Landau-Majorana-Stuckelberg-Zener dynamics driven by coupling for two interacting qudit systems

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A time-dependent two interacting spin-qutrit model is analysed and solved. The two interacting qutrits are subjected to a longitudinal field linearly varying over time as in the Landau-Majorana-Stuckelberg-Zener (LMSZ) scenario. Although a transverse field is absent, we show the occurrence of LMSZ transitions assisted by the coupling between the two spin-qutrits. Such a physical effects permits to estimate experimentally the coupling strength between the spins and allows, moreover, the generation of entangled states of the two qutrits by appropriately setting the slope of the ramp. Effects stemming from a noisy surrounding environment are also taken into account by introducing a random fluctuating field component as well as non-Hermitian terms in the Hamiltonian model.

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I. INTRODUCTION

Spin chains are the reference experimental scenario for quantum technology applications thanks to the possibility of entanglement generation also over long distances. Entanglement, indeed, is the key resource for quantum information tasks and its manipulation by field application is of course of fundamental importance.

In this context, a growing interest in qutrits - three-state quantum systems - should be emphasized. Besides the obvious exponential increase of their Hilbert space, qutrits, and qutdis in general, offer several advantages over qubits. For example, among the most important applications of qutrit systems we find: optimization of the Hilbert space dimensionality vs. control complexity, larger violations of nonlocality, new types of quantum protocols, more secure quantum communication, Bell inequalities resistant to noise. Moreover, efficient protocols and methods have been developed for the manipulation of qutrits and qutdis.

In this respect the possibility of realizing a local application of fields on a single qutrit while it interacts with other ones is of basic interest to generate physical effects in the spin chain by manipulating the single spin dynamics. Through the Scanning Tunneling Microscopy (STM), for example, it is possible to construct atom by atom a chain of interacting nanomagnets and to manipulate the state of a single spin by applying a local magnetic field on atomic scale with a STM tip. More precisely, the field created on the single spin is an effective magnetic field stemming from the tunable exchange interaction between the target spin we wish to manipulate and the spin present on the STM tip. Such an effective field may be also time-dependent thanks to the possibility of varying the distance between the tip spin and the one in the chain. It is possible, for example, to create a field varying linearly in time and changing its direction, as in the well known Landau-Majorana-Stuckelberg-Zener (LMSZ) scenario. Thus, STM makes experimentally possible, by atomic manipulation, to control the quantum state and the quantum dynamics of a single spin while the latter is interacting with other neighboring spins and to generate, then, delocalized effects by local field application.

The LMSZ scenario is one of the most famous and important exactly solvable time-dependent single-spin models thanks to the fact that, though its unphysical nature (infinite time duration of the physical process, implying divergence of the instantaneous energy separation as time goes on), it furnishes accurate predictions also for more realistic situations (finite times). However, the exact solutions of the LMSZ dynamical problem exist and may be given in terms of parabolic cylinder functions. Its popularity is confirmed also by a lot of studies, both theoretical and experimental, which have been developed aiming at generalizing the LMSZ scenario considering N-level systems and/or the presence of classical and quantum noise stemming from sources of incoherence.

In this paper we analyse, instead, the quantum dynamics of two interacting qutrits subjected to a LMSZ ramp along the quantization axis. Both the manipulation of the quantum dynamics of a single high spin-value magnetic atom or molecular magnet and the control of the interaction between qutdis in a chain offer, indeed, the experimental background in which such systems turn out to be actual powerful building blocks for quantum information and computation tasks.

The physical interest of the work is twofold. Firstly, we bring to light the existence of a physical effect consisting in the possibility of generating LMSZ transitions in the two-qutrit system, though a transverse constant field is absent. This fact results possible thanks to the coupling existing between the two spin-1’s.

Secondly, we show how such an effect
may be exploited for two relevant applications: the estimation of the strength of the coupling parameters and the possibility of generating asymptotically entangled states of the two qutrits by appropriately setting the slope of the ramp. Our symmetry-based analysis of the Hamiltonian model, usefully used in several problems,47–50 allows us to consider also effects stemming from the presence of a noisy field component.

The structure of the paper is the following. The model and the symmetry-based dynamical reduction are presented in Sec. II. In Sec. III and IV the quantum dynamics of the two qutrits is investigated in the four- and five-dimensional dynamically invariant subspace, respectively. In both sections we report the formal general solution of the dynamical problem and the LMSZ transition probabilities when a linearly varying ramp is applied on just one spin as well as on both the spins. We bring to light moreover the existence of dark states, that is states not evolving independently of the time-dependence of the applied fields. Finally we discuss the modification of the LMSZ probabilities when a random fluctuating field component is present. In Sec. V the study of the negativity as measure of Entanglement between the two qutrits is developed and the possibility of generating entangled states of the two spin-1’s through a LMSZ process is analysed. Finally, conclusive remarks and perspectives may be found in the last section VI.

