Universal Quantum (Semi)groups and Hopf Envelopes

Marco Andrés Farinati

Received: 16 June 2021 / Accepted: 7 February 2022 / Published online: 22 February 2022 © The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract

We prove that, in case \( A(c) \) = the FRT construction of a braided vector space \((V, c)\) admits a weakly Frobenius algebra \( B \) (e.g. if the braiding is rigid and its Nichols algebra is finite dimensional), then the Hopf envelope of \( A(c) \) is simply the localization of \( A(c) \) by a single element called the quantum determinant associated with the weakly Frobenius algebra. This generalizes a result of the author together with Gastón A. García in Farinati and García (J. Noncommutative Geom. 14(3), 879–911, 2020), where the same statement was proved, but with extra hypotheses that we now know were unnecessary. Along the way, we describe a concrete construction for a universal bialgebra associated to a finite dimensional vector space together \( V \) with some algebraic structure given by a family of maps \( \{f_i : V^{\otimes n_i} \rightarrow V^{\otimes m_i}\}_{i \in I} \). The Dubois-Violette and Launer Hopf algebra and the co-quasi triangular property of the FRT construction play a fundamental role in the proof.

Keywords Hopf algebras · Quantum groups · Universal bialgebra · FRT construction · Quantum determinant

1 Introduction

Given \( c : V \otimes V \rightarrow V \otimes V \) a solution of the braid equation, or equivalently, \( R : V \otimes V \rightarrow V \otimes V \) a solution of the Yang-Baxter equation, the FRT construction (Faddeev-Reshetikhin-Takhtajan) produces a co-quasi triangular bialgebra \( A(c) \), so that its comodule category is naturally braided, \( V \) is a comodule over \( A(c) \), and the map \( c \) is recovered as the categorical braiding. The FRT construction gives a standard way of constructing quantum semigroups, it is a bialgebra that is never a Hopf algebra (unless the trivial case \( V = 0 \)), and the problem of getting a Hopf algebra by inverting a quantum determinant is a classical one, for instance, this problem is present in Manin’s work [14]. In [8] we give a partial answer, motivated by the theory of finite dimensional Nichols algebras, we could exhibit very explicit examples generalizing quantum grassmannian algebras and other similar approaches to quantum determinants. The adjective “explicit” in [8] is double: we give explicit formulas for the
quantum determinant and explicit formulas for the antipode. However, the main results in [8] has hypothesis of two kind: the first main result has a theoretical assumption, that we know is not always satisfied, and the second main result has an ad hoc hypothesis, that one could check -using a computer- in concrete examples, but we couldn’t give a general statement where that hypothesis holds or not. This situation is solved in this work.

We emphasis that in [8] we give a framework that generalizes various previous situations, such as quantum grassmanian algebras (qga) or Frobenius quantum spaces (Fqs) introduced by Manin [14, 15], the quantum determinants constructed by Hayashi [11] for multi parametric quantum deformations of \( O(SL_n), O(GL_n), O(SO_n), O(O_n) \) and \( O(Sp_{2n}) \), the quantum exterior algebras (qea) in the work of Fiore [9] for \( SO_q(N), O_q(N) \), and \( S_{pq}(N) \). Also, qea’s appear in the work of Etingof, Schedler and Soloviev [6]. All these qga’s Fqs and qea’s, defined and considered above are quadratic algebras, and in [8] we give examples admitting no quadratic qga’s Fqs’s nor qea’s, but still there might be finite-dimensional Nichols algebras associated with it, hence quantum determinants, and calculation of Hopf envelopes.

It is convenient to see the FRT construction as a bialgebra satisfying a universal property, and by doing that, by the same price, one can define universal bialgebras for a big variety of natural and interesting situations, including the Hopf algebra introduced by Dubois-Violette and Launer, and many others. This construction is so natural that could be considered as folklore, it is almost present in Dubois-Violett and Launer’s original work, but in the author’s opinion, there is no explicit description in the literature, and since this universal point of view will be used intensively, we present it as the first section of this work. We notice that a recent work [1] deals with the same universal object, focused on the existence problem, but in our context, we exhibit a concrete construction, so the existence problem is trivial. Our presentation is different and very explicit so we decided to keep that section. Also, in [12] a similar notion is considered but in the weak-bialgebra context.

The paper is organized as follows: in Section 2 we define the universal bialgebra associated with a map \( f : V^\otimes n_1 \to V^\otimes n_2 \) where \( V \) is a finite dimensional vector space. Obvious generalization for families of maps is also presented. Hence, a bialgebra associated to an algebra is the case of a map \( m : V \otimes V \to V \), if one cares about the unit, one add to the family of maps the unit map \( u : k \to V \), viewing \( k = V^\otimes 0 \), for a coalgebra one may consider \( \Delta : V \to V \otimes V \), etc.

In Section 3 we recall the Hopf envelope of a bialgebra and study in detail the case of the Hopf algebra associated with a non-degenerate bilinear form. We prove that “half” of the set of the original Dubois-Violette and Launer defining relations for this Hopf algebra is redundant. This fact can be seen as the key point in proving that, in our setting, the antipode formula on the right implies the antipode formula on the left.

In Section 4 we recall the main construction in [8]: the definition of Weakly Graded Frobenius Algebra (WGFA) and the corresponding candidate for the explicit formula of the antipode. After that, using the co-quasi triangular property of the FRT construction and the universal construction developed in Section 2 for a modification of the the Dubois-Violete and Launer’s Hopf algebra, the main result (Theorem 4.10) is proved:

**Theorem** If the FRT construction admits a Weakly Graded Frobenius Algebra (WGFA, see Definition 4.1), then its Hopf envelope is the localization with respect to a single element, the quantum determinant associated with the WGFA.

Moreover, we exhibit the formula for the antipode.
In Section 5 we briefly comment on other applications of the universal idea of the first section, and comparison with other works.

## 2 Universal Construction

Let \( k \) be a field, \( V \) a finite dimensional vector space of dimension \( n \), let \( n_1, n_2 \in \mathbb{N}_0 \), and \( f : V^{\otimes n_1} \to V^{\otimes n_2} \) a linear map. Fix \( \{x_i\}_{i=1}^n \) a basis of \( V \) and consider \( C \) the coalgebra with basis \( \{t_i^j\}_{i,j=1}^n \) and comultiplication

\[
\Delta(t_i^j) = \sum_{k=1}^n t_k^i \otimes t_k^j
\]

We consider \( V \) as \( C \)-comodule via

\[
\rho(x_i) = \sum_{j=1}^n t_i^j \otimes x_j
\]

Denote \( TC \) the tensor algebra on \( C \), with bialgebra structure extending the comultiplication of \( C \). Since \( V \) is a \( C \)-comodule, then it is a \( TC \)-comodule, but because \( TC \) is a bialgebra, it follows that \( V^{\otimes \ell} \) is a \( TC \)-comodule for any \( \ell \in \mathbb{N}_0 \). The structure map is as follows: for multi-indices \( I = (i_1, \ldots, i_\ell), J = (j_1, \ldots, j_\ell) \in \{1, 2, \ldots, n\}^\ell \) denote

\[
x_I := x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_\ell} \in V^{\otimes \ell}
\]

\[
t_I^j := t_{i_1}^{j_1} t_{i_2}^{j_2} \cdots t_{i_\ell}^{j_\ell} \in TC
\]

In this notation, the \( TC \)-comodule structure of \( V^{\otimes \ell} \) is given by

\[
\rho(x_I) = \sum_{J \in \{1, \ldots, n\}^\ell} t_I^J \otimes x_J
\]

If \( f : V^{\otimes \ell_1} \to V^{\otimes \ell_2} \), \( f(x_I) = \sum_J f_I^J x_J \), it need not be \( TC \)-colinear, the condition \( \rho(f(x_I)) = (\text{id} \otimes f) \rho(x_I) \) is precisely the commutativity of the following diagram

That is, \( f \) is colinear if and only if

\[
\sum_{J,K} f_J^K t_I^J \otimes x_K = f_I^K t_J^K \otimes x_K \quad (\forall I,K)
\]

So, we define the two-sided ideal \( \mathcal{I}_f := \{ \sum_J (t_I^J f_J^K - f_J^K t_I^J) : \forall I,K \} \) and the algebra

\[
A(f) := TC/\mathcal{I}_f
\]
Theorem 2.1 \( A(f) \) is a bialgebra.

More precisely, \( \Delta I_f \subseteq I_f \otimes TC + TC \otimes I_f \) and \( I_f \subseteq \text{Ker}(\varepsilon) \), where \( \varepsilon : TC \to k \) is the algebra map determined by \( \delta(t^i_j) = \delta^i_j \). In other words, \( I_f \) is a bi-ideal.

