On the compactness of oscillation and variation of commutators

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Abstract
In this paper, we first establish the weighted compactness result for oscillation and variation associated with the truncated commutator of singular integral operators. Moreover, we establish a new CMO(\(\mathbb{R}^n\)) characterization via the compactness of oscillation and variation of commutators on weighted Lebesgue spaces.

Keywords Compactness · Commutator · Singular integral operators · Oscillation · Variation

Mathematics Subject Classification 42B20 · 42B25

1 Introduction
The singular integral operator with homogeneous kernel is defined by:
where $\Omega$ is a homogeneous function of degree zero and satisfies the following mean value zero property:

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $d\sigma$ is the spherical measure on the sphere $S^{n-1}$. Given a locally integrable function $b$ and a linear operator $T$, the commutator $[b, T]$ is defined by:

$$T_b(f)(x) := [b, T]f(x) := b(x)T(f)(x) - T(bf)(x)$$

for suitable functions $f$. The famous work of Coifman, Rochberg, and Weiss [9] gave a characterization of $L^p$-boundedness of $[b, R_j]$, for every Riesz transform $R_j$. This result was improved by Uchiyama in his remarkable work [25], in which he showed that the commutator $[b, T_\Omega]$ with $\Omega \in \text{Lip}_1(S^{n-1})$ is bounded (compact resp.) on $L^p(\mathbb{R}^n)$ if and only if the symbol $b$ is in $\text{BMO}(\mathbb{R}^n)$ (CMO($\mathbb{R}^n$) resp.), where CMO($\mathbb{R}^n$) denotes the closure of $C^\infty_c(\mathbb{R}^n)$ in the BMO($\mathbb{R}^n$) topology. Since then, the work on compactness of commutators of singular integral operators and its applications to PDEs have been paid more and more attention; see, for example, [4, 5, 7, 8, 13, 15, 24] and the references therein. Recently, inspired by Lerner et al. [17], the first, third, and fourth authors [12] gave some new characterizations of the compact commutators of singular integrals via CMO($\mathbb{R}^n$).

This paper is devoted to the weighted $L^p$-compactness of the oscillation and variation of the commutator of singular integral operator. It is known that the variation inequality was first proved by Lépingle [16] for martingales. Then, Bourgain [1] proved the variation inequality for the ergodic averages of a dynamic system. Since then, the oscillation and variation have been the active subject of recent research in the field of probability, ergodic theory, and harmonic analysis. In 2000, Campbell et al. [2] established the $L^p(\mathbb{R}^n)$-boundedness of variation for truncated Hilbert transform and then extended to higher dimensional case in [3]. For the weighted boundedness result, one can see [10, 18, 19].

To state our main results, we first recall some definitions and notations.

**Definition 1.1** The space of functions with bounded mean oscillation, denoted by BMO($\mathbb{R}^n$), consists of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, such that:

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \mathcal{O}(f; Q) < \infty,$$

where:

$$f_Q := \frac{1}{|Q|} \int_Q f(y)dy \text{ and } \mathcal{O}(f; Q) := \frac{1}{|Q|} \int_Q |f(x) - f_Q|dx.$$
The following class of $A_p$ was introduced by Muckenhoupt [20] to study the weighted norm inequalities of Hardy–Littlewood maximal operators.

**Definition 1.2** For $1 < p < \infty$, the Muckenhoupt class $A_p$ is the set of non-negative locally integrable functions $\omega$, such that:

$$[\omega]_{A_p}^{1/p} := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx \right)^{1/p'} < \infty,$$

where $1/p + 1/p' = 1$.

We say that a linear operator $T_K$ on $\mathbb{R}^n$ is a Calderón–Zygmund operator if $T_K$ is bounded on $L^2(\mathbb{R}^n)$ and it admits the following representation:

$$T_Kf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad \text{for all } x \not\in \text{supp}f \quad \text{(1.4)}$$

with kernel $K$ satisfying the size condition:

$$|K(x,y)| \leq \frac{C_K}{|x-y|^\gamma} \quad \text{(1.5)}$$

for $x, y \in \mathbb{R}^n$ with $x \neq y$ and a smoothness condition:

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C_K}{|x-y|^\gamma} \left( \frac{|x-x'|}{|x-y|} \right)^\gamma, \quad \text{(1.6)}$$

for all $x, y, x' \in \mathbb{R}^n$ with $|x-y| > 2|x-x'|$, where $C_K > 0$ and $\gamma > 0$.

Then, for a family of truncated Calderón–Zygmund operators $\{T_{K,\epsilon}\}_{\epsilon > 0}$ with $K$ given as (1.4), the $\rho$-variation with $\rho > 2$ for the family $\{T_{K,\epsilon}\}_{\epsilon > 0}$ is defined by:

$$\mathcal{V}_\rho(T_Kf)(x) := \sup_{\epsilon \downarrow 0} \left( \sum_{i=1}^{\infty} |T_{K,\epsilon_i,\epsilon}f(x) - T_{K,\epsilon_i,\epsilon}f(x)|^\rho \right)^{1/\rho},$$

where:

$$T_{K,\epsilon}(f)(x) := \int_{|x-y| > \epsilon} K(x,y)f(y)dy.$$

In general, the boundedness of variation operators may fail when $\rho \leq 2$, see the case of martingales in [22]. While the oscillation operators of $\{T_{K,\epsilon}\}_{\epsilon > 0}$ are given by:

$$\mathcal{O}(T_Kf)(x) := \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < \epsilon \leq t_i} |T_{K,\epsilon_i,\epsilon}f(x) - T_{K,\epsilon_i,\epsilon}f(x)|^2 \right)^{1/2},$$

where $\{t_i\}$ is a decreasing sequence of positive numbers converging to 0.

Our main results can be formulated as follows.
Theorem 1.3 Let $1 < p < \infty$, $\omega \in A_p$, $b \in \text{CMO}(\mathbb{R}^n)$, and $T^b_K$ be as in (1.3) with $T$ replaced by $T_K$. We have the following two statements.

(1) The $L^p_\omega(\mathbb{R}^n)$-boundedness of $\mathcal{O}(T^b_K)$ implies the $L^p_\omega(\mathbb{R}^n)$-compactness of $\mathcal{O}(T^b_K)$;
(2) The $L^p_\omega(\mathbb{R}^n)$-boundedness of $\mathcal{V}_\rho(T^b_K)$ implies the $L^p_\omega(\mathbb{R}^n)$-compactness of $\mathcal{V}_\rho(T^b_K)$.

To prove the necessity of compactness, we have to use some conclusions in [12] (see Lemmas 3.1–3.3 in Sect. 3 below), and hence, we consider oscillation and variation of the commutator of singular integral with homogeneous kernel. For a family of operators $\{T^b_{\Omega,e}\}_{e > 0}$ with $\Omega$ be given as (1.1), where:

$$
T^b_{\Omega,e}(f)(x) := \int_{|x-y| > \varepsilon} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.
$$

$\mathcal{V}_\rho(T^b_{\Omega})$ is given as $\mathcal{V}_\rho(T^b_{\Omega})$ with $T^b_K$ replaced by $T^b_{\Omega}$. While, to establish the necessity and equivalent characterization of compact oscillation operator, we need to modify oscillation by:

$$
\tilde{\mathcal{O}}(T^b_{\Omega})(x) := \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i \leq t_i} |T^b_{\Omega,\varepsilon_{i+1}} f(x) - T^b_{\Omega,\varepsilon_{i}} f(x)|^2 \right)^{1/2} + |T^b_{\Omega,t_1} f(x)|.
$$

This variant of oscillation is necessary for the following Theorem 1.4 and Corollary 1.5, since one can choose a function $b \notin \text{BMO}(\mathbb{R}^n)$, such that $\mathcal{O}(T^b_{\Omega})$ in Theorem 1.4 and Corollary 1.5 is a compact operator on $L^1_{\text{loc}}(\mathbb{R}^n)$. We put the details in Appendix A.

Theorem 1.4 Let $1 < p < \infty$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $\omega \in A_p$. Let $\Omega$ be a bounded measurable function on $\mathbb{S}^{n-1}$, which does not change sign and is not equivalent to zero on some open subset of $\mathbb{S}^{n-1}$. Then, we have the following two statements.

(1) Let $\{t_j\}_{j=1}^{\infty}$ be a sequence with $\sup_{i \in \mathbb{Z}} |\{j : 2^i \leq |t_j| < 2^{i+1}\}| < \infty$. Then, the $L^p_\omega(\mathbb{R}^n)$-compactness of $\tilde{\mathcal{O}}(T^b_{\Omega})$ implies $b \in \text{CMO}(\mathbb{R}^n)$;
(2) The $L^p_\omega(\mathbb{R}^n)$-compactness of $\mathcal{V}_\rho(T^b_{\Omega})$ implies $b \in \text{CMO}(\mathbb{R}^n)$.

We remark that truncated singular integral operators with smooth homogeneous kernel, for example, the truncated classical Riesz transforms, satisfy the assumptions in Theorem 1.4. Hence, we further have the following corollary.
Corollary 1.5 Let $1 < p < \infty$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega \in A_p$, and $\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})$ with $\Omega \not\equiv 0$. Then:

(1) $b \in \text{CMO}(\mathbb{R}^n) \iff \overline{O}(T^b_{\Omega})$ is compact on $L^p_{\omega}(\mathbb{R}^n)$;
(2) $b \in \text{CMO}(\mathbb{R}^n) \iff \mathcal{V}_\rho(T^b_{\Omega})$ is compact on $L^p_{\omega}(\mathbb{R}^n)$.

