Strong Markov property of determinantal processes with extended kernels

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Abstract

Noncolliding Brownian motion (Dyson’s Brownian motion model with parameter $β = 2$) and noncolliding Bessel processes are determinantal processes; that is, their space-time correlation functions are represented by determinants. Under a proper scaling limit, such as the bulk, soft-edge and hard-edge scaling limits, these processes converge to determinantal processes describing systems with an infinite number of particles. The main purpose of this paper is to show the strong Markov property of these limit processes, which are determinantal processes with the extended sine kernel, extended Airy kernel and extended Bessel kernel, respectively. We also determine the quasi-regular Dirichlet forms and infinite-dimensional stochastic differential equations associated with the determinantal processes.

Keywords  Determinantal processes · Correlation kernels · Random matrix theory · Infinite particle systems · Strong Markov property · Entire function and topology

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1 Introduction

In a system of $N$ independent one-dimensional diffusion processes, if we impose the condition that the particles never collide with one another, we obtain an interacting particle system with a long-range strong repulsive force between any pair of particles. We call such a system a noncolliding diffusion process. In 1962 Dyson [2] showed that when the individual diffusion process is one-dimensional Brownian motion (BM), it is related to a matrix-valued process. We call this stochastic process the Dyson model. The model solves the stochastic differential
equation (SDE)

\[ dX_j(t) = dB_j(t) + \sum_{k:k \neq j}^{N} \frac{dt}{X_j(t) - X_k(t)}, \quad j \in \mathbb{I}_N = \{1, 2, \ldots, N\}, \]  

where \( B_j(t), j = 1, 2, \ldots, N \) are independent one-dimensional BMs. When the individual diffusion process is a squared Bessel process with index \( \nu > -1 \), the noncolliding diffusion process is called a noncolliding squared Bessel process, satisfying the SDE

\[ dZ^\nu_j(t) = 2\sqrt{Z^\nu_j(t)}dB_j(t) + 2(\nu + 1)dt + \sum_{k:k \neq j}^{N} \frac{4Z^\nu_j(t)ds}{Z^\nu_j(t) - Z^\nu_k(t)}, \quad j \in \mathbb{I}_N \]  

and if \(-1 < \nu < 0\) having a reflection wall at the origin. These processes dynamically simulate the eigenvalue statistics of the Gaussian random matrix ensembles studied in random matrix theory \([17, 3]\).

Let \( \mathfrak{M} \) be the space of nonnegative integer-valued Radon measures on \( \mathbb{R} \). This space is a Polish space with the vague topology. The space can be regarded as a configuration space of unlabeled particles in \( \mathbb{R} \). A probability measure on the space \( \mathfrak{M} \) is called a determinantal point process (DPP) or Fermion point process, if its correlation functions are generally represented by determinants \([30, 31]\). In this paper we say that an \( \mathfrak{M} \)-valued process \( \Xi(t) \) is determinantal if the multitime correlation functions for any chosen series of times are represented by determinants. It has been shown that for any initial configuration \( \xi^N = \sum_{j=1}^{N} \delta_{x_j} \), the unlabeled process \( \Xi^N(t) = \sum_{j=1}^{N} \delta_{X_j(t)} \) is a determinantal process for the Dyson model in \([10]\), and for the noncolliding Bessel process in \([11]\).

Suppose that \( \langle \Xi(t), \mathbb{P}_\xi^N \rangle \) is a noncolliding diffusion process starting from the initial configuration \( \xi^N \) of a finite number of particles. Let \( \xi \) be the configuration of an infinite number of particles, and \( \xi_{[-L,L]} \) denote the the restriction of \( \xi \) on the set \([-L, L]\). When the sequence of processes \( \langle \Xi(t), \mathbb{P}_{\xi_{[-L,L]}} \rangle, \quad L \in \mathbb{N} = \{1, 2, \ldots\} \), converges to an \( \mathfrak{M} \)-valued process starting from the configuration \( \xi \), we denote the limit process by \( \langle \Xi(t), \mathbb{P}_\xi \rangle \), and say that process \( \langle \Xi(t), \mathbb{P}_\xi \rangle \) is well-defined. Sufficient conditions were given for configuration \( \xi \) that limit process is well defined, for noncolliding BM with bulk scaling \([10]\), noncolliding BM with soft edge scaling \([9]\), and noncolliding Bessel processes with hard-edge scaling \([11]\). We denote the associated limit processes by \( \langle \Xi(t), \mathbb{P}_{\xi_{\sin}} \rangle, \langle \Xi(t), \mathbb{P}_{\xi_{\mathrm{Ai}}} \rangle \) and \( \langle \Xi(t), \mathbb{P}_{\xi_{\mathrm{Airy}}} \rangle \), respectively.

Let \( \mu_{\sin}, \mu_{\mathrm{Ai}} \) and \( \mu_{\mathrm{J}_{\nu}} \) be the DPPs with the sine kernel \((2.1)\), the Airy kernel \((2.2)\) and the Bessel kernel \((2.3)\), respectively. They are the probability measures obtained in the bulk scaling and soft-edge scaling limit of the eigenvalue distribution \((3.10)\) in the Gaussian unitary ensemble (GUE), and in the hard-edge scaling limit of the eigenvalue distribution \((3.15)\) in the chiral Gaussian unitary ensemble (chGUE), respectively. It was shown in \([12]\) that processes \( \langle \Xi(t), \mathbb{P}_{\xi_{\sin}} \rangle, \langle \Xi(t), \mathbb{P}_{\xi_{\mathrm{Ai}}} \rangle \) and \( \langle \Xi(t), \mathbb{P}_{\xi_{\mathrm{Airy}}} \rangle \) have DPPs \( \mu_{\sin}, \mu_{\mathrm{Ai}} \) and \( \mu_{\mathrm{J}_{\nu}} \) as reversible measures, and the reversible processes coincide with determinantal processes \( \langle \Xi(t), \mathbb{P}_{\sin} \rangle \) with the extended sine kernel \((2.9)\), \( \langle \Xi(t), \mathbb{P}_{\mathrm{Ai}} \rangle \) with the extended Airy kernel \((2.10)\), and \( \langle \Xi(t), \mathbb{P}_{\mathrm{J}_{\nu}} \rangle \) with the extended Bessel kernel \((2.11)\), respectively. The main purpose of this paper is
to prove the strong Markov property of these infinite-dimensional determinantal processes 
\((\Xi(t), \mathbb{P}_\star)\), \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu : \nu > -1\}\) (Theorem 2.1).

For each \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu : \nu > -1\}\), the diffusion process \((\Xi(t), \mathbb{P}_\star^l)\) associated with DPP \(\mu_\star\) was constructed by a Dirichlet form technique presented by the first author [23, 24], and associated infinite-dimensional stochastic differential equation (ISDE) was derived [22, 26, 5]. The relation between processes \((\Xi(t), \mathbb{P}_\star^c)\) and \((\Xi(t), \mathbb{P}_\star^e)\) was first discussed in [8]; it was later proven that both of these are extensions of the same pre-Dirichlet form on the set of polynomial functions [25]. However, the coincidence of these two processes has long been an open problem. Using Theorem 2.1, it can be proved affirmatively (Theorem 2.2). The coincidence of processes from completely different approaches of construction enable us to examine them from various points of view. From the algebraic construction by virtue of their determinantal structure of their space-time correlation functions, we obtain quantitative information of the processes such as the moment generating functions of multitime distributions (2.8) given by the Fredholm determinant of space-time correlation functions [9, 10, 11]; while from the analytic construction through Dirichlet form theory, we deduce many qualitative properties of sample paths of the labeled diffusion by means of the ISDE representation, which makes us possible to apply the Ito calculus to the processes [20, 22, 26].

The proof is based on the following two facts:

(i) Determinantal process \((\Xi(t), \mathbb{P}_\star^c)\) solves the same ISDE as process \((\Xi(t), \mathbb{P}_\star^e)\).

(ii) The solutions of the ISDE are unique.

Fact (ii) was shown in [27]. Fact (i) is derived by the arguments in [22], in which the consistent families of diffusion processes \((X^k(t), H(t))\), \(k \in \mathbb{N}\) describing the joint processes of tagged particles and unlabeled infinite particles are introduced. To construct the processes, the strong Markov property of the unlabeled process \((\Xi(t), \mathbb{P}_\star^c)\) is crucial [21].

To prove the strong Markov property of the determinantal process \((\Xi(t), \mathbb{P}_\star^c)\) we introduce the subsets \(\mathfrak{X}\) in (2.5)–(2.7), and equip them with the topology defined by the inductive limit. One of the key points for the strong Markov property is to prove the Feller-like property of the process \((\Xi(t), \mathbb{P}_\star^c)\) restricted on \(\mathfrak{X}\). For showing the Feller property of the process it is necessary to have if \(\xi_n \to \xi\) in \(\mathfrak{X}\) with the topology, \(\mathbb{P}^\xi_{\infty} \to \mathbb{P}^\xi\) weakly on \(C([0, \infty], \mathfrak{X})\). In stead of the property, we prove that \(\mathbb{P}^\xi(\xi \in C([0, \infty], \mathfrak{X})) = 1\) for \(\mu_\star\)-a.s. \(\xi\) in Proposition 3.7, and combine it with the fact that if \(\xi_n \to \xi\) in \(\mathfrak{X}\) with the topology, \(\mathbb{P}^{\xi_n} \to \mathbb{P}^\xi\) weakly on \(C([0, \infty], \mathfrak{X})\), which was already proved in [9, 10, 11], and derive the strong Markov property of the reversible process \((\Xi(t), \mathbb{P}_\star)\).

Let \(\mu^N_{\sin}, \mu^N_{\text{Ai}}\) and \(\mu^N_{J_\nu}\) be probability measures on the configuration spaces of finite numbers of particles defined by (3.11), (3.13) and (3.16), respectively. For each \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu : \nu > -1\}\), \(\mu_\star\) is obtained by the limit of \(\mu^N_\star, N \to \infty\). It is interesting to study the relation between the Dirichlet forms related to \(\mu^N_\star\) and \(\mu\) with the same square field defined in (2.12). We denote by \((\Xi(t), \mathbb{P}^{\star N}_c)\) the Diffusion process associated the Dirichlet form related to \(\mu^{N}_\star\) and starting from any configuration of \(N\) particles. As one example of byproducts of Theorem 2.2, we obtain that \((\Xi(t), \mathbb{P}^{\star N}_c)\) converges to \((\Xi(t), \mathbb{P}_\star)_N \to \infty\), where \(\xi^N_{\infty}\) is the configuration given in (2.13) (Corollary 2.3).
The paper is organized as follows. In Section 2 preliminaries and main results are given. In section 3 we give the proof of the main theorems.

