GAPS OF SMALLEST POSSIBLE ORDER BETWEEN PRIMES IN AN ARITHMETIC PROGRESSION

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Abstract. Let $t \in \mathbb{N}, \eta > 0$. Suppose that $x$ is a sufficiently large real number and $q$ is a natural number with $q \leq x^{5/12 - \eta}$, $q$ not a multiple of the conductor of the exceptional character $\chi^*$ (if it exists). Suppose further that,

$$\max\{p : p|q\} < \exp\left(\frac{\log x}{C \log \log x}\right) \text{ and } \prod_{p|q} p < x^\delta,$$

where $C$ and $\delta$ are suitable positive constants depending on $t$ and $\eta$. Let $\mathcal{A} = \{n \in (x/2, x) : n \equiv a \pmod{q}\}$.

We prove that there are primes $p_1 < p_2 < \cdots < p_t$ in $\mathcal{A}$ with

$$p_t - p_1 \ll qt \exp\left(\frac{40t}{9 - 20\theta}\right).$$

Here $\theta = (\log q)/\log x$.

Key words and Phrases: GPY sieve, primes in arithmetic progressions, large values of Dirichlet polynomials, zeros of Dirichlet $L$-functions

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1. Introduction

Let $t \in \mathbb{N}$ and $0 \leq \eta < 1$ be given. Suppose that $x$ is a large positive real number, and that $q \in \mathbb{N}$ and $(a, q) = 1$, $q \leq x^{1 - \eta}$. Set

$$\mathcal{A} = \{n \in (x/2, x) : n \equiv a \pmod{q}\}.$$

It may be conjectured that there are primes $p_1 < p_2 < \cdots < p_t$ in $\mathcal{A}$ with

$$p_t - p_1 \ll t^{3/5} \exp(4t).$$

J. Maynard [13] has recently refined the Goldston-Pintz-Yıldırım sieve to prove this in the case of $q = 1$, showing that

$$p_t - p_1 \ll t^{3/5} \exp(4t).$$

In this paper, we prove (1.1) for a certain class of $q$’s. To describe this class, we first specify what is meant by the exceptional character. (See [9, p. 95].) For a certain positive absolute constant $c_1$, there is at most one primitive real character $\chi^*$ to a modulus not exceeding $x$ such that $L(\beta, \chi^*) = 0$ with $\beta \in \mathbb{R}$ and

$$\beta > 1 - \frac{c_1}{\log x}.$$ 

We shall write $w$ for the conductor of $\chi^*$ (if $\chi^*$ exists).

The key ingredient of our work, besides Maynard’s method (in the form of a general theorem of Baker and Zhao [2]), is Chang’s zero-free region [4]. She shows that for $\chi \neq \chi^*$ a primitive character with conductor $r \leq x$,

$$\mathcal{P} = \mathcal{P}(r) = \max\{p : p|r\}, \quad r' = r'(r) = \prod_{p|r} p$$

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1
and $K = \log r / \log r'$, there is a positive constant $c_2$ such that if $L(s, \chi) = 0$ for $|3s| < T$, then

$$\Re s < 1 - c_2 \min \left( \frac{1}{\log B}, \frac{\log r'}{\log 2K}, \frac{1}{(\log r')^{9/10}} \right).$$

We write $L = \log x$. Let $\varepsilon$ be a sufficiently small positive constant. Furthermore, we shall write $H(q, B, \delta)$ for the property

$$(H(q, B, \delta)) \quad \mathcal{P}(q) < \exp \left( \frac{c_2 L}{B \log L} \right); \quad q' = \prod_{p \mid q} p^\delta; \quad w \nmid q.$$

**Theorem 1.** Let $\eta > 0$, $t \geq 1$ and let $q = x^\theta$, $0 < \theta \leq 5/12 - \eta$, $(a, q) = 1$. Let

$$K(\theta) = \begin{cases} 
\frac{2}{1 - \theta} & \text{if } \theta < 2/5 - \varepsilon, \\
\frac{40\theta}{9 - 20\theta} & \text{if } \theta \geq 2/5 - \varepsilon.
\end{cases}$$

Suppose that $q$ satisfies $H(q, B, \delta)$ with

$$B = \frac{C_1}{\eta} \exp \left( \frac{4t}{K(\theta)} \right), \quad \delta = \frac{C_2 \eta}{t + \log(1/\eta)} \exp \left( - \frac{4t}{K(\theta)} \right)$$

for suitable absolute positive constants $C_1, C_2$. There are primes $p_1 < \cdots < p_t$ in $A$ with

$$p_t - p_1 < C_2 qt \exp (K(\theta)t).$$

Here $C_2$ is a positive absolute constant.

We shall deduce Theorem 1 from the following theorem of Bombieri-Vinogradov type. (See Section 6 for the details of this deduction.) Let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \mod q}} \Lambda(n),$$

where $\Lambda$ is the von Mangoldt function.

**Theorem 2.** Suppose that $A > 1$, $\eta > 0$, $q = x^\theta$ with $\theta < 5/12 - \eta$, $\varepsilon > 0$ and that $q$ satisfies $H(q, B, \delta)$ with $B$ and $\delta$ satisfying

$$B > \frac{5}{6\eta} (A + 2), \quad \delta \log \frac{1}{\delta} < \frac{c_2}{B}.$$ 

Let $v$ be the largest prime divisor of $w/(q, w)$ if $\chi^*$ exists and $v = 1$ otherwise. Then

$$\sum_{d \leq x^{L(\theta)}} \max_{(a, q) = 1} \left| \psi(x; qd, a) - \frac{x}{\varphi(qd)} \right| \ll \frac{x}{\varphi(q)} L^A.$$

Here $L(\theta) = 1/2 - \theta - \varepsilon$ if $\theta < 2/5 - \varepsilon$ and $L(\theta) = 9/20 - \theta - \varepsilon$ if $2/5 - \varepsilon < \theta \leq 5/12 - \eta$. The implied constant above depends on $\varepsilon$ and $A$.

Theorem 2 is a refinement of the work of P. D. T. A. Elliott [17]. In [17], $q$ is taken to be a power of a fixed integer while $\theta < 1/3$. Elliott used the work of H. Iwaniec [11] on the zero-free regions of $L$-functions; see [7] for the historical background on this topic.

For completeness, we also include an analog of the Barban-Davenport-Halberstam theorem ([5, Chapter 29]).

**Theorem 3.** Under the hypothesis of Theorem 2, we have

$$\sum_{d \leq Q/q} \sum_{a=1}^{\varphi(q)/d} \left( \psi(x; qd, a) - \frac{x}{\varphi(qd)} \right)^2 \ll \frac{xQL}{\varphi(q)}$$

whenever $xL^{-A} \leq Q \leq x$. The implied constant depends on $\eta$ and $A$. 
The reader will observe that the simpler condition
\[(1.6) \quad \prod_{p \mid q} \varphi(p) < \log C_3 x\]
with an absolute constant $C_3$, would give the conclusions of Theorems 1, 2 and 3 without any reference to the exceptional character. Moreover, a careful reading of our proof will show that the condition $(d, qv) = 1$ in Theorems 2 and 3 can be replaced by $(d, q) = 1$ when (1.6) holds.

2. Preliminary Lemmas

Let $v$ be as in Theorem 1 throughout. Unless otherwise stated, implied constants depend on $\varepsilon$ and, if $A$ is present, on $A$.

