REMARK ABOUT THE SPECTRUM OF THE p-FORM LAPLACIAN UNDER A COLLAPSE WITH CURVATURE BOUNDED BELOW

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Abstract. We give a lower bound on the number of small positive eigenvalues of the p-form Laplacian in a certain type of collapse with curvature bounded below.

1. Introduction

A general problem in spectral geometry is to estimate the eigenvalues of the p-form Laplacian on a closed Riemannian manifold $M$ in terms of the geometry of $M$. From Hodge theory, the number of zero eigenvalues is $b_p(M)$, the $p$-th Betti number of $M$. Hence the issue is to understand the positive eigenvalues. The papers [2], [7] and [8] study the case when one assumes an upper bound on the diameter of the manifold and double-sided bounds on the sectional curvatures. An important phenomenon is the possible appearance of positive eigenvalues of the $p$-form Laplacian that approach zero as a manifold collapses with bounded curvature.

The analysis of [7] and [8] uses the results of Cheeger, Fukaya and Gromov on the geometric structure of manifolds that collapse with double-sided curvature bounds. If one only assumes a lower sectional curvature bound then there are some structure results about collapsing in [4] and [12], but the theory is less developed than in the bounded curvature case.

In this paper we look at the small positive eigenvalues of the $p$-form Laplacian in an example of collapse with curvature bounded below. Namely, suppose that a compact Lie group $G$ acts isometrically on $M$ on the left. Give $G$ a left-invariant Riemannian metric. For $\epsilon > 0$, let $\epsilon G$ denote $G$ with its Riemannian metric multiplied by $\epsilon^2$. Let $M_\epsilon$ denote $M = G \setminus (\epsilon G \times M)$ equipped with the quotient Riemannian metric $g_\epsilon$, where $G$ acts diagonally on $\epsilon G \times M$ on the left. If $G$ is connected then $\lim_{\epsilon \to 0} M_\epsilon = G \setminus M$ in the Gromov-Hausdorff topology, and as $\epsilon$ goes to zero, the sectional curvatures of $M_\epsilon$ stay uniformly bounded below [12].

For notation, if $M$ is a smooth connected closed manifold with Riemannian metric $g$, let $\{\lambda_{p,j}(M, g)\}_{j=1}^{\infty}$ denote the eigenvalues (counted with multiplicity) of the Laplacian on $\operatorname{Im}(d) \subset \Omega^p_{L^2}(M)$.

Theorem 1. If $j = \dim (\ker (H^p(G \setminus M; \mathbb{R}) \to H^p(M; \mathbb{R})))$ then $\lim_{\epsilon \to 0} \lambda_{p,j}(M_\epsilon, g_\epsilon) = 0$.

In Section 2 we prove Theorem 1. The main points of the proof are the use of a certain variational expression for $\lambda_{p,j}(M, g)$, due to Cheeger and Dodziuk [3], and the avoidance of dealing with the detailed orbit structure of the group action. We then look at the example...
of an $S^1$-action on $S^{2n}$, which is the suspension of the Hopf action of $S^1$ on $S^{2n-1}$, and show that our results slightly improve those of Takahashi \cite{10}. In Section 3 we make some further remarks.

I thank Junya Takahashi for sending me a copy of his paper.

2. Proof of Theorem \[2\]

Let $\mathfrak{g}$ be the Lie algebra of $G$. It acquires an inner product from the left-invariant Riemannian metric on $G$. Given $\mathfrak{r} \in \mathfrak{g}$, let $\mathfrak{X}$ be the corresponding vector field on $M$. Let $i_{\mathfrak{X}}$ denote interior multiplication by $\mathfrak{X}$.

Let $\Omega^*(M)$ denote the smooth differential forms on $M$. Let $\Omega_{L^2}^*(M)$ be the $L^2$-completion of $\Omega^*(M)$. Put

$$\Omega_{\text{max}}^*(M) = \{ \omega \in \Omega_{L^2}^*(M) : d\omega \in \Omega_{L^2}^{*+1}(M) \},$$

where $d\omega$ is originally defined distributionally.

