A TWO-COMPONENT GEODESIC EQUATION ON A SPACE OF
CONSTANT POSITIVE CURVATURE

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Abstract. We propose a new two-component geodesic equation with the unusual
property that the underlying space has constant positive curvature. In the special
case of one space dimension, the equation reduces to the two-component Hunter-
Saxton equation.

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1. Introduction

Many basic equations in physics, including the Euler equations of motion of a rigid
body, and the Euler equations of fluid dynamics of an inviscid incompressible fluid, can
be viewed as geodesic equations on a Lie group endowed with a one-sided invariant
metric [1]. In the context of rigid bodies, the group is $SO(3)$; in the context of
incompressible fluids, it is the group $Diff_{\mu}(\mathbb{R}^3)$ of volume-preserving diffeomorphisms.
Other geodesic equations include Burgers’ equation ($Diff(S^1)$ with a right-invariant
$L^2$ metric), the KdV equation (Virasoro group with a right-invariant $L^2$ metric), and
the Camassa-Holm equation ($Diff(S^1)$ with a right-invariant $H^1$ metric), see [15] for
a survey.

In the recent work [7], the following equation was introduced and studied:

$$d \operatorname{div} u_t = -d \left( \iota_u d \operatorname{div}(u) + \frac{1}{2} \operatorname{div}(u)^2 \right),$$

where $u$ is a time-dependent vector field on a compact Riemannian manifold $M$,
$d$ denotes the exterior derivative, and $\iota$ denotes the interior product. Two of the
remarkable properties of (1) are: (a) It is the geodesic equation on a space of constant
positive curvature. Indeed, equation (1) is the equation for geodesic flow on the space
of right cosets $Diff(M)/Diff_{\mu}(M)$ endowed with the right-invariant metric given at
the identity by

$$\langle u, v \rangle = \frac{1}{4} \int_M \operatorname{div}(u) \operatorname{div}(v) d\mu,$$

and this space has constant curvature equal to $1/\mu(M) > 0$. (b) It is an integrable
evolution equation in any number of space dimensions. Indeed, it is shown in [7] that
(1) admits a complete set of independent integrals in involution.

In this paper we propose the following two-component generalization of (1):

$$\begin{cases} d \operatorname{div} u_t = -d \left( \iota_u d \operatorname{div}(u) + \frac{1}{2} \operatorname{div}(u)^2 - \frac{1}{2} \rho^2 \right), \\
\rho_t = -\operatorname{div}(\rho u), \end{cases}$$

where $\rho$ is a time-dependent real-valued function on $M$. We will show that (2) is the
geodesic equation on the homogeneous space $[Diff(M) \ast \mathcal{C}^\infty(M)]/Diff_{\mu}(M)$ endowed
with the right-invariant metric given at the identity by
\[
\langle\langle (u_1, u_2), (v_1, v_2) \rangle\rangle = \frac{1}{4} \int_M \left[ \text{div}(u_1) \text{div}(v_1) + u_2 v_2 \right] d\mu,
\]
and that this space has constant positive curvature.

Our search for a two-component generalization of (1) was motivated by the special case when \( M \) is the unit circle, i.e. \( M = S^1 \). In this case, equation (1) reduces to the Hunter-Saxton equation \[6\]
\[
u_{txx} = -\left( \nu u_{xx} + \frac{1}{2} \nu^2_x \right)_x,
\]
which admits the following integrable two-component generalization, see \[3, 16\]:
\[
\begin{align*}
\nu_{txx} &= -\left( \nu u_{xx} + \frac{1}{2} \nu^2_x - \frac{1}{2} \rho^2 \right)_x, \\
\rho_t &= -(\rho\nu)_x.
\end{align*}
\]
Equation (4) describes the geodesic flow on a sphere of constant positive curvature \[8, 12\] and it was recently observed that the two-component version (5) also is the geodesic equation on a space of constant positive curvature \[10\].

Since the three special cases (1), (4), and (5) of the system (2) are all integrable, we expect (2) to also be integrable, but the proof of this remains open.

Our main result is stated in section 2 and its proof is presented in section 3.

2. Main result

Let \( M \) be a compact connected \( n \)-dimensional Riemannian manifold and let \( \text{Diff}(M) \) denote the space of orientation-preserving diffeomorphisms of \( M \). We let \( G \) denote the semidirect product \( G = \text{Diff}(M) \circledcirc C^\infty(M) \) with multiplication given by
\[
(\varphi, f)(\psi, g) = (\varphi \circ \psi, g + f \circ \psi), \quad (\varphi, f), (\psi, g) \in G.
\]
Moreover, we let \( H = \text{Diff}_\mu(M) \) denote the subgroup of \( \text{Diff}(M) \) of volume-preserving diffeomorphisms. The group \( H \) acts on \( G \) by
\[
\psi \cdot (\varphi, f) = (\psi \circ \varphi, f), \quad \psi \in H, \ (\varphi, f) \in G,
\]
and our main interest lies in the homogeneous space \( G/H \) of right cosets.

We use right-invariance to extend (3) to a (degenerate) metric \( \langle\langle \cdot, \cdot \rangle\rangle \) on \( G \). Thus,
\[
\langle\langle U, V \rangle\rangle_{(\varphi, f)} = \frac{1}{4} \int_M \left[ \text{div}(U_1 \circ \varphi^{-1}) \text{div}(V_1 \circ \varphi^{-1}) + (U_2 \circ \varphi^{-1})(V_2 \circ \varphi^{-1}) \right] d\mu,
\]
where \( U = (U_1, U_2) \) and \( V = (V_1, V_2) \) are elements of \( T_{(\varphi, f)}G \) and \( d\mu \) denotes the volume form on \( M \) induced by the metric. It is straightforward to verify that this metric descends to a Riemannian metric on \( G/H \) (see section 3 for details), and we can now state our main result.

