A unified treatment of polynomial solutions and constraint polynomials of the Rabi models

Alexander Moroz

wave-scattering.com

E-mail: wavescattering@yahoo.com

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Abstract

General concept of a gradation slicing is used to analyze polynomial solutions of ordinary differential equations (ODE) with polynomial coefficients, \( L\psi = 0 \), where \( L = \sum_i p_i(z) d_i \), \( p(z) \) are polynomials, \( z \) is a coordinate, and \( d_i = d/dz \). It is not required that ODE is either (i) Fuchsian or (ii) leads to a usual Sturm–Liouville eigenvalue problem. General necessary and sufficient conditions for the existence of a polynomial solution are formulated involving constraint relations. The necessary condition for a polynomial solution of \( n \)th degree to exist forces energy to a \( n \)th baseline. Once the constraint relations on the \( n \)th baseline can be solved, a polynomial solution is in principle possible even in the absence of any underlying algebraic structure. The usefulness of theory is demonstrated on the examples of various Rabi models. For those models, a baseline is known as a Juddian baseline (e.g. in the case of the Rabi model the curve described by the \( n \)th energy level of a displaced harmonic oscillator with varying coupling \( g \)). The corresponding constraint relations are shown to (i) reproduce known constraint polynomials for the usual and driven Rabi models and (ii) generate hitherto unknown constraint polynomials for the two-mode, two-photon, and generalized Rabi models, implying that the eigenvalues of corresponding polynomial eigenfunctions can be determined algebraically. Interestingly, the ODE of the above Rabi models are shown to be characterized, at least for some parameter range, by the same unique set of grading parameters.

Keywords: exceptional polynomial solutions, gradation slicing, quasi-exactly solvable models

(Some figures may appear in colour only in the online journal)
1. Introduction

The Rabi model [1] describes the simplest interaction between a cavity mode with a bare frequency \( \omega \) and a two-level system with the levels separated by a frequency difference \( 2\Delta = \omega_0 \), where \( \hbar \) is the Planck constant and \( \omega_0 \) is a bare resonance frequency. The model is characterized by the Hamiltonian [1, 2]

\[
\hat{H}_R = \hbar \omega \hat{1} \hat{a}^\dagger \hat{a} + \hbar g \sigma_1 (\hat{a}^\dagger + \hat{a}) + \hbar \Delta \sigma_3,
\]

where \( \hat{1} \) is the unit matrix, \( \hat{a} \) and \( \hat{a}^\dagger \) are the conventional boson annihilation and creation operators of a boson mode with frequency \( \omega \), which satisfy commutation relation \( [\hat{a}, \hat{a}^\dagger] = 1 \), and \( g \) is a coupling constant. Here and elsewhere the standard representation of the Pauli matrices \( \sigma_l, l = 1, 2, 3 \), with \( \sigma_3 \) diagonal is assumed. The Rabi model applies to a great variety of physical systems, including cavity and circuit quantum electrodynamics, quantum dots, polaronic physics and trapped ions [3–8]. The Rabi model is not exactly solvable. Yet the model has been known for a long time to possess polynomial solutions, the so called Juddian isolated exact solutions [9, 10], at energy levels corresponding to those of a displaced harmonic oscillator, which is the \( \Delta = 0 \) limit of the Rabi model [2]. The latter has been known as the baseline condition for the Rabi model [2, 9, 10] (e.g. equations (38), (48), (57), (60) and (67) below).

In what follows we shall consider the (driven) Rabi model [11–15], together with its nonlinear two-photon [8, 11, 12, 16, 17] and nonlinear two-mode [8, 11, 12] versions, and the generalized Rabi model of [6–8]. A typical 2nd order linear ordinary differential equation (ODE) for the Rabi models turns out to be of the form \( \mathcal{L} \psi = 0 \), where \( \mathcal{L} = \sum_j p_j(z) d^{(2)}_j \), \( p_j(z) \) are polynomials, \( z \) is a one-dimensional coordinate, and \( d_j = d^j / dz \). What sets those equations apart from other common equations is that \( \mathcal{L} \) comprises energy \( E \) dependent terms \( \sim E d_n, E z, E^2 \) [2, 11, 12, 18] and, therefore, does not reduce to a standard eigenvalue problem. The physical problem is rather to find zero modes \( \phi \) of \( \mathcal{L} \). Differential equations for the Rabi models are not even Fuchsian, having an irregular singular point at infinity. Obviously in analyzing a given 2nd order ODE with polynomial coefficients one can always switch from the above form into a Schrödinger equation (SE) (also known as normal) form, where the first derivative term has been eliminated and the coefficient of \( \partial^2 \) has been set to one [19, 20]. Such a transformation leads to an energy dependent potential and to a non-Sturm–Liouville problem.

Inspired by 2nd order ODE which occur when trying to solve the Rabi models [1, 2, 6–8, 11, 12, 16, 17], we developed general necessary and sufficient conditions for the existence of a polynomial solution of \( n \)th degree of such equations. In recent years, the corresponding differential operator \( \mathcal{L} \) for a (driven) quantum Rabi model [11, 12, 18], the two-photon and two-mode quantum Rabi models [11, 12], and the generalized Rabi model [6] was shown to be expressible as a bilinear combination of \( sl_2 \) algebra generators, and hence to be an element of the enveloping algebra \( U(sl_2) \) for a certain choice of model parameters and of energy. For a typical eigenvalue problem, \( \mathcal{L} \psi = \lambda \psi \), one can always find \( n + 1 = 2j + 1 \) polynomial solutions corresponding to different eigenvalues \( \lambda_j \) of \( \mathcal{L} \) in a corresponding \( sl_2 \) module characterized by the spin \( j \). The constant term \( c_0 \) of \( \mathcal{L} \) is a free parameter that can always be absorbed to an eigenvalue \( \lambda_0 \). However, in the case of the Rabi models, the value of \( c_0 \) is fixed and only zero modes \( \phi \) of \( \mathcal{L} \), which satisfy \( \mathcal{L} \phi = 0 \), are physical solutions. One can easily show that the latter problem can have at most a single polynomial solution. We show that with \( \mathcal{L} \in U(sl_2) \), the ODE \( \mathcal{L} \psi = 0 \) need not have in general any polynomial solution. In particular, \( sl_2 \) algebra does not explain why the Juddian isolated exact solutions can be analytically computed [9, 10, 13–15], whereas the remaining part of the spectrum not. The Rabi models are thus an unusual example of quasi-exactly solvable (QES) models [18–27]. The QES models are distinguished by the fact that a finite number of their eigenvalues and corresponding eigenfunctions can be
determined algebraically [19–27]. Initially, QES was essentially a synonym for \( sl_2 \) algebraization in one dimensional quantum mechanical problems [18–27], up to the point that no difference was made between the terms Lie-algebraic and quasi-exactly solvable in the literature. The reason behind this was that \( sl_2 \) is the only algebra of first order differential operators with finite dimensional invariant modules. Burnside’s classical theorem ensures that every differential operator which leaves the space \( \mathcal{P}_n(z) \) invariant belongs to the enveloping algebra \( U(sl_2) \), since \( \mathcal{P}_n(z) := \text{span}\{1, z, \ldots , z^n\} \) is an irreducible module for the \( sl_2 \) action.

The article is organized as follows. On defining the grade of a term \( z^m d_z^l \) as integer \( m - l \), section 2 introduces other necessary definitions to perform a gradation slicing of a given ordinary linear differential equation (ODE) with polynomial coefficients,

\[
\mathcal{L}S_n(z) = \left\{A(z)\frac{d^2}{dz^2} + B(z)\frac{d}{dz} + C(z)\right\} S_n(z) = 0,
\]

where

\[
A(z) = \sum_{k=0} a_k z^k, \quad B(z) = \sum_{k=0} b_k z^k, \quad C(z) = \sum_{k=0} c_k z^k.
\]

In section 3 basic theorems are formulated that yield necessary and sufficient conditions for the existence of a polynomial solution of \( n \)th degree. The necessary condition in general constraints energy in a parameter space for each given \( n \) to a different baseline (e.g. equations (38), (48), (57), (60) and (67) below). Provided that the highest grade of the terms \( z^m d_z^l \) of ODE (2) is \( \gamma \), there are \( \gamma \) recursively defined constraints to be satisfied for a polynomial solution on the \( n \)th baseline to exist. It turns out that, with the exception of the generalized Rabi model [6], each of the Rabi models considered in this article is characterized by an ordinary linear differential equation comprising terms with highest grade \( \gamma \), the lowest grade \( \gamma_\text{min} \), gradation width \( w \), the highest grade slice \( F_\gamma \), and its induced multiplicator \( F_\gamma(n) \) as follows:

\[
\gamma = 1, \quad \gamma_\text{min} = -2, \quad w = 4, \quad F_\gamma = b_2 z^2 d_z + c_1 z, \quad F_\gamma(n) = nb_2 + c_1.
\]

For the generalized Rabi model one finds, depending on parameters, either \( \gamma = 1 \) or \( \gamma = 2 \). The corresponding constraints are shown to (i) reproduce known constraint polynomials for the usual and driven Rabi models [9, 10, 13–15] (see figures 1 and 2) and (ii) generate hitherto unknown constraint polynomials for the two-mode, two-photon, and the generalized Rabi models. Usual road to constraint polynomials required to reveal an ingenious Ansatz for the polynomial solutions. For example, the original Kus construction [10] consisted in an insightful observation that an exact polynomial solution of the Rabi model on the \( n \)th baseline can be constructed as a finite linear combination of the solutions \( \Phi^\pm_l \) of a displaced harmonic oscillator \( (\Delta = 0 \text{ limit of the Rabi model}) \) from all baselines \( l \leq n \) and for both even and odd parity. An analogous approach was attempted later by Emary and Bishop [17] in the case of two-photon Rabi model, and the others in the case of the driven Rabi model [13–15]. Yet the origin of constraint polynomials remained mysterious. It was not \textit{a priori} clear if they at all exist. In this regard theorem 3 of section 3 yields a recipe for determining constraint polynomials by a recurrence (11) and (14) with well defined coefficients for any \( \gamma > 0 \) problem, and in particular for any conceivable Rabi model generalization. In theorem 4 of section 3 we have succeeded to generalize an important result of Zhang (see equations (1.8)–(10) of [42]) obtained for 2nd order ODE’s to the case of arbitrary \( \gamma \geq 0 \). A \( sl_2 \) algebraization with spin \( j = ml/2 \) is shown in section 3.2 to be equivalent merely to the necessary condition for the existence of a polynomial solution of \( n \)th degree. A lemma is formulated which yields necessary
condition for a spectral problem \( T_2 \psi = \lambda \psi \), where \( \lambda \in \mathbb{C} \) is an eigenvalue, to have degenerate energy levels in an invariant \( \mathfrak{sl}_2 \) module of spin \( j \).

In section 4 our approach is illustrated in detail on the example of the usual quantum Rabi model. A driven Rabi model is considered in section 5. Nonlinear two-photon and two-mode Rabi models are dealt with in section 6, and the generalized Rabi model is the subject of section 7. In each of the above cases explicit expressions of the recurrence coefficients for the constraint polynomials are presented. Our results open a number of different avenues of further research which are discussed in section 8. We then conclude in section 9. For the sake or presentation, appendix A provides an overview of the basics of \( \mathfrak{sl}_2 \) algebraization. Relevant features of 2nd order linear differential equation with all solutions being polynomials are summarized in appendix B. Singular points at spatial infinity are dealt with in appendix C. Some other alternative forms of 2nd order ODE of [2, 18] for the Rabi model are examined in appendix D.

2. Gradation slicing

The subset of ODE (2) where \( A(z), B(z), C(z) \) are polynomial of degree at most 4, 3, 2, respectively, covers (i) all QES models within the context of \( \mathfrak{sl}_2 \) (see equations (24) and (A.5)) [19–27], (ii) all Fuchsian 2nd order ODE [28] and (iii) non-Fuchsian 2nd order ODE of the present article (such as equation (37) below, which has an irregular singular point at infinity (see appendix C)).

For the purpose of looking for (monic) polynomial solutions

\[
S_n(z) = \prod_{i=1}^{n} (z - z_i) = \sum_{k=0}^{n} a_{nk} z^k \quad (a_{nn} \equiv 1)
\]

of the ODE (2), or in general of

\[
\mathcal{L} S_n(z) = \left\{ \sum_{l=3}^{m} X_l(z) \frac{d^l}{dz^l} + A(z) \frac{d^2}{dz^2} + B(z) \frac{d}{dz} + C(z) \right\} S_n(z) = 0,
\]

where \( X_l(z) \) are also polynomials, it is expedient to rearrange \( \mathcal{L} \) into a more convenient form.

Obviously for any polynomial \( S_n(z) \) the image \( \mathcal{L} S_n(z) \) is also a polynomial. The basic idea is to characterize the terms of the operator \( \mathcal{L} \) which contribute to the same polynomial degree in the image \( \mathcal{L} S_n(z) \). In what follows we call (a positive or negative) integer \( g = m - l \) the grade of the term \( z^m d_z^l \). The grade describes a change of the degree of a monomial \( z^m \) under the action \( z^m d_z^l \). This is similar to the grading (A.2) of \( \mathfrak{sl}_2 \) generators (A.1) employed by Turbiner [25, 26].

We introduce the concept of a gradation slicing of an ODE (6), which comprises the following steps:

(S1) Consider a given differential equation as a linear combination of terms \( \sim z^m d_z^l \) and determine the grade of each term.

(S2) Rearrange all the terms of the ODE according to their grade. The subset \( \mathcal{F}_g(z^m d_z^l; m - l \equiv g) \) of the ODE with an identical grade \( g \) will be called a slice.

