**L∞-BMO BOUNDS FOR PSEUDO-MULTIPLIERS ASSOCIATED WITH THE HARMONIC OSCILLATOR**

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**Abstract.** In this note we investigate some conditions of Hörmander-Mihlin type in order to assure the $L^\infty$-BMO boundedness for pseudo-multipliers of the harmonic oscillator. The $H^1$-$L^1$ continuity for Hermite multipliers also is investigated. The final version of this paper will appear in Rev. Colombiana Mat.

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**1. Introduction**

The aim of this paper is to investigate the boundedness from $L^\infty(\mathbb{R}^n)$ into $\text{BMO}(\mathbb{R}^n)$ for pseudo-multipliers associated with the harmonic oscillator (see e.g. S. Thangavelu [21, 22]). As it was observed by M. Ruzhansky in [7], from the point of view of the theory of pseudo-differential operators, pseudo-multipliers would be the special case of the symbolic calculus developed in M. Ruzhansky and N. Tokmagambetov [17, 18] (see also Remark 2.2). Let us consider the (Hermite operator) quantum harmonic oscillator $H := -\Delta_x + |x|^2$, (where $\Delta_x$ is the standard Laplacian) which extends to an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$. It is a well known fact, that the Hermite functions\(^1\) $\phi_\nu$, $\nu \in \mathbb{N}_0^n$, are the $L^2$-eigenfunctions of $H$, with corresponding eigenvalues satisfying: $H\phi_\nu = (2|\nu|+n)\phi_\nu$. The system $\{\phi_\nu\}_{\nu \in \mathbb{N}_0^n}$, which is a subset of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, denotes the Hermite polynomial of order $\nu_j$.

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\(^1\)Each Hermite function $\phi_\nu$ has the form $\phi_\nu := \Pi_{j=1}^n \phi_{\nu_j}$, $\phi_{\nu_j}(x_j) = (2^{\nu_j} \nu^j \sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j) e^{-\frac{1}{2} x_j^2}$, where $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0^n$, and $H_{\nu_j}(x_j) := (-1)^{\nu_j} e^{x_j^2} \frac{d^{\nu_j}}{dx_j^{\nu_j}}(e^{-x_j^2})$.
provides an orthonormal basis of $L^2(\mathbb{R}^n)$. So, the spectral theorem for unbounded operators implies that

$$Hf(x) = \sum_{\nu \in \mathbb{N}_0^n} (2|\nu| + n) \hat{f}(\phi_\nu), \ f \in \text{Dom}(H), \quad (1.1)$$

where $\hat{f}(\phi_\nu)$ is the Fourier-Hermite transform of $f$ at $\phi_\nu$, which is given by

$$\hat{f}(\phi_\nu) = \int_{\mathbb{R}^n} f(x)\phi_\nu(x)dx. \quad (1.2)$$

If $G \subset \mathbb{R}^n$ is the complement of a subset of zero Lebesgue measure in $\mathbb{R}^n$, the pseudo-multiplier associated with a function $m : G \times \mathbb{N}_0^n \to \mathbb{C}$ is defined by

$$Af(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu)\hat{f}(\phi_\nu)\phi_\nu(x), \ x \in G, \ f \in \text{Dom}(A). \quad (1.3)$$

In this sense we say that $A$ is the pseudo-multiplier associated to the function $m$, and that $m$ is the symbol of $A$. In this paper the main goal is to give conditions on $m$ in order that $A$ can be extended to a bounded operator from $L^\infty$ to BMO. The problem of the boundedness of pseudo-multipliers is an interesting topic in harmonic analysis (see e.g. J. Epperson [10], S. Bagchi and S. Thangavelu [1], D. Cardona and M. Ruzhansky [7] and references therein). The problem was initially considered for multipliers of the harmonic oscillator

$$Af(x) = \sum_{\nu \in \mathbb{N}_0^n} m(\nu)\hat{f}(\phi_\nu)\phi_\nu(x), \ f \in \text{Dom}(A). \quad (1.4)$$

Indeed, an early result due to S. Thangavelu (see [20, 21]) states that if $m$ satisfies the following discrete Marcinkiewicz condition

$$|\Delta_\nu m(\nu)| \leq C_\alpha (1 + |\nu|)^{-|\alpha|}, \ \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq \left[ \frac{n}{2} \right] + 1, \quad (1.5)$$

where $\Delta_\nu$ is the usual difference operator, then the corresponding multiplier $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ extends to a bounded operator for all $1 < p < \infty$. In view of Theorem 1.1 of S. Blunck [3], (see also P. Chen, E. M. Ouhabaz, A. Sikora, and L. Yan, [8, p. 273]), if we restrict our attention to spectral multipliers $A = m(H)$, the boundedness on $L^p(\mathbb{R}^n)$, can be assured if $m$ satisfies the Hörmander condition of order $s$,

$$\|m\|_{L_u.H^s} := \sup_{r > 0} \|m(r \cdot)\eta(|\cdot|)\|_{H^s(\mathbb{R}^n)} = \sup_{r > 0} r^{s-n/2} \|m(\cdot)\eta(\cdot^{-1} |\cdot|)\|_{H^s(\mathbb{R}^n)} < \infty, \quad (1.6)$$

where $\eta \in \mathcal{D}(0, \infty)$ and $s > \frac{n+1}{2}$, for all $p \in \left[p_0, \frac{p_0}{p_0 - 1}\right]$, for some $p_0 \in (1, 2)$. If $|\nu| = \nu_1 + \cdots + \nu_n$, for spectral pseudo-multipliers

