ON PERIODIC APPROXIMATE SOLUTIONS OF DYNAMICAL SYSTEMS WITH QUADRATIC RIGHT-HAND SIDE

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We consider difference schemes for dynamical systems $\dot{x} = f(x)$ with quadratic right-hand side that have $t$-symmetry and are reversible. Reversibility is interpreted in the sense that a Cremona transformation is performed at each step of the computations using the difference scheme. The inheritance of periodicity and the Painlevé property by the approximate solution is investigated. In the computer algebra system Sage, values are found for the step $\Delta t$ for which the approximate solution is a sequence of points with period $n \in \mathbb{N}$. Examples are given, and conjectures about the structure of the sets of initial data generating sequences with period $n$ are formulated. Bibliography: 34 titles.

1. Introduction

At the end of the last century, a study of the inheritance by difference schemes of algebraic properties of exact solutions of ordinary differential equations, including dynamical systems, was begun. The most studied problem is the inheritance of algebraic integrals. Back in the 1990s, a family of symplectic Runge–Kutta schemes was discovered that preserve all linear and quadratic integrals of motion, e.g., all classical integrals in the problem of rotation of a top [1–4]. At the same time, the first difference scheme that preserves all algebraic integrals of the many-body problem was found by Greenspan [5–8] and independently by Simo and González [9,10]. Other conservative schemes for this problem, including high-order ones, can be constructed by introducing additional variables with respect to which all integrals of the many-body problem can be written as quadratic ones [11], thus combining the invariant energy quadratization method, suggested recently by Yang et al. [12–14], and the scalar auxiliary variable (SAV) approach, suggested by Jie Shen et al. [15]. Unfortunately, this method leads to implicit schemes, the use of which for numerical calculations turns out to be very resource-consuming [14].

Much less studied is the inheritance of reversibility by approximate solutions. In classical mechanics, it is believed that there is a one-to-one correspondence between the initial and final positions of bodies. In fact, an exact solution to a general nonlinear dynamical system has this property only locally. Moreover, as Painlevé noted [16–18], the possibility of integrating a dynamical system in Abelian functions and the reversibility of the solution are closely related to each other.

Any difference scheme defines a correspondence between the values of the solution at two times, separated by a step $\Delta t$. Therefore, it is natural to call it reversible if it defines a birational correspondence between these values (the definition is given below in Sec. 2). When considering birational correspondences, it is convenient to regard the phase space as a projective space. At the same time, we noticed that any dynamical system with quadratic right-hand side can be approximated by an invertible difference scheme (Sec. 3).

The theory of birational transformations of the plane was laid by L. Cremona [19], that is why birational transformations of projective spaces are called Cremona transformations. It should be noted that even two-dimensional Cremona transformations are a very complicated
object. In the theory of dynamical chaos, simple quadratic transformations, e.g., the integer Hénon transformation [20,21], are used to explain the origin of chaos. To relate this discrete model to Hamiltonian systems, Tabor [21, Sec. 4.2] noticed that models of this kind arise when discretizing differential equations with respect to $t$. However, the difference scheme described by him preserves the symplectic structure and, therefore, inherits quadratic integrals, but is not reversible. These properties cannot be preserved simultaneously [22], so we intend to sacrifice simplicity for reversibility. This property is extremely important not only for mechanics, but also for development of efficient numerical methods for studying dynamical systems, since it is free of the main drawback of conservative schemes, their implicitness.

Among the dynamical systems with quadratic right-hand side, there are models describing oscillations of a pendulum and rotation of a top and integrable in elliptic functions. The main qualitative property of this model is the periodicity of the solution. In this paper, we want to use these simple examples to understand how exactly the periodicity can be inherited by an approximate solution found using a reversible scheme.

2. Definitions

Let us consider a dynamical system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_m), \quad i = 1, 2, \ldots, m,$$

(1)

with polynomial right-hand side

$$f_i \in \mathbb{Q}[x_1, \ldots, x_m].$$

For brevity, we will use vector notation, meaning by $x$ the tuple $(x_1, \ldots, x_m)$. Within the framework of the finite difference method [4], the system of differential equations is replaced with a system of algebraic equations

$$g_i(x, \hat{x}, \Delta t) = 0, \quad i = 1, \ldots, n.$$  

(2)

In this case, $x$ is interpreted as the value of the solution at time $t$, and $\hat{x}$, as the value of the solution at time $t + \Delta t$.

