Subharmonic Configurations and
Algebraic Cauchy Transforms of Probability Measures

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Abstract
We study subharmonic functions whose Laplacian is supported on a null set $K \subset \mathbb{C}$ and in connected components of $\mathbb{C} \setminus K$ admit harmonic extensions to larger sets. We prove that if such a function has a piecewise holomorphic derivative then it is locally piecewise harmonic and in generic cases it coincides locally with the maximum of finitely many harmonic functions. Moreover, we describe $K$ when the holomorphic derivative satisfies a global algebraic equation. The proofs follow classical patterns and our methods may also be of independent interest.

Introduction
Let $\Omega$ be an open connected subset of the complex plane $\mathbb{C}$. Denote by $\text{SH}_0(\Omega)$ the class of subharmonic functions $V$ in $\Omega$ for which the support of the Laplacian $\Delta (V)$ has Lebesgue measure 0, where $\Delta (V)$ in the sense of distributions is a non-negative Riesz measure supported by the null set $\text{supp}(\Delta (V))$. As explained in Hörmander [7], every $V \in \text{SH}_0(\Omega)$ is identified with an element in $L^1_{\text{loc}}(\Omega)$ and can always be taken as an upper semi-continuous function. Moreover, the distribution derivatives $\partial V/\partial x$ and $\partial V/\partial y$ belong to $L^1_{\text{loc}}(\Omega)$. In particular, the distribution derivative

$$\partial V/\partial z = \frac{1}{2} (\partial V/\partial x - \partial V/\partial y)$$

is a holomorphic function in $\Omega \setminus \text{supp}(\Delta (V))$ which as a distribution is an element of $L^1_{\text{loc}}(\Omega)$. Therefore, if the holomorphic function $\partial V/\partial z$ defined in $\Omega \setminus \text{supp}(\Delta (V))$ extends to a holomorphic function $g$ defined in the whole set $\Omega$, then the distribution $\partial / \partial \bar{z} (\partial V/\partial z) = \Delta V / 4 = 0$, i.e., $V$ is harmonic in $\Omega$.

The aforementioned facts, already known to F. Riesz who laid the foundations of subharmonic functions in his famous article [7] from 1926, have led to the problems studied in the present paper. We call $V \in \text{SH}_0(\Omega)$ piecewise harmonic if there exists a finite set of harmonic functions $H_1, \ldots, H_k$ in $\Omega$ such that for every connected component $U$ of $\Omega \setminus \text{supp}(\Delta (V))$ one has $V = H_j$ in $U$ for some $1 \leq j \leq k$. In this case we refer to $V$ as a subharmonic configuration of the $k$-tuple $H_1, \ldots, H_k$. When $V$ is such a subharmonic configuration one easily shows the inclusion

$$\text{supp}(\Delta (V)) \subset \bigcup_{i \neq j} \{ H_i = H_j \}.$$
In general a $k$-tuple of harmonic functions $H_1, \ldots, H_k$ gives rise to several subharmonic configurations, see §2.11. An obvious subharmonic configuration is the maximum function $V^* = \max(H_1, \ldots, H_k)$. In Theorem 1.4 we show that $V^*$ is locally the unique subharmonic configuration of $H_1, \ldots, H_k$ in a neighborhood of a point $p \in \Omega$ when the $k$-tuple of gradient vectors $\nabla(H_1)(p), \ldots, \nabla(H_k)(p)$ are extreme points of their convex hull. An essential role in proving this as well as our other results is played by the Key Lemma 1.1 in Section 1.

The next issue in this article is to study functions $V$ in $\text{SH}_0(\Omega)$ for which the analytic function in $\Omega \setminus \text{supp}(\Delta(V))$ defined by $\partial V/\partial z$ is piecewise holomorphic. This means that there exists a finite set of holomorphic functions $g_1, \ldots, g_k$ in $\Omega$ and for every connected subset $U$ of $\Omega \setminus \text{supp}(\Delta(V))$ some $1 \leq j \leq k$ such that $\partial V/\partial z = g_j$ in $U$. Let us remark that if $V$ is piecewise harmonic with respect to $H_1, \ldots, H_k$ then $\partial V/\partial z$ is piecewise holomorphic with respect to the $k$-tuple $\{\partial H_i/\partial z\}_{i=1}^k$ in $\mathcal{O}(\Omega)$. Thus, if $V$ is piecewise harmonic then $\partial V/\partial z$ is piecewise holomorphic. A major result in this paper is the following converse: if $V$ is a subharmonic function such that $\partial V/\partial z$ is piecewise holomorphic then $V$ is locally piecewise harmonic. More precisely, we prove:

**Theorem 1.** Let $\partial V/\partial z$ be piecewise holomorphic with respect to a $k$-tuple $\{g_i\}_{i=1}^k$ in some open set $\Omega$. For each simply connected open subset $U$ of $\Omega$ one can choose a $k$-tuple of harmonic functions $H_1, \ldots, H_k$ such that $\partial H_i/\partial z = g_i$ in $U$, $1 \leq i \leq k$. Moreover, if $U_0$ is a relatively compact subset of $U$ there exists a finite number of constants $c_\nu = c_\nu(U_0)$, $1 \leq \nu \leq m$, such that the restriction $V|_{U_0}$ is piecewise harmonic with respect to a subfamily of the $m \cdot k$ many harmonic functions $\{H_i + c_\nu\}$.

The proof of Theorem 1 requires several steps. It is based upon the results of §1 and §2 and will be completed only at the end of §3. Let us point out that the difficulty in proving Theorem 1 stems from the fact that no special assumption is imposed on the open set $\Omega \setminus \text{supp}(\Delta(V))$, i.e., for a general null set $K$ of $\Omega$ there may à priori exist a relatively compact subset $U$ of $\Omega$ such that the number of connected components of $\Omega \setminus K$ which intersect $U$ is infinite. The main burden in the proof of Theorem 1 is then to show that this cannot occur when $V \in \text{SH}_0(\Omega)$ and $K = \text{supp}(\Delta(V))$.

Our final topic is about algebraic functions. In §4 we make use of the previously developed material to prove a result about non-negative Riesz measures supported by compact null sets in $\mathbb{C}$ whose Cauchy transforms satisfy an algebraic equation. More precisely, let $\mu$ be such a measure, denote by $K$ the support of $\mu$ and set

$$\hat{\mu}(z) = \int_K \frac{d\mu(\zeta)}{z - \zeta}.$$  

We say that the Cauchy transform $\hat{\mu}$ satisfies an algebraic equation if there exist some $k \geq 1$ and polynomials $p_0(z), \ldots, p_k(z)$ such that

$$p_k(z) \cdot \hat{\mu}(z) + \ldots + p_1(z) \cdot \hat{\mu}(z) + p_0(z) = 0, \quad z \in \mathbb{C} \setminus K.$$  

Note that à priori $K$ is just a null set and in general one can hardly say more than that. However, assuming (*) we can substantially improve this and get the following description of $K$:

**Theorem 2.** If (*) holds then the support of $\mu$ is a real analytic set of dimension at most one.
Finally, in §5 we discuss some further directions, open problems and conjectures inspired by the topics treated in this paper.

1. A Key Lemma

Let $Ω$ be an open and connected set in $C$ and $V ∈ SH_0(Ω)$. Set $K = supp(Δ(V))$ and decompose $Ω \setminus K = ∪_{α ∈ C} ω_α$ into open connected components. Suppose that $V = 0$ in an open subset $U = ∪_{α ∈ A} ω_α$ of $Ω \setminus K$ and furthermore that $Re(∂V/∂z) < 0$, $z ∈ W := Ω \setminus (K ∪ U) = ∪_{β ∉ A} ω_β$.

Observe that $∂V/∂z$ is a holomorphic function in each component $ω_α$.

1.1. Lemma. Let $z_0 ∈ ω_α ⊂ U$ and assume that $ℓ = \{z_0 + s : 0 ≤ s ≤ s_0\}$ is a line segment contained in $Ω$. If $0 < δ < dist(ℓ,∂Ω)$ and the open disk $D_δ(z_0)$ of radius $δ$ centered at $z_0$ is contained in $ω_α$, then

$$\{z : dist(z,ℓ) < δ\} = ∪_{0 ≤ s ≤ s_0} D_δ(z_0 + s) ⊂ ω_α.$$

Remark. The subsequent proof uses methods similar to those of [1, Lemma 2], in particular the idea to use the $Ψ$-function below. However, the new (and general) situation in Lemma 1.1 is that no finiteness condition is imposed on the range of $∂V/∂z$.

Proof of Lemma 1.1. By the choice of $δ$, the set $\{z : dist(z,ℓ) < δ\}$ is a relatively compact subset of $Ω$. Let $ε > 0$ and define the holomorphic function

$$Ψ(z) = Log(−ε + ∂V/∂z), \quad z ∈ Ω \setminus K,$$

where the single-valued branch of the complex Log-function is chosen so that

$$\pi/2 < \Im Ψ < 3π/2.$$

This is clearly possible, since by assumption $−ε + ∂V/∂z ≤ −ε$ in $Ω \setminus K$. Furthermore, since $∂V/∂z$ is locally integrable, $Ψ ∈ L^1_{loc}(Ω)$.