Hence the differential equation can be recast as
\[ L_{\gamma}(z) = \sum_{g = g_{\text{min}}}^{g_{\text{max}}} F_g \delta \gamma \delta \gamma = 0, \]  

where the sum runs over all grades \( g \). In what follows we will use an abbreviation \( \gamma = g_{\text{max}} \) for the highest grade and \( \gamma_\ast = g_{\text{min}} \) for the lowest grade.

**Definition.** A decomposition of original ordinary linear differential equations (6) into (7) will be called gradation slicing. We call the grade of an ordinary linear differential equation the highest grade \( \gamma \). A width \( w \) of the gradation slicing will be called the integer \( w := \gamma - \gamma_\ast + 1 \).

Define a function \( F_g(k) \) by

\[ F_{g}_{\delta \lambda} = F_{g}(k)z^{k+1}. \]

We shall call the function \( F_{g}(k) \) an induced multiplicator corresponding to the slice \( F_g \).

The width counts the number of possible slices with the grade between the minimal and maximal grades, \( \gamma_\ast \) and \( \gamma \), respectively. Unless \( C(z) \) is identically zero, one has always \( \gamma \geq 0 \). In what follows, we shall assume that \( \gamma_\ast \leq 0 \). The case \( \gamma_\ast > 0 \) can always be reduced to the case \( \gamma_\ast = 0 \) by factorizing \( z^{\gamma_\ast} \) out of the polynomial coefficients of the differential equation (6).

**Remark 1.** A hypergeometric equation is characterized by \( \gamma = 0, \gamma_\ast = -2 \), and \( w = 3 \). A typical Heine–Stieltjes problem [29–35], where \( A(z), B(z), C(z) \) are polynomials of exact degree \( N + 2, N + 1, N \), respectively, is grade \( \gamma = N, \gamma_\ast = -2, w = N + 3 \) problem.

It turns out that each of the Rabi models considered in this article is described by an ordinary linear differential equation characterized by \( \gamma = 1, \gamma_\ast = -2, w = 4, F_{\gamma} \), and induced multiplicator \( F_{\gamma}(n) \) as summarized by (4). The lowest grade and width are not absolute invariants of an ODE, because they may depend on the origin of coordinates.

Obviously the condition that the slice with the highest grade \( \gamma \) annihilates a monomial of degree \( n \), \( F_{\gamma}z^n = 0 \), provides a necessary condition for the existence of a polynomial solution of degree \( n \),

\[ F_{\gamma}(n) = 0. \]  

(8)

Provided that \( F_{\gamma}(n) \) depends on energy \( E \) one can consider the condition (8) as equation for \( E \). (For the Rabi models considered here this will be typically a linear equation.) Its solution \( E = E(n) \) constraints energy \( E \) in the parameter space (e.g. equations (38), (48), (57), (60) and (67) below). Therefore, we will refer to the condition \( F_{\gamma}(n) = 0 \) also as the baseline condition, although it defines a line only in the case of the original Rabi model, where it depends on a single parameter \( g \) [9, 10].

For the ensuing analysis of Rabi models we need both necessary and sufficient conditions for the existence of a polynomial solution of \( n \)th degree. In what follows we shall distinguish two main alternative types of differential equation (6):

\[ (A1) \] a conventional one when the highest derivative term (e.g. \( A(z)d^2_{z} \) in (2)) does contribute to the slice \( F_{\gamma} \) with the highest grade \( \gamma \). Applied to (2), the alternative occurs if

\[ \deg B \leq \deg A - 1, \quad \deg C \leq \deg A - 2, \]  

where strict equality applies in at least one of the above cases.

\[ 1 \] A translation of this often quoted passage can be found in [33].
(A2) An anomalous one, when the highest derivative term (e.g. $A(z)d_z^2$ in (2)) does not contribute to $F_\gamma$. Applied to (2), the alternative occurs if
\[ \deg B \geq \deg A, \quad \deg C = \deg B - 1. \] (10)

An example of the alternative (A1) are the Fuchsian equations, which include a hypergeometric one, and the Heine–Stieltjes problem [29–35]. The alternative (A2) is usually omitted in the analysis of polynomial solutions. Yet for all Rabi model examples which follow, the alternative (A2) will be the only relevant one. Anomalous alternative (A2) can be encountered also in other problems (see equation (5) of [36]; equation (31) of [37] for relative motion of two electrons in an external oscillator potential). Obviously, one has automatically $B \neq -A'$ for the alternative (A2). Therefore, the necessary condition (B.3) for a 2nd order ODE (2) with fixed polynomial coefficients $A(z), B(z), C(z)$ to have two linearly independent (and hence to possess only) polynomial solutions is always violated. Consequently if equation (2) has a polynomial solution, such a polynomial solution is necessarily unique.

3. Basic theorems

In this section general necessary and sufficient conditions for the existence of a polynomial solution of $n$th degree are formulated.

3.1. General theory

The condition that $S_n(z)$ solves (7) is equivalent to that all the coefficients of respective powers of $z$ of the image $LS_n(z)$ of $S_n(z)$ vanish. The latter brings us to the linear system of equations
\[
\begin{align*}
F_{\gamma-1}(n) + a_{n,\gamma-1}F_\gamma(n-1) &= 0, \\
F_{\gamma-2}(n) + a_{n,\gamma-2}F_{\gamma-1}(n-1) + a_{n,n-2}F_\gamma(n-2) &= 0, \\
&\vdots \\
a_{n,\gamma+2-\gamma_*}F_{\gamma_*}(\gamma + 2 - \gamma_*) + \ldots + a_{n,1}F_{\gamma-1}(3) + a_{n,2}F_\gamma(2) &= 0, \\
a_{n,\gamma+w}F_{\gamma_*}(w) + \ldots + a_{n,2}F_{\gamma-1}(2) + a_{n,1}F_\gamma(1) &= 0, \\
a_{n,n-1}F_{\gamma_*}(w-1) + \ldots + a_{n,1}F_{\gamma-1}(1) + a_{n,0}F_\gamma(0) &= 0,
\end{align*}
\] (11)

where each line summarizes all the terms contributing to the same power of $z$, beginning from $n-1+\gamma$ of the first equation down to $\gamma$ of the last equation. We recall that $w = \gamma - \gamma_* + 1$ is the gradation slicing width. If one tries to determine the coefficients $a_{nk}$ of $S_n(z)$ in the expansion (5) by direct substitution into underlying differential equation, the width $w$ thus yields the length of a downward recurrence. For $\gamma = 0$ both $S_n(z)$ and its image $LS_n(z)$ are polynomials of the same degree. Moreover, one has necessarily $\gamma_* \leq 0$. We have the following theorem.

**Theorem 1.** A necessary and sufficient conditions for the ODE (6) with the grade $\gamma = 0$ to have a unique polynomial solution of $n$th degree is that
\[ F_\gamma(n) = 0, \quad F_\gamma(k) \neq 0, \quad 0 \leq k < n, \] (12)

where the second condition applies for $n \geq 1$. □
Proof. The condition \( F_0(n) = 0 \) is nothing but the baseline condition (8) in the special case \( \gamma = 0 \), and is obviously necessary. In order to demonstrate sufficiency, note that the second condition \( F_0(k) \neq 0 \) ensures that each subsequent line in the system (11) of \( n \) equations, when progressing from the very top down, enables one to uniquely determine newly appearing coefficient (i.e. \( a_{0,n-l} \) in the \( l \)th line) and thus to determine at the end a unique set of coefficients \( a_{0,k}, 0 \leq k < n \). The initial condition \( a_{00} = 1 \) is used here to simply fix an arbitrary irrelevant multiplication factor. The point of crucial importance is that for (and only for) \( \gamma = 0 \) the image \( L S_0(z) \) and \( S_0(z) \) are polynomials of the same degree. Hence on summing up the lines of the above system one recovers the original system of \( n \) equations for \( n \) unknown coefficients \( a_{nk}, 0 \leq k < n \). Indeed, because by definition, and on substituting \( \gamma = 0 \),
\[
F_\gamma + m(c^{\gamma} - m - l) = F_{s + n(-\gamma - m - l)} \equiv 0, \quad 0 \leq m \leq w - 1, \ l > 0.
\]
(13)
Thereby the theorem is proven.

Corollary 1. If we drop ‘unique’ in theorem 1, then the condition (8) is both necessary and sufficient condition for the existence of a polynomial solution of an ODE of grade zero.

Proof. Apply theorem 1 to the smallest nonnegative zero of \( F_0 \).

Corollary 2. If \( F_\gamma(k) \) is a linear function of \( k \), there is always at most a single unique polynomial solution, because a linear function can have at most a single root.

Remark 2. For the hypergeometric equation characterized by \( \gamma = 0 \) and \( w = 3 \), the system of equations (11) reduces to the three-term recurrence relation (TTRR) studied exhaustively by Lesky [38]. \( F_0(k) \) is a quadratic function of \( k \) and there are, in principle, possible two linearly independent polynomial solutions, because quadratic function has in general two roots (see appendix B).

Remark 3. In the case of the Heine–Stieltjes problem [29–35], the usual condition \( B = -A' \) for ODE (2) to have only polynomial solutions (see appendix B for more detail) requires that for some \( n \)
\[
F_2(n) = n(n - 1)a_N + nb_{N-1} + c_{N-2} = n(n - 1 - N)a_N + c_{N-2} = 0.
\]

Remark 4. Theorem 1 does not rely on, and is independent of, the Frobenius analysis of a regular singular point of 2nd order ODE (see for instance chapter 10.3 of Whittaker and Watson [39], or chapter 5.3 of Hille [40]). Yet there are many parallels between the two approaches. In theorem 1 the condition \( F_0(n) = 0 \) gives an entry point to a downward recurrence. In the Frobenius analysis, one needs instead an entry point for an upward recurrence. Such an entry point is provided by the solutions of the so-called indicial equation, which can be viewed as \( F_\gamma(s) = 0, s \in \mathbb{C} \). Hence in the Frobenius analysis one is instead of the highest grade slice \( F_\gamma \) concerned with the lowest grade slice \( F_\gamma \). If \( s \) is a solution, i.e. \( F_\gamma(s) = 0 \), then the condition \( F_\gamma(n + s) \neq 0, n \geq 1 \), guarantees that a unique, in general infinite, set of coefficients of a solution at a regular singular point can be determined by the relevant recurrence.
As explained above, we assume that \( \gamma_s \leq 0 \), which can always be achieved by a suitable factorization of the polynomial coefficients of differential equation (6).

**Theorem 2.** A necessary and sufficient conditions for the ODE (6) with the grade \( \gamma > 0 \) to have a unique polynomial solution is that, in addition to the conditions (12) which determine the unique set of coefficients \( \{a_{nk}\}_{k=0}^n \) by the recurrence (11) of theorem 1, the subset \( \{a_{d0}, a_{d1}, \ldots, a_{d,\gamma-2}\} \) of the coefficients \( \{a_{nk}\}_{k=0}^n \) satisfies additional \( \gamma \) constraints:

\[
P_1 := a_{n-\gamma} + \ldots + a_{1} F_{n-1} + a_{0} F_{0} = 0, \quad P_2 := a_{n-1-\gamma} F_{n-1} + \ldots + a_{0} F_{0} = 0, \quad P_0 := a_{n} F_{n} = 0.
\]  

(14)

**Proof.** According to the definition, \( F_{n} z^{\gamma} \sim z^{\gamma} \). Therefore, whenever \( \gamma > 0 \), the recurrence (11) does not take into account the terms \( z^{\gamma} \) of degree \( k < \gamma \) of the image \( \mathcal{L} S_{0}(z) \). There are exactly \( \gamma \) of such polynomial terms with \( k = 0, \ldots, \gamma - 1 \). One can verify that the vanishing of the coefficients of \( z^{\gamma} \), \( 0 \leq k < \gamma \) amounts to solving the system (14). The vanishing of the coefficients of \( z^{\gamma} \), \( k < \gamma \) thus imposes \( \gamma \) constraints on the (up to a multiplication by a constant) unique set of coefficients \( a_{nk} \).

**Remark 5.** Grade \( \gamma < 0 \) problem has always a polynomial solution, because it leads to a system of \( (n - \lceil |\gamma| \rceil) < n \) equations for \( n \) unknowns. Obviously, whenever \( \gamma > 0 \), the differential operator \( \mathcal{L} \) is not exactly solvable. The image \( \mathcal{L} S_{0}(z) \) of \( S_{0}(z) \) is a polynomial of \( (n + \gamma) \) th degree. The vanishing of all the polynomial coefficients of the image \( \mathcal{L} S_{0}(z) \) then imposes \( (n + \gamma) \) different conditions.

**Remark 6.** After imposing on the energy \( E \) one of the baseline conditions \( F_{c}(n) = 0 \), the energy is expressed as a function of model parameters (e.g. equations (38), (48), (57) and (60), (67) below). Therefore after imposing the baseline condition, the coefficients \( F_{\gamma}(k) \) of the recurrence (11) and of (14) cease to depend on \( E \). When solving for the expansion coefficients \( a_{nk} \) by the \( w \)-term downward recurrence (11), each subsequent \( a_{nk} \), beginning from \( k = n - 1 \) in the first equation down to \( k = 0 \) in the last equation, is obtained by dividing the corresponding equation line by \( F_{\gamma}(k) \). The necessary and sufficient conditions for the ODE (6) with the grade \( \gamma > 0 \) to have a unique polynomial solution ensures that the product \( \prod_{k=n-1}^{0} F_{\gamma}(k) \neq 0 \). Provided that the coefficients \( F_{\gamma}(k) \) are polynomials in model parameters (e.g. examples (39), (50), (59), (62) and (68) below), each \( P_{\gamma} \) is of degree \( \gamma > 0 \), when multiplied by the product \( \prod_{k=n-1}^{0} F_{\gamma}(k) \neq 0 \), is necessarily a polynomial in model parameters. The resulting constraint polynomials are defined by the \( w \)-term recurrence (11), which yields the unique solution for \( a_{d2}, a_{d3}, a_{d0} \), that is substituted into the constraints (14) and each of the constraints (14) is multiplied by \( \prod_{k=n-1}^{0} F_{\gamma}(k) \neq 0 \).

