§1. Introduction.

Throughout this paper, the phrase *canonically-embedded curve of genus g* will be used to refer to any pure 1-dimensional, non-degenerate subscheme $C$ of $\mathbb{P}^{g-1}$ over an algebraically closed field $k$, for which

$$\mathcal{O}_C(1) \cong \omega_C,$$

the dualizing sheaf

and

$$h^0(C, \mathcal{O}_C) = 1, \quad h^0(C, \omega_C) = g.$$ 

The singularities of $C$ (if any) are Gorenstein, and $C$ is connected of degree $2g - 2$ and arithmetic genus $g$.

In a recent paper ([S]), Schreyer has proved that Petri’s normalization of the homogeneous ideal $I(C)$ of a smooth canonically-embedded curve (see also [P], [M], [SD], [ACGH] for the case of smooth curves) can be also carried out for singular curves, provided that the curve has a simple $(g - 2)$-secant (i.e. a linear $W \cong \mathbb{P}^{g-3}$ intersecting $C$ transversely at exactly $g - 2$ (smooth) points). We will call such $C$ Petri-general. Moreover, he shows that the variety defined by any system of quadrics and cubics of Petri’s form is a Petri-general canonical curve (Theorem 1.4 of [S]). The Groebner basis techniques used in [S] also lead naturally to a construction of a *Petri scheme* $\mathcal{P}_g$ parametrizing all ideals of Petri’s form for a fixed $g$.

In the present paper (in effect an extended footnote to [S]) we will use the map from $\mathcal{P}_g$ to the Hilbert scheme of curves of degree $2g - 2$ and arithmetic genus $g$ in $\mathbb{P}^{g-1}$ to study the low-genus cases $g = 5, 6$. Of course, the situation for smooth curves is completely understood for all $g$. Since the moduli space of smooth curves is irreducible, and the canonical embedding is determined by a choice of basis in $H^0(C, \omega_C)$, the points of the Hilbert scheme corresponding to smooth canonically-embedded curves all lie on one irreducible component of dimension

$$\dim(\mathcal{M}_g) + \dim(PGL(g)) = 3g - 3 + g^2 - 1.$$
(The Petri-general curves obtained as in [S] correspond to a certain subscheme of dimension $7g - 7$, defined by incidence conditions with the fixed $(g - 2)$-secant used in Petri’s construction.) However, there can be other components of the Hilbert scheme whose general points correspond to singular curves. Our main results (see Theorem 2.1, Theorem 3.1, Theorem 3.5) are that this possibility does not arise (yet) for Petri-general curves of $g = 5$ or $g = 6$: If $C$ is Petri-general, then $[C]$ lies on the component of the Hilbert scheme whose general point corresponds to a smooth curve.

The $g = 5$ result, Theorem 2.1, is quite easy (and certainly not new, though we could not find an explicit statement in the literature). It should follow, for example, from the algebraic structure theorems for Gorenstein ideals of codimension 3 in [BE]. However, we will use the extension of the Enriques-Petri theorem to singular curves given in §3 of [S]. A canonical curve of genus 5 is either a complete intersection of three quadrics in $\mathbb{P}^4$, or else it lies on a surface of degree 3 in $\mathbb{P}^4$ (a rational normal cubic scroll, or a degeneration) cut out by the quadrics in $I(C)$. Each of these two possibilities yields an irreducible family of curves, the “non-trigonal” complete intersections $\mathcal{H}_5'$ and the “trigonal” curves for which $I(C)$ is not generated by quadrics $\mathcal{H}_5''$, and $\mathcal{H}_5''$ is contained in the closure of $\mathcal{H}_5'$. In the genus 6 case, the situation is very similar, but somewhat more complicated. The most difficult part of the proof, in fact, is to show that the family of Petri-general canonical curves of genus 6, whose ideals are generated by quadrics, is irreducible. We do this by showing that each of these curves is the complete intersection of a surface of degree 5 (a quintic Del Pezzo surface or a degeneration) and a quadric in $\mathbb{P}^5$. The existence of such a surface for smooth $C$ is, of course, classical (see the comments at the start of §3). Our proof here uses many of the ideas of Schreyer’s study of the 2nd syzygy module of $I(C)$ in §4 of [S].

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§2. The Genus 5 Case.

Let $\mathcal{H}_5$ be the open subvariety of the Hilbert scheme of curves of degree 8 and arithmetic genus 5 in $\mathbb{P}^4$ consisting of points corresponding to Petri-general curves $C$. Recall,
this means that \( C \) contains 3 smooth points in spanning a 2-plane which intersects \( C \) transversely exactly in those points. (However, \( C \) may have any other singularities, non-reduced components, etc. consistent with this requirement.) In this section we will prove that \( \mathcal{H}_5 \) is irreducible. Although this fact is certainly well-known in a sense, we include it for completeness, and for the way that some of the ideas we use in this proof will reappear in the proofs for genus 6 curves later in the paper.

**Theorem 2.1.** \( \mathcal{H}_5 \) is irreducible of dimension 36, contained in one irreducible component of the Hilbert scheme.

**Proof.** The proof is in three sections. First we deal with the curves \( C \) for which \( I(C) \) is generated by quadrics (the “non-trigonal” case). Since \( \dim I(C)_2 = 3 \) for any genus 5 canonical curve, we have that \( C \) is a complete intersection, and the corresponding open subvariety \( \mathcal{H}_5' \) of \( \mathcal{H}_5 \) is isomorphic to a Zariski-open dense subset of the Grassmannian of three-dimensional subspaces of \( H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \). Hence \( \mathcal{H}_5' \) is irreducible, of dimension 36.

