A NOTE ON GALOIS THEORY FOR BIALGEBROIDS

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ABSTRACT. In this note we reduce certain proofs in [3, 5, 6] to depth two quasibases from one side only. This minimalistic approach leads to a characterization of Galois extensions for finite projective bialgebroids without the Frobenius extension property: a proper algebra extension is a left T-Galois extension for some right finite projective left bialgebroid T over some algebra R if and only if it is of left depth two and left balanced. Exchanging left and right in this statement, we have also a characterization of right Galois extensions for left finite projective right bialgebroids. As a corollary, we obtain insights into split monic Galois mappings and endomorphism ring theorems for depth two extensions.

1. INTRODUCTION AND PRELIMINARIES

Hopf algebroids arise as the endomorphisms of fiber functors from certain tensor categories to a bimodule category over base algebra. For example, Hopf algebroids over a one-dimensional base algebra are Hopf algebras while Hopf algebroids over a separable K-algebra base are weak Hopf algebras. Galois theory for right or left bialgebroids were introduced recently in [3, 5, 6] based on the theory of Galois corings [2] and ordinary definitions of Galois extensions [10, 4] with applications to depth two extensions. In [6] Frobenius extensions that are right Galois over a left finite projective right bialgebroid are characterized as being of depth two and right balanced. Then a Galois theory for Hopf algebroids, especially of Frobenius type, was introduced in [2, 1] with applications to Frobenius extensions of depth two and weak Hopf-Galois extensions over finite dimensional quantum groupoids (among other things, showing that these are Frobenius extensions). Although they break with the tradition of defining Galois extensions over bialgebras and have a more complex definition, Galois extensions over Hopf algebroids have more properties in common with Hopf-Galois extensions. However, several of these properties will follow from any Galois theory for bialgebroids which is in possession of two Galois mappings equivalent due to a bijective antipode, sometimes denoted by \( \beta \) and \( \beta' \), as is the case for finite Hopf-Galois theory [10, ch. 8], finite weak Hopf-Galois theory [7, section 5], or possibly some future, useful weakening of Hopf-Galois theory to Hopf algebroids over a symmetric algebra, Frobenius algebra or some other type of base algebra.

In [1] a characterization similar to [6] of depth two Frobenius one-sided balanced extensions are given in terms of Galois extensions over Hopf algebroids with integrals. This shows in a way that the main theorem in [6] makes no essential use of the hypothesis of Frobenius extension (only that a Frobenius extension is of left depth two if and only if it is of right depth two), and it would be desirable at some
time to remove the Frobenius hypothesis. This is then the objective of this paper: to show that Galois extensions over one-sided finite projective bialgebroids are characterized by one-sided depth two and balance conditions on the extension (Theorem 2.1). This requires among other things some care in re-doing the two-sided arguments in [2] to show that the structure $T := (A \otimes B A)^B$ on a one-sided depth two extension $A \mid B$ with centralizer $R$ is still a one-sided finite projective right bialgebroid (proposition 1.1). In an appendix, we do a similar one-sided derivation of the left bialgebroid structure on the $R$-dual $S := \text{End}_B A B$. These two sections may be read as an introduction to depth two theory.

Let $K$ be any commutative ground ring in this paper. All algebras are unital associative $K$-algebras and modules over these are symmetric unital $K$-modules. We say that $A \mid B$ is an extension (of algebras) if there is an algebra homomorphism $B \to A$, proper if this is monic. This homomorphism induces the natural bimodule structure $B A B$ which is most important to our set-up. The extension $A \mid B$ is left depth two (left D2) if the tensor-square $A \otimes_B A$ is centrally projective w.r.t. $A$ as natural $B$-$A$-bimodules: i.e.,

$$B A \otimes_B A_A \oplus_* \cong \oplus^n B A_A.$$  

This last statement postulates the existence then of a split $B$-$A$-epimorphism from $A \otimes_B A$ to direct product of $A$ with itself $n$ times.

