W-INFINITY AND STRING THEORY

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ABSTRACT

We review some recent developments in the theory of $W_\infty$. We comment on its relevance to lower-dimensional string theory.
1. INTRODUCTION

The advent of modern string theories in recent years has brought forth a tremendous amount of investigations in two-dimensional physics. In particular a great deal of progress has been made in two-dimensional conformal field theories (CFTs) [1,2,3]. These theories possess the celebrated Virasoro algebra as their underlying symmetry.

Virasoro symmetry has been well studied in both physics and mathematics literature. When the central charge $c$ vanishes, it is essentially the diffeomorphism group on a circle. The central extension furnishes the algebra with many non-trivial features such as its complicated space of representations. It is well known that for the case when $0 < c < 1$, a full classification of the unitary representations of Virasoro has been given [4]. In general for the case when $c > 1$, there exists no complete classification for the unitary representations.

The reason why there exists such a classification of the unitary representations of the Virasoro algebra with $0 < c < 1$ is due, to a large extent, to the tight structure imposed by the symmetry itself. In particular, all unitary theories with $0 < c < 1$ fall into the so-called minimal models of the Virasoro algebra, whose simplicity is a non-trivial feature of Virasoro symmetry with its central charge in such a range. When $c \geq 1$, such a concept of minimality is lost in general, and the space of representations of Virasoro is much more complex. One way to attack this problem is to introduce a larger symmetry, which contains Virasoro as its subalgebra, so that a new concept of minimality with respect to the larger symmetry emerges. These symmetries are generically referred to as extended conformal symmetries. Theories that possess such symmetries are referred to as the Rational Conformal Field Theories (RCFTs) [3].

Virasoro symmetry is generated by the spin-2 stress tensor field $T(z)$. Extended conformal symmetries include additional generators. For example, in the case of the superconformal algebras, there exist additional fermionic fields with appropriate conformal spins, plus additional bosonic fields necessary to form supermultiplets. Another class of extended conformal algebras is the famous $W$-algebras [5,6], the first of which, the $W_3$ algebra, was discovered by Zamolodchikov [5]. A characteristic of this class of algebras is that they contain fields with integral higher-spin. In this terminology, Virasoro may be referred to as the $W_2$ algebra. In general, the existence of additional generators indeed invokes more refined notions of minimality so as to render the space of representations more manageable.

The motivation discussed above to search and study extended conformal symmetries evolves around the quest to understand and classify the two-dimensional CFTs, whose ap-
Applications range from two-dimensional critical phenomena to string theory as a candidate for the unifying theory. From the viewpoint of string theory, there is also a need to understand symmetries other than the Virasoro algebra. In general, the underlying symmetry structure of string theory is not known, which makes a more coherent formulation of the theory still inaccessible. Since a symmetry on the world-sheet of the string is often reflected in the target space-time, investigations into algebraic structures on the two-dimensional world-sheet may shed light on the symmetry structure of the target space-time of the string. This strategy in understanding string theory differs from that of string field theory. Nonetheless it may be a useful way to probe string theory and lead to a successful formulation of string field theory.

Recently in the study of lower-dimensional string theories, and in the $c = 1$ (bosonic) model in particular, there emerges evidence for the existence of elegant symmetry structures [7,8,9,10]. These developments have made it more imperative to better understand symmetries larger than the Virasoro algebra.

One of the symmetries that has emerged in these investigations into lower-dimensional string theories is some $W_\infty$ symmetry. From the viewpoint of the world-sheet, on which conformal symmetry (Virasoro) plays an important role, $W_\infty$ symmetry can be viewed as the $N \to \infty$ limit of the extended conformal algebra $W_N$. It is important to emphasise that such a viewpoint does not necessarily imply that the emerging $W_\infty$-like symmetry exists on the world-sheet. Recent analysis seems to suggest that it should more likely be a symmetry in the configuration space of the theory.

In view of the above motivations, we shall review some recent developments in the theory of $W_\infty$ with an eye towards its application in studying string theories. Some of the topics of $W_\infty$ theory are interesting in their own right, and potentially may become relevant to string theory. They include topics such as $W_\infty$ gravity as a higher-spin extension of ordinary two-dimensional gravity, the $W_\infty$ string as an extension of ordinary string theory, and the concept of universal $W$-algebra that encompasses all finite-$N$ $W_N$ algebras. Hopefully such an effort may be of use in making some technology developed more available.

The paper is organized as follows. In Section 2 we shall first give the algebraic structures of various $W_\infty$ algebras, and make comments on issues pertaining to these algebras, such as their relationships to the area-preserving diffeomorphism of a two-surface, their subalgebras, their relationship to other algebras e.g. the algebra of differential operators of arbitrary degree on a circle.

Sections 3, 4, 5, 6 and 7 are about the field theory of $W_\infty$. We shall start in Section 3 with
some known global realizations of $W_\infty$. Section 4 covers the classical formulation of $W_\infty$ gravity and the $W_\infty$ string, where the introduction of $W_\infty$ gauge fields make the symmetry locally realized. Sections 5, 6 and 7 cover three different topics in the quantization of $W_\infty$ gravity and the $W_\infty$ string. In the first topic, we demonstrate that a classical $w_\infty$ gravity model is quantum mechanically inconsistent and deforms into a quantum $W_\infty$ gravity upon quantization. The second topic concerns the BRST analysis of $W_\infty$. The third contains the work that demonstrates the existence of an $SL(\infty, R)$ Kac-Moody symmetry in $W_\infty$ gauge theories.

In Section 8, we shall look into the recent investigation into the lower-dimensional string models, and draw some parallels between the symmetries in these models and the $W_\infty$ symmetry discussed in the previous sections. We summarize the paper in Section 9, and make brief reference to other topics left out of this review in the field of $W$ gravity and the $W$ string.

2. THE ALGEBRAIC STRUCTURE OF THE $W_\infty$ ALGEBRAS

The $W_\infty$ algebras are bosonic extensions of the Virasoro algebra. They can be viewed as the $N \to \infty$ limits of finite-$N W_N$ that contains generators of conformal spin $2,3,\ldots,N$. Thus a generic $W_\infty$ algebra contains an infinite number of generating currents of conformal-spin $3,4,\ldots,\infty$, in addition to the spin-2 stress tensor of Virasoro.

Since the procedure of taking $N \to \infty$ limit is rather subtle, it is believed that there exist more than one $W_\infty$ algebra that correspond to the same $W_N$ algebra. In particular there is a non-linear $W_\infty$ algebra [11] as a limit as well as the linear $W_\infty$ algebras that have been discovered [12,13,14]. In this paper we shall only be concerned with the linear versions of $W_\infty$.

A striking feature of the linear $W_\infty$ algebra is its resemblance to Virasoro, which will become more and more evident as we move on. Essentially what can be achieved in the case of Virasoro can be applied and generalized rather straightforwardly to the case of $W_\infty$. Since it is much less demanding technically to work with Virasoro than $W_\infty$, we shall be content as often as possible to illustrate basic ideas in the case of Virasoro, and then give the answers for the case of $W_\infty$.

Let us first start with the Virasoro algebra. In its Fourier modes $L_n$ of the spin-2 stress tensor field $T(z)$ given by

$$T(z) \equiv \sum \frac{L_n}{z^{n+2}},$$

(1)
the algebra reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m-1)m(m+1)\delta_{m+n,0}. \quad (2)$$

In terms of the Operator Product Expansion (OPE) of the stress tensor $T(z)$, the algebra is given by

$$T(z)T(w) \sim \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}. \quad (3)$$

Note that there exists an $SL(2, R)$ subalgebra formed by $\{L_{-1}, L_0, L_{-1}\}$. As will become clear presently when the $W_\infty$ algebras are given, the covariance property under this subalgebra of Virasoro dictates their algebraic structures to a large extent.

The centerless Virasoro algebra can be viewed as the algebra of a vector differential on a circle parametrized by $\theta$ in the following way:

$$L_m = e^{im\theta} \frac{d}{d\theta}, \quad \text{(4)}$$

which generates the diffeomorphism group $DiffS^1$. From this viewpoint, Virasoro is the centrally extended algebra of $DiffS^1$. This observation finds its natural generalization to the case of $W_\infty$.