II. THE MODEL

Let us consider the following model of two interacting qutrits subjected to local time-dependent fields

\[ H = \hbar \omega_1 \hat{\Sigma}_1 + \hbar \omega_2 \hat{\Sigma}_2 + \gamma_x \hat{\Sigma}_1 \hat{\Sigma}_2 + \gamma_y \hat{\Sigma}_1^\dagger \hat{\Sigma}_2^\dagger + \gamma_z \hat{\Sigma}_1^\dagger \hat{\Sigma}_2^\dagger \]  

(1)

where \( \omega_i (i = 1, 2) \) are the characteristic frequencies of the two qutrits, \( \gamma_k \) are the different energy contributions stemming from the coupling between the two three-level systems. The Pauli operators \( \hat{\Sigma}_k \) \((k = x, y, z)\) for a spin-1 system are related with the spin-1 operator components as

\[ \hat{\Sigma}_x = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_x, \quad \hat{\Sigma}_y = \frac{\hbar}{\sqrt{2}} \hat{\Sigma}_y, \quad \hat{\Sigma}_z = \hbar \hat{\Sigma}_z. \]  

(2)

Our scope is to study a Landau-Majorana-Stuckelberg-Zener (LMSZ) scenario for the two qutrits and analyse how the coupling between them and a noisy component of the magnetic field affect their dynamics.

In Ref.48 it was shown that two dynamically invariant Hilbert subspaces exist, one of dimension four spanned by \{ |10\>, |01\>, |0−1\>, |−10\> \} and the other one of dimension five spanned by \{ |11\>, |1−1\>, |00\>, |−11\>, |−1−1\> \}. They are related to the two eigenvalues (±1) of the constant of motion

\[ K = \cos(\pi \hat{\Sigma}_{\text{tot}}), \]

(3)

where \( \hat{\Sigma}_{\text{tot}} = \hat{\Sigma}_1 + \hat{\Sigma}_2 \) is the total spin of the composed system along the \( z \) direction.

In a48 an important property of the two-qutrit system was discovered, which is of basic importance for our analysis. The Hamiltonian governing the two-qutrit dynamics in the four-dimensional subspace may be written in terms of two non-interacting qubits as follows

\[ H_- = H_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes H_2, \]  

(4)

with

\[ H_1 = \frac{\hbar \Omega_+}{2} \hat{\Sigma}_1^+ + \gamma \hat{\Sigma}_1^z, \quad H_2 = \frac{\hbar \Omega_-}{2} \hat{\Sigma}_2^+ + \gamma \hat{\Sigma}_2^z \]  

(5)

where \( \hat{\Sigma}_k \) \((k = x, y, z)\) are the standard Pauli operators and we set \( \Omega_\pm = \omega_1 \pm \omega_2 \) and \( \gamma_\pm = \gamma_x \pm \gamma_y \). The mapping at the basis of such a rewriting is

\[ |10\> \leftrightarrow |++\>, \quad |01\> \leftrightarrow |+-\>, \quad |0−1\> \leftrightarrow |−+\>, \quad |−10\> \leftrightarrow |−−\>. \]  

(6)

The Hamiltonian governing the five dimensional subspace, instead, under the following conditions

\[ \gamma_x = 0, \quad \gamma_y = \gamma_\gamma = \gamma/2, \]  

(7)

is reduced to the following block-diagonal form

\[ H_+ = \begin{pmatrix} h \Omega_+ & 0 & 0 & 0 \\ 0 & h \Omega_- & \gamma & 0 \\ 0 & \gamma & -h \Omega_- & 0 \\ 0 & 0 & 0 & -h \Omega_+ \end{pmatrix}. \]  

(8)

The three-dimensional middle block possesses an su(2) structure and hence can be written in terms of spin variables of a fictitious spin-1, namely

\[ H_3 = \gamma \hat{\Sigma}^x + h \Omega_+ \hat{\Sigma}^z. \]  

(9)

We want now to study the two interacting qutrits when they are subjected to time-dependent fields, \( \omega_0(t) \) and \( \omega_2(t) \). To this end we stress that the results and the analysis reported before in Ref.48 are still valid also when we consider time-dependent fields and, more generally, when all the Hamiltonian parameters depend on time. This is due to the fact that the Hamiltonian structurally commutes with the constants of motion independently of its time-dependence. In the following we show that we are able to construct formally the time evolution operator for both four- and five-state subdynamics. In particular we are interested in analysing the case in which the \( z \)-magnetic field is a ramp as in the LMSZ scenario and in revealing intriguing dynamical effects stemming from the homogeneity or heterogeneity of the coupling parameters and the two fields and also from the consideration of a random fluctuating behaviour of the field, representing the influence of a surrounding environment.