This paper is organized as follows. Section 2 is devoted to the proof of the sufficiency of compactness, i.e., Theorem 1.3. It is well known that the Fréchet–Kolmogorov theorem is a powerful tool in the study of compactness of commutators of singular integral operators, see, for example, [25]. In the proof of Theorem 1.3, we also use the weighted Fréchet–Kolmogorov theorem (see Lemma 2.2) to prove the compactness of $O(T^b_K)$ and $\mathcal{V}_\rho(T^b_{\Omega})$. However, due to the special structures of oscillation and variation, the argument here is more complicated. Moreover, compared to the known case of singular integral operators, the regularity of oscillation or variation of commutator generated by a singular integral operator comes from not only the regularity of symbol $b$ and the kernel $K$, but also the smallness of corresponding measurable sets degenerated by the annuluses in the definition of oscillation or variation.

The necessity conditions of compactness will be dealt with in Sect. 3. By establishing two claims A and B, we reduce our cases to the known cases in [12]. Then, Theorem 1.4 can be proved. Appendix A is used to clarify the reasonableness of the modified oscillation in our results for the necessity.

We remark that all conclusions of this article can de extended to the high-order commutator case (oscillation and variation of high-order commutators) as in [12]. We omit such more complicated expression form just for conciseness, and leave the details to the interested readers.

Throughout this paper, we will adopt the following notations. Let $C$ be a positive constant which is independent of the main parameters. The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, $X \sim Y$ means $X \lesssim Y \lesssim X$. For a given cube $Q$, we use $c_Q$, $l_Q$, and $\chi_Q$ to denote the center, side length and characteristic function of $Q$. We also denote $(A \setminus B) \cup (B \setminus A)$ by $A \bigtriangleup B$. For any point $x_0 \in \mathbb{R}^n$ and sets $E, F \subset \mathbb{R}^n$, $E + x_0 := \{y + x_0 : y \in E\}$ and $E - F := \{x - y : x \in E, y \in F\}$.

2 Compactness of the oscillation and variation operators

In this part, we study the compactness property of oscillation and variation. Thanks to the Cauchy integral formula, the commutator:

$$[b, T]f = \frac{d}{dz} e^{zb} T(f e^{-zb}) \bigg|_{z=0} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{T_z(f)}{z^2} \, dz, \quad \epsilon > 0,$$
where:

\[ z \to T_z(f) := e^{zb}T \left( \frac{f}{e^{zb}} \right), z \in \mathbb{C}. \]

Applying Minkowski’s inequality, one can get the boundedness of \([b, T]\) via the corresponding boundedness of \(T\), which is the so-called conjugation method (see, for example, \([21, p.289]\)). In \([6]\), Chen et al. obtained the boundedness of \(\mathcal{V}_\rho(T^b_K)\) via the conjugation method. Similarly, we can also obtain the boundedness of \(\mathcal{O}(T^b_K)\). Precisely, we have the following lemma.

**Lemma 2.1** Let \(1 < p < \infty\), \(\omega \in A_p\) and \(b \in BMO(\mathbb{R}^n)\). We have the following two statements.

1. The \(L^p_\omega(\mathbb{R}^n)\)-boundedness of \(\mathcal{V}_\rho(T^b_K)\) implies the \(L^p_\omega(\mathbb{R}^n)\)-boundedness of \(\mathcal{V}_\rho(T^b_K)\) with:

\[
\| \mathcal{V}_\rho(T^b_K) \|_{L^p_\omega(\mathbb{R}^n) \to L^p_\omega(\mathbb{R}^n)} \lesssim \| b \|_{BMO(\mathbb{R}^n)}.
\]

2. The \(L^p_\omega(\mathbb{R}^n)\)-boundedness of \(\mathcal{O}(T^b_K)\) implies the \(L^p_\omega(\mathbb{R}^n)\)-boundedness of \(\mathcal{O}(T^b_K)\) with:

\[
\| \mathcal{O}(T^b_K) \|_{L^p_\omega(\mathbb{R}^n) \to L^p_\omega(\mathbb{R}^n)} \lesssim \| b \|_{BMO(\mathbb{R}^n)}.
\]

Now, we turn to the proof of Theorem 1.3. We recall the weighted Fréchet–Kolmogorov theorem \([8]\) as follows.

**Lemma 2.2** Let \(p \in (1, \infty)\) and \(\omega \in A_p\). A subset \(E\) of \(L^p_\omega(\mathbb{R}^n)\) is precompact (or totally bounded) if the following statements hold:

(a) \(E\) is bounded, i.e., \(\sup_{f \in E} \| f \|_{L^p_\omega(\mathbb{R}^n)} \lesssim 1\);

(b) \(E\) uniformly vanishes at infinity, that is:

\[
\lim_{N \to \infty} \int_{|x| > N} |f(x)|^p \omega(x) dx = 0, \text{ uniformly for all } f \in E.
\]

(c) \(E\) is uniformly equicontinuous, that is:

\[
\lim_{b \to 0} \sup_{y \in B(0, \rho)} \int_{\mathbb{R}^n} |f(x + y) - f(x)|^p \omega(x) dx = 0, \text{ uniformly for all } f \in E.
\]
Then, we collect some basic properties of the Muckenhoupt class $A_p$. One can see [11] for the proofs of (i)–(iii) of Lemma 2.3.

**Lemma 2.3** Let $1 < p < \infty$.

(i) $\omega \in A_p \iff \omega^{1-p'} = \omega^{-p'/p} \in A_{p'}$.

(ii) If $\omega \in A_p$, there exists a small constant $\epsilon$ depending only on $n$, $p$, and $[\omega]_{A_p}$, such that: $\omega \in A_{p-\epsilon}$.

(iii) For all $\lambda > 1$, and all cubes $Q$:

$$\omega(\lambda Q) \leq \lambda^{np}[\omega]_{A_p} \omega(Q).$$

(iv) If $\omega \in A_p$, we have:

$$\lim_{N \to +\infty} \int_{B(0,N)} \frac{\omega(x)}{|x|^{np}} \, dx = 0, \quad \lim_{N \to +\infty} \int_{B(0,N)} \frac{\omega(x)^{-p'/p}}{|x|^{np'}} \, dx = 0.$$

**Proof** We only give the proof of (iv). Since $\omega \in A_p$, there exists $\epsilon > 0$, such that $\omega \in A_{p-\epsilon}$. Write:

$$\int_{B(0,N)} \frac{\omega(x)}{|x|^{np}} \, dx = \sum_{j=0}^{\infty} \int_{2N \leq |x| < 2^{j+1}N} \frac{\omega(x)}{|x|^{np}} \, dx \lesssim \sum_{j=0}^{\infty} (2^j N)^{-np} \omega(B(0, 2^j N)) \lesssim \sum_{j=0}^{\infty} (2^j N)^{-np} (2^j N)^{p-p'} = N^{-\epsilon} \sum_{j=0}^{\infty} 2^{-jn} \to 0$$

as $N \to +\infty$, where we use property (iii) in the second inequality. Similarly, by property (i), we get $\omega^{1-p'} \in A_{p'}$, then the second equality of (iv) follows.

We now present the proof of Theorem 1.3. We point out that since there is some essential difference between the arguments for oscillation and variation, we give the proofs of (1) and (2), respectively.

**Proof of (1) in Theorem 1.3** Assume that $O(T_K)$ is bounded on $L^p_{\omega}(\mathbb{R}^n)$ and $b \in $CMO$(\mathbb{R}^n)$. Using Lemma 2.1 (1), we see that $O(T_K)$ is also bounded on $L^p_{\omega}(\mathbb{R}^n)$. Moreover, by the definition of $\text{CMO}(\mathbb{R}^n)$, it suffices to show $O(T_K)$ is compact on $L^p_{\omega}(\mathbb{R}^n)$ for $b \in C^\infty_c(\mathbb{R}^n)$. To this end, we follow the idea in [15] and consider smooth truncated singular integral operators. Take $\varphi \in C^\infty_c(\mathbb{R}^n)$ supported on $B(0, 1)$, such that $\varphi = 1$ on $B(0, 1/2)$, $0 \leq \varphi \leq 1$. Let $\delta > 0$ be a small constant:
\[ \varphi_\delta(x) := \varphi\left(\frac{x}{\delta}\right), \quad K^{\delta}(x, y) := K(x, y) \cdot (1 - \varphi_\delta(x - y)), \]

and

\[ T_{K}\!(f)(x) := b(x) \cdot \int_{\mathbb{R}^n} K^{\delta}(x, y)f(y)\,dy - \int_{\mathbb{R}^n} K^{\delta}(x, y)b(y)f(y)\,dy. \]

We first claim that for any \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \):

\[ \| \mathcal{O}(T_{K}^{b})(f) - \mathcal{O}(T_{K}^{b})(f) \|_{L^p_{\text{loc}}(\mathbb{R}^n)} \lesssim \| f \|_{L^p_{\text{loc}}(\mathbb{R}^n)}, \]  

(2.1)

where the implicit constant is independent of \( f \). In fact, when \(|x - y| > 2|x - x'|\), since

\[
\begin{align*}
|K^{\delta}(x, y) - K^{\delta}(x', y)| &+ |K^{\delta}(y, x) - K^{\delta}(y, x')| \\
&\leq |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \\
&+ (|K(x, y)| + |K(y, x)|) |\varphi_\delta(x - y) - \varphi_\delta(x' - y)| \\
&\lesssim \frac{1}{|x - y|^n} \left( \frac{|x - x'|^n}{|x - y|^n} + |\varphi_\delta(x - y) - \varphi_\delta(x' - y)| \right),
\end{align*}
\]

and

\[ |\varphi_\delta(x - y) - \varphi_\delta(x' - y)| \lesssim \frac{|x - x'|}{|x - y|}, \]

then we see that:

\[
|K^{\delta}(x, y) - K^{\delta}(x', y)| + |K^{\delta}(y, x) - K^{\delta}(y, x')| \lesssim \frac{1}{|x - y|^n} \left( \frac{|x - x'|}{|x - y|} \right)^{\min\{n, 1\}}.
\]

