Scheme-Independent Series Expansions at an Infrared Zero of the Beta Function in Asymptotically Free Gauge Theories

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Published in:
Physical Review D

DOI:
10.1103/PhysRevD.94.125005

Publication date:
2016

Document version
Final published version

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Citation for published version (APA):
Ryttov, T. A., & Shrock, R. (2016). Scheme-Independent Series Expansions at an Infrared Zero of the Beta Function in Asymptotically Free Gauge Theories. Physical Review D, 94(12), 1-25. [125005]. https://doi.org/10.1103/PhysRevD.94.125005

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Download date: 27. Apr. 2021
I. INTRODUCTION

The evolution of an asymptotically free gauge theory from the ultraviolet (UV) to the infrared is of fundamental importance. The evolution of the running gauge coupling $g = g(\mu)$, as a function of the Euclidean momentum scale, $\mu$, is described by the renormalization-group (RG) beta function \[ \beta = \frac{d\alpha}{dt} = \frac{g}{2\pi} \beta_g, \] (1.1)

where $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $dt = d\ln \mu$ (the argument $\mu$ will often be suppressed in the notation). Here we consider an asymptotically free (AF) vectorial gauge theory with non-Abelian, Yang-Mills gauge group $G$ and $N_f$ copies (flavors) of fermions $\psi_j$, $j = 1, \ldots, N_f$ transforming according to the representation $R$ of $G$. We take the fermions to be massless, since a massive fermion with mass $m_0$ would be integrated out of the effective field theory at scales $\mu < m_0$ and hence would not affect the infrared limit $\mu \to 0$ that we study here.

In an asymptotically free theory with sufficiently large fermion content, the beta function has an infrared zero at $\alpha_{IR}$ that controls the UV to IR evolution. Here we consider vectorial theories of this type. As the scale $\mu$ decreases from large values in the UV to small values in the IR, $\alpha(\mu)$ approaches $\alpha_{IR}$ as $\mu \to 0$. The properties of the theory at this IR zero of the beta function are of considerable interest. If this IR zero of the beta function occurs at sufficiently weak coupling so that the gauge interaction does not produce any spontaneous chiral symmetry breaking (S\(\chi\)SB), then it is an exact IR fixed point (IRFP) of the renormalization group. The theory thus exhibits scale invariance with anomalous dimensions for various (gauge-invariant) operators. In this infrared limit, the theory is in a chirally symmetric, deconfined, non-Abelian Coulomb phase (NACP). If, on the other hand, as $\mu$ decreases and $\alpha(\mu)$ increases toward $\alpha_{IR}$, there is a scale $\mu = \Lambda$ at which $\alpha(\mu)$ exceeds a critical value, denoted $\alpha_{cr}$, then the gauge interaction produces a nonzero chiral condensate, with associated spontaneous chiral symmetry breaking and dynamical mass generation for the fermions. These fermions are thus integrated out of the low-energy effective field theory that is operative for $\mu < \Lambda$. In this case, $\alpha_{IR}$ is only an approximate IRFP. We define $N_{f,cr}$ to be the critical value of $N_f$ such that if $N_f > N_{f,cr}$, then the (asymptotically free) theory does not undergo spontaneous chiral symmetry breaking.

At the two-loop $(2\ell)$ level, $\alpha_{2\ell} = -4\pi b_1/b_2$, where $b_\ell$ denotes the $\ell$-loop coefficient in the beta function [see Eqs. (2.1) and (2.5) below], and since $b_1$ [3] and $b_2$ [4] are independent of the scheme used for regularization and renormalization of the theory [5], it follows that $\alpha_{2\ell}$ is also scheme-independent.

Physical quantities evaluated at an infrared fixed point of the renormalization group at $\alpha = \alpha_{IR}$ are of basic interest. Since these are physical, their exact values must be scheme-independent. In conventional computations of these quantities, first, one expresses them as series expansions in powers of $\alpha$, calculated to $n$-loop order; second, one
computes the IR zero of the beta function, denoted $a_{IR,n}$, to the same $n$-loop order; and third, one sets $\alpha = a_{IR}$ in the series expansion for the given quantity to obtain its value at the IR zero of the beta function to this $n$-loop order. However, these conventional series expansions in powers of $\alpha$, calculated to a finite order, are scheme-dependent beyond the leading one or two terms. Specifically, the terms in the beta function are scheme-dependent at loop order $\ell \geq 3$ and the terms in an anomalous dimension are scheme-dependent at loop order $\ell \geq 2$. Indeed, as is well known, the presence of scheme-dependence in higher-order perturbative calculations is a general property in quantum field theory.

Clearly, it would be very valuable to have a calculational framework in which these physical quantities evaluated at $\alpha = a_{IR}$ are expressed as a series expansion such that at every order in this expansion the result is scheme-independent. A key point that was noted early on [3,4,6] is that $a_{IR}$ becomes small as the number $N_f$ of fermions increases toward the value $N_{f,b1z}$ [given below in Eq. (2.4)] at which the one-loop term in the beta function, $b_1$, passes through zero. At the two-loop level, $a_{IR} \propto \Delta_f$, where

$$\Delta_f = N_{f,b1z} - N_f.$$  \hspace{1cm} (1.2)

Indeed, in a theory with $G = SU(N_c)$ and fermions in the fundamental representation, in the limit $N_c \to \infty$ and $N_f \to \infty$ with $N_f/N_c$ fixed, $a_{IR}$ can be made arbitrarily small. Hence, one can envision reliable perturbative calculations of the series expansions for physical quantities at this IRFP [4,6] and, in particular, series expansions of these quantities in powers of $\Delta_f$ for reasonably small $\Delta_f$ [7]. Because $\Delta_f$ is obviously scheme-independent, it follows that this perturbative series expansion in powers of $\Delta_f$ is scheme-independent. Some early work on this was reported in [7,8]. Recently, in [9], a procedure for calculating the coefficients of this scheme-independent expansion was given for the anomalous dimension of the (gauge-invariant) fermion bilinear at the IR zero of the beta function, and the coefficients in this expansion were calculated up to order $\Delta_f^3$ in a vortical asymptotically free gauge theory with gauge group $G$ and $N_f$ fermions in a representation $R$. This work also presented an analogous calculation for a theory with $\mathcal{N} = 1$ supersymmetry to order $\Delta_f^2$. The results were then evaluated in the case of $SU(N_c)$ with fermions in the fundamental ($F$) representation, $R = F$, with Young tableau $\Box$. In [10], for $G = SU(3)$ and $R = F$, we calculated the $n$-loop value of the squared coupling, $a_{IR,n}$, and the resultant value of $\gamma_{\bar{\psi}\psi}$ to five-loop order, and in [11] we calculated the scheme-independent expansion of $\gamma_{\bar{\psi}\psi}$ for the representations $R$ is theory to order $\Delta_f^4$, using five-loop inputs, and performed an extrapolation to infinite order in $\Delta_f$ to estimate the exact value of $\gamma_{\bar{\psi}\psi}$ as a function of $N_f$. The improvement in the knowledge of the anomalous dimension $\gamma_{\bar{\psi}\psi}$ obtained from the scheme-independent series expansions in $[9,11]$ is valuable not only for general field-theoretic purposes, but also because theories with large anomalous dimensions of fermion bilinears may be relevant for ultraviolet completions of the Standard Model. Indeed, there has been considerable interest in theories that might produce large $\gamma_{\bar{\psi}\psi} \sim O(1)$ associated with an IR zero of the beta function and resultant quasi-conformal behavior [12]. In [11] we also compared our results with recent lattice measurements of $\gamma_{\bar{\psi}\psi}$.

In this paper we report a number of new results on scheme-independent series expansions in powers of $\Delta_f$. As noted, we consider an asymptotically free vectorial gauge theory with gauge group $G$ and $N_f$ fermions in the representation $R$. First, we present general formulas for coefficients in the scheme-independent expansion in powers of $\Delta_f$ of an arbitrary (gauge-invariant) physical quantity evaluated at $a_{IR}$. We calculate the scheme-independent expansion of the derivative of the beta function, $\beta = \alpha/\beta$, evaluated at $a_{IR}$, denoted $\beta_{IR}$, to order $\Delta_f^4$. As a consequence of the trace anomaly relation, in a theory with massless fermions, $\beta_{IR}$ is equivalent to $\gamma_{F^2,IR}$, the anomalous dimension, evaluated at $a_{IR}$, of the operator $\text{Tr}(F_{\mu\nu}F_{\nu\rho})$, where $F_{\mu\nu}$ is the non-Abelian Yang Mills field-strength tensor. For the special case where the gauge group is SU(3) and the fermions are in the fundamental (triplet) representation, we compute this expansion to order $\Delta_f^5$. This SU(3) theory corresponds to quantum chromodynamics (QCD) with $N_f$ massless quarks. For general $G$ and $R$, we calculate the scheme-independent expansion coefficients to order $\Delta_f^3$ for the anomalous dimension, evaluated at $a_{IR}$, of the flavor-nonsinglet and flavor-singlet fermion bilinear Dirac tensor operators. Since the $\Delta_f$ expansion starts at the upper end of the non-Abelian Coulomb phase (NACP) at $\Delta_f = 0$, i.e., $N_f = N_{f,b1z}$, and extends downward in $N_f$ with increasing $\Delta_f$, we focus mainly on the infrared behavior in the NACP. We show that our scheme-independent calculations of the anomalous dimensions of $\text{Tr}(F_{\mu\nu}F_{\nu\rho})$ and fermion bilinear operators in the non-Abelian Coulomb phase obey respective rigorous upper bounds for conformally invariant theories. As part of our analysis, we compare results for various quantities calculated via the scheme-independent expansion with results calculated via a conventional higher-loop scheme-dependent expansion. Further, for the case with $G = SU(N_c)$ and fermions in the fundamental representation, we discuss the limit $N_c \to \infty$ and $N_f \to \infty$ with $N_f/N_c$ fixed and finite. From ratios of scheme-independent expansion coefficients for $\beta_{IR}$ and the anomalous dimension of the fermion bilinear antisymmetric Dirac tensor operator, we show, in agreement with, and extending [9], that the scheme-independent $\Delta_f$ expansion should be reasonably accurate in the non-Abelian Coulomb phase. As with our earlier
work, the present study is motivated by the value of the new results for a basic understanding of the renormalization-group evolution of asymptotically free gauge theories, and also may be relevant to ultraviolet completions of the Standard Model.

The paper is organized as follows. Some relevant background and methods are discussed in Sec. II. In Sec. III we present explicit formulas for the calculations of certain coefficients [\(a_n\) and \(k_n\) in Eqs. (3.1) and (3.3)] that are needed for the rest of our work. General scheme-independent expansion formulas for anomalous dimensions of operators are given in Sec. IV. In this section we also discuss rigorous upper bounds on anomalous dimensions in a conformally invariant theory and their application here. We give our new results on scheme-independent calculations of \(\beta_R^{\text{IR}}\) in Sec. V. In Sec. VI we extend the analysis of the scheme-independent expansion of the anomalous dimension for the \(m = 0\) fermion bilinear previously studied in [9] and [11] with several new results. These include calculations for the limit \(N_c \to \infty, N_f \to \infty\) with \(N_f/N_c\) fixed and analyses of Padé approximants, with comparison to scheme-dependent higher-loop conventional calculations. Section VII presents scheme-independent calculations of the anomalous dimension for the fermion bilinear (flavor-nonsinglet and flavor-singlet) antisymmetric rank-2 Dirac tensor operator. Our conclusions are given in Sec. VIII and some auxiliary formulas are listed in Appendix.

II. BACKGROUND AND METHODS

The beta function of this theory has the series expansion

\[
\beta = -2\alpha \sum_{\ell=1}^{\infty} b_{\ell} \alpha^{\ell} = -2\alpha \sum_{\ell=1}^{\infty} \tilde{b}_{\ell} \alpha^{\ell},
\]

where

\[
a = \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi},
\]

\(b_{\ell}\) is the \(\ell\)-loop coefficient, \(\tilde{b}_{\ell} = b_{\ell}/(4\pi)^{\ell}\), and we extract a minus sign for convenience, so \(b_1 > 0\) for asymptotic freedom. For analysis of an IR zero of \(\beta\), it is convenient to extract the \(\alpha^2\) factor that gives rise to the UV zero at \(\alpha = 0\) and define a reduced \((r)\) beta function

\[
\beta_r = \frac{\beta}{(\alpha/b_1)^2} = 1 + \frac{1}{b_1} \sum_{\ell=2}^{\infty} b_{\ell} \alpha^{\ell-1}.
\]

The \(n\) (\(n\ell\)) beta function, denoted \(\beta_{nf}\) and reduced beta function, denoted \(\beta_{r,nf}\) are obtained from the respective Eqs. (2.1) and (2.3) by changing the upper limit on the \(\ell\)-loop summation from \(\infty\) to \(n\). As noted above, \(b_1\) and \(b_2\) are scheme-independent (SI), while the \(b_{\ell}\) with \(\ell \geq 3\) are scheme-dependent (SD) [5]. For a general gauge group \(G\) and fermion representation \(R\), the coefficients \(b_1\) and \(b_2\) were calculated in [3] and [4], and \(b_3\) and \(b_4\) were calculated in [13] and [14] (and checked in [15]) in the commonly used mass-independent \(\overline{\text{MS}}\) scheme [16]. Recently, for \(G = \text{SU}(3)\) and \(R = F\), \(b_2\) was calculated in [17]. For reference and to show our normalizations explicitly, \(b_1\) and \(b_2\) are listed in Appendix. As \(N_f\) increases, \(b_1\) decreases through positive values and vanishes with sign reversal at \(N_f = N_{f,blz}\), where

\[
N_{f,blz} = \frac{11C_A}{4T_f}.
\]

(the subscript \(blz\) means “\(b_1\) zero”), where \(C_A\) and \(T_f\) are group-theoretic invariants [18,19]. The asymptotic freedom condition therefore implies the upper bound \(N_f < N_{f,blz}\). We denote the interval \(0 \leq N_f < N_{f,blz}\) as \(I_{AF}\).

For \(N_f\) close to, but less than, \(N_{f,blz}\), \(b_2\) passes through zero to positive values as \(N_f\) passes through the value

\[
a_{IR,2\ell} = -\frac{\tilde{b}_1}{b_2} = -\frac{4\pi b_1}{b_2}.
\]

In general, the \(n\)-loop beta function has a double UV zero at \(\alpha = 0\) and \(n-1\) zeros away from the origin. Among the latter, the smallest (real, positive) zero, if such a zero occurs, is the physical IR zero, denoted \(a_{IR,nf}\). As \(N_f\) decreases from \(N_{f,blz}\), \(b_2\) passes through zero to positive values as \(N_f\) passes through the value

\[
N_{f,b2z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}.
\]

Hence, with \(N_f\) formally extended from nonnegative integers to nonnegative real numbers [19], \(\beta_{2\ell}\) has an IR zero (IRZ) for \(N_f\) in the interval

\[
I_{IRZ}: N_{f,b2z} < N_f < N_{f,blz}.
\]

We denote this interval as \(I_{IRZ}\).

