Linear-Time Compression of Bounded-Genus Graphs into Information-Theoretically Optimal Number of Bits

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January 11, 2014

Abstract

A compression scheme $A$ for a class $G$ of graphs consists of an encoding algorithm $Encode_A$ that computes a binary string $Code_A(G)$ for any given graph $G$ in $G$ and a decoding algorithm $Decode_A$ that recovers $G$ from $Code_A(G)$. A compression scheme $A$ for $G$ is optimal if both $Encode_A$ and $Decode_A$ run in linear time and the number of bits of $Code_A(G)$ for any $n$-node graph $G$ in $G$ is information-theoretically optimal to within lower-order terms. Trees and plane triangulations were the only known non-trivial graph classes that admit optimal compression schemes. Based upon Goodrich’s separator decomposition for planar graphs and Djidjev and Venkatesan’s planarizers for bounded-genus graphs, we give an optimal compression scheme for any hereditary (i.e., closed under taking subgraphs) class $G$ under the premise that any $n$-node graph of $G$ to be encoded comes with a genus-$o(\frac{n}{\log^2 n})$ embedding. By Mohar’s linear-time algorithm that embeds a bounded-genus graph on a genus-$O(1)$ surface, our result implies that any hereditary class of genus-$O(1)$ graphs admits an optimal compression scheme. For instance, our result yields the first-known optimal compression schemes for planar graphs, plane graphs, graphs embedded on genus-1 surfaces, graphs with genus 2 or less, 3-colorable directed plane graphs, 4-outerplanar graphs, and forests with degree at most 5. For non-hereditary graph classes, we also give a methodology for obtaining optimal compression schemes. From this methodology, we give the first known optimal compression schemes for triangulations of genus-$O(1)$ surfaces and floorplans.

*Accepted to SIAM Journal on Computing. A preliminary version appeared in SODA [65].
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1 Introduction

Compact representation of graphs are fundamentally important and useful in many applications, including representing the meshes in finite-element analysis, terrain models of GIS, and 3D models of graphics [80, 82, 81, 92, 85, 64, 89, 48], VLSI design [84, 56], designing compact routing tables of computer networks [94, 37, 66, 77, 35, 95, 1, 16, 3, 36], and compressing the link structure of the Internet [15, 2, 88, 7, 5, 21]. Let $G$ be a class of graphs. Let $\text{num}(G, n)$ denote the number of distinct $n$-node graphs in $G$. The information-theoretically optimal number of bits to encode an $n$-node graph in $G$ is $\lceil \log \text{num}(G, n) \rceil$.

For instance, if $G$ is the class of rooted trees, then $\text{num}(G, n) \approx 2^{2n/3}$ and $\log \text{num}(G, n) = 2n - O(\log n)$; if $G$ is the class of plane triangulations, then $\log \text{num}(G, n) = \log \frac{256}{27} n + o(n) \approx 3.2451 n + o(n)$ [97]. A compression scheme $A$ for $G$ consists of an encoding algorithm $\text{Encode}_A$ that computes a binary string $\text{Code}_A(G)$ for any given graph $G$ in $G$ and a decoding algorithm $\text{Decode}_A$ that recovers graph $G$ from $\text{Code}_A(G)$. A compression scheme $A$ for a graph class $G$ with $\log \text{num}(G, n) = O(n)$ is optimal if the following three conditions hold.

Condition C1: The running time of algorithm $\text{Encode}_A(G)$ is linear in the size of $G$.
Condition C2: The running time of algorithm $\text{Decode}_A(\text{Code}_A(G))$ is linear in the bit count of $\text{Code}_A(G)$.
Condition C3: For all positive constants $\beta$ with $\log \text{num}(G, n) \leq \beta n + o(n)$, the bit count of $\text{Code}_A(G)$ for an $n$-node graph $G$ in $G$ is no more than $\beta n + o(n)$.

Condition C3 basically says that the bit count of $\text{Code}_A(G)$ is information-theoretically optimal to within lower-order terms. Although there has been considerable work on compression schemes, trees (see e.g., [72, 50, 67, 11]) and plane triangulations [79] were the only known nontrivial graph classes that admit optimal compression schemes. A graph class is hereditary if it is closed under taking subgraphs. Below is the main result of the paper.

Theorem 1.1. Any hereditary class $G$ of graphs with $\log \text{num}(G, n) = O(n)$ admits an optimal compression scheme, as long as each input $n$-node graph in $G$ to be encoded comes with a genus-$o(\frac{n}{\log^2 n})$ embedding.

By Theorem 1.1 and Mohar’s linear-time genus-$O(1)$ embedding algorithm for genus-$O(1)$ graphs [70, 54] (see Lemma 2.5), any hereditary class of genus-$O(1)$ graphs admits an optimal compression scheme. For instance, our result yields the first-known optimal compression schemes for planar graphs, plane graphs, graphs embedded on genus-1 surfaces, graphs with genus 2 or less, 3-colorable directed plane graphs, 4-outerplanar graphs, and forests with degree at most 5. For non-hereditary graph classes, we also give an extension (see Corollary 5.1) of Theorem 1.1. As summarized in the following theorem, we show two classes of genus-$O(1)$ graphs whose optimal compression schemes

1All logarithms throughout the paper are to the base of two.
are obtainable via this extension, where the class of floorplans is defined in related work below.

**Theorem 1.2.** The following classes of graphs admit optimal compression schemes:

1. Triangulations of a genus-\( g \) surface for any integral constant \( g \).
2. Floorplans.

**Technical overview** The kernel of the proof of Theorem 1.1 is a linear-time disjoint partition \( G_0, \ldots, G_p \) of an \( n \)-node graph \( G \) embedded on a genus-\( o\left(\frac{n}{\log n}\right) \) surface. Let \( \text{poly}(n) \) denote \( O(n^{O(1)}) \). Based upon Goodrich’s separator decomposition of planar graphs and Djidjev and Venkatesan’s planarizer, partition \( G_0, \ldots, G_p \) satisfies the following conditions, where \( n_i \) is the number of nodes of \( G_i \) and \( d_i \) is the number of times that the nodes of \( G_i \) are duplicated in some \( G_j \) with \( j \neq i \):

- (a) \( n_0 = o\left(\frac{n}{\log n}\right) \)
- (b) \( n_i = \text{poly}(\log n) \)
- (c) \( \sum_{i=1}^{p} d_i = O\left(\frac{n}{\log n}\right) \)
- (d) \( \sum_{i=0}^{p} n_i = n + o\left(\frac{n}{\log n}\right) \).

By Condition (a), \( G_0 \) can be encoded in \( O(n) \) bits. By Conditions (b) and (c), the information required to recover \( G \) from \( G_0, G_1, \ldots, G_p \) can be encoded into \( o(n) \) bits (see Lemma 4.1). By Condition (d), we have \( \log \text{num}(G, n) \leq o(n) + \sum_{i=0}^{p} \log \text{num}(G, n_i) \). Therefore, the disjoint partition reduces the problem of encoding an \( n \)-node graph in \( G \) to the problem of encoding a \( \text{poly}(\log n) \)-node graph in \( G \). Applying such a reduction for one more level, it remains to encode a \( \text{poly}(\log \log n) \)-node graph in \( G \) into an information-theoretically optimal number of bits, which can be resolved by the standard technique (see, e.g., [47, 72, 78]) of precomputation tables (see Lemma 2.3).

**Related work** The compression scheme of Turán encodes an \( n \)-node plane graph that may have self-loops into \( 12n \) bits. Keeler and Westbrook improved this bit count to \( 10.74n \). They also gave compression schemes for several families of plane graphs. In particular, they used \( 4.62n \) bits for plane triangulation, and \( 9n \) bits for connected plane graphs free of self-loops and degree-one nodes. For plane triangulations, He et al. improved the bit count to \( 4n \). For triconnected plane graphs, He et al. also improved the bit count to at most \( 8.585n \) bits. This bit count was later reduced to at most \( \frac{9}{2} \log_2 3 \approx 7.134n \) by Chuang et al. For any given \( n \)-node graph \( G \) embedded on a genus-\( g \) surface, Deo and Litow showed an \( O(n^2g^2) \)-bit encoding for \( G \). These compression schemes all take linear time for encoding and decoding, but Condition C3 does not hold for them. The compression schemes of He et al. (respectively, Blelloch et al.) for planar graphs, plane graphs, and plane triangulations (respectively, separable graphs) satisfy Condition C3 but their encoding algorithms require \( \Omega(n \log n) \) time on \( n \)-node graphs.

