On a Koolen – Park inequality and Terwilliger graphs

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July 21, 2010

Abstract

J.H. Koolen and J. Park have proved a lower bound for intersection number $c_2$ of a distance-regular graph $\Gamma$. Moreover, they showed that the graph $\Gamma$ which attains the equality in this bound is a Terwilliger graph. We prove that $\Gamma$ is the icosahedron, the Doro graph or the Conway-Smith graph, if equality is attained and $c_2 \geq 2$.

Key Words: Terwilliger graphs, distance-regular graphs

*Partially supported by RFFI grant (project no. 08-01-00009).
1 Introduction

Let $\Gamma$ be a distance-regular graph with degree $k$ and diameter at least 2. Let $c$ be maximal such that for each vertex $x \in \Gamma$ and every pair of nonadjacent vertices $y, z$ of $\Gamma_1(x)$, there exists a $c$-coclique in $\Gamma_1(x)$ containing $y, z$. In [1], J.H. Koolen and J. Park have shown that the following bound holds:

$$c^2 - 1 \geq \max\left\{ \frac{c'(a_1 + 1) - k}{\binom{c}{2}} \mid 2 \leq c' \leq c \right\},$$

and equality implies $\Gamma$ is a Terwilliger graph. (For definitions see sections 2 and 3.)

The similar inequality for a distance-regular graph with $c$-claw was proved by C.D. Godsil, see [2]. J.H. Koolen and J. Park [1] have noted that the bound (1) is met exactly for all known examples of Terwilliger graphs. We recall that only three examples of distance-regular Terwilliger graphs with $c^2 \geq 2$ are known: the icosahedron, the Doro graph and the Conway-Smith graph.

In this paper, we will show that the distance-regular graph $\Gamma$ with $c^2 \geq 2$ which attains the equality in (1) is a known Terwilliger graph.

2 Definitions and preliminaries

We consider only finite, undirected graphs without loops or multiple edges. Let $\Gamma$ be a connected graph. The distance $d(u, w)$ between any two vertices $u$ and $w$ of $\Gamma$ is the length of a shortest path from $u$ to $w$ in $\Gamma$. The diameter $\text{diam}(\Gamma)$ of $\Gamma$ is the maximal distance occurring in $\Gamma$.

For a subset $A$ of the vertex set of $\Gamma$, we will also write $A$ for the subgraph of $\Gamma$ induced by $A$. For a vertex $u$ of $\Gamma$, define $\Gamma_i(u)$ to be the set of vertices which are at distance $i$ from $u$ ($0 \leq i \leq \text{diam}(\Gamma)$). The subgraph $\Gamma_1(u)$ is called the local graph of a vertex $u$ and the degree of $u$ is the number of neighbours of $u$, i.e. $|\Gamma_1(u)|$.

For two vertices $u, w \in \Gamma$ with $d(u, w) = 2$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ is called $\mu$-subgraph of vertices $u, w$. We say the number $\mu(\Gamma)$ is well-defined, if each $\mu$-subgraph occurring in $\Gamma$ contains the same number of vertices which is equal to $\mu(\Gamma)$.

Let $\Delta$ be a graph. A graph $\Gamma$ is locally $\Delta$, if, for all $u \in \Gamma$, the subgraph $\Gamma_1(u)$ is isomorphic to $\Delta$. A graph is regular with degree $k$, if the degree of each its vertex is $k$. 

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A connected graph $\Gamma$ with diameter $d = \text{diam}(\Gamma)$ is distance-regular, if there are integers $b_i, c_i$ ($0 \leq i \leq d$) such that for any two vertices $u, w \in \Gamma$ with $d(u, w) = i$, there are exactly $c_i$ neighbours of $w$ in $\Gamma_{i-1}(u)$ and $b_i$ neighbours of $w$ in $\Gamma_{i+1}(u)$ (we assume that $\Gamma_{-1}(u)$ and $\Gamma_{d+1}(u)$ are empty sets). In particular, distance-regular graph $\Gamma$ is regular with degree $b_0 = 0$ and $c_1 = 1$ and $c_2 = \mu(\Gamma)$. For each vertex $u \in \Gamma$ and $0 \leq i \leq d$, the subgraph $\Gamma_i(u)$ is regular with degree $a_i = b_0 - b_i - c_i$. The numbers $b_i, c_i$ ($0 \leq i \leq d$) are called the intersection numbers and the array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, is called the intersection array of distance-regular graph $\Gamma$.