2 Preliminaries and main results

Let $\mathcal{M} = \mathcal{M}(\mathbb{R})$ be the space of nonnegative integer-valued Radon measures on $\mathbb{R}$, which is a Polish space with the vague topology. We say $\xi_n, n \in \mathbb{N}$ converges to $\xi$ vaguely if $\lim_{n \to \infty} \int_{\mathbb{R}} \varphi(x)\xi_n(dx) = \int_{\mathbb{R}} \varphi(x)\xi(dx)$ for any $\varphi \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the set of all continuous real-valued functions with compact supports. A subset $\mathcal{N}$ of $\mathcal{M}$ is relatively compact if and only if $\sup_{\xi \in \mathcal{N}} \xi(K) < \infty$, for any compact set $K$ in $\mathbb{R}$. Each element $\xi$ of $\mathcal{M}$ can be represented as $\xi(\cdot) = \sum_{j \in \Lambda} \delta_{x_j}(\cdot)$ with an index set $\Lambda$ and a sequence of points $x = (x_j)_{j \in \Lambda}$ in $\mathbb{R}$ satisfying $\xi(K) = \sharp\{x_j : x_j \in K\} < \infty$ for any compact subset $K \subset \mathbb{R}$. We introduce the subspace $\mathcal{M}_0$ of $\mathcal{M}$ defined as $\mathcal{M}_0 = \{\xi \in \mathcal{M} : \xi(\{x\}) \leq 1, \ x \in \mathbb{R}\}$. Any element $\xi \in \mathcal{M}_0$ is identical to its support, and can be regarded as a countable subset $\{x_j\}_{j \in \Lambda}$. For an element $\xi \in \mathcal{M} \setminus \mathcal{M}_0$, there is a point $x \in \text{supp} \xi$ such that $\xi(\{x\}) \geq 2$. Such a point is called a multiple point. We call an element $\xi$ of $\mathcal{M}$ an unlabeled configuration, and a sequence of points $x = (x_j)_{j \in \Lambda}$ a labeled configuration. For $\xi(\cdot) = \sum_{j \in \Lambda} \delta_{x_j}(\cdot) \in \mathcal{M}$, we introduce the following operations

(restriction) $\xi_A = \xi \cap A = \sum_{x \in A} \delta_x(\cdot)$, for $A \subset \mathbb{R}$,

(shift) $\tau_u \xi(\cdot) = \sum_{j \in \Lambda} \delta_{x_j+u}(\cdot)$, for $u \in \mathbb{R}$.

We also introduce the configuration space $\mathcal{M}^+ = \{\xi \in \mathcal{M} : \xi((-\infty, 0)) = 0\}$.

A probability measure on the space $\mathcal{M}$ is called a DPP or Fermion point process, if its correlation functions are generally represented by determinants [30, 31]. In this paper we consider the DPPs with the following three continuous kernels on $\mathbb{R} \times \mathbb{R}$.

(i) Sine kernel $K_{\text{sine}}(x, y)$:

$$K_{\text{sine}}(x, y) = \frac{1}{2\pi} \int_{|k| \leq 1} dk \ e^{v-1k(x-y)} = \begin{cases} \frac{\sin(x-y)}{\pi(x-y)} & \text{if } x \neq y, \\ \frac{1}{\pi} & \text{if } x = y. \end{cases} \quad (2.1)$$

(ii) Airy kernel $K_{\text{Airy}}(x, y)$:

$$K_{\text{Airy}}(x, y) = \begin{cases} \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y} & \text{if } x \neq y, \\ (\text{Ai}'(x))^2 - x(\text{Ai}(x))^2 & \text{if } x = y. \end{cases} \quad (2.2)$$
where \( \text{Ai}(x) = \frac{1}{2\pi} \int_{\rho} dk e^{-k^2/2} \), \( x \in \mathbb{R} \), is the Airy function and \( \text{Ai}'(x) = d\text{Ai}(x)/dx \).

(iii) Bessel kernel \( K_{J_{\nu}}(x, y) \):

\[
K_{J_{\nu}}(x, y) = \begin{cases} 
\frac{J_{\nu}(2\sqrt{x})\sqrt{y}J_{\nu}'(2\sqrt{y}) - \sqrt{x}J_{\nu}'(2\sqrt{x})J_{\nu}(2\sqrt{y})}{x - y} & \text{if } x \neq y, \\
J_{\nu}(2\sqrt{x})^2 - J_{\nu+1}(2\sqrt{x})J_{\nu-1}(2\sqrt{x}) & \text{if } x = y,
\end{cases}
\]

(2.3)

where \( J_{\nu}(\cdot) \) is the Bessel function with index \( \nu > -1 \) defined as

\[
J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\nu)} \left( \frac{z}{2} \right)^{2n+\nu}, \quad z \in \mathbb{C}
\]

with the gamma function \( \Gamma(x) = \int_{0}^{\infty} e^{-u u x^{-1}} du \) and \( J_{\nu}'(x) = dJ_{\nu}(x)/dx \). We denote by \( \mu_{\sin}, \mu_{\text{Ai}}, \mu_{J_{\nu}} \), DPPs with the sine, Airy and Bessel kernels, respectively. We note that DPPs \( \mu_{\sin}, \mu_{\text{Ai}} \) and \( \mu_{J_{\nu}} \) are the limit distributions of DPPs \( \mu_{\sin}^N, \mu_{\text{Ai}}^N \), and \( \mu_{J_{\nu}}^N \) defined, respectively, in (3.11), (3.13), (3.16), which are related to the eigenvalue distributions of random matrices [17, 1].

Let \( \rho(dx) = \rho(x)dx \) be a Radon measure on \( \mathbb{R} \) with density \( \rho(x) \) such that \( \rho(\mathbb{R}) = \int_{\mathbb{R}} \rho(x)dx \in \{0\} \cup \mathbb{N} \cup \{\infty\} \). For \( \varepsilon, \kappa > 0 \) and \( m_0, L_0 \in \mathbb{N} \), we denote by \( X_{L_0,m_0}^{\varepsilon,\kappa}(\rho) \) the set of configurations \( \xi \in \mathcal{M} \) satisfying \( \xi(\mathbb{R}) = \rho(\mathbb{R}) \),

\[
|\rho([0, L]) - \varepsilon([0, L])| \leq L^\varepsilon, \quad |\rho([-L, 0]) - \varepsilon([-L, 0])| \leq L^\varepsilon, \quad L \geq L_0,
\]

and

\[
\max_{k \in \mathbb{Z}} \left\{ g^\kappa(k), g^\kappa(k+1) \right\} \leq m_0,
\]

where

\[
g^\kappa(x) = \text{sign}(x)|x|^\kappa, \quad x \in \mathbb{R}.
\]

(2.4)

The set \( X_{L_0,m_0}^{\varepsilon,\kappa}(\rho) \) is then relatively compact with the vague topology. We introduce the following three configurations associated with DPPs \( \mu_*, * \in \{\sin, \text{Ai}\} \cup \{J_{\nu} ; \nu > -1\} \):

\[
X_{\sin} = \bigcup_{L_0,m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (0,\kappa_{\sin})} X_{L_0,m_0}^{\varepsilon,\kappa}(\rho_{\sin}),
\]

(2.5)

\[
X_{\text{Ai}} = \bigcup_{L_0,m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (0,\kappa_{\text{Ai}})} X_{L_0,m_0}^{\varepsilon,\kappa}(\rho_{\text{Ai}}),
\]

(2.6)

\[
X_{J_{\nu}} = \bigcup_{L_0,m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (0,\kappa_{J_{\nu}})} X_{L_0,m_0}^{\varepsilon,\kappa}(\rho_{J_{\nu}}),
\]

(2.7)

where \( \kappa_{\sin} = 1, \kappa_{\text{Ai}} = 2/3, \kappa_{J_{\nu}} = 2 \), and \( \rho_* \) is the first order correlation function of DPPs \( \mu_* \) for each \( * \in \{\sin, \text{Ai}\} \cup \{J_{\nu} ; \nu > -1\} \). It is readily verified that \( \mu_*(X_*) = 1 \), for \( * = \sin, \text{Ai} \). This completes the proof of Theorem 2.8.
Ai or $J_{\nu}$. We equip space $\mathfrak{X}_*$ with the topology defined by the inductive limit. Note that this is a locally compact space with the topology. We denote by $C([0, \infty), \mathfrak{X}_*)$ the set of $\mathfrak{X}_*$-valued continuous functions with the topology. Then $\Xi(\cdot) \in C([0, \infty), \mathfrak{X}_*)$ implies that $\Xi(\cdot)$ is continuous with the vague topology, and for any $T > 0$ there exist $M_0, L_0 \in \mathbb{N}$, $\varepsilon \in (0, 1)$ and $\kappa \in (0, \kappa_*]$ such that $\Xi(t) \in \mathfrak{X}^{\varepsilon, \kappa}_{L_0, m_0}(\rho_*)$ for $t \in [0, T]$.

In this paper we say that an $\mathcal{M}$-valued process $\Xi(t)$ is determinantal, if the multitime correlation functions for any chosen series of times are represented by determinants. In other words, a determinantal process is an $\mathcal{M}$-valued process such that, for any integer $M \in \mathbb{N}$, $\mathbf{f} = (f_1, f_2, \ldots, f_M) \in C_0(\mathbb{R})^M$, and sequence of times $t = (t_1, t_2, \ldots, t_M)$ with $0 < t_1 < \cdots < t_M < \infty$, the moment generating function of multitime distribution, $\Psi^t[\mathbf{f}] \equiv \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M} \int_{\mathbb{R}} f_m(x) \Xi(t_m, dx)\right\}\right]$, is given by a Fredholm determinant

$$
\Psi^t[\mathbf{f}] = \text{Det}_{(s,t) \in \{(t_1, t_2, \ldots, t_M)\}^2} \left[\delta_{st}(y - x) + \mathbb{K}(s, x; t, y)\chi_t(y)\right],
$$

(2.8)

where $\chi_{tm} = e^{f_m} - 1$, $1 \leq m \leq M$, and $\mathbb{K}$ is a locally integrable function called a (space-time) correlation kernel [8, 9, 10, 11]. In this paper we study the determinantal processes with the following three correlation kernels:

(i) **Extended sine kernel** $\mathbb{K}_{\text{sine}}(s, x; t, y)$, $s, t \in \mathbb{R}^+ \equiv \{x \in \mathbb{R} : x \geq 0\}$, $x, y \in \mathbb{R}$:

$$
\mathbb{K}_{\text{sine}}(s, x; t, y) = \begin{cases} 
\frac{1}{\pi} \int_0^1 du e^{u^2(t-s)/2} \cos\{u(y - x)\} & \text{if } s < t, \\
\mathbb{K}_{\text{sine}}(x, y) & \text{if } s = t, \\
-\frac{1}{\pi} \int_1^\infty du e^{u^2(t-s)/2} \cos\{u(y - x)\} & \text{if } s > t.
\end{cases}
$$

