Maximal Summability Operators On the Dyadic Hardy Spaces

Ushangi Goginava, Salem Ben Said

Abstract. It is proved that the maximal operators of subsequences of Nörlund logarithmic means and Cesàro means with varying parameters of Walsh-Fourier series is bounded from the dyadic Hardy spaces $H^p$ to $L^p$. This implies an almost everywhere convergence for the subsequences of the summability means.

1. Walsh System

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $I := [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{l}{2^k}$ for some $l, p \in \mathbb{N}, \ 0 \leq p < 2^k$. By a dyadic interval in $I$ we mean one of the form $\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}, 0 \leq l < 2^k$. Denote $I_n := [0, 2^{-n}), I_n(x) := x + I_n$. For $0 < n \in \mathbb{N}$ denote by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. The $\sigma$-algebra generated by the dyadic intervals $\{I_n(x) : x \in I\}$ will be denoted by $\mathcal{A}_n (n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or $1$ and if $x$ is a dyadic rational number we choose the expansion which terminate in $0$’s.

Denote the dyadic expansion of $n \in \mathbb{N}$ by

$$n = \sum_{j=0}^{\infty} \varepsilon_j (n) 2^j, \varepsilon_j (n) = 0, 1.$$ 

Denote by $+$ the logical addition on $I$. That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$ 

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Define the binary operator \( \oplus : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) by
\[
k \oplus n = \sum_{i=0}^{\infty} |\varepsilon_i (k) - \varepsilon_i (n)| 2^i.
\]
(1)

It is well-known (see, e.g. [18], p. 5) that
\[
w_{m,n} (x) = w_m (x) w_n (x), \quad x \in [0, 1), n, m \in \mathbb{N}.
\]
(2)

The Rademacher system is defined by
\[
\rho_n (x) := (-1)^{n x} \quad (x \in \mathbb{I}, n \in \mathbb{N}).
\]

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:
\[
w_n (x) := \prod_{k=0}^{\infty} (\rho_k (x))^{n_k} = (-1)^{\sum_{k=0}^{\infty} n_k x_k} \quad (x \in \mathbb{I}, n \in \mathbb{N}).
\]

The Walsh-Dirichlet kernel is defined by
\[
D_n (x) = \sum_{k=0}^{n-1} \rho_k (x) (n \in \mathbb{N}).
\]

Recall that (see [18])
\[
D_{2^n} (x) = \begin{cases} 2^n, & \text{if } x \in I_n (0) \\ 0, & \text{if } x \in \mathbb{I} \setminus I_n (0) \end{cases}.
\]
(3)

Let \( f \in L_1 (\mathbb{I}) \). The partial sums of the Walsh-Fourier series are defined as follows:
\[
S_M (x, f) := \sum_{i=0}^{M-1} \hat{f} (i) w_i (x),
\]
where the number
\[
\hat{f} (i) = \int_I f (t) w_i (t) \, dt
\]
is said to be the \( i \)th Walsh-Fourier coefficient of the function \( f \). Set \( E_n (x, f) = S_{2^n} (x, f) \). The maximal function is defined by
\[
E^* (x, f) = \sup_{n \in \mathbb{N}} |E_n (x, f)|.
\]

2. Dyadic Hardy Spaces

The norm (or quasinorm) of the space \( L_p (\mathbb{I}) \) is defined by
\[
\|f\|_p := \left( \int_I |f (x)|^p \, dx \right)^{1/p} \quad (0 < p < +\infty).
\]
In case $p = \infty$, by $L^p(I)$ we mean $L^\infty(I)$, endowed with the supremum norm. The space weak-$L_1(I)$ consists of all measurable functions $f$ for which

$$
\|f\|_{\text{weak-}L_1(I)} := \sup_{\lambda > 0} \lambda \left( \|f > \lambda\| \right) < +\infty.
$$

The notation $a \lesssim b$ in the proofs stands for $a < c \cdot b$, where $c$ is an absolute constant.

Let $f \in L_1(I)$. For $0 < p < \infty$ the Hardy space $H_p(I)$ consists all functions for which

$$
\|f\|_{H_p} := \|E^p(f)\| < \infty.
$$

A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic interval $I$, such that

a) $\int_I a = 0$;

b) $\|a\|_\infty \leq |I|^{-1/p}$;

c) $\text{supp } a \subset I$.

An operator $T$ be called $p$-quasi-local if there exist a constant $c_p > 0$ such that for every $p$-atom $a$

$$
\int_I |Ta|^p \leq c_p < \infty,
$$

where $I$ is the support of the atom. We shall need the following

**Theorem W1 1 (Weisz [23]).** Suppose that the operator $T$ is $\sigma$-sublinear and $p$-quasi-local for each $0 < p \leq 1$. If $T$ is bounded from $L_\infty(I)$ to $L_\infty(I)$, then

$$
\|Tf\|_p \leq c_p \|f\|_p \quad (f \in H_p(I))
$$

for every $0 < p < \infty$. In particular for $f \in L_1(I)$, it holds

$$
\|Tf\|_{\text{weak-}L_1(I)} \leq C \|f\|_1.
$$

3. **Nörlund Logarithmic means**

In the literature, there is the notion of Riesz’s logarithmic means of a Fourier series. The $n$-th Riesz’s logarithmic means of the Fourier series of an integrable function $f$ is defined by

$$
R_n(x, f) := \frac{1}{I_n} \sum_{k=1}^n \frac{S_k(x, f)}{k},
$$

where $I_n := \sum_{k=1}^n (1/k)$.

Riesz’s logarithmic means with respect to the trigonometric system was studied by a lot of authors. This means with respect to the Walsh and Vilenkin systems was discussed by Simon [19], Blahota, Gát [4], Gát [7], Gát, Goginava [9], Tephnadze [20], Person, Tephnadze and Wall [17].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The $n$th Nörlund means for the Fourier series of $f$ is defined by

$$
\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k(f),
$$

where $Q_n := \sum_{k=0}^n q_k$. 

