Epistemic Łukasiewicz logic of partial knowledge

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Abstract
We offer a new logic, called Epistemic Łukasiewicz logic of partial knowledge that is represented as multimodal epistemic Łukasiewicz logic $K\mathcal{L}P(n)$ with $n$ knowledge operators $\Box_i$ ($1 \leq i \leq n$) interpreted in a non-archimedean monadic $MV$-algebra. We choose knowledge operators, which can be estimated by some grading (different kinds of knowledge): absolute knowledge or partial knowledge. We consider a very special type of partial knowledge. Actually, we take infinitesimal elements (the radical) of perfect $MV$-algebras as a range of this estimation. The choice of infinitesimal elements seems suitable for actual situations like a measure of partial information.

Keywords Agent · Many valued logic · Epistemic logic

1 Introduction
In many studies of distributed systems, a multiagent model is used. An agent can be a processor, sensor or finite state machine, interconnected by a communication network with other ‘agents’. Typically each agent has a local state that is a function of its initial state, the messages received from other agents, observations of the external environment and possible internal actions. It has become customary when using formal models of distributed systems to use modal epistemic logic as one of the tools for studying the knowledge of such systems. We recall that a similar link between formal systems and distributed system can be encountered, for example, also between Łukasiewicz logic and artificial neural networks, see (Di Nola et al. 2016; Di Nola and Vitale 2020) or Łukasiewicz logic and decision theory, see (Vitale 2020).

The basic logic for handling a system with $n$-agents is known as $S5_n$, introduced in Porter (2003). The logic $S5_n$ is obtained from ordinary classical propositional logic by adding ‘knowledge operators’ $\Box_i$. It models a community of ideal knowledge agents who have the properties of veridical knowledge (everything they know is true), positive introspection (they know what they know) and negative introspection (they know what they do not know).

Very often the agent is encountered with the uncertainty of information about some problem that needs a solution, and this solution depends on the degree of uncertainty. In other words, we need an algorithm based on partial (incomplete, not exact) information. But, the difference between incompletely and partially is that: incompletely in an incomplete manner; partially is a partial degree, i.e., related to only a part, not general or complete. In order to model a distributed system that is sensitive with respect to infinitesimal variations of information in a communication network, we use infinitesimal elements of a special type of MV-algebras—perfect MV-algebras that are non-archimedean MV-algebras. Summarizing saying above, we choose MV-algebra giving a possibility to interpret the valuation of Łukasiewicz sentences into some degree of uncertainty belonging to some MV-algebras.

Taking as a motivation mentioned above facts, in distinct from the classical case we suggest a new logic—multimodal epistemic Łukasiewicz logic $K\mathcal{L}P(n)$ with $n$ knowledge operators $\Box_i$ ($1 \leq i \leq n$) that are interpreted in a non-
archimedean monadic MV-algebra. Epistemic logic is a modal logic extended by some additional axioms, where the modal operator is interpreted as “knowledge”, which can be estimated by some grading (different kinds of knowledge): absolute knowledge or partial knowledge. We consider a very special type of partial knowledge. Actually, we take infinitesimal elements, the radical, of perfect MV-algebras as a range of this estimation. The choice of infinitesimal elements seems suitable for actual situations like a measure of partial information.

The logic $KL_P(n)$ is obtained extending the language of the logic $L_P$, the algebraic models of which are perfect MV-algebras, by adding $n$ ‘knowledge operators’ with corresponding axioms. The knowledge operators model a community of ideal knowledge agents who have the properties of veridical knowledge (everything they know is true), fuzzy knowledge (everything they know is quasi-true), positive introspection (they know what they know) and negative introspection (they know what they do not know) and so on. The knowledge operators permit the following interpretation:

$$\square_i \alpha \quad \text{“i knows proposition } \alpha \text{”};$$
$$\Diamond_i \alpha \quad \text{“i does not know that proposition } \alpha \text{ is false”}.$$

where $i$ belongs to $s$ set $A$ of agents.

There are MV-algebras which are not semisimple, i.e., the intersection of their maximal ideals (the radical of $A$) is different from $\{0\}$. Nonzero elements in the radical of $A$ are called infinitesimals. It is worth stressing that the existence of infinitesimals in some MV-algebras is due to the remarkable difference of behavior between Boolean algebras and MV-algebras.

Perfect MV-algebras, that were introduced by B. Belluce, A. Di Nola, and A. Lettieri in Belluce et al. (1993), are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical (Belluce et al. 2007). They generate the smallest non-locally finite subvariety of the variety MV of all MV-algebras. An important example of a perfect MV-algebra is the subalgebra $S$ of the Lindenbaum algebra $\mathbb{L}$ of first-order Łukasiewicz logic generated by the classes of formulas which are valid when interpreted in $[0, 1]$ but non-provable. Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Łukasiewicz first-order logic (see Belluce and Chang 1963; Scarpellini 1962). Infinitesimal elements of perfect MV-algebra spring to mind the idea of quasi-false and quasi-truth. Following this idea, A. Di Nola, R. Grigolia, and E. Turunen have been published the monograph Fuzzy Logic of Quasi-Truth: An Algebraic Treatment (Di Nola et al. 2016).