A graph $\Gamma$ is amply regular with parameters $(v, k, \lambda, \mu)$, if $\Gamma$ has $v$ vertices, it is regular with degree $k$ and the following two conditions hold:

i) for each pair of adjacent vertices $u, w \in \Gamma$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ contains exactly $\lambda$ vertices;

ii) $\mu = \mu(\Gamma)$ is well-defined.

An amply regular graph with diameter 2 is called a strongly regular graph and it is a distance-regular graph. A distance-regular graph is an amply regular graph with parameters $k = b_0$, $\lambda = b_0 - b_1 - 1$ and $\mu = c_2$. 

Recall that a $(c-)\text{clique}$ (or complete graph) is a graph (on $c$ vertices) in which every pair of its vertices is adjacent. A $(c-)\text{coclique}$ is a graph (on $c$ vertices) in which every pair of its vertices is not adjacent.

Let $\Gamma$ be a strongly regular graph with parameters $(v, k, \lambda, 1)$. There are integers $r$ and $s$ such that the local graph of each vertex of $\Gamma$ is the disjoint union of $r$ copies of $s$-clique. Furthermore, $v = 1 + rs + s^2r(r - 1)$, $k = rs$ and $\lambda = s - 1$. We denote the set of strongly regular graphs with such parameters by $F(s, r)$.

Any graph of $F(1, r)$, i.e. a strongly regular graph with $\lambda = 0$ and $\mu = 1$, is called a Moore strongly regular graph. It is well known (see Chapter 1 [3]) that any Moore strongly regular graph has degree 2, 3, 7 or 57. The graphs with degree 2, 3 and 7 are the pentagon, the Petersen graph and the Hoffman-Singleton graph, respectively. Whether a Moore graph with degree 57 exists is an open problem.

**Lemma 2.1** Suppose that $F(s, r)$ is nonempty set of graphs. Then $s+1 \leq r$.

**Proof.** Let $\Gamma$ be a graph of $F(s, r)$. We may choose vertices $u$ and $w$ of $\Gamma$ with $d(u, w) = 2$. Let $x$ be a vertex of $\Gamma_1(u) \cap \Gamma_1(w)$. Then the subgraph $\Gamma_1(w) - (\Gamma_1(x) \cup \{x\})$ contains a coclique of size at most $r-1$. Let us consider a
s-clique of $\Gamma_1(u) - \Gamma_1(w)$ on vertices $y_1, y_2, ..., y_s$. The subgraph $\Gamma_1(w) \cap \Gamma_1(y_i)$ ($1 \leq i \leq s$) contains a single vertex $z_i$. The vertices $z_1, z_2, ..., z_s$ are mutually nonadjacent and distinct. Hence, $s \leq r - 1$. The lemma is proved.

3 Terwilliger graphs

In this section we give a definition of Terwilliger graphs and some useful facts concerning them.

A Terwilliger graph is a connected noncomplete graph $\Gamma$ such that $\mu(\Gamma)$ is well-defined and each $\mu$-subgraph occurring in $\Gamma$ is a complete graph (hence, there are no induced quadrangles in $\Gamma$). If $\mu(\Gamma) > 1$, then, for each vertex $u \in \Gamma$, the local graph of $u$ will also be a Terwilliger graph with diameter 2 and $\mu(\Gamma_1(u)) = \mu(\Gamma) - 1$.

For an integer $\alpha \geq 1$, a $\alpha$-clique extension of a graph $\bar{\Gamma}$ is the graph $\Gamma$ obtained from $\bar{\Gamma}$ by replacing each vertex $\bar{u} \in \bar{\Gamma}$ by a clique $U$ of $\alpha$ vertices, where for any $\bar{u}, \bar{w} \in \bar{\Gamma}$, $u \in U$ and $w \in W$, $\bar{u}$ and $\bar{w}$ are adjacent if and only if $u$ and $w$ are adjacent.

**Lemma 3.1** Let $\Gamma$ be an amply regular Terwilliger graph with parameters $(v, k, \lambda, \mu)$, where $\mu > 1$. There is the number $\alpha$ such that the local graph of each its vertex is the $\alpha$-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$, where

$$\bar{v} = k/\alpha, \quad \bar{k} = (\lambda - \alpha + 1)/\alpha, \quad \bar{\mu} = (\mu - 1)/\alpha,$$

and $\alpha \leq \bar{\lambda} + 1$. In particular, if $\bar{\lambda} = 0$, then $\alpha = 1$.

