Small Locally Compact Linearly Lindelöf Spaces*

Kenneth Kunen†‡

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Abstract

There is a locally compact Hausdorff space of weight $\aleph_\omega$ which is linearly Lindelöf and not Lindelöf.

We shall prove:

Theorem 1 There is a compact Hausdorff space $X$ and a point $p$ in $X$ such that:

1. $\chi(p, X) = w(X) = \aleph_\omega$.
2. For all regular $\kappa > \omega$, no $\kappa$-sequence of points distinct from $p$ converges to $p$.

As usual, $\chi(p, X)$, the character of $p$ in $X$, is the least size of a local base at $p$, and $w(X)$, the weight of $X$, is the least size of a base for $X$. This theorem with “$\beth_\omega$” replacing “$\aleph_\omega$” was proved in [11]. Arhangel’skii and Buzyakova [11] point out that if $X, p$ satisfy (2) of the theorem, then the space $X \setminus \{p\}$ is linearly Lindelöf and locally compact; if in addition $\chi(p, X) > \aleph_0$, then $X \setminus \{p\}$ is not Lindelöf. (2) requires $\text{cf}(\chi(p, X)) = \omega$, because there must be a sequence of type $\text{cf}(\chi(p, X))$ converging to $p$. Thus, in (1) of the theorem, $\aleph_\omega$ is the smallest possible uncountable value for $\chi(p, X)$ and $w(X)$.

As in [11], the $X$ of the theorem will be constructed as an inverse limit, using the following terminology:

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†University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu

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Definition 2  An inverse system is a sequence \( \{X_n, \pi^{n+1}_n : n \in \omega\} \), where each \( X_n \) is a compact Hausdorff space, and each \( \pi^{n+1}_n \) is a continuous map from \( X_{n+1} \) onto \( X_n \).

Such an inverse systems yields a compact Hausdorff space, \( X_\omega = \lim_i X_n \), and maps \( \pi^n_m : X_\omega \to X_m \) for \( m < \omega \) and \( \pi^n_m : X_n \to X_m \) for \( m \leq n < \omega \).

Exactly as in \([11]\), one easily proves:

Lemma 3  Suppose that \( \{X_n, \pi^{n+1}_n : n \in \omega\} \) is an inverse system and \( p \in X = X_\omega \), with the \( p_n = \pi^n_\omega(p) \in X_n \) satisfying:

A. Each \( p_n \) is a weak \( P_{\aleph_n} \)-point in \( X_n \).
B. Each \( w(X_n) < \aleph_\omega \).
C. Each \( (\pi^n_0)^{-1}\{p_0\} \) is nowhere dense in \( X_n \).

Then \( X, p \) satisfies Theorem [10].

As usual, \( y \in Y \) is a weak \( P_\kappa \)-point iff \( y \) is not in the closure of any subset of \( Y \setminus \{y\} \) of size less than \( \kappa \), and \( y \) is a \( P_\kappa \)-point iff the intersection of fewer than \( \kappa \) neighborhoods of \( y \) is always a neighborhood of \( y \). These properties are trivial for \( \kappa = \aleph_0 \). The terms “\( P \)-point” and “weak \( P \)-point” denote “\( P_{\aleph_1} \)-point” and “weak \( P_{\aleph_1} \)-point”, respectively.

Every \( P_\kappa \)-point is a weak \( P_\kappa \)-point, but as pointed out in \([11]\), one cannot have each \( p_n \) being a \( P_{\aleph_n} \)-point, as that would contradict (C). In the construction we describe, it will be natural to make every \( p_n \) fail to be a \( P \)-point in \( X_n \).

We shall build the \( X_n \) and \( p_n \) inductively using the following:

Lemma 4  Assume that \( y \in F \subseteq Y \), where \( Y \) is compact Hausdorff, \( w(Y) \leq \aleph_n \), and \( \text{int}(F) = \emptyset \). Then there is a compact Hausdorff space \( X \), a point \( x \in X \), and a continuous \( g : X \to Y \) such that:

1. \( g(X) = Y \) and \( g(x) = y \).
2. \( g^{-1}(F) \) is nowhere dense in \( X \).
3. \( w(X) = \aleph_n \).
4. In \( X \), \( x \) is a weak \( P_{\aleph_n} \)-point and not a \( P \)-point.

