NOTE ON THE EULER NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper we investigate the properties of the Euler functions. By using the Fourier transform for the Euler function, we derive the interesting formula related to the infinite series. Finally we give some interesting identities between the Euler numbers and the second kind stirling numbers.

§1. Introduction/Definition

The constants $E_k$ in the Taylor series expansion

\begin{equation}
F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ for } |t| \leq \pi, \text{ (see [1-31]),}
\end{equation}

are known as the Euler numbers. The first few are $1, -\frac{1}{2}, 0, \frac{1}{4}, \cdots$, and $E_{2k} = 0$ for $k = 1, 2, 3, \cdots$. These numbers arise in the series expansions of trigonometric functions, and are extremely important in number theory and analysis. The Euler polynomials, $E_n(x)$, are defined as

\begin{equation}
F(x, t) = F(t)e^{xt} = \frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ for } x \in \mathbb{R}.
\end{equation}

For $x \in \mathbb{R}$ with $0 \leq x < 1$, the Euler polynomials are called the Euler functions. From (1) and (2) we can derive

\begin{equation}
E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l}, \text{ where } \binom{n}{l} = \frac{n(n-1) \cdots (n-l+1)}{l!}.
\end{equation}

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Thus, we obtain the distribution relation for the Euler polynomials as follows. For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), we have

\[
\sum_{k=0}^{d-1} (-1)^k E_n \left( \frac{x+k}{d} \right) = d^{-n} E_n(x).
\]

By (1), it is easy to see that the recurrence relation for the Euler numbers is given by

\[
E_0 = 1, \quad \text{and,} \quad \sum_{l=0}^{n} \binom{n}{l} E_l + E_n = 2\delta_{0,n} \quad \text{where} \quad \delta_{0,n} \text{ is Kronecker symbol.}
\]

From (3) and (4), we note that

\[
E_n(1) = \sum_{l=0}^{n} \binom{n}{l} E_l = -E_n, \quad \text{for} \quad n \geq 1.
\]

Thus, we obtain the following lemma.

**Lemma 1.** For \( n \in \mathbb{N} \), we have \( E_n(1) = -E_n \).

From (3) we can easily derive

\[
\frac{dE_n(x)}{dx} = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (n-k) E_k x^{n-k-1}
\]

\[
= n \sum_{k=0}^{n} \frac{(n-1)!}{(n-k-1)!k!} E_k x^{n-1-k} = n \sum_{k=0}^{n-1} \binom{n-1}{k} E_k x^{n-1-k} = nE_{n-1}(x).
\]

By (6), we obtain the following proposition.

**Proposition 2.** For \( n \geq 0 \), we have

\[
\int_0^x E_n(t) dt = \frac{1}{n+1} E_{n+1}(x).
\]

In this paper we investigate the properties of the Euler functions. By using the Fourier transform for the Euler function, we derive the interesting formula related to the infinite series. Finally we give some interesting identities between the Euler numbers and the second kind stirling numbers.

\[\text{§2. Euler Functions}\]
In this section, we assume that $E_n(x)$ is the Euler function. Let us consider the Fourier transform for the Euler function, $E_n(x)$, as follow. For $m \in \mathbb{N}$, the Fourier transform on the Euler function is given by
\begin{equation}
E_m(x) = \sum_{n=-\infty}^{\infty} a_n^{(m)} e^{(2n+1)\pi i x}, \quad (a_n^{(m)} \in \mathbb{C}),
\end{equation}
where
\begin{equation}
a_n^{(m)} = \int_0^1 E_m(x) e^{-(2n+1)\pi i x} dx.
\end{equation}
From (9), we note that
\begin{equation}
a_n^{(m)} = \int_0^1 E_m(x) e^{-\pi i (2n+1)x} dx
= \frac{(2n+1)\pi i}{m+1} \int_0^1 E_{m+1}(x) e^{-(2n+1)\pi i x} dx = \frac{(2n+1)\pi i}{m+1} a_n^{(m+1)}.
\end{equation}
Thus, we have
\begin{equation}
a_n^{(m)} = \frac{m}{(2n+1)\pi i} a_n^{(m-1)} - \frac{m(m-1)}{(2n+1)\pi i^2} a_n^{(m-2)} = \ldots = \frac{m!}{((2n+1)\pi i)^{m-1}} a_n^{(1)}.
\end{equation}
It is easy to see that
\begin{equation}
a_n^{(1)} = \int_0^1 E_1(x) e^{-(2n+1)\pi i x} dx = \int_0^1 (x-\frac{1}{2}) e^{-(2n+1)\pi i x} dx
= \frac{2}{((2n+1)\pi i)^2}.
\end{equation}
From (11) and (13), we can derive
\begin{equation}
a_n^{(m)} = 2 \frac{m!}{((2n+1)\pi i)^{m+1}}, \quad m \in \mathbb{N}, \text{ and } a_n^{(0)} = \frac{2}{(2n+1)\pi i}.
\end{equation}
By (8) and (13), we see that
\begin{equation}
E_m(x) = m!2 \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi i x}}{((2n+1)\pi i)^{m+1}}, \quad \text{for } 0 \leq x < 1.
\end{equation}
Therefore, we obtain the following theorem.
Theorem 3. For \( m \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\} \), \( x \in \mathbb{R} \) with \( 0 \leq x < 1 \), we have

