Generalisation of DGLAP equations to massive partons

C. Pascaud
LAL, Université Paris-Sud, CNRS/IN2P3, Orsay, France
pascaud@lal.in2p3.fr

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Abstract

DGLAP evolution equations are modified in order to use all the quark families in the full scale range, satisfying kinematical constraints and sumrules, thus having complete continuity for the pdfs and observables. Some consequences of this new approach are shown.

Comments: 12 Pages and 5 Figures

1 Introduction

As it is well known heavy quarks present a challenge in the phenomenological description of deep inelastic scattering of leptons against nucleons: DGLAP [1] evolution equations are essential ingredients to this description but they consider only massless partons. With the advent of HERA results, the increase of the statistical precision of the measurements and the perspectives open by LHC, the necessity to have a way to predict the observables from the lowest to the highest $Q^2$ scale became clear and several so-called variable flavor number schemes (VFNS) have appeared:

- In the massless approach a flavor $h$ is usually considered to be active when its quark mass $m_h^2$ is smaller than the scale $Q^2$. Up to this limit its parton distributions (for quark and antiquark) are null, after the number of flavors is increased by one and they start to evolve from zero.

- In the massive approach the number of flavors is considered to be constant but the heavy quark is produced by the existing partons thus it have structure functions coming from $Q^2$ dependent coefficient functions and no (pdf) distributions.

- VFNS approaches make use usually of the two preceding approaches according to predefined $Q^2$ regions: At some arbitrary fixed value the number of quark flavors is increased by one and the pdfs are rearranged according to theory inspired rules before resuming evolution. All this is a still an ongoing activity.

The path of this work is different: It will consider always six quark species but modify coherently DGLAP equations, splitting functions, coefficient functions, flavor number and running $\alpha_s$ in order to have all the parton distributions and structure functions continuous in the full kinematical range.
This will be done furthermore in such a way that when the phase space for an heavy quark is quasi null \((m_h^2 \gg Q^2)\) its distributions are quasi null and its influence on the full system is also quasi null. On the contrary when its phase space is quasi complete \((m_h^2 \ll Q^2)\) it will behave like a light quark. The key to this will be kinematic constraints and sumrules.

To fulfill these commitments, section 2 presents DGLAP equations in an appropriate manner, emphasizing the components which will need modification. Section 3 will present the modified (extended) equations defining the method thereafter called cfns (continuous flavor number scheme), section 4 brings arguments to justify the method, section 5 shows a possible program implementation and finally Section 6 will present a comparison between massless scheme and cfns at NNLO in \(\alpha_s\).

## 2 Kinematics and DGLAP equations

The electron proton reaction is:

\[
e(l) + P(p) \rightarrow e(l - q) + X(p + q)
\]

Within the symbols ( ) are quadrimomenta.

Kinematic variables are:

\[S = (l + p)^2 \quad Q^2 = S xy = -q^2 > 0 \quad x = \frac{q^2}{2p.q} \quad y = \frac{p.q}{p.t} \quad W^2 = (p + q)^2 = M^2 + Q^2(\frac{1}{x} - 1)\]

\(M\) and \(W\) are the initial and final hadronic masses.

Parton \(o\) may be kicked out of the target if the final hadronic mass is \(W > 2m_o\) where \(m_o\) is the parton \(o\) mass. This translate into a kinematic limit \(x < l_o\) with \(l_o = (1 + 4m_0^2Q^{-2})^{-1}\)

Light partons fulfill always this condition but heavy quarks do only for \(Q^2 \rightarrow \infty\).

Usually the flavor number \(N_f\) is taken as the number of quark families such that \(Q^2 > m_o^2: N_f = N_f\) integer. \(N_f\) is the main concern of the overall approach.