(2.9)

(ii) **Extended Airy kernel** $\mathbb{K}_{\text{Ai}}(s, x; t, y)$, $s, t \in \mathbb{R}^+, x, y \in \mathbb{R}$:

$$
\mathbb{K}_{\text{Ai}}(s, x; t, y) = \begin{cases} 
\int_0^\infty du e^{-u(t-s)/2} \text{Ai}(u + x)\text{Ai}(u + y) & \text{if } s < t, \\
\mathbb{K}_{\text{Ai}}(x, y) & \text{if } s = t, \\
-\int_0^\infty du e^{-u(t-s)/2} \text{Ai}(u + x)\text{Ai}(u + y) & \text{if } s > t.
\end{cases}
$$

(2.10)

(iii) **Extended Bessel kernel** $\mathbb{K}_{J_\nu}(s, x; t, y)$, $s, t \in \mathbb{R}^+, x, y \in \mathbb{R}^+$:

$$
\mathbb{K}_{J_\nu}(s, x; t, y) = \begin{cases} 
\int_0^1 du e^{-2u(s-t)} J_{\nu}(2\sqrt{ux})J_{\nu}(2\sqrt{uy}) & \text{if } s < t, \\
\mathbb{K}_{J_\nu}(x, y) & \text{if } s = t, \\
-\int_1^\infty du e^{-2u(s-t)} J_{\nu}(2\sqrt{ux})J_{\nu}(2\sqrt{uy}) & \text{if } s > t.
\end{cases}
$$

(2.11)

We denote the determinantal process with the extended sine kernel $\mathbb{K}_*, (\Xi(t), \mathbb{P}_*)$, $* = \text{sine, Ai, } J_{\nu}$. The determinantal process $(\Xi(t), \mathbb{P}_*)$ is reversible with the DPP $\mu_*$, for $* = \text{sine, Ai and } J_{\nu}$.

The main theorem of this present paper is as follows.
Theorem 2.1 Let \( \star \in \{\sin, \Ai\} \cup \{J_\nu : \nu > -1\} \). The determinantal process \((\Xi(t), \mathbb{P}_\star)\) is a reversible diffusion process.

Diffusion processes with reversible probability measures \( \mu_\star, \star \in \{\sin, \Ai\} \cup \{J_\nu : \nu > -1\} \) were constructed by means of the Dirichlet form technique by the first author \([23, 24]\) and associated ISDEs were derived in \([22, 26, 5]\). The second theorem of this paper is the coincidence of the process constructed by the Dirichlet form technique and the determinantal process \((\Xi(t), \mathbb{P}_\star), \star \in \{\sin, \Ai\} \cup \{J_\nu : \nu > -1\}\). To state the theorem we prepare some notation and review the related results. A function \( f \) on \( \mathfrak{M} \) is called local if \( f(\xi) = f(\xi_K) \) for some compact set \( K \). For a local function \( f \) with \( f(\xi) = f(\xi_K) \) we introduce the functions \( \tilde{f}_k \) on \( \mathbb{R}^k \), \( k \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N} \) defined as \( \tilde{f}_0 = f(\emptyset) \), where \( \emptyset \) is the null configuration, and for \( k \in \mathbb{N} \), as

\[
\tilde{f}_k(x_k) = f \left( \sum_{j=1}^{k} \delta_{x_j} \right) \quad \text{for} \; x_k \in K^k.
\]

We extend the domain of \( \tilde{f}_k(x_k) \) to \( \mathbb{R}^k \setminus K^k \) by the consistency arising from \( f(\xi) = f(\xi_K) \). Hence \( \tilde{f}_k, k \in \mathbb{N}_0 \), satisfy the consistency relation

\[
\tilde{f}_{k+1}(x_k, y) = \tilde{f}_k(x_k), \quad x_k \in \mathbb{R}^k, \; y \notin K.
\]

A local function \( f \) is called smooth if the \( \tilde{f}_k \) are smooth for \( k \in \mathbb{N}_0 \). We denote by \( \mathcal{D}_\infty \) the set of all local smooth functions on \( \mathfrak{M} \).

Let \( x_k = (x_i)_{i=1}^k \in \mathbb{R}^k \), \( k \in \mathbb{N}_0 \), and \( f, g \in \mathcal{D}_\infty \); then

\[
\mathbb{D}^a(f, g)(x_k) = \frac{1}{2} \sum_{i=1}^{k} a(x_i) \frac{\partial \tilde{f}_k(x_k)}{\partial x_i} \frac{\partial \tilde{g}_k(x_k)}{\partial x_i},
\]

where \( a \) is a smooth function on \( \mathbb{R} \). For given \( f, g \in \mathcal{D}_\infty \), the right-hand side is a permutation invariant function, and the square field \( \mathbb{D}(f, g) \) can be regarded as a local function with variable \( \xi = \sum_{i \in \mathbb{N}} \delta_{x_i} \in \mathfrak{M} \). For probability \( \mu \) on \( \mathfrak{M} \), we denote by \( L^2(\mathfrak{M}, \mu) \) the space of square integrable functions on \( \mathfrak{M} \) with the inner product \( \langle \cdot, \cdot \rangle_\mu \) and the norm \( \| \cdot \|_{L^2(\mathfrak{M}, \mu)} \). We consider the bilinear form \((\mathcal{E}_\mu^a, \mathcal{D}_\infty^\mu)\) on \( L^2(\mathfrak{M}, \mu) \) defined as

\[
\mathcal{E}_\mu^a(f, g) = \int_{\mathfrak{M}} \mathbb{D}^a(f, g) d\mu,
\]

\[
\mathcal{D}_\infty^\mu = \{ f \in \mathcal{D}_\infty : \| f \|^2_1 < \infty \},
\]

where \( \| f \|^2_1 = \mathcal{E}_\mu^a(f, f) + \| f \|^2_{L^2(\mathfrak{M}, \mu)} \). We consider three cases \((\mu, a(x)) = (\mu_{\sin}, 1), (\mu_{\Ai}, 1)\) and \((\mu_{J_\nu}, 4x)\). For each case it is proved in \([23, 24]\) that the closure \((\mathcal{E}_\star, \mathcal{D}_\star)\) of the bilinear form \((\mathcal{E}_\mu^a, \mathcal{D}_\infty^\mu)\) is a quasi-Dirichlet form, associated with the diffusion process \((\Xi(t), \mathbb{P}_\star), \star \in \{\sin, \Ai\} \cup \{J_\nu : \nu > -1\}\). One can naturally lift each unlabeled path \( \Xi \) to the labeled path \( \mathbf{X} \) by a labeled map \( 1 = (l)_{l \in \mathbb{N}} \) (see for example \([21, \text{Theorem 2.4}]\)). It was also proved that a labeled process \( \mathbf{X} = (X_j)_{j \in \mathbb{N}} \) with \( \Xi(t) = \sum_{j \in \mathbb{N}} \delta_{X_j(t)} \) solves the ISDE

\[
dX_j(t) = dB_j(t) + \lim_{r \to \infty} \sum_{k : k \neq j, |X_k(t)| < r} \frac{dt}{X_j(t) - X_k(t)},
\]

\( \text{(sin)} \)
under $P_{\sin}^\xi$, a labeled process $Y = (Y_j)_{j \in \mathbb{N}}$ with $\Xi(t) = \sum_{j \in \mathbb{N}} \delta_{Y_j(t)}$ solves the ISDE

$$dY_j(t) = dB_j(t) + \lim_{r \to \infty} \left\{ \sum_{k: k \neq j \atop |Y_k(t)| < r} \frac{1}{Y_j(t) - Y_k(t)} - \int_{-r}^r \hat{\rho}(x) dx \right\} dt,$$  \hspace{1cm} (Ai)

under $P_{\Airy}^\xi$ with $\hat{\rho}(x) \equiv (\sqrt{-1/x}/\pi)1(x < 0)$, and a labeled process $Z^\nu = (Z_j^\nu)_{j \in \mathbb{N}}$ with $\Xi(t) = \sum_{j \in \mathbb{N}} \delta_{Z_j^\nu(t)}$ solves the ISDE

$$dZ_j^\nu(t) = 2\sqrt{Z_j^\nu(t)} dB_j(t) + 2(\nu + 1) dt + \sum_{k: k \neq j \atop k = 1} \frac{4Z_j^\nu(t) dt}{Z_j^\nu(t) - Z_k^\nu(t)}.$$  \hspace{1cm} (J_\nu)

under $P_{\jnu}^\mu$, $\nu > -1$ [22, 26, 5].

Set $P_{\mu}^\ast = \int_{\mathfrak{M}} \mu_\ast(d\xi) P_{\xi}^\mu$, for $\ast \in \{\sin, \Airy\} \cup \{J_\nu : \nu > -1\}$. $(\Xi(t), P_{\mu}^\ast)$ is then a reversible diffusion process with reversible measure $\mu_\ast$. The second main result is the following theorem.

**Theorem 2.2** Let $\ast \in \{\sin, \Airy\} \cup \{J_\nu : \nu > -1\}$.

(i) A labeled process $X = (X_j)_{j \in \mathbb{N}}$ associated with the determinantal process $(\Xi(t), P_\ast)$ solves the ISDE $(\ast)$.

(ii) Process $(\Xi(t), P_\ast)$ coincides with process $(\Xi(t), P_{\mu}^\ast)$ in distribution. In particular, process $(\Xi(t), P_\ast)$ is associated with the Dirichlet form $(\mathcal{E}_\ast, \mathcal{D}_\ast)$.

We remark that Tsai [33] recently constructed non-equilibrium dynamics for the Dyson model with $0 < \beta < \infty$ by ISDE and obtained the result related to (i) in the above theorem.

