Discrete Time Quasi-Crystals

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Between space crystals and amorphous materials there exists a third class of aperiodic structures which lack translational symmetry but reveal long-range order. They are dubbed quasi-crystals and their formation, similarly as the formation of space crystals, is related to spontaneous breaking of translation symmetry of underlying Hamiltonians. Here, we investigate spontaneous emergence of quasi-crystals in periodically driven systems. We consider a quantum many-body system which is driven by a harmonically oscillating force and show that interactions between particles result in spontaneous self-reorganization of the motion of a quantum many-body system and in the formation of a quasi-crystal structure in time.

Quasi-crystals are related to spatial structures which can not be reproduced by translation of an elementary cell [1]. Nevertheless they form diffraction patterns but with spots forbidden by crystallographic point-group symmetries [2, 3]. Discovery of quasi-crystals led to a paradigm shift within chemistry for which Dan Shechtman was awarded the Nobel Prize in 2011. Quasi-crystals are a subject of research not only in solid state physics. They are investigated in the field of optics where quasi-crystal photonic materials can be engineered [4–7] and in ultra-cold atomic gases by means of a suitable arrangement of laser beams which create an effective potential for atoms of a quasi-crystal structure [8].

Recently research of crystalline structures has migrated to the time domain [9] (for phase-space crystals see [10–13]). Indeed, a quantum many-body system can spontaneously self-organize its motion and start moving periodically forming a crystalline structure in the time domain. While the first idea of such time crystals turned out to be impossible for the realization [14–17] another type of spontaneous formation of crystalline structures in time was proposed. These are the so-called discrete time crystals that are periodically driven quantum many-body systems which break spontaneously discrete time translation symmetry of Hamiltonians and start moving with a period different from the driving period [18–21]. Discrete time crystals have been already realized in laboratories [22–26] and they draw considerable attention in the literature [27–46] (see also [47–52] for classical version of time crystals). In the field of time crystals, quasi-crystal structures have been investigated in classical systems [53], quantum systems [40, 54, 55] and in an experiment on magnon condensation [56]. In Refs. [57, 58] quasi-crystal response of systems which are driven quasi-periodically in time was demonstrated.

In the present letter we analyze how a quasi-crystal structure forms due to spontaneous breaking of translation symmetry of a many-body Hamiltonian. Periodically driven many-body systems turn out to be suitable systems for such investigations. Our results indicate that novel structures in time can be formed spontaneously: discrete time quasi-crystals where discrete time translation symmetry of the time-periodic Hamiltonian is spontaneously broken into an aperiodic structure in time but with long-time order.

One-dimensional (1D) quasi-crystals can be generated by a cut of a square lattice with the help of a line whose gradient is an irrational number [59–61]. For the Fi-
bonacci quasi-crystal the gradient is the golden ratio and the successive cuts of vertical and horizontal lines of the square lattice produce a sequence \textit{LRLLRLRL...} of two elementary cells which we denote by \textit{L} and \textit{R}, see Fig. 1. This sequence can be also generated with the help of the substitution rule \textit{L} \rightarrow \textit{LR} and \textit{R} \rightarrow \textit{L} \ [1]. After each application of the rule the sequence grows and its length is given by the successive Fibonacci numbers. The sequence corresponds to a quasi-crystal structure where there is no translation symmetry but two elementary cells are not distributed randomly so that the sequence reveals long-range order indicated by the presence of peaks in its Fourier transform \ [1]. A finite Fibonacci quasi-crystal can be obtained by cutting the square lattice with a line whose gradient is a rational number that approximates the golden ratio, see Fig. 1. In the following we show how any finite Fibonacci quasi-crystal structure can spontaneously emerge in the time evolution of a periodically driven quantum many-body system if interactions between particles are sufficiently strong.

We focus on ultra-cold atoms bouncing between two orthogonal harmionically oscillating mirrors in a 2D model (similar phenomena can be realized in the 3D space and in other systems). Such a system can be realized experimentally \ [62] (for the stationary mirror experiments see \ [63–70]). The single-particle problem, i.e. an atom bouncing between the mirrors is described, in the frame oscillating with the mirrors \ [80] \ [71], by the Hamiltonian \ [72]

\[
H = \frac{p_x^2 + p_y^2}{2} + x + y + \lambda_x x \cos(\omega t + \Delta \phi) + \lambda_y y \cos(\omega t),
\]

where \(\omega\) is the frequency of the mirrors’ oscillations, \(\Delta \phi\) the relative phase and \(\lambda_x, \lambda_y\) determine the amplitudes of the oscillations. The mirrors are located at \(x = 0\) and at \(y = 0\) and the gravitational force \(F_g\) points in the \(-(x + y)\) direction, see inset of Fig. 1. We assume that in the many-body case, \(N\) bosons are bouncing between the mirrors and interact via Dirac-delta potential \(g_0 \delta(r)\) \ [73]. Such contact interactions are determined by the s-wave scattering length of atoms which is assumed to be negative \(g_0 < 0\). The system is periodically driven, thus, we may look for a kind of stationary states which evolve periodically in time. They are eigenstates of the Floquet Hamiltonian \(\hat{H}(t) = \hat{H} - i\partial_t\), where \(\hat{H}\) is a many-body version of (9) with the contact interactions between particles included, see \ [71]. The corresponding eigenvalues are called quasi-energies of the system \ [72, 74]. The discrete time translation symmetry of the time-periodic Hamiltonian, \(\hat{H}(t + 2\pi/\omega) = \hat{H}(t)\), implies that all Floquet eigenstates must evolve with the driving period \(2\pi/\omega\). In the following we show that in the limit when \(N \rightarrow \infty\) but \(g_0 N = \text{const.}\), there are subspaces of the Hilbert space of the many-body system where low-lying quasi-energy eigenstates are extremely fragile because they form macroscopic superposition. Consequently even an infinitesimally weak perturbation, e.g. a measurement of a position of one atom, is sufficient to induce collapse of the many-body state to one of the superimposed states. It results in breaking of the discrete time translation symmetry of the Hamiltonian \ [18]. Interestingly an evolving symmetry broken state can reveal a Fibonacci sequence in the time domain and therefore can form a quasi-crystal in time.

