Discussing the Spectrum of the Kalb-Ramond Field Coupled to 3D-Gravity

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Abstract

The mechanism of dynamical mass generation for the gauge field is studied through 1-loop. We find out that torsion is an obstruction to the appearance of a 1-loop mass correction. Contrary, if torsion is not present, a mass gap is generated for the 2-form field.

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1 Introduction

The Abelian 2-form gauge potential, usually referred to as Kalb-Ramond field (K-R), exhibits a number of interesting features in four dimensional space-time (4D) [1]; special emphasis is given to the fact that it mixes up with the electromagnetic gauge potential so as to yield a massive spin-1 excitation based on a $[U(1)]^2$-symmetry, without the need for Higgs scalars [2]. More recently, the coupling between the K-R and Maxwell fields has been reconsidered in connection with the issue of Dirac-like monopoles with massive photons [3]. Also, interesting discussions on the possible non-Abelian extension of the K-R field are set in the works of Ref. [4].

Reassessing the K-R field in three dimensional space-time (3D) brings about some peculiarities that show up as a by-product of three dimensional space-time. For example, the Abelian gauge field dual to the K-R potential in 3D is subject to a particular gauge symmetry that selects its longitudinal component as the physical propagating mode, whereas its transverse part appears as a compensating mode, and so it can always be gauged away. Once the mass-shell condition is imposed, no physical propagating degree of freedom survives and we have the peculiarity that the K-R gauge field carries no on-shell degrees of freedom in 3D [5]. We understand that this is a consequence of setting its dynamics on a flat space-time. So, our propose is to set out this paper in order to illustrate how the coupling of the K-R field to the 3D counterpart of the Einstein-Hilbert gravity excites a massive scalar mode in the spectrum of the theory. This is a result that already emerges at classical level.

The main motivation of our work is to compute 1-loop corrections to the K-R field self-energy, so as to understand how these effects change the mass spectrum set up at tree-level. We pursue our investigation in the cases of ordinary (torsionless) gravity (Section 2) and, later on, we go over to the general case of gravity with torsion (Section 3) [6]. We find some peculiarities at the end of our analysis and we shall comment on them in our Conclusive Comments (Section 4).
2 The Riemannian Case

The model we contemplate accounts for the non-minimal coupling of 3D-gravity to the K-R field according to the Lagrangian density given below:

\[ \mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{gauge} + \mathcal{L}_{int}, \]  

(2.1)

where the first term is the usual Einstein-Hilbert Lagrangian:

\[ \mathcal{L}_{EH} = \frac{1}{\kappa^2} \sqrt{g} R. \]  

(2.2)

Notice the absence of the overall minus sign that appears in 4D-gravity: in 3D, this must be the choice in such a way to avoid that the graviton becomes a ghost.

The second term describes the Lagrangian for the antisymmetric gauge field \( B_{\mu\nu} \),

\[ \mathcal{L}_{gauge} = \frac{1}{6} \sqrt{g} G_{\mu\nu\lambda} G^{\mu\nu\lambda}, \]  

(2.3)

where \( G_{\mu\nu\lambda} \) is a 3-form written in terms of the potential \( B_{\mu\nu} \) as follows:

\[ G_{\mu\nu\lambda} = \nabla_\mu B_{\nu\lambda} + \nabla_\lambda B_{\mu\nu} + \nabla_\nu B_{\lambda\mu}. \]  

(2.4)

The last term accounts for the interaction between gravity and the gauge field:

\[ \mathcal{L}_{int} = \xi \sqrt{g} \epsilon^{\mu\nu\lambda} \nabla_\mu B_{\nu\lambda} R, \]  

(2.5)

with \( \epsilon^{\mu\nu\lambda} = \frac{\epsilon_{\mu\nu\lambda}}{\sqrt{g}} \), \( \epsilon_{\mu\nu\lambda} \) being the totally antisymmetric symbol, while \( \nabla_\mu \) denotes the covariant derivative. The coupling constant \( \xi \) carries dimension of \((\text{mass})^{-\frac{1}{2}}\).

