Some problems on the boundary of fractal geometry and additive combinatorics

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Abstract

This paper is an exposition, with some new applications, of our results from [5, 6] on the growth of entropy of convolutions. We explain the main result on \( \mathbb{R} \), and derive, via a linearization argument, an analogous result for the action of the affine group on \( \mathbb{R} \). We also develop versions of the results for entropy dimension and Hausdorff dimension. The method is applied to two problems on the border of fractal geometry and additive combinatorics. First, we consider attractors \( X \) of compact families \( \Phi \) of similarities of \( \mathbb{R} \). We conjecture that if \( \Phi \) is uncountable and \( X \) is not a singleton (equivalently, \( \Phi \) is not contained in a 1-parameter semigroup) then \( \dim X = 1 \). We show that this would follow from the classical overlaps conjecture for self-similar sets, and unconditionally we show that if \( X \) is not a point and \( \dim \Phi > 0 \) then \( \dim X = 1 \). Second, we study a problem due to Shmerkin and Keleti, who have asked how small a set \( \emptyset \neq Y \subseteq \mathbb{R} \) can be if at every point it contains a scaled copy of the middle-third Cantor set \( K \). Such a set must have dimension at least \( \dim K \) and we show that its dimension is at least \( \dim K + \delta \) for some constant \( \delta > 0 \).

1 Introduction

1.1 Attractors of infinite compact families of similarities

Let \( G \) denote the group of similarities (equivalently, affine maps) of the line and \( S \subseteq G \) the semigroup of contracting similarities. Given a family \( \Phi \subseteq S \), a set \( X \subseteq \mathbb{R} \) is called the attractor of \( \Phi \) if it is a compact non-empty set satisfying

\[
X = \bigcup_{\varphi \in \Phi} \varphi(X).
\]

We then say that \( \Phi \) generates \( X \).

Hutchinson’s theorem \([1]\) tells us that when \( \Phi \) is finite, an attractor exists, and is unique. In this case \( X \) is said to be self-similar. These sets represent the simplest “fractal” sets, and their small-scale geometry has been extensively studied over the years. It is also natural to ask what happens if we allow \( \Phi \) to be infinite. This opens the door to various “pathologies”, including the possibility that no attractor exists at all (i.e. the only compact set satisfying \( \emptyset \) is the empty set), and there is an extensive literature devoted to the case...
that $\Phi$ is countable, primarily concerning the case when $\Phi$ is unbounded (see e.g. [14]). But there is another generalization, perhaps closer in spirit to the finite case, in which one takes $\Phi \subseteq S$ to be compact. As in the finite case, existence and uniqueness of the attractor is proved by showing that the map $Y \mapsto \bigcup_{\varphi \in \Phi} \varphi(Y)$ is a contraction on the space of compact subsets of $\mathbb{R}$, endowed with the Hausdorff metric, and hence has a unique fixed point.

We are interested in the dimension of the attractors of compact families $\Phi$. For finite $\Phi$ this problem has a long history, which we shall not recount here, but we note that even in this case our understanding is still incomplete (see below and [5]). However, there is some reason to believe that when $\Phi$ is compact and uncountable the situation may be, in a sense, simpler.

**Conjecture 1.** If $X \subseteq \mathbb{R}$ is the attractor of an uncountable compact family of contracting similarities, then either $X$ is a single point, or $\dim X = 1$.

Some evidence for the conjecture is the fact that it is implied by another well-known conjecture about the dimension of self-similar sets. Recall that given a finite set $\Phi \subseteq S$, the similarity dimension $s = s(\Phi)$ of $\Phi$ (and, by convention, of its attractor) is the solution to

$$\sum_{\varphi \in \Phi} ||\varphi||^s = 1,$$

where $||\varphi||$ denotes the unsigned contraction, or optimal Lipschitz constant, of $\varphi$. The similarity dimension is an upper bound on the dimension of the attractor $X$, and we always have $\dim X \leq 1$ (because $X \subseteq \mathbb{R}$) so there is the upper bound

$$\dim X \leq \min\{1, s(\Phi)\}. \quad (2)$$

The inequality can be strict, but it is believed that this can happen only for algebraic reasons. Specifically, let us say that $\Phi \subseteq G$ is free if the elements of $\Phi$ freely generate a free semigroup, that is, if $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \in \Phi$ then $\varphi_1 \cdot \ldots \cdot \varphi_m = \psi_1 \cdot \ldots \cdot \psi_n$ implies $m = n$ and $\varphi_i = \psi_i$ for all $i$.

**Conjecture 2** (e.g. [15]). If $\Phi \subseteq S$ is finite and free then its attractor $X$ has dimension $\dim X = \min\{1, s(\Phi)\}$.

**Theorem 1.** Conjecture 2 implies Conjecture 1.

The proof of this implication relies on algebraic considerations, namely, that for any “large” enough $\Phi \subseteq G$ there are infinite free subsets of $\Phi$ (or of $\Phi^k$ for some $k$). This result is in the same spirit as the classical Tits alternative, which asserts that if a subgroup of a linear group is not virtually solvable, then it contains free subgroups. Of course, we are working in the affine group of the

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1If we work in a complete metric space and general contractions, taking the topology of uniform convergence of maps on compact sets, then existence and uniqueness of the attractor is still true if we assume that all $\varphi \in \Phi$ contract by at least some $0 < r < 1$. Without uniformity, existence can fail already for a single map.
line, which is itself solvable, and so cannot contain free subgroups at all; but this does not preclude the existence of free semigroups, and this is what we need. Once we have a large free semigroup, Conjecture 2 ensures that the attractor has large dimension. For a precise statement and proof see Section 7.

We are not able to prove Conjecture 1, but we give the following result in its direction, where uncountability is replaced by positive dimension:

Theorem 2. Let $X \subseteq \mathbb{R}$ be the attractor of a compact family $\Phi \subseteq S$. If $\dim \Phi > 0$, then either $X$ is a point, or $\dim X = 1$.

We say a little about the proof later in this introduction, but before moving on, let us mention a related and intriguing variant of the conjecture, where uncountability is downgraded to ordinary infinity:

Problem 1. If $\Phi \subseteq S$ is an infinite compact family of similarities, and its attractor $X$ is not a single point, is $\dim X = 1$?

Of course if this were true, it would imply Conjecture 1. But this problem has the advantage that one can restrict it to algebraic $\Phi$, that is, families $\Phi \subseteq S$ all of whose coordinates in the standard parametrization lie in a common algebraic field. Note that Conjecture 2 was shown in [5] to hold under a similar assumption, and the methods of that paper reduce the problem above to one about the random-walk entropy of families of similarities of bounded contraction in a given algebraic group. It seems possible that either a proof or counterexample can be found in this setup.

There are other strengthenings of Conjecture 1, for example, instead of $\dim X = 1$ one may ask if $X$ has positive Lebesgue measure, or even non-empty interior. These problems are quite natural, but seem out of reach of current methods.

1.2 Sets containing many copies of the Cantor set

Our second subject concerns the following problem. Let $K$ denote the middle-1/3 Cantor set, translated so that it is symmetric around the origin.

Problem 2. Given $s > 0$, how large must a set $Y$ be if there is a compact $C \subseteq \mathbb{R}$ with $\dim C = s$ and $Y$ contains a scaled copy of $K$ centered at each $c \in C$? In particular, for $C = \mathbb{R}$, must we have $\dim Y = \min \{1, \dim C + \dim K\}$?

I first learned of this problem from P. Shmerkin and T. Keleti. It is related to problems on maximal operators on fractal sets, studied by Laba and Pramanik [12]. It is also a relative of the Furstenberg “$\alpha$-set” problem.

Theorem 3. Let $C \subseteq \mathbb{R}$ be compact and of positive dimension. If $Y \subseteq \mathbb{R}$ contains a scaled copy of $K$ centered at $c$ for every $c \in C$, then $\dim Y > \dim K + \delta$ where $\delta > 0$ depends only on $\dim C$ and $\dim K$.

There is nothing special about the middle-third Cantor set; our argument works when $K$ is any porous set. We note that A. Máthé recently observed
that, using a projection theorem due to Bourgain, one can deduce that \( \dim Y \geq \dim C/2 \) \[11\] Corollary 3.5. This gives better bounds than the theorem above in some cases, though never when \( \dim K > 1/2 \). Also worth noting is that for general sets \( K \), the last part of the problem (for the case \( C = \mathbb{R} \)) has a negative answer, as shown by recent examples by A. Máthé \[11\] Theorem 3.2. But for self-similar sets such as the middle-1/3 Cantor set the question remains open and little seems to be known. For a discussion of the history and related results see \[11\].

1.3 The role of additive combinatorics

Both of the problems above involve analysis of “product” sets, where the product operation is the action of \( G \) on \( \mathbb{R} \). Specifically, let \( \varphi \cdot x \) denote the image of \( x \in \mathbb{R} \) under \( \varphi \in G \), and for \( X \subseteq \mathbb{R} \) and \( \Phi \subseteq G \) denote

\[ \Phi \cdot X = \{ \varphi \cdot x : x \in X, \varphi \in \Phi \} \]

A large part of this paper is devoted to studying how the “size” of \( \Phi \cdot X \) is related to the “sizes” and structure of \( \Phi \) and \( X \). This subject belongs to the field of additive combinatorics, but we will not go into its history here. Rather, in the coming paragraphs we outline, in an informal way, the main ideas that we will encounter in the formal development later on. We emphasize that the discussion below is heuristic and contains several half-truths, which will be corrected later.

The leading principle is that \( \Phi \cdot X \) should be substantially larger than \( X \), unless there is some compatibility between the structure of \( X \) and \( \Phi \). To explain the phenomenon we begin with the analogous problem for sums of sets in the line, and choose Hausdorff dimension as our measure of size. Thus, suppose that \( \emptyset \neq X, Y \subseteq \mathbb{R} \) and consider their sum

\[ X + Y = \{ x + y : x \in X, y \in Y \}. \]

It is clear that \( \dim(Y + X) \geq \dim X \), since \( Y + X \) contains a translate of \( X \). Equality of the dimensions can occur in two trivial ways: (a) if the dimension of \( X \) is maximal (that is, \( \dim X = 1 \)), or (b) if the dimension of \( Y \) is minimal (that is, \( \dim Y = 0 \)). Besides the trivial cases there are many other non-trivial examples in which \( \dim(Y + X) = \dim X \) occurs, see e.g. \[3\]. However, when this happens, it turns out that the lack of dimension growth can be explained by the approximate occurrence of (a) and (b) for “typical” small “pieces” of the sets. To make this a little more precise, define a scale-\( r \) piece of \( X \) to be a set of the form \( X \cap B_r(x) \) for some \( x \in X \). The statement is then that, if \( \dim(Y + X) = \dim X \), then, roughly speaking, for typical scales \( 0 < r < 1 \), either (a) holds approximately for typical scale-\( r \) pieces of \( X \), or (b) holds approximately for typical scale-\( r \) pieces of \( Y \). The precise version of this, which is stated for measures rather than sets, was proved in \[4\]; we state a variant of it in Theorem\[5\] below, and use it as a black box. We remark that closely related results appear in the work of Bourgain, e.g. \[1\].
Returning now to the action of $G$ on $\mathbb{R}$, suppose that $\emptyset \neq X \subseteq \mathbb{R}$ and $\emptyset \neq \Phi \subseteq G$. Then we again always have $\dim(\Phi_.X) \geq \dim X$, and equality can be explained by the same global reasons (a) and (b) above. But there is also a third possibility, namely, (c) that $X$ is a point and $\Phi$ is contained in the group of similarities fixing that point. As with sumsets, $\dim(\Phi_.X) = \dim X$ can also occur in other ways, but it again turns out that if this happens then the trivial explanations still apply to typical “pieces” of the sets; thus at typical scales $0 < r < 1$, either (a) holds approximately for typical scale-$r$ pieces of $X$, or (b) holds approximately for typical scale-$r$ pieces of $\Phi$, or (c) holds approximately for typical pairs of scale-$r$ pieces of $X$ and $\Phi$.

But possibility (c) does not in reality occur unless $X$ is extremely degenerate. For suppose in the situation above that (c) holds at some scale $r$. Then for typical pairs $\varphi \in \Phi$ and $x \in X$ we would have that $\Phi \cap B_r(\varphi)$ is approximately contained in the stabilizer of $x$. But the $G$-stabilizers of different $x \in X$ are transverse (as submanifolds of $G$) so, assuming $X$ is infinite (or otherwise large so as to ensure that no single point in it is “typical”), by ranging over the possible values of $x$, we would find that $\Phi \cap B_r(\varphi)$ is approximately contained in the intersection of many transverse manifolds, hence is approximately a point; and we are in case (b) again. In summary, if $X$ is large enough, case (c) can be deleted from the list, leaving only (a) and (b).

We shall prove the statements in the last two paragraph (in their correct, measure formulation) in Section 4. But we note here that they are derived from the aforementioned result about sumsets, using a linearization argument. To give some idea of how this works, let $f : G \times \mathbb{R} \to \mathbb{R}$ denote the action map $f(\varphi,x) = \varphi.x$, so that $\Phi_.X = f(\Phi \times X)$. Consider small pieces $X' = X \cap B_r(x)$ and $\Phi' = \Phi \cap B_r(\varphi)$ of $X, \Phi$, respectively. Then $\Phi'.X' = f(\Phi' \times X')$ is a subset of $\Phi_.X$, and one can show that if we assume that $\dim \Phi_.X = \dim X$, then typical choices of $X', \Phi'$ also satisfy $\dim \Phi'.X' = \dim X'$, approximately. But, since $f$ is differentiable, for small $r$, the map $f$ is very close to linear on the small ball $B_r(\varphi) \times B_r(x)$, and hence $f(\Phi' \times X')$ is very close to the sumset $(\frac{\partial f}{\partial \varphi} \Phi')X' + (\frac{\partial f}{\partial x} \Phi')X'$ (here the subscript is the point at which the derivative is evaluated, and $\frac{\partial f}{\partial \varphi} \Phi' \times x$ is a $1 \times 2$-matrix, so the sum is a sum of sets in the line). To this sum we can apply our results on sumsets. We remark that the first term $\frac{\partial f}{\partial \varphi} \Phi' \times x$ in the sum may be substantially smaller than $\Phi'$, but if this is the case it is because $\Phi'$ is essentially contained in the kernel of $\frac{\partial f}{\partial \varphi} \Phi'$, and this corresponds to the case (c) above.

All this can be used to prove that, under mild assumptions, $\Phi_.X$ is substantially larger than $X$. To be concrete, for $0 < c < 1$ let us say that $X$ is $c$-porous if every interval $I \subseteq \mathbb{R}$ contains a sub-interval $J \subseteq I \setminus X$ of length $|J| = c|I|$. We then have

**Theorem 4.** For any $0 < c < 1$ there exists a $\delta = \delta(c) > 0$ such that for any $c$-porous set $X \subseteq \mathbb{R}$ of positive dimension, and any $\Phi \subseteq G$ with $\dim \Phi > c$, we have

$$\dim \Phi_.X \geq \dim X + \delta.$$
Unlike the previous discussion this theorem is true as stated, see Section 6.5. Nevertheless let us explain how it follows from our heuristic discussion. Suppose that \( \dim Y \cdot X = \dim X \); then for typical scales \( r \), either (a) applies to typical scale-\( r \) pieces of \( X \), or (b) applies to typical scale-\( r \) pieces of \( \Phi \). Suppose that \( X \) is porous; then no scale-\( r \) piece \( X \cap B_r(x) \) of \( X \) can be close to a set of full dimension, since porosity means that it contains a hole proportional in size to \( r \), and at every smaller scale. This rules out (a), so the remaining possibility is that \( Y \cap B_r(y) \) is approximately zero dimensional for typical \( y \in Y \). But one can show that if this is true for typical pieces of \( Y \) at typical scales, then it is true globally, i.e. \( \dim Y = 0 \), as desired.

To conclude this section let us explain how Theorem 4 is related to Theorems 2 and 3. In the first of these, the assumption is that \( X = \Phi \cdot X \) and \( \dim \Phi > 0 \). If \( \dim X < 1 \) implied that \( X \) were porous, then the theorem above would imply \( \dim X > \dim X + \delta \), which is the desired contradiction. In our setting \( X \) need not be porous (and a-posteriori cannot be), but by working with suitable measures on \( X \) we will be able to apply an analog of Theorem 4, which gives the result.

To see the connection with Theorem 3, suppose \( Y \subseteq \mathbb{R} \) contains a scaled copy of \( K \) centered at every point in a set \( C \subseteq \mathbb{R} \). Assume that \( \dim C > 0 \). Let \( \Phi \) denote the set of similarities \( \varphi_{r,c}(x) = rx + c \) for which \( c \in C \) and \( \varphi(K) \subseteq Y \), so that \( \Phi \cdot K \subseteq Y \). Since \( Y \) is closed, also \( \Phi \) is closed, and by assumption for every \( c \in C \) there exists at least one \( 0 < r < 1 \) such that \( \varphi_{r,c} \in \Phi \). The map \( \varphi_{r,c} \mapsto c \) is a Lipschitz map taking \( \Phi \) onto \( C \), so \( \dim \Phi \geq \dim C > 0 \). Finally, \( K \) is porous, so by Theorem 4 \( \dim Y \geq \dim (\Phi \cdot K) > \dim K + \delta \) for some \( \delta > 0 \), as claimed.

