A CLUSTER REALIZATION OF $U_q(\mathfrak{sl}_n)$ FROM QUANTUM CHARACTER VARIETIES.

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Abstract. We construct an algebra embedding of the quantum group $U_q(\mathfrak{sl}_{n+1})$ into a quantum torus algebra $D_n$. The algebra $D_n$ arises as a quantum cluster chart on a certain quantum character variety associated to a marked punctured disk. We obtain a description of the coproduct of $U_q(\mathfrak{sl}_{n+1})$ in terms of a quantum character variety associated to the marked twice punctured disk, and express the action of the $R$-matrix in terms of a mapping class group element corresponding to the half-Dehn twist rotating one puncture about the other. As a consequence, we realize the algebra automorphism of $U_q(\mathfrak{sl}_{n+1})$ given by conjugation by the $R$-matrix as an explicit sequence of cluster mutations, and derive a refined factorization of the $R$-matrix into quantum dilogarithms of cluster monomials.

Introduction

In [Fad99], an intriguing realization of the quantum group $U_q(\mathfrak{sl}_2)$ and the Drinfeld double of its Borel subalgebra was presented in terms of a quantum torus algebra $D$. Explicitly, the algebra $D$ has generators $\{w_1, w_2, w_3, w_4\}$, with the relations

$$w_i w_{i+1} = q^{-2} w_{i+1} w_i, \quad \text{and} \quad w_i w_{i+2} = w_{i+2} w_i \quad (0.1)$$

where $i \in \mathbb{Z}/4\mathbb{Z}$. In terms of the standard generators $E, F, K, K'$ of the Drinfeld double (see Section 3 for the definitions), the embedding proposed in [Fad99] takes the form

$$E \mapsto i(w_1 + w_2), \quad K \mapsto qw_2 w_3,$$
$$F \mapsto i(w_3 + w_4), \quad K' \mapsto qw_4 w_1, \quad (0.2)$$

where $i = \sqrt{-1}$.

The embedding (0.2) has some striking properties. Firstly, as proposed in [Fad99], one can use the Weyl-type relations (0.1) to define a modular double of $U_q(\mathfrak{sl}_2)$ compatible with the regime $|q| = 1$. Next, the image of the quasi $R$-matrix under this embedding admits a remarkable factorization into the product of four quantum dilogarithms:

$$R = \Psi^q (w_1 \otimes w_3) \Psi^q (w_1 \otimes w_4) \Psi^q (w_2 \otimes w_3) \Psi^q (w_2 \otimes w_4). \quad (0.3)$$

These properties have been exploited in [PT99, PT01, BT03] to define and study a new continuous braided monoidal category of ‘principal series’ representations of $U_q(\mathfrak{sl}_2)$. 

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On the other hand, factorizations of the $U_q(\mathfrak{sl}_2)$ $R$-matrix of the form $$(0.3)$$ have also appeared in quantum Teichmüller theory. In [Kas01], the action of the $R$-matrix is identified, up to a simple permutation, with an element of the mapping class group of the twice punctured disc. The mapping class group element in question corresponds to the half-Dehn twist rotating one puncture about the other. After triangulating the punctured disc, this transformation can be decomposed into a sequence of four flips of the triangulation, as shown in Figure 8. One is thus led to interpret each dilogarithm in the factorization $(0.3)$ as corresponding to a flip of a triangulation. In [HI14], this observation was used to re-derive Kashaev’s knot invariant.

In this paper, we explain how to generalize Faddeev’s embedding $(0.2)$ to the case of the quantum group $U_q(\mathfrak{sl}_{n+1})$ using the language of quantum cluster algebras. The key to our construction is the quantum cluster structure associated to moduli spaces of $PGL_{n+1}$-local systems on a decorated, marked surface, see [FG06a]. Cluster charts on these varieties are obtained from an ideal triangulation of the surface by ‘gluing’ certain simpler cluster charts associated to each triangle. In the case of moduli spaces of $PGL_{n+1}$-local systems, a flip of a triangulation can be realized as sequences of $\binom{n+2}{3}$ cluster mutations.

Taking a particular cluster chart on the moduli space associated to a triangulation of the punctured disk (defined precisely in Section 2), we obtain by this gluing procedure a quiver and a corresponding quantum torus algebra $D_n$. Our first main result, Theorem 4.4, is to describe an explicit embedding of $U_q(\mathfrak{sl}_{n+1})$ into $D_n$. Our embedding has the property that each Chevalley generator $E_i, F_i$ of $U_q(\mathfrak{sl}_{n+1})$ is a cluster monomial in some cluster torus mutation equivalent to $D_n$. In the simplest case $n = 1$, our result reproduces Faddeev’s realization $(0.2)$ of $U_q(\mathfrak{sl}_2)$ in terms of the quantum torus $D_1$ associated to the cyclic quiver with four nodes (see Figure 4). Moreover, our cluster embedding turns out to be compatible with the action of $U_q(\mathfrak{sl}_{n+1})$ in its positive representations [FI13], which are higher rank generalizations of the principal series representations of $U_q(\mathfrak{sl}_2)$.

Next, we turn to the problem of describing the coproduct and $R$-matrix of $U_q(\mathfrak{sl}_{n+1})$ in terms of our embedding. We formulate this description in terms of a quantum cluster chart $Z_n$ from another quantum cluster variety, this time corresponding to a quiver built from a triangulation of the twice punctured disk. As we explain in Remark 6.4, the coproduct admits a simple description in terms of the cluster variables of $Z_n$.

Finally, we prove in Theorem 6.1 that the automorphism $P \circ \text{Ad}_R$ of $U_q(\mathfrak{sl}_{n+1})^\otimes 2$, given by conjugation by the $R$-matrix followed by the flip of tensor factors, restricts to $Z_n$ and coincides with a cluster transformation given by the composite of the half-Dehn twist and a certain permutation. In the course of the proof, we obtain (Theorem 7.4) a refined factorization of $R$ with $4\binom{n+2}{3}$ quantum dilogarithm factors, one for each mutation required to achieve the half-Dehn twist realized as a sequence of four flips. In the
case of $U_q(\mathfrak{sl}_2)$, when each flip can be achieved by a single cluster mutation, we again recover Faddeev’s factorization \((0.3)\).

The article is organized as follows. In Section 1, we recall some basic facts about quantum cluster algebras and the quantum dilogarithm function. Section 2 reviews the quantum character varieties as defined in [FG06a]. We also recall the procedure of quiver amalgamation and explain how a flip of the triangulation can be realized as a sequence of cluster transformations. In Section 3, we fix our notations and conventions regarding the quantum group $U_q(\mathfrak{sl}_{n+1})$. We state our first main result in Section 4: an explicit embedding of $U_q(\mathfrak{sl}_{n+1})$ into a quantum cluster torus $\mathcal{D}_n$ built from the punctured disk. Section 5 recalls the combinatorial description of the half-Dehn twist on a twice punctured disk. In Section 6, we prove our main result on the cluster nature of the $R$-matrix, while its refined factorization appears in Section 7. We conclude the paper by comparing our results to those of Faddeev in Section 8.

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1. QUANTUM CLUSTER $X$-TORI

In this section we recall a few basic facts about cluster $X$-tori and their quantization following [FG06a]. We shall need only skew-symmetric exchange matrices, and we incorporate this in the definition of a cluster seed. A seed $i$ is triple $(I, I_0, \varepsilon)$ where $I$ is a finite set, $I_0 \subset I$ is a subset and $\varepsilon = (\varepsilon_{ij})_{i,j \in I}$ is a skew-symmetric $\mathbb{Q}$-valued matrix, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$. To a seed $i$ we associate an algebraic torus $X_i = (\mathbb{C}^\times)^{|I|}$, equipped with a set of coordinates $\{X_1, \ldots, X_{|I|}\}$ and a Poisson structure defined by

$$\{X_i, X_j\} = 2\varepsilon_{ij}X_iX_j, \quad i, j \in I.$$ 

We refer to the torus $X_i$ as the cluster torus and to the matrix $\varepsilon$ as the exchange matrix. The coordinates $X_i$ are called cluster variables and they are said to be frozen if $i \in I_0$.

Given a pair of seeds $i = (I, I_0, \varepsilon), i' = (I', I'_0, \varepsilon')$, and an element $k \in I \setminus I_0$ we say that an isomorphism $\mu_k: I \to I'$ is a cluster mutation in direction $k$.
if $\mu_k(I_0) = I_0'$ and

$$\varepsilon'_{\mu_k(i),\mu_k(j)} = \begin{cases} 
-\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\
\varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \\
\varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0.
\end{cases} \quad (1.1)$$

A mutation $\mu_k$ induces an isomorphism of cluster tori $\mu^*_k : X_i \to X_i'$ as follows:

$$\mu^*_k X_{\mu_k(i)} = \begin{cases} X_{k}^{-1} & \text{if } i = k, \\
X_i \left(1 + X_k^{-\text{sgn}(\varepsilon_{ki})}\right)^{-\varepsilon_{ki}} & \text{if } i \neq k.
\end{cases}$$

Note that the data of a cluster seed can be conveniently encoded by a quiver with vertices $\{v_i\}$ labelled by elements of the set $I$ and with adjacency matrix $\varepsilon$. The arrows $v_i \to v_j$ between a pair of frozen variables are considered to be weighted by $\varepsilon_{ij}$. Then the mutation $\mu_k$ of the corresponding quiver can be performed in three steps:

1. reverse all the arrows incident to the vertex $k$;
2. for each pair of arrows $k \to i$ and $j \to k$ draw an arrow $i \to j$;
3. delete pairs of arrows $i \to j$ and $j \to i$ going in the opposite directions.

