Compactification of closed preordered spaces

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Abstract

A topological preordered space admits a Hausdorff $T_2$-preorder compactification if and only if it is Tychonoff and the preorder is represented by the family of continuous isotone functions. We construct the largest Hausdorff $T_2$-preorder compactification for these spaces and clarify its relation with Nachbin’s compactification. Under local compactness the problem of the existence and identification of the smallest Hausdorff $T_2$-preorder compactification is considered.

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1. Introduction

A topological preordered space is a triple $(E, \mathcal{T}, \leq)$ where $(E, \mathcal{T})$ is a topological space and $\leq$ is a preorder on $E$, namely a reflexive and transitive relation on $E$. The preorder is an order if it is antisymmetric. There are many possible compatibility conditions between topology and preorder that can be added to this basic structure. We shall mainly consider the $T_2$-preordered spaces (closed preordered spaces), namely those spaces for which the graph

$$G(\leq) = \{(x, y) : x \leq y\},$$

is closed in the product topology $\mathcal{T} \times \mathcal{T}$ of $E \times E$. In this work we shall follow Nachbin’s terminology [22] but we remark that in computer science $T_2$-ordered spaces are very much studied and called pospaces.

A $T_2$-preordered space $E$ is a $T_1$-preordered space in the sense that for every $x \in E$, $i(x)$ and $d(x)$ are closed where $i(x) = \{y \in E : x \leq y\}$ is the increasing hull and $d(x) = \{y \in E : y \leq x\}$ is the decreasing hull.
We recall that an isotone function \( f : E \to \mathbb{R} \) is a function such that \( x \leq y \Rightarrow f(x) \leq f(y) \). We shall mostly work with continuous isotone functions with value in \([0,1]\), although we could equivalently work with bounded continuous isotone functions.

In this work we shall consider the problem of compactification for \( T_2 \)-preordered spaces. It is understood here that the compactification \( cE \) must be endowed with a preorder \( \leq_c \) which induces \( \leq \) on \( E \), namely if \( x, y \in E \), then \( x \leq y \) if and only if \( x \leq_c y \). The extended preorder is also demanded to be closed.

In the ordered case this problem has been solved by Nachbin who proved [4, 22, 23] that a topological ordered space admits a \( T_2 \)-order compactification if and only if it is a completely regularly ordered space, where a completely regular preorder is a topological preordered space for which the following two conditions hold

\begin{itemize}
  \item[(i)] \( \mathcal{T} \) coincides with the initial topology generated by the set of continuous isotone functions \( f : E \to [0,1] \),
  \item[(ii)] \( x \leq y \) if and only if for every continuous isotone function \( f : E \to [0,1] \), \( f(x) \leq f(y) \).
\end{itemize}

For future reference let us introduce the equivalence relation \( x \sim y \) on \( E \), given by \( "x \leq y \) and \( y \leq x" \). Let \( E/\sim \) be the quotient space, \( \mathcal{T}/\sim \) the quotient topology, and let \( \leq_\sim \) be defined by, \([x] \leq_\sim [y] \) if \( x \leq y \) for some representatives (with some abuse of notation we shall denote with \([x]\) both a subset of \( E \) and a point on \( E/\sim \)). The quotient preorder is by construction an order. The triple \((E/\sim, \mathcal{T}/\sim, \leq_\sim)\) is a topological ordered space and \( \pi : E \to E/\sim \) is the continuous quotient projection.

Nachbin proves [22, Prop. 8] that the completely regularly preordered spaces can be characterized as those topological preordered spaces \((E, \mathcal{T}, \leq)\) which come from a quasi-uniformity \( \mathcal{U} \), in the sense that \( \mathcal{T} = \mathcal{T}(\mathcal{U}^*) \) and \( G(\leq) = \bigcap \mathcal{U} \) (see [4, 22] for details on quasi-uniformities). Note that for these spaces, by (i) above, \((E, \mathcal{T})\) is completely regular but not necessarily Hausdorff (equivalently \( T_1 \)). Nevertheless, from (ii) it follows that \( E \) is a \( T_2 \)-preordered space, hence \( T_1 \)-preordered thus \([x] = d(x) \cap t(x) \) is closed. We conclude that in a completely regularly preordered space, \( \mathcal{T} \) is \( T_1 \), and hence \((E, \mathcal{T})\) is a Tychonoff space, if and only if \( \leq \) is an order [22].

In this work we look for topological preordered spaces that admit a Hausdorff \( T_2 \)-preordered compactification. Since the \( T_2 \)-preorder property is hereditary, and every topological space that admits a Hausdorff compactification is Tychonoff, the class that we are considering is contained in the family of \( T_2 \)-preordered Tychonoff spaces. In fact we shall see that all these spaces admit a \( T_2 \)-preorder compactification provided the family of continuous isotone functions determines the preorder. We shall then look for the largest Hausdorff \( T_2 \)-preorder compactification and we shall clarify its connection with Nachbin’s \( T_2 \)-order compactification. We will end the paper with a discussion of the smallest Hausdorff \( T_2 \)-preorder compactification.
2. A MOTIVATION: THE SPACETIME BOUNDARY

Since the next sections will be particularly abstract, it will be convenient to motivate this study mentioning an application. This author is particularly interested in general relativity, but the reader will easily find other applications in closely related fields, for instance, in dynamical systems theory.

This author’s interest for the compactifications of closed preordered spaces comes from the well-known problem of attaching a boundary to a spacetime (physicists term boundary what is known as remainder in topology). We recall that a spacetime is a connected, Hausdorff, time oriented Lorentzian manifold and is denoted \((M, g)\), where \(g\) is the Lorentzian metric. In relativity theory the concept of singularity has proved to be quite elusive. One would like to attach a boundary to spacetime so as to distinguish between points at infinity and singularities, where the distinction is made considering the behavior of the Riemann tensor near the boundary point (e.g. diverging or not).

There have been numerous attempts to construct such a boundary. We mention Penrose’s conformal boundary \([24]\), Geroch, Kronheime and Penrose’s causal boundary \([6]\), Scott and Szekeres’ abstract boundary \([28]\), and various other proposals by Budic and Sachs \([1]\), Racz \([25, 26]\), Szabados \([30, 31]\), Harris \([7]\), Flores \([5]\) and others. Apart for the case of Penrose’s conformal boundary, which cannot be applied in general, one does not demand that spacetime plus the boundary be still a manifold. In general, one wishes just to preserve some notion of continuity and provide a way of extending the causal relation to the boundary.

The above constructions are often quite involved. I propose a strategy which takes advantage of the fact that any spacetime is a topological preordered space. Let us clarify this point. The causal relation \(J^+\) on \(M\) is given by the pairs \((x, y)\) of points of \(M\) for which there is a \(C^1\) curve \(\gamma : [0, 1] \to M\), \(\gamma(0) = x\), \(\gamma(1) = y\), which is causal, in the sense that its tangent vector at any point stays in the future causal cone of \(g\). In general \(J^+\) might be non-closed, however, there is another relation, intimately connected with \(J^+\), which is always closed: the Seifert’s relation \(J^+_S\) \([18, 29]\). The Seifert relation turns spacetime into a topological space endowed with a closed relation and, provided some topological conditions are satisfied, it is indeed possible to compactify spacetime along the lines suggested in this work.

