Lines on Calabi–Yau complete intersections, mirror symmetry, and Picard Fuchs equations

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Introduction and statement of the result. It was suggested (cf. [COGP], [GP]) that (in some circumstances) if \(V\) is a Calabi–Yau threefold then one can relate to \(V\) a family \(W(V)_t, t \in \mathbb{C}\) of Calabi–Yau manifolds which are “mirrors” of \(V\), such that one has the following relation between their Euler characteristics: \(\chi(V) = -\chi(W(V)_t)\). One of the properties of this correspondence should be the following: the coefficients of the expansion of certain integrals attached to \(W(V)_t\) (so called Yukawa couplings) relative to an appropriately chosen parameter are integers from which one may calculate the numbers \(r_d\) of rational curves of degree \(d\) on a generic Calabi–Yau manifold which is a deformation of \(V\).

This was verified in [COGP] and [M1],[M2] in the case when \(V\) is the quintic hypersurface in \(\mathbb{CP}^4\) for rational curves of a small degree. Other authors ([CL],[S]) have suggested a large list of mirrors of hypersurfaces in weighted projective spaces. The purpose of this note is to verify the above predictions for the remaining types of Calabi–Yau complete intersections in complex projective space when \(d = 1\) i.e. the case of lines.

A Calabi–Yau threefold \(W\) is a Kahler manifold such that \(\dim W = 3\), the canonical bundle of \(W\) is trivial, and the Hodge numbers satisfy \(h^{1,0} = h^{2,0} = 0\). Let \(W_t\) be a family of such manifolds and let \(\omega_t\) be a family of holomorphic 3-forms on \(W_t\) (unique up to constant for each \(t\) because \(h^{3,0}(W_t) = 1\)). According to Griffiths transversality ([G]):

\[
\kappa_t(k) = \int_W \omega \wedge \frac{d^k \omega_t}{dt^k}
\]

is equal to zero for \(k \leq 2\). Let \(\kappa_{ttt} = \kappa_t(3)\). We assume that the monodromy \(T\) about \(t = \infty\) acting on \(H_3(W_t, \mathbb{Z})\) is maximally unipotent i.e. that \((T - I)^3 \neq 0\) and \((T - I)^4 = 0\). If this is the case then ([M1],[M2]) for \(N = \log T\) one has \(\dim(ImN^3) \otimes \mathbb{C} = 1\) (as a consequence of \(h^{3,0} = 1\)). Let \(\gamma_1, \gamma_0 \in H_3(W_t, \mathbb{Z})\) be a basis of \((ImN^2) \otimes \mathbb{C}\) such that \(\gamma_0 \in (ImN^3)\) is an indivisible element and \(\gamma_1 = 1/\lambda N^2 \gamma_1\) where \(\gamma_1\) is indivisible and the intersection index of \(\gamma_1\) and \(\gamma_0\) is 1. Let \(m\) be defined from the relation \(N \gamma_1 = m \cdot \gamma_0\) and let

\[
s = \frac{1}{m} \int_{\gamma_1} \omega, \quad q = e^{2\pi i s}.
\]

Then \(q\) is independent of a choice of the basis \(\gamma_0, \gamma_1\) and the form \(\omega\) up to root of unity of degree \(|m|\) (cf. [M1]).

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In this paper, we use an \textit{a priori} different normalization for the parameter $s$ determined by specifying the asymptotic behavior of $s$ as $t \to \infty$. This normalization is described in (18); it is analogous to that exploited in [COGP] and [M2].

