STUDYING UNIFORM THICKNESS II: TRANSVERSALLY NON-SIMPLE ITERATED TORUS KNOTS

DOUGLAS J. LAFOUNTAIN

Abstract. We prove that an iterated torus knot type fails the uniform thickness property (UTP) if and only if all of its iterations are positive cablings, which is precisely when an iterated torus knot type supports the standard contact structure. We also show that all iterated torus knots that fail the UTP support cabling knot types that are transversally non-simple.

1. Introduction

In this paper, we continue our general study of the uniform thickness property (UTP) in the context of iterated torus knots that are embedded in $S^3$ with the standard tight contact structure. As stated in a previous paper, Studying uniform thickness I [L], our goal in this study is to determine the extent to which iterated torus knot types fail to satisfy the UTP, and the extent to which this failure leads to cablings that are Legendrian or transversally non-simple. Motivation for this study is due to the work of Etnyre and Honda [EH1], who showed that the failure of the UTP is a necessary condition for transversal non-simplicity in the class of iterated torus knots. They also established that the $(2,3)$ torus knot fails the UTP and supports a transversally non-simple cabling. In [L] we extended this study of the UTP by establishing new necessary conditions for both the failure of the UTP and transversal non-simplicity in the class of iterated torus knots; in so doing we obtained new families of Legendrian simple iterated torus knots.

The specific goal of this note is to fully answer the first motivating question of our study by providing a complete UTP classification of iterated torus knots, that is, determining which iterated torus knot types satisfy the UTP, and which fail the UTP. We will also address the second motivating question of our study by proving that failure of the UTP for an iterated torus knot type is a sufficient condition for the existence of transversally non-simple cablings of that knot. Specifically, we have the following two theorems and corollary:

**Theorem 1.1.** Let $K_r = ((P_1, q_1), ..., (P_i, q_i), ..., (P_r, q_r))$ be an iterated torus knot, where the $P_i$’s are measured in the standard preferred framing, and $q_i > 1$ for all $i$. Then $K_r$ fails the UTP if and only if $P_i > 0$ for all $i$, where $1 \leq i \leq r$.

In the second theorem, $\chi(K)$ is the Euler characteristic of a minimal genus Seifert surface for a knot $K$:

**Theorem 1.2.** If $K_r$ is an iterated torus knot that fails the UTP, then it supports infinitely many transversally non-simple cablings $K_{r+1}$ of the form $(-\chi(K_r), k + 1)$, where $k$ ranges over an infinite subset of positive integers.

---

*Key words and phrases.* Legendrian, transversal, convex, uniform thickness property.

*2000 Mathematics Subject Classification.* Primary 57M25, 57R17; Secondary 57M50.
To state our corollary to Theorem [1.1], recall that if $K$ is a fibered knot, then there is an associated open book decomposition of $S^3$ that supports a contact structure, denoted $\xi_K$ (see [TW]). Iterated torus knots are fibered knots, and Hedden has shown that the subclass of iterated torus knots where each iteration is a positive cabling, i.e. $P_i > 0$ for all $i$, is precisely the subclass of iterated torus knots where $\xi_K$ is isotopic to $\xi_{std}$ [He1]. We thus obtain the following corollary:

**Corollary 1.3.** An iterated torus knot $K_r$ fails the UTP if and only if $\xi_{K_r} \cong \xi_{std}$.

We make a few remarks about these theorems. First, it will be shown that these transversally non-simple cablings will all have two Legendrian isotopy classes at the same rotation number and maximal Thurston-Bennequin number $\overline{tb}$, and thus they will exhibit Legendrian non-simplicity at $\overline{tb}$. Second, in the class of iterated torus knots there are certainly more transversally non-simple cablings than those in Theorem [1.2] as evidenced by Etnyre and Honda’s example of the transversally non-simple $(2,3)$-cabling of a $(2,3)$-torus knot. However, we present just the class of transversally non-simple cablings in Theorem [1.2] and leave a more complete Legendrian and transversal classification of iterated torus knots as an open question.

We now present a conjectural generalization of the above two theorems and corollary. To this end, recall that Hedden has shown that for general fibered knots $K$ in $S^3$, $\xi_K \cong \xi_{std}$ precisely when $K$ is a fibered strongly quasipositive knot [He3]; he also shows that for these knots, the maximal self-linking number is $\overline{sl}(K) = -\chi(K)$ [He2]. Furthermore, from the work of Etnyre and Van Horn-Morris [EV], we know that for fibered knots $K$ in $S^3$ that support the standard contact structure there is a unique transversal isotopy class at $\overline{sl}$. In the present paper, all of these ideas are brought to bear on the class of iterated torus knots, and this motivates the following conjecture concerning general fibered knots:

**Conjecture 1.4.** Let $K$ be a fibered knot in $S^3$; then $K$ fails the UTP if and only if $\xi_K \cong \xi_{std}$, and hence if and only if $K$ is fibered strongly quasipositive. Moreover, if a topologically non-trivial fibered knot $K$ fails the UTP, then it supports cablings that are transversally non-simple.

We also ask the following question of non-fibered knots in $S^3$:

**Question 1.5.** If $K$ is a non-fibered strongly quasipositive knot, does $K$ fail the UTP and support transversally non-simple cablings?

We will be using tools developed by Giroux, Kanda, and Honda, and used by Etnyre and Honda in their work, namely convex tori and annuli, the classification of tight contact structures on solid tori and thickened tori, and the Legendrian classification of torus knots. Most of the results we use can be found in [HI1], [EH1], [H2], or [L], and if we use a lemma, proposition, or theorem from one of these works, it will be specifically referenced.

The plan of the note is as follows. In §2 we recall definitions, notation, and identities used in [L] and [EH1]. In §3 we outline a strategy of proof of Theorem [1.1] that yields the statement of two key lemmas. In §4 and §5 we prove the first lemma. In §6 we prove the second lemma and complete the proof of Theorem [1.1]. In §7 we prove Theorem [1.2].

2. Definitions, notation, and identities

2.1. **Iterated torus knots.** *Iterated torus knots,* as topological knot types, can be defined recursively. Let 1-iterated torus knots be simply torus knots $(p_1, q_1)$ with $p_1$ and $q_1$ co-prime
nonzero integers, and \(|p_1|, q_1 > 1\). Here \(p_1\) is the algebraic intersection with a longitude, and \(q_1\) is the algebraic intersection with a meridian in the preferred framing for a torus representing the unknot. Then for each \((p_1, q_1)\) torus knot, take a solid torus regular neighborhood \(N((p_1, q_1))\); the boundary of this is a torus, and given a framing we can describe simple closed curves on that torus as co-prime pairs \((p_2, q_2)\), with \(q_2 > 1\). In this way we obtain all 2-iterated torus knots, which we represent as ordered pairs, \(((p_1, q_1), (p_2, q_2))\). Recursively, suppose the \((r - 1)\)-iterated torus knots are defined; we can then take regular neighborhoods of all of these, choose a framing, and form the \(r\)-iterated torus knots as ordered \(r\)-tuples \(((p_1, q_1), \ldots, (p_{r-1}, q_{r-1}), (p_r, q_r))\), again with \(p_r\) and \(q_r\) co-prime, and \(q_r > 1\).

For ease of notation, if we are looking at a general \(r\)-iterated torus knot type, we will refer to it as \(K_r\); a Legendrian representative will usually be written as \(L_r\). Note that we will use the letter \(r\) both for the rotation number (see below) and as an index for our iterated torus knots; context will distinguish between the two uses.

We will study iterated torus knots using two framings. The first is the standard framing for a torus, where the meridian bounds a disc inside the solid torus, and we use the preferred longitude which bounds a surface in the complement of the solid torus. We will refer to this framing as \(C\). The second framing is a non-standard framing using a different longitude that comes from the cabling torus. More precisely, to identify this non-standard longitude on \(\partial N(K_r)\), we first look at \(K_r\) as it is embedded in \(\partial N(K_{r-1})\). We take a small neighborhood \(N(K_r)\) such that \(\partial N(K_r)\) intersects \(\partial N(K_{r-1})\) in two parallel simple closed curves. These curves are longitudes on \(\partial N(K_r)\) in this second framing, which we will refer to as \(C'\). Note that this \(C'\) framing is well-defined for any cabled knot type. Moreover, for purpose of calculations there is an easy way to change between the two framings, which will be reviewed below.

Given a simple closed curve \((\mu, \lambda)\) on a torus, measured in some framing as having \(\mu\) meridians and \(\lambda\) longitudes, we will say this curve has slope of \(\frac{\lambda}{\mu}\); i.e., longitudes over meridians. Therefore we will refer to the longitude in the \(C'\) framing as \(\infty'\), and the longitude in the \(C\) framing as \(\infty\). The meridian in both framings will have slope 0.

We will also use a convention that meridians in the standard \(C\) framing, that is, algebraic intersection with \(\infty\), will be denoted by upper-case \(P\)’s. On the other hand, meridians in the non-standard \(C'\) framing, that is, algebraic intersection with \(\infty'\), will be denoted by lower-case \(p\)’s. Given a curve \((P, q)\) on a torus \(\partial N(K)\), there is then a relationship between the framings \(C'\) and \(C\) on \(\partial N(K)\). In terms of a change of basis, we get from \(C'\) to \(C\) by multiplying on the left by the matrix \[\begin{pmatrix} 1 & Pq \\ 0 & 1 \end{pmatrix}.\]

Given an iterated torus knot type \(K_r = ((p_1, q_1), \ldots, (p_r, q_r))\) where the \(p_i\)’s are measured in the \(C'\) framing, we define two quantities. The two quantities are:

\[
A_r := \sum_{\alpha=1}^{r} p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta \prod_{\beta=\alpha}^{r} q_\beta \\
B_r := \sum_{\alpha=1}^{r} \left(p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta\right) + \prod_{\alpha=1}^{r} q_\alpha
\]

Note here we use a convention that \(\prod_{\beta=r+1}^{r+1} q_\beta := 1\). Also, if we restrict to the first \(i\)
iterations, that is, to $K_i = ((p_1, q_1), ..., (p_i, q_i))$, we have an associated $A_i$ and $B_i$. For example, $A_i := \sum_{\alpha=1}^{\infty} p_\alpha \prod_{\beta=\alpha+1}^{\infty} q_\beta$.