$$Ef(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, 2|\nu| + n)\hat{f}(\phi_\nu)\phi_\nu(x), \ f \in \text{Dom}(E), \quad (1.7)$$

under one of the following conditions

\[\text{Dom}(A) = \{f \in L^2(\mathbb{R}^n) : \sum_{\nu \in \mathbb{N}_0^n} |m(\nu)\hat{f}(\phi_\nu)|^2 < \infty\}\] is a dense subset of $L^2(\mathbb{R}^n)$. Indeed, note that $\{\phi_\nu\}_\nu \subset \text{Dom}(A)$, and consequently $L^2(\mathbb{R}^n) = \text{span}(\{\phi_\nu\}_\nu) \subset \text{Dom}(A)$.
Let us assume that

\begin{equation}
|\Delta^\gamma_m(x, 2\nu + 1)| \leq C_n(2\nu + 1)^{-\gamma}, \quad 0 \leq \gamma \leq 5,
\end{equation}

(1.8)

\[ S. \text{Bagchi and S. Thangavelu, [1]: } n \geq 2, \ E \text{ bounded on } L^2(\mathbb{R}^n) \text{ and }
\]

\[ |\Delta^\gamma_m(x, 2|\nu| + 1)| \leq C_n(2|\nu| + 1)^{-\gamma}, \quad 0 \leq |\gamma| \leq n + 1,
\]

(1.9)

the operator \( E \) extends to an operator of weak type \((1,1)\). This means that \( E : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n) \) admits a bounded extension (we denote by \( L^{1,\infty}(\mathbb{R}^n) \) the the weak \( L^1 \)-space\(^3\)). In view of the Marcinkiewicz interpolation Theorem it follows that \( E \) extends to a bounded linear operator on \( L^p(\mathbb{R}^n) \), for all \( 1 < p \leq 2 \).

We can now that in the previous results the \( L^2 \)-boundedness of pseudo-multipliers is assumed. The problem of finding reasonable conditions for the \( L^2 \)-boundedness of spectral pseudo-multipliers, was proposed by S. Bagchi and S. Thangavelu in [1]. To solve this problem, it was considered in [7], the following Hörmander conditions,

\[
\|m\|_{L_{u,H^s}} := \sup_{r > 0, y \in \mathbb{R}^n} r^{(s-\frac{n}{2})} \|\langle x \rangle^s \mathcal{F}_{m(y, \cdot)} \psi(r^{-1} | \cdot |)(x)\|_{L^2(\mathbb{R}^n)} < \infty, \quad (1.10)
\]

\[
\|m\|_{L_{u,H^s}} := \sup_{k \geq 0} \sup_{y \in \mathbb{R}^n} 2^{k(s-\frac{n}{2})} \|\langle x \rangle^s \mathcal{F}_{H^{-1}}[m(y, \cdot) \psi(2^{-k} | \cdot |)](x)\|_{L^2(\mathbb{R}^n)} < \infty, \quad (1.11)
\]

defined by the Fourier transform \( \mathcal{F} \) and the inverse Fourier-Hermite transform \( \mathcal{F}_{H^{-1}} \). More precisely, the Hörmander condition (1.10) of order \( s > \frac{3n}{2} \), uniformly in \( y \in \mathbb{R}^n \), or the condition (1.11) for \( s > \frac{3n}{2} - \frac{1}{2} \), uniformly in \( y \in \mathbb{R}^n \), guarantee the \( L^2 \)-boundedness of the pseudo-multiplier (2.2). As it was pointed out in [7], in (1.10) we consider functions \( m \) on \( \mathbb{R}^n \times \mathbb{R}^n \), but to these functions we associate a pseudo-multiplier with symbol \( \{m(x, \nu)\}_{x \in \mathbb{R}^n, \nu \in \mathbb{N}_0^n} \). On the other hand, (see Corollary 2.3 of [7]) if we assume the condition,

\[
|\Delta^\gamma_m(x, \nu)| \leq C_\alpha(1 + |\nu|)^{-|\alpha|}, \quad \alpha \in \mathbb{N}_0^n, \quad |\alpha| \leq \rho,
\]

(1.12)

for \( \rho = [3n/2] + 1 \), then the pseudo-multiplier in (2.2) extends to a bounded operator on \( L^p(\mathbb{R}^n) \), and for \( p = 2n + 1 \) we have its \( L^p(\mathbb{R}^n) \)-boundedness for all \( 1 < p < \infty \). Now, we record the main theorem of [7]:

**Theorem 1.1.** Let us assume that \( 2 \leq p < \infty. \) If \( A = T_m \) is a pseudo-multiplier with symbol \( m \) satisfying (1.10), then under one of the following conditions,

- \( n \geq 2, \ 2 \leq p < \frac{2(n+3)}{n+1}, \) and \( s > s_{n,p} := \frac{3n}{2} + \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}) \),
- \( n \geq 2, \ p = \frac{2(n+3)}{n+1}, \) and \( s > s_{n,p} := \frac{3n}{2} + \frac{n-1}{2(n+3)} \),
- \( n \geq 2, \ \frac{2(n+3)}{n+1} < p \leq \frac{2n}{n-2}, \) and \( s > s_{n,p} := \frac{3n}{2} - \frac{1}{6} + \frac{2n}{3} (\frac{1}{2} - \frac{1}{p}) \),
- \( n \geq 2, \ \frac{2n}{n-2} \leq p < \infty, \) and \( s > s_{n,p} := \frac{3n-1}{2} + n(\frac{1}{2} - \frac{1}{p}) \),
- \( n = 1, \ 2 \leq p < 4, \ s > s_{1,p} := \frac{3}{2} \),
- \( n = 1, \ p = 4, \ s > s_{1,4} := 2 \),
- \( n = 1, \ 4 < p < \infty, \ s > s_{1,p} := \frac{5}{3} + \frac{2}{3}(\frac{1}{2} - \frac{1}{p}) \),

the operator \( T_m \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \). For \( 1 < p \leq 2 \), under one of the following conditions