For a Dirichlet character $\chi$, we use $\hat{\chi}$ to denote the primitive character that induces $\chi$. Moreover, let
\[
\sum'_{\chi \mod r} \quad \text{and} \quad \sum^*_{\chi \mod r}
\]
stand for, respectively, a sum restricted to nonprincipal characters modulo $r$ and a sum restricted to primitive characters modulo $r$. As usual, let
\[
\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n).
\]

**Lemma 1.**

(i) We have, for $r < x$,
\[
\max_{(a, r) = 1} \left| \psi(x; r, a) - \frac{x}{\varphi(r)} \sum_{\chi \mod r} \psi(x, \chi) \right| \ll \frac{x}{\varphi(r)} \sum_{\chi \mod r} |\psi(x, \chi)| + \frac{x}{\varphi(r)} L A.
\]

(ii) For each of the characters $\chi$ in the above, we have
\[
|\psi(x, \chi)| - |\psi(x, \hat{\chi})| \ll L^2.
\]

(iii) We have
\[
\sum_{a=1}^{r} \left( \psi(x; r, a) - \frac{x}{\varphi(r)} \right)^2 \ll \frac{1}{\varphi(r)} \sum_{\chi \mod r} |\psi(x, \chi)|^2 + \frac{x^2}{\varphi(r)} L A.
\]

**Proof.** These are standard results. See, for example, pp. 162-163 and 169-170 in [5].

**Lemma 2.**

(i) For any natural number $r$ and any complex-valued function $F$ defined on Dirichlet characters, we have
\[(2.1) \quad \sum'_{\chi \mod r} F(\chi) = \sum_{r_1 | r} \sum^*_{\chi \mod r_1} F(\chi_1).
\]

(ii) Suppose further that $F \geq 0$ and $qD < x$. There exists $D_1 \leq D$ such that
\[(2.2) \quad \sum_{d \leq D} \sum_{(d, qv) = 1} \sum'_{\chi \mod qd} F(\chi) \ll \frac{LD}{D_1} \sum_{q_1 | q} \sum_{d_1 \leq D_1} \sum_{(d_1, qv) = 1} \sum^*_{\chi_1 \mod q_1 d_1} F(\chi_1).
\]

**Proof.** The equation (2.1) is immediate from allocating the conductors of $\hat{\chi}$ into classes corresponding to divisors of $r$. For (2.2), the left-hand side is
\[
\sum_{d \leq D} \sum_{(d, qv) = 1} \sum_{q_1 | q} \sum_{d_1 \leq D_1} \sum_{(d_1, qv) = 1} \sum_{\chi_1 \mod q_1 d_1} F(\chi_1) \ll \frac{LD}{D_1} \sum_{q_1 | q} \sum_{D_1 < d_1 \leq 2D_1} \sum_{\chi_1 \mod q_1 d_1} F(\chi_1)
\]
Lemma 3. Let $L = L(\theta)$ as in Theorem 2 and

$$R(x; r, a) = \sum_{n \leq x \atop n \equiv a \mod r} \Lambda(n) \log \frac{x}{n}.$$  

Suppose that for $qD \ll x^L$ and some $A > 0$,

$$\sum_{D < d \leq 2D \atop (d, qv) = 1} \max_{(a, qd) = 1} \left| \frac{R(x; qd, a) - x}{\varphi(qd)} - \frac{x}{\varphi(q)^2 L^A + 1} \right| \ll \frac{x}{\varphi(q)^2 L^A + 1}.$$  

Then for $qD \ll x^L$,

$$\sum_{D < d \leq 2D \atop (d, qv) = 1} \max_{(a, qd) = 1} \left| \frac{x}{\varphi(q)} - \frac{R(x; qd, a) - x}{\varphi(qd)} \right| \ll \frac{x}{\varphi(q)^2 L^A}.$$  

Proof. We start with the identity

$$R(x; r, a) = \int_1^x \psi(y; r, a) \frac{dy}{y}.$$  

This, together with the fact that $\psi(y; r, a)$ is nondecreasing in $y$, gives that for all $\lambda > 0$,

$$\frac{R(x; r, a) - R(x e^{-\lambda}; r, a)}{\lambda} = \frac{1}{\lambda} \int_{e^{-\lambda} x}^x \psi(y; r, a) \frac{dy}{y} \leq \psi(x; r, a) \leq \frac{1}{\lambda} \int_{e^{-\lambda} x}^x \psi(y; r, a) \frac{dy}{y} = \frac{R(x e^{-\lambda}; r, a) - R(x; r, a)}{\lambda}.$$  

This leads to

$$\psi(x; r, a) - \frac{x}{\varphi(r)} \leq \frac{R(x; r, a) - x}{\lambda} - \frac{R(x; r, a) - x/\varphi(r)}{\lambda} + \left( \frac{e^\lambda - 1}{\lambda} - 1 \right) \frac{x}{\varphi(r)}$$  

and

$$\psi(x; r, a) - \frac{x}{\varphi(r)} \geq \frac{R(x; r, a) - x}{\lambda} - \frac{R(x; r, a) - e^{-\lambda} x/\varphi(r)}{\lambda} + \left( 1 - \frac{e^\lambda - 1}{\lambda} \right) \frac{x}{\varphi(r)}.$$  

Take $\lambda = L^{-A-1}$ so that

$$\frac{e^\lambda - 1}{\lambda} - 1 \ll L^{-A-1} \quad \text{and} \quad \frac{1 - e^{-\lambda}}{\lambda} - 1 \ll L^{-A-1}.$$  

We get, taking $D \ll x^L/q$, $r = qd$ and summing over $d \in (D, 2D]$, there is $\mu \in \{1, 0, -1\}$ for which

$$\sum_{D < d \leq 2D \atop (d, qv) = 1} \max_{(a, qd) = 1} \left| \psi(x; qd, a) - \frac{x}{\varphi(qd)} \right| \ll L^{A+1} \sum_{D < d \leq 2D \atop (d, qv) = 1} \max_{(a, qd) = 1} \left| R(e^\mu x; qd, a) - \frac{e^\mu x}{\varphi(qd)} \right| + \frac{e^\mu x}{\varphi(q)} \sum_{1 \leq d \leq 2D} \frac{1}{\varphi(d)} \ll \frac{x}{\varphi(q) L^A},$$  

using (2.4).
In the following lemma, let \( \beta + i\gamma \) denote a zero of any of the Dirichlet \( L \)-functions \( L(s, \chi) \) with \( \chi \) a non-principal character modulo \( r \).

**Lemma 4.** Let \( r < x \). Then

\[
\sum_{\chi \mod r} \sum_{\beta \geq 1/2, |\gamma| < x^{1/2}} \left| \frac{x^\beta L^{(1+1)} + x^{1/2} r L^2 + x}{L^A} \right| \ll \sum_{\chi \mod r} \sum_{\beta \geq 1/2, |\gamma| < x^{1/2}} \left| \frac{x^\beta L^{(1+1)} + x^{1/2} r L^2 + x}{L^A} \right|
\]

**Proof.** This is a very slight variant of a result established by Elliott [7, pp. 248-249]. \( \square \)

Let \( N(\sigma, T, \chi) \) denote the number of zeros of \( L(s, \chi) \) in the rectangle \([\sigma, 1) \times [-T, T]\). We shall need the following zero density result.