For our main aim, as an algebraic model of multi-monadic Łukasiewicz logic $KL_P(n)$ we use multi-monadic MV-algebras, that is a generalization of monadic MV-algebras, which are algebraic models of monadic Łukasiewicz logic that is a special kind of modal logic. Monadic MV-algebras (monadic Chang algebras by Rutledge’s terminology) were introduced and studied by Rutledge in Rutledge (1959), using a functional approach, as an algebraic model for the predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed the study of monadic Boolean algebras investigated by Halmos (1962). Extending the signature of MV-algebra by unary monadic (modal) operation, A. Di Nola and R. Grigolia in Di Nola and Grigolia (2004) define and study monadic MV-algebras as pairs of MV-algebras one of which there is a special case of relatively complete subalgebra named $m$-relatively complete. An $m$-relatively complete subalgebra determines a unique monadic operator. A necessary and sufficient condition is given for a subalgebra to be $m$-relatively complete.

We also mention the papers similar to this paper— (Hansoul and Teheux 2006) concerning the modal Łukasiewicz logic, (Di Nola et al. 2020; Harel et al. 2000; Parikh 1978; Pratt 1980, 1991; Segerberg 1977) concerning the multi-modal case.

The paper is organized in the following way. In Sect. 1, an introduction and some preliminaries and motivation are given. Let us note that Sects. 2, 3, 4 represent preliminaries for the concepts that are necessary for the main results. Section 2 represents a definition of MV-algebras and perfect MV-algebras. Section 3 represents a definition of a variety of monadic MV-algebras and its subvariety of monadic perfect MV-algebras. Section 4 represents multi-monadic perfect $KL_P(n)$-algebra that are algebraic models of multi-monadic Łukasiewicz logic $KL_P(n)$. In Sect. 5, we introduce multi-monadic Łukasiewicz logic and its algebraic counterpart. It is proved deduction theorem and completeness theorem. Conclusion is given in Sect. 6.

## 2 MV-algebras and perfect MV-algebras

In this section, we give a definition of MV-algebras and perfect MV-algebras, which are algebraic models of Łukasiewicz logic $\mathbb{L}$ and the logic $L_P$, respectively.

Infinite-valued logic has been introduced by Łukasiewicz (1920); Łukasiewicz and Tarski (1930). Taking into account Łukasiewicz’s idea on infinite-valued logic, afterward C. C. Chang has developed its algebraic counterpart, i.e., the variety of MV-algebras (Chang 1958), and proved the completeness theorem for Łukasiewicz logic with respect to the variety MV of MV-algebras.

An MV-algebra $A = (A, \oplus, \odot, \neg, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold: $x \oplus 1 = 1, \neg x = x, \neg 0 = 1, x \odot y = \neg(x \oplus \neg y), \neg(\neg x \oplus y) \oplus y = \neg(y \oplus x) \odot x$ (see Chang (1958)). We shall write
ab for \( a \odot b \) and \( a^n \) for \( \underbrace{a \odot \cdots \odot a}_n \), for given \( a, b \in A \). Every MV-algebra has an underlying ordered structure defined by

\[ x \leq y \text{ iff } \neg x \odot y = 1. \]

\((A, \leq, 0, 1)\) is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

\[ x \odot y \leq x \land y \leq x \lor y \leq x \odot y. \]

The unit interval of real numbers \([0, 1]\) endowed with the following operations: \( x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x \), becomes an MV-algebra that is named standard MV-algebra. Note that standard MV-algebra is an archimedean chain. It is well known that the unit interval of real numbers \([0, 1]\) endowed with the following operations:

\[ x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x, \]

is a bounded distributive lattice. Moreover, the functional description of the predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in Rutledge (1959). Scarpellini in Scarpellini (1962) has proved that the set of valid formulas of predicate calculus defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in Rutledge (1959), Scarpellini in Scarpellini (1962) has proved that the set of valid formulas is not recursively enumerable. Monadic MV-algebras were introduced and studied by Rutledge in Rutledge (1959) as an algebraic model for the predicate calculus \( QL \) of Łukasiewicz (infinitely valued) logic, in which only a single individual variable occurs. Rutledge followed the study of monadic Boolean algebras made by P.R. Halmos. In view of the incompleteness of the predicate calculus, the result of Rutledge in Rutledge (1959), showing the completeness of the monadic predicate calculus, has been of great interest.

Let \( L \) denote a first-order language based on \( \cdot, +, \rightarrow, \neg, \exists \) and let \( L_m \), denote a propositional language based on \( \cdot, +, \rightarrow, \neg, \exists \). Let \( Form(L) \) and \( Form(L_m) \) be the set of all formulas of \( L \) and \( L_m \), respectively. We fix a variable \( x \) in \( L \), associate with each propositional letter \( p \) in \( L_m \) a unique monadic predicate \( p^x(\cdot) \) in \( L \) and define by induction a translation \( \Psi : Form(L_m) \rightarrow Form(L) \) by putting:

\[ \Psi(p) = p^x(\cdot) \]

\[ \Psi(\Phi \circ \Psi(\Psi(\Phi))) = \begin{cases} 1 & \text{if } x = 1 - nc \\ 0 & \text{otherwise} \end{cases} \]

The algebra \( C \) has remarkable properties:

(1) \( C \) is generated by its radical,
(2) \( C = \text{Rad}(C) \cup \neg \text{Rad}(C) \),
(3) \( C/\text{Rad}(C) \cong [0, 1] \),
(4) \( 2x = 1 \) for every \( x \in \neg \text{Rad}(C) \),
(5) \( x^2 = 0 \) for every \( x \in \text{Rad}(C) \).

