NAMBU STRUCTURES AND INTEGRABLE 1-FORMS

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Abstract. Some years ago Moshé Flato pointed out that it could be interesting to develop the Nambu’s idea ([13]) to generalize Hamiltonian mechanic. An interesting new formalism in that direction was proposed by L. Takhtajan ([14]). His theory gave new perspectives concerning deformation quantization, and many authors have developed its mathematical features ([2], [3]).

The purpose of this paper is to show that this theory, at first dedicated to physic, gives a new point of view for the study of singularities of integrable 1-forms.

Namely, we will prove that any integrable 1-form which vanishes at a point and has a non-zero linear part at this point is, up to multiplication by a non-vanishing function, the formal pull-back of a two dimensional 1-form. We also obtain a classification of quadratic integrable 1-forms.

Key words: generalized Poisson structures, singular foliations, integrable differential forms, normal forms

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To the memory of Moshé Flato

1. Generalities

In the reference [14], L. Takhtajan, in 1994, proposed a formalism which generalizes the Poisson bracket. Let \( M \) be a manifold and \( A \) the algebra of smooth functions on \( M \). A Nambu structure of order \( r \) on \( M \) is an \( r \)-linear skew-symmetric map

\[
A \times \cdots \times A \to A : \\
(f_1, \ldots, f_r) \mapsto \{f_1, \ldots, f_r\}
\]

which satisfies the following properties:

\[
\{f_1, \ldots, f_{r-1}, gh\} = \{f_1, \ldots, f_{r-1}, g\}h + g\{f_1, \ldots, f_{r-1}, h\} \tag{L}
\]

\[
\{f_1, \ldots, f_{r-1}, \{g_1, \ldots, g_r\}\} =
\]

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\[ \sum_{i=1}^{r} \{ g_{i}, \ldots, g_{i-1}, \{ f_{1}, \ldots, f_{r-1}, g_{i} \}, g_{i+1}, \ldots, g_{r} \} \] 

(FI)

for any \( f_{1}, \ldots, f_{r-1}, g, h, g_{1}, \ldots, g_{r} \) in \( A \).

In this definition \((L)\) stands for Leibniz property, \((FI)\) for fundamental identity or for Filippov’s identity (see [9]). For \( r = 2 \), \((FI)\) is just Jacobi’s identity, so a Nambu structure of order 2 is a Poisson structure.

The identity \((L)\) implies that \( X_{f_{1}, \ldots, f_{r-1}} : g \mapsto \{ f_{1}, \ldots, f_{r-1}, g \} \) is a derivation of \( A \), hence a vector field on \( M \): It is, by definition, the Hamiltonian vector field associated to \( f_{1}, \ldots, f_{r-1} \).

The identity \((L)\) also implies that there is an \( r \)-vector field \( \Lambda \) such that

\[ \{ f_{1}, \ldots, f_{r} \} = \Lambda(df_{1}, \ldots, df_{r}). \]

This \( \Lambda \) is called a Nambu tensor. We can also consider the usual vector fields as Nambu structures of order 1.

The identity \((FI)\) implies that Hamiltonian vector fields define an integrable distribution, like in Poisson’s case. So, we have on \( M \) a singular foliation which generalizes symplectic foliations of Poisson manifolds.

Since 1996 appeared three proofs of the following surprising result ([7], [1], [12]).

**Theorem 1.1** (Local Triviality Theorem). Let \( \Lambda \) be a Nambu tensor of order \( r > 2 \). Near any point at which \( \Lambda \) does not vanish there are local coordinates \( x_{1}, \ldots, x_{n} \) such that

\[ \Lambda = \frac{\partial}{\partial x_{1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{r}}. \]

In particular this theorem shows that there are only two types of leaf for the foliation associated to \( \Lambda \): Either it reduces to a point (zero of \( \Lambda \)) or it is \( r \)-dimensional.

This theorem leads to a “covariant” presentation of Nambu tensors. Suppose that we have a volume form \( \Omega \) on our manifold \( M \). Set \( \omega := i_{\Lambda}\Omega \). Then we have the following result ([8]).

**Theorem 1.2.** Suppose \( \Lambda \) is a \( r \)-vector on \( M \) such that either \( r > 2 \) or \( r = 2 \) but, in this case, maximal rank of \( \Lambda \) is 2. If \( r \) is equal to the dimension \( n \) of \( M \), then \( \Lambda \) is always a Nambu tensor. When \( r < n \), \( \Lambda \) is a Nambu tensor if and only if we have

\[ i_{A}\omega \wedge \omega = 0 \]

\[ i_{A}\omega \wedge d\omega = 0 \]

for every \((n-r-1)\)-vector \( A \).