Proof Recall, for \( t^j_i = t^{j_1}_{i_1} \cdots t^{j_a}_{i_a} \), the comultiplication is given by
\[
\Delta(t^j_i) = \Delta(t^{j_1}_{i_1}) \cdots \Delta(t^{j_a}_{i_a})
= \left( \sum_{\ell_1} t^{\ell_1}_{i_1} \otimes t^{j_1}_{i_1} \right) \cdots \left( \sum_{\ell_a} t^{\ell_a}_{i_a} \otimes t^{j_a}_{i_a} \right)
= \sum_{\ell_1, \ldots, \ell_a} t^{\ell_1}_{i_1} \cdots t^{\ell_a}_{i_a} \otimes t^{j_1}_{i_1} \cdots t^{j_a}_{i_a}
= \sum_{L \in \{1, \ldots, n\}^a} t^j_i \otimes t^j_L
\]
So,
\[
\Delta \left( \sum_J (t^j_i f^K_j - f^j_i t^K_j) \right) = \sum_{J, L} (t^j_i \otimes t^j_L f^K_j - f^j_i t^j_L \otimes t^K_j)
= \sum_{J, L} \left( t^j_i \otimes (t^j_L f^K_j - f^j_i t^K_j) + t^j_L f^j_i \otimes t^K_j - f^j_i t^j_L \otimes t^K_j \right)
= \sum_{J, L} \left( t^j_i \otimes (t^j_L f^K_j - f^j_i t^K_j) + t^j_L f^j_i \otimes t^K_j - f^j_i t^j_L \otimes t^K_j \right)
= \sum_L t^j_i \otimes \left( \sum_J t^j_L f^K_j - f^j_i t^K_j \right) + \sum_J \left( \sum_L t^{j_L} f^j_i - f^j_i t^{j_L} \right) \otimes t^j_J
\]
Also
\[
\varepsilon \left( \sum_J (t^j_i f^K_j - f^j_i t^K_j) \right) = \sum_J \delta^j_i f^K_j - f^j_i \delta^K_j = f^K_j - f^K_j = 0
\]
We conclude that the ideal generated by \( \sum_J (t^j_i f^K_j - f^j_i t^K_j) \) is a coideal, contained in \( \text{Ker} \varepsilon \).

Remark 2.2 (Variation: families of maps) The above construction generalizes easily for a family of maps. Given \( F := \{ f_i : V \otimes^{n_i} \to V \otimes^{m_i} \}_{i \in I} \), if one defines \( \mathcal{I}_F := \sum_{i \in I} \mathcal{I}_{f_i} \), then \( \mathcal{I}_F \) is a sum of bi-ideals, so, it is a bi-ideal and \( A(F) := TC/\mathcal{I}_F \) is a bialgebra

2.1 Universal Property

Proposition 2.3 The construction \( A(F) \) satisfies the following universal property: If \( V \) is a comodule over a bialgebra \( A \), with structure map \( \rho_A : V \to A \otimes V \), and \( F = \{ f_i : V \otimes^{n_i} \to V \otimes^{m_i} \}_{i \in I} \) is a family of \( A \)-colinear maps, then there exists a unique bialgebra morphism \( \pi : A(F) \to A \) such that the \( A \)-comodule structure on \( V \) is the one coming from \( A(F) \) via \( \pi \), that is, the following diagram is commutative
\[
\begin{array}{c}
V \\
\downarrow \rho_A \quad \nearrow \rho \quad \downarrow \pi \otimes \text{id}_V \\
A \otimes V \\
\end{array}
\]
\[ \xymatrix{ V \ar[r]^-{\rho_A} & A \otimes V \ar[d]^-{\rho} \\
A(F) \otimes V \ar[u]_-{\pi \otimes \text{id}_V} & } \]

Springer
Proof Using the basis \( x_1, \ldots, x_n \) of \( V \), the comodule structure \( \rho_A(x_i) = \sum_j a_{ij} \otimes x_j \) determines uniquely the family of elements \( \{a_{ij}\}_{i,j} \) in \( A \). Define the map \( \pi : A(F) \to A \) via \( t_i^1 \mapsto a_i^1 \), since all \( f_i \) are \( A \)-colinear it follows that \( \pi \) is well defined.

Example 2.4 Let \( V = \mathbb{k}[x]/(x^2) \) be considered as unital algebra. That is, we have two maps, the multiplication and the unit \( F = \{m, u\} \):

\[
m : V \otimes V \to V \\
u : \mathbb{k} \to V
\]

Notice \( \mathbb{k} = V^{\otimes 0} \). Consider \( \{1, x\} \) as a basis of \( \mathbb{k}[x]/(x^2) = \mathbb{k} \oplus \mathbb{k}x \). If

\[
\rho(1) = a \otimes 1 + b \otimes x \\
\rho(x) = c \otimes 1 + d \otimes x
\]

Using \( \rho(1_\mathbb{k}) = 1 \otimes 1_\mathbb{k} \) and \( u(1_\mathbb{k}) = 1 \), the condition of \( u \) being colinear forces \( a = 1 \) and \( b = 0 \). Now because \( x^2 = 0 \), the colinearity of \( m \) gives

\[
0 = \rho(x^2) = \rho(m(x \otimes x)) = (\text{id} \otimes m)(\rho(x \otimes x)) = c^2 \otimes m(1 \otimes 1) + cd \otimes m(1 \otimes x) + dc \otimes m(x \otimes 1) + d^2 \otimes m(x \otimes x) = c^2 \otimes 1 + (cd + dc) \otimes x
\]

We get in \( A(m) \) the relations \( c^2 = 0 \) and \( cd = -dc \). Notice the (co)matrix comultiplication

\[
\Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d \\
\Delta c = c \otimes a + d \otimes c, \quad \Delta d = c \otimes b + d \otimes d
\]

with \( a = 1 \) and \( b = 0 \) gives

\[
\Delta c = c \otimes 1 + d \otimes c, \quad \Delta d = d \otimes d
\]

That is, \( d \) is group-like and \( c \) is a skew primitive. We conclude

\[
A = A((m, u)) = \mathbb{k}[c, d]/(c^2, cd + dc) = \mathbb{k}[N_0]#\mathbb{k}[c]/c^2
\]

It is a very pleasant exercise to show (see also the classical work [16]) that a comodule structure over \( \mathbb{k}[N_0]#\mathbb{k}[c]/c^2 \) is precisely a d.g. structure (with non-negative grading). In this example, the universal construction gives the natural grading \( |1| = 0, |x| = 1 \), together with the differential \( \partial x = 1, \partial 1 = 0 \). Notice that (at least if \( \frac{1}{2} \in \mathbb{k} \)) the abelianization \( A_{ab} := A/(\{A, A\}) = \mathbb{k}[d] = \mathbb{k}[N_0] \). That is, in the “classical” setting one gets only the natural grading on \( \mathbb{k}[x]/(x^2) \) (or the Torus action), but in this non-commutative or “quantum semigroup” action, one gets the differential structure due to the element \( c \).

Remark 2.5 Usual classical objects that are invariant under the group of automorphisms do not need to be automatically “quantum invariant”. For example, if \( A \) is a finite dimensional algebra, one can consider the trace map \( \text{tr} : A \to \mathbb{k} \) given by

\[
\text{tr}(a) = \text{tr}(x \mapsto a \cdot x) \quad (a, x \in A)
\]

In the above example, \( \text{tr}(1) = 2 \) and \( \text{tr}(x) = 0 \), but (assume \( \frac{1}{2} \in \mathbb{k} \))

\[
(id \otimes \text{tr}) \rho(x) = (id \otimes \text{tr})(c \otimes 1 + d \otimes x) = 2c \neq 0 = \rho(\text{tr}x)
\]

So, \( \text{tr} \) is not \( A(m, u) \)-colinear. That is, \( A \) is naturally a d.g. object, but \( \text{tr} \) is not a d.g. map. Similarly, the Killing form of a finite dimensional Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) is not necessarily
colinear with respect to the algebra $A([,])$. Nevertheless, one can always add the relations associated with that operation, for instance, $\text{tr}$ is always colinear with respect to $A(m, u, \text{tr})$, and the Killing form will be colinear using $A([,], \kappa)$.

### 3 The Hopf Algebra of Dubois-Violette and Launer

In [5], the authors define a Hopf algebra associated with a non-degenerate bilinear form in the following way. For a non degenerate bilinear map $b : V \otimes V \rightarrow \mathbb{k}$ and a basis $\{x_i\}_{i=1}^n$ of $V$, write

$$b(x_i \otimes x_j) = b_{ij} \in \mathbb{k}$$

and denote $b^{ij}$ the matrix coefficients of the inverse of the matrix $(B)_{ij} = b_{ij}$. Dubois-Violette and Launer define the $\mathbb{k}$-algebra with generators $\{t^i_j\}_{i,j=1}^n$ and relations (sum over repeated indices).

$$b_{\mu\nu}t^\mu_i t^\nu_j = b_{\lambda\rho} 1$$

and

$$b^{\mu\nu}t^\mu_i t^\nu_j = b^{\lambda\rho} 1$$

In our setting, $V$ is a comodule over the free bialgebra with generators $t^i_j$, and $V \otimes^2$ is a comodule via (sum over repeated indices)

$$\rho(x_i \otimes x_j) = t^i_k t^j_l \otimes x_k \otimes x_l,$$

Considering $\mathbb{k}$ as trivial comodule, the colinearity of $b$ requires, for all $i, j$:

$$\rho(b(x_i \otimes x_j)) = 1 \otimes b(x_i \otimes x_j) = 1 \otimes b_{ij} = (\text{id} \otimes b)(\rho(x_i \otimes x_j)) = t^i_k t^j_l \otimes b_{kl}$$

hence, the equations for $A(b)$ are

$$t^i_k t^j_l b_{kl} = b_{ij}$$

This is exactly the same as Eq. 1, one may wonder about Eq. 2. We have the following:

**Lemma 3.1** Equation 2 is redundant and so, Dubois-Violette and Launer’s construction agree with $A(b)$.

Before the proof, notice that $A(b)$ makes sense for any bilinear form -degenerate or not- and it is always a bialgebra. After [5], Lemma 3.1 gives the following:

**Corollary 3.2** Let $b : V \otimes^2 \rightarrow \mathbb{k}$ be a bilinear form. If $b$ is non-degenerate, then $A(b)$ is a Hopf algebra.

