Schrödinger Operators with Complex Sparse Potentials

Jean-Claude Cuenin

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK. E-mail: J.Cuenin@lboro.ac.uk

Received: 29 March 2021 / Accepted: 22 February 2022
Published online: 5 April 2022 – © The Author(s) 2022

Abstract: We establish quantitative upper and lower bounds for Schrödinger operators with complex potentials that satisfy some weak form of sparsity. Our first result is a quantitative version of an example, due to S. Bögli (Commun Math Phys 352:629–639, 2017), of a Schrödinger operator with eigenvalues accumulating to every point of the essential spectrum. The second result shows that the eigenvalue bounds of Frank (Bull Lond Math Soc 43:745–750, 2011 and Trans Am Math Soc 370:219–240, 2018) can be improved for sparse potentials. The third result generalizes a theorem of Klaus (Ann Inst H Poincaré Sect A (N.S.) 38:7–13, 1983) on the characterization of the essential spectrum to the multidimensional non-selfadjoint case. The fourth result shows that, in one dimension, the purely imaginary (non-sparse) step potential has unexpectedly many eigenvalues, comparable to the number of resonances. Our examples show that several known upper bounds are sharp.

1. Introduction and Main Results

1.1. Introduction. Many examples of Schrödinger operators with “strange” spectral properties involve sparse potentials. In his seminal work [60] Pearson constructed examples of real-valued potentials (on the half-line) leading to singular continuous spectrum. The potentials consists of an infinite sequence of “bumps” of identical profile, and the separation between these bumps increases rapidly. The physical interpretation is that a quantum mechanical particle will ultimately be reflected from a bump. These ideas were further developed in several directions, see e.g. [43,47,49,63,71,81], and the references therein. Scattering from sparse potentials in higher dimensions was studied by Molchanov and Vainberg [54,55]; see also [38,39,46,53,66]. The discrete spectrum for multidimensional lattice Schrödinger operators was investigated by Rozenblum and Solomyak [64]. They constructed examples of sparse potentials whose number of negative eigenvalues grows like an arbitrary given polynomial power in the large coupling limit. In the recent work [4] Bögli constructed a complex-valued sparse potential with
arbitrary small $L^q$ norm ($q > d$) that has infinitely many non-real eigenvalues accumulating at every point of the essential spectrum. Since the proof is based on compactness arguments, there is no quantitative bound on the rate of separation between the bumps, and hence no estimate on the pointwise decay of the potential or the accumulation rate of the eigenvalues is possible.

1.2. A Quantitative Version of Bögli’s Example. Our first result provides quantitative decay bounds for the example in [4]. Perhaps more importantly, the construction can be used to produce a potential together with an infinite number of eigenvalues (possibly not all of them) satisfying given upper and lower bounds on their accumulation rate. We formulate our result for the most interesting spectral region

$$\Sigma_0 = \{ z \in \mathbb{C} : |\text{Im } z| \leq \epsilon_0 |\text{Re } z| \},$$

where $\epsilon_0 > 0$ is small but fixed.

**Theorem 1.** Let $d \geq 1$, $q > d$, $\epsilon_1, \epsilon_2 \in (0, 1]$, $\gamma > 0$, and let $(\zeta_n)_n \subset \Sigma_0$ be a sequence satisfying

$$\left( \sum_{n \in \mathbb{N}} |\zeta_n|^{\frac{d}{2}} |\text{Im } \zeta_n|^{\frac{d}{q} - d} |\text{log}^d |\text{Im } \zeta_n/\zeta_n|| \right)^\frac{1}{q} \leq \epsilon_1.$$  

(2)

Then there exists a complex sparse potential $V$ such that the following hold:

(a) For each $n \in \mathbb{N}$ there exists a discrete eigenvalue $z_n$ of $H_V = -\Delta + V$ which is exponentially close to $\zeta_n$, in the sense $|z_n - \zeta_n| \leq \exp(-|\text{Im } \zeta_n|^{-\gamma})$.

(b) The potential satisfies $\|V\|_{L^q(\mathbb{R}^d)} \lesssim \epsilon_1$.

(c) The potential decays polynomially, i.e. there exists a positive constant $\beta = \beta(\gamma, d, q)$ such that $|V(x)| \lesssim \epsilon_2 (x)^{-\beta}$.

**Remark 1.** (i) In particular, for any $\lambda \in (0, \infty)$ there exists a sequence $(\zeta_n)_n \subset \Sigma_0$ satisfying (2) such that $\lim_{n \to \infty} \zeta_n = \lambda$. In this way one can find a sequence accumulating to every point of the essential spectrum. This yields a constructive proof of the result of Bögli [4].

(ii) One can remove the logarithm in (2) at the expense of replacing the $L^q$ norm of $V$ by the “Davies–Nath norm” [see (29)].

(iii) We will give explicit bounds on the polynomial decay $\beta$.

(iv) Substituting the trivial lower bound $|\zeta_n| \geq |\text{Im } \zeta_n|$ into (2) shows $\text{Im } \zeta_n \to 0$. This is the reason why we say that $z_n$ is exponentially close to $\zeta_n$.

(v) After acceptance of this article, Bögli and the author constructed a counterexample to the Laptev–Safronov conjecture [5]. Replacing the single bump potential in Sect. 7 by the (compactly supported) potential used in the counterexample, one can relax the summability condition (2) to the following,

$$\left( \sum_{n \in \mathbb{N}} |\zeta_n|^{\frac{1}{2}} |\text{Im } \zeta_n|^{|\frac{d}{q} - d| + 1} \right)^\frac{1}{q} \leq \epsilon_1, \quad (q > (d + 1)/2).$$

In view of the upper bounds of Frank [29] (see also Theorem 2 for comparison), which were shown to be sharp in [5], the above condition is natural and optimal. This remark does not affect the remainder of the article.
We believe that the pointwise condition c) is more natural than the $L^q$ condition b) for the phenomenon that takes place in Theorem 1. This is because complex analogues of classical phase space bounds, which motivate the consideration on $L^q$ norms in the first place, lack many of the features that make them so useful for real potentials (more on that in Sect. 1.5 below). Put simply, the $L^q$ norm does not see the separation between the bumps, while the pointwise bound does. We will nevertheless work with $L^q$ norms since we allow the bumps to have singularities. In the case where they are bounded the pointwise decay of the whole potential can easily be estimated by comparing the $L^\infty$ norms of the bumps to their spatial separation from the origin. In his fundamental work on non-selfadjoint Schrödinger operators, Pavlov [58,59] showed that the number of eigenvalues in one dimension is finite if $|V(x)| \lesssim \exp(-c|x|^{1/2})$, and that this exponential rate is best possible. This means that the potential in Theorem 1 cannot decay too fast. The $L^q$ bound imposes no decay whatsoever, but we can at least establish polynomial decay. For recent quantitative improvements of Pavlov’s bound we refer to Borichev–Frank–Volberg [7] and Sodin [73].

The proof of the example in [4] is based on “soft” methods like weak convergence, compact embedding and the notion of the limiting essential spectrum. In contrast, our proof uses “hard” estimates for the resolvent and the Birman–Schwinger operator, combined with tools from complex analysis such as Rouché’s theorem, Jensen’s formula and Cartan type estimates. This allows us to obtain more precise results that those in [4]. Rouché’s theorem and Jensen’s formula are among the most ubiquitous albeit simple tools in non-selfadjoint spectral theory, where such machinery as the variational principle or the spectral theorem is not available. In the present paper Cartan type estimates are crucial to bound a certain Fredholm determinant from below and get upper bounds on the norm of the resolvent. This opens the way to proving existence of eigenvalues by means of quasimode construction. We are then in a setting similar to the selfadjoint case where a quasimode of size $\epsilon$ guarantees the existence of a spectral point in an $\epsilon$-neighborhood of the quasi-eigenvalue. This follows from the inequality $\| (HV - z)^{-1} \| \leq 1/d(z, \sigma(HV))$, where $\sigma(HV)$ is the spectrum. In the non-selfadjoint case the inequality may fail dramatically. This phenomenon gives rise to the notion of pseudospectrum, which we will not discuss here (see e.g. the monograph [20]). The upper bounds obtained by Cartan type estimates generally grow exponentially in $1/d(z, \sigma(HV))$. In order to beat this, we are forced to construct exponentially small quasimodes, a challenging task in all but the simplest models. The strategy is reminiscent of the proof of existence of resonances close to the real axis due to Tang–Zworski [78] and Stefanov [74] (see also the recent book by Dyatlov–Zworski [25]). Our method is perhaps closest to that of Dencker–Sjöstrand–Zworski [23, Sect. 6] for non-selfajoint dissipative Schrödinger operators. The difference is that we consider decaying potentials and do not assume, as these authors do, that the quasi-eigenvalue is real (see [23, Proposition 6.4]). This means that the amplification of the exponential upper bound through the maximum principle (see [23, Proposition 6.2]) is in general not possible in our case. Another crucial difference is that we need a more quantitative version of the Cartan type estimate (Lemma 34) as well as of the conformal transformations between the spectral region and the model domain (the unit disk). The Riemann mapping theorem is notoriously non-quantitative. Instead, we use Cayley and Schwarz–Christoffel transformations, which have previously been used in other contexts related to non-selfadjoint spectral theory, especially in connection with Lieb–Thirring type inequalities. The combination with Rouché’s theorem and the Cartan type bounds is new and leads to results with an inverse problem flavor, as in Theorem 1.
1.3. Magnitude Bounds. The second result gives precise bounds on the magnitude of eigenvalues of Schrödinger operators with complex sparse potentials, or more generally, potentials of the form $V = \sum_{j=1}^{N} V_j$, where the $V_j$ have disjoint support and separate rapidly from each other. We will call these “separating” potentials. The Schrödinger operator $H_V = -\Delta + V$ behaves like an almost orthogonal sum, due to the rapid decoupling between the $N$ “channels”. This enables us to improve upon the bounds for general complex potentials due to Frank [28,29]. For simplicity we state the result here for $d \geq 3$. The general case along with further refinements can be found in Sect. 3.4. We refer to Sect. 2 for a more in-depth explanation of the terminology.

**Theorem 2.** Assume that $d \geq 3$ and $d/2 \leq q \leq (d+1)/2$. If $V$ is separating at scale $\eta^{-1}$, then every eigenvalue $z$ of $H_V$ with $\text{Im} \sqrt{z} \geq (d+1)\eta$ satisfies

$$|z|^q - d^2 \lesssim \sup_{j \in [N]} \|V_j\|_{L_q(R^d)}^q. \quad (3)$$

If $q > (d+1)/2$, then

$$|z|^\frac{1}{2} d(z, \mathbb{R}_+)^{q - \frac{d+1}{2}} \lesssim \sup_{j \in [N]} \|V_j\|_{L_q(R^d)}^q. \quad (4)$$

**Remark 2.** For $q < (d+1)/2$, the restriction $\text{Im} \sqrt{z} \geq (d+1)\eta$ can be removed if one assumes a stronger separation condition on the potential (see Sect. 3.5).

The bound (3) follows from (27) by a Birman–Schwinger argument. It could also be proved by using the eigenvalue bounds of the author [15], which are inspired by a method of Davies and Nath [21] in one dimension. For $N = 1$ the estimates (3), (4) coincide with those of Frank [28,29], respectively. The difference is that here $V$ might decay very slowly or not at all. Nevertheless, on the $\eta^2$ energy (spectral) scale the estimate is of the same quality as for $N = 1$.

We make a short remark about the connection with the Laptev–Safronov conjecture [48], which stipulates that

$$\sup_{V \in L^q(R^d)} \sup_{z \in \sigma(-\Delta + V) \setminus \mathbb{R}_+} \frac{|z|^q - d^2}{\|V\|_{L_q}^q} < \infty \quad \text{for all} \quad q \in [d/2, d]. \quad (5)$$

For the range $q \in [d/2, (d+1)/2]$ the conjecture was proven by Frank [28]. The question whether (5) is true for $q \in ((d+1)/2, d]$ is still open. The expectation, based on intuition from counterexamples to Fourier restriction (see e.g. [14,15] for more explanations) is that the conjecture is false in this range. Incidentally, Theorem 1 clearly implies the necessity of $q > d$ in the conjecture, but this already follows from Bögli’s result (without the pointwise bound). In fact, a single bump of the sparse potential used in Bögli’s construction (and in Theorem 1) already provides a counterexample. Since there seems to be some confusion about this issue we use the results of Sect. 7 to show necessity of the condition $q > d$. Indeed, Lemma 23 implies that for small $\epsilon > 0$ there is a potential $V(\epsilon)$ and an eigenvalue $z(\epsilon)$ such that $|z(\epsilon)|^{q-d}/\|V(\epsilon)\|_{L_q}^q \gtrsim \epsilon^{q-d} \log^q (1/\epsilon)$. The example (a complex step potential) is simple but, quite amazingly, generic enough to show optimality of several estimates in the literature (see [15]). In one dimension, the step potential can be tuned to essentially saturate any of the known magnitude bounds. For example, the last inequality also shows that the Davies–Nath bound [21] is sharp and, since $\text{Im} z(\epsilon) \asymp \epsilon$, that Frank’s bound [29, Theorem 1.1] is sharp up to a logarithm. In Sect. 1.5 we will show how the complex step potential also implies sharpness of another bound in [29].
1.4. A Generalization of Klaus’ Theorem. The following is a generalization of a result due to Klaus [44] on the characterization of the essential spectrum. The generalization is twofold: first, we admit complex potentials and second, we prove it for any dimension (whereas Klaus only proved the one-dimensional case). In this introduction we again focus on the case $d \geq 3$, but the statement is valid in $d = 1, 2$ for $q$ in the range (15).

**Theorem 3** (Klaus’ theorem [44] for complex potentials). Assume that $d \geq 3$ and $d/2 \leq q \leq (d+1)/2$. If $V$ is a separating potential and $\sup_{j \in [N]} \|V_j\|_{L^q(\mathbb{R}^d)} < \infty$, then

$$\sigma_e(H) = [0, \infty) \cup S,$$

where $S$ is the set of all $z \in \mathbb{C} \setminus [0, \infty)$ such that there exist infinite sequences $(i_n)_n, (z_n)_n$ with $z_n \in \sigma(H_{V_{i_n}})$, $i_n \to \infty$ and $z_n \to z$ as $n \to \infty$.

An alternative proof (also in one dimension) of Klaus’ theorem can be found in [17]. The role of Theorem 3 in this paper will be an auxiliary one, and we will only use it to argue that the essential spectrum is invariant under the perturbations we consider in Theorem 1. Although our proof follows the general strategy of [44] it is still worth emphasizing that some parts of it require somewhat novel techniques.

1.5. Weyl’s Law and Locality. Recently, Bögli and Štampach [6] disproved a conjecture by Demuth, Hansmann and Katriel [22] for one-dimensional Schrödinger operators with complex potentials by establishing a lower bound on a certain Riesz means of eigenvalues. More precisely, consider $H_{\alpha V}$, where $V = i1_{[-1, 1]}$ is a purely imaginary step potential and $\alpha$ is a large semiclassical parameter. Bögli and Štampach prove that, for any $p \geq 1$,

$$\alpha^{-p} \sum_{z \in \sigma_d(H_{\alpha V})} \left( \frac{\text{Im } z}{|z|^{1/2}} \right)^p \geq C_p \log \alpha. \quad (6)$$

The interesting feature of this bound is that it shows a logarithmic violation of Weyl’s law. To recall Weyl’s law, consider a self-adjoint operator, with a smooth real-valued potential. Note that if we set $h = 1/\sqrt{\alpha}$, then $\alpha^{-1} H_{\alpha V}$ takes the form of a semiclassical Schrödinger operator, $-h^2 \partial_x^2 + V(x)$. Semiclassical asymptotics (Weyl’s law) yield, for a suitable class of functions $f$,

$$\text{Tr } f(H_{\alpha V}) = \frac{\sqrt{\alpha}}{2\pi} \left( \int f(\alpha(\xi^2 + V(x)))\text{d}x\text{d}\xi + o(1) \right) \quad (7)$$

as $\alpha \to \infty$. In particular, if $f$ is homogeneous of degree $\gamma$, then

$$\lim_{\alpha \to \infty} \alpha^{-1/2 - \gamma} \text{Tr } f(H_{\alpha V}) = (2\pi)^{-1} \int f(\xi^2 + V(x))\text{d}x\text{d}\xi. \quad (8)$$

For $f(\lambda) := \lambda^\gamma$ and $\gamma \geq 1/2$ (since we are considering $d = 1$) the Lieb-Thirring inequality

$$\sum_{\lambda \in \sigma_d(H_{\alpha V})} \lambda^\gamma \leq C_\gamma \alpha^{1/2+\gamma} \int V_-(x)^{1/2+\gamma} \text{d}x \quad (9)$$
captures the semiclassical behavior (8), but is valid for any \( \alpha > 0 \), not only asymptotically. Returning to the complex potential \( V = i1_{[-1,1]} \) and noticing that \( f(z) := \frac{(\text{Im } z)^p}{|z|^{\gamma/2}} \) is homogeneous of degree \( \gamma = p - 1/2 \), we observe that (6) implies that the formal analogue of (8) cannot hold, i.e. that

\[
\liminf_{\alpha \to \infty} \alpha^{-1/2-\gamma} \sum_{z \in \sigma_d(H_{\alpha V})} \frac{(\text{Im } z)^{1/2+\gamma}}{|z|^{1/2}} = \infty,
\]

hence violating Weyl’s law (8). The comparison with Weyl’s law is formal because \( f(H_{\alpha}) \) does not make sense in general for a non-normal operator. However, (6) also shows that the complex analogue of the Lieb-Thirring inequality (9),

\[
\sum_{z \in \sigma_d(H_{\alpha V})} \frac{(\text{Im } z)^{1/2+\gamma}}{|z|^{1/2}} \leq C' \alpha^{1/2+\gamma} \int V_-(x)^{1/2+\gamma} \mathrm{d}x,
\]

cannot be true, thus disproving the conjectured bound in [22].

For a non-selfadjoint (pseudo)-differential operator with analytic symbol and in one dimension the eigenvalues typically lie on a complex curve, hence violating Weyl’s law (in terms of complex phase space). Small random perturbations typically restore the Weyl law (in real phase space). The literature on the subject is vast and we merely refer the interested reader to the book of Sjöstrand [72] for an overview of recent developments. In contrast, a classical result of Markus and Macaev [52] implies that Weyl’s law holds for the real part of the eigenvalues, as long as the bumps are disjoint. This feature of locality is connected with the notion of “locality”. In the semiclassical limit, in the self-adjoint case, each state occupies a volume of \( \sqrt{\alpha} \text{ Vol}(SE,V)(1 + o(1)) \),

\[
N(H_{\alpha V}; (-\infty, -E]) = \sqrt{\alpha} \text{ Vol}(SE,V)(1 + o(1)),
\]

where \( N(H; \Sigma) \) denotes the number of eigenvalues of an operator \( H \) in a set \( \Sigma \) and \( SE,V := \{(x, \xi) \in T^*\mathbb{R} : \xi^2 + V(x) \leq -E \} \) is the relevant part of phase space. If we consider a sum of disjoint bumps \( V = \sum_{j=1}^N V_j \), say with \( V_j(x) = W(x - x_j) \), then \( SE,V = N SE,W \), and hence

\[
\frac{N(H_{\alpha V}; (-\infty, -E])}{N(H_{\alpha W}; (-\infty, -E])} = N(1 + o(1)).
\]

This means that in the semiclassical limit each bump is responsible for an equal number of eigenvalues. In particular, two distinct realizations of \( V \) as a sum of bumps have the same number of eigenvalues, as long as the bumps are disjoint. This feature of locality is also captured by the Lieb-Thirring inequalities since the bound involves the integral linearly.