**Theorem 1.** The sectional curvature of \( G/H \) equipped with the metric \( \langle\langle \cdot, \cdot \rangle\rangle \) is constant and equal to \( 1/\mu(M) > 0 \). Moreover, the corresponding Euler equation for the geodesic flow is the two-component equation (2).

**Remark 2.** As far as regularity assumptions and function spaces are concerned, the geometry of equation (2) can be developed in a number of different settings, including the \( H^s \) and \( C^m \) settings (see \[5\] for further details in a similar situation). Since these issues are of no consequence for the considerations here, we have chosen for simplicity to use the notation of the smooth category.
3. Proof

We will use the following notations:
- \( e = (\text{id}, 0) \) will denote the identity element of \( G \).
- \( \mathfrak{X}(M) \) will denote the space of vector fields on \( M \).
- For \( p \geq 0 \), \( \Omega^p(M) \) will denote the space of \( p \)-forms on \( M \).
- \( \langle \cdot, \cdot \rangle \) will denote the metric on \( M \). \( \langle \cdot, \cdot \rangle \) will also denote the induced inner product on \( p \)-forms, as well as the natural pairing between elements of \( TM \) and \( T^*M \).
- If \( X,Y \) are vector fields on \( M \), \([X,Y]\) will denote their Lie bracket, given locally by \([X,Y] = DY \cdot X - DX \cdot Y\).
- If \( u = (u_1,u_2) \) and \( v = (v_1,v_2) \) are elements of \( T_eG \simeq \mathfrak{X}(M) \times C^\infty(M) \), \([u,v] \in T_eG \) will denote the following commutator:
  \[
  [u,v] = \left( \frac{\partial}{\partial v_2}(u_1) - \frac{\partial}{\partial u_2}(v_1) \right).
  \]

3.1. The metric on \( G/H \). We will first show that the metric \( \langle \cdot, \cdot \rangle \) descends to \( G/H \). According to the general theory of homogeneous spaces (see Proposition 3.1 in Chapter X of \[9\]), \( \langle \cdot, \cdot \rangle \) descends to \( G/H \) provided that
  \[
  \langle [\text{ad}_w u], v \rangle + \langle u, [\text{ad}_w v] \rangle = 0 \quad \text{for all} \quad u, v \in T_eG \text{ and } w \in T_eH,
  \]
where we view \( H \simeq \text{Diff}_\mu(M) \times \{1\} \subset G \) as a subgroup of \( G \). Using that the adjoint action in \( T_eG \) is given by
  \[
  \text{ad}_u v = -[u, v], \quad u, v \in T_eG,
  \]
it is straightforward to verify (6): Given elements \( u = (u_1,u_2), v = (v_1,v_2), \) and \( w = (w_1,w_2) \), we have
  \[
  \langle [\text{ad}_w u], v \rangle + \langle u, [\text{ad}_w v] \rangle = \frac{1}{4} \int_M \left( \text{div} [u_1, w_1] \text{ div } v_1 + \text{div } u_1 \text{ div } [v_1, w_1] \right) d\mu 
  \]
  \[
  + \frac{1}{4} \int_M \left( (\text{div}(w_2(u_1) - \text{div}(w_2(u_1))) v_2 + u_2(\text{div}(w_2(u_1) - \text{div}(w_2(u_1))) \right) d\mu.
  \]
If \( w \in T_eH \) then \( \text{div } w_1 = 0 \) and \( w_2 = 0 \), and the right-hand side of (7) equals
  \[
  \frac{1}{4} \int_M \left[ \langle [u_1, \text{grad } w_1] - \langle w_1, \text{grad } u_1 \rangle \rangle \text{ div } v_1 
  \right.
  \]
  \[
  + \text{div } u_1 \left( \langle [v_1, \text{grad } w_1] - \langle w_1, \text{grad } v_1 \rangle \rangle \right) d\mu - \frac{1}{4} \int_M \text{div}(w_2v_2u_1) d\mu
  \]
  \[
  = -\frac{1}{4} \int_M \text{div}(w_1 \text{ div } v_1 \text{ div } u_1) d\mu = 0.
  \]
This shows that \( \langle \cdot, \cdot \rangle \) descends to \( G/H \).

3.2. The Euler equation. We next show that the Euler equation associated with the metric \( \langle \cdot, \cdot \rangle \) on \( G/H \) is the two-component equation (2). It follows from the general theory developed by Arnold \[1\] that the Euler equation is given by
  \[
  u_t = B(u, u),
  \]
where \( u(t) \) is a curve in \( T_eG \) and the bilinear operator \( B : T_eG \times T_eG \to T_eG \) is defined by the condition
  \[
  \langle B(u, v), w \rangle = \langle [v, w], u \rangle, \quad u, v, w \in T_eG.
  \]
In order to determine $B$, we define the operator $A : \mathfrak{X}(M) \to \Omega^1(M)$ by $Av = d\delta v^\flat$, where $\delta$ denotes the codifferential on $M$ and $\delta : TM \to \mathfrak{X}(M)$ denotes the musical isomorphism with inverse $\sigma : T^*M \to TM$ induced by the metric $\langle \cdot, \cdot \rangle$. Recalling that $\text{div}(X) = -\delta X^\flat$ for a vector field $X$, the condition (10) can be written as

$$\int_M (AB_1(u, v), w_1)d\mu + \int_M B_2(u, v)w_2d\mu = \int_M (Au_1, [v, w]_1)d\mu + \int_M u_2[v, w]_2d\mu,$$

where $B_1$ and $B_2$ denote the two components of $B$. Using the general formula

$$[v_1, w_1] = \text{div}(w_1)v_1^\flat - \text{div}(v_1)w_1^\flat - \delta(v_1 \wedge w_1^\flat),$$

we see that the right-hand side of (10) equals

$$\int_M (Au_1, \text{div}(w_1)v_1^\flat) - \text{div}(v_1)w_1^\flat - \delta(v_1 \wedge w_1^\flat)d\mu + \int_M u_2(\text{div}(w_1) - \delta(v_1 \wedge w_1^\flat))d\mu$$

$$= \int_M (-d(Au_1, v_1) - \text{div}(v_1)Au_1 - \delta(v_1, Au_1, w_1)d\mu - \int_M (w_2 \text{div}(u_2v_1) + \langle u_2dv_2, w_1 \rangle)d\mu.$$  

Thus, since $w$ is arbitrary and $dAu_1 = 0$, we find

$$AB_1(u, v) = -d(Au_1, v_1) - \text{div}(v_1)Au_1 - u_2dv_2,$$

$$B_2(u, v) = -\text{div}(u_2v_1).$$

Substituting this expression for $B = (B_1, B_2)$ into (8), we find the two-component system (2).