The algebra of functions $O(X_i)$ admits a quantization $X^q_i$ called the quantum torus algebra associated to the seed $i$. It is an associative algebra over $\mathbb{C}(q)$ defined by generators $X^\pm_i$, $i \in I$ subject to relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i.$$ 

The cluster mutation in the direction $k$ induces an automorphism $\mu^q_k$ of $X^q_i$ called the quantum cluster mutation, defined by

$$\mu^q_k(X_i) = \begin{cases} X_k^{-1}, & \text{if } i = k, \\
X_i \prod_{r=1}^{\varepsilon_{ki}} \left(1 + q^{2r-1}X_k^{-1}\right)^{-\varepsilon_{ki}} , & \text{if } i \neq k \text{ and } \varepsilon_{ki} \geq 0, \\
X_i \prod_{r=1}^{-\varepsilon_{ki}} \left(1 + q^{2r-1}X_k\right), & \text{if } i \neq k \text{ and } \varepsilon_{ki} \leq 0.
\end{cases}$$

The quantum cluster mutation $\mu^q_k$ can be written as a composition of two homomorphisms, namely

$$\mu^q_k = \mu^q_k \circ \mu^q_k$$

where $\mu^q_k$ is a monomial transformation defined by

$$X_i \mapsto \begin{cases} X_k^{-1}, & \text{if } i = k, \\
q^{\varepsilon_{ik}\varepsilon_{ki}} X_i X_k^{\varepsilon_{ki}} , & \text{if } i \neq k \text{ and } \varepsilon_{ki} \geq 0, \\
X_i, & \text{if } i \neq k \text{ and } \varepsilon_{ki} \leq 0.
\end{cases}$$

and

$$\mu^q_k = \text{Ad}_{\Psi^q(X_k)}$$
is a conjugation by the quantum dilogarithm function
\[ \Psi^q(x) = \frac{1}{(1 + qx)(1 + q^2x)\ldots} \]
Mutation of the exchange matrix is incorporated into the monomial transformation \( \mu^q_k \). The following lemma will prove very useful.

**Lemma 1.1.** A sequence of mutations \( \mu^q_{i_k} \ldots \mu^q_{i_1} \) can be written as follows
\[ \mu^q_{i_k} \ldots \mu^q_{i_1} = \Phi_k \circ M_k \]
where
\[ \Phi_k = \text{Ad}_{\Psi^q(x)_{i_1}} \text{Ad}_{\Psi^q(\mu'_{i_1}(X)_{i_2})} \ldots \text{Ad}_{\Psi^q(\mu'_{i_{k-1}} \ldots \mu'_{i_1}(X)_{i_k})} \]
and
\[ M_k = \mu'_{i_k} \ldots \mu'_{i_2} \mu'_{i_1} \]

**Proof.** We shall prove the lemma by induction. Assume the statement holds for some \( k = r - 1 \). Then
\[ \mu^q_{i_r} \ldots \mu^q_{i_1} = \text{Ad}_{\Psi^q(\Phi_{r-1}(M_{r-1}(X)_{i_r}))} \mu'_{i_r} \Phi_{r-1} M_{r-1} \]
Now the proof follows from the fact that the homomorphisms \( \mu'_{i_r} \) and \( \Phi_{r-1} \) commute and the following relation:
\[ \text{Ad}_{\Psi^q(\Phi_{r-1}(M_{r-1}(X)_{i_r}))} \Phi_{r-1} = \Phi_{r-1} \text{Ad}_{\Psi^q(M_{r-1}(X)_{i_r})} = \Phi_r. \]

We conclude this section with the two properties of the quantum dilogarithm which we will use liberally throughout the paper. For any \( u \) and \( v \) such that \( uv = q^{-2}vu \) we have
\[ \Psi^q(u)\Psi^q(v) = \Psi^q(uv) \quad (1.2) \]
\[ \Psi^q(v)\Psi^q(u) = \Psi^q(u)\Psi^q(qvu)\Psi^q(v) \quad (1.3) \]
The first equality is nothing but a \( q \)-analogue of the addition law for exponentials, while the second one is known as the pentagon identity.

### 2. Quantum character varieties

We now recall some elements of the theory of quantum character varieties as defined in \[\text{[FG06a]}\]. Let \( \hat{S} \) be a decorated surface — that is, a topological surface \( S \) with boundary \( \partial S \), equipped with a finite collection of marked points \( x_1, \ldots, x_r \in \partial S \) and punctures \( p_1, \ldots, p_s \). In \[\text{[FG06a]}\], the moduli space \( X_{\hat{S},PGL_m} \) of \( PGL_m \)-local systems on \( S \) with reductions to Borel subgroups at each marked point \( x_i \) and each puncture \( p_i \), was defined and shown to admit the structure of a cluster \( \mathcal{X} \)-variety. In particular, suppose that \( T \) is an ideal triangulation of \( S \): recall that this means that all vertices of \( T \) are at either marked points or punctures. Then it was shown in \[\text{[FG06a]}\] that for each such ideal triangulation, one can produce a cluster \( \mathcal{X} \)-chart on
Moreover, the Poisson algebra of functions on such a chart admits a canonical quantization, whose construction we shall now recall.

The first step is to describe the quantum cluster $\mathcal{X}$-chart associated to a single triangle. To do this, consider a triangle $ABC$ given by the equation $x + y + z = m$, $x, y, z \geq 0$ and intersect it with lines $x = p$, $y = p$, and $z = p$ for all $0 < p < m$, $p \in \mathbb{Z}$. The resulting picture is called the $m$-triangulation of the triangle $ABC$. Let us now color the triangles of the $m$-triangulation in black and white, as in Figure 1 so that triangles adjacent to vertices $A$, $B$, or $C$ are black, and two triangles sharing an edge are of different color. We shall also orient the edges of white triangles counterclockwise. Finally, we connect the vertices of the $m$-triangulation lying on the same side of the triangle $ABC$ by dashed arrows in the clockwise direction. The resulting graph is shown in Figure 1. Note that the vertices on the boundary of $ABC$ are depicted by squares. Throughout the text we will use square vertices for frozen variables. All dashed arrows will be of weight $\frac{1}{2}$, that is a dashed arrow $v_i \rightarrow v_j$ denotes the commutation relation $X_iX_j = q^{-1}X_jX_i$.

![Figure 1. Cluster $\mathcal{X}$-coordinates on the configuration space of 3 flags and 3 lines.](image)

Now, let us recall the procedure of *amalgamating* two quivers by a subset of frozen variables, following [FG06b]. In simple words, amalgamation is nothing but the gluing of two quivers by a number of frozen vertices. More formally, let $Q_1$, $Q_2$ be a pair of quivers, and $I_1$, $I_2$ be certain subsets of frozen variables in $Q_1$, $Q_2$ respectively. Assuming there exists a bijection $\phi: I_1 \rightarrow I_2$ we can amalgamate quivers $Q_1$ and $Q_2$ by the subsets $I_1$, $I_2$ along $\phi$. The result is a new quiver $Q$ constructed in the following two steps:

1. for any $i \in I_1$ identify vertices $v_i \in Q_1$ and $v_{\phi(i)} \in Q_2$ in the union $Q_1 \sqcup Q_2$;
2. for any pair $i, j \in I_1$ with an arrow $v_i \rightarrow v_j$ in $Q_1$ labelled by $\varepsilon_{ij}$ and an arrow $v_{\phi(i)} \rightarrow v_{\phi(j)}$ in $Q_2$ labelled by $\varepsilon_{\phi(i),\phi(j)}$, label the arrow between corresponding vertices in $Q$ by $\varepsilon_{ij} + \varepsilon_{\phi(i),\phi(j)}$.
Amalgamation of a pair of quivers $Q_1, Q_2$ into a quiver $Q$ induces an embedding $\mathcal{X} \to \mathcal{X}_1 \otimes \mathcal{X}_2$ of the corresponding cluster $\mathcal{X}$-tori:

$$
X_i \mapsto \begin{cases} 
X_i \otimes 1, & \text{if } i \in Q_1 \setminus I_1, \\
1 \otimes X_i, & \text{if } i \in Q_2 \setminus I_2, \\
X_i \otimes X_{\phi(i)}, & \text{otherwise.}
\end{cases}
$$

An example of amalgamation is shown in Figure 2. There, the left quiver is obtained by amalgamating a triangle $ABC$ from Figure 1 with a similar triangle along the side $BC$ (or more precisely, along frozen vertices 10, 11, and 12 on the edge $BC$). Another example is shown in Figure 7 where a triangle $ABC$ is now amalgamated by 2 sides. Finally, the process of amalgamation is best shown in Figure 6.

As explained in [FG06a], in order to construct the cluster $\mathcal{X}$-coordinate chart on $\mathcal{X}^\ast_{\tilde{S}, PGL_m}$ corresponding to an ideal triangulation $T$ of $\tilde{S}$, one performs the following procedure:

1. $m$-triangulate each of the ideal triangles in $T$;
2. for any pair of ideal triangles in $T$ sharing an edge, amalgamate the corresponding pair of quivers by this edge.

In general, different ideal triangulations of an $\tilde{S}$ result in different quivers, and hence different cluster $\mathcal{X}$-tori. However, any triangulation can be transformed into any other by a sequence of flips that replace one diagonal in an ideal 4-gon with the other one. Each flip corresponds to the following sequence of cluster mutations that we shall recall on the example shown in Figure 2. There, a flip is obtained in three steps. First, mutate at vertices 10, 11, 12, second, mutate at vertices 7, 8, 14, 15, and third, mutate at vertices 4, 11, 18. Note, that the order of mutations within one step does not matter. In general, a flip in an $m$-triangulated 4-gon consists of $m - 1$ steps. On the $i$-th step, one should do the following. First, inscribe an $i$-by-$(m - i)$ rectangle in the 4-gon, such that vertices of the rectangle coincide with boundary vertices of the $m$-triangulation and the side of the rectangle of length $m - i$ goes along the diagonal of a 4-gon. Second, divide the rectangle into $i(m - i)$ squares and mutate at the center of each square. As in the example, the order of mutations within a single step does not matter.