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2 Precisely, the disjoint partition \( G_0, \ldots, G_p \) of the edges of the embedded graph \( G \) in the proof of Theorem 1.1 is \( G(V_0), G(V_1), \ldots, G(V_p) \), where \( [V_0, \ldots, V_p] \) is both (i) a 1-separation \( S_1 \) of an arbitrary triangulation \( \Delta \) of \( G \) and (ii) a refinement of the 0-separation \( S_0 = [\emptyset, \text{Node}(\Delta)] \) of \( \Delta \).

3 As a matter of fact, in our construction, all duplicated nodes of \( G_i \) with \( i \geq 1 \) belong to \( G_0 \).

4 For brevity, we omit all lower-order terms of bit counts in our discussion of related work.
Figure 1: Three floorplans with 14 nodes, 6 internal faces, and 19 edges. Floorplans (a) and (b) are equivalent, floorplans (b) and (c) are not equivalent.

Floorplanning is a fundamental issue in circuit layout [106, 43, 69, 51, 108, 82, 8, 17, 58, 24, 57, 91, 68, 84, 4]. Motivated by VLSI physical design, various representations of floorplans were proposed [110, 109, 33]. Designing a floorplan to meet a certain criterion is NP-complete in general [87, 44, 100], so heuristic techniques such as simulated annealing [102, 101, 17] are practically useful. The length of the encoding affects the size of the search space. A floorplan, which is also known as rectangular drawing, is a division of a rectangle into rectangular faces using horizontal and vertical line segments. Two floorplans are equivalent if they have the same adjacency relations and relative positions among the nodes. For instance, Figure 1 shows three floorplans: Floorplans (a) and (b) are equivalent. Floorplans (b) and (c) are not equivalent. Let $G$ be the input $n$-node floorplan. Under the conventional assumption that each node of $G$, other than the four corner nodes, has exactly three neighbors (see, e.g., [45, 107]), one can verify that $G$ has $0.5n$ faces and $1.5n - 2$ edges. Yamanaka and Nakano [103] showed how to encode $G$ into $2.5n$ bits. Chuang [19] reduced the bit count to $2.293n$. Takahashi et al. [90] further reduced bit count to $2n$. All these compression schemes for floorplans satisfy Conditions C1 and C2, but not Condition C3. Takahashi et al. [90] also showed that the number of distinct $n$-node floorplans is no more than $3.375^{n+o(n)} \approx 2^{1.755n+o(n)}$. Therefore, our Theorem 1.2 encodes an $n$-node floorplan into at most $1.755n$ bits.

For applications that require query support, Jacobson [50] gave a $\Theta(n)$-bit encoding for a connected and simple planar graph $G$ that supports traversal in $\Theta(\log n)$ time per node visited. Munro and Raman [71] improved this result and gave schemes to encode binary trees, rooted ordered trees, and planar graphs. For a general $n$-node $m$-edge planar graph $G$, they used $2m + 8n$ bits while supporting adjacency and degree queries in $O(1)$ time. Chuang et al. [20] reduced this bit count to $2m + (5 + \frac{1}{k})n$ for any constant $k > 0$ with the same query support. The bit count can be further reduced if only $O(1)$-time adjacency queries are supported, or if $G$ is simple, triconnected or triangulated [20]. Chiang et al. [18] reduced the number of bits to $2m + 2n$. Yamanaka and Nakano [105] showed a $6n$-bit encoding for plane triangulations with query support. The succinct encodings of Blandford et al. [13] and Blelloch et al. [14] for separable graphs support queries. Yamanaka et al. [104] also gave a compression scheme for floorplans with query support. For labeled planar graphs, Itai and Rodeh [49] gave an encoding.
of \( \frac{3}{2}n \log n \) bits. For unlabeled general graphs, Naor [74] gave an encoding of \( \frac{1}{2}n^2 \) bits. For certain graph families, Kannan et al. [52] gave schemes that encode each node with \( O(\log n) \) bits and support \( O(\log n) \)-time testing of adjacency between two nodes. Galperin and Wigderson [34] and Papadimitriou and Yannakakis [75] investigated complexity issues arising from encoding a graph by a small circuit that computes its adjacency matrix. Related work on various versions of succinct graph representations can be found in [73, 6, 38, 42, 38, 76, 83, 30, 29, 28, 9, 53] and the references therein.

Outline The rest of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 shows our algorithm for computing graph separations. Section 4 gives our optimal compression scheme for hereditary graph classes. Section 5 shows a methodology for obtaining optimal compression schemes for non-hereditary graph classes and applies this methodology on triangulations of genus-\( O(1) \) graphs and floorplans. Section 6 concludes the paper with a couple of open questions.

2 Preliminaries

2.1 Segmentation prefix

Let \( \|X\| \) denote the number of bits of binary string \( X \). A binary string \( X_0 \) is a segmentation prefix of binary strings \( X_1, \ldots, X_d \) if (a) it takes \( O(\sum_{i=1}^{d} \|X_i\|) \) time to compute \( X_0 \) from \( X_1, \ldots, X_d \) and (b) given the concatenation of \( X_0, X_1, \ldots, X_d \), it takes \( O(\sum_{i=0}^{d} \|X_i\|) \) time to recover all \( X_i \) with \( 1 \leq i \leq d \).

**Lemma 2.1** (See, e.g., [10, 27]). Any binary strings \( X_1, \ldots, X_d \) with \( d = O(1) \) have a segmentation prefix with \( O(\log \sum_{i=1}^{d} \|X_i\|) \) bits.

**Lemma 2.2.** Any binary strings \( X_1, \ldots, X_d \) have an \( O(\min\{m, d \log m\}) \)-bit segmentation prefix, where \( m = \|X_1\| + \cdots + \|X_d\| \).

**Proof.** Let \( X \) be the concatenation of \( X_1, \ldots, X_d \). If \( m \leq d \log m \), let \( X' \) be the \( m \)-bit binary string with exactly \( d \) copies of 1-bits such that the \( j \)-th bit of \( X' \) is 1 if and only if \( j = \|X_1\| + \cdots + \|X_i\| \) holds for some \( i = 1, \ldots, d \). Otherwise, let \( X' \) store the \( O(\log m) \)-bit numbers \( \|X_1\| + \cdots + \|X_i\| \) for all \( i = 1, \ldots, d \). Let \( X_0' \) be the segmentation prefix of \( X' \) and \( X \) as ensured by Lemma 2.1. The concatenation of \( X_0' \) and \( X' \) is a segmentation prefix \( X_0 \) of \( X_1, \ldots, X_d \) with \( O(\min\{m, d \log m\}) \) bits. The lemma is proved.

For the rest of the paper, let \( X_1 \circ \cdots \circ X_d \) denote the concatenation of \( X_0, X_1, \ldots, X_d \), where \( X_0 \) is the segmentation prefix of \( X_1, \ldots, X_d \) as ensured by Lemma 2.2.
Proof. Straightforward by Lemma 2.3. Let elements of $\mathcal{S}$ be a graph class satisfying $\log \text{num}(G,n) = O(n)$. Given positive integers $\ell$ and $n$ with $\ell = \text{poly}(\log \log n)$, it takes overall $o(n)$ time to compute (i) a labeling $\text{Label}(H)$ and a $\ceil{\log \text{num}(G, \text{node}(H))}$-bit binary string $\text{Optcode}(H)$ for each distinct graph $H \in \mathcal{G}$ with at most $\ell$ nodes and (ii) an $o(n)$-bit string $\text{Table}(G, \ell)$ such that the following statements hold.