**Lemma 5.** We have, for \( T \geq 1 \) \( 1/2 \leq \sigma < 1 \)

\[
\sum_{\chi \mod r} N(\sigma, T, \chi) \ll (rT)^{12/5+\epsilon}(1-\sigma).
\]

**Proof.** This is obtained by combining the results of M. N. Huxley [10] and M. Jutila [12]. \( \square \)

**Lemma 6.** Let \( a_n (n = 1, \ldots, N) \) be complex numbers and

\[
T(\chi) = \sum_{n=1}^N a_n \chi(n).
\]

For any natural numbers \( r \) and \( D \), we have

\[
\sum_{r_1|r} \sum_{d \leq D} \sum_{(d, r_1) = 1} \frac{r_1 d}{\varphi(r_1 d)} \sum_{\chi \mod r_1 d} |T(\chi)|^2 \ll (N + rD^2) \sum_{n=1}^N |a_n|^2.
\]

**Proof.** This is a variant of Lemma 6.5 in [6]. Set

\[
S(x) = \sum_{n=1}^N a_n e(nx),
\]

where \( e(z) = \exp(2\pi i z) \). Let

\[
S = \left\{ \frac{j}{dr_1} : 1 \leq j \leq dr_1, (j, dr_1) = 1, d \leq D, (d, r_1) = 1, r_1|r \right\}.
\]

It is easy to see that

\[
|s - s'| \geq \frac{1}{rD^2}
\]

for all distinct \( s \) and \( s' \) in \( S \). From the classical large sieve inequality (see [5, Chapter 27]), we get

\[
\sum_{s \in S} |S(s)|^2 \ll (N + rD^2) \sum_{n=1}^N |a_n|^2.
\]

Now by standard techniques that relate multiplicative characters to additive ones (see (10) on page 160 of [5]), we get

\[
\sum_{\chi \mod r_1 d} \frac{r_1 d}{\varphi(r_1 d)} |T(\chi)|^2 \leq \sum_{j=1}^{r_1 d} \left| S \left( \frac{j}{r_1 d} \right) \right|^2.
\]

Now the lemma follows by summing over pairs of \( r_1 \) and \( d \) with \( r_1|r \) and \( d \leq D \) with \( (d, r_1) = 1 \) in (2.6). \( \square \)
Lemma 7. Let $N \leq x$, $qD \leq x$ and $\mathcal{U}$ be a set of non-principal characters to moduli $qd$ with $d \leq D$, $(d, q) = 1$ and $q_1|q$. Suppose that, with $T(\chi)$ as in (2.4),

$$|T(\chi)| \geq V > 0$$

whenever $\chi \in \mathcal{U}$ and that $G = \sum_{n=1}^{N} |a_n|^2$. Then

$$\# \mathcal{U} \ll x^{5/20} \left(GV^{-2} N + G^3 V^{-6} NqD^2\right).$$

Proof. The contribution to $\# \mathcal{U}$ from a fixed $q_1|q$ is

$$\ll x^{5/20} \left(GV^{-2} N + G^3 V^{-6} Nq_1D^2\right)$$

by virtue of [10] Theorem 1. The lemma follows on summing over $q_1$ with $q_1|q$. ☐

Lemma 8. For $r \geq 3$ and $T \geq 1$,

$$\sum_{\chi \mod r}^{*} \left| L \left(\frac{1}{2} + it, \chi \right) \right|^4 dt \ll \varphi^*(r) T (\log rT)^4.$$ 

Proof. See [9] for a more precise form of this result. ☐

Lemma 9. Let $qD < x$, $N \leq x$, $|t| \leq x^2$ and

$$N(x, \chi) = \sum_{N < n \leq N'} \chi(n)n^{-s}$$

where $N$ and $N'$ are natural numbers with $N' \leq 2N$. Then

$$\sum_{q_1|q} \sum_{d \leq D} \sum_{\chi \mod q_1d}^{*} \left| N \left(\frac{1}{2} + it, \chi \right) \right|^4 \ll \varphi(q)D^2 \mathcal{L}^5 \left(1 + |t|\right).$$

Proof. Using Perron’s formula ([16] Lemma 3.12), we see that

$$N \left(\frac{1}{2} + it, \chi \right) = \int_{2 - ix^2}^{2 + ix^2} L \left(\frac{1}{2} + it + w, \chi \right) \left(\frac{(N' + 1/2)^w - (N + 1/2)^w}{w}\right) \mathrm{d}w + O(1).$$

By using the standard bounds for $L(s, \chi)$, we can move the line of integration to $[-ix^2, ix^2]$ at the cost of an error of size $O(1)$. By a splitting-up argument, it suffices to show that for $1 \leq T \leq x^2$ that

(2.7) $$\sum_{q_1|q} \sum_{d \leq D} \sum_{\chi \mod q_1d}^{*} \left(\frac{1}{T} \int_{T-1}^{2T} \left| L \left(\frac{1}{2} + it + iu, \chi \right) \right| \mathrm{d}u \right)^4 \ll \varphi(q)D^2 \mathcal{L}^4 \left(1 + |t|\right).$$

By Hölder’s inequality,

$$\left(\frac{1}{T} \int_{T-1}^{2T} \left| L \left(\frac{1}{2} + it + iu, \chi \right) \right| \mathrm{d}u \right)^4 \ll \frac{1}{T} \int_{T-1}^{2T} \left| L \left(\frac{1}{2} + it + iu, \chi \right) \right|^4 \mathrm{d}u.$$

Recalling Lemma 8 and (2.3), the left-hand side of (2.7) is

$$\ll \frac{1}{T} \sum_{q_1|q} \sum_{d \leq D} \sum_{\chi \mod q_1d}^{*} \int_{T-1+t}^{2T+t} \left| L \left(\frac{1}{2} + iv, \chi \right) \right|^4 \mathrm{d}v$$

$$\ll \frac{1}{T} \sum_{q_1|q} \sum_{d \leq D} \varphi^*(q_1d)\mathcal{L}^4(T + |t|) = \frac{\varphi(q)}{T} \sum_{d \leq D} \varphi^*(d)\mathcal{L}^4(T + |t|).$$

Now the lemma follows at once from this. ☐
GAPS OF SMALLEST POSSIBLE ORDER BETWEEN PRIMES IN AN ARITHMETIC PROGRESSION

Next, we have the Heath-Brown decomposition of the von Mangoldt function.

**Lemma 10.** Let \( f(n) \) be an arbitrary complex-valued function and \( k \in \mathbb{N} \). We can decompose the sum

\[
\sum_{n \leq x} \Lambda(n) f(n)
\]

into \( O(L^{2k}) \) sums of the form

\[
\sum_{n_i \in \{N_i, 2N_i\}} \log n_1 \mu(n_{k+1}) \cdots \mu(n_{2k}) f(n_1 \cdots n_{2k})
\]

in which \( N_i \geq 1 \), \( \prod_i N_i < x \) and \( 2N_i \leq x^{1/k} \) if \( i > k \).

**Proof.** This is from [8]. \( \square \)

3. The Small Values of \( d \) in Theorem 2

Let \( C_4 \) be a suitable positive absolute constant. We deal with the natural numbers \( d \leq L^{2A+C_4} \) in Theorem 2 by proving the following lemma.

**Lemma 11.** Let \( q \) be as in Theorem 2. Then

\[
\sum_{\chi \mod q}^\prime |\psi(x, \chi)| \ll \frac{x}{L^{A+1}}.
\]

**Proof.** By Lemma 4, the left-hand side of (3.1) is

\[
\ll \frac{x}{L^{A+1}} + x^\sigma L^{A+3} \sum_{\chi \mod q}^\prime \sum_{\sigma \leq \beta < \sigma + L^{-1}} \frac{1}{|\beta + i\gamma|^2}
\]

for some \( \sigma \) with \( 1/2 \leq \sigma < 1 \).

Chang’s zero-free region evidently does not require primitive \( \chi \), but only that \( \chi \) is not induced by \( \chi^* \). We use this to bound \( \sigma \). We clearly have, recalling \((H(q, B, \delta))\),

\[
B \log \frac{L}{\mathcal{L}} < \frac{c_2}{\log P}
\]

from the first inequality in (1.2). Moreover, if \( T = x^{1/2} \), then

\[
B \log \frac{L}{\mathcal{L}} < \frac{c_2}{(\log qT)^{9/10}}.
\]

Now we write \( q' = x^\gamma \) and \( q = x^\alpha \). Of course we have

\[
\frac{\log 2}{\mathcal{L}} \leq \gamma < \delta \quad \text{and} \quad \alpha < \frac{5}{12}.
\]

So, mindful that \( \delta \) is small and \( x \) is large, we get

\[
\gamma \log \left( \frac{2\alpha}{\gamma} \right) < \gamma \log \frac{1}{\gamma} < \delta \log \frac{1}{\delta} < \frac{c_2 \log \gamma \mathcal{L}}{B \log \mathcal{L}},
\]

which implies that

\[
\frac{B \log \mathcal{L}}{L} < \frac{c_2 \log \log q'}{(\log q') \log 2K}.
\]

