Manifolds of Projective Shapes

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Abstract

The projective shape of a configuration of $k$ points or “landmarks” in $\mathbb{RP}^d$ consists of the information that is invariant under projective transformations and hence is reconstructable from uncalibrated camera views. Mathematically, the space of projective shapes for these $k$ landmarks can be described as the quotient space of $k$ copies of $\mathbb{RP}^d$ modulo the action of the projective linear group $\text{PGL}(d)$. The main purpose of this paper is to give a detailed examination of the topology of projective shape space, and it is shown how to derive subsets that are in a certain sense maximal, differentiable Hausdorff manifolds which can be provided with a Riemannian metric. A special subclass of the projective shapes consists of the Tyler regular shapes, for which geometrically motivated pre-shapes can be defined, thus allowing for the construction of a natural Riemannian metric.

1 Introduction

The space of projective shapes $\mathbb{A}_k^d$ of $k$ landmarks in $d$-dimensional real projective space $\mathbb{RP}^d$ is of interest in computer vision. It is commonly defined as the topological quotient of $k$ copies of $\mathbb{RP}^d$ modulo the landmark-wise action of the projective linear group $\text{PGL}(d)$. This space arises naturally in the single view uncalibrated pinhole camera model: when taking a $d$-dimensional picture in $\mathbb{R}^{d+1}$ of a $d$-dimensional object without knowledge of any camera parameters such as focal length, angle between the object hyperplane and film hyperplane, etc., then the original object can only be reconstructed up to a projective transformation. Similarly, it arises in the multiple view uncalibrated pinhole camera model: when taking multiple $d$-dimensional pictures of an object in $\mathbb{R}^{d+1}$, the original configuration of landmarks can only be reconstructed up to a projective transformation. For details, we refer the reader to the literature, e.g. [3, 5].

Other space of interest in computer vision include similarity and affine shape spaces. In shape spaces, one would often like to make metric comparisons, which requires e.g. the structure of a Riemannian manifold. For affine or similarity shapes, the topology of the shape space is well understood and there are natural choices for a Riemannian metric. Similarity shape space is a CW complex after removing the trivial shape [6], while affine shape space has a naturally ordered stratification with each stratum being diffeomorphic to a Grassmannian [3, 10]. In both cases, the topological subspace of shapes with trivial isotropy group, i.e. the shape space of the configurations on which the group action is free, has a natural structure of a Riemannian manifold.

In the case of projective shapes, it turns out that the topological subspace of shapes with trivial isotropy group cannot be given the structure of a Riemannian manifold since it is only a differentiable $\text{T}1$ manifold, but not Hausdorff. Hence, we have to look for other topological subspaces, which can be endowed with a Riemannian metric. This search is the main purpose of this article.

Besides the quest for a Riemannian structure, there are more desirable properties for a “good” topological subspace:

(a) it should be a manifold with complete Riemannian metric;

(b) it should be closed and the Riemannian metric invariant under reordering of the landmarks in the configuration $p = (p_1, \ldots, p_k) \in (\mathbb{RP}^d)^k$ (relabeling);

(c) when containing a degenerate shape, i.e. a shape with non-trivial projective subspace constraints (see Section 2), it should also contain all less degenerate shapes; we will then say that the topological subspace respects the hierarchy of projective subspace constraints;

(d) it should contain as many shapes as possible in the sense that adding further shapes results in the violation of at least one of the properties (maximality).
To our knowledge, there are only two established ways to obtain topological subspaces fulfilling some of these properties, which will be discussed in Section 3. Firstly, one can take only those shapes whose first $d + 2$ landmarks are in general position and thus form a so-called projective frame. This topological subspace is homeomorphic to $k - d - 2$ copies of $\mathbb{R}P^d$ \cite{9}; in particular it respects the hierarchy of projective subspace constraints while being maximal, Hausdorff and a differentiable manifold, i.e. locally Euclidean with smooth transition maps and second-countable. Unfortunately, it is not closed under relabeling. Secondly, one can take all those shapes whose projective subspace constraints fulfill a certain regularity condition, called Tyler (fully-)regular \cite{2}. This topological subspace is Hausdorff, closed under relabeling, respects the hierarchy of projective subspace constraints and, as we show in Section 3 a differentiable manifold. Recall that a Riemannian metric can be defined on any differentiable Hausdorff manifold \cite{8}. However, these topological subspaces have been constructed in an ad hoc fashion. As of now there is no systematic approach to obtain “good” topological subspaces based on the geometrical and topological properties of projective shape space.

In this paper, we therefore analyze the topology of projective shape space in detail. After recalling some basic facts, fixing our notation in Section 2 and discussing prior approaches in Section 3 we show which shapes can be separated from each other in the T1 sense, i.e., either one is not contained in some open neighborhood of the other, in Section 4. In particular, we will show that the subspace of shapes with trivial isotropy group is T1 and a differentiable manifold. We thus generalize the notion of a frame to obtain charts. In Section 5, we show that two shapes which cannot be separated in the Hausdorff sense are already degenerate in a particular way. This allows us to characterize a reasonable family of differentiable Hausdorff manifolds in Section 6 which additionally possess properties (b), (c), and (d). In Section 7 we give a geometric justification for Tyler standardization of Tyler regular shapes introduced by Kent and Mardia \cite{2} and a Riemannian metric on this topological subspace.

## 2 Preliminaries and notation

For $d > 0$, real projective space $\mathbb{R}P^d$ is defined as the topological quotient of $\mathbb{R}^{d+1}\setminus\{0\}$ modulo the multiplicative group $\mathbb{R}\setminus\{0\}$, so it can be seen as the space of lines through the origin in $\mathbb{R}^{d+1}$. A projective subspace of $\mathbb{R}P^d$ of dimension $n < d$ is then the set of lines lying in an $(n+1)$-dimensional linear subspace of $\mathbb{R}^{d+1}$. Analogously, one can define the projective span of points in $\mathbb{R}P^d$ as the set of lines lying in the linear span of some representatives of the points in $\mathbb{R}^d$.

There is a natural, well-defined action of the general linear group $\text{GL}(d + 1)$ on $\mathbb{R}P^d$ by letting it act on representatives in $\mathbb{R}^{d+1}$. Since the action of a matrix on $\mathbb{R}P^d$ does not change when multiplying the matrix by a non-zero scalar, the action of $\text{GL}(d + 1)$ is identical with the action of the projective linear group $\text{PGL}(d) = \text{GL}(d + 1) / (\mathbb{R}\setminus\{0\})$. This action is naturally carried forward to the product space of configurations

$$\mathcal{A}^d_k = (\mathbb{R}P^d)^k = \mathbb{R}P^d \times \cdots \times \mathbb{R}P^d$$

by letting it act component-wise. Note that projective transformations, i.e. the elements of $\text{PGL}(d)$, map projective subspaces of $\mathbb{R}P^d$ to projective subspaces of the same dimension, i.e. points to points, lines to lines etc. So, if $p \in \mathcal{A}^d_k$ is a configuration with three landmarks on a line, then the images of these three landmarks under a projective transformation also lie on a line.

For $d \geq 1$ and $k \geq d + 3$, the space of projective shapes of $k$ landmarks in $\mathbb{R}P^d$ is defined to be the quotient space

$$a^d_k = (\mathbb{R}P^d)^k / \text{PGL}(d)$$

with the quotient topology. Since the projection map $\pi : \mathcal{A}^d_k \to a^d_k$ is open, the topology of $a^d_k$ is also second countable; it thus can be characterized by sequences, just like $\mathcal{A}^d_k$. Further, we can represent a configuration $p \in \mathcal{A}^d_k$ in homogeneous coordinates: up to left-multiplication with a diagonal $k \times k$-matrix with non-zero real entries, the $k$ landmarks in $\mathbb{R}P^d$ can be represented as a real $(d+1)\times k$-matrix $P$ whose non-trivial rows $P_i \in \mathbb{R}^{d+1}, i = 1, \ldots, k$, represent the landmarks in $\mathbb{R}P^d$. The corresponding equivalence class $[P]$, i.e. the shape, consists of all matrices of the form $DPB$ with $D$ being a non-singular diagonal $k \times k$-matrix, $B$ a non-singular $(d+1) \times (d+1)$-matrix, i.e.,

$$[P] = \{DPB : D \in \text{GL}(k) \text{ diagonal}, B \in \text{GL}(d+1)\}.$$
Throughout this article, we denote a configuration \( p \in (\mathbb{RP}^d)^k \) by a lower case letter, its matrix representation \( P \in \mathbb{R}^{k \times (d+1)} \) by the corresponding upper case letter and the shape of \( p \) resp. \( P \) by \([p]\) resp. \([P]\). In abuse of language, we will call \( P \) a configuration, too. Further, we define the rank \( \text{rk} \) of a configuration \( p \) to be the rank of any corresponding matrix \( P \). Note that the rank is invariant under \( \text{PGL}(d) \).