\(^3\)which consists of those functions \( f \) such that \( \|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \text{meas}\{x \in \mathbb{R}^n : |f(x)| > \lambda\} < \infty. \)
1.1 implies that under one of the following conditions,
\[ n \geq 2, \quad \frac{2(n+3)}{n+5} \leq p \leq 2, \text{ and } s > s_{n,p} \equiv \frac{3n}{2} + \frac{n-1}{2} \left( 1 - \frac{1}{p} \right), \]
\[ n \geq 2, \quad \frac{2n}{n+2} \leq p \leq \frac{2(n+3)}{n+5}, \text{ and } s > s_{n,p} \equiv \frac{3n}{2} + \frac{2n}{3} \left( 1 - \frac{1}{p} \right), \]
\[ n \geq 2, \quad 1 < p \leq \frac{2n}{n+2}, \text{ and } s > s_{n,p} \equiv \frac{3n}{2} + n \left( 1 - \frac{1}{p} \right), \]
\[ n = 1, \quad \frac{4}{3} \leq p < 2, \text{ and } s > s_{1,p} \equiv \frac{7}{4}. \]
\[ n = 1, \quad 1 < p < \frac{4}{3}, \text{ and } s > s_{1,p} \equiv \frac{11}{4} + \frac{2n}{3} \left( 1 - \frac{1}{p} \right), \]
the operator $T_m$ extends to a bounded operator on $L^p(\mathbb{R}^n)$. However, in general:

- for every $\frac{4}{3} < p < 4$ and every $n$, the condition $s > \frac{3n}{2}$ implies the $L^p$-boundedness of $T_m$.

If the symbol $m$ of the pseudo-multiplier $T_m$ satisfies the Hörmander condition (1.11), in order to guarantee the $L^p$-boundedness of $T_m$, in every case above we can take $s > s_{n,p} - \frac{1}{12}$. Moreover, the condition $s > \frac{3n}{2} - \frac{1}{12}$ implies the $L^p$-boundedness of $T_m$ for all $\frac{4}{3} < p < 4$.

Now we present our main result. We will provide a version of Theorem 1.1 for the critical case $p = \infty$. Because, in harmonic analysis the John-Nirenberg class BMO (see [11]) is a good substitute of $L^\infty$ we will investigate the boundedness of pseudo-multipliers from $L^\infty(\mathbb{R}^n)$ to BMO(\mathbb{R}^n).

**Theorem 1.2.** Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a continuous linear operator such that its symbol $m = \{m(x, \nu)\}_{x \in G, \nu \in \mathbb{N}_0^n}$ (see (2.1)) satisfies one of the following conditions,

(CI): $m$ satisfies the Hörmander-Mihlin condition
\[ \|m\|_{L^\infty, H^n} := \sup_{r > 0, y \in \mathbb{R}^n} r^{(s-\frac{1}{2})} \|\langle x \rangle^s \mathcal{F}[m(y, \cdot)\psi(r^{-1}|\cdot|)](x)\|_{L^2(\mathbb{R}^n)} < \infty, \quad (1.13) \]
where $s > \max\{\frac{7n}{4} + \frac{1}{2}, \alpha\}$, and $\alpha$ is defined as in (3.2),

(CII): $m$ satisfies the Marcinkiewicz type condition,
\[ |\Delta_0^\alpha m(x, \nu)| \leq C_\alpha (1 + |\nu|)^{-|\alpha|}, \quad |\alpha| \leq [7n/4 - 1/12] + 1. \quad (1.14) \]

Then the operator $A = T_m$ extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into BMO(\mathbb{R}^n).

Now, we will discuss some consequences of our main result.

**Remark 1.3.** In relation with the results of Epperson [10] and Bagchi and Thangavelu [1] mentioned above, Theorem 1.2 implies that under one of the following conditions,

- $n = 1$, $|\Delta_0^\gamma m(x, 2\nu + 1)| \leq C_\gamma (2\nu + 1)^{-|\nu|}, \quad 0 \leq \gamma \leq 2$,
- $n \geq 2$, $|\Delta_0^\gamma m(x, 2|\nu| + n)| \leq C_\gamma (2|\nu| + n)^{-|\nu|}, \quad 0 \leq |\gamma| \leq [7n/4 - 1/12] + 1$,

the spectral pseudo-multiplier
\[ Ef(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, 2|\nu| + n) \hat{f}(\phi_\nu) \phi_\nu(x), \quad f \in \text{Dom}(E), \quad (1.15) \]
extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into BMO(\mathbb{R}^n).
Remark 1.4. For $n = 1$, Theorem 1.1 implies that the symbol inequalities
\[ |\Delta^n_x m(x, \nu)| \leq C_{\gamma}(1 + \nu)^{-\alpha}, \quad 0 \leq \gamma \leq 2, \tag{1.16} \]
are sufficient conditions for the $L^p(\mathbb{R})$-boundedness of pseudo-multipliers with $4/3 < p < 4$, and also under the estimates
\[ |\Delta^n_x m(x, \nu)| \leq C_{\gamma}(1 + \nu)^{-\alpha}, \quad 0 \leq \gamma \leq 3, \tag{1.17} \]
we obtain the $L^p(\mathbb{R})$-boundedness of $T_m$ for all $p \in (1, 4/3) \cup (4, \infty)$. However, we can improve the conditions on the number of derivatives imposed in (1.17) to discrete derivatives up to order 2 in order to assure the $L^p(\mathbb{R})$-boundedness of $T_m$ for all $4/3 < p < \infty$. Indeed, from Theorem 1.2, the hypothesis (1.16) implies the boundedness of $T_m$ from $L^\infty(\mathbb{R})$ to BMO($\mathbb{R}$) and also its $L^p(\mathbb{R})$-boundedness for $4/3 < p < \infty$, in view of the Stein-Fefferman interpolation theorem applied to the $L^2$-$L^2$ and $L^\infty$-BMO boundedness results.