We have thus proven the following fundamental result:

**Theorem 3.** Provided that each \( F_{\gamma}(k) \) in (11) and (14) is a polynomial in model parameter(s) and the hypotheses of theorem 2 are satisfied, each recursively determined \( P_{\gamma} \) is a polynomial in model parameter(s).

□
Remark 7. The Rabi models in this article are all characterized by the same grading parameters summarized by (4). Given that \( w = 4 \), the recurrence system (11) and (14) reduces to a downward four-term recurrence relation (FTRR) for the coefficients \( a_{nk}, k < n \), of \( S_n(z) \). The necessary condition (8) for the existence of a polynomial solution becomes in view of (4)
\[
 nb_2 + c_1 = 0, \quad n \geq 0.
\]
(15)
For \( \gamma = 1 \) there is a single constraint (14) to be satisfied,
\[
P_1 := a_{n2}F_{-2}(2) + a_{n1}F_{-1}(1) + a_{n0}F_0(0) = 0,
\]
(16)
to guarantee the existence of a unique polynomial solution. For the Rabi models considered here all the coefficients \( F_{g}(k) \) are polynomials in physical parameters such as the coupling strength \( g \), \( \Delta \), and frequency \( \omega \) in equation (1) (e.g. examples (39), (50), (59), (62) and (67) below).

Corollary 3. For the Rabi model (1) the constraint (16) for each baseline is equivalent to the corresponding Kus polynomial [10]. For the driven Rabi model, the constraint (16) on a given baseline is equivalent to the corresponding generalized Kus polynomial [13–15].

The polynomial has to be equivalent to either the Kus polynomial [10], or generalized Kus polynomial [13–15], respectively, because the respective polynomials, which express the necessary and sufficient conditions for the existence of a unique polynomial solution, have the same zeros and multiplicity. The equivalence will be illustrated on examples and numerically in sections 4 and 5.

Remark 8. For the grade \( \gamma = 2 \) there would be two constraint polynomials. Common zeros of two different polynomials are the zeros of the so-called resultant [41]. This is the case of the generalized Rabi model discussed in section 7 below.

Theorem 4. Let the 2nd order ODE (2) be of grade \( \gamma \geq 0 \). General necessary conditions on the coefficients of the polynomial \( C(z) \) for the ODE (2) to have a polynomial solution \( S_n(z) \) of degree \( n \) with all zeros \( z_i \) simple are
\[
c_\gamma = -n(n-1)a_{\gamma+2} - nb_{\gamma+1},
\]
(17)
\[
c_{\gamma-1} = \left[2(n-1)a_{\gamma+2} + b_{\gamma+1}\right] \sum_{i=1}^{n} z_i - n(n-1)a_{\gamma+1} - nb_{\gamma},
\]
(18)
\[
c_{\gamma-2} = \left[2(n-1)a_{\gamma+2} + b_{\gamma+1}\right] \sum_{i=1}^{n} z_i^2 - 2a_{\gamma+2} \sum_{i<l}^{n} z_iz_l
- \left[2(n-1)a_{\gamma+1} + b_{\gamma}\right] \sum_{i=1}^{n} z_i - n(n-1)a_{\gamma} - nb_{\gamma-1}.
\]
(19)

Note in passing that if a zero \( z_i \) of \( S_n(z) \) were not simple, then the Wronskian of \( S_n(z) \) with any other (not necessary polynomial) nonsingular function would be zero at any multiple root \( z_i \) of \( S_n(z) \) (see appendix B for more detail). Here and below the coefficients \( a_k, b_k, c_k \) with a negative subscript are assumed to be identically zero.
Proof. According to the hypothesis we have
\[
F_{\gamma} = a_{\gamma}z^{\gamma} + b_{\gamma+1}z^{\gamma+1}d_{\gamma} + c_{\gamma}z, \\
F_{\gamma-1} = a_{\gamma}z^{\gamma} + b_{\gamma+1}z^{\gamma+1}d_{\gamma} + c_{\gamma}z^{\gamma-1}, \\
F_{\gamma-2} = a_{\gamma}z^{\gamma} + b_{\gamma+1}z^{\gamma+1}d_{\gamma} + c_{\gamma-2}z^{\gamma-2}.
\]
In particular, \(F_{\gamma}(n) = n(n-1)a_{\gamma+2} + nb_{\gamma+1} + c_{\gamma}\), and he necessary condition (8) for the existence of a polynomial solution of degree \(n\) constrains the coefficient \(c_{\gamma}\) immediately to (17).

The first of the recurrences of the system (11) requires
\[
n(n-1)a_{\gamma+1} + nb_{\gamma} + c_{\gamma-1} + a_{n-1}[(n-1)(n-2)a_{\gamma+2} + (n-1)b_{\gamma+1} + c_{\gamma}] = 0.
\]
On substituting the above expressions into (17) and solving for \(c_{\gamma-1}\) yields
\[
F_{\gamma-1}(n) = [2(n-1)a_{\gamma+2} + b_{\gamma+1}]a_{n-1} - n(n-1)a_{\gamma+1} - nb_{\gamma},
\]
On taking into account that \(a_{n-1} = -\sum z_{0i}\), this leads immediately to (18).

We now continue with the second of the recurrences of the system (11),
\[
F_{\gamma-2}(n) + a_{n-1}F_{\gamma-1}(n-1) + a_{n-2}F_{\gamma}(n-2) = 0.
\]
One has
\[
F_{\gamma-2}(n) = n(n-1)a_{\gamma} + nb_{\gamma-1} + c_{\gamma-2}, \\
F_{\gamma-1}(n-1) = n(n-1)(n-2)a_{\gamma+1} + (n-1)b_{\gamma} + c_{\gamma-1} \\
= [2(n-1)a_{\gamma+2} + b_{\gamma+1}]a_{n-1} + (n-1)[n-2-n]a_{\gamma+1} + [n-1-n]b_{\gamma} \\
= [2(n-1)a_{\gamma+2} + b_{\gamma+1}]a_{n-1} - 2(n-1)a_{\gamma+1} + b_{\gamma}, \\
F_{\gamma}(n-2) = (n-2)(n-3)a_{\gamma+2} + (n-2)b_{\gamma+1} + c_{\gamma} \\
= [(n-2)(n-3) - n(n-1)]a_{\gamma+2} - 2b_{\gamma+1} \\
= -2[2(n-1)a_{\gamma+2} + b_{\gamma+1}] + 2a_{\gamma+2},
\]
where (17) and (21) were used in arriving at the final results. On substituting the above expressions back into (22) one arrives at
\[
n(n-1)a_{\gamma} + nb_{\gamma-1} + c_{\gamma-2} \\
+ a_{n-1}[2(n-1)a_{\gamma+2} + b_{\gamma+1}]a_{n-1} - 2(n-1)a_{\gamma+1} - b_{\gamma} \\
- 2a_{n-2}[2(n-1)a_{\gamma+2} + b_{\gamma+1}] + 2a_{n-2}a_{\gamma+2} = 0.
\]
Solving for \(c_{\gamma-2}\) yields
\[
c_{\gamma-2} = [2(n-1)a_{\gamma+2} + b_{\gamma+1}]\left(2a_{n-2} - a_{n-1}^2\right) - 2a_{\gamma+2}a_{n-2} \\
+ [2(n-1)a_{\gamma+1} + b_{\gamma}]a_{n-1} - n(n-1)a_{\gamma} - nb_{\gamma-1}.
\]
To this end one makes use of
\[
a_{n-1} = \sum z_{0i}, \quad a_{n-2} = \sum_{i<j} z_{i}z_{j}, \\
a_{n-1}^2 = \left(\sum z_{0i}\right)^2 = \sum z_{i}^2 + 2\sum_{i<j} z_{i}z_{j}.
\]
Thereby one recovers (19).
Remark 9. In the special case of $\gamma = 2$, the conditions (17)–(19) reduce to those of theorem 1.1 of Zhang (see equations (1.8)–(10) of [42]), where they were derived by means of a functional Bethe Ansatz. Yet in the latter case the level of complexity increases significantly with $\gamma$ and each $\gamma$-case has to be treated separately. In contrast to that, the gradation slicing approach enables one to prove the formulas of theorem 4 for the coefficients $c_{\gamma-1}$, $l = 0, 1, 2$, in one go. Theorem 4 applies to both alternatives (9) and (10). Note in passing that if $a_{\gamma+2} = b_{\gamma+1} = 0$, the condition (18) for $c_{\gamma-1}$ reduces to the necessary condition (38) for the existence of a polynomial solution of degree $n$. Similarly for the condition (19) for $c_{\gamma-2}$, provided that additionally $a_{\gamma+1} = b_{\gamma} = 0$.

Remark 10. Provided that the $n$ simple roots $z_i$ are required to satisfy the set of the Bethe Ansatz algebraic equations (see equations (1.11) and (2.5) of [42]),

\[ \sum_{l \neq i}^{n} \frac{2}{z_i - z_l} + \frac{B(z_i)}{A(z_i)} = \sum_{l \neq i}^{n} \frac{2}{z_i - z_l} + \frac{b_3z_i^3 + b_2z_i^2 + b_1z_i + b_0}{a_4z_i^4 + a_3z_i^3 + a_2z_i^2 + a_1z_i + a_0} = 0, \quad i = 1, 2, \ldots, n, \]  

(23)

then the conditions together with those of theorem 4 provide necessary and sufficient conditions for the coefficients of $C(z)$ of the ODE (6) with grade $\gamma = 2$ to have a polynomial solution of $n$th degree with zeros $z_i$. Yet this does not answer the question under which conditions has the system of the Bethe Ansatz algebraic equations a solution. Theorem 1.1 of Zhang [42] is rather a set of general compatibility conditions between the polynomial zeros $z_i$ that satisfy the Bethe Ansatz algebraic equations for a given $A(z)$ and $B(z)$ on one hand, and the coefficients of $C(z)$ of the ODE (6) on the other hand. Similarly to the Kus recipe [10], the Bethe Ansatz equations for a polynomial of $n$th degree have a solution only for a discrete set of model parameters, which corresponds to zeros of a certain polynomial in model parameters [6].

3.2. $sl_2$ algebraization

The condition of a $sl_2$ algebraization is that a corresponding differential operator $\mathcal{L}$ can be expressed as a normally ordered bilinear combination of the $sl_2$ generators $J_\pm$, $J_0$,

\[ T_2 = C_+J_+J_+ + C_0J_+J_0 + C_0J_0J_0 + C_0J_0J_- + C_-J_-J_- + C_+J_+ + C_0J_0 + C_-J_- + C_+, \]

(24)

where $C_{\alpha\beta}, C_\alpha, C_+ \in \mathbb{R}$ [19–27]. Strictly speaking $T_2$ belongs to the central extension of $sl_2$ (see theorem 2 of [20]). With a slight abuse of notation we continue writing $T_2 \in \mathfrak{d}(sl_2)$. The properties of the $sl_2$ generators $J_\pm$, $J_0$ are summarized in appendix A.

Let us consider an anomalous $\mathcal{L}$ characterized by the grading parameters as in equation (4). In order that such a $\mathcal{L}$ reproduces $T_2$ in equation (A.4), one has to have $\deg P_4 = \deg P_3 = 2$ and $\deg P_2 = 1$ in equation (A.5). The latter immediately requires $C_{++} = C_{+0} \equiv 0$ in equations (24) and (A.5), whereas $C_+$ there has to satisfy

\[ C_+ = b_2 \quad \text{and} \quad 2j b_2 + c_1 = 0. \]

(25)

The above two conditions on $C_+$ require that the coefficients $b_2$ and $c_1$ of the terms of the highest grade satisfy

\[ 2j b_2 + c_1 = 0. \]

(26)
The necessary condition (26) for \( sl_2 \) algebraization reproduces the necessary condition (15) for the existence of a polynomial solution of \( n \)th degree if and only if the spin \( j \) of an irreducible \( sl_2 \) representation satisfies \( 2j = n \).

There are at most \( n \) different polynomial solutions of \( n \)th degree on each \( n \)th baseline [10]. The latter would not be surprising if we had an identical \( T_2 \) along whole given \( n \)th baseline. Yet each of those polynomial solutions corresponds to a different \( T_2 \in U(sl_2) \) (e.g. because \( C_+ \) depends on \( g \) and the polynomial solutions are nondegenerate on a given (base)line in the \( (E, g) \)-plane [2, 10, 43–45]).

The remaining conditions for the \( sl_2 \) algebraization in the anomalous deg \( P_4 = \deg P_3 = 2 \) and \( \deg P_2 = 1 \) case are

\[
\begin{align*}
C_{00} &= a_2, \\
C_0 &= b_1 + (2j - 1)a_2, \\
C_+ &= c_0 - j^2 C_{00} + jC_0 = c_0 + jb_1 + j(j - 1)a_2.
\end{align*}
\]  

(27)

Because \( C_+ \) is an arbitrary constant in equation (24), the conditions (27) can be always satisfied. Therefore condition (26) is both necessary and sufficient condition for \( sl_2 \) algebraization.

