Electrodynamics of balanced charges

Anatoli Babin and Alexander Figotin
University of California at Irvine

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Abstract

In this work we modify the wave-corpuscle mechanics for elementary charges introduced by us recently. This modification is designed to better describe electromagnetic (EM) phenomena at atomic scales. It includes a modification of the concept of the classical EM field and a new model for the elementary charge which we call a balanced charge (b-charge). A b-charge does not interact with itself electromagnetically, and every b-charge possesses its own elementary EM field. The EM energy is naturally partitioned as the interaction energy of pairs of different b-charges. We construct EM theory of b-charges (BEM) based on a relativistic Lagrangian with the following properties: (i) b-charges interact only through their elementary EM potentials and fields; (ii) the field equations for the elementary EM fields are exactly the Maxwell equations with proper currents; (iii) a free charge moves uniformly preserving up to the Lorentz contraction its shape; (iv) the Newton equations with the Lorentz forces hold approximately when charges are well separated and move with non-relativistic velocities. The BEM theory can be characterized as neoclassical one which covers the macroscopic as well as the atomic spatial scales, it describes EM phenomena at atomic scale differently than the classical EM theory. It yields in macroscopic regimes the Newton equations with Lorentz forces for centers of well separated charges moving with nonrelativistic velocities. Applied to atomic scales it yields a hydrogen atom model with a frequency spectrum matching the same for the Schrodinger model with any desired accuracy.

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1 Introduction

It is well recognized that the classical electromagnetic (CEM) theory formulated in the form of Maxwell-Lorentz equations provides an excellent description of electromagnetic phenomena at the macroscopic length scales. It also well known that CEM theory is inadequate in explaining electromagnetic (EM) phenomena at the atomic scales including spectroscopic data of the hydrogen atom (HA).

We develop here an EM theory which accounts for all classical EM phenomena at the macroscopic scales as well at least some EM phenomena at the atomic scale including the HA spectral lines. Our theory is classical, though we can apply it to some phenomena (HA energy spectrum in particular) at spatial scales compared with Bohr radius, that is spatial scales of order \(0.1\) nm, and our theory produces the same type of results as the quantum mechanics which is commonly used at such scales. We think that expanding the classical theory down to smallest possible spatial scales is important because the classical description of physical systems allows (at least in principle) more details compared with the probabilistic quantum-mechanical description. Since our neoclassical description applied to HA does not generate contradictions at atomic scales of order \(0.1\) nm (as CEM theory did), we expect it to be applicable to describe details of electromagnetic processes at nanometer scales. Note that at such scales the electron cannot be considered a point or a charged ball and we have to use a complete description of an electron which is presented in our model as a wave-corpuscle.

When attempting to change the CEM theory we want: (i) to stay on solid ground of the Lagrangian mechanics and the relativity principle; (ii) to recover in this new EM theory all well established experimental facts described by the CEM theory. The foundational pillars of the CEM theory - the Maxwell equations and the Lorentz force expression - remain to be key elements in proposed here new EM theory, and before we proceed with the new developments let us briefly recall the CEM theory fundamentals. First of all, the EM fields driven by prescribed currents in vacuum are described by the Maxwell equations

\[
\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{1}
\]

\[
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad \nabla \cdot \mathbf{E} = 4\pi \rho, \tag{2}
\]

where \(\mathbf{E}\) and \(\mathbf{B}\) are respectively electric field and the magnetic induction, and \(\rho(t,x), \mathbf{J}(t,x)\) are respectively prescribed charge and current densities. In particular, when the fields are generated by point charges, the sources \(\rho, \mathbf{J}\) are written in the form

\[
\rho = \sum_{\ell} q^\ell \delta \left( \mathbf{x} - \mathbf{r}^\ell (t) \right), \quad \mathbf{J} = \sum_{\ell} q^\ell \delta \left( \mathbf{x} - \mathbf{r}^\ell (t) \right) \mathbf{v}^\ell (t), \tag{3}
\]
where $q^\ell$ is charge value of the $\ell$-th point charge, $\mathbf{r}^\ell$ and is $\mathbf{v}^\ell(t) = \frac{d\mathbf{r}^\ell}{dt}$ are respectively its position and the velocity, and $\delta$ is the Dirac delta-function. Second, for a given EM field the motion of every point charge in the field is determined from the equation

$$\frac{d}{dt} \left[ m^\ell \mathbf{v}^\ell(t) \right] = q^\ell \left[ \mathbf{E}(t, \mathbf{r}^\ell(t)) + \frac{1}{c} \mathbf{v}^\ell(t) \times \mathbf{B}(t, \mathbf{r}^\ell(t)) \right],$$

(4)

where $m^\ell$ is the $\ell$-th point charge mass and the right hand side of (4) is the Lorentz force. The CEM theory essentially treats three types of problems: (i) studies of EM fields for prescribed charge and current densities described by the Maxwell equations (1), (2); (ii) the motion of charges in a prescribed external field; (iii) interaction of charges and their EM fields. The classical Maxwell-Lorentz system though very successful in describing many EM phenomena has well known problems including infinite self-energy which are discussed widely in the literature. As to important for the CEM theory concept of point charges or particles F. Rohrlich writes, [Roh, Section 2.1, p.9; Section 6.1, p.123]: “... the classical theory of charged particles as first conceived by Lorentz emerged as a hybrid theory of particles and fields: charged particles are interacting via an electromagnetic field”, "Macroscopic Maxwell electrodynamics knows only charge distributions. In this theory electrostatic charge is derived from a charge density (linear, surface, or volume density). The concept of charge as an aggregate of elementary charged particles is foreign to it.” There is also a fundamental thermodynamical problem for the classical Maxwell theory at atomic scale namely the lack of an "elementary process of absorption" as it was formulated by A. Einsteins in his seminal paper [Ein1909b]. He wrote there: "The fundamental property of the oscillation theory that engenders these difficulties seems to me the following. In the kinetic theory of molecules, for every process in which only a few elementary particles participate (e.g., molecular collisions), the inverse process also exists. But that is not the case for the elementary processes of radiation. ... The elementary process of emission is not invertible. In this, I believe, our oscillation theory does not hit the mark. Newton’s emission theory of light seems to contain more truth with respect to this point than the oscillation theory since, first of all, the energy given to a light particle is not scattered over infinite space, but remains available for an elementary process of absorption.” In particular, the proposed here theory does have an elementary process of asbroption as A. Einstein suggested. We call this process ”negative radiation” and its essence is that the EM energy is moving toward two closely located elementary charges if they oscillate with the same frequency but opposite phases.

The proposed here EM is based on a new concept for elementary charge which we call balanced charge or $b$-charge for short, and we refer the theory itself as balanced electromagnetic (BEM) theory. A key element of the BEM theory is a concept of an elementary EM field assigned to every single $b$-charge. A single $b$-charge is described by a pair $(\psi, A^\mu)$, where $\psi$ is its wave function and $A^\mu = (\varphi, \mathbf{A})$ is its 4-vector elementary potential with the corresponding elementary EM field defined by the familiar formula $F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu$. So, $b$-charge is a field over 4-dimensional space-time continuum for which the wave function $\psi$ represents its matter properties and the elementary potential $A^\mu$ mediates its EM interactions with all other $b$-charges. Importantly, (i) all internal forces of a $b$-charge are exclusively of non-electromagnetic origin; (ii) every $b$-charge is a source of its elementary EM field which represents force exerted by this charge on any other $b$-charge but not upon itself. The later allows to view a single $b$-charge as truly elementary one with respect to the electromagnetic interactions.

An idea to introduce an extended charge instead of the point one is not new, and the
most known models for it are the Abraham rigid charge model and the Lorentz relativistically covariant model. These models are studied and advanced in many papers, see [Jac, Sections 16], [Kie2], [Pea1], [Roh, Sections 2, 6], [Schwin], [Spon], [Yaghjian]. In contrast to those models, here and in [BabFig1], [BabFig2] we do not prescribe to an elementary charge a certain geometry, but instead the elementary charge has a wave function governed by a nonlinear Klein-Gordon or a nonlinear Schrödinger equation in the relativistic and nonrelativistic cases respectively. An idea to eliminate self-interaction is also, of course, not new. The latest to our best knowledge attempt to have this feature in the electrodynamics is due J. Wheeler and R. Feynman, [WF1], [WF2], but the EM theory proposed here is very different from it.

The BEM theory is constructed based on a relativistic Lagrangian with the following properties: (i) b-charges interact only through their elementary EM potentials and fields; (ii) the field equations for the elementary EM fields are exactly the Maxwell equations with proper conserved currents; (iii) a free charge moves uniformly preserving up to the Lorentz contraction its shape; (iv) the Newton equations with the Lorentz forces hold approximately when charges are well separated and move with non-relativistic velocities. Since an overwhelming number of EM phenomena are explained within the CEM theory by the Maxwell equations and the Lorentz forces the BEM theory is equally successful in explaining the same phenomena.

A system of \(N\) elementary charges in the BEM theory is modeled by \(N\) pairs \((\psi^\ell, A^{\ell\mu})\), \(1 \leq \ell \leq N\), and every b-charge \((\psi^\ell, A^{\ell\mu})\) is naturally assigned via the Lagrangian its elementary conserved 4-current \(J^{\ell\nu}\). The later property provides additional justification for calling the elementary field \(F^{\ell\mu\nu}\) electromagnetic. The classical (total) EM field \(A^\mu = (\varphi, A)\) is recovered in this theory as the sum of all elementary EM fields, namely

\[
\varphi = \sum_{1 \leq \ell \leq N} \varphi^\ell, \quad A = \sum_{1 \leq \ell \leq N} A^{\ell},
\]

but, importantly, this total field is not an independent entity with its degrees of freedom. Notice then that since in the BEM theory there is no EM self-interaction the action on \(\ell\)-th charge by EM fields of other charges is described by a field \(A^{\mu}_{\neq \ell} = (\varphi_{\neq \ell}, A_{\neq \ell})\) which is the total field \((\varphi, A)\) "balanced" by the removal from it the self-interaction, namely

\[
\varphi_{\neq \ell} = \sum_{\nu \neq \ell} \varphi^\nu, \quad A_{\neq \ell} = \sum_{\nu \neq \ell} A^{\nu}.
\]

Use of EM fields similar to ones in (6) is, of course, not a discovery and they can be found in many textbooks, but the BEM theory goes further than that and removes the elementary EM self-actions for the CEM Lagrangian and consequently the elementary self-energies from the classical EM energy-momentum tensor. Only with this removal of the elementary self-actions from the classical EM Lagrangian one gets the field equations in the form of the elementary Maxwell equations \(\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^{\mu\nu}\) for the elementary EM \(F^{\mu\nu}\) fields with the elementary conserved 4-currents \(J^{\mu\nu}\).

In Section 2 we study differences and similarities between the CEM and BEM theories. The differences and similarities stem from the differences and similarities between their Lagrangians and with consequent differences and similarities in the energy-momentum tensor definition. Since the both theories are based on the Maxwell equations they have exactly the same EM fields for prescribed currents but their total actions and the energy-momentum tensors differ with consequent differences in radiation phenomena. We show below that the
CEM theory is a limit of the BEM theory. That can be already seen by comparing formulas \( (5) \) and \( (6) \) for a large \( N \) since these large sums differ just by one term. The classical charge can be treated within the BEM theory as a cluster of many b-charges and it is shown, in particular, that such a classical charge does interact with itself electromagnetically and its interaction with other classical charges can be described approximately through the interaction with the single effective EM field. Notice also that the classical single EM field is defined by \( (5) \) as the sum of all elementary EM fields, but since the energy is a quadratic function of fields, \emph{the total single classical EM field has energy equal to the combined energy of interaction between all the pairs of elementary EM fields only approximately}. Most of EM phenomena at the macroscopic scale can be described based on the single EM field with astounding precision and the relative difference with the BEM theory since the inverse of the Avogadro constant, that is \( 10^{-23} \), is very small. But the differences between the CEM and BEM theories become more pronounced for smaller systems with fewer b-charges. The BEM theory predictions can significantly deviate from those of the CEM theory in the following situations: (i) there are just a few b-charges which are in close proximity; (ii) there is a large but highly coherent system of b-charges similar to those \emph{collective, coherent systems} (superconducting ring, laser and more) described by C. Mead \[ \text{Mead, p.5} \] in his ”collective electrodynamics”. In the BEM theory the EM energy is the energy of EM interaction of pairs of b-charges and consequently it is naturally partitioned between the pairs of charges. We show that for every pair of b-charges their EM interaction energy satisfies elementary energy-momentum conservations governed by the relevant Lorentz force densities.

In Section 3 we study particle-like properties of b-charges \( (\psi^\ell, A^{\mu}) \) which are captured by the concept of wave-corpuscle similarly to the wave-corpuscle mechanics (WCM), \[ \text{BabFig1}, \text{BabFig2} \]. A key ingredient providing for particle-like behavior of a b-charge are self-interaction nonlinearities \( G_{a,\ell} \) where size parameter \( a = a^\ell \) determines the size of a free particle which though small is non-zero. The shape of the wave-corpuscle \( \psi^\ell \) is intimately related to the the nonlinearity \( G_{a,\ell} \) via the rest charge equation - a nonlinear Klein-Gordon equation- which in non-relativistic case turns into a nonlinear Schrödinger equation. We also derive in this section the Newton equations with the Lorentz forces as an approximation when charges are well separated and move with non-relativistic velocities.

In Section 5 we provide a detailed sketch of the hydrogen atom model. At atomic scales when there are just a few elementary charges in a close proximity the BEM theory differs significantly from the CEM theory. \emph{It yields, in particular, the BEM hydrogen atom model with a frequency spectrum matching the same for the Schrödinger hydrogen atom with any desired accuracy.} The difference between the two HA models depends on on the size of the free electron as a parameter in the self-interaction nonlinearity \( G_a \), and an analysis suggests the size of a free electron to be of about 100 times of the Bohr radius.

## 2 Relativistic theory

In this section we introduce BEM Lagrangian for a system of bounced charges and derive the corresponding field equations, currents, energy-momentum tensors (EnMT) for system components and more. An important objective of this section is to study the EM interactions in the BEM theory and find out common and different features of the BEM theory and the CEM theory.
2.1 Lagrangian, field equations and currents

Let us consider a system of elementary charges \((\psi^\ell, A_\mu^\ell), 1 \leq \ell \leq N\). In what follows the \(\ell\)-th charge potential \(A_\mu^\ell\) and its EM field \(F_{\mu\nu}^\ell = \partial_\mu A^\ell_\nu - \partial_\nu A^\ell_\mu\) completely account for its action upon all other charges \(\ell' \neq \ell\). Consequently, the action upon \(\ell\)-th charge by all other charges is described by \(\ell\)-th exterior potential \(A_{\ell}^\mu\) and its EM field \(F_{\mu\nu}^\ell\) defined by

\[
A_{\ell}^\mu = \sum_{\ell' \neq \ell} A_{\ell'}^\mu, \quad A_{\ell}^\mu = (\varphi_{\ell}^\mu, A_{\ell}^\mu), \quad F_{\mu\nu}^\ell = \sum_{\ell' \neq \ell} F_{\mu\nu}^{\ell'}. \tag{7}
\]

We also introduce for the total potential \(A^\mu\) and the corresponding total EM field \(F_{\mu\nu}\) by the following formulas

\[
A^\mu = \sum_{1 \leq \ell \leq N} A_{\ell}^\mu, \quad F_{\mu\nu} = \sum_{1 \leq \ell \leq N} F_{\mu\nu}^\ell, \tag{8}
\]

\[
F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu.
\]

We furnish now the system of \(N\) b-charges with the following Lagrangian

\[
\mathcal{L} \left( \{ \psi^\ell, \psi^\ell_\mu \}, \{ \psi^\ell_\ast, \psi^\ell_\ast_\mu \}, A^\mu \right) = \sum_{1 \leq \ell \leq N} L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^\ell_\ast, \psi^\ell_\ast_\mu \right) + \mathcal{L}_{\text{BEM}}, \tag{9}
\]

\[
\mathcal{L}_{\text{BEM}} = \mathcal{L}_{\text{CEM}} - \mathcal{L}_e, \quad \mathcal{L}_{\text{CEM}} = \frac{F_{\mu\nu}^\ell F_{\mu\nu}^\ell}{16\pi}, \quad \mathcal{L}_e = - \sum_{1 \leq \ell \leq N} \frac{F_{\mu\nu}^\ell F_{\mu\nu}^\ell}{16\pi}. \tag{10}
\]

where \(L^\ell\) is the Lagrangian of the \(\ell\)-th bare charge, and the covariant derivatives are defined by the following formulas

\[
\psi^\ell_\mu = \bar{\partial}^\mu \psi^\ell, \quad \psi^\ell_\ast_\mu = \bar{\partial}^\ast_\mu \psi^\ell_\ast,
\]

\[
\bar{\partial}^\mu = \partial^\mu + \frac{iq^\ell A_{\ell}^\mu}{\chi c}, \quad \bar{\partial}^\ast_\mu = \partial^\mu - \frac{iq^\ell A_{\ell}^\mu}{\chi c}.
\]

where \(\psi^\ast\) is complex conjugate to \(\psi\).

Observe that EM part \(\mathcal{L}_{\text{BEM}}\) of the Lagrangian \(\mathcal{L}\) according to (10) is obtained by the removal from the classical EM Lagrangian \(\mathcal{L}_{\text{CEM}}\) all self-interaction contributions \(\mathcal{L}_e\) of the elementary EM fields and it can be recast as

\[
\mathcal{L}_{\text{BEM}} = - \sum_{\{\ell, \ell': \ell' \neq \ell\}} \frac{F_{\mu\nu}^{\ell'} F_{\mu\nu}^{\ell'}}{16\pi} = - \sum_{1 \leq \ell \leq N} \frac{F_{\mu\nu}^\ell F_{\mu\nu}^\ell}{16\pi}. \tag{12}
\]

The "bare" charge Lagrangians \(L^\ell\) are defined by exactly same expressions as in [BabFig1], [BabFig2], namely

\[
L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^\ast_\mu, \psi^\ast_\mu \right) = \frac{\chi^2}{2m^\ell} \left\{ \psi^\ast_\mu \psi^\ell_\mu - \kappa^\ell \psi^\ast_\mu \psi^\ell_\mu - G^\ell \left( \psi^\ast_\mu \psi^\ell_\mu \right) \right\}, \tag{13}
\]

where (i) \(G^\ell\) is a nonlinear self-interaction function of the \(\ell\)-th charge described below; (ii) \(m^\ell > 0\) is the charge mass; (iii) \(q^\ell\) is the value of the charge; (iv) \(\chi > 0\) is a constant similar to the Planck constant \(h = \hbar/2\pi\) and

\[
\kappa^\ell = \frac{\omega^\ell}{c}, \quad \omega^\ell = \frac{m^\ell c^2}{\chi}. \tag{14}
\]
The system Lagrangian $\mathcal{L}$ defined by (12)-(14) is manifestly Lorentz and gauge invariant with respect to the gauge transformations of the first kind (118). The gauge invariance via the Noether’s theorem allows to introduce elementary conserved currents, $[\text{BabFig1}, \text{BabFig2}].$

$$J^{\ell\nu} = -\frac{q^\ell}{\chi} \left( \frac{\partial L^\ell}{\partial \bar{\psi}_\nu} \psi^\ell - \frac{\partial L^\ell}{\partial \bar{\psi}_\nu} \psi^\ell \right) = -c \frac{\partial L^\ell}{\partial A_{\#\nu}^\ell},$$

with the conservation law

$$\partial_\nu J^{\ell\nu} = 0, \partial_\mu \rho^\ell + \nabla \cdot \mathbf{J}^\ell = 0, \ J^{\ell\nu} = (\rho^\ell c, \mathbf{J}^\ell).$$

The Euler-Lagrange field equations for the above Lagrangian $\mathcal{L}$ are (i) elementary wave equations

$$\left[ \tilde{\partial}_\mu \tilde{\partial}^{\ell\mu} + \kappa \ell^2 + G^\ell \left| \psi^\ell \right|^2 \right] \psi^\ell = 0, \ \tilde{\partial}^{\ell\mu} = \partial^{\ell\mu} + \frac{i q^\ell A_{\#}^{\ell\mu}}{\chi c},$$

together with the conjugate equation for $\psi^{*\ell}$

$$\left[ \tilde{\partial}_\mu \tilde{\partial}^{*\ell\mu} + \kappa \ell^2 + G^\ell \left| \psi^{*\ell} \right|^2 \right] \psi^{*\ell} = 0, \ \tilde{\partial}^{*\ell\mu} = \partial^{\ell\mu} - \frac{i q^\ell A_{\#}^{\ell\mu}}{\chi c},$$

and (ii) the Maxwell equations for the elementary EM fields

$$\partial_\mu F^{\ell\mu\nu} = \frac{4\pi c}{\chi} J^{\ell\nu},$$

with the familiar vector form

$$\nabla \cdot \mathbf{E}^\ell = 4\pi \rho^\ell, \quad \nabla \cdot \mathbf{B}^\ell = 0,$$

$$\nabla \times \mathbf{E}^\ell + \frac{1}{c} \partial_t \mathbf{B}^\ell = 0, \quad \nabla \times \mathbf{B}^\ell - \frac{1}{c} \partial_t \mathbf{E}^\ell = \frac{4\pi c}{\chi} \mathbf{J}^\ell.$$

We will refer to the equations (19), (20) as the elementary Maxwell equations and to the field equations (17)-(19) as field equations for b-charges. Using (13)-(15) we obtain the following representation for the $\ell$-th elementary current

$$J^{\ell\nu} = -\frac{q^\ell \chi}{2m^\ell} \left( \tilde{\partial}^{*\ell\nu} \psi^{\ell} \psi^\ell - \psi^{\ell*} \tilde{\partial}^{*\ell\nu} \psi^\ell \right) =$$

$$= -\frac{q^\ell \chi}{m^\ell} \left( \frac{\partial^\nu \psi^\ell}{\psi^\ell} + \frac{q^\ell A_{\#}^{\ell\nu}}{\chi c} \right),$$

or its vector form

$$\rho^\ell = \frac{q^\ell \left| \psi^\ell \right|^2}{m^\ell c^2} \left( \chi \text{Im} \frac{\partial \psi^\ell}{\psi^\ell} + q^\ell \varphi^\ell_{\#} \right),$$

$$\mathbf{J}^\ell = \frac{q^\ell \left| \psi^\ell \right|^2}{m^\ell} \left( \chi \text{Im} \frac{\nabla \psi^\ell}{\psi^\ell} - \frac{q^\ell A_{\#}^\ell}{c} \right).$$
The Maxwell equations for elementary EM fields \( \text{(19)} \) combined with the equalities \( \text{(8)-(7)} \) readily imply that total and exterior fields also satisfy the Maxwell equations

\[
\partial_\mu F^\ell\mu\nu = \frac{4\pi}{c} J^\nu, \quad \text{where} \quad J^\nu = \sum_{1 \leq \ell \leq N} J^{\ell\nu},
\]

\[
\partial_\mu F^{\neq\mu\nu} = \frac{4\pi}{c} J^{\nu}_{\neq}, \quad \text{where} \quad J^{\nu}_{\neq} = \sum_{\ell' \neq \ell} J^{\ell'\nu}.
\]

Observe that: (i) the field equations \( \text{(19)} \) for the elementary EM fields are exactly the Maxwell equations with the corresponding elementary currents; (ii) every elementary wave equation \( \text{(17)-(19)} \) indicates that the \( \ell \)-th charge is driven by its exterior potential \( A^{\ell\nu}_{\neq} \) indicating that there is no self-interaction. We can see also from equation \( \text{(25)} \) that the total EM field satisfies the Maxwell equations for the total currents as in the CEM theory.