**Proof of Lemma 3.1.** Let us write the defining relations of $A(b)$ in terms of matrices. Consider $B \in M_n(\mathbb{k}) \subset M_n(A(b))$ and $t \in M_n(A(b))$ given by $(B)_{ij} = b_{ij}$ and $(t)_{ij} = t^i_j$. Relation (1) is:

$$t \cdot B \cdot t = B$$

or equivalently

$$B^{-1} \cdot t \cdot B \cdot t = \text{Id}$$

because $B$ is invertible in $M_n(\mathbb{k})$ (and so it is invertible in $M_n(A(b))$). We see that $t$ has a left inverse $B^{-1} \cdot t \cdot B$, we will show that $t$ also has $B^{-1} \cdot t \cdot B$ as right inverse. For that denote $U := t \cdot (B^{-1} \cdot t \cdot B)$
We want to show that $U = \text{id}$. We compute $B^{-1} t B U$ and get

$$B^{-1} t B U = B^{-1} t B t B^{-1} t B$$

and using $t \cdot B \cdot t = B$ we get

$$B^{-1} (t B t) B^{-1} t B = B^{-1} B B^{-1} t B = B^{-1} t B$$

But from $B^{-1} t B U = B^{-1} t B$ it follows that

$$t B U = t B$$

and so

$$B^{-1} t B t B U = B^{-1} t B t B$$

Recall $t B t = B$, so $B^{-1} t B t = \text{id}$ and we conclude from the above equation that

$$B U = B$$

which clearly implies $U = \text{Id}$. Notice that $U = \text{Id}$ means

$$t \cdot B^{-1} \cdot t \cdot B = \text{id}$$
or equivalently

$$t \cdot B^{-1} \cdot t = B^{-1}$$

and the components of this equation is precisely equation Eq. 2. □

In particular, $A(b)$ is a Hopf algebra, the antipode is given by

$$S(t_i^j) := b^{jk} t_i^j b_{li}$$

We mention that the equations

$$S(h_1) h_2 = \varepsilon(h) \quad \text{and} \quad h_1 S(h_2) = \varepsilon(h)$$

for $h = t_i^j$ (and for all $i, j$) mean, respectively,

$$(B^{-1} t B)t = \text{id} \quad \text{and} \quad t(B^{-1} t B) = \text{id}$$

The fact that Dubois-Violette and Launer construction gives a Hopf algebra is well-known, but for completeness we include the following:

**Proof that the antipode is well-defined** Recall the relation $t_k^l t_j^l b_{kl} = b_{ij}$; we want to see that the opposite relations are valid for $S(t_i^j)$, that is

$$S(t_i^j) S(t_k^l) b_{kl} = b_{ij}$$

Let us denote

$$\tilde{B}_{ji} := S(t_j^i) S(t_k^l) b_{kl} = b_{ij} t_k^l b_{cj} b^{kd} t_d^e b_{ei} b_{kl}$$

We have

$$\tilde{B}_{ji} = t_k^l b_{cj} b^{kd} t_d^e b_{ei}$$

So

$$\tilde{B}_{ji} t_i^j = t_k^l b_{cj} b^{kd} t_d^e b_{ei} t_i^j = t_k^l b_{cj} b^{kd} (t_d^e b_{ei} t_i^j) = t_k^l b_{cj} b^{kd} b_{du} = t_k^l b_{cj} g_k^u = t_u^c b_{cj} = b_{ij} t_i^j$$

In matrix notation, $\tilde{B} t = B^t t$. Since $t$ is invertible, it follows that $\tilde{B}_{ji} = b_{ij}$ as desired. □

 Springer
Remark 3.3 Other situations where the universal bialgebra is already Hopf, outside non-degenerate bilinear forms, are possible. See for instance [3] or [4]. However, the Dubois-Violette and Launer’s Hopf algebra will be enough for our purpose.

3.1 The Hopf Envelope

It is well-known that the forgetful functor from Hopf algebras to bialgebras has a left adjoint, the general construction is due to Takeuchi [18]. In other words, if \( B \) is a bialgebra, then there exists a Hopf algebra \( H(B) \) together with a bialgebra map \( \iota_B : B \to H(B) \) such that every bialgebra map \( f : B \to H \) from \( B \) into a Hopf algebra \( H \) factors in a unique way through \( H(B) \):

\[
\begin{array}{ccc}
B & \xrightarrow{\forall f} & H \\
\downarrow \iota_B & & \downarrow \exists \tilde{f} \\
H(B) & & \\
\end{array}
\]

The general construction can be complicated: if \( B \) is given by generators \( b_1, \ldots, b_m \) and relations, one should add extra generators \( b'_1, \ldots, b'_m \) so that \( S(b_i) = b'_i \), and relations in order to get the Antipode axiom, but also one should add \( S^2(b_i) = S(b'_i) \) as generators, say \( b''_i \), and so on. It is not clear in general if the Hopf envelope of a finitely generated bialgebra is finitely generated. In some cases, very few elements are really necessary in order to get a Hopf algebra. For example, if one knows a priori that \( S^2 = \text{id} \) (e.g. if \( B \) is commutative or cocommutative), then the double of the original generators will be enough.

A particularly simple example is \( B = O(M_n(k)) \), whose Hopf envelope is \( O(GL_n(k)) = O(M_n(k))[\det^{-1}] \). That is, we only add a single commuting generator \( D^{-1} \), with the relation \( D^{-1} \cdot \det = \det \cdot D^{-1} = 1 \).

In the framework of the universal biagebra \( A(F) \) associated with a family of maps \( F \), one can also consider \( H(F) := H(A(F)) \), that is, the Hopf envelope of \( A(F) \). It will have the analogous universal property as \( A(F) \) but within Hopf algebras. It is not clear how many generators are really necessary to add, but very different things can happen. From this point of view, the FRT construction (see Section 4.1) is an opposite example of the Dubois-Violette and Launer’s Hopf algebra: The universal bialgebra associated with a non degenerate bilinear form is already a Hopf algebra, while the FRT construction is never Hopf (unless the trivial case \( V = 0 \)). However, the main result of this paper is to show a general circumstance where the Hopf envelope of the FRT construction is the (in general non-commutative - though normal) localization with respect to a single element, that we call the quantum determinant.

Notice that if \( H \) is a Hopf algebra and \( I \) a bi-ideal such that \( S(I) \subseteq I \), then clearly \( H/I \) is a Hopf algebra. This can be applied to the following situation, that will be useful later:

**Theorem 3.4** Let \( V \) be a finite dimensional vector space and let us fix a linear isomorphism \( \Phi : V \to V^{**} \) (not necessary the canonical one). Consider the maps

\[
\begin{align*}
\text{ev}_l : V^* \otimes V & \to \mathbb{k} \\
\phi \otimes x & \mapsto \phi(x) \\
\text{ev}_r : V \otimes V^* & \to \mathbb{k} \\
x \otimes \phi & \mapsto \Phi(x)(\phi)
\end{align*}
\]

\( \Phi \) Springer
If $W := V \oplus V^*$, then the universal bialgebra on $W$ such that the decomposition $W = V \oplus V^*$ and the bilinear maps $ev_1$ and $ev_r$ are colinear, is already a Hopf algebra. We will denote it by $H(ev_1, ev_r)$.

Proof If $b : W \otimes W \to k$ is the bilinear map determined by $b(v, w) = 0 = b(\phi, \psi)$, $b(\phi, v) = ev_1(\phi, v)$, $b(v, \phi) = ev_r(\phi, v)$, then $b$ is non-degenerate. Its universal bialgebra is Hopf because it coincides with Dubois-Violette and Launer’s one. Let us call it $H(b)$.

