Two loop $O(N_f \alpha_s^2)$ corrections to the decay width of the Higgs boson to two massive fermions.

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Abstract

We present an analytical calculation of additional real or virtual radiation of the light fermion pair in the fermionic decay of the Higgs boson $H \to f_1 \bar{f}_1$ for arbitrary ratios of the Higgs boson mass to the $f_1$ fermion mass. This result gives us a value of the $O(N_f \alpha_s^2)$ radiative correction to the inclusive decay rate $H \to f_1 \bar{f}_1$. Using this result in the framework of the Brodsky-Mackenzie-Lepage scheme, we discuss the scale setting in the one-loop QCD correction to the decay width $H \to f_1 \bar{f}_1$ for arbitrary relation between the Higgs boson and fermion masses.

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1 Introduction

Fermionic decay channels of the Standard Model (SM) Higgs boson are important channels for both discovery and investigation of this particle [1], [2]. Direct observation of such decays can give important information about Higgs–fermion Yukawa couplings and hence provide still absent check of the symmetry breaking mechanism in the fermion sector of the SM.

In this paper we discuss the $O(N_f \alpha^2)$ correction to the decay of the Higgs boson to the pair of massive fermions $H \to f_1 \bar{f}_1$ for arbitrary relation between the Higgs mass and the mass of the fermion $f_1$. This decay is studied in the literature good enough. The one-loop QCD radiative correction to the fermionic partial width of the Higgs boson was calculated long ago [3]. Since then, the studies of this decay were concentrated on the analyses of the limit $m_H \gg m_1$ (from the phenomenological point of view this is definitely a good approximation for the decay $H \to b \bar{b}$). In this limit the renormalization group methods were applied to this partial decay width [4] and the exact results on the complete $O(\alpha^2)$ correction including the power suppressed terms $O(m_1^2/m_H^2)$ were obtained analytically [4], [6]. Recently in the paper [7] these results were rederived and some new, previously missed, contributions were calculated.

Our work is motivated by the known fact that the QCD radiative corrections to a number of processes involving Higgs boson - quark interactions appear to be large. It is possible to attribute a bulk of them to the running of the Yukawa coupling or equivalently to the running of the fermion mass. In this respect it is important to calculate next-to-leading order QCD radiative corrections in order to reduce the scale ambiguity of the leading order results and check our understanding of the resummation procedures based on the renormalization group equations.

The complete analyses of the problem requires complete two-loop calculation of the QCD radiative correction to the $H \to f_1 \bar{f}_1$ decay channel for arbitrary relation between Higgs and quark masses. This task is too complicated at present.

A more easy way is provided by the Brodsky-Mackenzie-Lepage (BLM) method [8]. This method gives a possibility to obtain a value of the correct scale in the one-loop QCD correction (and hence a good idea of a two-loop contribution) by considering the two-loop diagrams, which arise due to the light fermion loop insertions into the gluon propagator in the one-loop QCD correction (see Fig.1).

Similar arguments and techniques were used for the analyses of the scale setting in the QCD radiative corrections to the $\rho$-parameter and to the top quark decay width [9].

In some sense, the results presented here are complementary to the results presented in [4], [6], [7] – we calculate the part of the two-loop QCD radiative correction, which corresponds to the running of the QCD coupling constant, however keeping a relation between the Higgs boson mass and the mass of the fermion arbitrary. We
mention here, that our approach is similar to the one of Ref. [10].

The subsequent part of the paper is organized as follows: in the next section we discuss real radiation of the light fermion pair in \( H \to f_1 \bar{f}_1 \); in the section 3 we analyse virtual radiation of the fermion pair in two cases \( m_1 \gg m_2 \) and \( m_1 = m_2 \); in the section 4 we discuss the \( O(N_f \alpha_s^2) \) correction to the total decay width \( H \to f_1 \bar{f}_1 \); finally we present some remarks and conclusions.

Some comments on our notations are in order. It is clear that the major part of our discussion applies to the QED case as well. Hence, in the first two sections we use the QED terminology. While discussing the total decay width \( H \to f_1 \bar{f}_1 \) in the section 4, we switch to the QCD notations, explicitly indicate \( N_f \) dependence of the result and use appropriate colour factors.

2 Real decay rate \( H \to f_1 \bar{f}_1 f_2 \bar{f}_2 \).

The Higgs boson couples to fermions proportionally to their masses. Hence, we consider only diagrams where the Higgs boson is connected with the heavy fermion lines. The generic graphs are shown in the Fig.1.

