LOWER Bounds of Martingale Measure Densities in the Dalang-Morton-Willinger Theorem

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Abstract. For a $d$-dimensional stochastic process $(S_n)_{n=0}^N$ we obtain criteria for the existence of an equivalent martingale measure, whose density $z$, up to a normalizing constant, is bounded from below by a given random variable $f$. We consider the case of one-period model $(N = 1)$ under the assumptions $S \in L^p$; $f, z \in L^q$, $1/p + 1/q = 1$, where $p \in [1, \infty]$, and the case of $N$-period model for $p = \infty$. The mentioned criteria are expressed in terms of the conditional distributions of the increments of $S$, as well as in terms of the boundedness from above of an utility function related to some optimal investment problem under the loss constraints. Several examples are presented.

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, endowed with a discrete-time filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^N$, $\mathcal{F}_N = \mathcal{F}$. Consider a $d$-dimensional stochastic process $S = (S_n)_{n=0}^N$, adapted to the filtration $\mathbb{F}$, and a $d$-dimensional $\mathbb{F}$-predictable process $\gamma = (\gamma_n)_{n=1}^N$. In the customary securities market model $S^i_n$ describes the discounted price of $i$th stock and $\gamma^i_n$ corresponds to the number of stock units in investor’s portfolio at time moment $n$. The gain process is given by

$$G_n^\gamma = \sum_{k=1}^n (\gamma_k, \Delta S_k), \quad \Delta S_k = S_k - S_{k-1}, \quad n = 1, \ldots, N,$$  \hspace{1cm} (0.1)

where $(a, b)$ is the scalar product of $a, b \in \mathbb{R}^d$.

Let’s recall the classical Dalang-Morton-Willinger theorem [3], [13] (ch.V, §2e). As usual, we say that the No Arbitrage (NA) condition is satisfied if the inequality $G_N^\gamma \geq 0$ a.s. (with respect to the measure $\mathbb{P}$) implies that $G_N^\gamma = 0$ a.s. A probability measure $Q$ on $\mathcal{F}$ is called a martingale measure if the process $S$ is a $Q$-martingale. The measures $\mathbb{P}$ and $Q$ are called equivalent if their null sets are the same. Denote by $\kappa_{n-1}(\omega)$ the support of the regular conditional distribution $P_{n-1}(\omega, dx)$ of the random vector $\Delta S_n$ with respect to $\mathcal{F}_{n-1}$. Theorem 0.1 (Dalang-Morton-Willinger). The following conditions are equivalent:

(i) NA;

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(ii) there exists an equivalent to $P$ martingale measure $Q$ with a.s. bounded density $z = dQ/dP$;
(iii) the relative interior of the convex hull of $z_{n-1}$ contains the origin a.s., $n = 1, \ldots, N$.

The question concerning the existence of an equivalent martingale measure $Q$, whose density $z$ satisfies the lower bound $z \geq c$ (where $c$ is a positive constant) was posed in [8] (Remark 7.5), [4] (Remark 6.5.2). In general, the answer to this question is negative. An evident necessary condition is the integrability of $S$ with respect to $P$. Moreover, the example of [4] shows that a measure $Q$ with the above properties need not exist even for a uniformly bounded process $S$. A sufficient condition was obtained in [8]. In particular it is satisfied for a process $S$ with independent increments, if the random vectors $\Delta S_n$ have finite moments.

Following [12], let us formulate the problem concerning the existence of an equivalent martingale measure, whose density (up to a normalization constant) is bounded from below by a random variable $f$, in a more general context. Denote by $E X$ the expectation with respect to $P$, by $L^p = L^p(\mathcal{F}) = L^p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$ the Banach spaces of equivalence classes of $\mathcal{F}$-measurable functions with the norms $\|X\|_p = E |X|^p$ and by $L^\infty$ the Banach space of essentially bounded functions with the norm $\|X\|_\infty = \text{ess sup} |X|$. The cone $L^p_+$ of non-negative elements induces the partial order on $L^p$.

Consider the subspace $K \subset L^p$, $p \in [1, \infty)$ of investor’s gains (discounted wealth increments). Denote by $q$ the conjugate exponent, that is, $1/p + 1/q = 1$. The condition $K \cap L^p_+ = \{0\}$ corresponds to NA. An element $f \in L^q_+$ induces the functional on $L^p$ by the formula $\langle X, f \rangle = E(Xf)$, $X \in L^p$. It turns out that the existence of an element $g$, satisfying the conditions

$$\langle X, g \rangle = 0, \quad X \in K; \quad g \geq f, \quad g \in L^q$$

is equivalent to the boundedness of $f$ form above on a certain subset $K_1$ of the subspace $K$:

$$v_p := \sup_{X \in K_1} \langle X, f \rangle < \infty, \quad K_1 = \{X \in K : \|X^-\|_p \leq 1\},$$

where $X^- = \max\{-X, 0\}$. For $p = \infty$, $q = 1$ this statement is not true in general, see [12], Examples 1 and 3. It becomes true under the assumption that $f$ is bounded from above on the subset $\{X \in K : X^- \in V\}$, where $V$ is a neighborhood of zero in the Mackey topology $\tau(L^\infty, L^1)$, or if $L^1$ is replaced by the topological dual space $(L^\infty)^*$ of $L^\infty$. These results are contained in Theorem 1 of [12].

It should be mentioned that the problems, equivalent to (0.3) when $f = 1$, were considered in the recent paper [6]. From the financial point of view they correspond to the maximization of expected gain under the loss constraint, if the loss value is measured either by $p$th moment $E |X^-|^p$ for $p \in [1, \infty)$ or by $\text{ess sup} |X^-|$ for $p = \infty$. The equivalence of (0.2) and (0.3) for $p \in (1, \infty)$ follows from the results of the cited paper as well ([6], Theorem 4.1). Unfortunately, the
related statement for \( p = \infty \) ([6], Theorem 6.1) is incorrect: a counterexample is, in fact, contained in [12] (Example 3) and its another version is given below (Example 5.4).

Turning back to the finite securities market model, assume that \( S \in L^p \) and denote by \( K \) the set of random variables \( G^\gamma_N \), where \( \gamma \) is a bounded predictable process. Then the elements \( g \), satisfying (0.2), up to a normalization constant, coincide with the \( P \)-densities of martingale measures: \( dQ/dP = g/Eg \).

The aim of the present paper is to establish effective criteria for the fulfilment of (0.2), (0.3) for a market model with finite discrete time and a finite collection of stocks. Such criteria, expressed in terms of the regular conditional distributions of the increments \( \Delta_{\omega} \), are obtained for a one-period model under the assumptions \( S \in L^p, f, g \in L^q, p \in [1, \infty] \) (Theorem 1.3), as well as for \( N \)-period model in the case \( p = \infty \) (Theorem 4.1). These results show also that in the case under consideration the equivalence of (0.2) and (0.3) for \( p = \infty \) is nevertheless true! Thereby, we give the negative answer to the question, raised in the end of the paper [12].

In the last part of the paper we give some examples, illustrating the effectiveness of the obtained criteria, and a counterexample to the mentioned statement of [6]. Also, it is interesting to note that the case \( p = 1 \) of Theorem 1.2 leads to a new proof of the key implication (iii) \( \implies \) (ii) of the Dalang-Morton-Willinger theorem (Remark 1.5).

### 1. One-period model

Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( \mathcal{H} \) be a sub-\( \sigma \)-algebra of \( \mathcal{F} \). A set-valued mapping \( F \), assigning some set \( F(\omega) \subset \mathbb{R}^d \) to each \( \omega \in \Omega \), is called \( \mathcal{H} \)-measurable, if \( \{ \omega : F(\omega) \cap V \neq \emptyset \} \in \mathcal{H} \) for any open set \( V \subset \mathbb{R}^d \). A function \( \eta : \Omega \mapsto \mathbb{R}^d \) is called a selector of \( F \), if \( \eta(\omega) \in F(\omega) \) for all \( \omega \in \text{dom} F := \{ \omega' : F(\omega') \neq \emptyset \} \). An \( \mathcal{H} \)-measurable set-valued mapping \( F \) with non-empty closed values \( F(\omega) \) is measurable if and only if there exists a sequence \( (\eta_i)_{i=1}^\infty \) of \( \mathcal{H} \)-measurable selectors of \( F \) such that the sets \( \{ \eta_i(\omega) \}_{i=1}^\infty \) are dense in \( F(\omega) \) for all \( \omega \) ([10], Theorem 1B). Such a sequence is called a Castaing representation of \( F(\omega) \).

Denote by \( \mathcal{B}(\mathbb{R}^d) \) the Borel \( \sigma \)-algebra of \( \mathbb{R}^d \). A function \( \varphi : \Omega \times \mathbb{R}^d \mapsto \mathbb{R} \) called a Carathéodory function if (a) \( \varphi(\cdot, x) : \Omega \mapsto \mathbb{R} \) is \( (\mathcal{H}, \mathcal{B}(\mathbb{R})) \)-measurable for all \( x \in \mathbb{R}^d \), (b) \( \varphi(\omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R} \) is continuous for all \( \omega \in \Omega \).