Now the combination of (3.3), (3.4) and (3.5) gives that

\[
\frac{B \log \mathcal{L}}{L} < c_2 \min \left( \frac{1}{\log P^*}, \frac{\log \log q'}{(\log q') \log 2K}, \frac{1}{(\log qT)^{9/10}} \right).
\]
From (1.2), and since no \( \chi \pmod{q} \) is induced by \( \chi^* \),

\[
\sum'_{\chi \pmod{q}} \sum_{\sigma \leq \beta < \sigma + L^{-1}} \frac{1}{|\beta + i\gamma|^2}
\]

is empty if

\[\sigma \geq 1 - B \log L \frac{L}{L}.\]

Suppose now that \( \sigma < 1 - B \log L / L \) in (3.2). It suffices to show that

\[
S := \sum'_{\chi \pmod{q}} \sum_{\sigma \leq \beta < \sigma + L^{-1}} \frac{1}{|\beta + i\gamma|^2} \ll x^{1 - \sigma} L^{2A + 4}. 
\]

Using Lemma 5,

\[
S \leq \sum'_{\chi} \sum_{j \geq 2, 2^{j-2} \leq x^{1/2}} 2^{-2j} N(\sigma, 2j+1, \chi) \ll \sum_{j \geq 0} 2^{-j/2} q^{(12/5 + \varepsilon)(1 - \sigma)} \ll x^{(12/5 + \varepsilon)(5/12 - \eta)(1 - \sigma)}. 
\]

Therefore,

\[
S \ll x^{1 - \sigma} L^{2A + 4} \ll x^{-(12\eta/5 - \varepsilon)(1 - \sigma)} L^{2A + 4} \ll L^{2A + 4 - (12\eta/5 - \varepsilon)B} \ll 1.
\]

by (1.4). This completes the proof of the lemma.

\[\square\]

**Lemma 12.** Let \( q \) be as in Theorem 2. Then

\[
\sum_{d \leq L^{2A + C_4}} \max_{(d,q) = 1} \left| \psi(x; dq, a) - \frac{x}{\varphi(qd)} \right| \ll \frac{x \log L}{\varphi(q) L^A}. 
\]

**Proof.** Note that since \( d \leq L^{2A + C_4} \) and \( q \) satisfies the bounds in Theorem 2 we have the bounds

\[
\max\{p : p|(dq)\} < \exp\left( \frac{c_2L}{B' \log L} \right)
\]

and

\[
\prod_{p|(dq)} p < x^{\delta'}
\]

with some positive constants \( B' \) and \( \delta' \) satisfying

\[B' > \frac{5}{6\eta'}(A + 2) \text{ with } 0 < \eta' < \eta, \quad \delta' \log \frac{1}{\delta'} < \frac{c}{B'} \]

We apply Lemma 11 with \( qd \) in place of \( q \) for \( d \leq L^{2A + C_4} \) and \( (d, qv) = 1 \). Note that the condition \( (d, qv) = 1 \) implies that no character to modulus \( dq \) is induced by \( \chi^* \). Moreover, \( qd \leq x^{5/12 - \eta'} \). Hence

\[
(3.6) \quad \sum'_{\chi \pmod{qd}} |\psi(x; \chi)| \ll \frac{x}{L^{A+1}}.
\]

Now

\[
\max_{(a,qd) = 1} \left| \psi(x; dq, a) - \frac{x}{\varphi(qd)} \right| \ll \frac{1}{\varphi(qd)} \sum'_{\chi \pmod{qd}} |\psi(x; \chi)| + \frac{x}{\varphi(qd) L^{A+1}} \ll \frac{x}{\varphi(q) \varphi(d) L^{A+1}}.
\]

Since

\[
\sum_{d \leq L^{2A + C_4}} \frac{1}{\varphi(q)} \ll \log L,
\]

we get the lemma by summing over \( d \).

\[\square\]
Recalling Lemma 3 it remains to show that
\[
\sum_{D < d \leq 2D \atop (d, q_1) = 1} \left| R(x; q_1d, a(d)) - \frac{x}{\varphi(q_1)} \right| \ll \frac{x}{\varphi(q_1) L^{2A + \varepsilon}}
\]
whenever \( L^{2A + C_4} < D \ll X^{L(0) - \varepsilon} \), for any sequence \( a(d) \) with \( (a(d), dq_1) = 1 \).

Now
\[
R(x; q_1d, a(d)) = \frac{1}{\varphi(q_1)} \sum_{n \equiv y \mod q_1d} \Lambda(n) \sum_{n \leq x} \Lambda(n) \chi(n) \log \frac{x}{n},
\]
The contribution of \( \chi_0 \) to the last expression is
\[
\frac{1}{\varphi(q_1)} \int_1^x \sum_{n \equiv y \mod q_1d} \Lambda(n) \frac{dy}{y} = \frac{x}{\varphi(q_1)} \left( 1 + O \left( L^{-(2A + 4)} \right) \right).
\]
Therefore,
\[
R(x; q_1d, a(d)) - \frac{x}{\varphi(q_1)} = \frac{1}{\varphi(q_1)} \sum'_{n \equiv y \mod q_1d} \chi_0(a(d)) \int_1^x \psi(y, \chi) \frac{dy}{y} + O \left( \frac{x}{\varphi(q_1) L^{2A + 4}} \right).
\]
By replacing \( \psi(y, \chi) \) by \( \psi(y, \hat{\chi}) \) in (4.1), we incur an error of size
\[
\ll L^3 \ll \frac{x}{\varphi(q_1) L^{2A + 4}}.
\]
Therefore, it suffices to show that
\[
\sum_{D < d \leq 2D \atop (d, q_1) = 1} \sum'_{n \equiv y \mod q_1d} \left| \int_1^x \psi(y, \hat{\chi}) \frac{dy}{y} \right| \ll \frac{x D}{L^{2A + 3}}.
\]
Now by virtue of (ii) of Lemma 2 the last sufficiency can be further reduced to showing for \( D_1 \leq D \) that
\[
S(D_1) := \sum_{q_1 \mid q} \sum_{D_1 < d \leq 2D_1 \atop (d, q_1) = 1} \sum^*_{n \equiv y \mod q_1d} \left| \int_1^x \psi(y, \hat{\chi}) \frac{dy}{y} \right| \ll \frac{x D_1}{L^{2A + 4}}.
\]
For brevity, we write \( \sum^\dagger \) in place of
\[
\sum_{q_1 \mid q} \sum_{D_1 < d \leq 2D_1 \atop (d, q_1) = 1} \sum^*_{n \equiv y \mod q_1d}.
\]
Recasting the absolute value signs as coefficients, we have
\[
S(D_1) = \sum^\dagger b(\chi) \int_1^x \psi(y, \chi) \frac{dy}{y} = \sum^\dagger b(\chi) \sum_{n \leq x} \Lambda(n) \chi(n) \log \frac{x}{n},
\]
Now applying Lemma 12 with \( k = 6 \) and
\[
f(n) = \sum^\dagger b(\chi) \chi(n) \log \frac{x}{n},
\]
we see that it suffices to show for each tuple \( N_1, \ldots, N_12 \) that
\[
\sum^\dagger b(\chi) \sum_{n_i \in [N_i, 2N_i] \atop n_1 \cdots n_{12} \leq x} a_1(n_1) \cdots a_{12}(n_{12}) \chi(n_1 \cdots n_{12}) \log \frac{x}{n_1 \cdots n_{12}} \ll \frac{x D_1}{L^{2A + 16}}.
\]
Using the formula
\[
\int_{1/2-i\infty}^{1/2+i\infty} y^s \frac{ds}{s^2} = \begin{cases} \log y, & \text{if } y > 1 \\ 0, & \text{if } 0 < y \leq 1 \end{cases}
\]
(cf. [14] p. 143), we need to show that
\[
\sum^{|t|} b(x) \int_{1/2-i\infty}^{1/2+i\infty} \frac{a_1(n_1) \cdots a_{12}(n_{12}) \chi(n_1 \cdots n_{12}) x^s ds}{s^2} \leq xD_1 \frac{L^{2A+16}}{2^A}.
\]
Now the condition \(n_1 \cdots n_{12} \leq x\) can be removed, since the integral vanishes otherwise. We also use a trivial estimate to discard the part of the integral with \(|3s| > x^2\). Thus our task is further reduced to showing that
\[
\sum^{|t|} b(x) \int_{-x^2} x^2 N_1 \left(\frac{1}{2} + it, \chi\right) \cdots N_{12} \left(\frac{1}{2} + it, \chi\right) \frac{x^{1/2+it}}{(1/2 + it)^2} dt \leq \frac{xD_1}{L^{2A+16}},
\]
where
\[
N_j(s, \chi) = \sum_{N_j < n \leq 2N_j} \frac{a_j(n) \chi(n)}{n^s}.
\]
To this end, it suffices to prove that
\[
\sum^{|t|} \left| N_1 \left(\frac{1}{2} + it, \chi\right) \cdots N_{12} \left(\frac{1}{2} + it, \chi\right) \right| \leq \frac{x^{1/2} D_1 (1 + |t|)}{L^{2A+16}}
\]
for \(|t| \leq x^2\). It is convenient to recall here that \(qD_1 \ll x^{1/2 - \epsilon}\) for all \(\theta\) and \(qD_1 \ll x^{9/20 - \epsilon}\) for \(\theta \geq 2/5 - \epsilon\).

