A new three-loop sum-integral of mass dimension two

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Abstract: We evaluate a new 3-loop sum-integral which contributes to the Debye screening mass in hot QCD. While we manage to derive all divergences analytically, its finite part is mapped onto simple integrals and evaluated numerically.
1 Introduction

Motivated by the need to compute a number of unknown three-loop vacuum sum-integrals of mass dimension 2, in order to complete the evaluation of the 3-loop Debye screening mass \( m_E^2 \) in hot QCD [1], we turn our attention towards one particular open case. The present paper is hence rather technical in practice, but serves to establish one of the few missing building blocks for the determination of the Debye mass at NNLO.

In Ref. [1], the bosonic contribution to \( m_E^2 \) has been reduced to a sum of ten sum-integrals times pre-factors which are rational functions in \( d \), by systematic [2] use of integration-by-parts (IBP) relations [3], generalized to finite temperatures [4]. Two out of the basis of those ten master sum-integrals were triple products of 1-loop cases \( I \) and hence trivial
(cf. Eq. (B.2)), while only two further non-trivial three-loop representatives of that basis (containing basketball-type sum-integrals) are presently known, namely $S_1$ of [5] (see also [6, 7]) and $B_{3,2}$ of [1], however not deeply enough in their $\epsilon$-expansion since the corresponding pre-factors develop single poles as $d \to 3$.

It can be shown that, by a change of basis, the eight non-trivial three-loop basketball-type masters can be traded for only three spectacles-type (plus some trivial factorized) sum-integrals, with pole-free pre-factors [8]. Of those three new masters, it turns out that one has already appeared in a study of the $g^7$ contributions to the pressure of massless thermal scalar $\phi^4$ theory in the framework of screened perturbation theory. It has originally been computed in Ref. [9] and been re-derived in Ref. [10], where it was named $M_{1,0}$. This leaves us with two elements of the new basis to be evaluated, one of which we tackle in the present paper.

Let us define the massless bosonic 3-loop vacuum sum-integral $V_2$ in terms of the 1-loop 2-point sum-integrals $\Pi, \bar{\Pi}$ as

$$V_2 \equiv \oint \frac{1}{[P^2]^2} \Pi(P) \bar{\Pi}(P), \quad \Pi(P) = \oint \frac{1}{Q^2 (P - Q)^2}, \quad \bar{\Pi}(P) = \oint \frac{R_0^2}{R^2 (P - R)^2}. \quad (1.1)$$

We use (Euclidean) bosonic four-momenta $P = (P_0, \mathbf{p}) = (2\pi n_T, \mathbf{p})$ with $P^2 = P_0^2 + \mathbf{p}^2$; the temperature of the system is denoted by $T$; and the sum-integral symbol is a shorthand for

$$\oint \equiv T \sum_{n_p \in \mathbb{Z}} \int \frac{d^d \mathbf{p}}{(2\pi)^d}, \quad \text{with} \quad d = 3 - 2\epsilon. \quad (1.2)$$

In the remainder of this paper, we evaluate the new sum-integral $V_2$ analytically as far as possible, utilizing methods from [5, 7, 10, 11]. In Sec. 2 we split up the sum-integral into nine pieces that are either calculable analytically, or explicitly finite such that they can be evaluated numerically in $d=3$. Sec. 3 treats each of these nine terms in turn, which are then summed to obtain our final result in Sec. 4. We conclude in Sec. 5, while some technicalities are relegated to the appendices.

## 2 Decomposition of $V_2$

Inspired by [11] and guided by our experience from [10], $V_2$ can be identically re-written as

$$V_2 = \oint \frac{\delta_{P_0}}{[P^2]^2} \Pi(P) \bar{\Pi}(P) + \oint \frac{1}{[P^2]^2} \left\{ [\Pi - \Pi_B] [\bar{\Pi} - \bar{\Pi}_B] + \Pi_D [\bar{\Pi} - \bar{\Pi}_C] + [\Pi_B - \Pi_D] [\bar{\Pi} - \bar{\Pi}_C] \Pi_D + [\Pi - \Pi_C] [\bar{\Pi}_B - \bar{\Pi}_D] + [\Pi_C - \Pi_B] \bar{\Pi}_B + \Pi_B [\bar{\Pi}_C - \bar{\Pi}_B] + \Pi_B \bar{\Pi}_B \right\}, \quad (2.1)$$

$$\equiv \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 + \nu_7 + \nu_8 + \nu_9, \quad (2.2)$$

where $\delta_{P_0}$ picks out the Matsubara zero-mode, the primed sum excludes the zero-mode and we have suppressed the argument $(P)$ of all functions in curly brackets. Our strategy amounts
to choosing \((\Pi_B: \text{zero-T part of } \Pi; \left[ \Pi_C - \Pi_B \right]: \text{large-P behavior of remainder})\)

\[
\Pi_B = \int_Q \frac{1}{Q^2(P - Q)^2} = \frac{G(1, 1, d + 1)}{(P^2)^{(3-d)/2}}, \quad \Pi_C = \Pi_B + \frac{2I_0^0}{P^2}, \quad \Pi_D = \frac{G(1, 1, d + 1)}{(\alpha_1 T^2)^{(3-d)/2}}, \quad (2.3)
\]

where the functions \(G\) and \(I\) are known analytically as given in App. B while \(\alpha_1\) and \(\alpha_2\) below are constants to be fixed later, as well as (with \(U = (1, 0)\))

\[
\bar{\Pi}_B = \int_R \frac{R_0^2}{R^2(P - R)^2} = U_{\mu}U_{\nu} \int_R \frac{R_{\mu}R_{\nu}}{R^2(P - R)^2} = U_{\mu}U_{\nu} \left\{ g_{\mu\nu}A(P^2) + P_{\mu}P_{\nu}B(P^2) \right\}
\]

\[
= A(P^2) + P_0^2B(P^2) = \frac{(d + 1)P_0^2 - P^2}{4d} \Pi_B, \quad (2.4)
\]

\[
\bar{\Pi}_C = \bar{\Pi}_B + \frac{P_0^2I_0^0 + 2I_1^2}{P^2}, \quad \bar{\Pi}_D = \frac{(d + 1)P_0^2 - P^2}{4d} \frac{G(1, 1, d + 1)}{(\alpha_2 T^2)^{(3-d)/2}}. \quad (2.5)
\]

3 Evaluation of the nine terms of Eq. (2.1)

We will now turn to each term of Eq. (2.1) in sequence, evaluating it up to the constant part.