The first relation in this theorem says that \( \omega \) is decomposable at each point, the second is an “integrability” condition. In the case \( r = n - 1 \), \( \omega \) is just an integrable 1-form, i.e., a 1-form such that \( \omega \wedge d\omega = 0 \). In the case \( r < n - 1 \), \( \omega \) can be called an integrable \((n-r)\)-form, see [10]. Roughly speaking, this theorem says that a Nambu structure (or a Poisson structure of maximal rank 2) is exactly the “dual” of an integrable \( p \)-form.

For Nambu structures there is an analogous of the so called modular vector field ([13], [14]) which can be defined as follows.
Definition 1.3. Let $\Lambda$ be a Nambu tensor of order $r$ and $\Omega$ be a volume form on the manifold $M$. The modular tensor of $\Lambda$ with respect to $\Omega$ is the tensor field $D_\Omega \Lambda$ defined by the formula

$$i_{D_\Omega \Lambda} \Omega = d(i_\Lambda \Omega).$$

Using the local triviality theorem we can prove the following results.

Theorem 1.4. The modular tensors are also Nambu tensors. If $\Lambda$ is a Nambu tensor of order $r$ with $r > 2$ or with $r = 2$, but with maximal rank 2, then we have, for any volume form $\Omega$, for every $s, s = 0, 1, \ldots, r - 2$, and for any smooth functions $g_1, \ldots, g_s$, the following properties

1) $i_{(dg_1 \land \ldots \land dg_s)} D_\Omega \Lambda \land \Lambda = 0$,

2) $[i_{(dg_1 \land \ldots \land dg_s)} D_\Omega \Lambda, \Lambda] = 0$, where the bracket $[,]$ is the Schouten bracket.

Note that the property 2)(with $s = 0$) remains valid for any Poisson tensor, even if its maximal rank is more than 2.

2. The Kupka phenomenon

The Kupka phenomenon ([8]) is the following: If $\omega$ is an integrable 1-form such that $d\omega$ is non zero at a point, then near this point there are local coordinates $x_1, \ldots, x_n$ such that $\omega$ depends only on two variables, i.e., we have

$$\omega = a(x_1, x_2) dx_1 + b(x_1, x_2) dx_2.$$

Using the fact that integrable 1-forms are the “duals” of Nambu tensors of order $n - 1$ ($n$ is the dimension of the ambiant manifold), we could rewrite this result in terms of Nambu tensors, but, hereafter, we will give a generalization of this result. For this we will use the following vocabulary.

Definition 2.1. Let $A$ be a Nambu tensor. We will say that $A$ is of type $2.r$ if there are $r$ commuting and everywhere linearly independent vector fields $X_1, \ldots, X_r$ such that we have

$$X_i \land A = 0$$

$$[X_i, A] = 0$$

for every $i = 1, \ldots, r$([,] is the Schouten bracket).

Remark 2.2. Locally this means that there are local coordinates $x_1, \ldots, x_n$ such that

$$A = \partial/\partial x_1 \land \cdots \land \partial/\partial x_r \land B$$

where $B$ is a Nambu tensor independent of the coordinates $x_1, \ldots, x_r$.

Theorem 2.3 (generalized Kupka phenomenon). Let $\Lambda$ be a Nambu tensor and $\Omega$ a volume form. If $D_\Omega \Lambda$ is a.e. non zero and is of type $2.r$ in a neighborhood of a point $m$ then $\Lambda$ is also of type $2.r$ in a (possibly different) neighborhood of $m$. 
Proof. We can choose local coordinates \((x_1, \ldots, x_n)\) such that \(X_i = \partial/\partial x_i\) for \(i = 1, \ldots, r\) and \(\Omega = dx_1 \wedge \cdots \wedge dx_n\). Then we have

\[
D_\Omega \Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_r \wedge Y
\]

where

\[
Y = \sum Y_{i_1 \cdots i_{q-1-r}} \partial/\partial x_{i_1} \wedge \cdots \wedge \partial/\partial x_{i_{q-1-r}}
\]

is a \((q-1-r)\)-tensor field independent of \(x_1, \ldots, x_r\). Since \(D_\Omega \Lambda\) is a.e. non zero we can suppose that \(Z := Y_{r+1} \cdots (q-1)\) is a.e. non zero.

Set \(\nu = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{q-1}\). We have \(i_\nu(D_\Omega \Lambda) = Z \partial/\partial x_i\).

The relation 1) of theorem 1.4 implies \(\partial/\partial x_i \wedge \Lambda = 0\). The latter relation holds for \(i = 1, \ldots, r,\) so we obtain

\[
\Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_r \wedge P,
\]

where \(P\) is a \((q-r)\)-tensor field.

