Projectors, matrix models and noncommutative instantons

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Abstract

We deconstruct the finite projective modules for the fuzzy four-sphere, described in a previous paper, and correlate them with the matrix model approach, making manifest the physical implications of noncommutative topology. We briefly discuss also the $U(2)$ case, being a smooth deformation of the celebrated BPST $SU(2)$ classical instantons on a sphere.
1 Introduction

The search for instantons on noncommutative manifolds has recently received growing attention in the physical literature \[1\]-\[7\]. The prospective we take in this paper is to describe nontrivial configurations in terms of finite projective modules \[8\]-\[9\], which can be easily constructed in the noncommutative case \[10\]-\[12\]. Then by deconstructing the projectors we can identify the associated nontrivial connections satisfying the $Y - M$ equations of motion, following the spirit of Refs. \[13\]-\[14\]. The explicit example we consider is given in Ref. \[11\], where a finite projective module description of $U(1)$ instantons on a fuzzy four-sphere \[15\]-\[18\] has been presented, based on the Hopf principal fibration $\pi : S^7 \rightarrow S^4$. We will briefly treat also the $U(2)$ case which is physically more interesting since it is a smooth deformation of the classical BPST $SU(2)$ instanton. Since the classical limit of the fuzzy four-sphere is more subtle than the fuzzy two-sphere \[19\]-\[22\], we need to introduce the problematic related to the fuzzy four-sphere case in the first sections. The main difference from the fuzzy two-sphere case is that the coordinates do not form a closed algebra, but we have to generalize the algebra from five hermitian operators to fifteen operators. In practice one is obliged to promote as extra coordinates the commutators of the real coordinates of the sphere. Therefore when one writes the action of $Y - M$ theory on a fuzzy four-sphere using a matrix model \[15\]-\[18\], one has problems in recognizing the classical limit since the derivatives of the extra coordinates enter into the game and there is no warranty that their contribution can be decoupled from the physical coordinates of the sphere $S^4$.

Despite such difficulties we are able to reach some interesting results. First of all, by introducing a simple link between projectors and the basic matrix variable $X_i$, we are able to recognize the class of matrix models for which the projectors presented in Ref. \[11\] are solutions to the $Y - M$ equations of motion, although we have no warranty that the contribution of the extra coordinates can be decoupled in the classical limit. This is probably true for the $U(2)$ case, which is a smooth deformation of the BPST $SU(2)$ instantons, in which case however the link with matrix models becomes cumbersome.

The last part of the paper is dedicated to recognize the gauge connection from the projectors \[11\], by decomposing the projectors in terms of more fundamental vector-valued operators. We have to correct a naive decomposition in terms of oscillators, to avoid a discontinuity problem in the background action, by dressing this decomposition with quasi-unitary operators. Finally we find a simple interpretation of the topologically non-trivial configurations at a level of the matrix model \[15\]-\[18\].
2 Properties of the fuzzy four-sphere

The fuzzy four-sphere is built in order to match the following two conditions:

$$\epsilon^{\mu\nu\lambda\rho\sigma} \hat{x}_\mu \hat{x}_\nu \hat{x}_\lambda \hat{x}_\rho = C \hat{x}_\sigma$$
$$\hat{x}_\mu \hat{x}_\mu = R^2$$

(2.1)

where $R$ is the radius of the four-sphere. These two conditions are invariant under a $SO(5)$ group, mixing the coordinates together.

We solve these two conditions introducing auxiliary finite matrices $\hat{G}_\mu$ as follows:

$$\hat{x}_\mu = \rho \hat{G}_\mu$$

(2.2)

where $\hat{G}_\mu$ can be built from the $n$-fold symmetric tensor product of the usual Dirac matrices $\gamma_\mu$, see [15] for details. The finite dimension of $\hat{G}_\mu$, $N$, is determined to be:

$$N = \frac{(n + 1)(n + 2)(n + 3)}{6}$$

(2.3)

and the constant $C$ must be consistently adjusted.

The matrices $\hat{G}^{(n)}_\mu$ automatically satisfy the following relations:

$$\hat{G}^{(n)}_\mu \hat{G}^{(n)}_\mu = n(n + 4) = c$$
$$\epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(n)}_\mu \hat{G}^{(n)}_\nu \hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\rho = \epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(n)}_\mu \hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\rho = (8n + 16) \hat{G}^{(n)}_\sigma$$

(2.4)

where

$$\hat{G}^{(n)}_{\mu\nu} = \frac{1}{2} [\hat{G}^{(n)}_\mu, \hat{G}^{(n)}_\nu]$$

(2.5)

Another way to write (2.4) is

$$\hat{G}^{(n)}_{\mu\nu} = -\frac{1}{2(n + 2)} \epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\rho \hat{G}^{(n)}_\sigma = -\frac{1}{2(n + 2)} \epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(n)}_\rho \hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\sigma.$$

(2.6)

From (2.4) we easily deduce that the conditions (2.1) are met if we pose $C$ as
\[ C = (8n + 16)\rho^3 \quad \leftrightarrow \quad R^2 = \rho^2 n(n + 4) = \rho^2 c. \quad (2.7) \]

We have also the relations

\[
\begin{align*}
\hat{G}^{(n)}_{\mu \nu} \hat{G}^{(n)}_{\nu \rho} &= 4 \hat{G}^{(n)}_{\mu} \\
\hat{G}^{(n)}_{\mu \nu} \hat{G}^{(n)}_{\nu \mu} &= 4n(n + 4) = 4c \\
\hat{G}^{(n)}_{\mu \nu} \hat{G}^{(n)}_{\nu \lambda} &= c\delta_{\mu \lambda} + \hat{G}^{(n)}_{\mu} \hat{G}^{(n)}_{\lambda} - 2\hat{G}^{(n)}_{\lambda} \hat{G}^{(n)}_{\mu}. \quad (2.8)
\end{align*}
\]

The combination of \( \hat{G}^{(n)}_{\mu} \) and \( \hat{G}^{(n)}_{\mu \nu} \) matrices form a closed SO(5,1) algebra,

\[
\begin{align*}
[\hat{G}^{(n)}_{\mu}, \hat{G}^{(n)}_{\nu \lambda}] &= 2(\delta_{\mu \nu} \hat{G}^{(n)}_{\lambda} - \delta_{\mu \lambda} \hat{G}^{(n)}_{\nu}) \\
[\hat{G}^{(n)}_{\mu \nu}, \hat{G}^{(n)}_{\lambda \rho}] &= 2(\delta_{\nu \lambda} \hat{G}^{(n)}_{\mu \rho} + \delta_{\mu \rho} \hat{G}^{(n)}_{\nu \lambda} - \delta_{\mu \lambda} \hat{G}^{(n)}_{\nu \rho} - \delta_{\nu \rho} \hat{G}^{(n)}_{\mu \lambda}). \quad (2.9)
\end{align*}
\]

Therefore we have to enlarge the coordinates of the sphere from 5 to 15

\[
\begin{align*}
\hat{x}_\mu &= \rho \hat{G}^\mu \\
\hat{w}_\mu &= \frac{i\rho}{2} [\hat{x}_\mu, \hat{G}^{(n)}_{\nu \lambda}]. \quad (2.10)
\end{align*}
\]

On the fuzzy four-sphere we have the following non-commutativity

\[
\begin{align*}
[\hat{x}_\mu, \hat{x}_\nu] &= -2i\rho \hat{w}_\mu \hat{w}_\nu \\
\epsilon^{\mu \nu \lambda \sigma} \hat{w}_\mu \hat{w}_\lambda \hat{w}_\rho &= -\rho(8n + 16) \hat{x}_\sigma. \quad (2.11)
\end{align*}
\]