From the point of view of mathematical modeling of mechanical phenomena, the inheritance of two properties by the difference scheme is of greatest importance, namely, of $t$-symmetry and reversibility. We say that the difference scheme (2) has $t$-symmetry if it is invariant under the transformation

$$\Delta t \to -\Delta t, \quad x \to \hat{x}, \quad \hat{x} \to x.$$

By reversibility we should understand our ability to uniquely determine, for any fixed value of the step $\Delta t$, the final data $\hat{x}$ from the initial data $x$ and vice versa using the system (2). Since Eqs. (2) are algebraic, this means that $\hat{x}$ must be a rational function of $x$, and $x$ must be a rational function of $\hat{x}$. We will regard $x, \hat{x}$ as two points of the projective space $\mathbb{P}_m$ and say that the difference scheme (2) is invertible if for any fixed value of $\Delta t$ this scheme defines a Cremona transformation. The combination of $t$-symmetry and reversibility means that the difference scheme defines a one-parameter family $C$ of Cremona transformations such that $\hat{x} = C(\Delta t)x$ and $C(\Delta t)^{-1} = C(-\Delta t)$.

In mechanics, the property of reversibility has to be introduced in a more complicated way, while the Cauchy theorem makes it possible to justify reversibility locally, in the vicinity of a nonsingular point. However, globally, a dynamical system may not have this property. Indeed, let us consider the following Painlevé [16] initial value problem

$$\frac{dx}{dt} = f(x), \quad x|_{t=t_0} = x_0$$

(3)
on a segment \([t_0, t_0 + \Delta t]\) of the real axis \(t\). For some values of \(t_0\), the procedure for the analytic continuation of the solution obtained in the Cauchy theorem along the segment encounters no singular points other than poles, and in this case the final value of \(x(t_0 + \Delta t)\) is uniquely determined by the initial value \(x_0\). However, if the path encounters a branch point, then the final value depends on the way it goes around this point. Therefore, \(x(t_0 + \Delta t)\) is in general a multivalued function of the initial value \(x_0\). Thus, if a dynamical system has the global reversibility property, then it also has the Painlevé property [23, Sec. 3.5]: the singular points of the solution are not branch points.

Classical completely integrable models, including pendulums and tops, are integrable in elliptic functions and, as can be seen from the solution, have the Painlevé property [24]. Usually, the general solution to a completely integrable nonlinear model having the Painlevé property defines a birational transformation not on the entire phase space, but on integral manifolds determined by algebraic integrals.

**Example 1.** The dynamical system
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= 6x^2 - a
\end{align*}
\]
has an algebraic integral
\[
y^2 - 4x^3 + 2ax = C_1,
\]
which has the meaning of the total mechanical energy. Therefore, this system can be integrated in Weierstrass elliptic functions:
\[
\begin{align*}
x &= \wp(t + C_2, 2a, C_1), \\
y &= \wp'(t + C_2, 2a, C_1),
\end{align*}
\]
where \(C_1, C_2\) are integration constants, and for this reason below it is referred to as the \(\wp\)-oscillator. By the addition theorem for elliptic functions, the general solution can be expressed rationally in terms of
\[
\wp(t, 2a, C_1), \wp'(t, 2a, C_1) \quad \text{and} \quad \wp(C_2, 2a, C_1), \wp'(C_2, 2a, C_1).
\]
Therefore, the general solution can be expressed rationally in terms of the initial data \(x_0 = \wp(C_2, 2a, C_1), y_0 = \wp'(C_2, 2a, C_1)\), and the coefficients of this expression depend on \(t\) and \(C_1\) transcendentally. It follows from the \(t\)-symmetry that these expressions define a birational transformation of the elliptic curve (5). Thus, on the integral curve (5), the Cauchy problem defines a birational correspondence between initial and final data. However, this correspondence cannot be extended to a Cremona transformation of the entire \(xy\) plane, although at first it seemed quite surprising [25, Chap. 7].

### 3. Equations with quadratic right-hand side

At present, there are well-developed Painlevé tests which make it possible to find out in practice whether a given dynamical system has the Painlevé property. In general, such a test is a set of necessary conditions for the absence of moving branch points, which can be checked algorithmically [23, Sec. 3.9].