Since \(Z\) is independent of \(x_1, \ldots, x_r\), the relation 2) of theorem 1.4 implies that \([\partial/\partial x_i, P] = 0\). It follows that \(P\) is independent of \(x_1, \ldots, x_r\). This ends the proof of our theorem. \(\triangle\)

Let \(\Lambda\) be the Nambu tensor of order \(n-1\) associated with an integrable 1-form \(\omega\), such that \(d\omega \neq 0\) at a point \(m\). Then the modular tensor of \(\Lambda\) is non-zero at \(m\) and the local triviality theorem for regular Nambu structures says that it is locally of the form \(\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-2}\), so it is of type 2.\((n-2)\). The theorem above says that \(\Lambda\) is also of type 2.\((n-2)\). According to the preceding remark we have, locally,

\[
\Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-2} \wedge B
\]

where \(B\) is independent of the coordinates \(x_1, \ldots, x_{n-2}\). Therefore, up to multiplication by a non-vanishing function, \(\omega\) depends only on 2 coordinates. It is easy to see that the latter remains true without multiplication by a nonvanishing function under a suitable choice of the involved volume form. Therefore our theorem can be thought as a generalization of the Kupka phenomenon.

For example, the formulated theorem has the following corollary (which can be proved directly).

**Theorem 2.4.** Let \(\omega\) be an integrable 1-form on \(\mathbb{R}^n\) or \(\mathbb{C}^n\). If \(d\omega\) is a.e. non zero and depends on less than \(s\) coordinates in a neighborhood of 0 then we have the same for \(\omega\).

### 3. Nambu tensors of order \(n-1\) with a non-zero linear part

In this section we give a formal normal form for Nambu tensors of order \(n-1\), vanishing at a point \(m\), but with a non-zero linear part at that point; it generalizes the one we gave in [6] for the 3 dimensional case.

We will distinguish the following two cases.

The simple case is the one where the modular tensor doesn’t vanish: In that case by “Kupka phenomenon” our Nambu tensor has the local form

\[
\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-2} \wedge X
\]
where $X$ is a vector field independent of the coordinates $x_1, \ldots, x_{n-1}$. Thus the local classification of Nambu tensors reduces to that of 2-dimensional vector fields, up to orbital equivalence.

The difficult case is the one where the modular tensor vanishes at $m$. In this case we have the following theorem.

**Theorem 3.1.** Let $\Lambda$ be a Nambu tensor of order $n-1$ on an $n$-dimensional manifold with $n \geq 3$. Suppose that $\Lambda$ vanishes at a point $m$, but has a non-zero linear part at this point. Suppose also that the modular tensors of $\Lambda$ vanish at $m$. Then there are local coordinates $x_1, \ldots, x_n$, in a neighborhood of $m$ such that

$$\{x_1, \ldots, x_{n-1}\} = x_n$$

$$\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\} = (-1)^{n-i}(\partial f/\partial x_i + x_n \partial g/\partial x_i) + \epsilon_i,$$

for $i = 1, \ldots, n-1$, where $f$ and $g$ are smooth functions, independent of $x_n$, such that $df \wedge dg = 0$, and $\epsilon_i$ is a smooth flat function at the origin (i.e., his Taylor expansion vanishes at $m$.)

The sequel of this section is dedicated to the proof of this theorem.

**Study of the linear part of $\Lambda$.** According to [5] the linear part $\Lambda^{(1)}$ has, in a suitable coordinates system, one of the following normal forms.

**Type 1:**

$$\Lambda^{(1)} = \sum_{i=1}^{r} \pm x_i \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_n,$$

which corresponds to a linear integrable 1-form of type $d(\sum_{i=1}^{r} \pm x_i^2/2)$.

**Type 2:**

$$\Lambda^{(1)} = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-2} \wedge X^{(1)},$$

where $X^{(1)}$ is a zero-trace linear vector field depending only on $x_{n-1}$ and $x_n$. This normal form corresponds to a linear integrable 1-form depending only on $x_{n-1}$ and $x_n$.

An elementary calculation shows that, in each of the cases, there are (possibly) new linear coordinates with

$$(1) \quad \{x_1, \ldots, x_{n-1}\}^{(1)} = x_n$$

for the linear Nambu structure determined by $\Lambda^{(1)}$. This means that the associated 1-form is of type $x_n dx_n + \sum_{i=1}^{n-1} t_i dx_i$.

**Remark 3.2.** In fact the preceding theorem is true for every case where one can find coordinates satisfying $(1)$. The only case where this is not so is the type 2 case with $X^{(1)}$ equivalent to $x_{n-1} \partial/\partial x_{n-1} + x_n \partial/\partial x_n$.

In the sequel of the proof of theorem 3.1 we will use following notations:

$$x := (x_1, \ldots, x_{n-1}), \quad y := x_n.$$ 

We also develop the function $h(x_1, \ldots, x_n) =: h(x, y)$ in the form

$$h^{(0)} + h^{(1)} + \cdots + h^{(p)} + \cdots,$$
where \( h^{(p)} \) is a \( p \)-homogeneous polynomial in \( x_1, \ldots, x_{n-1} \) with coefficients depending smoothly on \( y \) (\( y \) is considered as a parameter).

