BQO IS $\Pi^1_2$-COMPLETE

ALBERTO MARCONE

Abstract. In this paper we give a proof of the $\Pi^1_2$-completeness of the set of countable bqos (viewed as a subset of the Cantor space). This result was conjectured by Clote in [2] and proved by the author in his Ph.d. thesis ([7], see also [8]): here we prove it using Simpson’s definition of bqo ([16]) and as little bqo theory as possible.

1. Introduction

Let $C$ be the Cantor space $2^\omega$ of the infinite sequences of 0’s and 1’s with the product topology (where $2 = \{0, 1\}$ is given the discrete topology). Similarly $N$ is the Baire space $\mathbb{N}^\omega$ with the product topology (again $\mathbb{N}$ is endowed with the discrete topology). $C$ and $N$ are Polish spaces (complete separable metric spaces). The basic reference for these spaces from the viewpoint of descriptive set theory is Moschovakis’ monograph ([11]).

Definition 1.1. A subset of a Polish space $X$ is $\Sigma^1_1$ if it is the projection of a Borel subset of $X \times Y$ where $Y$ is a Polish space, it is $\Pi^1_n$ if it is the complement of a $\Sigma^1_n$ set and it is $\Sigma^1_{n+1}$ if it is the projection of a $\Pi^1_n$ subset of $X \times Y$.

Definition 1.2. For $n \geq 1$ we say that a set $C \subseteq X$ is $\Pi^1_n$-complete if it is $\Pi^1_n$ and for every $\Pi^1_n$ set $P \subseteq N$ there exists a continuous function $f : N \to X$ such that for every $x \in N$ we have $x \in P \iff f(x) \in C$. $f$ is called a reduction of $P$ to $C$.

A $\Pi^1_n$-complete set is true $\Pi^1_1$, i.e. it is $\Pi^1_1$ but not $\Sigma^1_1$. For an excellent survey on true $\Pi^1_n$ and $\Sigma^1_n$ sets see [4].

In this paper we will give an example of a subset of the Cantor space $C$ which is $\Pi^1_2$-complete. This set arises from combinatorics and is particularly interesting because it is a generalization of the canonical example of a $\Pi^1_2$-complete set, the set of all countable well-orderings. For other examples of $\Pi^1_2$-complete sets occurring in different branches of mathematics see [1] and the references quoted there.

A quasi-ordering (hereafter qo) consists of a set together with a binary relation on its elements which is reflexive and transitive: this definition is slightly more general than that of partial ordering (which requires also anti-symmetry) and is useful whenever there is no canonical way of choosing a representative among the elements that are equivalent under the given relation. The most natural generalization of the concept of well-ordering to qos is the notion of well quasi-ordering (hereafter wqo): a qo is a wqo if it is well founded and contains no infinite set of mutually incomparable elements. It is immediate that the notion of wqo is $\Pi^1_1$; moreover the

1991 Mathematics Subject Classification. Primary 04A15; Secondary 03E15, 06A07.

Key words and phrases. Better quasi ordering, $\Pi^1_2$-completeness.
linear orders that are wqos are exactly the well-orderings and hence the notion of wqo is $\Pi^1_1$-complete.

The concept of wqo is very natural but it does not enjoy nice closure properties: this was discovered in the 1950’s and in the 1960’s Nash-Williams ([12], [13]) proposed the stronger, but less natural, notion of better quasi-ordering (hereafter bqo). Since then bqos have become an interesting topic of research and very often a qo is proved to be a wqo by showing that it is actually a bqo (one of the most famous results of this kind is Laver’s proof of Fraïssé’s conjecture, see [6] and [16]). For surveys of wqo and bqo theory see [10] and [14]. An alternative definition of bqo, equivalent to Nash-Williams’ original one but without its combinatorial flavor, has been given by Simpson ([16]) and has proved to be very useful. Here, in contrast with [7] and [8], we will use Simpson’s definition because its more descriptive set theoretic flavor may be more appealing to the intended readers of this paper. We postpone the rather technical definition of bqo to section 2.

An immediate Tarski-Kuratowski computation shows that the set of all countable bqos (viewed as a subset of $\mathcal{C}$) is $\Pi^1_2$ and a natural conjecture made by Clote ([2]) stated that it is indeed $\Pi^1_2$-complete. We proved this conjecture in [6] and [13] using techniques originally devised to answer some questions dealing with the fine analysis of the notion of bqo. Feeling that the extraction from [6] or [13] of all details of the proof of the $\Pi^1_2$-completeness of bqo might be arduous for the reader interested more in descriptive set theory than in bqo theory, in this paper we give an exposition of the proof which uses the least possible amount of bqo theory.

In section 2 we give Simpson’s definition of bqo and prove some basic results of bqo theory which will be needed in the proof of the main theorem. In particular, using Simpson’s definition, we give a proof of Pouzet’s theorem (theorem 2.13) which allows us, as far as bqo theory is concerned, to substitute arbitrary binary relations in place of qos. In section 3 we introduce the notion of smooth subset of $[\mathbb{N}]^{<\omega}$ and prove some basic facts about it. Section 4 contains the proof of the main result of the paper, i.e. theorem 4.2.

I am very much indebted to Stephen G. Simpson, who introduced me to bqo theory, supervised my Ph.d. thesis where these results were originally proved and suggested several improvements to a previous version of this paper. I wish also to thank Maurice Pouzet, who explained to me his theorem which is an important tool in the proof of the main result of this paper, and Alessandro Andretta, whose interest in this result stimulated the writing of this paper.

