On saturation of Berge hypergraphs

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Abstract
A hypergraph $H = (V(H), E(H))$ is a Berge copy of a graph $F$, if $V(F) \subset V(H)$ and there is a bijection $f : E(F) \rightarrow E(H)$ such that for any $e \in E(F)$ we have $e \subset f(e)$. A hypergraph is Berge-$F$-free if it does not contain any Berge copies of $F$. We address the saturation problem concerning Berge-$F$-free hypergraphs, i.e., what is the minimum number $\text{sat}_r(n, F)$ of hyperedges in an $r$-uniform Berge-$F$-free hypergraph $H$ with the property that adding any new hyperedge to $H$ creates a Berge copy of $F$. We prove that $\text{sat}_r(n, F)$ grows linearly in $n$ if $F$ is either complete multipartite or it possesses the following property: if $d_1 \leq d_2 \leq \cdots \leq d_{|V(F)|}$ is the degree sequence of $F$, then $F$ contains two adjacent vertices $u, v$ with $d_F(u) = d_1$, $d_F(v) = d_2$. In particular, the Berge-saturation number of regular graphs grows linearly in $n$.

1 Introduction

Given a family $\mathcal{F}$ of (hyper)graphs, we say that a (hyper)graph $G$ is $\mathcal{F}$-free if $G$ does not contain any member of $\mathcal{F}$ as a subhypergraph. The obvious question is how large an $\mathcal{F}$-free (hyper)graph can be, i.e. what is the maximum number $\text{ex}(n, \mathcal{F})$ of (hyper)edges in an $\mathcal{F}$-free $n$-vertex (hyper)graph is called the extremal/Turán problem. A natural counterpart to this well-studied problem is the so-called saturation problem. We say that $G$ is $\mathcal{F}$-saturated if $G$ is $\mathcal{F}$-free, but adding any (hyper)edge to $G$ creates a member of $\mathcal{F}$. The question is how small an $\mathcal{F}$-saturated (hyper)graph can be, i.e. what is the minimum number $\text{sat}(n, \mathcal{F})$ of (hyper)edges in an $\mathcal{F}$-saturated $n$-vertex (hyper)graph.

In the graph case, the study of saturation number was initiated by Erdős, Hajnal, and Moon \cite{erdos1964}. Their theorem on complete graphs was generalized to complete uniform hypergraphs by Bollobás \cite{bollobas1979}. Kászonyi and Tuza \cite{kaszonyi1991} showed that for any family $\mathcal{F}$ of graphs, we have $\text{sat}(n, \mathcal{F}) = O(n)$. For hypergraphs, Pikhurko \cite{pikhurko2012} proved the analogous result that for any family $\mathcal{F}$ of $r$-uniform hypergraphs, he proved that we have $\text{sat}(n, \mathcal{F}) = O(n^{r-1})$. \vspace{1cm}

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For some further types of saturation (“strongly $F$-saturated” and “weakly $F$-saturated” hypergraphs) the exact exponent of $n$ was determined in [11] for every forbidden hypergraph $F$.

In this paper, we consider some special families of hypergraphs. We say that a hypergraph $H$ is a Berge copy of a graph $F$ (in short: $H$ is a Berge-$F$) if $V(F) \subset V(H)$ and there is a bijection $f : E(F) \to E(H)$ such that for any $e \in E(F)$ we have $e \subset f(e)$. We say that $F$ is a core graph of $H$. Note that there might be multiple core graphs of a Berge-$F$ hypergraph and $F$ might be the core graph of multiple Berge-$F$ hypergraphs.

Berge hypergraphs were introduced by Gerbner and Palmer [8], extending the notion of hypergraph cycles in Berge’s definition [1]. They studied the largest number of hyperedges in Berge-$F$-free hypergraphs (and also the largest total size, i.e. the sum of the sizes of the hyperedges). English, Graber, Kirkpatrick, Methuku and Sullivan [3] considered the saturation problem for Berge hypergraphs. They conjectured that $\text{sat}_r(n, \text{Berge-} F) = O(n)$ holds for any $r$ and $F$, and proved it for several classes of graphs. Here and throughout the paper the parameter $r$ in the index denotes that we consider $r$-uniform hypergraphs, and we will denote $\text{sat}_r(n, \text{Berge-} F)$ by $\text{sat}_r(n, F)$ for brevity. The conjecture was proved for $3 \leq r \leq 5$ and any $F$ in [4]. In this paper we gather some further results that support the conjecture.