Let $V_\lambda$ be given in $\mathbb{CP}^5$ by
\begin{align}
Q_1 &= x_1^3 + x_2^3 + x_3^3 - 3\lambda x_4 x_5 x_6 = 0 \\
Q_2 &= x_4^3 + x_5^3 + x_6^3 - 3\lambda x_1 x_2 x_3 = 0
\end{align}
This is a complete intersection which is a Calabi–Yau threefold for generic $\lambda$. Let $G_{81} \subset PGL(5, \mathbb{C})$ be the subgroup (of order 81) of transformations $g_{\alpha, \beta, \delta, \epsilon, \mu}$ where $\alpha, \beta, \delta, \epsilon, \mu \in \mathbb{Z}$ mod $3 \mu \in \mathbb{Z}_9$ and $3 \cdot \mu = \alpha + \beta = \delta + \epsilon$ mod 3. These transformations act as:
\begin{align}
g_{\alpha, \beta, \delta, \epsilon, \mu} : (x_1, x_2, x_3, x_4, x_5, x_6) &\mapsto (c_3^\alpha \cdot c_9^\mu \cdot x_1, c_3^\delta \cdot c_9^\epsilon \cdot x_2, c_3^\beta \cdot x_3, c_3^{-\delta} \cdot c_9^{-\epsilon} \cdot x_4, c_3^{-\beta} \cdot x_5, c_3^{-\mu} \cdot x_6)
\end{align}
and preserve both hypersurfaces $\hat{Q}_i$ given by the equations $Q_i = 0$ ($i = 1, 2$).

**Theorem.** The resolution of singularities $W(V_\lambda)$ of the quotient of $V_\lambda$ by the action of $G_{81}$ which is a Calabi–Yau manifold satisfies: $\chi(V_\lambda) = -\chi(W(V_\lambda))$. The monodromy of $W(V_\lambda)$ about infinity is maximally unipotent. For $q$ defined for the family $W(V_\lambda)$ by the asymptotic normalization (18), the coefficient of $q$ in the $q$-expansion of $\kappa_{sss}$ is equal to the number of lines on a generic non-singular complete intersection of two cubic hypersurfaces in $\mathbb{CP}^5$.

**A calculation of the Euler characteristic.** The statement on Euler characteristic (as well as the statement on the number of lines) are verified by direct calculation of the quantities involved. The total Chern class $c = 1 + c_1 + c_2 + c_3 \in H^*(V_\lambda, \mathbb{Z})$, of the tangent bundle of $V_\lambda$ satisfies $c \cdot (1 + 3 \cdot h)^2 = (1 + h)^6$ where $h$ is the generator $H^2(V_\lambda, \mathbb{Z})$ (here $(1 + 3 \cdot h)^2$ and $(1 + h)^6$ respectively the total Chern class of the normal bundle to $V_\lambda$ in $\mathbb{CP}^5$ and the pullback on $V_\lambda$ of the total Chern class of $\mathbb{CP}^5$). The Euler characteristic of $V_\lambda$ is $c_3$ evaluated on its fundamental class which (using the fact that $h^3$ evaluated on the fundamental class is 9) gives $\chi(V_\lambda) = -144$.

On the other hand according to the “physicist’s formula” (cf. [DHVW]) or rather to its reformulation due to Hirzebruch and Hofer (cf. [HH]) the Euler characteristic of a Calabi–Yau resolution of the quotient $V_\lambda/G_{81}$ can be found as
\begin{equation}
\Sigma_{[g]} \chi(V_\lambda^g / C(g))
\end{equation}
where the summation is over all conjugacy classes $[g]$ of elements of $G$, $C(g)$ denotes the centralizer of $g$ and $X^g$ is the fixed point set of an element $g$. Because $G_{81}$ is abelian the formula reduces to $\Sigma_g \chi(V_\lambda^g / G_{81})$ where the summation is over all elements of the group. There are 6 curves $C_{i,j}$ having non-trivial stabilizer corresponding to the vanishing of two variables in either of the two sets $(x_1, x_2, x_3)$ or $(x_4, x_5, x_6)$. The Euler characteristic
of such a curve, which is a complete intersection of two cubic surfaces in $\mathbb{P}^3$, is $-18$. The stabilizer of each curve contains 3 elements since for each curve there are 2 elements which have this curve as the fixed point set. Hence the number of elements which have one dimensional fixed point set is 12. The Euler characteristic of the quotient of the one dimensional fixed point set is 2. The zero dimensional fixed point sets $D_{i,j,k}$ on a curve $C_{i,j}$ ($i, j$ are in the same group of variables, and $k$ in another) are obtained by equating to zero a variable in another group. The stabilizer of such zero dimensional fixed point set has order 27. Each zero dimensional fixed point set $D_{i,j,k}$ belongs to 3 curves $C_{i,j}$. Hence the number of elements stabilizing $D_{i,j,k}$ is $27 - 3 \times 2 - 1 = 20$. The number of zero dimensional fixed point sets $D_{i,j,k}$ is 6 and each element with zero dimensional fixed point set stabilizes 2 sets $D_{i,j,k}$. Hence the number of elements with zero dimensional stabilizer is $20 \times 6/2 = 60$ and each such element stabilizes 6 points. The quotient of a zero dimensional fixed point set of an element by the group has the Euler characteristic equal to 2. The contribution in (5) from the identity element is