Put

$$\Omega_G^*(M) = \{ \omega \in \Omega^*(M) : g \cdot \omega = \omega \text{ for all } g \in G \}$$

and

$$\Omega_{\text{basic}}^*(M) = \{ \omega \in \Omega_G^*(M) : i_{\mathfrak{X}}\omega = 0 \text{ for all } \mathfrak{r} \in \mathfrak{g} \}.$$  \(\text{(2.3)}\)

Let $\Omega_{G,L^2}^*(M)$ and $\Omega_{\text{basic},L^2}^*(M)$ be the $L^2$-completions of $\Omega_G^*(M)$ and $\Omega_{\text{basic}}^*(M)$, respectively. Put

$$\Omega_{\text{basic,max}}^*(M) = \{ \omega \in \Omega_{\text{basic},L^2}^*(M) : d\omega \in \Omega_{\text{basic},L^2}^{*+1}(M) \},$$

where $d\omega$ is originally defined distributionally. Then $\Omega_{\text{basic,max}}^*(M)$ is a complex.

From \cite{8} and \cite{11}, the cohomology of the complex $\Omega_{\text{basic}}^*(M)$ is isomorphic to $H^*(G\backslash M; \mathbb{R})$.

Lemma 1. The cohomology of the complex $\Omega_{\text{basic,max}}^*(M)$ is isomorphic to $H^*(G\backslash M; \mathbb{R})$.

Proof. The proof is essentially the same as that of \cite{11}. The only point to note is that the homotopy operator $A$ used in the Poincaré lemma in \cite{11} sends $\Omega_{\text{basic,max}}^*$ to itself. \qed

The quotient map $p : \epsilon G \times M \to M_x$ defines a principal $G$-bundle. Pullback gives an isomorphism $p^* : \Omega^*(M_x) \cong \Omega^*_{\text{basic}}(\epsilon G \times M)$. The parallelism of $G$ gives an isomorphism

$$\Omega^*(\epsilon G \times M) \cong C^\infty(G) \otimes \Lambda^*(\mathfrak{g}^*) \otimes \Omega^*(M).$$

Taking $G$-invariants gives isomorphisms

$$\Omega_G^*(\epsilon G \times M) \to (C^\infty(G) \otimes \Lambda^*(\mathfrak{g}^*) \otimes \Omega^*(M))^G \xrightarrow{\beta} \Lambda^*(\mathfrak{g}^*) \otimes \Omega^*(M),$$

where $\beta$ comes from the map which sends $\sum_k f_k \otimes \eta_k \otimes \omega_k \in C^\infty(G) \otimes \Lambda^*(\mathfrak{g}^*) \otimes \Omega^*(M)$ to $\sum_j f_j(e) \eta_j \otimes \omega_k \in \Lambda^*(\mathfrak{g}^*) \otimes \Omega^*(M)$.

Let $\{ \mathfrak{r}_j \}_{j=1}^{\dim(G)}$ be a basis of $\mathfrak{g}$. For $\mathfrak{r} \in \mathfrak{g}$, let $e(\mathfrak{r}^*)$ denote exterior multiplication by $\mathfrak{r}^*$ on $\Lambda^*(\mathfrak{g}^*)$. 

Lemma 2. There is an isomorphism of complexes \( \mathcal{I} : \Omega^* (M) \to \Omega^*_{\text{basic}} (\epsilon G \times M) \subset \Lambda^* (\mathfrak{g}^*) \otimes \Omega^* (M) \) given by

\[
\mathcal{I} (\sigma) = \left( \prod_{j=1}^{\dim(G)} (1 - e (r_j^*) \otimes i_{x_j}) \right) (1 \otimes \sigma)
\]

\[
= \sum_{k=0}^{\dim(G)} (-1)^k \sum_{1 \leq j_1 < \ldots < j_k \leq \dim(G)} (r_{j_1}^* \wedge \ldots \wedge r_{j_k}^*) \otimes i_{x_{j_1}} \ldots i_{x_{j_k}} \sigma. \tag{2.7}
\]

