HURWITZ-HODGE INTEGRAL IDENTITIES FROM THE CUT-AND-JOIN EQUATION

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ABSTRACT. In this paper, we present some Hurwitz-Hodge integral identities which are derived from the Laplace transform of the cut-and-join equation for the orbifold Hurwitz numbers. As an application, we prove a conjecture on Hurwitz-Hodge integral proposed by J. Zhou in 2008.

1. INTRODUCTION

The Gromov-Witten theory for symplectic orbifolds has been developed by Chen-Ruan [4]. The algebraic part, the Gromov-Witten theory for smooth DM stacks, was established by Abramovich-Graber-Vistoli [1]. By the virtual localization, all the orbifold Gromov-Witten invariants for a toric DM stack descend to the computation of the Hurwitz-Hodge integrals on the moduli space $\overline{M}_{g,\gamma}(BG)$ of twisted stable maps to the classifying stack of a finite group $G$ [11]. In [15], J. Zhou described an effective algorithm to calculate the Hurwitz-Hodge integrals. By applying Tseng's orbifold quantum Riemann-Roch theorem [13], Hurwitz-Hodge integrals can be reconstructed from the descendant Hurwitz-Hodge integrals on $\overline{M}_{g,\gamma}(BG)$.

Recall that the descendant Hodge integrals on moduli space of curves can be computed by the DVV recursion which is equivalent to the Witten-Kontsevich theorem [14, 10]. An easy approach to DVV recursion is by studying the cut-and-join equation for simple Hurwitz numbers. Combining the famous ELSV formula, one can obtain a polynomial identity for linear Hodge integrals. Then it is direct to derive the DVV recursion by looking at the highest terms in this identity [3, 12]. Similarly, collecting the lowest terms, one obtains a formula for $\lambda_g$-integrals [8, 18].

In [9], Johnson-Pandharipande-Tseng established the ELSV-type formula for orbifold Hurwitz numbers which also satisfy the cut-and-join equation. With the same technique used in [5], Bouchard-Serrano-Liu-Mulase [2] established the Laplace transform of the cut-and-join equation for orbifold Hurwitz numbers. Starting from this formula, with the same method used in [8, 3, 19, 12], we obtain the following Hurwitz-Hodge integrals identities.

Theorem 1.1. When $r \geq 1$ and $0 \leq k_i \leq r - 1$, for $1 \leq i \leq l$, the Hurwitz-Hodge integrals satisfy the following orbifold-DVV recursion:

\begin{equation}
\langle \tau_{b_L} \rangle_g^{r,k_L} = r \sum_{j=2}^{l} \frac{C_{b_1,b_j}^{k_1,k_j}}{(2b_1 + 1)!!(2b_j - 1)!!} \langle \tau_{b_1 + b_j - 1} \tau_{b_L \setminus \{1,j\}} \rangle_g^{r,(k_1 + k_j,k_L \setminus \{1,j\})} \\
+ \frac{r^2}{2} \sum_{m+n=2}^{a+b=k_1} \frac{(2m + 1)!!(2n + 1)!!}{(2b_1 + 1)!!} \langle \tau_{m\tau_n} \tau_{b_L \setminus \{1\}} \rangle_g \langle \tau_{(a,b,k_L \setminus \{1\})} \rangle_g^{-1}
\end{equation}

1991 Mathematics Subject Classification. Primary 57N10.
Key words and phrases. Orbifold Hurwitz numbers, cut-and-join equation, Hurwitz-Hodge integrals.
\[
\begin{aligned}
&+ \sum_{g_1+g_2=g} \sum_{J=L \setminus \{1\} \atop a+|k_i|=0} \frac{(2m+1)!!(2n+1)!!}{(2b_1+1)!!} \langle \tau_m \tau_{b_1} \rangle_{g_1} \langle \tau_n \tau_{b_2} \rangle_{g_2} \left( \sum_{r, k_1, k_2} C^{k_1, k_2}_{b_1, b_2} \right).
\end{aligned}
\]

Where the constant \( C^{k_1, k_2}_{b_1, b_2} \) can be calculated from the formula (37) in Lemma 3.2. We have not written down an explicit formula for it in general. However, when \( r = 1 \), then \( k_i = 0 \), for \( i = 1, ..., l \). It is easy to compute
\[
C^{0,0}_{b_1, b_2} = (2b_1 + 2b_2 - 1)!!.
\]

Therefore, we have

**Corollary 1.2.** When \( r = 1 \), the orbifold-DVV recursion is reduced to the ordinary DVV recursion which is equivalent to the original Witten conjecture.

