Research Article

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Compactness and hypercyclicity of co-analytic Toeplitz operators on de Branges-Rovnyak spaces

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Abstract: We study the compactness and the hypercyclicity of Toeplitz operators $T_{\bar{\varphi}, b}$ with co-analytic and bounded symbols on de Branges-Rovnyak spaces $H(\mathbf{b})$. For the compactness of $T_{\bar{\varphi}, b}$, we will see that the result depends on the boundary spectrum of $b$. We will prove that there are non-trivial compact operators of the form $T_{\bar{\varphi}, b}$, with $\varphi \in H^\infty \cap C(\mathbf{T})$, if and only if $m(\sigma(b) \cap \mathbf{T}) = 0$. We will also show that, when $b$ is non-extreme, then $T_{\bar{\varphi}, b}$ is hypercyclic if and only if $\varphi$ is non-constant and $\varphi(\mathbf{T}) \cap \mathbf{T} \neq \emptyset$.

Keywords: Toeplitz operators, de Branges-Rovnyak spaces, compactness, hypercyclicity

MSC: 30J05, 30H10, 46E22, 47A16

1 Introduction

We shall mostly be discussing co-analytic Toeplitz operators $T_{\bar{\varphi}}$ with symbol $\bar{\varphi}$ where $\varphi \in H^\infty$, that are naturally defined on the de Branges-Rovnyak space into itself. These operators have been introduced by Lotto-Sarason in [13, Lemma 2.6], see also [14, Section II.7]. Some special cases have long ago appeared in literature for $\varphi \in L^\infty(\mathbf{T})$, most notably as standard Toeplitz operators $T_{\varphi} : H^2 \to H^2$ studied by A. Brown and P. Halmos in the paper [5] and as the adjoints of truncated Toeplitz operators $A_{\Theta}^\phi$ on model spaces $K_{\Theta}$ introduced by Sarason in [15]. We will consider Toeplitz operators with different domains and different ranges. To avoid confusion, we adopt different notations. We will denote by $T_{\bar{\varphi}}$ the Toeplitz operator defined from $H^2$ into itself, by $T_{\bar{\varphi}, b}$ the Toeplitz operator defined from $H(\mathbf{b})$ into itself, and by $T_{\bar{\varphi}, b}$ the Toeplitz operator defined from $H(\mathbf{b})$ into $H^2$.

It turns out that de Branges-Rovnyak spaces, which are a family of subspaces $H(\mathbf{b})$ of the Hardy space $H^2$, parametrized by elements $\mathbf{b}$ of the closed unit ball of $H^\infty$ are invariant under $T_{\bar{\varphi}}$, where $\varphi \in H^\infty$. We shall give the precise definition in section 2. In general $H(\mathbf{b})$ is not closed in $H^2$, but it carries its own norm $||-||_{H(\mathbf{b})}$ making it a hilbert space. The spaces $H(\mathbf{b})$ were introduced by de Branges and Rovnyak in the appendix of [6] and further studied in their book [7].

The general theory of $H(\mathbf{b})$-spaces generally splits into two cases, according to whether $\mathbf{b}$ is an extreme point or a non-extreme point of the unit ball of $H^\infty$. The dichotomy $\mathbf{b}$ extreme/non-extreme will also often appear in this paper. The general idea is that the extreme case has many features that are not far from the case of $\mathbf{b} = \Theta$ inner (the classical model space $K_{\Theta}$), while the non-extreme case has several properties that are similar to the case where $\mathbf{b} = 0$ (the Hardy space $H^2$).

This paper treats two properties related to the restricted Toeplitz operators $T_{\bar{\varphi}, b}$ when $\varphi \in H^\infty$. One of these properties is based on the particular operator $X_b = T_{z,b}$ that plays a central role in the theory and

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2 Preliminaries

2.1 Toeplitz operators and de Branges-Rovnyak spaces

We first recall some basic facts on Toeplitz operators on the Hardy space $H^2$ of the open unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$.

Given $\varphi \in L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, m)$ where $\mathbb{T} = \partial D$ and $m$ is the normalized Lebesgue measure on $\mathbb{T}$, the corresponding Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined by

$$T_\varphi f := P_+(\varphi f) \quad (f \in H^2),$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal projection of $L^2(\mathbb{T}) = L^2(\mathbb{T}, m)$ onto $H^2$. Clearly $T_\varphi$ is a bounded operator on $H^2$ with $\| T_\varphi \| = \| \varphi \|_{L^\infty(\mathbb{T})}$, moreover it is compact if and only if $\varphi = 0$ (Brown–Halmos, [5]). If $\varphi \in H^\infty$ the algebra of the analytic and bounded functions on $\mathbb{T}$, then $T_\varphi$ is simply the operator of multiplication by $\varphi$ and its adjoint is $T_\bar{\varphi}$. Consequently, if $\varphi, \psi \in H^\infty$, then $T_\varphi T_\psi = T_{\varphi \psi} = T_{\psi \varphi} = T_\psi T_\varphi$.

Moreover, if $\varphi \in H^\infty$ we have

$$T_\varphi k_\lambda = \overline{\varphi(\lambda)} k_\lambda,$$

where $k_\lambda(z) = (1 - \lambda z)^{-1}$ is the reproducing kernel of $H^2$ (see [8, Section 12.4]).
If $\varphi \in L^\infty(\mathbb{T})$ satisfies $\|\varphi\|_\infty \leq 1$, then the corresponding Toeplitz operator $T_\varphi$ is a contraction on the Hilbert space $H^2$. The associated de Branges-Rovnyak space $\mathcal{H}(T_\varphi)$ is defined by

$$\mathcal{H}(T_\varphi) = (I - T_\varphi T_\varphi)_{1/2} H^2.$$ 

For simplicity, we denote the complementary space $\mathcal{H}(T_\varphi)$ by $\mathcal{H}(\varphi)$ (see [8, Section 17.3]). Therefore, the definition of an $\mathcal{H}(\varphi)$-space uses the defect of the contraction $T_\varphi$ [8]. Hence, no doubt, the Toeplitz operators are extremely important in this context. Our main concern is when $\varphi$ is a nonconstant analytic function in the closed unit ball of $H^\infty$. In this case, by tradition, we use $b$ instead of $\varphi$.

We recall an alternative and equivalent definition based on reproducing kernels. Namely, $\mathcal{H}(b)$ is the Hilbert space of analytic functions on $\mathbb{D}$ whose reproducing kernel is given by

$$k^b_\lambda(z) = \frac{1 - b(\lambda) \overline{b(z)}}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D}.$$

That is,

$$f(\lambda) = \langle f, k^b_\lambda \rangle_b, \forall f \in \mathcal{H}(b), \forall \lambda \in \mathbb{D}.$$

For $b = 0$, we see that $k^b_\lambda$ coincides with $k_\lambda$ the reproducing kernels of $H^2$, given by $k_\lambda(z) = (1 - \lambda z)^{-1}$, whence $\mathcal{H}(0) = H^2$.

