Local convergence analysis of a proximal Gauss-Newton method under a majorant condition

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Abstract

In this paper, the proximal Gauss-Newton method for solving penalized nonlinear least squares problems is studied. A local convergence analysis is obtained under the assumption that the derivative of the function associated with the penalized least square problem satisfies a majorant condition. Our analysis provides a clear relationship between the majorant function and the function associated with the penalized least squares problem. The convergence for two important special cases is also derived.

Keywords: Penalized nonlinear least squares problems; Proximal Gauss-Newton method; Majorant condition; Local convergence.

1 Introduction

We consider the penalized nonlinear least squares problem

\[
\min_{x \in \Omega} \frac{1}{2} \|F(x)\|^2 + J(x),
\]

where $X$ and $Y$ are real or complex Hilbert spaces, $\Omega \subseteq X$ an open set, $F : \Omega \to Y$ is a continuously differentiable nonlinear function and $J : \Omega \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. A wide variety of applications can be found in mathematical programming literature, see for example [4, 15, 16]. In particular, if $J(x) = 0$, for all $x \in \Omega$, the problem (1) becomes the classical nonlinear least squares problem studied in [5, 6, 8, 9]. In this case, a

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generalization of the Newton method called the Gauss-Newton method, can be used. This iterative algorithm computes the sequence

\[ x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \ldots, \]

where \( F'(x_k)^\dagger \) denotes the Moore-Penrose inverse of the linear operator \( F'(x_k) \).

In this paper, we consider the proximal Gauss-Newton method, introduced in [15], for solving (1). This method extends the classical Gauss-Newton approach. It is defined as

\[ x_{k+1} = \text{prox}_{H(x_k)}^{J}(x_k - F'(x_k)^\dagger F(x_k)), \quad k = 0, 1, \ldots, \]

where \( \text{prox}_{H(x_k)}^{J} \) is the proximity operator associated to \( J \) (see [12, 13, 14, 15]) with respect to the metric defined by the operator \( H(x_k) := F'(x_k)^* F'(x_k) \). It shall be mentioned that the computation of the proximity operator is in general not straightforward and it may require an iterative algorithm itself, since, in general, a closed form is not available.

The aim of this paper is to present a new local convergence analysis of proximal Gauss-Newton method under a majorant condition. This majorant formulation follows the ideas used in [5, 6, 7, 8, 9]. This analysis provides a clear relationship between the majorant function, which relaxes the Lipschitz continuity of \( F' \), and the nonlinear operator \( F \) associated with the penalized nonlinear least squares problem. Two majorant functions are also considered. In the first case, which corresponds to functions with Lipschitz derivative, the classical convergence results are recovered. The convergence analysis for analytical operators is discussed for the first time.

The convergence of the sequence generated by the proximal Gauss-Newton method was also studied in [15]. There, instead of majorant function, the Wang’s condition, introduced in [19, 20], is used for the analysis. In fact, it can be shown that these conditions are equivalent. However, the formulation as a majorant condition is better, due to it provides a clear relationship between the majorant function and the nonlinear function \( F \) under consideration. Furthermore, the majorant condition simplifies the proof of the obtained results.

The organization of the paper is as follows. Next, we list some notations and a basic result used in our presentation. In Section 2 some results on Moore-Penrose inverse, proximity operators and the proximal Gauss-Newton algorithm are discussed. In Section 3 we state the main result and, for a better organization of the results, it is divided in three parts. First, some properties of the majorant function are established. Then in Subsection 3.2, we present the relationships between the majorant function and the nonlinear function \( F \). Finally, in the last part our main result is proven. Section 4 is devoted to show the consequences of this result in particular cases.

1.1 Notation and auxiliary results

The following notations and results are used throughout our presentation. Let \( X \) and \( Y \) be Hilbert spaces. The open and closed balls in \( X \) with center \( a \) and radius \( r \) are denoted, respectively by
For simplicity, given $x \in X$, we use the short notation
\[
\sigma(x) := \|x - x_*\|.
\]
From now on, $\Omega \subseteq X$ an open set, $J : \Omega \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional and $F : \Omega \to Y$ is a continuously differentiable function such that $F'$ has a closed image in $\Omega$. We use $\mathcal{L}(X, Y)$ to denote the space of bounded linear operators from $X$ to $Y$ and $I_X$ corresponds to the identity operator on $X$. Finally, if $A \in \mathcal{L}(X, Y)$, then $\text{Ker}(A)$ and $\text{im}(A)$ are the kernel and image of $A$, respectively, and $A^*$ its adjoint operator.

The following auxiliary results of elementary convex analysis will be needed:

**Proposition 1.** Let $\epsilon > 0$ and $\tau \in [0,1]$. If $\varphi : [0, \epsilon) \to \mathbb{R}$ is convex, then
- The function $l : (0, \epsilon) \to \mathbb{R}$ defined by
  \[
  l(t) = \frac{\varphi(t) - \varphi(\tau t)}{t},
  \]
  is monotone increasing.
- $D^+\varphi(0) = \lim_{u \to 0^+} \frac{\varphi(u) - \varphi(0)}{u} = \inf_{0<u} \frac{\varphi(u) - \varphi(0)}{u}$.

**Proof.** See Theorem 4.1.1 and Remark 4.1.2, pp. 21 of [11].

## 2 Preliminary

In this section some results on Moore-Penrose inverse and proximity operators will be presented. Then, the algorithm to solve problem (I) and some properties related to it will be introduced.

### 2.1 Generalized inverses

In this section some results on Moore-Penrose inverse, will be presented. More details can be found in [18, 21].

Let $A \in \mathcal{L}(X, Y)$ with a closed image. The Moore-Penrose inverse of $A$ is the linear operator $A^\dagger \in \mathcal{L}(Y, X)$ which satisfies:
\[
AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.
\]
From the definition of the Moore-Penrose inverse, it is easy to see that
\[
A^\dagger A = I_X - \Pi_{\text{Ker}(A)}, \quad AA^\dagger = \Pi_{\text{im}(A)}, \quad (2)
\]
where $\Pi_E$ denotes the projection of $X$ onto subspace $E$.

If $A$ is injective, then
\[
A^\dagger = (A^*A)^{-1}A^*, \quad A^\dagger A = I_X, \quad \|A^\dagger\|^2 = \|(A^*A)^{-1}\|.
\]
We end this part with a result concerning the variation of the pseudo-inverse, see [15] [18] [21].
Lemma 2. Let $A, B \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ with closed images. If $A$ is injective and $\|A^\dagger\|A - B\| < 1$, then $B$ is injective and
\[
\|B^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\|\|A - B\|}, \quad \|B^\dagger - A^\dagger\| \leq \frac{\sqrt{2}\|A^\dagger\|^2\|A - B\|}{1 - \|A^\dagger\|\|A - B\|}.
\]

2.2 Proximity operators

Proximity operators were introduced by Moreau and their use in signal theory goes back to [12]. We briefly recall some essential facts below and refer the reader to [12, 13, 15] for more details.