3.3. The operator $A$. Equation (12) only fixes the value of $B_1$ up to an element of ker $A$. This is a reflection of the fact that the inner product (3) is degenerate along $T_eH$ in $T_eG$. Accordingly, the solution of the Euler equation (8) is not uniquely determined in $T_eG$ by the initial data, but descends to a well-defined curve in $T_e(G/H)$.

The following lemma, which will be needed for the computation of the curvature, clarifies some properties of $A$ and $A^{-1}$. In particular, it shows that the kernel of $A$ equals $T_eH$ and consists of all divergence-free vector fields on $M$.

**Lemma 3.** The operator $A : \mathfrak{X}(M) \to \Omega^1(M)$ defined by $Av = d\delta v^\flat = -d \text{div} v$ satisfies

$$\text{im } A = \{df \mid f \in C^\infty(M)\}$$

and

$$\ker A = \{v \in \mathfrak{X}(M) \mid \text{div } v = 0\}.$$  

In particular, the map $\text{div } A^{-1}d : C^\infty(M) \to C^\infty(M)$ is well-defined and is given by

$$\text{div } A^{-1}d f = -f + \frac{1}{\mu(M)} \int_M f d\mu.$$  

**Proof.** The inclusion $\subset$ in (13) is obvious. On the other hand, if $f \in C^\infty(M)$, then standard properties of the Laplacian $\Delta = d\delta + \delta d$ on a compact manifold (see [14]) imply that $\ker \Delta$ consists of the constant functions on $M$ and that there exists $g \in C^\infty(M)$ such that $\Delta g = f - \frac{1}{\mu(M)} \int_M f d\mu$. Letting $v = (dg)^\flat$, we find

$$-\text{div } v = f - \frac{1}{\mu(M)} \int_M f d\mu$$

and so

$$Av = -d \text{div } v = df.$$
This proves (13). In order to prove (14), we first note that if \( \text{div} \, v = 0 \), then \( A v = 0 \). Conversely, suppose \( A v = 0 \). Then

\[
0 = \int_M \langle d\delta v^\flat, v^\flat \rangle d\mu = \int_M \langle \delta v^\flat, \delta v^\flat \rangle d\mu
\]

and we find \( \delta v^\flat = - \text{div} \, v = 0 \). This proves (14).

It is clear from (13) and (14) that the map \( \text{div} A^{-1} : C^\infty(M) \to C^\infty(M) \) is well-defined. Letting \( h = \text{div} A^{-1} df \), we find

\[
dh = -AA^{-1} df = -df.
\]

Thus, \( h = -f + c \), where \( c \) is a constant. Noting that

\[
0 = \int_M h d\mu = - \int_M f d\mu + c \mu(M),
\]

we infer that

\[
c = \frac{1}{\mu(M)} \int_M f d\mu.
\]

This proves (15). \( \square \)

**Remark 4.** In terms of the Hodge decomposition

\[
T^*_e \text{Diff}(M) = d\Omega^0(M) \oplus \delta\Omega^2(M) \oplus \mathcal{H}^1,
\]

where \( \mathcal{H}^k \) denotes the space of harmonic \( k \)-forms on \( M \), we have \( \ker A = T_eH = (\delta\Omega^2(M) \oplus \mathcal{H}^1)^\flat \), see Figure 1. It follows that the value of \( B_1 \) can be fixed by requiring that \( B_1(u,v) \in \{ (df)^\flat | f \in C^\infty(M) \} \). However, we will not need to do this.

### 3.4. Constant positive curvature

It remains to prove that \( (G/H, \langle \cdot, \cdot \rangle) \) has constant sectional curvature equal to \( 1/\mu(M) \). Letting \( R \) denote the curvature tensor on \( G/H \), this is equivalent to proving that

\[
\langle R(u,v)v, u \rangle = \frac{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2}{\mu(M)}, \quad u, v \in T_eG.
\]
We will use the following curvature formula which is derived in the appendix:

$$\langle\langle R(u,v)w, w \rangle\rangle = \langle\langle \delta, \delta \rangle\rangle + \langle\langle [u,v], \beta \rangle\rangle - \frac{3}{4} \langle\langle [u,v], [u,v] \rangle\rangle - \langle\langle B(u,u), B(v,v) \rangle\rangle,$$  \hspace{1cm} (18)

where $\delta$ and $\beta$ are defined by

$$\delta = \frac{1}{2}(B(u,u) + B(v,v)), \hspace{1cm} \beta = \frac{1}{2}(B(u,v) - B(v,u)).$$

**Remark 5.** 1. Formula \((18)\) coincides formally with the curvature formula for a Lie group equipped with a left-invariant metric given in Appendix 2 of [2]. In the appendix, we show that this formula remains valid in the case of a right-invariant metric on a homogeneous space.

2. Although $B_1$ is only defined up to an element of $\ker A$, Lemma 3 implies that the right-hand side of \((18)\) is well-defined. Indeed, we will see that $A^{-1}$ only enters \((18)\) via the combination $\mathrm{div} A^{-1}df$ and by \((15)\) this operator is uniquely defined.

