Superconducting Proximity Effect on a Two-Dimensional Dirac Electron System

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The superconducting proximity effect on two-dimensional massless Dirac electrons is usually analyzed using a simple model consisting of the Dirac Hamiltonian and an energy-independent pair potential. Although this conventional model is plausible, it is questionable whether it can fully describe the proximity effect from a superconductor. Here, we derive a more general proximity model starting from an appropriate microscopic model for the Dirac electron system in planar contact with a superconductor. The resulting model describes the proximity effect in terms of the energy-dependent pair potential and renormalization term. Within this framework, we analyze the density of states, the quasiparticle wave function, and the charge conservation of Dirac electrons. The result reveals several characteristic features of the proximity effect, which cannot be captured within the conventional model.

KEYWORDS: graphene, strong topological insulator, Dirac equation, superconductor

1. Introduction

Since the isolation of monolayer graphene,\textsuperscript{1} two-dimensional (2D) massless Dirac electron systems have attracted considerable attention in the condensed matter community. This attention has been intensified by the discovery of 2D Dirac electrons on the surface of strong topological insulators.\textsuperscript{2–4} Low-energy electrons in monolayer graphene are referred to as Dirac electrons as they obey the massless Dirac equation.\textsuperscript{5} A three-dimensional strong topological insulator is insulating in the bulk but hosts metallic electron states on its surface. The surface electrons of strong topological insulators are confined in a thin surface region and obey the massless Dirac equation,\textsuperscript{6} so they are also regarded as 2D Dirac electrons.

In this paper, we focus on the proximity effect on 2D Dirac electrons coupled with a bulk superconductor. Such a setup is most naturally created by depositing a superconductor on top of a graphene sheet\textsuperscript{7–9} or a topological insulator.\textsuperscript{10,11} Consequently, the resulting hybrid system has a 2D planar structure. The simplest way to describe electron states in the region covered by a superconductor is to add an effective pair potential $\Delta_{\text{eff}}$ to the Hamiltonian of the Dirac electron system on the $xy$ plane governed by the following $2 \times 2$ Dirac Hamiltonian:

$$\hat{H}_D^0 = \begin{bmatrix} -\mu & v\hat{k}_- \\ v\hat{k}_+ & -\mu \end{bmatrix},$$

where $v$ and $\mu$ are the velocity and chemical potential, respectively, and $\hat{k}_\pm \equiv -i\partial_x \pm \partial_y$. To theoretically treat quasiparticle states under the proximity effect, we need to use the Bogoliubov-de Gennes (BdG) equation,\textsuperscript{13} which can describe the mixing of electron and hole states induced by an effective pair potential. The corresponding $4 \times 4$ Hamiltonian for the covered region becomes

$$\hat{H}_\text{BdG}^0 = \begin{bmatrix} \hat{H}_D^0 & \Delta \\ \Delta^\dagger & -\hat{H}_D^0 \end{bmatrix},$$

where

$$\Delta = \begin{bmatrix} \Delta_{\text{eff}} & 0 \\ 0 & \Delta_{\text{eff}} \end{bmatrix}.$$
eral proximity model. Starting from an appropriate microscopic model for the 2D Dirac electron system in planar contact with a bulk superconductor, we derive an effective model for Dirac electrons by exactly integrating out the electron degrees of freedom in the superconductor. In the resulting effective model, the proximity effect is represented by the energy-dependent pair potential $\phi$ and renormalization term $\eta$. In the Matsubara representation with $\omega$ being the fermion Matsubara frequency, the effective Hamiltonian is given by

$$H_{\text{BdG}}(\omega) =$$

$$
\begin{bmatrix}
-\mu - \eta(\omega) & v\hat{k}_- & \phi(\omega) & 0 \\
v\hat{k}_+ & -\mu - \eta(\omega) & 0 & \phi(\omega) \\
\phi(\omega) & 0 & -\mu - \eta(\omega) & -v\hat{k}_- \\
0 & \phi(\omega) & -v\hat{k}_+ & \mu - \eta(\omega)
\end{bmatrix}
$$

with

$$\eta(\omega) = \frac{i\Gamma}{\Omega(\omega)},$$

$$\phi(\omega) = \frac{\Gamma\Delta_0}{\Omega(\omega)},$$

where $\Delta_0$ is the pair potential of the bulk superconductor, $\Gamma$ characterizes the coupling strength of Dirac electrons to the superconductor, and

$$\Omega(\omega) = \sqrt{\omega^2 + \Delta_0^2}.$$  

The effective Hamiltonian with real energy $\epsilon$ is simply obtained by carrying out the analytic continuation of $i\omega \rightarrow \epsilon + i\delta$, where $\delta$ is a positive infinitesimal. Within this framework, we analyze the density of states, the quasiparticle wave function, and the charge conservation of Dirac electrons to reveal the characteristic features of the proximity effect.

Here, it is fair to mention that Sau et al. have derived a model essentially equivalent to Eq. (4) by adapting the argument of McMillan to a microscopic model for a superconductor-topological insulator junction. However, they focused on the low-frequency limit of $\omega \rightarrow 0$ and did not explicitly examine the role of the $\omega$-dependence of the effective Hamiltonian. Furthermore, the derivation of Eq. (4) given below is simpler and more transparent than that in Ref. 23. The same model was also proposed in Refs. 25 and 26 for a superconductor-graphene junction.

