Quantifying alpha clustering in light nuclei from binding energies

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(Dated: May 25, 2020)

What is the origin of nuclear clustering and how does it emerge from the nuclear interaction? While there is ample experimental evidence for this phenomenon, its theoretical characterization directly from nucleons as degrees of freedom remains a challenge. In this work, it is shown that the degree of one- and two-alpha (\(^4\)He) clustering in a given nucleus can be quantified empirically using only the binding energies of its partition subsystems. The proposed clustering measures are parameter-free and correctly identify alpha clustering features in light nuclei and long-lived excited states such as the Hoyle state in \(^{12}\)C and the \(0^+_2\) state of \(^{14}\)C at 6.59 MeV. It is revealed that in light nuclei ranging from \(^6\)Li to \(^{14}\)C, state-of-the-art density functional theory and \textit{ab initio} approaches fail to adequately capture alpha clustering. Stringent constraints on binding energies are then provided by back-propagating 10% relative uncertainties on the experimental one-alpha clustering measure using a parallel Markov chain Monte Carlo algorithm. It is demonstrated that the nuclei \(^6\)Li, \(^7\)Be, \(^{10,11}\)B, and \(^{11}\)C are particularly sensitive to alpha clustering despite not being the most clustered systems identified. Using results on \(^{10}\)B, a strong case is made for a link between three-body forces and alpha clustering. This study provides the first quantification of alpha clustering based on binding energies only, as well as new and practical constraints for future optimizations of nuclear forces, potentially helping with the current issues in medium-mass nuclei.

I. INTRODUCTION

Nuclear clustering \cite{1,2} refers to an emergent phenomenon by which the atomic nucleus effectively acquires a molecular-like structure and dynamics due to the formation of tightly bound clusters of protons and neutrons at low energy \cite{3,4}. This phenomenon is fairly common and particularly salient in light nuclei close to cluster emission thresholds \cite{10,11}, as summarized in the famous Ikeda diagrams \cite{12}.

Light clusters are an important ingredient for the equation of state of nuclear matter \cite{13} and the symmetry energy at low temperatures and densities \cite{14,15,16}, effectively impacting supernovae explosions and pasta phases in the crust of neutron stars \cite{17}.

A proper understanding of nuclear clustering is still incomplete and calls for a closer link between experiment and theory \cite{9}. Fundamental approaches based on nucleons as degrees of freedom play a special role in this enterprise since they offer the possibility to compute the degree of clustering of a \(A\)-body system defined by the probability \(|\langle A|P(A)\rangle|^2\) where \(P(A)\) is a partition of the system.

Those approaches, given a model of nuclear forces, can theoretically solve the many-body problem exactly. In practice, only numerically controlled solutions can be provided due to finite computational resources, generating systematic errors. They include \textit{ab initio} approaches and, it can be argued, density functional theory.

At low energy, calculating the degree of clustering precisely can prove challenging due to the presence of competing emergent phenomena like pairing, deformation, or collective motion, but also single-particle effects responsible for shell closures and the “magic numbers”.

Fortunately, in light nuclei, precise calculations are feasible, and modern many-body forces adjusted on two- and three-body data are expected to give satisfactory results, while in medium-mass and heavier systems it is unclear whether or not it should be the case. However, are modern nuclear forces really describing clustering reasonably well?

There are deep reasons to believe that alpha (\(^4\)He) clustering specifically is an important feature of the nuclear interaction. Nuclear matter is near a phase transition between a nuclear liquid and a Bose-condensed gas of alpha particles \cite{18,19,20,21}, largely controlled by the strength of the interaction and its locality \cite{22}. A similar conclusion was reached in an earlier work \cite{23} connecting clustering with the depth of the nuclear mean-field, which itself depends on the strength of the interaction.

From a different perspective, it was also shown that the Wigner SU(4) symmetric part of the interaction, which controls the ground state of the alpha particle, dominates in large nuclear systems \cite{24}. Moreover, it was demonstrated that adjusting a simplistic interaction on alpha-alpha scattering \cite{25} improved binding energies beyond light nuclei.

Thus, it could be argued that nuclear forces must capture alpha clustering properly to be valid beyond light nuclei, where the mean-field dominates and its symmetries and geometry affect properties of clustered states \cite{26,27}.