III. FOUR-DIMENSIONAL SUBDYNAMICS

A. General Solution

We may formally write the time evolution operator \( U_j \) \((j = 1, 2)\) related to \( H_j \), that is solution of the Schrödinger equation
$i\hbar \dot{U}_j = H_j U_j$ as follows

$$U_j = \begin{pmatrix} a_j & b_j \\ -b_j^* & a_j^* \end{pmatrix}$$  \hspace{1cm} (10)

where $a_j$ and $b_j$ are time-dependent Cayley-Klein parameters satisfying $|a_j|^2 + |b_j|^2 = 1$. The time evolution operator $U_-$, satisfying the Schrödinger equation $i\hbar \dot{U}_- = H_- U_-$, then reads

$$U_- = U_1 \otimes U_2 = \begin{pmatrix} a_1 a_2 & a_1 b_2 & b_1 a_2 & b_1 b_2 \\ -a_1 b_2^* & a_1 a_2^* & -b_1 b_2^* & b_1 a_2^* \\ -b_1 a_2^* & -b_1 b_2^* & a_1 a_2 & a_1 b_2 \\ b_1^* a_2^* & b_1^* b_2^* & -a_1^* b_2 & a_1^* a_2 \end{pmatrix}.$$  \hspace{1cm} (11)

The mathematical expressions of $a_j(t)$ and $b_j(t)$ depend on the time-dependence of the two local magnetic fields $\omega_1(t)$ and $\omega_2(t)$.

**B. STM Scenario**

1. **Local Dynamics**

We analyse firstly the case of a single local $z$-magnetic field $B_z(t)$ applied on the first spin, consisting in a LMSZ ramp, such that

$$\hbar \omega_1(t) = \alpha t, \quad t \in (-\infty, \infty),$$  \hspace{1cm} (12)

where $\alpha$ is considered a positive real number and rules the adiabaticity of the process since $B_z \propto \alpha$. Let us consider the case of an excitation present in the system and localized in one of the two qutrits, say the second spin; in this case the initial state of the two qutrits (fictitious qubits) is $|−10⟩(|−⟩−)$. In this instance, each fictitious spin-1/2 is subjected to a LMSZ scenario with $\omega_1(t)$ as longitudinal magnetic field and a constant (effective) transverse magnetic field determined by the coupling parameters, which is easy to see by the expressions of the Hamiltonians in Eq. (5). In this way, the first and second fictitious spin-1/2 have the probability to make the transition to the up-state, respectively

$$P_1 = 1 - \exp\{-2\pi \beta_−\},$$  \hspace{1cm} (13)

and

$$P_2 = 1 - \exp\{-2\pi \beta_+\},$$  \hspace{1cm} (14)

with $\beta_± = \gamma_±^2 / \hbar \alpha$. Thus, the joint probability for the two fictitious spin-1/2’s to be found in the state $|++⟩, |+-⟩$ and $|-+⟩$, starting from $|−⟩−$, are respectively

$$P_1 P_2, \quad P_1 (1 - P_2), \quad (1 - P_1) P_2,$$  \hspace{1cm} (15)

being nothing but the probability of finding the two qutrits in the state $|10⟩, |01⟩$ and $|0 −1⟩$, respectively. The three curves are reported in Fig. 1 against the parameter $\beta = \beta_+$ for $\beta_+/\beta_− = 2$. For a complete LMSZ transition, the first spin accomplishes a LMSZ transition between the analogous states while the second spin state does not change.

Analogously, we may consider the excitation initially localized in the first spin, so that the two qutrits start from the state $|0 −1⟩$. In this instance the two-qutrit system is asymptotically driven to the state $|01⟩$ and the probability of the related transition acquires the same expression as the previous one in Eq. (15). It is worth noticing that in this case we generate a LMSZ transition from $|−1⟩$ to $|1⟩$ in the second spin, by applying a local magnetic field only on the first qutrit which, instead, remains in its initial state.

![Figure 1: (Color online) a) Asymptotic LMSZ probabilities [Eq. (15)] of finding the two qutrits in the state $|10⟩$ (blue dotted line), $|01⟩$ (magenta dot-dashed line), $|0 −1⟩$ (red dashed line) and $|−1⟩$ (green full line), when they start from the state $|−1⟩$ for $\gamma_− \neq \gamma_+, \beta = \beta_+$ and $\beta_+/\beta_− = 2$.](image)

2. **State Transfer between the Qutrits**

Another interesting effect to be highlighted is the possibility of realizing a state transfer between the two qutrits. Indeed, if the two qutrits (fictitious qubits) are initialized in the state $|−10⟩(|−⟩−)$ and we have $\gamma_− = \gamma_+$, the transition probability of the first fictitious spin-1/2 is forbidden, while the second one pass to $|+⟩$ with a probability $P = P_2$. In this way, the two qutrits (fictitious qubits) reach the state $|0 −1⟩(|−⟩+) having interchanged their initial state. The same effect is present if the two qutrits are initially prepared in $|10⟩$ passing to $|01⟩$. In such a case the transitions between the states in the four dimensional subspace of the two qutrit system are different since the condition $\gamma_− = \gamma_+$ introduces a further symmetry in the model related to the commutation of $H$ with $S_{tot}^z$. This generates the existence in the four dimensional Hilbert subspaces of other two dynamically invariant subspaces related to the eigenvalues of $S_{tot}^z$. This time, thus, the two qutrits starting from $|−10⟩$ can be found asymptotically only in the state $|0 −1⟩$.

At the light of the STM scenario, the physical effects previously discussed and analytically derived are of relevant interest. They show, indeed, that the presence of the coupling between the two qutrits allows us to manipulate the dynamics of the whole two-qutrit chain by the application of a single local magnetic field on one of the two spins, being exactly one of the task of the application of the STM technique. Moreover, the previous examples brought to light that, by studying the kind of transitions occurring in the two-qutrit system, we
may get information about the coupling parameters determin-
ing the symmetries of the Hamiltonian.

