STABLE AUSLANDER-REITEN COMPONENTS OF MONOMORPHISM CATEGORIES

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ABSTRACT. Let $\Lambda$ be an Artin algebra and $\text{Gprj-}\Lambda$ the subcategory of Gorenstein projective $\Lambda$-modules. In this paper, we study components of the stable Auslander-Reiten quiver of the monomorphism category $\Sigma(\text{Gprj-}\Lambda)$ containing certain vertices, which are called boundary vertices. This kind of the components is linked to the orbits of an auto-equivalence on the stable category $\text{Gprj-}\Lambda$. As a result, for when $\Lambda$ is a CM-finite algebra, we apply our results to study the Auslander-Reiten translation of simple modules of the associated stable Cohen-Macaulay Auslander algebra $A$ of $\Lambda$, and those components of the stable Auslander-Reiten quiver $A$ containing a simple module. In particular, we give some applications for the components of the stable Auslander-Reiten quiver over a preprojective algebra.

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1. Introduction

The origin of monomorphism categories maybe found in the study of the representation type of a submodule category which was first introduced by G. Birkhoff [B] in 1934. Around 70 years later, this concept attracted the attention of some researchers such as Ringel, Schmidmeier, Simson and Zhang [RS1, RS2, RS3, RZ, S, Z], who investigated the Auslander-Reiten theory from the representation theory of algebras in terms of monomorphism categories. After that, the importance of monomorphism categories has been investigated from various points of view.

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Chen [CH] proved that a monomorphism category of a Frobenius abelian category is a Frobenius exact category. Xiong, Zhang, and Zhang [XZZ] generalized Ringel and Schmidmeiers results on the Auslander-Reiten translation of the monomorphism category to a general case. Hu, Luo, Xiong and Zhou [HLXZ] studied Gorenstein projective bimodules over the tensor product of two algebras by monomorphism categories. Monomorphism categories associated to arbitrary species have also been studied in [GKSP]. In [H1], the first author investigated the monomorphism categories associated to a subcategory. Quite recently, Wang and Liu [WL] showed that there are recollements induced by the monomorphism category of Gorenstein and Ding projective modules.

For an Artin algebra $\Lambda$, denote by $\mathcal{S}(\Lambda)$ the monomorphism (submodule) category over $\Lambda$. For the special case $\Lambda_n = k[x]/(x^n)$, where $k[x]$ is the polynomial ring in one variable $x$ with coefficients in a field $k$, the monomorphism category $\mathcal{S}(n) := \mathcal{S}(k[x]/(x^n))$ is also called the category of invariant subspaces of nilpotent operators with nilpotency index at most $n$.

Let $\Pi_n$ be a preprojective algebra of the type $\mathbb{A}_n$. The main object in the paper [RZ] due to Ringel and Zhang was to investigate a connection between $\mathcal{S}(n)$ and $\text{mod-}\Pi_{n-1}$, the category of finitely generated (right) $\Pi_{n-1}$-modules, via two functors, which one has been constructed quite a long time ago by Auslander and Reiten, and recently another one by Li and Zhang [LZ]. This approach was later generalized by Eiriksson [E] to the module categories over self-injective algebras.

One of the functors applied in [RZ, E] is unified to a wide setting in [H1] as follows. Assume $\mathcal{X}$ is a resolving subcategory of $\text{mod-}\Lambda$. Denote by $\mathcal{S}(\mathcal{X})$ (denote by $\mathcal{S}_X(\Lambda)$ in [H1]) the subcategory of $\mathcal{S}(\Lambda)$, consisting of all monomorphisms $(X \xrightarrow{f} Y)$ such that $\Lambda$-modules $X, Y$ and $\text{Cok}(f)$ belong to $\mathcal{X}$. Associated to the subcategory $\mathcal{X}$, the functor $\Psi_{\mathcal{X}}$ from $\mathcal{S}(\mathcal{X})$ to the category $\text{mod-}\mathcal{X}^\perp$ of finitely presented functors over the stable category $\mathcal{X}$ is defined in [H1, Construction 3.1], see Construction 2.2 for the definition. It is also shown that the representation theory of $\mathcal{S}(\mathcal{X})$ is highly related to that of $\text{mod-}\mathcal{X}^\perp$ [H1].

In this paper we specialize such a connection to the case that the subcategory $\mathcal{X}$ coincides with the subcategory $\text{Gprj-}\Lambda$ Gorenstein projective modules over $\Lambda$. In particular, when $\Lambda$ is a CM-finite algebra, $\text{mod-}\Lambda$ admits finitely isomorphism classes of indecomposable Gorenstein-projective modules, which makes it possible to study the module category of the stable Cohen-Macaulay Auslander algebra of $\Lambda$ by the monomorphism category $\mathcal{S}(\text{Gprj-}\Lambda)$. We recall that the stable Cohen-Macaulay Auslander algebra of $\Lambda$ is the endomorphism algebra $\text{End}_\Lambda(G)$ of a (basic) additive generator $G$, i.e., $\text{add-}\Lambda = \text{Gprj-}\Lambda$, in the stable category $\text{Gprj-}\Lambda$.

Along the line of [RZ], we study the module category of the associated stable Cohen-Macaulay Auslander algebra $\Lambda$ of a CM-finite algebra $\Lambda$ via the monomorphism category $\mathcal{S}(\text{Gprj-}\Lambda)$ and the connection provided by the functor $\Psi_{\text{Gprj-}\Lambda}$. It is proved in [H1, Theorem 6.2] that the Auslander-Reiten quiver $\Gamma_\Lambda$ is a full valued subquiver of the Auslander-Reiten quiver $\Gamma_{\mathcal{S}(\text{Gprj-}\Lambda)}$ of the monomorphism category $\mathcal{S}(\text{Gprj-}\Lambda)$. Based on this embedding, we obtain an epimorphism from the set of components of the stable Auslander-Reiten quiver $\Gamma_\mathcal{S}(\text{Gprj-}\Lambda)$ containing a boundary vertex to the set of the components of the stable Auslander-Reiten $\Gamma_\Lambda$ quiver containing a simple vertex (Proposition 4.4). Particularly, we have a bijection when the infinite components are involved and we also prove that the shape of those infinite components is a stable tube. The vertices of $\Gamma_{\mathcal{S}(\text{Gprj-}\Lambda)}$ corresponding to isomorphism classes of indecomposable objects of either form $(0 \rightarrow G), (G \rightarrow G)$ and $(\Omega_\Lambda(G) \xrightarrow{\ell} P_G)$ are called boundary vertices. Among such a connection, we apply the almost split sequences in $\mathcal{S}(\text{Gprj-}\Lambda)$ to get some information of the Auslander-Reiten
translations of the simple modules over a stable Cohen-Macaulay Auslander algebra (Theorem 4.1).

Motivated by the works of Dlab, Ringel, Zhang [Dr, Rz], we can realize the preprojective algebra of type $A_{n-1}$ as the stable (Cohen-Macaulay) Auslander algebra of $k[x]/(x^n)$. The natural question may arise here is whether the same realization holds for other types of preprojective algebras. The case is true even for more general situation by a recent work due to Crawford [C]. Indeed, for a given weight $\lambda \in k^{\Omega}$ for Dynkin quiver $\Omega$, the corresponding deformed preprojective algebra is denoted by $\Pi^4(\Omega)$, see [C, Definition 2.12]. In view of [C, Lemma 2.15 and Theorem 3.6], we observe that $\Pi^4(\Omega)$ is Morita equivalent to the stable Cohen-Macaulay Auslander algebra of a CM-finite algebra. Especially, if $\lambda = 0$, then $\Pi(\Omega) := \Pi^4(\Omega)$ is the (undeformed) preprojective algebra of Gelfand and Ponomarev by [C, Theorem 2.13]. In this case, the candidate stable Cohen-Macaulay algebra is precisely the corresponding coordinate ring of the Kleinian singularity.

Therefore, for the deformed preprojective algebra, applying Theorem 4.1 on the stable Cohen-Macaulay Auslander algebras, we have the following corollary based on the the above observation.

**Corollary 1.1.** Let $S$ be a simple module in mod-$\Pi^4(\Omega)$ and set $A := \Pi^4(\Omega)$.

1. For $\varepsilon \in \{-2, -1, 1, 2\}$, $\tau^\varepsilon_A S = \Omega^1_{\tau^\varepsilon_A} S'$, for some simple module $S'$.
2. $\tau^3_A S$ and $\tau^{-3}_A S$ are simple modules.

We will provide more information concerning the position of simple modules over preprojective algebra of the type $A$ on the associated stable Auslander-Reiten quiver. Since in this case the algebra $k[x]/(x^n)$ is quite well-understood. We will count in Proposition 5.2 the number of stable tubes containing simple modules and also we show in Proposition 5.1 that simple module over this kind of preprojective algebra have $\tau$-periodicity 4 or 6.

**Conventions and notation** Throughout this paper all rings are always assumed to be an Artin algebras. By a module we always mean a right module unless otherwise stated. For an Artin algebra $\Lambda$, we denote by mod-$\Lambda$ (resp. prj-$\Lambda$) the category of finitely generated right $\Lambda$-modules (resp. finitely generated projective right $\Lambda$-modules).

For an additive category $\mathcal{C}$, which make sense the Auslander-Reiten theory for that, we denote by $\Gamma_\mathcal{C}$ the Auslander-Reiten quiver of $\mathcal{C}$ and by $\tau_\mathcal{C}$ the Auslander-Reiten translation of an object $C$ in $\mathcal{C}$. The stable Auslander-Reiten quiver $\Gamma^s_\mathcal{C}$ is obtained by removing all projective vertices, when $C$ is assumed to be an exact category. For the case $\mathcal{C} = \text{mod-}\Lambda$, we simply denote $\Gamma_\Lambda$ and $\Gamma^s_\Lambda$ instead of $\Gamma_{\text{mod-}\Lambda}$ and $\Gamma^s_{\text{mod-}\Lambda}$, respectively; moreover, we denote by $\tau_\Lambda M$, resp. $\tau_{\text{prj}} M$, the Auslander-Reiten translation of $M$ in mod-$\Lambda$, resp. prj-$\Lambda$. We ignore to write the valuations of the arrows in the Auslander-Reiten quivers if they are not essential. We often identify the isomorphism class of an indecomposable object with its representative, or with corresponding vertex in the Auslander-Reiten quiver. For the terminology about the Auslander-Reiten theory we refer the reader to [Ars, As, Li] and see [Ha] for a triangulated version of the theory. For an object $C \in \mathcal{C}$, we denote by add-$C$ the subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sums of objects in $\mathcal{C}$.

For a $\Lambda$-module $M$, consider a short exact sequence $0 \to \Omega_\Lambda(M) \xrightarrow{i_M} P_M \xrightarrow{\text{pr}_M} M \to 0$ with $P_M$ a projective cover of $M$ in mod-$\Lambda$. For simplicity, we often omit subscript “$M$” in $i_M$. The module $\Omega_\Lambda(M)$ (sometimes only $\Omega_\Lambda M$ without parenthesis) is then called a syzygy module of $M$. An n-th syzygy of $M$ will be defined inductively and denoted by $\Omega^n_\Lambda(M)$ for $n \geq 2$. For when $M$ is a Gorenstein projective module, we can define inductively $\Omega^n_\Lambda(M)$ for any negative integer.
n by using the notion of minimal left prj-Λ-approximation. In particular, for an indecomposable non-projective Gorenstein projective module $M$, we have $\Omega^{-1}_\Lambda \Omega_\Lambda M = M = \Omega_\Lambda \Omega^{-1}_\Lambda M$. Note that when $\Lambda$ is a self-injective algebra, $\Omega^2_\Lambda(M)$ for a negative integer $n$ is the same as $n$-th cosyzygy of $M$ in the usual sense, defining by using injective envelops.

2. Preliminaries

In this section, we recall some basic facts that we need throughout the paper. Firstly, we recall the notion of Gorenstein-projective modules.

For any projective module $Q \in \text{prj-} \Lambda$, a totally acyclic complex of projectives is an acyclic complex $P$ over prj-$\Lambda$ such that the induced complex $\text{Hom}_\Lambda(P, Q)$ is acyclic. The syzygies of a totally acyclic complex of projectives are called (finitely generated) Gorenstein projective modules by Enochs and Jenda [EJ]. The class of all Gorenstein projective modules in mod-$\Lambda$ is denoted by Gprj-$\Lambda$. Obviously, all projective modules are Gorenstein projective. Over self-injective algebras, i.e., Artin algebras $\Lambda$ that are injective as a $\Lambda$-module, all modules are Gorenstein projective, i.e., mod-$\Lambda = \text{Gprj-} \Lambda$.

2.1. Functor category. Let $\mathcal{C}$ be an essentially small additive category. The Hom sets will be denoted either by $\text{Hom}_\mathcal{C}(-, -)$, $\mathcal{C}(-, -)$ or even just $(-, -)$, if there is no risk of ambiguity. For the special case of $\mathcal{C} = \text{mod-} \Lambda$, we write it by $\text{Hom}_\Lambda(-, -)$. By the definition, a (right) $C$-module is a contravariant additive functor $F : \mathcal{C} \to \text{Ab}$, where $\text{Ab}$ denotes the category of abelian groups. The $\mathcal{C}$-modules and natural transformations between them, called morphisms, form an abelian category denoted by $\text{Mod-} \mathcal{C}$. An $\mathcal{C}$-module $F$ is called finitely presented if there exists an exact sequence

$$\mathcal{C}(-, C) \to \mathcal{C}(-, C') \to F \to 0,$$

with $C$ and $C'$ in $\mathcal{C}$. All finitely presented $\mathcal{C}$-modules form a full subcategory of $\text{Mod-} \mathcal{C}$, denoted by mod-$\mathcal{C}$. If $\mathcal{C}$ admits weak kernels, then mod-$\mathcal{C}$ is an abelian subcategory of $\text{Mod-} \mathcal{C}$ ([A2, §III, Section 2]). A particular example of $\mathcal{C}$ admits weak kernel is when it is a contravariantly finite subcategory of an abelian category $\mathcal{A}$, i.e., for any object $A \in \mathcal{A}$ the restricted functor $\mathcal{A}(-, A) |_{\mathcal{C}}$ is a finitely generated functor in $\text{Mod-} \mathcal{C}$.

