SEIDEL’S CONJECTURES IN HYPERBOLIC 3-SPACE

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Abstract. We prove, in the case of hyperbolic 3-space, a couple of conjectures raised by J. J. Seidel in On the volume of a hyperbolic simplex, Stud. Sci. Math. Hung. 21, 243-249, 1986. These conjectures concern expressing the volume of an ideal hyperbolic tetrahedron as a monotonic function of algebraic maps. More precisely, Seidel’s first conjecture states that the volume of an ideal tetrahedron in hyperbolic 3-space is determined by (the permanent and the determinant of) the doubly stochastic Gram matrix $G$ of its vertices; Seidel’s fourth conjecture claims that the mentioned volume is a monotonic function of both the permanent and the determinant of $G$.

1. Introduction

It is a typical phenomenon in hyperbolic geometry\(^1\) that explicit formulae for calculating even simple geometric invariants involve transcendental functions. These functions can be quite sophisticated and such is usually the case when it comes to volume-related problems which are our main concern in this paper. A way to deal with this difficulty is to express the geometric invariants in question as monotonic functions of algebraic maps: in a certain sense, this allows us to ‘replace’ complicated geometric invariants by much simpler ones.

As a toy example, let us consider the distance function in the projective model of hyperbolic $n$-space. Take an $(n+1)$-dimensional $\mathbb{R}$-linear space $V$ equipped with a bilinear symmetric form of signature $- - \cdots - +$. The hyperbolic $n$-space is the open $n$-ball of positive points $\mathbb{H}^n := \{ p \in PV \mid (p, p) > 0 \}$. (We denote respectively by $p$ and $P$ a point $p \in PV$ and a representative $P \in V$.) The ideal boundary of $\mathbb{H}^n$, also known as the absolute, is the $(n-1)$-sphere $\partial \mathbb{H}^n := \{ p \in PV \mid (p, p) = 0 \}$. Hyperbolic $n$-space is endowed with the distance function

$$d(p, q) := \arccosh \sqrt{\frac{(p, q)(q, p)}{(p, p)(q, q)}}, \quad p, q \in \mathbb{H}^n.$$ 

Clearly, distance is a monotonic function of the tance $\text{ta}(p, q) := \frac{(p, q)(q, p)}{(p, p)(q, q)}$. Due to its algebraic nature, it is usually much simpler to use the tance instead of the distance in applications (see, for instance, [AGG], [AGr2], [Ana1], [Ana2]). J. J. Seidel’s conjectures [Sei] concern applying a similar idea to the case of the volume of an ideal simplex in $\mathbb{H}^n$.

A (labelled) ideal simplex in $\mathbb{H}^n$ is an $(n+1)$-tuple $(v_1, \ldots, v_{n+1})$ of ideal points $v_i \in \partial \mathbb{H}^n$ called the vertices of the ideal simplex. The volume of an ideal simplex is the hyperbolic volume of the convex hull of the points $v_1, \ldots, v_{n+1}$. In dimension 3, for example, the volume of an ideal tetrahedron — an ideal tetrahedron — can be expressed (see Section 5) in terms of its dihedral angles and the Lobachevsky function $\Lambda : \mathbb{R} \to \mathbb{R}$,

$$\Lambda(\theta) := -\int_0^\theta \log |2\sin t| \, dt.$$ 

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\(^1\)As well as in several other classic geometries (see [AGr2]).
Let $S := (v_1, \ldots, v_{n+1})$ be an ideal simplex in $\mathbb{H}^{n}$. Choosing representatives $v_i \in V$ we obtain a Gram matrix $G$ of the vertices of $S$, where $G := [g_{ij}]$, $g_{ij} := \langle v_i, v_j \rangle$. Among all the Gram matrices of the vertices of $S$, there is a single one that is doubly stochastic (a square matrix is called doubly stochastic if all its entries are non-negative and the sum of entries in every row and in every column equals 1). Let $G_{ds}$ stand for the doubly stochastic Gram matrix of the vertices of $S$.

J. J. Seidel’s first conjecture is that the volume of $S$ should be determined by (some natural algebraic functions of) the entries of $G_{ds}$. The fourth conjecture states that the volume of $S$ is a monotonic function of both the determinant and the permanent of $G_{ds}$. We prove these conjectures in the case of hyperbolic 3-space. Two observations are in order here. First, we establish in Theorem 4.1 a stronger version of the first conjecture by showing that permanent and determinant actually serve as coordinates of the space of ideal tetrahedra modulo isometries (we also provide in Theorem 5.1 an explicit formula for the volume of an ideal tetrahedron in terms of the permanent and the determinant of $G_{ds}$). Secondly, Seidel’s original fourth conjecture says that the volume is a decreasing function of the permanent, while it is in fact an increasing one (see Theorem 6.10). Trying to understand what may have led Seidel to believe that the volume is decreasing in the permanent, we proved his third conjecture (see Theorem 6.11 and the related Proposition 6.12) which is a little bit technical and, in the particular low-dimensional case we consider, quite simple to show.

It should be emphasized that taking the doubly stochastic Gram matrix of the vertices of $S$ seems to be crucial to Seidel’s conjectures. In fact, N. V. Abrosimov [Abr] proved that the variant of the first conjecture where one considers, instead of $G_{ds}$, a normalized Gram matrix of the points polar to the faces of $S$ (polar points are discussed in Sections 2.1 and 2.2), actually does not hold in full generality: in this case, the volume of an ideal tetrahedron can be expressed as a function of determinant and permanent if and only if the tetrahedron is either acute-angled or obtuse-angled [Abr, Theorem 3 and Example 1]. Moreover, when we tried to consider Gram matrices of the vertices of $S$ which were not doubly stochastic (there are other apparently ‘natural’ choices of representatives), the fourth conjecture turned out to be false.

Seidel cites some results about doubly stochastic Gram matrices of vertices of simplices that are intended to justify this particular choice of Gram matrices in his conjectures [Sei, Facts 1, 4, 6]. Some of these facts involve relationships between the exterior and symmetric algebras of a linear space. Curiously, determinant and permanent are particular cases of what Littlewood-Richardson called the immanants of a matrix [LiR]. Immanants are closely related to Schur functors which, in their turn, seem to play an important role in hyperbolic geometry as well as in other classic geometries. (Exterior and symmetric powers are the Schur functors corresponding respectively to the determinant and permanent.) For instance, calculating the dihedral angles of an ideal tetrahedron in terms of the doubly stochastic Gram matrix of its vertices involves the use of the exterior power functor and of the Hodge star operator (see Section 2.2 and Proposition 2.2.4).

In Section 7 we briefly discuss the relationship between immanants and Schur functors; it is reasonable to expect that other immanants (besides the determinant and permanent) will be involved in possible generalizations of J. J. Seidel’s conjectures to higher dimensions and/or to more general polyhedra (not necessarily ideal ones).

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2 Some particular (very) degenerate ideal simplices do not admit a doubly stochastic Gram matrix (see Remark 3.2.3).
3 The permanent of an $m \times m$ matrix $A = [a_{ij}]$ is defined by the expression $\text{per} \ A := \sum_{\sigma \in S_m} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{m\sigma(m)}$, where $S_m$ stands for the symmetric $m$-group.
4 Well, it is possible to take $DG_{ds}D$ in place of $G_{ds}$, where $D$ is a (fixed) diagonal matrix whose diagonal entries are non-null constants, as this only multiplies the determinant and permanent of every $G_{ds}$ by the same positive number $(\det D)^2$. 
2. Preliminaries

2.1. Hyperbolic 3-space. The approach to hyperbolic geometry in this section follows [AGr2].

Let $V$ be an $\mathbb{R}$-linear space equipped with a bilinear symmetric form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ of signature $- - +$. This form divides the projective 3-space $\mathbb{P}V$ into positive, null (or ideal), and negative points:

$$ B V := \{ p \in \mathbb{P}V \mid \langle p, p \rangle > 0 \}, \quad S V := \{ p \in \mathbb{P}V \mid \langle p, p \rangle = 0 \}, \quad E V := \{ p \in \mathbb{P}V \mid \langle p, p \rangle < 0 \}. $$

(As stated in the introduction, we use the notation $p$ and $P$ respectively for a point $p \in \mathbb{P}V$ and a representative $p \in V$.) Note that $B V$ is an open 3-ball whose boundary is the 2-sphere $S V$, called the absolute.

Let $p \in \mathbb{P}V$ be a non-ideal point. We have the well-known natural identification

$$ (2.1.1) \quad T_p \mathbb{P}V = \text{Lin}(\mathbb{R} p, p^\perp) $$

of the tangent space to $\mathbb{P}V$ at $p$ and the $\mathbb{R}$-linear space of linear maps from the subspace generated by $p$ to its orthogonal $p^\perp$ with respect to $\langle \cdot, \cdot \rangle$. Given tangent vectors $\varphi_1, \varphi_2 \in \text{Lin}(\mathbb{R} p, p^\perp)$ at a point $p \in \mathbb{P}V$, we define

$$ (2.1.2) \quad \langle \varphi_1, \varphi_2 \rangle := - \frac{\langle \varphi_1(p), \varphi_2(p) \rangle}{\langle p, p \rangle}. $$

This endows $B V$ with a Riemannian metric known as the hyperbolic metric and $E V$ with a Lorentzian metric sometimes called the de Sitter metric. We call $B V$ the hyperbolic 3-space.

Any 2-dimensional linear subspace $W \subseteq V$ of signature $- +$ gives rise to a geodesic $\mathbb{P}W \cap B V$ in $B V$ and every geodesic in $B V$ is obtained in this way. (An analogous statement holds for de Sitter space.) Any 3-dimensional linear subspace $W \subseteq V$ of signature $- + +$ gives rise to a totally geodesic surface $\mathbb{P}W \cap B V$ in $B V$ known as a plane. There is a simple duality between planes in $B V$ and points in $E V$: the negative point $\mathbb{P}W^\perp \in E V$ corresponds to the plane $\mathbb{P}W \cap B V$, where $W \subseteq V$ is a 3-dimensional linear subspace of signature $- + +$. In other words, the Lorentzian manifold $E V$ is simply the space of all planes in $B V$. The point $\mathbb{P}W^\perp \in E V$ is called the polar point to the plane $\mathbb{P}W$.

In what follows, we will denote the hyperbolic 3-space $B V$ by $\mathbb{H}^3$. The projective space $\mathbb{P}V$ will be referred to as the extended hyperbolic 3-space.

2.2. Angle. Let us apply the duality between points and planes in the extended hyperbolic space in order to find an expression for the angle between two planes. This expression will be used later to find the dihedral angles of an ideal tetrahedron in terms of the entries of a certain Gram matrix of the tetrahedron (see Remark 3.2.4). First, we remind the reader a few basic facts about the Hodge star operator. The results in this subsection are related to those in [Moh, Section 2.2].