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where
\[ Q_n := \sum_{k=1}^{n} q_k. \]

If \( q_k = k \), then we get the Nörlund logarithmic means
\[ t_n (x, f) := \frac{1}{l_n} \sum_{k=0}^{n-1} S_k (x, f) \frac{n-k}{n}. \]

In this paper we call it logarithmic mean although it is a kind of “reverse” Riesz’s logarithmic mean.

It is easy to see that
\[ t_n (x, f) = \int_{\frac{x}{2}}^{x} f (t) F_n (x + t) \, dt, \]

where by \( F_n (t) \) we denote \( n \)th logarithmic kernel, i.e.
\[ F_n (t) := \frac{1}{n} \sum_{k=0}^{n-1} D_k (t) \frac{n}{n-k} I_n = \sum_{k=1}^{n} \frac{1}{k}. \]

The Fejér kernel is defined by
\[ K_n (t) := \frac{1}{n} \sum_{k=1}^{n} D_k (t). \]

For \( n = \sum_{j=0}^{\infty} \epsilon_j (n) 2^j, \epsilon_j (n) = 0, 1 \) we define
\[ n (k) := \sum_{j=0}^{k} \epsilon_j (n) 2^j. \]

It is easy to see that \( n (|n|) = n. \)

For a non-negative integer \( n \) let us denote the dyadic variation
\[ V (n) := \sum_{j=0}^{\infty} |\epsilon_j (n) - \epsilon_{j+1} (n)| + \epsilon_0 (n). \]

We define the weighted version of variation of an \( n \in \mathbb{N} \) with binary coefficients \( (\epsilon_k (n) : k \in \mathbb{N}) \) by
\[ L (n) := \sum_{k=1}^{n} |\epsilon_k (n) - \epsilon_{k+1} (n)| L_n (k). \]

Set for positive reals \( K \) the subset of natural numbers
\[ L_K := \left\{ n \in \mathbb{N} : \frac{L(n)}{|n|} \leq K \right\}. \]

It is known [17] that if \( n_j < n_{j+1} \) and
\[ \sup_j V (n_j) < \infty, \quad (4) \]

then a.e. \( S_{n_j} (f) \to f. \) On the other hand, Konyagin [14] proved that the condition (4) is not necessary for a.e. convergence of subsequence of partial sums. Moreover, he gave negative answer to the question of Balashov and proved the validity of the following theorem.
Theorem K 1 (Konyagin [14]). Suppose \( \{n_A\} \) is an increasing sequence of natural numbers, \( k_A := \log_2 n_A \) + 1, and \( 2^{k_A} \) is a divider of \( n_{A+1} \) for all \( A \). Then \( S_{n_A}(f) \to f \) a. e. for any function \( f \in L_1(\mathbb{I}) \).

For instance, for the sequence \( \{n_A\}, n_A := 2^{k_A} \sum_{i=0}^{4^i-1} \), such that \( \sup \|V(n_A)\| = \infty \), satisfies the hypotheses of the theorem.

Almost everywhere convergence of \( \{t_{2^{k_A}}(f) : A \geq 1\} \) with respect to Walsh-Paley system was studied by first author [11]. In particular, the following is proved

Theorem G1. Let \( f \in L_1(\mathbb{I}) \). Then \( t_{2^{k_A}}(x, f) \to f(x) \) as \( A \to \infty \) a. e. \( x \in \mathbb{I} \).

In [16], Nagy established a similar result for the Walsh-Kaczmarz system. Memi´c [15] improved Theorem G1. However, a divergence on the set with positive measure for the whole sequence \( \{t_n(f) : n \geq 1\} \) was proved by Gát and Goginava [8].

In [12] the following is proved.

Theorem G2. Let \( f \in L_1(\mathbb{I}) \) and \( K > 0 \). Then \( \lim_{L_k \to \infty} t_{n}(x, f) = f(x) \) for a. e. \( x \in \mathbb{I} \).

We define the maximal operator

\[
t_\ast(x, f) := \sup_{n \in \mathbb{N}} \left( |f * F_n(x)| \right).
\]

In this section it is proved that the maximal operator of subsequences of Nörlund logarithmic means of Walsh-Fourier series is bounded from the dyadic Hardy spaces \( H_p \) to \( L_p \). This implies an almost everywhere convergence for the subsequences of the summability means.

Theorem 3.1. Let \( p > 0 \). Then there exists a positive constant \( c_p \) such that

\[
\|t_\ast(f)\|_p \leq c_p \|f\|_{H_p} \quad (|f| \in H_p, p > 0)
\]

and

\[
\|t_\ast(f)\|_{\text{weak}-L_1(\mathbb{I})} \leq \|f\|_1.
\]

Corollary 3.2 (see [12]). Let \( f \in L_1(\mathbb{I}) \). Then

\[
\lim_{L_k \to \infty} t_{n}(x, f) = f(x) \quad \text{for a. e. } x \in \mathbb{I}.
\]

Theorem 3.3. Let \( \{m_A : A \in \mathbb{N}\} \) be a subsequence for which there does not exist \( K \) such that \( \{m_A : A \in \mathbb{N}\} \not\subseteq L_K \) for all \( K \in \mathbb{N} \), i. e. the condition

\[
\sup_A \frac{1}{|m_A|} \sum_{k=1}^{m_A} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| l_{m_A(k)} = \infty
\]

holds. The operator \( t_{m_A}(f) \) is not bounded from the dyadic Hardy spaces \( H_1(\mathbb{I}) \) to the space \( L_1(\mathbb{I}) \).