3. Effects of Environment

We wish to show now that the mapping of the two-qutrit dynamics into that of two decoupled spin-1/2’s in the four-dimensional subspace is useful not only to solve exactly the problem in ideal conditions, but also to take into account possible external influences due to the action of a surrounding environment. Such external influences may be regarded, for example, as noise in the magnetic field component. In this way the authors study the dynamics of a spin $S$ subjected to a noisy LMSZ scenario. The noisy time-dependent magnetic field $\eta(t)$ is considered only in the $z$ direction and characterized by a time correlation function of the form $\langle \eta(t)\eta(t') \rangle = 2\Gamma \delta(t-t')$. In such a way the noisy component cannot generates transitions between the different states but it leads only to loss of coherence. The authors show how the LMSZ transition probability is affected by the presence of such a noisy magnetic field in the case of a spin-1/2, a spin-1 and a spin-3/2. For a spin-1/2 and for large values of $\Gamma$ we have asymptotically

$$P(\tau) = 1 - \exp\left(-\frac{2\gamma^2}{\hbar \alpha}\right), \quad (16)$$

where $g$ is the energy contribution due to the coupling of the spin-1/2 with the constant transverse magnetic field. We see that the transition probability does not depend on the specific value of $\Gamma$, provided that $\Gamma$ is large. Moreover, it is important to note that the effect of the noise is to hinder the transition. Indeed, in the most convenient case, that is for $g^2/\hbar \alpha \gg 1$, at most the system reaches an equally populated condition of the two states. This is of particular interest for us since we have shown that the transition of the two qutrits studied before can be reduced to the LMSZ transition of a spin-1/2. Then, it means that the result previously reported, in our case, can be exploited to find the corrected LMSZ transition probability for the two qutrits when the field is affected by a noisy component. For example, if $\gamma_1 = \gamma_2$, the probability in Eq. (15) becomes $P_{1/2}/4$, reasonably meaning that, under the effect of noise, we reach an equally populated condition of the four states involved in the subdynamics under scrutiny. Analogously, if $\gamma_1 = \gamma_2 = \gamma_0$, the two qutrits started form $|\overrightarrow{-}\rangle$ we get the probability $P_{1/2}/2$ of transition to the state $|0\rangle - |1\rangle$, reaching this time an equally populated condition between these two states.

Such observation is based on the fact that, adding the noisy component $\eta(t)$ to the field applied to the first qutrit, nothing changes in the dynamics-decoupling procedure. The Hamiltonian transformation is completely unaffected since the only difference consists in a redefinition of the longitudinal field. In this way, what we obtain is an effective $z$-field for the two fictitious spin-1/2’s supplemented by a random field component. Thus, also in this case, we may reduce the two-qutrit dynamical problem into the analysis of the quantum dynamics of two decoupled spin-1/2’s.

In this respect, it is worth pointing out that the argument previously exposed continues to be valid also when we consider the possibility that the exited states $|0\rangle$ and $|1\rangle$ of the two qutrits decay irreversibly out of the system by some mechanism. Let us suppose that the spontaneous emission from the exited states to the ground one is negligible and that the two decay rates for the state $|0\rangle$ and $|1\rangle$ are $\Gamma (\Gamma')$ and $2\Gamma (2\Gamma')$, respectively, for the first (second) qutrit. It is easy to see that the analysis of such a scenario is equivalent, up to add a constant imaginary term, to phenomenologically introduce the non-Hermitian terms $i\Gamma \tilde{\sigma}_1$ and $i\Gamma \tilde{\sigma}_2$ in our Hamiltonian model. Also this time we have a simple redefinition of the parameters in front of the operators $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ without altering the symmetries possessed by the Hamiltonian $H$. Therefore, in such a case, within the four-dimensional subspace the two-qutrit dynamics may be described in terms of two decoupled two-level systems subjected to effective external fields and characterized by decaying states. Several results have been reported for a single qubit with a decaying state subjected to the LMSZ scenario$^{30-32}$, precisely, it has been proved that, on the one hand in the standard (ideal) LMSZ scenario the asymptotic transition probability does not depend on the decay rate at all (only its time-history, but not its asymptotic value)$^{30}$; on the other hand, in the more realistic LMSZ scenario characterized by a limited time-window, the exited state population exhibits a dependence on the decay rate$^{31}$. Therefore, also these results may be exploited and reread in terms of the two-qutrits to make quantitative prevision on the LMSZ transition probability for the system under scrutiny.

C. Local Fields

Now, we want to discuss the possibility of applying local fields on both the qutrits. Let us consider, firstly, the LMSZ case

$$\omega_1(t) = \omega_2(t) = \alpha \tau/2, \quad (17)$$

with $\tau$ going from $-\infty$ to $+\infty$.