As \(N_f\) decreases in this interval, \(a_{IR,2\ell}\) increases toward strong coupling. Hence, to study the IR zero for \(N_f\) toward the middle and lower part of \(I_{IRZ}\) with reasonable accuracy, one requires higher-loop calculations. These were performed in [20–27] for \(a_{IR,nf}\) and for the anomalous dimension of the fermion bilinear operator. Clearly, a perturbative calculation of the IR zero of \(\beta_{nf}\) is only reliable if the resultant \(a_{IR,nf}\) is not excessively large. Moreover, since the \(b_{\ell}\) with \(\ell \geq 3\) are scheme-dependent, it is also incumbent upon one to ascertain the degree of sensitivity of the value obtained for \(a_{IR,nf}\) for \(n \geq 3\) to the scheme used for the calculation. This task was carried out in [28–31]. One way to do this is to perform the calculation...
of $\alpha_{IR,ne}$ in one scheme, say $\overline{MS}$, apply a scheme transformation to obtain the value of $\alpha_{IR,ne}$ in another scheme, and compare how close the two values are. As we discussed in [28,29], an acceptable scheme transformation function must satisfy a set of conditions, and although these are automatically satisfied in the local vicinity of the origin, $\alpha = 0$ (as in optimized schemes for perturbative QCD calculations), they are not automatically satisfied, and indeed, are quite restrictive conditions, when one applies the scheme transformation at an IR zero away from the origin. Anomalous dimensions of composite fermion operators for $G = \text{SU}(3)$ have been calculated in [32].

The one-loop coefficient $b_1$ is a polynomial of degree 1 in $N_f$ and the higher-loop coefficients $b_\ell$ with $\ell \geq 2$ are polynomials of degree $\ell - 1$ in $N_f$. Let us define

$$b_\ell^{(0)} = b_\ell|_{N_f = N_{f,b1z}}$$

and, for $r \geq 1$,

$$b_\ell^{(r)} = \left. \frac{d^r b_\ell}{(dN_f)^r} \right|_{N_f = N_{f,b1z}} = (-1)^r \left. \frac{d^r b_\ell}{(d\Delta_f)^r} \right|_{\Delta_f = 0}.$$  (2.9)

Then one has the scheme-independent results

$$b_1^{(0)} = 0$$  (2.10)

(which is equivalent to the definition of $N_{f,b1z}$),

$$b_1^{(1)} = \frac{4T_f}{3},$$  (2.11)

$$b_2^{(0)} = -C_A(7C_A + 11C_f) \equiv -C_AD,$$  (2.12)

where

$$D = 7C_A + 11C_f,$$  (2.13)

and

$$b_2^{(1)} = -\frac{4}{3}(5C_A + 3C_f)T_f.$$  (2.14)

It is convenient to introduce the definition (2.13), since powers of $D$ occur in the denominators of the scheme-independent expansion coefficients of anomalous dimensions of bilinear fermion Dirac tensor operators and of $d\beta / d\alpha$ evaluated at the IR zero of the beta function.

Thus, one has the finite Taylor series expansions

$$b_1 = b_1^{(1)}(N_f - N_{f,b1z}) = -b_1^{(1)}\Delta_f$$  (2.15)

and, for $\ell \geq 2$,

$$b_\ell = b_\ell^{(\ell)}(N_f - N_{f,b1z})^{\ell} = \sum_{r=0}^{\ell-1} \frac{(-1)^r b_\ell^{(r)} \Delta_f^r}{r!}.$$  (2.16)

We write Eqs. (2.15) and (2.16) in a unified manner as

$$b_\ell = \sum_{r=0}^{r_{\max}(\ell)} \frac{(-1)^r b_\ell^{(r)} \Delta_f^r},$$  (2.17)

where $r_{\max}(1) = 1$ and $r_{\max}(\ell) = \ell - 1$ if $\ell \geq 2$.

It will also be useful to recall some basic properties of the theory regarding global flavor symmetries. Because the $N_f$ fermions are massless, the Lagrangian is invariant under the classical global chiral flavor ($fl$) symmetry $G_{fl,cl} = U(N_f)_L \otimes U(N_f)_R$, or equivalently,

$$G_{fl,cl} = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_V \otimes U(1)_A$$  (2.18)

(where $V$ and $A$ denote vector and axial-vector). The $U(1)_V$ represents fermion number, which is conserved by the bilinear operators that we consider. The $U(1)_A$ symmetry is broken by instantons, so the actual nonanomalous global flavor symmetry is

$$G_{fl} = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_V.$$  (2.19)

This $G_{fl}$ symmetry is respected in the (deconfined) non-Abelian Coulomb phase, since there is no spontaneous chiral symmetry breaking in this phase. As noted before, we focus on this phase in the present work, since the (scheme-independent) $\Delta_f$ expansion starts from the upper end of the interval $I_{IRZ}$ in this phase where $a_{IR} \rightarrow 0$ as $\Delta_f \rightarrow 0$. In contrast, in the phase with confinement and spontaneous chiral symmetry breaking, the gauge interaction produces a bilinear fermion condensate, which can be written as $\sum_{j=1}^{N_f} \bar{\psi}_j f_j$, and this breaks $G_{fl}$ to $SU(N_f)_V \otimes U(1)_V$.

**III. CALCULATION OF THE SERIES EXPANSION COEFFICIENTS $k_n$ AND $a_n$**

We know that the exact $a_{IR}$ (and also the $n$-loop approximation to this exact $a_{IR}$) vanishes (linearly) as a function of $\Delta_f$ and that it is analytic at $\Delta_f = 0$, so we can expand it, or equivalently, $a_{IR} = a_{IR}/(4\pi)$, as a series expansion in this variable, $\Delta_f$. We write

$$a_{IR} = \sum_{j=1}^{\infty} a_j \Delta_f^j.$$  (3.1)

[Note that $a_j$ as defined here is equal to $a_j/2$ in terms of the $a_j$ in Eq. (8) of [9].]
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One calculates the coefficients \(a_j\) in two steps. First, one evaluates \(\beta_\epsilon\) in Eq. (2.3) at \(a = a_{1R}\), where it vanishes. Since the prefactor \(-8\pi a_{1R}\) in Eq. (2.1) is nonzero in general (although it does vanish at \(\Delta_f = 0\)), it follows that

\[
\sum_{\ell=1}^{\infty} b_\ell (a_{1R})^{\ell-1} = 0. \tag{3.2}
\]

One then substitutes the finite Taylor series expansions for \(b_\ell\) and \(a_{1R}\), in Eqs. (2.15), (2.17), and (3.1), in Eq. (3.2) and thereby obtains the equation

\[
\beta_{1,\alpha=a_{1R}} = 0 = \sum_{\ell=1}^{\infty} \left[ \sum_{r=0}^{\ell} b_\ell (a_{1R})^{\ell-r} \left( \sum_{j=1}^{\ell} a_j \Delta_f^j \right)^\ell \right]
= \sum_{n=1}^{\infty} k_n \Delta_f^n. \tag{3.3}
\]

The results for the first three \(k_n\) were given in [9] and are

\[
k_1 = a_1 b_2^{(0)} - b_1^{(1)}, \tag{3.4}
\]

\[
k_2 = a_2 b_2^{(0)} + a_1^2 b_3^{(0)} - a_1 b_2^{(1)}, \tag{3.5}
\]

and

\[
k_3 = a_3 b_2^{(0)} + 2a_1 a_2 b_3^{(0)} + a_1^3 b_4^{(0)} - a_2 b_2^{(1)} - a_1^2 b_3^{(1)}. \tag{3.6}
\]

From Eq. (3.3), it follows that the coefficient \(a_n\) occurs linearly in the expression for \(k_n\), in the single term \(a_n b_2^{(0)}\) [9]. To further show the structural forms of the \(k_n\), we give \(k_4\) and \(k_5\) here:

\[
k_4 = a_4 b_2^{(0)} + (a_2^2 + 2a_1 a_3) b_3^{(0)} + 3a_1^2 a_2 b_4^{(0)} + a_1^3 b_5^{(0)}
- a_3 b_2^{(1)} - 2a_1 a_2 b_3^{(1)} - a_1^3 b_4^{(1)} + \frac{1}{2} a_1^2 b_3^{(2)}. \tag{3.7}
\]

\[
k_5 = a_5 b_2^{(0)} + 2(a_1 a_4 + a_2 a_3) b_3^{(0)} + 3a_1 (a_2^2 + a_1 a_3) b_4^{(0)}
+ 4a_1^2 a_2 b_5^{(0)} + a_1^3 b_6^{(0)} - a_4 b_2^{(1)} + (a_2^2 + 2a_1 a_3) b_3^{(1)}
- 3a_1^2 a_2 b_4^{(1)} - a_1^3 b_5^{(1)} + a_1 a_2 b_3^{(2)} + \frac{1}{2} a_1^3 b_4^{(2)}. \tag{3.8}
\]

In addition to the property that \(k_n\) contains a term \(a_n b_2^{(0)}\), we remark on two other general properties of the \(k_n\): (i) \(k_n\) contains a term \(-a_{n-1} b_2^{(1)}\) [which coincides with the term \(a_n b_2^{(0)}\) if \(n = 1\)] and (ii) if \(n \geq 2\), then \(k_n\) contains a term \(-a_{n-1} b_2^{(1)}\). Next, one observes that in Eq. (3.3), since \(\Delta_f\) is variable, this implies that the coefficients \(k_n\) of each power \(\Delta_f^n\) must vanish individually. One can solve the equations \(k_n = 0\) for the \(a_n\). The solutions are unique because of the property that \(a_n\) occurs linearly in \(k_n\). The solutions for the \(a_n\) with \(1 \leq n \leq 3\) were given in [9]. Thus, the equation \(k_1 = 0\) yields

\[
a_1 = \frac{b_1^{(1)}}{b_2^{(0)}}. \tag{3.9}
\]

One then substitutes this into the equation \(k_2 = 0\) and solves for \(a_2\), obtaining

\[
a_2 = -\frac{b_1^{(1)}}{(b_2^{(0)})^3} (b_2^{(1)} b_2^{(1)} - b_1^{(1)} b_3^{(0)}). \tag{3.10}
\]

One then proceeds iteratively in the manner, substituting the solutions for the \(a_k\) with \(1 \leq k \leq n - 1\) in the equation \(k_n = 0\) and solving for \(a_n\). For \(a_3\), this yields

\[
a_3 = \frac{b_1^{(1)}}{(b_2^{(0)})^3} \left[ (b_2^{(0)} b_2^{(1)})^3 - 3b_1^{(1)} b_2^{(0)} b_2^{(1)} b_3^{(0)} + 2(b_3^{(0)})^2 \right.
+ b_1^{(1)} (b_2^{(0)})^2 b_3^{(1)} - (b_1^{(1)})^2 b_2^{(1)} b_4^{(0)}]. \tag{3.11}
\]

In general, \(a_n\) depends on the \(b_\ell\) coefficients for \(1 \leq \ell \leq n + 1\). The \(a_n\) with \(1 \leq n \leq 3\) were all the coefficients of this type that were needed in [9] since the \(b_\ell\) have only been computed for a general gauge group \(G\) and fermion representation \(R\) up to \(\ell = 4\) loop order. These \(a_n\) have a factorized structure with a prefactor

\[
a_n \propto \frac{b_1^{(1)}}{(b_2^{(0)})^{2n-1}}. \tag{3.12}
\]

In [11] we also calculated and presented the result for \(a_4\) for the specific case \(G = SU(3)\) and fermion representation \(R = F\), since we were using the recent calculation of the five-loop coefficient \(b_4\) for this case in [17]. Here we give the general result for \(a_4\) for arbitrary gauge group \(G\) and fermion representation \(R\):

\[
a_4 = \frac{b_1^{(1)}}{(b_2^{(0)})^7} \left[ (b_2^{(0)} b_2^{(1)})^3 - \frac{1}{2} b_1^{(1)} (b_2^{(0)})^4 b_3^{(2)} \right.
+ (b_1^{(1)})^2 (b_2^{(0)})^3 b_4^{(1)} - 4(b_1^{(1)})^2 (b_2^{(0)})^2 (b_2^{(1)} b_4^{(0)} + b_3^{(0)} b_3^{(1)})
+ 3(b_1^{(1)})^3 (b_2^{(0)})^2 b_4^{(1)} (b_2^{(0)} b_3^{(1)} - 2b_2^{(1)} b_3^{(0)})
+ 10(b_1^{(1)})^2 b_2^{(0)} b_2^{(1)} (b_3^{(0)})^2
+ 5(b_1^{(1)})^3 b_3^{(0)} (b_2^{(0)} b_3^{(0)} - (b_3^{(0)})^2)
- (b_1^{(1)})^3 (b_2^{(0)})^2 b_3^{(0)}]. \tag{3.13}
\]

In the same manner, we have calculated \(a_5\) by substituting our solutions for the \(a_k\) with \(1 \leq k \leq 4\) in Eq. (3.8), and so forth for higher \(a_k\).
IV. SCHEME-INDEPENDENT SERIES EXPANSION FOR ANOMALOUS DIMENSIONS AT $\Delta_f^{\text{IR}}$

Let us consider a (gauge-invariant) operator $O$. Because of the interactions, the full scaling dimension of this operator, denoted $D_O$, differs from its free-field value, $D_{O,\text{free}}$:

$$D_O = D_{O,\text{free}} - \gamma_O,$$

where $\gamma_O$ is the anomalous dimension of the operator [33]. Since $\gamma_O$ arises from the gauge interaction, it can be expressed as a power series about $a = 0$:

$$\gamma_O = \sum_{\ell=1}^\infty c_{O,\ell} a^\ell,$$  (4.2)

where $c_{O,\ell}$ is the $\ell$-loop coefficient.

The exact anomalous dimension $\gamma_O$ evaluated at a zero of the exact beta function, denoted $\gamma_{O,\text{IR}}$, is a physical quantity and hence is scheme-independent. This was shown formally for the fermion bilinear operator $O = \bar{\psi} \psi$ in [5], and the proof given there can be straightforwardly extended to other (gauge-invariant) operators $O$. However, this scheme independence is not preserved in a finite-order perturbative calculation, owing to the scheme dependence of the $b_\ell$ for $\ell \geq 3$ and of the $c_{O,\ell}$ for $\ell \geq 2$.