3.1 Evaluation of \(\nu_1\)

To tame the infrared behavior of the first term in Eq. (2.1), we transform it via the integration-by-parts (IBP) relation of Eq. (E.1) into

\[
\nu_1 = \int_P \delta_P \Pi(P) \bar{\Pi}(P) = \frac{1}{d - 6} \int_P \delta_P \frac{\Pi_2}{P^2} \left\{ \bar{\Pi}_B + \Pi \bar{\Pi} - I_0^0 \frac{1}{P^2} \bar{\Pi} - I_2^0 \frac{1}{P^2} \Pi \right\} \quad (3.1)
\]

with

\[
\bar{\Pi} = \int_Q \frac{1}{Q^2(P - Q)^2}, \quad \bar{\Pi} = \int_R \frac{R_0^2}{R^2(P - R)^2}. \quad (3.2)
\]

Using \(\Pi_A = \Pi_B + \frac{dG(1, 1, d)}{(P^2)^{(3-d)/2}}\) and \(\bar{\Pi}_B = \int_R \frac{R_0^2}{R^2(P - R)^2}\) as well as the functions \(G, I\) and \(A\) that are listed in App. B, this is then identically re-written as

\[
\nu_1 = \frac{1}{d - 6} \int_P \delta_P \left\{ \Pi_B \bar{\Pi} + \left( \Pi - \Pi_B \right) \int_Q \frac{\delta_{Q_0}}{(Q^2)^2} \right\} + \left( \Pi - 2\Pi_B \right) \int_Q \left( Q^2 \right)^2 (P - Q)^2 + \left[ \Pi - \Pi_A \right] \left( \bar{\Pi} - \bar{\Pi}_B \right) + \Pi \bar{\Pi}_B + \Pi_A \bar{\Pi} - \Pi_A \bar{\Pi}_B - I_0^0 \frac{1}{P^2} \bar{\Pi} - I_2^0 \frac{1}{P^2} \Pi \right\} \quad (3.3)
\]

\[
= \frac{1}{d - 6} \left\{ - \frac{G(1, 1, d + 1)}{4d} \right\} A(\frac{d}{2}, 2, 1; 0) + \left[ T G(2, 1, d) A(\frac{d}{2}, 1, 1; 2) - 0_{\text{scalefree}} \right] + \nu_1a + \nu_1b + \frac{2G(1, 1, d + 1)}{4d} A(\frac{d}{2}, 1, 1; 0) + \left[ G(1, 1, d + 1) A(\frac{d}{2}, 2, 1; 2) + T G(1, 1, d) A(\frac{d}{2}, 2, 1; 2) \right] - 0_{\text{scalefree}} - I_0^0 A(2, 1, 1; 2) - I_2^0 A(2, 1, 1; 0) \quad (3.4)
\]

\[
\approx \frac{T^2}{(4\pi)^4} \left( \frac{\pi}{4e^\gamma} \right) \frac{1}{T^6} \left\{ \frac{1}{48e^2} + 10 \frac{1}{48e} + \text{const} + O(\epsilon) \right\}, \quad (3.5)
\]
where the const also includes the numerical contributions from \( V_{1a} \) and \( V_{1b} \), which are defined in Eqs. (3.6), (3.7) below. For the first term of Eq. (3.3), we have used that \( \delta P_0 \bar{\Pi}_B = \frac{G(1,d+1)}{4d} \frac{\delta P_0}{(P^2)^{1-d}/2} \) is a massless bubble, while for the second term we have used that the \( Q \)-integration gives a massless bubble,

\[
\delta P_0 \frac{\delta Q_0}{[Q^2]^2(P-Q)^2} = T \int \frac{d^d q}{(2\pi)^d} \frac{1}{|q|^2(P-q)^2} = \delta P_0 T \frac{G(2,1,d)}{(P^2)^{3-d}/2},
\]

leaving exactly solvable 2-loop tadpoles \( A \). The next two (finite) sum-integrals \( V_{1a} \) and \( V_{1b} \) will be treated numerically in coordinate space, as discussed below. For the fifth and seventh terms in Eq. (3.3), we have used that (with \( U = (1,0) \))

\[
\delta P_0 \bar{\Pi}_B = \delta P_0 U \mu U_{\nu} \int \frac{R_{\mu R_{\nu}}}{[R^2]^2(P-R)^2} = \delta P_0 U \mu U_{\nu} \left\{ g_{\mu \nu} A(P^2) + P_{\mu} P_{\nu} B(P^2) \right\} = \delta P_0 A(P^2)
\]

\[
= \frac{\delta P_0 P^2 g_{\mu \nu} - \frac{P_{\mu} P_{\nu}}{P^2}}{d P^2} \int \frac{R_{\mu R_{\nu}}}{[R^2]^2(P-R)^2} = \frac{\delta P_0}{4d P^2} \int \frac{P^2 (2R^2 - P^2) + (P-R)^2}{[R^2]^2(P-R)^2} \]

\[
= \frac{\delta P_0}{4d P^2} \int \frac{2R^2 - P^2}{[R^2]^2 (P+R)^2} + o_{\text{scalefree}} = \frac{\delta P_0}{4d} \frac{2G(1,1,d+1) - G(2,1,d+1)}{(P^2)^{3-d}/2}
\]

while the remaining three terms of Eq. (3.3) are immediately seen to be 2-loop tadpoles \( A \).

