QUADRATIC CONGRUENCES ON AVERAGE AND RATIONAL POINTS ON CUBIC SURFACES

by

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Abstract. — We investigate the average number of solutions of certain quadratic congruences. As an application, we establish Manin’s conjecture for a cubic surface whose singularity type is $A_5 + A_1$.

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1. Introduction

Given a (possibly singular) del Pezzo surface $S$ defined over the field $\mathbb{Q}$ of rational numbers and containing infinitely many rational points, we would like to study the distribution of these points more precisely. We will be most interested in the cubic surface of singularity type $A_5 + A_1$ defined in $\mathbb{P}^3$ by

$$x_1^3 + x_2 x_3^2 + x_0 x_1 x_2 = 0.$$  \hfill (1.1)

Let $H : S(\mathbb{Q}) \to \mathbb{R}$ be an anticanonical height function. The number of rational points of bounded height on $S$ is dominated by the number of points lying on the lines on (an anticanonical model of) $S$. Therefore, it is more interesting to study rational points of height bounded by $B$ on the complement $U$ of the lines on $S$, i.e., the number

$$N_{U,H}(B) = \# \{ x \in U(\mathbb{Q}) \mid H(x) \leq B \}.$$

Manin’s conjecture [FMT89] predicts that, as $B$ tends to $\infty$,

$$N_{U,H}(B) = c_{S,H} B (\log B)^{r-1}(1 + o(1)),$$

where $r$ is the rank of the Picard group of (a minimal desingularization of) $S$ and $c_{S,H}$ is a positive constant for which Peyre, Batyrev and Tschinkel have given a conjectural interpretation [Pey95, BT98b].

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If \( S \) is an equivariant compactification of an algebraic group \( G \), Manin’s conjecture can be proved in certain cases. For instance, see [BT98a] for the case of toric varieties (with \( G = \mathbb{G}_m^2 \)), [CLT02] for the case of the additive group \( G = \mathbb{G}_a \) and [TT12] for certain semidirect products \( G = \mathbb{G}_a \rtimes \mathbb{G}_m \). However, the equation (1.1) defines a cubic surface that is not covered by any of these results (see [DL10], [DL12]).

For general surfaces \( S \), one can approach Manin’s conjecture resorting to universal torsors. Using Cox rings, a universal torsor \( T \) of a minimal desingularization \( \tilde{S} \) of a del Pezzo surface \( S \) of degree \( d \) can be explicitly described as an open subset of an affine variety \( \text{Spec} \, \text{Cox}(\tilde{S}) \). The basic case is again the one of toric varieties [Sal98], where \( \text{Spec} \, \text{Cox}(\tilde{S}) \cong \mathbb{A}^{12 - d} \) is an affine space.

The next natural case is the one where \( \text{Spec} \, \text{Cox}(\tilde{S}) \subset \mathbb{A}^{13 - d} \) is a hypersurface, defined by one torsor equation in the variables \( \eta_1, \ldots, \eta_{13 - d} \). For example, for our surface of degree \( d = 3 \) and type \( A_5 + A_1 \), the torsor equation is

\[
\eta_2 \eta_6 \eta_9 + \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_8 = 0.
\]

All such del Pezzo surfaces are classified in [Der06], where is also given a detailed description of \( \text{Cox}(\tilde{S}) \).

The passage to a universal torsor translates the problem of counting rational points on \( S \) to the one of counting tuples \( (\eta_1, \ldots, \eta_{13 - d}) \) of integers satisfying the torsor equation and certain height and coprimality conditions.

This is basically done as follows. The coprimality conditions can be taken care of by Möbius inversions (in this introduction, we will simply ignore all auxiliary variables occurring because of this). Using a torsor equation such as (1.2), we may eliminate one variable \( \eta_{13 - d} \) that occurs linearly in it. Fixing \( \eta_1, \ldots, \eta_{11 - d} \), we are led to counting the number of integers \( \eta_{12 - d} \) satisfying a congruence condition modulo some integer \( q \) and lying in some range \( I \) given by the height conditions. In our example, the congruence condition is

\[
\eta_2 \eta_6^2 \equiv -\eta_4 \eta_5 \eta_7 \eta_8 \eta_{10} \eta_{11} \mod \eta_1.
\]
Note that both \(I\) and \(q\) may depend on \(\eta_1, \ldots, \eta_{11-d}\).

If \(\eta_{12-d}\) also occurs linearly in the torsor equation then the congruence is linear, so that the number of such \(\eta_{12-d}\) is basically \(q^{-1} \text{vol}(I) + E\), where \(E = O(1)\). Summing this over the remaining variables \(\eta_1, \ldots, \eta_{11-d}\), we must estimate the main term \(q^{-1} \text{vol}(I)\) and show that the contribution of the error term \(E\) is negligible.

The estimation of the error term of the first summation is sometimes straightforward and sometimes very hard. The estimation of the main term is expected to be often straightforward using the results of [Der09] Sections 4, 5, 7 in case of linear \(\eta_{12-d}\).

However, if \(\eta_{12-d}\) occurs with a square power in the torsor equation (such as \(\eta_1^2\) in (1.2)), the main term contains an extra factor of the shape

\[
\mathcal{N}(a, q) = \#\{q | 1 \leq q \leq (q, a) = 1, q^2 \equiv a \mod q\},
\]

where \(a\) and \(q\) are, basically, monomials in \(\eta_1, \ldots, \eta_{11-d}\) (for instance \(q = \eta_1\) and \(a = -\eta_2\eta_7\eta_8\) in our example; see also [Der09] Proposition 2.4). Our experience is that the presence of \(\mathcal{N}(a, q)\) usually makes the treatment of the error term in the next summation over \(\eta_{11-d}\) (over some interval \(J\)) much harder.

Following the most natural order of summation (which is guided by the requirement to start with the \(\eta_i\) that may be the largest), a term of the shape \(\mathcal{N}(a, q)\) appears in the treatment of the following singular del Pezzo surfaces (with one torsor equation):

- quartic del Pezzo surfaces of types \(D_5\) and \(A_4\),
- cubic surfaces of types \(E_6, D_5, A_5 + A_1\),
- del Pezzo surfaces of degree 2 of types \(E_7, E_6, D_6 + A_1\),
- del Pezzo surfaces of degree 1 of types \(E_8, E_7 + A_1\).

Let us sketch the effects of \(\mathcal{N}(a, q)\) in the summation of the main term over \(\eta_{11-d}\) in an interval \(J\). To avoid complications which are irrelevant to our point, we replace \(q^{-1} \text{vol}(I)\) by 1 for the moment; this can be restored by using partial summation. If \(\eta_{11-d}\) occurs linearly in \(a\), we can switch the order of the summations over \(q\) and \(\eta_{11-d}\). Then the summation over \(\eta_{11-d}\) subject to the linear congruence modulo \(q\) gives the main term \(q^{-1} \text{vol}(J)\) and an error term \(F = O(1)\), which we must sum over \(q\) subject to \(1 \leq q \leq q\) and \((q, q) = 1\) and over the remaining variables \(\eta_1, \ldots, \eta_{10-d}\).

The most naive estimation \(\sum_{q=1}^{q} F = O(q)\) is usually not good enough. This problem has been approached in several different ways.

- For the quartic \(A_4\) case [BD09b], it is enough to obtain an extra saving by using different orders of summation over \(\eta_{11-d}\) and \(\eta_{10-d}\), depending on their relative size.
- Alternatively, one can get an extra saving by making \(F\) explicit, improving \(O(q)\) to \(O(q^{1/2+\varepsilon})\) as in [BD09] Lemma 3 using Fourier series and quadratic Gauss sums, which is sufficient for the second summation for the quartic surface of type \(D_5\) [BB07] and for the cubic surface of type \(E_6\) [BBD07].
- For the cubic surface of type \(D_5\) [BD09a], the previous two approaches are combined and slightly improved.
- For the degree 2 del Pezzo surface of type \(E_7\) [BB11], the first two summations over \(\eta_{11-d}, \eta_{12-d}\) are treated simultaneously.

Furthermore, Manin’s conjecture is known for some smooth and singular del Pezzo surfaces of degree greater or equal to 3 for which the factor \(\mathcal{N}(a, q)\) does not appear.

However, for other cases such as the cubic surface \(S\) of type \(A_5 + A_1\), different ideas seem to be needed. In our approach, the main novelty is that we get cancellation effects from its summation over \(q\), several variables \(\eta_i\) occurring linearly in \(a\) and, most importantly, a variable \(\eta_i\) occurring in \(q\), while using the trivial \(O(1)\)-bound for \(F\). This is done in Section 2 using the Polya-Vinogradov bound for character sums and Heath-Brown’s large sieve for real character sums [HB95].
In what follows, for $X > 0$, the notation $x \sim X$ indicates that $X < x \ll 2X$. Let $K_2, K_4, K_7, K_8, Q \geq 1/2$ and $K = K_2K_4K_7K_8$. Applied to the cubic surface of type $\mathbf{A}_5 + \mathbf{A}_1$, the most basic case of our result gives the asymptotic formula

$$\sum_{\eta_i \sim K_i} \sum_{\eta_i = 1} N((-\eta_2\eta_4\eta_7\eta_8, \eta_1)) = cKQ + O(K^{4-\delta}(\log Q)^{4+\varepsilon}),$$

for some explicit $c, \delta > 0$ and for any fixed $\varepsilon > 0$.