The classical sphere \( S^4 \) is obtained as a large \( n \) limit keeping fixed the radius of the sphere \( R \), or in other words as a limit \( \rho \to 0 \) with \( R \) fixed. From (2.11) we can see that the coordinates become commuting in this limit:

\[
\begin{align*}
[\hat{x}_\mu, \hat{x}_\nu] &= -2i\rho \hat{w}_\mu \hat{w}_\nu \sim O(\rho R) \to 0 \\
[\hat{x}_\mu, \hat{w}_{\nu \lambda}] &= 0 \\
[\hat{w}_\mu, \hat{w}_{\nu \lambda}] &= 0. \quad (2.12)
\end{align*}
\]

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The non-commutativity is caused by the presence of the extra coordinates $\hat{w}_{\mu\nu}$; in practice we can state that an extra fuzzy two-sphere is attached to every point of fuzzy four-sphere. To realize that let’s diagonalize the matrix $\hat{x}_5 = \rho \hat{G}_5$. Then there is a subalgebra $SU(2) \times SU(2)$ generated by $\hat{G}_{\mu\nu}(\mu, \nu = 1, \ldots, 4)$ of the $SO(5)$ algebra generated by $\hat{G}_{\mu\nu}(\mu, \nu = 1, \ldots, 5)$, commuting with $\hat{x}_5$:

\[
\begin{align*}
[\hat{N}_i, \hat{N}_j] &= i \epsilon_{ijk} \hat{N}_k \\
[\hat{M}_i, \hat{M}_j] &= i \epsilon_{ijk} \hat{M}_k \\
[\hat{N}_i, \hat{M}_j] &= 0
\end{align*}
\] (2.13)

where

\[
\begin{align*}
\hat{N}_1 &= -\frac{i}{4} (\hat{G}_{23} - \hat{G}_{14}) \\
\hat{N}_2 &= \frac{i}{4} (\hat{G}_{13} + \hat{G}_{24}) \\
\hat{N}_3 &= -\frac{i}{4} (\hat{G}_{12} - \hat{G}_{34}) \\
\hat{M}_1 &= -\frac{i}{4} (\hat{G}_{23} + \hat{G}_{14}) \\
\hat{M}_2 &= \frac{i}{4} (\hat{G}_{13} - \hat{G}_{24}) \\
\hat{M}_3 &= -\frac{i}{4} (\hat{G}_{12} + \hat{G}_{34}).
\end{align*}
\] (2.14)

Conversely, $\hat{G}_{\mu\nu}(\mu, \nu = 1, \ldots, 4)$ can be written as:

\[
\begin{align*}
\hat{G}_{23} &= 2i(\hat{N}_1 + \hat{M}_1) \\
\hat{G}_{14} &= -2i(\hat{N}_1 - \hat{M}_1) \\
\hat{G}_{13} &= -2i(\hat{N}_2 + \hat{M}_2) \\
\hat{G}_{24} &= -2i(\hat{N}_2 - \hat{M}_2) \\
\hat{G}_{12} &= 2i(\hat{N}_3 + \hat{M}_3) \\
\hat{G}_{34} &= -2i(\hat{N}_3 - \hat{M}_3).
\end{align*}
\] (2.15)

The Casimir of every $SU(2)$ algebra is computed as follows:

\[
\begin{align*}
\hat{N}_i \hat{N}_i &= \frac{1}{16} (n + G_5)(n + 4 + G_5) \\
\hat{M}_i \hat{M}_i &= \frac{1}{16} (n - G_5)(n + 4 - G_5).
\end{align*}
\] (2.16)

Fixing the value of $\hat{G}_5 = G_5$ the $SU(2)$ algebra generated by $\hat{N}_i$ is realized by a $(\frac{n+G_5+2}{2})$ dimensional representation, while the $SU(2)$ algebra generated by $\hat{M}_i$ is realized by a $(\frac{n-G_5+2}{2})$ dimensional representation, resulting in $\frac{(n+G_5+2)(n-G_5+2)}{4}$ possible eigenvalues. If
we sum up the contributions of $G_5 = -n, -n + 2, \ldots, n - 2, n$ we end up with the dimension $N$ of the matrix.

We notice that an $SU(2)$ algebra decouples at the north pole, $G_5 = n$, since the Casimir of $\hat{N}_i$ and $\hat{M}_i$ are given by:

$$\hat{N}_i \hat{N}_i = \frac{n(n + 2)}{4}$$

$$\hat{M}_i \hat{M}_i = 0.$$  \hspace{1cm} (2.17)

We have reached the result that a fuzzy two-sphere, given by the $(n + 1)$-dimensional representation of $SU(2)$ is attached to the north pole. The radius of the fuzzy two-sphere $\sigma^2 = \rho^2 \frac{n(n+4)}{4}$ is comparable with the radius of the fuzzy four-sphere, $R^2 = \rho^2 n(n + 4)$.

By using the $SO(5)$ symmetry we can generalize this observation from the north pole to every point of the fuzzy four-sphere. This fuzzy two-sphere is a sort of internal space, in the sense that the fields on the fuzzy four-sphere carry internal quantum numbers corresponding to the $SU(2)$ angular momentum and therefore these extra degrees of freedom have the nice interpretation as spin.

## 3 Matrix model for the fuzzy four-sphere

To introduce a gauge theory on a fuzzy four-sphere we consider the following matrix model:

$$S = -\frac{1}{g^2} Tr \left[ \frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu] + \frac{k}{5} \epsilon^{\mu\nu\lambda\rho\sigma} X_\mu X_\nu X_\lambda X_\rho X_\sigma \right]$$  \hspace{1cm} (3.1)

where the indices $\mu, \nu, \ldots, \sigma$ take the values $1, \ldots, 5$ and are contracted with the Euclidean metric. $\epsilon^{\mu\nu\lambda\rho\sigma}$ is the $SO(5)$ invariant totally antisymmetric tensor. $X_\mu$ are hermitean $N \times N$ matrices and $k$ is a dimensional constant depending on $N$. The second term is known as Myers term and it has a consistent brane interpretation [IK].

This model has the global $SO(5)$ symmetry and the following unitary symmetry

$$X_\mu = U^\dagger X_\mu U \quad \quad U U^\dagger = U^\dagger U = 1$$  \hspace{1cm} (3.2)

apart from an extra translational symmetry $X_\mu \rightarrow X_\mu + c_\mu 1$.

The constant $k$ is determined by the conditions that the matrix model (3.1) has as a classical solution the fuzzy four-sphere:
\[ X_\mu = \hat{x}_\mu = \rho \hat{G}_\mu^{(n)}. \]  

(3.3)

Since the equations of motion of the action (3.1) are of the form

\[ [X_\nu, [X_\mu, X_\nu]] + k \epsilon^{\mu \nu \lambda \rho \sigma} X_\nu X_\lambda X_\rho X_\sigma = 0 \]  

(3.4)

the constant \( k \) is determined in terms of the label \( n \):

\[ k = \frac{2}{\rho(n + 2)}. \]  

(3.5)

It is also possible to introduce a Y-M action with mass term having the fuzzy four-sphere as a classical solution:

\[ S = -\frac{1}{g^2} Tr \left[ \frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu] + 8 \rho^2 X_\mu X_\mu \right]. \]  

(3.6)

The construction of the noncommutative gauge theory on the fuzzy four-sphere is obtained by expanding the general matrices \( X_\mu \) around the classical background \( \hat{x}_\mu \):

\[ X_\mu = \hat{x}_\mu + \rho \hat{a}_\mu. \]  

(3.7)

The functional space on which the fields \( X_\mu \) live is determined by the background. While the fields on the sphere can be developed in terms of the spherical harmonics, the field theory on a fuzzy sphere is realized by truncating the angular momentum with a cutoff parameter. Many papers deal with such construction for higher dimensional fuzzy spheres, see for example [23]. In four dimensions the basis is classified by irreducible representations of \( SO(5) \). The corresponding \( SO(5) \) Young diagram is labelled by two labels \( (r_1, r_2) \). Only the representations with \( r_2 = 0 \) correspond to the classical sphere \( (w_{\mu \nu} = 0) \).

The fuzzy four-sphere functional space is therefore obtained by the \( SO(5) \) irreducible representation with the cutoff \( r_1 \leq n \), i.e. with \( 0 \leq r_2 \leq r_1 \leq n \). By summing the dimensions of all these irreducible representations we obtain the square of \( N \), the rank of the matrix \( \hat{G}_\mu \).