Denote by \( L^p(\mathcal{H}, F) \), \( 1 \leq p < \infty \) the set of equivalence classes of \( \mathcal{H} \)-measurable functions \( \eta \) satisfying the conditions \( \int |\eta|^p \, dP < \infty, \eta \in F \text{ a.s.} \), where \( |x| = (x, x)^{1/2} \). We introduce also the sets of equivalence classes of essentially bounded functions \( L^\infty(\mathcal{H}, F) \) and of all \( \mathcal{H} \)-measurable functions \( L^0(\mathcal{H}, F) \), taking values in \( F \) a.s. In accordance with the above notation we put \( L^p(\mathcal{H}) = L^p(\mathcal{H}, \mathbb{R}) \). By \( L^p_+(\mathcal{H}) \) and \( L^p_{++}(\mathcal{H}) \) we denote the sets of non-negative and strictly positive elements of \( L^p(\mathcal{H}) \) respectively. Let \( \|X\|_p \) be the norm of an element \( X \) of the Banach space \( L^p(\mathcal{H}) \), \( 1 \leq p \leq \infty \).
The completion of the $\sigma$-algebra $\mathcal{H}$ with respect to the measure $P$ is denoted by $\mathcal{H}^P$. Note that $L^p(\mathcal{H}^P) = L^p(\mathcal{H})$ in the sense that any $\mathcal{H}^P$-measurable function possesses an $\mathcal{H}$-measurable modification.

In the sequel we use the customary notation of convex analysis for the polar $A^o = \{x \in \mathbb{R}^d : (x, y) \leq 1, y \in A\}$ of a set $A \subset \mathbb{R}^d$ and also for its Minkowski function and the support function:

$$\mu(x|A) = \inf\{\lambda > 0 : x \in \lambda A\}, \quad s(x|A) = \sup_{y \in A}(x, y).$$

Denote by $\text{conv} A$, $\text{ri} A$ the convex hull and the relative interior of $A$.

Consider the one-period model (0.1) (that is, $N = 1$). Put $\xi = \Delta S_1$, $\mathcal{H} = \mathcal{F}_0$. Let $P_\xi(\omega, dx)$ be the regular conditional distribution of $\xi$ with respect to $\mathcal{H}$ and let $\kappa_\xi(\omega)$ be the support of the measure $P_\xi(\omega, \cdot)$. By $D_\xi(\omega) \subset \mathbb{R}^d$ we denote the linear span of $\kappa_\xi(\omega)$. Define the functions

$$\psi_p(\omega, h) = \left(\int_{\mathbb{R}^d} [(h, x)^-]^p P_\xi(\omega, dx)\right)^{1/p}, \quad p \in [1, \infty);$$

$$\psi_\infty(\omega, h) = s(-h|\kappa_\xi(\omega))$$

from $\Omega \times \mathbb{R}^d$ to $[0, \infty]$, and the set-valued mappings

$$\omega \mapsto T_p(\omega) = \{h \in D_\xi(\omega) : \psi_p(\omega, h) \leq 1\}.$$  \hspace{1cm} (1.1)

**Lemma 1.1.** Assume that $0 \in \text{ri} (\text{conv} \kappa_\xi(\omega))$ a.s. Then $T_p$ is an $\mathcal{H}^P$-measurable set-valued mapping with a.s. compact values, $p \in [1, \infty]$.

**Proof.** The set-valued mapping $\omega \mapsto \kappa_\xi(\omega)$ is $\mathcal{H}$-measurable:

$$\{\omega : \kappa_\xi(\omega) \cap V \neq \emptyset\} = \{\omega : P_\xi(\omega, V) > 0\} \in \mathcal{H}$$

for any open set $V \subset \mathbb{R}^d$. Its values $\kappa_\xi(\omega)$ are closed. It follows from the formula

$$\psi_\infty(\omega, h) = \sup_{i \geq 1} (-h, \eta_i(\omega)),$$

where $(\eta_i)_{i=1}^\infty$ is a Cauchy representation of $\kappa_\xi$, that the function $\omega \mapsto \psi_\infty(\omega, h)$ is $\mathcal{H}$-measurable. The same property of $\psi_p$ for $p \in [1, \infty)$ is evident.

Put $\Omega_p = \{\omega : \int |x|^p \, dP_\xi(\omega, dx) < \infty\}$ for $p \in [1, \infty)$ and let $\Omega_\infty$ be the set of $\omega$, for which the set $\kappa_\xi(\omega)$ is compact. Note that $\Omega_\infty = \{\omega : \sup_{h \in \mathbb{D}} \psi_\infty(\omega, h) < \infty\}$, where $\mathbb{D}$ is a countable dense subset of $\mathbb{R}^d$. Consequently, $\Omega_p \in \mathcal{H}$, $p \in [1, \infty]$ and $P(\Omega_p) = 1$. Put $\Omega_p' = \Omega_p \cap \{\omega : 0 \in \text{ri} (\text{conv} \kappa_\xi(\omega))\}$. Clearly, $\Omega_p' \in \mathcal{H}^P$ and $P(\Omega_p') = 1$.

Assume that $\omega \in \Omega_p'$. It follows from continuity of $\psi_p$ with respect to $h$ that the set $T_p(\omega)$ is closed. From the codition $0 \in \text{ri} (\text{conv} \kappa_\xi(\omega))$ we see that for $h \in D_\xi(\omega)\backslash 0$ the set $\kappa_\xi(\omega)$ is not contained in the half-space $\{x \in D_\xi(\omega) : (h, x) \geq 0\}$. Therefore, $\psi_p(\omega, h) > 0$, $p \in [1, \infty]$ and the set $T_p(\omega)$ is compact, because $\psi_p(\omega, h) \to \infty$ when $|h| \to \infty$, $h \in D_\xi(\omega)$.

Consider the trace of the $\sigma$-algebra $\mathcal{H}$ on $\Omega_p'$: $\mathcal{H}_p = \{A \cap \Omega_p' : A \in \mathcal{H}\}$. To complete the proof it is sufficient to check that the set-valued mappings $\omega \mapsto T_p(\omega)$, $\omega \in \Omega_p'$ are $\mathcal{H}_p$-measurable. We make use of the representation
Proposition 1H of [10], and the measurability of $T$ of set-valued mappings, whose intersection is $T$.

Theorem 1.3. □

Let us recall the "measurable maximum theorem" ([1], Theorem 18.19).

Lemma 1.2. Let $F$ be an $\mathcal{H}$-measurable set-valued mapping with non-empty compact values $F(\omega) \subset \mathbb{R}^d$, and let $\varphi : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ be a Carathéodory function. Put

$$ m(\omega) = \max_{x \in F(\omega)} \varphi(\omega, x), \quad G(\omega) = \{ x \in F(\omega) : \varphi(\omega, x) = m(\omega) \}. $$

Then (a) the function $m$ and the set-valued mapping $G$ are $\mathcal{H}$-measurable; (b) there exists an $\mathcal{H}$-measurable selector $\eta^*$ of $G$.

Our first main result is the following.

Theorem 1.3. Let $\xi \in L^p(\mathcal{F}, \mathbb{R}^d)$, $f \in L^q(\mathcal{F})$, where $p \in [1, \infty]$ and $1/p + 1/q = 1$. If $0 \in \text{ri} (\text{conv} \, \mathcal{X}_\xi)$ a.s., then the following conditions are equivalent:

(i) $v_p := \sup \{ E(fX) : \|X^{-}\|_p \leq 1, \, X \in K \} < \infty$, where

$$ K = \{ (\gamma, \xi) : \gamma \in L^\infty(\mathcal{H}, D_\xi) \}; $$(1.2)

(ii) there exists a random variable $g \in L^q(\mathcal{F})$, satisfying the conditions

$$ E(g \xi | \mathcal{H}) = 0, \quad g \geq f; $$ (1.3)

(iii) $s(a | T_p) \in L^q(\mathcal{H})$, where $a = E(f \xi | \mathcal{H})$ and $T_p$ is defined by the formula (1.1).

Let us make some remarks before the proof of this theorem (sect. 2 and 3).

Remark 1.4. If $0 \in \text{ri} (\text{conv} \, \mathcal{X}_\xi)$ and $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$ does not depend on $\mathcal{H}$, then there exists $g \in L^\infty(\mathcal{F})$:

$$ E(g \xi | \mathcal{H}) = 0, \quad g \geq 1. $$

Actually, in this case $s(a | T_1) = s(E \xi | T_1)$ does not depend on $\omega$ and thus belongs to $L^\infty(\mathcal{H})$.

Remark 1.5. If $0 \in \text{ri} (\text{conv} \, \mathcal{X}_\xi)$ and $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$, then there exists $g \in L^\infty_{++}(\mathcal{F})$:

$$ E(g \xi | \mathcal{H}) = 0. $$

To prove this statement it is sufficient to note that there exists an $\mathcal{H}$-measurable function $f \in L^\infty_{++}(\mathcal{H})$ such that

$$ s(E(f \xi | \mathcal{H}) | T_1) = s(E(\xi | \mathcal{H}) | T_1) f \in L^\infty(\mathcal{H}). $$

A function $g \in L^\infty(\mathcal{F})$, satisfying (1.3), is the desired one.

