Axial anomaly: the modern status.

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Abstract

The modern status of the problem of axial anomaly in QED and QCD is reviewed. Two methods of the derivation of the axial anomaly are presented: 1) by splitting of coordinates in the expression for the axial current and 2) by calculation of triangle diagrams, where the anomaly arises from the surface terms in momentum space. It is demonstrated, that the equivalent formulation of the anomaly can be given, as a sum rule for the structure function in dispersion representation of three point function of AVV interaction. It is argued, that such integral representation of the anomaly has some advantages in the case of description of the anomaly by contribution of hadronic states in QCD. The validity of the t’Hooft consistency condition is discussed. Few examples of the physical application of the axial anomaly are given.

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1 Introduction

The phenomenon of anomaly plays an important role in quantum field theory: in many cases it determines whether or not the theory is selfconsistent and can be realized in the physical world and, therefore, allows one to select the acceptable physical theories. In the given theory the anomalies often are related to appearance of new quantum numbers (topological quantum numbers), result in emerging of mass scale, determine the spectrum of physical states. So, despite of its denomination, the anomaly is a normal and significant attribute of any quantum field theory.

The term “anomaly” has the following meaning. Let the classical action of the theory to obey some symmetry, i.e. it is invariant under certain transformations. If this symmetry is violated by account of quantum corrections, such a phenomenon is called an “anomaly”. (The reviews of anomalies are given in [1]-[3].) There are two types of anomalies – internal and external. In the first case the gauge invariance of the classical Lagrangian is destroyed at the quantum level. The theory becomes nonrenormalizable and cannot be considered as a selfconsistent theory. The standard method to solve this problem is the special choice of fields in the Lagrangian in such a way, that all internal anomalies are cancelled. (The approach is used in the Stanford Model of electroweak interaction – it is the Glashow, Illiopoulos, Maiani
mechanism.) The external anomaly corresponds to violation of symmetry of interaction with external sources, not related to gauge symmetry of the theory. Just such anomalies take place in QCD and are considered below. There are two anomalies in QCD: the axial (chiral) anomaly and the scale anomaly. Both are connected with singularities of the theory at small distances (at large momenta) and with the necessity of regularization: the regularization procedure, which respects to the symmetry, does not exist and the symmetry is violated by the anomaly. In QCD the evidence of anomalies came from perturbation theory, but, in fact, their occurrence follows from general principles.

2 The derivation the axial anomaly by coordinate splitting

The axial anomaly in QCD is very similar to those in massless QED. For this reason let us first consider the latter. The equations of motion of QED in the external electromagnetic field $A_\mu(x)$ have the form:

$$i\gamma_\mu \frac{\partial \psi(x)}{\partial x_\mu} = m\psi(x) - e\gamma_\mu A_\mu(x)\psi(x).$$

(1)

In massless QED, classically, i.e. without the account of radiation corrections, the axial current $j_{\mu5}(x)$ is conserved like the vector current,

$$\partial_\mu j_{\mu5}(x) = \partial_\mu J_\mu(x) = 0.$$  

(2)

However, it appears, that in quantum theory with the account of radiation corrections, it is impossible to keep the conservation of both currents – vector and axial. The origin for this comes from singular character of the currents. Vector and axial currents are composite operators built from local fermion fields and the products of local operators are singular, when their points coincide, as it is in the cases of $V$ and $A$ currents. In order to consider the problem correctly, define the axial current by placing two fermion fields at distinct points, separated by the distance $\varepsilon$, and go to the limit $\varepsilon \to 0$ in the final result,

$$j_{\mu5}(x,\varepsilon) = \bar{\psi}(x + \frac{\varepsilon}{2})\gamma_5\psi(x - \frac{\varepsilon}{2}) - ie\varepsilon \alpha \bar{\psi}(x + \frac{\varepsilon}{2})\gamma_\mu\gamma_5\psi(x - \frac{\varepsilon}{2})F_{\alpha\mu},$$

(3)

where $F_{\alpha\mu}$ is the electromagnetic field strength. For simplicity assume that $F_{\mu\nu}=$const. Take the vacuum average of Eq. (4). In the r.h.s. of (4) it can be used the expression for the electron propagator in the constant external electromagnetic field. The electron propagator

$$S_{\alpha\beta}(x) = \langle 0 | T\{\psi_\alpha(x), \bar{\psi}_\beta(0)\} | 0 \rangle$$

(5)
satisfies the equation:

\[
[i\gamma^\mu(\partial_\mu - ieA_\mu(x)) - m]S(x) = i\delta^4(x).
\]  

(6)

It is convenient to choose the fixed point gauge for the electromagnetic field: \(x_\mu A_\mu(x) = 0\). Then \(a_\mu(x)\) is expressed through the field strength tensor \(F^\mu_\nu\) by

\[
A_\mu(x) = \frac{1}{2}x_\nu F^\nu_\mu.
\]  

(7)

The solution of Eq.(6) up to linear in \(F^\mu_\nu\) terms is equal

\[
S(x) = \frac{i}{2\pi^2} \left[ \frac{\not{k}}{x^4} + \frac{i m}{2 x^2} + \frac{1}{16x^2}\varepsilon F^\mu_\nu(\not{\sigma}^\mu_\nu + \sigma^\mu_\nu \not{k}) \right],
\]  

(8)

where \(\sigma^\mu_\nu = (i/2)(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)\). The vacuum averaging corresponds to account of the first order \(e^2\) corrections. In massless QED the first term in the r.h.s. of (1) is absent and we get

\[
\langle 0 | \partial_\mu j^\mu_5 | 0 \rangle = \frac{e^2}{4\pi^2} F^\alpha_\mu F^\lambda_\sigma \varepsilon^\alpha\varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta, \quad \varepsilon_{0123} = 1.
\]  

(9)

Since there is no specific direction in space-time the limit \(\varepsilon \to 0\) should be taken symmetrically,

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon^\alpha\varepsilon^\beta}{\varepsilon^2} = \frac{1}{4} \delta_{\alpha\beta}.
\]  

(10)

The substitution of (10) into (9) gives

\[
\partial_\mu j^\mu_5 = \frac{e^2}{8\pi^2} F^\alpha_\beta \tilde{F}^\beta_\alpha,
\]  

(11)

where

\[
\tilde{F}^\alpha_\beta = \frac{1}{2} \varepsilon^\alpha_\beta\varepsilon^\lambda_\sigma F^\lambda_\sigma
\]  

(12)

is the dual field strength tensor. The symbol of vacuum averaging is omitted in (11), because in order \(e^2\) Eq.(11) can be considered as an operator equation. The relation (11) is called the Adler-Bell-Jackiw anomaly [5]-[8].