Let us write \(x_0 = \prod_{i=1}^{12} N_i\) and \(N_i = x_0^{\alpha_i}\) so that \(\alpha_i \geq 0, \alpha_1 + \cdots + \alpha_{12} = 1\) and \(x_0 \leq x\).

For a Dirichlet polynomial
\[
N(s) = \sum_{N < n \leq zN} a_n \chi(n)n^{-s}
\]
for some absolute constant \(z \in \mathbb{R}\), we use the abbreviation, for \(p > 1\),
\[
\|N\|_p = \left( \sum^{|s|} \left| N \left(\frac{1}{2} + it, \chi\right) \right|^p \right)^{1/p}
\]
and
\[
\|N\|_\infty = \max \left\{ \left| N \left(\frac{1}{2} + it, \chi\right) \right| : \chi \text{ appears in } \sum^{|s|} \right\}.
\]
Lemma \[3\] possibly in conjunction with a partial summation to incorporate a \(\log n\) factor, gives that
\[
\|N_j\|_4^4 \ll qD_1^2 L^6 (1 + |t|) \ll D_1 x^{1/2 - 2\epsilon/3} (1 + |t|)
\]
if \(N_j > x^{1/6}\). If \(N_j \leq x^{1/6}\), we obtain similar bounds from Lemma \[4\] applied to \(T = N_j^2\). Indeed, in this case,
\[
\|N_j\|_4^4 \ll (N_j^2 + qD_1^2) L^2 \ll \begin{cases} \frac{D_1 x^{1/2 - 2\epsilon/3}}{qD_1^2} \text{ if } \theta \geq 1/3. 
\end{cases}
\]

From now on, it is convenient to arrange \(N_1, \cdots, N_{12}\) so that
\[
N_1 \geq \cdots \geq N_{12}.
\]
The proof of (4.2) is divided into three cases.

**Case 1.** Suppose that \(N_1 N_2 \geq x_0^{1/2}\). Let \(M = N_3 \cdots N_{12}\). Then the left-hand side of (4.2) is
\[
\|MN_1 N_2\|_4 \leq \|M\|_2 \|N_1\|_4 \|N_2\|_4 \ll (M + qD_1^2)^{1/2} D_1^{1/2} x^{1/4 - \epsilon/3} (1 + |t|)^{1/2} \ll x^{1/4} D_1 x^{1/4 - \epsilon/3} (1 + |t|)^{1/2}.
\]
by Hölder’s inequality, Lemma \[3\] and (4.3). So (4.2) holds in Case 1.
Case 2. $N_1 N_2 < x_0^{1/2}$ and some sub-product $\prod_{i \in S} N_i$ (with $S \subseteq \{1, \ldots, 12\}$) satisfies

\begin{equation}
\label{eq:4.4}
x_0^{1/2} \leq N = \prod_{i \in S} N_i < x^{1 - \theta - \varepsilon}.
\end{equation}

Hence

\begin{equation}
M = \prod_{1 \leq i \leq 12} N_i \leq x_0^{1/2}.
\end{equation}

The left-hand side of (4.4) is, using Lemma 6 and with $C_4$ suitably chosen,

$$\|MN\|_1 \leq \|M\|_2 \|N\|_2 \ll (M + qD_1^{1/2})^{1/2} (N + qD_2^{1/2})^{1/2} L^{C_4/2} \ll (x_0^{1/2} + qD_2^2 + N^{1/2} q^{1/2} D_1) L^{C_4/2}.$$  

We clearly have $x_0^{1/2} L^{C_4/2} \ll x^{1/2} D_1 L^{-2A-17}$ as $D \geq L^{2A+C_4}$ and $q D_2^2 L^{C_4/2} \ll x^{1/2} D_1 L^{-2A-17}$. Lastly, using (4.4),

$$N^{1/2} q^{1/2} D_1 L^{C_4/2} \ll x^{1/2-\varepsilon/2} D_1 L^{C_4/2} \ll x^{1/2} D_1 L^{-2A-17}.$$  

So (4.2) also holds in Case 2.

We claim that if $\theta \leq 2/5 - \varepsilon$, then Case 1 or Case 2 must occur. Suppose not, then

$$\alpha_1 + \alpha_2 < \frac{1}{2}$$

and there is no sub-sum with

$$\frac{2}{5} \leq \sum_{i \in S} \alpha_i \leq \frac{3}{5}.$$

One can easily verify that this is impossible. See the details in Lemma 14 of [1].

Now we suppose that $2/5 - \varepsilon < \theta < 5/12$ and it still remains to consider

Case 3. $N_1 N_2 < x_0^{1/2}$ and no sub-product $\prod_{i \in S} N_i$ satisfies (4.4). Since $1 - \theta - \varepsilon \geq 7/12$, no sub-product $\prod_{i \in S} N_i$ lies in $[x_0^{5/12}, x_0^{7/12}]$. We start with a combinatorial lemma.

Lemma 13. Suppose that $\alpha_1 \geq \cdots \geq \alpha_{12} \geq 0$, $\alpha_1 + \cdots + \alpha_{12} = 1$, $\alpha_1 + \alpha_2 < 1/2$ and no sub-sum $\sum_{i \in S} \alpha_i$ for a set $S \subseteq \{1, \cdots, 12\}$ is in $[5/12, 7/12]$. Then $\alpha_5 > 1/6$ and

\begin{equation}
\alpha_1 + \alpha_2 + \alpha_6 + \alpha_7 + \cdots + \alpha_{12} < \frac{5}{12}.
\end{equation}

Proof. Clearly $\alpha_1 + \alpha_2 < 5/12$. Suppose that

\begin{equation}
\alpha_1 + \alpha_2 + \sum_{\alpha_i \leq 1/6} \alpha_i \geq \frac{5}{12}.
\end{equation}

Let $s$ be the least sum $\alpha_1 + \alpha_2 + \sum_{i \in B} \alpha_i$, for some set $B \subseteq \{i : \alpha_i \leq 1/6\}$, that is greater than $5/12$. This implies that $5/12 \leq s < 5/12 + 1/6 = 7/12$, contradicting one of the conditions of the lemma. So (4.6) must be false.

