Rényi entropy for monodromy defects of higher derivative free fields on even–dimensional spheres

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Explicit polynomial forms for Rényi and entanglement entropies are given on even –dimensional spheres which possess a spherical codimension–2 U(1) monodromy defect. Free scalar and Dirac fields are treated and higher-derivative propagation operators employed. The central charge, \(C_T\), is also calculated.

The results are compared with existing ones for a planar defect. It is shown that these can be obtained from the values in the present paper.

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1. Introduction.

This paper is a continuation of an earlier one, [1], which was concerned with a free scalar field propagating according to the Penrose–Yamabe conformally covariant Laplacian, $\mathcal{Y}_d$. It is of some technical interest to extend the analysis to spinors and also to the GJMS–type higher derivative operators which, conveniently, factorise on the sphere, $S^d$, (as well as some other, related, manifolds) leading to amenable spectral problems on $S^d$ and its factors such as conical deformations.

The availability of higher derivative expressions provides useful information, as demonstrated in Beccaria and Tseytlin, [2], for example.

For fields which are simply periodic on the sphere an account of the formalism and results is given in [3]. This is here generalised to a (complex) field that suffers a phase change due to an Aharonov–Bohm type flux through the sphere constituting, in present terminology, a $U(1)$ monodromy defect. The previous paper, [1], gives some history of the topic with a few relevant references, old and new.

The quantities of interest are the free energy (effective action) and its various derivatives such as the entanglement entropy. The aim is to provide explicit formulae for as many dimensions, $d$, as feasible, which, despite being for a somewhat restricted situation, should be helpful for comparison and checking purposes.

2. The operators

For completeness, it is necessary to give a few details, firstly of the propagating operators which here have the Branson structure, on the sphere,

$$\Omega_k(d) = \frac{\Gamma(B + k + 1/2)}{\Gamma(B - k + 1/2)},$$

where $k$ is a parameter, either an integer or a half–integer, to begin with. The (pseudo) operator $B$ depends on the field theory. For scalars $B = \sqrt{\mathcal{Y}_d} + 1/4$ and for Dirac spinors $B = (\nabla^2)^{1/2} = |\nabla|$.

When $k$ is integral or half integral, the form (1) reduces to a finite product of operators. To accord with the ordinary field case, $k$ is an integer for integer spin and a half–integer for half–integer spin. These give, respectively, an even or an odd number of factors. More generally, $k$ has a conformal field theory significance in terms of a scaling dimension, cf. [4–6].

The operator (1) for integer $k$ reads,

$$\Omega_k(d) = \prod_{j=0}^{k-1} (B^2 - (j + 1/2)^2) = \prod_{j=0}^{k-1} (B - (j + 1/2))(B + (j + 1/2)),$$

1
expressed as a product of $2k$ linear factors.

3. The spectra

As in [1], there is a choice of methods. One involves only the eigenvalue structure, and the other the propagating Green function. Here I will concentrate on the former as being more directly global, avoiding construction of complicated local field theory quantities, like the energy–momentum tensor, and their integration.

It was shown in [7] that the spectrum on a $q$-deformed sphere for a charged scalar undergoing a phase change of $2\pi \delta$ on circling the flux is the union of two sets of eigenvalues,

$$\left\{ \left( \frac{d-1}{2} + q|\delta| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right\} \cup \left\{ \left( \frac{d-1}{2} + q - q|\delta| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right\},$$

(3)

where $\mathbf{\omega}$ is the set of $d$ integers, $(q, 1_{d-1})$, but can be considered as real non-negative and referred to as the parameters. $\mathbf{n}$ is a set of $d$ integers, $(n_1, n_2, \ldots, n_d)$ each ranging from 0 to $\infty$. If the propagating operator is conformal, i.e. equals $Y_d$, then $\alpha = 1/2$. Degeneracies arise simply from coincidences between the eigenvalues as displayed in (3) and need not be specifically introduced being encoded in the sums over $\mathbf{n}$.

The Dirac eigenvalues are, likewise,

$$\left\{ \left( \frac{d-1}{2} + q + q|\delta| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right\} \cup \left\{ \left( \frac{d-1}{2} + q - q|\delta| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right\},$$

(4)

where $\alpha = 0$ for conformal invariance i.e. propagating operator $\nabla$. (The eigenvalues are then for the square, $\nabla^2$.)

