NNLO compatibility between pQCD theory and phenomenology in
determination of the $b$-quark pole and $\overline{\text{MS}}$ running masses

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Abstract

Measurements of open $b$-quark production in deep inelastic scattering (DIS) of $e^\pm p$ at HERA provide an important test of pQCD theory within the Standard Model and is used to constrain the proton PDFs. In this contribution, we attempt to determine the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$, using phenomenology of H1, ZEUS and $(\text{H1} + \text{ZEUS})$ $F_2^{b\bar{b}}$ beauty vertex production data sets and pQCD theory. Then we discuss about the compatibility between pQCD theory results and phenomenology approach in determination of the $b$-quark pole and $\overline{\text{MS}}$ running masses at the NNLO corrections. Also we investigate the role and influence of the $b$-quark mass as an extra degree of freedom added to the input parameters of the Standard Model Lagrangian, in the improvement of the uncertainty band of the gluon distribution. We show the compatibility between pQCD theory results and phenomenology approach in determination of the $b$-quark pole and $\overline{\text{MS}}$ running masses at the NNLO corrections and with precision of 1 part in $10^2$ are up to approximately: 99.78 %, 99.89 % and 99.98 % corresponding to H1, ZEUS and (H1 + ZEUS) $F_2^{b\bar{b}}$ beauty vertex data, respectively which show an excellent agreement between our phenomenology approach of experimental data with pQCD theory predictions.

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I. INTRODUCTION

Phenomenologically, the proton parton distribution functions as non-calculable part of the factorization theorem are classically extracted by QCD fitting of a parameterized standard functional form with experimental data from deep inelastic $e^\pm p$ scattering at HERA. The experimental data extracted from DIS of $e^\pm p$ play central role in probing of the internal structure of the proton. The inclusive neutral current (NC) and charged current (CC) cross sections at HERA contain contributions from all active quark and antiquark flavors and it should be noted that up to 45% of these contributions are originated from events with charm and beauty quarks in the final states [1–14].

Measurements of open $b$-quark production in DIS of $e^\pm p$ at HERA provide an important test of pQCD theory within the Standard Model and is used to constrain the proton PDFs. On the other hand, the $b$-quark mass as an extra degree of freedom added to the input parameters of the Standard Model Lagrangian play central role in high energy physics phenomenology [15–24]. The $b$-quark mass as an important pQCD parameter is significantly larger than the QCD scale parameter $\Lambda_{QCD} \sim 250$ MeV and accordingly it is known as a heavy-quark which is now kinematically accessible at HERA. At the leading order (LO) or 0-loop corrections, the dominant processes for the $b$-quark production at HERA are generally known as the boson-gluon fusion (BGF) reactions $\gamma g \rightarrow b\bar{b}$ which are strongly sensitive to the gluon content of the proton [25–36].

Within the pQCD framework, the ratio of photon couplings corresponding to a heavy-quark is given by $f(h) \sim \frac{Q_h^2}{\Sigma Q_q^2}$ where $Q_h^2$ denotes the electric charge squared of a heavy-quark and $Q_q^2 (q = u, d, s, c, b)$ refers to the electric charge squared of the kinematically accessible quark flavors at HERA. Accordingly for the $b$-quark we may write: $f(b) \sim \frac{Q_b^2}{\Sigma Q_q^2} = \frac{1}{3}/\frac{11}{9} = \frac{1}{11}$ or $f(b) \sim 0.09$, which clearly shows that up to 9% of the HERA inclusive cross sections are originated from processes with the $b$-quark in the final states. Therefore investigation of the role and influence of the $b$-quark experimental data on the proton PDFs and determination of the $b$-quark pole mass $M_b$ and $\overline{MS}$ running mass $\overline{m}_b$ as an extra degree of freedom in the input parameters of the Standard Model Lagrangian play central role in many of pQCD analysis [37–44].

It should be noted that in the pQCD framework the $b$-quark mass has been generated via a spontaneous symmetry breaking mechanism which in turn caused by a non-zero vacuum
expectation value of a Higgs boson field. However, the $b$-quark mass remains as an extra free parameter of the the Standard Model Lagrangian and should be determined by comparing of phenomenology of experimental data with theoretical predictions of the pQCD theory. These are our main motivations to develop this article. This contribution attempts to determine the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$, using phenomenology of experimental data and pQCD theory predictions at the next-to-next-to-leading order. Then we discuss about the compatibility between pQCD theory results and our phenomenology approach in determination of the $b$-quark pole and $\overline{\text{MS}}$ running masses at the NNLO corrections. Also we investigate the role and influence of the $b$-quark mass as an extra pQCD parameter on the improvement of the uncertainty band of the gluon distribution.

This paper is organized as follows. In Sec. II we describe the neutral and charged currents of deep inelastic $e^\pm p$ scattering cross sections and introduce the inclusive cross section of the $b$-quark production, as well. We explain and discuss about the $b$-quark pole and $\overline{\text{MS}}$ running masses based on the perturbative quantum chromodynamics theory predictions in Sec. III In Sec. IV the systematic uncertainties and our QCD analysis set-up are discussed. We present our results in Sec. V and finally, we conclude with a summary and conclusion in Sec. VI.

II. THE INCLUSIVE CROSS SECTION OF B-QUARK PRODUCTION

The HERA particle accelerator at the Deutsches Elektronen-Synchrotron (DESY) as a large QCD laboratory study both neutral and charged currents of $e^\pm p$ collisions and its data cover a wide range of phase space in Bjorken $x$ scale and negative four-momentum squared of the virtual photon $Q^2$ [1].