3. Quantum groups

In what follows, we consider the Lie algebra $\mathfrak{sl}_{n+1} = \mathfrak{sl}_{n+1}(\mathbb{C})$ equipped with a pair of opposite Borel subalgebras $\mathfrak{b}_+ \subset \mathfrak{b}_-. \quad \mathfrak{b}_+ \cap \mathfrak{b}_- = 0$. The corresponding root system $\Delta$ is equipped with a polarization $\Delta = \Delta_+ \sqcup \Delta_-$, consistent with the choice of Borel subalgebras $\mathfrak{b}_\pm$, and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\} \subset \Delta_+$. We denote by $(\cdot, \cdot)$ the unique symmetric bilinear form on $\mathfrak{h}^*$ invariant under the Weyl group $W$, such that $(\alpha, \alpha) = 2$ for all roots $\alpha \in \Delta$. Entries of the Cartan matrix are denoted $a_{ij} = (\alpha_i, \alpha_j)$. 
Let \( q \) be a formal parameter, and consider an associative \( \mathbb{C}(q) \)-algebra \( \mathcal{D}_n \) generated by elements
\[
\{ E_i, F_i, K_i, K'_i \mid i = 1, \ldots, n \},
\]
subject to the relations
\[
\begin{align*}
K_i E_j &= q^{a_{ij}} E_j K_i, & K'_i E_j &= q^{-a_{ij}} E_j K'_i, & K_i K_j &= K_j K_i, \\
K_i F_j &= q^{-a_{ij}} F_j K_i, & K'_i F_j &= q^{a_{ij}} F_j K'_i, & K_i K'_j &= K'_j K_i,
\end{align*}
\] (3.1)
the relation
\[
[E_i, F_j] = \delta_{ij} \left( q - q^{-1} \right) \left( K_i - K'_i \right),
\] (3.2)
and the quantum Serre relations
\[
\begin{align*}
E_i^2 E_{i \pm 1} - (q + q^{-1}) E_i E_{i \pm 1} E_i + E_{i \pm 1} E_i^2 &= 0, \\
F_i^2 F_{i \pm 1} - (q + q^{-1}) F_i F_{i \pm 1} F_i + F_{i \pm 1} F_i^2 &= 0, \\
[E_i, E_j] = [F_i, F_j] &= 0 \quad \text{if} \quad |i - j| > 1.
\end{align*}
\] (3.3)

The algebra \( \mathcal{D}_n \) is a Hopf algebra, with the comultiplication
\[
\begin{align*}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta(K_i) &= K_i \otimes K_i, \\
\Delta(F_i) &= F_i \otimes K'_i + 1 \otimes F_i, & \Delta(K'_i) &= K'_i \otimes K'_i,
\end{align*}
\]
the antipode
\[
\begin{align*}
S(E_i) &= -K_i^{-1} E_i, & S(K_i) &= K_i^{-1}, \\
S(F_i) &= -F_i K_i, & S(K'_i) &= (K'_i)^{-1},
\end{align*}
\]
and the counit
\[
\epsilon(K_i) = \epsilon(K'_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.
\]

The quantum group \( U_q(\mathfrak{sl}_{n+1}) \) is defined as the quotient
\[
U_q(\mathfrak{sl}_{n+1}) = \mathcal{D}_n / \langle K_i K'_i = 1 \mid i = 1, \ldots, n \rangle.
\]

Note that the quantum group \( U_q(\mathfrak{sl}_{n+1}) \) inherits a well-defined Hopf algebra structure from \( \mathcal{D}_n \). The subalgebra \( U_q(\mathfrak{b}) \subset \mathcal{D}_n \) generated by all \( K_i, E_i \) is a
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Hopf subalgebra in \( \mathfrak{D}_n \). The algebra \( U_q(\mathfrak{b}) \) is isomorphic to its image under the projection onto \( U_q(\mathfrak{sl}_{n+1}) \) and is called the quantum Borel subalgebra of \( U_q(\mathfrak{sl}_{n+1}) \). Note that \( \mathfrak{D}_n \) is nothing but the Drinfeld double of \( U_q(\mathfrak{b}) \).

Let us fix a normal ordering \( \prec \) on \( \Delta^+ \), that is a total ordering such that \( \alpha \prec \alpha + \beta \prec \beta \) for any \( \alpha, \beta \in \Delta^+ \). We set \( E_{\alpha_i} = E_i \), \( F_{\alpha_i} = F_i \), and define inductively

\[
E_{\alpha + \beta} = \frac{E_\alpha E_\beta - q^{-\langle \alpha, \beta \rangle} E_\beta E_\alpha}{q - q^{-1}}, \tag{3.4}
\]

\[
F_{\alpha + \beta} = \frac{F_\beta F_\alpha - q^{\langle \alpha, \beta \rangle} F_\alpha F_\beta}{q - q^{-1}}. \tag{3.5}
\]

Then the set of all normally ordered monomials in \( K_\alpha, K'_\alpha, E_\alpha, \) and \( F_\alpha \) for \( \alpha \in \Delta^+ \) forms a Poincaré-Birkhoff-Witt (PBW) basis for \( \mathfrak{D}_n \) as a \( \mathbb{C}(q) \)-module. In what follows, we denote

\[
E_{ij} = E_{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j} \quad \text{and} \quad F_{ij} = F_{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j}.
\]

Finally, let us introduce for future reference the automorphism \( \theta \) of the Dynkin diagram of \( U_q(\mathfrak{sl}_{n+1}) \) defined by

\[
\theta(i) = n + 1 - i, \quad 1 \leq i \leq n. \tag{3.6}
\]

4. AN EMBEDDING OF \( U_q(\mathfrak{sl}_{n+1}) \)

Let us now explain how to embed \( U_q(\mathfrak{sl}_{n+1}) \) into a quantum cluster \( \mathcal{X} \)-chart on the quantum character variety of decorated \( PGL_{n+1} \)-local systems on an disk \( \hat{S} \) with a single puncture \( p \), and with two marked points \( x_1, x_2 \) on its boundary.

We consider the ideal triangulation of \( \hat{S} \) in which we take the pair of triangles from Figure 1 and amalgamate them by two sides as in Figure 6. The resulting quiver is shown on Figure 3. Note that the vertices in the central column used to be frozen before amalgamation. We shall refer to this quiver as the \( \mathcal{D}_n \)-quiver and denote the corresponding quantum torus algebra by \( \mathfrak{D}_n \). The \( \mathcal{D}_n \)-quivers for \( n = 1, 2, \) and 3 are shown on Figures 4, 5, and 7 respectively.

Let us explain our convention for labelling the vertices of the \( \mathcal{D}_n \)-quiver. We denote frozen vertices in the left column by \( V_{i,-i} \) with \( i = 1, \ldots, n \) counting South to North. Now, choose a frozen vertex \( V_{i,-i} \) and follow the arrows in the South-East direction until you hit one of the vertices in the central column. Each vertex along the way is labelled by \( V_{i,r}, r = -i, \ldots, 0 \). Then, start from the central vertex \( V_{i,0} \) and follow arrows in the North-East direction labelling vertices \( V_{i,r}, r = 0, \ldots, i \), on your way until you hit a frozen vertex in the right column, which receives the label \( V_{i,i} \). This way we label all the vertices except for the upper half of those in the central column. Now, let us rotate the \( \mathcal{D}_n \)-quiver by 180°, and label the image of the vertex \( V_{i,r} \) by \( \Lambda_{i,r} \). Now, we have labelled every vertex twice by some \( \Lambda \) and some \( \Lambda \) except for those in the central column. This way to label
vertices, although redundant, will prove very convenient in the sequel. The following relation is easy to verify:

\[ V_{i,\pm r} = \Lambda_{\theta(\pm r), \pm \theta(i)}, \quad 1 \leq r < i \leq n. \]

In the above formula, \( \theta \) denotes the diagram automorphism defined in (3.6). Finally, we refer to the subset of vertices \( \{ V_{i,r} \mid -i \leq r < i \} \) as the \( V_i \)-path. Similarly, the \( \Lambda_i \)-path is \( \{ \Lambda_{i,r} \mid -i \leq r < i \} \).

**Example 4.1.** Let us refer to the \( i \)-th vertex in Figure 3 by \( X_i \). Then the labelling suggested above is as follows:

\[
\begin{align*}
V_{1,-1} &= X_1, & V_{1,0} &= X_2, & V_{1,1} &= X_3, \\
A_{1,-1} &= X_3, & A_{1,0} &= X_4, & A_{1,1} &= X_1.
\end{align*}
\]
Example 4.2. Similarly, we refer to the $i$-th vertex in Figure 5 by $X_i$. Then, one has

$$
V_{1,-1} = X_1, \quad V_{1,0} = X_2, \quad V_{1,1} = X_3, \quad V_{2,-2} = X_4,
$$
$$
V_{2,-1} = X_5, \quad V_{2,0} = X_6, \quad V_{2,1} = X_7, \quad V_{2,2} = X_8,
$$
$$
\Lambda_{1,-1} = X_8, \quad \Lambda_{1,0} = X_9, \quad \Lambda_{1,1} = X_4, \quad \Lambda_{2,-2} = X_3,
$$
$$
\Lambda_{2,-1} = X_7, \quad \Lambda_{2,0} = X_{10}, \quad \Lambda_{2,1} = X_5, \quad \Lambda_{2,2} = X_1.
$$

Remark 4.3. As shown in [SS15], for any semisimple Lie algebra $\mathfrak{g}$ the algebra $U_q(\mathfrak{g})$ can be embedded into the quantized algebra of global functions on the Grothendieck-Springer resolution $G \times_B B$, where $B \subset G$ is a fixed Borel subgroup in $G$. On the other hand, the variety $G \times_B B$ is isomorphic to the moduli space of $G$-local systems on the punctured disc, equipped with reduction to a Borel subgroup at the puncture, as well as a trivialization at one marked point on the boundary. Classically, this moduli space is birational to $X_{\tilde{S},G}$; and it would be interesting to understand the precise relation between the corresponding quantizations.

We now come to the first main result of the paper.

Theorem 4.4. There is an embedding of algebras $\iota: \mathcal{D}_n \to \mathcal{D}_n$ defined by the following assignment for $i = 1, \ldots, n$:

$$
\hat{E}_i \mapsto i \sum_{r=-i}^{i-1} q^{i+r} V_{i,-r} V_{i,1-i} \cdots V_{i,r}, \quad (4.1)
$$
$$
K_i \mapsto q^{2i} V_{i,-i} V_{i,1-i} \cdots V_{i,i}, \quad (4.2)
$$
$$
\hat{F}_{\iota(i)} \mapsto i \sum_{r=-i}^{i-1} q^{i+r} \Lambda_{i,-r} \Lambda_{i,1-i} \cdots \Lambda_{i,r}, \quad (4.3)
$$
$$
K'_{\iota(i)} \mapsto q^{2i} \Lambda_{i,-i} \Lambda_{i,1-i} \cdots \Lambda_{i,i}. \quad (4.4)
$$

Remark 4.5. The algebra embedding (4.1)–(4.4) turns out to be equivalent to the homomorphism from $U_q(\mathfrak{sl}_n)$ into an algebra of difference operators used to construct the positive representations introduced in [FI13]. We thank I. Ip for pointing this out to us.