From the general necessary condition (18) for the existence of a polynomial solution of \( n \)th degree of theorem 4 we know that for an anomalous \( L \) characterized by the grading parameters as in equation (4), i.e. with \( a_{\gamma+2} = a_3 = 0 \), we have to have

\[
c_0 = -b_2 \sum_{i=1}^{n} z_i - n(n - 1)a_2 - nb_1.
\]  

(28)

The general necessary condition (28) for the existence of a polynomial solution of \( n \)th degree of theorem 4 for \( 2j = n \) can be then recast in terms of \( C_+ \),

\[
C_+ = -b_2 \sum_{i=1}^{n} z_i - j(3j - 1)a_2 - jb_1.
\]  

(29)

3.2.1. Nondegenerate energy levels. Let us examine the conditions under which \( T_2 \) cannot have degenerate energy levels in an invariant \( sl_2 \) module. A degeneracy can only occur if the necessary condition (B.3),

\[
B = -A',
\]

is satisfied. The latter requires

\[
b_3 + 4a_4 = 0 \iff (4j - 3)b_3 = 0,
\]

which is impossible to satisfy for \( b_3, a_4 \neq 0 \). Hence

\[
b_3 = a_4 \equiv 0 \iff C_{++} = 0.
\]  

(30)

If \( b_2, a_3 \neq 0 \), the condition (B.3) requires

\[
b_2 + 3a_3 = 0 \iff -(3j - 4)C_{++} + C_+ = 0,
\]

or the constraint

\[
C_+ = (3j - 4)C_{++}.
\]  

(31)

If \( b_1, a_2 \neq 0 \), the condition (B.3) requires

\[
b_1 + 2a_2 = 0 \iff -(2j - 3)C_{00} + C_0 = 0,
\]

or the constraint

\[
C_0 = (2j - 3)C_{00}.
\]  

(32)

Eventually,
\[ b_0 + a_1 = 0 \iff C_- - (j - 1)C_{0-} = 0. \]  

(33)

We have thus proven the following result:

**Lemma.** If any of the conditions (30)–(33) is violated, the spectral problem \( T_2 \psi = \lambda \psi \), where \( \lambda \in \mathbb{C} \) is an eigenvalue, cannot have degenerate energy levels in an invariant \( sl_2 \) module of spin \( j \).

We remind here that such a nondegeneracy is, as shown by Lesky [38]), an intrinsic feature of any 2nd order ODE spectral problem defining an orthogonal polynomials system (OPS) [31, 46].

Usual \( sl_2 \) algebraization means that the corresponding spectral problem possesses \((n + 1)\) eigenfunctions in the form of polynomials of degree \( n = 2j \) for any given irreducible representation of \( sl_2 \) of spin \( j \). The foregoing analysis of anomalous grade \( \gamma = 1 \) problems implies for the Rabi model problems that any given irreducible representation of \( sl_2 \) of spin \( j \) can in the most optimal case add only at most a single new polynomial eigenfunction of degree \( n = 2j \) relative to a lower dimensional irreducible representation of \( sl_2 \) of spin \( j - (1/2) \)—a kind of onion algebraization. The forthcoming examples will demonstrate that more often than not no new polynomial eigenfunction will be added to the spectrum.

### 4. The Rabi model

For a theoretical investigation of the Rabi model it is expedient to work in an equivalent single-mode spin-boson picture, which amounts to interchanging \( \sigma_1 \) and \( \sigma_3 \) in (1). The latter is realized by unitary transformation \( \hat{H}_R = U_{13} \hat{H}_R U_{13} \), where \( U_{13} = (\sigma_1 + \sigma_3)/\sqrt{2} \). Assuming \( h = 1 \), one arrives at

\[ \hat{H}_R = \omega a^\dagger a + \Delta \sigma_z + g \sigma_x (a^\dagger + a), \]  

(34)

where \( \sigma_z = \sigma_1 \), and \( \sigma_x = \sigma_3 \) becomes diagonal. The \( 2 \times 2 \) matrix \( \hat{H}_R \) possesses the parity symmetry \( \hat{P}_{FG} = \hat{R} \sigma_1 \), where unitary \( \hat{R} = e^{i \pi a^\dagger a} \) induces reflections of the annihilation and creation operators: \( \hat{a} \rightarrow -\hat{a}, \hat{a}^\dagger \rightarrow -\hat{a}^\dagger \), and leaves the boson number operator \( a^\dagger a \) invariant [2, 8, 10, 47]. \( \hat{H}_R \) is immediately recognized to be of the Fulton–Gouterman form [8, 48], where the Fulton–Gouterman symmetry operation is realized by \( \hat{P}_{FG} \). The projected parity eigenstates \( \Phi^\pm \) have generically two independent components. The advantage of the Fulton and Gouterman form is that the projected parity eigenstates \( \Phi^\pm \) are characterized by a single independent component. In the Bargmann realization [49]: \( \alpha^\dagger \rightarrow z, \alpha \rightarrow d/dz, \) and the Hamiltonian \( \hat{H}_R \) of equation (34) becomes a matrix differential operator. After \( \hat{H}_R \) is diagonalized in the spin subspace, the corresponding 1D differential operators are found to be of Dunkl type [8]. The Fulton–Gouterman form and the 1D differential operators of Dunkl type can also be determined for all the remaining Rabi models discussed here [8].

In terms of the two-component wave function \( \psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} \), the time-independent Schrödinger equation gives rise to a coupled system of two first-order differential equations (see equations (2.4a) and (2.4b) of Kus [10])

\[
(\omega z + g) \frac{d}{dz} \psi_+(z) = (E - gz) \psi_+(z) - \Delta \psi_-(z),
\]

\[
(\omega z - g) \frac{d}{dz} \psi_-(z) = (E + gz) \psi_-(z) - \Delta \psi_+(z).
\]

(35)
If $\Delta = 0$ these two equations decouple and reduce to the differential equations of two uncoupled displaced harmonic oscillators which can be exactly solved separately [2, 11]. For this reason we will concentrate on the $\Delta \neq 0$ case in the following.

With the substitution
$$\psi_{\pm}(z) = e^{-z^2/\omega} \phi_{\pm}(z),$$
and on eliminating $\phi_-(z)$ from the system we obtain the uncoupled 2nd order differential equation for $\phi_+(z)$ [11, 12],

$$\begin{align*}
\left[(\omega z + g) \frac{d}{dz} - \left(\frac{g^2}{\omega} + E\right)\right] \phi_+(z) &= -\Delta \phi_-(z), \\
\left[(\omega z - g) \frac{d}{dz} - \left(2gz - \frac{g^2}{\omega} + E\right)\right] \phi_-(z) &= -\Delta \phi_+(z). 
\end{align*}$$

(The substitution $\psi_{\pm}(z) = e^{z^2/\omega} \phi_{\pm}(z)$ merely interchanges the roles of $\phi_{\pm}$ and does not bring anything new.)

Eliminating $\phi_-(z)$ from the system we obtain the uncoupled differential equation for $\phi_+(z)$,

$$\left[(\omega z - g) \frac{d}{dz} - \left(2gz - \frac{g^2}{\omega} + E\right)\right] \left[(\omega z + g) \frac{d}{dz} - \left(\frac{g^2}{\omega} + E\right)\right] \phi_+ = \Delta^2 \phi_+.$$

Explicitly,

$$\begin{align*}
(\omega z - g)(\omega z + g) \frac{d^2 \phi_+}{dz^2} + \left[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2)\right] \frac{d \phi_+}{dz} \\
+ \left[2g \left(\frac{g^2}{\omega} + E\right)z + E^2 - \Delta^2 - \frac{g^2}{\omega^2}\right] \phi_+ \equiv \mathcal{L} \phi_+ = 0.
\end{align*}$$

Because the polynomial coefficients $b_1, c_1, c_0$ are all energy dependent, this is obviously not a standard eigenvalue problem. As announced earlier in equation (4), the grading characteristics of the differential equation are $\gamma = 1, w = 4$, and

$$\begin{align*}
\mathcal{F}_1 &= b_2 d_z + c_1 = -2\omega g z^2 d_z + 2g \left(\frac{g^2}{\omega} + E\right)z, \\
F_1(n) &= -2\omega g + 2g \left(\frac{g^2}{\omega} + E\right), \\
\mathcal{F}_0 &= \omega^2 z^2 d_z^2 + (\omega^2 - 2g^2 - 2E\omega)zd_z + E^2 - \Delta^2 - \frac{g^2}{\omega^2}, \\
\mathcal{F}_{-1} &= \frac{g}{\omega}(2g^2 - \omega^2)d_z, \quad \mathcal{F}_{-2} = -g^2 d_z^2.
\end{align*}$$

$F_1(n)$ is a linear function of $n$ and, according to corollary 2, only one of any two linearly independent solution can be a polynomial. One can arrive at the same conclusion by noting independently that, in the case of equation (37), we have $\deg A = \deg B = 2, \deg C = 1$, and the anomalous alternative (A2) applies. Hence any polynomial solution of equation (37) is necessarily unique (assuming $A(z), B(z), C(z)$ being fixed polynomials). The other (nonpolynomial) solution is not normalizable (e.g. does not belong to a corresponding Bargmann space [49]), which follows from non-degeneracy of spectrum in each of the even and odd parity subspaces [43–45].
One can verify that the necessary condition (8) for the existence of a polynomial solution of degree \( n \), \( F_i(n) = 0 \), reduces to (15) and leads to

\[
E_n = n\omega - \frac{\mu^2}{\omega}.
\]

(38)

This is the familiar baseline condition for the Rabi model [2, 10]. Each baseline corresponds to an exact energy level of a displaced harmonic oscillator, which is the \( \Delta = 0 \) limit of the Rabi model [2]. The baseline is to be understood as the curve (38) in the parameter space \((E, g)\) (with a definite sign of \( g \neq 0 \)).

Focusing now on the \( n \)th baseline, \( b_1(E_n) = (1 - 2n)\omega^2 \), \( c_0(E_n) = n^2\omega^2 - \Delta^2 - 2ng^2 \), and

\[
\begin{align*}
F_1(k) &= 2\omega g(n - k), \\
F_0(k) &= k(k - 2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2, \\
F_{-1}(k) &= \frac{k\omega}{\omega}(2g^2 - \omega^2), \\
F_{-2}(k) &= -k(k - 1)g^2.
\end{align*}
\]

(39)

As for any \( \gamma = 1 \) problem, there is an additional single constraint (16) to be satisfied, which, when combined with the baseline condition (38), yields by theorem 2 the necessary and sufficient conditions for the existence of a unique polynomial solution of equation (37). On substituting (39) into (16), the latter becomes

\[
P_1(n) := -2g^2a_{n2} + \frac{g}{\omega}(2g^2 - \omega^2)a_{n1} + \left(n^2\omega^2 - \Delta^2 - 2ng^2\right)a_{n0} = 0.
\]

(40)

Because all \( F_i(k) \) are polynomials in model parameters, the constraint relation (16) is according to theorem 3 equivalent to a polynomial identity. For instance, in the special case of \( n = 1 \) we have \( a_{n2} = 0, a_{n1} = 1 \). From the system (11) one gets \( a_{10} = -F_0(1)/F_1(0) = (\Delta^2 + 2g^2)/2\omega \).

The 1st baseline constraint (40) then becomes

\[
P_1(1) := \frac{g}{\omega}(2g^2 - \omega^2) - (\omega^2 - \Delta^2 - 2g^2) \frac{F_0(1)}{F_1(0)} = -\frac{\omega\Delta^2}{2g} \left( \frac{4g^2}{\omega^2} + \frac{\Delta^2}{\omega^2} - 1 \right) = 0.
\]

(41)

In agreement with corollary 3, the term in parenthesis is nothing but the Kus polynomial \( K_{11} \) for the 1st baseline. Indeed, the Kus polynomial \( K_{nn} \) associated with the \( n \)th baseline (which is indicated by the left subscript) is defined by its own finite three-term recurrence for each \( n \geq 0 \) [10],

\[
K_{n0} = 1, \quad K_{n1} = 4\kappa^2 + \mu^2 - 1, \\
K_{nl} = (4l\kappa^2 + \mu^2 - 1)K_{n,l-1} - 4l(l - 1)(n - 1 + 1)\kappa^2K_{n,l-2}.
\]

\( K_{nn} \) is of \( n \)th degree (which is indicated by the right subscript) in rescaled variables \( 4\kappa^2 \) and \( \mu^2 \), where \( \kappa = g/\omega, \mu = \Delta/\omega \) [10]. Figure 1 then illustrates the validity of corollary 3 about equivalence of \( P_1(n) \) and \( K_{nn} \) for \( n = 5 \). The only computational difference is that the coefficients \( a_{n2}, a_{n1}, a_{n0} \) entering the constraint polynomial \( P_1(n) \) in (40) are determined by a four-term recurrence, whereas \( K_{nn} \) is obtained by a three-term recurrence.

Given that all zeros of \( K_{nn} \) are simple [10], an obvious consequence of the Kus construction [10] is that for a given fixed \( 0 < \mu < 1 \) there are:

(i) in total exactly \( n \) different polynomial solutions of the Rabi model on each baseline
(ii) there are no other polynomial solutions of the Rabi model.

By varying \( \mu \) one selects another set on \( n \) points on the \( n \)th baseline. For \( k < \mu < k + 1 \), \( 0 < k \leq n \), the number of real zeros of \( K_{nn} \) becomes \( n - k \) [10]. The zeros of \( K_{nn} \) correspond
to the points where the energies of positive and negative parity eigenstates cross each other as a function of $g$ [10]. There are no other degeneracy points of the Rabi model [10]. In particular there is, according to Kus, no $n = 0$ baseline solution for $\mu \not\in \mathbb{N}$. Obviously equation (37) has a $n = 0$ solution, but the latter requires $\Delta = \mu \equiv 0$, in which case $C(z) \equiv 0$.