### 1.4 Organization of the paper

In Section 2 we set up some notation, defining dyadic partitions on \( \mathbb{R} \) and \( G \), and discussing Shannon entropy and its properties. In Section 3 we define component measures and their distribution, and formulate the theorem on dimension growth of convolutions of measures on \( \mathbb{R} \). In Section 4 we give the linearization argument which leads to the analogous growth theorem for convolutions \( \nu \cdot \mu \) of \( \nu \in \mathcal{P}(G) \) and \( \mu \in \mathcal{P}(\mathbb{R}) \). In Section 5 we prove Theorem 8. In Section 6 we develop results for the Hausdorff dimension of convolutions, proving Theorem 4 (and in so doing, completing the proof of Theorem 3). In Section 7 we prove the implication between Conjecture 2 and Conjecture 1. Finally, in Section 8 we discuss another variant of Conjecture 1 in the non-linear setting.

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2 Measures, dyadic partitions, components and entropy

We begin with some background on entropy which will be used in our analysis of convolutions.

2.1 Probability measures

For a measurable space $X$ we write $P(X)$ for the space of probability measures on $X$. We always take the Borel structure when the underlying space is metric. Given a measurable map $f : X \to Y$ between measurable spaces and $\mu \in P(X)$, let $f_\mu \in P(Y)$ denote the push-forward measure, $\nu = \mu \circ f^{-1}$. For a probability measure $\mu$ and set $A$ with $\mu(A) > 0$ we write

$$\mu_A = \frac{1}{\mu(A)} \mu|_A$$

for the conditional measure on $A$.

2.2 Dyadic partitions

The level-$n$ dyadic partitions $D_n$ of $\mathbb{R}$ is given by

$$D_n = \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) : k \in \mathbb{Z} \right\},$$

and the level-$n$ dyadic partition of $\mathbb{R}^d$ by

$$D_n^d = \{ I_1 \times \ldots \times I_d : I_i \in D_n \}.$$  

The superscript is often suppressed.

We parametrize $G$ as $\mathbb{R}^2$, identifying $(s, t) \in \mathbb{R}^2$ with $x \mapsto e^s x + t$, and define a metric on $G$ by pulling back the Euclidean metric on $\mathbb{R}^2$. The importance of this choice of parametrization is that if $\varphi, \psi \in G$ and $d(\varphi, \psi) < C$ then the translation parts of $\varphi, \psi$ differ by an additive constant $C'$, and their contractions by a multiplicative constant $C''$, with $C', C''$ depending only on $C$. Note that, locally, this metric is equivalent (in fact diffeomorphic) to any Riemmanian metric on $G$ so the notion of dimension in $G$ is not affected by this choice of parametrization.

We equip $G$ with the dyadic partition $D_n^G = D_n^2$ induced from $\mathbb{R}^2$.

When $t$ is not an integer, we write $D_t = D_{[t]}$ and $D_t^G = D_{[t]}^G$.

2.3 Entropy

The Shannon entropy of a probability measure $\mu$ with respect to a finite or countable partition $\mathcal{E}$ is defined by

$$H(\mu, \mathcal{E}) = - \sum_{E \in \mathcal{E}} \mu(E) \log \mu(E),$$

\[\text{We use another parametrizations in Section 7 but only there.}\]
The logarithm is in base 2 and by convention \( 0 \log 0 = 0 \). This quantity is non-negative and we always have
\[
H(\mu, \mathcal{E}) \leq \log \#\{ E \in \mathcal{E} : \mu(E) > 0 \}.
\]
(4)

The conditional entropy with respect to another countable partition \( \mathcal{F} \) is
\[
H(\mu, \mathcal{E}|\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu(F) \cdot H(\mu_F, \mathcal{E}),
\]
(5)

where \( \mu_F \) is the conditional measure on \( \mathcal{F} \) (see (3)), which is undefined when \( \mu(F) = 0 \) but in that case its weight in the sum is zero and it is ignored. Writing \( \mathcal{E} \cap \mathcal{F} = \{ E \cap F : E \in \mathcal{E}, \ F \in \mathcal{F} \} \) for the smallest common refinement of \( \mathcal{E}, \mathcal{F} \), it is a basic identity that
\[
H(\mu, \mathcal{E} \cap \mathcal{F}) = H(\mu, \mathcal{E}|\mathcal{F}) + H(\mu, \mathcal{F}).
\]

Note that when \( \mathcal{E} \) refines \( \mathcal{F} \) (i.e. when every atom of \( \mathcal{E} \) is a subset of an atom of \( \mathcal{F} \)) we have
\[
H(\mu, \mathcal{E}|\mathcal{F}) = H(\mu, \mathcal{E}) - H(\mu, F).
\]

In general, we always have
\[
H(\mu, \mathcal{E}|\mathcal{F}) \leq H(\mu, \mathcal{E}),
\]
hence
\[
H(\mu, \mathcal{E} \cap \mathcal{F}) \leq H(\mu, \mathcal{E}) + H(\mu, \mathcal{F}).
\]

Entropy is concave, and almost convex: If with \( \mu_1, \mu_2 \) probability measures and \( \nu = \alpha \mu_1 + (1 - \alpha) \mu_2 \) for some \( 0 \leq \alpha \leq 1 \), then
\[
\alpha H(\mu_1, \mathcal{E}) + (1 - \alpha)H(\mu_2, \mathcal{E}) \leq H(\nu, \mathcal{E}) \leq \alpha H(\mu_1, \mathcal{E}) + (1 - \alpha)H(\mu_2, \mathcal{E}) + H(\alpha),
\]
where \( H(\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha) \). The same holds when all entropies above are conditional on a partition \( \mathcal{F} \). More generally if \( \mu = \mu^\omega \) is a random measure (\( \omega \) denoting the point in the sample space), then\footnote{We require that \( \omega \rightarrow \mu^\omega \in \mathcal{P}(X) \) be measurable in the sense that \( \omega \rightarrow \int f d\mu^\omega \) is measurable for all bounded measurable \( f : X \rightarrow \mathbb{R} \), and the expectation \( E(\mu) \) is understood the probability measure \( \nu \) determined by \( \nu(A) = E(\mu(A)) \) for all measurable \( A \), or equivalently, \( \int f d\nu = E(\int f d\mu) \) for bounded measurable \( f \).}
\[
H(\mathcal{E}(\mu), \mathcal{E}) \geq E(H(\mu, \mathcal{E})),
\]
and similarly for conditional entropies.

2.4 Translation, scaling and their effect on entropy

Define the translation map \( T_u : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
T_u(x) = x + u,
\]

\( \log \) and \( \log(x) \) denote the logarithm in base 2 and by convention \( 0 \log 0 = 0 \). This quantity is non-negative and we always have
\[
H(\mu, \mathcal{E}) \leq \log \#\{ E \in \mathcal{E} : \mu(E) > 0 \}.
\]
(4)

The conditional entropy with respect to another countable partition \( \mathcal{F} \) is
\[
H(\mu, \mathcal{E}|\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu(F) \cdot H(\mu_F, \mathcal{E}),
\]
(5)

where \( \mu_F \) is the conditional measure on \( \mathcal{F} \) (see (3)), which is undefined when \( \mu(F) = 0 \) but in that case its weight in the sum is zero and it is ignored. Writing \( \mathcal{E} \cap \mathcal{F} = \{ E \cap F : E \in \mathcal{E}, \ F \in \mathcal{F} \} \) for the smallest common refinement of \( \mathcal{E}, \mathcal{F} \), it is a basic identity that
\[
H(\mu, \mathcal{E} \cap \mathcal{F}) = H(\mu, \mathcal{E}|\mathcal{F}) + H(\mu, \mathcal{F}).
\]

Note that when \( \mathcal{E} \) refines \( \mathcal{F} \) (i.e. when every atom of \( \mathcal{E} \) is a subset of an atom of \( \mathcal{F} \)) we have
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In general, we always have
\[
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\]
hence
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H(\mu, \mathcal{E} \cap \mathcal{F}) \leq H(\mu, \mathcal{E}) + H(\mu, \mathcal{F}).
\]

Entropy is concave, and almost convex: If with \( \mu_1, \mu_2 \) probability measures and \( \nu = \alpha \mu_1 + (1 - \alpha) \mu_2 \) for some \( 0 \leq \alpha \leq 1 \), then
\[
\alpha H(\mu_1, \mathcal{E}) + (1 - \alpha)H(\mu_2, \mathcal{E}) \leq H(\nu, \mathcal{E}) \leq \alpha H(\mu_1, \mathcal{E}) + (1 - \alpha)H(\mu_2, \mathcal{E}) + H(\alpha),
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where \( H(\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha) \). The same holds when all entropies above are conditional on a partition \( \mathcal{F} \). More generally if \( \mu = \mu^\omega \) is a random measure (\( \omega \) denoting the point in the sample space), then\footnote{We require that \( \omega \rightarrow \mu^\omega \in \mathcal{P}(X) \) be measurable in the sense that \( \omega \rightarrow \int f d\mu^\omega \) is measurable for all bounded measurable \( f : X \rightarrow \mathbb{R} \), and the expectation \( E(\mu) \) is understood the probability measure \( \nu \) determined by \( \nu(A) = E(\mu(A)) \) for all measurable \( A \), or equivalently, \( \int f d\nu = E(\int f d\mu) \) for bounded measurable \( f \).}
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H(E(\mu), \mathcal{E}) \geq E(H(\mu, \mathcal{E})),
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and similarly for conditional entropies.

2.4 Translation, scaling and their effect on entropy

Define the translation map \( T_u : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
T_u(x) = x + u,
\]
and the scaling map $S_t : \mathbb{R} \to \mathbb{R}$ by

$$S_t x = 2^t x.$$ 

Note our choice of parametrization, and that $S_{s+t} = S_s S_t$.

It is clear that if $k \in \mathbb{Z}$ then

$$H(S_k \mu, D_{n-k}) = H(\mu, D_n),$$

because $S_k$ maps the atoms of $D_n$ to the atoms of $D_{n-k}$. When $t$ is not a power of 2 the same relation holds, but with an error term:

$$H(S_t \mu, D_{n-t}) = H(\mu, D_n) + O(1). \quad (6)$$

Translation affects entropy in a similar way: if $u = m/2^n$ for $m,n \in \mathbb{Z}$ then

$$H(T_u \mu, D_n) = H(\mu, D_n),$$

and for general $u \in \mathbb{R}$,

$$H(T_u \mu, D_n) = H(\mu, D_n) + O(1). \quad (7)$$

Combining all this we find that if $\varphi$ is a similarity and $\parallel \varphi \parallel$ is its unsigned contraction constant (it optimal Lipschitz constant), then

$$H(\varphi \mu, D_{n-\log ||\varphi||}) = H(\mu, D_n) + O(1). \quad (8)$$

If $\mu$ is supported on a set of diameter $O(1)$, then by $(6)$, $H(\mu, D_1) = O(1)$. It follows from the above that if $\mu$ is supported on a set of diameter $2^{-(n+c)}$. Then

$$H(\mu, D_n) = O_c(1), \quad (9)$$

and in particular, for $m > n$,

$$H(\mu, D_m | D_n) = H(\mu, D_m) - O_c(1). \quad (10)$$

Finally, although entropy is not quite continuous under small perturbations of the measure, it almost is. Specifically, let $\mu \in \mathcal{P}(\mathbb{R})$. If a function $f$ satisfies $c^{-1}d(x,y) \leq d(f(x), f(y)) \leq c d(x,y) \leq c$, then then

$$H(\varphi \mu, D_n) = H(\mu, D_n) + O(\log c). \quad (11)$$

and if $f,g : \mathbb{R} \to \mathbb{R}$ are $2^{-n}$-close to in the sup-distance (i.e. $|f(x) - g(x)| < 2^{-n}$ for all $x$), then

$$|H(f \mu, D_n) - H(g \mu, D_n)| = O(1). \quad (12)$$

### 3 Entropy growth for Euclidean convolutions

The convolution $\nu * \mu$ of $\nu, \mu \in \mathcal{P}(\mathbb{R})$ is the push-forward of $\nu \times \mu$ by the map $(x, y) \mapsto x + y$. In this section we state a result from [5] saying that convolution increases entropy, except when some special structure is present. The statement is in terms of the multi-scale structure of the measures, and we first develop the language necessary for describing it.
3.1 Component measures

For \( x \in \mathbb{R} \) let \( D_n(x) = D_n^d(x) \) denote the unique element of \( D_n^d \) containing it, and for a measure \( \mu \in \mathcal{P}(\mathbb{R}^d) \) define the level-\( n \) component of \( \mu \) at \( x \) to be the conditional measure on \( D_n(x) \):

\[
\mu_{x,n} = \frac{1}{\mu(D_n(x))} \mu|_{D_n(x)}.
\]

This is defined for \( \mu \)-a.e. \( x \).

We define components of a measure \( \nu \in \mathcal{P}(G) \) in the same way, using the dyadic partitions \( D^n \), so \( \nu_{g,n} = \frac{1}{\nu(D^n(g))} \nu|_{D^n(g)} \).

3.2 Random component measures

We often view \( \mu_{x,n} \) as a random variable, with \( n \) chosen uniformly within some specified range, and \( x \) chosen according to \( \mu \), independently of \( n \). This is the intention whenever \( \mu_{x,n} \) appears in an expression \( \mathbb{P}(\ldots) \) or \( \mathbb{E}(\ldots) \).

An equivalent way of generating \( \mu_{x,i} \) is to choose \( i \in \{0,\ldots,n\} \) uniformly, and independently choose \( I \in D_i \) with probability \( \mu(I) \). Then the random measure \( \mu_I \) has the same distribution as \( \mu_{x,i} \), and if we further then choose \( x \in I \) using the measure \( \mu_I \), then the distribution of \( \mu_{x,i} \) generated in this way agrees with the previous procedure.

For example, if \( \mathcal{U} \) is a set of measures then \( \mathbb{P}_{0 \leq i \leq n}(\mu_{x,i} \in \mathcal{U}) \) is the probability that \( \mu_{x,i} \in \mathcal{U} \) when \( i \in \{0,\ldots,n\} \) is chosen uniformly, and \( x \) is independently chosen according to \( \mu \).

Similarly, \( \mathbb{E}_{i=n}(H(\mu_{x,i},D_{i+m})) \) denotes the expected entropy of a component at level \( n \) (note that we took \( i = n \), so the level is deterministic), measured at scale \( n + m \), and by (3),

\[
H(\mu,D_{n+m}|D_n) = \mathbb{E}_{i=n}(H(\mu_{x,i},D_{n+m})).
\]

As another example, we have the trivial identity

\[
\mu = \mathbb{E}_{i=n}(\mu_{x,i}).
\]

We view components of measures on \( G \) as random variables in the same way as above and adopt the same notational conventions.

Our notation defines \( x \) and \( i \) implicitly as random variables. For example we could write

\[
\mathbb{P}_{0 \leq i \leq n}(H(\mu_{x,i},D_{i+1}) = 1 \text{ and } i \geq n_0)
\]

for the probability that a random component has full entropy at one scale finer, and the scale is at least \( n_0 \).

When several random components are involved, they are assumed to be chosen independently unless otherwise specified. Thus the distribution of \( \nu_{g,i} \times \mu_{x,i} \) is obtained by choosing \( i \) first and then choosing \( g \) and \( x \) independently according to \( \nu \) and \( \mu \), respectively. Note the resulting random measure has the same distribution as \( (\nu \times \mu)_{(g,x),i} \).
The distribution on components has the convenient property that it is almost invariant under repeated sampling, i.e. choosing components of components. More precisely, for a probability measure \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( m, n \in \mathbb{N} \), let \( P_n^\mu \) denote the distribution of components \( \mu_{x,i}, 0 \leq i \leq n \), as defined above; and let \( Q_{n,m}^\mu \) denote the distribution on components obtained by first choosing a random component \( \mu_{x,i}, 0 \leq i \leq n \), as above, and then, conditionally on \( \nu = \mu_{x,i} \), choosing a random component \( \nu_{y,j}, i \leq j \leq i + m \) in the usual way (note that \( \nu_{y,j} = \mu_{y,j} \) is indeed a component of \( \mu \)).

**Lemma 1.** Given \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( m, n \in \mathbb{N} \), the total variation distance between \( P_n^\mu \) and \( Q_{n,m}^\mu \) satisfies
\[
\| P_n^\mu - Q_{n,m}^\mu \| = O\left( \frac{m}{n} \right).
\]

In particular let \( A_i, B_i \subseteq \mathcal{P}([0,1]^d) \), write \( \alpha = \mathbb{P}_{0 \leq i \leq n}(\mu_{x,y} \in A_i) \), and suppose that \( \nu \in A_i \) implies \( \mathbb{P}_{i \leq j \leq i + m}(\nu_{x,j} \in B_j) \geq \beta \). Then
\[
\mathbb{P}_{0 \leq i \leq n}(\mu_{x,i} \in B_i) > \alpha \beta - O\left( \frac{m}{n} \right).
\]

These are essentially applications of the law of total probability, for details see [8].

### 3.3 Multiscale formulas for entropy

Let us call \( \frac{1}{n} H(\mu, D_n) \) the scale-\( n \) entropy of \( \mu \). A simple but very useful property of scale-\( n \) entropy of a measure is that when \( m \ll n \) it is roughly equal to the average of the scale-\( m \) entropies of its components, and for convolutions a related bound can be given. The proofs can be found in [5, Section 3.2].