Our other task in this study is to determine the relationship between $H_{\text{BdG}}(\omega)$ and $H^0_{\text{BdG}}$. In Ref. 26, an expression for the Josephson current through the Dirac electron system is derived on the basis of $H_{\text{BdG}}(\omega)$. It is shown that, at $T = 0$, the expression in the strong coupling limit of $\Gamma \gg \Delta_0$ reproduces that of Titov and Beenakker derived on the basis of $H^0_{\text{BdG}}$ with $\Delta_{\text{eff}} = \Delta_0$, suggesting some relationship between the two Hamiltonians. We show that, in spite of the apparent difference between them, the behavior of quasiparticles described by $H_{\text{BdG}}(\omega)$ becomes nearly identical to that described by $H^0_{\text{BdG}}$ under the condition of $\mu \gg \Gamma \gg \Delta_0$. This accounts for the correspondence of the two expressions for the Josephson current in the strong coupling limit.

In the next section, we introduce a simple microscopic model for the Dirac electron system in planar contact with a bulk superconductor, and derive the effective Hamiltonian $H_{\text{BdG}}$ for Dirac electrons fully taking account of the proximity effect from a superconductor. In Sect. 3, we calculate the density of states in the Dirac electron system as a simple application of $H_{\text{BdG}}$. In Sect. 4, we obtain the wave function of quasiparticle states by solving the BdG equation for $H_{\text{BdG}}$. It is shown that the resulting wave function is nearly identical to that obtained from the conventional model Hamiltonian $H^0_{\text{BdG}}$ when $\mu \gg \Gamma \gg \Delta_0$, indicating that $H_{\text{BdG}}$ and $H^0_{\text{BdG}}$ describe the same physics under this condition. The charge conservation in the system described by $H_{\text{BdG}}$ is considered in Sect. 5. The last section is devoted to a summary. We set $\hbar = k_B = 1$ throughout this paper.

2. Model and Formulation

Let us consider the 2D Dirac electron system on the $xy$ plane in planar contact with a bulk superconductor. For convenience of analysis, we assume that the superconductor consists of an infinite number of 2D superconducting layers stacked in the $z$-direction, and that the first layer is coupled with the Dirac electron system (see Fig. 1). We assume that the system is translationally invariant in the $x$- and $y$-directions, and treat the in-plane wave vector $k_{\parallel} \equiv (k_x, k_y)$ as a conserved quantity.

The action $S$ for this system is decomposed into $S = S_D + S_S + S_T$, where $S_D$ and $S_S$ respectively describe the
2D Dirac electron system and the bulk superconductor, and $S_T$ corresponds to the coupling between them. We consider only the $k_0$-component (and its time-reversed partner) of $S$ in the following argument since $k_\parallel$ is a good quantum number, and use the Matsubara representation. Firstly, we present an expression for $S_S$. Let $\psi_{j\sigma}$ be the electron field with spin $\sigma$ for the $j$th layer of the superconductor ($j = 1, 2, 3, \ldots$). Here and hereafter, we do not explicitly indicate the $k_\parallel$-dependence of the electron field. The adjacent layers are coupled by the transfer integral $t$, and the in-plane energy dispersion in each layer is $\epsilon(k_\parallel) \equiv k_\parallel^2/(2m)$ in the normal state. We assume that the pair potential $\Delta_0$ and the chemical potential $\mu_S$ are constant over all layers. Taking all these into account, $S_S$ is given by

$$
S_S = T \sum_\omega \sum_{j \geq 1} \left[ \sum_{\sigma = \uparrow, \downarrow} \left( \psi_{j\sigma}^\dagger(\omega) (-i\omega - \mu_S(k_\parallel)) \psi_{j\sigma}(\omega) 
- \frac{t}{2} \psi_{j\sigma}(\omega) \psi_{j+1\sigma}(\omega) - \frac{t}{2} \psi_{j+1\sigma}(\omega) \psi_{j\sigma}(\omega) \right) + \Delta_0 \psi_{j\uparrow}^\dagger(\omega) \psi_{j\downarrow}^\dagger(-\omega) + \Delta_0 \psi_{j\downarrow}^\dagger(-\omega) \psi_{j\uparrow}(\omega) \right],
$$

where $\mu_S(k_\parallel)$ is the effective chemical potential defined by $\mu_S(k_\parallel) = \mu_S - \epsilon(k_\parallel)$. Next we present an expression for $S_D$. We assume that the Dirac electron system is simply described by the Hamiltonian (1). Let $\psi_{D\sigma}$ be the Dirac electron field with spin $\sigma$. To treat the superconducting proximity effect, we must simultaneously consider electrons and holes. This is achieved by employing the four-component field $\Psi_D(\omega)$:

$$
\Psi_D(\omega) \equiv \begin{pmatrix} \psi_{D\uparrow}(\omega), \psi_{D\downarrow}(\omega), \psi_{D\downarrow}^\dagger(-\omega), -\psi_{D\uparrow}^\dagger(-\omega) \end{pmatrix}. 
$$

The corresponding action $S_D$ is written as

$$
S_D = T \sum_\omega \frac{1}{2} \Psi_D^\dagger(\omega) \left( -i\omega 1_{4 \times 4} + H_D \right) \Psi_D(\omega),
$$

where $1_{4 \times 4} = \text{diag}\{1, 1, 1, 1\}$,

$$
H_D = \begin{bmatrix} H_D^0 & 0 \\
0 & -H_D^0 \end{bmatrix},
$$

and the factor $1/2$ is attached to avoid double counting. Finally, the coupling term is expressed as

$$
S_T = T \sum_\omega \sum_{\sigma = \uparrow, \downarrow} \left[ -\gamma \psi_{\uparrow\sigma}^\dagger(\omega) \psi_{D\sigma}(\omega) - \gamma \psi_{D\sigma}^\dagger(\omega) \psi_{\uparrow\sigma}(\omega) \right],
$$

where $\gamma$ is the transfer integral connecting the Dirac electron system and the first layer of the bulk superconductor.