Our proof of \((17)\) will be a long computation using \((18)\) together with the expression \((12)\) for $B$. First note that

$$\delta = -\frac{1}{2} \left( A^{-1} [d(Au_1, v_1) + \mathrm{div} v_1 Au_1 + u_2 dv_2 + d(Av_1, u_1) + \mathrm{div} u_1 Av_1 + v_2 du_2] \right)$$

$$= -\frac{1}{2} \left( A^{-1} [d(Au_1, v_1) + \langle Av_1, u_1 \rangle - \mathrm{div} v_1 \mathrm{div} u_1 + v_2 u_2, u_1] \right).$$ \hspace{1cm} (19)

and

$$\beta = -\frac{1}{2} \left( A^{-1} [d(Au_1, v_1) + \mathrm{div} v_1 Au_1 + u_2 dv_2 - d(Av_1, u_1) - \mathrm{div} u_1 Av_1 - v_2 du_2] \right).$$ \hspace{1cm} (20)

We will consider the four terms on the right-hand side of \((18)\) in turn. Let $\delta_1$ and $\delta_2$ denote the two components of $\delta$. Equations \((19)\) and \((15)\) yield

$$\mathrm{div} \delta_1 = \frac{1}{2} (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \mathrm{div} v_1 \mathrm{div} u_1 + v_2 u_2)$$

$$= -\frac{1}{2} \mu(M) \int_M (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \mathrm{div} v_1 \mathrm{div} u_1 + v_2 u_2) d\mu.$$

Thus the first term on the right-hand side of \((18)\) is given by

$$\langle\langle \delta, \delta \rangle\rangle = \frac{1}{4} \int_M (\mathrm{div} \delta_1)^2 d\mu + \frac{1}{16} \int_M (\mathrm{div}(u_2 v_1 + v_2 u_1))^2 d\mu$$

$$= \frac{1}{16} \int_M (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \mathrm{div} v_1 \mathrm{div} u_1 + v_2 u_2)^2 d\mu$$

$$- \frac{1}{16} \mu(M) \left[ \int_M (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \mathrm{div} v_1 \mathrm{div} u_1 + v_2 u_2) d\mu \right]^2$$

$$+ \frac{1}{16} \int_M (\mathrm{div}(u_2 v_1 + v_2 u_1))^2 d\mu.$$ \hspace{1cm} (21)

Using the expression \((20)\) for $\beta$, we deduce that the second term on the right-hand side of \((18)\) is given by

$$\langle\langle [u,v], \beta \rangle\rangle = -\frac{1}{8} \int_M \langle [u_1, v_1], d(Au_1, v_1) + \mathrm{div} v_1 Au_1 + u_2 dv_2$$

$$- d(Av_1, u_1) - \mathrm{div} u_1 Av_1 - v_2 du_2 \rangle d\mu$$ \hspace{1cm} (22)
Thus the fourth term on the right-hand side of (18) is given by

\[-\frac{1}{8} \int_M (\langle u_1, dv_2 \rangle - \langle v_1, du_2 \rangle) \, \text{div}(u_2v_1 - v_2u_1) \, d\mu,\]

while the third term is given by

\[-\frac{3}{4} \langle [u,v], [u,v] \rangle = -\frac{3}{16} \int_M (\text{div} [u_1,v_1])^2 \, d\mu - \frac{3}{16} \int_M (\langle u_1, dv_2 \rangle - \langle v_1, du_2 \rangle)^2 \, d\mu.\]

(23)

Moreover,

\[B(u,u) = -\left[A^{-1}d \left[ \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div} u_1)^2 + \frac{1}{2} u_2^2 \right] \right],\]

so that, by (15),

\[\text{div } B_1(u,u) = \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div} u_1)^2 + \frac{1}{2} u_2^2 - \frac{1}{\mu(M)} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div} u_1)^2 + \frac{1}{2} u_2^2 \right) \, d\mu.\]

Thus the fourth term on the right-hand side of (18) is given by

\[-\langle B(u,u), B(v,v) \rangle = -\frac{1}{4} \int_M \text{div } B_1(u,u) \text{ div } B_1(v,v) \, d\mu - \frac{1}{4} \int_M B_2(u,u) B_2(v,v) \, d\mu \]
\[= -\frac{1}{4} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div} u_1)^2 + \frac{1}{2} u_2^2 \right) \left( \langle Av_1, v_1 \rangle - \frac{1}{2} (\text{div} v_1)^2 + \frac{1}{2} v_2^2 \right) \, d\mu \]
\[+ \frac{1}{4 \mu(M)} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div} u_1)^2 + \frac{1}{2} u_2^2 \right) \, d\mu \]
\[\times \int_M \left( \langle Av_1, v_1 \rangle - \frac{1}{2} (\text{div} v_1)^2 + \frac{1}{2} v_2^2 \right) \, d\mu - \frac{1}{4} \int_M \text{div } (u_1u_2) \text{ div } (v_1v_2) \, d\mu.\]

(24)

