BLACK HOLE ENTROPY AND ENTROPY OF ENTANGLEMENT

Daniel Kabat

Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855–0849

kabat@physics.rutgers.edu

Abstract

We compare the one-loop corrections to the entropy of a black hole, from quantum fields of spin zero, one-half, and one, to the entropy of entanglement of the fields. For fields of spin zero and one-half the black hole entropy is identical to the entropy of entanglement. For spin one the two entropies differ by a contact interaction with the horizon which appears in the black hole entropy but not in the entropy of entanglement. The contact interaction can be expressed as a path integral over particle paths which begin and end on the horizon; it is the field theory limit of the interaction proposed by Susskind and Uglum which couples a closed string to an open string stranded on the horizon.
1. Introduction and Overview

Black holes have thermal properties. This remarkable fact can be established by analogy with an ordinary thermodynamic system, where temperature is the inverse periodicity in Euclidean time, and entropy is the variation of free energy with respect to temperature. The temperature of a black hole is fixed by requiring that the Schwarzschild metric yield a smooth solution of the Einstein equations when continued to imaginary time. This forces the periodicity of the Euclidean time coordinate to be the inverse Hawking temperature. The entropy of a black hole is then found by varying the periodicity in Euclidean time while holding the geometry on a spatial slice fixed, a procedure which introduces a conical singularity at the horizon. The classical action for gravity evaluated on such a space leads to the usual expression for black hole entropy, while quantum corrections to the classical entropy result from fluctuations of the metric or matter fields in the background with the conical singularity. We work in the limit of infinitely massive black holes, so that curvature vanishes except possibly on the horizon, and the inverse Hawking temperature becomes the $2\pi$ periodicity of the plane in polar coordinates.

Quantum corrections to the entropy from matter fields have been extensively studied. It has been realized that, in some cases, these corrections have a state counting interpretation as entropy of entanglement, or equivalently, as Rindler thermal entropy. A quantum field in its Minkowski vacuum state has correlations between degrees of freedom located on opposite sides of an imaginary boundary, so measurements made only on one side of the boundary see a mixed state, with a corresponding entropy of entanglement. The density matrix which describes the mixed state is a thermal density matrix in Rindler coordinates, and entropy of entanglement may be equivalently understood as the thermal entropy which the field carries in Rindler space.

In this paper we explore the relationship between black hole entropy and entropy of entanglement for matter fields of spin zero, one-half, and one. We distinguish between black hole entropy, given by the response of the field to a conical singularity, and entropy of entanglement, obtained from the density matrix which describes the vacuum state of the field as observed from one side of a boundary in Minkowski space. For spins zero and one-half, there is no need for the distinction: the one loop correction to the black hole entropy is equal to the entropy of entanglement. At spin one, the black hole entropy is equal to the entropy of entanglement plus a contact interaction with the horizon. The contact interaction cannot be interpreted as entropy of entanglement in quantum field
theory; indeed we will see that it makes the one loop correction to the black hole entropy negative.

The first hint that something non-trivial happens for spin one comes from its one-loop renormalization of Newton’s constant. The effective action from integrating out a matter field in a curved background may be expanded in derivatives of the metric.

\[
\beta F = \int d^d x \sqrt{g} \mathcal{L}_{\text{eff}}
\]

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{2} \int^\infty_{\epsilon^2} \frac{ds}{(4\pi s)^{d/2}} e^{-s/m^2} \left( \frac{c_0}{s} + c_1 R + \mathcal{O}(s) \right)
\]

We must evaluate this effective action on a space which is a cone of deficit angle \(2\pi - \beta\) times a transverse flat \((d-2)\)-dimensional space with area \(A_\perp\). Susskind and Uglum have argued that for infinitesimal deficit angles only the Einstein-Hilbert term in the effective action contributes, so that we only need to know the coefficient \(c_1\).

\[
c_1 = \begin{cases} 
\frac{1}{6} & \text{minimally coupled scalar} \\
\frac{1}{12} 2^{[d/2]} & \text{Dirac fermion (with } 2^{[d/2]} \text{ components)} \\
\frac{d-2}{6} - 1 & \text{abelian gauge field including ghosts}
\end{cases}
\]

The integral of the scalar curvature is proportional to the deficit angle of the cone, \(\int d^d x \sqrt{g} R = 2A_\perp (2\pi - \beta)\). At the on-shell temperature \(\beta = 2\pi\), the contribution of the field to the entropy of a black hole is given by

\[
S = \left. \left( \beta \frac{\partial}{\partial \beta} - 1 \right) \right|_{\beta=2\pi} (\beta F) = 2\pi c_1 A_\perp \int^\infty_{\epsilon^2} \frac{ds}{(4\pi s)^{d/2}} e^{-s/m^2}.
\]

Note that the contribution to the entropy is proportional to the horizon area (set \(A_\perp = 1\) in \(d = 2\)). It is ultraviolet divergent; \(\epsilon\) is an ultraviolet cutoff put in by hand. In \(d = 2\) it also diverges in the infrared and the mass \(m\) must be kept non-zero to provide a cutoff.

For \(d < 8\) a spin one field makes a negative contribution to the coefficient of the Einstein-Hilbert term and also, therefore, to the black hole entropy. This negative contribution is responsible for the non-renormalization of Newton’s constant in certain supersymmetric theories. It means, however, that the contribution to the black hole entropy from a spin one field cannot be identified solely with entropy of entanglement or Rindler thermal entropy, both of which are intrinsically positive quantities.
To understand this better we turn to the work of Susskind and Uglum, who advocate that black hole entropy can be understood within string theory. The string diagrams they claim are responsible for black hole entropy are shown in Fig. 1. The genus zero diagram if time sliced with respect to the polar angle around the horizon describes the propagation of an open string with its ends stuck on the horizon. There are two classes of diagrams at one loop. The first class describes the propagation of a closed string around the horizon, and has a thermal interpretation as a string at a position-dependent proper temperature. The second class describes an interaction between a closed string and an open string stranded on the horizon. It does not have a thermal interpretation, and can only be viewed as an interaction correction to the entropy which is present at genus zero. Unfortunately it is difficult to give a precise meaning to these string diagrams, since they involve defining string theory in an off-shell background.

At low energies, the string diagrams reduce to the particle diagrams of Fig. 2. The physical interpretation of the classical entropy as counting configurations of stranded strings is lost. At one loop we have a thermal interpretation of a particle encircling the horizon, but also expect to find a contact interaction with the horizon. Susskind and Uglum have suggested that this contact term is responsible for the non-renormalization of Newton’s constant in certain supersymmetric theories.

