SUPERSTABILITY OF ADJOINTABLE MAPPINGS ON HILBERT \( C^* \)-MODULES

Michael Frank, Pasc Găvruța and Mohammad Sal Moslehian

Dedicated to the Memory of Professor D. S. Mitrinović

We define the notion of \( \varphi \)-perturbation of a densely defined adjointable mapping and prove that any such mapping \( f \) between Hilbert \( A \)-modules over a fixed \( C^* \)-algebra \( A \) with densely defined corresponding mapping \( g \) is \( A \)-linear and adjointable in the classical sense with adjoint \( g \). If both \( f \) and \( g \) are everywhere defined then they are bounded. Our work concerns with the concept of Hyers–Ulam–Rassias stability originated from the Th.M. Rassias’ stability theorem that appeared in his paper [On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300]. We also indicate interesting complementary results in the case where the Hilbert \( C^* \)-modules admit non-adjointable \( C^* \)-linear mappings.

1. INTRODUCTION

We say a functional equation \( (E) \) is stable if any function \( g \) approximately satisfying the equation \( (E) \) is near to an exact solution of \( (E) \). The equation \( (E) \) is called superstable if every approximate solution of \( (E) \) is indeed a solution (see [5] for another notion of superstability namely superstability modulo the bounded functions). More than a half century ago, S.M. Ulam [23] proposed the first stability problem which was partially solved by D.H. Hyers [10] in the framework of Banach spaces. Later, T. Aoki [3] proved the stability of the additive mapping and Th.M. Rassias [20] proved the stability of the linear mapping for mappings \( f \) from a normed space into a Banach space such that the norm of the Cauchy difference \( f(x + y) - f(x) - f(y) \) is bounded by the expression \( \varepsilon(\|x\|^p + \|y\|^p) \) for some \( \varepsilon \geq 0 \), for some \( 0 \leq p < 1 \) and for all \( x, y \). The terminology “Hyers–Ulam–Rassias stability” was indeed originated from Th.M. Rassias’s paper [20]. In 1994, a further generalization was obtained by P. Găvruţa [9], in which he replaced the bound \( \varepsilon(\|x\|^p + \|y\|^p) \) by a general control function \( \varphi(x, y) \). This terminology can be applied to functional equations and mappings on various generalized notions of Hilbert spaces; see [1] [2] [9]. We refer the interested reader to monographs [6] [7] [11] [13] [19] [22] and references therein for more information.

The notion of Hilbert \( C^* \)-module is a generalization of the notion of Hilbert space. This object was first used by I. Kaplansky [14]. Interacting with the theory of

2000 Mathematics Subject Classification: Primary 46L08; Secondary 47B48, 39B52, 46L05
Keywords and Phrases: Hyers–Ulam–Rassias stability, superstability, Hilbert \( C^* \)-module, \( C^* \)-algebra, \( \varphi \)-perturbation of an adjointable mapping.
operator algebras and including ideas from non-commutative geometry it progresses and produces results and new problems attracting attention, see [8, 15, 18].

Let $A$ be a $C^\ast$-algebra and $X$ be a complex linear space, which is a right $A$-module with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in X$, $a \in A$, $\lambda \in \mathbb{C}$. The space $X$ is called a (right) pre-Hilbert $A$-module if there exists an $A$-inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow A$ satisfying

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;
(ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$;
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$;
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in X$, $\lambda \in \mathbb{C}$, $a \in A$. The pre-Hilbert module $X$ is called a (right) Hilbert $A$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. Left Hilbert $A$-modules can be defined in a similar way. Two typical examples are

(I) Every inner product space is a left pre-Hilbert $\mathbb{C}$-module.

(II) Let $A$ be a $C^\ast$-algebra. Then every norm-closed right ideal $I$ of $A$ is a Hilbert $A$-module if one defines $\langle a, b \rangle = a^* b$ ($a, b \in I$).