Proof of Theorem [12]  Inductively construct the inverse system as in Lemma [19] with each \( w(X_n) = \aleph_n \). \( X_0 \) can be the Cantor set. When \( n > 0 \) and we are given \( X_{n-1}, p_{n-1} \), we apply Lemma [14] with \( F = (\pi^{n-1}_0)^{-1}\{p_0\} \).
Of course, we still need to prove Lemma 4. We remark that we do not assume that $F$ is closed, although that was true in our proof of Theorem 1. Even if $F$ is dense in $Y$ in Lemma 4, we still get (2) — that is $\text{int}(\text{cl}(g^{-1}(F))) = \emptyset$.

When $n = 0$ in Lemma 4 the “weak $P_{\aleph_0}$-point” is trivial, and the lemma is easily proved by an Aleksandrov duplicate construction. A more convoluted proof is: Let $D \subseteq Y \setminus F$ be dense in $Y$ and countable. Let $g$ map $\omega$ onto $D$ and extend $g$ to a map $\beta g : \beta \omega \to Y$. Choosing $x$ to be any point in $(\beta g)^{-1}(\{y\})$ yields (1)(2)(4), but $\beta \omega$ has weight $2^{\aleph_0}$. Now, we can take a countable elementary submodel of the whole construction to get an $X$ of weight $\aleph_0$. Our proof for a general $n$ will follow this pattern.

As usual, $\beta \kappa$ denotes the Čech compactification of a discrete $\kappa$, and $\kappa^* = \beta \kappa \setminus \kappa$. Equivalently, $\beta \kappa$ is the space of ultrafilters on $\kappa$ and $\kappa^*$ is the space of nonprincipal ultrafilters. If $g : \kappa \to Y$, where $Y$ is compact Hausdorff, then $\beta g$ denotes the unique extension of $g$ to a continuous map from $\beta \kappa$ to $Y$. Our weak $P_{\kappa}$-points in Lemma 4 will be good ultrafilters in the sense of Keisler [9]:

**Definition 5** An ultrafilter $x$ on $\kappa$ is good iff for all $H : [\kappa]^{<\omega} \to x$, there is a $K : \kappa \to x$ such that for each $s = \{\alpha_1, \ldots, \alpha_n\} \in [\kappa]^{<\omega}$, $K(\alpha_1) \cap \cdots \cap K(\alpha_n) \subseteq H(s)$.

The following is well-known.

**Lemma 6** Let $\kappa$ be any infinite cardinal.

1. There are ultrafilters $x$ on $\kappa$ which are both good and countably incomplete.
2. Any $x$ as in (1) is a weak $P_{\kappa}$ point and not a $P$-point in $\beta \kappa$.

In (2), $x$ is not a $P$-point by countable incompleteness, and proofs that it is a weak $P_{\kappa}$ point can be found in [2, 3, 5]. For (1), see [4], Theorem 6.1.4; also, [2, 3] construct good ultrafilters with various additional properties.

We first point out (Lemma 4) that taking $x$ to be a good ultrafilter on $\omega_n$ will give us (1)(2)(4) of Lemma 4. Unfortunately, $w(\beta \omega_n) = 2^{\aleph_n}$, so we shall take an elementary submodel to bring the weight down. Omitting the elementary submodel, our argument is as in [11], which obtained the $X$ of Theorem 4 with $w(X) = \beth_\omega$, rather than $\aleph_\omega$. A related use of elementary submodels to reduce the weight occurs in [7].

Before we consider the weight problem, we explain how to map the good ultrafilter onto the given point $y$. This part of the argument works for any regular ultrafilter.
Definition 7 An ultrafilter \( x \) on \( \kappa \) is regular iff there are \( E_\alpha \in x \) for \( \alpha < \kappa \) such that \( \{ \alpha : \xi \in E_\alpha \} \) is finite for all \( \xi < \kappa \).

Such an \( x \) is countably incomplete because \( \bigcap_{n<\omega} E_n = \emptyset \). For the following, see Exercise 6.1.3 of [4] or the proof of Lemma 2.1 in Keisler [10]:

Lemma 8 If \( x \) is a countably incomplete good ultrafilter on \( \kappa \), then \( x \) is regular.