There are many ways to enlarge the Virasoro algebra to various extended conformal algebras. Among them is the famous $W_3$ algebra of Zamolodchikov [5], in which there exists a spin-3 generator $W(z)$ in addition to the spin-2 $T(z)$. This can be generalized to the finite-$N$ $W_N$ algebra containing fields with spin 2, 3, $\cdots$, $N$ [6], and their supersymmetric generalizations [7]. The unique characteristic of these algebras, which makes them very interesting but technically very difficult to work with, is the non-linearity inherent in the structure, due to the introduction of higher-spin generators. For example, in the case of $W_3$, apart from the OPE between $T$ and $W$ given as

$$T(z)W(w) \sim \frac{\partial W(w)}{z-w} + \frac{3W(w)}{(z-w)^2}, \quad \text{(5)}$$

the OPE between $W$ reads

$$W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{16}{22+5c}\left(\frac{2\Lambda}{(z-w)^2} + \frac{\partial \Lambda}{z-w}\right) + \frac{1}{15}\frac{\partial^3 T}{z-w} + \frac{3}{10}\frac{\partial^2 T}{(z-w)^2} + \frac{\partial T}{(z-w)^3} + \frac{2T}{(z-w)^4}. \quad \text{(6)}$$
where the Λ field is defined by

\[ \Lambda(z) \equiv (TT) - \frac{3}{10} \partial^2 T \]

\[ (TT)(z) \equiv \oint \frac{dw}{2\pi i} \frac{T(w)T(z)}{w-z}. \]  

(7)

The appearance of this composite spin-4 field Λ is the source of non-linearity. However this new feature also drastically increases the magnitude of difficulty in e.g. formulating field theory exhibiting such a symmetry at both classical and quantum levels. In fact, for arbitrary \( N \), due to the complexity arising from the non-linearity, the structure constants of \( W_N \) are in general not known explicitly.

In order to circumvent the situation of having to deal with non-linearity, it is conceivable that if the composite higher-spin terms such as the Λ\((z)\) in \( W_3 \), which is necessary for the closure, are replaced by new fundamental fields, non-linear terms may disappear. The consequence is that one may need to introduce more and more fundamental higher-spin fields. Thus it may only be possible once one has introduced sufficiently many independent fields. A reasonable set of fields, for example, includes one field for each spin \( s \geq 2 \). Indeed, as has been shown in [12,13], such a choice of field content does yield a consistent algebra, which preserves most of the important features of Virasoro and \( W_N \), such as non-trivial central extensions for higher-spin generators. This algebra is thus naturally called the \( W_\infty \) algebra.

Historically the original discovery of the \( W_\infty \) algebras employed a method in which a particularly natural form for the sought-after algebra was assumed and then the structure constants were calculated by imposing Jacobi identities. Apart from the naturalness of the assumptions that went into the construction, it was quite a miracle that solutions were found at all. It has only become clear later by considerations from other angles that there must exist algebras as such. We shall make remarks to make this point more explicit in the relevant context.

The ansatz in [12,13] is of the following form:

\[ [V^i_m, V^j_n] = \sum_{\ell \geq 0} g_{ij}^{\ell}(m,n) V^{i+j-2\ell}_{m+n} + c_i(m) \delta^{ij} \delta_{m+n,0}, \]  

(8)

where \( V^i_m \) denotes the \( m \)'th Fourier mode of a spin-\((i+2)\) field, and the central terms are assumed to take the form of that of the \( W_N \) algebras given by

\[ c_i(m) = (m^2 - 1)(m^2 - 4) \cdots (m^2 - (i+1)^2)c_i, \]  

(9)
The solutions to the Jacobi identities are thus given by

\[ g^{ij}_\ell(m,n) = \frac{\phi^{ij}_\ell}{2(\ell+1)!} N^{ij}_\ell(m,n), \]

where the \( N^{ij}_\ell(m,n) \) are given by

\[ N^{ij}_\ell(m,n) = \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} [i+1+m][\ell+1-k][i+1-m][j+1+n][j+1-n][\ell+1-k]. \]

In (11), \([a]_n\) denotes the descending Pochhammer symbol given by

\[ [a]_n \equiv a(a-1) \cdots (a-n+1) = a!/(a-n)! \]

The functions \( \phi^{ij}_\ell \) can be expressed as

\[ \phi^{ij}_\ell = \, {}_4F_3 \left[ \begin{array}{c} -\frac{1}{2}, \frac{3}{2}, -\ell - \frac{1}{2}, -\ell \\ -i - \frac{1}{2}, -j - \frac{1}{2}, i + j - 2\ell + \frac{5}{2}, 1 \end{array} \right], \]

where the right-hand side is a Saalschützian \( {}_4F_3(1) \) generalized hypergeometric function [16].

The central charges \( c_i \) are given by

\[ c_i = \frac{2^{2i-3}i!(i+2)!}{(2i+1)!!(2i+3)!!}. \]

REMARKS:

* **SL(2, R) Covariance and “Wedge” Algebra**

  The functions \( N^{ij}_{2\ell}(m,n) \) are related to the Clebsch-Gordan coefficients of \( SU(2) \) or \( SL(2, R) \), while the functions \( \phi^{ij}_{2\ell} \) are related (in a formal sense, at least) to Wigner 6-j symbols. These facts are indicative of an underlying \( SL(2, R) \) structure of the \( W_\infty \) algebra, and indeed this is the case. As indicated previously for Virasoro, the generators \( L_{-1}, L_0 \) and \( L_1 \) give an anomaly-free \( SL(2, R) \) subalgebra. Since the Virasoro algebra is itself a subalgebra of \( W_\infty \), this implies that we also have an \( SL(2, R) \) subalgebra in \( W_\infty \), generated by \( V_{-1}, V_0^0 \) and \( V_1^0 \). This \( SL(2, R) \) in fact forms the bottom “rung” of a wedge of generators, comprising the \( V_m^i \) with \(-i-1 \leq m \leq i+1\) for a given value of \( i \); the generators on that rung transform as the \((2i+3)\)-dimensional representation of \( SL(2, R) \). The set of all wedge generators, which we shall call the “wedge” algebra, give rise to an \( SL(\infty, R) \) algebra [13].
The Wedge Algebra as Tensor algebra of $SL(2, R)$

The $SL(\infty, R)$ wedge subalgebra of $W_\infty$ can be constructed as a tensor algebra of $SL(2, R)$, modded out by the ideal generated by $C_2 - s(s+1)$, where $C_2$ is the Casimir operator of $SL(2, R)$, and $s$ is a constant that must be chosen to be zero in this case [17]. One obtains inequivalent $SL(\infty, R)$ algebras by taking values other than $s(s+1) = 0$ for the quadratic Casimir. Specifically the generators $V^i_m$ with $-1 - i \leq m \leq i+1$, transforming as the $(2i+3)$-dimensional representation of $SL(2, R)$, are constructed from appropriate polynomials of degree $i+1$ in the generators of $SL(2, R)$. If one starts with $V^i_{i+1} \equiv (V^0_1)^{i+1}$, then constructs the generators in the same representation of $SL(2, R)$ by acting on $V^i_{i+1}$ with the lowering operator $V^0_1$, modulo the identification imposed by the ideal given above, one obtains the entire $SL(\infty, R)$ algebra. This family of $SL(\infty, R)$ algebras parametrized by $s$ will be explicitly given in Eqs.(118-120), when they are an important part of the discussion on the existence of $SL(\infty, R)$ Kac-Moody symmetry in quantum $W_\infty$ gravity.

The Tensor Algebras and Area-preserving Diffeomorphism

There exists an intriguing relationship between this type of tensor algebras and the algebra of an area-preserving diffeomorphism on a two-dimensional surface [18,19], first discovered in [18]. Here we only sketch the basic idea of such an identification.

Consider the area-preserving algebra for the two-dimensional sphere, which can be viewed as being embedded in a three-dimensional Euclidean space with coordinates $x^i, i = 1, 2, 3$, defined by the following constraint

$$x^i x^i = r^2,$$

(15)

where $r$ is a constant. Let us introduce a Lie bracket given by

$$\{x^i, x^j\} = \epsilon_{ijk} x^k,$$

(16)

so that for functions $A(x)$ and $B(x)$ on the sphere the Lie bracket takes the form

$$\{A(x), B(x)\} = \epsilon_{ijk} x^i \partial_j A(x) \partial_k B(x).$$

(17)

The transformation law on $x^i$ generated by the function $f$ is given by

$$\delta f x^i = \epsilon_{ijk} x^j \frac{\partial f}{\partial x_k}.$$  

(18)

It is easy to check that the set of all functions generates the algebra of area-preserving diffeomorphism of the sphere.
In order to identify this algebra with a specific tensor algebra discussed above, one chooses the spherical harmonics given by

$$Y_{1,1} \sim x_1 + ix_2, \quad Y_{1,0} \sim x_3, \quad Y_{1,-1} \sim x_1 - ix_2,$$

from which one can construct the basis for polynomials of higher degree by taking $Y_{\ell,\ell}$ to be $(Y_{1,1})^{\ell}$ and acting on $Y_{\ell,\ell}$ with $Y_{1,-1}$ in the sense of the above Lie bracket. Numerologically one sees that if the following identification

$$V_0^1 \rightarrow Y_{1,1}, \quad V_0^0 \rightarrow Y_{1,0}, \quad V_{-1}^0 \rightarrow Y_{1,-1}$$

is made, then there exists a one-to-one correspondence between the generators of the tensor algebra and polynomials in the basis for generating the area-preserving diffeomorphism on the sphere. Indeed, it can be made rigorous that the area-preserving diffeomorphism on the two-dimensional sphere is isomorphic to the $SL(\infty, R)$ with the parameter $s = \infty$ [19].