The NC interactions cross sections have been published for $4.5 \cdot 10^{-4} \leq Q^2 \leq 5.0 \cdot 10^4 \text{GeV}^2$ and $6.0 \cdot 10^{-7} \leq x \leq 6.5 \cdot 10^{-3}$ at values of the inelasticity $5.0 \cdot 10^{-3} \leq y = \frac{Q^2}{s \times} \leq 9.5 \cdot 10^{-1}$. The reduced NC unpolarized deep inelastic $e^\pm p$ scattering cross sections at the centre-of-mass energies up to $\sqrt{s} \simeq 320 \text{GeV}$ after correction for Quantum Electro Dynamics (QED) radiative effects can be expressed in terms of the NC generalized structure functions $\widetilde{F}_2$, $x\widetilde{F}_3$.
and $F_L$ as follows [1]:

$$
\sigma_{r,NC}^{\pm} = \frac{d^2\sigma_{NC}^{e^\pm p}}{dx dQ^2} \frac{Q^4 x}{2\pi\alpha^2(1 + (1 - y)^2)}
= \tilde{F}_2 + \frac{(1 - (1 - y)^2)}{(1 + (1 - y)^2)^2} x \tilde{F}_3 - \frac{y^2}{(1 + (1 - y)^2)} \tilde{F}_L.
$$

(1)

Similarly the CC interactions cross sections have been published for $2.0 \cdot 10^2 \leq Q^2 \leq 5.0 \cdot 10^4 \text{ GeV}^2$ and $1.3 \cdot 10^{-2} \leq x \leq 4.0 \cdot 10^{-1}$ at values of the inelasticity $3.7 \cdot 10^{-4} \leq y = \frac{Q^2}{s \cdot x} \leq 7.6 \cdot 10^{-3}$. The reduced cross sections for inclusive unpolarized CC $e^\pm p$ scattering are defined in terms of CC structure functions $W^\pm_2$, $W^\pm_3$ and $W^\pm_L$ as follows [1]:

$$
\sigma_{r,CC}^{\pm} = \frac{2\pi x}{G_F^2} \left[ \frac{M_W^2 + Q^2}{M_W^2} \right]^2 \frac{d^2\sigma_{CC}^{e^\pm p}}{dx dQ^2}
= \frac{(1 + (1 - y)^2)}{2} W^\pm_2 + \frac{(1 - (1 - y)^2)}{2} x W^\pm_3 - \frac{y^2}{2} W^\pm_L.
$$

(2)

In Eqs. (1) and (2), the quantity $\alpha$ refers to the fine-structure constant which is defined at zero momentum transfer frame and $G_F$ refers to the Fermi constant which is related to the weak coupling constant $g$ and electromagnetic coupling constant $e$ by $G_F^2 = \frac{e^2}{4\sqrt{2} \sin^2 \theta_W M_W^2}$. More details about proton NC and CC generalized structure functions can be found in Ref. [45].

The NC measurements of deep inelastic $e^\pm p$ scattering at HERA for beauty contribution to the inclusive proton structure function $F_2^{b\bar{b}}$ have been studied by H1 vertex detector in the range of virtuality of the exchanged photon $5.0 \leq Q^2 \leq 2.0 \cdot 10^3 \text{ GeV}^2$ and Bjorken $x$ scale $2.0 \cdot 10^{-4} \leq x \leq 5.0 \cdot 10^{-2}$ based on a dataset with an integrated luminosity of $189 \text{ pb}^{-1}$ [26].

The production of the $b$-quark in deep inelastic $e^\pm p$ interactions at HERA has been studied with the ZEUS vertex detector in the range of virtuality of the exchanged photon $5.0 \leq Q^2 \leq 10^3 \text{ GeV}^2$ and the inelasticity $2.0 \cdot 10^{-2} \cdot 10^{-4} \leq y \leq 7.0 \cdot 10^{-2}$ based on a dataset with an integrated luminosity of $354 \text{ pb}^{-1}$ [25].

In analogy to the inclusive NC and CC deep inelastic $e^\pm p$ scattering cross sections, the reduced cross sections for the $b$-quark production in deep inelastic $e^\pm p$ scattering measurements can be expressed in terms of the $b$-quark contributions to the inclusive structure functions $F_2^{b\bar{b}}$, $xF_3^{b\bar{b}}$ and $F_L^{b\bar{b}}$ as follows:

$$
\sigma_{r,red}^{b\bar{b}} = \frac{d\sigma_{b\bar{b}}^{(e^\pm p)}}{dx dQ^2} \frac{Q^4 x}{2\pi\alpha^2(1 + (1 - y)^2)}
= F_2^{b\bar{b}} + \frac{(1 - (1 - y)^2)}{(1 + (1 - y)^2)^2} x F_3^{b\bar{b}} - \frac{y^2}{(1 + (1 - y)^2)} F_L^{b\bar{b}}.
$$

(3)
Within the quark parton model (QPM) framework where $Q^2 \ll M_Z^2$ the parity-violating structure function $xF_3$ can be neglected and accordingly the reduced cross sections for the $b$-quark contributions can be expressed by

$$\sigma_{red}^{\bar{b}b} = \frac{d\sigma^{\bar{b}b}(e^\pm p)}{dx dQ^2} \frac{Q^4 x}{2\pi\alpha^2(1 + (1 - y)^2)}$$

$$= F_2^{\bar{b}b} - \frac{y^2}{(1 + (1 - y)^2)} F_{L}^{\bar{b}b}. \quad (4)$$

More details may be found in Ref. [26].

It should be noted that in this pQCD analysis we perform six different fits entitled: H1BPoleMass, H1BRunMass, ZBPoleMass, ZBRunMass, TPoleMass and TRunMass so that in throughout of this article the words H1BPoleMass, H1BRunMass, ZBPoleMass, ZBRunMass, TPoleMass and TRunMass refer as follows:

- **H1BPoleMass**: Determination of the $b$-quark pole mass $M_b$ using HERA I and II combined and H1 beauty production data sets.

- **H1BRunMass**: Determination of the $b$-quark MS running mass $m_b$ using HERA I and II combined and H1 beauty production data sets.

- **ZBPoleMass**: Determination of the $b$-quark pole mass $M_b$ using HERA I and II combined and ZEUS beauty production data sets.

- **ZBRunMass**: Determination of the $b$-quark MS running mass $m_b$ using HERA I and II combined and ZEUS beauty production data sets.

- **TPoleMass**: Determination of the $b$-quark pole mass $M_b$ using HERA I and II combined and H1 + ZEUS beauty production data sets.