2. BETTER QUASI-ORDERINGS

If $s$ is a finite sequence we denote by $\text{lh}(s)$ its length and, for every $i < \text{lh}(s)$, by $s(i)$ its $(i + 1)$-th element. We also write $s = (s(0), \ldots, s(\text{lh}(s) - 1))$, so that $\langle \rangle$ denotes the empty sequence. If $s$ and $t$ are finite sequences we write $s \sqsubseteq t$ if $s$ is an initial segment of $t$, i.e. if $\text{lh}(s) \leq \text{lh}(t)$ and $\forall i < \text{lh}(s)\; s(i) = t(i)$. $s \sqsubset t$ has the obvious meaning and we extend this notation also to the case where $t$ is an infinite sequence.

We write $s \ast t$ for the concatenation of $s$ and $t$, i.e. the sequence $u$ such that $\text{lh}(u) = \text{lh}(s) + \text{lh}(t)$, for every $i < \text{lh}(s)\; u(i) = s(i)$ and for every $i < \text{lh}(t)\; u(\text{lh}(s) + i) = t(i)$. If $s$ is a finite sequence and $i \leq \text{lh}(s)$ we denote by $s[i]$ the initial segment of $s$ of length $i$, i.e. the unique sequence $t$ such that $t \sqsubseteq s$ and $\text{lh}(t) = i$. 

2
If $A$ is an infinite set we denote by $[A]^{\omega}$ the set of all countable infinite subsets of $A$, by $[A]^{<\omega}$ the set of all finite subsets of $A$ and by $[A]^n$ the set of elements of $[A]^{<\omega}$ with $n$ elements. We identify members of $[N]^{<\omega}$ and $[N]^\omega$ with the sequences (finite or infinite) which enumerate them in increasing order. With this identification $[N]^\omega$ can be viewed as a closed subspace of $\mathcal{N}$ and is actually homeomorphic to $\mathcal{N}$ via the map which sends $n \in [N]$ to $\{ k + \sum_{i=0}^{k} \alpha(i) | k \in N \} \in [N]^\omega$. For any $A \in [N]^\omega$ we will always consider $[A]^{\omega}$ endowed with the topology arising from this identification. A basis for this topology is given by the collection of all sets of the form $N_s = \{ X \in [A]^{\omega} | s \subseteq X \}$ where $s \in [A]^{<\omega}$.

**Definition 2.1.** If $X \subseteq \mathbb{N}$ is nonempty we denote by $X^-$ the set obtained from $X$ by removing its least element.

The map $X \mapsto X^-$ is continuous from $[\mathbb{N}]^{\omega}$ in itself.

**Definition 2.2.** Let $(Q, \preceq)$ be a quasi-ordering and equip $Q$ with the discrete topology. A $Q$-array is a Borel measurable function $f : [A]^{\omega} \to Q$, where $A \in [N]^{\omega}$. $f$ is good (with respect to $\preceq$) if there exists $X \in [A]^{\omega}$ such that $f(X) \preceq f(X^-)$. If $f$ is not good we say that it is bad. $f$ is perfect if for every $X \in [A]^{\omega}$ we have $f(X) \preceq f(X^-)$.

**Definition 2.3.** $(Q, \preceq)$ is bbo if every $Q$-array is good.

In this definition no role is played by the fact that $\preceq$ is a bqo and we can replace it by any binary relation on $Q$, which we usually denote by $R$; in this case we say that $R$ is a better binary relation or bbr.

In the sequel if $f : [A]^{\omega} \to Q$ is a $Q$-array and $B$ is an infinite subset of $A$ we will call the restriction of $f$ to $B$ the $Q$-array that should be more precisely called the restriction of $f$ to $[B]^{\omega}$.

We will need the following classical result of descriptive set theory, known as the Galvin-Prikry theorem ([8]): for its proof see e.g. [10].

**Theorem 2.4.** Let $A \in [N]^{\omega}$ and suppose $B$ is a Borel subset of $[A]^{\omega}$. Then there exists $B \in [A]^{\omega}$ such that either $[B]^{\omega} \subseteq B$ or $[B]^{\omega} \cap B = \emptyset$.

**Corollary 2.5.** Let $f : [A]^{\omega} \to Q$ be a $Q$-array and $R$ be a binary relation on $Q$. Then there exists $B \in [A]^{\omega}$ such that $f$ restricted to $B$ is either bad or perfect with respect to $R$.

*Proof.* Let $B = \{ X \in [A]^{\omega} | \{(X) R (X^-) \} \}$. Since $f$ is Borel measurable and $X \mapsto X^-$ is continuous $B$ is a Borel set. By theorem 2.4 there exists $B \in [A]^{\omega}$ such that either $[B]^{\omega} \subseteq B$ or $[B]^{\omega} \cap B = \emptyset$: in the first case $f$ restricted to $B$ is perfect with respect to $R$, in the second case it is bad with respect to $R$. \qed

**Lemma 2.6.** If $R$ and $R'$ are two bbrs on the same set $Q$ then the relation $S = R \cap R'$ is also bbr.

*Proof.* Let $f$ be a $Q$-array with dom($f$) = $[A]^{\omega}$: since $R$ is bbr by corollary 2.7 $f$ restricted to some $B \in [A]^{\omega}$ is perfect with respect to $R$. Since $R'$ is bbr there exists $X \in [B]^{\omega}$ such that $f(X) R' f(X^-)$ and hence $f(X) S f(X^-)$. \qed

A consequence of the Galvin-Prikry theorem is the following result, which follows from the results of section 6 of [8] and is proved in [10].
Theorem 2.7. Let $A \in [N]^\omega$ and suppose $Y$ is a metric space and $f: [A]^\omega \to Y$ is a Borel measurable function. Then there exists $B \in [A]^\omega$ such that the restriction of $f$ to $B$ is continuous.

Corollary 2.8. A binary relation $R$ on a set $Q$ is bbr if and only if every continuous $Q$-array is good with respect to $R$.

Proof. One direction of the equivalence is trivial. For the other, if $f: [A]^\omega \to Q$ is a $Q$-array by theorem 2.7 (since $Q$ with the discrete topology is metrizable) there exists $B \in [A]^\omega$ such that the restriction of $f$ to $B$ is continuous. By hypothesis $f$ restricted to $B$ is good and hence $f$ is good.