![Fig. 1. String diagrams responsible for the black hole entropy up to one loop.](image)
Motivated by these particle diagrams we calculate the partition function on a cone for fields of spin zero, one-half, and one. We now summarize our main results. For a scalar field the relevant one loop determinant may be expressed in terms of a single particle path integral, which can be explicitly evaluated.

$$\beta F_{\text{scalar}} = \frac{1}{2} \log \det(-\Box + m^2)$$

$$= -\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \int_{\text{closed loops}} \mathcal{D}x(\tau) \exp -\int_0^s d\tau \left( \frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + m^2 \right)$$

$$= -\frac{\pi^2}{3\beta} A_\perp \left( 1 - \left( \frac{\beta}{2\pi} \right)^2 \right) \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} e^{-sm^2}$$

In the final line we drop a divergent cosmological constant, which does not affect the entropy. In two dimensions $A_\perp = 1$, and the mass must be kept non-zero as an infrared regulator. Note that the on-shell entropy agrees with the result obtained above from the renormalization of Newton’s constant; this shows that the argument of Susskind and Uglum that only the Einstein-Hilbert term contributes is valid. We will show that this entropy is equal to the entropy of entanglement, as well as the Rindler thermal entropy, of a scalar field.

For an Abelian gauge field we have the following one loop determinant. We use a covariant gauge, neglect the cosmological constant, and introduce an infrared regulating
mass which may be set to zero in greater than two dimensions.

\[ \beta F_{\text{gauge}} = \frac{1}{2} \log \det \left( \nabla^\mu \nabla_\nu \right) - \log \det(\nabla^2 - \nabla^2) \]

\[ \beta F_{\text{gauge}} = (d - 2) \beta F_{\text{scalar}} + \beta \int_{c^2}^{\infty} ds \int_{x(0) = x(s) \in \text{horizon}} Dx(\tau) \exp - \int_0^s d\tau \left( \frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + m^2 \right) \]

\[ - \beta A_\perp \int_{c^2}^{\infty} ds \frac{ds}{(4\pi s)^{d/2}} e^{-sm^2} \]

The first term is the free energy of \( d - 2 \) scalar fields, corresponding to the \( d - 2 \) physical polarizations of a gauge field. The next term is a surface term at the horizon, expressed as an integral over particle paths which begin and end on the horizon. The last term is actually a surface term at infinity; it will cancel the surface term at the horizon when \( \beta = 2\pi \). The path integrals may be explicitly evaluated.

\[ \beta F_{\text{gauge}} = (d - 2) \beta F_{\text{scalar}} + A_\perp (2\pi - \beta) \int_{c^2}^{\infty} ds \frac{ds}{(4\pi s)^{d/2}} e^{-sm^2} \]

This gives the same on-shell entropy as was obtained above from the renormalization of Newton’s constant. We will show that the bulk term in the black hole entropy is equal to the entropy of entanglement of the field, and can also be understood as bulk thermal entropy in Rindler space. The surface term, on the other hand, does not admit a thermal or state counting interpretation; it is not present in the entropy of entanglement.

In two dimensions a Dirac fermion has the same partition function on a cone as a scalar field, \( \beta F_{\text{Dirac}} = \beta F_{\text{scalar}} \) (modulo a cosmological constant). This gives the same on-shell entropy as was obtained above from the renormalization of Newton’s constant. It is equal to the entropy of entanglement, as well as the Rindler thermal entropy, of the Dirac field.

The rest of this paper develops these claims in detail. In the next section we consider scalar fields; many of the results exist in the literature but we will need them for reference. The following two sections treat Abelian gauge fields and Dirac fermions. In the final section we summarize our results and speculate about their implications.
2. Scalar Fields

We wish to show that the entropy of entanglement of a scalar field is identical to the entropy which it contributes to a black hole. We’ll begin from the definition of entropy of entanglement, and show that it can be calculated from the partition function on a cone. For simplicity we work in two dimensions.

First, we must find the Minkowski vacuum state of the field. The vacuum wavefunctional is given by a Euclidean path integral on the upper half plane.

\[ \psi[\phi_0(x)] = \int D\phi \exp \left( -\int_0^\infty d\tau \int_{-\infty}^\infty dx \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 \right) \right) \]

A classical field \( \phi_{cl}(\tau,x) = \phi_0(x) \) and \( \phi_{cl} \to 0 \) as \( \tau \to \infty \), may be given in terms of the Dirichlet Green’s function on the upper half plane, which can be constructed by the method of images.

\[ \phi_{cl}(\tau,x) = \int_{-\infty}^{\infty} dx' \partial_{\tau'}|_{\tau'=0} G_D(\tau,x|\tau',x') \phi_0(x') \]

\[ G_D(\tau,x|\tau',x') = G(\tau,x|\tau',x') - G(\tau,x|-\tau',x') \]

The Gaussian path integral is easily evaluated.

\[ \psi[\phi_0(x)] = \det^{1/2} G_D \exp \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \phi_0(x) \partial_\tau|_{\tau=0} \partial_{\tau'}|_{\tau'=0} G(\tau,x|\tau',x') \phi_0(x') \]

Next we express this vacuum wavefunctional in a more convenient basis. In polar coordinates on the plane, the quantum mechanical rotation generator (Rindler Hamiltonian)

\[ H_R = \sqrt{-\tau \partial_\tau r \partial_r + m^2 r^2} \]

is self-adjoint with respect to the inner product \( < \phi_1|\phi_2 > = \int_0^\infty \frac{dr}{\tau} \phi_1^* \phi_2 \); its eigenfunctions obey the expected completeness relations.

\[ \phi_E(r) = \frac{1}{\pi} \sqrt{2E \sinh \pi E} K_i \epsilon(mr) \]

\[ H_R \phi_E(r) = E \phi_E(r) \quad 0 < E < \infty \]

\[ \int_0^\infty \frac{dr}{\tau} \phi_E(r) \phi_E'(r) = \delta(E - E') \]

The Green’s function takes on a canonical thermal form when expressed in terms of these eigenfunctions \[13\].

\[ G(r, \theta|\tau', \theta') = \frac{1}{2\pi} K_0 \left( m \sqrt{r^2 + \tau'^2 - 2r\tau' \cos(\theta - \theta')} \right) \]

\[ = \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{dE}{2E} e^{-E|\theta-\theta'| + 2\pi n} |\phi_E(r)\phi_E(r')| \]