Our result shall be compared with the work of Heath-Brown [HB03, Section 5]. In order to obtain an upper bound for $N_{U,H}(B)$ in the case of Cayley’s cubic surface, Heath-Brown proved that the left-hand side of (1.4) is $\ll KQ$. However, to obtain an asymptotic formula for $N_{U,H}(B)$ for the cubic surface defined by the equation (1.1), we need an asymptotic formula for the left-hand side of (1.4), but also for the more complicated expression $\Sigma$ defined in (2.7).

Comparing the proof of the asymptotic formula for $\Sigma$ stated in Theorem 2 and its application in Section 3.4 with Heath-Brown’s work, we notice that our result involves several extra difficulties. In particular, we have to isolate the main term, work out the application in Section 3.4 with Heath-Brown’s work, we notice that our result involves complicated expression $\Sigma$ defined in (2.7).

As an application of our general estimate for the average number of solutions of our quadratic congruence, we prove Manin’s conjecture for the cubic surface $S$ of singularity type $\mathbf{A}_5 + \mathbf{A}_1$ defined by the equation (1.1). The complement of the lines is $U = S \setminus \{x_1 = 0\}$. We use the anticanonical height function defined by $H(x) = \max(|x_0|, \ldots, |x_3|)$ for $x = (x_0 : \cdots : x_3)$, where $(x_0, \ldots, x_3) \in \mathbb{Z}^4$ is such that $(x_0, \ldots, x_3) = 1$. See Section 3.1 for more information on the geometry of $S$.

Theorem 1. — Let $\varepsilon > 0$ be fixed. As $B$ tends to $+\infty$, we have the estimate

$$N_{U,H}(B) = c_{S,H} B(\log B)^6 + O(B(\log B)^{4+\varepsilon}),$$

where

$$c_{S,H} = \frac{1}{172800} \cdot \omega_{\infty} \cdot \prod_p \left(1 - \frac{1}{p}\right) 7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right),$$

and

$$\omega_{\infty} = \int_{0 \leq |(x_1x_2)|^{-1}(x_1^2+x_2^2), |x_1|, |x_2|, |x_3| \leq 1} \frac{1}{x_1x_2} \, dx_1 \, dx_2 \, dx_3.$$

We will check in Section 3.6 that this agrees with Manin’s conjecture and that the constant $c_{S,H}$ is the one predicted by Peyre, Batyrev and Tschinkel.

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2. Quadratic congruences on average

As explained in the introduction, our motivation to study quadratic congruences in this section is their appearance in proofs of Manin’s conjecture.

2.1. Counting solutions of quadratic congruences. — To evaluate the main term of the first summation over a variable occurring non-linearly in the torsor equation (such as $\eta_0$ in (1.2) in our example; see Lemma 3 below for the result of the first summation in our case and [Der09] Proposition 2.4 for the result in a more general situation), we need to count solutions of quadratic congruences on average. To this end, we consider the following general situation.

Let $b \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}_{>0}$ with $(k, b) = 1$, $r \in \mathbb{Z}_{>0}$ with $r \geq 2$ and $K_1, ..., K_r, Q, V$ be positive real numbers. Throughout, for $X > 0$, we use the notation $x \sim X$ to indicate that $X < x \leq 2X$. Let $b \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}_{>0}$ with $(k, b) = 1$, $r \in \mathbb{Z}_{>0}$ with $r \geq 2$ and $K_1, ..., K_r, Q, V$ be positive real numbers. We assume that $\Phi$ is a continuous real-valued function defined on $\mathbb{R}^k$ such that $0 \leq \Phi \leq V$

and, in each of the variables, can be divided into finitely many continuously differentiable and monotone pieces whose number is bounded by an absolute constant.

We further assume that $Q^-$ and $Q^+$ are continuous real-valued functions defined on $\mathbb{R}^k$ such that $0 < Q^- \leq Q^+ \leq Q$.

Moreover, for any given $i \in \{1, ..., r\}$, for $x_j \sim K_j$ for $j \in \{1, ..., r\} \setminus \{i\}$, and for $0 < y \leq Q$, we assume that the set

$$
\mathcal{A}_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_r, y) = \{ x_i \sim K_i \mid Q^-(x_1, ..., x_r) < y \leq Q^+(x_1, ..., x_r) \}
$$

is the union of finitely many intervals whose number is bounded by an absolute constant. Throughout the sequel, for brevity, we write

$$
K = 2^{r+1}K_1 \cdots K_r,
$$

$$
Q^\pm = Q^\pm(a_1, ..., a_r),
$$

and

$$
\mathcal{A}_i(y) = \mathcal{A}_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_r, y).
$$

Finally, for any integer $n \in \mathbb{Z}_{>0}$, we set

$$
\text{rad}(n) = \prod_{p \mid n} p.
$$

Our goal is to evaluate asymptotically the expression

$$
\Sigma = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{Q^- < q \leq Q^+} \Phi(a_1, ..., a_r, q) N(-a_1 \cdots a_r, b, kq),
$$

where $N(-a_1 \cdots a_r, b, kq)$ is defined in (2.3).

We begin by splitting $\Sigma$ into a main term and an error term. Let $kq = 2^{v(\ell)}h$, where $v(\ell)$ is the 2-adic valuation of $\ell \in \mathbb{Z}_{>0}$ and $h$ is odd. Thus, for any $n \in \mathbb{Z}$, we have

$$
\sum_{\overline{q}^2 \equiv n \mod kq} 1 = \left( \sum_{\overline{q}^2 \equiv n \mod 2^{v(\ell)}h} 1 \right) \left( \sum_{\overline{q}^2 \equiv n \mod h} 1 \right).
$$
In the following, for \( j \geq 0 \), we set

\[
\left\{ \frac{n}{2^j} \right\} = \sum_{\varrho \equiv n \mod 2^j} 1.
\]

It is well-known that if \( (n, 2^j) = 1 \), then

\[
\left\{ \frac{n}{2^j} \right\} = \begin{cases} 1 & \text{if } j = 0, \\ 1 & \text{if } n \equiv 1 \mod 2 \text{ and } j = 1, \\ 2 & \text{if } n \equiv 1 \mod 4 \text{ and } j = 2, \\ 4 & \text{if } n \equiv 1 \mod 8 \text{ and } j \geq 3, \\ 0 & \text{otherwise}. \end{cases}
\]

(2.9)

Moreover, if \( h \) is odd and \( (n, h) = 1 \), then

\[
\sum_{\varrho \equiv n \mod h} \mu^2(d) \left( \frac{n}{d} \right) = \sum_{d|kh, (d, 2) = 1} \mu^2(d) \left( \frac{-a_1 \cdots a_r b}{d} \right).
\]

(2.10)

The equalities (2.8), (2.9) and (2.10) imply that if \( (a_1, \cdots, a_r b, kq) = 1 \) then

\[
\Sigma = M + E,
\]

(2.11)

where the main term \( M \) is defined by

\[
M = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{Q^{-} < q \leq Q^{+}} \Phi(a_1, \ldots, a_r, q) \left\{ \frac{-a_1 \cdots a_r b}{2^v(kq)} \right\},
\]

(2.13)

and the error term \( E \) is defined by

\[
E = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{Q^{-} < q \leq Q^{+}} \Phi(a_1, \ldots, a_r, q) \left\{ \frac{-a_1 \cdots a_r b}{2^v(kq)} \right\} \times \\
\sum_{d|kh, (d, 2) = 1} \mu^2(d) \left( \frac{-a_1 \cdots a_r b}{d} \right).
\]

(2.14)

In the following sections, we estimate the error term by generalizing the method used by Heath-Brown in [HB03, Section 5]. We shall not evaluate the main term any further since this is not needed in our application. Our result is as follows.

**Theorem 2.** — Let \( \varepsilon > 0 \) be fixed. Set \( L = \log(2 + Q) \). We have the estimate

\[
\Sigma - M \ll E',
\]

(2.15)

where

\[
E' = V K^{1/2 + \varepsilon} Q L^\varepsilon \left( K^{1/2 - 1/2r} \rad(k)^{1/4} + |b|^{r/2(1+\varepsilon)} \omega(k) + 2^{\omega(k)} L \right).
\]

(2.16)
The term $\Sigma$ is not exactly the one we need it in our application. Let $\Sigma'$ be defined like $\Sigma$ in (2.17), but with some additional coprimality conditions included, namely

$$
(2.15) \quad \Sigma' = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{1 \leq i < j \leq r} \Phi(a_1, ..., a_r, q) N(-a_1 \cdots a_r b, kq),
$$

where $t_1, ..., t_r, u \in \mathbb{Z}_{>0}$. Accordingly, we set

$$
(2.16) \quad M' = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{1 \leq i < j \leq r} \Phi(a_1, ..., a_r, q) \frac{-a_1 \cdots a_r b}{2q(kq)}.
$$

Removing the additional coprimality conditions using Möbius inversions, we shall deduce from Theorem 2 the following asymptotic formula for $\Sigma'$.

