VOLUME MINIMIZATION AND ESTIMATES FOR CERTAIN ISOTROPIC SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACES

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Abstract. In this note we show the following result using the integral-geometric formula of R. Howard: Consider the totally geodesic $\mathbb{R}P^m$ in $\mathbb{C}P^n$. Then it minimizes volume among the isotropic submanifolds in the same $\mathbb{Z}/2$ homology class in $\mathbb{C}P^n$ (but not among all submanifolds in this $\mathbb{Z}/2$ homology class). Also the totally geodesic $\mathbb{R}P^{m-1}$ minimizes volume in its Hamiltonian deformation class in $\mathbb{C}P^n$. As a corollary we’ll give estimates for volumes of Lagrangian submanifolds in complete intersections in $\mathbb{C}P^n$.

1. Introduction

On a Kähler $n$-fold $M$ there is a class of isotropic submanifolds. Those are submanifolds of $M$ on which the Kähler form $\omega$ of $M$ vanishes. The maximal dimension of such a submanifold is $n$ (the middle dimension) in which case it is called Lagrangian.

In this papers we’ll exhibit global volume-minimizing properties among isotropic competitors for certain submanifolds of the complex projective space. In general global volume-minimizing properties of minimal/Hamiltonian stationary Lagrangian/isotropic submanifolds in Kähler (particularly Kähler-Einstein) manifolds are still poorly understood. In dimension 2 there is a result of Schoen-Wolfson [ScW] (extended to isotropic case by Qiu in [Qiu]) which shows existence of Lagrangian cycles minimizing area among Lagrangians in a given homology class. Still it is not clear whether a given minimal Lagrangian has any global volume-minimizing properties.

The only instance where we have a clear cut answer to global volume-minimizing problem is Special Lagrangian submanifolds which are homologically volume-minimizing in Calabi-Yau manifolds [H]. In Kähler-Einstein manifolds of negative scalar curvature, besides geodesics on Riemann surfaces of negative curvature, we have some examples [Lee] of minimal Lagrangian submanifolds which are homotopically volume-minimizing. The author has a program for studying homotopy volume-minimizing properties for Lagrangians in Kähler-Einstein manifolds of negative scalar curvature [Gold], but so far there are no satisfactory results.

In positive curvature case there is a result of Givental-Kleiner-Oh which states that the canonical totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$ minimizes volume in its Hamiltonian deformation class, [GK]. The proof uses integral geometry and Floer homology to study intersections for Hamiltonian deformations of $\mathbb{R}P^n$. Those arguments can be generalized to products of Lagrangians in a product of symmetric Kähler manifolds, [LOS]. There is a related conjecture due to Oh that the Clifford torus minimizes volume in its Hamiltonian deformation class in $\mathbb{C}P^n$, [Oh]. Some progress towards
this was obtained in [Gold2]. Also general lower bounds for volumes of Lagrangians in a given Hamiltonian deformation class in \( \mathbb{C}^n \) were obtained in [Vit].

In this note we extend and improve the result of Givental-Kleiner-Oh to isotropic totally geodesic \( \mathbb{R}P^k \) sitting canonically in \( \mathbb{C}P^n \). Our main result is the following theorem:

**Theorem 1.** Consider the totally geodesic \( \mathbb{R}P^{2m} \) in \( \mathbb{C}P^n \). Then it minimizes volume among the isotropic submanifolds in the same \( \mathbb{Z}/2 \) homology class in \( \mathbb{C}P^n \) (but not among all submanifolds in this \( \mathbb{Z}/2 \) homology class). Also consider the totally geodesic \( \mathbb{R}P^{2m-1} \) in \( \mathbb{C}P^n \). Then it minimizes volume in its Hamiltonian deformation class.

A corollary of this is:

**Corollary 1.** Let \( f_1, \ldots, f_k \) be real homogeneous polynomials of odd degree in \( n+1 \) variables with \( 2m+k = n \). Let \( N \) be the zero locus of \( f_i \) in \( \mathbb{C}P^n \) and \( L \) be their real locus. Then \( \text{vol}(L) \leq \Pi \text{deg}(f_i) \text{vol}(\mathbb{R}P^{2m}) \) and if \( L' \) is a Lagrangian submanifold of \( N \) homologous mod 2 to \( L \) in \( N \) then \( \text{vol}(L') \geq \text{vol}(\mathbb{R}P^{2m}) \).