Our aim is to find topological subspaces of \( \mathcal{A}_d^k \) that can be given the structure of a Riemannian manifold. Topologically speaking, these topological subspaces need to be differentiable Hausdorff manifolds, as those can be given the structure of a Riemannian manifold \cite{8}.

Unfortunately, the space of all projective shapes \( \mathcal{A}_d^k \) is not a differentiable Hausdorff manifold, and indeed it is not even \( T1 \). This is easily seen by considering the open neighborhoods of the trivial shape where all landmarks coincide. Any open neighborhood of the trivial shape is actually already the full space \( \mathcal{A}_d^k \). This phenomenon occurs in similarity and affine shape space as well.

Before we turn to analyze \( \mathcal{A}_d^k \) in detail, we define some topological subspaces of \( \mathcal{A}_d^k \) (resp. \( \mathcal{A}_d^k \)):

\( \mathcal{G}_d^k \), which contains a configuration \( p = (p_1, \ldots, p_k) \) \( \in \mathcal{A}_d^k \) if and only if the landmarks \( p_1, \ldots, p_k \in \mathbb{RP}^d \) are in general position, i.e., no \( m \)-dimensional projective subspace of \( \mathbb{RP}^d \) with \( 0 \leq m < d \) contains more than \( m + 1 \) of the landmarks, i.e., any \( d \) of the landmarks in \( p \) span \( \mathbb{RP}^d \). An element of \( \mathcal{G}_d^{d+2} \) is called a (projective) frame. Note that \( \mathcal{G}_d^k \) is dense in \( \mathcal{A}_d^k \).

\( \mathcal{B}_d^k \), which contains a configuration \( p \in \mathcal{A}_d^k \) if and only if the first \( d + 2 \) landmarks in \( p \) form a frame, i.e., if and only if \( \{p_1, \ldots, p_{d+2}\} \in \mathcal{G}_d^{d+2} \), hence \( \mathcal{G}_d^k \subset \mathcal{B}_d^k \). The frames allow us to define the equivalent of Bookstein coordinates for similarity shapes, see Lemma 3.1 and [9, p. 1672; \( \mathcal{B}_d^k \) being called \( G(k, d) \) there].

\( \mathcal{P}_d^k \), which contains a configuration \( p \in \mathcal{A}_d^k \) if and only if it contains at least one frame, i.e., if and only if there exists a permutation \( \sigma \in S_k \) of the landmarks such that \( \sigma(p) \in \mathcal{B}_d^k \), thus \( \mathcal{B}_d^k \subset \mathcal{P}_d^k \) \( \mathcal{P}_d^k \) being called \( FC_{d} \) there].

\( \mathcal{F}_d^k \), which contains a configuration \( p \in \mathcal{A}_d^k \) if and only if it has trivial isotropy group, i.e., \( \{g \in \text{PGL}(d) : gp = p\} = \{e\} \). Elements with trivial isotropy group are called free or regular. Note that \( \mathcal{P}_d^k \subset \mathcal{P}_d^k \) as shown by Mardia and Patrangenaru [9].

\( \mathcal{S}_d^k \), which contains a configuration \( p \in \mathcal{A}_d^k \) if and only if it is splittable, i.e., there is a subset \( I \subseteq \{1, \ldots, k\} \) s.t. \( \text{rk} p_I + \text{rk} p_{I^c} = d + 1 \) where \( I^c = \{1, \ldots, k\} \backslash I \) and \( p_I \) denotes the restriction of \( p \) to landmarks with index \( i \in I \).

\( \mathcal{R}_d^k \), which contains a configuration if and only if it is of full rank, i.e., there is no projective subspace of dimension \( m < d \) which contains all landmarks. Note that \( \mathcal{A}_d^k \backslash \mathcal{R}_d^k \subset \mathcal{S}_d^k \) (take \( I \) = \( \{1\} \)).

\( \mathcal{I}_d^j \), which contains a configuration if and only if any \( j \)-dimensional projective subspace of \( \mathbb{RP}^d \), \( j = 0, \ldots, d - 1 \) contains fewer than \( \frac{k+1}{j+1} \) landmarks. These configurations are called Tyler (fully-regular) by Kent and Mardia [7].

Note that we always denote the set of equivalence classes by a lower case letter, the corresponding set of configurations by an upper case letter, for example \( \mathcal{A}_d^k \), \( \mathcal{B}_d^k \) etc. for the configuration spaces, \( \mathcal{A}_d^k \), \( \mathcal{B}_d^k \) etc. for the corresponding shape spaces.

We say that a configuration \( p \in \mathcal{A}_d^k \) fulfills the projective subspace constraint \((I,j)\) for a subset \( I \subseteq \{1, \ldots, k\} \) of size \(|I| \geq j\), \( 1 \leq j \leq d + 1 \), if and only if there is a projective subspace \( S \) of dimension \( j - 1 \) such that \( p_i \in S \) for all \( i \in I \), i.e., \( \text{rk} p_I \leq j \). We denote the collection of projective subspace constraints fulfilled by a configuration \( p \in \mathcal{A}_d^k \) by \( C(p) = \{(I,j) : p \text{ fulfills } (I,j)\} \). We call a projective subspace constraint \((I,j) \in C(p)\) trivial if \( I \subseteq \{1, \ldots, k\} \) is a subset of size \(|I| = j\), and non-trivial otherwise. Further, we call \((I,j) \in C(p)\) splittable in \( C(p)\) if there are \((I_1, j_1), (I_2, j_2) \in C(p)\) with \( j_1 + j_2 = j \), \( I_1 \cup I_2 = I \), \( I_1 \cap I_2 = \emptyset \). Thus a configuration \( p \) is splittable, i.e., \( p \in \mathcal{S}_d^k \) if and only if \( (d + 1, \{1, \ldots, k\}) \) is splittable (slightly generalizing our notation). We noted before that \( C(p) \) is invariant under \( \text{PGL}(d) \), i.e., \( C(p) = C(ap) \) for all \( a \in \text{PGL}(d) \), whence \( C(p) \) is a property of the projective shape \([p]\).

3 Previous approaches

The first statistical approach to projective shape space is via frames, which are a well-known concept in projective geometry. As mentioned before, a frame is an ordered set of \( d + 2 \) landmarks
in general position. The group action of \( \text{PGL}(d) \) is both transitive and free on the space \( \mathcal{G}_{d+1} \) of frames, i.e., for any two frames there is a unique projective transformation mapping one frame to the other \([2]\). This quickly leads to the following result by mapping the frame in the first \( d+2 \) landmarks to a fixed frame:

**Lemma 3.1** \((9)\) \( \hat{B}^k_d \) is homeomorphic to \((\mathbb{RP}^d)^{k-d-2} \).

So, \( \hat{B}^k_d \) is a differentiable Hausdorff manifold and respects the hierarchy of projective subspace constraints, but is not closed under relabeling. The closure of \( \hat{B}^k_d \) under permutations is—by definition—the topological subspace \( \overline{p}_d \) of shapes with a frame. \( \overline{p}_d \) is a differentiable manifold, but not Hausdorff for any \( d \geq 1, \ k \geq d+3 \) as we will see in Proposition \( 5.11 \).

**Corollary 3.2** \( p^k_d \) is homeomorphic to a \((d-k-d-2)\)-dimensional differentiable TI manifold.

**Proof** Lemma \( 3.1 \) gives homeomorphisms from the topological subspaces of shapes with a frame in a fixed subset of \( d+2 \) landmarks to \((\mathbb{RP}^d)^{k-d-2} \). Further, note that these topological subspaces of shapes with a frame in a fixed subset of \( d+2 \) landmarks are open in \( \mathcal{A}_d \) and \( \mathcal{G}_d^k \). Hence, these homeomorphism are “manifold-valued” charts on \( \mathcal{G}_d^k \). Ordinary charts on \( \mathcal{G}_d^k \) can easily be obtained by composition with charts on the manifold \((\mathbb{RP}^d)^{k-d-2} \), e.g., inhomogeneous coordinates. These charts are compatible since the transition maps are just multiplications with non-singular matrices as well as division by non-vanishing parameters depending smoothly on the representation matrix. \( \square \)

Alternatively, one can consider the space of shapes in general position \( \mathcal{G}_d^k \subset \hat{B}^k_d \) which is also a differentiable Hausdorff manifold, respects the hierarchy of projective subspace constraints, and is closed under relabeling. The drawback of \( \mathcal{G}_d^k \) is that it is not maximal for any \( d \geq 1, \ k \geq 4 \).