Remark 1.5. Let us consider a multiplier $T_m$ of the harmonic oscillator. Theorem 1.2 assures that under one of the following conditions,

(CI)': $m$ satisfies the Hörmander-Mihlin condition
\[ \|m\|_{t, a, H^s} := \sup_{r > 0} r^{(s - \frac{n}{2})/2} \langle x \rangle^s |m(\cdot)\psi(r^{1/2} \cdot)\rangle_{L^2(\mathbb{R})} < \infty, \tag{1.18} \]
where $s > \max\{\frac{n}{4} + \alpha, \frac{n}{2}\}$, and $\alpha$ is defined as in (3.2),

(CII)': $m$ satisfies the Marcinkiewicz type condition,
\[ |\Delta^\alpha m(\nu)| \leq C_{\alpha}(1 + |\nu|)^{-|\alpha|}, \quad |\alpha| \leq [7n/4 - 1/12] + 1, \tag{1.19} \]
the operator $T_m$ extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into BMO($\mathbb{R}^n$). Moreover, the duality argument shows the boundedness of $T_m$ from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

For certain spectral aspects and applications to PDE of the theory of pseudo-multipliers we refer the reader to the works [2, 4, 5, 6] and [20]. This paper is organised as follows. Section 2 introduces the necessary background of harmonic analysis that we will use throughout this work. Finally, in Section 3 we prove our main theorem.

2. Preliminaries

2.1. Pseudo-multipliers of the harmonic oscillator. To motivate the definition of pseudo-multipliers we will prove that these operators arise, for example, as bounded linear operators on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Theorem 2.1. Let us consider the set $G := \{z \in \mathbb{R}^n : \phi_{\nu}(z) \neq 0, \text{ for all } \nu\}$, and let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a continuous linear operator. Then, the function $m : G \times N_0^n \to \mathbb{C}$, defined by
\[ m(x, \nu) := \phi_{\nu}(x)^{-1} A\phi_{\nu}(x), \quad x \in G, \nu \in N_0^n, \tag{2.1} \]

4The symbol $m$ is defined a.e. $(x, \nu) \in \mathbb{R}^n \times N_0^n$. Indeed, note that $D = \{z : \phi_{\nu}(z) = 0 \text{ for some } \nu\}$ is a countable set, has zero measure and that $m$ is defined on $G \times N_0^n$, where $G = \mathbb{R}^n - D$.
satisfies the property
\[ Af(x) = \sum_{\nu \in \mathbb{N}^n_0} m(x, \nu) \hat{f}(\phi_\nu) \phi_\nu(x), \ x \in G, \ f \in \mathcal{S}(\mathbb{R}^n). \quad (2.2) \]

**Proof.** Let us assume that \( A \) is a continuous linear operator \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \).
Because, for every \( \nu \in \mathbb{N}^n_0 \), \( \phi_\nu \in \mathcal{S}(\mathbb{R}^n) = \text{Dom}(A) \), define for every \( x \in G \), and \( \nu \in \mathbb{N}^n_0 \), the function
\[ m(x, \nu) := \phi_\nu(x)^{-1} A \phi_\nu(x). \quad (2.3) \]
Let \( f \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \) and let us consider its Hermite series
\[ f = \sum_{\nu \in \mathbb{N}^n_0} \hat{f}(\phi_\nu) \phi_\nu. \quad (2.4) \]
Because \( \| f \|^2_{L^2(\mathbb{R}^n)} = \sum_{\nu} |\hat{f}(\phi_\nu)|^2 < \infty \), by Simon Theorem (see Theorem 1 of B. Simon [19]), the series
\[ f_N = \sum_{|\nu| \leq N} \hat{f}(\phi_\nu) \phi_\nu, \ N \in \mathbb{N}, \quad (2.5) \]
converges to \( f \) in the topology of the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \). Because, \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \), is a continuous linear operator, we have that \( Af_n \) converges to \( Af \) in the topology of \( \mathcal{S}(\mathbb{R}^n) \). Consequently, we have proved that
\[ Af = \sum_{\nu \in \mathbb{N}^n_0} \hat{f}(\phi_\nu) A \phi_\nu. \quad (2.6) \]
By observing that \( m(x, \nu) := \phi_\nu(x)^{-1} A \phi_\nu(x) \), we obtain the identity,
\[ Af(x) = \sum_{\nu \in \mathbb{N}^n_0} m(x, \nu) \hat{f}(\phi_\nu) \phi_\nu(x), \ x \in G, \ f \in \mathcal{S}(\mathbb{R}^n). \]
So, we end the proof. \( \square \)