Let $H : \mathbb{X} \to \mathbb{X}$ be a continuously, positive and selfadjoint, bounded from below and, therefore, invertible operator. Then we have a new scalar product on $\mathbb{X}$ by setting
\[
\langle x, z \rangle_H = \langle x, Hz \rangle.
\]
Hence, the corresponding induced norm $\|\cdot\|_H$ is equivalent to the given norm on $\mathbb{X}$, since the following inequalities hold
\[
\frac{1}{\|H^{-1}\|}\|x\|^2 \leq \|x\|_H^2 \leq \|H\|\|x\|^2.
\]

The Moreau-Yosida approximation of $J$ with respect to the scalar product induced by $H$ is the functional $M_J : \mathbb{X} \to \mathbb{R}$ defined by setting
\[
M_J(z) = \inf_{x \in \mathbb{X}} \left\{ J(x) + \frac{1}{2}\|x - z\|_H^2 \right\}. \tag{4}
\]
Recalling that $J$ is a convex, lower semicontinuous and proper function $J : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, it is easy to prove that the infimum of the last equation is attained at a unique point. Therefore, let us call $\text{prox}_J^H(z)$, the proximity operator associated to $J$ and $H$
\[
\text{prox}_J^H : \mathbb{X} \to \mathbb{X} \quad z \mapsto M_J(z) = \arg\min_{x \in \mathbb{X}} \left\{ J(x) + \frac{1}{2}\|x - z\|_H^2 \right\}. \tag{5}
\]
Writing the first order optimality conditions for (4), we obtain that
\[
p = \text{prox}_J^H(z) \iff 0 \in \partial J(p) + H(p - z) \iff Hz \in (\partial J + H)(p),
\]
which using that the minimum in (4) is attained at a unique point leads to
\[
\text{prox}_J^H(z) = (\partial J + H)^{-1}(Hz).
\]
This part ends with an important property of proximity operator.

Lemma 3. Let $H_1$ and $H_2$ be two continuous positive selfadjoint operators on $\mathbb{X}$, both bounded from below. Then,
\[
\|\text{prox}_{H_1}^J(z_1) - \text{prox}_{H_2}^J(z_2)\| \leq \sqrt{\|H_1\|\|H_1^{-1}\|\|z_1 - z_2\|} + \|H_1^{-1}\|\|(H_1 - H_2)(z_2 - \text{prox}_{H_2}^J(z_2))\|,
\]
for every $z_1, z_2 \in \mathbb{X}$.

Proof. See Remark 4 in [15].
2.3 The proximal Gauss-Newton method

In this section we present the algorithm to solve (1) as well as some related properties.

The goal of this method, introduced in [15], is to find stationary points of problem (1) as follows:

\[ x_{k+1} = \text{prox}_{H(x_k)}(x_k - F'(x_k)^\dagger F(x_k)), \quad k = 0, 1, \ldots \]

(6)

where \( H(x_k) = F'(x_k)^* F'(x_k) \) and \( \text{prox}_{H(x_k)} \) is the proximity operator associated to \( J \) and \( H(x_k) \) as defined in (5).

Remark 1. As proved in Proposition 6 of [15], given \( x_k \in X \), if \( F'(x_k) \) is injective with closed image, then \( x_{k+1} \) satisfies

\[ x_{k+1} = \arg \min_{x \in X} \frac{1}{2} \| F(x_k) + F'(x_k)(x - x_k) \|^2 + J(x). \]

This problem can be solved using first order methods for the minimization of nonsmooth convex functions, such as bundle methods or forward-backward methods (see [1, 10]). We will use the proximal point formulation because the theoretical results of this area will be very useful for the proof of the convergence of the method.

In the following, we establish the connection between the stationary point of the function defined in (1) and the fixed points of proximal point operator.

**Proposition 4.** Let \( x \in \Omega \) such that \(-F'(x)F(x) \in \partial J(x)\), i.e., \( x \) satisfies the first order conditions for local minimizers of (1). Assume that \( F'(x) \) is injective and \( \text{im}(F'(x)) \) is closed, then \( x \) satisfies the fixed point equation

\[ x = \text{prox}_{H(x)}(x - F'(x)^\dagger F(x)), \]

where \( H(x) = F'(x)^* F'(x) \).

**Proof.** The proof follows the same ideas of the proof of Proposition 5 in [15]. \( \square \)

3 Local analysis for the Gauss-Newton method

Our goal is to state and prove a local theorem for the proximal Gauss-Newton method defined in (6). First, we show some results regarding the scalar majorant function, which relaxes the Lipschitz condition to \( F' \). Then, we establish the main relationships between the majorant function and the nonlinear function \( F \). Finally we obtain that the Gauss-Newton method is well-defined and converges. The statement of the theorem is as follows:
Theorem 5. Let $\Omega \subseteq \mathbb{X}$ be an open set, $J : \Omega \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous functional and $F : \Omega \to \mathbb{Y}$ a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_* \in \Omega$, $R > 0$ and

$$c := \|F(x_*)\|, \quad \beta := \|F'(x_*)\|, \quad \kappa := \beta \|F'(x_*)\| \quad \delta := \sup \{ t \in [0,R) : B(x_*, t) \subseteq \Omega \}. $$

Suppose that $-F'(x_*)^*F(x_*) \in \partial J(x_*)$, $F'(x_*)$ is injective and there exists a continuously differentiable function $f : [0, R) \to \mathbb{R}$ such that

$$\beta \|F(x) - F'(x_*) + \tau(x - x_*)\| \leq f'(\tau \sigma(x)) - f'(\tau \sigma(x)),$$

where $x \in B(x_*, \delta)$, $\tau \in [0, 1]$ and $\sigma(x) = \|x - x_*\|$, and

**h1)** $f(0) = 0$ and $f'(0) = -1$;

**h2)** $f'$ is convex and strictly increasing;

**h3)** $(1 + \sqrt{2}) \kappa + 1 \beta D^+ f'(0) < 1.$

Let be given the positive constants $\nu := \sup \{ t \in [0,R) : f'(t) < 0 \}$,

$$\rho := \sup \left\{ t \in (0, \nu) : \left[ \frac{f'(t) + 1 + \kappa}{f'(t) - f(t) + c\beta (1 + \sqrt{2})(f'(t) + 1)} + c\beta f'(t) + 1 \right] < 1 \right\},$$

$$r := \min \{ \rho, \delta \}.$$

Given $H(x_k) = F'(x_k)^*F'(x_k)$, define $\text{prox}_J^{H(x_k)}$ as the proximity operator associated to $J$ and $H(x_k)$, see [5]. Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x_*, r)/\{x_*\}$

$$x_{k+1} = \text{prox}_J^{H(x_k)}(x_k - F'(x_k)^*F'(x_k)), \quad k = 0, 1, \ldots,$$

is well defined, the generated sequence $\{x_k\}$ is contained in $B(x_*, r)$, converges to $x_*$ and

$$\|x_{k+1} - x_*\| \leq \frac{f'(\sigma(x_0)) + 1 + \kappa}{\sigma(x_0)f'(\sigma(x_0))}\|x_k - x_*\|^2 + \frac{(1 + \sqrt{2})\beta c [f'(\sigma(x_0)) + 1]}{\sigma(x_0)f'(\sigma(x_0))}\|x_k - x_*\|^2 + \frac{c\beta (1 + \sqrt{2})\kappa + 1}{\sigma(x_0)f'(\sigma(x_0))}\|x_k - x_*\|, \quad (9)$$

for all $k = 0, 1, \ldots$.