In the one-dimensional case \((m = 1)\), only the Riccati equation
\[
\frac{dx}{dt} = a + bx + cx^2
\]
has the Painlevé property for any values of the constants \(a, b, c\), including zero. Moreover, the Cauchy problem defines a Mőbius transformation of the projective line. It is not difficult to construct a difference scheme that inherits this property:
\[
\hat{x} - x = \left( a + b \frac{x + \hat{x}}{2} + cx \hat{x} \right) \Delta t.
\]
Fig. 1. The solution of the system (4) for $a = 1/2$ with initial conditions $(x, y) = (1, 2)$. The solid line plots the exact solution, while the dots represent an approximate one.

Since any birational transformation of the projective line is a Möbius transformation, it is easy to prove the converse: in the one-dimensional case, an invertible difference scheme can be constructed only for the Riccati equation [26].

However, for $m > 1$ the continuous and the discrete cases are no longer similar. For any dynamical system with quadratic right-hand side, a $t$-symmetric reversible difference scheme can be constructed:

$$\hat{x}_i - x_i = F_i(x, \hat{x})\Delta t, \quad i = 1, \ldots, m,$$

(8)

where $F_i$ is obtained from $f_i$ by replacing monomials as follows: $x_j$ with $(\hat{x}_j + x_j)/2$, $x_j x_k$ with $(\hat{x}_j + x_j)(\hat{x}_k + x_k)/4$, and $x_j^2$ with $x_j \hat{x}_j$. However, only few dynamical systems with quadratic right-hand side have the Painlevé property. This phenomenon was studied long ago: the dynamical system describing the rotation of a rigid body around a fixed point always has a quadratic right-hand side, but has the Painlevé property only in three special cases found by S. V. Kovalevskaya [24].

Remark. It should be noted that any dynamical system can be reduced to a system with quadratic right-hand side via Appelroth’s theorem [27]. Furthermore, there are efficient algorithms for monomial quadratization for ODE systems [28].

We investigated how the Painlevé property is inherited by an approximate solution using two examples, the Riccati equation (6) and the $\wp$-oscillator (4). In both cases, it turned out that the computations using a reversible difference scheme can be continued beyond a pole without noticeable accumulation of errors. For the Riccati equation, this property was justified in [26]. It seems to us that this property of an approximate solution effectively translates the concept of the Painlevé property to finite difference equations, which by no means excludes other interpretations [29–31].

Example 2. Figure 1 shows the solution of the system (4) for $a = 1/2$ and initial conditions $(x, y) = (1, 2)$. The computation of an approximate solution according to a reversible scheme does not encounter any difficulties on the entire interval $0 < t < 12$ under consideration, which contains two poles of the exact solution.
An approximate solution is a sequence \( x_0, x_1, \ldots \) where each element is obtained from the previous one by applying the Cremona transformation \( C \):

\[ x_{n+1} = Cx_n. \]

This sequence will have period \( n \) if \( x_n = x_0 \), i.e., if \( x_0 \) is a fixed point of \( C^n \).

Let a positive integer \( n \) and an initial value \( x_0 \in \mathbb{Q}^m \) be given. Then a step \( \Delta t \) for which the sequence has period \( n \) can be selected in the following way. Regarding \( \Delta t \) as a symbolic variable, we calculate \( C^n x_0 \). We get \( m \) rational functions from \( \mathbb{Q}(\Delta t) \). Equating them to \( x_0 \), we obtain \( m \) algebraic equations whose common roots are the required step values. In general, several equations for one variable may not have common roots.

We have considered three examples: i) the linear oscillator, which can easily be investigated analytically [32], ii) the \( \wp \)-oscillator, and iii) the Jacobi oscillator, i.e., the dynamical system

\[ \dot{p} = qr, \quad \dot{q} = -pr, \quad \dot{r} = -k^2 pq, \quad (9) \]

which is integrable in elliptic Jacobi functions. We chose different initial data and considered \( n \) in the interval from 2 to 10. All computations were performed in the Sage computer algebra system [33] on an office PC. The degrees of the polynomials whose common roots give the desired step values increase exponentially with \( n \), which significantly limited our ability to increase \( n \).

**Example 3.** For the \( \wp \)-oscillator with the same initial conditions as were used in Example 2, there are no values of \( \Delta t \) for which the solution has period \( n = 2, 3, 6 \). For \( n = 4 \), the step is independent of the starting point. Table 1 contains all suitable positive values of \( \Delta t \) for \( n \) from 2 to 10.