As mentioned above, a method for calculating $\gamma_{\bar{\psi}\psi,\text{IR}}$ as a perturbative series expansion in powers of $\Delta_f$ was presented in [9], with the important property that at each order of the expansion the resulting approximation to $\gamma_{\bar{\psi}\psi,\text{IR}}$ is scheme-independent. We can calculate a scheme-independent series expansion in powers of $\Delta_f$ for the anomalous dimension $\gamma_O$ of a general (gauge-invariant) operator $O$, evaluated at $a_{\text{IR}}$ by taking the series (4.2) and inserting the expansions of $c_{O,\ell}$ and $a_{\text{IR}}$ as functions of $\Delta_f$. An advantage of this type of series expansion is that since $\Delta_f$ is scheme-independent, so is the expansion for $\gamma_O$, in contrast to the expression of $\gamma_O$ as a series in powers of $a_{\text{IR},\ell}$.

We proceed to give a generalization of the results of [9] for the anomalous dimension of an arbitrary (gauge-invariant) operator $O$ in an asymptotically free gauge theory with gauge group $G$ and $N_f$ fermions in the representation $R$, evaluated at $a_{\text{IR}}$. We denote this anomalous dimension as $\gamma_{O,\text{IR}}$. Specifically, we present a general method for calculating a series expansion of $\gamma_{O,\text{IR}}$ in powers of $\Delta_f$.

We begin with the series expansion (4.2) and substitute the series expansions for the $c_{O,\ell}$ and for $a_{\text{IR}}$. Let

$$c_{O,\ell}^{(0)} = c_{O,\ell}|_{N_f=N_f,\text{IR}},$$

and, for $r \geq 1$,

$$c_{O,\ell}^{(r)} = d^r c_{O,\ell} \left|_{N_f=N_f,\text{IR}} \right.,$$

and

$$\gamma_{O,\text{IR}} = \sum_{j=1}^\infty \left( \sum_{r=1}^\infty c_{O,\ell}^{(r)} \Delta_j^r \right) \left( \sum_{j=1}^\infty a_j \Delta_j^r \right)^\ell,$$

with

$$\gamma_{O,\text{IR}} = \sum_{n=1}^{\infty} \kappa_{O,n} \Delta_j^n.$$  (4.5)

We denote the value of $\gamma_{O,\text{IR}}$ obtained from this series calculated to order $O(\Delta_f^r)$, i.e., from the last line of Eq. (5.7) with the upper limit of the summand changed from $\infty$ to $p$, as $\gamma_{O,\text{IR},\Delta_f^p}$.

We calculate

$$\kappa_{O,1} = a_1 c_{O,1}^{(0)},$$  (4.6)

$$\kappa_{O,2} = a_2 c_{O,2}^{(0)} + a_1^2 c_{O,2}^{(1)},$$  (4.7)

$$\kappa_{O,3} = a_3 c_{O,3}^{(0)} + 2a_1 a_2 c_{O,2}^{(0)} + a_1 a_2 c_{O,2}^{(0)} + a_3^2 c_{O,3}^{(0)} + a_2^2 c_{O,2}^{(1)} + a_3 c_{O,3}^{(1)} + a_1 a_3 c_{O,3}^{(1)},$$  (4.8)

$$\kappa_{O,4} = a_4 c_{O,4}^{(0)} + (2a_1 a_3 + a_2^2) c_{O,2}^{(0)} + 3a_2^2 a_3 c_{O,3}^{(0)} + a_4 c_{O,4}^{(0)} + 2a_1 a_2 c_{O,2}^{(1)} + a_2^2 c_{O,3}^{(1)} + a_4 c_{O,4}^{(1)} + a_1 a_2 c_{O,2}^{(1)} + a_2^2 c_{O,3}^{(1)}.$$  (4.9)

etc. for $\kappa_{O,n}$ with $n \geq 5$. To calculate $\kappa_{O,n}$, one needs to know the $a_j$ and $c_j$ for $1 \leq j \leq n$. These $\kappa_{O,n}$ have the following general properties: (i) $\kappa_{O,n}$ contains the term $a_n c_{O,1}^{(0)}$ and (ii) $\kappa_{O,n}$ contains the term $a_1^n c_{O,n}^{(0)}$ (which coincides with (i) if $n = 1$).

A relevant question concerns the range of applicability of the scheme-independent series expansion (4.5). We address this question here. As noted above, our analysis in this paper is focused on the non-Abelian Coulomb phase, since there is no spontaneous symmetry breaking in this phase, and hence a zero of the beta function is an exact IR fixed point of the renormalization group. This means that the theory at this fixed point is scale-invariant. A number of studies have concluded that in this case of an exact IRFP in this asymptotically free gauge theory, scale invariance implies the larger symmetry of conformal invariance [34,35].

We will use several methods to assess the range of validity of the (scheme-independent) small-$\Delta_f$ expansion. A general comment is that the properties of the theory change qualitatively as $N_f$ decreases through the value $N_{f,\text{cr}}$ and spontaneous chiral symmetry breaking occurs and the fermions gain dynamical masses. The (chirally symmetric) non-Abelian Coulomb phase with $N_{f,\text{cr}} < N_f < N_{f,\text{IR}}$ is clearly qualitatively different from the confined phase with spontaneous chiral symmetry breaking.
at smaller $N_f$ below $N_{f,cr}$. Therefore, one does not, in general, expect the small-$\Delta_f$ series expansion to hold below $N_{f,cr}$. Estimating the range of applicability of this expansion is thus connected with estimating the value of $N_{f,cr}$.

For this purpose, as in our previous work [9,22,24], we can apply a rigorous upper bound on the anomalous dimension of an operator from the unitarity of a conformal field theory. If the approximate calculation of the anomalous dimension of a given quantity at a fixed value of $\Delta_f$, computed up to order $\Delta_f^0$, yields a value that exceeds this upper bound, then we can infer that the calculation is not applicable at this value of $\Delta_f$ or equivalently, $N_f$. In particular, this can give information on the extent of the non-Abelian Coulomb phase and the value of $N_{f,cr}$. This bound is applicable whether or not the coefficients $k_{\Delta_f,n}$ are all of the same sign, but it is most useful if these coefficients do have the same sign, since in this case for a fixed $\Delta_f$ the anomalous dimension is a monotonic function of the order to which the small-$\Delta_f$ series expansion is calculated.

A second method that we shall use to estimate the range of applicability of the series expansions in powers of $\Delta_f$ is the ratio test. If a function $f(z)$ has a Taylor series $\sum_{n=0}^{\infty} s_n z^n$, then the ratio test states that the series is (absolutely) convergent if $|z| < z_0$, where

$$z_0 = \lim_{n \to \infty} \frac{|s_n|}{|s_{n+1}|}.$$  (4.10)

Our application of the ratio test here is only intended to give a rough estimate of this range of applicability of the $\Delta_f$ series expansion since (i) we do not assume that the $\Delta_f$ expansion is a Taylor series expansion, and (ii) with only a few terms in the series for a given quantity, we can compute only a few ratios of adjacent coefficients.

Finally, a third method that we shall use is to calculate $[p,q]$ Padé approximants to the $\Delta_f$ series expansions. As rational functions of $\Delta_f$, the approximants with $q \geq 1$ have poles, and the nearest poles to the origin give one estimate of the range of validity of the expansions.

**A. Upper bound on anomalous dimensions**

We now state and apply the upper bound on the anomalous dimension of an operator in a theory with scale invariance or conformal invariance. Recall that a (finite-dimensional) representation of the Lorentz group is specified by the set $(j_1,j_2)$, where $j_1$ and $j_2$ take on nonnegative integral or half-integral values [36]. A Lorentz scalar operator thus transforms as $(0,0)$, a Lorentz vector as $(1/2,1/2)$, an antisymmetric tensor like the field-strength tensor $F_{\mu\nu}$ as $(1,0) \oplus (0,1)$, etc. Then the requirement of unitarity in a scale-invariant theory (in four spacetime dimensions) requires that the full dimension $D_O$ of an operator (other than the identity) must satisfy the lower bound [35]

$$D_O \geq j_1 + j_2 + 1.$$  (4.11)

With the definition (4.1), this is equivalent to the upper bound on the anomalous dimension

$$\gamma_O \leq D_O - (j_1 + j_2 + 1).$$  (4.12)

The case $(j_1,j_2) = (0,0)$ includes the Lorentz scalar operators $F_\mu^a F^{a\mu}$, and the flavor-nonsinglet and flavor-singlet fermion bilinear operators $\bar{\psi} T_{b\gamma} \psi$ and $\bar{\psi} \gamma \psi$, where here $T_b$ is an element of the Lie algebra of the global flavor symmetry group SU($N_f$). Hence, first, since $D^{f^2\text{free}} = 4$, it follows from (4.12) that the anomalous dimension of the $F_\mu^a F^{a\mu}$, evaluated at $\alpha_{IR}$, must satisfy

$$\gamma_{F^2,IR} \leq 3.$$  (4.13)

Second, (4.12) implies that the (equal) anomalous dimensions of the flavor-nonsinglet and flavor-singlet fermion bilinear operators $\bar{\psi} T_{b\gamma} \psi$ and $\bar{\psi} \gamma \psi$ evaluated at $\alpha_{IR}$, denoted $\gamma_{\bar{\psi} \psi,IR}$, must satisfy

$$\gamma_{\bar{\psi} \psi,IR} \leq 2.$$  (4.14)

The flavor-nonsinglet and flavor-singlet fermion bilinear antisymmetric rank-2 Dirac tensor operators proportional to $\bar{\psi} T_{\sigma\mu\nu} \gamma \psi$ and $\bar{\psi} \sigma_{\mu\nu} \gamma \psi$ to be analyzed below correspond to the case $(j_1,j_2) = (1,0) \oplus (0,1)$ (as is clear from the fact that they can couple to the non-Abelian field-strength tensor to form a Lorentz scalar). Hence, with $j_1 + j_2 = 1$ for $(j_1,j_2) = (1,0)$ or (0,1), the bound (4.12) implies that the (equal) anomalous dimensions of these operators evaluated at $\alpha_{IR}$, denoted $\gamma_{T,IR}$, must satisfy

$$\gamma_{T,IR} \leq 1.$$  (4.15)

We have applied the upper bound (4.14) in our previous calculations of $\gamma_{\bar{\psi} \psi,IR}$ at the $n$-loop level, up to $n = 4$ loops [9–11,22,26,27]. We have also applied a corresponding upper bound in [9,24,27] on the anomalous dimension of the (gauge-invariant) bilinear chiral superfield operator $\Phi \Phi$ in a vectorial asymptotically free gauge theory with gauge group $G$, $N' = 1$ supersymmetry, and $N_f$ pairs of chiral superfields $\Phi_j$ and $\Phi_j$, $1 \leq j \leq N_f$, transforming according to the representations $R$ and $\bar{R}$ of $G$ [24,27]. A theory of particular interest is the case $R = F$; here, $N_{f,b1} = 3N_c$ and the lower end of the conformal phase is known, namely $N_{f,cr} = (3/2)N_c$ [37,38] (which is integral and hence physical if $N_c$ is even). This theory corresponds to supersymmetric QCD with massless matter fields, and is often denoted SQCD. In this case, the upper bound is $\gamma_{\bar{\psi} \psi} \leq 1$, and this is saturated at the lower end of the non-Abelian Coulomb phase. The scheme-independent
expansion in [9] exhibited excellent agreement with this exact result.

V. SCHEME-INDEPENDENT CALCULATION OF $\beta'_{IR}$

A. Calculation to order $\Delta^4_f$ for general $G$ and $R$

An important property of an asymptotically free theory at an IR zero of the beta function (IRFP of the renormalization group) is the derivative of this beta function evaluated at $\alpha = \alpha_{IR}$,

$$\beta'_{IR} = \frac{d\beta}{d\alpha}\bigg|_{\alpha = \alpha_{IR}}.$$  

(5.1)

This is scheme-independent, as was proved in [5] [39]. In a theory with massless fermions, as considered here, the trace of the energy-momentum tensor, $T^\mu_\mu$, satisfies the relation [40]

$$T^\mu_\mu = \frac{\beta}{4\alpha} F^\mu_\mu F^{\mu\nu},$$  

(5.2)

where $F^\mu_\mu = \partial_\mu A^\mu_a - \partial_\nu A^\nu_a + g\epsilon^{abc} A^\mu_b A^\nu_c$ is the gluon field strength tensor [41]. Since the energy-momentum tensor is conserved, its anomalous dimension is zero, and its full dimension is equal to its free-field dimension, 4. Consequently, the full scaling dimension of the rescaled operator $F^\mu_\mu F^{\mu\nu}$, denoted $D_{F^2}$, satisfies

$$D_{F^2} = 4 + \beta' - \frac{2\beta}{\alpha},$$  

(5.3)

where we use the shorthand notation $F^2 \equiv F^\mu_\mu F^{\mu\nu}$ [42,43]. We denote the anomalous dimension of the operator $F^\mu_\mu F^{\mu\nu}$ as $\gamma_{F^2}$ and its evaluation at $\alpha_{IR}$ as $\gamma_{F^2, IR}$. From Eq. (5.3), it follows that at a zero of the beta function away from the origin, in particular, the IR zero of an asymptotically free gauge theory of interest here at $\alpha = \alpha_{IR}$, the derivative $\beta'_{IR}$ is equivalent to the anomalous dimension [33] of the operator $F^\mu_\mu F^{\mu\nu}$:

$$\beta'_{IR} = -\gamma_{F^2, IR}.$$  

(5.4)

From Eq. (2.1), one obtains the conventional series expansion for $\beta'_{IR}$ in powers of $\alpha$, or equivalently, $\alpha$:

$$\beta'_{IR} = -2\sum_{\ell=1}^\infty (\ell + 1)b_\ell \alpha^{\ell}_{IR}.$$  

(5.5)

We denote $\beta'_{IR,n, \alpha}$ as the $n$-loop truncation of this infinite series. The two-loop value is scheme-independent [26]:

$$\beta'_{IR, 2\alpha} = -\frac{2b_1^2}{b_2} = \frac{(11C_A - 4T_fN_f)^2}{3[2(5C_A + 3C_f)T_fN_f - 17C_A^2]},$$  

(5.6)

which is positive for $N_f \neq IRZ$. However, at the level of $n \geq 3$ loops, the quantity $\beta'_{IR,n, \alpha}$ is scheme-dependent. This quantity was calculated up to the four-loop level in [26,27], using $b_3$ and $b_4$ computed in the $\overline{MS}$ scheme from [13,14] (for SU(3), see also the four-loop study [44]).