English, Gerbner, Methuku and Tait [4] extended this conjecture to hypergraph-based Berge hypergraphs. Analogously to the graph-based case, we say that a hypergraph $H$ is a Berge copy of a hypergraph $F$ (in short: $H$ is a Berge-$F$) if $V(F) \subset V(H)$ and there is a bijection $f : E(F) \to E(H)$ such that for any $e \in E(F)$ we have $e \subset f(e)$. We say that $F$ is a core hypergraph of $H$. The conjecture in this case states that if $F$ is a $u$-uniform hypergraph, then $\text{sat}_r(n, F) = O(n^{u-1})$.

For a hypergraph $H = (V(H), E(H))$ and a family of hypergraphs $\mathcal{F}$ we say that $H$ is $\mathcal{F}$-oversaturated if for any hyperedge $h \subset V(H)$ that is not in $H$, there is a copy of a hypergraph $F \in \mathcal{F}$ that consists of $h$ and $|E(F)| - 1$ hyperedges in $E(H)$. Let $\text{osat}_r(n, \mathcal{F})$ denote the smallest number of hyperedges in an $\mathcal{F}$-oversaturated $r$-uniform hypergraph on $n$ vertices.

**Proposition 1.1.** For any $u$-uniform hypergraph $F$ and any $r > u$, we have $\text{osat}_r(n, F) = O(n^{u-1})$. Moreover, there is an $r$-uniform hypergraph $H$ with $O(n^{u-1})$ hyperedges such that adding any hyperedge to $H$ creates a Berge-$F$ such that its core hypergraph $F_0$ (which is a copy of $F$) is not a core hypergraph of any Berge-$F$ in $H$.

We remark that in the case $u = 2$, the linearity of $\text{osat}_r(n, \mathcal{F})$ follows from either of the next two theorems, as they imply $\text{sat}_r(n, K_k) = O(n)$. Indeed, if $v$ is the number of vertices of any graph in $\mathcal{F}$, then any Berge-$K_v$-saturated hypergraph is obviously Berge-$\mathcal{F}$-oversaturated.
Theorem 1.2. For any \( r, s \geq 2 \) and any sequence of integers \( 1 \leq k_1 \leq k_2 \leq \cdots \leq k_{s+1} \) we have
\[
\text{sat}_r(n, K_{k_1, k_2, \ldots, k_{s+1}}) = O(n).
\]

Let \( F \) be a fixed graph on \( v \) vertices with degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_v \). Set \( \delta := d_2 - 1 \). We say that \( F \) is of type I if there exist vertices \( u_1, u_2 \) with \( d_F(u_1) = d_1, d_F(u_2) = d_2 \) that are joined with an edge. Otherwise \( F \) is called of type II. Observe that any regular graph is of type I.

**Theorem 1.3.** For any graph \( F \) of type I and any \( r \geq 3 \) we have \( \text{sat}_r(n, F) = O(n) \).

2 Proofs

**Proof of Proposition** \[1.4\] We define \( V(H) \) as the disjoint union of a set \( R \) of size \( r - u \) and a set \( L \) of size \( n - r + u \). We take an \( F \)-saturated \( u \)-uniform hypergraph \( G \) with vertex set \( L \) that contains \( O(n^{u-1}) \) hyperedges. Such a hypergraph exists by the celebrated result of Pikhurko \[10\]. As another alternative, one may take an oversaturated hypergraph with \( O(n^{u-1}) \) hyperedges, whose existence is guaranteed by \[11\], Theorem 1. Then we let the hyperedges \( h \) of \( H \) be the \( r \)-sets with the property that \( h \cap L \) is a hyperedge of \( G \) or has at most \( u - 1 \) vertices from \( L \).