Finally, from [L] we obtain four useful identities which we will apply extensively throughout this note:

\begin{align*}
A_r &= q_r^2 A_{r-1} + p_r q_r \\
B_r &= q_r B_{r-1} + p_r \\
P_r &= q_r A_{r-1} + p_r \\
A_r &= P_r q_r
\end{align*}

### 2.2. Legendrian knots, convex tori, and the UTP

Recall that for Legendrian knots embedded in $S^3$ with the standard tight contact structure, there are two classical invariants of Legendrian isotopy classes, namely the Thurston-Bennequin number, $tb$, and the rotation number, $r$. For a given topological knot type, if the ordered pair $(r, tb)$ completely determines the Legendrian isotopy classes, then that knot type is said to be Legendrian simple. For transversal knots there is one classical invariant, the self-linking number $sl$; for a given topological knot type, if the value of $sl$ completely determines the transversal isotopy classes, then that knot type is said to be transversally simple. For a given topological knot type, if we plot Legendrian isotopy classes at points $(r, tb)$, we obtain a plot of points that takes the form of a Legendrian mountain range for that knot type.

We will be examining Legendrian knots which are embedded in convex tori. Recall that the characteristic foliation induced by the contact structure on a convex torus can be assumed to have a standard form, where there are $2n$ parallel Legendrian divides and a one-parameter family of Legendrian rulings. Parallel push-offs of the Legendrian divides gives a family of $2n$ dividing curves, referred to as $\Gamma$. For a particular convex torus, the slope of components of $\Gamma$ is fixed and is called the boundary slope of any solid torus which it bounds; however, the Legendrian rulings can take on any slope other than that of the dividing curves by Giroux’s Flexibility Theorem [G]. A standard neighborhood of a Legendrian knot $L$ will have two dividing curves and a boundary slope of $\frac{1}{tb(L)}$.

We can now state the definition of the uniform thickness property as given by Etnyre and Honda [EH1]. For a knot type $K$, define the contact width of $K$ to be

\begin{equation}
 w(K) = \sup \frac{1}{\text{slope}(\Gamma_{\partial N})}
\end{equation}

In this equation the $N$ are solid tori having representatives of $K$ as their cores; slopes are measured using the preferred framing where the longitude has slope $\infty$; the supremum is taken over all solid tori $N$ representing $K$ where $\partial N$ is convex. A knot type $K$ then satisfies the UTP if the following hold:

1. $\overline{tb}(K) = w(K)$, where $\overline{tb}$ is the maximal Thurston-Bennequin number for $K$.

2. Every solid torus $N$ representing $K$ can be thickened to a standard neighborhood of a maximal $tb$ Legendrian knot.

For a topological knot type $K$, if $N$ is a solid torus having a representative of $K$ as its core and convex boundary, then $N$ fails to thicken if for all $N' \supset N$, we have $\text{slope}(\Gamma_{\partial N'}) = \text{slope}(\Gamma_{\partial N})$.

If we define $t$ to be the twisting of the contact planes along $L$ with respect to the $C'$ framing on $\partial N(L)$, equation 2.1 in [EH1] gives us:
Observe that $t(L)$ is also the twisting of the contact planes with respect to the framing given by $\partial N$, and so is equal to $-\frac{1}{2} \times$ the geometric intersection number of $L$ with $\Gamma_{\partial N}$. We will denote the maximal twisting number with respect to this framing.

We also had two definitions introduced in [L] that will be useful in this note.

**Definition 2.1.** Let $N$ be a solid torus with convex boundary in standard form, and with slope$(\Gamma_{\partial N}) = \frac{a}{b}$ in some framing. If $|2b|$ is the geometric intersection of the dividing set $\Gamma$ with a longitude ruling in that framing, then we will call $\frac{a}{b}$ the *intersection boundary slope*.

Note that when we have an intersection boundary slope $\frac{a}{b}$, then $2\gcd(a, |b|)$ is the number of dividing curves.

**Definition 2.2.** For $r \geq 1$ and positive integer $k$, define $N^k_r$ to be any solid torus representing $K_r$ with intersection boundary slope of $\frac{k+1}{A_k k + B_r}$, as measured in the $C'$ framing. Also define the integer $n^k_r := \gcd((k+1), (A_k k + B_r))$.

Note that $N^k_r$ has $2n^k_r$ dividing curves. Note also that the above definition is only for $k \geq 1$. However, we will also define $N^0_r$ to be a standard neighborhood of a $tb(K_r)$ representative, and thus have this as the $k = 0$ case.

Finally, recall that if $A$ is a convex annulus with Legendrian boundary components, then dividing curves are arcs with endpoints on either one or both of the boundary components. Dividing curves that are boundary parallel are called *bypasses*; an annulus with no bypasses is said to be *standard convex*.

**2.3. Universally tight contact structures.** Recall that a contact structure $\xi$ on a 3-manifold $M$ is said to be *overtwisted* if there exists an embedded disc $D$ which is tangent to $\xi$ everywhere along $\partial D$, and a contact structure is *tight* if it is not overtwisted. Moreover, one can further analyze tight contact 3-manifolds $(M, \xi)$ by looking at what happens to $\xi$ when pulled back to the universal cover $\tilde{M}$ via the covering map $\pi : \tilde{M} \to M$. In particular, if the pullback of $\xi$ remains tight, then $(M, \xi)$ is said to be *universally tight*.

The classification of universally tight contact structures on solid tori is known from the work of Honda. Specifically, from Proposition 5.1 in [HI], we know there are exactly two universally tight contact structures on $S^1 \times D^2$ with boundary torus having two dividing curves and slope $s < -1$ in some framing. These are such that a convex meridional disc has boundary-parallel dividing curves that separate half-discs all of the same sign, and thus the two contact structures differ by $-id$. (If $s = -1$, there is only one tight contact structure, and it is universally tight.)

Also from the work of Honda, we know that if $\xi$ is a contact structure which is everywhere transverse to the fibers of a circle bundle $M$ over a closed oriented surface $\Sigma$, then $\xi$ is universally tight. This is the content of Lemma 3.9 in [H2], and such a transverse contact structure is said to be *horizontal*.

**2.4. Transverse push-offs of Legendrian knots.** Given a Legendrian knot $L$, recall that there are well-defined *positive and negative transverse push-offs*, denoted by $T_+(L)$ and $T_-(L)$, respectively. Moreover, the self-linking numbers of these transverse push-offs are given by the formula

\[(4) \quad tb(L) = Pq + t(L)\]
In this section we present a strategy of proof for Theorem 1.1. We begin with a theorem that in previous works has in effect been proved, but not stated. In this theorem \( K \) is a knot type and \( K(P,q) \) is the \((P,q)\)-cabling of \( K \).

**Theorem 3.1** (Etnyre-Honda, L.). If \( K \) satisfies the UTP, then \( K(P,q) \) also satisfies the UTP.

**Proof.** The case where the cabling fraction \( \frac{P}{q} < w(K) \) is the content of Theorem 1.3 in [EH1]. For the case where \( \frac{P}{q} > w(K) \), the proof follows from examining the proofs of Theorem 3.2 [EH1] and Theorem 1.1 in [L] and observing that Legendrian simplicity of \( K \) is not needed to preserve the UTP. \( \square \)

With this theorem in mind, we will prove Theorem 1.1 by way of two lemmas, one of which uses induction. For this purpose we make the following inductive hypothesis, which from here on we will refer to as the inductive hypothesis. We will need to justify its veracity for the base case of positive torus knots.

**Inductive hypothesis:** Let \( K_r = ((P_1,q_1),\ldots,(P_r,q_r)) \) be an iterated torus knot, as measured in the standard \( C \) framing. Assume that the following hold:

1. \( P_i > 0 \) for all \( i \), where \( 1 \leq i \leq r \). (Thus \( A_i > 0 \) for all \( i \) as well.)
2. \( 0 < \overline{tb}(K_r) = w(K_r) \leq A_r \). (Thus \( -A_r < \overline{t}(K_r) \leq 0 \).)
3. Any solid torus \( N_r \) representing \( K_r \) thickens to some \( N_r^k \) (including \( N_r^0 \) which is a standard neighborhood of a \( \overline{tb} \) representative).
4. If \( N_r \) fails to thicken then it is an \( N_r^k \), and it has at least \( 2n_r^k \) dividing curves.
5. The candidate non-thickenable \( N_r^k \) exist and actually fail to thicken for \( k \geq C_r \), where \( C_r \) is some positive integer that varies according to the knot type \( K_r \). Moreover, these \( N_r^k \) that fail to thicken have contact structures that are universally tight, with convex meridian discs containing bypasses all of the same sign. Also, a Legendrian ruling preferred longitude on these \( \partial N_r^k \) has rotation number zero for \( k > 0 \).

Our first key lemma used in proving Theorem 1.1 is the following, which along with the base case of positive torus knots, will show that if \( K_r = ((P_1,q_1),\ldots,(P_r,q_r)) \) is such that \( P_i > 0 \) for all \( i \), then \( K_r \) fails the UTP.

**Lemma 3.2.** Suppose \( K_r \) satisfies the inductive hypothesis, and \( K_{r+1} \) is a cabling where \( P_{r+1} > 0 \); then \( K_{r+1} \) satisfies the inductive hypothesis.

The main idea in the argument used to prove this lemma will be that since \( K_r \) satisfies the inductive hypothesis, there is an infinite collection of non-thickenable solid tori whose boundary slopes form an increasing sequence converging to \( -\frac{1}{A_r} \) in the \( C' \) framing (which is \( \infty \) in the \( C \) framing). As a consequence, it will be shown that cabling slopes with \( P_{r+1} > 0 \) in the \( C \) framing will have a similar sequence of non-thickenable solid tori.
Our second key lemma is the following, which along with Theorem 3.7 and the fact that negative torus knots satisfy the UTP, will show that if at least one of the $P_i < 0$, then $K_r$ satisfies the UTP.

**Lemma 3.3.** Suppose $K_r$ satisfies the inductive hypothesis, and $K_{r+1}$ is a cabling where $P_{r+1} < 0$; then $K_{r+1}$ satisfies the UTP.

Our outline for the next three sections is as follows. In the next section, §4, we establish the truth of the inductive hypothesis for the base case of positive torus knots. In §5 we prove Lemma 3.2 and in §6 we prove Lemma 3.3.

4. Positive torus knots fail the UTP

In this section we show that positive torus knots $(p_1, q_1)$ satisfy the inductive hypothesis described in §3. From Lemma 4.3 in [1], we know that items 1-4 of the inductive hypothesis are satisfied; it remains to establish item 5, that each solid torus candidate $N^k_i$ actually exists with a universally tight contact structure and the appropriate complement in $S^3$, and that these $N^k_i$ indeed fail to thicken (for all $k \geq 0$ in this case of positive torus knots).

To establish item 5, we employ arguments similar to those used in [EH1] for solid tori representing the $(2,3)$ torus knot, specifically Lemmas 5.2 and 5.3 in [EH1]. From Lemma 4.3 in [1], we know that if $N^k_i$ fails to thicken, its complement $M^k_i := S^3 \setminus N^k_i$ must be contactomorphic to the manifold obtained by taking a neighborhood of a Hopf link $N(L_1) \cup N(L_2)$ and a standard convex annulus $\mathcal{A}$ joining the two neighborhoods of the Hopf link, where $\mathcal{A}$ has boundary components that are Legendrian ruling representatives of $K_1 = (p_1, q_1)$. Moreover, we know that the two components of the Hopf link must have $tb$ values equal to $-(p_1 k + 1)$ and $-(q_1 k + 1)$, respectively, for $k \geq 0$.