\[
E_m(x) = m!2 \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\pi ix}}{((2n+1)\pi i)^{m+1}}.
\]

If we take \( x = 1 \), then we have

\[
E_m(1) = -m!2 \sum_{n=-\infty}^{\infty} \frac{1}{((2n+1)\pi i)^{m+1}}.
\]  \( \text{(14)} \)

By (14) and Lemma 1, we obtain the following corollary.

Corollary 4. For \( m \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\} \), we have

\[
E_m = m!2 \sum_{n=-\infty}^{\infty} \frac{1}{((2n+1)\pi i)^{m+1}}.
\]

From Corollary 4, we note that

\[
E_{2m+1} = (-1)^{m+1}2(2m+1)! \frac{\pi}{2^{2m+2}} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{2m+2}}.
\]

Thus, we have

\[
\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^{2m+2}} = (-1)^{m+1} \frac{E_{2m+1}}{2(2m+1)! \pi^{2m+2}}.
\]  \( \text{(15)} \)

By (15), we obtain the following corollary.

Corollary 5. For \( m \in \mathbb{Z}^+ \), we have

\[
\sum_{n=1}^{\infty} \frac{1}{(2n+1)^{2m+2}} = (-1)^{m+1} \frac{E_{2m+1}}{4(2m+1)! \pi^{2m+2}}.
\]

Note that

\[
\frac{1}{1+e^{-x}} = \sum_{n=0}^{\infty} \frac{e^{-nx}(-1)^n}{n!} = \sum_{n=0}^{\infty} (e^{-x})^n (-1)^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \right)^n (-1)^n
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \left( \sum_{a_1+a_2+\cdots=n} \frac{n!}{a_1!a_2!\cdots (1!)^{a_1} (2!)^{a_2} \cdots} \right) x^{a_1+2a_2+\cdots}.
\]  \( \text{(16)} \)
Let \( p(i, j) : a_1 + 2a_2 + \cdots = i, a_1 + a_2 + \cdots = j \). From (16), we note that

\[
\frac{1}{1 + e^{-x}} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} (-1)^n \sum_{p(m,n)} \frac{n!}{a_1!a_2! \cdots a_m! (1!)^{a_1} \cdots (m!)^{a_m}} \right) x^m/m!
\]

(17)

\[
= \sum_{m=0}^{\infty} (-1)^m \sum_{n=0}^{m} n!(-1)^n \sum_{p(m,n)} \frac{m!}{a_1!a_2! \cdots a_m! (1!)^{a_1} (2!)^{a_2} \cdots (m!)^{a_m}} \frac{(-1)^m}{m!} x^m/m!
\]

= \sum_{m=0}^{\infty} (-1)^m \sum_{n=0}^{m} n!(-1)^n s_2(m, n) \frac{x^m}{m!},

where \( s_2(m, n) \) is the second kind stirling number.

By the definition of Euler number, we easily see that

\[
\frac{1}{1 + e^{-x}} = \frac{1}{2} \left( \frac{2}{1 + e^{-x}} \right) = \frac{1}{2} \sum_{m=0}^{\infty} (-1)^m E_m \frac{x^m}{m!}, \quad \text{see [5] }.
\]

(18)

By comparing the coefficients of \( \frac{x^n}{n!} \) on the both sides of (17) and (18), we obtain

\[
E_m = 2 \sum_{n=0}^{\infty} (-1)^n n! s_2(m, n),
\]

where \( s_2(m, n) \) is the second kind stirling number.

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