Consider a non-negative cut-off function $ρ$ supported by the unit disk with integral 1. Let $δ > 0$, define $ρ_δ(z) = δ^{-2}ρ(z)$, and set

$$Ψ_δ := Log(−ε + ρ_δ * ∂V/∂z).$$

Taking a derivative, we get

$$\partialΨ_δ/∂z = \frac{1}{4} ∙ \frac{ρ_δ * ΔV}{−ε + ρ_δ * ∂V/∂z} \implies$$

$$Re(∂Ψ_δ/∂z) = \frac{4[ε - ρ_δ * ∂V/∂z]^2}{4|ε - ρ_δ * ∂V/∂z|^2} Re(∂V/∂z).$$

Since $ΔV$ is a non-negative Riesz measure and $Re(∂V/∂z)$ is a non-positive function, we deduce from (1) that $Re(∂Ψ_δ/∂z)$ is a non-positive function. Passing to the limit as $δ → 0$ we conclude that the distribution derivative $Re(∂Ψ/∂z)$ is a non-positive Riesz measure. Next, we can write

$$Ψ(z) = σ(z) + iτ(z), \quad π/2 < τ(z) < 3π/2,$$
functions we conclude that $L$ almost everywhere zero. Since subharmonic functions appear as a subspace of $V$ such that
\[ \partial V/\partial z \leq 0 \] the inequality \[ \Re(\partial V/\partial z) \leq 0 \] gives that
\[ (2) \quad \partial_z (\phi * \sigma) = \partial_y (\phi * \tau). \]
Since $\pi/2 \leq \tau \leq 3\pi/2$, the absolute value of the right-hand side is majorised by $M = \frac{3\pi}{2} \cdot ||\partial_y (\phi)||_1$, where $||\partial_y (\phi)||_1$ denotes the $L^1$-norm. Next, consider the function $s \mapsto \phi * \sigma(z_0 + s)$, where $0 \leq s \leq s_0$. Applying (2) and setting $z_1 = z_0 + s_0$ we obtain
\[ (3) \quad \frac{d}{ds} (\phi * \sigma(z_0 + s)) \leq M \implies \phi * \sigma(z_1) \leq \phi * \sigma(z_0) + M \cdot s_0. \]
Since $K = \text{supp}(\Delta(V))$ is a null set we can identify $\sigma$ with the following $L^1_{\text{loc}}$-function
\[ (4) \quad \sigma(z) = \log|\epsilon| \cdot \chi_U + \log|\epsilon - \partial V/\partial z| \cdot \chi_W, \quad W = \Omega \setminus (K \cup U). \]
Set $f_\epsilon = \log|\epsilon - \partial V/\partial z| \cdot \chi_W$. From now on $\epsilon < 1$ so that $\log|\epsilon| < 0$. Since the support of $\phi$ is small enough (i.e., less than the distance $\delta$ from $z_0$ to the boundary) $\phi * \sigma(z_0) = \log|\epsilon| (\phi * \chi_U)(z_0) = \log|\epsilon|$. Inserting in (4) the expression $f_\epsilon$, inequality (3) gives
\[ (5) \quad 1 \leq \phi * \chi_U(z_1) + \frac{1}{\log|\epsilon|} \cdot [-\phi * f_\epsilon(z_1) + M \cdot s_0]. \]
At this stage we perform a limit as $\epsilon \to 0$. For this note first that the function $-\Re(\partial V/\partial z) \cdot \chi_W$ belongs to $L^1_{\text{loc}}$ and is $> 0$ in $W$. Moreover, the disk $D_\delta(z_1)$ is relatively compact in $\Omega$. Elementary measure theory shows that for any $h \in L^1_{\text{loc}}(\Omega)$ such that $\Re(h) \geq 0$ in $W$ and $\{\Re(h) = 0\} \cap W$ is a null set one has
\[ (6) \quad \lim_{\epsilon \to 0} \frac{1}{\log|\epsilon|} \cdot \iint_{D_\delta(z_1) \cap W} |\log(|\epsilon + h|)| dx dy = 0. \]
Apply this with $h = -\partial V/\partial z$. Since the test function $\phi$ has support in $|z| \leq \delta$, we have the inequality
\[ (7) \quad |\phi * f_\epsilon(z_1)| \leq ||\phi||_{\infty} \cdot \iint_{D_\delta(z_1)} |f_\epsilon(z)| dx dy. \]
By (6) the quotient of this by $\log|\epsilon|\frac{1}{\log|\epsilon|}$ tends to zero as $\epsilon \to 0$. So after a passage to the limit as $\epsilon \to 0$, it follows from (5) and (7) that
\[ (8) \quad 1 \leq \phi * \chi_U(z_1). \]
Finally, since $\phi(z) > 0$ when $|z| < \delta$, inequality (8) implies that $D_\delta(z_1) \setminus U$ is a null set. Hence the restriction of the subharmonic function $V$ to this open disk is almost everywhere zero. Since subharmonic functions appear as a subspace of $L^1_{\text{loc}}$-functions we conclude that $D_\delta(z_1) \subset U$. This completes the proof of Lemma 1.1. ∎

Lemma 1.1 suggests defining the following notion:

**Definition.** For every $z \in \Omega$ set
\[ \rho^*(z) = \max\{a \in (0, \infty) : z + t \in \Omega \text{ for all real } 0 < t < a\}. \]
If $U$ is an open subset of $\Omega$ we define the *forward star domain* of $U$ by
\[ s^1(U) = \{z \in \Omega : \exists \zeta \in U \text{ such that } z = \zeta + t \text{ for some } 0 \leq t < \rho(\zeta)\}. \]
A more concise formulation of Lemma 1.1 is then as follows:

**1.2. Theorem.** Let \( V \in SH_0(\Omega) \), \( K = \text{supp}(\Delta(V)) \) and assume that \( \Omega \setminus K \) is the disjoint union \( U \cup W \) of two open sets such that \( \Re(\partial V/\partial z) < 0 \) in \( W \) and \( V = 0 \) in \( U \). Then \( U = s^1(U) \).

Notice that Theorem 1.2 applies to an arbitrary subharmonic function in \( SH_0(\Omega) \), not necessarily piecewise harmonic. It will be crucial for our study of the piecewise holomorphic case in §3 as well as for our next result that we proceed to describe.

**1.3. Local subharmonic configurations.** Let \( V \in SH_0(D) \) and assume that \( \partial V/\partial z \) is piecewise holomorphic with respect to some \( k \)-tuple \( g_1, \ldots, g_k \) in \( \mathcal{O}(D) \), where \( D \) is an open disk centered at the origin. With \( K = \text{supp}(\Delta(V)) \) we further define the open subset \( U \) of \( D \setminus K \) as the union of those connected components of \( D \setminus K \) where \( \partial V/\partial z = g_\nu \). We assume that the origin belongs to the closure of every \( U_\nu \). In the simply connected disc \( D \) we choose the unique \( k \)-tuple of harmonic functions \( \{ H_1, \ldots, H_k \} \) satisfying

\[
\partial H_\nu/\partial z = g_\nu \quad \text{and} \quad H_\nu(0) = 0, \quad 1 \leq \nu \leq k.
\]

Next, consider the \( k \)-tuple \( (g_1(0), \ldots, g_k(0)) \) and the convex set \( P \) generated by these complex numbers. Assume that \( g_k(0) \) is an extreme point of \( P \). This gives some \( \theta_* \) such that

\[
\Re(e^{i\theta_\nu} \cdot g_\nu(0)) < \Re(e^{i\theta_{\nu-1}} \cdot g_{\nu-1}(0)), \quad 1 \leq \nu \leq k - 1.
\]

After a rotation if necessary we may further assume that \( \theta_* = 0 \) and thus (by continuity) there exists \( \delta > 0 \) such that

\[
\Re(e^{i\theta} \cdot g_\nu(0)) < \Re(e^{i\theta} \cdot g_{\nu-1}(0)), \quad 1 \leq \nu \leq k - 1, \quad -\delta < \theta < \delta.
\]

We can apply Theorem 1.2 to the subharmonic function \( e^{i\theta}(V - H_k) \) for \( -\delta < \theta < \delta \), and setting \( U = \{ V = H_k \} \) we conclude:

**1.4 Proposition.** If \( \square = \{(x, y) \in \mathbb{R}^2 : -a < x, y < a\} \) and \( a > 0 \) is sufficiently small then the domain \( \square \cap U \) is connected and given by

\[
\square \cap U = \{(x, y) \in \square : x > \rho(y)\},
\]

where \( \rho(0) = 0 \) and \( \rho \) is a Lipschitz continuous function of norm \( \leq \frac{\cos \delta}{\sin \delta} \).

A similar conclusion holds for other indices as well. Indeed, if \( g_\nu(0) \) is an extreme point of \( P \) for every \( 1 \leq \nu \leq k \) then we obtain open connected sets \( U_1, \ldots, U_k \) as above after suitable rotations. This leads to the following result.

**1.5. Theorem.** Let \( V \in SH_0(D) \) and assume that \( \partial V/\partial z \) is piecewise holomorphic with respect to some \( k \)-tuple \( g_1, \ldots, g_k \) in \( \mathcal{O}(D) \), where \( D \) is an open disk centered at the origin. Assume further that each \( g_\nu(0) \) is an extreme point of the convex hull \( P \) of \( (g_1(0), \ldots, g_k(0)) \). Then there exists \( c \in \mathbb{R} \) such that in a neighborhood of the origin one has \( V = \max(H_1, \ldots, H_k) + c \).