It is instructive to see how the BEM system Lagrangian \( \mathcal{L} \) defined by \( \text{(9)-(12)} \) can be obtained by a modification of a similar WCM Lagrangian involving the classical single EM field introduced in \( \text{[BabFig1], [BabFig2]} \). The modification consists of two actions: (i) alteration of the covariant derivatives \( \text{(11)} \) which removes the self-action by using exterior potentials \( A^{\ell\mu}_{\neq} \) instead of the same total field potential \( A^{\mu} \); (ii) subtracting from the classical action of the total EM field the sum \( \mathcal{L}_e \) of the classical actions of the individual EM fields as in \( \text{(9)} \). It is the two described actions combined yield an EM theory with the Maxwell equations and the Lorentz force densities as its exact components. The natural alteration of the covariant derivatives alone is not sufficient since then the field equations will be not exactly the Maxwell equations nor there will be force densities described exactly by the Lorentz formula. Importantly, an additional action - the subtraction of the Lagrangian component \( \mathcal{L}_e \) defined in \( \text{(9)} \) - is necessary to have the Maxwell equations and the Lorentz forces as fundamentally exact parts of the new EM theory. One can see already significant differences brought into the EM theory by many elementary EM fields. It is evident from the formula \( \text{(12)} \) for \( \mathcal{L}_{\text{BEM}} \) that the EM action is the sum of elementary EM actions \( \mathcal{L}^{\ell\ell}_{\text{BEM}} \) associated with all pairs \( \{\ell, \ell'\} : \ell' \neq \ell \) of elementary charges and every elementary action \( \mathcal{L}^{\ell\ell}_{\text{BEM}} \) depends on the fields \( F^{\ell\mu\nu} \) and \( F^{\ell'\mu\nu} \) only. Continue this line we observe that though the EM Lagrangian \( \mathcal{L}_{\text{BEM}} \) according to formula \( \text{(12)} \) is a simple and natural summatory function of the elementary EM fields, it can not be reduced exactly to any function of a single EM field as in the CEM Lagrangian.

In what follows we often use a vector form of the system Lagrangian \( \mathcal{L} \) defined by \( \text{(9), (10)} \), that is

\[
\mathcal{L} \left( \left\{ \psi^{\ell} \right\}_{\ell=1}^N, \left\{ (\phi^{\ell}, A^{\ell}) \right\}_{\ell=1}^N \right) = \mathcal{L}_{\text{BEM}} + \sum_{1 \leq \ell, \ell' \leq N} \frac{\chi^2}{2m^{\ell'}} \left\{ \frac{\bar{\psi}^{\ell}\psi^{\ell'}^2}{c^2} - \left| \nabla \psi^{\ell} \right|^2 - \kappa_0^2 \left| \psi^{\ell} \right|^2 - G^\ell \left( \psi^{\ell *} \psi^{\ell} \right) \right\}
\]
where

\[ \mathcal{L}_{\text{BEM}} = \mathcal{L}_{\text{CEM}} - \mathcal{L}_e, \]
\[ \mathcal{L}_{\text{CEM}} = -\frac{1}{8\pi} \left[ (\nabla \varphi + \frac{\partial_t A}{c})^2 - (\nabla \times A)^2 \right], \]
\[ \mathcal{L}_e = \frac{1}{8\pi} \sum_{1 \leq \ell \leq N} \left[ (\nabla \varphi^\ell + \frac{\partial_t A^\ell}{c})^2 - (\nabla \times A^\ell)^2 \right]. \]

The corresponding field Euler-Lagrange field equations are, first of all, the Maxwell equations for all elementary EM potentials \( \varphi^\ell, A^\ell \)

\[ \nabla \cdot \left( \frac{1}{c} \partial_t A^\ell + \nabla \varphi^\ell \right) = -4\pi \rho^\ell, \]
\[ \nabla \times (\nabla \times A^\ell) + \frac{1}{c} \partial_t \left( \frac{1}{c} \partial_t A^\ell + \nabla \varphi^\ell \right) = \frac{4\pi}{c} J^\ell, \quad \ell = 1, \ldots, N, \]

where the charge densities and currents are defined by (23), (24), and, second of all, the equations for the wave functions \( \psi^\ell \) in the form of the nonlinear Klein-Gordon equations

\[ -\frac{1}{c^2} \tilde{\partial}_t^2 \tilde{\psi}^\ell + \tilde{\nabla}^2 \tilde{\psi}^\ell - G^\ell \left( \psi^{\ell*} \psi^\ell \right) \psi^\ell - \kappa_0^2 |\psi^\ell|^2 = 0, \quad \ell = 1, \ldots, N, \]

and similar equations for the complex conjugate variables \( \psi^{\ell*} \). Note that equations (31) for \( \psi^\ell \) are coupled with the equations for EM potentials via the covariant derivatives (9). If we choose for all elementary potentials the Lorentz gauge

\[ \frac{1}{c} \partial_t \varphi^\ell + \nabla \cdot A^\ell = 0, \]

then equations (29)-(30) turn into the wave equations

\[ \nabla^2 \varphi^\ell - \frac{1}{c^2} \tilde{\partial}_t^2 \varphi^\ell = -4\pi \rho^\ell, \]
\[ \frac{1}{c^2} \tilde{\partial}_t^2 A^\ell - \tilde{\nabla}^2 A^\ell = \frac{4\pi}{c} J^\ell, \quad \ell = 1, \ldots, N, \]

as in the CEM theory.

Importantly, as in the WCM, [BabFig1], [BabFig2], the nonlinearities \( G^\ell \) are determined based on the single \( \ell \)-th charge equations (29)-(31) under the assumptions that it is isolated and is at rest. Namely, let us consider a single b-charge, set \( N = 1 \) in (29)-(31) and simplify notations \( \psi = \psi^1, \varphi = \varphi^1, A = A^1, G^1 = G \). Then according to (9) the covariant derivatives are \( \tilde{\partial}_t = \partial_t, \tilde{\nabla}^\ell = \nabla \) and the Lagrangian for a single b-charge is

\[ \mathcal{L}_0 = \frac{\chi^2}{2m} \left\{ \left| \frac{\partial_t \psi}{c^2} \right|^2 - \left| \nabla \psi \right|^2 - \kappa_0^2 |\psi|^2 - G (\psi^{\ell*} \psi) \right\}. \]

Observe that the Lagrangian \( \mathcal{L}_0 \) above does not depend on the potentials \( \varphi, A \) since there is no self-interaction. Though we can still find the potentials based on the elementary Maxwell
equations (33), (34) they have no role to play and carry no energy. Let us consider now the rest state of the b-charge which we set as in the WCM, [BabFig1], [BabFig2], to be of the form

\[ \psi(t, x) = e^{-i\omega_0 t} \hat{\psi}(x), \quad \omega_0 = \frac{mc^2}{\chi} = c\kappa_0, \quad \] (36)

\[ \varphi(t, x) = \hat{\varphi}(x), \quad A(t, x) = 0, \]

where \( \hat{\psi}(|x|) \) and \( \hat{\varphi}(|x|) \) are real-valued radial functions. Substituting the \( \psi, \varphi \) and \( A \) defined by the relations (36) into the field equations (31) we obtain the following rest charge equations:

\[ -\nabla^2 \hat{\psi} + G'(\|\hat{\psi}\|^2) \hat{\psi} = 0, \quad \] (37)

\[ -\nabla^2 \hat{\varphi} = 4\pi |\hat{\psi}|^2. \quad \] (38)

The quantities \( \hat{\psi} \) and \( \hat{\varphi} \) are fundamental for our theory and we refer to them, respectively, as form factor and form factor potential. The equation (37) signifies a complete balance of the two forces acting upon the resting charge: (i) internal elastic deformation force associated with the term \( -\Delta \hat{\psi} \); (ii) internal nonlinear self-interaction of the charge associated with the term \( G'(\|\hat{\psi}\|^2) \hat{\psi} \). We refer to the equation (37), which establishes an explicit relation between the form factor \( \hat{\psi} \) and the self-interaction nonlinearity \( G \), as the charge equilibrium equation. Hence, if the form factor \( \hat{\psi} \) is given we can find from the equilibrium equation (37) the self-interaction nonlinearity \( G \) which exactly produces this factor under assumptions that \( \hat{\psi}(r) \) is a nonnegative, monotonically decaying and sufficiently smooth function of \( r \geq 0 \).

Now we pick the form factor \( \hat{\psi} \) considering it as the model parameter and then the nonlinear self-interaction function \( G \) is determined based on the charge equilibrium equation (37). As in the WCM, [BabFig1], [BabFig2], such a determination of the nonlinearity is a key feature of our approach: it allows to choose the form factor \( \hat{\psi} \) and then to determine matching self-interaction nonlinearity \( G \) rather than to deal with solving a nontrivial nonlinear partial differential equation.

To explicitly integrate the size of the resting b-charge into its model we introduce size parameter \( a > 0 \) into \( G = G_a \) through its derivative as follows

\[ G'_a(s) = a^{-2}G'_1(a^3s), \quad \text{where} \quad G'(s) = \partial_s G(s). \quad \] (39)

In the following Section 3.2 we give examples of the nonlinearity \( G \) and discuss its properties.

### 2.2 Energy-Momentum tensors and Lorentz forces

The conservation laws, particularly the energy-momentum conservation, play an important role in the physics of EM phenomena. A general source of conservation laws in Lagrangian theories is the Noether’s theorem, [Gold], Section 13.7, which yields canonically conservations laws based on the Lagrangian symmetries, that is its invariance with respect to continuous groups of transformations. The conservation laws are not independent equations, and they hold only if the fields satisfy the field Euler-Lagrange equations. The energy-momentum and
Charge conservations are two important laws in any EM theory and they have special significance in our studies for several reasons. First of all, the conservation of energy-momentum describes their transport in the space and is directly related to the point charge approximations. Second of all, the Lorentz force density expression which the one of the most important components of any EM theory arises in the energy-momentum conservation equations (219) and not in the field Maxwell equations. Third of all, the elementary charge and energy-momentum conservation equations (16), (58) account for charges individuality. In fact, the BEM Lagrangian is invariant with respect to a wider group of elementary gauge transformations (120), (121), and it is due to this symmetry the "two-way" representation (15) holds for every elementary current: one that involves the differentiation with respect to the wave functions derivatives and another one that involves the differentiation with respect to the elementary potentials. It is due to this two-way representation the conserved Noether's elementary current is exactly the source current in the corresponding elementary Maxwell equations (19), (20).

To get further insight into b-charges properties we need to find the symmetric EnMT $T^{\mu\nu}$ of the system Lagrangian $L$ defined by (12)-(14) making use of a general method used in BabFig1, BabFig2. The method yields the following representation

$$T^{\mu\nu} = \sum_{1 \leq \ell \leq N} T^{\mu\nu}_{\ell} + \Xi^{\mu\nu}, \quad \Xi^{\mu\nu} = \sum_{\ell' \neq \ell} \Xi^{\ell\ell'\mu\nu}, \quad (40)$$

where the individual EnMT $T^{\mu\nu}_{\ell}$ of the bare $\ell$-th charge and the EnMT components $\Xi^{\ell\ell'\mu\nu}$ for the EM fields are as follows

$$T^{\mu\nu}_{\ell} = \frac{\partial L^{\ell}}{\partial \psi^{\ell}_{\mu}} \psi^{\ell}{}_{\nu} + \frac{\partial L^{\ell}}{\partial \psi^{\ell*}_{\mu}} \psi^{\ell*}{}_{\nu} - g^{\mu\nu} L^{\ell} = \frac{\lambda^2}{2m^\ell} \left\{ \left( \psi^{\ell*}_{\mu} \psi^{\ell*}_{\nu} + \psi^{\ell*}_{\mu} \psi^{\ell*}_{\nu} \right) - [\psi^{\ell*}_{\mu} \psi^{\ell*}_{\mu} - \kappa^\ell \psi^{\ell*}_{\mu} \psi^{\ell*}_{\mu} - G^\ell \left( \psi^{\ell*}_{\mu} \psi^{\ell*}_{\mu} \right)] \delta^{\mu\nu} \right\}, \quad (41)$$

$$\Xi^{\ell\ell'\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F^{\ell}_{\gamma\xi} F^{\ell'\xi} F^{\ell'\xi} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F_{\ell'\gamma\xi} \right), \quad \ell' \neq \ell, \quad (42)$$

Notice that the expression (41) is the same as in BabFig1, BabFig2 with the only difference that the covariant derivatives here are defined by equalities (11). To emphasize notationally the new meaning of the EnMT of the EM fields we use for it the symbol $\Xi^{\mu\nu}$ and we continue to use the symbol $\Theta^{\mu\nu}$ for the classical EnMT of the the EM field as in formula (212). As to the expression (42) for the individual components of the symmetric EnMT it is obtained from the representation (9) and evidently is similar to the expression (212) for the classical EnMT $\Theta^{\mu\nu}$. An alternative and more elementary way to derive the expression (42) is based on the expression (210) for the canonical EnMT $\hat{\Theta}^{\mu\nu}$ as it is done in Jac, Section 12.10. Indeed, in our case the canonical EnMT takes the form

$$\hat{\Xi}^{\ell\ell'\mu\nu} = -\frac{F^{\ell\mu\xi} \partial^\nu A^{\ell}_{\xi}}{4\pi} + \frac{g^{\mu\nu} F^{\ell}_{\gamma\xi} F^{\ell'\gamma\xi}}{16\pi}, \quad \ell' \neq \ell, \quad (43)$$

The identity $\partial^\nu A^{\ell\xi} = -F^{\xi\nu} + \partial^\xi A^{\ell\nu}$ allows to recast the expression (43) into

$$\hat{\Xi}^{\ell\ell'\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F^{\ell}_{\gamma\xi} F^{\ell'\xi} F^{\ell'\xi} + \frac{1}{4} g^{\mu\nu} F^{\ell}_{\gamma\xi} F^{\ell'\gamma\xi} \right) - \frac{1}{4\pi} g^{\mu\gamma} F^{\ell}_{\gamma\xi} \partial^\xi A^{\ell\nu}. \quad (44)$$
Then using the Maxwell equations (19) we can recast the second term as

$$-\frac{1}{4\pi} g^{\mu\nu} F_{\gamma\xi} \partial^\xi A^\ell^\nu = \partial^\xi \left( -\frac{1}{4\pi} g^{\mu\gamma} F^\ell_\gamma A^\ell^\nu \right) - J^\ell_\gamma A^\ell^\nu. \quad (45)$$

The term $-J^\ell_\gamma A^\ell^\nu$ combined with the canonical EnMT $\tilde{T}^{\ell\mu\nu}$ yields the the symmetric $T^{\ell\mu\nu}$ and hence

$$\mathcal{T}^{\mu\nu} = \sum_{1 \leq \ell \leq N} \tilde{T}^{\ell\mu\nu} + \sum_{\ell' \neq \ell} \Xi^{\ell\ell'}^{\mu\nu} = \sum_{1 \leq \ell \leq N} T^{\ell\mu\nu} + \sum_{\ell' \neq \ell} \Xi^{\ell\ell'}^{\mu\nu} - \frac{1}{4\pi} \partial^\xi \left( \sum_{\ell' \neq \ell} g^{\mu\gamma} F^\ell_\gamma A^\ell'_{\gamma\xi} \right). \quad (46)$$

The later by the standard argument implies the conservation law

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0 \quad (47)$$

for the symmetric EnMT $\mathcal{T}^{\mu\nu}$ defined by the expressions (40)-(42).

Observe that using representations (42) we can also recast expression (40) for EnMT $\Xi^{\mu\nu}$ as follows

$$\Xi^{\mu\nu} = \sum_{\ell' \neq \ell} \Xi^{\ell\ell'}^{\mu\nu} = \sum_{1 \leq \ell \leq N} \Xi^{\ell\mu\nu}, \quad (48)$$

where

$$\Xi^{\ell\ell'}^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F^\ell_\gamma F^{\ell'}_{\gamma\xi} + \frac{1}{4} g^{\mu\nu} F^\ell_\gamma F^{\ell'}_{\gamma\xi} F^\ell_\gamma F^{\ell'}_{\gamma\xi} \right), \quad (49)$$

$$\Xi^{\ell\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F^\ell_\gamma F^{\ell}_{\gamma\xi} + \frac{1}{4} g^{\mu\nu} F^\ell_\gamma F^{\ell}_{\gamma\xi} F^\ell_\gamma F^{\ell}_{\gamma\xi} \right).$$

The relation between the classical and the new EnMTs $\Theta^{\mu\nu}$ and $\Xi^{\mu\nu}$ in view of their representations (40), (42) and (212) is as follows

$$\Xi^{\mu\nu} = \sum_{\ell' \neq \ell} \Xi^{\ell\ell'}^{\mu\nu} = \Theta^{\mu\nu} \left( \sum_{1 \leq \ell \leq N} F^{\ell\mu\nu} \right) - \sum_{1 \leq \ell \leq N} \Theta^{\mu\nu} (F^{\ell\mu\nu}) = \Theta^{\mu\nu} (\mathcal{T}^{\mu\nu}) - \sum_{1 \leq \ell \leq N} \Theta^{\mu\nu} (F^{\ell\mu\nu}). \quad (50)$$

Evidently the formula (50) reads that the new expression of the EnMT equals to the classical EnMT of the total EM field minus the sum of the the classical EnMTs for the elementary EM fields.