**Proof.** The result follows from [3, Theorem 1.16.3].

We know only three examples of amply regular Terwilliger graphs with $\mu \geq 2$. All of them are unique distance-regular locally Moore graphs:

1. the icosahedron with intersection array $\{5, 2, 1; 1, 2, 5\}$ is locally pentagon.

2. the Doro graph with intersection array $\{10, 6, 4; 1, 2, 5\}$ and the Conway-Smith graph with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ are locally Petersen graphs.

In [4], A. Gavrilyuk and A. Makhnev have shown that a distance-regular locally Hoffman – Singleton graph has intersection array $\{50, 42, 9; 1, 2, 42\}$.
or \{50, 42, 1; 1, 2, 50\} and hence it is a Terwilliger graph. Whether the graphs with these intersection arrays exist is an open question.

**Lemma 3.2** Let \( \Gamma \) be a Terwilliger graph. Suppose that, for an integer \( \alpha \geq 1 \), the local graph of each its vertex is the \( \alpha \)-clique extension of a Moore strongly regular graph \( \Delta \). Then \( \alpha = 1 \) and one of the following holds:

1. \( \Delta \) is the pentagon and \( \Gamma \) is the icosahedron;
2. \( \Delta \) is the Petersen graph and \( \Gamma \) is the Doro graph or the Conway-Smith graph;
3. \( \Delta \) is the Hoffman – Singleton graph or a graph with degree 57, in both cases diameter of \( \Gamma \) is at least 3.

**Proof.** It is easy to see that the graph \( \Gamma \) is amply regular. By Lemma 3.1, we have \( \alpha = 1 \). The statements (1) and (2) follow from [3, Proposition 1.1.4] and [3, Theorem 1.16.5], respectively.

If the graph \( \Delta \) is the Hoffman – Singleton graph and diameter of \( \Gamma \) is 2, then \( \Gamma \) is strongly regular with parameters \((v, k, \lambda, \mu)\), where \( k = 50, \lambda = 7 \) and \( \mu = 2 \). By [3, Theorem 1.3.1], the eigenvalues of \( \Gamma \) are \( k \) and the roots of the quadratic equation \( x^2 + (\mu - \lambda)x + (\mu - k) = 0 \). The roots of the equation \( x^2 - 5x - 48 = 0 \) are not integers, that is impossible. In the remained case, when \( \Delta \) is regular with degree 57, we will get the same contradiction. The lemma is proved.

The next lemma is useful in the proof of Theorem 4.2 (see Section 4).

**Lemma 3.3** Let \( \Gamma \) be a strongly regular Terwilliger graph with parameters \((v, k, \lambda, \mu)\). Suppose that, for an integer \( \alpha \geq 1 \), the local graph of each its vertex is the \( \alpha \)-clique extension of a strongly regular graph \( \Delta \) with parameters \((\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})\). Then \( \bar{k} - \bar{\lambda} - \bar{\mu} > 1 \) implies that \( k - \lambda - \mu > 1 \).

**Proof.** We have \( k = \alpha(1 + \hat{k} + \tilde{k}(\tilde{k} - \tilde{\lambda} - 1)/\tilde{\mu}), \lambda = \alpha\hat{k} + \alpha - 1 \) and \( \mu = \alpha\bar{\mu} + 1 \). If \( \hat{k} - \bar{\lambda} - \bar{\mu} > 1 \), then \( \bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} > \tilde{k} \) and this implies that \( k - \lambda - \mu = \alpha(\tilde{k}(\tilde{k} - \tilde{\lambda} - 1)/\tilde{\mu} - \tilde{\mu}) > \alpha(\tilde{k} - \tilde{\mu}) > \alpha(\tilde{\lambda} + 1) \geq 1 \). \( \blacksquare \)

## 4 Koolen – Park inequality

In this section, we consider the bound (11) and classify distance-regular graphs with \( c_2 \geq 2 \) which attain this bound.
The next proposition is a slight generalization of [1, Proposition 3]. J.H. Koolen and J. Park [1, Proposition 3] formulated the next proposition for distance-regular graphs. We generalize it to amply regular graphs. (Our proof is similar to the one in J.H. Koolen and J. Park [1], but we give it for convenience of the reader.)