Let $U$ stand for an $\mathbb{R}$-linear space equipped with a non-degenerate bilinear symmetric form $\langle \cdot, \cdot \rangle : U \times U \to \mathbb{R}$ of arbitrary signature $(n, m)$, where $N := \dim U = n + m$ and $n$ denotes the negative part of the signature. Let $\sigma = 1$ if $n = 0 \mod 2$ and $\sigma = -1$ otherwise.

The $k$-th exterior power $\bigwedge^k U$, $1 \leq k \leq N$, is equipped with the bilinear symmetric form defined by

$$ (v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) := \det[g_{ij}], \quad g_{ij} := \langle v_i, w_j \rangle. $$

This form on $\bigwedge^k U$ is non-degenerate. Indeed, let $(b_1, \ldots, b_N)$ be an orthonormal basis for $U$, that is, $\langle b_i, b_i \rangle = \sigma_i$ and $\langle b_i, b_j \rangle = 0$ for $i \neq j$, where $\sigma_i = \pm 1$. Then $(b_{i_1} \wedge \cdots \wedge b_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq N)$ is an orthonormal basis for $\bigwedge^k U$. 

Fix \( \omega \in \Lambda^N U \) such that \( \langle \omega, \omega \rangle = \sigma \). The Hodge star operator is the \( \mathbb{R} \)-linear map \( * : \Lambda^k U \to \Lambda^{N-k} U, \ b \mapsto *b \), defined by requiring that \( a \wedge *b = \langle a, b \rangle \omega \) for every \( a \in \Lambda^k U \). Clearly, \( *\omega = \sigma \) since \( \omega \wedge *\omega = \langle \omega, \omega \rangle \omega = \sigma \omega \).

The following proposition is an adaptation of [Huy, Proposition 1.2.20] to the case of a non-degenerate form of arbitrary signature. The properties of the Hodge star operator in the proposition seem to be well-known but we could not find a direct reference. The proofs are elementary and provided only for the sake of completeness.

2.2.1. Proposition. The Hodge star operator is injective. It satisfies the following identities:

- \( a \wedge *b = b \wedge *a \) for every \( a, b \in \Lambda^k U \)
- \( \langle a, b \rangle = \sigma \cdot *\langle a \wedge b \rangle \) for every \( a, b \in \Lambda^k U \)
- \( \langle a, *b \rangle = (-1)^{(N-k)} \langle *a, b \rangle \) for every \( a \in \Lambda^k U \) and \( b \in \Lambda^{N-k} U \)
- \( *\langle a \rangle = (-1)^{(N-k)} \sigma \langle a \rangle \) for every \( a \in \Lambda^k U \)

Proof. If \( *b = 0 \), then \( a \wedge *b = \langle a, b \rangle \omega = 0 \) for every \( a \in \Lambda^k U \) which implies \( b = 0 \) because the form on \( \Lambda^k U \) is non-degenerate.

The first identity is obvious. The second follows from \( *\langle a \wedge b \rangle = \langle a, b \rangle \cdot *\omega = \sigma \langle a, b \rangle \). The third follows from the second:

\[
\langle a, *b \rangle = \sigma \cdot *\langle a \wedge b \rangle = (-1)^{(N-k)} \sigma \cdot *\langle a \wedge *b \rangle = (-1)^{(N-k)} \langle *a, b \rangle.
\]

As for the last equality, take an orthonormal basis \( (b_1, \ldots, b_N) \) in \( U \), that is, \( (b_i, b_i) = \sigma_i = \pm 1 \) and \( \langle b_i, b_j \rangle = 0 \) for \( i \neq j \). We can assume that \( \omega = b_1 \wedge \cdots \wedge b_N \). Fix \( i_1 < i_2 < \cdots < i_k \) and let \( j_1 < j_2 < \cdots < j_{N-k} \) be the indices complementary to \( i_1 < i_2 < \cdots < i_k \). Let us show that

\[
*\langle b_{i_1} \wedge \cdots \wedge b_{i_k} \rangle = \text{sgn}(h) \cdot \sigma_{i_1} \ldots \sigma_{i_k} \cdot b_{j_1} \wedge \cdots \wedge b_{j_{N-k}},
\]

where \( h \) is the permutation \( (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_{N-k}) \mapsto (1, 2, \ldots, N) \). Indeed,

\[
(b_{i_1} \wedge \cdots \wedge b_{i_k}) \wedge (\text{sgn}(h) \cdot \sigma_{i_1} \ldots \sigma_{i_k} \cdot b_{j_1} \wedge \cdots \wedge b_{j_{N-k}}) = \begin{cases} 0, & \text{if } (l_1, \ldots, l_k) \neq (i_1, \ldots, i_k) \\ \sigma_{i_1} \ldots \sigma_{i_k} \cdot \omega, & \text{if } (l_1, \ldots, l_k) = (i_1, \ldots, i_k) \end{cases}
\]

\[
= \langle b_{i_1} \wedge \cdots \wedge b_{i_k}, b_{j_1} \wedge \cdots \wedge b_{j_{N-k}} \rangle \omega
\]

for every \( l_1 < \cdots < l_k \). It remains to observe that

\[
*\langle *\langle b_{i_1} \wedge \cdots \wedge b_{i_k} \rangle \rangle = \text{sgn}(h) \cdot \sigma_{i_1} \ldots \sigma_{i_k} \cdot *\langle b_{j_1} \wedge \cdots \wedge b_{j_{N-k}} \rangle = (-1)^{(N-k)} \sigma \cdot b_{i_1} \wedge \cdots \wedge b_{i_k},
\]

where \( (-1)^{(N-k)} = \text{sgn}(h) \cdot \text{sgn}(h') \) and \( h' \) stands for the permutation \( (j_1, \ldots, j_{N-k}, i_1, \ldots, i_k) \mapsto (1, 2, \ldots, N) \).

2.2.2. Remark. [Moh, Remark 1, Section 2.2]. Let \( v_1, v_2, v_3 \in S V \) be pairwise distinct ideal points and let \( P := \mathbb{P} W, W := \mathbb{R} v_1 + \mathbb{R} v_2 + \mathbb{R} v_3 \), be the plane generated by \( v_1, v_2, v_3 \). Then \( u \) is the polar point to \( P \), where \( u := *(v_1 \wedge v_2 \wedge v_3) \).

Proof. By Proposition 2.2.1, \( \langle v_i, *(v_1 \wedge v_2 \wedge v_3) \rangle \omega = v_i \wedge *(v_1 \wedge v_2 \wedge v_3) = v_i \wedge v_1 \wedge v_2 \wedge v_3 = 0 \) for \( i = 1, 2, 3 \).
2.2.3. Lemma. Let \( P \) be a plane with polar point \( u \in EV \) and let \( p \in P \). Let \( v \in BV \cup SV \) be a point that does not belong to \( P \). The tangent vector \( n := \langle -, p \rangle u, x \mapsto \langle x, p \rangle u \) (see (2.1.1)), points towards the half-space of \( BV \cup SV \) determined by \( P \) and containing \( v \) if and only if \( \langle u, v \rangle \langle v, p \rangle < 0 \).

Proof. By [AGG, Lemma 4.2.2], the tangent vector \( n \) is normal to \( P \) at \( p \). It follows from [AGGr2, Lemma 5.2] that \( \langle -, p \rangle \pi[p]v / (v, p) \) is tangent to the oriented segment of geodesic from \( p \) to \( v \), where \( \pi[p]v := v - \langle v, p \rangle (p, p) p \). Then \( n \) points towards the half-space determined by \( P \) containing \( v \) if and only if

\[
\langle \langle -, p \rangle u, \langle -, p \rangle \pi[p]v / (v, p) \rangle = - \langle p, p \rangle \langle u, v \rangle / (v, p) > 0,
\]

where the above product is taken with respect to the Riemannian metric introduced in (2.1.2).

Let \( v_1, v_2, v_3, v_4 \in SV \) be pairwise distinct ideal points. Let \( P_1 \) and \( P_2 \) be the planes respectively generated by \( v_1, v_2, v_3 \) and \( v_1, v_2, v_4 \). The geodesic \( \gamma \) joining \( v_1, v_2 \) is common to \( P_1 \) and \( P_2 \). Our intention is to measure the (non-oriented) angle in \([0, \pi]\) between the half-plane in \( P_1 \) which contains \( v_3 \) and is determined by \( \gamma \) and the half-plane in \( P_2 \) which contains \( v_4 \) and is determined by \( \gamma \). This angle is called the dihedral angle between \( P_1 \) and \( P_2 \) at the edge \( \gamma \).

2.2.4. Proposition. Let \( v_1, v_2, v_3, v_4 \in SV \) be pairwise distinct ideal points and let \( g_{ij} := \langle v_i, v_j \rangle \). There exist representatives \( v_i \in V \) so that \( g_{ij} > 0 \) for \( i \neq j \). Moreover, with such a choice of representatives, the dihedral angle \( \angle(P_1, P_2) \) between the plane \( P_1 \) generated by \( v_1, v_2, v_3 \) and the plane \( P_2 \) generated by \( v_1, v_2, v_4 \) is

\[
\angle(P_1, P_2) = \arccos \frac{g_{13}g_{24} + g_{14}g_{23} - g_{12}g_{34}}{2(g_{23}g_{31}g_{24}g_{41})^{1/2}}.
\]

Proof. Clearly, we can assume that \( g_{12}, g_{23}, g_{34} > 0 \). Since \( v_1, v_2, v_3 \) are ideal and pairwise distinct, the form on the subspace generated by \( v_1, v_2, v_3 \) has signature \(-,-,+\); the same holds for the subspace generated by \( v_1, v_2, v_4 \) and by \( v_2, v_3, v_4 \). Hence, the Gram matrices of \( v_1, v_2, v_3 \), of \( v_1, v_2, v_4 \), and of \( v_2, v_3, v_4 \) have positive determinant. This implies that \( g_{ij} > 0 \) for \( i \neq j \).

Assume that \( v_4 \) does not belong to \( P_1 \). Let \( \gamma \) be the geodesic joining \( v_1, v_2 \). Then \( p := v_1 + v_2 \) gives rise to a point \( p \in \gamma \) since \( \langle v_1 + v_2, v_1 + v_2 \rangle = 2g_{12} > 0 \). We define

\[
\omega := \frac{1}{\sqrt{-\det G}} v_1 \wedge v_2 \wedge v_3 \wedge v_4 \in \bigwedge^4 V,
\]

where \( G := [g_{ij}] \) stands for the Gram matrix of \( v_1, v_2, v_3, v_4 \) (its determinant is negative due to the signature \(-,-,-\) of the form on \( V \)). Note that \( \langle \omega, \omega \rangle = -1 \). Let \( u_1 := * (v_1 \wedge v_2 \wedge v_3) \) and \( u_2 := *(v_1 \wedge v_2 \wedge v_4) \). By Remark 2.2.2, \( u_1 \) and \( u_2 \) are respectively the polar points to \( P_1 \) and \( P_2 \).

