PRICING VULNERABLE OPTIONS UNDER A MARKOV-MODULATED JUMP-DIFFUSION MODEL WITH FIRE SALES

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Abstract. In this paper, we consider the valuation of vulnerable options under a Markov-modulated jump-diffusion model, where the option writer’s asset value is subject to price pressure from other financial institutions due to distressed selling. A change of numéraire technique, proposed by Geman et al. [14], is employed to obtain a semi-analytical pricing formula for a vulnerable European option in the presence of regime switching effect. The method is numerically implemented using the multinomial approach in Costabile et al. [6]. We study the impacts of distressed selling and regime switching on the European option prices via numerical experiments.

1. Introduction. Financial derivatives subject to counterparty risk, such as the Over-The-Counter (OTC) options, have experienced tremendous growth since the Global Financial Crisis of 2007-2008, caused mainly by subprime mortgages in the United States. Since these derivatives are traded in the OTC markets and there are no organized exchanges to guarantee the promised payments in these markets, the option holder is vulnerable to the counterparty risk due to a possible default of the option writer. Besides, as pointed out by Cont and Wagalath (2014) [5], the presence of distressed selling for large losses, triggered endogenously by market shocks, in a reference financial institution may lead to further endogenous volatility and correlation in asset prices, and hence influence the asset values of any other financial institutions with overlapping holdings. An example was the bankruptcy of Lehman Brothers on 15th, September 2008. The collapse of Lehman Brothers generated liquidation and deleveraging in many asset classes all over the world, resulting in major equity indices all losing around 10% on that day, see, for example Wiggins et al. (2014) [39], and Cont and Wagalath (2014) [5].

For modeling counterparty risk, two classes of models exist: structural and reduced form models. Structural models originated with Black and Scholes (1973) [2],
Merton (1974) [25] and reduced-form models originated with Jarrow and Turnbull (1992) [19], and subsequently studied by Jarrow and Turnbull (1995) [20], Duffie and Singleton (1999) [8] among others. The pricing formula of a vulnerable European option was first developed by Johnson and Stulz (1987) [21] under the condition that the option is the only liability of a counterparty. Consequently, many approaches have been proposed to study the valuation of vulnerable options. Under the structural approach for firm’s default, Klein (1996) [22] extended Johnson and Stulz’s (1985) [21] result by allowing the counterparty to have other liabilities in the capital structure and obtained a formula taking account of the correlation between the option writer’s assets and the asset underlying the option. Following Klein (1996) [22], Hung and Liu (2005) [18] studied the pricing of vulnerable options in an incomplete market. Later Yang et al. (2014) [40] studied the pricing of vulnerable European options under a fast mean-reverting stochastic volatility model. Fard (2015) [12] proposed a completely random generalized jump-diffusion asset model for pricing European vulnerable options. Using the Esscher transform and the Esscher-Girsanov transform, a closed-form pricing formula for the vulnerable option is obtained. Cheang and Teh (2014) [3] adopted Geman et al.’s (1995) [14] change of numéraire technique, and derived a European call option pricing formula where the stock price dynamics follow the Merton (1974) [25] jump-diffusion model, under a stochastic interest rate that was described by an Heath-Jarrow-Morton (HJM) type model with jumps. Elliott and Siu (2007) [9] discussed option pricing using the Esscher transform, related models were considered in, for example, Siu et al. (2008) [36], Shen and Siu (2013) [32], under a Markovian regime-switching jump-diffusion model. Elliott and Siu (2013) [10] discussed the pricing of derivatives in a continuous-time hidden Markov-modulated pure-jump asset model. They employed a version of the Esscher transform to select a pricing kernel for valuation and derived a valuation formula for an European option by using a Fourier transform. Some other works which concern the pricing of options under hidden Markov modulated models with jumps are, for example, Siu (2013, 2015) [37, 38], Elliott and Siu (2015) [11], Cont and Tankov (2004) [4] and Pascucci (2011) [28] amongst others. Niu and Wang (2016) [27] presented an improved method of pricing vulnerable options by allowing both the underlying stock price and the counterparty’s asset value to follow Markov-modulated jump-diffusion processes. They developed a parsimonious equivalent martingale representation for the two joint processes with correlated jump risks. A closed-form pricing formula for an vulnerable European option was obtained in the presence of regime switching effect using two-dimensional Laplace transforms, which can be numerically implemented through some Laplace transform inversion methods.

All the above studies assume the OTC option writer is an independent individual, not subject to price pressure triggered by other financial institutions’ fire sales. However, as Shleifer and Vishny (1992, 2011) [33, 34] have shown, in the presence of distressed selling, losses by financial institutions with overlapping holdings became self-reinforcing, leading to downward spirals for asset prices and, ultimately,  

1The term comes from the sale of inventory at a sharp discount rate due to fire damage. A fire sale is said to occur if the sales of goods occur at a heavy discount rate, for example, the seller is facing distressed selling for a big loss or even bankruptcy. The sharp drop of price can cause even some further serious consequences as people may perceive it as a negative signal of the company [33, 34].
to a default risk. The empirical link between fire sales and an increase in correlated default risks has been documented in several recent studies. For example, Coval and Stafford (2007) [7] examined institutional price pressure in equity markets by studying mutual fund transactions caused by capital flows. They showed that assets experiencing large outflows tended to decrease existing positions, which created price pressure in the securities held in common by distressed assets. Anton and Polk (2014) [1] connected stocks through their common active mutual fund owners. Greenwood and Thesmar (2011) [15] proposed a framework for modeling price dynamics by incorporating the ownership structure of financial assets, which is assumed to be given. Cont and Wagalath (2014) [5] proposed a tractable framework for modeling and estimating the impact of fire sales in multiple institutions on systemic risk in a multi-asset setting.

In this paper, we consider the valuation problems of vulnerable options in a more general setting. Specifically, the reference financial institutions’ asset values, the underlying stock prices, the OTC option writer’s asset value and the interest rates (hence bond prices) are assumed to follow a generalized two-point distribution, corresponding to upward and downward jumps, respectively. The jump-diffusion process is modulated by an $m$-state continuous-time Markov chain. Here the model parameters are allowed to change over time according to the transition state of the chain. All processes have different sources of Brownian motion shocks but share a common source of Poisson jump shocks in our model. Moreover, besides the basic model, we describe quantitatively the effect of the reference financial institutions’ price pressure on the OTC option writer’s asset value by distressed selling. By adopting the change of numéraire techniques [14], we are able to provide a semi-analytical pricing formula for an vulnerable option in the presence of regime switching, which can be numerically implemented through the multinomial approach proposed in Costabile et al. [6]. The impacts of distressed selling and regime switching on the option prices are then studied numerically.

The rest of the paper is organized as follows. Section 2 presents a modeling framework for the price impact of fire sales on the OTC option writer’s asset value. Section 3 formulates an underlying model for stochastic interest rates (hence bond prices). The issuance prices of European vulnerable options are discussed in Section 4. In Section 5, we present the change of numéraire technique proposed in Geman et al. [14]. The forward measure and the reciprocal forward measure are discussed in this section. In section 6, we derive an analytical formula for the price of a European vulnerable option. Simulation results are presented in Section 7. Finally, Section 8 concludes the paper.

2. The framework. As pointed out by Cont and Wagalath (2014) [5], the presence of distressed selling for large losses, triggered endogenously by capital requirements set by regulators or target leverage ratios set by asset managers, in a reference financial institution (RFI) may lead to endogenous volatility and correlation in asset prices. This may increase default risks faced by other financial institutions with overlapping holdings. We do not attempt to model the underlying mechanism at a micro-level but focus on its aggregate effect on the valuation of European vulnerable options issued by these (affected) financial institutions. This aggregating effect might be modeled briefly by introducing a deleveraging schedule, represented by a concave, increasing function $f$ which measures the systematic supply/demand
generated by the RFI as a function of the return of the fund\(^2\): When, due to market shocks at time \(t\), the fund value moves from \(X(t^-)\) to \(X(t)\), \(f(X(t^-)/X(0)) - f(X(t)/X(0))\) portion of the fund will be liquidated at that time. Here we consider a quantitative model to describe the endogenous risk incurred by the RFI’s fire sales. In the subsequent sections, this model will be adopted to discuss the valuation of a European vulnerable option issued by a small institution.

**Deleveraging Schedule \(f\)**

![Deleveraging Schedule Diagram]

We assume that all the stochastic processes in our model are defined on a given complete probability space \(\Omega, F, \{F_t\}_{t \geq 0}, P\). We describe the flow of information by a filtration \(\{F_t\}_{t \leq T}\) over \(0 \leq t \leq T\) for some fixed \(T\), the expiry time of the option. Furthermore, we assume that the underlying economic state evolves over time according to a finite-state continuous-time Markov chain \(\{\chi(t)\}_{t \geq 0}\) with state space \(\mathcal{M} = \{1, 2, \ldots, m\}\). We write \(Q = (q_{ij}(t))_{m \times m}\) for the generator of \(\{\chi(t)\}_{t \geq 0}\) under \(P\), with the entries satisfying the standard conditions for rate matrices:

1. \(q_{ij}(t) \geq 0\) if \(i \neq j\);
2. \(q_i(t) := -q_{ii}(t) = \sum_{j \neq i} q_{ij}(t)\) for \(i = 1, 2, \ldots, m\).

