Bose-Einstein condensates in a double well: mean-field chaos and multi-particle entanglement

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(Dated: submitted: 06 October 2008, to be published in the proceedings of the Laser physics conference in Trondheim 2008)

A recent publication [Phys. Rev. Lett. 100, 140408 (2008)] shows that there is a relation between mean-field chaos and multi-particle entanglement for BECs in a periodically shaken double well. “Schrödinger-cat” like mesoscopic superpositions in phase-space occur for conditions for which the system displays mean-field chaos. In the present manuscript, more general highly-entangled states are investigated. Mean-field chaos accelerates the emergence of multi-particle entanglement; the boundaries of stable regions are particularly suited for entanglement generation.

I. INTRODUCTION

Systems of ultra-cold atoms with time-periodic potential differences have recently started to be implemented experimentally both on the single-particle level \([1]\) and for Bose-Einstein condensates (BECs) \([2, 3]\). These experiments realise schemes of tunnelling-control devised in recent years \([4, 5]\). Very recently, even the control of the Mott-insulator transition via periodic shaking \([6, 7]\) has been realised experimentally \([8]\). This would, roughly speaking, correspond to making a glass of water freeze by turning on a loud-speaker. Proposals for shaking-induced tunnelling control for ultra-cold atoms not realised so far include schemes for entanglement generation \([9, 10, 11]\). The tunnelling control based on shaking the potentials uses concepts successfully implemented in the dressed-atom picture \([12]\).

The double-well potentials \([13]\) used for the single-particle experiments \([1]\) would promise further exciting experimental results for BECs: a simple mean-field model of this system is known to be chaotic for not too small interactions \([14, 15, 16, 17, 18]\). For BECs in a double-well potential with periodic shaking, we demonstrated numerically the emergence of “Schrödinger-cat” like states in phase-space if the parameters are chosen such that the mean-field system displays a coexistence of regular and chaotic dynamics \([19]\). In this paper, more general entangled states are investigated. As would be the authors of Ref. \([20]\), the quantum Fisher information \([21, 22]\) can be used to identify multi-particle entanglement. This leads to a a greater variety of highly entangled states than we investigated in Ref. \([19]\).

This manuscript is organised as follows: Section II introduces the models used to describe the BEC in a double well both on the mean-field (Gross-Pitaevskii) level and on the N-particle quantum level. Section III explains the way entangled states are detected in the numerics. Before concluding the paper, Sec. IV shows the numerical results for the emergence of entanglement.

II. MODEL

Bose-Einstein condensates in double-well potentials are often described via a model originally developed in nuclear physics \([23]\), a multi-particle Hamiltonian in two-mode approximation \([24]\):

\[
\hat{H} = -\frac{\hbar \Omega}{2} (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2^\dagger \hat{a}_1^\dagger) + \hbar \kappa (\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger) + \hbar (\mu_0 + \mu_1 \sin(\omega t)) (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1),
\]

where the operator \(\hat{a}_j^{(1)}\) creates (annihilates) a boson in well \(j\): \(\hbar \mu_0\) is the tilt between well 1 and well 2 and \(\hbar \mu_1\) is the driving amplitude. The interaction between a pair of particles in the same well is denoted by \(2\hbar \kappa\). Applications of such Hamiltonians include multi-particle entanglement \([9, 11, 25, 26]\), high precision measurements, many-body quantum coherence \([27, 28]\) and spin systems \([29]\).

Within the approximation on which the Gross-Pitaevskii equation is based, a multi-particle wave-function is characterised by two variables, \(\theta\) and \(\phi\). The fraction of atoms found in well 1 (well 2) is given by \(\cos^2[\theta/2] (\sin^2[\theta/2])\); the phase-factor between both wells is given by \(\exp(i\phi)\). All atoms occupy this single particle wave-function. One can also find corresponding \(N\)-particle wave-functions known as “atomic coherent states” \([30]\). In an expansion in the Fock-basis \(|n, N-n\rangle\) with \(n\) atoms in well 1 and \(N-n\) atoms in well 2 these wave-functions read:

\[
|\theta, \phi\rangle = \sum_{n=0}^{N} \binom{N}{n}^{1/2} \cos^n(\theta/2) \sin^{N-n}(\theta/2) \times e^{i(N-n)\phi} |n, N-n\rangle.
\]

The Gross-Pitaevskii dynamics can be mapped to that of a nonrigid pendulum \([31]\). Including the term describing the periodic shaking, the Hamilton function is given.
III. MULTI-PARTICLE-ENTANGLEMENT & QUANTUM FISHER INFORMATION

Multi-particle-entanglement is a hot topic of current research; to experimentally realise “Schrödinger-cat” like superpositions of BECs is still a challenge of fundamental research. For the periodically driven double-well potential, the relation between emergence of “Schrödinger-cat” like mesoscopic superpositions in phase-space and mean-field chaos has been discovered in Ref. [19]. An ideal example of such a mesoscopic superposition is shown in Fig. 1; the fidelity discovered in Ref. [19]. An ideal example of such a mesoscopic superposition is shown in Fig. 1; the fidelity discovered in Ref. [19].

In our case, $z/2$ is the experimentally measurable population imbalance which can be used to characterise the mean-field dynamics. The dynamics for this system are known to become chaotic [22].