Obviously \( H \) has \( O(n^{u-1}) \) hyperedges. Clearly we have that every \( r \)-set \( h \) that is not a hyperedge of \( H \) contains a \( u \)-element subset \( e \) of \( L \) that is not a hyperedge of \( G \). Then \( e \) creates a copy of \( F \). We let \( f(e) = h \), and for each other edge \( e' \) of that copy of \( F \), we let \( f(e') = e' \cup R \). This shows that this copy of \( F \) is the core of a Berge-\( F \). \( \square \)

**Proof of Theorem** \[1.5\] We consider two cases according to how large the uniformity \( r \) is compared to the sum of class sizes \( k_1, k_2, \ldots, k_{s+1} \). We set \( N := \sum_{i=1}^s k_i - 1 \). For brevity, we write \( K \) for \( K_{k_1, k_2, \ldots, k_{s+1}} \).

**Case I.** \( r \leq \sum_{i=1}^{s+1} k_i - 3 \)

Let \( (C, B_1, B_2, \ldots, B_m, R) \) be a partition of \( [n] \) with \( |C| = N \), \( |B_i| = k_{s+1} \) for all \( i \leq m \) where \( m = \lceil \frac{n-N}{k_{s+1}} \rceil \), and \( |R| \equiv n - N \pmod{k_{s+1}} \), \( |R| < k_{s+1} \). Consider the family \( \mathcal{G} = \{ A \in \binom{[n]}{k_i} : A \subset C \cup B_i \text{ for some } i \} \). Observe that \( \mathcal{G} \) is Berge-\( K \)-free. Indeed, a copy of a Berge-\( K \) must contain a vertex \( v \) in the smallest \( s \) classes of the core from outside \( C \). But then, if \( v \in B_i \), either the whole copy is in \( C \cup B_i \) or \( C \) must contain all classes of the core of the copy. As none of these are possible, \( \mathcal{G} \) is indeed Berge-\( K \)-free.

Next observe that adding any \( r \)-set \( G \) to \( \mathcal{G} \) that contains two vertices \( u \) and \( v \) from different \( B_i \)s, say \( u \in B_i, v \in B_j \) (\( i \neq j \)), would create a copy of a Berge-\( K \). Indeed, the assumption \( r \leq \sum_{i=1}^{s+1} k_i - 3 \) ensures that there exist bijections \( f_i : \binom{C \cup B_i}{2} \to \binom{C \cup B_j}{r} \) with \( e \in f(e) \). Then vertices of \( C \) and \( u \) can play the role of the \( s \) smallest classes of \( K \), and \( \{v\} \cup B_i \setminus \{u\} \) can play the role of the largest class of \( K \).
This shows that the additional hyperedges of any \( K \)-saturated family that contains \( G \) are subsets of \( C \cup B_1 \cup R \) for some \( i \), and hence there is only a linear number of them. As \( G \) also contains a linear number of hyperedges, the total size of such \( K \)-saturated families is \( O(n) \).

**Case II.** \( r \geq \sum_{i=1}^{s+1} k_i - 2 \)

Once again, we define a partition \( (C, B_1, B_2, \ldots, B_m, R) \) of \( [n] \) with \( |C| = N \), but now with \( |B_i| = r - 2 \) and \( |R| \equiv n - N \pmod{r - 2} \), \( |R| < r - 2 \). Let \( x_1, x_2, \ldots, x_N \) be the elements of \( C \), let \( e_1, e_2, \ldots, e_{\binom{N}{2}} \) be the edges of the complete graph on \( C \), and finally let \( \pi_1, \pi_2, \ldots, \pi_{\binom{N}{2}} \) be permutations of \( C \) such that the endvertices of \( e_i \) are the values \( \pi_i(j), \pi_i(j+1) \) for some \( j \).

Then let us define the family \( G \) as

\[
G = \bigcup_{i=1}^{m} \{ \{ x_{\pi_i(j)}, x_{\pi_i(j+1)} \} \cup B_i : 1 \leq j \leq N, \ h \equiv i \pmod{\binom{N}{2}} \}.
\]

First, we claim that \( G \) is Berge-\( K \)-free. Indeed, there are only \( N \) vertices with degree at least \( N + 1 \).