By [AGG, Lemma 4.2.2], the linear map \( n_i := \langle -, p \rangle u_i, v \mapsto \langle v, p \rangle u_i \), corresponds, via the natural isomorphism in (2.1.1), to a normal vector to \( P_i \) at \( p \), \( i = 1, 2 \). Let us show that \( n_1 \) points towards the half-space determined by \( P_1 \) containing \( v_4 \). Indeed, on one hand, \( \langle p, v_4 \rangle = \langle v_1 + v_2, v_4 \rangle = g_{14} + g_{24} > 0 \). On the other hand, by Proposition 2.2.1,

\[
\langle v_4, u_1 \rangle \omega = \langle v_4, * (v_1 \wedge v_2 \wedge v_3) \rangle \omega = v_4 \wedge * (v_1 \wedge v_2 \wedge v_3) = -v_1 \wedge v_2 \wedge v_3 \wedge v_4 = -\sqrt{-\det G} \omega.
\]
This implies that $\langle v_4, u_1 \rangle < 0$ and, by Lemma 2.2.3, $n_1$ points towards the desired direction. Similarly, one shows that $n_2$ points towards the half-space determined by $P_2$ not containing $v_3$. Therefore (see the picture above),

$$\cos \angle(P_1, P_2) = \frac{\langle n_1, n_2 \rangle}{\sqrt{\langle n_1, n_1 \rangle \langle n_2, n_2 \rangle}} = \frac{-\langle u_1, u_2 \rangle}{\sqrt{-\langle u_1, u_1 \rangle \langle u_2, u_2 \rangle}}.$$

It follows from Proposition 2.2.1 that

$$-\langle u_1, u_2 \rangle = -\langle * (v_1 \wedge v_2 \wedge v_3), * (v_1 \wedge v_2 \wedge v_4) \rangle = \det \begin{bmatrix} 0 & g_{12} & g_{14} \\ g_{21} & 0 & g_{24} \\ g_{31} & g_{32} & g_{34} \end{bmatrix} = g_{12}(g_{13}g_{24} + g_{14}g_{23} - g_{12}g_{34}) ,$$

$$-\langle u_1, u_1 \rangle = -\langle * (v_1 \wedge v_2 \wedge v_3), * (v_1 \wedge v_2 \wedge v_3) \rangle = \det \begin{bmatrix} 0 & g_{12} & g_{13} \\ g_{21} & 0 & g_{23} \\ g_{31} & g_{32} & 0 \end{bmatrix} = 2g_{12}g_{23}g_{31} ,$$

and

$$-\langle u_2, u_2 \rangle = -\langle * (v_1 \wedge v_2 \wedge v_4), * (v_1 \wedge v_2 \wedge v_4) \rangle = \det \begin{bmatrix} 0 & g_{12} & g_{14} \\ g_{21} & 0 & g_{24} \\ g_{41} & g_{42} & 0 \end{bmatrix} = 2g_{12}g_{24}g_{41} ,$$

which gives the desired formula.

Finally, if $v_4$ belongs to $P_1$, then the determinant of $G$ vanishes. This means that

$$g_{14}^2g_{23}^2 + (g_{13}g_{24} - g_{12}g_{34})^2 - 2g_{14}g_{23}(g_{13}g_{24} + g_{12}g_{34}) = 0 .$$

So,

$$\left( \frac{g_{13}g_{24} + g_{14}g_{23} - g_{12}g_{34}}{2(g_{23}g_{31}g_{24}g_{41})} \right)^2 = \frac{g_{14}^2g_{23}^2 + (g_{13}g_{24} - g_{12}g_{34})^2 + 2g_{14}g_{23}(g_{13}g_{24} - g_{12}g_{34})}{4g_{23}g_{31}g_{24}g_{41}} = \frac{2g_{14}g_{23}(g_{13}g_{24} + g_{12}g_{34}) + 2g_{14}g_{23}(g_{13}g_{24} - g_{12}g_{34})}{4g_{23}g_{31}g_{24}g_{41}} = 1 .$$

In other words, $\frac{g_{13}g_{24} + g_{14}g_{23} - g_{12}g_{34}}{2(g_{23}g_{31}g_{24}g_{41})} = \pm 1$. It is easy to verify (say, by considering particular cases in homogeneous coordinates), that the value 1 occurs exactly when $v_3, v_4$ lie on a same half-plane determined by $\gamma$ and that the value $-1$ occurs exactly when $v_3, v_4$ lie on distinct half-planes determined by $\gamma$.

In the next section we will calculate the dihedral angle $\angle(P_1, P_2)$ in terms of some specific representatives $v_1, v_2, v_3, v_4 \in V$.

### 3. Ideal tetrahedra

#### 3.1. Generalities. A labelled ideal tetrahedron is a 4-tuple $(v_1, v_2, v_3, v_4)$ of ideal points $v_i \in S V$. Each $v_i$ is a vertex of the labelled ideal tetrahedron. An ideal tetrahedron is a labelled ideal tetrahedron modulo permutations of the vertices. A (labelled) ideal tetrahedron is non-degenerate when its vertices do not belong to a same plane.

A Gram matrix of a labelled ideal tetrahedron $T = (v_1, v_2, v_3, v_4)$ is the Gram matrix $G := [\langle v_i, v_j \rangle]$ of representatives $v_1, v_2, v_3, v_4 \in V$ of the vertices. Clearly, two Gram matrices $G_1, G_2$ of $T$ are related by the expression $G_1 = DG_2D$, where $D$ is a diagonal matrix whose diagonal entries are non-null.
In this paper, we are mostly interested with Gram matrices that are doubly stochastic. (A doubly stochastic matrix is a square matrix $A = [a_{ij}]$ with non-negative entries satisfying $\sum_j a_{ij} = 1$ for every $j$ and $\sum_i a_{ij} = 1$ for every $i$.) A labelled ideal tetrahedron admits at most one doubly stochastic Gram matrix (see Theorem 3.2.1). It turns out that the doubly stochastic Gram matrix seems to be, in several circumstances, a natural choice among all the Gram matrices of a given labelled ideal tetrahedron; these matrices are used in the formulation of Seidel’s conjectures.

Let $T$ stand for the space of labelled ideal tetrahedra that admit a doubly stochastic Gram matrix modulo isometries preserving the order of the vertices. As we will shortly see, $T$ is made up of the classes of all labelled ideal tetrahedra with pairwise distinct vertices plus three particular classes of degenerate labelled ideal tetrahedra. Only a few labelled ideal tetrahedra do not admit a doubly stochastic Gram matrix; they are presented in Remark 3.2.3.

### 3.2. Classifying triangles

The main result in this subsection is Theorem 3.2.1. It gives a particular and explicit identification of the space $T$ with a plane equilateral triangle $\Delta$. In Proposition 3.2.5 we show that the symmetric 3-group $S_3$ acts faithfully on $\Delta$ giving rise to the space of (non-labelled) ideal tetrahedra modulo isometries.

#### 3.2.1. Theorem. The space $T$ can be identified with the equilateral triangle

$$\Delta := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1, x \leq y + z, y \leq z + x, z \leq x + y\}.$$

Explicitly, given a labelled ideal tetrahedron $T = (v_1, v_2, v_3, v_4)$ admitting a doubly stochastic Gram matrix, there exists a unique triple $(r, s, t) \in \Delta$ such that the doubly stochastic matrix

$$G := \begin{bmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{bmatrix}$$

is a Gram matrix of $T$ (it is the only Gram matrix of $T$ that is doubly stochastic). The triple $(r, s, t) \in \Delta$ depends only on the class of $T$ modulo isometries preserving the order of the vertices.

Conversely, given $(r, s, t) \in \Delta$, there exists, up to isometry preserving the order of the vertices, a unique labelled ideal tetrahedron admitting the above doubly stochastic Gram matrix.

The interior of $\Delta$ (respectively, the boundary of $\Delta$) corresponds to the non-degenerate (respectively, the degenerate) labelled ideal tetrahedra admitting a doubly stochastic Gram matrix.

**Proof.** Let $T = (v_1, v_2, v_3, v_4)$ be a labelled ideal tetrahedron with pairwise distinct vertices and let $v_1, v_2, v_3, v_4$ be representatives of $v_1, v_2, v_3, v_4$. The signature of the bilinear symmetric form $\langle \cdot, \cdot \rangle$ implies that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$. Rechoosing representatives, we assume that $\langle v_1, v_2 \rangle = \langle v_3, v_4 \rangle > 0$. The Gram matrix of $v_1, v_2, v_3$ has positive determinant because $v_1, v_2, v_3$ span a space of signature $- - +$. Hence, $\langle v_1, v_3 \rangle \langle v_2, v_3 \rangle > 0$. Simultaneously changing, if necessary, the signs of $v_1$ and $v_2$, we can assume that $\langle v_1, v_3 \rangle, \langle v_2, v_3 \rangle > 0$. Considering the determinants of the Gram matrices of $v_1, v_3, v_4$ and of $v_1, v_2, v_4$ we obtain $\langle v_1, v_4 \rangle, \langle v_2, v_4 \rangle > 0$. Now, scaling $v_1$ and $v_3$ by a same positive factor allows us to consider $\langle v_1, v_3 \rangle = \langle v_2, v_4 \rangle$; an analogous reasoning involving $v_1$ and $v_4$ leads to $\langle v_1, v_4 \rangle = \langle v_2, v_3 \rangle$. In other words, $T$ admits $G := \begin{bmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{bmatrix}$ as a Gram matrix. Dividing each representative $v_1, v_2, v_3, v_4$ by $\sqrt{r + s + t}$ we can assume that $G$ is doubly stochastic. Looking for a diagonal matrix $D$ whose diagonal entries are non-null and such that $DGD$ is doubly stochastic, we obtain only two possibilities, namely $D = \pm I$. Hence, $G$ is the unique Gram matrix of $T$ which is doubly stochastic.
Due to the signature of the form, \( \det G = -(r + s + t)(r - s + t)(r - s - t)(r + s - t) < 0 \) if \( T \) is non-degenerate. This means that \((r + s + t), (r - s + t), (r - s - t), (r + s - t)\) are positive numbers because, otherwise, one would obtain \( r < 0 \) or \( s < 0 \) or \( t < 0 \). Clearly, \( \det G = 0 \) if \( T \) is degenerate. Therefore, the obtained triple \((r, s, t)\) lies in the interior of \( \Delta \) if \( T \) is non-degenerate and in the boundary of \( \Delta \) if \( T \) is degenerate. The vertices of \( \Delta \) are \((0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0)\); so, \((r, s, t)\) cannot be a vertex of \( \Delta \).