Remark 4.6. Formulas (4.1) and (4.3) can be rewritten as follows:

$$
E_i \mapsto i \text{Ad}_{\Psi(V_{i,i-1})} \cdots \text{Ad}_{\Psi(V_{i,1-i})} V_{i,-i},
$$
$$
F_{\iota(i)} \mapsto i \text{Ad}_{\Psi(\Lambda_{i,i-1})} \cdots \text{Ad}_{\Psi(\Lambda_{i,1-i})} \Lambda_{i,-i}.
$$

Note, that the right hand side of the formula (4.1) coincides with the cluster $X$-variable corresponding to the vertex $V_{i,-i}$, in the cluster obtained from the initial one by consecutive application of mutations at variables $V_{i,r}$, where $r$ runs from $i-1$ to $1-i$. Similarly, the right hand side of the formula (4.3) coincides with the cluster $X$-variable for vertex $\Lambda_{i,-i}$ in the cluster obtained from the initial one by consecutive application of mutations at variables $\Lambda_{i,r}$, where $r$ runs from $i-1$ to $1-i$. 

Example 4.7. For \( n = 1 \), in the notations of Figure 4, the embedding \( \iota \) reads
\[
E \mapsto iX_1(1 + qX_2), \quad K \mapsto q^2X_1X_2X_3, \\
F \mapsto iX_3(1 + qX_4), \quad K' \mapsto q^2X_4X_3X_2.
\]

![Figure 4. \( D_1 \)-quiver.](image)

Example 4.8. For \( n = 2 \), in the notations of Figure 5, the embedding \( \iota \) reads
\[
E_1 \mapsto iX_1(1 + qX_2), \quad K_2 \mapsto q^4X_3X_4X_5X_6X_7, \\
E_2 \mapsto iX_4(1 + qX_5(1 + qX_6(1 + qX_7))), \quad K \mapsto q^2X_1X_2X_3, \\
F_1 \mapsto iX_3(1 + qX_7(1 + qX_{10}(1 + qX_5))), \quad K_1' \mapsto q^2X_8X_9X_4, \\
F_2 \mapsto iX_8(1 + qX_9), \quad K_1 \mapsto q^4X_7X_4X_{10}X_5X_1.
\]

![Figure 5. \( D_2 \)-quiver.](image)

The proof of Theorem 4.4 will follow from Propositions 4.9 and 4.10 stated below.

Proposition 4.9. The formulas (4.1) – (4.4) define a homomorphism of algebras.
Proof. In what follows we abuse notations and denote an element of the algebra $\mathcal{D}_n$ and its image under $\iota$ the same. For any $1 \leq i \leq n$ and $-i \leq r < i$, let us define

$$w^r_i = iq^{i+r}v_{i,-i} \cdots v_{i,r},$$
$$m^r_i = iq^{i+r}\lambda_{i,-i} \cdots \lambda_{i,r}.$$

Then, the formulas (4.1) – (4.4) can be rewritten as follows:

$$\hat{E}_i = w^{-i}_i + \cdots + w^{i-1}_i, \quad K_i = -qw^{-1}_i m^{-\theta(i)}_i,$$
$$\hat{F}_{\theta(i)} = m^{-i}_i + \cdots + m^{i-1}_i, \quad K'_{\theta(i)} = -qm^{-1}_i w^{-\theta(i)}_i.$$

It is immediate from inspecting the quiver that the relations (3.1) hold, as well as $[E_i, E_j] = [F_i, F_j] = 0$ for $|i - j| > 1$. To verify (3.2) it suffices to notice that $i < \theta(j)$ implies $w^r_i m^s_j = m^s_j w^r_i$, while

$$i = \theta(j) \implies w^r_i m^s_j = \begin{cases} q^2 m^s_j w^r_i & \text{if } r = -i, s = j - 1, \\ q^{-2} m^s_j w^r_i & \text{if } r = i - 1, s = -j, \\ m^s_j w^r_i & \text{otherwise}, \end{cases}$$
$$i > \theta(j) \implies w^r_i m^s_j = \begin{cases} q^2 m^s_j w^r_i & \text{if } r = \pm \theta(j), s = \mp \theta(i) - 1, \\ q^{-2} m^s_j w^r_i & \text{if } s = \pm \theta(i), r = \mp \theta(j) - 1, \\ m^s_j w^r_i & \text{otherwise}. \end{cases}$$

Let us now check the Serre relation

$$E^2_{i+1} E_i + E_i E^2_{i+1} = (q + q^{-1}) E_{i+1} E_i E_{i+1}.$$

Suppose $-i \leq t \leq i - 1$ and $-i - 1 \leq r \leq i$. We write

$$t \lessdot r \quad \text{if} \quad w^r_{i+1} w^t_i = q^{-1} w^t_i w^r_{i+1},$$
$$t \gtrdot r \quad \text{if} \quad w^r_{i+1} w^t_i = qw^t_i w^r_{i+1}.$$

It is easy to verify that

$$t \lessdot r \iff \begin{cases} t \leq r & \text{if } r < 0, \\ t < r & \text{if } r \geq 0 \end{cases} \quad \text{and} \quad t \gtrdot r \iff \begin{cases} t > r & \text{if } r < 0, \\ t \geq r & \text{if } r \geq 0. \end{cases}$$
We can now express
\[ E_{i+1}^2 E_i = \sum_{r,s,t} w_{i+1}^r w_i^s w_i^t \]
\[ = \sum_{t>r, t>s} w_{i+1}^r w_i^s w_i^t + \sum_{t>r, t<s} w_{i+1}^r w_i^s w_i^t \]
\[ + \sum_{t<r, t>s} w_{i+1}^r w_i^s w_i^t + \sum_{t<r, t<s} w_{i+1}^r w_i^s w_i^t \]
\[ = q \sum_{t>r, t>s} w_{i+1}^r w_i^t w_i^{s+1} + q^{-1} \sum_{t>r, t<s} w_{i+1}^r w_i^t w_i^{s+1} \]
\[ + q \sum_{t<r, t>s} w_{i+1}^r w_i^t w_i^{s+1} + q^{-1} \sum_{t<r, t<s} w_{i+1}^r w_i^t w_i^{s+1} \]

Analogously, we have
\[ E_i E_{i+1}^2 = q \sum_{t<r, t>s} w_{i+1}^r w_i^t w_i^{s+1} + q^{-1} \sum_{t>r, t<s} w_{i+1}^r w_i^t w_i^{s+1} \]
\[ + q \sum_{t<r, t>s} w_{i+1}^r w_i^t w_i^{s+1} + q^{-1} \sum_{t<r, t<s} w_{i+1}^r w_i^t w_i^{s+1} \]

Observe that if \( t > r \) and \( t < s \), then one necessarily has \( r < s \), which in turn implies \( w_{i+1}^r w_i^t w_i^{s+1} = q^{-2} w_{i+1}^{s+1} w_i^t w_i^{r+1} \). Similarly, if \( t < r \) and \( t > s \), it follows that \( r > s \) and \( w_{i+1}^r w_i^t w_i^{s+1} = q^2 w_{i+1}^t w_i^r w_i^{s+1} \). Hence
\[ \sum_{t>r, t<s} w_{i+1}^r w_i^t w_i^{s+1} = \sum_{t>r, t<s} w_{i+1}^r w_i^t w_i^r, \]
\[ \sum_{t<r, t>s} w_{i+1}^r w_i^t w_i^{s+1} = \sum_{t<r, t>s} w_{i+1}^r w_i^t w_i^r. \]

It therefore follows that
\[ E_{i+1}^2 E_i + E_i E_{i+1}^2 \]
\[ = (q - q^{-1}) \left( \sum_{t>r, t<s} + \sum_{t>r, t<s} + \sum_{t<r, t>s} + \sum_{t<r, t<s} \right) w_{i+1}^r w_i^t w_i^{s+1} \]
\[ = E_{i+1} E_i E_i. \]

The other nontrivial Serre relations are proved in an identical fashion. □

**Proposition 4.10.** The homomorphism \( \iota : \mathfrak{D}_n \rightarrow \mathcal{D}_n \) is injective.

**Proof.** It will be convenient to choose a different PBW basis of \( \mathfrak{D}_n \) from the one we considered in Section 3. Namely, for any simple root \( \alpha \) we set \( F'_\alpha = F_\alpha \), then define inductively
\[ F'_{\alpha + \beta} = \frac{F'_\alpha (q^{-1}) F'_\beta - q^{-1} (q, \beta) F'_\alpha F'_\beta}{q - q^{-1}}. \]
By the PBW theorem, the set $\text{Mon}_{PBW}$ of all normally ordered monomials in $K_\alpha, K'_\alpha, E_\alpha$, and $F'_\alpha$, $\alpha \in \Delta_+$, forms a basis for $\mathcal{D}_n$ over $\mathbb{C}(q)$. Let us now fix a degree-lexicographic order on the set of all monomials in the quantum torus $\mathcal{D}_n$, taken with respect to any total order on the generators $\{X_i\}$. To establish injectivity of $\iota$, it will suffice to show that there are no two PBW monomials $m, m'$ in $\text{Mon}_{PBW}$, such that $\iota(m)$ and $\iota(m')$ have the same leading term with respect to our chosen monomial order for $\mathcal{D}_n$. Indeed, if this is true, our monomial order induces a total order on $\text{Mon}_{PBW}$ with respect to which the map $\iota$ becomes triangular. In fact, given a monomial $\vec{X} \in \mathcal{D}_n$ that arises as the leading term of some PBW monomial, one can reconstruct the unique PBW monomial $m_{\vec{X}}$ such that the leading term of $\iota(m_{\vec{X}})$ is $\vec{X}$ as follows. In the cluster monomial $\vec{X}$, let $n_{ij}, s_{ij}, e_{ij},$ and $w_{ij}$ denote respectively the degrees of the cluster variables corresponding to North, South, East, and West nodes of the rhombus labelled by $ij$ in the right triangle in Figure 6. Let us also declare $w_{1n} = 0$. Then the degree of $E_{ij}$ in $m_{\vec{X}}$ is equal to $n_{ij} + s_{ij} - e_{ij} - w_{ij}$ and the degree of $K_i$ is equal to $e_{ii} - n_{in}$.