This together with \( |K^{\delta}(x, y)| \lesssim \frac{2}{|x - y|^n} \) shows that the kernel \( K^{\delta} \) also satisfies (1.5) and (1.6) with \( C_K \) replaced by certain constant \( C \) therein. Moreover, \( K^{\delta} \) is a bounded function, since for any \( x, y \) with \( x \neq y \):

\[
|K^{\delta}(x, y)| \lesssim \frac{1}{|x - y|^n} (1 - \varphi_\delta(x - y)) \lesssim \frac{1}{|x - y|^n} X_{(x, y):|x - y|\geq\delta/2} \lesssim \frac{1}{\delta^n}. \]  

(2.2)

By the sub-linearity of oscillation, we have:

\[ |\mathcal{O}(T_{K}^{b})(f) - \mathcal{O}(T_{K}^{b})(f)| \leq \mathcal{O}(T_{K}^{b} - T_{K}^{b})(f). \]

From this and the fact that for any \( x \) and \( y \):

\[ b(x) \cdot \int_{\mathbb{R}^n} K^{\delta}(x, y)f(y)\,dy - \int_{\mathbb{R}^n} K^{\delta}(x, y)b(y)f(y)\,dy. \]
\[ |b(x) - b(y)| \leq \|b\|_{L^{\infty}(\mathbb{R}^d)} |x - y|, \quad \text{and} \quad |\varphi_\delta(x - y)| \leq \chi_{\{x, y\colon |x - y| \leq \delta\}}(x, y), \]

we further deduce that for any \( x \):

\[
\begin{align*}
|\mathcal{O}(T_K^b)(f)(x) - \mathcal{O}(T_{K_n}^b)(f)(x)|
\leq & \left( \sum_{i=1}^{\infty} \sup_{l_{i+1} < \delta \leq l_i} \left| \int_{l_{i+1} < |x - y| \leq l_i} (b(x) - b(y)) \times \varphi_\delta(x - y)K(x, y)f(y)dy \right|^2 \right)^{1/2} \\
\leq & \left( \sum_{i=1}^{\infty} \sup_{l_{i+1} < \delta \leq l_i} \left| \int_{l_{i+1} < |x - y| \leq l_i} \varphi_\delta(x - y) \frac{|f(y)|}{|x - y|^{n-1}}dy \right|^2 \right)^{1/2} \\
\leq & \left( \int_{l_{i+1} < |x - y| \leq l_i} \varphi_\delta(x - y) \frac{|f(y)|}{|x - y|^{n-1}}dy \right)^{1/2} \\
\leq & \int_{B(x, \delta)} \frac{|f(y)|}{|x - y|^{n-1}}dy \leq \delta M(f)(x),
\end{align*}
\]

where \( M(f) \) is the Hardy–Littlewood maximal function of \( f \), and the implicit constant is independent of \( x, \delta \) and \( f \). This via the boundedness of \( M(f) \) on \( L^p_\omega(\mathbb{R}^n) \) implies that:

\[
\|\mathcal{O}(T_K^b)(f) - \mathcal{O}(T_{K_n}^b)(f)\|_{L^p_\omega(\mathbb{R}^n)} \leq \delta \|M(f)\|_{L^p_\omega(\mathbb{R}^n)} \leq \delta \|f\|_{L^p_\omega(\mathbb{R}^n)},
\]

and shows the claim (2.1).

Now, observe that to show \( \mathcal{O}(T_K^b) \) is compact on \( L^p_\omega(\mathbb{R}^n) \), we only need to show that the subset

\[
\{ \mathcal{O}(T_K^b)(f) : \|f\|_{L^p_\omega(\mathbb{R}^n)} \leq 1 \}
\]

de \( L^p_\omega(\mathbb{R}^n) \) is precompact. Then, by (2.1), it suffices to show that for \( \delta \) small enough, the set

\[
A(\mathcal{O}(T_{K_n}^b)) := \{ \mathcal{O}(T_{K_n}^b)(f) : \|f\|_{L^p_\omega(\mathbb{R}^n)} \leq 1 \}
\]

is precompact.
We now use Lemma 2.2 to show that $A(\mathcal{O}(\mathcal{T}^b_K))$ is precompact. First, note that (2.1) and the boundedness of $\mathcal{O}(\mathcal{T}^b_K)$ yield the $L^p_\omega(\mathbb{R}^n)$-boundedness of $\mathcal{O}(\mathcal{T}^b_K)$. Then, we see that $A(\mathcal{O}(\mathcal{T}^b_K))$ is a bounded set in $L^p_\omega(\mathbb{R}^n)$, and (a) of Lemma 2.2 is true for $A(\mathcal{O}(\mathcal{T}^b_K))$.

To show (b), without loss of generality, we assume that $b$ is supported in a cube $Q$ centered at the origin. For $x \in (2Q)^c$, by (1.5) for $K^\delta$, the Hölder inequality and $\|f\|_{L^p_\omega(\mathbb{R}^n)} \leq 1$, we have:

\[
|\mathcal{O}(\mathcal{T}^b_K)(f)(x)| = \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i \leq t_i} \left| \int_{\epsilon_i \leq |x-y| \leq \epsilon_i} b(y)K^\delta(x,y)f(y)dy \right|^2 \right)^{1/2} \\
\leq \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i \leq t_i} \left| \int_{\epsilon_i \leq |x-y| \leq \epsilon_i} |f(y)|\chi_Q(y) \frac{dy}{|x-y|^n} \right|^2 \right)^{1/2} \\
\leq \int_Q \frac{|f(y)|}{|x-y|^n} dy \leq \frac{1}{|x|^n} \int_Q |f(y)|dy \leq \frac{1}{|x|^n} \left( \int_Q \omega(y)^{-\nu'/p'} dy \right)^{1/p'}.
\]

Taking $N > 2$, we then have:

\[
\left( \int_{(2^nQ)^c} |\mathcal{O}(\mathcal{T}^b_K)(f)(x)|^p \omega(x)dx \right)^{1/p} \leq \left( \int_{(2^nQ)^c} \frac{\omega(x)}{|x|^\nu} dx \right)^{1/p} \left( \int_Q \omega(y)^{-\nu'/p'} dy \right)^{1/p'},
\]

which tends to zero as $N$ tends to infinity, where we use (iv) in Lemma 2.3. Thus, Lemma 2.2 (b) holds.

It remains to prove that $A(\mathcal{O}(\mathcal{T}^b_K))$ satisfies Lemma 2.2 (c). Taking $z \in \mathbb{R}^n$ with $|z| \leq \frac{\delta}{8}$, then we see that for any $x \in \mathbb{R}^n$:

\[
|\mathcal{O}(\mathcal{T}^b_K)(f)(x+z) - \mathcal{O}(\mathcal{T}^b_K)(f)(x)| \\
\leq \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i \leq t_i} |T^b_{K^\delta,\epsilon_i} f(x+z) - T^b_{K^\delta,\epsilon_i} f(x+z) \right. \\
- \left. (T^b_{K^\delta,\epsilon_i} f(x) - T^b_{K^\delta,\epsilon_i} f(x))|^2 \right)^{1/2}.
\]

Moreover, for each $i$, we write:
\begin{align*}
|T^b_{K^a, \epsilon_i} f(x + z) - T^b_{K^a, \epsilon_i} f(x + z) - (T^b_{K^a, \epsilon_{i+1}} f(x) - T^b_{K^a, \epsilon_i} f(x))| \\
= \left| \int_{\epsilon_{i+1} < |x-z-y| \leq \epsilon_i} (b(x+z) - b(y))K^\delta(x+z,y)f(y)dy \right. \\
- \left. \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} (b(x) - b(y))K^\delta(x,y)f(y)dy \right| \\
\leq \int_{\epsilon_{i+1} < |x-z-y| \leq \epsilon_i} (b(x+z) - b(y))K^\delta(x+z,y)f(y)dy \\
- \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} (b(x+z) - b(y))K^\delta(x,y)f(y)dy \\
+ \int_{\epsilon_{i+1} < |x-z-y| \leq \epsilon_i} (b(x+z) - b(x))K^\delta(x,y)f(y)dy \\
+ \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} (b(x+z) - b(y))(K^\delta(x+z,y) - K^\delta(x,y))f(y)dy \\
=: I_1(i) + I_2(i) + I_3(i).
\end{align*}

We first estimate \( I_2(i) \). Since \( b \in C^\infty_c(\mathbb{R}^n) \), we have:
\[
I_2(i) \leq |z| \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K^\delta(x,y)f(y)dy \right|,
\]
which yields that:
\[
\left( \sum_{i=1}^\infty \sup_{t_{i+1} \leq \epsilon_{i+1} < \epsilon_i \leq t_i} |I_2(i)|^2 \right)^{1/2} \lesssim |z| O(T_{K^a})(f)(x).
\]