**Proof.** The hypothesis implies that for each given \( 1 \leq \nu \leq k \) there exists some \( \theta \) such that \( \Re(e^{i\theta} \cdot g_\nu) < \ldots < \Re(e^{i\theta} \cdot g_m) \), \( \nu \neq m \). Theorem 1.2 applies after a rotation. It follows that \( U_m \cap D(\delta) \) is connected for a sufficiently small \( \delta \). Since this holds for every \( m \) it follows that \( V \) is piecewise harmonic with respect to the \( k \)-tuple \( H_1, \ldots, H_k \) in \( D(\delta) \). There remains to see that \( V \) is the maximum function. For
this we may consider without loss of generality the index \( m = 1 \). After a rotation we find that there exists a function \( \rho(y) \) such that
\[
U_1 = \{(x, y) \in D(\delta) : x > \rho(y)\} \quad \text{and} \quad \partial_x H_{\nu} < \partial_x H_1, \ \nu \geq 2.
\]
We have to show that \( H_1(x, y) < V(x, y) \) when \( x < \rho(y) \). To do this we fix \( y_0 \) and consider the function \( x \mapsto V(x, y_0) \). When \( x < \rho(y_0) \) the partial derivative \( \partial_x V \) is equal to \( \partial_x (H_{\nu}) \) for some \( \nu \geq 2 \) on intervals outside some finite set where \( V \) may shift from one \( H \)-function to another when a level curve \( \{H_i = H_k\} \) intersects the line \( y = y_0 \). By the strict inequalities above \( x \mapsto V(x, y_0) - H_1(x, y_0) \) is strictly decreasing and since it is zero when \( x = \rho(y_0) \) Theorem 1.5 follows.

1.6. A relaxed assumption. Let us drop the hypothesis that the origin belongs to \( \hat{U}_\nu \) for every \( \nu \) and suppose instead that there is some \( 1 \leq \ell \leq k - 1 \) such that the extreme points of \( P \) are \( g_i(0), \ 1 \leq i \leq \ell \). Without loss of generality we may assume that the origin belongs to \( \hat{U}_i, \ 1 \leq i \leq \ell \), and that the vertices of \( P \) are labelled consecutively \( g_1(0), \ldots, g_\ell(0) \) in say counter-clockwise order. The example in §2.11 below shows that in this case we cannot conclude that \( V \) is given by the maximum of \( H_1, \ldots, H_\ell \) up to a constant. However, the following extension of Theorem 1.5 holds:

1.7 Theorem. Suppose as above that \( \{g_i(0)\}_{i=1}^\ell \) are the extreme points of \( P \) and that for \( i \in \{\ell + 1, \ldots, k\} \) one has
\[
g_i(0) \notin \bigcup_{j=1}^\ell \{(1 - \alpha)g_j(0) + \alpha g_{j+1}(0) : 0 \leq \alpha \leq 1\},
\]
where \([j] = j\) for \( 1 \leq j \leq \ell \) and \([\ell+1] = 1\). Then in a sufficiently small neighborhood of the origin one has \( V = \max(H_1, \ldots, H_\ell) \) up to a constant.

2. Subharmonic Configurations:
The General Piecewise Harmonic Case

We begin with some preliminary observations which follow from the maximum principle for subharmonic functions and Stokes’ Theorem. We then study harmonic level sets and give a local description of arbitrary subharmonic configurations.

Let \( H_1, \ldots, H_k \) be harmonic functions in \( \Omega \) and \( V \in \text{SH}_0(\Omega) \) be piecewise harmonic function with respect to this \( k \)-tuple. In \( \Omega \) we get the real analytic set
\[
\Gamma = \bigcup_{i \neq j} \{H_i = H_j\}.
\]

Let \( \{U_\alpha\} \) be the connected components of \( \Omega \setminus \Gamma \). Then we have:

2.1. Lemma. For each \( \alpha \) there exists \( 1 \leq i(\alpha) \leq k \) such that \( V = H_{i(\alpha)} \) in \( U_\alpha \).

Proof. Given \( U_\alpha \) there is some permutation of the indices such that
\[
H_{j(1)} < \ldots < H_{j(k)}.
\]
Set \( K = \text{supp}(\Delta(V)) \). For each \( 1 \leq i \leq k \) we define
\[
U_\alpha(i) = \{z \in U_\alpha \setminus K : V = H_i \text{ in some neighborhood of } z\}.
\]
By assumption one has \( \bigcup_i U_\alpha(i) = U_\alpha \setminus K \). Let \( m \) be the largest integer such that \( U_\alpha(j(m)) \) is non-empty. Then we have:
Sublemma. The set \( U_\alpha(j(m)) \) is dense in \( U_\alpha \).

Proof. Assume the contrary and set \( U_* = U_\alpha \setminus \bar{U}_\alpha(j(m)) \). Since \( U_\alpha \) is connected we cannot have \( U_\alpha(j(m)) \cup U_* = U_\alpha \) and hence there exists a point \( p_* \in \partial(U_\alpha(j(m))) \cap U_\alpha \).

Consider a point \( p \in U_\alpha(j(m)) \cap D \) very close to \( p_* \). Let \( D \) be a disc centered at \( p \) of some radius \( r \) whose closure stays in \( U_\alpha \). With \( p \) sufficiently close to \( p_* \), the set \( D \cap U_* \) is non-empty. The mean value inequality for subharmonic functions gives

\[
H_{j(m)}(p) = V(p) \leq \frac{1}{\pi r^2} \int_D V(x, y) \cdot dx \, dy.
\]

Set \( H_*(z) = \max(H_{j(1)}, \ldots, H_{j(m-1)}) \). Since \( D \cap U_* \) is non-empty and \( K \) is a null set we have \( V(z) \leq H_*(z) \) almost everywhere in \( D \cap U_* \). But then (9) cannot hold since we have the strict inequality \( H_* < H_{j(m)} \).

Proof of Lemma 2.1, continued. By the Sublemma \( U_\alpha(j(m)) \) is dense in \( U_\alpha \) and since all the sets \( U_\alpha(i) \) are open we have

\[
U_\alpha(j(m)) = U_\alpha \setminus K.
\]

This means that the \( L^1_{loc} \)-function \( V \) equals \( H_{j(m)} \) in the whole set \( U_\alpha \) and then Lemma 2.1 follows with \( i(\alpha) = j(m) \).

\[\square\]

Remark. Note that Lemma 2.1 gives the inclusion

\[
\text{supp}(\Delta(V)) \subset \Gamma.
\]

Another way of proving Lemma 2.1 is by means of Grishin’s Lemma [5], see also [4]. In fact, using [5] one can show that if \( V \in SH_0(\Omega) \) is piecewise harmonic then \( \text{supp}(\Delta(V)) \) is a null set, so the latter property need not be assumed already from the start (which we did for the reader’s convenience).

2.2. A description of \( \Delta(V) \). Consider some pair \( (U_\alpha, U_\beta) \) with \( i(\alpha) \neq i(\beta) \) and such that \( \partial U_\alpha \cap \partial U_\beta \neq \emptyset \). As explained in §2.5 below, the portion of this common boundary set that avoids the closed union of the remaining \( U \)-sets is a smooth real analytic curve \( \gamma \) possibly up to a discrete set. Let \( ds_\gamma \) be arclength measure on \( \gamma \) and suppose \( H_{i(\alpha)} > H_{i(\beta)} \) holds in \( U_\alpha \) while \( H_{i(\alpha)} < H_{i(\beta)} \) in \( U_\beta \). Along \( \gamma \) we choose the normal \( n_\gamma \) directed into \( U_\alpha \). Hence the normal derivatives satisfy

\[
\partial_{n_\gamma} H_{i(\alpha)} > 0 \quad \text{and} \quad \partial_{n_\gamma} H_{i(\beta)} < 0
\]

outside the discrete set of possible singularities for the level curve \( \{H_{i(\alpha)} = H_{i(\beta)}\} \).

With these notations Stokes’ Theorem gives:

2.3. Proposition. One has

\[
\Delta(V)|_\gamma = \left[ \partial_{n_\gamma} H_{i(\alpha)} - \partial_{n_\gamma} H_{i(\beta)} \right] \cdot ds_\gamma.
\]

2.4. Remark. Let \( G, H \) be a pair of harmonic functions defined in some domain \( \Omega \), set \( \Gamma = \{G = H\} \) and let \( p \in \Gamma \) be a regular point, i.e., \( \nabla(G)(p) - \nabla(H)(p) \neq 0 \). Consider a small disk \( D \) centered at \( p \) and the two domains

\[
U_+ = \{G > H\} \quad \text{and} \quad U_- = \{G < H\}.
\]

Then \( V = \max(G, H) \) is subharmonic while the opposed function \( \min(G, H) \) fails to be subharmonic. The lesson of this observation is that when the pair \( G, H \) appears in a configuration of a subharmonic function \( V \) their normal derivatives satisfy

\[
\partial_{n} G \geq \partial_{n} H,
\]
where \( n \) is the normal to \( \Gamma \) directed into \( U_+ \). This simple – but essential – observation will be frequently used later on.

### 2.5. Harmonic level sets.

Let \( H(x, y) \) be a harmonic function defined in some open disk \( D \) centered at the origin in \( \mathbb{C} \) and \( z = x + iy \) be the complex variable. Now \( H = \Re(g) \) for some \( g \in \mathcal{O}(D) \). If \( g \) vanishes of some order \( m \geq 1 \) at \( z = 0 \) there exists a conformal map \( \rho(\zeta) \) from a disk in the complex \( \zeta \)-plane such that \( g \circ \rho(\zeta) = \zeta^m \). The zero set of \( \Re(\zeta^m) \) is the union of lines \( \arg(\zeta) = \frac{\pi}{2} + \nu \pi m \), \( 0 \leq \nu \leq m - 1 \). Passing to the \( z \)-disk and shrinking \( D \) if necessary we get that \( \{ H = 0 \} \) is the union of \( m \) smooth real analytic curves \( \gamma_1, \ldots, \gamma_m \) and \( D \setminus \{ H = 0 \} \) consists of \( 2m \) pairwise disjoint open sets \( U_1, \ldots, U_{2m} \), each \( U_\nu \) being bordered by a pair of \( \gamma \)-curves intersecting at the origin where the angle between their tangential vectors is \( \frac{\pi}{m} \). Thus, every \( U_\nu \) is a simply connected real analytic sector.