These spectra show that the eigenvalues of the product $\Omega_k$, (2), involve the quantities,

$$\lambda(\mathbf{n}, a, q, \delta)^2 - \alpha^2,$$

where

$$\lambda(\mathbf{n}, a, q, \delta) \equiv (a + \mathbf{n}\cdot\mathbf{\omega}), \quad \mathbf{\omega} = (q, 1, \ldots, 1),$$

(5)

associated with each factor ($2k$ in number) labelled by $j$ with, explicitly, $\alpha \equiv \alpha_j = j + 1/2$, $j = 0, 1, \ldots k - 1$. There are two values of the parameter $a$ to take into account for each spin, and it is useful to write them out,

$$a_1^S = \frac{d-1}{2} + q\delta, \quad a_2^S = \frac{d-1}{2} + q(1 - \delta),$$

(6)
for scalars, and,

\[ a_1^F = \frac{d-1}{2} + q\delta, \quad a_2^F = \frac{d-1}{2} - q\delta, \]

for spinors. \( a^F \) are got from \( a^S \) by the replacement \( \delta \to \delta + 1/2 \). The shift of 1/2 reflects the spinor sign change encountered on a complete circuit of the flux. Note that periodicity is then with respect to the unit cell \(-1/2 \leq \delta \leq 1/2\).\(^2\)

4. Conformal anomaly

In even dimensions, the conformal anomaly provides the universal coefficient of the logarithmic UV–divergent term in the effective action and equals the value of the propagating \( \zeta \)–function, here denoted by \( Z_k(s, d, q, \delta) \), at \( s = 0 \), up to a multiplicative constant (and assuming no zero modes).

The evaluation of \( Z_k(0, d, q, \delta) \) is described in [8] based on [9]. It relies on the linear factorisation of the product \( \Omega_k \) as shown in (2). For convenience, I outline the procedure.

The \( \zeta \)–function corresponding to the product (2) is given by,

\[
Z_k(s, d, q, \delta) = S \sum_i \sum_{n=0}^{\infty} \prod_{j=0}^{k-1} \left( \lambda(n, a_i - \alpha_j, q) \right)^s \left( \lambda(n, a_i + \alpha_j, q) \right)^s
\]

where \( S \) is the field dimension. For complex scalars \( S = 2 \) and for Dirac spinors, \( S = 2^{d/2+1} \).

Although the even product, (2), applies, as it stands, just to scalar fields, it can still be employed for spinors by continuing \( k \) to a half–integer. (See (1).)

The \( \zeta \)–function of a linear factor in (7) can be recognised as a Barnes \( \zeta \)–function, each one contributing additively to the value at \( s = 0 \), \( Z_k(0, d, q, \delta) \), which, when multiplied by \( k \), gives the required conformal anomaly,

\[
C(d, q, \delta, k) = (-1)^{2s} k Z_k(0, d, q, \delta) \quad (s = \text{spin}).
\]

The value of the Barnes \( \zeta \)–function at 0 is given by a generalised Bernoulli polynomial, \( B_v^{(d)}(*) \) which is easily accessible.

\(^2\) This can be seen by remarking that \( 1/2 + \delta \) for spinors plays the role that \( \delta \) plays for scalars i.e. it is \( 1/2 + \delta \) that has unit cell 0 to 1. Periodicity then implies that \( 1 - \delta \) is equivalent to \( \delta \).
For each pair of linear factors (i.e. value of \( j \) in (7)) there are four values of the argument of the polynomials and so,

\[
C(d, \delta, q, k) = \frac{(-1)^{2s} S}{2! q} \sum_{j=0}^{k-1} \left( B^{(d)}_d (a_1 \pm \alpha_j \mid q, 1_{d-1}) + B^{(d)}_d (a_2 \pm \alpha_j \mid q, 1_{d-1}) \right).
\]  

(8)

Symmetry properties of the \( B^{(d)}_\nu \) allow some simplification. For example for even \( \nu \),

\[
B^{(d)}_\nu (a_2^S \pm \alpha_j \mid q, 1_{d-1}) = B^{(d)}_\nu (d-1)/2 + q - a_2^S \mp \alpha_j \mid q, 1_{d-1}) = B^{(d)}_\nu (a_1^S \mp \alpha_j \mid q, 1_{d-1}),
\]

using the expressions for \( a_1^S \) and \( a_2^S \) in (6). The same result holds for the spinor arguments. Hence the second term in (8) can be expunged with a compensating overall factor of 2.