\[
(3)
\begin{aligned}
&\langle \tau_{b_L} \rangle_g = \sum_{j=2}^{l} \frac{(2b_1 + 2b_j - 1)!!}{(2b_1 + 1)!!(2b_j - 1)!!} \langle \tau_{b_1 + b_j - 1} \rangle_{g} \langle \tau_{b_L \setminus \{1\}} \rangle_g \\
&+ \frac{1}{2} \sum_{m+n=b_1-2 \atop m \geq 0, n \geq 0} \left( \frac{(2m+1)!!(2n+1)!!}{(2b_1+1)!!} \langle \tau_m \tau_n \rangle_{g} \langle \tau_{b_L \setminus \{1\}} \rangle_{g} \right) \\
&+ \sum_{g_1+g_2=g} \frac{(2m+1)!!(2n+1)!!}{(2b_1+1)!!} \langle \tau_m \tau_{b_1} \rangle_{g_1} \langle \tau_n \tau_{b_2} \rangle_{g_2} \left( \sum_{r, k_1, k_2} C^{k_1, k_2}_{b_1, b_2} \right).
\end{aligned}
\]

The rank of the Hodge bundle \( \mathbb{E}^U \) (see Section 2 for the detail definition) on \( \overline{M}_{g, \gamma}(BZ_{1r}) \) depends on the monodories at the marking points. When all the monodories are trivial, \( \text{rk} \mathbb{E}^U = g \). Otherwise, the rank is given by formula (19). By looking at the lowest terms in the formula (36), we obtain the following theorem.

**Theorem 1.3.** When all the monodromies at the marking points are trivial, i.e. \( k_i = 0 \), for all \( 1 \leq i \leq l \). If \( b_i \geq 0 \) for \( 1 \leq i \leq l \), and \( \sum_{i=1}^{l} b_i = 2g-3+1 \), we have the following closed formula for \( \lambda_g^U \)-integrals:

\[
(4)
\begin{aligned}
\langle \tau_{b_L} \lambda_g^U \rangle_g = \left( \frac{2g-3+1}{b_1, ..., b_l} \right) \langle \tau_{2g-2} \lambda_g^U \rangle_g.
\end{aligned}
\]

where the one-point integral \( \langle \tau_{2g-2} \lambda_g^U \rangle_g \) is determined by the following formula given in \[9\] (See the formula before Section 4):

\[
(5)
\begin{aligned}
&\frac{1}{r} + \sum_{g \geq 0} t^{2g} \langle \tau_{2g-2} \lambda_g^U \rangle_g = \frac{1}{r} \frac{t/2}{\sin(t/2)}.
\end{aligned}
\]

Otherwise, the \( \lambda_g^U \)-integral satisfies the following identity:

\[
(6)
\begin{aligned}
&Z \langle \tau_{b_L} \lambda_g^U \rangle_g = \left( \sum_{i \in Z} \sum_{j \in k_L \setminus Z} C_{b_1+b_2, j}^{k_1} \langle \tau_{b_1+b_j-1} \rangle_{g} \langle \tau_{k \setminus \{1\}} \rangle_g \right) \\
&+ \sum_{i,j \in Z, i < j} C_{b_1, b_2}^{0} \langle \tau_{b_1+b_j-1} \rangle_{g} \langle \tau_{(0, k \setminus \{1\})} \rangle_g.
\end{aligned}
\]
Corollary 1.4. When \( b_i \geq 0 \) for \( 1 \leq i \leq l \), and \( \sum_{i=1}^{l} b_i = 2g - 3 + l \),

\[
\langle \tau_{b_L} \lambda_g \rangle_g = \frac{1}{r} \langle \tau_{b_L} \lambda_g \rangle_g
\]

where the right hand side is the ordinary Hodge integral on moduli space of curves:

\[
\langle \tau_{b_L} \lambda_g \rangle_g = \int_{\mathcal{M}_{g,l}} \psi_{b_1} \cdots \psi_{b_l} \lambda_g.
\]

Proof. The ordinary \( \lambda_g \)-integral satisfies the following formula which was firstly obtained by Faber-Pandharipande [7]

\[
\langle \tau_{b_L} \lambda_g \rangle_g = \left( \frac{2g - 3 + l}{b_1, \ldots, b_l} \right) \langle \tau_{2g-2} \lambda_g \rangle_g
\]

where the one-point integral \( \langle \tau_{2g-2} \lambda_g \rangle_g \) is determined by

\[
1 + \sum_{g>0} t^{2g} \langle \tau_{2g-2} \lambda_g \rangle_g = \frac{t/2}{\sin(t/2)}.
\]

Hence, by formula (5),

\[
\langle \tau_{2g-2} \lambda_g \rangle_g = \frac{1}{r} \langle \tau_{2g-2} \lambda_g \rangle_g.
\]

Therefore, Corollary 1.4 is derived from the formula (4). \( \square \)

Remark 1.5. The formula (7) in Corollary 1.4 was first conjectured by J. Zhou [17]. We would like to thank Prof. Kefeng Liu and Hao Xu for pointing out it to us.