6
Entropy of entanglement, relative to a division of space at $x = 0$, is defined as the entropy of the density matrix obtained by tracing out the degrees of freedom located at $x < 0$ from the vacuum density matrix $|0><0|$. Set $\phi_0(x) = \phi_-(x)\theta(-x) + \phi_+(x)\theta(x)$. By expressing $\phi_-$ and $\phi_+$ in terms of the Rindler eigenfunctions one finds that the vacuum wavefunctional is an infinite product of harmonic oscillator propagators,

$$\psi[\phi_-, \phi_+] = \det^{1/2} G_D \exp \left( -\frac{1}{2} \int_0^\infty \frac{EdE}{\sinh \pi E} \left[ \left( \phi_+^2(E) + \phi_-^2(E) \right) \cosh \pi E - 2\phi_+(E)\phi_-(E) \right] \right)$$

where $H_E$ is the quantum mechanical Hamiltonian for a harmonic oscillator of frequency $E$. The oscillator is evolved through $\pi$ in imaginary time, from an initial coordinate $\phi_+(E)$, to a final coordinate $\phi_-(E)$. A precise definition of the product over $E$ could be given by putting the system in a box to make the spectrum discrete.

This shows that the entanglement density matrix is a thermal ensemble with respect to the Rindler Hamiltonian at an inverse temperature $\beta = 2\pi$. This is of course the Unruh effect, that the Minkowski vacuum state of a quantum field is a thermal state in Rindler space. It means that entropy of entanglement may be calculated as thermal entropy in Rindler space. Although the explicit calculation we have performed is specific to a scalar field, the formal proof that entropy of entanglement and Rindler thermal entropy are the same is quite general.

To calculate the entropy of entanglement we are lead to construct thermal ensembles with respect to the Rindler Hamiltonian at an arbitrary temperature ($\beta \neq 2\pi$). The next step in making contact with black hole entropy is to write the Rindler thermal partition function as a determinant, which in turn becomes a functional integral on ‘optical space’.

$$Z = \det^{-1/2} \left( -\partial_\theta^2 + H_R^2 \right)$$

$$= \int_{\phi(\theta + \beta, r) = \phi(\theta, r)} D\phi_{\text{optical}} \exp \left( -\int_0^\beta d\theta \int_0^\infty \frac{dr}{r} \frac{1}{2} \phi \left( -\partial_\theta^2 + H_R^2 \right) \phi \right)$$

The eigenfunctions entering in the determinant are to be periodic in $\theta$ with period $\beta$. The action in the path integral is exactly the action for a scalar field on a cone of deficit angle
$2\pi - \beta$. The integration measure is different, however, since it must be defined by

$$
\int \mathcal{D}\phi_{\text{optical}} \exp - \frac{1}{2} \int d^2x \sqrt{g_{\text{optical}}} \phi^2 = 1
$$

$$
d s_{\text{optical}}^2 = \frac{1}{r^2} (dr^2 + r^2d\theta^2) \quad 0 \leq \theta \leq \beta
$$

This optical measure is what one would obtain by writing the action for a scalar field on a cone in first order formalism, with coordinates $\phi(x)$ and momenta $\pi(x)$, and adopting the canonical measure $\mathcal{D}\pi \mathcal{D}\phi = \prod_x \frac{d\pi}{2\pi}$. Integrating out the momenta leaves the measure $\mathcal{D}\phi_{\text{optical}}$ for the integration over coordinates [20].

Rather than use these optical/canonical measures, we would like to do our calculations on a cone using a manifestly coordinate covariant measure.

$$
\int \mathcal{D}\phi_{\text{covariant}} \exp - \frac{1}{2} \int d^2x \sqrt{g_{\text{cone}}} \phi^2 = 1
$$

$$
d s_{\text{cone}}^2 = dr^2 + r^2d\theta^2 \quad 0 \leq \theta \leq \beta
$$

Evidently there is a conflict between manifest covariance and the canonical measure. As the conical and optical metrics are related by a conformal transformation, $d s_{\text{cone}}^2 = r^2 d s_{\text{optical}}^2$, with a conformal factor that is independent of $\theta$, the corresponding integration measures differ by the exponential of a term of $O(\beta)$ (a Liouville action in the case of a massless scalar field [20]). This difference does not affect the thermodynamics, which establishes that the entropy of entanglement of a scalar field may be computed from its covariant partition function on a cone.

We conclude this section with a calculation of the scalar partition function [21]. We work in arbitrary dimension on a space which is a cone of deficit angle $2\pi - \beta$ times a transverse flat $(d - 2)$ dimensional space with area $A_{\perp}$. The free energy can be written with an integral representation for the logarithm,

$$
\beta F = \frac{1}{2} \log \det (-\Box + m^2)
$$

$$
= - \frac{1}{2} \int d^2x \sqrt{g} \int_{\epsilon^2}^{\infty} \frac{ds}{s} e^{-sm^2} K(s, x, x),
$$

(2.2)

where the heat kernel $K(s, x, x)$ may be expressed as a single particle path integral.

$$
K(s, x, x') = \langle x | e^{-s(-\Box)} | x' \rangle
$$

$$
= \int_{\substack{x(t_0)=x' \\ x(t) = x}} \mathcal{D}x(\tau) \exp - \int_0^s d\tau \frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu
$$
To evaluate the heat kernel we diagonalize the Laplacian.

\[ \phi_{kl}(r, \theta) = \frac{e^{ik_\perp \cdot x_\perp}}{(2\pi)^{(d-2)/2}} (k/\beta)^{1/2} e^{i2\pi l \theta / \beta} J_{|2\pi l / \beta|}(kr) \]

\[ - \sum l \phi_{kl} = (k^2 + k_\perp^2) \phi_{kl} \quad l \in \mathbb{Z}, \quad k \in \mathbb{R}^+, \quad k_\perp \in \mathbb{R}^{d-2} \quad (2.3) \]

\[ \int d^dx \sqrt{g} \phi_{kl}^* \phi_{k'l'\perp} = \delta_{l'l'} \delta(k - k') \delta^{(d-2)}(k_\perp - k_\perp') \]

We have chosen eigenfunctions which are regular at the origin, corresponding to the Friedrichs extension of the Laplacian on a cone [22]. The heat kernel is given in terms of these eigenfunctions as

\[ K(s, x, x') = \sum_{l = -\infty}^{\infty} \int_0^\infty dk \int d^{d-2}k_\perp e^{-s(k^2 + k_\perp^2)} \phi_{klk_\perp}(x) \phi_{klk_\perp}^*(x') \]