Making the clear-cut identifications $\text{Hom}_B (B A \otimes_B A_A, B A_A) \cong \text{End}_B A B$ and $\text{Hom}_B (B A B, B A \otimes_B A_A) \cong (A \otimes_B A)^B$, we see that left D2 is characterized by there being left D2 quasibases $t_i \in (A \otimes_B A)^B$ and $\beta_i \in \text{End}_B A B$ such that for all $a, a' \in A$

$$a \otimes_B a' = \sum_{i=1}^n t_i \beta_i(a) a'.$$

(1)

The algebras $\text{End}_B A B$ and $(A \otimes_B A)^B$ (note that the latter is isomorphic to $\text{End}_A A \otimes_B A_A$ and thus receives an algebra structure) are so important in depth two theory that we fix (though not unbendingly) brief notations for these:

$$S := \text{End}_B A B \quad T := (A \otimes_B A)^B.$$

Similarly, a right depth two extension $A \mid B$ is defined by switching from the natural $B$-$A$-bimodules in the definition above to the natural $A$-$B$-bimodules on the same structures. Thus an extension $A \mid B$ is right D2 if $A A \otimes_B A_B \oplus_* \cong \oplus^m A A_B$. Equivalently, if there are $m$ paired elements $u_j \in T$, $\gamma_j \in S$ such that

$$a \otimes a' = \sum_{j=1}^m a \gamma_j(a') u_j$$

(2)

for all $a, a' \in A$.

A depth two extension is one that is both left and right D2. These have been studied in [9,5,6] among others, but without a focus on left or right D2 extensions.

Let $t, t'$ be elements in $T$, where we write $t$ in terms of its components using a notation that suppresses a possible summation in $A \otimes_B A$: $t = t^1 \otimes t^2$. Then the algebra structure on $T$ is simply

$$tt' = t'^1 t^1 \otimes t^2 t^2, \quad 1_T = 1_A \otimes 1_A$$

(3)

There is a standard “groupoid” way to produce right and left bialgebroids, which we proceed to do for $T$. There are two commuting embeddings of $R$ and its opposite.
algebra in \( T \). A “source” mapping \( s_R : R \to T \) given by \( s_R(r) = 1_A \otimes r \), which is an algebra homomorphism. And a “target” mapping \( t_R : R \to T \) given by \( t_R(r) = r \otimes 1_A \) which is an algebra anti-homomorphism and clearly commutes with the image of \( s_R \). Thus it makes sense to give \( T \) an \( R\)-\( R \)-bimodule structure via \( s_R \), \( t_R \) from the right: \( r \cdot t \cdot r' = t s_R(r') t_R(r) = t (r \otimes r') = r t^1 \otimes t^2 r' \), i.e., \( R T_R \) is given by

\[
(4) \quad r \cdot t^1 \otimes t^2 \cdot r' = r t^1 \otimes t^2 r'
\]

**Proposition 1.1.** Suppose \( A \mid B \) is either a right or a left \( D^2 \) extension. Then \( T \) is a right \( R \)-bialgebroid, which is either left f.g. \( R \)-projective or right f.g. \( R \)-projective respectively.

**Proof.** First we suppose \( A \mid B \) is left \( D^2 \) with quasibases \( t_i \in T, \beta_i \in S \). The proof that \( T \) is a right \( R \)-bialgebroid in [9, 5.1] carries through verbatim except in one place where right \( D^2 \) quasibases made a brief appearance, where coassociativity of the coproduct needs to be established through the introduction of an isomorphism. Thus we need to see that

\[
T \otimes_R T \otimes_R T \xrightarrow{\cong} (A \otimes_B A \otimes_B A \otimes_B A)^B
\]

via \( t \otimes t' \otimes t'' \mapsto t^1 \otimes t^2 t'^1 \otimes t'^2 t''^1 \otimes t''^2 \). The inverse is given by

\[
a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto \sum_{i,j} t_i \otimes_R t_j \otimes_R (\beta_j(a_1)a_2) a_3 \otimes_B a_4.
\]

for all \( a_i \in A \) (\( i = 1, 2, 3, 4 \)).

In the case that we only use right \( D^2 \) quasibases, this inverse is given by

\[
a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto \sum_{j,k} a_1 \otimes a_2 \gamma_k(a_3 \gamma_j(a_4)) \otimes_R u_k \otimes_R u_j.
\]

Both claimed inverses are easily verified as such by using the right and left \( D^2 \) quasibase equations repeatedly.