![Generic diagrams for the \( O(N_f \alpha_s^2) \) correction.](image)

The decay rate for the process \( H \to f_1 \bar{f}_1 f_2 \bar{f}_2 \), normalized to the lowest order width \( \Gamma(H \to f_1 \bar{f}_1) \), can be written as a two-dimensional integral:

\[
\frac{\Gamma(H \to f_1 \bar{f}_1 f_2 \bar{f}_2)}{\Gamma(H \to f_1 \bar{f}_1)} = -\frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 F_R
\]

\[
F_R = \frac{1}{\beta^3} \int_{4r_1}^{(1-\sqrt{17})^2} \int_{4r_2}^{(1-\sqrt{7})^2} dy dz \frac{(1 + 2r_2)}{z} \sqrt{1 - \frac{4r_2}{z}} \times \left\{ \frac{1}{1 - y + z + \beta(y) \Lambda^{1/2}(1, y, z)} \log \frac{1 - y + z + \beta(y) \Lambda^{1/2}(1, y, z)}{1 - y + z - \beta(y) \Lambda^{1/2}(1, y, z)} \right\}
\]
\[+ \frac{4 \beta(y) \Lambda^{1/2}(1, y, z)(1 - 4r_1)(2r_1 + z)}{(1 - y + z)^2 - \beta(y)^2 \Lambda(1, y, z)} \]  

where

\[r_1 = \frac{m_1^2}{m_H^2}, \quad r_2 = \frac{m_2^2}{m_H^2}, \quad \beta = \sqrt{1 - 4r_1} \]

and

\[\beta(y) = \sqrt{1 - \frac{4r_1}{y}}, \quad \Lambda(1, y, z) = 1 + y^2 + z^2 - 2(y - z - yz) \]

The integration variables have the following meaning: \(m_H^2 y\) is the invariant mass of the pair of heavy fermions \(f_1 \bar{f}_1\) while \(m_H^2 z\) gives the invariant mass of the pair of light fermions \(f_2 \bar{f}_2\).

In the limit of interest \((m_1 \gg m_2)\) the integration can be split into the “soft” and “hard” regions, depending on the energy of the light fermion pair. Integrations over these regions are performed separately.

Our result for the decay rate \(H \to f_1 \bar{f}_1 f_2 \bar{f}_2\) can be written in the following way:

\[F_R = f_R^{(2)} \log \frac{m_2^2}{m_H^2} + f_R^{(1)} \log \frac{m_2^2}{m_H^2} + f_R^{(0)} \]

Below we present the functions \(f_R^{(2)}, f_R^{(1)}, f_R^{(0)}\), obtained by direct integration of the Eq. (1).²

\[f_R^{(2)} = \frac{1 + p^2}{2(1 - p^2)} \log(p) + \frac{1}{2} \]

\[f_R^{(1)} = \frac{1 + p^2}{1 - p^2} \left\{ 4 \text{Li}_2(-p) + 6 \text{Li}_2(p) - 4 \zeta(2) + 2 \log(1 - p) \log(p) \right\} + 4 \log(1 + p) \log(p) - \frac{1}{2} \log^2(p) \}

\[+ \left( -11p^4 + 4p^3 + 20p^2 - 44p + 13 \right) \frac{\log(p)}{6(1 - p)^2(1 - p^2)} \log(p) + \frac{59(1 + p^2) - 136p}{12(1 - p)^2} \]

\[f_R^{(0)} = \frac{1 + p^2}{1 - p^2} \left\{ 8 \text{Li}_3(\frac{1}{1 + p}) - 4 \text{Li}_3(p^2) - 8 \text{Li}_3(1 - p) - 10 \text{Li}_3(1 - p^2) - 8 \text{Li}_3(p) \right\} + \frac{5 \zeta(3)}{3} \log^3(1 + p) + \frac{1}{6} \log^3(p) - 16 \log(p) \log^2(1 - p) - 16 \log(1 + p) \log(1 - p) \log(p) + 2 \log^2(p) \log(1 - p) \]

\[- 12 \text{Li}_2(p)(2 \log(1 - p) - \log(p)) - 8 \text{Li}_2(-p)(2 \log(1 - p) - \log(p)) \]

²Note, that the double logarithmic form factor is proportional to the infra-red divergent part of the partial decay width \(H \to f_1 \bar{f}_1 g\).
Here \( p \) is defined through the equation:

\[
\frac{m_H^2}{m_1^2} = \frac{(1+p)^2}{p}.
\]

\( \text{Li}_2 \) and \( \text{Li}_3 \) are di- and trilogarithms, defined in accordance with [11].