The sum and integral may be performed [23,24], with the result that at coincident points

\[ K(s, x, x) = \frac{1}{(4\pi s)^{d/2}} \frac{1}{2\beta} \int_\infty^{-\infty} dy e^{-y^2 \cosh^2(y/2)} \left( \cot \frac{\pi}{\beta} (\pi + iy) + \cot \frac{\pi}{\beta} (\pi - iy) \right) \quad (2.4) \]

The first term \(1/(4\pi s)^{d/2}\) in the heat kernel gives rise to a divergence in the free energy, which can be absorbed in a renormalization of the cosmological constant, and which does not affect the entropy. From the remainder of the heat kernel we find

\[ \beta F_{\text{scalar}} = -\frac{\pi^2}{3\beta} A_\perp \left( 1 - \frac{\beta}{2\pi} \right)^2 \int_{\epsilon^2}^\infty ds \frac{d}{(4\pi s)^{d/2}} e^{-sm^2} \quad (2.5) \]

This is the result given in the Introduction.

3. Vector Fields

We now show that the entropy of entanglement of an Abelian gauge field in two dimensions is identically zero, while the black hole entropy is given by a non-zero surface term at the tip of the cone.

It is easiest to see that the entropy of entanglement vanishes in Coulomb gauge. In Cartesian coordinates on the plane, Coulomb gauge corresponds to setting \(A_\tau = 0\) and \(\partial_x A_x = 0\). There are no dynamical degrees of freedom in this gauge, so there can be no
correlations present across a boundary, and the entropy of entanglement unambiguously vanishes.

Defining the Rindler thermal entropy of a gauge field is more subtle. Introduce polar coordinates on the plane. It is tempting to adopt Coulomb gauge, by setting $A_\theta = 0$ and $\partial_r A_r = 0$. It seems that there are no degrees of freedom in this gauge, so the Rindler thermal entropy must vanish. But Coulomb gauge breaks down at the origin, where $A_\theta$ is ill-defined, so this argument is only valid in the bulk: it shows that the bulk Rindler thermal entropy of a two dimensional gauge field vanishes.

In order to include the origin in our treatment, we must use a different gauge. We now re-analyze the problem in covariant gauge. What we shall find is that, instead of unambiguously vanishing, the Rindler thermal entropy is ill-defined, because the Rindler Hamiltonian does not have normalizeable eigenstates.

In covariant gauge we diagonalize the quantum mechanical Rindler Hamiltonian as follows. The equations of motion in covariant gauge $\nabla_\mu F^{\mu\nu} + \frac{1}{\xi} \nabla^\nu \nabla_\mu A^\mu = 0$ are solved by the ansatz

$$A_L^\mu = \partial_\mu \phi$$
$$A_T^\mu = \sqrt{g} \epsilon_{\mu\nu} \partial^\nu \phi,$$

where $\phi$ is a solution of the scalar equation of motion $\Box \phi = 0$. We seek solutions of the form $A_\mu(r, \theta) = e^{-E \theta} A_{E \mu}(r)$. Given an eigenfunction $\phi_E(r)$ of the scalar Rindler Hamiltonian (2.1) we can construct a zero mode of the Laplacian, $\phi_E(r, \theta) = e^{-E \theta} \phi_E(r)$, and in turn a pair of eigenfunctions of the vector Rindler Hamiltonian.

$$e^{-E \theta} A_{L_\mu}^E(r) = \partial_\mu \phi_E(r, \theta)$$
$$e^{-E \theta} A_{T_\mu}^E(r) = \sqrt{g} \epsilon_{\mu\nu} \partial^\nu \phi_E(r, \theta)$$

These vector eigenfunctions are not acceptable, however. As $r \to 0$, they behave like $1/r$, and this makes them too singular to be $\delta$-function normalizeable in the inner product

$$< A | B > = \int_0^\infty \frac{dr}{r} g^{\mu\nu} A^*_\mu B_\nu.$$

We see that the quantum mechanical Rindler Hamiltonian is ill-defined in covariant gauge, and that a thermal partition function cannot be constructed, due to the bad behavior of the eigenfunctions near the origin. In this way a vector field escapes the formal proof that entropy of entanglement is equal to black hole entropy [3,7,8,9,15].

\[1\] A possible constant background electric field does not change this conclusion [25].
One can nevertheless define the partition function for a gauge field on a cone using a path integral \[26\]. In the absence of a Hamiltonian description of this path integral, the entropy obtained by varying with respect to the deficit angle of the cone can and does come out negative. We begin in two dimensions and work in covariant gauge. The free energy may be expressed as a determinant.

\[
\beta F_{\text{gauge}} = \beta F_{\text{vector}} + \beta F_{\text{ghosts}} = \frac{1}{2} \log \det \left( g^{\mu\nu}(-\Box) - R^{\mu\nu} + \left( 1 - \frac{1}{\xi} \right) \nabla^\mu \nabla^\nu \right) - \log \det(-\Box)
\]

To calculate the determinant we diagonalize the vector wave operator. The explicit curvature term \( R^{\mu\nu} \) in the wave operator is a delta function at the tip of the cone; to treat this in a well defined way we delete the point at the tip, and work on the space \( \mathbb{R}^2 - \{0\} \). Eigenfunctions of the vector wave operator may then be generated from eigenfunctions of the scalar Laplacian \( 2.3 \), using the same ansatz we used above to solve the equations of motion.

\[
A_{kl}^L(r, \theta) = \frac{1}{\xi} \partial_\mu \phi_{kl}(r, \theta) \quad \text{with eigenvalue} \quad \frac{1}{\xi} k^2 \quad k \in \mathbb{R}^+, \; l \in \mathbb{Z}
\]

\[
A_{kl}^T(r, \theta) = \frac{1}{\sqrt{g}} \epsilon_{\mu\nu} \partial^\nu \phi_{kl}(r, \theta) \quad \text{with eigenvalue} \quad k^2
\]

\[
\int d^2 x \sqrt{g} g^{\mu\nu} A_{kl \mu}^L A_{k' l' \nu}^L = \int d^2 x \sqrt{g} g^{\mu\nu} A_{kl \mu}^T A_{k' l' \nu}^T = \delta_{ll'} \delta(k - k') \tag{3.1}
\]