In this work, a new and simple model-independent way to quantify indirectly the degree of nuclear clustering using only binding energies is introduced. Then, the quality of various state-of-the-art density functional theory (DFT) calculations and \textit{ab initio} results based on modern nuclear forces is assessed against alpha clustering in light nuclei. Finally, a Markov chain Monte Carlo algorithm is used to provide constraints on binding energies.
by back-propagating uncertainties on the surrogate clustering measure.

II. FROM BINDING ENERGIES TO ALPHA CLUSTERING

Intuitively, the degree of clustering, measured by the probability $|\langle A | P(A) \rangle|^2$, should be high (low) when the energy required to break the $A$-body system into its partition $P(A)$ is low (high), but also when the energy required to break the individual clusters in $P(A)$, in the presence of the other clusters, is high (low).

In the particular case of one-alpha clustering, i.e. $P(A) = \alpha + a$ where symbolically $a = A - \alpha$ is treated as a cluster, the energy to break $A$ into $P(A)$ is the $Q$-value defined as $Q_\alpha(A) = E(A) - E(\alpha) - E(\alpha)$.

The energy necessary to break the $a$-body cluster in the presence of the alpha is approximately the energy $\Sigma_\alpha(A)$ necessary to remove the $a$ nucleons one-by-one from the $A$-body system. In practice, proton and neutron separation energies cancel each other along any path connecting a given nucleus to the alpha particle, giving $\Sigma_\alpha(A) = E(A) - E(\alpha)$.

The goal here is not to determine the exact behavior of the probability $p_\alpha(A) = |\langle A | \alpha + a \rangle|^2$ with $Q_\alpha(A)$ and $\Sigma_\alpha(A)$, but only to find a quantity varying roughly in the same way and which is rather sensitive when approaching complete factorization or $p_\alpha(A) = 1$. With those requirements, the simplest surrogate measure of one-alpha clustering proposed is:

$$ r_1(A, \alpha) = \left| \frac{E(A) - E(\alpha)}{Q_\alpha(A)} \right| = 1 + \frac{E(A - \alpha)}{Q_\alpha(A)}, \quad (1) $$

where the absolute value ensures that the measure stays positive for unbound systems such as $^8\text{Be}$ which are clustered by definition. One notes that, unlike $p_\alpha(A)$ which cannot exceed one by definition, $r_1(A, \alpha)$ diverges when $Q_\alpha(A)$ tends to zero, making this measure sensitive near complete factorization as desired.

The dependence on $Q_\alpha(A)$ of this simple form can be approximately justified using perturbation theory for two weakly interacting clusters as shown in Appendix A.

The clustering ratio $r_1$ is model-independent in the sense that there are no parameters and different many-body approaches using the same interaction should provide the same energy spectra and hence the same clustering ratios.

Long-lived excited states can also be approximately considered by removing their excitation energy $E^*(A)$ to $\Sigma_\alpha(A)$. Moreover, the clustering ratio for two identical clusters in $A$ can be defined similarly to the one-cluster ratio:

$$ r_2(A, \alpha) = \left| \frac{E(A) - E(^8\text{Be})}{2Q_\alpha(A)} \right|, \quad (2) $$

where $E(A) - E(^8\text{Be})$ is the energy to remove all the nucleons above $2\alpha$ one-by-one. The main interest in these two extensions is to study (excited) systems with two alpha particles such as the Hoyle state in $^{12}\text{C}$ which is critical for nucleosynthesis in stars. Results for the one- and two-alpha clustering ratios are given in Tab. I for all nuclei where an estimate can be provided.