1. Given any graph $H \in \mathcal{G}$ with $\text{node}(H) \leq \ell$, it takes $O(\text{node}(H))$ time to obtain $\text{Optcode}(H)$ and $\text{Label}(H)$ from $\text{Table}(G, \ell)$.

2. Given $\text{Optcode}(H)$ for any graph $H \in \mathcal{G}$ with $\text{node}(H) \leq \ell$, it takes $O(\text{node}(H))$ time to obtain $H$ and $\text{Label}(H)$ from $\text{Table}(G, \ell)$.

Proof. Straightforward by $O(1)^{\text{poly}(\ell)} = o(n)$.

2.3 Separator decomposition of planar graphs

Sets $S_1, \ldots, S_d$ form a disjoint partition of set $S$ if $S_1, \ldots, S_d$ are pairwise disjoint and $S = S_1 \cup \cdots \cup S_d$. A subset $S$ of $\text{Node}(G)$ is a separator of graph $G$ with respect to $S_1$ and $S_2$ if

1. $S_1 \cup S_2 \cup S$ is a maximal subset of $\text{Node}(G)$,
2. $S_1$ and $S_2$ are not adjacent in $G$,
3. $|S| = \mathcal{O}(\text{node}(G)^{1/2})$, and
4. $\max\{|S_1|, |S_2|\} \leq \frac{2}{3} \cdot \text{node}(G)$.

A separator decomposition of $G$ is a rooted binary tree $T$ on a disjoint partition of $\text{Node}(G)$ such that the following two statements hold, where “nodes” specify elements of $\text{Node}(G)$ and “vertices” specify elements of $\text{Node}(T)$. Statement 1: Each leaf vertex of $T$ consists of a single node of $G$. Statement 2: Each interior vertex of $T$ has a subset of $\text{Node}(T)$ as its child nodes.
Figure 3: (a) A 9-node plane graph with a separation \([V_0, \ldots, V_3]\). (b) \(G[V_0], G(V_1), G(V_2),\) and \(G(V_3)\) form a disjoint partition of the edges of \(G\).

Statement 2: Each internal vertex \(S\) of \(\mathcal{T}\) is a separator of \(G[\text{Offspring}(S)]\) with respect to \(\text{Offspring}(S_1)\) and \(\text{Offspring}(S_2)\), where \(S_1\) and \(S_2\) are the child vertices of \(S\) in \(\mathcal{T}\) and \(\text{Offspring}(S)\) (respectively, \(\text{Offspring}(S_1)\) and \(\text{Offspring}(S_2)\)) is the union of all the vertices in the subtree of \(\mathcal{T}\) rooted at \(S\) (respectively, \(S_1\) and \(S_2\)). See Figure 2 for an illustration.

**Lemma 2.4** (Goodrich [40]). It takes \(O(n)\) time to compute a separator decomposition for any given \(n\)-node planar graph.

### 2.4 Planarizers for non-planar graphs

The **genus** of a graph \(G\) is the smallest integer \(g\) such that \(G\) can be embedded on an orientable surface with \(g\) handles without edge crossings [41]. For example, the genus of a planar graph is zero. By Euler’s formula (see, e.g., [39]), an \(n\)-node genus-\(O(n)\) graph has \(O(n)\) edges. Determining the genus of a general graph is NP-complete [93], but Mohar [70] showed that it takes linear time to determine whether a graph is of genus \(g\) for any \(g = O(1)\). Mohar’s algorithm is simplified by Kawarabayashi et al. [54].

**Lemma 2.5** (Mohar et al. [70, 54]). It takes \(O(n)\) time to compute a genus-\(O(1)\) embedding for any given \(n\)-node genus-\(O(1)\) graph.

Gilbert et al. [39] gave an \(O(n + g)\)-time algorithm to compute an \(O((gn^{0.5})\)-node separator of an \(n\)-node genus-\(g\) graph, generalizing Lipton and Tarjan’s classic separator theorem for planar graphs [63]. Our result relies on the following planarization algorithm.

**Lemma 2.6** (Djidjev and Venkatesan [26]). Given an \(n\)-node graph \(G\) embedded on a genus-\(g\) surface, it takes \(O(n + g)\) time to compute a subset \(V\) of \(\text{Node}(G)\) with \(|V| = O((gn^{0.5})\) such that \(G \setminus V\) is planar.

### 3 Separation and refinement

We say that \([V_0, \ldots, V_p]\) with \(p \geq 1\) is a separation of graph \(G\) if the following properties hold.
Property S1: $V_0, \ldots, V_p$ form a disjoint partition of $\text{Node}(G)$.

Property S2: Any two $V_i$ and $V_{i'}$ with $1 \leq i \neq i' \leq p$ are not adjacent in $G$.

For instance, Figure 4(a) shows a separation $[V_0, V_1, V_2, V_3]$ of graph $G$ and Figure 4(a) shows another separation $[U_0, U_1, U_2]$ of $G$. For any subset $V$ of $\text{Node}(G)$, let $G(V)$ be the subgraph of $G$ induced by $V \cup \text{Nbr}_G(V)$ excluding the edges of $G[\text{Nbr}_G(V)]$. If $[V_0, \ldots, V_p]$ is a separation of $G$, then $G[V_0], G[V_1], \ldots, G[V_p]$ form a disjoint partition of the edges of $G$. See Figures 3(b) and 4(b) for illustrations. Let $\log^{(0)} n = n$. For any positive integer $k$, let $\log^{(k)} n = \log (\log^{(k-1)} n)$. For notational brevity, for any nonnegative integer $k$, let

$$\ell_k = \max\{1, \log^{(k)} n\}.$$ 

For a nonnegative integer $k$, separation $[V_0, \ldots, V_p]$ of an $n$-node graph $G$ is a $k$-separation of $G$ if the following three properties hold.

Property S3: $|V_0| = o\left(\frac{n}{\ell_k}\right)$ and $p = o\left(\frac{n}{\ell_k}\right) + 1$.

Property S4: $|V_i| + \text{nbr}_G(V_i) = \text{poly}(\ell_k)$ holds for each $i = 1, \ldots, p$.

Property S5: $\sum_{i=1}^p \text{nbr}_G(V_i) = o\left(\frac{n}{\ell_k}\right)$.

One can verify that $[\emptyset, \text{Node}(G)]$ is a 0-separation of $G$\(^5\). Let $[V_0, \ldots, V_p]$ and $[U_0, \ldots, U_q]$ be two separations of graph $G$. We say that $[V_0, \ldots, V_p]$ is a refinement of $[U_0, \ldots, U_q]$ if the following three properties hold.

Property R1: $U_0 \subseteq V_0$.

Property R2: For each index $i = 1, \ldots, p$, there is an index $j$ with $1 \leq j \leq q$ and $V_i \subseteq U_j$.

Property R3: For any indices $i, i', i''$ with $1 \leq i < i' < i'' \leq p$, if $V_i \cup V_{i''} \subseteq U_j$, then $V_{i'} \subseteq U_j$.

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\(^5\)The “+1” in Property S3 is redundant for $k \geq 1$. However, we need it so that $[\emptyset, \text{Node}(G)]$ is a 0-separation of $G$, since $1 \neq o\left(\frac{n}{\ell_0}\right)$.
For instance, in Figure 4(a), \([V_0, V_1, V_2, V_3]\) is a refinement of \([U_0, U_1, U_2]\). Below is the main lemma of the section.

**Lemma 3.1.** Let \(k\) be a positive integer. Let \(G\) be an \(n\)-node connected graph embedded on a genus-\(o(n/\ell_k^2)\) surface. Given a \((k-1)\)-separation \(S_{k-1}\) of \(G\), it takes \(O(n)\) time to compute a \(k\)-separation \(S_k\) of \(G\) that is a refinement of \(S_{k-1}\).

The proof of Lemma 3.1 needs the following lemma, which can be proved by Lemmas 2.4 and 2.6.