In fact, this proves the implication (iii) $\implies$ (ii) of Theorem 0.1 for $N = 1$ and $S \in L^1$. As is known, this is the key point of the proof of the Dalang-Morton-Willinger theorem.
Remark 1.6. Note that
\[ a = E\{E(f | \mathcal{H} \vee \sigma(\xi)) | \mathcal{H}\} = \int b(\omega, x) x P_\xi(\omega, dx) \in D_\xi(\omega) \text{ a.s.} \quad (1.4) \]

The existence of an \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable function \( b(\omega, x) \), satisfying the condition \( E(f | \mathcal{H} \vee \sigma(\xi)) = b(\omega, \xi) \), follows from the fact that the \( \sigma \)-algebra \( \mathcal{H} \vee \sigma(\xi) \) is generated by the mapping \( \omega \mapsto (\omega, \xi(\omega)) \) from \( \Omega \) to the measurable space \( (\Omega \otimes \mathbb{R}^d, \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)) \).

Remark 1.7. We have the following convenient representation of the random variable \( s(a(\omega)|T_\omega(\omega)) \) for \( a \in D_\xi \) a.s.:
\[
s(a|T_\infty) = \sup\{(h, a) : h \in D_\xi \setminus \varepsilon_0^\circ = s(-a|\varepsilon_0^\circ) = \mu(-a|\text{conv } \varepsilon_0) \text{ a.s.} \}
\]

It the last equality we have used the formula
\[
\mu(x|A^\circ) = \inf\{\lambda > 0 : \lambda^{-1}x \in A^\circ\} = \inf\{\lambda > 0 : s(\lambda^{-1}x|A) \leq 1\} = s(x|A),
\]

which is true under the assumption \( 0 \in A \). We have also used the bipolar theorem: \( A^\circ = \text{cl}(\text{conv } A) \) and the compactness property of the convex hull of a compact set.

2. Proof of Theorem 1 for \( p \in [1, \infty) \)

Denote by \( U^p \) the unit ball of the space \( L^p = L^p(\Omega, \mathcal{F}, P) \) and put \( U^p_+ = \{X \in L^p_+ : X \in U^p\} \).

Lemma 2.1. For any element \( X \in L^p, p \in [1, \infty] \) we have
\[
\|X^+\|_p = \sup\{\langle X, z \rangle : z \in U^q_+\}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Proof. Consider the elements
\[
\zeta_q = \frac{(X^+)^{p/q}}{\|X^+\|^p_{L^p}} \in U^q_+, \quad q \in (1, \infty); \quad \zeta_\infty = I_{\{X \geq 0\}} \in U^\infty_+; \quad \zeta_1^n = \frac{I_{A_n}}{P(A_n)} \in U^1_+,
\]

where \( A_n = \{\omega : X(\omega) \geq \|X^+\|_\infty - 1/n\} \). If \( X \in L^p \) and \( q \) is the conjugate exponent, then
\[
\langle X, \zeta_q \rangle = \|X^+\|_p, \quad q \in (1, \infty]; \quad \langle X, \zeta_1^n \rangle \geq \|X^+\|_\infty - \frac{1}{n}.
\]

On the other hand,
\[
\langle X, z \rangle \leq \langle X^+, z \rangle \leq \|X^+\|_p, \quad z \in U^q_+ \quad \Box
\]

Though the next result follows from Theorem 1 of \cite{12}, it seems convenient to give its direct proof. The idea of this proof is contained also in the paper \cite{11} (Lemma 2.5).

Recall that the closure of a convex set \( A \subset L^p, p \in [1, \infty) \) in the weak topology \( \sigma(L^p, L^q) \), \( 1/p + 1/q = 1 \) coincides with its norm closure in \( L^p \).

Lemma 2.2. For a subspace \( K \subset L^p, p \in [1, \infty) \) and an element \( f \in L^q_+, 1/p + 1/q = 1 \) the following conditions are equivalent:
(a) \[ \sup_{X \in K_1} \langle X, f \rangle < \infty, \] where \( K_1 = \{X \in K : \|X^-\|_p \leq 1\} \);
(b) there exists \( g \in L^q \), satisfying the conditions
\[ \langle X, g \rangle = 0, \ X \in K; \ g \geq f. \] (2.1)

**Proof.** (b) \(\implies\) (a). If \( X \in K_1 \) then
\[ \langle X, f \rangle = \langle X, g \rangle + \langle X, f - g \rangle = -\langle X, g - f \rangle \leq \|g - f\|_q. \]
(a) \(\implies\) (b). Put \( \lambda = \sup_{X \in K_1} \langle X, f \rangle \). If the assertion (b) is false then
\[ (f + \lambda U^q_2) \cap K^o = \emptyset, \ K^o = \{z \in L^q : \langle X, z \rangle \leq 0, \ X \in K\}. \]

By applying the separation theorem ([P], Theorem 5.79) to the \(\sigma(L^q, L^p)\)-compact set \( f + \lambda U^q_2 \) and to the \(\sigma(L^q, L^p)\)-closed set \( K^o \), we conclude that there exists \( Y \in L^p \) such that
\[ \sup_{z \in K^o} \langle Y, z \rangle < \inf \{\langle Y, \zeta \rangle : \zeta \in f + \lambda U^q_2 \}. \]

Since \( K \) is a subspace it follows that \( \langle Y, z \rangle = 0, \ z \in K^o \) and \( Y \in K^{oo} = \text{cl}_pK \) by the bipolar theorem ([P], Theorem 5.103), where \( \text{cl}_pK \) is the closure of \( K \) in the norm topology of \( L^p \). Moreover,
\[ \langle Y, f \rangle + \lambda \inf \{\langle Y, \eta \rangle : \eta \in U^q_2 \} > 0. \] (2.2)

By Lemma 2.1 we have
\[ \inf \{\langle Y, \eta \rangle : \eta \in U^q_2 \} = -\sup \{-\langle Y, \eta \rangle : \eta \in U^q_2 \} = -\|Y^-\|_p. \] (2.3)

If \( Y^- = 0 \) then \( \langle Y, f \rangle > 0 \) and \( \alpha Y \in L^p_+ \cap \text{cl}_pK \) for any \( \alpha > 0 \). Hence, the functional \( X \mapsto \langle X, f \rangle \) is unbounded from above on the ray \( \{\alpha Y : \alpha > 0\} \), which lies in the set
\[ \text{cl}_pK_1 \supset \text{cl}_p \left( \{X : \|X^-\|_p < 1\} \cap K \right) \supset \{X : \|X^-\|_p < 1\} \cap \text{cl}_pK. \]

Here we have used the elementary inclusion \( \text{cl}_p(A \cap B) \supset A \cap \text{cl}_pB \), which holds true when the set \( A \) is open ([2], chap.1, §1, Proposition 5).

Thus, \( \|Y^-\|_p > 0 \). It follows from (2.2), (2.3) that
\[ \langle Y/\|Y^-\|_p, f \rangle > \lambda \]
in contradiction with the definition of \( \lambda \) since \( Y/\|Y^-\|_p \in K_1 \). \(\square\)

Lemma 2.2 implies that the conditions (i) and (ii) of Theorem 1.3 are equivalent. Indeed, for the subspace (1.2) condition \( \langle X, g \rangle = 0, \ X \in K \) means that
\[ E[g(\gamma, \xi)] = E(\gamma, E(g(\xi|\mathcal{H})) = 0, \ \gamma \in L^\infty(\mathcal{H}, D_\xi). \] (2.4)

In turn, (2.4) is reduced to the equality \( E(g(\xi|\mathcal{H}) = 0 \): putting
\[ \gamma = E(g(\xi|\mathcal{H}) I_{\{|E(g(\xi|\mathcal{H})| \leq M\}} \in L^\infty(\mathcal{H}, D_\xi) \]
and passing in (2.4) to the limit as \( M \to \infty \) we conclude that \( E(g\xi|\mathcal{H}) = 0 \) by the monotone convergence theorem.

The equivalence of the conditions (i) and (iii) for all \( p \in [1, \infty] \) follows from the equality \( v_p = \|s(a|T_p)|\|_q \), which is proved in Lemma 2.4 below.
Lemma 2.3. Let $\xi \in L^0(\mathcal{F}, \mathbb{R}^d)$ and $0 \in \mathfrak{H}(\mathfrak{X})$. If $(\gamma, \xi) \geq 0$ a.s. for some $\gamma \in L^0(\mathcal{H}, D_\xi)$, then $\gamma = 0$ a.s.