**Remark 2.** If $J = 0$, the proximal Gauss-Newton method becomes the classical Gauss-Newton method. However, with respect to the radius of the convergence ball this result does not correspond with the classical approach, see Theorem 7 of [6]. The reason is that the upper bound given in Lemma 3 is not affected if $J = 0$.  

Remark 3. If the inequality in (7) holds only for $\tau = 0$, an analogous theorem is true. In fact, if the definition of $\rho$ is replaced

$$\rho = \sup \left\{ t \in (0, \nu) : \frac{[f'(t) + 1 + \kappa][f'(t) + f(t) + 2t + c\beta(1 + \sqrt{2})(f'(t) + 1)] + c\beta[f'(t) + 1]}{t[f'(t)]^2} < 1 \right\},$$

the well definition of proximity operator, the inclusion of the computed sequence in $B(x^*, r)$ and its convergence are guaranteed. In particular:

$$\|x_{k+1} - x^*\| \leq \frac{[f'(\sigma(x_0)) + 1 + \kappa][f'(\sigma(x_0))\sigma(x_0) + f(\sigma(x_0)) + 2\sigma(x_0)]}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x^*\|^2 + \frac{(1 + \sqrt{2})\beta[f'(\sigma(x_0)) + 1]^2}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x^*\|^2 + \frac{c\beta[(1 + \sqrt{2})\kappa + 1][f'(\sigma(x_0)) + 1]}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x^*\|,$$

for all $k = 0, 1, \ldots$.

As before, $H(x_k) = F'(x_k)^*F'(x_k)$ and $\text{prox}^H_{J(x_k)}$ is the proximity operator defined in (5). For the zero-residual problems, i.e., $c = 0$, Theorem 5 becomes:

Corollary 6. Let $\Omega \subseteq \mathbb{Y}$ be an open set, $J : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous functional and $F : \Omega \rightarrow \mathbb{Y}$ a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_\ast \in \Omega$, $R > 0$ and

$$\beta := \|F'(x_\ast)^\dagger\|, \quad \kappa := \beta \|F'(x_\ast)\| \quad \delta := \sup \{ t \in [0, R) : B(x_\ast, t) \subset \Omega \}.$$

Suppose that $F(x_\ast) = 0$, $0 \in \partial J(x_\ast)$, $F'(x_\ast)$ is injective and there exists a continuously differentiable function $f : [0, \ R) \rightarrow \mathbb{R}$ such that

$$\beta \|F'(x) - F'(x_\ast + t(x - x_\ast))\| \leq f'(\sigma(x)) - f'(\tau \sigma(x)),$$

where $x \in B(x_\ast, \delta)$, $\tau \in [0, 1]$ and $\sigma(x) = \|x - x_\ast\|$, and

\begin{itemize}
  \item[h1)] $f(0) = 0$ and $f'(0) = -1$;
  \item[h2)] $f'$ is convex and strictly increasing.
\end{itemize}

Let be given the positive constants $\nu := \sup \{ t \in [0, R) : f'(t) < 0 \}$,

$$\rho := \sup \left\{ t \in (0, \nu) : \frac{[f'(t) + 1 + \kappa][tf'(t) - f(t)]}{t[f'(t)]^2} < 1 \right\}, \quad r := \min \{ \rho, \delta \}.$$

Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x_\ast, r)/\{x_\ast\}$

$$x_{k+1} = \text{prox}^H_{J(x_k)}(x_k - F'(x_k)^\dagger F(x_k)), \quad k = 0, 1, \ldots,$$

is well defined, the generated sequence $\{x_k\}$ is contained in $B(x_\ast, r)$, converges to $x_\ast$ and

$$\|x_{k+1} - x_\ast\| \leq \frac{[f'(\sigma(x_0)) + 1 + \kappa][f'(\sigma(x_0))\sigma(x_0) - f(\sigma(x_0))]^2}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x_\ast\|^2, \quad k = 0, 1, \ldots.$$
In order to prove Theorem 5, we need some results. From now on, we assume that all the assumptions of Theorem 5 hold.

### 3.1 The majorant function

Our first goal is to show that the constant $\delta$ associated with $\Omega$ and the constants $\nu$ and $\rho$ associated with the majorant function $f$ are positive. Also, we will prove some results related to the function $f$.

We begin by noting that $\delta > 0$, because $\Omega$ is an open set and $x_* \in \Omega$.

**Proposition 7.** The constant $\nu$ is positive and $f'(t) < 0$ for all $t \in (0, \nu)$.

**Proof.** As $f'$ is continuous in $(0, R)$ and $f'(0) = -1$, there exists $\epsilon > 0$ such that $f'(t) < 0$ for all $t \in (0, \epsilon)$. Hence, $\nu > 0$. Now, using $h2$ and definition of $\nu$ the last part of the proposition follows.

**Proposition 8.** The following functions are positive and increasing:

i) $[0, \nu) \ni t \mapsto -1/f'(t)$;

ii) $[0, \nu) \ni t \mapsto -[f'(t) + 1 + \kappa]/f'(t)$;

iii) $(0, \nu) \ni t \mapsto [tf'(t) - f(t)]/t^2$;

iv) $(0, \nu) \ni t \mapsto [f'(t) + 1]/t$.

As a consequence,

$(0, \nu) \ni t \mapsto [f'(t) + 1 + \kappa][tf'(t) - f(t)]/[tf'(t)]^2$,

$(0, \nu) \ni t \mapsto [f'(t) + 1]^2/[tf'(t)]^2$,

$(0, \nu) \ni t \mapsto f'(t) + 1/t[f'(t)]^2$,

are also positive and increasing functions.

**Proof.** Items i and ii are immediate, because $h1$, $h2$ and Proposition 7 imply that $f'$ is strictly increasing and $-1 \leq f'(t) < 0$ for all $t \in [0, \nu)$.

Now, note that after some simple algebraic manipulations we have

$$
\frac{tf'(t) - f(t)}{t^2} = \int_0^1 \frac{f'(t) - f'(\tau t)}{t} d\tau.
$$

Hence, as $f'$ is strictly increasing ($h2$), we obtain that the function of item iii is positive. Moreover, combining the last equation and Proposition 7 with $f' = \varphi$ and $\epsilon = \nu$, we conclude function of item iii is increasing. So, item iii is proved.