Therefore a general \( N \times N \) matrix \( \hat{a}_\mu \) can be developed in an abstract form:

\[ \hat{a}(\hat{x}, \hat{w}) = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \sum_{m_i} a_{r_1 r_2 m_i} \hat{Y}_{r_1 r_2 m_i}(\hat{x}, \hat{w}) \]  

(3.8)
where $m_i$ denote all the relevant quantum numbers. In the case of $\hat{w}_{\mu\nu} = 0$, $\hat{Y}_{r_1m_i}$ are the usual spherical harmonics. However in the fuzzy four-sphere case we need to assume that the fields depend also on the extra coordinates $\hat{w}_{\mu\nu}$.

The symbol corresponding to the matrix $\hat{a}(\hat{x}, \hat{w})$ is easily obtained as

$$a(x, w) = \sum_{r_1=0}^{n} \sum_{r_2=0}^{m_i} \sum_{m_i} a_{r_1r_2m_i} Y_{r_1r_2m_i}(x, w)$$

(3.9)

and the product of matrices can be mapped to a noncommutative and associative star product of symbols. The non-commutativity of the star product is produced by the existence of $\hat{w}_{\mu\nu}$.

To define the action of a noncommutative gauge theory on a fuzzy four-sphere we need to introduce derivative operators such as

$$Ad(\hat{G}_{\mu}) \rightarrow -2i \left( w_{\mu\nu} \frac{\partial}{\partial x_{\nu}} - x_{\nu} \frac{\partial}{\partial w_{\mu\nu}} \right)$$

$$Ad(\hat{G}_{\mu\nu}) \rightarrow 2 \left( x_{\mu} \frac{\partial}{\partial x_{\nu}} - x_{\nu} \frac{\partial}{\partial x_{\mu}} - w_{\mu\lambda} \frac{\partial}{\partial w_{\lambda\nu}} + w_{\nu\lambda} \frac{\partial}{\partial w_{\lambda\mu}} \right).$$

(3.10)

In the case of $Ad(\hat{G}_{\mu\nu})$ we can isolate the first two terms corresponding to the orbital parts while the last two have the meaning of isospin parts.

To make clear that the fields $\hat{a}(\hat{x}, \hat{w})$ are spin-dependent representations of the $SO(4)$ internal Lorentz group generated by $G_{ab}$, $(a, b = 1, ..., 4)$, we define its action as:

$$e^{i\hat{G}_{ab}\omega_{ab}} \hat{a}(\hat{x}, \hat{w}) e^{-i\hat{G}_{ab}\omega_{ab}} \simeq \hat{a}(\hat{x}, \hat{w}) + i\omega_{ab} Ad(G_{ab}) \hat{a}(\hat{x}, \hat{w})$$

$$\rightarrow a(x, w) + 2i\omega_{ab} \left( x_{a} \frac{\partial}{\partial x_{b}} - x_{b} \frac{\partial}{\partial x_{a}} - w_{ac} \frac{\partial}{\partial w_{cb}} + w_{bc} \frac{\partial}{\partial w_{ca}} \right) a(x, w).$$

(3.11)

The last term shows that the fields $a(x, w)$ have spin angular momentum, taking only integer values and with a maximum spin limited by the dimension of the matrices $\hat{N}_i$, which is $(n+1)$.

To isolate the spin $m$ ($m < n$) contribution, it is necessary to develop the field $a(x, w)$, for example at the north pole, in terms of the coordinates $N_i$:

$$a(x, w) = a(x, 0) + N_{i_1} \frac{\partial a(x, N)}{\partial N_{i_1}} |_{N=0} + ... + \frac{1}{n!} N_{i_1}N_{i_2}...N_{i_n} \frac{\partial^n a(x, N)}{\partial N_{i_1}\partial N_{i_2}...\partial N_{i_n}} |_{N=0}. $$

(3.12)
Each term in this development has a definite spin, i.e. \((m + 1)\)-th term represents an \(m\) spin field. This \(N\) dependence on the internal spin produces non-commutativity. It is also possible to remove the fuzzy two-sphere \((N_i)\) from the fuzzy four-sphere, however the product of fields, although commutative, becomes non-associative.

The gauge symmetry of the non-commutative gauge theory is a direct consequence of the unitary symmetry of the matrix model. By taking an infinitesimal transformation \(U \simeq 1 + i \hat{\lambda}\), a fluctuation around the fixed background transforms, similarly to a gauge field, as

\[
\delta \hat{a}_\mu(\hat{x}, \hat{w}) = -\frac{i}{R} [\hat{G}_\mu, \hat{\lambda}(\hat{x}, \hat{w})] + i [\hat{\lambda}(\hat{x}, \hat{w}), \hat{a}_\mu(\hat{x}, \hat{w})]
\]  

(3.13)

and for the corresponding symbol

\[
\delta a_\mu(x, w) = \frac{2}{R} \left( w_{\mu\nu} \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial w_{\mu\nu}} \right) \lambda(x, w) + i [\lambda(x, w), a_\mu(x, w)]^+.
\]  

(3.14)

By developing \(\hat{\lambda}\) as

\[
\hat{\lambda} = \lambda_0 + \lambda^\mu \hat{G}_\mu + \lambda^{\mu\nu} \hat{G}_{\mu\nu} + 0(G^2)
\]  

(3.15)

this gauge transformation contains many extra degrees of freedom which have no equivalence in a standard gauge theory.

The integration in the classical gauge theory is replaced by the trace in the action of the matrix model. However in the correspondence one has to take into account the presence of the extra internal two-dimensional space \(N_i\), apart from the fuzzy four-sphere.

Finally the Laplacian on the sphere has two possible extensions on the fuzzy four-sphere, \(Ad(\hat{G}_{\mu\nu})^2\) the quadratic Casimir of \(SO(5)\) and \(Ad(\hat{G}_\mu)^2\). The natural choice for a matrix model, in which we develop the matrices \(X_\mu\) around the background \(\hat{x}_\mu = \rho \hat{G}_\mu\), is given by \(Ad(\hat{G}_\mu)^2\), whose action is given by

\[
\frac{1}{4} \left[ \frac{\hat{G}_\mu}{R}, \left[ \frac{\hat{G}_\nu}{R}, \right] \right] = \frac{\partial^2}{\partial x_\mu \partial x_\nu} - \frac{x_\mu x_\nu}{R^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu} - \frac{4 x_\mu}{R^2} \frac{\partial}{\partial x_\nu} - \frac{2 w_{\mu\nu} w_{\mu\lambda}}{R^2} \frac{\partial^2}{\partial x_\mu \partial w_{\mu\lambda}} + \frac{w_{\mu\lambda}}{R^2} \frac{\partial}{\partial w_{\mu\lambda}}.
\]  

(3.16)

We recognize the usual Laplacian on a classical four-sphere in the first three terms. The action of the two Laplacians on a spherical harmonics is as follows:
\[
\frac{1}{4} \left[ \hat{G}_\mu, \left[ \hat{G}_\mu, \hat{Y}_{r_1r_2} \right] \right] = (r_1(r_1 + 3) - r_2(r_2 + 1))\hat{Y}_{r_1r_2} \\
\frac{1}{8} \left[ \hat{G}_{\mu\nu}, \left[ \hat{G}_{\mu\nu}, \hat{Y}_{r_1r_2} \right] \right] = (r_1(r_1 + 3) + r_2(r_2 + 1))\hat{Y}_{r_1r_2}.
\] (3.17)

4 Projectors for the fuzzy four-sphere

A general procedure to characterize instantons configurations on the classical sphere \( S^4 \) is starting from the algebra of \( N \times N \) matrices whose entries are elements of the smooth function algebra \( C^\infty(S^4) \) on the base space \( S^4 \), i.e. \( M_N(C^\infty(S^4)) \). The section module of the bundle on which the instanton lives can be identified with the action of a global projector \( p \in M_N(C^\infty(S^4)) \) on the trivial module \( (C^\infty(S^4))^N \), i.e. the right module \( p(C^\infty(S^4))^N \), where an element of \( (C^\infty(S^4))^N \) is simply the vector

\[
||f >> = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}
\]

with \( f_1, \ldots, f_N \) elements of \( C^\infty(S^4) \).