Furthermore, the GJMS sum over \( j \) can be performed to give the more compact expression, with a bulk, \((d + 1)\)-dimensional holographic appearance,

\[
C(d, q, \delta, k) = \frac{(-1)^{2s} S}{(d + 1)! q} \left( B^{(d+1)}_{d+1} (a_1 + k + 1/2 \mid q, 1_d) - B^{(d+1)}_{d+1} (a_1 - k + 1/2 \mid q, 1_d) \right).
\]  

(9)

It is to be noted that \( k \) can now be continued off the integers, in particular to \( k = 1/2 \) which delivers the conventional Dirac value with flux \( \delta \) if \( \delta \) in (9) is replaced by \( \delta + 1/2 \). This is a useful calculational shortcut.\(^4\)

The evaluation of the polynomials is straightforward and rapid by machine but, as the expressions become unwieldy, I give only a single example which exhibits the general structure. Omitting the \((-1)^{2s} S\) factor, there results, in 4 dimensions,

\[
C(4, q, \delta, k) = k \frac{11 - 20k^2 + 6k^4}{q} - kq^3(30\sigma^2 - 1) - 10kq(1 - k^2)(6\sigma - 1),
\]

where \( \sigma = \delta(1 - \delta) \) is a more convenient variable.

\(^3\) For odd \( \nu \) there is a sign change and the expressions would cancel to give zero conformal anomaly in odd dimensions, as is correct.

\(^4\) Of course the odd product (1) obtained when \( k \) is half–integral can be used. This would mean extra work, but can be helpful as a check. As functions of \( k \), the quantities can be regarded as giving a sort of interpolation between scalar and spinor, up to a sign and spin factor.
I list below a few examples of the defect contribution, for the full sphere, to the conformal anomaly for a Dirac field, defined by,

\[ \Delta C(d, 1, \delta, k) \equiv C(d, 1, 1/2 + \delta, k) - C(d, 1, 1/2, k), \quad k \in \mathbb{Z}_{n+1/2}, \]

again omitting \( S \).

For a standard Dirac field,

\[
\begin{align*}
\Delta C(4, 1, \delta, 1/2) & = \frac{1}{24} \delta^2 (2 - \delta^2), \\
\Delta C(6, 1, \delta, 1/2) & = - \frac{1}{1440} \delta^2 (24 - 15 \delta^2 + 2 \delta^4), \\
\Delta C(8, 1, \delta, 1/2) & = \frac{1}{120960} \delta^2 (432 - 294 \delta^2 + 56 \delta^4 - 3 \delta^6), \\
\Delta C(10, 1, \delta, 1/2) & = - \frac{1}{7257600} \delta^2 (5760 - 4100 \delta^2 + 910 \delta^4 - 75 \delta^6 + 2 \delta^8), \\
\end{align*}
\]

and for a cubic Dirac field,

\[
\begin{align*}
\Delta C(4, 1, \delta, 3/2) & = - \frac{1}{8} \delta^2 (2 + \delta^2), \\
\Delta C(6, 1, \delta, 3/2) & = \frac{1}{480} \delta^2 (16 - 5 \delta^2 - 2 \delta^4), \\
\Delta C(8, 1, \delta, 3/2) & = - \frac{1}{13440} \delta^2 (80 - 42 \delta^2 + \delta^6), \\
\Delta C(10, 1, \delta, 3/2) & = \frac{1}{2419200} \delta^2 (2880 - 1780 \delta^2 + 210 \delta^4 + 15 \delta^6 - 2 \delta^8) .
\end{align*}
\]

5. The term proportional to \( q \)

This section and the next contain observations on aspects of the form of the conformal anomaly not needed for the main calculation which continues in section 7.