![Figure 6. A pair of triangles amalgamated by 2 sides.](image)

Now, let $n_{ij}, s_{ij}, e_{ij},$ and $w_{ij}$ denote the degrees in $m_{\vec{X}}$ of the cluster variables corresponding the North, South, East, and West nodes of the corresponding rhombus in the left triangle in Figure 6 where we set $e_{1n} = 0$. Then the degree of $F'_{\theta(i)\theta(j)}$ equals $n_{ij} + s_{ij} - e_{ij} - w_{ij}$ in the left triangle where we set $e_{1n} = 0$ and the degree of $K'_{\theta(i)}$ equals $w_{i,i} - s_{i,n}$.

**Corollary 4.11.** The homomorphism $\iota$ induces an embedding of the quantum group $U_q(\mathfrak{sl}_{n+1})$ into the quotient of the algebra $\mathcal{D}_n$ by relations

$$q^{2n+2}V_{i,-i} \cdots V_{i,i} \cdot \Lambda_{\theta(i),-\theta(i)} \cdots \Lambda_{\theta(i),\theta(i)} = 1$$

for all $1 \leq i \leq n$. 

5. The Dehn twist on a twice punctured disk

In order to describe the coalgebra structure of $U_q(\mathfrak{sl}_{n+1})$, we will need to consider the moduli space $\mathcal{X}_{\hat{S}_2, PGL_{n+1}}$ of $PGL_{n+1}$-local systems on $\hat{S}_2$, a disk with two punctures $p_1, p_2$, and two marked points $x_1, x_2$ on its boundary. To obtain a quantum cluster chart on this moduli space, we consider the quiver corresponding to the $(n+1)$-triangulation of the left-most disk in Figure 8. Note that this quiver is formed by amalgamating two $D_n$-quivers by one column of frozen variables, see Figure 3. An example of two amalgamated $D_2$-quivers is shown in Figure 9, where one should disregard the gray arrows. We refer to the result of this amalgamation as the $Z_n$-quiver and denote the corresponding quantum torus algebra by $Z_n$.

Figure 8 shows four different ideal triangulations of a twice punctured disk with two marked points on the boundary; the arrows correspond to flips of ideal triangulations. Note that the right-most disk may be obtained from the left-most one by applying the half-Dehn twist rotating the left puncture clockwise about the right one. Hence this half-Dehn twist may be decomposed into a sequence of 4 flips. Let $Z'_n$ be the quiver obtained from the $(n+1)$-triangulation of the right-most disk. It is evident from inspecting the corresponding $(n+1)$-triangulations that there exists an isomorphism $\sigma$ between the $Z_n$- and the $Z'_n$-quivers that preserves all frozen variables. On the other hand, since there is no nontrivial automorphism of the $Z_n$-quiver fixing its frozen variables, we conclude that the isomorphism $\sigma$ is unique.

Let us now describe $\sigma$ explicitly. Recall that each $(n+1)$-triangulated triangle contains exactly $n$ solid oriented paths parallel to each of its sides. For example, in the 4-triangulation shown in Figure 1, one sees paths $1 \to 2$, $3 \to 4 \to 5$, and $6 \to 7 \to 8 \to 9$, parallel to the side $BC$. Now, consider the second disk in Figure 8: recall that the $(n+1)$-triangulation of the pair of triangles in the middle is shown in the right part of Figure 2. For $i = 1, \ldots, n$ we define the $i$-th permutation cycle to
• follow the \(i\)-th solid path parallel to the side \(a\) in the triangle \(\Delta_{abc}\) along the orientation,
• follow the \(i\)-th solid path parallel to the side \(d\) in the triangle \(\Delta_{bde}\) in the direction opposite to the orientation,
• follow the \(i\)-th solid path parallel to the side \(g\) in the triangle \(\Delta_{efg}\) along the orientation,
• follow the \(i\)-th solid path parallel to the side \(d\) in the triangle \(\Delta_{cdf}\) in the direction opposite to the orientation.

Now, the isomorphism \(\sigma\) is defined as follows: each vertex in the \(i\)-th permutation cycle is moved \(i\) steps along the cycle, frozen variables are left intact, the rest of the vertices are rotated by 180°. In Figure 9, the 2 cycles in the quiver \(\mathcal{Z}_2\) and the rotation of vertices 9 and 11 are shown by gray arrows; the action of \(\sigma\) reads

\[
\sigma = (2\ 7\ 15\ 17\ 13\ 4)\ (3\ 16\ 18)\ (8\ 10\ 12)\ (9\ 11),
\]

where the 2nd permutation cycle breaks into \((3\ 16\ 18)\ (8\ 10\ 12)\).

6. Cluster realization of the \(R\)-matrix

Recall that the universal \(R\)-matrix of the quantum group \(U_q(\mathfrak{sl}_{n+1})\) is an element

\[
\mathcal{R} \in U_q(\mathfrak{sl}_{n+1}) \otimes U_q(\mathfrak{sl}_{n+1})
\]

of a certain extension of its tensor square, and gives rise to a braiding on the category of finite dimensional \(U_q(\mathfrak{sl}_{n+1})\)-modules. The universal \(R\)-matrix admits decomposition

\[
\mathcal{R} = \bar{\mathcal{R}}\mathcal{K}.
\]

where

\[
\mathcal{K} = q^{\sum_{i,j} c_{ij} H_i \otimes H_j},
\]

\((c_{ij})\) is the inverse of the Cartan matrix, and \(H, H'\) are defined from the relations

\[
K = q^{H} \quad \text{and} \quad K' = q^{H'}.
\]

The tensor \(\bar{\mathcal{R}}\) is called the quasi \(R\)-matrix and is given by the formula

\[
\bar{\mathcal{R}} = \prod_{\alpha \in \Delta^+} \Psi^q (E_{\alpha} \otimes F_{\alpha}),
\]

(6.1)

\[\text{Figure 8. The half Dehn twist as a sequence of 4 flips.}\]
where the product is ordered consistently with the previously chosen normal ordering $\prec$ on $\Delta_+$. Let $\text{Ad}_K$ and $\text{Ad}_{\bar{R}}$ denote the automorphisms of $\mathcal{D}_n \otimes \mathcal{D}_n$ that conjugate by $K$ and $\bar{R}$ respectively. It is clear that both $\text{Ad}_K$ and $\text{Ad}_{\bar{R}}$ extend to automorphisms of $\mathcal{D}_n \otimes \mathcal{D}_n$ defined in the same way. We write $P$ for the automorphism of $\mathcal{D}_n \otimes \mathcal{D}_n$ permuting the tensor factors:

$$P(X \otimes Y) = Y \otimes X.$$ 

Recall the isomorphism of quivers described in the previous section. It defines a permutation of cluster variables $X_i \mapsto X_{\sigma(i)}$ which we also denote by $\sigma$ with a slight abuse of notation. Note that each of the 4 flips shown in Figure 8 corresponds to a sequence of $\binom{n+2}{3}$ cluster mutations, as explained at the end of Section 2. Let

$$N = 4 \cdot \binom{n+2}{3}$$

and $\mu_N \ldots \mu_1$ be the sequence of quantum cluster mutations constituting the half-Dehn twist. Now we are ready to formulate the next main result of the paper.

**Theorem 6.1.** The composition

$$P \circ \text{Ad}_{\bar{R}} : \mathcal{D}_n \otimes \mathcal{D}_n \rightarrow \mathcal{D}_n \otimes \mathcal{D}_n$$

restricts to the subalgebra $\mathcal{Z}_n$. Moreover, the following automorphisms of $\mathcal{Z}_n$ coincide:

$$P \circ \text{Ad}_{\bar{R}} = \mu_N \ldots \mu_1 \circ \sigma,$$

where the sequence of quantum cluster mutations $\mu_N \ldots \mu_1$ constitutes the half Dehn twist.
Proof. By Lemma 1.1 we have

\[ \mu_N \ldots \mu_1 = \Phi_N \circ M_N, \]

where \( M_N \) is a monomial transformation, and \( \Phi_N \) is a conjugation by a sequence of \( N \) quantum dilogarithms. The result of the theorem then follows from Propositions 6.2 and 6.3 below.

\[ \square \]

Proposition 6.2. The following automorphisms of \( Z_n \) coincide:

\[ P \circ \text{Ad}_K = M_N \circ \sigma. \]

Proof. We define the \( \Lambda V_i \)-path in the \( Z_n \)-quiver as the concatenation of the \( \Lambda \theta(i) \)-path in the left \( D_n \)-quiver with the \( V_i \)-path in the right \( D_n \)-quiver. For example, in the notations of Figure 9, the \( \Lambda V_1 \)-path consists of vertices 1, 7, 16, 9, 3, 4, 5. Each mutation from the sequence \( \mu_N \ldots \mu_1 \) happens at a vertex that belongs to a certain \( \Lambda V_i \)-path, has exactly two outgoing edges within this path, and has exactly two incoming edges from vertices that do not belong to the path. This claim can be easily verified by inspecting the \( Z_n \)-quiver and the sequence of mutations under discussion. In turn, it implies that the monomial transformation \( M_N \) restricts to each \( \Lambda V \)-path. The action of \( M_N \) on the \( \Lambda V_i \)-path is shown in Figure 10, where

\[
Z_- = q^{2\theta(i)} \cdot X_1 X_2 \ldots X_{2\theta(i)+1} \cdot Y_1, \\
Z_0 = q^{-2n} \cdot X_{2\theta(i)}^{-1} \ldots X_2^{-1} X_1^{-1} \cdot Y_{2i}^{-1} \ldots Y_{2i+1}^{-1} Y_1^{-1}, \\
Z_+ = q^{2i} \cdot X_1 \cdot Y_1 Y_2 \ldots Y_{2i+1}.
\]