The module \( T_R \) is finite projective since eq. (4) implies a dual bases equation \( t = \sum t_i f_i(t) \), for each \( t \in T \subseteq A \otimes_B A \), where \( f_i(t) := \beta_i(t^1) t^2 \) define \( n \) maps in \( \text{Hom}(T_R, R_R) \).

Suppose \( A \mid B \) is right \( D^2 \) with quasibases \( u_j \in T, \gamma_j \in S \). The algebra structure on \( T \) is given in the introduction above as is the \( R\)-\( R \)-bimodule structure. What remains is specifying the \( R \)-comodule structure on \( T \) and checking the five axioms of a right bialgebroid. The coproduct \( \Delta : T \to T \otimes_R T \) is given by

\[
(6) \quad \Delta(t) := \sum_j (t^1 \otimes_B \gamma_j(t^2)) \otimes_R u_j,
\]

which is clearly left \( R \)-linear, and right \( R \)-linear as well since

\[
\Delta(t r) = \sum_j t^1 \otimes \gamma_j(t^2 r) \otimes u_j \xrightarrow{\cong} t^1 \otimes 1 \otimes t^2 r
\]

under the isomorphism \( T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B \) given by \( t \otimes t' \mapsto t^1 \otimes t^2 t'^1 \otimes t'^2 \), which is identical to the image of

\[
\Delta(t) r = \sum_j t^1 \otimes \gamma_j(t^2) \otimes u_j r \mapsto t^1 \otimes 1 \otimes t^2 r.
\]
In detail, the definition is equivalent to the following.

This completes the proof that $(T, R, \Delta, \varepsilon)$ is a right bialgebroid. Finally, $R^T$ is finite projective via an application of the right D2 quasibase eq. \[4\].

A right comodule algebra is an algebra in the tensor category of right $R$-comodules. In detail, the definition is equivalent to the following.
Definition 1.2. Let $T$ be any right bialgebroid $(T, R, \tilde{s}, \tilde{t}, \Delta, \varepsilon)$ over any base algebra $R$. A right $T$-comodule algebra $A$ is an algebra $A$ with algebra homomorphism $R \to A$ (providing the $R$-$R$-bimodule structure on $A$) together with a coaction $\delta : A \to A \otimes_R T$, where values $\delta(a)$ are denoted by the Sweedler notation $a_{(0)} \otimes a_{(1)}$, such that $A$ is a right $T$-comodule over the $R$-coring $T$ [3, 18.1], $\delta(1_A) = 1_A \otimes 1_T$, $r_{a_{(0)}} a_{(1)} = a_{(0)} \otimes \tilde{t}(r) a_{(1)}$ for all $r \in R$, and $\delta(aa') = \delta(a) \delta(a')$ for all $a, a' \in A$. The subalgebra of coinvariants is $A^{\text{co}T} := \{a \in A | \delta(a) = a \otimes 1_T\}$. We also call $A$ a right $T$-extension of $A^{\text{co}T}$.

Lemma 1.3. For the right $T$-comodule $A$ introduced just above, $R$ and $A^{\text{co}T}$ commute in $A$.

Proof. We note that

$$\rho(rb) = b \otimes_R \tilde{s}(r) = \rho(br)$$

for $r \in R$, $b \in A^{\text{co}T}$. But $\rho$ is injective by the counitality of comodules, so $rb = br$ in $A$ (suppressing the morphism $R \to A$).

Definition 1.4. Let $T$ be any right bialgebroid over any algebra $R$. A $T$-comodule algebra $A$ is a right $T$-Galois extension of its coinvariants $B$ if the (Galois) mapping $\beta : A \otimes B \to A \otimes_R T$ defined by $\beta(a \otimes a') = aa'_{(0)} \otimes a'_{(1)}$ is bijective.

Left comodule algebras over left bialgebroids and their left Galois extensions are defined similarly, the details of which are in [7]. The values of the coaction is in this case denoted by $a_{(-1)} \otimes a_{(0)}$ and the Galois mapping by $a \otimes a' \mapsto a_{(-1)} \otimes a_{(0)} a'$.