**Remark 2.2.** It is a well known fact that several classes of pseudo-differential operators
\[ T_\sigma f(x) = \int_{\mathbb{R}^n} e^{i2\pi x \xi} \sigma(x, \xi) \hat{f}(\xi)d\xi, \ f \in C_0^\infty(\mathbb{R}^n), \quad (2.7) \]
are continuous linear operators on the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \). For example, if \( \sigma \) is a tempered and smooth function (i.e. that \( \sigma \in C^\infty(\mathbb{R}^{2n}) \) satisfies \( \int |\sigma(x, \xi)|(1 + |x| + |\xi|)^{-\kappa}dxd\xi < \infty \) for some \( \kappa > 0 \)) then \( T_\sigma : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \), extends to a continuous linear operator. More interesting cases arise with pseudo-differential operators with symbols \( \sigma \) in the Hörmander classes, or with more generality, in the Weyl-Hörmander classes (see L. Hörmander [15, 16]). From Theorem 2.1 we have that continuous pseudo-differential operators on \( \mathcal{S}(\mathbb{R}^n) \) also can be understood as pseudo-multipliers of the harmonic oscillator.
2.2. Functions of bounded mean oscillation BMO. We will consider in the following two subsection the necessary notions for introducing the BMO and $H^1$ spaces. For this, we will follow Fefferman and Stein [13]. Let $f$ be a locally integrable function on $\mathbb{R}^n$. Then $f$ is of bounded mean oscillation (abbreviated as $f \in \text{BMO}(\mathbb{R}^n)$), if

$$
\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx := \|f\|_* < \infty,
$$

(2.8)

where the supremum ranges over all finite cubes $Q$ in $\mathbb{R}^n$, $|Q|$ is the Lebesgue measure of $Q$, and $f_Q$ denote the mean value of $f$ over $Q$, $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$. It is a well-known fact that $L^\infty(\mathbb{R}^n) \subset \text{BMO}$. Moreover $\ln(|x|) \in \text{BMO}$. The class of functions of bounded mean oscillation, modulo constants, is a Banach space with the norm $\| \cdot \|_*$, defined above. According to the John-Nirenberg inequality, $f \in \text{BMO}(\mathbb{R}^n)$ if and only if, the inequality

$$
|\{ x \in Q : |f(x) - f_Q| > \alpha \}| \leq e^{-\frac{\alpha}{\|f\|_*}|Q|},
$$

(2.9)

holds true for every $\alpha > 0$. For understanding the behaviour of a function $f \in \text{BMO}(\mathbb{R}^n)$, it can be checked that

$$
\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+1}} \, dx < \infty.
$$

(2.10)

Moreover, a function $f \in \text{BMO}(\mathbb{R}^n)$, if and only if (2.10) holds and

$$
\iint_{|x-x_0|<\delta;0<t<\delta} t|\nabla u(x,t)|^2 \, dxdt \lesssim \delta^n,
$$

(2.11)

for all $x_0 \in \mathbb{R}^n$ and $\delta > 0$. Here, $u(x,t)$ is the Poisson integral of $f$ defined on $\mathbb{R}^n \times (0, \infty)$ by (see Fefferman [12]),

$$
u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y) \, dy, \quad P_t(x) := \frac{c_n t}{(t^2 + |x|^2)^{(n+1)/2}}.
$$

(2.12)

2.3. The space $H^1$. The Hardy spaces $H^p(\mathbb{D})$, $0 < p < \infty$, were first studied as part of complex analysis by G. H. Hardy [14]. An analytic function $F$ on the disk $\mathbb{D}$ is in $H^p(\mathbb{D})$, if

$$
\sup_{0<r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta < \infty.
$$

(2.13)

For $1 < p < \infty$, we can identify $H^p(\mathbb{D})$, with $L^p(\mathbb{T})$, where $\mathbb{T}$ is the circle. This identification does not hold, however, for $p \leq 1$. Unfortunately, these results cannot be extended to higher dimensions using the theory of functions of several complex variables. So, let us introduce the Hardy space $H^1(\mathbb{R}^n)$. Let $R_1, \ldots, R_n$, be the Riesz transform on $\mathbb{R}^n$,

$$
R_j f(x) = \lim_{\varepsilon \to 0} \int_{|\xi| > \varepsilon} e^{i2\pi x \cdot \xi_j} \xi_j / |\xi| \hat{f}(\xi), \quad f \in \text{Dom}(R_j),
$$

(2.14)
where \( \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx \) is the Fourier transform of \( f \) at \( \xi \). Then, \( H^1(\mathbb{R}^n) \) consists of those functions \( f \) on \( \mathbb{R}^n \), satisfying,

\[
\|f\|_{H^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)}.
\]  

(2.15)

The main remark in this subsection is that the dual of \( H^1(\mathbb{R}^n) \) is \( \text{BMO}(\mathbb{R}^n) \) (see Fefferman and Stein [13]). This can be understood in the following sense:

(a) If \( \phi \in \text{BMO}(\mathbb{R}^n) \), then \( \Phi : f \mapsto \int_{\mathbb{R}^n} f(x) \phi(x) dx \), admits a bounded extension on \( H^1(\mathbb{R}^n) \).

(b) Conversely, every continuous linear functional \( \Phi \) on \( H^1(\mathbb{R}^n) \) arises as in (a) with a unique element \( \phi \in \text{BMO}(\mathbb{R}^n) \).