Such an identification lends a geometrical flavor to the wedge algebra of $W_\infty$ ($s = 0$), or more precisely to the $SL(\infty, R)$ algebra with $s = \infty$. However, the full $W_\infty$ algebra with its Fourier modes “extending beyond the wedge” does not share this link to geometry. Although there are interesting and tantalizing ideas on how to establish such a link, none of them seems to be solidly demonstrated at this point. Thus $W_\infty$ is different from the algebra of area-preserving diffeomorphism on a two-dimensional surface, and is certainly by no means included in the class of area-preserving diffeomorphism algebras.

* The $w_\infty$ Algebra and Area-preserving Diffeomorphism

An alternative link to geometry that has been demonstrated is the following. If we perform the rescaling $V_m^i \rightarrow q^{-i}V_m^i$ for $W_\infty$, then one can easily see that in the limit $q \rightarrow 0$, all the lower-order terms on the right-hand side of (3) disappear, as do all the central terms apart from the one in the Virasoro subsector. So we see that in this limit the $W_\infty$ algebra contracts down to an algebra with the same content of generators, which is commonly referred to as the $w_\infty$ algebra, first discovered in [20]. It can be written in the following simple form

$$[v_m^i, v_n^j] = (j+1)m - (i+1)n) v_m^{i+j},$$

where we use $v_m^i$ to denote the generators so as to distinguish them from the $V_m^i$ of $W_\infty$.

Now the $w_\infty$ algebra (21) can be enlarged to $w_{1+\infty}$, with conformal spins $s = i+2 \geq 1$ simply by allowing the indices $i$ and $j$ to take the value $-1$ as well as the non-negative
integers. The resulting algebra admits a geometrical interpretation [20,21,22], as the algebra of smooth symplectic (i.e. area-preserving) diffeomorphisms of a cylinder [22]. To see this, consider the set of functions \( u^\ell_m = -ie^{mx}y^{\ell+1} \) on a cylinder \( S^1 \times R \), with coordinates \( 0 \leq x \leq 2\pi, -\infty \leq y \leq \infty \). These functions form a complete set if \( -\infty \leq m \leq \infty \) and \( \ell \geq -1 \). The symplectic transformations of the cylinder, which preserve the area element \( dx \wedge dy \), are generated by \( \delta x^\mu = \{ \Lambda, x^\mu \} \), where \( \Lambda \) is an arbitrary function and \( \{ f, g \} \) is the Poisson bracket

\[
\{ f, g \} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}.
\]

One can easily see, by expanding \( \Lambda \) in terms of the the basis \( u^\ell_m \), that the commutator of symplectic transformations satisfies precisely the algebra (1), with \( v^i_m \rightarrow u^\ell_m \) and \( [f, g] \rightarrow \{ f, g \} \).

So far we have illustrated two different ways of making contact with the area-preserving diffeomorphism. Although both ways employ Poisson brackets, there exists a subtle difference between them. In the first approach, the \( m \) index of \( V^i_m \) is confined with the wedge, while the \( m \) index of \( v^i_m \) has to cover the whole range in order to form a complete basis of functions on a cylindrical surface. This difference becomes important in the context of the \( c = 1 \) string coupled to two-dimensional gravity, where there have been discussions on algebras similar to the area-preserving diffeomorphism on two-dimensional surface.

* The \( W_{1+\infty} \) Algebra

Having extended \( w_\infty \) to include a spin-1 generator, we are naturally led to consider if it is possible to implement such an extension for the uncontracted \( W_\infty \). The answer is positive [14]. Explicitly, the \( W_{1+\infty} \) algebra has the same form as (8) and (9) except that the structure constants are now

\[
\tilde{g}^{ij}_\ell(m,n) = \frac{\tilde{\phi}^{ij}_\ell}{2(\ell+1)!} N^{ij}_\ell(m,n),
\]

\[
\tilde{\phi}^{ij}_\ell = 4F_3 \left[ \begin{array}{cccc} \frac{1}{2}, & \frac{1}{2}, & -\ell - \frac{1}{2}, & -\ell \\ -i - \frac{1}{2}, & -j - \frac{1}{2}, & i + j - 2\ell + \frac{5}{2}, & 1 \end{array} \right],
\]

the \( N^{ij}_\ell(m,n) \) remain the same, and the central charges \( \tilde{c}_i \) are given by

\[
\tilde{c}_i = \frac{2^{2i-2}((i+1)!!)^2}{(2i+1)!!!(2i+3)!!} c
\]

The wedge subalgebra in this case can be enlarged to include the single generator \( V^0_0 \) at the apex, giving the algebra \( GL(\infty, R) \), which again can be constructed as a tensor algebra
of $SL(2, R)$ modulo the same ideal with $s = -\frac{1}{2}$. A contraction of $W_{1+\infty}$, analogous to that described above for $W_\infty$, yields $w_{1+\infty}$ as a classical limit.

Note that in this paper we shall uniformly adopt the notation that structure constants, central charges, currents, gauge fields, etc., for the case of $W_\infty$ gravity will be denoted (as we have been doing so far) by untilded quantities, as opposed to the tilded quantities of $W_{1+\infty}$.

Since the spin content of $W_{1+\infty}$ contains that of $W_\infty$, with an additional spin-1 field, one might think that it should be possible to view $W_\infty$ as a subalgebra of $W_{1+\infty}$. Indeed, this is the case; the details may be found in [17]. An interesting feature of this inclusion is that the central charge of the $W_\infty$ subalgebra is $-2$ times the $W_{1+\infty}$ central charge.

* $W_\infty$ and the Algebra of All Differential Operators on $S^1$

To close this section, we shall present another viewpoint for $W_\infty$, which extends the well-known interpretation of the Virasoro algebra as the centrally-extended $Diff S^1$. Intuitively it is very natural to expect that the centerless $W_\infty$ algebra be realized in terms of all differential operators on a circle, given the fact that the centerless Virasoro algebra is represented by the vector differential on a circle. This is indeed the case, as was demonstrated in [17], for the centerless $W_\infty$ algebra. The question that remains is whether there exists a unique central extension or cocycle structure for the algebra of all differentials on a circle that coincides with the one given in the $W_\infty$ algebra. Recently an affirmative answer has been given in [23,59].

This standpoint to view $W_\infty$ as an extension of Virasoro was suggested some time ago [24] in the context of fermion bilinears. From this angle, the existence of an algebra as such is very transparent. The more difficult part is to explicitly obtain the structure constants as well as the structure for central charges. The original brute-force method [12] that has produced beautiful yet inexplicable algebraic structure has been nicely complemented by the geometrical understanding in ref.[23,59].

3. REALIZATIONS OF $W_\infty$

Firstly we shall summarize the known global realizations of $W_\infty$. In comparison to the many known classes of realizations of Virasoro, realizations of $W_\infty$ in field theory are relatively scarce, despite the kinship that we have stressed between the two. So far, there exist essentially only two realizations for $W_\infty$. The first one discovered in [25] is a bosonic realization, while the second [26,27] is a fermionic realization. A common feature shared
by the two realizations is that the generating currents of $W_\infty$ are built out of bilinears of either a boson or a fermion. Thus the transformations of $W_\infty$ currents on either a boson or a fermion in these two realizations are always linear. Non-linear realizations of $W_\infty$ may appear by bosonizing either fermion [28] or boson [29]. A more detailed summary of these realizations and their interrelationships can be found in [29].

Consider a free, complex scalar field $\phi$, with OPE

$$\phi^*(z)\phi(w) \sim -\log(z-w).$$

(25)

Virasoro can be realized in terms of $\phi$ simply as follows.

$$T(z) = -\partial \phi^* \partial \phi.$$  

(26)

To realize higher-spin currents, one considers bilinears of $\phi$ with higher numbers of derivatives. One can easily establish that at each order in the total number of derivatives distributed over the two fields, there is exactly one independent current; any other combination of the same number of derivatives distributed over the two fields can always be expressed as a linear combination of the independent current at that order together with derivatives of the lower-spin independent currents. Thus it is clear that the OPEs of all possible higher-spin independent currents will form a closed algebra. It turns out that these bilinears indeed give a realization of $W_\infty$. From this viewpoint, the existence of $W_\infty$ is very transparent. The currents $V^i(z)$ of $W_\infty$, related to the Fourier-mode components $V^i_m$ by

$$V^i(z) \equiv \sum_m V^i_m z^{-m-i-2},$$

(27)

are given by [25]

$$V^i(z) = \sum_{k=0}^i \alpha^i_k : \partial^{i-k+1} \phi^* \partial^{k+1} \phi :,$$

(28)

where the constants $\alpha^i_k$ are given by

$$\alpha^i_k = (-)^{i+k+1} \frac{2^{i-1}(i+2)!}{(2i+1)!!(i+1)} \binom{i+1}{k} \binom{i+1}{k+1}.$$  

(29)