A mapping $f : X \rightarrow Y$ between Hilbert $A$-modules is called adjointable if there exists a mapping $g : Y \rightarrow X$ such that $\langle f(x), y \rangle = \langle x, g(y) \rangle$ for all $x \in D(f) \subseteq X$, $y \in D \subseteq Y$. Throughout the paper, we assume that $f$ and $g$ are both everywhere defined or both densely defined. The unique mapping $g$ is denoted by $f^*$ and is called the adjoint of $f$.

An $A$-linear bounded operator $K$ on a Hilbert $A$-module $X$ is called "compact" if it belongs to the norm-closed linear span of the set of all elementary operators $\theta_{x,y}(z) = x\langle y, z \rangle$ ($z \in X$).

In this paper, we prove the superstability of adjointable mappings on Hilbert $C^\ast$-modules in the spirit of Hyers–Ulam–Rassias and indicate interesting complementary results in the case where the Hilbert $C^\ast$-modules admit non-adjointable $C^\ast$-linear mappings.

2. MAIN RESULTS

Throughout this section, $A$ denotes a $C^\ast$-algebra, $X$ and $Y$ denote Hilbert $A$-modules, and $\varphi : X \times Y \rightarrow [0, \infty)$ is a function. We start our work with the following definition.

Definition 2.1. A (not necessarily linear) mapping $f : X \rightarrow Y$ is called a $\varphi$-perturbation of an adjointable mapping if there exists a (not necessarily linear) corresponding mapping $g : Y \rightarrow X$ such that

\begin{equation}
(2.1) \|\langle f(x), y \rangle - \langle x, g(y) \rangle\| \leq \varphi(x, y) \quad (x \in D(f) \subseteq X, y \in D(y) \subseteq Y).
\end{equation}

To prove our main result, we need the following known lemma (cf. [15, p. 8]) that we prove it for the sake of completeness.
Lemma 2.2. Every densely defined adjointable mapping between Hilbert $C^*$-modules over a fixed $C^*$-algebra $A$ is $A$-linear. If the adjointable mapping is everywhere defined then it is bounded.

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{X}$ be a pair of densely defined adjointable mappings between two Hilbert $C^*$-modules $\mathcal{X}$ and $\mathcal{Y}$. For every $x_1, x_2, x_3 \in D(f) \subseteq \mathcal{X}$, every $y \in D(g) \subseteq \mathcal{Y}$, every $\lambda \in \mathbb{C}$, every $a \in A$ the following equality holds:

\[
(f(\lambda x_1 + x_2 + x_3 a), y) = \langle \lambda x_1 + x_2 + x_3 a, g(y) \rangle \\
= \lambda \langle x_1, g(y) \rangle + \langle x_2, g(y) \rangle + a^* \langle x_3, g(y) \rangle \\
= \lambda f(x_1, y) + (f(x_2, y) + a^* (f(x_3, y)) \\
= \langle \lambda f(x_1) + f(x_2) + f(x_3) a, y \rangle.
\]

By the density of the domain of $g$ in $\mathcal{Y}$ the equality yields the $A$-linearity of $f$.

Now, suppose $f$ and $g$ to be everywhere defined on $\mathcal{X}$ and $\mathcal{Y}$, respectively. For each $x$ in the unit sphere of $\mathcal{X}$, define $\tau_x : \mathcal{Y} \to A$ by $\tau_x(y) = (f(x), y) = \langle x, g(y) \rangle$. Then $\|\tau_x(y)\| \leq \|x\| \|g(y)\| \leq \|g(y)\|$ for any $x$ from the unit ball. By the Banach–Steinhaus theorem we conclude that the set $\{\|\tau_x\| : x \in \mathcal{X}, \|x\| \leq 1\}$ is bounded. Due to the equality $\|f(x)\| = \sup_{\|y\| \leq 1} \|\langle f(x), y \rangle\| = \sup_{\|y\| = 1} \|\tau_x(y)\| = \|\tau_x\|$ the mapping $f$ has to be bounded. □