Second, we consider the cases for which \( I(C) \) is not generated by quadrics (the “trigonal” case). Our main tool will be the extension of the Enriques-Petri Theorem to Petri-general, but possibly singular or reducible curves in §3 of [S]. Choosing coordinates à la Petri, we have the Petri coefficient \( \rho_{123} = 0 \), and the quadrics in \( I(C) \) have a basis of the form:

\[
\begin{align*}
    f_{12} &= x_1 x_2 - a_{112} x_1 - a_{212} x_2 - q_{12}(x_4, x_0) \\
    f_{13} &= x_1 x_3 - a_{113} x_1 - a_{313} x_3 - q_{13}(x_4, x_0) \\
    f_{23} &= x_2 x_3 - a_{223} x_2 - a_{323} x_3 - q_{23}(x_4, x_0)
\end{align*}
\]

Again since \( \rho_{123} = 0 \), the Petri syzygies on \( I(C) \) (see Corollary 1.5 of [S]) read as follows:

\[
x_j f_{ik} - x_k f_{ij} - \sum_{s=1, s \neq j}^3 a_{sik} f_{sj} - \sum_{s=1, s \neq k}^3 a_{sij} f_{sk} = 0.
\]

It follows that \( F = \{ f_{12}, f_{13}, f_{23} \} \) forms a Groebner basis for the ideal \( J \) they generate (using the graded reverse lexicographic order, with the variables ordered \( x_1 > x_2 > x_3 > x_4 > x_0 \) as in [S]).

The variety \( V(J) \) is a reduced, arithmetically Cohen-Macaulay surface \( S \) of degree 3 in \( \mathbb{P}^4 \). (In the case that \( C \) is a smooth curve, \( S \) is always a smooth rational normal scroll. However, for singular \( C \), in addition to smooth scrolls, cones over twisted cubic curves and
reducible surfaces can and do appear (see the examples in §3 of [S]). We claim that the family $S$ of all such surfaces in $\mathbb{P}^4$ is irreducible.

One proof of this (well-known) fact follows directly from the Petri form (1) for the generators of $J$. Using a linear change of coordinates, we can put the equations of any surface $S$ of the family $S$ into this form. The fact that $F$ forms a Groebner basis implies that $S$ lies in a 1-parameter family flatly deforming to the surface $S_0 = V(x_1x_2, x_1x_3, x_2x_3)$. The Zariski tangent space to Hilb at $[S_0]$ has dimension equal to $\dim(H^0(N_{S_0}\vert\mathbb{P}^4))$, which is equal to 18 by a direct calculation. This is exactly equal to the dimension of the component of the Hilbert scheme of surfaces of degree 3 containing the rational normal scrolls. It follows that $[S_0]$ is a smooth point of this Hilbert scheme, and hence that all of our surfaces correspond to points on one irreducible component.

Using the results of §3 of [S], on each surface $S$, the canonically-embedded “trigonal” curves form a 17-dimensional irreducible system.

By the discussion on p. 102 of [S], in addition to the smooth rational normal scrolls, the other possible surfaces that can occur here are:

a) $S$ = a cone over a twisted cubic curve,

b) $S = Q \cup P$, where $Q$ is a quadric in a $\mathbb{P}^3 \subset \mathbb{P}^4$, and $P$ is a $\mathbb{P}^2 \subset \mathbb{P}^4$ meeting $Q$ along a line (which meets $C$ in three points), or

c) $S = P_1 \cup P_2 \cup P_3$, where $P_i$ are $\mathbb{P}^2$’s intersecting in the following way (after suitably numbering the components): $P_1$ and $P_3$ meet $P_2$ along lines, while $P_1 \cap P_3$ is a point.

Given a canonically-embedded “trigonal” curve $C$, the complete intersection of the surface $S$ with one cubic not containing $S$ (e.g. one of the $G_{kl}$ in Petri’s normalization) will be $C$ union a line $L$ (a line of the ruling on a smooth scroll or in case a), in $Q$ in case b), in $P_2$ in case c) above). Conversely, if we fix such a line $L \subset S$, and consider the linear system of cubics in $\mathbb{P}^4$ containing $L$ but not containing $S$, the residual intersection will be a canonically-embedded curve of genus 5 on $S$. The dimension of the space of cubics containing $L$, modulo cubics in $I(S)$ is:

$$\left(\binom{4+3}{3} - 4 - 13\right) - 1 = 17.$$  

The conclusion of this analysis is that the family $\mathcal{H}_5''$ of all “trigonal” canonically-embedded curves of genus 5 is irreducible of dimension $18 + 17 = 35$.

Finally, we must show how $\mathcal{H}_5'$ and $\mathcal{H}_5''$ are related. It is easy to see that the smooth trigonal curves correspond to a dense open subset of $\mathcal{H}_5''$, since the general surface $S$ in our family $S$ is a smooth scroll, and the general element of the linear system $|3b + 5f|$ on a smooth scroll is smooth. Every smooth trigonal curve is a limit of smooth non-trigonal
curves. Hence, an open dense subset of the irreducible $H_5''$ is contained in the Zariski closure of $H_5''$. Hence all of $H_5''$ is contained in the closure of $H_5'$, and this completes the proof of the proposition. △

§3. The Genus 6 Case.

We now turn our attention to canonically-embedded curves $C$ of genus 6. Recall that the Enriques-Petri theorem for smooth curves of genus 6 says that $I(C)$ is generated by quadrics unless either $C$ is trigonal, or $C$ is isomorphic to a smooth plane quintic. In the first case, the quadrics cut out a rational normal scroll in $P^5$ containing $C$; in the second case, the quadrics cut out a Veronese surface. In this section, we will begin by proving the following theorem about the Hilbert scheme of (possibly singular) canonically-embedded curves of genus 6.

**Theorem 3.1.** Let $H_6'$ be the subset of the Hilbert scheme of curves of arithmetic genus 6 and degree 10 in $P^5$ parametrizing Petri-general curves $C$ for which $I(C)$ is generated by quadrics. Then $H_6'$ is contained in one irreducible component of Hilb.