Proof. If \( \sum_k f_k \otimes \eta_k \otimes \omega_k \in (\mathcal{O}^\infty (G) \otimes \Lambda^* (\mathfrak{g}^*) \otimes \Omega^* (M))^G \) is \( G \)-basic then for \( r \in \mathfrak{g} \), we also have

\[
\sum_k (f_k \otimes i_r \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes i_{x^*} \omega_k) = 0. \tag{2.8}
\]

Then

\[
\sum_k (f_k (e) \otimes i_r \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k (e) \otimes \eta_k \otimes i_{x^*} \omega_k) = 0, \tag{2.9}
\]

i.e. if \( \sum_k \eta_k \otimes \omega_k \) lies in the image of \( \beta \) restricted to \( \Omega^*_{\text{basic}} (\epsilon G \times M) \) then

\[
\sum_k (i_r \eta_k \otimes \omega_k + (-1)^{|\eta_k|} \eta_k \otimes i_{x^*} \omega_k) = 0. \tag{2.10}
\]

It follows that \( \sum_k \eta_k \otimes \omega_k \) can be written as \( \mathcal{I} (\sigma) \) for some \( \sigma \in \Omega^* (M) \). Thus \( \mathcal{I} \) is surjective. It is clearly injective.

It remains to show that \( \mathcal{I} \) is a morphism of complexes. Let \( d^{\text{inv}} \) denote the (finite-dimensional) differential on \( \Lambda^* (\mathfrak{g}^*) \). If an element of \( \Omega^*_{\text{basic}} (\epsilon G \times M) \) is represented as \( \sum_k f_k \otimes \eta_k \otimes \omega_k \in C^\infty (G) \otimes \Lambda^* (\mathfrak{g}^*) \otimes \Omega^* (M) \) then the \( G \)-invariance implies that for \( r \in \mathfrak{g} \),

\[
\sum_k (x f_k \otimes \eta_k \otimes \omega_k + f_k \otimes \eta_k \otimes \mathcal{L}_{x^*} \omega_k) = 0. \tag{2.11}
\]

The differential of \( \sum_k f_k \otimes \eta_k \otimes \omega_k \) is represented by

\[
\sum_k \left( \sum_{j=1}^{\dim(G)} x_j f_k \otimes e (r_j^*) \eta_k \otimes \omega_k + f_k \otimes d^{\text{inv}} \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes d \omega_k \right). \tag{2.12}
\]

From (2.11), this equals

\[
\sum_k \left( - \sum_{j=1}^{\dim(G)} f_k \otimes e (r_j^*) \eta_k \otimes \mathcal{L}_{x_j} \omega_k + f_k \otimes d^{\text{inv}} \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes d \omega_k \right). \tag{2.13}
\]

Using \( \beta \), it follows that the induced differential on \( \Lambda^* (\mathfrak{g}^*) \otimes \Omega^* (M) \) sends \( \sum_k \eta_k \otimes \omega_k \) to

\[
\sum_k \left( - \sum_{j=1}^{\dim(G)} e (r_j^*) \eta_k \otimes \mathcal{L}_{x_j} \omega_k + d^{\text{inv}} \eta_k \otimes \omega_k + (-1)^{|\eta_k|} \eta_k \otimes d \omega_k \right). \tag{2.14}
\]

One can check that when this acts on \( \mathcal{I} (\sigma) \), the result is \( \mathcal{I} (d \sigma) \). Thus \( \mathcal{I} \) is an isomorphism of complexes. \( \square \)
In fact, under our identifications, \( \mathcal{I} \) is the same as \( p^* \).

Let \( M^{\text{reg}} \) be the union of the principal orbits for the \( G \)-action on \( M \). It is a dense open subset of \( M \) with full measure. If \( m \in M^{\text{reg}} \), let \( H \subset G \) be its isotropy subgroup, with Lie algebra \( \mathfrak{h} \). Define \( \alpha : \mathfrak{g} \to T_m M \) by \( \alpha(r) = x_m \). It passes to an injection \( \overline{\alpha} : \mathfrak{g}/\mathfrak{h} \to T_m M \).