More generally when $||b||_\infty < 1$, $\mathcal{H}(b)$ coincides with the Hardy space $H^2$ with an equivalent norm.

For $b = \Theta$, with $\Theta$ an inner function (that is a function in the closed unit ball of $H^\infty$ such that $|\Theta(z)| = 1$ almost everywhere on $\mathbb{T} = \partial \mathbb{D}$), the space $\mathcal{H}(\Theta)$ is a closed subspace of $H^2$, and we have

$$\mathcal{H}(\Theta) = \Theta H^2 \perp := \{f \in H^2 : \langle f, \Theta g \rangle_2 = 0, \forall g \in H^2\}.$$

The space $\mathcal{H}(\Theta)$ is also called the model space and is denoted by $K_\Theta = \mathcal{H}(\Theta)$. By Beurling’s theorem, the spaces $K_\Theta$ correspond to the lattice of closed, non trivial, invariant subspaces for the backward shift operator $S^* = T_2$ on $H^2$.

In the general case, the spaces $\mathcal{H}(b)$ are Hilbert spaces that are contained contractively in $H^2$. Moreover, it is well-known that there are relations between the inner products of $\mathcal{H}(b)$ and its cousin $\mathcal{H}(\bar{b})$ since these relations are special cases of the Lotto–Sarason theorem [8, Theorem 16.18 and corollary 16.19]. For further reference, we restate this result below.

**Theorem 2.1** ([8], Theorem 17.8). Let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_b f \in \mathcal{H}(\bar{b})$ and

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle T_b f_1, T_b f_2 \rangle_b, \quad (f_1, f_2 \in \mathcal{H}(b)).$$

It is now a well-known fact that the general theory of $\mathcal{H}(b)$-spaces splits into two cases, according to whether $b$ is an extreme point or a non-extreme point of the unit ball of $H^\infty$ (recall that, according to De Leeuw-Rudin’s Theorem, $b$ is a non-extreme point of the closed unit ball of $H^\infty$ if and only if $\log(1 - |b|) \in L^1(\mathbb{T})$, in particular every inner function $b = \Theta$ is an extreme point).

For example,

$$\forall \lambda \in \mathbb{D}, k_\lambda \in \mathcal{H}(b) \iff b \text{ is non–extreme}, \quad (2)$$

(see [8, Theorem 23.23 and corollary 25.8]).

Furthermore from the above characterization of a non-extreme point it follows that, if $b$ is non-extreme, then there is an outer function $a$ such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$ [14]. The function $a$ is uniquely determined by $b$. We shall call $(a, b)$ an euclidian pair. The following result gives a useful characterization of $\mathcal{H}(b)$ in this case.
Theorem 2.2 ([8], Theorem 23.8). Let $b$ be a non-extreme point of the closed unit ball of $H^\infty$, let $(a, b)$ be an euclidian pair and let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_b f \in T_b(H^2)$. In this case, for $f_1, f_2 \in \mathcal{H}(b)$ there exists a unique $f_1', f_2' \in H^2$ such that $T_b f_i = T_b f_i'$ for $i = 1, 2$ and
\[
< f_1, f_2 >_b = < f_1, f_2 >_2 + < f_1', f_2' >_2.
\]
In particular, for each $f \in \mathcal{H}(b)$,
\[
||f||^2 = ||f||^2_2 + ||f||^2_2.
\]

An important operator in the theory of model spaces is the compression of Toeplitz operators on $K_\Theta$: for $\varphi \in L^\infty$ and $\Theta$ an inner function, one defines the truncated Toeplitz operator $A_{\varphi}^\Theta$ by
\[
A_{\varphi}^\Theta : K_\Theta \to K_\Theta
\]
\[
f \mapsto A_{\varphi}^\Theta f = P_\Theta(T_{\varphi} f),
\]
with $P_\Theta$ the orthogonal projection of $H^2$ to $K_\Theta$. It turns out that when $\varphi$ is in $H^\infty$, then $K_\Theta$ is invariant for $T_\varphi$ and the adjoint of the truncated Toeplitz operator with symbol $\varphi$ is $(A_{\varphi}^\Theta)^* = T_{\varphi, \Theta}$ [8, Section 14.7]. More generally, for $\varphi \in H^\infty$, $\mathcal{H}(b)$ is invariant for $T_{\varphi}$. We will denote by
\[
T_{\varphi, b} : \mathcal{H}(b) \to \mathcal{H}(b)
\]
\[
f \mapsto (T_{\varphi} f, \varphi f, \bar{\varphi} f, P_{\varphi, \Theta} f),
\]
and we have, $||T_{\varphi, b}||_{\mathcal{H}(b)} \leq ||\varphi||_\infty$ [8]. In particular, $\mathcal{H}(b)$ is invariant for the backward shift $S^* = T_2$.

When $b$ is a non-extreme point of the closed unit ball of $H^\infty$, it follows from (1) and (2) that
\[
T_{\varphi, b} k_\lambda = \varphi(\lambda) k_\lambda, \quad \lambda \in \mathbb{D}.
\]

3 Compactness of $T_{\varphi, b}$.

Ahern and Clark [1] have given a necessary and sufficient condition for the truncated Toeplitz operator $A_{\varphi}^\Theta$ to be compact, when the symbol $\varphi$ is continuous on the boundary. See also an alternative proof by Garcia-Ross-Wogen in [9]. The characterization of Ahern-Clark involves the notion of the spectrum of an inner function.

Recall that the spectrum of a function $b$ in the closed unit ball of $H^\infty$ [8, Section 5.2 and 22.6], denoted by $\sigma(b)$ is defined as follows
\[
\sigma(b) = \{ \zeta \in \mathbb{T} : \liminf_{z \to \zeta} |b(z)| < 1 \} \cup \mathcal{Z}(b),
\]
where $\mathcal{Z}(b) = \{ \lambda \in \mathbb{D} : b(\lambda) = 0 \}$.

A generalization of Livsic-Moeller’s result shows that $b$ and every element in $\mathcal{H}(b)$ can be analytically continued accross any arc $I \subset \mathbb{T} \setminus \text{clos}(\sigma(b))$, and $|b| = 1$ on $I$ [8, Theorem 20.13].