**Corollary 1.** — Let $\varepsilon > 0$ be fixed. We have the estimate

$$
\Sigma' - M' \ll (1 + \varepsilon)^{\omega(t_1) + \cdots + \omega(t_r) + \omega(u)} E'.
$$

**Remark 3.** — Theorem 2 and Corollary 1 remain true if the left half-open $q$-summation interval $(Q^-, Q^+)$ is replaced by an arbitrary interval $I(Q^-, Q^+)$ (left half-open, right half-open, open, closed) with endpoints $Q^-$ and $Q^+$. The proof is the same, with the relevant summation intervals being altered accordingly.

Theorem 2 and Corollary 1 trivially hold if $K_i < 1/2$ for some $i \in \{1, ..., r\}$ or $Q < 1$ since in this case we have $\Sigma = M = 0$. Therefore, we shall assume that $K_i \geq 1/2$ for any $i \in \{1, ..., r\}$ and $Q \geq 1$ throughout the following proofs of these results. Therefore, recalling the definition (2.14) of $K$, we note that $K \geq 2$.

### 2.2. Application of the Polya-Vinogradov bound

Let us write $d = fg$, where $g = (d, k)$. It follows that $(f, k/g) = 1$ and so the condition $d|kq$ is equivalent to $f|q$. Thus, we can write $q = ef$. Let us set

$$
Q^-(e, g) = \max\{1/g, Q^-/e\},
$$

$$
Q^+(e) = Q^+/e.
$$

Reordering the summations and noting that $\mu^2(fg) = 1$ if and only if $(f, g) = 1$ and $\mu^2(f) = \mu^2(g) = 1$, the error term $E$ defined in (2.14) can be rewritten as

$$
(2.17) \quad E = \sum_{g|k} \mu^2(g) \sum_{e \leq Q} E(e, g),
$$

where

$$
E(e, g) = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \Phi(a_1, ..., a_r, ef) \times \sum_{-a_1 \cdots a_r b > 0} \frac{1}{2q(kq)} \sum_{Q^-(e, g) < f \leq Q^+(e)} \Phi(e, f, 2k). \quad (2.18)
$$

In the following sections, we will estimate $E(e, g)$ in three different ways. We start with an application of the Polya-Vinogradov bound for character sums.
the summation over $a_1$, we get
\[
E(e, g) = \sum_{a_2 \in K_2} \cdots \sum_{a_r \in K_r} \sum_{1 \leq f \leq Q/e} \sum_{(f, 2k) = 1} \mu^2(f) \left( \frac{-a_2 \cdots a_r b}{fg} \right) \times
\]
\[
\sum_{\delta = 1}^{8} \left\{ \frac{-ha_2 \cdots a_r b}{2^{\delta}(ke)} \right\} \sum_{a_1 \in \mathcal{A}(ef)} \Phi(a_1, \ldots, a_r, ef) \left( \frac{a_1}{fg} \right),
\]
\[(2.19)\]

where $\mathcal{A}(ef)$ is defined in (2.3) and (2.5). In the following, we estimate the inner-most sum over $a_1$ under the assumption $\mu^2(fg) = 1$. Using partial summation and the assumptions on $\Phi$ in Section 2.1 (in particular, (2.1)), we get
\[
\sum_{a_1 \in \mathcal{A}(ef)} \Phi(a_1, \ldots, a_r, ef) \left( \frac{a_1}{fg} \right) \ll V \cdot \sup_{L_1 < L_2} \sum_{L_1 < a_1 \leq L_2} \frac{a_1}{fg}. \]
\[(2.20)\]

Removing the coprimality condition $(a_1, ke) = 1$ using a M"{o}bius inversion, we obtain
\[
\sum_{L_1 < a_1 \leq L_2} \frac{a_1}{fg} = \sum_{d | ke} \mu(d) \left( \frac{d}{fg} \right) \sum_{L_1 < a_1 \leq L_2/d} \frac{a_1}{fg}. \]
\[(2.21)\]

Recalling the assumption that $\mathcal{A}(ef)$ is the union of finitely many intervals whose number is bounded by an absolute constant, the Polya-Vinogradov bound for character sums gives
\[
\sum_{L_1 < a_1 \leq L_2/d} \frac{a_1}{fg} \ll f^{1/2}g^{1/2} \log fg,
\]
\[(2.22)\]

where we note that $fg$ is not a perfect square since $fg > 1$ and $\mu^2(fg) = 1$. Combining (2.19), (2.20), (2.21) and (2.22), we get
\[
E(e, g) \ll V K_2 \cdots K_r Q^{3/2} e^{-3/2} f^{1/2} g^{1/2} \log(2gQe^{-1}) 2^{\omega(ke)}.
\]

Similarly, for every $i \in \{1, \ldots, r\}$, we obtain
\[
E(e, g) \ll V \cdot \frac{K_1 \cdots K_r}{K_i} Q^{3/2} e^{-3/2} f^{1/2} g^{1/2} \log(2gQe^{-1}) 2^{\omega(ke)}.
\]

Hence, on taking $K_i$ as the maximum of $K_1, \ldots, K_r$, it follows that
\[
E(e, g) \ll V K^{1-1/r} Q^{3/2} e^{-3/2} f^{1/2} g^{1/2} \log(2gQe^{-1}) 2^{\omega(ke)}.
\]
\[(2.23)\]

where $K$ is defined in (2.4).

### 2.3. Application of the Polya-Vinogradov bound II.

In this section, we set $a = a_1 \cdots a_r$. Alternatively, we may use the Polya-Vinogradov bound to treat the
inner-most sum over \( f \) in \((2.18)\) non-trivially if \(-ab\) is not a perfect square, which we assume in the following. Using partial summation and the bound \((2.1)\), we deduce

\[
\sum_{Q^{-}(e,g)<f\leq Q^{+}(e) \atop (f,2k)=1} \Phi(a_{1},...,a_{r},ef)\mu^{2}(f) \left( \frac{-ab}{fg} \right)
\]

\((2.24)\)

\[
\ll V \cdot \sup_{Q^{-}(e,g)\leq F_{1}<F_{2}\leq Q^{+}(e)} \left| \sum_{f_{1}<f\leq F_{2} \atop (f,2k)=1} \mu^{2}(f) \left( \frac{-ab}{f} \right) \right|.
\]

Using the well-known formula

\[
\mu^{2}(f) = \sum_{d|f} \mu(d),
\]

and writing \( f = d^{2}\bar{f} \), we get

\[
\sum_{F_{1}<f\leq F_{2} \atop (f,2k)=1} \mu^{2}(f) \left( \frac{-ab}{f} \right) = \sum_{d\in F_{1}/2 \atop (d,2nk) = 1} \mu(d) \sum_{F_{1}/d^{2}<f\leq F_{2}/d^{2} \atop (f',2k)=1} \left( \frac{-ab}{f'} \right).
\]

Removing the coprimality condition \((\bar{f},k) = 1\) using a Möbius inversion, we obtain

\[
\sum_{F_{1}/d^{2}<f\leq F_{2}/d^{2} \atop (f',2k)=1} \left( \frac{-ab}{f'} \right) = \sum_{\bar{d}|k \atop (\bar{d},2k) = 1} \mu(\bar{d}) \sum_{F_{1}/(\bar{d}^{2}\bar{d})<f\leq F_{2}/(\bar{d}^{2}\bar{d}) \atop (f',2k)=1} \left( \frac{-ab}{f'} \right).
\]

The Polya-Vinogradov bound gives

\[
\sum_{F_{1}/(\bar{d}^{2}\bar{d})<f\leq F_{2}/(\bar{d}^{2}\bar{d}) \atop (f',2k)=1} \left( \frac{-ab}{f'} \right) \ll (a|b|)^{1/2} \log(2a|b|),
\]

where we recall our assumption that \(-ab\) is not a perfect square.

Let \( E'(e,g) \) be the contribution to \( E(e,g) \) of those \( a_{1},...,a_{r} \) for which \(-ab\) is not a perfect square. Then, combining \((2.22), (2.23), (2.25), (2.26)\) and \((2.27)\), we get

\[
E'(e,g) \ll VK^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2} \log(K|b|)^{2ε(k)}.
\]

The remaining contribution \( E^{\Box}(e,g) \) of perfect squares \(-ab\) is trivially calculated to be

\[
E^{\Box}(e,g) \ll VK^{1/2+ε}Qe^{-1}.
\]

Combining \((2.28)\) and \((2.29)\), we obtain

\[
E(e,g) \ll VK^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2} \log(K|b|)^{2ε(k)} + VK^{1/2+ε}Qe^{-1}.
\]

2.4. Application of Heath-Brown’s large sieve. — Finally, we may make use of Heath-Brown’s large sieve for real character sums to bound \( E(e,g) \), which we shall do in the following. Let us set

\[
u_{f} = \Phi(a_{1},...,a_{r},ef)\mu^{2}(f) \left( \frac{-a_{1}\cdots a_{r}b}{fg} \right).
\]
To make the summation ranges independent, we first remove the summation condition $Q^{-}(e, g) < f \leq Q^{+}(e)$ using Perron’s formula, getting

\[ (2.31) \quad \sum_{Q^{-}(e, g) < f \leq Q^{+}(e) \atop (f, 2k) = 1} u_{f} = \]