Assumption $h1$ and $h2$ imply that the function of item iv is positive. Hence, to conclude item iv use $h2$, $f'(0) = -1$ and Proposition 7 with $f' = \varphi$, $\epsilon = \nu$ and $\tau = 0$.

To prove that the functions in the last part are positive and increasing combine items i, ii and iii for the first function and items i and iv for the second and third functions.
Proposition 9. The constant $\rho$ is positive and there holds

$$\frac{[f'(t) + 1 + \kappa] [tf'(t) - f(t) + c\beta(1 + \sqrt{2})(f'(t) + 1)] + c\beta [f'(t) + 1]}{t[f'(t)]^2} < 1, \quad \forall \ t \in (0, \rho).$$

Proof. First, using $h_1$ and some algebraic manipulation gives

$$\frac{tf'(t) - f(t)}{t} = \left[ f'(t) - \frac{f(t) - f(0)}{t - 0} \right], \quad \frac{f'(t) + 1}{t} = \frac{f'(t) - f'(0)}{t - 0}.$$

Since $f'(0) = -1$ and $f'$ is convex, last equations and Proposition 1 lead to

$$\lim_{t \to 0} \frac{[tf'(t) - f(t)]/t}{t} = 0, \quad \lim_{t \to 0} \frac{f'(t) + 1}{t} = D^+ f'(0),$$

which, combined with $f'(0) = -1$ and simples calculus yields

$$\lim_{t \to 0} \frac{[f'(t) + 1 + \kappa] [tf'(t) - f(t) + c\beta(1 + \sqrt{2})(f'(t) + 1)] + c\beta [f'(t) + 1]}{t[f'(t)]^2} \leq -\beta f'(\sigma(x)).$$

Now, using $h_3$, i.e., $[(1 + \sqrt{2})\kappa + 1]c\beta D^+ f'(0) < 1$, we conclude that there exists an $\epsilon > 0$ such that

$$\frac{[f'(t) + 1 + \kappa] [tf'(t) - f(t) + c\beta(1 + \sqrt{2})(f'(t) + 1)] + c\beta [f'(t) + 1]}{t[f'(t)]^2} < 1, \quad t \in (0, \epsilon),$$

So, $\epsilon \leq \rho$, which proves the first statement.

To conclude the proof, we use the definition of $\rho$ and Proposition 8. \hfill $\square$

3.2 Relationship of the majorant function with the non-linear function $F$

In this part we will present the main relationships between the majorant function $f$ and the function $F$ associated with the problem (P). As usual $\sigma(x) = \|x - x_*\|.$

Lemma 10. Let $x \in \Omega$. If $\sigma(x) < \min\{\nu, \delta\}$, then $H(x) = F'(x)^* F'(x)$ is invertible and the following inequalities hold

$$\|F'(x)^*\| \leq \frac{-\beta}{f'(\sigma(x))}, \quad \|F'(x)^* - F'(x_*)^*\| < \frac{-\sqrt{2}\beta [f'(\sigma(x)) + 1]}{f'(\sigma(x))}.$$

In particular, $H(x) = F'(x)^* F'(x)$ is invertible in $B(x_*, r)$. 

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Proof. Let \( x \in \Omega \) such that \( \sigma(x) < \min\{\nu, \delta\} \). Since \( \sigma(x) < \nu \), by Proposition \( \ref{prop} \) \( f'(\sigma(x)) < 0 \). Using the definition of \( \beta \), the inequality \( \ref{ineq1} \) and \( h1 \) we have

\[
\|F'(x_*)\|^\dagger\|F'(x) - F'(x_*)\| = \beta \|F'(x) - F'(x_*)\| \leq f'(\sigma(x)) - f'(0) = f'(\sigma(x)) + 1 < 1.
\]

Taking account that \( F'(x_*) \) is injective, in view of Lemma \( \ref{lemma2} \) \( F'(x) \) is injective. So, \( H(x) \) is invertible. Moreover, again by Lemma \( \ref{lemma2} \)

\[
\|F'(x)\|^\dagger \leq \frac{\|F'(x_*)\|^\dagger}{1 - \|F'(x_*)\|^\dagger\|F'(x_*) - F'(x)\|}, \quad \|F'(x_*) - F'(x)\| \leq \frac{\sqrt{2\|F'(x_*)\|^\dagger^2\|F'(x_*) - F'(x)\|}}{1 - \|F'(x_*)\|^\dagger\|F'(x_*) - F'(x)\|}.
\]

Now, using the definition of \( \beta \), inequality \( \ref{ineq1} \), \( h1 \) and that \( \sigma(x) < \nu \), we have

\[
\frac{1}{1 - \|F'(x_*)\|^\dagger\|F'(x) - F'(x_*)\|} \leq \frac{1}{1 - (f'(\sigma(x)) - f'(0))} = \frac{1}{f'(\sigma(x))}.
\]

Thus, combining the last inequalities, we obtain the desired bounds for \( \|F'(x)\|^\dagger \) and \( \|F'(x_*) - F'(x)\| \). The last part follows by noting that \( r \leq \min\{\nu, \delta\} \).

To prove the convergence of sequence \( \{x_k\} \) on Theorem \( \ref{theorem} \) the following relations will be needed.

**Lemma 11.** Let \( x \in \Omega \). If \( \sigma(x) < \min\{\nu, \delta\} \), then

\begin{itemize}
  \item[i)] \( \|H(x)\|^{1/2} \leq \lfloor f'(\sigma(x)) + 1 + \kappa \rfloor / \beta \);
  \item[ii)] \( \|H(x)^{-1}\|^{1/2} \leq -\beta / |f'(\sigma(x))| \);
  \item[iii)] \( \beta \| (H(x) - H(x_*) F'(x_*)^\dagger \| \leq (f'(\sigma(x)) + 2 + \kappa) (f'(\sigma(x)) + 1) \).
\end{itemize}

**Proof.** First, simple calculus, inequality in \( \ref{ineq1} \) and definitions of \( \beta \) and \( \kappa \) gives

\[
\beta \|F'(x)\| = \|F'(x_*)\|^\dagger \|F'(x)\| \leq \beta \|F'(x) - F'(x_*)\| + \beta \|F'(x_*)\| \leq f'(\sigma(x)) + 1 + \kappa. \tag{10}
\]

As \( \|H(x)\|^{1/2} = \|F'(x)^{-1} F'(x)\|^{1/2} = \|F'(x)\| \), the first statement follows.

Now, to show item ii, use definition \( H \), last inequality in \( \ref{ineq2} \) and Lemma \( \ref{lemma10} \).

For iii, note that the definition \( H \), some algebraic manipulations and \( \ref{ineq1} \) gives

\[
\beta \|(H(x) - H(x_*) F'(x_*)^\dagger \| = \beta \|F'(x)^{-1} (F'(x) - F'(x_*) F'(x_*)^\dagger F'(x) - F'(x_*)^\dagger) \| + \beta \|F'(x) - F'(x_*)\|^\dagger \| \leq (\|F'(x)\|^\dagger \|F'(x_*)^\dagger \| + 1) \beta \|F'(x) - F'(x_*)\|,
\]

which, combined with \( \ref{ineq10} \) and inequality in \( \ref{ineq1} \) imply the desired statement.