The stable category of mod-$\Lambda$, denoted by mod-$\Lambda$, is a category whose objects are the same as those of mod-$\Lambda$, but the Hom-set mod-$\Lambda(M, M')$ of $M, M' \in \text{mod-} \Lambda$ is defined as

$$\text{mod-} \Lambda(M, M') := \text{Hom}_\Lambda(M, M') / P(M, M'),$$

where $P(M, M')$ consists of all morphisms from $M$ to $M'$ that factor through a projective module. We often write $\text{Hom}_\Lambda(M, M')$ in place of mod-$\Lambda(M, M')$. We have the canonical functor $\pi : \text{mod-} \Lambda \to \text{mod-} \Lambda$, defined by identity on objects but morphism $f : M \to M'$ will be sent to the residue class $\overline{f} = f + P(M, M')$. Once we intend to consider a module $M$ in mod-$\Lambda$ as an object in mod-$\Lambda$, the notation $\overline{M}$ will often be applied. Moreover, for a subcategory $\mathcal{X}$ of mod-$\Lambda$, the notation $\mathcal{X}$ denotes the image of $\mathcal{X}$ under the canonical functor $\pi$.

The canonical functor $\pi$ induces the fully faithful functor $\pi^* : \text{mod-} \text{(mod-} \Lambda) \to \text{mod-} \text{(mod-} \Lambda)$. Under this embedding we may identify mod-(mod-$\Lambda$) with the subcategory of mod-(mod-$\Lambda$) formed by those functors vanishing on projective modules. We have the similar connection between mod-(mod-$\mathcal{C}$) and mod-$\mathcal{C}$, whence $\mathcal{C}$ is a contravariantly finite subcategory of mod-$\Lambda$ containing prj-$\Lambda$. Keep the same $\mathcal{C}$ and assume it has almost split sequences. The indecomposable modules in $\mathcal{C}$ are in bijection with the simple functor in mod-$\mathcal{C}$, by sending an indecomposable module $M$
in \( \mathcal{C} \) to the simple functor \( S_M := (-, M)/\text{rad}(-, M) \); moreover, the associated simple functor to the indecomposable non-projective module \( M \) gets the following minimal projective resolution

\[
0 \rightarrow (-, N) \overset{(-, f)}{\rightarrow} (-, K) \overset{(-, g)}{\rightarrow} (-, M) \rightarrow S_M \rightarrow 0
\]

such that \( 0 \rightarrow N \overset{f}{\rightarrow} K \overset{g}{\rightarrow} M \rightarrow 0 \) is an almost split sequence in \( \mathcal{C} \) ([A4, §2]).

Assume \( \mathcal{C} \) is of finite representation type, i.e., there exists \( C \in \mathcal{C} \) such that \( \text{add-}C = \mathcal{C} \). It is known that the evaluation functor \( \xi_C : \text{mod-}\mathcal{C} \rightarrow \text{mod-End}_C(C) \) defined by \( \xi_C(F) = F(C) \), for any \( F \in \text{mod-}\mathcal{C} \), gives an equivalence of categories, see [A3, Proposition 2.7 (c)]. The equivalence is restricted to the equivalence \( \xi_C : \text{mod-}\mathcal{C} \rightarrow \text{mod-End}_\Lambda(C) \), where \( \text{End}_\Lambda(C) \) is the endomorphism algebra of the object \( C \) in the stable category \( \mathcal{C}^{\text{stab}} \).

\[
\begin{array}{ccc}
\text{mod-}\mathcal{C} & \xrightarrow{\xi_C} & \text{mod-End}_\Lambda(C) \\
\downarrow & & \downarrow \\
\text{mod-}\mathcal{C} & \xrightarrow{\xi_C} & \text{mod-End}_\Lambda(C)
\end{array}
\]

Therefore, by the above diagram we may identify \( \text{mod-}\mathcal{C} \) and \( \text{mod-}\mathcal{C} \) by the module categories \( \text{End}_\Lambda(C) \) and \( \text{mod-End}_\Lambda(C) \), respectively. We will apply freely this useful functorial approach through our paper especially for the case \( \mathcal{C} = \text{mod-}\Lambda \) or \( \text{Gprj-}\Lambda \).

2.2. (Mon)morphism category. The morphism category of \( \text{H}(\Lambda) \) takes \( \Lambda \)-homomorphisms in \( \text{mod-}\Lambda \) as the objects, and morphisms are given by commutative diagrams. In fact, it is equivalent to the category of finitely generated modules over the lower triangular matrix ring

\[
T(\Lambda) = \begin{bmatrix}
\Lambda & \Lambda \\
0 & \Lambda
\end{bmatrix}
\]

We denote an object in \( \text{H}(\Lambda) \) by \( (A_B)_{f, g} \), where \( f : A \rightarrow B \) is a map in \( \text{mod-}\Lambda \). A morphism in \( \text{H}(\Lambda) \) between \( (A_B)_{f, g} \) and \( (A_B')_{f', g'} \) is denoted by \( (\alpha_{g'})_{f} \), where \( \alpha : A \rightarrow C \) and \( \beta : B \rightarrow D \) are in \( \text{mod-}\Lambda \) such that \( \beta f = g \alpha \).

Let \( \mathcal{S}(\Lambda) \) denote the subcategory of all monomorphisms in \( \text{mod-}\Lambda \). For simplicity, especially in the diagrams or figures we write \( AB_{f} \) instead of \( (A_B)_{f, g} \), moreover if the morphism \( f \) is clear from the context we only write \( AB \), especially for the case \( f \) is either the identity morphism or the zero map. Denote by \( \mathcal{S}(\text{Gprj-}\Lambda) \) the subcategory of \( \mathcal{S}(\Lambda) \) consisting of all objects \( (A_B)_{f, g} \) such that \( A, B, \text{Cok}(f) \in \text{Gprj-}\Lambda \). Under the equivalence \( \text{H}(\Lambda) \simeq \text{mod-}T_2(\Lambda) \), the subcategory \( \mathcal{S}(\text{Gprj-}\Lambda) \) is mapped into the subcategory \( \text{Gprj-}T_2(\Lambda) \) of Gorensten projective modules over \( T_2(\Lambda) ([\text{EHS}, \text{Dual of Theorem 3.5.1}] \) and [LuZ, Theorem 5.1]).

2.3. From monomorphism categories to functor categories. We first remind from [H1] the structure of certain almost split sequences in \( \text{Gprj-}\Lambda \) for our later use.

Lemma 2.1. [H1, Lemma 6.3] Assume \( \delta : 0 \rightarrow A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \rightarrow 0 \) is an almost split sequence in \( \text{Gprj-}\Lambda \). Then

1. The almost split sequence in \( \mathcal{S}(\text{Gprj-}\Lambda) \) ending at \( (0 \rightarrow C) \) has the form

\[
\begin{array}{cccc}
0 & \overset{(A_B)_{1}}{\longrightarrow} & \overset{(f)}{\longrightarrow} & \overset{(g)}{\longrightarrow} \overset{(0)}{\longrightarrow} C \rightarrow 0
\end{array}
\]
(2) Let \( e : A \to I \) be a minimal left prj-\( \Lambda \)-approximation of \( A \). Then the almost split sequence in \( S(Gprj-\Lambda) \) ending at \((C \to C)\) has of the form

\[
0 \to (\varepsilon)_e \xrightarrow{f}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\xrightarrow{h}
\begin{pmatrix}
B \\
0
\end{pmatrix}
\xrightarrow{g}
\begin{pmatrix}
C \\
1
\end{pmatrix}
\to 0,
\]

where \( h \) is the map \((e')^T \) with \( e' : B \to I \) is an extension of \( e \). Indeed, it is induced from the following push-out diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{e} & & \downarrow{g} \\
I & \xrightarrow{I \oplus C} & C
\end{array}
\]

\[
\begin{array}{ccc}
\Omega_{\Lambda}(C) & \xrightarrow{\Omega_{\Lambda}(C)} & \Omega_{\Lambda}(C) \\
\downarrow{h} & & \downarrow{h} \\
A & \xrightarrow{(f \ b')} & P \\
\downarrow{f} & & \downarrow{b} \\
A & \xrightarrow{(f \ b')} & B \\
\downarrow{g} & & \downarrow{g} \\
A & \xrightarrow{h'} & C
\end{array}
\]

(3) Let \( b : P \to C \) be a projective cover of \( C \). Then the almost split sequence in \( S(Gprj-\Lambda) \) starting at \((0 \to A)\) has of the form

\[
0 \to (\varphi)_A \xrightarrow{\Omega_{\Lambda}(C)_{A \oplus P}} \Omega_{\Lambda}(C)_{A \oplus P} \xrightarrow{h} \Omega_{\Lambda}(C)_{F} \to 0,
\]

where \( h \) is the kernel of morphism \((f \ b') : A \oplus P \to B\), here \( b' \) is a lifting of \( b \) to \( g \). Indeed, it is induced from the following pull-back diagram

In the next section we will provide more information in concern of the middle terms appearing in the almost split sequences of the above lemma.

We specialize [H1, Construction 3.1] to define the functor \( \Psi_{Gprj-\Lambda} : S(Gprj-\Lambda) \to \text{mod-}(Gprj-\Lambda) \) as follows:

**Construction 2.2.** Take an object \((A \xrightarrow{f} B)\) of \( S(Gprj-\Lambda) \), then we have the following short exact sequence

\[
0 \to A \xrightarrow{f} B \to \text{Cok}(f) \to 0
\]

in \( Gprj-\Lambda \), this in turn gives the following short exact sequence

\[
0 \to (-, A) \xrightarrow{(-, f)} (-, B) \to (-, \text{Cok}(f)) \to F \to 0
\]

in \( \text{mod-}(Gprj-\Lambda) \). In fact, the above exact sequence corresponds to a projective resolution of \( F \) in \( \text{mod-}(Gprj-\Lambda) \). We define \( \Psi_{Gprj-\Lambda}(A \xrightarrow{f} B) := F \).
For morphism: let $(\frac{r_1}{s_1}, \frac{r_2}{s_2}) : (\frac{g}{h})_f \to (\frac{r_1'}{s_1'})_g$ be a morphism in $S(\text{Gprj-}\Lambda)$. It gives the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \text{Cok}(f) & \longrightarrow & 0 \\
\downarrow{\sigma_1} & & \downarrow{\sigma_2} & & \downarrow{\sigma_3} & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \longrightarrow & \text{Cok}(f') & \longrightarrow & 0,
\end{array}
$$

that $\sigma_3$ is determined uniquely by $\sigma_1$ and $\sigma_2$. By applying the Yoneda functor on the above diagram, we obtain the following diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & (-, A) & \xrightarrow{(-, f)} & (-, B) & \longrightarrow & (-, \text{Cok}(f)) & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow{(-, \sigma_1)} & & \downarrow{(-, \sigma_2)} & & \downarrow{(-, \sigma_3)} & \\
0 & \longrightarrow & (-, A') & \xrightarrow{(-, f') \circ (-, \sigma_1)} & (-, B') & \longrightarrow & (-, \text{Cok}(f')) & \longrightarrow & F' & \longrightarrow & 0
\end{array}
$$

in mod-$\mathcal{C}$. Define $\Psi_{\text{Gprj-}\Lambda}(\sigma) := (-, \sigma_3)$.

It is shown in [H1, Theorem 3.2] that the functor $\Psi := \Psi_{\text{Gprj-}\Lambda}$ induces an equivalence of categories

$$S(\text{Gprj-}\Lambda)/\mathcal{V} \simeq \text{mod-}(\text{Gprj-}\Lambda),$$

where the left side is the quotient category of $S(\text{Gprj-}\Lambda)$ modulo by the subcategory $\mathcal{V}$, that is, all finite direct sums of objects of the form $(G \xrightarrow{1} G)$ and $(0 \to G)$, which $G$ runs through all objects in $\text{Gprj-}\Lambda$. Hence by the equivalence we see that the study of mod-$\text{Gprj-}\Lambda$ is closely related to the one of $S(\text{Gprj-}\Lambda)$.

As discussed in [H1, Section 5], the functor $\Psi$ behaves well with respect to the Auslander-Reiten theory. More precisely, the image of any almost split sequence in $S(\text{Gprj-}\Lambda)$ with the right ending term not isomorphic to either $(\frac{G}{G})_0, (\frac{G}{G})_1, (\frac{\Omega_2(G)}{P_G})_1$, for some indecomposable module $G \in \text{Gprj-}\Lambda$, under the functor $\Psi$ remains an almost split sequence in mod-$\text{Gprj-}\Lambda$.

3. Stable Auslander-Reiten Components of Monomorphism Categories

In this section, we first record some results for the middle terms of the almost split sequences appeared in Lemma 2.1, which play a key role for our next computation. We then define an equivalence relation on the set of indecomposable non-projective Gorenstein projective $\Lambda$-modules. Next, we establish a correspondence between the equivalence classes and certain components of the stable Auslander-Reiten quiver of $S(\text{Gprj-}\Lambda)$. We get a bijective correspondence when we restrict to infinite components. Finally, we study those components which are finite.

3.1. The middle terms.

**Lemma 3.1.** Let $P$ be an indecomposable projective module in mod-$\Lambda$ with the radical $\text{rad}P$ and $i : \text{rad}P \to P$ the canonical inclusion. Then

1. The objects $(\frac{p}{h})_0$ and $(\frac{p}{h})_1$ are indecomposable projective-injective objects in $\text{Gprj-}\Lambda$. Each indecomposable projective-injective object arises in this way.

