Bloch oscillations in Fermi gases

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The possibility of Bloch oscillations for a degenerate and superfluid Fermi gas of atoms in an optical lattice is considered. For a one-component degenerate gas the oscillations are suppressed for high temperatures and band fillings. For a two-component gas, Landau criterion is used for specifying the regime where Bloch oscillations of the superfluid may be observed. We show how the amplitude of Bloch oscillations varies along the BCS-BEC crossover.

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interactions [14]. We define the limits of the one-band approximation for the physical potential Eq. (4) by demanding the lowest band gap to be bigger than the effective interaction $U$ (note that $U > |g|$ for the parameters of interest). The band gap can be estimated by approximating the cosine potential well by a quadratic one. Demanding the corresponding harmonic oscillator energy to be greater than $U$ gives the condition $\frac{V}{E_R} < \frac{1}{4\pi^2} \left( \frac{a}{|a_S|} \right)^2$.

Since $a > |a_S|$ imposed by considering on-site interactions only, the condition is easily valid in general, and for the parameters of Fig. 2 in particular. Estimates made using exact numerical band gaps in 1D support this argument. One-band approximation is sufficient because larger $V_0$ means steeper optical potential wells which not only increase the effective interaction $U$ but also the band gaps.

Bloch oscillations for a single atom can be characterized considering the mean velocity of a particle in a Bloch state $v(n, k) = \langle n, k | \mathbf{r} | n, k \rangle$ given by

$$v(n, k) = \frac{1}{\hbar} \nabla_k \varepsilon_n(k). \quad (2)$$

When a particle in the Bloch state $| n, k_0 \rangle$ is adiabatically affected by a constant external force $F = F_x \hat{\mathbf{x}}$ weak enough not to induce interband transitions, it evolves up to a phase factor into the state $| n, k(t) \rangle$ according to $k(t) = k_0 + F_x t a / \hbar$. The time evolution has a period $\tau_B = \hbar / (|F_x| a)$, corresponding to the time required for the quasimomentum to scan the whole Brillouin zone. If the force is applied adiabatically, it provides momentum to the system but not energy because the effective mass (given by $m(\varepsilon)^{-1} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial k^2}$) is not always positive. For optical lattices the force (or tilt: $V = \mathbf{F} \cdot \mathbf{r}$ term in the Hamiltonian) can be realized by accelerating the lattice [2, 3]. Using the tight-binding dispersion relation the velocity of an atom oscillates like

$$v_x(t) = \frac{J a}{\hbar} \sin(k_0 a + F_x t a / \hbar). \quad (3)$$

For cold bosonic atoms and condensates [2, 3] nearly all of the population is in the lowest mode of the optical potential, Eq. (3) therefore describes the oscillation of the whole gas. We generalize the result for the case when many momentum states of the band (at $T=0$, the states with wave vector $k \leq k_F$) are occupied. We calculate the velocity of the whole gas as the average over the normalized temperature-dependent distribution function (the Fermi distribution $f$) of the particles:

$$\langle v_x(t) \rangle = \frac{1}{\hbar} \sum_{k_0} f(k_0) \nabla_{k_0} \varepsilon(k_0 + Ft / \hbar). \quad (4)$$

Using the tight-binding dispersion relation for the Bloch energies we obtain the oscillations shown in Fig. 1. At $T=0$, Eq. (4) reduces to

$$\langle v_x(t) \rangle = \frac{Ja}{\hbar} \sin(k_F a) \sin \left( \frac{F \tau a}{\hbar} \right). \quad (5)$$

This shows that macroscopic coherent oscillation effect can still be observed if the band is not full, but the amplitude is suppressed by the band filling $k_F a$. The effect of the temperature can be seen in Fig. 1. The amplitude starts to decrease at temperatures of the order $T \geq 0.1J$ but is still non-negligible at half J. These results are valid for the one-component degenerate Fermi gas at low temperatures. In the two-component Fermi gas, atoms in the different hyperfine states interact with each other which may lead to a superfluid state. Above $T_c$, weak interactions can be described by a mean field shift in the chemical potential, leading to no qualitative changes in Bloch oscillations. Inelastic scattering and consequent damping of Bloch oscillations can be described e.g. by balance equations [15]. In the following we consider the superfluid case where qualitative changes are expected.