DGLAP equations read:

\[
\frac{\partial o(Q^2)}{\partial \ln(Q^2)} = \sum_i P_{oi} \otimes i(Q^2)
\]

\(P\) are splitting functions, \(i\) and \(o\) are parton distributions (pdf) and run on the \(1 + 2N_f\) partons species, they will be noted also by the name of their species \(p = g,d,\bar{d},u,\bar{u},...\) and for the quarks \(d^\pm = d \pm \bar{d},...\) will be introduced.

All these are \(x\) and \(Q^2\) functions. \(\otimes\) note the convolution between two functions of \(x\) defined by:

\[
[A \otimes B](x) = \int_x^1 A(z)B\left(\frac{x}{z}\right) \frac{dz}{z} = \int_x^1 A\left(\frac{x}{t}\right)B(t) \frac{dt}{t}
\]

The following properties hold:

\[
A \otimes B = B \otimes A \quad [A \otimes B] \otimes C = A \otimes [B \otimes C] \quad x^n[A \otimes B] = [x^nA] \otimes [x^nB]
\]

\[
\int_0^1 [A \otimes B] dx = \int_0^1 A(x) dx \int_0^1 B(x) dx
\]

\[
\int_0^1 \delta(x - l) dx = 1 \quad \int_0^1 x\delta(x - l) dx = l \quad \delta(x - l) \otimes A = \frac{1}{l}A\left(\frac{x}{l}\right)
\]

2
DGLAP equations separate into two independent subsystems the first one being:

\[
\frac{\partial g(Q^2)}{\partial \ln(Q^2)} = \mathcal{P}_{gg} \otimes g(Q^2) + \sum_{q=d}^{t} \mathcal{P}_{qq} \otimes q^+(Q^2) \tag{2}
\]

\[
\frac{\partial q^+(Q^2)}{\partial \ln(Q^2)} = \tilde{\mathcal{P}}_{qq} \otimes g(Q^2) + \mathcal{P}^+_{NS} \otimes q^+(Q^2) + \sum_{r=d}^{t} \tilde{\mathcal{P}}^+_S \otimes r^+(Q^2) \tag{3}
\]

Using definitions inspired by [2]: \( \mathcal{P}^+_S = \mathcal{P}^V_{qq} \pm \mathcal{P}^V_{qq}, \mathcal{P}^+_S = \mathcal{P}^S_{qq} \pm \mathcal{P}^S_{qq}, \tilde{\mathcal{P}}_{qq} = \frac{\mathcal{P}_{qq}}{N_f}, \tilde{\mathcal{P}}^+_S = \frac{\mathcal{P}^+_S}{N_f} \).

The second system may be deduced from the first by replacing superscript + by − and suppressing all references to gluon \( g \). So it will not be mentioned anymore and even superscript + will not be written in the following.

Kernels \( \mathcal{P} \) are polynomials in \( a_s = \frac{\alpha_s}{\pi} \) and \( N_f \) as follows:

\[
\begin{align*}
\mathcal{P}_{gg} &= a_s(\mathcal{P}^{00}_{gg} + N_f \mathcal{P}^{01}_{gg}) + a_s^2(\mathcal{P}^{10}_{gg} + N_f \mathcal{P}^{11}_{gg}) + a_s^3(\mathcal{P}^{20}_{gg} + N_f \mathcal{P}^{21}_{gg} + N_f^2 \mathcal{P}^{22}_{gg}) \\
\mathcal{P}_{qq} &= a_s N_f \mathcal{P}^{00}_{gg} \\
\mathcal{P}^{V}_{qq} &= a_s \mathcal{P}^{V}_{qq} \\
\mathcal{P}^+_{S} &= a_s^2(\mathcal{P}^{V10}_{qq} + N_f \mathcal{P}^{V11}_{qq}) + a_s^3(\mathcal{P}^{V20}_{qq} + N_f \mathcal{P}^{V21}_{qq} + N_f^2 \mathcal{P}^{V22}_{qq}) \\
\mathcal{P}^+_{S} &= a_s^2(N_f \mathcal{P}^{+11}_{S}) + a_s^3(N_f \mathcal{P}^{+21}_{S} + N_f^2 \mathcal{P}^{+22}_{S}) \\
\mathcal{P}^+_{S} &= a_s^2(N_f \mathcal{P}^{+11}_{S}) + a_s^3(N_f \mathcal{P}^{+21}_{S} + N_f^2 \mathcal{P}^{+22}_{S})
\end{align*}
\]

Notice that \( \mathcal{P}^{01}_{gg} \) existence comes from the \( \beta_0 \) term in \( \mathcal{P}_{gg} \).