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \sum_{1 \leq f \leq Q/e \atop (f, 2k) = 1} u_{f} f^{-s} \right) \left( Q^{+}(e)^{s} - Q^{-}(e, g)^{s} \right) \frac{ds}{s} + O \left( V + \frac{VQ \log 2Q}{eT} \right),
\]

where we have set $c = 1/\log 2Q$ and we have used (2.1). Set

\[ T = 2Q(\log 2Q)e^{-1}, \]

\[ A(a_{1}, \ldots, a_{r}; s) = \left( Q^{+}(e)^{s} - Q^{-}(e, g)^{s} \right) \left( \frac{-a_{1} \cdots a_{r} b}{2^{e(k_{e})}} \right) \]

\[ \quad \left( \frac{-a_{1} \cdots a_{r} b}{g} \right), \]

and

\[ I(s) = \sum_{a_{1} \sim K_{1}} \cdots \sum_{a_{r} \sim K_{r}} \sum_{1 \leq f \leq Q/e \atop (f, 2) = 1} \Phi(a_{1}, \ldots, a_{r}, e) A(a_{1}, \ldots, a_{r}; s) B(f; s) \left( \frac{a_{1} \cdots a_{r}}{f} \right). \]

Then it follows from (2.31) that

\[ (2.32) \quad E(e, g) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} I(s) \frac{ds}{s} + O(VK) \ll (\log T) \sup_{-T \leq t \leq T} |I(c + it)| + VK \]

\[ = (\log T) |I(c + it_{0})| + VK \]

for a particular $t_{0} \in [-T, T]$. From [HB95 Corollary 4], a version of Heath-Brown’s large sieve for real character sums, we have

\[ (2.33) \quad \sum_{a_{1} \sim K_{1}} \cdots \sum_{a_{r} \sim K_{r}} \sum_{1 \leq f \leq F \atop (f, 2) = 1} A'(a_{1}, \ldots, a_{r}) B'(f) \left( \frac{a_{1} \cdots a_{r}}{f} \right) \ll \left( KF^{1/2} + K^{1/2}F \right) (KF)^{\varepsilon} \]

whenever $A'(a_{1}, \ldots, a_{r}), B'(f) \ll 1$ and $F \geq 1$, and where we note that

\[ \left| \sum_{a_{1} \sim K_{1}} \cdots \sum_{a_{r} \sim K_{r}} A'(a_{1}, \ldots, a_{r}) \right| \ll \tau_{r}(a) \ll a^{\varepsilon} \]

for any given $a \in \mathbb{Z}_{>0}$ and where $\tau_{r}$ denotes the Dirichlet convolution of the constant arithmetic function equal to 1 by itself $r$ times. Using the bound (2.33) together with partial summation in $f$ to remove the weight function $\Phi(a_{1}, \ldots, a_{r}, e)$, we deduce that

\[ (2.34) \quad |I(c + it_{0})| \ll V \left( KQ^{1/2}e^{-1/2} + K^{1/2}Qe^{-1} \right) (KQe^{-1})^{\varepsilon}, \]

where we take into account that

\[ A(a_{1}, \ldots, a_{r}; t_{0}) \ll 1, \]

\[ B(f; t_{0}) \ll 1. \]
Combining (2.32) and (2.34), and noting that
\[
\log T = \log \frac{2Q \log 2Q}{e} = \log \left( \frac{2Q}{e} \right) + \log \log(2Q) \ll \left( \frac{Q}{e} \right)^{\varepsilon} \log^\varepsilon (2 + Q),
\]
we deduce that
\[
(2.35) \quad E(e, g) \ll V \left( KQ^{1/2}e^{-1/2} + K^{1/2}Q^{-1} \right) (KQ^{-1})^\varepsilon \log^\varepsilon (2 + Q).
\]

2.5. Proofs of Theorem 2 and Corollary 1 — We start by proving Theorem 2

Proof. — Combining the three bounds (2.23), (2.30) and (2.35), we obtain
\[
(2.36) \quad E(e, g) \ll \left( V (KQ^{-1})^\varepsilon \log^\varepsilon (2 + Q) \right) m + VK^{1/2+}Q^{-1},
\]
where
\[
m = \min \left\{ K^{-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon}, K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^\omega(k), KQ^{1/2}e^{-1/2} + K^{1/2}Q^{-1} \right\}
\]
\[
\ll \min \left\{ K^{-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon}, KQ^{1/2}e^{-1/2} \right\} + \min \left\{ K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^\omega(k), K^{1/2}Q^{-1} \right\}
\]
\[
\ll \left( K^{-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon} \right)^{\mu} \left( KQ^{1/2}e^{-1/2} \right)^{1-\mu} + \left( K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^\omega(k) \right)^{\nu} \left( K^{1/2}Q^{-1} \right)^{1-\nu}
\]
\[
\ll K^{-1/r}Q^{1/2}e^{-1/2}g^{1/2+\varepsilon} + K^{1/2+\nu}Q^{-1}e^{-1/(1-\nu)}|b|^{1/2+\varepsilon}2^\omega(k)
\]
for any \( \mu, \nu \in [0, 1] \). Choosing \( (\mu, \nu) = (1/2 - 3\varepsilon, 4\varepsilon) \), recalling (2.17) and (2.30), and summing over \( g \) and \( e \) now gives
\[
E \ll V K^{1-1/(2r)+}Q \log^\varepsilon (2 + Q) + VK^{1/2+4\varepsilon}Q|b|^{3r/2(1+4\varepsilon)}2^\omega(k) \log^\varepsilon (2 + Q) + VK^{1/2+\varepsilon}Q \log^\varepsilon (2 + Q) 2^\omega(k),
\]
which ends the proof of Theorem 2.

We can now deduce Corollary 1

Proof. — Since the technique is standard, we shall be brief. Removing the additional coprimality conditions using Möbius inversions, we are led to
\[
(2.37) \quad \Sigma' = \sum_{(d_{\alpha, \beta}) \in \mathbb{Z}^{(r-1)/2} \cap \{1 \leq \alpha < \beta \leq r \}} \sum_{d_1 | t_1} \cdots \sum_{d_r | t_r} \sum_{d | u} \left( \prod_{1 \leq i < j \leq r} \mu(d_{i,j}) \right) \left( \prod_{l=1}^r \mu(d_l) \right) \mu(d) \times
\]
\[
\Sigma((d_{i,j})_{1 \leq i < j \leq r}, d_1, \ldots, d_r, d),
\]
where the summations over \( d_{\alpha, \beta}, d_\gamma \) and \( d \) are subject to a suitable set of coprimality conditions, and
\[
(2.38) \quad \Sigma((d_{i,j})_{1 \leq i < j \leq r}, d_1, \ldots, d_r, d) = \sum_{a_1 \sim K_1/(d_1d_2 \cdots d_1r)} \cdots \sum_{a_r \sim K_r/(d_r \cdots d_{r-1}r)} \Phi(a_1d_1d_2 \cdots d_1r, a_2d_2d_3 \cdots d_2r, \ldots, a_rd_r \cdots d_{r-1}r, qd) N(-aDb, kdq),
\]
where \( a = a_1 \cdots a_r \) and \( D = (d_{1,2} d_{1,3} \cdots d_{r-1,r})^2 d_1 \cdots d_r \). Using Theorem 2 we obtain
\[
\Sigma((d_{i,j})_{1 \leq i < j \leq r}, d_1, \ldots, d_r, d) - M((d_{i,j})_{1 \leq i < j \leq r}, d_1, \ldots, d_r, d) \ll \\
V \left( \frac{K}{D} \right)^{1/2+\varepsilon} \frac{Q}{d} \frac{L^r}{d} \left( \left( \frac{K}{D} \right)^{1/2-1/2r} d^{1/4} \text{rad}(k)^{1/4} + |Db|^{\sigma(1+\varepsilon)\omega(dk)} + 2^{\omega(dk)} L \right),
\]
where \( L = \log(2 + Q) \) and
\[
M((d_{i,j})_{1 \leq i < j \leq r}, d_1, \ldots, d_r, d) = \\
\sum_{a_1 \sim K_1/(d_1 d_2 \cdots d_r)} \cdots \sum_{a_r \sim K_r/(d_1 \cdots d_{r-1,r})} \prod_{1 \leq i < j \leq r} \mu(d_{i,j}) \left( \prod_{l=1}^r \mu(d_l) \right) \mu(d) \times \\
\Phi(a_1 d_1, a_2 d_2, \cdots, d_r, a_r d_r),
\]
where the summations on the right-hand side are restricted in the same way as in (2.37) and \( M' \) is defined in (2.16). Summing up the error term in (2.39) over \( d_{a,b}, d_7 \), and \( d \), we get the error term claimed in Corollary 1 which ends the proof. \( \square \)

3. Counting rational points on a singular cubic surface

In this part, we give a proof of Manin’s conjecture (Theorem 1) for the singular cubic surface with \( A_5 + A_1 \) singularity type. We will apply our result on quadratic congruences (Corollary 1).

3.1. Geometry. — Our cubic surface \( S \) defined by (1.1) over the field \( \mathbb{Q} \) has singularities only in \( (0 : 0 : 1 : 0), \) of type \( A_1 \), and \((1 : 0 : 0 : 0)\) of type \( A_5 \). It contains precisely two lines \( \{x_1 = x_2 = 0\} \) and \( \{x_1 = x_3 = 0\} \). The complement of the lines is \( U = \{x \in S \mid x_1 \neq 0\} \). It is rational, as one can see by projecting to \( \mathbb{P}^2 \) from one of the singularities.