For standard conformally invariant scalar fields, the term proportional to \( q \) in the conformal anomaly vanishes. For higher derivative propagation, and no flux, it is a polynomial in \( k \) that vanishes when \( k \) is integral and satisfies the GJMS existence (‘subcritical’) condition \( k < d/2 \), as described in [3]. Calculation shows that the same holds true in the presence of flux, the coefficient being,

\[
\frac{(-1)^{d/2-1}}{(d-1)(d-2)!} \left( \frac{1}{6} - \sigma \right) k(k^2 - 1^2)(k^2 - 2^2) \ldots (k^2 - (d/2 - 1)^2).
\]

The derivation of this formula is outlined in an appendix.

It is interesting to note that the coefficient, (12), is zero for all \( d \) and \( k \) when \( \sigma = 1/6 \) i.e. a scalar flux of \( \delta = (\sqrt{3} - 1)/2\sqrt{3} \approx 0.2113249 \) and a Dirac flux of \( 1/\sqrt{12} \approx 0.288675 \). I do not know what, if anything, this signifies.
6. The small $q$ limit

As a check of the numerics, and as an interesting fact, the coefficient of the $1/q$ term in the conformal anomaly is found to equal minus twice the vacuum energy on the Einstein universe, $\mathbb{R} \times S^{d-1}$. An analytical proof of this can be given that relies on simple properties of the Bernoulli polynomials. Indeed, the coefficient of the $1/q$ term is a polynomial in $k$ for which a general Bernoulli formula is given in [10] realised as the vacuum energy of a Rac $k$-lineton on the Einstein universe.

The limit $q \to 0$ corresponds to an infinitely sheeted cover of the sphere, $S^d$, and is independent of $\delta$, which is the flux through the entire covering. A different result is obtained if the flux, $\mu$, through one sheet is required to remain constant by setting $\mu = q\delta$. This can be thought of as a Euclidean chemical potential. The earlier calculations, [11,7] then become more relevant.\footnote{Standard fields only were treated here.}

7. Other quantities. Entropies and conformal charge.

From the effective action (‘free energy’), \textit{i.e.} here just the conformal anomaly, other quantities of interest can be derived, such as the Rényi and entanglement entropies and various central charges. I will not repeat the standard definitions.

All these quantities are polynomial in $q, k$ and $\delta$. I will generally display only for specific values of $k$. For example, the Rényi entropy, $\mathcal{G}(d, q, \delta, k)$, for standard Dirac is ($q = 1/n$),

$$
\mathcal{G}(4, q, \delta, 1/2) = \frac{1 + q}{5760} \left(37 - 480\delta^2 + 240\delta^4 + q^2(7 - 120\delta^2 + 240\delta^4)\right).
$$

I list some entanglement entropies, $\mathcal{E}(d, \delta, k)$, of a standard Dirac spinor ($k = 1/2$ and flux $\delta$) on the full sphere (the physical spin factor has now been included),

$$
\begin{align*}
\mathcal{E}(4, \delta, 1/2) &= \frac{11}{90} - \frac{1}{3}\delta^2(5 - 4\delta^2), \\
\mathcal{E}(6, \delta, 1/2) &= -\frac{191}{3780} + \frac{1}{90}\delta^2(64 - 65\delta^2 + 12\delta^4), \\
\mathcal{E}(8, \delta, 1/2) &= \frac{2497}{113400} - \frac{1}{3780}\delta^2(1188 - 1323\delta^2 + 350\delta^4 - 24\delta^6), \\
\mathcal{E}(10, \delta, 1/2) &= -\frac{14797}{1496880} + \frac{8}{14175}\delta^2(16128 - 18860\delta^2 + 5824\delta^4 - 615\delta^6 + 20\delta^8).
\end{align*}
$$

(13)
The constant term is minus the usual conformal anomaly. The remainder is the effect of the monodromy.

The energy–momentum two–point central charge, \(C_T\), follows from the second derivative (with respect to \(1/q\)) of the free energy (conformal anomaly) using the Perlmutter factor. For free scalar and Dirac fields, and no flux, general formulae exist as functions of the dimension \(d\) and the derivative order \(2k\).

For standard \((k = 1)\) scalar fields some expressions for \(C_T\) were given in [1]. Here I extend these to higher derivatives and to spinors. The notation is \(C_T(d, \delta, k)\).