Consider that the DPPs $\mu_{\sin}^N$, $\mu_{\Airy}^N$ and $\mu_{J_\nu}^N$ are probability measures on the configuration spaces of a finite number of particles defined by (3.11), (3.13) and (3.16), respectively, and the bilinear forms $(\mathcal{E}_\mu^N, \mathcal{D}_\mu^N)$ with $(\mu, a(x)) = (\mu_{\sin}^N, 1), (\mu_{\Airy}^N, 1)$ and $(\mu_{J_\nu}^N, 4x)$. The closure $(\mathcal{E}_\ast^N, \mathcal{D}_\ast^N)$ of the bilinear form $(\mathcal{E}_\mu^N, \mathcal{D}_\mu^N)$ is a quasi-regular Dirichlet form, associated with the diffusion process $(\Xi(t), P_{\mu}^\ast)$, $\ast \in \{\sin, \Airy\} \cup \{J_\nu : \nu > -1\}$. We see that labeled processes associated with these diffusion processes solve SDEs (3.9), (3.12) and (3.14), respectively. We take a labeled map $I : \mathfrak{M} \to \mathbb{R}^N \oplus \bigoplus_{n=0}^{\infty} \mathbb{R}^n$ such that for $\xi \in \mathfrak{M}$

$$|I_{j-1}(\xi)| \leq |I_j(\xi)|, \hspace{1cm} j = 1, 2, \ldots, \xi(\mathbb{R}),$$

and set

$$\xi_N^I = \sum_{j=1}^N \delta_{I_j(\xi)} \hspace{1cm} \text{for} \hspace{0.5cm} \xi \in \mathfrak{M}. \hspace{1cm} (2.13)$$

Let $X = (X_j)_{j \in \mathbb{N}}$ and $X^N = (X_j^N)_{j=1}^N$ be the labeled processes associated with $(\Xi(t), P_{\sin}^\xi)$ and $(\Xi(t), P_{\mu}^\ast)$, respectively. Note that $X(0) = I(\xi) \equiv x$ and $X^N(0) = (I_j(\xi))^N_{j=1} \equiv x^N$. We have then the following as a corollary of Theorem 2.2.
Corollary 2.3 Let \( \star \in \{ \sin, A_i \} \cup \{ J_\nu : \nu > -1 \} \). (i) For \( \mu_* \) a.s. \( \xi \),

\[
(\Xi(t), P^\xi_N) \to (\Xi(t), P^\xi), \quad N \to \infty,
\]

weakly on the path space \( C([0, \infty), \mathfrak{M}) \).

(ii) For \( \mu_* \circ \Gamma^{-1} \) a.s. \( x \), and \( m \in \mathbb{N} \)

\[
(X_1^N(t), X_2^N(t), \ldots, X_m^N(t)) \to (X_1(t), X_2(t), \ldots, X_m(t)), \quad N \to \infty,
\]

weakly on the path space \( C([0, \infty), \mathbb{R}^m) \).

Suppose that \( \mu \) and \( \mu_N \) (\( N \in \mathbb{N} \)) are probability measures on \( \mathfrak{M} \), and \( (\Xi(t), P) \) and \( (\Xi(t), P^N) \) are diffusion processes associated with the Dirichlet spaces given by the closures of \( (\mathcal{E}^\alpha_\mu, D^\alpha_\infty, L^2(\mathfrak{M}, \mu)) \) and \( (\mathcal{E}^\alpha_{\mu_N}, D^\alpha_\infty, L^2(\mathfrak{M}, \mu^N)) \), respectively. Let us consider the problem on the weak convergence of stationary processes. That is,

\[
\mu_N \to \mu, \quad N \to \infty \Rightarrow (\Xi(t), P^N) \to (\Xi(t), P), \quad N \to \infty.
\]

If the measures \( \mu_N \) and \( \mu \) are singular each other, then such a convergence is not covered by a general theorem of convergence of diffusions associated with Dirichlet forms. We remark that Corollary 2.3 (i) gives examples of such a convergence even if the measures \( \mu_N \) and \( \mu \) are singular each other. Recently, Kawamoto-Osada [14] also showed Corollary 2.3 (ii) using a different method.

3 Proof of theorems

3.1 Some properties of determinantal point processes

Let \( \mu \) be a probability measure on \( \mathfrak{M} \) with correlation functions \( \rho_m(x_m), x_m \in \mathbb{R}^m, m \in \mathbb{N} \). Then, for \( f \in C_0(\mathbb{R}) \) and \( \theta \in \mathbb{R} \)

\[
\Psi(f, \theta) \equiv \int_{\mathfrak{M}} \mu(d\xi) e^{\theta \int_{\mathbb{R}} f(x) \xi(dx)} = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^m} dx_m \prod_{j=1}^{m} \left( e^{\theta f(x_j)} - 1 \right) \rho_m(x_m),
\]

where \( dx_m = \prod_{j=1}^{m} dx_j \). Let \( \mathcal{K} \) be a symmetric linear operator with kernel \( K \). In this section we assume that operator \( \mathcal{K} \) satisfies Condition A in [30]. Probability measure \( \mu \) is called a DPP with correlation kernel \( K \), if its correlation functions are given by

\[
\rho_m(x_m) = \det_{1 \leq j, k \leq m} [K(x_j, x_k)].
\]

We often write \( \rho \) for \( \rho_1 \), and \( \rho(A) \) for \( \int_A \rho(x)dx, A \in \mathcal{B}(\mathbb{R}) \). The following lemmas are [12, Lemma 3.1] and a modification of [12, lemma 3.3].

Lemma 3.1 Let \( \mu \) be a DPP. For any bounded closed interval \( D \) of \( \mathbb{R} \), we then have

\[
\int_{\mathfrak{M}} \mu(d\eta) \| \eta(D) - \int_D \rho(x)dx \|^2 \leq (3\rho(D))^k, \quad k \in \mathbb{N}.
\]
Lemma 3.2 Let $\mu$ be a DPP and $(\Xi(t), P)$ be a stationary process with stationary measure $\mu$. If

$$\sum_{k \in \mathbb{Z}} |k|^\ell \rho(|g^\kappa(k), g^\kappa(k+1)|)^m < \infty,$$  

(3.1)

for some $\ell, m \in \mathbb{N}$, and $\kappa > 0$, then for $P$-a.s. $\Xi$ there exists $m_0 = m_0(\Xi) \in \mathbb{N}$ such that

$$\Xi(t, [g^\kappa(k), g^\kappa(k+1)]) \leq m_0, \quad t = \left\lfloor \frac{j}{|k|^\ell} \right\rfloor, j = 1, 2, \ldots, |k|^\ell, k \in \mathbb{Z},$$  

(3.2)

where $g^\kappa$ is the function in (2.4). In particular

$$\lim_{m \to \infty} P\left(\Xi(t, [g^\kappa(k), g^\kappa(k+1)]) > m, \ t = \left\lfloor \frac{j}{|k|^\ell} \right\rfloor, j = 1, 2, \ldots, |k|^\ell, k \in \mathbb{Z}\right) = 1.$$  

(3.3)

Proof. Since a DPP has the repulsive property, from condition (3.1) we derive

$$\sum_{k \in \mathbb{Z}} |k|^\ell \int [g^\kappa(k), g^\kappa(k+1)]^m \rho_m(x_m)dx_m < \infty,$$

which implies

$$\sum_{k \in \mathbb{Z}} |k|^\ell \mu\left(\xi(g^\kappa(k), g^\kappa(k+1)) > m\right)$$

$$= \sum_{k \in \mathbb{Z}} P\left(\Xi(t, [g^\kappa(k), g^\kappa(k+1)]) > m, \ \text{for some } t \in \left\{ \left\lfloor \frac{j}{|k|^\ell} \right\rfloor, j = 1, 2, \ldots, |k|^\ell \right\} \right) < \infty.$$

By Borel-Cantelli’s lemma we obtain (3.1), and then (3.2). 

3.2 Non-equilibrium dynamics

In this subsection we review the results in [9, 10, 11] on non-equilibrium dynamics of non-colliding diffusion processes. For $\xi^N \in \mathfrak{M}$ with $\xi^N(\mathbb{R}) = N \in \mathbb{N}$ and $p \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, we consider the product

$$\Pi_p(\xi^N, w) = \prod_{x \in \text{supp } \xi^N} G\left(\frac{w}{x}, p\right)^{\xi^N(x)}, \quad w \in \mathbb{C},$$

where

$$G(u, p) = \begin{cases} 1 - u, & \text{if } p = 0 \\ (1 - u) \exp \left[u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right], & \text{if } p \in \mathbb{N}. \end{cases}$$

The functions $G(u, p)$ are called the Weierstrass primary factors [15]. We then set

$$\Phi_p(\xi^N, z, w) = \Pi_p(\tau_z \xi^N \cap \{0\}^c, w - z) = \prod_{x \in \text{supp } \xi^N \cap \{z\}^c} G\left(\frac{w - z}{x - z}, p\right)^{\xi^N(x)}, \quad w, z \in \mathbb{C},$$
where \( \tau_z \xi(\cdot) \equiv \sum_{j \in \Lambda} \delta_{x_j + z}(\cdot) \) for \( z \in \mathbb{C} \) and \( \xi(\cdot) = \sum_{j \in \Lambda} \delta_{x_j}(\cdot) \).

Let \( u \) be the unlabeled map from \( \cup_{n=0}^{\infty} \mathbb{R}^n \) to \( \mathcal{M} \) defined as \( u(x_1, \ldots, x_n) = \sum_{j=1}^n \delta_{x_j} \). We consider the process \( u(X(t)), \ t \in [0, \infty), \) for the solution \( X(t) = (X_j(t))_{j \in \mathbb{N}} \) of (1.1). We denote by \( \mathcal{P}_\sin^N \) the distribution of the process starting from any fixed configuration \( \xi^N \) with a finite number of particles. It was proved in Proposition 2.1 of [10] that process \( (\Xi(t), \mathcal{P}_\sin^N) \) is determinantal with the correlation kernel \( \mathbb{K}^\xi^N \) given by

\[
\mathbb{K}^\xi^N(s, x; t, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} du \int_{C_\sin(\xi^N)} dz p_{\sin}(s, x|z) \Pi_0(\tau_{-z} \xi^N, iu - z) p_{\sin}(-t, iu|y) - 1(s > t) p_{\sin}(s - t, x|y),
\]

where \( C_\sin(\xi^N) \) denotes a closed contour on the complex plane \( \mathbb{C} \) encircling the points in \( \text{supp} \ xi^N \) on the real line \( \mathbb{R} \) once in the positive direction but not point \( w \), and \( p_{\sin}(t, x|y) \) is the generalized heat kernel:

\[
p_{\sin}(t, x|y) = \frac{1}{\sqrt{2\pi|t|}} \exp \left\{ -\frac{(x - y)^2}{2t} \right\} 1(t \neq 0) + \delta(y - x) 1(t = 0), \quad t \in \mathbb{R}, \ x, y \in \mathbb{C}.
\]

In case \( \xi^N \) is a configuration without any multiple points, i.e. \( \xi^N \in \mathcal{M}_0 \), (3.4) is rewritten as

\[
\mathbb{K}^\xi^N(s, x; t, y) = \int_{\mathbb{R}} \xi^N(dx') \int_{\mathbb{R}} du p_{\sin}(s, x|x') \Phi_0(\xi^N, x', iu)p_{\sin}(-t, iu|y) - 1(s > t) p_{\sin}(s - t, x|y).
\]