**Lemma 2.** For compactly supported \( \mu \in \mathcal{P}(\mathbb{R}) \) or \( \mu \in \mathcal{P}(G) \), for every \( m, n \in \mathbb{N} \),
\[
\frac{1}{n} H(\mu, D_n) = \mathbb{E}_{1 \leq i \leq n} \left( \frac{1}{m} H(\mu_{x,i}, D_{i+m}) \right) + O\left( \frac{m}{n} \right).
\]

The error term depends only on the diameter of the support of \( \mu \).

For convolutions in \( \mathbb{R} \) we have a lower bound:

**Lemma 3.** For compactly supported \( \mu, \nu \in \mathcal{P}(\mathbb{R}) \), for every \( m, n \in \mathbb{N} \),
\[
\frac{1}{n} H(\nu \ast \mu, D_n) \geq \mathbb{E}_{1 \leq i \leq n} \left( \frac{1}{m} H(\nu_{y,i} \ast \mu_{x,i}, D_{i+m}) \right) - O\left( \frac{1}{m} + \frac{m}{n} \right).
\]

The error term depends only the diameter of the supports of \( \nu, \mu \).

In the expectations above, the random variables \( \mu_{x,i} \) and \( \nu_{y,i} \) are independent components of level \( i \).

Before we state the analogous formula for convolutions \( \nu, \mu \) where \( \nu \in \mathcal{P}(S) \) and \( \mu \in \mathcal{P}(\mathbb{R}) \), we first explain how contraction enters the formula. When \( \varphi \in S \)
acts on a measure $\mu \in \mathcal{P}(\mathbb{R})$, it contracts $\mu$ by $\|\varphi\|$. By (7) this implies that for any $i$,

$$H(\mu, D_i) = H(\varphi \mu, D_{i - \log \|\varphi\|}) + O(1)$$

(note that $\log \|\varphi\| < 0$ when $\varphi$ is a contraction). Thus if $\nu$ is a measure supported on a small neighborhood of $\varphi$, then the entropy of $\nu \cdot \mu$ should be measured at a resolution adjusted by $\log \|\varphi\|$-scales relative to the resolution at which we consider $\mu$. The analog of Lemma 3 now has the following form (see also [6, Lemma 5.7]):

**Lemma 4.** For compactly supported $\mu \in \mathcal{P}(\mathbb{R})$ and $\nu \in \mathcal{P}(S)$, for every $\varphi_0 \in \text{supp} \nu$ and for every $m, n$,

$$\frac{1}{n}H(\nu \cdot \mu, D_{n - \log \|\varphi_0\|}) \geq \mathbb{E}_{1 \leq i \leq n}(\frac{1}{m}H(\nu \varphi, \mu x, i, D_{i - \log \|\varphi_0\| + m})) - O(\frac{1}{m} + \frac{m}{n}).$$

The error term depends only the diameter of the supports of $\mu, \nu$.

In our application of this inequality, the support of $\nu$ will lie in a fixed compact set, and we can drop the scale-shift of $\log \|\varphi_0\|$ and absorb the change in the error term; that is we can replace $D_{n - \log \|\varphi_0\|}$ by $D_n$ and $D_{i - \log \|\varphi_0\| + m}$ by $D_{i + m}$.

### 3.4 Entropy porosity

For a general measure, the entropy of components may vary almost arbitrarily from scale to scale and within a fixed scale. The following definition imposes some degree of regularity, specifically, it prevents too many components from being too uniform.

Let $\mu \in \mathcal{P}(\mathbb{R})$. We say that $\mu$ is $(h, \delta, m)$-entropy porous from scale $n_1$ to $n_2$ if

$$\mathbb{P}_{n_1 \leq i \leq n_2} \left(\frac{1}{m}H(\mu x, i, D_{i + m}) \leq h + \delta\right) > 1 - \delta. \quad (13)$$

We say that it is $h$-entropy porous if for every $\delta > 0$, $m > m(\delta)$ and $n > n(\delta, m)$ the measure is $(h, \delta, m)$-entropy porous from scale 0 to $n$.

Note that if $\mu$ is $(h, \delta, m)$-entropy porous from scale 0 to $n$ then by Lemma 2 we have $H(\mu, D_n)/n \leq h + 2\delta + O(m/n)$.

We will use the fact that entropy porosity passes to components. More precisely,

---

4Entropy porosity in the sense above is weaker than porosity, since it allows the measure to be fully supported on small balls (i.e. there do not need to be holds in its support). But the upper bound on the entropy of components means that most components are far away from being uniform at a slightly finer scale.
Lemma 5. Let $0 < \delta < 1$, $m, k \in \mathbb{N}$ and $n > n(\delta, k)$. If $\mu \in \mathcal{P}(\mathbb{R})$ is $(h, \delta^2/2, m)$-entropy porous from scale 0 to $n$, then

$$
\mathbb{P}_{0 \leq i \leq n} \left( \mu_{x,i} \text{ is } (h, \delta, m)\text{-entropy porous from scale } i \text{ to } i + k \right) > 1 - \delta.
$$

(14)

Proof. By assumption,

$$
\mathbb{P}_{0 \leq i \leq n} \left( \frac{1}{m} H(\mu_{x,i}, \mathcal{D}_{i+m}) \leq h + \frac{\delta^2}{2} \right) > 1 - \frac{\delta^2}{2}.
$$

(15)

Let $\mathcal{B}_i \subseteq \mathcal{P}(\mathbb{R})$ denote the set of measures $\nu$ with $\frac{1}{m} H(\nu, \mathcal{D}_{i+m}) > h + \delta$, and $\mathcal{A}_i \subseteq \mathcal{P}(\mathbb{R})$ the set of $\nu$ such that $\mathbb{P}_{1 \leq j \leq i+k} (\nu_{x,j} \in \mathcal{B}_j) > \delta$. It suffices for us to show that $\mathbb{P}_{0 \leq i \leq n} (\mu_{x,i} \in \mathcal{A}_i) \leq 2\delta/3$. Indeed, if we had $\mathbb{P}_{0 \leq i \leq n} (\mu_{x,i} \in \mathcal{A}_i) > 2\delta/3$, then Lemma 1 would imply $\mathbb{P}_{0 \leq i \leq n} (\mu_{x,i} \in \mathcal{B}_i) \geq 2\delta^2/3 - O(k/n)$, which, assuming as we may that $n$ large relative to $k, \delta$, contradicts (15).

3.5 Entropy growth under convolution: Euclidean case

Recall that $\nu * \mu$ denotes the convolution of measures $\nu, \mu$ on $\mathbb{R}$. The entropy of a convolution is generally at least as large as each of the convolved measures, although due to the discretization involved there may be a small loss:

Lemma 6. For every $\mu, \nu \in \mathcal{P}(\mathbb{R})$,

$$
\frac{1}{n} H(\nu * \mu, \mathcal{D}_n) \geq \frac{1}{n} H(\mu, \mathcal{D}_n) - O(\frac{1}{n}).
$$

Proof. Let $X$ be a random variable with distribution $\nu$. Then

$$
\nu * \mu = \mathbb{E}(\delta X * \mu) = \mathbb{E}(T_X \mu).
$$

By concavity of entropy and (7),

$$
H(\nu * \mu, \mathcal{D}_n) \geq \mathbb{E}(H(T_X \mu, \mathcal{D}_n)) \geq \mathbb{E}(H(\mu, \mathcal{D}_n) + O(1)) = H(\mu, \mathcal{D}_n) + O(1).
$$

The lemma follows.

In general one expects the entropy to grow under convolution but this is not always the case. Theorem 2.8 of [5] provides a verifiable condition under which some entropy growth occurs.

Theorem 5. For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every $m > m(\varepsilon, \delta)$ and $n > n(\varepsilon, \delta, m)$, the following holds:
Let \( \mu, \nu \in \mathcal{P}((0, 1)) \) and suppose that \( \mu \) is \((1 - \varepsilon, \delta, m)\)-entropy porous from scale 0 to \( n \). Then
\[
\frac{1}{n} H(\nu, D_n) > \varepsilon \implies \frac{1}{n} H(\nu * \mu, D_n) > \frac{1}{n} H(\mu, D_n) + \delta.
\]

More generally, if \( \mu, \nu \) are supported on sets of diameter \( 2^{-i} \), and if \( \mu \) is \((1 - \varepsilon, \delta, m)\)-entropy porous from scale \( i \) to \( i + n \), then
\[
\frac{1}{n} H(\nu, D_{i+n}) > \varepsilon \implies \frac{1}{n} H(\nu * \mu, D_{i+n}) > \frac{1}{n} H(\mu, D_{i+n}) + \delta.
\]

The second statement follows from the first by re-scaling by \( 2^i \).

4 Linearization and entropy growth

We now consider \( \nu \in \mathcal{P}(G) \) and \( \mu \in \mathcal{P}(\mathbb{R}) \) and the convolution \( \nu * \mu \) obtained by pushing \( \nu \times \mu \) forward through \( (\varphi, x) \mapsto \varphi(x) = \varphi(x) \). Our goal is to extend the results of the last section to this case: namely, that under some assumptions on \( \nu, \mu \) the entropy of \( \nu * \mu \) is substantially larger than that of \( \mu \) alone.

It will be convenient to extend the notation and write \( \nu * x \) for the push-forward of \( \nu \in \mathcal{P}(G) \) via \( \varphi \mapsto \varphi(x) \), or equivalently, \( \nu * x = \nu * \delta_x \).

4.1 Linearization and entropy

Let \( f : \mathbb{R}^{d_1 + d_2} \to \mathbb{R}^{d_3} \), let \( \nu \in \mathcal{P}(\mathbb{R}^{d_1}), \mu \in \mathcal{P}(\mathbb{R}^{d_2}) \), and \( \lambda = f(\nu \times \mu) \in \mathcal{P}(\mathbb{R}^{d_3}) \). First suppose that \( f \) is affine, so that there exists \( y_0 \in \mathbb{R}^2 \) and matrices \( A, B \) of appropriate dimensions such that
\[
f(x, y) = y_0 + Ax + By = T_{y_0}(Ax + By).
\]

It follows that
\[
\lambda = f(\nu \times \mu) = T_{y_0}(A \nu * B \mu),
\]
and by (7),
\[
H(\lambda, D_n) = H(A \nu * B \mu, D_n) + O(1).
\]

Now suppose instead that \( f \) is twice continuously differentiable\(^5\) rather than affine, so at every point \( z_0 = (x_0, y_0) \in \mathbb{R}^{d_1 + d_2} \) there are matrices \( A = A_{x_0} \) and \( B = B_{x_0} \) such that
\[
f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + O(|x - x_0|^2 + |y - y_0|^2). \tag{16}
\]

\(^5\)Differentiability would be enough for most purposes, but then the error term in (16) would be merely \( o(|x - x_0| + |y - y_0|) \) instead of the quadratic error, and later on we will want the quadratic rate.
Fix $m$, and suppose further that $r > 0$ and that $\nu$ is supported on an $O(r)$-neighborhood $U$ of $x_0$ and $\mu$ is supported on an $O(r)$-neighborhood $V$ of $y_0$. Then, assuming $r$ is small enough that the error term in (16) is less than $2^{\log r - m}$ for all $(x, y) \in U \times V$, by (12) we have

$$H(f(\nu \times \mu), D_{-\log r + m}) = H(A\nu * B\mu, D_{-\log r + m}) + O(1). \quad (17)$$

The last equation shows that, in order to bound the entropy of the image of a product measure, we can apply results about convolutions, provided we control the error term. But the dependence between the parameters is crucial: We have controlled it for a given $m$ by requiring that $\nu \times \mu$ be supported close enough to $z_0$. In (17), we cannot take $m \to \infty$, because as we increase $m$, the supports of the measures may be required to shrink to a point.

This issue can be avoided by using multiscale formula for entropy, though this gives only a lower bound rather than equality. We specialize at this point to the linear action of the similarity group $G$ on $\mathbb{R}$, though the same ideas work in greater generality. Recall that we parametrize $G$ as $\mathbb{R}^2$, identifying $(s, t)$ with $x \mapsto e^s x + t$. In order to conform with the notation in previous sections, we denote the coordinates of $G \times \mathbb{R}$ by $(\varphi, x)$. Let

$$f : G \times \mathbb{R} \to \mathbb{R}$$

$$\varphi, x \mapsto \varphi, x$$

denote the action map, which we think of this as a smooth map defined on $\mathbb{R}^2 \times \mathbb{R}$. Note that by definition, $f(\mu \times \nu) = \mu \nu$. Also note that the derivative $A = A(\varphi, x) = \frac{\partial}{\partial x} f(\varphi, x)$ is a $1 \times 2$ matrix and $B = B(\varphi, x) = \frac{\partial}{\partial \varphi} f(\varphi, x)$ is a $1 \times 1$ matrix, which we identify simply with a scalar. Given $(u, v) \in \mathbb{R}^2 \times \mathbb{R}$ and matrices $A, B$ of these dimensions we have (recall that $S_t(x) = 2^t x$)

$$Au + Bv = S_{\log B}(B^{-1}Au + v).$$

Therefore,

$$A\nu * B\mu = S_{\log B}(B^{-1}A\nu * \mu).$$

**Proposition 1.** Let $I \times J \subseteq G \times \mathbb{R}$ be compact. Then for every $\nu \in \mathcal{P}(I)$ and $\mu \in \mathcal{P}(J)$, as $m \to \infty$ and $n/m \to \infty$, we have

$$\frac{1}{n} H(\nu \times \mu, D_n) \geq \mathbb{E}_{0 \leq i \leq n} \left( \frac{1}{m} H(B_{(\varphi, x)}^{-1} A(\varphi, x) \nu_{\varphi, i} * \mu_{x, i}, D_{i+m}) \right) + o(1).$$

**Proof.** Since $I \times J$ is compact and $f$ is smooth, the error term in (16) holds uniformly in $(\varphi_0, x_0) \in I \times J$. Given components $\nu_{\varphi, i}$ and $\mu_{x, i}$ of $\nu$ and $\mu$, respectively, each is supported on a set of diameter $O(2^{-i})$, so by (17),

$$\frac{1}{m} H(f(\nu_{\varphi, i} * \mu_{x, i}), D_{i+m}) = \frac{1}{m} H(A_{(\varphi, x)} \nu_{\varphi, i} * B_{(\varphi, x)} \mu_{x, i}, D_{i+m}) + o(1),$$

as $m, i \to \infty$ (uniformly in $(\varphi, x) \in I \times J$). By compactness, $B_{(\varphi, x)}$ is bounded for $(\varphi, x) \in I \times J$, and by (6), changing a measure by a bounded scaling affects
entropy by $O(1)$, which, upon division by $m$, is $o(1)$. Thus the last equation can be replaced by

$$\frac{1}{m} H(f(\nu, x) \times \mu_x(i), D_{i+m}) = \frac{1}{m} H(B^{-1}(\nu, x) A(\nu, x)(\nu, x) \times \mu_x(i), D_{i+m}) + o(1),$$

as $m, i \to \infty$. Finally, by Lemma 4 and the remark following it, and the last equation,

$$\frac{1}{n} H(\nu, \mu, D_n) \geq \mathbb{E}_{0 \leq i \leq n} \left( \frac{1}{m} H(H(f(\nu, x) \times \mu_x(i), D_{i+m}) \right) + O\left( \frac{1}{m} + \frac{m}{n} \right),$$

which gives the claim (we can move the error term outside the expectation because it is uniform). \qed

### 4.2 Entropy growth for the action

We now prove an analogue of Theorem 5 for the action of $G$ on $\mathbb{R}$.

**Theorem 6.** For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the following holds:

Let $\nu \in \mathcal{P}(G)$, $\mu \in \mathcal{P}(\mathbb{R})$ be compactly supported, and suppose that $\mu$ is non-atomic and $(1 - \varepsilon)$-entropy porous. Then for every $n > n(\varepsilon, \delta, \mu)$

$$\frac{1}{n} H(\nu, \mu, D_n) > \varepsilon \implies \frac{1}{n} H(\nu, D_n) > \frac{1}{n} H_n(\mu, D_n) + \delta.$$

We remark that $n$ is required to be large relative to $\mu$, but in fact the only dependence involves the modulus of continuity of $\mu$ (in the proof the dependence appears in Lemma 9), and on a choice of the parameter $m$ in the definition of entropy porosity for $\mu$.

To begin the proof, fix $\varepsilon > 0$. Apply Theorem 5 with parameter $\varepsilon' = \varepsilon/10$, obtaining a corresponding $\delta' > 0$. We will choose $\delta$ later to be small both compared to $\delta'$ and $\varepsilon$.

Fix parameters $m, k, n \in \mathbb{N}$. All the $o(1)$ error terms below are to be understood as becoming arbitrarily small if $m$ is large, $k$ is large enough depending on $m$, and $n$ is large enough in a manner depending on $m, k$.