**Lemma 3.2.** Let \(k\) be a positive integer. Given an \(n\)-node graph \(G\) embedded on a genus-\(o(n/\ell_k^2)\) surface, it takes \(O(n)\) time to compute an \(o(n/\ell_k)\)-node subset \(V\) of \(\text{Node}(G)\) such that each node of \(\text{Node}(G) \setminus V\) has degree at most \(\ell_k^2\) in \(G\) and each connected component of \(G \setminus V\) has at most \(\ell_k^4\) nodes.

**Proof.** We first apply Lemma 2.6 to compute in \(O(n)\) time an \(o(n/\ell_k)\)-node subset \(V'\) of \(\text{Node}(G)\) such that \(G \setminus V'\) is planar. We then apply Lemma 2.4 to compute in \(O(n)\) time a separator decomposition \(T\) of \(G \setminus V'\). For each vertex \(S\) of \(T\), let \(\text{Offspring}(S)\) denote the union of all the vertices in the subtree of \(T\) rooted at \(S\) and let \(\text{offspring}(S) = |\text{Offspring}(S)|\). Let \(r = \ell_k^2\). Let \(V''\) consist of the nodes of \(G\) with degree more than \(r\) in \(G\). Let \(V'''\) be the union of all the vertices \(S\) of \(T\) with \(\text{offspring}(S) > r^2\). Let \(V = V' \cup V'' \cup V'''\). By \(V' \cup V'' \subseteq V\) and the definition of \(T\), each connected component of \(G \setminus V\) has at most \(r^2\) nodes. By \(V'' \subseteq V\), each node of \(\text{Node}(G) \setminus V\) has degree at most \(r\) in \(G\). Since \(G\) has \(O(n)\) edges, \(|V'''| = O(n/r) = o(n/\ell_k)\). It remains to show \(|V'''| = o(n/\ell_k)\). For each index \(i \geq 1\), let \(I_i\) consist of the vertices of \(T\) with \(r^2 \cdot (\frac{3}{2})^{i-1} < \text{offspring}(S) \leq r^2 \cdot (\frac{3}{2})^i\). By \(r^2 \geq 1\) and \(i \geq 1\), each \(S \in I_i\) is an internal vertex of \(T\). By definition of \(T\), we know that \(\text{Offspring}(S)\) and \(\text{Offspring}(S')\) are disjoint for any two distinct elements \(S\) and \(S'\) of \(I_i\), implying that \(\sum_{S \in I_i} \text{offspring}(S) \leq n\) holds. Since \(\text{offspring}(S) > r^2 \cdot (1.5)^{i-1}\) holds for each \(S \in I_i\), we have \(|I_i| < \frac{n}{r^2 \cdot (1.5)^{i-1}}\). Since each \(S \in I_i\) is an internal vertex of \(T\), \(S\) is a separator of \(G[\text{Offspring}(S)]\). Therefore, \(|S| = O(r \cdot (1.5)^{i/2})\) holds for each vertex \(S\) in \(I_i\). We have \(|V'''| = \sum_{i \geq 1} \sum_{S \in I_i} |S| = \sum_{i \geq 1} O\left(\frac{n}{r^2 \cdot (1.5)^{i/2}}\right) = O\left(\frac{n}{r}\right) = o\left(\frac{n}{\ell_k}\right)\). The lemma is proved.

**Proof of Lemma 3.1** Suppose that \([U_0, \ldots, U_q]\) is the given \((k-1)\)-separation \(S_{k-1}\). Let \(V_0\) be the \(O(n)\)-time computable subset of \(\text{Node}(G)\) ensured by Lemma 3.2. We have \(|V_0| = o\left(\frac{n}{\ell_k}\right)\). Let \(V_0 = U_0 \cup V_0'\). Let \(C\) consist of the connected components of \(G \setminus V_0\). By \(V_0' \subset V_0\), each element of \(C\) has at most \(\ell_k^4\) nodes. By \(U_0 \subset V_0\) and Properties 1 and 2 of \(S_{k-1}\), each element of \(C\) is contained by some \(U_j\) with \(1 \leq j \leq q\). For each \(j = 1, \ldots, q\), let \(C_j\) consist of the elements of \(C\) with \(C \subset U_j\). We run Algorithm 1 to obtain (a) a disjoint partition \(V_1, \ldots, V_q\) of \(G \setminus V_0\) and (b) \(p\) nodes \(\text{hook}_1, \ldots, \text{hook}_p\) of \(V_0\), which may not be distinct. Let \(S_k = [V_0, \ldots, V_q]\). Since \(G\) is connected, each element of \(C\) is adjacent to \(V_0\). The first statement of the outer repeat-loop is well defined. Since each element of \(C\) has at most \(\ell_k^4\) nodes, the first statement of the inner repeat-loop is well defined. See Figure 5 for an illustration: Suppose that all nodes are in \(U_1\). All nodes are initially
Algorithm 1

Let $p = 0$ and let all elements of $C$ be initially unmarked.

For each $j = 1, \ldots, q$, perform the following repeat-loop.

Repeat the following steps until all elements of $C_j$ are marked.

Let $v_0$ be an arbitrary node of $V_0$ that is adjacent to some unmarked element of $C_j$.

Let $U$ consist of the unmarked elements of $C_j$ that are adjacent to $v_0$ in $G$.

Let $C_{i_1}, \ldots, C_{i_3}$ be the elements of $U$ in clockwise order around $v_0$ in $G$.

Mark all $i_3 - i_1 + 1$ elements of $U$.

Repeat the following four steps until $i_3 - i_1 = 1$.

Let $i_2$ be the largest index with $i_1 \leq i_2 \leq i_3$ and $|C_{i_1}| + \cdots + |C_{i_2}| \leq \ell_k^4$.

Let $p = p + 1$.

Let $hook_p = v_0$ and $V_p = C_{i_1} \cup \cdots \cup C_{i_2}$.

Let $i_1 = i_2 + 1$.

Output $V_1, \ldots, V_p$ and $hook_1, \ldots, hook_p$.

Figure 5: An illustration for Algorithm 1

unmarked. Let $V_0$ consist of the nine unlabeled nodes, including the three gray nodes. For each $i = 1, \ldots, 6$, let $C_i$ consist of the nodes with label $i$. That is, $C_1, \ldots, C_6$ are the six connected components of $G \setminus V_0$. Suppose that $\ell_k^4 = 7$ and the first two iterations of the outer repeat-loop obtain $V_1 = C_1$ and $V_2 = C_2$. In the third iteration of the outer repeat-loop, $C_3, \ldots, C_6$ are the unmarked elements of $C$ that are adjacent to $hook_3$ in clockwise order around $hook_3$. By $|C_3| + |C_4| + |C_5| = 7$, the two iterations of the inner repeat-loop obtain $V_3 = C_3 \cup C_4 \cup C_5$ and $V_4 = C_6$.

By definition of Algorithm 1, one can verify that Properties $R1$, $R2$, and $R3$ hold for $S_{k-1}$ and $S_k$ (that is, $S_k$ is a refinement of $S_{k-1}$) and Properties $S1$, $S2$ hold for $S_k$. By Property $S2$ of $S_{k-1}$, we have $|U_0| = o\left(\frac{n}{\ell_{k-1}}\right) = o\left(\frac{n}{\ell_k}\right)$. By $|V'_0| = o\left(\frac{n}{\ell_k}\right)$, we have $|V_0| \leq |U_0| + |V'_0| = o\left(\frac{n}{\ell_k}\right)$. Let $I_{\text{small}}$ consist of the indices $i$ with $1 \leq i \leq p$ and $|V_i| \leq \frac{1}{2} \cdot \ell_k^4$. Let $I_{\text{large}}$ consist of the indices $i$ with $1 \leq i \leq p$ and $|V_i| > \frac{1}{2} \cdot \ell_k^4$. We show $p = |I_{\text{small}}| + |I_{\text{large}}| = o\left(\frac{n}{\ell_k}\right)$ as follows. By Property $S1$ of $S_k$, we have $|I_{\text{large}}| = o\left(\frac{n}{\ell_k}\right)$. To show $|I_{\text{small}}| = o\left(\frac{n}{\ell_k}\right)$, we categorize the indices $i$ in $I_{\text{small}}$ with $1 \leq i \leq p$ into the the following types, where $j$ is the index with $V_i \subseteq U_j$:
Property S4 holds for
$i$ holds for each $i$.