**Proposition 4.1** Let $\Gamma$ be an amply regular graph with parameters $(v, k, \lambda, \mu)$ and $c \geq 2$ be maximal such that for each vertex $x \in \Gamma$ and every pair of non-adjacent vertices $y, z$ of $\Gamma_1(x)$, there exists a $c$-coclique in $\Gamma_1(x)$ containing $y, z$. Then

$$\mu - 1 \geq \max\left\{\frac{c'(\lambda + 1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\right\},$$

and if equality is attained, then $\Gamma$ is a Terwilliger graph.

**Proof.** Let $\Gamma_1(x)$ contain a coclique $C'$ on vertices $y_1, y_2, \ldots, y_{c'}$, $c' \geq 2$. Since $d(y_i, y_j) = 2$, $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| \leq \mu - 1$ holds for all $i \neq j$. Then by the principle of inclusion and exclusion,

$$k = |\Gamma_1(x)| \geq \left| \bigcup_{i=1}^{c'} (\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\})) \right|\geq \sum_{i=1}^{c'} |\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\})| - \sum_{1 \leq i < j \leq c'} |\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| \geq c'(\lambda + 1) - \binom{c'}{2}(\mu - 1).$$

So,

$$\mu - 1 \geq \frac{c'(\lambda + 1) - k}{\binom{c'}{2}}. \quad (2)$$

Note that equality in $(2)$ implies that $\Gamma_1(x) \subseteq \bigcup_{i=1}^{c'} (\Gamma_1(y_i) \cup \{y_i\})$ holds and we have $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| = \mu - 1$ for all $i \neq j$.

Let $c$ be maximal satisfying the condition of the Proposition 4.1. Then

$$\mu - 1 \geq \max\left\{\frac{c'(\lambda + 1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\right\}. \quad (3)$$

We may assume that for an integer $c''$, where $2 \leq c'' \leq c$, equality holds in $(2)$, i.e.

$$\mu - 1 = \frac{c''(\lambda + 1) - k}{\binom{c''}{2}} = \max\left\{\frac{c'(\lambda + 1) - k}{\binom{c'}{2}} \mid 2 \leq c' \leq c\right\}. \quad (4)$$

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We will show \( c = c'' \). For a vertex \( x \in \Gamma \) and nonadjacent vertices \( y, z \in \Gamma_1(x) \), there exists a \( c \)-coclique \( C \) in \( \Gamma_1(x) \) containing \( y, z \). The equality (1) implies that, for any subset of vertices \( \{y_1, y_2, \ldots, y_{\alpha c}\} \subseteq C \), \( \Gamma_1(x) \subseteq \bigcup_{i=1}^{\alpha c} (\Gamma_1(y_i) \cup \{y_i\}) \) holds. But if \( c'' < c \), then \( C \not\subseteq \bigcup_{i=1}^{\alpha c} (\Gamma_1(y_i) \cup \{y_i\}) \), which is the contradiction.

Hence, \( c = c'' \) and we have \( |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = \mu - 1 \) for every pair of nonadjacent vertices \( y, z \in \Gamma_1(x) \) and for all \( x \in \Gamma \). This implies that each \( \mu \)-subgraph occuring in \( \Gamma \) is a clique of size \( \mu \) and \( \Gamma \) is a Terwilliger graph.

We call the inequality (3) \( \mu \)-bound.

Let \( \Gamma \) be an amply regular Terwilliger graph with parameters \((v, k, \lambda, \mu)\). If \( \mu = 1 \), then the local graph of each its vertex is the disjoint union of \( k/(\lambda + 1) \) copies of \((\lambda + 1)\)-clique, so equality in \( \mu \)-bound is attained. If \( \mu \geq 2 \), then we know only three examples of \( \Gamma \) (see Section 3) with \( \mu = 2 \) and each of them attains equality in \( \mu \)-bound:

1. \( \Gamma \) is the icosahedron. The pentagon contains a 2-coclique and is regular with degree 2, i.e. \( c = 2 \) and \( \lambda = 2 \), hence we have \( (2 \cdot (2+1) - 5)/(\binom{2}{2}) = 1 = \mu - 1 \).

2. \( \Gamma \) is the Doro graph or the Conway-Smith graph. The Petersen graph contains a 4-coclique and is regular with degree 3, hence we have \( (4 \cdot (3 + 1) - 10)/(\binom{4}{2}) = (16 - 10)/6 = 1 = \mu - 1 \).