| Nucleus | $r_1(A, \alpha)$ | Nucleus | $r_1(A, \alpha)$ | $r_2(A, \alpha)$ |
|---------|----------------|---------|----------------|----------------|
| $^5\text{He}$ | 0.927 | $^9\text{Be}$ | 12.10 | 0.336 |
| $^6\text{He}$ | 1.000 | $^{10}\text{Be}$ | 4.948 | 0.572 |
| $^7\text{He}$ | 1.057 | $^{11}\text{Be}$ | 4.747 | 0.540 |
| $^8\text{He}$ | 1.000 | $^{12}\text{Be}$ | 4.505 | 0.678 |
| $^7\text{Li}$ | 1.166 | $^9\text{B}$ | 16.58 | 0.056 |
| $^6\text{Li}$ | 2.503 | $^{10}\text{B}$ | 8.171 | 0.927 |
| $^7\text{Li}$ | 4.439 | $^{11}\text{B}$ | 5.269 | 1.137 |
| $^9\text{Li}$ | 2.885 | $^{12}\text{B}$ | 5.127 | 1.153 |
| $^7\text{Be}$ | 1.007 | $^{10}\text{C}$ | 6.278 | 0.374 |
| $^8\text{Be}$ | 5.860 | $^{11}\text{C}$ | 5.983 | 1.123 |
| $^7\text{B}$ | 313.3 | $^{12}\text{C}$ | 8.668 | 2.420 |
| $^8\text{B}$ | 5.476 | $^{13}\text{C}$ | 6.461 | 1.906 |
| $^{14}\text{C}$ | 6.411 | 2.032 |

As expected, the helium isotopes do not exhibit any particular pattern since the excess neutrons around the alpha core are not clustered. In the other isotopic chains, the ratio $r_1$ is maximal for $^{7}\text{Li}$, $^8\text{Be}$, $^9\text{B}$, and $^{12}\text{C}$, while the ratio $r_2$ is maximal for $^{12}\text{Be}$, $^{12}\text{B}$, and $^{12}\text{C}$. This means that the latter have a pronounced $2\alpha + (A - 2\alpha)$ structure, except perhaps for the ground state of $^{12}\text{C}$ for which $r_1$ and $r_2$ are maximal simultaneously, indicating a $3\alpha$ structure as expected. These results are consistent with what is already known about the structure of light nuclei and can serve as a reliable basis to test nuclear models.

The Hoyle state at 7.65 MeV above the ground state of $^{12}\text{C}$, yields one- and two-alpha clustering ratios of $r_1 = 193.8$ and $r_2 = 48.29$, respectively. These large values are consistent with the characteristic clustered nature of the Hoyle state. However, the fact that $r_2$ is significantly different than $r_1$ suggests a two-plus-one-alpha structure rather than a three-alpha structure. This seems consistent with previous ab initio calculations [29,32], claiming that the Hoyle state has a "bent-arm" shape made of two-plus-one alphas.

In Ref. [33], it was suggested that the 0$^+_2$ state of $^{14}\text{C}$ at 6.59 MeV could have a similar structure than the Hoyle state. This hypothesis is supported by the clustering ratios $r_1 = 12.97$ and $r_2 = 3.887$ of this state, but the magnitude of clustering seems quite reduced in comparison.

Binding energies given by many-body approaches can be used as well. In this work, both DFT and ab initio results are used. The former are based on energy density functionals adjusted over thousands of ground state energies across the nuclear chart and in some cases other ob-
servables as well. They are not designed to give accurate results in light nuclei, but they are efficient at capturing bulked properties of nuclei such as deformation which are affected by clustering. The latter are based on modern nuclear forces usually adjusted on two- and three-body data only (but not always), and they should be precise in the light sector, but they can miss emergent properties of nuclei if those are not fully encoded into the interaction.

There are, of course, other well-known methods based on nucleons as degrees of freedom that could not be used in this work but should be mentioned, such as the antisymmetrized molecular dynamics [34, 35] and the fermionic molecular dynamics [36, 37], both being particularly well suited for studying clustering and with a long history summarized in Ref. [9].

In Fig. 1, the DFT results from Refs. [38, 39], denoted SLy4, UNEDF0, UNEDF1, SkP, and SkM* were used to compute the one-alpha clustering ratios of several Li, Be, B, and C isotopes and compared to experiment [27, 28] (see Tab. 1). Only the results from the SV-min energy density functional are not shown as they were not of sufficient quality. Also shown are the \textit{ab initio} Green's function Monte Carlo (GFMC) results based on the so-called AV18+IL7 high-precision two- and three-body potentials compiled in Ref. [32], as well as the GFMC results in Ref. [40] based on the local chiral two- and three-body potentials including intermediate ∆-excitations and denoted NV2+3Ia; and the no-core shell model (NCSM) results based on the JISP16 [41] and Daejon16 [42–44] two-body interactions, as well as the LENPIC interactions of 2018 [45] (two-body) and 2019 [46, 47] (two- and three-body).

