A Few Exactly Solvable Models For Riccati Type Equations

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Abstract

We consider the Ricatti equation in the context of population dynamics, quantum scattering and a more general context. We examine some exactly solvable cases of real life interest.

1 Introduction

The Riccati Equation [1, 2, 3]

\[ y' = p(t) + q(t)y + r(t)y^2 \]  \hspace{1cm} (1)

appears in several branches of applicable mathematics, for example population dynamics and mathematical physics, as in quantum scattering. It is known that it exhibits chaotic behavior [4]. It is also well known that, it reduces to a second order linear differential equation by the substitution, 

\[ y = \frac{u'}{u}, \]

when we get,

\[ u'' + g(t)u' + f(t)u = 0 \] \hspace{1cm} (2)

(1) and (2) have been studied in detail for a long time [5, 6, 7]. It may be mentioned that if two solutions of the Riccati equation are equal at a point, then they coincide. The reason lies in the continuity of the logarithmic derivative \( \frac{u'}{u} \) given in the substitution that lead from (1) to (2).
We will now consider exact solutions of the Riccati equation in two different contexts, from the field of population dynamics and the field of quantum scattering.

2 Some exact solutions

2.1 A Problem In Population Dynamics

We now examine coupled Riccati type equations of relevance to a population model.

Gause’s Model

Consider two-species populations occurring together, and assume that the growth of each is inhibited by members, both of its own and of the other species. Denoting the number of individuals in species 1 as \( N_1 \) and species 2 as \( N_2 \), we have the Gause’s competition equations:

\[
\frac{1}{N_1} \frac{dN_1}{dt} = r_1 - a_1 N_1 - a_2 N_2
\]

\[
\frac{1}{N_2} \frac{dN_2}{dt} = r_2 - a_3 N_1 - a_4 N_2
\]

where \( r_1, r_2, a_1, a_2 \) are defined below. Thus we are assuming that the per capita growth of each population at an instant is a linear function of the sizes of the two competing populations at that instant. Each population would grow logistically if it were alone with logistic parameters \( r_1 \) and \( a_1 \) for species 1 and \( r_2 \) and \( a_2 \) for species 2.

In general the simultaneous differential equations cannot be explicitly solved. We now consider a particular set of circumstances in which they can be solved. Thus for (3) and (4) we specialize to, with a more convenient notation,

\[
\frac{dy}{dt} = 1 - xy^2
\]

\[
\frac{dx}{dt} = 1 + yx^2
\]
From (5) and (6) we get,

\[
\frac{dy}{dx} = \frac{1 - xy^2}{1 + yx^2}
\]

Integrating (7) we have

\[
y - x = -x^2y^2 + C
\]

From (8) when \( y = 0, C = -x_0 \) so that \( y < x \)

Reverting back to \( N_1 \) and \( N_2 \), this is

\[
N_1 - N_2 = -N_2^2N_1^2 + C
\]

From (9) when \( N_1 = 0, C = -N_0 \) so that \( N_1 < N_2 \).

The Figure below illustrates a particular case of the above solution with the two populations \( N_1 \) and \( N_2 \) along Y and X axes.

2.2 Problems from Scattering Theory

By reversing the method used above, the quantum mechanical radial Schrödinger equation

\[
u'' - H(r)u = 0
\]

(10)
can be reduced to a Riccati equation

\[
v' + v^2 = H(r)
\]

(11)

by the substitution \( v = \frac{u'}{u} \). The form (10) or (11) is used in phase shift analysis for example in Calogero’s variable phase approach [9]. It is also possible to
use the form (11) for building up an iterative procedure. We would now like to point out that (11) can be used in a different context for providing exact solutions for specified classes of the potential function $f(r)$. Let us write in (11)

$$v \equiv \frac{1}{f}$$

$$f' = 1 - gf$$

Whence we get

$$H = \frac{g}{f} = \frac{ge\int gd\rho}{\int e\int gd\rho + C}$$

(12)

1. We put $g=\alpha$ in (12)

$$H = \frac{\alpha^2}{1 + c\alpha e^{-\alpha r}}$$

The potential $H$ is now of the form

$$H = \frac{D}{A + Be^{\alpha r}}$$

This is the well-known Wood-Saxon potential. The solution $u$ is given by

$$u = C_1(e^{\alpha r} + C_2\alpha)$$

2. Putting $g=r$ in (12), the potential $H$ is given by

$$H = \frac{re^{r^2/2}}{\int e^{r^2/2}dr + C_1}$$

This is a modified Gaussian potential and the solution $u$ is given by

$$u = K \int e^{-r^2} dr + C_3$$

The Graph is shown in the figure below and is seen to fall very steeply indicative of a confined state or particle.
3. Putting \( g = 1/r \) in (11), the potential \( H \) is given by

\[
H = \frac{2}{r^2 + c_4}
\]

This is a shifted inverse square potential and the solution \( u \) is given by

\[
u = K'(r^2 + C_4)
\]

### 3 Other Exact and Asymptotic Solutions

We now consider some asymptotic solutions of (11), which we write as,

\[
u' = u^2 + f
\]

Let

\[
\int fdr \equiv g,
\]

\( f \) and \( g \) being bounded functions. So (13) can be written as,

\[
z' = (z + g)^2 > 0, z = u - g,
\]

which shows that \( z \) is an increasing function of \( r \). Suppose \( z \) is unbounded. So for large \( r \), we should have from (14)

\[
z \approx z^2
\]

whence

\[
z = \frac{1}{c - r}
\]
which $\to 0$ as $r \to \infty$.

This is a contradiction. Therefore $z$ is bounded and so also $u$, that is $u \to M$ as $r \to \infty$.

Therefore $u' \approx 0$ (asymptotically)

Therefore for large $r$, (13) becomes

$u^2 = -f$, whence

$u = \pm (-f)^{1/2}$

(15)

By way of verification of (15), let us consider (13) with

$f = -\frac{\alpha^2}{r^2}$

So we expect that for large $r$, $u \sim \pm \frac{\alpha}{r}$ by (15). Let us put

$u = \frac{\beta}{r}$

(16)

in (13). So

$-\frac{\beta}{r^2} = \frac{\beta^2}{r^2} - \frac{\alpha^2}{r^2}$

or

$\beta = \frac{-1 \pm \sqrt{1 + 4\alpha^2}}{2} \approx \pm \alpha$ if $\alpha \gg 1$

That is

$u = \pm \frac{\alpha}{r}$

(everywhere, and so also for large $r$). Using this example, with transformations of the independent variable, we can generate similar solutions. For example, if we substitute for $r$, $t = t(r)$, (13) becomes

$t' \dot{u} = u^2 + f$,

where the dot denotes the derivative with respect to $t$. The choice $t' = f(r)$, leads to a similar equation, and one can verify that for

$f(r) = \frac{a}{3} r^{-2/3} - a^2 r^{2/3}$, \hspace{1em} u = ar^{1/3}$

is a solution. More generally as can be easily verified $ar^n$ is a solution for $f(r) = n a r^{n-1} - a^2 r^{2n}$ and so on. However in these examples, neither $u$ nor $u'$ are asymptotically bounded.

Finally it maybe observed that if one solution of the Riccati equation (13) is known, then others could be derived therefrom[3].

6
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