Furthermore, assume that there exists \( i_0 \in \mathbb{N} := \{1, 2, \ldots\} \), such that \( t_{i_0+1} < \delta \leq t_{i_0} \). Then, we see that for a. e. \( x \):
\[
O(T_{K^a})(f)(x)
\leq \left( \sum_{i=1}^{i_0-1} \sup_{t_{i+1} \leq \epsilon_{i+1} < \epsilon_i \leq t_i} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K^\delta(x,y)f(y)dy \right|^2 \right)^{1/2}
+ \sup_{t_{i_0+1} \leq \epsilon_{i_0+1} < \epsilon_0 \leq t_0} \left| \int_{\epsilon_{i_0+1} < |x-y| \leq \epsilon_0} K^\delta(x,y)f(y)dy \right|
+ \left( \sum_{i=i_0+1}^\infty \sup_{t_i \geq \epsilon_i \leq t_i} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K^\delta(x,y)f(y)dy \right|^2 \right)^{1/2}.
\]
Observe that for a. e. \( x \):

\[
\left( \sum_{i=1}^{t_i - 1} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K^\delta(x,y)f(y)dy \right|^2 \right)^{1/2}
\]

\[
= \left( \sum_{i=1}^{t_i - 1} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K(x,y)f(y)dy \right|^2 \right)^{1/2}
\]

\[
\leq O(T_K)(f)(x),
\]

and by (2.2):

\[
\left( \sum_{i=0}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K^\delta(x,y)f(y)dy \right|^2 \right)^{1/2}
\]

\[
\leq \sum_{i=0}^{\infty} \int_{t_{i+1} < |x-y| \leq t} |K^\delta(x,y)||f(y)|dy
\]

\[
\leq \int_{\delta/2 < |x-y| \leq \delta} |K^\delta(x,y)||f(y)|dy \lesssim M(f)(x).
\]

Moreover, take \( \tilde{\epsilon}_{i_0+1}, \tilde{\epsilon}_{i_0} \in [t_{i_0+1}, t_{i_0}] \), such that:

\[
\sup_{t_{i_0+1} \leq \epsilon_{i_0+1} < \epsilon_{i_0} \leq t_{i_0}} \left| \int_{\epsilon_{i_0+1} < |x-y| \leq \epsilon_{i_0}} K^\delta(x,y)f(y)dy \right|
\]

\[
\leq 2 \left| \int_{\tilde{\epsilon}_{i_0+1} < |x-y| \leq \tilde{\epsilon}_{i_0}} K^\delta(x,y)f(y)dy \right|.
\]

We then have:

\[
\sup_{t_{i_0} \leq \epsilon_{i_0+1} < \epsilon_{i_0} \leq t_{i_0}} \left| \int_{\epsilon_{i_0+1} < |x-y| \leq \epsilon_{i_0}} K^\delta(x,y)f(y)dy \right|
\]

\[
\leq 2 \left| \int_{\tilde{\epsilon}_{i_0+1} < |x-y| \leq \tilde{\epsilon}_{i_0}} K^\delta(x,y)f(y)dy \right|
\]

\[
\lesssim \int_{\tilde{\epsilon}_{i_0+1} < |x-y| \leq \delta} K^\delta(x,y)f(y)dy + \int_{\delta < |x-y| \leq \tilde{\epsilon}_{i_0}} K(x,y)f(y)dy
\]

\[
\lesssim M(f)(x) + O(T_K)(f)(x).
\]

Therefore, we conclude that for a. e. \( x \):

\[
O(T_{K^\delta})(f)(x) \lesssim O(T_K)(f)(x) + M(f)(x),
\]
where the implicit constant is independent of \(i_0, \delta, f, \) and \(x.\) From the above two estimates, the boundedness of \(\mathcal{O}(\mathcal{T}_K)(f)\) and \(M(f)\) on \(L^p_{\infty}(\mathbb{R}^n)\) and \(\|f\|_{L^p_{\infty}(\mathbb{R}^n)} \leq 1,\) we get:

\[
\left\| \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \xi < t_i} |I_2(i)|^2 \right)^{1/2} \right\|_{L^p_{\infty}(\mathbb{R}^n)} \lesssim |z| \left( \|\mathcal{O}(\mathcal{T}_K)(f)\|_{L^p_{\infty}(\mathbb{R}^n)} + \|M(f)\|_{L^p_{\infty}(\mathbb{R}^n)} \right) \lesssim |z|. \tag{2.4}
\]

Next, we turn to the estimate of \(I_3(i)\). Observe that \(K^\delta(x + z, y)\) and \(K^\delta(x, y)\) vanish when \(|x - y| \leq \frac{\delta}{4}\). Then, by (1.6), for \(K^\delta:\)

\[
I_3(i) \lesssim \int_{t_{i+1} < |x-y| \leq t_i} \frac{|z|^\gamma}{|x-y|^{n+\gamma}} \chi_{\{|x-y| > \delta/4\}}(y)|f(y)|dy,
\]

where \(\gamma\) is as in (1.6). From this, we further have:

\[
\left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \xi < t_i} |I_3(i)|^2 \right)^{1/2} \lesssim \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \xi < t_i} \left| \int_{t_{i+1} < |x-y| \leq t_i} \frac{|z|^\gamma}{|x-y|^{n+\gamma}} \chi_{\{|x-y| > \delta/4\}}(y)|f(y)|dy \right|^2 \right)^{1/2} \lesssim \sum_{i=1}^{\infty} \int_{t_{i+1} < |x-y| \leq t_i} \frac{|z|^\gamma}{|x-y|^{n+\gamma}} \chi_{\{|x-y| > \delta/4\}}(y)|f(y)|dy \lesssim \int_{|x-y| > \delta/4} \frac{|z|^\gamma}{|x-y|^{n+\gamma}} |f(y)|dy \lesssim \frac{|z|^\gamma}{\delta^\gamma} M(f)(x),
\]

where the implicit constant is independent of \(f, x, \delta, \) and \(z.\) Thus:

\[
\left\| \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \xi < t_i} |I_3(i)|^2 \right)^{1/2} \right\|_{L^p_{\infty}(\mathbb{R}^n)} \lesssim \frac{|z|^\gamma}{\delta^\gamma} \|M(f)\|_{L^p_{\infty}(\mathbb{R}^n)} \lesssim \frac{|z|^\gamma}{\delta^\gamma}. \tag{2.5}
\]

Finally, we proceed to the estimate of \(I_1(i)\). Again, observe that \(K^\delta(x + z, y)\) vanishes when \(|x - y| \leq \frac{\delta}{4}\) and \(|x + z - y| < \frac{\delta}{4}\). Since \(t_i\) tends to zero as \(i \to \infty,\) there exist only finite \(i \in \Gamma = \{1 \leq i \leq i_1\},\) such that \(I_1(i)\) is non-zero. Here and after, denote \(E_i(x, z) := \{y \in \mathbb{R}^n : \xi_{i+1} < |x + z - y| \leq \xi_i\}.\) It follows by an elementary calculation that:

\[
E_i(x, z) \triangle E_i(x, 0) \subset (B_{x, \xi_i} \Delta B_{x+z, \xi_i}) \cup (B_{x, \xi_{i+1}} \Delta B_{x+z, \xi_{i+1}}), \tag{2.6}
\]
where \( B_{x,r} \) means the ball of radius \( r \) centered at \( x \). Moreover, for \( |z| < \epsilon_i \), we have:

\[
|B_{x,\epsilon_i} \triangle B_{x+z,\epsilon_i}| \leq 2(|B_{x,\epsilon_i}| - |B_{x,\epsilon_i} - |z||) \\
\lesssim \epsilon_i^{n-1} |z| \leq t_i |z| \lesssim |z|, \quad i = 1, 2, \ldots i_1 + 1.
\]

Thus:

\[
|B_{x,\epsilon_i} \triangle B_{x+z,\epsilon_i}| \lesssim |z| \quad \text{holds for all } z. \tag{2.7}
\]

The combination of (2.6) and (2.7) yields that:

\[
|E_i(x, z) \triangle E_i(x, 0)| \\
\lesssim |B_{x,\epsilon_i} \triangle B_{x+z,\epsilon_i} + B_{x,\epsilon_i+1} \triangle B_{x+z,\epsilon_i+1}| \lesssim |z|, \quad i = 1, 2, \ldots i_1.
\]

Recall that \( b \) is supported in \( Q \) and \( |z| \leq \delta/8 \). Denote \( R := t_i + \sqrt{n \ell_Q} + \delta \). We have:

\[
x + z \in Q \implies |x| \leq R, |y| \leq R
\]

and

\[
y \in Q \implies |x| \leq R.
\]

Then:

\[
|b(x + z) - b(y)| = |b(x + z) - b(y)|\chi_{B(0,R)}(x)\chi_{B(0,R)}(y),
\]

when \( |x - y| \leq t_i \) or \( |x + z - y| \leq t_i \). Observe that if \( y \in E_i(x, z) \triangle E_i(x, 0) \), we have:

\[
|x - y| \leq \epsilon_i \leq t_i, \quad |x + z - y| \leq \epsilon_i \leq t_i.
\]

Hence:

\[
I_1(i) = \int_{E_i(x, z) \triangle E_i(x, 0)} (b(x + z) - b(y))K^\delta(x + z, y)f(y)dy \\
\leq \int_{E_i(x, z) \triangle E_i(x, 0)} |b(x + z) - b(y)| \cdot |K^\delta(x + z, y)| \cdot |f(y)| dy \\
= \left( \int_{E_i(x, z) \triangle E_i(x, 0)} |b(x + z) - b(y)|\chi_{B(0,R)}(y) \right. \\
\times |K^\delta(x + z, y)| \cdot |f(y)| dy \cdot \chi_{B(0,R)}(x) \\
\lesssim \left( \int_{E_i(x, z) \triangle E_i(x, 0)} |f(y)|\chi_{B(0,R)}(y)dy \right) \chi_{B(0,R)}(x),
\]

\begin{align*}
\end{align*}
where in the last inequality, we use the boundedness of $K^\delta$ [see (2.2)]. This implies that:

\[
\left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} |I_1(i)|^2 \right)^{1/2} = \left( \sum_{i \in \Gamma} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} |I_1(i)|^2 \right)^{1/2} \\
\leq \sup_{i \in \Gamma} \left( \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} I_1(i) \right) \cdot X_{B(0,R)}(x) \\
\leq \sup_{i \in \Gamma} \left( \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} \int_{E_i(x,z) \triangle E_i(x,0)} |f(y)| X_{B(0,R)}(y)dy \right) \cdot X_{B(0,R)}(x) \\
\leq \sup_{i \in \Gamma} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} \left( \int_{E_i(x,z) \triangle E_i(x,0)} \omega(y)^{-p'/p} X_{B(0,R)}(y)dy \right)^{1/p'} \\
\times \left( \int_{\mathbb{R}^n} |f(y)|^p \omega(y)dy \right)^{1/p} \cdot X_{B(0,R)}(x) \\
\leq \sup_{i \in \Gamma} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} \left( \int_{E_i(x,z) \triangle E_i(x,0)} \omega(y)^{-p'/p} X_{B(0,R)}(y)dy \right)^{1/p'} \\
\times \omega(B(0,R))^{1/p} \\
\]  