Suppose the RFI’s fund value evolves over time according to the following jump-diffusion model (JDM):

\[
\frac{dX(t)}{X(t^-)} = \mu^X(\chi(t))dt + \alpha^X(\chi(t)) \cdot dW(t) + \left[Y^X_{N(t)}(\chi(t^-)) - 1\right] dN(t) \tag{1}
\]

where \(t^-\) denotes the time just prior to time \(t\),

\(\{W(t) = (W_1(t), \ldots, W_n(t))\}_{t \geq 0}\)

is a standard \(n\)-dimensional Brownian motion under \(P\),

\(\alpha^X(\chi(t)) = (\alpha_1^X(\chi(t)), \ldots, \alpha_n^X(\chi(t)))\)

is the volatility vector, the diffusion part:

\[
\mu^X(\chi(t))dt + \alpha^X(\chi(t)) \cdot dW(t)
\]

\(^2\)The argument can be easily extended to the case when the deleveraging schedule, \(f\), is related to the economic state, i.e., \(f^X(t)\).
Suppose the price of the optioned stock follows $Y(t)$, which ensures a non-negative fund value. If $X(t)$ describes normal fluctuations in $X(t)$, we further assume that $Y(t)$ accounts for sudden extraordinary market shocks with large price impacts. Here $\{N(t)\}_{t \geq 0}$ is a Markov-modulated Poisson process with intensity process $\{\lambda(t, \chi(t))\}_{t \geq 0}$, adopted here to model the arrivals of the sudden extraordinary market shocks. If a market shock emerges at time $t$, i.e., $dN(t) = 1$, then according to Eq. (1), we have

$$X(t)/X(-) = Y^X_{N(t)}(\chi(-)).$$

We further assume that $Y^X_{N(t)}(\chi(-))$ is square integral, and that $Y^X_{N(t)}(\chi(-)) > 0$, which ensures a non-negative fund value. If $Y^X_{N(t)}(\chi(-)) > 1$, the arrival of the market shock will incur an upward jump in $X$, otherwise, a downward jump will be incurred.

For simplicity, we make the following assumptions:

(i): The RFI, the small institution, the optioned stock, and the associated zero-coupon bond (we will introduce it later) share a common source of market shocks.

(ii): When a market shock emerges at certain time, all the values as mentioned above vary.

Consider a small institution who is in the position to write options on a stock $S$. Suppose the price of the optioned stock follows

$$\frac{dS(t)}{S(t-)} = \mu^S(\chi(t))dt + \alpha^S(\chi(t)) \cdot dW(t) + \left[Y^S_{N(t)}(\chi(t)) - 1\right] dN(t). \tag{2}$$

Here $\alpha^S(\chi(t)) = (\alpha^S_1(\chi(t)), \ldots, \alpha^S_n(\chi(t)))$ is the volatility vector of the optioned stock price, and $Y^S_{N(t)}(\chi(t)) > 0$, square integral, is the jump triggered by a market shock at time $t$. Finally, we assume that, without the RFI’s price impact, the small institution’s fund value $V(t)$ follows the following jump-diffusion process:

$$\frac{dV(t)}{V(t-)} = \mu^V(\chi(t))dt + \alpha^V(\chi(t)) \cdot dW(t) + \left[Y^V_{N(t)}(\chi(t)) - 1\right] dN(t). \tag{3}$$

Here $\alpha^V(\chi(t)) = (\alpha^V_1(\chi(t)), \ldots, \alpha^V_n(\chi(t)))$ represents the volatility vector of the small institution’s fund value, and $Y^V_{N(t)}(\chi(t)) > 0$ (square integral) represents the involved jump in $V$ at time $t$. The instantaneous correlation between relative changes in $X(t)$ and $V(t)$ is

$$\frac{dX(t)}{X(t-)} \cdot \frac{dV(t)}{V(t-)} = \alpha^X(\chi(t)) \cdot \alpha^V(\chi(t)) dt$$

$$+ \left[Y^X_{N(t)}(\chi(t-)) - 1\right] \cdot \left[Y^V_{N(t)}(\chi(t-)) - 1\right] dN(t).$$

Assumption (ii) implies that the emergence of market shocks strengthens the relationship between $X$ and $V$, since the jump component is nonnegatively valued.
Let \( \{ \tau_1, \tau_2, \cdots \} \) denote the sequence of arrival times of the market shocks. The corresponding jump amplitudes in \( X, S, V \) and the later defined \( B \) are

\[
\begin{align*}
\{ Y^X_1(\chi(\tau_1 -)), & \ Y^X_2(\chi(\tau_2 -)), \cdots \}, \\
\{ Y^S_1(\chi(\tau_1 -)), & \ Y^S_2(\chi(\tau_2 -)), \cdots \}, \\
\{ Y^V_1(\chi(\tau_1 -)), & \ Y^V_2(\chi(\tau_2 -)), \cdots \}, \\
\{ Y^B_1(\chi(\tau_1 -)), & \ Y^B_2(\chi(\tau_2 -)), \cdots \},
\end{align*}
\tag{4}
\]

respectively. To simplify the discussion, we assume that they are four sequences of independent random variables (r.v.s). Furthermore, all three sources of uncertainty: (i) the standard Brownian motion \( \{ W(t) \} \); (ii) the Markov-modulated Poisson process; and (iii) the jump amplitudes in Eq. (4) are supposed to be independent.

We now analyze the potential impact of fire sales\(^3\) on the small institution’s fund value. When analyzing the potential impact of a set a macroeconomic and financial shocks, one should take into account systemic interactions and feedback effects. The financial crisis has demonstrated that these effects have the capacity to transform isolated stress events into global crisis threatening even large, well capitalized institutions, as well as systemic stability.

To model the impact, as suggested in Cont and Wagalath (2014) \(^5\), we introduce a \textit{market impact} function \( \phi(\cdot) \). We assume that \( \phi : \mathcal{R} \rightarrow \mathcal{R} \) is increasing and \( \phi(0) = 0 \). In any small time interval from \( t \) to \( t + \Delta t \), the impact of the RFI's behavior on the return of the small institution is then given by

\[
\phi \left( \rho_{VX}(\chi(t)) \left[ f \left( \frac{X(t + \Delta t)}{X(0)} \right) - f \left( \frac{X(t)}{X(0)} \right) \right] \right),
\tag{5}
\]

where, for any \( j \in M \),

\[
\rho_{VX}(j) = \frac{\alpha^X(j) \cdot \alpha^V(j)}{\sqrt{\| \alpha^X(j) \|^2 \sqrt{\| \alpha^V(j) \|^2}}}
\]

is the correlation between the small institution and the RFI. To study the market impact within an infinitesimal time interval, we assume that \( \phi(\cdot) \in C^3(\mathcal{R}) \). Let

\[
h(u; t) := \phi \left( \rho_{VX}(\chi(t)) \left[ f \left( \frac{X(t + u)}{X(0)} \right) - f \left( \frac{X(t)}{X(0)} \right) \right] \right).
\tag{6}
\]

Then, \( \lambda'(t) := \lim_{u \to 0} h(u; t)/u \) denotes the \textit{influence rate} at which the small institution’s asset value is affected by the RFI’s distressed behaviors.

**Theorem 2.1.** Under the above assumptions, the influence rate, given the initial state \( X(t) = x, \chi(t) = j \), is (denote \( \dot{x} = x/X(0) \))

\[
\lambda'(t) = I^{(1)}(x, j) + \sum_{i=1}^{n} I^{(2,i)}(x, j)e_i(t) + I^{(3)}(x, j)e(t),
\tag{7}
\]

where \( \{ e(t) = (e_1(t), \cdots, e_n(t)) \}_{t \geq 0} \) is a Gaussian white noise process, a generalized derivative of \( \{ W(t) \}_{t \geq 0} \) (see, for example Hida et al. (1988) \(^{17}\)), \( \{ e(t) \}_{t \geq 0} \) is a

\(^3\)We only consider the RFI’s fire sales here.
process defined by \( \frac{N(t+dt) - N(t)}{dt} \), and

\[
I^{(1)}(x, j) = \phi'(0) \rho_{\nu X}(j) \left[ f'(\hat{x}) \hat{x} \mu^X(j) + \frac{1}{2} f''(\hat{x}) \hat{x}^2 ||\alpha^X(j)||^2 \right] + \frac{1}{2} \phi''(0) \rho_{\nu X}(j) \left| f'(\hat{x}) \hat{x} \right|^2 \left| \alpha^X(j) \right| \|
\]

\[
I^{(2, i)}(x, j) = \phi'(0) \rho_{\nu X}(j) f'(\hat{x}) \hat{x} \alpha^X_i(j) \]

\[
I^{(3)}(x, j) = \phi \left( \rho_{\nu X}(j) \left[ f(\hat{x} \cdot Y^X_{N(t)}(j)) - f(\hat{x}) \right] \right).
\]

Proof. The proof is obtained by a direct application of \( \text{Itô}'s \) formula to the function \( h(u; t) \). It is included in Appendix A for the sake of completeness.

Therefore, in the presence of market impact, the small institution’s fund value can be described by the following stochastic process:

\[
\frac{dV(t)}{V(t-)} = [\mu^V(\chi(t)) + \chi^V(t)]dt + \alpha^V(\chi(t)) \cdot dW(t) + \left[ Y^V_{N(t)}(\chi(t)) - 1 \right] dN(t).
\]

Combining the results in Theorem 1 and Eq. (9), we find that the small institution’s fund value, \( V(t) \), depends on the RFI’s distressed behaviors only through the price impact function \( \phi(x) \) as well as its first and second derivatives evaluated at the point 0, say \( \phi'(0) \) and \( \phi''(0) \). The effect on the volatility (continuous component), one of the model parameters which will eventually enter the price formula of the vulnerable option, depends only on the slope \( \phi'(0) \) of the price impact function at the point 0. Following Cont and Wagala (2014) [5], we adopt the linear function

\[
\phi(x) = \frac{x}{L}
\]

(10)
to describe the price impact of the RFI’s distressed behaviors, where \( L > 0 \) represents market capacity and can be interpreted as the portion of fund the RFI has to liquidate in order to decrease the asset value of the small institution by 1%. To derive analytical solutions for several option-pricing problems, the deleveraging schedule \( f \) is assumed to take the following form:

\[
f(x) = \frac{\log(x)}{\eta}, \quad x > 0 \quad \text{and} \quad \eta > 0,
\]

where \( \eta \) represents the maximal leverage ratio (MLR). The greater the ratio, the slower the deleveraging rate. These assumptions, however, may be dropped.