Its minimal desingularization \( \tilde{S} \) is a blow-up of \( \mathbb{P}^2 \) in six points, so \( \text{Pic}(\tilde{S}) \) is free of rank 7. The Cox ring of \( \tilde{S} \) has been determined in [Der06]. It has 10 generators \( \eta_1, \ldots, \eta_{10} \) satisfying the relation (1.2). The configuration of the rational curves on \( \tilde{S} \) corresponding to the generators of \( \text{Cox}(\tilde{S}) \) is described by the extended Dynkin diagram in Figure 2 where each vertex corresponds to a curve \( E_i \) of \( \eta_i \), and an edge indicates that two curves intersect.

\[\text{Figure 2. Configuration of curves on } \tilde{S}.\]
3.2. Passage to a universal torsor. — Let
\[ \eta = (\eta_1, \ldots, \eta_{10}), \quad \eta' = (\eta_1, \ldots, \eta_8), \quad \eta^{(k_1, \ldots, k_8)} = \eta_1^{k_1} \cdots \eta_8^{k_8}, \]
for any \((k_1, \ldots, k_8) \in \mathbb{R}^8\).

For \(i = 1, \ldots, 10\), let
\[
(\mathbb{Z}_i, J_i, J'_i) = \begin{cases} 
(\mathbb{Z} > 0, \mathbb{R} \geq 1, \mathbb{R} \geq 1), & i \in \{1, \ldots, 6\}, \\
(\mathbb{Z} > 0, \mathbb{R} \geq 1, \mathbb{R} > 0), & i = 7, \\
(\mathbb{Z} \neq 0, \mathbb{R} \leq 1 \cup \mathbb{R} > 1, \mathbb{R}), & i = 8, \\
(\mathbb{Z}, \mathbb{R}, \mathbb{R}), & i \in \{9, 10\}. 
\end{cases}
\]

In the course of our argument, we estimate summations over \(\eta_9 \in \mathbb{Z}_i\) by integrations over \(\eta_9 \in J_i\), which we enlarge to \(\eta_9 \in J'_i\) in (3.24).

Lemma 4. — We have
\[
\mathcal{N}_{U;H}(B) = \# \{\eta \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_{10} \mid (3.2) - (3.2) \text{ hold}\}
\]
with the torsor equation
\[
(3.2) \quad \eta_1 \eta_{10} + \eta_2 \eta_9^2 + \eta_4 \eta_5^2 \eta_6^2 \eta_9^2 \eta_8 = 0,
\]
the height condition
\[
(3.3) \quad h(\eta', \eta_9; B) = B^{-1} \max \left\{ |\eta_1^{-1}(\eta_2 \eta_9 \eta_9^2 + \eta_4 \eta_5^2 \eta_6^2 \eta_9^2 \eta_8^2)|, |\eta^{(1,1,2,2,2,2,1)}|, |\eta^{(3,2,4,3,2,0,1,0)}|, |\eta^{(0,1,1,1,1,1,1)}| \eta_9 \right\} \leq 1
\]
and the coprimality conditions
\[
(3.4) \quad (\eta_{10}, \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7) = (\eta_9, \eta_1 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7) = 1
\]
\[
(3.5) \quad (\eta_8, \eta_1 \eta_2 \eta_3 \eta_4 \eta_7) = 1
\]
\[
(3.6) \quad (\eta_1, \eta_2 \eta_3 \eta_4) = (\eta_6, \eta_1 \eta_2 \eta_3 \eta_4) = (\eta_5, \eta_1 \eta_2 \eta_3) = (\eta_4, \eta_1 \eta_2) = 1.
\]

Proof. — Based on the birational projection \(S \dashrightarrow \mathbb{P}^2\) from the \(\mathbb{A}^1\)-singularity and the structure of \(\tilde{S}\) as a blow-up of \(\mathbb{P}^2\) in six points, we prove as in [DT07, Section 4] that the map
\[
\psi : \eta \mapsto (\eta_9 \eta_{10}, \eta^{(1,1,2,2,2,2,2,1)}, \eta^{(3,2,4,3,2,0,1,0)}, \eta^{(0,1,1,1,1,1,1,1)} \eta_9),
\]
gives a bijection between the rational points on \(U\) and the set of \(\eta \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_{10}\) satisfying (3.2) and the coprimality conditions encoded in the extended Dynkin diagram in Figure 2, which are (3.4) - (3.6).

We note that the coprimality conditions imply that the image of such \(\eta\) under \(\psi\) has coprime coordinates, so that the height of \(\psi(\eta)\) is simply the maximum of their absolute values. Using (3.2), we eliminate \(\eta_{10}\) and obtain (3.3).

3.3. Counting points. — Recalling the definition (3.1) of \(J_i\), let
\[
\mathcal{R}(B) = \{(\eta', \eta_9) \in J_1 \times \cdots \times J_9 \mid h(\eta', \eta_9; B) \leq 1\}
\]
be the set whose number of lattice points we want to compare with its volume (both under the torsor equation (3.2) and the coprimality conditions (3.4) - (3.6)).

Recall the definition (1.3) of \(\mathcal{N}(q,a)\). Summing over \(\eta_9\), with \(\eta_{10}\) as a dependent variable, we get:
Lemma 5. — We have

\[ N_{U,H}(B) = \sum_{\eta' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1(\eta') V_1(\eta'; B) + O(B(\log B)^3), \]

where

\[ (3.7) \quad V_1(\eta'; B) = \int_{(\eta', \eta_8) \in \mathbb{R}(B)} \eta_8^{-1} \, d\eta_9 \]

and

\[ \theta_1(\eta') = \sum_{\substack{k, \eta_3 \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8 \mid (k, \eta_3) = 1}} \frac{\mu(k) \varphi^*(\eta_3 \eta_4 \eta_8 \eta_9)}{k \varphi^*(\eta_3, k \eta_1)} \mathcal{N}(-\eta_2 \eta_4 \eta_7 \eta_8, k \eta_1) \]

if \( \eta' \) satisfies the coprimality conditions (3.5)–(3.6), while \( \theta_1(\eta') = 0 \) otherwise.

Proof. — Essentially because Figure 2 describing the coprimality conditions and the torsor equation (3.2) have the right shape, we are in the position to apply the general result of [Der09 Proposition 2.4]. This gives the main term as above after we simplify the condition \((k, \eta_2 \eta_4 \eta_7 \eta_8) = 1\) in the summation over \( k \) to \((k, \eta_2 \eta_4) = 1\), which is allowed because of \( k \mid \eta_3 \) and (3.5)–(3.6).

The sum of the error term over all relevant \( \eta' \) is bounded by

\[ \sum_{\eta'} 2^{\omega(\eta_3) + \omega(\eta_3 \eta_4 \eta_7 \eta_8) + \omega(\eta_1 \eta_8)} \ll \sum_{\eta_1, \ldots, \eta_7} \frac{2^{\omega(\eta_3) + \omega(\eta_3 \eta_4 \eta_7 \eta_8) + \omega(\eta_1 \eta_8)}}{\eta^{(1, 1, 1, 2, 2, 2, 2, 0)}} B \ll B(\log B)^3, \]

where we use the second part of (3.3) for the summation over \( \eta_8 \).

3.4. Application of Corollary 1 — Using Corollary 1, we now want to prove that Lemma 5 still holds when we replace the error term by \( O(B(\log B)^{4+\varepsilon}) \) and \( \theta_1 \) in the main term by \( \theta_1' \) with

\[ \theta_1'(\eta') = \sum_{\substack{k, \eta_3 \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8 \mid (k, \eta_3) = 1}} \frac{\mu(k) \varphi^*(\eta_3 \eta_4 \eta_8 \eta_9)}{k \varphi^*(\eta_3, k \eta_1)} \left\{ -\frac{\eta_2 \eta_4 \eta_7 \eta_8}{2^{\omega(k \eta_1)}} \right\} \]

if (3.5)–(3.6) hold and \( \theta_1'(\eta') = 0 \) otherwise. Hence, we want to show the following.

Lemma 6. — Let \( \varepsilon > 0 \) be fixed. We have

\[ N_{U,H}(B) = \sum_{\eta' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1'(\eta') V_1(\eta'; B) + O(B(\log B)^{4+\varepsilon}). \]

Proof. — First, we write

\[ \sum_{\eta' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1(\eta') V_1(\eta'; B) = F^+(B) + F^-(B), \]

where

\[ F^+(B) = \sum_{\eta' \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}_{>0}} \theta_1(\eta') V_1(\eta'; B), \]

and

\[ F^-(B) = \sum_{\eta' \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}_{<0}} \theta_1(\eta') V_1(\eta'; B). \]

The term \( F^-(B) \) can be treated similarly as \( F^+(B) \). Therefore, we confine ourselves to the treatment of the term \( F^+(B) \), which we now transform in such a way that Corollary 1 can be applied.
We break the summation ranges of \( \eta_1, \eta_2, \eta_4, \eta_7 \) and \( \eta_8 \) into dyadic intervals, i.e., we write
\[
F^{\star}(B) = \sum_{\eta'' \in \mathbb{Z}^*_2, k|\eta_3} \frac{\mu(k)}{k} \sum_{L_1, L_2, L_4, L_7, L_8} W(\eta'', k, L_1, L_2, L_4, L_7, L_8),
\]
where \( \eta'' = (\eta_3, \eta_4, \eta_6) \) satisfies the coprimality conditions \( (\eta_3, \eta_5 \eta_6) = 1 \). Hence, we may write
\[
W(\eta'', k, L_1, L_2, L_4, L_7, L_8) = \sum_{\eta_1 \sim L_1, \eta_4 \sim L_4} \varphi^*((\eta_3, k\eta_1))^{-1} \sum_{\eta_7 \sim L_7} \sum_{\eta_8 \sim L_8} V_1(\eta; B) \mathcal{N}(-\eta_2 \eta_4 \eta_7 \eta_8, k\eta_1).
\]
Here we note that the coprimality condition \( (\eta_2 \eta_4 \eta_7 \eta_8, k\eta_1) = 1 \) is contained in the definition of \( \mathcal{N}(-\eta_2 \eta_4 \eta_7 \eta_8, k\eta_1) \).