Next, observe that if we add a family \( F \) to \( G \) that contains a Berge-\( S_{k+1}(r-2) \) (a star with \( k+1(r-2) \) leaves) with core completely in \( \bigcup_{i=1}^{m} B_i \), then \( G \cup F \) contains a Berge-\( K \). Indeed, if \( v \) is the center of the star, then \( C \cup \{ v \} \) plays the role of the smallest classes of \( K \), and \( k+1 \) leaves that belong to distinct \( B_i \)'s can play the role of the largest class of \( K \). Here, we use the facts that every edge in \( \binom{C}{2} \) is contained in an unbounded number of hyperedges of \( G \) as \( n \) tends to infinity and that for any vertex \( u \in \bigcup_{i=1}^{m} B_i \), \( G \) contains a Berge-star with center \( u \) and core \( C \cup \{ u \} \); and if \( u \) and \( u' \) belong to different \( B_i \)'s, then the hyperedges of these Berge-stars are distinct.

Let \( F \) be such that \( F \cup G \) is Berge-\( K \)-free. Then by the above, \( F' = \{ F \cap (\bigcup_{i=1}^{m} B_i) : F \in F \} \) is Berge-\( S_{k+1}(r-2) \)-free. Note that \( |F'| \leq 2^{\lceil C \cup R \rceil} |F| \), thus showing that \( |F'| = O(n) \) finishes the proof. It is well-known that forbidding a Berge-star (or any Berge-tree) results in \( O(n) \) hyperedges, but for sake of completeness we include a proof for stars.

Observe that \( F' \) being Berge-\( S_{k+1}(r-2) \)-free is equivalent to the condition that for every \( x \in \bigcup_{i=1}^{m} B_i \), the family \( \{ F \setminus \{ x \} : x \in F \in F' \} \) is not disjointly \( k+1(r-2) \)-representable, i.e. there do not exist \( y_1, y_2, \ldots, y_{k+1(r-2)} \) and sets \( F_1, F_2, \ldots, F_{k+1(r-2)} \in F' \) with \( y_a \in F_{x, y_a, \ldots, j-1 \neq \alpha} \cup F_{\alpha} \setminus \bigcup_{j=1, j \neq \alpha}^{k+1(r-2)} F' \). By a well-known result of Frankl and Pach [7] if all sets of a family \( \mathcal{H} \) with this property have size at most \( r \), then \( |\mathcal{H}| \) is bounded by a constant depending only on \( r \) and \( k+1(r-2) \). That is, \( |F'_{x, y_a, \ldots, j-1 \neq \alpha}| \) is bounded by the same constant independently of \( x \), and therefore the size of \( F' \), and thus the size of \( G \) is linear. We obtained that any \( k \)-saturated family \( G' \) with \( G \subset G' \) has \( O(n) \) hyperedges.

**Proposition 2.1.** Let \( F \) be a graph with no isolated vertex and with an isolated edge \( (u_1, u_2) \). Then for any \( r \geq 3 \) we have \( \operatorname{sat}_r(n, F) = O(n) \).

**Proof.** Let \( U \) be a set of size \( n \). Let \( v \) denote the number of vertices of \( F \), let \( F' \) be the graph obtained from \( F \) by removing the edge \( (u_1, u_2) \) and let \( C \) be a \( (v-2) \)-subset of \( U \). Suppose
first $r \leq v - 1$, and let $G_0$ be a Berge copy of $F'$ with core $C$ and $G_0 \subseteq G_{C,1} \subseteq \binom{V}{r}$, where $G_{C,1}$ is the set of $r$-sets that contain at most one vertex from $U \setminus C$. Note that $G_{C,1}$ contains a linear number of $r$-subsets. Then let $G$ be an $r$-graph with $G_0 \subseteq G \subseteq G_{C,1}$ such that any $H \in G_{C,1} \setminus G$ creates a Berge copy of $F$ with $G$. Then $G$ has linearly many hyperedges and is clearly $F$-saturated since if $G$ contains at least two vertices from $U \setminus C$, then $G$ can play the role of $(u_1, u_2)$ and together with the Berge copy of $F'$ they form a Berge-$F$.

If $r \geq v$, then any $G$ with $e(F) - 1$ $r$-subsets sharing $v - 2$ common elements (denote their set by $C$) is $F$-saturated. Indeed, any additional $r$-set $G$ contains at least 2 vertices not in $C$, so those two vertices can play the role of $u_1$ and $u_2$, $G$ can play the role of the edge $(u_1, u_2)$, and the $r$-sets of $G$ form a Berge copy of $F'$ with core $C$.