The vector determinant can be expressed in terms of these eigenfunctions via a heat kernel. It is necessary to introduce both ultraviolet and infrared cutoffs in \( d = 2 \).

\[
\beta F_{\text{vector}} = -\frac{1}{2} \int d^2 x \sqrt{g} \left( \int_{\epsilon_T^2}^{\infty} \frac{ds}{s} e^{-s m_L^2} g_{\mu\nu} K_{L}^{\mu\nu}(s, x, x) + \int_{\epsilon_L^2}^{\infty} \frac{ds}{s} e^{-s m_T^2} g_{\mu\nu} K_{T}^{\mu\nu}(s, x, x) \right)
\]

\[
K_{L}^{\mu\nu}(s, x, x') = \sum_{l=-\infty}^{\infty} \int_0^{\infty} dk e^{-sk^2/\xi} A_{L}^{\mu}(x) A_{L}^{\nu}(x')
\]

\[
K_{T}^{\mu\nu}(s, x, x') = \sum_{l=-\infty}^{\infty} \int_0^{\infty} dk e^{-sk^2} A_{T}^{\mu}(x) A_{T}^{\nu}(x')
\]

We fix the remaining ambiguity in these expressions by imposing some physical requirements.

First, the vector wave operator must be self-adjoint. When a vector field is expressed in terms of a scalar field using the ansatz \( 3.1 \), this becomes the requirement that the scalar
field be non-singular at the tip of the cone. So in constructing the vector determinant, we
must use the non-singular scalar eigenfunctions (2.3).

Second, we must impose BRST invariance,

\[ \delta_\chi A_\mu = i\chi \partial_\mu \eta_2 \quad \delta_\chi \eta_1 = \frac{1}{\xi} \chi \nabla_\mu A^\mu \quad \delta_\chi \eta_2 = 0 \]

where \( \eta_1, \eta_2 \) are real scalar ghosts and \( \chi \) is a Grassmann parameter. This requires that
the ghost determinant be constructed from the same scalar eigenfunctions that enter in
the vector determinant. Note that BRST invariance plus self-adjointness implies that
\( \beta F_{\text{ghost}} = -2\beta F_{\text{scalar}} \), with the scalar determinant given in (2.5).

Third, we’d like the partition function to be independent of the gauge fixing parameter
\( \xi \). The partition function is gauge invariant; it will be independent of \( \xi \) provided we use
a gauge invariant cutoff. Suppose we regulate the ghost determinant as in (2.2), with an
infrared cutoff \( m \) and an ultraviolet cutoff \( \epsilon \). If we set the cutoffs on the vector determinant
(3.2) according to

\[ \epsilon_L^2 = \xi \epsilon^2, \quad m_L^2 = m^2/\xi \]
\[ \epsilon_T^2 = \epsilon^2, \quad m_T^2 = m^2 \]

then \( \xi \) will drop out of the free energy, upon rescaling the proper time \( s \to \xi s \). No matter
what choice of covariant gauge one makes initially, with this choice of cutoffs one ends up
in Feynman gauge, with \( \xi = 1 \).

In fact this choice of cutoffs is not arbitrary. We have constructed regulated determinants, defined by

\[ \log \det L \equiv \text{Tr} \left( e^{-\epsilon^2 L} \log \epsilon^2 (L + m^2) \right). \]

A BRST transformation relates a ghost mode with eigenvalue \( k^2 \) to a longitudinal mode
with a \textit{different} eigenvalue \( k^2/\xi \). To respect BRST invariance the regulator must modify
these two eigenvalues in the same way, which fixes the above relationship between the
ghost and longitudinal cutoffs.

We continue with the calculation of the partition function, by expressing the spin-
traced vector heat kernel at coincident points in terms of the eigenfunctions (3.1). We set
\( \xi = 1 \).

\[ g_{\mu\nu} K_{\text{vector}}(s, x, x) = 2 \sum_{l=-\infty}^{\infty} \int_0^\infty \frac{dk}{k} e^{-sk^2} \left( (\partial_r J_{2\pi l/\beta}^1(kr))^2 + \left( \frac{2\pi l}{\beta r} J_{2\pi l/\beta}^1(kr) \right)^2 \right) \]
This is equal to the scalar heat kernel plus a total derivative.

\[ g_{\mu\nu}K_{\text{vector}}^{\mu\nu}(s, x, x) = 2K_{\text{scalar}}(s, x, x) + \sum_{l=-\infty}^{\infty} \int_{0}^{\infty} \frac{dk}{\beta k} e^{-sk^2} \frac{1}{r} \partial_r r \partial_r (J_{2\pi l/\beta}(kr))^2 \]

\[ = 2K_{\text{scalar}}(s, x, x) + \frac{1}{r} \int_{s}^{\infty} ds' r \partial_r K_{\text{scalar}}(s', x, x) \]

\[ = 2K_{\text{scalar}}(s, x, x) + \frac{2}{r} \partial_r s K_{\text{scalar}}(s, x, x) \]

The last equality may be checked from the explicit expression for \( K_{\text{scalar}} \) given in (2.4).
It is a consequence of dimensional analysis; on dimensional grounds the scalar heat kernel takes the form \( \frac{1}{s} f(r^2/s') \), so we can replace \( r \partial_r \) with \( -2s' \partial_s' \), making the \( s' \) integral trivial.

The free energy becomes

\[ \beta F_{\text{vector}} = -\int d^2x \sqrt{g} \int_{\epsilon^2}^{\infty} \frac{ds}{s} e^{-sm^2} K_{\text{scalar}}(s, x, x) \]

\[ + \beta \int_{\epsilon^2}^{\infty} ds e^{-sm^2} \left( K_{\text{scalar}}(s, 0, 0) - \frac{1}{4\pi s} \right) \]

The term \( 1/4\pi s \) is a surface term from \( r = \infty \), which will cancel the surface term from \( r = 0 \) if \( \beta = 2\pi \). The bulk contribution from the vector cancels against the ghosts in \( d = 2 \).

