The relativistic Calogero model
in an external field*

J.F. van Diejen

Department of Mathematical Sciences, University of Tokyo,
Komaba 3-8-1, Meguro-ku, Tokyo 153, Japan

Abstract

Recent results are surveyed regarding the spectrum and eigenfunctions of the inverse square Calogero model with harmonic confinement and its relativistic analogue.

---

*Submitted to the Proceedings of the 4th Wigner Symposium, August 7-11, 1995, Guadalajara, México.
1 Introduction

The Calogero model is a dynamical system that consists of $N$ particles on the line interacting pairwise through an inverse square potential and coupled to a harmonic external field. Both at the level of classical and quantum mechanics the dynamics of the system has been studied in considerable detail in the literature \[1, 2\]. Key property of the Calogero model is that, despite the nontrivial interaction between the particles, the equations of motion describing the dynamics can be solved in closed form. The exact solubility of the system stems from the fact that it is integrable, i.e., that there are as many independent integrals of motion (in involution) as degrees of freedom (viz. $N$). For the classical model the equations of motion were integrated by Olshanetsky and Perelomov using a Lax pair representation, whereas for the quantum system the spectrum and the structure of the eigenfunctions had already been determined before by Calogero \[3\].

More recently, Ruijsenaars and Schneider introduced a relativistic generalization of the classical Calogero model without harmonic external field and solved the corresponding equations of motion \[4, 5\]. It was furthermore shown that also the relativistic system is integrable and that this integrability is preserved after quantization \[6\]. At the quantum level, the Hamiltonian of the relativistic model is given by a difference operator rather than a differential operator; the nonrelativistic limit then corresponds to sending the step size of this difference operator to zero.

Very recently the author has introduced a similar integrable relativistic analogue of the (quantum) Calogero model with harmonic external field \[7\], and computed the corresponding spectrum and eigenfunctions also for this case \[8\]. The present contribution intends to provide an overview of these results as well as to describe some new developments that have led to a more explicit construction for the eigenfunctions of the nonrelativistic model \[9\].

2 The Quantum Calogero Model

The quantum Calogero model is characterized by a Hamiltonian of the form

\[
H_C = \sum_{1\leq j\leq N} \left( -\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + \sum_{1\leq j\neq k\leq N} \frac{g(g-1)}{(x_j - x_k)^2}.
\]
It is immediate from the Hamiltonian that the model may be viewed as a system of $N$ coupled harmonic oscillators. Should the inverse square coupling term be absent ($g = 0$), then it is of course very easy to solve the corresponding eigenvalue problem. The (boson) eigenfunctions are in that case the product of a Gaussian ground state wave function and symmetrized products of Hermite polynomials, and the spectrum is that of an $N$-dimensional harmonic oscillator. The remarkable observation by Calogero, to date around 25 years ago, is that much of this picture is preserved after switching on the inverse square interaction ($g > 0$)

Specifically, for the interacting system the ground state wave function becomes

$$\Psi_0(\vec{x}) = \prod_{1 \leq j < k \leq N} |x_j - x_k|^g \exp\left(-\frac{\omega}{2} \sum_{1 \leq j \leq N} x_j^2\right)$$

and the ground state energy reads

$$E_0 = \omega N (1 + (N - 1)g).$$

The wave functions of the excited states are again products of the ground state wave function and certain symmetric polynomials:

$$\Psi_{\vec{n}}(\vec{x}) = \Psi_0(\vec{x}) P_{\vec{n}}(\vec{x}),$$

where $\vec{n} = (n_1, \ldots, n_N)$ denotes a vector of (integer) quantum numbers with $n_1 \geq n_2 \geq \cdots \geq n_N \geq 0$ labeling the eigenfunctions. The corresponding eigenvalues are given by

$$E_{\vec{n}} = E_0 + 2\omega \sum_{1 \leq j \leq N} n_j.$$
3 Creation and Annihilation Operators

It is well known that the wave functions for a system of independent harmonic oscillators can be constructed by means of creation and annihilation operators. Interestingly enough, it was recently discovered that the same classical technique may also be applied to the case with an inverse square coupling between the oscillators [9].