Before beginning the proof (which is somewhat involved) we want to describe the main idea and illustrate it with an example that led us to the general statement.

By a classical result (see [AH] for a modern treatment), a smooth curve of genus 6, neither trigonal nor isomorphic to a smooth plane quintic, lies on a surface of degree 5 in $P^5$ (a possibly degenerate quintic Del Pezzo surface, or a cone over an elliptic normal curve in $P^4$). The curve is then the complete intersection of the surface and one further quadric. For instance, projecting a general genus 6 canonical curve $C$ from a general 4-secant 2-plane to $C$ (spanned by the points of one divisor in one of the $g_1^4$'s on $C$) yields a plane curve of degree 6 with four double points, birational to $C$. The canonical divisors on the plane model are cut by cubics passing through the four double points. This linear system of cubics maps $P^2$ to a quintic Del Pezzo surface containing the original canonical curve $C$. This fact was also used in [KS] to show that the moduli space of smooth curves with $g = 6$ is stably rational.

What we will show is that even for singular curves, we still have a surface of degree 5 containing $C$ that plays the same role. To prove Theorem 3.1 we will use the fact that the family of such surfaces is irreducible. Since the canonical curves on any one such surface are simply cut by the linear system of quadrics from $P^5$, we will get the irreducibility statement to be proved.
The main step of the proof will be to isolate the quadrics defining the surface of degree 5. The whole space of quadrics $I(C)_2$ is 6-dimensional, and as before we can take a basis in Petri form:

$$f_{ij} = x_i x_j - a_{1ij} x_1 - a_{2ij} x_2 - a_{3ij} x_3 - a_{4ij} x_4 - q_{ij}(x_0, x_5),$$

where $1 \leq i < j \leq 4$, and if $i, j, k$ are distinct, $a_{kij} = \rho_{kij} \alpha_k$ for some constants $\rho_{kij}$, and non-zero linear forms $\alpha_k = \alpha_k(x_0, x_5)$. We will prove shortly that if $I(C)$ is generated by quadrics, then at least two of the $\rho_{ijk}$ are different from zero. By renumbering the variables $x_1, x_2, x_3, x_4$ if necessary, we may assume $\rho_{123}, \rho_{124}$ are non-zero. If this is the case, then we will show that the (5-dimensional) subspace of $I(C)_2$ spanned by

\begin{align*}
F_1 &= \rho_{134} f_{12} - \rho_{123} f_{14} \\
F_2 &= \rho_{134} f_{23} - \rho_{123} f_{34} \\
F_3 &= \rho_{124} f_{13} - \rho_{123} f_{14} \\
F_4 &= \rho_{124} f_{23} - \rho_{123} f_{24} \\
F_5 &= \rho_{234} f_{12} - \rho_{123} f_{24} \\
F_6 &= \rho_{234} f_{13} - \rho_{123} f_{34}
\end{align*}

(2)

generates the ideal of a surface of degree 5 and sectional genus 1.

**Remarks.** Readers of [S] will note a clear parallel between (2) and the quadrics in Claim 1 in the proof of Theorem 4.1 of that paper. However, this initial $g = 6$ case seems to have some different features from the $g \geq 7$ cases treated there, and we were not able to deduce our desired result directly from the techniques of §4 of [S]. We should note that it is probably also possible to derive this result from the purely algebraic structure theorem for Gorenstein ideals of codimension 4 and deviation 2 in [HM], but the proof using the Petri machinery is appealing in its own right, so we proceed this way.

Here is an amusing singular example that gives one possible type of singular surface that appears in this context.

**Example 3.2.** Consider the ideal $I$ generated by the Petri-form quadrics:

$$f_{ij} = x_i x_j - (x_k + x_l)(x_0 + x_5) - x_0 x_5,$$

where as usual $1 \leq i < j \leq 4$, and here $\{i, j, k, l\} = \{1, 2, 3, 4\}$. For this example, we can take $\rho_{ijk} = 1$, all $i, j, k$, and $\alpha_k = x_0 + x_5$, all $k$. Using a computer algebra system, it is
easy to see that $I$ has a Groebner basis of the form described in Theorem 1.4 of [S], so $C = V(I)$ is a canonical curve of arithmetic genus 6. It is not difficult to see that $C$ is a union of five smooth conics $C_i$ in planes $P_i$ defined by

$$
\begin{align*}
P_1 &= V(x_2 + x_5 + x_0, x_3 + x_5 + x_0, x_4 + x_5 + x_0) \\
P_2 &= V(x_1 + x_5 + x_0, x_3 + x_5 + x_0, x_4 + x_5 + x_0) \\
P_3 &= V(x_1 + x_5 + x_0, x_2 + x_5 + x_0, x_4 + x_5 + x_0) \\
P_4 &= V(x_1 + x_5 + x_0, x_2 + x_5 + x_0, x_3 + x_5 + x_0) \\
P_5 &= V(x_1 - x_4, x_2 - x_4, x_3 - x_4)
\end{align*}
$$

The five planes all contain the line

$$
L = V(x_1 + x_5 + x_0, x_2 + x_5 + x_0, x_3 + x_5 + x_0, x_4 + x_5 + x_0)
$$

and the conics $C_i$ all meet $L$ in the same two points $p, q$. (The tangents to the $C_i$ at $p$ all lie in a hyperplane, so the singularity has $\delta$-invariant 5; the situation at $q$ is the same.) In this case the surface of degree 5 defined by the combinations of the $f_{ij}$ given in (2) above is $S = P_1 \cup \cdots \cup P_5$. Note that the general hyperplane section $S \cap H$ is a union of 5 concurrent lines spanning $H \cong P^4$, a curve of arithmetic genus 1.