**Lemma 3.3.** Let \( r \geq 0 \). Suppose that there are coordinates \( x = (x_1, \ldots, x_{n-1}) \) and \( y \) such that

\[
\begin{align*}
\{x_1, \ldots, x_{n-1}\} &= y + c^{(r+2)}(x, y) + c^{(r+3)}(x, y) + \cdots \\
\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, y\} &= (-1)^{n-i}(a_i^{(0)}(x, y) + a_i^{(1)}(x, y) + \cdots)
\end{align*}
\]

where \( a_i^{(0)}, \ldots, a_i^{(r-1)} \) are affine with respect to \( y \) (vacuous hypothesis for \( r = 0 \)). There is a coordinates transformation of the form

\[
\begin{align*}
x'_1 &= x_1 + \mu^{(r+2)}(x, y) \\
x'_2 &= x_2, \ldots, x'_{n-1} = x_{n-1} \\
y' &= y + \gamma^{(r+1)}(x, y) + \gamma^{(r+2)}(x, y)
\end{align*}
\]

which gives

\[
\begin{align*}
\{x'_1, \ldots, x'_{n-1}\} &= y' + C^{(r+3)}(x', y') + C^{(r+4)}(x', y') + \cdots \\
\{x'_1, \ldots, x'_{i-1}, x'_{i+1}, \ldots, x'_{n-1}, y'\} &= (-1)^{n-i}(a_i^{(0)}(x', y') + \cdots + a_i^{(r-1)}(x', y') + A_i^{(r)}(x', y') + A_i^{(r+1)}(x', y') + \cdots)
\end{align*}
\]

where \( A_i^{(r)} \) is affine in \( y' \).

**Proof of the lemma.** Make a coordinates transformation of the form \( \tilde{x} = x, \tilde{y} = y(1 + c^{(r+1)}(x, y)) \). We obtain

\[
\begin{align*}
\{\tilde{x}_1, \ldots, \tilde{x}_{n-1}\} &= \tilde{y} + \tilde{y}(c^{(r+1)}(\tilde{x}, \tilde{y}) + c^{(r+2)}(\tilde{x}, \tilde{y}) + \cdots \\
\{\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_{n-1}, \tilde{y}\} &= (-1)^{n-i}(a_i^{(0)}(\tilde{x}, \tilde{y}) + \cdots + a_i^{(r-1)}(\tilde{x}, \tilde{y}) + A_i^{(r)}(\tilde{x}, \tilde{y}) + \cdots)
\end{align*}
\]

with

\[
A_i^{(r)} = a_i^{(r)} - y^2 \partial e^{r+1}/\partial x_i.
\]

Now denoting \( \Omega = dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy \), we have \( \omega := i_\Lambda \Omega = \Gamma dy + \sum_i \Delta_i dx_i \) with

\[
\Gamma = \{x_1, \ldots, x_{n-1}\}, \quad \Delta_i = (-1)^{n-i}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}\}.
\]

Recall that we have \( \omega \wedge d\omega = 0 \). The terms with \( dx_i \wedge dx_j \wedge dy \) in this last equation give

\[
\Gamma(\partial \Delta_i/\partial x_j - \partial \Delta_j/\partial x_i) + \Delta_i(\partial \Delta_j/\partial y - \partial \Gamma/\partial x_j) - \Delta_j(\partial \Delta_i/\partial y - \partial \Gamma/\partial x_i) = 0.
\]

Express \( \Delta_k \) in the form \( \Delta_k = \alpha_k(x) + y\beta_k(x) + y^2\delta_k(x, y) \). Our hypothesis says that \( \delta_k \) have developments \( \delta_k^{(r)} + \delta_k^{(r+1)} + \cdots \). Now, if we compare terms with \( y^3 \) and of order \( r-1 \) in the preceding equation, we get

\[
\partial \delta_i^{(r)}/\partial x_j - \partial \delta_j^{(r)}/\partial x_i = 0.
\]

Equation (2) can be rewritten in the form

\[
A_i^{(r)} = a_i^{(r)} + \beta_i^{(r)} + y^2(\delta_i^{(r)} - \partial e^{r+1}/\partial x_i).
\]
By the Poincaré lemma we can choose $e^{(r+1)}$ such that $A_i^{(r)}$ are affine in $y$ (we erase $\delta_i^{(r)}$).

Now (after this coordinates transformation) we can suppose
$$\Gamma = \{x_1, \ldots, x_{n-1}\} = y + y e^{(r+1)}(x, y) + e^{(r+2)}(x, y) + \cdots$$
$$\Delta_i = (-1)^{n-i}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, y\} = a_i^{(0)}(x, y) + a_i^{(1)}(x, y) + \cdots,$$
where $a_i^{(s)}$ are affine in $y$ for $s = 0, \ldots, r$.