Adding the four contributions from (21)-(24), we find an expression for \(\langle R(u,v)v, u \rangle\) in terms of \(u_1, v_1, u_2,\) and \(v_2\). It is convenient to divide this expression into two terms,

\[\langle R(u,v)v, u \rangle = I_1 + I_2,\]

where \(I_1\) consists of all the terms that contain neither \(u_2\) nor \(v_2\), whereas \(I_2\) consists of the remaining terms that contain \(u_2\) or \(v_2\). Equations (21)-(24) yield

\[I_1 = \frac{1}{16} \int_M \left( \langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \text{div } v_1 \text{ div } u_1 \right)^2 \, d\mu \]
\[\frac{1}{16\mu(M)} \left\{ \int_M (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \text{div } v_1 \text{ div } u_1) \, d\mu \right\}^2 \]
\[\frac{1}{8} \int_M \langle [u_1, v_1], d(Au_1, v_1) + \text{div } v_1 Au_1 - d(Av_1, u_1) - \text{div } u_1 Av_1 \rangle \, d\mu \]
\[\frac{3}{16} \int_M (\text{div} [u_1, v_1])^2 \, d\mu \]
\[-\frac{1}{4} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div } u_1)^2 \right) \left( \langle Av_1, v_1 \rangle - \frac{1}{2} (\text{div } v_1)^2 \right) \, d\mu \]
\[\frac{1}{4 \mu(M)} \left\{ \int_M (\langle Au_1, u_1 \rangle - \frac{1}{2} (\text{div } u_1)^2) \, d\mu \right\} \left\{ \int_M (\langle Av_1, v_1 \rangle - \frac{1}{2} (\text{div } v_1)^2) \, d\mu \right\}.\]

Simplification using the integration by parts identity

\[\int_M (d\alpha, \beta) d\mu = \int_M (\alpha, d\beta) d\mu\]
yields

\[ I_1 = \frac{1}{16} \int_M \left( \langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \operatorname{div} v_1 \operatorname{div} u_1 \right)^2 d\mu \\
- \frac{1}{16\mu(M)} \left( \int_M \operatorname{div} v_1 \operatorname{div} u_1 d\mu \right)^2 \\
+ \frac{1}{8} \int_M \left( \langle Au_1, v_1 \rangle - \langle Av_1, u_1 \rangle \right) \operatorname{div} [u_1, v_1] d\mu \\
- \frac{1}{8} \int_M \langle [u_1, v_1], \operatorname{div} v_1 Au_1 - \operatorname{div} u_1 Av_1 \rangle d\mu \\
- \frac{3}{16} \int_M \langle \operatorname{div} [u_1, v_1] \rangle^2 d\mu \\
- \frac{1}{4} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2} (\operatorname{div} u_1)^2 \right) \left( \langle Av_1, v_1 \rangle - \frac{1}{2} (\operatorname{div} v_1)^2 \right) d\mu \\
+ \frac{1}{16\mu(M)} \left( \int_M (\operatorname{div} u_1)^2 d\mu \right) \left( \int_M (\operatorname{div} v_1)^2 d\mu \right) .
\]

Employing the identity [11] as well as the identity

\[ \operatorname{div} [u_1, v_1] = \langle u_1, \operatorname{grad} \operatorname{div} v_1 \rangle - \langle v_1, \operatorname{grad} \operatorname{div} u_1 \rangle = \langle Au_1, v_1 \rangle - \langle Av_1, u_1 \rangle , \]

we can write this as

\[ \frac{1}{16} \int_M \langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle d\mu - \frac{1}{8} \int_M \langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle \operatorname{div} v_1 \operatorname{div} u_1 d\mu \\
+ \frac{1}{16} \int_M (\operatorname{div} u_1)^2 (\operatorname{div} v_1)^2 d\mu - \frac{1}{16\mu(M)} \left( \int_M \operatorname{div} v_1 \operatorname{div} u_1 d\mu \right)^2 \\
- \frac{1}{16} \int_M \langle Au_1, v_1 \rangle - \langle Av_1, u_1 \rangle \rangle^2 d\mu \\
- \frac{1}{8} \int_M \langle u_1 \operatorname{div} v_1 - v_1 \operatorname{div} u_1 - (\delta (u_1^2 \wedge v_1^2))^2, \operatorname{div} v_1 Au_1 - \operatorname{div} u_1 Av_1 \rangle d\mu \\
- \frac{1}{4} \int_M \langle Au_1, u_1 \rangle \langle Av_1, v_1 \rangle d\mu + \frac{1}{8} \int_M \langle Au_1, u_1 \rangle (\operatorname{div} v_1)^2 d\mu \\
+ \frac{1}{8} \int_M \langle Av_1, v_1 \rangle (\operatorname{div} u_1)^2 d\mu - \frac{1}{16} \int_M (\operatorname{div} u_1)^2 (\operatorname{div} v_1)^2 d\mu \\
+ \frac{1}{16\mu(M)} \left( \int_M (\operatorname{div} u_1)^2 d\mu \right) \left( \int_M (\operatorname{div} v_1)^2 d\mu \right) .
\]

Finally, using that

\[ \int_M \langle (\delta (u_1^2 \wedge v_1^2)), \operatorname{div} v_1 Au_1 - \operatorname{div} u_1 Av_1 \rangle d\mu = 2 \int_M \langle u_1^2 \wedge v_1^2, Au_1 \wedge Av_1 \rangle d\mu \\
= 2 \int_M \left[ \langle u_1, Au_1 \rangle \langle v_1, Av_1 \rangle - \langle u_1, Av_1 \rangle \langle v_1, Au_1 \rangle \right] d\mu ,
\]

we arrive after several cancellations at the following expression for \( I_1 \):

\[ I_1 = \frac{1}{16\mu(M)} \left( \int_M (\operatorname{div} u_1)^2 d\mu \right) \left( \int_M (\operatorname{div} v_1)^2 d\mu \right) - \frac{1}{16\mu(M)} \left( \int_M \operatorname{div} v_1 \operatorname{div} u_1 d\mu \right)^2 . \]
On the other hand, the terms in equations (21)-(24) that contain \( u_2 \) or \( v_2 \) yield

\[
I_2 = \frac{1}{16} \int_M \left( 2(\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \text{div} v_1 \text{div} u_1)u_2v_2 + (u_2v_2)^2 \right) d\mu
\]

\[
- \frac{1}{8\mu(M)} \left( \int_M (\langle Au_1, v_1 \rangle + \langle Av_1, u_1 \rangle - \text{div} v_1 \text{div} u_1) d\mu \right) \left( \int_M u_2v_2 d\mu \right)
\]