We first show that the candidate $N^k_i$ have universally tight contact structures.

**Lemma 4.1.** If $N^k_i$ fails to thicken, then its contact structure is universally tight; moreover, for $k > 0$, a convex meridian disc contains bypasses that all bound half-discs of the same sign. Also, a Legendrian ruling preferred longitude on $\partial N^k_i$ has rotation number zero for $k > 0$.

**Proof.** The lemma is immediately true for $k = 0$, so we may assume that $k > 0$. To fix notation, let $L_1$ be the Legendrian unknot with $tb = -(p_1 k + 1)$ and let $L_2$ be the unknot with $tb = -(q_1 k + 1)$. Then $N(L_1)$ thickens outward to $S^3 \setminus N(L_2)$; we denote $T_1 := \partial N(L_1)$ and $T_2 := \partial (S^3 \setminus N(L_2))$. Since $T_1$ and $T_2$ are convex, we can take $[0,1]$-invariant neighborhoods of each; our convention will be that the two $T_i \times \{0\}$ will bound a thickened torus that contains the two $T_i \times \{1\}$.

Now $T_2 \times \{0\}$ is a convex torus with dividing curves that divide the torus into two annuli, $\mathcal{A}_+$ and $\mathcal{A}_-$. We locate a (topological) meridian curve $\mu$ on $T_2 \times \{0\}$ that intersects each dividing curve efficiently ($q_1 k + 1$) times, and so that $\mu \setminus \partial \mathcal{A}$ consists of $q_1$ arcs which intersect $\mathcal{A}_+$ and $\mathcal{A}_-$ at least $k$ times each. We then can realize $\mu$ as a Legendrian ruling using Theorem 3.7 in [1].

We then examine a horizontal convex annulus $\mathcal{A}_H$ in the space $(S^3 \setminus N(L_2)) \setminus N(L_1)$, bounded by meridian rulings on $T_i \times \{0\}$. This horizontal convex annulus $\mathcal{A}_H$ has two dividing curves that connect its two boundary components; the other $q_1 k$ bypasses have endpoints on $T_2 \times \{0\}$. By Lemma 4.14 in [1], we may assume that all of these bypasses
are boundary compressible, meaning there are no nested bypasses. The two dividing curves connecting the two boundary components of $A_H$ thus divide $A_H$ into two discs, one containing all bypasses of positive sign, the other disc containing all negative bypasses. We will show that in fact all bypasses on $A_H$ must be of the same sign.

To this end, let $A$ be the standard convex annulus with $(p_1, q_1)$ Legendrian rulings as its boundary components on $T_i \times \{1\}$. We first examine $A \cap A_H$, which is $q_1$ arcs with endpoints on the two $T_i \times \{1\}$. At first glance, it is possible that there may be points of intersection between these $q_1$ arcs and the boundary-parallel dividing curves on $A_H$. However, up to a choice of contact vector field for the convex annulus $A_H$, we may assume that all boundary-parallel dividing curves for $A_H$ are in a collar neighborhood of $T_2 \times \{0\}$ and avoid $A$. This contact vector field may also be chosen so that the two non-separating dividing curves on $A_H$ intersect $A$ transversely.

Now $(T_2 \times \{1\}) \setminus \partial A$ is one of the annuli that forms $\partial N_1^k$, and the intersection of this annulus with $A_H$ will be $q_1$ arcs, which we denote as $\gamma_j$ for $1 \leq j \leq q_1$. By the above considerations we thus have that, as a collection, the $\gamma_j$ have support that intersects all of the $q_1 k$ bypasses on $A_H$. See Figure 1.

![Figure 1](image_url)

**Figure 1.** Shown is the horizontal convex annulus $A_H$. The thick gray arcs represent the intersection of $A$ with $A_H$. Some of the $q_1 k$ bypasses are shown; in the figure, $q_1 = 3$.

We next perform edge-rounding for the curves of intersection of $\partial N(A)$ and the two annuli coming from $(T_i \times \{1\}) \setminus \partial A$; after edge-rounding we obtain $\partial N_1^k$. Thus $\partial N_1^k$ intersects $A_H$ in $q_1$ (topological) meridian curves for $N_1^k$; this set of curves, call it $C$, is *nonisolating* in the sense of section 3.3.1 in [H1], meaning that each curve is transverse to the dividing set of $A_H$ and every component of $A_H \setminus (\Gamma \cup C)$ has boundary that intersects $\Gamma$. Moreover, $C$ is also a nonisolating set of curves on $\partial N_1^k$. Then by Theorem 3.7 in [H1], we can realize the $q_1$ topological meridian curves as Legendrian meridian curves for $\partial N_1^k$.

Now these Legendrian meridian curves may not have efficient geometric intersection with $\Gamma_{\partial N_1^k}$. However, by the construction of $\partial N_1^k$, any holonomy of dividing curves on the two annuli coming from the two sides of $N(A)$ cancels each other out. Thus we can destabilize these Legendrian meridian curves on the surface $\partial N_1^k$ so that they do have
geometric intersection \(2(k + 1)\) with \(\Gamma_{\partial N_1^{k}}\), and we can do so away from the \(\gamma_j\). These destabilizations can thus be accomplished by attachment of bypasses off of the \(q_1\) convex meridian discs, but these (attached) bypasses will avoid the original \(q_1 k\) bypasses along the \(\gamma_j\). The resulting \(q_1\) convex meridian discs therefore inherit the bypasses of \(A_H\).

By construction, it is possible that one of the \(q_1\) convex meridian discs may inherit \(k + 1\) bypasses from \(A_H\); if this is the case, however, these bypasses must all be of the same sign, and we have the desired conclusion. So we may assume that each of the \(q_1\) meridian discs intersects \(k\) bypasses of \(A_H\). So suppose, for contradiction, that the \(q_1 k\) bypasses on \(A_H\) have mixed sign, meaning some are negative and some are positive. Since each of the \(q_1\) meridian discs is a convex meridian disc for \(N_1^{k}\), then by the classification of tight contact structures on solid tori we know that if one of the discs has a negative bypass, then all of them must; the same is true for positive bypasses. But since the negative bypasses on \(A_H\) are grouped in succession, and since we may assume \(q_1 \geq 3\), this forces one of the discs to inherit only negative bypasses, contradicting the fact that it is supposed to also have positive bypasses. Thus all of the bypasses on \(A_H\) must be of the same sign, as must be all of the bypasses on a convex meridian disc for \(N_1^{k}\). As a result the contact structure for \(N_1^{k}\) is universally tight.

We can now calculate the rotation number for the \((p_1, q_1)\) ruling on \(N(L_1)\). Since \((S^3 \setminus N(L_2)) \setminus N(L_1)\) is universally tight, one can show that if \(\Sigma_1\) is a convex Seifert surface for the longitude on \(\partial N(L_1)\), we must have \(r(\partial \Sigma_1) = \pm (p_1 k)\). By Lemma 2.2 in [EH], we have that the \((p_1, q_1)\) ruling on \(\partial N(L_1)\) has rotation number equal to

\[
(5) \quad r((p_1, q_1)) = p_1 r(\partial D_1) + q_1 r(\partial \Sigma_1)
\]

This yields \(r((p_1, q_1)) = \pm (p_1 q_1 k)\). We now let \(D\) be a meridian disc for \(N_1^{k}\) and \(\Sigma\) be a Seifert surface for the preferred longitude on \(\partial N_1^{k}\). We know \(r(\partial D) = \pm k\), and we know that the \((p_1, q_1)\) torus knot, which is \(\infty\) on \(\partial N_1^{k}\), is actually a \((p_1 q_1, 1)\) knot on \(\partial N_1^{k}\) in the preferred framing. So using a similar equation from above, we obtain that \(r((p_1 q_1, 1)) = \pm (p_1 q_1 k) = \pm (p_1 q_1 k) + q_1 r(\partial \Sigma)\). Thus \(r(\partial \Sigma) = 0\).

We note that there are two universally tight contact structures, diffeomorphic by \(-id\), which satisfy the conditions set by the above lemma. We now show that these appropriate \(N_1^{k}\) and associated \(M_1^{k}\) actually exist in \(S^3\).

**Lemma 4.2.** The standard tight contact structure on \(S^3\) splits into a universally tight contact structure on \(N_1^{k}\) and \(M_1^{k}\).

**Proof.** The idea is to build \(S^3\). To begin, choose one of the above two universally tight candidates for \(N_1^{k}\). We then claim we can join \(N_1^{k}\) to itself by a standard convex annulus \(A'\) with boundary \(\infty'\) rulings so that \(R := N_1^{k} \cup N(A')\) is (universally tight) thickened torus with boundary \(T_2 - T_1\) having associated boundary slopes of \(-\frac{q_1 k + 1}{1}\) and \(-\frac{1}{p_1 k + 1}\) and two dividing curves. One way to see this is that we can think of \(\partial N_1^{k}\) as being composed of four annuli, one from \(T_2 \setminus A'\), one from \(-T_1 \setminus A'\), and two from \(\partial A' \times [\epsilon, \epsilon]\). Since we are constructing the thickened torus, with a suitable choice of holonomy of \(A'\), we can assure that the dividing curves on \(-T_1\) have only one longitude, and two components. Since we know the twisting of \(\infty'\) on \(N_1^{k}\) is equal to \(- (p_1 q_1 k + p_1 + q_1)\), a calculation shows that the dividing curves on \(-T_1\) must have slope \(-\frac{1}{p_1 k + 1}\). But then the slopes of the dividing
curves on \(-T_1\) and \(\partial N^k_1\) are determined, making the slope of dividing curves on \(T_2\) equal to \(-\frac{q+1}{1}\) based on equation 8 in Lemma 4.3 in [L].

Now as in the proof of Lemma 5.2 in [EHII], the contact structure on \(N^k_1 \cup N(\mathcal{A}')\) can be isotoped to be transverse to the fibers of \(N^k_1 \cup N(\mathcal{A}')\), which are parallel copies of \(K_1\), while preserving the dividing set on \(\partial(N^k_1 \cup N(\mathcal{A}'))\). Such a horizontal contact structure is universally tight.

We then use the classification of tight contact structures on \(S^3\), solid tori, and thickened tori to conclude that any tight contact structure on \(R = T^2 \times [1, 2]\) with boundary conditions being tori with two dividing curves and slopes \(-\frac{q+1}{1}\) and \(-\frac{1}{p+1}\) glues together with standard neighborhoods of unknots with those boundary slopes to give the tight contact structure on \(S^3\).

We now show that these \(N^k_1\) with complements \(M^k_1\) fail to thicken.