On the other hand, it is easy to see that the automorphism \( P \circ \text{Ad}_K \) acts as \( P \) on all nonfrozen variables in the product \( D_n \otimes D_n \). It is a matter of a straightforward calculation to verify that \( P \circ \text{Ad}_K \) acts on the frozen variables of \( Z_n \), and on those variables that used to be frozen before the
amalgamation, as follows:

\[
P(\text{Ad}_K(X_{2g(r)+1})) = q^{2g(r)} \cdot X_1 X_2 \cdots X_{2g(r)+1} \cdot Y_1, \\
P(\text{Ad}_K(X_1 Y_1)) = q^{-2n} \cdot X_{2g(r)}^{-1} \cdots X_1^{-1} \cdot Y_1^{-1} \cdots Y_2^{-1} Y_1^{-1}, \\
P(\text{Ad}_K(Y_{2r+1})) = q^{2r} \cdot X_1 \cdot Y_1 Y_2 \cdots Y_{2r+1}.
\]

Now we can see that under the action of \( P \circ \text{Ad}_K \), the initial cluster \( \mathcal{X} \) of the quiver \( \mathcal{Z}_n \) is transformed into a different cluster \( \mathcal{X}' \) with the underlying quiver isomorphic to \( \mathcal{Z}_n \). At the same time, \( M_N \) also turns \( \mathcal{X} \) into \( \mathcal{X}' \), but the corresponding quiver is \( \mathcal{Z}_n' \). Since there are no nontrivial automorphisms of the quiver \( \mathcal{Z}_n \) fixing the frozen variables, we conclude that the permutation \( \sigma \) satisfies (6.2).

\[\square\]

**Proposition 6.3.** The following automorphisms of \( \mathcal{Z}_n \) coincide:

\[\text{Ad}_P(\mathcal{R}) = \Phi_N.\]

**Proof.** Consider the factorization (7.18) of the quasi \( R \)-matrix obtained in Theorem 7.4. On the other hand, we have a different factorization of the \( R \)-matrix from inspecting the sequence of flips realizing the Dehn twist along with the corresponding sequence of mutations. The latter factorization reads

\[
\mathcal{R} = \prod_{k=0}^{n-1} \prod_{j=\theta(k)}^{n+1} \prod_{i=1}^{\theta(k+1)} \psi \left( m_{i-j}^{-\theta(k)} \otimes w_{i+j}^{-i} \right) \\
\cdot \prod_{k=0}^{n-1} \prod_{j=\theta(k)}^{n+1} \prod_{i=1}^{\theta(k+1)} \psi \left( m_{i-j}^{-\theta(j)} \otimes w_{i+j}^{\theta(j)} \right) \\
\cdot \prod_{k=1}^{n} \prod_{j=k+1}^{n+1} \prod_{i=1}^{k} \psi \left( m_{i-j}^{k-j} \otimes w_{j-i}^{-j} \right) \\
\cdot \prod_{k=1}^{n} \prod_{j=k+1}^{n+1} \prod_{i=1}^{k} \psi \left( m_{i-j}^{k-j} \otimes w_{j-i}^{k-j} \right),
\]

(6.3)

where all three products are taken in ascending order and expanded from left to right. Now, it suffices to show that formulas (7.18) and (6.3) coincide.

Let us write \((a_1, \ldots, a_N)\) for the sequence of dilogarithm arguments appearing in the factorization (7.18), read from left to right. Similarly, we write \((b_1, b_2, \ldots, b_N)\) for the sequence of dilogarithm arguments appearing in the factorization (6.3), again read from left to right. It is easy to see that the underlying sets \((a_1, \ldots, a_N)\) and \((b_1, \ldots, b_N)\) coincide. Moreover, we claim that for every pair \((b_i, b_j)\) with \(i < j\) such that \((b_i, b_j) = (a_k, a_l)\) for
some \( k > l \), we have \( [b_i, b_j] = 0 \). This follows from commutation relations
\[
\begin{align*}
    w_i^r w_i^s &= q^{2 \text{sgn}(r - s)} w_i^s w_i^r, \\
    w_i^r w_j^s &= w_j^s w_i^r & \text{if } |i - j| > 1, \\
    w_i^r w_{i+1}^s &= \begin{cases} 
        qw_{i+1}^s w_i^r & \text{if } r < s, \\
        q^{-1} w_{i+1}^s w_i^r & \text{if } r > s,
    \end{cases}
\end{align*}
\]
and similar relations for variables \( m_i^r \), all of which can be read from the \( D_n \)-quiver. Hence one can freely re-order the dilogarithms \( \psi(b_i) \) to match the order arising in (7.18), and the Proposition is proved. □

**Remark 6.4.** The homomorphism \((\iota \otimes \iota) \circ \Delta : D_n \to D_n \otimes D_n\) given composition of the comultiplication map with the tensor square of \( \iota \) factors through the subalgebra \( Z_n \): we have
\[(\iota \otimes \iota) \circ \Delta : D_n \to Z_n \subset D_n \otimes D_n.\]

Let us refer to the concatenation of the two \( V \)-paths in a pair of amalgamated \( D_n \)-quivers as a \( VV \)-path. Then, the formula for \( \Delta(E_i) \) is obtained by conjugating the first (frozen) variable in the \( VV \)-path by quantum dilogarithms with arguments running over consecutive vertices in the \( VV \)-path not including the last (frozen) vertex, and multiplying the result by \( i \). In particular, in the notations of Figure 9 one gets
\[
\begin{align*}
    \Delta(E_1) &= iX_1(1 + qX_2(1 + qX_3(1 + qX_4))), \\
    \Delta(E_2) &= iX_6(1 + qX_7(\ldots(1 + qX_{12}(1 + qX_{13})\ldots))).
\end{align*}
\]
The coproduct \( \Delta(K_i) \) is equal to the product of all the variables along the \( VV \)-path multiplied by \( q^{di} \). Again, in the notations of Figure 9 one gets
\[
\begin{align*}
    \Delta(K_1) &= q^4X_1X_2X_3X_4X_5, \\
    \Delta(K_2) &= q^8X_6X_7X_8X_9X_{10}X_{11}X_{12}X_{13}X_{14}.
\end{align*}
\]
Formulas for \( \Delta(F_\theta(i)) \) and \( \Delta(K_\theta'(i)) \) can be obtained from those for \( \Delta(E_i) \) and \( \Delta(K_i) \) via rotating the \( Z_n \)-quiver by 180°. Similarly, one can get formulas for iterated coproducts \( \Delta^k(A), A \in D_n \), by amalgamating \( k + 1 \) copies of the \( D_n \)-quiver.

7. **Factorization of the R-matrix**

In this section, we show that the embedding (15.14) gives rise to the refined factorization of the \( R \)-matrix of \( U_q(\mathfrak{sl}_{n+1}) \) used in the proof of Theorem 6.1.

We begin with some preparatory lemmas and remarks. It follows from formulas (3.4) and (14.11) that for every \( -i \leq r < i \) and \( -j \leq s < j \) there exist unique decompositions
\[
E_{i+1, j} = E_{i+1, j}^{r\pm} + E_{i+1, j}^{r\mp}, \quad E_{i, j+1} = E_{i, j+1}^{s\pm} + E_{i, j+1}^{s\mp}, \quad E_{r, i} = E_{r, i}^{t\pm} + E_{r, i}^{t\mp}, \quad E_{s, j} = E_{s, j}^{t\pm} + E_{s, j}^{t\mp}
\]
where the summands satisfy
\[
\begin{align*}
    w_i^r E_{i+1, j}^{r\pm} &= q^{\pm 1} E_{i+1, j}^{r\pm} w_i^r, \\
    w_j^s E_{i, j+1}^{s\pm} &= q^{\pm 1} E_{i, j+1}^{s\pm} w_j^s.
\end{align*}
\]
In a similar fashion, formulas (3.5) and (4.3) imply decompositions
\[ F_{i+1,j} = F_{i+1,j}^{r-} + F_{i+1,j}^{r+}, \]
\[ F_{i,j-1} = F_{i,j-1}^{s-} + F_{i,j-1}^{s+}, \]
where the summands are defined by
\[ m_{\theta(i)}^{r} F_{i+1,j}^{r-} = q^{\pm 1} F_{i+1,j}^{r+} m_{\theta(i)}, \]
\[ m_{\theta(j)}^{s} F_{i,j-1}^{s-} = q^{\pm 1} F_{i,j-1}^{s+} m_{\theta(j)}. \]

It is also evident that \( E_{ij} \) and \( F_{ij} \) can be decomposed as
\[ E_{ij} = -q \sum_{s=-j}^{j-1} E_{i,j-1}^{s+} w_{s} = q^{2} \sum_{s=-r}^{r-1} E_{i,r-1}^{s+} w_{s} E_{r+1,j}^{s-} = -q \sum_{s=-i}^{i-1} w_{s} E_{i+1,j}, \]
(7.1)
\[ F_{ij} = \sum_{s=-\theta(j)}^{\theta(j)-1} F_{i,j-1}^{s+} m_{\theta(j)}^{s} = \sum_{s=-\theta(r)}^{\theta(r)-1} F_{i,r-1}^{s+} m_{\theta(r)}^{s} F_{r+1,j}^{s-} = \sum_{s=-\theta(i)}^{\theta(i)-1} m_{\theta(i)}^{s} F_{i+1,j}^{s-}, \]
(7.2)
for any \( i < r < j \). We say that formulas (7.1) show decompositions of \( E_{ij} \) with respect to the \( V_{i}^{-}, V_{r}^{-}, \) and \( V_{j}^{-} \)-paths. Similarly, formulas (7.2) show decompositions of \( F_{ij} \) with respect to the \( \Lambda_{\theta(i)}^{-}, \Lambda_{\theta(r)}^{-}, \) and \( \Lambda_{\theta(j)}^{-} \)-paths.