Corollary 1. (Linear Market Impact) Assume that

\[
f(x) = \frac{\log(x)}{\eta} \quad \text{and} \quad \phi(x) = \frac{x}{L}
\]

where \( \eta \) and \( L \) are both positive. The small institution’s fund value, under the effect of the RFI’s fire sales, evolves over time according to

\[
\frac{dV(t)}{V(t-)} = \left\{ \mu^V(\chi(t)) + \rho_{\nu X}(\chi(t)) \left[ \frac{\mu^X(\chi(t))}{L\eta} - \frac{1}{2} ||\alpha^X(\chi(t))||^2 \right] \right\} dt
\]

\[
+ \left[ \alpha^V(\chi(t)) + \rho_{\nu X}(\chi(t)) \frac{\alpha^X(\chi(t))}{L\eta} \right] \cdot dW(t)
\]

\[
+ \left[ Y^V_{N(t)}(\chi(t-)) - 1 + \frac{\rho_{\nu X}(\chi(t-))}{L\eta} \log \left( Y^X_{N(t)}(\chi(t-)) \right) \right] dN(t).
\]

Some adjustments have been made w.r.t the jump component according to the arguments in proof of Theorem 1.
According to Assumption (ii), the RFI’s fund value strengthened by the RFI’s distressed behaviors. Note that the relationship between $X$ and $V$ is established through their relationships with the fund value of the small institution only through its model parameters. This property further simplifies the problem. In addition, under the effect of the RFI’s distressed behaviors, the instantaneous correlation between relative changes in $X$ and $V$ is

$$
\frac{dX(t)}{X(t-)} \cdot \frac{dV(t)}{V(t-)} = \left[ \alpha^X(\chi(t)) \cdot \alpha^V(\chi(t)) + \frac{\rho_{VX}(\chi(t))}{\eta_t} \cdot ||\alpha^X(\chi(t))||^2 \right] dt
+ \left[ Y_{N(t)}^X(\chi(t-)) - 1 \right] \left[ Y_{N(t)}^V(\chi(t-)) - 1 + \frac{\rho_{VX}(\chi(t-))}{\eta_t} \cdot \log \left( \frac{Y_{N(t)}^X(\chi(t-))}{1} \right) \right] dN(t).
$$

According to Assumption (ii),

$$
\begin{cases}
Y_{N(t)}^X(\chi(t-)) > 1 & \iff \ Y_{N(t)}^V(\chi(t-)) > 1 \quad \text{(upward jumps)} \\
Y_{N(t)}^X(\chi(t-)) \in (0, 1) & \iff \ Y_{N(t)}^V(\chi(t-)) \in (0, 1) \quad \text{(downward jumps)}.
\end{cases}
$$

Hence, we have

$$
\frac{dX(t)}{X(t-)} \cdot \frac{dV(t)}{V(t-)} \geq \left[ \alpha^X(\chi(t)) \cdot \alpha^V(\chi(t)) dt + \left[ Y_{N(t)}^X(\chi(t-)) - 1 \right] \left[ Y_{N(t)}^V(\chi(t-)) - 1 \right] dN(t). \right.
$$

It may be seen that the quantitative relationship between $X(t)$ and $V(t)$ is further strengthened by the RFI’s distressed behaviors. Note that the relationship between the RFI’s fund value $X(t)$ and the price of the underlying share $S(t)$ is established through their relationships with the fund value of the small institution $V(t)$ in the presence of the RFI’s fire sales effect.

3. **Stochastic interest rate with jumps.** Real markets do not have a single interest rate. Instead, there are bonds of different maturities, issued by different institutions, some paying coupons and others not. Rather than having a single interest rate, real markets have yield curves.

To discount the payoff of the option, we consider a zero-coupon treasury bond $5$ that pays 1 unit of currency at maturity $T$, the same time as the option. Define the discount process

$$
D(t) = e^{-\int_0^t R(s)ds}.
$$

---

5 We assume throughout the paper that the government bond has no default risk.
By Feynman-Kac formula and some standard risk-neutral pricing arguments, under a risk-neutral measure \( Q^\ast \), the value of this bond at time \( t \in [0, T] \) is given by

\[
B^{\chi(t)}(t, T, R(t)) = \frac{1}{D(t)} \mathbb{E}^{Q^\ast}[D(T) | \mathcal{F}_t].
\]

In particular, \( B^{\chi(t)}(T, T, R(T)) = 1 \).

Since the additional source of uncertainty introduced by the jump-diffusion regime-switching model makes the market incomplete, there in general, exist more than one risk-neutral measure. The discussion on how to select a feasible risk-neutral measure in an incomplete market is an interesting theoretical issue. We assume, here, that a risk-neutral measure \( Q^\ast \) has already been specified. Under \( Q^\ast \), all discounted portfolio processes adapted to the filtration \( \mathcal{F} \) are martingales.

To simplify the notation, we denote

\[
B(t, T) := B^{\chi(t)}(t, T, R(t)).
\]

Since a zero-coupon bond is an asset, the discounted bond price \( D(t) B(t, T) \) must be a martingale under \( Q^\ast \). We assume that the discounted bond price evolves over time according to the following dynamics:

\[
d\left( \frac{D(t) B(t, T)}{D(t-1) B(t-, T)} \right) = \alpha^{B}(\chi(t)) \cdot d\tilde{W}(t) + d\tilde{M}^{B}(t). \tag{13}
\]

Here

\[
\tilde{M}^{B}(t) = \int_0^t \left[ \tilde{Y}_{\tilde{N}(u)}^{B}(\chi(u-)) - 1 \right] d\tilde{N}(u) - \int_0^t \tilde{\lambda}(u, \chi(u-)) [\tilde{\pi}^{B}(\chi(u-)) - 1] du,
\]

is a compensated compound Poisson process\(^6\), \( \{\tilde{N}(t)\}_{t \geq 0} \) is exactly the Markov-modulated Poisson process adopted in the previous sections to model the arrivals of large market shocks. The additional tildes in Eq. (13) indicate that they are martingales or compensated processes under \( Q^\ast \). That is, under the risk-neutral measure \( Q^\ast \),

1. \( \{\tilde{W}(t)\}_{t \geq 0} \) is a standard \( n \)-dimensional Brownian motion;
2. \( \{\tilde{N}(t)\}_{t \geq 0} \) is a Markov-modulated Poisson process with intensity \( \{\tilde{\lambda}(t, \chi(t-))\}_{t \geq 0} \);
3. \( \tilde{Y}_{\tilde{N}(t)}^{B}(\chi(t-)) \) is the involved jump at time \( t \in [0, T] \) with average \( \tilde{\pi}^{B}(\chi(t-)) \).

Eq. (13) also implies that regime-switching risk is not priced, and hence the generator matrix \( Q \) is the same under both the real-world measure and the risk-neutral measure.

4. **The issuance price of an option on stock.** In our study, we impose some standard assumptions for capital markets imposed in the literature. For example, we assume that the market is frictionless and trading takes place continuously over time. There are no tax, no transaction cost, and no restriction on borrowing and short sales. Denote \( T \) as the option maturity, and \( K \) as the strike price. Following Klein (1996) [22], we use the Merton structural firm value model to explicitly describe the relationship between the asset value of a firm and the default of the firm. The default would occur if the firm’s asset value at maturity \( V(T) \) is less than an

\(^6\{\tilde{M}^{B}(t)\}_{t \geq 0} \) is a martingale under \( Q^\ast \).
exogenous default barrier \(d^*\), and a proportion of the nominal claim is paid out at the expiration date \(T\). The payoff of a vulnerable European call option is given by

\[
C(T) = (S(T) - K)^+ \left( 1_{\{V(T) \geq d^*\}} + 1_{\{V(T) < d^*\}} \frac{1 - a}{d} V(T) \right), \quad (14)
\]

where \(d\) is the value of total liabilities of the firm at the expiry date and \(a\) is the corresponding dead-weight cost related to the bankruptcy or reorganization process of the firm expressed as a percentage of the firm’s asset value.

Using standard risk-neutral pricing arguments, the price of the vulnerable European call option at time \(t\) is given by

\[
C(t, S(t), V(t), \chi(t)) = \frac{1}{D(t)} \mathbb{E}^{Q^*} \left[ D(T) C(T) \mid \mathcal{F}_t \right]. \quad (15)
\]

The computation of the right-hand side of this formula requires knowledge about the dependence between the discount factor \(D(T)\) and the payoff \(C(T)\) of the option. This, however, can be difficult to model. The change of numéraire result in Geman et al. [14] is frequently used to decompose the conditional expectation on the right-hand side of \((15)\) into four terms which can be interpreted as the conditional probabilities/expectations of the option being in-the-money at maturity (the definition of the probability measures \(\mathbb{P}^{Q^*.S}\) and \(\mathbb{P}^{Q^*.T}\) will be discussed later):

\[
C(t, S(t), V(t), \chi(t)) = \frac{1}{D(t)} \mathbb{E}^{Q^*} \left[ D(T) (S(T) - K)^+ \left( 1_{\{V(T) \geq d^*\}} + 1_{\{V(T) < d^*\}} \frac{1 - a}{d} V(T) \right) \mid \mathcal{F}_t \right] = S(t) \left\{ \mathbb{E}^{Q^*.S} \left[ (A \cap B) \mid \mathcal{F}_t \right] + \frac{1 - a}{d} \mathbb{E}^{Q^*.S} \left[ V(T) \mathbb{1}_{A \cap B^c} \mid \mathcal{F}_t \right] \right\} - KB(t, T) \left\{ \mathbb{E}^{Q^*.T} \left[ (A \cap B) \mid \mathcal{F}_t \right] + \frac{1 - a}{d} \mathbb{E}^{Q^*.T} \left[ V(T) \mathbb{1}_{A \cap B^c} \mid \mathcal{F}_t \right] \right\}, \quad (16)
\]

where \(A = \{S(T) < K\} \text{ and } B = \{V(T) \geq d^*\}.\) If we can find the evolution of the price of the underlying assets \(S\) and \(V\) over time under the probability measures \(\mathbb{P}^{Q^*.S}\) and \(\mathbb{P}^{Q^*.T},\) respectively. We can apply Eq. \((16)\), in which we only need to estimate \(S(T)\) and \(V(T),\) instead of using Eq. \((15)\), which requires us to estimate \(S(T), V(T), D(T)S(T)\) and \(D(T)V(T).\) We will further discuss the change of numéraire in the next section.

5. Change of Numéraire. A numéraire is the unit of account in which other assets are denominated. In principle, one can take any positively priced asset as a numéraire and denominate all other assets in terms of the chosen numéraire. Associated with each numéraire, we shall have an equivalent risk-neutral measure. In this section, we will discuss the change of numéraire under \(Q^*\) within an \(n\)-dimensional \((n \geq 4)\) market model.