To analyse the spectrum of the model, we consider the linearised approximation of the full theory. This consists in the expansion of the metric field around the flat background as below:

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \]  

(2.6)

where \( h_{\mu\nu} \) is the perturbation associated to the graviton field.

As for the 2-form gauge potential, it is more convenient to parametrise it in terms of its dual, \( B_\mu = \frac{1}{3!} \epsilon_{\mu\nu\lambda} B^{\nu\lambda} \). Doing so, the gauge transformation for \( B_{\mu\nu} \) can be rephrased as

\[ \delta B_\lambda = \epsilon_\lambda^{\mu\nu} \nabla_\mu \chi_\nu, \]  

(2.7)
where $\chi_\mu$ is the 3-vector gauge parameter.

In order to invert the wave operator that mixes $h_{\mu\nu}$ and $B_\mu$ in the bilinear piece of the action, we fix the De Donder gauge for $h_{\mu\nu}$,

$$ \mathcal{L}_{\text{Donder}} = \frac{1}{2\alpha} F_\mu F^\mu, $$

where

$$ F_\mu = \partial_\lambda \left( h^\lambda_\mu - \frac{1}{2} \delta^\lambda_\mu h \right). $$

On the other hand, the gauge transformations (2.7) for $B_\mu$ suggests the following gauge-fixing term for the Abelian symmetry associated to $B_{\mu\nu}$:

$$ \mathcal{L}_{\text{gf}} = \beta \left( \epsilon^{\mu\nu\lambda} \nabla_\nu B_\lambda \right)^2. $$

With these elements, the bilinear piece of the action can be cast under the form:

$$ \mathcal{L} = \frac{1}{2} \sum_{\alpha\beta} \phi^\alpha \mathcal{O}_{\alpha\beta} \phi^\beta, $$

where $\phi = (h_{\mu\nu}, B_\mu)$ and $\mathcal{O}$ is the differential wave operator to be inverted to give us the graviton and the gauge field propagators. This operator can be decomposed in four sectors, namely,

$$ \mathcal{O} = \begin{pmatrix} A_{\mu\nu,\kappa\lambda} & B_{\mu\nu,\kappa} \\ C_{\mu,\kappa\lambda} & D_{\mu,\nu} \end{pmatrix}, $$

where

$$ A_{\mu\nu,\kappa\lambda} = \frac{\Box}{2} P_{\mu\nu,\kappa\lambda}^{(2)} - \frac{\Box}{2\alpha} P_{\mu\nu,\kappa\lambda}^{(1)} + \frac{(\alpha + 1)}{2\alpha} P_{\mu\nu,\kappa\lambda}^{(0-s)} - \frac{\Box}{4\alpha} P_{\mu\nu,\kappa\lambda}^{(0-w)} $$

$$ + \frac{\sqrt{2}}{4\alpha} \left( P_{\mu\nu,\kappa\lambda}^{(0-sw)} + P_{\mu\nu,\kappa\lambda}^{(0-ws)} \right), $$

$$ B_{\mu\nu,\kappa} = -2\kappa \xi \Box \theta_{\mu\nu} \partial_\kappa, $$

$$ C_{\mu,\kappa\lambda} = 2\kappa \xi \Box \partial_\mu \theta_{\kappa\lambda}, $$