**Lemma 4.3.** The \(N^k_1\) with complement \(M^k_1\) fail to thicken.

**Proof.** By inequality 14 in [L], it suffices to show that \(N^k_1\) does not thicken to any \(N^{k'}_1\) for \(k' < k\). So to this end, observe that the \((p_1, q_1)\) positive torus knot is a fibered knot over \(S^1\) with fiber a Seifert surface \(\Sigma\) of genus \(g = \frac{(p_1 - 1)(q_1 - 1)}{2}\) (see [Mi]). Moreover, the monodromy is periodic with period \(p_1 q_1\). Thus, \(M^k_1\) has a \(p_1 q_1\)-fold cover \(\widetilde{M}^k_1 \cong S^1 \times \Sigma\). If one thinks of \(M^k_1\) as \(\Sigma \times [0, 1]\) modulo the relation \((x, 0) \sim (\phi(x), 1)\) for monodromy \(\phi\), then one can view \(\widetilde{M}^k_1\) as \(p_1 q_1\) copies of \(\Sigma \times [0, 1]\) cyclically identified via the same monodromy. Now note that downstairs in \(M^k_1\), \(\infty'\) intersects any given Seifert surface \(p_1 q_1\) times efficiently. It is therefore evident that we can view \(M^k_1\) as a Seifert fibered space with base space \(\Sigma\) and two singular fibers (the components of the Hopf link). The regular fibers are topological copies of \(\infty'\), which itself is a Legendrian ruling on \(\partial N^k_1\) with twisting \(-(A_1 k + B_1)\). In fact, the regular fibers can be assumed to be Legendrian isotopic to the \(\partial N^k_1\)-fibers except for small neighborhoods around the singular fibers.

We claim the pullback of the tight contact structure to \(\widetilde{M}^k_1\) admits an isotopy where the \(S^1\) fibers are all Legendrian and have twisting number \(-(A_1 k + B_1)\) with respect to the product framing. This isotopy can be accomplished because in \(\widetilde{M}^k_1\), the lifts of the singular fibers have tight neighborhoods with convex boundary tori which have dividing curves with one longitude and where \(\infty'\) has twisting \(-(A_1 k + B_1)\). Thus these neighborhoods of the lifts of the singular fibers are in fact standard neighborhoods of a Legendrian fiber with twisting \(-(A_1 k + B_1)\); the contact structure can then be isotoped so that every fiber inside these neighborhoods is Legendrian with twisting \(-(A_1 k + B_1)\).

So, if \(N^k_1\) can be thickened to \(N^{k'}_1\), then there exists a Legendrian curve topologically isotopic to the regular fiber of the Seifert fibered space \(M^k_1\) with twisting number greater than \(-(A_1 k + B_1)\), measured with respect to the Seifert fibration. Pulling back to the \(p_1 q_1\)-fold cover \(\widetilde{M}^k_1\), we have a Legendrian knot which is topologically isotopic to a fiber but has twisting greater than \(-(A_1 k + B_1)\). We will obtain a contradiction, thus proving that \(N^k_1\) cannot be thickened to \(N^{k'}_1\).

To obtain our contradiction, we let \(\pi : S^1 \times \Sigma \rightarrow \Sigma\) be the projection map onto the base space. Thus the hypothesis that \(N^k_1\) can be thickened to \(N^{k'}_1\) yields a knot \(\gamma \subset S^1 \times \Sigma\) which is isotopic to \(\pi^{-1}(p_0)\) for some \(p_0 \in \Sigma\), but where \(t(\gamma) > -(A_1 k + B_1)\). Thus there is a continuous isotopy \(F : S^1 \times I \rightarrow S^1 \times \Sigma\) where \(F(S^1 \times \{0\}) = \pi^{-1}(p_0)\) and \(F(S^1 \times \{1\}) = \gamma\).
Now look at \( \pi \circ F : S^1 \times I \to \Sigma \). Then this is a continuous map, and since \( \pi \circ F(S^1 \times \{0\}) = p_0 \), we actually obtain \( \pi \circ F : D^2 \to \Sigma \). This means that \( \gamma \) is contained inside a tight \( S^1 \times D^2 \) that is fibered by Legendrian fibers with twisting \(-(A_1 k + B_1)\), and is thus a solid torus neighborhood of a Legendrian knot with twisting \(-(A_1 k + B_1)\). By the classification of tight contact structures on solid tori, such a \( \gamma \) cannot exist. This is our contradiction. \( \square \)

5. Positive cablings that fail the UTP

Now that we know that the base case holds for positive torus knots, we begin to prove Lemma 3.2— for the bulk of this section we will thus have that \( P_{r+1} > 0, K_r \) satisfies the inductive hypothesis, and we work to show that \( K_{r+1} \) satisfies the inductive hypothesis. We will need to break the proof of Lemma 3.2 into two cases, Case I being where \( \text{Lemma 5.2} \), and Case II being where \( w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0 \). However, before we do that, we prove two general lemmas concerning iterated cablings that begin with positive torus knots.

**Lemma 5.1.** If \( K_r \) is an iterated torus knot with \( P_1 > 0 \), then \( A_r > B_r \).

**Proof.** We use induction. \( A_1 > B_1 \) is evident from equation 1 above. Then inductively, \( A_r = q_r^2 A_{r-1} + p_r q_r > q_r A_{r-1} + p_r > q_r B_{r-1} + p_r = B_r \).

We now use the above lemma to prove the following.

**Lemma 5.2.** Let \( K_r \) be an iterated torus knot with \( P_1 > 0 \). If \( k_1 < k_2 \) and both \( (A_r k_1 + B_r) \), \( (A_r k_2 + B_r) > 0 \), then \(-\frac{k_1 + 1}{A_r k_1 + B_r} < -\frac{k_2 + 1}{A_r k_2 + B_r} \).

**Proof.** We have that \(-\frac{k_1 + 1}{A_r k_1 + B_r} < -\frac{k_2 + 1}{A_r k_2 + B_r} \) if and only if \((k_1 + 1)(A_r k_2 + B_r) > (k_2 + 1)(A_r k_1 + B_r) \). But this is true if and only if \((A_r - B_r)k_2 > (A_r - B_r)k_1 \), which is true. \( \square \)

We now directly address the two different cases in two different subsections.

### 5.1. Case I: \( \frac{P_{r+1}}{q_{r+1}} > w(K_r) \)

We work through proving items 2-5 in the inductive hypothesis via a series of lemmas. The following lemma begins to address item 2.

**Lemma 5.3.** If \( \frac{P_{r+1}}{q_{r+1}} > w(K_r) \), then \( \overline{tb}(K_{r+1}) = A_{r+1} - (P_{r+1} - q_{r+1} w(K_r)) > 0 \).

**Proof.** The proof is similar to that of Lemma 3.3 in [EH]. We first claim that \( \overline{7}(K_{r+1}) < 0 \). If not, there exists a Legendrian \( L_{r+1} \) with \( t(L_{r+1}) = 0 \) and a solid torus \( N_r \) with \( L_{r+1} \) as a Legendrian divide. But then we would have a boundary slope of \( \frac{P_{r+1}}{q_{r+1}} > w(K_r) \) in the \( C \) framing, which cannot occur.

So since \( \overline{7}(K_{r+1}) < 0 \), any Legendrian \( L_{r+1} \) must be a ruling on a convex \( \partial N_r \) with slope \( s \geq \frac{1}{\overline{7}(K_r)} \) in the \( C' \) framing. But then if \( s = -\frac{\lambda}{\mu} > \frac{1}{\overline{7}(K_r)} \), we have that \( t(L_r) = -(p_{r+1} \lambda + q_{r+1} \mu) < -\lambda (p_{r+1} - \overline{7}(K_r) q_{r+1}) \leq -(p_{r+1} - \overline{7}(K_r) q_{r+1}) \). This shows that \( \overline{tb}(K_{r+1}) \) is achieved by a Legendrian ruling on a convex torus having slope \( \frac{1}{w(K_r)} \) in the standard \( C \) framing.

Finally, note that \( A_{r+1} - (P_{r+1} - q_{r+1} w(K_r)) = A_{r+1} - (q_{r+1} (A_r - w(K_r)) + p_{r+1}) > A_{r+1} - (q_{r+1} A_r + p_{r+1} q_{r+1}) = 0. \) \( \square \)
With the following lemma we prove that items 3 and 4 of the inductive hypothesis hold for \( K_{r+1}\).

**Lemma 5.4.** If \( \frac{p_{r+1}}{q_{r+1}} > \text{w}(K_r) \), let \( N_{r+1} \) be a solid torus representing \( K_{r+1} \), for \( r \geq 1 \). Then \( N_{r+1} \) can be thickened to an \( N_{r+1}^{k'} \) for some nonnegative integer \( k' \). Moreover, if \( N_{r+1} \) fails to thicken, then it has the same boundary slope as some \( N_{r+1}^{k'} \), as well as at least \( 2n_{r+1} \) dividing curves.

**Proof.** In this case, for the \( C' \) framing, we have either \( p_{r+1} > 0 \) or \( \frac{q_{r+1}}{p_{r+1}} < \frac{1}{t(K_r)} \) (the latter being relevant only if \( t(K_r) < 0 \)). The proof in this case is nearly identical to the proof of Lemma 4.4 in [L]; we will include some of the details, however, as certain particular calculations differ. Moreover, we will use modifications of this argument in Case II and thus will be able to refer to the details here.

Let \( N_{r+1} \) be a solid torus representing \( K_{r+1} \). Let \( L_r \) be a Legendrian representative of \( K_r \) in \( S^3\setminus N_{r+1} \) and such that we can join \( \partial N(L_r) \) to \( \partial N_{r+1} \) by a convex annulus \( \tilde{A}_{(p_{r+1}, q_{r+1})} \) whose boundaries are \( (p_{r+1}, q_{r+1}) \) and \( \infty' \) rulings on \( \partial N(L_r) \) and \( \partial N_{r+1} \), respectively. Then topologically isotopy \( L_r \) in the complement of \( N_{r+1} \) so that it maximizes \( \text{tb} \) over all such isotopies; this will induce an ambient topological isotopy of \( \tilde{A}_{(p_{r+1}, q_{r+1})} \), where we still can assume \( \tilde{A}_{(p_{r+1}, q_{r+1})} \) is convex. A picture is shown in (a) in Figure 2. In the \( C' \) framing we will have \( \text{slope}(\Gamma_{\partial N(L_r)}) = -\frac{1}{m} \) where \( m \geq 0 \), since \( t(K_r) \leq 0 \). Now if \( m = t(K_r) \), then there will be no bypasses on the \( \partial N(L_r) \)-edge of \( A_{(p_{r+1}, q_{r+1})} \); since the \( (p_{r+1}, q_{r+1}) \) ruling would be at maximal twisting. On the other hand, if \( m < t(K_r) \), then there will still be no bypasses on the \( \partial N(L_r) \)-edge of \( A_{(p_{r+1}, q_{r+1})} \), since such a bypass would induce a destabilization of \( L_r \), thus increasing its \( \text{tb} \) by one – see Lemma 4.4 in [H1]. To satisfy the conditions of this lemma, we are using the fact that either \( p_{r+1} > 0 \) or \( \frac{q_{r+1}}{p_{r+1}} < \frac{1}{t(K_r)} \). Furthermore, we can thicken \( N_{r+1} \) through any bypasses on the \( \partial N_{r+1} \)-edge, and thus assume \( A_{(p_{r+1}, q_{r+1})} \) is standard convex.