In a second step we use a coordinates transformation of the form $\tilde{x}_1 = x_1 + \theta^{(r+2)}(x, y)$, $\tilde{x}_2 = x_2, \ldots, \tilde{x}_{n-1} = x_{n-1}, \tilde{y} = y$ with $\partial\theta^{(r+2)}/\partial x_1 = -e^{(r+1)}$. Then we obtain
$$\{\tilde{x}_1, \ldots, \tilde{x}_{n-1}\} = \tilde{y} + +e^{(r+2)}(\tilde{x}, \tilde{y}) + \cdots$$
$$\{\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_{n-1}, \tilde{y}\} = (-1)^{n-i}(a_i^{(0)}(\tilde{x}, \tilde{y}) + \cdots + a_i^{(r)}(\tilde{x}, \tilde{y}) +$$
$$A_i^{(r+1)}(\tilde{x}, \tilde{y}) + \cdots).$$

Now we can suppose
$$\Gamma = y + e^{(r+2)} + \cdots$$
$$\Delta_i = a_i^{(0)} + \cdots,$$
where the $a_i^{(s)}$ are affine in $y$ for $s = 0, \ldots, r$.

Finally, to achieve the proof of the lemma, it suffices to perform a coordinates transformation $\tilde{x} = x$, $\tilde{y} = y + e^{(r+2)}$. \Halmos

We continue the proof of theorem 3.1.

Since we have (1), we can take $\{x_1, \ldots, x_{n-1}\}$ as a new variable $y$ to get
$$\{x_1, \ldots, x_{n-1}\} = y.$$ Then the hypothesis of lemma 3.3 holds for $r = 0$. We can apply inductively this lemma to show that, after a formal coordinates transformation (the formal composition of the coordinates transformations given by the lemma), we obtain
$$\{x_1, \ldots, x_{n-1}\} = y$$
$$\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, y\} = (-1)^{n-i}A_i,$$
where the functions $A_i$ have formal developments $A_i^{(0)} + A_i^{(1)} + \cdots$ with all terms here being affine in $y$. Therefore we can suppose that we have formally
$$\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, y\} = (-1)^{n-i}(a_i(x) + y\beta_i(x))$$
for $i = 1, \ldots, n - 1$.

Set $\Omega = dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy$. Then the associated integrable 1-form $\omega$ has the form
$$\omega = \sum_{i=1}^{n-1} (a_i(x) + y\beta_i(x)) dx_i + ydy.$$
The equation $\omega \wedge d\omega = 0$ implies
$$\alpha_i\beta_j - \alpha_j\beta_i \pm (\partial\alpha_i/\partial x_j - \partial\alpha_j/\partial x_i) \pm (\partial\beta_i/\partial x_j - \partial\beta_j/\partial x_i) = 0.$$ So we obtain, for every $i$ and $j$,
$$\partial\alpha_i/\partial x_j = \partial\alpha_j/\partial x_i, \ \partial\beta_i/\partial x_j = \partial\beta_j/\partial x_i, \ \alpha_i\beta_j = \alpha_j\beta_i.$$
The Poincaré lemma gives $\alpha_i = \partial f/\partial x_i$, $\alpha_i = \partial g/\partial x_i$, for every $i$.

Therefore the latter equations leads to $df \wedge dg = 0$. This ends the proof of theorem $\ref{thm3.1}$.\hfill\triangle$

Theorem $\ref{thm3.1}$ has the following consequence concerning integrable 1-forms.

**Theorem 3.4.** Let $\omega$ be an integrable 1-form which vanishes at a point $m$ and has a non-zero linear part at this point. Then, up to multiplication by a non-vanishing function, $\omega$ is, formally, the pullback of an integrable 1-form depending only on 2 variables.

**Proof.** If $d\omega$ is non-zero, we can apply the Kupka phenomenon. If $d\omega$ vanishes at $m$ then the Nambu vector associated to $\omega$ has the formal form of theorem $\ref{thm3.1}$. So we can suppose that

$$\Lambda = y\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-1} + (-1)^{n-1}(\partial f/\partial x_i + \sum y\partial g/\partial x_i)\partial/\partial x_1 \wedge \cdots$$

$$\cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_{n-1} \wedge \partial/\partial y.$$

Therefore

$$\omega = df + ydg + ydy$$

up to multiplication by a non-vanishing function (the Jacobian of the change of coordinates).

Since we also have $df \wedge dg = 0$ we can apply the result of $[11]$ to exhibit a function $h(x)$ such that

$$f = a \circ h, \quad g = b \circ h$$

(at least at the level of formal series; here $a$ and $b$ are functions in one variable). Then we have

$$\omega = (a'(h) + yb'(h))dh + ydy = \phi^*\omega_2$$

with $\omega_2 = (a'(u) + vb'(u))du + vdv$ and $\phi : (x, y) \mapsto (h(x), y)$. This ends the proof of the theorem.\hfill\triangle$

**Remark 3.5.** Theorems $\ref{thm3.1}$ and $\ref{thm3.4}$ give only formal normal forms for $(n-1)$ order Nambu structures or integrable 1-forms. We do not know if there are smooth or analytic versions.