Let us abbreviate

$$C(\nu, x) = B^{-1}(\nu, x) A(\nu, x),$$

so $C(\nu, x)$ is a $2 \times 1$ matrix, which we identify with a linear map $\mathbb{R}^2 \to \mathbb{R}$. By Proposition 4 we have

$$\frac{1}{n} H(\nu, \mu, D_n) \geq \mathbb{E}_{0 \leq i \leq n} \left( \frac{1}{k} H(C(\nu, x)(\nu, x) \times \mu_x(i), D_{i+k}) \right) - o(1). \quad (18)$$
Suppose that for some $c = c(\varepsilon) > 0$ it were true that
\[ P_{0 \leq i \leq n} \left( \frac{1}{k} H(C_{(\varphi,x)} \nu_{\varphi,i} \ast \mu_{x,i}, D_{i+k}) > \frac{1}{k} H(\mu_{x,i}, D_{i+k}) + \delta' \right) > c. \] (19)

Splitting the expectation in (18) by conditioning on the event in (19) and its complement, using Lemma 6 to control the expectation on the complement, and using Lemma 2, we would have
\[
\frac{1}{n} H(\nu, \mu_n) \geq \mathbb{E}_{0 \leq i \leq n} \left( \frac{1}{k} H(\mu_{x,i}, D_{i+k}) \right) + c\delta' - o(1) \\
= \frac{1}{n} H(\mu, D_n) + c\delta' - o(1),
\]
as claimed.

Now, by our choice of $\varepsilon'$ and $\delta'$, equation (19) will follow if we show that
\[ P_{0 \leq i \leq n} \left( \mu_{x,i} \text{ is } (1 - \varepsilon', \delta', m)\text{-entropy porous at scales } i \text{ to } i + k, \right. \]
\[ \left. \text{and } \frac{1}{k} H(C_{(\varphi,x)} \nu_{\varphi,i}, D_{i+k}) > \varepsilon' \right) \geq c. \] (20)

This is the probability of an intersection of two events. The first, involving $\mu_{x,i}$, can be dealt with using Lemma 5. Indeed, by the hypothesis, if $m$ is large enough and $n$ suitably large, then $\mu$ is $\left(1 - \varepsilon, \delta, m\right)$-porous, and hence $(1 - \varepsilon', \delta, m)$-porous, at scales 0 to $n$, so (assuming as we may that $\delta < (\delta')^2/2$) Lemma 5 implies
\[ P_{0 \leq i \leq n} \left( \mu_{x,i} \text{ is } (1 - \varepsilon', \delta', m)\text{-entropy porous at scales } i \text{ to } i + k \right) = 1 - o(1). \]

Thus, in order to prove (20), it remains to show that $\frac{1}{k} H(C_{(\varphi,x)} \nu_{\varphi,i}, D_{i+k}) > \varepsilon'$ with probability bounded away from 0, as $(\varphi, x)$ are chosen according to $\nu \times \mu$ and $0 \leq i < n$. Observe that if the expression involved the entropy of $\nu_{\varphi,i}$ instead of that of $C_{(\varphi,x)} \nu_{\varphi,i}$, we would be done, because by Lemma 2 and our hypothesis,
\[ \frac{1}{n} H(\nu, D_n) = \frac{1}{n} H(\nu, D_n) - o(1) > \varepsilon - o(1), \]
from which it follows that
\[ P_{0 \leq i \leq n} \left( \frac{1}{k} H(\nu_{\varphi,i}, D_{i+k}) \geq \frac{\varepsilon}{3} \right) > \frac{\varepsilon}{3} - o(1). \] (21)

The problem is that $C_{(\varphi,x)}$ is a linear map $\mathbb{R}^2 \to \mathbb{R}$, and has a 1-dimensional kernel, and if $\nu_{\varphi,i}$ happens to be supported on (or close to) a translate of the kernel, then $C_{(\varphi,x)} \nu_{\varphi,i}$ is a Dirac measure (at least approximately), and has entropy (essentially) equal zero no matter how large the entropy of $\nu_{\varphi,i}$ is.

The way to get around this problem is to note that the kernels of these transformations are generally transverse to each other, and intersect at a point; so if $\nu_{\varphi,i}$ has substantial entropy it cannot be supported on or near $\ker B_{(\varphi,x)}^{-1} A_{(\varphi,x)}$.
for too many values of $x$. Consequently, we shall show that conditioned on $\varphi$ and $i$, with high $\mu$-probability over the choice of $x$, $B_{(\varphi,x)}^{-1}(\varphi,i)\nu_{\varphi,i}$ must have at least a constant fraction of the entropy at scale $i+k$ as $\nu_{\varphi,i}$ itself. We prove this in the following sequence of lemmas.

A map $f$ between metric spaces has bi-Lipschitz constant $c > 0$ if $c^{-1}d(x, y) \leq d(f(x), f(y)) \leq cd(x, y)$ for every $x, y$.

**Lemma 7.** Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be such that the map $g(y) = (g_1(y), g_2(y))$ is bi-Lipschitz with constant $c$. Then for any $\mu \in \mathcal{P}(\mathbb{R}^2)$ and any $i$, some $j \in \{1, 2\}$ satisfies

$$H(\pi_j(\mu), D) > \frac{1}{2} H(\mu, D) - O(\log c).$$

**Proof.** Since $g$ is bi-Lipschitz, by (11),

$$H(g\mu, D) = H(\mu, D) + O(\log c).$$

Let $\pi_j$ be projection from $\mathbb{R}^2$ to the $j$-th coordinate. Then $D_i^2 = \pi_1^{-1}D_i \vee \pi_2^{-1}D_i$, so

$$H(g\mu, D) = H(g(\pi_1\mu), D_i) + H(g(\pi_2\mu), D_i) \leq H(g\mu, \pi_1^{-1}D_i) + H(g\mu, \pi_2^{-1}D_i) \leq H(\mu, D) + O(\log c),$$

where in the last step we used the identity $\pi_j \circ g = g_j$. Combining the last two equations gives the lemma. \qed

For $t > 0$ let

$$\Sigma_t = \{(x, y) : |x - y| \geq t\}.$$

Recall the definition of the matrix $A_{\varphi,x}$ preceding Proposition 1

**Lemma 8.** Let $\varphi \in G$ and $x \neq y \in \mathbb{R}$. Then the map $g : \mathbb{R}^2 \to \mathbb{R}$, $g(z) = (A_{\varphi,x}z, A_{\varphi,y}z)$, is bi-Lipschitz, and for $t > 0$, for $(x, y) \in \Sigma_t \cap (\text{supp } \mu)^2$ and $\varphi \in \text{supp } \nu$, its bi-Lipschitz constant is bounded uniformly by $O_R(1 + t^{-1})$, where $R$ is the smallest radius for which $\mu, \nu$ are supported on the $R$-ball at the origin.

Note that the first statement follows easily by observing that $\varphi \in G$ is determined by its action on any two points.

**Proof.** Suppose that $\varphi$ is represented by $(s, t)$ in coordinates, so $f(\varphi, x) = \varphi(x) = e^s x + t$. A direct calculation yields $A_{\varphi,x} = (\frac{\partial}{\partial x} f(\varphi, x), \frac{\partial}{\partial y} f(\varphi, x)) = (e^s x, 1)$, hence the linear map $g$ in question is represented by the matrix $\left( \begin{array}{cc} e^s x & 1 \\ e^s y & 1 \end{array} \right)$, which is invertible and has bi-Lipschitz with constant $O_\star(1 + |x - y|^{-1})$. The second statement is immediate since $\Sigma_t \cap (\text{supp } \mu)^2$ and $\text{supp } \nu$ are compact. \qed
Lemma 9. Let \( \nu, \mu \) be as in Theorem 6. Let \( h > 0 \), fix \( \varphi, i \), and write \( \theta = \nu_{\varphi,i} \). Then, assuming that \( \frac{1}{k} \mathcal{H}(\theta, D_{i+k}) > h \),

\[
\mu \left( x \in \mathbb{R} : \frac{1}{k} \mathcal{H}(B_{(\varphi,x)}^{-1} A_{(\varphi,x)} \theta, D_{i+k}) > \frac{1}{3} h \right) = 1 - o_h(1)
\]

as \( k \to \infty \), uniformly in \( \varphi \in \text{supp} \nu \) and \( i \in \mathbb{N} \). Furthermore if there are constants \( a, \alpha > 0 \) such that \( \mu(B_r(x)) < ar^\alpha \) for all \( x \) then the error term is \( o_{h,a,\alpha}(1/k^d) \) for every \( d \).

Proof. By compactness \( B_{(\varphi,x)} \) is bounded on the support of \( \nu \times \mu \), and scaling by a bounded constant changes entropy by \( O(1) \); so, after dividing by \( k \), it changes by \( o(1) \) as \( k \to \infty \). Thus we may omit the factor \( B_{(\varphi,x)}^{-1} \) in the statement.

Let \( \rho > 0 \). Since we have assumed that \( \mu \) is non-atomic, we can fix \( t > 0 \) such that \( \mu(B_t(x)) < \rho \) for all \( x \).

Suppose for some \( x' \in \text{supp} \mu \) we have

\[
\frac{1}{k} \mathcal{H}(A_{(\varphi,x') \theta}, D_{i+k}) \leq \frac{1}{3} h
\]

(if no such \( x' \) exists then we are done). Let \( c \) denote the uniform bound on the bi-Lipschitz constant associated to \( t \) in Lemma 8. By the previous two lemmas, if \( (x, x') \in \Sigma_n |t \cap (\text{supp} \mu)^2 \) then necessarily

\[
\frac{1}{k} \mathcal{H}(A_{(\varphi,x) \theta}, D_{i+k}) \geq \frac{1}{2} h - O\left( \frac{\log c}{k} \right) > \frac{1}{3} h,
\]

assuming \( k \) large enough relative to \( t \) (and hence \( \rho \)). This implies that the event in the statement of the lemma contains \( \mathbb{R} \setminus B_t(x') \) (up to a nullset). This set has \( \mu \)-measure at least \( 1 - \rho \) by our choice of \( t \). Thus we have shown that given \( \rho \), if \( k \) is large enough, then

\[
\mu \left( x \in \mathbb{R} : \frac{1}{k} \mathcal{H}(A_{(\varphi,x) \theta}, D_{i+k}) > \frac{1}{3} h \right) > 1 - \rho.
\] (22)

For the second statement, fix \( d \) and \( \rho = \rho_k = 1/k^d \). By assumption, \( \mu(B_r(x)) < ar^\alpha \) so in order for \( t = k \) to satisfy \( \mu(B_t(x)) < \rho \) it suffices to take \( t = O_k(\rho^{1/\alpha}) = O_k(1/k^{d/\alpha}) \). Then by Lemma 8 we have \( c = c_k = O(1 + t^{-1}) = O(k^{d/\alpha}) \) and since the error term \( \log c_k / k \) in (22) tends to zero as \( k \to \infty \), the analysis above holds and the conclusion of the proposition is valid with error term \( o_{h,a,\alpha}(k^d) \).

We return to the proof of Theorem 6. Taking \( h = \varepsilon/3 \) in the last lemma and combining it with equation (21), we find that for \( k \) large enough, with probability at least \( \varepsilon/4 \) (and hence probability at least \( \varepsilon' \)) over our choice of \( 0 \leq i \leq n \) and of \( (\varphi, x) \) (chosen with respect to \( \nu \times \mu \)), we will have \( \frac{1}{k} \mathcal{H}(B_{(\varphi,x)}^{-1} A_{(\varphi,x)} \theta, D_{i+k}) > \varepsilon/9 > \varepsilon' \). This completes the proof.
4.3 Entropy dimension

Define the entropy dimension of \( \mu \in \mathcal{P}(\mathbb{R}) \) to be

\[
\dim_e \mu = \lim_{n \to \infty} \frac{1}{n} H(\mu, D_n).
\]

if the limit exists, otherwise define the upper and lower entropy dimensions \( \dim_{e, \mu}, \dim_{e, \mu} \) by taking a limsup or liminf, respectively. We also note that if \( \mu \) is supported on a set \( Y \) then by (4), \( \dim_{B, Y} \geq \dim_{e, \mu} \), where \( \dim_{B, Y} \) is the upper box dimension, and a similar relation holds for lower entropy and box dimensions.

**Theorem 7.** For every \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that the following holds:

Let \( \nu \in \mathcal{P}(G) \), \( \mu \in \mathcal{P}(\mathbb{R}) \) be compactly supported, and suppose that \( \mu \) is non-atomic and \((1 - \varepsilon)\)-entropy porous. Then

\[
\begin{align*}
\dim_{e, \nu} > \varepsilon & \implies \dim_{e, \nu \cdot \mu} \geq \dim_{e, \mu} + \delta \\
\dim_{e, \nu} > \varepsilon & \implies \dim_{e, \nu \cdot \mu} \geq \dim_{e, \mu} + \delta \\
\dim_{e, \nu} > \varepsilon & \implies \dim_{e, \nu \cdot \mu} \geq \dim_{e, \mu} + \delta.
\end{align*}
\]

The proof is trivial from Theorem 6 upon taking \( n \to \infty \) and considering the definitions of the upper and lower entropy dimensions. We leave the verification to the reader.

Note that the case of convolutions \( \nu \ast \mu \) for \( \nu, \mu \in \mathbb{R} \) is contained in this theorem as a special case, since we can lift \( \nu \) to \( \nu' \in \mathcal{P}(G) \) by identifying \( t \in \mathbb{R} \) with the corresponding translation map \( x \mapsto x + t \). Then \( \nu \ast \mu = \nu' \cdot \mu \), and \( \dim \nu = \dim \nu' \), so we get conditions for entropy-dimension growth of Euclidean convolutions.

5 Proof of Theorem 8

5.1 Stationary measures

Let \( \Phi \subseteq \mathcal{S} \) be a compact set with \( \dim \Phi > 0 \) and attractor \( X \). Proving Theorem 2 requires us to find suitable measures on \( \Phi \) and \( X \) to work with. For \( \Phi \) we can take any measure \( \nu \) of positive dimension, which exists by Frostman’s lemma (see e.g. [13]). There then exists a unique measure \( \mu \) on \( \mathbb{R} \), called the \( \nu \)-stationary measure, satisfying

\[
\mu = \nu \cdot \mu.
\]  

The existence and uniqueness of \( \mu \) is proved by showing that \( \tau \mapsto \nu \cdot \tau \) is a contraction on \( \mathcal{P}(\mathbb{R}) \) when endowed with a suitable metric. This is again the same argument as the one establishing the existence of self-similar measures, and this is not surprising, since self-similar measures are special cases of stationary ones: when \( \nu = \sum_{\varphi \in \Phi} p_\varphi \cdot \delta_\varphi \) is finitely supported, the relation (23) becomes

\[
\mu = \sum_{\varphi \in \Phi} p_\varphi \cdot \varphi \mu,
\]  

(24)
which is the definition of a self similar measure (as usual, \( \varphi \mu = \mu \circ \varphi^{-1} \) is the push-forward of \( \mu \) by \( \varphi \)). We note that if a stationary measure is not a single atom, then it is continuous (has no atoms). The proof is standard and we omit it.

Recall the definition of entropy dimension from Section 4.3. We show below (Proposition 2) that if \( \mu \) satisfies (23) then its entropy dimension exists\(^6\) We also show that it is \( \dim_e \mu \)-entropy porous (Proposition 3). Then Theorem 7 has the following consequence:

**Theorem 8.** Let \( \mu \in \mathcal{P}(\mathbb{R}) \) be a \( \nu \)-stationary measure for a compactly supported \( \nu \in \mathcal{P}(S) \). If \( \dim_e \nu > 0 \) then either \( \mu \) is a Dirac measure, or \( \dim_e \mu = 1 \).

**Proof.** Suppose that \( \mu \in \mathcal{P}(\mathbb{R}) \) is not a Dirac measure. Write \( \alpha = \dim_e \mu \) and \( \beta = \dim_e \nu \). We assume that \( \alpha < 1 \) and \( \beta > 0 \), and wish to derive a contradiction. Set \( \varepsilon = \frac{1}{2} \min\{\beta, 1-\alpha\} > 0 \) and let \( \delta = \delta(\varepsilon) \) be as in Theorem 6. Then \( \mu \) is \((1-\varepsilon)\)-entropy porous and continuous, so by Theorem 6 \( \dim_e \nu, \mu > \dim_e \mu + \delta \), which is impossible.

To complete the proof of Theorem 2 we must show that \( \dim_e \mu = 1 \) implies \( \dim X = 1 \). This is simple: we have already noted that \( \dim_e \mu = 1 \) implies that \( \dim X = 1 \), and finally, this implies \( \dim X = 1 \) because \( X \) has equal box and Hausdorff dimensions. This last property is proven the same manner as for self-similar sets, see e.g. [4, Theorem 4].

It remains for us to show that \( \dim_e \mu \) exists and that it is entropy porous. We do this in the couple of sections.

### 5.2 Cylinder decomposition of stationary measures and entropy dimension

Let \( \Phi \subseteq S \) be compact, let \( r_0 = \min_{\varphi \in \Phi} \| \varphi \| \) and given \( n \in \mathbb{N} \) let \( \Phi_n \) denote the set

\[
\Phi_n = \left\{ (\varphi_1, \ldots, \varphi_k) \in \bigcup_{\ell=1}^{\infty} \Phi^\ell : r_0 2^{-n} \leq \| \varphi_1 \ldots \varphi_k \| < 2^{-n} \right\}
\]

Suppose that \( \Phi = \{ \varphi_1, \ldots, \varphi_m \} \) is finite, so also \( \Phi_n \) is finite, and that \( \mu = \sum_{i=1}^{m} p_i \varphi_i \mu \) is a self-similar measure. For \( \varphi_{i_1}, \ldots, \varphi_{i_k} \in \Phi \) write \( \varphi_{i_1 \ldots i_k} = \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_k} \) and \( (p_i)_{i=1}^{m} \) write \( p_{i_1 \ldots i_k} = p_{i_1} p_{i_2} \ldots p_{i_k} \). Then one can iterate the definition of \( \mu \) to get

\[
\mu = \sum_{(\varphi_{i_1 \ldots i_k})_{i=1}^{k} \in \Phi_n} p_{i_1 \ldots i_k} \cdot \varphi_{i_1 \ldots i_k} \mu.
\]

This “decomposes” \( \mu \) into finitely many images of itself, each by a map which contracts by roughly \( 2^{-n} \).