Let us start with the single-particle problem (9) which consists of the independent motion along \(x\) and \(y\) directions. We are interested in a resonant driving of the system, i.e. the frequencies \(\Omega_x\) and \(\Omega_y\) of the unperturbed particle motion along the \(x\) and \(y\) directions fulfill \(s_x \Omega_x = \omega\) and \(s_y \Omega_y = \omega\) with integer \(s_x\) and \(s_y\). The description of a resonantly driven particle can be reduced to an effective tight-binding Hamiltonian identical to a problem of an electron in a space crystal, see Refs. \ [28, 72, 76, 77] and \ [71]. When we switch from a single particle to many bosons resonantly driven, the single-

![FIG. 2: Density of atoms bouncing between two orthogonal oscillating mirrors at \(t = 2\pi/3\omega\). The left (\textit{L}) mirror is located at \(x = 0\) and the right (\textit{R}) mirror at \(y = 0\) and the gravitational force \(F_g\) points in the \(-(x + y)\) direction, see inset of Fig. 1. Left panel is related to the symmetry preserving state which evolves periodically with the driving period \(2\pi/\omega\) — the left and right mirrors are visited by atoms alternately: \textit{LRLLR}. The presented density consists of \(s_x s_y\) localized Wannier-like wavepackets (\(s_x = 2, s_y = 3\)). The trajectory of the Wannier wavepackets are moving along is drawn in the panels. Right panel corresponds to a symmetry broken state where interactions between atoms result in spontaneous breaking of discrete time translation symmetry of the Hamiltonian and emergence of a quasi-crystal structure in time. Atoms are visiting the left and right mirrors in an order that matches the Fibonacci sequence \textit{LRLLR} — a finite Fibonacci quasi-crystal is presented, i.e. the golden ratio gradient of the line in Fig. 1 is approximated by the rational number \(\Omega_x/\Omega_y = s_y/s_x = 3/2\). The parameters of the system are the following: \(\lambda_x = 0.094\), \(\lambda_y = 0.030\), \(\omega = 1.1\), \(\Delta \phi = 2\pi/3\) and the interaction strength \(g_0 N = 0\) (left) and \(g_0 N = -0.022\) (right). The latter results in \(U_{\text{HF}}/J = -81\), with \(J = 4.8 \times 10^{-6}\), in the Bose-Hubbard Hamiltonian (23) that describes an effective \(s_x \times s_y\) lattice. The results are obtained within the quantum secular approach \ [75].]
particle tight-binding Hamiltonian becomes the Bose-Hubbard Hamiltonian,

\[ \mathcal{H}_{\text{eff}} = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{i,j} U_{ij} \hat{a}_i^\dagger \hat{a}_j \hat{a}_j^\dagger \hat{a}_i, \]  

(2)

which is the many-body Floquet Hamiltonian written in a basis of time-periodic functions \( W_i = \psi_{\text{eff}}(r,t) = w_{L,R}(x,t)w_{L,R}(y,t) \) which are localized wavepackets propagating periodically along the classical resonant orbit with the period \( T = s_x s_y 2\pi / \omega \) and which play a role of Wannier functions known in condensed matter physics [78], see Fig. 2. There are \( s_x s_y \) Wannier functions \( W_i \) which are products of localized wavepackets \( w_{L,R}(x,t) \) and \( w_{L,R}(y,t) \) moving along the \( x \) and \( y \) directions with the periods \( 2\pi / \Omega_x \) and \( 2\pi / \Omega_y \), respectively. In (23), \( \hat{a}_i \)'s are the standard bosonic annihilation operators, the nearest neighbor tunneling amplitudes \( J_{ij} = -\frac{(2/\pi)}{\int_0^\pi dt \int d^2 r W_i^\dagger(r,t) [H - \partial_t] W_j(r,t) \) and the coefficients of the effective interactions \( U_{ij} = \frac{(2/\pi)}{\int_0^\pi dt \int d^2 r \mid W_i^\dagger W_j \mid ^2 \) for \( i \neq j \) and similar \( U_{ii} \) but by factor two smaller [28, 40, 71]. In the present Letter we choose the amplitudes of the mirrors' oscillations, \( \lambda_x \) and \( \lambda_y \), so that the resulting amplitudes for nearest neighbor tunnelings along the \( x \) and \( y \) directions are the same, \( J = J_{ii} \). Typically, the coefficient for the on-site interactions \( \lambda_{ii} \) is at least an order of magnitude larger than \( U_{ij} \) for long-range interactions (\( i \neq j \)) and therefore values of the former will be used to provide information about the strength of the interactions.

To conclude this part, the description of the resonantly driven many-body system is reduced, in the time-periodic basis \( W_i(r,t) \), to the Bose-Hubbard Hamiltonian (23) [71].