In particular if $b = \Theta$ is a non constant inner function, and since $\Theta$ is unimodular a.e. on $\mathbb{T}$ then
\[
\sigma(\Theta) = \left\{ \zeta \in \mathbb{D} : \liminf_{z \to \zeta} |\Theta(z)| = 0 \right\} = \text{clos}(\mathcal{Z}(\Theta)) \cup \text{supp}(\nu),
\]
where $\mathcal{Z}(\Theta) = \{ \lambda \in \mathbb{D} : \Theta(\lambda) = 0 \}$ and $\nu$ is the measure representing the singular part of $\Theta$.

Now Ahern and Clark’s result says:

Theorem 3.1 (Ahern-Clark, [1]). Let $\varphi \in C(\mathbb{T})$, then $A_{\varphi}^\Theta$ is compact if and only if $\varphi|_{\sigma(\varphi) \cap \mathbb{T}} = 0$.  

The compactness property of the operators $T_{\bar{\psi},b}$ will depend on the boundary spectrum of $b$ and it is a consequence of the generalization of Ahern and Clark's result.

For this reason we begin by this generalization, and we study the compactness of the general operator $T_{\bar{\psi},b}$ with $\varphi \in C(\mathbb{T})$, using the same technique used by Garcia, Ross and Wogen [9] to prove the Ahern-Clark result on compactness of $A_0^\varphi$.

### 3.1 Compactness of $T_{\bar{\psi},b}$

Recall that the notation $T_{\bar{\psi},b}$ represents the Toeplitz operator defined from $\mathcal{H}(b)$ into $H^2$.

**Theorem 3.2.** Let $b$ be a point of the closed unit ball of $H^\infty$ and let $\varphi \in C(\mathbb{T})$. Then the operator,

$$T_{\bar{\psi},b} : \mathcal{H}(b) \to H^2 \quad f \mapsto P_+(\bar{\psi}f),$$

is compact if and only if $\varphi|_{\sigma(b) \cap \mathbb{T}} = 0$.

**Proof.** ($\Leftarrow$) Suppose that $\varphi|_{\sigma(b) \cap \mathbb{T}} = 0$. Let $\varepsilon > 0$ and pick $\psi \in C(\mathbb{T})$; $\psi = 0$ on an open set containing $\text{clos}(\sigma(b) \cap \mathbb{T})$ and $||\psi - \varphi||_{\infty} < \varepsilon$. Since $||T_{\bar{\psi},b} - T_{\bar{\psi},b}||_{(\mathcal{H}(b),H^2)} \leq ||\psi - \varphi||_{\infty} < \varepsilon$, it suffices to show that $T_{\bar{\psi},b}$ is compact.

Let $K = \overline{\bar{\psi}^{-1}(\mathcal{C} \setminus \{0\})}$ then $K \subset \mathbb{T} \setminus \text{clos}(\sigma(b))$. And consider $(f_n)_n$ a sequence of $\mathcal{H}(b)$ such that $(f_n)_n$ weakly converges to zero.

We know that for each $\zeta \in K$, the function

$$k^b_\zeta(z) = \frac{1 - b(\overline{\zeta})b(z)}{1 - \overline{\zeta}z},$$

belongs to $\mathcal{H}(b)$ and for every $f \in \mathcal{H}(b)$,

$$f(\zeta) = \langle f, k^b_\zeta \rangle_b,$$

and

$$||k^b_\zeta||_b^2 = \frac{1 - |b(\zeta)|^2}{1 - |\zeta|^2} = |b'(\zeta)|.$$

(see [8, Theorem 21.1]).

In particular, since $(f_n)_n$ weakly converges to zero in $\mathcal{H}(b)$, we have $f_n(\zeta) = \langle f_n, k^b_\zeta \rangle_b \to 0$, as $n \to \infty$, and for every $n \in \mathbb{N}$, $||f_n||_b \leq C$.

Therefore, since $b$ is analytic on a neighborhood of the compact set $K$ we obtain

$$\forall \zeta \in K, |f_n(\zeta)| = |\langle f_n, k^b_\zeta \rangle_b| \leq ||f_n||_b ||k^b_\zeta||_b \leq C \sup_{\zeta \in K}\sqrt{b'(\zeta)} < \infty. \quad (4)$$

By the dominated convergence theorem, and using $(4)$ it follows that

$$||T_{\bar{\psi},b}f_n||_b^2 = ||P_+(\bar{\psi}f_n)||_b^2 \leq ||\bar{\psi}f_n||_b^2 = \int_\mathbb{T} |\psi|^2 |f_n|^2 d\zeta = \int_\mathbb{T} |\psi|^2 |f_n|^2 d\zeta \to 0,$$

whence $T_{\bar{\psi},b}$ is compact and therefore $T_{\bar{\psi},b}$ is compact.

($\Rightarrow$) Suppose that $\varphi \in C(\mathbb{T}), \zeta \in \sigma(b) \cap \mathbb{T}$ and $T_{\bar{\psi},b}$ is compact. Let

$$F_A(z) = \frac{1 - |\lambda|^2}{1 - |b(\lambda)|^2} \left| \frac{1 - \overline{b(\lambda)}b(z)}{1 - \lambda z} \right|^2,$$

which is the square of the absolute value of the normalized reproducing kernel for $\mathcal{H}(b)$. Observe that
\( F_\lambda(z) \geq 0. \)

Since \( \zeta \in \sigma(b) \cap \mathbb{T} \) then there is a sequence \( \lambda_n \) in \( \mathbb{D} \) such that \( \lambda_n \to \zeta \) and \( |b(\lambda_n)| \to c \) with \( c < 1 \) (by the definition of the spectrum of \( b \) already mentioned). Suppose that \( \zeta = e^{it} \) and note that if \( |t - a| \geq \delta \), then

\[
F_{\lambda_n}(e^{it}) \leq C_\delta \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2},
\]

for some absolute constant \( C_\delta > 0 \). Thus since \( |b(\lambda_n)| \to c \) with \( c < 1 \), we get that

\[
\sup_{|t - a| \geq \delta} F_{\lambda_n}(e^{it}) \to 0 \quad \text{as } n \to \infty.
\]

Write,

\[
\varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k^b_{\lambda_n}\|_b^2} k^b_{\lambda_n}, T_{\varphi,b} k^b_{\lambda_n} > 2
\]

\[
= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k^b_{\lambda_n}\|_b^2} k^b_{\lambda_n}, P_\ast(\bar{\varphi} k^b_{\lambda_n}) > 2
\]

\[
= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k^b_{\lambda_n}\|_b^2} k^b_{\lambda_n}, \bar{\varphi} k^b_{\lambda_n} > 2 \quad \text{(see [8, lemma 4.8])}
\]

\[
= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k^b_{\lambda_n}\|_b^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(e^{it})| k^b_{\lambda_n}(e^{it})|^2 dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varphi(\zeta) \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt
\]

\[
= \frac{1}{2\pi} \int_{|t - a| \geq \delta} (\varphi(\zeta) \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt + \frac{1}{2\pi} \int_{|t - a| \geq \delta} (\varphi(\zeta) \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt.
\]