Proof. Put $A = \{\gamma \neq 0\}$. For any $\omega \in A$ there exists $y \in \mathfrak{X}(\omega)$ such that $(\gamma(\omega), y) < 0$ and hence $\int (\gamma(\omega), x)^- P_\xi(\omega, dx) > 0$. If $P(A) > 0$ then we obtain the contradiction:

$$E(\gamma, \xi)^- \geq EE(I_A(\gamma, \xi)^- | \mathcal{H}) = E \left( I_A \int_{\mathbb{R}^d} (\gamma(\omega), x)^- P_\xi(\omega, dx) \right) > 0. \square$$

Lemma 1.1 together with the measurable maximum theorem (Lemma 1.2) imply the existence of an element $h_\gamma^\ast \in L^0(\mathcal{H}, T_p)$ such that

$$s(a(\omega)|T_p(\omega)) = (h_\gamma^\ast(\omega), a(\omega)) \text{ a.s.}$$

Lemma 2.4. Under the assumptions of Theorem 1.3 we have

$$v_p = \sup_{\gamma} \{E(\gamma, a) : \|(\gamma, \xi)^-\|_p \leq 1, \gamma \in L^\infty(\mathcal{H}, D_\xi) \} = \|s(a|T_p)\|_q, \quad p \in [1, \infty].$$

Proof. (a) The case $1 \leq p < \infty$. Put $U^p_+(\mathcal{H}) = \{g \in L^p_+(\mathcal{H}) : \|g\|_p \leq 1\}$. We have

$$U^p_+(\mathcal{F}) = \{g \in L^p_+(\mathcal{F}) : E(E(g^p|\mathcal{H})) \leq 1\}
\quad = \bigcup_{w \in U^p_+(\mathcal{H})} \{g \in L^p_+(\mathcal{F}) : (E(g^p|\mathcal{H}))^{1/p} \leq w\}.$$

Consequently,

$$v_p = \sup_{\gamma} \{E(\gamma, a) : (\gamma, \xi)^- \in U^p_+(\mathcal{F}), \gamma \in L^\infty(\mathcal{H}, D_\xi) \}
\quad = \sup_{w \in U^p_+(\mathcal{H})} \sup_{\gamma} \{E(\gamma, a) : (E([\gamma, \xi]^p|\mathcal{H}))^{1/p} \leq w, \gamma \in L^\infty(\mathcal{H}, D_\xi) \}.$$

On the set $\{w = 0\}$ we have the equality $E([\gamma, \xi]^p|\mathcal{H}) = 0$. Therefore,

$$E((\gamma I_{(w \neq 0)}, \xi)^-|\mathcal{H}) = 0$$

and $\gamma I_{(w = 0)} = 0$ by Lemma 2.3. Putting $\gamma = w\theta$, where $\theta$ is an $\mathcal{H}$-measurable vector, we obtain

$$v_p = \sup_{w \in U^p_+(\mathcal{H})} \sup_{\theta} \{Ew(\theta, a) : E((\theta I_{(w \neq 0)}, \xi)^-|\mathcal{H}) \leq 1, \theta \in L^\infty(\mathcal{H}, D_\xi) \}.$$

Since the values of $\theta$ on the set $\{w = 0\}$ do not affect $Ew(\theta, a)$, by the definition of $T_p$ and the equality $E((\theta, \xi)^-|\mathcal{H}) = \psi_p(\omega, \theta(\omega))$ a.s., we get

$$v_p = \sup_{w \in U^p_+(\mathcal{H})} \sup_{\theta} \{Ew(\theta, a) : \theta \in L^0(\mathcal{H}, T_p), \theta \in L^\infty(\mathcal{H}, D_\xi) \}.$$

But $(\theta, a) \leq s(a|T_p)$ a.s. for $\theta \in L^0(\mathcal{H}, T_p)$. This yields that

$$v_p \leq \sup_{w \in U^p_+(\mathcal{H})} E(s(a|T_p)w) = \|s(a|T_p)\|_q. \quad (2.5)$$

We have used Lemma 2.1 in the last equality.
To obtain the inequality, converse to (2.5), put \( \theta = h_p^* I_{\{w|\|h_p^*|\leq M\}} \), \( M > 0 \). Clearly, \( w \theta \in L^\infty(\mathcal{H}, D_\xi) \) and

\[
v_p \geq \sup_{w \in U_p^r(\mathcal{H})} E[w(h_p^*, a)I_{\{w|\|h_p^*|\leq M\}}] = \sup_{w \in U_p^r(\mathcal{H})} E(s(a|T_p)wI_{\{w|\|h_p^*|\leq M\}})
\]

\[
= \|s(a|T_p)I_{\{w|\|h_p^*|\leq M\}}\|_q.
\]

By the monotone convergence theorem it follows that \( v_p \geq \|s(a|T_p)\|_q \).

(b) The case \( p = \infty \). It follows from

\[
P((\gamma, \xi) \geq -1) = EP((\gamma, \xi) \geq -1)|\mathcal{H}) = EP_\xi(\omega, \{x : (\gamma(\omega), x) \geq -1\})
\]

that the condition \( \|(\gamma, \xi)\|_\infty \leq 1 \), meaning that \( P((\gamma, \xi) \geq -1) = 1 \), can be represented in the form \( P_\xi(\omega, \{x : (\gamma, x) \geq -1\}) = 1 \) a.s. In other words, \( \gamma(\omega) \in -x_\xi^2(\omega) \) a.s.

Since \( T_\infty = (-x_\xi^2) \cap D_\xi \) this implies that

\[
v_\infty = \sup_{\gamma} \{E(\gamma, a) : \gamma \in L^\infty(\mathcal{H}, (-x_\xi^2) \cap D_\xi)\} \leq Es(a|T_\infty).
\]

On the other hand, \( h_\infty^* I_{\{|h_\infty| \leq M\}} \in L^\infty(\mathcal{H}, (-x_\xi^2) \cap D_\xi) \) for all \( M > 0 \). Therefore,

\[
v_\infty \geq E((h_\infty^*, a)I_{\{|h_\infty| \leq M\}}) = E(s(a|T_\infty)I_{\{|h_\infty| \leq M\}})
\]

and \( v_\infty \geq Es(a|T_\infty) \) by the monotone convergence theorem. \( \square \)

3. PROOF OF THEOREM 1 FOR \( p = \infty \)

As we have already mentioned, Lemma 2.4 yields that conditions (i) and (iii) of Theorem 1.3 are equivalent. Assume that (ii) is satisfied and put \( X = (\gamma, \xi) \), \( \gamma \in L^\infty(\mathcal{H}, D_\xi) \). The implication (ii) \( \implies \) (i) is a consequence of the inequality

\[
E(fX) = E(gX) - E((g - f)X) \leq E(\gamma, E(g\xi|\mathcal{H})) + E((g - f)X^-)
\]

\[
\leq \|g - f\|_1\|X^-\|_\infty.
\]

(3.1)

Let us prove that (ii) follows from (iii). We look for \( g \) of the form \( g = f + \varphi(\omega, \xi(\omega)) \), where \( \varphi \in L_+^0(\mathcal{H} \otimes \mathcal{B}([\mathbb{R}^d])) \). Firstly, the desired function \( \varphi \) should satisfy (1.3):

\[
E(\varphi\xi|\mathcal{H}) = \int \varphi(\omega, x)P_\xi(\omega, dx) = -a(\omega) \text{ a.s.}
\]

Secondly, the function \( \omega \mapsto \varphi(\omega, \xi(\omega)) \) should be \( P \)-integrable. We construct a function \( \varphi \) with these properties in Lemma 3.3 after some preliminary work.

Lemma 3.1. Consider a probability measure \( Q \) on \((\mathbb{R}^d, \mathcal{B}([\mathbb{R}^d]))\) with the support \( \mathcal{X} \). If \( 0 \in \text{ri}(\text{conv} \mathcal{X}) \) then for all \( y \) in the linear span \( D \) of \( \mathcal{X} \) the following equality holds true:

\[
w(y) := \inf \left\{ \int \varphi(x)Q(dx) : \int \varphi(x)xQ(dx) = y, \ \varphi \in L_+^\infty(Q) \right\} = \mu(y|\text{conv} \mathcal{X}).
\]
Proof. It is easy to check that the epigraph of $w$: $\text{epi } w = \{(y, \alpha) \in D \times \mathbb{R} : w(y) \leq \alpha\}$ is a convex set (see [9], Lemma 2). Following the general scheme of duality theory (see e.g. [9], [7]) let us find the conjugate function (Young-Fenchel transform) of $w$:

$$w^*(\lambda) = \sup_{y \in D}\{(y, \lambda) - w(y)\} = \sup_{\varphi, y}\{(y, \lambda) - \int \varphi(x)Q(dx) : \int \varphi(x)Q(dx) = y, \varphi \in L^\infty_+(Q)\} = \sup_{\varphi}\int \varphi((x, \lambda) - 1)Q(dx) : \varphi \in L^\infty_+(Q)\} = \delta(\lambda \rhd \varnothing), \ \lambda \in D.$$

Here $\delta$ is the indicator function: $\delta(\lambda \rhd \varnothing) = 0, \ \lambda \in \varnothing; \ \delta(\lambda \rhd \varnothing) = +\infty, \ \lambda \notin \varnothing$. The Young-Fenchel transform of $w^*$ is of the form:

$$w^{**}(y) = \sup_{\lambda \in D}\{(y, \lambda) - w^*(\lambda)\} = (y|\varnothing) = \mu(y|\text{conv } \varnothing), \ y \in D.$$

We claim that $\text{dom } w := \{y \in D : w(y) < \infty\} = D$. Clearly, this is the case iff the set $A = \{\int \varphi(x)Q(dx) : \varphi \in L^\infty_+(Q)\}$ coincides with $D$.

Assume that $z \in D$ does not belong to the convex set $A$. Then there exists a non-zero vector $h \in D$, separating $A$ and $z$:

$$\left(\int \varphi(x)Q(dx), h\right) = \int \varphi(x)(x, h)Q(dx) \leq (z, h), \ \varphi \in L^\infty_+(Q).$$

Putting $\varphi(x) = cI_{\{(h, x) \geq 0\}}$, where $c \in \mathbb{R}_+$, we conclude that the inequality

$$c \int (x, h)^+Q(dx) \leq (z, h)$$

should hold true for all $c > 0$. Consequently $(x, h)^+ = 0$ $Q$-a.s. Then $(h, x) \leq 0$, $x \in \varnothing$ and $\varnothing$ is contained in the subspace orthogonal to $h$, since $0 \in \text{ri } (\text{conv } \varnothing)$. This means that the linear span of $\varnothing$ does not coincide with $D$, a contradiction.