Type 1: $i \in I_{\text{small}}$ and $i + 1 \in I_{\text{large}}$. The number of such indices $i$ is at most $|I_{\text{large}}| = o\left(\frac{n}{\ell_k}\right)$.

Type 2: $i \in I_{\text{small}}$ and $i + 1 \in I_{\text{small}}$.

Type 2a: $V_{i+1} \subseteq U_{j+1}$. The number of such indices $i$ is at most $q = o\left(\frac{n}{\ell_{k-1}}\right) = o\left(\frac{n}{\ell_k}\right)$.

Type 2b: $V_{i+1} \subseteq U_j$ and $\text{hook}_i \in V_0 \setminus U_0$. By Properties S1 and S2 of $S_{k-1}$, we know that $\text{hook}_i \notin U_j$. By definition of Algorithm 1, $\text{hook}_i \neq \text{hook}_{i'}$ holds for all indices $i'$ with $i < i' \leq p$. The number of such indices $i$ is at most $|V_0 \setminus U_0| \leq |V_0| = o\left(\frac{n}{\ell_k}\right)$.

Type 2c: $V_{i+1} \subseteq U_j$ and $\text{hook}_i \in U_0$. We have $\text{hook}_i \in \text{Nbr}_G(U_j)$. By definition of Algorithm 1, $\text{hook}_i \neq \text{hook}_{i'}$ holds for all indices $i' > i$ with $V_{i'} \subseteq U_j$. By Property S5 of $S_{k-1}$, the number of such indices $i$ is at most $\sum_{j=1}^{q} \text{nbr}_G(U_j) = o\left(\frac{n}{\ell_{k-1}}\right) = o\left(\frac{n}{\ell_k}\right)$.

We have $p = o\left(\frac{n}{\ell_k}\right)$. Property S3 holds for $S_k$. By definition of Algorithm 1, $|V_i| \leq \ell_k^2$ holds for each $i = 1, \ldots, p$. By $V'_0 \subseteq V_0$, each node of $\text{Node}(G) \setminus V_0$ has degree at most $\ell_k^2$. Property S4 holds for $S_k$.

To see Property S5 of $S_k$, we obtain a contracted graph from $G$ by performing the following two steps for each $i = 1, \ldots, p^6$

Step 1: Let $C_1, \ldots, C_{p^6}$ be the elements of $C$ with $V_i = C_{i_1} \cup C_{i_1+1} \cup \cdots \cup C_{i_2}$ in clockwise order around $\text{hook}_i$ in $G$. Split $\text{hook}_i$ into two adjacent nodes $\text{hook}_i$ and $v_i$ and let $v_i$ take over the neighbors of $\text{hook}_i$ in clockwise order around $\text{hook}_i$ from the first neighbor of $\text{hook}_i$ in $C_{i_1}$ to the first neighbor of $\text{hook}_i$ in $C_{i_2}$.

Step 2: Contract all nodes of $V_i$ into node $v_i$ and delete multiple edges and self-loops. See Figure 6(a) for an illustration: For each $i = 3, \ldots, 6$, let $C_i$ consist of the nodes with labels $i$ in Figure 6(a). Suppose that $i_1 = 3$, $i_2 = 5$, and $V_i = C_3 \cup C_4 \cup C_5$. The unlabeled circle nodes belong to $V_0$. The square nodes are two previously contracted nodes $v_i$ and $v_{i''}$ from $V_{i'}$ and $V_{i''}$ for some indices $i'$ and $i''$ with $1 \leq i' \neq i'' < i$. Figure 6(b) shows the result of Step 1. Figure 6(c) shows the result of Step 2. Observe that each node that is adjacent to $V_i$ becomes a neighbor of $v_i$ after applying Steps 1 and 2. Also, each neighbor of $\text{hook}_i$ that is not in $V_i$ either remains a neighbor of $\text{hook}_i$ or becomes a neighbor of $v_i$ after applying

---

6The contraction procedure is only for proving Property S5 of $S_k$, not needed for computing $S_k$. 

Figure 6: The operation that contracts all nodes of $V_i$ into a node $v_i$, which takes over some neighbors of $\text{hook}_i$. 

(a) 

(b) 

(c)
Lemma 4.1. Let $k$ be a positive integer. Let $G$ be an $n$-node graph embedded on a genus-$o(n/k)$ surface. Let $\Delta$ be a triangulation of $G$. Let $S_k = [V_0, \ldots, V_p]$ be a given $k$-separation of $\Delta$ and $S_{k-1} = [U_0, \ldots, U_q]$ be a given $(k-1)$-separation of $\Delta$ such that $S_k$ is a refinement of $S_{k-1}$. For any given labeling $L_{k,i}$ of $G(V_i)$ for each $i = 1, \ldots, p$, the following statements hold.

1. It takes overall $O(n)$ time to compute a labeling $L_{k-1,j}$ of subgraph $G(U_j)$ for each $j = 1, \ldots, q$.

2. Given the above labelings $L_{k-1,j}$ of subgraphs $G(U_j)$ with $1 \leq j \leq q$, it takes $O(n)$ time to compute an $o(n)$-bit string $Rec_k$ such that $G(U_j)$ and $L_{k-1,j}$ for all $j = 1, \ldots, q$ can be recovered in overall $O(n)$ time from $Rec_k$ and $G(V_i)$ and $L_{k,i}$ for all $i = 1, \ldots, p$.

Proof. Since $\Delta$ is a subgraph $G$ with $Node(\Delta) = Node(G)$, one can easily verify that $S_{k-1}$ (respectively, $S_k$) is also a $(k-1)$-separation (respectively, $k$-separation) of $G$. For each $j = 1, \ldots, q$, let $I_j$ consist of the indices $i$ with $V_i \subseteq U_j$. Let $W_j$ consist of the nodes of $G(U_j)$ that are not in any $V_i$ with $i \in I_j$. By Properties 11 and 12 of $S_k$, $W_j \subseteq V_0$. For instance, if $G$ is as shown in Figure 7(a), where $v_t$ with $0 \leq t \leq 8$ denotes the node with label $t$. We have $I_1 = \{1\}$, $I_2 = \{2, 3\}$, $W_1 = \{v_2, v_3\}$, and $W_2 = \{v_0, v_1, v_2, v_6\}$. Let the labeling $L_{k-1,j}$ for $G(U_j)$ be defined as follows.
For the nodes of $G(U_j)$ in $W_j$, let $L_{k-1,j}$ be an arbitrary one-to-one mapping from $W_j$ to $\{0, 1, \ldots, |W_j| - 1\}$. In Figure 7(c), we have $L_{k-1,1}(v_2) = 1$, $L_{k-1,1}(v_3) = 0$, $L_{k-1,2}(v_0) = 2$, $L_{k-1,2}(v_1) = 3$, $L_{k-1,2}(v_2) = 0$, and $L_{k-1,2}(v_6) = 1$.

For the nodes of $G(U_j)$ not in $W_j$, let $L_{k-1,j}$ be the one-to-one mapping from $\bigcup_{i \in I_j} V_i$ to $\{|W_j|, |W_j| + 1, \ldots, \text{node}(G(U_j)) - 1\}$ obtained by sorting $(i, L_{k,i}(v))$ for all indices $i \in I_j$ and all nodes $v \in V_i$ such that $L_{k-1,j}(v) < L_{k-1,j}(v')$ holds for a node $v$ of $V_i$ and a node $v'$ of $V_i'$. If and only if (a) $i < i'$ or (b) $i = i'$ and $L_{k,i}(v) < L_{k,i}(v')$. For instance, if $L_{k,1}, L_{k,2}$, and $L_{k,3}$ are as shown in Figure 7(b), then $L_{k-1,1}$ and $L_{k-1,2}$ can be as shown in Figure 7(c) and $L_{k-1,1}$ can be as shown in Figure 7(a).