The proximity correction $S_\Sigma$ to the Dirac electron system is expressed as

$$
\exp(-S_\Sigma) = \frac{\prod_{j,\sigma,\omega} D\psi_{j\sigma}(\omega) D\psi_{j\sigma}^\dagger(\omega) \exp(-S_S - S_T)}{\prod_{j,\sigma,\omega} D\psi_{j\sigma}(\omega) D\psi_{j\sigma}^\dagger(\omega) \exp(-S_S)}.
$$

Hence, the effective action for the Dirac electron system is given by $S_{\text{eff}} \equiv S_D + S_\Sigma$. We obtain an expression for $S_\Sigma$ by integrating out $\psi_{j\sigma}$ and $\psi_{j\sigma}^\dagger$ for all $j$. The derivation of $S_\Sigma$ is presented in Appendix A. The result is

$$
S_\Sigma = T \sum_\omega \frac{1}{2} \Psi_D^\dagger(\omega) \begin{pmatrix} V(\omega) \mathbb{I} & \chi(\omega) \mathbb{I} \\
\chi^\dagger(\omega) \mathbb{I} & -V(-\omega) \mathbb{I} \end{pmatrix} \Psi_D(\omega),
$$

where $\mathbb{I} = \text{diag}\{1, 1\}$, $\chi(\omega)$ is defined in Eq. (6), and

$$
V(\omega) = \frac{\Gamma \mu_S(k_\parallel)}{2t} - \chi \frac{i\Gamma \omega}{\Omega(\omega)}.
$$

with

$$
\chi = 1 - \frac{1}{2} \left( \frac{\mu_S(k_\parallel)}{2t} \right)^2.
$$

Here, the coupling strength $\Gamma$ is defined by

$$
\Gamma = \frac{\gamma^2}{t}.
$$

As $2t$ is the largest energy scale of the system under consideration, we can safely ignore the first term of $V(\omega)$ and set $\chi = 1$. With this reduction, the effective action is simplified to

$$
S_{\text{eff}} = T \sum_\omega \frac{1}{2} \Psi_D^\dagger(\omega) \left( -i\omega 1_{4 \times 4} + H_{\text{BDG}}(\omega) \right) \Psi_D(\omega),
$$

where $H_{\text{BDG}}$ is the $\omega$-dependent effective Hamiltonian presented in Eq. (4). That is, Dirac electrons under the proximity effect are described by this effective Hamiltonian. It should be emphasized that, in its derivation, we do not rely on a perturbative treatment with respect to the coupling term $S_T$ (see Appendix A). Hence, this approach is applicable even when the coupling is very strong.

In the above argument, we properly take account of the proximity effect on the Dirac electron system but completely ignore the reverse effect on the superconductor. Generally, the strength of the proximity effect is determined by the density of states in the partner to which the system under consideration is coupled. As the density
of states of Dirac electrons is much smaller than that of
the bulk superconductor, the reverse effect can be safely
ignored.

3. Density of States

As a simple application of the effective Hamiltonian

\[ H_{\text{BDG}}(\omega) \]

derived in the previous section, we analyze the
density of states in the Dirac electron system under the
proximity effect. The detailed behavior of quasiparticle
states is analyzed in the next section.

Let us introduce the thermal Green’s function defined
as

\[ \mathcal{G}(k_{||},\omega) = \left[ i\omega 1_{4\times 4} - H_{\text{BDG}}(\omega) \right]_{k_{||} - i\varepsilon \rightarrow k_{||} - i\delta \varepsilon}^{-1}, \]

in terms of which the density of states \( D(\epsilon) \) at energy \( \epsilon \) is expressed as

\[ D(\epsilon) = -\frac{1}{\pi} \int \frac{dk_{||}^2}{(2\pi)^2} \text{Im} \left\{ \text{tr} \left\{ \mathcal{G}(k_{||},\omega) \right\} \big|_{\omega \rightarrow \epsilon + i\delta} \right\}, \]

where \( \delta \) is a positive infinitesimal. It is easy to show that

\[ \text{tr} \left\{ \mathcal{G}(k_{||},\omega) \right\} = \frac{-2i\delta}{\sqrt{\epsilon^2 + \omega^2}} + \frac{-2\delta\omega}{\epsilon^2 + \omega^4}, \]

where \( \delta = (1 + \Gamma/\Omega)\omega \) and \( k_{||} = |k_{||}| \). To avoid the
unphysical divergence of the integral over \( k_{||} \), we introduce the cutoff energy \( \epsilon_c \) that is equivalent to half of the band
width. After analytic continuation, the energy-dependent pair potential becomes

\[ \phi(\epsilon) = \frac{\Gamma\Delta_0}{\sqrt{\Delta_0^2 - \epsilon_c^2}}, \]

where

\[ \epsilon_c = \epsilon + \delta\omega. \]

Note that the behavior of \( \phi(\epsilon) \) markedly changes depending on whether \( \epsilon \) is greater or smaller than \( \Delta_0 \) as follows:

\[ \phi(\epsilon) = \begin{cases} \frac{\Gamma\Delta_0}{\sqrt{\Delta_0^2 - \epsilon^2}} & (\Delta_0 > \epsilon \geq 0) \\ \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta_0^2}} & (\epsilon > \Delta_0). \end{cases} \]

It is convenient to introduce the renormalization factor

\[ Z(\epsilon) = 1 + \frac{\Gamma}{\sqrt{\Delta_0^2 - \epsilon_c^2}}, \]

and the effective pair potential

\[ \Delta(\epsilon) = \frac{\phi(\epsilon)}{Z(\epsilon)}. \]

As we see below, this effective pair potential determines the proximity-induced energy gap of Dirac electrons. Obviously, the \( \epsilon \)-dependence of the pair potential is completely ignored in the conventional model.