\[
- \frac{1}{16\mu(M)} \left( \int_M u_2v_2 d\mu \right)^2 + \frac{1}{16} \int_M (\text{div} (u_2v_1 + v_2u_1))^2 d\mu
\]

\[
- \frac{1}{8} \int_M \langle [u_1, v_1], u_2v_2 - v_2du_2 \rangle d\mu
\]

\[
- \frac{1}{8} \int_M (\langle u_1, dv_2 \rangle - \langle v_1, du_2 \rangle) \text{div}(u_2v_1 - v_2u_1) d\mu
\]

\[
- \frac{3}{16} \int_M (\langle u_1, dv_2 \rangle - \langle v_1, du_2 \rangle)^2 d\mu
\]

\[
- \frac{1}{8} \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2}(\text{div} u_1)^2 \right) v_2^2 d\mu
\]

\[
- \frac{1}{8} \int_M \left( \langle Av_1, v_1 \rangle - \frac{1}{2}(\text{div} v_1)^2 \right) u_2^2 d\mu - \frac{1}{16} \int_M (u_2v_2)^2 d\mu
\]

\[
+ \frac{1}{8\mu(M)} \left\{ \int_M \left( \langle Au_1, u_1 \rangle - \frac{1}{2}(\text{div} u_1)^2 \right) d\mu \right\} \left( \int_M v_2^2 d\mu \right)
\]

\[
+ \frac{1}{8\mu(M)} \left\{ \int_M \left( \langle Av_1, v_1 \rangle - \frac{1}{2}(\text{div} v_1)^2 \right) d\mu \right\} \left( \int_M u_2^2 d\mu \right)
\]

\[
+ \frac{1}{16\mu(M)} \left( \int_M u_2^2 d\mu \right) \left( \int_M v_2^2 d\mu \right) - \frac{1}{4} \int_M \text{div}(u_1u_2) \text{div}(v_1v_2) d\mu.
\]

We use the identity

\[
\text{div}(u_2v_1 + v_2u_1) = u_2 \text{div} v_1 + \langle dv_2, v_1 \rangle + v_2 \text{div} u_1 + \langle dv_2, u_1 \rangle
\]

in the third line, and the identity (11) in the fourth line of (26). Moreover, we use the identity

\[
\text{div}(u_2v_1 - v_2u_1) = u_2 \text{div} v_1 + \langle dv_2, v_1 \rangle - v_2 \text{div} u_1 - \langle dv_2, u_1 \rangle
\]

in the fifth line of (26) and combine the result with the sixth line. After simplification this leads to

\[
I_2 = \frac{1}{8} \int_M u_2v_2 \langle Au_1, v_1 \rangle d\mu + \frac{1}{8} \int_M u_2v_2 \langle Av_1, u_1 \rangle d\mu
\]

\[
- \frac{1}{8} \int_M u_2v_2 \text{div} v_1 \text{div} u_1 d\mu + \frac{1}{16} \int_M (u_2v_2)^2 d\mu
\]

\[
- \frac{1}{8\mu(M)} \left( \int_M \text{div} v_1 \text{div} u_1 d\mu \right) \left( \int_M u_2v_2 d\mu \right) - \frac{1}{16\mu(M)} \left( \int_M u_2v_2 d\mu \right)^2
\]

\[
+ \frac{1}{16} \int_M \left\{ u_2^2(\text{div} v_1)^2 + (\langle dv_2, v_1 \rangle)^2 + v_2^2(\text{div} u_1)^2 + (dv_2, u_1)^2
\]

\[
+ 2u_2 \text{div} v_1 \langle dv_2, v_1 \rangle + 2u_2 \text{div} v_1 v_2 \text{div} u_1 + 2u_2 \text{div} v_1 \langle dv_2, u_1 \rangle
\]

\[
+ 2\langle dv_2, v_1 \rangle v_2 \text{div} u_1 + 2\langle dv_2, v_1 \rangle \langle dv_2, u_1 \rangle + 2v_2 \text{div} u_1 \langle dv_2, u_1 \rangle \right\} d\mu
\]

\[
- \frac{1}{8} \int_M \left\{ \langle u_1 \text{div} v_1, u_2dv_2 \rangle - \langle u_1 \text{div} v_1, v_2du_2 \rangle - \langle v_1 \text{div} u_1, u_2dv_2 \rangle
\]

\[
+ \langle v_1 \text{div} u_1, v_2du_2 \rangle \right\} d\mu.
\]
This proves (17) and hence completes the proof of Theorem 1.

Using the identity
\[
\langle (\delta(u_1^2 \wedge v_1^2), u_2 dv_2 - v_2 du_2) \rangle d\mu
\]

and

\[
\frac{1}{8} \int_M \left\{ u_2 \div v_1(u_1, dv_2) - v_2 \div u_1(u_1, dv_2) - u_2 \div v_1(u_1, du_2) + v_2 \div u_1(v_1, du_2) \right\} d\mu
\]

as well as integration by parts, we arrive at the following expression for \( I_1 \):

\[
I_1 = -\frac{1}{8\mu(M)} \left( \int_M \div v_1 du_1 d\mu \right) \left( \int_M u_2 dv_2 d\mu \right) - \frac{1}{16\mu(M)} \left( \int_M u_2 v_2 d\mu \right)^2
\]

\[
+ \frac{1}{16\mu(M)} \left( \int_M (\div u_1)^2 d\mu \right) \left( \int_M v_2^2 d\mu \right)
\]

\[
+ \frac{1}{16\mu(M)} \left( \int_M (\div v_1)^2 d\mu \right) \left( \int_M u_2^2 d\mu \right)
\]

\[
+ \frac{1}{16\mu(M)} \left( \int_M u_2^2 d\mu \right) \left( \int_M v_2^2 d\mu \right) \tag{27}
\]

\[
+ \frac{1}{16\mu(M)} \left( \int_M u_2 v_2 d\mu \right)^2
\]