## 3 Modified DGLAP equations

The idea is to modify the kernels, keeping them functions of a single argument (apart from \( Q^2 \) dependence coming from \( N_f \) and \( a_s \)) in order to satisfy simultaneously the three kinematical constraints \( x_o < l_0, x_i < l_i, x_o < x_i \).

\( \mathcal{P}_{oi} \) gives the change to outgoing parton \( o \) at Bjorken \( x \) radiated by incoming parton \( i \) at Bjorken \( \bar{z} \).

Problematic cases are when parton \( o \) is heavier than parton \( i \) like for \( c \rightarrow b \) as shown Figure\[\text{Figure}\] where the corresponding term in equation\[\text{equation}\] is depicted. For a given value of \( x \) the convolution integral variable \( z \) runs vertically inside the big triangle at least for a massless parton. But for a massive quark the rightmost triangle has to be removed and if one wants to keep the splitting nature of the convolution only the leftmost hached triangle has to be kept. Notice that this will also supress an unwanted discontinuity of that term at \( x = l_o \) and bring it gently to 0.

An other way to present the modification is to replace \( \mathcal{P} \) by \( \mathcal{S} \) in the problematic changing term:

\[
\int_x^{x_0} \mathcal{S}_{oi}(\frac{x_0}{t})i(t)\frac{dt}{t} \tag{4}
\]

Requesting this term to be null for \( x \geq l_0 \) means that \( \mathcal{S}(u) = 0 \) for \( u \geq l_0 \) which is satisfied by:

\[
\mathcal{S}_{oi} = \mathcal{P}_{oi} \otimes \mathcal{K}_{oi} \tag{5}
\]

With the definition \( \mathcal{K}_{oi} = f_{oi} \delta(x - l_o) \), where \( f_{oi}(Q^2) \) is not decreasing and goes to 1 for \( Q^2 \rightarrow \infty \)

Notice that the effect of the \( \delta \) function is to replace \( \mathcal{P}(x) \) by \( \mathcal{P}(\xi) \) with \( x = \xi l_o \)
With this modification the subsystem becomes:

\[
\frac{\partial g(Q^2)}{\partial \ln(Q^2)} = P_{gg} \otimes K_{gg} \otimes g(Q^2) + \sum_{q=d}^t P_{gq} \otimes K_{gq} \otimes q(Q^2) \quad (6)
\]

\[
\frac{\partial q(Q^2)}{\partial \ln(Q^2)} = \tilde{P}_{gg} \otimes K_{qq} \otimes g(Q^2) + P_{NS} \otimes K_{qq}^{NS} \otimes q(Q^2) + \sum_{r=d}^t \tilde{P}_S \otimes K_{qr} \otimes r(Q^2) \quad (7)
\]

### 3.1 Sumrule constraints

In the above equations \( P \)'s are functions of the \( N_f \) to be defined now. The momentum sumrule imposes constraints which have to be satisfied (and are for the standard DGLAP). They are obtained by requesting that the first momentum of the sum of all parton distributions is constant and equal to 1 for any \( Q^2 \) value. Introducing \( \Delta = \int_0^1 K x dx \) and \( Q = \int_0^1 P x dx \), taking the \( Q^2 \) derivative of this momentum, using the DGLAP equations and the property that the n-momentum of a convolution product is equal to the product of the n-momenta of its components, one find easily:

\[
Q_{gg} \Delta_{gg} + \frac{Q_{gg}}{N_f} \sum_{q=d}^t \Delta_{qg} = 0 \quad (8)
\]

\[
Q_{gq} \Delta_{gq} + Q_{NS} \Delta_{qg}^{NS} + \frac{Q_{NS}}{N_f} \sum_{r=d}^t \Delta_{rq} = 0 \quad (9)
\]