Let $x_1, \ldots, x_n$ be a basis of $V$, $x^1, \ldots, x^n$ its dual basis, that for convenience we will denote $x_{n+1}, \ldots, x_{2n}$. Recall $H(b)$ has generators $t^i_j$:

$$S(t^i_j) = b^{ij} t^i_j b_{il}$$

Since $b(V, V) = b(V^*, V^*) = 0$, the matrix $B \in k^{2n \times 2n}$ of $b$ has a structure of $2 \times 2$ blocks of size $n \times n$ of the form

$$B = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

Similar block structure for its inverse. One may write, for $i, j = 1, \ldots, n$

$$S(t_{i,n+j}^j) = \sum_{k,l=1}^{2n} b^{n+j,k} t^j_k b_{li} = \sum_{k,l=1}^n b^{n+j,k} t^j_k b_{li} + \sum_{k,l=1}^n b^{n+j,k} t^j_k b_{n+i,l}$$

and similarly

$$S(t_{n+i}^j) = \sum_{k,l=1}^{2n} b^{j,k} t^j_k b_{i,n+i} = \sum_{k,l=1}^n b^{j,k} t^j_k b_{i,n+i} + \sum_{k,l=1}^n b^{j,n+k} t^j_{n+i} b_{i,n+i}$$

If we add the condition “the decomposition $W = V \oplus V^*$ is colinear”, this is the same as require that the projector

$$\pi_V : V \oplus V^* \to V \oplus V^*$$

$$x_i \mapsto x_i, \quad (i = 1, \ldots, n)$$

$$x_{n+i} \mapsto 0, \quad (i = 1, \ldots, n)$$

should be colinear. (Notice $\pi_{V^*} = id_W - \pi_V$, so we don’t need to add the other projector to the family of maps). We have, for $i = 1, \ldots, n$:

$$\rho(x_i) = \sum_{j=1}^{2n} t^i_j \otimes x_j = \sum_{j=1}^n t^i_j \otimes x_j + \sum_{j=1}^n t^{n+j}_i \otimes x_{n+j}$$

$$\Rightarrow (id \otimes \pi_V)(\rho(x_i)) = \sum_{j=1}^n t^i_j \otimes x_j$$

$$\rho(x_{n+i}) = \sum_{j=1}^{2n} t^j_{n+i} \otimes x_j = \sum_{j=1}^n t^j_{n+i} \otimes x_j + \sum_{j=1}^n t^{n+j}_{n+i} \otimes x_{n+j}$$

$$\Rightarrow (id \otimes \pi_V)(\rho(x_{n+i})) = \sum_{j=1}^n t^j_{n+i} \otimes x_j$$

If one ask $\pi_V$ to be colinear then the relations needed are

$$t^{n+j}_i = 0 = t^j_{n+i} \ \forall i, j = 1, \ldots, n$$
We see from the previous computation that the ideal generated by \( \{ t_i^{n+j}, t_j^{n+i}, i, j = 1, \ldots, n \} \) is stable by the antipode and so, the quotient bialgebra

\[
A(b, \pi_V) = H(b)/\langle t_i^{n+j} = 0 = t_j^{n+i}, i, j = 1, \ldots, n \rangle
\]
is a Hopf algebra. Since \( \pi_V : W \to W \) is \( A(b, \pi_V) \)-colinear, then \( \pi_{V^*} = \text{id}_W - \pi_V \) is colinear too, hence, \( V \subset W \) and \( V^* \subset W \) are subcomodules. Because \( b \) is colinear, we conclude that \( \text{ev}_l \) and \( \text{ev}_r \) are colinear maps too, because they can be computed using \( b \) and restrictions from \( W \) into subcomodules, that is, using compositions with colinear inclusions.

\[\square\]

As a corollary, we can give a proof of Theorem 4.10, that is the same statement as in [8] but almost without hypothesis.

**4 Main Result**

After recalling the main objects of interest here: the FRT construction and Weakly Graded Frobenius Algebras, we prove our main result: Theorem 4.10.

**4.1 FRT Construction: \( A(c) \)**

A *braided vector space* is a pair \((V, c)\), where \( V \) is \( k \)-vector space and \( c \in \text{End}(V \otimes V) \) is a solution of the braid equation:

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c) \quad \text{in } \text{End}(V \otimes V \otimes V),
\]

In [7], the authors define a bialgebra associated with an \( R \)-matrix, but to have an \( R \)-matrix is equivalent to have a braiding considering \( c := \tau \circ R \), where \( \tau : V \otimes V \to V \otimes V \) is the usual flip. In terms of the matrix coefficients of \( c \), the FRT construction is the \( k \)-algebra generated by \( t_i^j \) with relations

\[
\sum_{k, \ell} c_{ij}^{k\ell} t_k^j t_o^{r \ell} = \sum_{k, \ell} t_i^k t_j^\ell c_{rs}^{k\ell} \quad \forall 1 \leq i, j, r, s \leq n.
\]

It turns out that the FRT construction is *exactly* \( A(c) \).

The fact that \( c \) is a solution of the braid equation implies the very important fact that \( A(c) \) is a co-quasi-triangular bialgebra. That is, there exists a convolution-invertible bilinear map \( r : A \times A \to k \) satisfying

\[
\begin{align*}
(CQT1) \quad r(ab, c) &= r(a, c(1))r(b, c(2)) \\
(CQT2) \quad r(a, bc) &= r(a(2), b)r(a(1), c) \\
(CQT3) \quad r(a(1), b(1))a(2)b(2) &= b(1)a(1)r(a(2), b(2))
\end{align*}
\]

This map is uniquely determined by CQT1-2-3 and

\[
r(t_i^k, t_j^\ell) = c_{ji}^{k\ell} \quad \text{for all } 1 \leq i, j, k, \ell \leq n.
\]

(Notice the indices \( ij \) and \( ji \) in the definition of \( r \).) In particular, the category of \( A(c) \)-comodules is braided.

**4.2 Weakly Graded Frobenius Algebras**

Let us recall the main definitions and results of [8]. We begin with the definition of weakly graded Frobenius algebra, that extends the notion of Frobenius quantum space introduced
by Manin in [15, §8.1]. The motivation is to produce quantum determinants together with quantum Cramer-Lagrange identities, hence a formula for the antipode. The paradigmatic examples are finite dimensional Nichols algebras associated with rigid solutions of the braid equation.

**Definition 4.1** [8, 2.1] Let \( \mathcal{A} \) be a bialgebra and \( V \in \mathcal{A} \mathcal{M} \). An algebra \( \mathcal{B} \) is called a weakly graded-Frobenius (WGF) algebra for \( \mathcal{A} \) and \( V \) if the following conditions are satisfied:

1. **WGF1** \( \mathcal{B} \) is an \( \mathbb{N} \)-graded \( \mathcal{A} \)-comodule algebra, that is \( \mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}^n \), where \( \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \) is the structure map, and \( \mathcal{B}^n \cdot \mathcal{B}^m \subseteq \mathcal{B}^{n+m} \) for all \( n, m \geq 0 \);
2. **WGF2** \( \mathcal{B} \) is connected (i.e. \( \mathcal{B}^0 = k \)) and \( \mathcal{B}^1 = V \) as \( \mathcal{A} \)-comodules;
3. **WGF3** \( \dim_k \mathcal{B} < \infty \) and \( \dim_k \mathcal{B}^{top} = 1 \), where \( \mathcal{B}^{top} = \max\{n \in \mathbb{N} : \mathcal{B}^n \neq 0\} \);
4. **WGF4** the multiplication induces non-degenerate bilinear maps

\[ \mathcal{B}^1 \times \mathcal{B}^{top-1} \rightarrow \mathcal{B}^{top}, \quad \mathcal{B}^{top-1} \times \mathcal{B}^1 \rightarrow \mathcal{B}^{top}. \]

**Example 4.2** If \( \mathcal{B} = \mathcal{B}(c) \) is a finite dimensional Nichols algebra associated to a finite dimensional vector space with rigid braiding \((V, c)\), then \( \mathcal{B} \) is WGF algebra for \( \mathcal{A}(c) \).

We notice that conditions in (WGF4) appeared in [15], related to involutive solutions of the QYBE (thus the corresponding \( c \) is a symmetry) and in [10], related to Hecke-type solutions. It is known that in both cases the quantum exterior algebras are Nichols algebras, thus this Definition generalizes [10, 15].

**Definition 4.3** Let \( \mathcal{B} \) be a WGF algebra for \( \mathcal{A} \) and write \( \mathcal{B}^{top} = kb \) for some \( 0 \neq b \in \mathcal{B} \). We call such an element a volume element for \( \mathcal{B} \). Since \( \mathcal{B}^{top} \) is an \( \mathcal{A} \)-subcomodule, \( \rho(b) = D \otimes b \) for some group-like element \( D \in \mathcal{A} \). We call this element \( D \) the quantum determinant in \( \mathcal{A} \) associated with \( \mathcal{B} \).

**Notation 4.4** Let \( \{x_1, \ldots, x_n\} \) be a basis of \( V \). Since by assumption the multiplication \( \mathcal{B}^1 \times \mathcal{B}^{top-1} \rightarrow \mathcal{B}^{top} = kb \) is non-degenerate, there exists a basis of \( \mathcal{B}^{top-1} \), say \( \{\omega^1, \ldots, \omega^n\} \in \mathcal{B}^{top-1} \), such that

\[ x_i \omega^j = \delta^j_i b \in \mathcal{B}^{top} \quad (6) \]

For \( 1 \leq i, j \leq n \), we define the elements \( T_{ij} \in \mathcal{A} \) by the equality

\[ \rho(\omega^j) = \sum_{1 \leq i \leq n} T_{ij} \otimes \omega^i \]

It is easy to check that \( \Delta(T_{ij}) = \sum_{k=1}^n T_{ik} \otimes T_{kj} \) and \( \varepsilon(T_{ij}) = \delta^i_j \) for all \( 1 \leq i, j \leq n \).