For \( \epsilon \geq 0 \), put \( \rho_\epsilon(m) = \det^{1/2} (\epsilon^2 \text{Id}_\mathfrak{g} + \overline{\alpha}^* \overline{\alpha}) \). If \( m \notin M^{\text{reg}} \), put \( \rho_\epsilon(m) = 0 \). Note that for \( \epsilon > 0 \), \( \rho_\epsilon^{-1}(m) < \rho_0^{-1}(m) \).

**Lemma 3.** \( \rho_0^{-1} \in L^1(M,d\text{vol}). \)

**Proof.** If \( m \in M^{\text{reg}} \), then up to an overall constant, \( \rho_0(m) \) is the volume of the orbit \( G \cdot m \). Then \( \int_{M^{\text{reg}}} \rho_\epsilon^{-1}(m) \; d\text{vol}(m) \) is proportionate to the volume of \( G \backslash M^{\text{reg}} \subset G \backslash M \), which is seen to be finite. \( \square \)

Let \( \{ \mathbf{r}_j \}_{j=1}^{\dim(G)} \) be an orthonormal basis of \( \mathfrak{g} \).

**Lemma 4.** For \( \epsilon > 0 \), there is a positive constant \( C(\epsilon) \) such that \( \Omega^*(M_\epsilon) \) is isometrically isomorphic to \( \Omega^*(M) \) with the new norm

\[
\| \omega \|_\epsilon^2 = C(\epsilon) \int_M \rho_\epsilon^{-1}(m) \left( \| \omega(m) \|^2_M + \sum_{k=1}^{\dim(G)} \epsilon^{-2k} \sum_{1 \leq j_1 < \ldots < j_k \leq \dim(G)} |i_{\mathbf{x}_{j_1}} \ldots i_{\mathbf{x}_{j_k}} \omega(m)|^2_M \right) \; d\text{vol}(m). \tag{2.15}
\]

**Proof.** We can compute the norm squared of \( \omega \in \Omega^*(M_\epsilon) \) by taking the local norm squared of \( p^*\omega \) on \( \epsilon G \times M^{\text{reg}} \), dividing by the function which assigns to \( (g,m) \in \epsilon G \times M^{\text{reg}} \) the volume of the orbit \( \epsilon G \cdot (g,m) \), and integrating over \( \epsilon G \times M^{\text{reg}} \). If \( m \in M^{\text{reg}} \) then the relative volume of \( \epsilon G \cdot (g,m) \) is

\[
\det^{1/2} (\epsilon^2 \text{Id}_\mathfrak{g} + \alpha^* \alpha) = \epsilon^\dim(H) \rho_\epsilon(m). \tag{2.16}
\]

The map \( \beta \) of (2.6) is an isometry, up to a constant. As \( \{ \epsilon^{-1} \mathbf{r}_j \}_{j=1}^{\dim(G)} \) is an orthonormal basis for \( T_\epsilon(\mathcal{I}) \), the lemma follows from Lemma 3. \( \square \)

**Proof of Theorem [4]:** Put \( \lambda_{p,j}(\epsilon) = \lambda_{p,j}(M_\epsilon,g_\epsilon) \). From [3],

\[
\lambda_{p,j}(\epsilon) = \inf_V \sup_{\eta \in V - 0} \sup_{\theta \in d^{-1}(\eta)} \frac{\| \eta \|_\epsilon^2}{\| \theta \|_\epsilon^2}, \tag{2.17}
\]

where \( V \) ranges over \( j \)-dimensional subspaces of \( \text{Im}(d : \Omega^{p-1}(M) \to \Omega^p(M)) \), and \( \theta \in d^{-1}(\eta) \subset \Omega^{p-1}(M) \).