Our third result shows that this kind of locality can be violated in the non-selfadjoint case. We adopt the same notation \( N(H; \Sigma) \) for the number of eigenvalues of \( H \) in \( \Sigma \) as in the self-adjoint case, but emphasize that these are counted according to their \textit{algebraic} multiplicity. We consider one sparse and one non-sparse (or non-separating) realization of \( V \) and denote these by \( V_s \) and \( V_n \), respectively. For simplicity, we will consider the same potential as in [6], i.e. the purely imaginary step potential \( W = i|W_0|1_{[-R_0,R_0]} \) of size \( |W_0| \) and width \( R_0 \). For simplicity we fix these scales to be of order one. Then \( V_n \)
is the single well of width $R = NR_0$, while $V_s$ is a sum of $N$ disjoint wells $W(x - x_j)$ of width $R_0$. We will fix the coupling strength $\alpha$ and consider the limit $N \to \infty$. For the non-sparse operator this is in fact still a semiclassical limit, as can easily be seen by rescaling.

**Theorem 4.** Let $d = 1$, $N \gg 1$ and consider the rectangular set

$$
\Sigma := \{ z \in \mathbb{C} : C^{-1} \frac{N^2}{\log^2 N} \leq \Re z \leq C \frac{N^2}{\log^2 N}, \quad C^{-1} \leq \Im z \leq C \},
$$

where $C$ is a large constant. Then we have

$$
N(HV_n; \Sigma) \gtrsim \frac{N^2}{\log N}. \quad (10)
$$

Moreover, there exists a sequence $(x_j)_j$ such that

$$
N(HV_s; \Sigma) \lesssim N. \quad (11)
$$

Note that, by power counting (dimensional analysis), the constants in (10), (11) only depend on the dimensionless quantity $|W_0|^{1/2}R_0$. We also point out that our regime (the asymptotic shape of the set $\Sigma$) differs from that of Bögli and Stampach [6].

In the selfadjoint case, i.e. when $W$ is replaced by $|W_0| 1_{(-R_0,R_0]}$, the number of negative eigenvalues of $V_n$ is of order $N$, in agreement with semiclassics. Also note that, since in one dimension each $HV_J$ has at least one negative eigenvalue, we have $N(HV_s; \mathbb{R}_-) \asymp N$ in this case. Hence the left hand sides of (10) and (11) are equal in magnitude, which may be seen as a manifestation of locality. However, this locality is violated if one takes into account not only eigenvalues but also resonances. Zworski [82] proved that, for a compactly supported, bounded, complex potential, the number $n(r)$ of resonances $\lambda_j^2$ in a disk $|\lambda_j| \leq r$ asymptotically satisfies

$$
n(r) = \frac{2 |\text{ch supp}(V)|}{\pi} r(1 + o_V(1)) \quad (12)
$$

as $r \to \infty$ (the subscript means that the remainder depends on $V$). Moreover, for any $\epsilon_0 > 0$, the number of resonances in $|\lambda_j| \leq r$ but outside the sector $\Sigma_0$ [see (1)] is $o(r)$. This and the fact that eigenvalue bounds outside $\Sigma_0$ are “trivial” (in the sense that they can be proved by the same standard estimates as for real potentials, with the provision that the constants blow up as the implicit constant in (1) becomes small) motivates us to often restrict attention to the spectral set $\Sigma_0$. The result (12) was obtained earlier by Regge [62] in some special cases. Different proofs were given by Froese [32] and Simon [69]. Froese’s proof also works for complex potentials. Note that eigenvalues are included in the definition of resonances. Formula (12) can be seen as a Weyl law for resonances but is nonlocal as it includes the convex hull of the support of the potential. An obvious corollary of Zworski’s formula is that the right hand side of (12) is an upper bound for the number of eigenvalues $\lambda_j^2$ in the disk $|\lambda_j| \leq r$ (resonances in the upper half plane). For the potential $V_n$, taking $r \asymp N / \log N$ and observing that $\text{ch supp}(V_n) \asymp N$, we find that the leading term in (12) is of order $N^2 / \log N$, in agreement with the lower bound (10). Note, however, that the asymptotics are not uniform, i.e. the error may depend on
N (recall that $V_n$ depends on $N$). Korotyaev [45, Theorem 1.1] proved an upper bound which yields

$$n(r) \leq \frac{8N}{\pi \log 2} r + O(N \log N)$$

for $r \approx N/\log N$. Hence, (10) shows that, at the scale considered here, a substantial fraction of resonances are actual eigenvalues. In the special case of a step potential Stepanenko [77] proved that the total number of eigenvalues is bounded by $N^2/\log N$. The lower bound (10) shows that this is sharp up to constants; this was observed independently by Stepanenko [76].

In dimension $d \geq 3$ it may of course happen that all the $H_{V_j}$, and thus also $H_{V_s}$, have no eigenvalues at all if $|W_0|^{1/2}R_0$ is small (by the CLR bound), while $H_{V_n}$ has of the order $N^d$ eigenvalues. This is not the kind of phenomenon that takes place in Theorem 4. Indeed, there we allow $|W_0|^{1/2}R_0$ to be of unit size. We also note that in higher dimensions the results on the asymptotics of the resonance counting function are weaker, and there are example of complex potentials with no resonances at all (see Christiansen [9]). On the other hand, Christiansen [8] and Christiansen–Hislop [10,11] proved that $n(r)$ has maximal growth rate $r^d$ for “most” potentials in certain families.

A final comment regarding the implications of Theorem 4, also related to locality (or the lack thereof), concerns a general observation on Lieb–Thirring type inequalities for complex potentials. It is a fact that all known upper bounds for eigenvalue sums have a superlinear dependence on $N$. For example, in $d = 1$, [29, Theorem 1.3] yields the bound

$$\sum_j d(z_j, \mathbb{R}^+)^a \lesssim \left( \int_{\mathbb{R}} |V|^b dx \right)^c$$

with $(a, b, c) = (1, 1, 2)$. The lower bound (10) shows that the power $c$ cannot be decreased, while preserving the homogeneity condition $-2a = (-2b + 1)c$. Indeed,

$$\sum_j d(z_j, \mathbb{R}^+)^a \geq \sum_{z_j \in \Sigma} d(z_j, \mathbb{R}^+)^a \gtrsim |W_0|^a N^2 \frac{\log N}{\log N},$$

while

$$\left( \int_{\mathbb{R}} |V_n|^b dx \right)^c = (2NR_0|W_0|^b)^c,$$

so that the ratio of the first to the second is bounded below by $(|W_0|^{1/2}R_0)^{-c N^2 - c \log N}$, which tends to infinity as $N \to \infty$, for every $c < 2$. On the other hand, for sufficient rapid separation of the bumps in $V_n$, we can prove Frank’s bound (13) with a linear dependence on $N$, at least locally in the spectrum.

**Theorem 5.** Let $V \in \ell^2L^1(\mathbb{R})$ (see Sect. 2.1) and $N \gg 1$. For any $\eta > 0$ there exists a sequence $(x_j)_{j=1}^{N}$, $x_j = x_j(\eta, \|V\|_{\ell^2L^1})$, such that

$$\sum_{\text{Im} \sqrt{z_j} > (d+1)\eta} d(z_j, \mathbb{R}^+) \lesssim N \sum_{j=1}^{N} \left( \int_{\mathbb{R}} |V_j|dx \right)^2.$$
Notation. We write $\sigma(H)$, $\rho(H)$ for the spectrum, respectively the resolvent set of a closed linear operator $H$. The free and the perturbed resolvent operators are denoted by $R_0(z) = (-\Delta - z)^{-1}$ and $R_V(z) = (H_V - z)^{-1}$, respectively. We denote by $\mathcal{S}^p$ the Schatten spaces of order $p$ over the Hilbert space $L^2(\mathbb{R}^d)$ and by $\| \cdot \|_p$ the corresponding Schatten norms. We also write $\| \cdot \| = \| \cdot \|_{\infty}$ for the operator norm. To distinguish it from $L^p$ norms of functions we denote the latter by $\| \cdot \|_{L^p}$. We will use the notation $V^{1/2} = V/|V|^{1/2}$ and $\langle x \rangle := 2 + |x|$. The statement $a \lesssim b$ means that $|a| \leq C|b|$ for some absolute constant $C$. We write $a \asymp b$ if $a \lesssim b \lesssim a$. If the estimate depends on a list of parameters $\tau$, we indicate this by writing $a \lesssim_{\tau} b$. The dependence on the dimension $d$ and the Lebesgue exponents $p, q$ is usually suppressed. We write $a \ll b$ if $|a| \leq c|b|$ with a small absolute constant $c$, independent of any parameters. By an absolute constant we always understand a dimensionless constant $C = C(d, p, q)$. Here we choose units of length $l$ such that position, momentum and energy have dimensions $l, l^{-1}$ and $l^{-2}$, respectively. We chose the branch of the square root $\sqrt{z}$ on $\mathbb{C}\setminus\{0, \infty\}$ such that $\sqrt{z} \in \mathbb{H}$, where $\mathbb{H} = \{ \kappa \in \mathbb{C} : \Im \kappa > 0 \}$ denotes the upper half plane. The open unit disk in $\mathbb{C}$ is denoted by $\mathbb{D}$.

2. Definitions and Preliminaries

2.1. Separating and Sparse Potentials. We consider sparse potentials of the form

$$V(x) = V_j(x), \quad x \in \Omega_j,$$

where $j \in [N] = \{1, 2, \ldots, N\}, N \in \mathbb{N} \cup \{\infty\}$, and $\Omega_j \subset \mathbb{R}^d$ are mutually disjoint (not necessarily bounded) sets. We then set

$$L_j := d(\Omega_j, \bigcup_{i \in [N]} \Omega_i \setminus \Omega_j).$$

We assume that $V_j \in \ell^p L^q$, where the norms are defined by

$$\|V\|_{\ell^p L^q} := \left( \sum_{j \in [N]} \|V_j\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}$$

for $p \in [1, \infty)$ and $\|V\|_{\ell^p \infty L^q} := \sup_{j \in [N]} \|V_j\|_{L^q(\mathbb{R}^d)}$. Here $q$ will be in the range

$$q \in [1, \infty] \quad \text{if } d = 1, \quad q \in (1, \infty] \quad \text{if } d = 2, \quad q \in \left[\frac{d}{2}, \infty\right] \quad \text{if } d \geq 3. \quad \text{(15)}$$

In particular, we have $\|V\|_{L^q} = \|V\|_{\ell^q L^q}$. We sometimes write $V = V(L)$ or $V = V(L, \Omega)$ to emphasize the dependence of $V$ on the sequences $L = (L_j)_{j=1}^N$ or $\Omega = (\Omega_j)_{j=1}^N$.

Definition 1. We say that $V = V(L)$ is separating if

$$\text{sep}(L, \eta) := \sum_{j \in [N]} \exp(-\eta L_j) < \infty$$

for every $\eta > 0$. We call $V$ separating at scale $\eta^{-1}$ if $\text{sep}(L, \eta) \leq 1$. We say that $V$ is strongly separating if $\text{sep}(L, \delta \eta) \lesssim_\delta \text{sep}(L, \eta)$ for every $\delta, \eta > 0$. 

The constant $\text{sep}(L, \eta)$ only depends on $L$ and not on $V$ itself. We will sometimes abuse terminology and call the sequence $L$ separating. Note that since $\eta L_j$ is dimensionless, $\eta$ has the dimension of inverse length. We shall always assume that the sequence $L$ is increasing.

**Definition 2.** We say that $V = V(L, \Omega)$ is *sparse* if it is separating, the supports $\Omega_n$ are bounded, and $\lim_{n \to \infty} \text{diam}(\Omega_n)/L_n = 0$.

Most of our results hold for separating potentials. The strong separation condition is convenient and facilitates some of the proofs. Typical examples of strongly separating sequences are $(\eta \ll \eta_0)$:

- a) If $\eta_0 L_k \gtrsim k^\alpha$ for $\alpha > 0$, then $\text{sep}(L, \eta) \lesssim (\eta/\eta_0)^{-1/\alpha}$.
- b) If $\eta_0 L_k \gtrsim \exp(k)$, then $\text{sep}(L, \eta) \lesssim \log(\eta_0/\eta)$.
- c) If $\eta_0 L_k \gtrsim \exp(\exp(k))$, then $\text{sep}(L, \eta) \lesssim \log \log(\eta_0/\eta)$.

See Sect. 8.2 for a proof. The explicit example used to prove Theorem 1 turns out to be sparse. Note that, by the disjoint supports, the definition (14) is equivalent to $V = \sum_{j=1}^{N} V_j$.

### 2.2. Comparison with a Direct Sum.

In Sect. 5 we will compare the two operators

$$H_V = -\Delta + V, \quad H_{\text{diag}} = \bigoplus_{j \in [N]} (-\Delta + V_j).$$

Note that the point spectrum (eigenvalues) of $H_{\text{diag}}$,

$$\sigma_p(H_{\text{diag}}) = \bigcup_{j=1}^{N} \sigma_p(H_{V_j}) = \bigcup_{j=1}^{N} \sigma_p(H_{W_j}),$$

is independent of the sequence $L$. We will consider $\sigma_p(H_{W_j})$ as part of the data and seek to prove lower bounds on $L_j$ or $|x_i - x_j|$ that imply that $\sigma_p(H_V)$ is close to $\sigma_p(H_{\text{diag}})$. In fact, we will consider a subset of the point spectrum, the discrete spectrum. We will consider the $V_j$ as given only up to translations, i.e. we stipulate that

$$V_j(x) = W_j(x - x_j),$$

where $W_j \in L^q$ contains the origin in its support. Using the triangle inequality, it is easy to see that $|x_i - x_j| \geq L_j$ for $i \neq j$, and therefore

$$\sup_{i \in [N]} \sum_{j \in [N] \setminus \{i\}} \exp(-\eta|x_i - x_j|) \leq \text{sep}(L, \eta).$$

Straightforward arguments also show that

$$L_i \geq |x_i - x_j| - \text{diam}(\Omega_i) - \text{diam}(\Omega_j)$$

for all $j \in [N] \setminus \{i\}$. In particular, for sparse potentials,

$$L_i(1 + o(1)) \geq \sup_{j < i} |x_i - x_j|. \quad (16)$$

Note that $\text{diam}(\Omega_j) = \text{diam}(\text{supp } W_j)$ is part of the data. We can thus obtain a lower bound for $|x_i - x_j|$ in terms of $L_j$ and vice versa. For this reason we restrict attention to $L_j$ here.
2.3. Truncations. For technical reasons, it will turn out to be convenient to consider finite truncations. For \( n \in \mathbb{N} \) we define
\[
V^{(n)} := \sum_{j \in [n]} V_j, \quad H^{(n)} := -\Delta + V^{(n)}.
\]

2.4. Abstract Birman–Schwinger Principle. We mostly disregard operator theoretic discussions here and refer e.g. to [29] for the (standard) definition of \( H_{V_j} \) as \( m \)-sectorial operators. The rigorous definition of \( H_V \) is a bit more subtle since \( V \) need not be decaying. However, a classical construction of Kato [41] produces a closed extension \( H_V \) of \( -\Delta + V \) via a Birman–Schwinger type argument. This approach works as soon as one can find point \( z_0 \) in the resolvent set of \( H_0 = -\Delta \) at which the Birman–Schwinger operator
\[
BS_V(z) := |V|^{1/2}(-\Delta^2 - z)^{-1}V^{1/2}
\]
has norm less than one. Such bounds are provided by Lemma 6, but to avoid technicalities it is useful to think of the potential as being bounded by a large cutoff (and all of the bounds will be independent of that cutoff). By iterating the second resolvent identity,
\[
R_V(z) = R_0(z) - R_0(z)V R(z),
\]
it is then easy to see that
\[
R_V(z) - R_0(z) = -R_0(z)V^{1/2}(I + BS_V(z))^{-1}|V|^{1/2}R_0(z).
\]
For more background on the abstract Birman–Schwinger principle in a nonselfadjoint setting we refer to [3,29,33,37].

2.5. The Essential and Discrete Spectrum. We briefly recall some facts about the essential and discrete spectrum of a closed operator \( H \). There are several inequivalent definitions of essential spectrum for non-selfadjoint operators (but these all coincide for Schrödinger operators with decaying potentials [29, Appendix B]). We use the following standard definition.
\[
\sigma_e(H) := \{ z \in \mathbb{C} : H - z \text{ is not a Fredholm operator} \}.
\]
The discrete spectrum is defined as
\[
\sigma_d(H) := \{ z \in \mathbb{C} : z \text{ is an isolated eigenvalue of } H \text{ of finite multiplicity} \}.
\]
Note that, if \( H \) is not selfadjoint, then, in general, \( \sigma(H) \) is not the disjoint union of \( \sigma_e(H) \) and \( \sigma_d(H) \). However, by [34, Theorem XVII.2.1], if every connected component of \( \mathbb{C} \setminus \sigma_e(H) \) contains points of \( \rho(H) \), then
\[
\sigma(H) \setminus \sigma_e(H) = \sigma_d(H).
\]
In the situations we consider here (20) will always be true for \( H = H_V \) and \( H_{\text{diag}} \). In fact, Corollary 11 tells us that \( \sigma_e(H) = [0, \infty) \), just as for decaying potentials.
3. Universal Bounds for Separating Potentials

In this section we consider $H_V$ as a perturbation of $-\Delta$. We will thus only make assumptions about $V$ and not about $H_{\text{diag}}$.