This particular choice for the coefficients is the unique possibility (up to overall $i$-dependent scalings) that ensures that the central terms in the algebra are diagonal, i.e. that they arise only between fields of the same spin. Together with the specific $i$-dependence in (29), this choice gives precisely the standard form of the $W_\infty$ algebra.
There is a straightforward procedure for re-expressing the Fourier-mode forms of the $W_\infty$ algebra (8) and (9) in terms of operator-product expansions of the corresponding currents defined by (27). To do that, we first define $f_{\ell}^{ij}(m,n)$ given by

$$f_{\ell}^{ij}(m,n) = \frac{\phi_{\ell}^{ij}}{2(\ell+1)!} M_{\ell}^{ij}(m,n),$$

(30)

where the $M_{\ell}^{ij}(m,n)$ are given by

$$M_{\ell}^{ij}(m,n) = \sum_{k=0}^{\ell+1} (-1)^k \frac{(\ell+1)}{k} (2i-\ell+2)k[2j+2-k]_{\ell+1-k} m^{\ell+1-k} n^k,$$

(31)

and $\phi_{\ell}^{ij}$ is the same as given in (13). Here $[a]_n$ is the same as that given in (12) and $(a)_n$ is given by

$$(a)_n = a(a+1) \cdots (a+n-1).$$

(32)

Thus in terms of the polynomials $f_{\ell}^{ij}(m,n)$, the $W_\infty$ algebra in Fourier mode form in (8) and (9) can be expressed concisely in OPE by the following:

$$V^i(z)V^j(w) \sim -c_1 \delta^{ij}(\partial_z)^{2i+3} \frac{1}{z-w} - \sum_{\ell} f_{2\ell}^{ij}(\partial_z,\partial_w) \frac{V^{i+j-2\ell}(w)}{z-w}.$$  

(33)

Now one can verify that indeed the OPEs for the currents defined in (28) agree with those for the $W_\infty$ algebras with $c = 2$.

Another characteristically similar realization of $W_\infty$ is obtained in terms of bilinears of a free, complex fermion $\psi$, with OPE

$$\overline{\psi}(z)\psi(w) \sim \frac{1}{z-w}.$$  

(34)

The spin-2 stress tensor $T$ is realized as

$$T(z) = \partial \overline{\psi} \psi,$$

(35)

with $c = -2$, which is often known as a $(0,1)$ ghost realization of Virasoro. The higher-spin currents of $W_\infty$ are now given by

$$V^i(z) = \sum_{k=0}^{i+1} \beta_k z^k \overline{\psi} \partial^{i+1-k} \psi,$$

(36)
where $\beta^i_k$ is given by

$$
\beta^i_k = \binom{i+1}{k} \frac{(i+3-k)k(-i)_{i+1-k}}{(i+2)_{i+1}}.
$$

(37)

Since the transformations of $W_\infty$ on a generic field $\Phi$ are given by

$$
\delta_{k_i} \Phi \equiv \oint \frac{dz}{2\pi i} k_i(z) V^i(z) \Phi,
$$

(38)

where $k_i$ are transformation parameters of the $i$-th current $V^i$, the fields $\psi$, $\phi$ and their conjugates transform linearly, owing to the fact that the currents of $W_\infty$ are all bilinears. This linearity of transformation laws will be important when we quantize field theories with local $W_\infty$ symmetry and analyze their anomaly structures, which will be discussed in the first topic of quantization.

A non-linear realization of $W_\infty$ is obtained by bosonizing the fermion $\psi$, as was first done in [28]. With the identification $\psi \rightarrow e^\phi$ and $\psi \rightarrow e^{-\phi}$, and the OPE for $\phi$ given by

$$
\phi(z)\phi(w) \sim \log(z-w),
$$

(39)

the currents $V^i(z)$ of $W_\infty$ in terms of the scalar field $\phi$ can be obtained by making the following replacement in (36) :

$$
\partial^k \overline{\psi} \partial^{i+1-k} \psi \equiv \sum_{\ell=k+1}^{i+2} \frac{(-1)^{\ell+k+1}}{\ell} \binom{i+1-k}{\ell-1-k} \partial^{i+2-\ell} P(\ell)(z),
$$

(40)

where $P(\ell)(z)$ is given by

$$
P(\ell)(z) = e^{-\phi(z)} \partial^\ell e^{\phi(z)}.
$$

(41)

Since the currents are non-linear in the new scalar field $\phi$, they induce non-linear transformations on $\phi$. For example, the leading order in $\phi$ of the current $V^i$ is $1/2 \partial^i \partial \phi$, and the corresponding transformations on $\phi$ resulting from such a term in the generating current are given by

$$
\delta \phi = k_i (\partial \phi)^{i+1},
$$

(42)

which is manifestly non-linear in $\phi$.

One can easily check that the transformations given in (42) in fact close to form an algebra,

$$
[\delta_{k_i}, \delta_{k_j}] \phi = \delta_{k_{i+j}} \phi,
$$

(43)

where $k_{i+j}$ is given by

$$
k_{i+j} = (j+1) k_j \partial k_i - (i+1) k_i \partial k_j.
$$

(44)
One sees that this is nothing but the $w_\infty$ algebra. Algebraically $w_\infty$ arises as a contraction of $W_\infty$, as discussed in the previous section. In the context of field theory, the procedure of extracting a realization of $w_\infty$ from a bilinear-fermion realization of $W_\infty$ involves a rather subtle intermediate step of bosonization. The leading order terms in the bosonized field $\varphi$ generate transformations that close to form $w_\infty$. In the language of OPEs among the currents containing only the leading order terms in $\varphi$, the closure of $w_\infty$ is maintained if only single contractions of the field $\varphi$ are allowed, which are equivalent to taking the classical Poisson bracket of $\varphi$. Thus $w_\infty$ is realized classically in the scalar field $\varphi$, with its currents $v^i(z)$ given by

$$v^i(z) = \frac{1}{i+2} (\partial \varphi)^i + 2.$$ (45)

We shall end our discussion on realizations of $W_\infty$ with some comments about $w_\infty$. Algebraically $w_\infty$ does not retain one of the non-trivial properties of Virasoro, namely the central extensions. In the last topic of quantum $W_\infty$ gravity, we shall show that quantum $w_\infty$ theory seems to have little dynamics, due to the lack of central terms in the algebra. However, owing to the simplicity in its structure constants, it is often much easier to find a classical realization of $w_\infty$, such as the one given above. Having such a classical realization, one can proceed to build a classical $w_\infty$ gauge theory that is simple enough to illustrate the basic points, a task which we shall undertake presently. Such a theory, in the process of quantization, picks up additional terms in order to achieve quantum consistency in such a way that the underlying symmetry deforms into the full $W_\infty$. Therefore classical realizations of $w_\infty$ are important and can serve as convenient starting points for addressing many issues of $W_\infty$.

4. $W_\infty$ GRAVITY AND $W_\infty$ STRING

Two-dimensional gravity can be thought of as a gauging of the Virasoro algebra. An analogous gauging of the $W$ algebras will therefore give higher-spin generalizations of two-dimensional gravity. Such theories are known generically as $W$ gravity theories. They have been discussed in the context of a chiral gauging of $W_3$ [30]; a non-chiral gauging of $W_3$ [31]; chiral and non-chiral $w_\infty$ [32], $W_\infty$ and super $W_\infty$ [33].

The starting point for our $w_\infty$ gravity is a free Lagrangian for the scalar field $\varphi$, of the form

$$\mathcal{L} = \frac{1}{2} \partial \varphi \bar{\partial} \varphi,$$ (46)
where $z$ and $\bar{z}$ are the coordinates of the two-dimensional space-time. This action has many global symmetries. In fact, because of the factorization into left-moving and right-moving sectors in two dimensions, a “global” symmetry typically means one that has a parameter that depends on only $z$ or $\bar{z}$, but not both.

One particular global symmetry of (46) consists of transformations given in (42), which form the $w_\infty$ algebra. This symmetry can be made local by introducing gauge fields $A_\ell$ for each of the spin-$(\ell + 2)$ conserved currents $(\partial \varphi)^{\ell + 2}$ corresponding to the symmetries given in (42) with parameter $k_i$ dependent on both $z$ and $\bar{z}$, and writing

$$\mathcal{L} = \frac{1}{2}(\partial \varphi \partial \varphi) - \sum_{\ell \geq 0} \frac{1}{\ell + 2} A_\ell (\partial \varphi)^{\ell + 2}. \tag{47}$$

One finds that this is invariant under (42) provided that the gauge fields transform as

$$\delta A_\ell = \bar{\partial} k_\ell - \sum_{j=0}^{\ell+1} [(j+1)A_j \partial k_{\ell-j} - (\ell-j+1)k_{\ell-j} \partial A_j]. \tag{48}$$

In particular, if we focus attention on the spin-2 sector only, we recover two-dimensional gravity in the chiral gauge.