2. A characterization of Galois extensions for bialgebroids

We recall that a module $AM$ is balanced if all the endomorphisms of the natural module $M_E$ where $E = \text{End}_A M$ are uniquely left multiplications by elements of $A$: $A \xrightarrow{\approx} \text{End}_E M$ via $\lambda$. In particular, $AM$ must be faithful.

Theorem 2.1. Let $A | B$ be a proper algebra extension. Then

1. $A | B$ is a right $T$-Galois extension for some left finite projective right bialgebroid $T$ over some algebra $R$ if and only if $A | B$ is right $D_2$ and right balanced.

2. $A | B$ is a left $T$-Galois extension for some right finite projective left bialgebroid $T$ over some algebra $R$ if and only if $A | B$ is left $D_2$ and left balanced.

Proof. ($\Rightarrow$) Suppose $T$ is a left finite projective right bialgebroid over some algebra $R$. Since $RT \oplus \ast \cong R^t$ for some positive integer $t$, we apply to this the functor $A \otimes_R -$ from left $R$-modules into $A$-$B$-bimodules which results in $A A_B A_B \oplus \ast \cong A A_B$, after using the Galois $A$-$B$-isomorphism $A \otimes_B A \cong A \otimes_R T$. Hence, $A | B$ is right $D_2$.

Let $E := \text{End}_A B$. We show $A B$ is balanced by the following device. Let $R$ be an algebra, $M_R$ and $R V$ modules with $R V$ finite projective. If $\sum_j m_j \phi(v_j) = 0$ for all $\phi$ in the left $R$-dual $^*V$, then $\sum_j m_j \otimes_R v_j = 0$. This follows immediately by using dual bases $f_i \in ^*V$, $w_i \in V$.

Given $F \in \text{End}_E A$, it suffices to show that $F = \rho_b$ for some $b \in B$. Since $\lambda_a \in E$, $F \circ \lambda_a = \lambda_a \circ F$ for all $a \in A$, whence $F = \rho_{F(1)}$. Designate $F(1) = x$. If we show that $x_{(0)} \otimes x_{(1)} = x \otimes 1$ after applying the right $T$-valued coaction
on $A$, then $x \in A^{\otimes T} = B$. For each $\alpha \in \text{Hom}(\rho T, \rho R)$, define $\overline{\alpha} \in \text{End} AB$ by 
$$\overline{\alpha}(y) = y(0)\alpha(y(1)).$$
Since $\rho_r \in \mathcal{F}$ for each $r \in R$ by lemma,
$$x\alpha(1T) = F(\overline{\alpha}(1A)) = \overline{\alpha}(F(1A)) = x(0)\alpha(x(1))$$
for all $\alpha \in ^* T$. Hence $x(0) \otimes_R x(1) = x \otimes 1T$.

($\Leftarrow$) It follows from the proposition that a right D2 extension $A \mid B$ has a left finite projective right bialgebroid $T := (A \otimes_B A)^B$ over the centralizer $R$ of the extension. Let $R \mapsto A$ be the inclusion mapping. We check that $A$ is a right $T$-comodule algebra via the coaction $\rho_R : A \to A \otimes_R T$ on $A$ given by

$$\rho_R(a) = a(0) \otimes a(1) := \sum_j \gamma_j(a) \otimes u_j. \quad (8)$$

First, we demonstrate several properties by using the isomorphism $\beta^{-1} : A \otimes_R T \xrightarrow{\cong} A \otimes_B A$ given by $\beta^{-1}(a \otimes t) = at = a1 \otimes t^2$ [3.12(iii)] with inverse $\beta(a \otimes a') = \sum_j a a'_j \otimes a'_j (1)$ (cf. right D2 quasibase eq. (2)). This shows straightaway that the Galois mapping $\beta : A \otimes_B A \to A \otimes_R T$ is bijective. Then $A \otimes_R T \otimes_R T \cong A \otimes_B A \otimes_B A$ via $\Phi := (id_A \otimes \beta^{-1})(\beta^{-1} \otimes id_T)$, so coassociativity $(id_A \otimes \Delta)\rho_R = (\rho_R \otimes \text{id}_T)\rho_R$ follows from

$$\Phi((id \otimes \Delta_T) \circ \rho_R) = \sum_{j,k} \gamma_j(a)u_j^1 \otimes_B \gamma_k(u_j^2)u_k^1 \otimes_B u_k^2 = \sum_k 1 \otimes \gamma_k(a)u_k^1 \otimes_B u_k^2 = 1 \otimes 1 \otimes a = \Phi((id \otimes \text{id})\rho_R(a)).$$