- **TRunMass**: Determination of the $b$-quark MS running mass $m_b$ using HERA I and II combined and H1 + ZEUS beauty production data sets.

The reduced NC unpolarized deep inelastic $e^\pm p$ scattering cross sections $\sigma_{r,NC}^{+}$ as a function of $x$ for seven set of HERA run I and II combined data [1] and consistency of these experimental data with theory predictions are shown in Fig. [1].

In Fig. [2] we show the reduced cross sections for inclusive unpolarized CC $e^\pm p$ scattering double-differential cross sections $\frac{d^2\sigma_{CC}^{+}}{dx dQ^2}$ as a function of $x$ for seven set of HERA run I and II
Figure 1: The reduced NC unpolarized deep inelastic $e^\pm p$ scattering cross sections $\sigma_{r,NC}^\pm$ as a function of $x$ for seven set of HERA run I and II combined data [1] and also we show the consistency of these experimental data with theory predictions.

combined data [1] and also we show the consistency of these experimental data with theory predictions.

Total reduced beauty production cross sections, $\sigma_{\gamma p}(\gamma p \rightarrow b\bar{b})$ as a function of $x$ for (H1 + ZEUS) $F_2^{\gamma p}$ beauty vertex data (TB) and consistency of H1 + ZEUS reduced deep inelastic $e^\pm p$ scattering data with theory predictions as a function of $x$ for different values of $Q^2$ are
Figure 2: The reduced cross sections for inclusive unpolarized CC $e^\pm p$ scattering double-differential cross sections $\frac{d^2\sigma^{e^\pm p}}{dx^dQ}$ as a function of $x$ for seven set of HERA run I and II combined data and also we show the consistency of these experimental data with theory predictions.

Figure 3: Total reduced beauty production cross section, $\sigma^{\gamma p}(\gamma g \rightarrow b\bar{b})$ as a function of $x$ for (H1 + ZEUS) $F_2^{\gamma p}$ beauty vertex data (TB) and consistency of H1 + ZEUS reduced deep inelastic $e^\pm p$ scattering data with theory predictions as a function of $x$ for different values of $Q^2$. shown in Fig. 3
III. PERTURBATIVE QUANTUM CHROMODYNAMICS THEORY PREDICTIONS

One of the main features of QCD theory is color confinement postulate (the QCD short range feature) which says all hadron states and physical observables such as currents, energies, momenta and masses are color-singlets [46–52]. It should be noted that the above postulate is just a kinematical constraint to eliminate colored states. There is, however, a hope that the quark confinement may be the natural dynamical consequence of quantum chromodynamics theory.

Because of color confinement feature of QCD, free quarks are unobservable and accordingly there are different definitions for the $b$-quark mass such as pole mass and $\overline{\text{MS}}$ running mass [53–56].

Physically, the definition of the $b$-quark mass comes from its contribution as an extra free parameter in QCD Lagrangian as a one degree of freedom of non-Abelian gauge field theory and its exact value depends on the specific renormalization scheme [57, 58].

In the on-shell renormalization scheme, the $b$-quark mass is defined as the pole of the $b$-quark propagator and known as the $b$-quark pole mass $M_b$. The $b$-quark pole mass definition is same as the typical definition of the lepton mass [59, 60].

In the $\overline{\text{MS}}$ scheme, the $b$-quark mass is defined as a scale-dependent perturbative running parameter and is called the $b$-quark $\overline{\text{MS}}$ running mass. The $b$-quark $\overline{\text{MS}}$ running mass definition is same as the definition of the running strong coupling $\alpha_s(M_Z^2)$ [61, 62].

Each of these definitions for the $b$-quark mass has own advantages and disadvantages. The $b$-quark pole mass is a gauge invariant quantity and is well defined in each finite order of pQCD theory but this definition as the pole of the $b$-quark propagator involves some contribution from the non-perturbative region and accordingly suffers from an intrinsic uncertainty of order $\Lambda_{QCD}/m_b$, where as we previously mentioned the $\Lambda_{QCD} \sim 250$ MeV refers to the QCD scale parameter. On the other hand, since the $\overline{\text{MS}}$ running mass is defined at the renormalization scale $\mu_r$ where $\mu_r \gg \Lambda_{QCD}$, therefor the $b$-quark $\overline{\text{MS}}$ running mass definition avoids the intrinsic uncertainty of pole mass definition.

Within the pQCD framework the connection between renormalized and unrenormalized (bare) quark mass is given by
\[ m_0 = Z_{\overline{\text{MS}}}^m m_b, \quad (5) \]
\[ m_0 = Z_{\text{OS}}^m M_b, \quad (6) \]

where \( m_0 \) is the unrenormalized or bare \( b \)-quark mass and \( Z_{\overline{\text{MS}}}^m \) and \( Z_{\text{OS}}^m \) are renormalization factors for the \( b \)-quark mass in the \( \overline{\text{MS}} \) and the on-shell schemes, respectively.

From Eqs. (5), (6) we may write the relation between the \( b \)-quark pole mass \( M_b \) and \( \overline{\text{MS}} \) running mass \( \overline{m}_b \) as follows:

\[ \frac{\overline{m}_b}{M_b} = \frac{Z_{\text{OS}}^m}{Z_{\overline{\text{MS}}}^m}. \quad (7) \]

Now, it is clear from Eq. (7) to extract relation between the \( b \)-quark pole mass \( M_b \) and its \( \overline{\text{MS}} \) running mass \( \overline{m}_b \) at the NNLO, it is enough to determine \( Z_{\overline{\text{MS}}}^m \) and \( Z_{\text{OS}}^m \) renormalization factors in both the \( \overline{\text{MS}} \) and the on-shell schemes at the NNLO.