Take \( j = \dim(\text{Ker}(H^p(G\backslash M; \mathbb{R}) \to H^p(M; \mathbb{R}))) \). From Lemma 1, the inclusion of complexes \( \Omega^*_{\text{basic}}(M) \to \Omega^*_{\text{basic,max}}(M) \) induces an isomorphism on cohomology. Then there is a \( j \)-dimensional subspace \( V \) of

\[
\text{Ker}(d : \Omega^p_{\text{basic}}(M) \to \Omega^{p+1}_{\text{basic}}(M)) \cap \text{Im}(d : \Omega^{p-1}_{\text{basic}}(M) \to \Omega^p_{\text{basic}}(M)) \tag{2.18}
\]

such that if \( \eta \in V - 0 \) then \( \eta \notin \text{Im}(d : \Omega^{p-1}_{\text{basic,max}}(M) \to \Omega^p_{\text{basic,L}_\epsilon}(M)) \). We claim that

\[
\lim_{\epsilon \to 0} \sup_{\eta \in V - 0} \sup_{\theta \in d^{-1}(\eta)} \frac{\| \eta \|_\epsilon^2}{\| \theta \|_\epsilon^2} = 0. \tag{2.19}
\]
This will suffice to prove the theorem.

Suppose that (2.19) is not true. Then there are a constant $c > 0$, a sequence $\{\epsilon_r\}_{r=1}^{\infty}$ in $\mathbb{R}^+$ approaching zero, a sequence $\{\eta_r\}_{r=1}^{\infty}$ in $V - 0$ and a sequence $\{\theta_r\}_{r=1}^{\infty}$ in $\Omega^{p-1}(M)$ such that for all $r$, $d\theta_r = \eta_r$ and

$$\frac{\|\eta_r\|_{\epsilon_r}^2}{\|\theta_r\|_{\epsilon_r}^2} \geq c. \quad (2.20)$$

Doing a Fourier decomposition of $\theta_r$ with respect to $G$, the ratio in (2.20) does not decrease if we replace $\theta_r$ by its $G$-invariant component. Thus we may assume that $\theta_r$ is $G$-invariant.

Without loss of generality, we can replace the norm $\|\cdot\|_\epsilon$ of (2.15) by the same norm divided by $C(\epsilon)$, which we again denote by $\|\cdot\|_\epsilon$. As $\eta_r$ is smooth on $M$, it follows from Lemma 3 that the function $\rho^{-1}_0(m) |\eta_r(m)|^2_M$ is integrable on $M$. Without loss of generality, we may assume that

$$\int_M \rho^{-1}_0(m) |\eta_r(m)|^2_M d\text{vol}(m) = 1. \quad (2.21)$$

As $\{\eta_r\}_{r=1}^{\infty}$ lies in the sphere of a finite-dimensional space, there will be a subsequence, which we relabel as $\{\eta_r\}_{r=1}^{\infty}$, that converges smoothly to some $\eta_\infty \in V - 0$.

From (2.20),

$$\|\theta_r\|_{\epsilon_r}^2 \leq c^{-1} \|\eta_r\|_{\epsilon_r}^2 = c^{-1} \int_M \rho^{-1}_\epsilon(m) |\eta_r(m)|^2_M d\text{vol}(m) \leq c^{-1} \int_M \rho^{-1}_0(m) |\eta_r(m)|^2_M d\text{vol}(m) = c^{-1}. \quad (2.22)$$

For large $r$,

$$\int_M |\theta_r(m)|^2_M d\text{vol}(m) \leq (\inf \rho^{-1}_\epsilon) \int_M |\theta_r(m)|^2_M d\text{vol}(m) \leq (\inf \rho^{-1}_0) c^{-1}. \quad (2.23)$$

We now work with respect to the metric $g$ on $M$. By weak-compactness of the unit ball in $L^2$, there is a subsequence of $\{\theta_r\}_{r=1}^{\infty}$, which we relabel as $\{\theta_r\}_{r=1}^{\infty}$, that converges weakly in $L^2$ to some $\theta_\infty \in \Omega^{p-1}_{G,L^2}(M)$. Then for $\sigma \in \Omega^p(M)$,

$$\langle \sigma, \eta_\infty \rangle_M - \langle d^* \sigma, \theta_\infty \rangle_M = \lim_{r \to \infty} (\langle \sigma, \eta_r \rangle_M - \langle d^* \sigma, \theta_r \rangle_M) = \lim_{r \to \infty} \langle \sigma, \eta_r - d\theta_r \rangle_M = 0. \quad (2.24)$$

Thus $\theta_\infty \in \Omega^{p-1}_{\text{basic,max}}(M)$ and $d\theta_\infty = \eta_\infty$.