Let us now consider a finite family of (distinct) harmonic functions \( H_1, \ldots, H_k \) in \( D \) satisfying \( H_\nu(0) = 0 \) for all \( \nu \). Set

\[
\Gamma = \bigcup_{i \neq \nu} \{ H_i - H_\nu = 0 \}.
\]

Applying the previous observation to all pairs \( (H_i, H_\nu) \) it follows that \( \Gamma \) is a finite union of smooth real analytic curves \( \gamma_1, \ldots, \gamma_M \) such that they all pass through the origin and are pairwise disjoint in the punctured disk

\[
\hat{D} = \{ (x, y) \}.
\]

Of course, in general one must shrink \( D \) to achieve this. Thus, provided that \( D \) is sufficiently small, \( D \setminus \Gamma \) is a union of pairwise disjoint real analytic sectors, each of which is bordered by two "half-curves" coming from the above family of \( \gamma \)-curves.

Notice that no special assumptions are imposed on the gradient vectors of the \( H \)-functions at the origin. For example, they may all be zero. It may therefore occur that some of the real analytic sectors \( \Omega \) are bordered by a pair of \( \gamma \)-curves which do not intersect transversally at the origin. Up to a conformal map a typical topological picture is that a real analytic sector is given by

\[
\Omega = \{ (x, y) : 0 < x < \delta, 0 < y < \rho(x) \},
\]

where \( \rho(x) \) is a positive real analytic function on \((0, \delta)\) and there exists a holomorphic function \( g \in D \) such that \( \Re(g(x, \rho(x))) = 0 \).

### 2.6. Local subharmonic configurations.

Given an open disk \( D \) centered at the origin and a \( k \)-tuple of harmonic functions \( H_1, \ldots, H_k \) as above, we consider some \( V \in \text{SH}_0(D) \) which is piecewise harmonic with respect to this \( k \)-tuple. Lemma 2.1 implies that \( \text{supp}(\Delta(V)) \) is contained in the set \( \Gamma \) defined at the beginning of this section. Hence, if \( \omega_1, \ldots, \omega_N \) are the real analytic sectors whose union is \( D \setminus \Gamma \), we find for each \( \omega_\nu \) some \( 1 \leq j(\nu) \leq k \) such that \( V = H_{j(\nu)} \) in \( \omega_\nu \).

Next we describe the positive measure \( \Delta(V) \). Outside the origin it is supported by (a subset of) \( \Gamma \) and Proposition 2.3 shows that if one has two adjacent \( \omega \)-sectors, say \( \omega_1, \omega_2 \) with \( j(1) \neq j(2) \), then the portion of \( \Delta(V) \) supported by the real analytic curve \( \gamma = \partial \omega_1 \cup \partial \omega_2 \) is the positive measure

\[
[\partial_{n_\gamma} H_{j(1)} - \partial_{n_\gamma} H_{j(2)}] \cdot ds_\gamma,
\]

where \( ds_\gamma \) is arc-length measure and \( n_\gamma \), is the normal to \( \gamma \) directed into \( \omega_1 \) when \( H_{j(1)} > H_{j(2)} \) holds in \( \omega_1 \) while \( n_\gamma \), changes sign and is directed into \( \omega_2 \) if it happens
that $H_{j(2)} > H_{j(1)}$ holds in $\omega_1$, see Remark 2.4. There remains to show that $\Delta(V)$ cannot contain a point mass at the origin. For this, we construct the logarithmic potential $W$ of $\mu = \Delta(V)\partial\Omega$. Note that since $\Delta(V)\partial\Omega$ is a locally real-analytic density on real-analytic curves $W$ is a continuous and bounded subharmonic function and $V - W$ is harmonic outside the origin. So if $\Delta(V)$ has a point mass at the origin, there exists a constant $a > 0$ such that $V = a\log(|z|) + W + G$, where $G$ is harmonic in $D$. This is impossible since $V$ is a bounded function in the punctured open disk $D$. We conclude that $V$ can be taken as a continuous function, i.e., we have proved:

2.7. Theorem. Every piecewise harmonic subharmonic function is continuous.

Let us summarize our results so far. Given (distinct) harmonic functions $H_i$ and a subharmonic function $V$ which is piecewise harmonic with respect to this family, the following holds if the disk $D$ (centered at the origin) is sufficiently small:

2.8. Theorem. There exists a finite family of disjoint real analytic sectors, say $\omega_1, \ldots, \omega_m$ such that for each $1 \leq i \leq m$ one has

$$V|_{\omega_i} = H_{j(i)}, \quad 1 \leq j(i) \leq k.$$ 

Moreover, when $1 \leq i \leq m - 1$ a half-arc $\gamma_i$ from the level set $\{H_{j(i+1)} = H_{j(i)}\}$ borders $\omega_{i+1} \cap \omega_i$ outside the origin and here one has the strict inequality

$$\partial_{n_i} H_{j(i+1)} > \partial_{n_i} H_{j(i)},$$

where $n_i$ is the normal to $\gamma_i$ directed into $\omega_{i+1}$. When $i = m$ one returns from $\omega_m$ to $\omega_1$ and here one has $\partial_{n_m} H_{j(1)} > \partial_{n_m} H_{j(m)}$, where $n_m$ is the normal to a half-arc $\gamma_m$ of the level set $\{H_{j(m)} = H_{j(1)}\}$ which is directed into $\omega_1$. Finally, the measure $\Delta(V)$ is given by

$$\Delta(V) = \sum_{i=1}^{m-1} [\partial_{n_i} H_{j(i+1)} - \partial_{n_i} H_{j(i)}] \cdot d_{\gamma_i} s + [\partial_{n_m} H_{j(1)} - \partial_{n_m} H_{j(m)}] \cdot d_{\gamma_m} s.$$

2.9. A non-transversal case. Let $H_1, \ldots, H_k$ be harmonic in an open disk $D$ centered at the origin. Assume that they are all zero at the origin and their gradients there satisfy

$$\nabla(H_i) = (0, b_i), \quad b_1 < \ldots < b_k.$$ 

In this case the above results give a transparent description of all subharmonic configurations with respect to this $k$-tuple in a small neighborhood of the origin. Indeed, let $V$ be such a subharmonic configuration. Assume that the closure of the two sets $U_1 = \{V = H_1\}$ and $U_k = \{V = H_k\}$ both contain the origin. Theorem 2.8 implies that $U_1$ contains a sector of the form

$$\Omega_+ = \{(x, y): \quad 0 < x < \delta, \quad |y| < ax\}$$ 

for some appropriate $a, \delta > 0$. Similarly, $U_k$ contains a sector $\Omega_-$ where $x < 0$ and $y < a|x|$. In the upper semi-disk $D_+$ where $y > 0$ we have smooth half-arcs

$$\gamma^+_i = \{(x, y): \quad H_i(x, y) = H_1(x, y), \quad y > 0\}$$

and similar half-arcs $\gamma^-_i$ in the lower semi-disk $D_-$. With these notations we have:

2.10. Theorem. Let $V$ be a subharmonic configuration such that $U_1$ and $U_k$ contain $(0,0)$. There exist integers $m, n \geq 2$, a pair of sequences $1 = j_1^+ < \ldots < j_m^+ = k,$
1 = j_1^- < \ldots < j_n^- = k, and some \( \delta > 0 \) such that
\[
V|_{D_{+}(\delta)} = \max(H_{j_1^+}, \ldots, H_{j_n^+}) \quad \text{and} \quad V|_{D_{-}(\delta)} = \max(H_{j_1^-}, \ldots, H_{j_n^-}).
\]
Conversely, every such pair of \( j \)-sequences yields a subharmonic configuration.

2.11. Example. Let \( k = 3 \), \( H_1(x, y) = 0 \), \( H_2(x, y) = 4x + x^2 - y^2 \), \( H_3(x, y) = -x \).

There are three level curves through \((0, 0)\) to functions of the form \( H_i - H_j \) with \( i \neq j \). These are depicted in Figure 1 below.

![Figure 1. Maximal and non-maximal subharmonic configurations.](image)

Here we get three different subharmonic configurations (when the origin also belongs to the closure of \( \{V = H_2\} \)), one of these configurations being \( \max(H_1, H_2, H_3) \).

The function in the figure closest to the origin in each sector is the restriction of \( V \) to that sector.

### 3. Piecewise Holomorphic Functions

It suffices to prove Theorem 1 locally, i.e., we can restrict our attention to an open neighborhood of the origin where \( \partial V/\partial z \) is piecewise holomorphic with respect to some \( k \)-tuple \( g_1, \ldots, g_k \).

The neighborhood in question is chosen as an open square
\[
\square = \{(x, y) : -a < x, y < a\}.
\]

We shall first consider the case when the following holds in \( \square \):

\[
\Re(g_1) < \ldots < \Re(g_k).
\]

Given a \( k \)-tuple of constants \( c_1, \ldots, c_k \) there exist harmonic functions \( H_1, \ldots, H_k \) in \( \square \) such that \( \partial H_\nu/\partial z = g_\nu \) and \( H_\nu(0) = c_\nu \), \( 1 \leq \nu \leq k \).