In some \textit{ab initio} approaches, the occasional absence of results in very light nuclei (\(A \leq 5\)) was compensated using experimental data assuming they would be described correctly anyway.

The first observation is that DFT models do capture alpha clustering to a large extent but, like most other approaches presented here, they tend to underestimate it except in \(^{10}\text{C}\) where they overestimate it. Those models are efficient at describing bulk properties of nuclei and so they represent the minimum that more precise approaches should be able to achieve.

The second observation is that the \textit{ab initio} approaches based on two-body forces only, \textit{i.e.} the NCSM results using the JISP16, Daejon16, and LENPIC(18) interactions, give very different results depending on how they have been optimized.

The LENPIC(18) interaction is fitted only on two-body data and gives mixed results, with alpha clustering overestimated in \(^{7}\text{Li}\) and \(^{9}\text{Be}\) but vastly underestimated in \(^{8}\text{Be}\).

The JISP16 and Daejon16 interactions were fitted on \(B, C\) isotopes and compared to experiment [27, 28] (see Tab. 1). Only the results from the SV-min energy density functional are not shown as they were not of sufficient quality. Also shown are the \textit{ab initio} Green’s function Monte Carlo (GFMC) results based on the so-called AV18+IL7 high-precision two- and three-body potentials compiled in Ref. [32], as well as the GFMC results in Ref. [40] based on the local chiral two- and three-body potentials including intermediate ∆-excitations and denoted NV2+3Ia; and the no-core shell model (NCSM) results based on the JISP16 [41] and Daejon16 [42–44] two-body interactions, as well as the LENPIC interactions of 2018 [45] (two-body) and 2019 [46, 47] (two- and three-body).

For that reason, the JISP16 results stand out and do not match experiment. This outcome can seem puzzling because this interaction can describe \((n,\alpha)\) and \((p,\alpha)\) scattering data [48], as well as alpha-alpha scattering as shown in Ref. [49] using a new \textit{ab initio} method tailored for clustering.

The Daejon16 interaction, on the other hand, gives a near-perfect match to experiment. This interaction, unlike JISP16, is based on chiral effective field theory potentials but should be able to describe the same scattering data because of the way it was built. The only real difference is the quality of its optimization on binding energies. It is thus possible to build a pure two-body interaction compatible with the one-alpha clustering measure as well as \((n,\alpha), (p,\alpha), \) and alpha-alpha scattering data, providing that some information beyond few-body systems is included in the optimization.

Finally, the \textit{ab initio} results including two- and three-body forces adjusted on few-body data only, \textit{i.e.} the NCSM-LENPIC(19) and both GFMC results, give the correct trends and are consistent with each other even if they underestimate clustering significantly in \(^{8}\text{B}\) and \(^{12}\text{C}\) (outside the 10% relative uncertainty band). The comparison between the LENPIC interactions of 2018 and 2019 in \(^{9}\text{Be}\) provides clear evidence of the crucial role played by three-body forces when fitting only few-body data, despite an underperformance of the latter in \(^{13,14}\text{C}\).

Overall, the results in Fig. 1 present the first evidence of a systematic deficiency in most modern nuclear forces.
with respect to alpha clustering. On the one hand, ab initio approaches based on two-body forces give satisfactory results solely when adjusted on binding energies of $A \lesssim 16$ nuclei, at odd with the ab initio rationale. On the other hand, those including three-body forces and adjusted on $A \leq 3$ data show consistency but underestimate alpha clustering in critical nuclei.

From a strict ab initio perspective, either current forces are not constrained enough to encode clustering effects, or, as was suggested in Ref. [9], those effects come at higher orders in chiral effective field theory, or the nuclear interaction should be redesigned as to explicitly account for the existence of clusters in nuclear matter.

Can alpha clustering be used to improve nuclear forces? When experimental $Q_\alpha$ values are used in place of the theoretical ones when calculating $r_1$, all models fall into the 10% relative uncertainty band, except for the DFT results which are not expected to perform well in light nuclei by design. This suggests that only a marginal improvement on binding energies is required. Which energies in particular need to be improved is the topic of the next section.

### III. FROM ALPHA CLUSTERING TO BINDING ENERGIES

The one-alpha clustering measure introduced in the previous section offers the opportunity to identify which nuclei’s ground-state energies are the most important to constraint many-body approaches with respect to alpha clustering.