Here we calculate a scheme-independent expansion of $\beta'_{IR}$ in powers of $\Delta_f$ to order $\Delta^4_f$ for general $G$ and $R$ and to the five-loop level, i.e., order $\Delta^5_f$, for SU(3). For general $G$ and $R$, we substitute the expansions of $b_\ell$ and $a_{IR}$, as series in $\Delta_f$, in Eq. (5.5) to obtain

$$\beta'_{IR} = -2\sum_{\ell=1}^\infty (\ell + 1) \left[ \left( \sum_{r=0}^{\infty} b_\ell^{(r)} \Delta_f^r \right) \left( \sum_{j=1}^{\infty} a_j \Delta_f^j \right) \right]$$

$$= \sum_{n=1}^{\infty} d_n \Delta_f^n.$$  

(5.7)

We denote the value of $\beta'_{IR}$ obtained from this series calculated to order $\Delta^4_f$ as $\beta'_{IR, \Delta^4_f}$. The calculation of $d_n$ contains explicit dependence on the $b_\ell$ for $1 \leq \ell \leq n$ and on the $a_j$ for $1 \leq j \leq n - 1$; since $a_j$ depends on $b_\ell$ for $1 \leq \ell \leq j + 1$, it follows that the calculation of $d_n$ requires knowledge of $b_\ell$ for $1 \leq \ell \leq n$. Since the $b_\ell$ have been calculated for general gauge group $G$ and fermion representation $R$ up to four-loop level, we can thus calculate explicit expressions for the $d_n$ up to $n = 4$. For our calculation, in addition to the scheme-independent results for $b_1$ and $b_2$ [3,4], we have used the expressions for $b_3$ and $b_4$ calculated in the $\overline{MS}$ scheme in [13,14]. However, we stress that it does not matter which scheme we use for $b_3$ and $b_4$, because the resulting series expansion for $\beta'_{IR}$ in powers of $\Delta_f$ is scheme-independent.

Substituting the $b_\ell^{(r)}$ and $a_j$ into these equations, we find the following results. First,

$$d_1 = 0,$$  

(5.8)

so that $\beta'_{IR}$ vanishes quadratically with $\Delta_f$ as $\Delta_f \to 0$, i.e., as $N_f \to N_f, b_{1z}$. For $n \geq 2$, with the denominator factor $D = 7C_A + 11C_f$ as defined in Eq. (2.13), we calculate

$$d_2 = \frac{2^5T_f^3}{3^2C_A^2},$$  

(5.9)

$$d_3 = \frac{2^7T_f^3N_f}{3^{17}C_A^3D^2}.$$  

(5.10)
and

\[
d_4 = -\frac{2^3 T^2 \zeta}{3^{33} C^4 A^4} \left[ C^4 T^2 (-121335 + 1241856 \zeta^3) + C^4 T^2 C_f (-310800 + 2661120 \zeta^3) \right.
\]

\[
+ C^3 T^2 C_f (-217848 - 836352 \zeta^3) + C^3 d_{abcd, A} d_{abcd, A} (-2385152 + 5203968 \zeta^3) + C^2 T^2 (2855424 - 3066624 \zeta^3) \]

\[
+ C^2 T^2 (630784 - 6150144 \zeta^3) + C^2 C_f d_{abcd, A} d_{abcd, A} (-3748096 + 8177664 \zeta^3) \]

\[
+ 191664 C_A T^2 C_f + C_A T^2 (630784 - 6150144 \zeta^3) + C_A T^2 C_f (991232 - 9664512 \zeta^3) \]

\[
\left. + T^2 C_f \frac{d_{abcd, A} d_{abcd, A}}{d_A} (-56320 + 148684 \zeta^3) \right].
\]

(5.11)

Here,

\[
\zeta_s = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

(5.12)

is the Riemann zeta function, the quantities \(C_A, C_f, T_f\) are group invariants, and the contractions \(d_{abcd, A} d_{abcd, A}, d_{R} d_{R}, d_{A} d_{A}\) are additional group-theoretic quantities given in [14], and \(d_A\) is the dimension of the adjoint representation of \(G\). These calculations thus determine the quantity \(\beta_{IR}\) to order \(\Delta_f^2\) for an arbitrary gauge group \(G\) and fermion representation \(R\). We have also calculated \(d_5\), but the expression is sufficiently lengthy that we do not include it here; however, we shall use it below.

We note a general result on the signs of the first two nonzero coefficients in the scheme-independent expansion for \(\beta_{IR}\):

\[
d_n > 0 \quad \text{for } n = 2, 3 \quad \text{and arbitrary } G, R.
\]

(5.13)

TABLE I. Signs of expansion coefficients discussed in the text for gauge group \(G = SU(N_c)\) and fermion representation \(R = F\) (fundamental) and \(R = adj\) (adjoint). Several results on signs actually apply more generally for arbitrary \(G\) and \(R\); see text for details. For \(G = SU(3)\), we have also calculated \(d_{5,F}\) in Eq. (5.20) and find that it is negative. The entry for \(\kappa_{A,F}\) applies for \(G = SU(3)\) [see Eq. (6.14)], as calculated in [11], and this is indicated by the (+). The entry NA means “not available,” i.e., the coefficient has not yet been calculated.

| \(n\) | \(d_{n,F}\) | \(d_{n,adj}\) | \(\kappa_{n,F}\) | \(\kappa_{n,adj}\) | \(\kappa_{T,n,F}\) | \(\kappa_{T,n,adj}\) |
|------|---------|-------------|-------------|-------------|-------------|-------------|
| 1    | 0       | 0           | +           | +           | -           | -           |
| 2    | +       | +           | +           | +           | -           | -           |
| 3    | +       | +           | +           | +           | +           | +           |
| 4    | -       | + (+)       | NA          | NA          | NA          | NA          |

These positivity results are clear from Eqs. (5.9) and (5.10). In contrast, there are terms of both signs in the large square bracket in the expression for \(d_4\), Eq. (5.11); for example, in the large square bracket in Eq. (5.11), the coefficients of the \(C^4 T^2\) and \(C^4 T^2 C_f\) terms are positive while the coefficient of the \(C^2 T^2 C_f^2\) term is negative, etc. Indeed, we will show below in Eqs. (5.16) and (5.61) that for \(G = SU(N_c)\), \(d_4\) is negative if \(R = F\) and positive if \(R = adj\). A summary of the sign results for these coefficients and others is given in Table I for the case where \(G = SU(N_c)\).

In Table II we list the (scheme-independent) values that we calculate for \(\beta'_{IR, \Delta_f^2}\) with \(2 \leq p \leq 4\) for the illustrative gauge groups \(G = SU(2), SU(3), \) and \(SU(4)\), as functions of \(N_f\) in the respective intervals \(I_{IR}\) given in Eq. (2.7). For comparison, we list the \(n\)-loop values of \(\beta'_{IR, \Delta_f}\) with the \(2 \leq n \leq 4\), where \(\beta'_{IR, \Delta_f^2}\) and \(\beta'_{IR, \Delta_f^2}\) are computed in the \(\overline{\text{MS}}\) scheme. Values that exceed \(\beta'_{IR} = 3\) are marked as such. In the case of \(SU(3)\), we also include our calculation of \(\beta'_{IR, \Delta_f^2}\).

B. Evaluation for \(G = SU(N_c)\) and \(R = F\)

We proceed to evaluate our general formulas for the \(d_n\) coefficients for a case of particular interest, namely that in which the gauge group is \(G = SU(N_c)\) with \(N_f\) fermions in the fundamental representation, \(R = F\). In addition to Eq. (5.8), our general results (5.9)–(5.11) yield

\[
d_{2,SU(N_c),F} = \frac{2^4}{3^2(25N_c^2 - 11)},
\]

(5.14)

\[
d_{3,SU(N_c),F} = \frac{2^5(13N_c^2 - 3)}{3^3N_c(25N_c^2 - 11)^2},
\]

(5.15)

and
TABLE II. Scheme-independent values of $\beta'_{IR,\Delta f}$ with $2 \leq p \leq 4$ for $G = SU(2)$, SU(3), and SU(4), as functions of $N_f$ in the respective intervals $I_{BRZ}$ given in Eq. (2.7) with (2.4) and (2.6). For comparison, we list the $n$-loop values of $\beta'_{IR,\Delta f}$ with $2 \leq n \leq 4$, where $\beta'_{IR,\Delta f}$ and $\beta'_{IR,\Delta f}$ are computed in the $\overline{MS}$ scheme. Values that exceed the upper bound (4.13) are marked as such. In the case of SU(3), we also include our calculation of $\beta'_{IR,\Delta f}$. The notation $ae-n$ means $a \times 10^{-n}$. The notation $-$ means that the entry has not been calculated.

| $N_c$ | $N_f$ | $\beta'_{IR,\Delta f}$ | $\beta'_{IR,\Delta f}$ | $\beta'_{IR,\Delta f}$ | $\beta'_{IR,\Delta f}$ | $\beta'_{IR,\Delta f}$ | $\beta'_{IR,\Delta f}$ |
|-------|-------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| 2     | 6     | >3                      | 1.620                   | 0.975                   | 0.499                   | 0.957                   | 0.734                   |
| 2     | 7     | 1.202                   | 0.728                   | 0.677                   | 0.320                   | 0.554                   | 0.463                   |
| 2     | 8     | 0.400                   | 0.318                   | 0.300                   | 0.180                   | 0.279                   | 0.250                   |
| 2     | 9     | 0.126                   | 0.115                   | 0.110                   | 0.0799                  | 0.109                   | 0.1035                  |
| 2     | 10    | 0.0245                  | 0.0239                  | 0.0235                  | 0.0200                  | 0.0236                  | 0.0233                  |
| 3     | 9     | >3                      | 1.475                   | 1.464                   | 0.467                   | 0.882                   | 0.7355                  | 0.602                   |
| 3     | 10    | 1.523                   | 0.872                   | 0.853                   | 0.351                   | 0.621                   | 0.538                   | 0.473                   |
| 3     | 11    | 0.720                   | 0.517                   | 0.498                   | 0.251                   | 0.415                   | 0.3725                  | 0.344                   |
| 3     | 12    | 0.360                   | 0.2955                  | 0.282                   | 0.168                   | 0.258                   | 0.239                   | 0.228                   |
| 3     | 13    | 0.174                   | 0.1556                  | 0.149                   | 0.102                   | 0.144                   | 0.137                   | 0.134                   |
| 3     | 14    | 0.0737                  | 0.0699                  | 0.0678                  | 0.0519                  | 0.0673                  | 0.0655                  | 0.0649                  |
| 3     | 15    | 0.0227                  | 0.0223                  | 0.0220                  | 0.0187                  | 0.0220                  | 0.0218                  | 0.0217                  |
| 3     | 16    | 2.21e-3                 | 2.20e-3                 | 2.20e-3                 | 2.08e-3                 | 2.20e-3                 | 2.20e-3                 | 2.20e-3                 |
| 4     | 11    | >3                      | 2.189                   | 2.189                   | 0.553                   | 1.087                   | 0.898                   |
| 4     | 12    | >3                      | 1.430                   | 1.429                   | 0.457                   | 0.858                   | 0.729                   |
| 4     | 13    | 1.767                   | 0.965                   | 0.955                   | 0.370                   | 0.663                   | 0.578                   |
| 4     | 14    | 0.984                   | 0.655                   | 0.639                   | 0.292                   | 0.498                   | 0.445                   |
| 4     | 15    | 0.581                   | 0.440                   | 0.424                   | 0.224                   | 0.362                   | 0.331                   |
| 4     | 16    | 0.348                   | 0.288                   | 0.276                   | 0.1645                  | 0.251                   | 0.234                   |
| 4     | 17    | 0.204                   | 0.180                   | 0.1725                  | 0.114                   | 0.164                   | 0.156                   |
| 4     | 18    | 0.113                   | 0.105                   | 0.101                   | 0.0731                  | 0.0988                  | 0.0955                  |
| 4     | 19    | 0.0558                  | 0.0536                  | 0.0522                  | 0.0411                  | 0.0520                  | 0.0509                  |
| 4     | 20    | 0.0222                  | 0.0218                  | 0.0215                  | 0.0183                  | 0.0215                  | 0.0213                  |
| 4     | 21    | 5.01e-3                 | 4.99e-3                 | 4.96e-3                 | 4.57e-3                 | 4.97e-3                 | 4.96e-3                 |

As is evident, the coefficients $d_{2S,U(N_c),F}$ and $d_{3S,U(N_c),F}$ are positive-definite for all physical values of $N_c$. We find that $d_{4S,U(N_c),F}$ is negative-definite for all physical values of $N_c \geq 2$.

### C. Calculation to $O(\Delta_f^5)$ for $G = SU(3)$ and $R = F$

For the special case where the gauge group is $G = SU(3)$ and the $N_f$ fermions are in the fundamental representation, $R = F$, we can make use of the recent calculation of $b_6$ in the $\overline{MS}$ scheme in [17] to carry out the scheme-independent calculation of $\beta'_{IR}$ to one order higher than for general $G$ and $R$, namely to order $\Delta_f^4$. We first give the special cases of our results in Eqs. (5.8)–(5.11) for this theory. In addition to $d_{1S,USU(3),F} = 0$, we find

\[
d_{2S,USU(3),F} = \frac{8}{3^2 \cdot 107} = 0.830737 \times 10^{-2},
\]

\[
d_{3S,USU(3),F} = \frac{304}{3^3 \cdot (107)^2} = 0.983427 \times 10^{-3},
\]

and

\[
d_{4S,USU(3),F} = \frac{633325687}{2 \cdot 3^6 \cdot (107)^3} - \frac{682880}{3^8 \cdot (107)^6} \xi_3
\]

\[= -(0.463417 \times 10^{-4}).
\]

For $d_{5S,USU(3),F}$ we calculate

\[
d_{5S,USU(3),F} = \frac{-66670528901419}{2 \cdot 3^9 \cdot (107)^4} - \frac{122882810048}{3^8 \cdot (107)^6} \xi_3
\]

\[+ \frac{196275200}{3^6 \cdot (107)^3} \xi_5
\]

\[= -(0.564349 \times 10^{-5}).
\]

In these equations we have indicated the simple factorizations of the denominators that were already evident in the general analytic expressions (5.8)–(5.11).
The numerators do not, in general, have such simple systematic factorizations; for example, in $d_{4, SU(3), F}$, the number 633325687 = 227 · 311 · 8971, etc. We will also use this factorization format, indicating the factorizations of the denominators, in later equations. Substituting these coefficients into Eq. (5.7), we have, to $O(\Delta_f^5)$,

$$
\beta'_{\text{IR}} = \Delta_f^3 \left[ (0.830737 \times 10^{-2}) + (0.983427 \times 10^{-3})\Delta_f^2 - (0.463417 \times 10^{-4})\Delta_f^4 - (0.564349 \times 10^{-5})\Delta_f^5 \right],
$$

(5.21)

to the indicated floating-point accuracy.