**Example 3.3** For \( d = 1 \), a frame consists of three distinct landmarks. Hence, \( \mathcal{B}^1_d \) consists of all configurations with distinct first three landmarks and arbitrary fourth landmark. \( \hat{B}^1_d \) is then homeomorphic to the real projective line \( \mathbb{RP}^1 \) or—equivalently—the circle. Meanwhile, \( \mathcal{T}^1_d \) consists of all configurations with at least three of its landmarks distinct and thus forming a frame. \( \mathcal{P}^1_d \) is homeomorphic to a circle with three double points corresponding to the single pair coincidences, which cannot be separated in the Hausdorff sense \([7]\). Finally, \( \mathcal{G}^1_d \) consists of configurations with no landmark coincidences, hence \( \mathcal{G}^1_d \) is homeomorphic to the circle with three points removed.

A different approach was developed by Kent and Mardia \([7]\). The space \( \mathcal{T}^k_d \) of Tyler regular configurations comprises configurations \( p \) all of whose projective subspace constraints \((I, j) \in C(p)\) satisfy the inequality \( |I| < \frac{k}{d+1} \). It was shown that any Tyler regular configuration \( p \in \mathcal{T}^k_d \) has a matrix representation \( P \) fulfilling

\[
P_iP_i^T = \frac{d+1}{k}
\]

and

\[
P^TP = \mathbf{I}_{d+1}
\]

with \( \mathbf{I}_{d+1} \) denoting the \((d+1) \times (d+1)\)-dimensional identity matrix. This so-called Tyler standardization \( P \) is unique up to multiplication of the rows \( P_i \) by \( \pm 1 \) and right-multiplication by an orthogonal matrix, i.e. unique up to a compact group action, and can be viewed as a projective pre-shape. By considering \( PP^T \in \mathbb{R}^{k \times k} \), one can even remove the ambiguity of the \( \mathbf{O}(d+1) \)-action. This gives a covering space of the space \( t^k_d \) of Tyler regular shapes. The covering space is Hausdorff, whence \( t^k_d \) is Hausdorff.

We show in Section \([1]\) that \( t^k_d \) is a differentiable Hausdorff manifold; it is obviously closed under relabeling and respects the hierarchy of projective subspace constraints. Additionally, we show that \( t^k_d \) is maximal for some, but not all \( k \) and \( d \). Note that the approach via frames differs from the approach via Tyler regularity since, for \( d \geq 3 \), there are Tyler regular shapes without a frame, see Figure \([1]\).

**Example 3.4** In the case \( d = 1 \) and \( k = 4 \), \( t^1_d \) consists of shapes with projective subspace constraints \((I, j) \) with \( j = 1 \) and \( |I| = 1 < \frac{k}{d+1} \), i.e. the shapes in general position. Hence, \( t^1_d = \mathcal{G}^1_d \) with \( \mathcal{G}^1_d \) being homeomorphic to the circle with three points removed, as we have seen before.

Neither of these approaches discusses the topological background of these choices. The goal of this article is to shed light on the topology of these topological subspaces of projective shapes.
4 The manifold of the free

To understand a topology of a topological space $M$, it is vital to know which elements of $M$ cannot be separated from another by open neighborhoods. It is common to use the well-known separation axioms to describe the degree of separation. Two of those will be discussed here.

A topological space $M$ is said to be

**T1** if for any two points $p, q \in M$ there are open neighborhoods $U_p$ and $U_q$ of $p$ and $q$ respectively not containing the other point, i.e., $q \notin U_p$ and $p \notin U_q$.

**Hausdorff or T2** if for any two points $p, q \in M$ there are disjoint open neighborhoods of $p$ and $q$.

The intersection of all open neighborhoods to a point $p \in M$ is a useful tool towards understanding the separation properties of a space $M$. This set was introduced as the blur $\text{Bl}(p)$ of $p$ in $M$ by Groisser and Tagare in their discussion of affine shape space [4]. We will call a point $p \in M$ unblurry if $\text{Bl}(p) = \{p\}$, and blurry in the case that its blur is a strict superset of $\{p\}$.

Equivalently, the blur could also be defined via sequences.

**Lemma 4.1** Let $M$ be a topological space and $p, q \in M$. Then, $p \in \text{Bl}(q)$ if and only if the constant sequence $(p)_n \in \mathbb{N}$ converges to $q$.

**Proof** $p \in \text{Bl}(q)$ if and only if $p$ is in every neighborhood of $q$ which happens if and only if the sequence $(p)_n \in \mathbb{N}$ converges to $q$. \hfill \Box

This concept is closely related to the more familiar concept of closure which has also been pointed out by Groisser and Tagare [4].

**Lemma 4.2** [4] Lemma 5.2] Let $M$ be a topological space and $p, q \in M$. Then, $p \in \text{Bl}(q)$ if and only if $q \in \text{Cl}(p)$, the latter denoting the closure of $\{p\}$ in $M$.

In particular, every point is unblurry if and only if every point is closed, which in turn is equivalent to the space being T1 [1]. This motivates us to take a closer look at the unblurry shapes.

As it turns out, a shape is blurry if it is splittable; the converse is also true as we will show after Theorem [4.3].

**Proposition 4.3** Let $[p] \in \mathcal{S}_d^k$ be a splittable shape. Then $[p]$ is blurry.

**Proof** We will use Lemma [4.1]. First, consider an arbitrary shape $[P]$ with $\text{rk} \, P < d + 1$. There is a non-singular matrix $B \in \text{GL}(d + 1)$ such that $PB = (P_1, 0_k)$ for some $P_1$ and $0_k$ being a column vector of $k$ zeroes. Of course, $PB$ is still of shape $[P]$. Then, the sequence $((P_1, z)B_n)_{n \in \mathbb{N}}$ with $B_n = \text{diag}(1, \ldots, 1, \frac{1}{n})$ and arbitrary $z \in \mathbb{R}^k$ has limit $PB$. Hence, $\{(P_1, z)\} \in \text{Bl}([P])$ for any $z \in \mathbb{R}^k$, while there is a $z \in \mathbb{R}^k$ such that $\text{rk}(P_1, z) > \text{rk} \, P$. Therefore, $\text{Bl}([P]) \neq \{[P]\}$, whence $[P]$ is blurry.

Now, let $[P] \in \mathcal{S}_d^k$ be of rank $d + 1$ with $(I, j), (I^c, d + 1 - j) \in C(P)$. W.l.o.g. $j < |I^c|$, else $d + 1 - j < |I|$. Then there is a suitable permutation $\sigma$ of the rows of $P$ and a suitable non-singular matrix $B \in \text{GL}(d + 1)$ such that the matrix $\hat{P} = \sigma(P)B$ is a block diagonal matrix

$$\hat{P} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

for some matrices $P_1 \in \mathbb{R}^{|I| \times j}$ and $P_2 \in \mathbb{R}^{|I^c| \times (d + 1 - j)}$. The sequence given by

$$\left( n^I_{|I|} 0 \right) \left( \begin{array}{cc} P_1 & 0 \\ Z & P_2 \end{array} \right) \left( n^I_{|I^c|} 0 \right) \left( \begin{array}{cc} I_j & 0 \\ 0 & I_{d+1-j} \end{array} \right) = \left( \frac{1}{n} I \right) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

has limit $\hat{P}$ for any $Z \in \mathbb{R}^{|I^c| \times j}$. Hence,

$$\left[ \begin{array}{c} P_1 \\ Z \end{array} \right] \in \text{Bl}([P]).$$

Again, there is a $Z \in \mathbb{R}^{|I^c| \times j}$ which breaks a projective subspace constraint of $[\hat{P}]$, whence $\text{Bl}([P]) \neq \{[P]\}$ and consequently $\text{Bl}([P]) \neq \{[P]\}$ and $[P]$ is blurry. \hfill \Box
The same argument works when removing the landmark on the intersection point. Analogously, a free configuration of 7 landmarks is free, i.e., \( q \) with \( q_1 = q_2 \) and \( r \) with \( r_3 = r_4 \).

Due to Proposition 4.3, we henceforth limit ourselves to the analysis of those configurations (resp. shapes) which are not splittable. Those can be characterized algebraically via the group action.

**Proposition 4.5** A configuration is free if and only if it is not splittable, i.e. \( \mathcal{I}_d^A = \mathcal{A} \cup \mathcal{S}_d^A \).

**Proof** If \( \text{rk} P < d + 1 \), then \( P \) is obviously splittable, but not free. Hence, we will focus on configurations with \( \text{rk} P = d + 1 \).