To make Corollary \( \text{II} \) applicable, it is necessary to remove the arithmetic factors \( \varphi^*((\eta_3, k\eta_1))^{-1} \) and \( \varphi^*(\eta_2 \eta_4 \eta_7 \eta_8) \). We write
\[
\varphi^*((\eta_3, k\eta_1))^{-1} = \varphi^*(k \cdot (\eta_3/k, \eta_1))^{-1} = \varphi^*(k)^{-1} \prod_{\mu(p | (\eta_3/k, \eta_1)} \left( 1 + \frac{1}{p - 1} \right)
\]
and
\[
\varphi^*(\eta_2 \eta_4 \eta_7 \eta_8) = \prod_{\mu(p | \eta_4)} \left( 1 - \frac{1}{p} \right) \prod_{\mu(p | \eta_7)} \left( 1 - \frac{1}{p} \right)
\]
where we use the fact that \( (\eta_4, \eta_7) = 1 \). Hence, we may write
\[
W(\eta'', k, L_1, L_2, L_4, L_7, L_8) = \varphi^*(k)^{-1} \sum_{d_1 | \eta_4} \frac{\mu^2(d_1)}{\varphi(d_1)} \sum_{d_4 | \eta_4} \frac{\mu(d_4)}{d_4} \sum_{d_7 | \eta_7} \frac{\mu(d_7)}{d_7},
\]
where
\[
W(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) = \sum_{\eta_1 \sim L_1, \eta_4 \sim L_4} \sum_{\eta_7 \sim L_7} \sum_{\eta_8 \sim L_8} \sum_{\eta_9 \sim L_9} V_1(d_1 \eta_1, \eta_2, \eta_3, d_4 \eta_4, \eta_5, \eta_6, d_7 \eta_7, \eta_8, B) \mathcal{N}(-\eta_2 \eta_4 \eta_7 \eta_8 d_7, k\eta_1 d_1^2).
\]
Now we observe that for \( \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8 > 0 \), the set
\[
\{ y > 0 | V_1(d_1 y, \eta_2, \eta_3, d_4 \eta_4, \eta_5, \eta_6, d_7 \eta_7, \eta_8; B) > 0 \}
\]
is an interval. To evaluate $W(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)$, we shall apply Corollary 1 and Remark 3 with

\[ k \text{ replaced by } kd_1, \quad b = d_4d_7, \quad r = 4, \]
\[ a_1 = \eta_4, \quad a_2 = \eta_7', \quad a_3 = \eta_2, \quad a_4 = \eta_8, \quad q = \eta_1', \]
\[ t_1 = d_7\eta_6, \quad t_2 = d_4\eta_3, \quad t_3 = d_4d_7\eta_5\eta_6, \quad t_4 = d_4d_7\eta_3\eta_5, \quad u = \eta_5\eta_6, \]
\[ K_1 = L_4/d_4, \quad K_2 = L_7/d_7, \quad K_3 = L_2, \quad K_4 = L_8, \quad Q = 2L_1/d_4, \]
\[ \mathcal{I}(Q^-, Q^+) = \mathcal{I}(Q^- (\eta_4', \eta_7', \eta_2, \eta_8), Q^+(\eta_4', \eta_7', \eta_2, \eta_8)) = (L_1, 2L_1) \cap \{ y > 0 \mid V_1(d_1, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) > 0 \}, \]
\[ V = \sup_{\eta_1 \sim L_1, \eta_2 \sim L_2, \eta_3 \sim L_4, \eta_4 \sim L_7, \eta_5 \sim L_8} V_1(\eta; B), \]
\[ \Phi(\eta_4', \eta_7', \eta_2, \eta_8, y) = \begin{cases} V_1(d_1, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } Q^- < y < Q^+, \\ \lim_{\eta \rightarrow Q^-} V_1(d_2\eta, \eta_2, \eta_3, \eta_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y < Q^-, \\ \lim_{\eta \rightarrow Q^+} V_1(d_2\eta, \eta_2, \eta_3, \eta_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y \geq Q^+. \end{cases} \]

It is easy to check that the so-defined functions $\Phi, Q^-$ and $Q^+$ satisfy the conditions in Section 2.1. Therefore, applying Corollary 1 and Remark 3 gives

\[ W(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) = M(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) \]
\[ + E(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7), \]

where

\[ M(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) = \sum_{\eta_1 \sim L_1} \sum_{\eta_2 \sim L_2} \sum_{\eta_3 \sim L_4} \sum_{\eta_4 \sim L_7} \sum_{\eta_5 \sim L_8} \sum_{d_1 \sim L_1} \sum_{d_4 \sim L_4} \sum_{d_7 \sim L_7} \frac{1}{2^q(kd_1\eta_1')} \]

\[ V_1(d_1\eta_4', \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) \]

\[ E(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) \ll \sup_{\eta_1 \sim L_1} V_1(\eta; B) \cdot \left( \left( L_1L_2L_4L_7L_8 \right)^{7/8 + \varepsilon} d_1^{-3/4} (d_4d_7)^{-7/8} k_1^{1/4} \right) \]

\[ + L_1L_2L_4L_7L_8^{1/2 + 4\varepsilon} d_1^{-1} (d_4d_7)^{-1/2} (\log 4L_4)^{2(1+4\varepsilon)\omega(kd_1)} \times \]

\[ (1 + \varepsilon)^{\omega(d_3) + \omega(d_3)} \]

\[ \sum_{d_1 \leq L_1} \sum_{d_4 \leq 2L_4} \sum_{d_7 \leq L_7} M(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7), \]

\[ \Phi(\eta_4', \eta_7', \eta_2, \eta_8, y) = \begin{cases} V_1(d_1, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } Q^- < y < Q^+, \\ \lim_{\eta \rightarrow Q^-} V_1(d_2\eta, \eta_2, \eta_3, \eta_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y < Q^-, \\ \lim_{\eta \rightarrow Q^+} V_1(d_2\eta, \eta_2, \eta_3, \eta_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y \geq Q^+. \end{cases} \]

Summing these contributions over $k$, $L_i$ and $d_i$, we deduce from (3.8), (3.11), (3.12), (3.13) and (3.14) that

\[ F^+(B) = M^+(B) + E^+(B), \]

where

\[ M^+(B) = \sum_{\eta'' \in Z_{L_0}^{+\alpha}} \varphi''(\eta_3\eta_5\eta_6) \sum_{k \mid \eta_3} \sum_{d_1 \mid \eta_4} \frac{\mu(k)}{kd_1^2} M(\eta'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7), \]
and

\[(3.16) \quad E^+ (B) \ll \sum_{\eta' \in \mathbb{Z}_{>0}} (1 + \varepsilon)^{\omega(\eta_5) + \omega(\eta_6) + \omega(\eta_8)} \sum_{L_1, L_2, L_4, L_7, L_8} \mathbf{L} \sup_{\eta_i \sim L_i} V_1 (\eta; B),\]

where we have set

\[\mathbf{L} = L_1 (L_2 L_4 L_7 L_8)^{8/9} (\log 4 L_1)^{1 + \varepsilon} .\]

Reverting the decompositions of the arithmetic functions in (3.9) and (3.10), combining the \(\eta_1\)-, \(\eta_2\)-, \(\eta_4\)-, \(\eta_7\)- and \(\eta_8\)-ranges, and noting that if \(k | \eta_3\) then the conditions \((\eta_2 \eta_3 \eta_7 \eta_8, k \eta_4) = 1\) and \((k, \eta_2 \eta_4) = 1\) are equivalent, we simplify the main term \(M^+ (B)\) into

\[(3.17) \quad M^+ (B) = \sum_{\eta' \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} \theta'_i (\eta') V_1 (\eta'; B),\]

where \(\theta'_i (\eta')\) is defined in Lemma 5.1.