It was proved in [10, Theorem 2.4] with [13, Theorem 1.4] that if \( \xi \in \mathcal{X}_{\sin} \), process \( (\Xi(t), \mathcal{P}_\sin^N) \) converges to the determinantal process with correlation kernel \( \mathbb{K}^\xi \) as \( N \to \infty \), weakly on \( C([0, \infty), \mathcal{M}) \). In particular, when \( \xi \in \mathcal{X}_{\sin} \cap \mathcal{M}_0 \), \( \mathbb{K}^\xi_{\sin} \) is given by

\[
\mathbb{K}^\xi_{\sin}(s, x; t, y) = \int_{\mathbb{R}} \xi(dx') \int_{\mathbb{R}} du p_{\sin}(s, x|x') \Phi_0(\xi, x', iu)p_{\sin}(-t, iu|y) - 1(s > t) p_{\sin}(s - t, x|y).
\]

Let \( \hat{\rho}^N, N \in \mathbb{N} \) be a sequence of nonnegative functions on \( \mathbb{R} \) defined as

\[
\hat{\rho}^N(x) = \frac{1}{\pi} \int_{-4N^{2/3}}^{4N^{2/3}} \sqrt{x - (1 + \frac{x}{4N^{2/3}})} dx.
\]

Then, \( \int_{\mathbb{R}} dx \hat{\rho}^N(x) = N, \int_{\mathbb{R}} dx \frac{\hat{\rho}^N(x)}{x} = N^{1/3} \), and

\[
\hat{\rho}^N(x) \nearrow \hat{\rho}(x) = (\sqrt{-x/\pi}) 1(x < 0), \quad N \to \infty.
\]

Consider the process \( Y(t) = (Y_j(t))_{j \in \mathbb{N}} \) given by

\[
Y_j(t) = X_j(t) + \frac{t^2}{4} + t \int_{\mathbb{R}} \hat{\rho}^N(x) dx, \quad j \in \mathbb{N},
\]
with the solution \( X(t) = (X_j(t))_{j \in \mathbb{N}} \) of (1.1). In other words, \( Y(t) \) satisfies the following SDE:

\[
dY_j(t) = dB_j(t) + \left( \frac{t}{2} - N^{1/3} \right) dt + \sum_{k=1}^{N} \frac{dt}{Y_j(t) - Y_k(t)}, \quad j \in \mathbb{N}.
\]

(3.5)

We denote by \( \mathbb{P}_{\xi}^{N} \) the distribution of the process \( u(Y(t)), t \in [0, \infty) \) starting from any fixed configuration \( \xi^{N} \) with a finite number of particles. It was proved in Proposition 2.4 in [9] that process \( (\Xi(t), \mathbb{P}_{\xi}^{N}_{\mathbb{A}i}) \) is determinantal with correlation kernel \( K_{\rho}^{N} \) given by

\[
K_{\rho}^{N}(s, x \cap \{0\}, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} du \int_{C_{iu}(\xi^{N})} dz q(0, s, x - z) \frac{\Pi_{0}(\tau_{-z} \xi, iu - z)}{iu - z} \times \exp \left[ (iu - z) \int_{\mathbb{R}} \frac{\rho^{N}(v)dv}{v} \right] q(t, 0, iu - y) - \mathbf{1}(s > t) q(t, s, x - y),
\]

where \( q(s, t, y - x), s, t \in \mathbb{R}, s \neq t, x, y \in \mathbb{C} \), is given by

\[
q(s, t, y - x) = p_{\sin} \left( t - s, \left( y - \frac{t^{2}}{4} \right) - \left( x - \frac{s^{2}}{4} \right) \right).
\]

Note that \( q(s, t, y - x), 0 \leq s < t, x, y \in \mathbb{R} \) is the transition density function of process \( B(t) + t^{2}/4 \), where \( B(t), t \in [0, \infty) \) is the one-dimensional standard BM.

Let \( M_{A}(\xi) \) be the function defined as

\[
M_{A}(\xi) = \lim_{L \to \infty} \int_{0 < |x| < L} \frac{\hat{\rho}(x)dx - \xi(dx)}{x}.
\]

For \( \xi \in \mathcal{M} \) with \( M_{A}(\xi) < \infty \) and \( z \in \mathbb{C} \), we define

\[
\Phi_{Ai}(\xi, w) \equiv \exp \left[ wM_{A}(\xi) \right] \Pi_{1}(\xi \cap \{0\}, w), \quad w \in \mathbb{C},
\]

\[
\Phi_{Ai}(\xi, z, w) \equiv \Phi_{Ai}(\tau_{-z} \xi, w - z), \quad w, z \in \mathbb{C}.
\]

We note that \( \Phi_{Ai}(\xi, z, w) \) exists finitely and \( \Phi_{A}(\xi, z, w) \neq 0 \), if \( \xi \in \mathcal{X}_{Ai} \).

If \( \xi \in \mathcal{X}_{Ai} \), sequence of the processes \( (\Xi(t), \mathbb{P}_{\xi}^{N}_{\mathbb{A}i}) \) converges to the determinantal process \( (\Xi(t), \mathbb{P}_{\xi}^{N}_{\mathbb{A}i}) \) with correlation kernel \( \mathbb{K}_{Ai}^{N} \) as \( N \to \infty \), weakly on \( C([0, \infty), \mathcal{M}) \). In particular, when \( \xi \in \mathcal{X}_{Ai} \cap \mathcal{M}_{0} \), \( \mathbb{K}_{Ai}^{N} \) is given by

\[
\mathbb{K}_{Ai}^{N}(s, x \cap \{0\}, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} q(0, s, x - x') \Phi_{Ai}(\xi, x', iu)q(t, 0, iu - y) - \mathbf{1}(s > t) q(t, s, x - y).
\]

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This result was stated in [9, Section 2.4], in which the weak convergence is in the sense of finite dimensional distributions. The convergence on $C([0, \infty), \mathcal{M})$ is derived using the same argument as [13, Theorem 1.4].

We consider a one-parameter family of $\mathcal{M}^\nu$-valued processes with parameter $\nu > -1$, $u(Z^\nu(t))$, $t \in [0, \infty)$, for the solution $Z^\nu(t) = (Z^\nu_j(t))_{j \in \mathbb{N}}$ of SDE (1.2). For a given configuration $\xi^N \in \mathcal{M}^\nu$ with a finite number of particles, we denote by $\mathbb{P}^\xi_N$ the distribution of the process starting from $\xi^N$. In [11, Theorem 2.1] it was proved that $(\Xi(t), \mathbb{P}^\xi_N)$ is determinantal with correlation kernel

$$
K^\xi_N(s, x; t, y) = \frac{1}{2\pi i} \int_{-\infty}^0 du \int_{\mathcal{C}_0(\xi^N)} dz \, p^{(\nu)}(s, x|z) \left( \prod_{\alpha=1}^n (\tau_{-\alpha}^N, u - z) \right) p^{(\nu)}(-t, u|y)
$$

where for $t \in \mathbb{R}$ and $x, y \in \mathbb{C}$

$$
p^{(\nu)}(t, y|x) = \frac{1}{2|t|} (\frac{y}{x})^{\nu/2} \exp \left( -\frac{x + y}{2t} \right) I_\nu \left( \frac{\sqrt{xy}}{|t|} \right) 1(t \neq 0, x \neq 0) + \frac{y^{\nu}}{(2|t|)^{\nu+1} \Gamma(\nu + 1)} \exp \left( -\frac{y^2}{2t} \right) 1(t \neq 0, x = 0) + \delta(y - x) 1(t = 0). \tag{3.7}
$$

Note that for $t \geq 0$ and $x, y \in \mathbb{R}_+$, $p^{(\nu)}(t, y|x)$ is the transition density function of a $2(\nu + 1)$-dimensional squared Bessel process. If $\xi \in \mathbb{X}_{J_\nu}$, process $(\Xi(t), \mathbb{P}^\xi_N)$ converges to the determinantal process with correlation kernel $K^\xi_N$ as $N \rightarrow \infty$ weakly on $C([0, \infty), \mathcal{M})$. In particular, when $\xi \in \mathbb{X}_{J_\nu} \cap \mathcal{M}_0$, $K^\xi_N$ is given by

$$
K^\xi_N(s, x; t, y) = \int_0^\infty \xi(dx') \int_{-\infty}^0 du \int p^{(\nu)}(s, x|x') \Phi_0(\xi, x', u) p^{(\nu)}(-t, u|y)
$$

$$
-1(s > t)p^{(\nu)}(s - t, x|y). \tag{3.8}
$$

The result was proved in [11, Theorem 2.2] in which the weak convergence is in the sense of finite dimensional distributions. The weak convergence on $C([0, \infty), \mathcal{M})$ can be proved from the tightness of $\{\mathbb{P}^\xi_N\}_{N \in \mathbb{N}}$, which is derived by the same procedure to show [13, Theorem 1.4] using determinatal martingales representation in [7, Theorem 1.2] instead of complex Brownian motion representation in [13, Theorem 1.1].

The following proposition is a combination of the results in [9, 10, 11, 12] and the tightness of $\{\mathbb{P}^\xi_\star\}_{\star \in \{\sin, \operatorname{Ai}\} \cup \{J_\nu : \nu > -1\}}$, which is derived by the same argument as that of $\{\mathbb{P}^\xi_{J_\nu}\}_{\nu \in \mathbb{N}}$.

**Proposition 3.3** Let $\star \in \{\sin, \operatorname{Ai}\} \cup \{J_\nu : \nu > -1\}$.

(i) Suppose that $\xi, \xi_n \in \mathbb{X}_\star, n \in \mathbb{N}$. If $\xi_n$ converges to $\xi$ in $\mathbb{X}_\star$, then

$$
(\Xi(t), \mathbb{P}^\xi_n) \rightarrow (\Xi(t), \mathbb{P}^\xi), \quad n \rightarrow \infty \tag{3.8}
$$

weakly on $C([0, \infty), \mathcal{M})$.