2. Preliminaries

2.1. Hurwitz-Hodge integrals. Let \( \overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_r) \) be the moduli space of stable maps to the classifying space \( B\mathbb{Z}_r \) where \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a vector elements in \( \mathbb{Z}_r \). In particular, when \( a = 1 \), the moduli space of stable maps \( \overline{\mathcal{M}}_{g,(0,0)}(B\mathbb{Z}_r) \) is specialized to \( \overline{\mathcal{M}}_{g,n} \). Let \( U \) be the irreducible \( \mathbb{C} \)-representation of \( \mathbb{Z}_r \) given by

\[
\phi^U : \mathbb{Z}_r \to \mathbb{C}^*, \quad \phi^U(1) = \exp(2\pi \sqrt{-1}/r).
\]

We have the corresponding Hodge bundle \( E_{g,\gamma}^U \to \overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_r) \) and the Hodge classes \( \lambda_{i}^{U,\mathbb{Z}_r,\gamma} = c_i(E_{g,\gamma}^U) \). More generally, for any irreducible representation \( R \) of \( \mathbb{Z}_r \), we denote the corresponding Hodge bundle and Hodge classes as \( E_{g,\gamma}^R \) and \( \lambda_{i}^{R,\mathbb{Z}_r,\gamma} \) respectively. In the following, for brevity, we
will also use the notations $E^R$ and $\lambda_i^R$ to denote the Hodge bundle and Hodge classes without confusion.

The $i$-th cotangent line bundle $L_i$ on the moduli space of curves has fiber $L_i|_{(C, p_1, \ldots, p_n)} = T^*_p C$. The $\psi$-classes on $\overline{M}_{g,n}$ are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}, \mathbb{Q}).$$

The $\psi$-classes on $\overline{M}_{g,\gamma}(B\mathbb{Z}_r)$ are defined by pull-back via the morphism

$$\epsilon : \overline{M}_{g,\gamma}(B\mathbb{Z}_r) \to \overline{M}_{g,n}$$

as

$$\widetilde{\psi}_i = \epsilon^*(\psi_i) \in H^2(\overline{M}_{g,\gamma}(B\mathbb{Z}_r), \mathbb{Q}).$$

Hurwitz-Hodge integrals over $\overline{M}_{g,\gamma}(B\mathbb{Z}_r)$ are the top intersection products of the classes $\{\lambda_i^R\}$ and $\{\psi_j\}_{1 \leq j \leq n}$:

$$\int_{\overline{M}_{g,\gamma}(B\mathbb{Z}_r)} \overline{\psi}_1^{b_1} \cdots \overline{\psi}_n^{b_n} (\lambda_i^R)^{k_1} \cdots (\lambda_j^R)^{k_j} (\lambda^R_{rkE^U})^{k_{rkE^U}}.$$

We let

$$\Lambda^R(t) = \sum_{j \geq 0} (-t)^j \lambda_j^R,$$

where $rkE^R$ is the rank of $E^R$ determined by the orbifold Riemann-Roch formula.

2.2. Orbifold ELSV formula. In [9], Johnson-Pandharipande-Tseng established the following ELSV-type formula for orbifold Hurwtiz number $H_{g,l}^r(\mu_1, \ldots, \mu_l)$.

Theorem 2.1. The orbifold Hurwitz number has an expression in terms of linear Hurwitz-Hodge integrals as follows:

$$H_{g,l}^r(\mu) = r^{1-g+\sum_{i=1}^l (\frac{\mu_i}{r})} \prod_{i=1}^l \left( \frac{\mu_i}{r} \right)! \int_{\overline{M}_{g,\mu}(B\mathbb{Z}_r)} \frac{\Lambda}{\prod_{i=1}^l (1 - \mu_i \overline{\psi}_i)}.$$

where $\Lambda = \Lambda^U(r) = \sum_{j \geq 0} (-r)^j \lambda_j^U$, $\mu = (\mu_1, \ldots, \mu_l)$ is a partition with length $l$. The floor and fractional part of a $q \in \mathbb{Q}$ is denoted by $q = [q] + \langle q \rangle$.

The rank of the Hodge bundle $E^U$ can be calculated by the orbifold Riemann-Roch formula. When all the monodromies are trivial, i.e. $\langle \mu_i \rangle = 0$, for all $i = 1, \ldots, l$, the rank of $E^U_{g,\mu}$ is $g$, otherwise, the rank is (see formula (3.17) in [2]) given by

$$rkE^U_{g,\mu} = g - 1 + \sum_{i=1}^l \left( \frac{\mu_i}{r} \right).$$

2.3. The Laplace transform of the cut-and-join equation for orbifold Hurwitz numbers. The orbifold Hurwitz number $H_{g,l}^{(r)}(\mu)$ also satisfies the cut-and-join equation, the following formula was established in Bouchard-Serrano-Liu-Mulase [2].

Theorem 2.2. (Cut-and-join equation [2])

$$sH_{g,l}^{(r)}(\mu) = \frac{1}{2} \sum_{i \neq j} (\mu_i + \mu_j) H_{g,l-1}^{(r)}(\mu_i + \mu_j, \mu(\hat{i}, \hat{j})).$$
where

\[ s = 2g - 2 + l + \sum_{i=1}^{l} \frac{\mu_i}{r} \]

is the number of the simple ramification point given by the Riemann-Hurwitz formula. The notation \( \hat{i} \) indicates that the parts \( \mu_i \) is erased.