Equations (5)-(7) give the exact result for the real decay rate \( H \to f_1 \bar{f}_1 f_2 \bar{f}_2 \) in the limit \( m_H \gg m_2, m_1 \gg m_2 \).

Let us consider now the limit of the heavy Higgs boson, which is given by the conditions \( r_1 \ll 1, \quad r_2 \ll r_1 \). In this case we expand the complete formulae up to the terms of the order of \( O(r_1) \) and get:

\[
F_R \to \frac{1}{2} \log(r_2)^2 (\log(r_1) + 1) + \log(r_2) \left( -\frac{1}{2} \log^2(r_1) + \frac{13}{6} \log(r_1) - 4\zeta(2) + \frac{59}{12} \right) + \frac{1}{6} \log^3(r_1) - \frac{13}{12} \log^2(r_1) + \left( \frac{133}{36} + 2\zeta(2) \right) \log(r_1) - 5\zeta(3) - \frac{55}{6} \zeta(2) + \frac{433}{36} \\
+ r_1 \left[ \log(r_2)^2 + \log(r_2) (-3 \log(r_1) + \frac{53}{6}) + \frac{3}{2} \log^2(r_1) \right] - \frac{7}{2} \log(r_1) - \zeta(2) + \frac{403}{18} \tag{8}
\]

The opposite limit is realized when the mass of the Higgs boson is close to two fermion masses: \( r_1 \approx 1 \). In this case the velocity of the fermion is small. The photon couples to a slow fermion proportionally to the velocity of the latter. Hence we expect, that in the limit \( r_1 \approx 1 \) the emission of the pair should be suppressed as
the square of the velocity. Calculating this limit from the complete expression, one finds:

\[
F_R \to \beta^2 \left[ -\frac{2}{3} \log^2(r_2) + \log(r_2) \left( \frac{16}{3} \log(2\beta) - \frac{799}{12} \right) - \frac{32}{3} \log^2(2\beta) + \frac{799}{18} \log(2\beta) + 8\zeta(2) - \frac{27425}{432} \right] \quad (9)
\]

3 Virtual radiative correction.

3.1 General formulas.

In this section we discuss virtual radiation of the additional fermion pair. First we present some general formulas which are valid for arbitrary relation between fermionic masses \(m_1\) and \(m_2\). Later we analyse two cases of practical importance: \(m_1 = m_2\) and \(m_1 \gg m_2\).

Additional virtual radiation of the fermion pair corresponds to the insertion of the fermion loop to the gluon line in the one-loop QCD correction (Fig.1a). The first step in this consideration is to write the contribution of the light fermion pair to the gluon polarization operator through dispersion integral subtracted at zero momentum transfer, which corresponds to the QED-like normalization of the coupling constant \[3\] :

\[
\frac{1}{k^2} \to \frac{\alpha}{3\pi} \int \frac{d\lambda^2}{4m_2^2} \frac{1}{\lambda^2 - k^2 + i\epsilon} \left( 1 + \frac{2m^2}{\lambda^2} \right) \sqrt{1 - \frac{4m^2}{\lambda^2}} \quad (10)
\]

Due to the vector current conservation, \(k_\mu k_\nu\) part of the polarization operator does not contribute to the physical amplitude.

To evaluate the \(O(N_f\alpha_s^2)\) correction to the Yukawa coupling we consider both bare radiative correction to the triangle graph (Fig.1a) and the counterterms. In both we insert the fermion loop into the gluon line. After writing polarization operator through dispersion integral (Eq.10), integration over “gluon mass” \(\lambda\) factorizes. Hence as the first step we evaluate corrections to the Yukawa coupling coming from the massive vector boson exchange between heavy fermions and then integrate this result over the masses of the vector boson with the spectral density given by the fermion contribution to the imaginary part of the gluon polarization operator (see Eq.(10)). It is clear then, that we can discuss renormalization already at the first step of our calculation.

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3 As far as we are concerned with the QED-like graphs such subtraction is evidently possible. Technically, it is more convenient to change the scale of the coupling constant in the final result, than to subtract gluon vacuum polarization at the arbitrary scale.
The counterterms Lagrangian is known from the one-loop QCD radiative correction \[3\] and can be written as

\[ L_{ct} = g_Y(\Sigma_S(m, \lambda) + 2m_1^2(\Sigma'_V(m, \lambda) + \Sigma'_S(m, \lambda)))\bar{\psi}\psi H. \tag{11} \]

Here \( g_Y \) is the Yukawa coupling, \( \Sigma_V, \Sigma_S \) are defined through the quark mass operator:

\[ \Sigma(p, \lambda) = -i\left(\hat{p}\Sigma_V(p, \lambda) + m_1\Sigma_S(p, \lambda)\right) \tag{12} \]

and \( \Sigma'_V, \Sigma'_S \) is the derivative of the corresponding quantity with respect to \( p^2 \). In accordance with the preceding discussion we explicitly indicate a dependence of the quark mass operator on the gluon mass \( \lambda \).