Carrying out the integration over \( k_{||} \), we finally obtain

\[ D(\epsilon) = \frac{1}{\pi^2 v^2} \text{Im} \left\{ \epsilon Z(\epsilon) \ln \left( \frac{\epsilon^2 - \mu^2}{Z(\epsilon)^2 (\Theta(\epsilon))^2 + \mu^2} \right) \right\} \]

\[ + \frac{i\mu}{\Theta(\epsilon)} \text{ln} \left( \frac{iZ(\epsilon)\Theta(\epsilon) + \mu}{iZ(\epsilon)\Theta(\epsilon) - \mu} \right), \]

where

\[ \Theta(\epsilon) = \sqrt{\Delta(\epsilon)^2 - \epsilon_c^2}. \]

As \( Z(\epsilon) \) and hence \( \Delta(\epsilon) \) are real numbers when \( \epsilon < \Delta_0 \),
it is easy to see that

\[ \Theta(\epsilon) = \begin{cases} \sqrt{\Delta(\epsilon)^2 - \epsilon^2} & (\Delta(\epsilon) > \epsilon \geq 0) \\ -i\sqrt{\epsilon^2 - \Delta(\epsilon)^2} & (\Delta_0 > \epsilon > \Delta(\epsilon)). \end{cases} \]

Accordingly, the density of states vanishes when \( \Delta(\epsilon) > \epsilon \geq 0 \) because the function in the square brackets of
Eq. (27) has no imaginary part. This indicates that the
proximity-induced energy gap \( \epsilon_g \) in the Dirac electron
system is determined by

\[ \epsilon_g = \Delta(\epsilon_g). \]

It is easy to show that \( \epsilon_g \) in the weak coupling limit of \( \Delta_0 \gg \Gamma \) is approximated as

\[ \epsilon_g = \frac{\Delta_0\Gamma}{\Delta_0 + \Gamma}, \]

while, in the opposite strong coupling limit, we find

\[ \epsilon_g = \Delta_0 - \frac{2\Delta_0^3}{\Gamma^2}. \]

Equation (32) indicates that the energy gap approaches \( \Delta_0 \) in the limit of \( \Gamma/\Delta_0 \rightarrow \infty \). The energy gap numerically
determined as a function of \( \Gamma/\Delta_0 \) is shown in Fig. 2,
where the dotted (dashed) line represents the approximate
expression for the weak coupling (strong coupling)
limit.

In the region of \( \Delta_0 > \epsilon > \epsilon_g \), we can easily extract the imaginary part of the function in the square brackets and simply express the density of states as

\[ D(\epsilon) = \frac{\epsilon}{\pi v^2} \left[ Z(\epsilon) \theta \left( Z(\epsilon)\sqrt{\epsilon^2 - \Delta(\epsilon)^2} - \mu \right) \right. \]

\[ + \left. \frac{\mu}{\sqrt{\epsilon^2 - \Delta(\epsilon)^2}} \theta \left( \mu - Z(\epsilon)\sqrt{\epsilon^2 - \Delta(\epsilon)^2} \right) \right], \]

(33)
where $\theta(x)$ is the Heaviside step function. It is interesting to note that the familiar square-root singularity [i.e., the second term of Eq. (33)] disappears when $\mu \approx 0$. This should be regarded as a characteristic feature of the Dirac electron system.

The density of states for arbitrary $\epsilon$ is numerically obtained from Eq. (27). As an example, we calculate $D(\epsilon)$ at $\Gamma/\Delta_0 = 0.2$ in the cases of $\mu/\Delta_0 = 0$ and $5$ with $\epsilon_c/\Delta_0 = 2000$. To simulate inelastic scattering effects, which are inevitably present in actual experimental situations, we set the positive infinitesimal $\delta$ to be $\delta/\Delta_0 = 0.01$. The result is shown in Fig. 2. The peak structure at $\epsilon/\Delta_0 = 1$ reflects the square-root singularity of the density of states in the bulk superconductor. Obviously, this cannot be captured within the conventional model with the energy-independent pair potential $\Delta_{\text{eff}}$ since information on the bulk superconductor is not adequately encoded in it. We observe that, at the gap edge (i.e., $\epsilon = \epsilon_c$), there is no singular behavior in the case of $\mu/\Delta_0 = 0$, while a sharp enhancement appears in the case of $\mu/\Delta_0 = 5$, in accordance with Eq. (33).

### 4. Quasiparticle States

We introduce an effective BdG equation with real energy $\epsilon$ and obtain its exact eigenstates. Considering the resulting eigenstate wave function, we demonstrate that our model becomes essentially equivalent to the conventional model in a certain limit.

The effective Hamiltonian for the BdG equation is obtained by carrying out the analytic continuation $i\omega \rightarrow \epsilon + i\delta$ in $H_{\text{BdG}}(\omega)$. Then the BdG equation is given in the following form:

$$H_{\text{BdG}}(\epsilon)\Psi(x,y) = \epsilon\Psi(x,y),$$  \hspace{1cm} (34)

where

$$H_{\text{BdG}}(\epsilon) = \begin{pmatrix}
-\mu - \eta(\epsilon) & v\hat{k}_- & \phi(\epsilon) & 0 \\
v\hat{k}_+ & -\mu - \eta(\epsilon) & 0 & \phi(\epsilon) \\
\phi(\epsilon) & 0 & \mu - \eta(\epsilon) & -v\hat{k}_- \\
0 & \phi(\epsilon) & -v\hat{k}_+ & \mu - \eta(\epsilon)
\end{pmatrix}$$  \hspace{1cm} (35)

with $\eta(\epsilon) = (Z(\epsilon) - 1)\epsilon$.