3.1. Birman–Schwinger Analysis. Since the $V_j$ have mutually disjoint supports, we can write the Birman–Schwinger operator (18) as

$$BS_V(z) = \sum_{i=1}^{N} BS_{ii}(z) + \sum_{i \neq j} BS_{ij}(z),$$

where

$$BS_{ij}(z) = |V_i|^{1/2} R_0(z) V_j^{1/2}.$$

The first term is unitarily equivalent to the orthogonal sum

$$BS_{\text{diag}}(z) := \bigoplus_{i=1}^{N} BS_{ii}(z).$$

on the Hilbert space $\mathcal{H} \simeq \bigoplus_{i=1}^{N} \mathcal{H}_i$, where $\mathcal{H} = L^2(\mathbb{R}^d)$, $\mathcal{H}_i = L^2(\Omega_i)$. By abuse of notation we will always identify these two Hilbert spaces and the corresponding operators. The off-diagonal contribution is

$$BS_{\text{off}}(z) := \sum_{i \neq j} BS_{ij}(z).$$

In the following we will use the notation

$$\omega_q(z) := \begin{cases} |z|^{\frac{d}{2q} - 1} & \text{if } q \leq q_d, \\ |z|^{-\frac{d}{2q}} d(z, \mathbb{R}_+) \frac{q_d}{q} - 1 & \text{if } q \geq q_d, \end{cases}$$

(21)

where $q_d = (d + 1)/2$. Note that for $z \in \Sigma_0$ [see (1)] we have $d(z, \mathbb{R}_+) = |\text{Im } z|$. We use the abbreviation

$$s(L, z) := \text{sep}(L, \text{Im } \sqrt{z}/(d + 1))$$

(22)

and set $\alpha(q) := 2 \max(q, q_d)$.

Lemma 6. Assume $q$ is in the range (15) and $p \in [\alpha(q), \infty]$. Then the following hold.

(i) For any $i \in [N]$,

$$\| BS_{ii}(z) \|_{\alpha(q)} \lesssim \omega_q(z) \| V_i \|_{L^q}.$$ (23)

(ii) For any $i, j \in [N], i \neq j$,

$$\| BS_{ij}(z) \|_{\alpha(q)} \lesssim \exp\left(-\frac{2}{d+1} \text{Im } \sqrt{z} d(\Omega_i, \Omega_j) \omega_q(z) \right) \| V_i \|_{L^q}^{1/2} \| V_j \|_{L^q}^{1/2}.$$ (24)

(iii) The diagonal part satisfies

$$\| BS_{\text{diag}}(z) \|_p \lesssim \omega_q(z) \| V \|_{\ell^p L^q}.$$ (25)
(iv) **The off-diagonal part satisfies**

\[ \|BS_{\text{off}}(z)\|_{\alpha(q)} \lesssim s(L, z)^2 \omega_q(z) \|V\|_{L^\infty L^q}, \quad (26) \]

(v) **The full Birman–Schwinger operator satisfies**

\[ \|BS_V(z)\|_p \lesssim \omega_q(z) \langle s(L, z) \rangle^2 \|V\|_{L^p L^q}. \quad (27) \]

**Proof.** It follows from

\[
\|BS_{\text{diag}}(z)\|_p = \left( \sum_{i \in [N]} \|BS_{ii}(z)\|^p_p \right)^{1/p}
\]

that (iii) is a consequence of (i). In view of the embeddings \( \mathcal{S}^{p_1} \subset \mathcal{S}^{p_2}, \ell^{p_1} \subset \ell^{p_2} \) for \( p_1 \leq p_2 \), (v) follows from (iii) and (iv). Moreover, (iv) follows from (ii) and the triangle inequality, using the estimate \( d(\Omega_i, \Omega_j) \geq \frac{1}{2} L_i + \frac{1}{2} L_j \) to sum the double series. The estimate (i) is the same as for the \( N = 1 \) case and follows from known results in the literature: For \( q \leq q_d \), from [31, Theorem 12] for \( d \geq 3 \), from [13, Theorem 4.1] for \( d = 2 \) and from [1, Theorem 4] for \( d = 1 \), for \( q \geq q_d \) and \( d \geq 1 \) from [29, Proposition 2.1].

The bound (ii) is proved by complex interpolation as in [31, Theorem 12] in the case \( q \leq q_d \) and in [29, Proposition 2.1] for \( q \geq q_d \). The only difference is that we include the (second) exponential in the pointwise bound

\[ |(-\Delta - z)^{-(a+it)}(x - y)| \leq C_1 e^{C_2 t^2} e^{-\text{Im} \sqrt{z} |x-y|} |x-y|^{d+1+a} \quad (28) \]

for \( a \in [(d-1)/2, (d+1)/2] \) and \( d \geq 2 \), see e.g. [50, (2.5)]. For \( d = 1 \) one can use the explicit formula for the resolvent kernel. \( \square \)

**Remark 3.** Using the results of [15] (or [21] in one dimension) in the proof of Lemma 6 we could replace the bounds (23), (24) by the following. For \( q \leq q_d \) and \( i, j \in [N], \)

\[
\|BS_{ij}(z)\|_{\alpha(q)} \lesssim \exp(-\frac{2}{d+1} \text{Im} \sqrt{z} d(\Omega_i, \Omega_j)))|z|^\frac{d}{d+1} F_{V, q}^{1/2}(\text{Im} \sqrt{z}) F_{V, q}^{1/2}(\text{Im} \sqrt{z}),
\]

where \( F_{V, q}(s) \) is the “Davies–Nath norm”

\[
F_{V, q}(s) := \left( \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |V(x)|^q \exp(-s|x-y|) dx \right)^{1/q}.
\]  

This implies the bounds

\[
\|BS_{\text{diag}}(z)\|_p \lesssim |z|^\frac{d}{d+1} \sup_{i \in [N]} F_{V, q},
\]

\[
\|BS_{\text{off}}(z)\|_{\alpha(q)} \lesssim s(L, z)^2 |z|^\frac{d}{d+1} \sup_{i \in [N]} F_{V, q},
\]

\[
\|BS_V(z)\|_{\alpha(q)} \lesssim (s(L, z))^2 |z|^\frac{d}{d+1} \sup_{i \in [N]} F_{V, q}.
\]
3.2. Norm Resolvent Convergence.

Lemma 7. Under the assumptions of Lemma 6 we have

\[ \| |V|^{1/2} R_0(z)\|_{2p} \lesssim |\text{Im } z|^{-1} \omega_q(z) (s(L, z))^2 \|V\|^{1/2}_{\ell^p L^q}. \]

Proof. The claim readily follows from (27), the identity

\[ R_0(z) R_0(\bar{z}) = \frac{1}{2 \text{Im } z} (R_0(\bar{z}) - R_0(z)) \]

and a TT* argument. \( \square \)

Proposition 8. Under the assumptions of Lemma 6 the Schrödinger operators \( H_{V^{(n)}} \) with truncated potentials converge in norm resolvent sense to \( H_V \).

Proof. We first note that all the bounds of Lemma 6 also hold for \( H_{V^{(n)}} \), uniformly in \( n \). Since \( V^{(n)} \) converges to \( V \) in \( \ell^p L^q \), it follows that the Birman–Schwinger operator associated to \( H_{V^{(n)}} \) converges to \( BS_V(z) \). Moreover, by Lemma 7, the operators \( |V^{(n)}|^{1/2} R_0(z) \) converge in \( S_{2p} \)-norm. Note that the square root of \( V \) is trivial to compute, owing to the disjointness of supports of the \( V_j \). We choose \( z = it, t \gg 1 \). For such \( z \) the norm of the Birman–Schwinger operator is \( <1 \), hence a Neumann series argument and the resolvent identity (19) yield the claim. \( \square \)

Proposition 9. As \( s(L, z) \to 0 \) for fixed \( z \), the Schrödinger operators \( H_{V} \) (depending on \( L \)) converge in norm resolvent sense to \( H_{\text{diag}} \).

Proof. By (26), it follows that \( BS_{\text{off}}(z) \to 0 \) in norm as \( s(L, z) \to 0 \). The remainder of the proof is similar to that of Proposition 8. \( \square \)

3.3. Proof of Klaus’ Theorem. The proof is an adaptation of Klaus’ original argument, which is based on the Birman–Schwinger principle. In the non-selfadjoint setting one may use e.g. [20, Lemma 11.2.1] as a substitute for Weyl sequences. This yields an easy proof of the inclusion \( \sigma_e(H_V) \supset [0, \infty) \cup S \). For the converse inclusion we prove an analogue of [44, Proposition 2.3].

Proposition 10. For \( V \in \ell^\infty L^q \) and \( z \in \mathbb{C} \setminus [0, \infty) \) we have

\[ \sigma(BS_{\text{diag}}(z)) = \bigcup_i \sigma(BS_{ii}(z)). \]

Proof. Only the inclusion \( \subset \) is nontrivial. The resolvent set of a direct sum of bounded operators \( A = \bigoplus_i A_i \) is known to be

\[ \rho(A) = \{ \lambda \in \bigcap_i \rho(A_i) : \sup_i \| (A_i - \lambda)^{-1} \| < \infty \}. \]

We set \( A_i = BS_{ii}(z) \), whence \( A = BS_{\text{diag}}(z) \). Assume that \( \lambda \in \mathbb{C} \setminus \bigcup_i \sigma(BS_{ii}(z)) \). This clearly implies \( \lambda \in \bigcap_i \rho(A_i) \). It remains to prove that \( \sup_i \| (A_i - \lambda)^{-1} \| < \infty \). By assumption, there exists \( \delta > 0 \) such that \( d(\lambda, \sigma(A_i)) \geq \delta \). By [2, Theorem 4.1],

\[ \| (A_i - \lambda)^{-1} \| \leq \delta^{-1} \exp(a \| A_i \|_a^{\alpha} \delta^{-\alpha} + b), \]

where \( \alpha = \alpha(q) \) and \( a = a(q), b = b(q) \) are constants. Since, by (25), \( \| A_i \|_a \lesssim \| V \|_{\ell^\infty L^q} \), it follows that \( \sup_i \| (A_i - \lambda)^{-1} \| < \infty \). \( \square \)
With Proposition 10 in hand is immediate that [44, Lemma 2.4] \((K_i = A_i\) in our notation) holds in the generality needed here. The Schatten bound (26) provides a substitute for the compactness arguments [44, Proposition 2.1–2.2]. For the remainder of the proof one can follow the arguments in [44] verbatim.

Corollary 11. If \(V \in \ell^p L^q\) with \(q\) in the range (15) and \(p < \infty\), then we have \(S = \emptyset\), i.e. \(\sigma_e(H_V) = [0, \infty)\). The same holds for \(H_{\text{diag}}\).

**Proof.** By (27), we have that \(\lim_{t \to \infty} \|B_{SV}(it)\| = 0\), which means that the inverse exists as a bounded operator for \(z = it\) and \(t \gg 1\). Corollary 7 implies that \(R_0(z)V^{1/2}\) is compact, whence the resolvent difference (19) is compact. The claim now follows by Weyl’s theorem [26, Theorem IX.2.4]. \(\square\)

3.4. Magnitude Bounds. The following universal bounds generalize those of Theorem 2 in the introduction. They are an immediate consequence of (27), Remark 3 and the Birman–Schwinger principle.

**Theorem 12.** Let \(q\) be in the range (15). If \(V\) is separating, then every eigenvalue \(z\) of \(H_V\) satisfies

\[
\omega_q(z)^{-1} \langle s(L, z) \rangle^{-2} \lesssim \|V\|_{\ell^\infty L^q},
\]

as well as

\[
|z|^{q - \frac{d}{2}} \langle s(L, z) \rangle^{-2} \lesssim \sup_{j \in [N]} F_{V_j, q}(\text{Im} \sqrt{z}),
\]

where \(\omega_q(z)\) is defined in (21).

**Remark 4.** (i) In the case of a single “bump” \((N = 1)\) the bound (30) was proved by Frank in [30] for \(q \leq q_d\) and in [29] for \(q > q_d\). In the latter case it was observed that the inequality implies \(\text{Im} z \to 0\) as \(\text{Re} z \to +\infty\) for eigenvalues \(z\) of \(H_V\). More precisely, if we fix the norm of the potential, then

\[
|\text{Im} z|^{1 - \frac{q d}{q}} \lesssim (\text{Re} z)^{-\frac{1}{q d}}.
\]

In the case \(N \neq 1\) (30) implies that the above holds with an additional factor \(\langle s(L, z) \rangle^2\) on the right. If \(L\) grows at least polynomially, \(\eta_0 L_k \gtrsim k^\alpha\), then we obtain

\[
|\text{Im} z|^{1 - \frac{q d}{q} + \frac{2}{\alpha}} \lesssim (\text{Re} z)^{-\frac{1}{q d} + \frac{1}{\alpha} \eta_0^{-\frac{2}{\alpha}}},
\]

see Example a) after Definition 2. Hence, for sufficiently large \(\alpha\) (depending on \(q\) and \(d\)) the exponent of \(|\text{Im} z|\) remains positive, while that of \(\text{Re} z\) remains negative, and we still get the conclusion that \(\text{Im} z \to 0\) as \(\text{Re} z \to +\infty\).

(ii) The \(N = 1\) case of (31) was proved in one dimension by Davies and Nath [21] and in higher dimensions by the author [15]. The inequality is similar to (30) for \(q > q_d\). Both are relevant for “long-range” potentials. In the case of the step potential (31) is sharp, while (30) (both for \(N = 1\)) loses a logarithm [see (63) and (64)].
3.5. Alternative Magnitude Bound. Next, we state an alternative estimate to the magnitude bound (30). The difference is that the alternative estimate is valid up to the essential spectrum, whereas the bounds of Theorem 2 require a lower bound on $\text{Im} \sqrt{z}$. On the other hand, the alternative estimate is only valid under a much stronger sparsity assumption on the potential. For the remainder of this subsection only, we define the symmetric matrix

$$L_{ij} := \delta_{ij} + \text{dist}(\Omega_i, \Omega_j)$$

and assume that

$$\sup_{i \in [N]} \sum_{j \in [N]} L_{ij}^{-\epsilon} < \infty \quad \text{for every } \epsilon > 0. \quad (32)$$

As before, we assume that $q$ satisfies (15). Moreover, we restrict ourselves to the case $N \in \mathbb{N}$ here.

**Proposition 13.** Assume that $q < (d+1)/2$ and that (32) holds. Then there exists $C_q > 0$ such that every eigenvalue $z$ of $-\Delta + V$ satisfies

$$|z|^{q-d} \leq C_q \|V\|_{L^\infty L^q}^q. \quad (33)$$

**Proof.** One possibility to prove this would be to modify the argument in the proof of Lemma 6. However, instead of working with the Birman–Schwinger operator, we prefer to give a different proof here. We use the multilinear expansion

$$[R_0 V]^n = \sum_{i_1, \ldots, i_n} R_0 V_{i_1} R_0 V_{i_2} \ldots R_0 V_{i_n},$$

where we sum over $i_k \in [N]$, and $n \in \mathbb{N}$ is arbitrary. In the above formula we omitted the spectral parameter $z$ in the free resolvent $R_0$. Throughout the proof, we will assume that $z$ lies on the unit circle and $z \neq 1$ (this is sufficient, by scaling). We also assume that $V$ is bounded, so that $R_0 V$ is bounded as well. Since the resulting estimates are independent of the $L^\infty$-norm of $V$, this restriction is easily removed by a limiting argument. More precisely, one uses that the Schrödinger operators with truncated potentials $V_k = V1_{|V| \leq k}$ converge in norm resolvent sense to $-\Delta + V$. The proof is routine (similar to that of Proposition 8) and is omitted.

By the triangle inequality and Cauchy–Schwarz, we have

$$\| [R_0 V]^n \| \leq \sum_{i_1, \ldots, i_n} \| R_0 \| V_{i_1}^{\frac{1}{2}} \| \| V_{i_1}^{\frac{1}{2}} R_0 \| V_{i_2}^{\frac{1}{2}} \| \ldots \| V_{i_{n-1}}^{\frac{1}{2}} R_0 \| V_{i_n}^{\frac{1}{2}} \| \| V_{i_n}^{\frac{1}{2}} \|. \quad (33)$$

Here we are again assuming, as we may, that $V$ is bounded. The operator norm $\| V_{i_n}^{\frac{1}{2}} \|$ (equal to the $L^\infty$ norm) will be annihilated by taking the $n$-th root at the end and letting $n$ tend to infinity. We claim that

$$\| V_{i}^{\frac{1}{2}} R_0 V_{j}^{\frac{1}{2}} \| \leq C_q L_{ij}^{1-\frac{d+1}{2q}} \| V \|_{L^\infty L^q}. \quad (34)$$

To prove this, one again uses the pointwise bound (28). The difference to Lemma 6 is that we now ignore the (second) exponential, but we borrow an epsilon of polynomial
decay to achieve summability in $i, j$. More precisely, consider the standard analytic family $V_i^{\zeta/2} R_0^{\zeta/2} |V_j|^{\zeta/2}$. Then (28) implies that, for $\text{Re} \, \zeta = q$, the kernel is bounded by

$$|V_i(x)^{\zeta/2} R_0^{\zeta/2}(x - y)|V_j(y)|^{\zeta/2}| \leq C_1 e^{C_2|\text{Im} \, \zeta|^2} L_{i,j}^{-\delta} |V_i(x)|^q |V_j(y)|^q,$$

where $\delta = (d + 1)/2 - q > 0$, leading to the Hilbert–Schmidt bound

$$\|V_i^{\zeta/2} R_0^{\zeta/2} |V_j|^{\zeta/2}\| \leq C_3 L_{i,j}^{-\delta} \|V_i\|_q^{\zeta/2} \|V_j\|_q^{\zeta/2}$$

for some constant $C_3$. Interpolating this with the trivial bound $\|V_i^{\zeta/2} R_0^{\zeta/2} |V_j|^{\zeta/2}\| \leq C_1 e^{C_2|\text{Im} \, \zeta|^2}$ for $\text{Re} \, \zeta = 0$ yields (34). Summing (33) first over $i_1$, then over $i_2$, etc., and using (24), we get

$$\|R_0 V\|_n \| \leq \|R_0|V_i|^{1/2} \|C_n^{n-1} \|V\|_p^{n-1} L_{n,q}^\infty \left( \sup_{i \in N} \sum_{j \in N} L_{i,j}^{-\delta/q} \right)^{n-1} N \|V_{i_n}\|,$$

where the factor $N$ comes from the final sum over $i_n$. Taking the $n$-th root and letting $n \to \infty$, we find that

$$\text{spr}(R_0 V) \leq C_3 \|V\|_p^{\infty} L_{\infty,q} \left( \sup_{i \in N} \sum_{j \in N} L_{i,j}^{-\delta/q} \right).$$

where $\text{spr}(\cdot)$ denotes the spectral radius. By the resolvent identity

$$-\Delta + V - z = (-\Delta - z) (I + R_0(z) V)$$

it follows that $z$ is not in the spectrum of $-\Delta + V$ if $\text{spr}(R_0 V) < 1$. Thus, (35) and the sparsity assumption (32) yield the claim. \hfill $\Box$

4. Determinant Bounds

**Assumption 1.** Let $q$ be in the range (15), $p \in [2 \max(q, q_d), \infty)$, $V = V(L)$ strongly separating and $\|V\|_{\ell_p L_q} \lesssim 1$.

The assumption $\|V\|_{\ell_p L_q} \lesssim 1$ is for convenience only and can easily be removed by power counting arguments (since the estimates are scale-invariant).