The above chiral gauging can be extended to a non-chiral one by observing that the free action (46) is also invariant under a second copy of $w_\infty$, where $\partial$ in (42) is replaced by $\bar{\partial}$, and the parameters of the transformations are taken to depend upon $\bar{z}$ only. The two copies of $w_\infty$ are made into local symmetries by introducing two sets of gauge fields, $A_\ell$ and $\bar{A}_\ell$, where the $A_\ell$ gauge the original copy of $w_\infty$, and the $\bar{A}_\ell$ gauge the second copy. The action in this case becomes more complicated, and is most conveniently written by introducing auxiliary fields $J$ and $\bar{J}$. The required action is then given by [32]

$$\mathcal{L} = -\frac{1}{2} \partial \varphi \bar{\partial} \varphi - J \bar{J} + \partial \varphi J + \partial \varphi \bar{J} - \sum_{\ell \geq 0} \frac{1}{\ell + 2} \left( \bar{A}_\ell \bar{J}^{\ell + 2} + A_\ell J^{\ell + 2} \right). \tag{49}$$

The equations of motion for the auxiliary fields $J$ and $\bar{J}$ give

$$J = \partial \varphi - \sum_{\ell=0} \bar{A}_\ell J^{\ell + 1},$$

$$\bar{J} = \bar{\partial} \varphi - \sum_{\ell=0} A_\ell J^{\ell + 1}. \tag{50}$$
It is straightforward to check that this Lagrangian is invariant under the $k_\ell$ and $\bar{k}_\ell$ gauge transformations

$$
\delta \varphi = \sum_{\ell \geq -1} \left( k_\ell \bar{J}^{\ell+1} + \bar{k}_\ell J^{\ell+1} \right)
$$

$$
\delta A_\ell = \bar{\partial} k_\ell - \sum_{j=0}^{\ell+1} [(j+1) A_j \bar{\partial} k_{\ell-j} - (\ell-j+1) k_{\ell-j} \partial A_j]
$$

$$
\delta A_\ell = \partial \bar{k}_\ell - \sum_{j=0}^{\ell+1} [(j+1) \bar{A}_j \partial \bar{k}_{\ell-j} - (\ell-j+1) \bar{k}_{\ell-j} \partial \bar{A}_j]
$$

$$
\delta J = \sum_{\ell \geq -1} \partial (k_\ell J^{\ell+1})
$$

$$
\delta \bar{J} = \sum_{\ell \geq -1} \partial (\bar{k}_\ell \bar{J}^{\ell+1}).
$$

(51)

Note that $J$ is inert under the $\bar{k}$ transformations while $\bar{J}$ is inert under $k$ transformations.

For $W_\infty$, one proceeds in a similar manner to the one described above, starting, for example, from a free Lagrangian of a complex fermion $\psi$ given by

$$
\mathcal{L} = \bar{\psi} \partial \psi.
$$

(52)

One sees that there exist globally conserved currents of $W_\infty$ given in (36), which induce the following global transformations on $\psi$ and $\bar{\psi}$ according to (38):

$$
\delta_{k_i} \psi = \sum_{k=0}^{i+1} (-1)^{k+1} \beta_i^k \partial^k (k_i \partial^{i+1-k} \psi)
$$

$$
\delta_{\bar{k}_i} \bar{\psi} = \sum_{k=0}^{i+1} (-1)^{i+1-k} \beta_i^k \partial^{i+1-k} (k_i \partial^k \bar{\psi})
$$

(53)

To gauge the chiral $W_\infty$ symmetry, we now allow the parameters $k_i$ to depend on $\bar{z}$ as well as $z$. Now the free action $\mathcal{L}$ is not invariant, and the remaining term in the variation of $\mathcal{L}$ arising from the local parameters $k_i(z, \bar{z})$ reads

$$
\delta_{k_i} \mathcal{L} = -\bar{\partial} k_i \sum_{k=0}^{i+1} \beta_i^k \partial^k \bar{\psi} \partial^{i+1-k} \psi = -\bar{\partial} k_i V_i(z),
$$

(54)

so we must also introduce gauge fields $A_i$ and add the (gauge field)-(current) coupling terms to the Lagrangian:

$$
\mathcal{L} = \bar{\psi} \partial_- \psi + A_i V^i.
$$

(55)
The Noether procedure now goes as follows. Now the variation in (54) is cancelled by the leading-order transformation of the gauge field which is of the form \( \delta A_i = \bar{\partial} k_i + \cdots \). We next vary the currents by using the formula (38) with \( \Phi \) taken to be \( V^j \). The result can conveniently be expressed in the form

\[
\delta k_j V^i(z) = \sum_{\ell} \int dw k_j(w) f_{2\ell}^{ij}(\partial_z, \partial_w) \left( \delta(z-w)V^{i+j-2\ell} \right),
\]

where \( f_{2\ell}^{ij}(m,n) \) is given in (30). We see that this variation is cancelled by adding terms to the transformation rule of the gauge field \( A_i \) so that its total variation \( \delta A_i = \bar{\partial} k_i + \hat{\delta} A_i \) is given by

\[
\delta A_i = \bar{\partial} k_i + \sum_{\ell=0}^{\infty} \sum_{j=0}^{i+2\ell} f_{2\ell}^{j,i-j+2\ell}(\partial_A, \partial_k) A_j k_{i-j+2\ell},
\]

where \( \partial_A \) and \( \partial_k \) are the \( \partial \) derivatives acting on \( A \) and \( k \), respectively. The term \( \hat{\delta} A_i \) is a co-adjoint transformation of \( A_i \), while the \( V^i \) transforms in the adjoint representation of \( W_\infty \); i.e. \( \int (\hat{\delta} A_i V^i + A_i \delta V^i) = 0 \). This model of a complex fermion coupled to \( W_\infty \) gauge fields is in fact intimately related to the model with a single scalar coupled to \( w_\infty \) gauge fields we also just discussed, as will be shown presently.

5. QUANTIZATION DEFORMS \( w_\infty \) TO \( W_\infty \)

The programme of quantization in the case of \( W_\infty \) again follows closely that of Virasoro. Consider an action with local \( W_\infty \) symmetry on a two-dimensional “world-sheet,” which is a functional of \( W_\infty \) gauge fields denoted collectively by \( A \) and generic matter fields \( \Phi \). In the path-integral formalism, the partition function can be formally written as the following:

\[
Z = \int [DA][D\Phi] e^{\frac{1}{\hbar} \int \mathcal{L}(A,\Phi)}.
\]

One can now proceed in two stages. Firstly one integrates out the matter fields to arrive at an effective action of the gauge fields \( A \), defined by

\[
\Gamma[A] \equiv \log \int [D\Phi] e^{\frac{1}{\hbar} \int \mathcal{L}}.
\]

The second stage consists of quantizing the gauge fields in the effective action \( \Gamma \) that results from integrating out the matter fields in the first stage.

The main issue that will be discussed is the quantum consistency of \( W_\infty \) gravity and the \( W_\infty \) string. We shall investigate this issue in the model of \( w_\infty \) gravity coupled with a
single scalar that we discussed above in the classical picture, and demonstrate perturbatively that the process of quantizing the matter field $\varphi$ coupled to the gauge fields of $w_\infty$ gravity “renormalizes” the theory to be $W_\infty$ gravity in order to maintain quantum consistency [34]. This is the first indication that the theory of $w_\infty$ makes sense only at the classical level. This statement will be strengthened later when quantum $w_\infty$ gravity (coupled to some matter system, in which case its quantization does not lead to inconsistency) is shown to have little dynamics [35].

Quantum consistency requires that, after the completion of the first stage of quantization, the effective action $\Gamma[A]$ be invariant under gauge transformations on $A$. In general this is often not the case; an anomaly may occur. For a symmetry realized non-linearly in matter fields such as the one of $w_\infty$ realized in a single scalar $\varphi$ given in (45), there are two types of anomalies. The first type is called universal anomaly, which is only dependent on the gauge fields themselves. The second is called matter-dependent anomaly, which explicitly depends on the matter fields that are integrated out. It is necessary to remove the matter-dependent anomaly at once, because it is illogical for the effective action to be dependent on matter fields that are supposed to have been integrated out in obtaining the effective action. To achieve this, one adds finite local counter-terms order by order in loops to the classical Lagrangian. The addition of these terms changes the couplings between the matter field and the gauge fields, thus modifying currents of the symmetry in such a way that $w_\infty$ deforms into $W_\infty$. Consequentially the transformation laws for the gauge fields are modified in response to the change of the symmetry.

To illustrate precisely how this procedure is implemented, we shall go over the discussion on the following 1-loop diagrams that give rise to a matter-dependent anomaly, whose removal deforms the symmetry structure of the theory.

Fig. 1
The first diagram that can generate matter-dependent anomalies in the \( w_\infty \) algebra is given in Fig. 1. Its contribution to the effective action is

\[
\Gamma_{01}\varphi = \frac{1}{\pi^2} \int d^2z d^2w A_0(z) A_1(w) \frac{1}{(z-w)^4} \partial \varphi(w)
\]

\[
= -\frac{1}{6\pi} \int d^2z d^2w A_0(z) A_1(w) \frac{\partial^3}{\partial z} \delta^2(z-w) \partial \varphi(w)
\]

\[
= -\frac{1}{6\pi} \int d^2z \left( \frac{\partial^3}{\partial A_0(z)} \right) A_1(z) \partial \varphi(z). \tag{60}
\]

Under the leading order inhomogeneous terms in the gauge transformations (2.3) (\( \delta A_0 = \bar{\partial} k_0 + \cdots \), \( \delta A_1 = \bar{\partial} k_1 + \cdots \)) the anomalous variation of \( \Gamma_{01}\varphi \) is

\[
\delta \Gamma_{01}\varphi = -\frac{1}{6\pi} \int d^2z (A_1 \partial^3 k_0 - k_1 \partial^3 A_0) \partial \varphi. \tag{61}
\]

Note that in the derivation of this result, one may drop terms proportional to the \( \varphi \) field equation, since these cancel in the quantum Ward identity [34] against terms involving operator insertions of the \( \varphi \) transformations into the relevant one-loop diagrams.