**Remark 4.** Note that in the Lemmas \( \ref{lemma10} \) and \( \ref{lemma11} \) we have used the fact that condition \( \ref{ineq1} \) holds only for \( \tau = 0 \).
It is convenient to study the linearization error of $F$ at a point in $\Omega$, for which we define

$$ E_F(x, y) := F(y) - \left[ F(x) + F'(x)(y - x) \right], \quad y, x \in \Omega. \quad (11) $$

We will bound this error by the error in the linearization on the majorant function $f$

$$ e_f(t, u) := f(u) - \left[ f(t) + f'(t)(u - t) \right], \quad t, u \in [0, R). \quad (12) $$

**Lemma 12.** Let $x \in \Omega$. If $\sigma(x) < \delta$, then $\beta \| E_F(x, x_*) \| \leq e_f(\sigma(x), 0)$.

**Proof.** Since $B(x_*, \delta)$ is convex, we obtain that $x_* + \tau(x - x_*) \in B(x_*, \delta)$, for $0 \leq \tau \leq 1$. Thus, as $F$ is continuously differentiable in $\Omega$, the definition of $E_F$ and some simple manipulations yield

$$ \beta \| E_F(x, x_*) \| \leq \int_0^1 \beta \| F'(x) - F'(x_* + \tau(x - x_*)) \| \| x_* - x \| \, d\tau. $$

From the last inequality and the assumption (7), we obtain

$$ \beta \| E_F(x, x_*) \| \leq \int_0^1 \left[ f'(\sigma(x)) - f'(\tau \sigma(x)) \right] \sigma(x) \, d\tau. $$

Evaluating the above integral and using the definition of $e_f$, the statement follows. \qed

**Remark 5.** If the inequality in (7) holds only for $\tau = 0$, then the upper bound of $\beta \| E_F(x, x_*) \|$ in the previous Lemma becomes $e_f(\sigma(x), 0) + 2(f(\sigma(x)) + \sigma(x))$.

In particular, Lemma 10 guarantees that $H(x) = F'(x)^*F'(x)$ is invertible in $B(x_*, r)$ and consequently, $F'(x)^\dagger$ and $\text{prox}_H(x)$ are well defined in this region. Hence, the proximal Gauss-Newton iteration map is also well defined. Let us call $G_F$, the proximal Gauss-Newton iteration map for $F$ in that region:

$$ G_F : B(x_*, r) \rightarrow \mathbb{X}, \quad x \mapsto \text{prox}_H(G_F(x)), \quad (13) $$

where

$$ H(x) = F'(x)^*F'(x), \quad G_F(x) = x - F'(x)^\dagger F(x). \quad (14) $$

Take $x \in B(x_*, r)$. Note that the point computed by the proximal Gauss-Newton iteration, $G_F(x)$, may not be an element of $B(x_*, r)$, or may not even belong to the domain of $F$. To ensure that the Gauss-Newton iterations may be repeated indefinitely, this is enough to guarantee that the method is well defined for one iteration, as we will show in the following result.
Lemma 13. Let $x \in \Omega$. If $\sigma(x) < r$, then $G_F$ is well defined and there holds

$$\|G_F(x) - x_*\| \leq \frac{\|f'(\sigma(x)) + 1 + \kappa\| f'(\sigma(x)) \sigma(x) - f(\sigma(x))\|}{\|\sigma(x) f'(\sigma(x))\|^2} \|x - x_*\|^2 + \frac{(1 + \sqrt{2}) \sigma(x) f'(\sigma(x)) + \beta}{\|\sigma(x) f'(\sigma(x))\|^2} \|x_k - x_*\|^2 + \frac{c\beta(1 + \sqrt{2}) \sigma(x) f'(\sigma(x)) + \beta}{\|\sigma(x) f'(\sigma(x))\|^2} \|x - x_*\|. \quad (15)$$

In particular,

$$\|G_F(x) - x_*\| < \|x - x_*\|. \quad (16)$$

Proof. First, as $\|x - x_*\| < r$, it follows from Lemma 10 that $H(x) = F'(x)^* F'(x)$ is invertible; then $G_F(x)$ and $G_F(x)$ are well defined. Now, as $-F'(x)^* F(x) \in \partial J(x)$ and $F'(x)$ is injective, it follows from Proposition 13 and 14 that $x_* = \text{prox}_{H(x)}(G_F(x))$. Hence,

$$\|G_F(x) - x_*\| = \|\text{prox}_{H(x)}(G_F(x)) - \text{prox}_{H(x)}(G_F(x))\|,$$

which, combined with Lemma 3 yields

$$\|G_F(x) - x_*\| \leq (\|H(x)\| \|H(x)^{-1}\|)^{1/2} \|G_F(x) - G_F(x)\| + \|H(x)^{-1}\| \|(H(x) - H(x_*))(H(x) - H(x_*))(G_F(x) - \text{prox}_{H(x)}(G_F(x)))\|.$$

Using $x_* = \text{prox}_{H(x)}(G_F(x))$ and (14), the last inequality becomes

$$\|G_F(x) - x_*\| \leq (\|H(x)\| \|H(x)^{-1}\|)^{1/2} \|G_F(x) - G_F(x)\| + \|H(x)^{-1}\| \|(H(x) - H(x_*))F'(x)\| ||F'(x)||.$$

For simplicity, the notation defines the following terms:

$$A(x, x_*) = (\|H(x)\| \|H(x)^{-1}\|)^{1/2} \|G_F(x) - G_F(x)\| \quad (17)$$

and

$$B(x, x_*) = \|H(x)^{-1}\| \|(H(x) - H(x_*))F'(x)\| ||F'(x)||. \quad (18)$$

So, from the three latter inequalities we have

$$\|G_F(x) - x_*\| \leq A(x, x^*) + B(x, x^*). \quad (19)$$

Now, we will obtain upper bounds of $A(x, x^*)$ and $B(x, x^*)$. First, some algebraic manipulations and definitions in (11) and (14) yield

$$\|G_F(x) - G_F(x_*)\| = \|F'(x)\| \|F'(x)(x - x_*) - F(x) + F(x_*)\| + \|F'(x)\| \|F'(x)\| ||F(x)||.$$
Combining last inequality, Lemmas 10 and 12 and definition of \( c \), we have

\[
\|G_F(x) - G_F(x_*)\| = \frac{e_f(\sigma(x), 0)}{-f(\sigma(x))} + \frac{\sqrt{2}c\beta[f'(\sigma(x)) + 1]}{-f'(\sigma(x))}.
\]

So, the definition in (17), last inequality and Lemma 11-ii imply

\[
A(x, x^*) \leq \frac{f'(\sigma(x)) + 1 + \kappa}{[f'(\sigma(x))]^2} \left( e_f(\sigma(x), 0) + \sqrt{2}c\beta[f'(\sigma(x)) + 1] \right).
\]

On the other hand, from definition in (18), items ii and iii of Lemma 11 we have

\[
B(x, x_*) \leq \frac{c\beta}{[f'(\sigma(x))]^2} (f'(\sigma(x)) + 2 + \kappa)(f'(\sigma(x)) + 1).
\]

Hence, (19), (20) and (21) imply

\[
\|G_F(x) - x_*\| \leq \frac{f'(\sigma(x)) + 1 + \kappa}{[f'(\sigma(x))]^2} \left( e_f(\sigma(x), 0) + \sqrt{2}c\beta[f'(\sigma(x)) + 1] \right) + \left( 1 + \sqrt{2} \right)c\beta \left[ f'(\sigma(x)) + 1 \right] \frac{\sigma(x)}{[f'(\sigma(x))]^2}.
\]

which, combined with (12), h1 and simple manipulation yields to (15).