Now let \( \Phi \subseteq S \) be a general compact set, \( \nu \in \mathcal{P}(S) \) a compactly supported, and \( \mu \) a \( \nu \)-stationary, \( \nu \cdot \mu = \mu \). We want to have a similar representation of

---

\(^6\)One can also show that \( \mu \) is exact-dimensional, but we do not need this fact here.
µ, but now instead of a sum we will have an integral, the family \( \Phi_n \) generally being uncountable, and a suitable measure replacing the weights \( p_i \) in the sum. The way to do this is to consider the Markov chain obtained by repeatedly applying to \( \mu \) a random map, chosen according to \( \nu \). Indeed the relation \( \mu = \nu \cdot \mu \) just means that, if \( \phi \) denotes a random similarity chosen according to \( \nu \), then

\[
\mu = \mathbb{E}(\phi \mu).
\]

Thus let \( (\phi_i)_{i=1}^{\infty} \) be an independent sequence of similarities with common distribution \( \nu \) and consider the measure-valued random process

\[
\mu_n = \phi_1 \phi_2 \cdots \phi_n \mu.
\]

This is a martingale with respect to the filtration \( \mathcal{F}_n = \theta(\phi_1, \ldots, \phi_n) \), since, writing \( \Omega \) for the sample space of the process and \( \mu_\omega \) to indicate the dependence on \( \omega \in \Omega \),

\[
\mathbb{E}(\mu_{n+1} | \mathcal{F}_n)(\omega) = \mathbb{E}(\phi_1(\omega) \cdots \phi_n(\omega) \cdot \phi_{n+1} \mu) = (\phi_1(\omega) \cdots \phi_n(\omega)) \mathbb{E}(\phi_{n+1} \mu) = \mu_\omega = \mathbb{E}(\mu_\tau) = \mathbb{E}(\mu_0) = \mu.
\]

Recall that a random variable \( \tau \) is a stopping time for \( \mathcal{F}_n \) if the event \( \{ \tau \leq k \} \) belongs to \( \mathcal{F}_k \) for all \( k \in \mathbb{N} \). Given a bounded stopping time, Doob’s optional stopping theorem [10, Theorem 7.12] asserts that

\[
\mathbb{E}(\mu_\tau) = \mathbb{E}(\mu_0) = \mu.
\]

We apply this to the stopping time

\[
\tau_n = \min\{ k \in \mathbb{N} : \| \phi_1 \cdots \phi_k \| < 2^{-n} \}.
\]

Since supp\( \nu \) is compact, there exist \( 0 < r_0 < r_1 < 1 \) such that \( r_0 \leq \| \phi \| \leq r_1 \) for all \( \phi \in \Phi \), which implies that \( 2^{-n} r_0 \leq \| \phi_1 \cdots \phi_{\tau_n} \| < 2^{-n} \), and also that \( \tau_n \leq n/\log(1/r_1) \), so \( \tau_n \) is bounded. Therefore the identity

\[
\mu = \mathbb{E}(\mu_\tau) = \mathbb{E}(\phi_1 \cdots \phi_\tau \mu)
\]

is the desired analog of (25).

**Proposition 2.** \( \dim_a \mu = \lim_{n \to \infty} \frac{1}{n} H(\mu, D_n) \) exists.

---

7To derive this from the sampling theorem for real-valued random variables, note that we need to show that \( \int f d\mathbb{E}(\mu_\tau) = \int f d\mu \) for all bounded functions \( f \), and this follows since \( \xi_n = \int f d\mu_n \) is easily seen to be a martingale for \( (\mathcal{F}_n) \), and by Fubini \( \int f d\mathbb{E}(\mu_\tau) = \mathbb{E}(\int f d\mu_\tau) = \mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_0) = \int f d\mu \), where we used the real-valued optional stopping theorem in the second to last equality.
Proof. For any \( m, n \), by (8) and by the fact that \( \varphi_1 \ldots \varphi_\tau \) contracts by \( 2^{-(n+O(1))} \), we see that \( \mu_{\tau_n} \) is supported on a set of diameter \( 2^{-n+O(1)} \). Therefore

\[
H(\mu_{\tau_n}, D_{n+m}) = H(\mu, D_m) + O(1).
\]

and for the same reason, by (10),

\[
H(\mu_{\tau_n}, D_{n+m}|D_n) = H(\mu_{\tau_n}, D_{n+m}) + O(1).
\]

Write \( a_n = H(\mu, D_n) \). Then by concavity of conditional entropy and the discussion above,

\[
a_{m+n} = H(\mu, D_{m+n})
\]

\[
= H(\mu, D_m) + H(\mu, D_{m+n}|D_n)
\]

\[
= H(\mu, D_m) + H(\mu_{\tau_n}, D_{m+n}|D_n)
\]

\[
\geq H(\mu, D_m) + H(\mu_{\tau_n}, D_{m+n}|D_n)
\]

\[
= a_m + a_n + O(1).
\]

It follows that up to an \( O(1) \) error \( (a_n) \) is super-additive, so \( \lim_{n \to \infty} \frac{1}{n} a_n \) exists, as desired. \( \square \)

5.3 Entropy porosity of stationary measures

Returning to our stationary measures, our next goal is to show that they are entropy-porous. The argument is essentially the same as in [5, Section 5.1], with some additional minor complications due to continuity of \( \nu \).

Let \( \mu = \nu \cdot \mu \) be a stationary measure for a compactly supported \( \nu \in \mathcal{P}(S) \), and assume \( \mu \) is not a Dirac measure. By a change of coordinates \( x \mapsto 2^{-N} (x+k) \) for suitable choice of \( N, k \in \mathbb{N} \), we may assume that \( \mu \) is supported on \([0, 1/2)\).

Write

\[
\alpha = \dim_e \mu.
\]

Our goal is to prove the following:

**Proposition 3.** For every \( \varepsilon > 0 \) and \( m > m(\varepsilon) \), for all large enough \( n \),

\[
P_{0 \leq i \leq n} \left( \frac{1}{m} H(\mu_{x,i}, D_{i+m}) - \alpha < \varepsilon \right) > 1 - \varepsilon.
\]

In particular, \( \mu \) is \( \alpha \)-entropy porous, and satisfies the conclusion of Lemma 5.

To prove this we need only prove that for every \( \varepsilon > 0 \), \( m > m(\varepsilon) \) and all \( n \),

\[
P_{0 \leq i \leq n} \left( \frac{1}{m} H(\mu_{x,i}, D_{i+m}) > \alpha - \varepsilon \right) > 1 - \varepsilon.
\]

\[\text{One way to see this is by adapting the proof of the classical Fekete lemma. Alternatively consider } b_n = a_n - \sqrt{n}, \text{ which after dividing by } n \text{ has the same asymptotics as } a_n, \text{ but satisfies } b_{m+n} \geq b_m + b_n \text{ for all } m, n \text{ large enough, so that Fekete’s lemma applies to it.} \]
Indeed, by Lemma 2 and the fact that \( \frac{1}{m}H(\mu, D_n) \to \alpha \), for large enough \( n \),

\[
E_{0 \leq i \leq n}(\frac{1}{m}H(\mu_{x,i}, D_{i+m})) \leq \alpha + \varepsilon. \tag{28}
\]

This is an average of a non-negative quantity which, by (27), with probability \( 1 - \varepsilon \) is not more than \( 2\varepsilon \) less than its mean, so

\[
P_{0 \leq i \leq n}(\frac{1}{m}H(\mu_{x,i}, D_{i+m}) > \alpha + \sqrt{2\varepsilon}) < \sqrt{2\varepsilon}.
\]

Starting from \( \varepsilon^2/8 \) instead of \( \varepsilon \), and combining with (27), this proves the proposition.

We turn to the proof of (27). Let \( \delta > 0 \) be a parameter to be determined later. Since \( \mu \) is not a Dirac measure it is continuous (has no atoms), so there is a \( \rho > 0 \) such that \( \mu(B_{\rho}(x)) < \delta \) for all \( x \in \mathbb{R} \) (here and throughout, balls are open). We can assume that \( \rho < \frac{1}{4}. \)

Let \( \varphi_1, \varphi_2, \ldots \) be an i.i.d. sequence with marginal \( \nu \), defined on some sample space \( \Omega \). Let \( \tau_i \) be the stopping time defined in (26).

Denote \( r_0 = \inf\{|\varphi| : \varphi \in \text{supp} \nu\} \). Fix \( i \) and let \( V_i \subseteq \mathbb{R} \) denote the set of points \( x \) whose distance from \( Z/2^i \) is less than \( 2^{-i}\rho r_0 \), that is, \( V_i = \bigcup_{k \in \mathbb{Z}} B_{2^{-i}\rho r_0}(k/2^i) \).

**Lemma 10.** \( \mu(V_i) < \delta. \)

*Proof.* Since \( \mu = E(\varphi_1 \ldots \varphi_{\tau_i} \mu) \), it is enough to show that \( (\varphi_1 \ldots \varphi_{\tau_i} \mu)(V_i) < \delta \) a.s. over the choice of the maps. Writing \( r_i = |\varphi_1 \ldots \varphi_{\tau_i}| \), for some \( t_i \in \mathbb{R} \) we have

\[
(\varphi_1 \ldots \varphi_{\tau_i} \mu)(V_i) = \mu((\varphi_1 \ldots \varphi_{\tau_i})^{-1}V_i) = \mu(\frac{1}{r_i}V_i + t_i).
\]

But by definition of \( \tau_i \) we have \( 2^{-i}r_0 < r_i \leq 2^{-i} \), so

\[
\frac{1}{r_i}V_i + t_i = \bigcup_{k \in \mathbb{Z}} B_{2^{-i}r_0/r_i}(k/(r_i 2^i)) + t_i \subseteq \bigcup_{k \in \mathbb{Z}} B_{r}(t_i + k/(r_i 2^i)).
\]

On the other hand, \( \{t_i + k/(r_i 2^i)\}_{k \in \mathbb{Z}} \) is a periodic sequence with gap size at least 1, and since \( \rho < 1/4 \) and \( \mu \) is supported on a set of diameter \( 1/2 \), at most one of the balls \( B_{\rho}(t_i + k/(r_i 2^i)) \) intersects the support of \( \mu \). The \( \mu \)-mass of this ball is less than \( \delta \) by our choice of \( \rho \), and the claim follows. \( \square \)

**Lemma 11.** \( \mu(V_i) < \sqrt{\delta} \mu(D_i(x)) \).

*Proof.* Elementary, using \( \mu(V_i) < \delta. \) \( \square \)
Let \( \ell \in \mathbb{N} \) be large enough that the diameter of \( \text{supp} \, \mu \) is less than \( 2^\ell \rho r_0 \). Assume that \( D \in D_i \) and \( \mu(D) > 0 \). Then \( \mu = \mathbb{E}(\varphi_1 \ldots \varphi_{\tau_{t+\ell}}) \) implies
\[
\mu|_D = \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})|_D \right).
\]

Let \( A_D \) denote the event that \( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}}) = 1 \) and \( B_D \) the event that \( 0 < (\varphi_1 \ldots \varphi_{\tau_{t+\ell}}) < 1 \). Then we have
\[
\mu|_D = \mathbb{P}(A_D) \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})|_D \big| A_D \right) + \mathbb{P}(B_D) \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})|_D \big| B_D \right) \tag{29}
\]

(the missing term, where the expectation is conditioned on the complement of \( A_D \cup B_D \), is zero). Dividing the equation by \( \mu(D) \), and dividing and multiplying each integrand by \( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}}) \) and using the fact that this is 1 on \( A_D \), we obtain
\[
\mu_D = \frac{\mathbb{P}(A_D)}{\mu(D)} \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})|_D \big| A_D \right) + \frac{\mathbb{E}(B_D)}{\mu(D)} \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})|_D \big| B_D \right). \tag{30}
\]

Evaluating this measure-valued equation on \( D \) shows that
\[
\frac{\mathbb{P}(A_D)}{\mu(D)} + \frac{\mathbb{P}(B_D)}{\mu(D)} \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(D) \big| B_D \right) = 1. \tag{31}
\]

**Lemma 12.** If \( D \in D_i \) and \( \mu(D \cap V_i) < \sqrt{\delta} \mu(D) \) then \( \mathbb{P}(A_D)/\mu(D) > 1 - \sqrt{\delta} \).

**Proof.** Suppose that \( 0 < (\varphi_1 \ldots \varphi_{\tau_{t+\ell}}) < 1 \). Then \( \varphi_1 \ldots \varphi_{\tau_{t+\ell}} \) gives positive mass to both \( D \) and \( \mathbb{R} \setminus D \). On the other hand the diameter of this measure is at most \( 2^{-(i+\ell)} \) times the diameter of \( \text{supp} \, \mu \), which by choice of \( \ell \) is at most \( \rho 2^{-i} \), so \( \varphi_1 \ldots \varphi_{\tau_{t+\ell}} \) must be supported within \( \rho 2^{-i} \) of \( \partial D \), and hence it is supported on \( V_i \). We have found that on the event \( B_D \), if \( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(D) > 0 \) then \( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(V_i) = 1 \), and therefore also \( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(D)(V_i) = 1 \). Consequently, by our hypothesis and (30),
\[
\sqrt{\delta} > \mu_D(V_i)
\]
\[
\geq \frac{\mathbb{P}(B_D)}{\mu(D)} \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(D) \big| (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(V_i) \big| B_D \right)
\]
\[
= \frac{\mathbb{P}(B_D)}{\mu(D)} \cdot \mathbb{E} \left( (\varphi_1 \ldots \varphi_{\tau_{t+\ell}})(D) \big| B_D \right) .
\]

The claim follows using (31). \( \square \)

We now prove (27), proving Proposition 3. Let \( \varepsilon > 0 \), and continue with the previous notation, eventually taking \( \delta \) small relative to \( \varepsilon \), and \( m \) large relative to \( \varepsilon, \delta \) (and hence relative to \( \rho \) and \( \ell \), since they are determined by \( \delta \)).
Suppose that $D \in D_i$ and $\mu(D \cap V_i) < \sqrt{\delta} \mu(D)$. By (30) we can write

$$\mu_D = \frac{\mathbb{P}(A_D)}{\mu(D)} \mathbb{E}((\varphi_1 \cdots \varphi_{\tau_++} \mu)_D | A_D) + (1 - \frac{\mathbb{P}(A_D)}{\mu(D)}) \nu$$

for some probability measure $\nu$. By concavity of entropy and the last lemma,

$$\frac{1}{m} H(\mu_D, D_{i+m}) \geq \frac{\mathbb{P}(A_D)}{\mu(D)} \cdot \frac{1}{m} H(\mathbb{E}((\varphi_1 \cdots \varphi_{\tau_++} \mu)_D | A_D), D_{i+m}) \geq (1 - \sqrt{\delta}) \mathbb{E} \left( \frac{1}{m} H((\varphi_1 \cdots \varphi_{\tau_++} \mu)_D, D_{i+m}) | A_D \right).$$

Conditioned on the event $A_D$ we have $(\varphi_1 \cdots \varphi_{\tau_++} \mu)_D = \varphi_1 \cdots \varphi_{\tau_++} \mu$, and since $\varphi_1 \cdots \varphi_{\tau_++} \mu$ contracts by at most $2^{-(i+\ell) r_0}$, we have

$$\frac{1}{m} H(\varphi_1 \cdots \varphi_{\tau_++} \mu, D_{i+m}) = \frac{1}{m} H(\mu, D_m) + O_{r_0, \ell}(\frac{1}{m}).$$

Combined with the previous inequality we obtain

$$\frac{1}{m} H(\mu_D, D_{i+m}) \geq \frac{(1 - \sqrt{\delta})}{m} \cdot H(\mu, D_m) + O_{r_0, \ell}(\frac{1}{m}) \geq \alpha - \varepsilon,$$

assuming $\delta$ is small and $m$ large.

The analysis above holds for $D \in D_i$ such that $\mu(D \cap V_i) < \sqrt{\delta} \mu(D)$. By Lemma 11 and assuming as we may that $\delta < \varepsilon^2$ and $m$ is large enough, this implies the proposition.

6 Growth of Hausdorff dimension under convolution

So far we have analyzed the growth of entropy at fixed small scales, which in the limit leads to results for entropy dimension. We now turn the growth of the Hausdorff dimension of measures. Technically, involves replacing the “global” distribution of components $\mathbb{P}_{\eta_x,i}$, in which $\eta_{x,i}$ is selected by randomizing both $x$ and $i$, with “pointwise” distributions of components, where $x$ is fixed and we average only over the scales. This requires us to modify some of the definitions and slightly strengthen the hypotheses. It also calls for some additional analysis, based to a large extent on the local entropy averages method.