Regarding (degenerate) labelled ideal tetrahedra whose vertices are not pairwise distinct, it is not difficult to see that only those of the forms \((vvv, vvv, vvv, vvv), (vvv, vvv', vvv'), v ← v',\) admit a doubly stochastic Gram matrix. Such Gram matrices correspond respectively to the listed vertices of \( \Delta \).

Conversely, given \((r, s, t) \in \Delta \), there exists a labelled ideal tetrahedron admitting the doubly stochastic Gram matrix \( G \) by Sylvester’s criterion.\(^5\)

Finally, it follows from [AGr1, Lemma 4.8.1] that two labelled ideal tetrahedra admit a same Gram matrix if and only if they differ by an isometry preserving the order of the vertices.

\(^{3.2.3.} \textbf{Remark.} \text{ The labelled ideal tetrahedra that do not admit a doubly stochastic Gram matrix are the following: exactly 2 vertices coincide (as in } (vvv, vvv, vvv, vvv), v v \neq v', \text{ for example) or at least 3 vertices coincide (as in } (vvv, vvv, vvv), \text{ for example). In what follows, we only deal with labelled ideal tetrahedra that admit a doubly stochastic Gram matrix. So, whenever we refer to labelled ideal tetrahedra, we are actually referring to those that admit a doubly stochastic Gram matrix. The same goes for (non-labelled) ideal tetrahedra, as we only consider those arising from labelled ideal tetrahedra that admit a doubly stochastic Gram matrix.} \)

In view of the identification \( \Delta \simeq T \) we will also refer to a point in \( \Delta \) as a ‘labelled ideal tetrahedron.’

\(^{3.2.4.} \textbf{Remark.} \text{ It is curious to note that } \Delta \text{ is nothing but the space of Euclidean triangles with ordered vertices and fixed perimeter (= 1) modulo isometries of the plane that preserve the order of the vertices. Label } (r, s, t) \in \Delta \text{ be a point that is not a vertex of } \Delta. \text{ Then } (r, s, t) \text{ corresponds to an Euclidean triangle with ordered vertices whose vertices are pairwise distinct. Let } (\theta_1, \theta_2, \theta_3) \text{ be the interior angles of this Euclidean triangle}^6 \text{ (the angle } \theta_1 \text{ is opposite to a side of length } r, \text{ the angle } \theta_2 \text{ is opposite to a side of length } s, \text{ and the angle } \theta_3 \text{ is opposite to a side of length } t). \text{ Hence,}

\[
\cos \theta_1 := \frac{r^2 - s^2 + t^2}{2st}, \quad \cos \theta_2 := \frac{r^2 - s^2 + t^2}{2rt}, \quad \cos \theta_3 := \frac{r^2 + s^2 - t^2}{2rs}.
\]

Applying Proposition 2.2.4 to the doubly stochastic Gram matrix (3.2.2) of the labelled ideal tetrahedron \( T = (r, s, t) \in \Delta \), it is straightforward to see that the angles \( \theta_1, \theta_2, \theta_3 \) are precisely the dihedral angles of \( T \). More specifically, the dihedral angles at the edge joining the vertices \( v_1, v_2 \) and at the edge joining \( v_3, v_4 \) are both \( \theta_1 \); the dihedral angles at the edge joining the vertices \( v_1, v_3 \) and at the edge joining \( v_2, v_4 \) are both \( \theta_2 \); the dihedral angles at the edge joining the vertices \( v_1, v_4 \) and at the edge joining \( v_2, v_3 \) are both \( \theta_3 \).

In particular, opposite dihedral angles of an ideal tetrahedron are equal and dihedral angles incident to a same vertex sum \( \pi \) — a couple of well-known facts (see, for instance, [Mil] or [Sei]).

\(^5\text{Explicitly, if } (r, s, t) \in \Delta \text{ is not a vertex of } \Delta, \text{ then } v_1 := b_1 + b_2, v_2 := \frac{b_2}{2} b_1 - \frac{b_1}{2} b_2, v_3 := \left( \frac{1}{2} + \frac{1}{2} \right) b_1 + \left( \frac{1}{2} - \frac{1}{2} \right) b_2 + \sqrt{\frac{2st}{2s}} b_3, v_4 := \left( \frac{1}{2} + \frac{1}{2} \right) b_1 + \left( \frac{1}{2} - \frac{1}{2} \right) b_2 + \frac{-r^2 + s^2 + t^2}{2rt} b_3 + \sqrt{-\frac{\det G}{2srt}} b_4 \text{ (where } b_1, b_2, b_3, b_4 \text{ is an orthonormal basis of } V \text{ satisfying } (b_1, b_1) = 1, (b_2, b_2) = (b_3, b_3) = (b_4, b_4) = -1 \text{ are the vertices of a labelled ideal tetrahedron having the required Gram matrix.} \)

\(^6\text{Here, we exclude the degenerate Euclidean triangles corresponding to the vertices of } \Delta \text{ because, in this case, two vertices of the Euclidean triangle coincide and the internal angles are not well-defined.} \)
3.2.5. Proposition. The kernel of the natural action of the symmetric 4-group $S_4$ on $\Delta$ is isomorphic to the Klein four group $H$. The symmetric 3-group $S_3 = S_4/H$ acts on $\Delta$ by permutations of coordinates or, equivalently, by reflections on the altitudes of $\Delta$.

Proof. Clearly, $S_4$ acts on $\Delta$ and every permutation of coordinates in $\Delta$ is induced by the action of an element of $S_4$. Let $p \in H$ be a permutation in the Klein four group

$$H := \{ I, (12)(34), (13)(24), (14)(23) \} \leq S_4.$$ 

A direct calculation shows that the labelled ideal tetrahedra $(v_1, v_2, v_3, v_4)$ and $(v_{p(1)}, v_{p(2)}, v_{p(3)}, v_{p(4)})$ have the same doubly stochastic Gram matrix (3.2.2). Hence, $H$ is contained in the kernel of the $S_4$-action. Taking a point $(r, s, t) \in \Delta$ with pairwise distinct $r, s, t$, there are six points in $\Delta$ corresponding to the permutations of $r, s, t$ (they are the reflections of $(r, s, t)$ on the altitudes $x = y, y = z$, and $z = x$ of $\Delta$) and, therefore, the action of $S_3 = S_4/H$ on $\Delta$ is faithful.

By Remark 3.2.4, points in the altitudes of $\Delta$ (except the vertices of $\Delta$) correspond to the 'isosceles' tetrahedra (i.e., two dihedral angles are equal); the tetrahedron $r = s = t = \frac{1}{3}$ is the 'equilateral' or 'regular' one (i.e., all of its dihedral angles are equal). These tetrahedra have fewer copies by the $S_3$ action on $\Delta$ because, in comparison to the generic case, there are more permutations of their vertices that can be achieved by means of isometries of the ambient space (after all, they are more regular). In particular, in the equilateral case, every permutation of vertices can be accomplished through an isometry.

The altitudes of $\Delta$ divide this triangle into six congruent triangles $\Delta_i$, $i = 1, \ldots, 6$; we put

$$\Delta_1 := \{ (x, y, z) \in \Delta \mid x \leq y \leq z \}.$$ 

3.2.7. Corollary. Each $\Delta_i$ is a copy of the space of (non-labelled) ideal tetrahedra modulo isometries.

In order to prove Seidel’s fourth conjecture it will be easier to take, in place of $\Delta$, an equilateral triangle centred at the origin of a plane:

3.2.8. Remark. We identify the affine plane $\{(x, y, z) \mid x + y + z = 1\} \subset \mathbb{R}^3$ with $\mathbb{R}^2$ by taking $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as the origin and $(0, -\frac{1}{2}, \frac{1}{2})$, $\frac{\sqrt{3}}{2} (-1, \frac{1}{2}, \frac{1}{2})$ as a basis. In these coordinates, $\Delta$ becomes the equilateral triangle

$$\tilde{\Delta} := \left\{ (c, d) \in \mathbb{R}^2 \mid d \geq \frac{\sqrt{3}}{6}, d \leq -\sqrt{3}c + \frac{\sqrt{3}}{3}, d \leq \sqrt{3}c + \frac{\sqrt{3}}{3} \right\} \subset \mathbb{R}^2$$

centred at the origin. The point $(c, d) \in \tilde{\Delta}$ corresponds to the point $\left(1 - \frac{\sqrt{3}d}{3}, \frac{2-3c+\sqrt{3}d}{6}, \frac{2+3c+\sqrt{3}d}{6}\right) \in \Delta$. Each $\Delta_i$ will be denoted by $\tilde{\Delta}_i$ in the new coordinates; in particular, the triangle $\tilde{\Delta}_1$ in (3.2.6) becomes

$$\tilde{\Delta}_1 := \left\{ (c, d) \in \mathbb{R}^2 \mid c \geq 0, d \geq \frac{\sqrt{3}}{3}, c, d \leq -\sqrt{3}c + \frac{\sqrt{3}}{3} \right\}. \quad \blacksquare$$

4. Proof of Seidel’s first conjecture

We now show that determinant and permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra can be taken as coordinates of the space of ideal tetrahedra modulo isometries. In other words, these algebraic functions uniquely determine an ideal tetrahedron modulo isometries. This is a
stronger version of Seidel’s first conjecture which claims that volume is a function of determinant and permanent.

We remind the reader that the permanent of an \(m \times m\) matrix \(A = [a_{ij}]\) is defined by the expression

\[
\text{per } A := \sum_{\sigma \in S_m} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{m\sigma(m)},
\]

where \(S_m\) stands for the symmetric \(m\)-group.

**4.1. Theorem [Seidel’s Speculation 1].** Let \(\alpha, \beta \in \mathbb{R}\) be respectively the determinant and permanent of the doubly stochastic Gram matrix of a labelled ideal tetrahedron \(T \in \Delta\). Then the only tetrahedra in \(\Delta\) whose doubly stochastic Gram matrices have determinant and permanent respectively equal to \(\alpha, \beta\) are those that differ from \(T\) by the \(S_3\)-action in Proposition 3.2.5. In other words, the pair \((\alpha, \beta)\) determines a unique (non-labelled) ideal tetrahedron.

**Proof.** Let \(G := \begin{bmatrix} 0 & r & s & t \\ r & 0 & t & s \\ s & t & 0 & r \\ t & s & r & 0 \end{bmatrix}\) be the doubly stochastic Gram matrix of \(T\). Then \(r, s, t\) satisfy the equations

\[
\begin{aligned}
x + y + z &= 1 \\
-(x + y + z)(-x + y + z)(x - y + z)(x + y - z) &= \alpha \\
(x^2 + y^2 + z^2)^2 &= \beta
\end{aligned}
\]

which are equivalent to

\[
\begin{aligned}
x + y + z &= 1 \\
xy + yz + zx &= \frac{1 - \sqrt[3]{\beta}}{2} \\
xyz &= \frac{\alpha - 2\sqrt[3]{\beta} + 1}{8}
\end{aligned}
\]

So, \(r, s, t\) are the roots of the polynomial

\[
p(w) := (w-r)(w-s)(w-t) = w^3-(r+s+t)w+(rs+st+tr)w-rst = w^3-w^2+\frac{1-\sqrt[3]{\beta}}{2}w-\frac{\alpha - 2\sqrt[3]{\beta} + 1}{8}.
\]

If a triple \((r', s', t') \in \Delta\) satisfies (4.3), then \(r', s', t'\) are the roots of the polynomial \((w-r')(w-s')(w-t') = p(w)\), that is, \((r', s', t')\) is a permutation of \((r, s, t)\).