Within the pQCD framework the mass renormalization factor \( Z_{\overline{\text{MS}}}^m \) in the \( \overline{\text{MS}} \)-scheme is given by:

\[ Z_{\overline{\text{MS}}}^m = 1 + \sum_{i=1}^{\infty} C_i \left( \frac{\alpha_s(\mu)}{\pi} \right)^i, \quad (8) \]

with

\[ C_1 = -\frac{1}{\varepsilon}, \]
\[ C_2 = \frac{1}{\varepsilon^2} \left( \frac{15}{8} - \frac{1}{12} N_f \right) + \frac{1}{\varepsilon} \left( -\frac{101}{48} + \frac{5}{72} N_f \right), \quad (9) \]

where \( N_f \) is the number of different fermion flavors and \( \varepsilon \) is the dimensional regularization parameter which is related to the the space-time dimension \( D \) by \( \varepsilon = \frac{4-D}{2} \).

To determine the renormalization factor \( Z_{\text{OS}}^m \) in the on-shell scheme we start from the perturbative quark propagator which is defined as follows:

\[ \hat{S}_F(p) = \frac{i}{\hat{p} - m_0 + \Sigma(p, M_b)}, \quad (10) \]

where \( \Sigma(p, M_b) \) is the one particle irreducible \( b \)-quark self-energy which is parameterized as follows:

\[ \hat{\Sigma}(p, M_b) = M \Sigma_1(p^2, M_b) + (\hat{p} - M_b) \Sigma_2(p^2, M_b). \quad (11) \]
Since the $b$-quark pole mass $M_b$ corresponds to the position of the pole of the $b$-quark propagator we may write

$$Z_m^{\text{OS}} = 1 + \Sigma_1(p^2, M_b)|_{p^2=M_b^2},$$

which is the simplest formula for the renormalization factor $Z_m^{\text{OS}}$ in the on-shell scheme.

Now, having computed the NNLO contribution to $Z_m^{\text{OS}}$ and using Eqs. (7), (8), we can obtain a NNLO relation between the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ in terms of the color factors as follows:

$$\overline{m}_b(M_b) = M_b \left[ 1 - C_F \left( \frac{\alpha_s}{\pi} \right) + C_F \left( \frac{\alpha_s}{\pi} \right)^2 \left( C_F d_1^{(2)} + C_A d_2^{(2)} + T_R N_L d_3^{(2)} + T_R N_H d_4^{(2)} \right) \right],$$

where:

- $C_F$ is the Casimir operator of the fundamental representation of the color gauge SU(3) group.
- $C_A$ is the Casimir operator of the adjoint representation of the color gauge SU(3) group.
- $T_R$ denotes the trace normalization of the fundamental representation.
- $N_L$ refers to the number of massless quark flavors.
- $N_H$ refers to the number of quark flavors with a pole mass equal to $M_b$.
- $\alpha_s \equiv \alpha_s^{(N_L+N_H)}(M_b)$ refers to the $\overline{\text{MS}}$ strong coupling which is renormalized at the scale of the pole mass $\mu = M_b$ in the pQCD theory with $N_L + N_H$ active flavors.

From Eqs. (13), we may obtain the following results for the coefficients $d_k^{(n)}$:

$$d_1^{(2)} = \frac{7}{128} - \frac{3}{4} \zeta_3 + \frac{1}{2} \pi^2 \log 2 - \frac{5}{16} \pi^2,$$
$$d_2^{(2)} = \frac{-1111}{384} + \frac{3}{8} \zeta_3 - \frac{1}{4} \pi^2 \log 2 + \frac{1}{12} \pi^2,$$
$$d_3^{(2)} = \frac{71}{96} + \frac{1}{12} \pi^2,$$
$$d_4^{(2)} = \frac{143}{96} - \frac{1}{6} \pi^2.$$
Now, if we insert the standard values of the pQCD color factors: $C_F = 4/3$, $C_A = 3$, $T_R = 1/2$ and setting the number of heavy flavors to $N_H = 1$, one may finds the following result:

$$m_b(M_b) = M_b \left[ 1 - \frac{4}{3} \left( \frac{\alpha_s}{\pi} \right) + \left( \frac{\alpha_s}{\pi} \right)^2 \left( N_L \left( \frac{71}{144} + \frac{\pi^2}{18} \right) - \frac{3019}{288} + \frac{1}{6} \zeta_3 - \frac{\pi^2}{9} \log 2 - \frac{\pi^2}{3} \right) \right],$$

or numerically we may find:

$$m_b(M_b) = M_b \left[ 1 - \frac{4}{3} \left( \frac{\alpha_s}{\pi} \right) + \left( \frac{\alpha_s}{\pi} \right)^2 (1.0414 N_L - 14.3323) \right],$$

or:

$$M_b = \bar{m}_b(m_b) \left[ 1 + \frac{4}{3} \left( \frac{\bar{\alpha}_s}{\pi} \right) + \left( \frac{\bar{\alpha}_s}{\pi} \right)^2 (-1.0414 N_L + 13.4434) \right],$$

with $\bar{\alpha}_s \equiv \alpha_s(m_b)$.

It should be noted that at the leading order (LO) or 0-loop calculation, $(\frac{\alpha_s}{\pi}) \to 0$ and accordingly the difference between the $b$-quark pole mass and its $\overline{\text{MS}}$ running mass vanishes [63–76].

IV. QCD ANALYSIS SET-UP AND SYSTEMATIC UNCERTAINTIES

In this contribution, we use three different data sets. To control the $u$-valence and $d$-valence distributions parameters in the generic HERAPDF approach [45], we use the seven sets of HERA run I and II combined NC and CC deep $e^\pm p$ scattering cross sections data [1] as our central pQCD analysis data set and to determine the most precise $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\bar{m}_b$, we use the H1 [26] and ZEUS [25] beauty vertex production data sets.

Using these experimental data sets we made six different fits to determine the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\bar{m}_b$ based on the following QCD set-up:

- To choose the PDFStyle and parametrize the PDFs, we use generic HERAPDF functional form:

$$x f(x) = A x^B (1 - x)^C (1 + Dx + Ex^2),$$

(17)
with 14 free central fit parameters and 1 extra free parameter $m_b$ at the starting scale of the QCD evolution $Q_0^2 = 1.9 \text{ GeV}^2$.

- To include the $b$-quark contribution, we use two different variants of very recently updated of FONLL scheme, FONLL-C and FONLL-C RUNM ON.