Hence \( C \) is just made by infinitesimal elements and co-infinitesimal elements. Let us take note that if we take \( C \) as truth values of logical formulas, then the values from \( \neg \text{Rad}(C) \) are considered as truth values. In other words, if the valuation \( v(\alpha) \in \neg \text{Rad}(C) \), then \( v(\neg \alpha) = 1 \).

We say that an MV-algebra \( A \) is perfect if for each element \( x \in A \), \( \text{ord}(x) < \infty \) iff \( \text{ord}(\neg x) = \infty \), where the order of an element \( x \), in symbols \( \text{ord}(x) \), is the least integer \( m \) such that \( mx = 1 \); if no such integer \( m \) exists, then \( \text{ord}(x) = \infty \).

The unit interval of real numbers \([0, 1]\) endowed with the following operations:

\[ x \odot y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x, \]

generate the variety \( \text{MV} \) of all MV-algebras, i.e., \( V(S) = \text{MV} \).

### 3 Monadic perfect MV-algebras

In this section, we define a variety of monadic MV-algebras and its subvariety of monadic perfect MV-algebras which are algebraic models of the logic \( \mathcal{L}_p \).

The propositional calculi, which have been described by Łukasiewicz and Tarski in Łukasiewicz and Tarski (1930), are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely valued) logic \( QL \) is defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in Rutledge (1959), Scarpellini in Scarpellini (1962) has proved that the set of valid formulas is not recursively enumerable. Monadic MV-algebras were introduced and studied by Rutledge in Rutledge (1959) as an algebraic model for the predicate calculus \( QL \) of Łukasiewicz (infinitely valued) logic, in which only a single individual variable occurs. Rutledge followed the study of monadic Boolean algebras made by P.R. Halmos. In view of the incompleteness of the predicate calculus, the result of Rutledge in Rutledge (1959), showing the completeness of the monadic predicate calculus, has been of great interest.

Let \( L \) denote a first-order language based on \( \cdot, +, \rightarrow, \neg, \exists \) and let \( L_m \) denote a propositional language based on \( \cdot, +, \rightarrow, \neg, \exists \). Let \( Form(L) \) and \( Form(L_m) \) be the set of all formulas of \( L \) and \( L_m \), respectively. We fix a variable \( x \) in \( L \), associate with each propositional letter \( p \) in \( L_m \) a unique monadic predicate \( p^x(\cdot) \) in \( L \) and define by induction a translation \( \Psi : Form(L_m) \rightarrow Form(L) \) by putting:

\[ \Psi(p) = p^x(\cdot) \]

\[ \Psi(\Phi \circ \Psi(\Psi(\Phi))) = \begin{cases} 1 & \text{if } x = 1 - nc \\ 0 & \text{otherwise} \end{cases} \]
\begin{itemize}
  \item $\Psi(p) = p^*(x)$ if $p$ is propositional variable,
  \item $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$,
  \item $\Psi(\exists \alpha) = \exists x \Psi(\alpha)$.
\end{itemize}

Through this translation $\Psi$, we can identify the formulas of $L_m$ with monadic formulas of $L$ containing the variable $x$.

An algebra $A = (A, \oplus, \ominus, \rightarrow, \exists, 0, 1)$ is said be a monadic MV-algebra (Di Nola and Grigolia 2004) (MMV-algebra for short) if $A = (A, \oplus, \ominus, \rightarrow, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities:

\begin{align*}
\text{E1. } x &\leq \exists x, \\
\text{E2. } \exists(x \vee y) &\leq \exists x \vee \exists y, \\
\text{E3. } \exists(\neg(\exists x)) &\leq \neg(\exists x), \\
\text{E4. } \exists(x \oplus \exists y) &\leq \exists x \oplus \exists y, \\
\text{E5. } \exists(x \odot y) &\leq \exists x \odot \exists y, \\
\text{E6. } \exists(x \odot y) &\leq \exists y \odot \exists x, \\
\text{E7. } \exists(x \odot \exists y) &\leq \exists x \odot \exists y.
\end{align*}

Sometimes we shall denote a monadic MV-algebra $A = (A, \leftrightarrow, \ominus, \rightarrow, \exists, 0, 1)$ by $(A, \exists)$, for brevity. We can define a unary operation $\forall x = \neg(\exists x)$ corresponding to the universal quantifier.

Let $A_1$ and $A_2$ be any MV-algebras. A mapping $h : A_1 \to A_2$ is an MMV-homomorphism if $h$ is an MV-homomorphism and for every $x \in A_1$ $h(\exists x) = \exists h(x)$. Denote by MMV the variety and the category of MMV-algebras and MMV-homomorphisms.

As it is well known, MV-algebras form a category that is equivalent to the category of abelian lattice ordered groups (groupoids, for short) with strong unit (Mundici 1986). Let us denote by $\Gamma$ the functor implementing this equivalence. If $G$ is an $\ell$-group, then for any element $u \in G, u > 0$ we let $[0, u] = \{x \in G : 0 \leq x \leq u\}$ and for each $x, y \in [0, u] x \odot y = u \land (x + y)$ and $\neg x = u - x$.