If success is defined as being compatible with the experimental clustering ratios $r_1^{\text{exp}}(A)$ provided in Tab. I within 10% relative uncertainties, the question is: what are the constraints on binding energies?

This is a standard Bayesian inference problem where uncertainties on a fixed output must be back-propagated on the input so that the latter is compatible with the former. In this work, this is done using a Markov chain Monte Carlo (MCMC) algorithm based on the Metropolis-Hastings method as commonly used in different fields of physics.

The problem is analogous to a network calculation in nuclear astrophysics [50], albeit computationally much less expensive. To compute one set of $r_1(A)$ values, all the binding energies of all the nuclei in Tab. I as well as those of their partition subsystems, are required. Those energies are initialized randomly around the corresponding experimental values ±30% to start relatively far from the ideal values. Then they are updated by the MCMC algorithm at each step, which is itself guided by whether or not the $r_1(A)$ values are approaching the experimental ones within the accepted uncertainties, as defined by the $\chi^2$ measure at a step $i$:

$$
\chi^2(i) = \frac{1}{N_{\text{dof}}} \sum_{A} (r_1^{\text{exp}}(A) - r_1^{(i)}(A))^2 / (\Delta r_1(A))^2,
$$

where $\Delta E$ controls the "speed" at which the parameter space is explored.

The MCMC run is stopped when the energies are stable and the $\chi^2(i)$ does not decrease significantly. The two hyper-parameters controlling these criteria are the number of MCMC steps $N_{\text{steps}}$ and $\Delta E$. In the present work, sufficient convergence was found for $N_{\text{steps}} = 5000$ and $\Delta E = 10$ keV.

At the end of one MCMC run, one set of binding energies (input) compatible with the experimental one-alpha clustering ratios (output) is obtained. The principle of parallel MCMC is to repeat this process many times to sample the distribution of the input compatible with the output, and then to extract mean values and standard distribution of individual binding energies, assuming Gaussian distributions.

In practice, the binding energies of $^2,^3\text{H}$ and $^3,^4\text{He}$ were kept fixed to their experimental values since they are very well reproduced theoretically. One additional exception was considered for $^8\text{Be}$ to ensure the stability of the MCMC algorithm. Its binding energy was allowed to change, but its clustering ratio was not included in the $\chi^2$ because of its very large value.

The results from the averaging of 20 runs of 50 independent MCMC runs with 5000 steps each are shown in Tab. II. In principle, a single run of 50 independent MCMC runs would be enough, but the additional averaging procedure ensures a better precision and allows to check that the error on the mean energies and their standard deviations is very small.

The 30 binding energies are well constrained by the optimization of the 24 clustering ratios $r_1$ as indicated by the mean values $\mu(E)$. When looking at standard deviations, nuclei fall into two groups defined by $\sigma(E) \approx 100 – 250$ keV and $\sigma(E) < 30$ keV. The latter contains only the six nuclei $^6,^7\text{Li}, ^7\text{Be}, ^{10},^{11}\text{B}$, and $^{11}\text{C}$ whose ground-state energies are the most constrained by the one-alpha clustering measure. They are all made of one or two alphas and either $^2\text{H}$, $^3\text{H}$, or $^3\text{He}$; and except for $^7\text{Li}$, they are surprisingly not the most clustered nuclei identified earlier ($^7\text{Li}, ^8\text{Be}, ^9\text{B}$, and $^{12}\text{C}$) but their direct neighbors.

Knowing that the binding energies of $^5\text{He}$, $^5\text{Li}$, and $^8\text{Be}$ do not exhibit a strong sensitivity to alpha clustering makes the situation of the JISP16 interaction less puzzling. Also, the reason why the Daejon16 interaction matches experimental $r_1$ values is because it reproduces the binding energies of the nuclei $^6\text{Li}$ and $^{10}\text{B}$.