$$\chi(V_{\lambda}/G_{s1}) = \frac{1}{|G_{s1}|} \sum_{g} \chi(V_{\lambda}^g)$$

which is equal to $1/81(-144 + 60 \times 6 + 12 \times (-18)) = 0$. Hence using (5) the Euler characteristic of a Calabi–Yau resolution is $0 + 2 \times 60 + 2 \times 12 = +144$.

**A method for constructing the Picard Fuchs equations.** To find the Picard Fuchs equations for the periods of $W(V_{\lambda})$ we shall extend to complete intersections the Griffiths description of cohomology classes of hypersurfaces using meromorphic forms on the ambient space. Let $T(\tilde{Q}_1 \cap \tilde{Q}_2)$ be a small tubular neighbourhood of $\tilde{Q}_1 \cap \tilde{Q}_2$ in $\mathbb{CP}^5$ and $\partial T(\tilde{Q}_1 \cap \tilde{Q}_2)$ be the boundary of $T(\tilde{Q}_1 \cap \tilde{Q}_2)$. Then

$$H^3(\tilde{Q}_1 \cap \tilde{Q}_2) = H^3(\tilde{Q}_1 \cap \tilde{Q}_2)^* = H^7(T(\tilde{Q}_1 \cap \tilde{Q}_2, \partial T(\tilde{Q}_1 \tilde{Q}_2)) = H^7(\mathbb{CP}^5, \mathbb{CP}^5 - T(\tilde{Q}_1 \cap \tilde{Q}_2))$$

( use Poincare duality, retraction combined with Lefschetz duality, and excision). The latter group is isomorphic to $H^6(\mathbb{CP}^5 - \tilde{Q}_1 \cap \tilde{Q}_2)$ as follows from the exact sequence of the pair. The Mayer Vietoris sequence combined with these isomorphisms gives the identification:

$$H^5(\mathbb{CP}^5 - (\tilde{Q}_1 \cup \tilde{Q}_2)/Im \ (H^5(\mathbb{CP}^5 - \tilde{Q}_1) \oplus H^5(\mathbb{CP}^5 - \tilde{Q}_2)) = H^3(\tilde{Q}_1 \cap \tilde{Q}_2) \quad (7)$$

An alternative description of this isomorphism can be obtained by interpreting a meromorphic 5-form on $\mathbb{CP}^5$ having poles along $\tilde{Q}_1 \cup \tilde{Q}_2$ as a functional on $H_3(\tilde{Q}_1 \cap \tilde{Q}_2)$ which is given by assigning to a 3-cycle $\gamma$ representing a homology class in the latter group the integral over a 5-cycle in $\mathbb{CP}^5 - (\tilde{Q}_1 \cup \tilde{Q}_2)$; This 5-cycle is the restriction to $\gamma$ of a torus fibration on which $T(\tilde{Q}_1 \cap \tilde{Q}_2) - (\tilde{Q}_1 \cup \tilde{Q}_2) \cap T(\tilde{Q}_1 \cap \tilde{Q}_2)$ retracts as a consequence of the non-singularity of $\tilde{Q}_1 \cap \tilde{Q}_2$. Moreover in the isomorphism (7) the filtration by the total order of the pole corresponds to the Hodge filtration on $H^3(\tilde{Q}_1 \cap \tilde{Q}_2)$ (details of this will appear elsewhere). The residues of the meromorphic 5-forms which are $G_{s1}$-invariant give
the forms on \( V_{\lambda} \) which descend to \( V_{\lambda}/G_{81} \); The pull–back of these forms, which give a basis of \( H^3(W(V_{\lambda})) \), are

\[
(x_1x_2x_3)^{i-1}(x_4x_5x_6)^{n-i-1}\Omega
\]