We do not claim that the compactification constructed in this way, denoted \(\beta(E)\), will be the most physical. Indeed, it will add many more points than intuitively required. Nevertheless, it will provide an important step since it will dominate any other possible compactification which, therefore, will be obtainable from \(\beta(E)\) through a suitable identification of the boundary points. The possibility of adding a boundary and extending the preorder so as to keep its closure is not known among physicists. It suffices to say that the boundary constructions mentioned above, either apply to very special spacetimes, or do not share this property.
We could also try a different approach by first showing that the spacetime
is not only a topological preordered space, but in fact a quasi-pseudo-metric
space, and then completing it with a preorder generalization of the Cauchy
completion. Unfortunately, although we could prove, using the results of [20],
that most interesting spacetimes are quasi-pseudo-metrizable, the completion
would depend on the chosen quasi-pseudo-metric. Therefore, this strategy is
not entirely viable unless we prove the existence of some natural spacetime
quasi-pseudo-metric.

Let us end this section explaining why we have to generalize Nachbin’s comp-
pactification to the preordered case, even in those cases in which \( E \) is ordered.
A key example is provided by Misner’s spacetime, a 2-dimensional spacetime
which retains several features of the Taub-NUT spacetime [8]. This spacetime
has topology \( S^1 \times \mathbb{R} \) and metric \( g = 2d\theta dt + t^2 d\theta^2 \). The line \( t = 0 \) of topology
\( S^1 \) is a closed lightlike geodesic. Through any point of the region \( t \leq 0 \) passes
a closed causal curve.

The topological space \( E \) given by the region \( t \geq 0 \) of Misner’s spacetime can
be endowed with a preorder given by the causal relation. This relation is closed,
and the subset \( t > 0 \) with the induced topology and preorder is a completely
regularly ordered space (indeed it can be shown to be convex and it is normally
preordered due to the results of [19]). The set \( t = 0 \) represents a natural con-
ected piece which bounds the region \( t > 0 \), but Nachbin’s compactification
cannot dominate a compactification with this piece of boundary since Nach-
bin’s compactification would be ordered while the set \( t = 0 \) is a closed null
geodesic, and hence any pair of points in this set violates antisymmetry. In
summary, although the region \( t > 0 \) is ordered, its most natural compactifica-
tions are not ordered. Evidently, Nachbin’s compactification is too restrictive
for applications, and the order condition on the compactified space must be
relaxed.

3. Hausdorff \( T_2 \)-preorder compactifications

Given two topological preordered spaces \((E_1, \mathcal{R}_1, \leq_1)\) and \((E_2, \mathcal{R}_2, \leq_2)\) the function \( H : E_1 \to E_2 \) is a preorder homeomorphism if \( H \) is bijective, continuous
and isotone and so is its inverse. We speak of preorder embedding if \( H \) is a preorder homeomorphism of \( E_1 \) on its image \( H(E_1) \subset E_2 \), where \( H(E_1) \) is
given the induced topology and induced preorder.

We are interested in establishing under which conditions a topological pre-
ordered space \((E, \mathcal{R}, \leq)\) admits a preorder compactification, namely a pre-
order embedding \( c : E \to cE \) into a compact topological preordered space
\((cE, \mathcal{R}_c, \leq_c)\) in such a way that \( c(E) \) is a dense subset of \( cE \). We shall often iden-
tify \( E \) with \( c(E) \) because \( c \) is a preorder homeomorphism between \( E \) and \( c(E) \).
We shall be especially interested in Hausdorff \( T_2 \)-preordered compactifications,
that is, in those preorder compactifications for which \((cE, \mathcal{R}_c, \leq_c)\) is also a
Hausdorff \( T_2 \)-preordered space. Sometimes we shall write that \((cE, \mathcal{R}_c, \leq_c)\) is
a preorder compactification by meaning with this that the map \( c : E \to cE \) is
a preorder compactification.
Definition 3.1. If $c_1E, c_2E$, are two preorder compactifications of $E$ we write $c_1 \leq c_2$ if there is a continuous isotone map $C : c_2E \rightarrow c_1E$ such that $C \circ c_2 = c_1$ ($c_1 \leq c_2$ reads “$c_2$ dominates over $c_1$”). The map $C$ is just an extension to $c_2E$ of the preorder homeomorphism $c_1 \circ c_2^{-1} : c_2(E) \rightarrow c_1(E)$. Two preorder compactifications are equivalent if $c_1 \leq c_2$ and $c_2 \leq c_1$.

We remark that two compactifications may be such that $c_1E = c_2E$, $C = Id$, but correspond to different preorders on $c_1E$. In this case $c_1 \leq c_2$ means that, because $Id$ must be isotone, $G(\leq_{c_2}) \subset G(\leq_{c_1})$ (in our conventions the set inclusion is reflexive). Intuitively, to enlarge the compactification means to enlarge the domain $cE$ or to narrow the preorder $\leq_e$ or both. From the definition it follows that the relation of domination on the set of all the compactification is a preorder. The next result establishes that it is actually an order provided we pass to the quotient made by the classes of compactifications related by preorder homeomorphisms.

Proposition 3.2. If two Hausdorff preorder compactifications $c_1, c_2$, are equivalent, then there is a preorder homeomorphism $H : c_2E \rightarrow c_1E$ such that $H \circ c_2 = c_1$.

Proof. Since $c_1 \leq c_2$ there is a continuous isotone map $C_{12} : c_2E \rightarrow c_1E$ such that $C_{12} \circ c_2 = c_1$ and since $c_2 \leq c_1$ there is a continuous isotone map $C_{21} : c_1E \rightarrow c_2E$ such that $C_{21} \circ c_1 = c_2$. Applying $C_{12}$ to the latter equation and using the former equation we get $C_{12} \circ C_{21} \circ c_1 = C_{12} \circ c_2 = c_1$ which implies that $C_{12} \circ C_{21}|_{c_1(E)} = Id_{c_1(E)}|_{c_1(E)}$. Since $c_1(E)$ is dense in $c_1E$ and $c_1E$ is a Hausdorff space we have that $C_{12} \circ C_{21} = Id_{c_1E}$ (e.g. [32, Cor. 13.14]). Arguing with the roles of 1 and 2 exchanged we get $C_{21} \circ C_{12} = Id_{c_2E}$ thus $C_{12}$ and $C_{21}$ are one the inverse of the other. But they are both isotone thus $H := C_{12}$ is a preorder homeomorphism. □

Proposition 3.3. If $c_1, c_2$ are two Hausdorff preorder compactifications of $E$ and $c_1 \leq c_2$ then the continuous isotone map $C : c_2E \rightarrow c_1E$ such that $C \circ c_2 = c_1$ satisfies $C(c_2(E)) = c_1E, C(c_2(E)) = c_1E$ and $C(c_2E \cap c_2(E)) = c_1E \cap c_1(E)$.

Proof. The map $C$ is necessarily onto because $C(c_2E)$ is compact and hence closed and the image of $C$ includes $C(c_2(E)) = c_1E$ which is dense in $c_1E$. The preorder compactifications are compactifications so that the last equation follows from [3, Theor. 3.5.7]. □

Let $f : E \rightarrow [0, 1]$ be a continuous function on a topological space $(E, \mathcal{F})$, we shall denote by $\leq_f$ the total preorder given by “$x \leq_f y$ if $f(x) \leq f(y)$”. Its graph will be denoted with $G_f$.