Theorem 2.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\varphi$-perturbation of an adjointable mapping with corresponding mapping $g : \mathcal{Y} \to \mathcal{X}$. Suppose that the mappings $f$ and $g$ are everywhere defined on the respective Hilbert $C^*$-modules. Furthermore, suppose that for some sequence $\{c_n\}$ of non-zero complex numbers either both the conditions (2.2) and (2.3) or both the conditions (2.4) and (2.5) below hold for the perturbation bound mapping $\varphi(x, y)$:

\[
(2.2) \quad \lim_{n \to \infty} |c_n|^{-1} \varphi(c_n x, y) = 0 \quad (x \in \mathcal{X}, y \in \mathcal{Y})
\]

\[
(2.3) \quad \lim_{n \to \infty} |c_n|^{-1} \varphi(x, c_n y) = 0 \quad (x \in \mathcal{X}, y \in \mathcal{Y}),
\]

\[
(2.4) \quad \lim_{n \to \infty} |c_n| \varphi(c_n^{-1} x, y) = 0 \quad (x \in \mathcal{X}, y \in \mathcal{Y})
\]

\[
(2.5) \quad \lim_{n \to \infty} |c_n| \varphi(x, c_n^{-1} y) = 0 \quad (x \in \mathcal{X}, y \in \mathcal{Y}).
\]

Then $f$ is adjointable. In particular, $f$ is bounded, continuous and $A$-linear, as well as its adjoint is $g$.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Replacing $x$ by $\lambda x$ in (2.1), we get

\[
\|\langle f(\lambda x), y \rangle - \langle \lambda x, g(y) \rangle\| \leq \varphi(\lambda x, y),
\]

and since a multiplication of (2.1) by $|\lambda|$ yields

\[
\|\langle \lambda f(x), y \rangle - \langle \lambda x, g(y) \rangle\| \leq |\lambda| \varphi(x, y)
\]
we obtain
\[(2.6) \quad \|\langle f(\lambda x), y \rangle - \langle \lambda f(x), y \rangle\| \leq \varphi(\lambda x, y) + |\lambda| \varphi(x, y)\]
If (2.3) holds, we take \(c_n y\) instead of \(y\) in (2.6) to get
\[(2.7) \quad \|\langle f(\lambda x), y \rangle - \langle \lambda f(x), y \rangle\| \leq |c_n|^{-1} \varphi(\lambda x, c_n y) + |\lambda||c_n|^{-1} \varphi(x, c_n y)\]
and, as \(n \to \infty\), we obtain
\[(2.8) \quad \langle f(\lambda x), y \rangle = \langle \lambda f(x), y \rangle \quad (x \in X, \lambda \in \mathbb{C}).\]

If (2.5) holds, we take \(c_n^{-1} x\) instead \(x\) in (2.6) and we arrive also at (2.7). Therefore,
\[(2.9) \quad f(\lambda x) = \lambda f(x) \quad (x \in X, \lambda \in \mathbb{C}).\]

Using the sequence \(c_n = 2^n\) we get the following results.

**Corollary 2.4.** If \(f : X \to Y\) is an everywhere defined \(\varphi\)-perturbation of an adjointable mapping, where \(\varphi(x, y) = \varepsilon \|x\|^p \|y\|^q\) \((\alpha > 0, p \neq 1, q \neq 1)\), then \(f\) is adjointable and hence a bounded \(C^*\)-linear mapping.

**Corollary 2.5.** If \(f : X \to Y\) is an everywhere defined \(\varphi\)-perturbation of an adjointable mapping, where \(\varphi(x, y) = \varepsilon_1 \|x\|^p + \varepsilon_2 \|y\|^q\) \((\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, p \neq 1, q \neq 1)\), then \(f\) is adjointable and hence a bounded \(C^*\)-linear mapping.