Proof. [Proof of Theorem 3.1] The following representation is known (see [12])

\[
l_n F_n(t) = H_n^{(1)}(t) + H_n^{(2)}(t),
\]
where

\[ H_n^{(1)}(t) = w_n(t) \left( \sum_{j=1}^{[n]} \varepsilon_j(n) D_2(0) \rho_j(t) u_n(j) \right) , \]

\[ H_n^{(2)}(t) = \left( \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{k=1}^{2^j} D_k(t) \frac{D_k(t)}{k+n(j-1)} \right) \prod_{j=0}^{[n]} (\rho_j(t))^{r_n(j)} . \]

Hence, we have

\[ f * F_n(x) = \left( f * \frac{H_n^{(1)}}{l_n} \right)(x) + \left( f * \frac{H_n^{(2)}}{l_n} \right)(x) . \] (5)

It is easy to see that

\[ w_n(t) H_n^{(1)}(t) = \sum_{j=1}^{[n]} \varepsilon_j(n) (D_{2^j-1}(t) - D_{2^j}(t)) u_n(j) \]

\[ = \sum_{j=1}^{[n]-1} \left( \varepsilon_j(n) u_n(j) - \varepsilon_{j+1}(n) u_n(j+1) \right) D_{2^j-1}(t) \]

\[ + \varepsilon_{[n]}(n) u_{[n]} D_{2^{[n]}}(t) - \varepsilon_1(n) u_1 D_2(t) \]

\[ = \sum_{j=1}^{[n]-1} \left( \varepsilon_j(n) - \varepsilon_{j+1}(n) \right) u_n(j) D_{2^j-1}(t) \]

\[ + \sum_{j=1}^{[n]-1} \varepsilon_{j+1}(n) \left( u_n(j) - u_n(j+1) \right) D_{2^j-1}(t) \]

\[ + \varepsilon_{[n]}(n) u_{[n]} D_{2^{[n]}}(t) - \varepsilon_1(n) u_1 D_2(t) . \]

Consequently,

\[ \frac{|H_n^{(1)}(t)|}{l_n} \ll \frac{1}{l_n} \sum_{j=1}^{[n]-1} \left| \varepsilon_j(n) - \varepsilon_{j+1}(n) \right| u_n(j) D_{2^j-1}(t) \]

\[ + \sum_{j=1}^{[n]-1} \varepsilon_{j+1}(n) \left( u_n(j+1) - u_n(j) \right) D_{2^j-1}(t) \]

\[ + \varepsilon_{[n]}(n) u_{[n]} D_{2^{[n]}}(t) - \varepsilon_1(n) u_1 D_2(t) . \]

\[ = : p_n(t) \] (6)
Let \( n \in L_K \). Then we can write
\[
\| (f * P_n) (x) \| \leq \frac{1}{l_n} \sum_{j=1}^{[n]-1} |\epsilon_j (n) - \epsilon_{j+1} (n)| \| l_{n(j)} (f * D_{2^j}) (x) \|
+ \frac{1}{l_n} \sum_{j=1}^{[n]} \epsilon_j (n) \left( l_{n(j+1)} - l_{n(j)} \right) (f * D_{2^{2j}}) (x)
+ \frac{\epsilon_{[n]} (n) l_{n[0][n]} (f * D_{2^{[n][n]}}) (x)}{l_n} 
+ \frac{\epsilon_1 (n) l_{n(1)} (f * D_2) (x)}{l_n}.
\]

\[
\leq E^* (x, f) \left\{ \frac{1}{l_n} \sum_{j=1}^{[n]-1} |\epsilon_j (n) - \epsilon_{j+1} (n)| \| l_{n(j)} \|
+ \frac{1}{l_n} \sum_{j=1}^{[n]} \epsilon_j (n) \left( l_{n(j+1)} - l_{n(j)} \right) + 2 \right\}
\leq L_K E^* (x, f).
\]

Since (see [18, 23])
\[
\| E^* (f) \|_p \leq c_p \| f \|_{H_p} \quad (p > 0),
\] (7)

and
\[
\| E^* (f) \|_{\text{weak-}L^1 (I)} \leq c_p \| f \|_1 ,
\] (8)

we have
\[
\left\| \sup_{n \in L_K} |(f * P_n) (x)| \right\|_p \leq c_p \| f \|_{H_p} \quad (f \in H_p, p > 0)
\] (9)

and
\[
\left\| \sup_{n \in L_K} |(f * P_n) | \right\|_{\text{weak-}L^1 (I)} \leq \| f \|_1 .
\] (10)

Now, we can write
\[
\frac{H_{2^{j}} (t)}{l_n} \leq \frac{1}{l_n} \sum_{j=1}^{[n]} \epsilon_j (n) \sum_{k=1}^{2^{j}} \frac{D_k (t)}{k + n (j - 1)}
\]

Using Abel’s transformation we obtain
\[
\sum_{k=1}^{2^{j}} \frac{D_k (t)}{k + n (j - 1)}
= \sum_{k=1}^{2^{j+1}} \left( \frac{1}{k + n (j - 1)} - \frac{1}{k + 1 + n (j - 1)} \right) kK_k (t)
+ \frac{2^j}{2^j + n (j - 1)} K_{2^j} (t).
\]
Consequently,

\[
\left| H_n^{(2)}(x) \right| \leq \frac{1}{l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) \sum_{k=1}^{2^{n-1}} \left( \frac{1}{k + n (j - 1)} - \frac{1}{k + 1 + n (j - 1)} \right) k |K_k(x)| \\
+ \frac{1}{l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) \frac{2^i}{2^i + n (j - 1)} K_{2^i}(x) \\
= : H_n^{(21)}(x) + H_n^{(22)}(x).
\]

Since (see [18], p. 46)

\[
|K_l(x)| \leq 3 \cdot 2^{-s} \sum_{i=0}^{s-1} \sum_{j=0}^{\sigma} 2^i D_{2^i} \left( x + 2^{-r-1} \right)
\]
when \(2^{-1} \leq l < 2^s\). We have

\[
\left| H_n^{(21)}(x) \right| \leq \frac{3}{l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) \sum_{k=0}^{s-1} \sum_{r=0}^{k} 2^i D_{2^i} \left( x + 2^{-r-1} \right)
\]

\[
= \frac{3}{l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) \sum_{k=0}^{s-1} \sum_{r=0}^{k} 2^i D_{2^i} \left( x + 2^{-r-1} \right).
\]