Now, assume there are projective subspace constraints \((I, j), (I', d+1-j)\) such that \( \text{rk} P_I + \text{rk} P_{I'} = \text{rk} P = d + 1 \). Then there is a permutation \( \sigma \) of the rows of \( B \) and a matrix \( B \in \text{GL}(d+1) \) such that \( \sigma(P)B \) is a block diagonal matrix \( \begin{pmatrix} \hat{P}_I & 0 \\ 0 & \hat{P}_{I'} \end{pmatrix} \). Hence, \( \sigma(P)B \) is not free since

\[
\begin{pmatrix} \hat{P}_I & 0 \\ 0 & \hat{P}_{I'} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \hat{P}_I & 0 \\ 0 & \hat{P}_{I'} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} I \end{pmatrix} + \begin{pmatrix} I_{d+1-j} \end{pmatrix}
\]

Therefore, \( \sigma(P) \) is not free, henceforth neither is \( P \).

For the opposite direction, assume \( P \) is not free. Then there exists a diagonal matrix \( D \) and some \( B \in \text{GL}(d+1) \), \( B \neq \lambda I_{d+1} \), \( \lambda \in \mathbb{R} \setminus \{0\} \), such that \( DPB = P \). Hence, the rows of \( P \) are eigenvectors of \( B^d \) with corresponding eigenvalues \( \lambda_1, \ldots, \lambda_k \), say (taking at most \( d + 1 \) distinct eigenvalues). There are at least two distinct eigenvalues, else \( B = \lambda I_{d+1} \) contradicting the assumption. Then, \((I, \text{rk} P_I), (I', \text{rk} P_{I'}) \in C(P) \) with \( I = \{i : \lambda_i = \lambda_1\} \), while \( \text{rk} P_I + \text{rk} P_{I'} = d + 1 \), whence \( P \) is splittable.

From Propositions 4.3 and 4.5 we conclude that the subspace \( f_d^k \) of the free shapes is the largest subspace, which is \( T_1 \) and respects the hierarchy of subspace constraints.

In the case \( d = 1 \), the splittable shapes are those comprising of at most two distinct landmarks as we have seen before. Thus, Proposition 4.3 states that a shape \( [p] \in A_1^k \) is free if and only if it has at least three distinct landmarks. Three distinct landmarks always form a frame for \( d = 1 \). Indeed, Mardia and Patrangenaru [9] have shown for any \( d \geq 1 \) that shapes which include a frame are free, i.e. \( p_d^k \subseteq f_d^k \). However, the other inclusion does not hold for \( d \geq 3 \): e.g. for \( d = 3 \), take three lines, which are not coplanar, but have a common intersection point, and put two landmarks on each line, and another on the intersection point. Such a configuration of 7 landmarks is free, but does not contain a frame since there are no 5 landmarks in general position, see Figure 1(b).

The same argument works when removing the landmark on intersection point. Analogously, a free shape without a frame can be constructed for any \( d > 3 \).

Hence, having a frame is not essential for a shape to be free. While frames can be used as charts on \( p_d^k \), this is not possible for \( f_d^k \) for \( d \geq 3 \) since the charts associated with frames do not cover \( f_d^k \) for \( d \geq 3 \). However, the notion of a frame can be generalized to obtain charts on \( f_d^k \) as follows.

A free configuration contains at least \( d + 1 \) landmarks in general position since a free configuration is of full rank. Now, a configuration \( P = (\bar{P}_I) \), whose first \( d + 1 \) landmarks \( P_0 \), say, are in general position, i.e. \( P_0 \in G_{d+1} \), is equivalent to a matrix of the form

\[
\tilde{P} = \begin{pmatrix} I_{d+1} \\ P_0 \end{pmatrix},
\]
and thus $\lambda$-tree $G$ is a tree on graph. This generalizes the idea of a “frame” since a frame is a pseudo-frame with a connected and gets disconnected if an edge is removed whence it is a minimal substructure of a connected graph.

For any two connected columns $i, j$ of $D$, let $D = \text{diag}(\lambda_1, \ldots, \lambda_k)$ and $B \in \text{GL}(d+1)$ such that $DPB = P$. Then $B = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_k^{-1})$, since Equation (4) implies

$$\text{diag}(\lambda_1, \ldots, \lambda_{d+1}) I_{d+1} B = I_{d+1}$$

for the first $d+1$ rows of $P$. For any two connected columns $i, j$ of $P$, there is a row $P_l$ such that both $P_{li} \neq 0$ and $P_{lj} \neq 0$. Hence,

$$P_l = \lambda_l P_l B = \lambda_l \sum_{n=1}^{d+1} \lambda_n^{-1} P_{ln} e_n,$$

where $e_n$ is the $n$-th row vector of the standard basis of $\mathbb{R}^{d+1}$. From this we conclude

$$\lambda_l = \lambda_j = \lambda_{d+1}^{-1}$$

and thus $\lambda_1 = \ldots = \lambda_{d+1}$, since all columns are connected, so $D = \lambda_1 I_k$ and $B = \lambda_1^{-1} I_{d+1}$, i.e., $P$ is free.

In the following, we will call $d+1$ landmarks in general position together with a connected tree $G$ with edges labeled with the remaining landmarks a pseudo-frame. So $G$ contains no cycles and gets disconnected if an edge is removed whence it is a minimal substructure of a connected graph. This generalizes the idea of a “frame” since a frame is a pseudo-frame with a connected tree on $d+1$ landmarks in general position where all edges are labeled with the same landmark (see Figure 2), i.e., a uni-colored tree gives rise to a frame. We will say that a configuration $p$ (resp. shape $[p]$) contains a pseudo-frame $((i_1, \ldots, i_{d+1}), G)$ if $p_{i_1}, \ldots, p_{i_{d+1}}$ are in general position.

\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad G(P) \]

Figure 2: A frame $P$ and its graph $G(P)$ which is a complete graph. All spanning trees of $G(P)$ give a pseudo-frame.
and the corresponding graph to this configuration (resp. shape) has the tree $G$ as a subgraph. We conclude from Proposition 4.4 that every free shape contains a pseudo-frame.

Since pseudo-frames are a generalization of frames, we obtain a topological Hausdorff subspace when considering all shapes containing a fixed pseudo-frame, thus generalizing the definition of $\mathcal{B}_d^k$ and Lemma 3.1 denote the number of edges in the tree $G = (\{i_1, \ldots, i_{d+1}\}, E)$ labeled with the landmark $l$ by $|E_l|$, and define $\#E = \{|l : E_l \neq \emptyset\}$.

**Proposition 4.7** The topological subspace of all shapes containing a certain pseudo-frame $(\{i_1, \ldots, i_{d+1}\}, G)$ is homeomorphic to the $d(k - d - 2)$-dimensional differentiable Hausdorff manifold

$$\left(\mathbb{RP}^d\right)^{k-d-1-\#E} \times \prod_{l=d+2}^k \mathbb{R}^{d-|E_l|}.$$  \hspace{1cm} (5)

**Proof** The final factor of the product in Equation (5) has dimension $d(\#E-1)$ since $\sum_{l=d+2}^k |E_l| = d$ is the number of edges in the tree $G$ with $d + 1$ vertices. This explains the dimension of the manifold.

To show the homeomorphy, consider for a shape $[P]$ (after reordering the rows) a representative of the form in Equation (4). Obviously, the rows of $P_*$ which are not used for the graph give us the first factor of the product in Equation (5). By rescaling of rows and columns the non-zero entries determined by the labeled tree are w.l.o.g. equal to 1, and the rest of the row may be filled with any real number, hence we obtain $\mathbb{R}^{d+1-|E_{d+1}|} = \mathbb{R}^{d-|E_l|}$ for row $l$ if $|E_l| \neq 0$.

Now, Proposition 4.7 gives us finitely many, manifold-valued charts for $f^k_d$ whence it is a differentiable manifold.

**Theorem 4.8** $f^k_d$ is a $d(k - d - 2)$-dimensional differentiable $T1$ manifold.

**Proof** From Proposition 4.7 we obtain homeomorphisms from open subsets of $f^k_d$ to a differentiable manifold. When composing those with charts of the differentiable manifold, we obtain charts on $f^k_d$ whose domains cover the full space. Since the transition maps between these charts are just multiplications with non-singular diagonal and non-singular matrices depending smoothly on the representation matrix, the manifold is indeed differentiable.