It takes $O(\text{node}(G(U_j))) = O(|U_j| + \text{nbr}_G(U_j))$ time to compute $L_{k-1,j}$ from all $L_{k,i}$ with $i \in I_j$. By Property 5 of $S_{k-1}$, it takes overall $O(n)$ time to compute all $L_{k-1,j}$ with $1 \leq j \leq q$ from all $L_{k,i}$ with $1 \leq i \leq p$. Statement 4 is proved.

By Property 4 of $S_{k-1}$, the label of each node of $G(U_j)$ assigned by $L_{k-1,j}$ can be represented by $O(\log \text{poly}(\ell_{k-1})) = O(\ell_k)$ bits. By Property 4 of $S_k$, the label of each node of $G(V_i)$ assigned by $L_{k,i}$ can be represented by $O(\log \text{poly}(\ell_k)) = O(\ell_{k+1})$ bits. For each index $j = 1, \ldots, q$,

- string $Rec'_{k,j}$ stores the adjacency list of the embedded subgraph of $G(V_j)$ induced by $W_j$ via the labeling $L_{k-1,j}$ of $W_j$,
- string $Rec''_{k,j}$ stores the information required to recover $L_{k-1,j}$ from all $L_{k,i}$ with $i \in I_j$, and
- string $Rec'''_{k,j}$ stores the information required to recover the embedding of $G(U_j)$ from the embeddings of all $G(V_i)$ with $i \in I_j$ and the embedding of the subgraph of $G(U_j)$ induced by $W_j$.

By definition of $W_j$, we have $|W_j| = |V_0 \cap U_j| + \text{nbr}_G(U_j)$. It follows from Property 4 of
$S_k$ and Property S3 of $S_{k-1}$ that
\[
\sum_{j=1}^{q} |W_j| \leq |V_0| + \sum_{j=1}^{q} \text{nbr}_G(U_j) = o\left(\frac{n}{\ell_k}\right) + o\left(\frac{n}{\ell_{k-1}}\right) = o\left(\frac{n}{\ell_k}\right).
\]

Let $W = \bigcup_{j=1}^{q} W_j$. Since $G[V_0], G(V_1), \ldots, G(V_q)$ form a disjoint partition of the edges of $G$, the overall number of edges in the subgraphs of $G(V_j)$ induced by $W_j$ for all $j = 1, \ldots, q$ is no more than the number of edges in $G[W]$, which is $O(|W| + o\left(\frac{n}{\ell_k}\right)) \leq O(\sum_{j=1}^{q} |W_j|) + o\left(\frac{n}{\ell_k}\right) = o\left(\frac{n}{\ell_k}\right)$. Therefore,
\[
\sum_{j=1}^{q} \|\text{Rec}_{k,j}'\| = o\left(\frac{n}{\ell_k}\right) \cdot O(\ell_k) = o(n).
\]

It suffices for $\text{Rec}_{k,j}'$ to store the list of $(i, L_{k-1}(v), L_{k-1,j}(v))$ for all $i \in I_j$ and all $v \in \text{nbr}_G(V_i)$. By Property R3 of $S_{k-1}$ and $S_k$ and Property S4 of $S_{k-1}$, index $i$ can be represented by an $O(\ell_k)$-bit offset $t$ such that $i$ is the $t$-th smallest index in $I_j$. Thus, $\|\text{Rec}_{k,j}'\| = \sum_{i \in I_j} \text{nbr}_G(V_i) \cdot O(\ell_k)$. By Property S5 of $S_k$, we have $\sum_{j=1}^{q} \sum_{i \in I_j} \text{nbr}_G(V_i) = \sum_{i=1}^{n} \text{nbr}_G(V_i) = o\left(\frac{n}{\ell_k}\right)$. Therefore,
\[
\sum_{j=1}^{q} \|\text{Rec}_{k,j}'\| = o\left(\frac{n}{\ell_k}\right) \cdot O(\ell_k) = o(n).
\]

It suffices for $\text{Rec}_{k,j}''$ to store the list of $(L_{k-1}(v), L_{k-1,j}(v''), L_{k-1,j}(v'''))$ for all pairs of edges $(v, v')$ and $(v, v''')$ of $G(U_j)$ such that (a) $v'''$ is the neighbor of $v$ that immediately succeeds $v'$ in clockwise order around $v$ in $G(U_j)$ and (b) nodes $v'$ and $v'''$ are not in the same partition of $\text{Node}(G(U_j))$ formed by the $|I_j| + 1$ disjoint sets $W_j$ and $V_i$ with $i \in I_j$. By Property S2 of $S_k$, node $v$ belongs to $W_j \subseteq V_0$. Since $\Delta$ is a triangulation of $G$, the neighbors of $v$ in $\Delta$ form a cycle that surrounds $v$ in $\Delta$. Let $P$ be the path of the cycle from $v'$ to $v'''$ in clockwise order around $v$. At least one node, say $u$ of $P$ belongs to $V_0$, since otherwise Property S2 of $S_k$ would imply that all nodes of $P$ belong to the same $V_i$ for some $i \in I_j$, contradicting with the choices of $v'$ and $v'''$. Edge $(v, u)$ belongs to $\Delta[V_0]$. Observe that each edge of $\Delta[V_0]$ can be identified by at most four such edge pairs $(v, v')$ and $(v, v''')$. Since the edges of $G(U_j)$ and $G(U_{j'})$ with $1 \leq j \neq j' \leq q$ are disjoint, the number of edge pairs stored in $\text{Rec}_{k,j}'''$ is at most four times the number of edges in $\Delta[V_0]$. By Property S3 of $S_k$ and the fact that $\Delta$ has genus $o\left(\frac{n}{\ell_k}\right)$, the number of edge pairs stored in $\text{Rec}_{k,j}'''$ is $o\left(\frac{n}{\ell_k}\right)$. Therefore,
\[
\sum_{j=1}^{q} \|\text{Rec}_{k,j}'''\| = o\left(\frac{n}{\ell_k}\right) \cdot O(\ell_k) = o(n).
\]

Let
\[
\begin{align*}
\text{Rec}_k' &= \text{Rec}'_{k,1} \circ \cdots \circ \text{Rec}'_{k,q}, \\
\text{Rec}_k'' &= \text{Rec}''_{k,1} \circ \cdots \circ \text{Rec}''_{k,q}, \\
\text{Rec}_k''' &= \text{Rec}'''_{k,1} \circ \cdots \circ \text{Rec}'''_{k,q}, \\
\text{Rec}_k &= \text{Rec}_k' \circ \text{Rec}_k'' \circ \text{Rec}_k'''.
\end{align*}
\]
By Equations (1), (2), and (3) and Lemma 2.2 we have \(|Rec_k| = o(n)|. It takes \(O(n)\) time to compute \(Rec_k\) from all labelings \(L_{k,j}\) and all embedded graphs \(G(U_j)\) with \(1 \leq j \leq q\) and all labelings \(L_{k-1,j}\) and all embedded graphs \(G(V_i)\) with \(1 \leq i \leq p\). It also takes \(O(n)\) time to recover all labelings \(L_{k,j}\) and all embedded graphs \(G(U_j)\) with \(1 \leq j \leq q\) from \(Rec_k\) and all labelings \(L_{k-1,i}\) and all embedded graphs \(G(V_i)\) with \(1 \leq i \leq p\). Statement [2] holds. The lemma is proved.

4.2 Proving Theorem 1.1

We are ready to prove the main theorem of the paper.

Proof of Theorem 1.1 Let \(G \in \mathcal{G}\) be the \(n\)-node input graph embedded on a genus-\(o(\frac{n}{\log^2 n})\) surface. The encoding algorithm \(Encode_A\) performs the following four steps on \(G\).

\(E1\): Triangulate the embedded graph \(G\) into a triangulation \(\Delta\) of \(G\). Let \(S_0\) be the 0-separation \([\emptyset, Node(\Delta)]\) of \(\Delta\). For each \(k = 1, 2\), apply Lemma 3.1 to obtain a \(k\)-separation \(S_k\) of \(\Delta\) that is a refinement of \(S_{k-1}\).