One of the basic tools for showing that a qo is bqo is a theorem known as the minimal bad array lemma. It is implicit in Nash-Williams’ work and its present formulation (in terms of Borel measurable functions) was given by Simpson ($[18]$). The proof we give here is due to van Engelen, Miller and Steel ($[3]$). Before proving the minimal bad array lemma we prove a combinatorial lemma that is needed in its proof and make two preliminary definitions which are needed for its statement.

Lemma 2.9. Let $\omega_1$ denote the first uncountable ordinal and $\{ A_\alpha \mid \alpha < \omega_1 \}$ be a sequence of elements of $[N]^\omega$ such that $\alpha < \beta < \omega_1$ implies that $A_\beta \setminus A_\alpha$ is finite. Then there exist $B \in [N]^\omega$ and $I \in [\omega_1]^\omega$ such that $B \subseteq \bigcap_{\alpha \in I} A_\alpha$.

Proof. Notice that the hypothesis of the lemma implies that if $s \in [\omega_1]^{< \omega}$ and $\beta = \max s$ then

$$\bigcap_{\alpha \in s} A_\alpha = A_\beta \setminus \bigcup_{\alpha \in s \setminus \{ \beta \}} (A_\beta \setminus A_\alpha)$$

is infinite.

We define by induction $I_n \in [\omega_1]^n$, $B_n \in [N]^n$ and $V_n \subseteq \omega_1$ uncountable such that

$$I_n \subseteq I_{n+1}, B_n \subseteq B_{n+1}, B_n \subseteq \bigcap_{\alpha \in I_n} A_\alpha, B_n \subseteq \bigcap_{\alpha \in V_n} A_\alpha$$

and $I_n \cap V_n = \emptyset$.

To complete the proof it will then suffice to let $I = \bigcup_n I_n$ and $B = \bigcup_n B_n$.

We start by setting $I_0 = B_0 = \emptyset$ and $V_0 = \omega_1$. Supposing $I_n$, $B_n$ and $V_n$ have already been defined let $\alpha_n$ be the least element of $V_n$ and set $I_{n+1} = I_n \cup \{ \alpha_n \}$. For every $\beta \in V_n$ let $C_\beta = A_\beta \cap \bigcap_{\alpha \in I_{\beta+1}} A_\alpha$. By the remark at the beginning of the proof $C_\beta$ is infinite and hence there exists $m_\beta \in C_\beta \setminus B_n$. Since $V_n$ is uncountable there exist $m \in N$ and an uncountable $V_{n+1} \subseteq V_n \setminus \{ \alpha_n \}$ such that for every $\beta \in V_{n+1}$ we have $m_\beta = m$. Let $B_{n+1} = B_n \cup \{ m \}$. It is easy to check that $I_{n+1}$, $B_{n+1}$ and $V_{n+1}$ satisfy all the conditions.

Definition 2.10. Let $(Q, \leq')$ be a qo: if $f$ and $g$ are $Q$-arrays with domains $[A]^\omega$ and $[B]^\omega$ respectively, we write $f \preceq' g$ if $B \subseteq A$ and for all $X \in [B]^\omega$ we have that $g(X) \preceq' f(X)$. We write $f \prec' g$ if $B \subseteq A$ and for all $X \in [B]^\omega$ we have that $g(X) \prec' f(X)$ (i.e. $g(X) \preceq' f(X)$ and not $f(X) \preceq' g(X)$).

Definition 2.11. Let $(Q, \leq)$ be a qo: another qo $\leq'$ on $Q$ is compatible with $\leq'$ if it is well founded and $q_0 \preceq' q_1$ implies $q_0 \leq q_1$. In this setting we say that the $Q$-array $f$ is minimal bad if it is bad (with respect to $\leq'$) and every $Q$-array $g$ satisfying $g \prec' f$ is good.
Now we can state and prove the minimal bad array lemma.

**Theorem 2.12.** Let \( \preceq \) and \( \preceq' \) be qos on \( Q \) such that \( \preceq' \) is compatible with \( \preceq \). If \( f \) is a \( Q \)-array which is bad (with respect to \( \preceq \)) then there exists a minimal bad \( Q \)-array \( g \) such that \( g \preceq' f \).

**Proof.** Suppose the theorem fails: we define by induction a sequence of bad \( Q \)-arrays \( \{ f_\alpha : [A_\alpha]^\omega \to Q \mid \alpha < \omega_1 \} \) such that if \( \alpha < \beta < \omega_1 \) then \( f_\alpha \preceq' f_\beta \), \( A_\beta \setminus A_\alpha \) is finite and for every \( X \in [A_\alpha \cap A_\beta]^\omega \) we have \( f_\beta(X) \prec f_\alpha(X) \). An application of lemma 2.9 then easily shows that \( \preceq' \) is not well founded, a contradiction.

Start by letting \( f_0 = f \). The definition of \( f_\alpha \) is immediate when \( \alpha \) is a successor ordinal: if \( \alpha = \beta + 1 \) it suffices to take \( f_\alpha \prec f_\beta \) bad.