To end the proof first, note that the right-hand side of (15) is equivalent to

\[
\left[ f'(\sigma(x)) + 1 + \kappa \right] \frac{\sigma(x)f'(\sigma(x)) - f(\sigma(x)) + c\beta [1 + \sqrt{2}] (f'(\sigma(x)) + 1) + c\beta f'(\sigma(x)) + 1 }{\sigma(x)[f'(\sigma(x))]^2} \right] \sigma(x).
\]

On the other hand, as \( x \in B(x_*, r)/\{x_*\} \), i.e., \( 0 < \sigma(x) < r \leq \rho \) we apply the Proposition 9 with \( t = \sigma(x) \) to conclude that the quantity in the bracket above is less than one. So, (16) follows.

\[ \square \]

Remark 6. If the inequality in (7) holds only for \( \tau = 0 \), then (15) becomes

\[
\|G_F(x) - x_*\| \leq \frac{f'(\sigma(x)) + 1 + \kappa}{[\sigma(x)f'(\sigma(x))]^2} \left| f'(\sigma(x)) \right| \left| x - x_* \right| \left| x - x_* \right| + \frac{1 + \sqrt{2}c\beta[f'(\sigma(x)) + 1]}{[\sigma(x)f'(\sigma(x))]^2} \left| x - x_* \right| \left| x - x_* \right| + \frac{c\beta[(1 + \sqrt{2})\kappa + 1]}{\sigma(x)[f'(\sigma(x))]^2} \left| x - x_* \right|.
\]
3.3 Proof of Theorem 5

First of all, note that the equation in (8) together with (13) and (14) imply that the sequence \( \{x_k\} \) satisfies
\[
x_{k+1} = G_F(x_k), \quad k = 0, 1, \ldots
\] (22)

**Proof.** Since \( x_0 \in B(x_*, r)/\{x_*\} \), i.e., \( 0 < \sigma(x_0) < r \), by a combination of Lemma 10, the last inequality in Lemma 13 and an induction argument it is easy to see that \( \{x_k\} \) is well defined and remains in \( B(x_*, r) \).

Now, our goal is to show that \( \{x_k\} \) converges to \( x_* \). As \( \{x_k\} \) is well defined and contained in \( B(x_*, r) \), (22) and Lemma 13 leads to
\[
\|x_{k+1} - x_*\| \leq \frac{[f'(\sigma(x_k)) + 1 + \kappa][f'(\sigma(x_k))\sigma(x_k) - f(\sigma(x_k))]}{[\sigma(x_k)f'(\sigma(x_k))]^2}\|x_k - x_*\|^2 + \frac{(1 + \sqrt{2})c\beta[f'(\sigma(x_k))] + 1}{\sigma(x_k)f'(\sigma(x_k))^2}\|x_k - x_*\|,
\]
for all \( k = 0, 1, \ldots \). Using again (22) and the second part of Lemma 13, it is easy to conclude that
\[
\sigma(x) = \|x_k - x_*\| < \|x_0 - x_*\| = \sigma(x_0), \quad k = 1, 2, \ldots
\] (23)

Hence, by combining the last two inequalities with the last part of Proposition 8 we obtain that
\[
\|x_{k+1} - x_*\| \leq \frac{[f'(\sigma(x_0)) + 1 + \kappa][f'(\sigma(x_0))\sigma(x_0) - f(\sigma(x_0))]}{[\sigma(x_0)f'(\sigma(x_0))]^2}\|x_k - x_*\|^2 + \frac{(1 + \sqrt{2})c\beta[f'(\sigma(x_0))] + 1}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x_*\|,
\]
for all \( k = 0, 1, \ldots \), which is the inequality (9). Now, combining last inequality with (23) we obtain
\[
\|x_{k+1} - x_*\| \leq \left[\frac{[f'(\sigma(x_0)) + 1 + \kappa][f'(\sigma(x_0))\sigma(x_0) - f(\sigma(x_0))]}{\sigma(x_0)f'(\sigma(x_0))^2}\right] + \frac{c\beta[f'(\sigma(x_0))] + 1}{\sigma(x_0)f'(\sigma(x_0))^2}\|x_k - x_*\|,
\]
for all \( k = 0, 1, \ldots \). Applying Proposition 9 with \( t = \sigma(x_0) \) it is straightforward to conclude from the latter inequality that \( \{\|x_k - x_*\|\} \) converges to zero. So, \( \{x_k\} \) converges to \( x_* \). \( \square \)

4 Special cases

In this section, we present two special cases of Theorem 5. The convergence theorem for proximal Gauss-Newton method under Lipschitz condition and Smale’s theorem on proximal Gauss-Newton for analytic functions are included.
4.1 Convergence result for Lipschitz condition

In this section we show a correspondent theorem for Theorem 5 under Lipschitz condition, instead of the general assumption (7).

Theorem 14. Let $\Omega \subseteq \mathbb{X}$ be an open set, $J : \Omega \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous functional and $F : \Omega \to \mathbb{Y}$ a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_0 \in \Omega$, $R > 0$ and

$$c := \|F(x_0)\|, \quad \beta := \|F'(x_0)\|, \quad \kappa := \beta \|F'(x_0)\|, \quad \delta := \sup\{t \in [0, R] : B(x_0, t) \subset \Omega\}.$$ 

Suppose that $-F'(x_0)^*F(x_0) \in \partial J(x_0)$, $F'(x_0)$ is injective and there exists a $L > 0$ such that

$$h := [(1 + \sqrt{2})\kappa + 1]c \beta L < 1,$$

$$\beta \|F'(x) - F'(x_0 + \tau(x - x_0))\| \leq L(1 - \tau)\sigma(x), \quad (24)$$

where $x \in B(x_0, \delta)$, $\tau \in [0, 1]$ and $\sigma(x) = \|x - x_0\|$. Let

$$r := \min \left\{ \frac{4 + \kappa + 2c(1 + \sqrt{2})\beta L - \sqrt{(4 + \kappa + 2c(1 + \sqrt{2})\beta L)^2 - 8(1 - h)}}{2L}, \delta \right\}.$$ 

Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x_0, r) \setminus \{x_0\}$

$$x_{k+1} = \text{prox}_J^{H(x_k)}(x_k - F'(x_k)^*F(x_k)),$$ 

$k = 0, 1, \ldots,$

is well defined, the generated sequence $\{x_k\}$ is contained in $B(x_0, r)$, converges to $x_*$ and

$$\|x_{k+1} - x_*\| \leq \frac{\kappa L + 2c(1 + \sqrt{2})\beta L^2 + L^2\sigma(x_0)}{2[1 - L\sigma(x_0)]^2}\|x_k - x_*\|^2 + \frac{[(1 + \sqrt{2})\kappa + 1]c \beta L}{[1 - L\sigma(x_0)]^2}\|x_k - x_*\|, \quad (25)$$

for all $k = 0, 1, \ldots$.