\( Q_1^iQ_2^{n-i} \)

where \( n = 2, 3, 4, 5 \) and \( \Omega \) is the Euler form:

\[
\Omega = \sum (-1)^i x_i dx_1 \wedge \cdots \wedge \hat{d}x_i \wedge \cdots \wedge dx_6.
\]

**Calculating the Picard–Fuchs Equation.** A cohomology class in \( H^3(V_{\lambda}) \) by (7) is represented by a differential form

\[
\eta = \sum_{i=1}^{n} \frac{P_i}{Q_1^iQ_2^{n-i}}\Omega
\]

where \( \deg(P_i) = 3(n-2) \) and \( n \geq 2 \). Relations among forms of this type arise from consideration of forms \( d\phi \), where

\[
\phi = \sum (x_iA_j - A_ix_j) dx_1 \cdots \hat{d}x_i \cdots \hat{d}x_j \cdots dx_6.
\]

The relations take the form

\[
\frac{i \sum A_i \partial Q_1}{Q_1^{i+1}Q_2^j} + j \sum A_i \partial Q_2 = \sum \frac{\partial A_i}{\partial x_i} \equiv \frac{\partial A_i}{\partial x_i} \pmod{\text{exact}} (9)
\]

In addition, a form with poles along only one of the forms \( Q_i \) is equivalent to zero:

\[
\frac{P}{Q_i^j} \Omega \equiv 0 \pmod{\text{exact}} (10)
\]

We will now describe a procedure for finding canonical representations for meromorphic forms modulo the relations (1) and (2), by constructing an explicit representation of these relations.

Let \( J_1 \) and \( J_2 \) represent the rows of the jacobian matrix of \((Q_1, Q_2)\):

\[
J_i = \begin{pmatrix}
\frac{\partial Q_1}{\partial x_1} & \cdots & \frac{\partial Q_1}{\partial x_6}
\end{pmatrix}
\]

If \( n > 2 \) is an integer, we construct an \((n-1) \times 6(n-2)\) matrix \( B_n \) as follows:

\[
B_n = \begin{pmatrix}
(n-2)J_1 & 0 & 0 & \cdots & 0 & 0 \\
J_2 & (n-3)J_1 & 0 & \cdots & 0 & 0 \\
0 & 2J_2 & (n-4)J_1 & \cdots & 0 & 0 \\
0 & 0 & 3J_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & (n-3)J_2 & J_1 \\
0 & 0 & 0 & 0 & (n-2)J_2 & \end{pmatrix}
\]
Let $I_{n-1}$ denote the $(n-1) \times (n-1)$ identity matrix. We consider the module presented by the $(n-1) \times (8n-14)$ matrix $K_n = (B_n \ Q_1 I_{n-1} \ Q_2 I_{n-1})$:

$$S^{8n-14} \rightarrow S^{n-1} \rightarrow M_n^* \rightarrow 0,$$

where $S$ is the graded polynomial ring in the variables $x_1, \ldots, x_6$.

When $n > 2$ let $M_n$ denote the part of $M_n^*$ which is homogeneous of degree $3(n-2)$, and let $M_2 = \mathbb{C}$.

To see the relationship between $M_n$ and $H^3(V_\lambda)$, suppose that $\omega$ belongs to $\text{Fil}^i H^3(V_\lambda)$ i.e. the Hodge filtration of $\omega$ is $i$. We may represent $\omega$ in the form

$$\omega = \left( \sum_{k=1}^{i+1} \frac{p_k}{Q_1^i Q_2^{i+1-k}} \right) \Omega.$$ 

Define a homogeneous map $\phi_i$ from $\text{Fil}^i H^3(V_\lambda)$ to $S^{i+1}$ by setting $\phi_i(\omega) = (p_1, \ldots, p_{i+1})$, and we let $\overline{\phi}_i$ denote the composition of $\phi_i$ with the projection map to $M^*_{i+2}$. It follows from our description of the relations (9) and (10) that

$$0 \rightarrow \text{Fil}^i H^3(V_\lambda) \rightarrow \text{Fil}^i H^3(V_\lambda) \rightarrow M^*_{i+2} \rightarrow 0 \quad (11)$$

is exact. We may now briefly describe an algorithm for putting a cohomology class $\omega$, presented as above, into a standard form. This standard form will consist of elements $m_k(\omega) \in M_k$ for $k = 2, \ldots, n+2$ with the property that two forms $\omega$ and $\omega'$ represent the same cohomology class if and only if $m_k(\omega) = m_k(\omega')$ for all $k$ in this range.