Let us introduce some notations: let $C_0 = \Gamma(\mathbb{Z}, 1), C_1 = C \cong \Gamma(\mathbb{Z} \times \ell \mathbb{G}, \mathbb{Z}, (0, 0))$ with generator $(0, 1) = c_1(= c)$, where $\times \ell \mathbb{G}$ is the lexicographic product. Let us denote $\text{Rad}(A) \cup \neg \text{Rad}(A)$ through $R^*(A)$, where $\neg \text{Rad}(A) = \{\neg x : x \in \text{Rad}(A)\}$.

Let $(A, \oplus, \ominus, \rightarrow, \exists, 0, 1)$ be a monadic MV-algebra. Let $\exists A = \{x \in A : x = \exists x\}$. By Di Nola and Grigolia (2004), $(\exists A, \oplus, \ominus, \rightarrow, 0, 1)$ is an MV-subalgebra of the MV-algebra $(A, \oplus, \ominus, \rightarrow, 0, 1)$.

A subalgebra $A_0$ of an MV-algebra $A$ is said to be $m$-relatively complete (Di Nola and Grigolia 2004), if $A_0$ is relatively complete and two additional conditions hold:

\begin{align*}
\text{(1) } & (\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \land v \odot v \leq x), \\
\text{(2) } & (\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \lor v \odot v \leq x).
\end{align*}

Notice that a two-elements Boolean subalgebra of the standard MV-algebra $S = ([0, 1], \oplus, \ominus, \rightarrow, 0, 1)$ is relatively complete, but not $m$-relatively complete.

**Proposition 1** (Di Nola and Grigolia 2004). Let $A$ be a monadic MV-algebra. Then $\exists A$ is $m$-relatively complete subalgebra of the monadic MV-algebra $A$.

**Proposition 2** Let $A$ be an MV-algebra.

1. (Di Nola and Grigolia (2004); Rutledge (1959). If $A_0$ is $m$-relatively complete totally ordered MV-subalgebra of the MV-algebra $A$, then $A_0$ is a maximal totally ordered subalgebra of $A$.

2. (Di Nola and Grigolia (2004); Rutledge (1959). If $(A, \exists)$ is a totally ordered monadic MV-algebra, then $A = \exists A (= A_0)$.

3. (Di Nola and Grigolia (2004); Rutledge (1959). $(A, \exists)$ is a subdirectly irreducible monadic MV-algebra if and only if $\exists A (= A_0)$ is totally ordered.

4. Rutledge (1959). Any monadic MV-algebra $(A, \exists)$ is isomorphic to a subdirect product of monadic MV-algebras $(A_i, \exists)$ such that $\exists A_i$ is totally ordered.

A monadic MV-algebra $A = (A, \oplus, \ominus, \rightarrow, \exists, 0, 1)$ is said to be MMV(C)-algebra if in addition satisfies the identity (Di Nola and Lettieri 1999):

\[ (\forall \exists) \quad 2(x^2) = (2x)^2, \]

that is $\text{MMV}(C) = \text{MMV} + (\forall \exists)$.

Denote the variety of perfect MV-algebras by MMV(C), which is a subvariety of the variety MMV.

Let

\[ \text{Alt}_{C}^2 = \forall(2x_1^2) \lor \forall(2x_1^2 \to 2x_2^2) \lor \forall(2x_1^2 \land 2x_1^2 \to 2x_2^3). \]

Let $\text{MMV}(C) \uparrow$ (Di Nola et al. 2018) be the subvariety of $\text{MMV}(C)$ defined by the identity $\text{Alt}_{C}^2 = 1$.

**Theorem 3** (Di Nola et al. 2018). The identity $\text{Alt}_{C}^2 = 1$ is true in finitely generated subdirectly irreducible algebra $A \in \text{MMV}(C)$ if and only if $A$ contains as a maximal homomorphic image the monadic Boolean algebra $(2^2, \exists)$. 

\[ \odot \text{ Springer} \]
Let us consider the identity

\((K\mathcal{L}_P) \quad (\exists x)^2 \wedge (\exists \neg x)^2 = 0.\)

It holds

**Theorem 4** (Di Nola et al. 2018). The identity \((\exists x)^2 \wedge (\exists \neg x)^2 = 0\) is satisfied in the subdirectly irreducible MMV(C) algebra \((A, \exists)\) if and only if the MV-algebra reduct of that is perfect MV-algebra.

\[MMV(C)^1\text{-algebra} A \text{ is said to be } MMV(C)^1\text{-algebra if it satisfies the identity (}K\mathcal{L}_P\text{). Denote by \(MMV(C)^1 \_K\) the variety of all } MMV(C)^1\text{-algebras.}\]

The main interest for us is the \(MMV(C)^1\text{-algebra}\)

\[K = (\oplus, \ominus, \vee, \wedge, \neg, \exists), 0, 1) = \mathcal{R}^C = \text{Rad}(C^2) \cup \neg\text{Rad}(C^2),\]

where for any \((x_1, x_2) \in K\) we have \(\bigvee(x_1, x_2) = (\max(x_1, x_2), \max(x_1, x_2))\) and \(\bigwedge(x_1, x_2) = (\min(x_1, x_2), \min(x_1, x_2)).\]

In other words, \(\exists K = \{(x, x) : x \in C\}\), i.e. \(\exists K\) is a diagonal of \(C^2\). Notice, that \(\exists K\) is a subalgebra of \(K = \mathcal{R}^C\), where \(\bigvee = x = \bigwedge x = x\) (as well) for every \(x \in \exists K\). So, the MV-reduct \(\exists K\) is isomorphic to \(C\).

