THE BOUNDARIES OF GOLDEN-MEAN SIEGEL DISKS IN THE COMPLEX QUADRATIC HÉNON FAMILY ARE NOT SMOOTH

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ABSTRACT. As was recently shown by the first author and others in [GaRYa], golden-mean Siegel disks of sufficiently dissipative complex quadratic Hénon maps are bounded by topological circles. In this paper we investigate the geometric properties of such curves, and demonstrate that they cannot be $C^1$-smooth.

1. Introduction

Up to a biholomorphic conjugacy, a complex quadratic Hénon map can be written as

$$H_{c,a}(x, y) = (x^2 + c + ay, ax) \quad \text{for} \quad a \neq 0;$$

this form is unique modulo the change of coordinates $(x, y) \mapsto (x, -y)$, which conjugates $H_{c,a}$ with $H_{c,-a}$. In this paper we will always assume that the Hénon map is dissipative, $|a| < 1$. Note that for $a = 0$, the map $H_{c,a}$ degenerates to

$$(x, y) \mapsto (f_c(x), 0),$$

where $f_c(x) = x^2 + c$ is a one-dimensional quadratic polynomial. Thus for a fixed small value of $a_0$, the one parameter family $H_{c,a_0}$ can be seen as a small perturbation of the quadratic family.

As usual, we let $K^\pm$ be the sets of points that do not escape to infinity under forward, respectively backward iterations of the Hénon map. Their topological boundaries are $J^\pm = \partial K^\pm$. Let $K = K^+ \cap K^-$ and $J = J^- \cap J^+$. The sets $J^\pm$, $K^\pm$ are unbounded, connected sets in $\mathbb{C}^2$ (see [BS1]). The sets $J$ and $K$ are compact (see [HOV1]). In analogy to one-dimensional dynamics, the set $J$ is called the Julia set of the Hénon map.

Note that a Hénon map $H_{c,a}$ is determined by the multipliers $\mu$ and $\nu$ at a fixed point uniquely up to changing the sign of $a$. In particular,

$$\mu \nu = -a^2,$$

the parameter $c$ is a function of $a^2$ and $\mu$:

$$c = (1 - a^2) \left( \frac{\mu}{2} - \frac{a^2}{2\mu} \right) - \left( \frac{\mu}{2} - \frac{a^2}{2\mu} \right)^2.$$

Hence, we sometimes write $H_{\mu,\nu}$ instead of $H_{c,a}$, when convenient.
When \( \nu = 0 \), the Hénon map degenerates to
\[
H_{\mu,0}(x, y) = (P_{\mu}(x), 0), \quad \text{where} \quad P_{\mu}(x) = x^2 + \mu/2 - \mu^2/4.
\]

We say that a dissipative Hénon map \( H_{c,a} \) has a semi-Siegel fixed point (or simply that \( H_{c,a} \) is semi-Siegel) if the eigenvalues of the linear part of \( H_{c,a} \) at that fixed point are \( \mu = e^{2\pi i \theta} \), with \( \theta \in (0,1) \setminus \mathbb{Q} \) and \( \nu \), with \( |\nu| < 1 \), and \( H_{c,a} \) is locally biholomorphically conjugate to the linear map
\[
L(x, y) = (\mu x, \nu y).
\]

The classic theorem of Siegel states, in particular, that \( H_{\mu,\nu} \) is semi-Siegel whenever \( \theta \) is Diophantine, that is \( q_{n+1} < cq_n^d \), where \( p_n/q_n \) are the continued fraction convergents of \( \theta \). The existence of a linearization is a local result, however, in this case there exists a linearizing biholomorphism \( \phi: \mathbb{D} \times \mathbb{C} \to \mathbb{C}^2 \) sending \((0,0)\) to the semi-Siegel fixed point,
\[
H_{\mu,\nu} \circ \phi = \phi \circ L,
\]
such that the image \( \phi(\mathbb{D} \times \mathbb{C}) \) is maximal (see [MNTU]). We call \( \phi(\mathbb{D} \times \mathbb{C}) \) the Siegel cylinder; it is a connected component of the interior of \( K^+ \) and its boundary coincides with \( J^+ \) (see [BS2]). We let
\[
\Delta = \phi(\mathbb{D} \times \{0\}),
\]
and by analogy with the one-dimensional case call it the Siegel disk of the Hénon map. Clearly, the Siegel cylinder is equal to the stable manifold \( W_s(\Delta) \), and \( \Delta \subset K \) (which is always bounded). Moreover, \( \partial \Delta \subset J \), the Julia set of the Hénon map.

**Remark 1.1.** Let \( q \) be the semi-Siegel fixed point of the Hénon map. Then \( \Delta \subset W^c(q) \), the center manifold of \( q \) (see e.g. [S] for a definition of \( W^c \)). The center manifold is not unique in general, but all center manifolds \( W^c(q) \) coincide on the Siegel disk. This phenomenon is nicely illustrated in [O], Figure 5.

In a recent paper [GaRYa] it was shown that:

**Theorem 1.2 (GaRYa).** There exists \( \delta > 0 \) such that the following holds. Let \( \theta_\ast = (\sqrt{5} - 1)/2 \) be the inverse golden mean, \( \mu_\ast = e^{2\pi i \theta_\ast} \), and let \( |\nu| < \delta \). Then the boundary of the Siegel disk of \( H_{\mu_\ast,\nu} \) is a homeomorphic image of the circle.

Furthermore, the linearizing map
\[
\phi: \mathbb{D} \times \{0\} \to \Delta
\]
extends continuously and injectively to the boundary. However, the restriction
\[
\phi: S^1 \times \{0\} \to \partial \Delta
\]
is not \( C^1 \)-smooth.
This is the first result of its kind on the structure of the boundaries of Siegel disks of complex Hénon maps. It is based on a renormalization theory for two-dimensional dissipative Hénon-like maps, developed in [GaYa2]. Below, we will briefly review the relevant renormalization results.

Theorem 1.2 raises a natural question whether the boundary $\partial \Delta$ can ever lie on a smooth curve. Classical results (see [War]) imply that the smoothness of $\partial \Delta$ must be less than $C^{1+\epsilon}$ – otherwise, $\phi$ would have a $C^{1+\epsilon}$ extension to the boundary, contradicting Theorem 1.2. However, we can ask, whether $\partial \Delta$ can be a $C^1$-smooth curve. In the present note we answer this in the negative:

**Main Theorem.** Let $\delta > 0$ be as in Theorem 1.2 and $|\nu| < \delta$. Then the boundary of the Siegel disk of $H_{\mu,\nu}$ is not $C^1$-smooth.