It is well known (see [18], p. 47) that if \(j \in \mathbb{N}\) then

\[
K_{2^j}(x) = \frac{1}{2} \left( 2^{-j} D_{2^j}(x) + \sum_{i=0}^{j} 2^{i-j} D_{2^j} \left( x + \frac{1}{2^{j+1}} \right) \right).
\]

In particular, \(K_{2^j} \geq 0\) everywhere on \(I\). Then we have

\[
H_n^{(22)}(x) \leq \frac{1}{2l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) 2^{-j} D_{2^j} \left( x \right) \\
+ \frac{1}{2l_n} \sum_{j=1}^{\lfloor \ln \mid n \mid \rfloor} \varepsilon_j(n) \sum_{i=0}^{j} 2^{i-j} D_{2^j} \left( x + \frac{1}{2^{j+1}} \right).
\]
Combining (11), (13) and (15) we have

\[
|H^{(2)}_n(x)| \leq \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{s=1}^{j} \left( \frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^s + n(j-1)} \right)
\times \sum_{k=0}^{j} \sum_{r=0}^{k} 2^r D_{2^r} (x + 2^{-r-1})
\]

\[
=: Q_n(x).
\]

We can write

\[
(f * Q_n)(x) = f \left( \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{s=1}^{j} \left( \frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^s + n(j-1)} \right)
\times \sum_{k=0}^{j} \sum_{r=0}^{k} 2^r D_{2^r} (x + 2^{-r-1}) \right)(x).
\]

First, we prove that the operator \(f * Q_n\) is bounded from \(L_\infty(I)\) to \(L_\infty(I)\). Indeed, since

\[
\sup_{n \in \mathbb{N}} \|Q_n\|_1
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{s=1}^{j} 2^s \sum_{k=2^{s-1}}^{2^s} \left( \frac{1}{k + n(j-1)} - \frac{1}{k + 1 + n(j-1)} \right)
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{s=1}^{j} 2^{s-1} \sum_{k=2^{s-1}}^{2^s} \left( \frac{1}{k + n(j-1)} - \frac{1}{k + 1 + n(j-1)} \right)
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{k=2}^{k + n(j-1)} \frac{k}{(k + n(j-1))^2}
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \sum_{k=2}^{k + n(j-1)} \left( \frac{1}{k + n(j-1)} + \frac{n(j-1)}{(k + n(j-1))^2} \right)
\]

\[
\leq \sup_{n \in \mathbb{N}} \frac{1}{l_n} \sum_{j=1}^{[n]} \varepsilon_j(n) \left( l_{n(j)} - l_{n(j-1)} + 1 \right)
\]

\[
\leq c < \infty.
\]

we obtain that

\[
\sup_{n \in \mathbb{N}} \|f * Q_n\|_\infty \leq c \|f\|_\infty.
\]

Hence, the operator \(f * Q_n\) is bounded from \(L_\infty(I)\) to \(L_\infty(I)\).

We suppose that \(f \in H_p(I)\). Let function \(a\) be an \(H_p\) atom. It means that either \(a\) is constant or there is an interval \(I_a(u)\) such that \(\text{supp}(a) \subset I_a(u)\), \(\|a\|_\infty \leq 2^{N/p}\) and \(\int a = 0\). Without lost of generality we can suppose that \(u = 0\). Consequently, for any function \(g\) which is \(\mathcal{A}_{\mathcal{N}}\)-measurable we have that \(\int ag = 0\). We prove that
the operator \( \sup_{n>N} (f \ast Q_n)(x) \) is \( H_p \)-quasi local. That is,

\[
    \int_{I_N} \left( \sup_{n>N} |a \ast Q_n| \right)^p \leq c_p. \tag{17}
\]

Let \( x \in I_N \). Then we can write

\[
    |(a \ast Q_n)(x)|
    = \frac{1}{I_n} \int_{I_n} a(t) \left( \sum_{j=N+1}^{[n]} \epsilon_j(n) \sum_{s=N+1}^{j} \right)
    \left( \frac{1}{2^{s-1} + n(j-1)} - \frac{1}{2^{s} + n(j-1)} \right)
    \times \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r D_2 \left( x + t + 2^{-r-1} \right) dt
    \leq \frac{2^{N/p}}{I_n} \sum_{j=N+1}^{[n]} \epsilon_j(n) \sum_{s=N+1}^{j} \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r D_2 \left( x + t + 2^{-r-1} \right) dt
    \leq \frac{2^{N/p}}{I_n} \sum_{j=N+1}^{[n]} \epsilon_j(n) \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right)
    \times \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r D_2 \left( x + t + 2^{-r-1} \right) dt
    \leq \frac{2^{N/p}}{I_n} \sum_{j=N+1}^{[n]} \epsilon_j(n) \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right)
    \times \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r D_2 \left( x + t + 2^{-r-1} \right) dt
    = \frac{2^{N/p}}{I_n} \sum_{j=N+1}^{[n]} \epsilon_j(n) \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right)
    \times \left( \sum_{j=N+1}^{i} \sum_{k=r}^{j} + \sum_{k=N+1}^{j} \sum_{r=0}^{k} 2^r D_2 \left( x + t + 2^{-r-1} \right) dt \right).}

Since

$$\sum_{r=N+1}^{j} \sum_{k=r}^{j} 2^{r} \int_{I_{N}} D_{2r}(x + t + 2^{-r-1}) dt = 0 \quad (x \in \overline{I_{N}}),$$

we have

$$|a \ast Q_{n}| \leq \frac{2^{N/p}}{I_{n}} \sum_{j=N+1}^{N/p} \varepsilon_{j}(n) \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right) \times \sum_{r=0}^{N} 2^{r} 1_{I_{n}(2^{-r-1})}(x).$$