For negligible interactions between particles the ground state of \( \mathcal{H}_{\text{eff}} \) is a Bose-Einstein condensate \( \psi_0(r_1, \ldots, r_N, t) = \prod_{j=1}^N \psi(r_j, t) \), i.e. all atoms occupy a condensate wavefunction \( \psi(r, t) \propto \sum_i W_i(r, t) \) which evolves with the driving period \( 2\pi / \omega \). Indeed, despite the fact that each \( W_i \) evolves with the period \( T = s_x s_y 2\pi / \omega \), after each period \( 2\pi / \omega \), the Wannier wavefunctions \( W_i \) exchange their positions so that the condensate wavefunction \( \psi(r, t) \) propagates with the driving period, see Fig. 2. When the interactions between atoms are present and they are attractive and sufficiently strong we may expect that it is energetically favourable to group all atoms in a single localized wavepacket \( W_i(r, t) \) [18]. Then, we expect the ground state to be of the form \( \psi_0 = \prod_{j=1}^N W_i(r_j, t) \) where \( i \) can be arbitrary. However, such a state cannot be a Floquet eigenstate of the system because it evolves with the period \( T = s_x s_y 2\pi / \omega \) while the discrete time translation symmetry of the Hamiltonian requires that all Floquet eigensates must evolve with the period of the driving \( 2\pi / \omega \). In order to reconcile the energy and symmetry requirements, the ground state of \( \mathcal{H}_{\text{eff}} \) takes the form \( \psi_0 \propto \sum_i \prod_{j=1}^N W_i(r_j, t) \) which is macroscopic superposition of Bose-Einstein condensates. However, such a macroscopic superposition is extremely fragile and it is sufficient, e.g., to measure position of one atom and the ground state collapses to one of the Bose-Einstein condensates which form the macroscopic superposition, \( \Psi_0 \to \Psi \approx \prod_{j=1}^N W_i(r_j, t) \) — which \( W_i \) is chosen depends on a result of the measurement. In the limit when \( N \to \infty \) but \( U_{ii} N = \text{const.} \), the latter state is robust and evolves with the period \( T = s_x s_y 2\pi / \omega \) and thus breaks time translation symmetry of the many-body Hamiltonian. The described scenario is an example of a process which is responsible for spontaneous breaking of time translation symmetry in the quantum many-body system. We will see that if we choose properly the frequencies \( \Omega_x \) and \( \Omega_y \), time evolution of a symmetry broken state can reveal a Fibonacci quasi-crystal.

In order to get quantitative prediction for critical interaction strength that leads to spontaneous breaking of time translation symmetry and to obtain exact symmetry broken states we apply the mean-field approach [44, 71, 79]. The mean-field approximation is valid because the ground state of (23) for negligible interactions and also symmetry broken states, \( \Psi \approx \prod_{j=1}^N W_i(r_j, t) \), in the regime of the quasi-crystal formation are Bose-Einstein condensates. The simplest way to switch to the mean-field description is to substitute the operators \( \hat{a}_i \) by complex numbers \( a_i \) in (23) [44, 71]. Then, the energy of the system per particle reads \( E = -\langle J / 2 \rangle \sum_{\langle i,j \rangle} a_i^* a_j + \langle N / 2 \rangle \sum_i (U_{ii} |a_i|^2 |a_i|^2 \) and we are looking for a condensate wavefunction \( \psi(r, t) = \sum_i a_i W_i(r, t) \) which minimizes \( E \). In the left panel of Fig. 2 we show such a wavefunction \( \psi(r, t) \) obtained for negligible interactions, \( U_{ii} N / J \approx 0 \), and for \( \Omega_x = \omega / 2 \) and \( \Omega_y = \omega / 3 \) (i.e. \( s_x = 2, s_y = 3 \) ). The wavefunction \( \psi(r, t) \) is a uniform superposition of \( s_x s_y = 6 \) localized Wannier wavepackets, it evolves with the period \( 2\pi / \omega \) and describes atoms bouncing periodically once from the left mirror \( (L) \) then from the right mirror \( (R) \) and so on. If we plot probabilities for the measurement of atoms close to the left mirror, \( \rho_L(t) = \int dy \psi(x \approx 0, y, t)^2 \), and close to the right mirror, \( \rho_R(t) = \int dx |\psi(x, y \approx 0, t)|^2 \), we can see that maximal values of \( \rho_L, \rho_R \) appear alternately and form a periodic sequence \( LRLR \ldots \), see Fig. 3. However, if the interactions are sufficiently strong, i.e. \( U_{ii} N / J \lesssim -6.5 \), the system chooses spontaneously motion with the period \( T = s_x s_y 2\pi / \omega \). That is, the mean-field approach shows that the ground state energy is degenerate and the corresponding wavefunctions are not uniform superposition of Wannier wavepackets. The system prepared in a lowest energy mean-field state breaks discrete time translation symmetry of the many-body Hamiltonian because it evolves with the period different from the driving period. For \( U_{ii} N / J \lesssim -25 \) the symmetry broken degenerate ground states reduce to single Wannier wavefunctions \( \psi(r, t) \approx W_i(r, t) \) with accuracy better than...
99% — which \( W_1 \) is chosen by the system is determined in a spontaneous symmetry breaking process. In Fig. 2 we show an example of such a ground state wavefunction \( \psi(r, t) \) where a single localized wavepacket bouncing between the mirrors is visible. The corresponding probabilities \( \rho(t) \) form a sequence \( LRLLR \), whose length is \( s_x + s_y = 5 \), which is repeated with the period \( T \), see Fig. 3. The sequence is actually a Fibonacci quasi-crystal corresponding to the fifth Fibonacci number [53].

It becomes now clear how to realize conditions where any finite Fibonacci quasi-crystal emerges due to spontaneous breaking of discrete time translation symmetry of the time-periodic Hamiltonian: (i) One has to choose a rational number \( s_x/s_y \) which approximates the golden ratio and reproduces a given Fibonacci sequence when it is taken as the gradient of the line in Fig. 1. (ii) Then, we know which resonant subspace of the periodically driven many-body system is able to realize such a quasi-crystal, i.e. the subspace corresponding to the frequencies of unperturbed single-particle motion \( \Omega_x = \omega/s_x \) and \( \Omega_y = \omega/s_y \). (iii) If the many-body system is prepared in a low-lying eigenstate within this subspace, then atoms are bouncing off the left and right mirrors either in a periodic way, if the interactions are negligible, or in an aperiodic manner if the interactions are sufficiently strong. In the latter case a quasi-crystal structure is formed due to spontaneous breaking of the discrete time translation symmetry. In the right panels of Fig. 3 we illustrate these two situations for \( s_x = 8 \) and \( s_y = 13 \). In the symmetry preserving case, the probabilities for detection atoms close to the left and right mirrors, \( \rho_L(t), \rho_R(t) \), show a periodic sequence of maxima \( LRRL \ldots \). However, when the attractive interactions are sufficiently strong, the discrete time translation symmetry is spontaneously broken and the Fibonacci sequence \( LRRLRLRL \ldots \) is formed which corresponds to the eighth Fibonacci number equal \( s_x + s_y = 21 \) [53].