To proceed further to a more precise discussion let us fix the notations. We write the Yukawa coupling of the Higgs boson to the fermions in the following way:

\[ -ig_YT_{V}^{(2)}\bar{\psi}\psi H \tag{13} \]

where \( T_V^{(2)} \) is the two-loop form factor. The sum of the bare radiative corrections to the vertex and the counterterms, gives the following representation for the \( O(N_f\alpha_s^2) \) correction to the \( Hf\bar{f}_1 \) vertex:

\[ T_{V}^{(2)}(\lambda^2) = \frac{\alpha}{3\pi}\left\{ 4\pi\alpha\int\frac{d^4k}{(2\pi)^4}\left(\frac{-i(4m_1\hat{k} - 4p_1p_2)}{(k^2 - \lambda^2)((p_1 - k)^2 - m_1^2)((p_2 + k)^2 - m_1^2)}\right. \right. \]
\[ + 2m_1^2(\Sigma'_V(m_1, \lambda) + \Sigma'_S(m_1, \lambda)) \right\} \tag{14} \]

\[ T_{V}^{(2)} = \int_{4m_2^2}^{\infty}\frac{d\lambda^2}{\lambda^2}\frac{2m_2^2}{\lambda^2}\sqrt{1 - \frac{4m_2^2}{\lambda^2}}T_{V}^{(2)}(\lambda^2) \tag{15} \]

We assume here that the matrix element of the \( \gamma \)-matrices should be evaluated with respect to the on-shell fermion and anti-fermion spinors.

This formulae is valid for arbitrary relation between the masses \( m_1 \) and \( m_2 \). Below we consider two special cases \( m_1 = m_2 \) and \( m_1 \gg m_2 \).

In the Ref. \[10\] it was suggested to calculate first the one-loop integrals with the arbitrary gluon mass and than to integrate this result with the gluon spectral density. Here we choose a different way, which, in our opinion, is more suitable for the two special cases we are interested in.

To demonstrate it, we discuss below the calculation of the contribution of the scalar three-point function (term proportional to \( 4p_1p_2 \) in the Eq.(14)) to the two-loop form factor \( T_V^{(2)} \).

\[ \text{This counterterms Lagrangian corresponds to the on-shell subtraction of the quark mass operator. Hence } m_1 \text{ below is the pole mass of the quark.} \]
3.2 Contribution of the scalar three-point function.

In this subsection we indicate all the steps which are necessary to evaluate the following integral:

\[ I = \int_{\frac{4m_1^2}{\lambda^2}}^{\infty} \frac{d\lambda^2}{\lambda^2} (1 + \frac{2m_1^2}{\lambda^2}) \sqrt{1 - \frac{4m_2^2}{\lambda^2}} C(m_H^2, \lambda^2). \]  \hspace{1cm} (16)

Here \( C(m_H^2, \lambda^2) \) is the scalar three-point function:

\[ C(m_H^2, \lambda^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \lambda^2)((p_1 - k)^2 - m_1^2)((p_2 + k)^2 - m_1^2)} \]  \hspace{1cm} (17)

Clearly, \( C(m_H^2, \lambda^2) \) is an analytic function in the complex plane of the \( s \)-variable with the cut going from \( 4m_1^2 \) to \( \infty \) along the real axis. We write dispersion representation:

\[ C(m_H^2, \lambda^2) = -\frac{i}{(4\pi)^2} \frac{1}{\pi 4m_1^2} \int_{\frac{4m_1^2}{\lambda^2}}^{\infty} \frac{ds'}{s' - s - i\epsilon} C_I(s', \lambda^2) \]

\[ C_I(s', \lambda^2) = \frac{\pi}{s'\beta(s')} \log \left( \frac{s' - 4m_1^2 + \lambda^2}{\lambda^2} \right) \]  \hspace{1cm} (18)

Using this representation in the Eq.(16) and changing the order of integration we get:

\[ I = -\frac{i}{16\pi^2} \int_{\frac{4m_1^2}{\lambda^2}}^{\infty} \frac{ds'}{(s' - s - i\epsilon) s'\beta(s')} \int_{\frac{4m_2^2}{\lambda^2}}^{\infty} \frac{d\lambda^2}{\lambda^2} \left(1 + \frac{2m_1^2}{\lambda^2}\right) \sqrt{1 - \frac{4m_2^2}{\lambda^2}} \log \left( \frac{s' - 4m_1^2 + \lambda^2}{\lambda^2} \right) \]

It can be seen, that in both cases of interest this representation is very convenient for further integration.