The norm of \( \phi \) as a linear functional on \( H^1(\mathbb{R}^n) \) is equivalent with the \( \text{BMO} \) norm. Important properties of the \( \text{BMO} \) and the \( H^1 \) norm are the followings,

\[
\|f\|_* = \sup_{\|g\|_{\text{BMO}} = 1} \left| \int_{\mathbb{R}^n} f(x) g(x) dx \right|, \quad \|g\|_{H^1} = \sup_{\|f\|_{H^1} = 1} \left| \int_{\mathbb{R}^n} f(x) g(x) dx \right|.
\]  

(2.16)

For our further analysis we will use the following fact (see Fefferman and Stein [13, pag. 183]): if \( f \in H^1(\mathbb{R}^n) \), and \( \phi \in \mathcal{S}(\mathbb{R}^n) \) satisfies \( \int \phi(x) dx = 1 \), let us define

\[
u^+ f(x) := \sup_{t > 0} |\phi_t * f(x)| = \sup_{t > 0} \left| \int_{\mathbb{R}^n} \phi_t(x - y) f(y) dy \right|, \quad \phi_t(x) = t^{-n} \phi\left(\frac{x}{t}\right).
\]  

(2.17)

Then, \( u^+ f \in L^1(\mathbb{R}^n) \), \( f(x) = \lim_{t \to 0} \phi_t * f(x), \ a.e. x \), and there exist positive constants \( A \) and \( B \) satisfying

\[
A \|f\|_{H^1} \leq \|u^+ f\|_{L^1} \leq B \|f\|_{H^1}.
\]  

(2.18)

The duals of the \( H^p(\mathbb{R}^n) \) spaces, \( 0 < p < 1 \), are Lipschitz spaces. This is due to P. Duren, B. Romberg and A. Shields [9] on the unit circle, and to T. Walsh [23] in \( \mathbb{R}^n \).

2.4. The Hörmander-Mihlin condition for pseudo-multipliers. As we mentioned in the introduction, if \( m \) is a function on \( \mathbb{R}^n \), we say that \( m \) satisfies the Hörmander condition of order \( s > 0 \), if

\[
\|m\|_{l,u,H^s} := \sup_{r > 0} \|m(r^\cdot) \eta(\cdot | \cdot)\|_{H^s(\mathbb{R}^n)} = \sup_{r > 0} r^{s - \frac{n}{2}} \|m(\cdot) \eta(r^{-1} | \cdot)\|_{H^s(\mathbb{R}^n)} < \infty,
\]  

(2.19)

where \( H^s(\mathbb{R}^n) \) is the usual Sobolev space of order \( s \). Indeed, we also can use the following formulation for the Hörmander-Mihlin condition,

\[
\|m\|_{l,u,H^s} := \sup_{j \in \mathbb{Z}} \|m(2^j | \cdot) \eta(\cdot)\|_{H^s(\mathbb{R}^n)} = \sup_{j \in \mathbb{Z}} 2^{j(s - \frac{n}{2})} \|m(\cdot) \eta(2^{-j} | \cdot)\|_{H^s(\mathbb{R}^n)} < \infty.
\]  

(2.20)
In particular, if we choose \( \eta \in \mathcal{D}(0, \infty) \) with compact support in \([1/2, 2]\), and by assuming that \( m \) has support in \( \{ \xi : |\xi| > 2 \} \), we have that \( m(\cdot)\eta(2^{-j}|\cdot|) = 0 \) for \( j \leq 0 \). So, for a such symbol \( m \), we have

\[
\|m\|_{L^1, H^s} := \sup_{j \geq 1} \|m(2^j \cdot |\cdot|)\eta(\cdot)\|_{H^s(\mathbb{R}^n)} = \sup_{j \geq 1} 2^{j(s-\frac{n}{2})}\|m(\cdot)\eta(2^{-j}|\cdot|)\|_{H^s(\mathbb{R}^n)} < \infty.
\]

Because we define multipliers by associating to \( T \) the Hörmander condition to \( S \), we can always split \( T_m = T_0 + S_m \), where \( T_0 \) has symbol supported in \( \{ \nu : |\nu| \leq 2 \} \) and the pseudo-multiplier \( S_m \) has symbol supported in \( \{ \nu : |\nu| > 2 \} \). We will apply the Hörmander condition to \( S_m \) in order to assure its \( \ell^\infty \)-BMO boundedness, and later we will conclude that \( T_m \) is \( \ell^\infty \)-BMO bounded, by observing that the \( \ell^\infty \)-BMO boundedness of \( T_0 \) is trivial. This analysis will be developed in detail in the next section, in the context of pseudo-multipliers by employing the Hörmander type condition

\[
\|m\|_{L^1, H^s} := \sup_{j \geq 1, x \in \mathbb{R}^n} 2^{j(s-\frac{n}{2})}\|m(x, \cdot)\eta(2^{-j}|\cdot|)\|_{H^s(\mathbb{R}^n)} < \infty,
\]

for \( s \) large enough which follows from (1.13).

3. \( \ell^\infty \)-BMO continuity for pseudo-multipliers

In this section we present the proof of our main result. The main strategy in the proof of Theorem 1.2 will be a suitable Littlewood-Paley decomposition of the symbol together with some suitable estimates for the operator norm of pseudo-multipliers associated to each part of this decomposition. Our starting point is the following lemma. We use the symbol \( X \lesssim Y \) to denote that there exists a universal constant \( C \) such that \( X \leq CY \).