Now let \( N_r := N_{r+1} \cup N(A_{(p_{r+1}, q_{r+1})}) \cup N(L_r) \). Inductively we can thicken \( N_r \) to an \( N_r^k \) with intersection boundary slope \( -\frac{k+1}{A_r k + B_r} \) where \( k \) is minimized over all such thickenings (if we have \( k = 0 \), then we will have \( N_{r+1} \) thickening to a standard neighborhood of a knot at \( \text{tb} \) – see the proof of Theorem 1.1 in [L]; so we can assume \( k > 0 \)). Then consider a convex annulus \( \tilde{A} \) from \( \partial N(L_r) \) to \( \partial N_r \), such that \( \tilde{A} \) is in the complement of \( N_r \) and \( \partial \tilde{A} \) consists of \( (p_{r+1}, q_{r+1}) \) rulings. A picture is shown in (b) in Figure 2. By an argument identical to that used in Lemma 4.4 in [L], \( \tilde{A} \) is standard convex; in brief, if \( \tilde{A} \) was not standard convex, either a bypass would occur on its \( \partial N(L_r) \)-edge, or \( k \) would not be minimized, neither of which is true.

Now four annuli compose the boundary of a solid torus \( \tilde{N}_{r+1} \) containing \( N_{r+1} \): the two sides of a thickened \( \tilde{A} \); \( \partial N_r \setminus \partial \tilde{A} \); and \( \partial N(L_r) \setminus \partial \tilde{A} \). We can compute the intersection boundary slope of this solid torus. To this end, recall that \( \text{slope}(\Gamma_{\partial N(L_r)}) = -\frac{1}{m} \) where \( m > 0 \) (\( m = 0 \) would be the \( \tilde{T} \) case which we have take care of above). To determine \( m \) we note that the geometric intersection of \( (p_{r+1}, q_{r+1}) \) with \( \Gamma \) on \( \partial N_r^k \) and \( \partial N(L_r) \) must be equal, yielding the equality

\[
p_{r+1} + m q_{r+1} = p_{r+1} k + p_{r+1} + q_{r+1}(A_r k + B_r)
\]
These equal quantities are greater than zero, since \( \frac{q_{r+1}}{p_{r+1}} < \frac{1}{m} \) — we note here that this will yield \( (A_{r+1}k' + B_{r+1}) > 0 \) for the calculations below. In the meantime, however, the above equation gives

\[
m = p_{r+1} \frac{k}{q_{r+1}} + A_r k + B_r
\]

We define the integer \( k' := \frac{k}{q_{r+1}} \). We now choose \((p'_{r+1}, q'_{r+1})\) to be a curve on these two tori such that \( p_{r+1}q'_{r+1} - p'_{r+1}q_{r+1} = 1 \), and we change coordinates to a framing \( C'' \) via the map \( ((p_{r+1}, q_{r+1}), (p'_{r+1}, q'_{r+1})) \mapsto ((0,1), (-1,0)) \). Under this map we obtain

\[
slope(\Gamma_{\partial N_{r+1}}) = \frac{q'_{r+1}(A_r k + B_r) + p'_{r+1}(q_{r+1} k' + 1)}{A_{r+1} k' + B_{r+1}}
\]

\[
slope(\Gamma_{\partial N(L_r)}) = \frac{q'_{r+1}(p_{r+1}k' + A_r k + B_r) + p'_{r+1}}{A_{r+1} k' + B_{r+1}}
\]

We then obtain in the \( C' \) framing, after edge-rounding, that the intersection boundary slope of \( \tilde{N}_{r+1} \) is

\[
slope(\Gamma_{\partial \tilde{N}_{r+1}}) = \frac{q'_{r+1}(A_r k + B_r) + p'_{r+1}(q_{r+1} k' + 1)}{A_{r+1} k' + B_{r+1}} - \frac{q'_{r+1}(p_{r+1}k' + A_r k + B_r) + p'_{r+1}}{A_{r+1} k' + B_{r+1}} - \frac{1}{A_{r+1} k' + B_{r+1}} \frac{k' + 1}{A_{r+1} k' + B_{r+1}}
\]
This shows that any $N_{r+1}$ representing $K_{r+1}$ can be thickened to one of the $N_{r+1}^{k'}$, and if $N_{r+1}$ fails to thicken, then it has the same boundary slope as some $N_{r+1}^{k'}$. We now note that if $N_{r+1}$ fails to thicken, and if it has the minimum number of dividing curves over all such $N_{r+1}$ that fail to thicken and have the same boundary slope as $N_{r+1}^{k'}$, then $N_{r+1}$ is actually an $N_{r+1}^{k'}$, by an argument identical to that used in Lemma 5.4 in [L]. In brief, if $N_{r+1}$ fails to thicken and is at minimum number of dividing curves, then taking $N_{r+1} \cup N(L_r) \cup (A_{(p_{r+1}, q_{r+1})})$ gives an $N_{r+1}^{k'}$; one then concludes that $N_{r+1}$ is an $N_{r+1}^{k'}$. \□

We now finish the proof of item 2 of the inductive hypothesis.

**Lemma 5.5.** If $\frac{p_{r+1}}{q_{r+1}} > w(K_r)$, then $w(K_{r+1}) = \overline{tb}(K_{r+1})$.

**Proof.** We show that $\frac{1}{\ell(K_{r+1})} < -\frac{k'+1}{q_{r+1}k'+B_{r+1}}$ for any candidate $N_{r+1}^{k'}$. As a consequence, since any $N_{r+1}$ thickens to some $N_{r+1}^{k'}$ (including $k' = 0$), we have, to prevent overtwisting, that $w(K_{r+1}) = \overline{tb}(K_{r+1})$. Now note that our intended inequality is automatically true if $\ell = 0$; thus we may assume that $\overline{t}(K_{r+1}) < 0$.

We have that $\frac{1}{\ell(K_{r+1})} < -\frac{k'+1}{q_{r+1}k'+B_{r+1}}$ holds if and only if

\begin{equation}
A_{r+1}k' + B_{r+1} > (k' + 1)(p_{r+1} - q_{r+1}\overline{t}(K_r))
\end{equation}

Inductively we know that $\frac{1}{\ell(K_r)} < -\frac{k+1}{A_rk+B_r}$ where $k = k'q_{r+1}$. This implies that

\begin{equation}
A_rk + B_r > -(k'q_{r+1} + 1)\overline{t}(K_r)
\end{equation}

We can now prove inequality 11; we begin with $A_{r+1}k' + B_{r+1}$. We have:

\begin{align*}
A_{r+1}k' + B_{r+1} &= (q_{r+1}^2 A_r + q_{r+1}p_{r+1})k' + p_{r+1} + q_{r+1}B_r \\
&= q_{r+1}(A_rk + B_r) + p_{r+1}q_{r+1}k' + p_{r+1} \\
&> -q_{r+1}(k'q_{r+1} + 1)\overline{t}(K_r) + p_{r+1}q_{r+1}k' + p_{r+1} \\
&= (k' + 1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) + k'(q_{r+1} - 1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) \\
&> (k' + 1)(p_{r+1} - q_{r+1}\overline{t}(K_r))
\end{align*}

\(\ □\)

We conclude this subsection by proving item 5 of the inductive hypothesis.

**Lemma 5.6.** If $\frac{p_{r+1}}{q_{r+1}} > w(K_r)$, the candidate $N_{r+1}^{k'}$ exist and actually fail to thicken for $k' \geq C_{r+1}$, where $C_{r+1}$ is some positive integer. Moreover, these $N_{r+1}^{k'}$ have contact structures that are universally tight and have convex meridian discs whose bypasses bound half-discs all of the same sign. Also, the preferred longitude on $\partial N_{r+1}^{k'}$ has rotation number zero for $k' > 0$.

**Proof.** We first prove that the contact structure on a candidate $N_{r+1}^{k'}$ which fails to thicken is universally tight. To see this note that from Lemma 5.2 above, and the inductive hypothesis, such a candidate $N_{r+1}^{k'}$ is embedded inside a $N_{r}^{k}$ with a universally tight contact structure. Now there is a $q_{r+1}$-fold cover of $N_{r}^{k}$ that maps $q_{r+1}$ lifts $\tilde{N}_{r+1}^{k'}$ to $N_{r+1}^{k'}$, the lifts
we construct a universally tight $R_k$ and fails to thicken for $k/q$. This shows that $k$ admits an isotopy so that all the fibers with twisting $-c$ on $\partial N$ in particular, the neighborhood of a knot with twisting $-c$ on $\partial N$ must do so inside of $M$, that is a Seifert fibered space with one singular fiber, $L_r$, and with regular fibers that are topologically isotopic to the Legendrian copies of $K_{r+1}$ on the boundary of $M$. $M$ has a $q_{r+1}$-fold cover, $\tilde{M}$, that is a $q_{r+1}$-punctured disc times $S^1$, where the tight contact structure admits an isotopy so that all the $S^1$ fibers are Legendrian with twisting $-(A_{r+1}k' + B_{r+1})$ with respect to the product framing. We can then glue in $q_{r+1}$ standard neighborhoods of fibers with twisting $-(A_{r+1}k' + B_{r+1})$ to obtain an $S^1 \times D^2$ which itself is a standard neighborhood of a knot with twisting $-(A_{r+1}k' + B_{r+1})$. But then, if $N'_{r+1}$ thickens to a $N''_{r+1}$, where $k'' < k'$, that means that in this cover there will be a knot isotopic to one of the fibers, but with twisting greater than $-(A_{r+1}k' + B_{r+1})$, contradicting the classification of tight contact structures on solid tori.

To show that the preferred longitude on $\partial N_{r+1}'$ has rotation number zero, we use an argument similar to that used in Lemma 4.1. We call the meridian disc for $N_{r+1}'$, $\Sigma_r$. If we then look at the $(P_{r+1}, q_{r+1})$ cable on $\partial N_r$, we can calculate its rotation number as

$$r((P_{r+1}, q_{r+1})) = P_{r+1}r(\partial D_r) + q_{r+1}r(\partial \Sigma_r) = P_{r+1}(\pm q_{r+1}k')$$

But then since this same knot is a $(P_{r+1}q_{r+1}, 1)$ cable on $\partial N_{r+1}'$, we have that $r((P_{r+1}, q_{r+1})) = P_{r+1}q_{r+1}(\pm k') + q_{r+1}r(\partial \Sigma)$, where $\Sigma$ is a Seifert surface for the preferred longitude on $\partial N_{r+1}'$. This implies that $r(\partial \Sigma) = 0$.