We would like to point out that for $d = 1$ the concept of pseudo-frames adds no extra insight, since a pseudo-frame is already a frame in this case (any colored tree with $d + 1 = 2$ vertices is uni-colored). For $d = 2$, any shape with a pseudo-frame already contains a frame, i.e. $f^k_d = f^k_d$ for $d = 1, 2$. The critical shape to consider in the case $d = 2$ is (in the form of Equation (4))

$$[p] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
u & v & w \\
x & y & z \\
\vdots
\end{bmatrix}.$$  

Let w.l.o.g. there be a pseudo-frame in the first 5 rows of $[p]$. If either all of $u, v, w \neq 0$ or all of $x, y, z \neq 0$, then $[p]$ contains a frame. So, let there be a vanishing value in both of the rows. Since there is a pseudo-frame in the first 5 rows of $[p]$, there is at most one vanishing value in each row, and it cannot be in the same column. For the sake of argument, let $u, v = 0$. Then, $p(1,3,4,5)$ is a frame. Thus, charts stemming from frames suffice to cover $f^k_d$, while pseudo-frames give a larger atlas on $f^k_d$. From Theorem 4.8 follows that free shapes $[p] \in f^k_d$ are unblurry which is the converse direction of Proposition 4.3 $f^k_d$ is open in $a^k_d$ since $\mathcal{F}^k_d$ is open in $\mathcal{A}^k_d$ and $\pi : \mathcal{A}^k_d \to a^k_d$ is an open map. Hence, neighborhoods of $[p]$ in $f^k_d$ are already neighborhoods of $[p]$ in $a^k_d$. Now, $f^k_d$ is $T1$ by Theorem 4.8 whence the intersection of all neighborhoods of $[p]$ in $f^k_d$ is just $\{[p]\}$, so is the intersection of all neighborhoods of $[p]$ in $a^k_d$. Hence, $\text{Bl}([p]) = \{[p]\}$.

Unfortunately, the manifold $f^k_d$ of the free is never Hausdorff for $d \geq 1$ and $k \geq d + 3$. Even the subset $p^k_d$ is never Hausdorff for any $d \geq 1$ and $k \geq d + 3$ which will follow from Proposition 5.1. Note, however, that all open subsets of $f^k_d$ are differentiable manifolds by Theorem 4.8. In particular, all topological subspaces $y = \mathcal{Y}/\text{PGL}(d)$ of $f^k_d$ respecting the hierarchy of projective
subspace constraints are differentiable manifolds, since then any configuration \( p \in \mathcal{Y} \) has an open neighborhood \( U \ni p \) with \( U \subseteq \mathcal{Y} \) whence \( \mathcal{Y} \) is open in \( \mathcal{A}_d^k \) and \( y \) is open in \( \mathcal{A}_d^k \).

The situation in similarity resp. affine shape space is not as complicated: in both cases, the full shape space is not \( T_1 \). The largest \( T_1 \) space in similarity shape space is the full space without the trivial shape, while in affine shape space it is the subspace of the free just like in projective shape space. In both cases, the subspace of the free is a differentiable manifold and, in contrast to the projective situation, Hausdorff.

## 5 Hausdorff subsets

In applications, one is often interested in metric comparisons of different shapes. Therefore, the underlying shape space needs to be a metrizable topological space (e.g. a Riemannian manifold) which is—of course—at least Hausdorff. Hence, we are looking for topological Hausdorff subspaces of projective shape space.

Consider a shape \([P]\) which fulfills the projective subspace constraint \((\{1, \ldots, i\}, j)\), which may be trivial or non-trivial, i.e., \([P]\) has a representative

\[
P = \begin{pmatrix} P_1 & 0 \\ \frac{1}{n} Y & P_2 \end{pmatrix}
\]

for some matrices \(P_1 \in \mathbb{R}^{i \times j}, P_2 \in \mathbb{R}^{(k-i) \times (d+1-j)}, \) and \(Z \in \mathbb{R}^{(k-i) \times j} \) (\(Z\) possibly being zero). Additionally, consider the sequence \([P_n]_{n \in \mathbb{N}}\) with

\[
P_n = \begin{pmatrix} P_1 & \frac{1}{n} Y \\ \frac{1}{n} Z & P_2 \end{pmatrix}
\]

for some \(Y \in \mathbb{R}^{i \times (d+1-j)}\). This sequence converges to \([P]\) and to \([Q]\) with

\[
Q = \begin{pmatrix} P_1 & Y \\ 0 & P_2 \end{pmatrix}
\]

as \(n\) goes to infinity. But \([P] \neq [Q]\) for some choices for \(Y, Z\) as \(Y, Z\) may break some projective subspace constraint. Hence, a topological subspace of \(\mathcal{A}_d^k\) containing such \([P]\), \([Q]\) and \([P_n]\) for all \(n \in \mathbb{N}\) would not be Hausdorff since sequences in Hausdorff spaces have at most one limit point. Note that \(Q\) fulfills the projective subspace constraint \((\{i + 1, \ldots, k\}, d + 1 - j)\).

This observation can be strengthened to the following result for determining if a projective subspace \(\mathcal{A}_d^k\) is Hausdorff.

**Proposition 5.1** Let \(y \subseteq r_d^k\) be a topological subspace which contains \(g_d^k\) and is not Hausdorff. Then, there are two shapes \([p], [q] \in y\) with \([p] \neq [q]\), \((I, j) \in C(p)\) and \((I^c, d + 1 - j) \in C(q)\). More precisely, \(y\) is not Hausdorff if and only if there are two distinct shapes \([p], [q] \in y\) which after simultaneous reordering of rows have the form

\[
[p] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{i-1,1} & \cdots & P_{i-1,m} \\ 0 & \cdots & 0 & P_{im} \end{bmatrix}
\]

and

\[
[q] = \begin{bmatrix} D_1 P_{11} B_1 & 0 & \cdots & 0 \\ Q_{21} & \cdots & Q_{2,m-1} & \vdots \\ \vdots & \ddots & \vdots & 0 \\ Q_{l1} & \cdots & Q_{l,m-1} & D_l P_{lm} B_m \end{bmatrix}
\]

where \(P_{rs}, Q_{rs}\) are matrices of the same dimensions, and

(i) \(l, m > 1\) since \([p] \neq [q]\),
Figure 3: The form of the matrices in Equations (6) and (7) of Proposition 5.1. $P$ is zero in the blue, hatched area ($\clubsuit$) due to (iii), $Q$ is zero in the red, hatched area ($\heartsuit$) due to (iv). In the green area ($\diamondsuit$), the corresponding matrices are equivalent due to (ii).

(ii) if $P_{rs}, Q_{rs} \neq 0$, then $Q_{rs} = D_r P_{rs} B_s$ with $D_r$ diagonal and non-singular, $B_s$ non-singular,

(iii) $P_{rs} = 0$ if there is a pair $(a, b) \neq (r, s)$ with $a \leq r$, $b \geq s$ and $Q_{ab} \neq 0$,

(iv) $Q_{rs} = 0$ if there is a pair $(a, b) \neq (r, s)$ with $a \geq r$, $b \leq s$ and $P_{ab} \neq 0$.

Note that columns can be reordered by the right-action of $\text{GL}(d+1)$. The form of the matrices $P$ and $Q$ is illustrated in Figure 3.

Proof The strategy of the proof is as follows: first, we will show that a topological non-Hausdorff subspace contains two shapes of the described form. This will be demonstrated by using the definition of Hausdorff spaces via sequences in first-countable spaces: if $p, q \in M$ with $M$ a first-countable topological space do not possess disjoint open neighborhoods, then there is a sequence with limit points $p$ and $q$. In shape space, this gives us the sequences $(\{P_n\})_{n \in \mathbb{N}}, (\{Q_n\})_{n \in \mathbb{N}}$ with $D_n P_n = Q_n B_n$ for all $n \in \mathbb{N}$ and distinct limit points $[P], [Q]$. We will show that w.l.o.g. $B_n$ is diagonal for all $n \in \mathbb{N}$, and that the sequences $(B_n)_{n \in \mathbb{N}}, (D_n)_{n \in \mathbb{N}}$ converge to singular matrices. Different speeds of convergence lead to the described form of the limit points.

For the other direction, we will again use the idea of different speeds of convergence to construct, like in the proof of Proposition 1.3, a shape in any neighborhood of some $[p], [q] \in y$ of the described form.