**Proof of Theorem 3.1.** Let $C$ be a Petri-general canonically-embedded curve of arithmetic genus 6 in $P^5$, whose ideal is generated by quadrics. We will begin by proving the assertion above that at least two of the Petri coefficients $\rho_{ijk}$ must be non-zero. Let $T$ be the graph with vertices $V = \{1, 2, 3, 4\}$, and an edge $(i, j)$ if and only if there is some $k$ such that $\rho_{ijk} \neq 0$. By Proposition 3.2 of [S], in the minimal free resolution of the homogeneous coordinate ring of $C$, the graded Betti number $\beta_{13}$ (giving the number of cubics in a minimal basis for $I(C)$) satisfies

$$
\beta_{13} = \# \text{ connected components of } T - 1
$$

By our assumption, $\beta_{13} = 0$, so $T$ must be connected. By the definition and the symmetry of the $\rho_{ijk}$ in the indices, this implies that there is at most one edge of the complete graph on $V$ that is not contained in $T$. After renumbering if necessary, the potentially missing edge can be taken to be $(3, 4)$, and hence $\rho_{123} \neq 0$ and $\rho_{124} \neq 0$. Under this assumption, the quadrics $F_i$ given in (2) above always generate a 5-dimensional vector subspace of $I(C)_2$; there is exactly one linear dependence between them:

$$
\rho_{234}(F_1 - F_3) - \rho_{124}(F_2 - F_6) + \rho_{134}(F_4 - F_5) = 0 .
$$
**Step 1.**

Let $J = \langle F_1, \cdots, F_6 \rangle$. As explained above, the first step in the proof will be to show that $V(J)$ is a surface of degree 5 in $\mathbb{P}^5$. To do this, we will analyze the form of the unique reduced Groebner basis for $J$ with respect to the graded reverse lexicographic order with the variables ordered $x_1 > x_2 > x_3 > x_4 > x_5 > x_0$.

The results depend on whether $\rho_{134}$ and $\rho_{234}$ are zero; we will consider the case where all $\rho_{ijk} \neq 0$ first. (Note that this will be the case, for example, for a generic choice of $(g-2)$-secant in Petri’s construction if $C$ is irreducible.)

For simplicity, we begin by taking linear combinations of the $F_i$ to eliminate common terms and to isolate the possible leading monomials of quadrics in the ideal. The resulting basis for $J_2$ is:

\begin{align*}
F_1' &= \rho_{234} \rho_{134} f_{12} - \rho_{124} \rho_{123} f_{34} \\
F_2' &= \rho_{234} f_{13} - \rho_{123} f_{34} \\
F_3' &= \rho_{234} f_{14} - \rho_{124} f_{34} \\
F_4' &= \rho_{134} f_{23} - \rho_{123} f_{34} \\
F_5' &= \rho_{134} f_{24} - \rho_{124} f_{34}
\end{align*}

For instance, $F_1' = \rho_{234}(F_1 - F_3) + \rho_{124}F_6$. We omit the rest of the details in this calculation. For future reference, however, we note the following observation.

**Observation 3.3.** All quadrics of the forms

\[ \rho_{ijk} f_{ij} - \rho_{iik} f_{ij} \]

and

\[ \rho_{ikl} \rho_{jkl} f_{ij} - \rho_{ijk} \rho_{ijl} f_{kl}, \]

(where $\{i,j,k,l\} = \{1,2,3,4\}$ in each case) belong to $J$.

This may be seen directly by forming linear combinations as above. Now, applying Buchberger’s algorithm, we begin the Groebner basis computation on the $F_i'$. At the first step, the $S$-polynomial $S(F_1', F_2')$ yields

\[ x_3 F_1' - \rho_{134} x_2 F_2' \equiv \rho_{123}(\rho_{234} x_2 x_3 x_4 - \rho_{124} x_5^2 x_4) \mod \langle x_5, x_0 \rangle . \]

Replacing this last polynomial with its remainder on division by the $F_i'$ and adjusting constants, we obtain a new Groebner basis element

\[ G \equiv \rho_{123} x_3^2 x_4 - \rho_{124} x_3 x_4^2 \mod \langle x_5, x_0 \rangle . \]
We claim that $G = \{F_1', \ldots, F_5', G\}$ is the reduced Groebner basis for $J$. Indeed, working modulo $\langle x_5, x_0 \rangle$, it is easy to see that all further $S$-pairs reduce to zero, modulo $\langle x_5, x_0 \rangle$. Hence to prove the claim, it suffices to show that $x_5, x_0$ are a regular sequence in $k[x_1, \ldots, x_5, x_0]/J$, or that the resulting syzygies modulo $\langle x_5, x_0 \rangle$ all lift to syzygies on $G$. Using the following lemma, we will show that this is a consequence of the Petri syzygies on the generators of $I(C)$.

**Lemma 3.4.** Let $f_{ij}$ be a basis for the quadrics in the ideal of a canonical curve of genus $g \geq 6$ in Petri’s form, and let $\{i, j, k, l\}$ be any four distinct indices in $\{1, 2, \ldots, g - 2\}$. Then each syzygy on the leading terms of the $f_{ij}$ of the form:

$$
\rho_{i\ell n} x_k (\rho_{i\ell n} x_i x_l - \rho_{i\ell l} x_i x_n) + \rho_{i\ell n} x_l (\rho_{i\ell l} x_i x_n - \rho_{i\ell n} x_i x_k) + \\
\rho_{i\ell l} x_n (\rho_{i\ell n} x_i x_k - \rho_{i\ell k} x_i x_l) = 0
$$

(3)

lifts to a syzygy on the quadrics of the form

$$
\rho_{\alpha\beta\gamma} f_{\delta\epsilon} - \rho_{\alpha\beta\delta} f_{\gamma\epsilon}
$$

in $I(C)$.