### 3 The derivation of the axial anomaly by the calculation of triangle diagrams

In order to have the better understanding of the origin of the anomaly let us consider the same problem in the momentum space. In QED the matrix element for the transition of the axial current with momentum \(q\) into two real or virtual photons with momenta \(p\) and \(p'\) is represented by the diagrams of Fig.1. The matrix element is equal:

\[
T_{\mu\alpha\beta}(p,p') = \Gamma_{\mu\alpha\beta}(p,p') + \Gamma_{\mu\beta\alpha}(p',p),
\]  

(13)

\[
\Gamma_{\mu\alpha\beta}(p,p') = -e^2 \int \frac{d^4k}{(2\pi)^4} Tr \left[ \gamma_\mu \gamma_5(\not{k} + \not{p} - m)^{-1}\gamma_\alpha(\not{k} - m)^{-1}\gamma_\beta(\not{k} - \not{p}' - m)^{-1} \right].
\]  

(14)
Consider the divergence of the axial current $q \mu T_{\mu \alpha \beta}(p, p')$, $q = p + p'$. For $q \mu \Gamma_{\mu \alpha \beta}(p, p')$ we can write (at $m = 0$):

$$
q \mu \Gamma_{\mu \alpha \beta}(p, p') = -e^2 \int \frac{d^4k}{(2\pi)^4} Tr [ (p + k + p' - k) \gamma_5 (k + p) \gamma_\alpha (k - p')^{-1} ] =
$$

$$
= -e^2 \int \frac{d^4k}{(2\pi)^4} Tr [ -\gamma_5 \gamma_\alpha (k - p')^{-1} - \gamma_5 (k + p) \gamma_\alpha (k - p')^{-1} ]
$$

(15)

Each of the two terms in square brackets in the right-hand side (r.h.s.) of (15) after integration of $k$ depends on only one 4-vector – $p$ or $p'$. Each of these terms should be proportional to the unit totally antisymmetric tensor $\varepsilon_{\alpha \beta \gamma \delta}$ times the product of two different vectors. Since we have only one vector at our disposal, the result is zero. This fact looks to be in contradiction with the anomaly relation (11). However, we cannot trust in this result. The arguments are the following. The integral (14) is linearly divergent. In a linearly divergent integral it is illegitimate to shift the integration variable: such shift may result in appearance of the so-called “surface terms”. So, if the integration variable $k$ in (14),(15) would be changed to $k + cp + dp'$, where $c$ and $d$ are some numbers, $q \mu \Gamma_{\mu \alpha \beta}(p, p')$ would not be zero. The other argument against the calculation, performed in (15), is that $T_{\mu \alpha \beta}(p, p')$ must satisfy the conditions of the conservation of vector current: $p_\alpha T_{\mu \alpha \beta}(p, p') = 0$ and $p'_\beta T_{\mu \alpha \beta}(p, p') = 0$. The calculations, performed using the same integration variable, as in (15) show, that these conditions are not fullfilled. The question arises if it is possible to choose the integration variable in such a way, that $q \mu T_{\mu \alpha \beta} = 0$ and simultaneously $p_\alpha T_{\mu \alpha \beta} = 0$, $p'_\beta T_{\mu \alpha \beta} = 0$. Following Ref.8, consider $\Gamma_{\mu \alpha \beta}$, defined by (14), where the integration variable $k$ is shifted by an a arbitrary constant vector $a_\lambda$, $k_\lambda \rightarrow k_\lambda + a_\lambda$. We can write

$$
\Gamma_{\mu \alpha \beta}(p, p'; a) = \Gamma_{\mu \alpha \beta}(p, p') + \Delta_{\mu \alpha \beta}(p, p', a)
$$

(16)

$$
\Delta_{\mu \alpha \beta}(p, p'; a) = \Gamma_{\mu \alpha \beta}(p, p')_{k \rightarrow k + a} - \Gamma_{\mu \alpha \beta}(p, p')
$$

(17)

where $\Gamma_{\mu \alpha \beta}(p, p')$ is given by (14) and, therefore $q \mu \Gamma_{\mu \alpha \beta}(p, p') = 0$ according to (15). $\Gamma_{\mu \alpha \beta}(p, p')_{k \rightarrow k + a}$ is obtained from (14) by substituting $k_\lambda \rightarrow k_\lambda + a_\lambda$. $\Delta_{\mu \alpha \beta}(p, p'; a)$ is the surface term, the integral is convergent and its calculation gives [8]:

$$
\Delta_{\mu \alpha \beta} = -\frac{e^2}{8\pi^2} \varepsilon_{\mu \alpha \beta \gamma} a_\gamma.
$$

(18)
Generally, $a_\lambda$ is expressed in terms of two vectors, involved in the problem – $p$ and $p'$, $a_\lambda = (a + b)p_\lambda + bp'_\lambda$. Accounting the crossing diagram, we get:

$$T_{\mu\alpha\beta}(p, p', a) = T_{\mu\alpha\beta}(p, p') - \frac{e^2}{8\pi^2}a\varepsilon_{\mu\alpha\beta\gamma}(p_\gamma - p'_\gamma).$$  \hspace{1cm} (19)$$

The matrix element of the divergence of the axial current appears to be equal:

$$q_\mu T_{\mu\alpha\beta}(p, p'; a) = q_\mu T_{\mu\alpha\beta}(p, p') + \frac{e^2}{4\pi^2}a\varepsilon_{\alpha\beta\gamma\sigma}p_\gamma p'_\sigma.$$  \hspace{1cm} (20)$$

As it was demonstrated above, the first term in r.h.s. of (20), vanishes in the limit of massless quarks (the Sutherland-Veltman theorem [9, 10], see also Ref.[8]). As follows from (20) in order to ensure the conservation of the axial current it is necessary to choose $a = 0$. Such choice is just the repetition of the already obtained result in Eq.(15). Let us check now the conservation of vector current. The direct calculation gives:

$$p_\alpha T_{\mu\alpha\beta}(p, p'; a) = \frac{e^2}{4\pi^2}\varepsilon_{\mu\alpha\beta\gamma\sigma}p_\gamma p'_\sigma \left(1 + \frac{a}{2}\right)$$  \hspace{1cm} (21)$$

and the similar equality for $p'_\beta T_{\mu\alpha\beta}(p, p'; a)$. As follows from (21) the conservation of vector current can be achieved, if $a = -2$. That means, that it is impossible to have simultaneously the conservation of vector and axial currents in massless QED. Since we are sure, that the vector current is conserved, otherwise the photon would be massive and all electrodynamics would be ruined, we must choose $a = -2$. The substitution of $a = -2$ in (20) gives back Eq.(11).