If \( \alpha < \omega_1 \) is a limit ordinal and \( f_\beta \) has been defined for all \( \beta < \alpha \) we begin by constructing a bad \( Q \)-array \( g \) as follows. Since \( \alpha \) is countable we can reorder \( \{ A_\beta \mid \beta < \alpha \} \) as \( \{ B_n \mid n \in \mathbb{N} \} \) with \( B_0 = A_0 \). As in the proof of lemma 2.9 we have that for every \( k \in \mathbb{N} \) the set \( \bigcap_{n \leq k} B_n \) is infinite. Let \( m_0 \) be the least element of \( B_0 \) and \( m_{k+1} \) be the least element of \( \bigcap_{n \leq k+1} B_n \) which is larger than \( m_k \). Then \( A = \{ m_k \mid k \in \mathbb{N} \} \) satisfies \( A \subseteq A_0 \) and \( A \setminus A_\beta \) finite for every \( \beta < \alpha \). We define \( g : [A]^\omega \to Q \) by \( g(X) = f_\beta(X) \) where \( \beta \) is maximal such that \( X \in [A_\beta]^\omega \) (notice that for every \( X \in [\mathbb{N}]^\omega \) the set \( \{ \beta < \alpha \mid X \in [A_\beta]^\omega \} \) is finite because \( \preceq' \) is well founded).

**Sublemma 2.12.1.** \( g \) is a bad \( Q \)-array.

**Proof.** To see that \( g \) is Borel measurable it suffices to recall that \( Q \) is given the discrete topology and notice that for every \( q \in Q \) and \( X \in [A]^\omega \) we have \( g(X) = q \) if and only if

\[
\exists \beta < \alpha (X \in [A_\beta]^\omega \land X \in f_\beta^{-1}(\{q\}) \land \forall \beta' (\beta < \beta' < \alpha \land X \notin [A_{\beta'}]^\omega))
\]

and hence \( g^{-1}(\{q\}) \) is a Borel set in \([A]^\omega\).

To see that \( g \) is bad suppose that for some \( X \in [A]^\omega \) we have \( g(X) \preceq g(X^-) \): if \( g(X) = f_\beta(X) \) and \( g(X^-) = f_{\beta'}(X^-) \), from \( X^- \subseteq X \) follows \( \beta \leq \beta' \) and therefore \( f_{\beta'}(X^-) \preceq' f_\beta(X^-) \). Since \( \preceq' \) is compatible with \( \preceq \) we have \( f_{\beta'}(X^-) \preceq f_\beta(X^-) \) and hence \( f_\beta(X) \preceq f_{\beta'}(X^-) \), contradicting the badness of \( f_\beta \).

Applying the successor step to \( g \) we obtain \( f_\alpha : [A_\alpha]^\omega \to Q \) which is bad and such that \( f_\alpha \prec g \). Since for every \( \beta < \alpha \) and \( X \in [A \cap A_\beta]^\omega \) we have either \( g(X) = f_\beta(X) \) or \( g(X) \prec f_\beta(X) \) it follows that \( f_\alpha(X) \prec f_\beta(X) \) for every \( X \in [A_\alpha \cap A_\beta]^\omega \) and in particular \( f_\alpha \prec f_\beta \). This completes our construction and the proof of the theorem.

The following result allows us, whenever we are looking for a bqo, to look instead for a bbr knowing that inside it we will find a bqo. It is a consequence of a sharper theorem due to Pouzet ([3], see also [2] or [4]): here we translate Pouzet’s proof in the terminology of Simpson’s definition of bqo.

**Theorem 2.13.** Let \( R \) be a binary relation on a countable set \( Q \). Then there exists a partial ordering \( \preceq \) of \( Q \) such that \( \preceq \subseteq R \) (as subsets of \( Q \times Q \)) and such that \( \preceq \) is bbr if and only if \( R \) is bbr.

If we fix an enumeration of \( Q \) we can view \( R \) and \( \preceq \) as elements of \( \mathcal{C} \times \mathcal{C} \cong \mathcal{C} \): then the function \( R \mapsto \preceq \) is continuous.
Proof. Let \( \{q_n\} \) be an enumeration of \( Q \) and let \( \# : Q \to \mathbb{N} \) be the function such that \( \#(q_n) = n \). We define \( \preceq \) by primitive recursion as follows: if \( m \leq n \) suppose we have already established whether \( q_i \preceq q_j \) for all \( i < m \) and \( j \leq n \) and set \( q_m \preceq q_n \) if and only if \( g_m R q_n \) and \( \forall i < m (q_i \preceq q_m \implies q_i \preceq q_n) \). If \( m > n \) then \( q_m \preceq q_n \) does not hold.

It is clear that \( \preceq \) is a partial ordering and its definition is continuous (indeed uniformly primitive recursive) in \( Q \) and \( R \) and that \( \preceq \subseteq R \). If \( \preceq \) is bqo then (since \( \preceq \subseteq R \) \( R \) is bbr.

Now suppose that \( \preceq \) is not bqo: since \( \preceq \) is well founded it is compatible with itself and by theorem \( 2.12 \) there exists a minimal bad \( Q \)-array \( f \). We may assume that \( \text{dom}(f) = [N]^\omega \) (this is not restrictive since \( [A]^\omega \) is homeomorphic to \( [N]^\omega \) for every \( A \in [N]^\omega \)).

Define \( h : [N]^\omega \to [Q]^{<\omega} \) by \( h(X) = \{ q \in Q \mid q < f(X) \} \): we have \( h(X) \in [Q]^{<\omega} \) because it has at most \( \#(f(X)) \) elements. We define a partial ordering on \( [Q]^{<\omega} \) by setting \( a \preceq^* b \) if and only if \( \forall q \in a \exists q' \in b \ q \preceq q' \).

\textbf{Sublemma 2.13.1.} There exists \( A \in [N]^\omega \) such that \( h \) restricted to \( A \) is perfect with respect to \( \preceq^* \).