We note that $\rho_R$ is right $R$-linear, since

$$\rho_R(ar) = \sum_j \gamma_j(ar) \otimes u_j \xrightarrow{\beta^{-1}} 1 \otimes_B ar = \beta^{-1}(\rho_R(a)r)$$

since $\rho_R(a)r = \sum_j \gamma_j(a) \otimes u_j r$. Also, $a(0)\varepsilon_T(a(1)) = \sum_j \gamma_j(a)u_j^1u_j^2 = a$ for all $a \in A$.

Next,

$$\beta^{-1}(r \cdot a(0) \otimes a(1)) = \sum_j r \gamma_j(a)u_j = r \otimes_B a = \sum_j \gamma_j(a)u_j^1r \otimes_B u_j^2 = \beta^{-1}(a(0) \otimes t_R(r)a(1)).$$

Whence the statement $\rho_R(aa') = \rho_R(a)\rho_R(a')$ makes sense for all $a, a' \in A$. We check the statement:

$$\beta^{-1}(\rho_R(a)\rho_R(a')) = \sum_{j,k} \gamma_j(a)\gamma_k(a')u_ju_k = \sum_{j,k} \gamma_j(a)\gamma_k(a')u_j^1u_j^2 \otimes_B u_k^1u_k^2$$

$$= 1 \otimes aa' = \sum_j \gamma_j(aa')u_j = \beta^{-1}(\rho_R(aa')).$$

Also $\rho_R(1_A) = 1_A \otimes 1T$ since $\gamma_j(1_A) \in R$. Finally we note that for each $b \in B$

$$\rho_R(b) = \sum_j \gamma_j(b) \otimes_R u_j = b \otimes \sum_j \gamma_j(1)u_j = b \otimes 1T$$

so $B \subseteq A^{\otimes \rho R}$. Conversely, if $\rho_R(x) = x \otimes 1T = \sum_j \gamma_j(x) \otimes u_j$ applying $\beta^{-1}$ we obtain $x \otimes_B 1 = 1 \otimes_B x$. Let $f \in \text{End} AB$. Then applying $\mu(f \lambda(a) \otimes \text{id})$ to this we obtain $f(ax) = f(a)x$ since $\lambda(a) \in \text{End} AB$ for each $a \in A$. It follows that $f \rho(x) = \rho(x)f$ so $\rho(x) \in \text{End} \varepsilon A$. Since $AB$ is balanced, $\rho(x) = \rho(b)$ for some $b \in B$, whence $x = b \in B$. 

The second part of the theorem is proven similarly. In the $\Leftarrow$ direction, we convert the right $R$-bialgebroid $T$ to a left $R$-bialgebroid $T^{op}$ with $s_L = t_R$, $t_L = s_L$, the same $R$-coring structure and opposite multiplication, which leads to the left $R$-$R$-bimodule structure coinciding with the usual $R$-$R$-bimodule structure on $T$ in eq. (4). We then define a left $T^{op}$-comodule algebra structure on $A$ via $\rho_L : A \to T \otimes R A$ defined via left D2 quasibases by

$$\rho_L(a) = a_{(-1)} \otimes a_{(0)} := \sum_i t_i \otimes \beta_i(a).$$

The isomorphism $T \otimes_R A \cong A \otimes_B A$ given by $t \otimes a \mapsto t^1 \otimes t^2 a$ is inverse to the Galois mapping $\beta_L(a \otimes a') = a_{(-1)} \otimes a_{(0)} a'$ by the left D2 quasibase eq. (1). One needs the opposite multiplication of $T$ when showing $\rho_L(a a') = \rho_L(a)\rho_L(a')$ for $a, a' \in A$. 

Let $T$ be a left finite projective right bialgebroid over some algebra $R$ in the next corollary.