For the scalar, Paneitz–Tseytlin four–derivative operator,

\[
C_T(4, \delta, 2) = -\frac{64}{3} + 160\sigma(1 - \sigma),
\]

\[
C_T(6, \delta, 2) = -12 + 42\sigma^2(9 - 2\sigma),
\]

\[
C_T(8, \delta, 2) = -\frac{64}{7} + \frac{16}{3}\sigma^2(46 + 26\sigma - 3\sigma^2),
\]

\[
C_T(10, \delta, 2) = -\frac{70}{9} + \frac{55}{216}\sigma^2(776 - 6\sigma^3 + 69\sigma^2 + 640\sigma).
\]

As in the standard case, as the flux is turned on, \(C_T\) changes sign, here from negative to positive at a root, \(\sigma_0(d)\), in the unit cell \(0 \leq \delta \leq 1\), i.e. \(0 \leq \sigma \leq 1/4\) of the polynomial. Typically, \(\sigma_0(8) \approx 0.18391467\). The corresponding flux is \(\delta_0(8) \approx 0.24292933\).

For a standard \((k = 1/2)\) Dirac field, omitting the spin factor \(S\), \((\delta = \text{the flux})\),

\[
C_T(4, \delta, 1/2) = 2 - 30\delta^2 + 40\delta^4,
\]

\[
C_T(6, \delta, 1/2) = 3 - \frac{21}{4}\delta^2(\delta^2 - 3)(4\delta^2 - 3),
\]

\[
C_T(8, \delta, 1/2) = 4 - \frac{1}{3}(193\delta^2 + 350\delta^4 - 133\delta^6 + 12\delta^8),
\]

\[
C_T(10, \delta, 1/2) = 5 - \frac{55}{576}\delta^2(852 - 1615\delta^2 + 714\delta^4 - 99\delta^6 + 4\delta^8).
\]

\(C_T\) changes sign at \(\delta = \delta_0(d)\), with, typically, \(\delta_0(10) \approx \pm 0.2657463\). The defect contribution is obtained by dropping the first (standard) term, \(d/2\).

For cubic Dirac \((\nabla^3)\),

\[
C_T(4, \delta, 3/2) = -\frac{2}{3} - 10\delta^2 + 120\delta^4,
\]

\[
C_T(6, \delta, 3/2) = -\frac{18}{5} + \frac{63}{4}\delta^2(\delta^2 + 4\delta^4 - 3),
\]

\[
C_T(8, \delta, 3/2) = -\frac{36}{7} + 77\delta^2 - 100\delta^4 - 13\delta^6 + 12\delta^8,
\]

\[
C_T(10, \delta, 3/2) = -\frac{115}{18} + \frac{55}{1728}\delta^2(5065\delta^2 - 1022\delta^4 - 219\delta^6 + 36\delta^8 - 3140).
\]
and, \( e.g., \, \delta_0(10) \approx \pm 0.26872496 \).

It is interesting to note that, for higher derivatives, as a function of a \textit{complex} flux, \( C_T \) has zeros on the imaginary axis for fermions while for bosons the real part is shifted to 1/2. For example, for the Paneitz case, \( C_T(10, \delta, 2) \), the roots are \( \approx 1/2 \pm 4.19796i \) while for \( k = 3 \) there are four roots, \( \approx 1/2 \pm 2.0606i \) and \( \approx 1/2 \pm 7.41175i \). In the complex unit cell, \( 0 \leq \text{Re} \delta \leq 1 \). \(^6\) It is amusing to analyse the root structure a little further.

If \( k \) is extended to a real number, the central charge, for example, can vanish at more than two values of the flux. The graph below displays the variation, for dimension \( d = 6 \), of scalar \( C_T \) against \( \delta \) for \( k = 1.2 \), showing four real roots.

![Graph showing variation of \( C_T \) against \( \delta \) for \( k = 1.2 \).](image)

As \( k \) increases towards 2, say, the two inner roots come together until they coincide at \( \delta = 1/2 \) (\( \sigma = 1/4 \)), the value of \( k \) being approximately 1.2346900. Thereafter, as shown \( e.g. \) in the graph for \( k = 1.25 \), these two roots have disappeared to become complex conjugate ones. As \( k \) increases further, this process repeats itself with complex roots, of real part 1/2, being created as \( k \) passes through certain non–integer values. For each \( d \), the number of these \( k \)–values is finite, the total number, including 0 and negative values, is \( d - 1 \).