**Example 4.** For the Jacobi oscillator (9) with \( k = \frac{1}{5} \) and initial conditions

\[ p = 0, \quad q = 1, \quad r = 0, \]

there are positive values of \( \Delta t \) for any periods \( n > 2 \), see Table 2.

The approximate solution has period \( n = 5 \) for two values. In the \( pq \) plane, for the first value of the step, we obtain an almost regular pentagon, and for the second value, a pentagram (Fig. 2). In both cases, the integral of motion \( p^2 + q^2 = 1 \) is not exactly conserved. The approximate solution has period \( n = 6 \) also for two values of the step. The first of them coincides with the step for \( n = 3 \) and gives us a triangle, while the second one gives a hexagon (Fig. 3). The approximate solution has period \( n = 7 \) for three values of the step (Fig. 4).

| \( n \) | \( \Delta t \)                        |
|-------|-------------------------------------|
| 2     | \( \emptyset \)                     |
| 3     | \( \emptyset \)                     |
| 4     | 1.074                               |
| 5     | 6.908                               |
| 6     | \( \emptyset \)                     |
| 7     | 0.556, 5.870, 7.759                 |
| 8     | 0.535, 1.074, 6.843                 |
| 9     | 0.504, 9.187                       |
| 10    | 0.471, 0.559, 6.777, 6.908          |

Table 1. The values of \( \Delta t \) for which the solution from Example 3 is periodic.
Table 2. The values of $\Delta t$ for which the solution from Example 4 is periodic.

| $n$ | $\Delta t$          |
|-----|---------------------|
| 2   | $\emptyset$        |
| 3   | 3.609               |
| 4   | 2.041               |
| 5   | 1.47, 6.86          |
| 6   | 1.17, 3.60          |
| 7   | 0.97, 2.57, 10.85   |
| 8   | 0.83, 2.04, 5.18    |
| 9   | 0.73, 1.70, 3.60, 16.23 |

As $n$ grows, the number of step values for which the approximations to the solution of the Cauchy problem considered in Example 4 are periodic grows. In this case, the smallest possible $\Delta t$ for fixed $n$ corresponds to an almost regular $n$-gon in the $pq$ plane. These solutions return to their initial values in the times $n\Delta t$ collected in Table 3. These times seem to form a monotonically decreasing sequence converging to the exact period.

It is convenient to present the results of our experiments in the form of two conjectures:
Fig. 4. The approximate solution from Example 4 for two steps ensuring period $n = 7$.

| $n$ | $n\Delta t$ |
|-----|-------------|
| 3   | 10.827      |
| 4   | 8.164       |
| 5   | 7.379       |
| 6   | 7.022       |
| 7   | 6.827       |
| 8   | 6.706       |
| 9   | 6.627       |
| $\infty$ | 6.347 |

Table 3. The sequence of periods for the approximate solutions from Example 4.

(1) for any sufficiently large $n$ and any initial conditions, one can specify a finite number of positive values of the step $\Delta t$ for which the periodic sequences with period $n$ are obtained,

(2) if with each $n$ we associate the minimum period, we get a sequence converging to the period of the exact solution as $n \to \infty$.

By the first conjecture, any exact particular solution can be approximated by an approximate solution that inherits the periodic nature of the exact solution; and by the second conjecture, the approximation step $\Delta t$ can be taken arbitrarily small, so we can approximate the exact solution with any given accuracy.

5. Equiperiodic sets

In the previous section, we kept track of one solution, but changed $\Delta t$. Let us now look at the behavior of solutions in the phase space, but for a fixed $\Delta t$.

The set in the phase space formed by all the initial data generating approximate solutions with the same period $n$ is algebraic; we will call it the equiperiodic set of order $n$. It is easy to deduce from the first conjecture that equiperiodic sets of sufficiently large orders are not empty and have codimension 1.

To find them in the previous algorithm, we must regard $x_0$ as a tuple of $m$ symbolic variables. We managed to find these sets only for small $n$.