In this case, the Hamiltonians of the two fictitious spin-1/2’s, through which we describe effectively the dynamics of the two qutrits in the four dimensional subspace, read

$$H_1 = h\Omega_+ (t) \tilde{\sigma}_1^+ + \gamma_+ \tilde{\sigma}_1^z, \quad H_2 = \gamma_+ \tilde{\sigma}_2^z, \quad (18)$$

with $\Omega_+ (t) = \alpha \tau$. We see that the second fictitious spin-1/2 is subjected only to a magnetic field in the $x$-direction, while the first one is subjected to standard Landau-Zener scenario where the role of the external transverse constant magnetic field now results from the existence of a coupling between the two spins.

1. Determination of $\gamma_+$

We study now the instance in which only one excitation is present in the system, equally shared by the two qutrits. We consider, then, the entangled state $(|\overrightarrow{0}\rangle + |\overrightarrow{1}\rangle)/\sqrt{2}$ as
initial condition. By the mapping in Eq. (6), such a state, rewritten in terms of the two spin-1/2 states, acquires the form

$$|\rangle \propto |+\rangle + |-\rangle \over \sqrt{2}. \quad (19)$$

It is easy to see that the second spin does not change its state in time since the latter is an eigenvalue of $H_2$. The first spin, instead, evolves according to the LMSZ dynamics, so that the probability to find it in the opposite state $|+\rangle$ at very large time instants ($t \to \infty$) is $P_2$. Of course, it expresses too the probability of the two spin-1/2’s to be found in the state $|+\rangle \propto \frac{|+\rangle + |-\rangle}{\sqrt{2}}$. The relevant point is that, in view of Eq. (6), it provides the probability for the two qutrits of reaching the state

$$|10\rangle + |01\rangle \over \sqrt{2}. \quad (20)$$

Thus, if $\beta_+ \gg 1$, through the linear ramp we have created an excitation in the system. It is important to underline that such a transition depends strongly on the coupling parameters between the two qutrits, since their difference constitute the effective transverse magnetic field entering in the expression of the LMSZ parameter $\beta_z$. Indeed, if the two parameters are equal or very close, the transition is forbidden, while, if they are opposite, the transition probability reaches its maximum efficiency. This suggests us that, choosing at will $\alpha$ and studying the characteristic time of the transition, we may get information about the value of $\gamma_-$. 

If we now consider

$$\omega_1(t) = -\omega_2(t) = \alpha t / 2 \quad (21)$$

and the two qutrits initially prepared in the state $\frac{|-10\rangle + |0-1\rangle}{\sqrt{2}}$, we get a specular dynamics. That is, the first fictitious spin-1/2, subjected only to a static $x$-magnetic field ($H_1 = \gamma_c \sigma_3^x$), does not evolve, while the second fictitious spin-1/2 makes a transition from $|\rangle$ to $|+\rangle$ (being $H_2 = \hbar \alpha \sigma_z^x + \gamma_z \sigma_3^x$). Studying such a transition, this time, we get information about $\gamma_+$, since it rules the characteristic time of such a transition. Finally, by comparing the two values of $\gamma_+$ and $\gamma_-$ we may estimate the original coupling parameters of the two qutrits $\gamma_c$ and $\gamma_z$.

2. Dark States

We emphasize that, under the conditions $\gamma_c = \gamma_z = \gamma / 2$ and $\omega_1(t) = \omega_2(t) = \omega(t) / 2$ (unique homogeneous magnetic field), the following four states

$$|\psi_{1/2}^0\rangle = \frac{|10\rangle \pm |01\rangle}{\sqrt{2}}, \quad |\psi_{3/4}^0\rangle = \frac{|-10\rangle \pm |0-1\rangle}{\sqrt{2}} \quad (22)$$

result steady states independently of the time dependences of the magnetic field. This may be easily understood in terms of the two spin-1/2’s. Indeed, the second spin-1/2 is in an eigenstate $\{|\rangle \pm |-\rangle\}/\sqrt{2}$ of its constant Hamiltonian $H_2 = \gamma \sigma^x$ and evolves trivially only acquiring the phase factor $\exp\{-i\tau / \hbar\}$; the first fictitious spin-1/2, instead, (being in the state $|\rangle$) keeps only the phase factor $\exp\{-i \int_0^t \alpha(t) dt\}$ since its Hamiltonian $H_1 = \hbar \alpha \sigma_z^x \sigma_3^x$ does not mix the two standard basis states. This means that for these four states we have ($j = 1 \ldots 4$)

$$H(t)|\psi_j^0\rangle = E_j(t)|\psi_j^0\rangle, \quad E_{1/2}(t) = \omega(t) \pm \gamma, \quad E_{3/4}(t) = -E_{1/2}(t) \quad (23)$$

implying

$$|\psi_j(t)\rangle = \exp\left\{-i \int_0^t dt' E(t') / \hbar\right\}|\psi_j^0\rangle. \quad (24)$$

It is easy to see that, considering the time-independent case, such states result to be the eigenstates of the Hamiltonian $\hat{H}$. So, this model, in this specific case, presents a peculiar characteristic consisting in maintaining its steady states also when the Hamiltonian parameters are time-dependent. A remarkable consequence of this circumstance is that the following class of states $\rho(t) = \sum_j |\psi_j(t)\rangle \langle\psi_j(t)|$ (the Boltzman constant and the Temperature, respectively), do not evolve in time, that is

$$\rho(t) = \sum_j p_j |\psi_j(t)\rangle \langle\psi_j(t)| = \sum_j p_j |\psi_j^0\rangle \langle\psi_j^0| = \rho_0. \quad (25)$$

Therefore, any physical observable calculated for such class of states exhibit a constant value in time. We can call such states ‘dark states’ since, under the conditions written before, they are unaffected by the coupling and the time-dependent field also when the latter presents a random fluctuating behaviour stemming, for example, from a surroundings nuclear spin bath.