Note, that the first method of the derivation of the anomaly – by the use of coordinate splitting in the expression for axial current, is valid for constant external electromagnetic field, since Eq.(6.273) corresponds to such case. The second method of the derivation, based on consideration of the diagrams of Fig.1 is much more general – it is valid for arbitrary varying external electromagnetic fields, including the emission of real or virtual photons. In this case the anomaly condition has the form:

$$q_\mu T_{\mu\alpha\beta}(p, p') = \left[2mG(p, p') - \frac{e^2}{2\pi^2}\right]\varepsilon_{\alpha\beta\lambda\sigma}p_\lambda p'_\sigma.$$  \hspace{1cm} (22)$$

In (22) the term, proportional to electron mass is retained and $G(p, p')$ is defined by

$$\langle p, \varepsilon_\alpha; p', \varepsilon'_\beta | \bar{\psi}\gamma_5\psi | 0 \rangle = G(p, p')\varepsilon_{\alpha\beta\lambda\sigma}p_\lambda p'_\sigma,$$  \hspace{1cm} (23)$$

where $\varepsilon_\alpha, \varepsilon'_\beta$ are photon polarizations.

The proof of the axial anomaly – Eq.’s(11),(22) can be obtained also by other methods: by dimension regularization scheme, by Pauli – Villars regularization and by consideration of functional integral [11],[12],[13]. In the latter the axial anomaly arises due to noninvariance of the fermion measure in external gauge field at $\gamma_5$ transformations in functional integral.

Nevertheless, the axial current is not conserved in massless QED, there does exist a conserved, gauge invariant axial charge [5],[7],[8]. Define

$$\tilde{j}_{\mu 5} = j_{\mu 5} - \frac{e^2}{4\pi^2}F_{\mu\nu}A_\nu.$$  \hspace{1cm} (24)$$

5
The current $\tilde{j}_{\mu 5}$ is conserved, but is not gauge invariant. However, the axial charge

$$Q_5 = \int d^3 x \tilde{j}_{05}(x)$$  \hspace{1cm} (25)$$

is gauge invariant.

The axial anomaly in QED was considered till now in order of $e^2$. It was shown, that there are no corrections to Eq. (11) in order $e^4$ \cite{5, 6} the argument is that in this order all radiative corrections correspond to insertion of photon line inside the triangle diagrams of Fig.1. If the integration over the photon momentum is carried out after the integration over the fermion loop, then the fermion loop integral is convergent and there is no anomaly. (This argumentation was supported by direct calculation \cite{6}). In higher orders of perturbation theory any insertions of photon lines and fermion loops inside the triangle of Fig.1 diagrams do not give the corrections to anomaly \cite{5, 14}. The corrections to Adler-Bell-Jackiw anomaly arise from high order diagrams like shown in Fig.2 \cite{15}. The account of Fig.2 diagram results to renormalization of the anomaly term in (11), (22) of the order $e^6$ \cite{5, 14, 15}. In this order \cite{15}

$$\partial_{\mu}j_{\mu 5} = \frac{\alpha}{2\pi} (F_{\mu\nu} \tilde{F}_{\mu\nu})_{\text{ext}} \left( 1 - \frac{3}{4} \frac{\alpha^2}{\pi^2} \ln \frac{\Lambda^2}{q^2} \right),$$  \hspace{1cm} (26)$$

where $\Lambda$ is the ultraviolet cut-off, the axial vector vertex is renormalized and the axial vector current, unlike the vector current, acquires the anomalous dimension.

Turn now to QCD. Here $j_{\mu 5}$ can be identified with the current of light quarks. In case of interaction with external electromagnetic field if $j_{\mu 5}$ corresponds to the axial current of one quark flavour with electric charge $e_q$ the anomaly has the form of Eq.'s (11), (22) with the only difference, that the r.h.s. is multiplied by $e^2_q N_c$, where $N_c$ is the number of colours. In QCD there is also an another possibility, where the external fields are gluonic fields. In this case instead of (11) we have:

$$\partial_{\mu}j_{\mu 5} = \frac{\alpha_s N_c}{4\pi} G_{\mu\nu}^m \tilde{G}_{\mu\nu}^m,$$  \hspace{1cm} (27)$$

where $G_{\mu\nu}^m$ is the gluon field strength and $\tilde{G}_{\mu\nu}^m$ is its dual. Eq. (27) can be considered as an operator equation and the fields $G_{\mu\nu}^m, \tilde{G}_{\mu\nu}^m$ can be represented by virtual gluons. Note, that
due to the same argumentation as in case of radiation correction to the anomaly in QED, the perturbative corrections to (27) start from $\alpha_s^3$. Evidently, the flavour octet axial current

$$j_{\mu 5}^i = \sum_q \bar{\psi}_q \gamma_\mu (\lambda^i/2) \psi_q, \quad i = 1, \ldots 8$$  \hspace{1cm} (28)

is conserved in QCD. (Here $\lambda^i$ are Gell-Mann $SU(3)$ matrices and the sum is performed over the flavours of light quarks, $q = u, d, s$.) Neglecting $u, d, s$ quark masses, we have instead of (27):

$$\partial_\mu j_{\mu 5}^i = 0.$$  \hspace{1cm} (29)

However, the anomaly persists for singlet axial current

$$j^{(0)}_{\mu 5} = \sum_q \bar{\psi}_q \gamma_\mu \gamma_5 \psi_q,$$

$$\partial_\mu j^{(0)}_{\mu 5} = 3 \frac{\alpha_s N_c}{4\pi} G^m_{\mu\nu} \tilde{G}^m_{\mu\nu}.$$  \hspace{1cm} (31)

From (29), (31) it follows, that because of spontaneous chiral symmetry breaking the octet of pseudoscalar mesons ($\pi, K, \eta$) is massless – in the approximation $m_q \to 0$, they are Goldstone bosons, while the singlet pseudoscalar meson – the $\eta'$ – remains massive. Therefore, the occurrence of anomaly solve the so called $U(1)$ problem [16]. (The detailed exposition of this statement is given in [17], see also [18], [19] for review.)