In general at the zeros of $K_{nn}$ the corresponding wave function component $\psi_+ (z)$ is given by

$$\psi_+ (z) = e^{-\frac{g^2}{\omega}} \prod_{i=1}^{n} (z - z_i),$$

and the component $\psi_- (z) = e^{-g^2/\omega} \phi_- (z)$ with $\phi_- (z)$ is determined by the first equation of (36) for $\Delta \neq 0$. The differential operator

$$\mathcal{L}_R^1 \equiv (\omega z + g) \frac{d}{dz} - \left( \frac{g^2}{\omega} + E \right)$$

in the first equation of (36) is grade zero operator, and hence is exactly solvable, because $\mathcal{L}_R^1$ preserves $\mathcal{P}_n = \text{span}\{1, z, z^2, \ldots, z^n\}$ for any system parameters. Given that $\phi_- (z) = -\frac{1}{2} \mathcal{L}_R^1 \phi_+ (z)$, $\phi_- (z)$ automatically belongs to the same invariant subspace as $\phi_+ (z)$. Here the $n$ simple roots $z_i$ are required to satisfy the set of the Bethe Ansatz algebraic equation (23),

Figure 1. A comparison of the polynomial constraint relation (16) and the Kus polynomial for the Rabi model at the fifth baseline shows that they have coinciding zeros. The polynomials are plotted as a function of $g$ with fixed $\Delta = 0.1$ and $\omega = 0.4$. Each zero of the polynomials correspond to the parameter values for which a polynomial solution of the fifth degree exist. The Kus polynomial has much larger amplitude of oscillations in this case, e.g. decreasing below $-5 \times 10^5$ between the last two zeros. Therefore it looks as if crossing x axis vertically.
\[ \sum_{i \neq j}^{n} \frac{2}{z_i - z_j} = \frac{2\omega g z_i^2 + (2n - 1)\omega^2 z_i + g(\omega^2 - 2g^2)/\omega}{(\omega z_i - g)(\omega z_i + g)}, \quad i = 1, 2, \ldots, n. \]

When the constraint (18) on \( c_0 \) now being \( c_0(E_n) = n^2\omega^2 - \Delta^2 - 2ng^2 \) is substituted into theorem 4, the following constraint on \( \sum_{i=1}^{n} z_i \) is obtained,

\[ c_0(E_n) = n^2\omega^2 - 2ng^2 - \Delta^2 = -n [(n - 1)\omega^2 + \omega^2(1 - 2n)] + 2\omega g \sum_{i=1}^{n} z_i \]

or

\[ \Delta^2 + 2ng^2 + 2\omega g \sum_{i=1}^{n} z_i = 0. \] (42)

These relations are exactly the same results obtained by the method of Juddian isolated exact solution [9, 10].

### 4.1. sl\(_2\) algebraization

The necessary and sufficient condition (26) for sl\(_2\) algebraization with spin \( j \) satisfying \( 2j = n \) reproduces the necessary condition (38) for the existence of a polynomial solution of nth degree, which is the familiar baseline condition for the Rabi model [2, 10]. Algebraically, the necessary condition (38) ensures that the \( z^{n+1} \) term disappears and \( L \) preserves a finite dimensional subspace \( P_n \). Therefore \( L \) is quasi-exactly solvable with invariant subspace \( P_n \). However only zero modes of \( L \) are relevant physical solutions.

In virtue of (25), \( T_2 \in \mathcal{U}(sl_2) \) requires that \( C_+ = b_2 = -2\omega g \) in equation (24). Obviously, a different value of \( b_2 \) implies a different \( T_2 \in \mathcal{U}(sl_2) \). On substituting for \( a_2, b_1 \) from equation (37), and on using the baseline condition (38), the first two conditions in (27) are satisfied by

\[ C_{00} = \omega^2, \]
\[ C_0 = \omega^2 - 2g^2 - 2E\omega + (2j - 1)\omega^2 = -2j\omega^2. \]

The constant \( C_0 \) is then easily determined on substituting the above two expressions, together with \( c_0(E_n) = n^2\omega^2 - \Delta^2 - 2ng^2 \), into the last condition of (27). Hence the Rabi model Hamiltonian becomes an element of \( \mathcal{U}(sl_2) \) on each baseline - all that without the necessity of a gauge transformation usually required for the time-independent Schrödinger equation eigenvalue problems. The point of crucial importance is that with \( E \) being fixed by (38), all the coefficients of the linear 2nd order ODE (37) are fixed, and thus there are only two linearly independent solutions of equation (37), with at most one of them being polynomial one.

The same conclusion follows also by using the alternative 2nd ODE’s of Schweber [2] and Koc et al [18]. We find it necessary to check it, because the non Sturm–Liouville character of underlying equations prevents any kind of a residual sl\(_2\) gauge transformation as employed for conventional Sturm–Liouville problems in [20]. The results are summarized in appendix D.
5. Driven Rabi model

The Hamiltonian of the driven (also called an asymmetric) Rabi model (for more detail see [12–15, 50])

\[ \hat{H}_R = \omega 1a^\dagger a + \Delta \sigma_z + g \sigma_x (a^\dagger + a) + \delta \sigma_x \quad (43) \]

differs from \( \hat{H}_R \) in equations (1) and (34) by the addition of the driving term \( \delta \sigma_x \). The latter breaks the \( Z_2 \) symmetry of the Rabi model. The driven Rabi model (43) is relevant to the description of some hybrid mechanical systems (see e.g. [50]). With the substitution

\[ \psi_\pm(z) = e^{-\kappa z/\omega} \phi_\pm(z), \]

and on eliminating \( \phi_-(z) \) from the system we obtain the uncoupled differential equation for \( \phi_+(z) \),

\[ \hat{H}_R \phi_+(z) = \Delta^2 \phi_+(z), \quad (44) \]

where [12, 13]

\[ \hat{H}_R = (\omega z - g)(\omega z + g) \frac{d^2}{dz^2} + \left[ -2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z - g\omega \right. \]
\[ \left. + 2g \left( \frac{g^2}{\omega} - \delta \right) \frac{d}{dz} + 2g \left( \frac{g^2}{\omega} + E - \delta \right) z + E^2 - \left( \delta - \frac{g^2}{\omega} \right)^2 \right] \cdot (45) \]

In contrast to the non-driven Rabi model, the substitution

\[ \tilde{\psi}_\pm(z) = e^{+\kappa z/\omega} \tilde{\phi}_\pm(z) \]

leads to another set of solutions of the driven Rabi model. On elimination of \( \varphi_+(z) \) one obtains the uncoupled differential equation for \( \varphi_-(z) \),

\[ \hat{\tilde{H}}_R \varphi_-(z) = \Delta^2 \varphi_-(z), \quad (46) \]

where [12, 13]

\[ \hat{\tilde{H}}_R = (\omega z - g)(\omega z + g) \frac{d^2}{dz^2} + \left[ -2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + g\omega \right. \]
\[ \left. - 2g \left( \frac{g^2}{\omega} + \delta \right) \frac{d}{dz} - 2g \left( \frac{g^2}{\omega} + E + \delta \right) z + E^2 - \left( \delta + \frac{g^2}{\omega} \right)^2 \right] \cdot (47) \]

The above two cases are related by a symmetry \( \psi_+(z, \delta) = \tilde{\psi}_-(z, -\delta), \psi_-(z, \delta) = \tilde{\psi}_+(z, -\delta) \) in the wave function components [13].

Again one can easily verify that the grading characteristics of the respective differential operators \( \hat{H}_R \) and \( \hat{\tilde{H}}_R \) in equations (45) and (47), respectively, are as announced in equation (4): \( \gamma = 1, w = 4 \), and

\[ F_1 = \mp 2\omega g z^2 d \pm 2g \left( \frac{g^2}{\omega} + E \mp \delta \right) z, \quad F_1(n) = \mp 2n\omega g \pm 2g \left( \frac{g^2}{\omega} + E \mp \delta \right). \]

The baseline condition \( F_1(n) = 0 \) is the (i) necessary condition for the existence of a polynomial solution of degree \( n \) and (ii) necessary and sufficient condition for \( \mathfrak{sln} \) algebraization with spin \( j = n/2 \). In each case it implies that the exact (exceptional) energies of the driven Rabi model are restricted to the baselines [12, 13, 50],
\[ E_n = n\omega - \frac{k^2}{\omega} \pm \delta, \quad n = 0, 1, 2, \ldots \]  

(48)

Imposing the constraint (48) unambiguously fixes all the coefficients of the corresponding 2nd order linear differential operators \( H_k \) and \( \tilde{H}_k \). Therefore each of the respective spectral problems (44) and (46) can have at most two linearly independent solutions. The respective spectral problems (44) and (46) lead to the anomalous alternative (A2) of the ODE (2) with \( B \neq -A' \). Because the latter violates the necessary condition (B.3) for equation (2) with fixed polynomial coefficients \( A(z), B(z), C(z) \) to have two linearly independent (and hence to possess only) polynomial solutions, any polynomial solution the spectral problems (44) and (46) is necessarily unique. One arrives at the same conclusion also from that \( F_1(n) \) is a linear function of \( n \) and on applying corollary 2.

On the \( n \)th baseline,

\[
\begin{align*}
 b_1(E_n) &= \omega^2 - 2g^2 - 2E\omega = (1 - 2n)\omega^2 \mp 2\delta\omega, \\
c_0(E_n) &= E^2 - \Delta^2 - \left( \delta \mp \frac{g^2}{\omega} \right)^2 = n^2\omega^2 - \Delta^2 - 2ng^2 \pm 2n\omega\delta.
\end{align*}
\]

(49)

The driving term \( \delta\sigma_x \) in (43) does not change only the grade −2 term of the QRM. Therefore we can use only the result for \( F_{-2}(k) \) from (39), whereas the remaining coefficients of the recurrence system (11) become

\[
\begin{align*}
 F_1(k) &= \pm 2\omega g(n - k), \\
 F_0(k) &= k(k - 2n)\omega^2 + n^2\omega^2 - \Delta^2 - 2ng^2 \pm 2(n - k)\omega\delta, \\
 F_{-1}(k) &= \pm \frac{kg}{\omega} (2g^2 - \omega^2 \mp 2\omega\delta), F_{-2}(k) = -k(k - 1)g^2.
\end{align*}
\]

(50)

Because all \( F_g(k) \) are polynomials in model parameters, the constraint relation (51) is, according to theorem 3, equivalent to a polynomial identity. On substituting from (39) and (50) into (16), the constraint becomes

\[
P_1 := -2g^2a_{10} \pm \frac{g}{\omega}(2g^2 - \omega^2 \mp 2\omega\delta)a_{11} + (n^2\omega^2 - \Delta^2 - 2ng^2 \pm 2n\omega\delta) a_{10} = 0.
\]

(51)

For instance, in the special case of \( n = 1 \) one has \( a_{10} = -F_0(1)/F_1(0) = \pm (\Delta^2 + 2g^2)/2g\omega \).

The 1st baseline polynomial constraint becomes

\[ \mp \frac{\Delta^2}{2g\omega} (4g^2 + \Delta^2 - \omega^2 \pm 2\omega\delta) = 0, \]

(52)

which differs from (41) in the usual Rabi case by the \( \delta \)-dependent term. The term in parenthesis coincides with the generalized Kus polynomials on the 1st baseline (see recursions (B.1) and (B.11) of [13]; recursions (5.1) and (5.2) for \( \omega = 1 \) of [14]). Figure 2 then illustrates validity of corollary 3 for \( n = 9 \).

With \( a_{12} = \omega^2 \), the constant \( C_\ast \) required for the \( sl_2 \) algebraization is easily determined on substituting the expressions for \( b_1(E_n) \) and \( c_0(E_n) \) from (49) into the last condition of (27). Contrary to claims made in [12, 13, 50], we want to emphasize that the \( sl_2 \) algebraization does not automatically mean that the corresponding spectral problems (44) and (46) possess \((n + 1)\) eigenfunctions, respectively, in the form of polynomials of degree \( n \). The coefficient \( C_\ast \) of \( T_2 \in \mathcal{U}(sl_2) \) in equation (24) has to satisfy (25). Because we have \( b_2 = \pm 2\omega g \) in equations (45) and (47), a different value of \( b_2 \) automatically implies a different \( T_2 \in \mathcal{U}(sl_2) \).
Let us consider two different nonlinear Rabi model generalizations. They are nonlinear in a quantum optics sense, because they describe a multi-photon or multi-mode interaction. A corresponding differential equation remains linear, which will allows us to apply the same strategy as before.

The Hamiltonian of the two-photon Rabi model reads \[ \hat{H}_{2p} = \omega a^\dagger a + \Delta \sigma_z + g \sigma_x [(a^\dagger)^2 + a^2] \].