Based on the formula (42) for the components $\Xi^{\ell\ell'}$ for $\ell' \neq \ell$ of the symmetric EnMT we can represent their entries in terms of the fields $E^\ell$ and $B^\ell$, namely

$$w^{\ell\ell'} = \Xi^{\ell\ell'}_{00} = \frac{\mathbf{E}^\ell \cdot \mathbf{E}^{\ell'} + \mathbf{B}^\ell \cdot \mathbf{B}^{\ell'}}{8\pi}, \quad (51)$$

$$c_g \Xi^{\ell\ell'} = \Xi^{\ell\ell'}_{0i} = \Xi^{\ell\ell'}_{00} = \frac{\mathbf{E}^\ell \times \mathbf{B}^{\ell'}}{4\pi}. \quad (51)$$
\[ \Xi_{ij}^{\ell \ell'} = -\frac{1}{8\pi} \left[ E_{i}^{\ell} E_{j}^{\ell'} + B_{i}^{\ell} B_{j}^{\ell'} - \frac{1}{2} \delta_{ij} \left( E^{\ell} \cdot E^{\ell'} + B^{\ell} \cdot B^{\ell'} \right) \right], \]  

(52)

\[ \Xi_{\alpha \beta}^{\ell \ell'} = \left[ w_{\ell \ell'}^{\ell \ell'} c g_{\ell \ell'}^{\ell \ell'} - \delta_{ij} \right], \]  

(53)

which are evidently similar to the same entries (213)-(215) for the classical EM theory. In particular, we have the following expression for the Poynting vectors similar to (218)

\[ S_{\ell \ell'}^{\ell} = c^2 g_{\ell \ell'}^{\ell \ell'} = \frac{c E^{\ell} \times B^{\ell'}}{4\pi}. \]  

(54)

To find expressions for the EM force densities acting upon b-charges let us examine their conservation laws. Combining the Maxwell field equations (19), (26), an elementary identity

\[ \partial^{\mu} F_{\xi \nu} - \partial_{\xi} F^{\mu \nu} = \partial^{\nu} F^{\mu \xi} - \partial^{\mu} F^{\nu \xi} = 0, \]  

and the antisymmetry of the EM field tensors \( F_{\ell \mu \nu} \) we obtain for \( \ell' \neq \ell \):

\[ \partial_{\mu} \Xi_{\ell \ell'}^{\ell \ell'} = \frac{1}{c} J_{\xi}^{\ell} F_{\ell \nu}^{\ell \xi} + \frac{1}{4} \left[ -\frac{1}{2} F_{\mu \xi}^{\ell} F_{\nu \xi}^{\ell'} + \frac{1}{4} \partial_{\nu} \left( F_{\gamma \xi}^{\ell} F_{\ell \gamma \xi}^{\ell'} \right) \right], \]  

(56)

Let us introduce an elementary EM field interaction energy for the pair \( \{ \ell, \ell' \} \) of b-charges

\[ \Xi^{\ell \ell'}_{\mu \nu} = \Xi_{\ell \ell'}^{\ell \ell \mu \nu} + \Xi_{\ell \ell'}^{\ell' \ell \mu \nu}, \]  

(57)

which in view of (56) satisfies the following elementary energy-momentum conservation laws

\[ \partial_{\mu} \Xi^{\ell \ell'}_{\mu \nu} = -\frac{1}{c} \left( J_{\xi}^{\ell} F_{\ell \nu}^{\ell \xi} + J_{\xi}^{\ell'} F_{\ell \nu}^{\ell' \xi} \right) \]  

(58)

with the right-hand side being the negative of the sum of the corresponding Lorentz force density.

It readily follows from relations (57) and (19) that the interaction EnMT \( \Xi^{\ell \ell'}_{\mu \nu} \) has the following representation in terms of the classical EnMT \( \Theta^{\mu \nu} \):

\[ \Xi^{\ell \ell'}_{\mu \nu} = \Theta^{\mu \nu} \left( F_{\ell \mu \nu} + F_{\ell' \mu \nu} \right) - \Theta^{\mu \nu} \left( F_{\ell' \mu \nu} \right) - \Theta^{\mu \nu} \left( F_{\ell \mu \nu} \right). \]  

(59)

The representation (59) is evidently similar to (50).

Notice that in view of (51), (52) and (54) the entries of the tensor \( \Xi^{\ell \ell'}_{\mu \nu} \) are as follows

\[ w_{\ell \ell'}^{\ell \ell} = \Xi^{\ell \ell'}_{\mu \nu}^{00} = \frac{E_{i}^{\ell} E_{j}^{\ell'} + B_{i}^{\ell} B_{j}^{\ell'}}{4\pi}, \]  

\[ c g_{i}^{\ell \ell'} = \Xi^{\ell \ell'}_{\mu \nu}^{0i} = \Xi^{\ell \ell'}_{\mu \nu}^{0i} = \frac{E_{i}^{\ell} \times B^{\ell'} + E_{i}^{\ell'} \times B^{\ell}}{4\pi}, \]  

\[ S_{\ell \ell'}^{\ell} = \frac{c E^{\ell} \times B^{\ell'}}{4\pi}, \]  

\[ \tau_{ij}^{\ell \ell'} = \Xi^{\ell \ell'}_{\mu \nu}^{ij} = \]  

\[ = -\frac{1}{4\pi} \left[ E_{i}^{\ell} E_{j}^{\ell'} + B_{i}^{\ell} B_{j}^{\ell'} - \frac{1}{2} \delta_{ij} \left( E^{\ell} \cdot E^{\ell'} + B^{\ell} \cdot B^{\ell'} \right) \right]. \]  

(61)
The expressions (60) and (61) can be alternatively derived from their classical counterparts (213), (214) and (218) based on relation (59).

The relations (40) together with (58) readily imply the total energy-momentum conservation law

\[ \partial_\mu \xi_{\mu\nu} = 1 \sum_{\ell \neq \ell'} J_\ell F_{\ell\ell'}^\xi, \]  

with the sum of the negative of the Lorentz forces in the right-hand side.

Using expression (41) for \( T_{\ell\mu\nu} \) and the same transformations as in the similar case considered in BabFig2 we obtain the following elementary conservation laws

\[ \partial_\mu T_{\ell\mu\nu} = 1 J_\ell \sum_{\ell' \neq \ell} F_{\ell'\ell'}^\nu = 1 J_\ell F_{\ell\ell'}^\nu. \]  

Observe that the expression on the right-hand side of the above equality is the Lorentz force density 4-vector acting upon the 4-vector current \( J_\ell \) by its exterior field \( F_{\ell\ell'}^\nu \), and the same vectors with the minus sign arise in the conservation laws for the EM fields (58) and (62). Notice also that the natural partition (40)-(42) of the system EnMT into charges EnMTs \( T_{\ell\mu\nu} \) and the EnMTs \( \Xi^{(\ell,\ell')\mu\nu} \) for pairs \( \{\ell, \ell'\} \) of interacting elementary EM fields produces exactly Lorentz force densities in the elementary conservation laws (58), (62) and (63). This provides a solid justification for the energy-momentum partition (40)-(42), and it seems there is no simple way to alter it when preserving the Lorentz forces expression.

The vector form of the elementary conservation law (58) is similar to the CEM theory conservation law [220, Jac1, Section 6.8], namely

\[ \frac{1}{c} \left[ \partial_t w^{(\ell,\ell')} + \nabla \cdot S^{(\ell,\ell')} \right] = -\frac{1}{c} \left[ J^{\ell} \cdot E^{\ell'} + J^{\ell'} \cdot E^{\ell} \right], \]  

(64)

\[ \frac{\partial g_i^{(\ell,\ell')}}{\partial t} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} g_{ij}^{(\ell,\ell')} = f_i^{(\ell,\ell')}, \quad i = 1, 2, 3, \]  

(65)

where \( f_i^{(\ell,\ell')} \) is the Lorentz force density satisfying

\[ f_i^{(\ell,\ell')} = f_i^{\ell\ell'} + f_i^{\ell'\ell}, \quad f_i^{\ell\ell'} = \rho E_i^{\ell'}, \quad f_i^{(\ell,\ell')} = \frac{1}{c} \left( J^{\ell} \times B^{\ell'} \right)_i. \]  

(66)

**Remark 1** The conservation law (64) can be easily derived from the classical energy conservation law (the Poynting theorem) as follows. Let us consider the classical energy conservation law associated with the 3 Maxwell equations (19): (i) for \( \ell \)-th EM fields, (ii) for \( \ell' \)-th EM field and (iii) for their sum. Then subtracting from the conservation law for the sum of the fields the sum of conservation laws for \( \ell \)-th and \( \ell' \)-the fields we obtain exactly the conservation law (64).

### 2.3 Elementary EM fields for prescribed elementary currents

To study the properties of the elementary EM fields for balanced changes and the energy-momentum transfer in the space-time it is instructive to consider a situation of prescribed currents for b-charges similarly to the same for the case the CEM theory Lagrangian (207).
In the BEM theory the corresponding Lagrangian $L_{EMJ}$ is based on the representations (9) and is of the form

$$L_{BEMJ} = -\frac{F_{\mu\nu} F_{\mu\nu}}{16\pi} + \sum_{1 \leq \ell \leq N} \frac{F_{\mu\nu}^\ell F_{\mu\nu}^\ell}{16\pi} - \frac{1}{c} \sum_{1 \leq \ell \leq N} J_\mu^\ell A_\mu^{\ell}. \quad (67)$$

Based on the representation (12) it can be alternatively written as

$$L_{BEMJ} = -\sum_{\ell \neq \ell'} \left( \frac{F_{\mu\nu}^\ell F_{\mu\nu}^{\ell'}}{16\pi} - \frac{1}{c} \sum_{1 \leq \ell \leq N} J_\mu^\ell A_\mu^{\ell} \right) \quad (68)$$

The field equations for the Lagrangian $L_{EMJ}$ can be obtained directly from the variational principle. Indeed, taking into account the antisymmetry of every field tensor $F_{\mu\nu}$, the fact every pair $\ell, \ell'$ with $\ell' \neq \ell$ appear two times in the Lagrangian representation (68), and the usual assumption of the decay of all the fields at infinity we obtain

$$\delta \int L_{BEMJ} \, dx = \int \sum_{1 \leq \ell \leq N} \left( \frac{1}{4\pi} \partial_\mu F_{\mu\nu}^\ell - \frac{1}{c} J_\mu^\ell \right) \delta A_\mu^{\ell} \, dx \quad (69)$$

Since the variables $\delta A_{\neq \nu}^\ell$, $1 \leq \ell \leq N$ can vary independently (see relations (122) and (123)) the requirement for the above variation to vanish yield the field equations which are exactly the same Maxwell equations for the elementary EM fields as in (19), (20), (20).

Importantly, the system of elementary EM fields with prescribed currents satisfy the elementary conservation laws (64), (65) which can also be directly derived from the Maxwell equations (21). Indeed using the general vector identity (272) and the Maxwell equations for the indices $\ell$ and $\ell'$ we obtain

$$\nabla \cdot \left( E^\ell \times B^\ell \right) = B^{\ell'} \cdot \left( \nabla \times E^\ell \right) - E^\ell \cdot \left( \nabla \times B^{\ell'} \right) = -B^{\ell'} \cdot \frac{1}{c} \partial_\nu B^\ell \left( \frac{1}{c} \partial_\nu E^\ell + \frac{4\pi}{c} J^{\ell'} \right) \quad (70)$$

Adding then to the identity (70) a similar one obtained from it by swapping indices $\ell$ and $\ell'$ we obtain the elementary energy conservation (64). Similar direct derivation is possible for the elementary momentum conservation (65).

Observe also that according to the BEM conservation laws (58) the energy and the momentum are assigned not to the elementary EM fields by themselves but rather to their interacting pairs. This is a noticeable change compared to the CEM theory where the EM field has an energy and a momentum on its own.

### 2.3.1 Dipole elementary currents

For simplicity sake let us assume all prescribed elementary currents to be in the form of ideal electric dipoles. Recall that an ideal electric dipole source concentrated at a point $x_0$, with
the charge and current densities $J_d$ and $\rho_d$, is defined as follows, [vBl10, Section 7.10, (7.151), Appendix 8],

$$
J(t, x) = \partial_t \left[ p(t) \delta(x - x_0) \right] = \dot{p}(t) \delta(x - x_0),
$$

$$
\rho(t, x) = -\nabla \cdot \left[ p(t) \delta(x - x_0) \right] = -p(t) \cdot \nabla \delta(x - x_0)
$$

where $\delta(x - x_0)$ is the Dirac delta-function, and $p(t)$ and $\partial_t p(t)$ satisfy

$$
p_j(t) = \int x'_j \rho(t, x') \, dx', \quad j = 1, 2, 3, \quad p = (p_1, p_2, p_3),
$$

$$
\dot{p}(t) = \partial_t p(t) = \int J(t, x') \, dx'.
$$

It readily follows from (71) (237) that the potentials of the ideal electric dipole are

$$
\varphi(t, x) = 0, \quad A(t, x) = \frac{\dot{p}(t_0)}{c |R|},
$$

where

$$
R = x - x_0, \quad \hat{R} = \frac{R}{|R|}, \quad t_0 = t - \frac{|x - x_0|}{c}.
$$

Then applying the Jefimenko and the Panofsky-Phillips formulas (241) and (242) for the ideal electric dipole sources $\rho(t, x)$ and $J(t, x)$ defined by formulas (71) we obtain the following formulas for the EM field

$$
E(t, x) = \frac{3 \left( \hat{R} \cdot p(t_0) \right) \hat{R} - \dot{p}(t_0)}{|R|^3} + \frac{3 \left( \dot{p}(t_0) \cdot \hat{R} \right) \hat{R} - \ddot{p}(t_0)}{c^2 |R|^2} + \frac{\left( \dddot{p}(t_0) \times \hat{R} \right) \times \hat{R}}{e^2 |R|^2},
$$

$$
B(t, x) = \left[ \frac{\dot{p}(t_0)}{c R^2} + \frac{\dddot{p}(t_0)}{c^2 R} \right] \times \hat{R}, \quad \text{where} \quad \hat{p} = \partial^2_t p.
$$

When deriving formula (76) we used a vector identity (269).

In a simpler case when the dipole function $p(t)$ is time harmonic and complex valued of the form

$$
p(t) = p_\omega e^{-i\omega t}, \quad p(t_0) = p_\omega e^{i k |R| - i \omega t}, \quad \text{where} \quad k = \frac{\omega}{c},
$$

the general formulas (76), (77) yield the well known formulas for the ideal electric dipole fields, [Jac11, Section 9.2],

$$
E(t, x) = -\frac{k^2 e^{i k |R| - i \omega t}}{|R|} \times \left\{ \left( p \times \hat{R} \right) \times \hat{R} - \left[ 3 \left( \hat{R} \cdot p \right) \hat{R} - p \right] \left( \frac{1}{k^2 |R|^2} - \frac{i}{k |R|} \right) \right\}
$$

$$
B(t, x) = -k^2 e^{i k |R| - i \omega t} \frac{p \times \hat{R}}{|R|} \left( 1 + \frac{i}{k |R|} \right),
$$

16
implying in radiation zone $k |\mathbf{R}| \gg 1$ the following asymptotic formulas

$$
E(t, x) = k^2 e^{k |\mathbf{R}| - i \omega t} \frac{\hat{\mathbf{R}} \times (p_\omega \times \hat{\mathbf{R}})}{|\mathbf{R}|} \left[ 1 + O \left( \frac{1}{k |\mathbf{R}|} \right) \right],
$$

$$
B(t, x) = k^2 e^{k |\mathbf{R}| - i \omega t} \frac{\hat{\mathbf{R}} \times p_\omega}{|\mathbf{R}|} \left[ 1 + O \left( \frac{1}{k |\mathbf{R}|} \right) \right].
$$

(81)

(82)

For the case of a multi-frequency dipole function $p(t)$ we can introduce

$$
k_{\text{min}} = \min_{\omega \in \Lambda_p} \left\{ \frac{\omega}{c} \right\} > 0.
$$

(83)

Then radiation components decaying as $|\mathbf{R}|^{-1}$ dominate for $k_{\text{min}} |\mathbf{R}| \gg 1$ in formulas (76) (77) implying the following asymptotic expressions for the radiation fields, see also [Gri, Section 11.1.4],

$$
E(t, x) = \frac{\tilde{p}^\ell(t_0) \times \hat{\mathbf{R}}}{c^2 |\mathbf{R}|} \left[ 1 + O \left( \frac{1}{k_{\text{min}} |\mathbf{R}|} \right) \right],
$$

$$
B(t, x) = \frac{\tilde{p}^\ell(t_0) \times \hat{\mathbf{R}}}{c^2 |\mathbf{R}|} \left[ 1 + O \left( \frac{1}{k_{\text{min}} |\mathbf{R}|} \right) \right], \text{ for } k_{\text{min}} |\mathbf{R}| \gg 1.
$$

(84)

Energy flux for a system of arbitrary elementary dipoles Using expression (84) for the EM fields and the vector identities (270) and (271) we consequently obtain the following formulas for the energy flux $S^{\ell\ell'}$ and the corresponding total powers $P^{\ell\ell'}$ and $P^{(\ell, \ell')}$ radiated through a sphere centered at the dipole location

$$
S^{\ell\ell'} = \frac{c}{4\pi} E^\ell \times B^{\ell'} = \frac{\tilde{p}^\ell(t_0) \cdot \tilde{p}^{\ell'}(t_0) - \left( \tilde{p}^\ell(t_0) \cdot \hat{\mathbf{R}} \right) \left( \hat{\mathbf{R}} \cdot \tilde{p}^{\ell'}(t_0) \right)}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}} = \frac{\left[ \left( \tilde{p}^\ell(t_0) \times \hat{\mathbf{R}} \right) \cdot \left( \tilde{p}^{\ell'}(t_0) \times \hat{\mathbf{R}} \right) \right]}{4\pi c^3 |\mathbf{R}|^2} \hat{\mathbf{R}},
$$

$$
P^{\ell\ell'} = \int_{|x|=|\mathbf{R}|} S^{\ell\ell'} \, d\sigma = \frac{2}{3c^3} \tilde{p}^\ell(t_0) \cdot \tilde{p}^{\ell'}(t_0),
$$

$$
P^{(\ell, \ell')} = P^{\ell\ell'} + P^{\ell\ell'} = \frac{4}{3c^3} \tilde{\mathbf{p}}^\ell(t_0) \cdot \tilde{\mathbf{p}}^{\ell'}(t_0).
$$

(85)

(86)

Let us assume now that the dipole functions $p^\ell(t)$ depend on time $t$ almost periodically as in Section 6.4. Then the representation (86) together with relations (60), (75) and (265)
consequently imply the following formulas for the time-averaged radiated powers

\[
\langle S^{\ell \ell'} \rangle = \frac{\langle \vec{p}^{\ell} \cdot \vec{p}^{\ell'} \rangle - \langle \vec{p}^{\ell} \cdot \hat{R} \rangle \langle \vec{p}^{\ell'} \rangle \hat{R}}{4 \pi c^3 |\hat{R}|^2} = \frac{\left[ \langle (\vec{p}^{\ell} \times \hat{R}) \cdot (\vec{p}^{\ell'} \times \hat{R}) \rangle \right] \hat{R}}{4 \pi c^3 |\hat{R}|^2}
\]

\[
= \frac{\hat{R}}{8 \pi c^3 |\hat{R}|^2} \sum_{\omega \in \Lambda_{p^{\ell}} \cap \Lambda_{p^{\ell'}}} \omega^4 \text{Re} \left\{ (\vec{p}_\omega \times \hat{R}) \left( \vec{p}^{\ell*}_{\omega} \times \hat{R} \right) \right\},
\]

\[
\langle P^{\ell \ell'} \rangle = \frac{2 \langle \vec{p}^{\ell} \cdot \vec{p}^{\ell'} \rangle}{3c^3} = \sum_{\omega \in \Lambda_{p^{\ell}} \cap \Lambda_{p^{\ell'}}} \frac{\omega^4}{3c^3} \text{Re} \left\{ \vec{p}_\omega \cdot \vec{p}^{\ell*}_{\omega} \right\},
\]

\[
\langle P^{(\ell,\ell')} \rangle = \frac{4 \langle \vec{p}^{\ell} (t) \cdot \vec{p}^{\ell'} (t) \rangle}{3c^3} = \sum_{\omega \in \Lambda_{p^{\ell}}} \omega^4 |p^{\ell}_\omega|^2 > 0 \text{ if } p^{\ell'} (t) = p^{\ell} (t),
\]

\[
\langle P^{(\ell,\ell')} \rangle = -\frac{4 \langle \vec{p}^{\ell} (t) \cdot \vec{p}^{\ell'} (t) \rangle}{3c^3} = -\sum_{\omega \in \Lambda_{p^{\ell}}} \omega^4 |p^{\ell}_\omega|^2 < 0 \text{ if } p^{\ell'} (t) = -p^{\ell} (t).
\]

where \( \Lambda_{p^{\ell}} \) and \( \Lambda_{p^{\ell'}} \) are respectively the frequency spectra of \( p^{\ell} (t) \) and \( p^{\ell'} (t) \). It readily follows from the formula (89) that the time-averaged radiated power \( \langle P^{(\ell,\ell')} \rangle \) can take any real value: negative, zero or positive. Indeed, according to the formulas (89), (265) \( \langle P^{(\ell,\ell')} \rangle \) vanishes if the frequency spectra \( \Lambda_{p^{\ell}} \) and \( \Lambda_{p^{\ell'}} \) don’t have any common frequencies, i.e.

\[
\langle P^{(\ell,\ell')} \rangle = 0 \text{ if } \Lambda_{p^{\ell}} \cap \Lambda_{p^{\ell'}} = \emptyset.
\]

In particular, the relation (91) shows that if the both \( \ell \)-th and \( \ell' \)-th b-charges are monochromatic of different frequencies then the time-averaged radiated power is exactly zero or in other words there is no radiation. The formulas (89), (265) readily imply

\[
\langle P^{(\ell,\ell')} \rangle = \frac{4 \langle \vec{p}^{\ell} (t) \cdot \vec{p}^{\ell'} (t) \rangle}{3c^3} = \frac{2}{3c^3} \sum_{\omega \in \Lambda_{p^{\ell}}} \omega^4 |p^{\ell}_\omega|^2 > 0 \text{ if } p^{\ell'} (t) = p^{\ell} (t),
\]

\[
\langle P^{(\ell,\ell')} \rangle = -\frac{4 \langle \vec{p}^{\ell} (t) \cdot \vec{p}^{\ell'} (t) \rangle}{3c^3} = -\frac{2}{3c^3} \sum_{\omega \in \Lambda_{p^{\ell}}} \omega^4 |p^{\ell}_\omega|^2 < 0 \text{ if } p^{\ell'} (t) = -p^{\ell} (t).
\]

Evidently the relation (92) describes a situation when for a given pair \( \ell, \ell' \) the time-averaged radiated power \( \langle P^{(\ell,\ell')} \rangle \) is negative, that is the radiated energy propagates with the speed of light toward the source rather than away from it.