**Proof.** We consider the Petri syzygies in the form

$$(S_{i\ell k})
\quad x_k f_{i\ell} - x_l f_{i\ell} + \sum_{s \neq k} a_{s il} f_{sk} - \sum_{s \neq l} a_{s ik} f_{sl} + \rho_{i\ell k} G_{kl} = 0
$$

where the $G_{kl}$ are Petri’s cubics in $I(C)$, satisfying relations

$$
G_{kl} + G_{ln} + G_{nk} = 0 \quad (4)
$$

Consider the linear combination

$$(5)
\quad \rho_{i\ell n} \rho_{i\ell n} S_{i\ell k} + \rho_{i\ell n} \rho_{i\ell l} S_{i\ell n} + \rho_{i\ell n} \rho_{i\ell k} S_{i\ell n}
$$

Using (4), the $G$ terms appearing in (5) cancel, so we obtain a relation on quadrics only. The remaining terms are:

$$
0 = \rho_{i\ell n} \rho_{i\ell n} (x_k f_{i\ell} - x_l f_{i\ell} + \sum_{s \neq k} a_{s il} f_{sk} - \sum_{s \neq l} a_{s ik} f_{sl}) + \\
\rho_{i\ell n} \rho_{i\ell l} (x_l f_{i\ell} - x_n f_{il} + \sum_{s \neq l} a_{s in} f_{sl} - \sum_{s \neq n} a_{s il} f_{sn}) + \\
\rho_{i\ell n} \rho_{i\ell k} (x_n f_{ik} - x_k f_{in} + \sum_{s \neq n} a_{s ik} f_{sn} - \sum_{s \neq k} a_{s in} f_{sl})
$$

9
Rearranging and using the relations $a_{ijk} = \rho_{ijk} \alpha_i$ when $i, j, k$ are distinct, we have:

$$
\rho_{iln} x_k (\rho_{ikn} f_{il} - \rho_{ikl} f_{ln}) + \rho_{ikn} x_l (\rho_{ikl} f_{ln} - \rho_{iln} f_{ik}) + \rho_{ikl} x_n (\rho_{iln} f_{ik} - \rho_{ikn} f_{il})
$$

$$
= \rho_{ikl} \sum_{s \neq k, l} a_{sil} (\rho_{ikf_{sl} - \rho_{ikl} f_{sn}}) + \rho_{iln} \sum_{s \neq n, l} a_{sik} (\rho_{ikn f_{sl} - \rho_{ikl} f_{sn}})
$$

$$
+ \rho_{ikl} \sum_{s \neq k, l} a_{sln} (\rho_{ikn f_{sk} - \rho_{ikn f_{sl}}}) + \rho_{ikn} (\rho_{ikl} \alpha_k - \rho_{iln} \alpha_n) f_{kn}
$$

$$
+ \rho_{iln} (\rho_{ikn} \alpha_n - \rho_{ikl} \alpha_i) f_{ln} + \rho_{ikl} (\rho_{iln} \alpha_l - \rho_{ikn} \alpha_k) f_{lk}
$$

$$
= \rho_{ikl} \sum_{s \neq k, n} a_{sil} (\rho_{ikf_{sl} - \rho_{ikl} f_{sn}}) + \rho_{iln} \sum_{s \neq n, l} a_{sik} (\rho_{ikn f_{sl} - \rho_{ikl} f_{sn}})
$$

$$
+ \rho_{ikl} \sum_{s \neq k, l} a_{sln} (\rho_{ikn f_{sk} - \rho_{ikn f_{sl}}}) + \alpha_n \rho_{ikn} \rho_{iln} (\rho_{ikn f_{ln} - \rho_{iln f_{kn}}})
$$

$$
+ \alpha_l \rho_{ikl} \rho_{iln} (\rho_{ikl f_{lk} - \rho_{ikl f_{ln}}})
$$

$$
+ \alpha_k \rho_{ikl} \rho_{ikn} (\rho_{ikl f_{kn} - \rho_{ikn f_{lk}}})
$$

This gives the desired lifting. \(\triangle\)

We now return to Step 1 in the proof of Theorem 3.1. The lemma, combined with Observation 3.3, shows that any syzygy on the elements of $G$ modulo $\langle x_5, x_0 \rangle$ that can be expressed in terms of the syzygies (3) can be lifted to a syzygy on $G$, and hence that no new elements of the Groebner basis will be produced in those cases. In fact, all syzygies on $G$ modulo $\langle x_5, x_0 \rangle$ can be expressed in terms of syzygies of the form (3), so no new Groebner basis elements at all are introduced after $G$. For example, reducing $S(F_1', F_3')$ yields

$$
x_4 F_1' - \rho_{134} x_2 F_3' - \rho_{124} x_4 F_4' \equiv 0 \mod \langle x_5, x_0 \rangle,
$$

or combining the two terms with $x_4$,

$$
(6) \quad \rho_{134} (x_4 (\rho_{234} f_{12} - \rho_{124} f_{23}) - x_2 (\rho_{234} f_{14} - \rho_{124} f_{34})) \equiv 0 \mod \langle x_5, x_0 \rangle
$$

This relation is apparently of a different form than the ones in the Lemma, but the fact that it too lifts it can be deduced from the Lemma as follows. Modulo $\langle x_5, x_0 \rangle$, the first term on the left of (6) is the $\frac{\rho_{134}}{\rho_{124}}$ times the term with the factor of $x_4$ on the left of relation (3) with $i = 2$. Similarly, the second term on the left, modulo $\langle x_5, x_0 \rangle$, is exactly the $x_2$ term on the left of relation (3) with $i = 4$. We form the corresponding linear combination of those two relations of the form (3) and clear denominators of $\rho_{123}$ yielding that

$$
\rho_{123} \rho_{134} (x_4 (\rho_{234} f_{12} - \rho_{124} f_{23}) - x_2 (\rho_{234} f_{14} - \rho_{124} f_{34}))
$$

$$
- \rho_{234} \rho_{124} (x_1 (\rho_{123} f_{34} - \rho_{134} f_{23}) - x_3 (\rho_{134} f_{12} - \rho_{123} f_{14}))
$$

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equals the difference between the corresponding combination of the right hand sides of the two relations (3). But this shows that (6) lifts to a syzygy on the quadrics in \( G \). The other \( S \)-pairs are handled similarly using relations (3) directly, and relations of this last type.