Since

$$\sum_{j=N+1}^{N} \varepsilon_{j}(n) \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right) \leq \sum_{j=N+1}^{N} \left( \frac{1}{2^{N} + n(j-1)} - \frac{1}{n(j)} \right) \leq \frac{|n|}{2^{N}},$$

we have

$$|a \ast Q_{n}| \leq \frac{2^{N/p}}{2^{N}} \sum_{r=0}^{N} 2^{r} 1_{I_{n}(2^{-r-1})}(x),$$

where $1_{E}$ is characteristic function of the set $E$ and consequently,

$$\int_{I_{n}} \sup_{n \geq N} |a \ast Q_{n}|^{p} \leq \frac{2^{N}}{2^{Np}} \sum_{r=0}^{N} 2^{r} \int_{I_{n}} 1_{I_{n}(2^{-r-1})} \leq c_{p}.$$  

Hence,

$$\left\| \sup_{n \in \mathbb{N}} |f \ast Q_{n}| \right\|_{p} \leq c_{p} \left\| f \right\|_{H^{p}_{0}} \quad (f \in H^{p}, p > 0) \quad (18)$$

and

$$\left\| \sup_{n \in \mathbb{N}} |f \ast Q_{n}| \right\|_{\text{weak--}L^{1}(I)} \leq \left\| f \right\|_{1}. \quad (19)$$

Since

$$|f \ast F_{n}|(x) \leq |f| \ast P_{n} + |f| \ast Q_{n},$$

from (9), (10), (18) and (19) we have

$$\left\| \sup_{n \in \mathbb{N}} |f \ast F_{n}| \right\|_{p} \leq c_{p} \left\| f \right\|_{H^{p}_{0}} \quad (|f| \in H^{p}, p > 0).$$
and
\[ \left\| \sup_{n \in \mathbb{N}} \left| f \ast F_n \right| \right\|_{\text{weak} - L_1(0)} \leq \left\| f \right\|_1. \]

Which complete the proof of Theorem 3.1. □

Proof. [Proof of Theorem 3.3] Set
\[ f_A := D_{2^{m_A+1}} - D_{2^{m_A}}. \]

Then it is easy to see that
\[ \sup_{n \in \mathbb{N}} \left| S_{2^n} (f_A) \right| = D_{2^{m_A}} \]
and consequently,
\[ \left\| f_A \right\|_{H_1} = \left\| \sup_{n \in \mathbb{N}} \left| S_{2^n} (f_A) \right| \right\|_1 = \left\| D_{2^{m_A}} \right\|_1 = 1. \]

Set
\[ m_A = 2^{m_A} + q_A, \]
where
\[ q_A := \sum_{j=0}^{m_A} \varepsilon_j (m_A) 2^j. \]

Then we can write
\[ t_{m_A} (f_A) = \frac{1}{t_{m_A}} \sum_{k=2^{m_A+1}}^{2^{m_A+q_A-1}} \frac{S_k (f_A)}{m_A - k}. \]

It is easy to see that
\[ S_k (f_A) = S_k \left( D_{2^{m_A+1}} - D_{2^{m_A}} \right) = D_k - D_{2^{m_A}}. \]

Hence, we have
\[ t_{m_A} (f_A) = \frac{1}{t_{m_A}} \sum_{k=2^{m_A+1}}^{2^{m_A+q_A-1}} \frac{D_k - D_{2^{m_A}}}{m_A - k} \]

\[ = \frac{1}{t_{m_A}} \sum_{k=1}^{q_A-1} \frac{D_k - D_{2^{m_A}}}{m_A - k} \]

\[ = \frac{w_{2^{m_A}}}{t_{m_A}} \sum_{k=1}^{q_A-1} \frac{D_k}{q_A - k}. \]

From the condition of Theorem 3.3 we conclude that
\[ \sup_{A \in \mathbb{N}} \left\| t_{m_A} (f_A) \right\|_1 = \sup_{A \in \mathbb{N}} \left\| F_{m_A} \right\|_1 = \infty. \]

Theorem 3.3 is proved. □
4. Cesàro Means with Varying Parameters

The \((C, \alpha_n)\) means of the Walsh-Fourier series of the function \(f\) is given by

\[
\sigma_n^{\alpha_n}(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n-1} S_{j}(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha_n} \hat{f}(j) w_j(x),
\]

where

\[
A_{n}^{\alpha_n} := \frac{(1 + \alpha_n) \ldots (n + \alpha_n)}{n!}
\]

for any \(n \in \mathbb{N}, \alpha_n \neq -1, -2, \ldots\).

It is known that [26]

\[
A_{n}^{\alpha_n} = \sum_{k=0}^{n} A_{k}^{\alpha_k-1} A_{n-k}^{\alpha_n-1} = \frac{\alpha_n}{\alpha_n + n} A_{n}^{\alpha_n}.
\]

The \((C, \alpha_n)\) kernel is defined by

\[
K_{n}^{\alpha_n} = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n-1} D_{j} = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha_n} w_j.
\]

The following estimations was proved by Akhobadze [2, 3]: Let \(k, n \in \mathbb{N}\). Then

\[
c_1 (1 + \alpha_n) (2 + \alpha_n) k^{\alpha_n} < A_{k}^{\alpha_n} < c_2 (1 + \alpha_n) (2 + \alpha_n) k^{\alpha_n},
\]

when \(-2 < \alpha_n < -1\); \hspace{1cm} (21)

\[
c_1 (1 + \alpha_n) k^{\alpha_n} < A_{k}^{\alpha_n} < c_2 (1 + \alpha_n) k^{\alpha_n}, \text{ when } -1 < \alpha_n < 0;
\]

\[
c_1 (d) k^{\alpha_n} < A_{k}^{\alpha_n} < c_2 (d) k^{\alpha_n}, \text{ when } 0 < \alpha_n \leq d.
\]

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [13] and the introduction of these \((C, \alpha_n)\) means of Fourier series is due to Akhobadze (see [3] or [2]) who investigated the behavior of the \(L_1\)-norm convergence of \(\sigma_n^{\alpha_n}(f) \rightarrow f\) for the trigonometric system.