The behaviour in the Dirac case is similar except that the vertical axis is shifted by 1/2 to the right.

A further fact worth remarking is that the periodic extension of \( C_T \) displays cusps at the boundary of the unit cell.

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\(^6\) At an earlier stage, a similar behaviour occurs for the basic free energy. The root imaginary part would correspond to a real chemical potential while the real part turns bosons into fermions.
8. Comparisons. A tale of two defects

A prediction, when $d = 6$, is given in [12] for the effect of a codimension 2 monodromy defect on the Dirac entanglement entropy which I copy here, dropping the extra singular mode,\(^7\)

$$-\frac{1}{90}\delta^2(16 - 5\delta^2).$$

This differs from the value found here which is, from (13),

$$\frac{1}{90}\delta^2(64 - 65\delta^2 + 12\delta^4).$$

The sign difference is a consequence of conventions but the coefficients differ. In particular, (14) is a higher degree polynomial. As noted in [1], there is also a similar disparity in the scalar field quantities.

The origin of this difference can be traced to the geometric positioning, relative to the entangling surface, of the particular defects involved. In fact, the two defects could be thought of as being orthogonal. In the case discussed in the present paper, the defect is spherical and coincident with the entangling surface while that in [12] is planar and cuts through the (same) entangling surface which is centred on the defect.

The analytical consequences of this are to be found in a term in the expression for the entropy derived in [13] via an expansion of the replica formula about $n = 1$. The derivative part of this is the term in question and consists of an integral over the hyperbolic cylinder of a vacuum–averaged, ambient conformal energy density, $\langle T_\tau^\tau \rangle$, which is related by conformal transformation to the vacuum average of the energy–momentum tensor in flat space. From its general form, given in [13], the two energy densities, corresponding to the two defect geometries, are related by a factor of $-(d - 1).$\(^8\) In particular, $\langle T_\tau^\tau \rangle$ equals (to factors of $\sqrt{g}$ and anomalies) the conventional, flat space energy density, $\langle T_0^0 \rangle$, for the planar defect in [12] but equals the angular, or Rindler, average, $\langle T_\theta^\theta \rangle = -(d - 1)\langle T_0^0 \rangle$ for the spherical defect employed here. The other part of the expression for the defect entanglement entropy is the difference in free energies, and is the same for both defects.

These facts allow the entanglement entropy for the planar defect to be obtained easily by simple algebra from that for the spherical one computed in this paper.

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\(^7\) According to the authors of [12] there is a substitutional error in [12] and I have recalculated the expression using the forms of $a_C$ and $d_2$ as given in this reference.

\(^8\) I am grateful to the authors of [12] for explaining this particular point and also the geometry involved in the two cases.
I find that, for ordinary scalar fields the planar defect entanglement entropy difference, $\Delta \mathcal{E}_P(d, \delta, k)$, is, in my sign conventions,

$$
\Delta \mathcal{E}_P(4, \delta, 1) = 0,
\Delta \mathcal{E}_P(6, \delta, 1) = \frac{\sigma^2}{180},
\Delta \mathcal{E}_P(8, \delta, 1) = -\frac{\sigma^2(9 + 2\sigma)}{7560},
\Delta \mathcal{E}_P(10, \delta, 1) = \frac{\sigma^2(216 + 72\sigma + 5\sigma^2)}{907200}.
$$

The first two values are given in [14,12] which have thus been confirmed, and extended, using a distinct, global technique.

For the Paneitz case,

$$
\Delta \mathcal{E}_P(4, \delta, 2) = 4\sigma^3/3,
\Delta \mathcal{E}_P(6, \delta, 2) = -\frac{\sigma^2}{18},
\Delta \mathcal{E}_P(8, \delta, 2) = \frac{\sigma^2(6 + \sigma)}{945},
\Delta \mathcal{E}_P(10, \delta, 2) = -\frac{\sigma^2(444 + 128\sigma + 5\sigma^2)}{453600}.
$$

For ordinary spinors (flux = $\delta$ and omitting $S$),

$$
\Delta \mathcal{E}_P(4, \delta, 1/2) = -\frac{\delta^2}{24},
\Delta \mathcal{E}_P(6, \delta, 1/2) = \frac{\delta^2(16 - 5\delta^2)}{1440},
\Delta \mathcal{E}_P(8, \delta, 1/2) = -\frac{\delta^2(324 - 147\delta^2 + 144\delta^4)}{120960},
\Delta \mathcal{E}_P(10, \delta, 1/2) = \frac{\delta^2(4608 - 2460\delta^2 + 364\delta^4 - 15\delta^6)}{7257600}.
$$

Again, the first two values confirm those in [14,12].