The Fock–Bargmann Hilbert space \( \mathcal{B} \) based on the coherent states associated with the Heisenberg algebra [49] gets replaced by a more general Hilbert space of entire analytic functions of growth \((1, 1)\) associated with the so-called Barut–Girardello coherent states [51] of the annihilation operator \( K_{-} \) of the \( su(1, 1) \) Lie algebra. With the substitution

\[
\psi_{\pm}(z) = e^{-\frac{1}{2} \Omega^2} \phi_{\pm}(z), \quad \Omega = \sqrt{1 - \frac{4g^2}{\omega^2}},
\]

where \( |2g/\omega| < 1 \) [16], and on eliminating \( \phi_{-}(z) \) from the system, one obtains the 4th-order differential equation for \( \phi_{+}(z) \),

\[
\hat{H}_{2p} \phi_{+}(z) = -\Delta^2 \phi_{+}(z), \quad (53)
\]

where [12]

\[
\text{Figure 2. A comparison of the polynomial constraint relation (16) and the generalized Kus polynomial (generated by recursion (B.1) of [13]) for the driven Rabi model at the ninth baseline shows that they have coinciding zeros. The polynomials are plotted as a function of } g \text{ with fixed } \Delta = 0.1, \omega = 0.4, \text{ and } \delta = 0.02. \text{ Each zero of the polynomials corresponds to the parameter values for which polynomial solution of the ninth degree exist. Around the first three and the last crossings of the } x\text{-axis the respective polynomials appear to be overlapping.}
\]
\[ \hat{H}_{2p} = 16g^2z^2 \frac{d^4}{dz^4} + 64g^2 \left[ \frac{\omega}{4g} (\Omega - 1)z^2 + \left( q + \frac{1}{2} \right) z \right] \frac{d^3}{dz^3} \]

\[ + \left\{ 4\omega^2 (\Omega - 3\Omega + 1)z^2 + 16\omega g \left[ 3 \left( q + \frac{1}{2} \right) \Omega - 3q - 1 \right] z + 64g^2q \left( q + \frac{1}{2} \right) \right\} \frac{d^2}{dz^2} \]

\[ + \left\{ \frac{2}{g} \Omega (1 - \Omega)z^2 + \left[ 8\omega^2q(1 - \Omega) + 8\omega^2 \left( q + \frac{1}{2} \right) (1 - \Omega)^2 \right] + 4\omega \left( E - 2\omega \left( q + \frac{1}{4} \right) \right) z + 32\omega gq \left( q + \frac{1}{2} \right) \Omega - q \right\} \frac{d}{dz} \]

\[ + \frac{\omega^2}{g} (1 - \Omega) \left( 2q\omega\Omega - \frac{1}{2} \omega - E \right) z + 4\omega^2q^2(1 - \Omega)^2 - \left[ E - 2\omega \left( q - \frac{1}{4} \right) \right]^2 . \] (54)

In arriving at the equation, a single-mode bosonic realization of \( su(1, 1) \) has been used. The latter provides the infinite-dimensional unitary irreducible representation of \( su(1, 1) \) known as the positive discrete series \( D^+(q) \), with the quadratic Casimir operator taking the value \( \mathcal{C} = 3/16 \) and the so-called Bargmann index \( q = 1/4, 3/4 \) [11, 12].

The Hamiltonian of the nonlinear two-mode quantum Rabi model reads [8, 11, 12]

\[ \hat{H}_{2m} = \omega 1(\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2) + \Delta \sigma_z + g \sigma_z (\alpha_1^\dagger \alpha_2^\dagger + \alpha_1 \alpha_2) , \]

where one assumes the boson modes to be degenerate with the same frequency \( \omega \). Note in passing that the model is different from a linear two-mode quantum Rabi model studied in [52]. With the substitution

\[ \psi_\pm(z) = e^{-\frac{i}{\Omega}(1-\Lambda)} \varphi_\pm(z), \quad \Lambda = \sqrt{1 - \frac{g^2}{\omega^2}}, \]

where \( |g/\omega| < 1 \) [16], and on eliminating \( \varphi_-(z) \) from the system, one obtains the 4th-order differential equation for \( \varphi_+(z) \),

\[ \hat{H}_{2m}\varphi_+(z) = -\Delta^2 \varphi_+(z) , \] (55)

where [12]

\[ \hat{H}_{2m} = g^2z^2 \frac{d^4}{dz^4} + 4g^2 \left[ \frac{\omega}{g} (\Lambda - 1)z^2 + \left( q + \frac{1}{2} \right) z \right] \frac{d^3}{dz^3} \]

\[ + \left\{ 4\omega^2(\Lambda^2 - 3\Lambda + 1)z^2 + 4\omega g \left[ 3 \left( q + \frac{1}{2} \right) \Lambda - 3q - 1 \right] z + 4g^2q \left( q + \frac{1}{2} \right) \right\} \frac{d^2}{dz^2} \]

\[ + \left\{ \frac{8\omega^2}{g} \Lambda (1 - \Lambda)z^2 + \left[ 8\omega^2q(1 - \Lambda) + 8\omega^2 \left( q + \frac{1}{2} \right) (1 - \Lambda)^2 \right] + 4\omega(\omega - E)z + 8\omega gq \left( q + \frac{1}{2} \right) \Lambda - q \right\} \frac{d}{dz} \]

\[ + \frac{4\omega^2}{g} (1 - \Lambda) \left( 2q\omega\Lambda - \omega - E \right) z + 4\omega^2q^2(1 - \Lambda)^2 - \left[ E - 2\omega \left( q - \frac{1}{4} \right) \right]^2 . \] (56)

In arriving at the equation, a two-mode bosonic realization of \( su(1, 1) \) has been employed that requires the quadratic Casimir to take the value \( \mathcal{C} = q(1 - q) \), where \( q > 0 \) stands for the Bargmann index, which can be any positive integers or half-integers, i.e. \( q = 1/2, 1, 3/2, \ldots \) [11, 12].

The point of crucial importance is that those 4th order linear differential operators \( \hat{H}_{2p} \) and \( \hat{H}_{2m} \) have rather special form. From the grading point of view, the 4th derivative term has
grade $-2$, whereas the 3rd derivative contributes two terms with grade $-2$ and $-1$, respectively. The same as for the (driven) Rabi model, the highest grade terms of each of $\hat{H}_{2p}$ and $\hat{H}_{2m}$ are the terms $\sim z^2d_2$ and $\sim z$, both having a positive grade of $+1$. Consequently, on using the same notation as in (2) and (3) for the respective 2nd order linear differential parts of the operators $\hat{H}_{2p}$ and $\hat{H}_{2m}$ we have the usual anomalous ODE characterized by the grading parameters as summarized by (4). The necessary condition for the existence of a polynomial solution of each of equations (53) and (55) remains identical to (15).

In the case of equation (53) the necessary condition (15) becomes

$$F_1(n) := \frac{\omega}{g} \left(1 - \Omega\right) \left(2q\omega\Omega - \frac{1}{2}\omega - E + 2n\omega\Omega\right) = 0,$$

which yields the following baseline constraint on energy,

$$E_n = -\frac{1}{2}\omega + 2(n + q)\omega\Omega, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (57)

Here it was assumed that $\Omega \neq 1$; the $\Omega = 1$ case is trivial corresponding to $g = 0$. Equation (54) yields

$$a_2 = 4\omega^2(\Omega^2 - 3\Omega + 1),$$

$$b_1(E_n) = 8\omega^2 \left(q + \frac{1}{2}\right) (\Omega^2 - 3\Omega + 1) + 4\omega \left[E + \omega\Omega - \frac{1}{2}\omega\right]$$

$$= 8\omega^2 \left(q + \frac{1}{2}\right) (\Omega^2 - 3\Omega + 1) + 4\omega [-\omega + (2n + 2q + 1)\omega\Omega]$$

$$= 8\omega^2 \left[\left(q + \frac{1}{2}\right) \Omega^2 + (n - 2q - 1)\Omega + q\right],$$

$$c_0(E_n) = 4\omega^2q^2 \left(1 - \Omega\right)^2 - \left[E - 2\omega \left(q - \frac{1}{4}\right)\right]^2 = -4\omega^2 \left[n^2\Omega^2 + 2nq\Omega(\Omega - 1)\right],$$ \hspace{1cm} (58)

where in arriving at the last two expressions we have substituted the baseline condition (57) for $E_n$. On the $n$th baseline the coefficients of the recurrence system (11) are

$$F_1(k) = -2(n - k)(1 - \Omega) \frac{\omega^3\Omega}{g},$$

$$F_0(k) = k(k - 1)a_2 + kb_1(E_n) + c_0(E_n),$$

$$F_{-1}(k) = 16k(k - 1)(k - 2)\omega g(\Omega - 1) + 16k(k - 1)\omega g \left[3 \left(q + \frac{1}{2}\right) \Omega - 3q - 1\right]$$

$$+ 32k\omega qg \left[\left(q + \frac{1}{2}\right) \Omega - q\right],$$

$$F_{-2}(k) = 16k(k - 1)(k - 2)(k - 3)g^2 + 64k(k - 1)g^2(k - 2 + q) \left(q + \frac{1}{2}\right).$$ \hspace{1cm} (59)

The constraint polynomial of equation (16) is determined by

$$F_0(0) = c_0(E_n), \quad F_{-1}(1) = 32\omega qg \left[\left(q + \frac{1}{2}\right) \Omega - q\right],$$

$$F_{-2}(2) = 128g^2q \left(q + \frac{1}{2}\right).$$
The necessary condition (26) for \( sl_2 \) algebraization reproduces the necessary condition (15) for the existence of a polynomial solution of \( n \)th degree if and only if the spin \( j \) of an irreducible \( sl_2 \) representation satisfies \( 2j = n \). Thereby the energies of the 2-photon Rabi model are constraint by the baseline condition (57). The \( sl_2 \) algebraization conditions are satisfied by
\[
C_{00} = a_2 = 4\omega^2(\Omega^2 - 3\Omega + 1), \\
C_0 = b_1 + (2j - 1)a_2 = 8\omega^2(q + j)(\Omega^2 - 3\Omega + 1) + 4\omega^2[(2n + 2q + 1)\Omega - 1].
\]
The constant \( C \) is then easily determined on substituting the above two expressions, together with \( c_0 \) from equation (58), into the last condition of (27).

Similarly, in the case of equation (55) the necessary condition (15) imposes
\[
F_i(n) := \frac{4\omega^2}{g} (1 - \Lambda) (2q\omega\Lambda - \omega - E + 2\omega\Lambda) = 0,
\]
which yields the following baseline constraint on energy,
\[
E_n = -\omega + 2(n + q)\omega\Lambda.
\] (60)
Here \( \Lambda \neq 1 \), as the \( \Lambda = 1 \) case is trivial corresponding to \( g = 0 \).

Equation (56) yields
\[
a_2 = 4\omega^2(\Lambda^2 - 3\Lambda + 1), \\
b_1(E_n) = 8\omega^2 \left(q + \frac{1}{2}\right) (\Lambda^2 - 3\Lambda + 1) + 4\omega (E + \omega\Lambda), \\
= 8\omega^2 \left(q + \frac{1}{2}\right) (\Lambda^2 - 3\Lambda + 1) + 4\omega \left[ -\omega + (2n + 2q + 1)\omega\Lambda \right] \\
= 8\omega^2 \left(q + \frac{1}{2}\right) \Lambda^2 + (n - 2q - 1)\Lambda + q,
\]
\[
e_0(E_n) = 4\omega^2 q^2 (1 - \Lambda)^2 - \left[ E - 2\omega \left(q + \frac{1}{2}\right) \right]^2 = -4\omega^2 \left[ n^2\Lambda^2 + 2nq\Lambda(\Lambda - 1) \right], \] (61)
where in arriving at the last two expressions we have substituted the baseline condition (60) for \( E_n \). On the \( n \)th baseline the coefficients of the recurrence system (11) are
\[
F_i(k) = -8(n - k)(1 - \Lambda) \frac{\omega^3\Lambda}{g}, \\
F_0(k) = k(k - 1)a_2 + kb_1(E_n) + c_0(E_n), \\
F_{-1}(k) = 4k(k - 1)(k - 2)\omega g(\Lambda - 1) + 4k(k - 1)\omega g \left[ 3 \left(q + \frac{1}{2}\right) \Lambda - 3q - 1 \right] \\
+ 8k\omega g \left(q + \frac{1}{2}\right) \Lambda - q, \\
F_{-2}(k) = k(k - 1)(k - 2)(k - 3)g^2 + 4k(k - 1)g^2(k - 2 + q) \left(q + \frac{1}{2}\right). \] (62)
The constraint polynomial of equation (16) is determined by
\[ F_0(0) = c_0(E_n), \quad F_{-1}(1) = 8\omega g q \left[ \left( q + \frac{1}{2} \right) \Lambda - q \right], \]
\[ F_{-2}(2) = 8g^2 q \left( q + \frac{1}{2} \right). \]

As before, the necessary condition (26) for \( sl_2 \) algebraization reproduces the necessary condition (15) provided that \( 2j = n \). Thereby the energies of the two-mode Rabi model are constrained by the baseline condition (60). The \( sl_2 \) algebraization conditions are satisfied by
\[ C_{00} = a_2 = 4\omega^2(\Lambda^2 - 3\Lambda + 1), \]
\[ C_0 = b_1 + (2j - 1)a_2 = 8\omega^2 (q + j) (\Lambda^2 - 3\Lambda + 1) + 4\omega^2 [(2n + 2q + 1)\Lambda - 1]. \]

The constant \( C \) is then easily determined on substituting the above two expressions, together with \( c_0 \) from equation (61), into the last condition of (27). In virtue of (25), \( T_2 \in U(sl_2) \) requires that \( C_+ = b_2 \) in equation (24). A different value of \( b_2 \) in equations (54) and (56) thus automatically implies a different \( \hat{H}_2p \) or \( \hat{H}_2m \in U(sl_2) \).