Observe that if $F$ is of type I, then it cannot contain isolated vertices, and since graphs with an isolated edge are covered by Proposition 2.1, we may and will assume that $d_2 - 1 = \delta \geq 1$ holds.

Proof of Theorem 1.3. Let $F$ be a graph of type I on $v$ vertices and let $u_1, u_2$ be a pair of vertices of $F$ showing the type I property. Set $d := |N(u_1) \cap N(u_2)|$ and let $F'$ denote the subgraph of $F$ on $N(u_1) \cap N(u_2)$ spanned by the edges incident to $u_1$ or $u_2$ with the edge $(u_1, u_2)$ removed. Our strategy to prove the theorem is to construct a Berge-$F$-free $r$-graph $G$ with $O(n)$ hyperedges such that any $F$-saturated $r$-graph $G' \supseteq G$ contains at most a linear number of extra hyperedges.

Let us say that $G$ is $F$-good if its vertex set $V$ can be partitioned into $V = C \cup B_1 \cup B_2 \cup \cdots \cup B_m \cup R$ such that $|C| = v - 2$, all $B_i$’s have equal size $b$ at most $r$, $|R| < b$ and the following properties hold:

1. every hyperedge of $G$ not contained in $C$ is of the form $A \cup B_i$ for some $i = 1, 2, \ldots, m$ with $A \subset C$,
2. every vertex $u \in \cup_{i=1}^m B_i$ has degree $\delta$ in $G$,
3. for every $1 \leq i < j \leq m$ and $y \in B_i, y' \in B_j$, the sub-$r$-graph $\{G \in G : y \in G\} \cup \{G \in G : y' \in G\}$ contains a Berge-$F'$ with $y, y'$ being the only vertices of the core not in $C$ and $y, y'$ playing the role of $u_1, u_2$,
4. for any $1 \leq i < j \leq m$ there exist $(v-2)_r$ hyperedges $G_1, G_2, \ldots, G_{(v-2)_r} \in G$ that are disjoint from $B_i \cup B_j$ and if $e_1, e_2, \ldots, e_{(v-2)_r}$ is an enumeration of the edges of the complete graph on $C$, then $e_h \subset G_h$ for all $h = 1, 2, \ldots, (v-2)_r$, i.e., these hyperedges form a Berge-$K_{v-2}$ with core $C$.

Claim 2.2. If $G$ is $F$-good, then $G$ is Berge-$F$-free and any $F$-saturated supergraph $G'$ of $G$ contains at most a linear number of extra edges compared to $G$. 

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Proof of Claim. Observe first that \( G \) is Berge-F-free as the core of a copy of a Berge-F should contain at least two vertices not in \( C \), both of degree \( \delta < d_2 \).

Next, we claim that for any hyperedge \( H \) meeting two distinct \( B \)'s, say \( B_i \) and \( B_j \), the \( r \)-graph \( G \cup \{ H \} \) contains a Berge-F. Indeed, let \( y \in B_i \cap H, y' \in B_j \cap H \). Then by item 3 of the \( F \)-good property, \( y \) can play the role of \( u_1 \) and \( y' \) can play the role of \( u_2 \). \( H \) can play the role of the edge \((u_1u_2)\), and item 4 of the \( F \)-good property ensures that the other vertices of \( C \) can play the role of the rest of the core of \( F \).

Finally, let \( G' \) be any \( F \)-saturated \( r \)-graph containing \( G \). Then by the above, any hyperedge in \( G' \setminus G \) meets at most one \( B_i \), and thus is of the form \( P \cup Q \) with \( P \subset C \cup R, Q \subset B_i \) for some \( i \). The number of such sets is at most \( 2^b 2^{v-2+b} m = O(n) \).  

Claim 2.3. For any type I graph \( F \) on \( v \geq 7 \) vertices with \( \delta > 0 \) and any integer \( r \geq 6 \) there exists an \( F \)-good \( r \)-graph \( G \) with \( O(n) \) hyperedges.

Proof of Claim. We fix a set \( D \subset C \) of size \( d \).