Finally, addition of the expressions (25) and (27) for \( I_1 \) and \( I_2 \) yields

\[
\|R(u,v)v,u\| = \frac{1}{16\mu(M)} \left( \int_M (\div u_1)^2 d\mu + \int_M u_2^2 d\mu \right) \left( \int_M (\div v_1)^2 d\mu + \int_M v_2^2 d\mu \right)
\]

\[
- \frac{1}{16\mu(M)} \left( \int_M \div u_1 du_1 d\mu + \int_M u_2 v_2 d\mu \right)^2
\]

\[
= \frac{\|u,u\| \|v,v\| - \|u,v\|^2}{\mu(M)}.
\]

This proves (17) and hence completes the proof of Theorem 1.
Appendix A. The curvature of a homogeneous space

In this appendix we derive the curvature formula \([18]\) for a homogeneous space with a right-invariant metric.

Suppose that \(G\) is a Lie group and that \(H \subset G\) is a (closed) subgroup of \(G\). Let \(M = G/H\) denote the homogeneous space of right cosets. The Lie algebras of \(G\) and \(H\) are denoted by \(\mathfrak{g}\) and \(\mathfrak{h}\), and we let \(\mathfrak{m} \subset \mathfrak{g}\) be a linear subspace such that \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\). Letting \(\pi : G \to G/H\) denote the quotient map, we find that \(\pi_*\) maps \(\mathfrak{m}\) isomorphically onto \(T_{e \pi(e)}M\), where \(e\) denotes the identity element in \(G\). Suppose that \(\langle \cdot, \cdot \rangle\) is a right-invariant (degenerate) metric on \(G\) whose restriction to \(\mathfrak{m}\) is positive definite and whose kernel is \(\mathfrak{h}\), i.e. \(\langle u, v \rangle = 0\) for all \(v \in \mathfrak{g}\) iff \(u \in \mathfrak{h}\). We assume that the restriction of the metric \(\langle \cdot, \cdot \rangle\) to \(\mathfrak{m}\) is \(\text{Ad}(H)\)-invariant so that \(\langle \cdot, \cdot \rangle\) descends to a right-invariant metric on \(G/H\) which will be denoted by \(\langle \cdot, \cdot \rangle\).

**Remark 6.**

1. We do *not* require that \(\mathfrak{m}\) be an \(\text{Ad}(H)\)-invariant subspace. In particular, the homogeneous space \(G/H\) does not have to be reductive.

2. For the homogeneous space of Theorem 1, we have

\[ \mathfrak{g} = \mathfrak{X}(M) \times C^\infty(M), \quad \mathfrak{h} = \{(v, 0) \in \mathfrak{g} \mid \text{div} v = 0\}, \]

and we may choose \(\mathfrak{m} = \{(df)^2, g \mid f, g \in C^\infty(M)\}\).

For any \(X \in \mathfrak{g}\), we define the vector field \(\tilde{X} \in \mathfrak{X}(G/H)\) as the push-forward by \(\pi\) of \(X^L\), where \(X^L\) denotes the unique left-invariant vector field on \(G\) whose value at \(e\) is \(X\). Thus,

\[ \tilde{X}_{\pi(g)} = \left. \frac{d}{dt} \right|_{t=0} \pi(ge^{tX}) = \pi_*X^L_g, \quad g \in G. \]

where \(e^{tX}\) is the one-parameter subgroup in \(G\) generated by \(X\). The flow \(\Phi^t_X : M \to M\) of \(\tilde{X}\) is given by

\[ \Phi^t_X(\pi(g)) = \pi(ge^{tX}), \quad g \in G, \ t \in \mathbb{R}. \]

If \(U \in TG\), then \((\Phi^t_X)_*\pi_*U = \pi_*R_{e^{tX}}U\), where \(R_g\) denotes right multiplication by \(g \in G\). Hence, since \(\langle \cdot, \cdot \rangle\) is right invariant,

\[ \langle (\Phi^t_X)_*\pi_*U, (\Phi^t_X)_*\pi_*V \rangle = \langle R_{e^{tX}}U, R_{e^{tX}}V \rangle = \langle U, V \rangle = \langle \pi_*U, \pi_*V \rangle, \]

whenever \(U, V \in T_gG\), that is, the flow of \(\tilde{X}\) consists of isometries. This implies that \(\tilde{X}\) is a Killing field. In particular, \(\tilde{X}\) satisfies

\[ \tilde{X}(Y, Z) = ([\tilde{X}, Y], Z) + (Y, [\tilde{X}, Z]) \quad (A.1) \]

and

\[ \langle \nabla_Y \tilde{X}, Z \rangle + \langle \nabla_Z \tilde{X}, Y \rangle = 0 \quad (A.2) \]

for all vector fields \(Y, Z\) on \(G/H\).

Note that

\[ [\tilde{X}, \tilde{Y}] = -[\tilde{X}, Y], \quad X, Y \in \mathfrak{g}, \quad (A.3) \]

where the bracket on the left-hand side is the Lie bracket of vector fields on \(M\) and the bracket on the right-hand side is the Lie bracket in \(\mathfrak{g}\) induced by right-invariant vector fields (this bracket is minus the bracket induced by left-invariant vector fields).