In order to describe the Laplace transform of the above cut-and-join formula (20). We need to introduce the auxiliary functions showed in \([2]\) (see section 7.2) firstly:

\begin{equation}
\xi_{r,k}^r(\eta) = \begin{cases} 
\frac{1}{r} \eta^r, & k = 0, \\
\frac{1}{k r^{k/r}} \eta^k, & 0 < k < r.
\end{cases}
\end{equation}

and for \( m \geq -1 \), we let

\begin{equation}
\xi_{m+1}^r(\eta) = \frac{\eta}{1 - \eta^r} \frac{d}{d\eta} \xi_m^r(\eta).
\end{equation}

For the following exposition, we also need to fix some notations. let \( L = (1, \ldots, l) \) be an index set, we denote \( b_L = (b_1, \ldots, b_l) \) with \( b_i \geq 0 \), \( k_L = (k_1, \ldots, k_l) \) with \( 0 \leq k_i < r \), \( |k_L| = \sum_{i=1}^{l} k_i \) and \( \tau_{b_L} = \prod_{i=1}^{l} \tau_{b_i} \), \( \xi_{b_L}^r(\eta_L) = \prod_{i=1}^{l} \xi_{b_i}^r(\eta_i) \).

The Hurwitz-Hodge integrals are abbreviated as

\begin{equation}
\langle \tau_{b_L} \Lambda \rangle_g^{r,k} := \langle \tau_{b_1} \tau_{b_2} \cdots \tau_{b_l} \Lambda \rangle_g^{r,k} = \int_{\tau_{b_L} \subset \mathbb{B}^r} \prod_{i=1}^{l} \psi_{b_i}^{r} \Lambda.
\end{equation}

Now the Laplace transform of the cut-and-join equation can be described as follows (see formula (7.26) in \([2]\)).

\begin{equation}
\sum_{|k_L|=0}^{r} \frac{r^{k_L}}{|b_L|} \langle \tau_{b_L} \Lambda \rangle_g^{r,k_L} \left[ (2g - 2 + l) \xi_{b_L}^r(\eta_L) + \frac{1}{r} \sum_{i=1}^{l} (1 - \eta_i^r) \xi_{b_i+1}^r(\eta_i) \xi_{b_L \setminus \{i\}}^r(\eta_L \setminus \{i\}) \right]
\end{equation}

\begin{align*}
&= \sum_{1 \leq i < j \leq l \atop a + |k_L \setminus \{i,j\}| \geq 0 \atop m + b_L \setminus \{i,j\} \geq 3g - 4 + l} \frac{r^{a+|k_L \setminus \{i,j\}|}}{r^{a+|k_L \setminus \{i\}|}} \langle \tau_{m} \tau_{b_L \setminus \{i,j\}} \Lambda \rangle_g^{r,(a,k_L \setminus \{i,j\})} \frac{1}{\eta_i - \eta_j} \\
&\times \left[ \frac{\eta_i \xi_{m+1}^r(\eta_i)}{1 - \eta_i^r} - \frac{\eta_j \xi_{m+1}^r(\eta_j)}{1 - \eta_j^r} \right] \xi_{b_L \setminus \{i,j\}}^r(\eta_L \setminus \{i,j\}) \xi_{b_L \setminus \{i\}}^r(\eta_L \setminus \{i\}) \\
&+ \frac{r}{2} \sum_{i=1}^{l} \left( \sum_{a+b+|k_L \setminus \{i\}| \geq 0 \atop m+n+|b_L \setminus \{i\}| \geq 3g - 4 + l} \frac{r^{a+b+|k_L \setminus \{i\}|}}{r^{a+b+|k_L \setminus \{i\}|}} \langle \tau_{m+n} \tau_{n+1} \Lambda \rangle_g^{r,(a,b,k_L \setminus \{i\})} \xi_{m+1}^r(\eta_i) \xi_{n+1}^r(\eta_i) \xi_{b_L \setminus \{i\}}^r(\eta_L \setminus \{i\}) \right)
\end{align*}
3. Proof of the main results

In this section, we present the proofs for the results showed in Section 1. First, we introduce the new variable \( t \) as

\[
t = \frac{1}{1 - \eta^r}.
\]

It is easy to get

\[
\frac{\eta}{1 - \eta^r} \frac{d}{d\eta} = t^{r+1}(t^r - 1) \frac{d}{dt}.
\]