- To fitting with experimental data, we use the xFitter package as a powerful QCD open source framework.

- Evolution of the parametrized PDFs has been done based on the DGLAP collinear evolution with QCDNUM package and DGLAP collinear evolution with APFEL package corresponding to FONLL-C and FONLL-C RUNM ON, respectively.

- We choose the lower band of the $b$-quark mass as $M_b = 4.192 \text{ GeV}$ and then varied in steps of 0.01.

- To include light flavor contribution, we set the renormalization and factorization scales as $\mu_r = \mu_f = Q$, while for the $b$-quark contribution we use the typical definition as: $\mu_r = \mu_f = \mu_r = \sqrt{Q^2 + 4m_b^2}$.

- The strangeness suppression factor and the strong coupling constant fixed to $f_s = 0.4$ and $\alpha_s^{\text{NNLO}}(M_Z^2) = 0.118$, respectively.

- To extract the PDFs with the best fit quality, we impose the minimum and maximum value of $Q^2$ cut on the inclusive deep inelastic $e^\pm p$ scattering cross sections data as $Q^2_{\text{min}} = 3.5 \text{ GeV}^2$ and $Q^2_{\text{max}} = 10^6 \text{ GeV}^2$.

- To minimize the $\chi^2$—function as a measure of the compatibility between theory and experimental data, we set the running mode based on the standard MINUIT-minimization of central PDFs and one extra fit parameter $m_b$.

V. RESULTS

In Tables (I)—(III), we show data sets used in our QCD analysis, correlated $\chi^2$ and extracted $\chi^2_{\text{min}}$ corresponding to H1 $F_2^{\Upsilon}$ beauty vertex data (H1B), ZEUS $F_2^{\Upsilon}$ beauty vertex data (ZB) and (H1 + ZEUS) $F_2^{\Upsilon}$ beauty vertex data (TB), respectively.
According to the numerical results from Tables (I)−(III), the best fit qualities in determination of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ are: $\frac{\chi^2_{\text{pole}}}{\text{dof}} = 1.178$ and $\frac{\chi^2_{\text{run}}}{\text{dof}} = 1.170$ corresponding to (H1 + ZEUS) $F^b_2$ beauty vertex data (TB). Also, according to relative improvement of $\chi^2$−function which is defined by $\frac{\chi^2_{\text{pole}} - \chi^2_{\text{run}}}{\chi^2_{\text{pole}}}$, we get an improvement up to $1.178 - 1.170 \sim 0.7\%$ in the quality of the fit for determination of the $b$-quark $\overline{\text{MS}}$ running mass $\overline{m}_b$ relative to the $b$-quark pole mass $M_b$.

| Experiment | H1BPoleMass | H1BRunMass |
|------------|-------------|------------|
| HERA I+II CC $e^+p$ [1] | 51 / 39 | 50 / 39 |
| HERA I+II CC $e^-p$ [1] | 49 / 42 | 49 / 42 |
| HERA I+II NC $e^-p$ [1] | 218 / 159 | 217 / 159 |
| HERA I+II NC $e^+p$ 460 [1] | 215 / 204 | 215 / 204 |
| HERA I+II NC $e^+p$ 575 [1] | 212 / 254 | 211 / 254 |
| HERA I+II NC $e^+p$ 820 [1] | 62 / 70 | 62 / 70 |
| HERA I+II NC $e^+p$ 920 [1] | 419 / 377 | 413 / 377 |
| H1 $F^b_2$ beauty vertex [26] | 2.6 / 12 | 3.1 / 12 |
| Correlated $\chi^2$ | 123 | 122 |
| $\frac{\chi^2_{\text{total}}}{\text{dof}}$ | 1352 / 1142 | 1343 / 1142 |

Table I: Data sets, correlated $\chi^2$ and extracted $\frac{\chi^2_{\text{total}}}{\text{dof}}$ corresponding to H1 $F^b_2$ beauty vertex data (H1B).

In Tables (IV−VI), we present NNLO numerical values of 15 fit parameters and their uncertainties, including 14 free central PDF parameters and 1 extra $m_b$ parameter corresponding to H1 $F^b_2$ beauty vertex data (H1B), ZEUS $F^b_2$ beauty vertex data (ZB) and (H1 + ZEUS) $F^b_2$ beauty vertex data (TB), respectively.

According to the numerical results from Tables (IV−VI), the best improvement in the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ uncertainties are: pole mass $M_b = 4.65 \pm 0.17$ and $\overline{\text{MS}}$ running mass $\overline{m}_b = 4.39 \pm 0.16$, corresponding to (H1 + ZEUS) $F^b_2$ beauty vertex data (TB).

In Sec. III we extracted the relation between the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running
Table II: Data sets, correlated $\chi^2$ and extracted $\chi^2_{\text{total}}$ corresponding to ZEUS $F_2^{b\bar{b}}$ beauty vertex data (ZB).

mass $\overline{m}_b$ at the NNLO of pQCD framework as follows:

$$\overline{m}_b(M_b) = M_b \left[ 1 - \frac{4}{3} \left( \frac{\alpha_s}{\pi} \right) + \left( \frac{\alpha_s}{\pi} \right)^2 \left( 1.0414 \times N_L - 14.3323 \right) \right].$$

Now, if we insert our numerical results for the $b$-quark pole and $\overline{\text{MS}}$ running masses from Tables (IV–VI) into Eq. (18) which comes from pQCD theory predictions, we get:

for H1 $F_2^{b\bar{b}}$ beauty vertex data (H1B):

$$4.50 \sim 4.55 \left[ 1 - \frac{4}{3} \left( \frac{0.118}{3.141} \right) + \left( \frac{0.118}{3.141} \right)^2 \left( 1.0414 \times 3 - 14.3323 \right) \right], \quad (18)$$

$$4.50 \sim 4.55 \times 0.94, \quad (19)$$

$$4.50 \sim 4.28,$$

for ZEUS $F_2^{b\bar{b}}$ beauty vertex data (ZB):
Table III: Data sets, correlated $\chi^2$ and extracted $\frac{\chi^2_{\text{corr}}}{\text{dof}}$ corresponding to ($\text{H1} + \text{ZEUS}$) $F_2^{\text{bJ}}$ beauty vertex data (TB).