Analogously, if we have $\gamma_c = -\gamma_z$ and $\omega_1(t) = -\omega_2(t)$ the four dark states are

$$|10\rangle \pm |01\rangle \over \sqrt{2}, \quad |01\rangle \pm |10\rangle \over \sqrt{2}. \quad (26)$$

Finally, we emphasize that the previous results are not restricted to the LMSZ scenario, but they are valid whatever the time-dependence of the field is.

IV. FIVE-DIMENSIONAL SUBDYNAMICS

A. General Solution

In the second section we saw that the central block of $H_+$ in Eq. (8) has an $su(2)$ structure and then it is interpretable as the Hamiltonian of a (fictitious) spin-1 subjected to (fictitious as well) magnetic fields (see Eq. (9)). It is well known that the time evolution operator related to a 3x3 $su(2)$ Hamiltonian may be put in the following form

$$U_3 = \begin{pmatrix} a_3^2 & \sqrt{2}a_3b_3 & b_3^2 \\ -\sqrt{2}a_3b_3 & |a_3|^2 - |b_3|^2 & \sqrt{2}a_3b_3 \\ b_3^2 & -\sqrt{2}a_3b_3 & a_3^2 \end{pmatrix}. \quad (27)$$
where \( a_3 \) and \( b_3 \) are two time-dependent parameters, solution of the analogous dynamical problem for a single spin-1/2. In other words, \( a_3 \) and \( b_3 \) may be found by solving the dynamical problem of a single spin-1/2 subjected to the same magnetic field acting upon the fictitious spin-1. Thus, we may formally write the time evolution operator \( U_+ \), solution of the Schrödinger equation \( i\hbar U_+ = H_+ U_+ \), as follows

\[
U_+ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \alpha & \sqrt{2} \alpha b_3 & 0 \\
0 & -\sqrt{2} \alpha b_3 & |a_3|^2 - |b_3|^2 & \sqrt{2} \alpha b_3 \\
0 & b_3^2 & -\sqrt{2} \alpha b_3 & a_3^2 \\
0 & 0 & 0 & 0
\end{pmatrix} e^{-\frac{i}{\hbar} H_+ t}.
\]

(28)

\[ B. \text{ Dark States} \]

First of all, it is important to underline that also for the five-dimensional subdynamics we have dark states. Indeed, if the two qutrits are initially prepared in \( |11⟩ \) or \( |-1-1⟩ \), independently of the time-dependence of the z-magnetic field, the two-qutrit system remains in its initial state, also if the magnetic field component randomly fluctuates remaining along the \( z \)-direction. Moreover, if we consider the case \( \omega_1(t) = \omega_2(t) \), a generic state belonging to the three-dimensional subspace, namely

\[
c_1|1-1⟩ + c_2|00⟩ + c_3|-11⟩,
\]

is completely unaffected by the presence of time-dependent magnetic fields, since in this instance \( \Omega_-(t) = 0 \) and the Hamiltonian governing the three-dimensional dynamics is simply \( H_3 = \gamma \hat{S}^z \). Such states, then, evolves only under the action of the coupling between the two qutrits. It means then that the three eigenstates of \( \Sigma^z \) rewritten in terms of two-qutrit states

\[
|\psi^0⟩ = \frac{|1-1⟩ + \sqrt{2}|00⟩ + |-11⟩}{2},
|\psi^1⟩ = \frac{|1-1⟩ - |-11⟩}{\sqrt{2}}
|\psi^2⟩ = \frac{|1-1⟩ - \sqrt{2}|00⟩ + |-11⟩}{2}
\]

result steady state of the two-qutrit system also when a unique homogeneous magnetic field is applied on the two spin-1’s. Consequently, every classical mixture of these three states does not evolve and every physical quantity related to this state is constant in time. Given that the states in Eq. (25) have the same property under the same conditions (\( \omega_1(t) = \omega_2(t) \) and \( \gamma_1 = \gamma_2 \)), we may conclude that, in this scenario, the thermal state of the system and, more in general, every mixture involving the steady states \( |11⟩, |-1-1⟩ \) and the ones in Eqs. (25) and (30), namely

\[
\rho = k_1|11⟩⟨11| + \sum_{j=1}^2 p_j |\psi^j⟩⟨\psi^j| + k_2|-1-1⟩⟨-1-1|,
\]

such that \( k_1 + k_2 + \sum_j p_j = 1 \), is a stationary state of the two-qutrit system.

\[ C. \text{ STM Scenario and LMSZ Transition Probabilities} \]