Hence, in the new variable \( t \),

\[
\xi_{r,k}^{r,-1}(t) = \begin{cases} 
\frac{1}{r} \left( \frac{t^r - 1}{t^r} \right), & k = 0, \\
\frac{1}{kr^k/r} \left( \frac{t^r - 1}{t} \right)^k, & 1 \leq k \leq r - 1.
\end{cases}
\]

and for \( m \geq -1 \), we have

\[
\xi_{r,k}^{r,m+1}(t) = t^{r+1}(t^r - 1) \frac{d}{dt} \xi_{r,k}^{r,m}(t).
\]

**Lemma 3.1.** For \( 0 \leq k \leq r - 1 \), the function \( \xi_{r,k}^{r,m}(t) \) has the following expansion form:

\[
\xi_{r,k}^{r,m}(t) = \frac{1}{r^m} t^{(m+1)r-k}(t^r - 1) \left( c_{r,m}^k t^m + c_{r,m-1}^k t^{(m-1)r} + \cdots + c_{r,0}^k \right).
\]

where the coefficients \( c_{r,m,i}^k \), for \( 0 \leq i \leq m \), satisfy certain recursion relations. In particularly, we have

\[
c_{r,m,m}^k = (2m - 1)!! r^m, \quad c_{r,m,0}^k = \prod_{j=1}^{m} (k - jr).
\]

**Proof.** By definition, we have

\[
\xi_{r,k}^{r,1}(t) = t^{r+1}(t^r - 1) \frac{d}{dt} \left( \frac{1}{r^r} t^r \left( \frac{t^r - 1}{t} \right)^k \right) = \frac{1}{r^r} t^{2r-k}(t^r - 1)^k (rt^r - (r-k)).
\]

So \( \xi_{r,k}^{r,1}(t) \) has the expansion form (30) showed in Lemma 3.1, with

\[
c_{r,1,1}^k = r, \quad c_{r,1,0}^k = k - r.
\]

When \( m \geq 1 \), one has

\[
\xi_{r,k}^{r,m+1}(t) = t^{r+1}(t^r - 1) \frac{d}{dt} \xi_{r,k}^{r,m}(t).
\]
\[
\frac{1}{r^r} t^{r+1} (t^r - 1) \frac{d}{dt} \left( t^{m+1} r - k (t^r - 1)^{\frac{k}{r}} \sum_{j=0}^{m} c_{m,j}^{k} t^r \right) \\
= \frac{1}{r^r} t^{(m+2)r-k} (t^r - 1)^{\frac{k}{r}} ((2m+1)r c_{m,m} t^{(m+1)r} + \cdots + (k - (m+1)r) c_{m,0}^k).
\]

Therefore,
\[
(35) \quad c_{m+1,m+1}^k = (2m+1)r c_{m,m}^k, \quad c_{m+1,0}^k = (k - (m+1)r) c_{m,0}^k.
\]

It is direct to obtain the formula (31) by the initial values in formula (33). \[\square\]

In terms of the new auxiliary functions \(\xi_m^{r,k}(t)\), the formula (25) can be changed to
\[
(36) \quad \sum_{|k_L| \equiv 0} \sum_{|b_L| \leq 3g-3+l} \left( t_{j_1}^{r+1} (t_{j_1} - 1)^{\frac{1}{r}} \xi_{m+1}^{r,a}(t_{j_1}) - t_{j_2}^{r+1} (t_{j_2} - 1)^{\frac{1}{r}} \xi_{m+1}^{r,a}(t_{j_2}) \right).
\]

Lemma 3.2. For \(0 \leq a \leq r - 1\), we have
\[
(37) \quad \frac{t_{j_1}^{r+1} (t_{j_1} - 1)^{\frac{1}{r}} \xi_{m+1}^{r,a}(t_{j_1}) - t_{j_2}^{r+1} (t_{j_2} - 1)^{\frac{1}{r}} \xi_{m+1}^{r,a}(t_{j_2})}{t_{j_1}^{r+1} (t_{j_1} - 1)^{\frac{1}{r}} - t_{j_2}^{r+1} (t_{j_2} - 1)^{\frac{1}{r}}} = \frac{1}{r^r} \sum_{p=0}^{m+1} \sum_{s=0}^{a-2} (t_{j_1}^{s+1}(m+3+p)r - (a-1-s) t_j (m+3+p)r - (s+1))
\]

\[
- \sum_{q=0}^{m+3+p} \binom{m+3+p}{q} (-1)^q \sum_{s=0}^{q-2} \binom{q-2}{s} (t_j^{s+1}(m+3+p)r - (q-1-s) t_j (m+3+p)r - (s+1))
\]
Proof. By a direct calculation, we have

\[
\frac{t_j^{r+1}(t_j^r - 1)^{\frac{1}{r}} \xi_m(t_i)}{t_j(t_j^r - 1)^{\frac{1}{r}} - t_i(t_i^r - 1)^{\frac{1}{r}}}
\]