2. **Review of renormalization theory for Siegel disks**

In this section we give a brief summary of the relevant statements on renormalization of Siegel disks; we refer the reader to [GaYa2] for the details.

2.1. **One-dimensional renormalization: almost-commuting pairs.** For a domain $Z \subset \mathbb{C}$, we denote $\mathcal{H}(Z)$ the Banach space of bounded analytic functions $f : Z \to \mathbb{C}$ equipped with the norm

$$\|f\| = \sup_{x \in Z} |f(x)|. \quad (3)$$

Denote $\mathcal{H}(Z,W)$ the Banach space of bounded pairs of analytic functions $\zeta = (f,g)$ from domains $Z \subset \mathbb{C}$ and $W \subset \mathbb{C}$ respectively to $\mathbb{C}$ equipped with the norm

$$\|\zeta\| = \frac{1}{2}(\|f\| + \|g\|). \quad (4)$$

Henceforth, we assume that the domains $Z$ and $W$ contain 0.

For a pair $\zeta = (f,g)$, define the *rescaling map* as

$$\Lambda(\zeta) := (s_\zeta^{-1} \circ f \circ s_\zeta, s_\zeta^{-1} \circ g \circ s_\zeta), \quad (5)$$

where

$$s_\zeta(x) := \lambda_\zeta x \quad \text{and} \quad \lambda_\zeta := g(0).$$

**Definition 2.1.** We say that $\zeta = (\eta,\xi) \in \mathcal{H}(Z,W)$ is a *critical pair* if $\eta$ and $\xi$ have a simple unique critical point at 0. The space of critical pairs is denoted by $\mathcal{C}(Z,W)$.

**Definition 2.2.** We say that $\zeta = (\eta,\xi) \in \mathcal{C}(Z,W)$ is a *commuting pair* if

$$\eta \circ \xi = \xi \circ \eta.$$

**Definition 2.3.** We say that $\zeta = (\eta,\xi) \in \mathcal{C}(Z,W)$ is an *almost commuting pair* (cf. [Bur, Stir]) if

$$\frac{d^i(\eta \circ \xi - \xi \circ \eta)}{dx^i}(0) = 0 \quad \text{for} \quad i = 0, 2,$$
and
\[ \xi(0) = 1. \]
The space of almost commuting pairs is denoted by \( \mathcal{B}(Z,W) \).

**Proposition 2.4** (cf. [GaYa2]). The spaces \( \mathcal{C}(Z,W) \) and \( \mathcal{B}(Z,W) \) have the structure of an immersed Banach submanifold of \( \mathcal{H}(Z,W) \) of codimension 2 and 5 respectively.

Denote
\[ c(x) := \bar{x}. \]

**Definition 2.5.** Let \( \zeta = (\eta, \xi) \in \mathcal{B}(Z,W) \). The pre-renormalization of \( \zeta \) is defined as:
\[ pR((\eta, \xi)) := (\eta \circ \xi, \eta). \]
The renormalization of \( \zeta \) is defined as:
\[ R((\eta, \xi)) := \Lambda(c \circ \eta \circ \xi \circ c, c \circ \eta \circ c). \]

It is easy to see that

**Proposition 2.6.** The renormalization of an (almost) commuting pair is an (almost) commuting pair (on different domains).

It is convenient to introduce the following multi-index notation. Let \( \mathcal{I} \) be the space of all finite multi-indexes
\[ \varpi = (a_0, \ldots, a_n) \in (\{0\} \cup \mathbb{N})^n \text{ for some } n \in \mathbb{N}, \]
with the partial ordering relation defined as follows:
\[ (a_0, a_1, \ldots, a_k, b) \prec (a_0, a_1, \ldots, a_n, a_{n+1}) \]
if either \( k < n \) and \( b \leq a_{k+1} \), or \( k = n \) and \( b < a_{n+1} \). For a pair \( \zeta = (\eta, \xi) \) and a multi-index \( \varpi = (a_0, \ldots, a_n) \in \mathcal{I} \), denote
\[ \zeta^\varpi = \phi^{a_n} \circ \ldots \circ \xi^{a_1} \circ \eta^{a_0} \]
where \( \phi \) is either \( \eta \) or \( \xi \), depending on whether \( n \) is even or odd. Define a sequence \( \{\varpi_0, \varpi_1, \ldots\} \subset \mathcal{I} \) such that
\[ pR^\varpi(\zeta) = (\zeta^{\varpi_n}, \zeta^{\varpi_{n-1}}). \]

The following is shown in [GaYa2]:

**Theorem 2.7.** There exist topological disks \( \hat{Z} \ni Z \) and \( \hat{W} \ni W \), and an almost commuting pair \( \zeta_* = (\eta_*, \xi_*) \in \mathcal{B}(Z,W) \) such that the following holds:

1. There exists a neighbourhood \( \mathcal{N}(\zeta_*) \) of \( \zeta_* \) in the submanifold \( \mathcal{B}(Z,W) \) such that
\[ R : \mathcal{N}(\zeta_*) \rightarrow \mathcal{B}(\hat{Z},\hat{W}) \]
is an anti-analytic operator.

2. The pair \( \zeta_* \) is the unique fixed point of \( R \) in \( \mathcal{N}(\zeta_*) \).
The differential $D\mathcal{R}^2|_{\zeta}$ is a compact linear operator. It has a single, simple eigenvalue with modulus greater than 1. The rest of its spectrum lies inside the open unit disk $\mathbb{D}$ (and hence is compactly contained in $\mathbb{D}$ by the spectral theory of compact operators).

2.2. Renormalization of two-dimensional maps. For a domain $\Omega \subset \mathbb{C}^2$, we denote $O(\Omega)$ the Banach space of bounded analytic functions $F : \Omega \to \mathbb{C}^2$ equipped with the norm

$$\|F\| = \sup_{(x,y)\in \Omega} |F(x,y)|.$$  

Define

$$\|F\|_y := \sup_{(x,y)\in \Omega} |\partial_y F(x,y)|.$$  

Moreover, for

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

define

$$\|F\|_{\text{diag}} := \sup_{(x,y)\in \Omega} |f_1(x,y) - f_2(x,y)|.$$  

Denote $O(\Omega, \Gamma)$ the Banach space of bounded pairs of analytic functions $\Sigma = (F,G)$ from domains $\Omega \subset \mathbb{C}^2$ and $\Gamma \subset \mathbb{C}^2$ respectively to $\mathbb{C}^2$ equipped with the norm

$$\|\Sigma\| = \frac{1}{2} (\|F\| + \|G\|).$$  

Define

$$\|\Sigma\|_y := \frac{1}{2} (\|F\|_y + \|G\|_y).$$  

Moreover,

$$\|\Sigma\|_{\text{diag}} := \frac{1}{2} (\|F\|_{\text{diag}} + \|G\|_{\text{diag}}).$$  