To complete the evaluation of \( V_1 \), we need the two (finite) sum-integrals \( V_{1a} \) and \( V_{1b} \),

\[
V_{1a} \equiv \int_P \frac{\delta P_0}{P^2} \left[ \bar{\Pi} - \bar{\Pi}_B \right] \int_Q \frac{1}{[Q^2]^2(P-Q)^2} \approx \frac{T^2}{(4\pi)^4} V_{1a} + O(\epsilon), \quad (3.6)
\]

\[
V_{1b} \equiv \int_P \frac{\delta P_0}{P^2} \left[ \bar{\Pi} - \bar{\Pi}_A \right] \left[ \bar{\Pi} - \bar{\Pi}_B \right] \approx \frac{T^2}{(4\pi)^4} V_{1b} + O(\epsilon). \quad (3.7)
\]

For \( V_{1a} \), using the 3d spatial Fourier transforms of \([\bar{\Pi} - \bar{\Pi}_B]\) and \( \bar{\Pi}' \) as given in App. A (used at \( P_0 = 0 \)); integrating over \( p \) via Eq. (C.5); letting \(|r| = x/(2\pi T)\) and \(|r'| = y/(2\pi T)\); and using Eq. (C.9) for the angular integral:

\[
V_{1a} = \frac{1}{6} \int_0^\infty dx \int_0^\infty dy \left[ \bar{f}(x,0) - \bar{f}_B(x,0) \right] \bar{f}'(y,0) \frac{x+y-|x-y|}{xy} \]

\[
= -\frac{1}{36} \int_0^\infty \frac{dx}{x} \left[ \bar{f}(x,0) - \bar{f}_B(x,0) \right] \left[ 3Li_3(e^{2x}) - 3\zeta(3) - 2\pi^2 x + 6i\pi x^2 + 4x^3 \right] \]

\[
\approx -0.0285014376988(1), \quad (3.8)
\]

where the functions \( \bar{f}, \bar{f}_B \) and \( \bar{f}' \) are listed in Eqs. (A.10), (A.11) and (A.17), respectively.

For \( V_{1b} \), re-writing \( \delta P_0 \left[ \bar{\Pi} - \bar{\Pi}_A \right] = \delta P_0 \left[ \bar{\Pi} - \bar{\Pi}_B \right] - \frac{1}{8 \pi} \times \int \frac{d^4 p}{(2\pi)^4} e^{ipr} \frac{\delta p}{(2\pi)^d} + O(\epsilon) \), where \( \frac{1}{8} = G(1,1,3) \) while the extra integral is unity and introduced here for notational simplicity; using the 3d spatial Fourier transforms of \([\bar{\Pi} - \bar{\Pi}_B]\) and \([\bar{\Pi} - \bar{\Pi}_A]\) (at \( P_0 = 0 \)); integrating
over \( p \) via Eq. (C.5); letting \(|r| = x/(2\pi T)\) and \(|r'| = y/(2\pi T)\); and using Eq. (C.9) for the angular integral:

\[
V_{1b} = \frac{1}{4} \int_0^\infty dx \int_0^\infty dy \left[ f(x, 0) - f_B(x, 0) - 1 \right] \left[ \tilde{f}(y, 0) - \tilde{f}_B(y, 0) \right] \frac{x + y - |x - y|}{xy} \tag{3.10}
\]
\[
= -\frac{1}{2} \int_0^\infty dx \left[ f(x, 0) - f_B(x, 0) - 1 \right] \left[ x + \ln (x \cosh(x)) \right] \tag{3.11}
\]
\[
\approx +1.197038271143294592038702(1) \tag{3.12}
\]

### 3.2 Evaluation of \( V_2 \)

This integral is finite, so we write

\[
V_2 \equiv \frac{f}{f_B} \left[ \frac{1}{(P^2)^2} \left[ \Pi - \Pi_B \right] \left[ \Pi - \Pi_B \right] \right] \approx \frac{T^2}{(4\pi)^2} V_{2a} + O(\epsilon) \tag{3.13}
\]

Using the 3d spatial Fourier transforms of \([\Pi - \Pi_B]\) and \([\Pi - \Pi_B]\) from App. A; using Eq. (C.4); and letting \(|r| = x/(2\pi T)\) and \(|r'| = y/(2\pi T)\),

\[
V_{2a} = \frac{1}{3} \int_0^\infty dx \int_0^\infty dy \frac{1}{2} \int_1^{-1} du \frac{e^{-n\sqrt{x^2+y^2+2xy}}}{n} \left[ f(x, n) - f_B(x, n) \right] \left[ \tilde{f}(y, n) - \tilde{f}_B(y, n) \right] \tag{3.14}
\]
\[
= \frac{1}{6} \int_0^\infty dx \int_0^x dy \frac{1}{2} \int_1^{-1} du \frac{e^{-2n x}}{n^3} \left[ 1 + nx - ny - e^{-2ny}(1 + nx + ny) \right] \times \left[ f(x, n) - f_B(x, n) \right] \left[ \tilde{f}(y, n) - \tilde{f}_B(y, n) \right] + x \leftrightarrow y \tag{3.15}
\]

where in the last step we have integrated over angles via Eq. (C.8), and split the integration region into two to avoid taking absolute values. Finally, summing via Eq. (D.1) (which gives a large expression containing polylogarithms) and solving the remaining double integral over \( x \) and \( y \) numerically with the help of Mathematica [12],

\[
V_{2a} \approx +0.0143560494342(1) \tag{3.16}
\]

### 3.3 Evaluation of \( V_3 \)

Noting from Eq. (2.3) that \( \Pi_D \) is \( P \)-independent, we re-write the third term of Eq. (2.1) as

\[
V_3 \equiv \int \frac{1}{P^4} \Pi_D \left[ \Pi - \Pi_C \right] = \Pi_D \int \frac{1}{P^4} \left\{ \Pi - \delta P_0 \Pi - \Pi_C + \delta P_0 \Pi_C \right\} \tag{3.17}
\]

Now the first term is a regular 2-loop sum-integral, which reduces via the IBP relation Eq. (E.5) to a product of 1-loop tadpoles; the second term is a special 2-loop tadpole \( A \); the third term is immediately recognized to be a sum of 1-loop tadpoles by noticing that the \( P \)-dependence of \( \Pi_C \) is trivial (powers of \( P_0 \) and \( P^2 \) only); and, for the same reason, the fourth term is scale-free and hence vanishes in dimensional regularization. We hence get the exact expression

\[
V_3 = -\frac{G(1, 1, d + 1)}{(\alpha_1 T^2)^{(3-d)/2}} \left\{ \frac{1}{d - 5} \left( 2P_3^2 P_1^0 + P_2^0 P_0^0 \right) + A(2, 1, 1; 2) + \ldots \right\} \tag{3.18}
\]
\[ + \left[ \frac{G(1,1,d+1)}{4d} \left( (d+1)P_2^2 - P_0^2 \right) + \frac{I_2^0I_1^0}{2} \right] - 0_{\text{scalefree}} \] (3.17)
\[ \approx \frac{T^2}{(4\pi)^4} \left( \frac{1}{\alpha_1 T^2} \right)^\epsilon \left( \frac{z_0}{\epsilon} + \text{const} + O(\epsilon) \right), \quad z_0 = \frac{1}{24} \left( \frac{\zeta(3)}{5} + \frac{\zeta(-1)}{\zeta(-1)} - \gamma_E - \frac{3}{2} \right), \] (3.18)

where \text{const} is somewhat lengthy, so we do not display it here.