First, in the case \( m_1 = m_2 \) this is the one-scale integral. Integration over \( \lambda^2 \) provides a simple expression and the subsequent integration over \( s' \) is cumbersome but trivial.

The second case, \( m_1 \gg m_2 \) is more tricky. If \( s' - 4m_1^2 \gg 4m_2^2 \) the integration over \( \lambda^2 \) can be simply performed providing a possibility to make subsequent integration over \( s' \). The region of integration, questioning this opportunity is the region \( s' - 4m_1^2 \sim 4m_2^2 \). It is easy to see however, that the contribution of this region is suppressed as \( O(m_2) \). Hence it can be completely neglected as far as we are not interested in the light mass power corrections.

We hope that after this discussion all the steps necessary to evaluate the virtual radiation become clear.
3.3 Results for the virtual corrections

Finally we present the result of our calculation of the virtual radiative corrections. In the case $m_1 \gg m_2$ we write the $O(N_f\alpha_s^2)$ correction in the form:

$$T_{V}^{(2)} = \frac{1}{6} \left( \frac{\pi}{\alpha} \right)^2 F_{V}^{(2)}$$

and

$$F_{V}^{(2)} = f_{V}^{(2)} \log^2 \frac{m_2^2}{m_H^2} + f_{V}^{(1)} \log \frac{m_2^2}{m_H^2} + f_{V}^{(0)}$$

The expressions for the quantities $f_{V}^{(i)}$ are:

$$f_{V}^{(2)} = \frac{1 + p^2}{2(1 - p^2)} \log(p) + \frac{1}{2}$$

$$f_{V}^{(1)} = \frac{1 + p^2}{1 - p^2} \left\{ -2 \text{Li}_2(p) - 4\zeta(2) + 2 \log(1 + p) \log(p) - 2 \log(1 - p) \log(p) - \frac{1}{2} \log^2(p) \right\} + 2 \log(1 + p) + \frac{2(1 + 6p + 4p^2)}{3(1 - p^2)} \log(p) + \frac{8}{3}$$

$$f_{V}^{(0)} = \frac{1 + p^2}{1 - p^2} \left\{ -2 \text{Li}_3(p) - 4\text{Li}_3(1 - p) + 2\zeta(3) + 2\text{Li}_2(p) \left[ \log(p) - 2 \log(1 + p) \right] \right\} + 2 \zeta(2) \left( \log(p) + 6 \log(1 - p) - 4 \log(1 + p) \right) - 4 \log(p) \log(1 + p) \log(1 - p) + \log^2(p) \log(1 - p) + \log^2(p) \log(1 + p) + \frac{1}{6} \log^3(p) \right\} + \frac{-1}{3(1 - p^2)} \left\{ (10 + 24p + 10p^2)(\text{Li}_2(p) + \log(1 - p) \log(p)) \right\} + (14 + 48p + 26p^2) \zeta(2) + (1 + 6p + 4p^2) \log^2(p) - 4 \log(p) \log(1 + p) - \frac{4}{3} \log(1 + p) + \frac{16}{3} \log(1 + p) + \frac{77}{18} \right\}$$

In the case when the Higgs boson is much heavier than the fermion $f_1$, the expression for the $F_{V}^{(2)}$ reads:

$$F_{V}^{(2)} \to \frac{1}{2} \log(r_2)^2(\log(r_1) + 1) + \log(r_2) \left( -\frac{1}{2} \log^2(r_1) + \frac{2}{3} \log(r_1) - 4\zeta(2) + \frac{8}{3} \right) + \frac{1}{6} \log^3(r_1) - \frac{1}{3} \log^2(r_1) + \left( \frac{4}{9} + 2\zeta(2) \right) \log(r_1) - 2\zeta(3) - \frac{14}{3} \zeta(2) + \frac{77}{18}$$

$$+ \ r_1 \left[ \log(r_2)^2 + \log(r_2)(6 \log(r_1) + \frac{4}{3}) - 3 \log^2(r_1) \right] + 16 \log(r_1) - 28\zeta(2) + \frac{8}{9}$$
In the opposite case, when the Higgs boson mass is close to the threshold of two heavy fermions, the following result can be obtained:

$$F_V^{(2)} \rightarrow -\frac{\pi^2}{2\beta} \left( \log \left( \frac{m_2^2}{m_H^2\beta^2} \right) + \frac{11}{3} \right) + \log \left( \frac{4m_2^2}{m_H^2} \right) - \frac{5}{6} \quad (26)$$