We hereafter assume that $\Psi(x,y)$ varies as $e^{ik_y y}$ in the $y$ direction with a real $k_y$, and hence $\Psi(x,y)$ is rewritten as

$$\Psi(x,y) \equiv e^{ik_y y}\Psi(x).$$  \hspace{1cm} (36)

Note that the BdG equation has evanescent solutions. That is, $\Psi(x)$ can be an exponentially decreasing or increasing function of $x$. These solutions are necessary in analyzing the scattering problem in the Dirac electron system partially covered by a bulk superconductor when the interface between covered and uncovered regions is located along the $y$-axis.\textsuperscript{14} In terms of the wave number in the $x$-direction,

$$k_x^\pm = \sqrt{\left(\frac{\mu \pm iZ(\epsilon)\Theta(\epsilon)}{v}\right)^2 - k_y^2},$$  \hspace{1cm} (37)

the solutions of Eq. (34) are expressed as

$$\Psi_x^\pm(x) = C \begin{pmatrix}
\left(\frac{\epsilon + i\Theta(\epsilon)}{\Delta(\epsilon)}\right)^{\frac{\pm}{2}} & v\hat{k}_x^\pm \\
\left(\frac{\epsilon - i\Theta(\epsilon)}{\Delta(\epsilon)}\right)^{\frac{\pm}{2}} & v\hat{k}_x^\pm
\end{pmatrix} e^{i\zeta k_x^\pm x},$$  \hspace{1cm} (38)

where $C$ is the normalization constant, $\zeta(=\pm)$ specifies

Fig. 2. Energy gap $\epsilon_c$ as a function of $\Gamma/\Delta_0$.

Fig. 3. Density of states in the Dirac electron system in the cases of $\mu/\Delta_0 = 0$ and $5$ with $\Gamma/\Delta_0 = 0.2$. 

\[ D(\epsilon)/(\pi v^2)^{-1} \]

\[ 0.0 \quad 0.5 \quad 1.0 \quad 1.5 \]

\[ 0.0 \quad 1.0 \quad 2.0 \]

\[ \mu/\Delta_0=5 \]

\[ \mu/\Delta_0=0 \]
the direction of propagation, and
\[ k^\pm_x = \zeta k^\pm_y - ik_y. \]  
(39)

Unless \( Z(\epsilon) \Theta(\epsilon) \) is pure imaginary, \( k^\pm \) contains an imaginary part and hence \( \Psi^\pm_\pm(x) \) becomes an exponentially increasing or decreasing function of \( x \). When applying this to the scattering problem, we need to choose only evanescent solutions.

Below, we demonstrate that the eigenfunction (38) is reduced to that derived from the conventional model under the condition of \( \mu \gg \Gamma \gg \Delta_0 \). Let us focus on the strong coupling limit of \( \Gamma \gg \Delta_0 \). Note that, in this limit, the effective pair potential \( \Delta(\epsilon) \) becomes identical to \( \Delta_0 \), being independent of \( \epsilon \), as is evident in Eq. (26). Additionally, if \( \mu \) is much greater than \( \Gamma \), Eq. (38) can be approximated as
\[ \Psi^\pm_\pm(x) = C \begin{pmatrix} (\epsilon + \Theta_0(\epsilon))/\Delta_0 & \frac{1}{2} \mu \\ \frac{1}{2} \mu & \epsilon + \Theta_0(\epsilon)/\Delta_0 \end{pmatrix} \begin{pmatrix} v k^\pm_x \\ v k^\pm_y \end{pmatrix} e^{i\zeta k^\pm_x x}. \]
(40)
with \( \Theta_0(\epsilon) = \sqrt{\Delta_0^2 - \epsilon^2} \). It turns out that Eq. (40) is equivalent to the solution of the BdG equation for \( H_B^D \) with the energy-independent pair potential \( \Delta_0 \) in the case of \( \mu \gg \Delta_0 \). That is, under the condition of \( \mu \gg \Gamma \gg \Delta_0 \), the behavior of quasiparticles described by \( H_B^D(\epsilon) \) is equivalent to that described by \( H_B^D \). This statement is also supported by the fact that, in the limit of \( \mu \gg \Gamma \gg \Delta_0 \), the density of states \( (27) \) is reduced to
\[ D(\epsilon) = \frac{1}{\pi v^2} \frac{\mu \epsilon}{\sqrt{\epsilon^2 - \Delta_0^2}}, \]
(41)
which is equivalent to that obtained from \( H_B^D \).

The above argument accounts for the correspondence that, in the strong coupling limit of \( \Gamma \gg \Delta_0 \), the expression for the Josephson current derived on the basis of \( H_B^D(w) \) reproduces that of the conventional model, where \( \mu \gg \Delta_0 \) is assumed from the outset.

### 5. Charge Conservation

The evanescent solutions obtained above exponentially decay with increasing or decreasing \( x \), implying the disappearance of quasiparticle current. This missing current should be transferred to the bulk superconductor coupled with the Dirac electron system. In this section, we derive the charge conservation law for quasiparticle states from the BdG equation.