From (2.22), we also obtain that for each $1 \leq j \leq \dim(G)$,

$$\int_M |i_{x_j} \theta_r(m)|^2_M d\text{vol}(m) \leq (\inf \rho^{-1}_\epsilon) \int_M |i_{x_j} \eta_r(m)|^2_M d\text{vol}(m) \leq (\inf \rho^{-1}_0) c^{-1} \epsilon_r^2. \quad (2.25)$$

Then for all $\sigma \in \Omega^{p-2}(M)$,

$$\langle \sigma, i_{x_j} \theta_\infty \rangle_M = \langle (i_{x_j})^* \sigma, \theta_\infty \rangle_M = \lim_{r \to \infty} \langle (i_{x_j})^* \sigma, \theta_r \rangle_M = \lim_{r \to \infty} \langle \sigma, i_{x_j} \theta_r \rangle_M = 0. \quad (2.26)$$

Thus $i_{x_j} \theta_\infty = 0$ and $\theta_\infty \in \Omega^{p-1}_{\text{basic,max}}(M)$. Hence $\eta_\infty \in \text{Im} \left( d : \Omega^{p-1}_{\text{basic,max}}(M) \to \Omega^p_{\text{basic},L^2}(M) \right)$, which is a contradiction. \qed
Example 1. Let $G = U(1)$ act on $M = S^{2n}$ by the suspension of the Hopf action of $U(1)$ on $S^{2n-1}$. Then $G\backslash M = U(1)\backslash S^{2n}$ is the suspension of $\mathbb{C}P^{n-1}$. One finds that $\text{Ker}(H^p(G\backslash M; \mathbb{R}) \to H^p(M; \mathbb{R}))$ is nonzero if and only if $p \in \{3, 5, \ldots , 2n-1\}$. From Theorem 1, as $\epsilon \to 0$ there are small eigenvalues of the $p$-form Laplacian on $\text{Im}(d^*) \subset \Omega^p_{L^2}(M_\epsilon)$ for $p \in \{3, 5, \ldots , 2n-1\}$. From the Hodge decomposition, there will also be small eigenvalues of the $p$-form Laplacian on $\text{Im}(d^* \epsilon) \subset \Omega^p_{L^2}(M_\epsilon)$ for $p \in \{2, 4, \ldots , 2n-2\}$. Then using Hodge duality, one concludes that there are small eigenvalues on

1. $\text{Im}(d^*) \subset \Omega^1_{L^2}(M_\epsilon)$,
2. $\text{Im}(d^*) \subset \Omega^p_{L^2}(M_\epsilon)$ and $\text{Im}(d^*) \subset \Omega^p_{L^2}(M_\epsilon)$ for $p \in \{2, 3, 4, \ldots , 2n-3, 2n-2\}$, and
3. $\text{Im}(d^*) \subset \Omega^{2n-1}_{L^2}(M_\epsilon)$.

This slightly sharpens [10, Theorem 1.2]. Note that from eigenvalue estimates for the scalar Laplacian $\square$, there are no small eigenvalues on $\text{Im}(d^*) \subset \Omega^0_{L^2}(M_\epsilon)$, $\text{Im}(d) \subset \Omega^1_{L^2}(M_\epsilon)$, $\text{Im}(d^*) \subset \Omega^{2n-1}_{L^2}(M_\epsilon)$ or $\text{Im}(d) \subset \Omega^{2n}_{L^2}(M_\epsilon)$.

3. Remarks

1. In the case of a locally-free torus action, there is some intersection between Theorem 1 and the results of [2], [7] and [8]. In [2] one deals with the cohomology of a certain $\mathbb{R}$-graded sheaf $H^0(A_{[0]}^\infty)$ on the limit space $X$. In the case of a collapsing coming from a locally-free torus action, Theorem 1 is a statement about the case $* = 0$, when the sheaf $H^0(A_{[0]}^\infty)$ is the constant $\mathbb{R}$-sheaf on $X$. Of course, the result of Theorem 1 will generally not give all of the small positive eigenvalues that arise in a collapse. As seen in the Example, one can obtain more small eigenvalues just from Hodge duality.