It is instructive to combine this result with the threshold $O(\alpha)$ correction which can be found in [3]. The result for the threshold form factor is then:

$$T_V = 1 + \frac{\alpha}{2\pi} \left[ \pi^2 \left( 1 - \frac{\alpha}{\beta} \right) \left( \log \left( \frac{m_2^2}{m_H^2\beta^2} \right) + \frac{11}{3} \right) \right] - 1 + \frac{\alpha}{3\pi} \left( \log \left( \frac{4m_2^2}{m_H^2} \right) - \frac{5}{6} \right)$$

Now it is clear, that we can eliminate large logarithms, appearing in the next-to-leading order calculation by appropriate choice of the coupling constant. The $\overline{MS}$ coupling constant, renormalized at the arbitrary scale $\mu$, can be related to the on-shell renormalized coupling constant by the following equation:

$$\alpha = \alpha_{\overline{MS}}(\mu) \left( 1 + \frac{\alpha_{\overline{MS}}(\mu^2)}{3\pi} \log \left( \frac{m_2^2}{\mu^2} \right) \right) + O(\alpha_{\overline{MS}}^3)$$

Substituting this expression to the equation for the threshold form factor, one finds:

$$T_V = 1 + \frac{\alpha_{\overline{MS}}(\mu^2)}{2\pi} \left[ \frac{\pi^2}{2\beta} \left( 1 - \frac{\alpha}{3\pi} \left( \log \left( \frac{m_2^2}{m_H^2\beta^2} \right) + \frac{11}{3} \right) \right) \right] - 1 + \frac{\alpha}{3\pi} \left( \log \left( \frac{4m_2^2}{m_H^2} \right) - \frac{5}{6} \right) \quad (27)$$

To eliminate large logarithms appearing in this expression we have to choose (see also [10]) two different scales for the $\overline{MS}$ coupling constant: in the terms exhibiting Coulomb singularity we set $\mu^2 = m_H^2/2$ (hence the scale is given by the value of the non-relativistic three momenta of the corresponding particles) while for the part of the correction which does not exhibit Coulomb singularity the reasonable scale is $\mu^2 = \frac{1}{4} m_2^2$. Hence the relevant expression for the threshold form factor reads:

$$T_V = 1 + \frac{\pi \alpha_{\overline{MS}}(m_2^2\beta^2)}{4\beta} \left( 1 - \frac{11\alpha}{9\pi} \right) - \frac{\alpha_{\overline{MS}}(m_H^2/4)}{2\pi} \left( 1 + \frac{5}{18\pi} \right) \quad (28)$$

Our discussion of the threshold region given above, is quite similar to the discussion given in the Ref. [10] for the threshold behaviour of the vector current form factors (for a more detailed discussion see Ref. [12]). This similarity is definitely in accord with the universality of the threshold region where dynamics is defined by a long-range Coulomb force.

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Exactly the same technique as above can be used in the **equal mass case** \( m_1 = m_2 \). In this case the relevant integrations can be performed very quickly providing a simple results. The effective form factor in this case reads:

\[
f_V^{(0)} = \frac{1 + p^2}{1 - p^2} \left( \frac{1}{6} \log^3(p) - 2 \log(p) \zeta(2) \right) + \frac{(5 + 4p + 5p^2)(1 - 6p + p^2)}{6(1 - p)^4} \log^2(p) \\
+ \frac{4}{9(1 + p)(1 - p)^3} \log(p) \\
- 14\zeta(2) + \frac{407(1 + p^3) - 1389p(1 + p^3) + 982p^2(1 + p)}{18(1 + p)(1 - p)^4}
\]  

(29)

The threshold expansion can be obtained from the previous equation:

\[ f_V^{(0)} \to \frac{137}{6} - 12\zeta(2) \approx 3.09412 \]  

(30)

High energy expansion in this case is:

\[
F_V^{(2)} \to \frac{1}{6} \log^3(r_1) + \frac{5}{6} \log^2(r_1) + \left( \frac{28}{9} - 2\zeta(2) \right) \log(r_1) \\
- 14\zeta(2) + \frac{407}{18} \\
- r_1 \left( 10 \log(r_1) + 4\zeta(2) + \frac{28}{9} \right)
\]  

(31)

### 4 Decay rate \( H \to f_1 \bar{f}_1 \).

Let us now discuss the total decay rate \( H \to f_1 \bar{f}_1 \) which is obtained by summing virtual and real corrections calculated so far. In doing so, we find that the double logarithms of the ratio of the square of the mass of the light fermion to the Higgs boson mass cancel, while the single logarithmic term survives. The coefficient of this logarithmic term is proportional to the one-loop QCD correction to the total decay rate of \( H \to f_1 \bar{f}_1 \). Hence, we can eliminate this large logarithm by expressing the total decay rate \( H \to f_1 \bar{f}_1 \) through \( \alpha_s(\mu^2) \) evaluated at the scale \( \mu^2 = m_H^2 = s \).