**Radiated power of a system of identical dipoles** A basis for simple comparison of radiative properties in the BEM and CEM theories let us consider a system of \( N > 1 \) b-charges described by one and the same dipole moment \( p (t) \). Then in view of (91) we have for the system the following time-average radiated power

\[
\langle P_{\text{BEM}} \rangle = \sum_{\ell' \neq \ell} \langle P^{(\ell,\ell')} \rangle = N (N - 1) \frac{2 \langle \vec{p}^{\ell} (t) \cdot \vec{p}^{\ell} (t) \rangle}{3c^3} = N (N - 1) \frac{1}{3c^3} \sum_{\omega \in \Lambda_p} \omega^4 |p_\omega|^2.
\]
In the CEM theory a similar system of $N$ identical dipoles $\mathbf{p}$ has an effective dipole moment $N\mathbf{p}$. Substituting this number for $\mathbf{p}$ in the expression \(88\) we get

$$
\langle P_{\text{CEM}} \rangle = N^2 \frac{2}{3c^3} \langle \dot{\mathbf{P}}(t) \cdot \dot{\mathbf{P}}(t) \rangle = N^2 \frac{1}{3c^3} \sum_{\omega \in \Delta p} \omega^4 |\mathbf{p}_\omega|^2.
$$

(94)

Relating representations \(93\) and \(94\) we readily obtain

$$
\langle P_{\text{BEM}} \rangle = \left(1 - \frac{1}{N}\right) \langle P_{\text{CEM}} \rangle.
$$

(95)

### 2.3.2 Elementary currents for point charges

It is curious to see how the BEM theory describes fields and radiations phenomena when the elementary currents are caused by a system of point charges as, for instance, in the Rutherford atom model. In the case of point charges we use the acceleration fields $\mathbf{E}_a(t, \mathbf{x})$ and $\mathbf{B}_a(t, \mathbf{x})$ defined by formulas \(253\), \(254\) to find the time-averaged radiation power. Consequently, for every pair $\{\ell, \ell'\}$ of b-charges we have

$$
S_{\ell\ell'} = \frac{c}{4\pi} \mathbf{E}_a^{\ell'} \times \mathbf{B}_a^{\ell'} \quad \text{and} \quad S_{\ell'\ell} = \frac{c}{4\pi} \mathbf{E}_a^{\ell} \times \mathbf{B}_a^{\ell}.
$$

(96)

Suppose that one of the two b-charges, say $\ell$-th charge, is at rest or moves uniformly implying that is $\dot{\mathbf{\beta}}^{\ell} = 0$. Then it follows from formulas \(253\), \(254\) that $\mathbf{E}_a^{\ell} = 0$ and $\mathbf{B}_a^{\ell} = 0$ implying that $S_{\ell'\ell}$ and $S_{\ell\ell'}$ vanish, namely

$$
S_{\ell'\ell} = S_{\ell\ell'} = 0 \quad \text{if} \quad \dot{\mathbf{\beta}}^{\ell} = 0 \quad \text{or} \quad \dot{\mathbf{\beta}}^{\ell'} = 0.
$$

(97)

Suppose that the position functions $\mathbf{r}^{\ell}(t)$ and $\mathbf{r}^{\ell'}(t)$ of the corresponding b-charges are almost periodic functions as described in Section \(6.4\) and that mod $(\mathbf{r}^{\ell}(t))$ and mod $(\mathbf{r}^{\ell'}(t))$ have no common frequencies. Then in this case the time-averaged flux $\langle S_{\ell'\ell} \rangle$ is exactly zero, i.e.

$$
\langle S_{\ell'\ell} \rangle = 0 \quad \text{if} \quad \text{mod}(\mathbf{r}^{\ell}(t)) \cap \text{mod}(\mathbf{r}^{\ell'}(t)) = \emptyset.
$$

(98)

Now let us compare this estimate with a similar computation in the framework of the CEM theory with the total field $\mathbf{E} = \mathbf{E}_a^{\ell} + \mathbf{E}_a^{\ell'}$, $\mathbf{B} = \mathbf{B}_a^{\ell} + \mathbf{B}_a^{\ell'}$ or in the framework of the BEM theory with two prescribed charges and a test charge. In the framework of the BEM theory if we consider action of the system of two charges onto a test charge or onto a distant system of test charges which does not noticeably affect the radiation of two charges, the field of two charges turns into an external field given by the same formula $\mathbf{E} = \mathbf{E}_a^{\ell} + \mathbf{E}_a^{\ell'}$, $\mathbf{B} = \mathbf{B}_a^{\ell} + \mathbf{B}_a^{\ell'}$. The radiation power at large distances generated by two charges in accordance with \(252\) is expressed by the formula

$$
\mathbf{S} = \frac{c}{4\pi} \left( \mathbf{E}_a^{\ell} + \mathbf{E}_a^{\ell'} \right) \times \left( \mathbf{B}_a^{\ell} + \mathbf{B}_a^{\ell'} \right).
$$

(99)

If $\ell$-th charge is at rest $\mathbf{B}_a^{\ell} = 0$ and $\mathbf{E}_a^{\ell} = 0$ and we obtain the leading part and its time average

$$
\mathbf{S} = \frac{c}{4\pi} \mathbf{E}_a^{\ell'} \times \mathbf{B}_a^{\ell'}, \quad \langle \mathbf{S} \rangle = \frac{c}{4\pi} \langle \mathbf{E}_a^{\ell'} \times \mathbf{B}_a^{\ell'} \rangle,
$$

(100)

which differs from \(98\) and \(97\).
Remark 2: Note that formulas (98) and (97) describe the radiation of EM interaction energy between the two charges, and it is not the same as the EM radiation of the fields which act on test charges. One may ask why we are interested in the interaction energy? The answer is that this is the energy which is derived from the Lagrangian and the interaction energy is conserved when the system is closed. Of course the system where the motion of sources is prescribed is not closed, and there must be an external source of energy to provide for the prescribed currents and for the radiation which acts onto test charges.

Let us consider in conclusion the case when the both charges move slowly and hence $|\beta^\ell|, |\beta^{\ell'}| \ll 1$. Notice that that relations (255) defining the point charge EM fields are similar to the dipole expressions (84) and can be obtained from them by a substitution $\ddot{p}^\ell (t) \to cq \ddot{v}^\ell (t)$. Consequently, expressions for the energy flux and the radiated power for the pair of slowly moving charges can be readily obtained from the dipole expressions (85)-(85) by a substitution $\ddot{p}^\ell (t) \to cq \ddot{v}^\ell (t)$ with an additional correcting asymptotic factor

$$\delta^\ell = (1 + O(|\beta^\ell|)) \left(1 + O\left(|\beta^{\ell'}|\right)\right).$$

### 2.4 Comparison with the classical EM theory

We would like to show here that the CEM theory can be viewed as a limit case of the BEM theory. Particularly, we are interested in seeing: (i) how the single classical EM field is modeled by elementary EM fields in BEM theory; (ii) how classical EM phenomena including the radiation are modeled within the BEM theory.

As an example of relation between the classical EM field and elementary EM fields we consider clusters of many tightly bound identical b-charges, every cluster is labeled by index $\ell$. Namely, for every $\ell$ we introduce $N_\ell$ b-charges having identical wave functions and elementary EM potentials with charges $q^\ell = N_\ell q^\ell$:

$$\psi^{(\ell,s)} = \psi^{(\ell,s)} = \psi^\ell, \quad A^{(\ell,s)\mu} = A^{(\ell,s)\mu} \quad \text{where } 1 \leq \ell \leq N, \quad 1 \leq s \leq N_\ell, \quad (101)$$

and identical EM fields $F^{\ell\mu\nu} = F^{(\ell,s)\mu\nu}$ and currents $J^{\ell,\nu} = J^{(\ell,s)\nu}$. Now we compare these fields with classical EM fields $F^{\mu\nu}$ defined by (8) with the same currents by comparing the electromagnetic parts of the Lagrangians in BEM and CEM theories. We have for classical theory

$$\mathcal{L}_\text{CEM} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} \sum_{\ell'} N_{\ell'} N_\ell F^{\ell'\mu\nu} F_{\mu\nu}^{\ell'} = -\frac{1}{16\pi} \sum_{\ell' \neq \ell} N_{\ell'} N_\ell F^{\ell'\mu\nu} F_{\mu\nu}^{\ell'} - \frac{1}{16\pi} \sum_{\ell=1}^{N} N_\ell^2 F^{\ell\mu\nu} F_{\mu\nu}^{\ell},$$

20
For the BEM theory (12) takes the form
\[ L_{\text{BEM}} = -\frac{1}{16\pi} \sum \left\{ F^{(\ell,s)\mu\nu} F_{\mu\nu}^{(\ell',s')} \right\} \]
\[ = -\frac{1}{16\pi} \sum_{\ell' \neq \ell} N_{\ell} F^{\ell\mu\nu} F_{\mu\nu}^{\ell'} - \frac{1}{16\pi} \sum_{\ell} N_{\ell} (N_{\ell} - 1) F^{\ell\mu\nu} F_{\mu\nu}^{\ell}. \]

The difference between \( L_{\text{CEM}} \) and \( L_{\text{BEM}} \) can be attributed to interactions inside every cluster. In particular, we have for both theories
\[ L_{\text{CEM}} = N_{\ell}^{2} F^{\ell\mu\nu} F_{\mu\nu}^{\ell}, \quad L_{\text{BEM}} = N_{\ell} (N_{\ell} - 1) F^{\ell\mu\nu} F_{\mu\nu}^{\ell}. \]
readily implying the following expression for the relative difference
\[ L_{\text{BEM}} / L_{\text{CEM}} - 1 = 1/N_{\ell} \quad (102) \]
The difference (102) evidently becomes small as the number \( N_{\ell} \) of particles in the cluster becomes large.

To summarize, the CEM theory with its single EM field can be formally derived from the BEM theory as a limit obtained by binding together \( N_{\ell} \) identical particles with \( N_{\ell} \to \infty \).

2.4.1 Lagrangians for clusters of charges
Here we make a comparison of two Lagrangians for \( N \) clusters of charges with \( N_{\ell} \) particles within every cluster which are formally derived from WCM and BEM theories respectively. To this end we consider first a Lagrangian as in WCM theory with \( N_{1} + \ldots + N_{N} \) charges and a fixed nonlinearity \( G_{w}^{\ell} \).

To this end we consider first a Lagrangian as in WCM theory with \( N_{1} + \ldots + N_{N} \) charges and a fixed nonlinearity \( G_{w}^{\ell} \)
\[ L_{w}^{\ell}(\{\psi_{\ell,s}\}, \{\psi_{\mu,\ell}\}) = \sum_{1 \leq \ell \leq N} \sum_{1 \leq s \leq N_{\ell}} L^{\ell}(\psi_{\ell,s}, \psi_{\mu,\ell}) - \frac{F_{\mu\nu}^{\ell} F_{\mu\nu}^{\ell}}{16\pi}, \quad (103) \]
and to introduce clusters we impose additional constraints as in (101), namely
\[ \psi_{\ell,s} = \psi_{w}^{\ell}, \quad q_{\ell,s} = q_{\ell}, \quad s = 1, \ldots, N_{\ell}. \quad (104) \]
A cluster in the equilibrium is described by the real valued functions \( \psi_{\ell} \) and \( \varphi_{\ell,s}^{s} \) which satisfy the following system of equations:
\[ -\Delta \varphi_{\ell}^{w} = 4\pi \sum_{s=1}^{N_{\ell}} q_{\ell,s}^{s} \left( 1 - \frac{q_{\ell,s}^{s} \varphi_{w}^{s}}{m^{s,c}^{2}} \right) \left( \psi_{\ell,s} \right)^{2}, \quad (105) \]
\[ -\Delta \psi_{\ell,s} + \frac{m^{s,c}^{2} \varphi_{w}^{s} q_{\ell,s}^{s}}{\chi^{2}} \left( 2 - \frac{q_{\ell,s}^{s} \varphi_{w}^{s}}{m^{s,c}^{2}} \right) \varphi_{\ell,s}^{s} + G'_{w} \left( \left| \psi_{\ell,s} \right|^{2} \right) \psi_{\ell,s} = 0. \quad (106) \]
Note that the WCM theory developed in [BabFig1], [BabFig2] coincides with the case where every cluster contains exactly one charge. In this case the charge equilibrium equation has the form
\[ -\Delta \varphi_{\ell} = 4\pi q_{\ell} \left( 1 - \frac{q_{\ell} \varphi_{\ell}^{w}}{mc^{2}} \right) \psi_{\ell}^{2}, \quad (107) \]
\[ -\Delta \psi_{\ell} + \frac{m^{\ell,c}^{2} \varphi_{\ell}^{w}}{\chi^{2} q_{\ell}^{2}} \left( 2 - \frac{q_{\ell} \varphi_{\ell}^{w}}{mc^{2}} \right) \psi_{\ell} + G'_{w} \left( \left| \psi_{\ell} \right|^{2} \right) \psi_{\ell} = 0. \quad (108) \]
Comparing with (105) and (106) and setting
\[
\varphi_w = N_\ell \varphi_w^\ell, \quad q_w = N_\ell q^\ell, \quad m_w = N_\ell^2 m^\ell
\] (109)
we find then that \( \varphi_w^\ell \) satisfies the following equation
\[
- \Delta \varphi_w^\ell = 4\pi q_w \left( 1 - \frac{q_w^\ell \varphi_w^\ell}{m_w^\ell c^2} \right) \varphi_w^\ell,
\] (110)
which has exactly the same form as (107) in the WCM theory. The charge normalization condition is also fulfilled. Equations (105) and (108) can be rewritten in the form
\[
- \Delta \varphi^\ell + \frac{m_w^\ell \varphi^\ell}{N_\ell^2 \chi^2} 2 \varphi^\ell \left( 2 - \frac{q_w \varphi^\ell}{m_w^\ell c^2} \right) \varphi^\ell + G_w' \left| \varphi^\ell \right|^2 \varphi^\ell = 0.
\] (111)
This equation has the form of (106) for \( \varphi^\ell = \varphi_w^\ell \) if we set
\[
\chi_w = N_\ell^2 \chi.
\] (112)
Observe that relations (114)- (109) readily imply the following identities for the model constants
\[
\kappa_w^\ell = \frac{m_w^\ell c}{\chi_w} = \frac{m^\ell c}{\chi} = \kappa^\ell,
\] (113)
If we introduce Lagrangian \( L_w \left( \{ \psi^\ell \}, \{ \psi_w^\ell \} \right) \) with (i) constants defined by (109), (112) and (ii) EM potential defined by
\[
A_\mu^w = \sum_{\ell,s} A_\mu^{\ell,s} = \sum_\ell A_\mu^\ell, \quad A_\mu^w = N_\ell A_\mu^\ell, \quad 1 \leq \ell \leq N,
\] (114)
then the Euler-Lagrange Equations for Lagrangian \( L \left( \{ \psi^{(\ell,s)} \}, \{ \psi_w^{(\ell,s)} \} \right) \) defined by (103) with restrictions (104) are equivalent to the Euler-Lagrange Equations for \( L_w \). Hence, we can conclude that introduction of clusters of charges is equivalent to a proper rescaling of the constants of the WCM theory. Note that the quadratic dependence of \( m_w^\ell \) on the number \( N_\ell \) of charges in the cluster is natural in the relativistic theory since the energy of interactions depends on the number of particles quadratically, and relativistic mass of the cluster is proportional to its energy.

Now let us relate the above treatment with the BEM theory. To make the comparison more transparent we consider a generalized version of the BEM theory, namely the theory where the basic equilibrium object is a cluster of \( N_\ell \) b-charges, \( N_\ell \geq 1 \). The BEM theory for elementary b-charges corresponds to the case \( N_\ell = 1 \) for all \( \ell \) in this generalized setting. The Lagrangian \( L_{\text{BEMC}} \left( \{ \psi^{(\ell,s)} \}, \{ \psi_w^{(\ell,s)} \} \right) \) for clusters is given by an obvious modification of (9) and the variables are subjected to additional constraints as in (104). The equilibrium state for a generalized cluster of \( n = N_\ell \) identical b-charges is denoted by \( \psi_{b,n}^{\ell,s}, \varphi_{b,n}^{\ell,s}, \psi_{w,n}^{\ell,s}, s = 1, ..., n \). The equilibrium condition takes the following form similar to (105) and (106)
\[
- \Delta \psi_{w,n}^{\ell,s} = 4\pi \sum_{s=2}^{N_\ell} q_{b,n}^{\ell,s} \left( 1 - \frac{q_{b,n}^{\ell,s} \psi_{b,n}^{\ell,s}}{m_{b,n}^{\ell,s} c^2} \right) \left| \psi_{b,n}^{\ell,s} \right|^2,
\] (115)
\[
- \Delta \psi_{b,n}^{\ell,s} + \frac{q_{b,n}^{\ell,s} \varphi_{b,n}^{\ell,s}}{\chi_{b,n}^2} \psi_{b,n}^{\ell,s} \left( 2 - \frac{q_{b,n}^{\ell,s} \psi_{b,n}^{\ell,s}}{m_{b,n}^{\ell,s} c^2} \right) \psi_{b,n}^{\ell,s} + G_{b,n}' \left| \psi_{b,n}^{\ell,s} \right|^2 \psi_{b,n}^{\ell,s} = 0.
\] (116)
Note that the only difference between (115) and (116) is that the first term with \( s = 1 \) in the sum is omitted. Now \( \dot{\psi}_{b,2} \), \( \dot{\varphi}_{b,2} \) is the basis for comparison and once again we fix the nonlinearity \( G'_{b,n} = G'_{b,2} = G'_{b} \) and set similarily to (109), (112):

\[
\begin{align*}
\varphi_{b,n}^\ell &= (n - 1) \varphi_{b,2}^\ell, \\
q_{b,n}^\ell &= (n - 1) q_{b,2}^\ell, \\
m_{b,n}^\ell &= (n - 1)^2 m_{b,2}^\ell, \\
\chi_{b,n}^\ell &= (n - 1)^2 \chi_{b,2}^\ell.
\end{align*}
\]

Let us introduce Lagrangian \( \mathcal{L}_{BEM,n}(\{\psi_{b,n}^\ell\},\{\varphi_{b,n}^\ell\}) \) with so defined constants. Observe that the Euler-Lagrange equations for \( \mathcal{L}_{BEM}(\{\psi_{\ell,s}^\ell\},\{\psi_{\mu}^\ell,s\}) \) with restrictions (104) are equivalent to the Euler-Lagrange equations for \( \mathcal{L}_{BEM,n}(\{\psi_{b,n}^\ell\},\{\varphi_{b,n}^\ell\}) \). Obviously if all \( N_\ell \geq 2 \) then \( \mathcal{L}_{BEM,n} \) has the same form as \( \mathcal{L}_w \). In addition to that, if

\[
G'_{b,2} = G'_w, \quad q_{b,2}^\ell = q^\ell, \quad m_{b,2}^\ell = m^\ell, \quad \chi_{b,2}^\ell = \chi,
\]

then the relative difference of coefficients of Lagrangians \( \mathcal{L}_{BEM,n} \) and \( \mathcal{L}_w \) with the same \( n \geq 2 \) is of order \( \frac{1}{n} \). For example, \( (q_w^\ell - q_{b,2}^\ell) / q_w^\ell = 1/n \). Notice that the case \( n = 1 \) (which is the primary one considered in this article) is special, and there is a non-vanning difference between \( \dot{\psi}_{b,2}^\ell, \dot{\varphi}_{b,2}^\ell \) and the fundamental equilibrium \( \dot{\psi}_{b,1}^\ell, \dot{\varphi}_{b,1}^\ell = \dot{\psi}^\ell, \dot{\varphi}^\ell \) determined by (37), (38).

Thus, based on the above analysis we conclude that the WCM theory can be considered as an approximation for the generalized BEM theory if the WCM charge is identified with a cluster of a large number of b-charges.

### 2.5 New EM features of the BEM theory

In this section we discuss new EM features of BEM theory not presented in the CEM theory. Looking at the b-charge Lagrangian \( \mathcal{L} \) with its field equations and conserved currents in Section 2.1 we can already see new important features of the theory of interacting charges and EM fields. The first striking feature of the new theory compare to the classical one is that the single EM field as independent entity is no more, instead we have elementary EM potentials \( A^\mu \) and fields \( F^{\mu\nu} \) defined by the classical formulas and satisfying individually Maxwell equations (19). Though we still can naturally define the total potential \( A^\mu \) and the corresponding total EM field \( F^{\mu\nu} \) as the sum of individual potentials and fields by formulas (3)- (7), and though the total EM field \( F^{\mu\nu} \) satisfies the Maxwell equations (25) this total field is not an independent entity.