The next proposition establishes some necessary conditions for the existence of a Hausdorff $T_2$-preorder compactification.

Proposition 3.4. If $(E, \mathcal{F}, \leq)$ is a subspace of a Hausdorff $T_2$-preordered compact space, then $E$ is a $T_2$-preordered Tychonoff space and the family of continuous isotone functions $\mathcal{F}$, $f : E \rightarrow [0, 1]$, is such that $x \leq y$ if and only if for every $f \in \mathcal{F}$, $f(x) \leq f(y)$ (equivalently $G(\leq) = \bigcap_{f \in \mathcal{F}} G_f$).
Proof. Let $E$ be a subspace of a Hausdorff $T_2$-preordered compact space which we denote $(E', \mathcal{P}', \leq')$. Since every compact Hausdorff space is Tychonoff and this property is hereditary, we have that $E$ is Tychonoff. The $T_2$-preorder space property is also hereditary thus $E$ is $T_2$-preordered. Finally, since every $T_2$-preordered compact space is normally preordered [19], for $x', y' \in E'$, $x' \leq y'$ if $F(x') \leq F(y')$ where $F : E' \to [0,1]$ is any continuous and isotone function on $E'$ (see e.g. [21, Prop. 1.1]). Let $G$ be the family of continuous isotone functions, $f : E \to [0,1]$, which come from the restriction of some continuous isotone function $F : E' \to [0,1]$. Evidently, for $x, y \in E$, $x \leq y$ iff for every $f \in G$, $f(x) \leq f(y)$. Since $F$ includes $G$ and is made of isotone functions the last claim follows. □

3.1. The largest Hausdorff $T_2$-preorder compactification. The next result establishes that the previous necessary conditions are actually sufficient and that there is a Hausdorff $T_2$-preordered compactification which dominates over all the other Hausdorff $T_2$-preordered compactifications. The locally compact $\sigma$-compact Hausdorff $T_2$-preordered spaces satisfy these necessary and sufficient conditions [19].

**Theorem 3.5.** Let $(E, \mathcal{F}, \leq)$ be a $T_2$-preordered Tychonoff space, let $F$ be the family of continuous isotone functions $f : E \to [0,1]$, and assume that the preorder is represented by the continuous isotone functions i.e. $G(\leq) = \bigcap_{f \in F} G_f$.

Let $\beta : E \to \beta E$ be the Stone-Čech compactification and let $\tilde{F}$ be the set of continuous functions over $\beta E$ obtained from the (unique) extension\(^1\) of the elements of $F$. There is a largest Hausdorff $T_2$-preordered compactification of $(E, \mathcal{F}, \leq)$ given by $(\beta E, \mathcal{F}_\beta, \leq_\beta)$ where $G(\leq_\beta) = \bigcap_{f \in \tilde{F}} G_f$. Every continuous isotone function on $E$ extends to a continuous isotone function on $\beta E$.

Proof. Each graph $G_f$ is closed because the functions $\tilde{f} : \beta E \to [0,1]$ are continuous, thus $G(\leq_\beta)$ being the intersection of closed sets is closed. Further the graphs $G_f$ contain the diagonal of $\beta E$, thus $G(\leq_\beta)$ contains the diagonal. Moreover, $\leq_\beta$ is transitive which implies that $\leq_\beta$ is transitive and hence a closed preorder on $\beta E$. For every $f \in F$, if $x, y \in E$ then $f(x) \leq f(y)$ if $\tilde{f}(x) \leq \tilde{f}(y)$ thus $G(\leq) = G(\leq_\beta) \cap (E \times E)$ which proves that $(\beta E, \mathcal{F}_\beta, \leq_\beta)$ is a preorder compactification.

If $f : E \to [0,1]$ is a continuous isotone function on $E$ then its continuous extension to $\beta E$, $\tilde{f}$, is such that $\tilde{f} \in \tilde{F}$ and by definition of $\leq_\beta$, $G(\leq_\beta) \subset G_f$ which means that $\tilde{f}$ is isotone.

Let $(\varepsilon E, \mathcal{F}_c, \leq_c)$ be another preorder compactification then, since $(\beta E, \mathcal{F}_\beta)$ is the largest Hausdorff compactification [32, Theor. 19.9] there is a continuous map $H : \beta E \to \varepsilon E$ such that $H \circ \beta = c$. The relation on $\beta E$, $R := (H \times H)^{-1}G(\leq_c)$ which is clearly reflexive and transitive is also closed in $\beta E \times \beta E$ because $H$ is continuous.

\(^1\)Note that the extension $\tilde{F}$ is really the extension of $f \circ \beta^{-1}$.
The map $H$ extends into a continuous function on $\beta E$ the preorder homeomorphism $c \circ \beta^{-1} : \beta(E) \to c(E)$ thus $R \cap (\beta(E) \times \beta(E)) = G(\leq) \cap (\beta(E) \times \beta(E))$, that is, $(\beta \times \beta)^{-1} R = G(\leq)$. If a function $g : \beta E \to [0,1]$ is continuous and $R$-isotone then $g \circ \beta : E \to [0,1]$ is continuous and isotone which means that $g \in \mathcal{F}$ (the extension of a continuous function to a continuous function on $\beta E$ is unique because $\beta(E)$ is dense in $\beta E$), that is $g$ is also $\beta$-isotone.

Since $(\beta E, \mathcal{F}, R)$ is a compact $T_2$-preordered space it is normally preordered [19, Theor. 2.4] thus $R = \bigcap_{g \in \mathcal{G}} G_g$ where the intersection is with respect to the family $\mathcal{G}$ of all the continuous $R$-isotone functions on $\beta E$. As we have just proved, this family is a subset of $\mathcal{F}$ thus $G(\leq) \subset R$. Since $G(\leq) \subset (H \times H)^{-1}G(\leq)$ we conclude that $H$ is isotone and hence that $c \leq \beta$. \hfill \Box

**Theorem 3.6.** A Hausdorff $T_2$-preorder compactification $(cE, \mathcal{F}, \leq_c)$ which shares the properties

(a) every continuous function $f : E \to [0,1]$ can be extended to a continuous function on $cE$,

(b) every continuous isotone function $f : E \to [0,1]$ can be extended to a continuous isotone function on $cE$,

is necessarily equivalent to $(\beta E, \mathcal{F}_\beta, \leq_\beta)$.

**Proof.** We already know that $c \leq \beta$ because $\beta E$ is the largest Hausdorff $T_2$-preorder compactification. Since the compactification $(cE, \mathcal{F}, \leq_c)$ shares property (a) it is equivalent with the Stone-Čech compactification $(\beta E, \mathcal{F}_\beta)$, in particular there is a continuous map $D : cE \to \beta E$ such that $D \circ c = \beta$. The relation on $cE$, $R := (D \times D)^{-1}G(\leq)$ which is clearly reflexive and transitive is also closed in $cE \times cE$ because $D$ is continuous.