Now we know inductively that there exists a $C_r$ such that if $k \geq C_r$, then the $N_r$ exist and fail to thicken. So suppose $k/q_{r+1} \in \mathbb{N}$ for some $r \geq C_r$. We will show that $N_{r+1}'$ exists and fails to thicken for $k'' := k/q_{r+1}$. Then $C_{r+1}$ will be the least such $k/q_{r+1} \in \mathbb{N}$.

We take one of the two universally tight candidate $N'_{r+1}$, and as in Lemma 4.2 above we construct a universally tight $R$ and glue in an appropriate solid torus neighborhood of a Legendrian knot $L_r$ to obtain a universally tight $N_r$, which then glues into $S^3$ inductively. This shows that $N_{r+1}'$ exists.

To show that $N_{r+1}'$ fails to thicken, by Lemma 5.2 it suffices to show that $N_{r+1}'$ does not thicken to any $N_{r+1}'$, where $k'' < k'$. Inductively, we can assume $N_r$ fails to thicken; in particular, the $N_{r+1}'$ that contains $N_{r+1}'$ fails to thicken. Thus, if $N_{r+1}'$ admits a non-trivial thickening, it must do so inside of $N_{r+1}'$. Define $M := N_{r+1}' \setminus N_{r+1}'$, then $M$ is a Seifert fibered space with one singular fiber, $L_r$, and with regular fibers that are topologically isotopic to the Legendrian copies of $K_{r+1}$ on the boundary of $M$. $M$ has a $q_{r+1}$-fold cover, $\tilde{M}$, that is a $q_{r+1}$-punctured disc times $S^1$, where the tight contact structure
5.2. Case II: \( w(K_r) > \frac{p_{r+1}}{q_{r+1}} > 0 \). As in Case I, we work through proving items 2-5 in the
inductive hypothesis via a series of lemmas.

We begin by proving item 2 in the inductive hypothesis.

Lemma 5.7. If \( w(K_r) > \frac{p_{r+1}}{q_{r+1}} > 0 \), then \( \overline{b}(K_{r+1}) = w(K_{r+1}) = A_{r+1} \).

Proof. The proof is almost identical to that of step 1 in Theorem 1.5 in [L]; we will include the
details, though, since certain key aspects differ. We first examine representatives of
\( K_{r+1} \) at \( \overline{b} \). Since there exists a convex torus representing \( K_r \) with Legendrian divides that
are \((p_{r+1}, q_{r+1})\) cablings (inside of the solid torus representing \( K_r \) with \( \text{slope}(\Gamma) = \frac{1}{t(K_r)} \)),
we know that \( \overline{b}(K_{r+1}) \geq P_{r+1}q_{r+1} = A_{r+1} \). To show that \( \overline{b}(K_{r+1}) = A_{r+1} \), we show
that \( \overline{T}(K_{r+1}) \) is a convex torus with \( \text{slope}(\Gamma_{\partial N_{r+1}}) = s > 0 \), as measured in the \( C' \) framing, and two dividing
curves. After shrinking \( N_{r+1} \) if necessary, we may assume that \( s \) is a large positive integer.
Then let \( \mathcal{A} \) be a convex annulus from \( \partial N_{r+1} \) to itself having boundary curves with slope \( \infty' \). Taking a neighborhood of \( N_{r+1} \cup \mathcal{A} \) yields a thickened torus \( R \) with boundary tori \( T_1 \) and \( T_2 \), arranged so that \( T_1 \) is inside the solid torus \( N_r \) representing \( K_r \) bounded by \( T_2 \).

Now there are no boundary parallel dividing curves on \( \mathcal{A} \), for otherwise, we could pass
through the bump and increase \( s \) to \( \infty' \), yielding excessive twisting inside \( N_{r+1} \). Hence
\( \mathcal{A} \) is in standard form, and consists of two parallel nonseparating arcs. We now choose a
new framing \( C'' \) for \( N_r \) where \((p_{r+1}, q_{r+1}) \mapsto (0, 1) \); then choose \((p'', q'') \mapsto (1, 0) \) so that
\( p''q_{r+1} - q''p_{r+1} = 1 \) and such that \( \text{slope}(\Gamma_{T_1}) = -s \) and \( \text{slope}(\Gamma_{T_2}) = 1 \). As mentioned in
[EH1], this is possible since \( \Gamma_{T_1} \) is obtained from \( \Gamma_{T_2} \) by \( s + 1 \) right-handed Dehn twists.
Then note that in the \( C' \) framing, we have that \( \frac{q_{r+1}}{p_{r+1}} > \text{slope}(\Gamma_{T_2}) = \frac{q''}{p''} > \frac{q''}{p''} \); thus we can thicken \( N_r \) to one of the solid tori with
\( \text{slope}(\Gamma) = \frac{-k+1}{p_k+k+1} \), which fails to thicken. Then, just as in Claim 4.2 in [EH1], we have
(i) inside \( R \) there exists a convex torus parallel to \( T_1 \) with \( \frac{q_{r+1}}{p_{r+1}} \); (ii) \( R \) can thus be
decomposed into two layered basic slices; (iii) the tight contact structure on \( R \) must have
mixing of sign in the Poincaré duals of the relative half-Euler classes for the layered basic
slices; and (iv) this mixing of sign cannot happen inside the universally tight solid torus
which fails to thicken. This last statement is due to the proof of Proposition 5.1 in [HI],
where it is shown that mixing of sign will imply an overtwisted disc in the universal cover
of the solid torus. Thus we have contradicted \( s > 0 \). So \( \overline{b}(K_{r+1}) = P_{r+1}q_{r+1} = A_{r+1} \). \( \Box \)

With the following lemma we prove that items 3 and 4 of the inductive hypothesis
hold for \( K_{r+1} \).

Lemma 5.8. If \( w(K_r) > \frac{p_{r+1}}{q_{r+1}} > 0 \), let \( N_{r+1} \) be a solid torus representing \( K_{r+1} \), for \( r \geq 1 \).
Then \( N_{r+1} \) can be thickened to an \( N_{r+1}^{k'} \) for some nonnegative integer \( k' \). Moreover, if \( N_{r+1} \)
fails to thicken, then it has the same boundary slope as some \( N_{r+1}^{k'} \), as well as at least \( 2n_{r+1}^{k'} \)
dividing curves.

Proof. This is the case where \( p_{r+1} < 0 \) but \( \frac{q_{r+1}}{p_{r+1}} \in (\frac{1}{t(K_r)}, -\frac{1}{t(K_r)}) \); we have that \( \overline{T}(K_{r+1}) = 0 \).
We begin as we did in Case I. If \( N_{r+1} \) is a solid torus representing \( K_{r+1} \), as before choose
STUDYING UNIFORM THICKNESS II: TRANSVERSALLY NON-SIMPLE ITERATED TORUS KNOTS

Let $L_r$ in $S^3 \setminus N_{r+1}$ such that $\partial N(L_r)$ is joined to $\partial N_{r+1}$ by an annulus $A_{(p_{r+1}, q_{r+1})}$, and with $tb(L_r)$ maximized over topological isotopies in the space $S^3 \setminus N_{r+1}$.

Now suppose slope($\Gamma_{\partial N(L_r)}$) = $-\frac{1}{m}$ where $-\frac{1}{m} < \frac{q_{r+1}}{p_{r+1}}$. Then inside $N(L_r)$ is an $N_r$ with boundary slope $\frac{q_{r+1}}{p_{r+1}}$. But then we can extend $A_{(p_{r+1}, q_{r+1})}$ to an annulus that has no twisting on one edge, and we can thus thicken $N_{r+1}$ so it has boundary slope $\infty'$. Moreover, since there is twisting inside $N(L_r)$, we can assure there are two dividing curves on the thickened $N_{r+1}$. So this situation yields no nontrivial solid tori $N_{r+1}$ which fail to thicken.

Alternatively, suppose $-\frac{1}{m} > \frac{q_{r+1}}{p_{r+1}}$. Furthermore, for the moment suppose $-\frac{1}{m+1} > \frac{q_{r+1}}{p_{r+1}}$. Then we can use Lemma 4.4 in [III] to conclude that there are no bypasses on the $\partial N(L_r)$-edge of $A_{(p_{r+1}, q_{r+1})}$, and so we can thicken $N_{r+1}$ through bypasses so that $A_{(p_{r+1}, q_{r+1})}$ is standard convex. Then the calculation of the boundary slope goes through as above in Lemma 5.4, and we conclude that $N_{r+1}$ thicken to some $N_{r+1}'$. The $N_{r+1}'$ that is used for this will have $\frac{q_{r+1}}{p_{r+1}} < -\frac{k+1}{A_r k + B_r}$; note that such $N_{r+1}'$ exist since $-\frac{k+1}{A_r k + B_r} \rightarrow -\frac{1}{m}$ as $k$ increases.

For the remaining case, suppose $-\frac{1}{m} > \frac{q_{r+1}}{p_{r+1}}$ and $m$ is the least positive integer satisfying this inequality. Thus $-\frac{1}{m+1} < \frac{q_{r+1}}{p_{r+1}}$. Again look at the $\partial N(L_r)$-edge of $A_{(p_{r+1}, q_{r+1})}$. We claim this edge has no bypasses. So, for contradiction, suppose it does. Then we can thicken $N(L_r)$ to a solid torus where the (efficient) geometric intersection of $(p_{r+1}, q_{r+1})$ with dividing curves is less than $p_{r+1} + m q_{r+1}$. Suppose the slope of this new solid torus is $\frac{-\lambda}{m} < -\frac{1}{m}$, where $\lambda > 1$ since $m$ is minimized in the complement of $N_{r+1}$.

We do some calculations. Note first that if $\frac{m}{\mu} > 1$, then $m > \mu$, which means $m-1 \geq \mu$, which implies $-\frac{1}{m-1} \geq -\frac{1}{\mu} > -\frac{1}{m}$, which cannot happen, again since $m$ is minimized in the complement of $N_{r+1}$. Thus we must have $\frac{m}{\mu} \leq 1$. But then the geometric intersection of $(p_{r+1}, q_{r+1})$ with $(-\mu, \lambda)$ is $\lambda p_{r+1} + \mu q_{r+1} > \frac{m}{\mu} p_{r+1} + \mu q_{r+1} \geq \frac{m}{\mu} [\mu p_{r+1} + \mu q_{r+1}] = p_{r+1} + m q_{r+1}$. This is a contradiction.