The first integral can be made small by the continuity of \( \varphi \). Once \( \delta > 0 \) is fixed the second term goes to zero since \( \sup_{|t - a| \geq \delta} F_{\lambda_n}(e^{it}) \to 0 \) as \( n \to \infty \). In addition

\[
\int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt = \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2} \int_{-\pi}^{\pi} \frac{1 - b(\lambda_n)}{1 - \lambda_n e^{it}}^2 dt
\]

\[
\approx \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2} (1 - |b(\lambda_n)|)^2 \int_{-\pi}^{\pi} \frac{1}{1 - \lambda_n e^{it}}^2 dt
\]

\[
= \frac{(1 - |\lambda_n|^2)(1 - |b(\lambda_n)|)^2}{(1 - |b(\lambda_n)|)(1 + |b(\lambda_n)|)} \frac{1}{1 - |\lambda_n|^2} 
\]

\[
= \frac{1 - |b(\lambda_n)|}{1 + |b(\lambda_n)|} \geq \frac{1 - c}{2} > 0.
\]

Furthermore, on one hand

\[
\frac{\|k^b_{\lambda_n}\|_b^2}{\|k^b_{\lambda_n}\|_b^2} \leq 1.
\]

And on the other hand the sequence \( \frac{k^b_{\lambda_n}}{\|k^b_{\lambda_n}\|_b} \) converges weakly to 0, because \( |\lambda_n| \to 1 \) and \( |b(\lambda_n)| \to c \) with \( c < 1 \). Indeed, using that

\[
\|k^b_{\lambda_n}\|_b^2 = \frac{1 - |b(\lambda_n)|^2}{1 - |\lambda_n|^2}.
\]
We deduce that for $f \in H^\infty \cap \mathcal{H}(b)$,

$$| \langle f, \frac{k_{\lambda_n}^b}{||k_{\lambda_n}^b||_b} \rangle > b | = \frac{|f(\lambda_n)| \sqrt{1 - |\lambda_n|^2}}{\sqrt{1 - |b(\lambda_n)|^2}} \leq \frac{|f|_\infty \sqrt{1 - |\lambda_n|^2}}{\sqrt{1 - |b(\lambda_n)|^2}} \to 0 \quad \text{as} \quad n \to \infty.$$  

Furthermore, $H^\infty \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$, since for every $A \in \mathbb{D}$, $k_A^b \in H^\infty \cap \mathcal{H}(b)$. Thus the sequence $\frac{k_{\lambda_n}^b}{||k_{\lambda_n}^b||_b}$ converges weakly to 0 in $\mathcal{H}(b)$, with $T_{\bar{\varphi},b}$ considered compact. We deduce that

$$\frac{1}{||k_{\lambda_n}^b||_b^2} | \langle k_{\lambda_n}^b, T_{\bar{\varphi},b}k_{\lambda_n}^b > 2 | \leq \frac{||k_{\lambda_n}^b||_2 ||T_{\bar{\varphi},b}k_{\lambda_n}^b||_2}{||k_{\lambda_n}^b||_b} \leq \frac{||T_{\bar{\varphi},b}k_{\lambda_n}^b||_2}{||k_{\lambda_n}^b||_b} \to 0 \quad \text{as} \quad n \to \infty.$$  

After all these computations, we see that

$$\varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{||k_{\lambda_n}^b||_b} | \langle k_{\lambda_n}^b, T_{\bar{\varphi},b}k_{\lambda_n}^b > 2 | \to 0 \quad \text{as} \quad n \to \infty.$$  

with

$$\int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt > 0 \quad \text{and} \quad \frac{1}{||k_{\lambda_n}^b||_b} | \langle k_{\lambda_n}^b, T_{\bar{\varphi},b}k_{\lambda_n}^b > 2 | \to 0 \quad \text{as} \quad n \to \infty.$$  

Finally, $\varphi(\zeta) = 0$.\hfill $\Box$

We now present consequences of this result.

### 3.2 Compactness of $T_{\varphi,b}$

Recall that the notation $T_{\varphi,b}$ represents the Toeplitz operator defined from $\mathcal{H}(b)$ into itself.

**Corollary 3.3.** Let $b$ be a point of the closed unit ball of $H^\infty$ such that $m(\sigma(b) \cap \mathbb{T}) > 0$. Let $\varphi \in C(\mathbb{T}) \cap H^\infty$. Then the operator

$$T_{\varphi,b} : \mathcal{H}(b) \to \mathcal{H}(b) \quad \text{such that} \quad f \mapsto P_b(\overline{\varphi f}),$$

is compact if and only if $\varphi = 0$.

**Proof.** Assume that $T_{\varphi,b}$ is compact. Then $T_{\bar{\varphi},b}$ is also compact and by Theorem 3.2,

$$\varphi(\sigma(b) \cap \mathbb{T}) = 0.$$  

However, since $\varphi \in H^\infty$ with $m(\sigma(b) \cap \mathbb{T}) > 0$ then $\varphi = 0$.\hfill $\Box$

**Corollary 3.4.** Let $b$ be a point of the closed unit ball of $H^\infty$. The following assertions are equivalent:

1. $\exists \varphi \in H^\infty \cap C(\mathbb{T})$, $\varphi \neq 0$ such that $T_{\varphi,b}$ is compact.
2. $m(\sigma(b) \cap \mathbb{T}) = 0$. 

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This text is a direct transcription of the mathematical content from the document, preserving the structure and notation as closely as possible. The document is focused on the compactness and hypercyclicity in de Branges-Rovnyak spaces, with specific theorems and corollaries proving the compactness of certain operators under specific conditions.
Proof. (2) ⇒ (1) First note that (2) implies that $b$ is an inner function. Indeed, assume on the contrary that $b$ is not inner. Then, the set

$$E = \{ \zeta \in \mathbb{T} : |b(\zeta)| \neq 1 \}$$

has a positive Lebesgue measure. Moreover, it turns out that $E \subset \text{clos}(\sigma(b)) \cap \mathbb{T}$. Indeed, if $\zeta \in \mathbb{T} \setminus \text{clos}(\sigma(b))$ then $b$ admits an analytic continuation across a neighborhood $D(\zeta, r) = \{ w : |w - \zeta| < r \}$ of $\zeta$ with $|b| \equiv 1$ on the arc $D(\zeta, r) \cap \mathbb{T}$.