$D$ extends into a continuous function on $cE$ the preorder homeomorphism $\beta \circ c^{-1} : c(E) \to \beta(E)$ thus $R \cap (c(E) \times c(E)) = G(\leq) \cap (c(E) \times c(E))$, that is, $(c \times c)^{-1} R = G(\leq)$. If a function $g : cE \to [0,1]$ is continuous and $R$-isotone then $g \circ c : E \to [0,1]$ is continuous and isotone which means by property (b) that $g$ is also $G_c$-isotone (the extension of a continuous function to a continuous function on $cE$ is unique because $c(E)$ is dense in $cE$).

Since $(cE, \mathcal{F}, R)$ is a compact $T_2$-preordered space it is normally preordered [19, Theor. 2.4] thus $R = \bigcap_{g \in \mathcal{G}} G_g$ where the intersection is with respect to the family $\mathcal{G}$ of all the continuous $R$-isotone functions on $cE$. As we have just proved, this family is contained in the family of continuous $G_c$-isotone functions $\mathcal{C}, \bigcap_{g \in \mathcal{G}} G_g \subset R$. Finally, note that $(cE, \mathcal{F}_c, \leq_c)$ is also a compact $T_2$-preordered space hence normally preordered and hence with a preorder represented by the continuous $G_c$-isotone functions, $G(\leq_c) = \bigcap_{g \in \mathcal{G}} G_g$, which implies $G(\leq_c) \subset R$. The inclusion $G(\leq_c) \subset (D \times D)^{-1}G(\leq)$ proves that $D$ is isotone and hence that $\beta \leq c$. \hfill \Box

Adapting the terminology of Fletcher and Lindgren [4] for ordered compactifications we can say that the next result proves that $(\beta E, \mathcal{F}_\beta, \leq_\beta)$ is a strict preorder compactification.
Theorem 3.7. On \((\beta E, \mathcal{T}_\beta)\) the closed preorder \(\leq_\beta\) is the smallest closed preorder inducing \(\leq\) on \(E\).

Proof. Let \(\leq_R\) be another closed preorder such that \(R \cap (E \times E) = G(\leq)\). The map \(\beta' : E \to \beta E, \beta' = \beta\), where \(\beta E\) is regarded as the preordered space \((\beta E, \mathcal{T}_\beta, R)\) is a preorder compactification. Since \(\beta\) is the largest \(\beta' \leq \beta\), which means that there is a continuous isotone function \(B : \beta E \to \beta' E\) such that \(B \circ \beta = \beta'\). On \(\beta(E)\) the map \(B\) coincides with \(\beta' \circ \beta^{-1} = \beta \circ \beta^{-1} = Id\), thus \(B\) is the identity over \(\beta E\). The fact that it is isotone means \(G(\leq_\beta) \subset R\) which is the thesis. \(\square\)

Theorem 3.8. If \((E, \mathcal{T}, \leq)\) is a compact Hausdorff \(T_2\)-preordered space, then its Hausdorff \(T_2\)-preorder compactification \(\beta : E \to \beta E\) constructed in Theorem 3.5 is equivalent with the identity \(Id : E \to E\).

Proof. The map \(c : E \to E\) where \(c = Id_E\) and \((cE, \mathcal{T}_c, \leq_c) = (E, \mathcal{T}, \leq)\) is a preorder compactification which satisfies both conditions (a) and (b) of Theorem 3.6, thus the preorder compactification \(Id\) is equivalent to \(\beta\). \(\square\)

The discrete preorder is that preorder for which the increasing hull of a point is made only by the point (thus it is actually an order). The indiscrete preorder is that preorder for which the increasing hull of a point is the whole space. The indiscrete preorder is closed while the discrete preorder requires the Hausdorffness of the space, which we assume.

Corollary 3.9. If \(\leq\) is the discrete (indiscrete) preorder then \((\beta E, \mathcal{T}_\beta, \leq_\beta)\) is the Stone-\v{C}ech compactification endowed with the discrete (resp. indiscrete) preorder.

Proof. The discrete preorder \(\leq_d\) on \(\beta E\) is clearly the smallest closed preorder inducing the discrete preorder \(\leq\), thus \(\leq_d \leq \leq_\beta\).

For the indiscrete case let \(x, y \in \beta E\) and let \(O_x, O_y\) be neighborhoods of \(x\) and \(y\) respectively. Since \(\beta(E)\) is dense there are points \(x', y' \in E\) such that \(x' \in \beta(E) \cap O_x, y' \in \beta(E) \cap O_y\), from \(\beta^{-1}(x') \leq \beta^{-1}(y')\) since \(\beta\) is isotone we get \(x' \leq_\beta y'\) and since \(\leq_\beta\) is closed we conclude \(x \leq y\). \(\square\)

3.2. The relation with Nachbin’s \(T_2\)-order compactification. In this section we wish to study the relation between the compactification \(\beta : E \to \beta E\) and the Nachbin’s compactification \(n : E \to nE\) in those cases in which \(E\) is a completely regularly ordered space so that the latter compactification applies. In this case, although \(\leq\) is an order, \(\leq_\beta\) need not be an order. We want to prove that the Nachbin’s compactification is obtained by taking the quotient with respect to \(\sim_\beta\).

Let \((E/\sim, \mathcal{T}/\sim, \leq)\) be the quotient topological preordered space and let \(\pi : E \to E/\sim\) be the continuous quotient projection. Every open (closed) increasing (decreasing) set on \(E\) projects to an open (resp. closed) increasing (resp. decreasing) set on \(E/\sim\) and all the latter sets can be regarded as such
projections. As a consequence, \((E, \mathcal{T}, \leq)\) is a normally preordered space (\(T_1\)-preordered space) if and only if \((E/\sim, \mathcal{T}/\sim, \leq)\) is a normally ordered space (resp. \(T_1\)-ordered space). Using this fact it is easy to prove (see [19, Cor. 4.3])

**Theorem 3.10.** If \((E, \mathcal{T}, \leq)\) is a compact \(T_2\)-preordered space, then \((E/\sim, \mathcal{T}/\sim, \leq)\) is a compact \(T_2\)-order compactification.

We are ready to establish the connection with the Nachbin \(T_2\)-order compactification.

**Theorem 3.11.** Let \((E, \mathcal{T}, \leq)\) be a \(T_2\)-preordered Tychonoff space such that \(E/\sim\) is a completely regularly ordered space, then the preorder \(\leq\) is represented by the continuous isotone functions on \(E\). Let \(\beta : E \to \beta E\) be the Hausdorff \(T_2\)-preorder compactification constructed in Theorem 3.5 and let \(\Pi : \beta E \to \beta E/\sim\beta\) be the quotient projection on the \(T_2\)-ordered space \((\beta E/\sim\beta, \mathcal{J}_\beta/\sim\beta, \leq\beta)\), then\(^2\) \(\Pi \circ \beta \circ \pi^{-1} : E/\sim \to \beta E/\sim\beta\) is a \(T_2\)-order compactification equivalent to the Nachbin \(T_2\)-order compactification \(n : E/\sim \to n(E/\sim)\). In particular, up to equivalences, the following diagram commutes

\[
\begin{array}{ccc}
E & \xrightarrow{\beta} & \beta E \\
\downarrow \pi & & \downarrow \Pi \\
E/\sim & \xrightarrow{n} & n(E/\sim)
\end{array}
\]

**Proof.** The order \(\leq\) on \(E/\sim\) is represented by the continuous isotone functions because \(E/\sim\) is completely regularly ordered. Since for \(x, y \in E\), \(x \leq y\) iff \(\pi(x) \leq \pi(y)\), and the continuous isotone functions on \(E\) pass to the quotient, the continuous isotone functions on \(E\) represent \(\leq\).