Henceforth, we assume that

$$\Omega = Z \times Z \quad \text{and} \quad \Gamma = W \times W,$$

where $Z$ and $W$ are subdomains of $\mathbb{C}$ containing 0. For a function

$$F(x,y) := \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

from $\Omega$ or $\Gamma$ to $\mathbb{C}^2$, we denote

$$p_1 F(x) := f_1(x,0) \quad \text{and} \quad p_2 F(x) := f_2(x,0).$$

For a pair $\Sigma = (F,G)$, define the rescaling map as

$$\Lambda(\Sigma) := (s^{-1}_\Sigma \circ F \circ s_\Sigma, s^{-1}_\Sigma \circ G \circ s_\Sigma),$$  

where

$$s_\Sigma(x,y) := (\lambda_\Sigma x, \lambda_\Sigma y) \quad \text{and} \quad \lambda_\Sigma := p_1 G(0).$$
Definition 2.8. For $0 < \kappa \leq \infty$, we say that $\Sigma \in O(\Omega, \Gamma)$ is a $\kappa$-critical pair if $p_1 A$ and $p_1 B$ have a simple unique critical point which is contained in a $\kappa$-neighbourhood of 0. The space of $\kappa$-critical pairs in $O(\Omega, \Gamma)$ is denoted by $C^*_2(\Omega, \Gamma, \kappa)$.

Definition 2.9. We say that $\Sigma = (A, B) \in C^*_2(\Omega, \Gamma, \kappa)$ is a commuting pair if $A \circ B = B \circ A$.

Definition 2.10. We say that $\Sigma = (A, B) \in C^*_2(\Omega, \Gamma, \kappa)$ is an almost commuting pair if $p_1 A [A, B] dx(0) := p_1 (A \circ B - B \circ A) dx(0) = 0$ for $i = 0, 2$, and $p_1 B(0) = 1$.

Proposition 2.11 (cf. [GaYa2]). The space $B^*_2(\Omega, \Gamma, \kappa)$ has the structure of an immersed Banach submanifold of $O(\Omega, \Gamma)$ of codimension 3.

For $0 < \epsilon, \delta \leq \infty$, let $O(\Omega, \Gamma, \epsilon, \delta)$ be the open subset of $O(\Omega, \Gamma)$ consisting of pairs $\Sigma = (A, B)$ such that the following holds:

1. $\|\Sigma\|_y < \epsilon$, and
2. $\|\Sigma\|_{\text{diag}} < \delta$.

We denote $C^*_2(\Omega, \Gamma, \epsilon, \delta, \kappa) := O(\Omega, \Gamma, \epsilon, \delta) \cap C^*_2(\Omega, \Gamma, \kappa)$, and $B^*_2(\Omega, \Gamma, \epsilon, \delta, \kappa) := O(\Omega, \Gamma, \epsilon, \delta) \cap B^*_2(\Omega, \Gamma, \kappa)$.

Proposition 2.12 (cf. [GaYa2]). If $\epsilon, \delta, \kappa$ are sufficiently small, then there exists an analytic map $\Pi_{ac} : C^*_2(\Omega, \Gamma, \epsilon, \delta, \kappa) \rightarrow B^*_2(\Omega, \Gamma, \epsilon, \delta, \kappa)$ such that $\Pi_{ac}|_{B^*_2(\Omega, \Gamma, \epsilon, \delta, \kappa)} \equiv \text{Id}$.

Lemma 2.13. Consider the sequence of multi-indexes $\{\overline{\alpha}_0, \overline{\alpha}_1, \ldots\} \subset \mathcal{I}$ defined by (6). Let $\zeta_0 = (\eta_0, \xi_0) \in \mathcal{B}(Z, W)$ be a four times 1D renormalizable pair. There exists a neighbourhood $\mathcal{N}(\zeta_0) \subset \mathcal{H}(Z, W)$ of $\zeta_0$ such that for any pair $\zeta = (\eta, \xi) \in \mathcal{N}(\zeta_0)$, the pair $R^4(\zeta) = \Lambda(pR^4(\zeta)) := (s^{-1}_{pR^4(\zeta)} \circ \zeta_{R^4(\zeta)}, s^{-1}_{pR^4(\zeta)} \circ \zeta_{R^4(\zeta)}, s^{-1}_{pR^4(\zeta)} \circ \zeta_{R^4(\zeta)}, s^{-1}_{pR^4(\zeta)} \circ \zeta_{R^4(\zeta)})$, is a well defined element of $\mathcal{H}(Z, W)$.

It is instructive to note that $pR^4(\zeta) = (\zeta_{R^4(\zeta)}) = (\eta \circ \xi \circ \eta^2 \circ \eta \circ \eta \circ \eta \circ \xi, \eta \circ \xi \circ \eta \circ \xi \circ \eta \circ \eta^2 \circ \xi)$. Let $\mathcal{D}(\Omega, \Gamma, 0)$ be the subset of $O(\Omega, \Gamma)$ consisting of pairs $\Sigma = (A, B)$ such that the following holds:
The functions $A : \Omega \to \mathbb{C}^2$ and $B : \Gamma \to \mathbb{C}^2$ are of the form
\[ A(x, y) = \begin{bmatrix} \eta(x) \\ h(x) \end{bmatrix} \quad \text{and} \quad B(x, y) = \begin{bmatrix} \xi(x) \\ g(x) \end{bmatrix}. \]

The pair $\zeta := (\eta, \xi)$ is contained in $B(Z, W)$ and is four-times 1D renormalizable.

The function $g$ is conformal on $\eta^2 \circ \xi \circ \eta \circ \xi(U)$ and $\eta^2 \circ \xi(U)$, where $U := \lambda_{p_R^4(\zeta)} Z \cup W$.

Let $D(\Omega, \Gamma, \epsilon) \subset O(\Omega, \Gamma, \epsilon, \infty)$ be a neighbourhood of $D(\Omega, \Gamma, 0)$ consisting of pairs $\Sigma = (A, B)$ such that $\Lambda(\tilde{\Sigma})$, where
\[ \tilde{\Sigma} := (\Sigma^4_1, \Sigma^3_1), \] (17)
is a well-defined element of $O(\Omega, \Gamma)$, and for $V := \lambda_{\Sigma} \Omega \cup \Gamma$:

1. $p_1A$ is conformal on $(p_1A)^{-1}(V)$,
2. $p_1A \circ B$ is conformal on $(p_1A \circ B)^{-1}(V)$, and
3. $p_2B$ is conformal on $p_1A^2 \circ B \circ A \circ B \circ A(V)$ and $p_1A^2 \circ B(V)$.