3.1 Proposition. If (**) holds there exists an open disk \( D \) centered at \((0, 0)\) such that \( V|_{D} \) is piecewise harmonic with respect to \( H_1, \ldots, H_k \) up to additive constants.

The proof requires several steps. Set \( K = \text{supp}(\Delta(V)) \) and let \( U_k \) be the subset of \( \square \setminus K \) where \( \partial V/\partial z = g_k \). Without loss of generality we may assume that all \( g \)-functions are active in the sense that the sets \( \{\partial V/\partial z = g_\nu\} \) contain points arbitrarily close to the origin for each \( \nu \). Theorem 1.2 applied to the subharmonic function \( V - H_k \) gives \( U_k = s^\dagger(U_k) \), where \( s^\dagger(U_k) \) is the forward star domain of \( U_k \).
as defined in Section 1. Since the origin by assumption belongs to $\bar{U}_k$ this equality yields

$$U_k = \{(x, y) \in \Box : x > \rho^*(y)\},$$

where $\rho^*(0) = 0$. Moreover, as explained in Proposition 1.4, $\rho^*$ is Lipschitz continuous if we from the start shrink $\Box$ a bit so that the inequality

$$\Re(e^{i\theta}g_{\nu}) < \Re(e^{i\theta}g_k), \quad -\theta_0 < \theta < \theta_0,$$

holds in $\Box$ for some $\theta_0 > 0$ and every $1 \leq \nu \leq k - 1$.

Since $U_k$ is connected $V - H_k$ is a constant function in $U_k$. We choose $c_k$ above so that $V = H_k$ holds in $U_k$. Reversing signs in the Key Lemma 1.1 and considering the subharmonic function $H_1 - V$ it follows that if we set $U_1 = \{\partial V/\partial z = g_1\}$ then

$$U_1 = \{(x, y) \in \Box : x < \rho_*(y)\},$$

where $\rho_*$ is also Lipschitz continuous. Since $U_1$ is connected $V - H_1$ is a constant function in $U_1$ and we choose $c_1$ such that $V = H_1$ holds in $U_1$.

To complete the proof of Proposition 3.1 we proceed by induction over $k$. Consider first the case $k = 2$. Then $\rho^* = \rho_*$ and by Lipschitz continuity the curve $x = \rho^*(y)$ is a null set. We conclude that $V = \max(H_1, H_2)$ in a small disk centered at the origin, as required.

The case $k \geq 3$. Since we only assume that $K = \text{supp}(\Delta(V))$ is a null set it is \textit{à priori} not clear why the open subsets $U_\nu$ of $\Box \setminus K$ where $\partial V/\partial z = g_\nu$ have a finite number of connected components when $2 \leq \nu \leq k - 1$. To prove this we shall consider the real analytic curve $\Gamma = \{H_1 = H_k\}$. Since $\partial_x H_1 < \partial_x H_k$ this curve is defined by an equation of the form $x = \rho(y)$, where $\rho$ is real analytic. Moreover, it is obvious that

$$\rho_*(y) \leq \rho(y) \leq \rho^*(y)$$

in some sufficiently small interval $-y_0 < y < y_0$.

The $\Gamma$-curve is oriented by increasing $y$. The tangential derivatives $\partial_T H_\nu$ are real analytic functions on $\Gamma$ for each $\nu$. We shall first consider these tangential derivatives along the portion of $\Gamma$ where $y > 0$. Since the zero set of a real analytic function is discrete, it follows that if $\Box$ is if necessary decreased a bit (i.e., for a small enough) then for every $2 \leq \nu \leq k - 1$ the function

(i) \hspace{1cm} $y \mapsto \partial_T H_\nu(\rho(y), y) - \partial_{\nu} H_\nu(\rho(y), y), \quad 0 < y < a,$

is either identically zero or else strictly monotone, i.e., strictly increasing or decreasing. Similarly, there exists a permutation of $\{2, \ldots, k - 1\}$ such that

(ii) \hspace{1cm} $\partial_T H_\nu(\rho(y), y) \leq \ldots \leq \partial_{\nu} H_\nu(\rho(y), y), \quad 0 < y < a,$

where $\leq$ means that we either have equality on the whole portion of $\Gamma$ or a strict inequality.

3.2. The Non-Return Lemma. If there exists some $\delta > 0$ such that $\rho_*(y) = \rho^*(y)$ holds for $0 \leq y < \delta$ then $V$ restricted to the rectangle $\{-a < x < a, 0 < y < \delta\}$ is equal to $\max(H_k, H_1)$ and we are done. Next, we consider the situation when no such $\delta$ exists.

3.3. Lemma. Assume that $\rho^* - \rho_*$ is not identically zero on some interval $[0, \delta)$, that strict inequalities hold in (ii) and that the function in (i) is strictly monotone for some $\delta > 0$. Then there exists $0 < \delta_0 < \delta$ such that $\rho_*(y) < \rho^*(y)$ for $0 < y < \delta_0$. 
Proof. If the assertion is not true there exists a sequence of disjoint intervals \( J_\nu = (\alpha_\nu(\nu), \alpha^*(\nu)) \) on the positive \( y \)-axis which decrease to \( y = 0 \) as \( \nu \to \infty \) and such that at the end-points one has

\[ \rho_\nu(\alpha_\nu(\nu)) = \rho^*(\alpha_\nu(\nu)) \quad \text{and} \quad \rho_\nu(\alpha^*(\nu)) = \rho^*(\alpha^*(\nu)) \]

while \( \rho_\nu(y) < \rho^*(y) \) holds inside every \( J \)-interval. For any \( \nu \) we consider the domain

\[ \Omega_\nu = \{(x, y) : \rho_\nu(y) < x < \rho^*(y) \text{ and } \alpha_\nu(\nu) < y < \alpha^*(\nu)\} \]

Inside each \( \Omega_\nu \) we notice that \( \partial V/\partial z \) is piecewise holomorphic with respect to \( g_2, \ldots, g_k \). By the induction assumption we may assume that \( V \) is locally piecewise harmonic in \( \Omega_\nu \) with respect to \( H_2, \ldots, H_{k-1} \) up to additive constants. Therefore, when \( \alpha_\nu(\nu) < y < \alpha^*(\nu) \) is kept fixed the function \( x \mapsto V(x, y) \) is piecewise real analytic and \( \partial_x V \) is equal to some \( \partial_x H_{\nu} \) with \( 2 \leq \nu \leq k - 1 \) outside a discrete set. Since \( \partial_x H_{\nu} < \partial_x H_1 \) for each such \( \nu \), we conclude that the function

\[ x \mapsto H_1(x, y) - V(x, y) \]

is strictly increasing. In the same way we find that

\[ x \mapsto H_k(x, y) - V(x, y) \]

is strictly decreasing. Using this we conclude that the \( \Gamma \)-curve passes through \( \Omega_\nu \), i.e., we must have

\[ \rho_\nu(y) < \rho(y) < \rho^*(y) \quad \text{and} \quad \alpha_\nu(\nu) < y < \alpha^*(\nu). \]

Here \( \rho^* = \rho = \rho_\nu \) at the end-points of \( J_\nu \). Set

\[ p_\nu = (\rho(\alpha_\nu(\nu)), \alpha_\nu(\nu)) \quad \text{and} \quad q_\nu = (\rho(\alpha^*(\nu)), \alpha^*(\nu)). \]

Sublemma. There cannot exist a single index \( 2 \leq j \leq k - 1 \) and a constant \( c_j \) such that \( V = H_j + c_j \) holds along \( \Omega_\nu \cap \Gamma \).

Proof. If this occurs we get a contradiction as follows. First, by the induction over \( k \) the restriction of \( V \) to \( \Omega_\nu \) is piecewise harmonic which yields a uniform bound for \( \partial_k(V) \). Moreover, when \( \alpha_\nu(\nu) < y < \alpha^*(\nu) \) we encounter the point \( p = (\rho^*(y), y) \) where there exists an open neighborhood such that \( \partial V/\partial z \) is piecewise holomorphic with respect to \( g_2, \ldots, g_k \). So by an induction over \( k \) it follows that \( V \) is piecewise harmonic in a neighborhood of this point hence also continuous by Theorem 2.7.