The fact that \((\beta E/\sim\beta, \mathcal{J}_\beta/\sim\beta, \leq\beta)\) is \(T_2\)-ordered follows from Theorem 3.10.

The expression \(\varphi := \Pi \circ \beta \circ \pi^{-1}\) gives a well defined function, indeed suppose \(x, y \in E\) project on the same element \([x] \in E/\sim\), then \(x \sim y\) and since \(\beta\) is a preorder embedding \(\beta(x) \sim_\beta \beta(y)\) which implies \(\Pi(\beta(x)) = \Pi(\beta(y))\).

The function \(\varphi\) is continuous, indeed let \(O \subset \beta E/\sim\beta\) be an open subset then \(\beta^{-1}(\Pi^{-1}(O))\) is open and if \(x \in \beta^{-1}(\Pi^{-1}(O))\) and \(y \sim x\) then as \(\beta\) is a preorder embedding \(\beta(y) \sim_\beta \beta(x)\), \(\beta(x) \in \Pi^{-1}(O)\) which implies \(\beta(y) \in \Pi^{-1}(O)\) and hence \(y \in \beta^{-1}(\Pi^{-1}(O))\). The open set \(\beta^{-1}(\Pi^{-1}(O))\) is \(E\), being projectable has an open projection by definition of quotient topology which implies that \(\varphi^{-1}(O)\) is open.

Let us prove that \(\varphi\) is isotope. Let \([x] \leq [y], x, y \in E\), then \(x \leq y\) and, since \(\beta\) is a preorder embedding, \(\beta(x) \leq_\beta \beta(y)\), and finally \(\Pi(\beta(x)) \leq_\beta \Pi(\beta(y))\) by definition of quotient order.

Let us prove that \(\varphi\) is injective. Let \([x], [y] \in E/\sim\) be such that \(\varphi([x]) = \varphi([y])\), that is, \(\Pi(\beta(x)) = \Pi(\beta(y))\). This equality implies \(\beta(x) \sim_\beta \beta(y)\), and since \(\beta\) is a preorder embedding \(x \sim y\), that is, \([x] = [y]\).

\(^2\)The inverse \(\pi^{-1}\) is multivalued but the composition \(\Pi \circ \beta \circ \pi^{-1}\) is a well defined function.
Let us prove that \( \varphi^{-1} |_{\omega(E/\sim)} : \varphi(E/\sim) \to E/\sim \) is isotone. Let \( x, y \in E \) and \( \Pi(\beta(x)) \sim_{\beta} \Pi(\beta(y)) \) then \( \beta(x) \leq_{\beta} \beta(y) \) and, since \( \beta \) is a preorder embedding, \( x \leq y \) which implies \( \{x\} \subseteq \{y\} \).

Let us prove that \( \varphi \) is an embedding. Since \( \pi \) is continuous, given an open set \( N \subset E/\sim \) we have that \( \pi^{-1}(N) \) is open, thus we have only to prove that \( \Pi \circ \beta \) sends open sets on \( E \) of the form \( \pi^{-1}(N) \) to open sets on \( \beta \circ \beta(E) \) with the topology induced from \( \beta(E)/\sim_{\beta} \). Let \( O \subset E \) be an open set of the form \( O = \pi^{-1}(N) \) with \( N \) open set on \( E/\sim \) and let \( x \in O \) (thus \( \{x\} \subset N \)). Since \( E/\sim \) is completely regularly ordered space there are \([22]\) a continuous isotone function \( f : E/\sim \to [0,1] \) and a continuous anti-isotone function \( \tilde{f} : E/\sim \to [0,1] \) such that \( \tilde{f}(\{x\}) = \delta(\{x\}) = 1 \) and \( \min(f(\{y\}), \delta(\{y\})) = 0 \) for \( \{y\} \subset E \setminus N \).

Let us define \( f = \tilde{f} \circ \pi \), \( g = g \circ \pi \), so that they are respectively continuous isotone and continuous anti-isotone and such that \( f(x) = g(x) = 1 \) and \( \min(f(y), g(y)) = 0 \) for \( \{y\} \subset E \setminus O \).

The functions \( f, g \circ \beta^{-1} \) extend to functions \( \tilde{f}, \tilde{g} : \beta \circ \beta(E) \to [0,1] \) respectively isotone and anti-isotone (extend \( g \) in place of \( g \) and take minus the extended function). Since they are isotone or anti-isotone there are continuous functions \( F, G : \beta \circ \beta(E)/\sim_{\beta} \to [0,1] \), respectively isotone and anti-isotone, such that \( \tilde{f} = F \circ \Pi \), \( \tilde{g} = G \circ \Pi \) (continuity follows from the universality property of the quotient map \([32], \text{Theor. 9.4}\)).

The function \( h = \min(\tilde{f}, \tilde{g}) = \min(F, G) \circ \Pi \) is continuous and vanishes on \( \beta(E) \setminus O \) and hence \( \min(F, G) \) vanishes on \( (\Pi \circ \beta)(E) \setminus O = \varphi((E/\sim) \setminus N) \) and equals \( 1 \) on \( \beta(x) \). Since \( \varphi \) is injective the open set \( Q = \{[w]_{\beta} : \min(F([w]_{\beta}), G([w]_{\beta})) > 0\} \) contains \( \varphi(x) \) and is such that \( Q \cap \varphi(E/\sim) \) \( \subseteq \varphi(N) \) which proves, due to the arbitrariness of \( \{x\} \), that \( \varphi(N) \) is open in the topology induced on \( \varphi(E/\sim) \) by \( \beta \circ \beta(E)/\sim_{\beta} \). We infer that \( \varphi \) is an embedding and since it is isotone with its inverse it is a preorder embedding.

If \( [z]_{\beta} \in (\beta \circ \beta(E)/\sim_{\beta}) \setminus \varphi(E/\sim) \) and \( W \) is an open set containing \( [z]_{\beta} \) then \( \Pi^{-1}(W) \) is open and since \( \beta \) is a dense embedding there is some \( r \in E \) such that \( \beta(r) \in \Pi^{-1}(W) \). Thus \( \{r\} \subset E/\sim \) is such that \( \varphi([r]) \in W \), that is, \( \varphi(E/\sim) \) is dense in \( \beta \circ \beta(E)/\sim_{\beta} \) and hence \( \varphi \) is a \( T_2 \)-order compactification.