We define an isometric embedding $\iota$ of the space $\mathcal{H}(Z)$ to $O(\Omega)$ as follows:
\[ \iota(f)(x, y) = \iota(f)(x) := \begin{bmatrix} f(x) \\ f(y) \end{bmatrix}. \] (18)

We extend this definition to an isometric embedding of $\mathcal{H}(Z, W)$ into $O(\Omega, \Gamma)$ as follows:
\[ \iota((\eta, \xi)) := (\iota(\eta), \iota(\xi)). \] (19)

Note that
\[ \iota(B(Z, W)) = B_2(\Omega, \Gamma, 0, 0, 0). \]

Consider the fixed point $\zeta_* = (\eta_*, \xi_*) \in B(Z, W)$ of the 1D renormalization operator $\mathcal{R}$ given in Theorem 1.2. Fix $\epsilon > 0$, and let $\tilde{\mathcal{N}}(\iota(\zeta_*)) \subset \mathcal{D}(\Omega, \Gamma, \epsilon)$ be a neighbourhood of $\iota(\zeta_*)$ whose closure is contained in $\mathcal{D}(\Omega, \Gamma, \epsilon)$.

Let $\Sigma = (A, B) = \left( \begin{bmatrix} a \\ h \end{bmatrix}, \begin{bmatrix} b \\ g \end{bmatrix} \right)$ be a pair contained in $\tilde{\mathcal{N}}(\iota(\zeta_*))$. Denote
\[ \eta_i(x) := p_iA(x) \quad \text{and} \quad \xi_i(x) := p_iB(x) \quad \text{for} \quad i \in \{1, 2\}, \]
and let
\[ \zeta := (\eta_1, \xi_1). \]

Denote
\[ a_y(x) := a(x, y), \]
and consider the following non-linear changes of coordinates:
\[ H(x, y) := \begin{bmatrix} a_y(x) \\ y \end{bmatrix} \quad \text{and} \quad V(x, y) := \begin{bmatrix} x \\ \eta_1 \circ \xi_1 \circ \xi_2^{-1}(y) \end{bmatrix}. \] (20)
Observe that
\[ A \circ H^{-1}(x, y) = \left[ a_y \circ a_y^{-1}(x) g(a_y^{-1}(x), y) \right] = \left[ x g(a_y^{-1}(x), y) \right]. \]
Furthermore,
\[ V \circ H \circ B = \left[ a_g \circ b \eta_1 \circ \xi_1 \circ \xi_2^{-1} g \right]. \]
Thus, we have
\[ \| A \circ H^{-1} \|_y < O(\epsilon) \quad \text{and} \quad \| V \circ H \circ B - \iota(\eta_1 \circ \xi_1) \| < O(\epsilon) \]
where defined.

Let
\[ A_1 := V \circ H \circ A^{-1} \circ \Sigma^4 \circ A \circ H^{-1} \circ V^{-1}, \]
and
\[ B_1 := V \circ H \circ A^{-1} \circ \Sigma^3 \circ A \circ H^{-1} \circ V^{-1}. \]
Define the pre-renormalization of \( \Sigma \) as
\[ pR(\Sigma) = \Sigma_1 := (A_1, B_1). \] (21)
By the definition of \( D(\Omega, \Gamma, \epsilon) \), the pair \( pR \) is a well-defined element of \( O(\lambda_\Sigma \Omega, \lambda_\Sigma \Gamma) \).

From the above inequalities, it follows that
\[ \| pR(\Sigma) - \iota(pR^4(\zeta)) \| < O(\epsilon) \quad \text{and} \quad \| pR(\Sigma) \|_y < O(\epsilon^2). \] (22)

By the argument principle, if \( \epsilon \) is sufficiently small, then the function \( p_1 B_1 \circ A_1 \) has a simple unique critical point \( c_a \) near 0. Set
\[ T_a(x, y) := (x + c_a, y), \] (23)
Likewise, the function \( p_1 T_a^{-1} \circ A_1 \circ B_1 \circ T_a \) has a simple unique critical point \( c_b \) near 0. Set
\[ T_b(x, y) := (x + c_b, y). \] (24)
Note that if \( \Sigma \) is a commuting pair (i.e. \( A \circ B = B \circ A \)), then \( T_b \equiv \text{Id} \).

Define the critical projection of \( pR(\Sigma) \) as
\[ \Pi_{\text{crit}} \circ pR(\Sigma) = (A_2, B_2) := (T_b^{-1} \circ T_a^{-1} \circ A_1 \circ T_a, T_a^{-1} \circ B_1 \circ T_a \circ T_b). \] (25)
Note that
\[ 0 = p_1(B_2 \circ A_2)'(0) = (p_1 A_2)'(0) + O(\epsilon^2), \]
and likewise
\[ 0 = p_1(A_2 \circ B_2)'(0) = (p_1 B_2)'(0) + O(\epsilon^2). \]
Hence,
\[ (p_1 A_2)'(0) = O(\epsilon^2) \quad \text{and} \quad (p_1 B_2)'(0) = O(\epsilon^2). \] (26)
It follows that there exists a uniform constant \( C > 0 \) such that the rescaled pair \( \Lambda \circ \Pi_{\text{crit}} \circ pR(\Sigma) \) is contained in \( C_2(\Omega, \Gamma, C\epsilon^2, C\epsilon^2) \) (recall that this means \( \Lambda \circ \Pi_{\text{crit}} \circ pR(\Sigma) \) is a \( C\epsilon^2 \)-critical pair with \( C\epsilon^2 \) dependence on \( y \) that is \( C\epsilon \) away from the diagonal; see (14)).
Finally, define the 2D renormalization of $\Sigma$ as

$$R(\Sigma) := \Pi_{ac} \circ \Lambda \circ \Pi_{crit} \circ pR(\Sigma),$$

(27)

where the projection map $\Pi_{ac}$ is given in proposition 2.12.

**Proposition 2.14.** If $\Sigma = (A, B) \in \mathcal{D}(\Omega, \Gamma, \epsilon)$ is a commuting pair (i.e. $A \circ B = B \circ A$), then $R(\Sigma)$ is a conjugate of $(\Sigma^{\alpha_4}, \Sigma^{\alpha_3})$.