Case I. \( r \leq v-4 \)

Then putting all \( r \)-subsets of \( C \) into \( G \) ensures item 4 of the \( F \)-good property. We set \( b = r - 2 \), so all further sets will meet \( C \) in \( 2 \) vertices. Observe that \( d_1 - 1 - d + \delta - d_2 \leq v - 2 - d \) is equivalent to \( 2(d_1 - 1 - d) + \delta - d_1 + 1 \leq v - 2 - d \). Let \( D = \{ x_1, x_2, \ldots, x_d \} \), and \( y_1, 1, y_2, 1, y_2, 2, \ldots, y_{d_1-1-d_1}, y_{d_1-1-d_2}, z_1, z_2, \ldots, z_{\delta-d_1+1} \) be distinct vertices in \( C \). Note that \( d_1 - 1 \leq \delta \) implies that \( \delta - d_1 + 1 \) is non-negative. Then let \( G_0 := \{ \{ x_\ell, x_{\ell+1} \} : 1 \leq \ell \leq d \} \cup \{ \{ y_{\ell,1}, y_{\ell,2} \} : 1 \leq \ell \leq d_1 - 1 - d \} \cup \{ \{ z_\ell, z_{\ell+1} \} : 1 \leq \ell \leq \delta - d_1 + 1 \} \), where addition is always modulo the underlying set, i.e., \( G_0 \) consists of two cycles and a matching. Let us put all sets of the form \( A \cup B_h \) with \( A \in G_0 \) and \( 1 \leq h \leq m \) into \( G \). Then items 1 and 2 of the \( F \)-good property are satisfied, thus we need to check item 3.

For any \( 1 \leq i < j \leq m, D \) plays the role of \( N(u_1) \cap N(u_2) \) and the hyperedges \( B_j \cup \{ x_\ell, x_{\ell+1} \}, B_j \cup \{ x_\ell, x_{\ell+1} \} \) play the role of the edges connecting \( u_1, u_2 \) to vertices of \( D \), respectively. Vertices \( y_{\ell,1} \) play the role of vertices in \( N(u_1) \setminus (N(u_2) \cup \{ u_2 \}) \), while vertices \( y_{\ell,2} \) with \( \ell = 1, 2, \ldots, d_1 - 1 - d \) and \( z_\ell \) with \( \ell' = 1, 2, \ldots, \delta - d_1 + 1 \) play the role of \( N(u_2) \setminus (N(u_1) \cup \{ u_1 \}) \). The use of hyperedges as edges is straightforward.

Case II. \( r > v-4 \)

Then we set \( b = r - (v-4) \) and thus every hyperedge meets \( C \) in \( c := v - 4 = |C| - 2 \) vertices. Consequently, \( |R| \) is the residue of \( n - v + 2 \) modulo \( b \). By \( v \geq 7 \), we obtain \( c \geq 3 \). Let \( e_1, e_2, \ldots, e_{\binom{n-2}{2}} \) be an enumeration of the edges of the complete graph on \( C \). Then for any \( 1 \leq h \leq m \), we will put a hyperedge of the form \( A_{1,h} \cup B_h \) with \( e_\alpha \subset A_{1,h} \subset C \) where \( \alpha \equiv h (mod(\binom{n-2}{2})) \). As \( n \) tends to infinity, so does \( m \), and this will ensure item 4 of the \( F \)-good property.

Suppose first \( d > 0 \) and observe that \( (d_1 - 1) + \delta \leq v - 2 - d \), as \( d_1 - 1 \) is the size of \( N(u_1) \setminus (N(u_2) \cup \{ u_2 \}) \) and \( \delta \) is the size of \( N(u_2) \setminus (N(u_1) \cup \{ u_1 \}) \). For any \( 1 \leq h \leq m \),
we define \(A_{1,h}, A_{2,h}, \ldots, A_{d,h}\) and put \(A_{\ell,h} \cup B_h\) into \(G\) for all \(1 \leq \ell \leq \delta\) as follows. Let \(x_1, x_2, \ldots, x_d\) be the elements of \(D\), and \(A_{1,h}\) be a \((v-4)\)-element set containing \(x_1\) and \(e_\alpha\) (with \(\alpha\) defined in the previous paragraph), and for \(2 \leq \ell \leq d\) let \(A_{\ell,h}\) be an arbitrary \((v-4)\)-element subset of \(C\) containing \(x_1, x_\ell\). (We need \(v-4 \geq 3\) to be able to make the choice of \(A_{1,h}\).) Finally, let \(A_{d+1,h}, A_{d+2,h}, \ldots, A_{d,h}\) be distinct \((v-4)\)-element subsets of \(C \setminus \{x_1\}\). There are \(v-3\) such subsets, each missing one element of \(C \setminus \{x_1\}\). We take them one by one, starting with those that miss an element from \(D \setminus \{x_1\}\). The choice of \(A_{1,h}\) verifies item 4 of the \(F\)-good property and items 1 and 2 hold by definition.