**Lemma 3.1.** Let \( \phi_\nu, \nu \in \mathbb{N}_0^n \) be a Hermite function. Then, there exists \( \kappa \leq -1/12 \), such that

\[
\|\phi_\nu\|_{\text{BMO}} \lesssim |\nu|^\kappa. \tag{3.1}
\]

**Proof.** By using that \( \ell^\infty \subset \text{BMO} \), we have \( \|\phi_\nu\|_{\text{BMO}} \lesssim \|\phi_\nu\|_{\ell^\infty} \). Now, from Remark 2.5 of [7] we can estimate \( \|\phi_\nu\|_{\ell^\infty} \lesssim |\nu|^{-1/12} \) which implies the desired estimate. Indeed, if

\[
\kappa := \inf \{ \omega \in \mathbb{R} : \|\phi_\nu\|_{\text{BMO}} \lesssim |\nu|^\omega \}, \tag{3.2}
\]

we have that \( \kappa \leq -1/12 \). \( \square \)

**Proof of Theorem 1.2.** We will prove that if \( m \) satisfies the condition (CI), then \( A = T_m \) can be extended to a bounded operator from \( \ell^\infty(\mathbb{R}^n) \) to \( \text{BMO}(\mathbb{R}^n) \). Let us consider the operator

\[
\mathcal{R} := \frac{1}{2}(H - n),
\]

where \( H \) is the harmonic oscillator on \( \mathbb{R}^n \), and let us fix a dyadic decomposition of its spectrum: we choose a function \( \psi_0 \in C_0^\infty(\mathbb{R}) \), \( \psi_0(\lambda) = 1 \), if \( |\lambda| \leq 1 \), and
\(\psi(\lambda) = 0\), for \(|\lambda| \geq 2\). For every \(j \geq 1\), let us define \(\psi_j(\lambda) = \psi_0(2^{-j}\lambda) - \psi_0(2^{-j+1}\lambda)\). Then we have
\[
\sum_{l \in \mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0. \tag{3.4}
\]

Let us consider \(f \in L^\infty(\mathbb{R}^n)\). We will decompose the symbol \(m\) as
\[
m(x, \nu) = m(x, \nu)(\psi_0(|\nu|) + \psi_1(|\nu|)) + \sum_{k=2}^\infty m_k(x, \nu), \quad m_k(x, \nu) := m(x, \nu) \cdot \psi_k(|\nu|). \tag{3.5}
\]

Let us define the sequence of pseudo-multipliers \(T_{m(j)}, \ j \in \mathbb{N}\), associated to every symbol \(m_j\), for \(j \geq 2\), and by \(T_0\) the operator with symbol \(\sigma \equiv m(x, \nu)(\psi_0 + \psi_1)\). Then we want to show that the operator series
\[
T_0 + S_m, \quad S_m := \sum_k T_{m(k)}, \tag{3.6}
\]
satisfies,
\[
\|T_m\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \leq \|T_0\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} + \sum_k \|T_{m(k)}\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))}, \tag{3.7}
\]
where the series in the right hand side converges. Because, \(f \in L^\infty(\mathbb{R}^n)\) and for every \(j\), \(T_{m(j)}\) has symbol with compact support, \(T_{m(j)} : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)\) is bounded, and consequently \(T_{m(j)} f \in L^\infty(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n)\). Now, because \(T_{m(j)} f \in \text{BMO}(\mathbb{R}^n)\), we will estimate its BMO norm \(\|T_{m(j)} f\|_\ast\). By using that every symbol \(m_k\) has variable \(\nu\) supported in \(\{\nu : 2^{k-1} \leq |\nu| \leq 2^{k+1}\}\), we have
\[
T_{m(k)} f(x) = \sum_{2^{k-1} \leq |\nu| \leq 2^{k+1}} m_k(x, \nu) \phi_\nu(x) \widehat{f}(\phi_\nu), \quad x \in \mathbb{R}^n.
\]
Consequently,
\[
\|T_{m(k)} f\|_\ast \leq \sum_{2^{k-1} \leq |\nu| \leq 2^{k+1}} \|m_k(\cdot, \nu) \phi_\nu(\cdot)\|_\ast |\widehat{f}(\phi_\nu)|. \tag{3.8}
\]