Using the fact that those equations are satisfied by the original DGLAP equations where all the \( \Delta \)s are 1, one gets:

\[
N_f = \frac{\sum_{q=d}^t \Delta_{gg}}{\Delta_{gg}} = \frac{Q_{gg} + Q_{NS}}{Q_{gq} \Delta_{gg} + Q_{NS} \Delta_{qg}^{NS}} \sum_{r=d}^t \Delta_{rq} \quad (10)
\]

Leading to the final extended DGLAP equations \[1\]

\[
\frac{\partial g(x, Q^2)}{\partial \ln(Q^2)} = P_{gg} \otimes g(x, Q^2) + \sum_{q=d}^t P_{gq} \otimes q^+(x, Q^2) \quad (11)
\]

\[
\frac{\partial q^+(x, Q^2)}{\partial \ln(Q^2)} = K_q \otimes \left[ \tilde{P}_{gg} \otimes g(x, Q^2) + \sum_{r=d}^t \tilde{P}_S^+ \otimes r^+(x, Q^2) \right] + P_{NS}^+ \otimes q^+(x, Q^2) \quad (12)
\]

Where \( K_q = \delta(x - l_q) \). Furthermore \( Q(P) \) being polynomials in the flavor number, the only solution is to redefine the latter as being \( N_f = \sum_{q=d}^t l_q \).

Quark kinematical limit or longitudinal phase space \( l_q \) then may be also viewed as quark activity going smoothly from 0 to 1.

\[1\] The simplest hypothesis has been used: \( f_{io} = \Delta_{gg} = \Delta_{qg}^{NS} = 1 \).
3.2 Subsystem decoupling

As it is well known the usual DGLAP equations decouple in a $g, \Sigma$ system of two coupled equations and several non-singlet independent equations. It is now not so simple. The $g, \Sigma$ singlet system still decouple but the non-singlets cannot all decouple. In fact there is not anymore a unique way to simplify the system. The following solution is one of the both which have been realised in the program implementation section 5.

Define:
\[ K = \sum_{q=d}^{t} K_q \]
\[ \Sigma_L = \sum_{q=d}^{s} q \]
\[ l_{LN} = l - \frac{\Sigma L}{3} \]

Subscript $LN$ is used do distinguish these non singlets from the usual ones which may not be used here with $N_f$ varying continuously. With these one get the following subsystem:

\[ \frac{\partial g(Q^2)}{\partial \ln(Q^2)} = P_{gg} \otimes g(Q^2) + P_{gq} \otimes \Sigma(Q^2) \]
(13)

\[ \frac{\partial \Sigma(Q^2)}{\partial \ln(Q^2)} = K \otimes \tilde{P}_{gg} \otimes g(Q^2) + (P_{NS} + K \otimes \tilde{P}_S) \otimes \Sigma(Q^2) \]
(14)

\[ \frac{\partial l_{LN}(Q^2)}{\partial \ln(Q^2)} = P_{NS} \otimes l_{LN}(Q^2) \]
(15)

\[ \frac{\partial h(Q^2)}{\partial \ln(Q^2)} = K_h \otimes \tilde{P}_{gg} \otimes g(Q^2) + P_{NS} \otimes h(Q^2) + K_h \otimes \tilde{P}_S \otimes \Sigma(Q^2) \]
(16)

After evolution of the seven pdfs $g, \Sigma, c, b, t, d_{LN}, u_{LN}$ the full system may be recovered using:

\[ \Sigma_L = \Sigma - c - b - t \]
\[ d_{LN} + u_{LN} + s_{LN} = 0 \]

Note that for $l_h \rightarrow 0$ the corresponding DGLAP equation will get decoupled (Appelquist-Carazzone theorem) and the kinematical constraint automatically verified.