In particular, $|b(\zeta)| = 1$ and $\zeta \in \mathbb{T} \setminus E$. We deduce that

$$0 < m(E) \leq m(\text{clos}(\sigma(b)) \cap \mathbb{T}).$$

Now according to Rudin’s theorem, we can find a function $\varphi \in A(\mathbb{D}) = Hol(\mathbb{D}) \cap C(\mathbb{T})$, $\varphi \neq 0$ such that $\varphi(\sigma(b) \cap \mathbb{T}) = 0$. Then we apply Ahern-Clark’s result (see Theorem 3.1) to get that $T_{\varphi, b} = (A^b_\varphi)^*$ is compact. (1) ⇒ (2) Follows from Corollary 3.3.

In the case where $b$ is a non-extreme point of the closed unit ball of $H^\infty$, we can get a more general result without the hypothesis that the symbol $\varphi$ is continuous.

**Theorem 3.5.** Let $b$ be a non-extreme point of the closed unit ball of $H^\infty$ and let $\varphi \in H^\infty$. Then the operator

$$T_{\varphi, b} : \mathcal{H}(b) \to \mathcal{H}(b), \quad f \mapsto P_N(\varphi f),$$

is compact if and only if $\varphi = 0$.

**Proof.** Let $a$ be the unique outer function such that $(a, b)$ is an euclidian pair. Note that since $b$ is non-extreme then $k_z \in \mathcal{H}(b)$, for all $z \in \mathbb{D}$ (see (2)).

Suppose that $T_{\varphi, b}$ is compact. Notice that for every $(\lambda_n)_n \subset \mathbb{D}$ such that $|\lambda_n| \to 1$, the sequence $(\frac{k_{\lambda_n}}{|k_{\lambda_n}|})_n$ converges weakly to 0 in $\mathcal{H}(b)$.

Indeed, let $f \in \mathcal{H}(b)$ such that $f$ and $f^+ \in H^\infty$. Recall that $f^+$ is defined in Theorem 2.2. Then, using that

$$T_b k_{\lambda_n} = \overline{b(\lambda_n)} k_{\lambda_n} = T_{\overline{b(\lambda_n)}} \left( \frac{b(\lambda_n)}{a(\lambda_n)} k_{\lambda_n} \right) \quad \text{(see (3))},$$

we see that

$$k^+_{\lambda_n} = \frac{b(\lambda_n)}{a(\lambda_n)} k_{\lambda_n}.$$  

Whence by Theorem 2.2, we have

$$< f, \frac{k_{\lambda_n}}{||k_{\lambda_n}||_b} >_b = < f, \frac{k_{\lambda_n}}{||k_{\lambda_n}||_b} >_2 + < f^+, \frac{k^+_{\lambda_n}}{||k^+_{\lambda_n}||_b} >_2 = \left( f(\lambda_n) + \frac{b(\lambda_n)}{a(\lambda_n)} f^+(\lambda_n) \right) \frac{1}{||k_{\lambda_n}||_b}.$$  

On the other hand, it is known that in the non-extreme case:

$$||k_{\lambda_n}||^2_b = \frac{1}{1 - |\lambda_n|^2} \left( 1 + \frac{|b(\lambda_n)|^2}{|a(\lambda_n)|^2} \right)$$

(see [8, Corollary 23.25]). Hence

$$\frac{|b(\lambda_n)|^2}{|a(\lambda_n)|^2 ||k_{\lambda_n}||^2_b} = \frac{(1 - |\lambda_n|^2)|b(\lambda_n)|^2}{|a(\lambda_n)|^2 + |b(\lambda_n)|^2} \leq 1 - |\lambda_n|^2.$$
Using this inequality and the inequality
\[ \| k_{\lambda_n} \|_b^2 \geq \frac{1}{1 - |\lambda_n|^2}, \]
we deduce that
\[
| < f, \ \frac{k_{\lambda_n}}{\| k_{\lambda_n} \|_b} > b | \leq (|f(\lambda_n)| + |f^*(\lambda_n)|) \sqrt{1 - |\lambda_n|^2} \\
\leq (\| f \|_\infty + \| f^* \|_\infty) \sqrt{1 - |\lambda_n|^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Furthermore, the set \( \{ f \in \mathcal{H}(b); f & f^* \in H^\infty \} \) is dense in \( \mathcal{H}(b) \), since \( \{ k_\lambda^b; \lambda \in \mathbb{D} \} \subset \{ f \in \mathcal{H}(b); f & f^* \in H^\infty \} \). Indeed, for every \( \lambda \in \mathbb{D} \),
\[
(k_\lambda^b)^* = (k_\lambda - \overline{b(\lambda)} b k_\lambda)^* = k_\lambda^* - \overline{b(\lambda)} (b k_\lambda)^* = \overline{b(\lambda)} a k_\lambda \in H^\infty,
\]
with
\[
(b k_\lambda)^* = \frac{k_\lambda}{a(\lambda)} - a k_\lambda
\]
(see [8, Theorem 23.23]).
Hence, the sequence \( (\frac{k_{\lambda_n}}{\| k_{\lambda_n} \|_b})_n \) converges weakly to 0 in \( \mathcal{H}(b) \).

Now by compactness of \( T_{\phi,b} \) it follows that for every \( (\lambda_n)_n \subset \mathbb{D} \) such that \( |\lambda_n| \rightarrow 1 \),
\[
\| T_{\phi,b} \frac{k_{\lambda_n}}{\| k_{\lambda_n} \|_b} \|_b \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{5}
\]
But \( T_{\phi,b} k_{\lambda_n} = \overline{\phi(\lambda_n)} k_{\lambda_n} \) (see (3)). Thus (5) is equivalent to
\[
\forall (\lambda_n)_n \subset \mathbb{D}; \lim_{n \rightarrow \infty} |\lambda_n| = 1, |\phi(\lambda_n)| \rightarrow 0.
\]
Which implies that \( \phi = 0 \). \( \square \)

The proof of Theorem 3.5 obviously doesn’t work in the case when \( b \) is an extreme point of the closed unit ball of \( H^\infty \), since in that case, the Cauchy kernels \( k_\lambda \) do not belong to \( \mathcal{H}(b) \) when \( b(\lambda) \neq 0, \lambda \in \mathbb{D} \).

## 4 Hypercyclicity of \( T_{\phi,b} \)

### 4.1 Hypercyclic and frequently hypercyclic operators

Let \( X \) be a complex infinite-dimensional separable Banach space. An operator \( T \in L(X) \) is said to be hypercyclic if there is some vector \( x \in X \) such that the orbit
\[
O(x, T) := \{ T^n(x); n \in \mathbb{N} \}
\]
is dense in \( X \). Such a vector \( x \) is said to be hypercyclic for \( T \), and the set of all hypercyclic vectors for \( T \) is denoted by \( HC(T) \).