We investigate now the STM experimental scenario characterized by a single local magnetic field on the first spin-1, namely \( \omega_1(t) = \alpha t \) and the two qutrits initialized in the state \( |1-1⟩ \). In this case the two-qutrit system behaves effectively like a three-level system (spin-1) subjected to a LMSZ ramp with an effective constant transverse magnetic field related to the coupling constant \( \gamma \). For such a time-dependent scenario, the transition probabilities, from \( |1-1⟩ \) to the other two states \( |00⟩ \) and \( |-11⟩ \), may be found analytically. Indeed, at the light of the spin-1 - spin-1/2 transition probability relationship based on the SU(2) group structure, for large time instants, we have

\[
P^+ = P^2 = 2P_3(1 - P_3), \quad P^- = (1 - P_3)^2,
\]

(32)

where \( P_3 = 1 - e^{-2 \pi \beta} \) and \( \beta = 2 \gamma^2 / \hbar \alpha \). In the previous expressions we have labelled with -1, 0 and 1 the states \( |1-1⟩, |00⟩ \) and \( |-11⟩ \), respectively. The plots of the asymptotic probabilities are reported in Fig. 2 against the coupling-dependent LMSZ parameter \( \beta' \). We see that the interplay be-

![Figure 2: (Color online) Asymptotic LMSZ probabilities [Eq. (32)] of finding the two qutrits in the state |1-1⟩ (blue dot-dashed line), |00⟩ (red dashed line) and |-11⟩ (green full line), when they start from the state |-11⟩ for \( \gamma_1 = \gamma_2 \).](image)
D. Noise Effects

We consider now the field along the $z$ axis affected by the random fluctuating contribution we saw in the previous section. We may exploit again the results reported in Ref.\textsuperscript{34} where the authors solved the dynamical problem of a noisy ramp in a LMSZ scenario also for a spin-1. In such a case, the transition probabilities affected by a noisy field component along the $z$-axis and characterized by the following time-correlation function $\langle \eta(t')\eta(t) \rangle = 2\Gamma\delta(t-t')$, become

$$P_{\pm 1}^+ = \frac{1}{6}(2 + e^{-3\pi\beta'} - 3e^{-\pi\beta'})$$

$$P_{\pm 1}^0 = \frac{1}{3}(1 - e^{-3\pi\beta'})$$

$$P_{\pm 1}^- = \frac{1}{6}(2 + e^{-3\pi\beta'} + 3e^{-\pi\beta'})$$

(33)

Also these expressions, valid for large values of $\Gamma$, are independent of the value of the same $\Gamma$. We see that, also this time, the main effect of the noise is to hinder the transition generating at most equally populated states when $\beta' \gg 1$. In this way, we brought to light how the symmetry-based analysis of the model reported in the second sections plays a key role for disclosing the exact quantum dynamics of the two interacting qutrits subjected to time-dependent magnetic fields, both in ideal and more realistic conditions.

V. ENTANGLEMENT

The negativity, introduced by G. Vidal and R. F. Werner in\textsuperscript{51}, of a two-qutrit system described by the density matrix $\rho$ reads\textsuperscript{52}

$$\mathcal{N}_\rho = \frac{||\rho_{TB}||_1 - 1}{2},$$

(34)

where $\rho_{TB}$ is the partial transpose of the matrix $\rho$ with respect to the subsystem $B$. The symbol $|||\cdot|||_1$ is the trace norm which, for a hermitian matrix, results in the sum of the absolute values of the negative eigenvalues of $\rho_{TB}$ which is hermitian and such that $\text{Tr}\{\rho_{TB}\} = 1$. The range of values of $\mathcal{N}_\rho$ is $[0, 1]$\textsuperscript{52} and its calculation is independent of the factorized orthonormal basis in which the matrix $\rho$ is represented as well as of the subsystem with respect to which we calculate the partial transpose, since $(\rho_{TB})^\dagger = \rho_{TB}$ and $||X||_1 = ||X^T||_1$ for any operator $X$.

A. Four-Dimensional Sub-Dynamics

For our two-qutrit system, it has been proved\textsuperscript{48} that the negativity for a generic pure as well as mixed state belonging to the four dimensional subspace possesses the upper bound $\mathcal{N} = 1/2$. In case of a generic pure state $|\Psi\rangle = w_1|10\rangle + w_2|01\rangle + w_3|0-\rangle + w_4|-1\rangle$, the Negativity acquires indeed the simple form\textsuperscript{48}

$$\mathcal{N} = \sqrt{x(1-x)}, \quad x = |w_1|^2 + |w_4|^2.$$  

(35)

If we consider as initial condition the two-qutrit state $|-10\rangle$, through the exact form of the time evolution operator in Eq. (28), it is easy to verify that

$$x(t) = |w_1(t)|^2 + |w_4(t)|^2 = |a_1|^2|a_2|^2 + |b_1|^2|b_2|^2$$

(36)