Now, let \( f : E/\sim \to [0,1] \) be a continuous isotone function, and let \( f = \hat{f} \circ \pi \). The function \( f : E \to [0,1] \) is a continuous isotone function and we know that there is a continuous isotone function \( \hat{f} : \beta \circ \beta(E) \to [0,1] \) which extends \( f \circ \beta^{-1} : \beta(E) \to [0,1] \). Since \( \hat{f} \) is isotone there is some continuous isotone function \( F : \beta \circ \beta(E)/\sim_{\beta} \to [0,1] \) (continuity follows from the universality property of the quotient map) such that \( \hat{f} = F \circ \Pi \), thus \( F \) extends \( f \circ \varphi^{-1} : \varphi(E/\sim) \to [0,1] \).

Since the Nachbin \( T_2 \)-order compactification is characterized by this extension property \([4, 22]\) it follows that \( \varphi \) is equivalent to \( \pi \).

Finally, \( \varphi \circ \pi = (\Pi \circ \beta \circ \pi^{-1}) \circ \pi = \Pi \circ \beta \) which proves that, up to equivalences, the diagram commutes. \( \square \)
Corollary 3.12. Let $E$ be a completely regularly ordered space, let $\beta : E \to \beta E$ be the Hausdorff $T_2$-preorder compactification constructed in Theorem 3.5 and let $\Pi : \beta E \to \beta E/\sim_\beta$ be the quotient projection on the $T_2$-ordered space $(\beta E/\sim_\beta, \mathcal{T}_\beta/\sim_\beta, \leq_\beta)$, then $\Pi \circ \beta : E \to \beta E/\sim_\beta$ is a $T_2$-order compactification equivalent to the Nachbin $T_2$-order compactification $n : E \to nE$.

Proof. It follows from the previous theorem noting that a completely regularly ordered space is a $T_2$-preordered Tychonoff space.

If $E$ is a completely regularly ordered space the preorder compactification $\beta$ need not be equivalent with the Nachbin compactification. Consider for instance the interval $[0, 1]$ with the usual topology and order. The Nachbin compactification is given by $[0, 1]$ but $\beta([0, 1])$ includes many more points.

3.3. The smallest Hausdorff $T_2$-preorder compactification. In this section we make some progress in the problem of finding the smallest Hausdorff $T_2$-preorder compactification of a topological preordered space in those cases in which it exists. The problem of identifying and characterizing the smallest $T_2$-order compactification was considered in [13, 15–17, 27].

In this section $(E, \mathcal{T}, \leq)$ is a locally compact $T_2$-preordered Tychonoff space and $\mathcal{F}$ is the family of continuous isotone functions $f : E \to [0, 1]$. Accordingly with the necessary conditions singled out in Prop. 3.4, we shall assume that the preorder is represented by the continuous isotone functions i.e. $G(\leq) = \bigcap_{f \in \mathcal{F}} G_f$.

Let $\mathcal{C}$, $\mathcal{C}^-$ and $\mathcal{C}^+$ be the families of continuous functions in $[0, 1]$ which are constant outside a compact set, which have compact support and which have value 1 outside a compact set, respectively.

For every $\mathcal{H} \subset \mathcal{F}$ such that $G(\leq) = \bigcap_{h \in \mathcal{H}} G_h$ we can construct a $T_2$-preorder compactification $(cE, \mathcal{F}_c, \leq_c)$, which we call $\mathcal{H}$-compactification, through the embedding $c : E \to [0, 1]^{\mathcal{H}\times\mathcal{C}}$ identifying $cE$ with the closure of the image. Indeed, the family $\mathcal{H}\cup\mathcal{C}$ separates points and has an initial topology coincident with $\mathcal{T}$ (thanks to local compactness and the inclusion of $\mathcal{C}$ in the family) thus $c$ is an embedding [32, Theor. 8.12]. The topology $\mathcal{T}_c$ is that induced from the product topology in $[0, 1]^{\mathcal{H}\times\mathcal{C}}$ on $cE$.

We define the $T_2$-preorder $\preceq$ on $[0, 1]^{\mathcal{H}\times\mathcal{C}}$ as that given by $x \preceq y$ if $x_h \leq_h y_h$ for every $h \in \mathcal{H}$, where $\leq_h$ is the usual order on the $h$-th factor $[0, 1]$. This preorder is closed because the projections $\pi_h : [0, 1]^{\mathcal{H}\times\mathcal{C}} \to \mathbb{R}$ are continuous, and hence $G(\preceq) = \bigcap_{h \in \mathcal{H}} (\pi_h \times \pi_h)^{-1} G(\leq_h)$ is closed. It is a preorder rather than an order because two points can have the same $h$-components while being different. The $T_2$-preorder $\leq_c$ on $cE$ is that induced by $\preceq$ and is again closed because of the hereditarity of the $T_2$-preorder property. Finally, $c : E \to c(E)$ is isotone with its inverse because $G(\preceq) = \bigcap_{h \in \mathcal{H}} G_h$.

Observe that $h \circ c^{-1} : c(E) \to [0, 1]$ extends to the continuous isotone function $\pi_h|_{cE}$, that is, the continuous isotone functions belonging to $\mathcal{H}$ are extendible to the $\mathcal{H}$-compactification $cE$ keeping the same properties.
Proof. Let \( (\mathcal{C}, \leq) \) be a locally compact \( T_2 \)-preordered Hausdorff space then every \( T_2 \)-preordered Hausdorff compactification \( c : E \to cE \) dominates a \( \mathcal{H} \)-compactification for a family \( \mathcal{H} \subseteq \mathcal{F} \) where \( \mathcal{H} \) is such that \( G(\leq) = \bigcap_{h \in \mathcal{H}} G_h \). The family \( \mathcal{H} \) is made by those continuous isotone function with value in \([0,1]\) in \( E \) that extend with the same properties to \( cE \).

**Proposition 3.13.** The just defined \( \mathcal{H} \)-compactification gives back the usual one-point compactification if the preorder \( \leq \) is indiscrete and \( \mathcal{H} \) is chosen empty (the additional point is that of coordinates \( f_e, c \in \mathcal{C} \), where \( f_e \) is the constant value taken by \( c \) outside a compact set).

If the preorder \( \leq \) is discrete and \( \mathcal{H} \) is chosen to coincide with \( \mathcal{C} \) then the compactified space is still the one-point compactification but endowed with the discrete preorder. If \( \mathcal{H} \) is chosen equal to \( \mathcal{C}^+ \), then the added point is less than any other point. If \( \mathcal{H} \) is chosen equal to \( \mathcal{C}^- \), then the added point is greater than any other point.

In the next proofs we shall often identify \( c(E) \) with \( E \) especially when referring to the extension of functions.

**Proposition 3.14.** Let \( c : E \to cE \) be a \( \mathcal{H} \)-compactification. The remainder \( cE \setminus c(E) \) endowed with the preorder induced from \( \leq_c \) is a \( T_2 \)-ordered space.

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a compact set disjoint from \(c_1E \setminus c_1(E)\). The restriction of the elements of the family \(C_{c_1}\) to \(c_1(E)\) gives back \(C\). By definition, the map \(C\) sends \(x \in c_1E\) to the point of \([0,1]^\mathcal{H}_{c_1 \cup C_{c_1}}\) whose \(f\) coordinate is the value \(f(x), f \in \mathcal{H}_{c_1 \cup C_{c_1}}\). This map is continuous [32, Theor. 8.8] and isotone, where we define the preorder on \([0,1]^{\mathcal{H}_{c_1 \cup C_{c_1}}}\) as that determined by the family \(\mathcal{H}_{c_1}\). Let us prove that its image is included in \(c_2E\). From the definitions we have that if \(x \in c_1(E)\) then \(C(x)\) belongs to \(c_2(E)\). As \(C\) is continuous, and \(c_1(E)\) is dense in \(c_1E\), if \(x \in c_1E\) its image \(C(x)\) belongs to the closure of \(c_2(E)\) namely to \(c_2E\).