We will use the regularized Fredholm determinants (see for instance [35, IV.2], [70, Chapter 9] or [24, XI.9.21])

$$f_{\text{diag}}(z) := \det_p(I + BS_{\text{diag}}(z)), \quad f_V(z) := \det_p(I + BS_{V}(z)),$$

where $p \in [2 \max(q, q_d), \infty)$ is assumed to be an integer. The main property that we will use is that the $f_{\text{diag}}, f_V$ are analytic functions in $\mathbb{C} \setminus [0, \infty)$ and have zeros (counted with multiplicity) exactly at the eigenvalues of $H_{\text{diag}}, H_V$, respectively.
4.1. Upper Bounds. We collect some useful estimates that we will repeatedly use (these follow from [70, Theorem 9.2]):

\[ |f(z)| \leq \exp(\mathcal{O}(1)\langle \|BS(z)\|_p \rangle^P), \quad (38) \]

\[ |f_{\text{diag}}(z) - f_V(z)| \leq \|BS_{\text{off}}(z)\|_p \exp(\mathcal{O}(1)\langle \|BS_{\text{diag}}(z)\|_p + \|BS_V(z)\|_p \rangle^P), \quad (39) \]

where \( f = f_{\text{diag}} \) or \( f_V \) and \( BS(z) = BS_{\text{diag}}(z) \) or \( BS_V(z) \). Formulas (38), (39), together with the bounds of Lemma 6 motivates the following definitions,

\[ \psi_p(t) := \exp(\mathcal{O}_p(1)(t)^P), \quad \varphi_p(t) := t \exp(\mathcal{O}_p(1)(t)^P), \quad t \geq 0, \]

and for \( z \in \mathbb{C} \setminus [0, \infty) \),

\[ \Psi_{p,q}(z) := \psi_p(\omega_q(z)\|V\|_{\ell_p\ell_q}), \quad \Phi_{p,q}(z) := \varphi_p(\omega_q(z)\|V\|_{\ell_p\ell_q}), \quad \Phi_{p,q}(L, z) := \varphi_p(s(L, z)^2\omega_q(z)\|V\|_{\ell_p\ell_q}), \quad (40) \]

where we suppressed the dependence on \( V \). We recall that \( \omega_q(z) \) and \( s(L, z) \) were defined in (21) and (22). The constant \( \mathcal{O}_p(1) \) is allowed to vary from one occurrence to another. Thus, for example, the inequality \( \varphi_p(t)\psi_p(t) \lesssim \varphi_p(t) \) holds, but \( \psi_p(t) \lesssim 1 \) need not be true.

4.2. Lower Bounds Away from Zeros. The following lemma can be considered as one of the key technical results. To state it we introduce the notation

\[ M_{p,q}(z) := \frac{\langle z \rangle}{|\text{Im} z|} \left( \frac{|z|}{\langle z \rangle} \right)^{\frac{5p(\frac{2q}{q} - 1) - 4}{2pq}} (\omega_q(z))^p, \]

\[ M_{p,q}(L, z) := M_{p,q}(z)s(L, \left( \frac{|z|}{\langle z \rangle} \right)^5)^{2p}. \quad (41) \]

Moreover, we set

\[ \delta_H(z) := \min(\frac{1}{2}, d(z, \sigma(H))). \quad (42) \]

In the following \( H \) denotes either \( H_{\text{diag}} \) or \( H_V \). We then write \( \delta_{\text{diag}}(z) := \delta_{H_{\text{diag}}}(z) \) and \( \delta_V(z) := \delta_{H_V}(z) \).

**Lemma 14.** Suppose Assumption 1 holds. Then for all \( z \in \Sigma_0 \),

\[ |f_{\text{diag}}(z)|^{-1} \leq \exp(\mathcal{O}(1)M_{p,q}(z)\log \frac{1}{\delta_{\text{diag}}(z)}), \quad (43) \]

\[ |f_V(z)|^{-1} \leq \exp(\mathcal{O}(1)M_{p,q}(L, z)\log \frac{1}{\delta_V(z)}). \quad (44) \]
Proof. In the following, $f$ denotes either $f_{\text{diag}}$ or $f_{V}$. We are going to apply Lemma 35 with parameters (in the notation of that lemma)

$$
\begin{align*}
    r_1^2 &= c|z|, & r_2^2 &= c^2|z|, & r_3^2 &= c^3|z|, \\
    R_1^2 &= c^{-1}|z|, & R_2^2 &= c^{-2}(z), \\
    2\varphi_1 &= \epsilon|\arg(z)|, & 2\varphi_2 &= \epsilon^2|\arg(z)|, & 2\varphi_3 &= c\epsilon^2|\arg(z)|, \\
    \theta_1 &= \pi - \varphi_1, & \theta_2 &= \pi - \varphi_2, & \theta_3 &= \pi - \varphi_3,
\end{align*}
$$

and with $U_j$, $j = 1, 2, 3$, the wedges defined in (92). Here $c \ll 1$ is a small absolute constant and $\epsilon = \epsilon(z)$ is a small parameter that will be chosen momentarily. Note that $|\arg(z)| \ll 1$ since $z \in \Sigma_0$ and thus $\sin(2\varphi_j) \approx \varphi_j \approx \tan(2\varphi_j)$, which will be used repeatedly in the proof. We will now verify the assumptions of Lemma 35. The first condition in (93) is satisfied by definition. Using that, for $j = 1, 2$,

$$
d(\partial U_j, \partial U_3) = \sin(2\varphi_j - 2\varphi_3)r_j^2 \approx (\epsilon)^j|\text{Im } z|,
$$

we find that the second and third condition in (93) become

$$
c\epsilon \ll (c^2(|z|/\langle z \rangle)^{1/2})^{2\pi/(3-\varphi_3)} + 2, \quad c^3\epsilon|\text{Im } z|/\langle z \rangle (c^2(|z|/\langle z \rangle)^{1/2})^{2\pi/(3-\varphi_3)} \ll 1,
$$

respectively, which means that the third condition is trivially satisfied. Since $c^2(|z|/\langle z \rangle)^{1/2} \ll 1$ and $2\pi/(3-\varphi_3) \leq 3$, the second condition is satisfied if we choose e.g. $\epsilon = c^{10}(|z|/\langle z \rangle)^{5/2}$, which we do. We will now show that, for a suitable choice of $z_2 \in U_2$,

$$
\max_{w \in U_3} \log |f(w)| - \log |f(z_2)| \lesssim \epsilon \left( (s(L, c^2z))^2 \epsilon \exp \left( -2\frac{d_{\Phi}}{d} - \right) \langle \omega_q(z) \rangle \right)^p,
$$

where $p = 0$ if $f = f_{\text{diag}}$ and $p = 1$ if $f = f_{V}$. Since in the present case

$$
\frac{R_2^2}{d(\partial U_2, \partial U_3)} \left( \frac{R_2}{r_2} \right)^{\frac{2\pi}{3-\varphi_3}} \lesssim c^{-28} \frac{\langle z \rangle}{|\text{Im } z|} \left( \frac{\langle z \rangle}{|z|} \right)^8,
$$

we see that (94) holds. Lemma 35 thus implies that (43), (44) hold. To prove (45) we first observe that, by the maximum principle, $|f|$ attains its maximum on the boundary of the wedge $U_3$. We estimate this on the boundary component corresponding to the ray $w\rho^2e^{2i\varphi_3}$, $\rho > r_3$, the estimates for the other two boundary components being similar. By (38) and the bounds of Lemma 6, this maximum is bounded by the right hand side of (45), where we used that $\sup_{\rho > r_3} \omega_q(\rho^2e^{2i\varphi_3}) \lesssim \epsilon \exp \left( -2\frac{d_{\Phi}}{d} - \right) \langle \omega_q(z) \rangle$ and that $L$ is strongly separating. This proves (45) for the first term on the left. The other part follows by selecting, for instance, $z_2 = \frac{1}{4} R_2^2$ and using the estimates (similar to (39), see [70, Theorem 9.2])

$$
|f_{\text{diag}}(z_2) - 1| \lesssim \Phi_{p,q}(z_2), \quad |f_{V}(z_2) - 1| \lesssim \Phi_{p,q}(L, z_2),
$$

where we once again used Lemma 6 and $\omega_q(z_2) \lesssim \omega_q(z)$. Note that in the case of $f_{V}$ we can absorb the factor $(s(L, z_2))^2$ in the definition of $\Phi_{p,q}(L, z_2)$ into the $O(1)$ term since $\text{Im } \sqrt{z_2} \gtrsim 1$ and $L$ is strongly separating. Since $\omega_q(z_2) \ll 1$ for $R_2 \gg 1$ (which is true whenever $c \ll 1$), it follows that $|f(z_2)| \geq 1/2$ for $c$ sufficiently small. This finishes the proof of (45). \qed
4.3. Upper Bound on the Resolvent Away from Eigenvalues. As a consequence of Lemma 14 we also obtain an upper bound for the resolvent of $H_V$ away from the spectrum. The idea is to use the following infinite-dimensional analogue of Cramer’s rule (see [68, (7.10)]),

$$
\|(I + BS(z))^{-1}\| \leq \frac{\exp(O(1)\|BS(z)\|^p)}{|f(z)|^p}.
$$

(46)

**Proposition 15.** Suppose Assumption 1 holds. Then for all $z \in \Sigma_0$,

$$
\|(H_{\text{diag}} - z)^{-1}\| \leq \exp(O(1)M_{p,q}(z) \log \frac{1}{\delta_{\text{diag}}(z)}),
$$

(47)

$$
\|(H_V - z)^{-1}\| \leq \exp(O(1)M_{p,q}(L, z) \log \frac{1}{\delta_V(z)}),
$$

(48)

where $M_{p,q}(z)$, $M_{p,q}(L, z)$ are given by (4.2).

**Proof.** In view of the trivial bound

$$
\|(-\Delta - z)^{-1}\| \leq |\text{Im} z|^{-1},
$$

the claim follows from Lemma 16, the resolvent identity (19), Corollary 7 and the fact that $|\text{Im} z|^{-1} \lesssim M_{p,q}(z)$.

\[\square\]

**Lemma 16.** The operator norms of $(I + BS_{\text{diag}}(z))^{-1}$ and $(I + BS_V(z))^{-1}$ are bounded by the right hand side of (47) and (48), respectively.

**Proof.** We only prove the claim for $BS_{\text{diag}}(z)$; the other part is similar. By (46), (27) and (40), we have $\|(I + BS_{\text{diag}}(z))^{-1}\| \lesssim |f_{\text{diag}}(z)|^{-1}\Psi_{p,q}(z)$. Since $\Psi_{p,q}(z) \lesssim M_{p,q}(z)$, Lemma 14 implies that the latter is bounded by (47).

\[\square\]

5. Comparison Between $H_{\text{diag}}$ and $H_V$

5.1. Gershgorin Type Upper Bounds. We record the following Gershgorin type bound. We temporarily restore the norm of the potential and define

$$
\omega_{q,i}(z) := \omega_q(z)\|V_i\|_{L^q},
$$

and $M_{p,q,i}(z)$ is defined by (4.2) with $\omega_q(z)$ replaced by $\omega_{q,i}(z)$.

**Proposition 17.** Under Assumption 1 (Sect. 4) the discrete spectrum of $\sigma(H_V)$ in $\Sigma_0$ is contained in the set

$$
\bigcup_{i \in [N]} \{z \in \mathbb{C} : \text{Im} \sqrt{z}L_i - \log(s(L, z)) \lesssim -M_{p,q,i}(z) \log \delta_{H_V}(z) + \log \omega_{q,i}(z)\}. 
$$

(49)
Proof. Assume first that \( N < \infty \), and consider the Hilbert spaces \( \mathcal{H}_n = L^2(\Omega_n) \), \( \mathcal{H} = \bigoplus_{n \in [N]} \mathcal{H}_n \), with operators \( A_{ij} = \delta_{ij} I + B S_{ij}(z) \) and \( A = (A_{ij})_{i,j=1}^{N} \). Applying the Gershgorin theorem for bounded block operator matrices due to Salas [65] (see also [79, Theorem 1.13.1]) yields

\[
\sigma(A) \subset \bigcup_{i=1}^{N} \{ \lambda \in \mathbb{C} : \| (A_{ii} - \lambda)^{-1} \|^{-1} \leq \sum_{j \in [N] \setminus \{i\}} \| A_{ij} \| \}.
\]

Note that here we are using the convention that \( \| (A_{ii} - \lambda)^{-1} \| = \infty \) if \( \lambda \in \sigma(A_{ii}) \). By the Birman–Schwinger principle, this implies that

\[
\sigma(H_V) \subset \bigcup_{i=1}^{N} \{ z \in \mathbb{C} : \| (I + B S_{ii}(z))^{-1} \|^{-1} \leq \sum_{j \in [N] \setminus \{i\}} \| B S_{ij}(z) \| \}.
\]

Again, we include the spectrum of \( A_{ii} \) in the set on the right. By Lemma 16, we have

\[
\| (I + B S_{ii}(z))^{-1} \| \leq \exp(\mathcal{O}(1) \mu_{p,q,i}(z) \log \frac{1}{\delta_{H_V}(z)}).
\]

On the other hand, by (24) and the strong separation property,

\[
\sum_{j \in [N] \setminus \{i\}} \| B S_{ij}(z) \| \lesssim s(L, z) \exp(-\frac{1}{\mu_{p,q,i}} \text{Im} \sqrt{z} L_i) \omega_q(z).
\]

The claim for \( N < \infty \) follows. Similarly, it follows for the truncated operators \( H^{(n)} \). Since the set (49) is independent of \( n \) the claim for the case \( N = \infty \) then follows from the norm resolvent convergence of the truncated operators (Proposition 8).

5.2. Lower Bounds: Qualitative Results. In the following we establish criteria on the sequence \( (L_j)_j \) that guarantee proximity of \( \sigma_d(H_V) \) to \( \sigma_d(H_{\text{diag}}) \) in various regions of the spectral plane. Let us first discuss some standard facts from perturbation theory ([42, Chapter IV]). It is well known that the spectrum of a closed operator \( H \) is upper semicontinuous under perturbations, and the same is true for each separated part of the spectrum [42, Theorem 3.16]. Moreover, a finite system of eigenvalues \( \{\xi_1, \ldots, \xi_n\} \) changes continuously, just as in the finite-dimensional case. This follows from the fact that the Riesz projection

\[
\frac{1}{2\pi i} \int_{\Gamma} (\zeta - H)^{-1} d\zeta
\]

is continuous in \( H \) in the uniform topology (in the sense of generalized convergence of operators). Here \( \Gamma \) is a closed contour (a piecewise smooth curve) in the resolvent set of \( H \) and encircling the eigenvalues \( \xi_1, \ldots, \xi_n \) (and no other point of the spectrum) once in the counterclockwise direction. Then

\[
\frac{1}{2\pi i} \text{Tr} \int_{\Gamma} (\zeta - H)^{-1} d\zeta = n.
\]

Here we consider a finite system of eigenvalues of \( H_{\text{diag}} \) in some compact subset \( \Sigma \subset \mathbb{C} \setminus [0, \infty) \). By Corollary 11, Assumption 1 (Sect. 4) implies that each point in \( \Sigma \) is either
in the resolvent set or a discrete eigenvalue of $H_{\text{diag}}$. By compactness, $\Sigma \cap \sigma(H_{\text{diag}})$ is a finite set. We then have
\[ \delta_0(\Sigma) := \min\{d(\zeta, \sigma(H_{\text{diag}}) \setminus \{\zeta\}) : \zeta \in \Sigma \cap \sigma(H_{\text{diag}})\} > 0. \] (50)

For $\delta \in (0, \frac{1}{3} \min(1, \delta_0(\Sigma)))$ we set
\[ U_\delta := \bigcup_{\zeta \in \Sigma \cap \sigma(H_{\text{diag}})} D(\zeta, \delta), \quad \Gamma_\delta := \partial U_\delta. \] (51)

In general it is hard to determine $\delta_0(\Sigma)$, but we still have $\Gamma_\delta \subset \rho(H_{\text{diag}})$ for generic $\delta$. This is all that is needed for a lower bound on the number of eigenvalues in $U_\delta$. The norm resolvent convergence (Proposition 9) implies the following proposition. In Sect. 5.4 we will give an alternative proof using the argument principle.

Let us first state our assumptions for the remainder of this section.

**Assumption 2.** Let $\Sigma \subset \Sigma_0 \cap \mathbb{C} \setminus [0, \infty)$ be a compact subset, let $\delta_0(\Sigma)$, $U_\delta, \Gamma_\delta$ be defined by (50), (51) and let $\delta \in (0, \delta_0(\Sigma))$.

**Proposition 18.** Suppose Assumptions 1 (Sect. 4), 2 hold. Then for any $\delta \in (0, \delta_0(\Sigma))$ there exists a constant $C = C(\delta, \Sigma)$ such that, if $s(L, \zeta) \leq 1/C$, then
\[ N(H_V; U_\delta) = N(H_{\text{diag}}; U_\delta). \]

5.3. **Argument Principle.** The argument in the previous subsection involved compactness and continuity and is obviously non-quantitative. The issue is of course the need for a quantitative estimate of the resolvent on the curve $\Gamma_\delta$. We will prove such estimates in Proposition 15. Here we argue in a slightly different (albeit closely related) manner. By the generalized argument principle (see e.g. [29, Theorem 3.2] or [3, Theorem 6.7]),
\[ N(H; U_\delta) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{d}{d\zeta} \log f(\zeta) d\zeta, \]
where $H = H_{\text{diag}}$ or $H_V$ and $f = f_{\text{diag}}$ or $f_V$ (these functions were defined in (36), (37)). This suggests a comparison between $H_{\text{diag}}$ and $H_V$ via Rouché’s theorem (see e.g. [67] for related ideas). We set
\[ r_\delta := \sup_{z \in \Gamma_\delta} \frac{|f_{\text{diag}}(z) - f_V(z)|}{|f_{\text{diag}}(z)|}. \] (52)

We will show that $r_\delta < 1$ if $\max_{j \in [n]} s(L, \xi_j)$ is sufficiently small. Rouché’s theorem then asserts that $f_{\text{diag}}$ and $f$ have the same number of zeros in $U_\delta$.

5.4. **Alternative Proof of Proposition 18.** Without loss of generality we may assume that $\Sigma$ contains exactly one eigenvalue $\zeta$ of $H_{\text{diag}}$. We are going to use Lemma 33. For this purpose we set $U_1 = U_\delta$ and let $U_2 \subset \mathbb{C} \setminus [0, \infty)$ be a precompact simply connected neighborhood of $U_1$ containing a point $\zeta_2 \notin \sigma(H_{\text{diag}})$. This is possible by (3) applied to $V_j$ (i.e. with $N = 1$) since we can take $\zeta_2 = -A$, where $A \gg 1$. By (39) we find that
\[ \sup_{z \in \Gamma_\delta} |f_{\text{diag}}(z) - f_V(z)| \leq C_1 s(L, \zeta)^2, \] (53)
where $C_1 = C_1(\delta, \Sigma)$. We take $A$ so large that
\[
\Phi_{p,q}(z_0) \leq \frac{1}{2}.
\]
This is possible since $\lim_{A \to \infty} \omega_q(-A) = 0$ by (25). By Lemma 33 there exists a constant $C_2 = C_2(\delta, \Sigma)$ such that
\[
\log |f_{\text{diag}}(z)| \geq -C_2 \quad \text{for all} \quad z \in \Gamma_\delta.
\]
Here we used that $\log |f_{\text{diag}}(z_0)| \geq -\log 2$, which follows from (54), (25) and (38). Combining (53) and (55), we infer that $r_\delta < 1$ if $s(L, \zeta)$ is sufficiently small.