Thus, $\text{dom } w = D$, $w$ is continuous on $D$ and $w = w^{**}$ by the Fenchel-Moreau theorem [7]. □

Lemma 3.2. There exists a function $\chi : [0, 1] \times \mathbb{R}^d \mapsto \mathbb{R}$, measurable with respect to $\mathcal{B}([0, 1]) \otimes \mathcal{B}(\mathbb{R}^d)$ and possessing the following property: for any probability measure $Q$ on $\mathcal{B}(\mathbb{R}^d)$ and for any $\mathcal{B}(\mathbb{R}^d)$-measurable real-valued function $f$ there exists $r \in [0, 1]$ such that $\chi(r, x) = f(x) Q$-a.s.

Lemma 3.2 is borrowed from the paper [5] (Theorem A.3).

Lemma 3.3. If $\xi \in L^1(\mathcal{F}, \mathbb{R}^d)$, $0 \in \text{ri } (\text{conv } \varnothing_\xi)$ $a.s.$, $a \in L^0(\mathcal{H}, D_\xi)$ and $\nu = \mu(-a|\text{conv } \varnothing_\xi)$, then there exists a function $\varphi \in L^1_+(\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d))$ such that

$$\int \varphi(\omega, x)P_\xi(\omega, dx) = -a(\omega) \ a.s.,$$

$$\int \varphi(\omega, x)P_\xi(\omega, dx) \in [\nu(\omega), \nu(\omega) + \varepsilon(\omega)] \ a.s.$$ for any $\mathcal{H}$-measurable function $\varepsilon > 0$. 


Proof. Consider the trace $\mathcal{H}' = \Omega' \cap \mathcal{H}$ of the $\sigma$-algebra $\mathcal{H}$ on the set $\Omega' = \{ \omega : 0 \in \text{ri} (\text{conv } \mathcal{X}(\omega)) \} \in \mathcal{H}^p$. Let $\chi$ be some function, mentioned in Lemma 3.2. We fix an $\mathcal{H}'$-measurable function $\varepsilon > 0$ and introduce the set-valued mapping $G : \Omega' \mapsto [0,1]$ by the formula

$$G(\omega) = \{ y \in [0,1] : \int \chi(y,x) P_\xi(\omega, dx) \in [\nu(\omega), \nu(\omega) + \varepsilon(\omega)],$$

$$\int \chi(y,x) P_\xi(\omega, dx) = -a(\omega), \int \chi^-(y,x) P_\xi(\omega, dx) = 0 \}.$$

Applying Lemma 3.1 to $Q(dx) = P_\xi(\omega, dx)$ and Lemma 3.2, we conclude that $G(\omega) \neq \emptyset$ for all $\omega \in \Omega'$. The functions

$$\int \chi^-(y,x) P_\xi(\omega, dx), \int \chi(y,x) P_\xi(\omega, dx), \int \chi(y,x) P_\xi(\omega, dx),$$

depending on $(\omega,y)$, are measurable with respect to $\mathcal{H} \otimes \mathcal{B}([0,1])$: see [3], Lemma 2.2(a). Hence,

$$\text{gr} G = \{ (\omega,y) \in \Omega' \times [0,1] : y \in G(\omega) \} \in \mathcal{H}' \otimes \mathcal{B}([0,1])$$

and by Aumann’s measurable selection theorem there exists an $\mathcal{H}'$-measurable function $r : \Omega' \mapsto [0,1]$, satisfying the condition $r(\omega) \in G(\omega)$ a.s. on $\Omega'$ ([1], Corollary 18.27). The function $\varphi(\omega,x) = \chi(\tilde{r}(\omega), x)$, where $\tilde{r}$ is an $\mathcal{H}$-measurable modification of $r$, has the desired properties. □

The end of the proof of Theorem 1.3. Let us prove that condition (iii) implies (ii) ($p = \infty$). According to the assumption,

$$s(a|T_\infty) = \mu(-a|\text{conv } \mathcal{X}_\xi) \in L^1(\mathcal{H}), \quad a = E(f\xi|\mathcal{H}).$$

Let $\varepsilon > 0$ be some constant. Using the notation of Lemma 3.3, we put $g(\omega) = f(\omega) + \varphi(\omega, \xi(\omega))$. The function $g \geq f$ is $\mathcal{F}$-measurable, $\mathbb{P}$-integrable since

$$E(\varphi \wedge M) = \mathbb{E}(\varphi \wedge M|\mathcal{H}) = \mathbb{E} \int (\varphi(\omega,x) \wedge M) P_\xi(\omega, dx)$$

$$\leq \mathbb{E} \mu(-a|\text{conv } \mathcal{X}_\xi) + \varepsilon, \quad M > 0,$$

and satisfies the equality (3.2):

$$E(g\xi|\mathcal{H}) = a(\omega) + \int \varphi(\omega,x) x P_\xi(\omega, dx) = 0 \quad \text{a.s.}$$

4. N-period model

We turn to N-period market model on a filtered probability space, presented in the introductory section. In addition to the introduced notation denote by $D_{n-1}(\omega)$ the linear span of $\mathcal{X}_{n-1}(\omega)$.

Our second main result is the following.

Theorem 4.1. If the process $S_n \in L^\infty(\mathcal{F}_n, \mathbb{R}^d)$, $n = 0, \ldots, N$ satisfies the NA property, then for an element $f \in L^1_{++}(\mathcal{F}, \mathbb{P})$ the following conditions are equivalent:
(i) \( v := \sup \{ \mathbb{E}(fX) : \|X\|_\infty \leq 1, \ X \in K \} < \infty \), where
\[
K = \{ \gamma_n \in L^\infty(\mathcal{F}_{n-1}, D_{n-1}) ; \ n = 1, \ldots, N \};
\]
(ii) there exist an equivalent to \( \mathbb{P} \) martingale measure \( Q \), whose density satisfies the inequality \( dQ/d\mathbb{P} \geq cf \) with some constant \( c > 0 \);
(iii) the recurrence relation
\[
\beta_N = f, \quad \beta_n = \mathbb{E}(\beta_{n+1} | \mathcal{F}_n) + \mu(-a_n \text{conv } \mathcal{K}_n), \quad a_n = \mathbb{E}(\beta_{n+1} \Delta S_{n+1} | \mathcal{F}_n)
\]
specifies the \( \mathbb{P} \)-integrable sequence \( (\beta_n)_{n=0}^\infty \).

**Proof.** (ii) \( \Rightarrow \) (i). This statement follows from an estimate, similar to (3.1).
(i) \( \Rightarrow \) (iii). Consider the process \( X^\gamma = 1 + \gamma \): 
\[
X^\gamma_{n+1} = X^\gamma_n + (\gamma_{n+1}, \Delta S_{n+1}), \quad X^\gamma_0 = 1.
\]
If the random variable \( \beta_n \in L^1_+ (\mathcal{F}_n) \) is well-defined, put 
\[
u_n = \sup_\gamma \{ \mathbb{E}(\beta_n X^\gamma_n) : X^\gamma_k \geq 0, \ \gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1}), \ 1 \leq k \leq n \}.
\]
By virtue of assumption (i) we have 
\[
u_N \leq \sup_\gamma \{ \mathbb{E}(\beta_N X^\gamma_N) : X^\gamma_N \geq 0, \ \gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1}), \ 1 \leq k \leq n \} = \mathbb{E}f + v < \infty.
\]
If \( u_{m+1} < \infty \) and the process \( \gamma \) satisfies the conditions of the definition of \( u_{m+1} \), then \( \beta_{m+1} \in L^1(\mathcal{F}_{m+1}) \) and 
\[
\mathbb{E}(\beta_{m+1} X^\gamma_{m+1}) = \mathbb{E}(X^\gamma_m \mathbb{E}(\beta_{m+1} | \mathcal{F}_m)) + \mathbb{E}(\gamma_{m+1}, a_m).
\]
Consequently, 
\[
u_{m+1} \geq \mathbb{E}(X^\gamma_m \mathbb{E}(\beta_{m+1} | \mathcal{F}_m)) + t_{m+1}, \quad (4.1)
\]
\[
t_{m+1} = \sup_\gamma \{ \mathbb{E}(\gamma_{m+1}, a_m) : X^\gamma_{m+1} \geq 0, \ \gamma_{m+1} \in L^\infty(\mathcal{F}_m, D_m) \}.
\]
The condition \( X^\gamma_{m+1} = X^\gamma_m + (\gamma_{m+1}, \Delta S_{m+1}) \geq 0 \) a.s. can be rephrased as 
\[
(\gamma_{m+1}(\omega), x) \geq -X^\gamma_m(\omega), \quad x \in \mathcal{K}_m(\omega) \text{ a.s.,}
\]
that is, \( \gamma_{m+1} \in -X^\gamma_m \mathcal{K}_m(\omega) \) a.s. (see the proof of Lemma 2.4 for \( p = \infty \)). Here we take into account that \( \gamma_{m+1} = 0 \) a.s., if \( (\gamma_{m+1}, \Delta S_{m+1}) \geq 0 \) and \( \gamma_{m+1} \in D_m \) a.s. (Lemma 2.3). Thus,
\[
t_{m+1} = \sup_\gamma \{ \mathbb{E}(\gamma_{m+1}, a_m) : \gamma_{m+1} \in L^\infty(\mathcal{F}_m, -X^\gamma_m \mathcal{K}_m) \}.
\]
The measurability of the set-valued mapping \( \mathcal{K}_m(\omega) \) with respect to \( \mathcal{F}_m \) follows from \( \mathcal{K}_m(\omega) = \bigcap_{i=1}^\infty \{ h : (h, \eta_i(\omega)) \leq 1 \} \), where \( (\eta_i)_{i=1}^\infty \) is a Castaing representation of \( \mathcal{K}_m \) and from Theorem 1M of [10], concerning the measurability of a countable intersection. Owing to the compactness of \( \mathcal{K}_m(\omega) \) a.s., which follows from \( 0 \in \text{ri(\text{conv } \mathcal{K}_m)} \), by the measurable maximum theorem there exists an element \( \gamma^*_{m+1} \in L^0(\mathcal{F}_m, -X^\gamma_m \mathcal{K}_m) \) such that 
\[
(\gamma_{m+1}, a_m) \leq (\gamma^*_{m+1}, a_m) = s(a_m - X^\gamma_m \mathcal{K}_m) = X^\gamma_m \mu(-a_m \text{conv } \mathcal{K}_m).
\]
In particular, \( t_{m+1} \leq \mathbb{E}(\gamma_{m+1}^*, a_m) \). On the other hand, by approximation of \( \gamma_{m+1}^* \) by the elements \( \gamma_{m+1}^* I_{|\gamma_{m+1}^*| \leq M} \in L^\infty(\mathcal{F}_m, -X^\gamma_m) \), \( M \to \infty \), we deduce that