To this end it is convenient to introduce the concept of the so-called Dunkl derivative [12]:

\[ D_j = \frac{\partial}{\partial x_j} + g \sum_{1 \leq k \leq N, k \neq j} (x_k - x_j)^{-1} S_{j,k}, \] (6)

where \( S_{j,k} \) denotes the transposition operator interchanging the particles \( j, k \)

\[ S_{j,k} \Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = \Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N). \] (7)

The Dunkl derivative \( D_j \) is a deformation of the ordinary derivative \( \partial_j = \partial/\partial x_j \) with the coupling constant \( g \) acting as a deformation parameter. The essential point is now that it is possible to incorporate the inverse square coupling between the oscillators in the standard construction of the eigenfunctions by means of creation and annihilation operators if one replaces the ordinary derivative by the Dunkl derivative. More specifically, after setting

\[ A_j^\pm = \frac{1}{\sqrt{2}}(\mp D_j + \omega x_j), \] (8)

one has

\[ \tilde{H}_C \equiv \sum_{1 \leq j \leq N} (A_j^+ A_j^- + A_j^- A_j^+) = \sum_{1 \leq j \leq N} \left( -\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right) + \sum_{1 \leq j \neq k \leq N} \frac{g(g - S_{j,k})}{(x_j - x_k)^2} \] (9)

and

\[ [\tilde{H}_C, A_j^\pm] = \pm 2\omega A_j^\pm. \] (10)

When restricted to the boson sector (i.e., the space of permutation invariant wave functions), the operator \( \tilde{H}_C \) coincides with the Calogero Hamiltonian \( H_C \) (cf. Eq. [1]). The ground state wave function in this sector is determined by its permutation-invariance and the fact that it is annihilated by the lowering operators

\[ A_j^- \Psi_0(\vec{x}) = 0, \quad j = 1, \ldots, N. \] (11)
The excited states are obtained by acting with the creation operators on the ground state
\[ \Psi_{\vec{n}}(\vec{x}) = \sum_{\sigma \in S^N} \left( \prod_{1 \leq j \leq N} (A_{\sigma(j)}^+)^{n_j} \right) \Psi_0(\vec{x}) \] (12)

(where we have, in order to obtain boson eigenfunctions, symmetrized over all permutations). The shift of the spectrum as compared to the model without inverse square coupling between oscillators originates from a small change in the usual commutation relations satisfied by the modified creation and annihilation operators \( A_j^\pm \):\[ [A_j^\pm, A_k^\pm] = 0, \quad [A_j^-, A_k^+] = \omega(1 + g \sum_{1 \leq l \leq N} S_{j,l}) \delta_{j,k} - \omega g S_{j,k}. \] (13)

(So \( H_C \Psi_0 = \sum_j ([A_j^-, A_j^+] + 2A_j^+ A_j^-) \Psi_0 = \sum_j \omega(1 + (N - 1)g) \Psi_0 = E_0 \Psi_0 \), with \( E_0 \) given by Eq. 3.)

4 Relativistic Analogue

A few years ago, Ruijsenaars introduced an integrable quantum \( N \)-particle system characterized by a (rather unorthodox) Hamiltonian given by the second order difference operator \( \sum_{1 \leq j \leq N} \left( V_j^{1/2} e^{\frac{i}{\omega_j} \nabla_j^{1/2}} + V_j^{1/2} e^{-\frac{i}{\omega_j} \nabla_j^{1/2}} \right) \), \( \nabla_j \)\( = \prod_{1 \leq k \leq N, k \neq j} v(x_j - x_k) \), \( v(z) = 1 + g/(iz) \) (15)

(with \( \nabla_j \) denoting the complex conjugate of \( V_j \)). As it turns out, Ruijsenaars’ difference model may be interpreted as a system composed of \( N \) relativistic particles in \( (1+1)D \) that interact with each other by means of the coefficients \( V_j \). For \( g = 0 \) the particles are independent \( (V_j = 1) \), whereas for \( g > 0 \) each particle feels the presence of the remaining \( N - 1 \) particles as a change of its dynamical mass. A more detailed discussion of the model (both at the classical and quantum level), with an emphasis on matters involving...
integrability and Poincaré invariance, can be found in Ruijsenaars’ papers [4, 5, 6].