**Step 1.** Compute Grobner bases for the modules $M_n$ for $n = 0, \ldots, i+1$. Such a calculation provides a canonical form for elements of $M_{i+2}$ represented as vectors in $S^{i+1}$.

**Step 2.** Reduce the vector $(p_1, \ldots, p_{i+1})$ to canonical form modulo the image of $I_{i+2}$ using Step 1. Suppose that $m_{i+2}(\omega)$ is this canonical form. In the reduction process, compute a vector $A$ so that

$$\left( \begin{array}{c} p_1 \\ \vdots \\ p_{i+1} \end{array} \right) = m_{i+2}(\omega) + K_{i+2} A.$$  

**Step 3.** Let $A_{i,j}$ denote the subvector of $A$ consisting of the entries $A_i, \ldots, A_j$. We denote by $\nabla \cdot A_{i,i+5}$ the usual “divergence” of the 6–vector $A_{i,i+5}$ relative to the $x_i$:

$$\nabla \cdot A_{i,i+5} = \Sigma_k \frac{\partial A_k}{\partial x_k}.$$ Construct a new vector $p'$ in $S^i$ representing $\omega - m_{i+2}(\omega)$ (which, by the lemma, belongs to $\text{Fil}^{i-1}$) by defining:

$$p_1 = \nabla \cdot A_{1,6} + A_{6i+1}$$
$$p_2 = \nabla \cdot A_{7,12} + A_{6i+2} + A_{7i+3}$$
$$\vdots$$
$$p_{i-1} = \nabla \cdot A_{6i-11,6i-6} + A_{7i-1} + A_{8i+1}$$
$$p_i = \nabla \cdot A_{6i-5,6i} + A_{8i+2}$$
Repeat Steps 2 and 3 for $p'$, and continue decreasing $i$ by one each time, until $i = 2$.

We must apply this algorithm in one concrete situation, which we now describe. Define 3-forms $\omega_i$, for $i = 2, 3, \ldots$ by the formula

$$\omega_n = (-1)^n(n-2)! \sum_{i=1}^{n-1} \frac{\lambda^n(x_1x_2x_3)^{i-1}(x_4x_5x_6)^{n-i-1}}{Q_1^n Q_2^{n-1}} \Omega.$$ 

These define forms on the complement of $\tilde{Q}_1 \cup \tilde{Q}_2$ which, by the residue construction, define cohomology classes on $V_\lambda$ invariant under the automorphism group $G_{81}$. In fact, these forms span the space of $G_{81}$-invariant three forms on $V_\lambda$, and therefore span $H^3(W_\lambda)$ for the mirror manifold.

Let $z = \lambda^{-6}$, so that $z$ is a uniformizing parameter at $\infty$ for the parameter space of $V_\lambda$. In terms of the derivation

$$\Theta = z \frac{d}{dz} = -\frac{1}{6} \frac{d}{d\lambda}$$

we have the following fundamental relation:

$$\Theta \omega_i = -\frac{i}{6} \omega_i + \omega_{i+1}.\quad (13)$$

It follows from this relation, $rk H^3(W(V_\lambda)) = 4$, and the $G_{81}$ invariance of the forms $\omega_i$ that $\omega_6$ is dependent on the forms $\omega_2, \ldots, \omega_5$. By analogy with Morrison ([M2]), we postulate a relationship of the following form:

$$\omega_6(z) = \sum_{i=2}^{5} \frac{a_i z + b_i}{z-1} \omega_i(z)$$

(14)

where the $a_i$ and $b_i$ are small rational numbers. Once the $a_i$ and $b_i$ are known, it is straightforward to compute the Picard–Fuchs equation as in [M2]. The most powerful tool available for carrying out the calculations described in the reduction algorithm and computing the relation (14) is the Macaulay program of Bayer and Stillman ([Mac]). It has one sizeable limitation which limits its direct application to our problem – it computes Grobner bases over a finite field, whereas at first glance our problem requires computing over the rational function field $\mathbb{C}(\lambda)$. However, if we assume the form of the relation we seek is as in (14), we may avoid this problem by exploiting the Chinese Remainder Theorem:

**Step 1.** Set the parameter value $\lambda$ to various constant values $\lambda_0$ in the finite field $\mathbb{F}_p$. Now use Macaulay to apply the reduction algorithm in the corresponding fiber of the family and find the relations:

$$\omega_6(\lambda_0^{-6}) = \sum h_i(\lambda_0^{-6}) \omega_i(\lambda_0^{-6}).$$

Here the $h_i$ are constants in $\mathbb{F}_p$, and these relations are the specializations of the relation (14).
Step 2. Knowledge of the values of the $h_i$ for, say, three distinct $\lambda_0$ determines the $a_i$ and $b_i \mod p$. Now repeat the calculation in Step 1 for various different choices of $p$ (again using Macaulay), then apply the Chinese remainder theorem. (This is not totally straightforward, since the $a_i$ and $b_i$ are rational numbers, not integers, and we have no proved a priori estimate on their denominators; we guessed that the denominators involved powers of two and three, found some reasonable $a_i$ and $b_i$, then verified that those coefficients worked for many choices of prime $p$.)

Using this method, we found the following relation:

$$\omega_6 = \frac{z - 7}{3(z - 1)}\omega_5 + \frac{z + 55}{36(z - 1)}\omega_4 + \frac{z - 65}{216(z - 1)}\omega_3 + \frac{1}{81(z - 1)}\omega_2. \quad (15)$$

The associated Picard–Fuchs equation, calculated using this relation and (13), is the generalized hypergeometric equation:

$$(\Theta^4 - z(\Theta + 1/3)^2(\Theta + 2/3)^2)F = 0 \quad (16)$$

In particular, this implies that the monodromy at $\lambda = \infty$ is maximally unipotent.

**Computing the Yukawa Coupling.** To determine the expansion of the Yukawa coupling from the equation, we again follow [M2]. The holomorphic solution $F_0$ to (5) is

$$F_0(z) = \sum_{n=0}^{\infty} \left(\frac{(3n)!}{(n!)^3}\right)^2 \left(\frac{z}{3^6}\right)^n.$$

We let $F_1$ denote the unique solution to (16) which involves $\log(z)$ (but no higher powers of $\log(z)$) and such that

$$s(z) = F_1(z)/F_0(z)$$

has the property

$$s(z) \sim \log(3^{-6}z) = -6 \log(3\lambda) \quad \text{as} \quad z \to 0. \quad (17)$$

(This is the asymptotic normalization mentioned at the beginning of the paper, in this special case.) If we let

$$W = F_0\Theta F_1 - F_1\Theta F_0$$

then the Yukawa potential $\kappa_{sss}$, expressed in the canonical parameter $q(z) = \exp(s(z))$, and normalized so that its leading term is 9 (=the degree of our Calabi–Yau family $V_\lambda$) is

$$\kappa_{sss} = -9\frac{F_0^4}{W^3(s(q) - 1)}.$$

To determine the predicted number of rational curves of given degree, we write $\kappa_{sss}$ in the form

$$\kappa_{sss} = 9 + \sum_{d} \frac{n_d d^3 q^d}{1 - q^d}.$$
With our choices of normalization, we obtain integral values for the $n_d$, and record them in Table 1.

**Extrapolations.** We know that the Picard–Fuchs equation associated to the quintic hypersurface is the generalized hypergeometric equation with parameters $\{1/5, \ldots, 4/5\}$, while that for the complete intersection of two cubics is the hypergeometric equation with parameters $\{1/3, 1/3, 2/3, 2/3\}$. It seems reasonable us to extrapolate from this that the equations for the remaining types of Calabi–Yau complete intersections are hypergeometric as well; with parameters as given in the following table:

| Description                        | Parameters          |
|------------------------------------|---------------------|
| 4 quadrics in $\mathbf{P}^7$       | $\{1/2, 1/2, 1/2\}$|
| 2 quadrics and cubic in $\mathbf{P}^6$ | $\{1/2, 1/2, 1/3, 2/3\}$ |
| 2 cubics in $\mathbf{P}^5$         | $\{1/3, 2/3, 1/3, 2/3\}$ |
| Quartic and quadric in $\mathbf{P}^5$ | $\{1/4, 1/2, 3/4, 1/2\}$ |
| Quintic in $\mathbf{P}^4$          | $\{1/5, 2/5, 3/5, 4/5\}$ |