Let us express the four components of \( \Psi(x) \) as
\[ \Psi(x) = \begin{pmatrix} u_\uparrow(x), v_\uparrow(x), v_\downarrow(x), u_\downarrow(x) \end{pmatrix}, \]
(42)
where \( u_\sigma(x) \) and \( v_\sigma(x) \) respectively are the electron and hole wave functions for spin \( \sigma \). In terms of the electron charge \( e \), we define the quasiparticle charge density \( Q(x) \) as
\[ Q(x) = e \left( |u_\uparrow(x)|^2 + |v_\uparrow(x)|^2 - |v_\downarrow(x)|^2 - |u_\downarrow(x)|^2 \right). \]
(43)
In the stationary state in which we are interested, it is obvious that the time derivative of \( Q(x) \) vanishes:
\[ \partial_t Q(x) = 0. \]
(44)
On the other hand, by noting that \( i\hat{\epsilon} \Psi(x) \) can be identified with \( e\Psi(x) \) in the stationary state with energy \( \epsilon \) and then using the BdG equation (34), we can show that the quasiparticle charge density \( Q(x) \) and quasiparticle current density \( J_Q(x) \) in the \( x \)-direction satisfy
\[ \partial_t Q(x) = \Lambda_Q(x) + \Lambda_S(x) - \partial_x J_Q(x), \]
(45)
where
\[ \Lambda_Q(x) = -2\text{Im} \{ Z(\epsilon) \} eQ(x), \]
\[ \Lambda_S(x) = 4e\text{Re} \{ \phi(\epsilon) \} \text{Im} \{ u_\uparrow(x)^* v_\downarrow(x) + u_\downarrow(x)^* v_\uparrow(x) \}. \]
(46)
(47)
The current density is expressed as
\[ J_Q(x) = \Psi(x)^T \hat{J}_Q \Psi(x) \]
(48)
with
\[ \hat{J}_Q = ev \begin{pmatrix} \hat{\sigma}_x & 0 \\ 0 & \hat{\sigma}_x \end{pmatrix}, \]
(49)
where \( \hat{\sigma}_x \) is the \( x \)-component of the Pauli matrices. Combining Eqs. (44) and (45), we arrive at the charge conservation law
\[ \Lambda_Q(x) + \Lambda_S(x) - \partial_x J_Q(x) = 0 \]
(50)
in the stationary state. It is obvious from Eqs. (46) and (47) that \( \Lambda_Q \) and \( \Lambda_S \) are drain terms describing charge tunneling into the superconductor: \( \Lambda_Q \) represents the contribution of quasiparticle tunneling, while \( \Lambda_S \) represents that of pair tunneling. It should be emphasized that the conventional model does not involve the drain term \( \Lambda_Q(x) \) due to quasiparticle tunneling, indicating its inadequacy in describing the proximity effect.

Let us examine charge transfer processes between the Dirac electron system and the superconductor on the basis of Eq. (50). In the region of \( \Delta_0 > \epsilon \geq 0 \), \( Z(\epsilon) \) has no imaginary part, resulting in \( \Lambda_Q = 0 \). This indicates that the quasiparticle tunneling plays no role, reflecting the fact that the quasiparticle density of states in the superconductor vanishes in this energy region. To contrast, \( \phi(\epsilon) \) becomes pure imaginary in the region of \( \epsilon > \Delta_0 \), resulting in \( \Lambda_S = 0 \). This indicates that pair tunneling
plays no role. To gain further insight into charge transfer processes, we evaluate $\Lambda_Q$ and $\Lambda_S$ by substituting the wave function given in Eq. (38) into Eqs. (46) and (47), and confirm that the charge conservation law (50) actually holds (see Appendix B). An important finding is that $\Lambda_S = 0$ in the region of $\Delta_0 > \epsilon > \epsilon_g$. Our result is summarized as

$$\Lambda_S(x) - \partial_x J_Q(x) = 0$$

(51)

for $\epsilon > \epsilon_g$, and

$$\partial_x J_Q(x) = 0$$

(52)

for $\Delta_0 > \epsilon > \epsilon_g$, and

$$\Lambda_Q(x) - \partial_x J_Q(x) = 0$$

(53)

for $\epsilon > \Delta_0$.

It is worth pointing out that quasiparticle states in the region of $\Delta_0 > \epsilon > \epsilon_g$ are decoupled from the superconductor in the sense that the corresponding quasiparticle current is conserved within the Dirac electron system. Supposing that the Dirac electron system is partially covered by a bulk superconductor, let us consider the electron transport from the uncovered region to the superconductor. Note that electrons inevitably pass through the covered region in the transport process. We expect that quasiparticle states in the covered region with $\Delta_0 > \epsilon > \epsilon_g$ do not contribute to the electron transport since they are decoupled from the superconductor. In contrast, quasiparticle states with $\epsilon_g > \epsilon > 0$ do contribute to that through the pair tunneling into superconducting condensate. It is of interest whether such contrasting behaviors can be experimentally observed.

6. Summary

We have studied the proximity effect on a two-dimensional massless Dirac electron system in planar contact with a bulk superconductor, starting from an appropriate microscopic model that explicitly takes account of the coupling of Dirac electrons to the superconductor. Integrating out the electron degrees of freedom in the superconductor, we have derived a general proximity model for Dirac electrons. The resulting effective model takes account of the proximity effect in terms of the energy-dependent pair potential and renormalization term, and is applicable regardless of the strength of coupling of Dirac electrons to the superconductor. From the analysis of the density of states, the quasiparticle wave function, and the charge conservation of Dirac electrons, it is shown that the effective model reveals several characteristic features of the proximity effect, which cannot be captured by the conventional model, implying its advantage over the conventional model.