The first result with respect to the a.e. convergence of the Walsh-Fejér means \(\sigma_n^{\alpha_n}(f)\) for all integrable function \(f\) with constant sequence \(\alpha_n = \alpha > 0\) is due to Fine [5] (see also Weisz [22]). On the rate of convergence of Cesàro means in this constant case see the paper of Yano [25], Fridli [?] .

For \(n := \sum_{i=0}^{\infty} \varepsilon_i(n) 2^i \varepsilon_i(n) = 0, 1, i \in \mathbb{N}\) set two variable function

\[
P(n, \alpha) := \sum_{i=0}^{\infty} \varepsilon_i(n) 2^{\alpha_i} \in \mathbb{N}, \alpha := \{\alpha_n : n \in \mathbb{N}\}.
\]

The function \(P(n, \alpha)\) was introduced by Abu Joudeh and Gát in [1]. Also set for sequence \(\alpha := \{\alpha_n : n \in \mathbb{N}\}\) and positive reals \(K\) the subset of natural numbers

\[
P_K(\alpha) := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha)}{n^{\alpha_n}} \leq K \right\}.
\]

The a.e. divergence of Cesàro means with varying parameters of Walsh-Fourier series was investigated by Tetunashvili [21]. Abu Joudeh and Gát in [1] proved the almost everywhere convergence (with some restrictions) of the Cesàro \((C, \alpha_n)\) means of integrable functions. In particular, the following is proved
Theorem JG 1. Suppose that \( \alpha_n \in (0,1) \). Let \( f \in L_1 (\mathbb{I}) \). Then we have the almost everywhere convergence
\[
\sigma_n^{\alpha_n} (f) \to f
\]
provided that \( P_K (\alpha) \equiv n \to \infty \).

In this section we define the weighted version of variation of an \( n \in \mathbb{N} \) with binary coefficients \((\epsilon_k (n) : k \in \mathbb{N})\) by
\[
V(n, \alpha) := \sum_{i=0}^{\infty} |\epsilon_i (n) - \epsilon_{i+1} (n)| 2^{i\alpha_i} \ (n \in \mathbb{N}).
\]

Set for sequence \( \alpha := \{\alpha_n : n \in \mathbb{N}\} \) and positive reals \( K \) the subset of natural numbers
\[
V_K (\alpha) := \left\{ n \in \mathbb{N} : \frac{V(n, \alpha)}{n^{\alpha_n}} \leq K < \infty \right\}.
\]

It is easy to see that \( P_K (\alpha) \subseteq V_{2K} (\alpha) \). On the other hand, if \( \alpha_n \to 0 \), then there exists \( K \) such that \( 2^n - 1 \in V_K (\alpha) \) for all \( n \), but there does not exists \( K \), such that \( 2^n - 1 \in P_K (\alpha) \) for all \( n \).

The boundedness of maximal operators of subsequences of \((C, \alpha_n)\) - means of partial sums of Walsh-Fourier series from the Hardy space \( H_p \) into the space \( L_p \) is studied in [10]. In particular, the following is proved.

Theorem GG2 1. Let \( p > 0 \). Then there exists a positive constant \( c_p \) such that
\[
\left\| \sup_{n \in \mathbb{N}} |f * K^\alpha_n| \right\|_p \leq c_p \|f\|_{H_p} \quad (f \in H_p).
\]

Weisz [24] generalized Theorem GG2 for both the Cesàro and Riesz means by taking the supremum over all indices \( n \in \mathbb{N}_c \). Here \( \mathbb{N}_c \) denotes the set of all \( n = 2^{n_1} + \cdots + 2^{n_v} \) with a fixed parameter \( v \). In particular, the following is proved.

Theorem W2 1. Let \( p > 0 \). Then there exists a positive constant \( c_p \) such that
\[
\left\| \sup_{\alpha \in \mathbb{V}_k (\alpha)} |f * K^\alpha_n| \right\|_p \leq c_p \|f\|_{H_p} \quad (|f| \in H_p).
\]

In this section we are going to improve Theorem W2. We prove that the maximal operator of subsequences of Cesàro means with varying parameters of Walsh-Fourier series is bounded from the dyadic Hardy spaces \( H_p \) to \( L_p \). This implies an almost everywhere convergence for the subsequences of the summability means.

Theorem 4.1. Let \( p > 0 \). Then there exists a positive constant \( c_p \) such that
\[
\left\| \sup_{\alpha \in \mathbb{V}_k (\alpha)} |f * K^\alpha_n| \right\|_p \leq c_p \|f\|_{H_p} \quad (|f| \in H_p)
\]

and
\[
\left\| \sup_{\alpha \in \mathbb{V}_k (\alpha)} |f * K^\alpha_n| \right\|_{\text{rank } J_1 (\mathbb{I})} \leq c \|f\|_1 \quad (f \in L_1).
\]

Corollary 4.2. Let \( f \in L_1 (\mathbb{I}) \). Then
\[
\lim_{\mathbb{V}_k (\alpha) n \to \infty} \sigma_n^{\alpha_n} (x, f) = f (x) \quad \text{for a.e. } x \in \mathbb{I}.
\]
Remark 4.3. We suspect that Theorem 3.1 and Theorem 4.1 will be valid in the case when \( f \in H_p (I) \) \((p > 0)\), but we could not proved these.

Now, we prove Theorem 4.1.