**Proposition 3.16.** If \(\mathcal{H}_2 \supset \mathcal{H}_1\) then the \(\mathcal{H}_2\)-compactification dominates over the \(\mathcal{H}_1\)-compactification.

**Proof.** Indeed, if \(c_2 : E \to c_2E \subset [0,1]^\mathcal{H}_2 \cup C\) is the former and \(c_1 : E \to c_1E \subset [0,1]^\mathcal{H}_1 \cup C\) is the latter preorder compactification, then there is a continuous isotone map \(C : c_2E \to c_1E\) such that \(C \circ c_2 = c_1\). This map is the restriction to \(c_2E\) of \(H : [0,1]^\mathcal{H}_2 \cup C \to [0,1]^\mathcal{H}_1 \cup C\) where \(H\) identifies points with the same coordinates belonging to the set \(\mathcal{H}_1 \cup C\).

Once a \(\mathcal{H}\)-compactification is given it is well possible that some \(f \in F \setminus \mathcal{H}\) could be extendable as a continuous isotone function to the whole compactification. Let \(i(\mathcal{H})\) be the subset of \(F\) of so extendable functions. This set being larger than \(\mathcal{H}\) has again the property that it represents \(\leq\).

**Proposition 3.17.** The \(\mathcal{H}\)-compactification and the \(i(\mathcal{H})\)-compactification are equivalent.

**Proof.** Since \(\mathcal{H} \subset i(\mathcal{H})\) the \(i(\mathcal{H})\)-compactification dominates over the \(\mathcal{H}\)-compactification. For the converse let \(c_2 : E \to c_2E \subset [0,1]^\mathcal{H} \cup C\) be the \(\mathcal{H}\)-compactification and let \(c_1 : E \to c_1E \subset [0,1]^\mathcal{H} \cup C\) be the \(i(\mathcal{H})\)-compactification. A continuous isotone map \(C : c_2E \to c_1E\) such that \(C \circ c_2 = c_1\) can be constructed as follows. All the functions of \(i(\mathcal{H}) \cup C\) extend (uniquely because \(c_2(E)\) is dense in \(c_2E\)) from \(E\) to \(c_2E\) thus to every \(x \in c_2E\) we assign the image \(C(x)\) given by the point of \([0,1]^{(\mathcal{H}) \cup C}\) having as coordinates the values taken by the functions belonging to \(i(\mathcal{H}) \cup C\). By construction \(C\) is continuous [32, Theor. 8.8]. Let us prove that the image is included in \(c_1E\). From the definitions we have that if \(x \in c_2(E)\) then \(C(x)\) belongs to \(c_1(E)\). As \(C\) is continuous, and \(c_2(E)\) is dense in \(c_2E\), if \(x \in c_2E\) its image \(C(x)\) belongs to the closure of \(c_1(E)\) namely to \(c_1E\). The fact that \(C\) is isotone follows immediately from the definition of preorder in \([0,1]^{(\mathcal{H}) \cup C}\) and from the fact that the extension of the function in \(i(\mathcal{H})\) to \(c_2E\) are, by assumption, continuous and isotone.

**Corollary 3.18.** Let \(P(F)\) denote the family of subsets of \(F\). The map \(i : P(F) \to P(F)\) is idempotent, namely \(i(i(\mathcal{H})) = i(\mathcal{H})\). Furthermore, if \(\mathcal{H}_1 \subset \mathcal{H}_2\) then \(i(\mathcal{H}_1) \subset i(\mathcal{H}_2)\).

**Proof.** If a continuous isotone function \(f : E \to [0,1]\) can be extended as a continuous isotone function to the \(i(\mathcal{H})\)-compactified space, i.e. \(f \in i(i(\mathcal{H}))\) then, as the \(\mathcal{H}\)-compactification and the \(i(\mathcal{H})\)-compactification are equivalent,
it can be extended as a continuous isotone function to the \( H \)-compactified space that is \( f \in i(H) \).

For the last statement, let \( f \in i(H_1) \) that is \( f : E \to [0, 1] \) can be extended as a continuous isotone function \( f_1 : c_1E \to [0, 1] \) to the \( H_1 \)-compactified space. But the \( H_2 \)-compactification dominates over the \( H_1 \)-compactification, that is if \( c_2 : E \to c_2E \) is the former and \( c_1 : E \to c_1E \) is the latter, there is a continuous isotone function \( C : c_2E \to c_1E \) such that \( C \circ c_2 = c_1 \). The pullback with \( C \) of the extension to \( c_1E \), namely \( f_2 = f_1 \circ C \), is a continuous isotone extension on \( c_2E \) of \( f \) thus \( f \in i(H_2) \).

\[ \Box \]

**Theorem 3.19.** The \( H \)-compactification is the smallest Hausdorff \( T_2 \)-preordered compactification for which the function belonging to \( H \) are extendable as continuous isotone functions to the compactified space.

**Proof.** Let \( c : E \to cE \) be a Hausdorff \( T_2 \)-preordered compactification for which the functions belonging to \( H \) are extendable. By Prop. 3.15 the compactification \( c \) dominates a \( G \)-compactification where \( G \) is the set of continuous isotone functions on \( E \) with value in \([0,1]\) which are extendable with these properties to \( cE \). Thus \( H \subset G \) and by Prop. 3.16 the \( G \)-compactification dominates over the \( H \)-compactification, thus \( c \) dominates the \( H \)-compactification. \[ \Box \]

**Definition 3.20.** The family of invariant sets \( I \) is the set of subsets \( H \subset F \) which satisfy \( G(\leq) = \bigcap_{h \in H} G_h \) and are left invariant by \( i \). The set \( I \) is ordered by inclusion.

The next theorem serves to define the family of continuous isotone functions \( S \) which characterizes the smallest compactification.

**Theorem 3.21.** If the smallest Hausdorff \( T_2 \)-preorder compactification exists then it is a \( S \)-compactification where \( G(\leq) = \bigcap_{h \in S} G_h \), \( i(S) = S \) and \( S = \bigcap I \).

**Proof.** Suppose that there is a Hausdorff \( T_2 \)-preorder compactification which is dominated by all the other Hausdorff \( T_2 \)-preorder compactifications, then by Prop. 3.15 it is equivalent to a \( S \)-compactification where \( S \subset F \) is such that \( G(\leq) = \bigcap_{h \in S} G_h \).