We can write $\{i : \alpha_i < 1/6\}$ as $\{i : \alpha_i > t\}$ for some $t$ with $1 \leq t \leq 12$. If $t \geq 6$, then by the previously-established falsehood of (4.6) and that the $\alpha_i$’s are in descending order,

$$\alpha_1 + \cdots + \alpha_{12} > \frac{t}{6} \geq 1$$

which is false. If $t \leq 4$, then

$$\alpha_1 + \cdots + \alpha_{12} \leq \left(\alpha_1 + \alpha_2 + \sum_{i > t} \alpha_i\right) + (\alpha_3 + \alpha_4) < \frac{5}{12} + \frac{5}{12} < 1$$

which is also false. Therefore, $t = 5$ and both claims of the lemma are proved. \qed
By Lemma 13 in Case 3, we can partition $N_1 \cdots N_{12}$ into three parts $M$, $N$ and $N_5$,

$$M(s, \chi) = N_1(s, \chi)N_2(s, \chi) \prod_{i \geq 6} N_i(s, \chi) = \sum_{M \leq m \leq M} \alpha_m \chi(m)m^{-s}$$

and

$$N(s, \chi) = N_3(s, \chi)N_4(s, \chi) = \sum_{N \leq n \leq N} \beta_n \chi(n)n^{-s},$$

where $M < \frac{5}{12}N$, $N < \frac{5}{12}N$, $N > \frac{1}{6}N$, $M \geq N$. So $MN \geq \frac{1}{2}N$.

We need the stronger assertion that

$$(4.7) \quad N_5 > x^{1/6-\epsilon}.$$  

If this does not hold, then

$$x^{1/2} \leq MN_5 < x^{5/12}x^{1/6-\epsilon} < x^{1-\theta-\epsilon},$$

an impossibility in Case 3.

The utility of (4.7) stems partly from the following lemma.

**Lemma 14.** Let $\chi$ be a character modulo $q_1d$ that appears in $\sum^1$. Then

$$\sum_{k \leq K} \chi(k) \ll K^{1-\epsilon/2}$$

whenever $K \geq x^{3/20}$.

**Proof.** By a theorem of D. A. Burgess [3], we have

$$\sum_{k \leq K} \chi(n) \ll (q_1d)^{1/9+\epsilon^2}K^{2/3} \ll x^{(9/20-\epsilon)(1/9+\epsilon^2)}K^{2/3} \ll K^{1-\epsilon/2},$$

completing the proof. \hfill $\square$

It is easy to obtain

$$(4.8) \quad \|N_5\|_\infty \ll N_5^{1/2}x^{-\epsilon/13}(1 + |t|)$$

from Lemma 14, (4.7) and a partial summation argument.

The contribution in (4.2) form $\chi$ with

$$\min \left\{ \left| M \left( \frac{1}{2} + it, \chi \right) \right|, \left| N \left( \frac{1}{2} + it, \chi \right) \right|, \left| N_5 \left( \frac{1}{2} + it, \chi \right) \right| \right\} < x^{-1}$$

is clearly

$$\ll \sum^\dagger 1 \ll x^{1/2-\epsilon}D_1.$$  

Therefore, by a splitting-up argument, it suffices to show, for any $U$, $V$ and $W$ with

$$U \leq \|N_5\|_\infty, \quad V \leq \|M\|_\infty \quad \text{and} \quad W \leq \|N\|_\infty,$$

that

$$UVW \#A(U, V, W) \ll (1 + |t|)x^{1/2}D_1L^{-2A-20}.$$  

Here

$$A(U, V, W) = \left\{ \chi : \chi \text{ appears in } \sum^\dagger, U < \left| N_5 \left( \frac{1}{2} + it, \chi \right) \right| \leq 2U, \right.$$

$$V < \left| M \left( \frac{1}{2} + it, \chi \right) \right| \leq 2V, \quad W < \left| N \left( \frac{1}{2} + it, \chi \right) \right| \leq 2W. \left. \right\}$$
Now let
\[ P = \min \left\{ \frac{M + qD_1^2}{V^2}, \frac{N + qD_1^2}{W^2}, \frac{qD_1^2 M}{V^2}, \frac{qD_1^2 N}{W^2}, \frac{qD_1^2 N_5^2}{U^4} \right\}. \]

It is a consequence of Lemmas 6, 7 and the first inequality in (4.3) that
\[ \#A(U, V, W) \ll P^{x_2/20}. \]

So it is enough to show that
\[ UVWP \ll x^{1/2-\varepsilon/13} D_1(1 + |t|). \]

To do this, we consider four sub-cases, according to the size of $P$ in comparison with those of $2V^{-2}M$ and $2W^{-2}N$.

(a) $P \leq 2V^{-2}M$ and $P \leq 2W^{-2}N$. In this case, (4.8) yields
\[ UVWP \ll UVW(V^{-2}M)^{1/2}(W^{-2}N)^{1/2} \ll (MN)^{1/2}\|N_5\|_\infty \ll x^{1/2-\varepsilon/13}(1 + |t|), \]
as desired for (4.9).

(b) $P > 2V^{-2}M$ and $P > 2W^{-2}N$. Here, we have
\[ P \leq 2 \min \left\{ qD_1^2 V^{-2}, qD_1^2 W^{-2}, qD_1^2 MV^{-6}, qD_1^2 NW^{-6}, (1 + |t|)qD_1^2 U^{-4}, N_5^2 U^{-4} \right\} + 2 \min \left\{ qD_1^2 V^{-2}, qD_1^2 W^{-2}, qD_1^2 MV^{-6}, qD_1^2 NW^{-6}, (1 + |t|)qD_1^2 U^{-4}, qD_1^2 N_5^2 U^{-12} \right\} \leq 2(qD_1^2 V^{-2})^{5/16}(qD_1^2 W^{-2})^{5/16}(qD_1^2 MV^{-6})^{1/16}(qD_1^2 NW^{-6})^{1/16} \left( \min \left\{ qD_1^2 U^{-4}, N_5^2 U^{-4} \right\} \right)^{1/4} (1 + |t|)^{1/4} + 2 \min \left\{ (qD_1^2 V^{-2})^{5/16}(qD_1^2 W^{-2})^{5/16}(qD_1^2 MV^{-6})^{1/16}(qD_1^2 NW^{-6})^{1/16}(qD_1^2 U^{-4})^{1/4}(1 + |t|)^{1/4}, (qD_1^2 V^{-2})^{7/16}(qD_1^2 W^{-2})^{7/16}(qD_1^2 MV^{-6})^{1/48}(qD_1^2 NW^{-6})^{1/48}(qD_1^2 N_5^2 U^{-12})^{1/12} \right\} \leq 2(1 + |t|)^{1/4}(UVW)^{-1}qD_1^2(MN)^{1/16} \left( \min \left\{ 1, (qD_1^2)^{-1/4}N_5^{1/2} \right\} + \min \left\{ 1, N_5^{1/6}(MN)^{-1/21} \right\} \right) \ll (1 + |t|)^{1/4}(UVW)^{-1} \left( x^{1/16}(qD_1^2)^{31/32} + x^{1/20}qD_1^2 \right). \]