\[
\mathbb{E}(\gamma_{m+1}^*, a_m) = \lim_{M \to \infty} \mathbb{E}(\gamma_{m+1}^* I_{|\gamma_{m+1}^*| \leq M}, a_m) \leq t_{m+1}
\]

by the monotone convergence theorem.

By plugging the obtained value \( t_{m+1} = \mathbb{E}[X^\gamma_n \mu(-a_m \mid \text{conv } \mathcal{X}_m)] \) in (4.1), we get

\[
v_{m+1} \geq \mathbb{E}\left(\left( \mathbb{E}(\beta_{m+1} | \mathcal{F}_m) + \mu(-a_m \mid \text{conv } \mathcal{X}_m) \right) X^\gamma_m \right) = \mathbb{E}(\beta_m X^\gamma_m).
\]

This inequality holds true under the assumption \( X^\gamma_k \geq 0 \), \( \gamma_k \in L^\infty(\mathcal{F}_{k-1}, D_{k-1}) \), \( k = 1, \ldots, m \). Hence, \( v_m \leq v_{m+1} < \infty \). By induction this implies (iii).

(iii) \( \implies \) (ii). Put \( \nu_n = \mu(-a_n \mid \text{conv } \mathcal{X}_n) \). Recall that \( a_n \in L^0(\mathcal{F}_n, D_n) \) (see (1.4)). By Lemma 3.3 for any \( n = 1, \ldots, N \) there exists a function \( \varphi_n \in L^0_+(\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R}^d)) \) such that

\[
\int \varphi_n(\omega, x) P_n(\omega, dx) = -a_n(\omega) \text{ a.s.},
\]

(4.2)

\[
\int \varphi_n(\omega, x) P_n(\omega, dx) \in [\nu_n(\omega), \nu_n(\omega) + \beta_n(\omega)] \text{ a.s.}
\]

(4.3)

Put \( \zeta_{n+1}(\omega) = \varphi_n(\omega, \Delta S_{n+1}(\omega)) \). The inequality

\[
\mathbb{E}(\zeta_{n+1} \wedge M) = \mathbb{E}\left(\left( \varphi_n(\omega, x) \wedge M \right) P_n(\omega, dx) \right) \leq \mathbb{E}(\nu_n + \beta_n),
\]

similar to (3.2), these functions are \( \mathbb{P} \)-integrable. We can rewrite (4.2), (4.3) as follows:

\[
\mathbb{E}(\zeta_{n+1} \Delta S_{n+1} | \mathcal{F}_n) = -a_n, \quad \mathbb{E}(\zeta_{n+1} | \mathcal{F}_n) = \nu_n + \varepsilon_n \beta_n,
\]

(4.4)

where \( \varepsilon_n \) is an \( \mathcal{F}_n \)-measurable function, taking values in \([0, 1]\). Put \( z_N = 1 + \zeta_N / f \),

\[
z_n = \frac{1}{1 + \varepsilon_n} \left( 1 + \frac{\zeta_n}{\beta_n} \right), \quad n = 1, \ldots, N - 1; \quad Z = f \prod_{n=1}^{N} z_n.
\]

We claim that the random variable \( Z \) is integrable and

\[
\mathbb{E}(z_{n+1} \ldots z_N f | \mathcal{F}_n) = \beta_n(1 + \varepsilon_n), \quad n = 0, \ldots, N - 1.
\]

(4.5)

By virtue of (4.4) and the definition of \( (\beta_n)_{n=0}^N \) we have

\[
\mathbb{E}(z_N f | \mathcal{F}_{N-1}) = \mathbb{E}(f | \mathcal{F}_{N-1}) + \mathbb{E}(\zeta_N | \mathcal{F}_{N-1})
\]

\[
= \mathbb{E}(\beta_N | \mathcal{F}_{N-1}) + \nu_N - 1 + \varepsilon_{N-1} \beta_{N-1} = (1 + \varepsilon_{N-1}) \beta_{N-1}.
\]

Assume that the random variable \( z_{m+1} \ldots z_N f \) is integrable and (4.5) holds true for \( n = m \). Then

\[
\mathbb{E}(I_{\{z_m \leq M\}} z_m z_{m+1} \ldots z_N f) = \mathbb{E}(I_{\{z_m \leq M\}} z_m \beta_m(1 + \varepsilon_m)) \leq \mathbb{E}(\beta_m + \zeta_m).
\]

Hence, \( z_m z_{m+1} \ldots z_N f \in L^1(\mathcal{F}) \). Moreover,

\[
\mathbb{E}(z_m z_{m+1} \ldots z_N f | \mathcal{F}_{m-1}) = \mathbb{E}(z_m \beta_m(1 + \varepsilon_m) | \mathcal{F}_{m-1}) = \mathbb{E}(\beta_m + \zeta_m | \mathcal{F}_{m-1})
\]

\[
= \mathbb{E}(\beta_m | \mathcal{F}_{m-1}) + \nu_{m-1} + \varepsilon_{m-1} \beta_{m-1} = (1 + \varepsilon_{m-1}) \beta_{m-1}.
\]
By induction (4.5) hold true for all \( n \). In particular, \( Z \in L^1(\mathcal{F}) \).

Consider a probability measure \( Q \) with the density \( dQ/dP = cZ, \ c = 1/EZ \). Evidently, \( dQ/dP \geq 2^{-N+1}c.f. \) Let us check that \( Q \) is a martingale measure. Put \( A_{n-1} \in \mathcal{F}_{n-1} \). We have

\[
\frac{1}{c}E_Q(I_{A_{n-1}}\Delta S_n) = E(E(Z|\mathcal{F}_n)I_{A_{n-1}}\Delta S_n) = E(z_1 \ldots z_n\beta_n(1 + \varepsilon_n)I_{A_{n-1}}\Delta S_n) = E(z_1 \ldots z_n\beta_nE((\beta_n + \zeta_n)\Delta S_n|\mathcal{F}_{n-1})) = 0
\]

since \( E(\zeta_n\Delta S_n|\mathcal{F}_{n-1}) = -a_{n-1} = -E(\beta_n\Delta S_n|\mathcal{F}_{n-1}). \)

\[\Box\]

5. Examples

In example 5.1 we concretize the formulas of condition (iii) of Theorem 1.3 for a scalar random variable \( \xi \) in the case of general probability space. In example 5.2 we consider a one-period model on a countable space.

Example 5.3 underlines the non-local character of the conditions of Theorem 4.1. Therein we construct a process \( (S_0, S_1, S_2) \) with no martingale measure, whose density is bounded from below by a positive constant, but, at the same time, for each of the processes \( (S_0, S_1), (S_1, S_2) \) such a measure exists.

At last, example 5.4 shows that conditions (0.2), (0.3) need not be equivalent for \( p = \infty \) even if there exists \( z \in L^1_{1+} \), satisfying the condition \( E(Xz) = 0 \), \( X \in K \) and the subspace \( K \) is generated by a countable collection of elements.

**Example 5.1.** Consider the case of scalar random variable \( \xi \). We use the notation of Theorem 1.3. Assume that \( \xi \in L^p(\mathcal{F}), \ 0 \in \text{ri}(\text{conv} \ \mathcal{H}) \) and \( f \in L^q(\mathcal{F}), \ 1/p + 1/q = 1, \ p \in [1, \infty] \).

