LIOUVILLE THEOREM FOR BOUNDED HARMONIC FUNCTIONS ON MANIFOLDS AND GRAPHS SATISFYING NON-NEGATIVE CURVATURE DIMENSION CONDITION

BOBO HUA

Abstract. Brighton [Bri13] proved the Liouville theorem for bounded harmonic functions on weighted manifolds satisfying non-negative curvature dimension condition, i.e. CD(0, ∞). In this paper, we provide a new proof of this result by using the reverse Poincaré inequality. Moreover, we adopt this approach to prove the Liouville theorem for bounded harmonic functions on graphs satisfying the CD(0, ∞) condition.

1. Introduction

Yau [Yau75] proved that positive harmonic functions are constant on a complete, noncompact Riemannian manifold with non-negative Ricci curvature. As a corollary, any bounded harmonic function is constant. These are called Liouville theorems for harmonic functions, regarded as the generalizations of classical Liouville’s theorem for bounded holomorphic functions on the complex plane. Later, Cheng and Yau [CY75] obtained a related quantitative result, so-called Cheng-Yau gradient estimate for positive harmonic functions. This type of gradient estimate was generalized to positive solutions of heat equations by Li and Yau [LY86]. The space of positive (bounded resp.) harmonic functions corresponds to the Martin (Poisson resp.) boundary at infinity for Brownian motion or random walks in the probability theory, see e.g. [Kai96, Woe09]. In this terminology, the Liouville theorem means the triviality of the corresponding boundary at infinity. Liouville theorems for harmonic functions have received much attention in the literature, to cite a few [KV83, Sui83, And83, LDS87, Lyo87, Gri90, Ben91, Gri91, SC92, Wan02, Ers04, Li12, Bri13]. In this paper, we study Liouville theorems for bounded harmonic functions on weighted manifolds and graphs.

We first recall some facts on Riemannian manifolds. Let (M, g) be a d-dimensional complete Riemannian manifold with the Riemannian metric g, ∆ be the Laplace-Beltrami operator on M and $C_0^\infty(M)$ be the space of compactly supported smooth functions on M. As is well-known, the Ricci curvature tensor of a manifold is bounded below by Kg and the dimension d is at most n, $K \in \mathbb{R}$ and $n \in (0, \infty]$, if and only if the following Bochner
inequality holds,
\[ \frac{1}{2} \Delta |\nabla f|^2 \geq \frac{1}{n} (\Delta f)^2 + \langle \nabla f, \nabla \Delta f \rangle + K |\nabla f|^2, \quad \forall f \in C_0^\infty(M), \]
where $\nabla \cdot$ is the Riemannian gradient of a function. In a general setting, we consider a Riemannian manifold $(M, g)$ equipped with a measure $e^{-V(x)} d\text{vol}(x)$, where $V$ is a smooth function on $M$, and call the triple $(M, g, e^{-V} d\text{vol})$ a weighted manifold. Weighted manifolds, also called smooth metric measure spaces, have been extensively studied in the literature, see e.g. [Mor05, WW09, Gri09, MW11, MW12]. The associated $V$-Laplacian is defined as
\[ \Delta_V f = \Delta f - \langle \nabla V, \nabla f \rangle, \quad \forall f \in C_0^\infty(M). \]
A function $u$ is called $V$-harmonic if $\Delta_V u = 0$ on $M$. The weighted Ricci curvature is defined as
\[ \text{Ric}_V = \text{Ric} + \nabla^2 V, \]
where $\nabla^2 \cdot$ is the Hessian of a function. Yau’s Liouville theorem for bounded harmonic functions was generalized to weighted manifolds by Brighton [Bri13].

**Theorem 1** ([Bri13]). Let $(M, g, e^{-V} d\text{vol})$ be a complete weighted manifold satisfying $\text{Ric}_V \geq 0$. Then any bounded $V$-harmonic function is constant.

Brighton adopted the techniques for Cheng-Yau’s gradient estimate [CY75] to establish the following: For any $q \in M$, $R \geq 1$ and any positive $V$-harmonic function $u$ on $B_R(q)$, the ball of radius $R$ centered at $q$, we have
\[ |\nabla u|(q) \leq \frac{C}{\sqrt{R}} \sup_{B_R(q)} u, \]
where $C$ is a constant independent of $R$. The theorem follows from passing to the limit $R \to \infty$.