| Experiment          | TPoleMass | TRunMass |
|---------------------|-----------|----------|
| HERA I+II CC $e^+p$ [1] | 51 / 39   | 50 / 39  |
| HERA I+II CC $e^-p$ [1] | 49 / 42   | 49 / 42  |
| HERA I+II NC $e^-p$ [1] | 218 / 159 | 217 / 159|
| HERA I+II NC $e^+p$ 460 [1] | 215 / 204 | 215 / 204|
| HERA I+II NC $e^+p$ 575 [1] | 212 / 254 | 211 / 254|
| HERA I+II NC $e^+p$ 820 [1] | 62 / 70   | 62 / 70  |
| HERA I+II NC $e^+p$ 920 [1] | 419 / 377 | 413 / 377|
| H1 $F_2^{\text{bJ}}$ beauty vertex [26] | 3.2 / 12   | 3.4 / 12  |
| ZEUS $F_2^{\text{bJ}}$ beauty vertex [25] | 12 / 17    | 12 / 17   |
| Correlated $\chi^2$ | 124       | 123      |
| $\chi^2_{\text{corr}}/\text{dof}$ | 1366 / 1159 | 1357 / 1159 |

\[ 4.43 \sim 4.60 \left[ 1 - \frac{4}{3} \left( \frac{0.118}{3.141} \right) + \left( \frac{0.118}{3.141} \right)^2 (1.0414 \times 3 - 14.3323) \right], \quad (20) \]
\[ 4.43 \sim 4.60 \times 0.94, \quad (21) \]
\[ 4.43 \sim 4.32, \quad (22) \]

for H1 and ZEUS $F_2^{\text{bJ}}$ beauty vertex data (TB):

\[ 4.39 \sim 4.65 \left[ 1 - \frac{4}{3} \left( \frac{0.118}{3.141} \right) + \left( \frac{0.118}{3.141} \right)^2 (1.0414 \times 3 - 14.3323) \right], \quad (22) \]
\[ 4.39 \sim 4.65 \times 0.94, \quad (23) \]
\[ 4.39 \sim 4.37, \quad (23) \]

where according to our methodology in Sec. [IV], we set the strong coupling constant to $\alpha_{\text{sNNLO}}(M_Z^2) = 0.118$. 

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Table IV: The NNLO numerical values of 15 fit parameters and their uncertainties, including 14 free central PDF parameters and 1 extra \(m_b\) parameter corresponding to H1 \(F_2^{b\bar{b}}\) beauty vertex data (H1B).

Based on the absolute error formula \(\Delta x = |x_f - x_i|\), we see that the differences of our numerical results extracted based on phenomenology of experimental data for the \(b\)-quark pole mass \(M_b\) and \(\overline{\text{MS}}\) running mass \(\overline{m}_b\) with the pQCD theory prediction at the NNLO approximation are less than: \(|4.50 - 4.28| = 0.22\), \(|4.43 - 4.32| = 0.11\) and \(|4.39 - 4.37| = 0.02\) corresponding to H1, ZEUS and \((\text{H1} + \text{ZEUS})\) \(F_2^{b\bar{b}}\) beauty vertex data, respectively. In the other words, the compatibility of our numerical results with the pQCD theory predictions at the NNLO corrections and with precision of 1 part in \(10^2\) are up to approximately: 99.78 \%, 99.89 \% and 99.98 \% corresponding to H1, ZEUS and \((\text{H1} + \text{ZEUS})\) \(F_2^{b\bar{b}}\) beauty vertex data (TB), respectively which show an excellent agreement between our phenomenology analysis results with pQCD theory predictions. Furthermore, the comparison our numerical results
Table V: The NNLO numerical values of 15 fit parameters and their uncertainties, including 14 free central PDF parameters and 1 extra $m_b$ parameter corresponding to ZEUS $F_2^{t\overline{b}}$ beauty vertex data (ZB).

| Parameter | H1BPoleMass | H1BRunMass |
|-----------|-------------|------------|
| $B_{u_3}$ | 0.846 ± 0.039 | 0.848 ± 0.039 |
| $C_{u_3}$ | 4.463 ± 0.076 | 4.466 ± 0.077 |
| $E_{u_3}$ | 11.4 ± 1.6 | 11.4 ± 1.6 |
| $B_{d_3}$ | 1.05 ± 0.10 | 1.04 ± 0.11 |
| $C_{d_3}$ | 4.26 ± 0.39 | 4.23 ± 0.41 |
| $C_{\overline{U}}$ | 7.24 ± 0.97 | 7.21 ± 0.97 |
| $D_{\overline{U}}$ | 8.6 ± 2.2 | 7.8 ± 2.1 |
| $A_{\overline{D}}$ | 0.1637 ± 0.0091 | 0.1793 ± 0.0095 |
| $B_{\overline{D}}$ | −0.1818 ± 0.0069 | −0.1703 ± 0.0066 |
| $C_{\overline{D}}$ | 5.39 ± 0.98 | 5.54 ± 1.00 |
| $B_g$ | 0.12 ± 0.12 | 0.13 ± 0.14 |
| $C_g$ | 5.41 ± 0.82 | 5.64 ± 0.89 |
| $A'_g$ | 2.29 ± 0.40 | 2.32 ± 0.45 |
| $B'_g$ | 0.006 ± 0.057 | 0.021 ± 0.068 |
| $m_b$ | pole mass $M_b = 4.60 ± 0.32$ MS RunMass $\overline{m}_b = 4.43 ± 0.30$ |

with the measurements from the PDG [94] world average, shows a very good agreement with the expected $b$-quark pole masses.

In Table VII we summarize all the numerical results from Tables (II–VI). As can be seen from Table VII, the best result and improvement in the $b$-quark pole mass $M_b$ and $\overline{MS}$ running mass $\overline{m}_b$ uncertainties and also the best results for QCD fit qualities with approximately 99.98% compatibility with pQCD theory with precision of 1 part in $10^2$ have been extracted based on the (H1 + ZEUS) $F_2^{t\overline{b}}$ beauty vertex data (TB) analysis.