2. Theorem 1 indicates that the relevant cohomology of the limit space is the ordinary cohomology, as opposed for example to the $L^2$-cohomology. This is consistent with the results of [2] in the bounded curvature case.

3. If $G$ has positive dimension and acts effectively on $M$ then Theorem 1 describes small positive eigenvalues in a collapsing situation. In some noncollapsing situations, one can show that small eigenvalues do not exist. Here is one such criterion.

Proposition 1. Let $\mathcal{M}$ be a collection of closed $n$-dimensional Riemannian manifolds, with $n > 0$. Give $\mathcal{M}$ the Lipschitz metric, coming from biLipschitz homeomorphisms. Suppose that $\mathcal{M}$ can be covered by a finite number of metric balls. For $p \in \mathbb{Z} \cap [0, n]$ and $j \geq 0$, there are positive numbers $a_{p,j}$ and $A_{p,j}$ so that if $(M, g) \in \mathcal{M}$ then $a_{p,j} \leq \lambda_{p,j}(M, g) \leq A_{p,j}$, and $\lim_{j \to \infty} a_{p,j} = \infty$.

Proof. Suppose first that for some $p$ and $j$, there is no upper bound on $\lambda_{p,j}(M, g)$ as $(M, g)$ ranges over $\mathcal{M}$. Then there is a sequence $\{(M_i, g_i)\}_{i=1}^\infty$ in $\mathcal{M}$ with the property that $\lim_{i \to \infty} \lambda_{p,j}(M_i, g_i) = \infty$. A subsequence of $\{(M_i, g_i)\}_{i=1}^\infty$, which we relabel as $\{(M_i, g_i)\}_{i=1}^\infty$, will have finite distance from some $(M_\infty, g_\infty) \in \mathcal{M}$. Then there are a number $\epsilon > 0$ and a sequence of biLipschitz homeomorphisms $h_i : M_\infty \to M_i$ so that for all $i$,

$$e^{-\epsilon} g_\infty \leq h_i^* g_i \leq e^\epsilon g_\infty.$$ (3.1)
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Here $h_i^*g_i$ is a Lipschitz metric on $M^\infty$. From Hodge theory,

$$\lambda_{p,j}(M_i, g_i) = \inf_{V} \sup_{\eta \in V - 0} \sup_{\theta \in d^{-1}(\eta)} \frac{\| \eta \|^2_{M_i}}{\| \theta \|^2_{M_i}}, \quad (3.2)$$

where $V$ ranges over $j$-dimensional subspaces of $\text{Im}(d : \Omega_{\text{max}}^{p-1}(M_i) \to \Omega_L^p(M_i))$, and $\theta \in d^{-1}(\eta) \subset \Omega_{\text{max}}^{p-1}(M_i)$. By naturality,

$$\lambda_{p,j}(M_i, g_i) = \inf_{V} \sup_{\eta \in V - 0} \sup_{\theta \in d^{-1}(\eta)} \frac{\| \eta \|^2_{h_i^*g_i}}{\| \theta \|^2_{h_i^*g_i}}, \quad (3.3)$$

where $V$ ranges over $j$-dimensional subspaces of $\text{Im}(d : \Omega_{\text{max}}^{p-1}(M^\infty) \to \Omega_L^p(M^\infty))$, and $\theta \in d^{-1}(\eta) \subset \Omega_{\text{max}}^{p-1}(M^\infty)$.

As in [3], it follows from (3.1) and (3.3) that there is a positive integer $J$ which only depends on $n$ so that

$$e^{-J\epsilon} \lambda_{p,j}(M^\infty, g_i) \leq \lambda_{p,j}(M_i, g_i) \leq e^{J\epsilon} \lambda_{p,j}(M^\infty, g_i). \quad (3.4)$$

This contradicts the assumption that $\lim_{i \to \infty} \lambda_{p,j}(M_i, g_i) = \infty$.