Quite recently, the present author observed that it is possible to introduce an external field coupling to the relativistic system without destroying its integrability [7]. The Hamiltonian of the corresponding difference model reads

\[ H = \sum_{1 \leq j \leq N} \left( V_j^{1/2} e^{\frac{\partial}{\partial x_j}} \nabla_j^{1/2} + \nabla_j^{1/2} e^{-\frac{\partial}{\partial x_j}} V_j^{1/2} - V_j - \nabla_j \right), \] (16)

with

\[ V_j = \psi(x_j) \prod_{1 \leq k \leq N, k \neq j} v(x_j - x_k), \] (17)

\[ v(z) = 1 + g/(iz), \quad w(z) = (a + iz)(b + iz). \] (18)

The function \( w \) encodes the external field. For \( w = 1 \) (this is achieved by sending the parameters \( a \) and \( b \) to infinity after having rescaled the Hamiltonian by division by \( ab \)), the Hamiltonian \( H \) reduces, up to an irrelevant additive constant, to the Ruijsenaars Hamiltonian \( H_R \). The constant is caused by the part \(-\sum_{1 \leq j \leq N}(V_j + \nabla_j)\), which does not depend on \( x_j \) if \( w = 1 \) (as is readily seen with the aid of Liouville’s theorem after having inferred that the expression is regular in \( x_j \) and bounded for \( x_j \to \infty \)).

Below, we will discuss the spectrum and eigenfunctions of the difference model with external field given by the Hamiltonian in Eqs. [16, 18] and describe its relation to the Calogero model discussed in Sections 2 and 3. Throughout it will be assumed that the coupling constant \( g \) be nonnegative and that (the real parts of) the parameters \( a \) and \( b \) be positive.

5 Spectrum and Eigenfunctions

The ground state wave function for the difference Hamiltonian \( H \) is given by

\[ \Psi_0(x) = \prod_{1 \leq j < k \leq N} \left| \frac{\Gamma(g + i(x_j - x_k))}{\Gamma(i(x_j - x_k))} \right| \prod_{1 \leq j \leq N} |\Gamma(a + ix_j)\Gamma(b + ix_j)|. \] (19)

Probably the simplest way to see that this is indeed an eigenfunction is to check that conjugation of the Hamiltonian with \( \Psi_0 \) yields (using the standard
difference equation $\Gamma(z + 1) = z \Gamma(z)$ for the gamma function)

$$
\mathcal{H} = \Psi_0^{-1} H \Psi_0 = \sum_{1 \leq j \leq N} \left( V_j \left(e^{\frac{\theta}{\sqrt{\gamma}}} - 1\right) + V_j \left(e^{-\frac{\theta}{\sqrt{\gamma}}} - 1\right) \right).
$$

(20)

The transformed operator $\mathcal{H}$ clearly annihilates constant functions, so $\Psi_0$ is an eigenfunction of $H$ with eigenvalue zero.

Just as for the nonrelativistic Calogero model, the wave functions corresponding to the excited states are a product of the ground state wave function and symmetric polynomials. More precisely, one has

$$
H \Psi_{\vec{n}} = E_{\vec{n}} \Psi_{\vec{n}}
$$

(21)

with

$$
E_{\vec{n}} = \sum_{1 \leq j \leq N} n_j(n_j + a + \bar{a} + b + 2(N - j)g)
$$

(22)

and

$$
\Psi_{\vec{n}}(\vec{x}) = \Psi_0(\vec{x}) P_{\vec{n}}(\vec{x}),
$$

(23)

where $P_{\vec{n}}(\vec{x})$ is the symmetric polynomial determined by the conditions:

1. $P_{\vec{n}}(\vec{x}) = m_{\vec{n}}(\vec{x}) + \sum_{\vec{n}' < \vec{n}} c_{\vec{n},\vec{n}'} m_{\vec{n}'}(\vec{x})$;
2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_{\vec{n}}(\vec{x}) m_{\vec{n}'}(\vec{x}) \Psi_0^2(\vec{x}) dx_1 \cdots dx_N = 0$ if $\vec{n}' < \vec{n}$.