To conclude we have shown that quasi-crystal structures can emerge in the time domain due to spontaneous breaking of discrete time translation symmetry of the time-periodic many-body Hamiltonian. By analogy to discrete time crystals, related to spontaneous breaking of discrete time translation symmetry into another discrete time translation one [18–21], we dub the considered phenomena discrete time quasi-crystals. They can be realized, e.g., in ultra-cold atoms bouncing between oscillating atom mirrors if atoms are loaded to a resonant classical orbit. The latter can be done if an atomic Bose-Einstein condensate is prepared in a trap located at a classical turning point of a resonant trajectory and afterwards the trapping potential is turned off at a proper moment of time synchronized with the oscillations of the mirrors [44]. It results in a quantum state where all atoms occupy a single localized Wannier-like wavepacket that evolves along a resonant orbit. For sufficiently strong attractive interactions between atoms, the localized atomic wavepacket will perform evolution with a quasi-crystal structure in time and will not decay. In contrast, the same experiment performed for non-interacting atoms will show tunneling of atoms to
other localized wavepackets evolving along the resonant orbit and consequently destruction of the quasi-crystal structure in time.

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SUPPLEMENTAL MATERIAL

In this Supplemental Material we describe details of the description of the many-body system of atoms bouncing resonantly between two orthogonal oscillating atom mirrors in the presence of the gravitational force. We begin with a single-particle problem and then generalize the approach to the many-body case.

SINGLE-PARTICLE PROBLEM

We start with a short introduction to the Floquet formalism [74] and then switch to the description of a single-particle system which we are interested in. We show how to obtain an effective Floquet Hamiltonian that describes resonant dynamics of the system.

Floquet states

Consider the following time-dependent Schrödinger equation

\[ i \frac{\partial}{\partial t} \psi(x, t) = H(t) \psi(x, t) \quad (3) \]

with Hamiltonian having a discrete time translation symmetry \( H(t+T) = H(t) \). According to the Floquet theorem, solution of the equation (3) may be written as a linear combination of the functions of the form

\[ \psi_k(x, t) = e^{-i\varepsilon_k t} \phi_k(x, t), \quad (4) \]

where \( \phi_k(x, t) \) are time-periodic with the same period \( T \) as the Hamiltonian [74]. Substituting the solution (4) into the Schrödinger equation (3) we obtain

\[ H \phi_k(x, t) = \varepsilon_k \phi_k(x, t), \quad (5) \]

where \( H = H(t) - i \frac{\partial}{\partial t} \) is termed the Floquet Hamiltonian and \( \phi_k(x, t) \) are so-called Floquet eigenstates. Since the function \( \phi_k(x, t) e^{i\frac{\varepsilon_k}{\omega} t} \), where \( n \) is integer, is also a solution of the eigenequation (5) corresponding to the eigenvalue \( \varepsilon_k + n \frac{2\pi}{T} \), the quasi-energy spectrum is periodic with the period \( \frac{2\pi}{\omega} \) and in the description of a system it is actually sufficient to restrict to a fragment of the spectrum, i.e. to a single Floquet zone of the width of \( \frac{2\pi}{T} \).

Single-particle Hamiltonian

Let us consider an atom bouncing between two orthogonal oscillating atom mirrors which form the angle \( \pi/4 \) with the gravitational force vector. The Hamiltonian, in the standard gravitational units [44] but with the gravitational acceleration \( g \to g/\sqrt{2} \), reads

\[ H = \frac{p_x^2 + p_y^2}{2} + x + y + F_x \left( x + \frac{\lambda_x}{\omega^2} \cos(\omega t + \Delta \phi) \right) \\
+ F_y \left( y + \frac{\lambda_y}{\omega^2} \cos(\omega t) \right), \quad (6) \]

where \( F_x(x) \) and \( F_y(y) \) describe the mirrors, i.e., the profile of the reflecting potentials along \( x \) and \( y \) directions, respectively. The mirrors oscillate harmonically with the frequency \( \omega \) around \( x = 0 \) and \( y = 0 \) with the amplitudes \( \lambda_x/\omega^2 \) and \( \lambda_y/\omega^2 \). Description of the system is more convenient if we switch from the laboratory frame to the frame oscillating with the mirrors. In the classical case it can be done by means of the canonical transformation,

\[ x' = x + \frac{\lambda_x}{\omega} \cos(\omega t + \Delta \phi), \quad y' = y + \frac{\lambda_y}{\omega} \cos(\omega t), \]

\[ p'_x = p_x - \frac{\lambda_x}{\omega} \sin(\omega t + \Delta \phi), \quad p'_y = p_y - \frac{\lambda_y}{\omega} \sin(\omega t), \quad (7) \]

while in the quantum case by the corresponding unitary transformation, i.e. \( U_y = e^{i\frac{\lambda_y}{\omega} \sin(\omega t)} e^{i(p'_y \frac{\lambda_y}{\omega} \cos(\omega t)} \) and a similar one for the motion along \( x \). The resulting Hamiltonian is the following

\[ H = \frac{p_x^2 + p_y^2}{2} + x' + y' + \lambda_x x' \cos(\omega t + \Delta \phi) \\
+ \lambda_y y' \cos(\omega t) + F_x(x') + F_y(y'). \quad (8) \]