In Ref. [11] we found \( U(1) \) instantons configurations for the fuzzy four-sphere. These have been obtained generalizing the construction of the non-commutative monopoles, based on the Hopf fibration \( \pi : S^7 \to S^4 \) to the case \( \pi : S^7 \to S^4 \).

Briefly speaking, the Hopf fibration \( \pi : S^7 \to S^4 \) is a classical map from four complex coordinates \( a_i \), constrained to live in \( S^7 \), to five real coordinates \( x_i \), constrained to live in \( S^4 \):

\[
\begin{align*}
x_1 &= \rho (a_1 + \overline{a}_1) \\
x_2 &= i\rho (a_1 - \overline{a}_1) \\
x_3 &= \rho (a_2 + \overline{a}_2) \\
x_4 &= i\rho (a_2 - \overline{a}_2) \\
x_5 &= \rho (a_0\overline{a}_0 + a_1\overline{a}_1 - a_2\overline{a}_2 - a_3\overline{a}_3) \\
\alpha_1 &= a_0\overline{a}_2 + a_3\overline{a}_1 \\
\alpha_2 &= a_0\overline{a}_3 - a_2\overline{a}_1 \\
\sum_i x_i^2 &= \rho^2 \\
\sum_i |a_i|^2 &= 1.
\end{align*}
\] (4.2)

The idea proposed in Ref. [11] is that promoting the complex coordinates \( a_i \) to four oscillators
the coordinates $\hat{x}_i$ are part of an algebra, coinciding with the fuzzy four-sphere algebra. For example the $SU(2) \times SU(2)$ subalgebra made by $N_i$ and $M_i$ can be represented in terms $a_i$ as follows:

\[
\begin{align*}
\hat{N}_3 &= \frac{1}{2}(a_3 a_3^\dagger - a_2 a_2^\dagger) \\
\hat{M}_3 &= \frac{1}{2}(a_0 a_0^\dagger - a_1 a_1^\dagger) \\
\hat{N}_+ &= \hat{N}_1 + i\hat{N}_2 = a_2 a_3^\dagger \\
\hat{M}_+ &= \hat{M}_1 + i\hat{M}_2 = a_1 a_0^\dagger \\
\hat{N}_- &= \hat{N}_1 - i\hat{N}_2 = a_3 a_2^\dagger \\
\hat{M}_- &= \hat{M}_1 - i\hat{M}_2 = a_0 a_1^\dagger.
\end{align*}
\] (4.4)

It is possible to define a total number operator $\hat{N}$, whose eigenvalue corresponds exactly with the label $n$ of the representation $\hat{G}_\mu^{(n)}$, introduced in formula (2.3):

\[
\hat{N} = a_0^\dagger a_0 + a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \quad \hat{N} \rightarrow n.
\] (4.5)

In terms of $\hat{N}$ the Casimir for $\hat{x}_i^2$ is:

\[
\sum_i \hat{x}_i^2 = \rho^2 \hat{N}(\hat{N} + 4) = R^2.
\] (4.6)

The idea of Ref. [11] is to construct the projective modules for the $k$-instanton, starting from a vector whose entries belong to the oscillator algebra (4.3):

\[
|\psi_k> = N_k(\hat{N}) \left( \begin{array}{c} (a_0)^k \\
\vdots \\
\sqrt{\frac{k!}{i_1 i_2 i_3 (k - i_1 - i_2 - i_3)!}} (a_0)^{k-i_1-i_2-i_3} (a_1)^{i_1} (a_2)^{i_2} (a_3)^{i_3} \\
\vdots \\
(a_1)^k \end{array} \right). \]
\] (4.7)

where $0 \leq i_1 \leq k$, $0 \leq i_2 \leq k - i_1$, $0 \leq i_3 \leq k - i_1 - i_2$.

In total the number of entries of the vector $|\psi_k>$ is given by summing on $i_1, i_2, i_3$ as follows:

\[
N_k = \frac{(k + 1)(k + 2)(k + 3)}{6}
\] (4.8)

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which can be identified with the matrix $\hat{\Gamma}_n^{(k)}$ (i.e. posing $n = k$).

Fortunately the normalization condition for $|\psi_k>$ can be solved by a function function $\hat{N}_k$ of the number operator $\hat{N}$ as follows:

$$<\psi_k|\psi_k> = 1$$
$$N_k = N_k(\hat{N}) = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(\hat{N} - i + k)}} = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(n - i + k)}}. \quad (4.9)$$

The projector for the $k$-instanton on the fuzzy four-sphere is defined as

$$P_k = |\psi_k><\psi_k|$$
$$P_k^2 = P_k$$
$$P_k^\dagger = P_k. \quad (4.10)$$

In the ket-bra product there appear only entries commuting with the number operator $\hat{N}$, and therefore those are polynomial functions of the sixteen combinations $a_i a_j^\dagger$, which produce the fuzzy four-sphere algebra and the number operator, equal to its eigenvalue $n$. Therefore we fulfill the requirement that the projector $P_k$ has as entries the elements of the basic operator algebra of the theory.

We can verify that the trace of the projector $P_k$ is always a positive integer, as it should be

$$Tr \ P_k = \frac{(n + k + 1)(n + k + 2)(n + k + 3)}{(n + 1)(n + 2)(n + 3)} \ Tr1$$
$$= \frac{(n + k + 1)(n + k + 2)(n + k + 3)}{6} < \frac{(n + 1)(n + 2)(n + 3)}{6} \frac{(k + 1)(k + 2)(k + 3)}{6} = Tr1_P$$

where $1_P$ is the identity projector.

With the knowledge of the vector-valued operator $<\psi_k|$ it is possible to compute in principle the corresponding 1-form connection for a $k$-instanton

$$A_k^\nabla = <\psi_k|d|\psi_k>. \quad (4.12)$$

To build the $(-k)$-instanton it is enough to take the analogous of the vector $|\psi_k>$ with adjoint entries, apart from a new normalization function $N_k(\hat{N})$. In fact we consider the vector
\begin{equation}
|\psi_{-k} > = N_k(\hat{N}) \left( \begin{array}{cccc}
(a_0^\dagger)^k \\
\vdots \\
(a_1^\dagger)^{k-i_1-i_2-i_3} \\
\vdots \\
(a_3^\dagger)^{k} 
\end{array} \right). \tag{4.13}
\end{equation}

Again we obtain the good property that the normalization condition is solved by a function $N_k$ of the number operator:

\begin{equation}
<\psi_{-k}|\psi_{-k}> = 1 \\
N_k = N_k(\hat{N}) = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(\hat{N} + i + 4 - k)}} = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(n + i + 4 - k)}}. \tag{4.14}
\end{equation}

The corresponding projector for the $(-k)$-instanton is

\begin{equation}
P_{-k} = |\psi_{-k}> <\psi_{-k}| \quad k < n + 4. \tag{4.15}
\end{equation}

We have to be careful with the trace of this projector

\begin{equation}
Tr P_{-k} = \frac{(n - k + 1)(n - k + 2)(n - k + 3)}{(n + 1)(n + 2)(n + 3)} Tr 1 = \\
\frac{(n - k + 1)(n - k + 2)(n - k + 3)}{6} < Tr 1_P \tag{4.16}
\end{equation}

since it is definite positive if and only if

\begin{equation}
k < n + 1. \tag{4.17}
\end{equation}

For the special cases $k = n + 1, n + 2, n + 3$, $P_{-k}$ is simply the null projectors.