After that, the expression for the decay width \( H \to f_1 \bar{f}_1 \) including the \( O(N_f \alpha_s^2) \) corrections reads:

\[
\Gamma(H \to f_1 \bar{f}_1) = \Gamma_0 \left\{ 1 + \frac{4}{3} \left( \frac{\alpha_s(s)}{\pi} \right) \delta_1 + \frac{2}{9} N_f \left( \frac{\alpha_s(s)}{\pi} \right)^2 \left( f_V^{(0)} - f_R^{(0)} \right) \right\}
\]  

(32)

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5 We remind the reader, that in this section we completely switch to the QCD terminology. Below \( \alpha_s \) always denotes the QCD coupling constant in the \( \overline{MS} \)–scheme.

6 In the Eq.(2.27) of the second reference in [3] there is a misprint. The term \(-3 \log \frac{1}{1 + \beta_0} \log \frac{1 + \beta_0}{1 - \beta_0}\) should be written as \(-3 \log \frac{2}{1 + \beta_0} \log \frac{1 + \beta_0}{1 - \beta_0}\).
Here $\delta_1$ is the one-loop QCD radiative correction to the decay width (see Ref. 3) and $f_V^{(0)}$, $f_R^{(0)}$ are given by the Eqs.(23) and (7) respectively. $\Gamma_0$ is the Born value for the decay width $H \to f_1\bar{f}_1$:

$$\Gamma_0 = \frac{3G_F m_H m_1^2}{4\pi\sqrt{2}} \beta^3$$  \hspace{1cm} (33)

We now apply the BLM scale fixing procedure for the decay width $H \to f_1\bar{f}_1$ including the full mass dependence of the radiative corrections. The general result for the BLM scale is than:

$$\mu_{BLM} = \sqrt{s} \exp \left\{ \frac{(f_V^{(0)} - f_R^{(0)})}{2\delta_1} \right\}$$  \hspace{1cm} (34)

The numerical results for the ratio $\mu_{BLM}/\sqrt{s}$ as a function of the ratio $m_1/\sqrt{s}$ are shown in the Figs. 2,3. The one-loop radiative correction goes to zero in the vicinity of the point $m_1/\sqrt{s} \approx 0.23$. Around this point the BLM analyses can not be applied. The threshold BLM scale approaches zero, in accordance with the discussion in the section 3. Note that the scale is quite low even sufficiently far from the threshold. Beyond the point $m_1/\sqrt{s} \approx 0.23$ the BLM scale for the coupling constant is extremely low being of the order of $\sim 0.01 - 0.02 m_H$.

We remind that our discussion was applied to the width expressed through the pole mass of the quark. As was recently pointed out (see for instance Ref. 5) the low value of the BLM scale usually encountered in such cases is connected with the fact that the pole quark mass receives large contributions from the region of the small loop momenta. The possible way to avoid this problem is to express the result for the width in terms of the running quark mass.

Usually, this substitution is used in the asymptotic regime for the radiative corrections. Here we want to check if it for the whole mass range. For this aim we express the result for the width $H \to f_1\bar{f}_1$ through the running mass keeping only the terms of the order of $O(N_f\alpha_s^2)$ in the $O(\alpha_s^2)$ corrections. The expression for the pole mass in terms of the running mass reads:

$$m^2 = \bar{m}^2(\mu^2) \left\{ 1 + \frac{\alpha_s(\mu^2)}{\pi} \left( -2 \log \frac{\bar{m}^2(\mu^2)}{\mu^2} + \frac{8}{3} \right) \right\} + N_f \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 \left\{ -\frac{1}{6} \log^2 \frac{\bar{m}^2(\mu^2)}{\mu^2} + \frac{13}{18} \log \frac{\bar{m}^2(\mu^2)}{\mu^2} - \frac{2}{3} \left( \zeta(2) + \frac{71}{48} \right) \right\}$$  \hspace{1cm} (35)

Using the expression for the width in terms of the running quark mass, we recalculate the BLM scale. The numerical results are presented in the Fig.4.