Note that $|E_i(x,z) \triangle E_i(x,0)| \leq |z| \to 0$, uniformly for all $x$ and $\epsilon_i(i \leq t_1)$, and that $\omega^{-p'/p} \in A_{p'} \subset L^1_{loc} (\mathbb{R}^n)$, we have:

\[
\left( \int_{E_i(x,z) \triangle E_i(x,0)} \omega(y)^{-p'/p} X_{B(0,R)}(y)dy \right)^{1/p'} \to 0, \quad (|z| \to 0)
\]

uniformly for all $x$ and $\epsilon_i$. Thus:

\[
\left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} |I_1(i)|^2 \right)^{1/2} \leq \beta_{|z|} X_{B(0,R)}(x),
\]

where $\beta_{|z|} \to 0$ as $|z| \to 0$. Hence:

\[
\left\| \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < \epsilon_i \leq t_i} |I_1(i)|^2 \right)^{1/2} \right\|_{L^p_{\infty}(\mathbb{R}^n)} \leq \beta_{|z|} \omega(B(0,R))^{1/p} \quad (2.8)
\]
tends to zero as $|z| \to 0$. The equicontinuity of $A(O(T^b_δ))$ follows by the combination of (2.4), (2.5) and (2.8). We have now completed this proof.

\[\square\]

**Proof of (2) in Theorem 1.3**

Assume that $\mathcal{V}_b(T^K)$ is bounded on $L^p_0(\mathbb{R}^n)$ and $b \in \text{CMO}(\mathbb{R}^n)$. Recall that by Lemma 2.1, we have that $\mathcal{V}_b(T^K)$ is bounded on $L^p_0(\mathbb{R}^n)$. Take $K^δ(x,y)$ with $δ > 0$ as in the proof of (1). Arguing as in (2.3), we also have that for any $f \in L^p_0(\mathbb{R}^n)$:

$$\|\mathcal{V}_b(T^K_b)(f) - \mathcal{V}_b(T^K)(f)\|_{L^p_0(\mathbb{R}^n)} \lesssim \delta \|M(f)\|_{L^p_0(\mathbb{R}^n)} \lesssim \delta \|f\|_{L^p_0(\mathbb{R}^n)},$$

from which we further obtain the $L^p_0(\mathbb{R}^n)$-boundedness of $\mathcal{V}_b(T^K_b)$ via this inequality and the boundedness of $\mathcal{V}_b(T^K)$. Moreover, by a similar argument, to show $\mathcal{V}_b(T^K_b)$ is compact, we only need to verify the set:

$$A(\mathcal{V}_b(T^K_b)) := \{\mathcal{V}_b(T^K_b)(f) : \|f\|_{L^p_0(\mathbb{R}^n)} \leq 1\}$$

is a precompact subset of $L^p_0(\mathbb{R}^n)$, where $b \in C^\infty_c(\mathbb{R}^n)$.

Without loss of generality, we assume that $b$ is supported in a cube $Q$ centered at the origin. By the boundedness of $\mathcal{V}_b(T^K_b)$, $A(\mathcal{V}_b(T^K_b))$ is a bounded set in $L^p_0(\mathbb{R}^n)$. Therefore condition (a) of Lemma 2.2 holds.

Again, by the same argument as in the proof of (1) in Theorem 1.3, we have that for any $\|f\|_{L^p_0(\mathbb{R}^n)} \leq 1$ and $x \not\in 2NQ$ with $N > 2$:

$$\mathcal{V}_b(T^K_b)(f)(x) \lesssim \frac{1}{|x|^n} \left(\int_Q \omega(y)^{-p'/p} dy\right)^{1/p'}.$$

And hence:

$$\left(\int_{(2^NQ)^c} |\mathcal{V}_b(T^K_b)(f)(x)|^p \omega(x) dx\right)^{1/p} \lesssim \left(\int_{(2^NQ)^c} \frac{\omega(x)}{|x|^m} dx\right)^{1/p} \left(\int_Q \omega(x)^{-p'/p} dx\right)^{1/p'}$$

which tends to zero as $N$ tends to infinity. This proves condition (b) of Lemma 2.2.

It remains to prove that $A(\mathcal{V}_b(T^K_b))$ is uniformly equicontinuous in $L^p_0(\mathbb{R}^n)$. Take $z \in \mathbb{R}^n$ with $|z| \leq \delta < l_Q/2$. By the sub-linearity of $\mathcal{V}_b$, we have:
\[
|\mathcal{V}_{\rho}(T_{K^\delta})(f)(x + z) - \mathcal{V}_{\rho}(T_{K^\delta})(f)(x)|
\leq \sup_{\epsilon_i \leq 0} \left( \sum_{i=1}^{\infty} \left| T_{K^\delta, \epsilon_i}^b f(x + z) - T_{K^\delta, \epsilon_i}^b f(x + z) - (T_{K^\delta, \epsilon_i}^b f(x) - T_{K^\delta, \epsilon_i}^b f(x)) \right|^\rho \right)^{1/\rho}.
\]

Write:
\[
|T_{K^\delta, \epsilon_i}^b f(x + z) - T_{K^\delta, \epsilon_i}^b f(x + z) - (T_{K^\delta, \epsilon_i}^b f(x) - T_{K^\delta, \epsilon_i}^b f(x))|
= \left| \int_{\epsilon_{i+1} \leq |x+y| \leq \epsilon_i} (b(x + z) - b(y)) K^\delta(x + z, y) f(y) dy \right| - \left| \int_{\epsilon_{i+1} \leq |x-y| \leq \epsilon_i} (b(x) - b(y)) K^\delta(x, y) f(y) dy \right|
\leq \left| \int_{\epsilon_{i+1} \leq |x+y| \leq \epsilon_i} (b(x + z) - b(y)) K^\delta(x + z, y) f(y) dy \right|
- \left| \int_{\epsilon_{i+1} \leq |x-y| \leq \epsilon_i} (b(x) - b(y)) K^\delta(x, y) f(y) dy \right|
+ \left| \int_{\epsilon_{i+1} \leq |x+y| \leq \epsilon_i} (b(x + z) - b(x)) K^\delta(x, y) f(y) dy \right|
+ \left| \int_{\epsilon_{i+1} \leq |x-y| \leq \epsilon_i} (b(x + z) - b(y))(K^\delta(x + z, y) - K^\delta(x, y)) f(y) dy \right|
=: J_1(i) + J_2(i) + J_3(i).
\]

Observe that \( J_2(i) \) is dominated by:
\[
||\nabla b||_{L^\infty} |z| \left| \int_{\epsilon_{i+1} \leq |x-y| \leq \epsilon_i} K^\delta(x, y) f(y) dy \right|,
\]
which yields that:
\[
\sup_{\epsilon_i \leq 0} \left( \sum_{i=1}^{\infty} J_2(i)^\rho \right)^{1/\rho} \lesssim |z| |\mathcal{V}_{\rho}(T_{K^\delta})(f)(x)|.
\]
Furthermore, for a.e. \( x \):

\[
\]
\[
\mathcal{V}_\rho(T_K)(f)(x) \leq \sup_{\varepsilon_i \leq \delta} \left( \sum_{\varepsilon_i < |x-y| \leq \varepsilon_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} K^\delta(x, y)f(y)dy \right)^{1/\rho} \\
+ \sup_{\varepsilon_i \leq \delta} \left( \sum_{\varepsilon_i < |x-y| \leq \varepsilon_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} K^\delta(x, y)f(y)dy \right)^{1/\rho} \\
= \sup_{\varepsilon_i \leq \delta} \left( \sum_{\varepsilon_i < |x-y| \leq \varepsilon_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} K(x, y)f(y)dy \right)^{1/\rho} \\
+ \sup_{\varepsilon_i \leq \delta} \left( \sum_{\varepsilon_i < |x-y| \leq \varepsilon_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} K^\delta(x, y)f(y)dy \right)^{1/\rho} \\
\leq \mathcal{V}_\rho(T_K)(f)(x) + \int_{\delta/2 < |x-y| \leq \delta} |K^\delta(x, y)||f(y)|dy \\
\leq \mathcal{V}_\rho(T_K)(f)(x) + M(f)(x).
\]

Here, the implicit constant is independent of the choice of \( \delta, f \) and \( x \). Thus, by the boundedness of \( \mathcal{V}_\rho(T_K)(f) \) and \( M(f) \), for any \( f \) with \( \|f\|_{L^\rho_n(\mathbb{R}^n)} \leq 1 \):

\[
\left\| \sup_{\varepsilon_i \leq \delta} \left( \sum_{i=1}^{\infty} J_2(i) \right)^{1/\rho} \right\|_{L^\rho_n(\mathbb{R}^n)} \lesssim |z| \|\mathcal{V}_\rho(T_K)(f) + M(f)\|_{L^\rho_n(\mathbb{R}^n)} \lesssim |z|.
\]