Now, let $[p], [q] \in y$ with $[p] \neq [q]$ such that there are no disjoint open neighborhoods of $[p]$ and $[q]$. Since the topology of $\mathbb{R}^d$ is determined by sequences, there is a sequence $([r_n])_{n \in \mathbb{N}}$ in $y$ with limits $[p], [q]$. W.l.o.g. $[r_n] \in \mathbb{R}^d$ for all $n \in \mathbb{N}$ since $\mathbb{R}^d$ is dense in $\mathbb{R}^d$ and contained in $y$. Thus, there are sequences $(P_n)_{n \in \mathbb{N}}$ with limit $P$ and $(Q_n)_{n \in \mathbb{N}}$ with limit $Q$ in the configuration space $\mathcal{A}_d$ such that $\pi(P_n) = \pi(Q_n) = [r_n]$ for all $n \in \mathbb{N}$ and $\pi(P) = [p], \pi(Q) = [q]$. Since $P_n$ and $Q_n$ have the same shape, there are non-singular diagonal matrices $D_n$ and matrices $B_n \in \text{GL}(d+1)$ such that

$$D_n P_n = Q_n B_n$$

for all $n \in \mathbb{N}$. Without loss of generality:

- $B_n$ is diagonal for all $n \in \mathbb{N}$: in fact, using a singular value decomposition for $B_n$, one obtains the existence of diagonal matrices $D_n, E_n$ and orthogonal matrices $U_n, V_n \in \mathbb{O}(d+1)$ such that $D_n P_n = Q_n V_n E_n U_n^t$ or equivalently $D_n P_n U_n = Q_n V_n E_n$. The sequences $(U_n)_{n \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ have common converging subsequences since $\mathbb{O}(d+1)$ is compact, so w.l.o.g. $U_n \rightarrow U, V_n \rightarrow V, P_n U_n \rightarrow PU$ and $Q_n V_n \rightarrow QV$. Since right-multiplication by an orthogonal matrix does not change the projective shape of $P_n$ resp. $Q_n$, we can choose $P_n, Q_n$ such that the corresponding $B_n$ is diagonal.

- $\|B_n\|_\infty = 1$ for all $n \in \mathbb{N}$; otherwise, consider the matrices $\|B_n\|_\infty^{-1} D_n$ and $\|B_n\|_\infty^{-1} B_n$ instead of $D_n$ and $B_n$.

- $(B_n)_{n \in \mathbb{N}}$ converges to some limit $B$ with $\|B\|_\infty = 1$ since $(B_n)_{n \in \mathbb{N}}$ is w.l.o.g. bounded in the infinity norm, hence possesses at least a converging subsequence. Thus $Q_n B_n \rightarrow QB$.

- $(D_n)_{n \in \mathbb{N}}$ converges to some limit $D$, hence $\|D_n\|_\infty \leq \rho, \rho > 0$, for all $n \in \mathbb{N}$; else, since $D_n P_n \rightarrow QB$ and $P_n \rightarrow P$, a row of $P$ would be the null vector which is impossible.

- $B$ and $D$ are singular, but non-trivial, i.e., $B, D \neq 0$: if $B$ is non-singular, so is $D$ since, otherwise, $QB$ and thus $Q$ would have a vanishing row which is impossible. If $D$ is non-singular, so is $B$ since, otherwise, $P$ would be of rank less than $d+1$ in contradiction to the
assumption $y \subseteq r_j$. If both are non-singular, then $P = D^{-1}QB$ in contradiction to $[p] \neq [q]$. $B$ is non-trivial since $\|B\|_\infty = 1$, while $D$ is non-trivial since $B$ is non-trivial and $P$ and $Q$ are of full rank.

By reordering of rows and columns, one may assume that $(D_n)_{ii}$ and $(B_n)_{jj}$ converge to a finite limit for all $i < j$, so $(D_n)_{ii}$ does not grow faster than $(D_n)_{jj}$ for all $i < j$. By merging of columns respectively rows of equal speed of convergence into a block labeled $(r, s)$, one derives the proposed block structure of $P$ and $Q$. Blocks of type (ii) may arise if $(D_n)_{ii}$ converges to a non-zero value for some, and hence all $(i, j)$ in block $(r, s)$. If the sequence $(D_n)_{ii}$ converges to 0, then $Q_{ij} = 0$ which explains type (iv). For the blocks of type (iii), consider the equalities $P_nF_n = G_nQ_n$ with $F_n = \|B_n\|^{-1}_\infty B_n^{-1}$ and $G_n = \|B_n\|^{-1}_\infty D_n^{-1}$ for all $n \in \mathbb{N}$. If the sequence $(G_n)_{jj}$ diverges, or equivalently, the sequence $(D_n)_{jj}$ converges to 0, then $P_{ij} = 0$ which explains type (iii). Recall that neither $P$ nor $Q$ may have trivial rows or columns by assumption.

Finally, we have to show that the upper left and bottom right blocks are of type (ii): since every row of $Q$ is non-trivial, $(D_n)_{kk}$ does not converge to 0. Since $P$ is of full rank, the sequence of inverses $(D_n)_{kk}$ does not converge to 0, whence it converges to a non-zero number. Analogously, $(D_n)_{kk}$ converges to a non-zero number since $P$ has no row of zeroes, and $Q$ is of full rank. This finishes the proof that $[p], [q]$ are of the described form.

Conversely, assume there exist $[P], [Q] \in y$ with $P, Q$ in the described form, and let $U[p]$ and $U[q]$ be open neighborhoods of $[p]$ resp. $[q]$. Then there is a $\delta > 0$ such that $B_\delta(P) \subseteq \pi^{-1}(U[p])$ and $B_\delta(Q) \subseteq \pi^{-1}(U[q])$ in the space of (matrix) configurations. We will construct a configuration $A$ which is an element of both $B_\delta(P)$ and $B_\delta(Q)$. For $n \in \mathbb{N}$, consider block diagonal matrices

$$\tilde{D}_n = \begin{pmatrix} n^{d_1} \tilde{D}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n^{d_s} \tilde{D}_s \end{pmatrix}$$

and

$$\tilde{B}_n = \begin{pmatrix} n^{-b_1} \tilde{B}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n^{-b_m} \tilde{B}_m \end{pmatrix}$$

with non-singular diagonal matrices $\tilde{D}_r$, non-singular matrices $B_s$ and speeds of convergence $d_r, b_s \in \mathbb{N}_0$ such that

- $b_r > b_s, d_r > d_s$ for all $r > s$;

- $d_r = b_s$ and $\tilde{D}_r = D_r, \tilde{B}_s = B_s$ for pairs $(r, s)$ with $P_{rs}, Q_{rs} \neq 0$, and thus $Q_{rs} = D_rP_{rs}B_s$;

- $b_s \neq d_r$ and $\tilde{D}_r = \text{Id}, \tilde{B}_s = \text{Id}$ else; more precisely, let $d_r < b_s$ for all $(r, s)$ with $P_{rs} \neq 0$, while $d_r > b_s$ for all $(r, s)$ with $Q_{rs} \neq 0$.

Next, define the matrix $A = (A_{rs})$ with the same block structure as $P, Q$ and entries

$$A_{rs} = \begin{cases} P_{rs} & \text{if } P_{rs} \neq 0, \\ n^{b_s-d_r} \tilde{D}_{-1} Q_{rs} \tilde{B}_{s} & \text{if } P_{rs} = 0. \end{cases}$$

Then

$$(\tilde{D}A\tilde{B})_{rs} = \begin{cases} Q_{rs} & \text{if } P_{rs} = 0, \\ n^{d_r-b_s} \tilde{D}_r P_{rs} \tilde{B}_s & \text{if } P_{rs} \neq 0. \end{cases}$$

Moreover,

$$\max \{ n^{b_r-d_r} : (r, s) \text{ with } Q_{rs} \neq 0, P_{rs} = 0 \} \leq n^{-1}. $$
and
\[
\max \{ n^{d-r-b_s} : (r, s) \text{ with } P_{rs} \neq 0, Q_{rs} = 0 \} \leq n^{-1}.
\]

Now, choose \( n \) large enough such that
\[
n^{-1} \cdot \max \{ \| \tilde{D}_r P_{rs} \tilde{B}_s \|_\infty, \| \tilde{D}_r^{-1} Q_{rs} \tilde{B}_s^{-1} \|_\infty \} < \delta,
\]
whence \( A \in B_\delta(P) \cap B_\delta(Q) \), i.e. \( B_\delta(P) \cap B_\delta(Q) \neq \emptyset \) as subsets of \( \mathcal{A}_d^k \). Since \( G_d^k \) is dense in \( \mathcal{A}_d^k \), there is an \( \tilde{A} \in G_d^k \) with \( \tilde{A} \in B_\delta(P) \cap B_\delta(Q) \) whence \( [\tilde{A}] \in U[p] \cap U[q] \). Therefore, \( y \) is not Hausdorff.

Proposition 5.1 shows that \( p_d^k \) is not Hausdorff: the configurations
\[
P = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
Q = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & 1 \\
1 & \cdots & 1 & 1
\end{pmatrix}
\]
are in \( T^{d+3} \). Thus, \( p_d^k \) is not Hausdorff since \([P] \) and \([Q] \) are of the described form of Proposition 5.1. For \( k > d + 3 \), some of the landmarks may be repeated.