We assume that the mirrors can be modeled by hard wall potentials located at \( x_t = 0 \) and \( y_t = 0 \) and therefore we may drop the \( F_x(x') \) and \( F_y(y') \) in (8) keeping in mind that motion of a particle takes place for \( x' \geq 0 \) and \( y' \geq 0 \). In the following we also drop primes and the final single-particle Hamiltonian reads

\[ H = \frac{p_x^2 + p_y^2}{2} + x + y + \lambda_x x \cos(\omega t + \Delta \phi) + \lambda_y y \cos(\omega t). \quad (9) \]

Description of resonant dynamics

Let us start with the classical mechanics. The single-particle problem described by the Hamiltonian (9) consists of independent motion along \( x \) and \( y \) directions. We are interested in a resonance driving and in order to describe resonant dynamics we perform canonical transformation to the so-called action-angle variables of the unperturbed problem (i.e. when \( \lambda_x = \lambda_y = 0 \) [76].
In these new canonically conjugate variables, the unperturbed Hamiltonian depends on the momenta (the actions \( I_x \) and \( I_y \) only),

\[
H_0(I_x, I_y) = \frac{p_x^2 + p_y^2}{2} + x + y = \frac{(3\pi)^{2/3}}{2} \left( \frac{p_x^2}{3} + \frac{p_y^2}{3} \right). \tag{10}
\]

If \( \lambda_x = \lambda_y = 0 \), the actions are constant of motion \((I_x, I_y) = \text{const})\) and the corresponding position variables (the angles \( \theta_x \) and \( \theta_y \)) evolve linearly in time, i.e. \( \theta_x, \theta_y = \Omega_{x,y} t + \theta_{x,y}(0) \) where

\[
\Omega_x(I_x) = \frac{dH_0(I_x, I_y)}{dI_x}, \quad \Omega_y(I_y) = \frac{dH_0(I_x, I_y)}{dI_y}, \tag{11}
\]

are frequencies of an unperturbed periodic bounding of a particle on the static mirrors. The total Hamiltonian (9) in the action-angle variables takes the form [72]

\[
H = H_0(I_x, I_y) + \lambda_x \cos(\omega t + \Delta \phi) \sum_n h_n(I_x) e^{i\theta_x} + \lambda_y \cos(\omega t) \sum_n h_n(I_y) e^{i\theta_y}, \tag{12}
\]

where \( h_0(I_{x,y}) = \left( \frac{\pi I_{x,y}}{\sqrt{3}} \right)^{2/3} \) and \( h_n(I_{x,y}) = (-1)^{n+1} \left( \frac{3\pi I_{x,y}}{\sqrt{3}} \right)^{2/3} \) if \( n \neq 0 \).

The resonant driving of a particle corresponds to the conditions

\[
s_x \Omega_x(I_{x_0}) = \omega, \quad s_y \Omega_y(I_{y_0}) = \omega, \tag{13}
\]

where \( s_x \) and \( s_y \) are integers and \( I_{x_0} \) and \( I_{y_0} \) are resonant values of the actions. In order to obtain an effective Hamiltonian that describes motion of a particle close to a resonant orbit we apply the classical secular approximation [72, 76]. First, we extend the phase space of the system by the time variable and its canonically conjugate momentum \( p_t = -\dot{H} \) which play a role of additional coordinates. The Hamiltonian in the extended phase space \( \mathcal{H} = H + p_t \) is the classical analogue of the quantum Floquet Hamiltonian, where \( p_t \to -i \frac{\partial}{\partial t} \). Next, we perform a canonical transformation to the frame moving along a resonant orbit,

\[
\Theta_x = \theta_x - \frac{\omega_t}{s_x} t, \quad \Theta_y = \theta_y - \frac{\omega_t}{s_y} t, \quad P_t = p_t + \frac{\omega I_x}{s_x} + \frac{\omega I_y}{s_y}, \tag{16}
\]

which results in

\[
\mathcal{H} = H_0(I_x, I_y) - \frac{\omega I_x}{s_x} - \frac{\omega I_y}{s_y} + P_t + \lambda_x \cos(\omega t + \Delta \phi) \sum_n h_n(I_x) e^{i\theta_x} e^{i\omega t/s_x} + \lambda_y \cos(\omega t) \sum_n h_n(I_y) e^{i\theta_y} e^{i\omega t/s_y}, \tag{17}
\]

and carry out averaging over time keeping all dynamical variables fixed. The latter is allowed because in the moving frame (16) both the actions and the angles and \( P_t \) are slowly varying quantities if we are close to the resonant orbit (i.e. \( P_x = I_x - I_{x_0} \approx 0 \) and \( P_y = I_y - I_{y_0} \approx 0 \)) and if the time-dependent perturbation is weak, i.e.

\[
\frac{d\Theta_x}{dt} = \Omega_x(I_x) - \frac{\omega}{s_x} + \mathcal{O}(\lambda_x) \approx 0, \quad \frac{d\Theta_y}{dt} = \Omega_y(I_y) - \frac{\omega}{s_y} + \mathcal{O}(\lambda_y) \approx 0,
\]

where \( V_x = \lambda_x h_{-s_x}(I_{x_0}), m_{-eff,x}^{-1} = \frac{d^2 H_0(I_{x_0}, I_{y_0})}{dI_{x_0}^2} \) and similar expressions for \( V_y \) and \( m_{eff,y} \).