Observe that \( K^\delta(x + z, y) \) and \( K^\delta(x, y) \) vanish when \( |x - y| \leq \frac{\delta}{4} \). Then, by (1.6) and the fact that \( |z| < \delta/8 \), \( J_3(i) \) is dominated by:

\[
\int_{\varepsilon_i < |x-y| \leq \varepsilon_i} \frac{|z|^\gamma}{|x-y|^n} \chi_{\{ |x-y| > \delta/4 \}}(y)|f(y)|dy \lesssim \left( \frac{|z|}{\delta} \right)^\gamma Mf(x).
\]

By the same argument as the estimate of \( I_3 \) in the proof of (1) of Theorem 1.3, we get that for any \( f \), such that \( \|f\|_{L^\rho_n(\mathbb{R}^n)} \leq 1 \):

\[
\left\| \sup_{\varepsilon_i \leq \delta} \left( \sum_{i=1}^{\infty} J_3(i) \right)^{1/\rho} \right\|_{L^\rho_n(\mathbb{R}^n)} \lesssim \left( \frac{|z|}{\delta} \right)^\gamma \|M(f)\|_{L^\rho_n(\mathbb{R}^n)} \lesssim \left( \frac{|z|}{\delta} \right)^\gamma.
\]

Finally, we turn to the estimate of \( J_1(i) \). Recalling that \( b \) is supported in \( Q \) and \( |z| \leq \delta/8 \leq l_Q/2 \), we have that for any \( x \in \mathbb{R}^n \):

\[
b(x + z) = b(x + z)\chi_{2Q}(x).
\]
This and the sub-linearity of $\mathcal{V}_\rho$ imply that:

$$
\left\| \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} J_i(i)^\rho \right)^{1/\rho} \right\|_{L_p^0(\mathbb{R}^n)} \\
= \left\| \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} (b(x + z) - b(y)) \times K^\delta(x + z, y)f(y)dy \right|^\rho \right)^{1/\rho} \right\|_{L_p^0(\mathbb{R}^n)} \\
\leq \left\| \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} b(x + z)K^\delta(x + z, y)f(y)dy \right|^\rho \right)^{1/\rho} \right\|_{L_p^0(2Q)} \\
+ \left\| \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} b(y)K^\delta(x + z, y)f(y)dy \right|^\rho \right)^{1/\rho} \right\|_{L_p^0(\mathbb{R}^n)} \\
=: L_1 + L_2.
$$

We start with the estimate of $L_1$. For some large positive constant $N$, denote $f_1 := f \chi_{(2^N Q)^c}, f_2 := f \chi_{2^N Q}$:

$$
L_1^1(x) := \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} K^\delta(x + z, y)f_1(y)dy \right|^\rho \right)^{1/\rho}
$$

and

$$
L_1^2(x) := \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} K^\delta(x + z, y)f_2(y)dy \right|^\rho \right)^{1/\rho}.
$$

Then, we write:

$$
L_1 \leq \|b\|_{L_\infty(\mathbb{R}^n)} \left\| \sup_{\epsilon, \lambda} \left( \sum_{i=1}^{\infty} \left| \int_{E_i(x, z) \triangle E_i(x, 0)} K^\delta(x + z, y)f(y)dy \right|^\rho \right)^{1/\rho} \right\|_{L_p^0(2Q)} \\
\leq \|L_1^1\|_{L_p^0(2Q)} + \|L_1^2\|_{L_p^0(2Q)}.
$$

Recall $|K^\delta(x + z, y)| \leq \frac{1}{|x + z - y|^n} \sim \frac{1}{|y|^n}$ for $x \in 2Q$ and $y \in (2^N Q)^c$. For $x \in 2Q$, we have:
\[ L_1^2(x) \leq \sup_{\epsilon, 10} \left( \sum_{i=1}^{\infty} \left( \int_{E(x, z) \Delta E_i(x, 0)} \left| f(y) \right|^p \, dy \right)^{\frac{1}{p'}} \right) \]

\[ \leq \sup_{\epsilon, 10} \sum_{i=1}^{\infty} \int_{E(x, z) \Delta E_i(x, 0)} \frac{\left| f(y) \right|^p}{|y|^n} \, dy \]

\[ \leq \int_{\mathbb{R}^n} \frac{\left| f(y) \right|^p}{|y|^n} \, dy = \int_{(2^NQ)^c} \frac{\left| f(y) \right|^p}{|y|^n} \, dy \]

\[ \leq \left( \int_{\mathbb{R}^n} |f(y)|^p \omega(y) \, dy \right)^{\frac{1}{p'}} \left( \int_{(2^NQ)^c} \omega(y)^{-p'/p} \, dy \right)^{\frac{1}{p'}} \leq \beta_N^{(1)}, \]

where \( \beta_N^{(1)} \rightarrow 0 \) as \( N \rightarrow \infty \) by (iv) in Lemma 2.3.

Next, recall \(|K^\delta(x + z, y)| \leq \frac{1}{\delta^n}\). For fixed \( N > 0, x \in 2Q \), we have:

\[ L_1^1(x) \leq \frac{1}{\delta^n} \sup_{\epsilon, 10} \left( \sum_{i=1}^{\infty} \left( \int_{E(x, z) \Delta E_i(x, 0)} \left| f_1(y) \right| \, dy \right)^{\frac{1}{p'}} \right) \]

\[ \leq \frac{1}{\delta^n} \sup_{\epsilon, 10} \left( \sum_{i=1}^{\infty} \left( \int_{E(x, z) \Delta E_i(x, 0)} \left| f_1(y) \right| \, dy \right)^{\frac{1}{p'}} \right) \]

\[ \leq \frac{1}{\delta^n} \sup_{\epsilon, 10} \left( \int_{E(x, z) \Delta E_i(x, 0)} \omega(y)^{-p'/p} \chi_{2^NQ}(y) \, dy \right)^{\frac{1}{p'}} \]

\[ \times \left( \int_{E(x, z) \Delta E_i(x, 0)} \left| f_1(y) \right|^p \omega(y) \, dy \right)^{\frac{1}{p'}} \]

\[ \leq \frac{1}{\delta^n} \sup_{\epsilon, 10} \left( \int_{E(x, z) \Delta E_i(x, 0)} \omega(y)^{-p'/p} \chi_{2^NQ}(y) \, dy \right)^{\frac{1}{p'}} \]

where in the last-to-second inequality, we use the fact that for any \( f \) with \( \|f\|_{L^p_\rho(\mathbb{R}^n)} \leq 1 \).

For the last term, we claim that:

\[ |2^NQ \cap (E(x, z) \triangle E_i(x, 0))| \leq \beta_{N, |z|}^{(2)} \]

uniformly for all \( x \in 2Q, \{\epsilon_i\} \) and \( i \in \mathbb{N} \), where \( \beta_{N, |z|}^{(2)} \rightarrow 0 \) as \( |z| \rightarrow 0 \) for any fixed \( N \).

In fact, for any \( \{\epsilon_i\} \), assume \( i_0 \in \mathbb{N} \) be such that:

\[ \epsilon_{i_0 + 1} < \sqrt{n(2 + 2^N)} \ell_0 + \delta \leq \epsilon_{i_0}. \]

Then, we see that for any \( i \), such that \( i \leq i_0 - 1 \):

\[ E_i(x, z) \cap 2^NQ = \emptyset, \quad E_i(x, 0) \cap 2^NQ = \emptyset, \]
which implies that:

\[ |2^N Q \cap (E_i(x, z) \triangle E_i(x, 0))| = 0. \]

Moreover, we may further assume that \( \epsilon_{i_0} \leq \sqrt{n(2 + 2^N)}l_Q + \delta \). Then, for all \( i \geq i_0 \), by (2.6), we get:

\[
|E_i(x, z) \triangle E_i(x, 0)| \leq |B_{x, \epsilon_i} \triangle B_{x+z, \epsilon_i}| + |B_{x, \epsilon_{i+1}} \triangle B_{x+z, \epsilon_{i+1}}|
\]
\[
\lesssim |\epsilon_i + |z|^{n-1}|z| + (|\epsilon_{i+1} + |z|)|^{n-1}|z|
\]
\[
\lesssim (\sqrt{n(2 + 2^N)}l_Q + 2\delta)^{n-1}|z| \leq C_N |z|. \]

Therefore, the claim follows.

This claim and the fact \( \omega^{-\rho'/p} x_{2^N Q} \in L^1(\mathbb{R}^n) \) yield that:

\[
\sup_{\epsilon, 1} \sup_{i} \left| \int_{E_{i}(x, z) \triangle E_{i}(x, 0)} \omega(y)^{-\rho'/p} x_{2^N Q}(y) dy \right| \rightarrow 0, \text{ as } |z| \rightarrow 0.
\]

Hence:

\[
L^1_1(x) \lesssim \frac{1}{\delta N, |z|}. \tag{2.9}
\]

Combination of the above estimates for \( L^1_1 \) and \( L^2_1 \) yields that:

\[
L_1 \lesssim \|L^1_1 \|_{L^p_0(2Q)} + \|L^2_1 \|_{L^p_0(2Q)} \lesssim \rho^{(1)} + \frac{1}{\delta \beta^{(2)}_{N, |z|}}.
\]

Taking sufficient large \( N \) and sufficient small \( |z| \), we can make \( L_1 \) arbitrary small.