5.5. Lower Bounds: Quantitative Results.

In the following we establish quantitative versions of Proposition 18.

We return to estimating the quantity $r_\delta$ in (52) featuring in Rouché’s theorem.

**Lemma 19.** Suppose Assumptions 1 (Sect. 4), 2 (Sect. 5.2) hold. Then
\[
r_\delta \leq \max_{\zeta \in \Sigma} s(L, \zeta) \exp \left( O(1) s(L, \zeta)^2 M_{p,q}(\zeta) \log \frac{1}{\delta} \right),
\]
where $\delta_{H_{\text{diag}}}(\zeta)$, $s(L, \zeta)$, $M_{p,q}(\zeta)$ are given by (42), (3.1), (4.2), respectively.

**Proof.** Again we may assume that $\Sigma$ contains exactly one eigenvalue $\zeta$ of $H_{\text{diag}}$, so that $U_\delta = D(\zeta, \delta)$ and $\Gamma_\delta = \partial D(\zeta, \delta)$. We clearly have $\delta(z) = \delta$ and $\omega_q(z) \asymp \omega_q(\zeta)$ for $z \in \Gamma_\delta$. It is easy to see that (39) and Lemma 6 imply
\[
\sup_{z \in \Gamma_\delta} |f_{\text{diag}}(z) - f_V(z)| \lesssim \Phi_{p,q}(L, \zeta) \Psi_{p,q}(\zeta).
\]
We have also used that $L$ is strongly separating, hence $s(L, z) \lesssim s(L, \zeta)$. In order to estimate $r_\delta$ it remains to bound $|f_{\text{diag}}(z)|$ from below using (43). \qed

As an immediate corollary we obtain an improvement of Proposition 18.

**Proposition 20.** Suppose Assumptions 1 (Sect. 4), 2 (Sect. 5.2) hold. Then
\[
N(H_V; U_\delta) = N(H_{\text{diag}}; U_\delta),
\]
provided that $L$ is so large that $r_\delta < 1$ in (56).

6. From Quasimodes to Eigenvalues

6.1. Existence of a Single Eigenvalue. We record a useful corollary of Proposition 15.

**Corollary 21.** Suppose Assumption 1 (Sect. 4) holds. Assume that there is a normalized $\psi \in L^2(\mathbb{R}^d)$ such that
\[
\|(H_V - \zeta)\psi\| \leq \epsilon,
\]
where $\epsilon < 2^{-M_{p,q}(\zeta)}$. Then $\sigma_d(H_V) \cap D(\zeta, \epsilon^{O(1)/M_{p,q}(\zeta)}) \neq \emptyset$.

**Proof.** By assumption and Proposition 15,
\[
\epsilon^{-1} \leq \|(H_V - \zeta)^{-1}\| \leq \exp(O(1) M_{p,q}(\zeta) \log \frac{1}{\delta_V(\zeta)}).
\]
The claim follows since the assumption $\epsilon < 2^{-M_{p,q}(\zeta)}$ implies $\delta_V(\zeta) = \text{dist}(z, \sigma(H_V))$. \qed
6.2. Existence of a Sequence of Eigenvalues. If instead of a single quasi-eigenvalue we consider a sequence \((\zeta_j)_j\) with \(\lim_{j \to \infty} \text{Im} \sqrt{\zeta_j} = 0\), an across-the-board assumption like the one in Corollary 21 is not feasible since the right hand side of (48) at \(z = \zeta_n\) tends to infinity as \(n \to \infty\). One possible solution would be to modify the previous arguments and select \(L\) as a function of the sequence \((\zeta_j)_j\). We will follow a similar, albeit slightly different strategy which we find more intuitive. It is also closer in spirit to the inductive argument in [4], which is based on strong resolvent convergence. Once more, the approach we will outline can be viewed as a quantitative version of that method.

The strategy will be to first construct quasimodes of \(H_V\) in a direct way (Lemma 22) and then use Corollary 21 to obtain existence of eigenvalues. The quasi-eigenvalues and quasimodes will be actual eigenvalues and eigenfunctions of \(H_{\text{diag}}\). We introduce a sequence of scales \(\varepsilon_j\) and \(a_j\), where \(\varepsilon_j\) has dimension of energy and \(a_j\) has dimension of length, and assume that the eigenfunctions \(\psi_j\) corresponding to \(\zeta_j\) decay exponentially away from \(\Omega_j\) in such a way that

\[
\|V \psi_j\| \leq C_q a_j^{-d/\tilde{q}} \exp(-c_0 \text{Im} \sqrt{\zeta_j} d (\Omega_i, \Omega_j)) \|V\|_{L\tilde{q}},
\]

where \(\tilde{q} \geq 2\) and \(c_0 > 0\). In the following applications we can take \(c_0 = 1\). We will then choose \(L\) such that

\[
\text{Im} \sqrt{\zeta_n} L_n \geq C \log \left( n \log^2 \langle n \rangle \sup_{j \in [n]} \varepsilon_j^{-1} a_j^{-d/\tilde{q}} \sup_{i \in [n]} \|V_i\|_{L\tilde{q}} \right),
\]

where \(C = C(d, \tilde{q})\) is a large constant.

**Lemma 22.** Assume that \(V \in \ell^\infty L\tilde{q}\) for some \(\tilde{q} \geq 2\) and that \(\zeta_j\) are eigenvalues of \(H_{V_j}\) with normalized eigenfunctions \(\psi_j\) satisfying (57). Then there exists an absolute constant \(C = C(d, \tilde{q})\) such that if \(V(L)\) is separating and satisfies (58), then \(H_V\) has a sequence of normalized quasimodes \(\tilde{\psi}_j\),

\[
\| (H_V - \zeta_j) \tilde{\psi}_j \| \leq \varepsilon_j.
\]

**Remark 5.** Lemma 22 could be seen as a quantitative version of Lemma 2 in [4].

**Proof.** Let \(\psi_j\) be the eigenfunctions of \(H_{V_j}\) corresponding to \(\zeta_j\), i.e. \((H_{V_j} - \zeta_j) \psi_j = 0\). For \(n \in [N]\) we make the following (stronger) induction hypothesis \(P(n)\):

\[
\| (H^{(n)} - \zeta_j) \psi_j \| \leq \varepsilon_j \left( 1 - \frac{1}{\log \langle n \rangle} \right), \quad j \in [n]
\]

(recall (17) for the definition of \(H^{(n)}\)). The base case \(n = 1\) is true by assumption. Assume now that \(P(n - 1)\) holds. By the exponential decay (57),

\[
\| (H^{(n)} - \zeta_n) \psi_n \| \leq \sum_{j=1}^{n-1} \|V_j \psi_n\| \leq n C_q a_n^{-d/\tilde{q}} \exp(-c_0 \eta_n L_n) \sup_{j \in [n]} \|V_j\|_{L\tilde{q}},
\]

where we have set \(\eta_n := \text{Im} \sqrt{\zeta_n}\). Moreover, by induction hypothesis, for \(j \in [n-1]\),

\[
\| (H^{(n)} - \zeta_j) \psi_j \| \leq \| (H^{(n-1)} - \zeta_j) \psi_j \| + \|V_n \psi_j\|
\leq \varepsilon_j \left( 1 - \frac{1}{\log \langle n - 1 \rangle} \right) + C_q a_j^{-d/\tilde{q}} \exp(-c_0 \eta_j L_n) \|V_n\|_{L\tilde{q}}.
\]
Hence $P(n)$ would hold if $L_n$ satisfied the estimates
\begin{align}
  n C q a_n^{-d/	ilde{q}} \exp\left(-c_0 \eta_n L_n \right) \sup_{j \in [n]} \|V_j\|_{L^{\tilde{q}}} &\leq \varepsilon_n, \tag{60} \\
  C q a_j^{-d/	ilde{q}} \exp\left(-c_0 \eta_j L_n \right) \|V_n\|_{L^{\tilde{q}}} &\leq \varepsilon_j \left(\frac{1}{\log (n - 1)} - \frac{1}{\log (n)}\right), \tag{61}
\end{align}
for $j \in [n]$. By the mean value theorem
\begin{align}
  \left(\frac{1}{\log (n - 1)} - \frac{1}{\log (n)}\right) \gtrsim \frac{1}{n \log^2 n},
\end{align}
and it is easy to check that (60)–(61) are satisfied for the choice (58). This completes the induction step. For $N < \infty$ the claim now follows from (59) with $n = N$. Now consider the case $N = \infty$. Since $L$ is separating,
\begin{align}
  \lim_{n \to \infty} \|(H - H^{(n)})\psi\| &\leq \lim_{n \to \infty} \| (V - V^{(n)})\psi\| = \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \|V_k\psi\|^2 \\
  &\leq a_j^{-d/\tilde{q}} \|V\|_{L^{\tilde{q}}}^2 \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \exp(-2 \eta_j L_k) = 0.
\end{align}
Together with (59) this yields the claim for $N = \infty$. \hfill \Box

**Remark 6.** The factor $n \log^2 n$ in (58) comes from the induction hypothesis and should not be taken too seriously. One could of course replace $\log (n)$ by any other slowly varying sequence tending to infinity. However, this would not change the bound (58) significantly.

### 6.3. Quasimode Construction

We now construct the potential $W_j$ having $\zeta_j$ as an eigenvalue.

**Lemma 23.** Given $\zeta \in \Sigma_0$ and $x_0 \in \mathbb{R}^d$ there exists a potential $W = W(\zeta, x_0) \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ such that the following hold.

1. $H_W$ has eigenvalue $\zeta$;
2. $\text{supp } W \subset B(x_0, R)$, where
\begin{align}
  R = R(\zeta) \asymp |\zeta|^{1/2} |\text{Im } \zeta|^{-1} |\log |\text{Im } \zeta||/|\zeta|., \tag{62}
\end{align}
3. For any $1 \leq q \leq \infty$
\begin{align}
  \|W\|_{L^q(\mathbb{R}^d)} \asymp |\zeta|^{d/2} |\text{Im } \zeta|^{1 - d/\tilde{q}} |\log^2 |\text{Im } \zeta||/|\zeta|., \tag{63}
\end{align}
4. For $1 \leq q \leq q_d$
\begin{align}
  F_{W,q}(\text{Im } \sqrt{\zeta}) \preceq |\zeta|^{d/2} |\text{Im } \zeta|^{1 - d/\tilde{q}} \tag{64}
\end{align}
5. The normalized eigenfunction $\psi = \psi(\zeta, x_0)$ of $H_W$ corresponding to $\zeta$ satisfies the exponential decay estimate
\begin{align}
  |\psi(x)| \leq C |\zeta|^{1/4} |x|^{-d/2} \exp(-\text{Im } \sqrt{\zeta} \ d(x, \text{supp } W)). \tag{65}
\end{align}
Proof. By scaling it suffices to prove this for $|\xi| \asymp 1$. In view of the results of Sect. 7 (and $\xi = E$ in the notation of that section) we can then simply choose $W$ as a shifted step potential. The shift of course does not affect the eigenvalues nor the $L^q$ norms. The latter are trivial to compute using the size bound $|W| = O(\epsilon)$ and the formula (62) for the width of the step. The estimate (64) follows from a direct computation. The exponential decay follows from Lemma 24 or the explicit form of the wavefunction for the step potential.

Remark 7. Similar results involving complex step potentials are contained in [4,15,16], albeit in a less quantitative form. A technical detail that distinguishes our proof from these is that we first pick the eigenvalue, then find the potential. This avoids the use of Rouché’s theorem in [15,16].

6.4. Exponential Decay. We prove that the exponential decay bound (57) holds for a class of compactly supported potentials that will be relevant in the next section. The important point here is that the constant $C$ in (66) is independent of $W$.

Lemma 24. Assume that $\text{supp} W \subset B(0, R)$ and $\xi \in [0, \infty)$, $\text{Im} \sqrt{\xi} \leq \frac{1}{2} R^{-1} \log R$, $|\xi|^{1/2} \geq K R^{-1}$ for a large absolute constant $K$. Assume that $\psi$ is a normalized eigenfunction of $H_W$ with eigenvalue $\xi$. Then there exists an absolute constant $C = C(d)$ such that for $|x| > R$,

$$|\psi(x)| \leq C|\xi|^{1/4}|x|^{-d-1/2} \exp(-\text{Im} \sqrt{\xi}|x|).$$

(66)

Proof. Since $\psi$ is normalized in $L^2$ it has units $l^{-d/2}$. By homogeneity, we may thus assume that $|\xi| = 1$. Since $\psi$ solves the Helmholtz equation

$$-\Delta \psi(x) = \kappa^2 \psi(x)$$

for $|x| > R$ and $\kappa^2 = \xi$, we have (see e.g. [80, Chapter 1, Sect. 2])

$$\psi(x) = A|x|^{-\nu} H^{(1)}_{\nu}(\kappa |x|)$$

in this region, where $H^{(1)}_{\nu}$ is the Hankel function, $\nu = (d - 2)/2$ and $A = A(d, W)$ is a normalization constant. By the well-known asymptotics of the Hankel function at infinity,

$$\psi(x) = Ac_d|x|^{-d-1/2} \exp(-\text{Im} \kappa |x|)(1 + O(|x|^{-1})).$$

This would imply (66) if we could show that $A$ has an upper bound independent of $W$. Since $\psi$ is normalized,

$$A^2 c_d^2 \int_{|x| > R} |x|^{-(d-1)} \exp(-2\text{Im} \kappa |x|)(1 + O(|x|^{-1})) dx \leq \|\psi\|^2 = 1.$$

For sufficiently large $K$ we estimate the integral from below by

$$(1 - O(K^{-1})) \int_{R}^{2R} \exp(-2\text{Im} \kappa r) dr \geq (1 - O(K^{-1})) R \exp(-\log R) \geq \frac{1}{4},$$

which proves that $A \leq 2/c_d$. \qed
Corollary 25. Assume that $V_j(x) = W_j(x - x_j)$ and that the assumptions of Lemma 24 are satisfied for $W_j$, $\xi$, $\psi_j$, $R_j$. Then (57) holds for any $\tilde{q} \geq 2$ and with

$$a^{-d/\tilde{q}} \lesssim |\xi_j|^{1/4} \left( \frac{L_j}{\text{Im} \sqrt{\xi_j}} \right)^{-\frac{d-1}{2}} \left( \frac{1}{\tilde{q}} \right). \ (67)$$

Proof. Let $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. By Hölder,

$$\|V_j\psi_j\| \leq \|V_j\|_{L^\tilde{q}} \|\psi_j\|_{L^r(\Omega_i)}$$

and by (66),

$$\|\psi_j\|_{L^r(\Omega_i)} \lesssim |\xi_j|^{1/4} \left( \int_{\Omega_i} |x - x_j|^{-\frac{d}{2}} \exp(-\text{Im} \sqrt{\xi_j} |x - x_j|)^r \, dx \right)^{1/r}.$$ 

Since $|x - x_j| \geq d(x, \Omega_j)$,

$$\|\psi_j\|_{L^r(\Omega_i)} \lesssim |\xi_j|^{1/4} \left( \frac{L_j}{\text{Im} \sqrt{\xi_j}} \right)^{-\frac{d-1}{2r}}.$$ 

The claim follows. $\square$

6.5. A Quantitative Version of Bögli’s Example. In view of Corollary 21, given $\xi_j \in \sigma_{H_{V_j}}$ and $\delta_j > 0$ we would like to choose $\varepsilon_j = \varepsilon_j(\xi_j, \delta_j, L)$ as

$$\varepsilon_j^{-1} = \exp(O(1)) M_{p,q}(L, \xi_j) \log \frac{1}{\delta_j}, \ (68)$$

and require that (58) holds with $a_j$ as in (67). This gives a sufficient condition on the sequence $L$ ensuring that $d(\xi_j, \sigma(H_V)) \leq \delta_j$. The following proposition follows immediately from Corollary 21, Lemma 22 and Corollary 25.

Proposition 26. Suppose Assumption 1 (Sect. 4) holds, $V_j \in \ell^\infty L^\tilde{q}$ for some $\tilde{q} \geq 2$ and that supp $V_j(\cdot + x_j) \subset B(0, R_j)$ for some positive $R_j$. Let $\xi_j, \delta_j$ be sequences satisfying $\text{Im} \sqrt{\xi_j} \leq \frac{1}{2} R_j^{-1} \log R_j, |\xi_j|^{1/2} \geq K R_j^{-1}$ for some large absolute constant $K$ and $\delta_j \in (0, 1/2)$. Assume that $L$ satisfies (58) with $a_j$ as in (67) and $\varepsilon_j$ as in (68). If $\xi_j \in \Sigma_0$ is an eigenvalue of $H_{V_j}$ of multiplicity $m_j$, then $D(\xi_j, \delta_j)$ contains at least $m_j$ eigenvalues of $H_V$, counted with multiplicity.

In the following we apply Proposition 26 with $V_j = W(\xi_j, x_j)$, where $W$ is the complex step potential in Lemma 22. Clearly, $V_j \in L^q(\mathbb{R}^d)$ for every $q \in [1, \infty]$, with

$$\|V\|_{L^q} \lesssim \left( \sum_n \left( |\xi_n|^{\frac{d}{q}} |\text{Im} \xi_n|^{1 - \frac{d}{q}} \log \frac{d}{|\text{Im} \xi_n/|\xi_n||} \right)^p \right)^{\frac{1}{p}}, \ (69)$$

$$\sup_{j \in [n]} \|V_j\|_{L^\infty} \lesssim \sup_{j \in [n]} |\text{Im} \xi_n|. \ (70)$$

We will also take $\tilde{q} = \infty$, so that $a_j^{-d/\tilde{q}} = 1$. For the remainder of this section we assume the following.
**Assumption 3.** Let $V_j = W(\xi_j, x_j)$ as above, $q > d$, and assume that (2) holds. We also assume (without loss of generality) that $\Im H_n$ is monotonically decreasing.

**Lemma 27.** Under Assumption 3 the following hold.

(i) $\|V\|_{L^q, \nu} \lesssim \varepsilon_1$, $\|V\|_{L^\infty} \lesssim 1$.

(ii) $|\Im H_n| \lesssim 1$.

(iii) $\langle \zeta \rangle \lesssim |\Im H_n|^{-\frac{2(q-d)}{d}}$.

(iv) $M_{p,q}(\zeta_n) \gtrsim |\Im H_n|^{p\left(\frac{q-d}{aq} + \frac{qd}{\nu n} - 1\right)}$.