At infinite time so we have

$$x(\infty) = P_1P_2 + (1 - P_1)(1 - P_2),$$

(37)

where the expressions of $P_1$ and $P_2$ are reported in Eq. (13) and (14), respectively. If we put the expression in Eq. (37) into Eq. (35), we get the asymptotic expression of the Negativity. In Fig. 3a such an Expression of the negativity is reported against the LMSZ parameter $\beta = \beta_+$, for $\beta_-/\beta_+ = 1/2$. We see that two maxima are present and they correspond to the values $\log(2)/2\pi \approx 0.11$ and $\log(2)/\pi \approx 0.22$. It means that, by appropriately setting the parameter $\beta$, when the two-qutrit system start from the state $|\!-10\rangle$, through the LMSZ process we may generate asymptotically an entangled state of the two spin-qutrits with the maximum level of entanglement possible in such a subspace. This fact is confirmed by Fig. 3b where the time behaviour of the Negativity is reported against the dimensionless parameter $\tau = \sqrt{\alpha/\hbar T}$ for $\beta = 0.11$. In this case, we used the expression of $x(t)$ in Eq. (36) with the exact solution of the LMSZ dynamical problem which read\textsuperscript{24}

$$a_{1/2} = \frac{\Gamma_L(1 - i\beta_+)}{\sqrt{2\pi}}$$

$$\times \left[D_{i\beta_+} \left(\sqrt{2}e^{i\pi/4}\right)D_{1+i\beta_+} \left(\sqrt{2}e^{3i\pi/4}\right)_\tau\right] + \left[D_{i\beta_-} \left(\sqrt{2}e^{i\pi/4}\right)_\tau\right]$$

$$b_{1/2} = \frac{\Gamma_L(1 - i\beta_-)}{\sqrt{2\pi}}$$

$$\times \left[D_{i\beta_-} \left(\sqrt{2}e^{-i\pi/4}\right)_\tau\right] + \left[D_{i\beta_+} \left(\sqrt{2}e^{3i\pi/4}\right)_\tau\right].$$

(38)

The functions $D_{\nu}(z)$ are the parabolic cylinder functions\textsuperscript{53} and $\tau_\nu$ identify the initial time instant. We emphasize that the parameter $\beta$, besides the asymptotic value, deeply influences the trend in time of the Negativity curve, as it can be appreciated by Figs. 3c and 3d, related to $\beta = 0.5$ and $\beta = 2$, respectively.

It is interesting to point out that the initial state $|\!-10\rangle + |0-1\rangle)/\sqrt{2}$ under the condition $\phi_0(t) = \phi_2(t)$, we have taken into account in Sec. III C, exhibits a constant maximum level of entanglement (1/2) during the evolution. Such a peculiar feature is independent of the specific time-dependence of the field and it may be understood at the light of the analysis reported in\textsuperscript{48}. There the authors analyse the same two-qutrit model but with time-independent fields. They have brought to light the existence of eight states with such a feature which is related to the symmetry property of the Hamiltonian. Being such property unaffected by a general time-dependence of the applied field as we showed before, we find of course the same feature also here.
The dynamical problem has been solved thanks to the reduction to two easier problems: one of two non-interacting fictitious spin-1/2’s and the other of a fictitious three-level system. Such a reduction relies on the symmetry-based analysis of the Hamiltonian model reported in Ref. \(^{48}\) which is unaffected by the time-dependences of the applied fields and, more generally, by the time-dependences of all Hamiltonian parameters. This means that the same analysis may be developed considering other possible time-dependences of the field leading to exactly solvable problems \(^{47,59-64}\).

The main result of the paper is the physical effect we called coupling-driven LMSZ transition. It consists in the fact that, though a transverse constant field is absent, LMSZ transitions between two-qutrit states are still possible thanks to the presence of the coupling between the two spin-1’s. Indeed, the fictitious dynamics of the two decoupled qubits and the one of a fictitious spin-1 are characterized by a LMSZ longitu-
and we have brought to light, moreover, how the LMSZ transition probabilities modify depending on the (an)isotropy of the coupling terms.

We have showed that the physical relevance of the coupling-driven LMSZ transitions is twofold. Firstly, by the knowledge of the transition probabilities we may estimate the coupling parameters of the two-qutrit model. Secondly, basing on such an estimation, we illustrated that an appropriate and specific choice of the slope of the LMSZ ramp can generate asymptotically entangled states of the two qutrits. We have analysed the level of entanglement by studying both the asymptotic Negativity against the LMSZ parameters and its determinants. In the latter case, we have used the exact solutions of the LMSZ dynamical problem and we have investigated the effects of the coupling determining the LMSZ parameter. We reported how such a parameter, depending on the ratio of the squared coupling and the slope of the ramp, determines not only the asymptotic value, but also the trend of the Negativity.

Finally, we have discussed also how the LMSZ transition probabilities are modified by the presence of a noisy field component stemming from the interaction of the the two-qutrit system with a surrounding environment. Such an analysis is based on the fact that the dynamical reduction is unaffected by the presence of the noise and so, also in this case, we may reduce the two-spin-1 problem to easier problems whose solutions are known in literature. Following the same philosophy, we have exposed the possibility of treating exactly the problem also by introducing the environment effects with non-Hermitian terms in the Hamiltonian model.

The main perspective of this work is to take into account also quantum degrees of freedom of the bath. In this case, the basic and fundamental symmetry-based dynamical reduction might be joined with recent approaches to reach a deeper understanding of the dynamics of two-qutrit systems in more realistic experimental situations.

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