From (2.16) and by using the Fourier inversion formula we have,
\[
\|m_k(\cdot, \nu) \phi_\nu(\cdot)\|_\ast = \sup_{|\Omega|_{h^1} = 1} \left| \int_{\mathbb{R}^n} m_k(x, \nu) \phi_\nu(x) \Omega(x) dx \right| = \sup_{|\Omega|_{h^1} = 1} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i2\pi \nu \cdot \xi} \hat{m}_k(x, \xi) d\xi \phi_\nu(x) \Omega(x) dx \right|
\leq \sup_{|\Omega|_{h^1} = 1} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{m}_k(x, \xi)| d\xi \times \int_{\mathbb{R}^n} |\phi_\nu(x)||\Omega(x)| dx.\]
By the Cauchy-Schwarz inequality, and the condition $s > n/2$, we have
\[
\int_{\mathbb{R}^n} |\hat{m}_k(x, \xi)| d\xi \leq \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{m}_k(x, \xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{-2s} d\xi \right)^{\frac{1}{2}}. \tag{3.9}
\]
Consequently, we claim that
\[
\int_{\mathbb{R}^n} |\hat{m}_k(x, \xi)| d\xi \leq C \|m\|_{L^1, H^s} \times 2^{-k(s-\frac{n}{2})}. \tag{3.10}
\]
Indeed, if $\hat{\psi}(\lambda) := \hat{\psi}_0(\lambda) - \hat{\psi}_0(2\lambda)$, then $\hat{\psi} \in \mathcal{D}(\mathbb{R})$ and,
\[
\int_{\mathbb{R}^n} |\hat{m}_k(x, \xi)| d\xi \lesssim \|m_k(x, \cdot)\|_{H^s(\mathbb{R}^n)} = \|m(x, \cdot)\hat{\psi}(2^{-k}\cdot)\|_{H^s(\mathbb{R}^n)}
\[
\lesssim \|m(x, \cdot)\|_{L^1, H^s} \times 2^{-k(s-\frac{n}{2})} \lesssim \|m\|_{L^1, H^s} \times 2^{-k(s-\frac{n}{2})}.
\]
So, we obtain
\[
\|m_k(\cdot, \nu)\phi_\nu(\cdot)\|_s \leq \|m\|_{L^1, H^s} \times 2^{-k(s-\frac{n}{2})} \sup_{\|\Omega\|_{H^1} = 1} \int_{\mathbb{R}^n} |\phi_\nu(x)| |\Omega(x)| dx
\[
= \|m\|_{L^1, H^s} \times 2^{-k(s-\frac{n}{2})} \sup_{\|\Omega\|_{H^1} = 1} \int_{\mathbb{R}^n} \text{sig}(\Omega(x)) |\phi_\nu(x)| |\Omega(x)| dx,
\]
where $\text{sig}(\Omega(x)) = -1$, if $\Omega(x) < 0$, and $\text{sig}(\Omega(x)) = 1$, if $\Omega(x) \geq 0$. By the duality relation (2.16), and by using that
\[
\|\text{sig}(\Omega(x))|\phi_\nu(x)||_{\text{BMO}} \leq 2\|\text{sig}(\Omega(x))|\phi_\nu(x)||_{\text{BMO}} = 2\|\phi_\nu(x)||_{\text{BMO}},
\]
we conclude that
\[
\|m_k(\cdot, \nu)\phi_\nu(\cdot)\|_s \lesssim \|m\|_{L^1, H^s} 2^{-k(s-\frac{n}{2})} \sup_{\|\Omega\|_{H^1} = 1} \|\phi_\nu\|_{\text{BMO}} \|\Omega\|_{H^1}.
\]
Returning to the estimate (3.8), we can write
\[
\|T_{m(k)} f\|_s \leq \sum_{2^{k-1} \leq |\nu| \leq 2^{k+1}} \|m\|_{L^1, H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\text{BMO}} \|\hat{f}(\phi_\nu)\|
\[
\leq \sum_{2^{k-1} \leq |\nu| \leq 2^{k+1}} \|m\|_{L^1, H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\text{BMO}} \|\phi_\nu\|_{L^1} \|f\|_{L^\infty}.
\]
Thus, the analysis above implies the following estimate for the operator norm of $T_{m(k)}$, for all $k \geq 2$,
\[
\|T_{m(k)}\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \lesssim \sum_{2^{k-1} \leq |\nu| \leq 2^{k+1}} \|m\|_{L^1, H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\text{BMO}} \|\phi_\nu\|_{L^1}.
\]
By using Lemma 2.2 of [7] we have $\|\phi_\nu\|_{L^1(\mathbb{R}^n)} \lesssim |\nu|^\frac{n}{2}$. Additionally, the inequality (3.1):
\[
\|\phi_\nu\|_{\text{BMO}} \lesssim |\nu|^\alpha,
\]
This analysis allows us to estimate the operator norm of $T$ and $\nu$ in the

Now, by using that $T_0$ is a pseudo-multiplier whose symbol has compact support in the $\nu$-variables, we conclude that $T_0$ is bounded from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$ and

$$
\|T_0\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \leq C \|m\|_{L^\infty}.
$$

This analysis allows us to estimate, the operator norm of $T_m$ as follows,

$$
\|T_m\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \leq \|T_0\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} + \sum_k \|T_m(k)\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))}
$$

$$
\lesssim \|m\|_{L^\infty} + \sum_{k=1}^{\infty} 2^{-k(s-\frac{7n}{4}+\alpha)} \|m\|_{l.a.H^s}
\leq C(\|m\|_{L^\infty} + \|m\|_{l.a.H^s}) < \infty,
$$

provided that $s > \frac{7n}{4} + \alpha$, for some $\alpha \leq -1/12$. So, we have proved the $L^\infty$-BMO boundedness of $T_m$. In order to end the proof we only need to prove that, under the condition (CII), the operator $T_m$ is bounded from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$. But, if $m$ satisfies (CII), then it also does to satisfy (CI), in view of the inequality,

$$
\|m\|_{l.a.H^s} \lesssim \sup_{|\alpha| \leq \frac{7n}{4} - 1/12 + 1} (1 + |\nu|)^{|\alpha|} \sup_{x,\nu} |\Delta^\alpha m(x, \nu)|, \quad (3.12)
$$

for $s > 0$ satisfying, $\frac{7n}{4} - \frac{1}{12} < s < \frac{7n}{4} - 1/12 + 1$, (see Eq. (2.29) of [7]).

**Remark 3.2.** According to the proof of Theorem 1.2, if $T_m$ satisfies the condition (CI), then we have

$$
\|T_m\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \leq C(\|m\|_{L^\infty} + \|m\|_{l.a.H^s}). \quad (3.13)
$$

On the other hand, if we assume (CII), the operator norm of $T_m$ satisfies

$$
\|T_m\|_{\mathcal{B}(L^\infty(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))} \leq C \sup_{|\alpha| \leq \frac{7n}{4} - 1/12 + 1} (1 + |\nu|)^{|\alpha|} \sup_{x,\nu} |\Delta^\alpha m(x, \nu)|. \quad (3.14)
$$

$\square$

**Acknowledgements.** I would like to thank Professor Michael Ruzhansky for several discussions on the subject.

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