(v) $M_{p,q}(\zeta_n) \lesssim |\Im H_n|^{-\frac{2(q-d)}{d} - p\left(\frac{qd}{\nu n} - 1\right)}$.

(vi) $|V(x)| \lesssim |\Im H_n|$ for $|x - x_n| \leq R(\zeta_n)$ and $V(x) = 0$ elsewhere.

**Proof.** Condition (2) states that the right hand side of (69) with $p = q$ is bounded by $\varepsilon_1$. Since $p > q$ and the embedding $\ell^q \subset \ell^p$ is contractive, the first claim in (i) follows. Since $|\Im H_n| \leq |\xi_n|$ and $|\log |\Im H_n/\xi|| \geq 1$ for $\xi_n \in \Sigma_0$, Condition (2) also implies

$$|\Im H_n|^{-\frac{d}{d}} \leq \varepsilon_1, \quad |\zeta_n|^\frac{q}{d} \leq \varepsilon_1 |\Im H_n|^{q-d}. \quad (71)$$

Since $q > d$, the first bound implies (ii) and thus the second claim in (i) follows from (70). The claim (iii) follows from the second bound in (71) and (ii). Using (iii), we find

$$\omega_q(\zeta_n) = |\zeta_n|^{\frac{1}{2}} |\Im H_n|^{\frac{q}{d}} - 1 \gtrsim |\Im H_n|^{(\frac{q-d}{aq}) + \frac{qd}{\nu n} - 1}. \quad (72)$$

This yields (iv) since $M_{p,q}(\zeta_n) \geq \omega_q(\zeta_n)^p$. It also follows from (ii) that

$$\frac{|\zeta_n|}{|\zeta_n|} \gtrsim |\Im H_n|. \quad (73)$$

Combining (iii), (72) and (73) with the trivial lower bound $|\zeta_n| \geq |\Im H_n|$ in (4.2) yields (v). The bound in (vi) follows from (63).

**Remark 8.** From the first equality in (72) and the definition of $M_{p,q}(\xi)$ in (4.2) it is easy to see that for $|\zeta_n| \asymp 1$, we have better bounds

$$|\Im H_n|^{p\left(\frac{qd}{\nu n} - 1\right)} \lesssim M_{p,q}(\zeta_n) \lesssim |\Im H_n|^{p\left(\frac{qd}{\nu n} - 1\right)} - 1.$$

**Lemma 28.** Suppose that $L_k \gtrsim k^\alpha$. Then, under Assumption 3,

$$\langle s(L, \left(\frac{|\zeta_n|}{\zeta_n}\right)^5 \xi_n\rangle \lesssim |\Im H_n|^{-\frac{1}{\alpha} + \frac{q-d}{aq} + \frac{d}{d}}, \quad M_{p,q}(L, \xi_n) \lesssim |\Im H_n|^{-\kappa_{\alpha}}.$$

where $\kappa_{\alpha} := 1 + \frac{2(q-d)}{d} + 5p\left(\frac{qd}{\nu n} - 1\right) + 8 - p\left(\frac{qd}{\nu n} + \frac{qd}{\nu n} - 1\right) + \frac{2p}{\alpha}\left(\frac{q}{2} + \frac{q-d}{d}\right)$.

**Proof.** Combining (73) with the estimate

$$\Im \sqrt{\zeta_n} \asymp \frac{|\Im H_n|}{|\zeta_n|^{1/2}} \lesssim |\Im H_n|^{1+\frac{q-d}{aq}}, \quad (74)$$

where the first bound holds since $\zeta_n \in \Sigma_0$ and the second bound follows from Lemma 27 (iii), we obtain

$$\langle s(L, \left(\frac{|\zeta_n|}{\zeta_n}\right)^5 \xi_n\rangle \lesssim \text{sep}(L, |\Im H_n|^{1+\frac{q-d}{aq}}).$$

The claim thus follows from Proposition 36 and Example a) following it. \qed
Remark 9. For $|\zeta_n| \approx 1$, we again have better bounds

$$
\langle s(L, \left(\frac{|\zeta_n|}{\langle \zeta_n \rangle}\right)^5 \zeta_n) \rangle \lesssim |\text{Im } \zeta_n|^{-\frac{1}{\alpha}},
$$

$$
M_{p,q}(L, \zeta_n) \lesssim |\text{Im } \zeta_n|^{\rho \left(\frac{q-d}{q} - 1\right) - \frac{2q}{\alpha}}.
$$

We assume now that

$$
\delta_n \geq \exp\left(-|\text{Im } \zeta_n|^{-\gamma}\right)
$$

for some $\gamma > 0$. This lower bound is motivated from the corresponding upper bound that results from the Gershgorin estimate (49) and a posteriori by (77).

Lemma 29. Fix a compact set $\Sigma \subset \Sigma_0 \cap \mathbb{C} \setminus [0, \infty)$. The there exists $c = c(\Sigma)$ such that

$$
\sigma(H_V) \cap \Sigma \subset \{z : \delta_{HV_n}(z) \leq \exp(-cL_n)\}.
$$

Proof. This follows immediately from (49) since all involved quantities depending on $z$ (other than $\delta_{HV_n}(z)$) are bounded from above and below on compact subsets.  

The following lemma is obvious.

Lemma 30. If $\epsilon_n$ is defined by (68), $L_k \gtrsim k^\alpha$ and $\delta_n$ satisfies (75), then under Assumption 3,

$$
\log \epsilon_n^{-1} \lesssim |\text{Im } \zeta_n|^{-\kappa - \gamma}.
$$

Lemma (27) (i) and Lemma 30 imply that the right hand side of (58) (with $\tilde{q} = \infty$) is bounded by $|\text{Im } \zeta_n|^{-\kappa - \gamma} \log(n)$. Since the series (2) is absolutely convergent, we may assume without loss of generality (by possibly reordering the series) that $|\text{Im } \zeta_n|$ is monotonically decreasing. Under this condition, we will show that (see Lemma 31 below)

$$
n \lesssim |\text{Im } \zeta_n|^{-\frac{d}{2} - q + 1},
$$

which will then give a sufficient condition for the choice of $L$ in Proposition 26, namely

$$
L_n \geq C|\text{Im } \zeta_n|^{-\kappa - \gamma - 1 - \frac{(q-d)}{d} \log(|\text{Im } \zeta_n|^{-1})}.
$$

Here we have used (74) to estimate $\text{Im } \sqrt{\zeta_n}$ from below. In order to be consistent with our assumption $L_k \gtrsim k^\alpha$ we actually choose

$$
L_n = C|\text{Im } \zeta_n|^{-\tilde{\kappa}},
$$

where, in view of (76), it suffices to take

$$
\tilde{\kappa} := \max(\kappa_\alpha + \gamma + \frac{1}{2} - \frac{(q-d)}{d}, \alpha(\frac{d}{2} + q - 1) + 1).
$$

The exact choice of $\alpha$ is not important for us and we choose $\alpha = 1$ for convenience. We recall that $\kappa_\alpha$ is given in Lemma 28.

The estimate (76) follows from the following simple lemma.
Lemma 31. Let \((b_n)_n \in \ell^1\) be a nonnegative monotonically decreasing sequence. Then 
\[ b_n = O(n^{-1}). \]

Proof. Without loss of generality \(\sum_{n \in \mathbb{N}} b_n = 1\). Given \(t > 0\), let \(n(t) \in \mathbb{N}\) be such that 
\[ b_{n(t)} \leq t \leq b_{n(t)-1}. \] 
Then 
\[ \#\{n \in \mathbb{N} : b_n \geq t\} \geq n(t) - 1. \]
By Markov’s inequality, 
\[ n(t) - 1 \leq t^{-1} \leq b_{n(t)-1}^{-1}. \]
Since the map \(t \mapsto n(t)\) is surjective, the claim follows. \(\square\)

6.6. Proof of Theorem 1. We now specialize Proposition 26 to the step potential \(V_j = W(\zeta_j, x_j)\) and the explicit choice (77), which will prove Theorem 1. Since we already know that the exponential decay bound is true for these potentials [see (65)] we do not need to check the conditions \(\text{Im} \sqrt{\zeta_j} \leq \frac{1}{2} R_j^{-1} \log R_j, |\zeta_j|^{1/2} \geq K R_j^{-1}\), but it is easy to see from (62) that they do hold.

Proposition 32. Suppose Assumption 3 (Sect. 6.5) holds, \(\delta_n > 0\) satisfies (75) for some \(\gamma > 0\), and let \(V = V(L)\) be the potential whose bumps \(V_n = W(\zeta_n, x_n)\) are separated by \(L_n\) in (77). Then \(\|V\|_{L^q} \lesssim \epsilon_1\) and \(D(\zeta_n, \delta_n)\) contains an eigenvalues of \(H_V\). Moreover, after possibly passing to a density one subsequence of \((V_n)_n\), the resulting potential \(\tilde{V}\) decays polynomially,
\[ |\tilde{V}(x)| \lesssim \langle x \rangle^{-\frac{1}{\tilde{\kappa}}}, \]
where \(\tilde{\kappa}\) is given by (78) for \(\alpha = 1\).

Proof. In view of Proposition 26 and Lemma 27 it suffices to prove the last claim. Since \(\tilde{\kappa} \geq 2\), a comparison between \(L_n\) and \(|\Omega_n| = R(\zeta_n)\) in (62) shows that \(V\) is sparse. Therefore, by (16), we have \(L_n \gtrsim |x_n|\). Hence, passing to a density one subsequence, (77) yields,
\[ |\tilde{V}(x_n)| \lesssim |\text{Im} \zeta_n| \lesssim L_n^{-\frac{1}{2}} \lesssim x_n^{-\frac{1}{\tilde{\kappa}}}, \]
from which the decay bound follows. \(\square\)

7. Complex Step Potential

In this section we will establish precise estimates for eigenvalues of the spherically symmetric complex step potential \(V = V_0 \mathbf{1}_{B(0,R)}\), where \(V_0 \in \mathbb{C}\) and \(R > 0\). The bound state problem for \(V_0 < 0\) and \(d = 1, 3\) is treated in virtually any quantum mechanics textbook (see e.g. Problem 25 and Problem 63 in [27]). We adopt the notation
\[ \chi = \sqrt{E}, \quad \kappa = \sqrt{\chi^2 - V_0}. \]
Here \(E \in \mathbb{C}\) is the eigenvalue parameter, i.e. we consider the stationary Schrödinger equation
\[ -\Delta \psi + (V - E) \psi = 0, \]
which becomes \(-\Delta \psi - \kappa^2 \psi = 0\) inside the step and \(-\Delta \psi - \chi^2 \psi = 0\) outside the step.
7.1. One Dimension. We start with one-dimensional case. The solution space to (80) then splits into even and odd functions, while in higher dimensions it splits into functions with definite angular momentum $\ell$. We consider odd functions as these also provide a solution for the case $d = 3$ and $\ell = 0$ ($s$-waves). The standard procedure to solving the square well problem reduces the task to finding zeros of the nonlinear scalar function $F(V_0, \kappa) := i\chi - \kappa \cot(\kappa R)$, where $\chi = \sqrt{\kappa^2 + V_0}$ by (79). A complete study of all the complex poles of this equation was initiated by Nussenzveig [57] for $V_0 \in \mathbb{R} \setminus \{0\}$. Subsequent articles in the physics literature [18,19,36,40] investigated the case of complex potentials. The solution $\kappa = \kappa(V_0)$ is not single-valued as there are branch points where $\partial F/\partial \chi = 0$. The viewpoint endorsed by [36] is to regard the equation $F(V_0, \kappa) = 0$ as the definition of a Riemann surface. This approach treats the complex variables $\kappa$ and $V_0$ on equal footing. In fact, it is easy to see that one can always use $\kappa$ as a coordinate, i.e. one can solve for $V_0$,

$$V_0 = -\kappa^2 \sec^2(\kappa R).$$  

(81)

For the purpose of the construction of the sparse potential in Sect. 6.5 we do not need to solve for $\kappa$. Instead, we pick $\kappa$ first and then define $V_0$ by (81). To get an eigenvalue (i.e. a resonance on the physical sheet) we simply need to take care of the condition $\Im \chi > 0$, i.e.

$$\Re (\kappa R \cot(\kappa R)) > 0.$$  

(82)

We are only interested in complex eigenvalues $E$ with $|E| \asymp 1$ (the general case can be obtained by scaling). We will try to make $V_0$ in (81) small, i.e. we postulate that $V_0 = \varepsilon \tilde{V}_0$, where $\varepsilon > 0$ is a small parameter and $\tilde{V}_0 \in \mathbb{C}$ is of unit size. By (79) this implies that $|\kappa| \asymp 1$, and (81) then reveals that $|\sin^2(\kappa R)| \asymp \varepsilon^{-1}$, which means that

$$e^{2\Im \kappa R} \asymp \varepsilon^{-1}, \quad e^{-2\Im \kappa R} \asymp \varepsilon.$$  

(83)

Since we are free to choose $\kappa$, set $\kappa = \pm 1 + i\varepsilon \sigma$ with $\sigma > 0$, which will yield an eigenvalue with $\Re E = 1 + \mathcal{O}(\varepsilon)$ and $|\Im E| = \mathcal{O}(\varepsilon)$. Going back to (83) we see that we must have

$$R = \frac{1}{2\sigma\varepsilon} \log \frac{C}{\varepsilon}$$  

(84)

for some constant $C$. It is quickly checked that this is consistent with the bound of Abramov et al. [1] since $\|V\|_{L^1} \gtrsim \log \frac{1}{\varepsilon}$ and $|E| \asymp 1$. In view of (83) we may write $e^{2i\kappa R} = \varepsilon u$, where $u = C e^{2i\Re \kappa R}$. Using the Taylor approximation

$$\sec^2(\kappa R) = -4\varepsilon u(1 + 2\varepsilon u + \mathcal{O}(\varepsilon^2))$$  

(85)

we obtain from (81) that

$$V_0 = 4\varepsilon u(\Re \kappa)^2 + \mathcal{O}(\varepsilon^2),$$  

(86)

which provides the desired smallness $|V_0| = \mathcal{O}(\varepsilon)$. As already mentioned, we need to make sure that (82) holds. Using the Taylor approximation

$$\cot(\kappa R) = i(1 + 2\varepsilon u + \mathcal{O}(\varepsilon^2)),$$
we find that (82) holds if
\[-(\text{Re } \kappa)(\text{Im } u) - (\text{Im } \kappa)(\text{Re } u) > 0.\]

(87)

In particular, for \( u \in i\mathbb{R}_+ \), we find that (87) forces us to choose \( \text{Re } \kappa = -1 \). Adopting this choice for \( u \), it is then easy to check that we get an eigenvalue with \( \text{Re } E = 1 + O(\epsilon) \) and \( \text{Im } E = \epsilon(4|u| - 2\sigma) + O(\epsilon^2) \) as desired. By simple scaling arguments this proves the one-dimensional case of Lemma 23. We observe that the result is consistent with the trivial numerical range bound \( \text{Im } E \leq \text{Im } V_0 \); in fact, by choosing \( \sigma \) small, \( E \) can be taken arbitrarily close to the boundary of the numerical range \( \text{Im } z = 4\epsilon|u| \), up to errors of order \( \epsilon^2 \).

Before we conclude the one-dimensional case we note that the same result could have been obtained with an even wavefunction, in which case sec is replaced by csc in (81) and cot is replaced by tan in (82). The Taylor approximations

\[
csc^2(\kappa R) = 4\epsilon u (1 + O(\epsilon)), \quad \tan(\kappa R) = i(1 - 2\epsilon u + O(\epsilon^2)),
\]

and the freedom to choose the signs and the imaginary part of \( u \) yields a proof of Lemma 23 using odd solutions.

7.2. Higher Dimensions. By symmetry reductions we are led to consider the radial Schrödinger equation

\[
(-\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{\ell(\ell + d - 2)}{r^2} + V(r) - E)\psi_\ell(r) = 0.
\]

It can be shown (see e.g. [11, (5.12)]) that an eigenvalue \( E \) corresponds to a zero of the function (Wronskian)

\[
F(V_0, \kappa) := \kappa J'_v(\kappa R)H^{(1)}_v(\chi R) - \chi J_v(\kappa R)H^{(1)'}_v(\chi R),
\]

where \( v = \ell + \frac{d-2}{2} \). We recall that \( \chi, \kappa, E, V_0 \) are related by (79). Computations of resonances for spherically symmetric potentials can be found in [11,56,75,83]. The last three papers use uniform asymptotic expansion of Bessel functions for large order. Here we only consider \( s \)-waves, i.e. \( \ell = 0 \). Then we have the asymptotics

\[
J_v(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - \frac{\pi v}{2} - \frac{\pi}{4}) (1 + O(|z|^{-1})),
\]

\[
H^{(1)}_v(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp(iz - \frac{i\pi v}{2} - \frac{i\pi}{4}) (1 + O(|z|^{-1})),
\]

\[
J'_v(z) = -\left(\frac{2}{\pi z}\right)^{1/2} \sin(z - \frac{\pi v}{2} - \frac{\pi}{4}) (1 + O(|z|^{-1})),
\]

\[
H^{(1)'}_v(z) = i\left(\frac{2}{\pi z}\right)^{1/2} \exp(iz - \frac{i\pi v}{2} - \frac{i\pi}{4}) (1 + O(|z|^{-1})).
\]

With the same choice of \( \kappa \) as in the one-dimensional case and with \( u \in \mathbb{C} \) such that \( e^{2i(\kappa R - \frac{\pi v}{2} - \frac{\pi}{4})} = \epsilon u \), we then obtain that the zeros of \( F(V_0, \kappa) \) coincide with the zeros of a function

\[
\kappa \sin(\omega(\kappa R)) - i\chi \cos(\omega(\kappa R)) + O(R^{-1}),
\]
where $\omega(\kappa R) = \kappa R - \frac{\pi \nu}{\omega} - \frac{\pi}{\omega}$ and the $\kappa$-derivative of the error term is $O(1)$. Recall that $\chi = \chi(V_0, \kappa)$ is given by (79). The zeros of the function without the error term are found exactly as in the one-dimensional case and can be parametrized by $\kappa$, v.i.z.

$$V_0 = -\kappa^2 \csc^2(\omega(\kappa R)).$$

(88)

This follows by dividing the above expression by $\cos(\omega(\kappa R))$ which has no zeros since $\text{Im} \kappa > 0$. Since $|\cos(\omega(\kappa R))^{-1}| = O(\epsilon^{1/2})$ we get from (84) that the error after dividing is $O(\epsilon^{3/2})$, i.e. we are looking for the zeros of a function

$$\tilde{F}(V_0, \kappa) = V_0 + \kappa^2 \csc^2(\omega(\kappa R)) + O(\epsilon^{3/2}),$$

where the derivative of the error is $O(\epsilon^{1/2})$. The implicit function theorem thus yields $\partial \tilde{F}(V_0, \kappa)/\partial V_0 = 1 + O(\epsilon^{1/2})$, which means that we can solve $\tilde{F}(V_0, \kappa)$ for $V_0$, and the solution satisfies (88) up to errors $O(\epsilon^{3/2})$. Hence we obtain that $|V_0| = O(\epsilon)$ as before.