\section{Projectors for the SU(2) instantons}

A nice paper \cite{8} deals with a finite projective module description of the non-trivial $SU(2)$ gauge configurations on the sphere $S^4$. In the case of $SU(2)$ instanton we must take care of the
vector space underlying the theory, i.e. the quaternion field \( H \). Let’s define \( A_H = C^\infty(S^4, H) \) the algebra of the smooth functions taking values in \( H \) on the base space \( S^4 \). The projector \( p \in M_N(A_H) \) for the quaternion valued functions can be built using the principal Hopf fibration \( \pi : S^7 \to S^4 \) on the sphere. It is convenient to introduce also \( B_H = C^\infty(S^7, H) \) the algebra of smooth functions with values in \( H \) on the total base space \( S^7 \).

The projector can be written as

\[
p = |\psi><\psi|
\]  

with

\[
|\psi> = \begin{pmatrix} 
\psi_1 \\
\vdots \\
\psi_N
\end{pmatrix}
\]  

a vector valued function on \( S^7 \), i.e. an element of \( (B_H)^N \). Imposing that the vector valued function \( |\psi> \) is normalized, i.e.

\[
<\psi|\psi> = 1
\]  

we obtain that \( p \) is a projector since

\[
p^2 = |\psi><\psi| |\psi><\psi| = p \quad p^\dagger = p.
\]  

We have already introduced the principal Hopf fibration \( SU(2) \cong Sp(1) \pi : S^7 \to S^4 \) on the four-dimensional sphere, but to prepare the \( SU(2) \) case we need to realize it in terms of a couple of quaternions, instead of four oscillators:

\[
S^7 = \{(a, b) \in H^2, |a|^2 + |b|^2 = 1\}
\]  

with right action

\[
S^7 \times Sp(1) \to S^7 \quad (a, b)w = (aw, bw)
\]

\[
w \in Sp(1) \leftrightarrow w\overline{w} = 1.
\]  

The right action \( 5.6 \) respects the \( S^7 \) constraint. In terms of the quaternions \( a, b \) the fiber bundle projector \( \pi : S^7 \to S^4 \), the Hopf fibration, is realized as \( \pi(a, b) = (x_1, x_2, x_3, x_4, x_5) \)
\[ x_1 = a\bar{b} + b\bar{a} \]
\[ \xi = a\bar{b} - b\bar{a} = -\bar{\xi} \]
\[ x_5 = |a|^2 - |b|^2 \]
\[ \sum_{\mu=1}^{5} (x_{\mu})^2 = (|a|^2 + |b|^2)^2 = 1. \]  \hfill (5.7)

The basic \( Sp(1) \) invariant functions on \( S^7 \) are obtained by inverting (5.7)

\[ |a|^2 = \frac{1}{2}(1 + x_5) \]
\[ |b|^2 = \frac{1}{2}(1 - x_5) \]
\[ \bar{a}b = \frac{1}{2}(x_1 + \xi) \]  \hfill (5.8)

from which we can build a generic (polynomial) invariant function on \( S^7 \) as a function of these variables.

The projector for \( k = 1 \) instanton is obtained by considering the following ket valued function:

\[ |\psi > = \begin{pmatrix} a \\ b \end{pmatrix} \]  \hfill (5.9)

satisfying the normalization condition \(<\psi|\psi > = |a|^2 + |b|^2 = 1 \) on \( S^7 \).

We can define a projector in \( M_2(A_H) \) as

\[ p = |\psi ><\psi| = \begin{pmatrix} |a|^2 & \bar{a}b \\ b\bar{a} & |b|^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + x_5 & x_1 + \xi \\ x_1 - \xi & 1 - x_5 \end{pmatrix}. \]  \hfill (5.10)

The right action \( Sp(1) : S^7 \times Sp(1) \rightarrow S^7 \) transforms the vector \( |\psi > \) in a multiplicative manner

\[ |\psi > \rightarrow |\psi^w > = \begin{pmatrix} aw \\ bw \end{pmatrix} = |\psi > w \quad \forall w \in Sp(1) \]  \hfill (5.11)

while the projector \( p \) remains invariant and therefore its elements belong to the algebra \( A_H \) instead of \( B_H \), as it should be.
The canonical connection associated with the projector $\nabla = p \cdot d$ has curvature given by

$$\nabla^2 = p(dp)^2 = |\psi><\psi|d\psi><\psi|d\psi><\psi| + |\psi><\psi|d\psi><\psi|. \quad (5.12)$$

Because of the fact that $<\psi|d\psi>$ is a 1-form with values in $H$, the first term is non-vanishing. The associated Chern classes are

$$C_1(p) = -\frac{1}{2\pi i} Tr(p(dp)^2)$$
$$C_2(p) = -\frac{1}{8\pi^2} [Tr(p(dp)^4) - C_1(p)^2]. \quad (5.13)$$

Since the two-form $p(dp)^2$ has values in the purely imaginary quaternions, its trace is vanishing:

$$C_1(p) = 0. \quad (5.14)$$

Instead for the second Chern class we obtain

$$C_2(p) = -\frac{3}{8\pi^2} d(vol(S^4)) \quad (5.15)$$

and the corresponding Chern number is given by

$$c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2} \int_{S^4} d(vol(S^4)) = -\frac{3}{8\pi^2} \frac{8\pi^2}{3} = -1. \quad (5.16)$$

To obtain a nonequivalent projector it is enough to take the transpose of $p$

$$q = \begin{pmatrix} |a|^2 & b\bar{a} \\ a\bar{b} & |b|^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + x_5 & x_1 - \xi \\ x_1 + \xi & 1 - x_5 \end{pmatrix} \quad (5.17)$$

but to write this projector in a ket-bra combination we need to introduce a trick, see [8] for details. In this case the Chern number is equal to 1. Having different topological charges the projectors $p$ and $q$ are inequivalent.

It is easy to compute the 1-form connection, associated with the projector $p$

$$A^\nabla = <\psi|d\psi> = \bar{a}da + \bar{b}db. \quad (5.18)$$
Since $A^\nabla$ is anti-hermitian, it has values in the purely imaginary quaternions that can be identified with the Lie algebra $Sp(1) \simeq SU(2)$.

Non-equivalent gauge connections are obtained acting on the vector $|\psi\rangle$ with an element $g \in GL(2, H)$ module $Sp(2) \simeq Spin(5)$:

$$|\psi\rangle \rightarrow |\psi^g\rangle = \frac{1}{[<\psi|g^\dagger g|\psi>]^1/2}g|\psi\rangle.$$  \hspace{1cm} (5.19)

Requiring that these transformations maintain the self-duality condition of the instanton, $GL(2, H)$ must be reduced to the pure conformal transformations. The (preserving orientation) conformal group of $S^4$ is $SL(2, H)$ and since

$$\text{dim}(SL(2, H)) - \text{dim}(Sp(2)) = 15 - 10 = 5$$ \hspace{1cm} (5.20)

we obtain a five parameter family of instantons, i.e. exactly the ADHM construction of instantons.

At a non-commutative level this construction must be modified since the coordinates of the fuzzy sphere do not form a closed algebra, and we must add the contribution of the $\hat{w}_{\mu\nu}$ coordinates. However we now show how to construct projectors which tend with continuity to the $SU(2)$ projectors on $S^4$, in which the contribution of the extra coordinates $\hat{w}_{\mu\nu}$ decouples in the classical limit.