In Fig. 1 we plot the values of $\beta'_{\text{IR}}$, calculated to order $\Delta_f^p$ with $2 \leq p \leq 5$. In the general calculations of $\gamma_{\psi\bar{\psi}, \text{IR}}$ as a series in powers of $\Delta_f$ to maximal power $p = 3$ (i.e., order $\Delta_f^3$) in [9] and, for $G = SU(3)$ and $R = F$, to maximal power $p = 4$ in [11], it was found that, for a fixed value of $N_f$, or equivalently, $\Delta_f$, in the interval $I_{\text{IRZ}}$, these anomalous dimensions increased monotonically as a function of $p$. This feature motivated our extrapolation to $p = \infty$ in [9] to obtain estimates for the exact $\gamma_{\psi\bar{\psi}, \text{IR}}$. In contrast, here we find that, for a fixed value of $N_f$, or equivalently, $\Delta_f$, in $I_{\text{IRZ}}$, as a consequence of the fact that different coefficients $d_n^{\gamma}$ do not all have the same sign, $\beta'_{\text{IR}, \Delta_f}$ is not a monotonic function of $p$. Because of this nonmonotonicity, we do not attempt to extrapolate our series to $p = \infty$. Lattice measurements of $\gamma_{F^2, \text{IR}}$ or $\beta'_{\text{IR}}$ would be useful here (see also [44]). In particular, for $G = SU(3)$ and fermions in the fundamental representation, the lattice measurements of $\gamma_{F^2, \text{IR}}$ could be compared with our scheme-independent calculation of $\beta'_{\text{IR}}$ to order $\Delta_f^3$, similar to the comparison of our scheme-independent calculation of $\gamma_{\psi\bar{\psi}, \text{IR}}$ to order $\Delta_f^4$ (which also used five-loop inputs [45]) with lattice results that we carried out in [11].

To get a rough estimate of the range of accuracy and applicability of the series expansion for $\beta'_{\text{IR}}$, we can compute ratios of coefficients, as discussed in connection with Eq. (4.10). Thus, we have

$$
\frac{d_{2, SU(3), F}}{d_{3, SU(3), F}} = 8.447,
$$

(5.22)

$$
\frac{d_{3, SU(3), F}}{d_{4, SU(3), F}} = 21.221,
$$

(5.23)

and

$$
\frac{|d_{4, SU(3), F}|}{|d_{5, SU(3), F}|} = 8.2115.
$$

(5.24)

Since $N_{f, b1z} = 16.5$ and $N_{f, b2z} = 153/19 = 8.053$ in this $SU(3)$ theory, the maximal value of $\Delta_f$ in the interval $I_{\text{IRZ}}$ is

$$
(\Delta_f)^{\text{max}} = \frac{321}{38} = 8.447 \quad \text{for } SU(3), \quad N_f \in I_{\text{IRZ}}.
$$

(5.25)

Therefore, these ratios suggest that the small-$\Delta_f$ expansion may be reasonably reliable in most of this interval, $I_{\text{IRZ}}$, and the associated non-Abelian Coulomb phase.

### D. Calculation in the LNN limit and comparison with conventional calculation

For theories having gauge the group $G = SU(N_c)$ with $N_f$ fermions in the fundamental representation of this group, i.e., $R = F$, it is of interest to consider the limit

$$
N_c \to \infty, \quad N_f \to \infty
$$

with $r \equiv \frac{N_f}{N_c}$ fixed and finite

and $\xi(\mu) \equiv \alpha(\mu)N_c$ is a finite function of $\mu$.

(5.26)

We will use the symbol $\lim_{LNN}$ for this limit, where “LNN” stands for “large $N_c$ and $N_f$” [with the constraints in Eq. (5.26) imposed]. In this LNN ( ‘t Hooft-Veneziano) limit we define the quantities

$$
x = \lim_{LNN} \frac{g^2 N_c}{16\pi^2} = \frac{\xi}{4\pi},
$$

(5.27)

$$
r_{b1z} = \lim_{LNN} \frac{N_{f, b1z}}{N_c},
$$

(5.28)
With values
\[ r_{b_{1z}} = \frac{11}{2} = 5.5 \quad (5.30) \]

and
\[ r_{b_{2z}} = \frac{34}{13} = 2.615 \quad (5.31) \]

it follows that \( \beta' \) is finite in the LNN limit (5.26). In terms of the variable \( x \) defined in Eq. (5.27), we have
\[ \beta' = -2 \sum_{\ell=1}^{\infty} (\ell + 1) \hat{b}_\ell x^\ell. \quad (5.39) \]

Because \( \beta'_{IR} \) is scheme-invariant and is finite in the LNN limit, one is motivated to calculate the LNN limit of the scheme-independent expansion (5.7). For this purpose, in addition to the rescaled quantities \( \Delta \), defined in Eq. (5.33), we define the rescaled coefficient
\[ \tilde{d}_n = N_c^n d_n, \quad (5.40) \]

which is finite in the LNN limit. Then each term
\[ \lim_{LNN} d_n \Delta^n = (N_c^n d_n) \left( \frac{\Delta}{N_c} \right)^n = \tilde{d}_n \Delta^n \quad (5.41) \]
is finite in this limit. Thus, writing \( \lim_{LNN} \beta'_{IR} \) as \( \beta'_{IR,LNN} \), we have
\[ \beta'_{IR,LNN} = \sum_{n=1}^{\infty} \tilde{d}_n \Delta^n = \sum_{n=1}^{\infty} \tilde{d}_n \Delta^n. \quad (5.42) \]

We denote the value of \( \beta'_{IR,LNN} \) obtained from this series calculated to order \( O(\Delta^n) \) as \( \beta'_{IR,LNN,\Delta^n} \).

From Eqs. (5.8)–(5.11), we find that the approach to the LNN limits for \( \tilde{d}_n \) involves correction terms that vanish like \( 1/N_c^2 \). This is the same property that was found in [26,27] and, in the same way, it means that the approach to the LNN limit for finite \( N_c \) and \( N_f \) with fixed \( r = N_f/N_c \) is rather rapid, as discussed in [27]. We calculate \( \tilde{d}_1 = 0 \) and
\[ \tilde{d}_2 = \frac{2^4}{3^2 \cdot 5^2} = 0.0711111, \quad (5.43) \]
\[ \tilde{d}_3 = \frac{416}{3^3 \cdot 5^4} = 2.465185 \times 10^{-2}, \quad (5.44) \]

and
\[ \tilde{d}_4 = \frac{5868512}{3^5 \cdot 5^{10}} - \frac{5632}{3^4 \cdot 5^8} \tilde{d}_3 = -(2.876137 \times 10^{-3}). \quad (5.45) \]

Thus, numerically
\[ \beta'_{IR,LNN} = \gamma_{F^2,IR} \]
\[ = \Delta^n [0.0711111 + (2.4652 \times 10^{-2}) \Delta] 
- (2.8761 \times 10^{-3}) \Delta^2 \]. \quad (5.46)
over which the small-\(\Delta_r\) expansion is reliable in this LNN limit. We find

\[
\frac{\hat{d}_2}{d_3} = 2.885 
\]

and

\[
\frac{\hat{d}_3}{|d_4|} = 8.571.
\]

For \(r \in I_{\text{IRZ}},\) the maximal value of \(\Delta_r\) is

\[
(\Delta_r)_{\text{max}} = \frac{75}{26} = 2.885 \quad \text{for} \ r \in I_{\text{IRZ}}.
\]

Therefore, these LNN ratios suggest, in agreement with our analysis for SU(3) and \(R = F\), that the small-\(\Delta_r\) expansion may be reasonably reliable over much of the interval \(I_{\text{IRZ}}\).

It is useful to compare these scheme-independent calculations of \(\beta_{\text{IR-LNN}}\) with the results of conventional \(n\)-loop calculations, denoted \(\beta_{\text{IR-n/NN}}\). These derivatives are computed from the \(n\)-loop truncation of the series in Eq. (5.39). As a special case of our remark below Eq. (5.5), we note that in calculating the \(n\)-loop truncation of the series (5.39) at the IR zero of the beta function, for \(n \geq 3\), one uses the property that

\[
\sum_{\ell=1}^{n} \hat{b}_\ell \chi_{\text{IR}}^{\ell-1} = 0,
\]

to eliminate the highest-loop term \(\hat{b}_n \chi_{\text{IR}}^{n-1}\), expressing it as \(\hat{b}_n \chi_{\text{IR}}^{n-1} = -\sum_{\ell=1}^{n-2} \hat{b}_\ell \chi_{\text{IR}}^{\ell-1}\). The two-loop result for \(x_{\text{IR}}\) is

\[
x_{\text{IR,2}} = \frac{11 - 2r}{13r - 34} \quad \text{for} \ r \in I_{\text{IRZ}}.
\]

The resultant two-loop for \(\beta_{\text{IR}}\) is

\[
\beta_{\text{IR,2}} = \frac{2(11 - 2r)^2}{3(13r - 34)}.
\]

Both \(x_{\text{IR,2}}\) and \(\beta_{\text{IR,2}}\) are scheme-independent. However, the higher-loop expressions for these quantities at loop level \(n \geq 3\) do not preserve the scheme-independence of the exact \(\beta_{\text{IR}}\). Let us define the polynomials (see Eqs. (3.9) and (2.26) in [27])

\[
C_{3\ell} = -52450 + 41070r - 7779r^2 + 448r^3
\]

and

\[
D_{3\ell} = -2857 + 1709r - 112r^2,
\]

both of which are positive for \(r \in I_{\text{IRZ}}\). The three-loop value of the IR zero of the beta function in the LNN limit, computed in the \(\overline{\text{MS}}\) scheme, is [27]

\[
x_{\text{IR,3}} = \frac{3[-3(13r - 34) + \sqrt{C_{3\ell}}]}{D_{3\ell}}.
\]

We calculate the three-loop result for \(\beta_{\text{IR}}\) or equivalently the anomalous dimension of \(\text{Tr}(F_{\mu\nu}F^{\mu\nu})\), in the LNN limit, again in the \(\overline{\text{MS}}\) scheme, to be

\[
\beta_{\text{IR,3}} = \frac{2[-3(13r - 34) + \sqrt{C_{3\ell}}]}{D_{3\ell}^2}
\]

\[
\times [-52450 + 41070r - 7779r^2 + 448r^3
\]

\[
-3(13r - 34)\sqrt{C_{3\ell}}].
\]

We compute the four-loop result \(\beta_{\text{IR,4}}\) in this scheme in a similar manner. In Table III we list the numerical values of these conventional \(n\)-loop calculations in comparison with our scheme-independent results calculated to \(O(\Delta^4_r)\) for \(2 \leq n \leq 4\) and \(1 \leq p \leq 3\). We see that, especially for \(r\) values in the upper part of the interval \(I_{\text{IRZ}}\), the results are rather close, and, furthermore, that, as expected, for a given \(r\), the higher the loop level \(n\) and the truncation order \(p\) in the respective calculations of \(\beta_{\text{IR}}\) in the \(\overline{\text{MS}}\) scheme and the scheme-independent \(\beta_{\text{IR,adj}}\), the better the agreement between these two results. All of the entries shown in Table III have \(\beta_{\text{IR}} < 3\) except for the two-loop values \(\beta_{\text{IR,2}}\) for \(r = 3.0\) and \(r = 2.8\) which are 3.333 and 8.100, respectively.

**E. Calculation of the \(d_n\) to \(O(\Delta^4_r)\)**

for \(G = \text{SU}(N_c)\) and \(R = \text{adj}\)

It is worthwhile to compare our results obtained for \(G = \text{SU}(N_c)\) with \(N_f\) fermions in the fundamental representation to the case in which the fermions are in the adjoint representation, denoted as \(\text{adj}\) for short. In this case, the general expressions for \(N_{f,b_{1z}}\) and \(N_{f,b_{2z}}\) are

\[
N_{f,b_{1z}} = \frac{11}{4} = 2.75 \quad \text{for} \ R = \text{adj}
\]

and

\[
N_{f,b_{2z}} = \frac{17}{16} = 1.0625 \quad \text{for} \ R = \text{adj}.
\]

so the interval \(I_{\text{IRZ}}\) only contains the single integer value \(N_f = 2\).

For this theory, our general expressions (5.9) and (5.10) reduce to pure numbers, independent of \(N_c\):
The large-$\Delta_f$ expansion may again be reasonably accurate in the $LNN$ limit (5.26) as functions of $r = 5.5 - \Delta_f$. For comparison, we also list the $n$-loop values $\beta^\prime_{\text{IR},ae}$ with $2 \le n \le 4$, where $\beta^\prime_{\text{IR},ae}$ and $\beta^\prime_{\text{IR},ae}$ are computed in the $\overline{\text{MS}}$ scheme [and values that exceed the upper bound (4.13) are marked as such]. The notation $ae-n$ means $\alpha \times 10^{-n}$.

| $r$ | $\beta^\prime_{\text{IR},ae}$ | $\beta^\prime_{\text{IR},ae,\overline{\text{MS}}}$ | $\beta^\prime_{\text{IR},ae,\overline{\text{MS}}}^\prime$ | $\beta^\prime_{\text{IR},ae}$ | $\beta^\prime_{\text{IR},ae}^\prime$ | $\beta^\prime_{\text{IR},ae}^\prime$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2.8 | $> 3$            | 1.918           | 1.949           | 0.518           | 1.004           | 0.851           |
| 3.0 | $> 3$            | 1.376           | 1.523           | 0.444           | 0.830           | 0.717           |
| 3.2 | 1.856            | 1.006           | 1.100           | 0.376           | 0.676           | 0.596           |
| 3.4 | 1.153            | 0.7395          | 0.72985         | 0.314           | 0.542           | 0.486           |
| 3.6 | 0.752            | 0.542           | 0.528           | 0.257           | 0.426           | 0.388           |
| 3.8 | 0.500            | 0.393           | 0.378           | 0.2055          | 0.327           | 0.303           |
| 4.0 | 0.333            | 0.279           | 0.267           | 0.160           | 0.243           | 0.229           |
| 4.2 | 0.219            | 0.193           | 0.185           | 0.120           | 0.174           | 0.166           |
| 4.4 | 0.139            | 0.128           | 0.122           | 0.0860          | 0.119           | 0.115           |
| 4.6 | 0.0837           | 0.0792          | 0.0766          | 0.0576          | 0.0756          | 0.0737          |
| 4.8 | 0.0460           | 0.0445          | 0.0435          | 0.0348          | 0.0433          | 0.0426          |
| 5.0 | 0.0215           | 0.0212          | 0.0208          | 0.0178          | 0.0209          | 0.0207          |
| 5.2 | 0.714e-2         | 0.710e-2        | 0.706e-2        | 0.640e-2        | 0.707e-2        | 0.704e-2        |
| 5.4 | 0.737e-3         | 0.736e-3        | 0.7356e-3       | 0.7111e-3       | 0.7358e-3       | 0.7355-3        |

The interval $I_{\text{IRZ}}$ and for the corresponding value $N_f = 2$ in this theory.