### 3.4 Evaluation of \( V_4 \)

Expanding \( [\Pi_B - \Pi_D] = G(1,1,d+1)[(P^2)^{-\epsilon} - (\alpha_1 T^2)^{-\epsilon}] \approx G(1,1,4-2\epsilon) \epsilon [\ln \frac{\alpha_1 T^2}{\epsilon} + O(\epsilon)] \) as well as \( G(1,1,4-2\epsilon) \approx \frac{(4\pi e^2/\gamma_0)^\epsilon}{(4\pi)^2 \epsilon} (1 + O(\epsilon^2)) \), we see that this integral is finite, so we write

\[ V_4 = \frac{C}{f_0} \int P^2 \left[ \Pi_B - \Pi_D \right] [\bar{\Pi} - \bar{\Pi}_C] \approx \frac{T^2}{(4\pi)^4} V_4 + O(\epsilon). \] (3.19)

Using the 3d spatial Fourier transform of \( [\bar{\Pi} - \bar{\Pi}_C] \) from App. A; integrating over angles via Eq. (3.7); letting \(|r| = x/(2\pi T), |p| = |P_0|y\) and \(|P_0| = 2\pi T n\); and using the exponential-integral Eq. (3.3):

\[ V_4 = \frac{1}{6} \sum_n \frac{1}{n} \int_0^\infty dx \left\{ e^{-2nx} \ln \left( \frac{2x \alpha_1 e^{\gamma-1}}{n \pi^2} \right) - Ei(-2nx) \right\} \left[ \bar{f}(x,n) - \bar{f}_C(x,n) \right] \] (3.20)
\[ = z_0 \ln \left( \frac{\alpha_1 e^{\gamma-1}}{16\pi^2} \right) + V_{4a} \] (3.21)
\[ V_{4a} = z_0 \ln(\alpha_6) + \frac{1}{6} \sum_n \frac{1}{n} \int_0^\infty dx \left\{ e^{-2nx} \ln \left( \frac{2x}{\alpha_6 n} \right) - Ei(-2nx) \right\} \left[ \bar{f}(x,n) - \bar{f}_C(x,n) \right] \] (3.22)

where the explicit \( \alpha_1 \)-dependence follows from \( \alpha_1 \)-independence of \( V_3 + V_4 \) term with \( z_0 \) from Eq. (3.18), and we have introduced an arbitrary parameter \( \alpha_6 \) to parameterize some freedom in evaluating the number \( V_{4a} \). Mathematica [12] actually gives a result for the sum for the first term in curly brackets of Eq. (3.20), but none for the Ei term. For obtaining the numerical approximation, we have truncated the sum over \( n \) at some \( n_{\text{max}} = 30000 \). In order to estimate the magnitude of the remainder, we have interpolated the one-dimensional integral in the range \( n \in [10^4, 10^5] \) with a simple power law \( n^{-a} \) and performed the summation from \( n = 30001 \ldots \infty \) analytically, from which we infer that the remainder is of \( O(10^{-10}) \).

### 3.5 Evaluation of \( V_5 \)

In close analogy to the treatment of \( V_3 \), we identically re-write the fifth term of Eq. (2.1) as

\[ V_5 = \frac{C}{f_0} \int P^2 \left[ \Pi - \Pi_C \right] \Pi_D = \frac{G(1,1,d+1)}{(\alpha_2 T^2)^{(3-d)/2}} \frac{1}{4d} \int f P^2 \left( \frac{(d+1)P_2^2}{P^2} - P^2 \right) \left\{ \Pi - \delta R_P \Pi - \Pi_C + \delta R_P \Pi_C \right\} \] (3.23)
Now the first term of Eq. (3.23) is a sum of two regular 2-loop sum-integrals, which both reduce to zero: first, from the IBP relation of Eq. (E.2) we get

\[ \int_P \frac{(d + 1)P_0^2 - P^2}{P^4} \Pi = (d + 1) I(211, 20) - I(111, 00) = -\frac{d(d - 2)}{3} \Pi(111, 00) , \]

while furthermore, the 2-loop sunset sum-integral \( I(111, 00) \) vanishes identically via IBP identity Eq. (E.3). The second term of Eq. (3.23) is a special 2-loop tadpole \( A \); the third term is immediately recognized to be a sum of 1-loop tadpoles by noticing that the \( P \)-dependence of \( \Pi_C \) is trivial (powers of \( P^2 \) only); and, for the same reason, the fourth term is scale-free and hence vanishes in dimensional regularization. We hence get the exact expression

\[ \mathcal{V}_5 = \frac{G(1, 1, d + 1)}{(\alpha_T T)^{3 - d/2}} \frac{1}{4d} \left\{ 0_{IBP} - 0_{IBP} \right\} - \left[ G(1, 1, d + 1) \left( (d + 1)P_0^2 - P^2 \right) + 2P_0^2 \left( (d + 1)P_3^2 - P_2^2 \right) \right] + 0_{\text{scalefree}} \]

\[ \approx \frac{T^2}{(4\pi)^4} \left( \frac{1}{\alpha_T^2 T^2} \right) \left\{ \frac{z_1}{\epsilon} + \text{const} + \mathcal{O}(\epsilon) \right\} , \quad z_1 = \frac{1}{12} \left( \frac{\zeta(-1)}{\zeta(-1)} - \ln(2\pi) - \frac{1}{6} \right) , \]

where const is somewhat lengthy, so we do not display it here.