To obtain that, we rewrite the quaternions $(a, b)$ in terms of the basic oscillators of the theory, see (4.3):

$$a = \begin{pmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{pmatrix}, \quad b = \begin{pmatrix} a_2 & -a_3^\dagger \\ a_3 & a_2^\dagger \end{pmatrix}$$ \hspace{1cm} (5.21)

from which it follows that the combination $a\overline{b}$ is a function of the fuzzy four-sphere coordinates, while the combination $\overline{ba}$ is outside from the algebra $a_ia_j^\dagger$. To succeed in obtaining combinations of the type $(a\overline{a}, a\overline{b}, b\overline{a}, b\overline{b})$ in the projectors we must start from a vector of the form

$$|\psi_0\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.$$ \hspace{1cm} (5.22)

Imposing the normalization condition
\[
< \psi_0 | \psi_0 > = (\bar{a} \bar{b}) \left( \begin{array}{c} a \\ b \end{array} \right) = \bar{a} a + \bar{b} b = \left( \begin{array}{cc} \hat{N} & 0 \\ 0 & \hat{N} + 4 \end{array} \right)
\] (5.23)

we obtain a diagonal matrix with elements dependent on the number operator \( \hat{N} \), hence we simply redefine \(| \psi_0 \rangle\) as:

\[
| \psi_0 \rangle \rightarrow | \psi \rangle = \left( \begin{array}{c} a' \\ b' \end{array} \right) \quad a' = a \sqrt{h(\hat{N})} \quad b' = b \sqrt{h(\hat{N})}
\] (5.24)

where

\[
h(\hat{N}) = \left( \begin{array}{cc} \frac{1}{\hat{N}} & 0 \\ 0 & \frac{1}{\hat{N}+4} \end{array} \right).
\] (5.25)

We can develop the projectors as

\[
p_N = | \psi \rangle \langle \psi | = \left( \begin{array}{c} a \\ b \end{array} \right) \left( \begin{array}{cc} h(\hat{N}) & 0 \\ 0 & h(\hat{N}) \end{array} \right) (\bar{a} \bar{b}) = \left( \begin{array}{cc} ah(\hat{N}) & ah(\hat{N}) \bar{b} \\ bh(\hat{N}) \bar{a} & bh(\hat{N}) \bar{b} \end{array} \right).
\] (5.26)

Now these elements are functions not only of the coordinates \( \hat{x}_\mu \) but also of the \( \hat{w}_{\mu\nu} \). For example let’s work out the first entry:

\[
ah(\hat{N}) \bar{a} = \left( \begin{array}{cc} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{array} \right) \left( \begin{array}{cc} \frac{1}{\hat{N}} & 0 \\ 0 & \frac{1}{\hat{N}+4} \end{array} \right) \left( \begin{array}{cc} a_0^\dagger & a_1^\dagger \\ -a_1 & a_0 \end{array} \right) = \left( \begin{array}{cc} \frac{1}{N+1} a_0 a_0^\dagger + \frac{1}{N+3} a_1^\dagger a_1 \\ \frac{1}{N+1} a_1 a_1^\dagger + \frac{1}{N+3} a_0 a_0^\dagger \end{array} \right).
\] (5.27)

At this level the projector is no more singular in the number operator \( \hat{N} \) and it is possible to substitute to \( \hat{N} \) its eigenvalue \( n \). In this way we obtain the final form of the projector \( p_N \).

Its trace is determined by the formula

\[
Tr \ p_N = \left( \frac{n+4}{n+1} + \frac{n}{n+3} \right) \quad TrI = \left( \frac{(n+2)(n+3)(n+4)}{6} + \frac{n(n+1)(n+2)}{6} \right) = \quad 2 \ TrI + (n+2) \quad < TrI_P = 4 \ TrI
\] (5.28)

and it is always an integer, for every value of \( n \).
6 Projectors and equations of motion

Let’s recall the most general Y-M action on the fuzzy four-sphere in terms of matrix models:

\[
S(\lambda) = -\frac{1}{g^2} Tr \left[ \frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu] + \frac{2\lambda}{5(n+2)\rho} \epsilon^{\mu\nu\lambda\rho\sigma} X_\mu X_\nu X_\lambda X_\rho X_\sigma + 8(1 - \lambda) \rho^2 X_\mu X_\mu \right].
\] (6.1)

The matrix variable

\[
X_\mu = \rho (\hat{G}_\mu + A_\mu)
\] (6.2)

is related to the fluctuation \(A_\mu\), which contains the degrees of freedom of a pure Y-M connection on a sphere \(S^4\). The corresponding equations of motion

\[
[X_\nu, [X_\mu, X_\nu]] + \frac{2\lambda}{(n+2)\rho} \epsilon^{\mu\nu\lambda\rho\sigma} X_\nu X_\lambda X_\rho X_\sigma + 16(1 - \lambda) \rho^2 X_\mu = 0
\] (6.3)

are of course solved by the background \(X_\mu = \rho \hat{G}_\mu\) due to the identity:

\[
[X_\nu, [G_\mu, G_\nu]] = -16G_\mu
\]

\[
G_\mu = \frac{1}{8(n+2)} \epsilon^{\mu\nu\lambda\rho\sigma} G_\nu G_\lambda G_\rho G_\sigma.
\] (6.4)

Our aim is to prove that the projectors, introduced in section 4, are solutions of a particular class of models \(S(\lambda)\).

We firstly attempt to link the projector \(p\) to the matrix variable \(X_\mu\) according to the formula

\[
\tilde{X}_\mu = pp\hat{G}_\mu p.
\] (6.5)

In this naive identification we obtain a gauge invariant matrix variable, which is not directly coincident with the standard one \(X_\mu\). However by using the property that the projector can be put in the form of a ket-bra function,
\[ p = |\psi \rangle \langle \psi| \]  

(6.6)

the passage from the gauge invariant formulation \( \tilde{X}_\mu \) to the usual gauge covariant one \( X_\mu \) is straightforward

\[ X_\mu = \langle \psi | \tilde{X}_\mu | \psi \rangle = \langle \psi | \rho \hat{G}_\mu | \psi \rangle = \rho \hat{G}_\mu + \rho < \psi | [\hat{G}_\mu, | \psi \rangle]. \]  

(6.7)

At this level we recognize the fluctuation \( A_\mu \) represented in terms of the vector valued function \( | \psi \rangle \):

\[ A_\mu = \langle \psi | [\hat{G}_\mu, | \psi \rangle]. \]  

(6.8)

In ref. [13] it was already noticed that introducing the link (6.5) directly in the classical action produces an ambiguity in the variational problem. It is clear that the variation of the classical action with respect to a generic projector \( p \), subject only to the conditions \( p^2 = p, p^\dagger = p \), is more general than the variation with respect to the connection \( \tilde{X}_\mu \).

To avoid such ambiguity we will discuss the link (6.5) only for the \( Y - M \) equations of motion, showing that the instanton projectors (4.10) and (4.15) are indeed solutions to them.

By introducing the link (6.5), the equation of motion (6.3) can be written, thanks to the identity

\[ [X_\mu, X_\nu] = \rho^2 p([G_\mu, p][G_\nu, p] - [G_\nu, p][G_\mu, p] + [G_\mu, G_\nu])p \]  

(6.9)

as

\[ p[G_\nu, (G_\mu, p][G_\nu, p] - (\mu \leftrightarrow \nu))]p + \]

\[ p[[G_\mu, p], [G_\nu, p]]p] + \]

\[ \frac{\lambda}{2(n + 2)} \epsilon^{\mu \nu \lambda \rho \sigma} (4p[G_\nu, p][G_\lambda, p][G_\mu, p][G_\rho, p][G_\sigma, p] + 2p[G_\nu, p][G_\lambda, p][G_\rho, G_\sigma]p + 2p[G_\nu, G_\lambda][G_\rho, p][G_\sigma, p]p + p[[G_\mu, G_\lambda], p][[G_\rho, G_\sigma], p]) = 0. \]  

(6.10)

To show that the equation (6.10) is solved by the projectors \( p_k \), it is simpler to compute directly \( \tilde{X}_\mu \).
\[ \tilde{X}^{(k)}_{\mu} = \rho p_k G_{\mu} p_k = \rho \frac{N}{N+k} |\psi_k > G_{\mu} < \psi_k| = f(k)(\rho)|\psi_k > G_{\mu} < \psi_k| \quad k > 0 \]

\[ \tilde{X}^{(-k)}_{\mu} = \rho p_{(-k)} G_{\mu} p_{(-k)} = \rho \frac{N+4}{N+4-k} |\psi_{(-k)} > G_{\mu} < \psi_{(-k)}| = f_{(-k)}(\rho)|\psi_{(-k)} > G_{\mu} < \psi_{(-k)}| \quad 0 < k < N + 1. \] (6.11)