Next, from the uniform bound of \( \partial_x V \) we have a constant \( C \) which can be taken as the maximum over sup-norms of \( \{\partial_x H_{\nu}\}_{2}^{k-1} \) in a fixed neighborhood of the origin and get

\[ |V(\rho(y), y) - V(\rho^*(y), y)| \leq C \cdot |\rho^*(y) - \rho(y)| \quad \text{for all } \alpha_\nu(\nu) < y < \alpha^*(\nu). \]

Here \( V(\rho^*(y), y) = H_k(\rho^*(y), y) \). So if \( V = H_j + c_j \) is valid on \( \Omega_\nu \cap \Gamma \) for some constant \( c_j \) we obtain

\[ |H_j(\rho(y), y) + c_j - H_k(\rho^*(y), y)| \leq C \cdot |\rho^*(y) - \rho(y)| \quad \text{whenever } \alpha_\nu(\nu) < y < \alpha^*(\nu). \]

Passing to the limit as \( y \to \alpha^*(\nu) \) or \( y \to \alpha_\nu(\nu) \) we conclude that one has the two equalities:

\[ H_j(\rho(\alpha_\nu(\nu)), \alpha_\nu(\nu)) + c_j = H_k(\rho(\alpha_\nu(\nu)), \alpha_\nu(\nu)), \]

\[ H_j(\rho(\alpha^*(\nu)), \alpha^*(\nu)) + c_j = H_k(\rho(\alpha^*(\nu)), \alpha^*(\nu)). \]
These two identities cannot hold if $H_k - H_j$ is strictly monotone along $\Gamma$. So there remains only the possibility that $H_k - H_j$ is constant along $\Gamma$. But this again gives a contradiction. For then we get

\begin{equation}
V(z) = H_j(z) + c_j = H_k(z), \quad z \in \Omega_\nu \cap \Gamma.
\end{equation}

Now the domain $\Omega_\nu$ is bordered by the two simple curves

\[ \gamma_* = \{(x,y) : x = \rho_*(y)\} \quad \text{and} \quad \gamma^* = \{(x,y) : x = \rho^*(y)\} \]

where the inequalities $\alpha_*(\nu) \leq y \leq \alpha^*(\nu)$ hold. Since $\partial_x H_k > \partial_z H_1$ we have

\begin{equation}
H_1(\rho_*(y), y) < H_k(\rho^*(y), y), \quad \alpha_*(\nu) < y < \alpha^*(\nu).
\end{equation}

The subharmonic function $V$ is equal to $H_1$ on $\gamma_*$ and it equals $H_k$ on $\gamma^*$, so it follows from (11) that we must have $V < H_k$ inside the domain $\Omega_\nu \cap \Gamma$. This contradicts equality (10) and the Sublemma is proved.

\begin{proof}[Proof of Lemma 3.3, continued.]
The Sublemma shows that the locally piecewise harmonic function $V$ inside $\Omega_\nu$ must have at least one jump along $\Gamma \cap \Omega_\nu$, say from from $H_j + c_j$ to $H_i + c_i$ for some indices $i < j$ in $\{2, \ldots, k-1\}$. In other words, for some $\alpha_*(\nu) < y_0 < \alpha^*(\nu)$ there exists a small $\epsilon > 0$ and constants $c_i, c_j$ such that

\begin{align*}
V(\rho(y), y) &= H_i(\rho(y), y) + c_i, \quad y' - \epsilon < y < y', \\
V(\rho(y), y) &= H_j(\rho(y), y) + c_j, \quad y' < y < y' + \epsilon.
\end{align*}

By Proposition 2.3 and the strict monotonicity of the sequence formed by the $\partial_\gamma$-derivatives of $H_2, \ldots, H_{k-1}$ in (ii) preceding the Non-return Lemma it follows that

\[ \partial_\gamma H_j > \partial_\gamma H_i \]

on the whole of $\Gamma$. This is true for $\partial_\gamma$-derivatives whenever a jump occurs in some domain $\Omega_\nu$. Hence, by the fact that the sequence in (ii) is strictly increasing we cannot return to some $H_i$-function at a later stage if this function appears in some $\Omega_\nu$-domain encountered previously. Therefore $V|_{\Omega_\nu \cap \Gamma}$ can jump for at most $k-2$ values of $\nu$. On the other hand, by the Sublemma a jump must always occur. We conclude that the infinite sequence of intervals $\{J_\nu\}$ tending to $y = 0$ as $\nu \to \infty$ cannot exist. This proves Lemma 3.3.
\end{proof}

### 3.4. Completing the proof of Proposition 3.1.

Ignoring the case when $\rho_*(y) = \rho^*(y)$ in some interval $(0, \delta)$, in which case the equality $V = \max(H_1, H_2)$ holds in a small rectangle

\[ \square_0 = \{(x,y) : -a < x < a, \ 0 < y < b\} \]

we have some positive $\delta_0$ from Lemma 3.3. Set

\[ \Omega_0 = \{(x,y) : -a < x < \rho^*(y), \ 0 < y < \delta_0\} \]

In this domain $\partial V/\partial z$ is piecewise holomorphic with respect to the $k-1$-tuple $g_1, \ldots, g_{k-1}$. By an induction over $k$ we may therefore assume that $V|_{\Omega_0}$ is locally piecewise harmonic with respect to $H_1, \ldots, H_{k-1}$ up to additive constants. Assume that $U_{k-1} \cap \Omega_0 \neq \emptyset$. Applying Theorem 1.2 it follows that $U_{k-1} \cap \Omega_0 = s^1(U_{k-1} \cap \Omega_0)$. This gives a function $\rho_1(y)$ such that

\[ U_{k-1} \cap \Omega_0 = \{(x,y) : x > \rho_1(y), \ 0 < y < \delta_0\} \]

and there exists some $\delta_1 \leq \delta_0$ such that

\[ \rho_*(y) \leq \rho_1(y) < \rho^*(y), \quad 0 < y < \delta_1. \]
Since \( U_{k-1} \cap \Omega_0 \) is connected and \( H_{k-1} \) only has to be determined up to a constant we can assume that \( V = H_{k-1} \) in \( U_{k-1} \) and then \( H_{k-1}(\rho^*(y), y) = H_k(\rho^*(y), y) \) must hold when \( 0 < y < \delta_1 \), which entails that \( H_{k-1}(0, 0) = H_k(0, 0) \). For the next step we consider the domain

\[
\Omega_1 = \{(x, y) : \ -a < x < \rho_1(y), 0 < y < \delta_1 \}.
\]

Again, if the closure of \( U_{k-2} \cap \Omega_1 \) contains the origin, then it is equal to its forward star domain, which gives a function \( \rho_2(y) \) and some \( 0 < \delta_2 < \delta_1 \) such that

\[
U_{k-2} \cap \Omega_1 = \{(x, y) : x > \rho_2(y), 0 < y < \delta_2 \}.
\]

If it happens that the \( g_{k-2} \)-function is non-active, i.e., the closure of \( U_{k-2} \cap \Omega_1 \) is empty, we get a similar conclusion by taking the largest integer \( m \leq k - 3 \) such that the closure of \( U_m \cap \Omega_1 \) contains the origin. We can continue in this way and arrive at the following result, where we use the notation

\[
\square^+(a, \delta) = \{(x, y) : \ -a < x < a, 0 < y < \delta \}.
\]

### 3.5. Proposition

There exist a strictly increasing sequence \( 1 = j_1 < \ldots < j_m = k \) and \( a, \delta > 0 \) such that if for each \( 2 \leq i \leq m - 1 \) we set

\[
\Omega_i = \{(x, y) : H_{j_{i-1}}(x, y) < H_{j_i}(x, y) < H_{j_{i+1}}(x, y) \} \cap \square^+(a, \delta)
\]

then \( V = H_{j_i} \) in \( \Omega_i \) for \( 2 < i \leq m - 1 \) while \( V = H_k \) when \( H_k(x, y) > H_{j_{m-1}}(x, y) \) and \( V = H_1(x, y) \) if \( H_1(x, y) < H_{j_2}(x, y) \).

**Remark.** A simpler way to express the above result is that in \( \square^+(a, \delta) \) we have the equality \( V = \max(H_{j_1}, \ldots, H_{j_m}) \).

We can then proceed in exactly the same way in the lower half-disk where \( y < 0 \) and obtain another J-sequence. From this we conclude that if the resulting \( \square^+ \) (which is a neighborhood of the origin) is sufficiently small, then \( U_{\nu} \cap \square^+ \) has at most two connected components when \( 2 \leq \nu \leq k - 1 \) and is connected when \( \nu = 1 \) or \( k \). This proves that \( V \) is piecewise harmonic with respect to \( H_1, \ldots, H_k \) in \( \square^+ \), and completes the whole proof of Proposition 3.1.

### 3.6. Proof of Theorem 1

Given a domain \( \Omega \) and some \( k \)-tuple \( g_1, \ldots, g_k \) in \( \mathcal{O}(\Omega) \) we have the discrete set

\[
\sigma = \bigcap_{\nu \neq j} \{g_\nu = g_j\}.
\]

If \( z_0 \in \Omega \setminus \sigma \) there exists some \( \theta \) such that the sequence \( \{\text{Re}(e^{i\theta} g_\nu)(z_0)\}_i \) consists of distinct real numbers. Up to a rotation we have the same local situation as in Proposition 3.1. Hence \( V \) is locally piecewise harmonic in \( \Omega \setminus \sigma \). There remains only to study \( V \) close to a single point \( z_0 \) in \( \sigma \) and establish that it is locally piecewise harmonic in a neighborhood of \( z_0 \). Working locally we may take \( z_0 \) as the origin and in a disk \( D \) centered at \( (0, 0) \) we have the open subsets \( U_\nu = \{\partial V / \partial z = g_\nu\} \) of \( D \setminus \text{supp}(\Delta(V)) \). Here the situation is more favorable than previously since we already know that \( V \) is locally piecewise harmonic in the punctured disk \( D \). Moreover, to prove that \( V \) is locally piecewise harmonic in a neighborhood of the origin it suffices to find some small \( \delta > 0 \) such that the number of connected components of each \( U_\nu \cap D(\delta) \) is finite for all \( 1 \leq \nu \leq k \), where \( D(\delta) \) is the open disk of radius \( \delta \) centered at \( (0, 0) \). To achieve this we will decompose small discs into a finite number of real analytic sectors \( \{\Omega_\alpha\} \) and prove that \( U_\nu \cap \Omega_\alpha \) is empty or connected for each \( \nu \) and \( \alpha \). For if this is done then we may remove the union
of real analytic curves which border these sectors without affecting the situation since this union is a null set, i.e., the locally integrable subharmonic function $V$ is not changed by such a removal.