The Hamiltonian (19) has been obtained within the classical approach. In order to switch to the quantum description one has to quantize (19), i.e. \( P_{x,y} = -i \frac{\partial}{\partial x,y} \) and \( P_t = -i \frac{\partial}{\partial t} \). The other option is to apply the quantum version of the secular approximation from the very beginning [75] that leads to the same result if we choose \( I_{x_0} \gg 1 \) and \( I_{y_0} \gg 1 \) [44]. The secular Hamiltonian (19) is time-independent which implies that \( P_t = \text{constant} \) and can be dropped. Actually in the quantum description, due to the time-periodicity of the system, eigenvalues \( k\omega \) of \( P_t \) are quantized (i.e. \( k \) is integer [72]) that makes the quasi-energy spectrum to repeat itself with the period \( \omega \) as expected, see the previous subsection. In the following we consider Floquet eigenstates corresponding to \( k = 0 \).

If we focus on a resonance where \( s_{x,y} \gg 1 \), the Hamiltonian (19) corresponds to a solid state problem of an electron moving in a two-dimensional space crystal. We will be considering the first energy band of the quantized version of (19) and therefore the description of a resonantly driven particle can still be simplified. Indeed, superposing the Bloch wave eigenfunctions of (19) corresponding to the first energy band we can construct Wannier functions \( W_{i_{x,y}}(\Theta_x, \Theta_y) = w_{i_{x}}(\Theta_x) w_{i_{y}}(\Theta_y) \) localized in different sites of the periodic potential in (19). These Wannier functions in the laboratory frame appear as localized wavepackets moving along a classical resonant orbit with the period \( T = s_x s_y 2\pi/\omega \), i.e. \( W_{i_{x,y}}(x,y,t) = w_{i_{x}}(x,t) w_{i_{y}}(y,t) \) where \( w_{i_{x}}(x,t) \) is periodic with the period \( s_x 2\pi/\omega \) and \( w_{i_{y}}(y,t) \) is periodic with
the period $s_y 2\pi/\omega$. In the basis of the Wannier functions, i.e. when we restrict to wavefunctions of the form $\psi = \sum_i a_i W_i$, the Hamiltonian (19) reads [18, 28]

$$H_{\text{eff}} \approx -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} a_i^* a_j,$$  \hspace{1cm} (20)

where the sum runs over nearest neighbour sites of the potential in (19) and

$$J_{ij} = -2\langle W_i | H_{\text{eff}} | W_j \rangle,$$  \hspace{1cm} (21)

stand for amplitudes of tunneling of a particle between neighbouring sites.

Equation (20) is a standard tight-binding model and it indicates that a resonantly driven particle is equivalent to a solid state problem if we use the basis of localized wavepackets $W_i(x,y,t)$ which are evolving periodically along a resonant classical trajectory [28].

**MANY-BODY PROBLEM**

**Many-body Hamiltonian**

In the present section we would like to switch from the single-particle problem to many ultra-cold atoms which are bosons and which are bouncing between two oscillating orthogonal mirrors. We focus on a resonant driving and restrict to the Hilbert subspace which is spanned by the localized Wannier wavepackets $W_i(x,y,t)$ introduced in the previous section. In other words we consider the subspace spanned by the Fock states $|n_{(i_1,1)}, \ldots, n_{(s_x,s_y)}\rangle$, where $n_{(i_x,i_y)}$ denotes numbers of bosons occupying a Wannier wavepacket $W_{i=(i_x,i_y)}$. Restricting to this subspace and expanding the bosonic field operator in the Wannier basis,

$$\hat{\psi}(x,y,t) \approx \sum_i W_i(x,y,t) \hat{a}_i,$$  \hspace{1cm} (22)

where $\hat{a}_i$ are the standard bosonic annihilation operators, and substituting (22) to the many-body Floquet Hamiltonian we obtain a many-body version of the tight-binding model (20),

$$\hat{\mathcal{H}} = \frac{1}{T} \int_0^T dx dy \, \hat{\psi}^\dagger \left[ H + g_0 \frac{\psi}{2} \hat{\psi} - i \partial_t \right] \hat{\psi},$$

$$\approx -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{ij} U_{ij} a_i^\dagger a_j^\dagger a_j a_i,$$  \hspace{1cm} (23)

where $H$ is given in (9), $T = s_x s_y 2\pi/\omega$ and $g_0$ is the parameter (proportional to the atomic s-wave scattering length) that characterizes the potential of contact interactions between atoms, $g_0 \delta(x) \delta(y)$. The effective interaction coefficients in (23) read

$$U_{ij} = \frac{2g_0}{T} \int_0^T dx dy \, |W_i|^2 |W_j|^2,$$  \hspace{1cm} (24)

for $i \neq j$ and similar $U_{ii}$ but by factor two smaller [28, 40]. Typically the on-site interaction coefficients $U_{ii}$ are at least an order of magnitude larger than $U_{ij}$ for long-range interactions ($i \neq j$). In the Letter we choose the parameters of the system so that all nearest neighbour tunneling amplitudes are the same,

$$J \equiv J_{ij}.$$  \hspace{1cm} (25)

The Hamiltonian (23) is actually the Bose-Hubbard model which is a many-body counterpart of the single-particle tight-binding Hamiltonian (20). It is valid if the interaction energy per particle is much smaller than the gap $E_{\text{gap}}$ between the first and the second energy bands of the single-particle problem (19) [28, 40]. For example for the parameters used in Figs. 2-3 in the Letter, the maximal on-site interaction energy per particle $|U_{ii}| N/J \approx 80$, where $N$ is the total number of atoms, that is much smaller than the energy gap $E_{\text{gap}}/J \approx 3000$.