**VI. ANALYSIS OF SCHEME-INDEPENDENT EXPANSION COEFFICIENTS FOR $\gamma_{\bar{\psi}\psi,1R}$**

A. Review of calculation to $O(\Delta_f^4)$ for general $G$ and $R$

We consider the (gauge-invariant) flavor-nonsinglet ($fns$) and flavor-singlet ($fs$) bilinear fermion operators

$$J_{0,fns} = \sum_{j,k=1}^{N_f} \bar{\psi}_j(T_b)_{jk} \psi_k,$$

where here $T_b$ with $b = 1, \ldots, N^2_f - 1$ is a generator of the global flavor group SU($N_f$), and

$$J_{0,fs} = \sum_{j=1}^{N_f} \bar{\psi}_j \psi_j.$$  

We will often suppress the flavor indices and write these simply as $\bar{\psi} T_b \psi$ and $\bar{\psi} \psi$. These have the same anomalous dimension (e.g., [46]), which we denote as $\gamma_{\bar{\psi}\psi}$. (Thus, one may simply consider the operator $\bar{\psi}_j \psi_j$ with no sum on $j$, but here we shall refer to $J_{0,fns}$ and $J_{0,fs}$.) The operator $J_{0,fs}$ has the chiral decomposition $\bar{\psi} T_b \psi = \bar{\psi}_L T_b \psi_R + \bar{\psi}_R T_b \psi_L$. Hence, in the non-Abelian Coulomb phase where the flavor symmetry is (2.19), one may regard the $T_b$ in the term $\bar{\psi}_L T_b \psi_R$ acting to the right as an element of SU($N_f$)$_R$ and acting to the left as an element of SU($N_f$)$_L$.

The usual series expansion of $\gamma_{\bar{\psi}\psi}$ in powers of $\alpha$, or equivalently, $\alpha$, is
where $c_\ell$ is the $\ell$-loop coefficient. For general $G$ and $R$ the coefficients $c_\ell$ have been calculated up to $\ell = 4$ loop level [47] (earlier work includes [48]) and for the special case $G = SU(3)$ and $R = F$, $c_4$ has been calculated [49]. The scheme-independent expansion of $\gamma_{\bar{\psi}\psi}$ can be written as

$$
\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell a^\ell, 
$$

(6.3)

We denote the truncation of this sum to maximal power $n = p$ as $\gamma_{\bar{\psi}\psi,\text{IR}}^{\Delta_n^p}$. For a general asymptotically free and vectorial gauge theory with gauge group $G$ and $N_f$ fermions in an arbitrary representation $R$, the coefficients $\kappa_n$ were given in [9] up to order $n = 3$, yielding the expansion of $\gamma_{\bar{\psi}\psi,\text{IR}}$ to order $\Delta^1_f$. For reference, we display the $\kappa_n$ coefficients from Ref. [9] [with the denominator factor $D$ given in Eq. (2.13)]:

$$
\kappa_1 = \frac{8T_f C_f}{C_A D},
$$

(6.5)

$$
\kappa_2 = \frac{4T_f^2 C_f (5C_f + 88C_f)(7C_f + 4C_f)}{3C_A^2 D^3},
$$

(6.6)

### B. Evaluation of $\kappa_n$ for $G = SU(N_c)$ and $R = F$

For the case where the $N_f$ fermions are in the representation $R = F$, these results (6.5)–(6.7) from [9] take the following forms:

$$
\kappa_{1, SU(N_c), F} = \frac{4(N_f^2 - 1)}{N_c(25N_f^2 - 11)},
$$

(6.8)

$$
\kappa_{2, SU(N_c), F} = \frac{4(N_f^2 - 1)(9N_f^2 - 2)(49N_f^2 - 44)}{3N_c^2(25N_f^2 - 11)^3},
$$

(6.9)

and

$$
\kappa_{3, SU(N_c), F} = \frac{8(N_f^2 - 1)}{3^3N_c^3(25N_f^2 - 11)^3} \left[ 274243N_f^8 - 455426N_f^6 - 114080N_f^4 + 47344N_f^2 + 35574 - 4224N_f^2(4N_f^2 - 11)(25N_f^2 - 11) \zeta_3 \right].
$$

(6.10)

We find that these coefficients $\kappa_{n, SU(N_c), F}$ with $1 \leq n \leq 3$ are positive-definite for all physical $N_c \geq 2$. This is obvious for $n = 1, 2$, and an examination of the polynomial in square brackets in Eq. (6.10), of degree 8 in $N_f$, proves the result for $n = 3$.

### C. Calculation of $\kappa_n$ coefficients to $O(\Delta_f^4)$ for $G = SU(3)$ and $R = F$

For comparison with the $\kappa_n$ with other values of $N_c$, we recall our calculation of the $\kappa_n$ to order $n = 4$, i.e., to order $O(\Delta_f^4)$ in [11]. We found
for an exact $\gamma_{\bar{\psi}\psi, IR}$ in the SU(3) theory with $R = F$. A
generalization of our conjecture in [11] that is motivated by our present results is that, in the notation of
Eqs. (6.11)-(6.14), $\kappa_{n,SU(N)_c,F} > 0$ for all $n \geq 1$ and all $N_c \geq 2$. Importantly, in [11] we noted that, if
this monotonicity property holds, then, combining it with the upper bound $\gamma_{\bar{\psi}\psi, IR} < 2$, one would infer that $\gamma_{IR}$
saturates its upper bound (4.14) as $N_f$ decreases and passes through the value $N_{f,cr}$ at the lower end of the non-
Abelian Coulomb phase, it would follow from our extrapolated values of $\gamma_{\bar{\psi}\psi, IR}$ that $8 < N_{f,cr} < 9$. Here
one must mention the caveat that it is not known if, in fact, $\gamma_{IR}$ saturates its upper bound in this way as $N_f \searrow N_{f,cr}$.
Indeed, the nature of the transition as $N_f$ decreases through $N_{f,cr}$ has not been definitely established.
Analyses via Schwinger-Dyson equations suggested that, as $N_f \nearrow N_{f,cr}$ from within the phase with confinement and
chiral symmetry breaking, the fermion condensate $\langle \bar{\psi}\psi \rangle$ could vanish with an essential zero [50]. Some
insight into this may be derived from the known results in SQCD. In SQCD, as noted above, the upper bound is
$\gamma_{\bar{\psi}\psi, IR} < 1$ and is saturated at the lower end of the non-
Abelian Coulomb phase [37,38].

In the case $G = SU(3)$ and $R = F$, one of the major values of the five-loop calculation of $\gamma_{\bar{\psi}\psi, IR}$ in [10] and
the scheme-independent calculations of $\gamma_{\bar{\psi}\psi, IR}$ to order $\Delta_f^3$ in [9] and to order $\Delta_f^4$ in [11], with the additional analysis
here, is the comparison of these results with fully non
perturbative lattice measurements of this anomalous dimension [51]. [Since our discussion here is on the operator $\bar{\psi}\psi$
and the gauge group SU(3), when there is no danger of confusion, we omit these subscripts in the $\bar{\psi}\psi$ anomalous dimension.] A number of lattice groups have obtained data and carried out analyses of these data for the SU(3) theory
with $N_f = 12$ fermions with $R = F$. These groups have reported the following values: $\gamma_{IR} = 0.414 \pm 0.016$ [52],
$\gamma_{IR} \sim 0.35$ [53], $\gamma_{IR} = 0.4$ [54], $\gamma_{IR} = 0.27(3)$ [55], $\gamma_{IR} = 0.25$ [56], $\gamma_{IR} = 0.235(46)$ [57], and $0.2 \lesssim \gamma \lesssim 0.4$ [58].
(For comparative discussions of these different results and estimates of overall uncertainties, the reader is advised to consult
the reviews in [51] and the original papers [52–54,57–59].) As we noted in [11], our value $\gamma_{IR,\Delta_f}$
$= 0.338$ and our extrapolated $\gamma_{IR} = 0.40$ are consistent with this range of lattice measurements, taking into account
the different methods of lattice data analysis used, and are somewhat higher than the five-loop value $\gamma_{IR,5f} = 0.255$ from the conventional $\alpha_s$ series that we obtained in
[10]. The $\gamma_{IR,5f} = 0.255$ value in [10] is in very good agreement with the measured values of $\gamma_{IR}$ reported in
[55,57,58,60–62].

There have also been lattice studies of the SU(3) theory with $N_f = 10$ [60] and $N_f = 8$ [51,61,62]. For the SU(3)
theory with $N_f = 10$ fermions, our scheme-independent calculation presented in [11] and discussed further here


$$\kappa_{SU(3), F, 1} = \frac{16}{3 \cdot 107} = 4.9844 \times 10^{-2}, \quad (6.11)$$

$$\kappa_{SU(3), F, 2} = \frac{125452}{(3 \cdot 107)^2} = 3.7928 \times 10^{-3}, \quad (6.12)$$

$$\kappa_{SU(3), F, 3} = \frac{972349306}{(3 \cdot 107)^3} - \frac{140800}{3^2 \cdot (107)^4} \delta \zeta^3$$

$$= 2.3747 \times 10^{-4}. \quad (6.13)$$

$$\kappa_{SU(3), F, 4} = \frac{33906710751871}{2^2 \cdot (3 \cdot 107)^4} - \frac{1684980608}{3^3 \cdot (107)^5} \delta \zeta^5$$

$$+ \frac{598400000}{(3 \cdot 107)^6} \delta \zeta^6$$

$$= 3.6789 \times 10^{-5}. \quad (6.14)$$

In Ref. [9] the ratio test was applied to the first three coefficients, $\kappa_{SU(3), F, n}$, $n = 1, 2, 3$ and the excellent convergence was noted. Here, using our calculation of $\kappa_{SU(3), F, 4}$ in [11], we calculate the next ratio,$\kappa_{SU(3), F, 4}/\kappa_{SU(3), F, 3}$. We have

$$\frac{\kappa_{SU(3), F, 1}}{\kappa_{SU(3), F, 2}} = 13.142, \quad (6.15)$$

$$\frac{\kappa_{SU(3), F, 2}}{\kappa_{SU(3), F, 3}} = 15.972, \quad (6.16)$$

and

$$\frac{\kappa_{SU(3), F, 3}}{\kappa_{SU(3), F, 4}} = 6.455. \quad (6.17)$$

Since the maximal value of $\Delta_f$ in the interval $l_{IRZ}$ is 8.447 [see Eq. (5.25)], these ratios suggest, as noted in [9] and
in agreement with our earlier calculation of coefficient ratios for $\beta_{IR}$, that the small-$\Delta_f$ expansion may be reasonably
reliable over much of the interval $l_{IRZ}$.

The positivity of the $\kappa_{SU(3), F, n}$ for $1 \leq n \leq 3$ is in
agreement with our more general positivity results given above, and, as we noted in [11], we also found that $\kappa_{SU(3), F, 4}$ is positive. These signs are recorded in Table I. The
positivity of all of these coefficients played an important role in our analysis in [11] because it meant that for a given value of $N_f$, or equivalently, $\Delta_f$, the value of $\gamma_{\bar{\psi}\psi}$ calculated to $O(\Delta_f^n)$, denoted $\gamma_{\bar{\psi}\psi, \Delta_f^n}$, is a monotonically increasing function of $n$ over the full range $1 \leq n \leq 4$ that we calculated. We then conjectured that this positivity would be true for all $n$, i.e., we conjectured that $\kappa_n > 0$ for all $n \geq 1$. Assuming the validity of this conjecture, we then computed the extrapolation to $n \rightarrow \infty$
obtained in [9] for the scheme-independent expansion of $\gamma_{\psi\psi,IR}$. There is not yet a consensus on the value of $\gamma_{\psi\psi,IR}$, approximate, since the theory flows away from this value as the anomalous dimension that describes the renormalization-group behavior as the theory is flowing near to the approximate zero of the beta function.

D. Evaluation of $\kappa_{n, SU(N_c), R}$ to $O(\Delta_f^3)$ for $R = \text{adj}$

In the case $R = \text{adj}$, the general results in [9] reduce as follows:

$$\kappa_{1, SU(N_c), \text{adj}} = \frac{4}{3^2} = 0.4444, \quad (6.18)$$

$$\kappa_{2, SU(N_c), \text{adj}} = \frac{341}{2 \cdot 3^6} = 0.23388, \quad (6.19)$$

$$\kappa_{3, SU(N_c), \text{adj}} = \frac{61873 N_c^2 - 42624}{2^3 \cdot 3^{10} N_c^2}. \quad (6.20)$$

This is positive for all physical $N_c$ and has the large-$N_c$ limit

$$\lim_{N_c \to \infty} \kappa_{3, SU(N_c), \text{adj}} = \frac{61873}{2^3 \cdot 3^{10}} = 0.130978. \quad (6.21)$$

The positive signs of these $\kappa_{n, SU(N_c), \text{adj}}$ coefficients are recorded in Table I.

E. Comparison of scheme-independent calculation of $\gamma_{\psi\psi, IR}$ with conventional calculations

It is of considerable interest to compare the results obtained in [9] for the scheme-independent expansion of $\gamma_{\psi\psi, IR}$ to order $O(\Delta_f^3)$ (using calculations of the $b_n$ to $n = 4$ loop order and $c_n$ to $n = 3$ loop order) with results obtained previously with the conventional calculation of the $n$-loop $\gamma_{\psi\psi,IR,n'}$ in powers of the $n$-loop $\gamma_{IR,n'}$ in [22] (using calculations of the $b_n$ and $c_n$ up to $n = 4$ loop order). Here and below, for specific calculations we take the gauge group to be $SU(N_c)$ with various values of $N_c$. For notational brevity, in this section we will often leave the subscript $\psi\psi$ implicit on these and other quantities and thus write $\gamma_{IR} \equiv \gamma_{\psi\psi, IR}, \gamma_{IR,n'} \equiv \gamma_{\psi\psi,IR,n'}, \kappa_n \equiv \kappa_{\psi\psi,n'}$, etc. in this and the next section. Since $\gamma_{IR,n'}$ is scheme-dependent beyond the lowest order, one must choose a scheme for this comparison. Here we choose the widely used $\overline{\text{MS}}$ scheme, for which $b_3$ and $b_4$ and $c_n$ for $2 \leq n \leq 4$ were calculated for a general gauge group $G$ and fermion representation $R$ [13–15] [47]. In the special case of $G = SU(3)$ and $R = F$, using the recent calculations of the five-loop coefficients $b_5$ and $c_5$ in the $\overline{\text{MS}}$ scheme, we computed $\gamma_{IR,n'}$ up to $n = 5$ loop level [10] in this $\overline{\text{MS}}$ scheme and performed a scheme-independent calculation up to order $\Delta_f^3$ [11]. For this special case we compared the results obtained via these two different approaches. Here we carry out a similar comparison for other $SU(N_c)$ theories. The scheme-independent expansion of $\gamma_{IR}$ has the form (6.4). We denote the value of $\gamma_{IR}$ obtained from this series calculated to order $O(\Delta_f^3)$ as $\gamma_{IR,\Delta_f^3}$.