## 2. A Formula from Integral Geometry

In this section we establish a formula from integral geometry for volumes of isotropic submanifolds of \( \mathbb{C}P^n \) following the exposition in R. Howard [How]. In our case the group \( SU(n+1) \) acts on \( \mathbb{C}P^n \) with a stabilizer \( K \simeq U(n) \). Thus we view \( \mathbb{C}P^n = SU(n+1)/K \) and the Fubini-Study metric is induced from the bi-invariant metric on \( SU(n+1) \). Let \( P^{2m} \) be an isotropic submanifold of \( \mathbb{C}P^n \) of dimension \( 2m \) and let \( Q \) be a linear \( \mathbb{C}P^{n-m} \subset \mathbb{C}P^n \). For a point \( p \in P \) and \( q \in Q \) we define an angle \( \sigma(p, q) \) between the tangent planes \( T_pP \) and \( T_qQ \) as follows: First we choose some elements \( g \) and \( h \) in \( SU(n+1) \) which move \( p \) and \( q \) respectively to the same point \( r \in \mathbb{C}P^n \). Now the tangent planes \( g_*T_pP \) and \( h_*T_qQ \) are in the same tangent space \( T_r\mathbb{C}P^n \) and we can define an angle between them as follows: take an orthonormal basis \( u_1 \ldots u_{2m} \) for \( g_*T_pP \) and an orthonormal basis \( v_1 \ldots v_{2n-2m} \) for \( h_*T_qQ \) and define

\[
\sigma(g_*T_pP, h_*T_qQ) = |u_1 \wedge \ldots \wedge v_{2n-2m}|
\]

The later quantity \( \sigma(g_*T_pP, h_*T_qQ) \) depends on the choices \( g \) and \( h \) we made. To mend this we’ll need to average this out by the stabilizer group \( K \) of the point \( r \). Thus we define:

\[
\sigma(p, q) = \int_K \sigma(g_*T_pP, k_*h_*T_qQ) dk
\]

Since \( SU(n+1) \) acts transitively on the Grassmanian of isotropic planes and the complex planes in \( \mathbb{C}P^n \) we conclude that this angle is a constant depending just on \( m \) and \( n \):

\[
\sigma(p, q) = C_{m,n}
\]

There is a following general formula due to R. Howard [How]:

\[
\int_{SU(n+1)} \#(P \cap gQ) dg = \int_{P \times Q} \sigma(p, q) dp dq = C_{m,n} \text{vol}(P) \text{vol}(Q)
\]

Here \( \#(P \cap gQ) \) is the number of intersection points of \( P \) with \( gQ \), which is finite for a generic \( g \in SU(n+1) \). To use the formula we need to have some control over the intersection pattern of \( P \) and \( gQ \). We have the following lemma:
Lemma 1. Let $P$ be the totally geodesic $\mathbb{RP}^{2m} \subset \mathbb{CP}^{n}$, let $Q = \mathbb{CP}^{n-m} \subset \mathbb{CP}^{n}$. Let $g \in SU(n+1)$ s.t. $P$ and $gQ$ intersect transversally. Then $\#(P \cap gQ) = 1$. Also let $f_{1}, \ldots, f_{k}$ be real homogeneous polynomials in $n+1$ variables with $2m+k = n$ and let $P'$ be their real locus. If $P'$ is transversal to $gQ$ then $\#(P' \cap gQ) \leq \Pi \deg(f_{i})$.

Proof: For the first claim we have $gQ$ is given by an $(n-m+1)$-plane $H \subset \mathbb{C}^{n+1}$ and hence it is a zero locus of $m$ linear equations on $\mathbb{C}^{n+1}$. Hence $(P \cap gQ)$ is cut out by $2m$ linear equations in $\mathbb{RP}^{2m}$.