Proof. We can write

\[
K_n^{α_n} = \frac{1}{A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) w_{(s+1)} \sum_{j=1}^{2^{s-1} - 1} A_n^{α_n - 2} j K_j,
\]

\[
- \frac{1}{A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) w_{(s+1)} A_n^{α_n - 1} 2^s K_{2s},
\]

\[
+ \frac{1}{A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) w_{(s+1)} A_n^{α_n - 1} D_{2s},
\]

\[
= : T_n^{(1)} + T_n^{(2)} + T_n^{(3)},
\]

and

\[
\sup_{n \in \mathbb{N}} \left\| f \ast T_n^{(3)} \right\| \leq c K E^∗ (x, f)
\]

Then from (7) and (8) we have

\[
\left\| \sup_{n \in \mathbb{N}} \left\| f \ast T_n^{(3)} \right\| \right\|_{p} \leq c_p \| f \|_{H_p} \quad (f \in H_p, \ p > 0)
\]

and

\[
\left\| \sup_{n \in \mathbb{N}} \left\| f \ast T_n^{(3)} \right\| \right\|_{weak L_1 (I)} \leq c \| f \|_{1} \quad (f \in L_1 (I)).
\]

It is easy to see that (see (20))

\[
\left| T_n^{(1)} \right| \leq \frac{2}{A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) \sum_{j=1}^{2^{s-1} - 1} A_n^{α_n - 1} |K_j|,
\]

then, from (12) we have

\[
\left| T_n^{(1)} \right| + \left| T_n^{(2)} \right| 
\]

\[
\leq \frac{6}{A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) \sum_{j=1}^{2^{s-1} - 1} \frac{1}{2^j} 
\]

\[
	imes \sum_{j=2}^{2^{s-1} - 1} A_n^{α_n - 1} \sum_{k=0}^{l-1} \sum_{r=0}^{k} 2^r D_{2r} \left( x + 2^{-r-1} \right)
\]

\[
+ \frac{1}{2A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) A_n^{α_n - 1} D_{2s} (x)
\]

\[
+ \frac{1}{2A_n^{α_n}} \sum_{s=0}^{[n]} \varepsilon_s(n) A_n^{α_n - 1} \sum_{j=0}^{s} D_{2j} \left( x + \frac{1}{2^{s-j}} \right)
\]

\[
= : \tilde{T}_n (x).
\]
Now, we discuss the operator \( \sup_{n \in \mathbb{N}} \left( f \ast \overline{T}_n \right) \). First, we show that the operator is bounded from \( L_{\infty}(I) \) to \( L_{\infty}(I) \). Indeed, since (see [18]) \( \sup_n \|K_n\| < 2 \) and from (22), (23) we have

\[
\sup_{n \in \mathbb{N}} \left\| T_n \right\|_1 \leq \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) \sum_{l=1}^{s} \frac{1}{2^l} \times \sum_{j=2^{l-1}}^{2^{l-1}} A_{n_{j+1}}^{s-1} \sum_{k=0}^{l-1} \sum_{r=0}^{k} 2^r \int_{\mathbb{I}} D_{2^r} \left( x + 2^{r-1} \right) \, dx + \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) A_{n_{j+1}}^{s-1} \int_{\mathbb{I}} D_{2^r} \left( x \right) \, dx + \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) A_{n_{j+1}}^{s-1} \int_{\mathbb{I}} D_{2^r} \left( x + \frac{1}{2^{r+1}} \right) \, dx \leq \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) \sum_{l=1}^{s} \frac{1}{2^l} \times \sum_{j=2^{l-1}}^{2^{l-1}} A_{n_{j+1}}^{s-1} + \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) A_{n_{j+1}}^{s-1} \int_{\mathbb{I}} D_{2^r} \left( x \right) \, dx + \sup_{n \in \mathbb{N}} \sum_{s=0}^{[n]} \| \varepsilon_s(n) A_{n_{j+1}}^{s-1} \int_{\mathbb{I}} D_{2^r} \left( x + \frac{1}{2^{r+1}} \right) \, dx \leq c < \infty,
\]

which implies the boundedness of operator \( \sup_{n \in \mathbb{N}} \left( f \ast \overline{T}_n \right) \) from the space \( L_{\infty}(I) \) to the space \( L_{\infty}(I) \).

We suppose that \( f \in H_p(I) \). Let function \( a \) be an \( H_p \) atom. It means that either \( a \) is constant or there is an interval \( J_N(u) \) such that \( \supp(a) \subset J_N(u), \|a\|_\infty \leq 2^{N/p} \) and \( \int a = 0 \). Without lost of generality we can suppose that \( u = 0 \). Consequently, for any function \( g \) which is \( \mathcal{A}_N \)-measurable we have that \( \int ag = 0 \). We prove that operator \( \sup_{n \in \mathbb{N}} \left( f \ast \overline{T}_n \right) \) is \( H_p \)-quasi local. That is,

\[
\int \left( \sup_{j > N} |a \ast \overline{T}_j| \right)^p \leq c_p.
\]
Let \( x \in \tilde{T}_N \). Then from (3) we can write

\[
\begin{align*}
\left| a \ast \tilde{T}_n \right| &= \left| \int_{\tilde{T}_n} a(t) \left( 6 \sum_{s=N+1}^{n} \varepsilon_s(n) \sum_{l=N+1}^{n} \frac{1}{2^l} \right) \right| \\
\times &\sum_{j=2^{n+1}}^{2^{n+1}-1} A_{n_0-1+j}^{\alpha_{n_0-1+j}} \sum_{k=N_0+1}^{l-1} \sum_{r=0}^{\beta_{n_0-1+k}} 2^{j} D_{2} (x + t + 2^{j-1}) \\
&+ \frac{1}{2^{n_0-1}} \sum_{s=N+1}^{n} \varepsilon_s(n) A_{n_0-1}^{\alpha_{n_0-1}} D_{2} (x + t) \\
&+ \frac{1}{2^{n_0-1}} \sum_{s=N+1}^{n} \varepsilon_s(n) A_{n_0-1}^{\alpha_{n_0-1}} \sum_{l=0}^{N-1} 2^{l} D_{2} (x + t + \frac{1}{2^{l+1}}) dt \\
\leq &\sum_{j=2^{n+1}}^{2^{n+1}-1} A_{n_0-1+j}^{\alpha_{n_0-1+j}} \sum_{k=N_0+1}^{l-1} \sum_{r=0}^{\beta_{n_0-1+k}} 2^{j} D_{2} (x + t + 2^{j-1}) dt \\
&+ \frac{2^{N/p}}{A_{n_0-1}} \sum_{s=N+1}^{n} \varepsilon_s(n) A_{n_0-1}^{\alpha_{n_0-1}} \sum_{l=0}^{N-1} 2^{l} D_{2} (x + t + \frac{1}{2^{l+1}}) dt.
\end{align*}
\]