After these preliminary remarks we begin to construct suitable $\Omega$-sectors. Consider the harmonic functions $\partial_x H_i - \partial_x H_\nu$ for pairs $i \neq \nu$. Notice that such a function is identically zero if and only if $H_i = H_\nu + cy$ for some constant $c$. After a rotation we may assume that this never happens and get the real analytic set

$$\Gamma = \bigcup_{i \neq \nu} \{ \partial_x H_i = \partial_x H_\nu \}$$

which is described in §2.5. So when $\delta > 0$ is sufficiently small then $D(\delta) \setminus \Gamma$ is a disjoint union of real analytic sectors $\{\Omega_\alpha\}$. It may occur that some sector contains a real line segment $0 < x < \delta$ or $-\delta < x < 0$. Apart from this case a typical sector is given by

$$\Omega = \{(x,y): \rho_\ast(y) < x < \rho^\ast(y), 0 < y < \delta\}$$

or by a similar sector in the lower half-disk where $-\delta < y < 0$. To handle sectors that may potentially contain a line segment on the $x$-axis we can simply replace $x$ by $y$ in the arguments above and start with the real analytic set

$$\Gamma_1 = \bigcup_{i \neq \nu} \{ \partial_y (H_i) = \partial_y (H_\nu) \}.$$  

Then we again obtain a finite number of real analytic sectors where those which contain a line segment on the $x$-axis are defined by

$$\{(x,y): 0 < x < \delta, \rho_\ast(x) < y < \rho^\ast(x)\}.$$  

Replacing $x$ by $y$ in Proposition 3.1 if necessary, we conclude that the proof of Theorem 1 is finished if we can show the following:

**3.7. Proposition.** Suppose $\partial_x H_1 < \ldots < \partial_x H_k$ holds in $\Omega$, where $\Omega$ is a real analytic sector of the form

$$\{(x,y): \rho_\ast(y) < x < \rho^\ast(y), 0 < y < \delta_0\}.$$  

Then there exists $0 < \delta < \delta_0$ such that if $\Omega(\delta) = \Omega \cap \{(x,y): 0 < y < \delta\}$ then $U_\nu \cap \Omega(\delta)$ is connected or empty for every $\nu$.

**Proof.** Arguments similar to those used in the proof of Proposition 3.1 yield the desired result. \qed

We have thereby completed the proof of Theorem 1.

### 4. On Algebraic Root Functions: Proof of Theorem 2

**4.1. A result inside sectors.** As preparation for the proof of Theorem 2 we first prove another result (Theorem 4.3 below) where the harmonic functions under consideration are only defined in a real analytic sector. Let

$$\Omega = \{(x,y): \rho_\ast(y) < x < \rho^\ast(y), 0 < y < \delta_0\}$$  

be a real analytic sector and suppose that there are functions $\rho(y) < \rho_\ast(y)$ and $\rho_1(y) > \rho^\ast(y)$ defining a larger sector

$$\Omega^* = \{(x,y): \rho(y) < x < \rho_1(y), 0 < y < \delta_0\}.$$
In $\Omega^*$ one is given a subharmonic function $V$ such that $\partial V/\partial z$ is piecewise holomorphic with respect to some $k$-tuple $g_1, \ldots, g_k$ in $\mathcal{O}(\Omega^*)$. Since $\Omega^*$ is simply connected we have also a corresponding $k$-tuple of harmonic functions $H_1, \ldots, H_k$ in $\Omega^*$. Next, assume that Theorem 1 holds in this situation, i.e., that $V$ is locally piecewise harmonic inside $\Omega^*$ with respect to the above $H$-functions plus constants. In addition to this assumption we impose the following:

4.2. Condition on $\partial_\Gamma$-derivatives. For each pair $i \neq \nu$ and each constant $c$ set

$$\Gamma(i, \nu, c) = \{H_i - H_\nu = c\} \cap \Omega.$$ 

For every such real analytic curve we require that there exists $\delta > 0$ and some permutation of $\{1, \ldots, k\}$ such that the inequalities

$$\partial_{\Gamma(i, \nu, c)} H_{j(1)} \leq \cdots \leq \partial_{\Gamma(i, \nu, c)} H_{j(k)}$$

hold in

$$\Gamma(i, \nu, c) \cap \{(x, y) : 0 < y < \delta\}.$$ 

Moreover, we require that there exist index permutations so that these inequalities hold for the tangential $H$-derivatives along the two real analytic curves $\{x = \rho_*(y)\}$ and $\{x = \rho^*(y)\}$.

4.3. Theorem. Under the aforementioned conditions there exist $\delta > 0$ and an increasing integer sequence $1 \leq j_1 < \cdots < j_m \leq k$ such that $V = \max(H_{j_1}, \ldots, H_{j_m})$ in $\Omega \cap \{(x, y) : 0 < y < \delta\}$.

Proof. Follows by repeated use of Theorem 1 and arguments similar to those used in the proof of Proposition 3.1. $\square$

4.4. As further preparation for the proof of Theorem 2 we need some results about root functions which arise as follows. Let

$$f(z, y) = q_k(z)y^k + \cdots + q_1(z)y + q_0(z)$$

be a polynomial in $y$ with coefficients $q_\nu \in \mathcal{O}(D)$, where $D$ is an open disk centered at the origin. We assume that $f$ has no multiple factors and get the factorization

$$f(z, y) = q_k(z) \cdot \prod_{\nu=1}^{k} (y - \alpha_\nu(z)),$$

where the $\alpha$-functions in general are multi-valued in the punctured disk $\bar{D}$. Set

$$\Gamma = \bigcup_{\nu \neq i} \{\text{Re}(\alpha_i) - \text{Re}(\alpha_\nu) = 0\}.$$ 

The real analytic set $\Gamma$ is to begin with only defined in $\bar{D}$. Nevertheless, it extends to the whole disk $D$ and becomes a union of smooth real analytic curves passing through the origin. To see this we recall the classical Normalisation Theorem saying that there exists an integer $M$ such that if $\rho: \zeta \mapsto \zeta^M$ then $\alpha^*_\nu := \alpha \circ \rho$ becomes meromorphic in a disk of the $\zeta$-plane. In this $\zeta$-disk we get the set

$$\Gamma^* = \bigcup_{\nu \neq i} \{\text{Re}(\alpha^*_i) - \text{Re}(\alpha^*_\nu) = 0\}.$$
which is a disjoint union of smooth real analytic curves, hence so is the image \( \Gamma = \rho(\Gamma^*) \). Next we consider the upper half-disk \( D^+ \) where \( y > 0 \). Here we find single-valued branches of the root functions and consider their primitives

\[
A_\nu(z) = \int_z^p \alpha_\nu(w)dw,
\]

where the complex line integrals start from some \( p = ai \) with a small \( a > 0 \). In the Puiseux expansions of root functions it may occur that \( z^{-1} \) appears. So in \( D^+ \) we have

\[
A_\nu(z) = \lambda_\nu \cdot \text{Log}(z) + \sum \psi_{i,\nu}(z) \cdot z^{\nu/M},
\]

where the \( \psi \)-functions are meromorphic in \( D \) and the \( \lambda_\nu \)'s are complex numbers.

Note that any difference \( A_i - A_\nu \) has a similar expansion. Given some constant \( c \) we use polar coordinates \( z = re^{i\theta} \) to express a level curve as

\[
\Gamma = \{ \text{Re}(A_i - \text{Re}(A_\nu) = c \} = \{(r, \theta): u \cdot \text{Log}(r) - v \cdot \theta + \text{Re}(\Phi)(r, \theta) = c \},
\]

where \( u, v \) are real constants and \( \Phi(r, \theta) = M^{-1} \sum_{j=0}^{M-1} r^{j/N} \cdot e^{j\theta/M} \cdot \phi_{\nu}(r, \theta) \)

with \( \phi_0, \ldots, \phi_{M-1} \) meromorphic in \( D \).

4.5. Tangential derivatives. In \( D^+ \) we get harmonic functions \( H_\nu = \text{Re}(A_\nu) \) which for each \( \nu \) give \( \partial H_\nu / \partial z = \alpha_\nu \). Along a level curve \( \Gamma \) as above we consider a difference

\[
\partial_\Gamma H_m - \partial_\Gamma H_\ell, \quad 1 \leq m, \ell \leq k.
\]

Now we want to prove:

4.6. Proposition. Unless \( \partial_\Gamma H_m - \partial_\Gamma H_\ell \) is identically zero, there exists \( \delta > 0 \) such that this difference is non-vanishing in \( \Gamma(\delta) = \Gamma \cap D(\delta) \).

Proof. If \( p \in \Gamma \) we notice that this difference is zero at \( p \) if and only if

\[
\Im \left( \frac{\alpha_m - \alpha_\ell}{\alpha_i - \alpha_\nu} \right) = 0, \quad i \neq \nu.
\]

The function \( G := \frac{\alpha_m - \alpha_\ell}{\alpha_i - \alpha_\nu} \) has a Puiseux series expansion in \( D^+ \):

\[
G(z) = \sum_{\nu=0}^{M-1} g_\nu(z) \cdot z^{\nu/M}.
\]

Hence there only remains to show:

Sublemma. There exists \( \delta > 0 \) such that \( p \in \Gamma(\delta) \Rightarrow \Im(G)(p) \neq 0 \).