**Lemma 7.1.** For all \( a < b \), we have
\[ \left( E_{i,j-1}^{a+} w_{j}^{a} \right) \left( E_{i,j-1}^{b+} w_{j}^{b} \right) = q^{-2} \left( E_{i,j-1}^{b+} w_{j}^{b} \right) \left( E_{i,j-1}^{a+} w_{j}^{a} \right), \]
(7.3)
\[ \left( E_{i,j-1}^{a+} m_{\theta(j)}^{a} \right) \left( E_{i,j-1}^{b+} m_{\theta(j)}^{b} \right) = q^{-2} \left( E_{i,j-1}^{b+} m_{\theta(j)}^{b} \right) \left( E_{i,j-1}^{a+} m_{\theta(j)}^{a} \right), \]
(7.4)
and
\[ \left( E_{i,j-1}^{a+} w_{j}^{a} E_{j+1,k}^{a} \right) \left( E_{i,j-1}^{b+} w_{j}^{b} E_{j+1,k}^{b} \right) = q^{-2} \left( E_{i,j-1}^{b+} w_{j}^{b} E_{j+1,k}^{b} \right) \left( E_{i,j-1}^{a+} w_{j}^{a} E_{j+1,k}^{a} \right). \]
(7.5)

**Proof.** We shall only prove the first relation as the proofs of the other two are similar. First, note that since \( a < b \) we have
\[ w_{j}^{a} E_{i,j-1}^{b+} = q E_{i,j-1}^{b+} w_{j}^{a} \quad \text{and} \quad w_{j}^{a} w_{j}^{b} = q^{-2} w_{j}^{b} w_{j}^{a}, \]
therefore
\[ w_{j}^{a} \left( E_{i,j-1}^{b+} w_{j}^{b} \right) = q^{-1} \left( E_{i,j-1}^{b+} w_{j}^{b} \right) w_{j}^{a}. \]
Let us set
\[ E_{i,j-1}^{a\downarrow b} = E_{i,j-1}^{a+} - E_{i,j-1}^{b+}. \]
Then by definition we have
\[ E_{i,j-1}^{b+} w_{j}^{b} = q^{-1} w_{j}^{b} E_{i,j-1}^{b+}, \]
and it only remains to commute \( E_{i,j-1}^{a\downarrow b} \) through \( E_{i,j-1}^{b+} w_{j}^{b} \). For this, is enough to show that
\[ E_{i,j-1}^{a\downarrow b} E_{i,j-1}^{b+} = q^{-2} E_{i,j-1}^{b+} E_{i,j-1}^{a\downarrow b}. \]
(7.6)
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since

$$E_{i,j}^{a \triangleright b} w_j^b = q w_j^b E_{i,j}^{a \triangleright b}.$$ 

We finish the proof by induction on $j$. Assume that equalities (7.3) and (7.6) hold for all $j < k$. To prove the base of induction, it is enough to note that if $j = i + 1$, the relation (7.6) follows readily from inspecting the quiver, which in turn implies (7.3). In order to make the step of induction, we decompose both $E_{i,k}^{a \triangleright b}$ and $E_{i,k}^{b,i}$ with respect to the $V_{k-1}$-path and apply (7.3) for $j = k - 2$. □

Lemma 7.2. For $i < j$ we have

$$E_{i,j} E_k = \begin{cases} 
q^{-1} E_k E_{i,j} & \text{if } k = j, \\
q E_k E_{i,j} & \text{if } k = i, \\
E_k E_{i,j} & \text{if } i < k < j. 
\end{cases} \quad (7.7)$$

Proof. The proof follows from the decomposition (7.1) and considerations similar to the proof of those in the proof of Lemma 7.1. □

For any $i < j$ let us declare

$$F_{i,j}^{\geq s} = \sum_{r \geq s} F_{i,j-1}^{r,i} m_{\theta(j)}^r.$$ 

Note that

$$F_{i,j}^{\geq s} = \begin{cases} 
F_{i,j}^{s,i} & \text{if } s < 0, \\
F_{i,j-1}^0 m_{\theta(j)}^0 + F_{i,j}^{s,i} & \text{if } s \geq 0. 
\end{cases} \quad (7.8)$$

In what follows we use the following shorthand:

$$\psi(x) = \Psi^q(-x)$$

Note that the pentagon identity (1.3) now reads

$$\psi(v) \psi(u) = \psi(u) \psi(-quv) \psi(v)$$

for any $u$ and $v$ satisfying $vu = q^2 uv$.

Lemma 7.3. We have

$$\psi \left(E_{i,j} \otimes F_{i,j}^{s,i} \right) \psi \left(E_{i,j+1} \otimes F_{i,j+1}^{s,i} \right) \psi \left(E_{j+1} \otimes m_{\theta(j+1)}^s \right)$$

$$= \psi \left(E_{j+1} \otimes m_{\theta(j+1)}^s \right) \psi \left(E_{i,j} \otimes F_{i,j}^{s,i} \right) \psi \left(E_{i,j+1} \otimes F_{i,j+1}^{s,i} \right). \quad (7.10)$$

Proof. By equality (1.2) and Lemma 7.1 there exists a factorization

$$\psi \left(E_{i,j+1} \otimes F_{i,j+1}^{s,i} \right) = \prod_{r \geq s} \psi \left(E_{i,j+1} \otimes F_{i,j}^{r,i} m_{\theta(j+1)}^r \right),$$

where the product is taken in ascending order. Note that by Lemma 7.2, the dilogarithm $\psi \left(E_{j+1} \otimes m_{\theta(j+1)}^s \right)$ commutes with all but the left-most factor.
in this product. Hence the left-hand side of (7.10) may be re-ordered so that we have a triple of adjacent factors

\[ \psi \left( E_{i,j} \otimes F_{i,j}^{\uparrow s^+} \right) \psi \left( E_{i,j+1} \otimes F_{i,j}^{\uparrow s^+} m_{\theta(j+1)}^s \right) \psi \left( E_{j+1} \otimes m_{\theta(j+1)}^s \right). \]

Now, using (7.1) we get similar factorizations

\[ \psi \left( E_{i,j} \otimes F_{i,j}^{\uparrow s^+} \right) = \prod_{r=-i}^{i-1} \psi \left( -qE_{i,j-1} w_j \otimes F_{i,j}^{\uparrow s^+} \right) \] (7.11)

and

\[ \psi \left( E_{i,j+1} \otimes F_{i,j}^{\uparrow s^+} m_{\theta(j+1)}^s \right) = \prod_{r=-i}^{i-1} \psi \left( q^2 E_{i,j-1} w_j^* E_{j+1} \otimes F_{i,j}^{\uparrow s^+} m_{\theta(j+1)}^s \right), \] (7.12)

with the products again being taken in ascending order. By Lemma 7.1, the rightmost factor \( \psi \left( -qE_{i,j-1} w_j \otimes F_{i,j}^{\uparrow s^+} \right) \) in (7.11) commutes with all but the rightmost factor in (7.12), so we can re-order again to get a triple of adjacent factors

\[ \psi \left( -qE_{i,j-1} w_j \otimes F_{i,j}^{\uparrow s^+} \right) \psi \left( q^2 E_{i,j-1} w_j^* E_{j+1} \otimes F_{i,j}^{\uparrow s^+} m_{\theta(j+1)}^s \right) \cdot \psi \left( E_{j+1} \otimes m_{\theta(j+1)}^s \right). \] (7.13)

On the other hand, we can factor

\[ \psi \left( E_{j+1} \otimes m_{\theta(j+1)}^s \right) = \psi \left( E_{j+1}^{\downarrow (i-1)^+} \otimes m_{\theta(j+1)}^s \right) \psi \left( E_{j+1}^{\downarrow (i-1)+} \otimes m_{\theta(j+1)}^s \right), \] (7.14)

and then apply the pentagon identity (7.9) to (7.13), yielding

\[ \psi \left( E_{j+1}^{\downarrow (i-1)^+} \otimes m_{\theta(j+1)}^s \right) \psi \left( -qE_{i,j-1}^{\downarrow (i-1)+} w_j \otimes F_{i,j}^{\uparrow s^+} \right) \cdot \psi \left( E_{j+1}^{\downarrow (i-1)^+} \otimes m_{\theta(j+1)}^s \right). \] (7.15)

Note that the right two factors in the product (7.15) commute, so it can be re-expressed as

\[ \psi \left( E_{j+1} \otimes m_{\theta(j+1)}^s \right) \psi \left( -qE_{i,j-1}^{\downarrow (i-1)+} w_j \otimes F_{i,j}^{\uparrow s^+} \right). \] (7.16)

Repeating the same procedure for each of the remaining factors in the product (7.11), one arrives at (7.10). □
Theorem 7.4. The quasi $R$-matrix of $U_q(\mathfrak{sl}_n)$ can be factored as follows:
\[
\tilde{\mathcal{R}}_n = \psi(E_1 \otimes m_1^{-n}) \psi(E_2 \otimes m_2^{-n+1}) \cdots \psi(E_n \otimes m_n^{-1}) \\
\cdot \psi(E_1 \otimes m_1^{-n}) \psi(E_2 \otimes m_2^{-n+1}) \cdots \psi(E_{n-1} \otimes m_{n-1}^{-1}) \\
\vdots \\
\cdot \psi(E_1 \otimes m_1^{-2}) \psi(E_2 \otimes m_2^{-1}) \\
\cdot \psi(E_1 \otimes m_1^{-1}) \\
\cdot \psi(E_2 \otimes m_2^0) \psi(E_1 \otimes m_1^0) \\
\vdots \\
\cdot \psi(E_{n-1} \otimes m_{n-1}^0) \psi(E_{n-2} \otimes m_{n-2}^1) \cdots \psi(E_1 \otimes m_n^{n-2}) \\
\cdot \psi(E_n \otimes m_n^1) \psi(E_{n-1} \otimes m_{n-1}^1) \cdots \psi(E_1 \otimes m_n^{n-1}).
\]
Equivalently, we have
\[
\tilde{\mathcal{R}} = \prod_{k=1}^n \prod_{j=1}^{\theta(k)} \prod_{i=-j}^{j-1} \psi(w_j^i \otimes m_{\theta(j)}^{-k+j-i}) \\
\cdot \prod_{k=1}^n \prod_{j=\theta(k)}^{\theta(j)-1} \prod_{i=-j}^{j-1} \psi(w_j^i \otimes m_{\theta(j)}^{j-i})
\] (7.18)
where in the above formula, the products are taken in ascending order and expanded from left to right, that is one should first expand the formula in $k$, then in $j$, and then in $i$.