Proof. It is immediate to prove that $F$, $x_0$ and $f : [0, \delta) \to \mathbb{R}$ defined by $f(t) = Lt^2/2 - t$, satisfy the inequality (7), conditions h1 and h2. Since $[(1 + \sqrt{2})\kappa + 1]c \beta L < 1$, the condition h3 also holds. In this case, it is easy to see that the constants $\nu$ and $\rho$ as defined in Theorem 5 satisfy

$$0 < \rho = \frac{4 + \kappa + 2c(1 + \sqrt{2})\beta L - \sqrt{(4 + \kappa + 2c(1 + \sqrt{2})\beta L)^2 - 8(1 - h)}}{2L} \leq \nu = 1/L,$$

as a consequence, $0 < r = \min\{\delta, \rho\}$. Therefore, as $F$, $J$, $r$, $f$ and $x_0$ satisfy all of the hypotheses of Theorem 5 taking $x_0 \in B(x_0, r) \setminus \{x_0\}$ the statements of the theorem follow from Theorem 5.  □
Remark 7. If the second inequality in (24) holds only for $\tau = 0$, then an analogous theorem holds true. More specifically, if we replace the definition of $r$ in above theorem by

$$r := \min \left\{ \frac{-4 + 3\kappa + 2c(1 + \sqrt{2})\beta L - \sqrt{4 + 3\kappa + 2c(1 + \sqrt{2})\beta L^2 + 8(1 - h)}}{2L}, \delta \right\},$$

then all the statements of the previous theorem are valid with exception of inequality (25), which in this case becomes

$$\|x_{k+1} - x_*\| \leq \frac{3\kappa L + 2c(1 + \sqrt{2})\beta L^2 + 3L^2\sigma(x_0)}{2[1 - L\sigma(x_0)]^2}\|x_k - x_*\|^2 + \frac{[(1 + \sqrt{2})\kappa + 1]c\beta L}{[1 - L\sigma(x_0)]^2}\|x_k - x_*\|,$$

for all $k = 0, 1, \ldots$.

For the zero-residual problems, i.e., $c = 0$, the Theorem 14 becomes:

Corollary 15. Let $\Omega \subseteq X$ be an open set, $J : \Omega \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous functional and $F : \Omega \to \mathbb{Y}$ a continuously differentiable function such that $F'$ has a closed image in $\Omega$. Let $x_* \in \Omega$, $R > 0$ and

$$\beta := \|F'(x_*)\|^1, \quad \kappa := \beta \|F'(x_*)\| \quad \delta := \sup \{t \in [0, R) : B(x_*, t) \subset \Omega\}.$$

Suppose that $F(x_*) = 0$, $0 \in \partial J(x_*)$, $F'(x_*)$ is injective and there exists a $L > 0$ such that

$$\beta \|F'(x) - F'(x_* + \tau(x - x_*))\| \leq L(1 - \tau)\sigma(x),$$

where $x \in B(x_*, \delta)$, $\tau \in [0, 1]$ and $\sigma(x) = \|x - x_*\|$. Let

$$r := \min \left\{ \frac{4 + \kappa - \sqrt{(4 + \kappa)^2 - 8}}{2L}, \delta \right\}.$$

Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x_*, r) \setminus \{x_*\}$

$$x_{k+1} = \text{prox}_J^{H(x_k)}(x_k - F'(x_k)\dagger F(x_k)), \quad k = 0, 1, \ldots,$$

is well defined, the generated sequence $\{x_k\}$ is contained in $B(x_*, r)$, converges to $x_*$ and

$$\|x_{k+1} - x_*\| \leq \frac{\kappa L + L^2\sigma(x_0)}{2[1 - L\sigma(x_0)]^2}\|x_k - x_*\|^2,$$

for all $k = 0, 1, \ldots$. 

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4.2 Convergence result under Smale’s condition

In this section we present a correspondent theorem to Theorem 5 under Smale’s condition. For more details, see [2, 3, 17]. First we will prove two auxiliary Lemmas.

Lemma 16. Let \( \Omega \subseteq X \) be an open set, \( F : \Omega \to Y \) an analytic function and
\[
\gamma := \sup_{n>1} \beta \left| \frac{F^{(n)}(x_*)}{n!} \right|^{1/(n-1)} < +\infty,
\]
where \( \beta := \| F'(x_*) \| \). Suppose that \( x_* \in \Omega \) and \( B(x_*, 1/\gamma) \subset \Omega \). Then, for all \( x \in B(x_*, 1/\gamma) \) there holds
\[
\beta \| F''(x) \| \leq (2\gamma)/(1 - \gamma \| x - x_* \|)^3.
\]

Proof. The proof follows the same pattern as the proof of Lemma 21 of [6]. \( \square \)

The next result provides a condition which is easier to check than (7), for two-times continuously differentiable functions.

Lemma 17. Let \( \Omega \subseteq X \) be an open set, \( x_* \in \Omega \) and \( F : \Omega \to Y \) be twice continuously differentiable on \( \Omega \). If there exists a twice continuously differentiable function \( f : [0, R) \to \mathbb{R} \) such that
\[
\beta \| F''(x) \| \leq f''(\| x - x_* \|),
\]
for all \( x \in B(x_*, R) \cap \Omega \), then \( F \) and \( f \) satisfy (7).