### 4 Multi-monadic perfect \(K\mathcal{L}_P(n)\)-algebra

In this section, we define a variety of multi-monadic perfect algebraic models of multi-monadic epistemic \(\mathcal{L}_P\)-logic \(K\mathcal{L}_P(n)\).

An algebra \(A = (A, \oplus, \ominus, \exists_1, ..., \exists_n, 0, 1)\) is said to be \(K\mathcal{L}_P(n)\)-algebra if it satisfies the following identities:

\[\begin{align*}
&\text{KE1. } x \leq \exists x, 0 < i \leq n, \\
&\text{KE2. } \exists_1(x \vee y) = \exists_1 x \vee \exists_1 y, 0 < i \leq n, \\
&\text{KE3. } \exists_1(\neg \exists x) = \neg \exists(x), 0 < i \leq n, \\
&\text{KE4. } \exists_1(x \oplus y) = \exists_1 x \oplus \exists_1 y, 0 < i \leq n, \\
&\text{KE5. } \exists_i(x \otimes x) = \exists_i x \otimes \exists_i x, 0 < i \leq n, \\
&\text{KE6. } \exists_i(x \otimes y) = \exists_i x \otimes \exists_i y, 0 < i \leq n, \\
&\text{KE7. } \exists_i(x \otimes y) = \exists_i x \otimes \exists_i y, 0 < i \leq n, \\
&\text{KE8. } \exists_i x \leq \forall_i \exists_i x, 0 < i \leq n, \\
&\text{KE9. } 2(x^2) = (2x)^2, \\
&\text{KE10. } \forall_i(2x_1^2 \vee \forall_i(2x_2^2 \to 2x_3^2) \vee \forall_i(2x_4^2 \wedge 2x_5^2 \to 2x_6^2)) = 1, 0 < i \leq n, \\
&\text{KE11. } (\exists x)^2 \wedge (\exists \neg x)^2 = 0.
\end{align*}\]

The algebra \(A = (A, \oplus, \ominus, \exists_1, ..., \exists_n, 0, 1)\) is said to be \(K(n)\)-algebra if \((A, \oplus, \ominus, \exists, 0, 1)\) is algebra \(K\) for \(0 < i \leq n\), and denote the algebra by \(K(n)\).

### 5 Multi-monadic epistemic \(\mathcal{L}_P\)-logic \(K\mathcal{L}_P(n)\)

In this section we introduce and investigate multi-monadic epistemic \(\mathcal{L}_P\), and prove deduction theorem and completeness theorem with respect to the multi-monadic perfect MV-algebras.

The formulas of Lukasiewicz logics are built from a countable set of propositional variables \(\text{Var} = \{p, q, \ldots\}\) using the connectives \& (conjunction), \(\to\) (implication) and \(\perp\) (falsity truth constant). We introduce the connectives \(\wedge, \vee, \leftrightarrow, \neg, \forall, \exists\) and \(\top\) (the semantics counterpart will be denoted, respectively, by \(\wedge, \vee, \leftrightarrow, \neg, \forall, \exists\) and \(\top\) for \&) as the following abbreviations:

\[
\begin{align*}
\varphi \land \psi & = \varphi \& \psi, \\
\varphi \lor \psi & = \varphi \vee \psi, \\
\varphi \leftrightarrow \psi & = \psi \leftrightarrow \varphi, \\
\neg \varphi & = \varphi \to \perp, \\
\phi \forall \psi & = \neg (\neg \phi \land \neg \psi). \quad \top = \neg \bot.
\end{align*}
\]

Infinite-valued Lukasiewicz logic \(L\) is axiomatized by the following axioms schemata:

\[
\begin{align*}
\text{L1. } \varphi & \to (\psi \to \varphi), \\
\text{L2. } (\varphi \to \psi) & \to ((\psi \to \chi) \to (\varphi \to \chi)), \\
\text{L3. } ((\varphi \to \psi) & \to \psi) \to ((\psi \to \varphi) \to \varphi), \\
\text{L4. } (\neg \varphi & \to \neg \psi) \to (\psi \to \varphi).
\end{align*}
\]

The inference rule is Modus Ponens: \(\varphi, \varphi \to \psi \to \psi\).

Łukasiewicz logic \(L\) is axiomatized by the axioms of \(L\) plus the schema:

\[
\text{L}_p \quad (\varphi \& \psi) \to (\varphi \& \varphi), \quad (\varphi \& \varphi) \to (\varphi \& \psi).
\]

We extend Łukasiewicz logic \(L\) to the multimodal Łukasiewicz logic \(K\mathcal{L}_P(n)\) by adding \(n\) unary knowledge (modal) operators \(\square_i\) and \(\bigcirc_i\) \((i = 1, ..., n)\) to the language of \(L\).