Example 7.5. In the case of $U_q(\mathfrak{sl}_3)$, formula (7.18) yields a factorization of the quasi $R$-matrix into the following 16 factors:
\[
\tilde{\mathcal{R}} = \psi(w_1^{-1} \otimes m_2^2) \psi(w_1^0 \otimes m_2^{-2}) \psi(w_2^{-2} \otimes m_1^{-1}) \psi(w_2^0 \otimes m_1^1) \\
\cdot \psi(w_2^0 \otimes m_1^{-1}) \psi(w_1^1 \otimes m_2^{-1}) \psi(w_1^{-1} \otimes m_2^{-1}) \psi(w_1^0 \otimes m_2^1) \\
\cdot \psi(w_1^{-1} \otimes m_2^0) \psi(w_1^0 \otimes m_2^0) \psi(w_2^{-2} \otimes m_1^0) \psi(w_2^{-1} \otimes m_1^0) \\
\cdot \psi(w_2^0 \otimes m_1^0) \psi(w_1^0 \otimes m_1^0) \psi(w_1^{-1} \otimes m_2^0) \psi(w_1^0 \otimes m_2^1).
\]
Proof. Choosing the normal ordering
\[
\alpha_1 \prec (\alpha_1 + \alpha_2) \prec (\alpha_1 + \cdots + \alpha_n) \prec \alpha_2 \prec \cdots \prec (\alpha_2 + \cdots + \alpha_n) \prec \cdots \prec \alpha_n
\]
in the formula (7.18), we can write
\[
\tilde{\mathcal{R}}_{n+1} = \psi(E_1 \otimes F_1) \psi(E_{1,2} \otimes F_{1,2}) \cdots \psi(E_{1,n+1} \otimes F_{1,n+1}) \cdot \tilde{\mathcal{R}}_n.
\] (7.19)
\[\text{In fact, one only needs to order the product over } k, \text{ for the reason that all factors with a fixed } k \text{ commute. However, it is slightly easier to check that formulas (7.18) and (6.3) coincide if all three products are ordered.}\]
where we may assume by induction that $\mathcal{R}_n$ factors as follows:

$$
\mathcal{R}_n = \psi \left( E_2 \otimes m_n^{-n} \right) \psi \left( E_3 \otimes m_{n-1}^{1-n} \right) \cdots \psi \left( E_{n+1} \otimes m_1^{-1} \right) \\
\vdots \\
\psi \left( E_2 \otimes m_n^{-2} \right) \psi \left( E_3 \otimes m_{n-1}^{-1} \right) \\
\psi \left( E_2 \otimes m_n^{-1} \right) \\
\psi \left( E_2 \otimes m_0^0 \right) \\
\psi \left( E_3 \otimes m_{n-1}^{0} \right) \psi \left( E_2 \otimes m_n^{1} \right) \\
\vdots \\
\psi \left( E_n \otimes m_0^0 \right) \psi \left( E_{n-1} \otimes m_n^{3} \right) \cdots \psi \left( E_2 \otimes m_n^{-2} \right) \\
\psi \left( E_{n+1} \otimes m_0^0 \right) \psi \left( E_n \otimes m_n^{2} \right) \cdots \psi \left( E_2 \otimes m_n^{-1} \right).
$$

(7.20)

By Lemma 7.2, we may shuffle the prefix of (7.19) and the first row of (7.20) into the following form:

$$
\psi \left( E_1 \otimes F_1 \right) \psi \left( E_{1,2} \otimes F_{1,2} \right) \psi \left( E_2 \otimes m_n^{-n} \right) \cdots \psi \left( E_{1,n} \otimes F_{1,n} \right) \psi \left( E_n \otimes m_2^{-2} \right) \\
\psi \left( E_{1,n+1} \otimes F_{1,n+1} \right) \psi \left( E_{n+1} \otimes m_1^{-1} \right).
$$

(7.21)

We can then apply Lemma 7.3 to write

$$
\psi \left( E_1 \otimes F_1 \right) \psi \left( E_{1,2} \otimes F_{1,2} \right) \psi \left( E_2 \otimes m_n^{-n} \right) \\
= \psi \left( E_1 \otimes m_n^{-n+1} \right) \psi \left( E_1 \otimes F_1^{\uparrow(-n)+} \right) \psi \left( E_{1,2} \otimes F_{1,2} \right) \psi \left( E_2 \otimes m_n^{-n} \right) \\
= \psi \left( E_1 \otimes m_n^{-n+1} \right) \psi \left( E_2 \otimes m_n^{-n} \right) \psi \left( E_1 \otimes F_1^{\uparrow(-n)+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{\uparrow(1-n)+} \right).
$$

After repeated applications of Lemma 7.3, the last of these being to write

$$
\psi \left( E_{1,n} \otimes F_1^{\uparrow(-n)+} \right) \psi \left( E_{1,n+1} \otimes F_{1,n+1} \right) \psi \left( E_{n+1} \otimes m_1^{-1} \right) \\
= \psi \left( E_{n+1} \otimes m_1^{-1} \right) \psi \left( E_{1,n} \otimes F_1^{\uparrow(-n)+} \right) \psi \left( E_{1,n+1} \otimes F_1^{\uparrow(1-n)+} \right).
$$

we arrive at the following form of (7.21):

$$
\psi \left( E_1 \otimes m_n^{-n+1} \right) \psi \left( E_2 \otimes m_n^{-n} \right) \cdots \psi \left( E_{n+1} \otimes m_1^{-1} \right) \\
\psi \left( E_1 \otimes F_1^{\uparrow(-n)+} \right) \cdots \psi \left( E_{1,n} \otimes F_1^{\uparrow(-1)+} \right) \psi \left( E_{1,n+1} \otimes F_1^{\uparrow(0)+} \right).
$$
We can now repeat this reasoning for each of the next $n - 1$ rows in the product (7.20). This results in an expression for $\mathcal{R}_{n+1}$ of the form

$$\mathcal{R}_{n+1} = \psi \left( E_1 \otimes m_{n+1}^{-n-1} \right) \psi \left( E_2 \otimes m_{n+1}^{-n} \right) \cdots \psi \left( E_{n+1} \otimes m_1^{-1} \right)$$

$$\cdot \psi \left( E_1 \otimes m_{n+1}^{-n} \right) \psi \left( E_2 \otimes m_{n+1}^{-n+1} \right) \cdots \psi \left( E_n \otimes m_2^{-1} \right)$$

$$\vdots$$

$$\cdot \psi \left( E_1 \otimes m_{n+1}^{-2} \right) \psi \left( E_2 \otimes m_n^{-1} \right)$$

$$\cdot \psi \left( E_1 \otimes m_{n+1}^{-1} \right)$$

$$\cdot \psi \left( E_1 \otimes m_{n+1}^{0} \right)$$

$$\cdot \psi \left( E_1 \otimes F_1^{0+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{0} \right) \cdots \psi \left( E_{1,n+1} \otimes F_{1,n+1}^{0} \right)$$

$$\cdot \psi \left( E_2 \otimes m_n^{0} \right)$$

$$\cdot \psi \left( E_3 \otimes m_{n-1}^{0} \right) \psi \left( E_2 \otimes m_1^{1} \right)$$

$$\vdots$$

$$\cdot \psi \left( E_{n+1} \otimes m_1^{0} \right) \psi \left( E_n \otimes m_2^{1} \right) \cdots \psi \left( E_2 \otimes m_n^{0} \right).$$

Note that the first $n + 2$ rows of factors in this product are now in the desired form. Now we need to focus on the following factor:

$$\psi \left( E_1 \otimes F_1^{0+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{0} \right) \cdots \psi \left( E_{1,n+1} \otimes F_{1,n+1}^{0} \right)$$

$$\cdot \psi \left( E_2 \otimes m_n^{0} \right)$$

$$\cdot \psi \left( E_3 \otimes m_{n-1}^{0} \right) \psi \left( E_2 \otimes m_1^{1} \right)$$

$$\vdots$$

$$\cdot \psi \left( E_{n+1} \otimes m_1^{0} \right) \psi \left( E_n \otimes m_2^{1} \right) \cdots \psi \left( E_2 \otimes m_n^{0} \right).$$

By Lemma (7.2) we can reshuffle this block so that it begins with an adjacent triple of terms

$$\psi \left( E_1 \otimes F_1^{0+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{0} \right) \psi \left( E_2 \otimes m_n^{0} \right)$$

$$= \psi \left( E_2 \otimes m_n^{0} \right) \psi \left( E_1 \otimes F_1^{0+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{0} \right)$$

$$= \psi \left( E_2 \otimes m_n^{0} \right) \psi \left( E_1 \otimes m_1^{1} \right) \psi \left( E_1 \otimes F_1^{1+} \right) \psi \left( E_{1,2} \otimes F_{1,2}^{0} \right),$$

where we once again used Lemma (7.3). Note that now this recovers the correct form of row $(n + 3)$ in (7.17), continuing in a similar fashion one arrives at the desired expression for $\mathcal{R}$.

8. Comparison with Faddeev’s results

We conclude by comparing the rank 1 case of our results with Faddeev’s embedding (0.2) as promised in the introduction. Consider the quiver
in Figure 4. The corresponding quantum cluster $D_1$ has initial variables $\langle X_1, X_2, X_3, X_4 \rangle$ subject to the relations

$$X_i X_{i+1} = q^{-2} X_{i+1} X_i \quad \text{and} \quad X_i X_{i+2} = X_{i+2} X_i \quad \text{where} \quad i \in \mathbb{Z}/4\mathbb{Z}.$$ 

In this case, the embedding (4.4) takes the form

$$\hat{E} \mapsto iX_1(1 + qX_2), \quad K \mapsto q^2 X_1 X_2 X_3,$$

$$\hat{F} \mapsto iX_3(1 + qX_4), \quad K' \mapsto q^2 X_3 X_4 X_1,$$

while our formula (7.17) for the universal $R$-matrix reads

$$R = \overline{\Psi}^q (X_1 \otimes X_3) \Psi^q (q X_1 \otimes X_3 X_4) \cdot \Psi^q (q X_3 \otimes X_3) \Psi^q (q^2 X_1 X_2 \otimes X_3 X_4).$$

Hence, Faddeev’s formulas (0.2) and (0.3) are recovered from ours under the monomial change of variables

$$w_1 \mapsto X_1, \quad w_2 \mapsto q X_1 X_2, \quad w_3 \mapsto X_3, \quad w_4 \mapsto q X_3 X_4. \quad (8.2)$$

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