In order to observe Bloch oscillations of the superfluid Fermi gas, the critical velocity of the superfluid should not be reached before the edge of the Brillouin zone. A BCS-superconductor can carry a persistent current $q$ until a critical velocity, $v_c = \frac{\Delta}{\hbar p_F}$. For higher current values, even at $T = 0$, it might be energetically favorable to break Cooper pairs and create a pair of quasiparticles [16]. This costs $2\Delta$ in binding energy and decreases the Bloch energy by $|\xi_{k+q} - \xi_{k-q}| \equiv 2|E_D|$. Therefore, for the current to be stable $|E_D| < \Delta$. This is the Landau criterion of superfluidity. For a tight binding lattice dispersion relation, we rewrite the condition as $J \sin(qa) \sin(k_F a) < \Delta$. To complete a Bloch oscillation, $\sin(qa)$ should achieve its maximum value 1, i.e.

$$\sin k_F a < \Delta / J. \quad (6)$$

For weak coupling, $\Delta / J$ is given by the BCS theory, and in the attractive Hubbard model in the strong coupling
limit the gap at \( T = 0 \) is given by \( \Delta = \frac{J}{2} \) for half filling \( \frac{1}{2} \). Using these estimates, we show in Fig. 2 the relation \( \alpha \) for a gas of \(^6\)Li atoms together with the transition temperature. To relate the criterion to the Cooper pair size, we rewrite Eq. (6) in terms of the BCS coherence length \( \xi = \frac{\hbar v_F}{2\pi a_s} \) and insert \( J \sin(k_F a) = \hbar v_F / a \) which yields \( \xi < a / \pi \). The observation of Bloch oscillations is thus restricted to superfluids with BCS coherence length smaller than the lattice periodicity. This is the intermediate – strong coupling regime. The length argument can be also understood by thinking that the pairs have to be smaller than the lattice sites in order to see it as a periodic potential.

For calculating the superfluid velocity a space dependent description of the superfluid has to be used. We combine the BCS ansatz with the Bloch ansatz for the lattice potential using the Bogoliubov – de Gennes (BdG) equations\(^{17}\). As given by the Landau criterion above, the interesting regime is the intermediate – strong coupling one. Note that even in the strong-coupling limit, the algebra of the BCS theory can be applied to all coupling strengths \( \xi \) together with an extra definition for the chemical potential which in the weak coupling limit is given just by the Fermi energy of the non-interacting gas. The BdG equations are:

\[
\begin{aligned}
\left( \frac{H(r) - \mu}{\Delta(r)} \right) \begin{pmatrix} \Delta(r) \phi(r) \\ \phi(r) \end{pmatrix} &= E \begin{pmatrix} \phi(r) \\ \phi(r) \end{pmatrix} \\
\end{aligned}
\]  \tag{7}

When the external potential is periodic one can use the Bloch ansatz for \( \phi \) and \( \psi \) because by self-consistency the Hartree and pairing fields are also periodic. We obtain

\[
\begin{align*}
\phi_k(r) &= e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\phi}_k(r) ; \quad \psi_k(r) &= e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\psi}_k(r) \\
\Delta(r) &= \sum_k |g|[1 - 2f(E_k)]\phi_k(r)\psi_k^*(r),
\end{align*}
\]  \tag{8}

where \( \phi_k \) are the Bloch enveloping functions, such that \( \langle H(r) - \mu \rangle \phi_k e^{i\mathbf{k} \cdot \mathbf{r}} = \xi_k \phi_k e^{i\mathbf{k} \cdot \mathbf{r}} \).