4 The spectral representation of the three point AVV function and the axial anomaly

Return now to QED and consider the matrix element of the transition of the axial current into two real or virtual photons, i.e. the function $T_{\mu\alpha\beta}(p, p')$ (13), described by the diagrams of Fig.3.2.1, where the internal lines correspond to propagators of electrons. The general expression for $T_{\mu\alpha\beta}(p, p')$, which satisfies the Bose symmetry of two photons, reads [5], [7], [20]:

$$T_{\mu\alpha\beta}(p, p') = A_1(p, p') S_{\mu\alpha\beta} - A_1(p', p) S'_{\mu\alpha\beta} + A_2(p, p') p_\beta R_{\mu\alpha} - A_2(p', p') p'_\alpha R_{\mu\beta} + A_3(p, p') p'_\beta R_{\mu\alpha} - A_3(p', p) p_\alpha R_{\mu\beta},$$  \hspace{1cm} (32)

where

$$R_{\mu\nu} = \varepsilon_{\mu\nu\sigma\rho} p_\rho p'_\sigma, \quad S_{\mu\alpha\beta} = \varepsilon_{\mu\alpha\beta\sigma} p_\sigma, \quad S'_{\mu\alpha\beta} = \varepsilon_{\mu\alpha\beta\sigma} p'_\sigma.$$  \hspace{1cm} (33)

The vector current conservation leads to

$$A_1(p, p') = (pp') A_2(p, p') + p^2 A_3(p, p')$$

$$A_1(p', p) = (pp') A_2(p', p) + p'^2 A_3(p', p)$$  \hspace{1cm} (34)

Using the identity

$$\delta_{\alpha\beta} \varepsilon_{\sigma\mu\nu\tau} - \delta_{\alpha\sigma} \varepsilon_{\beta\mu\nu\tau} + \delta_{\alpha\mu} \varepsilon_{\beta\nu\sigma\tau} - \delta_{\alpha\nu} \varepsilon_{\beta\sigma\mu\tau} + \delta_{\alpha\tau} \varepsilon_{\beta\sigma\mu\nu} = 0,$$  \hspace{1cm} (35)

we derive

$$p_\sigma R_{\mu\nu} - p_{\mu} R_{\sigma\nu} + p_{\nu} R_{\sigma\mu} + (pp') S_{\sigma\mu\nu} - p^2 S'_{\sigma\mu\nu} = 0.$$
The Lorenz structures $S_{\sigma\mu\nu}$ and $S^\prime_{\sigma\mu\nu}$ are retained in (32) in order to avoid kinematical singularities [20]. Let us put $p^2 = p'^2 \leq 0$. Using (34) and the identities (36) $T_{\mu\alpha\beta}(q,p,p')$ can be expressed in terms of two functions $- F_1(q^2, p^2)$ and $F_2(q^2, p^2)$ [21, 22]:

$$T_{\mu\alpha\beta}(p,p') = F_1(q^2,p^2)q_\mu \varepsilon_{\alpha\beta\rho\sigma} p_\rho p'_\sigma -$$

$$- \frac{1}{2} F_2(q^2,p^2) [\varepsilon_{\mu\alpha\beta\sigma}(p-p')_\sigma - \frac{p_\alpha}{p^2} \varepsilon_{\mu\beta\rho\sigma} p_\rho p'_\sigma + \frac{p'_\beta}{p^2} \varepsilon_{\mu\alpha\rho\sigma} p_\rho p'_\sigma].$$

(37)

If $p^2 \neq 0$, the form factors $F_1(q^2, p^2) = -A_2$ and $F_2(q^2, p^2) = 2A_1$ are free of kinematical singularities [22]. Consider now the divergence

$$q_\mu T_{\mu\alpha\beta}(p,p') = [F_2(q^2, p^2) + q^2 F_1(q^2, p^2)] \varepsilon_{\alpha\beta\rho\sigma} p_\rho p'_\sigma.$$

(38)

The substitution in the l.h.s. of (38) of the anomaly condition (22) gives the sum rule [21]

$$F_2(q^2, p^2) + q^2 F_1(q^2, p^2) = 2mG(q^2, p^2) - \frac{e^2}{2\pi^2}.\]$$

(39)

The functions $F_1(q^2, p^2), F_2(q^2, p^2)$ and $G(q^2, p^2)$ can be represented by the unsubtracted dispersion relations in $q^2$:

$$f_i(q^2, p^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{Im f_i(t, p^2)}{t-q^2} dt, \quad f_i = F_1, F_2, G.$$

(40)

The direct calculation of $Im F_1(q^2, p^2)$ gives [21, 23]

$$Im F_1(q^2, p^2) = \frac{e^2}{2\pi} \frac{2p^2}{q^2} \left\{ \frac{q^2 + 2p^2}{(q^2 - 4p^2)^2} \left( 1 - \frac{4m^2}{q^2} \right)^{1/2} + \frac{2p^2(q^2 - 2p^2)}{(q^2 - 4p^2)^{5/2}} \right\} \times$$

$$\times \left[ \frac{q^2 - p^2}{2q^2 - 2p^2} + m^2 \frac{q^2 - 4p^2}{2p^2} \ln \frac{q^2 - 2p^2 - [(q^2 - 4m^2)(q^2 - 4p^2)]^{1/2}}{q^2 - 2p^2 + [(q^2 - 4m^2)(q^2 - 4p^2)]^{1/2}} \right].$$

(41)

At $p^2 = 0$

$$Im F_1(q^2, 0) = \frac{e^2}{\pi} \frac{m^2}{q^4} \ln \frac{1 + \sqrt{1 - 4m^2/q^2}}{1 - \sqrt{1 - 4m^2/q^2}},$$

(42)

and at large $q^2$

$$Im F_1(q^2, p) \approx -\frac{e^2}{2\pi} \frac{2p^2}{q^4} \left[ 1 + \frac{m^2}{p^2} \ln \frac{m^2}{q^2} \right].$$

(43)

For imaginary parts of $F_1, F_2, G$ we have the relation:

$$Im F_2(q^2, p^2) + q^2 Im F_1(q^2, p^2) = 2m Im G(q^2, p^2).$$

(44)

As follows from (43) and (44) $F_2(q^2, p^2)$ and $G(q^2, p^2)$ are decreasing as $1/q^2$ at $q^2 \to \infty$. Therefore, the nonsubtracted dispersion relation (40) are legitimate. From (39)-(44) the sum rule

$$\int_{4m^2}^{\infty} Im f_i(t, p^2) dt = \frac{e^2}{2\pi}$$

(45)
follows. The sum rule (45) has been verified explicitly by Frishman et al [23] for \( p^2 < 0, m = 0 \), by Hořejší [21] at \( p^2 = p'^2 < 0 \) and by Veretin and Teryaev [24] in general case, \( p^2 \neq p'^2 \).