Indeed,

\[ [\tilde{X}, \tilde{Y}] = [\pi_*X^L, \pi_*Y^L] = \pi_*[X^L, Y^L] = -\pi_*[X, Y]^L = -[\tilde{X}, Y]. \]
Let $\nabla$ denote the Levi-Civita connection associated with the Riemannian metric $g$ on $M = G/H$. Then, for any vector fields $X, Y, Z$ on $M$,

$$2(\nabla_X Y, Z) = X(Y, Z) + Y(Z, X) - Z(X, Y) + (Z, [X, Y]) - (Y, [X, Z]) - (X, [Y, Z]).$$

In the particular case when $X, Y, Z$ are Killing vector fields, (A.1) implies that

$$2(\nabla_X Y, Z) = ([X, Y], Z) + ([X, Z], Y) + (X, [Y, Z]).$$

Lemma 7. Let $X, Y \in \mathfrak{m}$. Then

$$(\nabla_X \tilde{Y})_{\pi(e)} = -\frac{1}{2}\pi_*([X, Y] + B(X, Y) + B(Y, X)), \quad (A.5)$$

where $B(X, Y) \in \mathfrak{g}$ is defined by

$$\langle B(X, Y), Z \rangle = \langle X, [Y, Z] \rangle. \quad (A.6)$$

Remark 8. Equation (A.6) only determines $B(X, Y)$ up to addition by an element in $\mathfrak{h}$. We can fix this freedom by requiring that $B(X, Y) \in \mathfrak{m}$. Alternatively, we can just note that $\pi_* B(X, Y)$ is uniquely determined and so the right-hand side of (A.5) is well-defined.

Proof. Let $X, Y, Z \in \mathfrak{m}$. Then (A.4) and (A.3) give

$$2(\nabla_X \tilde{Y}, \tilde{Z}) = -([X, Y], \tilde{Z}) - ([X, Z], \tilde{Y}) - (X, [Y, Z]).$$

Evaluating this equation at the point $\pi(e)$, we find

$$2\left(\left(\nabla_X \tilde{Y}\right)_{\pi(e)} + \frac{1}{2}[X, Y]_{\pi(e)}, \tilde{Z}_{\pi(e)}\right) = -\langle [X, Z], Y \rangle - \langle X, [Y, Z] \rangle,$$

and this equality gives (A.5). \qed

Proposition 9. Let $X, Y \in \mathfrak{m} \simeq T_{\pi(e)} M$. The curvature tensor $R$ of $M = G/H$ satisfies

$$(R(X, Y)Y, X) = \langle \delta, \delta \rangle + \langle [X, Y], \beta \rangle - \frac{3}{4} \langle [X, Y], [X, Y] \rangle - \langle B(X, Y), B(Y, X) \rangle, \quad (A.7)$$

where $\delta$ and $\beta$ are defined by

$$\delta = \frac{1}{2}(B(X, Y) + B(Y, X)), \quad \beta = \frac{1}{2}(B(X, Y) - B(Y, X)).$$

Proof. We compute

$$(R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y}) = (\nabla_{\tilde{X}} \nabla_{\tilde{Y}} \tilde{X}, \tilde{Y}) - (\nabla_{\tilde{Y}} \nabla_{\tilde{X}} \tilde{X}, \tilde{Y}) - (\nabla_{[\tilde{X}, \tilde{Y}]} \tilde{X}, \tilde{Y}).$$

This gives, using (A.2) in the last term,

$$(R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y}) = \tilde{X}(\nabla_{\tilde{Y}} \tilde{X}, \tilde{Y}) - \langle \nabla_{\tilde{Y}} \tilde{X}, \nabla_{\tilde{X}} \tilde{Y} \rangle$$

$$- \langle \tilde{Y}(\nabla_{\tilde{X}} \tilde{X}, \tilde{Y}) + (\nabla_{\tilde{X}} \tilde{X}, \nabla_{\tilde{Y}} \tilde{Y}) + (\nabla_{\tilde{Y}} \tilde{X}, [\tilde{X}, \tilde{Y}]) \rangle.$$

Using the following two equations, which are consequences of (A.4),

$$2(\nabla_{\tilde{X}} \tilde{X}, \tilde{Y}) = ([\tilde{Y}, \tilde{X}], \tilde{Y}) + ([\tilde{Y}, \tilde{X}], \tilde{X}) + ([\tilde{Y}, \tilde{X}], \tilde{X}) = 0,$$

$$2(\nabla_{\tilde{X}} \tilde{Y}, \tilde{Y}) = ([\tilde{X}, \tilde{Y}], \tilde{Y}) + ([\tilde{X}, \tilde{Y}], \tilde{X}) + ([\tilde{X}, \tilde{Y}], \tilde{X}) = 2([\tilde{X}, \tilde{Y}], \tilde{X}),$$

as well as the fact that $\nabla$ is torsion-free, i.e. $\nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X} = [\tilde{X}, \tilde{Y}]$, we find

$$(R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y}) = -\tilde{Y}([\tilde{X}, \tilde{Y}], \tilde{X}) + (\nabla_{\tilde{X}} \tilde{X}, \nabla_{\tilde{Y}} \tilde{Y}) - (\nabla_{\tilde{Y}} \tilde{X}, \nabla_{\tilde{X}} \tilde{Y}).$$
Equation (A.1) implies that
\[ \tilde{Y}([\tilde{X}, \tilde{Y}], \tilde{X}) = ([\tilde{Y}, [\tilde{X}, \tilde{Y}]], \tilde{X}) + ([\tilde{X}, [\tilde{Y}, \tilde{X}]], \tilde{Y}) \]
so that, in view of (A.5),
\[ (R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y}) = -([\tilde{Y}, [\tilde{X}, \tilde{Y}]], \tilde{X}) - ([\tilde{X}, [\tilde{Y}, \tilde{X}]], \tilde{Y}) + \langle B(X, X), B(Y, Y) \rangle - \frac{1}{4} (\langle Y, X \rangle, [Y, X]) - \langle Y, X \rangle - \langle \delta, \delta \rangle. \]
Finally, the relation (A.3) yields
\[ (R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y}) = -\langle B(X, Y), [X, Y] \rangle + \langle [X, Y], [X, Y] \rangle + \langle B(X, X), B(Y, Y) \rangle - \frac{1}{4} (\langle [X, Y], [X, Y] \rangle + \langle [X, Y], \delta \rangle - \langle \delta, \delta \rangle. \]
This is (A.7). \qed

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