We see that the use of the running quark mass in the expression for the width makes the BLM scale for the coupling constant higher for arbitrary relation between
the Higgs and the fermion mass (excluding the region close to the threshold, where the use of the running mass is artificial). This does not make much difference for the mass region not far from the threshold, but for higher energies the difference is huge. The curve on the Fig.4 can be well approximated by the following equation:

$$\mu_{BLM} \approx 0.49 \ m_H - \bar{m}(m_H)$$  

(36)

The important check of our results can be performed by studying the limit $m_H \gg m_1$. As it was mentioned above, the $O(\alpha_s^2)$ corrections to the decay width $H \to f_1 \bar{f}_1$ are known in this limit up to the power suppressed terms $O(m_1^2/m_H^2)$. Expanding the expression for the width up to the terms of the order $O(m_1^2/s)$ and using the expression for the running mass, we get:

$$\Gamma = \frac{3 G_F m_H \bar{m}_f^2(s)}{4 \pi \sqrt{2}} \left[ 1 + \frac{17}{3} \left( \frac{\alpha_s(s)}{\pi} \right) + N_f \left( \frac{\alpha_s(s)}{\pi} \right)^2 \left( \frac{2}{3} \zeta(3) + \frac{1}{3} \zeta(2) - \frac{65}{24} \right) \right]$$

$$- \frac{\bar{m}_f^2(s)}{s} \left[ 6 + 40 \left( \frac{\alpha_s(s)}{\pi} \right) + N_f \left( \frac{\alpha_s(s)}{\pi} \right)^2 \left( 4 \zeta(3) + 4 \zeta(2) - \frac{313}{18} \right) \right]$$

(37)

Our result for the width in Eq.(36) is in complete agreement with the $N_f$-dependent part of the $O(\alpha_s^2)$ correction given in Ref. [4], [5], [6], [7].

5 Conclusion.

By no doubts Higgs interaction with massive quarks is quite important from phenomenological point of view. A vivid example is provided by the decay mode $H \to \bar{b}b$, which can be used for the detection of the light Higgs boson. Even leaving aside the problem of finding this particle, direct measurement of the coupling of the Higgs boson to quarks seems to be necessary. The measurement of such type can test the symmetry breaking mechanism in the Higgs-fermion sector of the Standard Model.

In this paper we present analytical results for the $O(N_f \alpha_s^2)$ correction to the decay width of the Higgs boson into the pair of massive fermions for the arbitrary relation between the mass of the Higgs boson and the mass of the fermion. We calculate both real and virtual radiation of the light fermion pair in this decay. As a byproduct of this analyses, we obtain the formulae (see Eq.(4) and below) for the width of the rare decay: $H \to f_1 \bar{f}_1 f_2 \bar{f}_2$ in the limit $m_1 \gg m_2$.

It seems that the most important phenomenological application of our analyses is connected with the Higgs boson decay to two top quarks. In this case the value of the top quark mass and the expected value of the Higgs boson mass suggests that there will be no small (or large) mass ratios in this problem. In this case any results on the next-to-leading order QCD radiative corrections are absent, and our results provide the first step in this direction.
Our analyses of the BLM scale indicates that the use of the running quark mass in the complete expression for the one-loop QCD radiative correction to $H \rightarrow f_1\bar{f}_1$ is definitely a good choice for arbitrary relation between the Higgs and the fermion masses. We hope that if both the running quark mass and the BLM scale for the coupling constant, evaluated in this paper (see Eq.(35)), are used for the description of the one-loop QCD corrected decay width $H \rightarrow f_1\bar{f}_1$, this should substitute quite reasonable approximation for the description of this process for arbitrary ratio of the Higgs boson mass to the fermion mass.

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Figure 2: The BLM scale for the one-loop QCD correction to the decay width $H \rightarrow f_1 \bar{f}_1$ expressed through the pole quark mass. The vertical axes is the ratio $\mu_{BLM}/\sqrt{s}$, the horizontal axes is the ratio $m/\sqrt{s}$. 
Figure 3: The BLM scale for the one-loop QCD correction to the decay width $H \rightarrow f_{1} \bar{f}_{1}$ expressed through the pole quark mass. The vertical axes is the ratio $\mu_{BLM}/\sqrt{s}$, the horizontal axes is the ratio $m/\sqrt{s}$. 
Figure 4: The BLM scale for the one-loop QCD correction to the decay width $H \to f_1\bar{f}_1$ expressed through the running mass. The vertical axes is the ratio $\mu_{BLM}/\sqrt{s}$, the horizontal axes is the ratio $\bar{m}(s)/\sqrt{s}$. 