3.3 Renormalisation equation

As seen above the momentum integral sumrule leads to a specific non integer value of $N_f$ and as a consequence also for $\beta_0$ and by extension for the full set of $\beta$ governing the $\alpha_s$ running. It is also natural that the coupling constant depends on the sum of flavor activities $l_q$ and not only on flavor number.

3.4 Coefficient functions

As it is the structure functions and not the parton distributions which are observable one has to find also a procedure to modify the coefficient functions.

- The transform parton distribution $\rightarrow$ structure function has exactly the same structure that the one of DGLAP equations, it is obtained by the changes:

\[ \frac{\partial o(Q^2)}{\partial \ln(Q^2)} \rightarrow F_o \]
\[ P \rightarrow \mathcal{C} \]
• This transform is in fact nothing more than a change of scheme, an example is going from $\overline{MS}$ to $DIS$ for $F_2$. For $F_1$ and $F_3$ the schemes are unnamed but they still exist.

From this one may infer that coefficient functions have to be modified in the same way that splitting functions.

In fact it is the importance of kinematic constraints stressed by one of the R.Thorne papers and its use in the latest schemes which started this work.

**Charged currents**

Exactley the same procedure will be used, the phase space only will change using:

$$l_o = (1 + m_s^2 Q^{-2})^{-1}$$

(17)

## 4 Theoretical Considerations

Extended DGLAP equations have been set up very close to the ordinary ones.

• They reduce to it when scale $Q^2$ is far away from any heavy quark mass squared.

• They are integro differential equations, linear in all the pdf's which they use in conjunction with convolution integrals.

• They satisfy the kinematic constraints of the heavy partons.

• The resulting pdf are continous in $x$ and $Q^2$ at least if the input pdfs at the initial scale are.

• The derivatives of the pdfs are also continous.

• The same properties apply for charged and neutral currents structure functions $F_1,F_2$ and $F_3$ (see Figure 4).

• When there is only active flavors (all $l_h$ close to 0 or 1) $S_{oi}(x,Q^2) \rightarrow P_{oi}(x,Q^2,N_f = N_f)$

• They are supported by works and concepts which are not really new:

  – $\xi$ the scaling variable was used in many papers, see [3] as an example.

  – [3] use also anomalous dimensions variable with $Q^2$. Anomalous dimensions leading to splitting functions their arguments should hold here. They advocate $l_i \approx (1 + 2m_i^2 Q^{-2})^{-1}$.

  – Moreover [3] presented a $\beta$-function variable with $Q^2$ (due to $\beta_0$ ). They advocate here $l_i \approx (1 + 5m_i^2 Q^{-2})^{-1}$.

  – [4] have considered $\mathcal{N}_f$ to be $Q^2$ and even $\alpha_s$ order dependent and there is still development on the $\beta$-function [5].

  – Small are the differences between massless and continous behavior for $\alpha_s$ (see Figure 2).

\footnote{they did not have the same name and used it differently however.}
The procedure leading to satisfaction of the kinematical constraints as been used lately for coefficient functions in GM-VFNS schemes [7].

Presented here is the simplest solution to the chosen goal but it is still possible to make modifications making use of the freedom brought by the $f_{ai}(Q^2)$s following [4] example.

Satisfying all the points it was designed for, it as some serious advantages:

- It does not mix up different $\alpha_s$ orders as do mixed schemes.
- Heavy quarks participate to the evolution when they start to appear, that is at the beginning and at very low $x$ and even at leading order in $\alpha_s$.
- There is only internal partons, no externals.
- It covers the charm-bottom region where they are both opening up (see $N_f$ graph Figure 2) which is not yet the case in VFNS.
- It should give a better treatment of the small $x$ region where heavy quark pdfs appear first.