Now, noting that
\[ x^{1/16}(qD_1^2)^{31/32} \ll x^{1/16+31/32-9/20} D_1^{31/32} \ll x^{1/2-\varepsilon} D_1 \]
and
\[ x^{1/20}qD_1^2 \ll x^{1/20+9/20-\varepsilon} D_1 \ll x^{1/2-\varepsilon} D_1, \]
we get that
\[ P \ll (1 + |t|)^{1/4}(UVW)^{-1}x^{1/2-\varepsilon} D_1, \]
which gives (4.9).
(c) \( P > 2V^{-2}M \) and \( P \leq 2W^{-2}N \). Now we have
\[
P \leq 2 \min \left\{ qD_1^2V^{-2}, NW^{-2}, qD_2^2MV^{-6}, qD_1^2U^{-4}(1 + |t|), N_5^2U^{-4} \right\} + 2 \min \left\{ qD_1^2V^{-2}, NW^{-2}, qD_2^2MV^{-6}, qD_1^2U^{-4}(1 + |t|), qD_1^2N_5^2U^{-12} \right\}
\]
\[
\leq 2(qD_1^2V^{-2})^{1/8}(NW^{-2})^{1/2}(qD_2^2MV^{-6})^{1/8} \left( \min \left\{ qD_1^2U^{-4}, N_5^2U^{-4} \right\} \right)^{1/4} (1 + |t|)^{1/4} + 2 \min \left\{ (qD_1^2V^{-2})^{1/8}(NW^{-2})^{1/2}(qD_2^2MV^{-6})^{1/8}(qD_1^2U^{-4})^{1/4} (1 + |t|)^{1/4}, \right.
\]
\[
\left. (qD_1^2V^{-2})^{3/8}(NW^{-2})^{1/2}(qD_2^2MV^{-6})^{1/24}(qD_1^2N_5^2U^{-12})^{1/12} \right\}
\]
\[
\leq 2(1 + |t|)^{1/4}(UVW)^{-1}(qD_1^2N_5^{1/2})^{1/2} M^{1/8} \left( \min \left\{ 1, (qD_1^2)^{-1/4} N_5^{1/2} \right\} + \min \left\{ 1, N_5^{-1/6}M^{-1/12} \right\} \right)
\]
\[
\ll (1 + |t|)(UVW)^{-1} \left( x^{1/8}(qD_1^2)^{7/16}N^{3/8} + x^{1/12}(qD_1^2)^{1/2}N^{5/12} \right).
\]
To estimate these last two terms, we have
\[
x^{1/8}(qD_1^2)^{7/16}N^{3/8} \ll x^{1/8}(qD_1)^{7/16}D_1^{7/16}(x^{5/12})^{3/8} \ll x^{1/8+9/20}7/16+5/12 \ll 2x^{1/2-\varepsilon}D_1
\]
and
\[
x^{1/12}(qD_1^2)^{1/2}N^{5/12} \ll x^{1/12}(qD_1)^{1/2}D_1^{1/2}x^{25/144} \ll x^{1/12+9/40+25/144}D_1^{1/2} \ll x^{1/2-\varepsilon}D_1.
\]
These bounds lead to
\[
P \ll (1 + |t|)^{1/4}(UVW)^{-1}x^{1/2-\varepsilon}D_1,
\]
giving (4.19).

(d) \( P > 2W^{-2}N \) and \( P \leq 2V^{-2}M \). We proceed the same way as in subcase (c), interchanging the roles of \( M \) and \( N \).

This completes the proof of Theorem 2.

5. Proof of Theorem 3

From (iii) of Lemma 11 we get
\[
\sum_{d \leq Q/q} \sum_{a=1 \atop (a,dq)=1}^{dq} \left( \psi(x; dq, a) - \frac{x}{\varphi(qd)} \right)^2 \ll \sum_{d \leq Q/q} \frac{1}{\varphi(qd)} \sum_{\chi \mod dq} \left| \psi(x, \chi) \right|^2 + \frac{x^2}{\varphi(q) \mathcal{L}^2A} \sum_{d \leq Q/q} \frac{1}{\varphi^2(d)}.
\]
As the second term is \( \ll Qx\varphi(q)^{-1} \), it suffices to prove that
\[
\sum_{d_1 \leq Q/q} \frac{1}{\varphi(qd)} \sum_{\chi \mod dq} \left| \psi(x, \chi) \right|^2 \ll \frac{Qx\mathcal{L}}{\varphi(q)}.
\]
and that
\[
\sum_{d \leq Q/q} \frac{1}{\varphi(qd)} \sum_{\chi \mod dq} \left( \left| \psi(x, \chi) \right|^2 - \left| \psi(x, \chi) \right|^2 \right) \ll \frac{Qx\mathcal{L}}{\varphi(q)}.
\]
It is easy to see that, in (5.2),
\[
\left| \psi(x, \chi) \right|^2 - \left| \psi(x, \chi) \right|^2 \ll \left( \sum_{p^k \leq x} \log p \right) \left( \sum_{p^k \mid dq} \log p \right).
\]
The contribution to (5.2) from \( k \geq 2 \) is
\[
\ll \sum_{d \leq Q/q \atop (d,qv)=1} x^{1/2+\varepsilon} \ll \frac{Q x^{1/2+\varepsilon}}{q}
\]
which is acceptable. The contribution from \( k = 1 \) to (5.2) is
\[
\ll \sum_{d \leq Q/q \atop p|dq} \frac{1}{\varphi(d)} \sum_{q \leq x \atop q|d} \sum_{p \leq x} \log p \ll \frac{Q x}{q} \sum_{p \leq x} \log p \ll \frac{xQ \varphi(q)}{p} \ll \frac{xQ \varphi(q)}{2}\]
which is also acceptable. (Incidentally, the error term corresponding to (5.2) is treated incorrectly on page 170 of [5]; the above discussion corrects this minor error.)

It remains to prove (5.1) in the form
\[(5.3) \quad \sum_{q_1|q \atop (d_1,qv)=1} \frac{1}{\varphi(d_1)} \sum_{q \leq x} \sum_{\chi \mod dq_1} \sum^{*} \psi(x,\chi)|^2 \ll Q x \mathcal{L}.
\]
We split the sum over \( d_1 \) in (5.3) into dyadic sub-sums of the form \( \sum_{D<d_1 \leq 2D} \) where \( D \) takes on the values \( 2^{-k}Q/q, k \geq 1 \) and \( 2^{-k}Q/q > 1/2 \). Let \( \Sigma_D \) denote the contribution to (5.3) from a given \( D \). Hence
\[\Sigma_D \ll \left( \log \frac{Q}{qD} \right) \sum_{D<d_1 \leq 2D \atop (d_1,qv)=1} \frac{1}{\varphi(d_1)} \sum_{q_1|q \atop q \mod dq_1} \sum^{*} \psi(x,\chi)|^2 .\]
We first deal with the contributions from \( D \leq \mathcal{L}^{2A} \):
\[
\sum_{D \leq \mathcal{L}^{2A}} \Sigma_D \ll \mathcal{L} x \sum_{d_1 \leq \mathcal{L}^{2A} \atop (d_1,qv)=1} \frac{1}{\varphi(d_1)} \sum_{q_1|q \atop q \mod dq_1} \sum^{*} \psi(x,\chi)|^2
\ll \mathcal{L} x \sum_{d_1 \leq \mathcal{L}^{2A} \atop (d_1,qv)=1} \frac{1}{\varphi(d_1)} \sum_{\chi \mod dq} \left( |\psi(x,\chi)| + \mathcal{L}^2 \right) \ll \frac{x^2 \log \mathcal{L}}{\mathcal{L}^A} + x \varphi(q) \mathcal{L}^{2A+3} \ll Q x \mathcal{L},
\]
where we have used (ii) of Lemma [1] and (3.6). (Note that (3.6) still holds if \( A \) is enlarged slightly without violating (1.4), so we may disregard the factor \( \log \mathcal{L} \) in the calculation above.)

Now for the remaining \( D \)'s with \( D > \mathcal{L}^{2A} \), we use Lemma [6] and get
\[\Sigma_D \ll \frac{1}{D} \log \frac{Q}{qD} (x + qD^2) \sum_{n \leq x} \Lambda^2(n) \ll \frac{xe \log \mathcal{L}}{qD} (x + qD^2).
\]
Now we observe easily that
\[\sum_{D > \mathcal{L}^{2A}} \frac{x^2 \mathcal{L}}{D} \log \frac{Q}{qD} \ll \frac{x^2}{\mathcal{L}^A} \ll Q x
\]
and
\[\sum_{D > \mathcal{L}^{2A}} q x \mathcal{L} D \log \frac{Q}{qD} \ll qx \mathcal{L} \sum_{k \geq 1} \frac{k Q}{q2^k} \ll Q x \mathcal{L}.
\]
This completes the proof of Theorem [3].
6. Proof of Theorem 1

We say that a set \( \mathcal{H} = \{ h_1, \cdots, h_k \} \) of distinct non-negative integers is admissible if for every prime \( p \), there is an integer \( a_p \) such that

\[
a_p \not\equiv h \pmod{p}
\]

for all \( h \in \mathcal{H} \).