We would like to point out that the proof of Theorem 2.3 works equally well in the case that the functions \(f\) and \(g\) are well-defined merely on norm-dense subsets of \(X\) and \(Y\), respectively. This case covers the situation of pairs of adjoint to each other, densely defined \(A\)-linear operators between pairs of Hilbert \(A\)-modules. However, since boundedness cannot be demonstrated, in general, in that case we arrive at the following statement:

**Theorem 2.6.** Let \(f : X \to Y\) be a \(\varphi\)-perturbation of an adjointable mapping with corresponding mapping \(g : Y \to X\). Suppose, that the mappings \(f\) and \(g\) are densely defined on the respective Hilbert \(C^*\)-modules. Furthermore, suppose that for the perturbation bound mapping \(\varphi(x, y)\) either both the conditions (2.2) and (2.3), or both the conditions (2.4) and (2.5) hold. Then \(f\) is adjointable. In particular, \(f\) is \(A\)-linear, as well as its adjoint is \(g\).
Corollary 2.7. The equation \( f(x)^* y = x g(y)^* \ (x \in I, y \in J) \) is superstable, where
\( f : I \to J \) and \( g : J \to I \) are adjoint to each other, densely defined \( A \)-linear mappings between right ideals \( I, J \) of \( A \).

The critical case of \( \varphi \)-perturbations is that one where the function \( \varphi \) satisfies neither the pair of conditions (i) and (ii), nor the pair of conditions (i’) and (ii’).

We demonstrate that there may exist \( \varphi \)-perturbed bounded \( C^* \)-linear mappings \( f \) on certain types of Hilbert \( C^* \)-modules \( X \) over certain \( C^* \)-algebras \( A \) which are not adjointable. Moreover, any non-adjointable bounded \( C^* \)-linear mapping \( f \) on suitably selected Hilbert \( C^* \)-modules \( X \) can be \( \varphi \)-perturbed by “compact” operators on \( X \) using this type of perturbation functions.

Proposition 2.8. Let \( X \) be a Hilbert \( A \)-module over a given \( C^* \)-algebra \( A \). Suppose there exists a non-adjointable bounded \( A \)-linear mapping \( f : X \to X \), (so \( X \) cannot be self-dual by [15, 8]). Then there exist (at least countably many) positive constants \( c_\alpha \), and respective “compact” \( A \)-linear operators \( K_\alpha : X \to X \) \( (\alpha \in I) \) such that \( f \) is \( \varphi \)-perturbed for a function \( \varphi(x, y) = c_\alpha \cdot \|x\| \cdot \|y\| \) and for \( g = K_\alpha^* \).

Proof. By results of Huaxin Lin [16, Thm. 1.5], the Banach algebra \( \text{End}_A(X) \) of all bounded \( A \)-linear mappings on \( X \) is the left multiplier algebra of the \( C^* \)-algebra \( K_A(X) \) of all “compact” \( A \)-linear operators on \( X \). Since \( \text{End}_A(X) \) is the completion of \( K_A(X) \) with respect to the left strict topology defined by the set of semi-norms \( \{\|K\| : K \in K_A(X)\} \), there exists a bounded net \( \{K_\alpha : \alpha \in I\} \) of “compact” operators such that the set \( \{K_\alpha K : \alpha \in I\} \) converges with respect to the operator norm to \( fK \) for any single “compact” operator \( K \). Therefore,
\[
0 = \lim_{\alpha \in I} \|((f - K_\alpha)K)(x), y\| = \lim_{\alpha \in I} \|((f - K_\alpha)K)(x), y\|
\]
for any “compact” operator \( K \). However, the set \( \{K(x) : K \in K_A(X), x \in X\} \) is norm-dense in \( X \), hence
\[
\|f(x, y) - (K_\alpha(x), y)\| \leq \|f - K_\alpha\| \cdot \|x\| \cdot \|y\|
\]
for any \( x, y \in X \) and any \( \alpha \in I \). Setting \( c_\alpha = \|f - K_\alpha\| \) for any fixed index \( \alpha \) and taking into account the adjointability of the operators \( \{K_\alpha\} \) we arrive at the desired result.