Consider now the transition of axial current into two real photons in QCD with one flavour of unit charge. Instead of (45) we have

\[
\int_{4m_q^2}^{\infty} \text{Im} F_1(t, 0) dt = 2\alpha N_c. \tag{46}
\]

\( F_2(q^2, p^2) \) should vanish at \( p^2 \Rightarrow 0 \) in order that \( T_{\mu\alpha\beta}(p, p') \) have no pole there, which would correspond to massless hadronic state in \( J^{PC} = 1^{--} \) channel. In the limit of massless quarks, \( m_q = 0 \), the r.h.s. of (39) is given by the anomaly and in QCD we have

\[
F_1(q^2, 0) = -\frac{2\alpha N_c}{\pi} \frac{1}{q^2}; \tag{47}
\]

\[
T_{\mu\alpha\beta}(p, p') = -\frac{2\alpha}{\pi} N_c \frac{q_\mu q_\nu}{q^2} \varepsilon_{\alpha\beta\lambda\sigma} p_\lambda p'_\sigma. \tag{48}
\]

The imaginary part of \( F_1(q^2, 0) \) at \( m_q^2 = 0 \) is proportional to \( \delta(q^2) \) [25]:

\[
\text{Im} F_1(q^2, 0)_{m_q^2 = 0} = 2\alpha N_c \delta(q^2) \tag{49}
\]

and the sum rule (46) is saturated by the contribution of zero-mass state. It is interesting to look how the limit \( m_q^2 \rightarrow 0, q^2 \rightarrow 0 \) proceeds. At \( m_q \neq 0 \) \( \text{Im} F_1(q^2, 0) \) is equal [25]

\[
\text{Im} F_1(q^2, 0) = 4\alpha N_c \frac{m_q^2}{q^4} \ln \frac{1 + \sqrt{1 - 4m_q^2/q^2}}{1 - \sqrt{1 - 4m_q^2/q^2}} \tag{50}
\]

and in the limit \( m_q^2 \rightarrow 0, q^2 \rightarrow 0 \) indeed we get (49). The most interesting case is, when the current \( j_{\mu5} \) is equal to the third component of the isovector current

\[
j_{(3)\mu5} = \bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d. \tag{51}
\]

Then at \( p^2 = p'^2 = 0, m_u^2 = m_d^2 = 0 \) [25]:

\[
T_{\mu\alpha\beta}(p, p') = -\frac{2\alpha}{\pi} N_c \frac{q_\mu q_\nu}{q^2} (e_u^2 - e_d^2) \varepsilon_{\alpha\beta\lambda\sigma} p_\lambda p'_\sigma. \tag{52}
\]

The amplitude \( T_{\mu\alpha\beta} \) (52) corresponds to the transition of isovector axial current into two photons. Eq.(52) is consistent with the fact that the transition proceeds through virtual \( \pi^0 \) and \( \pi^0 \) is massless at \( m_q = 0 \) (the pole in the amplitude at \( q^2 = 0 \) [25]. Then the process is described by the diagram like Fig.3. The use of relation \( \langle 0 \mid j_{(3)\mu5} \mid \pi^0 \rangle = \sqrt{2} f_\pi q_\mu \) determines the amplitude of \( \pi^0 \rightarrow 2\gamma \) decay

\[
M(\pi^0 \rightarrow 2\gamma) = A \varepsilon_{\alpha\beta\lambda\sigma} \varepsilon_{1\alpha} \varepsilon_{2\beta} p_{1\lambda} p_{2\sigma}, \tag{53}
\]

where \( \varepsilon_{1\alpha} \) and \( \varepsilon_{2\beta} \) are the polarizations of the first and the second photons. From (52) the constant \( A \) is found to be

\[
A = \frac{2\alpha}{\pi} \frac{1}{\sqrt{2} f_\pi}. \tag{54}
\]
and the $\pi^0 \to 2\gamma$ decay rate is equal

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2}{32\pi^3} \frac{m^3}{f^2}. \quad (55)$$

Eq. (55) gives the theoretical value of $\pi^0 \to 2\gamma$ decay width $\Gamma(\pi^0 \to 2\gamma)_{\text{theor}} = 7.7$ eV in very good agreement with experimental value $\Gamma(\pi^0 \to 2\gamma)_{\text{exp}} = 7.8 \pm 0.6$ eV [26]. (The accuracy of theoretical value (55) is 5-7%. The higher accuracy of theoretical prediction was achieved in [27] in the framework of CET and $1/N_c$ expansion.)

Despite of the fact, that the axial anomaly results in appearance of massless $\pi^0$ in the transition of isovector axial current into two photons and predicts well the $\pi^0$ decay rate, it is incorrect to say that the existence of massless pseudoscalar Goldstone bosons (at $m_q = 0$) are caused by the anomaly. The reasons are the following. $Im F_1(q^2, p^2)$ has $\delta(q^2)$ singularity at $p^2 = 0$. According to the Chiral Effective Theory (CET) it is expected that the same singularity persist in the case of $p^2 \neq 0$ – the diagram of Fig.3. contributes in this case as well. However, the examination of $Im F_1(q^2, p^2)$, Eq. (41) shows, that $Im F_1(q^2, p^2)$ is a regular function of $q^2$ near $q^2 = 0$ at $p^2 \neq 0$. The sum rules (45), (46) are satisfied by the triangle diagram contribution at $p^2 \neq 0$. Therefore, it is fulfilled the statement of CET, that the transition of $j^{\mu\nu}_{\text{iso}}$ into 2$\gamma$’s with $p^2 \neq 0$ is described by the diagram of Fig.3.2.3 with the same $\pi^0 \cdot 2\gamma$ coupling constant as at $p^2 = 0$, but only in the sense of integrals (45), (46), not locally. (It is assumed that $|p^2|$ is less than CET characteristic mass scale). Consider now the transition of 8-th component of octet axial current