Now, we turn to the estimate of \( L_2 \).

\[
L_2 \lesssim \sup_{\epsilon, 1} \left( \sum_{i=1}^{\infty} \left| \int_{E_{i}(x, z) \triangle E_{i}(x, 0)} b(y)K^\delta (x + z, y)f(y) dy \right| \right)^{1/p} \left\| L^p_0(2^N Q)^c \right\|
\]
\[
+ \sup_{\epsilon, 1} \left( \sum_{i=1}^{\infty} \left| \int_{E_{i}(x, z) \triangle E_{i}(x, 0)} b(y)K^\delta (x + z, y)f(y) dy \right| \right)^{1/p} \left\| L^p_0(2^N Q) \right\|
\]
\[
=: L^1_2 + L^2_2.
\]

First, we deal with \( L^1_2 \). Recall \( |K^\delta (x + z, y)| \lesssim \frac{1}{|x + z - y|^n} \sim \frac{1}{|x|^n} \) for \( y \in Q, x \in (2^N Q)^c \).

We have:
\[ L_2^1 \leq \left\| \sup_{\varepsilon, t_0} \left( \sum_{i=1}^{\infty} \int_{E_i(x, \varepsilon) \triangle E_i(x, 0)} \chi_{E_i(x, 0)} |f(y)| |y|^{\rho} \right)^{\frac{1}{\rho}} \left\| \frac{1}{|x|} \right\|_{L^p_\omega(\mathbb{R}^n)} \right\|_{L^p_\omega((2^8 Q)^r)} \]
\[ \leq \int_Q |f(y)| dy \cdot \left\| \frac{1}{|x|} \right\|_{L^p_\omega(\mathbb{R}^n)} \]
\[ \leq \left( \int_{(2^8 Q)^r} \frac{\omega(x)}{|x|^{np}} dx \right)^{\frac{1}{p}} \left( \int_Q \omega(y)^{-1/p} dy \right)^{1/p'} =: \beta_N^{(3)}, \]

where \( \beta_N^{(3)} \to 0 \) as \( \tilde{N} \to \infty \) by (iv) in Lemma 2.3. Then, for fixed \( \tilde{N} \), by the same technique as in the estimate of \( L_1^1 \):
\[ L_2^2 \leq \frac{1}{\delta^p} \left\| \sup_{\varepsilon, t_0} \left( \sum_{i=1}^{\infty} \int_{E_i(x, \varepsilon) \triangle E_i(x, 0)} |f(y)| \chi_{E_i(x, 0)} dy \right)^{\frac{1}{\rho}} \right\|_{L^p_\omega(2^8 Q)} \]
\[ \leq \frac{1}{\delta^n} \beta_N^{(4)}, \]

where \( \beta_N^{(4)} \to 0 \) as \( |z| \to 0 \) for any fixed \( \tilde{N} \). Hence:
\[ \left\| \sup_{\varepsilon, t_0} \left( \sum_{i=1}^{\infty} |J_1(0)|^{\rho} \right)^{\frac{1}{\rho}} \right\|_{L^p_\omega(\mathbb{R}^n)} \]
\[ \leq L_1 + L_2 \leq \beta_N^{(1)} + \frac{1}{\delta^p} \beta_N^{(2)} + \beta_N^{(3)} + \frac{1}{\delta^n} \beta_N^{(4)}. \]

Therefore, we conclude that the set \( A(V_\rho(T_{K^*}^p)) \) is uniformly equicontinuous in \( L^p_\omega(\mathbb{R}^n) \), and hence, Lemma 2.2 (c) holds, which finishes the proof of Theorem 1.3 (2).

\[ \square \]

3 Necessity of compact oscillation and variation of commutators

This section is devoted to the proof of Theorem 1.4. We follow the approach of [12]. For any measurable function \( f \), let \( f^* \) be the non-increasing rearrangement of \( f \), namely, for any \( t \in (0, \infty) \):
\[ f^*(t) := \inf \{ a \in (0, \infty) : |\{ x \in \mathbb{R}^n : |f(x)| > a \} | < t \}. \]

Recall the John–Strömberg equivalence (see [14] and [23]) of a function \( f \in BMO(\mathbb{R}^n) \):
\[ \|f\|_{BMO(\mathbb{R}^n)} \sim \|f\|_{BMO(\mathbb{R}^n)} := \sup_Q a_\lambda (f; Q), \]
where for \( 0 < \lambda < 1 \), the local mean oscillation of \( f \) over a cube \( Q \) is defined by:
\[ a_\lambda(f;Q) := \inf_{c \in \mathbb{C}} ((f - c)\chi_Q)^*(\lambda |Q|). \]

In [12], the following equivalent characterization of CMO(\(\mathbb{R}^n\)) in terms of the local mean oscillation was established.

**Lemma 3.1** ([12]) Let \( f \in \text{BMO}(\mathbb{R}^n) \). Then, \( f \in \text{CMO}(\mathbb{R}^n) \) if and only if the following three conditions hold:

1. \( \lim_{r \to 0} \sup_{|Q|=r} a_\lambda(f;Q) = 0, \)
2. \( \lim_{r \to \infty} \sup_{|Q|=r} a_\lambda(f;Q) = 0, \)
3. \( \lim_{d \to \infty} \sup_{Q \cap (-d,d)^n = \emptyset} a_\lambda(f;Q) = 0. \)

To deal with the necessity conditions for the compact oscillation and variation of commutators, we recall following two lemmas from [12].

**Lemma 3.2** (lower estimates) Let \( \omega \in A_p, \lambda \in (0, 1) \) and \( b \) be a real-valued measurable function. For a given cube \( Q \), there exists a cube \( P \) with the same side length of \( Q \) satisfying \( |c_Q - c_P| = k_0\lambda Q (k_0 > 10\sqrt{n}) \), and measurable sets \( E \subseteq Q \) with \( |E| = \frac{1}{2}|Q| \), and \( F \subseteq P \) with \( |F| = \frac{1}{2}|Q| \), such that for \( f := (\int_F \omega(x)dx)^{-1/p} \chi_F \), and any measurable set \( B \) with \( |B| \leq \frac{1}{8}|\lambda Q| \):

\[ \|T_{\Omega}^b(f)\|_{L^\infty(E \setminus B)} \geq Ca_\lambda(b;Q). \] (3.2)

**Lemma 3.3** (upper estimates) Let \( b \in \text{BMO}(\mathbb{R}^n) \), \( \Omega \in L^\infty(\mathbb{S}^{n-1}) \), and \( \omega \in A_p \). For a given cube \( Q \), denote by \( F \) the set associated with \( Q \) mentioned in Lemma 3.2. Let \( f := (\int_F \omega(x)dx)^{-1/p} \chi_F \). Then, there exists a positive constant \( \delta \), such that:

\[ \|T_{\Omega}^b(f)\|_{L^\infty(2\delta^{n+1}Q,2\delta^Q)} \leq 2^{-\delta d/p}d\|b\|_{\text{BMO}(\mathbb{R}^n)} \]

for sufficient large \( d \), where the implicit constant is independent of \( d \) and \( Q \).

**Claim A:** Under the assumptions of Theorem 1.4, Lemma 3.2 is also valid if we replace \( T_{\Omega}^b \) by \( \tilde{T}_{\Omega}^b \) or by \( V_\rho(T_{\Omega}^b) \).

**Proof of Claim A** Arguing as in the proof of [12, Proposition 4.2], we see that for any given cube \( Q \), the sets \( P, E, \) and \( F \) exist. Moreover, for \( f := (\int_F \omega(x)dx)^{-1/p} \chi_F \), the following function:

\[ (b(x) - b(y)) \frac{\Omega(x - y)}{|x - y|^n} f(y) \]

does not change sign on \( E \times F \). Hence, for \( x \in E \):
\[ \mathcal{O}(T^b_\Omega)(f)(x) = \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \epsilon_i < t_i} \left| \int_{\epsilon_i + 1 < |x-y| \leq \epsilon_i} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|^{1/2} \right)^2 + |T^b_{\Omega,t_i} f(x)| \]

(3.3)

Observe that for any \( x \in E, y \in F: \)
\[ x - y \in E - F \subset Q - P \subset \{ x \in \mathbb{R}^n : |c_Q - c_P|/2 \leq |x| \leq 2|c_Q - c_P| \}. \]

By this fact and the assumption:
\[ \sup_{j \in \mathbb{Z}} \{ j : 2^j \leq |t_j| < 2^{j+1} \} < \infty, \]

there are only finite terms which are non-zero in the series in (3.3). Thus:
\[ \mathcal{O}(T^b_\Omega)(f)(x) \sim \sum_{i=1}^{\infty} \left| \int_{t_{i+1} < |x-y| \leq t_i} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right| + |T^b_{\Omega,t_i} f(x)| \]
\[ \geq \left| \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right| = |T^b_\Omega f(x)|, \]

and the implicit constant depends on \( \{ t_i \}_i \) but not on \( x, Q, P, E, \) and \( F. \) Therefore, by this fact and Lemma 3.2, (3.2) with \( T^b_\Omega \) replaced by \( \mathcal{O}(T^b_\Omega) \) holds.

On the other hand, for any \( x \in E: \)
\[ \mathcal{V}_\rho(T^b_\Omega)(f)(x) = \sup_{\epsilon_i \downarrow 0} \left( \sum_{i=1}^{\infty} \left| \int_{\epsilon_i + 1 < |x-y| \leq \epsilon_i} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|^{\rho/2} \right)^{1/\rho} \]
\[ \leq \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy = |T^b_\Omega f(x)| \]
\[ = \int_{(k_0 - \sqrt{n})Q \leq |x-y| \leq (k_0 + \sqrt{n})Q} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \]
\[ \leq \mathcal{V}_\rho(T^b_\Omega)(f)(x). \]
It follows that $\mathcal{V}_\rho(T^b_\Omega(f))(x) = |T^b_\Omega(f)(x)|$ for $x \in E$, which implies that (3.2) with $T^b_\Omega$ replaced by $\mathcal{V}_\rho(T^b_\Omega)$ holds. □

We further have the following corollary directly follows from Lemma 3.2 and Claim A.