Recall that the Hoffman – Singleton graph contains a 15-coclique. If \( \Gamma \) is an amply regular locally Hoffman – Singleton graph and is a Terwilliger graph, then \( \mu = 2 \), but equality in \( \mu \)-bound is not attained.

\textbf{Theorem 4.2} Let \( \Gamma \) be an amply regular graph with parameters \((v, k, \lambda, \mu)\) and \( \mu > 1 \). If \( \Gamma \) attains equality in \( \mu \)-bound, then \( \mu = 2 \) and \( \Gamma \) is the icosahedron, the Doro graph or the Conway-Smith graph.

\textit{Proof.} By Proposition 3.1, the graph \( \Gamma \) is a Terwilliger graph and, by Lemma 3.1, there is an integer \( \alpha \geq 1 \) such that the local graph of each its vertex is the \( \alpha \)-clique extension of a strongly regular Terwilliger graph with parameters \((\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})\). By Lemma 3.1 we have \( k = \alpha \bar{v}, \lambda = \alpha \bar{k} + (\alpha - 1) \) and \( \mu = \alpha \bar{\mu} + 1 \).

By the assumption on \( \Gamma \), for a vertex \( u \in \Gamma \), the local graph of \( u \) contains a \( c \)-coclique which attains equality in \( \mu \)-bound, i.e.

\[
\mu(\Gamma) - 1 = \alpha \bar{\mu} = \frac{c(\alpha \bar{k} + (\alpha - 1) + 1) - \alpha \bar{v}}{\binom{2}{2}} = \alpha \frac{c(\bar{k} + 1) - \bar{v}}{\binom{2}{2}}
\]
and
\[ \tilde{\mu} = \frac{c(\bar{k} + 1) - \bar{v}}{c}. \]

Straightforward,
\[ c^2 \tilde{\mu} - c(\tilde{\mu} + 2(\bar{k} + 1)) + 2\bar{v} = 0, \]
so
\[ c = \frac{(\tilde{\mu} + 2(\bar{k} + 1)) \pm \sqrt{(\tilde{\mu} + 2(\bar{k} + 1))^2 - 8\bar{v}\tilde{\mu}}}{2\tilde{\mu}}, \]
and
\[ (\tilde{\mu} + 2(\bar{k} + 1))^2 \geq 8\bar{v}\tilde{\mu}. \]

Let the subgraph \( \Gamma_1(u) \) be isomorphic to the \( \alpha \)-clique extension of a strongly regular Terwilliger graph \( \Delta \). The cardinality of the vertex set of \( \Delta \) is equal to \( \bar{v} = 1 + \bar{k} + \bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} \), hence:
\[ (\bar{\mu} + 2(\bar{k} + 1))^2 \geq 8(\bar{\mu} + \bar{k}\bar{\mu} + \bar{k}^2 - 8\bar{k}\bar{\lambda} - 16\bar{k}). \]

Next,
\[ (\bar{\mu}/2)^2 + 1 \geq \bar{\mu} + \bar{k}\bar{\mu} + \bar{k}^2 - 2\bar{k}\bar{\lambda} - 4\bar{k};\]
\[ ((\bar{\mu}/2) - (\bar{k} + 1))^2 \geq 2\bar{k}(\bar{k} - \bar{\lambda} - 1). \quad (5) \]

At first, we may assume \( \bar{\mu} = 1 \). There are integers \( s, r \) such that \( \Delta \in \mathcal{F}(s, r) \) and \( \bar{k} = rs, \bar{\lambda} = s - 1 \). If \( \bar{k} - \bar{\lambda} - 1 \geq \bar{k}/2 + 1 \), then \( 2\bar{k}(\bar{k} - \bar{\lambda} - 1) \geq 2\bar{k}(\bar{k}/2 + 1) = \bar{k}^2 + 2\bar{k} \). It follows from (5) that \( (\bar{k} + 1/2)^2 \geq \bar{k}^2 + 2\bar{k} \) and hence \( 1/4 \geq \bar{k} \), that is impossible. Therefore, \( \bar{k} - \bar{\lambda} - 1 < \bar{k}/2 + 1 \), i.e. \( \bar{k} < 2(\bar{\lambda} + 2) \) holds. Substituting the expressions for \( \bar{k} \) and \( \bar{\lambda} \) into the previous inequality yields \( rs < 2(s + 1) \). By Lemma 2.1, we have \( s + 1 \leq r \). Hence, \( s + 1 \leq r < 2(s + 1)/s \) and this implies that \( s = 1, r \in \{2, 3\} \) and \( \Delta \) is the pentagon or the Petersen graph. In both cases Theorem 4.2 follows from Lemma 3.2.