\(E2\): Let \([V_0, \ldots, V_p] = S_2\). Apply Lemma 2.3 with \(\ell = \max_{1 \leq i \leq p} node(G(V_i))\) to compute (a) \(Label(H)\) and \(Optcode(H)\) for all distinct graphs \(H\) in class \(\mathcal{G}\) with \(node(H) \leq \ell\) and (b) \(Table(G, \ell)\). For each \(i = 1, \ldots, p\), apply Lemma 2.3(1) to compute from \(Table(G, \ell)\) the binary string \(Code(V_i) = Optcode(G(V_i))\) and the labeling \(L_{2,i} = Label(G(V_i))\).

\(E3\): For each \(k = 2, 1\), perform the following two substeps.

\(E3.1\): Let \([U_0, \ldots, U_q] = S_{k-1}\) and \([V_0, \ldots, V_p] = S_k\). For each \(j = 1, \ldots, q\), let binary string \(Code(U_j) = Code(V_{i_1}) \circ \cdots \circ Code(V_{i_2})\), where \(\{i_1, i_1 + 1, \ldots, i_2\}\) are the indices \(i\) with \(V_i \subseteq U_j\).

\(E3.2\): Apply Lemma 4.1(1) to obtain the labelings \(L_{k-1,j}\) of subgraphs \(G(U_j)\) for all \(j = 1, \ldots, q\). Apply Lemma 4.1(2) to obtain the o\((n)\)-bit binary string \(Rec_k\).

\(E4\): By \(S_0 = [\emptyset, Node(G)]\), now we have \(Code(Node(G))\) (and a labeling \(L_{0,1}\) for \(G = G(Node(G))\)). The output binary string \(Code_A(G) = Code(Node(G)) \circ Table(G, \ell) \circ Rec_1 \circ Rec_2\).

By Lemma 3.1 Step \(E1\) takes \(O(n)\) time. By Property 5 of \(S_2\), we have \(\sum_{i=1}^{p} node(G(V_i)) = n + o(n)\). By Lemma 2.3 Step \(E2\) takes \(O(n)\) time. By Lemmas 2.2 and 4.1 Step \(E3\) takes \(O(n)\) time. By Lemma 2.2 Step \(E4\) takes \(O(n)\) time. Therefore, the encoding algorithm \(Encode_A(G)\) runs in \(O(n)\) time. Condition C[1] holds.

The decoding algorithm \(Decode_A\) performs the following five steps on \(Code_A(G)\).

\(D1\): Obtain \(Code(Node(G)), Table(G, \ell), Rec_1,\) and \(Rec_2\) from \(Code_A(G)\).

\(D2\): Let \(S_0 = [\emptyset, Node(G)]\). For each \(k = 1, 2\), perform the following substep.

\(D2.1\): Let \([U_0, \ldots, U_q] = S_{k-1}\) and \([V_0, \ldots, V_p] = S_k\). For each \(j = 1, \ldots, q\), obtain all \(Code(V_i)\) with \(V_i \subseteq U_j\) from \(Code(U_j)\).
D3: Let \([V_0, \ldots, V_p] = S_2\). Apply Lemma 2.3 to obtain \(G(V_i)\) and \(L_{2,i} = \text{Label}(G(V_i))\) from \(\text{Code}(V_i) = \text{Optcode}(G(V_i))\) and \(\text{Table}(G, \ell)\) for each \(i = 1, \ldots, p\).

D4: For each \(k = 2, 1\), perform the following substep.

\[ D4.1: \text{Let} \ [U_0, \ldots, U_q] = S_{k-1} \text{ and} \ [V_0, \ldots, V_p] = S_k. \text{ Apply Lemma 4.1 to recover} \ G(U_j) \text{ and} \ L_{k-1,j} \text{ with} \ 1 \leq j \leq q \text{ from} \ G(V_i) \text{ and} \ L_{k,i} \text{ with} \ 1 \leq i \leq p \text{ and} \ \text{Rec}_k. \]

D5: Output \(G = G(\text{Node}(G))\).

By Lemma 2.2, Step D1 takes \(O(n)\) time. By Lemma 2.2, Step D2 takes \(O(n)\) time. By Property S5 of \(S_2\), we have \(\sum_{i=1}^{p} \text{node}(G(V_i)) = n + o(n)\). By Lemma 2.3, Step D3 takes \(O(n)\) time. By Lemma 4.1, Step D4 takes \(O(n)\) time. Therefore, the decoding algorithm \(Dekode_A(G)\) runs in \(O(n)\) time. Condition S2 holds. By \(S_0 = [\emptyset, \text{Node}(G)]\), graph \(G = G(\text{Node}(G))\) is correctly recovered from \(\text{Code}_A(G)\) at the end of Step D4. Therefore, \(A\) is a compression scheme for \(G\).

To show Condition S3 we first prove the following claim for each \(k = 1, 2, \ldots\).

**Claim 1.** Suppose that \([U_0, \ldots, U_q] = S_{k-1} \text{ and} \ [V_0, \ldots, V_p] = S_k. \text{ If} \ \sum_{i=1}^{p} \|\text{Code}(V_i)\| \leq \beta n + o(n) \text{ and} \ \|\text{Code}(U_j)\| = \text{poly}(\ell_k) \text{ holds for each} \ i = 1, \ldots, p, \text{ then} \ \sum_{j=1}^{q} \|\text{Code}(U_j)\| \leq \beta n + o(n) \text{ and} \ \|\text{Code}(U_j)\| = \text{poly}(\ell_k) \text{ holds for each} \ j = 1, \ldots, q. \)**

**Proof of Claim 1.** For each \(j = 1, 2, \ldots, q\), let \(I_j\) consist of the indices \(i\) with \(V_i \subseteq U_j\). By Property S4 of \(S_{k-1}\), we have \(|I_j| \leq |U_j| = \text{poly}(\ell_{k-1})\). Therefore, \(\sum_{i \in I_j} \|\text{Code}(V_i)\| = \text{poly}(\ell_{k-1})\), implying \(O(\log \sum_{i \in I_j} \|\text{Code}(V_i)\|) = O(\ell_k)\). By Property S5 of \(S_k\), \(\sum_{j=1}^{q} |I_j| = p = O(\frac{n}{\ell_k})\). By Lemma 2.2, we have

\[
\sum_{j=1}^{q} \|\text{Code}(U_j)\| = \sum_{j=1}^{q} O(|I_j| \cdot \ell_k) + \sum_{i=1}^{p} \|\text{Code}(V_i)\| \leq \beta n + o(n).
\]

We also have

\[
\|\text{Code}(U_j)\| = O(|I_j| \cdot \ell_k) + \sum_{i \in I_j} \|\text{Code}(V_i)\| = \text{poly}(\ell_{k-1}).
\]

The claim is proved.

Let \([V_0, \ldots, V_p] = S_2\). For each \(i = 1, \ldots, p\), let \(n_i = \text{node}(G(V_i)) = |V_i| + \text{nbr}_G(V_i)\). By Property S5 of \(S_2\), we have \(\sum_{i=1}^{p} n_i = n + o(n)\). By Step D2 of \(\text{Encode}_A(G)\) and Lemma 2.3, we have \(\|\text{Code}(V_i)\| = \|\text{Optcode}(G(V_i))\| = [\log \text{num}(G, n_i)] \leq \beta n_i + o(n_i)\). By Property S4 of \(S_2\) and the assumption that \(\log \text{num}(G, n) = O(n)\),

\[
\|\text{Code}(V_i)\| = \text{poly}(\ell_2)
\]

holds for each \(i = 1, 2, \ldots, p\). We have

\[
\sum_{i=1}^{p} \|\text{Code}(V_i)\| \leq \sum_{i=1}^{p} (\beta n_i + o(n_i)) = \beta \cdot (n + o(n)) + o(\beta n + o(n)) = \beta n + o(n).
\]
Combining Equations (4) and (5), Claim 1 for $k = 2, 1$, and $S_0 = [\emptyset, \text{Node}(G)]$, we have $\|\text{Code}(\text{Node}(G))\| \leq \beta n + o(n)$. By Lemma 2.2 and $\|\text{Table}(G, \ell)\| + \|\text{Rec}_1\| + \|\text{Rec}_2\| = o(n)$, we have $\|\text{Code}_A(G)\| \leq \beta n + o(n)$. Condition C3 holds. The theorem is proved.