(ii) Determinantal process $(\Xi(t), \mathbb{P}^\star)$ with the correlation kernel $K_\star$ is identical in distribution to process $(\Xi(t), \mathbb{P}^{\nu}_\star)$, where $\mathbb{P}^{\nu}_\star = \int_\mathbb{M}_\star \nu(d\xi)$ for probability measure $\nu$ on $\mathcal{M}$.
3.3 Path property of noncolliding processes

Let $X(t) = (X_1(t), X_2(t), \ldots, X_N(t))$ be the Dyson model defined by (1.1). We introduce the following version of the Dyson model

$$\tilde{X}_j(t) = e^{-\gamma_N t} X_j(\tau_N(t)),$$

where $\gamma_N = \frac{1}{2N}$ and $\tau_N(t) = (e^{2t\gamma_N} - 1)/2\gamma_N$. Process $\tilde{X}(t) = (\tilde{X}_j(t))_{j=1,\ldots,N}$ then solves the SDE

$$d\tilde{X}_j(t) = d\tilde{B}_j(t) - \frac{\tilde{X}_j(t) dt}{2N} + \sum_{k: k \neq j} \frac{dt}{2N} \tilde{X}_j(t) - \tilde{X}_k(t), \quad j \in \mathbb{I}_N, \tag{3.9}$$

where the $\tilde{B}_j(t)$’s are independent one-dimensional standard BMs. The eigenvalue distribution in GUE,

$$m_{\text{GUE}}^N(dx_N) = \frac{1}{Z} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp \left\{ -\frac{1}{2N} \sum_{k=1}^N x_k^2 \right\} dx_N \tag{3.10}$$

is a reversible probability measure of process $\tilde{X}(t)$. Hereafter, the notation $Z$ is used to denote the normalizing constant. We denote the distribution of $u(X(t))$ with $u(X(0)) = \xi^N$ by $\tilde{P}_{\text{sin}}^\xi$. Process $(\Xi(t), \tilde{P}_{\text{sin}}^\xi)$ has the probability measure

$$\mu_{\text{sin}}^N \equiv m_{\text{GUE}}^N \circ u^{-1} \tag{3.11}$$

as the reversible measure. Measure $\mu_{\text{sin}}^N$ is the DPP with the correlation kernel

$$\mathbb{K}_N(x, y) = \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \varphi_k \left( \frac{x}{\sqrt{2N}} \right) \varphi_k \left( \frac{y}{\sqrt{2N}} \right),$$

where $h_k = \sqrt{\pi} 2^k k!$ and

$$\varphi_k(x) = \frac{1}{\sqrt{h_k}} e^{-x^2/2} H_k(x),$$

with Hermite polynomials $H_k, k \in \mathbb{N}_0$.

We also introduce process $\tilde{Y}(t) = (\tilde{Y}_j(t))_{j \in \mathbb{I}_N}$ obtained from the Dyson model in (3.9) by the transformation given by $N^{-1/3} \tilde{X}_j(N^{2/3} t) - 2N^{2/3}$, which solves the SDE

$$d\tilde{Y}_j(t) = dB_j(t) - \frac{\tilde{Y}_j(t) + 2N^{2/3} dt}{2N^{1/3}} + \sum_{k: k \neq j} \frac{dt}{2N^{1/3}} \tilde{Y}_j(t) - \tilde{Y}_k(t), \quad j \in \mathbb{I}_N. \tag{3.12}$$

We denote the distribution of process $u(\tilde{Y}(t))$ with $u(\tilde{Y}(0)) = \xi^N$ by $\tilde{P}_{\text{Ai}}^\xi$. Process $(\Xi(t), \tilde{P}_{\text{Ai}}^\xi)$ has

$$\mu_{\text{Ai}}^N \equiv m_{\text{Ai}}^N \circ u^{-1} \tag{3.13}$$
with

\[ m_{\lambda_i}^N(dx_N) = \frac{1}{Z} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp \left\{ -\frac{1}{2} \sum_{k=1}^{N} (x_k - N^{1/3})^2 \right\} \, dx_N \]

as the reversible measure. For the noncolliding squared Bessel process in (1.2) we set \( \tilde{X}_j^\nu(t) = e^{-\gamma t} X_j^\nu(\tau_N(t)) \). Process \( \tilde{X}^\nu(t) = (\tilde{X}_j^\nu(t))_{j \in \mathbb{N}} \) then solves the SDE

\[
d\tilde{X}_j^\nu(t) = 2 \sqrt{X_j^\nu(t)} dB_j(t) - \tilde{X}_j^\nu(t) dt + 2(\nu + 1) dt + \sum_{k : k \neq j}^{N} \frac{4\tilde{X}_j^\nu(t) dt}{X_j^\nu(t) - X_k^\nu(t)}. \tag{3.14}
\]

The eigenvalue distribution in chGUE,

\[
m_{J_\nu}^N(dx_N) = \frac{1}{Z} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{j=1}^{N} x_j^{\nu+1/2} \exp \left\{ -\frac{1}{2} \sum_{k=1}^{N} x_k \right\} \, dx_N, \tag{3.15}
\]

is a reversible probability measure of \( \tilde{X}^\nu(t) \). We denote the distribution of process \( u(\tilde{X}^\nu(t)) \) by \( \tilde{P}_x^\nu \). Process \( (\Xi(t), \tilde{P}_x^\nu) \) has the probability measure

\[
\mu_{J_\nu}^N \equiv m_{J_\nu}^N \circ u^{-1} \tag{3.16}
\]

as the reversible measure. Measure \( \mu_{J_\nu}^N \) is the DPP with correlation kernel

\[
\mathbb{K}_N^{(\nu)}(x, y) = \frac{1}{2N} \sum_{k=0}^{N-1} \phi_k^\nu \left( \frac{x}{2N} \right) \phi_k^\nu \left( \frac{y}{2N} \right),
\]

where

\[
\phi_k^\nu(x) = \frac{\Gamma(k + 1)}{\Gamma(\nu + k + 1)} x^{\nu/2} L_k^\nu(x) e^{-x/2},
\]

with the Laguerre polynomials \( L_k^\nu(x), k \in \mathbb{N}_0 \) with parameter \( \nu > -1 \) [8]. From the construction of \( \tilde{P}_x^\nu, \star \in \{\sin, \Ai, J_\nu; \nu > -1\} \) the following is readily derived from Proposition 3.3.

**Proposition 3.4** Let \( \star \in \{\sin, \Ai\} \cup \{J_\nu; \nu > -1\} \). Suppose that \( \xi \in \mathbb{X}_\star \). Then

\[
(\Xi(t), \tilde{P}_x^\nu) \rightarrow (\Xi(t), \mathbb{P}_x^\nu), \quad N \rightarrow \infty,
\]

weakly on \( C([0, \infty), \mathcal{M}) \), where \( \xi_N^1 \) is the configuration with a finite number of particles given in (2.13).

**Remark** (i) Let \( \star \in \{\sin, \Ai\} \cup \{J_\nu; \nu > -1\} \). The reversible diffusion process \( (\Xi(t), \tilde{P}_x^\nu) \) is associated with the Dirichlet form \( (\mathcal{E}_\star^N, \mathcal{D}_\star^N) \) introduced in Section 2.
Lemma 3.5 (i) Let \( \star \in \{\sin, Ai\} \). For each \( T > 0 \), there exists a positive constant \( C_\star \) such that for any interval \( D \) of \( \mathbb{R} \) and \( \varepsilon > 0 \)

\[
\mathbb{P}_\star \left( \exists X \in \mathbb{E} \ s.t. \ X(0) \in D, \sup_{t \in [0,T]} |X(t) - X(0)| > \varepsilon \right) \leq C_\star (\rho_\star(D) \vee 1) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right),
\]

where \( \text{Erf}(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \).

(ii) Let \( \star \in \{J_\nu, \nu > -1\} \). For each \( T > 0 \), there exists a positive constant \( C_\star \) such that for any interval \( D \) of \( [0, \infty) \) with \(|D| \geq 1\) and \( \varepsilon > 0 \)

\[
\mathbb{P}_\star \left( \exists X \in \mathbb{E} \ s.t. \ X(0) \in D, \sup_{t \in [0,T]} \left| \sqrt{X(t)} - \sqrt{X(0)} \right| > \varepsilon \right) \leq C_\star (\rho_\star(D) \vee 1) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right),
\]

To prove Lemma 3.5 we prepare a lemma concerning on reversible diffusions in \( \mathbb{R}^N \).

Lemma 3.6 Let \( Y(t) = (Y_j(t))_{j \in \mathbb{N}} \) be a reversible diffusion process satisfying

\[
Y_j(t) = Y_j(0) + B_j(t) + \int_0^t b_j(Y(s)) ds, \quad j \in \mathbb{N}, \ t \in [0,T],
\]

with some measurable function \( b = (b_1, \ldots, b_N) \). Here the \( B_j(t), 1 \leq j \leq N \) are independent BMs, and \( Y(0) = (Y_j(0))_{j \in \mathbb{N}} \) is a random variable distributed by the reversible probability measure, which is independent of the BMs. Putting \( \rho_Y(A) = \sum_{j=1}^N P(Y_j(0) \in A) \) for \( A \in \mathcal{B}(\mathbb{R}) \), and

\[
c = \sum_{\ell=0}^\infty \frac{\rho_Y(A_\ell)}{\rho_Y(D) \vee 1} \frac{\text{Erf} \left( \{2(\ell \vee 1) - 1\} \varepsilon / \sqrt{T} \right)}{\text{Erf}(\varepsilon / \sqrt{T})}, \quad (3.18)
\]
we then obtain

\[
P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |Y_j(t) - Y_j(0)| > \varepsilon \text{ for some } j \right) \leq 4(1 + c)(\rho_Y(D) \vee 1) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right).
\]

(3.19)

Proof. We use a Lyons-Zheng decomposition for the proof (see for example Section 5.7 in [4]). We deduce from the Lyons-Zheng decomposition that \( \hat{Y}(t) = Y(T - t) \) satisfies

\[
\hat{Y}_j(t) = \hat{Y}_j(0) + \hat{B}_j(t) + \int_0^t b_j(\hat{Y}(s))ds, \quad j \in \mathbb{I}_N, \ t \in [0,T],
\]

with independent BMs \( \hat{B}_j(t) \), \( 1 \leq j \leq N \). Then

\[
Y_j(t) - Y_j(0) = \hat{Y}(T - t) - \hat{Y}(T) = \hat{B}_j(T - t) - \hat{B}_j(T) - \int_0^t b_j(Y(s))ds = \frac{1}{2} \left( B_j(t) + \hat{B}_j(T - t) - \hat{B}_j(T) \right).
\]

(3.20)

Putting \( A_\ell = A_\ell(\varepsilon) = \{ x \in \mathbb{R} : \inf_{y \in D} |x - y| \in [\ell \varepsilon, (\ell + 1)\varepsilon) \}, \ell \in \mathbb{N} \cup \{0\} \), we have

\[
P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |B_j(t)| \leq \varepsilon, \sup_{t \in [0,T]} |Y_j(t) - Y_j(0)| > \varepsilon \right)
\]

\[
\leq \sum_{\ell=0}^{\infty} P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |B_j(t)| \leq \varepsilon, Y_j(T) \in A_\ell, \sup_{t \in [0,T]} |Y_j(t) - Y_j(0)| \geq (\ell \vee 1)\varepsilon \right)
\]

\[
\leq \sum_{\ell=0}^{\infty} P \left( Y_j(0) \in D, Y_j(T) \in A_\ell, \sup_{t \in [0,T]} |\hat{B}_j(T - t) - \hat{B}_j(T)| \geq \{2(\ell \vee 1) - 1\}\varepsilon \right) \text{ by (3.20)}
\]

\[
\leq \sum_{\ell=0}^{\infty} P \left( Y_j(T) \in A_\ell, \sup_{t \in [0,T]} |\hat{B}_j(T - t) - \hat{B}_j(T)| \geq \{2(\ell \vee 1) - 1\}\varepsilon \right).
\]