Lemma 9 Let \( x \) be a regular ultrafilter on \( \kappa \). Assume that \( y \in F \subseteq Y \), where \( Y \) is compact Hausdorff, \( w(Y) \leq \kappa \), and \( \text{int}(F) = \emptyset \). Then there is a map \( g : \kappa \to Y \) such that

A. \( \beta g \) maps \( \beta \kappa \) onto \( Y \).
B. \( (\beta g)(x) = y \).
C. \( g(\xi) \notin F \) for all \( \xi \in \kappa \).
D. \( g^{-1}(F) \) is nowhere dense in \( \beta \kappa \).

Proof. Of course, (D) follows from (C) because \( g^{-1}(F) \subseteq \kappa^* \). Fix \( A \subseteq \kappa \) with \( A \notin x \) and \( |A| = \kappa \). Let \( \{ E_\alpha : \alpha < \kappa \} \) be as in Definition 7, with each \( E_\alpha \cap A = \emptyset \). Let \( \{ U_\alpha : \alpha < \kappa \} \) be an open base at \( y \) in \( Y \). Let \( D \subseteq Y \setminus F \) be dense in \( Y \). Choose \( g : \kappa \to Y \) such that \( g \) maps \( A \) onto \( D \) (ensuring (A)) and each \( g(\xi) \in \bigcap \{ U_\alpha : \xi \in E_\alpha \} \setminus F \) (ensuring (B)(C)).

To apply the elementary submodel technique (as in Dow [6]), we put the construction of Lemma 9 inside an \( H(\theta) \), where \( \theta \) is a suitably large regular cardinal. Let \( M \prec H(\theta) \), with \( \kappa \subset M \) and \( |M| = \kappa \), such that \( M \) contains \( Y \) and its topology \( T \), along with \( F, g, x, y \). Let \( B = \mathcal{P}(\kappa) \cap M \), let \( \text{st}(B) \) denote its Stone space, and let \( \Gamma : \beta \kappa \to \text{st}(B) \) be the natural map; so \( \Gamma(x) = x \cap B = x \cap M \). Since \( T \cap M \) is a base for \( Y \) (by \( w(Y) \leq \kappa \)), we have \( \Gamma(z_1) = \Gamma(z_2) \to (\beta g)(z_1) = (\beta g)(z_2) \), so that \( \beta g \) yields a map \( \tilde{g} : \text{st}(B) \to Y \) with \( \beta g = \tilde{g} \circ \Gamma \). Note that \( B \) contains all finite subsets of \( \kappa \), so that \( \text{st}(B) \) is some compactification of a discrete \( \kappa \). It is easily seen that we still have (A–D), replacing \( \beta g \) by \( \tilde{g} \) and \( \beta \kappa \) by \( \text{st}(B) \), and \( x \) by \( \Gamma(x) \). Note that \( \Gamma(x) \) must be countably incomplete by \( M \prec H(\theta) \), so that \( \Gamma(x) \) will not be a \( P \)-point in \( \text{st}(B) \). But to prove Lemma 4 (letting \( \kappa = \aleph_\omega \)), we also need \( \Gamma(x) \) to be a weak \( P_\kappa \)-point in \( \text{st}(B) \). We may assume that \( x \in \beta \kappa \) is good, so it is a weak \( P_\kappa \)-point there. But we need to show that in \( \text{st}(B) \), \( \Gamma(x) \) is not a limit point of any set of size \( \lambda < \kappa \). Our argument here needs to assume that \( M \) is \( \lambda \)-covering and that \( \lambda^+ \) is not a Jónsson cardinal. These two assumptions will cause no problem when \( \lambda < \aleph_\omega \).
As usual, \( M \prec H(\theta) \) is \( \lambda \)-covering iff for all \( E \in [M]^{\lambda}, \) there is an \( F \in [M]^{\lambda} \) such that \( E \subseteq F \) and \( F \in M. \) By taking a union of an elementary chain of type \( \lambda^+ \) (see [3], §3), we see that there is an \( M \prec H(\theta) \) with \( |M| = \lambda^+ \) such that \( M \) is \( \lambda \)-covering.