The anomalous variation (61) can be cancelled by adding the finite local counterterms \( L_{1/2} + L_1 \), given by

\[
L_{1/2} = \frac{1}{2} \left( A_0 \partial^2 \varphi + A_1 \partial \varphi \partial^2 \varphi \right), \tag{62}
\]

\[
L_1 = \frac{1}{12} A_1 \partial^3 \varphi, \tag{63}
\]

and by simultaneously correcting the \( \varphi \) transformation (42) by the extra terms \( \delta_{1/2}\varphi + \delta_1\varphi \) given by

\[
\delta_{1/2}\varphi = -\frac{1}{2} \left( \partial k_0 + \bar{\partial} k_1 \partial \varphi \right), \tag{64}
\]

\[
\delta_1\varphi = \frac{1}{12} \partial^2 k_1. \tag{65}
\]

One can check that \( \delta_{1/2}L_0 + \delta_0 L_{1/2} = 0 \), so up to 1-loop order, the remaining anomaly-cancelling terms as desired read

\[
\delta_0 L_1 + \delta_{1/2} L_{1/2} + \delta_1 L_0. \tag{66}
\]

These variations cancel the anomalies in (61) completely.

The occurrence of the counterterms (62) and (63) implies that the original spin-2 and spin-3 currents of the form (45) have received corrections, so that they now take the form

\[
V^0 = \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} \partial^2 \varphi, \tag{67}
\]

\[
V^1 = \frac{1}{3} (\partial \varphi)^3 + \frac{1}{2} \partial \varphi \partial^2 \varphi + \frac{1}{12} \partial^3 \varphi. \tag{68}
\]
The transformation rules for the matter field $\varphi$, including the corrections (64) and (65), are precisely those that follow from the standard expression (38) with $V^i$ replaced by the expressions given for spin-2 and spin-3 by (67) and (68).

One can in principle proceed, by looking at higher-order diagrams with higher-spin external gauge fields, to determine the appropriate modifications to all the higher-spin currents that are needed in order to remove matter-dependent anomalies. At the same time, the transformation rules for the $\varphi$ field will require higher-spin modifications too. As in the sample diagram studied above, the modifications to the $\varphi$ variation will be precisely those that follow by substituting the modified currents into (38). There are further kinds of matter-dependent anomalies, of types that are not illustrated by the diagram in Fig. 1, whose cancellation requires that the gauge-transformation rules (48) should also be modified. To build up the entire structure of the modifications to currents and transformation rules by these diagrammatic methods would clearly be a cumbersome procedure. We shall just consider one more diagram to illustrate the way in which the gauge-transformation rules (48) must receive corrections.

Fig. 2

The simplest diagram that gives rise to a matter-dependent anomaly whose removal requires making modifications to the gauge-field transformation rules is shown in Fig. 2. It produces a contribution to the effective action given by

$$\Gamma_{11\varphi\varphi} = -\frac{1}{6\pi} \int d^2 z A_1(z) \partial\varphi(z) \frac{\partial^3}{\partial^3} \left( A_1(z) \partial\varphi(z) \right).$$

(69)

This gives rise to an anomalous variation with respect to the leading-order inhomogeneous term in the $A_1$ variation, i.e. $\delta A_1 = \bar{\partial} k_1 + \cdots$:

$$\delta \Gamma_{11\varphi\varphi} = \frac{1}{3\pi} \int d^2 z A_1(z) \partial\varphi(z) \partial^3 \left( k_1(z) \partial\varphi(z) \right).$$

(70)
Cancellation of this anomaly requires, in addition to the modifications to the spin-2 and spin-3 currents in (67) and (68) and the modifications (64) and (65) to the \( \varphi \) transformation rules, a correction to the spin-2 gauge transformation rule:

\[
\delta_1 A_0 = \frac{1}{20} \left( 2\partial^3 A_1 k_1 - 3\partial^2 A_1 \partial k_1 + 3\partial A_1 \partial^2 k_1 - 2A_1 \partial^3 k_1 \right),
\]

(71)

and a new counterterm

\[
L_1 = A_2 \left( \frac{1}{3} \partial \varphi \partial^3 \varphi - \frac{1}{20} (\partial^2 \varphi)^2 \right).
\]

(72)

This counterterm implies that the spin-4 current receives quantum corrections. There are other anomaly diagrams that give rise to further quantum corrections to the spin-4 current. Indeed they turn out to be precisely the spin-4 current of \( W_\infty \) realized in terms of a single scalar.

We have now seen how the mechanism for cancelling the matter-dependent anomalies arising from the diagrams in Fig. 1 and Fig. 2 leads to quantum corrections to the currents, and to the matter and gauge-field transformation rules. These constructions can be carried out to arbitrary order in principle. For now we shall be content with just these two examples in drawing the conclusion that quantum consistency promotes \( w_\infty \) to \( W_\infty \) in the process of renormalization (see ref. [34] for more details).

There is in fact a much more elegant way of understanding the consistent theory of a matter field \( \varphi \) coupled to the \( W_\infty \) gauge fields, given in [34]. One can “fermionize” the field \( \varphi \) to obtain a matter system described by a complex fermion \( \psi \). The advantage in taking such a viewpoint is that the \( W_\infty \) currents (obtained from the \( w_\infty \) currents in terms of \( \varphi \) plus modifications from renormalization) realized non-linearly in terms of \( \varphi \) convert into bilinears in terms of the fermions, which implies transformations on \( \psi \). In this picture, 1-loop diagrams never have external \( \psi \) legs; \( \psi \) only appears in the internal loops, because of the fact that there exist only three-point couplings between \( \psi \) and the gauge fields \( A_i \). Hence there is no matter-dependent anomaly after the quantization of \( \psi \). Thus as far as the matter-dependent anomaly is concerned with regard to quantum consistency, the system of a complex fermion matter field coupled to \( W_\infty \) gravity makes perfect sense. In fact it is precisely the model of \( W_\infty \) gravity discussed in our previous section.

We now proceed to the second stage in quantizing the gauge fields. Although we have succeeded in getting rid of the matter-dependent anomaly, there remains the universal anomaly, which spoils the invariance of the effective action under \( W_\infty \) gauge transformations. However the universal anomaly, though undesirable, is strongly dictated by the structure of \( W_\infty \).
symmetry (hence the name “universal,” for it is not dependent on any particular matter system to realize the symmetry) and has a very simple form. As we shall show presently, in the light-cone gauge it is given by
\[
\delta_k \Gamma = \sum_{i \geq 0} c_i \int d^2 z k_i \partial^{2i+3} A_i,
\]  
where \(c_i\) are related to the total central charge \(c_{\text{total}}\) of the Virasoro sector of \(W_{\infty}\) (14).

We shall derive (73) in the operator formalism. Let us define
\[
\langle \mathcal{O} \rangle \equiv \int [D\Phi] e^{\frac{1}{\pi} \int \mathcal{L}_0 \mathcal{O}},
\]  
where \(\mathcal{L}_0\) is the free Lagrangian of a generic matter system with its fields denoted by \(\Phi\); \(\mathcal{O}\) denotes a generic operator. Thus the effective action can be written in this language as follows.
\[
e^{-\Gamma(A_i)} = \langle \exp(-\frac{1}{\pi} \int \sum_i A_i V_i) \rangle.\]  
(75)

Varying (75) with respect to \(A_i(z)\), one finds
\[
\frac{\delta \Gamma}{\delta A_i(z)} = \frac{1}{\pi} \langle V^i(z) \exp(-\frac{1}{\pi} \int \sum_j A_j V^j) \rangle e^\Gamma\]  
and hence
\[
\bar{\partial} \frac{\delta \Gamma}{\delta A_i(z)} = \frac{1}{\pi} \langle \bar{\partial} V^i(z) \exp(-\frac{1}{\pi} \int \sum_j A_j V^j) \rangle e^\Gamma.\]  
(77)

The occurrence of the \(\bar{\partial} = \partial_{\bar{z}}\) derivative in (77) means that the only non-zero contributions will come from \(\bar{\partial}\) acting on singular terms in the operator product expansion of the operators. Thus, we may calculate
\[
\bar{\partial} V^i(z) \exp(-\frac{1}{\pi} \int \sum_j A_j V^j) = \bar{\partial} V^i(z) \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{1}{\pi} \int A_j V^j\right)^n = -\frac{1}{\pi} \sum_{n \geq 1} \frac{1}{(n-1)!} \int d^2 w [\bar{\partial} V^i(z) V^j(w)] A_j(w) \left(-\frac{1}{\pi} \int A_k V^k\right)^{n-1},\]  
(78)

where the brackets around \(\bar{\partial} V^i(z) V^j(w)\) in (78) indicate that the operator product expansion should be taken just between these two operators.