**Theorem 2.15.** Let $\zeta^*$ be the fixed point of the 1D renormalization given in Theorem 1.2. For any sufficiently small $\epsilon > 0$, let $\hat{\mathcal{N}}(\iota(\zeta^*)) \subset \mathcal{D}(\Omega, \Gamma, \epsilon)$ be a neighbourhood of $\iota(\zeta^*)$ whose closure is contained in $\mathcal{D}(\Omega, \Gamma, \epsilon)$. Then there exists a uniform constant $C > 0$ depending on $\hat{\mathcal{N}}(\iota(\zeta^*))$ such that the 2D renormalization operator

$$R : \mathcal{D}(\Omega, \Gamma, \epsilon) \to \mathcal{O}(\Omega, \Gamma),$$

is a well-defined compact analytic operator satisfying the following properties:

1. $R|_{\hat{\mathcal{N}}(\iota(\zeta^*))} : \hat{\mathcal{N}}(\iota(\zeta^*)) \to \mathcal{B}_2(\Omega, \Gamma, C\epsilon^2, C\epsilon, C\epsilon^2)$.
2. If $\Sigma = (A, B) \in \hat{\mathcal{N}}(\iota(\zeta^*))$ and $\zeta := (p_1 A, p_1 B)$, then

$$\|R(\Sigma) - \iota(R^4(\zeta))\| < C\epsilon.$$

Consequently, if $\mathcal{N}(\zeta^*) \subset \mathcal{B}(Z, W)$ is a neighbourhood of $\zeta^*$ such that $\iota(\mathcal{N}(\zeta^*)) \subset \hat{\mathcal{N}}(\iota(\zeta^*))$, then

$$R \circ \iota|_{\mathcal{N}(\zeta^*)} \equiv \iota \circ R^4|_{\mathcal{N}(\zeta^*)}.$$

3. The pair $\iota(\zeta^*)$ is the unique fixed point of $R$ in $\hat{\mathcal{N}}(\iota(\zeta^*))$.
4. The differential $D_{\iota(\zeta^*)}R$ is a compact linear operator whose spectrum coincides with that of $D_{\zeta^*}R^4$. More precisely, in the spectral decomposition of $D_{\iota(\zeta^*)}R$, the complement to the tangent space $T_{\iota(\zeta^*)}(\iota(\mathcal{N}(\zeta^*)))$ corresponds to the zero eigenvalue.

We denote the stable manifold of the fixed point $\iota(\zeta^*)$ for the 2D renormalization operator $R$ by $W^s(\iota(\zeta^*)) \subset \mathcal{D}(\Omega, \Gamma, \epsilon)$.

Let $H_{\mu, \nu}$ be the Hénon map with a semi-Siegel fixed point $q$ of multipliers $\mu = e^{2\pi i \theta}$ and $\nu$, where $\theta = (\sqrt{5} - 1)/2$ is the inverse golden mean rotation number, and $|\nu| < \epsilon$. We identify $H_{\mu, \nu}$ as a pair in $\mathcal{D}(\Omega, \Gamma, \epsilon)$ as follows:

$$\Sigma_{H_{\mu, \nu}} := \Lambda(H_{\mu, \nu}^2, H_{\mu, \nu}).$$

(28)

The following is shown in [GaRYa]:

**Theorem 2.16.** The pair $\Sigma_{H_{\mu, \nu}}$ is contained in the stable manifold $W^s(\iota(\zeta^*)) \subset \mathcal{D}(\Omega, \Gamma, \epsilon)$ of the fixed point $\iota(\zeta^*)$ for the 2D renormalization operator $R$. 

3. Proof of Main Theorem

3.1. Preliminaries. Let

\[ \zeta_* = (\eta_*, \xi_*) \]

be the fixed point of the 1D renormalization operator \( R \) given in theorem \[ \text{1.2} \] By theorem \[ \text{2.15} \] the fixed point of the 2D renormalization operator

\[ \hat{\mathcal{N}}(\iota(\zeta_*)) \to \mathcal{B}_2(\Omega, \Gamma, C\varepsilon^2, C\varepsilon). \]

is the diagonal embedding \( \iota(\zeta_*) \) of \( \zeta_* \). Thus, we have

\[ \iota(\zeta_*) = R(\iota(\zeta_*)) = \left( (s_*^{-1} \circ \iota(\zeta) \pi_4 \circ s_*, s_*^{-1} \circ \iota(\zeta) \pi_3 \circ s_*) \right), \]

where

\[ s_*(x, y) := (\lambda_* x, \lambda_* y), \quad |\lambda_*| < 1. \]

Let \( \Sigma = (A, B) \) be a pair contained in the stable manifold \( W^s(\iota(\zeta_*)) \) of the fixed point \( \iota(\zeta_*) \). Assume that \( \Sigma \) is commuting, so that

\[ A \circ B = B \circ A. \]

Set

\[ \Sigma_n = (A_n, B_n) = \left( \begin{bmatrix} a_n \\ h_n \end{bmatrix}, \begin{bmatrix} b_n \\ g_n \end{bmatrix} \right) = R^n(\Sigma). \]

Let

\[ \eta_n(x) := p_1 A_n(x) = a_n(x, 0) \quad \text{and} \quad \xi_n(x) := p_1 B_n(x) = b_n(x, 0). \]

By theorem \[ \text{2.15} \] we may express

\[ A_n = \iota(\eta_n) + E_n \quad \text{and} \quad B_n = \iota(\xi_n) + F_n \]

where the error terms \( E_n \) and \( F_n \) satisfy

\[ \|E_n\| < C\varepsilon^{2^n-1} \quad \text{and} \quad \|F_n\| < C\varepsilon^{2^n-1}. \]

Hence, the sequence of pairs \( \{\Sigma_n\}_{n=0}^\infty \) converges to \( \mathcal{B}_2(\Omega, \Gamma, 0, 0, 0) \) super-exponentially.

Let

\[ H_n(x, y) := \begin{bmatrix} a_n(x, y) \\ y \end{bmatrix} \quad \text{and} \quad V_n(x, y) := \begin{bmatrix} x \\ \eta_n \circ \xi_n \circ (p_2 B_n)^{-1}(y) \end{bmatrix}. \]

be the non-linear changes of coordinates given in \[ \text{20} \], let

\[ T_n(x, y) := (x + d_n, y), \]

be the translation map given in \[ \text{23} \], and let

\[ s_n(x, y) := (\lambda_n x, \lambda_n y), \quad |\lambda_n| < 1 \]

be the scaling map so that if

\[ \phi_n := H_n^{-1} \circ V_n^{-1} \circ T_n \circ s_n, \]

then by proposition \[ \text{2.14} \] we have

\[ A_{n+1} = \phi_n^{-1} \circ A_n^{-1} \circ \Sigma_n \circ A_n \circ \phi_n. \]
Denote
\[ \Phi_n := \phi_n \circ \phi_{n+1} \circ \ldots \circ \phi_{k-1} \circ \phi_k, \quad \Omega_n^k := \Phi_n^k(\Omega) \quad \text{and} \quad \Gamma_n^k := \Phi_n^k(\Gamma). \]