Given a tetrahedron \(T \in \Delta\), let \(G_T\) denote the doubly stochastic Gram matrix of \(T\). We define

\[
S := \{ (\alpha, \omega) \mid \alpha = \det G_T \text{ and } \omega = \sqrt{\text{per } G_T} \text{ for } T \in \Delta \} \subset \mathbb{R}^2.
\]

Let \(\Delta_i\) and \(\tilde{\Delta}_i\) be as defined in the previous section (see Remark 3.2.8). We have the following corollary to Theorem 4.1:

**4.5. Corollary.** The function \(\Delta_i \to S, T \mapsto (\det G_T, \sqrt{\text{per } G_T})\), is a homeomorphism for each \(i = 1, 2, \ldots, 6\). (Of course, the same is true for \(\tilde{\Delta}_i\) in place of \(\Delta_i\).)

**Proof.** The map in question is clearly continuous; it is bijective by Theorem 4.1. Since \(\Delta_i\) is compact and \(S\) is Hausdorff, the map is a homeomorphism.
The space $S$, depicted on the left, can be seen as a reparameterization of the space $\Delta_i$ of (non-labelled) ideal tetrahedra modulo isometries; in this reparameterization, the determinant and (the square root of the) permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra are the coordinates. The space $S$ will be used in the proof of Seidel’s fourth conjecture (see Sections 5 and 6) and it is explicitly described in the proof of Theorem 6.10. The curves $c_1$ and $c_2$ correspond to the isosceles ideal tetrahedra; the curve $c_3$ lists the degenerate ideal tetrahedra. The vertices $(-\frac{1}{2\sqrt{3}}, \frac{1}{3})$, $(0, \frac{\sqrt{3}}{2})$, and $(0, \frac{3}{8})$ correspond respectively to the equilateral (or regular) ideal tetrahedron, to the degenerate ideal tetrahedron with two pairs of coinciding vertices, and to the ‘regular’ degenerate ideal tetrahedron.\footnote{The latter can be described as follows. Take four pairwise distinct points $v_1, v_2, v_3, v_4$ on the ideal boundary of a same plane. Assume that these points are oriented in the counterclockwise sense. Then the geodesic joining $v_1, v_3$ and the geodesic joining $v_2, v_4$ intersect orthogonally.}

\section*{5. A volume formula}

The volume of an ideal tetrahedron is the hyperbolic volume of the convex hull of its vertices. In this section we provide an explicit formula for the volume of an ideal tetrahedron in terms of the coordinates of the space $S$ of ideal tetrahedra modulo isometries (see Corollaries 3.2.7 and 4.5). In other words, Theorem 5.1 describes the volume of an ideal tetrahedron as a function of the determinant and the (square root of the) permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra.

Let $T$ be an ideal tetrahedron with dihedral angles $\theta_1, \theta_2, \theta_3$ (see Remark 3.2.4). Milnor’s beautiful volume formula \cite{Mil} states that the volume of $T$ is given by

$$\text{vol}(T) = \text{vol}(\theta_1) + \text{vol}(\theta_2) + \text{vol}(\theta_3),$$

where $\text{vol} : \mathbb{R} \to \mathbb{R}$ is the Lobachevsky function defined in (1.1). This is the volume formula we are going to use in the next theorem.

\begin{theorem}
\textbf{The volume function} \text{vol} : S \to \mathbb{R} \text{ is given by

$$\text{vol}(\alpha, \omega) = \begin{cases} 
\text{vol}(\theta_1) + \text{vol}(\theta_2) + \text{vol}(\theta_3), & (\alpha, \omega) \neq (0, \frac{1}{2}) \\
0, & (\alpha, \omega) = (0, \frac{1}{2}) 
\end{cases}$$

where

$$\theta_1 := \arccos \frac{-r^2 + s^2 + t^2}{2st}, \quad \theta_2 := \arccos \frac{r^2 - s^2 + t^2}{2rt}, \quad \theta_3 := \arccos \frac{r^2 + s^2 - t^2}{2rs},$$

$$(r, s, t) := \left(\frac{1 - \sqrt{3}d}{3}, \frac{2 - 3c + \sqrt{3}d}{6}, \frac{2 + 3c + \sqrt{3}d}{6}\right),$$

and

$$(c, d) := \begin{cases} 
(0, 0), & (\alpha, \omega) = (0, \frac{1}{2}) \\
\sqrt{2\omega - \frac{3}{4} \cdot (\sin \kappa, \cos \kappa)}, & (\alpha, \omega) \neq (0, \frac{1}{2}) \quad \kappa := \frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^2} 
\end{cases}$$

\end{theorem}
Proof. Let \((\alpha, \omega) \in S\). Writing the equations (4.2) in terms of the coordinates \((c, d) \in \tilde{\Delta}\) (see Remark 3.2.8) one obtains

\[
\begin{cases}
\frac{(2\sqrt{3}d + 1) \left(9c^2 - (\sqrt{3}d - 1)^2\right)}{27} = \alpha, \\
\frac{3c^2 + 3d^2 + 2}{6} = \omega.
\end{cases}
\]

By Theorem 4.1, these equations have a unique solution in \(\tilde{\Delta}_1\). If \((\alpha, \omega) = (-\frac{1}{27}, \frac{1}{3})\), it is easy to see that this solution is \((0, 0)\) \(\in \tilde{\Delta}_1\). Let us assume that \((\alpha, \omega) \neq (-\frac{1}{27}, \frac{1}{3})\).

It follows from (5.2) that \(2\sqrt{3}d\) satisfies the cubic equation

\[
x^3 + a_0 x + b_0 = 0
\]

where

\[
a_0 := 6(1 - 3\omega), \quad b_0 := 27\alpha - 18\omega + 7.
\]

A straightforward calculation shows that the cubic equation \(x^3 + ax + b = 0\) has the three roots

\[
x_k := 2\sqrt{-a} \cos \left(\frac{1}{3} \arccos \frac{-3\sqrt{3}b}{2\sqrt{-a^3} + 2k\pi} \right), \quad k = 0, 1, 2,
\]

when \(a, b \in \mathbb{R}\) are such that \(a \neq 0\) and \(\frac{b^2}{4} + \frac{a^3}{27} \leq 0\). Let us show that \(a_0, b_0\) satisfy the previous inequalities. Indeed,

\[
a_0 = 6 \left(1 - 3\frac{3c^2 + 3d^2 + 2}{6}\right) = -9 \left(c^2 + d^2\right) \neq 0
\]

because \((c, d) = (0, 0)\) leads to \((\alpha, \omega) = (-\frac{1}{27}, \frac{1}{3})\). Moreover,

\[
b_0 = 27 \frac{2\sqrt{3}d + 1} {27} \left(9c^2 - (\sqrt{3}d - 1)^2\right) - 18\frac{3c^2 + 3d^2 + 2}{6} + 7 = 6\sqrt{3}d \left(3c^2 - d^2\right)
\]

implies

\[
\frac{b_0^2}{4} + \frac{a_0^3}{27} = -27c^2 \left(3d^2 - c^2\right) \leq 0.
\]

Therefore, taking \(d_k := \frac{1}{2\sqrt{3}} \cdot x_k\) and defining \(c_k := \sqrt{2\omega - d_k^2 - \frac{2}{3}}\), \(k = 0, 1, 2\), provides all possible solutions \((\pm c_k, d_k) \in \tilde{\Delta}\) of equations (5.2). The solution \((c_0, d_0)\) lies in \(\tilde{\Delta}_1\) since \(0 \leq \kappa \leq \frac{\pi}{3}\) implies \(c_0 \geq 0\) and \(d_0 \geq \frac{\sqrt{3}}{3} \cdot c_0\).

It remains to apply Remarks 3.2.8, 3.2.4, and Milnor’s volume formula.

It is well-known that the regular ideal tetrahedron (the one whose dihedral angles are all equal to \(\pi/3\)) has maximal volume among all tetrahedra in hyperbolic 3-space [Mil]. This is the tetrahedron corresponding to the point \((\alpha, \omega) = (-1/27, 1/3)\) (see also the explicit description of \(S\) in the proof of Theorem 6.10).

6. Proof of Seidel’s fourth conjecture

This section contains the main result of the paper. In Theorem 6.10 it is shown that the volume function \(\text{vol} : S \to \mathbb{R}\) from the space \(S\) of ideal tetrahedra modulo isometries (see Corollaries 3.2.7
and 4.5) is monotonic both in the determinant and in the permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra. Theorem 6.10 follows from a long calculation (presented below) which had to be approached quite carefully. Indeed, in many places, we had to make some appropriate choices of coordinates in order to keep the expressions treatable.

It is curious to note that, at the end of the day, proving that the volume is a monotonic function of the determinant and of the permanent of doubly stochastic Gram matrices of labelled ideal polyhedra amounts to determining the sign of a single expression, the term $k'$ defined in (6.9). Even more intriguing is the fact that studying the sign of $k'$ is, in some sense, the most delicate part of the proof of the fourth conjecture.

Let $O := \{(r, s, t) \in \mathbb{R}^3 \mid r < s < t; r < s + t; s < r + t; t < r + s\}$ be an open set in $\mathbb{R}^3$. Using the notation from Theorem 5.1, we define the functions

$$h : S \rightarrow \bar{\Delta}_1, \quad (\alpha, \omega) \mapsto \sqrt{2\omega - \frac{2}{3}} \cdot (\sin \kappa, \cos \kappa),$$

where $\kappa := \frac{1}{3} \arccos \frac{-27\alpha + 18\omega - 7}{4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}}$,

$$g : \bar{\Delta}_1 \rightarrow \Delta_1, \quad (c, d) \mapsto \left(\frac{1 - \sqrt{3}d}{3}, \frac{2 - 3c + \sqrt{3}d}{6}, \frac{2 + 3c + \sqrt{3}d}{6}\right),$$

and

$$f : O \rightarrow \mathbb{R}, \quad (r, s, t) \mapsto \Pi(\theta_1) + \Pi(\theta_2) + \Pi(\theta_3),$$

where

$$\theta_1 := \arccos \frac{-r^2 + s^2 + t^2}{2st}, \quad \theta_2 := \arccos \frac{r^2 - s^2 + t^2}{2rt}, \quad \theta_3 := \arccos \frac{r^2 + s^2 - t^2}{2rs}.$$ 

Theorem 5.1 implies that the restriction $\bar{\circ} : S \rightarrow \mathbb{R}$ is given by $\bar{\circ} = f \circ g \circ h$.