We first notice that, under \(Q^*,\) the discounted processes \(\{D(t)S(t)\}_{t \geq 0}\) and \(\{D(t)V(t)\}_{t \geq 0}\) are martingales. Hence

\[
\frac{d(D(t)S(t))}{D(t-)} = \alpha^S(\chi(t)) \cdot d\tilde{W}(t) + d\tilde{M}^S(t) \quad \text{and} \quad \frac{d(D(t)V(t))}{D(t-)} = \frac{\alpha^V(\chi(t)) + \rho V X(\chi(t))}{L\bar{f}} \cdot d\tilde{W}(t) + d\tilde{M}^V(t), \quad (17)
\]
where

\[ \tilde{M}^S(t) = \int_0^t [\tilde{Y}^S_{N(u)}(\chi(u-)) - 1]d\tilde{N}(u) - \int_0^t \tilde{\lambda}(u, \chi(u-))[\tilde{\pi}^S(\chi(u-)) - 1]du \]

\[ \tilde{M}^V(t) = \int_0^t [\tilde{Y}^V_{N(u)}(\chi(u-)) - 1 + \tilde{Z}_{\tilde{N}}(\chi(u-))]d\tilde{N}(u) \]

\[ - \int_0^t \tilde{\lambda}(u, \chi(u-))[\tilde{\pi}^V(\chi(u-)) - 1 + \tilde{\kappa}^V(\chi(u-))]du \]

are two compensated compound Poisson processes, and

\[
\begin{align*}
\tilde{Z}_{\tilde{N}(t)}(\chi(t-)) &= \frac{\rho_{\chi}(\chi(t-))}{\tilde{L}_t} \log \left( \tilde{Y}^S_{\tilde{N}(t)}(\chi(t-)) \right) \\
\tilde{\kappa}^S(\chi(t-)) &= \mathbb{E}^{Q^*}[\tilde{Z}_{\tilde{N}(t)}(\chi(t-))].
\end{align*}
\]

Solving the stochastic differential equation (17) yields the dynamics of the optioned stock price:

\[
D(t)S(t) = S(0) \exp \left\{ \int_0^t \alpha^S(\chi(u)) \cdot d\tilde{W}(u) - \frac{1}{2} \int_0^t ||\alpha^S(\chi(u))||^2 du \right\} \times \exp \left\{ - \int_0^t \tilde{\lambda}(u, \chi(u-))[\tilde{\pi}^S(\chi(u-)) - 1]du + \sum_{\tau_i \leq t} \log \left( \tilde{Y}^S_i(\chi(\tau_i-)) \right) \right\}.
\]

5.1. \( S(t) \) as a Numéraire. Following Runsgalder [30], we let

\[
L_t^{(S)} = \exp \left\{ \int_0^t \alpha^S(\chi(u)) \cdot d\tilde{W}(u) - \frac{1}{2} \int_0^t ||\alpha^S(\chi(u))||^2 du \right\} \times \exp \left\{ - \int_0^t \tilde{\lambda}(u, \chi(u-))[\tilde{\pi}^S(\chi(u-)) - 1]du + \sum_{\tau_i \leq t} \log \left( \tilde{Y}^S_i(\chi(\tau_i-)) \right) \right\}.
\]

Then \( \mathbb{E}^{Q^*}[L_t^{(S)}] = 1 \) and \( L_t^{(S)} = \mathbb{E}^{Q^*}[L_t^{(S)}]F_t \geq 0 \). Thus, the \( F_t \)-adapted process \( \{L_t^{(S)}\}_{t \geq 0} \) is a Radon-Nikodym derivative. According to the standard Girsanov’s Theorem for jump-diffusion processes, we can use this process to change the measure.

\[ \text{Lemma 5.1. Let } \mathbb{P}^{Q^*} \text{ be a probability measure defined by:} \]

\[ L_t^{(S)} = \frac{d\mathbb{P}^{Q^*}}{d\mathbb{P}^{Q^*}} \bigg|_{F_t}. \]

Then, under \( \mathbb{P}^{Q^*} \),

\[ \tilde{W}^S(t) = - \int_0^t \alpha^S(\chi(u))du + \tilde{W}(t) \]

is an \( n \)-dimensional standard Brownian motion. In particular, under \( \mathbb{P}^{Q^*} \), the Brownian motions \( \tilde{W}^S(t), \cdots, \tilde{W}^n(t) \) are mutually independent. For any Markov-modulated compound Poisson process

\[ \tilde{N}(t) = \sum_{i=1}^{\tilde{N}(t)} J_i(\chi(\tau_i-)), \]

the arrival intensity rate under \( \mathbb{P}^{Q^*} \) is given by

\[ \tilde{\lambda}^S(t, \chi(t-)) = \tilde{\lambda}(t, \chi(t-)) \times \tilde{\pi}^S(\chi(t-)). \]
The distribution of the jump-size at time $t$, $J_{S(t)}(\chi(t-))$, is determined by the moment generating function

$$M^Q_{S(t)}(\chi(t-)) = \frac{\mathbb{E}^Q[e^{\theta J_{S(t)}(\chi(t-))}\tilde{Y}_{S(t)}(\chi(t-))]}{\hat{\pi}^S(\chi(t-))}. $$

Hence we have

$$\mathbb{E}^Q[S[J_{S(t)}(\chi(t-)))] = \frac{\mathbb{E}^Q[J_{S(t)}(\chi(t-))\tilde{Y}_{S(t)}(\chi(t-))]}{\hat{\pi}^S(\chi(t-))}. $$

Proof. Because of the independence assumption of the diffusion processes and the jump processes in the model, the distributions of the $n$-dimensional Brownian motion and the jump process under $\mathbb{P}^Q.S$ can be determined separately using the diffusion component and the jump component of the Radon-Nikodym derivative, respectively.

Using the standard Girsanov’s theorem,

$$\tilde{W}^S(t) = -\int_0^t \alpha^S(\chi(u))du + \tilde{W}(t)$$

is a standard $n$-dimensional Brownian motion under $\mathbb{P}^Q.S$.

To determine the distribution of the jump component under $\mathbb{P}^Q.S$, we derive its moment generating function under this measure first. By a version of the Bayes’ rule,

$$\mathbb{E}^Q.S\left[\exp\left\{\theta \sum_{i=1}^{\tilde{N}(t)} \tilde{Y}_{i}(\chi(\tau_i-))\right\}\right] =$$

$$\mathbb{E}^Q.S\left[\exp\left\{\sum_{i=1}^{\tilde{N}(t)} \left[\theta \cdot J_i(\chi(\tau_i-)) + \log \left(\tilde{Y}_{i}^S(\chi(\tau_i-))\right)\right]\right\} \right.$$ 

$$\times \exp\left\{-\int_0^t \tilde{\lambda}(u,\chi(u-))\hat{\pi}^S(\chi(u-)) - 1\right\}du\right\}$$

$$= \exp\left\{\int_0^t \tilde{\lambda}(u,\chi(u-))\left[\mathbb{E}^Q.S[e^{\theta J_{\tilde{N}(u)}(\chi(u-))}\tilde{Y}_{\tilde{N}(u)}^S(\chi(u-))\right] \right\}$$

$$\times \exp\left\{-\int_0^t \tilde{\lambda}(u,\chi(u-))\hat{\pi}^S(\chi(u-))\right\}du\right\}$$

$$= \exp\left\{\int_0^t \tilde{\lambda}(u,\chi(u-)) \times \hat{\pi}^S(\chi(u-))\left[\mathbb{E}^Q.S[e^{\theta J_{\tilde{N}(u)}(\chi(u-))}\tilde{Y}_{\tilde{N}(u)}^S(\chi(u-))] \right.\right.$$ 

$$\left.- 1\right]du\right\}. $$

The second-to-last equation follows the fact that, given any compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad N_t \sim \text{Poi}(\lambda t),$$

The moment generating function of $X_t$ is given by

$$M^{Q.S}_{X_t}(u) = \mathbb{E}^Q.S[e^{u X_t}] = \exp(\lambda t(M^{Q.S}_{X_t}(u) - 1)).$$
Theorem 5.2. (Change of numéraire). If we take $S(t)$ as the numéraire, then the process

$$\{V^{(S)}(t) = \frac{V(t)}{S(t)}\}_{t \geq 0}$$

is a martingale under $\mathbb{P}^{Q^{*, S}}$. Moreover,

$$dV^{(S)}(t) / V^{(S)}(t-1) = \left[ \alpha^V(\chi(t)) + \frac{\rho v X(\chi(t))}{L\eta} \alpha^X(\chi(t)) - \alpha^S(\chi(t)) \right] \cdot d\tilde{W}^S(t)$$

$$- \tilde{\lambda}(t, \chi(t-)) \left[ \tilde{\pi}^V(\chi(t-)) + \tilde{\alpha}^X(\chi(t-)) - \tilde{\pi}^S(\chi(t-)) \right] dt$$

$$+ \left[ \frac{\tilde{Y}_{\tilde{N}(t)}^V(\chi(t-)) + \tilde{Z}_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}_{\tilde{N}(t)}^S(\chi(t-))} - 1 \right] d\tilde{N}(t)$$

where $\{\tilde{W}^S(t)\}_{t \geq 0}$ is an $n$-dimensional Brownian motion under $\mathbb{P}^{Q^{*, S}}$, $\{\tilde{N}(t)\}_{t \geq 0}$ is a Markov-modulated Poisson process with intensity process under $\mathbb{P}^{Q^{*, S}}$ being $\{\tilde{\lambda}^{(S)}(t, \chi(t-))\}_{t \geq 0} = \tilde{\lambda}(t, \chi(t-)) \times \tilde{\pi}^S(\chi(t-))$.