**Corollary 2.2.** Suppose $A|B$ is a right $T$-extension. If the Galois mapping $\beta$ is a split monic, then $A|B$ is a right $(A \otimes_B A)^B$-Galois extension.

**Proof.** This follows from $A A \otimes_B A_B \oplus \star \cong A_B \otimes_R T$ and the arguments in the first few paragraphs of the proof above (the balance argument makes only use of $A|B$ being a right $T$-extension). Hence, $A|B$ is right D2 and right balanced. Whence $A|B$ is a right Galois extension w.r.t. the bialgebroid $(A \otimes_B A)^B$. 

Notice that $T$ is possibly not isomorphic to $(A \otimes_B A)^B$. For example, one might start with a Hopf algebra Frobenius extension with split monic Galois map and conclude it is a weak Hopf-Galois extension (if the centralizer is separable, the antipode being constructible from the Frobenius structure).

We also observe that putting Theorems 2.1 and [7, 2.6] together yields a type of endomorphism ring theorem for depth two extensions, without a Frobenius extension hypothesis (cf. [9, Theorem 6.6]).

**Corollary 2.3.** Suppose $A|B$ is a depth two algebra extension. Then $\text{End}_{A_B} A$ is a left $D2$ and left balanced extension of $A^{op}$.

**Proof.** In [7, Theorem 2.6] it is established that $\text{End}_{A_B} A$ is a left $S$-Galois extension of $\rho(A) = \{\rho(a)|a \in A\}$ where $\rho(a)(x) = xa$ for all $x, a \in A$. But $S$ is a left and right f.g. projective left bialgebroid by the D2 hypothesis (cf. proposition 3.1 below). It follows from the second statement in the theorem above that $\text{End}_{A_B} A \mid A^{op}$ is left $D2$ and left balanced.

By carefully checking the proof of [7, 2.6] for an airtight reliance on only right D2 quasibases, and referring to the proposition below, I believe it is likely that the corollary may be somewhat improved to show that a right $D2$ algebra extension has a left $D2$ left endomorphism algebra extension.

3. **Appendix**

In this section we answer some natural questions about the theory of one-sided depth two extensions. One of the apparent questions after a reading of proposition 1.1 would be if the endomorphism algebra $S$ is also a bialgebroid over the centralizer, to which the next proposition provides an answer in the affirmative.
Proposition 3.1. Suppose $A \mid B$ is either a right or a left $D2$ extension with centralizer $R$. Then $S$ is a left $R$-bialgebroid, which is either right f.g. $R$-projective or left f.g. $R$-projective respectively.

Proof. The algebra structure comes from the usual composition of endomorphisms in $S = \operatorname{End}_B A_B$. The source and target mappings are $s_L(r) = \lambda(r)$ and $t_L(r) = \rho(r)$, whence the structure $R_{SR}$ is given by

$$r \cdot \alpha \cdot r' = \lambda(r)\rho(r')\alpha = r\alpha(-)r'. $$

Suppose now we are given a right $D2$ structure on $A \mid B$ by quasibases $u_j \in T$, $\gamma_j \in S$. The $R$-coring structure on $R_{SR}$ is given by a coproduct $\Delta : S \rightarrow S \otimes_R S$ defined by

$$\Delta(\alpha) = \sum_j \gamma_j \otimes u_j^1\alpha(u_j^2^-), $$

and a counit $\varepsilon : S \rightarrow R$ given by

$$\varepsilon(\alpha) = \alpha(1_A). $$

Clearly $\varepsilon$ is an $R-R$-bimodule mapping with $\varepsilon(1_S) = 1_A$, satisfying the counitality equations and

$$\varepsilon(\alpha\beta) = \varepsilon(\alpha s_L(\varepsilon(\beta))) = \varepsilon(\alpha t_L(\varepsilon(\beta))). $$

Also $\Delta$ is right $R$-linear and $\Delta(1_S) = 1_S \otimes_R 1_S$. By making the identification

$$S \otimes_R S \cong \operatorname{Hom}(B_A \otimes_B A_B, B_A B_B), \quad \alpha \otimes \beta \mapsto (a \otimes a' \mapsto \alpha(a)\beta(a'))$$

with inverse $F \mapsto \sum_j \gamma_j \otimes u_j^1 F(u_j^2 \otimes -)$, we see that the coproduct is left $R$-linear, satisfies $\alpha(1_{(1)} t_L(r) \otimes \alpha(2)) = \alpha(1_{(1)} \otimes \alpha(2)) s_L(r)$ for all $r \in R$, and $\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta)$ for all $\alpha, \beta \in S$. For with the independent variables $x, x' \in A, \alpha, \beta \in S$ and $r \in R$, each of these expressions becomes equal to $r\alpha(xx'), \alpha(xx')$, and $\alpha(\beta(xx'))$ respectively.