Based on this hypothesis we calculated the Yukawa potential in each of these cases. There are two constants which must be chosen for each such calculation; one of these forces $\kappa_{sss}$ to have initial term the degree of the variety, while the other determines the asymptotic behavior of the coordinate $s$ in terms of the “hypergeometric” variable $z$ as in equation (16). In each case, we made the choice

$$s(z) \sim \log(z) - \sum d_i \log(d_i)$$

(18)

where the $d_i$ are the degrees of the hypersurfaces defining the complete intersection. With these choices, we obtained the correct values for the number of straight lines in each case, and integral values for the predicted number of rational curves. The results of our calculations are summarized in the Table 1.
Table 1.
Numerical Results

Predicted Number of Rational Curves of Given Degree
For Various Types of Complete Intersection Calabi–Yau Manifolds

| Degree | $V_{3,3} \subset \mathbb{P}^5$ | $V_{2,4} \subset \mathbb{P}^5$ |
|--------|-------------------------------|-------------------------------|
| 1      | 1053*                         | 1280*                         |
| 2      | 52812                         | 92288                         |
| 3      | 642432                        | 15655168                      |
| 4      | 1139448384                    | 3883902528                    |
| 5      | 249787892583                  | 1190923282176                 |
| 6      | 62660964509532                | 417874605342336               |
| 7      | 17256453900822009             | 160964588281789696             |
| 8      | 5088842568426162960           | 66392895625625639488           |
| 9      | 158125071797655787945         | 28855060316616488359936        |
| 10     | 512045241907209106828608      | 13069047760169269024822656     |

| Degree | $V_{2,2,2,2} \subset \mathbb{P}^7$ | $V_{2,2,3} \subset \mathbb{P}^6$ |
|--------|-------------------------------|-------------------------------|
| 1      | 512*                          | 720*                          |
| 2      | 9728                          | 22428                         |
| 3      | 416256                        | 1611504                       |
| 4      | 25703936                      | 168199200                     |
| 5      | 1957983744                    | 21676931712                   |
| 6      | 170535923200                  | 3195557904564                 |
| 7      | 16300354777600                | 517064870788848                |
| 8      | 1668063096387072             | 89580965599606752              |
| 9      | 179845756064329728           | 16352303769375910848           |
| 10     | 20206497983891554816         | 3110686153486233022944         |

(*) These numbers coincide with those given in [L] p. 52. The number of lines on $V_{2,2,2,2}$ (resp. $V_{2,2,3}$) is not given explicitly there (only as part of theorem 3). It is easy to
check that the lines belonging to a quadric in $\mathbf{P}^7$ form a cycle on the Grassmanian $Gr(1,7)$ of lines in $\mathbf{P}^7$ which is homologous to $4\Omega_{4,6}$ ($\Omega_{p,q}$ denotes the Schubert cycle consisting of lines in a generic $\mathbf{P}^q$ intersecting generic $\mathbf{P}^p \subset \mathbf{P}^q$). Its 4-fold self–intersection equals 512, which gives the number of lines on $V_{2,2,2,2}$. On the other hand, the lines in $\mathbf{P}^6$ which belong to a generic hypersurface of degree 3 (resp. 2) form the cycle in $Gr(1,6)$ homologous to $18\Omega_{2,5} + 27\Omega_{3,4}$ (resp. $4\Omega_{3,5}$). The intersection index: $(18\Omega_{2,5} + 27\Omega_{3,4})(4\Omega_{3,5})^2$ equals 720 which gives the number of lines on $V_{3,2,2,2}$.

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P.S. Besides calculation in [L] of the number of lines on generic complete intersection of arbitrary dimension on the case when it is finite the case of lines on three dimensional complete intersections with $K = 0$ also treated in S.Katz paper in Math. Zeit. 191 (1986) .293-296. We are thanking S.Katz for pointing this out.