Proof. Again solving the stochastic differential equation (17) yields

$$D(t)V(t) = V(0) \exp \left\{ \int_0^t \left[ \alpha^V(\chi(u)) + \frac{\rho v X(\chi(u))}{L\eta} \alpha^X(\chi(u)) \right] \cdot d\tilde{W}(u)$$

$$- \frac{1}{2} \int_0^t ||\alpha^V(\chi(u)) + \frac{\rho v X(\chi(u))}{L\eta} \alpha^X(\chi(u))||^2 du$$

$$- \int_0^t \tilde{\lambda}(u, \chi(u-)) \left[ \tilde{\pi}^V(\chi(u-)) + \tilde{\alpha}^X(\chi(u-)) - 1 \right] \cdot d\tilde{W}(u)$$

$$+ \frac{1}{2} \int_0^t ||\alpha^S(\chi(u))||^2 du$$

$$\times \prod_{i=1}^{\tilde{N}(u)} \tilde{Y}_{\tilde{N}(t)}^V(\chi(\tau_i-)) + \tilde{Z}_{\tilde{N}(t)}(\chi(\tau_i-)).$$

$$D(t)S(t) = S(0) \exp \left\{ \int_0^t \alpha^S(\chi(u)) \cdot d\tilde{W}(u) - \frac{1}{2} \int_0^t ||\alpha^S(\chi(u))||^2 du$$

$$- \int_0^t \tilde{\lambda}(u, \chi(u-)) \left[ \tilde{\pi}^S(\chi(u-)) - 1 \right] \cdot d\tilde{W}(u)$$

$$\times \prod_{i=1}^{\tilde{N}(u)} \tilde{Y}_{\tilde{N}(t)}^S(\chi(\tau_i-))$$

Consequently, we have

$$V^{(S)}(t) = \frac{V(0)}{S(0)} \exp \left\{ \int_0^t \left[ \alpha^V(\chi(u)) + \frac{\rho v X(\chi(u))}{L\eta} \alpha^X(\chi(u)) - \alpha^S(\chi(u)) \right] \cdot d\tilde{W}(u)$$

$$- \frac{1}{2} \int_0^t \left[ ||\alpha^V(\chi(u)) + \frac{\rho v X(\chi(u))}{L\eta} \alpha^X(\chi(u))||^2 - ||\alpha^S(\chi(u))||^2 \right] du$$

$$- \int_0^t \tilde{\lambda}(u, \chi(u-)) \left[ \tilde{\pi}^V(\chi(u-)) + \tilde{\alpha}^X(\chi(u-)) - \tilde{\pi}^S(\chi(u-)) \right] \cdot d\tilde{W}(u)$$

$$\times \prod_{i=1}^{\tilde{N}(u)} \frac{\tilde{Y}_{\tilde{N}(t)}^V(\chi(\tau_i-)) + \tilde{Z}_{\tilde{N}(t)}(\chi(\tau_i-))}{\tilde{Y}_{\tilde{N}(t)}^S(\chi(\tau_i-))}.$$
We define

\[ \mathcal{V}(t) = \int_0^t \left[ \alpha^V(\chi(u)) + \frac{\rho \nu \chi(\chi(u))}{\eta} \alpha^X(\chi(u)) - \alpha^S(\chi(u)) \right] \cdot d\tilde{W}(u) \]

\[ - \frac{1}{2} \int_0^t \left[ ||\alpha^V(\chi(u)) + \frac{\rho \nu \chi(\chi(u))}{\eta} \alpha^X(\chi(u))||^2 - ||\alpha^S(\chi(u))||^2 \right] du \]

\[ - \int_0^t \tilde{\lambda}(u,\chi(u)) [\tilde{\pi}^V(\chi(u-)) + \tilde{\kappa}^X(\chi(u-)) - \tilde{\pi}^S(\chi(u-))] du \]

\[ + \int_0^t \log \left( \frac{\tilde{Y}^V_{\tilde{N}(u)}(\chi(u-)) + \tilde{Z}_{\tilde{N}(u)}(\chi(u-))}{\tilde{Y}^S_{\tilde{N}(u)}(\chi(u-))} \right) d\tilde{N}(u). \]

Applying the standard Itô-Doeblin formula gives:

\[ d\mathcal{V}(t) = \left[ \alpha^V(\chi(t)) + \frac{\rho \nu \chi(\chi(t))}{\eta} \alpha^X(\chi(t)) - \alpha^S(\chi(t)) \right] \cdot d\tilde{W}(t) \]

\[ - \frac{1}{2} \left[ ||\alpha^V(\chi(t)) + \frac{\rho \nu \chi(\chi(t))}{\eta} \alpha^X(\chi(t))||^2 - ||\alpha^S(\chi(t))||^2 \right] dt \]

\[ - \tilde{\lambda}(t,\chi(t-)) [\tilde{\pi}^V(\chi(t-)) + \tilde{\kappa}^X(\chi(t-)) - \tilde{\pi}^S(\chi(t-))] dt \]

\[ + \log \left( \frac{\tilde{Y}^V_{\tilde{N}(t)}(\chi(t-)) + \tilde{Z}_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}^S_{\tilde{N}(t)}(\chi(t-))} \right) \tilde{N}(t), \]

\[ d\langle V^V, V^V \rangle(t) = ||\alpha^V(\chi(t)) + \frac{\rho \nu \chi(\chi(t))}{\eta} \alpha^X(\chi(t)) - \alpha^S(\chi(t))||^2 dt \]

\[ = \left\{ ||\alpha^V(\chi(t)) + \frac{\rho \nu \chi(\chi(t))}{\eta} \alpha^X(\chi(t))||^2 + ||\alpha^S(\chi(t))||^2 \right\}dt. \]

With \( f(x) = \frac{V^{(S)(0)}}{V^{(S)(0)}} e^x \), we have \( V^{(S)}(t) = f(V(t)) \). Directly applying Itô’s formula to \( f(V(t)) \) yields:

\[ dV^{(S)}(t) \\
= df(V(t)) \\
= f'(V(t))dV(t) + \frac{1}{2} f''(V(t))d\langle V^V, V^V \rangle(t) + [f(V(t)) - f(V(t-))]d\tilde{N}(t) \]

\[ = V^{(S)}(t-) \left\{ \alpha^V(\chi(t)) + \frac{\rho \nu \chi(\chi(t))}{\eta} \alpha^X(\chi(t)) - \alpha^S(\chi(t)) \right\} \cdot d\tilde{W}^{(S)}(t) \]

\[ - \tilde{\lambda}(t,\chi(t-)) [\tilde{\pi}^V(\chi(t-)) + \tilde{\kappa}^X(\chi(t-)) - \tilde{\pi}^S(\chi(t-))] dt + \frac{\tilde{Y}^V_{\tilde{N}(t)}(\chi(t-)) + \tilde{Z}_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}^S_{\tilde{N}(t)}(\chi(t-))} - 1) d\tilde{N}(t). \]

(Martingale) (18)

Since \( \tilde{W}^{(S)}(t) \) is an \( n \)-dimensional Brownian motion under \( P^{Q^{S}} \), using Lemma 1, we have

\[ \mathbb{E}^{Q^{S}} \left[ \int_0^t \left[ \frac{\tilde{Y}^V_{\tilde{N}(u)}(\chi(u-)) + \tilde{Z}_{\tilde{N}(u)}(\chi(u-))}{\tilde{Y}^S_{\tilde{N}(u)}(\chi(u-))} - 1 \right] d\tilde{N}(u) \right] = \mathbb{E}^{Q^{S}} \left[ \int_0^t \tilde{\lambda}(u,\chi(u-)) [\tilde{\pi}^V(\chi(u-)) + \tilde{\kappa}^X(\chi(u-)) - \tilde{\pi}^S(\chi(u-))] du \right] \]

\[ = \mathbb{E}^{Q^{S}} \left[ \int_0^t \lambda(u,\chi(u-)) [\pi^V(\chi(u-)) + \kappa^X(\chi(u-)) - \pi^S(\chi(u-))] du \right]. \]
Combing the result in Eq. (18), we assert that the process \( \{ V^{(S)}(t) \}_{t \geq 0} \) is a martingale under \( \mathbb{P}^{Q^*, S} \).

**Remark 1.** Similarly, we can prove that the reciprocal forward price process given by:

\[
\left\{ \frac{1}{F_{t, T}} = \frac{B(t, T)}{S(t)} \right\}_{t \geq 0}
\]

is also a martingale under \( \mathbb{P}^{Q^*, S} \). Moreover,

\[
d \left( \frac{1}{F_{t, T}} \right) = \frac{1}{F_{t, T}} \left\{ \left[ \alpha^B(\chi(t)) - \alpha^S(\chi(t)) \right] \cdot d\tilde{W}^S(t) - \tilde{\lambda}(t, \chi(t-)) \left[ \tilde{\pi}^B(\chi(t-)) - \tilde{\pi}^S(\chi(t-)) \right] dt \\
+ \left[ \frac{\tilde{Y}^B_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}^S_{\tilde{N}(t)}(\chi(t-))} - 1 \right] d\tilde{N}(t) \right\}.
\]

### 5.2. \( B(t, T) \) as a Numéraire.

Similarly, when can take the \( T \)-bond, \( B(t, T) \), as the numéraire. Let

\[
L^{(T)}_t = \exp \left\{ \int_0^t \alpha^B(\chi(u)) \cdot dW(u) - \frac{1}{2} \int_0^t \| \alpha^B(\chi(u)) \|^2 du \right\}
\]

\[
\times \exp \left\{ - \int_0^t \tilde{\lambda}(u, \chi(u-)) \left[ \tilde{\pi}^B(\chi(u-)) - 1 \right] du + \sum_{\tau_i \leq t} \log \left( \tilde{Y}^B_{i}(\chi(\tau_i-)) \right) \right\}.
\]

Denote

\[
L^{(B)}_t = \frac{d\mathbb{P}^{Q^*, T}}{d\mathbb{P}^{Q^*}} \bigg|_{\mathcal{F}_t}.
\]

Then, under the new defined probability measure \( \mathbb{P}^{Q^*, T} \),

\[
\tilde{W}^T(t) = - \int_0^t \alpha^B(\chi(u)) du + \tilde{W}(t)
\]

is a standard \( n \)-dimensional Brownian motion, and \( \{ \tilde{N}(t) \}_{t \geq 0} \) is a Markov-modulated Poisson process with intensity

\[
\tilde{\lambda}^T(t, \chi(t-)) = \tilde{\lambda}(t, \chi(t-)) \times \tilde{\pi}^B(\chi(t-)).
\]