In Figs. 4–6, we show the ratios of pQCD theory predictions for determination of the $b$-quark pole mass $M_b$ and $\overline{MS}$ running mass $\overline{m}_b$ at the NNLO corrections in two separate panels corresponding to H1 and ZEUS beauty vertex data and HERA I and II combined.
| Parameter | H1BPoleMass | H1BRunMass |
|-----------|-------------|------------|
| $B_{uv}$  | 0.847 ± 0.039 | 0.848 ± 0.039 |
| $C_{uv}$  | 4.463 ± 0.076 | 4.466 ± 0.077 |
| $E_{uv}$  | 11.4 ± 1.6 | 11.4 ± 1.6 |
| $B_{dv}$  | 1.05 ± 0.10 | 1.04 ± 0.11 |
| $C_{dv}$  | 4.26 ± 0.39 | 4.22 ± 0.41 |
| $C_{ar{U}}$ | 7.24 ± 0.97 | 7.21 ± 0.97 |
| $D_{ar{U}}$ | 8.5 ± 2.2 | 7.8 ± 2.1 |
| $A_{ar{D}}$ | 0.1640 ± 0.0091 | 0.1794 ± 0.0095 |
| $B_{ar{D}}$ | -0.1817 ± 0.0069 | -0.1702 ± 0.0066 |
| $C_{ar{D}}$ | 5.38 ± 0.98 | 5.52 ± 1.00 |
| $B_g$ | 0.12 ± 0.12 | 0.13 ± 0.14 |
| $C_g$ | 5.41 ± 0.82 | 5.64 ± 0.90 |
| $A'_g$ | 2.29 ± 0.40 | 2.32 ± 0.45 |
| $B'_g$ | 0.006 ± 0.057 | 0.021 ± 0.068 |
| $m_b$ | pole mass $M_b = 4.65 ± 0.17$ | MS RunMass $\overline{m}_b = 4.39 ± 0.16$ |

Table VI: The NNLO numerical values of 15 fit parameters and their uncertainties, including 14 free central PDF parameters and 1 extra $m_b$ parameter corresponding to (H1 + ZEUS) $F_2^{q\bar{q}}$ beauty vertex data (TB).

data as our central data set, respectively.

In Figs. (7–9), we show the compatibility between pQCD theory and the phenomenology of experimental data in determination of the $b$-quark pole mass $M_b$ and $\overline{MS}$ running mass $\overline{m}_b$ at the NNLO corrections in three separate panels, including pull, $\frac{\text{Theory+Shifts}}{\text{Data}}$ and $\frac{\text{Theory}}{\text{Data}}$, corresponding to H1 and ZEUS beauty vertex data and HERA I and II combined data as our central data set, respectively.

In Figs. (10–12), we show the pure impact of the $b$-quark pole mass $M_b$ and $\overline{MS}$ running mass $\overline{m}_b$ on the gluon distribution as a function of $x$ for H1 $F_2^{q\bar{q}}$ beauty vertex data (H1B), ZEUS $F_2^{q\bar{q}}$ beauty vertex data (ZB) and (H1 + ZEUS) $F_2^{q\bar{q}}$ beauty vertex data (TB), respectively.
| Experiment         | PoleMass | RunMass | $\chi^2_{pole\ dof}$ | $\chi^2_{Run\ dof}$ | Compatibility |
|--------------------|----------|---------|----------------------|----------------------|---------------|
| H1 $F_2^{b\bar{b}}$ | $M_b = 4.55 \pm 0.46$ | $\overline{m}_b = 4.50 \pm 0.42$ | 1.184 | 1.176 | 99.78 % |
| ZEUS $F_2^{b\bar{b}}$ | $M_b = 4.60 \pm 0.32$ | $\overline{m}_b = 4.43 \pm 0.30$ | 1.186 | 1.179 | 99.89 % |
| (H1 + ZEUS) $F_2^{b\bar{b}}$ | $M_b = 4.65 \pm 0.17$ | $\overline{m}_b = 4.39 \pm 0.16$ | 1.179 | 1.170 | 99.98 % |

Table VII: A summary of all the numerical results from Tables (I–VI). The best result and improvement in the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ uncertainties and also the best results for QCD fit qualities with approximately 99.98 % compatibility with pQCD theory with precision of 1 part in $10^2$ have been extracted based on the (H1 + ZEUS) $F_2^{b\bar{b}}$ beauty vertex data (TB) analysis.

Figure 4: Ratios of pQCD theory predictions for determination of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ at the NNLO corrections in two separate panels corresponding to H1 $F_2^{b\bar{b}}$ beauty vertex data.

As we expected from numerical results of Tables (IV–VI), the gluon distribution is sensitive to the $b$-quark mass, when it is considered as an extra free parameter in pQCD framework. Accordingly, in Figs. (10–12), we see that the shape of the gluon distribution is sensitive to the $b$-quark pole and $\overline{\text{MS}}$ running masses. Furthermore, as we expected from the numerical results of Table VII the best improvement in the uncertainty band of the gluon distribution is corresponding to pure impact of the $b$-quark $\overline{\text{MS}}$ running mass for (H1 + ZEUS) $F_2^{b\bar{b}}$ beauty vertex data (TB).

Figs. (13–15), show the pure impact of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ on the ratio of $\Sigma$-PDF over $g$-distribution (the sea quark $\Sigma$-PDF is defined by $\Sigma = 2x(\bar{u} + \bar{d} + \bar{s} + \bar{c})$ and $g$ stands to the gluon distribution) as a function of $x$ for H1 $F_2^{b\bar{b}}$ beauty vertex data (H1B), ZEUS $F_2^{b\bar{b}}$ beauty vertex data (ZB) and (H1 + ZEUS) $F_2^{b\bar{b}}$ beauty vertex data (TB).
Figure 5: Ratios of pQCD theory predictions for determination of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ at the NNLO corrections in two separate panels corresponding to ZEUS $F_2^{b\overline{b}}$ beauty vertex data.