Define
\[
U_n^k := \bigcup_{\alpha < \alpha_n} \Sigma_n^{-\alpha} (\Omega_n^k) \quad \text{and} \quad V_n^k := \bigcup_{\alpha < \alpha_n} \Sigma_n^{-\alpha} (\Gamma_n^k).
\]

It is not hard to see that \( \{U_n^k \cup V_n^k\}_{k=n}^{\infty} \) form a nested sequence. Define the renormalization arc of \( \Sigma_n \) as
\[
\gamma_n := \bigcap_{k=n}^{\infty} U_n^k \cup V_n^k.
\]

**Proposition 3.1.** The renormalization arc \( \gamma_n \) is invariant under the action of \( \Sigma_n \). Moreover, if
\[
p_n^k := \bigcup_{\alpha < \alpha_n} \Sigma_n^{-\alpha} (\Phi_n^k(\gamma_k \cap \Omega)) \quad \text{and} \quad q_n^k := \bigcup_{\alpha < \alpha_n} \Sigma_n^{-\alpha} (\Phi_n^k(\gamma_k \cap \Gamma)),
\]
then
\[
\gamma_n = p_n^k \cup q_n^k.
\]

Let \( \theta_* = (\sqrt{5} - 1)/2 \) be the golden mean rotation number, and let
\[
I_L := [-\theta_*, 0] \quad \text{and} \quad I_R := [0, 1].
\]

Define \( L : I_L \to \mathbb{R} \) and \( R : I_R \to \mathbb{R} \) as
\[
L(t) := t + 1 \quad \text{and} \quad R(t) := t - \theta_*.
\]

The pair \((R, L)\) represents rigid rotation of \( \mathbb{R}/\mathbb{Z} \) by angle \( \theta_* \).

The following is a classical result about the renormalization of 1D pairs.

**Proposition 3.2.** Suppose \( \|\Sigma\|_y = 0 \). Then for every \( n \geq 0 \), there exists a quasisymmetric homeomorphism between \( I_L \cup I_R \) and the renormalization arc \( \gamma_n \) that conjugates the action of \( \Sigma_n = (A_n, B_n) \) and the action of \( (R, L) \). Moreover, the renormalization arc \( \gamma_n \) contains the unique critical point \( c_n = 0 \) of \( \eta_n \).

The following is shown in [GaRYa].

**Theorem 3.3.** Let \( \Sigma = (A, B) \) be a commuting pair contained in the stable manifold \( W^s(\nu(\zeta_*)) \) of the 2D renormalization fixed point \( \iota(\zeta_*) \). Then for every \( n \geq 0 \), there exists a homeomorphism between \( I_L \cup I_R \) and the renormalization arc \( \gamma_n \) that conjugates the action of \( \Sigma_n = (A_n, B_n) \) and the action of \( (R, L) \). Moreover, this conjugacy cannot be \( C^1 \) smooth.

Theorem 1.2 follows from the above statement and the following:
Theorem 3.4 (GaRYa). Suppose 
\[ \Sigma = \Sigma_{H_{\mu,\nu}}; \]
where \( \Sigma_{H_{\mu,\nu}} \) is the renormalization of the Hénon map given in theorem 2.16. Then the linear rescaling of the renormalization arc \( s_0(\gamma_0) \) is contained in the boundary of the Siegel disc \( \Delta \) of \( H_{\mu,\nu} \). In fact, we have 
\[ \partial \Delta = s_0(\gamma_0) \cup H_{\mu,\nu} \circ s_0(\gamma_0). \]
Henceforth, we consider the renormalization arc of \( \Sigma_n \) as a continuous curve 
\[ \gamma_n = \gamma_n(t) \]
parameterized by \( I_L \cup I_R \). The components of \( \gamma_n \) are denoted 
\[ \gamma_n(t) = \begin{bmatrix} \gamma^x_n(t) \\ \gamma^y_n(t) \end{bmatrix}. \]
Lastly, denote the renormalization arc of \( \iota(\zeta_*) \) by 
\[ \gamma_*(t) = \begin{bmatrix} \gamma^x_*(t) \\ \gamma^y_*(t) \end{bmatrix}. \]
The following are consequences of Theorem 2.15.

Corollary 3.5. As \( n \to \infty \), we have the following convergences (each of which occurs at a geometric rate):
1. \( \eta_n \to \eta_* \),
2. \( \lambda_n \to \lambda_* \) (hence \( s_n \to s_* \)),
3. \( \phi_n \to \psi_* \), where 
\[ \psi_*(x,y) = \begin{bmatrix} \eta_*^{-1}(\lambda_* x) \\ \eta_*^{-1}(\lambda_* y) \end{bmatrix}, \] and
4. \( \gamma_n \to \gamma_* \) (hence \( |\gamma^x_*(0)| \to 0 \)).

3.2. Normality of the compositions of scope maps. Define 
\[ \psi_n(x,y) := \begin{bmatrix} \eta_n^{-1}(\lambda_n x) \\ \eta_n^{-1}(\lambda_n y) \end{bmatrix}. \]
For \( n \leq k \), denote 
\[ \Psi^k_n := \psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_{k-1} \circ \psi_k. \]
Let 
\[ \left[ \begin{array}{cc} \sigma_k & 0 \\ 0 & \sigma_k \end{array} \right] := (D_{(0,0)}\Psi^k_n)^{-1}. \]

Proposition 3.6. The family \( \{\sigma_n^k\Psi^k_n\}_{n=1}^{\infty} \) is normal.