The processes

\[
\left\{ V^{(T)}(t) = \frac{V(t)}{B(t, T)} \right\}_{t \geq 0} \quad \text{and} \quad \left\{ F_{t, T} = \frac{S(t)}{B(t, T)} \right\}_{t \geq 0},
\]

under \( \mathbb{P}^{Q^*, T} \), are martingales. Furthermore, we have (i)

\[
\frac{dV^{(T)}(t)}{V^{(T)}(t-)} = \left[ \alpha^V(\chi(t)) + \frac{\rho^\nu \lambda(\chi(t))}{L} \alpha^X(\chi(t)) - \alpha^B(\chi(t)) \right] \cdot d\tilde{W}^T(t) \\
- \tilde{\lambda}(t, \chi(t-)) \left[ \tilde{\pi}^V(\chi(t-)) + \tilde{\nu}^X(\chi(t-)) - \tilde{\pi}^B(\chi(t-)) \right] dt \\
+ \left[ \frac{\tilde{Y}^V_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}^S_{\tilde{N}(t)}(\chi(t-))} + \frac{\tilde{Z}_{\tilde{N}(t)}(\chi(t-))}{\tilde{Y}^S_{\tilde{N}(t)}(\chi(t-))} - 1 \right] d\tilde{N}(t);
\]
Recall that the two event \( A \) and \( B \) are defined as

\[
A = \{ S(T) > K \} = \{ F_{T,T} > K \} = \{ \frac{1}{F_{T,T}} < \frac{1}{K} \}
\]

and

\[
B = \{ V(T) \geq d^* \} = \{ V(T)(T) \geq d^* \} = \{ V(S)(T) \geq \frac{d^*}{F_{T,T}} \}.
\]

The price of the vulnerable European call option can be obtained as follows:

\[
C(t, S(t), V(t), \chi(t)) = S(t) \left\{ \mathbb{E}^{Q^{*,S}} \{ A \cap B | \mathcal{F}_t \} + \frac{1 - a}{d} \mathbb{E}^{Q^{*,S}} \{ \mathbb{I}_{A \cap B} | \mathcal{F}_t \} \right\}
\]

\[
- KB(t, T) \left\{ \mathbb{E}^{Q^{*,T}} \{ A \cap B | \mathcal{F}_t \} + \frac{1 - d}{d} \mathbb{E}^{Q^{*,T}} \{ V(T) | \mathbb{A} \cap B | \mathcal{F}_t \} \right\}.
\]

As we can see from Eq. (19), the option pricing formula requires not only the joint-density distribution of \( \frac{1}{F_{T,T}} \) and \( V(S) \) under \( \mathbb{P}^{Q^{*,S}} \), but also that of \( F_{T,T} \) and \( V(T) \) under \( \mathbb{P}^{Q^{*,T}} \). However, the joint-densities are path-dependent. They depend not only on the occupation time of the underlying economy \( \{ \chi(t) \}_{t \geq 0} \) in each of the economic regimes, i.e., \( (T_1, \cdots, T_m) \), but also on the number of jumps, \( (\tilde{N}_1(t_1), \cdots, \tilde{N}_m(t_m)) \), given \( (T_1, \cdots, T_m) = (t_1, \cdots, t_m) \), and the realized jump sizes, during these time periods.

For the sake of convenience in the computation, we make the following assumption:

**Assumption 1.** Let \( \nu_j^X(dy) \), \( \nu_j^S(dy) \) and \( \nu_j^B(dy) \) denote, respectively, the distribution measures of the jumps in \( X \), \( S \) and \( B \) under \( Q^* \), when the underlying economy is in regime \( j \in \mathcal{M} \). In this paper, we assume that

\[
\nu_j^X(dy) = p_j \cdot \delta_{d^*,j}(dy) + (1 - p_j) \cdot \delta_{d^*,j}(dy), \quad u^# \cdot j > 1 \quad \text{and} \quad d^# \cdot j \in (0,1),
\]

where \( # \in \{ X, S, B \} \).
Given in regime $j$ of the market with respect to these shocks. Whenever there comes a market shock i.e., whether the triggered event is “good” or “bad” for the market, and the trend of the market with respect to these shocks. Whenever there comes a market shock in regime $j$, with probability $p_j$, the market shock will incur an upward jump of amplitude $u^j$, probability $(1 - p_j)$, a downward jump of amplitude $d^j$. Under this assumption, the number of positive/negative market shocks can be described separately using two independent Markov-modulated Poisson processes
\[
\{\tilde{N}^u(t)\}_{t \geq 0} \quad \text{and} \quad \{\tilde{N}^d(t)\}_{t \geq 0},
\]
with intensities $\{p_x \xi(t -) \hat{\lambda}(t, \chi(t-))\}_{t \geq 0}$ and $\{(1 - p_x \xi(t -)) \hat{\lambda}(t, \chi(t-))\}_{t \geq 0}$, respectively (the superscript $u$ and $d$ represent up- and down-jumps).

**Lemma 6.1.** Given $(T_1, \ldots, T_m) = (t_1, \ldots, t_m)$. Let
\[
h^#(n^u_1, \ldots, n^u_m, n^d_1, \ldots, n^d_m | (t_1, \ldots, t_m))
\]
denote the joint conditional probability distribution function, under the probability measure $\mathbb{P}^{Q^*, #} \neq \{S, T\}$, of the number of positive/negative market shocks arrived during these time periods. Then
\[
h^#(n^u_1, \ldots, n^u_m, n^d_1, \ldots, n^d_m | (t_1, \ldots, t_m)) = \exp \left\{ - (\hat{\lambda}^u_1 \cdot t_1 + \cdots + \hat{\lambda}^u_m \cdot t_m) \right\} \cdot \prod_{j=1}^m \left( \frac{(n^u_j)! \cdots (n^d_m)!}{n^u_j! \cdots n^d_m!} \right) \cdot \left( \prod_{j=1}^m \frac{n^u_j}{n^d_j} \right) \cdot \left( \prod_{j=1}^m p_j^u (1 - p_j)^{n^d_j} \right)
\]
where, for any $j \in \mathcal{M} = \{1, 2, \ldots, m\}$, $n_j = n^u_j + n^d_j$ and $\hat{\lambda}^# = \hat{\lambda}^#(\cdot, j)$ (positive-valued constant).

Following [6, 26], we use the fact that the joint distribution of $\frac{1}{F_T}$ and $V_T^{(S)}$ under $\mathbb{P}^{Q^*, S}$, and the joint distribution of $F_T$ and $V_T^{(T)}$ under $\mathbb{P}^{Q^*, T}$ are conditionally bivariate normal given the occupation time in each regime and the number of positive/negative market shocks arrived in each occupation time period.

The following lemma can be directly deduced from the formulas.

**Lemma 6.2.** Given $(T_1, \ldots, T_m) = (t_1, \ldots, t_m)$, and $(\tilde{N}^u_1(t_1), \ldots, \tilde{N}^u_m(t_m)) = (n^u_1, \ldots, n^u_m)$, where $# \in \{u, d\}$. Let
\[
g^S(x, y | F_t, t_1, \ldots, t_m, n^u_1, \ldots, n^u_m, n^d_1, \ldots, n^d_m)
\]
denote the joint conditional probability density function of $(\log(\frac{1}{F_T}), \log(V_T^{(S)}))$ under $\mathbb{P}^{Q^*, S}$, and
\[
g^T(x, y | F_t, t_1, \ldots, t_m, n^u_1, \ldots, n^u_m, n^d_1, \ldots, n^d_m)
\]
that of $(\log(F_T), \log(V_T^{(T)}))$ under $\mathbb{P}^{Q^*, T}$, where, for each $i = T, S$,
\[
g^i(x, y | F_t, t_1, \ldots, t_m, n^u_1, \ldots, n^u_m, n^d_1, \ldots, n^d_m)
:= \left. g^i(x, y | F_t \cup \sigma\{T_1, \ldots, T_m\} \cup \sigma\{N^u_1, \ldots, N^u_m\} \cup \sigma\{N^d_1, \ldots, N^d_m\}) \right|_{(t_1, \ldots, t_m) = (t_1, \ldots, t_m), (\tilde{N}^u_1(t_1), \ldots, \tilde{N}^u_m(t_m)) = (n^u_1, \ldots, n^u_m), # \in \{u, d\}},
\]
and $A \cup B \cup C$ is the minimal $\sigma$-field containing the $\sigma$-fields $A$, $B$ and $C$. Therefore
(i): \( g^S(x, y | F_1, t_1, \cdots, t_m, n_1^u, \cdots, n_m^u, n_1^v, \cdots, n_m^v) \) is the joint probability density function of the following bivariate normal distributions:

\[
N_2 \left( \left( \begin{array}{c} d^1_{t_1, \cdots, t_m} \\ \vdots \\ d^m_{t_1, \cdots, t_m} \end{array} \right), \left( \begin{array}{c} c^1_{t_1, \cdots, t_m} \\ \vdots \\ c^m_{t_1, \cdots, t_m} \end{array} \right) \right), \sqrt{\sum_{i=1}^m \left| \alpha^B(i) - \alpha^S(i) \right|^2 t_i} 
\]

where \( \sum_{i=1}^m \left| \alpha^B(i) - \alpha^S(i) \right|^2 t_i \)

\[
\begin{aligned}
&d^2_{t_1, \cdots, t_m} = \log \left( \frac{V(t_{i-1})}{S(t_{i-1})} \right) - \sum_{i=1}^m \left[ \lambda_i [ \xi^B(t_i) - \xi^S(t_i)] + \frac{1}{2} |\alpha^B(t_i) - \alpha^S(t_i)|^2 \right] t_i \\
&\quad + \sum_{i=1}^m \log \left[ \frac{\xi^B(t_i)}{\xi^S(t_i)} \right] \\
&c^2_{t_1, \cdots, t_m} = \log \left( \frac{V(t_{i-1})}{S(t_{i-1})} \right) - \sum_{i=1}^m \left[ \lambda_i [ \xi^Y(t_i) + \xi^X(t_i) - \xi^S(t_i)] + \frac{1}{2} |\alpha^B(t_i) + \frac{\rho \xi^X(t_i)}{L_{\eta}} - \alpha^S(t_i)|^2 \right] t_i \\
&\quad + \sum_{i=1}^m \log \left[ \frac{\xi^Y(t_i) + \xi^X(t_i)}{-\rho \xi^X(t_i)} \right] \\
\end{aligned}
\]