7.3. Proof of Theorem 4. We return to one dimension. We first prove the upper bound (11). Since $\sqrt{|V_0|} R_0$ is of order one, the bound in [30] yields that the total number of eigenvalues of $HV_j$ is also of order one. By Proposition 18 (it is clear that the assumption on the norm of $V$ can be dropped), given $N \gg 1$, we can find $L = L(N)$ such that $H_s$ has the same number of eigenvalues in $\Sigma = \Sigma(N)$ as $H_{\text{diag}}$, which is just the $N$-fold orthogonal sum of the $HV_j$ and hence has less than $O(N)$ eigenvalues by the first part of the argument.

To prove the lower bound (10) we return to the formulas (81), (82), but we now fix $V_0 = i$. We also set $R = N R_0$ and $R_0 \gg 1$, so that the dimensionless parameter $\sqrt{|V_0|} R$ is of size $N$. We first solve an approximate equation and then use Rouché’s theorem to show that the exact equation (81) has solutions close to the approximate ones. Finally, we use (82) to check that we have found a pole on the physical plane (i.e. an eigenvalue).

The approximate equation is $G_1(\kappa) = 0$, where

$$G_1(\kappa) := V_0 - 4\kappa^2 e^{2i\kappa R},$$

and the approximation will be valid in the regime $\text{Im} \kappa R \gg 1$. Since $G_1$ can be factorized,

$$G_1(\kappa) = (\sqrt{V_0} - 2\kappa e^{i\kappa R})(\sqrt{V_0} + 2\kappa e^{i\kappa R}),$$

we only look for zeros of the first factor. These zeros $\kappa_n$ are expressed by means of the Lambert $W$ function,

$$\kappa_n R = -i W_n(i\sqrt{V_0} R/2),$$

where $n \in \mathbb{Z}$ and $W_n$ are the branches of the Lambert $W$ function. According to [12, (4.19)] the asymptotic expansion of $W_n(z)$ as $|z| \to \infty$ is

$$W_n(z) = \log(z) + 2\pi i n - \log(\log(z) + 2\pi i n) + O\left(\frac{\log(\log(z) + 2\pi i n)}{\log(z) + 2\pi i n}\right),$$

where log is the principal branch of the logarithm on the slit plane with the negative real axis as branch cut. For $z = i\sqrt{V_0} R/2$ this gives

$$\kappa_n R = 2\pi n - i \log(i\sqrt{V_0} R/2) + i \log(\log(i\sqrt{V_0} R/2) + 2\pi i n) + E_n(V_0, R),$$

where $E_n(V_0, R)$ is the error term.
where, for $N \gg 1$, the error satisfies $|E_n(V_0, R)| \lesssim \log(\log N + |n|)/\log N + |n|$, where we recalled that $\sqrt{|V_0|}R \asymp N$. For the assumption $\text{Im}\kappa R \gg 1$ made before to be consistent with the formula for $\kappa_n$ we require

\[
\text{Re} \log \left( \frac{\log(i\sqrt{|V_0|}R/2 + 2\pi in)}{i\sqrt{|V_0|}R/2} \right) \gg 1 \iff \left| \log(i\sqrt{|V_0|}R/2 + 2\pi in) \right| \gg 1.
\]

Since $N \gg 1$ we can neglect the logarithm in the second expression and deduce the condition $|n| \gg N$, which we will assume henceforth. This gives us the error bound $|E_n(V_0, R)| \lesssim \log |n|/|n|$, which implies that

\[
\text{Im}\kappa_n R \gtrsim \log \frac{|n|}{N},
\]

in agreement with the assumption $\text{Im}\kappa R \gg 1$. We also obtain the more precise formulas

\[
\text{Re}\kappa_n R = 2\pi n + \frac{5\pi}{4} + \mathcal{O}(\log |n|/|n|), \quad \text{Im}\kappa_n R = \log \frac{|n|}{N} + \mathcal{O}(1), \quad (89)
\]

where we assumed that $n < 0$. To justify this assumption, we recall from the discussion at the end of Sect. 7.1 that (86) and (87), together with (89) and the assumption $V_0 = i$ made at the beginning of this subsection, imply that $n$ must be negative.

Having found the large zeros of $G_1(\kappa)$ we proceed to find those of

\[
G_2(\kappa) := V_0 + \kappa^2 \sec^2(\kappa R),
\]

which determines the eigenvalues of the step potential (see the beginning of Sect. 7.1). We define $\epsilon_n := \exp(-2\text{Im}\kappa_n R)$, so that $e^{2i\kappa_n R} = \epsilon_n u_n$ for some $u_n$ on the unit circle. Note that, by (89), $\epsilon_n = \mathcal{O}(1)((N/|n|)^2$. Using (85) with $\epsilon = \epsilon_n, u = u_n$, we estimate, for $\tilde{\epsilon}_n \ll 1$,

\[
\sup_{|\kappa - \kappa_n| = \tilde{\epsilon}_n} |G_2(\kappa) - G_1(\kappa)| = \mathcal{O}(n^2\epsilon_n^2). \quad (90)
\]

Moreover, for $|\kappa - \kappa_n| = \tilde{\epsilon}_n$ we have

\[
|G_1'(\kappa)| \gtrsim |\kappa|^2 \text{Re}^{-2\text{Im}\kappa R} \gtrsim Nn^2\epsilon_n,
\]

and

\[
|G_1''(\kappa)| \lesssim |\kappa|^2 R^2 e^{-2\text{Im}\kappa R} \lesssim N^2n^2\epsilon_n.
\]

Using $G_1(\kappa_n) = 0$ and Taylor expanding, it follows that

\[
|G_1(\kappa)| \gtrsim Nn^2\epsilon_n \tilde{\epsilon}_n + \mathcal{O}(N^2n^2\epsilon_n^2 \tilde{\epsilon}_n^2).
\]

For this to be meaningful we must of course assume $\tilde{\epsilon}_n \ll 1/N$, which we do. Then we have $|G_1(\kappa)| \gtrsim Nn^2\epsilon_n \tilde{\epsilon}_n$. Comparing this with (90) we see that

\[
\sup_{|\kappa - \kappa_n| = \tilde{\epsilon}_n} |G_1(\kappa)|^{-1} |G_2(\kappa) - G_1(\kappa)| < 1,
\]

provided $\tilde{\epsilon}_n \gg \epsilon_n/N$. Adopting the choice $\tilde{\epsilon}_n = C \sqrt{n}/\epsilon_n$, where $C$ is a large constant, we see that there exists a zero $\tilde{\kappa}_n \in D(\kappa_n, C \sqrt{n}/\epsilon_n)$ of $G_2$. By the smallness of $N/n^2$, it follows that $\tilde{\kappa}_n$ also satisfies (89). We drop the tilde, i.e. we now denote the zeros of $G_2$ by $\kappa_n$. Summarizing what we have done so far, we have found infinitely many resonances $\kappa_n$, but...
$|n| \gg N$, of the step potential satisfying (89). The last step is to check which of the resonances lie on the physical sheet, i.e. are actual eigenvalues. For this we need to check condition (82). By (89),

$$\cot(\kappa_1 n R) = -i \left( 1 + i \left( \frac{N}{|n|} \right)^2 e^{O(1)} + i O(\log |n|/|n|) + O\left( \left( \frac{N}{|n|} \right)^4 \right) \right),$$

and

$$\Re (\kappa_1 n R \cot(\kappa_1 n R)) = 2\pi n N^2 e^{O(1)} \left( 1 + O(\log^2 |n|/|n|) \right) + \log \frac{|n|}{N} \left( 1 + O(\log |n|/|n|) \right).$$

Hence, recalling that $n < 0$, the condition (82) is nonvoid and is satisfied whenever $|n| \log |n|/N \ll N^2$; this holds for $|n| \ll N^2/\log N$. Recalling (79) we obtain the complex energies $E = E_n,$

$$\Re E_n \asymp \frac{n^2}{N^2}, \quad \Im E_n \asymp \frac{|n|}{N^2} \log \frac{|n|}{N},$$

and those energies with $c_{N^2 \log N} \leq |n| \leq C_{N^2 \log N}$ lie in the rectangle $\Sigma$ (see Theorem 4). This completes the proof of the lower bound (10).

### 8. Technical Tools

**8.1. Lower Bounds on Moduli of Holomorphic Functions.** We collect some well known results about the modulus of holomorphic functions away from zeros, based on Cartan’s bound for polynomials (see e.g. [51]).

Let $U_1 \Subset U_2 \Subset \mathbb{C}$, where $U_2$ is simply connected. Assume that $f$ is holomorphic in a neighborhood of $U_2$ and $\zeta_2 \in U_2$. Let $z_1, z_2, \ldots, z_n$, be the zeros of $f$ in $U_2$. Define

$$Z_{f, \delta, U_2} := \bigcup_{j=1}^{n} D(z_j, \delta).$$

The following version can be found in [25, Appendix D].

**Lemma 33.** There exists a constant $C = C(U_1, U_2, \zeta_2)$ such that for any sufficiently small $\delta > 0$,

$$\log |f(z)| \geq -C \log \frac{1}{\delta} \left( \max_{z \in U_2} \log |f(z)| - \log |f(\zeta_2)| \right)$$

for all $z \in U_1 \setminus Z_{f, \delta, U_2}$.

We need also use a more precise version, where $f$ is holomorphic in a neighborhood of $U_3$, where

$$U_j = D(0, r_j), \quad j = 1, 2, 3,$$

with $r_1 < r_2 < r_3$.

**Lemma 34.** Assume (91). Then there exists an absolute constant $C$ such that for any sufficiently small $\delta > 0$,

$$\log |f(z)| \geq -C \log \frac{1}{\delta} \left( (r_2 - r_1)^{-1} + \log^{-1} \left( \frac{r_3}{r_2} \right) \max_{|z|=r_3} |f(z)| - \log |f(\zeta_2)| \right)$$

for all $z \in U_1 \setminus Z_{f, \delta r_2, U_2}$.
Proof. The proof is a straightforward adaptation of [51, Chapter 1, Theorem 11], but we include it for completeness. In the following, \( r_1, r_2, r_3 \) play the roles of \( R, 2R, 2eR \) in [51]. Without loss of generality we may and will assume that \( \zeta_2 = 0 \) and that \( f(0) = 1 \); otherwise we could replace \( f(z) \) by \( \frac{f(z+\zeta_2)}{f(\zeta_2)} \). Consider the function

\[
\varphi(z) := \frac{(-r_2)^n}{z_1 z_2 \ldots z_n} \prod_{k=1}^{n} \frac{r_2(z - z_k)}{r_2 - z_k z}.
\]

We recall that \( z_1, z_2, \ldots, z_n \) are the zeros of \( f \) in \( D(0, r_2) \). Observe that \( \varphi(0) = 1 \) and

\[
\varphi(r_2 e^{i\theta}) = \frac{r_2^n}{|z_1 z_2 \ldots z_n|}
\]

for \( \theta \in \mathbb{R} \). The function

\[
\Psi(z) := \frac{f(z)}{\varphi(z)}
\]

has no zeros in \( D(0, r_2) \) and satisfies \( \psi(0) = 1 \); therefore, by Carathéodory’s theorem [51, Theorem 9], for \( |z| \leq r_1 \),

\[
\log |\psi(z)| \geq -\frac{2r_1}{r_2 - r_1} \left( \max_{|z|=r_2} \log |f(z)| + \log \frac{r_2^n}{|z_1 z_2 \ldots z_n|} \right)
\]

\[
\geq -\frac{2r_1}{r_2 - r_1} \max_{|z|=r_2} \log |f(z)|.
\]

To estimate \( \varphi \) from below for \( |z| \leq r_1 \) we use

\[
\prod_{k=1}^{n} |r_2(z - z_k)| < (2r_2^2)^n,
\]

\[
\prod_{k=1}^{n} |r_2(z - z_k)| > \left( \frac{\delta r_2}{e} \right)^n r_2^n, \quad z \notin Z_{f,2\delta r_2,U_2}.
\]

The second inequality follows from Cartan’s estimate [51, Theorem 10]. We thus obtain the lower bound

\[
|\varphi(z)| > (2r_2^2)^{-n} \left( \frac{\delta r_2}{e} \right)^n \frac{r_2^n}{|z_1 z_2 \ldots z_n|} > \left( \frac{\delta}{2e} \right)^n, \quad z \notin Z_{f,2\delta r_2,U_2}.
\]

By Jensen’s formula [51, Lemma 4], since \( f(0) = 1 \),

\[
n \leq \log^{-1} \left( \frac{r_3}{r_2} \right) \max_{|z|=r_3} \log |f(z)|,
\]

and consequently

\[
\log |\varphi(z)| > \log^{-1} \left( \frac{r_3}{r_2} \right) \max_{|z|=r_3} \log |f(z)| \log \left( \frac{\delta}{2e} \right), \quad z \notin Z_{f,2\delta r_2,U_2}.
\]

Together with the lower bound for \( \log |\psi| \) this leads to the claimed estimate, upon redefining \( \delta \) and absorbing an error into the constant \( C \). \( \square \)
Next we state a version of Lemma 34 for “wedges” of the form

\[ W(\varphi, \theta; r, R) := \{ z \in \mathbb{C} \setminus [0, \infty) : \arg(z) \in (2\varphi, 2\theta), |z| \in (r^2, R^2) \} \]  

(92)

and \( W(\varphi, \theta; R) := W(\varphi, \theta; R, \infty) \), where \( 0 \leq \varphi < \theta \leq \pi \). In the following we fix \( 0 \leq \varphi_3 < \varphi_2 < \varphi_1 < \theta_1 < \theta_2 < \theta_3 \leq \pi \) and \( 0 < r_3 < r_2 < r_1 < R_1 < R_2 \), and define

\[ U_1 := W(\varphi_1, \theta_1; r_1, R_1), \quad U_2 := W(\varphi_2, \theta_2; r_2, R_2), \quad U_3 := W(\varphi_3, \theta_3; r_3). \]

Lemma 35. Assume \( f \) is a bounded holomorphic function on \( U_3 \) and that

\[ r_3 \ll r_2, \quad \frac{d(\partial U_2, \partial U_3)}{d(\partial U_1, \partial U_3)} \ll \left( \frac{r_2}{R_2} \right)^{\frac{2\pi}{\varphi_3}} \cdot \frac{d(\partial U_1, \partial U_3)}{(\theta_3 - \varphi_3)R_2^2} \left( \frac{r_2}{R_2} \right)^{\frac{2\pi}{\varphi_3}} \ll 1. \]  

(93)

Then there exists an absolute constant \( C \) such that for any \( \zeta_2 \in U_2 \) and any sufficiently small \( \delta > 0 \),

\[ \log |f(z)| \geq -C \frac{R_2^2}{d(\partial U_2, \partial U_3)} \left( \frac{R_2}{r_2} \right)^{\frac{2\pi}{\varphi_3}} \log \frac{1}{\delta} \left( \max_{z \in U_3} |f(z)| - \log |f(\zeta_2)| \right) \]  

(94)

for all \( z \in U_1 \setminus Z_{f, \delta, U_2} \).

Remark 10. The constant \( C \) only depends on the implicit constants in (93). Hence, one can optimize the inequality with respect to \( \zeta_2 \), subject to the conditions above.

Proof. We map \( U_3 \) conformally onto the unit disk, using a composition of the following conformal maps (where, by abuse of notation, we denote the variable and the map by the same letter):

(i) \( U_3 \to \kappa(U_3) \subset \mathbb{H}, \kappa(z) := \sqrt{z}; \)

(ii) \( \kappa(U_3) \to S := \{ \sigma \in \mathbb{C} : 0 < \Im \sigma < \pi, \Re \sigma > 0 \}, \)

\[ \sigma(\kappa) := \log(e^{\frac{i\pi\varphi_3}{\varphi_3 - \varphi_3}} (\kappa/r_3)^{\frac{\pi}{\varphi_3 - \varphi_3}}), \]

where we select the principal branch of the logarithm on \( \mathbb{C} \setminus [0, -i \infty) \);

(iii) The Schwarz-Christoffel transformation \( S \to \mathbb{H}, \tau(\sigma) := \cosh(\sigma). \)

(iv) The Möbius transformation \( \mathbb{H} \to D(0, 1), w(\tau) := \frac{\tau - \tau_2}{\tau + \tau_2}, \) where \( \tau_2 := \tau(\sigma(\sqrt{\zeta_2})). \)

The choice of \( \tau_2 \) has been made in such a way that \( w(z) = 0 \) if \( z = \zeta_2 \). Here we again abuse notation and write \( w(z) = w(\tau(\sigma(\sqrt{z}))). \) Note that

\[ \tau(\sigma(\kappa)) = \frac{1}{2}(\alpha(\kappa) + \alpha(\kappa)^{-1}) \quad \alpha(\kappa) := e^{\frac{i\pi\varphi_3}{\varphi_3 - \varphi_3}} (\kappa/r_3)^{\frac{\pi}{\varphi_3 - \varphi_3}}. \]

By distortion bounds [61, Cor. 1.4],

\[ \left| \frac{dw(z)}{dz} \right| d(z, U_3) \leq 1 - |w(z)|^2 \leq 4 \left| \frac{dw(z)}{dz} \right| d(z, U_3). \]  

(95)
We compute the differential of \( w \) at \( z \in U_2 \) by the chain rule,

\[
\frac{dw}{dz} = \frac{dw}{d\tau} \frac{d\tau}{d\sigma} \frac{d\sigma}{d\kappa} \frac{d\kappa}{dz} = \frac{\pi |\tau_2| |\sinh(\sigma)|}{(\theta_3 - \varphi_3)|z|(|\tau + \tau_2|^2)}
\]

\[
= \frac{\pi}{(\theta_3 - \varphi_3)|z|} \frac{|\alpha(\kappa_2) + \alpha(\kappa_2)^{-1}| |\alpha(\kappa) + \alpha(\kappa)^{-1}|}{|\alpha(\kappa_2) + \alpha(\kappa_2)^{-1} + \alpha(\kappa) + \alpha(\kappa)^{-1}|^2}.
\]

Since \(|\kappa| \geq r_2 \gg r_3\), we have \(|\alpha(\kappa)| \gg 1\), whence

\[
\frac{|\alpha(\kappa_2) + \alpha(\kappa_2)^{-1}| |\alpha(\kappa) + \alpha(\kappa)^{-1}|}{|\alpha(\kappa_2) + \alpha(\kappa_2)^{-1} + \alpha(\kappa) + \alpha(\kappa)^{-1}|^2} \gg \frac{|\alpha(\kappa_2)| |\alpha(\kappa)|}{(|\alpha(\kappa_2)| + |\alpha(\kappa)|)^2},
\]

which, in view of \(|\alpha(\kappa)| = (|\kappa|/r_3)^{2\pi/3\nu_3}\), leads to

\[
\frac{1}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3} \lesssim \left| \frac{dw}{dz} \right| \lesssim \frac{1}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3}
\]

for \( z \in U_2 \). Denoting the numbers \( r_j \) in Lemma 34 by \( \rho_j \) instead (with \( \rho_3 = 1 \)), we then find, using (95),

\[1 - \rho_2 \lesssim \frac{d(\partial U_2, \partial U_3)}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3},\]

\[\rho_2 - \rho_1 \lesssim \frac{d(\partial U_1, \partial U_3)}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3} - \frac{d(\partial U_2, \partial U_3)}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3} \gtrsim \frac{d(\partial U_1, \partial U_3)}{(\theta_3 - \varphi_3)^2} \left( \frac{r_2}{R_2} \right)^{2\pi/3\nu_3},\]

where in the second line we used the triangle inequality and the second inequality in (93). By the third inequality in (93) we can Taylor expand

\[\log \left( \frac{1}{\rho_2} \right) = -\log(1 - (1 - \rho_2)) \asymp 1 - \rho_2.\]

Lemma 34 now yields the claim. \( \square \)

8.2. Distribution Function. For \( s > 0 \), we define

\[h_L(s) = |\{k \in [N] : \eta_0 L_k \leq 1/s\}| \in \mathbb{Z}_+,
\]

where \( \eta_0^{-1} \) is an arbitrary length scale. Note that \( h_L \) is decreasing and tends to infinity as \( s \to 0 \). In fact, \( h_L \) is the distribution function of the sequence \( (\eta_0 L_k)^{-1} \). Since we assume that \( L_k \) is increasing, we also have

\[h_L(s) = \min\{k \in \mathbb{Z}_+ : \eta_0 L_{k+1} > 1/s\}.
\]

We will show that, under the assumption

\[\exists \lambda \in (0, 1) \text{ such that } \limsup_{s \to 0^+} \frac{h_L(\lambda s)}{s h_L(s)} < 1, \tag{96}\]

the potential \( V(L) \) is strongly separating in the sense of Definition 2.
Proposition 36. Assume (96). Then

\[ \text{sep}(L, \eta) \lesssim \exp(-\eta L_1)(h_L(\eta/\eta_0)). \]  

(97)

In particular, this implies that the examples in Sect. 2.1 are strongly separating:

a) If \( \eta_0 L_k \gtrsim k^\alpha \) for \( \alpha > 0 \), then \( h_L(s) \lesssim s^{-1/\alpha} \).

b) If \( \eta_0 L_k \gtrsim \exp(k) \), then \( h_L(s) \lesssim \log(1/s) \).

c) If \( \eta_0 L_k \gtrsim \exp(\exp(k)) \), then \( h_L(s) \lesssim \log \log(1/s) \).