**Example 4.5** If \( V = \mathbb{k}^n \), \( c = -\tau \) on \( V \otimes V \), \( \mathcal{A}(c) = \overline{\mathcal{O}(M_n(\mathbb{k}))} \), \( \mathcal{B} = \Lambda V \), then \( b = x_1 \wedge \cdots x_n \) is the usual volume form, the elements \( w^j = (-1)^{i+1} x_1 \wedge \cdots \hat{x}_j \cdots \wedge x_n \) give a “dual basis” with respect to \( \{x_1, \ldots, x_n\} \). The elements \( T_{ij} \) are the minors of the generic matrix. The bialgebra \( \overline{\mathcal{O}(M_n(\mathbb{k}))} \) is not Hopf, but its localization \( \overline{\mathcal{O}(\mathcal{GL}(n, \mathbb{k}))} = \mathcal{O}(M_n(\mathbb{k}))[\det^{-1}] \) is a Hopf algebra.
One of the main goals in [8] was to generalize the Lagrange formula for expanding the determinant by rows, and hence to have a natural candidate for the antipode on the localization by quantum determinants. The general statement is:

**Proposition 4.6** [8, Proposition 2.6] The following formula holds in $A(c)$:

$$\sum_{k=1}^{n} t_i^k r_j^k = \delta_i^j D \quad \text{for all } 1 \leq i, j \leq n. \quad (7)$$

**Proof** Apply the comodule structure map $\rho$ to Eq. 6. \qed

We recall a result of Hayashi.

**Lemma 4.7** [11, Theorem 2.2] Let $A$ be a co-quasi triangular bialgebra. For any group-like element $g \in A$, there is a bialgebra automorphism $\mathfrak{J}_g : A \to A$ given by

$$\mathfrak{J}_g(a) = r(a_{(1)}, g)a_{(2)}r^{-1}(a_{(3)}, g)$$

such that

$$ga = \mathfrak{J}_g(a)g \quad \text{for all } a \in A.$$

In particular, $D \in A(c)$ is a group-like element in a co-quasi triangular bialgebra, so we have a bialgebra isomorphism $\mathfrak{J} : A(c) \to A(c)$ such that

$$Da = \mathfrak{J}(a)D \quad \text{for all } a \in A(c)$$

**Definition 4.8** Let $A(c)[D^{-1}]$ be the $\mathbb{k}$-algebra generated by $A(c)$ and a new element $D^{-1}$ satisfying the relations

$$DD^{-1} = 1 = D^{-1}D. \quad (8)$$

It is easy to see that $A(c)[D^{-1}]$ is indeed a (non commutative) localization of $A(c)$ in $D$. We denote by $\iota : A(c) \to A(c)[D^{-1}]$ the canonical map. Notice that, in virtue of Hayashi’s result, $D$ is a normal element, so, the general non-commutative localization can be computed in terms of left (or right) fractions. The next result follows from [11, Theorem 3.1].

**Lemma 4.9** $A(c)[D^{-1}]$ is a co-quasi triangular bialgebra.

### 4.3 Main Result

In this section

- $(V, c)$ is a finite dimensional braided vector space,
- $A(c)$ is the FRT construction,
- we assume that $A(c)$ admits a WGF algebra $\mathfrak{B}$ (see Definition 4.1), denote $D$ its associated quantum determinant.

Since $D \in A(c)$ is a group-like element, it must be invertible in the Hopf envelope of $A(c)$. The question is if this localization is enough in order to get a Hopf algebra. The answer is yes, when $A(c)$ admits a WGF algebra $\mathfrak{B}$:
Theorem 4.10 Let \((V, c)\) be a finite dimensional braided vector space and assume there exists \(B\) a WGF algebra for \(A(c)\). Denote \(D \in A(c)\) the associated quantum determinant. Then \(A(c)[D^{-1}]\) is a Hopf algebra, and the formula for the antipode on generators is given by

\[ S(D^{-1}) = D \quad \text{and} \quad S(t^i_j) := T^i_j D^{-1}, \quad \forall 1 \leq i, j \leq n \]

Proof The formula \(S(D^{-1}) = D\) is necessary because \(D\) is a group-like element. Notice also that Eq. 7 of Proposition 4.6 can be written in \(A(c)[D^{-1}]\) in the following form

\[ \sum_{k=1}^{n} t^i_k T^j_k D^{-1} = \delta^j_i = \varepsilon(t^i_j) = \varepsilon((\mathrm{id} \otimes S)(t^i_j)) = \sum_{k=1}^{n} t^i_k S(t^i_k) \quad (9) \]

So, \(S(t^i_j) := T^i_j D^{-1}\) is the natural candidate for the antipode formula. We need to see that \(S\), defined on generators as before, is well-defined and that it satisfies the antipode axiom.

Using that \(A(c)[D^{-1}]\) is co-quasi triangular, the category of comodules is braided, and so, the following two comodules are isomorphic:

\[ V^*: = \mathcal{B}_{\text{top}^{-1}} \otimes kD^{-1} \]

\[ *V := kD^{-1} \otimes \mathcal{B}_{\text{top}^{-1}} \]

Fix an isomorphism. Define \(W := *V \oplus V\) and \(b : W \times W \rightarrow k\) the bilinear map as follows:

\[ b(v, v') = 0 = b(\phi, \phi') \quad \forall \; v, v' \in V, \; \phi, \phi' \in *V \]

Define \(b(\phi, v)\) through the multiplication \(m : \mathcal{B}_{\text{top}^{-1}} \otimes V \rightarrow \mathcal{B}_{\text{top}}:\n\]

\[ *V \otimes V = kD^{-1} \otimes \mathcal{B}_{\text{top}^{-1}} \otimes V \overset{\mathrm{id} \otimes m}{\longrightarrow} kD^{-1} \otimes \mathcal{B}_{\text{top}} \cong kD^{-1} \otimes kD \cong k \]

\[ \phi \otimes v = (D^{-1} \otimes \eta_{\text{top}^{-1}}) \otimes v \mapsto D^{-1} \otimes \eta_{\text{top}^{-1}} v = D^{-1} \otimes (b(\phi, v)b \mapsto b(\phi, v)) \]

and finally, define \(b(v, \phi)\) through the fixed isomorphism \(*V \cong V^*:\n\]

\[ V \otimes *V \cong V \otimes V^* = V \otimes (\mathcal{B}_{\text{top}^{-1}} \otimes kD^{-1}) \overset{\mathrm{id} \otimes m}{\longrightarrow} \mathcal{B}_{\text{top}} \otimes kD^{-1} \cong k \]

It is clear that \(b\) is non degenerate, so \(H(b)\) is a Hopf algebra. But also, because the multiplication in \(\mathcal{B}\) is \(A(c)\) colinear, and the isomorphism \(kD^{-1} \otimes kD \cong k\) is \(A(c)[D^{-1}]\) colinear, we get that \(b\) is also \(A(c)[D^{-1}]\) colinear. Using the universal property for \(H(b)\) we get a map

\[ H(b) \rightarrow A(c)[D^{-1}] \]

Moreover, \(V^*\) and \(V\) are \(A(c)[D^{-1}]\) comodules, hence the projection \(\pi_V : V \oplus V^* \rightarrow V\) is colinear, and the above epimorphism factor through

\[ H(b) \rightarrow H(\mathrm{ev}_l, \mathrm{ev}_r) \rightarrow A(c)[D^{-1}] \]

(notation is as in Theorem 3.4) Recall the set of generators \(\{t^i_j, i, j = 1, \ldots, 2n\}\) of \(H(b)\) and \(H(\mathrm{ev}_l, \mathrm{ev}_r)\). Notice the subset \(\{t^i_j, i, j = 1, \ldots, n\}\) maps into the generators \(\{t^i_i, i, j = 1, \ldots, n\}\) of \(A(c)\).

We also know from Proposition 4.6 that the following equation holds:

\[ \sum_{k=1}^{n} t^i_k T^j_k = \delta^j_i D \quad \text{for all} \; 1 \leq i, j \leq n. \]
so, in $A(c)[D^{-1}]$ we have

$$
\sum_{k=1}^{n} t_{k}^{i} T_{k}^{j} D^{-1} = \delta_{i}^{j} \tag{10}
$$

or in matrix notation

$$
t \cdot \mathcal{I} = \text{id}_{n \times n}
$$

where $(t)_{ij} = t_{i}^{j}$ and $(\mathcal{I})_{ij} = T_{i}^{j} D^{-1}$ are elements of $M_{n}(A(c)[D^{-1}])$.