The scalar heat kernel was explicitly evaluated in (2.4); representing it as a single particle path integral leads to the result given in the Introduction, that the free energy for a gauge field in two dimensions is given solely by a surface term, which is an integral over particle paths beginning and ending on the horizon.

\[ \beta F_{gauge} = \beta F_{\text{vector}} + \beta F_{\text{ghosts}} \]

\[ = \beta \int_{\epsilon^2}^{\infty} ds \left[ \int_{x(0)=x(s)=0} Dx(\tau) \exp - \int_{0}^{s} d\tau (\frac{1}{4}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2) - \frac{e^{-sm^2}}{4\pi s} \right] \]

\[ = (2\pi - \beta) \int_{\epsilon^2}^{\infty} \frac{ds}{4\pi s} e^{-sm^2} \]

It is straightforward to extend this result to higher dimensions. We will only discuss Feynman gauge \( \xi = 1 \), since the gauge fixing parameter drops out of the free energy with a suitable choice of cutoffs, just as it did in two dimensions. It is convenient to introduce indices in the \( r-\theta \) plane, \( \alpha, \beta = r, \theta \), and indices in the transverse flat directions, \( i, j = 1, \ldots, d-2 \). The vector field \( A_\mu \) decomposes into a two dimensional vector \( A_\alpha \), and a
collection of $d-2$ scalar fields $A_i$. The vector wave operator is block diagonal in Feynman gauge.

\[
(-\Box A)\alpha = (-\nabla_\beta \nabla^\beta - \partial_j \partial^j) A_\alpha \\
(-\Box A)_i = (-\nabla_\beta \nabla^\beta - \partial_j \partial^j) A_i
\]

Note the distinction: in the first line, the covariant derivative acts on a vector, while in the second it acts on a scalar. The eigenfunctions of the vector wave operator can be expressed in terms of the scalar eigenfunctions (2.3).

\[
A_L^\alpha = \frac{1}{k} \partial_\alpha \phi_{kl\perp}, \quad A_L^i = 0 \\
A_T^\alpha = \frac{1}{k} \sqrt{g} \epsilon_{\alpha\beta} \partial_\beta \phi_{kl\perp}, \quad A_T^i = 0 \\
A_{(i)}^\alpha = 0, \quad A_{(i)}^i = \delta_{(i)}^i \phi_{kl\perp} \quad (i) = 1, \ldots, d-2
\]

This leads to an expression for the spin-traced vector heat kernel in terms of the scalar heat kernel.

\[
g_{\mu\nu} K_{\text{vector}}^{\mu\nu}(s, x, x) = (d-2)K_{\text{scalar}}(s, x, x) \\
+ 2 \sum_{l=-\infty}^{\infty} \int_0^\infty dk \int d^{d-2}k_\perp e^{-s(k^2+k_\perp^2)} \frac{1}{k^2} g^{\alpha\beta} \partial_\alpha \phi_{kl\perp} \partial_\beta \phi^*_{kl\perp} \\
= d K_{\text{scalar}}(s, x, x) \\
+ \frac{1}{r} \partial_r \sum_{l=-\infty}^{\infty} \int_0^\infty dk \int d^{d-2}k_\perp e^{-s(k^2+k_\perp^2)} \frac{1}{k^2} r \partial_r |\phi_{kl\perp}|^2 \\
= d K_{\text{scalar}}(s, x, x) + \frac{2}{r} \partial_r s K_{\text{scalar}}(s, x, x)
\]

The ghosts cancel the bulk free energy of two scalar fields, and one obtains the result given in the Introduction.

\[
\beta F_{\text{gauge}} = (d-2) \left(-\frac{1}{2}\right) \int_{\epsilon^2}^{\infty} \frac{ds}{s} \int_{\text{closed loops}} Dx(\tau) \exp - \int_0^s d\tau \frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\
+ \beta \int_{\epsilon^2}^{\infty} ds \int_{x(0)=x(s) \in \text{horizon}} Dx(\tau) \exp - \int_0^s d\tau \frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\
- \beta A_\perp \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \\
= (d-2) \beta F_{\text{scalar}} + A_\perp (2\pi - \beta) \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}}
\]

14
A few remarks on this result:

(i) The surface terms cancel at $\beta = 2\pi$, which is fortunate, because there should not be a contact interaction with the horizon when the curvature there vanishes.

(ii) There are different measures for the integral over proper time in the surface free energy compared to the bulk. The additional $1/s$ is present in the bulk because the group of global isometries of a circle, $\tau \rightarrow \tau + \text{const.}$, must be gauge fixed when performing a path integral over closed particle paths [27].

As we shall see in the next section, no surface term arises for spinor fields. Note that the vector surface term arises from the behavior of the fields in the two dimensions which contain the conical singularity. The reason gauge fields differ from spinors and scalars is that they are not conformally invariant in $d = 2$. All fields become effectively massless close to the tip of the cone, where the proper temperature goes to infinity. Scalar and spinor fields are conformally invariant in this limit, and one may use techniques from conformal field theory to obtain their entropy [28]. A conformal mapping from the cone to a cylinder shows that, for any conformal field theory, the leading divergence of the entropy is insensitive to the boundary conditions imposed at the tip of the cone. This argument does not apply to gauge fields, where a singular boundary term arises. For example, cutting out a small disc around the tip of the cone and imposing boundary conditions on the resulting edge would dramatically change the vector partition function by eliminating the surface term.

4. Spinor Fields

Having treated spins zero and one, it is natural to inquire as to what happens for spin one-half. We now show that no surface term arises, and that the contribution of a fermion to the entropy of a black hole is equal to its entropy of entanglement as well as its Rindler thermal entropy. We work in two dimensions.

We first ask whether there is a well defined Rindler Hamiltonian for fermions. Introduce the standard zweibein and spin connection on the cone.

$$ds^2 = dr^2 + r^2 d\theta^2 \quad 0 \leq \theta \leq \beta$$

$$e^1 = \cos \left( \frac{2\pi}{\beta} \theta \right) dr - r \sin \left( \frac{2\pi}{\beta} \theta \right) d\theta$$

$$e^2 = \sin \left( \frac{2\pi}{\beta} \theta \right) dr + r \cos \left( \frac{2\pi}{\beta} \theta \right) d\theta$$

$$\omega = \left( 1 - \frac{2\pi}{\beta} \right) d\theta$$

(4.1)
The spin connection has been chosen so that parallel transport around the tip of the cone generates a rotation through the deficit angle $2\pi - \beta$. At $\beta = 2\pi$ this zweibein reduces to the usual Cartesian basis of flat space, and the spin connection vanishes. An inequivalent spin structure on the cone is defined by

$$e^1 = dr, \quad e^2 = r d\theta, \quad \omega = d\theta.$$  
(4.2)

The two spin structures differ by a topologically non-trivial Lorentz rotation, which winds once around the Lorentz group as the origin is encircled. This will be important below.