With all that said about the new status of the total EM field its physical significance as the field sensed by a small test charge remains valid in the BEM theory. Indeed let us look at how different charges sense each other via the field equations (17)-(19). It is evident from the equations and the formulas (22) for individual currents that the entire evolution of \( \ell \)-th charge is completely determined by a single quantity, its exterior potential \( A^\mu_{\ell} \), and, consequently, charges \( (\psi^\ell, A^\mu_{\ell}) \) sense each other exclusively by their potentials \( A^\mu_{\ell} \). In other words, the state of every b-charge \( (\psi^\ell, A^\mu) \) and its current \( J^\mu \) are completely determined by the action upon it of all other charges potentials via the exterior potential \( A^\mu_{\ell} \) and every charge \( (\psi^\ell, A^\mu) \) acts upon other charges. So in this theory as we can see it is the individual \( A^\mu_{\ell} \) potentials as components of respective b-charges \( (\psi^\ell, A^\mu) \) that facilitate the EM interaction between charges. This is in contrast to the classical theory as well to
our own WCM theory in [BabFig1], [BabFig2] in which it is a single and independent EM
field interacting with every charge facilitates all EM interactions between different charges
$\psi^\ell$. In particular, in the BEM theory by its very set up there is no any EM self-interaction as
in the classical theory and our theory in [BabFig1], [BabFig2], where such interaction exists
because there is only a single EM field.

Based on the analysis in Section 2.4 one may expect noticeable differences between CEM
theory and BEM theory for example when $1/N_\ell$ in (102) is not small. These differences can
become more pronounced when $L_\ell$ in (28) is comparable with the classical EM Lagrangian
$L_{\text{CEM}}$ not to mention when it is a dominant term in $L_{\text{BEM}}$.

An important signature of the BEM theory differentiating it from the CEM theory is a
mechanism of negative radiation for certain prescribed currents, i.e. a situation when the EM
energy propagates with the speed of light toward the current source rather than away from it
as we have shown in Section 2.3. This mechanism can conceivably work for a limited time in a
system of several bound charges, such as an atom or a molecule, resulting in effective energy
gain coming from matching energy loss of b-charges outside of this system. Such energy
transfer is completely accounted for by interacting elementary EM fields as the components
of the involved b-charges including external ones which might be represented effectively by
an external EM field. The very possibility of the negative radiation indicates a significant
difference between the EM energy transport at the elementary atomic scale from the same
at the macroscopic scale. For the later most of the time one would observe well known
classical charge behavior including the EM self-interaction and exclusively normal (positive)
radiation of the EM energy away from the source. In other words the BEM theory allows for
a differentiation between macroscopic and atomic scales at the level of EM fields alone, and
that makes the concept of a b-charge with its elementary EM field to be truly elementary.
This is in noticeable contrast with some of the classical charge models which are based on the
concept of a single EM field, and we would like to quote here F. Rohrich, [Roh, Section 6-1]:
"This was the problem faced by Abraham, Lorentz, and Poincare in the first few years of
this century. The most obvious model of a charged particle is a sphere carrying a spherically
symmetrical charge wave function. While such a model is meant (and was indeed proposed)
as a picture of a charged elementary particle (an electron, for example), it is obvious that it
is basically a macroscopic charged body, only much smaller. There is nothing "elementary"
about it."

Notice also that as we can see from the BEM expression (60) for the interaction energy
density for a pair of b-charges the interaction energy density can be negative. The later is
analogous to the negative sign of the electrostatic energy for two classical point charges when
one of them is positive and another one is negative, and the interaction energy is defined to
be zero when charges are separated by infinite distance. This way to calibrate the interaction
energy in the CEM theory is in line with defining the interaction energy as work done to
assemble a system of charges from the state when they don’t interact, that is when they are
separated by infinite distance.

Let us look at the gauge properties of the BEM system Lagrangian $\mathcal{L}$ defined by (12), (10).
Recall that gauge transformation of the first or the second kind (known also as respectively
global and local gauge transformation) are described respectively by the following formulas,
[PauRF (17), (23a), (23b)], [Wen Section 11, (11.4)]
\begin{align}
\psi^\ell &\rightarrow e^{i\gamma^\ell} \psi^\ell, \quad \psi^{\ell *} \rightarrow e^{-i\gamma^\ell} \psi^{\ell *}, \quad \text{where } \gamma^\ell \text{ is any real constant,} \\
\psi^\ell &\rightarrow e^{\frac{i\theta^\mu(x)}{\chi_c}} \psi^\ell, \quad \psi^{\ell *} \rightarrow e^{-\frac{i\theta^\mu(x)}{\chi_c}} \psi^{\ell *}, \quad A^\mu \rightarrow A^\mu + \partial^\mu \lambda.
\end{align}
The significance of individual EM fields in the new theory is manifested in the existence of
a new much large group of gauge transformations than described by \((118)-(119)\). We have
already pointed out that the system Lagrangian \(L\) is invariant with respect to the gauge
transformation of the first kind \((118)\) and used that to construct an instrumental for the
theory conserved individual currents \(J^\mu_\ell\) defined in \((19)\). We introduce now a new gauge
transformation which we call \textit{elementary gauge transformations gauge or transformations of
the third kind}:

\[
A^\mu_\ell \rightarrow A^\mu_\ell + \partial^\mu \lambda^\ell (x),
\]
\[
\psi^\ell \rightarrow e^{\frac{i\int\lambda^\ell(x)}{x}} \psi^\ell, \quad \psi^{\ell*} \rightarrow e^{\frac{-i\int\lambda^\ell(x)}{x}} \psi^{\ell*},
\]

where functions \(\lambda^\ell (x), 1 \leq \ell \leq N\), are independent real-valued scalar functions of \(x\) and

\[
\lambda(x) = \sum_{1 \leq \ell \leq N} \lambda^\ell (x), \quad \lambda^\ell (x) = \lambda(x) - \hat{\lambda}^\ell (x),
\]

implying the independence of functions \(\hat{\lambda}^\ell (x), 1 \leq \ell \leq N\). A straightforward examination
shows that for \(N \geq 2\) the system Lagrangian \(L\) defined by \((12)-(14)\) is invariant with respect
to the gauge transformation of the third kind. As we have already pointed out it is due to the
gauge invariance with respect to elementary gauge transformations \((120), (121)\) the two-way
representation \((15)\) holds for the elementary currents that accounts for an important fact that
the conserved Noether’s elementary current is exactly the source current in the corresponding
elementary Maxwell equations \((19), (20)\).

Another feature of the BEM theory is the new expressions \((48), (49)\) and \((50)\) of the total
EM energy and the entire EM field EnMT compare with the classical expressions \((212)-(214)\).
Let us take a look at the representation \((50)\) which relates the EM EnMT to the classical
one. This identity shows that the EnMT can be effectively obtained by the removal from
the classical EnMT for the total EM field the sum of all classical EnMT’s for individual EM
fields. So if our intention was to remove the EM self-interaction in a consistent way keeping
the Lagrangian structure it is perfectly accomplished by the BEM theory.

3 \ Non-relativistic dynamics of localized charges

In this section based on our relativistic model we introduce a non-relativistic model for
the case where charges move slowly compared with speed of light. First we introduce field
equations and the Lagrangian and briefly describe their derivation from the relativistic model.

Using frequency-shifting substitution \((36)\) with more general \(\psi = \psi^\ell_\omega (t, x)\) which depends
on \((t, x)\) we observe that the second time derivative in the Klein-Gordon equation \((31)\) can be written in the form

\[
- \frac{1}{c^2} \tilde{\partial}_t \tilde{\partial}_t \psi^{\ell}_\omega = \frac{1}{c^2} \left( \partial_t + \frac{iq^\ell_\omega}{\chi} \varphi^{\ell*}_\omega \right)^2 \psi^{\ell}_\omega - \frac{2i m^\ell}{\chi} \left( \partial_t + \frac{iq^\ell_\omega}{\chi} \varphi^{\ell*}_\omega \right) \psi^{\ell}_\omega + \kappa^2_\omega \psi^{\ell}_\omega
\]

(124)
where

$$\varphi^\ell = \sum_{\ell' \neq \ell} \varphi^{\ell'}.$$  \hspace{1cm} (125)

To get a non-relativistic approximation, we neglect in (124) the term with the factor $\frac{1}{\epsilon^2}$ and substitute $-2i \frac{m}{\epsilon^2} \left( \partial_t + \frac{q}{m} \mathbf{A}_{\text{ex}}^\ell \right) \psi^\ell + \kappa_0^2 \psi^\ell$ instead of the term $-\frac{1}{\epsilon^2} \partial_t \partial_t^\ell \psi^\ell$ in (31). As a result, the nonlinear Klein-Gordon equation (31) is approximated by the following nonlinear Schrödinger equation

$$\chi i \partial_t \psi^\ell + \frac{\chi^2}{2m^\ell} \left( \nabla^\ell \right)^2 \psi^\ell - \frac{\chi^2}{2m^\ell} G^\ell \left( \psi^{\ell^*} \psi^\ell \right) \psi^\ell - q^\ell \varphi^\ell \psi^\ell = 0,$$  \hspace{1cm} (126)

with $\mathbf{A}_{\text{ex}}^\ell$ given by (11), where for notational simplicity we write $\psi^\ell$ in place of $\psi^\ell$. Since magnetic fields generated by moving charges also have coefficient $\frac{1}{\epsilon^2}$ and are small for small velocities, they can be also neglected, and consequently we preserve only the external magnetic fields and replace $\nabla^\ell$ by the covariant gradient

$$\mathbf{A}_{\text{ex}}^\ell = \nabla - \frac{i q^\ell \mathbf{A}_{\text{ex}}}{\chi c}.$$  \hspace{1cm} (127)

Notice that in the nonrelativistic case the $\ell$-th b-charge is described by a pair $\psi^\ell$, $\varphi^\ell$ where the elementary EM field is represented only by the scalar electric potential $\varphi^\ell$. The non-relativistic model has the the Lagrangian

$$\hat{\mathcal{L}} \left( \varphi, \{ \psi^\ell \}_{\ell = 1}^N, \{ \varphi^\ell \}_{\ell = 1}^N \right) = \frac{|\nabla \varphi|^2}{8 \pi} + \sum_{\ell} \hat{\mathcal{L}}^\ell \left( \psi^\ell, \psi^{\ell^*}, \varphi \right),$$

where

$$\hat{\mathcal{L}}^\ell = \frac{\chi}{2} \left[ \psi^{\ell^*} \partial_t \psi^\ell - \psi^\ell \partial_t \psi^{\ell^*} \right] - \frac{\chi^2}{2m^\ell} \left\{ \left| \mathbf{A}_{\text{ex}}^\ell \right|^2 + G^\ell \left( \psi^{\ell^*} \psi^\ell \right) \right\} - q^\ell \left( \varphi + \varphi_{\text{ex}} - \varphi^\ell \right) \psi^\ell \psi^{\ell^*} - \frac{|\nabla \varphi|^2}{8 \pi},$$

and

$$\varphi = \sum_{\ell} \varphi^\ell.$$  \hspace{1cm} (129)

Importantly, to total potential $\varphi$ defined by (129) is not an independent entity but just the sum of the elementary potentials $\varphi^\ell$, and $\psi^{\ell^*}$ a variable complex conjugate to $\psi^\ell$. The Euler-Lagrange equations for charge densities $\psi^\ell$ take the form similar to (126), namely

$$i \chi \partial_t \psi^\ell = -\frac{\chi^2}{2m^\ell} \left( \mathbf{A}_{\text{ex}}^\ell \right)^2 \psi^\ell + q^\ell \left( \varphi^\ell + \varphi_{\text{ex}} \right) \psi^\ell + \frac{\chi^2}{2m^\ell} \left[ G_0^\ell \right]' \left( |\psi^\ell|^2 \right) \psi^\ell,$$  \hspace{1cm} (130)

where $\varphi_{\text{ex}}$ is given by (125) and every potential $\varphi^\ell$ can be determined from the equation

$$\nabla^2 \varphi^\ell = -4 \pi q^\ell |\psi^\ell|^2,$$  \hspace{1cm} (131)

$\ell = 1, \ldots, N.$

The solution of the above equation (131) is given by the Green’s formula

$$\varphi^\ell (t, \mathbf{x}) = q^\ell \int_{\mathbb{R}^3} \frac{|\psi^\ell|^2(t, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y},$$  \hspace{1cm} (132)

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The properties and examples of the nonlinearities $G^\ell_a$ are provided in the following 3.2. As the result of charge conservation the norms $\|\psi^\ell\|^2$ remain constant on solutions of (130) and choose those constants to 1. In other words we impose the following charge normalization condition:

$$\|\psi^\ell\|^2 = \int_{\mathbb{R}^3} |\psi^\ell|^2 \, dx = 1, \; t \geq 0, \; \ell = 1, \ldots, N.$$  \hfill (133)

A motivation for this particular normalization is based on the formula (132) and requirement that the elementary potential $\varphi^\ell$ is asymptotically the Coulomb’s potential $q^\ell/|x|$ for large $|x|$.

Recall that in the classical electrodynamics the evolution of a point charge $q$ of a mass $m$ and and position vector $r(t)$ in an external electromagnetic (EM) field in the non-relativistic case is governed by the Newton’s equation (4). In our model a charge is described by a wave function $\psi^\ell(t, x)$ with dynamics described by a system of nonlinear Schrödinger equations (NLS) (130) coupled through corresponding electric potentials. Nevertheless we show below that in macroscopic regimes we can derive from the field equations as an approximation the Newton’s equations for centers $r^\ell(t)$ of localized waves defined by

$$r^\ell(t) = \int_{\mathbb{R}^3} x |\psi^\ell(t, x)|^2 \, dx.$$  

These center adequately describe the positions of wave functions $\psi^\ell(t, x)$ when they are localized. To introduce localized solutions we use the nonlinearity which depends on a size parameter $a > 0$ which, in turn, determines the spatial scale of a the charge when it is at the rest state. We consider below the macroscopic dynamics, with a macroscopic spatial scale $R \gg a$. We prove in Section 4 that in the case of the non-relativistic field equations when $a \to 0$ the centers of the interacting charges converge to solutions of the Newton’s equations with the Lorentz forces if $\psi^\ell$ remain localized. We also provide examples of exact solutions of the field equations in the form of accelerating solitons for which the localization assumption holds.

### 3.1 Single charge

In the case of a single charge the Lagrangian and the field equations are obtained by setting $N = 1$ in (128), (130), (131) and this case evidently $\varphi^\ell_\neq = 0$. In particular, for a single charge without the external field we have $\varphi^\ell_\neq = 0$, $\varphi_\text{ex} = 0$, $A_\text{ex} = 0$ and we obtain the following equations for the rest state $\psi$ with $\partial_t \psi = 0$:

$$\begin{align*}
-\frac{\lambda^2}{2m^\ell} \nabla^2 \psi + q \varphi_\text{ex} \psi + \frac{\lambda^2}{2m^\ell} \left[ G^\ell_a \right]' (|\psi|^2) \psi &= 0, \\
\nabla^2 \varphi &= -4\pi q |\psi|^2.
\end{align*}$$  \hfill (134)

Obviously, the equations coincide with (38), and therefore the rest solutions of relativistic and non-relativistic equations coincide, as it should be expected in the case of zero velocity.

Let us consider now a single charge, omitting the index $\ell$, in an external EM field and set $N = 1$ in (130), (131). For a special class of external fields we present explicit solutions to the field equations (130), (131) for a single charge in the form of wave-corpuscles (accelerating solitons). We assume here for simplicity a purely electric external EM field, i.e. when $A_\text{ex} = 0$, $E_\text{ex} (t, x) = -\nabla \varphi_\text{ex} (t, x)$ (see [BabFig1], [BabFig2] for a similar exact solution with non-zero
magnetic field). For the purely electric external field the field equations (130), (131) take the form

\[ i\chi \partial_t \psi = -\frac{\chi^2 \nabla^2 \psi}{2m} + q\varphi_{ex} \psi + \frac{\chi^2}{2m} G'_a (|\psi|^2) \psi, \]  
\[ \nabla^2 \varphi = -4\pi q |\psi|^2. \]  

We define then wave-corpuscle \( \psi, \varphi \) by the following formula:

\[ \psi (t, x) = e^{iS/\chi} \hat{\psi}, \quad S = m v \cdot (x - r) + s_p (t), \]  
\[ \hat{\psi} = \hat{\psi} (|x - r|), \quad \varphi = \hat{\varphi} (|x - r|), \quad r = r (t). \]

In the above formula \( \hat{\psi} \) is the form factor satisfying (141), \( \hat{\varphi} \) is a radial function determined from (131), and we refer to the function \( r (t) \) as wave-corpuscle center. Since \( \hat{\psi} \) is center-symmetric, this definition agrees with more general definition (163).

Suppose that \( \varphi_{ex} ((t, x)) \) is a continuous function which is linear with respect to \( x \). Then \( \psi \) defined by (137) provide an exact solution to (135), provided that \( r (t) \) is determined from the equation

\[ m \frac{d^2 r (t)}{dt^2} = qE_{ex} (t, r), \]  
and \( v (t), s_p (t) \) are determined by formulas

\[ v = \frac{dr}{dt}, \quad s_p = \int_0^t \left( \frac{mv^2}{2} - q\varphi_{ex} (t, r (t)) \right) dt'. \]

The verification of the fact that (137), (138) and (139) determine an exact solution is straightforward and details can be found in [BabFig1], [BabFig2] or [BabFig3].

Note that in a simpler case when the external fields \( \varphi_{ex} \) and \( A_{ex} \) vanish, a simpler solution of (135)-(136) is provided by (137) with \( r (t) = r_0 + vt \) with constant velocity \( v \). In this case the wave-corpuscle solution (137) of the field equations (135)-(136) can be obtained from the rest solution \( \hat{\psi}, \hat{\varphi} \) by certain Galilean-gauge transformations. Solutions of a similar form are known in the theory of nonlinear Schrödinger equations, see [Sul] and references therein. For the particular case of the logarithmic nonlinearity solutions of the form (137) were found in [Bia] in the form of accelerating gaussons. The exponential factor in wave-corpuscle solution of the form (137) can be identified with the de Broglie wave, for details see [BabFig1], [BabFig2].

**Remark 3** The construction of the solution (137) does not depend on a particular form of the nonlinearity \( G' = G'_a \) as long as (37) is satisfied. It is uniform with respect to \( a > 0 \), and the dependence on \( a \) in (137) is only through \( \psi (|x - r|) = a^{-3/2} \psi_1 (a^{-1} |x - r|) \). Obviously, if \( \psi (t, x) \) is defined by (137) then \( |\psi (t, x)|^2 \to \delta (x - r) \) as \( a \to 0 \).

**Remark 4** In [BabFig1], [BabFig2] we introduced a non-relativistic model where equations for a steady states coincide with (134). We derived there nonrelativistic equations from the corresponding relativistic one not only assuming that velocities are vanishingly small, but also assuming that the Sommerfeld’s fine structure constant is vanishingly small. If we would not neglect there the terms with Sommerfeld’s fine structure constant the steady states would satisfy more involved equations, see [BabFig1] for details. We also would like to note that though \( \psi \) would be defined differently, equations (138) would be the same. In the present paper the rest state equations (134) are simpler even if the Sommerfeld’s fine structure constant is not replaced by zero.
3.2 Determination of Nonlinearity,

As we have already mentioned, the nonlinear self interaction function $G$ is determined from the charge equilibrium equation (37) based on the form factor (ground state) $\hat{\psi}$. Important features of our nonlinearity include: (i) the boundedness or slow subcritical growth of its derivative $G'(s)$ for $s \to \infty$ with consequent boundedness from below of the wave energy; (ii) slightly singular behavior about $s = 0$, that is for small wave amplitudes.

In this section we consider construction of the function $G$, study its properties and provide examples for which the construction of $G$ is carried out explicitly. Throughout this section we have

$$\psi, \hat{\psi} \geq 0$$

and hence $|\psi| = \psi$.

We introduce explicitly the dependence of the free ground state $\hat{\psi}$ on the size parameter $a > 0$ through the following representation of the function $\hat{\psi}$:

$$\hat{\psi}(r) = \hat{\psi}_a(r) = a^{-3/2} \hat{\psi}_1(a^{-1}r),$$

where $\hat{\psi}_1(r)$ is a twice continuously differentiable function of the dimensionless parameter $r \geq 0$, and, as a consequence of (133), the function $\hat{\psi}_a(r)$ satisfies the charge normalization condition for every $a > 0$. Obviously, definition (140) is consistent with (37) and (39). The size parameter $a$ naturally has the dimension of length. A properly defined spatial size of $\hat{\psi}_a$, based, for instance, on the variance, is proportional to $a$ with a coefficient depending on $\hat{\psi}_1$.

The charge equilibrium equation (37) can be written in the following form:

$$\nabla^2 \hat{\psi}_a = G'_a(\hat{\psi}_a^2) \hat{\psi}_a.$$  

The function $\hat{\psi}_a(r)$ is assumed to be a smooth positive monotonically decreasing function of $r \geq 0$ which is square integrable with weight $r^2$ and we assume it to satisfy the charge normalization condition of the form (133); such a function is usually called in literature a ground state.