2. The following composition map is a sink in $S(\text{Gprj-}\Lambda)$

$$
(0) \circ (0) : (0)_G_0 \to (\text{rad}P)_0 \to (\frac{p}{h})_0,
$$

where $f : G \to \text{rad}P$ is a minimal right $\text{Gprj-}\Lambda$-approximation.
(3) The following composition map is a sink in $S(Gprj\Lambda)$

$$( i ) \circ ( \phi_2 ) : ( \frac{G'}{G} )_g \to ( \frac{\text{rad}_P}{P} )_i \to ( \frac{P'}{P} )_1,$$

where $\phi_1 : ( \frac{G'}{G} )_g \to ( \frac{\text{rad}_P}{P} )_i$ is a minimal right $S(Gprj\Lambda)$-approximation of $( \frac{\text{rad}_P}{P} )_i$. Moreover, $G$ belongs to prj-$\Lambda$.

**Proof.** (1) It follows from [HM, Proposition 1].

(2) Set $K := \text{Ker}(f)$. By Wakamatsu’s Lemma we have $K \in Gprj\Lambda^\perp$. For any $\left( \frac{G}{G'} \right)_e$ in $S(Gprj\Lambda)$, by applying $\text{Ext}^1_{H(\Lambda)}(-, (\frac{0}{K})_0)$ to the following short exact sequence

$$0 \to \left( \frac{G}{\text{Im}(e)} \right)_g \to \left( \frac{G}{G'} \right)_e \to \left( \frac{\text{coker}(e)}{0} \right)_0 \to 0,$$

where $g$ is an isomorphism, we obtain $\text{Ext}^1_{H(\Lambda)}(\left( \frac{G}{G'} \right)_e, (\frac{0}{K})_0) = 0$. Note that here we apply the classification of Gorenstein projective objects in $\text{H}(\Lambda)$, given in 2.2, and using the appropriate adjoint pairs between $H(\Lambda)$ and $\text{mod-}\Lambda$. Hence $(\frac{0}{K})_0 \in S(Gprj\Lambda)^\perp$. This implies that $(\frac{0}{G}) : (\frac{0}{G'})_g \to (\frac{\text{rad}_P}{P})_i$ is a right $S(Gprj\Lambda)$-approximation, an moreover minimal in $S(Gprj\Lambda)$ by using the (right) minimality of $f$ in $\text{mod-}\Lambda$. Now, we will show that $(\frac{0}{G'})_g$ is right minimal almost split (a sink). If $(\frac{0}{G'})_g$ is right minimal. To complete the task, we will show that it is right almost split. Assume $(\frac{0}{G'})_g : (\frac{X}{Y})_v \to (\frac{0}{P})_0$ is not a retraction. If $(\frac{0}{G'})_g$ is not a retraction, then as the inclusion $(\frac{0}{G'})_g$ is a sink [RS2, Lemma 3.1] in $S(\Lambda)$, there exists $(\frac{0}{G'})_g : (\frac{X}{Y})_v \to (\frac{\text{rad}_P}{P})_0$ such that $(\frac{0}{G'})_g = (\frac{0}{G'})_g \circ (\frac{0}{G'})_g$. As we proved $(\frac{0}{G'})_g$ is a right $S(Gprj\Lambda)$-approximation, so there is $(\frac{0}{G'})_g : (\frac{X}{Y})_v \to (\frac{0}{P})_0$ such that $(\frac{0}{G'})_g = (\frac{0}{G'})_g \circ (\frac{0}{G'})_g$. Hence $(\frac{0}{G'})_g$ factors through $(\frac{0}{G'})_g$ via $(\frac{0}{G'})_g$, as desired.

(3) The proof of the first part is similar to (2), with using here this fact that $(\frac{\text{rad}_P}{P})_i \to (\frac{P}{P})_1$ is a sink in $S(\Lambda)$, by [RS2, Lemma 3.1]. To prove the second part, letting $(\frac{K_i}{K_2})_d$ be the kernel of $(\frac{G}{G'})_g$. By Wakamatsu’s Lemma, $\text{Ext}^1_{H(\Lambda)}(\left( \frac{\text{rad}_P}{P} \right)_i, (\frac{K_i}{K_2})_d) = 0$ for any $(\frac{X}{Y})_v$ in $S(Gprj\Lambda)$. This property implies that we have the following vanishing of $\text{Ext}$ for any $G \in Gprj\Lambda$

$$\text{Ext}^1_{H(\Lambda)}((\frac{0}{G})_0, (\frac{K_i}{K_2})_d) = \text{Ext}^1_{H(\Lambda)}(G, K_2) = 0,$$

$$\text{Ext}^1_{H(\Lambda)}((\frac{G}{G'})_e, (\frac{K_i}{K_2})_d) = \text{Ext}^1_{H(\Lambda)}(G, K_1) = 0.$$

Hence we deduce $K_1, K_2 \in Gprj\Lambda^\perp$. Consider the following short exact sequence in $S(Gprj\Lambda)$

$$0 \to \left( \frac{K_i}{K_2} \right)_d \to \left( \frac{G'}{G} \right)_g \to (\frac{\text{rad}_P}{P})_i \to 0.$$

The above sequence gives the short exact sequence $0 \to K_2 \to G \to P \to 0$ in $\text{mod-}\Lambda$, which must split. Hence $K_2 \in Gprj\Lambda \cap Gprj\Lambda^\perp$. But we know that $Gprj\Lambda \cap Gprj\Lambda^\perp = \text{prj-}\Lambda$, so $G \in Gprj\Lambda$.

**Lemma 3.2.** Let $P$ be an indecomposable non-zero projective module in $Gprj\Lambda$.

(1) If $(\frac{0}{P})_0$ is a direct summand of the middle term of an almost split sequence with the left ending term $C$, then $C = (\frac{0}{G})_0$, where $G$ is an indecomposable non-projective module in $Gprj\Lambda$.

(2) If $(\frac{0}{P})_1$ is a direct summand of the middle term of an almost split sequence with the left ending term $C$, then $C = (\frac{\Omega_{H\Lambda}}{P_H})_1$, where $H$ is an indecomposable non-projective module in $Gprj\Lambda$. 

\[\square\]
Proof. Applying Lemma 3.1, we get the right minimal almost morphism in \( S(\mathrm{Gprj-}\Lambda) \) with the codomain \( (\frac{B}{C})_h \) is of the form \( (\frac{G}{P})_0 \). This means that the domain of an irreducible morphism with the codomain \( (\frac{B}{C})_h \) is a direct summand of \( (\frac{G}{P})_0 \). As a result, we get the desired form for the \( C \) in the first statement. The proof of (2) is similar. \( \square \)

Lemma 3.3. Keep in mind the notation in the assertion of Lemma 2.1(2). One of the following cases occurs for the middle term \( (B \xrightarrow{h} I \oplus C) \).

(a) it is indecomposable;

(b) otherwise, there is a decomposition as

\[
(\frac{B}{C})_h = (\frac{B'}{C'})_h, \oplus (\frac{I}{1})_1,
\]

where \( (\frac{B'}{C'})_h \) is an indecomposable non-projective object.

Proof. Assume the middle term is decomposable. We know from the sequence \( \delta, C \) is non-projective. Let \( \delta': 0 \to C \to B' \xrightarrow{g'} C' \to 0 \) be the almost split sequence in \( \text{Gprj-}\Lambda \) starting at \( C \). Applying the assertions (1) and (2) of Lemma 2.1 for the almost split sequences \( \delta \) and \( \delta' \) give us the following almost split sequences in \( S(\text{Gprj-}\Lambda) \)

\[
\begin{array}{c}
0 \to (\frac{C}{1})_1 \xrightarrow{(f)} (\frac{C}{C'})_0 \xrightarrow{(0)} (\frac{C}{1})_0 \to 0, \\
0 \to (\frac{A}{1})_e \xrightarrow{(f)} (\frac{B}{1})_{I \otimes C} \xrightarrow{(g)} (\frac{C}{C'})_0 \to 0,
\end{array}
\]

where \( h \) is the map \( (\frac{e'}{g}) \) with \( e': B \to I \) is an extension of \( e \). Since \( (\frac{C}{C'})_0 \) is indecomposable, because of the almost split sequence \( \delta' \), the middle term \( (\frac{B}{I \otimes C})_h \) can be written as \( X \oplus Y \), where \( X = (\frac{B'}{C'})_h \), an indecomposable non-projective object and \( Y \) is a projective (or injective) object in the Frobenius exact category \( S(\text{Gprj-}\Lambda) \). Using the characterization of projective-injective objects given in Lemma 3.1, we can write \( Y = (I_1 \to 1) \oplus (0 \to I_2) \) for some projective modules \( I_1, I_2 \). If there were a non-zero direct summand of \( I_2 \), say \( J \), then, by Lemma 3.1, \( (\frac{1}{1})_e \) would be a direct summand of \( (\frac{0}{G})_h \), where \( G \) is a minimal right \( \text{Gprj-}\Lambda \)-approximation of \( \text{rad} J \). But this means that \( A = 0 \), a contradiction. Hence \( (\frac{B}{I \otimes C})_h = (\frac{B'}{C'})_h \oplus (\frac{I_1}{1})_1 \). This implies that \( B = B' \oplus I_1 \) and \( I \oplus C = C' \oplus I_1 \). The second equality along with this fact \( C \) being non-projective yields \( C = C' \) and \( I = I' \). This completes the proof. \( \square \)

By the above lemma, we see when the middle term is decomposable, there is the following mesh in \( \Gamma_{S(\text{Gprj-}\Lambda)} \)

\[
\begin{array}{c}
B' C_h' \\
\downarrow \text{AI}_e \downarrow \downarrow \text{CC}_1 \\
\downarrow \text{II}_1 \\
A \oplus P
\end{array}
\]

Lemma 3.4. Keep in mind the notation in the assertion of Lemma 2.1(3). One of the following cases occurs for the middle term \( (\Omega_{\Lambda}(C) \xrightarrow{h} A \oplus P) \).

(a) it is indecomposable;
(b) otherwise, there is a decomposition as
\[
\left( \Omega_{\text{A}(\text{C})} \right)_{b'} = \left( \Omega_{\text{A}(\text{C})} \right)_{b'} \oplus \left( \frac{0}{P} \right)_0
\]
where \( \left( \Omega_{\text{A}(\text{C})} \right)_{b'} \) is an indecomposable non-projective object.

**Proof.** The proof runs along similar lines of the proof of Lemma 3.3. More explanation, this turn we need to exclude those projective indecomposable of the form \( \left( \frac{0}{P} \right)_1 \). If the case held, then \( \left( \frac{0}{\lambda} \right)_0 \) would be a direct summand of \( \left( \frac{C}{\lambda} \right)_0 \), which is a minimal right \( \mathcal{S} \)-approximation of \( \left( \frac{\text{rad} P}{P} \right)_0 \). It follows from Lemma 3.1(3) that \( G \) is projective. Hence \( A \) is projective as well, that is a contradiction. \( \Box \)

Analogously, when the middle term of the almost split sequence is decomposable, by the above lemma, there is the following mesh in

\[
\begin{array}{ccc}
\Omega_{\text{A}(\text{C})}A_{b'} & \rightarrow & \Omega_{\text{A}(\text{C})}P_i \\
\downarrow \quad \downarrow & & \downarrow \\
0A & \rightarrow & \Omega_{\text{A}(\text{C})}P_i
\end{array}
\]

3.2. **Stable components.** Let \( M \) be an object in \( \mathcal{S}(\mathcal{Gprj} \backslash \Lambda) \). For simplicity, we write \( \tau_\mathcal{S} M \) in place of \( \tau_\mathcal{S}(\mathcal{Gprj} \backslash \Lambda) M \). The \( \tau_\mathcal{S} \)-orbit of \( M \) is the set of all objects \( \tau_\mathcal{S}^n M \), with \( n \in \mathbb{Z} \). The module \( M \) is called \( \tau_\mathcal{S} \)-periodic if \( \tau_\mathcal{S}^m M \cong M \) for some \( m \geq 1 \).

We recall that a vertex \( x \) in a valued translation quiver \( (\Gamma, \rho, \nu) \) is called stable if \( \rho^i x \) is defined for every integer \( i \). We refer to [Li] for the notion of the Auslander-Reiten quiver \( \Gamma_c \) of a Krull-Schmidt category \( \mathcal{C} \) and relevant combinatorial background, such as the valued translation quivers.

Denote by \( \vartheta_\Lambda := \tau_\mathcal{S} \Omega^{-1}_\Lambda \tau_\mathcal{S}^2 \). In fact, \( \vartheta := \vartheta_\Lambda \) is an auto-equivalence on \( \mathcal{Gprj} \backslash \Lambda \). Similarly, for any \( G \) in \( \mathcal{Gprj} \backslash \Lambda \), the \( \vartheta \)-orbit of \( G \), denoted by \( [G]_{\vartheta} \), is the set of all modules \( \vartheta^i G \) with \( i \in \mathbb{Z} \).

Let us give an important remark in concern of the auto-equivalence \( \vartheta \). That is the equalities \( \vartheta = \Omega^{-1}_\Lambda \tau_\mathcal{S}^2 = \tau_\mathcal{S}^2 \Omega^{-1}_\Lambda \) on the level of objects. The equalities may be functional but for our computation to know only having the equalities on the level of objects is enough. The equalities come from these facts that \( \Omega^{-1}_\Lambda \) is an auto-equivalence on the triangulated category \( \mathcal{Gprj} \backslash \Lambda \) and also an auto-equivalence preserves the Auslander-Reiten triangles. Note that an Auslander-Reiten triangle in \( \mathcal{Gprj} \backslash \Lambda \) is induced by an almost split sequence in the exact category \( \mathcal{Gprj} \backslash \Lambda \), one can easily see it only by following the definitions.

**Example 3.5.**

1. First, we recall that a triangulated \( k \)-category \( \mathcal{T} \), Hom-finite with split idempotents, is \( d \)-Calabi-Yau if there is a bifunctorial isomorphism
\[
\text{Hom}_\mathcal{T}(X, Y) \cong D\text{Hom}_\mathcal{T}(Y, X[d])
\]
for all \( X, Y \in \mathcal{T} \). If we assume that the triangulated category \( \mathcal{Gprj} \backslash \Lambda \) is \( d \)-Calabi-Yau, then there is a functorial isomorphism \( \tau_\mathcal{S} = \Omega^d_\Lambda^{-1} \). Therefore, in this case \( \vartheta_\Lambda = \Omega^{3d-4}_\Lambda \). We point out here that a weaker condition of \( d \)-Calabi-Yau property works for our purpose to compute \( \vartheta_\Lambda \)-orbits. Namely, it doesn’t require the isomorphism included in the definition of a \( d \)-Calabi-Yau triangulated category to be compatible with the triangulated structure.