Let $h_1, h_2$ and $g_1, g_2, g_3$ stand respectively for the coordinate functions of $h$ and $g$. In what follows, we will calculate the derivatives

$$\frac{\partial \text{vol}}{\partial \alpha} = \left(\frac{\partial f}{\partial r} \frac{\partial g_1}{\partial c} + \frac{\partial f}{\partial s} \frac{\partial g_2}{\partial c} + \frac{\partial f}{\partial t} \frac{\partial g_3}{\partial c}\right) \frac{\partial h_1}{\partial \alpha} + \left(\frac{\partial f}{\partial r} \frac{\partial g_1}{\partial d} + \frac{\partial f}{\partial s} \frac{\partial g_2}{\partial d} + \frac{\partial f}{\partial t} \frac{\partial g_3}{\partial d}\right) \frac{\partial h_2}{\partial \alpha} =$$

$$= \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) \frac{\partial h_1}{\partial \alpha} + \frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) \frac{\partial h_2}{\partial \alpha}$$

and

$$\frac{\partial \text{vol}}{\partial \omega} = \frac{1}{2} \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) \frac{\partial h_1}{\partial \omega} + \frac{\sqrt{3}}{6} \left(-2\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) \frac{\partial h_2}{\partial \omega}$$

at each point $(\alpha, \omega) \in \bar{\circ}$.

6.3. Lemma. At each point $(\alpha, \omega) \in \bar{\circ}$ we have

$$\frac{\partial h_1}{\partial \alpha} = \frac{\sqrt{3} h_2}{2h_1(3h_2^2 - h_1^2)}, \quad \frac{\partial h_2}{\partial \alpha} = -\frac{\sqrt{3}}{2(3h_2^2 - h_1^2)},$$

$$\frac{\partial h_1}{\partial \omega} = \frac{3h_2^2 - 3h_1^2 - \sqrt{3} h_2}{3h_1(3h_2^2 - h_1^2)}, \quad \frac{\partial h_2}{\partial \omega} = \frac{6h_2 + \sqrt{3}}{3(3h_2^2 - h_1^2)}.$$
Proof. Note that \( \frac{\partial h_1}{\partial \alpha} = \frac{3\sqrt{6(3\omega - 1)}}{l} \cos \kappa = \frac{9}{l} \cdot h_2 \), where

\[
l := \sqrt{\left(4\sqrt{2}(3\omega - 1)^{\frac{3}{2}}\right)^2 - (-27\alpha + 18\omega - 7)^2}.
\]

Let \( a_0, b_0 \) be as in (5.3). Then

\[
l = \sqrt{32(3\omega - 1)^3 - (-27\alpha + 18\omega - 7)^2} = \sqrt{32 \left(-\frac{a_0}{6}\right)^3 - (-b_0)^2} = 2\sqrt{-\left(\frac{b_0^2}{4} + \frac{a_0^3}{27}\right)}
\]

because \((h_1(\alpha, \omega), h_2(\alpha, \omega))\) satisfies equations (5.2). So, (5.4) implies that

\[
l = 2\sqrt{27h_1^2 (3h_2^2 - h_1^2)}.
\]

It follows from \((h_1(\alpha, \omega), h_2(\alpha, \omega))\) ∈ \( \hat{\Delta}_1 \) that \( h_1 > 0 \) and \( 3h_2^2 > h_1^2 \); hence,

\[(6.4) \quad l = 6\sqrt{3h_1(3h_2^2 - h_1^2)}.
\]

The expression for \( \frac{\partial h_2}{\partial \alpha} \) is obtained in a similar fashion.

Concerning the derivatives with respect to \( \omega \), we have

\[
\frac{\partial h_1}{\partial \omega} = \frac{\sqrt{3}}{\sqrt{2\sqrt{3} - 1}} \sin \kappa - \frac{3\sqrt{3}(9\alpha - 2\omega + 1)}{l\sqrt{2\sqrt{3} - 1}} \cos \kappa = \frac{3}{2(3\omega - 1)} h_1 - \frac{9(9\alpha - 2\omega + 1)}{2(3\omega - 1)l} h_2.
\]

But \( 2(3\omega - 1) = 3(h_1^2 + h_2^2) \) and

\[
9\alpha - 2\omega + 1 = \frac{9}{27} \left(2\sqrt{3}h_2 + 1\right) \left(9h_2^2 - (\sqrt{3}h_2 - 1)^2\right) - 2\left(3h_1^2 + 2\right) + 2 = 2 \left(3\sqrt{3}h_2 + h_1^2 - \sqrt{3}h_2^2 + h_2^2\right)
\]

because \((h_1(\alpha, \omega), h_2(\alpha, \omega))\) satisfies equations (5.2). It remains to use the expression for \( l \) in (6.4). The derivative \( \frac{\partial h_2}{\partial \omega} \) is obtained analogously.

\[\blacksquare\]

6.5. Lemma. At each point \((r, s, t) \in \hat{\Delta}_1 = \{(r, s, t) \in O \mid r + s + t = 1\}\) we have

\[
\frac{\partial f}{\partial r} = \frac{2}{\sqrt{-\alpha}} \left(-r \log r + \cos \theta_3 s \log s + \cos \theta_2 t \log t\right),
\]

\[
\frac{\partial f}{\partial s} = \frac{2}{\sqrt{-\alpha}} \left(\cos \theta_3 r \log r - s \log s + \cos \theta_1 t \log t\right),
\]

\[
\frac{\partial f}{\partial t} = \frac{2}{\sqrt{-\alpha}} \left(\cos \theta_2 r \log r + \cos \theta_1 s \log s - t \log t\right).
\]
**Proof.** A straightforward calculation shows that, at each point \((r, s, t) \in O\),

\[
\frac{\partial}{\partial r} \pi(\theta_1) = -\frac{2r}{k} \log \left( \frac{k}{st} \right), \quad \frac{\partial}{\partial r} \pi(\theta_2) = \frac{r^2 + s^2 - t^2}{rk} \log \left( \frac{k}{rt} \right), \quad \frac{\partial}{\partial r} \pi(\theta_3) = \frac{r^2 - s^2 + t^2}{rk} \log \left( \frac{k}{rs} \right),
\]

where \(k := \sqrt{(r+s+t)(-r+s+t)(r-s+t)(r+s-t)}\). Adding these equations, one obtains

\[
\frac{\partial f}{\partial r} = \frac{1}{k} \left( -2r \log r + \frac{r^2 + s^2 - t^2}{r} \log s + \frac{r^2 - s^2 + t^2}{r} \log t \right) = \frac{2}{k} (-r \log r + \cos \theta_3 s \log s + \cos \theta_2 t \log t).
\]

The second equation in (4.2) says that \(k = \sqrt{-\alpha} \) for \((r, s, t) \in \tilde{\Delta} \). The remaining derivatives follow by symmetry.

For notational simplicity, in what follows, we will write \(c = h_1(\alpha, \omega), \ d = h_2(\alpha, \omega), \ r = g_1(h(\alpha, \omega)), \ s = g_2(h(\alpha, \omega)), \) and \(t = g_3(h(\alpha, \omega)) \).

**6.6. Proposition.** At each point \((\alpha, \omega) \in \tilde{S}\) we have

\[
\frac{\partial \text{vol}}{\partial \alpha} = \frac{\sqrt{3}}{648c(3d^2 - c^2)rst\sqrt{-\alpha}} \left( M \log \frac{st}{r^2} + N \log \frac{t}{s} \right),
\]

where

\[
M := c(3d - \sqrt{3})(21d^2 + 4\sqrt{3}d + 9c^2 - 2),
\]

\[
N := \sqrt{3}(-27c^4 + 9c^2(3d^2 + \sqrt{3}d + 1) + d(18d^3 + 9\sqrt{3}d^2 - 9d - 2\sqrt{3})).
\]

Moreover,

\[
\frac{\partial \text{vol}}{\partial \omega} = \frac{\sqrt{-\alpha}}{12c(3d^2 - c^2)rst} \left( P \log \frac{st}{r^2} + Q \log \frac{t}{s} \right),
\]

where

\[
P := 2c(\sqrt{3}d - 1)
\]

\[
Q := -3c^2 + d(3d + 2\sqrt{3}).
\]

**Proof.** By Equations (6.1) and (6.2),

\[
\frac{\partial \text{vol}}{\partial \alpha} = \frac{1}{2} \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial h_1}{\partial \alpha} + \frac{\sqrt{3}}{6} \left( -2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial h_2}{\partial \alpha}
\]

and

\[
\frac{\partial \text{vol}}{\partial \omega} = \frac{1}{2} \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial h_1}{\partial \omega} + \frac{\sqrt{3}}{6} \left( -2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) \frac{\partial h_2}{\partial \omega}.
\]

On one hand, it follows from Lemma 6.5 that

\[
\frac{1}{2} \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) = \frac{1}{\sqrt{-\alpha}} \left( -\frac{1}{2r} - \frac{1}{2s} \right) \log r + \frac{1}{2t} \log s - \frac{1}{2s} \log t
\]

since

\[
\cos \theta_1 = \frac{1 - 2r}{2st} - 1, \quad \cos \theta_2 = \frac{1 - 2s}{2rt} - 1, \quad \cos \theta_3 = \frac{1 - 2t}{2rs} - 1
\]
due to \( r + s + t = 1 \). So,

\[
\frac{1}{2} \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) = \frac{A \log r + B \log s + C \log t}{2rst\sqrt{-\alpha}}
\]

where \( A := r(s(1 - 2s) - t(1 - 2t)) \), \( B := rs(1 - 2r) \), \( C := -rt(1 - 2r) \). The equalities \( r = \frac{1 - \sqrt{3}d}{3} \), \( s = \frac{2 - 3c + \sqrt{3}d}{6} \), and \( t = \frac{2 + 3c + \sqrt{3}d}{6} \) imply

\[
A = \frac{c}{3} \left( -6d^2 + \sqrt{3}d + 1 \right),
\]

\[
B = \frac{1}{54} \left( 18cd^2 - 3\sqrt{3}cd - 3c - 6\sqrt{3}d^3 - 9d^2 + 3\sqrt{3}d + 2 \right),
\]

\[
C = \frac{1}{54} \left( 18cd^2 - 3\sqrt{3}cd - 3c + 6\sqrt{3}d^3 + 9d^2 - 3\sqrt{3}d - 2 \right).
\]

On the other hand,

\[
\frac{\sqrt{3}}{6} \left( -2 \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) = \frac{\sqrt{3}}{3\sqrt{-\alpha}} \left( \frac{1 - 2t}{2s} + \frac{1 - 2s}{2t} \right) \log r + \left( \frac{1 - 2t}{r} + \frac{1 - 2r}{2t} \right) \log s + \left( \frac{1 - 2s}{r} + \frac{1 - 2r}{2s} \right) \log t = \frac{D \log r + E \log s + F \log t}{6rst\sqrt{-\alpha}},
\]

where \( D := \sqrt{3}(t(1 - 2t) + s(1 - 2s)) \), \( E := \sqrt{3}s(r(1 - 2r) - 2t(1 - 2t)) \), \( F := \sqrt{3}t(r(1 - 2r) - 2s(1 - 2s)) \).