7. The generalized Rabi model

For the sake of comparison with the results by Tomka et al [6] their notation is largely adopted here. The generalized, or also known as an asymmetric, Rabi model described by the Hamiltonian
\[ \hat{H}_{gr} = \omega a^\dagger a + \Delta \sigma_z + g_1 (a^\dagger \sigma_- + a \sigma_+) + g_2 (a^\dagger \sigma_+ + a \sigma_-) \]
interpolates between the Jaynes–Cummings (JC) model [53] (for \( g_2 = 0 \)) and the original Rabi model (\( g_1 = g_2 \)). Numerous different motivations to consider this model have been summarized in [6, 8]. The substitution
\[ \psi_1(z) = e^{-\kappa z} S(z), \quad \kappa \equiv \sqrt{g_1 g_2 \omega}, \]
leads to the following 2nd order ordinary differential equation [6]
\[ \left[ \frac{d^2}{dz^2} + \left( \sum_{s=1}^{3} \frac{\nu_s}{z - \rho_s} - 2\kappa \right) \frac{d}{dz} + \frac{C_2(z)}{\prod_{s=1}^{3}(z - \rho_s)} \right] S(z) = 0, \quad (63) \]
where \( C_2(z) = \sum_{l=0}^{2} c_l z^l \) is a polynomial of degree 2. Obviously, the differential equation (63) has grade \( \gamma = 2 \). The constants in equation (63) are
\[ \nu_1 = -\epsilon + 1, \quad \nu_2 = -\epsilon, \quad \nu_3 = -1, \]
where
\[ \mu \equiv \frac{\Delta}{\omega}, \quad \lambda_+ \equiv \frac{g_1^2 + g_2^2}{2\omega^2}, \quad e \equiv \frac{E}{\omega}, \quad \epsilon \equiv e + \lambda_+ \quad (64) \]
are dimensionless quantities,
\[ \rho_1 = \kappa, \quad \rho_2 = -\kappa, \quad \rho_3 = \nu, \]
are the zeros of \( A_3(z) \), and
\[ \nu = \frac{2\Delta \sqrt{g_1 g_2}}{g_1^2 - g_2^2} = \frac{\mu}{\lambda_+ \kappa} \quad (65) \]
The coefficients of $C_2(z)$ are given by [6]
\[
\begin{align*}
    c_0 &= \kappa \left( \delta^2 - \epsilon^2 + 2\epsilon \lambda_+ - \lambda_+^2 + \lambda_+ + \kappa^2 + \kappa^4 \right) \\
    &\quad + \kappa \left( \epsilon - \lambda_+ - \kappa^2 \right), \\
    c_1 &= e(\epsilon + 1) - \delta^2 + \delta \lambda_+ + \nu \kappa - \kappa^2 - 2\nu \epsilon - \kappa^4, \\
    c_2 &= 2\kappa \epsilon.
\end{align*}
\]

Given that $\sum_j r_1 = \nu$, $\sum_{j < l} r_1 r_2 = -\kappa^2$, and $\rho_1 \rho_2 \rho_3 = -\nu \kappa^2$, the differential equation (63) can be recast as
\[
A_3(z)S'' + B_3(z)S' + C_2(z)S = 0,
\]
\[
A_3(z) = (z^2 - \kappa^2)(z - \nu) = z^3 - \nu z^2 - \kappa^2 z + \nu \kappa^2,
\]
\[
B_3(z) = -2\kappa A_3(z) - 2\epsilon z^2 + (2\epsilon \nu + \kappa - \nu)z - \kappa(\nu - \kappa),
\]
where, in general, $\deg A_3 = \deg B_3 = 3$, $\deg C_2 = 2$, i.e. the usual anomalous alternative (A2). The necessary condition (8) for the existence of a polynomial solution $S_0(z)$ of degree $n$ reduces to $F_2(n) = nb_3 + c_2 = 0$, which for $b_3 = -2\kappa$ and $c_2 = 2\kappa \epsilon$ yields the baseline condition
\[
\epsilon = n.
\]

On the $n$th baseline
\[
\begin{align*}
    F_2 &= -2\kappa z^3 dz + 2\kappa \nu \kappa^2, \\
    F_1 &= z^2 d^2 z + (2\kappa \nu - n)z^2 dz + c_1(n)z, \\
    F_0 &= -\nu z^2 d^2 z + (2\kappa^3 + 2\nu \kappa)z dz + c_0(n), \\
    F_{-1} &= -\nu z^2 d^2 z - [\nu(\nu - \kappa) + 2\nu \kappa^2] dz, \\
    F_{-2} &= \nu \kappa^2 d^2 z,
\end{align*}
\]
where the argument of $c_0(n)$ and $c_1(n)$ indicates that the baseline condition (67) has been imposed. The coefficients of the recurrence system (11) are
\[
\begin{align*}
    F_2(l) &= 2(n - l)\kappa, \\
    F_1(l) &= l(l - 1 - 2n + 2\kappa \nu) + c_1(n), \\
    F_0(l) &= l[2\kappa^3 + (2n - l)\nu + \nu] + c_0(n), \\
    F_{-1}(l) &= -l[\nu(\nu - \kappa) + (l - 1)\kappa^2 + 2\nu \kappa^3], \\
    F_{-2}(l) &= l(l - 1)\nu \kappa^2.
\end{align*}
\]

The first constraint polynomial $P_1$ in (14), which takes on the form (16), is determined by
\[
F_0(0) = c_0(n), \quad F_{-1}(1) = -\kappa(\nu - \kappa) - 2\nu \kappa^3, \quad F_{-2}(2) = 2\nu \kappa^2.
\]

The second constraint polynomial $P_2$ of (14) is determined by
\[
P_2(n) := a_{n3} F_{-2}(3) + a_{n2} F_{-1}(2) + a_{n1} F_0(1) + a_{n0} F_1(0) = 0,
\]
where
\[
\begin{align*}
    F_1(0) &= c_1(n), \\
    F_0(1) &= 2\kappa^3 + 2\nu \kappa + \nu - c_0(n), \\
    F_{-1}(2) &= -2\kappa^2 - 2[\nu(\nu - \kappa) + 2\nu \kappa^3], \\
    F_{-2}(3) &= 6\nu \kappa^2.
\end{align*}
\]
In order to satisfy the constraints (14), one has to determine common zeros of the polynomials $P_1$, $P_2$. The latter task amounts to determining zeros of the resultants of the two polynomials, $R(P_1, P_2)$, also known as the $\text{eliminant}$ [41]. A resultant is an important joint invariant of two polynomials, which is expressed as a polynomials function of the coefficients of two
polynomials. Thereby the task of finding polynomial solutions of the generalized Rabi model is reduced again to an algebraic equation.

One determines easily that equation (66) cannot be viewed as a special case of normally ordered bilinear combination $T_2$ given by equation (24) of $sl_2$ generators (A.1). In order that $\deg P_4 = 3$ in equation (A.5), one has to set $C_{++} \equiv 0$. Yet with $C_{++} \equiv 0$ one can never have $\deg P_3 = 3$ in equation (A.5). Thus any polynomial solution of equations (63) and (66) does not belong to the $sl_2$ algebraic sector and is, by definition, exceptional [54, 55].

A $sl_2$ algebraic sector reappears only in the special case of $\kappa^2 = \nu^2$. The latter is possible if $\mu = \pm \nu - \kappa$ (see equation (65)). For instance on taking $\nu = -\kappa$, the three roots $\rho_{1,2,3}$ degenerate into two (namely to $\pm \kappa$), the polynomial coefficients in equation (66) factorize as $A_3(z) = (z + \kappa)^2(z - \kappa), C_2(z) = (z + \kappa)(2\kappa z + d_0/\kappa)$, where [6]

$$
\frac{d_0}{\kappa} = \epsilon^2 + \lambda_+ (\lambda_+ - 2) - \mu^2 - \epsilon(2\lambda_+ - 1) - 2\kappa^2 - \kappa^4.
$$

Because also [6]

$$
B_3(z) = (z + \kappa) \left[ -2\kappa z^2 - 2\epsilon z + 2\kappa(\kappa^2 + 1) \right],
$$

one can factorize out the monomial $(z + \kappa)$ from equations (63) and (66). Thereby the degree of the polynomial coefficients in equation (66) is reduced by one, so that $\deg A_3 = \deg B_3 = 2$, $\deg C_2 = 1$. We arrive back at the usual anomalous alternative (A2) with grade $\gamma = 1$, where the single constraint polynomial $P_1$ on the $n$th baseline in (16) is obtained with the recurrence coefficients (see equation (68))

$$
F_1(l) = 2(n - l)\kappa, \quad F_0(l) = l(l - 1 - 2n) + (d_0/\kappa),
$$

$$
F_{-1}(l) = 2l(n\kappa^2 + 1), \quad F_{-2}(l) = -l(l - 1)\kappa^2.
$$

The necessary and sufficient condition (26) for $sl_2$ algebraization with spin $j$ reproduces the necessary condition (67) for the existence of a polynomial solution of $n$th degree provided that $2j = \epsilon = n$. The remaining $sl_2$ algebraization conditions are satisfied by $C_{00} = a_2 = 1, C_0 = -2\epsilon + 2j - 1$, together with the relation (27) for $C_\nu$. The coefficient $C_\gamma$ of $T_2 \in U(sl_2)$ in equation (24) has to satisfy (25). Because we have $b_2 = -2\kappa$ in equation (70), a different value of $\kappa$ automatically implies a different $T_2 \in U(sl_2)$.

8. Discussion

The problems defined by equations (37), (45), (47), (D.1) and (D.2):

- are not a standard Heine–Stieltjes problem, because of $\deg P_4 = \deg P_3 = 2$, and $\deg P_2 = 1$, i.e. the degrees of $P_n$ are not strictly decreasing with $n$ and $\deg P_3 \neq \deg P_4$
- are not described by a Fuchsian equation
- are not a standard eigenvalue problem, because $L$ contains terms $\sim E_z d_z, E_z, E^2$
- do not lead to a Sturm–Liouville problem in an equivalent Schrödinger equation form, because then they are described by a nontrivially energy dependent potential.

In the case of a relative motion of two electrons in an external oscillator potential [36, 37] one had an ordinary eigenvalue problem $L\psi = \lambda\psi$ (see equation (5) of [36]; equation (31) of [37]). When the relevant 2nd order ODE of [36, 37] was recast into a Schrödinger equation, such an equation also did not reduce to a standard Sturm–Liouville eigenvalue problem. A solution of the problem involved a so-called coupling constant metamorphosis between energy parameter and other model parameters to have an algebraization. However, in the present case
of the Rabi model the situation is different in that we do not have an ordinary eigenvalue problem \( L \psi = \lambda \psi \). Rather the problem reduces to the determination of zero modes of \( L \). A \( sl_2 \) algebraization amounts merely to the necessary condition for the existence of polynomial solutions.

Original result of Zhang (see equations (1.8)–(10) of [42]) was limited to the 2nd order ODE’s which precluded its application to nonlinear Rabi model generalizations described by the 4th order ODE’s of section 6. In theorem 4 of section 3 we have succeeded to generalize the important result of Zhang to to the case of arbitrary \( \gamma \geq 0 \), which also covers the nonlinear Rabi model generalizations of section 6. With their simple Ansätze, which are close in spirit to that of Kus [10], Emary and Bishop [17] were not able to find polynomial solutions of two-photon Rabi Hamiltonian \( \hat{H}_2 \) that occur at the level-crossings between energy eigenstates that have different Bargmann indices. In the latter case one eigenstate is composed of odd number states, whilst the other is composed only of even number states. Consequently, no superposition of these states could lead to a reduction in the complexity of either wave function and one is unable to find polynomial solutions at these level-crossings. Using our constraint relations it is possible to give a definite answer if there are polynomial solutions at those level crossings and also investigate the case of two-mode Rabi Hamiltonian \( \hat{H}_{2m} \). This will be dealt with elsewhere.

In section 7 is has been shown that the task of finding polynomial solutions of the generalized Rabi model amounts to solving algebraic equation \( R(P_1, P_2) = 0 \), where \( R(P_1, P_2) \) is the resultant of the two constraint polynomials \( P_1, P_2 \). The conditions determining the locations of the exceptional solutions in parameter space were given in [6] through the Bethe Ansatz equations, which turned out to be the same as those of the reduced BCS (Richardson) model having three degenerate levels of energies \( \epsilon_{1,2,3} \) with degeneracies \( \nu_{1,2,3} \) respectively [6]. Similarly to the Kus recipe [10], the conditions of solvability of the Bethe Ansatz equations on a \( n \)th baseline reduced to looking for zeros of a certain \( n \)th order polynomial (see appendix C of [6]). It would be interesting to explore relation between our algebraic equation \( R(P_1, P_2) = 0 \) and that of [6].

Recent decade has witnessed a rapid development and understanding of QES which goes beyond the initial paradigm of the \( sl_2 \) algebraization. There are many other finite dimensional polynomial spaces which are not irreducible modules for the \( sl_2 \) action, and in these cases there might be non-Lie algebraic differential operators which leave the space invariant [54, 55]. In a direct approach to quasi-exact solvability [55] more general polynomial spaces are considered and the set of differential operators that preserve them are investigated without any reference to Lie algebras. Models were found possessing multiple algebraic sectors, which simultaneously allow for both \( sl_2 \) and exceptional polynomial sectors [57]. Every exceptional orthogonal polynomial system was proven to be related to a classical system by a Darboux–Crum transformation [35, 56, 58]. The existence of polynomial solutions in the absence of any apparent Lie algebra symmetry has been demonstrated for the generalized Rabi model by Tomka et al [6]. Hence, although the differential equation (63) is more general than the one corresponding to the usual Rabi model with \( g_1 = g_2 = g \), it still has a polynomial solution \( S_n(z) \) of degree \( n \) [6].

In a general case, the constraint relation(s) (14) need not to be a polynomial in model parameters. Therefore, there could, in principle, exist polynomial solutions which eigenenergy is not governed by an algebraic equation.

9. Conclusions

The idea of gradation slicing of ordinary differential equations with polynomial coefficients has been demonstrated to provide an efficient tool for the analysis of polynomial solutions of
such equations. The necessary condition for a polynomial solution of \( n \)th degree to exist forces energy to a \( n \)th baseline. Once the constraint relations (14) on the \( n \)th baseline can be solved, a polynomial solution is in principle possible even in the absence of any underlying algebraic structure. Theorem 3 can be viewed as a recipe for recursively generating constraint polynomials by a recurrence (11) and (14) with well defined coefficients for any \( \gamma > 0 \) problem. We have succeeded in theorem 4 to generalize the main result of Zhang for the functional Bethe Ansatz (see equations (1.8)–(10) of [42]) to the case of arbitrary \( \gamma \geq 0 \).