Proof. By corollary 3.5 there exists a domain \( U \subset \mathbb{C}^2 \) and a uniform constant \( c < 1 \) such that for all \( k \) sufficiently large, the map \( \psi_k \) is well defined on \( U \), and 
\[ \Omega \cup A_{k+1}(\Omega) \cup \Gamma \cup B_{k+1}(\Gamma) \subseteq cU. \]
Thus, by choosing a smaller domain $U$ if necessary, we can assume that $\psi_k$ and hence, $\Psi^k_k$ extends to a strictly larger domain $V \ni U$. It follows from applying Koebé distortion theorem to the first and second coordinate that $\{\sigma_n^k \Psi^k_n\}_{k=n}^\infty$ is a normal family. □

**Proposition 3.7.** There exists a uniform constant $M > 0$ such that
$$||\phi_n - \psi_n|| < M \epsilon^{2^{n-1}}.$$  

**Proof.** The result follows readily from (29) and (30). □

**Proposition 3.8.** There exists a uniform constant $K > 0$ such that
$$\sigma_n^k||\Phi^k_n - \Psi^k_n|| < K \epsilon^{2^{n-1}}.$$  

**Proof.** By proposition 3.7, we have
$$\phi_{k-1} = \psi_{k-1} + \tilde{E}_{k-1} \quad \text{and} \quad \phi_k = \psi_k + E_k,$$
where $||\tilde{E}_{k-1}|| < M \epsilon^{2^{k-2}}$ and $||E_k|| < M \epsilon^{2^{k-1}}$. Observe that
$$\phi_{k-1} \circ \phi_k = \phi_{k-1} \circ (\psi_k + E_k)$$
$$= \phi_{k-1} \circ \psi_k + \tilde{E}_k$$
$$= (\psi_{k-1} + \tilde{E}_{k-1}) \circ \psi_k + \tilde{E}_k$$
$$= \psi_{k-1} \circ \psi_k + \tilde{E}_{k-1} \circ \psi_k + \tilde{E}_k,$$
where $||\tilde{E}_k|| < L \epsilon^{2^{k-1}}$ for some uniform constant $L > 0$ by corollary 3.5. Let
$$E_{k-1} := \tilde{E}_{k-1} + \tilde{E}_k \circ \psi_{k-1}^{-1}.$$  

By corollary 3.5, $\psi_k^{-1}$ is uniformly bounded, and hence, we have
$$||E_{k-1}|| < M \epsilon^{2^{k-1}} + 2L \epsilon^{2^{k-1}} < 2M \epsilon^{2^{k-2}}.$$  

Thus, we have
$$\phi_{k-1} \circ \phi_k = \psi_{k-1} \circ \psi_k + E_{k-1} \circ \psi_k.$$  

Proceeding by induction, we obtain
$$\Phi^k_n = \Psi^k_n + E_n \circ \psi_{n+1} \circ \ldots \circ \psi_k,$$
where
$$||E_n|| < 2M \epsilon^{2^{n-1}}.$$  

By definition, we have
$$\sigma_n^k(\psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_k)'(0) = 1.$$  

Factor the scaling constant as
$$\sigma_n^k := \dot{\sigma}_n^k \sigma_{n+1}^k.$$
so that

$$|\dot{\sigma}_n^k(\psi_{n+1} \circ \ldots \circ \psi_k(0))| = 1,$$

and

$$|\sigma_{n+1}^k(\psi_{n+1} \circ \ldots \circ \psi_k)'(0)| = 1.$$  

Let

$$M := \sup_{x \in Z} \eta_n'(x).$$

Observe that $\dot{\sigma}_n^k$ is uniformly bounded by $\lambda_n^{-1}M$. Moreover, by proposition 3.6 we have that $\sigma_{n+1}^k(\psi_{n+1} \circ \ldots \circ \psi_k)'$ is also uniformly bounded. Therefore,

$$||\sigma_n^k(E_n \circ \psi_{n+1} \ldots \circ \psi_n)'|| = ||\dot{\sigma}_n^k E'_n(\psi_{n+1} \ldots \circ \psi_n)|| \cdot ||\sigma_{n+1}^k(\psi_{n+1} \circ \psi_n)'||$$

$$< K \epsilon^{2^n-1}$$

for some universal constant $K > 0$. \hfill $\square$

By proposition 3.6 and 3.8 we have the following theorem.

**Theorem 3.9.** The family $\{\sigma_n^k \Phi_n^k\}_{k=n}^\infty$ is normal.

3.3. The boundary of the Siegel disk is not smooth.

3.4. The boundary of the Siegel disk is not smooth. Let $[t_l, t_r] \subset \mathbb{R}$ be a closed interval, and let $C : [t_l, t_r] \rightarrow \mathbb{C}$ be a smooth curve. For any subset $N \subset \mathbb{C}$ intersecting the curve $C$, we define the angular deviation of $C$ on $N$ as

$$\Delta_{\text{arg}}(C, N) := \sup_{t,s \in C^{-1}(N)} |\text{arg}(C'(t)) - \text{arg}(C'(s))|,$$  \hspace{1cm} (33)

where the function $\text{arg} : \mathbb{C} \rightarrow \mathbb{R}/\mathbb{Z}$ is defined as

$$\text{arg}(re^{2\pi i \theta}) := \theta.$$  \hspace{1cm} (34)

**Lemma 3.10.** Let $\theta \in \mathbb{R}/\mathbb{Z}$, and let $C_\theta : [0, 1] \rightarrow \mathbb{C}$ be a smooth curve such that $C_\theta(0) = 0$ and $C_\theta(1) = e^{2\pi i \theta}$. Then for some $t \in [0, 1]$, we have

$$\text{arg}(C_\theta'(t)) = \theta.$$

**Lemma 3.11.** Let

$$q_2(x) := x^2 \quad \text{and} \quad A^R_r := \{ z \in \mathbb{C} \mid r < |z| < R \}. \hspace{1cm} (35)$$

Suppose $C : [t_l, t_r] \rightarrow \mathbb{D}_R$ is a smooth curve such that $|C(t_l)| = |C(t_r)| = R$, and $|C(t_0)| < r$ for some $t_0 \in [t_l, t_r]$. Then for every $\delta > 0$, there exists $M > 0$ such that if $\text{mod}(A^R_r) > M$, then either $\Delta_{\text{arg}}(C, \mathbb{D}_R)$ or $\Delta_{\text{arg}}(q_2 \circ C, \mathbb{D}_{R^2})$ is greater than $1/6 - \delta$. 

Proof. Without loss of generality, assume that $R = 1$, and $C(t_r) = 1$. We prove the case when $r = 0$, so that $C(t_0) = 0$. The general case follows by continuity.

Suppose that $\Delta \text{arg}(C, D) < 1/6$. Then by lemma 3.10, we have

$$1/3 < \text{arg}(C(t)) < 2/3.$$  

This implies that

$$-1/3 < 2 \text{arg}(C(t)) < 1/3.$$  

Hence, by lemma 3.10, we have $\Delta \text{arg}(q_2 \circ C, D_R^2) > 1/6$. □

Corollary 3.12. Let $W \subset \mathbb{C}$ be a simply connected neighbourhood of 0, let $C : [t_l, t_r] \to \overline{W}$ and $E : [t_l, t_r] \to \mathbb{C}$ be smooth curves, and let $f : W \to \mathbb{C}$ be a holomorphic function with a unique simple critical point at $c \in D_r$ for $r < 1$.

Consider the smooth curve $\tilde{C} := f \circ C + E$.