For the second claim we note that as before $gQ \cap \mathbb{RP}^{n}$ is the zero locus of $2m$ linear polynomials $h_{1}, \ldots, h_{2m}$ on $\mathbb{RP}^{n}$. Moreover $P'$ is a zero locus of $f_{1}, \ldots, f_{n-2m}$ on $\mathbb{RP}^{n}$. For generic $g \in SU(n+1)$ we’ll have that $gQ$ and $P'$ intersect transversally in $\mathbb{RP}^{n}$. By Bezout’s theorem (see [GH], p. 670) the common zero locus of $h_{1}, \ldots, h_{2m}$ and $f_{1}, \ldots, f_{n-2m}$ is $\mathbb{CP}^{n}$ is $\Pi \deg(f_{i})$ points. Now $P' \cap gQ$ is a part of this locus, hence $\#(P' \cap gQ) \leq \Pi \deg(f_{i})$.

3. Proof of the Volume Minimization

Now we can prove the result stated in the Introduction:

Theorem 1. Consider the totally geodesic $\mathbb{RP}^{2m}$ in $\mathbb{CP}^{n}$. Then it minimizes volume among the isotropic submanifolds in the same $\mathbb{Z}/2$ homology class in $\mathbb{CP}^{n}$ (but not among all submanifolds in this $\mathbb{Z}/2$ homology class). Also consider the totally geodesic $\mathbb{RP}^{2m-1}$ in $\mathbb{CP}^{n}$. Then it minimizes volume in its Hamiltonian deformation class.

Proof: Let $P$ be an isotropic submanifold homologous to $\mathbb{RP}^{2m}$ mod 2 and let $Q = \mathbb{CP}^{n-m}$. By Lemma 1, the intersection number mod 2 of $P$ and $gQ$ is 1. Hence the formula in the previous section tells that

$$C_{m,n} \vol(P) \vol(Q) = \int_{SU(n+1)} \#(P \cap gQ) dg \geq \vol(SU(n+1))$$

and

$$C_{m,n} \vol(\mathbb{RP}^{2m}) \vol(Q) = \int_{SU(n+1)} \#(\mathbb{RP}^{2m} \cap gQ) dg = \vol(SU(n+1))$$

and this proves the first part. We also note that that $\mathbb{CP}^{1}$ is homologous to $\mathbb{RP}^{2}$ mod 2 in $\mathbb{CP}^{n}$ but

$$\vol(\mathbb{CP}^{1}) < \vol(\mathbb{RP}^{2})$$

The second assertion will follow from the first one. Consider $\mathbb{C}^{n+1}$ and a unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. We have a natural circle action on $S^{2n+1}$ (multiplication by unit complex numbers). Let the vector field $u$ be the generator of this action. We have a 1-form $\alpha$ on $S^{2n+1}$,

$$\alpha(v) = u \cdot v$$

Also $d\alpha = 2\omega$ where $\omega$ is the Kähler form of $\mathbb{C}^{n+1}$. The kernel of $\alpha$ is the horizontal distribution. We have a Hopf map $p : S^{2n+1} \rightarrow \mathbb{CP}^{n}$. We have $\mathbb{RP}^{2m-1} \subset \mathbb{CP}^{n}$ and $S^{2m-1} \subset S^{2n+1}$ which is a horizontal double cover of $\mathbb{RP}^{2m-1}$.

Let $f$ be a (time-dependent) Hamiltonian function on $\mathbb{CP}^{n}$. Then we can lift it to a Hamiltonian function on $\mathbb{C}^{n+1} - \{0\}$ and its Hamiltonian vector field $H_{f}$ is horizontal on $S^{2n+1}$. Consider now the vector field

$$w = -2f \cdot u + H_{f}$$

The vector field $w$ is $S^{1}$-invariant. We also have:
Proposition 1. The Lie derivative \( L_w \alpha = 0 \)