**Mean-field approximation**

If the interaction between atoms is negligible (i.e. $g_0 \to 0$), the ground state of the Bose-Hubbard Hamiltonian (23) is a Bose-Einstein condensate where all atoms occupy a condensate wavefunction which is a uniform superposition of all Wannier functions

$$\psi(x,y,t) \propto \sum_i W_i(x,y,t).$$  \hspace{1cm} (26)

That is, the many-body ground state reads

$$\Psi(x_1, y_1, \ldots, x_N, y_N, t) = \prod_{j=1}^N \psi(x_j, y_j, t).$$  \hspace{1cm} (27)

On the other hand, if interactions between atoms are attractive and sufficiently strong it is energetically favorable to group all atoms in a single localized Wannier wavepacket because it decreases the energy. Then, we expect the ground state to be of the form $\Psi = \prod_{i=1}^N W_i(x_j, y_j, t)$ where $i = (i_x, i_y)$ is arbitrary. However, such a state cannot be a Floquet eigenstate of the system because it evolves with the period $T = s_x s_y 2\pi/\omega$ while the discrete time translation symmetry of the Hamiltonian requires that all Floquet eigenstates must evolve with the period of the driving $2\pi/\omega$. In order to reconcile the energy and symmetry requirements, the
ground state takes the form
\[ \Psi(x_1, y_1, \ldots, x_N, y_N, t) \propto \prod_{i=1}^{N} W_i(x_i, y_i, t), \] 
which in the Fock states basis reads
\[ |\Psi\rangle = \frac{1}{\sqrt{s_{x,y}}} (|N, 0, 0, \ldots, 0\rangle + |0, N, 0, \ldots, 0\rangle + \ldots + |0, 0, \ldots, 0, N\rangle). \]
Such a ground state is actually a Schrödinger cat-like state, i.e. a superposition of macroscopic states, and it is extremely fragile to any perturbation. For example if is sufficient to measure position of a single atom and the Schrödinger cat state collapses to one of the states which form the superposition,
\[ \Psi \rightarrow \prod_{j=1}^{N} W_i(x_j, y_j, t), \] 
in the Fock states basis it corresponds to
\[ |\Psi\rangle \rightarrow |0, \ldots, 0, N, 0, 0, \ldots, 0\rangle. \]
Which localized Wannier wavepacket \( W_i \) is chosen in (30) depends on the result of the measurement of the position of an atom. In other words this is an example of a spontaneous process which is responsible for spontaneous breaking of the discrete time translation symmetry of the Hamiltonian.

Note that both the ground state (27) of the weakly interacting system and a symmetry broken state (30) are Bose-Einstein condensates which can be described within the mean-field approximation. Thus, we may use the mean-field approach to describe the phenomenon we are after. In the mean-field description, the spontaneous time translation symmetry breaking will be indicated by appearance of a bifurcation where the symmetry-preserving ground state solution (27) looses its stability and new stable solutions are born which break the discrete time translation symmetry of the Hamiltonian and evolve with a period which is different from the driving period.

It is straightforward to obtain the mean-field equations. Indeed, the mean-field quasi-energy functional reads [18, 44]
\[ E = \frac{1}{T} \int_0^T dt \int dx dy \psi^* \left[ H + \frac{g_0 N}{2} |\psi|^2 - i \partial_t \right] \psi, \]
\[ \approx - \frac{1}{2} \sum_{(i,j)} J_{ij} a_i^* a_j + \frac{N}{2} \sum_{i,j} U_{ij} |a_i|^2 |a_j|^2, \] 
where we have restricted to the resonant subspace with a condensate wavefunction \( \psi(x, y, t) = \sum_i a_i W_i(x, y, t) \).

The ground state of (32) can be found by solving the corresponding Gross-Pitaevskii equation [79]. For negligible interactions, the ground state is of the form (26) while for sufficiently strong attractive interactions there are \( s_x s_y \) degenerate mean-field ground state solutions \( \psi(x, y, t) \approx W_i(x, y, t) \) corresponding to different values of \( i = (i_x, i_y) \).

The mean-field approach has been used in the Letter in order to obtain quantitative prediction for a range of the system parameters where time quasi-crystals form spontaneously.

[1] C. Janot, *Quasicrystals: A Primer* (Oxford University Press, 1994).
[2] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. 53, 1951 (1984), URL https://link.aps.org/doi/10.1103/PhysRevLett.53.1951.
[3] D. Levine and P. J. Steinhardt, Phys. Rev. Lett. 53, 2477 (1984), URL https://link.aps.org/doi/10.1103/PhysRevLett.53.2477.
[4] M. Kohmoto, B. Sutherland, and K. Iguchi, Phys. Rev. Lett. 58, 2436 (1987), URL https://link.aps.org/doi/10.1103/PhysRevLett.58.2436.
[5] W. Steurer and D. Sutter-Widmer, Journal of Physics D: Applied Physics 40, R229 (2007), URL http://stacks.iop.org/0022-3727/40/i=13/a=R01.
[6] E. Albuquerque and M. Cottam, Physics Reports 376, 225 (2003), ISSN 0370-1573, URL http://www.sciencedirect.com/science/article/pii/S0370157302005598.
[7] Vardeny Z. Valy, Nahata Ajay, and Agrawal Amit, Nature Photonics 7, 177 (2013).
[8] K. Viebahn, M. Sbroscia, E. Carter, J.-C. Yu, and U. Schneider, ArXiv e-prints (2018), 1807.00823.
[9] K. Sacha and J. Zakrzewski, Rep. Prog. Phys. 81, 016401 (2018), URL https://doi.org/10.1088/1361-6633/aa8b38.
[10] L. Guo, M. Marthaler, and G. Schön, Phys. Rev. Lett. 111, 205303 (2013), URL https://link.aps.org/doi/10.1103/PhysRevLett.111.205303.
[11] L. Guo and M. Marthaler, New Journal of Physics 18, 023006 (2016), URL http://stacks.iop.org/1367-2630/18/i=2/a=023006.
[12] L. Guo, M. Liu, and M. Marthaler, Phys. Rev. A 93, 053616 (2016), URL https://link.aps.org/doi/10.1103/PhysRevA.93.053616.
[13] L. Pengfei, M. Michael, and L. Guo, New Journal of Physics 20, 023043 (2018), ISSN 1367-2630, URL http://stacks.iop.org/1367-2630/20/i=2/a=023043.
[14] F. Wilczek, Phys. Rev. Lett. 109, 160401 (2012), URL http://link.aps.org/doi/10.1103/PhysRevLett.109.160401.
[15] P. Bruno, Phys. Rev. Lett. 111, 070402 (2013), URL http://link.aps.org/doi/10.1103/PhysRevLett.111.070402.
[16] H. Watanabe and M. Oshikawa, Phys. Rev. Lett. 114, 251603 (2015), URL http://link.aps.org/doi/10.1103/PhysRevLett.114.251603.
[17] A. Syrwid, J. Zakrzewski, and K. Sacha, Phys. Rev. Lett. 111, 205303 (2013), URL https://link.aps.org/doi/10.1103/PhysRevLett.111.205303.
(1986), ISSN 0973-7111, URL https://doi.org/10.1007/BF02845262.