Suppose $C(t_l), C(t_r) \in \partial W$, and $|C(t_0)| < r$ for some $t_0 \in [t_l, t_r]$. Then for every $\delta > 0$, there exists $\epsilon > 0$ and $M > 0$ such that if $\|E\| < \epsilon$ and $\text{mod}(W \setminus D_r) > M$, then either $\Delta \text{arg}(C, W)$ or $\Delta \text{arg}(\tilde{C}, f(W))$ is greater than $1/6 - \delta$.

Let $U \subset \mathbb{Z} \subset \mathbb{C}$ be a simply-connected domain containing the origin. For all $k$ sufficiently large, the unique critical point $c_k$ of $\eta_k$ is contained in $U$. Let $V_k := \eta_k(U)$. Then there exists conformal maps $u_k : (D, 0) \to (U, c_k)$ and $v_k : (D, 0) \to (V_k, \eta_k(c_k))$ such that the following diagram commutes:

$$\begin{array}{ccc}
D & \xrightarrow{u_k} & U \\
\downarrow{q_2} & & \downarrow{\eta_k} \\
D & \xrightarrow{v_k} & V_k 
\end{array}$$

By corollary 3.5, we have the following result:

Proposition 3.13. The maps $u_k : (D, 0) \to (U, c_k)$ and $v_k : (D, 0) \to (V_k, \eta_k(c_k))$ converge to conformal maps $u_* : (D, 0) \to (U, 0)$ and $v_* : (D, 0) \to (\eta_*(U), \eta_*(0))$. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
D & \xrightarrow{u_*} & U \\
\downarrow{q_2} & & \downarrow{\eta_*} \\
D & \xrightarrow{v_*} & \eta_*(U) 
\end{array}$$

Proof of Non-smoothness. By theorem 3.9, the sequence $\{\sigma^k_0 \Phi^k_0\}_{k=0}^\infty$ has a converging subsequence. By replacing the sequence by this subsequence if necessary,
assume that $\{\sigma_k^k\phi_0^k\}_{k=0}^{\infty}$ converges. Consider the following commutative diagrams:

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{u_k} & U \\
\downarrow q_2 & & \downarrow \eta_k \\
\mathbb{D} & \xrightarrow{v_k} & V_k
\end{array}
\quad
\begin{array}{ccc}
\Omega & \xrightarrow{\phi_0^k} & \Omega \\
\downarrow A_k & & \downarrow A_0 \\
A_k(\Omega) & \xrightarrow{\phi_0^k} & A_0(\Omega)
\end{array}
$$

Let $\delta > 0$. Since $\{\sigma_k^k\phi_0^k\}_{k=0}^{\infty}$ converges, we can choose $R > 0$ sufficiently small so that if $X_k := u_k(\mathbb{D}_R) \subset U_k$, and $Y_k := v_k(\mathbb{D}_{R^2}) \subset V_k$, then for any smooth curves $C_1 \subset \Omega := U \times U$ and $C_2 \subset A_k(\Omega)$ intersecting $X_k \times X_k$ and $Y_k \times Y_k$ respectively, we have

$$\kappa \Delta_{\text{arg}}(C_1, X_k \times X_k) < \Delta_{\text{arg}}(\phi_0^k \circ C_1, \phi_0^k(X_k \times X_k))$$

and

$$\kappa \Delta_{\text{arg}}(C_2, Y_k \times Y_k) < \Delta_{\text{arg}}(\phi_0^k \circ C_2, \phi_0^k(Y_k \times Y_k))$$

for some uniform constant $\kappa > 0$.

Consider the renormalization arc of $\Sigma_n$:

$$\gamma_n(t) = \begin{bmatrix} \gamma_n^x(t) \\ \gamma_n^y(t) \end{bmatrix}.$$  

Recall that we have

$$\gamma_n^x, \gamma_n^y \to \gamma_s \quad \text{as} \quad n \to \infty,$$

where $\gamma_s$ is the renormalization arc of the 1D renormalization fixed point $\zeta_s$. Now, choose $r > 0$ is sufficiently small so that the annulus $X_k \setminus \mathbb{D}_r$ satisfies the condition of Corollary 3.12 for the given $\delta$. Next, choose $K$ sufficiently large so that for all $k > K$, we have

$$|c_k|, |\gamma_k^x(0)| < r,$$

and

$$A_k = \begin{bmatrix} a_k \\ h_k \end{bmatrix} = \iota(\eta_k) + (e_x, e_y),$$

such that $\|E := (e_x, e_y)\| < \epsilon$, where $\epsilon$ is given in Corollary 3.12.

Now, suppose towards a contradiction that the renormalization arc $\gamma_0$ of $\Sigma_0$, and hence the renormalization arc $\gamma_k$ of $\Sigma_k$ for all $k \geq 0$, are smooth. By the above estimates, we can conclude:

$$\Delta_{\text{arg}}(\gamma_0, \phi_0^k(X_k \times X_k)) = \Delta_{\text{arg}}(\phi_0^k \circ \gamma_k, \phi_0^k(X_k \times X_k)) > \kappa \Delta_{\text{arg}}(\gamma_k, X_k \times X_k),$$

and

$$\Delta_{\text{arg}}(\gamma_k^x, X_k) > \kappa \Delta_{\text{arg}}(\gamma_k^x, X_k).$$
and
\[
\Delta_{\arg}(\gamma_0, \Phi_0^k(Y_k \times Y_k)) = \Delta_{\arg}(\Phi_0^k \circ \gamma_k, \Phi_0^k(Y_k \times Y_k))
\]
\[
> \kappa \Delta_{\arg}(\gamma_k, Y_k \times Y_k)
\]
\[
= \kappa \Delta_{\arg}(A_k \circ \gamma_k, Y_k \times Y_k)
\]
\[
> \kappa \Delta_{\arg}(a_k \circ \gamma_k, Y_k)
\]
\[
= \kappa \Delta_{\arg}(\eta_k \circ \gamma_k^x + e_x(\gamma_k), Y_k).
\]

By Lemma 3.12, either \(\Delta_{\arg}(\gamma_k^x, X_k)\) or \(\Delta_{\arg}(a_k \circ \gamma_k, Y_k)\) is greater than \(1/6 - \delta\).

Hence,
\[
\max\{\Delta_{\arg}(\gamma_0, \Phi_0^k(X_k \times X_k)), \Delta_{\arg}(\gamma_0, \Phi_0^k(Y_k \times Y_k))\} > l
\]
for some uniform constant \(l > 0\). Since \(\Phi_0^k(X_k \times X_k)\) and \(\Phi_0^k(Y_k \times Y_k)\) both converge to a point in \(\gamma_0\) as \(k \to \infty\), this is a contradiction. \(\square\)

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