For a set of natural numbers \( \mathcal{A} \), we write \( X(\mathcal{A}; n) \) for the indicator function of \( \mathcal{A} \). For a smooth function \( F \) supported on

\[
\mathcal{R}_k = \{ (x_1, \cdots, x_k) \in [0, 1]^k : \sum_{i=1}^{k} x_i \leq 1 \}
\]

and \( 1 \leq m \leq k \), let

\[
I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1, \cdots, t_k)^2 dt_1 \cdots dt_k
\]

and

\[
J^{(m)}_k(F) = \int_0^1 \cdots \int_0^1 \left( \int_0^1 F(t_1, \cdots, t_k) dt_m \right)^2 dt_1 \cdots dt_{m-1} dt_{m+1} \cdots dt_k.
\]

Furthermore, set

\[
M_k = \sup_F \frac{\sum_{m=1}^{k} J^{(m)}_k(F)}{I_k(F)}
\]

where the supremum is taken over \( F \) described above with \( I_k(F) \neq 0 \), \( J^{(m)}_k(F) \neq 0 \) for \( m = 1, \cdots, k \). It is shown in [13] that

\[
M_k \geq \log k - 2 \log \log k + O(1).
\]

This bound is strengthened slightly in [15] to

\[
(6.1) \quad M_k \geq \log k + O(1).
\]

We now state a special case of [2, Theorem 1] for the integers \( q \) and \( a \) in the introduction. Set

\[
D_0 = \frac{\log \log(x/2)}{\log \log \log(x/2)}.
\]

**Lemma 15.** Let \( t, k \) be natural numbers and \( K \) be a positive constant such that

\[
M_k > \frac{2t - 2}{K}.
\]

Let \( \mathcal{H} = \{ h_1, \cdots, h_k \} \) be an admissible set with \( h_1 < \cdots < h_k \), with \( q|h_j \) for \( j = 1, \cdots, k \). Suppose that \( p|h_i - h_j \) with \( i \neq j \), \( p > D_0 \) implies \( p|q \). Let \( x \) be large in terms of \( k \) and

\[
\mathcal{A} = \left\{ n : \frac{x}{2} < q \leq x, n \equiv a \pmod{q} \right\} \quad \text{and} \quad \mathbb{P} = \{ p : p \in \mathcal{A} \}.
\]

Set

\[
Y = \frac{x}{2\varphi(q)} \quad \text{and} \quad Y_1 = \frac{1}{\varphi(q)} \int_{x/2}^{x} \frac{dt}{\log t}.
\]

Suppose that

\[
\sum_{d \leq x^k} \mu^2(q) \tau_{3k}(q) \left| \sum_{n \equiv b \pmod{qd}} X(\mathcal{A}; n) - \frac{Y}{d} \right| \ll \frac{Y}{L^{k+\epsilon}}
\]

where \( \tau_{3k}(q) \) is the third divisor function of \( q \) and \( \mu(q) \) is the Möbius function.
for any \( b_d \equiv a \pmod{q} \), and

\[
(6.3) \quad \sum_{d \leq x^\varepsilon} \mu^2(q) \tau_3(q) \sum_{n \equiv b_d \pmod{qd}} X((A + h_m) \cap \mathbb{P}; n) - \frac{Y_1}{\phi(d)} \ll \frac{Y}{L^A}
\]

for every integer \( b_d \equiv a \pmod{q} \) with \((b_d, q) = 1\). Then there are primes \( p_1 < \cdots < p_k \) in \( A \) satisfying

\[ p_t - p_1 \leq h_k - h_1. \]

We remark that in Theorem 1 of [2], we have taken \( q_0 = q \) and \( q_1 = v \) if \( v > D_0 \); otherwise, \( v = 1 \). In both cases, the requirement \( \varphi(v) = v(1 + o(1)) \) is satisfied.

Proof of Theorem 1. We may suppose that \( t \) is sufficiently large. Suppose that \( q \) satisfies the hypothesis of Theorem 1. Let

\[ A = \left\{ n \in \left( \frac{x}{2}, x \right] : n \equiv a \pmod{q} \right\} \]

and \( 0 < h'_1 < \cdots < h'_k \) be an admissible set with

\[ h'_k \ll k \log k. \]

Then \( \mathcal{H} = \{ h'_1, \cdots, h'_k \} \) is an admissible set for which \( p > D_0 \), \( p|h_i - h_j (i \neq j) \) implies \( p|q \). Further,

\[ h'_k q - h'_1 q \ll qk \log k. \]

Here we choose the least \( k \) such that

\[ M_k > \frac{2t - 2}{K(\theta)} \]

and hence \( M_k > \frac{2t - 2}{K(\theta)} - \varepsilon \) for a small \( \varepsilon > 0 \). Mindful of (6.1), we get

\[ \log k \leq \frac{2t}{K(\theta)} + O(1), \]

from which we infer

\[ k \ll \exp\left( \frac{2t}{K(\theta)} \right). \]

It now remains to verify that the hypotheses of Lemma 15 are satisfied with \( K = K(\theta) \).

The bound (6.2) presents no difficulty, as

\[ \sum_{n \equiv b_d \pmod{dq}} X(A; n) = \frac{Y}{dq} + O(1). \]

To verify (6.3), we observe that for \((d, q) = 1\), \( b_d \equiv a \pmod{q} \) and \((b_d, dq) = 1\),

\[ \sum_{n \equiv b_d \pmod{q}} X((A + h_m) \cap \mathbb{P}; n) = \sum_{p \equiv b_d \pmod{dq} \atop x/2 + h_m \leq p \leq x} 1. \]

Let

\[ A = 10k^2 \], \[ B = \frac{5}{6\eta}(A + 3) \]

and define \( \delta \) by

\[ \delta = \frac{c_4}{B \log B} \]

where \( c_4 \) is a suitable absolute positive constant. Since

\[ B < \frac{C_1}{\eta} \exp\left( \frac{4t}{K(\theta)} \right), \frac{\delta}{t + \log(1/\eta)} \exp\left( \frac{-4t}{K(\theta)} \right) \]
if $C_1$, $c_3$ are suitably chosen, the hypotheses of Theorem 2 concerning $q$ are satisfied. Let

$$R_d = \left| \sum_{p \equiv b_d \mod dq, x/2 + h < p \leq x} 1 - \frac{Y_1}{\varphi(dq)} \right|.$$ 

We readily deduce from Theorem 2 that

$$\sum_{d \leq x^K} R_d \ll \frac{Y}{L^A};$$

compare the argument at the end of 7. Hence the Cauchy-Schwartz inequality together with the Brun-Titchmarsh inequality gives

$$\sum_{d \leq x^K} \mu^2(d)\tau_{3k}(d)R_d \leq \left( \sum_{d \leq x^K} \mu^2(d)\tau_{3k}(d)R_d \right)^{1/2} \left( \sum_{d \leq x^K} R_d \right)^{1/2} \ll Y \left( \sum_{d \leq x^K} \frac{\tau_{3k}(d)}{\varphi(d)} \right)^{1/2} L^{-A/2} \ll Y L^{(9k^2-A)/2} \ll Y L^{-(k+\varepsilon)}.$$ 

Now we may apply Lemma 15 and obtain primes $p_1 < \cdots < p_t$ in $\mathcal{A}$ with

$$p_t - p_1 \leq (h'_k - h'_1)q \ll qk \log k \ll qt \exp \left( \frac{2t}{K(\theta)} \right).$$

This completes the proof of Theorem 1.

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GAPS OF SMALLEST POSSIBLE ORDER BETWEEN PRIMES IN AN ARITHMETIC PROGRESSION

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