It seems that resummation is done, due to evolution, but not the forward divergence pole subtraction. In that case this method should be considered only as a phenomenological model which may be optimized by adjusting the $f_{ai}$ functions with the help of the rates given by [8]. The comparison between the $F^e_2$ of the two approaches would however be not that simple, having so different concepts (quark internal versus external).

An important point is that nature has 6 flavors and not 3,4,5 or 6 depending on physicist will, so neglecting that fact is making an approximation. This paper is also making approximations which are certainly valid when $N_f(Q^2)$ is close to an integer. Maybe renormalisation group theory could bring some light to this.

5 Program implementation

Modifications to be made to existing programs may be done without changing their conceptual building, however changes are not that small. The QCDFIT case ([4]) will be presented in some details as an example.

QCDFIT concept

It is a program which works in $x$ space. It includes an optimizing procedure (Minuit package [10]). It accept a variety of input distributions. It has a variety of outputs: Pdfs, cross section for lepto production, Drell-Yang mecanism... It precalculate the full evolution saving there a lot of CPU time.

Pdf representation

QCDFIT uses two grids: The $x$ grid has linear spacing in log($x$). There is much more liberty for the $Q^2$ grid. Present version uses a $Q^2$ grid approximately equally spaced in $\alpha_s$. All the functions of $x$ or $Q^2$ or $(x, Q^2)$ are calculated on these grids. To know values elsewhere a linear approximation has to be made (there is specialised routines for that). One consequence is that distributions are represented by vectors and that the evolution of the distributions from one $Q^2$ grid value to an other

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3 A maximum of 20000 $x$ nodes and 600 $Q^2$ nodes has been used
$Q^2$ grid value is given by matrices.

**Kernel representation**

They are linear operators applied to the pdfs and so after discretisation have a matrix representation. But due to the linear approximation used, the structure of convolution and more deeply to the concept of parton branching those matrices are upper triangular band matrices: their elements are: $M_{ij} = m_{i-j}$ with $i \geq j$.

As a consequence in QCDFIT they are represented by one dimensional arrays and a system of fast and simple routines have been developed to deal with kernel multiplication, inversion, exponentiation, square root and vector (pdf) multiplication. Notice that target mass correction, higher twist and renormalon a la M.Dasgupta and B.R. Webber are eventually computed in this kernel frame.

**QCDFIT Evolution**

Integration of the renormalisation group equation is made analytically for the needed flavors. Integration of $a_s^n$ needed for kernel integration is made at the same time and with the same technique. For the NS case it is easy to show that the solution of DGLAP involves an integration of the kernel followed by an exponentiation (easy and fast operation with QCDFIT concept). For the Singlet case things are more complicated and an additional calculation is needed.

**QCDFIT Modification**

The representation of the Pdfs, the kernels, the convolution algebra stay the same, but the evolution itself has to be modified: Integration of the renormalisation group equation is now made numerically as its $\beta$ parameters are functions of $N_f$ and so of $Q^2$. The transport matrices defined by $o(Q^2_{j+1}) = \sum_i T(o, i) \otimes i(Q^2_j)$ are obtained by integration of the subsystems using:

$$T(o, i) = \prod_{Q^2_j}^{Q^2_{j+1}} \left( 1 + \frac{\partial^2 o}{\partial i \partial \ln(Q^2)} \delta \ln(Q^2) \right)$$

In the product $\delta \ln(Q^2)$ has to be small enough to see only the rounding errors when increasing the number of $Q^2$ nodes.

**6 cfns massless comparaison at NNLO**

The aim of the exercise is to show what kind of new features might be seen on cfns and how far they extend away from the transition points of the other schemes.