2. If \( \Lambda \) is a finite dimensional symmetric \( k \)-algebra over a field \( k \), then we know that \( \tau_\Lambda = \Omega^2_\Lambda \), see [SY1, §IVCorolalry 8.6]. So, in this case we have \( \vartheta_\Lambda = \Omega^3_\Lambda \).
If $\Lambda$ is G-semisimple, which is defined in [H2], then $\tau_\Lambda = \Omega_\Lambda$. Hence, in this case, $\vartheta_\Lambda = \Omega_\Lambda^2$.

Let us in below compute all $\vartheta$-orbits of a concrete example.

**Example 3.6.** Let $\Lambda$ be the $k$-algebra given by the quiver

![Diagram of the quiver](image)

and bound by $abc = 0$, $bca = 0$, $cab = 0$. Then $\Gamma^*_\Lambda$ is given by

$$
\begin{array}{c}
(1) \rightleftharpoons (2) \rightleftharpoons (3) \rightleftharpoons (1) \\
(\frac{1}{3}) \rightleftharpoons (\frac{2}{3}) \rightleftharpoons (\frac{1}{3}) \rightleftharpoons (\frac{1}{3})
\end{array}
$$

where the vertices with the same label are identified, and are described via their composition series. Here $\Lambda$ is self-injective, so $\tau_\Lambda = \tau_\mathcal{G}$. There are 3 $\vartheta$-orbits as follows: $\{(1), (\frac{2}{3})\}$, $\{(3), (\frac{1}{3})\}$ and $\{(2), (\frac{1}{3})\}$.

For a $G \in \text{Gprj-}\Lambda$, we denote by $e : G \to I_G$ a minimal left prj-\Lambda-approximation of $G$.

**Proposition 3.7.** Assume $G$ is an indecomposable non-projective Gorenstein projective module, and $d = 3m + k$ with $m \geq 0$, $0 \leq k \leq 2$. Then the following hold.

1. If $k = 0$, then $\tau^d_S(0) = (0_{\vartheta_m\Lambda})_0$ and $\tau^d_S(\Lambda) = (0_{\vartheta_m\Lambda})_0$.

2. If $k = 1$, then $\tau^d_S(0) = (\tau_{\mathcal{G}}\vartheta_m\Lambda)_0$ and $\tau^d_S(\Lambda) = (\Omega_{\mathcal{G}}\tau_{\mathcal{G}}^1\vartheta_m\Lambda)_0$.

3. If $k = 2$, then $\tau^d_S(0) = (\tau_{\mathcal{G}}^2\vartheta_m\Lambda)_0$ and $\tau^d_S(\Lambda) = (\tau_{\mathcal{G}}^2\Omega_{\mathcal{G}}\tau_{\mathcal{G}}^1\vartheta_m\Lambda)_0$.

\textbf{Proof.} The results follow by repeating application of Lemma 2.1. We only give a proof for the first part of (1) by induction on $m$. Assume $\tau^{3m}_{\mathcal{S}}(0) = (0_{\vartheta_m\Lambda})_0$. Applying Lemma 2.1(1) for the almost split sequence $0 \to \tau_{\mathcal{G}}\vartheta^m\Lambda \to B \to \vartheta^m\Lambda \to 0$ in $\text{Gprj-}\Lambda$ gives the following almost split sequence in $\mathcal{S}(\text{Gprj-}\Lambda)$

$$
0 \to (\tau_{\mathcal{G}}\vartheta^m\Lambda) \to (\tau_{\mathcal{G}}\vartheta^m\Lambda)_1 \to (\tau_{\mathcal{G}}\vartheta^m\Lambda)_0 \to 0.
$$

Hence by the induction hypothesis and the above sequence $\tau^{3m+1}_{\mathcal{S}}(0) = (\tau_{\mathcal{G}}^1\vartheta^m\Lambda)_0$. Now by applying Lemma 2.1(2) for the almost split sequence $0 \to \tau_{\mathcal{G}}^1\vartheta^m\Lambda \to B' \to \tau_{\mathcal{G}}\vartheta^m\Lambda \to 0$ in $\text{Gprj-}\Lambda$, we obtain the following almost split sequence in $\mathcal{S}(\text{Gprj-}\Lambda)$

$$
0 \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda) \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda)_1 \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda)_0 \to 0.
$$

Thus $\tau^{3m+2}_{\mathcal{S}}(0) = (\tau_{\mathcal{G}}^2\vartheta^m\Lambda)_0$. Finally, by applying Lemma 2.1(3) for the almost split sequence $0 \to \tau_{\mathcal{G}}^1\tau_{\mathcal{G}}^1\vartheta^m\Lambda \to B'' \to \tau_{\mathcal{G}}^1\tau_{\mathcal{G}}^1\vartheta^m\Lambda \to 0$ in $\text{Gprj-}\Lambda$, we obtain the following almost split sequence in $\mathcal{S}(\text{Gprj-}\Lambda)$

$$
0 \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda) \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda)_1 \to (\tau_{\mathcal{G}}^2\vartheta^m\Lambda)_0 \to 0.
$$
Note that \( \text{Cok}(e) = \Omega_{-1}^{2} \phi^m A \). So \( \tau_{\Lambda}^{3(m+1)}(0)_{0} = \left( \tau_{\phi}^{3} \Omega_{0}^{2} \phi^m A \right)_{0} = \left( \phi \Omega_{0}^{2} \phi^m A \right)_{0} \). This finishes the proof. \( \square \)

Recall that an Artin algebra \( \Lambda \) is said to be CM-finite if there are finitely many isomorphism classes of indecomposable Gorenstein projective \( \Lambda \)-modules.

**Corollary 3.8.** Assume \( G \) is an indecomposable non-projective Gorenstein projective module. If \( \Lambda \) is CM-finite, then either indecomposable objects \( (0)_{G} \) and \( (\Omega_{G})_{i} \) is \( \tau_{G} \)-periodic.

**Proof.** We only prove \( (0)_{G} \) is \( \tau_{G} \)-periodic. One can show the others by the similar argument. From Proposition 3.7, we obtain \( \tau_{G}^{m+1}(0)_{G} = (0)_{G} \) for every \( m \in \mathbb{Z} \). Since \( \Lambda \) is CM-finite, the set \( \{ \phi^m G \mid m \in \mathbb{Z} \} \) is finite, and so is \( \{ \tau_{G}^{m} (0)_{G} \mid m \in \mathbb{Z} \} \). This ends the proof. \( \square \)

Denote by \( \Gamma_{s(Gprj-\Lambda)} \) the stable Auslander-Reiten quiver of \( \Gamma_{s(Gprj-\Lambda)} \) which is obtained by removing projective-injective vertices of \( \Gamma_{s(Gprj-\Lambda)} \). For simplicity, we often denote them respectively by \( \Gamma_{s} \) and \( \Gamma_{s} \). A vertex in \( \Gamma_{s} \) is said to be a boundary vertex if it is of the form \( (0)_{G} \), \( (G)_{1} \) and \( (\Omega_{G})_{i} \), for our convenience, we call them of the form \( (a) \), \( (b) \) and \( (c) \), respectively.

**Proposition 3.9.** Assume \( \Lambda \) is a CM-finite algebra and \( \Gamma \) a component of \( \Gamma_{s} \) containing a boundary vertex. Then the following assertions hold.

1. If \( \Gamma \) is finite, then \( \Gamma = \mathbb{Z} \Delta / G \), where \( \Delta \) is a Dynkin quiver and \( G \) is an automorphism group of \( \mathbb{Z} \Delta \) containing a positive power of the translation.
2. If \( \Gamma \) is infinite, then \( \Gamma \) is a stable tube.

In particular, if \( S(Gprj-\Lambda) \) is of finite representation type, then \( \Gamma_{s} \) is a disjoint union of the finite components containing a boundary vertex.

**Proof.** First of all since we remove the projective-injective vertices, the component \( \Gamma \) is stable. According to [Li, Theorem 5.3], for both (1) and (2) it suffices to show that the component contains a \( \tau_{G} \)-periodic object. Our assumption in conjunction with Corollary 3.8 guarantees such a vertex.

For the last part, we need to show that each component of \( \Gamma_{s} \) contains a boundary vertex. Without of loss generality we may assume that \( \Lambda \) is an indecomposable algebra, i.e., \( \text{prj-} \Lambda \) is a connected category. This implies that \( \text{prj-}T_{2}(\Lambda) \) is connected as well. Since \( S(Gprj-\Lambda) \) contains \( \text{prj-}T_{2}(\Lambda) \), thus \( S(Gprj-\Lambda) \) is a connected category. The condition of \( S(Gprj-\Lambda) \) being connected and of finite representation type follows \( \Gamma_{s} \) is connected, see [Li, Lemma 5.1]. Assume \( \Gamma \) is an arbitrary component of \( \Gamma_{s} \). Take a vertex \( x \) of \( \Gamma \). The connectedness of \( \Gamma_{s} \) yields a walk \( y = x_{0} \leftarrow x_{1} \leftrightarrow \cdots \leftrightarrow x_{t} = x \) with \( y \) a projective-injective vertex and none of \( x_{d} \), \( d > 0 \), to be projective. Each of \( x_{d} \leftrightarrow x_{d+1} \) means that there is an arrow \( x_{d} \rightarrow x_{d+1} \) or \( x_{d+1} \rightarrow x_{d} \) in \( \Gamma_{s} \).

The two cases hold: If \( y \) is isomorphic to \( (0)_{G} \), for some indecomposable projective module \( P \), then by Lemma 3.4 we get \( x_{1} \) to be of the forms \( (a) \) and \( (c) \). If \( y \) is isomorphic to \( (P)_{1} \), for some indecomposable projective module \( P \), then, by Lemma 3.3, \( x_{1} \) is of the forms \( (b) \) and \( (c) \). So for the both cases the vertex \( x_{1} \) is boundary, as desired. \( \square \)

Let \( G \) be an indecomposable non-projective Gorenstein projective \( \Lambda \)-module. Denote by \( \Gamma_{s}(G) \) the unique component of the stable Auslander-Reiten quiver \( \Gamma_{s} \) containing the vertex \( (0)_{G} \). To state our next result we need to introduce some notations as follows: let

\[
\mathcal{K} = \{ \Gamma_{s}(G) \mid G \in U \}, \quad \mathcal{R} = \{ [G]_{\nu} \mid G \in U \},
\]
where \( U \) is a complete set of pairwise non-isomorphic indecomposable non-projective Gorenstein projective modules.

One can define a map \( \delta : \mathcal{R} \to \mathcal{K} \), by sending \([G]\), to \( \Gamma_s^*(G) \). The map is well-defined. In fact, let \( G, G' \) belong to the equivalence class \([G]_\theta\). Hence there is some integer \( m \) such that \( \theta^m(G) = G' \). Proposition 3.7 implies that \( \tau^m_s(0)_{G_0} = (0)_{G_0} = (0)_{G_0} \). So \( (0)_{G_0} \) and \((0)_{G'}_0 \) lie in the same \( \tau_s \)-orbit, and consequently the same component of \( \Gamma_s^* \). Therefore, the vertices \((0)_{G_0} \) and \((0)_{G'}_0 \) are connected together by a path in \( \Gamma_s^* \). Since the almost split sequences in \( S(Gprj-\Lambda) \) with ending term in the \( \tau_s \)-orbit of \((0)_{G_0} \) or \((0)_{G'}_0 \) have a non projective-injective direct summand in their middle terms, by Lemmas 3.3, 3.4, hence we can find a path between the vertices \((0)_{G_0} \) and \((0)_{G'}_0 \) in \( \Gamma_s^* \). But this means that \( \Gamma_s^*(G) = \Gamma_s^*(G') \).

We also denote by \( \mathcal{K}^{\infty} \) the subset of \( \mathcal{K} \) consisting of all infinite components, and by \( \mathcal{R}^{\infty} \) the inverse image of \( \mathcal{K}^{\infty} \) under the map \( \delta \).

Recall that the set of vertices of a stable tube having exactly one immediate predecessor (equivalently, exactly one immediate successor) is called mouth of the tube.

**Proposition 3.10.** With the above notations, the following statements hold.

(1) The map \( \delta \) is surjective.

(2) The restricted map \( \delta | : \mathcal{R}^{\infty} \to \mathcal{K}^{\infty} \) is bijective.

**Proof.** (1) by definition is clear. For (2), assume that the component \( \Gamma_s^*(G) \) is infinite, for an indecomposable non-projective Gorenstein projective module \( G \). Hence by Proposition 3.9 \( \Gamma_s^*(G) \) is a stable tube. Because of Lemma 3.3, the middle terms of the almost split sequences with ending terms of vertices by the \( \tau_s \)-orbit of \((0)_{G_0} \) contain exactly one non-projective direct summand. Hence the \( \tau_s \)-orbit of \((0)_{G_0} \) generates all the vertices in the mouth of the stable tube. Since the mouth of any stable tube is unique, so \( \Gamma_s^*(G) \) is uniquely determined by the equivalence class \([G]_\theta\). Hence the restricted map \( \delta | \) is injective, consequently, it is bijective by the first part. \( \Box \)

In the case that \( S(Gprj-\Lambda) \) is of finite representation type, then it is clear that for an indecomposable non-projective module \( G \) in \( Gprj-\Lambda \) the associated component \( \Gamma_s^*(G) \) is finite. In contrast, for when that \( S(Gprj-\Lambda) \) is of infinite representation type, we may have both finite or infinite components \( \Gamma_s^*(G) \).