Now we may assume \( \bar{\mu} > 1 \). Since \( \bar{\mu} < \bar{k} \), the left side of (5) is at most \( \bar{k}^2 \). On the other hand, if \( \bar{k} - \bar{\lambda} - 1 > \bar{k}/2 \) holds, then the right side of (5) is more than \( 2\bar{k}\bar{k}/2 = \bar{k}^2 \), that is impossible. Hence, we have \( \bar{k} - \bar{\lambda} - 1 \leq \bar{k}/2 \), i.e. \( \bar{k} \leq 2(\bar{\lambda} + 1) \).
Since \( \bar{\mu} > 1 \), there is an integer \( \alpha_1 \geq 1 \) such that, for a vertex \( w \in \Delta \), the subgraph \( \Delta_1(w) \) is the \( \alpha_1 \)-clique extension of a strongly regular Terwilliger graph \( \Sigma \) with parameters \( (v_1, k_1, \lambda_1, \mu_1) \), where \( v_1 = \frac{\bar{k}}{\alpha_1}, k_1 = \frac{\bar{\lambda} - (\alpha_1 - 1)}{\alpha_1}, \mu_1 = \frac{\bar{\mu} - 1}{\alpha_1} \). Then the inequality \( \bar{k} \leq 2(\bar{\lambda} + 1) \) is equivalent to the inequality \( v_1 \leq 2(k_1 + 1) \) and the cardinality of the vertex set of \( \Sigma \) is equal to

\[
v_1 = 1 + k_1 + \frac{k_1(k_1 - \lambda_1 - 1)}{\mu_1}.
\]

Next, \( v_1 \leq 2(k_1 + 1) \) implies that

\[
\frac{k_1(k_1 - \lambda_1 - 1)}{\mu_1} \leq k_1 + 1,
\]

so

\[
k_1 - \lambda_1 - 1 \leq \mu_1(1 + 1/k_1) < \mu_1 + 1,
\]

and

\[
k_1 < \lambda_1 + \mu_1 + 2. \tag{6}
\]

If \( \mu_1 = 1 \), then, for certain \( s_1, r_1 \), we have \( k_1 = r_1s_1 \) and \( \lambda_1 = s_1 - 1 \). It follows from (6) that \( r_1s_1 < s_1 - 1 + 1 + 2 = s_1 + 2, r_1 < 1 + 2/s_1 \) and \( s_1 = 1, r_1 = 2 \). Hence, the graph \( \Delta_1(w) \) is the \( \alpha_1 \)-clique extension of the pentagon. By Lemma 3.2 the graph \( \Delta \) is the icosahedron and diameter of \( \Gamma_1(u) \) is 3, that is impossible because \( \Gamma \) is a Terwilliger graph.

Hence, \( \mu_1 > 1 \). Let us consider a sequence of strongly regular graphs \( \Sigma_1 = \Sigma, \Sigma_2, \ldots, \Sigma_h, h \geq 2 \) such that, for an integer \( \alpha_{i+1} \geq 1 \), the local graph of a vertex in \( \Sigma_i \) is the \( \alpha_{i+1} \)-clique extension of a strongly regular Terwilliger graph \( \Sigma_{i+1} \) with parameters \( (v_{i+1}, k_{i+1}, \lambda_{i+1}, \mu_{i+1}) \), \( 1 \leq i < h \) and \( \mu(\Sigma_h) = 1 \), i.e. \( \Sigma_h \in \mathcal{F}(s_h, r_h) \) for certain \( s_h, r_h \). The sequence exists by Lemma 3.1.

Assuming that \( s_h > 1 \), we may note \( k_h - \lambda_h - \mu_h = r_hs_h - (s_h - 1) - 1 = s_h(r_h - 1) > 1 \). According to Lemma 3.3 we have \( k_i - \lambda_i - \mu_i > 1 \) for all \( 1 \leq i \leq h - 1 \), which is the contradiction with (6). Hence, \( s_h = 1 \) and \( \Sigma_h \) is a Moore strongly regular graph. By Lemma 3.2 diameter of \( \Sigma_{h-1} \) is at least 3, which is the contradiction that completes the proof. \( \blacksquare \)
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