The case of $^{16}\text{B}$ is quite revealing. It was found in ab initio calculations [9, 51, 52] that this nucleus is particularly sensitive to three-body forces. It is also known that $^9\text{Be}$ and $^9\text{B}$, which are one proton and one neutron
TABLE II. Mean binding energies (in MeV) and standard deviations (in keV) of light nuclei from the back-propagation of 10% relative uncertainties on the clustering ratios \( r_1 \) given in Tab. I (except for \(^{8}\)Be). The experimental binding energies (in MeV) are shown for reference.

| Nucleus | \( E_{\text{exp}} \) | \( \mu(E) \) | \( \sigma(E) \) | Nucleus | \( E_{\text{exp}} \) | \( \mu(E) \) | \( \sigma(E) \) |
|---------|----------------|------------|------------|---------|----------------|------------|------------|
| \(^{5}\)He | -27.56 | -27.44 | 114 | \(^{12}\)Be | -68.65 | -68.65 | 181 |
| \(^{6}\)He | -29.27 | -29.27 | 96 | \(^{8}\)B | -37.74 | -37.75 | 215 |
| \(^{7}\)He | -28.86 | -28.86 | 88 | \(^{9}\)B | -56.31 | -56.32 | 152 |
| \(^{8}\)He | -31.4 | -31.39 | 141 | \(^{10}\)B | -64.75 | -64.77 | 8 |
| \(^{7}\)Li | -26.33 | -26.33 | 143 | \(^{11}\)B | -76.2 | -76.22 | 20 |
| \(^{6}\)Li | -31.99 | -32.00 | 3 | \(^{12}\)B | -79.57 | -79.58 | 169 |
| \(^{7}\)Li | -39.25 | -39.25 | 4 | \(^{13}\)B | -84.45 | -84.44 | 216 |
| \(^{8}\)Li | -41.28 | -41.28 | 136 | \(^{14}\)B | -85.42 | -85.42 | 217 |
| \(^{9}\)Li | -45.34 | -45.34 | 207 | \(^{8}\)C | -24.81 | -24.82 | 219 |
| \(^{6}\)Be | -26.92 | -26.91 | 138 | \(^{9}\)C | -39.04 | -39.04 | 210 |
| \(^{7}\)Be | -37.6 | -37.61 | 2 | \(^{10}\)C | -60.32 | -60.31 | 165 |
| \(^{8}\)Be | -56.5 | -56.49 | 141 | \(^{11}\)C | -73.44 | -73.45 | 15 |
| \(^{9}\)Be | -58.16 | -58.21 | 124 | \(^{12}\)C | -92.16 | -92.16 | 159 |
| \(^{10}\)Be | -64.98 | -64.99 | 120 | \(^{13}\)C | -97.11 | -97.17 | 146 |
| \(^{11}\)Be | -76.2 | -76.22 | 2 | \(^{14}\)C | -84.45 | -84.44 | 216 |

IV. CONCLUSIONS

In conclusion, the empirical one- and two-alpha clustering measures introduced in this work, based solely on binding energies, allow to correctly identify the nuclei \(^{7}\)Li, \(^{8}\)Be, \(^{8}\)B, and \(^{12}\)C as being the most clustered in their respective isotopic chains, and to characterize excited states of interest such as the Hoyle state.

It was shown that, according to the one-alpha clustering measure, state-of-the-art \textit{ab initio} approaches based on modern nuclear forces significantly underestimate alpha clustering in key nuclei such as \(^{8}\)Be and \(^{12}\)C.

Finally, relative uncertainties on the one-alpha clustering measure were back-propagated on binding energies using a MCMC method and revealed that the ground states energies of \(^{8,7}\)Li, \(^{7}\)Be, \(^{10,11}\)B, and \(^{11}\)C are particularly sensitive to alpha clustering.

By examining the literature on \(^{10}\)B together with the results of the present work, a strong case was made for a link between three-body forces and alpha clustering.

This work opens many new avenues ranging from the extension of the clustering measures beyond \(^{14}\)C and to other types of clusters, to the systematic study of clustered excited states, and to the testing of current and future DFT models and \textit{ab initio} interactions. Perhaps, the most interesting opportunity is the development of new two- and three-body forces in the spirit of the NNLO sat interaction \(^{50}\) including the binding energies of some of the six nuclei identified in their optimization. Such interactions could give satisfying results in medium-mass nuclei studied at the Facility for Rare Isotope Beams (FRIB) \(^{57}\). The results presented can also help experiments by telling which isotope to use in a reaction depending on the clusters of interest, and even which energy to choose by plotting the clustering measures as a function of the excitation energy to determine when clustering is maximal (not shown).

In all fairness, the measures introduced are empirical and mostly provide \textit{a posteriori} tests of the presence of clusters. For these reasons, a genuine attempt was made at quantifying alpha clustering to reveal real issues in nuclear models, but conclusions were mostly limited to qualitative statements.