As discussed above, our discussion is restricted to the interval $I_{\text{IRZ}}$ of values of $N_f$, given in Eq. (2.7), for which the (scheme-independent) two-loop beta function has an IR zero. Using the results for the lower and upper ends of this interval, $N_{f,b2z}$ and $N_{f,b1z}$ from Eqs. (2.4) and (2.6), one has, for $(N_{f,b1z}, N_{f,b2z})$, the respective values (5,5,11), (8,05,16,5), and (10,61,22) for $N_c = 2, 3, 4$ [19], and hence the physical intervals $I_{\text{IRZ}}$ with integral $N_f$: $6 \leq N_f \leq 10$ for $SU(2)$, $9 \leq N_f \leq 16$ for $SU(3)$, and $11 \leq N_f \leq 21$ for $SU(4)$. Our results for these three illustrative values of $N_c$ are listed in Table IV. For the special case $N_c = 3$, we have carried these calculations one order higher, namely to five-loop level and to order $\Delta_f^3$ in [10,11].

Since the calculation of $\kappa_n$ and the resultant $\gamma_{IR,\Delta_f^3}$ uses information from the $(n + 1)$-loop beta function from (2.1) and the $n$-loop expansion of $\gamma$ in (4.2), it is natural to compare the (SD) $\gamma_{IR,\Delta_f^3}$ with the (SD) $\gamma_{\psi\psi,IR,n'}$ for $n' = n$ and $n' = n + 1$. Since $\gamma_{IR,\Delta_f^3}$ includes $n$-loop information about $\gamma_{IR,n'}$, one would expect the closest agreement between $\gamma_{IR,\Delta_f^3}$ and $\gamma_{IR,n'}$, and our results confirm this expectation. In the upper and middle part of the interval $I_{\text{IRZ}}$ for a given $N_c$, we find that $\gamma_{IR,\Delta_f^3}$ is slightly larger than $\gamma_{IR,3n'}$, with the difference increasing as $N_f$ decreases below $N_{f,b1z}$, i.e., as $\Delta_f$ increases.

We recall the upper bound (4.14) that applies at an IRFP in the non-Abelian Coulomb phase, based on the scale invariance and inferred conformal invariance in this phase. The bound (4.14) also applies, for a different reason, in the
phase with confinement and spontaneous chiral symmetry breaking; in that phase it is a consequence of the physical requirement that the momentum-dependent dynamically generated effective fermion mass

\[ m(k) \sim \Lambda \left( \frac{\Lambda}{k} \right)^{2-\gamma_{\text{IR}}} \]  

must satisfy the constraint \( \lim_{k \to \infty} m(k) = 0 \), where \( k \) is the Euclidean momentum. In the upper and middle parts of the interval \( I_{\text{IRZ}} \) in the NACP, the values of \( \gamma_{\text{IR},nf} \) calculated in the conventional series expansion in powers of \( \alpha_{\text{IR},nf} \) obey this upper bound. However, for a given \( N_c \), toward the lower end of the respective intervals \( I_{\text{IRZ}} \), the IR coupling \( \alpha_{\text{IR},nf} \) become too large for the perturbative calculations to be applicable, and some resultant values of the anomalous dimensions exceed the bound (4.14). This occurs for the scheme-independent two-loop values \( \gamma_{\text{IR}} \), \( 2\ell' \) for \( N_f = 6, 7 \) if \( N_c = 2 \); for \( N_f = 9, 10 \) if \( N_c = 3 \), and for \( 11 \leq N_f \leq 14 \) if \( N_f = 4 \). In these cases, since it is not clear that the higher-order values \( \gamma_{\text{IR},nf} \) are reliable, we leave them unplotted (u), as we did in [22].

From these calculations and the entries in Table IV, one of the important advances achieved by the scheme-independent \( \Delta_f \) expansion is evident, namely that the values of \( \gamma_{\text{IR},nf} \) with \( 1 \leq p \leq 3 \) (and, for \( SU(3) \) also \( p = 4 \) in [11]) that we calculate via this method obey the upper bound (4.14) throughout all of the interval \( I_{\text{IRZ}} \) and associated non-Abelian Coulomb phase, in contrast with some of the values calculated via the conventional loop expansion toward the lower end of \( I_{\text{IRZ}} \). In general, for all of the \( N_c \) values considered, our results for \( \gamma_{\text{IR},nf} \) here satisfy the upper bound (4.14) and hence are consistent with the conclusion that the \( \Delta_f \) expansion is reasonably reliable throughout the interval \( I_{\text{IRZ}} \) and non-Abelian Coulomb phase. We regard this, together with the scheme-independence itself, as being a major advantage of the \( \Delta_f \) expansion.

### F. L.N N limit for \( \gamma_{\psi\psi,\text{IR}} \)

Here we consider theories with \( G = SU(N_c) \) and \( N_f \) copies of fermions in the representation \( R = F \) in the LNN limit ([27]). We recall that in this LNN limit, the interval \( I_{\text{IRZ}} \) is given by Eq. (5.32) and the scaled \( \Delta_f \), is defined by Eq. (5.33). We define rescaled coefficients \( \tilde{\kappa}_n \)

\[ \tilde{\kappa}_n = \lim_{N_c \to \infty} N_c \kappa_n \]

that are finite in this LNN limit. The anomalous dimension \( \gamma_{\psi\psi,\text{IR}} \) is also finite in this limit and is given by

\[ \lim_{L_{\text{NNN}}\gamma_{\psi\psi,\text{IR}}} = \sum_{n=1}^{\infty} \kappa_n \Delta_f^n = \sum_{n=1}^{\infty} \tilde{\kappa}_n \Delta_r^n. \]

From (5.32), it follows that as \( r \) decreases from \( r_{b1z} \) to \( r_{b2z} \), \( \Delta_r \) increases from 0 to its maximal value

\[ (\Delta_r)_{\text{max}} = \frac{75}{26} = 2.8846 \quad \text{for } r \in I_{\text{IRZ},r}. \]

From the results for \( \kappa_n, n = 1, 2, 3 \) in [9] or the special cases given above for \( G = SU(N_c) \) and \( R = F \) in Eqs. (6.8)-(6.10), we find

\[ \tilde{\kappa}_1 = \frac{4}{25} = 0.1600, \]

\[ \tilde{\kappa}_2 = \frac{588}{56} = 0.037632, \]

and
FIG. 2. Plot of $\gamma_{\psi\psi,IR,\Delta r}$ for $1 \leq p \leq 3$ as a function of $r \in I_{IRZ}$ in the LNN limit (5.26). From bottom to top, the curves (with colors online) refer to $\gamma_{\psi\psi,IR,\Delta r}$ (red), $\gamma_{\psi\psi,IR,\Delta r}$ (green) $\gamma_{\psi\psi,IR,\Delta r}$ (blue).

$k_3 = \frac{2193944}{3^3 \cdot 5^{10}} = 0.83207 \times 10^{-2}$, \hspace{1cm} (6.28)

where, as above, we indicate the factorization of the denominators. Numerically, to order $O(\Delta_r^3)$,

\[
\lim_{LNN} \gamma_{\psi\psi,IR} = \Delta_r \left[ 0.160000 + 0.037632 \Delta_r - 0.0083207 \Delta_r^2 + O(\Delta_r^3) \right]. \hspace{1cm} (6.29)
\]

We plot the value of $\gamma_{\psi\psi,IR}$ calculated to order $\Delta_r^p$, denoted $\gamma_{\psi\psi,IR,\Delta r}$, for $1 \leq p \leq 3$, as a function of $r \in I_{IRZ}$ in Fig. 2. As a consequence of the positivity of the $k_p$ in Eqs. (6.26)–(6.28), for a fixed $r$, $\gamma_{\psi\psi,IR,\Delta r}$ is a monotonically increasing function of the order of calculation, $p$. Interestingly, as $r$ decreases toward the lower end of the interval $I_{IRZ}$, at $r = r_{IRZ} = 13/34 = 2.6154$, the value of $\gamma_{\psi\psi,IR}$ calculated to the highest order in this LNN limit, namely $O(\Delta_r^3)$ is slightly less than 1. This is similar to the behavior that was found for the specific cases of SU(2) and SU(3) gauge groups and $R = F$ in [9] and for SU(3) with $\gamma_{\psi\psi,IR}$ calculated to the next order, $O(\Delta_r^4)$ in [11].

As discussed above, our calculations of $\gamma_{\psi\psi,IR}$ via the $\Delta_r$ expansion, both for specific values of $N_c$ and in the LNN limit, have yielded results satisfying the upper bound (4.14) throughout the interval $I_{IRZ}$. These results support the conclusion that the small-$\Delta_r$ series expansion is reliable throughout this interval $I_{IRZ}$ and associated non-Abelian Coulomb phase. It is also worthwhile to obtain an estimate of the range of applicability of the small-$\Delta_r$ series expansion via a different method, the aforementioned ratio test. From the coefficients $k_n$ that we have calculated with $1 \leq n \leq 3$, we compute the ratios

\[
\frac{k_1}{k_2} = 4.252 \hspace{1cm} (6.30)
\]

and

\[
\frac{k_2}{k_3} = 4.523. \hspace{1cm} (6.31)
\]

Recalling that the maximal value of $\Delta_r$ in the interval $I_{IRZ}$ is 2.885 [Eq. (5.49)], these ratios are again consistent with the inference that the small-$\Delta_r$ series expansion may be reasonably accurate in this interval $I_{IRZ}$. Since $r$ has a maximal value of 5.5 in this LNN limit, the above ratios also suggest that one could not reliably apply the small-$\Delta_r$ expansion down to small $r$ (see also [63]). This is in agreement with the fact that the properties of theory change qualitatively as $r$ decreases below $r_c$ in Eq. (5.34); in particular, there is spontaneous chiral symmetry breaking at small $r$.

G. Analysis with Padé approximants

To get further insight into the behavior of $\gamma_{\psi\psi,IR}$, we shall calculate and analyze Padé approximants (PAs) [64]. For this purpose, we shall use a reduced function normalized to unity at $\Delta_r = 0$, namely

\[
\tilde{\gamma}_{\psi\psi,IR} = \frac{\gamma_{\psi\psi,IR}}{k_1} = 1 + \sum_{n=1}^{\infty} \frac{\kappa_n}{\kappa_1} \frac{\Delta_r^{n-1}}{\Delta_r}. \hspace{1cm} (6.32)
\]

The calculation of $\gamma_{\psi\psi,IR}$ to order $\Delta_r^3$ yields $\tilde{\gamma}_{\psi\psi,IR}$ to order $\Delta_r^3$. In turn, from this we can compute three PAs: $[2, 0]_{\tilde{\gamma}_{\psi\psi,IR}}$, $[1, 1]_{\tilde{\gamma}_{\psi\psi,IR}}$, and $[0, 2]_{\tilde{\gamma}_{\psi\psi,IR}}$. Since the $[2, 0]$ PA is just $\tilde{\gamma}_{\psi\psi,IR}$ itself, to order $\Delta_r^2$, we focus on the $[1, 1]$ and $[0, 2]$ PAs. We calculate

\[
[1, 1]_{\tilde{\gamma}_{\psi\psi,IR}} = \frac{\left( \Delta_r + \frac{34957}{2480625} \Delta_r^2 \right)}{\left( \Delta_r - \frac{2480625}{548486} \right)}. \hspace{1cm} (6.33)
\]

and

\[
[0, 2]_{\tilde{\gamma}_{\psi\psi,IR}} = \frac{\left( \Delta_r + \frac{147}{625} \Delta_r^2 + \frac{85457}{10546875} \Delta_r^3 \right)}{\left( \Delta_r - \frac{2480625}{548486} \right)}. \hspace{1cm} (6.34)
\]

The $[1, 1]$ PA has no physical zero and a pole at

\[
(\Delta_r)_{pole,[1,1]_{\tilde{\gamma}_{\psi\psi,IR}}} = \frac{2480625}{548486} = 4.523. \hspace{1cm} (6.35)
\]

Since this value is well beyond the maximum value of $\Delta_r$ for $r \in I_{IRZ}$, namely 2.885, it follows that the $[1, 1]$ PA is finite for all $r \in I_{IRZ}$.

The $[0, 2]$ PA obviously has no zero, and has two poles, at
TABLE V. Values of \( \gamma^{\psi,\text{IR}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \) together with \( \gamma^{\psi,\text{IR,nF}} \) with \( n = 2, 3, 4 \) from Table V of [27] for comparison, as a function of \( r \) for \( r \in \text{IR}_{Z, R} \), and satisfying \( \gamma^{\psi,\text{IR}} < 2 \). Here, \( \Delta_\gamma = 5.5 - r \), as in Eq. (5.33). To save space, we omit the subscript \( \psi \) below, values that exceed the bound \( \gamma^{\psi,\text{IR}} < 2 \) from conformal invariance [see Eq. (4.14)] are marked as such.

| \( r \) | 2.8 | 3.0 | 3.2 | 3.4 | 3.6 | 3.8 | 4.0 | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 | 5.4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \gamma^{\text{IR,2}\psi} \) | 1.708 | 1.165 | 0.8540 | 0.6563 | 1.853 | 1.178 | 0.7847 | 0.5366 | 0.3707 | 0.2543 | 0.1696 | 0.1057 | 0.0562 | 0.01682 |
| \( \gamma^{\text{IR,3}\psi} \) | 0.1902 | 0.2254 | 0.2637 | 0.2933 | 0.5201 | 0.4197 | 0.3414 | 0.2771 | 0.2221 | 0.1735 | 0.1313 | 0.08886 | 0.05123 | 0.01637 |
| \( \gamma^{\text{IR,4}\psi} \) | 0.8701 | 0.7652 | 0.6683 | 0.5790 | 0.4969 | 0.4216 | 0.2877 | 0.2566 | 0.2173 | 0.1745 | 0.1333 | 0.09945 | 0.05156 | 0.01638 |
| \( \gamma^{\psi,\text{IR,a}} \) | 1.1127 | 0.9259 | 0.7731 | 0.6458 | 0.5383 | 0.4463 | 0.3528 | 0.2973 | 0.2326 | 0.1805 | 0.1338 | 0.09058 | 0.05163 | 0.01638 |
| \( \gamma^{\psi,\text{IR,nF}} \) | 1.1102 | 0.9244 | 0.7722 | 0.6453 | 0.5380 | 0.4461 | 0.3666 | 0.2972 | 0.2362 | 0.18205 | 0.1338 | 0.09058 | 0.05163 | 0.01638 |
| \( \gamma^{\psi,\text{IR,a}} \) | 0.0802 | 0.1057 | 0.1224 | 0.1497 | 0.1902 | 0.2254 | 0.2637 | 0.2933 | 0.5201 | 0.4197 | 0.3414 | 0.2771 | 0.2221 | 0.1735 |

The first of these, at \( \Delta_\gamma = 4.5425 \), is well beyond \( \Delta_\gamma^{\text{max}} = 2.885 \) so that the \{0,2\} PA is finite for all \( r \in \text{IR}_{Z, R} \), and the second is also irrelevant, since it corresponds to the value, \( r = 72 \), far beyond the AF interval, \( r \in [0, 34/13] \). The irrelevance of these poles in the Padé approximants is in agreement with the conclusion that we have reached from our other methods that the small-\( \Delta_\gamma \) expansion is reasonably reliable throughout the interval \( \text{IR}_{Z, R} \) and related non-Abelian Coulomb phase. In Table V we list our results for \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \) together with \( \gamma^{\psi,\text{IR,nF}} \) with \( n = 2, 3, 4 \) from [27] for comparison.