Moreover we say that \( T \) is frequently hypercyclic, if there exists a vector \( x \in X \) such that for every non-empty open subset \( U \) of \( X \), the set \( N(x, U) = \{ n \geq 0; T^n(x) \in U \} \) of instants when the iterates of \( x \) under \( T \) visit \( U \) has positive lower density, i.e.
\[
\overline{\text{dens}}(N(x, U)) = \lim_{N \rightarrow \infty} \inf \frac{\text{card}(N(x, U) \cap [1, N])}{N} > 0.
\]
We refer the reader to the recent book [4] for more information on these topics.

Frequent hypercyclicity is a much stronger notion than hypercyclicity, and some operators are hypercyclic without being frequently hypercyclic; an example is the Bergman backward shift [3].

Let us complete this section by recalling two criterions for hypercyclicity and frequent hypercyclicity that we will use to study the hypercyclicity properties of the Toeplitz operator $T_{\phi, b}$.

We start with the Godefroy-Shapiro Criterion [11], according to which a bounded operator having a large supply of eigenvectors associated to eigenvalues of modulus strictly larger than 1 and strictly smaller than 1 is hypercyclic.

**Theorem 4.1** (Godefroy-Shapiro Criterion, [11]). Let $T \in L(X)$. Suppose that $\bigcup |\lambda| < 1 \text{ Ker}(T - \lambda)$ and $\bigcup |\lambda| > 1 \text{ Ker}(T - \lambda)$ both span a dense subspace of $X$. Then $T$ is hypercyclic.

Then it was shown by S. Grivaux that an operator $T$ which has "sufficiently many" eigenvectors associated to eigenvalues of modulus 1 is automatically frequently hypercyclic.

**Theorem 4.2** (S. Grivaux, [12]). Let $T$ be a bounded operator on $X$. Suppose that there exists a sequence $(u_i)_{i \geq 1}$ of vectors of $X$ having the following properties:

1. For each $i \geq 1$, $u_i$ is an eigenvector of $T$ associated to an eigenvalue $\lambda_i$ of $T$ with $|\lambda_i| = 1$ and the $\lambda_i$'s all distinct;
2. $\text{span}\{u_i; i \geq 1\}$ is dense in $X$;
3. For any $i \geq 1$ and any $\varepsilon > 0$, there exists an $n \neq i$ such that $||u_n - u_i|| < \varepsilon$.

Then $T$ is frequently hypercyclic (in particular hypercyclic).

It is also a natural question, given a family of hypercyclic operators to ask if they have a common hypercyclic vector. The following result gives a sufficient condition for a family of multiples of an operator to have a dense $G_\delta$-set of common hypercyclic vectors.

**Theorem 4.3** (Shkarin, [17]). Let $X$ be a separable Fréchet space, $T \in L(X)$, $0 \leq a < c \leq \infty$. Assume also that for all $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < c$ there exists a dense subset $E$ of $X$ and a map $S : E \rightarrow E$ such that $TSx = x, a^{-n}T^nx \rightarrow 0$ and $\beta^nS^n x \rightarrow 0$ for each $x \in E$. Then

$$\cap \text{HC}(\lambda T : c^{-1} < |\lambda| < c^{-1})$$

is a dense $G_\delta$-set in $X$.

We finish by giving an example of a hypercyclic Toeplitz operator. Rolewicz's result [4] in 1960, says that the operator $\lambda S^* = T_{\lambda} : H^2 \rightarrow H^2$ for every $\lambda \in \mathbb{C}$, $|\lambda| > 1$, is hypercyclic, this was shown using Kitaï's Criterion (a particular case of the Hypercyclicity Criterion) [10]. This result of Rolewicz was generalized by Godefroy-Shapiro [11] in 1991.

**Theorem 4.4** (Godefroy-Shapiro). Let $\phi \in H^\infty$. The operator $T_{\phi} : H^2 \rightarrow H^2$ is hypercyclic if and only if $\phi$ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. 
4.2 Hypercyclicity of $T_{\phi,b}$.

Following the approach of Godefroy-Shapiro, we generalize Theorem 4.4 to operators $T_{\phi,b}$ when $b$ is a non-extreme point of the closed unit ball of $H^\infty$ and $\phi \in H^\infty$, even we get a better result. And according to this generalization we noticed that in the non-extreme case, for every $|\lambda| > 1$ the operator $\lambda X_b = \lambda T_{\phi,z,b}$ is frequently hypercyclic (in particular hypercyclic). On the contrary when $b$ is extreme, $\lambda X_b$ is never hypercyclic.

**Theorem 4.5.** Let $b$ be a non-extreme point of the closed unit ball of $H^\infty$ and let $\phi \in H^\infty$. Then the operator

$$T_{\phi,b} : \mathcal{H}(b) \rightarrow \mathcal{H}(b) \quad f \mapsto T_{\phi,b}f = P_*(\phi f).$$

is frequently hypercyclic if and only if $\phi$ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

**Proof.** Let $\phi$ be a non-constant analytic function on $\mathbb{D}$ and assume that $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Let $\zeta_0 \in \phi(\mathbb{D}) \cap \mathbb{T}$. Since $\phi(\mathbb{D})$ is an open set of $\mathbb{C}$ (open mapping theorem), there exists $r > 0$ such that $D(\zeta_0, r) \subset \phi(\mathbb{D})$ and let $I$ be a closed arc contained in $D(\zeta_0, r) \cap \mathbb{T} \subset \phi(\mathbb{D})$. Consider an exhaustive sequence of compacts $(K_n)_n$ associated to $\mathbb{D}$. Then

$$\phi^{-1}(I) = \bigcup_{n=1}^{\infty} \phi^{-1}(I) \cap K_n,$$

since $\phi^{-1}(I)$ is uncountable, there is, indeed, $n_0 \geq 1$ such that $\text{Card}(\phi^{-1}(I) \cap K_{n_0}) = +\infty$. We can therefore construct a sequence $(\lambda_n)_n \in \phi^{-1}(I) \cap K_{n_0}$ with $\lambda_n \neq \lambda_\ell$, $n \neq \ell$. Now $\lambda_\ell \in K_{n_0}-\text{compact}$, so there is a subsequence $(\lambda_{n_\ell})_\ell$ such that $\lambda_{n_\ell} \rightarrow \lambda \in K_{n_0} \subset \mathbb{D}$, $\ell \rightarrow +\infty$. Since $b$ is non-extreme, reproducing kernels of $H^2$, $k_\lambda$ for all $\lambda \in \mathbb{D}$, are elements of $\mathcal{H}(b)$ and they are eigenvectors of $T_{\phi,b}$, of eigenvalues $\overline{\phi(\lambda)}$ (see (3)). Thus the sequence of reproducing kernels of $H^2$ associated to the subsequence $(\lambda_{n_\ell})_\ell$ will be dense in $\mathcal{H}(b)$ (since $\lambda$ is the unique accumulation point of this subsequence):