Finally, it is worth mentioning that the approach developed in this paper can be straightforwardly applied to the hybrid system of multilayer graphene in planar contact with a bulk superconductor.\textsuperscript{26,32,33}

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Appendix A: Derivation of $S_S$

To obtain the expression for $S_S$, we perform the integration over $\psi_{j\sigma}$ and $\psi_{j\sigma}^\dagger$ in Eq. (13) following Ref. 29. It is convenient to rewrite the electron field as

$$\psi_{j\sigma}(\omega) = \frac{2}{\pi} \int_0^\pi dq\sin(qj)\psi_{\sigma}(q, \omega).$$

(A-1)

The substitution of this into the expression for $S_S$ yields

$$S_S = T \sum_\omega \frac{2}{\pi} \int_0^\pi dq \left( \psi_{\uparrow}(q, \omega), \psi_{\downarrow}(q, -\omega) \right)$$

$$\times \left( \begin{array}{cc} -i\omega + \xi_q & \Delta_0 \\ \Delta_0 & -i\omega - \xi_q \end{array} \right) \left( \begin{array}{c} \psi_{\uparrow}(q, \omega) \\ \psi_{\downarrow}(q, -\omega) \end{array} \right),$$

(A-2)

where $\xi_q = -2t \cos q - \mu_S(k_0)$. We can simplify Eq. (A-2) in terms of the Bogoliubov transformation:

$$\left( \begin{array}{c} \psi_{\uparrow}(q, -\omega) \\ \psi_{\downarrow}(q, -\omega) \end{array} \right) = \left( \begin{array}{cc} u_q & -v_q \\ v_q & u_q \end{array} \right) \left( \begin{array}{c} \varphi_{\uparrow}(q, \omega) \\ \varphi_{\downarrow}(q, -\omega) \end{array} \right),$$

(A-3)

where

$$u_q = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi_q}{E_q}},$$

(A-4)

$$v_q = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi_q}{E_q}}$$

(A-5)

with $E_q = \sqrt{\xi_q^2 + \Delta_0^2}$. Consequently, $S_S$ is reduced to

$$S_S = T \sum_\omega \frac{2}{\pi} \int_0^\pi dq \sum_{\tau=\pm} \varphi_{\tau}(q, \omega) \left( -i\omega + E_q \right) \varphi_{\tau}(q, \omega).$$

(A-6)

In terms of $\varphi_{\tau}$ and $\varphi_{\tau}^\dagger$, the coupling term is rewritten as

$$S_T = T \sum_\omega \frac{2}{\pi} \int_0^\pi dq (-\gamma) \sin q$$

$$\times \left[ \left( u_q \varphi_{\tau}^\dagger(q, \omega) - v_q \varphi_{-\tau}(q, -\omega) \right) \psi_{\downarrow}(\omega) + \text{H.c.} \right.$$}

$$+ \left( v_q \varphi_{\tau}(q, \omega) + u_q \varphi_{-\tau}^\dagger(q, -\omega) \right) \psi_{\downarrow}(\omega) + \text{H.c.} \right].$$

(A-7)
We substitute Eqs. (A-6) and (A-7) into Eq. (13) and replace the integration variables as

\[
\prod_{j, \sigma, \omega} D\psi_{j\sigma}(\omega)D\psi_{j\sigma}^\dagger(\omega) \rightarrow \prod_{q, \tau, \omega} D\varphi_\tau(q, \omega)D\varphi_\tau^\dagger(q, \omega).
\]

(A-8)

We can obtain \( S_\Sigma \) by integrating out \( \varphi_\tau \) and \( \varphi_\tau^\dagger \). It should be emphasized that this integration can be performed exactly without relying on a perturbative treatment with respect to \( S_T \). The result is

\[
S_\Sigma = T \sum_\omega \gamma^2 \frac{2}{\pi} \int_0^\infty dq \sin^2 q \times \left[ \left( \frac{v^2}{\omega + E_q} - \frac{v^2}{-i\omega + E_q} \right) \sum_{\sigma = \uparrow, \downarrow} \psi_{1\sigma}^\dagger(\omega)\psi_{1\sigma}(\omega) + 2\frac{u_q v_q E_q}{\omega^2 + E_q^2} (\psi_{1\downarrow}(-\omega)\psi_{1\uparrow}(\omega) + H.c.) \right].
\]

(A-9)

Finally, we carry out the integration over \( q \). Since \( 2t \) is the largest energy scale of the system under consideration, it is natural to assume that \( 2t \gg \mu_S(k_\parallel), \Delta_0, |\omega| \). We then arrive at

\[
S_\Sigma = T \sum_\omega \left[ \frac{\Gamma \mu_S(k_\parallel)}{2t} - \frac{i\Gamma \omega}{\Omega(\omega)} \right] \sum_{\sigma = \uparrow, \downarrow} \psi_{1\sigma}^\dagger(\omega)\psi_{1\sigma}(\omega) + \frac{\Gamma \Delta_0}{\Omega(\omega)} \left( \psi_{1\downarrow}(-\omega)\psi_{1\uparrow}(\omega) + H.c. \right),
\]

(A-10)

where \( \Gamma = \gamma^2 / t, \Omega(\omega) = \sqrt{\omega^2 + \Delta_0^2} \), and

\[
\chi = 1 - \frac{1}{2} \left( \frac{\mu_S(k_\parallel)}{2t} \right)^2.
\]

(A-11)

This expression is equivalent to Eq. (14).