In this paper, we will give a new proof of Theorem 1 using semigroup approaches. For general Markov semigroups, Bakry and Émery [BE85, Bak87, BGL14] introduced the so-called $\Gamma$-calculus, and defined the curvature dimension condition mimicking (1), denoted by $\text{CD}(K, n)$, and derived various interesting analytic properties of the semigroups under these conditions. For a weighted manifold $(M, g, e^{-V} d\text{vol})$, we denote by $P_t = e^{t \Delta_V}$ the associated heat semigroup. In this setting, the $\text{CD}(0, \infty)$ condition means that $\text{Ric}_V \geq 0$. It is well-known [BGL14, Wan11] that $\text{Ric}_V \geq 0$ is equivalent to
\[ |\nabla P_t f|^2 \leq P_t |\nabla f|^2, \quad \forall f \in C_0^\infty(M), t > 0, \]
or
\[ 2t |\nabla P_t f|^2 \leq P_t (f^2) - (P_t f)^2, \quad \forall f \in C_0^\infty(M), t > 0, \]
where the latter is called the reverse Poincaré inequality. We will use the reverse Poincaré inequality to give a proof of Theorem 1 in Section 2. The heuristic argument is as follows: Let $f$ be a bounded harmonic function on $M$. We may apply (3) to $f$, to be justified by proper approximation. By the semigroup property and the boundedness of $f$, the right hand side of
is bounded above by a constant $C$ independent of $t$. Moreover, by the harmonicity of $f$, $P_t f = f$, which implies that

$$|\nabla f|^2 = |\nabla P_t f|^2 \leq \frac{C}{2t} \to 0, \quad t \to \infty.$$  

This yields that $|\nabla f| \equiv 0$ and hence $f$ is constant.

In the following we introduce the setting of graphs, see [HL17] for notation, and adapt the previous argument to the discrete case. Let $(V, E)$ be a connected, undirected, combinatorial graph with the set of vertices $V$ and the set of edges $E$. We say $x, y \in V$ are neighbors, denoted by $x \sim y$, if there is an edge connecting $x$ and $y$. The graph is called locally finite if each vertex has finitely many neighbors. In this paper, all graphs we consider are locally finite. We assign a weight $m$ to each vertex, $m : V \to (0, \infty)$, and a weight $\mu$ to each edge, $\mu : E \to (0, \infty)$, $E \ni \{x, y\} \mapsto \mu_{xy} = \mu_{yx}$, and refer to the quadruple $G = (V, E, m, \mu)$ as a weighted graph. We denote by $C_0(V)$ the set of finitely supported functions on $V$, by $\ell^p(V, m)$, $p \in [1, \infty]$, the $\ell^p$ spaces of functions on $V$ with respect to the measure $m$, and by $\| \cdot \|_{\ell^p(V, m)}$ the $\ell^p$ norm of a function.

For any weighted graph $G = (V, E, m, \mu)$, we define the Laplacian $\Delta$ as

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V : y \sim x} \mu_{xy}(f(y) - f(x)), \quad \forall f : V \to \mathbb{R}.$$  

It is proven by Keller and Lenz [KL12] that $\Delta$ is a bounded linear operator on $\ell^2(V, m)$ if and only if

$$\sup_{x \in V} \frac{1}{m(x)} \sum_{y \in V : y \sim x} \mu_{xy} < \infty.$$  

From another point of view, for a weighted graph $G$, it associates with a Dirichlet form with respect to the Hilbert space $\ell^2(V, m)$,

$$Q : \quad D(Q) \times D(Q) \to \mathbb{R}$$  

$$(f, g) \mapsto \frac{1}{2} \sum_{x, y \in V : x \sim y} \mu_{xy}(f(y) - f(x))(g(y) - g(x)),$$  

where the form domain $D(Q)$ is defined as the completion of $C_0(V)$ under the norm $\| \cdot \|_Q$ given by

$$\| f \|_Q^2 = \| f \|^2_{\ell^2(V, m)} + \frac{1}{2} \sum_{x, y \in V : x \sim y} \mu_{xy}(f(y) - f(x))^2, \quad \forall f \in C_0(V),$$  

see [KL12]. We denote by $\mathcal{L}$ the infinitesimal generator of the Dirichlet form $Q$ and by $P_t = e^{t\mathcal{L}}$ the corresponding $C_0$-semigroup. For a locally finite graph, the generator $\mathcal{L}$ coincides with the Laplacian $\Delta$ on the domain of generator $D(\mathcal{L})$ which contains $C_0(V)$, i.e. $\mathcal{L} f = \Delta f$, for any $f \in D(\mathcal{L})$, see [KL12] Theorem 6 and 9].
Note that the weights $\mu$ and $m$ determine the properties of the Laplacian and the semigroup. Given the edge weights $\mu$, typical choices of $m$ are of particular interest:

- $m(x) = \sum_{y \sim x} \mu_{xy}$ for any $x \in V$ and the associated Laplacian is called the normalized Laplacian.
- $m \equiv 1$ and the Laplacian is called the physical (or combinatorial) Laplacian.

Now we introduce the $\Gamma$-calculus on graphs following [BGL14, LY10]. The “carré du champ” operator $\Gamma$ is defined as

$$\Gamma(f, g)(x) := \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)(x), \quad \forall f, g : V \to \mathbb{R}, x \in V.$$ 

For simplicity, we write $\Gamma(f) := \Gamma(f, f)$. The iterated $\Gamma$ operator, $\Gamma_2$, is defined as

$$\Gamma_2(f) := \frac{1}{2}\Delta\Gamma(f) - \Gamma(f, \Delta f).$$

A Riemannian manifold $(M, g)$ is complete if and only if there exists a non-decreasing sequence of non-negative functions $\{\eta_k\}_{k=1}^{\infty}$ in $C_0^\infty(M)$ such that

$$\lim_{k \to \infty} \eta_k = 1 \quad \text{and} \quad |\nabla \eta_k| \leq \frac{1}{k},$$

where $1$ is the constant function $1$ on $M$. Motivated by this, a weighted graph $G = (V, E, m, \mu)$ is called complete if there exists a non-decreasing sequence of non-negative finitely supported functions $\{\eta_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} \eta_k = 1 \quad \text{and} \quad \Gamma(\eta_k) \leq \frac{1}{k}. \quad (6)$$

Note that the weights $\mu$ and $m$ are crucial to the completeness of a weighted graph. This condition was defined for Markov diffusion semigroups in [BGL14, Definition 3.3.9] and adapted to graphs in [HL17]. A large class of graphs possessing appropriate intrinsic metrics have been shown to be complete, see [HL17, Theorem 2.8], where $\eta_k$ are constructed via distance functions to some vertices. In particular, graphs with bounded Laplacians are always complete.

In this paper, we say that a weighted graph $G$ satisfies the assumption $(A)$ if one of the following holds:

- $(A_1)$ The Laplacian $\Delta$ is bounded on $\ell^2(V, m)$, i.e. $[5]$ holds.
- $(A_2)$ $G$ is complete and $m$ is non-degenerate, i.e. $\inf_{x \in V} m(x) > 0$.

Note that normalized Laplacians are always bounded, hence satisfy $(A_1)$; physical Laplacians satisfy $(A_2)$ provided that they are complete.

For our purposes, we need the curvature dimension condition on graphs, which was initiated in [BE85, Bak87] on manifolds (or Markov diffusion semigroups) and introduced to graphs by [Sch99, LY10]. We say that a graph $G = (V, E, m, \mu)$ satisfies the CD$(K, n)$ condition, for $K \in \mathbb{R}$ and
Now we are ready to state our main result.

**Theorem 2.** Let $G = (V, E, m, \mu)$ be a weighted graph satisfying the assumption $(A)$ and the CD$(0, \infty)$ condition. Then any bounded harmonic function on $G$ is constant.

We remark that the Liouville theorem for bounded harmonic functions on graphs can be derived from the Li-Yau gradient estimate or the Harnack inequality. They had been obtained by [BHL+15, HLY14] under other curvature dimension conditions, CDE$(0, n)$ and CDE$'(0, n)$, for bounded Laplacians and finite $n$. It was proven by Münch [M17] that CDE$'(0, n)$ implies that CD$(0, \infty)$. So that our assumption of CD$(0, \infty)$ is much weaker.

The assumption $(A)$ is essential for the following result, which was proved for finite graphs by [LP14], for graphs with bounded Laplacians by [LL15], and partially for graphs with unbounded Laplacians satisfying $(A_2)$ by [HL17] and [GL17].