$$ D_{\mu,\nu} = -2 \Box (\beta \theta_{\mu\nu} + \omega_{\mu\nu}), $$

where the $P$'s are the Barnes-Rivers projector operators for symmetric second-rank tensors in D=3 (see Ref. [7] for details), while $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are respectively the transverse and longitudinal operators for vectors.
The propagators are given by
\[
\langle 0 | T [\Phi_\alpha (x) \Phi_\beta (y)] | 0 \rangle = i (\mathcal{O}^{-1})_{\alpha\beta} \delta^3 (x - y),
\]
(2.14)
where wave operator is inverted by using of the multiplicative relations satisfied by the projectors [7]. In so doing, the propagators in momentum space read as follows:
\[
\langle hh \rangle = \frac{2i}{p^2} \left\{ -P^{(2)} + \alpha P^{(1)} + \frac{2 [p^2 - m^2 (\alpha + 1)] P^{(0-w)}}{(p^2 - m^2)} \right. \\
\left. - \frac{m^2}{(p^2 - m^2)} \left[ P^{(0-s)} + \sqrt{2} (P^{(0-sw)} + P^{(0-vs)}) \right] \right\},
\]
\[
\langle h_{\mu\nu} B_\kappa \rangle = \frac{1}{4\kappa\xi p^2 (p^2 - m^2)} (\theta_{\mu\nu} + 2\omega_{\mu\nu}) p_\kappa,
\]
(2.15)
\[
\langle B_\mu B_\nu \rangle = \frac{i}{2\beta p^2} \theta_{\mu\nu} - \frac{im^2}{2p^2 (p^2 - m^2)} \omega_{\mu\nu},
\]
where \( m^2 = \frac{1}{8\kappa^2 \xi^2} \) and we have suppressed the indices in \( h_{\mu\nu} \) and in the \( P_{\mu\nu,\kappa\lambda} \) projectors.

The above propagators display poles at \( p^2 = 0 \) and \( p^2 = m^2 > 0 \), so that tachyons do not show up. The next step is a investigation of the possibility that negative-norm state might be present.

A necessary criterium for unitarity at tree-level is analysed by saturating the propagator with external currents \( J^{\mu\nu} \) and \( J^\mu \) (for the graviton and the gauge field, respectively), compatible with the gauge symmetries of the Lagrangian. Ghosts are absent in the model if the imaginary part of the residue of the current-current transition amplitude taken at the propagator poles is non-negative. For the graviton, this amplitude in momentum space is written as
\[
\mathcal{A} \equiv J^{*\mu\nu} (p) \langle T [h_{\mu\nu} (-p) h_{\kappa\lambda} (p)] J^{\kappa\lambda} (p),
\]
(2.16)
where, by virtue of the transversality of \( J^{\mu\nu} (p) \), only the spin projectors \( P^{(2)} \) and \( P^{(0-s)} \) survive. This fact enable us to write the imaginary part of the residue of the transition amplitude at the pole \( p^2 = 0 \) as
\[
Im (\mathcal{R}es \mathcal{A}) = 2 \left( |J^{\mu\nu}|^2 - |J^\nu|^2 \right).
\]
(2.17)
Now, defining the following set of independent vectors in momentum space,
\[ p^\mu \equiv (p^0, \mathbf{p}), \quad \bar{p}^\mu \equiv (p^0, -\mathbf{p}), \quad \epsilon^\mu \equiv (0, \epsilon), \]
(2.18)
that satisfy the conditions:
\[
\bar{p} \cdot p = (p^0)^2 + (\mathbf{p})^2 > 0,
\]
(2.19)
\[
p \cdot \epsilon = \bar{p} \cdot \epsilon = 0,
\]
\[
\epsilon^\mu \epsilon_\mu = -1,
\]
the symmetric tensor current \( J^{\mu\nu}(p) \) can be decomposed according to:
\[
J^{\mu\nu}(p) = a(p) p^\mu p^\nu + b(p) p(\mu) \bar{p}(\nu) + c(p) p(\mu) \epsilon(\nu) +
\]
\[
d(p) \bar{p}(\mu) \bar{p}(\nu) + e(p) \bar{p}(\mu) \epsilon(\nu) + f(p) \epsilon(\mu) \epsilon(\nu).
\]
(2.20)
Substituting (2.20) into (2.17), and making use the relations (2.18) and (2.19), we find that \( I m(\text{Res}\, A) = 0 \), showing that the massless pole does not propagate. For the pole at \( p^2 = m^2 \), one obtains
\[
I m(\text{Res}\, A) = |J^\mu_\mu|^2,
\]
(2.21)
that is always positive-definite. From this result, we find only one on-shell degree of freedom for the massive graviton.