Example 5. For the linear oscillator and the $\wp$-oscillator, the equiperiodic sets of orders 2 and 3 are empty. For $n = 3$, the equation of the curve degenerates into

$$3\Delta t^4 - 4 = 0,$$
which agrees with the aforementioned fact: for \( n = 4 \), the step is independent of the initial data (Example 2). For \( n = 5 \), the equiperiodic set appears to be an elliptic curve:

\[
27\Delta t^{10} x - 432\Delta t^{8} x y^2 + 432\Delta t^{8} x^2 + 1728\Delta t^{6} x^3 + 27\Delta t^{8} \\
-432\Delta t^{6} y^2 - 936\Delta t^{6} x + 168\Delta t^4 + 240\Delta t^2 x - 80 = 0.
\] (10)

This curve and the equiperiodic curves for \( n = 6, 7, \) and 8 are plotted in Fig. 5. The degrees of the polynomials \( F_n \in \mathbb{Q}[\Delta t][x,y] \) are presented in Table 4. Since the degrees grow, the equiperiodic curves do not belong to the same sheaf, linear or irrational.

| \( n \) | Degree of \( F_n \) |
|------|------------------|
| 4    | 0                |
| 5    | 3                |
| 6    | 3                |
| 7    | 6                |
| 8    | 6                |
| 9    | 9                |
| 10   | 12               |

Table 4. The degrees of equiperiodic curves for Example 5.

Fig. 5. The equiperiodic curves for Example 5 for \( \Delta t = 1 \) (left) and for \( \Delta t = 0.5 \) (right).

**Example 6.** In the case of the Jacobi oscillator, the equiperiodic set consists of some surface in the \( pqr \) space to which the coordinate lines should be added. The expressions obtained are very cumbersome.

The very definition implies the simplest properties of equiperiodic sets.

- If some point of an approximate solution belongs to an equiperiodic set, then this solution has period \( n \) and all its points belong to this set. Therefore, equiperiodic sets are integral sets for an approximate solution. Thus, on periodic solutions, conservation laws are satisfied, but these laws are different from those known for the continuous model.
• If equiperiodic sets of orders $n'$ and $n''$ intersect at a nonsingular point of the Gremona
transformation, then the solution that starts at the intersection point must have periods
$n'$ and $n''$ simultaneously. Therefore, the numbers $n'$ and $n''$ must have a common
divisor $n'''$, and the intersection point itself must lie on the equiperiodic set of order $n'''$.

If we consider, e.g., the $xy$ plane of the $\wp$-oscillator with a fixed value of the step $\Delta t$, we
will observe countably many equiperiodic curves $F_n(x,y,\Delta t) = 0$. A solution that starts at
a point on the curve $F_n$ passes through exactly $n$ points of this curve and returns. Solutions
that do not fall on these curves will be aperiodic.

6. Discussion and conclusion

In classical mechanics, attempts were repeatedly made to view Newton’s differential equa-
tions as difference equations, regarding $\Delta t$ as a small but finite quantity [34]. However, if we
replace the equation $\frac{dx}{dt} = f(x)$ with $\hat{x} - x = f(x)\Delta t$, all fundamental laws of nature are vi-
olated, including $t$-symmetry and conservation laws. As a result, the difference model loses the
well-known properties of the continuous model, so one has to use the continuous model, which
has the correct qualitative properties, while the discrete model is regarded as an imperfect one
suitable only for numerical calculations.

The ultimate goal is to create discrete models that have the most important properties of
mechanical models. These undoubtedly include the inheritance of algebraic conservation laws,
t-symmetry, reversibility, and periodicity. As we found out earlier, one cannot combine invert-
ibility and exact preservation of all algebraic integrals [22]. In this paper, as in the continuous
case, we restricted consideration to an integral manifold (Example 1) and considered difference
schemes on the manifold. However, in the discrete case, one can proceed differently, giving up
the exact preservation of precisely those expressions that are preserved in the continuous case.

The starting point for this article was the observation that dynamical systems with quadratic
right-hand side can be approximated by reversible difference schemes with $t$-symmetry. The
approximate solutions found using these schemes are birational functions of the initial data
on the entire phase space. This is surprising, since in the continuous case, for this property to
appear, one had to restrict the phase space to an algebraic integral manifold (see Example 1).

For any initial data and any $n \in \mathbb{N}$, computer algebra methods can find all possible values
for the step $\Delta t$ for which the approximate solution is a sequence of points with period $n$. Our
experiments demonstrate that for sufficiently large $n$, the set of such steps is not empty and
the minimum step tends to zero with increasing $n$. Thus, for any initial data, one can find a
periodic approximate solution arbitrarily close to the exact one in the uniform norm.

In the text, the set of points in the phase plane such that the solutions starting at these points
are sequences with period $n$ was called equiperiodic. The study of equiperiodic sets seems to
us to be a very promising and beautiful problem, which could clarify to what extent the
discretization according to a reversible difference scheme randomizes a completely integrable
continuous problem.

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