Let us look first at the case $a = 1$, $\hat{\psi}_a = \hat{\psi}_1$, $\hat{\varphi}_a = \hat{\varphi}_1$, for which the equation (141) yields the following representation for $G'(\hat{\psi}_1^2)$ from (141):

$$G'_1 \left(\hat{\psi}_1^2(r)\right) = \frac{\left(\nabla^2 \hat{\psi}_1(r)\right)}{\hat{\psi}_1(r)}.$$  

Since $\hat{\psi}_1(r)$ is a monotonic function, we can find its inverse $r = r(\hat{\psi}^2)$, yielding

$$G'_1(s) = \left[\frac{\nabla^2 \hat{\psi}_1}{\hat{\psi}_1}\right] \left(r(s)\right), \quad 0 = \hat{\psi}_1(\infty) \leq s \leq \hat{\psi}_1(0).$$

Since $\hat{\psi}_1(r)$ is smooth, $G'(|\psi|^2)$ is smooth for $0 < |\psi|^2 \leq \hat{\psi}_1(0)$. If we do not need $G'(s)$ to be smooth, we extend $G'(s)$ for $s \geq \hat{\psi}_1(0)$ as a constant, namely

$$G'_1(s) = G'_1(\hat{\psi}_1(0)) \quad \text{if} \quad s \geq \hat{\psi}_1(0).$$  

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First derivative of such an extension at \( s = \psi_1^2(0) \) has a discontinuity point. If \( \psi_a^2(r) \) is a smooth function of class \( C^m \), we always can define an extension of \( G'(s) \) for \( s \geq \psi_1^2(0) \) as a bounded function of class \( C^m \) for all \( r > 0 \) and

\[
G'_1(s) = G'_1\left(\psi_1^2(0)\right) - 1 \text{ if } s \geq \psi_1^2(0) + 1. \tag{145}
\]

Slowly growing functions \( G'(s) \) also can be used.

In the case of arbitrary size parameter \( a > 0 \) we define \( G'_a(s) \) by formula (39), this definition is consistent with (140) and (143).

Let us take a look at general properties of \( G'(s) \) as they follow from defining them relations (143)-(39). In the examples below the \( G'(s) \) is not differentiable at \( s = 0 \). But if \( \psi (r) \) decays exponentially or by a power law the nonlinearity \( g(\psi) = G'(|\psi|^2)\psi \) as it enters the field equations is differentiable for all \( \psi \) including zero, and hence it satisfies the Lipschitz condition. For a Gaussian \( \psi_1(r) \) which decays superexponentially \( G'(|\psi|^2) \) is unbounded at zero and \( g(\psi) \) is not differentiable at zero. Since \( \psi_1(|x|) > 0 \), the sign of \( G'_1(|\psi|^2) \) coincides with the sign of \( \nabla^2 \psi_1(|x|) \). At the origin \( x = 0 \) the function \( \psi_1(|x|) \) has its maximum and, consequently, \( G'_1(s) \leq 0 \) for \( s \) close to \( s = \psi_1^2(0) \). The Laplacian applied to radial functions \( \psi_1 \) takes the form \( \frac{1}{r^2} \frac{\partial^2}{\partial r^2} \left(r^2 \psi_1(|x|)\right) \). Consequently, if \( r\psi_1(r) \) is convex at \( r = |x| \) we have \( \nabla^2 \psi_1(|x|) \geq 0 \). Since \( r^2 \psi_1(r) \) is integrable, we naturally assume that \( |x| \psi_1(|x|) \to 0 \) as \( |x| \to \infty \). Then if the second derivative of \( r\psi_1(r) \) has a constant sign near infinity, it must be non-negative. For an exponentially decaying \( \psi_1(r) \) the second derivative of \( r\psi_a(r) \) is positive implying \( G'_1(s) > 0 \) for \( s \ll 1 \). In the examples we give below \( G'_1(s) \) has exactly one zero on the half-axis.

**Example 1.** Consider a form factor \( \psi_1(r) \) decaying as a power law, namely

\[
\psi_1(r) = \frac{c_{pw}}{(1 + r^2)^{5/4}}, \tag{146}
\]

where \( c_{pw} \) is the normalization factor, \( c_{pw} = 3^{1/2}/(4\pi)^{1/2} \). This function evidently is positive and monotonically decreasing. Let us find now \( G'(s) \) based on the relations (143). An elementary computation of \( \nabla^2 \psi_1 \) shows that

\[
G'(s) = \frac{15s^{2/5}}{4c_{pw}^{1/5}} - \frac{45s^{4/5}}{4c_{pw}^{3/5}}, \quad G(s) = \frac{75s^{7/5}}{28c_{pw}^{4/5}} - \frac{25s^{9/5}}{4c_{pw}^{8/5}}, \quad \text{for } 0 \leq s \leq c_{pw}^2. \tag{147}
\]

The extension for \( s \geq c_{pw}^2 \) can be defined as a constant or the same formula (147) can be used for all \( s \geq 0 \).

If we explicitly introduce size parameter \( a \) into the form factor using (140), we define \( G'_a(s) \) by (39). Notice that the variance of the form factor \( \psi_1^2(|x|) \) decaying as a power law (146) is infinite.

**Example 2.** Now we consider an exponentially decaying form factor \( \psi_1 \) of the form

\[
\psi_1(r) = c_e e^{-(r^2+1)^{1/2}}, \quad c_e = \left(4\pi \int_0^\infty r^2 e^{-2(r^2+1)^{1/2}} \, dr \right)^{-1/2}. \tag{148}
\]

Evidently \( \psi_1(r) \) is positive and monotonically decreasing as required. The dependence \( r(s) \) defined by the relation (148) is as follows:

\[
r = \left[\ln^2 \left(c_e/\sqrt{s}\right) - 1\right]^{1/2} \text{, if } \sqrt{s} \leq \psi_1(0) = c_e e^{-1}. \tag{149}
\]
An elementary computation shows that
\[
- \frac{\nabla^2 \hat{\psi}_1}{\hat{\psi}_1} = \frac{2}{(r^2 + 1)^{\frac{3}{2}}} + \frac{1}{(r^2 + 1)} + \frac{1}{(r^2 + 1)^{\frac{3}{2}}} - 1. \tag{150}
\]
Combining (149) with (150) we readily obtain the following function for \( s \leq c_2 e^{-2} \)
\[
G_1' (s) = \left[ 1 - \frac{4}{\ln (c_2^2/s)} - \frac{4}{\ln^2 (c_2^2/s)} - \frac{8}{\ln^3 (c_2^2/s)} \right]. \tag{151}
\]
We can extend it for larger \( s \) as follows:
\[
G_1' (s) = G_1' (c_2^2 e^{-2}) = -3 \text{ if } s \geq c_2^2 e^{-2}. \tag{152}
\]
or we can use a smooth extension as in (145). The function \( G_1' (s) \) is not differentiable at \( s = 0 \). At the same time if we set \( g (0) = 0 \) the function \( g (\psi) = G_1' (\psi (r)) \psi \) is continuous and \( g (\psi) \) is continuously differentiable with respect to \( \psi \) at zero and \( g (\psi) \) satisfies a Lipschitz condition. The variance of the exponential form factor \( \hat{\psi}_1 (r) \) is obviously finite. To find \( G_a' (s) \) for arbitrary \( a \) we use its representation (39).

**Example 3.** Now we define *Gaussian form factor* by the formula
\[
\hat{\psi} (r) = C g e^{-r^2/2}, C_g = \frac{1}{\pi^{3/4}}. \tag{153}
\]
Such a ground state is called *gausson* in \[Bia\]. Elementary computation shows that
\[
- \frac{\nabla^2 \hat{\psi} (r)}{\hat{\psi} (r)} = r^2 - 3 = -\ln \left( \frac{\hat{\psi}^2 (r) / C_g^2}{C_g^2} \right) - 3.
\]
Hence, we define the nonlinearity by the formula
\[
G' (|\psi|^2) = - \ln \left( |\psi|^2 / C_g^2 \right) - 3, \tag{154}
\]
we call this nonlinearity *logarithmic nonlinearity*. The nonlinear potential function has the form
\[
G (s) = \int_0^s \left( - \ln \left( s'/C_g^2 \right) - 3 \right) ds' = -s \ln s + s \left( \ln \frac{1}{\pi^{3/2}} - 2 \right). \tag{155}
\]
Dependence on the size parameter \( a > 0 \) is given by the formula
\[
G_a' (|\psi|^2) = -a^{-2} \ln \left( a^2 |\psi|^2 / C_g^2 \right) - 3. \tag{156}
\]
Obviously \( g (\psi) = G_1' (|\psi|^2) \psi \) is continuous for all \( \psi \in \mathbb{C} \) if at zero we set \( g (0) = 0 \) and is differentiable for every \( \psi \neq 0 \) but is not differentiable at \( \psi = 0 \) and does not satisfy Lipschitz condition.

### 4 Charges in remote interaction regimes

The primary focus of this section is to show that if the size parameter \( a \to 0 \) the dynamics of the centers of localized solutions is approximated by the Newton’s law of motion (the macroscopic limit \( a \to 0 \) assumes that there is a fixed macroscopic scale \( R \gg a \)).
is done in the spirit of the well known in quantum mechanics Ehrenfest Theorem, Schiff, Sections 7, 23. We also Wave-corpuscle solutions defined by [137], [138] provide an example of explicit solutions which have such a dynamics.

As we stated, the Lagrangian \( \hat{\mathcal{L}} \) in (27) is gauge invariant and every \( \ell \)-th charge has a 4-current \((\rho^\ell, J^\ell)\) defined by

\[
\rho^\ell = q \left| \psi^\ell \right|^2, \quad J^\ell = \left( \frac{\chi q^\ell}{m^\ell} \Im \frac{\nabla \psi^\ell}{\psi^\ell} - \frac{q^{\ell^2} A_{\text{ex}}}{m^\ell c} \right) \left| \psi^\ell \right|^2, \tag{157}
\]

which satisfies the continuity equations \( \partial_t \rho^\ell + \nabla \cdot J^\ell = 0 \) or explicitly

\[
\partial_t \left| \psi^\ell \right|^2 + \nabla \cdot \left( \frac{\chi}{m^\ell} \Im \frac{\nabla \psi^\ell}{\psi^\ell} \left| \psi^\ell \right|^2 - \frac{q^\ell}{m^\ell c} A_{\text{ex}} \left| \psi^\ell \right|^2 \right) = 0. \tag{158}
\]

(Note that \( J^\ell \) defined by (157) agrees with the definition of current (24) in Maxwell equations). Equations (158) can be obtained via multiplying (130) by \( \psi^\ell \) and taking imaginary part. Integrating the continuity equation we find that \( \left\| \psi^\ell \right\|^2 = \text{const} \) and we impose the normalization condition (133). The momentum density \( P^\ell \) for the Lagrangian \( \hat{\mathcal{L}}_0 \) in (128) is defined by the formula

\[
P^\ell = \frac{i\chi}{2} \left( \psi^\ell \nabla^\ell \psi^\ell - \psi^* \nabla^\ell \psi^\ell \right). \tag{159}
\]

Note that so defined momentum density \( P^\ell \) is related with the current \( J^\ell \) in (157) by the formula

\[
P^\ell (t, x) = \frac{m^\ell}{q^\ell} J^\ell (t, x). \tag{160}
\]

We introduce the total individual momenta \( P^\ell \) for \( \ell \)-th charge by

\[
P^\ell = \int_{\mathbb{R}^3} P^\ell \, dx, \tag{160}
\]

and obtain the following equations for the total individual momenta

\[
\frac{dP^\ell}{dt} = q^\ell \int_{\mathbb{R}^3} \left[ \left( \sum_{\ell' = \ell} E^{\ell'} + E_{\text{ex}} \right) \left| \psi^\ell \right|^2 + \frac{1}{c} v^\ell \times B_{\text{ex}} \right] \, dx, \tag{161}
\]

where

\[
v^\ell (t, x) = \frac{1}{m^\ell} P^\ell (t, x) = \frac{1}{q^\ell} J^\ell (t, x). \tag{162}
\]

The external EM fields \( E_{\text{ex}}, B_{\text{ex}} \) in (161) corresponding to the potentials \( \varphi_{\text{ex}}, A_{\text{ex}} \) are determined by standard formulas (201). Derivation of (161) is rather elementary. Indeed, in the simplest case where \( A_{\text{ex}} = 0 \) we multiply (130) by \( \nabla \psi^\ell \), take the real part and integrate the result over the entire space using integration by parts. To obtain (161) in more involved general case one can similarly multiply (130) by \( \nabla^\ell \psi^\ell \) and then integrate the result by parts using some vector algebra manipulation.

Let us show now that if the size parameter \( a \) is small compared to the typical scale of variation of EM fields, the charge evolution can be described approximately by Newton equations with Lorentz forces similar to (4). Here we skip technical details. Conditions under which this kind of derivation is justified are given in BabFig3.
We introduce the $\ell$-th charge position $\mathbf{r}^\ell(t)$ and velocity $\mathbf{v}^\ell(t)$ as the following spatial averages:

$$
\mathbf{r}^\ell(t) = \mathbf{r}_a^\ell(t) = \int_{\mathbb{R}^3} \mathbf{x} \left| \psi_a^\ell(t, \mathbf{x}) \right|^2 \, d\mathbf{x}, \quad \mathbf{v}^\ell(t) = \frac{1}{q^\ell} \int_{\mathbb{R}^3} \mathbf{J}^\ell(t, \mathbf{x}) \, d\mathbf{x},
$$

(163)

where the current density $\mathbf{J}^\ell$ is defined by (157). We show below that a combination of the continuity equation (158) with the momentum evolution equations (161) imply the following remarkable property: the positions $\mathbf{r}^\ell(t)$ satisfy with a high accuracy Newton’s equations of motion for the system of $N$ point charges if the size parameter $a$ is small.

Multiplying continuity equation (158) by $\mathbf{x}$ and integrating we find the following identities

$$
\frac{d\mathbf{r}^\ell(t)}{dt} = \int_{\mathbb{R}^3} \mathbf{x} \partial_t \left| \psi^\ell \right|^2 \, d\mathbf{x} = - \frac{1}{q^\ell} \int_{\mathbb{R}^3} \mathbf{x} \nabla \cdot \mathbf{J}^\ell \, d\mathbf{x} = \frac{1}{q^\ell} \int_{\mathbb{R}^3} \mathbf{J}^\ell d\mathbf{x} = \mathbf{v}^\ell(t),
$$

(164)

showing the positions and velocities defined by formulas (163) are related exactly as in the point charge mechanics. Then integrating (159) we obtain the following kinematic representation for the total momentum

$$
P^\ell(t) = \frac{m^\ell}{q^\ell} \int_{\mathbb{R}^3} \mathbf{J}^\ell(t, \mathbf{x}) \, d\mathbf{x} = m^\ell \mathbf{v}^\ell(t),
$$

(165)

which also is exactly the same as for point charges mechanics. Relations (164) and (165) yield

$$
m^\ell \frac{d^2\mathbf{r}^\ell(t)}{dt^2} = m^\ell \frac{d\mathbf{v}^\ell(t)}{dt} = \frac{dP^\ell}{dt},
$$

(166)

and we obtain from (161) the following system of equations of motion for $N$ charges:

$$
m^\ell \frac{d^2\mathbf{r}^\ell(t)}{dt^2} = q^\ell \int_{\mathbb{R}^3} \left[ \left( \sum_{\ell' \neq \ell} \mathbf{E}^\ell + \mathbf{E}_{\text{ex}} \right) \left| \psi_{\ell'} \right|^2 + \frac{1}{c} \mathbf{v}^\ell \times \mathbf{B}_{\text{ex}} \right] \, d\mathbf{x}, \quad \ell = 1, \ldots, N,
$$

(167)

where $\mathbf{E}^\ell(t, \mathbf{x}) = - \nabla \varphi^\ell(t, \mathbf{x})$, $\mathbf{E}_{\text{ex}}$ and $\mathbf{B}_{\text{ex}}$ are defined by (201).

The derivation of the above system is analogous to the well known in quantum mechanics Ehrenfest Theorem, [Schiff, Sections 7, 23] and [Bia]. Now we give a formal derivation of the Newton’s law of motion for charge centers.

Let us suppose that for every $\ell$-th charge density $\left| \psi^\ell \right|^2$ and the corresponding current density $\mathbf{J}^\ell$ are localized in $a$-vicinity of the position $\mathbf{r}^\ell(t)$, and that $\left| \mathbf{r}^\ell(t) - \mathbf{r}^\ell(t) \right| \geq \gamma > 0$ with $\gamma$ independent on $a$ on time interval $[0, T]$. Then if $a \to 0$ we get

$$
\left| \psi^\ell \right|^2(t, \mathbf{x}) \to \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right), \quad \mathbf{v}^\ell(t, \mathbf{x}) = \mathbf{J}^\ell/q^\ell \to \mathbf{v}^\ell(t) \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right),
$$

(168)

where the coefficients before the Dirac delta-functions are determined by the charge normalization conditions (133) and relations (163). Using potential representations (132) we infer from (168) the convergence of the potentials $\varphi^\ell$ to the corresponding Coulomb’s potentials, namely

$$
\varphi^\ell(t, \mathbf{x}) \to \varphi_0^\ell(t, \mathbf{x}) = \frac{q^\ell}{\left| \mathbf{x} - \mathbf{r}^\ell \right|}, \quad \nabla_r \varphi^\ell(t, \mathbf{x}) \to \frac{q^\ell \left( \mathbf{x} - \mathbf{r}^\ell \right)}{\left| \mathbf{x} - \mathbf{r}^\ell \right|^3} \text{ as } a \to 0.
$$

(169)

Hence, if we pass to the limit as $a \to 0$, we can recast the equations of motion (167) as the system

$$
m^\ell \frac{d^2\mathbf{r}^\ell}{dt^2} = \mathbf{f}^\ell + \epsilon_0,
$$

(170)
where
\[ f^\ell = \sum_{\ell' \neq \ell} q^\ell q^{\ell'} E_0 + q^\ell E_{\text{ex}}(r^\ell) + \frac{1}{c} v^\ell \times B_{\text{ex}}(r^\ell), \quad \ell = 1, \ldots, N, \]
and \( \epsilon_0 \to 0 \) as \( a \to 0 \). Notice that the terms \( f^\ell \) in equations (170) coincide with the Lorentz forces and we see that the limit equations of motion obtained from (170) coincide with Newton’s equations of motion for point charges interacting via the Coulomb forces and with the external EM field via corresponding Lorentz forces, namely
\[ m^\ell \frac{d^2 r^\ell}{dt^2} = -\sum_{\ell' \neq \ell} q^\ell q^{\ell'} \frac{(r^{\ell'} - r^\ell)}{|r^{\ell'} - r^\ell|^3} + q^\ell E_{\text{ex}}(r^\ell) + \frac{1}{c} v^\ell \times B_{\text{ex}}(r^\ell), \quad \ell = 1, \ldots, N. \quad (171) \]

Note that we essentially use the fact that the nonlinearity \( G''_a \), which, according to (39), singularly depends on \( a \) as \( a \to 0 \), does not enter the system (167) explicitly. For mathematical details of the above derivation see [BabFig3].

**Remark 5** In special case of one particle in an external EM field which depends only on time we presented explicit wave-corpuscle solution of (130), (131) given by formula (137). Note that if the EM fields have a general form of spatial dependence, as they do in the case of multiple particles we cannot find write solutions explicitly. Nevertheless, if the EM fields do not vary fast in space, which is the case of a system of charges in the regime of remote interaction considered in this section, the wave-corpuscle solutions of the form (137) produce approximate solutions to the field equations with high accuracy. The dynamics of charge centers \( r^\ell \) and their velocities \( v^\ell \) in general case is given by the Newton’s equations (171). Since the spatial extent of \( \psi \) is small (of order \( a \) ) and we can linearize potentials of external fields acting on the charge near the center of the wave-corpuscle, the substitution of wave-corpuscle into field equations produces relative discrepancy of order
\[ O \left( \frac{a^2}{R^2} + \frac{a |v|}{R c} \right) \]
where \( R \) is the typical spatial scale of variation of external fields near the charge trajectory. Consequently, we may expect the wave-corpuscle form of localized solutions to be preserved for long times. We would like to stress that even for very small \( a \) the moving charges are not reduced just to points and the oscillatory de Broglie wave factors \( e^{iSL/\chi} \) in (137) remain to be significant. More detailed studies of approximate wave-corpuscle solutions in similar situation in the framework of the WCM can be found in [BabFig1], [BabFig2].

### 5 Hydrogen atom model

In this section we provide a detailed sketch of our hydrogen atom (HA) model with an intention to write a separate detailed paper on this subject. In this model \( \psi_\ell \) are spinless, therefore one cannot expect spin related effects to be modeled here (though it is quite clear that extensions of this model to multi-component \( \psi_\ell \) which can possess spin are possible, this is a subject of future research).