To see item 3, let \(1 \leq i < j \leq m\). We need to create a copy of a Berge-\(F\). Vertices of \(D\) will play the role of \(N(u_1) \cap N(u_2), A_{1,i} \cup B_i, A_{2,i} \cup B_i, \ldots, A_{d,i} \cup B_i\) will play the role of the edges connecting \(u_1\) to all the vertices of \(D\) and similarly \(A_{1,j} \cup B_j, A_{2,j} \cup B_j, \ldots, A_{d,j} \cup B_j\) will play the role of the edges connecting \(u_2\) to all the vertices of \(D\). To finish the Berge copy of \(F\) we will connect both \(u_1\) and \(u_2\) to all the vertices in \(C \setminus D\) (thus in fact we present a Berge copy of \(K_{2,v-2}\), which clearly contains \(F\)).

We will use the hyperedges \(A_{i,d+1}, A_{i,d+2}, \ldots, A_{i,d+1}, A_{i,d+1}\) to connect \(u_1\) to the vertices in \(C \setminus D\). As they each contain \(u_1\), it is enough to show an injection \(f\) from \(A_{1,d+1}, A_{1,d+2}, \ldots, A_{i,d+1}, A_{i,d+1}, A_{j,d+2}, \ldots, A_{j,d+1}, A_{j,d+1}\) to \(C \setminus D\) such that \(f(H) \in H\) for all sets. All we need to check is whether Hall’s condition holds: as for any two distinct sets, their union contains \(C \setminus D\), the only problem can occur if \(A_{i,d} \cap (C \setminus D) = A_{i,d} \cap (C \setminus D) \neq C \setminus D\) and \(|C \setminus D| = 1\) or \(2\). But then by \(v-2 \geq 5\), we have \(d \geq 3\) and thus all choices of \(A_{i,d}, A_{j,d}\) contain \(C \setminus D\) by the assumption that we picked those such subsets first that miss another element of \(D\) apart from \(x_1\).

Suppose next \(d = 0\). Then for any \(1 \leq h \leq m\) let us fix \(\pi_h\), a permutation \(z_1, z_2, \ldots, z_{v-2}\) of vertices of \(C\) with \(z_1, z_2\) being the endvertices of the edge \(e_\alpha\). Now let \(A_{1,h}, A_{2,h}, \ldots, A_{d,h}\) be cyclic intervals of length \(v-4\) of \(\pi_h\) with \(e_\alpha \subset A_{1,h}\). Then putting the sets of the form \(A_{\ell,h} \cup B_h\) to \(G\) will satisfy items 1 and 2 by definition, item 4 by the choice of \(A_{1,h}\), and item 3 by a similar Hall-condition reasoning as in the case of \(d > 0\).

Now we are ready to prove the theorem. If \(\delta = 0\), then \(F\) contains an isolated edge, and we are done by Proposition 2.1. Otherwise by Claim 2.3 there exists an \(F\)-good hypergraph \(G\) with \(O(n)\) hyperedges, and by Claim 2.2 any \(F\)-saturated extension of \(G\) has a linear number of hyperedges.