$$\text{Span}(k_{\lambda_{n_\ell}} : n_\ell) = \mathcal{H}(b)$$

and $|\phi(\lambda_{n_\ell})| = 1$. Moreover this sequence of eigenvectors satisfies the property of "continuity" which is the third condition of Grivaux’s criterion (see Theorem 4.2) because the application $\mu \rightarrow k_\mu$ is continuous and the subsequence is convergent. Hence the Toeplitz operator $T_{\phi,b}$ is frequently hypercyclic.

Conversely, assume that $T_{\phi,b}$ is frequently hypercyclic then it is in particular hypercyclic (so that $\phi$ is certainly non-constant). And assume that $\phi(\mathbb{D}) \cap \mathbb{T} = \emptyset$. Since $\mathbb{D}$ is connected with $\phi$ continuous on $\mathbb{D}$, $\phi(\mathbb{D})$ is connected, hence $\phi(\mathbb{D}) \subset \mathbb{D}$ or $\phi(\mathbb{D}) \subset \subset \mathbb{D}$. If $\phi(\mathbb{D}) \subset \mathbb{D}$ then $\forall z \in \mathbb{D}$, $|\phi(z)| < 1$, it implies that $||\phi||_\infty \leq 1$ and $||T_{\phi,b}|| \leq ||\phi||_\infty \leq 1$, whence $T_{\phi,b}$ is non-hypercyclic (absurd). If $\phi(\mathbb{D}) \subset \subset \mathbb{D}$ then $\forall z \in \mathbb{D}$, $|\phi(z)| > 1$. In this case, $\frac{1}{\phi} \in H^\infty$ and $T_{\phi,b}$ is non-hypercyclic since $||T_{\phi,b}|| \leq ||\frac{1}{\phi}||_\infty \leq 1$. Seeing that, $T_{\phi,b}T_{\phi,b} = T_{\phi,b}T_{\phi,b} = I$, then $T_{\phi,b} = (T_{\phi,b})^{-1}$, consequently $T_{\phi,b}$ is non-hypercyclic (indeed an invertible operator is hypercyclic if and only if its inverse is hypercyclic [4, page 3]). We get also a contradiction. \hfill \square

**Remark 4.6.** Note that when $b = 0$, we recover Theorem 4.4 of Godefroy and Shapiro.

In the particular case when $\phi(z) = z$, corresponding to operator $X_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b)$, $X_b(f) = S^*f$, we get the following result:

**Corollary 4.7.** Let $b$ be a non-extreme point of the closed unit ball of $H^\infty$. Let the operator

$$X_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b) \quad f \mapsto S^*f.$$

For all $|\lambda| > 1$, $\lambda X_b$ is frequently hypercyclic (in particular hypercyclic).
As we saw in the previous Corollary, for all \(|\lambda| > 1\), \(\lambda X_b\) is hypercyclic, so this naturally raises the question of finding a common hypercyclic vector for \((\lambda X_b)_{|\lambda|>1}\). We will apply Shkarin’s Theorem 4.3 but we need to introduce another operator on \(\mathcal{H}(b)\).

It is well known that \(\mathcal{H}(b)\) is invariant under the unilateral forward shift operator \(S\) if and only if \(b\) is non-extreme [8, Corollary 20.20]. In that case, the mapping

\[
S_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b), \quad f \mapsto Sf = zf
\]

gives a well-defined operator. Moreover \(S_b\) is bounded on \(\mathcal{H}(b)\) with \(\|S_b\| = \sqrt{1 + |a(0)|^2}\|S\|_b\) (see [8, Section 24.1]). In particular, we see that except in the case when \(b = 0\) (corresponding to \(\mathcal{H}(b) = H^2\)), the operator \(S_b\) has a norm strictly greater than 1.

**Theorem 4.8.** Let \(b\) be a non-extreme point of the closed unit ball of \(H^\infty\), and let

\[
X_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b), \quad f \mapsto S^*f
\]

Then,

\[
\mathcal{G} = \bigcap HC(\lambda X_b; ||S_b|| < |\lambda| < \infty),
\]

is a dense \(G_\delta\)-set of \(\mathcal{H}(b)\).

**Proof.** We would like to apply Shkarin’s Theorem 4.3 with \(a = 0\), \(c = ||S_b||^{-1}\), and \(E = \mathcal{P}\), with \(\mathcal{P}\) the set of analytic polynomials, dense in \(\mathcal{H}(b)\) [8, Theorem 23.13]. Let

\[
S_b : \mathcal{P} \rightarrow \mathcal{P}; \quad S_b p = zp.
\]

It is clear that \(X_bS_b = I\). For all \(0 < \alpha < \beta < ||S_b||^{-1}\), and for all \(p \in \mathcal{P}\), we have on one hand \(\alpha^{-n}X_b^n p \rightarrow 0\) as \(n \rightarrow \infty\), since from a certain rank \(n_0 = deg(p) + 1\), \(X_b^n p = 0\), and on the other hand, \(\|\beta^n S_b^n p\|_b \leq (\beta \|S_b\|)^n \|p\|_b \rightarrow 0\) as \(n \rightarrow \infty\). Hence, using Theorem 4.3, we conclude that \(\mathcal{G}\) is a dense \(G_\delta\)-set of \(\mathcal{H}(b)\). \(\square\)

**Remark 4.9.** It remains the question of whether we can replace in the previous Theorem the lower bound

\[
||S_b|| < |\lambda| \quad \text{by} \quad 1 < |\lambda|.
\]

In other word, is the set

\[
\bigcap HC(\lambda X_b; 1 < |\lambda| < \infty)
\]

a dense \(G_\delta\)-set of \(\mathcal{H}(b)\)?

In the case where \(b\) is extreme, the operator \(X_b\) is no longer hypercyclic, which shows a significant difference in the \(\mathcal{H}(b)\) space theory following that \(\log(1 - |b|)\) is integrable or not on \(\mathbb{T}\). The proof of this result requires basic facts on the spectrum of hypercyclic operators, which we now briefly recall.