**Appendix B: Check of Charge Conservation**

In this Appendix, we check that the charge conservation law (50) actually holds for the quasiparticle wavefunction \( \Psi_\pm(x) \) obtained in Sect. 4. We examine only the case of \( \zeta = + \), for which \( \Psi_\pm \) is written as

\[
\Psi_\pm(x) = C \begin{pmatrix} \left( \frac{x + i\Theta(\epsilon)}{2\epsilon} \right)^{\chi/2} \left( \frac{v k^\pm_\epsilon}{\Delta(\epsilon)} \right) & \left( \mu \pm iZ(\epsilon)\Theta(\epsilon) \right) \\
\left( \frac{x - i\Theta(\epsilon)}{2\epsilon} \right)^{-\chi/2} & \left( \mu \pm iZ(\epsilon)\Theta(\epsilon) \right) \end{pmatrix} e^{ik^\pm_\epsilon x}.
\]

(B-1)

We separately treat the three energy regions \( \epsilon_q > \epsilon > 0 \), \( \Delta_0 > \epsilon > \epsilon_g \), and \( \epsilon > \Delta_0 \). Remember that, in the first two cases, \( \Delta_Q = 0 \) when \( \Im\{Z(\epsilon)\} = 0 \) for \( \Delta_0 > \epsilon > 0 \), while \( \Delta_S = 0 \) in the last case since \( \Re\{\phi(\epsilon)\} = 0 \) for \( \epsilon > \Delta_0 \).

In the subgap region of \( \epsilon_q > \epsilon > 0 \), both \( Z(\epsilon) \) and \( \Theta(\epsilon) \) are real numbers. Thus, Eq. (37) indicates that \( k^\pm_\epsilon \) has an imaginary part, so we set

\[
k^\pm_\epsilon = k^0_\epsilon + i\kappa^\pm.
\]

(B-2)

If \( \kappa^\pm \) is positive (negative), \( \Psi_\pm^0(x) \) is an exponentially decreasing (increasing) function of \( x \). From Eqs. (37) and (B-2), we can show that

\[
\kappa^\pm = \pm \frac{\mu Z(\epsilon)\Theta(\epsilon)}{v^2 k^0_\epsilon}.
\]

(B-3)

Since \( \Delta_Q = 0 \) in this case, we obtain \( \Delta_S \) and \( J_Q \) for \( \Psi_\pm^0(x) \) to check the charge conservation law. Substituting Eq. (B-1) into Eqs. (47) and (48) and then using Eq. (37), we readily find that

\[
\Delta_S = 8C^2 \left[ \mp Z(\epsilon)\Theta(\epsilon) \right] \left[ \mu^2 + v^2 k^\pm(\kappa^\pm - k_y) \right] e^{-2k^\pm x},
\]

(B-4)

\[
J_Q = 4C^2 v^2 \left[ k^0_\epsilon \mp (\kappa^\pm - k_y) Z(\epsilon)\Theta(\epsilon) \right] e^{-2k^\pm x}.
\]

(B-5)

Modifying the expression for \( \Delta_S \) using Eq. (B-3), we confirm the following charge conservation law:

\[
\Delta_S(x) - \partial_x J_Q(x) = 0.
\]

(B-6)

In the region of \( \Delta_Q > \epsilon > \epsilon_g \), again \( \Delta_Q = 0 \) and \( Z(\epsilon) \) is a real number. However, \( \Theta(\epsilon) \) becomes a pure imaginary number as \( \Theta(\epsilon) = -i\sqrt{\epsilon^2 - \Delta(\epsilon)^2} \). The substitution of Eq. (B-1) into Eq. (47) straightforwardly yields \( \Delta_S = 0 \). Correspondingly, \( J_Q \) does not depend on \( x \) as \( k^\pm_\epsilon \) has no imaginary part in this region. Taking these into account, we find that

\[
\partial_x J_Q = 0
\]

(B-7)

holds.

In the last case of \( \epsilon > \Delta_0 \), both \( Z(\epsilon) \) and \( \Theta(\epsilon) \) contain real and imaginary parts. Hence, \( k^\pm_\epsilon \) has an imaginary part and is expressed in the form of Eq. (B-2). It is convenient to decompose \( iZ(\epsilon)\Theta(\epsilon) \) into real and imaginary parts as

\[
iZ(\epsilon)\Theta(\epsilon) = \alpha + i\beta,
\]

(B-8)

in terms of which \( \kappa^\pm \) is expressed as

\[
\kappa^\pm = \pm \frac{(\mu + \alpha)\beta}{v^2 k^0_\epsilon}.
\]

(B-9)

Since \( \Delta_S = 0 \) when \( \epsilon > \Delta_0 \), we obtain \( \Delta_Q \) and \( J_Q \) for \( \Psi_\pm^0(x) \). Substituting Eq. (B-1) into Eqs. (46), and (48),
we find after calculations using Eqs. (37) and (B-8) that
\[
\Lambda_Q = - C^2 \frac{\mid \epsilon \mp i \Theta (\epsilon) \mid - \mid \epsilon \mp i \Theta (\epsilon) \mid}{\Delta (\epsilon)} 
\times \left[ (\mu + \alpha)^2 + \nu^2 (\kappa^z - k_y) \right] e^{- 2 \kappa^z x}, \quad \text{(B-10)}
\]
\[
J_Q = 2C^2 \frac{\epsilon}{\mid \epsilon \mp i \Theta (\epsilon) \mid} \left( \mu + \alpha \right) e^{- 2 \kappa^z x}. \quad \text{(B-11)}
\]
Modifying the expression for \( \Lambda_Q \) using Eq. (B-9) and the identity
\[
\left( \mid \epsilon \mp i \Theta (\epsilon) \mid - \mid \epsilon \mp i \Theta (\epsilon) \mid \right) \epsilon
\]
\[
= \pm \left( \mid \epsilon \mp i \Theta (\epsilon) \mid + \mid \epsilon \mp i \Theta (\epsilon) \mid \right) \frac{i \Theta (\epsilon) \beta}{\Gamma}, \quad \text{(B-12)}
\]
we can show that the charge conservation law, i.e.,
\[
\Lambda_Q (x) - \partial_x J_Q (x) = 0, \quad \text{(B-13)}
\]
actually holds.

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