Lemma 37. Assume (96). Then for any \( \delta > 0 \) and for all \( s > 0 \),

\[ \langle h_L(\delta s) \rangle \lesssim \delta \langle h_L(s) \rangle. \]

Proof. We may restrict our attention to the case \( \delta < 1 \) as the case \( \delta \geq 1 \) is trivial. By (96) there exist \( \lambda \in (0, 1) \) and \( s_0 > 0 \) such that

\[ h_L(\lambda s) < e^{h_L(s)} \]  

(98)

holds for all \( s \in (0, s_0] \). Now let \( n \) be the smallest integer such that \( \lambda^n \leq \delta \). Iterating (98) \( n \) times, we get

\[ h_L(\delta s) < e^n h_L(s) \]

for all \( s \in (0, s_0] \). For \( s > s_0 \), the inequality holds trivially. \( \Box \)

Proof of Proposition 36. Without loss of generality we may assume that \( \eta = \eta_0 \). We first consider the case \( \eta L_1 \leq 1 \), and hence \( h_L(1) \geq 1 \). Then

\[ \sum_{k=1}^{\infty} \exp(-\eta L_k) \leq h_L(1) \exp(-\eta L_1) + \sum_{k=h_L(1)}^{\infty} \exp(-\eta L_k). \]

It remains to show that the second term is bounded by the right hand side of (97). To this end, we decompose the sum into dyadic intervals \( I_j = [h_L(2^{-j}), h_L(2^{-j-1})] \), \( j \in \mathbb{Z}_+ \). Then

\[ \sum_{k \in I_j} \exp(-\eta L_k) \leq \exp(-2^j h_L(2^{-j-1})). \]

Summing over \( j \) and using Cauchy’s condensation test yields

\[ \sum_{k=h_L(1)}^{\infty} \exp(-\eta L_k) \lesssim \sum_{n=1}^{\infty} \exp(-n) h_L(\frac{1}{2^n}). \]  

(99)

By the quotient test, the series converges provided that

\[ \limsup_{n \to \infty} \frac{h_L(\lambda_n \frac{1}{2^n})}{e h_L(\frac{1}{2^n})} < 1, \]

where \( \lambda_n = \frac{n}{n+1} \). But this follows from assumption (96). Indeed, since \( \lambda_n \to 1 \), we have \( \lambda < \lambda_n \) for large \( n \) and hence \( h_L(\lambda_n \frac{1}{2^n}) \leq h_L(\frac{1}{2^n}) \). The series (99) is thus bounded by \( \langle h_L(1) \rangle \), where we have used Lemma 37 with \( \delta = 1/2 \). This proves (97) in the case \( \eta L_1 \leq 1 \). The case \( \eta L_1 > 1 \) is similar, but (99) is bounded by \( \exp(-n_0) h_L(\frac{1}{2n_0}) \), where \( n_0 \) is the least integer such that \( h_L(\frac{1}{2n_0}) \geq 1 \), i.e. \( n_0 = \lceil \eta L_1 \rceil \). Another application of Lemma 37 completes the proof.
Acknowledgements The author gratefully acknowledges correspondence with Sabine Bögli and comments of Rupert Frank, who pointed out the failure of Weyl’s law and the connection with nonlocality. Many thanks also go to Stéphane Nonnenmacher for useful discussions on resonances and to Alexei Stepanenko for explaining his recent preprint. Special thanks go to Tanya Christiansen for many helpful remarks on a preliminary version of the introduction.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Abramov, A.A., Aslanyan, A., Davies, E.B.: Bounds on complex eigenvalues and resonances. J. Phys. A 34(1), 57–72 (2001)
2. Bandtlow, O.F.: Estimates for norms of resolvents and an application to the perturbation of spectra. Math. Nachr. 267, 3–11 (2004)
3. Behrndt, J., ter Elst, A.F.M., Gesztesy, F.: The Generalized Birman-Schwinger principle. Trans. Amer. Math. Soc. 375(3), 799–845 (2022)
4. Bögli, S.: Schrödinger operator with non-zero accumulation points of complex eigenvalues. Commun. Math. Phys. 352(2), 629–639 (2017)
5. Bögli, S., Cuenin, J.-C.: Counterexample to the Laptev–Safronov Conjecture (2021)
6. Bögli, S., Stampach, F.: On Lieb-Thirring Inequalities for one-dimensional non-self-adjoint Jacobi and Schrödinger operators. J. Spectr. Theory 11(3), 1391–1413 (2021)
7. Borichev, A., Frank, R., Volberg, A.: Counting eigenvalues of Schrödinger operators with fast decaying complex potentials, Adv. Math. 397, Paper No. 108115 (2022)
8. Christiansen, T.: Several complex variables and the distribution of resonances in potential scattering. Commun. Math. Phys. 259(3), 711–728 (2005)
9. Christiansen, T.: Schrödinger operators with complex-valued potentials and no resonances. Duke Math. J. 133(2), 313–323 (2006)
10. Christiansen, T., Hislop, P.D.: The resonance counting function for Schrödinger operators with generic potentials. Math. Res. Lett. 12(5–6), 821–826 (2005)
11. Christiansen, T.J., Hislop, P.D.: Maximal order of growth for the resonance counting functions for generic potentials in even dimensions. Indiana Univ. Math. J. 59(2), 621–660 (2010)
12. Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., Knuth, D.E.: On the Lambert W function. Adv. Comput. Math. 5(4), 329–359 (1996)
13. Cuenin, J.-C.: Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. J. Funct. Anal. 272(7), 2987–3018 (2017)
14. Cuenin, J.-C.: Embedded eigenvalue bounds of generalized Schrödinger operators. J. Spectr. Theory 10(2), 415–437 (2020)
15. Cuenin, J.-C.: Improved eigenvalue bounds for Schrödinger operators with slowly decaying potentials. Commun. Math. Phys. 376(3), 2147–2160 (2020)
16. Cuenin, J.-C., Ibrogimov, O.O.: Sharp spectral bounds for complex perturbations of the indefinite Laplacian. J. Funct. Anal. 280(1), 108804 (2021)
17. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry. Texts and Monographs in PhysicsTexts and Monographs in Physics, study Springer, Berlin (1987)
18. Dabrowski, J.: Poles of the S-matrix for a complex square well potential. J. Phys. G: Nucl. Part. Phys. 23(11), 1539–1550 (1997)
19. Dabrowski, J.: Poles of the s matrix for a complex potential. Phys. Rev. C 53, 2004–2006 (1996)
20. Davies, E.B.: Linear Operators and Their Spectra. Cambridge Studies in Advanced Mathematics, vol. 106. Cambridge University Press, Cambridge (2007)
21. Davies, E.B., Nath, J.: Schrödinger operators with slowly decaying potentials. J. Comput. Appl. Math. 148(1), 1–28 (2002) (On the occasion of the 65th birthday of Professor Michael Eastham)
22. Demuth, M., Hansmann, M., Katriel, G.: Lieb–Thirring type inequalities for Schrödinger operators with a complex-valued potential. Integral Equ. Oper. Theory 75(1), 1–5 (2013)

23. Dencker, N., Sjöstrand, J., Zworski, M.: Pseudospectra of semiclassical (pseudo-) differential operators. Commun. Pure Appl. Math. 57(3), 384–415 (2004)

24. Dunford, N., Schwartz, J. T.: Linear Operators. Part II. Wiley Classics Library. Wiley, New York (1988). Spectral theory, Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication

25. Dyatlov, S., Zworski, M.: Mathematical Theory of Scattering Resonances. Graduate Studies in Mathematics, vol. 200. American Mathematical Society, Providence (2019)

26. Edmunds, D.E., Evans, W.D.: Spectral Theory and Differential Operators. Oxford Mathematical Monographs, Oxford University Press, New York (1987)

27. Flügge, S.: Practical Quantum Mechanics. Classics in Mathematics, English edn. Springer, Berlin (1999) (Translated from the German original)

28. Frank, R.L.: Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc. 43(4), 745–750 (2011)

29. Frank, R.L.: Eigenvalue bounds for Schrödinger operators with complex potentials. III. Trans. Am. Math. Soc. 370(1), 219–240 (2018)

30. Frank, R.L., Laptev, A., Safronov, O.: On the number of eigenvalues of Schrödinger operators with complex potentials. J. Lond. Math. Soc. (2) 94(2), 377–390 (2016)

31. Frank, R.L., Sabin, J.: Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates. Am. J. Math. 139(6), 1649–1691 (2017)

32. Froese, R.: Asymptotic distribution of resonances in one dimension. J. Differ. Equ. 137(2), 251–272 (1997)

33. Gesztesy, F., Latushkin, Y., Mitrea, M., Zinchenko, M.: Nonselfadjoint operators, infinite determinants, and some applications. Russ. J. Math. Phys. 13(4), 443–457 (2005)

34. Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators. V ol. I, volume 49 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel (1990)

35. Gohberg, I.C., Kre˘ in, M.G.: Introduction to the theory of linear selfadjoint operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence (1969)

36. Grama, C., Grama, N., Zamfirescu, I.: Riemann surface approach to bound and resonant states: exotic resonant states for a central rectangular potential. Phys. Rev. A 61, 032716 (2000)

37. Hansmann, M., Krejcirik, D.: The Abstract Birman–Schwinger Principle and Spectral Stability (2020)

38. Hundertmark, D., Kirsch, W.: Spectral theory of sparse potentials. In: Stochastic Processes, Physics and Geometry: New Interplays, I (Leipzig, 1999), volume 28 of CMS Conf. Proc., pp. 213–238. American Mathematical Society, Providence (2000)

39. Jakšić, V., Poulin, P.: Scattering from sparse potentials: a deterministic approach. In: Analysis and Mathematical Physics, Trends Math., pp. 205–210. Birkhäuser, Basel (2009)

40. Kato, T.: Wave operators and similarity for some non-selfadjoint operators. Math. Ann. 162, 258–279 (1965/1966)

41. Kato, T.: Perturbation theory for linear operators. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer, New York (1966)

42. Kiselev, A., Last, Y., Simon, B.: Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. Commun. Math. Phys. 194(1), 1–45 (1998)

43. Klaus, M.: On $−d^2/dx^2 + V$ where $V$ has infinitely many “bumps”. Ann. Inst. H. Poincaré Sect. A (N.S.) 38(1), 7–13 (1983)

44. Korotyaev, E.: Estimates of 1D resonances in terms of potentials. J. Anal. Math. 130, 151–166 (2016)

45. Krutikov, D., Remling, C.: Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure. Commun. Math. Phys. 223(3), 509–532 (2001)

46. Laptev, A., Safronov, O.: Eigenvalue estimates for Schrödinger operators with complex potentials. Commun. Math. Phys. 292(1), 29–54 (2009)

47. Last, Y., Simon, B.: Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators. Invent. Math. 135(2), 329–367 (1999)

48. Lee, Y., Seo, I.: A note on eigenvalue bounds for Schrödinger operators. J. Math. Anal. Appl. 470(1), 340–347 (2019)

49. Levin, B.J.: Distribution of zeros of entire functions, volume 5 of Translations of Mathematical Monographs. American Mathematical Society, Providence, R.I., revised edition (1980). Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman
52. Markus, A.S., Macaev, V.I.: Asymptotic behavior of the spectrum of close-to-normal operators. Funkt. Anal. i Prilozhen. 13(3), 93–94 (1979)
53. Molchanov, S.: Multiscattering on sparse bumps. In: Advances in differential equations and mathematical physics (Atlanta, GA, 1997), volume 217 of Contemp. Math., pp. 157–181. American Mathematical Society, Providence (1998)
54. Molchanov, S., Vainberg, B.: Scattering on the system of the sparse bumps: multidimensional case. Appl. Anal. 71(1–4), 167–185 (1999)
55. Molchanov, S., Vainberg, B.: Spectrum of multidimensional Schrödinger operators with sparse potentials. In: Analytical and Computational Methods in Scattering and Applied Mathematics (Newark, DE, 1998), Volume 417 of Chapman & Hall/CRC Res. Notes Math., pp. 231–254. CRC, Boca Raton (2000)
56. Newton, R.G.: Analytic properties of radial wave functions. J. Mathematical Phys. 1, 319–347; errata, 452 (1960)
57. Nussenzveig, H.: The poles of the s-matrix of a rectangular potential well or barrier. Nuclear Phys. 11, 499–521 (1959)
58. Pavlov, B.S.: On a non-selfadjoint Schrödinger operator. In: Problems of Mathematical Physics, No. 1, Spectral Theory, Diffraction Problems (Russian), pp. 133–157. Izdat. Leningrad. Univ, Leningrad (1966)
59. Pavlov, B.S.: On a non-selfadjoint Schrödinger operator. II. In: Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian), pp. 133–157. Izdat. Leningrad. Univ, Leningrad (1967)
60. Pearson, D.B.: Singular continuous measures in scattering theory. Commun. Math. Phys. 60(1), 13–36 (1978)
61. Pommerenke, C.: Boundary behaviour of conformal maps. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299. Springer, Berlin (1992)
62. Regge, T.: Introduction to complex orbital momenta. Nuovo Cimento 10(14), 951–976 (1959)
63. Remling, C.: A probabilistic approach to one-dimensional Schrödinger operators with sparse potentials. Commun. Math. Phys. 185(2), 313–323 (1997)
64. Rozenblum, G., Solomyak, M.: On the spectral estimates for the Schrödinger operator on $\mathbb{Z}^d$, $d \geq 3$. volume 159, pp. 241–263. (2009) (Problems in mathematical analysis. No. 41)
65. Salas, H.N.: Gershgorin’s theorem for matrices of operators. Linear Algebra Appl. 291(1–3), 15–36 (1999)
66. Shen, Z.: Completeness for sparse potential scattering. J. Math. Phys. 55(1), 012108 (2014)
67. Siedentop, H.K.H.: On a generalization of Rouché’s theorem for trace ideals with applications for resonances of Schrödinger operators. J. Math. Anal. Appl. 140(2), 582–588 (1989)
68. Simon, B.: Notes on infinite determinants of Hilbert space operators. Adv. Math. 24(3), 244–273 (1977)
69. Simon, B.: Resonances in one dimension and Fredholm determinants. J. Funct. Anal. 178(2), 396–420 (2000)
70. Simon, B.: Trace Ideals and Their Applications, Volume 120 of Mathematical Surveys and Monographs, second edn. American Mathematical Society, Providence (2005)
71. Simon, B., Stolz, G.: Operators with singular continuous spectrum. V. Sparse potentials. Proc. Am. Math. Soc. 124(7), 2073–2080 (1996)
72. Sjöstrand, J.: Non-self-adjoint differential operators, spectral asymptotics and random perturbations. Pseudo-Differential Operators, vol. 14. Theory and Applications. Springer, Cham (2019)
73. Sodin, S.: On the number of zeros of functions in analytic quasianalytic classes. Zh. Mat. Fiz. Anal. Geom. 16(1), 55–54 (2020)
74. Stefanov, P.: Quasimodes and resonances: sharp lower bounds. Duke Math. J. 99(1), 75–92 (1999)
75. Stefanov, P.: Sharp upper bounds on the number of resonances near the real axis for trapping systems. Am. J. Math. 125(1), 183–224 (2003)
76. Stepanenko, A.: Unpublished notes
77. Stepanenko, A.: Bounds for Schrödinger Operators on the Half-line Perturbed by Dissipative Barriers (2020)
78. Tang, S.-H., Zworski, M.: From quasimodes to resonances. Math. Res. Lett. 5(3), 261–272 (1998)
79. Tretter, C.: Spectral Theory of Block Operator Matrices and Applications. Imperial College Press, London (2008)
80. Yafaev, D.R.: Mathematical Scattering Theory, Analytic theory. Volume 158 of Mathematical Surveys and Monographs. American Mathematical Society, Providence (2010)
81. Zlatoš, A.: Sparse potentials with fractional Hausdorff dimension. J. Funct. Anal. 207(1), 216–252 (2004)
82. Zworski, M.: Distribution of poles for scattering on the real line. J. Funct. Anal. 73(2), 277–296 (1987)
83. Zworski, M.: Sharp polynomial bounds on the number of scattering poles of radial potentials. J. Funct. Anal. 82(2), 370–403 (1989)

Communicated by R.Seiringer