$$j^{(8)}_{\mu5} = \frac{1}{\sqrt{6}} (\bar{u}\gamma_\mu \gamma_5 u + \bar{d}\gamma_\mu \gamma_5 d - 2\bar{s}\gamma_\mu \gamma_5 s) \quad (56)$$

into two real photons at $m_u = m_d = m_s = 0$. The amplitude $F_1(q^2, 0)$ has a pole at $q^2 = 0$, which can be attributed to $\eta$-meson. The $\eta \to 2\gamma$ decay width is given by the relation, analogous to (55)

$$\Gamma(\eta \to 2\gamma) = \frac{\alpha^2}{32\pi^3} \frac{m^3}{f^2}. \quad (57)$$

However, (57) strongly disagrees with experiment: $\Gamma(\eta \to 2\gamma)_{\text{theor}} = 0.13$ keV (at $f_\eta = 150$ Mev) in comparison with $\Gamma(\eta \to 2\gamma)_{\text{exp}} = 0.510 \pm 0.026$ keV [26]. The possible explanation
of this discrepancy is strong nonperturbative interactions like instantons, which persist in pseudoscalar channel (see [28]). The η′′ mixing remarkably increases Γ(η → 2γ). Another discrepancy arises, if we consider the the transition \( j^{(0)}_{\mu 5} \rightarrow 2\gamma \), where \( j^{(0)}_{\mu 5} \) is the singlet axial current:

\[
\bar{u} \gamma_{\mu} \gamma_{5} u + \bar{d} \gamma_{\mu} \gamma_{5} d + \bar{s} \gamma_{\mu} \gamma_{5} s. \tag{58}
\]

Since at \( m_u = m_d = m_s = 0 \) there are the poles at \( q^2 = 0 \) in \( F_1(q^2, 0) \) for each quark flavour, the transition amplitude \( T^{(0)}_{\mu \alpha \beta}(q, p, p') \) has a pole at \( q^2 = 0 \). The corresponding pseudoscalar meson is η′′. But η′′ is not a Goldstone boson – it is massive! The possible explanation is the important role of instantons in η′ → 2γ decay [29] and that the contribution of the diagram similar to Fig.3 (with virtual gluons instead of virtual photons) and, may be, the ladder (or parquet) of box diagrams is of importance here [24]. (We do not touch the theoretical determination of η → 2γ and η′ → 2γ decay rates by using additional hypothesis, besides the anomaly condition – see [30] and references therein.)

Turn now back to Eq.'s (45), (46). These equations are equivalent to anomaly conditions. The integrals in the l.h.s. of these equations are convergent. \( (Im F_1(q^2, p^2)_{q^2 \to \infty} \sim 1/q^4) \). This means, that with such interpretation the anomaly arises from finite domain of \( q^2 \). Eq. (45) can be rewritten in another form [24]:

\[
\lim_{q^2 \to \infty} q^2 \pi F_1(q^2, p^2) = \frac{e^2}{2\pi} \tag{59}
\]

This form returns us to initial interpretation of the anomaly, as corresponding to the domain of infinitely large \( q^2 \). So, it is possible to speak about the double face of the anomaly: from one point of view it corresponds to large \( q^2 \), from the other – its origin is connected with finite \( q^2 \). As is clear from the discussion above, both points of view are correct. These two possibilities of interpretation of the anomaly are interconnected by analyticity of the corresponding amplitudes.

t’Hooft suggested the hypothesis, that the singularities of amplitudes, calculated in QCD on the level of quarks and gluons shall be reproduced on the level of hadrons (the so called t’Hooft consistency condition [31]). Of course, if it is possible to prove, that such singularity cannot be smashed out by perturbative and nonperturbative corrections, this statement is correct and, even more, it is trivial. But, as a rule, no such proof can be given. In the presented above examples of the realization of axial anomaly (with the exception of \( \pi^0 \to 2\gamma \) decay) t’Hooft conjecture was not realized. Much better chances are for the duality conditions, like Eq. (46), when the QCD amplitude, integrated over some duality interval, gives the same result, as the corresponding hadronic amplitude integrated over the same duality interval (the so called quark-hadron duality).

In QCD the case, when one of the photons in Fig.3.2.1 is soft, is of special interest [32]. If the momentum of the soft photon is \( p'_{\beta} \) and its polarization is \( \varepsilon'_{\beta} \), then, restricting ourselves by the linear terms in \( p'_{\beta} \), the amplitude \( T_{\mu \alpha \beta \varepsilon'_{\beta}} \) can be represented in terms of two structure functions:

\[
T_{\mu \alpha \beta \varepsilon'_{\beta}} = w_T(q^2)(-q^2 \tilde{f}_{\alpha \mu} + q_{\alpha} q_{\sigma} \tilde{f}_{\sigma \mu} - q_{\mu} q_{\sigma} \tilde{f}_{\sigma \alpha} + \tilde{w}_L(q^2)q_{\mu} q_{\sigma} \tilde{f}_{\sigma \alpha}), \tag{60}
\]

where

\[
\tilde{f}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \sigma} (p'_{\lambda} \varepsilon'_{\sigma} - p'_{\sigma} \varepsilon'_{\lambda}). \tag{61}
\]
The first structure is transversal with respect to axial current momentum $q_\mu$, while the second is longitudinal. From the triangle diagram the relation [24],[33]:

$$w_L(q^2) = 2w_T(q^2)$$

(62)

follows. The anomaly condition gives for massless quark:

$$w_L(q^2) = 2\frac{\alpha}{\pi} N_c \frac{1}{q^2}.$$  

(63)

Because of (62) this condition determines also the transverse structure function. According to the Adler-Bardeen nonrenormalization theorem, there are no perturbative corrections to the triangle diagram. But, as was demonstrated in [32] there are nonperturbative corrections, which at large $q^2$ can be expressed through OPE series in terms of vacuum condensates, induced by external electromagnetic field. In terms of OPE Eq. (63) represents the contribution of the dimension 2 operator $\hat{F}_{\mu\nu}$. The next in dimension vacuum condensate is the quark condensate magnetic susceptibility $\chi(d = 3)$, defined by

$$\langle 0 | \bar{q}\sigma_{\mu\nu}q | 0 \rangle_F = e_q F_{\mu\nu}\langle 0 | \bar{q}q | 0 \rangle \chi,$$