(38)

\[
= \frac{1}{r^2} \sum_{p=0}^{m+1} \eta^{a-1}_{m+1,p} \sum_{p=0}^{m+1} \eta^{a-1}_{m+1,p} \left( (1 - \eta_j)^{m+3+p} - \eta_j^m \right) \]

\[
= \frac{1}{r^2} \sum_{p=0}^{m+1} \eta^{a-1}_{m+1,p} \left( (1 - \eta_j)^{m+3+p} - \eta_j^m \right) \]

\[
= \frac{1}{r^2} \sum_{p=0}^{m+1} \eta^{a-1}_{m+1,p} \left( (1 - \eta_j)^{m+3+p} - \eta_j^m \right) \]

(39)

\[
\prod_{j=1}^{l} (t_j^r - 1)^{\frac{1}{r}} t_1^{(2b_1+2)r-k_1} \prod_{j=2}^{l} t_j^{(2b_j+1)r-k_j}.
\]

Proof of the Theorem 1.1:

Proof. When \( \sum_{j=1}^{l} b_j = 3g - 3 + l \), we consider the coefficient of the monomial
in the formula (36). This coefficient in the left hand side is equal to

\begin{equation}
\frac{1}{r} \prod_{j=2}^{l} \frac{k_j}{c_{b_j, b_j} (c \tau_{b_j})^r}.
\end{equation}

When \( m = b_1 + b_j - 1 \) and \( a \equiv k_1 + k_j \), for \( j = 2, \ldots, l \), by Lemma 3.2, we denote the coefficient of the term

\begin{equation}
(t_1^r - 1)^{k_1} t_j^{k_j} (t_j^r - 1)^{k_j} (2b_1 + 2)r - k_1 (2b_j + 1)r - k_j
\end{equation}

in the formula (37) as \( C_{b_1, b_j}^{k_1, k_j} \). Then the coefficient in the first term of the right hand side is

\begin{equation}
\sum_{j=2}^{l} C_{b_1, b_j}^{k_1, k_j} \prod_{i \neq 1, j} \frac{k_i}{c_{b_i, b_i} (c \tau_{b_i})^r} (r)(r_{(k_1^k + k_j^k)}, k_L \setminus (1, j))
\end{equation}

This coefficient in the second term of the right hand side is

\begin{equation}
\frac{r}{2} \prod_{j=2}^{l} \frac{k_j}{c_{b_j, b_j}} \sum_{\substack{a + b = k_1 \\text{stable} \\text{g}_1 + g_2 = g \\text{g}_1 + g_2 \equiv 0 \\text{g}_1 + g_2 \equiv 0 \\text{g}_1 + g_2 \equiv 0 \\text{g}_1 + g_2 \equiv 0}} \left( c_{m + 1, m + 1} c_{n + 1, n + 1} \left( c_{m + 1, m + 1} c_{n + 1, n + 1} \right) \left( c_{m + 1, m + 1} c_{n + 1, n + 1} \right) \right)
\end{equation}

Where the coefficient \( C_{b_1, b_j}^{k_1, k_j} \) are given by formula (31). Collecting the formulas (40),(42) and (43) together, we obtain the Theorem 1.1. \( \square \)

Proof of the Theorem 1.3:

Proof. Now we consider the lowest nonzero terms in the formula (36). The rank of the Hodge bundle \( \mathcal{E}^U_{g,k_L} \) is \( g \) if all the loop monodories are trivial, i.e. \( k_i = 0 \), for 1 \( \leq i \leq l \). Otherwise, the rank is

\begin{equation}
rk \mathcal{E}^U_{g,k_L} = g - 1 + \sum_{i=1}^{l} \left( - \frac{k_i}{r} \right).
\end{equation}

First, we consider the nontrivial case. So the smallest possible \( |b_L| \) is

\begin{equation} |b_L| = 3g - 3 + l - (g - 1 + \sum_{i=1}^{l} \left( - \frac{k_i}{r} \right))
\end{equation}

\begin{equation} = 2g - 2 + N(\{k_L\}) + \frac{|k_L|}{r},
\end{equation}

where we have used the notation \( N(\{k_L\}) \) to denote the number of the set \( \{i | k_i = 0, 1 \leq i \leq l\} \).

The lowest term of a \( \xi_n^r, k \) is

\begin{equation}
\frac{1}{r^{k/r}} \prod_{j=1}^{m} \left( t_j^r - 1 \right)^{k_j} (t_j^r - 1)^{k_j r}\n\end{equation}

where \( c_{m, 0}^k = \prod_{j=1}^{m} (k - j r) \). When \( |b_L| = \sum_{j=1}^{l} b_j = 2g - 2 + N(\{k_L\}) + \frac{|k_L|}{r} \), consider the coefficient of the monomial

\begin{equation}
\prod_{j=1}^{l} (t_j^r - 1)^{k_j/r} (t_j^r + 1)^{r - k_j}
\end{equation}
in the formula (36).