For the vector field, the transition amplitude is given by
\[
A = J^{*\mu}(p) \langle T [B_\mu (-p) B_\nu (p)] \rangle J^{\nu}(p).
\]
(2.22)
Expanding the current \( J^\mu(p) \) with respect to the basis (2.18),
\[
J^\mu(p) = a(p) p^\mu + b(p) \bar{p}^\mu + c(p) \epsilon^\mu,
\]
(2.23)
and by making use of its conservation law,
\[
\epsilon^{\mu\nu\lambda} p_\nu J_\lambda = 0,
\]
(2.24)
we can show that the massless pole is again non-dynamical, while the residue of the amplitude at the massive pole give us:
\[
\frac{1}{2} |a|^2 m^2,
\]
(2.25)
which ensures one physical degree of freedom. Therefore, the non-minimal coupling of the K-R field to gravity results in a appearance of a dynamical massive pole in the longitudinal sector of $B_{\mu\nu}$.

We now turn into the calculation of the 1-loop mass correction for the gauge field propagator. The relevant vertices to our discussion are shown in figure 1, while all possible self-energy corrections are depicted in figure 2. The vertices of the figures 1.a and 1.b come from the Lagrangians (2.3) and (2.5) by expanding the metric up to order $\kappa$, while the one of figure 1.c is obtained from (2.3) up to order $\kappa^2$.

![Figure 1](image1)

![Figure 2](image2)

From the Feynman rules displayed in figure 1, the contribution coming from the loop integrals can be evaluated. Due to algebraic complexity of the vertices and propagators, the explicit evaluation of these diagrams only was possible by means of the software FORM. Since we are concerned with the contribution to the vector field mass, and we checked that a Chern-Simons-like term is not generated, we can calculate the diagrams by setting all momenta appearing on the external legs to zero. One therefore, obtains the following contribution to the 1-loop shift in the gauge field mass:

$$\Delta m^2\big|_{1-loop} = \frac{3\pi^2}{16\sqrt{2}\kappa \xi^3}.$$  \hspace{1cm} (2.26)
3 The Non-Riemannian Case

Now, following the same procedure as the one in the previous section, we propose to extend our analysis to the case where torsion is non-vanishing. In three-dimensional space-time, the torsion field can be covariantly split according to its SO(1,2) irreducible components:

\[ T_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda}\varphi + \frac{1}{2}(\eta_{\nu\lambda}t_\mu - \eta_{\mu\lambda}t_\nu) + \varepsilon_{\mu\nu\alpha}X^{\alpha}\lambda, \quad (3.27) \]

namely, a trace part \( t_\mu = T^\nu_{\mu\nu} \), a totally antisymmetric part \( \varphi = \frac{1}{3!}\varepsilon^{\mu\nu\lambda}T_{\mu\nu\lambda} \) and a traceless rank-2 symmetric tensor \( X_{\mu\nu} \).

A possible gauge-invariant action may be so chosen that there is no coupling at all between the K-R field and torsion, i.e., the former “neither yields nor feels torsion”. Thus, let us consider the action

\[ S = \int d^3x \sqrt{g} \left( \frac{1}{\kappa^2} R + \frac{1}{6} G_{\mu\nu\lambda}G^{\mu\nu\lambda} + 2\xi \sqrt{g} \tilde{R} \tilde{\nabla}_\mu B^\mu \right), \quad (3.28) \]

where the tilde on \( \nabla_\mu \) means that we are considering only the Riemannian part of the covariant derivative. Varying the total action (3.28) with respect to the independent fields, we get the following result for the trace part of the torsion:

\[ t_\mu = -\frac{4\kappa^2\xi \tilde{\nabla}_\mu B^\mu}{1 + 2\kappa^2\xi \tilde{\nabla}_\mu B^\mu}, \quad (3.29) \]

while the variations with respect to the fields \( \varphi \) and \( X_{\mu\nu} \) give us

\[ \begin{cases} 
\varphi = 0, \\
X_{\mu\nu} = 0. 
\end{cases} \quad (3.30) \]