\textit{Proof.} By corollary \( 2.5 \) there exists \( A \) such that \( h \) restricted to \( A \) is either perfect or bad. Suppose the latter holds and notice that \( \preceq \) is well founded on \( [Q]^{<\omega} \) and compatible with \( \preceq^* \). By theorem \( 2.12 \) there exists \( h' \subseteq h \) which is minimal (with respect to \( \preceq \)) bad (with respect to \( \preceq^* \)). Let \( \text{dom}(h') = [A']^\omega \): since \( h' \) is bad \( h'(X) \neq \emptyset \) for every \( X \in [A']^\omega \) and we can define \( g(X) \) to be an element of \( h'(X) \) (e.g. the first one to appear in \( \{q_n\} \)) and \( h''(X) = h'(X) \setminus \{g(X)\} \). The minimality of \( h' \) and corollary \( 2.3 \) imply that \( h'' \) restricted to some \( A'' \in [A']^\omega \) is perfect. Therefore \( g \) restricted to \( A'' \) must be bad. But for all \( X \in [A'']^\omega \) we have \( g(X) \in h'(X) \subseteq h(X) \) and hence \( g(X) \prec f(X) \), that is \( g \prec f \) violating the minimality of \( f \).

Let \( A \) be given by the sublemma and consider \( f \) restricted to \( A \): suppose towards a contradiction that \( R \) is bbr we have by lemma \( 2.4 \) that also the intersection of \( R \) with \( \{ (q, q') \mid \#(q) \leq \#(q') \} \) is bbr, and therefore there exists \( X \in [A]^\omega \) such that \( f(X) R f(X^-) \) and \( \#(f(X^-)) \leq \#(f(X^-)) \). Since \( h(X^-) \preceq h(X^-) \) we have that for every \( q \prec f(X) \) there exists \( q' \prec f(X^-) \) such that \( q \preceq q' \) and hence
\[
\forall i < \#(f(X)) (q_i \preceq f(X) \implies q_i \preceq f(X^-))
\]
Therefore \( f(X) \preceq f(X^-) \), against the badness of \( f \).

3. Smooth subsets of \([N]^{<\omega}\)

In this section we study the notion of smooth subset of \([N]^{<\omega}\), which was introduced in \( 6 \) and \( 7 \) to solve some problems in the fine theory of bqos. Every \( C \subseteq [N]^{<\omega} \) can be viewed as a code for a collection of basic open (indeed clopen) subsets of \([N]^\omega \) (namely \( \{N_s\}_{s \in C} \)).

\textbf{Definition 3.1.} If \( C \subseteq [N]^{<\omega} \) let \( \text{base}(C) = \{ n \mid \exists s \in C \ n \in s \} \).

\( C \) is a block if \( \text{base}(C) \) is infinite, the elements of \( C \) are mutually incomparable under \( \supseteq \) and \( \forall X \in [\text{base}(C)]^\omega \exists s \in C \ s \supseteq X \).

It is immediate to check that \( C \) is a block if and only if \( \{N_s\}_{s \in C} \) is a partition of \([\text{base}(C)]^\omega \). The following lemma shows the connection between blocks and \( Q \)-arrays.

6
Lemma 3.2. If \( f : [A]^{\omega} \rightarrow Q \) is a continuous \( Q \)-array there exists a block \( B \) with base(\( B \)) = \( A \) such that \( f \) is constant on \( N_s \) for each \( s \in B \).

Proof. If \( X \in [A]^{\omega} \), since \( \{f(X)\} \) is open in \( Q \), there exists \( n > 0 \) such that \( f(Y) = f(X) \) for every \( Y \in N_{X[n]} \); denote by \( n(X) \) the least such \( n \) and let \( B = \{X[n(X)] \mid X \in [A]^{\omega}\} \).  

Definition 3.3. If \( s, t \in [N]^{<\omega} \) let \( t \subseteq s \) mean that \( \text{lh}(t) = \text{lh}(s) \) and for all \( i < \text{lh}(s) \) \( t(i) \leq s(i) \).

A set \( C \subseteq [N]^{<\omega} \) is smooth if for all \( s, t \in C \) such that \( \text{lh}(s) < \text{lh}(t) \) there exists \( i < \text{lh}(s) \) such that \( s(i) < t(i) \), i.e. \( t[\text{lh}(s)] \ll s \) does not hold.

We will use the following terminology for trees.

Definition 3.4. A set \( T \subseteq [N]^{<\omega} \) is a tree if it is closed under initial segments, i.e. \( s \in T \) and \( t \subseteq s \) imply \( t \in T \). \( T \) is well founded if for every \( X \in [\text{base}(T)]^{\omega} \) there exists \( n \) such that \( X[n] \notin T \).

If we have a subset of \([N]^{<\omega}\) we will turn it into a smooth subset by the following procedure.

Definition 3.5. If \( C \subseteq [N]^{<\omega} \) let \( A = \text{base}(C) \) and define
\[
T(C) = \{ s \in [A]^{<\omega} \mid \forall t \subseteq s \ t \notin C \} \\
T^*(C) = \{ s \in [A]^{<\omega} \mid \exists t \in T(C) \ t \ll s \} \\
C^* = \{ s \in [A]^{<\omega} \mid s \notin T^*(C) \land \forall t \subseteq s \ t \in T^*(C) \}
\]

Notice that both \( T(C) \) and \( T^*(C) \) are trees and \( T(C) \subseteq T^*(C) \).

Lemma 3.6. \( C^* \) is smooth for every \( C \subseteq [N]^{<\omega} \).

Proof. Assume, towards a contradiction, that there exist \( s, s' \in C^* \) such that \( \text{lh}(s) < \text{lh}(s') \) and \( s'[\text{lh}(s')] \ll s \). \( s'[\text{lh}(s') - 1] \in T^*(C) \) implies that for some \( t' \in T(C) \) we have \( t' \ll s'[\text{lh}(s') - 1] \). Let \( t = t'[\text{lh}(s)] \); then \( t \in T(C) \) and \( t \ll s \) which entail \( s \in T^*(C) \). This contradicts \( s \in C^* \).

Lemma 3.7. If \( C \) is a block then:
1. \( T(C) \) and \( T^*(C) \) are well founded trees.
2. For every \( t \in C \) there exists \( s \in C^* \) such that \( t \subseteq s \) and hence \( N_s \subseteq N_t \).
3. \( C^* \) is a block.