Our conclusion is that in the case that all \( \rho_{ijk} \neq 0 \), \( G \) is the reduced Groebner basis of \( J \). Computing the Hilbert function of \( J \) from this information, we see that \( S = V(J) \) has degree 5 and codimension 3. Indeed, \( V(J) \cap V(x_0, x_5) \) consists of the five points with homogeneous coordinates

\[
(1, 0, 0, 0, 0, 0), \ (0, 1, 0, 0, 0, 0), \ (0, 0, 1, 0, 0, 0), \ (0, 0, 0, 1, 0, 0),
\]

\[
\left( \frac{1}{\rho_{234}}, \frac{1}{\rho_{134}}, \frac{1}{\rho_{124}}, \frac{1}{\rho_{123}}, 0, 0 \right)
\]

so \( S \) is reduced.

The remaining cases to consider are those where one or both of \( \rho_{134}, \rho_{234} \) are zero. The arguments in those cases are basically similar to the ones given here, so we will omit most of the details and give only the form of the corresponding Groebner basis in each case. If both \( \rho_{134} = \rho_{234} = 0 \), then the quadrics in (2) reduce to

\[
\{f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}.
\]

Using the vanishing of the Petri coefficients, we see that the initial ideal of \( J \) has the form

\[
M_2 = \langle x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4, x_1^2 x_5 \rangle
\]

in this case. Computing the Hilbert function gives degree 5 and codimension 3.

Finally if just one of the Petri coefficients, say \( \rho_{234} \), is zero, then from (2), we see that \( J \) is generated by

\[
\rho_{134} f_{12} - \rho_{123} f_{14}, \ \rho_{124} f_{13} - \rho_{123} f_{14}, \ f_{23}, \ f_{24}, \ f_{34}
\]

The initial ideal of \( J \) has the form

\[
M_1 = \langle x_1 x_2, x_1 x_3, x_2 x_3, x_2 x_4, x_3 x_4, x_1^2 x_4 \rangle
\]

and the Hilbert function is the same as in the other cases.

This completes Step 1 of the proof of Theorem 3.1.

**Step 2.**
We now want to show that the family $S$ of all surfaces $S$ of degree 5 obtained in step 1 is irreducible. Looking at the minimal free resolution of the coordinate ring of $S$ in each case, we have that $J$ is a Gorenstein ideal of codimension three, since the Betti diagram (as in the “betti” command of the Macaulay system of Bayer and Stillman) is

\[
\begin{array}{cccc}
1 & - & - & - \\
- & 5 & 5 & - \\
- & - & - & 1
\end{array}
\]

That is, writing $R = k[x_1, \ldots, x_5, x_0]$, the minimal resolution has the form

(7) \[ 0 \to R(-5) \to R(-3)^5 \to R(-2)^5 \to R \to R/J \to 0 \]

We can use the structure theorem for Gorenstein ideals of codimension 3 ([BE]) to give a uniform description of the ideal $J$ valid in all cases. Namely, every ideal $J$ that appears here is generated by the $4 \times 4$ Pfaffians of a $5 \times 5$ skew-symmetric rank-4 matrix of linear forms $A = (a_{ij})$ (the “middle matrix” of the resolution (7) under a suitable choice of basis for $J$). Apart from the requirement that $\text{rank}(A) = 4$, the entries in $A$ are arbitrary. It follows that the surfaces $S$ obtained in Step 1 form one irreducible family, of dimension 35. (This may also be seen by a normal bundle calculation as in §2.)

**Step 3.**

A general $C$ is contained in exactly one surface $S$ of the type described above. To complete the proof, we complete the basis (2) of $J$ to a basis of $I(C)$. Recall that we are assuming that $I(C)$ is generated by quadrics, so any one further quadric in $I(C)$ not in $J$ (such as $f_{34}$ in the case that all $\rho_{ijk} \neq 0$) will do the job. Hence $C$ is the complete intersection of $S$ and a quadric hypersurface. (Conversely, given a surface $S$ of degree 5 of the form above and a general quadric $Q$ – not containing any component of $S$ for instance – then $Q \cap S$ will be a non-degenerate Gorenstein curve $C$ of degree 10 and arithmetic genus 6 in $\mathbb{P}^5$.). The family of surfaces $S$ is irreducible by Step 2, and given $S$, to obtain $C \subset S$, the additional quadric $Q$ can be chosen essentially arbitrarily in $H^0(C, \mathcal{O}_S(2))$ which is irreducible of dimension 15. Hence, we have that $H_6'$ is irreducible of dimension $35 + 15 = 50$. △

(Note that, as expected, this is the same as the dimension of the family of all smooth canonically-embedded curves in $\mathbb{P}^5$, which is

\[
\dim(M_6) + \dim(PGL(6)) = 3 \cdot 6 - 3 + 6^2 - 1 = 50 .
\]

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We may ask if there is a result for $g = 6$ including the curves for which $I(C)$ is not generated by quadrics, fully parallel to Proposition 2.1. The answer is yes, as we will now see. However, the situation is somewhat complicated by two new features. First, even for smooth curves, by the classical Enriques-Petri theorem, there are two different possibilities for the variety $V(f_{ij})$ when $I(C)$ is not generated by quadrics. Indeed, consider the family of ideals generated by quadrics in Petri’s form for which all $\rho_{ijk} = 0$ (so that $f_{ij}$ are a Groebner basis for the ideal they generate), and in which, for simplicity, the low-order terms in $f_{ij}$ are normalized to

$$q_{ij}(x_0, x_5) = b_{ij}x_0x_5.$$  (8)