Now suppose that it is not true that there is a uniform lower bound $\lambda_{p,j}$ on $\{\lambda_{p,j}(M, g)\}_{(M,g) \in \mathcal{M}}$ with the property that $\lim_{j \to \infty} \lambda_{p,j} = \infty$. Then there are a number $C > 0$, a sequence $\{(M_i, g_i)\}_{i=1}^\infty$ in $\mathcal{M}$ and a sequence of integers $\{j_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} j_i = \infty$ and for each $i$, $\lambda_{p,j_i}(M_i, g_i) \leq C$. Take a subsequence $\{(M_i, g_i)\}_{i=1}^\infty$ and an $(M^\infty, g^\infty)$ as before. Then for each $j_i$,

$$\lambda_{p,j}(M^\infty, g^\infty) \leq \sup_{i \to \infty} \lambda_{p,j_i}(M^\infty, g^\infty) \leq \sup_{i \to \infty} e^{J\epsilon} \lambda_{p,j_i}(M_i, g_i) \leq e^{J\epsilon} C. \quad (3.5)$$

This contradicts the discreteness of the spectrum of the $p$-form Laplacian on $(M^\infty, g^\infty)$. $\square$

Proposition 1 shows that in a certain sense, one has uniform eigenvalue bounds in the noncollapsing case. It seems possible that for a given $n \in \mathbb{Z}^+$, $K \in \mathbb{R}$ and $v, D > 0$, the collection $\mathcal{M}$ of connected $n$-dimensional Riemannian manifolds $(M, g)$ with sectional curvatures greater than $K$, volume greater than $v$ and diameter less than $D$ satisfies the hypotheses of Proposition 1. It is known that there is a finite number of homeomorphism types in $\mathcal{M}$ [3]. On the other hand, the analogous space of metrics defined with Ricci curvature, instead of sectional curvature, will generally not satisfy the hypotheses of Proposition 1 [4].

References

[1] P. Bérard, “From Vanishing Theorems to Estimating Theorems: the Bochner Technique Revisited”, Bull. Amer. Math. Soc. 19, p. 371-406 (1988)
[2] B. Colbois and G. Courtois, “A Note on the First Nonzero Eigenvalue of the Laplacian Acting on $p$-Forms”, Manuscripta Math. 68, p. 143-160 (1990)
[3] J. Dodziuk, “Eigenvalues of the Laplacian on Forms”, Proc. Amer. Math. Soc. 85, p. 437-443 (1982)
[4] K. Fukaya and T. Yamaguchi, “The Fundamental Groups of Almost Nonnegatively Curved Manifolds”, Ann. of Math. 136, p. 253-333 (1992)
[5] K. Grove, P. Petersen and J. Wu, “Geometric Finiteness Theorems via Controlled Topology”, Invent. Math. 99, p. 205-213 (1990)
[6] J. Koszul, “Sur Certains Groupes de Transformations de Lie”, in Géométrie différentielle. Colloques Internationaux du CNRS, Strasbourg, p. 137-141 (1953)
[7] J. Lott, “Collapsing and the Differential Form Laplacian: The Case of a Smooth Limit Space”, to appear, Duke Math. J., [http://www.math.lsa.umich.edu/~lott](http://www.math.lsa.umich.edu/~lott)

[8] J. Lott, “Collapsing and the Differential Form Laplacian: The Case of a Singular Limit Space”, preprint, [http://www.math.lsa.umich.edu/~lott](http://www.math.lsa.umich.edu/~lott)

[9] G. Perelman, “Construction of Manifolds of Positive Ricci Curvature with Big Volume and Large Betti Numbers”, in Comparison Geometry, Math. Sci. Res. Inst. Publ. 30, Cambridge Univ. Press, Cambridge, p. 157-163 (1997)

[10] J. Takahashi, “Small Eigenvalues on p-Forms for Collapsing of the Even Dimensional Spheres”, preprint (2001)

[11] A. Verona, “A de Rham Type Theorem for Orbit Spaces”, Proc. Amer. Math. Soc. 104, p. 300-302 (1988)

[12] T. Yamaguchi, “Collapsing and Pinching Under a Lower Curvature Bound”, Ann. of Math. 133, p. 317-357 (1991)

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