For cubic Dirac,

$$
\Delta \mathcal{E}_P(4, \delta, 3/2) = \frac{5\delta^2}{24},
\Delta \mathcal{E}_P(6, \delta, 3/2) = -\frac{\delta^2(4 - \delta^2)}{160},
\Delta \mathcal{E}_P(8, \delta, 3/2) = \frac{\delta^2(188 - 73\delta^2 + 2\delta^4)}{40320},
\Delta \mathcal{E}_P(10, \delta, 3/2) = -\frac{\delta^2(7056 - 3400\delta^2 + 308\delta^4 + 5\delta^6)}{7257600}.$$


Finally, in this section, I remark that the defect considered in the present paper is the same as that investigated in [15].

8. Comments and conclusion

The results of a machine evaluation of several quantities, based on the conformal anomaly, have been presented for free fields satisfying higher-derivative propagation equations on even-dimensional spheres in the presence of a spherical monodromy defect.

Comparison with existing results highlights the differing geometrical constructions of the defects and their analytical consequences.

The variation of a central charge with the monodromy flux is investigated.

In a further communication, I will discuss the case of \( p \)-forms which have a more complicated field theory. The analysis will also be extended to odd dimensions, part of which already exists in [16].

Acknowledgment

I thank the authors of [12] for their very helpful communications.

Appendix. A little Bernoulli-ology

In this appendix I derive the expression for the term proportional to \( q \) in the conformal anomaly. It is given by

\[
\frac{1}{2} \frac{\partial^2}{\partial q^2} q C(d, q, \delta, k) \bigg|_{q=0}
\]

which means, from (9) that one needs,

\[
\frac{\partial^2}{\partial q^2} B_{d+1}^{d+1}(d/2 + \delta q \pm k \mid q, 1_d) \bigg|_{q=0}.
\]

The required \( q \)-polynomial is provided by the standard formula (I give the general form, [17])

\[
B_{\nu}^{(n)}(x \mid \omega) = \sum_{s=0}^{\nu} \binom{\nu}{s} \omega_s B_{\nu-s}^{(n-1)}(x \mid \bar{\omega}),
\]
where $\mathring{\omega}$ is the set $\omega$ omitting the first element, $\omega_1$. For present application, $n = d + 1$, $\nu = d + 1$ and $\omega_1 = q$ with $\mathring{\omega} = 1_d$. This formula allows one to find the derivatives with respect to the $q$–dependence coming from the parameters $\omega$.

The derivative with respect to the $q$–dependence in the argument, $x = d/2 + \delta q \pm k$, is given by the basic recursion,

$$
\frac{\partial}{\partial x} B^{(n)}_{\nu}(x \mid \omega) = \nu B^{(n)}_{\nu-1}(x \mid \omega).
$$

Taking the second derivative produces four terms corresponding to the two sources of $q$–dependence. The derivatives with respect to the argument give factors of $\delta$ and simple algebra yields,

$$
\frac{\partial^2}{\partial q^2} B^{(d+1)}_{d+1}(d/2 + \delta q \pm k \mid q, 1_d) \bigg|_{q=0} = d(d + 1)(1/6 - \sigma) B^{(d)}_{d-1}(d/2 \pm k),
$$

where the parameters $1_d$ in the final Bernoulli polynomial have been conventionally omitted.

Inserting the factors to give the conformal anomaly, (9), gives for the coefficient of the $q$ term,

$$
\frac{1 - 6\sigma}{6(d - 2)!(d - 1)} B^{(d)}_{d-1}(d/2 + k),
$$

where (for even $d$) the antisymmetry of the Bernoulli polynomial in $k$ has been used.

Finally, the product form, (this follows from (16)),

$$
B^{(n+1)}_{n}(x) = (x - 1)(x - 2) \ldots (x - n),
$$

converts the above expression into the one in section 5.

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