To describe Bloch oscillations we impose the adiabatic condition, that is, momenta evolve according to \( \mathbf{k} \rightarrow \mathbf{k} + \mathbf{F}a/\hbar \equiv \mathbf{k} + \mathbf{q} \), i.e. we consider BCS state with a drift (again only in x-direction). The solutions of the BdG equations take the form

\[
\begin{align*}
\phi_k^0(r) &= e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\phi}_k^0(q+\mathbf{q})(r) ; \quad \psi_k^0(r) &= e^{i\mathbf{k} \cdot \mathbf{r}} e^{-i\mathbf{q} \cdot \mathbf{r}} \tilde{\psi}_k^0(q+\mathbf{q})(r), \\
\Delta^q(r) &= \sum_k |g|[1 - 2f(E_k^q)]\phi_k^q(r)\psi_k^{q*}(r)e^{2i\mathbf{q} \cdot \mathbf{r}},
\end{align*}
\]  \tag{10}

\( E_k^q = (\xi_{k+q} - \xi_{k-q})^2/2 \pm \sqrt{(\xi_{k+q} + \xi_{k-q})^2/4 + |\Delta^q(r)|^2} \equiv E_D \pm \sqrt{E_A^2 + |\Delta^q(r)|^2} \), where \( E_D \) is the energy difference and \( E_A \) the average energy. The \( \pm \) holds for the particle and hole branch, respectively, and the particle branch eigenfunctions are \( |\phi_k^0|^2, |\psi_k^0|^2 = (1 \pm E_D/\sqrt{E_A^2 + |\Delta^q(r)|^2})/2 \). The Hamiltonian transformed under the Bogoliubov transformation leading to Eq. (10) has to be positive definite. This means that one should use the solutions for which \( E_k^q > 0 \), i.e. \( \min(\sqrt{E_A^2 + |\Delta_q|^2}) = |\Delta^q(r)| > E_D \). Remarkably, this condition is the same as obtained using the Landau criterion.

In the BCS ansatz, a common momentum \( \mathbf{q} \) can be added to all particles, leading to correlations of the type \( \langle c_i^\dagger \mathbf{q}^\dagger c_i - c_{i+\mathbf{q}} \rangle \). The superfluid net momentum becomes \( 2\mathbf{q} \). One can formally calculate this obvious result also by using the plane wave ansatz \( u_k = |u_k|e^{i\mathbf{k} \cdot \mathbf{r}} \), \( v_k = |v_k|e^{i\mathbf{k} \cdot \mathbf{r}} \) (Eq. (10) with \( \phi = 0 \)) and introducing an (unnormalized) order parameter wave function \( \Delta^q(r) = e^{i2\mathbf{q} \cdot \mathbf{r}}C \), where \( C \) is given by Eq. (10) to be a constant in \( r \). Expectation values like momentum \( \langle \mathbf{p} = -i\partial/\partial \mathbf{r} \rangle \) can be calculated: \( \langle \mathbf{p} \rangle = \langle \Delta^q(r) \rangle = 2\mathbf{q} \). The order parameter wave function is defined in the spirit of (but not with a one-to-one correspondence to) the Ginzburg-Landau theory with a space dependent wave function whose absolute value equals the gap. In case of Fermionic atoms the Ginzburg-Landau approach has been used to describe harmonic confinement\(^{18}\) and vortices\(^{20}\). For the periodic potential we introduce the order parameter wave function in the form \( \Delta^q(r) = \sum_{k} \Delta^q_{k}(r) \), where using Eq. (10),

\[
|\Delta^q_{k}(r)| = F(k, q)|\phi_{k+q}e^{i(k+q)\mathbf{r}} |\phi_{k-q}e^{-i(k-q)\mathbf{r}}|
\]  \tag{11}

and \( F(k, q) = |g|[1 - 2f(E_k^q)]|\tilde{\phi}_k^0|^2 \tilde{\psi}_k^0 |^2 q^2 \). We calculate the superfluid velocity using \( \langle \mathbf{v}_S \rangle = \mathcal{N}(|\Delta^q(r)|^2) \mathbf{F} \delta^q \), where \( \mathcal{N} = (|\Delta^q(r)|^2)^{-1} \). Using \( \langle \mathbf{r} \rangle \phi_{k-q}e^{-i(k-q)\mathbf{r}} = \mathcal{N}(|\Delta^q(r)|^2)^{-1}(k+q)\mathbf{r} = \mathcal{N}^2 \mathbf{r} \).