(ii): \( g^T(x, y | F_1, t_1, \cdots, t_m, n_1^u, \cdots, n_m^u, n_1^v, \cdots, n_m^v) \) is the joint probability density function of the following bivariate normal distributions:

\[
N_2 \left( \left( \begin{array}{c} c^1_{t_1, \cdots, t_m} \\ \vdots \\ c^m_{t_1, \cdots, t_m} \end{array} \right), \left( \begin{array}{c} c^1_{t_1, \cdots, t_m} \\ \vdots \\ c^m_{t_1, \cdots, t_m} \end{array} \right) \right), \sqrt{\sum_{i=1}^m \left| \alpha^B(i) - \alpha^S(i) \right|^2 t_i} 
\]

where

\[
\begin{aligned}
&c^2_{t_1, \cdots, t_m} = \log \left( \frac{V(t_{i-1})}{S(t_{i-1})} \right) - \sum_{i=1}^m \left[ \lambda_i [ \xi^B(t_i) - \xi^S(t_i)] + \frac{1}{2} |\alpha^B(t_i) - \alpha^S(t_i)|^2 \right] t_i \\
&\quad + \sum_{i=1}^m \log \left[ \frac{\xi^B(t_i)}{\xi^S(t_i)} \right] \\
&c^2_{t_1, \cdots, t_m} = \log \left( \frac{V(t_{i-1})}{S(t_{i-1})} \right) - \sum_{i=1}^m \left[ \lambda_i [ \xi^Y(t_i) + \xi^X(t_i) - \xi^S(t_i)] + \frac{1}{2} |\alpha^B(t_i) + \frac{\rho \xi^X(t_i)}{L_{\eta}} - \alpha^S(t_i)|^2 \right] t_i \\
&\quad + \sum_{i=1}^m \log \left[ \frac{\xi^Y(t_i) + \xi^X(t_i)}{-\rho \xi^X(t_i)} \right] \\
\end{aligned}
\]

The next theorem presents the vulnerable European call price at any time \( t \) conditional on the initial economic regime \( \chi(t) \).

\( \tau u^{Z,t} \) and \( d^{Z,t} \) are computed through the relation (17).
Theorem 6.3. Given the initial economic regime \( \chi(t) \), the price of the vulnerable European call price at time \( t \) is given by

\[
C(t, S(t), V(t), \chi(t)) = \sum_{(n_1^u, \ldots, n_m^u)} \{ S(t) \int_{-\infty}^{\log(K)} dx \int_{-\infty}^{\log(K)} dy 
\left[ \frac{1 - a}{d} e^{y-x} \right] g^u(x, y| F_t, t_1, \ldots, t_m, n_1^u, \ldots, n_m^u) dy dx 
- K B(t, T) \left[ \int_{\log(K)}^{t+\infty} + \int_{-\infty}^{\log(K)} \frac{1 - a}{d} e^{y} \right] g^u(x, y| F_t, t_1, \ldots, t_m, n_1^u, \ldots, n_m^u) dy dx \big] dt_1 \cdots dt_m,
\]

where \( f(t_1, \ldots, t_m| F_t) \) is the joint conditional probability density function of the occupation times in each regime (refer to Sérica (2000) [31]), and \((n_1^u, \ldots, n_m^u) = (n_1^u, \ldots, n_m^u) \cup (n_1^d, \ldots, n_m^d)\).

We remark that the series inside the integral of the above theorem converges very fast especially for a small value \( T \). Thus this reduces substantially the computational time for the call option price.

7. Simulation results. In this section, we present numerical results for the vulnerable European call option under the impact of the RFI’s “distressed selling”, with the economic state driven by a finite-state, continuous-time Markov chain. To simplify the analysis, we assume that the parameters of the assets dynamics switch according to a two-state\(^8\), i.e. \( m = 2 \), continuous-time, time-homogeneous Markov chain with generator \( \mathbf{Q} \in \mathbb{R}^{2 \times 2} \),

\[
\mathbf{Q} = \begin{pmatrix} -\theta_1 & \theta_1 \\ \theta_2 & -\theta_2 \end{pmatrix}.
\]

The first and second regime can be interpreted as a good (Bull) and a bad (Bear) economic state, respectively. In Eq. (21), \( \theta_1 \) represents the transition rate from the good state to the bad one, and \( \theta_2 \) represents the transition rate from the bad state to the good one. The transition probability matrix over the time interval \([t, t + \Delta t]\) is given by

\[
P = e^{\mathbf{Q} \Delta t} = \sum_{n=0}^{\infty} \frac{(\mathbf{Q} \Delta t)^n}{n!} = I + \mathbf{Q} \Delta t + O(\Delta t),
\]

where \( I \) is the identity matrix. Hence, ignoring terms of order superior to \( \Delta t \), if at any time \( t \) the economic state is in regime 1, then with probability \( \theta_1 \Delta t \) there will be a switch to regime 2 at time \( t + \Delta t \); the probability of remaining in regime 1 is \( 1 - \theta_1 \Delta t \). Transition probabilities from regime 2 are determined similarly. Then, under the Markov-modulated regime switching framework, each model parameter has two possible values corresponding to the two different economic states. This may provide some flexibility to capture changes in economic states and the presence of distressed selling through regime switching and asset values’ correlation, respectively.

Here we consider the time zero vulnerable European call price, i.e. \( C(s_0, t_0, \chi(0)) \). As suggested in Costabile et al. (2014) [6], we present the discrete version of the

\(^8\)The algorithm can be extended to the case of more than one reference institution and more than two regimes.
continuous time framework, which is based on a multinomial representation of
the risky asset dynamics. We establish a multinomial grid based on \( N \) time steps of
length \( \Delta t = T/N \), with \( T \) being the option maturity, where the number of nodes,
amount of jumps in each regime, at each time step is bounded by fixing a tolerance
level \( \epsilon \). Let \( n_1^{(k,S)} \), \( n_2^{(k,S)} \), \( n_1^{(k,T)} \) and \( n_2^{(k,T)} \) be the smallest integers satisfying
\[
e^{-\tilde{\lambda}_1^{g} k \Delta t} \cdot \frac{\left( \tilde{\lambda}_1^{g} k \Delta t \right)^{n_1^{(k,S)}}}{n_1^{(k,S)}!} < \epsilon, \quad e^{-\tilde{\lambda}_2^{g} k \Delta t} \cdot \frac{\left( \tilde{\lambda}_2^{g} k \Delta t \right)^{n_2^{(k,S)}}}{n_2^{(k,S)}!} < \epsilon,
\]
\[
e^{-\tilde{\lambda}_1^{l} k \Delta t} \cdot \frac{\left( \tilde{\lambda}_1^{l} k \Delta t \right)^{n_1^{(k,T)}}}{n_1^{(k,T)}!} < \epsilon, \quad e^{-\tilde{\lambda}_2^{l} k \Delta t} \cdot \frac{\left( \tilde{\lambda}_2^{l} k \Delta t \right)^{n_2^{(k,T)}}}{n_2^{(k,T)}!} < \epsilon,
\]
respectively. During time interval \([t_{k-1}, t_k)\), the probability of having more than
\( n_1^{(k,S)} \) (\( n_1^{(k,T)} \)) jumps in the economic state 1, or \( n_2^{(k,S)} \) (\( n_2^{(k,T)} \)) jumps in
the economic state 2 under the measure \( \mathbb{P}^{Q^{g}:S} \) (\( \mathbb{P}^{Q^{l}:T} \)) is smaller than \( \epsilon \). Furthermore,
the conditional densities of the occupation time in the first regime given the two
possible values of the initial state \( \chi(0) \) are given by (Pedler (1971) [29] and Naik
(1993) [26]):
\[
f_{T_1}(t|\chi(0) = 1) = e^{-\theta_1 t - \theta_2 (T-t)} \cdot \left[ \delta_0(T - t) + \left[ \theta_1 \theta_2 t \right] \sum_{k=0}^{\infty} \frac{[h(t)]^{2k}}{k!(k+1)!} + \theta_1 \sum_{k=0}^{\infty} \frac{[h(t)]^{2k}}{k!k+1} \right],
\]
\[
f_{T_1}(t|\chi(0) = 2) = e^{-\theta_1 t - \theta_2 (T-t)} \cdot \left[ \delta_0(t) + \left[ \theta_1 \theta_2 (T - t) \right] \sum_{k=0}^{\infty} \frac{[h(t)]^{2k}}{k!(k+1)!} + \theta_2 \sum_{k=0}^{\infty} \frac{[h(t)]^{2k}}{k!k+1} \right],
\]
where \( \delta_0(\cdot) \) is Dirac’s delta function and \( h(t) = \sqrt{\theta_1 \theta_2 (T - t)} \).

For the purpose of illustration, we use the set of baseline parameters given in
Table 1. Preference parameters listed in Table 1 represent a special situation. In
the basic case, the vulnerable option is written by a firm influenced by a highly
leveraged reference institution with leverage ratio \( \eta = 10 \). Time to maturity \( T \) is
assumed to be one year.\(^9\) Since the risky asset values may be less volatile in the
good economy than in the bad economy, shocks to the risky asset values in the good
economic state are less than shocks in the bad economic state. Thus, we assume
that \( \tilde{\lambda}_1 = 2 \) and \( \tilde{\lambda}_2 = 3 \), i.e. the shocks to the risky asset values happen on average
2 times a year in the good state and 3 times a year in the bad state, respectively.
Beyond that, the parameters describing the jumps, which could be either upward
or downward, in the risky asset values under two economic states together with the
jump probabilities are also presented in Table 1. As illustrated in Costabile et al.
(2014) [6], the series inside the integral in Eq. (20) converges very fast especially for
small values of \( T \), and this reduces substantially the computational time for option
pricing.

\(^9\)Actually, we can also analyze the sensitivity of option prices with different time-to-maturities
to changes in other model parameters. However, the expression of the vulnerable option price
depends on the price of the zero-coupon treasury bond with the same maturity as the option.
Therefore, to examine the combining effects of the time-to-maturity and other model parameters
on the option prices, we should first set up a model for the forward curve of the real markets. This
is not the focus of our research in this paper and may be studied in our future research.
The presence of reference institutions’ distressed selling for large losses triggered endogenously by market shocks may affect the small companies’ asset values, and hence influence their issuance prices of options and other financial securities. To assess the impact of fire sales on the issuance prices of vulnerable European call options, we present a comparison among the vulnerable European call option prices implied by our model without market impacts, and the one obtained from our model subject to the market impact of distressed selling.