\section{Concluding remarks}

For any graph \(F\), integer \(r \geq 2\) and enumeration \(\pi: G_1, G_2, \ldots, G_{\binom{n}{r}}\) of \(\binom{n}{r}\) we can define a greedy algorithm that outputs a Berge-\(F\)-saturated \(r\)-uniform hypergraph \(G\) as follows: we let \(G_0 = \emptyset\), and then for any \(i = 1, 2, \ldots, \binom{n}{r}\) we let \(G_i = G_{i-1} \cup \{G_i\}\) if \(G_{i-1} \cup \{G_i\}\) is Berge-\(F\)-free, and \(G_i = G_{i-1}\) otherwise. Clearly, \(G_{\pi,r} = G_{\binom{n}{r}}\) is Berge-\(F\)-saturated.
Proof. Observe that no matter what $\delta$ at least Proposition 3.2. Suppose $e \in G$ the hyperedges of $G$ is a star with $e$. Let $B$ with $1 \leq i \leq j \leq m$, then $\{G\} \cup G_0$ contains a Berge-$F$. Indeed, $G$ can play the role of the cut-edge with $u_i$ and $u_j$ as its two endpoints. We need to show that $G_0$, contains a Berge-$F$, with $u$ played by $u_i$. (The proof for $G_{0,j}$ containing a Berge-$F$, with $w$ played by $u_j$ is identical.) The graph $F_u$ contains at most $e(F) - 1$ edges. We need to verify Hall’s condition in the auxiliary bipartite graph $B$ with one part the edges of $F_u$ and the other part the hyperedges of $G_{0,i}$ and an edge $e$ is connected to a hyperedge $G$ if and only if $e \in G$. Note that the degree of any edge $e$ is at least $e(F_u) - 2$ and if $e_1$ and $e_2$ are disjoint edges of $F_u$, then in $B$ their neighborhood is $G_{0,i}$. Therefore the only problem that can occur is if $F_u$ is a star with $e(F) - 1$ leaves. If the center of $F_u$ is $u$, then $F$ is also a star, contradicting our assumption. If the center $c$ of $F_u$ is not $u$, then a vertex $u' \in B_i$ that is contained in all hyperedges of $G_{0,i}$ can play the role of $c$. \qed

Proposition 3.2. Suppose $F$ is connected and contains a cut-edge $u,w$. Then $r \geq e(F)$ implies $sat_r(n,F) = O(n)$.

Proof. It is known that the saturation number of stars is linear, so we can assume that $F$ is not a star. Let $F_u$ denote the component of $u$ in $F \setminus \{(u,w)\}$ and $F_w$ denote the component of $w$ in $F \setminus \{(u,w)\}$. Let us consider a partition $B_0, B_1, B_2, \ldots, B_m$ of an $n$-element set $U$ with $|B_i| = |B_1| = \cdots = |B_m| = r + 1$ and $|B_0| \leq r$. For $i = 1, 2, \ldots, m$, let $G_{0,i}$ consist of $e(F) - 1$ $r$-subsets of $B_i$ and let $G_0 = \cup_{i=1}^m G_{0,i}$. Clearly, $G_0$ is Berge-$F$-free as its components contain $e(F) - 1$ hyperedges. We claim that if $G$ contains vertices $u_i \in B_i$ and $u_j \in B_j$ with $1 \leq i < j \leq m$, then $\{G\} \cup G_0$ contains a Berge-$F$. Indeed, $G$ can play the role of the cut-edge with $u_i$ and $u_j$ as its two endpoints. We need to show that $G_0$, contains a Berge-$F$, with $u$ played by $u_i$. (The proof for $G_{0,j}$ containing a Berge-$F$, with $w$ played by $u_j$ is identical.) The graph $F_u$ contains at most $e(F) - 1$ edges. We need to verify Hall’s condition in the auxiliary bipartite graph $B$ with one part the edges of $F_u$ and the other part the hyperedges of $G_{0,i}$ and an edge $e$ is connected to a hyperedge $G$ if and only if $e \in G$. Note that the degree of any edge $e$ is at least $e(F_u) - 2$ and if $e_1$ and $e_2$ are disjoint edges of $F_u$, then in $B$ their neighborhood is $G_{0,i}$. Therefore the only problem that can occur is if $F_u$ is a star with $e(F) - 1$ leaves. If the center of $F_u$ is $u$, then $F$ is also a star, contradicting our assumption. If the center $c$ of $F_u$ is not $u$, then a vertex $u' \in B_i$ that is contained in all hyperedges of $G_{0,i}$ can play the role of $c$. \qed

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