Now one can compute in $M_{n}(H(b))$ the equality given by the antipode property, for $i, j = 1, \ldots, n$:

$$
\delta_{i}^{j} = \varepsilon(t'_{i}^{j}) = (t'_{i}^{j}) S(t'_{i}^{j}) = \sum_{k=1}^{2n} t'^{k}_{i} S(t'_{k}^{j}) = \sum_{k,l,r=1}^{2n} t'^{k}_{i} b^{kl} t'^{l}_{r} b_{rj}
$$

But in $H(\text{ev}_{l}, \text{ev}_{r})$, using $t_{a+b}^{i} = 0 = t_{n+a}^{b}$, the above sum gives

$$
= \sum_{k=1}^{n} \sum_{l,r=1}^{n} t'^{k}_{i} b^{kl} t'^{l}_{r} b_{rj}
$$

And because the only possible nonzero coefficients of the bilinear form are $b_{k,n+l}$, or $b_{n+k,l}$ ($k, l = 1, \ldots, n$) this is the same as

$$
= \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{r=1}^{n} t'^{k}_{i} b^{kl} t'^{l}_{n+r} b_{n+r,j}
$$

But also, in $H(\text{ev}_{l}, \text{ev}_{r})$ we have $t_{l}^{n+r} = 0$ for $l, r = 1, \ldots, n$, so

$$
= \sum_{k,l,r=1}^{n} t'^{k}_{i} b^{k,n+l} t'^{l}_{n+r} b_{n+r,j}
$$

Similar equation of the left-axiom of the antipode, shows that the matrix $t' \in M_{n}(H(\text{ev}_{l}, \text{ev}_{r}))$ is invertible (and not only the $(t'_{i}^{j})_{i,j=1}^{2n}$-matrix in $M_{2n}(H(\text{ev}_{l}, \text{ev}_{r}))$).

Equation 10 says that the matrix $t \in M_{n}(A(c)[D^{-1}])$ has right inverse, but because $t'$ is the image of $t' \in M_{n}(H(\text{ev}_{l}, \text{ev}_{r}))$ and $t'$ is invertible we conclude that $t$ is also invertible, hence, it has both right and left inverse and they coincide. In other words, we conclude that

$$
\sum_{k=1}^{n} t_{k}^{i} (T_{k}^{j} D^{-1}) = \delta_{i}^{j} \implies \sum_{k=1}^{n} (T_{k}^{j} D^{-1}) t_{k}^{i} = \delta_{i}^{j} \tag{11}
$$

This was the hardest part. Now the computations follows easily: we are ready to prove that $S$ is well defined. Recall the relations in $A(c)[D^{-1}]$ are $DD^{-1} = D^{-1} D = 1$ and

$$
\sum_{k,\ell} c_{k\ell}^{i j} t_{k}^{i} t_{\ell}^{j} = \sum_{k,\ell} \frac{t_{k}^{i}}{t_{\ell}^{j}} c_{k\ell}^{rs}
$$

We need to prove

$$
\sum_{k,\ell} c_{k\ell}^{i j} S(t_{k}^{i}) S(t_{\ell}^{j}) = \sum_{k,\ell} S(t_{k}^{i}) S(t_{\ell}^{j}) c_{k\ell}^{rs} \tag{12}
$$
Let us call as before $T^j_i = T^j_iD^{-1}$, so that $\sum_k t^k_i T^j_k = \delta^j_i = \sum_k \bar{T}_i^k t^j_k$. Now, from

$$\sum_{k,\ell} c_{ij}^k t^r_k t^s_\ell = \sum_{k,\ell} l^r_i t^s_\ell c_{k\ell}$$

we get

$$\sum_{k,\ell,r,s,i,j} c_{ij}^k \bar{T}_s^t r^s_k \bar{T}_p^j s^t_i = \sum_{k,\ell,r,s,i,j} \bar{T}_s^t r^s_k \bar{T}_p^j s^t_i$$

Using that $\bar{T}$ is the inverse matrix of $t$ we get

$$\sum_{k,\ell,r,s,i,j} c_{ij}^k \bar{T}_s^t r^s_k \bar{T}_p^j s^t_i = \sum_{k,\ell,r,s,i,j} \bar{T}_s^t r^s_k \bar{T}_p^j s^t_i$$

or equivalently

$$\sum_{\ell,s,i,j} c_{ij}^\ell \bar{T}_s^t \delta^\ell_\ell \bar{T}_p^j s^t_i = \sum_{k,r,s,i} \bar{T}_s^t r^s_k \delta^\ell_\ell$$

and this is precisely Eq. 12. This equation shows that $S$ is well-defined as anti-multiplicative map. Equation 10 proves that $S$ satisfies the right antipode axiom on generators and Eq. 11 says that $S$ satisfies the left antipode axiom on generators.

In the next section we briefly comment on other applications of the universal idea of the first section, and comparison with other works.

5 The (Locally Finite) Graded Case and Comments on Other Related Works

5.1 $\mathbb{Z}$-Graded Vector Spaces and Graded Coactions

Assume $W = \bigoplus_{p \in \mathbb{Z}} W_p$ is a graded vector space such that $W_p$ is finite dimensional for every $p \in \mathbb{Z}$. Consider the coalgebras

$$C_p := \text{End}(W_p)^* \quad \text{and} \quad C_{gr} := \bigoplus_{p \in \mathbb{Z}} C_p$$

For every $p$, fix $\{x_1^{(p)}, \ldots, x_{\dim W_p}^{(p)}\}$ a basis of $W_p$, denote $\{t^{(p)}_i\}^{\dim W_p}_{i=1}$ the corresponding basis of $C_p$, then $W$ is a $C_p$-comodule by defining

$$\rho(x_i^{(p)}) := \sum_{j=1}^{\dim W_p} t^{(p)}_i \otimes x_j$$

It verifies $\rho(W_p) \subseteq C_p \otimes W_p \subseteq C_{gr} \otimes W_p$, that is, it is a graded $C_{gr}$-comodule. Define $T(C_{gr})$ the tensor algebra with comultiplication extending the comultiplication of $C_{gr}$. $W$ is also a (graded) $T(C_{gr})$-comodule, and hence $W^\otimes n$ is a $T(C_{gr})$-comodule for any $n$.  

\(\text{Springer}\)
If $F = \{ f_i : V \otimes n_i \to V \otimes m_i \}_{i \in I}$ is a family of graded linear maps, the ideal $\mathcal{I}_F$ can be defined exactly in the same way, $A_{gr}(F) = T C_{gr}/\mathcal{I}_F$ will be a bialgebra and $W$ will be a graded $A_{gr}(F)$-comodule.

**Remark 5.1** If $\dim V = \sum_{p \in \mathbb{Z}} \dim V_p < \infty$ then one can consider both $C_{gr}$ or $C = \text{End}(V)^*$, and $E = \{ e_p : V \to V \}$ the family of projectors corresponding the direct summands $V_p$ (i.e. $e_p e_q = \delta_{p,q} e_q$, $\text{Im}(e_p) = V_p$). We have $T(C_{gr}) = T C/\mathcal{I}_E$. If $F = \{ f_i : V \otimes n_i \to V \otimes m_i \}$ is a family of graded maps, then $A_{gr}(F)$ is a quotient of $A(F)$:

$$A(F) \to A(E \cup F) = A_{gr}(F)$$

In particular, if $\mathcal{B}$ a graded algebra, with unit $u$ and multiplication $m : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$, one may consider graded comodule structures on $\mathcal{B}$, and they will be governed by $A_{gr}(u, m)$.

**Example 5.2** Let $V = k[x]/x^2$ be considered as a unital algebra with grading $|1| = 0$ and $|x| = 1$. Every graded component is of dimension 1, so, the graded comodule structures are necessarily of the form

$$\rho(1) = a \otimes 1$$

$$\rho(x) = d \otimes 1$$

with $a$ and $d$ group-like elements. If the unit map $u : k \to k[x]/x^2$ is colinear then $a = 1$, and $A_{gr}(u, m) = k[d] \cong k[N_0]$. One lose the differential structure (compare with Example 2.4). However, we will see that in some cases the graded comodule structures may still be very interesting, since they will correspond, in the quadratic case, to Manin bialgebras.

An easy lemma is the following:

**Lemma 5.3** Let $\mathcal{B}$ be a connected associative graded algebra, namely $\mathcal{B}_{-n} = 0$ for $n > 0$ and $\mathcal{B}_0 = k$. Denote $\mathcal{B}_1 = V$ and assume $\mathcal{B}$ is generated by $V$. That is, $\mathcal{B} = TV/(R)$ where $R$ is homogeneous (but not necessarily concentrated in some specific degree). Then $A_{gr}(\mathcal{B}) := A_{gr}(u, m)$ ($u$ is the unit and $m$ the multiplication) is generated by $C_1 := \text{End}(V)^*$. That is, the map $T(C_1) \hookrightarrow T(C_{gr})$ induces a surjective map $T(C_1) \to A_{gr}(\mathcal{B})$.