**Remark 3.6.** In fact theorem $\ref{thm3.4}$ can be proven directly (without using Nambu formalism). The crucial point of the proof is that, up to multiplication by a non-vanishing function, an integrable 1-form $ydy + \sum A_i dx_i$ is formally equivalent to a form $ydy + \alpha_0 + y\alpha_1$, where $\alpha_0$ and $\alpha_1$ are 1-forms depending on $x_1, \ldots, x_{n-1}$ only. This result has the following generalization.

**Theorem 3.7.** Let $\omega = y^p dy + \sum_{i=1}^{n-1} A_i dx_i$ be an integrable 1-form on $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then, up to multiplication by a non-vanishing function, $\omega$ is formally equivalent to an integrable 1-form

$$\omega_0 = y^p dy + \sum_{i=0}^{p} y^i \alpha_i$$

where $\alpha_i$ are 1-forms depending only on $x_1, \ldots, x_{n-1}$. 
Proof. We consider the associated Nambu tensor
\[ \Lambda = y^p \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-1} + \sum (-1)^{n-i} A_i \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-1} \wedge \partial/\partial x_{n+1} \wedge \cdots \wedge \partial/\partial x_{n-1} \wedge \partial/\partial y. \]

If \( A_i^{(r)} \) are all polynomials of degree \( p \) in \( y \), with coefficients depending on \( x \), then we can apply exactly the same method as in the first step of the proof of lemma [3] to bring \( A_i^{(r)} \) to a polynomial in \( y \) of degree \( p \). In order to get this, we make a coordinates transformation \( \tilde{x} = x, \tilde{y} = y(1 + c^{(r+1)}) \) with the notations of the proof of this lemma. Then
\[
\{ \tilde{x}_1, \ldots, \tilde{x}_{n-1} \} = \tilde{y}^p(1 + \tilde{c}^{(r+1)}(\tilde{x}, \tilde{y}) + \cdots)^p
\]
\[
\{ \tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_{n-1}, \tilde{y} \} = (-1)^{n-i} A_i(1 + c^{(r+1)} - y\partial c^{(r+1)}/\partial y) - \]
\[
(-1)^{n-i} \tilde{y}^{p+1} \partial c^{(r+1)}/\partial x_i = (-1)^{n-i} (A_i^{(1)}(\tilde{x}, \tilde{y}) + \cdots + A_i^{(r-1)}(\tilde{x}, \tilde{y}) - \tilde{y}^{p+1} \partial c^{(r+1)}/\partial x_i + \tilde{A}_i^{(r+1)} + \cdots)
\]
Now we develop \( A_i^{(r)} \) in the form
\[
\alpha_{i,0}^{(r)} + y \alpha_{i,1}^{(r)} + \cdots + y^p \alpha_{i,p}^{(r)} + y^{p+1} \tilde{y}^{(r)}
\]
where \( \alpha_{i,j}^{(r)} \) depends only on \( x \) for \( j = 0, \ldots, p \).

The identity \( \omega \wedge d\omega = 0 \) implies that
\[
y^p(\partial A_i/\partial x_j - \partial A_j/\partial x_i) + A_i \partial A_j/\partial y - A_j \partial A_i/\partial y = 0
\]
and
\[
\partial \delta_i^{(r)}/\partial x_j - \partial \delta_j^{(r)}/\partial x_i = 0.
\]
Therefore we can choose \( c^{(r+1)} \) such that
\[
\delta_i^{(r)} = \partial c^{(r+1)}/\partial x_i
\]
for all \( i \) and then
\[
\{ \tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_{n-1}, \tilde{y} \} = (-1)^{n-i}(A_i^{(1)} + \cdots + A_i^{(r)} + \cdots)
\]
with \( A_i^{(s)} \) polynomial of degree \( p \) in \( y \) for \( s = 0, \ldots, r \).