### 3.6 Evaluation of \( \mathcal{V}_6 \)

In close analogy to the treatment of \( \mathcal{V}_4 \), expanding \[ \bar{\Pi}_B - \bar{\Pi}_D \approx \frac{(d+1)P_0^2 - P^2}{4d} \frac{1}{(4\pi)^2} \alpha_T T^2 + \mathcal{O}(\epsilon) \]
we see that the sixth term of Eq. (2.1) is finite, so we write

\[ \mathcal{V}_6 = \int_P \frac{1}{|P|^2} [\Pi - \Pi_C] \left[ \bar{\Pi}_B - \bar{\Pi}_D \right] \approx \frac{T^2}{(4\pi)^4} \mathcal{V}_6 + \mathcal{O}(\epsilon) . \]

Using the 3d spatial Fourier transform of \[ [\Pi - \Pi_C] \] from App. A; integrating over angles via Eq. (C.7); letting \[ |r| = x/(2\pi T) , \quad |p| = |P_0|y \] and \[ |P_0| = 2\pi T n; \]

\[ \mathcal{V}_6 = \frac{1}{3\pi} \sum_{n=1}^{\infty} \int_0^\infty dx \int_0^\infty dy \left( \frac{3 - y^2}{y^2 + 1} \right) \ln \left( \frac{\alpha_T/(2\pi n)}{y^2 + 1} \right) y \sin(nx y) \frac{e^{-nx}}{x} \left[ f(x, n) - f_C(x, n) \right] \]

(3.27)

Now, splitting \( (3 - y^2) = 4 - (y^2 + 1) \) and integrating over \( y \) using Eqs. (C.3), (C.2):

\[ \mathcal{V}_6 = \frac{1}{3} \sum_{n=1}^{\infty} \int_0^\infty dx \frac{\ln \left( \frac{\alpha_T/(2\pi n)}{y^2 + 1} \right)}{y^2 + 1} \left[ e^{-2nx} (L - 1) - \text{Ei}(-2nx) \right] \left[ f(x, n) - f_C(x, n) \right] - \frac{1}{6} \sum_{n=1}^{\infty} \int_0^\infty dx \frac{1}{x} \left[ e^{-2nx} L + \text{Ei}(-2nx) \right] \left[ f(x, n) - f_C(x, n) \right] , \]

(3.28)

where \( L \equiv \ln \left( \frac{2\pi \alpha_T/(2\pi n)}{y^2 + 1} \right) \). At \( \alpha_T = 16\pi^2 e^{-7} \), the last line of Eq. (3.28) equals \[ \frac{C_6}{3^{6}} \approx 0.003496 \] (see Eq. (2.14) of [10], where its numerical evaluation had cost some sweat; see also Eq. (D.27)
of [9], where 0.0034814 is quoted). Noting that the $\alpha_2$-dependence of Eq. (3.28) follows from $\alpha_2$-independence of $V_5 + V_6$, let us re-write it as (with $z_1$ from Eq. (3.25) above)

$$V_6 = z_1 \ln \left( \frac{\alpha_2 \varepsilon^\gamma}{16 \pi^2} \right) + V_{6a} + V_{6b} + \frac{C}{36} \approx z_1 \ln \left( \frac{\alpha_2 \varepsilon^\gamma}{16 \pi^2} \right) + 0.00037813(1), \quad (3.29)$$

$$V_{6a} = 3\alpha_3 z_1 + \int_0^\infty dx \sum_{n=1}^\infty e^{-2nx} \left[ f(x, n) - f_C(x, n) \right] \left[ \frac{\alpha_3}{2x} - n \left( \alpha_3 + \frac{1}{3} \right) \right] \approx +0.0025600539026700475010965(1), \quad (3.30)$$

$$V_{6b} = -V_{6a} \ln(\alpha_4) - \frac{1}{3} \int_0^\infty dx \sum_{n=1}^\infty n \left[ f(x, n) - f_C(x, n) \right] \left[ \text{Ei}(-2nx) - e^{-2nx} \ln \left( \frac{2x}{\alpha_4 n} \right) \right] \approx -0.002619354(1), \quad (3.31)$$

$$C = 36(V_{6a} + z_1) \ln(\alpha_5) - 6 \int_0^\infty dx \sum_{n=1}^\infty \frac{1}{x} \left[ f(x, n) - f_C(x, n) \right] \left[ \text{Ei}(-2nx) + e^{-2nx} \ln \left( \frac{2x}{\alpha_5 n} \right) \right] \approx +0.0034960718(1), \quad (3.32)$$

and where we have introduced three arbitrary parameters \{\alpha_3, \alpha_4, \alpha_5\} to parameterize some freedom we found in evaluating the three numbers \{V_{6a}, V_{6b}, C\}. Concerning numerical precision, see the discussion below Eq. (3.22). As mentioned above, the constant $C$ has already been evaluated in [10] (at $\alpha_5 = 1$; see App. B therein). Some further analytic work on these constants seems possible (for example, $f - f_C$ is $n$-independent and the corresponding sums can be performed; $V_{6a}$ seems to be a suitable candidate for analytic treatment), but the numeric result is at this point fully sufficient for our purposes.