Since

\[
\sum_{j=2^{n+1}}^{2^{n+1}-1} A_{n_0-1+j}^{\alpha_{n_0-1+j}} = \sum_{j=2^{n+1}}^{2^{n+1}-1} \left( A_{n_0-1+j}^{\alpha_{n_0-1+j}} - A_{n_0-1+j-1}^{\alpha_{n_0-1+j-1}} \right)
\]

\[
= A_{n_0-1+j-1}^{\alpha_{n_0-1+j-1}} - A_{n_0-1+j}^{\alpha_{n_0-1+j} - 1}
\]

we have

\[
\begin{align*}
\left| a \ast \tilde{T}_n \right| &\leq \frac{2^{N/p+1}}{A_{n_0-1}} \sum_{s=N+1}^{n} \varepsilon_s(n) \sum_{l=0}^{N-1} 2^{l} \left( A_{n_0-1+l}^{\alpha_{n_0-1+l} - 1} - A_{n_0-1+l+2^{l+1}-1}^{\alpha_{n_0-1+l+2^{l+1}-1}} \right) \\
\times &\sum_{k=0}^{l} \sum_{r=0}^{\beta_{n_0-1+k}} 2^{j} D_{2} (x + t + 2^{j-1}) dt \\
&+ \frac{2^{N/p}}{A_{n_0-1}} \sum_{s=N+1}^{n} \varepsilon_s(n) A_{n_0-1}^{\alpha_{n_0-1}} \sum_{l=0}^{N-1} 2^{l} b_{l+2^{l+1}-1} (x)
\end{align*}
\]
\[
\sum_{r=0}^{N-1} 2^r \sum_{k=r}^{l-1} \int_{I_k} D_{2^r} \left( x + t + 2^{-r-1} \right) dt = 0 \quad (x \in \overline{I}_N)
\]

we get

\[
\left| \tilde{a} * T_{n}^{(1)} \right| 
\leq \frac{2^{N/p}}{A_n^{\alpha_n}} \sum_{s=N+1}^{\infty} \varepsilon_s (n) \sum_{j=N+1}^{s} \frac{1}{2^j} \left( A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} - A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} \right) 
\times \sum_{r=0}^{N} 2^r \sum_{k=r}^{l-1} \int_{I_k} D_{2^r} \left( x + t + 2^{-r-1} \right) dt 
+ \frac{\alpha_n 2^{N/p}}{2^{N}} \sum_{j=0}^{N-1} 2^j 1_{I_n} (2^{-j-1}) (x) 
\]

\[
\leq \frac{2^{N/p}}{A_n^{\alpha_n}} \sum_{s=N+1}^{\infty} \varepsilon_s (n) \sum_{j=N+1}^{s} \frac{(l - N)}{2^j} 
\times \left( A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} - A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} \right) \sum_{r=0}^{N} 2^r 1_{I_n} (2^{-j-1}) (x) 
+ \frac{\alpha_n 2^{N/p}}{2^{N}} \sum_{j=0}^{N-1} 2^j 1_{I_n} (2^{-j-1}) (x) 
\]

\[
\leq \frac{2^{N/p}}{A_n^{\alpha_n}} \sum_{s=N+1}^{\infty} \varepsilon_s (n) \sum_{j=N+1}^{s} \left( A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} - A_{\alpha_n}^{(0)} n_{s-j} + 2^{-j-1} \right) 
\times \sum_{r=0}^{N} 2^r 1_{I_n} (2^{-j-1}) (x) 
+ \frac{\alpha_n 2^{N/p}}{2^{N}} \sum_{j=0}^{N-1} 2^j 1_{I_n} (2^{-j-1}) (x) 
\]
\[
\begin{align*}
\leq & \frac{2^{N/p}}{A_{n-1}^{\alpha_n} A_{n}^{\alpha_n}} \sum_{s=N+1}^{[n]} \varepsilon_s(n) \left( A_{n_{[n]}}^{\alpha_n} - A_{n_{[n]-1}}^{\alpha_n} \right) \\
& \times \sum_{r=0}^{N} 2^r 1_{I_n}((2^{-r-1}) (x)) \\
& + \frac{a_n 2^{N/p}}{2^N} \sum_{l=0}^{N-1} 2^l 1_{I_n}(2^{-l-1}) (x) \\
\leq & \frac{2^{N/p}}{2^N} \sum_{r=0}^{N} 2^r 1_{I_n}(2^{-r-1}) (x).
\end{align*}
\]

Therefore,
\[
\int_{I_n} \left( \sup_{n > N} \left| a * \tilde{T}_{l,n}^{(1)} (x) \right| \right)^p \, dx 
\leq \frac{2^N}{2^{N/p}} \sum_{r=0}^{N} 2^r \int_{I_n} 1_{I_n}(2^{-r-1}) (x) \, dx 
= \frac{1}{2^{N/p}} \sum_{r=0}^{N} 2^r \leq c_p < \infty.
\]

and consequently,
\[
\| \sup_{n \in \mathbb{N}} \left( f * \tilde{T}_{l,n} \right) \|_p \leq c_p \| f \|_{H_p} \quad (f \in H_p, \ p > 0) \quad (27)
\]

and
\[
\| \sup_{n \in \mathbb{N}} \left( f * \tilde{T}_{l,n} \right) \|_{\text{weak-L}_1(I)} \leq c \| f \|_1 \quad (f \in L_1(I)). \quad (28)
\]

Since
\[
\sup_{n \in V_{k}^{(n)}} \left| f * K_{n}^{\alpha_n} \right| \leq \sup_{n \in \mathbb{N}} \left( f * \tilde{T}_{l,n} \right) + \sup_{n \in \mathbb{N}} \left( f * \tilde{T}_{l,n}^{(3)} \right)
\]

Combining (24), (25), (26), (27) and (28) we complete the proof of Theorem 4.1.

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