To complete the proof, choose \( \tilde{\omega} = d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_{n-1} \wedge d\tilde{y} \). Then \( \omega = iA \tilde{\omega} \) is equal to \( \omega \) multiplied by a function of type \( 1 + u^{(r+1)} + \cdots \) and we have
\[
\tilde{\omega} = y^p(1 + \tilde{c}^{(r+1)} + \cdots)dy + \sum A_i dx_i.
\]
We can multiply \( \tilde{\omega} \) by the inverse of \( (1 + \tilde{c}^{(r+1)} + \cdots) \) to get
\[
\omega' = y^p dy + \sum A_i' dx_i,
\]
where \( A_i' = A_i^{(1)} + \cdots + A_i^{(r)} + A_i^{(r+1)} + \cdots \). So, step by step, we obtain the proof of our theorem. \( \square \)

In the case \( p = 2 \) the last theorem can be improved. The integrability condition \( \omega_0 \wedge d\omega_0 = 0 \) is equivalent to the system of equations:
\[
d\alpha_1 = d\alpha_2 = 0, \quad d\alpha_0 = \alpha_2 \wedge \alpha_1, \quad \alpha_0 \wedge \alpha_1 = \alpha_2 \wedge \alpha_0 = 0.
\]
So we can write, at the level of formal series, \( \alpha_1 = dg, \alpha_2 = dh \) and, since \( d(\alpha_0 - h\alpha_1) = 0 \), we have \( \alpha_0 = dk + hdg \) for some function \( k \). Now, the last two equations of our system give \( dk \wedge dg = 0 \) and \( dg \wedge dh = 0 \). Using Moussu’s result ([1]), we
can conclude that there is a function \( f \) whose formal series satisfies the relations 
\[ g = a \circ f, \quad h = b \circ f \quad \text{and} \quad k = c \circ f, \]
for some functions \( a, b \) and \( c \) in one variable. So we obtain
\[ \omega_0 = y^2 dy + (c'(f) + b(f)a'(f) + ya'(f) + y^2b'(f))df. \]
This can be interpreted as follows: \( \omega_0 \) is the formal pullback of a 2-dimensional 1-form \( y^2 dy + (\gamma_0(x) + y\gamma_1(x) + y^2\gamma_2(x))dx \) by a mapping of the form \((x_1, \ldots, x_{n-1}), x_n) \mapsto (f(x_1, \ldots, x_{n-1}), x_n).\)

It seems that \( \omega_0 \) is a formal pullback of a 2-dimensional 1-form for any value of \( p. \)

4. Quadratic integrable 1-forms

In this paragraph we will give a classification of quadratic integrable 1-forms or, equivalently, a classification of quadratic Nambu tensors of order \( n - 1 \), up to multiplication by a constant.

Let \( \Lambda \) be such a quadratic Nambu tensor of order \( n - 1 \). Its modular tensor \( DA \) relatively to any constant volume form is intrinsically defined and it is a linear Nambu tensor of order \( n - 2 \). The classification of linear Nambu tensors ([5]) says that we have the following two cases.

1- \( DA \) is of type 2:

This means that we have, in a suitable coordinates system,
\[ DA = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n \wedge X \]
where \( X \) is a vector field depending on coordinates \( x_1, x_2 \) and \( x_3 \) only.

With the notation introduced in definition 2.1, \( DA \) is of type 2. \((n - 3)\). So, due to the generalized Kupka phenomenon (theorem 2.3), \( \Lambda \) is also of type 2. \((n - 3)\). Then we have
\[ \Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n \wedge \Lambda_3, \]
where \( \Lambda_3 \) is a quadratic Poisson structure depending on the variables \( x_1, x_2 \) and \( x_3 \) only. We see that the classification of these Nambu structures reduces to the classification of quadratic 3-dimensional Poisson structures. The latter classification is known (see [4]).

2- \( DA \) is of type 1:

In this case it is easier to work with the associated quadratic integrable 1-form \( \omega \); \( DA \) is of type 1 if we have \( d\omega = dx \wedge dq \) where \( q \) is a quadratic form of type 
\[ q = \sum_{i=1}^r \pm y_i^2 / 2 + xz \]
in a system of coordinates \( x, y_1, \ldots, y_r, z, t_1, \ldots, t_s \) with \( r + s = n - 2 \) or \( q = \sum_{i=1}^r \pm y_i^2 / 2 \) in a system of coordinates \( x, y_1, \ldots, y_r, t_1, \ldots, t_s \) with \( r + s = n - 1 \). In the sequel we will consider the first case with \( r \geq 2 \). The other cases, with \( r = 0, r = 1 \) or without variable \( z \) are easier, and we let them to the reader.

Since we have \( d\omega = -d(qdx) \), we can express \( \omega \) in the form \( \omega = -qdx + df \), where \( f \) is a homogeneous function of degree 3. Denote \( \overline{q} = \sum_{i=1}^r \pm y_i^2 / 2 \). Then we have
\[ 0 = \omega \wedge d\omega = df \wedge dx \wedge dq = \]

\[
(\sum_i \frac{\partial f}{\partial y_i} dy_i + \frac{\partial f}{\partial t_j} dt_j + \sum_j \frac{\partial f}{\partial t_j} dt_j) \wedge dx \wedge (d\theta + x dz).
\]