### 3.7 Evaluation of $V_7 - V_9$

The last three terms of Eq. (2.1) are trivial tadpoles, and can immediately be evaluated analytically in terms of the functions $G$ and $I$ from App. B as

$$V_7 \equiv \sum_P \frac{1}{|P|^2} \left[ \Pi_C - \Pi_B \right] \Pi_B = 2G(1, 1, d + 1)I_{10} \left( \frac{d + 1}{4d} I_{3+\epsilon}^2 - \frac{1}{4d} I_{2+\epsilon}^0 \right),$$

$$\approx \frac{T^2}{(4\pi)^4} \left( \frac{1}{4\pi T^2} \right)^2 \left\{ \frac{1}{48\epsilon} + \text{const} + O(\epsilon) \right\}, \quad (3.33)$$

$$V_8 \equiv \sum_P \frac{1}{|P|^2} \Pi_B \left[ \Pi_C - \Pi_B \right] = G(1, 1, d + 1) \left( I_{10}^2 I_{3+\epsilon}^2 + 2I_{10} I_{2+\epsilon}^0 \right) \quad (3.34)$$

$$\approx \frac{T^2}{(4\pi)^4} \left( \frac{1}{4\pi T^2} \right)^2 \left\{ \frac{13}{96\epsilon} + \frac{\gamma_E + 2\zeta(-1)}{96\epsilon} - \frac{4}{\pi^2} \frac{\zeta(3)}{96\epsilon} + \text{const} + O(\epsilon) \right\}, \quad (3.35)$$

$$V_9 \equiv \sum_P \frac{1}{|P|^2} \Pi_B \Pi_B = G(1, 1, d + 1)^2 \left( \frac{d + 1}{4d} I_{2+2\epsilon}^2 - \frac{1}{4d} I_{1+2\epsilon}^0 \right) \quad (3.36)$$
\[
\approx \frac{T^2}{(4\pi)^4} \left( \frac{1}{4\pi T^2} \right)^{3\varepsilon} \left\{ -\frac{1}{48\varepsilon^2} + \frac{-5 + 3\gamma_E - 6\zeta'(-1)}{48\varepsilon} + \text{const} + O(\varepsilon) \right\}.
\] (3.38)

4 Result

Collecting all expressions that have been evaluated in the previous sections in order to reassemble \( V_2 \) according to Eq. (2.2), setting \( d = 3 - 2\varepsilon \) and expanding, we obtain our final result

\[
V_2 \approx \frac{T^2}{(4\pi)^4} \frac{(4\pi T^2)^{-3\varepsilon}}{96\varepsilon^2} \left[ 1 + v_{21} \varepsilon + v_{22} \varepsilon^2 + O(\varepsilon^3) \right],
\] (4.1)

\[
v_{21} = \frac{67}{6} + \gamma_E + 2\frac{\zeta'(-1)}{\zeta(-1)},
\] (4.2)

\[
v_{22} = \frac{443}{36} - \frac{39}{2} \gamma_E^2 + \gamma_E \left( \frac{23}{6} + 6\frac{\zeta'(-1)}{\zeta(-1)} - 8\ln(2\pi) + \frac{4}{5} \zeta(3) \right) - \frac{6}{5} \zeta(3) - \frac{8}{5} \zeta'(3) + \frac{143}{36} \pi^2 + 8\ln^2(2\pi) + 8\ln(\pi) \ln(4\pi) - 48\gamma_1 + 21\frac{\zeta'(-1)}{\zeta(-1)} - \frac{6}{5} \zeta''(-1) + 96 \left[ -\frac{1}{3} (V_{1a} + V_{1b}) + V_{2a} + V_{4a} + V_{6a} + V_{6b} + \frac{1}{36} C \right]
\approx 128.63807196263994701 \cdots + 96 \left[ -0.370298231(2) \right] = 93.0894417(2),
\] (4.3)

where the Stieltjes constant \( \gamma_1 \) is defined by \( \zeta(1+\varepsilon) = 1/\varepsilon + \gamma_E - \gamma_1 \varepsilon + O(\varepsilon^2) \), and the constant \( v_{22} \) contains the seven (sum-)integrals for which we do not have analytic representations. They are defined in Eqs. (3.8), (3.11), (3.14), (3.22), (3.30), (3.31) and (3.32), and were evaluated numerically in the above. The main uncertainty in the final numerical result Eq. (4.4) is dominated by the ones in \( V_{4a} \) and \( V_{6b} \).

5 Conclusions

We have successfully evaluated a new massless bosonic three-loop sum-integral of mass dimension two, which we have named \( V_2 \), and which represents one of the two remaining unknown ingredients for a 3-loop evaluation of the Debye screening mass in hot Yang-Mills theory, [1, 8].

Our strategy has relied heavily on the particular structure of \( V_2 \), containing two different one-loop two-point sub-integrals, whose known analytic properties we have repeatedly exploited. This particular line of attack has served to evaluate a number of similar (but somewhat less involved) cases of sum-integrals in the past.

We have obtained all divergent terms of \( V_2 \) analytically, which will be important when it is finally used in a concrete physics computation; for the constant part, we had to resort to a numerical treatment for some pieces thereof. While it is certainly possible to increase the precision of our numerics substantially, we feel that the numbers given above will be sufficiently accurate for all practical purposes. It would of course be nice to obtain all terms...
in $v_{22}$ analytically, but we have no systematic method to do so for now. In passing, there might be some relation of (parts of) $V^2$ to the sum-integral $M_{1,0}$ that was treated in \cite{9,10}, since the constant $C$ of our Eq. (3.32) had already contributed there. However, his might also be pure coincidence, and we have not pursued this relation further.

Finally, we hope that the last remaining sum-integral that is needed at dimension two is amenable to similar techniques, and leave its computation for future work.

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### A 3d Fourier transforms

All (inverse) 3d spatial Fourier transforms derive from Eqs. (22-24) in \cite{7}:

$$\frac{1}{(q^2 + Q_0^2)^s} = \frac{2\pi T^2}{(8\pi T^2)^s} \int \frac{d^3 r}{r} e^{iqr} e^{-|Q_0|r} \left(\frac{|\bar{r}|}{|Q_0|}\right)^{s-1} f_s(|Q_0|r), \quad (A.1)$$

where

$$f_s(m) = \sqrt{\frac{2}{\pi}} m e^{m K_{3/2-s}(m)}, \quad (A.2)$$

with special cases of the modified Bessel function of second kind evaluating to $f_1(m) = f_2(m) = 1$ and $f_0(m) = f_3(m) = (1 + 1/m)$. From this one gets a generic formula for e.g. (inverse) 3d spatial Fourier transforms of 1-loop $\Pi$'s:

$$\Pi_{ij}^k(P) = \sum_{Q} \frac{Q_0^k}{|Q|^2} |(P - Q)^2|^{i+j} \int \frac{d^3 r}{r^2} e^{i|P_0|r} \pi_{ij}^k(\bar{r}, |\bar{P_0}|) \quad (A.3)$$

$$\pi_{ij}^k(\bar{r}, |\bar{P_0}|) = \sum_{n=-\infty}^{\infty} n^k e^{-|n||P_0-n|} \left(\frac{|\bar{r}|}{|n|}\right)^{i-1} f_i(|n|\bar{r}) \left(\frac{|\bar{r}|}{|P_0-n|}\right)^{j-1} f_j(|\bar{P_0}-n|\bar{r}). \quad (A.4)$$