**Corollary 3.4** Let $1 < p < \infty$, $b \in L^1_{loc}(\mathbb{R}^n)$, and $\omega \in A_p$. Let $\Omega$ be a measurable function on $\mathbb{S}^{n-1}$, which does not change sign and is not equivalent to zero on some open subset of $\mathbb{S}^{n-1}$. Then:

1. Let $\{t_j\}_{j=1}^\infty$ be a sequence with $\sup_{j \in \mathbb{Z}} |\{j : 2^j \leq |t_j| \leq 2^{j+1}\}| < \infty$, and then, the $L^p_\omega(\mathbb{R}^n)$-boundedness of $\mathcal{O}(T^b_\Omega)$ implies $b \in \text{BMO}(\mathbb{R}^n)$;
2. the $L^p_\omega(\mathbb{R}^n)$-boundedness of $\mathcal{V}_\rho(T^b_\Omega)$ implies $b \in \text{BMO}(\mathbb{R}^n)$.

Moreover, by combining Corollary 3.4, [19, Corollary 1.4] and the boundedness of $T^b_\Omega$, we have another corollary on the characterization of bounded $\mathcal{O}(T^b_\Omega)$ and $\mathcal{V}_\rho(T^b_\Omega)$.

**Corollary 3.5** Let $1 < p < \infty$, $b \in L^1_{loc}(\mathbb{R}^n)$ and $\omega \in A_p$, $\Omega \in \text{Lip}_1(\mathbb{S}^{n-1})$ and $\Omega \neq 0$. Let $\{t_j\}_{j=1}^\infty$ be a sequence with $\sup_{j \in \mathbb{Z}} |\{j : 2^j \leq |t_j| \leq 2^{j+1}\}| < \infty$ in the definition of $\mathcal{O}$. Then:

1. $b \in \text{BMO}(\mathbb{R}^n) \iff \mathcal{O}(T^b_\Omega)$ is bounded on $L^p_\omega(\mathbb{R}^n)$;
2. $b \in \text{BMO}(\mathbb{R}^n) \iff \mathcal{V}_\rho(T^b_\Omega)$ is bounded on $L^p_\omega(\mathbb{R}^n)$.

**Claim B:** Lemma 3.3 is also valid if we replace $T^b_\Omega$ by $\mathcal{O}(T^b_\Omega)$ or by $\mathcal{V}_\rho(T^b_\Omega)$.

**Proof of Claim B** It follows from the definitions of $\mathcal{O}(T^b_\Omega)$ and $\mathcal{V}_\rho(T^b_\Omega)$ that for any cube $Q$, $d \in \mathbb{N}$ large enough, $x \in 2^{d+1}Q \setminus 2^dQ$ and the function $f := (\int_F \omega(x)dx)^{-1/p} x_F$:

$$
\mathcal{O}(T^b_\Omega)(f)(x), \mathcal{V}_\rho(T^b_\Omega)(f)(x) \leq \int_{\mathbb{R}^n} |(b(x) - b(y)) \frac{\Omega(x - y)}{|x - y|^n} f(y)| dy.
$$

Then, arguing as in the proof of [12, Proposition 4.4], we have that:

$$
\left\| \int_{\mathbb{R}^n} |(b(x) - b(y)) \frac{\Omega(x - y)}{|x - y|^n} f(y)| dy \right\|_{L^p_\omega(2^{d+1}Q \setminus 2^dQ)} \lesssim 2^{-\delta_{\text{d}}/2} d \|b\|_{\text{BMO}(\mathbb{R}^n)}.
$$
where the implicit constant is independent of \(d\) and \(Q\). Then, the desired conclusion follows. \qed

**Proof of Theorem 1.4** Using Lemma 3.1, Claims A and B, the proof of Theorem 1.4 is just a repetition of the proof of [12, Theorem 1.4]. We omit the details here. \qed

**Proof of Corollary 1.5** By a similar argument as in the proof of (2) of Theorem 1.3, one can verify that \(T_{\Omega, t_1}^b\) is compact on \(L^p_0(\mathbb{R}^n)\). Then, the sufficiency follows from [19, Corollary 1.4], Theorem 1.3 and the compactness of \(T_{\Omega, t_1}^b\). The necessity follows from Theorem 1.4. \qed

### 4 Appendix

In this section, we give an example of oscillation of \(O(T_{\Omega}^b)\), such that \(b \notin \text{CMO}(\mathbb{R}^n)\) and \(O(T_{\Omega}^b)\) is compact on \(L^p_0(\mathbb{R}^n)\).

To begin with, take \(b\) to be a smooth function on \(\mathbb{R}^n\), such that:

\[
b(x) := \begin{cases} 0 & |x| \leq 1; \\ |x|^{1/2}, & |x| \geq 2.
\end{cases}
\]

One can check that \(b \notin \text{BMO}(\mathbb{R}^n)\) by:

\[
\lim_{r \to \infty} \frac{1}{r^n} \int_{[0,r]^n} |b(y) - b_{[0,r]^n}| \, dy = \infty.
\]

However, assume \(\Omega \in \text{Lip}(\mathbb{R}^n)\). We find that \(O(T_{\Omega}^b)\) is a compact operator on \(L^p_0(\mathbb{R}^n)\).

In fact, let \(\varphi\) be a smooth bump function with \(0 \leq \varphi \leq 1\), supported in the ball \(\{\xi : |\xi| < 2\}\) and be equal to 1 on the ball \(\{\xi : |\xi| \leq 1\}\). For any positive number \(N \in \mathbb{N}\), we define \(\varphi_N(x) := \varphi(x/N)\). Define:

\[
O_N(T_{\Omega}^b) := \varphi_N O(T_{\Omega}^b).
\]

We claim that \(\{O_N(T_{\Omega}^b)\}_{N \in \mathbb{N}}\) is a sequence of compact operators on \(L^p_0(\mathbb{R}^n)\). In fact, for any fixed \(N\), we have:

\[
b(x) - b(y) = b(x)\varphi_{2N+t_1}(x) - b(y)\varphi_{2N+t_1}(y) = : b_{2N+t_1}(x) - b_{2N+t_1}(y)
\]

for every \(x, y\), such that \(\varphi_N(x) \neq 0\) and \(|x - y| \leq t_1\), where \(t_1\) is the first term of the sequence \(\{t_j\}_{j=1}^{\infty}\) in the definition of \(O(T_{\Omega}^b)\). From this and the definition of \(O_N(T_{\Omega}^b)\), we have:
\[ O_N(\mathcal{T}^b_\Omega)(f)(x) = \varphi_N(x) \left( \sum_{i=1}^{\infty} \sup_{t_i + \varepsilon_i \leq s_i \leq t_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} (b(x) - b(y)) \right) \times \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \]

Observe that \( b_{2N+1} \in C_c^\infty(\mathbb{R}^n) \subset \text{CMO}(\mathbb{R}^n) \). Then, the compactness of \( O(\mathcal{T}^b_\Omega) \) follows from Theorem 1.3. Since \( \varphi_N \) is a bounded operator on \( L^p_0(\mathbb{R}^n) \) as a point-wise multiplier, the operator \( O_N(\mathcal{T}^b_\Omega) = \varphi_N(x)O(\mathcal{T}^b_\Omega) \), as the product of a bounded operator and a compact operator, is also compact on \( L^p_0(\mathbb{R}^n) \).

Finally, we claim that \( O(\mathcal{T}^b_\Omega) \) is the limit of \( O_N(\mathcal{T}^b_\Omega) \) in the sense of operator norm, as \( N \to \infty \). Then, the compactness of \( O(\mathcal{T}^b_\Omega) \) follows.

Write:

\[ |O(\mathcal{T}^b_\Omega)(f)(x) - O_N(\mathcal{T}^b_\Omega)(f)(x)| = (1 - \varphi_N(x))O(\mathcal{T}^b_\Omega)(f)(x). \]

By the definition of \( b \) and the mean value theorem, we have:

\[ (1 - \varphi_N(x))|b(x) - b(y)| \leq (1 - \varphi_N(x)) \sup_{y \in B(x,t_i)} |\nabla b(y)| \cdot |x-y| \lesssim N^{-1/2} |x-y| \]

for all \( |x-y| \leq t_i \) and \( N \geq 2t \).

Hence, for any \( x \in \mathbb{R}^n \):

\[ |(1 - \varphi_N(x))O(\mathcal{T}^b_\Omega)(f)(x)| \lesssim N^{-1/2} \left( \sum_{i=1}^{\infty} \sup_{t_i + \varepsilon_i \leq s_i \leq t_i} \int_{\varepsilon_i < |x-y| \leq \varepsilon_i} |x-y| \right) \times \frac{|\Omega(x-y)|}{|x-y|^n} \left( \frac{|f(y)|}{|x-y|^n} dy \right)^{1/2} \lesssim N^{-1/2} \int_{|x-y| \leq t_i} \frac{|f(y)|}{|x-y|^n} dy \lesssim N^{-1/2} M(f)(x) . \]

It follows that:

\[ \| (1 - \varphi_N)O(\mathcal{T}^b_\Omega)(f) \|_{L^p_0(\mathbb{R}^n)} \to_{L^p_0(\mathbb{R}^n)} \lesssim N^{-1/2} \|M(f)\|_{L^p_0(\mathbb{R}^n)} \to_0 . \]
as $N \to \infty$. We have now completed this proof.

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