**Example 5.2** In the case \( d = 1 \) and \( k = 4 \), Proposition 5.1 states that e.g. the topological subspace \( G_1^4 \) (no coincidences) together with the single pair coincidences \([p] \) with three distinct landmarks \( p_1, p_2, p_3 \) but \( p_3 = p_4 \) and \([q] \) with three distinct landmarks \( q_2, q_3, q_4 \) but \( q_1 = q_2 \), though being T1, is not Hausdorff. In fact, then
\[
[p] = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad [q] = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
1 & 1
\end{pmatrix}
\]
with \( l = m = 2 \).

6 Topological subspaces bounded by projective subspace numbers

Proposition 5.1 shows again that a space of shapes with a fixed pseudo-frame is a Hausdorff manifold. However, these kind of spaces are not closed under relabeling, i.e., they do not fulfill requirement (b) of the introduction. As a remedy we introduce the idea of bounding the number of landmarks in a projective subspace depending on its dimension.

To a vector \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) with \( 1 \leq n_1 < n_2 < \cdots < n_d \) define the topological subspace
\[
\mathcal{N}_d^k(n) = \{ p \in \mathcal{A}_d^k : |I| \leq n_j \text{ for all } (I, j) \in C(p) \},
\]
i.e., \( \mathcal{N}_d^k(n) \subseteq \mathcal{A}_d^k \) comprises those configurations \( p \) for which there will be at most \( n_j \) landmarks in any \((j-1)\)-dimensional projective subspace of \( \mathbb{R}P^d \). We will then say the topological subspace \( \mathcal{N}_d^k(n) \) is bounded by the projective subspace numbers \( n \). Note that \( \mathcal{N}_d^k(n) \) is closed under permutations and respects the hierarchy of projective subspace constraints, i.e. requirement (c) in the introduction, and contains \( G_d^k \) since \( n_j \geq j \) for all \( 1 \leq i \leq d \), while \( \mathcal{N}_d^k(n) = G_d^k \) if and only if \( n_j = j \) for all \( 1 \leq j \leq d \), and \( \mathcal{N}_d^k(n) = \mathcal{A}_d^k \) if and only if \( n_j \geq k \) for all \( 1 \leq j \leq d \).

We are interested in projective subspace numbers \( n \) which lead to Hausdorff spaces \( n_d^k(n) \). From Proposition 5.1 we can infer conditions for feasible \( n \in \mathbb{N}^d \) under which the corresponding shape space \( n_d^k(n) \) is a Hausdorff manifold.

**Theorem 6.1** Consider projective subspace numbers \( n = (n_1, \ldots, n_d) \). The following statements are equivalent:

(i) \( n_d^k(n) \) is Hausdorff.
(ii) \( n^k_d(n) \subseteq f^k_d \);

(iii) \( n^k_d(n) \) is an open, Hausdorff submanifold of \( f^k_d \);

(iv) \( n_j + n_{d+1-j} < k \) for all \( 1 \leq j \leq d \).

**Proof** First, assume (iv) \( n_j + n_{d+1-j} < k \) for all \( 1 \leq j \leq d \). If \( n^k_d(n) \) were not Hausdorff, there would be shapes \([p],[q] \in n^k_d(n)\) as in Proposition 5.4 with \((I,j) \in C(p)\) and \((I',d+1-j) \in C(q)\) for some \( I \subseteq \{1, \ldots, k\} \) and some \( j \in \{1, \ldots, d\} \) with \(|I| \leq n_j\) and \(|I'| \leq n_{d+1-j}\). But then \( k = |I| + |I'| \leq n_j + n_{d+1-j} \) in contradiction to the assumption. Additionally, \( n^k_d(n) \subseteq f^k_d \) since \( f^k_d \) does not contain any splittable shapes. Further, \( n^k_d \) is an open subset of \( f^k_d \) since it respects the hierarchy of projective subspace constraints, see Section 4. Then, \( n^k_d \) is a submanifold of the differentiable manifold \( f^k_d \) (Theorem 4.8) and (iii) holds.

Conversely, assume that \( n_j + n_{d+1-j} \geq k \) for some \( 1 \leq j \leq d \). Then, there are shapes \([p],[q] \in n^k_d(n)\) with \((I,j),(I',d+1-j) \in C(p)\). But those shapes are splittable and \([p] \neq BL([p]) \subseteq n^k_d(n)\) whence \( n^k_d(n) \) is not even \( T_1 \).

Now, there is a canonical partial order on \( \mathbb{N}^d \) induced by the component-wise total order on \( \mathbb{N} \). We call a vector \( n \in \mathbb{N}^d \) maximal if \( n^k_d(n) \) is Hausdorff and \( n^k_d(m) \) is not Hausdorff for any \( m > n \) with respect to that partial order. This notion of maximality accords with requirement (d) of the introduction.

Note that \( g^k_d \) is bounded by projective subspace numbers \( n_j = j \) for \( j \in \{1, \ldots, d\} \) whence \( g^k_d \) is a Hausdorff manifold since \( k \geq d + 3 \). However, this topological subspace is not maximal unless \( d = 1 \) and \( k = 4 \), since then \( n_1 + n_d = d + 1 \), so \( n_d \) can be increased by 1 without violating Theorem 6.11(iv) if \( d > 1 \), or \( n_1 \) and \( n_d \) if \( k > d + 3 \).

7 Tyler regular shapes

The space \( t^k_d \) of Tyler regular shapes (cf. Section 3) is a differentiable Hausdorff manifold since \( t^k_d = n^k_d(t) \) is bounded by the projective subspace numbers \( t = (t_1, \ldots, t_d) \) with

\[
t_j = \left\lfloor \frac{jk}{d+1} \right\rfloor - 1 \quad \text{for all } j \in \{1, \ldots, d\},
\]

and thus for these values \( t_j + t_{d+1-j} < \frac{jk}{d+1} + \frac{(d+1-j)k}{d+1} = k \). In fact, \( t^k_d \) is maximal for some choices for \( k \) and \( d \).

**Proposition 7.1** The vector \( t \in \mathbb{N}^d \) in Equation (8) of projective subspace numbers of \( t^k_d \) is maximal if and only if the greatest common divisor of \( k \) and \( d+1 \) is either 1 or 2. In particular, \( t^k_d \) is maximal for

(i) \( d = 1 \) and arbitrary \( k \geq d + 3 \),

(ii) arbitrary \( d \) and \( k = d + 3 \), as well as

(iii) relatively prime \( k \) and \( d + 1 \).

**Proof** If \( k \) and \( d+1 \) are relatively prime, then \( t_j + t_{d+1-j} = k - 1 \) for all \( 1 \leq j \leq d \) due to rounding. More precisely, \( t_j + t_{d+1-j} = k - 1 \) if \( \frac{jk}{d+1} \) is not integral. Otherwise \( k \) and \( d \) have a greatest common divisor \( c > 1 \), so \( j < d + 1 \) needs to be a multiple of \( \frac{d+1}{c} \). However,

\[
t_{(d+1)/c} + t_{d+1-(d+1)/c} = \left\lfloor \frac{d+1}{c} \cdot \frac{k}{d+1} \right\rfloor - 1 + \left\lfloor \frac{(c-1)(d+1)}{c} \cdot \frac{k}{d+1} \right\rfloor - 1
\]

\[
= \frac{k}{c} + (c-1)\frac{k}{c} - 2
\]

\[
= k - 2,
\]

whence \( t_{(d+1)/c} \) or \( t_{d+1-(d+1)/c} \) can be increased by 1 for \( c \neq 2 \) without violating Theorem 6.11(iv). In case \( c = 2 \), though, \( j = \frac{d+1}{2} = d + 1 - \frac{d+1}{2} \) is the only projective subspace dimension for which \( \frac{jk}{d+1} \) is integral and \( t_{(d+1)/2} = t_{d+1-(d+1)/2} = k/2 \) cannot be increased.
For example, $t_D^k$ is not maximal. Here, $t = (1, 3)$ which is not maximal since both $n = (1, 4)$ and $m = (2, 3)$ are larger and do not violate Theorem 0.3iv.

Since $t_D^k$ is a differentiable Hausdorff manifold, it may be equipped with a Riemannian metric, for example in the following way.

Recall that any Tyler regular configuration $p \in \mathcal{T}_d^k$ has a matrix representation $P$ fulfilling

$$P^t P = I_{d+1}$$

and

$$P_i P_i^t = \frac{d+1}{k} \quad \text{for all } i \in \{1, \ldots, k\}$$

(see Section 3). Again, this Tyler standardization $P$ is only unique up to multiplication of the rows by $\pm 1$ and right-multiplication by an orthogonal matrix, i.e. unique up to a compact group action, and can be considered as a projective pre-shape. Even more, we can remove the action of the orthogonal group by passing to the $(k \times k)$-matrix $PP^t$.