For \( q \in (1, \infty] \) we have

\[
\psi_p(\omega, h) = \int [(hx)^{-}]^p P_\xi(\omega, dx) = (h^+)^pE((\xi^-)^p|\mathcal{H})(\omega) + (h^-)^pE((\xi^+)^p|\mathcal{H})(\omega)
\]

and condition (iii) shapes to

\[
s(a|T_p) = \sup_h \{E(f\xi|\mathcal{H})h : \psi_p(\omega, h) \leq 1\}
\]

\[
= \frac{(E(f\xi|\mathcal{H}))^+}{E((\xi^-)^p|\mathcal{H})^{1/p}} + \frac{(E(f\xi|\mathcal{H}))^-}{E((\xi^+)^p|\mathcal{H})^{1/p}} \in L^q(\mathcal{H}). \quad (5.1)
\]

For \( q = 1, p = \infty \) we have \( \text{conv} \ \mathcal{H}(\omega) = [\delta_1(\omega), \delta_2(\omega)] \), \( 0 \in (\delta_1, \delta_2) \) a.s. By virtue of Remark 1.7 condition (iii) becomes

\[
\mu(-a[\delta_1, \delta_2]) = \frac{(E(f\xi|\mathcal{H}))^+}{|\delta_1|} + \frac{(E(f\xi|\mathcal{H}))^-}{\delta_2} \in L^1(\mathcal{H}). \quad (5.2)
\]

**Example 5.2.** Here we slightly generalize the model of [4] (Remark 6.5.2), [12] (Example 2). Put \( \Omega = \mathbb{N} \). Consider a countable partition \( (A_i^j)_{j=1}^\infty \) of the set \( \Omega \):

\[
\mathbb{N} = \bigcup_{j=1}^\infty A_i^j, \ A_i^j \cap A_i^k = \emptyset, \ i \neq k.
\]
Denote by $\mathcal{H}$ the $\sigma$-algebra, generated by this partition. Let

$$A^j_0 = A^{2j-1}_1 \cup A^{2j}_1, \quad A^{2j-1}_1 \cap A^{2j}_1 = \emptyset, \quad j = 1, \ldots, \infty$$

and consider the $\sigma$-algebra $\mathcal{F}$ generated by the sets $(A^j_1)_{j=1}^{\infty}$. Assume that $P(A^1_1) > 0$, $j \in \mathbb{N}$ and let $\xi \in L^p(\mathcal{F})$, $1 \leq p \leq \infty$ be a random variable with $0 \in \text{ri} (\text{conv} \mathcal{X}_j)$:

$$\xi(\omega) > 0, \quad \omega \in A^{2j-1}_1, \quad \xi(\omega) < 0, \quad \omega \in A^{2j}_1, \quad j \in \mathbb{N}.$$ 

Let $f \in L^q_+(\mathcal{F})$, $1/p + 1/q = 1$, $p \in [0, \infty]$. For brevity, we put $\eta^j = \eta(\omega)$, $\omega \in A^1_1$ for any $\mathcal{F}$-measurable random variable $\eta$. Define the random variable $\rho$ by the formula

$$\rho(\omega) = \sum_{j=1}^{\infty} \left( f^{2j} \left| \frac{\xi^{2j}}{\xi^{2j-1}} \right| \frac{P(A^{2j}_1)}{P(A^{2j-1}_1)} I_{A^{2j-1}_1}(\omega) + f^{2j-1} \left| \frac{\xi^{2j-1}}{\xi^{2j}} \right| \frac{P(A^{2j-1}_1)}{P(A^{2j}_1)} I_{A^{2j}_1}(\omega) \right).$$

We claim that a necessary and sufficient condition for the existence of a random variable $g$, satisfying conditions (ii) of Theorem 1.3, is the following:

$$\rho \in L^q(\mathcal{F}). \quad (5.3)$$

We make use of conditions (5.1), (5.2), obtained in example 5.1. In our case

$$E(f\xi \mathcal{H})(\omega) = \sum_{j=1}^{\infty} \frac{f^{2j-1}\xi^{2j-1}P(A^{2j-1}_1) + f^{2j}\xi^{2j}P(A^{2j}_1)}{P(A^j_1)} I_{A^j_0}(\omega),$$

$$(E(f\xi \mathcal{H}))^+(\omega) = \sum_{j=1}^{\infty} \frac{|\xi^{2j}|P(A^{2j}_1)(\rho^{2j} - f^{2j})^+}{P(A^j_1)} I_{A^j_0}(\omega),$$

$$(E(f\xi \mathcal{H}))^-(\omega) = \sum_{j=1}^{\infty} \frac{\xi^{2j-1}P(A^{2j-1}_1)(f^{2j-1} - \rho^{2j-1})^-}{P(A^j_1)} I_{A^j_0}(\omega).$$

Let $q = 1$. Since $[\delta_1, \delta_2] = \sum_{j=1}^{\infty} [\xi^{2j}, \xi^{2j-1}] I_{A^j_0}$, condition (5.2) shapes to

$$E\mu(-a|[\delta_1, \delta_2]) = \sum_{j=1}^{\infty} \left( (\rho^{2j} - f^{2j})^+P(A^{2j}_1) + (f^{2j-1} - \rho^{2j-1})^-P(A^{2j-1}_1) \right)$$

$$= \| (\rho - f)^+ \|_1 < \infty,$$

which is equivalent to (5.3), as long as $f \in L^1(\mathcal{F})$.

For $q \in (1, \infty]$ we use (5.1). By virtue of the equalities

$$E((\xi^-)^p \mathcal{H}) = \sum_{j=1}^{\infty} \frac{|\xi^{2j}|^pP(A^{2j}_1)}{P(A^j_1)} I_{A^j_0}, \quad E((\xi^+)^p \mathcal{H}) = \sum_{j=1}^{\infty} \frac{(\xi^{2j-1})^pP(A^{2j-1}_1)}{P(A^j_1)} I_{A^j_0}$$

we get

$$s(a|T_p) = \sum_{j=1}^{\infty} \frac{(\rho^{2j} - f^{2j})^+P(A^{2j}_1)^{1-1/p} + (f^{2j-1} - \rho^{2j-1})^-P(A^{2j-1}_1)^{1-1/p}}{P(A^j_1)^{1-1/p}} I_{A^j_0}.$$
For \( q \in (1, \infty) \) condition (5.1) means that
\[
\|s(a|T_p)\|_q^q = \sum_{j=1}^{\infty} \left( [(\rho^{2j} - f^{2j})^+] q P(A^{2j}_1) + +[(f^{2j-1} - \rho^{2j-1})^-] q P(A^{2j-1}_1) \right) \\
= \|((\rho - f)^+) q\|_q < \infty,
\]
and is reduced to (5.3). At last, condition \( s(a|T_1) \in L^\infty(\mathcal{H}) \) for \( f \in L^\infty(\mathcal{F}) \) is equivalent to the boundedness of \( \rho \).

**Example 5.3.** Put \( \Omega = \mathbb{N} \) and consider the filtration \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \), where the \( \sigma \)-algebra \( \mathcal{F}_n \) is generated by the sets \( (A_n^j)_{j=1}^\infty \), \( n = 0, 1, 2 \),
\[
A^0_0 = \{4j - 3, 4j - 2, 4j - 1, 4j\}, \quad A^1_1 = \{2j - 1, 2j\}, \quad A^2_2 = \{j\}.
\]
Define the probability measure \( P \) on \( \mathcal{F}_2 = \mathcal{F} \) by \( P(A^{2j-1}_2) = P(A^{2j}_2) = 2^{-j-1} \).

Note that
\[
P(A^j_1) = P(A^{2j-1}_2) + P(A^{2j}_2) = 2^{-j},
\]
\[
P(A^j_0) = P(A^{2j-1}_1) + P(A^{2j}_1) = 2^{-2j-1} + 2^{-2j} = \frac{3}{2^{2j}}.
\]

We put
\[
\xi_1(\omega) = \Delta S_1(\omega) = \sum_{j=1}^{\infty} \left( I_{A^{2j-1}_1}(\omega) - \frac{1}{2^j} I_{A^{2j}_1}(\omega) \right),
\]
\[
\xi_2(\omega) = \Delta S_2(\omega) = \sum_{j=1}^{\infty} \left( I_{A^{2j-1}_2}(\omega) - \frac{1}{2^{j/2}} I_{A^{2j}_2}(\omega) \right).
\]

According to Example 5.2, for the existence of \( g_n \in L^1(\mathcal{F}_n) \), \( n = 1, 2 \), satisfying the conditions
\[
E(g_n \xi_n | \mathcal{F}_{n-1}) = 0, \quad g_n \geq 1,
\]
it is necessary and sufficient that the functions
\[
\rho_n(\omega) = \sum_{j=1}^{\infty} \left( \frac{1}{2^{j/n}} \frac{P(A^{2j};n)}{P(A^{2j-1};n)} I_{A^{2j-1}_n}(\omega) + \frac{P(A^{2j-1};n)}{P(A^{2j};n)} I_{A^{2j}_n}(\omega) \right), \quad n = 1, 2
\]
in the conditions of the form (5.3), are integrable. A simple calculation shows that it is the case:
\[
E \rho_1 = E \sum_{j=1}^{\infty} \left( \frac{1}{2^{j+1}} I_{A^{2j+1}_1} + \frac{2^j}{2^{j+1}} I_{A^{2j}_1} \right) = \sum_{j=1}^{\infty} \left( \frac{1}{2^{2j}} + \frac{2}{2^j} \right) < \infty,
\]
\[
E \rho_2 = E \sum_{j=1}^{\infty} \left( \frac{1}{2^{j/2}} I_{A^{2j+1}_2} + \frac{2^{j/2}}{2^{j+1}} I_{A^{2j}_2} \right) = \sum_{j=1}^{\infty} \left( \frac{1}{2^{2j/2+1}} + \frac{1}{2^{j/2+1}} \right) < \infty.
\]