**Theorem 3.** For a weighted graph $G = (V, E, m, \mu)$ satisfying the assumption $(A)$, the following are equivalent:

(a) $G$ satisfies CD$(0, \infty)$.
(b) $P_t(f^2) - P_t(f)^2 \leq 2tP_t(\Gamma(f))$, $\forall f \in C_0(V), t > 0$.
(c) $2tP_t(\Gamma(f)) \leq P_t(f^2) - P_t(f)^2$, $\forall f \in C_0(V), t > 0$.

d. $2t\Gamma(P_t f) \leq P_t(f^2) - P_t(f)^2$, $\forall f \in C_0(V), t > 0$.

The paper is organized as follows: We prove Theorem 1 in the next section. The last section is devoted to the proofs of Theorem 2 and Theorem 3.

2. Proof of Theorem 1

In this section, we prove Theorem 1.

**Lemma 4.** Let $(M, g, e^{-V}d\text{vol})$ be a complete weighted manifold satisfying $\text{Ric}_V \geq 0$, $f$ be a bounded $V$-harmonic function. Then $P_tf = f$.

*Proof.* By the gradient estimate (2), the semigroup $P_t$ is stochastically complete, see e.g. [BGL14 Theorem 3.3.23 and Corollary 3.2.2]. So that any bounded solution to the heat equation $\partial_t u = \Delta u$ is unique, see [Gri09 Theorem 8.18]. Note that by the harmonicity of $f$, $u(t, x) = f(x)$ solves the heat equation with initial data $f$ which is bounded for all $t \geq 0$. Hence $P_tf = u = f$. \qed

**Proof of Theorem 1.** Let $(M, g, e^{-V}d\text{vol})$ be a complete weighted manifold satisfying $\text{Ric}_V \geq 0$, $f$ be a bounded $V$-harmonic function. Let $\{\eta_k\}_{k=1}^\infty$ be a non-decreasing sequence of non-negative compactly supported smooth functions such that

$$\lim_{k \to \infty} \eta_k = 1.$$
Set $f_k := f \cdot \eta_k$, for $k \geq 1$. By the dominated convergence theorem, one can show that for any $t > 0$, 

$$P_t f_k \to P_t f, \quad k \to \infty.$$ 

By the reverse Poincaré inequality, (3),

$$2t|\nabla P_t f_k|^2 \leq P_t (f_k^2) - (P_t f_k)^2 \leq C^2,$$

where $C = \|f\|_\infty$. Hence

$$|\nabla P_t f_k| \leq \frac{C}{\sqrt{2t}}.$$ 

For any fixed $x, y \in M$, let $\gamma_{xy}(s), s \in [0, d(x, y)]$, be the shortest geodesic connecting $x$ and $y$ parametrized by the arc length. Then by (7),

$$|P_t f_k(y) - P_t f_k(x)| \leq \int_0^{d(x, y)} \left| \frac{d}{ds} P_t f_k(\gamma_{xy}(s)) \right| ds \leq d(x, y) \frac{C}{\sqrt{2t}}.$$ 

Passing to the limit $k \to \infty$, and applying Lemma 4, we have

$$|f(y) - f(x)| = |P_t f(y) - P_t f(x)| \leq d(x, y) \frac{C}{\sqrt{2t}}.$$ 

Moreover, letting $t \to \infty$, we conclude that $f(x) = f(y)$ for any $x, y \in M$. This proves the theorem.

3. Proof of Theorem 2

In this section, we will prove Theorem 2. For that purpose, we first prove Theorem 3.

Proof of Theorem 3. For bounded Laplacians, the theorem has been proved in [LL15, Theorem 3.1]. It suffices to prove the result under the assumption of $(A_2)$. The equivalence of $(a)$ and $(b)$ was obtained in [HL17, Theorem 1.1]. The equivalence of $(a)$ and $(c)$ was proved in [GL17, Theorem 3.2].

We prove $(a) \implies (d)$ following [Wan11, Theorem 1.1] or [LL15, Theorem 3.1]. The reverse direction is omitted here since it follows verbatim.