Using (33), the operator products in (78) may be evaluated, to give
\[
\bar{\partial} V^i(z) \exp(-\frac{1}{\pi} \int \sum_i A_i V^i) = \frac{1}{\pi} \int d^2 w \partial_z \left( \sum_{\ell \geq 0} f^{ij}_{\ell}(\partial_z, \partial_w) \frac{V^{i+j-2\ell}(w)}{z-w} + c_i \partial_z^{2i+3} \frac{1}{z-w} \right) A_j(w) \exp(-\frac{1}{\pi} \int A_k V^k).\]  
(79)
Since $\partial_z \frac{1}{z-w} = \pi \delta^{(2)}(z-w)$, we may perform the integration in (79). Thus we find, from (79), that

$$
\bar{\partial} \frac{\delta \Gamma}{\delta A_i} + \sum_{\ell \geq 0} f_{2\ell}^{ij}(\partial, -\partial_A) \left( \frac{\delta \Gamma}{\delta A_{i+j-2\ell}} A_j \right) = -\frac{c_i}{\pi} \partial^{2i+3} A_i. \tag{80}
$$

The subscript $A$ on the second derivative argument of $f^{ij}$ indicates that it should act only on the explicit $A_j$ in the parentheses that follows it, whilst the first derivative argument of $f^{ij}$ acts on all terms in the parentheses. Eq. (80) is the anomalous Ward identity for $W_{\infty}$ gravity.

It is easy now to obtain the variation of the effective action when the gauge fields $A_i$ are transformed according to laws of $W_{\infty}$ given in (57). If we now multiply (80) by the spin-$(i+2)$ transformation parameters $k_i$ and integrate, we find

$$
\int \frac{\delta \Gamma}{\delta A_i} \left( \bar{\partial} k_i + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} f_{2\ell}^{i-j+2\ell,j}(\partial, \partial_A) k_{i-j+2\ell} A_j \right) = \frac{c_i}{\pi} \int k_i \partial^{2i+3} A_i. \tag{81}
$$

Now we see that the left-hand side of this equation involves precisely the $W_{\infty}$ gauge-transformation rule for $A_i$ under $k_j$ given in (57), and so it can be written simply as $\delta_k \Gamma$, which is nothing but the desired result in (73).

Since the universal anomaly spoils the invariance of the effective action under $W_{\infty}$ symmetry, it has to be removed in order to have a fully consistent theory. There are two possibilities. The first one is that the total central charge vanishes, in which case the dynamics of the gauge fields decouples from the matter system involved. In string terminology, this is known to be the critical case. The other possibility in the light-cone gauge is to restrict the dynamics of the gauge fields by the following vanishing condition:

$$
\partial^{2i+3} A_i = 0. \tag{82}
$$

This case is often known in string theory as non-critical. Note that in ordinary string theory, the non-critical situation has the well-known elegant formulation in the conformal gauge, where the dynamics of two-dimensional gravity is described by the Liouville field [36]. The Liouville field contributes to the total stress tensor in such a way that the total central charge of both the matter and the Liouville (gravity) sectors amounts to the critical value. For the $W_{\infty}$ string, it is not clear how such an analogue should be formulated.

It is necessary now to take full account of the central charges. The contribution from $W_{\infty}$ matter is simple to account for, i.e. the central charge of its Virasoro sector. The other
important part is the ghost contribution to the total central charge that arises from the need to avoid over-counting in the path-integral over the gauge fields, due to the gauge degrees of freedom. Again this is basically a property of the gauge symmetry itself, irrespective of the matter system involved; thus it can be dealt with quite independently. The most convenient method of removing such a redundancy arising from gauge degrees of freedom is the well-known BRST formalism, which is our next topic.

6. THE BRST ANALYSIS OF $W_\infty$

Before plunging into the full BRST analysis of $W_\infty$, which is rather demanding technically, we shall start with some well-known results on this issue for the case of the finite-$N$ $W_N$ algebra, and try to grasp some ideas about what the result in the limit $N \to \infty$ would be. It is expected that there might not be a unique limiting procedure, which is the case for the algebraic structure of finite-$N$ $W_N$ in the large $N$ limit; nonetheless such a strategy may still prove to be instructive.

For a theory with local $W_N$ gauge symmetry, it is well known that the ghost contribution to the total anomaly in the Virasoro sector is given by

$$c_{gh} = \sum_{s=2}^{N} c_{gh}(s),$$ \hfill (83)

where

$$c_{gh}(s) = -2(6s^2 - 6s + 1)$$ \hfill (84)

is the contribution from a pair of ghosts with spins $(1-s, s)$ for the spin-$s$ gauge fields. The ghosts are necessary in order to remove the over-counting in the integration over the spin-$s$ gauge field. Thus, after the summation, (83) becomes

$$c_{gh} = -(N-1)(4N^2 + 4N + 2).$$ \hfill (85)

In addition, there will also be ghost contributions to the anomalies in all the higher-spin sectors. Of course the values of the central-charge contributions in the various spin sectors are all related to one another, since there is just one overall central-charge parameter in the $W_N$ algebra.

Naively, by setting $N = \infty$ in (85), one would think that the total ghost contribution in the Virasoro sector of $W_\infty$ would be $c_{gh} = -\infty$. Such a scenario, though logical, is arguably
less desirable and manageable. A more appealing approach is to treat the divergent sum (85) over the individual spin-$s$ contributions as a quantity that should be rendered finite by some regularization procedure [37]. Likewise, the ghost contributions in all the higher-spin sectors will be given by divergent sums, which can also be regularized. The regularization procedures for each spin must be consistent with one another, since there is just one overall central-charge parameter in the $W_\infty$ algebra. In [38], it was shown that a natural zeta-function regularization scheme gives the regularized result

$$c_{gh} = 2.$$  \hspace{1cm} (86)

The trick is to introduce the generalized zeta function defined through analytic continuation in $s$ of the sum

$$\zeta(s,a) = \sum_{k \geq 0} (k+a)^{-s}, \hspace{1cm} (87)$$

which converges for $s > 1$. Thus (83) can now be written as follows.

$$c_{gh} = -6\zeta(-2,\frac{3}{2}) + \frac{1}{2}\zeta(0,\frac{3}{2}), \hspace{1cm} (88)$$

which gives (86) in the sense of analytical continuation.

A consistent extension of this regularization scheme to all spin sectors was proposed in [38], where it was shown that it gave consistent results at least up to the spin-18 level. The fact that such a universal scheme exists is highly suggestive of an underlying interpretation and rigorous justification for the regularization procedure, possibly in terms of a higher-dimensional theory. Next we shall only review the mechanical procedure of this regularization scheme, leaving aside the more difficult question of finding an underlying reason.

The standard prescription for constructing the BRST charge for a Lie algebra with structure constants $f^{ab}_{c}$ is

$$Q = c_a T^a - \frac{1}{2} f^{ab}_{c} c_a c_b b^c,$$ \hspace{1cm} (89)

where $c_a$ and $b^a$ are the ghosts (anticommuting for a bosonic algebra) that satisfy $\{ c_a, b_b \} = \delta^b_a$, with the other anticommutators vanishing. $Q$ may be written as $Q = Q_T + Q_{gh}$, with $Q_T = c_a T^a$ and $Q_{gh} = \frac{1}{2} f^{ab}_{c} c_a T^a_{gh}$, where $T^a_{gh} = \{ Q, b^a \}$ gives a ghostly realisation of the algebra. The generic index in (89) for the case of an infinite-dimensional algebra such as Virasoro can be either the $z$ coordinate of the spin-2 stress tensor or its Fourier mode index. The former is the BRST analysis in OPE language, while the latter is somewhat conventional.
Here we shall present the analysis in the Fourier mode convention. For the \( W_\infty \) algebra, given by (8-14), \( Q \) in (89) becomes
\[
Q = \alpha_0 c_0 + \sum_{i,m} V^i_m c^m_i - \frac{1}{2} \sum_{i,j,\ell,m,n} g_{2\ell}(m,n) : c^{-m}_i c^{-n}_j b^{m+n}_{i+j-2\ell} :,
\]
where \( c^m_i \) and \( b^m_i \) are the \( m \)'th Fourier modes of ghosts and antighosts for spin \( i+2 \). In (90) we have allowed for an intercept \( \alpha_0 \), expected on general principles due to normal-ordering ambiguities in the remaining terms. Since \( W_\infty \) is a Lie algebra, \( Q \) is guaranteed to be nilpotent up to central terms. One finds that
\[
Q^2 = \sum_{m>0} c^m_i c^{-m}_j \left( R^{ij}_T(m) + R^{ij}_{\text{gh}}(m) \right),
\]
where
\[
R^{ij}_T(m) = \delta^{ij} (c_i(m) - \alpha_0 g^{ij}_{2\ell}(m, -m)),
\]
while the contribution from \( Q^2_{\text{gh}} \) reads
\[
R^{ij}_{\text{gh}}(m) = \sum_{r=0}^{(i+j)/2} \sum_{k=\max(0,2r-i)}^{\infty} \left\{ \sum_{p=1}^{m} g^{ik}_{2r}(m, -p) g^{j,k+i-2r}_{i+j-2r} (-m, m-p) \right\}
\]
when \( i+j \) is even, and zero otherwise (actually, it turns out that \( R^{ij}_{\text{gh}}(m) \) vanishes identically for \( i \neq j \), just as \( R^{ij}_T(m) \)).