The torsion irreducible components act as mere auxiliary fields and one can eliminate them by means of their respective algebraic field equations. Therefore, substituting this results back into the action, one obtain a Lagrangian expressed exclusively in terms of \( B_\mu \) and \( h_{\mu\nu} \):

\[ \mathcal{L} = \frac{1}{\kappa^2} \sqrt{g} \tilde{R} + \frac{1}{6} \sqrt{g} G_{\mu\nu\lambda}G^{\mu\nu\lambda} + 2\xi \sqrt{g} \tilde{R} \tilde{\nabla}_\mu B^\mu \\
+ 8\kappa^2\xi^2 \sqrt{g} \frac{\tilde{\nabla}_\lambda \tilde{\nabla}_\mu B^\mu \tilde{\nabla}^\lambda \tilde{\nabla}_\nu B^\nu}{1 + 2\kappa^2\xi \tilde{\nabla}_\mu B^\mu}. \quad (3.31) \]
From the bilinear sector of the Lagrangian above supplemented by the
gauge fixing terms (2.8) and (2.10), we obtain the following propagators:

\[ \langle hh \rangle = \frac{2i}{p^2} \left\{ -P^{(2)} + \alpha P^{(1)} + 2 \left[ (\alpha + 1) + \frac{p^2}{m^2} \right] P^{(0-w)} 
+ \left[ 1 + \frac{p^2}{m^2} \right] \left[ P^{(0-s)} + \sqrt{2} \left( P^{(0-sw)} + P^{(0-ws)} \right) \right] \right\}, \quad (3.32) \]

\[ \langle B_{\mu}B_{\nu} \rangle = \frac{i}{2p^2} \left( \frac{1}{\beta} \theta_{\mu\nu} + \omega_{\mu\nu} \right). \]

It is remarkable to notice now that the poles are located at \( p^2 = 0 \),
contrary to the previous case, where torsion was not present. It is just
the last term of the action (3.31), the responsible for the suppression of the
massive pole. The mixing between \( h_{\mu\nu} \) and \( B_{\mu} \) arising from the mentioned
term appears with the right coefficient that eliminates the massive pole.

Proceeding analogously to what we did in the previous Section, the mass-
less poles are shown to be non-dynamical for both fields. From this result,
one concludes that the coupling of the K-R field to gravity with non-vanishing
torsion may result in a theory without on-shell degrees of freedom at tree-
level as long as the action (3.31) is concerned. Moreover, by computing the
Feynman rules for this action and calculating the 1-loop self-energy graphs,
we checked that no mass is dynamically generated by radiative corrections.

4 Concluding Remarks

As a final result, we can state that the coupling of the 3D K-R field with
Einstein-Hilbert gravity excites a dynamical mode of the rank-2 gauge po-
tential, whenever torsion is not considered. The inclusion of torsion as a
non-dynamical field changes this result, as we checked in Section 3. How-
ever, had we considered non-minimal couplings with higher powers in the
curvature, the situation would be completely changed, as torsion would be-
come dynamical and its mixing with the graviton would for sure trigger
a dynamical mass for the gauge field on the basis of the results worked out
in Section 2.
To conclude, we would like to mention that, if gravity were described by a pure Chern-Simons term, even in the torsionless case the K-R field would remain non-dynamical on-shell, since no mass would be generated neither at the classical nor at the 1-loop level.

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