We suggest the following schemata of axioms for multi-modal epistemic logic \(K\mathcal{L}_P(n)\): to the schema of axioms of \(L(p) = (L + L(p))\) we add

\[
\begin{align*}
1) & \quad \square_i \varphi \to \varphi, \quad i = 1, ..., n, \\
2) & \quad \square_i \varphi \to \square_i \square_i \varphi, \quad i = 1, ..., n, \\
3) & \quad \square_i (\varphi \land \psi) \leftrightarrow (\square_i \varphi \land \square_i \psi), \quad i = 1, ..., n, \\
4) & \quad \square_i (\varphi \& \psi) \leftrightarrow (\square_i \varphi \& \square_i \psi), \quad i = 1, ..., n, \\
5) & \quad \square_i (\varphi \& \psi) \leftrightarrow (\square_i \varphi \& \square_i \psi), \quad i = 1, ..., n, \\
6) & \quad \diamond_i \varphi \to \square_i \square_i \varphi, \quad i = 1, ..., n, \\
7) & \quad (\square_i (\varphi_1 \& \varphi_1) \vee (\square_i (\varphi_1 \& \varphi_1)) \vee (\square_i (\varphi_1 \& \varphi_1) \vee (\square_i (\varphi_1 \& \varphi_1)))) \to \\
& \quad (\square_i (\varphi_2 \& \varphi_2)) \vee (\square_i (\varphi_2 \& \varphi_2)) \vee (\square_i (\varphi_2 \& \varphi_2)) \vee (\square_i (\varphi_2 \& \varphi_2)) \vee (\square_i (\varphi_2 \& \varphi_2)), \quad i = 1, ..., n; \\
8) & \quad \square_i (\varphi \& \psi) \to (\psi \& \square_i \varphi \& \square_i \varphi), \quad i = 1, ..., n.
\end{align*}
\]

Inference rules: \(\varphi, \varphi \to \psi / \psi, \varphi \to \psi / \psi\).

Notice that Axiom 7) is a logical analog of algebraic identity KE11.
Recall, that the logic KL_{P}(n) is obtained from Łukasiewicz propositional logic L_{P}, corresponding to perfect MV-algebras, by adding n ‘knowledge operators’, with corresponding axioms. The knowledge operators model a community of ideal knowledge agents who have the properties of veridical knowledge (everything they know is true), fuzzy knowledge (everything they know is quasi-true), positive introspection (they know what they know) and negative introspection (they know what they do not know) and so on. The knowledge operators permit the following interpretation:

\[ \square \alpha \] - “i knows proposition \( \alpha \)’;
\[ \Diamond \alpha \] - “i does not know that proposition \( \alpha \) is false”.

An evaluation is a mapping \( e_i : \text{Var} \to K(n) \) that is extended to the set of all formulas in the following way:

\[ e_i(\varphi \& \psi) = e_i(\varphi) \otimes e_i(\psi), \]
\[ e_i(\varphi \rightarrow \psi) = e_i(\varphi) \Rightarrow e_i(\psi), \]
\[ e_i(\bot) = 0, \]
\[ e_i(\square \varphi) = (\max(e_i(\varphi)), \max(e_i(\varphi))) \in K(n), \quad i = 1, \ldots, n. \]

Let \( e(\varphi) = \bigwedge_{i=1}^{n} e_i(\varphi) \). A formula \( \varphi \) is said to be valid when it is evaluated to 1 in all evaluations \( e_i, \quad i = 1, \ldots, n \). In other words, a formula \( \varphi \) is said to be valid when it is evaluated to 1 for any evaluation \( e \).

Semantically we can define the notion of “partial knowledge” \( \square \alpha \) - “agent i has partial knowledge about proposition \( \alpha \)” if \( e_i(\alpha) \in \neg \text{Rad}(K(n) - \{\alpha\}) \). Taking into account the completeness theorem (that we will prove below) we can say that “agent i has partial knowledge about proposition \( \alpha \)” if \( \not \exists i \alpha \) and \( \models \square \alpha \supset \square \alpha \).

**Lemma 5** (Hajek 1998). The following formulas are theorems in \( L \):

(1) \( \models (\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \& \psi) \rightarrow \chi) \);
(2) \( \models ((\varphi_1 \rightarrow \psi_1) \& (\varphi_2 \rightarrow \psi_2)) \leftrightarrow ((\varphi_1 \& \varphi_2) \rightarrow (\psi_1 \& \psi_2)) \);
(3) \( \models (\varphi \& (\varphi \rightarrow \psi)) \rightarrow \psi \);
(4) \( \models \varphi \leftrightarrow \neg \neg \varphi \);
(5) \( \models \varphi \rightarrow (\neg \psi \rightarrow \neg \psi) \).

**Lemma 6** The following formulas are theorems in \( L \):

(1) \( (\varphi^m \rightarrow \varphi) \rightarrow ((\neg \varphi)^m \rightarrow \varphi) \rightarrow ((\varphi^m \& (\neg \varphi)^m) \rightarrow \varphi) \rightarrow \alpha \), \( 0 < m \in \omega \);
(2) \( \neg \psi \rightarrow (\psi \rightarrow \chi) \);
(3) \( (\varphi \& \neg \varphi) \rightarrow ((\chi \& \neg \chi) \rightarrow ((\psi \rightarrow \chi) \& \neg (\psi \rightarrow \chi)) \);
(4) \( \chi \rightarrow ((\varphi \& \neg \varphi) \rightarrow ((\psi \rightarrow \chi) \& \neg (\psi \rightarrow \chi)) \).

(5) \( \neg \psi \rightarrow ((\chi \& \neg \chi) \rightarrow ((\psi \& \varphi) \rightarrow (\neg (\psi \rightarrow \chi)) \).

**Proof** The formulas \( (\varphi^m \rightarrow \varphi) \rightarrow ((\neg \varphi)^m \rightarrow \varphi) \rightarrow ((\varphi^m \& (\neg \varphi)^m) \rightarrow \varphi) \rightarrow \alpha \), \( 0 < m \in \omega \).