The terms with \(dt_j \wedge dx \wedge dy_i\) in this relation give \(\frac{\partial f}{\partial t_j} = 0\), therefore \(f\) is independent of \(t_j\). The terms with \(dy_j \wedge dx \wedge dy_i\) give
\[
\sum_i \frac{\partial f}{\partial y_i} dy_i \wedge d\theta = 0,
\]
and an elementary calculation leads to the relation
\[
f = (\lambda x + \mu z)\theta + b(x, z).
\]

Using this relation we obtain
\[
0 = ((\mu \theta + \partial b/\partial z) dz + (\lambda x + \mu z) d\theta) \wedge dx \wedge (d\theta + x dz)
= ((\lambda x + \mu z)x - \mu \theta - \partial b/\partial z) d\theta \wedge dx \wedge dz.
\]

Considering the terms with \(y_i\) in the latter relation, we obtain, step-by-step: \(\mu = 0\), \(\partial b/\partial z = \lambda x^2\), \(b = \lambda x^2 z + \alpha x^3\), and finally \(f = \lambda x + \alpha x^3\), where \(\alpha\) is a constant.

Returning to the expression of \(\omega\), we get
\[
\omega = \theta q dx + \beta x dq + \gamma x^2 dx
\]
where \(\theta\), \(\beta\) and \(\gamma\) are constants.

The preceding calculations are summarized in the following theorem.

**Theorem 4.1.** If \(\omega\) is a quadratic integrable 1-form, it is the pull-back of a 3-dimensional integrable 1-form. More precisely, if \(d\omega\) is of type 2, then \(\omega\) is a quadratic integrable 1-form depending only on three (well chosen) coordinates; if \(d\omega\) is of type 1, then, in a suitable system of coordinates,
\[
\omega = \phi^*((\gamma x^2 + \theta y) dx + \beta x dy)
\]
with
\[
\phi(x_1, \ldots, x_n) = (x_1, \sum_{i=1}^r \pm y_i^2 / 2 + \epsilon x)
\]
and \(\epsilon, \beta, \theta\) and \(\gamma\) being constants \((\epsilon = 0\) or \(\epsilon = 1\)). In this last case \(\omega\) is, in fact, the pull-back of a 2-dimensional 1-form.

**Conjecture.** The preceding section and the theorem above lead to the following conjecture: Every integrable 1-form on \(\mathbb{R}^n\) or \(\mathbb{C}^n\) with a non-zero 2-jet at 0 is, up to multiplication by a non-vanishing function, the pull-back of an integrable 1-form in dimension 3. More generally we can ask if every integrable 1-form on \(\mathbb{R}^n\) or \(\mathbb{C}^n\) with a non-zero \(q\)-jet at 0 is, up to multiplication by a non-vanishing function, the pull-back of an integrable 1-form in dimension \(q + 1\).

**References**

[1] D. Alekseevsky, P. Guha., *On decomposability of Nambu-Poisson tensor*, Acta Math. Univ. Comenianae, 65 (1996), 1-10
[2] G. Dito, M. Flato, *Generalized abelian deformations: application to Nambu mechanics*, Lett. Math. Phys., 39 (1997), 107-125.
[3] G. Dito, M. Flato, D. Sternheimer, L. Takhtajan, *Deformation quantization and Nambu mechanics*, Commun. Math. Phys., 183 (1997), 1-22.
[4] J. P. Dufour, A. Haraki, *Rotationnel et structures de Poisson quadratiques*, C.R. Acad. Sci. Paris, 312 I (1991), 137-140.
[5] J.P. Dufour, Nguyen T.Z., Linearisation of Nambu structures, Compositio Math., 117 (1999), 77-98.
[6] J. P. Dufour, M. Zhitomirskii, Singularities and bifurcations of 3-dimensional Poisson structures, Preprint math.DG/9802115. To appear in Israel journal of Mathematics.
[7] Ph. Gautheron, Some remarks concerning Nambu mechanics, Lett. Math. Phys., 37 (1996), 103-116.
[8] I. Kupka, The singularities of integrable structurally stable Pfaffian forms, Proc. Nat. Acad. Sci. USA, 52 (1964), 1431-1432.
[9] G. Marmo, G. Vilasi, A.M. Vinogradov, The local structure of n-Poisson and n-Jacobi manifolds, J. of Geom. and Phys. 25 (1998) 141-182.
[10] A. Medeiros, Structural stability of integrable differential forms, Lecture Notes in Math., 597 (1977), 395-428.
[11] R. Moussu, Sur l’existence d’intégrales premières pour un germe de forme de Pfaff, Ann. Inst. Fourier, 26 (2) (1976), 171-220.
[12] N. Nakanishi, On Nambu-Poisson manifolds, Reviews Math. Phys., 10 (1998), 499-510.
[13] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev., D7 (1973), 2405-2412.
[14] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys., 160 (1994), 295-315.
[15] A. Weinstein, The Modular Automorphism Group of a Poisson Manifold, J. Geom. Phys., 23 (1997), 379-374.

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