We can extract from (6.11) the contribution of the (gauge-invariant) connection, related to the charge \( k \) of the instanton according to

\[ |\psi_k > A^{(k)}_{\mu} < \psi_k| = -\frac{k}{N+k} |\psi_k > G_{\mu} < \psi_k| \quad k > 0 \]

\[ |\psi_{(-k)} > A^{(-k)}_{\mu} < \psi_{(-k)}| = \frac{k}{N+4-k} |\psi_{(-k)} > G_{\mu} < \psi_{(-k)}| \quad 0 < k < N + 1. \] (6.12)

We notice that the solution produced by the projectors \( p_k \) is a simple re-scaling of the background, as it happens in the case of non-commutative monopoles. By substituting the ansatz

\[ X^\pm_{\mu} = f^\pm_{\pm k}(\rho) \hat{G}_{\mu} \] (6.13)

in the equations of motion for the matrix variable \( X_{\mu} \) we obtain

\[ \lambda(f^\pm_{\pm k}(\rho)^2 + \rho f^\pm_{\pm k}(\rho) + \rho^2) = \rho(f^\pm_{\pm k}(\rho) + \rho) \] (6.14)

from which we can fix the coupling constant \( \lambda \) as a function of \( N \):

\[ \lambda = \frac{\rho(f^\pm_{\pm k}(\rho) + \rho)}{f^\pm_{\pm k}(\rho)^2 + \rho f^\pm_{\pm k}(\rho) + \rho^2} \]

\[ f_k(\rho) = \rho \frac{N}{N+k} = \rho(1 + O(\frac{k}{N})) \]

\[ f_{-k}(\rho) = \rho \frac{N+4}{N+4-k} = \rho(1 + O(\frac{k}{N})). \] (6.15)

The class of models, for which the noncommutative projectors \( p_k \) are solutions, is of the type:

\[ \lambda = \frac{2}{3} + O(\frac{k}{N}) \] (6.16)
i.e. they are situated around the classical value \( \lambda_{cl} = \frac{2}{3} \). With a mechanism similar to
the non-commutative monopoles, we notice that to define non-commutative soliton solutions
on a fuzzy four-sphere it is necessary to perturb the \( \lambda \) coupling constant in a form

\[
\lambda = \lambda_{cl} + \frac{c}{N}.
\] (6.17)

In the special case \( \lambda = \frac{2}{3} \) we should obtain soliton solutions related to the classical sphere
\( S^4 \). However the matrix model \( X_\mu \) contains many extra degrees of freedom with respect
to the pure \( Y - M \) theory, like the dependence on the extra coordinates \( w_{\mu\nu} \). We leave
as an open question to check if these extra degrees of freedom can be decoupled from the
true variables of the classical sphere \( S^4 \). We have already reached an important result i.e.
we know for what models our non-commutative projectors \( p_k \) are solutions of the \( Y - M \)
equations of motion.

7 Reconstruction of the gauge connection from the projectors

The results obtained in section 6 are still partial, since to reconstruct the gauge connection
we cannot use directly the vector \(< \psi_k | \) as in (6.11), function of the oscillator algebra which
is more general than the fuzzy four-sphere algebra \( a_i a_j^\dagger \). This fact produces a discontinuity
problem in the derivative action, since the adjoint action of the background \([ \hat{G}_\mu, . \] on the
oscillator has the form:

\[
\begin{align*}
[\hat{G}_1, . ] &= \bar{a}_2 \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_2} + \bar{a}_1 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_1} + \bar{a}_0 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial a_0} + \bar{a}_3 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_3} \\
[\hat{G}_2, . ] &= i(\bar{a}_2 \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_2} + \bar{a}_1 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_1} - \bar{a}_0 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_0} - \bar{a}_3 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_3} ) \\
[\hat{G}_3, . ] &= \bar{a}_3 \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_3} - \bar{a}_1 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_1} + \bar{a}_0 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_0} - \bar{a}_2 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2} \\
[\hat{G}_4, . ] &= i(\bar{a}_3 \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_3} - \bar{a}_1 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_1} - \bar{a}_0 \frac{\partial}{\partial a_3} + a_3 \frac{\partial}{\partial a_0} + \bar{a}_2 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_2} ) \\
[\hat{G}_5, . ] &= \bar{a}_0 \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_0} + \bar{a}_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_1} - \bar{a}_2 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_2} - \bar{a}_3 \frac{\partial}{\partial a_3} + a_3 \frac{\partial}{\partial a_3}
\end{align*}
\] (7.1)

and it contains a dependence on a spurious angle which is physically irrelevant because
the physical operator algebra is generated by the basic combinations \( a_i a_j^\dagger \).

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This problem has been solved in the case of non-commutative monopoles [24], since the vector $|\psi>$ has a gauge arbitrariness $|\psi> \rightarrow |\psi> U$, leaving the projector invariant.

Consider for example the vector valued operator $|\psi>$ in the case $k = -1$:

$$|\psi_{k=-1}> = \frac{1}{\sqrt{N + 3}} \left( \begin{array}{c} \bar{a}_0 \\ \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{array} \right) = \left( \begin{array}{c} \bar{a}_0 \\ \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{array} \right)$$

(7.2)

where

$$\bar{a}_0 = \sum_{n_1,n_2,n_3,n_4 = 0}^{\infty} \sqrt{\frac{n_1 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1 + 1, n_2, n_3, n_4 > < n_1, n_2, n_3, n_4|$$

$$\bar{a}_1 = \sum_{n_1,n_2,n_3,n_4 = 0}^{\infty} \sqrt{\frac{n_2 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1, n_2 + 1, n_3, n_4 > < n_1, n_2, n_3, n_4|$$

$$\bar{a}_2 = \sum_{n_1,n_2,n_3,n_4 = 0}^{\infty} \sqrt{\frac{n_3 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1, n_2, n_3 + 1, n_4 > < n_1, n_2, n_3, n_4|$$

$$\bar{a}_3 = \sum_{n_1,n_2,n_3,n_4 = 0}^{\infty} \sqrt{\frac{n_4 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1, n_2, n_3, n_4 + 1 > < n_1, n_2, n_3, n_4|$$

$$< \psi_{-1}|\psi_{-1}> = 1.$$  

(7.3)

Unfortunately we are facing with the problem that the action of $|\psi_{-1}>$ doesn’t commute with the number operator $\hat{N}$ and therefore it is not possible to restrict its action to a fixed number $N$, as instead it is required for the construction of a fuzzy four-sphere. We are going to redefine the vector $|\psi_{-1}>$, maintaining the projector $p_1$ invariant in form, with an operator acting on the right and non-commuting with the number operator, i.e. a quasi-unitary operator:

$$|\psi_{-1}> \rightarrow |\psi'_{-1}> = |\psi_{-1}> U \quad UU^\dagger = 1 \quad (U^\dagger U = 1 - P_0).$$

(7.4)

The condition $UU^\dagger = 1$ is enough to keep invariant the form of the non-commutative projectors $p_k$ .