Here the functions $m_{\vec{n}}(\vec{x})$, $\vec{n} = (n_1, \ldots, n_N)$ with $n_1 \geq n_2 \geq \cdots \geq n_N \geq 0$, denote the basis consisting of symmetrized monomials

$$
m_{\vec{n}}(\vec{x}) = \sum_{\vec{n}' \in S_N(\vec{n})} x_1^{n_1'} \cdots x_N^{n_N'},
$$

(24)

which is partially ordered by the definition

$$
\vec{n}' \leq \vec{n} \quad \text{iff} \quad n_1' + \cdots + n_k' \leq n_1 + \cdots + n_k \quad \text{for} \quad k = 1, \ldots, N
$$

(25)

($\vec{n}' < \vec{n}$ if $\vec{n}' \leq \vec{n}$ and $\vec{n}' \neq \vec{n}$). Thus, the polynomial $P_{\vec{n}}(\vec{x})$ boils down to the symmetrized monomial $m_{\vec{n}}(\vec{x})$ minus its orthogonal projection with respect to the $L^2$ inner product with weight function $\Psi_0^2(\vec{x})$ onto the finite-dimensional subspace spanned by the monomials $m_{\vec{n}'}(\vec{x})$ with $\vec{n}' < \vec{n}$. For $N = 1$ the resulting polynomials are well-studied in the literature and known as continuous Hahn polynomials [13, 14].
The proof of the above statements hinges on a standard technique going back (essentially) to Sutherland [13]. First it is shown that the transformed operator $\mathcal{H}$ is triangular with respect to the partially ordered monomial basis, i.e.

$$ (\mathcal{H} m_{\vec{n}})(\vec{x}) = \sum_{\vec{n}' \leq \vec{n}} [\mathcal{H}]_{\vec{n}, \vec{n}'} m_{\vec{n}'}(\vec{x}), \quad (26) $$

where the $[\mathcal{H}]_{\vec{n}, \vec{n}'}$ represent certain (complex) matrix elements. It is clear that acting with $\mathcal{H}$ on a monomial $m_{\vec{n}}$ yields a permutation invariant rational function. The permutation symmetry guarantees that the simple poles at $x_j = x_k$ (caused by the zero in the denominator of $v(z)$) all cancel each other. Hence, the rational function is actually a (permutation invariant) polynomial and can thus indeed be expanded in symmetrized monomials. To see that in this expansion only monomials $m_{\vec{n}'}$ with $\vec{n}' \leq \vec{n}$ occur (triangularity), one uses the asymptotics at infinity. Setting $x_j = R y_j$ with $y_1 > y_2 > \cdots > y_N > 0$ gives the following asymptotics for $R \to +\infty$:

$$ m_{\vec{n}'} = R^{\vec{n}' \cdot \vec{y}} + o(R^{\vec{n}' \cdot \vec{y}}), \quad (27) $$

$$ \mathcal{H} m_{\vec{n}} = O(R^{\vec{n} \cdot \vec{y}}). \quad (28) $$

By comparing the asymptotics of Eqs. 27 and 28, and using the fact that

$$ \vec{n}' \leq \vec{n} \quad \text{iff} \quad \vec{n}' \cdot \vec{y} \leq \vec{n} \cdot \vec{y} \quad \forall \vec{y} \quad \text{with} \quad y_1 > y_2 > \cdots > y_N > 0, \quad (29) $$

one infers that the matrix elements $[\mathcal{H}]_{\vec{n}, \vec{n}'}$ in the expansion of $\mathcal{H} m_{\vec{n}}$ in terms of symmetrized monomials $m_{\vec{n}'}$ can only be nonzero if $\vec{n}' \leq \vec{n}$, which gives Eq. 26.

Next one observes that since $H$ is Hermitian the transformed operator $\mathcal{H}$ is symmetric with respect to the $L^2$ inner product with weight function $\Psi_0^2(\vec{x})$:

$$ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\mathcal{H} m_{\vec{n}})(\vec{x}) \overline{m_{\vec{n}'}(\vec{x})} \Psi_0^2(\vec{x}) \, dx_1 \cdots dx_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} m_{\vec{n}}(\vec{x}) \overline{(\mathcal{H} m_{\vec{n}'}) (\vec{x})} \Psi_0^2(\vec{x}) \, dx_1 \cdots dx_N. \quad (30) $$

By combining the triangularity and symmetry of $\mathcal{H}$ it is not difficult to deduce that $\mathcal{H} P_{\vec{n}}$ is a linear combination of monomials $m_{\vec{n}'}$ with $\vec{n}' \leq \vec{n}$, which is orthogonal to $m_{\vec{n}'}$ with $\vec{n}' < \vec{n}$. Hence, $\mathcal{H} P_{\vec{n}}$ must be proportional to $P_{\vec{n}}$, i.e., $P_{\vec{n}}$ is an eigenfunction of $\mathcal{H}$. The corresponding eigenvalue $E_{\vec{n}}$ is obtained by computing the diagonal matrix element $[\mathcal{H}]_{\vec{n}, \vec{n}}$, i.e., by explicitly computing the leading coefficient in Expansion 26.
6 The Nonrelativistic Limit