The coproduct $\Delta$ is coassociative since

$$S \otimes_R S \otimes_R S \cong \operatorname{Hom}(B_A \otimes_B A_B \otimes_B A_B, B_A B_B B_B), \quad \alpha \otimes \beta \otimes \gamma \mapsto (x \otimes y \otimes z \mapsto \alpha(x)\beta(y)\gamma(z))$$

with inverse given by

$$F \mapsto \sum_{i,j,k} \gamma_i \otimes u_i^1 \gamma_j (u_j^2 \gamma_k(-)) \otimes u_j^1 F(u_j^2 u_k^1 \otimes u_k^2 \otimes -) $$

Applying this identification to $(\Delta \otimes \text{id}_S)\Delta(\alpha)$ and to $(\text{id}_S \otimes \Delta)\Delta(\alpha)$ then to $x \otimes_B y \otimes_B z$ both expressions equal $\alpha(xyz)$.

$S_R$ is f.g. projective since for each $\alpha \in S$, we have $\alpha = \sum_j \gamma_j h_j(\alpha)$ where $h_j \in \operatorname{Hom}(S_R, R_R)$ is defined by $h_j(\alpha) = u_j^1 \alpha(u_j^2)$. The proof that given left $D2$ quasibases $t_i \in T$, $\beta_i \in S$, we have left f.g. projective left bialgebroid $S$ with identical bialgebroid structure is similar and therefore omitted.

Suppose $A \mid B$ is right $D2$. Then we have seen that $S$ is a right finite projective left bialgebroid while $T$ is a left finite projective right bialgebroid. There is a nondegenerate pairing between $S$ and $T$ with values in the centralizer $R$ given by $\langle t | \alpha \rangle := t^1\alpha(t^2)$, since

$$\eta : R_T \xrightarrow{\cong} \operatorname{Hom}(S_R, R_R)$$

via $\eta(t) = \langle t | - \rangle$ with inverse $\phi \mapsto \sum_j \phi(\gamma_j) u_j$. By proposition [9, 2.5] a right f.g. projective left bialgebroid $S$ has a right $R$-bialgebroid $R$-dual $S^*$. The question is
then if the bialgebroid $S^*$ is isomorphic to the bialgebroid $T$ via $\eta$? The question is partly answered in the affirmative by corollary [9, 5.3], where it is shown without using left D2 quasibases that $T$ and $S^*$ are isomorphic via the pairing above as algebras and $R$-$R$-bimodules.

**Corollary 3.2.** Suppose $A \mid B$ is right D2. Then $T$ is isomorphic as right bialgebroids over $R$ to the right $R$-dual of $S$ via $\eta$. If $A \mid B$ is left D2, then $T$ is isomorphic to the bialgebroid left $R$-dual of $S$.

**Proof.** What remains to check in the first statement is that $\eta$ is a homomorphism of $R$-corings using right D2 quasibases. We compute:

$$
(t_1) \cdot \langle t_2 \mid \alpha' \rangle = \sum_j \langle t_1 \otimes_B \gamma_j(t_2) u_j \alpha'(u_j) \mid \alpha \rangle = t_1 \alpha'(t_2) = \langle t \mid \alpha \circ \alpha' \rangle,
$$

Whence $\Delta(\eta(t)) = \eta(t_1) \otimes \eta(t_2)$ by uniqueness [9, 2.5 (41)].

The proof of the last statement is similar to the first in using the pairing $[\alpha \mid t] := \alpha(t_1)t_2$ and the right bialgebroid of the left dual of a left bialgebroid in [9, 2.6]. The details are left to the reader. □

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