_{0,t} \psi_0(x) - i \lambda \int_0^T P_{s,t} |\psi|^{2\sigma} \psi(s, x) ds.
\]

We denote \( B_{R, L^q([0, T]; L^p(\mathbb{R}^d))} \) a closed ball of \( L^q([0, T]; L^p(\mathbb{R}^d)) \) of radius \( R > 0 \) that will be set later. For any \( \psi \in B_{R, L^q([0, T]; L^p(\mathbb{R}^d))} \), we apply the \( L^q([0, T]; L^p(\mathbb{R}^d)) \) norm to (3.5) and deduce, thanks to (1.5) and (1.6),

\[
\left\| \Gamma(\psi) \right\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \leq C_1 \left\| \psi \right\|_{L^2(\mathbb{R}^d)} + C_2 \left\| \psi \right\|_{L^{r'+(2\sigma+1)}((0, T]; L^{l'+(2\sigma+1)}(\mathbb{R}^d))},
\]

for any \( (r, l) \) sub-admissible. By choosing \( (q, p) = (r, l) = (a, 2\sigma + 2) \), \( \lambda \) such that

\[
d\sigma \frac{4}{d} < \frac{2(\sigma + 1)}{a} < 1,
\]

we have that \( (a, 2\sigma + 2) \) is sub-admissible and that

\[
l' = \frac{l}{l - 1} = \frac{2\sigma + 2}{2\sigma + 1}.
\]

Hence, we obtain, by Hölder’s inequality,

\[
\left\| \psi \right\|_{L^{r'+(2\sigma+1)}((0, T]; L^{l'+(2\sigma+1)}(\mathbb{R}^d))} = \left\| \psi \right\|_{L^{r'+(2\sigma+1)}((0, T]; L^{l'+(2\sigma+1)}(\mathbb{R}^d))} \leq T^{1 - \frac{2\sigma + 2}{a}} \left\| \psi \right\|_{L^r([0, T]; L^{l'}(\mathbb{R}^d))} \left\| \psi \right\|_{L^2(\mathbb{R}^d)}.
\]

11
which gives us
\[
\|\Gamma(\psi)\|_{L^q((0,T];L^p(\mathbb{R}^d))} \leq C_1\|\psi_0\|_{L^2(\mathbb{R}^d)} + C_2|\lambda|T^{1-\frac{2\sigma+2}{a}}\|\psi\|_{L^a((0,T];L^{2\sigma+2}(\mathbb{R}^d))}. \tag{3.7}
\]

By similar computations, we obtain that, \(\forall\psi_1, \psi_2 \in B_{R,L^q((0,T];L^p(\mathbb{R}^d))},\)
\[
\|\Gamma(\psi_1) - \Gamma(\psi_2)\|_{L^q((0,T];L^p(\mathbb{R}^d))} \leq C_2|\lambda|T^{1-\frac{2\sigma+2}{a}}R^{2\sigma}\|\psi_1 - \psi_2\|_{L^a((0,T];L^{2\sigma+2}(\mathbb{R}^d))}. \tag{3.8}
\]

We remark that, thanks to (3.6), we have
\[
1 - \frac{2\sigma + 2}{a} > 0.
\]

Hence, by setting
\[
R = 2C_1\|\psi_0\|_{L^2(\mathbb{R}^d)},
\]
and taking \(T > 0\) small enough to have
\[
C_2|\lambda|T^{1-\frac{2\sigma+2}{a}}R^{2\sigma} < 1,
\]
we can see that \(\Gamma\) is a contraction from \(B_{R,L^a((0,T];L^{2\sigma+2}(\mathbb{R}^d))}\) to itself. It follows from a Banach fixed point theorem that there exists a unique local solution to (3.4) in \(B_{R,L^q((0,T];L^p(\mathbb{R}^d))}\). Since \(R\) and \(T\) are independent of this solution, we can iterate this procedure to construct a solution in \(B_{R,L^a([0,\infty];L^{2\sigma+2}(\mathbb{R}^d))}\) which concludes the proof of Theorem 1.1.

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13