Nevertheless, as we shall see, in the two-period model under consideration, there is no equivalent martingale measure \( Q \) with the density \( dQ/dP \geq c > 0 \), where \( c \) is some constant.
Let \( \omega \in A_1^j \). With the notation of Theorem 4.1 we have \( \beta_2 = 1 \),
\[
a_1(\omega) = E(\xi_2 | F_1)(\omega) = \frac{E(\xi_2 1_{A_1^j})}{P(A_1^j)} = \frac{P(A_2^{2j-1}) - 2^{-j/2}P(A_2^{2j})}{P(A_1^j)} = \frac{1 - 2^{-j/2}}{2},
\]
\[
\mu(-a_1|\text{conv} \, \omega_1)(\omega) = \inf \{ \lambda > 0 : -a_1(\omega) \in \lambda[-2^{-j/2}, 1] \} = 2^{j/2}a_1(\omega),
\]
\[
\beta_1(\omega) = 1 + \mu(-a_1|\text{conv} \, \omega_1)(\omega) = 1 + 2^{j/2} \frac{(1 - 2^{-j/2})}{2} = \frac{2^{j/2} + 1}{2}
\]
and \( E\beta_1 = \sum_{j=1}^{\infty} (2^{j/2} + 1)P(A_1^j)/2 < \infty \).

Now assume that \( \omega \in A_0^j \). Then
\[
a_0(\omega)P(A_0^j) = \frac{E(1_{A_1^j} \xi_1 | F_0)(\omega)P(A_0^j)}{P(A_0^j)} = \frac{E(\beta_1 \xi_1 | F_0)(\omega)P(A_0^j)}{P(A_0^j)} = \frac{2^{j-1/2} + 1}{2}P(A_1^{2j-1})
\]
\[
- \frac{2^j + 1}{2} \frac{1}{2j} P(A_1^{2j}) = \frac{1}{2^{j+1}} \left( 2^{j-1/2} + \frac{1}{2} - \frac{1}{2^{j+1}} \right)
\]
In addition, \( a_0(\omega) > 0 \) and
\[
\mu(-a_0|\text{conv} \, \omega_0)(\omega) = \inf \{ \lambda > 0 : -a_0(\omega) \in \lambda[-2^{-j}, 1] \} = 2^j a_0(\omega).
\]
This yields that
\[
E\mu(-a_0|\text{conv} \, \omega_0) = \sum_{j=1}^{\infty} 2^j a_0^j P(A_1^j) = \infty, \quad a_0^j = a_0(\omega), \ \omega \in A_0^j.
\]
Therefore, \( \beta_0 = E(\beta | F_0) + \mu(-a_0|\text{conv} \, \omega_0) \notin L^1(F_0) \).

This result shows also that
\[
\sup_{\gamma} \{ E\gamma_n : \gamma_n \in L^\infty(F_{n-1}) \land \gamma_n \geq -1 \} = \infty,
\]
whereas
\[
\sup_{\gamma_n} \{ E(\gamma_n, \xi_n) : \gamma_n \in L^\infty(F_{n-1}) \land (\gamma_n, \xi_n) \geq -1 \} < \infty, \quad n = 1, 2.
\]

Let us present a strategy \( \gamma_n \in L^0(F_{n-1}) \land \gamma_n \geq -1 \), satisfying the conditions
\( E\gamma_1 = \infty \), \( G_2^\gamma \geq -1 \).

The strategy, constructed below, is "aggressive" and consists in buying of the maximal allowable amount of stocks in each step.

Put \( \gamma_1(\omega) = \sum_{j=1}^{\infty} 2^j I_{A_1^j} \). Then
\[
G_1^\gamma = \sum_{j=1}^{\infty} \left( 2^j I_{A_1^{2j-1}} - I_{A_1^{2j}} \right) \geq -1.
\]
Since \( A_1^{2j-1} = A_2^{4j-3} \cup A_2^{4j-2} \) and
\[
\xi_2(\omega) = 1, \quad \omega \in A_2^{4j-3}, \quad \xi_2(\omega) = -\frac{1}{2^{j-1/2}}, \quad \omega \in A_2^{4j-2},
\]
we see that the portfolio $\gamma_2(\omega) = \sum_{j=1}^{\infty} 2^{j-1/2}(2^j + 1)I_{A_{1j-1}}$ is admissible:

$$G_2^\gamma = \sum_{j=1}^{\infty} \left( 2^j I_{A_{1j-1}} - I_{A_{1j}} \right) + \sum_{j=1}^{\infty} \left( 2^{j-1/2}(2^j + 1)I_{A_{2j-3}} - (2^j + 1)I_{A_{2j-2}} \right) \geq -1$$

and $E G_2^\gamma = \infty$ as long as

$$P(A_{1j-1}^2) = 2^{-2j+1}, \ P(A_{1j}^2) = 2^{-2j}, \ P(A_{2j-3}^2) = P(A_{2j-2}^2) = 2^{-2j}.$$

**Example 5.4.** Let $\Omega = \mathbb{N}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $\mathcal{F}$ be generated by one-point subsets of $\mathbb{N}$. We put $A_j = \bigcup_{i=j}^{\infty} \{2i\}$, $B_j = \{4j + 1\}$,

$$\Delta S^i_1 = \xi^i = 2^j I_{B_{j-1}} - I_{A_j}, \ j \in \mathbb{N}$$

and define the probability measure $Q$ on $\mathcal{F}$ by $Q\{2j - 1\} = Q\{2j\} = 2^{-j-1}$. Clearly, $Q$ is a martingale measure for $S$:

$$Q(B_{j-1}) = Q\{2(2j - 1) - 1\} = \frac{1}{2^{2j}}, \ Q(A_j) = \sum_{i=j}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^j}$$

$$E_Q \xi^i_j = 2^j Q(B_{j-1}) - Q(A_j) = 0.$$

Put $B = \bigcup_{j=1}^{\infty} B_{j-1}$ and $B' = \Omega \setminus (A_1 \cup B) = \bigcup_{j=1}^{\infty} \{4j - 1\}$. The set $\Omega$ coincides with the union of disjoint sets $A_1, B, B'$. We note that

$$Q(A_1) = \frac{1}{2}, \ Q(B') = \sum_{j=1}^{\infty} Q\{2(2j) - 1\} = \sum_{j=1}^{\infty} \frac{1}{2^{2j+1}} = \frac{1}{6}$$

and define an equivalent to $Q$ ”market” measure $P$ by

$$P(C) = E_Q(\zeta I_C), \ \zeta = \sum_{i=1}^{\infty} 2^{i-1} I_{B_{i-1}} + \frac{3}{4} (I_{A_1} + I_{B'}) , \ C \in \mathcal{F}.$$

Let $J$ be a finite subset of $N$. Putting in the inequality

$$G_2^\gamma(\omega) := \sum_{j \in J} \gamma^j \xi^j(\omega) \geq -1, \ \omega \in \mathbb{N} \ \ (5.4)$$

$\omega = 2m > \max J$ and then $\omega = 4(m-1) + 1$, we get:

$$\sum_{j \in J} \gamma^j \leq 1, \ 2^m \gamma^m \geq -1.$$

As far as

$$E_Q(\zeta \xi^j) = E_Q(2^{j-1} I_{B_{j-1}} - \frac{3}{4} I_{A_j}) = \frac{1}{2} - \frac{3}{4} \frac{1}{2^j},$$

for $\gamma$ satisfying (5.4) we have

$$EG_2^\gamma = \sum_{j \in J} \gamma^j E_Q(\zeta \xi^j) = \frac{1}{2} \sum_{j \in J} \gamma^j - \frac{3}{4} \sum_{j \in J} \gamma^j 2^{-j} \leq \frac{1}{2} + \frac{3}{4} \sum_{j=1}^{\infty} \frac{1}{2^{2j}} = \frac{3}{4}.$$

On the other hand, if $g$ is the $P$-density of a martingale measure and $g$ is uniformly bounded from below by a constant $c > 0$, then

$$E(g \xi^j) = 2^j E(g I_{B_{j-1}}) - E(g I_{A_j}) = 0,$$
\[ E(qI_{A_j}) \geq c2^j P(B_{j-1}) = c2^{2j-1} Q(B_{j-1}) = \frac{c}{2}, \]

in contradiction to the dominated convergence theorem, since \( \lim_{j \to \infty} I_{A_j} = 0 \) a.s.

Summing up, for the subspace \( K \subset L^\infty(\mathcal{F}) \), generated by the countable collection of elements \( (\xi_j)_{j=1}^\infty \), condition (0.3) is satisfied for \( f = 1, p = \infty \). Moreover, there exists and element \( z = \zeta^{-1} \in L^1_{++}(\mathcal{F}) \) such that \( \langle X, z \rangle = E(Xz) = EQX = 0, \) \( X \in K \). However, there is no element \( g \) satisfying (0.2) for \( q = 1 \): a counterexample to the assertion of Theorem 6.1 of [6].

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