**Example 3.11.** Let \( \Omega \) be the quiver \( \begin{array}{ccc} a & \overset{\alpha}{\longrightarrow} & b \end{array} \) the ideal generated by \( \alpha \) and \( A = k\Omega/I \)

the associated bound quiver algebra, and moreover \( \Omega' \) the quiver \( \begin{array}{ccc} c & \overset{\beta}{\longrightarrow} & d \end{array} \) the ideal generated by \( \gamma \) and \( B = k\Omega'/I' \) the associated bound algebra. Assume \( \Lambda \) is a simple gluing algebra of \( A \) and \( B \), obtained by identifying the vertices \( b \) and \( c \), see \([L]\) for the precise definition. The stable Auslander-Reiten quiver \( \Gamma_s^*(S(Gprj-\Lambda)) \) contains a finite component \( \Gamma_s^*(G) \) and an infinite component \( \Gamma_s^*(G') \) for some indecomposable non-projective modules \( G, G' \) in \( Gprj-\Lambda \). To this end, first consider the following remark:

**Remark 3.12.** let \( W \) be a CM-finite algebra, and \( V \) the associated stable Cohen-Macaulay Auslander algebra. From \([H1, \text{ Theroem 6.2}]\) we have the embedding \( \Gamma_V \subseteq \Gamma_S(Gprj-\Lambda) \). One can see easily that the embedding can be restricted to the embedding \( \Gamma_V \subseteq \Gamma_S(Gprj-\Lambda) \) and further under the embedding each component \( \Delta \) of \( \Gamma_V \) is contained in exactly one component \( \Delta' \) of \( \Gamma_S \). In addition, \( \Delta \) is finite if and only if \( \Delta' \) so is, see \([H1, \text{ Theorem 6.1}]\).
We now turn back to prove that $\Lambda$ has the desired property. By [L, Theorem 4.4] we have $\text{Gprj}-\Lambda \simeq \text{Gprj}-A \oplus \text{Gprj}-B$. Hence the equivalence implies that $\Gamma_{V_1} = \Gamma_{V_2} \times \Gamma_{V_3}$, where $V_1, V_2$ and $V_3$ are respectively the stable Cohen Macaulay Auslander algebra of $\Lambda, A$ and $B$. Indeed, $\text{Gprj}-A \simeq \text{Gprj}-k[x]/(x^3)$, as the vertex $b$ is a sink, and being self-injective of $k[x]/(x^3)$ for any $i$, we can consider $V_1$ and $V_2$ as stable Auslander algebras. In view of the classification given in [RS3], we deduce that any component of $\Gamma^{\text{Gprj}}_{k[x]/(x^3)}$ is finite, and with the remark established in the above, we get any component of $\Gamma_{V_3}$ is also finite. In addition, any component of $\Gamma^{\text{Gprj}}_{k[x]/(x^6)}$, by [RS3], is infinite; so this is also true for any component of $\Gamma_{V_2}$, using the remark. Therefore, we get the indecomposable algebra $\Lambda$ having the desired property.

### 3.3. Finite components

In the rest of this section we study those components $\Gamma^{\text{Gprj}}_{\mathcal{S}}(G)$ which are finite. From now on, we assume that $\text{Gprj}-\Lambda$ is a contravariantly finite subcategory of $\text{mod-}\Lambda$. Hence, by [AS], it has almost split sequences.

**Definition 3.13.** Following [H2], an indecomposable non-projective module in $\text{Gprj}-\Lambda$ is called $\mathcal{S}$-semisimple if the canonical short exact sequence $0 \to \Omega^i_{\Lambda}(G) \to P_G \to G \to 0$ is an almost split sequence in $\text{Gprj}-\Lambda$.

In a recent work [H2] by the first named author those algebras which all indecomposable non-projective Gorenstein projective modules are $\mathcal{S}$-semisimple algebra, are studied. This type of algebras in there is called $G$-semisimple.

**Lemma 3.14.** Let $G$ be a $\mathcal{S}$-semisimple module. Then for any $i \in \mathbb{Z}$, $\Omega^i_{\Lambda}(G)$ is $\mathcal{S}$-semisimple. In particular, for any $i \in \mathbb{Z}$, $\tau^i_{\mathcal{S}} G = \Omega^i_{\Lambda}(G)$.

**Proof.** We inductively only prove for non-positive integers $i$. The case is true for $i = 0$ due to our assumption, so we assume $i > 0$. Consider an almost split sequence $0 \to \tau^i_{\mathcal{S}} \Omega^i_{\Lambda}(G) \to B \to \Omega^{i+1}_{\Lambda}(G) \to 0$ in $\text{Gprj}-\Lambda$, which exists because of the setup we fixed in the beginning of this subsection. But $B$ has no any non-projective indecomposable direct summand. Otherwise, if we assume that $B$ has a non-projective indecomposable direct summand, say $C$, then it implies that $P^{i-1}$ has the non-projective summand $\tau^i_{\mathcal{S}} C$. As by hypothesis induction $P^{i-1}$ is the middle term of the almost split sequence in $\text{Gprj}-\Lambda$ starting at $\Omega^i_{\Lambda}(G)$. This contradicts the projectivity of $P^{i-1}$. Hence $B$ is a projective module. Consequently, $\tau^i_{\mathcal{S}} \Omega^i_{\Lambda}(G) = \Omega^{i+1}_{\Lambda}(G)$, keeping in mind the morphisms involved in an almost split sequence are minimal. \hfill \Box

**Proposition 3.15.** Assume $\Lambda$ is a CM-finite algebra and $G$ a $\mathcal{S}$-semisimple module. Then the component $\Gamma^{\mathcal{S}}_{\mathcal{S}}(G)$ is a oriented cycle, as below,

\[
\begin{array}{ccc}
0G & \overset{0}{\rightarrow} & \Omega^i_{\Lambda}(G)
\downarrow & & \downarrow
\Omega^i_{\Lambda}(G) & \overset{P^0}{\rightarrow} & \Omega^i_{\Lambda}(G)
\end{array}
\]

and consists of the following vertices:

\[
\begin{pmatrix}
0
\end{pmatrix}_{\Omega^i_{\Lambda}(G)}, \begin{pmatrix}
\Omega^i_{\Lambda}(G)
\end{pmatrix}_{\Omega^{i+1}_{\Lambda}(G)}, \begin{pmatrix}
\Omega^{i+1}_{\Lambda}(G)
\end{pmatrix}_{P^0}, \text{and } \begin{pmatrix}
\Omega^{i+1}_{\Lambda}(G)
\end{pmatrix}_{u_i},
\]

where $0 \leq i \leq n - 1$ and $u_i$ denotes the canonical inclusion.
Proof. Since $\Lambda$ is CM-finite, the set $\{\Omega^i_\Lambda(G) \mid i \in \mathbb{Z}\}$ is finite. Hence we may choose a minimal positive number $n$ with $G = \Omega^i_\Lambda(G)$. Consider the following exact sequence induced by a minimal projective resolution of $G$

$$0 \to \Omega^i_\Lambda(G) \to P^{n-1} \to \cdots \to P^1 \to P^0 \to G \to 0.$$ 

We split the above sequence to the following short exact sequences

$$\varepsilon_i : 0 \to \Omega^{i+1}_\Lambda(G) \to P^i \to \Omega^i_\Lambda(G) \to 0,$$

where $0 \leq i \leq n - 1$. From Lemma 3.14 we know that all the sequences $\varepsilon_i$ are almost split sequences in $\text{Gprj-} \Lambda$.

Applying Lemmas 3.3, 3.4 and Lemma 2.1(1) for the two short exact sequences $\varepsilon_0, \varepsilon_1$ we reach the following full subquiver of $\Gamma^s_\zeta$

\[ \begin{array}{cccccc}
\Omega^2_\Lambda(G) & \Omega^2_\Lambda(G) & 0 & 0 & \Omega_\Lambda(G) & \Omega_\Lambda(G) \to P^0 \\
0 & 0 & \Omega^2_\Lambda(G) & \Omega^2_\Lambda(G) & 0 & \Omega_\Lambda(G) \\
0 & 0 & 0 & \Omega^2_\Lambda(G) & \Omega^2_\Lambda(G) & 0 \\
\end{array} \]

Notice that projective modules $P^0$ or $P^1$ might not be indecomposable. For our purpose, it doesn’t matter if they are indecomposable or not. Repeating the same construction for the pair of the short exact sequences $(\varepsilon_1, \varepsilon_2)$ until to the pair $(\varepsilon_{n-2}, \varepsilon_{n-1})$, we will obtain $n - 1$ full subquivers of $\Gamma^s_\zeta$ as the above such that one corresponding to $(\varepsilon_{n-2}, \varepsilon_{n-1})$ has the object $(0_G)_0$ in the leftmost side. Hence the construction will stop at $(n - 1)$-th step. By glowing the obtained full subquivers we get the full subquiver $\tilde{\Gamma}$ of $\Gamma^s_\zeta$ containing the $\tau_\zeta$-orbit of $(0_G)_0$. By deleting the projective-injective vertices of the full subquiver $\tilde{\Gamma}$, we then get the following component $\tilde{\Gamma}^a$ of $\Gamma^s_\zeta$ containing the vertex $(0_G)_0$

\[ \begin{array}{cccccc}
GG & \cdots & 0 & \cdots & \Omega_\Lambda(G) & \Omega_\Lambda(G) \to P^0 \\
0G & \cdots & GP^{n-1} & \cdots & \Omega^2_\Lambda(G) & \Omega^2_\Lambda(G) \\
\end{array} \]

which release all facts of the proposition. \hfill \square

Proposition 3.16. Assume that $G$ is an indecomposable non-projective Gorenstein projective module. If the associated component $\Gamma^s_\zeta(G)$ is finite, then $\Gamma^s_\zeta(G)$ contains at most two distinct $\tau_\zeta$-orbits containing a boundary vertex different than the $\tau_\zeta$-orbit of the boundary vertex $(0_G)_0$.

Proof. Assume

$$X_n \xrightarrow{a_n} \cdots \xrightarrow{a_2} X_1 \xrightarrow{a_1} X_0 = (0_G)_0$$

is the longest sectional path in $\Gamma^s(G)$. According to Proposition 3.9, we have $\Gamma^s_\zeta(G) = \mathbb{Z}\Delta/G$ for some valued Dynkin quiver $\Delta$. Therefore, we can have the following three forms of the meshes in $\Gamma^s_\zeta(G)$
Since \(X_1\) is a boundary vertex and \(X_n\) is the starting vertex of the longest sectional path, we get the meshes containing \(X_1, \tau_SX_1\) and \(X_n, \tau_SX_n\) are of form (1). But for each \(2 \leq i \leq n - 1\), the mesh \(\Xi_i\) containing \(X_i, \tau_SX_i\) are of the forms (2) or (3). More precisely, when \(\Delta\) has an underlying graph of one of the Dynkin types \(\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{F}\) and \(\mathbb{G}\), then all \(\Xi_i\) are of the form (2), but for the remaining types there exists exactly 1 different \(\Xi_i\) such that \(\Xi_j\) containing \(X_j, \tau_SX_j\) is of the form (3). Moreover, if exists, let \(Y_j\) be the middle vertex of the mesh \(\Xi_j\) other than \(X_{j+1}\) and \(\tau_SX_{j-1}\), the meshes containing \(Y_j\) as a leftmost vertex must be of the form (1). These facts follow easily from the covering of valued translation quiver \(\mathbb{Z}\Delta \to \mathbb{Z}\Delta / G\). Using these facts we observe that all the sectional paths ending at \(X_1\) as follows:

\[
2 \leq i \leq n \quad u_i : X_i \xrightarrow{a_i} X_{i-1} \rightarrow \cdots \xrightarrow{a_2} X_1,
\]

and if there exists the vertex \(j\)

\[
v_j : Y_j \xrightarrow{b_j} X_j \xrightarrow{a_j} \cdots \xrightarrow{a_2} X_1,
\]

where \(Y_j\) is the middle term of the mesh \(\Xi_j\) different from \(X_{j+1}\) and \(\tau_SX_{j-1}\). Following this way, we obtain the sectional path ending at \(\tau^m_SX_1\), for any \(m\), is obtained by applying the \(\tau^m_S\) on \(u_i\) or \(v_j\) (if exists). To complete the proof, we need this claim: if \(X\) and \(Y\) are indecomposable non-projective vertices lying in different \(\tau_S\)-orbits, then there is a sectional path in \(\Gamma^*_S(G)\) from \(X\) to \(\tau^m_SY\) for some integer \(m\). We refer to [SY1, §IV Lemma 15.5] for a proof of our claim. Although, the proof in there given for the stable Auslander-Reiten quiver over a self-injective algebra, but the same proof works here in our setting. Now the claim and the above facts concerning the sectional paths show that any vertex of \(\Gamma^*_S(G)\) lies in the \(\tau_S\)-orbit of \(X_i, 1 \leq i \leq n\), or \(Y_j\), if such a \(j\) exists. This implies that only possible vertices lying as the leftmost or rightmost vertex of a mesh with exactly one vertex in the middle are the vertices belonging to the \(\tau_S\)-orbits of \(X_1, X_{n+1}\), and \(Y_j\). Hence we get our result as boundary vertices satisfying such a property on the middle vertices. \(\Box\)

Therefore, by the above proposition, we deduce that the inverse image of every single subset of \(\mathcal{K}\) has cardinality at most 3.

In below example, we see that a finite component \(\Gamma^*_S(G)\) may contain three different \(\tau_S\)-orbits of boundary vertices.