Proof. (1) Let \( A = \text{base}(C) \). Since \( \forall X \in [A]^{\omega} \ \exists s : C \ s \ X \) it is obvious that \( T(C) \) is well founded. Suppose \( T^*(C) \) is not well founded and let \( X \in [A]^{\omega} \) be such that \( \forall i \ X[i] \in T^*(C) \). Then for every \( i \) there exists \( t \in T(C) \) such that \( t \ll X[i] \). Hence the tree \( \{t \in T(C) \mid t \subseteq X[\text{lh}(t)]\} \) is an infinite finitely branching tree (a node \( t \) has at most \( X[\text{lh}(t)] + 1 \) immediate successors) and by König’s lemma is not well founded, which is impossible because \( T(C) \) is well founded.

(2) If \( t \in C \) then, since the elements of \( C \) are incomparable under \( \subseteq \), \( t[\text{lh}(t) - 1] \in T(C) \subseteq T^*(C) \) by (1) there exists \( s \in C^* \) such that \( t \subseteq s \).

(3) By (1) \( \text{base}(C^*) = \text{base}(C) \) and hence \( \text{base}(C^*) \) is infinite. It is clear that two elements of \( C^* \) are incomparable under \( \subseteq \). Let \( X \in [A]^{\omega} \); by (1) there exists \( k \) such that \( X[k] \notin T^*(C) \). If \( k \) is minimal then \( X[k] \in C^* \).
The binary relation we will now introduce is basic to Nash-Williams’ combinatorial definition of bqo, but here will be employed only as a technical tool in the proof of theorem 4.1.

**Definition 3.8.** Let \( s, t \in \mathbb{N}^\omega \); we write \( s \prec t \) if there exists \( u \in \mathbb{N}^\omega \) such that \( s \subseteq u \) and \( t \subseteq u^- \) or, equivalently, if there exists \( X \in \mathbb{N}^\omega \) such that \( X \in N_s \) and \( X^- \in N_t \).

The following lemmas follow immediately from the definitions.

**Lemma 3.9.** \( s \prec t \) implies \( s(i) < t(i) \) and \( s(i + 1) = t(i) \) whenever these expressions make sense.

**Lemma 3.10.** If \( C \) is smooth and \( s, t \in C \) are such that \( s \prec t \) then \( \text{lh}(s) \leq \text{lh}(t) \).

In the proof of theorem 4.1 we will need the following result about \( C \)-arrays for smooth sets \( C \) with the binary relation \( \prec \).

**Lemma 3.11.** Suppose \( C \subseteq \mathbb{N}^\omega \) is smooth and \( f : [A]^\omega \to C \) is a continuous \( C \)-array which is perfect with respect to \( \prec \). By lemma 3.2 let \( B \) be a block with base(\( B \)) = \( A \) such that \( f \) has constant value \( \tilde{f}(s) \) on \( N_s \) for each \( s \in B \). Then for every \( s \in B \) we have:

1. \( \text{lh}(\tilde{f}(s)) \leq \text{lh}(s) \);
2. \( \forall i < \text{lh}(\tilde{f}(s)) \ s(i) \leq \tilde{f}(s)(i) \), i.e. \( s[\text{lh}(\tilde{f}(s))] \leq \tilde{f}(s) \).

**Proof.** Without loss of generality we may assume that \( A = \mathbb{N} \). Recalling the definition of \( \prec \) notice that \( \tilde{f} \) perfect implies \( \forall s, t \in B \ (s \prec t \implies \tilde{f}(s) \prec \tilde{f}(t)) \).

(1) Let \( \text{lh}(s) = k \): for every \( i \leq k \) there exist unique \( s_i, s'_i \in B \) such that

\[
\begin{align*}
  s_i &\subseteq \langle s(i), \ldots, s(k - 1), s(k - 1) + 1, s(k - 1) + 2, \ldots \rangle \\
  s'_i &\subseteq \langle s(i), \ldots, s(k - 1), s(k - 1) + 2, s(k - 1) + 3, \ldots \rangle
\end{align*}
\]

Therefore \( s = s_0 \), \( s_k \prec s'_k \) and for every \( i < k \) \( s_i \prec s_{i + 1} \) and \( s'_i \prec s'_{i + 1} \). Suppose that \( \text{lh}(\tilde{f}(s)) > k \) so that \( \tilde{f}(s)(k) \) exists: \( C \) is smooth and by lemma 3.10 (insuring that all the sequences involved are long enough) and lemma 3.3 we have

\[
\begin{align*}
  \tilde{f}(s)(k) &= \tilde{f}(s_1)(k - 1) = \cdots = \tilde{f}(s_k)(0) \\
  \tilde{f}(s)(k) &= \tilde{f}(s'_1)(k - 1) = \cdots = \tilde{f}(s'_k)(0)
\end{align*}
\]

Therefore \( \tilde{f}(s_k)(0) \not= \tilde{f}(s'_k)(0) \), contradicting \( \tilde{f}(s_k) \prec \tilde{f}(s'_k) \).