Before defining the quantum Fisher information, we note that the creation and annihilation operators can be used to define

$$J_x = \frac{1}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) ,$$  \hspace{1cm} (4)

$$J_y = -\frac{i}{2} (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) ,$$  \hspace{1cm} (5)

$$J_z = \frac{1}{2} (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) ,$$  \hspace{1cm} (6)

which satisfy angular momentum commutation rules. Except for a factor of $N$, the operator $\hat{J}_z$ is the operator of the (experimentally measurable) population imbalance.

While the quantum Fisher information $F_Q$ can be defined for statistical mixtures, it is particularly simple for pure states [20]:

$$F_Q = 4(\Delta \hat{J}_n)^2 ;$$  \hspace{1cm} (7)

where $\Delta \hat{J}_n$ are the mean-square fluctuations of $\hat{J}$ in direction $n$. In the following we choose the $z$-direction, thus

$$F_Q = 4(\Delta \hat{J}_z)^2 .$$  \hspace{1cm} (8)

For $N$ particles, a sufficient condition for multi-particle entanglement is given by [20]:

$$F_{ent} > 1 ,$$  \hspace{1cm} (9)

$$F_{ent} \equiv \frac{F_Q}{N} .$$  \hspace{1cm} (10)

For the ideal “Schrödinger-cat” state in real space,

$$|\psi_{NOON}\rangle = \frac{1}{\sqrt{2}} (|N,0\rangle + |0,N\rangle) ,$$  \hspace{1cm} (11)

this entanglement flag reaches a value of $N$. Thus, while Eq. (9) already indicates multi-particle entanglement, values of

$$F_{ent} \gg 1$$  \hspace{1cm} (12)

demonstrate highly entangled states.

IV. RESULTS

In order to characterise whether or not the mean-field dynamics are chaotic, Poincaré surface of sections are particularly suited: for a set of initial conditions, the mean-field dynamics of the periodically driven system characterised by the angular frequency $\omega$ is plotted at integer multiples of $2\pi/\omega$. Figure 2 shows a phenomenon which is very characteristic for the non-rigid pendulum on which the BEC in a double well was mapped within mean-field: the coexistence between chaotic and regular regions.

While each initial condition will, in general, lead to many dots in plots like Fig. 2, we proceed in the spirit...
FIG. 2: Poincaré surface of section for the mean-field system. Closed loops characterise stable orbits whereas chaos is represented by irregular dots. The parameters are chosen such that they correspond to a one-photon resonance: a tilt of $2\mu_0/\Omega = 3.0$, a driving frequency of $\omega = 3\Omega$, an interaction of $N\kappa/\Omega = 0.8$ and a driving amplitude of $2\mu_1/\Omega = 0.9$ (cf. Ref. [5]).

of Ref. [19] to characterise the $N$-particle dynamics: In Fig. 3 each initial condition leads to only one point: the entanglement flag (10) for this initial condition after a fixed time $t\Omega$. Highly entangled states (Eq. (12)) can be found on the boundary of stable regions; already for short times (Fig. 3a) such highly entangled states can occur. For larger times (Fig. 3b) many features of the Poincaré surface of section in Fig. 2 are visible in the entanglement generation which essentially spreads over the entire chaotic part of the initial conditions.

For parameters which display no chaotic parts in the Poincaré surface of section (Fig. 1), on short time-scales hardly any entanglement emerges (Fig. 5a). For larger time-scales, entanglement generation occurs on the boundaries of stable regions (Fig. 5b).

V. CONCLUSION

In Ref. [19] we discovered the relation between mesoscopic “Schrödinger-cat” like superpositions in phase space and mean-field entanglement. In the present paper, we demonstrated that also for more general entangled states, multi-particle entanglement can be a quantum signature of chaos. For regular systems, the general entangled states also occur. However, it is restricted to the boundaries of stable regions and only occurs on longer time-scales. While the focus in Ref. [19] was on finding particularly highly entangled states for each initial condition, in the present manuscript the entanglement production was shown for each initial condition at the same point in time, both for short and longer times between

FIG. 3: (Colour online) The entanglement flag (10) in a two-dimensional projection as a function of the initial condition for $N = 100$ particles. All other parameter as in Fig. 2. The $N$-particle wave-function which corresponds to the mean-field initial conditions $(z_0, \phi_0)$ can be found in Eq. (2). In the upper panel, the (experimentally measurable) entanglement flag (10) is shown after a time of $t\Omega = 10$, in the lower panel the time is $t\Omega = 100$. Highly entangled states (cf. Eq. (12)) occur in the entire chaotic regime.

FIG. 4: Poincaré surface of section for the mean-field system (cf. Fig. 2). The parameters are chosen such that they correspond to a 3/2-photon resonance [6] with $N\kappa/\Omega = 0.1$, $2\mu_0/\Omega = 3.0$, $\omega/\Omega = 2.08$ and $2\mu_1/\Omega = 1.8$. 
As an entanglement-flag, we apply the quantum Fisher information for pure states. In this paper we use it in a way which is particularly easy to measure experimentally. However, using Eq. (10) assumes a pure state and would thus only be valid in an ideal system without decoherence. For more realistic situations, experimental signatures as in Refs. [40, 41] should be investigated in the future.

Acknowledgments

We would like to thank A. Smerzi for useful discussions and M. Holthaus for his continuous support. N.T. acknowledges funding by the Studienstiftung des deutschen Volkes.

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