As we expected from numerical results of Tables (IV - VI), the shape of ratio of $\Sigma$-PDF over $g$-distribution is sensitive to the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ and the best improvement of this ratio is corresponding to pure impact of the $b$-quark $\overline{\text{MS}}$ running mass $\overline{m}_b$ for (H1 + ZEUS) $F_2^{b\overline{b}}$ beauty vertex data (TB) analysis.
Figure 6: Ratios of pQCD theory predictions for determination of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $m_b$ at the NNLO corrections in two separate panels corresponding to HERA run I and II combined data as our central data sets.
Figure 7: Compatibility between pQCD theory and the phenomenology of experimental data in determination of the $b$-quark pole mass $M_b$ and MS running mass $\bar{m}_b$ at the NNLO corrections in three separate panels, include of pulls, $\frac{\text{Theory+Shifts}}{\text{Data}}$ and $\frac{\text{Theory}}{\text{Data}}$ corresponding to H1 and ZEUS $F_2^{b\bar{b}}$ beauty vertex data sets.
Figure 8: Compatibility between pQCD theory and the phenomenology of experimental data in determination of the $b$-quark pole mass $M_b$ and MS running mass $\overline{m}_b$ at the NNLO corrections in three separate panels, include of pulls, $\frac{\sigma_{CC}^{Theory+Shifts}}{\sigma_{CC}^{Data}}$ and $\frac{\sigma_{CC}^{Theory}}{\sigma_{CC}^{Data}}$ corresponding to double-differential cross sections $\frac{d^2\sigma_{CC}^{p+\nu}}{dx dQ^2}$ as a function of $x$ for HERA run I and II combined data.
Figure 9: Compatibility between pQCD theory and the phenomenology of experimental data in determination of the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ at the NNLO corrections in three separate panels, include of pulls, $\frac{\text{Theory+Shifts}}{\text{Data}}$ and $\frac{\text{Theory}}{\text{Data}}$ corresponding to reduced NC unpolarized deep inelastic $e^\pm p$ scattering cross sections $\sigma_{e^\pm NC}^\pm$ as a function of $x$ for HERA run I and II combined data.
Figure 10: Comparison of pure impact of the $b$-quark pole mass $M_b$ (yellow color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the gluon distribution as a function of $x$ for H1 $F_2^{b\bar{b}}$ beauty vertex data (H1B).

Figure 11: Comparison of pure impact of the $b$-quark pole mass $M_b$ (red color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the gluon distribution as a function of $x$ for ZEUS $F_2^{b\bar{b}}$ beauty vertex data (ZB).
Figure 12: Comparison of pure impact of the $b$-quark pole mass $M_b$ (purple color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the gluon distribution as a function of $x$ for (H1 + ZEUS) $F_2^b$ beauty vertex data (TB).

Figure 13: Comparison of the pure impact of the $b$-quark pole mass $M_b$ (yellow color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the ratio of $\Sigma$ over $g$ distribution (the sea quark $\Sigma$-PDF is defined by $\Sigma = 2x(\bar{u} + \bar{d} + \bar{s} + \bar{c})$ and $g$ stands to the gluon distribution) as a function of $x$ for H1 $F_2^{b\overline{b}}$ beauty vertex data (H1B).
Figure 14: Comparison of the pure impact of the $b$-quark pole mass $M_b$ (red color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the ratio of $\Sigma$ over $g$ distribution as a function of $x$ for ZEUS $F_2^{\overline{\text{S}}} \text{ beauty vertex data (ZB).}$

Figure 15: Comparison of the pure impact of the $b$-quark pole mass $M_b$ (purple color) and $\overline{\text{MS}}$ running mass $\overline{m}_b$ (blue color) on the ratio of $\Sigma$ over $g$ distribution as a function of $x$ for (H1 + ZEUS) $F_2^{\overline{\text{MS}}} \text{ beauty vertex data (TB).}$
VI. SUMMARY AND CONCLUSION

• We determine the $b$-quark pole mass $M_b$ and $\overline{\text{MS}}$ running mass $\overline{m}_b$ with two different approaches at NNLO corrections. At the first approach, we derive a relation between the $b$-quark pole mass $M_b$ and its $\overline{\text{MS}}$ running mass $\overline{m}_b$ at the NNLO corrections based on the pQCD theory predictions. While at the second approach we extract numerical values of the $b$-quark pole and $\overline{\text{MS}}$ running masses based on the phenomenology of experimental data at the NNLO corrections.

• Comparison of numerical results for the $b$-quark pole mass $M_b$ and its $\overline{\text{MS}}$ running mass $\overline{m}_b$ at the NNLO corrections extracted from phenomenological base approach with pQCD theory prediction shows an excellent compatibility between these two different approaches.

• As we know, HERA as a large QCD laboratory is consist of two different colliders, H1 and ZEUS to test of the possible theoretical approaches to heavy-flavor production against experimental data sets. This paper not only provides a comparison between precisions of H1 and ZEUS detectors of HERA but also shows the best compatibility between pQCD theory and phenomenology of experimental data is corresponding to H1 and ZEUS combination (H1 + ZEUS or TB in our analysis) with up to 99.98% compatibility between pQCD theory and phenomenology of $F_2^{\overline{\text{MS}}}$ beauty vertex data sets.

• Heavy-quark production measurements may be used to constrain important QCD parameters, such as the $b$-quark pole and $\overline{\text{MS}}$ running masses. Also such measurements have some important consequences for the determination of other pQCD parameters like the strong coupling constant $\alpha_s(M_Z^2)$. This NNLO QCD analysis reveals the role and influence of the $b$-quark pole and $\overline{\text{MS}}$ running masses as an extra degree of freedom added to the input parameters of the Standard Model Lagrangian, in the improvement of the uncertainty band of the gluon distribution and some of its ratios.

• This NNLO QCD analysis has been developed based on the three experimental data sets and consists of six different fits to determine the $b$-quark pole and $\overline{\text{MS}}$ running masses. Full standard LHAPDF files corresponding to these six different fits are available and can be obtained from authors via e-mail.
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