**Example 3.17.** The triangular matrix algebra \(T_2(k[x]/(x^2))\) is a \(k\)-algebra given by quiver

\[
\begin{array}{c}
\bullet \\
\downarrow & \downarrow \\
\bullet & \bullet
\end{array}
\]

with relations \(\lambda_1^2, \lambda_2^2, \lambda_1\beta - \beta\lambda_2\). Firstly, we draw the Auslander-Reiten of \(S(k[x]/(x^2))\), remember
where $\Lambda = k[x]/(x^2)$ and $S = k[x]/(x)$. To compute the above Auslander-Reiten quiver we apply this fact that the canonical short exact sequence $0 \to S \to \Lambda \to S \to 0$ is an almost split sequence in $\text{Gprj}-\Lambda$. The vertices with the same label are identified. Hence by the above computation we get the stable Auslander algebra $\Delta$ of $T_2(\Lambda)$ is the following Nakayama algebra of Loewy length 2

$$
\begin{array}{cccc}
\tau & \tau & \tau & 0 \\
0 & S & 0 & S \\
S & 0 & S & \Lambda \\
\Lambda & \Lambda & \Lambda & 0
\end{array}
$$

that is, the path algebra of the above quiver modulo the square of the arrow ideal. The stable Auslander algebra of $T_2(\Lambda)$ is of finite representation type as it is a Nakayama algebra, hence by [H1, Theorem 3.2] we have $\mathcal{S}(\text{Gprj}-T_2(\Lambda))$ is of finite representation type, equivalently to say, $T_2(T_2(\Lambda))$ is CM-finite. Consider the objects $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $K = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathcal{S}(k[x]/(x^2))$.

The above Auslander-Reiten quiver helps us to compute the $\tau_3$-orbits in $\mathcal{S}(\text{Gprj}-T_2(\Lambda))$ as follows:

$$
\tau_3(0)^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tau_3(1)^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tau_3(2)^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Note that here $K = \Omega_{T_2(\Lambda)}(\Delta)$. Hence, by the above computations, we have three different $\tau_3$-orbits of the boundary vertices in $\mathcal{S}(\text{Gprj}-T_2(\Lambda))$. Since $\Delta$ is of finite representation type, whose Auslander-Reiten quiver $\Gamma_\Delta$ has only one component, and moreover by [H1, Theorem 6.2], $\Gamma_\Delta \subseteq \Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda))$. Under the embedding we can identify the vertices of $\Gamma_\Delta$ by ones of $\Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda))$, especially the projective vertices of $\Gamma_\Delta$ by the boundary vertices of type (c) in $\Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda))$. The quiver $\Gamma_\Delta$ is also contained in $\Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda))$ as we remove only projective-injective vertices of $\Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda))$. In addition, $\Gamma_\Delta$ contains the vertices $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This implies that $\Gamma_\mathcal{S}(\text{Gprj}-T_2(\Lambda)) = \Gamma_\mathcal{S}(\Delta) = \Gamma_\mathcal{S}(\Delta)$.

4. **Stable Auslander-Reiten components of stable Cohen-Macaulay Auslander algebras**

Throughout this section, let $\Lambda$ be a CM-finite algebra and $A$ the corresponding stable Cohen-Macaulay Auslander algebra. In this section we give some applications of our results in the preceding section for the components of the stable Cohen-Macaulay Auslander algebras. As a result, we will show that the components of $\Gamma_\mathcal{S}$ containing a boundary vertex are connected to those of $\Gamma_\Delta$ containing a simple vertex. We also show that computing the Auslander-Reiten translation of a simple $A$-module is related to doing a computation in $\text{Gprj}-\Lambda$.

Recall $A = \text{End}_A(M)$, the endomorphism algebra of $M$ in the stable category $\text{Gprj}-\Lambda$, where $M$ is a (basic) additive generator of $\text{Gprj}-\Lambda$, i.e., $\text{add}-M = \text{Gprj}-\Lambda$. Due to Subsection 2.1, there
is an equivalence mod-Gprj-Λ \cong \text{mod-}A$ given by the evaluation functor $\xi_M$. Hence we identify $A$-modules by finitely presented functors over $\text{Gprj}-\Lambda$. On the other hand, as discussed in there, mod-Gprj-Λ can be viewed as a subcategory of mod-Gprj consisting of all those functors vanish on projective modules. Under the identification, we can take into account a simple $A$-module as a simple functor $S_G$ associated with an indecomposable non-projective Gorenstein projective $A$-module $G$.

**Theorem 4.1.** Let $G$ be an indecomposable non-projective Gorenstein projective $A$-module and $S_G$ the corresponding simple $A$-module. If $G$ is not $\mathcal{G}$-semisimple, then the following statement hold.

1. $\tau_A S_G = \Omega_A^{-1}S_{\tau_G G}$ and $\tau_{\Lambda}^{-1}S_G = \Omega_{\Lambda} S_{\tau_G \Lambda G}$.
2. $\tau_A^2 S_G = \Omega_A S_{\Omega_{\Lambda}^{-1} \tau_G \Lambda G}$ and $\tau_{\Lambda}^2 S_G = \Omega_{\Lambda}^{-1} S_{\Omega_{\Lambda} \tau_G \Lambda G}$.
3. $\tau_A^3 S_G = S_{\theta G}$ and $\tau_{\Lambda}^3 S_G = S_{\theta^{-1} G}$.

In particular, for any $n \in \mathbb{Z}$, the $3n$-th Auslander-Reiten translation of a simple $A$-module is again simple.

**Proof.** We only prove the first part of each statement, as the rest is proved similarly.

Proof of (1): In view of Lemma 2.1, the almost split sequences $\gamma : 0 \to \tau_G G \to B \to G \to 0$ and $\gamma' : 0 \to \tau_G^2 G \to C \to \tau_G G \to 0$ in $\text{Gprj}-\Lambda$ induce respectively the following almost split sequences in $\mathcal{S}($Gprj-Λ$)$

\[
\begin{array}{c}
0 \to \left(\tau_G G\right)_1 \to \left(\tau_G G\right)_f \to \left(\tau_G G\right)_0 \to 0, \\
0 \to \left(\tau_G^2 G\right)_i \to \left(\tau_G^2 G\right)_C \to \left(\tau_G G\right)_0 \to 0.
\end{array}
\]

where the second sequence is obtained from the following push-out diagram

\[
\begin{array}{ccc}
0 & \to & \tau_G^2 G \\
\downarrow & & \downarrow \\
0 & \to & C
\end{array}
\]

\[
\begin{array}{ccc}
\tau_G^2 G & \to & \tau_G G \\
i & \quad \tau_G G & 0 \\
\downarrow & & \downarrow \\
0 & \to & \Omega_{\Lambda}^{-1} \tau_G^2 G \\
\end{array}
\]

\[
\begin{array}{ccc}
P_{\tau_G G} & \to & \Omega_{\Lambda} \tau_G G \\
h & \quad \tau_G G & 0 \\
\downarrow & & \downarrow \\
0 & \to & \Omega_{\Lambda}^{-1} \tau_G G \\
\end{array}
\]

From the almost split sequences ($\dagger$) and ($\dagger\dagger$) we get $\tau_S \left(\tau_G G\right)_f = \left(\tau_{\Lambda} C_{\tau_G \tau_G G}\right)$. Since $G$ is not $\mathcal{G}$-semisimple, $B$ is not projective. This implies that the object $\left(\tau_G G\right)_f$ is not boundary. In view of Subsection 2.3, we have $\tau_A \Psi \left(\tau_G G\right)_f = \Psi \left(\tau_G \tau_G G\right)$. Note that $\Psi \left(\tau_G G\right)_f = S_G$. See Subsection 2.3 for the definition of the functor $\Psi$. To complete the proof, it suffices to show
that $\Psi\left(p_{r_2 G}^{C} @ r_2 G\right)_h = \Omega_1^- S_{r_2 G}$. Setting $F := \Psi\left(p_{r_2 G}^{C} @ r_2 G\right)_h$. Applying the Yoneda functor on the middle column of the above diagram gives rise to the following exact sequence in mod-Gprj-$\Lambda$

$$(-, C) \rightarrow (-, P_{r_2 G}^{\tau_2 G} @ r_2 G) \rightarrow (-, \Omega^-_{\Lambda} r_2 G) \rightarrow \text{Ext}_A^1(-, C) \rightarrow \text{Ext}_A^1(-, P_{r_2 G}^{\tau_2 G} @ r_2 G) \rightarrow \text{Ext}_A^1(-, \Omega^-_{\Lambda} r_2 G)$$

where $F$ is located in the sequence as the kernel of $\text{Ext}_A^1(-, h)$. Moreover, since the second row in the above diagram splits, this allows us to present $h = \left[\begin{array}{c} g \\ d \end{array}\right]$, where $d : C \rightarrow P_{r_2 G}$. This presentation of $h$ and $P$ being projective follow that we may assume $\text{Ext}_A^1(-, h) = \text{Ext}_A^1(-, g)$. On the other hand, by applying the Yoneda functor on the almost split sequence $\gamma'$, we get the following exact sequence in mod-Gprj-$\Lambda$

$$(-, r_2 G) \rightarrow (-, C) \rightarrow (-, \tau G) \rightarrow \text{Ext}_A^1(-, r_2 G) \rightarrow \text{Ext}_A^1(-, C) \rightarrow \text{Ext}_A^1(-, \tau G).$$

From the second exact sequence in the functor category, we obtain the short exact sequence

$$0 \rightarrow S_{r_2 G} \rightarrow \text{Ext}_A^1(-, r_2 G) \rightarrow F \rightarrow 0.$$

But, by [AR, Theorem 7.5], we know that $\text{Ext}_A^1(-, r_2 G)$ is an injective envelop of $S_{r_2 G}$ in mod-Gprj-$\Lambda$. Notice that Gprj-$\Lambda$ is a dualizing variety as it is a functorially subcategory due to our assumption. Now the above short exact sequence completes the proof.

Proof of (2): keep the above notations. Similarly, we first need to compute the Auslander-Reiten translation of $\left(p_{r_2 G}^{C} @ r_2 G\right)_h$ in $\delta$(Gprj-$\Lambda$). Applying Lemma 2.1(3) to the almost split sequence $\gamma'' : 0 \rightarrow \tau_2 G \rightarrow E \rightarrow \Omega^-_{\Lambda} r_2 G \rightarrow 0$ in Gprj-$\Lambda$ leads to the following almost split sequence in $\delta$(Gprj-$\Lambda$), note that $\vartheta G = \tau_2 G \rightarrow \tau_2 G$,

$$0 \rightarrow \vartheta G \rightarrow \vartheta G \rightarrow \Omega^-_{\Lambda} r_2 G \rightarrow 0$$

where this sequence is obtained from the following pull-back diagram

$$\begin{array}{c}
0 & \rightarrow & \vartheta G \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\tau_2 G & \rightarrow & \tau_2 G \\
\downarrow & \uparrow & \downarrow \\
0 & \rightarrow & 0
\end{array}$$

In view of the almost split sequences ($\uparrow\uparrow$) and ($\uparrow\uparrow\uparrow$), we get $\tau_2 \left(p_{r_2 G}^{C} @ r_2 G\right)_h = \left(\tau_2 G \rightarrow \vartheta G \oplus P_{r_2 G} \rightarrow P_{r_2 G} \rightarrow 0\right)$.

Hence, we have

$$\tau_2^2 S_G = \Psi \tau_2^{\mathbb{S}} \left(\tau_2 G \right) = \Psi \tau_2^{\mathbb{S}} \left(p_{r_2 G}^{C} @ r_2 G\right)_h = \Psi \left(\tau_2 G \rightarrow \vartheta G \oplus P_{r_2 G} \rightarrow P_{r_2 G} \rightarrow 0\right).$$
So, by the above and definition of $\Psi$, to compute $\tau_A^3 S_G$ we need to apply the Yoneda functor to the middle column of the pull-back diagram. This gives us the following exact sequence in mod-$\text{Gprj-}\Lambda$

$$(-, \tau_G^2 G) \rightarrow (-, \partial G \oplus \tau_G^2 G) \rightarrow (-, E) \rightarrow \text{Ext}_A^1(-, \partial G \oplus \tau_G^2 G) \rightarrow \text{Ext}_A^1(-, \partial G \oplus \tau_G^2 G) \rightarrow \text{Ext}_A^1(-, E),$$

where $\tau_G^2 S_G$ is the kernel of $\text{Ext}_A^1(-, u)$. The above long exact sequence implies the following short exact sequence

$$0 \rightarrow \tau_A^2 S_G \rightarrow \text{Ext}_A^1(-, \tau_G^2 G) \rightarrow G \rightarrow 0,$$

where $G := \text{KerExt}_A^1(-, v)$. We can write $v = [t_r]$, where $r : \tau_G^2 G \rightarrow E$. Hence, we can consider $G$ as the kernel of $\text{Ext}_A^1(-, u)$ as $P_{\tau_G^2 G}$ is projective. On the other hand, by applying the Yoneda functor on the almost split sequence $\gamma''$, we obtain

$$0 \rightarrow (-, \partial G) \rightarrow (-, E) \rightarrow (-, \Omega^{-1} \tau_G^2 G) \rightarrow \text{Ext}_A^1(-, \partial G) \rightarrow \text{Ext}_A^1(-, E) \rightarrow \text{Ext}_A^1(-, \Omega^{-1} \tau_G^2 G).$$

Hence, $S_{\Omega^{-1} \tau_G^2 G} = \text{KerExt}_A^1(-, t) = G$. Then the sequence $\tau$ finishes the proof.

Proof of (3): we follow the same argument as (1) and (2). To do this, we need to compute the Auslander-Reiten translation of $(\tau_G^2 G)_{\partial G \oplus \tau_G^2 G}$ in $\text{S}(\text{Gprj-}\Lambda)$. Applying Lemma 2.1(1) to the almost split sequence $\zeta$:

$$0 \rightarrow \tau_G \partial G \rightarrow T \rightarrow \partial G \rightarrow 0$$

yields the following almost split sequence in $\text{S}(\text{Gprj-}\Lambda)$

$$0 \rightarrow (\tau_G \partial G) \rightarrow (\tau_G \partial G)_T \rightarrow \partial G \rightarrow 0.$$  

Therefore, by the above almost split sequence along with $(\dagger \dagger \dagger)$, we see that $\tau_G \left( (\tau_G \partial G)_T \right) = (\tau_G \partial G)_T$. Hence

$$\tau_A^3 S_G = \Psi \tau_A^3 \left( \tau_G \partial G \right)_T = \Psi \tau_A^3 \left( P_{\tau_G^2 G \oplus \tau_G^2 G} \right)_T = \Psi \tau_G \left( \tau_G \partial G \right)_T = \partial G.$$  

This completes the proof. \hfill \Box

Example 4.2. Let $\Lambda$ be the algebra given in Example 3.6. In view of Proposition 4.1 and using the corresponding computation in mod-$\Lambda$, we see that any simple module in $\text{Gprj-}\Lambda$ has no $\tau_A$-orbits in $\Gamma_A$ or $\Gamma_A$. More precisely, we have these three $\tau_A$-orbits in $\Gamma_A$ or $\Gamma_A$, containing the simple $A$-modules

$$S_{(1)}, \Omega_A S_{(2)}, \Omega^{-1}_A S_{(1)}, S_{(2)}, \Omega_A S_{(3)}, \Omega^{-1}_A S_{(2)}, S_{(3)}, \Omega_A S_{(3)}, \Omega^{-1}_A S_{(2)}, S_{(3)}$$

where each row is arranged by the i-th Auslander-Reiten translation. Using this fact $\Omega^{-1}_A S_{(3)} = S_{(3)}$ implies that all there orbits are the same. Indeed, all simple $A$-modules, as below, ordered with respect to the Auslander-Reiten translation:

$$S_{(1)}, S_{(2)}, S_{(2)}, S_{(2)}, S_{(3)}, S_{(3)}, S_{(3)}, S_{(3)}, S_{(1)}$$

Using our results in the preceding section in conjunction with 2.3, we get the next results in concern of those components of the stable Auslander-Reiten quiver $\Gamma_A$ containing simple vertices.