**Kinematical range used**

Very often the start of evolution $Q^2_{input}$ is chosen just below the charm mass squared in order to define pdf inputs only for light partons. But for cfns heavy quarks are always present if the kinematical range permits, so in order to have only light quarks present, a $Q^2_{input} = 0.6$ Gev$^2$ is used, low enough to justify neglecting all the heavy quarks at input. Needless to say that at so low a $Q^2$ the predictive power of the pdfs is completely absent but it is a parameterless way to get a sensible charm when out of the non perturbative region.

A data sample made of about 1850 F2 or cross section measurements extracted from NMC, BCDMS on protons and H1 preliminary is used to fit the input pdfs independently for cfns and for massless

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$^4$QCDFIT use presently grids from 10000 to 500000 $Q^2$ nodes
scheme. The fitted distributions are \( g, u_{val}, d_{val}, \bar{u} = \bar{d} = \lambda \bar{s} \) with \( \int_0^1 x \ dx \ \bar{s} = 0.53 \int_0^1 x \ dx \ \bar{d} \) at \( Q^2 = 10 Gev^2 \). [2]

Results

Figure 3 shows the momentum integral fractions for the partons species in both schemes as a function of \( \log(Q^2) \). Note for heavy quarks the earlier start and slower rise of cfns. Figures 4 shows the cfns and massless structure function \( F_{e^- p \rightarrow \nu X} \) at \( x = 0.01 \) versus \( Q^2 \). The three kinks at \( m_h^2 \) in massless are due to the alternative coming in of quarks and antiquarks with negative sign. In cfns slope continuity is restored because there in a single DGLAP system and coefficient functions with no discontinuities. Figures 5 shows the cfns and massless top distributions at \( Q^2 = 39800 \) where both momentum fractions are equal, cfns has a much steeper distribution with higher values at small \( x \) coming from its early development and from the kinematical constraint (not shown is the same effect for charm and beauty).

Conclusion

Effects shown are impressive and justify the idea of evolving heavy quarks even when they have to be considered as heavy.

Work will continue to assess on more firmer theoretical ground these ideas.

References

[1] V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. 15, 438 (1972); ibid. 15, 675 (1972).
L. N. Lipatov, Sov. J. Nucl. Phys. 20, 94 (1975).
G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).
Y. L. Dokshitzer, Sov. Phys. JETP 46, 641 (1977).
G. Parisi, Phys. Lett. 90B, 295 (1980).

[2] W.L. van Neerven and A. Vogt, hep-ph/9907472 and hep-ph/0006154

[3] H. Georgi and H.D. Politzer Physical Review D 15,7 (1976)

[4] S. Brodsky et al arXiv:hep-ph/9906324

[5] D.D. Dietrich arXiv:0908.1364 [hep-th]

[6] C. Pascaud and F. Zomer hep-ph/0104013

[7] R.S. Thorne arXiv:1006.5925 [hep-th]

[8] S. Riemersma, J. Smith, W.L. van Neerven Physics Letter B 347 (1995) 143-151

[9] A. D. Martin, W. J. Stirling, R. S. Thorne, G. Watt, Eur. Phys. J. C63 (2009) 189-285.
[arXiv:0901.0002 [hep-ph]].

[10] F. James, Cern Program Library, D506.
Figure 1: $C \rightarrow B$ a problematic changing term. Convolution domain of Equation [4]. Vertical thin spaced lines show the kinematically reduced domain.
Figure 2: scheme comparaison with $\alpha_s(M_Z^2) = .118$

a) $N_f$ versus log $Q^2$ for cfns and massless.

b) Relative difference between massless and cfns for $\alpha_s$ versus log $Q^2$.

Figure 3: From top to bottom $g, u_{val}, d_{val}, \bar{u} = \bar{d}, s, c, b, t$ momentum fractions versus log $Q^2$.

cfns: black full, massless: red dotted.
Figure 4: $\mathcal{F}_3^{e^- p \rightarrow \nu X}$ versus $\log Q^2$ cfns: black full, massless: red dotted.

Figure 5: Top pdf cfns: black full, massless: red dotted at $Q^2 = 39800 Gev^2$ where both momentum fractions are equal.