**FIG. 2:** The transition temperature, Landau criterion at \( T=0 \) and the amplitude of the velocity Bloch oscillations for \(^6\)Li atoms in hyperfine states with scattering length \( a_s = -2.5 \times 10^{-6} a_0 \) for a half filled 3D CO2 laser lattice (\( a = 10^{-6} a_0 \)) as a function of the lattice depth. The amplitudes of the oscillations at \( T = 2/3T_{c, max} \) (horizontal line) are denoted by \( \phi \) for the superfluid velocity at \( T = 0 \) Eq. (10) and in the boson limit Eq. (10) pair size \( l = a/3 \) by \( \times \) and \( + \) for pair size \( l = a/4 \). The Landau criterion condition Eq. (10) requires \( J/\Delta > 1 \) for the half filled band. Here \( E_D \) is the recoil energy and \( E_D^q \) is the Fermi energy for free fermions with the same density.
The superfluid velocity for selected parameters is shown in Fig. 3. We have also calculated the thermal quasiparticle contribution but is turns out to be negligible.

In the composite boson limit, one could describe the center-of-mass movement of the composite particle by defining $J^* = J(m \rightarrow 2m)$. In order to give a simple estimate for the effect of the Fermi statistics, we interpret $|F(k, q)|^2 \sim |F(k)|^2$ in Eq. (12) as reflecting the internal wavefunction of the pair in the composite boson limit, c.f. [5, 8]. The average velocity for the bosons becomes $\langle v_{sB} \rangle \propto J^* a / h \sin qa \sum_k |F(k)|^2 \cos k a$. If the pairs are extremely strongly bound, the internal wave function in real space is a delta-function, corresponding to a constant in k-space. This means $\langle v_{sB} \rangle = 0$ since the cosine integration in Eq. (12) would extend to the whole k-space with equal weight, i.e. there are no empty states in the Brillouin zone required for Bloch oscillations. For on-site pairs, we use $|F(r)|^2 \propto \exp(-r^2/l^2)$ leading to $|F(k)|^2 \propto N \exp(-k^2 l^2/4)$, therefore the suppression factor for the Bloch oscillations becomes $S \sim N \int dk \exp(-k^2 l^2/4) \cos ka$, where $l$ is the pair size. As a rough estimate for the average velocity we thus obtain

$$\langle v_{sB} \rangle \sim S J^* a / h \sin (F r a / h).$$

This is shown in Fig. 2 for pair sizes $l = a/3$ and $l = a/4$. It gives an order-of-magnitude estimate, approaching the results given by the BCS algebra.

Another way of treating the composite boson limit is to derive a Gross-Pitaevskii type of equation for the composite bosons with $M = 2m$ and with a repulsive non-linear interaction term $n_B U_B = n_B 4 \pi \hbar^2 a_B / M$, $a_B = 2a$, where $a$ is the renormalized s-wave scattering length [21]. If the non-linear term is small compared to the Bloch energy $E_B = \hbar^2 / (M a^2)$, the non-linearity leads only to a change in the band width $J^*$ [3]. Therefore, composite bosons oscillate but with a modified amplitude. Large non-linearity would not allow Bloch oscillations, corresponding to a large suppression factor in the above discussion. Note that the Landau criterion for a superfluid Bose gas gives the critical velocity $v_{\text{sound}} = \sqrt{\frac{U_{\text{pp}} a}{M}}$ which is orders of magnitude bigger than Eq. (13) for half filling and parameters in Fig. 2. Problems arise only in the extremely empty lattice limit.

In summary, we have defined a set of tools for qualitative and quantitative description of Bloch oscillations for the BCS-BEC crossover regime. The amplitude of the oscillations decreases when the crossover is scanned, in general due to the shrinking of the bandwidth. However, the change from the normal to the superfluid state description leads to a drastic change in the amplitude. This is due to smoothening of the Fermi-edge by pairing. Bloch oscillations could be used for exploring pairing correlations since any localization in space (pair size) leads to broadening in momentum which suppresses the amplitude in the same way as band filling in the non-interacting gas. Achievement of superfluidity is still a great challenge, but even at $T >> T_c$, the effect of collisions on Bloch oscillations can be studied producing information useful for applications of Bloch oscillations such as production of Terahertz radiation [15-22]. Observation of oscillating fermionic atoms in optical lattices would contribute to the quest for a steadily driven fermionic Bloch oscillator.

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