This coefficient in the left hand side of (36) is

\begin{equation}
(-r)^{r_k E_g, k_L} \langle \tau_{b_L} \lambda_{r_k E_g} \rangle_g^{r_k L} \prod_{j=1}^{l} c_{b_j, 0}^{k_j} (2g - 2 + l + \frac{1}{r} \sum_{i=1}^{l} (k_i - (b_i + 1)r))
\end{equation}

\begin{equation}
= (-r)^{r_k E_g, k_L} \langle \tau_{b_L} \lambda_{r_k E_g} \rangle_g^{r_k L} \prod_{j=1}^{l} c_{b_j, 0}^{k_j} (2g - 2 + \frac{|k_L|}{r} - |b_L|)
\end{equation}

\begin{equation}
= -N(\{k_L\})(-r)^{r_k E_g, k_L} \prod_{j=1}^{l} c_{b_j, 0}^{k_j} \langle \tau_{b_L} \lambda_{U} \rangle_g^{r_k L}
\end{equation}

To compute coefficients in the second term of right hand side of (36), first find that range of \(m, n\) with nonzero contribution in \(\langle \tau_m \tau_n \tau_{b_L \setminus \{i\}} \lambda_{g-1} \rangle_g^{r_{(a,b,k_L \setminus \{i\})}}\) is

\begin{equation}
m + n + |b_L \setminus \{i\}| \geq 2g - 4 + N(\{a, b, k_L \setminus \{i\}\}) + \frac{|k_L \setminus \{i\}| + a + b}{r} \iff
\end{equation}

\begin{equation}
m + n \geq b_i - 2 + N(\{a, b\}) - N(\{k_i\}) + \frac{a + b - k_i}{r}
\end{equation}

Range of \(m, n\) in \(\langle \tau_m \tau_{b_l} \lambda_{g-1} \rangle_g^{r_{(a,k_l)}} \langle \tau_n \tau_{b_J} \lambda_{g-1} \rangle_g^{r_{(b,k_J)}}\) is

\begin{equation}
m + |b_I| \geq 2g_1 - 2 + N(\{a, k_I\}) + \frac{|k_I| + a}{r}
\end{equation}

\begin{equation}
n + |b_J| \geq 2g_2 - 2 + N(\{b, k_J\}) + \frac{|k_J| + b}{r}
\end{equation}

Hence

\begin{equation}
m + n \geq b_i - 2 + N(\{a, b\}) - N(\{k_i\}) + \frac{a + b - k_i}{r}.
\end{equation}

Since \(0 \leq a, b, k_i < r\) and \(\frac{a + b - k_i}{r} = 0\), it is easy to obtain that \(a + b - k_i\) is equal to \(b_i - 1\) or \(b_i - 2\).

On the other hand, the lowest term of \(\xi^{r, a}_{m+1}(t_i) \xi^{r, b}_{m+1}(t_i)\) is

\begin{equation}
(t_i^r - 1) \frac{a+b}{r} t_i^{(m+n+4)r-a-b}
\end{equation}

Only when \(k_i = 0\) and \(a + b = r\), this term contributes a coefficient \(-1\) to the term

\begin{equation}
(t_i^r - 1) \frac{k_i}{r} t_i^{(b_i+1)r-k_i}.
\end{equation}

Hence, the coefficient of the monomial \(\prod_{j=1}^{l} t_j^{k_j/r} t_j^{(b_j+1)r-k_j}\) in the second term of right hand side is

\begin{equation}
- \frac{r}{2} \sum_{i \in \mathbb{Z}} \sum_{a+b=r, m+n=b_i-2} (-r)^{r_k E_g, (a,b,k_L \setminus \{i\})} \langle \tau_m \tau_n \tau_{b_L \setminus \{i\}} \lambda_{r_k E_g} \rangle_g^{r_k L} c_{a+m+1,0}^{a} c_{b+n+1,0}^{b} \prod_{j \neq i} c_{b_j, 0}^{k_j} + \sum_{\text{stable}} \sum_{\substack{g_1+g_2=g \\text{ and } \sum_{j \neq i} k_j \equiv 0 \mod r \\text{ and } \sum_{j \neq i} k_j \equiv 0 \mod r \\text{ and } m+|b_I|=2g_1-2+N(\{k_I\})+\frac{a+b}{r} \\text{ and } n+|b_J|=2g_2-2+N(\{k_J\})+\frac{b}{r} \\text{ and } a+b=k_i}} (-r)^{r_k E_g, (a,k_I)} + r_k E_g, (b,k_J)}
\end{equation}
By using the rank formula (19), it is easy to compute that the ranks appearing in the formulas (59) satisfy:

\[ \sum_{j \neq i} k_j \geq N(\{a\}) - N(\{k_i, k_j\}) + b_i + b_j + \frac{a - k_i - k_j}{r}. \]

From the formula (37), only when \(k_i, k_j = 0\) and \(a = 0\) or \(k_i = 0\) and \(a = k_j\), we take \(m = b_i + b_j - 1\), then the left hand side of the formula (37) can contribute a coefficient \(-\frac{1}{r} r^a b_i b_j, 0\)

to the term \((t_i^r - 1) t_i^{(b_i+1)r-k_i} (t_j^r - 1) t_j^{(b_j+1)r-k_j}\).