Moreover, the one-alpha clustering measure does not discriminate between a meaningful model and a fictitious one if both are perfectly fitting binding energies of light nuclei. However, to quote J. von Neumann "With four parameters I can fit an elephant, and with five I can make him wiggle his trunk", and so, as for all models, good judgment matters.

A picture is emerging where alpha clustering, three-body forces, and observables in medium-mass nuclei are interrelated. Alpha clustering is an important feature of the nuclear interaction and should be used when developing and evaluating modern nuclear interactions as was recently done in Ref. \(^{58}\) for three-body forces on \((n,\alpha)\) scattering.

Appendix A: One-alpha clustering measure approximation in perturbation theory

The isospin symmetric approximation of the one-alpha clustering measure \( r_1(A,\alpha) \approx |1 + E(A - \alpha)/Q_{\alpha}(A)| \) can be justified approximately using perturbation theory. Assuming two non-interacting clusters \( \alpha \) and \( a \) with their non-orthogonal ground and first excited states \( |\alpha, a^*\rangle \) and \( |\alpha, a^*\rangle \), respectively, the state of the interacting system \( A = \alpha + a \) writes in the first order of perturbation theory:

\[
|A\rangle = |\alpha, a\rangle + \varepsilon \frac{\langle \alpha, a^* | V | \alpha, a \rangle}{E(\alpha, a) - E(\alpha, a^*)} |\alpha, a^*\rangle,
\]  

(A1)
where $\varepsilon$ is a small parameter and $V$ is the interaction between the two clusters. The interaction matrix elements will be written using the short notation $\langle \alpha, a | V| \alpha, a \rangle = V_{\alpha, a}^{\ast}$ for convenience. The energy of the total system in its first excited state is:

$$E(A^\ast) = E(\alpha, a^\ast) + \varepsilon V_{\alpha, a^\ast},$$ (A2)

Using the fact that $E(\alpha, a) = E(\alpha) + E(a)$ (non-interacting clusters) and denoting the energy difference between the ground state and the first excited state $\delta E(A) = E(A^\ast) - E(A)$, The energy difference $E(\alpha, a) - E(\alpha, a^\ast)$ can be written as:

$$E(\alpha, a) - E(\alpha, a^\ast) = E(\alpha) + E(a) - E(A) + \delta E(A) + \varepsilon V_{\alpha, a^\ast},$$ (A3)

where one recognizes the $Q_\alpha$-value. The overlap between the interacting and factorized systems then writes:

$$\langle \alpha, a | A \rangle = 1 + \frac{\varepsilon V_{\alpha, a^\ast}}{\delta E(A) - Q_\alpha(A) + \varepsilon V_{\alpha, a^\ast}} \langle \alpha, a | \alpha, a^\ast \rangle.$$ (A4)

For small perturbations, i.e. $\varepsilon \ll 1$, one obtains:

$$\langle \alpha, a | A \rangle \approx 1 + \frac{C}{\delta E(A) - Q_\alpha(A)},$$ (A5)

where $C$ is a small constant. The degree of clustering is then given by the probability $p_\alpha(A) = |\langle \alpha | \alpha, a \rangle|^2$, indicating possible corrections in $1/Q_\alpha^2(A)$ to the one-alpha clustering measure $r_1(A, \alpha)$.

**ACKNOWLEDGMENTS**

I am grateful to M. Piarulli for sharing the binding energies obtained using the GFMC-NV2+3Ia interaction, as well as to P. Maris and J. Vary for sharing the last update of the binding energies obtained using the NCSM-JISP16, Daejon16, and LENPIC(19) interactions, and for the ensuing discussions. I also want to thank N. Vassh and S. König for clarifying the use of the parallel MCMC algorithm and for pushing me to develop the two-alpha clustering measure during their productive visits at Argonne National Laboratory. My appreciation also goes to R. Wirringa, H. Hergert, S. Bogner, and W. Nazarewicz for their patience and useful comments on clustering and nuclear forces. Finally, I would like to thank T. Li and S. Wang for pointing out errors in the original manuscript. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under the FRIB Theory Alliance award de-sc0013617 An award of computer time was provided by the Institute for Cyber-Enabled Research at Michigan State University.

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