Thus there are no bypasses on the $\partial N(L_r)$-edge of $A_{(p_{r+1}, q_{r+1})}$, and we can thicken $N_{r+1}'$ through any bypasses so that $A_{(p_{r+1}, q_{r+1})}$ is standard convex. The calculations that show $N_{r+1}$ thicken to $N_{r+1}'$ go through as above in Lemma 5.4.

This shows that any $N_{r+1}$ representing $K_{r+1}$ can be thicken to one of the $N_{r+1}'$ and if $N_{r+1}$ fails to thicken, then it has the same boundary slope as some $N_{r+1}'$. We now show that if $N_{r+1}$ fails to thicken, and if it has the minimum number of dividing curves over all such $N_{r+1}$ which fail to thicken and have the same boundary slope as $N_{r+1}'$, then $N_{r+1}$ is actually an $N_{r+1}'$.

To see this, as above we can choose a Legendrian $L_r$ that maximizes $tb$ in the complement of $N_{r+1}$ and such that we can join $\partial N(L_r)$ to $\partial N_{r+1}$ by a convex annulus $A_{(p_{r+1}, q_{r+1})}$ whose boundaries are $(p_{r+1}, q_{r+1})$ and $\infty'$ rulings on $\partial N(L_r)$ and $\partial N_{r+1}$, respectively. Now since $N_{r+1}$ fails to thicken, we can assume that $\frac{q_{r+1}}{p_{r+1}} < -\frac{1}{m}$ and that there are no bypasses on the $\partial N(L_r)$-edge, and in this case we have no bypasses on the $\partial N_{r+1}$-edge since $N_{r+1}$ fails to thicken and is at minimum number of dividing curves.

As above, let $N_r := N_{r+1} \cup N(A_{(p_{r+1}, q_{r+1})}) \cup N(L_r)$. We claim $N_r$ fails to thicken – the proof proceeds identically as above in Lemma 5.4, as does the proof that $N_{r+1}$ is in fact an $N_{r+1}'$. \qed
The following proof of item 5 of the inductive hypothesis is similar to that of Case I.

**Lemma 5.9.** If \( w(K_r) > \frac{p_{r+1}}{q_{r+1}} > 0 \), the candidate \( N_{r+1}^{k'} \) exist and actually fail to thicken for \( k' \geq C_{r+1} \), where \( C_{r+1} \) is some positive integer. Moreover, these \( N_{r+1}^{k'} \) have contact structures that are universally tight and have convex meridian discs whose bypasses bound half-discs all of the same sign. Also, the preferred longitude on \( \partial N_{r+1}^{k'} \) has rotation number zero for \( k' > 0 \).

**Proof.** The proof that the contact structure on a candidate \( N_{r+1}^{k'} \) which fails to thicken is universally tight is identical to the argument in Case I, as is the proof that their convex meridian discs have bypasses all of the same sign, as well as the proof that the rotation number of the preferred longitude is zero.

Now we know inductively that there exists a \( C_r \) such that if \( k \geq C_r \), then the \( N_r^k \) exist and fail to thicken. So suppose \( k/q_{r+1} \in \mathbb{N} \) for some \( k \geq C_r \). Also assume that \( -\frac{q_{r+1}}{p_{r+1}} < -\frac{k+1}{A_{r,k+B_r}} \); we know such a \( k \) exists since \( -\frac{k+1}{A_{r,k+B_r}} \to -\frac{1}{A_r} \) as \( k \) increases. Then \( N_{r+1}^{k'} \) exists and fails to thicken as in the argument for Case I for \( k' := k/q_{r+1} \), and \( C_{r+1} \) will be the least such \( k/q_{r+1} \in \mathbb{N} \).

**6. Negative cablings that satisfy the UTP**

We provide below the proof of Lemma 3.3, which is really just a matter of referencing a previous proof.

**Proof.** This is the case where \( -\frac{1}{A_r} < -\frac{q_{r+1}}{p_{r+1}} < 0 \), we know \( K_r \) satisfies the inductive hypothesis, and we wish to show that \( K_{r+1} \) satisfies the UTP. The proof is identical to that of steps 1 and 2 in the proof of Theorem 1.5 from \( [L] \), the key being that since \( -\frac{1}{A_r} < -\frac{q_{r+1}}{p_{r+1}} < 0 \), this cabling slope is shielded from any \( N_r^k \) that fail to thicken.

**7. Transversally non-simple iterated torus knots**

We have completed the UTP classification of iterated torus knots; it now remains to show that in the class of iterated torus knots, failing the UTP is a sufficient condition for supporting transversally non-simple cablings. To this end, in this section we prove Theorem [L2] we do so by working through a series of lemmas. These lemmas will first give us information about just a piece of the Legendrian mountain range for \( K_r = ((P_1, q_1), ..., (P_r, q_r)) \) where \( P_i > 0 \) for all \( i \); we will then use this information to obtain enough information about the Legendrian mountain ranges of certain cables \( K_{r+1} \) to conclude that these cables are transversally non-simple. We will therefore not be completing the Legendrian or transversal classification of these iterated torus knots.

**Lemma 7.1.** Suppose \( K_r = ((P_1, q_1), ..., (P_r, q_r)) \) is an iterated torus knot where \( P_i > 0 \) for all \( i \). Then \( -\chi(K_r) = A_r - B_r \).

**Proof.** A formula for \( \chi(K_r) \) is given at the end of the proof of Corollary 3 in \([BW]\). In the notation used in that paper, the formula is \( \chi(K_r) = \prod_{i=1}^{r} p_i - \sum_{i=1}^{r} q_i(p_i - 1) \prod_{j=i+1}^{r} p_j \); since in our case all the \( c_i = 1 \) as we are cabling positively at each iteration. However, note that our \( (P_i, q_i) \) corresponds to \( (q_i, p_i) \) in \([BW]\) for \( i > 1 \).

Examination of this formula for \( \chi(K_r) \) yields the following recursive expression using our \( P \)'s and \( q \)'s:
\[ \chi(K_r) = q_r \chi(K_{r-1}) - P_r q_r + P_r \]

Now for a positive torus knot \((P_i, q_i)\), we have \(\chi = -A_1 + B_1\), so we can inductively assume the lemma holds for \(K_{r-1}\). Thus using the recursive expression we have

\[
\begin{align*}
\chi(K_r) &= q_r \chi(K_{r-1}) - P_r q_r + P_r \\
&= q_r (A_{r-1} + B_{r-1}) - A_r + q_r A_{r-1} + p_{r-1} \\
&= -A_r + B_r
\end{align*}
\]

In the next lemma we establish the structure of just a piece of the Legendrian mountain range for \(K_r\):

**Lemma 7.2.** Suppose \(K_r = ((P_1, q_1), ..., (P_r, q_r))\) is an iterated torus knot where \(P_i > 0\) for all \(i\). Then there exists Legendrian representatives \(L_r^\pm\) with \(tb(L_r^\pm) = 0\) and \(r(L_r^\pm) = \pm(A_r - B_r)\); also, \(L_r^\pm\) destabilizes.

**Proof.** The lemma is true for positive torus knots [EH2], so we inductively assume it is true for \(K_{r-1}\). Then look at Legendrian rulings \(\tilde{L}_r^\pm\) on standard neighborhoods of the inductive \(L_{r-1}^\pm\). In the \(C'\) framing the boundary slope of these \(N(L_{r-1}^\pm)\) is \(-\frac{1}{A_{r-1}}\), and so a calculation shows that \(t(\tilde{L}_r^\pm) = -P_r\); hence \(tb(\tilde{L}_r^\pm) = A_r - P_r\).

To calculate the rotation number of \(\tilde{L}_r^\pm\), we use the following formula from [EH1], where \(D\) is a convex meridian disc for \(N(L_{r-1}^\pm)\) and \(\Sigma\) is a Seifert surface for the preferred longitude on \(\partial N(L_{r-1}^\pm)\):

\[
\begin{align*}
\ r(\tilde{L}_r^\pm) &= P_r r(\partial D) + q_r r(\partial \Sigma) \\
&= \pm q_r (A_{r-1} - B_{r-1}) \\
&= \pm (q_r A_{r-1} + p_r - q_r B_{r-1} - p_r) \\
&= \pm (P_r - B_r)
\end{align*}
\]

This gives us

\[
\begin{align*}
\ s.l.(\tilde{L}_r^-) &= (A_r - P_r) + (P_r - B_r) = A_r - B_r \\
\text{and} \\
\ s.l.(\tilde{L}_r^+) &= (A_r - P_r) - (P_r - B_r) = A_r - B_r
\end{align*}
\]

This, along with Lemma 7.1, shows us that \(\tilde{L}_r^+\) is on the right-most slope of the Legendrian mountain range of \(K_r\), and \(\tilde{L}_r^-\) is on the left-most edge. To the former we can perform positive stabilizations to reach \(L_r^+\) at \(tb = 0\) and \(r = A_r - B_r\); to the latter we can perform negative stabilizations to reach \(L_r^-\) at \(tb = 0\) and \(r = -(A_r - B_r)\) – we know such stabilizations can be performed since \(A_r - P_r > 0\). \(\square\)

So suppose \(K_r\) is an iterated torus knot that fails the UTP (which is precisely when \(P_i > 0\) for all \(i\)). Then we know that for \(k \geq C_r\) there exist non-thickenable solid tori \(N_k^r\)
having intersection boundary slopes of $-\frac{k+1}{A_r k + B_r}$, where these slopes are measured in the $C'$ framing. Switching to the standard $C$ framing, these intersection boundary slopes are $\frac{k+1}{A_r - B_r}$. Now as $k \to \infty$, there are infinitely many values of $k + 1$ which are prime and greater than $A_r - B_r$. As a consequence, there are infinitely many $N^k_r$ with two dividing curves. Based on this observation, we make the following definition:

**Definition 7.3.** Suppose $K_r = ((P_1, q_1), ..., (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all $i$. Let $\hat{K}_{r+1}$ be a cabling of $K_r$ with $C'$ slope $-\frac{k+1}{A_r k + B_r}$, where $-\frac{1}{A_r - 1} < -\frac{k+1}{A_r k + B_r} < -\frac{1}{A_r}$ and there is an $N^k_r$ with two dividing curves that fails to thicken.

So given $K_r$, there are infinitely many such cabling knot types $\hat{K}_{r+1}$, all of these being cabling of the form $(-\chi(K_r), k+1)$ as measured in the preferred framing. The following lemma will then prove Theorem 1.2.