Then from this we deduce that

\[
P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |Y_j(t) - Y_j(0)| > \varepsilon \text{ for some } j \right)
\]

\[
\leq \sum_{j=1}^{N} P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |Y_j(t) - Y_j(0)| > \varepsilon \right)
\]

\[
\leq \sum_{j=1}^{N} P \left( Y_j(0) \in D, \sup_{t \in [0,T]} |B_j(t)| > \varepsilon \right)
\]

\[
+ \sum_{\ell=0}^{\infty} \sum_{j=1}^{N} P \left( Y_j(T) \in A_\ell, \sup_{t \in [0,T]} |\hat{B}_j(T - t) - \hat{B}_j(T)| \geq \{2(\ell \vee 1) - 1\}\varepsilon \right).
\]
Note that \( \hat{Y}(0) \) is independent of \( \hat{B}_j(t), 1 \leq j \leq N \), since \( Y(0) \) is independent of \( B(t), 1 \leq j \leq N \). We then have that the right-hand side of the above inequality is bounded by

\[
\sum_{j=1}^{N} P(Y_j(0) \in D) P \left( \sup_{t \in [0,T]} |B_j(t)| > \varepsilon \right)
+ \sum_{\ell=0}^{\infty} \sum_{j=1}^{N} P(Y_j(T) \in A_{\ell}) P \left( \sup_{t \in [0,T]} |\hat{B}_j(T-t) - \hat{B}_j(T)| \geq \{2(\ell \lor 1) - 1\} \varepsilon \right)
\leq 4 \sum_{j=1}^{N} P(Y_j(0) \in D) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right) + 4 \sum_{\ell=0}^{\infty} \sum_{j=1}^{N} P(Y_j(0) \in A_{\ell}) \text{Erf} \left( \frac{2(\ell \lor 1) - 1}{\sqrt{T}} \varepsilon \right),
\]

\[
= 4\rho_Y(D) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right) + 4 \sum_{\ell=0}^{\infty} \rho_Y(A_{\ell}) \text{Erf} \left( \frac{2(\ell \lor 1) - 1}{\sqrt{T}} \varepsilon \right).
\]

Here we used the estimate \( P \left( \sup_{t \in [0,T]} |B_j(t)| > \varepsilon \right) \leq 4 \text{Erf}(\varepsilon/\sqrt{T}) \) and the reversibility of the process. (3.19) follows from this and (3.18) immediately.

**Proof of Lemma 3.5.** We apply Lemma 3.5 to SDEs (3.9), (3.12), and (3.17). By simple calculation, for the solutions of these SDEs, \( c \) is a constant independent of \( D \) and \( \varepsilon \). We then see that there exists a positive constant \( C^*_\star > 0 \) such that for \( \star \in \{\sin, \text{Ai}\}
\]

\[
\hat{\mathbb{P}}_{\star}^{\mu_N} \left( \exists X \in \Xi_{\star} \text{ s.t. } X(0) \in D, \sup_{t \in [0,T]} |X(t) - X(0)| > \varepsilon \right) \leq C^*_{\star}(\rho_{\star}^N(D) \lor 1) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right),
\]

and for \( \star \in \{J_{\nu}, \nu > -1\}
\]

\[
\hat{\mathbb{P}}_{\star}^{\mu_N} \left( \exists X \in \Xi_{\star} \text{ s.t. } X(0) \in D, \sup_{t \in [0,T]} |\sqrt{X(t)} - \sqrt{X(0)}| > \varepsilon \right) \leq C^*_{\star}(\rho_{\star}^N(D) \lor 1) \text{Erf} \left( \frac{\varepsilon}{\sqrt{T}} \right),
\]

where \( \rho_{\star}^N(D) = \int_D \rho_{\star}^N(x)dx \) with the density (the first correlation function) \( \rho_{\star}^N(x) \) of \( \mu_N \). Since \( \hat{\mathbb{P}}_{\star}^{\mu_N} \) converges to \( \mathbb{P}_{\star} \) weakly on \( \mathcal{C}([0,\infty), \mathfrak{M}) \), as \( N \to \infty \) (see for instance [8, Section 7]), we obtain the desire result by simple observation.

### 3.4 Proof of Theorem 2.1

Let \( \mathfrak{X} \) be a subset of \( \mathfrak{M} \) and \( (\Xi(t), P_\xi), \xi \in \mathfrak{X} \) be a stochastic process. Suppose that

\[
P_{\xi_n}(\cdot) \to P_\xi(\cdot), \quad \text{if } \xi_n \to \xi \text{ in } \mathfrak{X}.
\]  

(3.21)

Further suppose \( \xi \in \mathfrak{X} \) satisfying \( P_\xi(\Xi(t) \text{ is continuous in } \mathfrak{X}) = 1 \). If \( (\Xi(t), P_\xi) \) is Markovian, then it is strong Markovian.

Let \( \star \in \{\sin, \text{Ai}\} \cup \{J_{\nu} ; \nu > -1\} \). From Proposition 3.3 (1) we see that \( (\Xi(t), \mathbb{P}_\star \text{ satisfies (3.21). Hence from the Markov property of process } (\Xi(t), \mathbb{P}_\star \text{ given in [12] and Proposition 3.3 (ii), we deduce Theorem 2.1 from the following proposition.} \)
Proposition 3.7 Let $\star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}$. Then process $(\Xi(t), \mathbb{P}_\star)$ is continuous in $\mathcal{X}_\star$.

Proof. This proposition is derived from the following claims:
1) $\Xi(t)$ has a vaguely continuous path.
2) For each $T \in \mathbb{N}$, there exist $\varepsilon \in (0, 1), \kappa \in (1/2, \kappa_\star)$ such that
$$
\lim_{L \to \infty} \lim_{m \to \infty} \mathbb{P}_\star(\Xi(t) \in \mathcal{X}_{L,m}^{\varepsilon, \kappa}, t \in [0, T]) = 1.
$$

Claim 1) has already shown in Section 3.2. We remark that it is also derived by Kolmogorov’s criterion: for any polynomial function $f$, which is a smooth function on $\mathcal{M}$ given in (3.27),
$$
\mathbb{E}_\star[|f(\Xi(t)) - f(\Xi(s))|^\beta] \leq C|t - s|^\alpha, \quad 0 \leq s < t \leq T < \infty,
$$
for some $\alpha > 1, \beta > 0$, and $C > 0$, which is readily proved from Lemma 3.5.

Claim 2) is derived from two estimates
$$
\lim_{L \to \infty} \mathbb{P}_\star(|\rho_\star(D_L) - \Xi(t, D_L)| \leq L^\varepsilon, t \in [0, T]) = 1, \quad D_L = [0, L], [-L, 0],
$$
and
$$
\lim_{m \to \infty} \mathbb{P}_\star(\Xi(t, [g^\kappa(k), g^\kappa(k + 1)]) \leq m, t \in [0, T] k \in \mathbb{Z}) = 1. \quad (3.23)
$$

From Lemma 3.1, there exist $m' \in \mathbb{N}$ and $p < m' - 1$ such that
$$
\int_{\mathcal{M}} \mu_\star(d\xi)|\rho_\star(D_L) - \xi(D_L)|^m' = O(L^p), \quad L \to \infty. \quad (3.24)
$$

Taking $\varepsilon \in ((p + 1)/m', 1)$ and using Chebyshev’s inequality with (3.24), we can find a positive constant $C$ such that
$$
\mu_\star(|\rho_\star(D_L) - \Xi(D_L)| \geq L^\varepsilon) \leq CL^{p-m'\varepsilon}.
$$

Since $p - m'\varepsilon < -1$, we have
$$
\sum_{L=1}^{\infty} \mu(|\rho_\star(D_L) - \Xi(D_L)| \geq L^\varepsilon) < \infty.
$$

By Borel-Cantelli’s lemma and the stationarity of the process, for any $k \in \mathbb{N}$
$$
\lim_{L \to \infty} \mathbb{P}_\star\left(|\rho_\star(D_L) - \Xi(t, D_L)| \leq L^\varepsilon, t = \frac{j}{k}, j = 1, 2, \ldots, kT\right) = 1.
$$

(3.22) is then derived from Lemma 3.5. Estimate (3.23) is derived from Lemmas 3.2 and 3.5. In fact Lemma 3.5 (i) implies that for $\star \in \{\sin, \text{Ai}\}$ and $k \in \mathbb{Z},$
$$
\mathbb{P}_\star\left(\exists X \in \Xi \text{ s.t. } X(0) \in [g^\kappa(k), g^\kappa(k + 1)], \quad \sup_{u \in [0, 1/k]} |X(u) - X(0)| > |k|^\kappa-1\right)
\leq C_\star\{\rho_\star([g^\kappa(k), g^\kappa(k + 1)]) \vee 1\} \text{Erf}(|k|^\kappa/2 - 1).
Take $\ell \geq 2$ and choose $m \in \mathbb{N}$ such that (3.1) holds. Using Borel-Cantelli’s lemma with simple calculations, we see that for $\mathbb{P}^\star$-a.s. $\Xi$, there exists $k_0 = k_0(\Xi) \in \mathbb{N}$ such that for any $k \in \mathbb{Z}$ with $|k| \geq k_0$, if $X \in \Xi$ and $s \in \{J_{|k| \ell}; j = 1, 2, \ldots, |k| \ell T\}$ satisfy
\[ X(s) \in [g^\kappa(k), g^\kappa(k + 1)], \]
then
\[ \sup_{u \in [0, 1/k \ell]} |X(s + u) - X(s)| \leq |k|^{\kappa - 1}. \]
Combining this fact with (3.2), we see that for $\mathbb{P}^\star$-a.s. $\Xi$ there exists $m_1 = m_1(\Xi) \in \mathbb{N}$ such that
\[ \Xi(t, [g^\kappa(k), g^\kappa(k + 1)]) \leq m_1, \ t \in [0, T], \ k \in \mathbb{Z}. \quad (3.25) \]
We thus obtain (3.23). For $\star \in \{j_\nu, \nu > 1\}$ we can obtain the desire result by the same argument as above with Lemma 3.5 (ii). This completes the proof. 