Finally, we show that \(E^+ (B)\) is an error term. To estimate \(V_1\), an application of [Der09] Lemma 5.1 gives

\[(3.18) \quad V_1 (\eta'; B) \ll \min \left\{ \frac{B^{1/2}}{\eta_1^{1/2} \eta_2^{1/2} \eta_8^{1/2}}, \frac{B}{\eta_9 (0.1, 0.1, 2.1, 2.3, 3.3/2, 3/2)} \right\} \]

\[(3.19) \quad < \frac{B^{2/3}}{[\eta (1/1.2, 1.1, 1.1, 1.1)]} \left( \frac{B}{[\eta (1.1, 2.2, 2.2, 2.1)]} \right),\]

\[(3.20) \quad \ll \frac{B}{[\eta (1.1, 1.1, 1.1, 1.1)]} \left( \frac{B}{[\eta (1.1, 2.2, 2.2, 2.1)]} \right)^{-1/6},\]

where (3.19) is the weighted average of the two parts of (3.18), and (3.20) indicates how the second and third parts of the height condition 3.3 will be used below when summing over \(\eta_6, \eta_7\). Set

\[L' = L_1 (L_2 L_4 L_8)^{8/9} (\log 4 L_1)^{1 + \varepsilon}.\]

Then, starting from (3.16), we see that

\[E^+(B) \ll \sum_{L_1, L_2, L_4, L_7, L_8} \mathbf{L} \sup_{\eta_i \sim L_i} \left( \sum_{\eta_3, \eta_5, \eta_6} \frac{(1 + \varepsilon)^{\omega(\eta_3) + \omega(\eta_5) + \omega(\eta_6)} B^{2/3}}{[\eta (1/1.1, 2.0, 1/6.6, 2.2, 3/2, 3/2, 3/2)]} \right) \]

\[\ll \sum_{L_1, L_2, L_4, L_8} \mathbf{L}' \sup_{\eta_i \sim L_i} \left( \sum_{\eta_3, \eta_5, \eta_6, \eta_7} \frac{(1 + \varepsilon)^{\omega(\eta_3) + \omega(\eta_5) + \omega(\eta_6)} B^{2/3}}{[\eta (1/1.1, 2.0, 1/6.6, 2.2, 3/2, 3/2, 3/2, 3/2, 3/2, 3/2, 3/2)]} \right) \]

\[\ll \sum_{L_1, L_2, L_4, L_8} \mathbf{L}' \sup_{\eta_i \sim L_i} \left( \sum_{\eta_3, \eta_5, \eta_6, \eta_7} \frac{(1 + \varepsilon)^{\omega(\eta_3) + \omega(\eta_5) B (\log B)^\varepsilon}}{[\eta \eta_2 \eta_3 \eta_4 \eta_5 \eta_8]} \right) \]

\[\ll \sum_{L_1, L_2, L_4, L_8} \mathbf{L}' \sup_{\eta_i \sim L_i} \frac{B (\log B)^{2 + 3 \varepsilon}}{[\eta \eta_2 \eta_4 \eta_8]} \]

\[\ll \sum_{L_1, L_2, L_4, L_7} \frac{B (\log B)^{2 + 4 \varepsilon} (\log 4 L_1)}{(L_2 L_4 L_8)^{1/9}} \]

\[\ll B (\log B)^{1 + 4 \varepsilon}.\]

Combining this with (3.15) and (3.17), and treating \(F^- (B)\) similarly as \(F^+ (B)\), we obtain the desired result.
3.5. Completion of the proof of Theorem 1 — For the proof of Theorem 1 it remains to evaluate the main term in Lemma 6 asymptotically. To this end, we would like to apply [Der09, Proposition 4.3]. We note that \( \theta_i(n') \) is not of the form considered in [Der09, Section 7] because of the extra 2-adic factor. However, this factor turns out to be 1 on average, and the remaining part of \( \theta_i(n') \) has the necessary properties. As in [Der09, Definition 3.7], \( A(\theta_i(n'), \eta_h) \) denotes the average size of \( \theta_i \) when summed over \( \eta_h \).

**Lemma 7.** — We have \( \theta_i(n') \in \Theta_{2, s}(C) \) [Der09, Definition 4.2] for some \( C \in \mathbb{R}_{>0} \), with

\[
A(\theta_i(n'), \eta_h) = \Delta_2(\eta_h, \eta_f) = \prod_{p} \theta_2(p, \eta_f) \in \Theta_1(2),
\]

[Der09, Definition 7.8], where \( I_p(\eta_1, \ldots, \eta_f) = \{ i \in \{ 1, \ldots, 7 \} \mid p \mid \eta_i \} \) and

\[
\theta_2, p(I) = \begin{cases} 
1, & I = \emptyset, \\
1 - \frac{1}{p}, & I = \{1\}, \{2\}, \{6\}, \\
1 - \frac{1}{p^2}, & I = \{4\}, \{5\}, \{7\}, \{1, 3\}, \{2, 3\}, \\
\left(1 - \frac{1}{p}\right)^2, & I = \{3, 4\}, \{4, 5\}, \{5, 7\}, \{6, 7\}, \\
(1 - \frac{1}{p})(1 - \frac{2}{p^2}), & I = \{3\}, \\
0, & \text{otherwise}.
\end{cases}
\]

Proof. — We will see that

\[
\sum_{0 < \eta_h \leq t} \theta_i(n') = t\Delta_2(\eta_h, \eta_f) + O(2^{\nu(\eta_f)}(1 + |\mu|) + \omega(\eta_f)),
\]

where

\[
\theta_2(\eta_1, \ldots, \eta_f) = \sum_{k|\eta_1} \prod_{(k, \eta_2, \ldots, \eta_f) = 1} \phi^*(\eta_1 \eta_2 \ldots \eta_f)
\]

if (3.6) holds and \( \theta_2(\eta_1, \ldots, \eta_f) = 0 \) otherwise.

We observe that \( \theta_i(n') \in \Theta_{1, s}(3, \eta_h) \) [Der09, Definition 3.8] since we have \( \theta_i(n') \leq \prod_{i=1}^{\eta_h}(\phi^*(\eta_i))^2 \in \Theta_{0, s}(0) \) [Der09, Definition 3.2] by [Der09, Example 3.3], and because \( \theta_i(n') \) as a function in \( \eta_h \) lies in \( \Theta_{0, s}(0) \) [Der09, Definition 3.7] by (3.21), and because its average is \( \theta_2(\eta_1, \ldots, \eta_f) \leq \prod_{i=1}^{\eta_h}(\phi^*(\eta_i))^2 \in \Theta_{0, s}(0) \) as before, and because the error term is \( \ll \prod_{i=1}^{\eta_h} 4^{\nu(\eta_i)} \in \Theta_{0, s}(3) \) also as in [Der09, Example 3.3].

Furthermore, we see that \( \theta_2(\eta_1, \ldots, \eta_f) \) has the form of [Der09, Definition 7.8], and a computation shows that its local factors \( \theta_2, p \) are as in the statement of the result, so \( \theta_2(\eta_1, \ldots, \eta_f) \in \Theta_1(2) \), and \( \theta_2(\eta_1, \ldots, \eta_f) \in \Theta_{2, s}(C) \) for some \( C \geq 3 \) by [Der09, Corollary 7.9]. In total, this shows \( \theta_i(n') \in \Theta_{2, s}(C) \) [Der09, Definition 4.2].

It remains to prove (3.21). If (3.6) does not hold, both sides are 0. Otherwise,

\[
\sum_{0 < \eta_h \leq t} \theta_i(n') = \sum_{k|\eta_1} \frac{\mu(k)\phi^*(\eta_1 \eta_2 \ldots \eta_f)}{k\phi^*(\eta_1, \eta_2)} \sum_{0 < \eta_h \leq t} \left\{ \frac{-\eta_2 \eta_4 \eta_5 \eta_6 \eta_7}{2^\eta} \right\}.
\]

We must show that the inner sum over \( \eta_h \) is \( t\phi^*(\eta_1 \ldots \eta_f) + O(2^{\nu(\eta_1 \ldots \eta_f)}) \). Let \( n = \min(\nu(k\eta_1), 3) \). If \( n = 0 \), this holds by Möbius inversion. If \( n > 0 \), (3.6) implies that \( \eta_2, \eta_4, \eta_5, \eta_6, \eta_7 \) are odd. Then the inner sum equals (with \( \frac{-\eta_2 \eta_4 \eta_5 \eta_6 \eta_7}{2^n} \) the multiplicative inverse of \( -\eta_2 \eta_4 \eta_5 \eta_6 \eta_7 \mod 2^n \))

\[
\sum_{0 < \eta_h \leq t} 2^{n-1} = \sum_{\eta_h = \eta_2 \eta_4 \eta_5 \eta_6 \eta_7 \mod 2^n} \mu(t) \sum_{0 < \eta_h \leq t/1} 2^{n-1}.
\]
If \( l \) is even, the congruence is never fulfilled, so the inner sum over \( \eta_0 \) is 0. If \( l \) is odd, the inner sum over \( \eta_0 \) is
\[
\sum_{l | \eta_1 \cdots \eta_6} \frac{\mu(l)}{2l} t + O(2^{\omega(\eta_1 \cdots \eta_6 \eta_7)}) = \sum_{l | \eta_1 \cdots \eta_6} \frac{\mu(l)}{l} t + O(2^{\omega(\eta_1 \cdots \eta_6 \eta_7)})
\]
where removing the condition \( l \) odd works by the observation
\[
\frac{\mu(l)}{2l} = \frac{\mu(l)}{l} + \frac{\mu(2l)}{2l}.
\]
Summing the error term over \( k \) only gives another factor \( 2^{\omega(\eta_3)} \).