If we define the index set \(Z \subseteq L\) as

\[ Z = \{i | k_i = 0, 1 \leq i \leq l\}. \]

The coefficient of the monomial \(\prod_{j=1}^l (t_i^r - 1) t_i^{(b_i+1)r-k_i}\) in the first term of right hand side is

\[ - \left( \sum_{i \in Z} \sum_{j \in k_L \setminus Z} (-r)^{r_{k_E^{g,(k_j,k_L\{i,j\})}\{}}} (\tau_{b_i+b_j-1} \tau_{b_L\setminus\{i,j\}} \lambda_{r_{k_E^{g}}^{U}})^{r_{k_E^{g,(k_j,k_L\{i,j\})}\{}}} c_{b_i+b_j,0} \right) \prod_{q \neq i,j} c_{q,0}. \]

By using the rank formula (19), it is easy to compute that the ranks appearing in the formulas (48), (56) and (59) satisfy:

\[ r_{k_E^{g,(k_j,k_L\{i,j\})}} = r_{k_E^{g,k_L}}, \]
\[ r_{k_E^{g,(0,k_L\{i,j\})}} = r_{k_E^{g,k_L}}, \]
\[ r_{k_E^{g-1,(a,b,k_L\{i\})}} = r_{k_E^{g,k_L}} + 1, \]
\[ r_{k_E^{g,(a,k_j)}} + r_{k_E^{g,(b,k_L\{i\})}} = r_{k_E^{g,k_L}} + 1. \]

Finally, combining (48), (56) and (59) together, we obtain the following formula

\[ \sum_{j=1}^l c_{b_j,0}^r \sum_{i \in Z} \sum_{j \in k_L \setminus Z} (\tau_{b_i+b_j-1} \tau_{b_L\setminus\{i,j\}} \lambda_{r_{k_E^{g}}^{U}})^{r_{k_E^{g,(k_j,k_L\{i,j\})}\{}}} c_{b_i+b_j,0} \]

\[ = \left( \sum_{i \in Z} \sum_{j \in k_L \setminus Z} (\tau_{b_i+b_j-1} \tau_{b_L\setminus\{i,j\}} \lambda_{r_{k_E^{g}}^{U}})^{r_{k_E^{g,(k_j,k_L\{i,j\})}\{}}} c_{b_i+b_j,0} \right) \prod_{q \neq i,j} c_{q,0} \]

\[ - \frac{r^2}{2} \sum_{i \in Z} \left( \sum_{a+b=r}^{m+n=b-2} \sum_{m+n=b-2} (\tau_{m+n} \tau_{b_L\setminus\{i\}} \lambda_{r_{k_E^{g}}^{U}})^{r_{k_E^{g,(a,b,k_L\{i\})}} c_{m+1,0} r^b c_{b_j,0}} \right) \]
\[+ \sum_{g_1 + g_2 = g \atop \sum a_i = 0} \sum_{t \in \mathbb{Z}^2} \langle \tau_m \tau_{br} \lambda^U \rangle_{g_1}^{r(a,k_t)} \langle \tau_n \tau_{bj} \lambda^U \rangle_{g_2}^{r(b,k_j)} c_{m+1,0}^{t_n+1,0} \prod_{j \neq i} c_{b_j,0}^{k_j}\]

So we have the formula (6) in Theorem 1.3.

As to the trivial monodromies case, i.e. \( k_i = 0 \), for all \( 1 \leq i \leq l \). We have \( r k^U = g \). With the analogue analysis as above, when \( \sum_i b_i = 2g - 3 + l \), the coefficients of \( \prod_{i=1}^{l} t_i^{2b_i+1} \) at the left hand side of formula (36) takes the form

\[(62) \quad (1 - l) \langle \tau_{b_i} \lambda^U \rangle_g^{r} \prod_{i=1}^{l} c_{b_i,0}^{0}.\]

While at the right hand side, only the first term has the contribution

\[(63) \quad \sum_{1 \leq i < j \leq t} \langle \tau_{b_i+b_j} \lambda^U \rangle_g^{r} \prod_{q \neq i,j} c_{b_q,0}^{0} \]

where the coefficients \( c_{b_i,0}^{0} \) is given by formula (31). So we obtain

\[(64) \quad \langle \tau_{b_i} \lambda^U \rangle_g^{r} = \frac{1}{l - 1} \sum_{1 \leq i < j \leq t} \frac{(b_i + b_j)!}{b_i! b_j!} \langle \tau_{b_i+b_j} \rangle_g^{r}.\]

Then by induction, we obtain the formula (4) in Theorem 1.3.

\[\square\]

Acknowledgements. The authors would like to thank Hao Xu for bringing the paper [17] to their attentions. Thank Prof. Kefeng Liu for useful discussions. This research is supported by China Postdoctoral Science Foundation 2011M500986 and National Science Foundation of China grants No.11201417.

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