Since $V(f_{ij}, x_0, x_5)$ consists of the four points

$$(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0),$$

we see that $V(f_{ij})$ is a reduced, arithmetically Cohen-Macaulay surface of degree 4. By the classification of surfaces of degree $n − 1$ in $\mathbb{P}^n$, we see that there are two components. One, of dimension 11, has general point corresponding to the ideal of a quartic scroll, another, of dimension 9 has general point corresponding to the ideal of a Veronese surface. (Recall that these Petri quadrics are normalized so that the two additional points $(0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)$ lie on the variety they define. If we do not require these incidence conditions, the corresponding components of the Hilbert scheme of surfaces have dimensions $11 + 18 = 29$ (scrolls), and $9 + 18 = 27$ (Veroneses) respectively.)

The curves lying on scrolls, and their degenerations, and the plane quintics and their degenerations each form an irreducible family, whose general element is a smooth curve. This follows by an argument similar to that given above in Theorem 2.1. For instance, on a scroll or a degeneration of a scroll, the canonical curves are the residual intersections of the scroll and a cubic containing 2 fixed lines. This gives a

$$\left(\binom{5+3}{3} - 8 - 28\right) - 1 = 19$$

dimensional irreducible family of curves on each $S$. Similarly, on a Veronese surface or degeneration, the canonical curves belong to a 20-dimensional irreducible family (e.g. the 2-uple images of all the plane quintics on a smooth Veronese.) Hence, an argument similar to the one given in the last section of the proof of Theorem 1.2 shows that if $C$ is any singular canonically-embedded curve on a scroll or a Veronese, or one of their degenerations, then $[C]$ belongs to the same irreducible component of the Hilbert scheme as the points in $\mathcal{H}_6$.  

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There is a small second complication here as well. Namely, by §3 of [S], in addition to the two cases we have accounted for, in which the graded Betti number $\beta_{13} = 0$ ($C$ non-trigonal, non-plane quintic), or $\beta_{13} = 3$ ($C$ trigonal, or plane quintic), there is apparently another possibility when $g = 6$: $\beta_{13} = 1$. By Proposition 3.2 of [S], this would happen only if there were exactly one $\rho_{ijk} \neq 0$. The 4-vertex graph $T$ introduced in Step 1 of the proof of Theorem 3.1 would be composed of three edges forming a triangle together with one disconnected vertex. We follow the reasoning of Propositions 3.3 and 3.4 of [S]. Under the assumption $\rho_{124} = \rho_{134} = \rho_{234} = 0$, but $\rho_{123} \neq 0$, the quadrics $f_{12}, f_{13}, f_{23}$ depend only on $x_1, x_2, x_3, x_0, x_5$ and satisfy the Petri syzygy $S_{123}$. From the other Petri syzygies, $a_{114} = a_{224} = a_{334}$, and the linear form $x_4 - a_{114}$ must divide $f_{14}, f_{24}, f_{34}$. We see from Theorem 1.4 of [S] that $f_{12}, f_{13}, f_{23}$, together with the linear form $x_4 - a_{114}$ must generate the ideal of a non-trigonal canonically-embedded curve of genus 5: $C_1 \subset H = V(x_4 - a_{114}) \cong \mathbb{P}^4 \subset \mathbb{P}^5$. Furthermore, $V = V(f_{ij}) = C_1 \cup P$, where $P$ is a plane. By degree considerations, the other component $C_2$ of $C$ lying in $P$ must be a conic in $P$. (In order for $C$ to be Petri-general, $C_2$ must be reduced as well: By the Petri construction, $(0,0,0,1,0,0)$ must be a smooth point of $C$, but it cannot be contained in $C_1$. Hence it must be a smooth point of $C_2$.)

We claim that such a curve cannot be a canonically-embedded curve (the condition $\mathcal{O}_C(1) \cong \omega_C$ will fail), even though its Hilbert point almost certainly lies on the same component as those of canonically-embedded curves. The reason is the following. The hyperplane $H$ containing $C_1$ and the plane $P$ containing $C_2$ must intersect transversely along a line $L$ if $C$ is to span $\mathbb{P}^5$. In order for $C$ to be connected, we need $C_1 \cap C_2 \neq \emptyset$.

By a standard fact on dualizing sheaves ([C], Lemma 1.12),

$$\omega_C|_{C_2} \cong \omega_{C_2} \otimes (\mathcal{I}_{C_1} \otimes \mathcal{O}_{C_2})^{-1}$$

where $\mathcal{I}_{C_1}$ is the ideal sheaf of $C_1$. Since $C_1$ is a non-trigonal curve, not containing $L$ as a component, $C_1 \cap L$ contains at most two points. This implies that neither restriction $\omega_{C}|_{C_1}$ is correct. For example, in order for $C_2$ to embed as a conic by the dualizing sheaf, $(\mathcal{I}_{C_1} \otimes \mathcal{O}_{C_2})^{-1}$ would have to have degree 4 on $C_2$, but this is impossible.