Proof. We use the existence of a holomorphic map \( \gamma \) from a complex \( w \)-disk onto \( D \) such that \( G \circ \gamma(w) = w^N \) holds for some integer \( N \). Here \( \{\Im(G \circ \gamma) = 0\} \) is a union of lines given by \( \arg(w) = m\pi/N \), \( 0 \leq m \leq 2N - 1 \). At the same time \( \Gamma \) is the image of a curve \( \Gamma^* \) in the \( w \)-disk defined by an equation of the form

\[
\Gamma^* = \{ w: \Re[\lambda \cdot \text{Log}(w) + S(w)] = c \},
\]

where \( S(w) \) is a meromorphic function. In polar coordinates in the \( w \)-disk, \( \Gamma^* \) is given by

\[
u \cdot \log(r) + \Re(S(re^{i\theta})) - v\theta - c = 0,
\]
where \( u, v \) are real constants. The Sublemma follows since on each line in the zero set of \( G \circ \gamma \) where the \( \theta \)-angle is fixed, say \( \theta = \theta_0 \), it is obvious that the function
\[ r \mapsto u \cdot \log(r) + \mathfrak{Re}(S(re^{i\theta_0})) - v\theta_0 - c \]
is non-vanishing for \( 0 < r < \delta \) if \( \delta \) is small enough, unless the function happens to be identically zero. This finishes the proof of Proposition 4.6. \( \Box \)

4.7. Proof of Theorem 2. Denote by \( \mathfrak{M}_{\text{alg}}^+ \) the class of probability measures \( \mu \) such that \( \text{supp}(\mu) \) is a compact null set and the Cauchy transform \( \hat{\mu}(z) \) satisfies an algebraic equation
\[ p_k(z) \cdot \hat{\mu}(z)^k + \ldots + p_1(z) \cdot \hat{\mu}(z) + p_0(z) = 0, \quad z \in \mathbb{C} \setminus \text{supp}(\mu), \]
where \( p_0(z), \ldots, p_k(z) \in \mathbb{C}[z] \). Set \( P(z, y) = p_k(z) \cdot y^k + \ldots + p_1(z) \cdot y + p_0(z) \), which we assume to be irreducible in \( \mathbb{C}[z, y] \). When the leading polynomial \( p_k(z) \neq 0 \) we have a factorization
\[ P(z, y) = p_k(z) \cdot \prod_{i=1}^{k} (y - \alpha_{\nu}(z)). \]
We also get the rational discriminant
\[ \mathcal{D}(z) = \prod_{\nu \neq j} (\alpha_{\nu}(z) - \alpha_j(z)). \]
Let \( \Sigma \) be the union of \( p_k^{-1}(0) \) and the zeros of \( \mathcal{D}(z) \) in \( \mathbb{C} \setminus p_k^{-1}(0) \). Thus, if \( U \) is a simply connected subset of \( \mathbb{C} \setminus \Sigma \) then the \( k \)-tuple of distinct \( \alpha \)-roots are analytic functions in \( U \) and there exists some \( 1 \leq i \leq k \) such that \( \hat{\mu} = \alpha_i \) in \( U \).

Consider now the subharmonic function
\[ V(z) = \int \log |z - \zeta| \cdot d\mu(\zeta). \]
Since \( \partial V/\partial z = \hat{\mu} \) we can apply Theorem 1 in the complement of \( \Sigma \). More precisely, if \( U \) as above is simply connected we find harmonic functions \( H_1, \ldots, H_k \) in \( U \) such that \( \partial_z(H_\nu) = \alpha_\nu \), \( 1 \leq \nu \leq k \), and the restriction of \( V\mid_U \) is locally piecewise harmonic with respect to this \( k \)-tuple up to additive constants.

Next, using Proposition 4.6 we see that the conditions on \( \partial_\Gamma \)-derivatives in §4.2 are satisfied when we consider suitable simply connected sectors around each individual point in \( \Sigma \). It follows again that the restriction of \( V \) to each such sector is piecewise harmonic with respect to \( H_1, \ldots, H_k \) up to constants. Applying Theorem 2.8 one finally arrives at Theorem 2.

5. Further Directions and Open Problems

5.1. Existence of measures in \( \mathfrak{M}_{\text{alg}}^+ \). Recall the class of probability measures \( \mathfrak{M}_{\text{alg}}^+ \) defined in §4.7. Consider a polynomial of the form
\[ P(y) = y + c_2 y^2 + \ldots + c_k y^k, \]
where \( k \geq 2 \). With \( z \) as a new independent complex variable we study the algebraic equation
\[ P(y) = \frac{1}{z}. \]
From (12) we see that if \( R \) is sufficiently large then there exists a single-valued analytic function \( \alpha^*(z) \) defined in the exterior domain \(|z| > R\) whose Laurent expansion is

\[
\alpha^*(z) = \frac{1}{z} + a_2 \cdot \frac{1}{z^2} + \ldots.
\]

Let us assume that the zeros of \( P(y) \) are simple. In the complex \( z \)-plane we get the finite set

\[
\sigma = \left\{ z = \frac{1}{P(\alpha)} : P'(\alpha) = 0 \right\}.
\]

Clearly, \( \sigma \) consists of \( k - 1 \) points outside the origin. Now \( \alpha^*(z) \) extends to an (in general multi-valued) analytic function defined in \( C \setminus (\sigma \cup \{0\}) \). By an analytic tree in \( C \) we mean a connected set \( \Gamma \) which is a finite union of simple and closed real analytic Jordan arcs and the open complement \( C \setminus \Gamma \) is connected. So by adding the point at infinity the domain \( \Omega_\Gamma = C \setminus \Gamma \) is simply connected. For every such tree \( \Gamma \) which contains the set \( \sigma \cup \{0\} \) the function \( \alpha^*(z) \) extends from the exterior disk \(|z| > R\) to a single-valued analytic function in \( \Omega_\Gamma \). We also get the Riesz measure \( \mu_\Gamma \) supported by \( \Gamma \) such that

\[
\alpha^*(z) = \hat{\mu}_\Gamma(z), \quad z \in \Omega \setminus \Gamma.
\]

Since \( \frac{1}{z} \) is the leading term in the Laurent expansion of \( \alpha^*(z) \) we see that

\[
(13) \quad \int_\Gamma d\mu_\Gamma = 1.
\]

The measure \( \mu_\Gamma \) is in general complex-valued. In fact, consider some relatively open Jordan arc \( \gamma \subset \Gamma \) which stays outside \( \sigma \cup \{0\} \). Along the two opposite sides of \( \gamma \) we have two branches \( \alpha_1(z) \) and \( \alpha_2(z) \) of \( \alpha^*(z) \). By a classic formula from analytic function theory the restriction of \( \mu_\Gamma \) to the Jordan arc \( \gamma \) is expressed by

\[
\frac{i}{\pi} \cdot [\alpha_2(z) - \alpha_1(z)] \cdot dz.
\]

To be precise, if \( f(z) \) is a continuous function whose compact support is disjoint from \( \Gamma \setminus \gamma \) then

\[
\int f \cdot d\mu = i \cdot \int_\gamma f(z) \cdot [\alpha_2(z) - \alpha_1(z)] \cdot dz.
\]

Notice that we can choose many different analytic trees \( \Gamma \) as above. For every such tree the total variation of \( \mu_\Gamma \) is \( \geq 1 \) by (13). We propose the following:

**Conjecture.** There exists a unique analytic tree \( \Gamma \) such that \( \mu_\Gamma \) is a probability measure, i.e., \( \mu_\Gamma \in \mathcal{M}^+_{\text{alg}} \).

**Example.** Consider the case \( P(y) = y^2 + y \). Here \( P'(y) = 2y + 1 \) and \( \sigma = \{-4\} \). Let \( \Gamma \) be the (analytic) tree given by the real interval \(-4 \leq x \leq 0\). On this interval we define the non-negative measure

\[
d\mu(x) = \frac{1}{2\pi} \cdot \frac{\sqrt{4+x}}{\sqrt{-x}}.
\]

Then we have

\[
\alpha^*(z) = \hat{\mu}_\Gamma(z)
\]

in the complement of \( \Gamma \).
5.2. Combinatorics of subharmonic configurations. Given a \( k \)-tuple of harmonic functions \( H_1, \ldots, H_k \) defined in an open connected set \( \Omega \subset \mathbb{C} \) it is clear that there are locally only finitely many subharmonic configurations with respect to this \( k \)-tuple. It is natural to ask for the exact number of such configurations locally at a point \( z \in \Omega \) in terms of the geometry of the convex hull of the gradients \( \nabla H_\nu(z) \), \( 1 \leq \nu \leq k \). Note that when all these gradients are extreme points of their convex hull Theorem 1.7 shows that there is only one possible configuration, namely the maximum of these harmonic functions. However, as seen in Example 2.11, in general there might be several such configurations. (Compare also Theorem 2.10.)

5.3. Plurisubharmonic configurations. An obvious question in this context is to try to extend some of our results to several variables, i.e., to study plurisubharmonic configurations with respect to given pluriharmonic functions.

5.4. Configurations induced by fundamental solutions. A further interesting direction is to consider any partial differential operator of elliptic type for which one can define appropriate analogs of subharmonic functions. There are well known examples of possible such operators in the literature – see, e.g., the subsolutions to elliptic differential equations in [8, Chapter 11] as well as [9, 10]. It is known that the maximum of subsolutions is a subsolution (cf., e.g., [3]), and hence one might for instance ask for conditions – similar to the ones in Theorem 1.5 – under which this is the only subsolution.

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