Because of (3.20) and Lemma 7, we are in the position to apply Der09 Proposition 4.3, giving
\[
\sum_{\eta' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_6} \theta'_1(\eta')V_1(\eta'; B) = c_0 V_0(B) + O(B(\log B)^5(\log \log B)^2)
\]
with
\[
V_0(B) = \int_{\eta'} V_1(\eta'; B) \, d\eta' = \int_{(\eta', \eta_6) \in R(B)} \eta_6^{-1} \, \text{d} \eta_6 \, d\eta'
\]
and
\[
c_0 = A(\theta'_1(\eta'), \eta_8, \ldots, \eta_1) = A(\theta_2(\eta_1, \ldots, \eta_7), \eta_7, \ldots, \eta_1) = \prod_p \omega_p,
\]
whose local factors can be computed from the presentation of \( \theta_2 \) in Lemma 7 by Der09 Corollary 7.10 as
\[
\omega_p = \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right).
\]
Recall the definition (3.1) of \( J'_i \). We define
\[
R'_1(B) = \left\{(\eta_1, \ldots, \eta_6) \in J'_1 \times \cdots \times J'_6 \mid \eta^{(3,2,4,3,2,0,0,0)}_3 \leq B_1, \eta^{(5,3,6,4,2,2,0,0)}_5 \geq B_5\right\},
\]
\[
R'_2(\eta_7, \ldots, \eta_9; B) = \{(\eta_7, \eta_8, \eta_9) \in J_7' \times J_8' \times J_9' \mid b(\eta', \eta_9; B) \leq 1\},
\]
\[
\mathcal{R}'(B) = \left\{(\eta', \eta_9) \in \mathbb{R}^9 \mid (\eta_1, \ldots, \eta_6) \in R'_1(B), (\eta_7, \eta_8, \eta_9) \in R'_2(\eta_1, \ldots, \eta_9; B)\right\},
\]
\[
V'_0(B) = \int_{(\eta', \eta_9) \in \mathcal{R}'(B)} \eta_9^{-1} \, \text{d} \eta_9 \, d\eta',
\]
where the definition of \( \mathcal{R}'_1(B) \) is inspired by the description of the polytope whose volume is \( \omega(S) \) in (3.20).

We claim that
\[
V_0(B) = V'_0(B) + O(B(\log B)^5).
\]
Comparing their definitions, in particular \( J_i \) and \( J'_i \) for \( i \in \{6, 8\} \), we see that we must remove the conditions \( \eta_6 \geq 1 \) and \( |\eta_8| \geq 1 \) and add the two conditions from the definition of \( \mathcal{R}'_1(B) \), all with a sufficiently small error term. We do this in four steps as in Der09 Lemma 8.7; the order is important:

1. Add \( \eta^{(3,2,4,3,2,0,0,0)} \leq B \): This does not change anything because this condition follows from \( \eta_7 \geq 1 \) and \( \eta^{(3,2,4,3,2,0,1,0)} \leq B \) by (3.1).
2. Add \( \eta^{(5,3,6,4,2,-2,0,0)} \geq B \): Using [Der09] Lemma 5.1(3) for the integration over \( \eta_7, \eta_9 \), we see that the error term is
\[
\ll \int \frac{B^{5/6}}{\eta^{(1,6,1,2,0,1,3,2,3,4,3,0,7,6)}} \, d(\eta_1, \ldots, \eta_6, \eta_8).
\]
Using the opposite of our new condition for the integration over \( \eta_6 \) together with \( 1 \leq \eta_1, \ldots, \eta_5 \leq B \) and \( |\eta_6| \geq 1 \), we see that this is \( \ll B(\log B)^5 \).

3. Remove \( |\eta_8| \geq 1 \): Using [Der09] Lemma 5.1(1) for the integration over \( \eta_8 \), we see that the error term is
\[
\ll \int \frac{B^{1/2}}{\eta_1^{1/2} \eta_2^{1/2} |\eta_8|^{1/2}} \, d\eta.
\]
Using \( |\eta_8| \leq 1 \) and \( \eta^{(3,2,4,3,2,0,1,0)} \leq B \) for \( \eta_7 \), and \( \eta^{(5,3,6,4,2,-2,0,0)} \geq B \) for \( \eta_6 \), and finally \( 1 \leq \eta_1, \ldots, \eta_5 \leq B \), we see that this is \( \ll B(\log B)^5 \).

4. Remove \( \eta_7 \geq 1 \): Using [Der09] Lemma 5.1(2) for the integration over \( \eta_8, \eta_9 \), we see that the error is
\[
\ll \int \frac{B^{3/4}}{\eta^{(4,1,2,0,1,4,1,2,1,3,3,4,0)}} \, d(\eta_1, \ldots, \eta_7).
\]
Using \( 0 \leq \eta_7 \leq 1 \) and \( \eta^{(3,2,4,3,2,0,0,0)} \leq B \) for \( \eta_5 \) with \( 1 \leq \eta_1, \ldots, \eta_4, \eta_6 \leq B \), we see that this is \( \ll B(\log B)^5 \).

Next, we claim as in [Der09] Lemma 8.6] that
\[
(3.25) \quad V_0'(B) = \alpha(S) \omega_\infty B(\log B)^6.
\]
Indeed, substituting
\[
x_2 = B^{-1} \eta^{(3,2,4,3,2,0,1,0)}, \quad x_1 = B^{-1} \eta^{(1,1,2,2,2,2,1)}, \quad x_3 = B^{-1} \eta^{(0,1,1,1,1,1,1,1)} \eta_9
\]
to \( \omega_\infty \) as in Theorem [1] where \( \eta_1, \ldots, \eta_6 \) should be regarded as parameters and \( \eta_7, \eta_8, \eta_9 \) as the new integration variables, we see that
\[
\frac{B \omega_\infty}{\eta_1 \cdots \eta_6} = \int_{(\eta_7, \eta_8, \eta_9) \in \mathcal{R}_3(\eta_1, \ldots, \eta_6; B)} \eta_1^{-1} \, d(\eta_7, \eta_8, \eta_9).
\]
Finally, we see that
\[
\alpha(S)(\log B)^6 = \int_{\mathcal{R}_3(\eta_7, \eta_8, \eta_9)} \frac{1}{\eta_1 \cdots \eta_6} \, d(\eta_1, \ldots, \eta_6)
\]
by substituting \( \eta_i = B^{\nu_i} \) into \( \alpha(S) = \text{vol}(P') = \int_{t \in P'} dt \) (see (3.20) below).

Combining Lemma [6] with (3.22), (3.23), (3.24) and (3.25), completes the proof of Theorem [1].

3.6. Compatibility with Manin’s conjecture. — As the rank of \( \text{Pic}(\tilde{S}) \) is equal to 7 (see Section [3.1]), the exponent of \( \log B \) in Theorem [1] is as predicted by Manin’s conjecture. By [Pey95, BT98b], we have conjecturally \( e_{S,H} = \alpha(S) \cdot \omega_H(S) \).

We have
\[
\alpha(S) = \frac{\alpha(S_0)}{\#W(A_5) \cdot \#W(A_1)} = \frac{1}{180 \cdot 6! \cdot 2!} = \frac{1}{172800}
\]
by [Der07] Table [1] and [DJT08] Theorem [1.3], where \( S_0 \) is a split smooth cubic surface. Since
\[
[K_\tilde{S}] = [3E_1 + 2E_2 + 4E_3 + 3E_4 + 2E_5 + E_7],
\]
\[
[K_\tilde{S}] = [2E_1 + E_2 + 2E_3 + E_4 - 2E_6 - E_7],
\]
we also have $\alpha(S) = \text{vol}(P) = \text{vol}(P')$, where
\[
P = \left\{ (t_1, \ldots, t_7) \in \mathbb{R}_{\geq 0}^7 \mid \begin{array}{l}
3t_1 + 2t_2 + 4t_3 + 3t_4 + 2t_5 + t_7 = 1, \\
2t_1 + t_2 + 2t_3 + t_4 - 2t_6 - t_7 \geq 0
\end{array} \right\}
(3.26)
\]
\[
\cong P' = \left\{ (t_1, \ldots, t_6) \in \mathbb{R}_{\geq 0}^6 \mid \begin{array}{l}
3t_1 + 2t_2 + 4t_3 + 3t_4 + 2t_5 \leq 1, \\
5t_1 + 3t_2 + 6t_3 + 4t_4 + 2t_5 - 2t_6 \geq 1
\end{array} \right\}.
\]
Furthermore,
\[
\omega_H(S) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^7 \omega_p,
\]
where
\[
\omega_p = \frac{\# S(F_p)}{p^2} = 1 + \frac{7}{p} + \frac{1}{p^2}
\]
because the minimal desingularization $\tilde{S}$ of $S$ is a blow-up of $\mathbb{P}^2$ (which has $p^2 + p + 1$ points over $F_p$) in six points (each replacing one point by an exceptional divisor containing $\# \mathbb{P}^1(F_p) = p + 1$ points over $F_p$).

We check using the techniques of [Pey95, BT98b] that $\omega_\infty$ is as in Theorem 1 since the Leray form of $\tilde{S}$ is
\[
\omega_L(\tilde{S}) = (x_1 x_2)^{-1} \, dx_1 \, dx_2 \, dx_3
\]
(where $x_1 x_2$ is the derivative of (1.1) with respect to $x_0$) and by writing $x_0$ in terms of $x_1, x_2, x_3$ using the defining equation (1.1).

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