Hence, to obtain curves of this type, the only remaining possibility is that the line $L$ is a component of $C_1$. There are canonical curves of genus 5 of this type: $C_1 = C'_1 \cup C''_1$ where $C'_1$ has genus 3, $C''_1 \cong \mathbb{P}^1$, and $C'_1$ intersects $C''_1$ transversely in three points $\{p_1, p_2, p_3\}$. By (9), $\omega_C$ is very ample in this case, and embeds $C'_1$ as a curve of degree 7 in $\mathbb{P}^4$, with $C''_1$ as a trisecant line. See 3.6 below for a concrete example; if $p_1 + p_2 + p_3$ is not a divisor of one of the $g_3^1$’s on $C'_1$, we even obtain curves of this type which are not “trigonal.” However this case too leads to a situation where $C$ fails to be canonically-embedded. The reason
is the same as before. In a general such curve, the smooth conic $C_2$ will again meet the line $L$ in two points. Hence $\mathcal{O}_{C_i'}(1)$ and $\mathcal{O}_{C_2}(1)$ are incorrect for a canonically-embedded curve. If $C_2$ meets $L$ at one or two of the points $p_i$, and $C'_1$ is smooth, we obtain one or two non-planar triple points (with delta-invariant $\delta = 2$). These singularities are not even locally Gorenstein. More degenerate curves also occur but the conclusion is the same in all cases.

In sum, the case $\beta_{13} = 1$ does not actually occur for canonically-embedded curves when $g = 6$. We have proved the following.

**Theorem 3.5.** Let $\mathcal{H}_6$ be the open subscheme of the Hilbert scheme of curves of degree 10 and arithmetic genus 6 in $\mathbb{P}^5$ corresponding to Petri-general canonically embedded curves. Then $\mathcal{H}_6$ is irreducible.

We conclude with two examples illustrating the analysis of the $\beta_{13} = 1$ cases above.

**Example 3.6.** Consider the curve $C = C_1 \cup C_2$, where $C_i$ are defined as follows. Let $C_1$ be the non-trigonal genus 5 canonical curve defined by the Petri-form quadrics

$$
q_{12} = x_1x_2 - (x_5 - x_0)x_1 + (x_0 + x_5)x_3 \\
q_{13} = x_1x_3 + (x_0 + x_5)x_2 \\
q_{23} = x_2x_3 + (x_0 + x_5)x_1 + (4x_5 - x_0)x_3
$$

(noting the $x_4 = 0$. (We have $\rho_{123} = 1$, and $\alpha_1 = \alpha_2 = \alpha_3 = -(x_0 + x_5)$. $C_1$ contains $L = C_1'' = V(x_1, x_2, x_3, x_4)$ as a component. The other component $C'_1$ is a curve of arithmetic genus 3 as above.

Next, let $C_2$ be the conic $V(x_1, x_2, x_3, x_4x_5 - 2x_4x_0 + 4x_0x_5)$. Intersecting $I(C_1)$ and $I(C_2)$, we find a Groebner basis for $I(C)$ yielding the following information. The initial ideal of $I(C)$ is

$$
\langle x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1^2x_5, x_2^2x_5, x_3^2x_5, x_4^3x_5 \rangle
$$

(note the differences between this and the initial ideal given in Theorem 1.4 of [S]). However, $C$ still has degree 10 and arithmetic genus 6. The Betti diagram of the minimal resolution of $I(C)$ has the form:

| 1 | - | - | - | - |
|---|---|---|---|---|
| - | 6 | 6 | 1 | - |
| - | 1 | 6 | 6 | 1 |
| - | - | - | 1 | 1 |
We see that $\beta_{13} = 1$, but that the resolution is not self-dual, which confirms the fact that $C$ is not canonically-embedded. However, we note that by a calculation, $\dim H^0(N_{C|\mathbb{P}^5}) = 50$, which strongly suggests that the Hilbert point of $C$ lies on the same component of the Hilbert scheme as the canonical curves of genus 6.

In the case that $L$ is not a component of $C$, we obtain non-canonically-embedded curves of degree 10 and arithmetic genus 6 for which $\beta_{13} = 0$.

**Example 3.7.** Consider the curve $C = C_1 \cup C_2$, where $C_i$ are defined as follows. Let $C_1$ be the non-trigonal genus 5 canonical curve defined by the Petri-form quadrics

\[
q_{12} = x_1x_2 + (x_0 + x_5)x_3 + x_0x_5 \\
q_{13} = x_1x_3 + (x_0 + x_5)x_2 \\
q_{23} = x_2x_3 + (x_0 + x_5)x_1
\]

together with $x_4 = 0$. (We have $\rho_{123} = 1$, and $\alpha_1 = \alpha_2 = \alpha_3 = -(x_0 + x_5)$. $C_1$ meets $L = V(x_1, x_2, x_3, x_4)$ in the two points $(0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)$.) Next, let $C_2$ be the conic $V(x_1, x_2, x_3, x_4x_5 - 2x_4x_0 + 4x_0x_5)$, which meets $L$ in the same two points as $C_1$. Intersecting $I(C_1)$ and $I(C_2)$, we find that instead of the quadric $q_{12}$ above, $I(C)$ contains the quadric

\[
f_{12} = x_1x_2 + (x_0 + x_5)x_3 + (1/4x_5 - 1/2x_0)x_4 + x_0x_5
\]

This shows that we have a curve similar to the curves with $\rho_{123}, \rho_{124} \neq 0$ studied above in the proof of Theorem 3.1. (Indeed the reader will have no difficulty constructing a surface of degree 5 (a kind of “degenerate Del Pezzo surface”) containing $C$. The initial ideal of $I(C)$ is the same as in Example 3.6. However, now the Betti diagram for the minimal resolution of $I(C)$ has the form:

\[
\begin{array}{ccccccc}
1 & - & - & - & - & - & - \\
- & 6 & 5 & 1 & - & - & - \\
- & - & 6 & 6 & 1 & - & - \\
- & - & - & 1 & 1 & - & - \\
\end{array}
\]

We see that $\beta_{13} = 0$, but that as in the previous example, the resolution is not self-dual. By another calculation, $\dim H^0(N_{C|\mathbb{P}^5}) = 50$, which again strongly suggests that the Hilbert point of $C$ lies on the same component of the Hilbert scheme as the canonical curves of genus 6.
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