Proof. The proof follows the same pattern as the proof of Lemma 22 of [6]. \( \square \)

Theorem 18. Let \( \Omega \subseteq X \) be an open set, \( J : \Omega \to \mathbb{R} \cup \{ +\infty \} \) a proper, convex and lower semicontinuous functional and \( F : \Omega \to Y \) an analytic function such that \( F' \) has a closed image in \( \Omega \). Let \( x_* \in \Omega \), \( R > 0 \) and
\[
c := \| F(x_*) \|, \quad \beta := \| F'(x_*) \|, \quad \kappa := \beta \| F'(x_*) \| \quad \delta := \sup \{ t \in [0, R) : B(x_*, t) \subset \Omega \}.
\]
Suppose that \(-F'(x_*)^* F(x_*) \in \partial J(x_*)\), \( F'(x_*) \) is injective and
\[
h = 2c \gamma \beta[(1 + \sqrt{2})\kappa + 1] < 1,
\]
recall that \( \gamma := \sup_{n>1} \beta \left| \frac{F^{(n)}(x_*)}{n!} \right|^{1/(n-1)} < +\infty \). Let the constants \( a = \gamma c \beta, b = (1 + \sqrt{2})\gamma c \beta, \)
\[
\bar{\rho} := \inf \left\{ s \in (\sqrt{2}/2, 1) : p(s) := -4s^4 + (1 - \kappa + a + b(\kappa - 1))s^3 + (3 + \kappa + a + b(\kappa - 1))s^2 + (b - 1)s + b < 0 \right\},
\]
(28)
Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x_*, r) \setminus \{x_*\}$

$$
x_{k+1} = \operatorname{prox}_j^{H(x_k)}(x_k - F'(x_k)^t F(x_k)), \quad k = 0, 1, \ldots,
$$
is well defined, the generated sequence $\{x_k\}$ is contained in $B(x_*, r)$, converges to $x_*$ and

$$
\|x_{k+1} - x_*\| \leq \frac{1 + (\kappa - 1)(1 - \gamma\sigma(x_0))^2}{1 - 2(1 - \gamma\sigma(x_0))^2} \|x_k - x_*\|^2 + \frac{(1 + \sqrt{2})\beta c\gamma^2(2 - \gamma\sigma(x_0))^2}{[1 - 2(1 - \gamma\sigma(x_0))^2]^2} \|x_k - x_*\|^2 + \frac{c\beta[(1 + \sqrt{2})\kappa + 1](2 - \gamma\sigma(x_0))(1 - \gamma\sigma(x_0))^2}{[1 - 2(1 - \gamma\sigma(x_0))^2]^2} \|x_k - x_*\|,
$$
for all $k = 0, 1, \ldots$.

**Proof.** Consider the real function $f : [0, 1/\gamma) \to \mathbb{R}$ defined by

$$
f(t) = \frac{t}{1 - \gamma t} - 2t.
$$
It is straightforward to show that $f$ is analytic and that

$$
f(0) = 0, \quad f'(t) = 1/(1 - \gamma t)^2 - 2, \quad f'(0) = -1, \quad f''(t) = (2\gamma)/(1 - \gamma t)^3, \quad f^n(0) = n! \gamma^{n-1},
$$
for $n \geq 2$. It follows from the last equalities that $f$ satisfies h1 and h2. Since $h = 2\gamma c\beta[(1 + \sqrt{2})\kappa + 1] < 1$, the condition h3 also holds. Now, as $f''(t) = (2\gamma)/(1 - \gamma t)^3$ combining Lemmas 16 and 17 we conclude that $F$ and $f$ satisfy (7) with $R = 1/\gamma$. In this case,

$$
\nu = (2 - \sqrt{2})/2\gamma < 1/\gamma.
$$
Now, we will obtain the constant $\bar{\rho}$ as defined in Theorem 5. For simplicity, consider the following change of variable

$$
s = 1 - \gamma t.
$$

Then, $t = (1 - s)/\gamma$. Moreover, if $t$ satisfies $0 < t < \nu = (2 - \sqrt{2})/2\gamma$, then $\sqrt{2}/2 < s < 1$. Hence, determine the constant $\rho$ as defined in Theorem 5 is equivalent to determine the constant $s$ such that

$$
\bar{\rho} = \inf \left\{ s \in (\sqrt{2}/2, 1) : p(s) = -4s^4 + (1 - \kappa + a + b(\kappa - 1))s^3 + (3 + \kappa + a + b(\kappa - 1))s^2 + (b - 1)s + b < 0 \right\},
$$
where $a = \gamma c\beta$ and $b = (1 + \sqrt{2})\gamma c\beta$. Thus, taking in account the change of variable, we have $\rho = (1 - \bar{\rho})/\gamma$ and

$$
r = \min \left\{ (1 - \bar{\rho})/\gamma, \delta \right\}.
$$
Therefore, as $F$, $J$, $r$, $f$ and $x_*$ satisfy all hypothesis of Theorem 5, taking $x_0 \in B(x_*, r) \setminus \{x_*\}$, the statements of the theorem follow from Theorem 5.

\[\square\]
Remark 8. Fixed the numerical values of $a$, $b$ and $\kappa$, as $p(1) = h - 1 < 0$, it is easy to compute $\bar{\rho}$, defined in [28]. Moreover, as $f(t) = t/(1 - \gamma t) - 2t$ is the majorant function, by Proposition 8, it follows that $p$ is decreasing in $(\sqrt{2}/2, 1)$.

For the zero-residual problems, i.e., $c = 0$, the Theorem 18 becomes:

Corollary 19. Let $\Omega \subseteq \mathbb{X}$ be an open set, $J : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous functional and $F : \Omega \rightarrow \mathbb{Y}$ an analytic function such that $F'$ has a closed image in $\Omega$. Let $x^* \in \Omega$, $R > 0$ and

$$
\beta := \|F'(x^*)\|, \quad \kappa := \beta \|F'(x^*)\|, \quad \delta := \sup \{t \in [0, R) : B(x^*, t) \subset \Omega\}.
$$

Suppose that $F(x^*) = 0$, $0 \in \partial J(x^*)$, $F'(x^*)$ is injective and

$$
\gamma := \sup_{n>1} \beta \left\| \frac{F^{(n)}(x^*)}{n!} \right\|^{1/(n-1)} < +\infty.
$$

Let be given the positive constants

$$
\bar{\rho} := \inf \left\{ s \in (\sqrt{2}/2, 1) : p(s) := -4s^3 + (1 - \kappa)s^2 + (3 + \kappa)s - 1 < 0 \right\}, \quad r := \min \{(1 - \bar{\rho})/\gamma, \delta\}.
$$

Then, the proximal Gauss-Newton method for solving (1), with starting point $x_0 \in B(x^*, r)/\{x^*\}$

$$
x_{k+1} = \text{prox}_{J}^H(x_k)\left(x_k - F'(x_k)^\dagger F(x_k)\right), \quad k = 0, 1, \ldots,
$$

is well defined, the generated sequence $\{x_k\}$ is contained in $B(x^*, r)$, converges to $x^*$ and

$$
\|x_{k+1} - x^*\| \leq \frac{1 + (\kappa - 1)(1 - \gamma \sigma(x_0))^2}{[1 - 2(1 - \gamma \sigma(x_0))^2]^2} \|x_k - x^*\|^2, \quad k = 0, 1, \ldots.
$$

5 Final remark

Under a majorant condition, we present a new local convergence analysis of the proximal Gauss-Newton method for solving penalized nonlinear least squares problem. It would also be interesting to present a semi-local convergence analysis of the proximal Gauss-Newton method, under a majorant condition, for the problem on consideration. This local analysis will be performed in the future.

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