Matsubara sums can then be solved using Eq. (19) of \cite{7}, as reprinted in Eq. (D.3) below. In particular, we get the (inverse) 3d spatial Fourier transforms (see also \cite{10})

$$\Pi_i (P) = \frac{T}{(4\pi)^2} \int \frac{d^3 r}{r^2} e^{i|P_0|r} f_i(\bar{r}, |\bar{P_0}|) \quad (A.5)$$

$$f(x, n) = \coth(x) + n, \quad (A.6)$$

$$f_B(x, n) = n + \frac{1}{x}, \quad (A.7)$$

$$f_C(x, n) = n + \frac{1}{x} + \frac{x}{3}, \quad (A.8)$$

$$\bar{\Pi}_i (P) = \frac{T^3}{24} \int \frac{d^3 r}{r^2} e^{i|P_0|r} \bar{f}_i(\bar{r}, |\bar{P_0}|) \quad (A.9)$$
\[ f(x, n) = 3 \coth^3(x) - 3 \coth(x) + n(3 \coth^2(x) - 2) + 3n^2 \coth(x) + 2n^3, \quad (A.10) \]
\[ f_B(x, n) = \frac{3}{x^3} + \frac{3n}{x^2} + \frac{3n^2}{x} + 2n^3, \quad (A.11) \]
\[ f_C(x, n) = \frac{3}{x^3} + \frac{3n}{x^2} + \frac{3n^2}{x} + 2n^3 + n^2x - \frac{x}{5}, \quad (A.12) \]
\[ \bar{\Pi}_i(P) = \frac{T}{4(4\pi)^2} \int \frac{d^3r}{r^2} e^{i p r} e^{-|P| r} \bar{f} \left( \frac{r}{|P|} \right), \quad (A.13) \]
\[ f(x, n) = x \left[ \coth^2(x) - 1 + n \coth(x) + n^2 \right], \quad (A.14) \]
\[ f_B(x, n) = x \left[ \frac{1}{x^2} + \frac{n}{x} + n^2 \right], \quad (A.15) \]
\[ \delta_{P_0} \Pi'(P) = \frac{4}{T(4\pi)^4} \int \frac{d^3r}{r^2} e^{i p r} e^{-|P| r} f'(r, 0), \quad (A.16) \]
\[ f'(x, 0) = -x \ln \left( 1 - e^{-2x} \right). \quad (A.17) \]

B Standard integrals

For convenience, we collect here the functions used above, as defined in [7, 10]. They are the zero-temperature massless one-loop propagator

\[ G(s_1, s_2, d) \equiv (p^2)^{s_1-d/2-\frac{d}{2}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2]^s_1 [(q - p)^2]^s_2} = \frac{\Gamma(s - d/2) \Gamma(s_1 - d/2) \Gamma(s_2 - d/2)}{(4\pi)^{d/2} \Gamma(s_1) \Gamma(s_2) \Gamma(d - s_2)}; \quad (B.1) \]

the one-loop bosonic tadpoles

\[ I^0_s \equiv \oint_{Q} \frac{|Q_0|^a}{|Q|^2} = \frac{2T \zeta(2s - a - d)}{(2\pi T)^{2s-a-d}} \frac{\Gamma(s - d/2)}{(4\pi)^{d/2} \Gamma(s)}, \quad I_s \equiv \oint_{Q} \frac{1}{|Q|^2} = I^0_s; \quad (B.2) \]

and a specific two-loop tadpole

\[ A(s_1, s_2, s_3) \equiv A(s_1, s_2, s_3; 0), \quad (B.3) \]
\[ A(s_1, s_2, s_3; s_4) \equiv \oint \oint_{PQ} \frac{\delta_{Q_0} |P_0|^a}{|Q|^2 |P|^2 |P - Q|^2} \frac{1}{s_3} = \frac{2T^2 \zeta(2s_{123} - 2d - s_4)}{(2\pi T)^{2s_{123} - 2d - s_4}} \frac{\Gamma(s_{13} - d/2) \Gamma(s_{12} - d/2) \Gamma(s - d/2) \Gamma(s_{123} - d)}{(4\pi)^d \Gamma(s_2) \Gamma(s_3) \Gamma(d/2) \Gamma(s_{123} - d)}, \quad (B.4) \]

where \( s_{abc} \equiv s_a + s_b + s_c + \ldots \);

C Other integrals

In the main text, we have used the exponential integral, defined as

\[ \text{Ei}(x) \equiv - \int_{-\infty}^{\infty} \frac{df}{t} e^{-t} \quad (C.1) \]
for the following integrals (the first of which has already been used in Sec. 2.2 of [10]):

\[
\frac{2}{\pi} e^{|z|} \int_0^\infty \frac{y \sin(y |z|)}{y^2 + 1} \ln \frac{\alpha}{y^2 + 1} = e^{2|z|} \text{Ei}(-2|z|) + \gamma \ln \frac{|z|\alpha}{2}, \tag{C.2}
\]

\[
\frac{4}{\pi |z|} \int_0^\infty \frac{y \sin(y |z|)}{(y^2 + 1)^2} \ln \frac{\alpha}{y^2 + 1} = \ln \frac{|z|\alpha}{2e^{1-\gamma}} - e^{2|z|} \text{Ei}(-2|z|). \tag{C.3}
\]