From above expressions for \( r, s, t \) one obtains

\[
D = \frac{1}{27} \left( 27c^2d^2 - 9\sqrt{3}c^2 + 9d^2 + 2d^3 \right),
\]

\[
E = \frac{1}{54} \left( -27\sqrt{3}c^3 - 27c^2d + 9\sqrt{3}c^2 + 27\sqrt{3}cd^2 + 27cd + 9\sqrt{3}c - 9d^3 + 9d - 2\sqrt{3} \right),
\]

\[
F = \frac{1}{54} \left( 27\sqrt{3}c^3 - 27c^2d + 9\sqrt{3}c^2 - 27\sqrt{3}cd^2 - 27cd - 9\sqrt{3}c - 9d^3 + 9d - 2\sqrt{3} \right).
\]

Therefore, by Lemma 6.3,

\[
\frac{\partial \text{vol}}{\partial \alpha} = \frac{(3dA - cD) \log r + (3dB - cE) \log s + (3dC - cF) \log t}{4\sqrt{3}(3d^2 - c^2)rst\sqrt{-\alpha}} = \frac{\sqrt{3}}{12c(3d^2 - c^2)rst\sqrt{-\alpha}} \left( \frac{M}{27} \log r + \frac{M - N}{54} \log s + \frac{M + N}{54} \log t \right) = \frac{\sqrt{3}}{648c(3d^2 - c^2)rst\sqrt{-\alpha}} \left( M \log \frac{st}{r^2} + N \log \frac{t}{s} \right).
\]

Finally,

\[
\frac{\partial \text{vol}}{\partial \omega} = \frac{(3(3d^2 - 3c^2 - \sqrt{3}d)A + c(6d + \sqrt{3})D) \log r + (3(3d^2 - 3c^2 - \sqrt{3}d)B + c(6d + \sqrt{3})E) \log s + (3(3d^2 - 3c^2 - \sqrt{3}d)C + c(6d + \sqrt{3})F) \log t}{18c(3d^2 - c^2)rst\sqrt{-\alpha}}.
\]
Indeed, the factor\( (r, s, t) \) to verify the above equalities, one may substitute \( r, s, t \).

\[
\frac{(3(3d^2 - 3c^2 - \sqrt{3d})C + c(6d + \sqrt{3d})F)}{18c(3d^2 - c^2)rst\sqrt{-\alpha}}\log t = \\
\frac{-2(2\sqrt{3}d + 1)(9c^2 - (\sqrt{3}d - 1)^2)}{18c(3d^2 - c^2)rst\sqrt{-\alpha}} \left( -\frac{1}{9} P \log r + \frac{1}{18} (P - Q) \log s + \frac{1}{18} (P + Q) \log \right).
\]

Using the first equation in (5.2), we obtain

\[
\frac{\partial \text{vol}}{\partial \omega} = \frac{27(-a)}{324c(3d^2 - c^2)rst\sqrt{-\alpha}} \left( P \log \frac{st}{r^2} + Q \log \frac{t}{s} \right) = \frac{\sqrt{-\alpha}}{12c(3d^2 - c^2)rst} \left( P \log \frac{st}{r^2} + Q \log \frac{t}{s} \right).
\]

**6.7. Proposition.** Let \((\alpha, \omega) \in \hat{S}\). Then \(\frac{\partial \text{vol}}{\partial \alpha} (\alpha, \omega) < 0\) and \(\frac{\partial \text{vol}}{\partial \omega} (\alpha, \omega) > 0\).

**Proof.** By Proposition 6.6, the inequality \(\frac{\partial \text{vol}}{\partial \alpha} (\alpha, \omega) < 0\) follows from

\[
(6.8) \quad M \log \frac{st}{r^2} + N \log \frac{t}{s} < 0.
\]

Indeed, the factor \(\frac{\sqrt{-\alpha}}{648c(3d^2 - c^2)rst\sqrt{-\alpha}}\) in the expression for \(\frac{\partial \text{vol}}{\partial \alpha} (\alpha, \omega)\) is positive because \((\alpha, \omega) \in \hat{S}\) implies \(c > 0\) and \(3d^2 - c^2 > 0\) (as observed in the proof of Lemma 6.3).

Let us establish inequality (6.8). A straightforward calculation shows that

\[
M = -54\sqrt{3}r(t - s)(r^2 + s^2 + t^2 - 2r(s + t)), \\
N = -54\sqrt{3}(-2s(s - t)^2t + r^3(s + t) - 2r^2(s^2 + st + t^2) + r(s^3 + s^2t + st^2 + t^3))
\]

(to verify the above equalities, one may substitute \(r, s, t\) respectively by \(\frac{1 - \sqrt{3}d}{3}, \frac{2 - 3c + \sqrt{3}d}{6}, \frac{2 + 3c + \sqrt{3}d}{6}\)).

Since \(r, s, t\) satisfy the triangle inequalities and \(0 < r < s < t\) due to \((r, s, t) \in \Delta_1\), we have

\[
a := \frac{s}{r} > 1, \quad 0 < b := \frac{t - s}{r} < 1.
\]

Hence,

\[
M = -54\sqrt{3}r^4b(2a^2 + 2ab - 4a + b^2 - 2b + 1), \\
N = -54\sqrt{3}r^4(4a^3 - 2a^2b^2 + 6a^2b - 6a^2 - 2ab^3 + 4ab^2 - 6ab + 2a + b^3 - 2b^2 + b)
\]

(to verify the above equalities, it suffices to substitute \(a, b\) by the respective expressions in \(r, s, t\)).

It follows that

\[
M \log \frac{st}{r^2} + N \log \frac{t}{s} = (M + N) \log \frac{t}{r} + (M - N) \log \frac{2}{r} = (M + N) \log (a + b) + (M - N) \log a = -54\sqrt{3}r^4k
\]

where

\[
k := 2((a + b)(a - 1)(2a - b^2 + 2b - 1) \log (a + b) - a(a + b - 1)(2a - b^2 - 1) \log a) = \\
= 4b(a + b)(a - 1) \log (a + b) + 2(2a - b^2 - 1)(a + b)(a - 1) \log (a + b) - a(a + b - 1) \log a).
\]
Since \( 4b(a + b)(a - 1) \log(a + b) > 0 \) and \( 2a - b^2 - 1 > 0 \) (due to \( a > 1 \) and \( 0 < b < 1 \)), it remains to show that

\[
(6.9) \quad k' := (a + b)(a - 1) \log(a + b) - a(a + b - 1) \log a > 0
\]

Using the series

\[
\log x = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x - 1}{x} \right)^n
\]

which converges (absolutely) for each \( x \geq 1 \), we have

\[
k' = (a + b)(a - 1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a + b - 1)^n}{(a + b)^n} - a(a + b - 1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a - 1)^n}{a^n} =
\]

\[
= (a - 1)(a + b - 1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a + b - 1)^{n-1}}{(a + b)^{n-1}} - (a - 1)(a + b - 1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{(a - 1)^{n-1}}{a^{n-1}} =
\]

\[
= (a - 1)(a + b - 1) \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{a + b - 1}{a + b} \right)^{n-1} - \left( \frac{a - 1}{a} \right)^{n-1} > 0
\]

because \( \frac{a + b - 1}{a + b} - \frac{a - 1}{a} = \frac{b}{a(a + b)} > 0 \) and \( \frac{a + b - 1}{a + b}, -\frac{a - 1}{a} > 0 \).

Concerning the derivative with respect to \( \omega \), we need to show that

\[
P \log \frac{st}{r^2} + Q \log \frac{t}{s} > 0.
\]

The procedure is analogous to the one we just used. First, one can readily verify that

\[
P = -6r(t - s) = -6r^2b
\]

\[
Q = 6(2st - rs - rt) = 6r^2(2a^2 + 2ab - 2a - b).
\]

It follows that

\[
P \log \frac{st}{r^2} + Q \log \frac{t}{s} = 6r^2 \left( -b \log a(a + b) + (2a^2 + 2ab - 2a - b) \log \frac{a + b}{a} \right).
\]

Finally,

\[
-b \log a(a + b) + (2a^2 + 2ab - 2a - b) \log \frac{a + b}{a} = 2k' > 0,
\]

where \( k' \) is defined in (6.9).

6.10. Theorem [Seidel’s Speculation 4]. The volume \( \text{vol} : S \to \mathbb{R} \), \( \text{vol} = \text{vol}(\alpha, \omega) \), is decreasing \(^8\) in \( \alpha \) and increasing in \( \omega \) when \( \alpha \neq 0 \). (Tetrahedra corresponding to \( \alpha = 0 \) are degenerate and have vanishing volume.)

Proof. In view of Proposition 6.7, it suffices to show that the intersection of \( S \) with a horizontal (respectively, vertical) line is, if non-empty, a closed interval.