\( \kappa \) is called a Jónsson cardinal iff for all \( \psi : [\kappa]^{<\omega} \rightarrow \kappa, \) there is a \( W \in [\kappa]^\kappa \) such that \( \psi([W]^{<\omega}) \) is a proper subset of \( \kappa. \) By Tryba [12] (or see [8]):

**Lemma 10** No successor to a regular cardinal is Jónsson.

In particular, each \( \aleph_n \) is not a Jónsson cardinal; this fact is much older and is easily proved by induction on \( n. \)

**Lemma 11** Let \( \kappa \) be infinite and \( x \in \beta \kappa \) a good ultrafilter on \( \kappa. \) Fix an infinite \( \lambda < \kappa \) and let \( \theta > 2^\kappa \) be regular. Let \( M \prec H(\theta), \) with \( x, \kappa \in M \) and \( \kappa \in M. \) Assume that \( M \) is \( \lambda \)-covering and \( \lambda^+ \) is not a Jónsson cardinal. Let \( \mathcal{B} = \mathcal{P}(\kappa)^{<\omega}, \) and let \( \Gamma : \beta \kappa \rightarrow \text{st}(\mathcal{B}) \) be the natural map. Then \( \Gamma(x) \) is a weak \( P_{\lambda^+} \)-point of \( \text{st}(\mathcal{B}). \)

**Proof.** Fix \( S \subseteq \text{st}(\mathcal{B}) \setminus \{\Gamma(x)\} \) with \( |S| \leq \lambda. \) We shall show that \( \Gamma(x) \) is not in the closure of \( S. \) For each \( z \in S, \) choose \( F_z \in \Gamma(x) = x \cap \mathcal{B} = x \cap M \) such that \( F_z \notin z. \) Since \( M \) is \( \lambda \)-covering, we can get \( \langle G_\xi : \xi < \lambda \rangle \in M \) such that each \( G_\xi \in x \) and \( \forall z \in S \exists \xi < \lambda [G_\xi = F_z]. \) Since \( \lambda^+ \) is not Jónsson and \( \lambda^+ \in M, \) we can fix \( \psi \in M \) such that \( \psi : [\lambda^+]^{<\omega} \rightarrow \lambda \) and such that \( \psi([W]^{<\omega}) = \lambda \) for all \( W \in [\lambda^+]^{\lambda^+}. \) Define \( H(s) = G_{\psi(s)}. \) Then \( H \in M \) and \( H : [\lambda^+]^{<\omega} \rightarrow \Gamma(x). \) Since \( x \) is good, we can find \( \langle K_\alpha : \alpha < \lambda^+ \rangle \in M \) such that each \( K_\alpha \) is in \( x \) (and hence in \( \Gamma(x) = x \cap M \)), and such that \( K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subseteq H(\{\alpha_1, \ldots, \alpha_n\}) \) for each \( n \) and each \( \alpha_1, \ldots, \alpha_n \in \lambda^+. \)

Now (in \( V \)), we claim that \( \exists \alpha < \lambda^+ \forall z \in S [K_\alpha \notin z] \) (so that \( \Gamma(x) \notin \text{cl}(S)). \) If not, then we can fix \( W \in [\lambda^+]^{\lambda^+} \) and \( z \in S \) such that \( K_\alpha \notin z \) for all \( \alpha \in W. \) Fix \( \xi < \lambda \) such that \( G_\xi \notin z. \) Since \( \psi([W]^{<\omega}) = \lambda, \) fix \( s \in [W]^{<\omega} \) such that \( \psi(s) = \xi. \) Say \( s = \{\alpha_1, \ldots, \alpha_n\}. \) Then \( G_\xi = G_{\psi(s)} = H(s) \supseteq K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \in z, \) a contradiction. \( \square \)

**Proof of Lemma 4.** Use Lemmas 11 and 9 with \( \kappa = \lambda^+ = \aleph_n. \) \( \square \)

In view of Lemma 10, we can also prove Theorem 1 replacing \( \aleph_\omega \) with any other singular cardinal of cofinality \( \omega, \) since we can replace \( \aleph_n \) in Lemma 4 by any successor to a regular cardinal.
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