Proof: We have
\[
L_w \alpha = d(i_w \alpha) + i_w d\alpha = -2d\alpha + 2df
\]
Let now \( \Phi_t \) be the time \( t \) flow of \( w \) on \( S^{2m+1} \) and let \( \Xi_t \) be the Hamiltonian flow of \( f \) on \( \mathbb{C}P^n \). Then \( \Phi_t(S^{2m-1}) \) is horizontal and isotropic and it is a double cover of \( \Xi_t(\mathbb{R}P^{2m-1}) \). Hence
\[
\text{vol}(\Phi_t(S^{2m-1})) = 2 \text{vol}(\Xi_t(\mathbb{R}P^{2m-1}))
\]
Let \( S_t = \Phi_t(S^{2m-1}) \). We build a suspension \( \Sigma S_t \) of \( S_t \) in \( S^{2n+3} \subset \mathbb{C}P^{n+2} \);
\[
\Sigma S_t = \{(\sin \theta \cdot x, \cos \theta) \in \mathbb{C}P^{n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}|0 \leq \theta \leq \pi, \ x \in S_t\}
\]
One immediately verifies that \( \Sigma S_t \) is horizontal and it is a double cover of an isotropic submanifold \( L_t \) (with a conical singularity) of \( \mathbb{C}P^{n+1} \) with \( L_0 = \mathbb{R}P^{2m} \).

Also one readily checks that
\[
\text{vol}(\Sigma S_t) = \text{vol}(S_t) \cdot \int_{\theta=0}^{\pi} \sin^{2m-1} \theta \ d\theta
\]
Hence
\[
2 \text{vol}(L_t) = \text{vol}(\Sigma S_t) = 2 \text{vol}(\Xi_t(\mathbb{R}P^{2m-1})) \cdot \int_{\theta=0}^{\pi} \sin^{2m-1} \theta \ d\theta
\]
Now the first part of our theorem implies that \( \text{vol}(L_t) \geq \text{vol}(L_0) \). Hence we conclude that \( \text{vol}(\Xi_t(\mathbb{R}P^{2m-1})) \geq \text{vol}(\mathbb{R}P^{2m-1}) \). Q.E.D.

Remark: One notes from the proof that for \( \mathbb{R}P^{2m-1} \) it would be sufficient to use exact deformations by isotropic immersions of \( \mathbb{R}P^{2m-1} \). A family \( L_t \) of isotropic immersions of \( \mathbb{R}P^{2m-1} \) is called exact if the 1-form \( i_w \omega \) is exact when restricted to each element of the family. Here \( v \) is the deformation vector field and \( \omega \) is the symplectic form. Thus embeddedness is not important for the conclusion of the theorem.

The theorem has the following corollary:

Corollary 1. Let \( f_1, \ldots, f_k \) be real homogeneous polynomials of odd degree in \( n+1 \) variables with \( 2m+k = n \). Let \( N \) be the zero locus of \( f_i \) in \( \mathbb{C}P^n \) and \( L \) be their real locus. Then \( \text{vol}(L) \leq \Pi \text{deg}(f_i) \text{vol}(\mathbb{R}P^{2m}) \) and if \( L' \) is a Lagrangian submanifold of \( N \) homologous mod 2 to \( L \) in \( N \) then \( \text{vol}(L') \geq \text{vol}(\mathbb{R}P^{2m}) \).

Proof: We note that \( N \) is a complex 2m-fold and \( L \) is its Lagrangian submanifold. Since the degrees of \( f_i \) are odd, we have by adjunction formula that \( L \) and \( \mathbb{R}P^{2m} \) represent the same homology class in \( H_{2m}(\mathbb{R}P^n, \mathbb{Z}/2) \). Let \( Q \) be a linear \( \mathbb{C}P^{n-m} \) in \( \mathbb{C}P^n \) and \( g \in SU(n+1) \). The intersection number mod 2 of \( gQ \) with \( L' \) is 1. We have that
\[
C_{m,n} \text{vol}(\mathbb{R}P^{2m}) \text{vol}(Q) = \int_{SU(n+1)} 1 \ dg
\]
\[
C_{m,n} \text{vol}(L') \text{vol}(Q) = \int_{SU(n+1)} #(L' \cap gQ) \ dg
\]
Also using Lemma
\[
C_{m,n} \text{vol}(L) \text{vol}(Q) = \int_{SU(n+1)} #(L \cap gQ) \ dg \leq \Pi \text{deg}(f_i) \text{vol}(SU(n+1))
\]
and our claims follow. Q.E.D.

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