\(^8\)Or, as Seidel states, increasing in \( |\alpha| = -\alpha \).
Remark 3.2.8 and Equations (5.2) imply that the hypotenuse \( \{(c, d) \in \tilde{\Delta}_1 \mid c = 0, 0 \leq d \leq \frac{\sqrt{3}}{2}\} \) of \( \tilde{\Delta}_1 \) is sent to

\[
c_1 := \{ (\alpha, \omega) \in S \mid \alpha = -\frac{1}{27}(2(6\omega - 2)\frac{3}{2} - 3(6\omega - 2) + 1), \frac{1}{3} \leq \omega \leq \frac{1}{2} \}
\]

by the homeomorphism \( \tilde{\Delta}_1 \to S, (c, d) \mapsto \left( \det G_T, \sqrt{\text{per} G_T} \right) \) (see Corollary 4.5). Similarly, the legs \( \{(c, d) \in \tilde{\Delta}_1 \mid d = \frac{\sqrt{3}}{3}c, 0 \leq c \leq \frac{1}{4}\} \) and \( \{(c, d) \in \tilde{\Delta}_1 \mid d = -\sqrt{3}c + \frac{\sqrt{3}}{3}, 0 \leq c \leq \frac{1}{4}\} \) of \( \tilde{\Delta}_1 \), are respectively sent to

\[
c_2 := \{ (\alpha, \omega) \in S \mid \alpha = -\frac{1}{27}(-2(6\omega - 2)\frac{3}{2} - 3(6\omega - 2) + 1), \frac{1}{3} \leq \omega \leq \frac{3}{8} \},
\]

\[
c_3 := \{ (\alpha, \omega) \in S \mid \alpha = 0, \frac{3}{8} \leq \omega \leq \frac{1}{2} \}.
\]

It is not difficult to see that \( S \) is the closed region in \( \mathbb{R}^2 \) bounded by the curves \( c_1, c_2, c_3 \); in other words,

\[
S = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{3} \leq y \leq \frac{3}{8}, f_1(y) \leq x \leq f_2(y) \right\} \cup \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{3}{8} \leq y \leq \frac{1}{2}, f_1(y) \leq x \leq 0 \right\}
\]

where

\[
f_1(y) := -\frac{1}{27}(2(6y - 2)\frac{3}{2} - 3(6y - 2) + 1), \quad \frac{1}{3} \leq y \leq \frac{1}{2},
\]

\[
f_2(y) := -\frac{1}{27}(-2(6y - 2)\frac{3}{2} - 3(6y - 2) + 1), \quad \frac{1}{3} \leq y \leq \frac{3}{8}
\]

(see the picture at the end of Section 4). It follows that, for each \( y_0 \in \left[ \frac{1}{3}, \frac{1}{2} \right] \), the intersection of the horizontal line \( y = y_0 \) with \( S \) is a closed interval (it is a single point when \( y = \frac{1}{2} \)). As can be easily seen by taking derivatives, the functions \( f_1 \) and \( f_2 \) are invertible (in the indicated domains) and, therefore, a similar argument involving the inverses of \( f_1 \) and \( f_2 \) shows that the intersections of vertical lines with \( S \), if non-empty, are closed intervals.

It should be noted that the original fourth conjecture states that the volume is decreasing in the permanent while it is actually increasing. However, this is irrelevant; as discussed at the introduction, the whole point is to express the volume as a monotonic function of algebraic expressions. In what follows, we will see that, without an explicit description of the space of ideal tetrahedra modulo isometries in terms of the determinant and (the square root of the) permanent of doubly stochastic Gram matrices (that is, the region \( S \) in the above theorem), one may be easily led to believe that the volume is decreasing in the permanent.

Let \( I_n \) denote the \( n \times n \) identity matrix and let \( J_n \) denote \( n \times n \) matrix whose entries are all equal to 1. Let \( Z_n \) denote the set of all \( n \times n \) symmetric doubly stochastic matrices which have vanishing diagonal. Van der Waerden's conjecture, proved by Egoritsjev and Falikman, states that the permanent of a doubly stochastic \( n \times n \) matrix attains a unique minimum for the matrix \( \frac{1}{n} J_n \) (see [VaL]). In view of this, Seidel formulated a similar conjecture in the context of ideal hyperbolic tetrahedra. More specifically, Seidel's third conjecture states that the permanent in \( Z_n \) attains a unique minimum for the matrix \( \frac{1}{n-1}(J_n - I_n) \). This conjecture is very simple to prove in our case, i.e., when \( n = 4 \):

6.11. Theorem [Seidel’s Speculation 3]. The matrix \( \frac{1}{5}(J_4 - I_4) \) is the unique matrix for which the permanent of the matrices from \( Z_4 \) attains its minimal value.
Proof. Let \( M \in Z_4 \) be a symmetric doubly stochastic matrix with vanishing diagonal, that is,
\[
M = \begin{bmatrix}
0 & r & s & t \\
0 & 0 & t' & s' \\
st' & 0 & r' & t \\
t' & s' & r & 0
\end{bmatrix}
\]
and \( r, s, t, r', s', t' \) are non-negative numbers satisfying \( r + s + t = 1 \), \( r + s' + t' = 1 \),
\( r' + s + t = 1 \), and \( r' + s' + t = 1 \). One easily concludes that \( r = r' \), \( s = s' \), and \( t = t' \). Hence,
\[
\text{per } M = (r^2 + s^2 + t^2)^2 \quad \text{and} \quad r + s + t = 1 \implies \text{per } M \text{ is minimal when } r = s = t = \frac{1}{3}.
\]

Note that the space \( \Delta \) is strictly contained in \( Z_4 \) (since, in \( Z_4 \), it is not required that \( r, s, t \) satisfy
the triangle inequalities).

Similarly, we have the following

6.12. Proposition. The matrix \( \frac{1}{3}(J_4 - I_4) \) is the unique matrix for which the determinant of the
matrices from \( Z_4 \) attains its minimal value.

Proof. Let \( M \) be a matrix in \( Z_4 \). As in the proof of Theorem 6.11,
\[
\det M = \det \begin{bmatrix}
0 & r & s & t \\
0 & 0 & t & s \\
s & t & 0 & r \\
t & s & r & 0
\end{bmatrix} = -(r + s + t)(-r + s + t)(r - s + t)(r + s - t)
\]
where \( r, s, t \) are non-negative numbers such that \( r + s + t = 1 \).

If the numbers \( r, s, t \) do not satisfy the triangle inequalities, we have one and only one of the following:
\( r > s + t, s > r + t, \) or \( t > r + s \). Indeed, assuming (for example) that \( r > s + t \) and \( s > r + t \),
we obtain \( t < 0 \), a contradiction. Therefore, if \( r, s, t \) do not satisfy the triangle inequalities, then
\[
\det M = -(r + s + t)(-r + s + t)(r - s + t)(r + s - t) \geq 0.
\]
Assume that the numbers \( r, s, t \) satisfy the triangle inequalities. Let \( A \) denote the area of the Euclidean
triangle of sides of lengths \( r, s, t \). Then
\[
A = \frac{1}{4} \sqrt{(r + s + t)(-r + s + t)(r - s + t)(r + s - t)} = \frac{1}{4} \sqrt{-\det M}.
\]
So, \( \det M = -16A^2 \leq 0 \). Since the area of triangles with fixed perimeter \( r + s + t = 1 \) has a unique
maximum at the equilateral triangle of side lengths \( r = s = t = \frac{1}{3} \), the function \( Z_4 \to \mathbb{R}, M \mapsto \det M \),
has a unique minimum at the matrix \( \frac{1}{3}(J_4 - I_4) \).

The fact that the ideal tetrahedron of maximal volume corresponds to the point in \( S \) where the
permanent is minimal may be a source of the idea that the volume should be decreasing in the permanent.
However, being on the point \((-\frac{1}{27}, \frac{1}{3}) \) in \( S \) which corresponds to the ideal tetrahedron of maximal volume
it is not possible to vary (inside of \( S \)) the permanent while keeping the determinant constant (see the
picture at the end of Section 4).

At the end of the paper the reader may find a couple of graphs of the volume as a function of the
determinant and permanent.

7. Schur functors

We follow Weyl’s construction of Schur functors as presented in [Ful] (including the notation).

Let \textbf{FinLin} stand for the category of finite dimensional real (or complex) linear spaces. Let \( n \in \mathbb{N} \)
be a natural number and let \( \lambda = (\lambda_1, \ldots, \lambda_k) \), \( \lambda_1 \geq \cdots \geq \lambda_k \geq 1 \), \( \lambda_1 + \cdots + \lambda_k = n \), be a partition
of \( n \). Taking a Young tableau (say, the canonical one) related to \( \lambda \), we obtain the corresponding Young
symmetrizer \( c_\lambda \in A \), where \( A \) denotes the group algebra of the symmetric \( n \)-group \( S_n \). Given a finite
dimensional linear vector space $V$, the symmetric group acts on the right on the tensor power $V^\otimes n$ by permuting factors. The image of the Young symmetrizer $c_\lambda$ on $V^\otimes n$ is a linear subspace $S_\lambda V \subseteq V^\otimes n$.

The linear space $S_\lambda V$ is made up of finite linear combinations of *indecomposable* elements of the form

$$v_1 \ldots v_n := (v_1 \otimes \cdots \otimes v_n)c_\lambda.$$ 

For instance, in the case of the partition $\lambda = (1, \ldots, 1)$, one gets the exterior power $S_\lambda V = \wedge^n V$ whose indecomposable elements are typically denoted by $v_1 \wedge \cdots \wedge v_n := v_1 \ldots v_n$. In the case of the partition $\lambda = (n)$, the symmetric power $S_\lambda V = S^n V$ is obtained.

We have just arrived at the Schur functor $S_\lambda : \text{FinLin} \to \text{FinLin}$. At the level of objects, $V \mapsto S_\lambda V$; at the level of morphisms, given a linear map $f : V \to W$ between finite dimensional linear spaces, then $S_\lambda f : S_\lambda V \to S_\lambda W$ is defined, in terms of indecomposable elements, by $f(v_1 \ldots v_n) := f(v_1) \ldots f(v_n)$.

It is curious to observe that, if a finite dimensional linear space $V$ is equipped with a bilinear symmetric form $\langle -, - \rangle$ (or a hermitian form in the complex case), then there is an induced form on $S_\lambda V$. It is defined, in terms of indecomposable elements, by

$$\langle v_1 \ldots v_n, v'_1 \ldots v'_n \rangle := \sum_{\sigma \in S_n} \chi(\sigma) g_{1\sigma(1)}g_{2\sigma(2)} \cdots g_{n\sigma(n)},$$

where $g_{ij} := \langle v_i, v'_j \rangle$ and $\chi$ is the character of the representation of $S_n$ on its group algebra $A$ induced by the partition $\lambda$. In the exterior power case, this induced form is nothing but the determinant of the matrix $G := [g_{ij}]$; in the symmetric power case, it is the permanent of $G$. In general, $\langle v_1 \ldots v_n, v'_1 \ldots v'_n \rangle$ is called the *immanant* of $G$ (immanants of matrices were introduced in [LiR]).

It is quite common to find connections between the exterior power functor and hyperbolic geometry. Such connections can be seen, for instance, in Section 2.2 or in [AGr3]. Perhaps, Seidel’s conjectures provide, via the permanent of doubly stochastic Gram matrices of labelled ideal tetrahedra, the first link of the symmetric power functor with hyperbolic geometry. It is not unreasonable to expect the other immanants to play a role in hyperbolic geometry, say, in generalizations of Seidel’s conjectures to other (not necessarily ideal) polyhedra or to higher dimensions.

8. A couple of graphs

![Graph 1](image1.png)

Volume as a function of the determinant. The permanent is constant and its square root equals $7/16$. The determinant varies between $14 - 5\sqrt{10}/4$ (isosceles tetrahedron) and 0 (degenerate tetrahedron).

![Graph 2](image2.png)

Volume as a function of the square root of the permanent. The determinant is constant and equals $-\frac{1}{14}$. The square root of the permanent varies between $\frac{\sqrt{2} - 3\sqrt{3}}{12}$ (isosceles tetrahedron) and $\frac{1}{8}$ (isosceles tetrahedron).
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