6.1 Hausdorff and pointwise dimension

To start off, recall that the (lower) Hausdorff dimension of a measure $\eta \in \mathcal{P}(\mathbb{R})$ is given by

$$\dim_{\text{dim}} \eta = \inf \{ \dim E : \eta(E) > 0 \},$$

as $E$ ranges over Borel sets. Unlike entropy dimension, which averages the behavior of a measure over space, Hausdorff dimension is determined by the
pointwise behavior of a measure. Indeed, define the (lower, dyadic) pointwise dimension of $\eta$ at $x$ to be
\[ \dim(\eta, x) = \liminf_{n \to \infty} \frac{-\log \eta(D_n(x))}{n}. \]
(one may take the limit along integer or continuous parameter $n$). Then
\[ \dim \eta = \text{essinf}_{x \sim \eta} d(\eta, x). \]

It is elementary that if $n_i \to \infty$ and $n_{i+1}/n_i \to 1$ then in the definition of $\dim(\eta, x)$ we can take the limit along $n_i$. For reasons which will become apparent later we will want to take advantage of this freedom.

We mention a basic stability property of the local dimension:

**Lemma 13.** If $\eta \ll \theta$ are probability measures on $\mathbb{R}$ then $\dim(\eta, x) = \dim(\theta, x)$ for $\eta$-a.e. $x$.

This is a consequence of the martingale convergence theorem, according to which for $\eta, \theta$ as in the lemma, $\frac{\eta(D_n(x))}{\theta(D_n(x))} \to d\eta/d\theta(x) \in (0, \infty)$ at $\eta$-a.e. point $x$.

### 6.2 Local entropy averages

The connection of pointwise dimension and entropy is via the so-called local entropy averages method, introduced in [7]. This can be regarded as a pointwise analog of Lemmas 2 and 3. We give a version of the lemma along a sparse sequence of scales, specifically, of power growth. Let $[\cdot]$ denote the integer value function.

**Lemma 14.** Let $\tau > 0$ and let $n_i = [i^{1+\tau}]$. Then for any $\eta \in \mathcal{P}(\mathbb{R}^d)$ and $\eta$-a.e. $x$,
\[ \dim(\eta, x) \geq \liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \frac{1}{n_{i+1} - n_i} H(\eta_{x,n_i} \cdot D_{n_{i+1}}) - \tau, \]  
and if $\theta \in \mathcal{P}(\mathcal{S})$ and $\eta \in \mathcal{P}(\mathbb{R})$, then for $\theta \times \eta$-a.e. $(\varphi, x)$ and $y = \varphi(x)$,
\[ \dim(\theta, \eta, y) \geq \liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} \cdot \eta_{x,n_i} \cdot D_{n_{i+1}}) - \tau. \]  

**Proof.** We start with the first statement. Clearly $2^{-n_i} = 2^{-n_{i-1}(1+o(1))}$, so $\dim(\eta, x) = -\liminf \frac{1}{n_k} \log \mu(D_{n_k}(x))$. Set $w_{k,i} = (n_i - n_{i-1})/n_k$, so $(w_{k,1}, \ldots, w_{k,k})$ is a probability vector. From
\[ \eta(D_{n_k}(x)) = \sum_{i=0}^{k} \log \frac{\mu(D_{n_i}(x))}{\mu(D_{n_{i-1}}(x))}, \]
we find that

$$\dim(\eta, x) = -\liminf_{k \to \infty} \frac{1}{n_k} \sum_{i=1}^{k} \log \frac{\mu(D_{n_i}(x))}{\mu(D_{n_{i-1}}(x))}$$

By a variation on the law of large numbers for one-sided bounded uncorrelated $L^2$ random variables, we find that $\eta$-a.e. $x$ satisfies

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \left( -\frac{1}{n_i - n_{i-1}} H(\eta, D_{n_i} | D_{n_{i-1}}) \right) \geq 0,$$

so

$$\dim(\eta, x) \geq \liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} w_{k,i} \cdot \frac{1}{n_i - n_{i-1}} H(\eta, D_{n_i} | D_{n_{i-1}}).$$

Finally, writing $a_i = H(\eta, D_{n_i} | D_{n_{i-1}})/(n_i - n_{i-1})$, the proof is completed by showing that $\sum_{i=1}^{k} w_{k,i} a_i = \frac{1}{k} \sum_{i=1}^{k} a_i - \tau - o(1)$ as $k \to \infty$. Indeed, let

$$E_k = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq k, 1 \leq j \leq (1+\tau)k^\tau \}$$

$$F_k = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq k, 1 \leq j \leq i^{1+\tau} - (i-1)^{1+\tau} \}$$

$$\subseteq E_k$$

9Here is a proof sketch: Let $(X_i)$ be a martingale with $EX_i = 0,$ $\mathbb{E}(X_i^2) \leq a,$ and $X_i \geq -b$ for some constants $a, b > 0$. Let $w_{k,i}$ be as before, write $S_k = \sum_{i=1}^{k} w_{k,i} X_i$. We claim that $\lim inf S_k \geq 0$ a.s. Consider first the subsequence $S_{k^2}$. Using $w_{k,i} = (1 + \tau + o(1))k^{-(1+\tau)}i^\tau$ and $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$, we have

$$\mathbb{E}((S_{k^2})^2) = \sum_{i=1}^{k^2} w_{k^2,i}^2 \mathbb{E}(X^2_i) = O(k^{-4(1+\tau)} k^2 \sum_{i=1}^{k^2} i^{2\tau}) = O(k^{-2})$$

Hence by Markov’s inequality $\sum P(S_{k^2} > \epsilon) < \infty$, and by Borel-Cantelli, $S_{k^2} \to 0$ a.s. We now interpolate: for $k^2 \leq \ell < (k + 1)^2$ and using $w_{\ell,i} = (\ell \ell^{1+\tau})^{1/\ell} w_{k^2,i}$, we have

$$S_\ell = \sum_{i=1}^{k^2} w_{\ell,i} X_i + \sum_{i=k^2+1}^{\ell} w_{\ell,i} X_i$$

$$\geq \left( \frac{\ell}{k^2} \right)^{1+\tau} S_{k^2} - \sum_{i=k^2}^{\ell} w_{\ell,i} b$$

$$= (1 + o_\tau(1)) S_{k^2} - o_\tau(1),$$

from which the claim follows.
Evidently,
\[
\frac{1}{k} \sum_{i=1}^k a_i = \frac{1}{|E_k|} \sum_{(i,j) \in E_k} a_i
\]
\[
\sum_{i=1}^k w_{k,i} a_i = \frac{1}{|F_k|} \sum_{(i,j) \in F_k} a_i
\]
An elementary calculation also shows that \(|E_k|/|F_k| = 1 + \tau + o(1)|. This, together with \(|a_i| \leq 1|, implies that
\[
\sum_{i=1}^k w_{k,i} a_i = \frac{1}{|F_k|} \sum_{(i,j) \in F_k} a_i - \frac{1}{|F_k|} \sum_{(i,j) \in E_k \setminus F_k} a_i - o(1)
\]
\[
\geq \frac{|F_k|}{|E_k|} \sum_{(i,j) \in E_k} a_i - \frac{1}{|F_k|} (|E_k| - |F_k|) - o(1)
\]
\[
= (1 + \tau) \frac{1}{k} \sum_{i=1}^k a_i - (1 + \tau - 1) - o(1).
\]
\[
\geq \frac{1}{k} \sum_{i=1}^k a_i - \tau - o(1).
\]
as desired.

The second part of the lemma is a similar adaptation of the local entropy averages lemma to the action setting, similar to the projection case in [7]. We omit the details.

We need a variant for convolutions in the action setting, which may be regarded as a pointwise analog of Lemma 4. To control the error term in the linearization, we use the fact that \(n_i = [i^{1+\tau}]\) satisfies \(n_{i+1} - n_i \to \infty\).

**Lemma 15.** Let \(\tau > 0\) and let \(n_i = [i^{1+\tau}]\). Then for any \(\theta \in \mathcal{P}(G)\) and \(\eta \in \mathcal{P}(\mathbb{R})\), any \(\varphi \in \text{supp} \theta\) and \(x \in \text{supp} \eta\), and writing \((A,B) = (A\varphi,x,B\varphi,x)\) for the derivative of the action map at \((\varphi,x)\), and \(y = \varphi(x)\), we have

\[
\dim(\eta,\theta, y) \geq \liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \frac{1}{n_{i+1} - n_i} H(B^{-1} A\theta \varphi, n_i \ast \eta_{x,n_i}, D_{n_{i+1}}) - \tau. \tag{34}
\]

**Proof.** This is a combination of (33) and the linearization argument of Section 4.1 which, essentially, allows us to replace the term \(H(\theta \varphi, n_i \ast \eta_{x,n_i}, D_{n_{i+1}})\) in (33) with \(H(B^{-1} A\theta \varphi, n_i \ast \eta_{x,n_i}, D_{n_{i+1}})\). In more detail, let \(\theta' = \theta \varphi, n_i \ast \eta_{x,n_i}\). The supports of \(\theta', \eta'\) are of diameter \(O(2^{-n})\), making the error term in (4.1) of order \(O(2^{-2n})\). Then, as explained in the paragraph following (16), if \(m \ll n\) we will have \(\frac{1}{m} H(\theta', \eta', D_{n_{i+1}+m}) = \frac{1}{m} H(A\varphi \ast B\eta', D_{n_{i+1}+m}) + O(\frac{1}{m})\). Taking \(n = n_i\) and \(m = n_{i+1} - n_i\), and using \(n_{i+1}/n_i \to 1\), we obtain the bound (34), where we have moved \(B\) from one side of the convolution to the other by the same argument as before. \(\square\)
6.3 Pushing entropy from $G$ to $\mathbb{R}$ and pointwise porosity

Next, we need a pointwise version of Lemma 9 which says that large entropy of a component $\theta_{g,i}$ of $\theta \in \mathcal{P}(G)$ translates to large entropy of most push-forwards $\theta_{g,i} \circ x$.

**Lemma 16.** Let $(n_i)$ be an increasing integer sequence satisfying $\sum_{i=1}^{\infty} (n_{i+1} - n_i)^{-d} < \infty$ for some $d > 0$. Suppose that $\theta \in \mathcal{P}(G)$ and $\eta \in \mathcal{P}(\mathbb{R})$ are compactly supported and further that $\eta(A_i(x)) \leq a r^\alpha$ for some $a, \alpha > 0$. Then for $\theta \times \eta$-a.e. $(\varphi, x)$, we have

\[
\liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} \circ x, D_{n_{i+1}}) \geq \frac{1}{3} \liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} \circ x, D_{n_{i+1}}^G).
\]

**Proof.** Given $\varphi \in \text{supp } \theta$, for each $i$, let

\[ A_i = \left\{ x \in \mathbb{R} : \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} \circ x, D_{n_{i+1}}) \leq \frac{1}{3} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} \circ x, D_{n_{i+1}}^G) \right\}. \]

By Lemma 9, for every $d > 0$ we have $\eta(A_i) = O(1/(n_{i+1} - n_i)^d)$. Therefore by the assumption on $(n_i)$, there is a choice of $d$ so that $\sum \eta(A_i) < \infty$. By Borel-Cantelli, $\eta$-a.e. $x$ belongs to finitely many $A_i$, and for such $x$ the desired conclusion holds for the given $\varphi$. By Fubini, the conclusion holds for $\theta \times \eta$-a.e. pair $(\varphi, x)$. \qed

Finally, we need a notion of porosity at a point, in which, instead of describing the typical behavior of components over the whole measure, relates only to components containing a fixed point $x$ (i.e. the components $\eta_{x,i}$) and require that on average they exhibit porosity. We again do this relative to a subsequence of scales. For an integer sequence $n_i \to \infty$, we say that $\eta$ is $(h, \delta, m)$-entropy porous along $(n_i)$ at $x \in \text{supp } \eta$ if

\[
\liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} 1(\eta_{x,n_i} \text{ is } (h, \delta, m)\text{-entropy porous from scale } n_i \text{ to } n_{i+1}) > 1 - \delta.
\]

We say that $\eta$ is $h$-entropy porous along $(n_i)$ at $x$ if for every $\delta > 0$ and $m$ it is $(h, \delta, m)$-entropy porous along $(n_i)$ at $x$.

**Lemma 17.** Let $(n_i)$ be a sequence such that $n_{i+1}/n_i \to 1$ and $n_{i+1} - n_i \to \infty$, and suppose that $\eta$ is $(h, \delta, m)$-entropy porous along $(n_i)$ at $\eta$-a.e. $x$. If $\eta' \ll \eta$ then $\eta'$ is also $(h, \delta, m)$-entropy porous along $(n_i)$ at $\eta'$-a.e. $x$.

**Proof.** This follows from the fact that by the martingale convergence theorem, $\eta'_{x,i} : \eta_{x,i}$ are asymptotic in total variation (that is, $\|\eta'_{x,i} - \eta_{x,i}\| \to 0$) for $\eta'$-a.e. $x$. The details are left to the reader. \qed
6.4 Entropy growth of Hausdorff dimension under convolution

We can now state the main result of this section, an analog of Theorem 6 for Hausdorff dimension.

**Theorem 9.** For every $\varepsilon > 0$ there exists a $\delta' = \delta'(\varepsilon) > 0$ such that the following holds.

Let $\eta \in \mathcal{P}(\mathbb{R})$ be compactly supported with $\dim \eta > 0$, and for every $\tau > 0$ and $n_i = [i^{1+\tau}]$ suppose that $\eta$ is $(1 - \varepsilon)$-entropy porous along $(n_i)$ at $\eta$-a.e. $x$. Then for any $\theta \in \mathcal{P}(G)$,

$$\dim \theta > \varepsilon \implies \dim \theta \cdot \eta > \dim \eta + \delta.$$

**Proof.** Fix $\varepsilon > 0$, $\theta, \eta$. Let $\delta = \delta(\varepsilon/6)$ be as in Theorem 6, and also choose $m, n$ large enough for that theorem to hold. Write $\alpha = \dim \eta$ so we are assuming $\alpha > 0$.

Fix $0 < \tau < 1$ and $n_i = [i^{1+\tau}]$. We shall show that $\dim \theta \cdot \eta > \dim \eta + \delta \varepsilon/12 - \tau$, which is enough, since $\tau$ is arbitrary.

First, we claim that we can assume without loss of generality that there is an $a > 0$ and $\beta > 0$ such that $\eta(B_r(x)) \leq ar^\beta$ at every $x$. Indeed, given $0 < \beta < \alpha$, by Egorov’s theorem we can find disjoint sets $A_i$ whose union supports $\eta$, and such that $\eta(A_i \cap B_r(x)) \leq ar^\beta$ for each $i$. Then $\theta \cdot \eta = \sum \theta \cdot (\eta|_{A_i})$ and by Lemmas 13 and 17 it suffices to analyze a single $\eta|_{A_i}$, which puts us in the desired situation.

Let $(\varphi, x) \in G \times \mathbb{R}$ be $\theta \times \eta$-typical and set $y = \varphi(x)$, which is a $\theta \cdot \eta$-typical point. By the local entropy averages lemma (Lemma 14), it suffices for us to show that

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H((B^{-1}A\theta_{\varphi,i} \cdot x) \ast \eta_{x,n_i} D_{n_i+1}) \geq \alpha + \frac{\delta \varepsilon}{12}.$$  (36)

For this we shall analyze the behavior of the terms in the average and show that they are large for a large fraction of $i = 1, \ldots, k$, for all large enough $k$.

For the components $A^{-1}B \theta_{\varphi,i}$, we know that

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} D_{n_i+1}) = d(\theta, \varphi) \geq \varepsilon,$$

Because $(\varphi, x)$ is $\theta \times \eta$-typical, by Lemma 16

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H(\theta_{\varphi,n_i} D_{n_i+1}) \geq \frac{\varepsilon}{3},$$

which, since $B^{-1}A$ is bi-Lipschitz and $n_{i+1} - n_i \to \infty$, this implies

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H(B^{-1}A\theta_{\varphi,i} \cdot x, D_{n_i+1}) \geq \frac{\varepsilon}{3}.$$
Writing
\[ I_k = \left\{ 1 \leq i \leq k : H(B^{-1}A\theta, n_i, x, D_{n_i+1}) \geq \frac{\varepsilon}{6} \right\}, \]
this gives us
\[ \liminf_{k \to \infty} \frac{1}{k} |I_k| \geq \frac{\varepsilon}{6}. \] (37)

For the components \( \eta_{x,i} \), we also know that
\[ \liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H(\eta_{x,n_i}, D_{n_i+1}) = d(\eta, x) \geq \alpha. \]

Also, fixing a \( 0 < \gamma < \varepsilon/12 \) and some \( m \), write
\[ J_k = \{ 1 \leq i \leq k : \eta_{x,n_i} \text{ is not } (1 - \varepsilon, \gamma, m)\text{-entropy porous from scale } n_i \text{ to } n_i+1 \}. \]
Then by assumption
\[ \limsup_{k \to \infty} \frac{1}{k} |J_k| < \gamma. \] (38)

For \( i \in I_k \setminus J_k \), we can apply Theorem 3 and conclude that
\[ \frac{1}{n_{i+1} - n_i} H((B^{-1}A\theta, i, x) \ast \eta_{x,n_i}, D_{n_i+1}) \geq \frac{1}{n_{i+1} - n_i} H(\eta_{x,n_i}, D_{n_i+1}) + \delta. \]

Finally, by (37) and (38), for \( k \) large enough, \( \frac{1}{k} |I_k \setminus J_k| \geq \varepsilon/12 \), and so by the last inequality we can estimate (36) by
\[ \liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} H((B^{-1}A\theta, i, x) \ast \eta_{x,n_i}, D_{n_i+1}) \]
\[ \geq \liminf_{k \to \infty} \left( \frac{1}{k} \sum_{i \in I_k} \frac{1}{n_{i+1} - n_i} H(\eta_{x,n_i}, D_{n_i+1}) + \delta \cdot \frac{1}{k} |I_k \setminus J_k| \right) \]
\[ \geq \dim(\eta, x) + \delta \cdot \frac{\varepsilon}{12} \]
and we are done. \( \square \)

### 6.5 Proof of Theorem 4

Let \( X \subseteq \mathbb{R} \) be a compact \( c \)-porous set and \( \Phi \subseteq G \) compact with \( \dim \Phi > c \). We now prove that \( \dim \Phi, X > \dim X + \delta \) for some \( \delta = \delta(c) > 0 \).