Proposition 4.3. Assume $\Gamma$ is a component of $\Gamma_A$ containing a simple module. Then
If $\Gamma$ is finite, then $\Gamma = \mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver and $G$ is an automorphism group of $\mathbb{Z}\Delta$ containing a positive power of the translation.

If $\Gamma$ is infinite, then $\Gamma$ is a stable tube.

In particular, if $A$ is of finite representation type, then $\Gamma^s_A$ is a disjoint union of the finite components containing a simple vertex.

**Proof.** Assume $\Gamma$ contains the simple $A$-module $S_G$ for some indecomposable Gorenstein projective $A$-module $G$. In view of Subsection 2.3, see also [H1, Section 6], we have this observation that the component $\Gamma$ is obtained by removing the vertices of type $(a), (b)$ and $(c)$ of the component $\Gamma^s(G)$. If $G$ is $\mathcal{S}$-semisimple, we know from Proposition 3.15 the component $\Gamma^s(G)$ is a cyclic quiver of those types; so it remains nothing to prove, and $\Gamma^s$ becomes a single vertex. If $G$ is not $\mathcal{S}$-semisimple, then Proposition 3.9 implies that $\Gamma$ has a $\tau_A$-periodic vertex. Then the statements (1) and (2) follow from [Li, Theorem 5.5]. The last part of the statement follows from the last one of Proposition 3.9 and making use of the observation already mentioned from [H1].

Denote by $\mathcal{L}$ the set of all component of $\Gamma^s_A$ containing a simple module, and including the empty set. Similar to the map $\delta$, defined in the preceding section, there is a map $\lambda$ from $\mathcal{K}$ to $\mathcal{L}$ by sending the component $\Gamma^s_A(S_G)$ of $\Gamma^s$ containing a boundary vertex $(0_S)_{0}$ to the component $\Gamma^s_A(S_G)$ of $\Gamma^s$ containing the simple vertex $S_G$, when $G$ is not $\mathcal{S}$-semisimple, and otherwise to the empty set.

Our definition of $\lambda$ is well-defined. Let $(0_{G'})_0$ be another boundary vertex in the components $\Gamma^s_A(G)$. If $G$ is $\mathcal{S}$-semisimple, then we can deduce by Proposition 3.15 that $G'$ is also $\mathcal{S}$-semisimple. Hence $\Gamma^s_A(G')$ is sent to the empty set under $\lambda$. Otherwise, if $\Gamma^s_A(G)$ is finite, then based on the proof of Proposition 3.16, there exists a walk $\left(\tau_{g/G_S}\right)_{g} \leftarrow x_1 \leftarrow \cdots \leftarrow \left(\tau_{g'/G'_S}\right)_{g'}$ such that none of $x_i$ is a boundary vertex. Here $g$ and $g'$ denote the minimal almost split morphisms starting at $\tau_{g/G_S}$ and $\tau_{g'/G'_S}$, respectively. By applying the functor $\Psi$ on the walk we get a walk in $\Gamma^s_A$ linking the simple vertex $S_G$ and $S_{G'}$. Hence $\Gamma^s_A(S_G) = \Gamma^s_A(S_{G'})$. If $\Gamma^s_A(G)$ is infinite, then, by proposition 3.9, it is a tube. So $\Gamma^s_A(S_G)$ is a tube and obtained by removing the vertices of the mouth of the tube $\Gamma^s_A(G)$, due to [H1, Section 6], or see Remark 3.12. This implies $\Gamma^s_A(S_G) = \Gamma^s_A(S_{G'})$.

Denote by $\mathcal{L}^\infty$ the subset of $\mathcal{L}$ consisting of all infinite components.

**Proposition 4.4.** The following statements hold.

1. The map $\lambda$ is surjective.
2. The restricted map $\lambda : \mathcal{K}^\infty \rightarrow \mathcal{L}^\infty$ is bijective.

**Proof.** (1) follows only from the definition of $\lambda$. For (2) we only need to consider this fact: the mouth of the infinite component $\Gamma^s_A(G)$, resp. $\Gamma^s_A(G)$, is formed by the $\tau_S$-orbit, resp. $\tau_A$-orbit, of the boundary vertex $(0_{S})_{0}$, resp. the simple vertex $S_G$, where $G$ is an indecomposable non-projective Gorenstein projective $A$-module.

As a consequence of the above proposition and Proposition 3.10, the composition $\lambda \circ \delta$ gives us a bijection between $\mathcal{K}^\infty$ and $\mathcal{L}^\infty$.

**Proposition 4.5.** Assume that $G$ is an indecomposable non-projective Gorenstein projective $A$-modules. Then component $\Gamma^s_A(S_G)$ contains at most two distinct $\tau_A$-orbits containing a simple $A$-module different than the $\tau_A$-orbit of the simple vertex $S_G$.

**Proof.** This follows immediately from Proposition 3.16 and the observation given in Subsection 2.3. □
5. **Stable Auslander-Reiten components of preprojective of type A**

In this section we study those components of stable Auslander-Reiten quivers over preprojective of type $A$ which contain a simple module.

Throughout this section, let $n$ be a positive integer, $\Lambda = k[x]/(x^d)$ and $B$ the Auslander algebra of $\Lambda$. Then the Auslander-Reiten quiver of $\Lambda$ is

$$k \xrightarrow{a_1} k[x]/(x^2) \xrightarrow{a_2} k[x]/(x^3) \xrightarrow{a_3} \cdots \xrightarrow{a_{n-2}} k[x]/(x^{n-1}) \xrightarrow{a_{n-1}} k[x]/(x^n),$$

and the Auslander algebra $\Lambda$ is presented by the quiver

$$1 \xrightarrow{b_2} 2 \xrightarrow{b_3} 3 \xrightarrow{b_4} \cdots \xrightarrow{b_{n-1}} n-1 \xrightarrow{b_n} n$$

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-1$. Denote by $A$ the stable Cohen-Macaulay Auslander algebra of $\Lambda$, which is, indeed, the preprojective algebra of type $A_{n-1}$[DR]. In the rest, for any $1 \leq i \leq n-1$, we write $S_i$ instead of $S_{k[x]/(x^i)}$, the corresponding simple $A$-module to the indecomposable $\Lambda$-module $k[x]/(x^n)$. Keeping in mind, we identify here mod-$B$, resp. mod-$A$, with the category mod-mod-$\Lambda$, resp. mod-mod-$\Lambda$, of finitely presented functors over the category mod-$\Lambda$, resp. the stable category mod-$\Lambda$.

**Proposition 5.1.** Let $1 \leq i \leq n-1$. The following statements hold.

(a) If $i \neq \frac{n}{2}$, then

1. $\tau_A^i S_i = S_i$,
2. $\tau_A^{i+1} S_i = S_{n-i}$,
3. $\tau_A^i S_i = \Omega_A^{-1} S_i$ and $\tau_A^i S_i = \Omega_A^{-1} S_{n-i}$,
4. $\tau_A^i S_i = \Omega_A S_{n-i}$ and $\tau_A^i S_i = \Omega_A S_i$.

(b) If $i = \frac{n}{2}$, necessarily $n$ is even, then

1. $\tau_A^i S_i = S_i$,
2. $\tau_A^i S_i = \Omega_A^{-1} S_i$ and $\tau_A^i S_i = \Omega_A S_i$.

**Proof.** The Auslander-Reiten quiver provided in the beginning of the section yields the almost split sequences

if $i \neq 1$, $0 \to k[x]/(x^i) \to k[x]/(x^{i+1}) \oplus k[x]/(x^{i-1}) \to k[x]/(x^i) \to 0$, and

if $i = 1$, $0 \to k[x]/(x^1) \to k[x]/(x^2) \to k[x]/(x) \to 0$.

And the short exact sequence $0 \to k[x]/(x^{d-i}) \to k[x]/(x^d) \to k[x]/(x^i) \to 0$ implies that

$\Omega_A(k[x]/(x^i)) = k[x]/(x^{d-i})$. In view of these facts and Proposition 4.1 we obtain all the statements. \qed

Recall that $\mathcal{L}$ is the set of all stable components of $\Gamma^{\circ}_A$ containing a simple module.

**Proposition 5.2.** Let $n \geq 6$.

1. If $n$ is odd, the cardinality of $\mathcal{L} = \mathcal{L}^\infty$ is $\frac{n-1}{2}$;
2. If $n$ is even, the cardinality of $\mathcal{L} = \mathcal{L}^\infty$ is $\frac{n}{2}$.

**Proof.** First note that $n \geq 6$ guarantees that $\mathcal{S}(\Lambda)$ is of infinite representation type, see [RS3]. From Propositions 3.10 and 4.4, we see that there is a bijection between the sets $\mathcal{R}^\infty$ and $\mathcal{L}^\infty$. If $n$ is odd, then all $\theta$-orbits are as follows:

$$\{k[x]/(x), k[x]/(x^d), \cdots, k[x]/(x^d_{\theta^{-1}}), \cdots k[x]/(x^d_{\theta^{-1}}), \cdots k[x]/(x^{d_{\theta^{-1}}}), k[x]/(x^{d_{\theta^{-1}}})\};$$
and, if \( n \) is even, then all \( \theta \)-orbits are as follows:
\[
\{ k[x]/(x), k[x]/(x^{n-1}), \cdots, k[x]/(x^i), k[x]/(x^{n-i}), \cdots \{ k[x]/(x^2) \} \}.
\]
So the bijection completes the proof. \( \square \)

Denote by \( \mathcal{T}_i \) the component of \( \mathcal{T}_A \) containing the simple module \( S_i \). In view of Proposition 5.1 the mouth of \( \mathcal{T}_i \) for \( i \neq \frac{n}{2} \) is
\[
\{ \Omega_A S_i, \Omega_A^{-1} S_{n-i}, S_{n-i}, \Omega_A S_{n-i}, \Omega_A^{-1} S_i, S_i \},
\]
and for \( i = \frac{n}{2} \) is
\[
\{ \Omega_A S_i, \Omega_A^{-1} S_i, S_i \}.
\]
Since for every \( i, S_i \) is a simple \( A \)-module, we then, by applying the equivalence \( \Omega_A : \text{mod-}A \rightarrow \text{mod-}A \), conclude that all elements of the mouth of \( \mathcal{T}_i \) are brick, i.e. the endomorphism algebra in the stable category is a division algebra.

But, the elements of the mouth of \( \mathcal{T}_i \) are not pairwise orthogonal in the stable module category \( \text{mod-}A \). However, we have the following of vanishing of Hom-spaces of the (syzygies of) elements of the mouths.

The following remark is helpful for our next proofs.

**Remark 5.3.** Let \( C \) be an Artin algebra.

1. Let \( f : F_1 \rightarrow F_2 \) be a map in \( \text{mod-}C \) and \( (-, X_1) \rightarrow (-, X_0) \rightarrow F_1 \rightarrow 0 \) and \( (-, Y_1) \rightarrow (-, Y_0) \rightarrow F_2 \rightarrow 0 \) be projective presentations of \( F_1 \) and \( F_2 \), respectively. In view of the proof of [A1, Proposition 2.1], we can construct the following projective presentation of \( \text{Cok}(f) \),
\[
(-, Y_1 \oplus X_1) \rightarrow (-, Y_0) \rightarrow \text{Cok}(f) \rightarrow 0.
\]

2. Assume further \( C \) is self-injective. Let \( 0 \rightarrow V \rightarrow U \rightarrow W \rightarrow 0 \) be a short exact sequence in \( \text{mod-}C \). One has an induced long exact sequence in \( \text{mod-mod}C \) [MT]
\[
\cdots \rightarrow (-, \Omega_C(U)) \rightarrow (-, \Omega_C(W)) \rightarrow (-, V) \rightarrow (-, U) \rightarrow (-, W) \rightarrow (-, \Omega_C^{-1}(V)) \rightarrow (-, \Omega_C^{-1}(U)) \rightarrow \cdots
\]

**Proposition 5.4.** Let \( 1 \leq i \leq n \).

1. \( \text{Hom}_A(S_i, \Omega_A^{-1} S_i) = 0 \);
2. If \( i \neq \frac{n}{2} \) then \( \text{Hom}_A(S_i, \Omega_A S_i) = 0 \);
3. If \( i \neq \frac{n-1}{2} \) and \( \frac{n+1}{2} \), then \( \text{Hom}_A(S_i, \Omega_A^{-1} S_{n-i}) = 0 \).
4. If \( i \neq \frac{n}{2} \) then \( \text{Hom}_A(\Omega_A^2 S_i, S_{n-i}) = 0 \). In particular, \( \text{Ext}_B^2(S_i, S_{n-i}) = 0 \).

**Proof.** For the case \( i = 1 \), the middle term ending at \( k[x]/(x) \) has only one indecomposable direct summand, that is \( k[x]/(x^2) \). To simplify, we give our proof for the case \( i \neq 1 \), which also works for the \( i = 1 \) with an easy modification. Throughout the proof we assume that \( i \neq 1 \).