Let \( p_i \) be an evaluation for \( p_1, \ldots, p_k \). Then let \( p_i \) be such that \( e(p_1) = 1, p_i \notin p_i \) if \( e(p_1) = 0 \), \( p_i \equiv p_i \) if \( e(p_1) \in K(n) - \{0, 1\} \), and let \( p_i \notin p_1 \) if \( e(p_1) = 1, p_i \equiv p_1 \) if \( e(p_1) = 0, p_i \notin p_1 \).

**Lemma 9** Let \( \varphi \) be a formula containing propositional variables \( p_1, \ldots, p_k \) and let \( e(\varphi) \) be some evaluation for \( p_1, \ldots, p_k \). Then let \( p_i \) be such that \( e(p_1) = 1, p_i \notin p_i \) if \( e(p_1) = 0 \), \( p_i \equiv p_i \) if \( e(p_1) \in K(n) - \{0, 1\} \), and let \( p_i \notin p_1 \) if \( e(p_1) = 1, p_i \equiv p_1 \) if \( e(p_1) = 0, p_i \notin p_1 \). Then \( p_1, \ldots, p_n \).

**Proof** We prove this assertion by induction on the number \( n \) of connectives of the formula \( \varphi \). If \( n = 0 \), then the formula \( \varphi \)

\[ \square \] Springer
is a propositional variable $p_1$ and the assertion of the lemma is
to $p_1 \vdash p_1, \neg p_1 \vdash \neg p_1$ and $p_1 \wedge p_1 \vdash p_1 \wedge \neg p_1$.
Let us suppose that the lemma is true for any $j < n$.

**Case 1.** $\varphi = \neg \psi$. The number of connectives in $\psi$
less than $n$.

(a) Let $e(\varphi) = 1$. Then $e(\varphi) = 0$. Thus $\psi = \varphi$ and $\varphi' = \neg \psi$.
By the induction hypotheses we have $p_1', \ldots, p_k' \vdash \psi$.
Hence, by Lemma 5 (4) and modus ponens, $p_k', p_k' \vdash \neg \neg \psi$. But $\neg \neg \psi = \psi$.
(b) Let $e(\varphi) = 0$. Thus $\psi' = \neg \psi$ and $\varphi' = \varphi$. By the
induction hypotheses we have $p_1', \ldots, p_k' \vdash \neg \psi$. But $\varphi = \neg \psi$.
(c) Let $e(\varphi) \in K(n) - [0, 1]$. Thus $\psi' = \varphi \wedge \neg \psi$. Then
$p_1', \ldots, p_k' \vdash \psi \wedge \neg \psi$. But $\psi \wedge \neg \psi$ is equivalent to $\varphi \wedge \neg \varphi$.

**Case 2.** Let $\varphi = \psi \rightarrow \chi$. Then the number of connectives
in $\varphi$ and $\psi$ less than $n$. So, by the induction hypotheses
$p_1', \ldots, p_k' \vdash \psi$ and $p_1', \ldots, p_k' \vdash \chi'$.

(a) Let $e(\varphi) = 0$. Then $e(\varphi) = 1$, and $\psi' = \neg \psi$ and $\varphi' = \varphi$.
Thus, $p_1', \ldots, p_k' \vdash \psi$ and, by the Lemma 6 (2), and
modus ponens, $p_k', p_k' \vdash \psi \rightarrow \chi$. But $\varphi = \psi \rightarrow \chi$.
(b) Let $e(\varphi) = 1$. Then $e(\varphi) = 1$, and $\chi' = \chi$, and $\varphi' = \varphi$.
So, we have $p_1', \ldots, p_k' \vdash \chi$, and then, by Lemma 5 (5),
p_1', \ldots, p_k' \vdash \psi \rightarrow \chi$, where $\psi = \psi \rightarrow \chi$.
(c) Let $e(\varphi), e(\chi) \in \neg \neg \neg (K(n)) - [1] (e(\varphi), e(\chi) \in \neg \neg \neg (K(n)) - [1])$ and $e(\varphi) \leq e(\psi)$. Then $e(\varphi) = 1$.
Thus, $p_1', \ldots, p_k' \vdash \psi \wedge \neg \psi$, $p_1', \ldots, p_k' \vdash \chi \wedge \neg \chi$ and
by Lemma 6 (3), and modus ponens, $p_1', \ldots, p_k' \vdash \psi \rightarrow \chi$ and $\neg \neg \neg (\psi \rightarrow \chi)$.
But $((\psi \rightarrow \chi) \wedge \neg \psi) \rightarrow (\psi \rightarrow \chi)$ and $\neg \neg \neg (\psi \rightarrow \chi)$.
So, by modus ponens, $p_1', \ldots, p_k' \vdash \psi$.
(d) Let $e(\varphi) \in \neg \neg \neg (K(n)) - [0], e(\chi) \in \neg \neg \neg (K(n)) - [1], \neg \neg \neg (\varphi) = 1$. Then $e(\varphi) = 1$.
Thus, $p_1', \ldots, p_k' \vdash \psi \wedge \neg \psi$, $p_1', \ldots, p_k' \vdash \chi \wedge \neg \chi$ and $\neg \neg \neg (\varphi)$ and $p_1', \ldots, p_k' \vdash \psi \rightarrow \chi$.
But $\varphi' = (\psi \rightarrow \chi) \wedge \neg \psi$ and $\varphi' = (\psi \rightarrow \chi) \wedge \neg \psi$.
By the induction hypotheses, we have $p_1', \ldots, p_k' \vdash \psi \rightarrow \chi$.
(e) Let $e(\varphi) \in \neg \neg \neg (K(n)) - [0], e(\chi) \in \neg \neg \neg (K(n)) - [0].$ Then $e(\varphi) \in \neg \neg \neg (K(n)) - [0].$ Thus, $p_1', \ldots, p_k' \vdash \psi \wedge \neg \psi$, $p_1', \ldots, p_k' \vdash \chi \wedge \neg \chi$ and $\neg \neg \neg (\varphi)$ and $p_1', \ldots, p_k' \vdash (\psi \rightarrow \chi)$.
But $\varphi' = (\psi \rightarrow \chi) \wedge \neg \psi$.