Let \(X\) be a complex Banach space, and let \(T \in \mathfrak{L}(X)\) be hypercyclic. Then \(\sigma_p(T^*) = \emptyset\) and every connected component of the spectrum of \(T\) intersects the unit circle (see [4, Page 11]).

**Theorem 4.10.** Let \(b\) be an extreme point of the closed unit ball of \(H^\infty\) and let

\[
X_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b), \quad f \mapsto S^*f
\]

Then for every complex number \(\lambda\), \(\lambda X_b\) is not hypercyclic.

**Proof.** For all \(|\lambda| \leq 1\), \(||AX_b|| \leq 1\) hence \(AX_b\) is not hypercyclic. Now take \(\lambda \in \mathbb{C}, |\lambda| > 1\). By [8, Corollary 26.3], we have

\[
\sigma_p(AX_b^*) = \lambda \sigma_p(X_b^*) = \{\lambda \beta : \beta \in \mathbb{D} \quad \text{and} \quad b(\beta) = 0\}.
\]
By the preceding equality, we notice that, if \( b \) has a Blaschke factor then \( \sigma_p(\overline{\lambda X_b}) \neq \emptyset \), and thus \( \lambda X_b \) is not hypercyclic. Now if \( b \) does not admit a Blaschke factor we get from [8, Corollary 26.4] that \( \sigma(X_b) \subset \mathbb{T} \). That implies, since \( |\lambda| > 1 \), \( \sigma(\lambda X_b) \cap \mathbb{T} = (\sigma(X_b)) \cap \mathbb{T} = \emptyset \). Therefore one of the connected component of \( \sigma(\lambda X_b) \) do not intersect the unit circle, hence \( \lambda X_b \) is not hypercyclic. We conclude that for every complex number \( \lambda \), \( \lambda X_b \) is not hypercyclic.

In the case when \( b \) is extreme, it has not been possible to reach the non-hypercyclicity of the Toeplitz operator \( T_{\phi, b} \). However we give a necessary (but not sufficient) condition for such an operator to be non-hypercyclic.

**Proposition 4.11.** Let \( b \) be an extreme point of the closed unit ball of \( H^\infty \), and let \( \phi \in H^\infty \). If the Toeplitz operator \( T_{\phi, b} \) is hypercyclic then \( \sigma_p(T_{\phi, b}) = \emptyset \).

**Proof.** From [18] it follows that \( T_{\phi, b} \) is complex symmetric, and therefore, if it has an eigenvalue, the adjoint also has an eigenvalue and in this case \( T_{\phi, b} \) is not hypercyclic.

In particular, if \( b \) has a blashcke factor, then \( T_{\phi, b} \) is not hypercyclic, as shown in the following result.

**Corollary 4.12.** If \( \lambda \in \mathbb{D} \) such that \( b(\lambda) = 0 \) then \( T_{\phi, b} \) is not hypercyclic.

**Proof.** Suppose that \( \lambda \in \mathbb{D} \) such that \( b(\lambda) = 0 \), then \( k_{\lambda} = k_0^b \in \mathcal{H}(b) \). Moreover \( T_{\phi, b} k_{\lambda} = \overline{\phi(\lambda)} k_{\lambda} \) (see (3)). Hence \( \overline{\phi(\lambda)} \in \sigma_p(T_{\phi, b}) \). Thus by Proposition 4.11, \( T_{\phi, b} \) is not hypercyclic.

**Remark 4.13.** It turns out that the necessary condition in proposition 4.11 is not sufficient. Indeed, if \( b \) has no Blaschke factor, then \( \sigma_p(X_b) \) is empty, though by Theorem 4.10, we know that \( X_b \) is not hypercyclic.

**Remark 4.14.** Notice that

\[
\sigma_p(T_{\phi, b}) = \emptyset \iff \forall \lambda \in \mathbb{C}, \ K_{(\varphi - \lambda)} \cap \mathcal{H}(b) = \{0\},
\]

with \( \varphi - \lambda = (\varphi - \lambda)_i (\varphi - \lambda)_e \), and \( (\varphi - \lambda)_i \) and \( (\varphi - \lambda)_e \) are respectively the inner and outer part of \( \varphi - \lambda \).

**Proof.** Let \( \lambda \in \mathbb{C} \). Then \( \overline{\lambda} \in \sigma_p(T_{\phi, b}) \) if and only if there exists \( f \in \mathcal{H}(b), f \neq 0 \) such that

\[
T_{\phi, b} f = T_{\overline{\phi}} f = \overline{\lambda} f.
\]

This equation is equivalent to

\[
T_{\overline{\varphi - \lambda}} f = 0 \iff T_{(\varphi - \lambda)} f = 0 \iff T_{(\varphi - \lambda)} f = 0,
\]

because when \( a \) is outer \( T_a \) is one-to-one. The last equation is equivalent to

\[
f \in K_{(\varphi - \lambda)} \cap \mathcal{H}(b),
\]

where \( K_{(\varphi - \lambda)} \) denotes the model space associated to \( (\varphi - \lambda)_i \). Thus \( \overline{\lambda} \in \sigma_p(T_{\phi, b}) \) if and only if \( K_{(\varphi - \lambda)} \cap \mathcal{H}(b) \neq \{0\} \).

**Remark 4.15.** If \( b \) is extreme and outer, then \( \sigma_p(T_{\phi, b}) = \emptyset \).

**Proof.** Let \( \lambda \in \mathbb{C} \).

If \( (\varphi - \lambda)_i \equiv cte \) then \( K_{(\varphi - \lambda)} = \{0\} \) (see [8, Theorem 18.2]), which implies that \( K_{(\varphi - \lambda)} \cap \mathcal{H}(b) = \{0\} \).

On the other hand if \( (\varphi - \lambda)_i \neq cte \), and if \( f \in K_{(\varphi - \lambda)} \), then \( f \) is not a cyclic vector for \( S^* \) (since \( \text{span}(S^* f : n \geq 0) \subset K_{(\varphi - \lambda)} \neq H^2 \) because \( S^* K_{(\varphi - \lambda)} \subset K_{(\varphi - \lambda)} \), see section 2.1 and [8, Section 1.10]). Using [8, Theorem 25.17], it implies \( f \in K_{\Theta} \) where \( \Theta = b_i \) is the inner part of \( b \). But since \( b \) is considered outer, then \( b_i \equiv cte \).
thus $K_0 = \{0\}$. Hence $K_{(\phi - \lambda)} \cap \mathcal{H}(b) = \{0\}$.
Which gives by Remark 4.14 that $\sigma_p(T_{\phi,b}) = \emptyset$. 

**Remark 4.16.** In the case where $b$ is extreme and outer, it would be interesting to know if $T_{\phi,b}$ is hypercyclic or not.

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