**Proof** If $\{x_1, \ldots, x_n\}$ is basis of $V = \mathcal{B}_1$, then the elements of the form $x_{i_1} \ldots x_{i_p}$ generates $\mathcal{B}_p$. Denote $\rho(x_i) = \sum_j t^{(1)}_i j \otimes x_j$. Since $\rho : \mathcal{B} \to A_{gr}(\mathcal{B}) \otimes \mathcal{B}$ is an algebra map,

$$\rho(x_{i_1} \ldots x_{i_p}) = \sum_{j_1, \ldots, j_p} t^{(1)}_{i_1 j_1} \ldots t^{(1)}_{i_p j_p} \otimes x_{i_1} \cdots x_{i_p}$$

and we see that $C_1$ generates $A_{gr}(\mathcal{B})$. \hfill $\square$

### 5.2 The Manin Bialgebra

Recall a quadratic algebra is a $k$-algebra of the form $B = TV/(R)$ with $R \subseteq V^\otimes 2$. We will assume further that $V$ is finite dimensional. In the seminal work [14], Manin define operations $\bullet$, $\circ$ and $(-)^!$ on quadratic algebras. He proves that given a quadratic algebra $B = TV/(R)$, then $\text{end}(B) := B^* \cdot B$ is a bialgebra, $B$ is an $\text{end}(B)$ comodule-algebra, the structure map $\rho : B \to \text{end}(B) \otimes B$ satisfies $\rho(B) \subseteq \text{end}(B) \otimes B$ (Vp), and moreover, $\text{end}(B)$ is universal with respect to those properties (see [15, Section 6.6]). As a corollary we have:

$\mathcal{B}$ Springer
Proposition 5.4 Let $B$ be a finitely generated quadratic algebra: $B = TV/(R)$. If $u_B : k \to B$ denotes the unit map and $m_B : B \otimes B \to B$ its multiplication, then

$$A_{gr}(B) := A_{gr}(u_B, m_B) = B^1 \bullet B$$

Proof Both algebras are generated by $V^* \otimes V = \text{End}_k(V)^*$ and they share the same universal property. The unique isomorphism determined by the universal properties is the identity on generators.

Remark 5.5 In case of a quadratic algebras $B = TV/(R)$, the subspace of defining relations of $A_{gr}(B) = B^1 \bullet B$ is clear. It would be interesting to expand the class of algebras $B$ where the defining relations of $A(B)$ (or $A_{gr}(B)$ if $B$ is graded) can be made explicit.

5.3 $N$-Homogeneous Algebras

In [17], the author follows Manin construction for $N$-homogeneous algebras. That is, if $A = TV/R$ where $R \subseteq R^\otimes N$ for some $N \geq 2$. Define $R^1 \subseteq (V^*)^\otimes N \cong (V^\otimes N)^*$ the annihilator of $R$ and $A^1 := T(V^*)/R^1$. Notice $(A^1)^1 \cong A$. Similarly he defines the operation $\bullet$ as follows. For two $N$-homogeneous algebras $A = TV/(R)$ and $B = TW/(S)$ (where $R \subseteq V^\otimes N$ and $S \subseteq W^\otimes N$ for the same $N$), he define

$$A \bullet B := (T(V \otimes W)/\tau(R \otimes S))$$

where $\tau : (V^\otimes N) \otimes (W^\otimes N) \to (V \otimes W)^\otimes N$ is defined by

$$\tau(v_1 \otimes \cdots \otimes v_N \otimes w_1 \otimes \cdots \otimes w_N) = (v_1 \otimes w_1) \otimes (v_2 \otimes w_2) \otimes \cdots \otimes (v_N \otimes w_N) \in (V \otimes W)^\otimes N$$

Denoting $\text{end}(A) := A^1 \bullet A$, it is a bialgebra and $A$ is a left comodule algebra over it. Analogous considerations for the algebra $A^1$. Finally, he defines $e(A)$ as the quotient of $\text{end}(A)$ by the relations of $\text{end}(A^1)$ (or vice versa). We notice that in our setting, for any finitely generated graded algebra $A$, the universal bialgebra $A_{gr}(A)$ is defined, independently of the degree of the relations, and also works for homogeneous relations of eventualy different degrees, and the same for a pair of graded algebras $(A, A')$ with the same set of generators. For $N$-homogeneous graded algebras, we don’t say that our approach is easier or better than the one in [17], but we say that our approach is sufficiently general and flexible to adapt perfectly to the $N$-homogeneous case, and also for multigraded case, or even to the non-graded case (if the algebras are finite dimensional).

5.4 Path Algebras

Let $Q = (Q_0, Q_1)$ be a finite quiver. If $Q$ has no oriented cycles, then $kQ$ is a finite dimensional $k$-algebra and one may consider the multiplication map $m : kQ \otimes kQ \to kQ$ and unit $u : k \to kQ$, and consequently the universal bialgebra $A(Q) := A(m, u)$. But also, the path algebra $kQ$ is naturally graded by length of paths; that is, $|x_i| = 0 \ \forall i \in Q_0$ and $|x_{\alpha}| = 1 \ \forall \alpha \in Q_1$. So, even if $Q$ happens to have cycles, if $Q$ is finite, maybe $kQ$ is not finite dimensional, but $kQ$ is a locally finite graded vector space, generated as algebra in degree 0 and 1. Hence, the graded version $A_{gr}(Q)$ is defined, and generated by $\text{End}(V_0)^* \oplus \text{End}(V_1)^*$ where $V_0 = k[Q_0]$ and $V_1 = k[Q_1]$ are the vector spaces spanned by $Q_0$ and $Q_1$ respectively. Since $Q_0$ and $Q_1$ are sets, the vector spaces $V_0$ and $V_1$ have
cannonical bases \( \{ x_i : i \in Q_0 \}, \{ x_\alpha : \alpha \in Q_1 \} \), and \( \text{End}(V_0)^* \oplus \text{End}(V_1)^* \) is the coalgebra with basis
\[
\{ t_{ij}^i : i, j \in Q_0 \} \cup \{ t_\alpha^\beta : \alpha, \beta \in Q_1 \}
\]
and comultiplication
\[
\Delta t_{ij}^i = \sum_{k \in Q_0} t_{ik}^k \otimes t_{kj}^j
\]
\[
\Delta t_\alpha^\beta = \sum_{\gamma \in Q_1} t_\gamma^\alpha \otimes t_\gamma^\beta
\]

**Remark 5.6** If one consider the universal bialgebra associated to the graded object \( \mathbb{k} Q \) and the multiplication map \( m : \mathbb{k} Q \to \mathbb{k} Q \to \mathbb{k} Q \) (i.e. one ignores the unit of this algebra) then \( A_{sr}(m) \) is the bialgebra generated by \( \{ t_{ij}^i : i, j \in Q_0 \} \cup \{ t_\alpha^\beta : \alpha, \beta \in Q_1 \} \) and relations
\[
t_{ij}^i t_{kj}^j = \delta_{ik} t_{kj}^j
\]
\[
t_{ij}^{(\beta)} t_\alpha^\beta = \delta_{i,t(\alpha)} t_\alpha^\beta
\]
\[
t_\alpha^{s(\beta)} t_{ij}^j = \delta_{j,s(\alpha)} t_\alpha^\beta
\]
\((s \text{ and } t \text{ are the source and target maps } s, t : Q_1 \to Q_0) \) with comultiplication induced by
\[
\Delta t_{ij}^i = \sum_{k \in Q_0} t_{ik}^k \otimes t_{kj}^j
\]
\[
\Delta t_\alpha^\beta = \sum_{\gamma \in Q_1} t_\gamma^\alpha \otimes t_\gamma^\beta
\]

**Proof** It is straightforward form the relations defining \( k Q \):
\[
x_i x_j = \delta_{ij} x_i \ \forall \ i, j \in Q_0
\]
\[
x_i x_\alpha = \delta_{i,t(\alpha)} x_\alpha \ \forall \ i \in Q_0, \alpha \in Q_1
\]
\[
x_\alpha x_j = \delta_{j,s(\alpha)} x_\alpha \ \forall \ j \in Q_0, \alpha \in Q_1
\]
and the multiplicativity of the structure map. As an illustration we show the second set of relations. From
\[
x_i x_\alpha = \delta_{i,t(\alpha)} x_\alpha
\]
applying \( \rho \) we get
\[
\sum_{j \in Q_0} \sum_{\beta \in Q_1} t_{ij}^j t_\alpha^\beta \otimes x_j x_\beta = \delta_{i,t(\alpha)} \sum_{\beta \in Q_1} t_\alpha^\beta \otimes x_\beta
\]
But \( x_j x_\beta = x_\beta \) if \( j = t(\beta) \) and zero otherwise, so we get
\[
\sum_{\beta \in Q_1} t_{ij}^{t(\beta)} t_\alpha^\beta \otimes x_\beta = \delta_{i,t(\alpha)} \sum_{\beta \in Q_1} t_\alpha^\beta \otimes x_\beta
\]
Since \( \{ x_\beta \}_{\beta \in Q_1} \) are l.i. the result follows.

Changing notation \( t_{ij}^i \leftrightarrow y_{ij} \) and \( t_\alpha^\beta \leftrightarrow y_{pq} \), the universal bialgebra \( A_{gr}(m : \mathbb{k} Q^{\otimes 2} \to \mathbb{k} Q) \) is not the same (they consider weak bialgebras) but very similar to the ones considered in Lemma 4.2 of [12].
5.5 C*-Context

In the C*-algebra context, Wang define (see [19]) a C*-algebra associated with a finite dimensional C*-algebra. This definition includes that a natural state is colinear. The C*-algebra definition is more restrictive, for instance, for the C*-algebra given by the group algebra \( \mathbb{C}[G] \) of a finite group \( G \), it is showed in [13] that the corresponding C*-Hopf algebra is commutative, while it is a relatively easy exercise to show that the universal construction of Section 2 for the algebra \( M_2(\mathbb{k}) \) gives a non commutative nor cocommutative bialgebra. Using that \( \mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \) we see that our construction gives an object different from the one defined by Wang.