To model the hydrogen atom (HA) we set \( N = 2 \) in the non-relativistic system (130), (131) where the indices \( \ell \) take two values \( \ell = 1 \), for electron, and \( \ell = 2 \), for proton, and the charges values \( q_1 = -q = q_2 \). The electric fields in the resting hydrogen atom have to be
time-independent, hence $|\psi_{\ell}|^2$ in (131) must be time-independent too. Therefore we assume that only phase factors depend on time and consider the multi-harmonic solutions of this system, namely solutions of the form

$$\psi_{\ell} (t, \mathbf{x}) = e^{-i\omega_{\ell} t} \psi_{\ell} (\mathbf{x}), \quad \varphi_{\ell} (t, \mathbf{x}) = \varphi_{\ell} (\mathbf{x}), \quad \ell = 1, 2. \quad (172)$$

Plugging the expressions (172) in equations (130), (131) we find that the functions $\psi_{\ell} (\mathbf{x})$ satisfy the following nonlinear eigenvalue problem

$$\chi \omega_{\ell} \psi_{\ell} + \frac{\chi^2}{2m_{\ell}} \nabla^2 \psi_{\ell} - q_{\ell}^2 \varphi_{\ell} \psi_{\ell} = \frac{\chi^2}{2m_{\ell}} G'_{\ell} (|\psi_{\ell}|^2) \psi_{\ell}, \quad (173)$$

where, in accordance with (125),

$$\varphi_{\neq 1} = \varphi_2, \quad \varphi_{\neq 2} = \varphi_1 \quad \frac{1}{4\pi} \nabla^2 \varphi_{\ell} = -q_{\ell}^2 |\psi_{\ell}|^2. \quad (174)$$

Let us introduce

$$\Phi_{\ell} = \frac{\varphi_\ell}{a_{\ell}}, \quad a_{\ell} = \frac{\chi^2}{q_{\ell}^2 m_{\ell}}, \quad \ell = 1, 2, \quad (175)$$

where the quantity $a_1$ turns into the Bohr radius if $\chi$ equals to the Planck constant $\hbar$, and $m_1, q$ are the electron mass and charge respectively. Using (175) we rewrite the system (173), (174) as the following nonlinear eigenvalue problem

$$\frac{\chi}{q_{\ell}^2} \omega_{\ell} \psi_1 + \frac{a_{\ell}}{2} \nabla^2 \psi_1 + \Phi_1 \psi_1 = \frac{a_{\ell}}{2} G_1' (|\psi_1|^2) \psi_1, \quad (176)$$

$$\frac{\chi}{q_{\ell}^2} \omega_{\ell} \psi_2 + \frac{a_{\ell}}{2} \nabla^2 \psi_2 + \Phi_1 \psi_2 = \frac{a_{\ell}}{2} G_2' (|\psi_2|^2) \psi_2, \quad (177)$$

$$\nabla^2 \Phi_1 = -4\pi |\psi_1|^2, \quad \nabla^2 \Phi_2 = -4\pi |\psi_2|^2. \quad (178)$$

where $\psi_1$ and $\psi_2$ are respectively the wave functions for the electron and the proton with

$$\|\psi_1\| = 1, \quad \|\psi_2\| = 1, \quad (179)$$

according the charge normalization condition (133). The nonlinearities $G_1', G_2'$ are assumed to be logarithmic as defined by (156) where the size parameter $a = a^\ell$ is different for electron and proton. One can similarly consider other nonlinearities but in this paper we stay with the logarithmic ones.

Let us introduce an energy functional $\mathcal{E} (\psi_1, \psi_2)$ associated with the energy derived from the Lagrangian (128) by the following formula

$$\mathcal{E} (\psi_1, \psi_2) = \mathcal{E}_1 (\psi_1, \psi_2) + \mathcal{E}_2 (\psi_1, \psi_2), \quad (180)$$

where

$$\mathcal{E}_1 = \frac{q_{\ell}^2 a_{\ell}}{2} \int [\nabla |\psi_1|^2 + G_1 (|\psi_1|^2)] \, \mathbf{x} - 2\pi q_{\ell}^2 \int \left[ (-\nabla^{-1}) |\psi_2|^2 \right] |\psi_1|^2 \, \mathbf{x},$$

$$\mathcal{E}_2 = \frac{q_{\ell}^2 a_{\ell}}{2} \int [\nabla |\psi_2|^2 + G_2 (|\psi_2|^2)] \, \mathbf{x} - 2\pi q_{\ell}^2 \int \left[ (-\nabla^{-1}) |\psi_1|^2 \right] |\psi_2|^2 \, \mathbf{x},$$

and $(-\nabla^{-1}) |\psi_1|^2$ is defined by the Green’s formula (132). Observe then that equations (176), (177), (178) are the Euler equations for critical points of the functional $\mathcal{E}$ under the normalization constraint (179), if we set in (178)

$$\Phi_1 = 4\pi \left[ (-\nabla^{-1}) \psi_1^2 \right], \quad \Phi_2 = 4\pi \left[ (-\nabla^{-1}) \psi_2^2 \right].$$
and the frequencies $\omega_1, \omega_2$ are the corresponding Lagrange multipliers. Importantly, it turns out that for the logarithmic as in (155) nonlinearities $G_1, G_2$ the frequencies $\omega_1, \omega_2$ and critical values of $E_1, E_2$ satisfy Planck-Einstein formula $E = h \omega$ (see [BabFig3] for details and also [Bia] where a relation between the Planck-Einstein formula and the logarithmic nonlinearity was discovered in a different setting).

The problem of finding critical values for the energy functional $E$ defined by (180) can be reduced approximately to a simpler problem for a single wave function in a way similar to the Born-Oppenheimer approximation in the quantum mechanics. To that we introduce a change of variables

$$x = a_\ell y_\ell, \quad \ell = 1, 2,$$

where $a_\ell$ are defined by (175), and rescale the fields as follows:

$$\Phi_\ell (x) = \frac{\phi_\ell (y_\ell)}{a_\ell^{3/2}}, \quad \psi_\ell (x) = \frac{\Psi_\ell (y_\ell)}{a_\ell^{3/2}}, \quad \ell = 1, 2.$$

Then the equations (176), (177) turn into the following system

$$\frac{\chi}{q^2} \omega_1 \Psi_1 + \frac{1}{2a_1} \nabla^2 \Psi_1 + \frac{1}{a_2} \phi_2 \left( \frac{a_1}{a_2} y_1 \right) \Psi_1 = \frac{1}{2a_1} G'_1 (|\psi_1|^2) \Psi_1,$$

$$\frac{\chi}{q^2} \omega_2 \Psi_2 + \frac{1}{2a_2} \nabla^2 \Psi_2 + \frac{1}{a_1} \phi_1 \left( \frac{a_2}{a_1} y_2 \right) \Psi_2 = \frac{1}{2a_2} G'_2 (|\Psi_2|^2) \Psi_2,$$

$$\nabla^2 \phi_1 = -4\pi |\Psi_1|^2, \quad \nabla^2 \phi_2 = -4\pi |\Psi_2|^2,$$

where not to complicate notations we use the same letter $G$ in rescaled variables. Recall now that the electron/proton mass ratio is small, that is

$$b = \frac{m_1}{m_2} = \frac{a_2}{a_1} \simeq \frac{1}{1800} \ll 1.$$

Then we recast (183), (184) in the following system

$$\frac{\chi a_1}{q^2} \omega_1 \Psi_1 + \frac{1}{2} \nabla^2 \Psi_1 + \frac{1}{b} \phi_2 \left( \frac{1}{b} y \right) \Psi_1 = \frac{1}{2} G'_1 (|\psi_1|^2) \Psi_1,$$

$$\frac{\chi a_2}{q^2} \omega_2 \Psi_2 + \frac{1}{2} \nabla^2 \Psi_2 + b\phi_1 (by) \Psi_2 = \frac{1}{2} G'_2 (|\Psi_2|^2) \Psi_2.$$

Notice that equation (188) for the proton wave function $\Psi_2$ depends on $\Psi_1$ through the potential $b\phi_1 (by)$ which is small since $b$ is small. On the other hand, the electron wave function $\Psi_1$ depends on $\Psi_2$ through the potential $\frac{1}{b} \phi_2 \left( \frac{1}{b} y \right)$, and if we look for radial solutions we can use the formula

$$\frac{1}{b} \phi_2 \left( \frac{1}{b} r \right) = \frac{1}{r} - \frac{4\pi}{r} \int_{r/b}^{\infty} (r_1 - r/b) r_1 |\Psi_2 (r_1)|^2 \, dr_1.$$

Restricting ourselves to lower energy levels for $E (\psi_1, \psi_2)$ we can conclude using (189) that $\frac{1}{b} \phi_2 \left( \frac{1}{b} r \right)$ can be replaced by the Coulomb potential $\frac{1}{r}$ with an error of order $b^2$ (see [BabFig3]).
for details). With that in mind we introduce the following energy functional with the Coulomb potential for a single wave function:

\[ \mathcal{E}_{Cb}(\Psi_1) = \frac{q^2}{a_1} \int_{\mathbb{R}^3} \left[ \frac{1}{2} |\nabla \Psi_1|^2 + \frac{1}{2} G_1 (|\Psi_1|^2) - \frac{1}{|y|} |\Psi_1|^2 \right] \, dy. \]  

(190)

Then with a small and controlled error we substitute the original problem of finding frequencies \( \omega_1 \) based on critical points of \( \mathcal{E}(\psi_1, \psi_2) \) for lower energy levels with a simpler problem of finding critical points, lower critical levels and corresponding frequencies \( \omega_1 \) for the energy functional \( \mathcal{E}_{Cb}(\Psi_1) \) subjected to the constraint \( \| \Psi_1 \| = 1 \). The corresponding nonlinear eigenvalue problem for the electron wave function \( \Psi_1 \) and dimensionless spectral parameter \( \omega = \frac{\chi a_1}{q^2} \omega_1 \) is

\[ \omega \Psi_1 + \frac{1}{2} \nabla^2 \Psi_1 + \frac{1}{|y|} \Psi_1 = \frac{1}{2} G_1' (|\Psi_1|^2) \Psi_1. \]  

(191)

As we have already mentioned a similar reduction to a single Schrodinger equation with the Coulomb potential is made in the quantum mechanics via the Born-Oppenheimer approximation.

Let us exploit the dependence of the nonlinearity \( G_1' = G_1'_{1a} \) on the small parameter \( \kappa = \frac{a_1}{a} \) which is the the ratio of the electron Bohr radius \( a_1 \) to the size parameter \( a \). If \( \kappa \) is small then the nonlinearity \( G_1'_{1a} (s) = \kappa^2 G_1' (\kappa^{-3} s) \) is small and plays a role of a small perturbation in the eigenvalue problem \([191]\). A detailed analysis shows that lower energy levels of the functional \( \mathcal{E}_{Cb}(\Psi_1) \) are arbitrary close to the energy levels of the Schrodinger operator for HA provided that \( \kappa = \frac{a_1}{a} \) is sufficiently small. Consequently, based on estimates obtained in \([BabFig3]\) we can conclude that \( n \)-th lower frequency \( \omega_{1n} \) for solution of \([176]\), \([177]\), \([178]\) are given by the following approximate formula

\[ \chi \omega_{1n} = - \frac{1}{n^2} \frac{q^2}{2a_1} \left[ 1 + O \left( b^2 + \left( \frac{a_1}{a} \right)^2 |ln \left( \frac{a_1}{a} \right)| \right) \right], \quad n = 1, 2, ... \]  

(192)

The correction term \( O \left( b^2 + \left( \frac{a_1}{a} \right)^2 |ln \left( \frac{a_1}{a} \right)| \right) \) in \([192]\) is small if \( b \) given by \([186]\) and \( \frac{a_1}{a} \) are small. Observe that differences of energy levels of the nonlinear eigenvalue problem are very close for the same in the Rydberg formula with relative error of order \( 10^{-4} \) if \( \frac{a_1}{a} \) is of order \( 10^{-2} \). Hence, if we assume that the size \( a \) of a free electron is 100 larger than the Bohr radius, then the introduced here hydrogen atom model is a good quantitative agreement with the hydrogen spectroscopic data. We think that it is quite reasonable to assume that a free electron has much larger size then an electron bound in a hydrogen atom where it is naturally contracted by the electric force of the positively charged proton.

Remark 6 If we use the provisional HA model from \([BabFig2]\) we still obtain discrete energy levels, but as \( \frac{a_1}{a} \to 0 \) the limiting linear eigenvalue problem involves a potential \( -\frac{q^2}{|y|} + q \phi(y) \) where in addition to the Coulomb potential there is a term \( q \phi(y) \) due the electron EM self-interaction. So, if there is EM self-interaction the limiting eigenvalue problem as \( \frac{a_1}{a} \to 0 \) does not turn into the same for the linear Schrodinger operator for the HA and consequently the energy levels do not converge to the known expressions for HA as \( \frac{a_1}{a} \to 0 \).

Now we briefly compare the above non-relativistic treatment of the HA with the treatment in the framework of the full relativistic version of our model. We start directly from the
We consider the Maxwell equations for the EM fields and their covariant form following to [Jac, Section 11.9], [LanLif EM, Sections 23, 30], [Gri, Sections 7.4, 11.2], in CGS units

\[ \nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \cdot \mathbf{B} = 0, \]

as follows:

\[ \psi^\ell(t, \mathbf{x}) = e^{-i(\omega \tau + \omega_0)t} \psi^\ell(\mathbf{x}), \quad \varphi^\ell(t, \mathbf{x}) = \varphi^\ell(\mathbf{x}), \]

where \( \ell = 1, 2, \) \( \omega_0 = \frac{m c^2}{\chi} = c \kappa_0 \ell. \) We arrive then at a system similar to (173), (174)

\[ \frac{1}{c^2} \left( \frac{m c^2}{\chi} + \omega \ell - \frac{q^\ell \varphi^\ell}{\chi} \right)^2 \psi^\ell + \nabla^2 \psi^\ell - G^{\ell \ell'} (\psi^\ell \psi^{\ell'}) \psi^\ell - \frac{1}{c^2} \left( \frac{m c^2}{\chi} \right)^2 \psi^\ell = 0. \]  

Based on smallness of electron/proton mass ratio we similarly to the non-relativistic case arrive to the following eigenvalue problem for electron density which is a relativistic version of (191):

\[ \left( m_1 c^2 + \chi \omega_1 + \frac{q^2}{|y|} \right)^2 \psi_1 + c^2 \chi^2 \psi_1 - c^2 \chi^2 \psi_1 - c^2 \chi^2 \psi_1 - m_1^2 c^4 \psi_1 = 0. \]

If the ratio \( \kappa = \frac{a_p}{a_e} \) is small, the nonlinearity \( G^\ell (s) = \kappa^2 G^\ell (s) \) can be treated as a small perturbation, and the linear part of (193) essentially determines the lower energy levels. Note that if we set \( \chi = \hbar \) the linear part of (193) coincides with relativistic Schrodinger equation (see [Schiff] p.309), which has the form

\[ ( -\hbar^2 c^2 \nabla^2 + m_1^2 c^4 ) u = \left( E + \frac{q}{|x|} \right)^2 u \]

with energy level \( E = mc^2 + \chi \omega_1. \) According to [Schiff] the energy levels of the linear relativistic Schrödinger equation in a contrast to the non-relativistic hydrogen Schrödinger equation have a fine structure, the fine structure energy levels are given by Sommerfeld’s formula and the relative scale of the fine structure is controlled by \( \alpha = \frac{2}{3} \approx \frac{1}{137} \). This shows that at atomic scales in our relativistic model relativistic effects are present even in the case of zero velocities if the square of Sommerfeld’s fine structure constant is not assumed to be negligible.

**Remark 7** In our treatment of charges in Section 4 at macroscopic scales we assume the electron size \( a \) to be very small and in this section we assume \( \kappa = a_1/a \) to be very small. Since there is huge gap of scales between the macroscopic and atomic scales there is no contradiction if we take into account the small value of the Bohr radius \( a_1 \sim 5.3 \times 10^{-11} \text{ m} \) compared with the scale of variation of EM fields. Note that the error \( \epsilon_0 \) of approximation by Newtonian trajectory in (170) is of order \( a_1^2 / R_{\text{macr}}^2 \ll 1 \) where \( R_{\text{macr}} \) is the scale of spatial variation of EM fields which act on a charge. In the treatment of HA in this section we assume \( \kappa^2 = a_1^2/a^2 \ll 1. \) Taking \( a \sim 10^2 a_1 \) we arrive at the restriction \( R_{\text{macr}} \gg 5.3 \times 10^{-9} \text{m} \) which is an estimate of the scale of spatial variation of EM fields for which the Newton’s equations with Lorentz force holds with a good accuracy.

### 6 Appendix

#### 6.1 Classical electrodynamics

We consider the Maxwell equations for the EM fields and their covariant form following to [Jac, Section 11.9], [LanLif EM, Sections 23, 30], [Gri, Sections 7.4, 11.2], in CGS units

\[ \nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \cdot \mathbf{B} = 0, \]
\[
\n\nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} = \frac{4\pi}{c} \mathbf{J}.
\]

To represent Maxwell equations in a manifestly Lorentz invariant form it is common to introduce a four-vector potential \( A^\mu \) and a four-vector current density \( J^\nu \):

\[
A^\mu = (\phi, \mathbf{A}), \quad J^\mu = (c \rho, \mathbf{J}),
\]

\[
\partial_\mu \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, -\nabla \right),
\]

and, then, an antisymmetric second-rank tensor, the "field strength tensor,

\[
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,
\]

so that

\[
F^{\mu\nu} = \begin{bmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{bmatrix},
\]

\[
F_{\mu\nu} = \begin{bmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & B_2 & B_1 & 0
\end{bmatrix},
\]

and

\[
\mathbf{E} = -\nabla \phi - \frac{1}{c} \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}.
\]

Then the two inhomogeneous equations and the two homogeneous equations from the four Maxwell equations (196) take respectively the form

\[
\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu,
\]

\[
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \quad \alpha, \beta, \gamma = 0, 1, 2, 3.
\]

It follows from the asymmetry of \( F^{\mu\nu} \), the Maxwell equation (202) and (198)-(199) that the four-vector current \( J^\mu \) must satisfy the continuity equation

\[
\partial_\mu J^\mu = 0 \text{ or } \partial_t \rho + \nabla \cdot \mathbf{J} = 0.
\]

The Maxwell equations (202) turn into the following equations for the four-vector potential \( A^\mu \)

\[
\Box A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu,
\]

where

\[
\Box = \partial_\mu \partial^\mu = \frac{1}{c^2} \partial_t^2 - \nabla^2 \text{ (d’Alembertian operator).}
\]

The EM field Lagrangian is, [Jac, Section 12.7], [Bar, Section IV.1]

\[
L_{\text{em}}(A^\mu) = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_{\mu} A^\mu,
\]

where \( J_\mu \) is an external (prescribed) current. Using (200), (201) and (198) we can recast (207) as

\[
L_{\text{em}}(A^\mu) = \frac{1}{8\pi} \left( \mathbf{E}^2 - \mathbf{B}^2 \right) - \rho \phi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A}
\]

\[
= \frac{1}{8\pi} \left[ \left( \nabla \phi + \frac{1}{c} \partial_t \mathbf{A} \right)^2 - (\nabla \times \mathbf{A})^2 \right] - \rho \phi + \frac{1}{c} \mathbf{J} \cdot \mathbf{A}.
\]

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In particular, if there are no sources the above Lagrangians turn into

\[ L_{em} (A) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi} (E^2 - B^2) = \]

\[ = \frac{1}{8\pi} \left[ \left( \nabla \varphi + \frac{1}{c} \partial_t A \right)^2 - (\nabla \times A)^2 \right]. \tag{209} \]

The canonical energy-momentum (stress-, power-momentum) tensor \( \hat{\Theta}^{\mu\nu} \) for the EM field is as follows, \[ \text{Jac}, (12.104), \text{Bar}, \text{Section III.4.D} \]

\[ \hat{\Theta}^{\mu\nu} = -\frac{F^{\gamma\tau A_\gamma}}{4\pi} + g^{\mu\nu} \frac{F_{\xi\gamma} F_{\xi\gamma}}{16\pi}, \tag{210} \]

or, in particular, for \( i, j = 1, 2, 3 \)

\[ \hat{\Theta}^{00} = -\frac{E^2 - B^2}{8\pi} \]

\[ + \rho \varphi - \frac{1}{c} J \cdot A - \frac{\partial_0 A \cdot E}{4\pi}, \tag{211} \]

\[ \hat{\Theta}^{0i} = \frac{\partial_i A \cdot E}{4\pi}, \quad \hat{\Theta}^{i0} = -\frac{E_i \partial_0 \varphi}{4\pi} + \frac{(B \times \partial_0 A)_i}{4\pi}, \]

\[ \hat{\Theta}^{ij} = -\frac{E_i \partial_j \varphi}{4\pi} + \frac{(B \times \partial_j A)_i}{4\pi} + \frac{E^2 - B^2}{8\pi} - \rho \varphi - \frac{1}{c} J \cdot A, \]

whereas the symmetric one \( \Theta^{\alpha\beta} \) for the EM field is, \[ \text{Jac}, \text{Section 12.10}, (12.113), \text{Bar}, \text{Section III.3} \]

\[ \Theta^{\alpha\beta} = \frac{1}{4\pi} \left( g^{\alpha\mu} F_{\mu\nu} F^{\nu\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right), \tag{212} \]

implying the following formulas for the field energy density \( w \), the momentum density \( g \) and the Maxwell stress tensor \( \tau_{ij} \):

\[ w = \Theta^{00} = \frac{E^2 + B^2}{8\pi}, \quad c g_i = \Theta^{0i} = \Theta^{i0} = \frac{E \times B}{4\pi}, \tag{213} \]

\[ \Theta^{ij} = -\frac{1}{4\pi} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2) \right], \tag{214} \]

\[ \Theta^{\alpha\beta} = \begin{bmatrix} w & c g \\ c g & -\tau_{ij} \end{bmatrix}, \quad \Theta^{\alpha\beta} = \begin{bmatrix} w & -c g \\ -c g & -\tau_{ij} \end{bmatrix}, \tag{215} \]

\[ \Theta^{\alpha\beta} = \begin{bmatrix} w & -c g \\ c g & -\tau_{ij} \end{bmatrix}, \quad \Theta^{\alpha\beta} = \begin{bmatrix} w & c g \\ -c g & -\tau_{ij} \end{bmatrix}. \]

Note that in the special case when the vector potential \( A \) vanishes and the scalar potential \( \varphi \) does not depend on time using the expressions (201) we get the following representation for the canonical energy density defined by (211)

\[ \hat{\Theta}^{00} = -\frac{(\nabla \varphi)^2}{8\pi} + \rho \varphi \text{ for } A = 0 \text{ and } \partial_0 \varphi = 0, \tag{216} \]

whereas \( \Theta^{00} = \frac{(\nabla \varphi)^2}{8\pi}. \)
It is instructive to observe a substantial difference between the above expressions $\Theta^{00}$, which is the Hamiltonian density of the EM field, and the energy density $\Theta^{00}$ defined by (213).