### 3.5 Proof of Theorem 2.2

We now introduce Dirichlet forms describing $k$-labeled dynamics. For this we recall the definition of Palm and Campbell measures. Let $x_k = (x_1, \ldots, x_k) \in \mathbb{R}^k$. We set
\[ \mu_{x_k} = \mu(\cdot - \sum_{i=1}^k \delta_{x_i} | \xi(x_i) \geq 1 \text{ for } i = 1, \ldots, k), \]
which is called the (reduced) Palm measure of $\mu$. The Campbell measure for probability measure $\mu$ is then given by
\[ \nu^k(dx_k d\eta) = \mu_{x_k}(d\eta) \rho_k(x_k) dx_k. \]
Here $\rho_k : \mathbb{R}^k \to [0, \infty)$ is the $k$-th correlation function of $\mu$ and $dx_k = dx_1 \cdots dx_k$ is the Lebesgue measure on $\mathbb{R}^k$ as before. Let $\mathcal{D}_0^k = C_0^\infty(\mathbb{R}^k) \otimes D_\infty$. For $f, g \in \mathcal{D}_0^k$, let $\nabla^k[f, g]$ be such that
\[ \nabla^k[f, g](x_k, \xi) = \frac{1}{2} \sum_{j=1}^k a(x_j) \frac{\partial}{\partial x_j} f(x_k, \xi) \frac{\partial}{\partial x_j} g(x_k, \xi). \quad (3.26) \]
We set $\mathbb{D}^{a,k}$ as
\[ \mathbb{D}^{a,k}[f, g][x_k, \xi] = \nabla^k[f, g](x_k, \xi) + \mathbb{D}[f(x_k, \cdot), g(x_k, \cdot)](\xi). \]
We consider the bilinear form $(\mathcal{E}^{a,k}_{\nu^k}, \mathcal{D}_{\infty}^{a,k})$ on $L^2(\mathbb{R}^k \times \mathfrak{M}, \nu^k)$ defined as
\[ \mathcal{E}^{a,k}_{\nu^k}(f, g) = \int_{\mathfrak{M}} \mathbb{D}^{a,k}[f, g][x_k, \xi] d\nu^k, \]
\[ \mathcal{D}_{\nu^k} = \{ f \in \mathcal{D}_0^k : \mathcal{E}^{a,k}_{\nu^k}(f, f) + \|f\|_{L^2(\mathbb{R}^k \times \mathfrak{M}, \nu^k)}^2 < \infty \}. \]
It was proved that \((\mathcal{E}^{a,k}_\nu, \mathcal{D}^{a,k}_\nu)\) is closable and its closure \((\mathcal{E}^{a,k}_\nu, \mathcal{D}^{\nu}_\infty)\) is a quasi-regular Dirichlet form in case \(\mu\) is a quasi Gibbs measure [21]. Let \(\nu^k_\star\) be the Campbell measure associated with DPP \(\mu_\star\), \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}\). For the case where \((\nu^k, a(x)) = (\nu^k_{\text{Ai}}, 1)\), \((\nu^k_{\text{Ai}}, 1)\), and \((\nu^k_{J_\nu}, 4x)\), we denote the quasi-Dirichlet form by \((\mathcal{E}^k_\star, \mathcal{D}^k)\) and the associated diffusion process by \((\mathbf{X}^k(t), H(t), \mathbb{P}^{\nu^k, \eta}_\infty), \star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}\). We introduce the map \(u : \mathbb{R}^k \times \mathcal{M} \to \mathcal{M}\) defined as
\[
u(x, \xi) = \sum_{j=1}^k \delta_{x_j} + \xi, \quad x \in \mathbb{R}^k, \xi \in \mathcal{M}.
\]
It is shown in [21] that if \(u(x, \eta) = \xi\), process \((u(X^k(t), H(t)), \mathbb{P}^{x_\eta}_\infty)\) coincides with process \((\Xi(t), \mathbb{P}^\xi_\infty)\).

Let \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}\). Theorem 2.1 implies that the Dirichlet form \((\mathcal{E}_\star, \mathcal{D}_\star)\) associated with process \((\Xi(t), \mathbb{P}^\xi)\) is quasi-regular [16, 4]. A function \(f\) on the configuration space \(\mathcal{M}\) is said to be polynomial if it is written in the form
\[
\begin{align*}
f(\xi) &= F \left( \int_{\mathbb{R}} \phi_1(x) \xi(dx), \int_{\mathbb{R}} \phi_2(x) \xi(dx), \ldots, \int_{\mathbb{R}} \phi_k(x) \xi(dx) \right) \quad (3.27)
\end{align*}
\]
with polynomial function \(F\) on \(\mathbb{R}^k, k \in \mathbb{N}\), and smooth functions \(\phi_j, 1 \leq j \leq k\) on \(\mathbb{R}\) with compact supports. Let \(\mathcal{P}\) be the set of all polynomial functions on \(\mathcal{M}\). In [8, Proposition 7.2] we showed that
\[
\mathcal{E}_\star(f, g) = \mathcal{E}_\star(f, g), \quad f, g \in \mathcal{P}.
\]
\((\mathcal{E}_\star, \mathcal{D}_\star)\) and \((\mathcal{E}_\star, \mathcal{D}_\star')\) are then closed extensions of \((\mathcal{E}_\star, \mathcal{P}_\star)\), and the former is the smallest one [25]. These relations are generalized to \(k\)-labeled dynamics.

**Lemma 3.8** Let \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}\). For each \(k \in \mathbb{N}\), there exists a diffusion process \((\mathbf{X}^k(t), H(t), \mathbb{P}^{\nu^k, \eta}_\star)\) associated with quasi-regular Dirichlet form \((\mathcal{E}^k_\star, \mathcal{D}^k_\star)\) such that
\[
\begin{align*}
(u(X^k(t), H(t)), \mathbb{P}^{x_\eta}_\star) &= (\Xi(t), \mathbb{P}^\xi_\star), \quad \text{if } u(x, \eta) = \xi, \quad (3.29) \\
\mathcal{E}^k_\star(f, g) &= \mathcal{E}^k_\star(f, g), \quad f, g \in C^0(\mathbb{R}^k) \otimes \mathcal{P}_\star. \quad (3.30)
\end{align*}
\]

**Proof.** Let \(\star \in \{\sin, \text{Ai}\} \cup \{J_\nu; \nu > -1\}\). We introduce \((\mathcal{E}^k_\star, \mathcal{D}^k_{\star,0})\) defined as
\[
\mathcal{D}^k_{\star,0} = \{f(x_k, \eta) = g(x_k, u(x_k, \eta)) ; g \in C^\infty(\mathbb{R}^k) \otimes \mathcal{D}_\star\},
\]
\[
\mathcal{E}^k_\star(f, g) = \int_{\mathcal{M}} \nabla [f, g](x_k, \eta) + \mathbb{D}^k_\star [f(x_k, \cdot), g(x_k, \cdot)](\nu^k_\star(dx_k d\eta), \quad f, g \in \mathcal{D}^k_{\star,0},
\]
where the derivatives are taken in the sense of the Schwartz distribution, \(\mathbb{D}^k_\star = \mathbb{D}^{1,k}_\star\) if \(\star \in \{\sin, \text{Ai}\}\), and \(\mathbb{D}^k_\star = \mathbb{D}^{4,k}_\star\) if \(\star \in \{J_\nu, \nu > -1\}\). For \(\eta \in \mathcal{M}\) and \(r \in \mathbb{N}\), we set \(\eta_r = \eta_{[-r,r]} = \sum_{j=1}^r \delta_{y_j}\) and \(\zeta = \eta_{[-r,r]^c}\). For \(f \in \mathcal{D}^k_{\star,0}\) we set
\[
J_{r,\zeta}(x_k, \eta_r) = f(x_k, \eta) \mathbb{1}_{[-r,r]}(x_k)
\]

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and
\[
\mathbb{D}^k_{*,r}[f, g][x_k, \eta_r] = \nabla^k [f_{r, \zeta}, g_{r, \zeta}](x_k, \eta_r) + \mathbb{D}^k_{*}[f_{r, \zeta}(x_k, \cdot), g_{r, \zeta}(x_k, \cdot)](\eta_r)
\]
\[
\tilde{E}^k_{*,r}(f, g) = \int_{\mathbb{R}} \mathbb{D}^k_{*,r}(f, g) d \nu^k, \quad f, g \in \tilde{D}^k_{*,0}.
\]

It is readily seen that the bilinear forms \((\tilde{E}^k_{*,r}, \tilde{D}^k_{*,0})\), \(r \in \mathbb{N}\) are closable and increasing. From [19, Lemma 2.1 (1)] we see that \((\tilde{E}^k_{*}, \tilde{D}^k_{*})\) is closable. The quasi-regularity of the closure \((\tilde{E}^k_{*}, \tilde{D}^k_{*})\) is derived from that of the Dirichlet form \((\hat{E}^k_{*}, \hat{D}^k_{*})\). Hence, the associated diffusion process \(((X^k(t), H(t)), \mathbb{P}^{x_k,\eta})\) can be constructed. Equation (3.30) is derived from (3.28), while (3.29) is derived from the argument to show [21, Lemma 4.2]. This completes the proof.

**Proof of Theorem 2.2.** We show Theorem 2.2 (i) by applying [22, Theorem 26] to process \((\Xi(t), \mathbb{P}_*)\) associated with the Dirichlet form \((\hat{E}_*, \hat{D}_*)\), \(* \in \{\sin, \text{Ai}\} \cup \{J_\nu : \nu > -1\}\). Of the assumptions (A.1)–(A.5) of the theorem, (A.1), (A.2), and (A.5) are satisfied, because the related measures \(\mu_*\) are the same as those of the Dirichlet form \((E_*, D_*)\), which have already been verified in [22, 5, 26]. Assumption (A.4) is derived from the fact that the capacity related to \((E_*, D_*)\) is greater than that of \((\hat{E}_*, \hat{D}_*)\), since these Dirichlet forms are closed extensions of \((E_*, D_*)\), and the former is the smallest one.

Condition (A.3) is used in the proof of [22, Theorem 26] to construct a \(k\)-labeled process and to check that process \(X_j(t) - X_j(0), j = 1, 2, \ldots, k\) can be regarded as a Dirichlet process. These claims are derived from Lemma 3.8 together with the fact that \(D^k_* \subset \hat{D}^k_*\). (i) is thus proved.

Claim (ii) is derived from (i) and the uniqueness of the strong solutions of ISDEs \((\sin), (\text{Ai}), (J_\nu)\), which have been proved in [27].

**Proof of Corollary 2.3.** Process \((\Xi(t), \mathbb{P}^{x_{\xiN}}_*)\), \(* \in \{\sin, \text{Ai}\} \cup \{J_\nu : \nu > -1\}\) is identical in distribution to process \((\Xi(t), \tilde{\mathbb{P}}^{x_{\xiN}}_*)\). Corollary 2.3 (i) is thus derived from Theorem 2.2 and Proposition 3.4. Claim (ii) is readily derived from Claim (i).

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