[60] Bombieri, E. and Taylor, J. E., J. Phys. Colloques 47, C3 (1986), URL https://doi.org/10.1051/jphyscol:1986303.

[61] Z. Lin, H. Kubo, and M. Goda, Zeitschrift f"ur Physik B Condensed Matter 98, 111 (1995), ISSN 1431-584X, URL https://doi.org/10.1007/BF01318285.

[62] A. Steane, P. Szriftgiser, P. Desbiolles, and J. Dalibard, Phys. Rev. Lett. 74, 4972 (1995), URL http://link.aps.org/doi/10.1103/PhysRevLett.74.4972.

[63] T. M. Roach, H. Abele, M. G. Boshier, H. L. Grossman, K. P. Zetie, and E. A. Hinds, Phys. Rev. Lett. 75, 629 (1995), URL https://link.aps.org/doi/10.1103/PhysRevLett.75.629.

[64] A. I. Sidorov, R. J. McLean, D. C. Lau, J. E. Murphy, M. Walkiewicz, G. I. Opat, and P. Hannaford, Quantum and Semiclassical Optics: Journal of the European Optical Society Part B 8, 713 (1996), URL http://stacks.iop.org/1355-5111/8/i=3/a=030.

[65] N. Westbrook, C. I. Westbrook, A. Landragin, G. Labeyrie, L. Cognet, V. Savalli, G. Horvath, A. Aspert, C. Hendel, K. Moelmer, et al., Phys. Scr. T 78, 7 (1998).

[66] D. C. Lau, A. I. Sidorov, G. I. Opat, R. J. McLean, W. J. Rowlands, and P. Hannaford, Eur. Phys. J. D 5, 193 (1999), URL https://doi.org/10.1007/s100530050244.

[67] K. Rz¸ a˙ zewski, A. Sanpera, and M. Lewenstein, Phys. Rev. Lett. 83, 3577 (1999), URL https://link.aps.org/doi/10.1103/PhysRevLett.83.3577.

[68] A. Sidorov, R. McLean, F. Scharnberg, D. Gough, T. Davis, B. Sexton, G. Opat, and P. Hannaford, Acta Phys. Pol. B 33, 2137 (2002).

[69] J. Fiutowski, D. Bartoszek-Bober, T. Dohmalik, and T. Kawalec, Opt. Commun. 297, 59 (2013).

[70] T. Kawalec, D. Bartoszek-Bober, R. Pana´ s, J. Fiutowski, A. Plawecka, and H.-G. Rubahn, Opt. Lett. 39, 2932 (2014), URL http://ol.osa.org/abstract.cfm?URI=ol-39-10-2932.

[71] See Supplemental Material for the detailed description of the resonant motion of atoms bouncing on oscillating atom mirrors and for the description of the approach used in the analysis of the spontaneous emergence of quasicrystals in time.

[72] A. Buchleitner, D. Delande, and J. Zakrzewski, Physics reports 368, 409 (2002), URL http://www.sciencedirect.com/science/article/pii/S0370157302002703.

[73] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010), URL https://link.aps.org/doi/10.1103/RevModPhys.82.1225.

[74] J. H. Shirley, Phys. Rev. 138, B979 (1965), URL https://link.aps.org/doi/10.1103/PhysRev.138.B979.

[75] G. Berman and G. Zaslavsky, Physics Letters A 61, 295 (1977), ISSN 0375-9601, URL http://www.sciencedirect.com/science/article/pii/0375960177906181.

[76] A. Buchleitner and M. Lieberman, Regular and chaotic dynamics, Applied mathematical sciences (Springer-Verlag, 1992), ISBN 9783540977452, URL https://books.google.pl/books?id=2ssPAQAAMAAJ.

[77] K. Sacha and D. Delande, Phys. Rev. A 94, 023633 (2016), URL http://link.aps.org/doi/10.1103/PhysRevA.94.023633.

[78] Dutta, O. and Gajda, M. and Hauke, P. and Lewenstein, M. and Lüthmann, D.-S. and Malomed, B. A. and Sowi´ nski, T. and Zakrzewski, J., Reports on Progress in Physics 78, 066601 (2015), ISSN 0034-4885, URL http://stacks.iop.org/0034-4885/78/i=6/a=066601.

[79] C. Pethick and H. Smith, Bose-Einstein condensation in dilute gases (Cambridge University Press, Cambridge, England, 2002).

[80] In order to switch from the laboratory frame (where one mirror oscillates like $-\frac{\lambda}{\omega} \cos(\omega t + \Delta \phi)$ along the $x$ direction and the other like $-\frac{\lambda}{\omega} \cos \omega t$ along the $y$ direction) to the coordinate frame where the mirrors do not move, the unitary transformation $U_y = e^{i \frac{\lambda}{\omega} \sin \omega t e^{ip y \frac{\lambda}{2} \cos \omega t}}$, and a similar one for the motion along $x$, has been applied. We use the gravitational units but assume that the gravitational acceleration is given by $g/\sqrt{2}$.