The Euclidean Dirac equation $(i\nabla - m)\psi = 0$ on a cone may be presented in the form

$$\exp(-i\pi \gamma^5/\beta) \gamma^2 (\partial_\theta + H_R) \exp(i\pi \gamma^5/\beta) \psi = 0$$

where the Rindler Hamiltonian is explicitly given by

$$H_R = \begin{pmatrix} -i(\partial_r + \frac{1}{2}) & -imr \\ imr & i(\partial_r + \frac{1}{2}) \end{pmatrix}.$$  

The rotation operators $e^{i\pi \theta \gamma^5/\beta}$ are present because the zweibein (4.1) depends on $\theta$. The rotation operators make up for this, by rotating $\psi$ to the $\theta$-independent frame (4.2), where the rotational invariance of the cone is manifest, and rotations are generated by a Rindler Hamiltonian that is independent of $\theta$. $H_R$ is self-adjoint in the inner product $<\psi_1|\psi_2> = \int_0^\infty dr \psi_1^\dagger(r)\psi_2(r)$, and has a complete set of eigenfunctions.

$$\psi_E(r) = \frac{1}{\pi} \sqrt{m \cosh \pi E} \begin{pmatrix} K_{iE-1/2}(mr) \\ K_{iE+1/2}(mr) \end{pmatrix},$$

$$H_R \psi_E = E \psi_E \quad -\infty < E < \infty$$

$$\int_0^\infty dr \psi_E^\dagger(r) \psi_{E'} = \delta(E - E').$$

We see that there is no difficulty in constructing a Rindler Hamiltonian for fermions. The formal argument showing that entropy of entanglement is identical to Rindler thermal entropy applies to fermions [3,7,8,9,15]. To construct the entropy of entanglement for fermions directly from its definition one must introduce a functional state space for fermions [29]. The entropy of entanglement for fermions has been calculated directly from its definition by Larsen and Wilczek [30].

---

2 Conventions: $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^2$, $\gamma^5 = i\gamma^1\gamma^2 = \sigma^3$, $\bar{\psi} = \psi^\dagger \gamma^2$. $\Sigma_{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, $\nabla_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab} = \partial_\mu - \frac{i}{2}\omega^{ab}_\mu \gamma^b$.  

16
To make contact with black hole entropy, we write the Rindler thermal partition function as a determinant,

$$\beta F_{\text{Dirac}} = -\frac{1}{2} \log \det(-\partial_\theta^2 + H_R^2),$$

where the determinant is over functions which are anti-periodic in $\theta$. This determinant may be represented as a functional integral over anti-periodic Grassmann fields.

$$\beta F_{\text{Dirac}} = -\log \det i (\partial_\theta + H_R)$$

$$= -\log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \int_0^\beta d\theta \int_0^\infty dr \, \psi^\dagger \, i \, (\partial_\theta + H_R) \, \psi$$

$$= -\log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp -\int d^2 x \sqrt{g_{\text{cone}}} \, \bar{\psi} \, \frac{i}{r} \gamma^2 \, (\partial_\theta + H_R) \, \psi$$

$$= -\log \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp -\int d^2 x \sqrt{g_{\text{cone}}} \, \bar{\psi} \, (i \nabla - m) \psi$$

One gets the classical action for a fermion on a cone, but with the other choice of spin structure. That is, in the last line $\nabla$ is constructed from (4.2), which differs from the standard spin connection (4.1) by a Lorentz rotation which varies from zero to $2\pi$ as the origin is encircled. One may revert to the standard spin connection on a cone, at the price of changing the fermion boundary conditions from anti-periodic to periodic. So in the end we have an expression for entropy of entanglement in terms of a functional integral over fermions which are periodic on a cone.

There is also the question of the appropriate measure in the functional integral. As in the scalar case, the measure that reproduces the Rindler thermal partition function is the canonical measure on a cone, which differs from the covariant measure by a term of $O(\beta)$ in the free energy. This difference does not affect the thermodynamics, so we will neglect it, and proceed to calculate the fermion free energy using the covariant measure.

The free energy can be expressed in terms of a heat kernel.

$$\beta F = -\log \det (i \nabla - m)$$

$$= -\frac{1}{2} \log \det (\nabla^2 + m^2)$$

$$= \frac{1}{2} \int d^2 x \sqrt{g} \int_0^\infty ds \frac{ds}{s} e^{-sm^2} \text{Tr} \, K_{\text{Dirac}}(s, x, x)$$
The heat kernel is constructed from eigenfunctions of the Dirac operator. For $\beta < 2\pi$ the Dirac operator has a complete set of non-singular eigenfunctions \[31\].

$$\psi_{kl} = \sqrt{k/2\beta} \left( e^{i2\pi l \theta / \beta} J_{|\nu|}(kr) - i \text{sign}(\nu) e^{i2\pi (l+1) \theta / \beta} J_{|\nu|+\text{sign}(\nu)}(kr) \right)$$

$$\nu \equiv \frac{2\pi}{\beta} (l + \frac{1}{2}) - \frac{1}{2}, \quad k \in \mathbb{R}^+, \quad l \in \mathbb{Z}$$

$$\nabla \psi_{kl} = k \psi_{kl} \quad \nabla \gamma^5 \psi_{kl} = -k \gamma^5 \psi_{kl}$$

$$\int d^2x \sqrt{g} \psi_{kl}^\dagger \psi_{k'\ell'} = \delta_{ll'} \delta(k-k')$$

For $\beta > 2\pi$ these eigenfunctions develop a singularity at the origin. In fact there are no non-singular eigenfunctions of the Dirac operator when $\beta > 2\pi$, as using the other set of Bessel functions $Y_\nu(z)$ also leads to singular eigenfunctions. This necessitates a non-trivial self-adjoint extension of the Dirac operator \[31\]. The self-adjoint extension parameter must be chosen so as to make the free energy analytic across $\beta = 2\pi$. A simpler procedure, which we will adopt, is to calculate for $\beta < 2\pi$ and define the free energy for $\beta > 2\pi$ by analytic continuation. The heat kernel may be evaluated using the same techniques as in the scalar case \[23,24\]. At coincident points the spin traced heat kernel is given by

$$\text{Tr} K_{\text{Dirac}}(s, x, x) = \text{Tr} \sum_{l=-\infty}^{\infty} \int_0^\infty dk e^{-sk^2} \left( \psi_{kl}(x) \psi_{k'l}(x) + \gamma^5 \psi_{kl}(x) \psi_{k'l}(x) \gamma^5 \right)$$

$$= \frac{1}{2\pi s} + \frac{1}{\beta 2\pi s} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{2}} \cosh^2(y/2) \frac{i \sinh y/2}{\sin \frac{\pi}{\beta} (\pi + iy)}.$$

Substituting this into the expression for the free energy, we find the same free energy as a scalar field in two dimensions, aside from a difference in the cosmological constant which we neglect.

$$\beta F_{\text{Dirac}} = -\frac{\pi^2}{3\beta} \left( 1 - \left( \frac{\beta}{2\pi} \right)^2 \right) \int_{\epsilon^2}^{\infty} ds \frac{e^{-sm^2}}{4\pi s} e^{-sm^2}$$

This result was given in the Introduction; it leads to the same entropy at $\beta = 2\pi$ as was calculated from the renormalization of Newton’s constant in $d = 2$.