Figure 1 shows the issuance prices of vulnerable European call options in different economic states against spot-to-strike ratio, \( S_0/K \), for the cases of “subject to market impact of distressed selling” and “without market impact”. The figures above and below show, respectively, the issuance price of vulnerable European call option in the good economic state and in the bad economic state. The “triangles line” and “squares line” correspond to the cases of “without market impact” and “subject to market impact of distressed selling”, respectively. We can see from Figure 1, the vulnerable call option prices subject to market impact of distressed selling are obvious smaller than those without market impact. For each case, given the value of the underlying asset, for example, \( S_0 = 40 \), the vulnerable European call option price increases as the strike price \( K \) decreases. Meanwhile, with all else being equal, a stronger volatile in the initial economic state, i.e. the bad economic state, corresponding to larger jumps and larger volatilities, produces a positive effect on the vulnerable call option price, which is consistent with the results presented in some of the existing literature, see for example, \([27, 26]\).
To further assess the sensitivity of option prices to changes in the regime state, in Table 2, we provide a comparison for the vulnerable option prices subject to different levels of persistence of the underlying economic state process in the good and bad states.\textsuperscript{10}

\textsuperscript{10}The parameters $\theta_1$ and $\theta_2$ in the generator matrix determine the probability of a switch in the economic state over a small time interval. The parameter $\theta_1$ measures the likelihood of a switch of the economic state from good to bad, and the parameter $\theta_2$ measures how fast the economy environment reverts to its good state. By choosing the parameters $\theta_1$ and $\theta_2$ appropriately, we can describe different levels of persistence of the economic state process.
Table 2. We use the multinomial recombining grid approximation method to calculate the vulnerable European call option prices with different strike prices, initial states and state persistence under different setting of market impact, without market impact (No) and subject to market impact (Impact). The default choices are given by the basic parameters in Table 1. We present, in this table, a comparison for the vulnerable European call option prices subject to different levels of persistence of the underlying economic state process in the good and bad economic states.

| Q  | θ₀ | χ₀ | No | Impact | No | Impact | No | Impact | No | Impact |
|----|----|----|----|--------|----|--------|----|--------|----|--------|
| 2.0 | 0.8 | 1.7845 | 0.7617 | 0.2812 | 0.1200 | 0.4341 | 0.1853 | 0.3153 | 0.1346 |
| 1.0 | 1.0 | 0.5520 | 0.2354 | 0.0475 | 0.0203 | 0.0848 | 0.0362 | 0.3197 | 0.1363 |
| 1.25 | 1.0 | 0.5520 | 0.2354 | 0.0475 | 0.0203 | 0.0848 | 0.0362 | 0.3197 | 0.1363 |

Table 2 shows how the vulnerable call option prices behave under different generator matrices with other parameters taking on the values adopted in Figure 1. We can see from the table, in the bad initial economic state, i.e. χ(0) = 2, the prices of call options subject to a low level of reverting are smaller than the prices of those subject to a high level of reverting, see for example columns 3-and-4 and 9-and-10 and columns 5-and-6 and 7-and-8 (having the same departure rate, but different reverting rates), or columns 3-and-4 and 7-and-8 and columns 5-and-6 and 9-and-10 (having different departure rates, but the same reverting rate). The reverse is true in the good economic state. This results from the fact that, with a low level of reverting, any change in the economic state is transitory, and the fact that, with all else being equal, a stronger volatility in the economic state produces a positive effect on the option prices. An implication of this result is that, in the bad economic state, the prices of the call options implied by the models without regime-switching, i.e. θ₁ = 0 or θ₂ = 0, are greater than the prices implied by the models with regime-switching. The reverse seems to hold for the case of the good economic state.

8. Conclusions. A pricing model for the valuation of European vulnerable call options was discussed. In the model, the reference financial institutions’ asset values, the underlying stock prices, the OTC option writer’s asset value and the interest rates (thus bond prices) are supposed to follow Markov-modulated two-point distributed jump-diffusion processes, having different sources of Brownian motion shocks but sharing a common source of Poisson jump shocks. The effect of other financial institutions’ price pressure on the OTC option writer’s asset value by distressed selling is described quantitatively in our model. Based on this model, we provide a semi-analytical pricing formula for the vulnerable European call options in the presence of regime switching via the Geman et al. (1995) [14] change of numéraire technique. The method can be numerically implemented through the Costabile et al. (2014) [6] multinomial approach method. The price of a European
vulnerable call option, which is related to the economic state, subject to price pressure is confirmed to be less than the price of the corresponding non-price-pressure vulnerable option due to the extra default risk triggered by other financial institutions’ distressed selling.

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Appendix A.

Proof. Applying Ito’s formula to \( f(X(t)/X(0)) \) gives (denote \( \hat{X}(t) = X(t)/X(0) \)):

\[
df(\hat{X}(t)) = f'(\hat{X}(t))dX^c(t)/X(0) + \frac{1}{2}f''(\hat{X}(t))d(X^c, X^c)(t)/X^2(0)
+ [f(\hat{X}(t)) - f(\hat{X}(t^-))]dN(t)
\]

\[
= \left[ f'(\hat{X}(t))\hat{X}(t)\mu^X(\chi(t)) + \frac{1}{2}f''(\hat{X}(t))\hat{X}^2(t)||\alpha^X(\chi(t))||^2 \right] dt
\]

\[
+ f'(\hat{X}(t))\hat{X}(t)\alpha^X(\chi(t))dW(t)
\]

\[
+ [f(\hat{X}(t^-)+Y_{\hat{X}(t^-)(\chi(t^-))}) - f(\hat{X}(t^-))]dN(t),
\]

where \( \{X^c(t)\}_{t\geq 0} \) is the continuous part of \( \{X(t)\}_{t\geq 0} \), and \( \{\langle X^c, X^c \rangle \}_{t\geq 0} \) is the predictable quadratic variation of \( \{X^c(t)\}_{t\geq 0} \).

For any fixed \( t \), denote \( K(u; t) = f(\hat{X}(t + u)) - f(\hat{X}(t)) \). Then,

\[
dK(u; t) = \left[ f'(\hat{X}(t + u))\hat{X}(t + u)\mu^X(\chi(t + u))
\right.
\]

\[
+ \frac{1}{2}f''(\hat{X}(t + u))\hat{X}^2(t + u)||\alpha^X(\chi(t + u))||^2 \right] du
\]

\[
+ f'(\hat{X}(t + u))\hat{X}(t + u)\alpha^X(\chi(t + u))dW(t + u)
\]

\[
+ [f(\hat{X}(t + u)+Y_{\hat{X}(t+u)(\chi((t + u))}) - f(\hat{X}(t + u))]dN(t + u).
\]

Here and elsewhere in the following proof, \( d \) indicates the differential with respect to the variable \( u \). Note that \( K(0; t) = 0 \), \( h(u; t) = \phi\left(\rho_{\nu X}(\chi(t))K(u; t)\right) \), and \( h(0; t) = 0 \). According to Itô’s lemma, we obtain

\[
dh(u; t) = \phi'\left(\rho_{\nu X}(\chi(t))K(u; t)\right)\rho_{\nu X}(\chi(t))dK^c(u; t)
\]

\[
+ \frac{1}{2}\phi''\left(\rho_{\nu X}(\chi(t))K(u; t)\right)\rho_{\nu X}^2(\chi(t))d\langle K^c, K^c \rangle(u; t) + [h(u; t) - h(u^-; t)]dN(t + u).
\]

Integrating Eq. (22) from 0 to \( \Delta t \) yields

\[
h(\Delta t; t) = \int_0^\Delta t \phi'\left(\rho_{\nu X}(\chi(t))K(u; t)\right)\rho_{\nu X}(\chi(t))dK^c(u; t)
\]

\[
+ \frac{1}{2}\int_0^\Delta t \phi''\left(\rho_{\nu X}(\chi(t))K(u; t)\right)\rho_{\nu X}^2(\chi(t))d\langle K^c, K^c \rangle(u; t)
\]

\[
+ \int_0^\Delta t [h(u; t) - h(u^-; t)]dN(t + u).
\]
Note that, in any small time interval \((t, t + \Delta t]\), the probability of observing a jump is \(\lambda(t, \chi(t)) \Delta t\), and the probability of observing a regime switch from state \(\chi(t)\) to state \(i\), \(i \neq \chi(t)\), equals \(q_{\chi(t), i} \Delta t\). Given all the information up to time \(t\),

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{0}^{\Delta t} \left[ h(u; t) - h(u-; t) \right] dN(t + u) = \phi \left( \rho v_X(\chi(t)) \left[ f(\hat{X}(t)) Y_{X}^{\chi(t)}(\chi(t)) - f(\hat{X}(t)) \right] \right) e(t),
\]

where \(\{e(t)\}_{t \geq 0}\) is a process defined by \(\frac{N(t + dt) - N(t)}{dt}\). Similar arguments can then be applied to other terms in (23). Finally, we have

\[
X'(t) = \lim_{\Delta t \to 0} \frac{h(\Delta t ; t)}{\Delta t} = \phi'(0) \rho v_X(\chi(t)) \left[ f'(\hat{X}(t)) \hat{X}(t) \mu^X(\chi(t)) + \frac{1}{2} f''(\hat{X}(t)) \hat{X}^2(t) ||\alpha^X(\chi(t))||^2 \right] \\
+ f'(\hat{X}(t)) \hat{X}(t) \alpha^X(\chi(t)) \cdot e(t) + \frac{1}{2} \phi''(0) \rho v_X^2(\chi(t)) \left[ f'(\hat{X}(t)) \hat{X}(t) \right] ||\alpha^X(\chi(t))||^2 \\
+ \phi \left( \rho v_X(\chi(t)) \left[ f(\hat{X}(t)) Y_{X}^{\chi(t)}(\chi(t)) - f(\hat{X}(t)) \right] \right) e(t),
\]

where \(\{e(t) = (\epsilon_1(t), \ldots, \epsilon_n(t))\}_{t \geq 0}\) is the Gaussian white noise process, a generalized derivative of \(\{W(t)\}_{t \geq 0}\) (see, for example Hida et al. (1988) [17]).

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