Let $f \in C_0(V)$. We claim that $(P_t f)^2 \in D(\mathcal{L})$ for any $t \geq 0$. By the assumption $(A_2)$, [KL12, Theorem 5] yields that

$$D(\mathcal{L}) = \{ u \in \ell^2(V, m) : \Delta u \in \ell^2(V, m) \}.$$

On one hand, one can see that $(P_t f)^2 \in \ell^2(V, m)$ since

$$\|P_t f\|_{L^p(V, m)} \leq \|f\|_{L^p(V, m)}, \quad \forall p \in [1, \infty].$$

On the other hand,

$$\Delta (P_t f)^2 = 2P_t f \Delta P_t f + 2\Gamma(P_t f).$$
Note that \( \Delta P_t f \in \ell^2(V, m) \) since \( P_t f \in D(\mathcal{L}) \). Moreover, by Proposition 3.7, 
\[
\|\Gamma(P_t f)\|_{\ell^1(V, m)} \leq \|\Gamma(f)\|_{\ell^1(V, m)} < \infty.
\]
Hence \( \Gamma(P_t f) \in \ell^2(V, m) \) which follows from \( \ell^1(V, m) \subset \ell^2(V, m) \), derived from the assumption (A2). Combining these facts, we have \( \Delta(P_t f)^2 \in \ell^2(V, m) \) and prove the claim.

Given \( t > 0 \), set \( \phi(s, x) = P_s(P_t-s f)^2(x), 0 \leq s \leq t \). Direct calculation shows that
\[
\frac{d}{ds} \phi = \Delta P_s(P_t-s f)^2 - P_s(2P_t-s f \Delta P_t-s f) = P_s [\Delta(P_t-s f)^2 - 2P_t-s f \Delta P_t-s f] = 2P_s \Gamma(P_t-s f),
\]
where the second equality follows from that \( \Delta P_s(P_t-s f)^2 = P_s \Delta(P_t-s f)^2 \) since \( (P_t-s f)^2 \in D(\mathcal{L}) \), proved in the previous claim. Using (b) in Theorem 3 with the approximation of \( P_t-s f \) by finitely supported functions,
\[
\frac{d}{ds} \phi = 2P_s \Gamma(P_t-s f) \geq 2\Gamma(P_s P_t-s f) = 2\Gamma(P_t f).
\]
Integration over \( s \) on \([0, t]\) gives the result (d). This proves the theorem.

Lemma 5. Let \( G = (V, E, m, \mu) \) be a weighted graph satisfying the assumption (A) and the CD(0, \( \infty \)) condition. Then for any bounded harmonic function \( f \), \( P_t f = f \).

Proof. By (b) in Theorem 3 the semigroup \( P_t \) is stochastically complete, see [HL17, Theorem 1.2]. So that any bounded solution to the heat equation \( \partial_t u = \Delta u \) is unique, see [Hua11, Theorem 1.5.1] or [KL12, Theorem 1]. The lemma follows from the same argument as in the proof of Lemma 4.

Now we can prove the Liouville theorem for bounded harmonic functions for non-negatively curved graphs.

Proof of Theorem 4. By the heuristic argument (4) in the introduction and Lemma 5 it suffices to prove that (d) in Theorem 3 holds for any bounded function \( f \). By the completeness of the graph, there exists cut-off functions \( \{\eta_k\}_{k=1}^\infty \subset C_0(V) \) satisfying (5). Set \( f_k := f \cdot \eta_k \), for \( k \geq 1 \). Note that \( f_k \to f, k \to \infty \), pointwise. By (d) in Theorem 3
\[
2t\Gamma(P_t f_k) \leq P_t(f_k^2) - (P_t f_k)^2.
\]
Since \( f \) is bounded, the dominated convergence theorem yields that
\[
P_t((f_k)^a) \to P_t(f^a), \quad k \to \infty, \quad a = 1, 2.
\]
Moreover, noting that the graph is locally finite, for any \( x \in V \), we have
\[
\Gamma(P_t(f_k))(x) \to \Gamma(P_t f)(x), \quad k \to \infty,
\]
since both sides involve only finitely many summands. Passing to the limit, \( k \to \infty \), in (8), we prove that (d) in Theorem 3 holds for any bounded function \( f \). Hence the theorem follows.
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E-mail address: bobohua@fudan.edu.cn

School of Mathematical Sciences, LMNS, Fudan University, Shanghai 200433, China; Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China