The theory was illustrated on the examples of various Rabi models. For those models, a baseline is known as a Juddian baseline (e.g. in the case of the Rabi model the curve described by the \( n \)th energy level of a displaced harmonic oscillator with varying coupling \( g \)). The corresponding constraint relations were shown to (i) reproduce known constraint polynomials for the usual and driven Rabi models and (ii) generate hitherto unknown constraint polynomials for the two-mode, two-photon, and generalized Rabi models, implying that the eigenvalues of corresponding polynomial eigenfunctions can be determined algebraically. Interestingly, the ODE of the above Rabi models were shown to be characterized, at least for some parameter range, by the same unique set of grading parameters. We have not analyzed here a linear two-mode quantum Rabi model [52], intensity-dependent quantum Rabi models [59, 60], and a two-mode multi-photon intensity-dependent Rabi model of [61], yet one expects similar conclusions to apply. Although our main motivation came from anomalous problems (the alternative A2), the concept can be applied universally.

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Appendix A. \( sl_2 \) constraints

A projectivized differential operator realization of the \( 2j + 1 \) dimensional representation of the \( sl_2(\mathbb{R}) \) algebra of spin \( j \) is provided by the operators [19–22, 25, 26]

\[
J_+ = z^2 \partial_z - 2j z, \quad J_0 = z \partial_z - j, \quad J_- = \partial_z, \\
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0,
\]

(A.1)

where \( 2j = n \geq 0 \) is an integer parameter. \( sl_2(\mathbb{R}) \) is a graded algebra with the grading of generators (A.1) [25, 26]:

\[
\text{deg}(J_+) = +1, \quad \text{deg}(J_0) = 0, \quad \text{deg}(J_-) = -1.
\]

(A.2)

Hence

\[
\text{deg}(J_0^n J_+^m J_-^p) = n_+ - n_-.
\]

(A.3)

On using commutation relations, any product of the \( sl_2 \) generators can be brought into a normally ordered form, where the generators in the product are ordered in decreasing grade from the left to the right, plus a linear combination of the products with a lesser number of generators. The corresponding normally ordered quadratic quasi-exactly-solvable differential operator can be represented as in equation (24). In the normally ordered bilinear combination (24) one can always assume that \( C_{+-} \equiv 0 \). In a Lie algebra, any product \( XY \) can be recast as \( XY = \frac{1}{2} [X,Y] + \frac{1}{2} \{X,Y\} \), where the antisymmetric (commutator) part is equivalent
to a *redefinition* of the coefficients $C_\nu$. If $C_{++} \neq 0$, one can, on making use of the commutation relations, form a symmetric quadratic term, and then apply the Casimir identity, $J_3^2 - \frac{1}{2} \{J_+, J_-\} = j(j+1)$. The antisymmetric part is always equivalent to a *redefinition* of the coefficients $C_\nu$. Hence the number of free parameters is $\text{par} \, (T_2) = 8$ (or $\text{par} \, (T_2) = 7$ if the numerical constant term $C_\alpha$ is ignored).

The corresponding *normally ordered* $T_2$ in equation (24) can be recast as (see theorem 2 of [20])

$$T_2 = P_4(z) \frac{d^2}{dx^2} + P_3(z) \frac{d}{dx} + P_2(z), \quad (A.4)$$

where the polynomials $P_l(z)$ of the $\leqslant l$th degree are given by

$$P_4(z) = C_{++} z^4 + C_{+0} z^3 + C_{00} z^2 + C_{0-} z + C_{--},$$

$$P_3(z) = -2(2j - 1)C_{++} z^3 + [- (3j - 1)C_{+0} + C_+] z^2$$

$$- [(2j - 1)C_{00} - C_0] z - jC_{0-} + C_-, \quad (A.5)$$

Note in passing that the respective polynomials $P_l(z)$ (i) do not necessarily have different degree (e.g. if $C_{++} = C_{+0} = 0$), (ii) nor is their degree necessarily equal to $l$ (e.g. if $C_{++} = 0$). One has in general only that (iii) $\deg P_l \leqslant l$ and (iv) $\deg P_l \leqslant \deg P_m$ for $l < m$.

If one compares the polynomials $P_l(z)$ in equation (A.5) with the polynomials $A(z), B(z), C(z)$ in the ODE (2) and (3), then the $sl_2(\mathbb{R})$ realization (A.5) of equation (2) implies

$$a_4 = C_{++} = -\frac{1}{2} (2j - 1) b_1, \quad a_3 = C_{+0}, \quad a_2 = C_{00}, \quad a_1 = C_{0-}, \quad a_0 = C_{--}, \quad (A.6)$$

$$b_3 = -2(2j - 1)C_{++} = -2(2j - 1) a_4, \quad b_2 = -(3j - 1)C_{+0} + C_+, \quad b_1 = -[(2j - 1)C_{00} - C_0], \quad b_0 = -jC_{0-} + C_. \quad (A.7)$$

### Appendix B. Equations with all solutions being polynomials

For a second order differential equation in general form,

$$A(z)w''(z) + B(z)w'(z) + C(z)w(z) = 0, \quad (B.1)$$

the Wronski determinant of two solutions is either nonzero or zero everywhere (see chapter 5.2 of Hille [40]). For two linearly independent (not necessarily *polynomial*) solutions of (B.1)

$$W(u_1, u_2) = u_1(z)u_2'(z) - u_1'(z)u_2(z) \neq 0 \quad (B.2)$$

implies that:

(i) the respective solutions *cannot* have a common zero

(ii) if $u_l(z_0) = 0$ for some $l = 1, 2$ and $z = z_0$, then $u_l'(z_0) \neq 0$, and vice versa.

Otherwise the Wronski determinant (B.2) would be zero. If one of the solutions is a polynomial, the latter condition (ii) prohibits it from having a multiple zero.

The coefficients of (B.1) are proportional to the minors of the first row of the determinant,
\[
\begin{vmatrix}
  w'' & w' & w \\
  w_1'' & w_1' & w_1 \\
  w_2'' & w_2' & w_2 \\
\end{vmatrix} = 0,
\]
and can be determined as \([40, 62]\)
\[
A : B : C = \begin{vmatrix}
  w_1' & w_1 \\
  w_2' & w_2 \\
\end{vmatrix} : - \begin{vmatrix}
  w_1'' & w_1' \\
  w_2'' & w_2' \\
\end{vmatrix} : \begin{vmatrix}
  w_1'' & w_1' \\
  w_2'' & w_2' \\
\end{vmatrix}.
\]

An obvious consequence of the above relation is that the coefficients of (B.1) having two polynomial solutions \(w_i\) have to be polynomials. The other consequence is that
\[
\text{deg } B = \text{deg } A - 1.
\]

The consequence can be shown to hold for a linear ODE of any order. Its proof requires only an elementary formula for the derivative of a corresponding Wronskian.

Given a polynomial
\[
A(z) = \prod_{l=1}^{n} (z - z_l)
\]
isometric zeros, an equation of the form
\[
Ay'' - A' y' + Cy = 0
\]
has a non-trivial polynomial solution \(y\) if and only if
\[
C = \frac{-(Ay'' + A'y')}{y},
\]
with \(y\) given by
\[
y(z) = (z - z_1) \ldots (z - z_m), \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_m\},
\]
where \((z_1, \ldots, z_m)\) are satisfying
\[
2 \sum_{i \neq k} \frac{1}{z_k - z_i} - \sum_{i=1}^{n} \frac{1}{z_i - z_m} = 0.
\]

**Lemma ([63, Lemma 7]).** If equation (B.4) has one non-trivial polynomial solution, then all its solutions are polynomials (see Heine–Stieltjes problem \([29–35]\)).

**Appendix C. Singular points at spatial infinity**

With \(\xi = 1/z\) one has
\[
\begin{align*}
\frac{d}{dz} &= \frac{d}{d\xi} = -\frac{1}{\xi^2} \frac{d}{d\xi}, \\
\frac{d^2}{dz^2} &= 2 \frac{d}{d\xi} + \frac{1}{\xi^2} \frac{d^2}{d\xi^2} = 2\xi^3 d\xi + \xi^4 d^2\xi. 
\end{align*}
\]

Thereby equation (37) is transformed into
\[
(\omega - g\xi)(\omega + g\xi) \frac{d^2 \phi_+}{d\xi^2} + \left[ \frac{2(\omega - g\xi)(\omega + g\xi) - b_1}{\xi} - b_0 - b_2\xi^{-2} \right] \frac{d\phi_+}{d\xi} + \left[ c_1\xi^{-3} + c_2\xi^{-2} \right] \phi_+ = 0.
\]

Because of the singularities \(b_2\xi^{-2}\) and \(c_1\xi^{-3}\), equation (37) has an irregular singular point at \(z = \infty\). Analogous conclusion holds for equations (D.1) and (D.2), and for the respective 2nd order linear differential operators \(\hat{H}_R\) and \(\hat{\varphi}_R\) defined by equations (45) and (47).
Appendix D. Alternative 2nd order ODE

The alternative 2nd order ODE (3.23) of Schweber [2],

\[
\begin{aligned}
\zeta(z - \kappa) \frac{d^2 \phi}{dz^2} + \left[ -\kappa z^2 + \left( \kappa^2 - \frac{2\epsilon}{\omega} + 1 \right) z + \left( \frac{\kappa \epsilon}{\omega} - \kappa \right) \right] \frac{d\phi}{dz} \\
+ \left\{ \frac{\kappa \epsilon}{\omega} z + \left[ \left( \frac{\epsilon}{\omega} \right)^2 - \left( \frac{\Delta}{\omega} \right)^2 - \frac{\kappa^2 \epsilon}{\omega} \right] \right\} \phi = 0,
\end{aligned}
\]  
\[(D.1)\]

where \( \kappa = 2g/\omega \) and energy \( \epsilon \) in (D.1) corresponds to \( E + \omega \kappa^2 / 4 \). The grading characteristics of the 2nd order ODE are again the same as announced earlier in equation (4), and the equation corresponds to the anomalous alternative (A2) with \( \deg A = \deg B = 2, \deg C = 1 \), characterized by

\[
b_2 = -\kappa, \quad c_1 = \frac{\kappa \epsilon}{\omega}.
\]

Hence for equation (D.1) the very same \( sl_2 \) algebraization condition (26) applies, which for \( n = 2j \) coincides with the necessary condition (15) for the existence of a polynomial solution of equation (D.1). The condition (15) yields again the familiar baseline condition for the Rabi model,

\[
\frac{\kappa \epsilon}{\omega} = n\kappa \iff \epsilon = n\omega \iff E = n\omega - \frac{g^2}{\omega}.
\]

We have on the \( n \)th baseline \( a_2 = 1 \) and

\[
b_1(\epsilon_n) = \kappa^2 - 2n + 1, \quad c_0(\epsilon_n) = n^2 - \mu^2 - n\kappa^2.
\]

Equation (18) in the case of \( \gamma = 1 \) then yields

\[
\mu^2 + \kappa \sum_{i=1}^{n} z_i = 0,
\]

i.e. an \( n \)-independent constraint on \( \sum_{i=1}^{n} z_i \) (see equation (42)).

Let us now examine the second order differential equation (5) of [18],

\[
\begin{aligned}
\zeta(1 - z) \frac{d^2 \Re(z)}{dz^2} + \left[ 4\kappa^2 z^2 - (2\kappa^2 - 2\epsilon + 1) z + 1 - \epsilon - \kappa^2 \right] \frac{d\Re(z)}{dz} \\
+ \left[ -4\kappa^2 (\epsilon + \kappa^2) z + 3\kappa^4 + 2\kappa^2 \epsilon^2 - \epsilon^2 + \mu^2 \right] \Re(z) = 0,
\end{aligned}
\]  
\[(D.2)\]

where \( \kappa = g/\omega \). Equation (D.2) corresponds to a different choice of the independent variable in equation (37) and the dimensionless energy \( \epsilon \) in (D.2) corresponds to \( E/\omega \) in (37).

The grading characteristics of the 2nd order ODE are again the same as announced earlier in equation (4), and equation (D.2) corresponds again to the anomalous alternative (A2) with \( \deg A = \deg B = 2, \deg C = 1 \), characterized by

\[
b_2 = 4\kappa^2, \quad b_1 = 2\epsilon - 2\kappa^2 - 1, \\
c_1 = -4\kappa^4 - 4\kappa^2 \epsilon^2 = -4\kappa^2 (\epsilon + \kappa^2), \\
c_0 = 3\kappa^4 + 2\kappa^2 \epsilon^2 - \epsilon^2 + \mu^2.
\]  
\[(D.3)\]
The necessary and sufficient condition (26) for $sl_2$ algebraization with spin $j$, or the necessary condition (15) for the existence of a polynomial solution of $n = 2j$th degree, yields (see equation (12) of [18])

$$-4\kappa^2(\epsilon + \kappa^2) = -4n\kappa^2 \text{ or } \epsilon_n = n - \kappa^2, \quad n \geq 0.$$ 

On recalling the rescaling of energy, this is again the familiar baseline condition for the Rabi model.

We have on the $n$th baseline

$$c_0(\epsilon_n) = 4n\kappa^2 - n^2 + \mu^2, \quad b_1(\epsilon_n) = 2n - 4\kappa^2 - 1.$$ 

Equation (18) in the case of $\gamma = 1$ then, given $a_2 = -1$, yields

$$4n\kappa^2 - n^2 + \mu^2 = -4\kappa^2 \sum_{i=1}^{n} z_i + n(n - 1) - n(2n - 4\kappa^2 - 1),$$

i.e. again an $n$-independent constraint on $\sum_{i=1}^{n} z_i$ (see equation (42)),

$$\mu^2 + 4\kappa^2 \sum_{i=1}^{n} z_i = 0.$$ 

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