To relate the difference model to the Calogero model one needs to explicitly introduce a step size parameter by rescaling the positions $x_j$ and the parameters $a, b$. Substituting $x_j \to \beta^{-1} x_j$ (so $\partial_j \to \beta \partial_j$), $a \to (\beta^2 \omega)^{-1}$, $b \to (\beta^2 \omega')^{-1}$, and multiplying $H$ by $\beta^2 \omega \omega'$, leads to a Hamiltonian $H$ given by Eqs. 16, 17 with $\exp(\pm i \partial_j)$ replaced by $\exp(\pm i \beta \partial_j)$ and functions $v, w$ of the form

$$v(z) = 1 + \beta g/(iz), \quad w(z) = \beta^{-2} (1 + i \beta \omega z)(1 + i \beta \omega' z). \quad (31)$$

The step size parameter $\beta$ should be compared with the inverse of the light speed appearing in Ruijsenaars’ model [3].

For $\beta \to 0$ one now has

$$H \to \sum_{1 \leq j \leq N} \left( -\frac{\partial^2}{\partial x_j^2} + (\omega + \omega')^2 x_j^2 \right) + \sum_{1 \leq j \neq k \leq N} \frac{g(g-1)}{(x_j - x_k)^2} - \varepsilon_0, \quad (32)$$

with $\varepsilon_0 = (\omega + \omega') N (1 + (N-1) g)$, and $E_{\vec{n}} \to 2(\omega + \omega') \sum_{1 \leq j \leq N} n_j$. The wave functions go (after dividing by a divergent numerical factor arising from the gamma factors in $\Psi_0$, Eq. 19) over in $\Psi_{\vec{n}}(\vec{x}) = \Psi_0(\vec{x}) P_{\vec{n}}(\vec{x})$, where

$$\Psi_0(\vec{x}) = \prod_{1 \leq j < k \leq N} |x_j - x_k|^g \prod_{1 \leq j \leq N} e^{-\frac{1}{2} (\omega + \omega') x_j^2} \quad (33)$$

and $P_{\vec{n}}(\vec{x})$ is the polynomial determined by the Conditions i. and ii. of Section 5 with $\Psi_0$ now taken from Eq. 33. This way we recover for $\beta \to 0$ the Hamiltonian, the spectrum and the eigenfunctions of the Calogero model.

Acknowledgments

This work was made possible by financial support from the Japan Society for the Promotion of Science (JSPS).
References

[1] A.M. Perelomov, *Integrable systems of classical mechanics and Lie algebras, vol. I* (Birkäuser, Basel, 1990).

[2] M.A. Olshanetsky and A.M. Perelomov, *Phys. Reps.* 94, 313 (1983).

[3] F. Calogero, *J. Math. Phys.* 12, 419 (1971).

[4] S.N.M. Ruijsenaars and H. Schneider, *Ann. Phys. (N.Y.)* 170, 370 (1986).

[5] S.N.M. Ruijsenaars, *Commun. Math. Phys.* 115, 127 (1988).

[6] S.N.M. Ruijsenaars, *Commun. Math. Phys.* 110, 191 (1987).

[7] J.F. van Diejen, *J. Math. Phys.* 36, 1299 (1995).

[8] J.F. van Diejen, *J. Phys. A: Math. Gen.* 28, L369 (1995).

[9] L. Brink et al, *Phys. Lett.* B 286, 109 (1992).

[10] A.M. Perelomov, *Theor. and Math. Phys.* 6, 263 (1971).

[11] P.J. Gambardella, *J. Math. Phys.* 16, 1172 (1975).

[12] C.F. Dunkl, *Trans. Amer. Math. Soc.* 311, 167 (1989).

[13] N.M. Atakishiyev and S.K. Suslov, *J. Phys. A: Math. Gen.* 18, 1583 (1985).

[14] R. Askey, *J. Phys. A: Math. Gen.* 18, L1017 (1985).

[15] B. Sutherland, *Phys. Rev.* A 5, 1372 (1972).