**Lemma 7.4.** $\hat{K}_{r+1}$ is a transversally non-simple knot type.

**Proof.** We first calculate $\chi(\hat{K}_{r+1})$. Using the recursive expression we obtain

$$\chi(\hat{K}_{r+1}) = q_{r+1} \chi(K_r) - P_{r+1} q_{r+1} + P_{r+1} = (k + 1)(-A_r + B_r) - (A_r - B_r)(k + 1) + (A_r - B_r) = (2k + 1)(-A_r + B_r)$$

We now look at the two universally tight non-thickenable $N^k_r$ that have representatives of $\hat{K}_{r+1}$ as Legendrian divides. These Legendrian divides have $tb = A_{r+1} = q_{r+1} P_{r+1} = (k + 1)(A_r - B_r)$. To calculate rotation numbers, we have two possibilities, depending on which boundary of the two universally tight $N^k_r$ the Legendrian divides reside. Using the formula from [EH], we obtain

$$r(\hat{K}_{r+1}) = q_{r+1} r(\partial \Sigma) + P_{r+1} r(\partial D) = P_{r+1}(\pm(q_{r+1} - 1)) = \pm k(A_r - B_r)$$

We will call the two Legendrian divides corresponding to $r = \pm k(A_r - B_r)$, $L^+_{r+1}$ respectively. We can calculate the self-linking number for the negative transverse push-off of $L^+_{r+1}$ to be $sl = (2k+1)(A_r - B_r) = -\chi(\hat{K}_{r+1})$. This shows that $L^+_{r+1}$ is on the right-most edge of the Legendrian mountain range and is at $\delta \overline{b}$. Similarly, $L^-_{r+1}$ is on the left-most edge of the Legendrian mountain range and is at $\overline{tb}$.

We now look at solid tori $\hat{N}_r$ with intersection boundary slope $-\frac{k+1}{A_r k + B_r}$, but which thicken to solid tori with intersection boundary slopes $-\frac{1}{A_r - 1}$. Such tori $\partial \hat{N}_r$ are embedded in universally tight basic slices bounded by tori with dividing curves of slope $-\frac{1}{A_r - 1}$ and $-\frac{1}{A_r}$. Legendrian divides on such $\hat{N}_r$ have $tb = (k + 1)(A_r - B_r)$; to calculate possible rotation numbers for these Legendrian divides, we recall the procedure used in the proof of Theorem 1.5 in [L]. There we used a formula for the rotation numbers from [EH], where the range of rotation numbers was given by the following (substituting $A_r - 1$ for $n$):

$$r(L_{r+1}) \in \{\pm(p_{r+1} + (A_r - 1)q_{r+1} + q_{r+1} r(L_r))|tb(L_r) = A_r - (A_r - 1) = 1\}$$
Now from Lemma 7.2 we know that there is an $L_r$ with $tb(L_r) = 1$ and $r(L_r) = -(A_r - B_r) + 1$. Plugging this value of the rotation number into the expression above yields $r(L_r) = \pm k(A_r - B_r)$. We will call the Legendrian divides having these rotation numbers $\hat{L}_{r+1}^\pm$, respectively. Important for our purposes is that $\hat{L}_{r+1}^\pm$ have the same values of $tb$ and $r$ as $L_{r+1}^\pm$.

We focus in, for the sake of argument, on $L_{r+1}^-$ and $\hat{L}_{r+1}^-$, and we show that $T_+(L_{r+1}^-)$ is not transversally isotopic to $T_-(\hat{L}_{r+1}^-)$, despite having the same self-linking number.

Consider first $T_+(L_{r+1}^-)$. It is in fact one of the dividing curves on $\partial N_r^k$, and is also at maximal self-linking number for $\hat{K}_{r+1}$. Similarly, $T_+(\hat{L}_{r+1}^-)$ is one of the dividing curves on $\partial \hat{N}_r$, and is also at maximal self-linking number. Now from [He1] we know that $\hat{K}_{r+1}$ is a fibered knot that supports the standard contact structure, since it is an iterated torus knot obtained by cabling positively at each iteration. As a consequence, from [EV], we also know that $\hat{K}_{r+1}$ has a unique transversal isotopy class at $\hat{s}$. Hence we know that $T_+(L_{r+1}^-)$ and $T_+(\hat{L}_{r+1}^-)$ are transversally isotopic. Thus there is a transverse isotopy (inducing an ambient contact isotopy) that takes these two dividing curves on the two different tori to each other. Thus we may assume that $\partial N_r^k$ and $\partial \hat{N}_r$ intersect along one component of the dividing curves; we call this component $\gamma_+$.

Now suppose, for contradiction, that $T_-(L_{r+1}^-)$ is transversally isotopic to $T_-(\hat{L}_{r+1}^-)$. These transverse knots are represented by the other two non-intersecting dividing curves on $\partial N_r^k$ and $\partial \hat{N}_r$, respectively, and there is a transverse isotopy taking one to the other. We claim that this transverse isotopy can be performed relative to $\gamma_+$. To see this, note that associated to $S^3 \setminus N(\gamma_+)$ is an open book decomposition of $S^3$, with pages being Seifert surfaces $\Sigma$ for the knot $\gamma_+$. Moreover, the standard contact structure is supported by this open book decomposition. Thus the transverse isotopy taking $T_-(L_{r+1}^-)$ to $T_-(\hat{L}_{r+1}^-)$ will induce an ambient isotopy of open book decompositions supporting the standard contact structure, all with a transversal representative of $\gamma_+$ on the binding. Since $\partial N_r^k$ is incompressible in $S^3 \setminus N(\gamma_+)$, it is therefore evident that the isotopy taking $T_-(L_{r+1}^-)$ to $T_-(\hat{L}_{r+1}^-)$ can be accomplished simply as an isotopy of $\partial N_r^k$ relative to $\gamma_+$.

Thus we may assume that after a contact isotopy of $S^3$, $\partial N_r^k$ and $\partial \hat{N}_r$ intersect along their two dividing curves, which we denote as $\gamma_+$ and $\gamma_-$, and we observe that there is an isotopy (not necessarily a contact isotopy) of $N_r^k$ to $\hat{N}_r$ relative to $\gamma_+$ and $\gamma_-$. We claim that as a result $\hat{N}_r$ cannot thicken, thus obtaining our contradiction. We do this by noting that the isotopy of $N_r^k$ to $\hat{N}_r$ relative to $\gamma_+$ and $\gamma_-$ may be accomplished by the attachment of successive bypasses. Since these bypasses are attached in the complement of the two dividing curves, none of these bypass attachments can change the boundary slope. However, they may increase or decrease the number of dividing curves. Starting with $T = \partial N_r^k$, we make the following inductive hypothesis, which we will prove is maintained after bypass attachments:

1. $T$ is a convex torus which contains $\gamma_+$ and $\gamma_-$, and thus has slope $-\frac{k+1}{A_r k + B_r}$.
2. $T$ is a boundary-parallel torus in a $[0, 1]$-invariant $T^2 \times [0, 1]$ with slope($\Gamma_{T_0}$) = slope($\Gamma_{T_1}$) = $-\frac{k+1}{A_r k + B_r}$, where the boundary tori have two dividing curves.
3. There is a contact diffeomorphism $\phi : S^3 \rightarrow S^3$ which takes $T^2 \times [0, 1]$ to a standard $I$-invariant neighborhood of $\partial N_r^k$ and matches up their complements.
The argument that follows is similar to Lemma 6.8 in [EH1]. First note that item 1 is preserved after a bypass attachment, since such a bypass is in the complement of \( \gamma_+ \) and \( \gamma_- \), and thus cannot change the slope of the dividing curves. To see that items 2 and 3 are preserved, suppose that \( T' \) is obtained from \( T \) by a single bypass. Since the slope was not changed, such a (non-trivial) bypass must either increase or decrease the number of dividing curves by 2. Suppose first that the bypass is attached from the inside, so that \( T' \subset N \), where \( N \) is the solid torus bounded by \( T \). For convenience, suppose \( T = T_{0.5} \) inside the \( T^2 \times [0, 1] \) satisfying items 2 and 3 of the inductive hypothesis. Then we form the new \( T^2 \times [0.5, 1] \) by taking the old \( T^2 \times [0.5, 1] \) and adjoining the thickened torus between \( T \) and \( T' \). Now \( T' \) bounds a solid torus \( N' \), and, by the classification of tight contact structures on solid tori, we can factor a nonrotative outer layer which is the new \( T^2 \times [0, 0.5] \).

Alternatively, if \( T' \subset (S^3 \setminus N) \), then we know that \( N' \) thickens to an \( N'_r \), and thus there exists a nonrotative outer layer \( T^2 \times [0.5, 1] \) for \( S^3 \setminus N' \), where \( T_1 \) has two dividing curves. Thus the proof is done, for after enough bypass attachments we will obtain \( T = \partial N'_r \), with \( N'_r \) non-thickenable. But this is a contradiction, since \( N'_r \) does thicken. \( \square \)

University at Buffalo, Buffalo, NY
E-mail address: djl2@buffalo.edu

References

[BW] J. Birman and N. Wrinkle, On transversally simple knots, Journal of Differential Geometry 55 (2000), 325-354.
[EFM] J. Epstein, D. Fuchs, and M. Meyer, Chekanov-Eliashberg invariants and transverse approximations of Legendrian knots, Pac. J. Math. 201 (1) (2001), 89-106.
[E] J. Etnyre, Lectures on open book decompositions and contact structures, e-print at arXiv:math/0409402v3.
[EH1] J. Etnyre and K. Honda, Cabling and transverse simplicity, Ann. of Math. (2) 162 (2005), no. 3, 1305-1333.
[EH2] J. Etnyre and K. Honda, Knots and Contact Geometry I: Torus Knots and the Figure Eight Knot, Journal of Symplectic Geometry, Vol. 1, No. 1 (2001), 63-120.
[EV] J. Etnyre and J. Van Horn-Morris, Fibered transverse knots and the Bennequin bound, (2008), e-print at arXiv:0803.0758v2.
[G] E. Giroux, Convexité en topologie de contact, Comm. Math. Helv. 66 (1991), 615-689.
[He1] M. Hedden, Some remarks on cabling, contact structures, and complex curves, Proc. of 13th Gokova Geometry-Topology Conference (2008), 1-11.
[He2] M. Hedden, An Ozsváth-Szabó Floer homology invariant of knots in contact manifolds, e-print at arXiv:0708.0448.
[He3] M. Hedden, Notions of positivity and the Ozsváth-Szabó concordance invariant, e-print at arxiv:math/0509490.
[H1] K. Honda, On the classification of tight contact structures I, Geometry & Topology 4 (2000), 309-368.
[H2] K. Honda, On the classification of tight contact structures II, Journal of Differential Geometry 55 (2000), 83-143.
[L] D. LaFountain, Studying uniform thickness I: Legendrian simple iterated torus knots, e-print at arXiv:0905.2760.
[Mi] J. Milnor, Singular points of complex surfaces, Princeton Univ. Press, Princeton 1968.
[TW] W.P. Thurston and H.E. Winkelnkemper, On the existence of contact forms, Proc. Amer. Math. Soc. 52, (1975), 345-347.