By Prop. 3.17 \( S \) can be chosen such that \( S = i(S) \), thus belonging to \( I \). Clearly, \( \bigcap I \subset S \) because \( S \in I \). Suppose that \( H \in I \) and that \( f \in F \), \( f \notin H = i(H) \). This means that \( f \) is not extendable as a continuous isotone function to the \( H \)-compactified space. If \( C \) is the continuous isotone map from the \( H \)-compactified space to the \( S \)-compactified space (as the \( S \)-compactification is dominated by all the other compactifications) one has that if \( f \) were extendable to the \( S \)-compactified space then by pullbacking the extension to the \( H \)-compactified space through \( C \) one would get an extension in the \( H \)-compactified space. The contradiction proves that \( f \notin i(S) = S \) thus \( S \subset H \), and finally \( S \subset \bigcap I \). \[ \Box \]
Remark 3.22. The smallest compactification does not necessarily exist. For instance, if $E$ is non-compact and endowed with the discrete preorder, the $C$-compactification dominates over the $C^-$-compactification and the $C^+$-compactification (see Remark 3.13), indeed $C^+ \subset C$ see Prop. 3.16. Stated in another way, the one-point compactification endowed with the discrete preorder dominates over that in which the added point is less (resp. greater) than any other point (indeed, the former has a smaller preorder). However, $C^+$ is not contained in $i(C^-)$ and conversely, thus the $C^-$ and $C^+$ compactifications differ. Actually, it is easy to realize that they are minimal, thus there is no smallest compactification.

4. Conclusions

We have investigated the compactification of topological preordered spaces, showing the existence of a largest Hausdorff $T_2$-preorder compactification for every $T_2$-preordered Tychonoff space for which the preorder is represented by the continuous isotone functions. An interesting subclass of this family is that of locally compact $\sigma$-compact Hausdorff $T_2$-preordered spaces [19]. It turns out that this largest compactification is essentially the Stone-Čech compactification endowed with a suitable preorder. It can be characterized as the Hausdorff $T_2$-preorder compactification for which all the continuous function can be continuously extended and the continuous isotone function do so preserving the isotone property. If the preorder is an order or the quotient space is a completely regularly ordered space it is also possible to show a clean relation with Nachbin’s $T_2$-order compactification.

We have considered the problem of identifying the smallest Hausdorff $T_2$-preorder compactification whenever it exists. We have shown that it corresponds necessarily to the compactification obtained demanding the extendibility of a suitable set of continuous isotone functions. Generically, this set $S$ is expected to be strictly included in the full set $F$ of continuous isotone functions with value in $[0,1]$.

The approach followed in this work relies on the study of continuous isotone functions and their extension properties. We close noting that filter approaches are also possible. For instance Choe and Park [2] have constructed a Wallman type preorder compactification which has been subsequently extensively investigated in [9–12, 14] together with some variations. For instance, in [10] the authors show that it is possible to obtain the Nachbin compactification from the Wallman compactification by identifying the points that take the same value on continuous isotone functions. We have followed a similar procedure to show that the Nachbin compactification $nE$ can be obtained from the same functional quotient starting from $\beta E$. 
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REFERENCES

[1] R. Budic and R. K. Sachs, Causal boundaries for general relativistic spacetimes, J. Math. Phys. 15 (1974), 1302–1309.
[2] T. H. Choe and Y. S. Park, Wallman’s type order compactification, Pacific J. Math. 82 (1979), 339–347.
[3] R. Engelking, General Topology, Berlin: Helderman Verlag (1989).
[4] P. Fletcher and W. Lindgren, Quasi-uniform spaces, vol. 77 of Lect. Notes in Pure and Appl. Math., New York: Marcel Dekker, Inc. (1982).
[5] J. L. Flores, The causal boundary of spacetimes revisited, Commun. Math. Phys. 276 (2007), 611–643.
[6] R. Geroch, E. H. Kronheimer and R. Penrose, Ideal points in spacetime, Proc. Roy. Soc. Lond. A 237 (1972), 545–567.
[7] S. G. Harris, Universality of the future chronological boundary, J. Math. Phys. 39 (1998), 5427–5445.
[8] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge: Cambridge University Press (1973).
[9] D. C. Kent, On the Wallman order compactification, Pacific J. Math. 118 (1985), 159–163.
[10] D. C. Kent, D. Liu and T. A. Richmond, On the Nachbin compactification of products of totally ordered spaces, Internat. J. Math. & Math. Sci. 18 (1995), 665–676.
[11] D. C. Kent and T. A. Richmond, Separation properties of the Wallman ordered compactification, Internat. J. Math. & Math. Sci. 13 (1990), 209–222.
[12] D. C. Kent and T. A. Richmond, A new ordered compactification, Internat. J. Math. & Math. Sci. 16 (1993), 117–124.
[13] H.-P. A. Künzi, Minimal order compactifications and quasi-uniformities, Berlin: Akademie-Verlag, vol. Recent Developments of General Topology and its Applications of Mathematical Research 67 (1992), pages 181–186.
[14] H.-P. A. Künzi, A. E. McCluskey and T. A. Richmond, Ordered separation axioms and the Wallman ordered compactification, Publ. Math. Debrecen 73 (2008), 361–377.
[15] D. M. Liu and D. C. Kent, Ordered compactifications and families of maps, Internat. J. Math. & Math. Sci. 20 (1997), 105–110.
[16] T. McCallion, Compactifications of ordered topological spaces, Proc. Camb. Phil. Soc. 71 (1972), 463–473.
[17] S. D. McCartan, Separation axioms for topological ordered spaces, Proc. Camb. Phil. Soc. 64 (1968), 965–973.
[18] E. Minguzzi, The causal ladder and the strength of K-causality. II, Class. Quantum Grav. 25 (2008), 015010.
[19] E. Minguzzi, Normally preordered spaces and utilities, Order, to appear (DOI: 10.1007/s11083-011-9230-4. arXiv:1106.4457v2).
[20] E. Minguzzi, Quasi-pseudo-metrization of topological preordered spaces, Topol. Appl. 159 (2012), 2888–2898.
[21] E. Minguzzi, Topological conditions for the representation of preorders by continuous utilities, Appl. Gen. Topol. 13 (2012), 81–89.
[22] L. Nachbin, Topology and order, Princeton: D. Van Nostrand Company, Inc. (1965).
[23] S. Nada, *Studies on Topological Ordered Spaces*, Ph.D. thesis, Southampton (1986).
[24] R. Penrose, *Conformal treatment of infinity*, New York: Gordon and Breach, vol. Relativity, Groups and Topology, pages 563–584 (1964).
[25] I. Rácz, *Causal boundary of space-times*, Phys. Rev. D 36 (1987), 1673–1675.
[26] I. Rácz, *Causal boundary for stably causal spacetimes*, Gen. Relativ. Gravit. 20 (1988), 893–904.
[27] T. A. Richmond, *Posets of ordered compactifications*, Bull. Austral. Math. Soc. 47 (1993), 59–72.
[28] S. Scott and P. Szekeres, *The abstract boundary: a new approach to singularities of manifolds*, J. Geom. Phys. 13 (1994), 223–253.
[29] H. Seifert, *The causal boundary of space-times*, Gen. Relativ. Gravit. 1 (1971), 247–259.
[30] L. B. Szabados, *Causal boundary for strongly causal spacetimes*, Class. Quantum Grav. 5 (1988), 121–134.
[31] L. B. Szabados, *Causal boundary for strongly causal spacetimes II*, Class. Quantum Grav. 6 (1989), 77–91.
[32] S. Willard, *General topology*, Reading: Addison-Wesley Publishing Company (1970).

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