Now, the space of Tyler standardized configurations of Tyler regular shapes is a submanifold of $\mathbb{R}^{k \times (d+1)}$ and therefore naturally inherits a Riemannian metric from $\mathbb{R}^{k \times (d+1)}$. Since every element of the remaining group action acts as an isometry on $\mathbb{R}^{k \times (d+1)}$, the push-forward of the Riemannian metric on the space of Tyler standardized configurations to $t_D^k$ is a Riemannian metric on $t_D^k$.

This standardization suggests itself through the following geometric reasoning: consider a shape $[P] \in \mathcal{T}_d^k$ of full rank and one of its matrix configurations $P$. By definition, $P$ is only unique up to left-multiplication with non-singular diagonal $(k \times k)$-matrices and right-multiplication of non-singular $(d+1) \times (d+1)$-matrices. Indeed, we can view the columns of the $(k \times (d+1))$-matrix $P$ as a basis of a $(d+1)$-dimensional linear subspace of $\mathbb{R}^k$, and the action of $\text{GL}(d+1)$ as a change of basils. In particular, we can choose an orthonormal basis of the column space as a representation, i.e. a matrix $P$ with orthonormal columns. Then, $P^t P = I_{d+1}$ with $P$ being unique up to the action of $O(d+1)$ from the right.

Following this line of thought, we can think of the left-action of diagonal matrices as an action on the Grassmannian manifold $\text{Gr}(k, d+1)$ of $(d+1)$-dimensional linear subspaces of $\mathbb{R}^k$. Of course, elements of the Grassmannian $\text{Gr}(k, d+1)$ can be represented by the corresponding projection matrices

$$\mathcal{P}_P = P (P^t P)^{-1} P^t$$

onto the column space of $P$. This is the so-called Veronese-Whitney embedding of $\text{Gr}(k, d+1)$ into $\mathbb{R}^{k \times k}$. $\mathcal{P}_P$ is then a $(k \times k)$-matrix of rank and trace $d+1$. In this representation, the action of diagonal matrices on the Grassmannian acts infinitesimally like certain rotations in $\mathbb{R}^k$: for a non-singular diagonal matrix $D = \text{diag}(D_i)_{i=1, \ldots, k}$ in a sufficiently small neighborhood of $I_k$ use

$$(P^t D^2 P)^{-1} = (I_{d+1} - (I_{d+1} - P^t D^2 P))^{-1}$$

$$= \sum_{n=0}^{\infty} (I_{d+1} - P^t D^2 P)^n$$

and $\frac{\partial}{\partial D_{ij}} P_{ij} = e_i e_j^t$ with $e_i$ being the $i$-th canonical basis vector of $\mathbb{R}^k$ to obtain

$$\frac{\partial}{\partial D_{ij}} P_{ij} = \frac{\partial}{\partial D_{ij}} P (P^t D^2 P)^{-1} P^t D$$

$$= e_i e_j^t P (P^t D^2 P)^{-1} P^t D + DP (P^t D^2 P)^{-1} P^t e_i e_j^t$$

$$+ DP \left[ \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{i=1}^{n} (I_{d+1} - P^t D^2 P)^{n-1} (-2 D_i P^t e_i e_j^t P) (I_{d+1} - P^t D^2 P)^{n-1} \right] P^t D.$$

For $D = I_k$, $P^t P = I_{d+1}$, and consequently $P^t D^2 P = I_{d+1}$, $D_i = 1$, $\mathcal{P}_P = PP^t$, we conclude

$$\frac{\partial}{\partial D_{ij}} P_{ij} = e_i e_j^t PP^t + PP^t e_i e_j^t - 2PP^t e_i e_j^t PP^t$$

antisymmetric

while the infinitesimal action of the orthogonal group $O(k)$ acting by conjugation is given by

$$\frac{\partial}{\partial t_{\text{lin}} \circ O(t)} \mathcal{P}_P O(t)^t = \dot{O}(t) \mathcal{P}_P + \mathcal{P}_P \dot{O}(t)^t$$

$$= \dot{O}(t) \mathcal{P}_P - \mathcal{P}_P \dot{O}(t)$$
for a differentiable curve $R \ni t \mapsto O(t) \in O(k)$, $O(0) = I_k$ with antisymmetric $\hat{O}(0) \in so(k) = \{ M \in R^{k \times k} : M = -M^t \}$. Hence, the diagonal matrices act infinitesimally like certain rotations. In fact, $\frac{d}{dt}PDP$ is an infinitesimal rotation in the plane spanned by $Ppe_i$ and $e_i$. This suggests to fix the angle

$$\langle e_i, Ppe_i \rangle = e_i^tPpe_i = e_i^tPpPpe_i = \|Ppe_i\|^2 = P_i P_i^t$$

for all $1 \leq i \leq k$ in order to standardize the projection matrix $P$ and thus the configuration $P$. Of course, we require invariance under permutations whence all directions $e_i$, resp. landmarks $P_i$ have to be treated equally, i.e.

$$P_i P_i^t = C \in R$$

for all $1 \leq i \leq k$. The constant $C$ has to be $\frac{4 k}{d}$, since the values $P_i P_i^t$ are the diagonal elements of $P_P$ and $P_P$ has trace $d + 1$ as it is the orthogonal projection onto a $(d + 1)$-dimensional linear subspace. We thus obtain Equation (10).

This discussion of Tyler standardization shows that the topological subspace of Tyler regular shapes is a topological subspace of the quotient of a Grassmannian with a finite group action (multiplication of the rows by $\pm 1$) whence we can obtain a Riemannian metric on this space by considering one on the Grassmannian: the tangent space at the point $P = P (P^t P)^{-1} P^t$ is

$$\{ [P, A] : A \in so(k), \text{diag} [P, A] = 0 \},$$

with the standard Riemannian metric $\langle A, B \rangle = \text{tr}(A^t B)$ on $R^{k \times k}$ which up to a constant induces the very metric given above.

A result by Tyler [11], cf. [7], shows that Tyler standardization is possible for the Tyler regular shapes defined in Section 3, the only other ones for which it is possible are those splittable shapes $[p]$ for which $|I| = \frac{d}{d+1}$ and $|I^t| = \frac{(d+1)(d+2)}{dk}$, for any projective subspace constraint $(I, j) \in C(p)$ with $(I^t, d + 1 - j) \in C(p)$ and $|I| < \frac{d+1}{k}$ otherwise. The latter can obviously only exist when $d + 1$ and $k$ have a common divisor. The space of projective shapes which allow Tyler standardization then does not respect the hierarchy of projective subspace constraints if there exists such a splittable Tyler standardizable shape. However, it can be shown to be closed under permutations and a differentiable manifold by identifying these splittable configurations with those in its blur. Unfortunately, it is unclear if the Riemannian metric given above can be extended to this subspace since the remaining discrete group action is not free on the splittable Tyler standardized configurations. Even worse, the metric on $t^k_d$ given above is not complete if splittable Tyler standardizable shapes exist.

If $t^k_d$ is not maximal, i.e., if and only if $d + 1$ and $k$ have a common divisor greater than 2, then $t^k_d$ is a submanifold of a larger feasible topological subspace bounded by projective subspace numbers. However, the Riemannian metric on $t^k_d$ given above cannot be extended to the larger topological subspace since the elements lying in the blur of a splittable Tyler standardizable shape would have distance 0 in this extension, i.e., the extension cannot be a metric.

**Example 7.2** In the case $d = 1$ and $k = 4$, the Tyler standardizable shapes are the Tyler regular ones, i.e., those in general position, and the three splittable shapes with double pair coincidences

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

**8 Discussion**

The subject of this article was to find a reasonable differentiable Hausdorff submanifold of projective shape space. It turns out that the topological subspace comprising shapes of configurations with trivial isotropy group is only a differentiable T1 manifold, but not Hausdorff in contrast to the situation in similarity and affine shape spaces, cf. [2, 6] resp. [4, 10]. Charts were constructed by introducing the concept of pseudo-frames generalizing the well-known notion of projective frames.

Additionally, by bounding the number of landmarks per projective subspace of $RP^d$, a new class of reasonable topological subspaces, namely those bounded by projective subspace numbers, was introduced. For this class, a criterion was given for deciding whether these topological subspaces are differentiable Hausdorff manifolds. Indeed, one of these topological subspaces has been
considered in literature before, namely the space of Tyler regular shapes. By Tyler standardization, for which we presented new, geometric arguments, this topological subspace can be endowed with a Riemannian metric. When it is maximal in the class of topological subspaces bounded by projective subspace numbers, one could say that it fulfills all of the requirements except that the Riemannian metric might not be complete.

However, it remains unclear how to endow other topological subspaces with a complete Riemannian metric, in particular in cases where the topological subspace of Tyler regular shapes is not maximal.

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