5. Conclusions

In exploring the relationship between black hole entropy and entropy of entanglement, we have seen that, for scalar and spinor fields, the two are identical, while for gauge fields,
they differ by a contact interaction with the horizon which appears in the black hole entropy but not in the entropy of entanglement. The contact interaction makes a negative contribution to the entropy of a black hole, and is responsible for the non-renormalization of Newton’s constant in certain supersymmetric theories. It does not have a thermal or state counting interpretation within quantum field theory. It is the field theoretic residue of the interaction proposed by Susskind and Uglum which couples a closed string to an open string stranded on the horizon.

The classical entropy of a black hole also arises from a contact term localized on the horizon \[1,2\]. A state counting interpretation of these contact terms is only possible if quantum gravity introduces a degree of non-locality, which resolves the point-like contact terms into extended interactions for which a notion of state counting can exist. It seems plausible that such an effect does occur, as one would expect quantum gravity to delocalize the horizon by at least a Planck length. A concrete proposal has been put forth by Susskind and Uglum, in which the extended nature of fundamental strings provides a resolution of the contact interactions and leads to a state counting interpretation of the classical black hole entropy \[4\].

We are led to investigate these issues in theories which are non-local at short distances. One way to obtain such behavior within ordinary quantum field theory is to study the behavior of a composite field, constructed in a non-local way from elementary fields. Such constructions arise naturally in the low energy description of certain large \(N\) field theories \[32\].

Acknowledgments

I am grateful to my collaborators Steve Shenker and Matt Strassler for their many suggestions, and to Roberto Emparan and Leonard Susskind for valuable discussions. This work was supported by the D.O.E. under contract #DE-FG05-90ER40559. Part of this work was completed at the Aspen Institute for Physics.
References

[1] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **72**, 957 (1994), gr-qc/9309026.
[2] S. Carlip and C. Teitelboim, gr-qc/9312002.
[3] W. Israel, *Phys. Lett.* **A57**, 107 (1976).
[4] G. ’t Hooft, *Nucl. Phys.* **B256**, 727 (1985).
[5] L. Bombelli, R. K. Koul, J. Lee, R. D. Sorkin, *Phys. Rev.* **D34**, 373 (1986).
[6] M. Srednicki, *Phys. Rev. Lett.* **71**, 666 (1993).
[7] L. Susskind, hep-th/9309143.
  L. Susskind and J. Uglum, *Phys. Rev.* **D50**, 2700 (1994), hep-th/9401070.
[8] C. Callan and F. Wilczek, *Phys. Lett.* **B333**, 55 (1994), hep-th/9401072.
[9] D. Kabat and M. Strassler, *Phys. Lett.* **B329**, 46 (1994), hep-th/9401123.
[10] J. S. Dowker, *Class. Quant. Grav.* **11**, L55 (1994), hep-th/9401159.
[11] S. Solodukhin, *Phys. Rev.* **D51**, 609 (1995), hep-th/9407700.
  D. Fursaev, hep-th/9408066.
  J. G. Demers, R. Lafrance, and R. Myers, gr-qc/9503003.
[12] M. Bordag and A. Bytsenko, gr-qc/9412054.
[13] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge, 1982) section 6.2.
[14] Orbifolds may be used at a discrete infinity of temperatures. A. Dabholkar, hep-th/9408098.
[15] See for example G. L. Sewell, *Ann. Phys.* **141**, 201 (1982);
  S. Fulling and S. Ruijsenaars, *Phys. Rep.* **152**, 135 (1987).
[16] W. G. Unruh, *Phys. Rev.* **D14**, 870 (1976).
[17] P. Ramond, *Field Theory: A Modern Primer* (Addison-Wesley, 1989) section 3.7.
[18] J. L. F. Barbon, *Phys. Rev.* **D50**, 2712 (1994), hep-th/9402004.
[19] R. Emparan, hep-th/9407064.
[20] S. P. de Alwis and N. Ohta, hep-th/9412027.
[21] G. Cognola, K. Kirsten, and L. Vanzo, *Phys. Rev.* **D49**, 1029 (1994), hep-th/9308106.
  D. Fursaev, *Class. Quant. Grav.* **11**, 1431 (1994), hep-th/9309050.
[22] B. S. Kay and U. M. Studer, *Commun. Math. Phys.* **139**, 103 (1991).
[23] J. Dowker, *J. Phys.* **A 10**, 115 (1977);
  J. Dowker, *Phys. Rev.* **D36**, 3742 (1987).
[24] S. Deser and R. Jackiw, *Commun. Math. Phys.* **118**, 495 (1988).
[25] A. P. Balachandran, L. Chandar, and E. Ercolessi, hep-th/9411164.
[26] Gauge fields on non-trivial space-times have been discussed recently by D. V. Vassilevich, gr-qc/9404052.
  D. V. Vassilevich, gr-qc/9411036.
[27] A. M. Polyakov, *Gauge Fields and Strings* (Harwood 1987), sections 9.1 – 9.3.
[28] C. Holzhey, F. Larsen, and F. Wilczek, *Nucl. Phys.* **B424**, 443 (1994), hep-th/9403108.

[29] R. Floreanini and R. Jackiw, *Phys. Rev.* **D37**, 2206 (1988); R. Jackiw, in *Field Theory and Particle Physics*, O. Eboli, M. Gomez, and A. Santoro, eds. (World Scientific, 1990).

[30] F. Larsen and F. Wilczek, hep-th/9408089.

[31] P. de Sousa Gerbert and R. Jackiw, *Commun. Math. Phys.* **124**, 229 (1989).

[32] D. Kabat, S. Shenker, and M. Strassler, hep-th/9506182.