(64)

which was introduced in [34],[35]. The index $F$ in (64) means that the vacuum expectation value is taken in the presence of the constant weak electromagnetic field $F_{\mu\nu}$. It was assumed in Ref.[32], that only the lowest hadronic state – the pion contributes to the anomaly and $q^2$ in the denominator was substituted by $q^2 - m_\pi^2$. Then the expansion in $m_\pi^2$ in the first order and identification of this term with dimension 3 term of OPE (the proportional to $1/q^4$ term in $w_L$) allows one to find the quark condensate magnetic susceptibility

$$\chi = -\frac{N_c}{2\pi^2 f_\pi^2} = -8.9 \text{ GeV}^{-2}$$

(65)

The value of $\chi$, determined from QCD sum rules is equal [36]

$$\chi_{1 \text{ GeV}} = -4.4 \pm 0.4 \text{ GeV}^{-2}$$

(66)

The disagreement of (65) and (66) cannot be considered as a strong discrepancy. $\chi$ has anomalous dimension ($d = -16/27$). No $\alpha_s$-corrections were accounted in (65), therefore it is not clear to what scale the value (65) refers. The saturation by pion contribution can be valid at low scale, where $\alpha_s$-corrections are large. The contribution of excited state are of the same order as the pion contribution – there are no small parameter there. For all these reasons (65) can be considered as the order of magnitude estimation of $\chi$ and the comparison of (65) and (66) is merely an argument in favour of the used approach.

5 The axial anomaly and the scattering of polarized electron (muon) on polarized gluon

Consider the scattering of longitudinally polarized electron (muon) on longitudinally polarized gluon. The first moment of the forward scattering amplitude is proportional to the diagonal matrix element

$$\langle g_{\text{polar}} | J_{\mu5} | g_{\text{polar}} \rangle,$$

(67)
where the gluons are on mass shell. The corresponding Feynman diagrams are the same as in Fig.1 with the only difference, that the wavy lines represent now the polarized gluons and the lower vertices are the vertices of quark-gluon interaction. Put \( q = 0, p = -p', p^2 < 0 \).

It is convenient to use the light-cone kinematics, where \( p_0 = p_+ + p_-/2, p_z = p_+ - p_-/2, p^2 = 2p_+p_- < 0 \) and work in the infinitely fast moving system along the \( z \)-direction. The calculation of (68) is performed for gluon helicity +1. The contribution of the crossing diagram Fig.3.2.1b is equal to the last propagator in (68), what results in the integration domain in \( k \)-space, 

\[
\Gamma_\mu(p) = 2ig^2N_f \left( \frac{\lambda^n}{2} \right)^2 \int \frac{d^4k}{(2\pi)^4} Tr \{ \not{k}(k + m)\gamma_\mu\gamma_5(k + m) \not{\gamma}(- \not{p} + \not{k} + m) \}
\]

\[
\times \frac{1}{(k^2 - m^2 + i\varepsilon)^2} \frac{1}{((p - k)^2 - m^2 + i\varepsilon)}. \tag{68}
\]

Here \( N_f \) is the number of flavours, \( \lambda^n, n = 1,...8 \) are the Gell-Mann SU(3) matrix in colour space, \( m \) are the quark masses, which are assumed to be equal for all flavours, \( \varepsilon_\mu \) is the gluon polarization vector,

\[
\varepsilon_\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0) \tag{69}
\]

for gluon helicity +1. The contribution of the crossing diagram Fig.3.2.1b is equal to the direct one and is accounted in (68) by the factor 2. The calculation of (68) is performed using the dimensional regularization in \( n \neq 4 \) dimensions. According to the ’t Hooft-Veltman recipe (see, e.g.[2]) it is assumed, that \( \gamma_5 \) is anticommuting with \( \gamma_\mu \) at \( \mu = 0, 1, 2, 3 \), and is commuting with \( \gamma_\mu \) at \( \mu \neq 0, 1, 2, 3 \). After integration over \( k_- \) it was found for the component \( \Gamma_{5+} [37] \):

\[
\Gamma_{5+} = -\frac{\alpha_s N_f p_+}{\pi^2} \int_0^1 dx \int \frac{d^{n-2}k_T}{[k_T^2 + m^2 + P^2x(1-x)]^2} \left\{ k_T^2(1-2x) - m^2 - 2\left(\frac{n-4}{n-2}\right)k_T^2(1-x) \right\}, \tag{70}
\]

where \( P^2 = -p^2 \). In the integration over \( k_- \) it was enough to take the residue at the pole of the last propagator in (68), what results in the integration domain in \( k_+ \): \( 0 < k_+ < p_+ \), and allowed to put \( k_+ = xp_+ \). The last term in (70) arised from \( n - 4 \) regulator dimensions and is proportional to \( \hat{k}^2 \), where \( \hat{k} \) is the projection of \( k \) into these dimensions. The azimuthal average gives \( \overline{k}^2 = k_T^2\frac{n-4}{n-2} \).

The first term in the curly brackets in (70) vanishes after integration over \( x \). (In fact, the equal to zero result after integration over \( x \) is multiplied by divergent integral over \( k_T \). So, strictly speaking, this term is uncertain. This problem will be discussed later.) After integration over \( k_T \), using the rules of dimensional regularization and going to \( n = 4 \), we get [33]:

\[
\Gamma_{5+} = -\frac{\alpha_s N_f p_+}{\pi} \left\{ 1 - \int_0^1 \frac{2m^2(1-x)dx}{m^2 + P^2x(1-x)} \right\}. \tag{71}
\]

The first term in the r.h.s. of (71) arises from the last term in (70) and is of ultraviolet origin. As was stressed by Gribov [38] and in Ref. [37] it cannot be addressed to any definite set of quark-gluon configurations and is a result of collective effects in QCD vacuum. In other words, this term can be considered as a local probe of gluon helicity.

where the gluons are on mass shell. The corresponding Feynman diagrams are the same as in Fig.1 with the only difference, that the wavy lines represent now the polarized gluons and the lower vertices are the vertices of quark-gluon interaction. Put \( q = 0, p = -p', p^2 < 0 \).