(2) We prove this simultaneously for all \( s \in B \) by induction on \( i \). If \( i = 0 \), denoting again \( \text{lh}(s) \) by \( k \), for every \( j \leq s(0) \) let \( s_j \in B \) be such that

\[
\begin{align*}
  s_j &\subseteq \langle s(0) - j, s(0) - j + 1, \ldots, s(0) - 1 \rangle \prec s^- \langle s(k - 1) + 1, s(k - 1) + 2, \ldots \rangle
\end{align*}
\]

Therefore \( s_0 = s \) and \( s_{j + 1} \prec s_j \) for every \( j < s(0) \). By lemma 3.9 we have

\[
\begin{align*}
  \tilde{f}(s)(0) &\geq \tilde{f}(s_1)(0) + 1 \geq \cdots \geq \tilde{f}(s_{s(0)})(0) + s(0) \geq s(0)
\end{align*}
\]

Now suppose \( i + 1 < \text{lh}(\tilde{f}(s)) \) and pick \( t \in B \) such that \( s \prec t \) (such a \( t \) exists because \( B \) is a block): we have \( \tilde{f}(s) \prec \tilde{f}(t) \) and, by lemma 3.10, \( i < \text{lh}(\tilde{f}(t)) \). By lemma 3.3 and the induction hypothesis \( \tilde{f}(s)(i + 1) = \tilde{f}(t)(i) \geq t(i) \) \( = s(i + 1) \). □
4. Proof of the $\Pi^1_1$-completeness of $\mathbf{bqo}$

**Theorem 4.1.** Let $P \subset \mathcal{N}$ be a $\Pi^1_2$ set. For each $x \in \mathcal{N}$ we can define a countable set $Q_x$ and a reflexive binary relation $R_x$ on $Q_x$ such that $x \in P$ if and only if $R_x$ is a bbr. Moreover, if we view $(Q_x, R_x)$ as an element of $\mathcal{C} \times \mathcal{C} \cong \mathcal{C}$, the map $x \mapsto (Q_x, R_x)$ is continuous.

**Proof.** By the representation theorem for $\Sigma^1_1$ sets ([11]) and using the fact that $[\mathbb{N}]^\omega$ is isomorphic to $\mathcal{N}$ there exists $C \subset \bigcup_n \omega^n \times 2^n \times [\mathbb{N}]^n$ such that

\[
x \in P \iff \forall y \in C \exists \mathcal{X} \in [\mathbb{N}]^\omega \forall (y \setminus \{\}, \{\}, \mathcal{X} \setminus \{\}) \notin \mathcal{C}
\]

We suppose that $(\{\}, \{\}, \{\}) \notin \mathcal{C}$ (otherwise $P = \emptyset$ and for every $x$ we can take $R_x$ to be a fixed non-bbr relation). Moreover we can suppose that $C$ consists of sequences incomparable under $\sqsubset \sqsubset \sqsubset$.

Throughout this proof $\sigma$ and $\tau$ will denote sequences in $2^{<\omega}$, while $s$ and $t$ will denote elements of $[\mathbb{N}]^{<\omega}$.

Let $x \in \mathcal{N}$ be fixed and define $C_x = \{ (\sigma, s) \mid (x[\text{lh}(\sigma)], \sigma, s) \in C \}$ and

\[
D_x = \{ (\sigma, s) \mid \text{lh}(\sigma) = \text{lh}(s) \land \forall t \ll s \exists i \leq \text{lh}(s) \ (\sigma[i], t[i]) \in C_x \}
\]

$D_x$ is countable and can be viewed as an element of $\mathcal{C}$: then the map $x \mapsto D_x$ is continuous because for every $s$ the set $\{ t \mid t \ll s \}$ is finite. Let

\[
Q_x = \{ (\sigma, s) \in D_x \mid \forall i < \text{lh}(s) \ (\sigma[i], s[i]) \notin D_x \}
\]

and define a binary relation $R_x$ on $Q_x$ by:

\[
(\sigma, s) R_x (\tau, t) \iff \sigma \nsubseteq \tau \land s \nsubseteq t
\]

Notice that the map $x \mapsto (Q_x, R_x)$ is continuous.

To complete the proof of the theorem by showing that $x \in P$ if and only if $R_x$ is bbr we need a couple of definitions and a sublemma.

For any $y \in \mathcal{C}$ let $C_{x,y} = \{ s \mid (y[\text{lh}(s)], s) \in C_x \}$. Notice that the elements of $C_{x,y}$ are incomparable under $\sqsubset \sqsubset \sqsubset$ and, using definition 3.5, construct $C_{x,y}^*$. 

**Sublemma 4.1.1.** For every $y \in \mathcal{C}$ and $s \in [\mathbb{N}]^{<\omega}$ we have $(y[\text{lh}(s)], s) \in Q_x$ if and only if $s \in C_{x,y}^*$.

**Proof.** If $(y[\text{lh}(s)], s) \in Q_x$ then $(y[\text{lh}(s)], s) \in D_x$ and $\forall i < \text{lh}(s) (y[i], s[i]) \notin D_x$: therefore for every $t \ll s$ there exists $i \leq \text{lh}(s)$ such that $(y[i], t[i]) \in C_x$ and for at least one $t$ this $i$ is $\text{lh}(s)$. Hence $\forall t \ll s \exists i \leq \text{lh}(s) t[i] \in C_{x,y}$ and $\exists t \ll s \in C_x$. Therefore $\forall t \ll s \in T(C_{x,y})$ and $\exists t \ll s \in C_{x,y}$.

If $s \in C_{x,y}^*$ we have $s \nsubseteq T^*(C_{x,y})$. Hence if $t \ll s$ we have $t \notin T(C_{x,y})$ and there exists $i \leq \text{lh}(s)$ such that $t[i] \in C_{x,y}$: this means $(y[i], t[i]) \in C_x$. On the other hand for all $i < \text{lh}(s)$ we have $s[i] \in T^*(C_{x,y})$ and hence there exists $t \ll s[i]$ such that $t \in T(C_{x,y})$: for every $j \leq i$ we have $t[j] \notin C_{x,y}$ and hence $(y[j], t[j]) \notin C_x$. This shows that $(y[i], s[i]) \notin D_x$ and completes the proof that $(y[\text{lh}(s)], s) \in Q_x$.