5 Extension

This section proves Theorem 1.2. The only place in our proof of Theorem 1.1 requiring $G$ to be hereditary is Step H2. We need $G(V_i) \in \mathcal{G}$ so that Optcode($G(V_i)$) and Label($G(V_i)$) can be obtained from Table($G$, $\ell$). For a non-hereditary class $\mathcal{G}$, we can substitute $G(V_i)$ by a graph $H_i \in \mathcal{G}$ that is close to $G(V_i)$ for each $i = 1, 2, \ldots, p$ as long as the overall number of bits required to encode the overall difference between $G(V_i)$ and $H_i$ is $o(n)$. The following corollary is an example of such an extension.

**Corollary 5.1.** Let $\mathcal{G}$ be a class of graphs satisfying $\log \text{num}(G, n) = O(n)$ and that any input $n$-node graph $G \in \mathcal{G}$ to be encoded comes with a genus- $o(\frac{n}{\log^* n})$ embedding. If for any 2-separation $[V_0, \ldots, V_p]$ of any graph $G \in \mathcal{G}$, there exist graphs $H_1, \ldots, H_p$ in $\mathcal{G}$ such that each $G(V_i)$ with $1 \leq i \leq p$ can be obtained from $H_i$ by first deleting $O(n\text{br}_G(V_i))$ nodes (together with their incident edges) and then updating (adding or deleting) $O(n\text{br}_G(V_i))$ edges, then $\mathcal{G}$ admits an optimal compression scheme.

**Proof.** We revise algorithm Encode$_A$ by updating Steps H2 and H4 as follows.

**E2**: Let $[V_0, \ldots, V_p] = S_2$. Compute $H_1, \ldots, H_p$ from $G(V_1), \ldots, G(V_p)$. Apply Lemma 2.3 with $\ell = \max_{1 \leq i \leq p} \text{node}(H_i)$ to compute (a) Label($H$) and Optcode($H$) for each distinct graph $H \in \mathcal{G}$ with node($H$) $\leq \ell$ and (b) Table($G$, $\ell$). Apply Lemma 2.3 to compute Code($V_i$) = Optcode($H_i$) and $L_{2,i} = \text{Label}(H_i)$ from Table($G$, $\ell$) for all indices $i = 1, \ldots, p$. Let $L_{2,i}$ be the labeling of $G(V_i)$ obtained from the labeling $L_{2,i}$ of $H_i$ such that if $v$ and $v'$ are two distinct nodes of $G(V_i)$ with $L_{2,i}(v) < L_{2,i}(v')$, then we have $L_{2,i}(v) < L_{2,i}(v')$. Let $F_i$ be the binary string storing the difference between $H_i$ and $G(V_i)$ via labeling $L_{2,i}$. Let $\text{Fix} = F_{i1} \circ \cdots \circ F_{ip}$.

**E4**: By $S_0 = [\emptyset, \text{Node}(G)]$, now we have Code($\text{Node}(G)$) and a labeling $L_{0,1}$ for $G = G(\text{Node}(G))$. The output binary string Code$_A(G)$ for $G$ is \mbox{Code($\text{Node}(G)$)} \circ \text{Table} \circ \text{Rec}_1 \circ \text{Rec}_2 \circ \text{Fix}.

By $O(1)^{\text{poly} (\ell)} = o(n)$, it takes $o(n)$ time to compute an $o(n)$-bit string $\text{Table'}$ such that graphs $H_1, \ldots, H_p$, satisfying the above conditions can be obtained from $G(V_1), \ldots, G(V_p)$ and $\text{Table'}$ in $O(n)$ time. By Property S3 of $S_2$ and the conditions of $H_1, \ldots, H_p$, we have $\sum_{i=1}^p \text{node}(G(V_i)) \leq \sum_{i=1}^p \text{node}(H_i) = n + o(n)$. By Lemmas 2.2 and 2.3, Step H2 takes $O(n)$ time. By Lemma 2.2, Step H4 takes $O(n)$ time. Therefore, Condition C3 holds for the revised Encode$_A$. We revise algorithm Decode$_A$ by updating Steps D1 and D3 as follows.

**D1**: Obtain Code($G$), Table($G$, $\ell$), Rec$_1$, Rec$_2$, and Fix from Code$_A(G)$.
Both revised steps take $O(n)$ time. Condition $\Box$ holds for the revised $Decode_A$. Subgraph $G(V_i)$ can be obtained from $H_i$ by first deleting $O(nbr_G(V_i))$ nodes (and their incident edges) and then updating $O(nbr_G(V_i))$ edges. It follows from Property $\mathbf{4}$ of $S_2$ that $||Fix_i|| = O(nbr_G(V_i) \cdot \ell_2)$. By Property $\mathbf{5}$ of $S_2$, we have $\sum_{i=1}^{p} ||Fix_i|| = o(\frac{n^2}{\ell_2}) \cdot O(\ell_2) = o(n)$. By Lemma $2.2$ we have $||Fix|| = o(n)$. Condition $\Box$ holds the revised $Code_A(G)$. The corollary is proved. \hfill $\square$

We use Corollary $5.1$ to prove Theorem $1.2$.\hfill $\Box$

Proof of Theorem $1.2$. Let $\Delta$ be an $n$-node triangulation of a genus-$g$ surface. Let $[V_0, \ldots, V_p]$ be a 2-separation of $\Delta$. Let $F$ consist of the non-triangle faces of $\Delta[V_i]$. Let $H_i$ be the plane triangulation obtained from $\Delta[V_i]$ by performing the following two steps for each face $F \in F$: (1) Add a node $v_F$ in $F$. (2) For each node $u$ on the boundary of $F$, add an edge $(u, v_F)$. See Figure 8 for an illustration. Since $\Delta$ is a triangulation, the boundary of $F$ contains at least two nodes $u$ with $Nbr_{\Delta}(u) \not\subseteq Node(\Delta(V_i))$. Therefore, at least two nodes of $Nbr(V_i)$ belong to the boundary of $F$. Let $e_F$ be an edge between two arbitrary nodes of $Nbr(V_i)$ that belong to the boundary of $F$. The union of $e_F$ over all faces $F \in F$ has genus no more than $g = O(1)$. Therefore, the number of added nodes to triangulate $\Delta[V_i]$ is $O(nbr_{\Delta}(V_i))$. The number of edges in $\Delta[V_i] \setminus \Delta(V_i)$ is also $O(nbr_{\Delta}(V_i))$. Thus, $\Delta(V_i)$ can be obtained from $H_i$ by first deleting $O(nbr_{\Delta}(V_i))$ nodes together with their incident edges and then deleting $O(nbr_{\Delta}(V_i))$ edges. By Corollary $5.1$, Statement $1$ is proved.

Let $G$ be an $n$-node floorplan. Since each node of $G$ has at most three neighbors in $G$, one can easily obtain a floorplan $H_i$ from $G(V_i)$ by adding $O(nbr_G(V_i))$ nodes and edges. See Figure 9 for an example. Statement $2$ follows from Corollary $5.1$.\hfill $\square$
Figure 9: (a) A floorplan $G$, where $V_i$ consists of the gray nodes. (b) The subgraph $G(V_i)$. (c) A floorplan $H_i$ obtained from $G(V_i)$ by adding $O(nbr_G(V_i))$ nodes and edges.

6 Concluding remarks

Our optimal compression schemes rely on a linear-time obtainable embedding. Can this requirement be dropped? It would be of interest to extend our compression schemes to support efficient queries and updates. We leave open the problems of obtaining optimal compression schemes for $O(1)$-connected genus-$O(1)$ graphs and 3D floorplans [23, 60, 86, 59, 98, 99, 61, 22].

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