5.6 Lie/Leibniz Algebras

In [2], the authors define a commutative bialgebra associated with a Lie or Leibniz algebra by studying the adjoint of the functor

\[
A \rightarrow \mathfrak{h} \otimes A
\]

where \( A \) is a commutative \( k \)-algebra and \( \mathfrak{h} \) is a fixed Leibniz (or Lie) algebra. The bracket in the current algebra is given by

\[
[x \otimes a, y \otimes a'] := [x, y] \otimes aa'
\]

The Leibniz (resp. Lie) algebra \( A \otimes \mathfrak{h} \) is called the current algebra. From the adjoint functor of the current algebra they define what they call the universal algebra of \( \mathfrak{h} \). This (necessary commutative) algebra turns out to be a quotient of a polynomial algebra in \( n^2 \) variables (\( n = \dim \mathfrak{h} \)) that in fact is a bialgebra, with a universal property among commutative bialgebras coacting on \( \mathfrak{h} \). Recall that a Leibniz algebra is a generalization of a Lie algebra in the sense that the operation is not required to be antisymmetric, but it satisfies a choice of the Jacobi identity:

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y]
\]

Notice that there is a permutation of the letters \( x, y, z \), so, if \( A \) is not commutative, it is not clear how to define a Leibniz structure on \( \mathfrak{h} \otimes A \), and the point of view in [2] do not generalizes to non-commutative algebras. However, from the point of view of our universal construction (Section 2), the non-commutative universal bialgebra coacting on \( \mathfrak{h} \) is clear: just consider the bracket operation as a map \( [-, -]_\mathfrak{h} : \mathfrak{h}^2 \rightarrow \mathfrak{h} \), and \( A(\mathfrak{h}) := A([-,-]_\mathfrak{h}) \) will give the universal (in general non-commutative) bialgebra such that \( \mathfrak{h} \) is a comodule and \([-,-]_\mathfrak{h} \) is colinear. The abelianization

\[
A(\mathfrak{h})_{ab} := A(\mathfrak{h})/([A(\mathfrak{h}), A(\mathfrak{h})])
\]

will be of course commutative, and since the ideal generated by brackets is always a bi-ideal, this is also a bialgebra, and satisfies the same universal property of \( A(\mathfrak{h}) \) but among commutative bialgebras. We conclude that the universal commutative bialgebra constructed in [2] is the abelianization of \( A(\mathfrak{h}) \). Similar comments for the Hopf envelope \( H(\mathfrak{h}) \) and \( H(\mathfrak{h})_{ab} \). However, the advantage of having a noncommutative universal bialgebra/Hopf coacting on \( \mathfrak{h} \) is clear. For instance skew-derivations, (e.g. differential graded structures) are detected by non-commutative bialgebras. In a similar way to the example \( k[x]/(x^2) \), the smallest non-abelian Lie algebra already gives a nontrivial (non commutative, nor cocommutative) universal object:
Example 5.7 Let $\mathfrak{h}$ be the non-commutative 2-dimensional Lie algebra $\mathfrak{h} = kx \oplus ky$ with antisymmetric bracket $[x, y] = x$. Writing $C$ as the 4-dimensional coalgebra with basis $a, b, c, d$,

$$
\rho(x) = a \otimes x + b \otimes y, \quad \rho(y) = c \otimes x + d \otimes y,
$$

$$
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,
$$

$$
\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.
$$

The structure map on $x \otimes y$ is computed using the standard diagonal structure:

$$
\rho(x \otimes y) = (a \otimes x + b \otimes y) (c \otimes x + d \otimes y) = ac \otimes (x \otimes x) + ad \otimes (x \otimes y) + bc \otimes (y \otimes x) + bd \otimes (y \otimes y)
$$

The requirement $(\text{id} \otimes [,]) \rho(x \otimes y) = \rho([x, y])$ gives

$$
ac \otimes [x, x] + ad \otimes [x, y] + bc \otimes [y, x] + bd \otimes [y, y] = (ad - bc) \otimes x
$$

$$
= \rho([x, y]) = \rho(x) = a \otimes x + b \otimes y
$$

that is,

$$
(ad - bc) = a, 0 = b
$$

or equivalently $b = 0$ and $ad = a$. Before checking the other conditions for colinearity using $x \otimes y, y \otimes x$ and $y \otimes y$, we see that $b = 0$ implies that $a$ is group-like. In the Hopf envelope $H(\mathfrak{h})$, $a$ must be invertible, and $ad = a$ forces $d = 1$. It is an easy exercise that these conditions are enough to get the bracket colinear: the bialgebra freely generated by $a^{\pm 1}$ and $c$ with

$$
\Delta(a) = a \otimes a, \quad \Delta(c) = c \otimes a + 1 \otimes c
$$

is the universal Hopf algebra, coacting as

$$
\rho(x) = a \otimes x,
$$

$$
\rho(y) = c \otimes x + 1 \otimes y
$$

In other words, the Universal Hopf construction detects a (unique) non trivial d.g. structure on $\mathfrak{h}$.

If we don’t require $d$ to be invertible, it is easy to check that the other conditions on the colinearity of the bracket are $ad = da = a, cd = dc$. So the universal bialgebra is

$$
A(\mathfrak{h}) = k(a, c, d)/(ad = da = a, cd = dc)
$$

with comultiplication

$$
\Delta(a) = a \otimes a, \quad \Delta(d) = d \otimes d, \quad \Delta(c) = c \otimes a + d \otimes c
$$

and coaction

$$
\rho(x) = a \otimes x, \quad \rho(y) = c \otimes x + d \otimes y
$$

Remark 5.8 If $G$ is a group and $\mathfrak{h}$ is a $G$-graded Lie algebra, because of the antisymmetry of the bracket, one may assume that $G$ is abelian. But for Leibniz algebras, grading over non-commutative groups makes perfect sense, and grading over a non-commutative group $G$ is the same as a coaction over the non-commutative Hopf algebra $k[G]$.

Acknowledgements I wish to thank Gastón A. García for fruitful discussions on the developments of this work. Research Partially supported by the projects UBACyT 2018-2021 “K-teoría y bialgebras en álgebra, geometría y topología” and PICT 2018-00858 “Aspectos algebraicos y analíticos de grupos cuánticos”.

Funding Partially supported by UBACyT 2018-2021 “K-teoría y bialgebras en álgebra, geometría y topología” and PICT 2018-00858 “Aspectos algebraicos y analíticos de grupos cuánticos”.

Springer
Data Availability  This manuscript has no associated data.

References

1. Agore, A., Gordienko, A., Vercruysee, J.: V-Universal Hopf Algebras (co)acting on Ω-algebras. arXiv:2005.12954 (2020)
2. Agore, A., Militaru, G.: A new invariant for finite dimensional Leibniz/Lie algebras. arXiv:2006.00711 (2020)
3. Bichon, J., Dubois-Violette, M.: The quantum group of a preregular multilinear form. Lett. Math. Phys. 103(4), 455–468 (2013)
4. Chirvasitu, A., Walton, C., Wang, X.: On quantum groups associated to a pair of preregular forms, to appear in J. Noncommutative Geom. 13(1), 115–159 (2019)
5. Dubois-Violette, M., Lauener, G.: The quantum group of a non-degenerate bilinear form. Phys. Lett. B 245(2), 175–177 (1990)
6. Etingof, P., Schedler, T., Soloviev, A.: Set-theoretical solutions to the quantum Yang-Baxter equation. Duke Math. J. 100(2), 169–209 (1999)
7. Faddeev, L.D., Reshetikhin, N.Y., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. Leningrad Math. J. 1, 193 (1990)
8. Farinati, M., García, G.A.: Quantum function algebras from finite-dimensional Nichols algebras. J. Noncommutative Geom. 14(3), 879–911 (2020)
9. Fiore, G.: Quantum groups SO_q(N), Sp_q(n) have q-determinants, too. J. Phys. A 27(11), 3795–3802 (1994)
10. Gurevich, D.: Algebraic aspects of the quantum Yang-Baxter equation. Leningrad J. Math. 2, 801–828 (1991)
11. Hayashi, T.: Quantum groups and quantum determinants. J. Alg. 152, 146–165 (1992)
12. Huang, H., Walton, C., Wicks, E., Won, R.: Universal Quantum Semigroups. arXiv:2008.00606 (2020)
13. Kasprzak, P., Soltan, P., Woronowicz, S.: Quantum automorphism groups of finite quantum groups are classical. J. Geom. Phys. 89, 32–37 (2015)
14. Manin, Y.: Some remarks on Koszul algebras and quantum groups. Annales de l’institut Fourier 37(4), 191–205 (1987)
15. Manin, Y.: Quantum Groups and Noncommutative Geometry. Université De Montréal Centre De Recherches Mathématiques, Montreal, QC (1988)
16. Pareigis, B.: A non-commutative non-cocommutative Hopf algebra in “Nature”. J. Alg. 70, 356–374 (1981)
17. Popov, T.: Automorphisms of Regular Algebras. arXiv:math/0601264 (2006)
18. Takeuchi, M.: Free Hopf algebras generated by coalgebras. J. Math. Soc. Japan 23, 561–582D (1971)
19. Wang, S.: Quantum symmetry groups of finite spaces. Comm. Math. Phys. 195(1), 195–211 (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.