If there are no external currents the with differential conservation laws takes the form

$$\partial_{\alpha} \Theta^{\alpha\beta} = 0,$$

and, in particular, the energy conservation law

$$0 = \partial_{\alpha} \Theta^{\alpha\beta} = \frac{1}{c} \left( \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} \right),$$

where $w$ is the energy density, and

$$\mathbf{S} = c^2 \mathbf{g} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

is the Poynting vector.

In the presence of external currents the conservation laws take the form, \cite{Jac, Section 12.10]

$$\partial_{\alpha} \Theta^{\alpha\beta} = -f^\beta,$$

and the time and space components of the equations (219) are the conservation of energy $w$ and momentum $g$ which can be recast

$$\frac{1}{c} \left( \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} \right) = -\frac{1}{c} \mathbf{J} \cdot \mathbf{E},$$

$$\frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \tau_{ij} = - \left[ \rho E_i + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right].$$

The energy conservation law (220) is often called the Poynting’s theorem, \cite{Jac, Section 6.7]. The 4-vector $f^\beta$ in the conservation law (219) is known as the Lorentz force density

$$f^\beta = \frac{1}{c} F^{\beta\nu} J_\nu = \left( \frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right).$$

### 6.2 Potentials and fields for prescribed currents

In this section we describe EM fields $F^{\mu\nu}$ arising from prescribed (external) currents $J^\nu$ following mostly to \cite{Jac, Section 6.4, 6.5 and 12.11]. Namely, the EM fields $F^{\mu\nu}$ satisfy the inhomogeneous Maxwell equation

$$\partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

which take the following form for the potentials $A^\nu$

$$\Box A^\nu - \partial^\nu \partial_{\mu} A^\mu = \frac{4\pi}{c} J^\nu.$$

If the potentials satisfy the Lorentz condition, $\partial_{\mu} A^\mu = 0$, they are then solutions of the four-dimensional wave equation,

$$\Box A^\nu = \frac{4\pi}{c} J^\nu.$$
The solution of the inhomogeneous wave equation (225) is accomplished by finding a Green function $G(x, x')$ for the equation

$$\Box G(z) = \delta^{(4)}(z), \quad G(x, x') = G(x - x'),$$

(226)

where $\delta^{(4)}(z) = \delta(z_0) \delta(z)$ is a four-dimensional delta function. One can introduce then the so-called retarded or causal Green function solving the above equation (226), namely

$$G^{(+)}(x - x') = \frac{\theta(x_0 - x'_0) \delta(x_0 - x'_0 - R)}{4\pi R}, \quad R = |x - x'|,$$

(227)

where $\theta(x_0)$ is the Heaviside step function. The name causal or retarded is justified by the fact that the source-point time $x'_0$ is always earlier than the observation-point time $x_0$. Similarly one can introduce the advanced Green function

$$G^{(-)}(x - x') = \frac{\theta[-(x_0 - x'_0)] \delta(x_0 - x'_0 + R)}{4\pi R}, \quad R = |x - x'|.$$

(228)

These Green functions can be written in the following covariant form

$$G^{(+)}(x - x') = \frac{1}{2\pi} \theta(x_0 - x'_0) \delta[(x - x')^2],$$

(229)

$$G^{(-)}(x - x') = \frac{1}{2\pi} \theta(x'_0 - x_0) \delta[(x - x')^2],$$

where

$$(x - x')^2 = (x_0 - x'_0)^2 - |x - x'|^2,$$

$$\delta[(x - x')^2] = \frac{1}{2R} [\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)].$$

(230)

The more explicit form of the Green functions $G^{(\pm)}$ in terms of time space variable is

$$G^{(\pm)}(\tau, R) = \frac{1}{R} \delta\left(\tau \mp \frac{R}{c}\right)$$

(231)

where

$$R = |x - x'|, \quad \tau = t - t',$$

(232)

or

$$G^{(\pm)}(t, x; t', x') = \frac{1}{|x - x'|} \delta\left(t - \left[t' \mp \frac{|x - x'|}{c}\right]\right).$$

(233)

The solution to the wave equation (225) can be written in terms of the Green functions

$$A^\nu(x) = A^\nu_{in}(x) + \frac{4\pi}{c} \int G^{(+)}(x - x') J^\nu(x') \, dx'$$

(234)

or

$$A^\nu(x) = A^\nu_{out}(x) + \frac{4\pi}{c} \int G^{(-)}(x - x') J^\nu(x') \, dx'$$

(235)

where $A^\nu_{in}(x)$ and $A^\nu_{out}(x)$ are solutions to the homogeneous wave equation. In (234) the retarded Green function is used. In the limit $x_0 \to -\infty$, the integral over the sources vanishes, assuming the sources are localized in space and time, because of the retarded
nature of the Green function, and \( A_\nu^\nu (x) \) can be interpreted as "incident" or "incoming" potential, specified at \( x_0 \to -\infty \). Similarly, in (235) with the advanced Green function, the homogeneous solution \( A_\nu^{\nu \text{out}} (x) \) is the asymptotic "outgoing" potential, specified at \( x_0 \to +\infty \). The radiation fields are defined as the difference between the "outgoing" and "incoming" fields, and their 4-vector potential is, \[ \text{DirCE} \]

\[
A_\nu^{\nu \text{rad}} (x) = A_\nu^{\nu \text{out}} (x) - A_\nu^{\nu \text{in}} (x) = \frac{4\pi}{c} \int G(x-x') \, J^\nu (x') \, dx', \quad \text{where} \]

\[
G(x-x') = G(+) (x-x') - G(-) (x-x').
\]

More explicit form of the potential \( A_\nu (x) \) solving the inhomogeneous wave equation (225) based on the retarded Green function \( G(+) \) and with \( A_\nu^{\nu \text{in}} (x) = 0 \) is

\[
\varphi(t,x) = \int \left[ \frac{\rho(t',x')}{R} \right]_{\text{ret}} \, d\mathbf{x}', \quad \mathbf{A}(t,x) = \frac{1}{c} \int \left[ \frac{\mathbf{J}(t',x')}{cR} \right]_{\text{ret}} \, d\mathbf{x}', \quad \text{with} \quad R = \mathbf{x} - \mathbf{x}', \quad \text{and} \quad R = |\mathbf{x} - \mathbf{x}'|,
\]

and the symbol \([\cdot]_{\text{ret}}\) means that the quantity in the square brackets is to be evaluated at the retarded time

\[
t' = t_{\text{ret}} = t - \frac{R}{c} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}.
\]

The corresponding to the potentials (237) EM fields can be represented by Jefimenko formulas, \[ \text{Jef, Section 15.7}, \quad \text{Jac, Section 6.5} \]

\[
\mathbf{E}(t,x) = \int \left[ \frac{\rho(t',x')}{R^2} \right]_{\text{ret}} \, d\mathbf{x}' + \int \left[ \frac{\partial_t \rho(t',x')}{cR} \right]_{\text{ret}} \, d\mathbf{x}' - \int \left[ \frac{\partial_t \mathbf{J}(t',x')}{c^2 R} \right]_{\text{ret}} \, d\mathbf{x}', \quad \text{with} \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|},
\]

\[
\mathbf{B}(t,x) = \int \left\{ \left[ \frac{\mathbf{J}(t',x')}{cR^2} \right]_{\text{ret}} + \left[ \frac{\partial_t \mathbf{J}(t',x')}{c^2 R} \right]_{\text{ret}} \right\} \times \mathbf{R} \, d\mathbf{x}',
\]

where \( \hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|} \). An essentially equivalent form of the Jefimenko equation (240) for the electric field \( \mathbf{E} \) is due Panofsky and Phillips, \[ \text{PanPhi, Section 14.3} \]

\[
\mathbf{E}(t,x) = \int \left[ \frac{\rho(t',x')}{R^2} \right]_{\text{ret}} \, d\mathbf{x}' + \int \left[ \frac{\mathbf{J}(t',x')}{cR^2} \right]_{\text{ret}} \, d\mathbf{x}' - \int \left[ \frac{\partial_t \mathbf{J}(t',x')}{c^2 R} \right]_{\text{ret}} \, d\mathbf{x}',
\]

It was pointed out by McDonald in \[ \text{McDo} \] that the combination of equations (241) and (242) has a certain advantage since it "manifestly displays the mutually transverse character of the radiation fields (those that vary as \( 1/R \))". Since the radiation fields \( \mathbf{E}_{\text{rad}} \) and \( \mathbf{B}_{\text{rad}} \)
decay as $1/R$ for large $R$ we can extract them from the expressions (242), (241) for the entire EM fields obtaining

$$E_{\text{rad}}(t, x) = \int \frac{[\partial_t J(t', x')]_{\text{ret}} \times \hat{R}}{c^2 R} \, dx', \quad (243)$$

$$B_{\text{rad}}(t, x) = \int [\partial_t J(t', x')]_{\text{ret}} \times \frac{\hat{R}}{c^2 R} \, dx' \quad (244)$$

### 6.3 Point charge and the Liénard-Wiechert Potential

If the particle is a point charge $q$ whose position and velocity in an inertial frame are respectively $r(t)$ and $v(t) = \partial_t r(t)$ the corresponding charge and current densities in that frame are, [Jac, Section 12.11, (12.138)]

$$\rho(t, x) = q \delta(x - r(t)) \quad \text{and} \quad J(t, x) = q v(t) \delta(x - r(t)). \quad (245)$$

Using the formulas (257) for the charge density and the current as in the equations (245) we obtain the Liénard-Wiechert Potential, [Jac, Section 14.1]

$$\phi(t, x) = \left[ \frac{q}{1 - \beta \cdot \hat{R}} \right]_{\text{ret}}, \quad \mathbf{A}(t, x) = \left[ \frac{q \beta}{1 - \beta \cdot \hat{R}} \right]_{\text{ret}} \quad (246)$$

where

$$\mathbf{R} = x - r(t), \quad R = |x - r(t)|, \quad \hat{R} = \frac{R}{|R|}, \quad \beta = \frac{v(t)}{c}, \quad (247)$$

and the retarded time $t_r = t_r(x, t)$ is defined implicitly by the following equation

$$t_r = t - \frac{|x - r(t_r)|}{c}. \quad (248)$$

Then with the help of the Jefimenko formulas (240), (241) applied for the point charge density and the current (245) one can derive the Heaviside-Feynman formulas (first discovered by Heaviside (1902) and rediscovered by Feynman (1950)), [Heav1], [Heav2, Subsection 510], [Fey, Vol. I, Section 28; Vol II, Section 21], [Jac, Section 6.5], [Jan], [Mona]:

$$\mathbf{E}(t, x) = q \left\{ \left[ \frac{\hat{R}}{R^2} \right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c} \partial_t \left[ \frac{\hat{R}}{R^2} \right]_{\text{ret}} + \frac{1}{c^2} \partial_t^2 \left[ \frac{\hat{R}}{R^2} \right]_{\text{ret}} \right\}, \quad (249)$$

$$\mathbf{B}(t, x) = \frac{q}{c} \left\{ \left[ \frac{\mathbf{v} \times \hat{R}}{|\mathbf{v}|^2 R^2} \right]_{\text{ret}} + \frac{1}{c [R]_{\text{ret}}} \partial_t \left[ \frac{\mathbf{v} \times \hat{R}}{|\mathbf{v}|} \right]_{\text{ret}} \right\} = \quad (250)$$

$$= \frac{q}{c} \left\{ \left[ \frac{\hat{R}}{R} \right]_{\text{ret}} \times \partial_t \left[ \frac{\hat{R}}{R} \right]_{\text{ret}} + \left[ \frac{\hat{R}}{c} \right]_{\text{ret}} \times \partial_t^2 \left[ \frac{\hat{R}}{c} \right]_{\text{ret}} \right\} = \mathbf{E}(t, x) \times \left[ \frac{\hat{R}}{R} \right]_{\text{ret}} \quad (250b)$$
\[ \kappa = 1 - \frac{v \cdot \hat{R}}{c}. \]  

\( \text{(251)} \)

In view of the implicit relations (248) between the retarded time \( t_r \) and the time-space variables \( (x, t) \) it is important to keep in mind that there is evidently a difference between \( \partial_t [\cdot] \) and \( \partial_t [\cdot] \). The Heaviside-Feynman formulas (249), (250) imply the following formulas for the radiation fields of the moving point charge

\[ E_{\text{rad}} (t, x) = \frac{q}{c^2} \partial_t^2 \left[ \frac{\hat{R}}{R^2} \right] \ret, \quad B_{\text{rad}} (t, x) = \frac{q}{c^2} \partial_t \left[ \frac{v \times \hat{R}}{\kappa} \right] \ret. \]  

\( \text{(252)} \)

Another important representation of the EM fields of an arbitrary moving charge is their decomposition into the velocity and acceleration fields, \[ \text{Jac1, Section 14.1} \]

\[ E (t, x) = E_v (t, x) + E_a (t, x), \quad \text{where for } \beta = \frac{v}{c} \]  

\( \text{(253)} \)

\[ E_v (t, x) = q \left[ \frac{\left( \hat{R} - \beta \right) (1 - \beta^2)}{x^2 R^2} \right] \ret, \]  

\( \text{(254)} \)

\[ B (t, x) = \hat{R} \times E (t, x) = B_v (t, x) + B_a (t, x), \]  

\( \text{(255)} \)

\[ B_v (t, x) = \hat{R} \times E_v (t, x), \]  

\[ B_a (t, x) = B_{\text{rad}} (t, x) = \hat{R} \times E_a (t, x). \]

The velocity fields are essentially static fields falling off as \( R^{-2} \), whereas the acceleration fields are typical radiation fields, both \( E_a \) and \( B_a \) being transverse to the radius vector \( R \) and varying as \( R^{-1} \). For low velocities the formulas (253), (251) turn into the following simpler asymptotic expressions

\[ E_{\text{rad}} (t, x) = \frac{q}{c^2} \left[ \hat{R} \times \left( \hat{R} \times \dot{v} \right) \right] \frac{1}{R} \left( 1 + O (|\beta|) \right), \]  

\( \text{(256)} \)

\[ B_{\text{rad}} (t, x) = \frac{q}{c} \left[ \frac{\dot{v} \times \hat{R}}{R} \right] \left( 1 + O (|\beta|) \right), \quad |\beta| \ll 1. \]

\subsection*{6.4 Almost periodic functions and their time-averages}

We provide here very basic information on real and complex valued almost periodic (a.p.) functions following \[ \text{Cor1, Section 3.1} \]. In fact, we consider for simplicity sake a class \( \mathcal{A}_1 (\mathbb{R}, \mathbb{C}) \) of a.p. function of the form

\[ f (t) = \sum_{n=1}^{\infty} f_n e^{-i\omega_n t}, \quad -\infty < t < \infty, \]  

\( \text{(257)} \)
where the exponents \( \omega_s \) are real valued numbers and amplitudes \( f_s \) are complex valued numbers and such that
\[
|f| = \sum_{s=1}^{\infty} |f_s| < \infty. \tag{257}
\]

We refer to the set of exponents \( \Lambda_f = \{ \omega_s \} \) as the Fourier spectrum of the function \( f \) and to the numbers \( f_s \) as its Fourier coefficients. The class \( A_1(\mathbb{R}, \mathbb{C}) \) is a Banach algebra, that is (i) it is linear space and (ii) for any \( f \) and \( g \) in \( A_1(\mathbb{R}, \mathbb{C}) \) the product \( fg \) is in \( A_1(\mathbb{R}, \mathbb{C}) \) as well and \( |fg| \leq |f| |g| \). The derivatives of a.p. \( f(t) \) satisfy
\[
\partial^r_t f(t) = \sum_{s=1}^{\infty} (-i\omega_s)^r f_s e^{-i\omega_st}, \tag{258}
\]
provided
\[
\sum_{s=1}^{\infty} |\omega_s|^r |f_s| < \infty. \tag{259}
\]

Every a.p. function \( f \) in \( A_1(\mathbb{R}, \mathbb{C}) \) is assigned the time average (mean) value \( \langle f \rangle \) defined by
\[
\langle f \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) \, dt. \tag{260}
\]
The mean value has the fundamental property
\[
\langle f(t) e^{i\theta t} \rangle = \begin{cases} f_s & \text{if } \theta = \omega_s \\ 0 & \text{otherwise} \end{cases}. \tag{261}
\]

In some applications it is convenient to view a real a.p. function \( f(t) \) as the real part of a complex valued function \( f(t) \) written in the following form
\[
f(t) = \sum_{\omega \in \Lambda_f} f_\omega e^{-i\omega t}, \tag{262}
\]
where \( f_\omega = \langle f(t) e^{i\omega t} \rangle = \langle f \cos(\omega t) \rangle + i \langle f \sin(\omega t) \rangle \).

The set \( \Lambda_f \) in (262) is at most countable set of non-negative frequencies \( \omega \geq 0 \), and we refer to it as the frequency spectrum of \( f \). Then the corresponding real valued a.p. function has the following representation
\[
f(t) = \text{Re} \{ f(t) \} = \frac{1}{2} \sum_{\omega \in \Lambda_f} (f_\omega e^{-i\omega t} + f_\omega^* e^{i\omega t}). \tag{263}
\]

Evidently, any real a.p. function can be represented in the form (262), (263). Observe that for any a.p. complex valued function \( f \) and \( g \) of the form (262) we have
\[
\langle \text{Re} \{ f(t) \} \text{Re} \{ g(t) \} \rangle = \begin{cases} \frac{1}{2} \sum_{\omega \in \Lambda_f \cap \Lambda_g} \text{Re} \{ f_\omega g_\omega^* \} & \text{if } \Lambda_f \cap \Lambda_g \neq \emptyset, \\ 0 & \text{if } \Lambda_f \cap \Lambda_g = \emptyset \end{cases}. \tag{264}
\]

In particular, if \( f \) and \( g \) are real valued the relation (264) imply
\[
\langle f(t) g(t) \rangle = \begin{cases} \frac{1}{2} \sum_{\omega \in \Lambda_f \cap \Lambda_g} \text{Re} \{ f_\omega g_\omega^* \} & \text{if } \Lambda_f \cap \Lambda_g \neq \emptyset, \\ 0 & \text{if } \Lambda_f \cap \Lambda_g = \emptyset \end{cases}. \tag{265}
\]
The identity (264) readily follows from formula (262). It reads that if the frequency spectra $\Lambda^+_f$ and $\Lambda^+_g$ have no common frequencies the time-average $\langle \text{Re} \{ f \} \text{Re} \{ g \} \rangle$ is identically zero. A particular case of the formula (264) when the frequency spectra of $f$ and $g$ consist of a the same single frequency $\omega$ is customary used in electrodynamics for time harmonic fields, [PanPhin Section 11.2], [Strat Section 2.20].

For any a.p. function $f$ with the frequency spectrum $\Lambda_f$ one can introduce the smallest additive group in the set of all real numbers that contains all frequencies $\omega$ from $\Lambda_f$. Such a smallest group is called the module of $f$ and is denoted by $\text{mod} (f)$, [Cor Section 4.6]. It is easy to see that $\text{mod} (f)$ consists of all real numbers of the form

$$\sum_{j=1}^{s} m_j \omega_j$$

where $m_j$ are integers, $\omega_j \in \Lambda_f$ and $s$ is a natural number.

6.5 Vector Identities

Here is the fist set of commonly used vector identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \quad (266)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}, \quad (267)$$

$$\mathbf{(a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \quad (268)$$

Using the above identities we readily obtain for any vectors $\mathbf{a}, \mathbf{b}$ and any unit vector $\mathbf{u}$

$$\mathbf{u} \times (\mathbf{a} \times \mathbf{u}) = \mathbf{a} - (\mathbf{a} \cdot \mathbf{u}) \mathbf{u}, \quad |\mathbf{u}| = 1, \quad (269)$$

$$\left[ \mathbf{u} \times (\mathbf{a} \times \mathbf{u}) \right] \times (\mathbf{u} \times \mathbf{b}) = [(\mathbf{a} \times \mathbf{u}) \times \mathbf{u}] \times (\mathbf{b} \times \mathbf{u}) = \left[ (\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{b}) \right] \mathbf{u} = \left[(\mathbf{a} \times \mathbf{u}) \cdot (\mathbf{b} \times \mathbf{u}) \right] \mathbf{u}, \quad |\mathbf{u}| = 1. \quad (270)$$

$$\frac{1}{4\pi} \int_{|x|=1} \left[ \mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \hat{x})(\hat{x} \cdot \mathbf{b}) \right] d\sigma = \frac{2}{3} \mathbf{a} \cdot \mathbf{b}, \quad \hat{x} = \frac{x}{|x|}. \quad (271)$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}). \quad (272)$$

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