First, choose \( \nu \in \mathcal{P}(\Phi) \) with \( \dim \nu > c \), which, since \( \dim \Phi > c \), exists by Frostman’s lemma.

Second, note that by porosity of \( X \), any \( \mu \in \mathcal{P}(X) \) is \( (1 - c') \)-entropy porous for some \( c' \) depending only on \( c \), and furthermore \( \mu \) is \( (1 - c') \)-entropy porous at
every $x \in \text{supp} \mu$ (along any sequence of scales).

Let $\delta > 0$ be the parameter $\delta'$ supplied by in Theorem 9 for $\varepsilon = \min\{c', c\}$ (so $\delta$ depends only on $c$), and use Frostman’s lemma again to find $\mu \in \mathcal{P}(X)$ with $\dim \mu > \dim X - \delta/2$.

Now, by Theorem 9

$$\dim \nu \cdot \mu > \dim \mu + \delta > \dim X + \frac{\delta}{2}.$$ 

Since, $\nu \cdot \mu$ is supported on $\Phi \cdot X$, so we get $\dim \Phi \cdot X > \dim X + \delta/2$, as desired.

7 Conjecture 2 implies Conjecture 1

7.1 A Tits-like alternative for semigroups

In this section we prove Theorem 1, which asserts that Conjecture 2 implies Conjecture 1. The main idea is use largeness of $\Phi$ to show that $\Phi$, or some power of it, contains an infinite free set (i.e. a set freely generating a semigroup).

The largeness we require is expressed both in terms of the cardinality of $\Phi$ and its algebraic properties; specifically, we require that it not be contained in too small a subgroup of $G$. Recall that a group is said to be virtually abelian if it contains a finite-index abelian subgroup. It is not too hard to show that every virtually abelian subgroups of $G$ is contained either in the isometry group, or in the stabilizer group of some point (this can be derived from Lemma 18 below). With these assumptions we will prove:

Proposition 4. Suppose that $\Phi \subseteq G$ is uncountable and is not contained in a virtually abelian subgroup. Then there exists a $k \in \mathbb{N}$ such that $\Phi^k = \{\varphi_1 \circ \ldots \circ \varphi_k : \varphi_i \in \Phi\}$ contains an infinite free set.

The fact that all generators lie in the same power $\Phi^k$ is important (it is much simpler to show that $\bigcup_{k=1}^{\infty} \Phi^k$ contains an infinite free set). Related (and much deeper) statements exist in the context of the classical Tits alternative, see e.g. [2], but they do not seem to give what we need here.

Assuming this proposition, we can prove the implication between the conjectures:

Proof of Theorem 1. Fix a compact uncountable $\Phi \subseteq S$ whose attractor $X$ is not a single point. Using compactness of $\Phi$ we can find $0 < r_0 < 1$ such that $\|\varphi\| \geq r_0$ for all $\varphi \in \Phi$. Now, $\Phi$ is not contained in the $G$-stabilizer of a point $x_0$ (since otherwise we would have $X = \{x_0\}$ contrary to assumption), nor in the isometry group (since $\Phi$ consists of contractions), so $\Phi$ is not contained in a virtually abelian subgroup. By Proposition 4 there exists a $k$ such that $\Phi^k$

10To see this note that for $m$ such that $2^{-m} < c/2$, any dyadic interval of length $2^{-i}$ contains a dyadic interval of length $2^{-(i+m)}$ disjoint from $X$. Therefore, for any component any $\mu_{x,i}$ of $\mu \in \mathcal{P}(X)$, we have $H(\mu_{x,i}, D_{i+m}) \leq \log(2^m-1)/m < 1$. The porosity statements follow from this.
contains an infinite free set. In particular for \( \ell = \left\lfloor \frac{1}{r_0^{2k}} \right\rfloor \) there is a free subset \( \Phi_0 \subseteq \Phi^k \) of size \( \ell \). Since \( \| \phi_i \| \geq r_0^{2k} \) for all \( \phi \in \Phi^k \), for any \( s \leq 1 \) we have

\[
\sum_{\phi \in \Phi_0} \| \phi \|^s \geq \sum_{\phi \in \Phi_0} r_0^{2ks} \geq \ell r_0^{2ks} > \ell r_0^{2k} \geq 1,
\]

showing that \( s(\Phi_0) \geq 1 \). By Conjecture 2 the attractor \( X_0 \) of \( \Phi_0 \) satisfies \( \dim X_0 = \min\{1, s(\Phi_0)\} = 1 \). Since \( X_0 \subseteq X \) we have \( \dim X = 1 \), giving conjecture 1.

We present the proof of the proposition, which is elementary but not short, over the next few sections.

Throughout, we parametrize \( G \) as a subset of \( \mathbb{R}^2 \), identifying \( \varphi(x) = sx + t \) with \( (s, t) \in \mathbb{R}^2 \). This parametrization differs from that used in previous sections but it simplifies some of the algebraic considerations.

### 7.2 Subgroups of \( G \)

For most of the proof we work in the group \( G^+ \) of orientation-preserving similarities of \( \mathbb{R} \). In parameter space, this is the subset \((0, \infty) \times \mathbb{R} \).

A one-parameter subgroup of \( G^+ \) is the image of a continuous injective homomorphism \( \mathbb{R} \to G \). There are two types of examples: First, the group of translations \( x \mapsto x + t \) for \( t \in \mathbb{R} \); and second, for each \( x_0 \), the \( G^+ \)-stabilizer of \( x_0 \), consisting of maps \( x \mapsto s(x - x_0) + x_0 \), \( s > 0 \).

Observe that a similarity has no fixed point if and only if it is a non-trivial translation, and if it is not a translation, then the fixed point is unique (this is just because the equation \( sx + t = x \) has no solution if \( s = 1 \) and \( t \neq 0 \), and precisely one solution if \( s \neq 1 \)). Thus every non-trivial element of \( G^+ \) belongs either to the translation group, or to a stabilizer group, but not both. Also, by uniqueness of the fixed point, the stabilizer groups of different points can intersect only in the identity. This shows that the translation and stabilizer groups cover all of \( G^+ \) but any two meet only at the identity.

**Lemma 18.** If \( H \leq G^+ \) is a 1-parameter subgroup then it is either the translation group or a stabilizer group, and in the latter case, \( \varphi H \varphi^{-1} \cap H = \{ \text{id} \} \) for all \( \varphi \in G^+ \setminus H \).

**Proof.** Let \( H \leq G^+ \) be a 1-parameter subgroup not contained in the translation group. Then there is some \( \psi \in H \) with a fixed point \( y_0 \). If \( \varphi \in H \) then \( \varphi \psi \varphi^{-1} \) fixes \( \varphi(y_0) \), but since \( H \) is abelian, \( \varphi \psi \varphi^{-1} = \psi \), so it fixes \( y_0 \). By uniqueness of the fixed point we have \( \varphi(y_0) = y_0 \), so \( \varphi \) belongs to the stabilizer group \( H' \) of \( y_0 \). Since \( \varphi \in H \) was arbitrary this shows that \( H \leq H' \), and since \( H' \) is isomorphic to \( \mathbb{R} \) it has no nontrivial closed subgroups, so \( H = H' \). This proves the first statement.

For the second statement, let \( H \) be the stabilizer of \( y_0 \), and \( \varphi \in G^+ \setminus H \), so by definition \( \varphi(y_0) \neq y_0 \). Given any \( \text{id} \neq \psi \in H \), the unique fixed point of \( \varphi \psi \varphi^{-1} \) is \( \varphi(y_0) \neq y_0 \), which shows \( \varphi \psi \varphi^{-1} \notin H \). Since \( \psi \in H \) was arbitrary, this implies \( \varphi H \varphi^{-1} \cap H = \{ \text{id} \} \).
Lemma 19. Every 1-parameter subgroup of $G^+$ is given in parameter space by the intersection of a line with $(0, \infty) \times \mathbb{R}$.

Proof. Writing $\varphi(x) = sx + t$ for a general element of $G^+$, the translation group is given by the equation $s = 1$, and the stabilizer of $x_0$ by the equation $sx_0 + t = x_0$ (and $s > 0$). These are the only 1-parameter groups by the previous lemma.

7.3 A class of curves and their stabilizers

Let $C$ denote the collection of subsets $\Gamma \subseteq G^+$ which are either singletons, lines (i.e. in coordinates they are determined by a linear equation), or in coordinates have the form $\{(s, p(s)/q(s)) : s > 0, q(s) \neq 0\}$ for some real polynomials $p, q$. An easy computation shows that $C$ is closed under the action of $G^+$ by pre- and post-composition. It is also easy to check that if $\Gamma_1, \Gamma_2 \in C$, then either $\Gamma_1 \cap \Gamma_2 = \Gamma_1 = \Gamma_2$ or else $\Gamma_1 \cap \Gamma_2$ is finite.

By Lemma 19 every 1-parameter subgroup of $G^+$ is in $C$. Given $\Gamma \in C$, set

$$G_\Gamma = \{g \in G^+ : \Gamma g \subseteq \Gamma\}.$$ 

Lemma 20. If $\Gamma \in C$ then either $G_\Gamma = \{id\}$ or $G_\Gamma \in C$ is a 1-parameter group and $\Gamma = \gamma G_\Gamma$ is a coset.

Proof. Suppose $id \neq g \in G_\Gamma$ and let $H \subseteq G^+$ be the 1-parameter subgroup containing $g$, so $H \in C$. Fix $\gamma \in \Gamma$, so that $\gamma g^n \in \Gamma$ for all $n \in \mathbb{N}$ and all these elements are distinct. Hence $\{\gamma g^n\} \subseteq \gamma H \cap \Gamma$, so $\gamma H \cap \Gamma$ is infinite. Since $\gamma H, \Gamma \in C$, we conclude that $\gamma H \cap \Gamma = \Gamma = \gamma H$.

Finally, if $id \neq g' \in G_\Gamma$ and $H'$ is the 1-parameter group containing $g'$, then by the same argument, $\Gamma = \gamma H'$. Thus $\gamma H = \gamma H'$, so $H = H'$ and in particular $g' \in H$. Since $g' \in G_\Gamma$ was arbitrary we conclude that $G_\Gamma = H$, and $\Gamma = \gamma G_\Gamma$.

Corollary 1. If $\Gamma \in C$ then $G_\Gamma = \{g \in G^+ : \Gamma g = \Gamma\}$.

7.4 Relations

A word $w(z_1, \ldots, z_n)$ over the letters $z_1, \ldots, z_n$ is a finite formal product of the letters, $z_i, z_{i_2}, \ldots, z_{i_N}$ in which all variables appear. For a sequence of elements $\varphi_1, \ldots, \varphi_n \in G^+$ we write $w(\varphi_1, \ldots, \varphi_n) = \varphi_{i_1} \varphi_{i_2} \cdots \varphi_{i_n}$ for the group element obtained by substituting $\varphi_i$ for $z_i$ in the formal product. We say that $\Phi_0 \subseteq G^+$ is free if $w(\varphi_1, \ldots, \varphi_m) = w'(\psi_1, \ldots, \psi_n)$ and $\varphi_i, \psi_i \in \Phi_0$ implies $w = w'$ (this implies that the semigroup generated by $\Phi_0$ is free, not necessarily the group; for groups, we would need to allow inverses and consider reduced words).

Given words $w, w'$ and $\varphi_i, \psi_i \in \Phi_0$, we are interested in describing the set of $\gamma \in G^+$ which satisfy the relation $w(\varphi_0, \ldots, \varphi_m, \gamma) = w'(\psi_0, \ldots, \psi_n, \gamma)$. If such an equality holds for some $w \neq w'$ we say that $\gamma$ satisfies a relation over $\Phi_0$. We begin by considering words in a certain canonical form.

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Proposition 5. Let \( \{\varphi_0\}_{i=0}^m \) and \( \{\psi_2\}_{j=0}^n \) be sequences of elements of \( G^+ \) and let \( \Gamma \) denote the set of all \( \gamma \in G^+ \) satisfying

\[
\varphi_0 \varphi_2 \varphi_1 \cdots \varphi_{2m-2} \varphi_{2m} = \psi_0 \psi_2 \psi_1 \cdots \psi_{2n-2} \psi_{2n}.
\]  

(39)

Then either \( \Gamma \) is empty, or it is a finite union of elements of \( C \), or \( \Gamma = G^+ \); and the latter occurs if and only if \( m = n \) and \( \varphi_i = \psi_i \) for all \( i = 0, \ldots, n \).

Proof. Suppose that \( \gamma(x) = sx + t \) is a solution and set \( \varphi_{2i+1} = \psi_{2i+1} = \gamma \), so that the assumption is that \( \varphi_0 \varphi_1 \cdots \varphi_{2m} = \psi_0 \psi_1 \cdots \psi_{2n} \). Write \( \varphi_i(x) = a_i x + b_i \) and \( \psi_i(x) = c_i x + d_i \), in particular \( a_{2i+1} = c_{2i+1} = s \) and \( b_{2i+1} = d_{2i+1} = t \). We compute the product explicitly:

\[
\varphi_0 \varphi_1 \cdots \varphi_{2m}(x) = b_2m + a_{2m} (b_{2m-1} + a_{2m-1} (b_{2m-2} + a_{2m-2} (\ldots (x))))
\]

\[
= \left( \prod_{i=0}^{2m} a_i \right) x + \sum_{i=0}^{2m} b_i \cdot \left( \prod_{j=i+1}^{2m} a_i \right)
\]

\[
= s^m \left( \prod_{i=0}^{m} a_{2i} \right) x + \left( t \sum_{i=1}^{m} s^{m-i} \prod_{\ell=i}^{m} a_{2\ell} \right) + \sum_{i=0}^{m} \left( b_{2i} \cdot s^{m-i} \cdot \prod_{\ell=i+1}^{m} a_{2\ell} \right),
\]

(40)

where in the last equality we simply separated out the term containing \( x \), the terms containing \( t \), and the rest. The corresponding formula holds for \( \psi_0 \cdots \psi_{2n} \).

Thus, in order for \( \varphi_0 \cdots \varphi_{2m} = \psi_0 \cdots \psi_{2n} \), we must have agreement between the coefficient of \( x \) and the constant term in each product. The first of these conditions translates to

\[
s^m \left( \prod_{i=0}^{m} a_{2i} \right) = s^n \left( \prod_{i=0}^{n} c_{2i} \right).
\]

(41)

There is either a unique positive solution \( s \), or, if \( m = n \) and \( \prod_{i=0}^{n} a_{2i} = \prod_{i=0}^{n} c_{2i} \), every \( s \) is a solution.

The equality of the constant terms (those not involving \( x \)) yields an equation \( e(s)t + f(s) = 0 \) in which the coefficients \( e(s), f(s) \) are given by

\[
e(s) = \sum_{i=1}^{m} \left( s^{m-i} \prod_{\ell=i}^{m} a_{2\ell} \right) - \sum_{i=1}^{n} \left( s^{n-i} \prod_{\ell=i}^{n} c_{2\ell} \right)
\]

\[
f(s) = \sum_{i=0}^{m} \left( b_{2i} \cdot s^{m-i} \cdot \prod_{\ell=i+1}^{m} a_{2\ell} \right) - \sum_{i=0}^{n} \left( d_{2i} \cdot s^{n-i} \cdot \prod_{\ell=i+1}^{n} c_{2\ell} \right),
\]

which are polynomial in \( s \).

If every \( s \) solves (39), we distinguish two cases.
Case 1 \(e(s)\) is not identically zero. Then for every \(s\) such that \(e(s) \neq 0\), the equation \(e(s)t + f(s) = 0\) has the unique solution \(t = -f(s)/e(s)\), and we have found the curve \((s, -f(s)/e(s))\) in solution space. There may also be finitely many values of \(s\) for which \(e(s) = 0\). For such \(s\), if \(f(s) \neq 0\) there is no solution, while if \(f(s) = 0\) any \(t\) is a solution, and we have found a line in solution space.

Case 2 \(e(s)\) is identically zero. Since \(a_i, c_i \neq 0\) this can happen only if \(m = n\). Then by comparing coefficients we find by induction \(a_{2i} = c_{2i}\), for all \(i = 1, \ldots, n\). Since we are assuming \((\prod)\) is a trivial equation, \(\prod_{i=0}^n a_{2i} = \prod_{i=0}^n c_{2i}\), and the corresponding terms are equal and non-zero for \(i \geq 1\), they are equal also for \(i = 0\), and we find that \(a_{2i} = c_{2i}\) for all \(i = 0, \ldots, n\). Next, if \(b_{2i} = d_{2i}\) for all \(i\), then we would have \(\varphi_{2i} = \psi_{2i}\) for all \(i\), and the solution space is all of \(G^+\). Otherwise there is an \(i\) with \(b_{2i} \neq d_{2i}\), and this, together with \(a_{2i} = c_{2i}\) for all \(i\), implies that \(f(s)\) is not the zero polynomial. Recalling that \(e(s) = 0\) for all \(s\), our equation has become \(0t + f(s) = 0\), which can be solved only when \(f(s) = 0\). This occurs for finitely many values of \(s\), and when it does, any \(t\) solves the equation, giving a line in solution space.