Corollary 2.9. Let \( X \) be a Hilbert \( A \)-module over a given \( C^* \)-algebra \( A \). Suppose there exists a non-adjointable bounded \( A \)-linear mapping \( f : X \to X \). Then there does not exist any \( \varphi \)-perturbation of \( f \) such that \( \varphi(x, y) \) satisfies either both the conditions (2.2) and (2.3) or both the conditions (2.4) and (2.5).

References
[1] M. Amyari: Stability of \( C^* \)-inner products, J. Math. Anal. Appl. 322 (2006), 214–218.
[2] M. Amyari, M. S. Moslehian: Stability of derivations on Hilbert C*-modules, Topological Algebras and Applications, 31–39, Contemp. Math. 724, 2007.
[3] T. Aoki: On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950), 64–66.
[4] C. Baak, H. Y. Chu, M. S. Moslehian: On linear n-inner product preserving mappings, Math. Inequal. Appl., 9 (2006), no. 3, 453–464.
[5] J. Baker: The stability of the cosine equation, Proc. Amer. Math. Soc. 74 (1979), 242–246.
[6] S. Czerwik (ed.): Stability of Functional Equations of Ulam–Hyers–Rassias Type, Hadronic Press Inc., Palm Harbor, Florida, 2003.
[7] S. Czerwik: Functional equations and inequalities in several variables, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.
[8] M. Frank: Geometrical aspects of Hilbert C*-modules, Positivity 3 (1999), 215–243.
[9] P. Gavruta: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431–436.
[10] D. H. Hyers: On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222–224.
[11] D. H. Hyers, G. Isac, Th. M. Rassias: Stability of Functional Equations in Several Variables, Birkhauser, Boston, Basel, Berlin, 1998.
[12] D. H. Hyers, Th. M. Rassias: Approximate homomorphisms, Aequationes Math. 44 (1992), no. 2-3, 125–153.
[13] S.-M. Jung: Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, Florida, 2001.
[14] I. Kaplansky: Modules over operator algebras, Amer J. Math. 75 (1953), 839–858.
[15] E. C. Lance: Hilbert C*-Modules, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
[16] H. Lin: Hilbert C*-modules and their bounded module maps, Sci. China, Ser. A 34 (1991), no. 12, 1409–1421.
[17] H. Lin: Bounded module maps and pure completely positive mappings, J. Operator Theory 26 (1991), 121–138.
[18] V. M. Manuilov, E. V. Troitsky: Hilbert C*-modules, Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.
[19] D. S. Mitrinović: Analytic inequalities, In cooperation with P. M. Vasić, Die Grundlehren der mathematischen Wissenschaften, Band 165, Springer-Verlag, New York-Berlin, 1970.
[20] Th. M. Rassias: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300.
[21] Th. M. Rassias: On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000), no. 1, 23–130.
[22] Th. M. Rassias (ed.): Functional Equations, Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
[23] S. M. Ulam: Problems in Modern Mathematics, Chapter VI, Science Editions, Wiley, New York, 1964.

Michael Frank: Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig, Fachbereich Informatik, Mathematik und Naturwissenschaften (FbIMN), PF 301166, D-04251 Leipzig, Germany.
mfrank@imn.htwk-leipzig.de

Pasc Gavruta: Department of Mathematics, University ‘Politehnica’ of Timișoara, Piata Victoriei, No. 2, 300006 Timișoara, Romania.
pgavruta@yahoo.com
Mohammad Sal Moslehian: Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran; Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran. moslehian@ferdowsi.um.ac.ir and moslehian@ams.org