It is convenient to use the light-cone kinematics, where \( p_0 = p_+ + p_-/2, p_z = p_+ - p_-/2, p^2 = 2p_+p_- < 0 \) and work in the infinitely fast moving system along the \( z \)-direction. The calculation of (68) is performed for gluon helicity +1. The contribution of the crossing diagram Fig.3.2.1b is equal to the last propagator in (68), what results in the integration domain in \( k \)-space, 

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\times \frac{1}{(k^2 - m^2 + i\varepsilon)^2} \frac{1}{((p - k)^2 - m^2 + i\varepsilon)}. \tag{68}
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\Gamma_{5+} = -\frac{\alpha_s N_f p_+}{\pi^2} \int_0^1 dx \int \frac{d^{n-2}k_T}{[k_T^2 + m^2 + P^2x(1-x)]^2} \left\{ k_T^2(1-2x) - m^2 - 2\left(\frac{n-4}{n-2}\right)k_T^2(1-x) \right\}, \tag{70}
\]

where \( P^2 = -p^2 \). In the integration over \( k_- \) it was enough to take the residue at the pole of the last propagator in (68), what results in the integration domain in \( k_+ \): \( 0 < k_+ < p_+ \), and allowed to put \( k_+ = xp_+ \). The last term in (70) arised from \( n - 4 \) regulator dimensions and is proportional to \( \hat{k}^2 \), where \( \hat{k} \) is the projection of \( k \) into these dimensions. The azimuthal average gives \( \overline{k}^2 = k_T^2\frac{n-4}{n-2} \).

The first term in the curly brackets in (70) vanishes after integration over \( x \). (In fact, the equal to zero result after integration over \( x \) is multiplied by divergent integral over \( k_T \). So, strictly speaking, this term is uncertain. This problem will be discussed later.) After integration over \( k_T \), using the rules of dimensional regularization and going to \( n = 4 \), we get [33]:

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\Gamma_{5+} = -\frac{\alpha_s N_f p_+}{\pi} \left\{ 1 - \int_0^1 \frac{2m^2(1-x)dx}{m^2 + P^2x(1-x)} \right\}. \tag{71}
\]

The first term in the r.h.s. of (71) arises from the last term in (70) and is of ultraviolet origin. As was stressed by Gribov [38] and in Ref. [37] it cannot be addressed to any definite set of quark-gluon configurations and is a result of collective effects in QCD vacuum. In other words, this term can be considered as a local probe of gluon helicity.
The magnitude of $\Gamma_{5^+}$ strongly depends on the ratio $m^2/P^2$. At $m^2/P^2 \ll 1$

$$\Gamma_{5^+} = -\frac{\alpha_s N_f p_+}{\pi}. \tag{72}$$

In the opposite case, $m^2/P^2 \gg 1$ the second term in r.h.s. of (71) almost entirely cancels
the first one and approximately

$$\Gamma_{5^+} \approx 0. \tag{73}$$

The real physical situation corresponds to the first case. Gluons do not exist as free particles,
they are confined in hadrons and their virtualities are of order of inverse confinement radius squared, $P^2 \sim R_c^{-2} \gg m^2$.

Turn now to a more detailed discussion of the contribution of the first term in the
curly brackets in (70). At fixed $k_T$ this contribution is zero, because the integration over
$x$: the denominator is symmetric under interchange $x \leftrightarrow (1-x)$, while the numerator is
antisymmetric under such interchange. For the same reason the contribution of this term
vanishes at dimensional regularization at $n \neq 4$. However, the domain of low $k_T^2 \leq P^2$
contributes to the integral over $k_T^2$ here. In this domain we cannot be sure, that the integrand
in the first term in (70) has the same form as it is presented there. If this form is different –
we know nothing about it – and if it is not symmetric under interchange $x \rightarrow (1-x)$, then
nonvanishing infrared contribution to $\Gamma_{5^+}$ can arise from this term [22]. Consider the simple
model with infrared cut-off in $k_{1\perp}^2$,

$$k_T^2 > M^2(x, P^2), \tag{74}$$

where $M^2(x, P^2)$ is some function of $x, P^2$. In the first term of (68) the integral over $k_T^2$
can be written as the integral in the limits $(0, \infty)$, which vanishes after integration over $x$, as
before, minus the integral in the limits $(0, M^2)$. As a result, neglecting the term, proportional
to $m^2$, instead of (72) we get [22]:

$$\Gamma_{5^+} = -\frac{\alpha_s N_f p_+}{\pi} \left\{ 1 - \int_0^1 dx (1-2x)[\ln r(x) - r(x)] \right\}, \tag{75}$$

where

$$r(x) = \frac{x(1-x)P^2}{x(1-x)P^2 + M^2(x, P^2)}. \tag{76}$$

Eq.(75) demonstrates, that the matrix element $\langle g_{polar} | j_{\mu 5} | g_{polar} \rangle$ is not entirely contributed
by ultraviolet domain, connected with the anomaly, but can get the contribution from infrared region.

Note, that in the parton model $\Gamma_{5^+}$ is related to the part of hadron spin carried by gluons
in polarized hadron:

$$\Delta g_1^{h\perp} = \int_0^1 g_{1,gl}^h(x)dx = (\Gamma_{5^+}/2p_+)\[ g_{gl^+} - g_{gl^-} \]. \tag{77}$$

Here $g_{gl^+}$ and $g_{gl^-}$ are the numbers of gluons in hadron, with helicities $+1$ and $-1$ correspondingly.
$g_{1,gl}^h(x)$ is the contribution of gluons to the structure function $g_1^h(x)$. For polarized proton it was found [39,40,41]

$$\Delta g_1 = -\frac{\alpha_s N_f}{2\pi}(g_{gl^+} - g_{gl^-}). \tag{78}$$
6 Summary

The analysis of the axial anomaly was performed not only by the standard methods, but also by the use of the less well known method – the method of the spectral representation of the three point AVV amplitude [21], [23], [24]. It was shown, that the latter has some advantages in comparison with the formers. E.g. it allows one to describe not only the decay of $\pi^0$ into two real photons, as was done earlier [25], but also into two virtual ones. It was argued that in cases of octet and singlet axial currents in QCD nonperturbative effects are important. The t’Hooft consistency condition was discussed and it was demonstrated, that it is not universal: the nonperturbative effects can spoil condition.

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