We will see that the presence of the quasi-unitary operator $U$ not only adjusts the classical limit but it reveals also the topological character of the solution, at the matrix model level. Many choices of $U$ are possible, for example:
\[ U_1 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} |n_1,n_2,n_3,n_4 > < n_1 + 1, n_2, n_3, n_4| \] \hspace{1cm} (7.5)

Interchanging \( n_i \leftrightarrow n_j \) we obtain equivalent gauge connections, related by a pure gauge transformation. Let us compute

\[ |\psi'_{-1} > = |\psi_{-1} > U_1 = \begin{pmatrix} \tilde{a}'_0 \\ \tilde{a}'_1 \\ \tilde{a}'_2 \\ \tilde{a}'_3 \end{pmatrix} \]

\[ \tilde{a}'_0 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_1 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1 + 1, n_2, n_3, n_4 > < n_1 + 1, n_2, n_3, n_4| \]

\[ \tilde{a}'_1 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_2 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1 + 1, n_2, n_3, n_4 > < n_1 + 1, n_2, n_3, n_4| \]

\[ \tilde{a}'_2 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_3 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1, n_2, n_3, n_4 > < n_1 + 1, n_2, n_3, n_4| \]

\[ \tilde{a}'_3 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_4 + 1}{n_1 + n_2 + n_3 + n_4 + 4}} |n_1, n_2, n_3, n_4 + 1 > < n_1 + 1, n_2, n_3, n_4| \] \hspace{1cm} (7.6)

Since, by construction, \([\hat{N}, |\psi'_{-1} >] = 0\) the action of \( |\psi'_{-1} > \) is well defined at a fixed \( N = n \), i.e. for a particular fuzzy four-sphere:

\[ \tilde{a}'_0 |_N = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-k_1-1} \sum_{k_3=0}^{N-k_1-k_2-1} \sqrt{\frac{k_1 + 1}{N + 3}} \]

\[ |k_1 + 1, k_2, k_3, N - k_1 - k_2 - k_3 - 1 > < k_1 + 1, k_2, k_3, N - k_1 - k_2 - k_3 - 1| \]

\[ \tilde{a}'_1 |_N = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-k_1-1} \sum_{k_3=0}^{N-k_1-k_2-1} \sqrt{\frac{k_2 + 1}{N + 3}} \]

\[ |k_1, k_2 + 1, k_3, N - k_1 - k_2 - k_3 - 1 > < k_1 + 1, k_2, k_3, N - k_1 - k_2 - k_3 - 1| \]

\[ \tilde{a}'_2 |_N = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-k_1-1} \sum_{k_3=0}^{N-k_1-k_2-1} \sqrt{\frac{k_3 + 1}{N + 3}} \]

\[ |k_1, k_2, k_3 + 1, N - k_1 - k_2 - k_3 - 1 > < k_1 + 1, k_2, k_3, N - k_1 - k_2 - k_3 - 1| \]

\[ \tilde{a}'_3 |_N = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-k_1-1} \sum_{k_3=0}^{N-k_1-k_2-1} \sqrt{\frac{N - k_1 - k_2 - k_3}{N + 3}} \]
\[|k_1, k_2, k_3, N - k_1 - k_2 - k_3 > < k_1 + 1, k_2, k_3, N - k_1 - k_2 - k_3 - 1|. \quad (7.7)\]

These actions now belong to the functional space of the fuzzy four-sphere and they can be recast in terms of the generalized spherical harmonics.

Having solved the discontinuity problem in the derivative action, it is straightforward to prove that \(|\psi'_{-1} > \) gives rise to connections satisfying the \(Y - M\) equations of motion (in the gauge-covariant formulation) since

\[\tilde{X}_\mu = |\psi_{-1} > U f_{-1}(\rho)(U^\dagger G_\mu U)U^\dagger < \psi_{-1}| = |\psi'_{-1} > X_\mu < \psi'_{-1}|. \quad (7.8)\]

Now we can look at the physical solution \(X_\mu\), of the form:

\[X_\mu = f_{-1}(\rho)U^\dagger G_\mu U \quad (7.9)\]

that satisfies the matrix model equations of motion, if \(\lambda\) satisfies to (6.15). The generalization of these results to the case \(|\psi_{-k} > \) with \(k\) generic is direct.

Summarizing all these results, the solution of the matrix model \(X_\mu\) for charge \(-k\) is therefore obtained in two steps:

i) Firstly by re-scaling the background solution \(X_\mu^{(0)} = f(\rho)G_\mu\);

ii) secondly by dressing \(X_\mu^{(0)}\) as \(X_\mu = U^\dagger f(\rho)G_\mu U\); the quasi-unitary operator \(U\) maps the background to a reducible representation of the fuzzy four-sphere algebra, making clear the topological nature of the solution.

It remains to be investigated if this construction can be repeated for the case \(|\psi_k > \) \((k > 0)\), for example \(k = 1\)

\[|\psi_{k=1} > = \frac{1}{\sqrt{N + 1}} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \tilde{a}_0 \\ \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix}\]

\[\tilde{a}_0 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_1 + 1}{n_1 + n_2 + n_3 + n_4 + 1}} |n_1, n_2, n_3, n_4 > < n_1 + 1, n_2, n_3, n_4|\]

\[\tilde{a}_1 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_2 + 1}{n_1 + n_2 + n_3 + n_4 + 1}} |n_1, n_2, n_3, n_4 > < n_1 + 1, n_2 + 1, n_3, n_4|\]

\[\tilde{a}_2 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \sqrt{\frac{n_3 + 1}{n_1 + n_2 + n_3 + n_4 + 1}} |n_1, n_2, n_3, n_4 > < n_1, n_2, n_3 + 1, n_4|\]
\[\tilde{a}_3 = \sum_{n_1, n_2, n_3, n_4=0}^{\infty} \sqrt{\frac{n_4 + 1}{n_1 + n_2 + n_3 + n_4 + 1}} |n_1, n_2, n_3, n_4 > | n_1, n_2, n_3, n_4 + 1\]

< \psi_{k-1} | = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \frac{1}{\sqrt{N + 1}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)

< \psi_1 | \psi_1 >= 1 - < 0, 0, 0 | 0, 0, 0 >= 1 - P_0. \tag{7.10}

In the last normalization condition we can forget the presence of \(P_0\), when verifying that \(p_k\) is a projector, since

\[|\psi_1 > P_0 = P_0 < \psi_1 | = 0 \tag{7.11}\]

the action of \(|\psi_1 >\) on the projector \(P_0\) is null.

However to redefine \(|\psi_1 >\) in order that \([N, \psi'_1]\) is satisfied, it is necessary to use the adjoint of the quasi-unitary operator \(U_1\):

\[U_1^\dagger = \sum_{n_1, n_2, n_3, n_4=0}^{\infty} |n_1 + 1, n_2, n_3, n_4 > | n_1, n_2, n_3, n_4|. \tag{7.12}\]

Unfortunately the dressing with \(U_1^\dagger\) changes the form of the projector \(p_k\) since

\[|\psi'_1 > | \psi'_1 | = |\psi_1 > U^\dagger U < \psi_1 | = |\psi_1 > \left(1 - \sum_{n_2, n_3, n_4=0}^{\infty} |0, n_2, n_3, n_4 > | 0, n_2, n_3, n_4|\right) < \psi_1 |. \tag{7.13}\]

There is an extra contribution in parenthesis not cancelled by the presence of \(|\psi_1 >\). We conclude that it is impossible for the charge \(k\)-instantons to define a connection satisfying the \(Y - M\) equations of motion while the corresponding projectors do it, similarly to what we have found for the noncommutative monopoles in Ref. [24].

8 Conclusions

In Ref. [11] a finite module description of \(U(1)\) instantons on a fuzzy four-sphere has been presented. In this work we have investigated the relationship between these projectors and the matrix model, which defines the physical dynamics from a connection point of view. Basically we have reached two results, i.e. finding the physical models for which the projectors of Ref. [11] are solution to the corresponding equations of motion and characterizing the nontrivial topology at the level of the matrix model.
However it remains open the more interesting case of $U(2)$ gauge theory, leading in the classical limit to the BPST $SU(2)$ instanton, which is complicated by the structure of the quaternions. Already in the $U(1)$ case we have identified the main ingredients to build a noncommutative topology on a fuzzy four-sphere:

i) the presence of a scaling factor, which allows for the smooth limit to a commutative solution;

ii) the presence of quasi-unitary operators acting on the background of the matrix model, which map it, an irreducible representation of the $SO(5, 1)$ algebra, to reducible representations.

In this sense we are able to find connections satisfying the $Y - M$ equations of motion only for negative charge projectors $p_{-k}(0 < k < N + 1)$, since in a matrix variable $X_i$ with fixed dimension we can insert only a (reducible) representation with rank less than that of $X_i$ but not bigger than $X_i$.

We believe that the finite module description is a very powerful method in noncommutative geometry, but we hope with this work to make a bridge with the more familiar physical language of connections, which is encoded in the matrix model approach.

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