1. We have the isomorphism \( \text{Hom}_A(S_i, \Omega_A^{-1} S_i) \approx \text{Ext}_A^1(S_i, S_i) \), using this fact that \( A \) is self-injective. So we must show that \( \text{Ext}_B^1(S_i, S_i) = 0 \), or equivalently \( \text{Ext}_B^1(S_i, S_i) = 0 \). The almost split sequence \( \theta : 0 \rightarrow k[x]/(x^i) \rightarrow k[x]/(x^{i+1}) \oplus k[x]/(x^{i-1}) \rightarrow k[x]/(x) \rightarrow 0 \) induces the following minimal projective resolution of \( S_i \) as an object in \( \text{mod-}B \)
\[
\varepsilon : 0 \rightarrow (-, k[x]/(x^i)) \rightarrow (-, k[x]/(x^{i+1}) \oplus (-, k[x]/(x^{i-1})) \rightarrow (-, k[x]/(x^i)) \rightarrow S_i \rightarrow 0.
\]
As $\text{Ext}^1_i(S_i, S_i) \simeq \text{Hom}_{\mathbb{D}^b(\text{mod-}B)}(S_i, S_i[1])$, where $\mathbb{D}^b(\text{mod-}B)$ denotes the bounded derived category over $\text{mod-}B$, hence it suffices to prove that any chain map between the deleted minimal projective resolution of $S_i$ and whose one shifting, as below,

$$
\begin{array}{cccc}
0 & \to & (-, k[x]/(x')) & \to & (-, k[x]/(x')^i) \\
& \downarrow \eta & \downarrow \eta & \downarrow \eta \\
& (-, k[x]/(x')^i) & \to & (-, k[x]/(x')^i) \\
\end{array}
$$

is null-homotopic. By the Yoneda lemma we can write $\eta_0 = (-, f)$ for some morphism $f : k[x]/(x')^i \to k[x]/(x')$. But since $\lambda$ is clearly non-retraction, $f$ must be factored through the epimorphism lying in the almost split sequences $\theta$. Consequently, by applying the Yoneda functor on the obtained factorization in mod-$\Lambda$, we get the desired factorization in mod-$B$. Then, it is standard to see that this factorization implies that the chain map is null-homotopic.

(2) The short exact sequence $0 \to \Omega^2_\Lambda(S_i) \to P \to \Omega_\Lambda(S_i) \to 0$, where $P$ is a projective cover of $\Omega_\Lambda(S_i)$ in mod-$\Lambda$, induces the isomorphism $\text{Hom}_\Lambda(S_i, \Omega_\Lambda S_i) \simeq \text{Ext}^1_\Lambda(S_i, \Omega^2_\Lambda S_i)$. Hence it is enough to show $\text{Ext}^1_\Lambda(S_i, \Omega^2_\Lambda S_i) = 0$.

From now on, setting $D := k[x]/(x')$, $C := k[x]/(x')^i \oplus k[x]/(x')$ in the short exact sequence $\theta$. Applying Remark 5.3 to the short exact sequence $\epsilon$ implies the following diagram in mod-$\Lambda$

$$
\begin{array}{cccc}
(-, \Omega_\Lambda(D)) & \to & (-, D) & \to & (-, C) & \to & (-, D) \\
\Omega^2_\Lambda(S_i) & \to & \Omega_\Lambda(S_i) & \to & S_i \\
\end{array}
$$

Note that $\Omega_\Lambda(D) = k[x]/(x'^i)$. By the above diagram one gets a projective resolution of $S_i$ and $\Omega^2(S_i)$, respectively, in mod-$\Lambda$. To prove the result, in view of the above isomorphism, we need to show that any chain map between the corresponding projective resolutions, as described in below, is null-homotopic,

$$
\begin{array}{cccc}
\ldots(-, \Omega_\Lambda(D)) & \to & (-, D) & \to & (-, C) & \to & (-, D) \\
\downarrow \kappa_1 & \downarrow \kappa_0 & \downarrow \kappa_1 & \downarrow \kappa_0 & \downarrow \kappa_1 & \downarrow \kappa_0 \\
\ldots(-, \Omega_\Lambda(C)) & \to & (-, \Omega_\Lambda(D)) & \to & (-, D) & \to & 0 \\
\end{array}
$$

By the Yoneda lemma we can write $\kappa_1 = (-, f)$ for some $f : D \to \Omega_\Lambda(D)$ in mod-$\Lambda$. Our assumption on $i$ guarantees $D \neq \Omega_\Lambda(D)$. Then, as $\epsilon$ is an almost split sequence, we get a factorization of $f$ through $g$ in mod-$\Lambda$. This implies a factorization of $\kappa_1$ through $(-, g)$. Next, in a standard way we can obtain the desired homotopic morphism.

(3) The proof is analogous to the proof of the statement (1). However, we sketch the proof, leaving details to the reader. Consider the isomorphism $\text{Hom}_\Lambda(S_i, \Omega^1_{\Lambda S_{n-i}}) = \text{Ext}^1_\Lambda(S_i, S_{n-i})$. So it suffices to show that $\text{Ext}^1_\Lambda(S_i, S_{n-i}) = 0$, or equivalently $\text{Ext}^1_B(S_i, S_{n-i}) = 0$. The almost split sequence $\theta' : 0 \to k[x]/(x'^i) \to k[x]/(x'^{i+1}) \oplus k[x]/(x'^{i-1}) \to k[x]/(x'^i) \to 0$ induces the following minimal projective resolution of $S_{n-i}$ as an object in mod-$B$

$$
\epsilon' : 0 \to (-, k[x]/(x'^i)) \to (-, k[x]/(x'^{i+1}) \oplus (-, k[x]/(x'^{i-1})) \to (-, k[x]/(x'^i)) \to S_{n-i} \to 0.
$$
To get the result, we need to show that any chain map between the deleted minimal projective resolutions of \( S_i \) and the one shifting of that of \( S_{n-i} \), as described in \( \varepsilon \) and \( \varepsilon' \),

\[
\begin{array}{cccc}
0 & \longrightarrow & (-, D) & \longrightarrow & (-, C) & \longrightarrow & (-, D) \\
\downarrow & & \downarrow \kappa_1 & & \downarrow \kappa_0 & & \downarrow \\
(-, k[x]/(x^{n-i})) & \longrightarrow & (-, k[x]/(x^{n-i+1}) \oplus k[x]/(x^{n-i-1})) & \longrightarrow & (-, k[x]/(x^{n-i})) & \longrightarrow & 0
\end{array}
\]

is null-homotopic. Since \( D \) is not a direct summand of \( k[x]/(x^{n-i+1}) \oplus k[x]/(x^{n-i-1}) \), thanks to our assumption on \( i \), this implies that the morphism \( \kappa_1 \) factors through the morphism \( (-, D) \rightarrow (-, C) \). Then by use of such a factorization we obtain the desired homotopy morphism in an obvious way.

(4) Based on the proof of the statement (2), we obtain the following exact sequence in mod-\( A \) (or in mod-\( B \))

\[
(-, k[x]/(x^{n-i})) \rightarrow (-, k[x]/(x^{n})) \rightarrow \Omega^3_A S_i \rightarrow 0.
\]

Next, by applying Remark 5.3 and using this fact that the global dimension of \( B \) is at most of 2, we get the following projective resolution

\[
0 \rightarrow (-, P) \rightarrow (-, k[x]/(x^{n-i}) \oplus k[x]/(x^n)) \rightarrow (-, k[x]/(x^n)) \rightarrow \Omega^2_A S_i \rightarrow 0
\]

in mod-\( B \). Take \( f \in \text{Hom}_A(\Omega^2_A S_i, S_{n-i}) \), where \( f : \Omega^2_A S_i \rightarrow S_{n-i} \) in mod-\( A \), (which can be considered as a morphism in mod-\( B \)). The morphism \( f \) is lifted to a chain map between the projective resolution of \( \Omega^2_A S_i \), as given in the above, and the one of \( S_{n-i} \) as given in \( (\varepsilon') \),

\[
\begin{array}{cccc}
(-, P) & \longrightarrow & (-, D' \oplus \Lambda) & \longrightarrow & (-, D) & \longrightarrow & \Omega^2_A S_i \\
\downarrow & & \downarrow \lambda_2 & & \downarrow \lambda_1 & & \downarrow \lambda_0 & & \downarrow f \\
(-, D') & \longrightarrow & (-, C') & \longrightarrow & (-, D') & \longrightarrow & S_{n-i}
\end{array}
\]

where \( D' = k[x]/(x^{n-i}) \) and \( C' = k[x]/(x^{n-i+1}) \oplus k[x]/(x^{n-i-1}) \). Our assumption on \( i \) guarantees that \( \lambda_0 \) is not an isomorphism, so by the almost split property we get \( \lambda_0 \) factors through the \( (-, C') \rightarrow (-, D') \). This means that \( f = 0 \), so \( f \), as required. \( \Box \)

The next result states that the elements of each mouth \( \mathcal{T}_i \) are not pairwise orthogonal.

**Lemma 5.5.** Let \( 1 \leq i \leq n \). Then \( \text{Hom}_A(S_i, \Omega_A S_{n-i}) \neq 0 \).

**Proof.** By Proposition 5.1, we have \( \tau_A \Omega^{-1}_A S_i = \tau_A^2 S_i = \Omega_A S_{n-i} \). Hence, we get the almost split sequence \( 0 \rightarrow \Omega_A S_{n-i} \rightarrow B \rightarrow \Omega^{-1}_A S_i \rightarrow 0 \) in mod-\( A \). This induces the Auslander-Reiten triangle

\[
\Omega_A S_{n-i} \rightarrow B \rightarrow \Omega^{-1}_A S_i \xrightarrow{h} S_{n-i}
\]

in the triangulated category mod-\( A \). But \( h \neq 0 \), otherwise, the triangle splits, a contradiction. After applying the equivalence \( \Omega_A \) on \( h \) we get the non-zero morphism \( \Omega_A h \) in \( \text{Hom}_A(S_i, \Omega_A S_{n-i}) \), as desired. \( \Box \)

From now on, let \( n \geq 6 \) and \( 1 \leq i \leq n \). Thus, the associated component \( \mathcal{T}_i \) to the simple module \( S_i \) is infinite. Hence by Proposition 4.3, \( \mathcal{T}_i \) is a tube. Following the terminology in [Ri], in this case \( \mathcal{T}_i \) is a stably quasi-serial component of \( \Gamma_A \). Now suppose that \( X \) is an element in the mouth of \( \mathcal{T}_i \). In \( \mathcal{T}_i \) there is a unique infinite sectional path

\[
X = X[1] \rightarrow X[2] \rightarrow \cdots X[m] \rightarrow X[m+1].
\]
The elements of the mouth is also called quasi-simple. Moreover, the quasi-length of a module $M$ in $\mathcal{T}_i$ is the number of the row containing $M$, so $X[r]$ has quasi-length $r$.

If $i \neq \frac{n}{2}$, denote for each $r \in \{1, 2, \cdots, 6\}$, $X'_r = \tau^{-1}_A S_i$ and $X'_7 = X^1_i$. Similarly, if $i = \frac{n}{2}$ is even, denote for each $r \in \{1, 2, 3\}$, $X'_r = \tau^{-1}_A S_i$ and $X'_4 = X^1_i$.

In below we establish a result of independent interest which determines the Auslander-Reiten translations of objects in the component $\mathcal{T}_i$ via their (co)syzygies.

**Proposition 5.6.** Then the following statements hold.

1. If $i \neq \frac{n}{2}$, then $\tau^{-1}_A (X'_r[m]) = \Omega^{-1}_A (X'_r[m])$ for $j \in \{1, 2, 3, 4\}$, and $\tau^{-1}_A (X'_7[m]) = \Omega_A (X'_7[m])$, where $r \in \{1, 2, 3\}$.

2. If $i = \frac{n}{2}$, then $\tau^{-1}_A (X'_r[m]) = \Omega^{-1}_A (X'_r[m])$ and $\tau^{-1}_A (X'_7[m]) = \Omega_A (X'_7[m])$, where $r \in \{1, 2, 3\}$.

**Proof.** We only prove (1) as the proof of (2) is similar. Since the equivalences $\tau_A, \Omega_A : \text{mod}-A \to \text{mod}-A$ preserve the almost split sequences, they induce naturally an automorphism of translation quivers of $\Gamma_A$. The automorphisms are restricted to the component $\mathcal{T}_i$. Using the automorphisms, it is enough to prove the result for $r = m = 1$. To this end, we first claim $\Omega^{-1}_A S_i = S_{n-i}$, note that $S_i = X^1_i$. By Proposition 5.1, we have the almost split sequence $0 \to \Omega^{-1}_A S_i \to B \to S_i \to 0$ in $\text{mod}-A$, and so the triangle $\Omega^{-1}_A S_i \to B \to S_i \to 0$ in the stable category $\text{mod}-A$. By applying the equivalence $\Omega^{-1}_A$, we get the Auslander-Reiten triangle $\Omega^{-2}_A S_i \to \Omega^{-1}_A B \to \Omega^{-1}_A S_i \to 0$ in $\text{mod}-A$, hence $\tau^2_A S_i = \Omega^{-2}_A S_i$. On the other hand, by Proposition 5.1, we know that $\tau^2_A S_i = \Omega S_{m-i}$. Thus, $\tau^2_A S_i = \Omega^{-2}_A S_i = \Omega S_{m-i}$, so the claim follows. Then using the claim in conjunction with Proposition 5.1 we obtain the result. \[\square\]

**Remark 5.7.** As we have observed the component $\mathcal{T}_i$ is closed under (co)syzygies. The natural question may arise here is whether the additive closure $\text{add}\mathcal{T}_i$ of $\mathcal{T}_i$ is a sub-triangulated category of $\text{mod}-A$. This can be considered as a triangulated version of the similar result in the setting of the module categories over hereditary algebras, see [SY2, §VII Theorem 3.7].

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