**Case 3.** Let $\varphi = \bigcirc_i, \psi$. Then the number of connectives in
$\psi$ less than $n$. So, by the induction hypotheses $p_1', \ldots, p_k' \vdash \psi$.

(a) Let $e(\varphi) = 1$. Then $e(\varphi) = 1$. Thus, $\psi = \psi$.
But $\psi = \psi$. But $\bigcirc_i = \bigcirc_i$.
(b) Let $e(\varphi) = 0$. Then $e(\varphi) = 0$. Thus, $p_1', \ldots, p_k' \vdash \neg \psi$ and, according to 1) $\vdash \bigcirc_i \psi \rightarrow \psi$
and by Lemma 5 (6) ($\bigcirc_i \psi \rightarrow \psi$) $\rightarrow (\neg \psi \rightarrow \neg \bigcirc_i \psi$), and modus ponens, we have $p_1', \ldots, p_k' \vdash \neg \bigcirc_i \psi$.
But $\psi' = \neg \bigcirc_i \psi$.
(c) Let $p_1', \ldots, p_k' \vdash \bigcirc_i \psi$. Then $p_1', \ldots, p_k' \vdash \psi \wedge \neg \psi$.
But $\psi \wedge \neg \bigcirc_i \psi \rightarrow (\psi \wedge \neg \bigcirc_i \psi)$ and $p_1', \ldots, p_k' \vdash \bigcirc_i \psi$. Then, $p_1', \ldots, p_k' \vdash (\psi \wedge \neg \bigcirc_i \psi)$.
So, $p_1', \ldots, p_k' \vdash \bigcirc_i \psi \wedge \neg \bigcirc_i \psi$.

**Lemma 10** ($\alpha \wedge \neg \alpha$) $\alpha \wedge \neg \alpha$ and ($\alpha \wedge \neg \alpha$) $\alpha \wedge \neg \alpha$ are tautologies in $L_P$.

**Proof** ($\alpha \wedge \neg \alpha$) and ($\alpha \wedge \neg \alpha$) are tautologies in $L_P$.

**Theorem 11** ($\alpha \wedge \neg \alpha$) ($\alpha \wedge \neg \alpha$) is a theorem of $K\alpha\beta(\alpha \wedge \neg \alpha$).

**Proof** For simplicity we will also write $e(\varphi)$ instead of $\bigcirc_i e(\varphi)$.
Let us suppose that $\psi$ is valid and it contains
propositional variables $p_1, \ldots, p_k$. So, for every evaluation
1, according to Lemma 9, $p_1', \ldots, p_k' \vdash \psi$, since $e(\varphi) = 1$ and,
$\varphi = \psi$. So, when $e(\varphi) = 1$,
and, according to Lemma 9, $p_1', \ldots, p_k' \vdash \psi$, and when $e(\varphi) = 0$, according
to the same lemma, $p_1', \ldots, p_k' \vdash \neg \psi$. From here, according to the deduction
theorem, we obtain (1) $p_1', \ldots, p_k' \vdash (\bigcirc_i \psi)$.
Then, (2) $p_1', \ldots, p_k' \vdash (\bigcirc_i \psi)$.
From (3) we have $p_1', \ldots, p_k' \vdash \neg \psi$. From here, according to the deduction
theorem, we obtain (1) $p_1', \ldots, p_k' \vdash (\bigcirc_i \psi)$.
Then, (2) $p_1', \ldots, p_k' \vdash (\bigcirc_i \psi)$.
Applying Lemma 6 (1), (1), (2), (1), (2), (1), and modus ponens we obtain $p_1', \ldots, p_k' \vdash \psi$.
Applying the same procedures for other variables finally after $k$ steps we come to $\vdash \psi$. $\square$

**6 Conclusion**

We have proposed new epistemic logics ($K\alpha\beta(n)$) aimed to
model the partial knowledge of a distributed finite system of
agents acting independently. This situation is represented by
the axiomatization of the proposed logic. The semantics of
$K\alpha\beta(n)$ is given by multimodal perfect MV-algebras. Indeed,
an agent partially knows about a proposition (a formula of
$K\alpha\beta(n)$) with some degree of information. This degree is
expressed, via the evaluation map, by an element of its
algebraic model.
Among the remarkable algebraic properties of the model is that it is sensitive to infinitesimal variations of the degree of information. By this, we can consider distributed finite systems of agents whose partial knowledge can infinitesimally vary, but it always is infinitesimally close to a crisp (complete) knowledge.

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