Furthermore, we needed one special 3d Fourier transform and its generalization

\[
\int \frac{d^3 p}{(2\pi)^3} e^{ipr} \frac{1}{(p^2 + P_0^2)^2} = \frac{e^{-|P_0| r}}{8\pi |P_0|}, \tag{C.4}
\]

\[
\int \frac{d^3 p}{(2\pi)^3} e^{ipr} \frac{1}{(p^2 + P_0^2)^{n+1}} = \frac{1}{n!} \partial^n P_0^2 \frac{e^{-|P_0| r}}{4\pi r} = \frac{e^{-|P_0| r}}{4\pi} \frac{2|P_0|}{(2|P_0|)^2} f_n(|P_0| r), \quad \text{with} \quad f_{\{0,1,2,3\}}(x) = \{1/(2x), 1 + x, 2 + 2x + 2x^2/3\}, \tag{C.5}
\]

as well as the following angular averages:

\[
\frac{1}{2} \int_{-1}^1 du e^{ipru} = \frac{\sin(pr)}{pr}, \tag{C.7}
\]

\[
\frac{1}{2} \int_{-1}^1 du e^{-u \sqrt{x^2 + y^2 + 2xyu}} = \frac{1}{2xy n^2} \left[(1+n|x-y|)e^{-n|x-y|} - (1+n(x+y))e^{-n(x+y)}\right], \tag{C.8}
\]

\[
\frac{1}{2} \int_{-1}^1 du \frac{1}{\sqrt{x^2 + y^2 + 2xyu}} = \frac{x + y - |x - y|}{2xy}. \tag{C.9}
\]

D Summation formulae

When dealing with discrete sums (see also [7]), Zeta functions and polylogarithms enter through

\[
\zeta(s) \equiv \sum_{n=1}^\infty n^{-s}, \quad \text{Li}_s(x) \equiv \sum_{n=1}^\infty \frac{x^n}{n^s} \quad \tag{D.1}
\]

with \(\text{Li}_1(x) = -\ln(1-x)\) (one can use Eq. (D.1) at \(s = 1, x = e^{-r}\)) and then get the cases \(s < 1\) via \(\partial_r\), while hyperbolic functions appear via \((m \in \mathbb{Z})\)

\[
\sum_{n=-\infty}^{\infty} e^{-(|n|+|n-m|-|m|)r} = |m| + \coth(r), \quad \tag{D.2}
\]

\[
\sum_{n=-\infty}^{\infty} e^{-(|n|+|n-m|-|m|)r} p(|n|) = \sum_{n=0}^{|m|} p(n) + p(-\partial_{2r}) \frac{1}{e^{2r} - 1} + e^{2|m|r} p(-\partial_{2r}) \frac{e^{-2|m|r}}{e^{2r} - 1}, \tag{D.3}
\]

where \(p(x)\) can be a polynomial. Note that Eq. (D.2) is just a special case of Eq. (D.3).
E  IBP relations

We need one 3-loop IBP relation for Sec. 3.1, which can be derived as

\[ 0 = \partial_p p \circ V_1 \]
\[ = d V_1 + \int_{PQ} \delta_{Pb} \frac{R_0^2}{[P^2] Q^2 (P - Q)^2 R^2(P - R)^2} \]
\[ = d V_1 + \int_{PQ} \frac{\delta_{Pb} R_0^2 \cdot N}{[P^2]^3 Q^2 [(P - Q)^2]^2 R^2[(P - R)^2]^2} \]
\[ \delta_{Pb} N = \delta_{Pb} \{ -4p p (P - Q)^2 (P - R)^2 - 2p(p - q)P^2(P - R)^2 - 2p(p - r)P^2(P - Q)^2 \} \]
\[ \text{let } pp \rightarrow P^2, \ 2p(p - q) \rightarrow (P - Q)^2 - Q^2 + P^2, \ 2p(p - r) \rightarrow (P - R)^2 - R^2 + P^2 \]
\[ = \delta_{Pb} P^2 \{ -6(P - Q)^2 (P - R)^2 + (Q^2 - P^2)(P - R)^2 + (R^2 - P^2)(P - Q)^2 \} \]
shift \( Q \rightarrow P - Q \) and \( R \rightarrow P - R \)
\[ = (d - 6) V_1 + \int_{P} \frac{\delta_{Pb}}{[P^2]^2} \left( \int_{Q} \frac{(P - Q)^2 - P^2}{[Q^2]^2 (P - Q)^2} \bar{\Pi} + \Pi \int_{R} \frac{R_0^2 ((P - R)^2 - P^2)}{[R^2]^2 (P - R)^2} \right) \ . \]  \hspace{1cm} (E.1)

The 2-loop IBP relations that we need for Sec. 3.3 and Sec. 3.5 can be obtained in the same way. Letting \( I(abc, de) \equiv \sum_{PQ} p Q \frac{[Pb]^4 (Qc)^4}{[P^2]^2 [Q^2]^2 [(P - Q)^2]^2} \), we can derive e.g.

\[ 0 = \partial_p p \circ I(111,00) \]
\[ = d I(111,00) + \int_{PQ} \frac{1}{[P^2] Q^2 (P - Q)^2} \]
\[ = d I(111,00) + \int_{PQ} \left( \frac{-2 p p}{[P^2]^2 Q^2 (P - Q)^2} + \frac{-2p(p - q)}{P^2 Q^2 [(P - Q)^2]^2} \right) \text{ do } P \rightarrow Q - P \text{ in last term} \]
\[ = d I(111,00) + \int_{PQ} \frac{-4p^2 + 2pq}{[P^2]^2 Q^2 (P - Q)^2} \]
\[ = d I(111,00) + \int_{PQ} \frac{-3(P^2 - P_0^2) - p(p - 2q)}{[P^2]^2 Q^2 (P - Q)^2} \text{ last term odd under } Q \rightarrow P - Q \]
\[ = (d - 3) I(111,00) + 3 I(211,20) \ . \]  \hspace{1cm} (E.2)

More conveniently, using our computer-algebraic [13] implementation of a Laporta-type [2] algorithm to systematically derive and solve such finite-temperature IBP relations [4], we get the special cases

\[ I(111;00) = 0 \ , \quad \text{(for explicit IBP derivation see [6])} \]  \hspace{1cm} (E.3)
\[ I(211;20) = 0 \ , \]  \hspace{1cm} (E.4)
\[ I(211;02) = - \frac{1}{d - 5} \left( 2I_0^0 I_3^0 + I_2^0 I_0^0 \right) \ , \]  \hspace{1cm} (E.5)

where the second relation confirms Eq. (E.2).
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