On the other hand, suppose \((\prod)\) has a unique solution \(s_0\). Then the solution set \(\Gamma\) of the original relation consists of those \((s_0, t) \in (0, \infty) \times \mathbb{R}\) for which \(t\) satisfies \(e(s_0)t + f(s_0) = 0\). This equation either has no solutions, one solution \(t_0\) (in which case \(\Gamma = \{(s_0, t_0)\}\), or else every \(t\) solves it, in which case \(\Gamma\) is the line \(s = s_0\).

Examining the result in each of the cases, we find we have proved the proposition.

\[\square\]

Corollary 2. Let \(w(z_0, \ldots, z_m, z)\) and \(w'(z_0, \ldots, z_n, z)\) be words, let \(\Phi_0 \subseteq G^+\) be a free set, and let \(\varphi_0, \ldots, \varphi_m, \psi_0, \ldots, \psi_n \in \Phi_0\). Let \(\Gamma\) be the set of \(\gamma \in G^+\) such that \(w(\varphi_0 \ldots \varphi_m, \gamma) = w(\psi_0 \ldots \psi_n, \gamma)\). Then either \(\Gamma = G^+, n, w = w', or \Gamma\) is a finite union of elements of \(C\).

Proof. By multiplying together consecutive occurrences of the \(\varphi_i, \psi_i\), breaking occurrences of \(\gamma^k\) into \(\gamma \Id \gamma \Id \ldots \Id \gamma\), and inserting if necessary the identity at the beginning and end of the product, \(\Gamma\) becomes the set of \(\gamma\) satisfying a relation

\[\varphi_0' \gamma \varphi_2' \gamma \ldots \varphi_{2(m'-1)}' \gamma \varphi_{2m'}' = \psi_0' \gamma \psi_{2}' \gamma \ldots \psi_{2(n'-1)}' \gamma \psi_{2n'}',\]

with each \(\varphi_i', \psi_i'\) either the identity or a product of the original \(\varphi_i, \psi_i\). By the proposition, either \(\Gamma\) is a finite union of elements of \(C\) or \(\Gamma = G^+, n, m' = n'\) \(\varphi_i' = \psi_i'\). In the latter case, because \(\Phi_0\) is free, this means that each \(\varphi_i' = \psi_i'\) decomposes uniquely as a product of the original \(\varphi_i, \psi_i\), and we conclude that \(m = n\) and \(\varphi_i = \psi_i\) as claimed.

\[\square\]

7.5 Proof of Proposition 4: Cosets of the translation group

We first prove a special case of Proposition 4 in which \(\Phi \subseteq G\) is contained in a coset of the translation group, or equivalently, there is some common \(a \neq 0\)
such that all \( \varphi \in \Phi \) are of the form \( x \mapsto \alpha x + \beta \) for some \( \beta \).

If \( \varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \in \Phi \) satisfy \( \varphi_1 \cdots \varphi_m = \psi_1 \cdots \psi_n \), then, writing \( \varphi_1(x) = \alpha x + \beta \) and \( \psi_j(x) = \alpha x + \delta \), by a similar calculation to the one in (40) we have
\[
a^m + \sum_{i=1}^{m} a^{m-i} \beta_i = a^n + \sum_{i=1}^{n} a^{n-i} \delta_i. \tag{42}
\]

Let \( E = 1 \cup \{ \beta_i \} \cup \{ \delta_i \} \) and note that the union may not be disjoint. Grouping together the coefficients of each \( e \in E \) in the last equation, we obtain an equation of the form
\[
\sum_{e \in E} p_e(a) \cdot e = 0,
\]
where \( p_e(\cdot) \) is a polynomial with coefficients \( \pm 1 \) and \( 0 \). If \( a \) is not the root of any polynomial of this kind, this implies each \( \beta_i \) is in the field generated by the other \( \beta_j, \delta_j \) and \( a \). Thus we can produce an infinite free set in \( \Phi \) by choosing the \( i \)-th map \( x \mapsto \alpha x + \beta_i \) in such a way that \( \beta_i \in \mathbb{R} \setminus \mathbb{Q}(a, b_1, \ldots, b_{i-1}) \), which we can do because \( \Phi \) is uncountable.

However, when \( a \) is the root of a polynomial with coefficients \( \pm 1, 0 \), this argument and its conclusion fail. For example, consider either of the roots \( a = (1 \pm \sqrt{5})/2 \) of the equation \( x^2 - x - 1 = 0 \), and the words
\[
w(z_1, z_2) = z_1 z_2 z_2 \quad \text{and} \quad w'(z_1, z_2) = z_2 z_1 z_1.
\]
Then for any \( \varphi(x) = \alpha x + b \) and \( \psi(x) = \alpha x + d \), the relation \( w(\varphi, \psi) = w'(\psi, \varphi) \) is equivalent to
\[
b + ad + a^2 d + a^3 = d + ab + a^2 b + a^3.
\]
Rearranging we get
\[
b(a^2 - a - 1) = d(a^2 - a - 1).
\]
Since \( a^2 - a - 1 = 0 \), every \( b, d \) satisfy this, so \( \{ \varphi, \psi \} \) is not free for any choice of \( b, d \).

We can avoid this problem by taking finitely many powers.

**Proposition 6.** If \( \Phi \subseteq G \) is uncountable and contained in a non-trivial coset of the translation group, then there exists a \( k \in \mathbb{N} \) such that \( \Phi^k \) contains an infinite free subset.

**Proof.** Suppose \( a \in \mathbb{R} \) and all elements of \( \Phi \) are of the form \( x \mapsto \alpha x + b \) for some \( b \). All non-zero roots of a polynomial with coefficients \( \pm 1, 0 \) have modulus in the range \( (\frac{1}{2}, 2) \), so if \( |a| \geq 2 \) or \( |a| \leq \frac{1}{2} \) we can use the construction discussed above to obtain an infinite free subset of \( \Phi \). Otherwise set \( k = || \log_2 |a| \|| \) and note that every \( \varphi \in \Phi^k \) is of the form \( x \mapsto \alpha^k x + b \) for some \( b \), and \( |a| \notin (\frac{1}{2}, 2) \) by choice of \( k \), so by the same argument we can find a free subset of \( \Phi^k \). \( \square \)
7.6 Proof of Proposition 4: Other cosets

**Proposition 7.** If $\Phi \subseteq G^+$ is uncountable and contained in a non-trivial coset $\varphi_0 H$ of a 1-parameter subgroup $H$ other than the translation group. Then $\Phi^2$ contains an infinite free subset.

**Proof.** Write $F = \varphi_0 H$; we first claim that the collection $\{\varphi F\}_{\varphi \in \Phi}$ is pairwise disjoint. If not, then $\varphi_1 f_1 = \varphi_2 f_2 \neq \emptyset$ for some distinct pair $\varphi_1, \varphi_2 \in \Phi$ and some $f_1, f_2 \in F$. Since $\Phi \subseteq \varphi_0 H = F$ we can write $\varphi_i = \varphi_0 h_i$ for some distinct $h_1, h_2 \in H$, and $f_i = \varphi_0 \overline{h_i}$ for some $\overline{h_1}, \overline{h_2} \in H$. Thus

$$\varphi_0 h_1 \varphi_0 \overline{h_1} = \varphi_0 h_2 \varphi_0 \overline{h_2}. \quad (43)$$

Since $h_1 \neq h_2$ we conclude that $\overline{h_1} \neq \overline{h_2}$. But rearranging (43) gives $\varphi_0^{-1} h_2^{-1} h_1 \varphi_0 = \overline{h_2 \overline{h_1}} \neq \text{id}$, showing that $\varphi_0^{-1} H \varphi_0 \cap \overline{H} \neq \{\text{id}\}$. By Lemma 18 this can occur only if $H$ is the translation group, contrary to our assumption.

It suffices for us to show that given a finite free subset $\Delta \subseteq G^+$ we can find $\gamma \in \Phi^2$ such that $\Delta \cup \{\gamma\}$ is free, since we can then build an infinite free set by induction. Fix $\Delta$. By Corollary 2 the set of all $\gamma \in G^+$ such that $\Delta \cup \{\gamma\}$ is not free is a countable union of sets $\Gamma_1, \Gamma_2, \ldots \subseteq C$, so $\Delta \cup \{\gamma\}$ is free for any $\gamma \in \Phi^2 \setminus \bigcup_{i=1}^{\infty} \Gamma_i$. Thus, our goal is to show that $\Phi^2 \setminus \bigcup_{i=1}^{\infty} \Gamma_i \neq \emptyset$. Now, if $\varphi F \cap \Gamma_i$ is infinite for some $\varphi$ and $i$, then $\varphi F = \Gamma_i$ (because both sets are in $C$), hence, since $\{\varphi F\}_{\varphi \in \Phi}$ is pairwise disjoint, for each $i$ there is at most one $\varphi \in \Phi$ such that $\varphi F \cap \Gamma_i$ is infinite. Therefore, since $\Phi$ is uncountable, there must be some $\varphi \in \Phi$ such that $\varphi F \cap \Gamma_i$ is finite for all $i$. Since $\Phi \subseteq F$ also $\varphi \Phi \cap \Gamma_i$ is finite, and since $\Phi$ is uncountable, $\varphi \Phi \setminus \bigcup_{i=1}^{\infty} \Gamma_i \neq \emptyset$, as desired. \qed

7.7 Proof of Proposition 4: Orientation-preserving case

**Proposition 8.** If $\Phi \subseteq G^+$ is uncountable and is not contained in a 1-parameter subgroup then either $\Phi$ or $\Phi^2$ contains an infinite free subset.

**Proof.** It suffices to show that $\Phi \cup \Phi^2$ contains an infinite free subset. To do this it suffices to show that, given a finite free set $\Delta \subseteq G^+$, there is a $\gamma \in \Phi \cup \Phi^2$ such that $\Delta \cup \{\gamma\}$ is free. Fix $\Delta$, and define $\Gamma_0, \Gamma_1, \Gamma_2, \ldots \subseteq C$ as in the proof of the previous proposition, so we must show that there is a $k$ with $\Phi \cup \Phi^2 \not\subseteq \bigcup_{i=1}^{\infty} \Gamma_i$.

If $\Phi \cap \Gamma_i$ is countable for all $i$ we are done, since $\Phi$ is uncountable. So suppose one of the intersections is uncountable; without loss of generality it is $\Phi \cap \Gamma_0$, and write $\Phi_0 = \Phi \cap \Gamma_0$.

If for some $\varphi \in \Phi_0$ we have $\Phi_0 \varphi \not\subseteq \bigcup \Gamma_i$ then we are done, so assume the contrary. Then for each $\varphi \in \Phi$ we have $\Phi_0 \varphi \not\subseteq \bigcup_{i=0}^{\infty} \Gamma_i$ and by another cardinality argument there is some $i = i(\varphi)$ such that $\Phi_0 \varphi \cap \Gamma_i$ is uncountable, and since $\Phi_0 \varphi \cap \Gamma_i \subseteq \Gamma_0 \varphi \cap \Gamma_i$ also $\Gamma_0 \varphi \cap \Gamma_i$ is uncountable. Since both sets are in $C$ we conclude that $\Gamma_0 \varphi = \Gamma_{i(\varphi)}$.

Since $\Phi_0$ is uncountable, there must be distinct $\varphi, \psi \in \Phi_0$ such that $i(\varphi) = i(\psi)$, i.e. $\Gamma_0 \varphi = \Gamma_0 \psi$, or equivalently, $\Gamma_0 = \Gamma_0 \psi^{-1}$. Thus $\psi \varphi^{-1}$ is a non-trivial element of $G_{\Gamma_0}$, so by Lemma 20 $\Gamma_0$ is a coset $\varphi_0 H$ of the 1-parameter group $H = G_{\Gamma_0}$.

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If \( \varphi_0 \notin H \) (i.e., \( \Gamma_0 \neq H \)) we are done, since we have \( \Phi_0 \subseteq \varphi_0 H \) and we can apply Proposition \( 6 \) or \( 7 \) to \( \Phi_0 \).

Otherwise \( \varphi_0 \in H \) and \( \Phi_0 \subseteq H \). But by hypothesis \( \Phi \not\subseteq H \), and we can choose \( \psi \in \Phi \setminus H \). Then \( \psi \Phi_0 \subseteq \psi H \), and we can again apply Proposition \( 6 \) or \( 7 \) to \( \psi \Phi_0 \). Since \( \psi \Phi_0 \subseteq \Phi \), this gives the claim.

### 7.8 Proof of Proposition 4: general case

Suppose that \( \Phi \subseteq G \) is uncountable and not virtually abelian. We shall show that \( \Phi^2 \cap G^+ \) is uncountable and not contained in a 1-parameter group. This is enough, since we can then apply the results of the previous section to \( \Phi^2 \cap G^+ \).

Observe that the group \( H \) of all \( \varphi \in G \) fixing a given \( x_0 \in \mathbb{R} \) is virtually abelian, since \( H \cap G^+ \) is abelian and \( H \cap G^+ \) is the kernel of the homomorphism \( H \to \{\pm 1\} \) mapping \( \varphi(x) = ax + b \) to \( \text{sgn} \, a \), implying that \( H \cap G^+ \) has index two in \( H \). Similarly, the isometry group of \( \mathbb{R} \) contains the group of translations as an index two subgroup, so it is also virtually abelian. In particular, we conclude that \( \Phi \) is neither contained in the isometry group, nor in the stabilizer in \( G \) of any \( x_0 \in \mathbb{R} \).

Since either \( \Phi \cap G^+ \) or \( \Phi \cap (G \setminus G^+) \) must be uncountable, and the square of each of these sets is contained in both \( \Phi^2 \) and \( G^+ \), certainly \( \Phi^2 \cap G^+ \) is uncountable. Let \( \text{id} \neq \varphi \in \Phi^2 \cap G^+ \).

Suppose that \( \varphi \) is a translation. Since \( \Phi \) is not contained in the isometry group, there is a \( \psi \in \Phi \) which is not an isometry. But then \( \psi^2 \in \Phi^2 \) also is not an isometry, and \( \psi^2 \in G^+ \). Thus \( \Phi^2 \cap G^+ \) contains both translations and nontrivial elements that are not translations, so \( \Phi^2 \cap G^+ \) is not contained in a 1-parameter group.

Otherwise, since \( \varphi \) is not a translation or the identity, it fixes some point \( x_0 \). Since \( \Phi \) is not contained in the stabilizer group of \( x_0 \), there is some \( \psi \in \Phi \) that does not fix \( x_0 \). Then \( \psi^2 \) also does not fix \( x_0 \) (for either \( \psi \) fixes another point, and \( \psi^2 \) does as well, or else \( \psi \) was already a translation without fixed points, and then \( \psi^2 \) is too). Also, \( \psi^2 \in G^+ \). Thus \( \Phi^2 \cap G^+ \) cannot be contained in a 1-parameter subgroup.

This completes the proof of Proposition 4.

### 8 One more variation

We conclude with a variation on Conjecture 1 in the non-linear setting, where we as yet are unable to prove even the analog of Theorem 2:

**Problem 3.** Let \( \Phi \subseteq C^\omega([0,1]) \) be a compact set of contracting real-analytic maps of \([0,1]\). Let \( X \) denote the attractor of \( \Phi \). If \( \dim \Phi > 0 \) and \( X \) is not a singleton, is \( \dim X = 1 \)?

This question is not well posed because \( C^\omega \) does not carry a canonical metric through which to define the condition \( \dim \Phi > 0 \). One can easily imagine suitable definitions, though, for example we could ask the dimension to be positive in
the $C^1$-metric, or as a subset of $L^2$. Note that formulating the problem for $C^\alpha$, $1 \leq \alpha \leq \infty$, poses some difficulty, since any proper compact subset $X \subseteq [0,1]$ admits a positive dimension set of $C^\alpha$-maps preserving it - namely maps which are the identity on $X$ and act only in its complement. Working with analytic maps eliminates this problem and is in any case the most likely case to be true.

Certain aspects of our proof carry over to this setting, in particular, one can linearize the action, and to some extent obtain an analog of Proposition 1. One technical difficulty here is the absence of a dyadic-like partition of the ambient vector space; one can introduce refining partitions which at each stage consist of cells comparable to a ball, but each cell will split into countably many sub-cells at each stage. This means that the iterated entropy formulas from Section 3.3 become rather useless, because, while formally correct, all the entropy of components could be concentrated at a negligible fraction of levels. It may be possible to overcome this by looking for a partition of $\Phi$ rather than the whole space, but this does not completely solve the problem.

Another crucial issue is that, even when a suitable partition of $\Phi$ can be found, the analogue of Lemma 8 (and consequently Lemma 9) may be false. That lemma was based on the fact that the map $f \mapsto (f(x_1), \ldots, f(x_k))$ determines $f$ when $x_i$ are well-separated points and $k$ is large enough. This occurs when $\Phi$ is contained in a finite-dimensional parameter space, but in general it will fail. The only remedy we know of at present is to make some finite-dimensionality assumptions which are rarely satisfies. In fact the only nontrivial application of these ideas at present appears in [8], where this strategy was applied to stationary measures on the projective line under the (projective) action of $SL_2(\mathbb{R})$. In general Problem 3 remains open.

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