First assume that $x \notin P$: for some $y \in \mathcal{C}$ we have $\forall X \in [\mathbb{N}]^\omega \exists n X[n] \in C_{x,y}$. Hence $C_{x,y}$ is a block and, by lemma 3.3, $C_{x,y}^*$ is also a block. If $X \in [\mathbb{N}]^\omega$ denote by $s_X$ the unique element of $C_{x,y}^*$ such that $s_X \sqsubseteq X$, so that $s_X \nless s_{X'}$. Define $f : [\mathbb{N}]^\omega \to Q_x$ by $f(X) = (y[\text{lh}(s_X)], s_X)$. By the sublemma $f(X) \in Q_x$ and it is
immediate, using lemma 3.10, to check that \( f \) is a bad continuous \( Q_x \)-array with respect to \( R_x \) and hence \( R_x \) is not bbr.

Suppose \( x \in P \) but \( R_x \) is not bbr: by corollary 2.8 there exists a continuous bad \( Q_x \)-array \( f : [\mathbb{N}]^\omega \rightarrow Q_x \). Let \( B \) and \( \tilde{f} \) be as in the statement of lemma 2.11 so that \( \tilde{f} : B \rightarrow Q_x \). Let us write \( \tilde{f}(s) = (g(s), h(s)) \). The badness of \( f \) implies that whenever \( s, t \in B \) are such that \( s \not< t \) we have (a) \( g(s) \subseteq g(t) \) and (b) \( h(s) < h(t) \).

If \( \not<^* \) is the transitive closure of \( \not< \) and \( s, t \in B \) there exists \( u \in B \) such that \( s \not<^* u \) and \( t \not<^* u \). By (a) we have \( g(s) \subseteq g(u) \) and \( g(t) \subseteq g(u) \) so that either \( g(s) \subseteq g(t) \) or \( g(t) \subseteq g(s) \). It follows that there exists \( y \in C \) such that \( \forall s \in B \ g(s) \subseteq y \) and hence \( \tilde{f}(s) = (y[\text{lh}(h(s))], h(s)) \in Q_x \).

By the sublemma the range of \( h \) is a subset of \( C_{x,y} \). Since \( x \in P \) there exists \( X \in [\mathbb{N}]^\omega \) such that \( \forall n (y[n], X[n]) \notin C_x \). If \( s \in B \) is such that \( s \subseteq X \), by (b) and since \( C_{x,y} \) is smooth by lemma 3.3, lemma 3.11 applies and \( X[\text{lh}(h(s))] \subsetneq h(s) \); it is then obvious that \( f(X) = (y[\text{lh}(h(s))], h(s)) \notin D_x \), a contradiction.

The following theorem is the main result of the paper.

**Theorem 4.2.** Let \( BQO \) be the set of all (codes for) bqsos on \( \mathbb{N} \). \( BQO \) is a \( \Pi^1_2 \)-complete subset of \( C \).

**Proof.** By lemma 3.2 a \( \mathbb{N} \)-array \( f : [A]^\omega \rightarrow \mathbb{N} \) is continuous if and only if there exists a block \( B \) with base \( B = A \) such that \( f \) is constant on \( N_s \) for every \( s \in B \): the set of all blocks is a \( \Pi^1_1 \) subset of \( C \) and using corollary 2.8 it is immediate to see that \( BQO \) is a \( \Pi^1_1 \) subset of \( C \).

To show that \( BQO \) is \( \Pi^1_2 \)-complete, given a \( \Pi^1_2 \) set \( P \subseteq \mathbb{N} \) it suffices to apply theorem 4.1 followed by theorem 2.13 to obtain a reduction of \( P \) to \( BQO \). \( \square \)

**References**

[1] H. Becker, *Descriptive set theoretic phenomena in analysis and topology*, in *Set Theory of the Continuum*, edited by H. Judah, W. Just and H. Woodin, Springer-Verlag, Berlin, 1992, pp. 1–25.

[2] P. Clote, *The metamathematics of Fraïssé’s order type conjecture*, in *Recursion Theory Week*, edited by K. Ambos-Spies, G. H. Müller and G. E. Sacks, Springer-Verlag, Berlin, 1990, pp. 41–56.

[3] F. van Engelen, A. W. Miller and J. Steel, *Rigid Borel sets and better quasiorder theory*, in *Logic and Combinatorics*, edited by S. G. Simpson, American Mathematical Society, Providence, 1987, pp. 199–222.

[4] F. Galvin and K. Prikry, *Borel sets and Ramsey’s theorem*, J. Symbolic Logic 38 (1973), 193–198.

[5] *Graphs and Orders*, edited by I. Rival, D. Reidel, Boston, 1985.

[6] R. Laver, On Fraïssé’s order type conjecture, *Ann. of Math.* 93 (1971), 89–111.

[7] A. Marcone, *Foundations of bqo theory and subsystems of second order arithmetic*, Ph.d. thesis, The Pennsylvania State University, 1993.

[8] A. Marcone, *Foundations of bqo theory*, Trans. Amer. Math. Soc., to appear.

[9] A. R. D. Mathias, *Happy families*, Ann. Math. Logic 12 (1977), 59–111.

[10] E. C. Milner, *Basic uqa- and bqo-theory*, in [11], pp. 487–502.

[11] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[12] C. St. J. A. Nash-Williams, On well-quasi-ordering infinite trees, Proc. Cambridge Philos. Soc. 61 (1965), 697–720.

[13] C. St. J. A. Nash-Williams, On better-quasi-ordering transfinite sequences, Proc. Cambridge Philos. Soc. 64 (1968), 273–290.

[14] M. Pouzet, Applications of well quasi-ordering and better quasi-ordering, in *Graphs and posets with no infinite independent set*, in *Finite and Infinite Combinatorics in Sets and Logic*, edited by N. W. Sauer et al., Kluwer, 1993, pp. 313–335.
[16] S. G. Simpson, Bqo-theory and Fraïssé’s conjecture, chapter 9 of: R. Mansfield and G. Weitkamp, Recursive Aspects of Descriptive Set Theory, Oxford University Press, New York, 1985.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10, 10123 TORINO, ITALY
E-mail address: marcone@dm.unito.it