We find that if \( r \) is in the upper part of the interval \( \text{IR}_{Z, R} \), then there is excellent agreement between our higher-loop calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and the present calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \). As \( r \) decreases in this interval \( \text{IR}_{Z, R} \), the values of the anomalous dimension calculated in the various different ways begin to exhibit small deviations from each other, and, as expected, these deviations become larger as \( r \) descends toward the lower end of the interval \( \text{IR}_{Z, R} \).

\[
(\Delta_\gamma)_{\text{poles,0.2}}^{\psi,\text{IR}} = \frac{1875}{69914} (1323 \pm 17\sqrt{4605})
= 4.5425, 66.420. \quad (6.36)
\]

The calculation of the anomalous dimension \( \gamma^{\psi,\text{IR}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \) is done via the Padé approximants, and the values are consistent with the conformal invariant predictions. The small-\( \Delta_\gamma \) expansion is reliable throughout the interval \( \text{IR}_{Z, R} \), and related non-Abelian Coulomb phase. In Table V we list our results for \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \) together with \( \gamma^{\psi,\text{IR,nF}} \) with \( n = 2, 3, 4 \) from [27] for comparison.

We find that if \( r \) is in the upper part of the interval \( \text{IR}_{Z, R} \), then there is excellent agreement between our higher-loop calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and the present calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \). As \( r \) decreases in this interval \( \text{IR}_{Z, R} \), the values of the anomalous dimension calculated in the various different ways begin to exhibit small deviations from each other, and, as expected, these deviations become larger as \( r \) descends toward the lower end of the interval \( \text{IR}_{Z, R} \).

\[
(\Delta_\gamma)_{\text{poles,0.2}}^{\psi,\text{IR}} = \frac{1875}{69914} (1323 \pm 17\sqrt{4605})
= 4.5425, 66.420. \quad (6.36)
\]

The first of these, at \( \Delta_\gamma = 4.5425 \), is well beyond \( \Delta_\gamma^{\text{max}} = 2.885 \) so that the \{0,2\} PA is finite for all \( r \in \text{IR}_{Z, R} \), and the second is also irrelevant, since it corresponds to the value, \( r = 72 \), far beyond the AF interval, \( r \in [0, 34/13] \). The irrelevance of these poles in the Padé approximants is in agreement with the conclusion that we have reached from our other methods that the small-\( \Delta_\gamma \) expansion is reasonably reliable throughout the interval \( \text{IR}_{Z, R} \) and related non-Abelian Coulomb phase. In Table V we list our results for \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \) together with \( \gamma^{\psi,\text{IR,nF}} \) with \( n = 2, 3, 4 \) from [27] for comparison.

We find that if \( r \) is in the upper part of the interval \( \text{IR}_{Z, R} \), then there is excellent agreement between our higher-loop calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and the present calculations of \( \gamma^{\psi,\text{IR,a}}; \{1, 1\}^{\psi,\text{IR,a}} \), and \{1, 1\}^{\psi,\text{IR,a}} \). As \( r \) decreases in this interval \( \text{IR}_{Z, R} \), the values of the anomalous dimension calculated in the various different ways begin to exhibit small deviations from each other, and, as expected, these deviations become larger as \( r \) descends toward the lower end of the interval \( \text{IR}_{Z, R} \).

\[
(\Delta_\gamma)_{\text{poles,0.2}}^{\psi,\text{IR}} = \frac{1875}{69914} (1323 \pm 17\sqrt{4605})
= 4.5425, 66.420. \quad (6.36)
\]
\[
\kappa_{T,3} = \frac{4C_f T_f}{3C_A^4 D_s^5} \left[ 3C_A T_f \left\{ C_A^4 \left( -11319 + 188160 \zeta_3 \right) + C_A^3 C_f \left( -337204 + 64512 \zeta_3 \right) + C_A^2 C_f^2 \left( 83616 - 890112 \zeta_3 \right) + C_f^3 \left( 1385472 - 354816 \zeta_3 + 743424 \zeta_3 \right) \right\} - 512 T_f^2 D \left( -5 + 312 \zeta_3 \right) \frac{d_{\alpha \beta \gamma \delta} d_{\alpha \beta \gamma \delta}^*}{d_A} \right] - 15488 C_A^2 D \left( -11 + 24 \zeta_3 \right) \frac{d_{\alpha \beta \gamma \delta} d_{\alpha \beta \gamma \delta}^*}{d_A} + 11264 C_A T_f D \left( -4 + 39 \zeta_3 \right) \frac{d_{\alpha \beta \gamma \delta} d_{\alpha \beta \gamma \delta}^*}{d_A} \right].
\] (7.8)

We note that
\[
\kappa_{T,1} = -\frac{1}{3} \kappa_1. \tag{7.9}
\]

**B. Evaluation for \( G = SU(N_c) \) and \( R = F \)**

As we did with the \( \kappa_n \) coefficients, we exhibit the reduction of these general formulas for the gauge group \( G = SU(N_c) \) with \( N_f \) fermions in the representation \( R = F \). In accordance with Eq. (7.9), we obtain

\[
\kappa_{T,3,SU(N_c),F} = \frac{8(N_c^2 - 1)}{3^4 N_c^3 (25 N_c^2 - 11)^5} \times \left[ 23057 N_c^8 - 557686 N_c^6 + 1084692 N_c^4 - 354200 N_c^2 - 13310 + 192(25 N_c^2 - 11)(163 N_c^4 - 225 N_c^2 - 22) \zeta_3 \right]. \tag{7.12}
\]

The coefficient \( \kappa_{T,1,SU(N_c),F} \) is manifestly negative for all \( N_c \geq 2 \), and this is also true of \( \kappa_{T,2,SU(N_c),F} \), while we find that \( \kappa_{T,3,SU(N_c),F} \) is positive for all \( N_c \geq 2 \).

**C. LNN I for \( \gamma_{T,IR} \)**

Here we evaluate the \( \kappa_{T,n} \) and \( \gamma_{T,IR} \) in the LNN limit. The rescaled quantities that are finite in this limit are the analogues of those that we defined and studied for \( \gamma_{\hat{\psi}_\psi,IR} \) in Sec. VI F. We calculate

\[
\hat{\kappa}_{T,n} = \lim_{N_c \to \infty} N_c^n \kappa_{T,n} \tag{7.13}
\]

have the values

\[
\hat{\kappa}_{T,1} = -\frac{4}{3 \cdot 5^2} = -0.053333, \tag{7.14}
\]

\[
\hat{\kappa}_{T,2} = -\frac{1364}{3^2 \cdot 5^6} = -\left( 0.969956 \times 10^{-2} \right), \tag{7.15}
\]

and

\[
\hat{\kappa}_{T,3} = \frac{184456}{3^4 \cdot 5^{10}} = 2.3319 \times 10^{-4}. \tag{7.16}
\]

Hence, to third order in the rescaled quantity \( \Delta_r \), defined in Eq. (5.33), we have the following scheme-independent expansion for \( \gamma_{T,IR} \) in the LNN limit:

\[
\gamma_{T,IR} = \frac{4(N_c^2 - 1)}{3N_c (25 N_c^2 - 11)^3} \times \left[ 23057 N_c^8 - 557686 N_c^6 + 1084692 N_c^4 - 354200 N_c^2 - 13310 + 192(25 N_c^2 - 11)(163 N_c^4 - 225 N_c^2 - 22) \zeta_3 \right]. \tag{7.17}
\]

In Fig. 3 we plot \( \gamma_{T,IR,\Delta_r} \) for \( 1 \leq p \leq 3 \) as a function of \( r \) in the interval \( I_{IRZ,r} \). As a consequence of the fact that both \( \hat{\kappa}_{T,1} \) and \( \hat{\kappa}_{T,2} \) are negative, for a fixed value of \( r \), \( \gamma_{T,IR,\Delta_r} \) is negative and larger in magnitude than \( \gamma_{T,IR,\Delta_r'} \). Although

![Graph](image_url)
\( \kappa_{T,3} \) is positive, it is sufficiently small that for a given \( r \), the value of \( r, \gamma_{T,IR}^r, \Delta_r^\gamma \) is close to the value of \( r, \gamma_{IR}^r, \Delta_r^\gamma \).

D. Calculation of \( \gamma_{T,IR} \) to \( O(\Delta^3) \)

As another interesting comparison, we evaluate our general expressions for the \( \kappa_{T,n} \) in the special case where the gauge group is \( G = \text{SU}(3) \) and the fermion representation is \( R = F \). We find

\[
\kappa_{T,\text{SU}(3),F,1} = -\frac{16}{3^2 \cdot 107} = -(1.6615 \times 10^{-2}),
\]

\[
\kappa_{T,\text{SU}(3),F,2} = -\frac{37252}{3 \cdot 107^3} = -(1.12625 \times 10^{-3}),
\]

and

\[
\kappa_{T,\text{SU}(3),F,3} = -\frac{341234350}{3^2 \cdot (107)^7} + \frac{2855936}{3^6 \cdot (107)^7 \cdot 3^3} = 2.480155 \times 10^{-5}.
\]

Thus, the leading two terms in the \( \Delta_f \) expansion for \( J_2 \) are negative, with the coefficient of \( \Delta_f^3 \) being positive but smaller in magnitude. These results may be contrasted to those obtained in [9] for \( \kappa_n = \gamma_{\bar{\psi} \psi,IR}^n \) with \( 1 \leq n \leq 3 \) and in [11] for \( n = 4 \) for this \( \text{SU}(3) \) theory with \( R = F \), which are listed above in (6.11)–(6.14). We have computed ratios of the magnitudes of successive coefficients as before and again infer that the small-\( \Delta_f \) expansion can be reliable in the interval \( I_{\text{IRZ}} \).

E. Evaluation for \( R = \text{adj} \)

For \( G = \text{SU}(N_c) \) and \( R = \text{adj} \), our general results above reduce to

\[
\kappa_{T,1,\text{SU}(N_c),adj} = -\frac{4}{3^3} = -0.05333,
\]

\[
\kappa_{T,2,\text{SU}(N_c),adj} = -\frac{53}{2 \cdot 3^3} = -(1.2117 \times 10^{-2}),
\]

and

\[
\kappa_{T,3,\text{SU}(N_c),adj} = \frac{N_c^2(34799 - 9216\zeta_3)}{2^3 \cdot 3^{11} N_c^2} + 42624.
\]

This is positive for all physical \( N_c \) and has the large-\( N_c \) limit

\[
\lim_{N_c \to \infty} \kappa_{T,3,\text{SU}(N_c),adj} = \frac{34799 - 9216\zeta_3}{2^3 \cdot 3^{11}} = 0.0167381.
\]
APPENDIX: SERIES COEFFICIENTS FOR $\beta_\xi$ AND $\gamma_{\bar{\psi} \psi}$ IN THE LNN LIMIT

For reference, we list here the rescaled series coefficients for $\beta_\xi$ and $\gamma_{\bar{\psi} \psi}$ in the LNN limit (5.26). First, we recall that [3]

$$b_1 = \frac{1}{3} (11C_A - 4T_fN_f)$$  \hspace{1cm} (A1)

and [4]

$$b_2 = \frac{1}{3} [34C_A^2 - 4(5C_A + 3C_f)T_fN_f],$$  \hspace{1cm} (A2)

where $C_A$, $C_f$, and $T_f$ are group invariants [18]. It follows that in the LNN limit the $\hat{b}_\xi$ with $\varepsilon' = 1, 2$ are

$$\hat{b}_1 = \frac{1}{3} (11 - 2r)$$  \hspace{1cm} (A3)

and

$$\hat{b}_2 = \frac{1}{3} (34 - 13r).$$  \hspace{1cm} (A4)

The coefficients $b_3$ and $b_4$ have been calculated in the MS scheme [13,14]. With these inputs, one obtains [27]

$$\hat{b}_3 = \frac{1}{54} (2857 - 1709r + 112r^2)$$  \hspace{1cm} (A5)

and

$$\hat{b}_4 = \frac{150473}{486} - \left(\frac{485513}{1944}\right)r + \left(\frac{8654}{243}\right)r^2$$

$$+ \left(\frac{130}{243}\right)r^3 + \frac{4}{9} (11 - 5r + 21r^2)\zeta_3.$$  \hspace{1cm} (A6)

For the coefficients $\hat{c}_\xi$ in Eq. (6.24), one has [47] and references therein

$$\hat{c}_1 = 3,$$  \hspace{1cm} (A7)

$$\hat{c}_2 = \frac{203}{12} - \frac{5}{3} r.$$  \hspace{1cm} (A8)

$$\hat{c}_3 = \frac{11413}{108} - \left(\frac{1177}{54} + 12\zeta_3\right)r - \frac{35}{27} r^2.$$  \hspace{1cm} (A9)

and

$$\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81} r + \frac{899}{162} r^2 - \frac{83}{81} r^3$$

$$+ \left(\frac{1157}{9} - \frac{889}{3} r + 20r^2 + \frac{16}{9} r^3\right)\zeta_3$$

$$+ r(66 - 12r)\zeta_4 + (-220 + 160r)\zeta_5.$$  \hspace{1cm} (A10)

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The Casimir invariants $C_{ij}(R)$ and $T(R)$ are defined as $\sum_{\alpha} \delta_{ij}D_{\alpha}(T_{\alpha}^{a}) = C_{ij}(R)\delta_{ab}$ and $\sum_{\alpha} D_{\alpha}(T_{\alpha}^{a})^{i}D_{\alpha}(T_{\alpha}^{b}) = T(R)\delta_{ab}$, where $R$ is the representation, $T_{\alpha}$ are the generators of $G$, normalized according to $\text{Tr}(T_{\alpha}T_{\beta}) = (1/2)\delta_{\alpha\beta}$ and $D_{\alpha}$ is the matrix representation (Darstellung) of $R$. For the adjoint representation, we denote $C_{ij}(ad) = C_{ij}$, and for fermions transforming according to the representation $R$, we denote $C_{ij}(R) = C_{ij}$ and $T(R) = T_{ij}$.

Here we discuss the case $\alpha \geq 2$.

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In terms of the rescaled $F_{x,\nu}^{\mu}$, Eq. (5.2) is expressed equivalently as $T_{\nu}^{\mu} = (\beta/(16\pi r^{2}))F_{x,\nu}^{\mu}$. (The color in this paper.

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[65] We note the following misprint in [44]: in the last line of Eq. (7) of that paper, the term $-144T^2_F C_F^2$ should read $-144T^2_F N^2_F$. We thank J. A. Gracey for confirming this (private communication).