At this point one may think that it is straightforward to use the generalized zeta-function to extract finite answers to these expressions. However, a little manipulation with them reveals that there exist many ways of expressing them in terms of the zeta-functions, which may give different answers, and thus correspond to different regularization schemes. Thus, \textit{a priori}, one could obtain any result that one wishes by choosing the regularization scheme appropriately. However, we know that the generators \( V^i_m(\text{gh}) \) in \( Q_{\text{gh}} = \frac{1}{2} c^m_i V^i_m(\text{gh}) \) should provide a ghost realization of the algebra. This means that since the central terms in \( W_\infty \) are uniquely determined up to an overall scale, it follows that all the central terms in \( Q^2_{\text{gh}} \) must be regularized in a self-consistent way in order that their regularized values be consistent with the Jacobi identities for the algebra.

In [38] it was shown that there is in fact a natural-looking, and easily specifiable, scheme for regularizing all the central terms in a consistent manner. It amounts to first performing a constant shift \( \Delta_r \) of the \( k \) parameter in (93), for each value of \( r \), in order to make the
summand into an even function of the shifted parameter. The fact that this can be done is non-trivial. The shift $\Delta_{ir}$, which turns out to depend upon $r$ and $i$ (but not upon $j$), is given by $\Delta_{ir} = \frac{1}{2}(i+3) - r$. When $r \leq i/2$, the summand in (93) will now take the general form

$$\sum_p \sum_{k \geq 0} A_p(k+\Delta_{ir})^{2p} + \sum_{k \geq 0} F(k),$$

where $F(k)$ is an absolutely-convergent sum of simple fractions of the form $1/(k+b)^q$. The divergent polynomial sums are then regularized using the generalized zeta function defined in (87).

For the $W_\infty$ algebra, one finds that the coefficients $R_{gh}^{ij}(m)$ in (93) are zero unless $i = j$. The non-zero coefficients are precisely of the form $R_{gh}^{ij}(m)$ (with $\alpha_0 = 0$) determined by the central terms in the $W_\infty$ algebra, with a (regularized) central charge $c_{gh} = 2$. For example, for $i = j = 0$ one has

$$R_{gh}^{00}(m) = \frac{1}{12}c(m^3 - m) - 2\alpha_0 m,$$

where $c$ is the usual central charge in the matter sector, and

$$R_{gh}^{00}(m) = \sum_{k \geq 0} \left(-m^3\left((k+1)^2 + (k+1) + \frac{1}{3}\right) + \frac{1}{6}m\right)^2$$

$$= \sum_{k \geq 0} \left(-m^3(k+\frac{3}{2})^2 + \frac{1}{12}m^3 + \frac{1}{6}m\right)$$

$$= -m^3\zeta(-2,\frac{3}{2}) + \left(\frac{1}{12}m^3 + \frac{1}{6}m\right)\zeta(0,\frac{3}{2})$$

$$= \frac{1}{6}(m^3 - m).$$

Thus, requiring that the coefficient of $c_0^{-m}c_0^m$ in $Q^2$ vanish leads to the anomaly-freedom conditions $c = -2$ and $\alpha_0 = 0$.

One can carry out a similar analysis for $W_{1+\infty}$, in which case the anomaly-freedom condition is given by [38]

$$c = 0, \quad \alpha_0 = 0.$$  

(97)

This means in particular that $W_{1+\infty}$ gravity is consistent by itself without coupling to matter.

7. $SL(\infty, R)$ KAC-MOODY SYMMETRY IN $W_\infty$ GRAVITY

Having analyzed the ghost contribution to the total central charge in the universal anomaly, we are in a position to obtain a fully consistent quantum theory of $W_\infty$ gravity. Since the (regularized) ghost contribution $c_{gh} = 2$, one needs a matter system with
\( c_{\text{matter}} = -2 \) in order to have \( c_{\text{total}} = 0 \) for the critical \( W_\infty \) string. As a matter of fact, the model with a single scalar \( \varphi \) renormalized to realize local \( W_\infty \) symmetry discussed previously is precisely one of them, since the matter sector has central charge \(-2\). In the corresponding fermion language, the matter system is viewed as a pair of spin \((0,1)\) \( b-c \) systems, which is especially useful, for example in the bosonization of bosons [39] and two-dimensional topological gravity [40].

When the central charge of a matter system is not the critical value \(-2\) of the \( W_\infty \) string, it is still possible to have a consistent theory, in which case the \( W_\infty \) gauge fields will not decouple. This is because the gauge fields can be tuned to make up for the difference between the necessary critical value and the actual value of the central charge of matter. In the light-cone gauge, it amounts to restricting the configuration space of the gauge fields in such a way that the universal anomaly vanishes, thus giving rise to a consistent quantum theory. In the case of two-dimensional gravity coupled some matter system, such a strategy has proved to be rather fruitful in that an \( SL(2,R) \) Kac-Moody symmetry was discovered by Polyakov [41], from which the authors of ref. [42] were able to extract some non-perturbative information about the system.

Naturally there arises the question whether such a strategy can also be applied and generalized to the case of \( W_\infty \) gravity coupled to some \( W_\infty \) matter. The most obvious question is what the analogue of the \( SL(2,R) \) Kac-Moody algebra of two-dimensional quantum gravity is for quantum \( W_\infty \) gravity. In ref. [35], it has been shown that it is the \( SL(\infty,R) \) Kac-Moody algebra.

To prove such a statement, let us first recall the way Polyakov has shown the existence of \( SL(2,R) \) Kac-Moody symmetry in light-cone two-dimensional gravity. Firstly a set of recursion relations for the spin-2 gauge field was derived from the anomalous Ward identity of two-dimensional gravity. Secondly since the spin-2 gauge field is restricted by the anomaly-freedom condition given in (82) with \( i = 0 \), there are only three dynamical components \( j^a \) \((a = -1, 0, 1)\) in powers of \( z \) as follows.

\[
A_0 = j^{(1)} - 2j^{(0)}z + j^{(-1)}z^2.
\]  

One next deduces a set of recursion relations for \( j^a(\bar{z}) \), which turns out to be precisely that dictated by an \( SL(2,R) \) Kac-Moody symmetry of those \( j^a(\bar{z}) \). Thus one proves the existence of such a symmetry.

This line of logic proceeds essentially unaltered in the case of \( W_\infty \). Firstly we set up
our notations. For a generic operator $\mathcal{O}$ that is a functional of the gauge fields $A_i$ only, its expectation value is defined by

$$\langle \langle \mathcal{O} \rangle \rangle \equiv \int \mathcal{D}A e^{-\Gamma} \mathcal{O}. \quad (99)$$

Here the double-angle brackets are used to be distinguishable from the single-angle brackets introduced previously in the first stage of quantization that correspond to integration over the configuration space of the matter fields only. Note that while the single-angle brackets are defined for operators that can be a functional of both matter fields and the gauge fields, the double-angle brackets only make sense for operators of the gauge fields, because matter fields are supposed to have been integrated out at the second stage.

Consider the $(n+1)$-point correlation function $\langle \langle A_i(z) A_{j_1}(x_1) \cdots A_{j_n}(x_n) \rangle \rangle$ for the gauge fields $A_i$. Applying the operator $\partial^2_{z} + 3z$ to it, and recalling the $W_\infty$ anomalous Ward identity given in (80), we can now write down recursion relations for the correlation functions of the gauge fields $A_i$. Thus we have

$$\begin{align*}
- \frac{c_i}{\pi} \partial^2_{z} & \langle \langle A_i(z) A_{j_1}(x_1) \cdots A_{j_n}(x_n) \rangle \rangle = \langle \langle \bar{\partial}_z \frac{\delta \Gamma}{\delta A_i(z)} A_{j_1}(x_1) \cdots A_{j_n}(x_n) \rangle \rangle \\
& + \sum_{k \geq 0} \sum_{\ell=0}^{[(i+k)/2]} f_{2\ell}^{ik} (\partial_z - \partial A_i) \langle \langle \frac{\delta \Gamma}{\delta A_{i+k-2\ell}(z)} A_k(z) A_{j_1}(x_1) \cdots A_{j_n}(x_n) \rangle \rangle \\
& - \sum_{p=1}^{n} \partial \delta(z - x_p) \delta^{ip} \langle \langle A_{j_1}(x_1) \cdots A_{j_p}(x_p) \cdots A_{j_n}(x_n) \rangle \rangle \\
& + \sum_{k \geq 0} \sum_{\ell=0}^{[(i+k)/2]} \sum_{p=0}^{n} f_{2\ell}^{ik} \partial_{z} \delta(z - x_p) \delta_{i+k-2\ell, j_p} \langle \langle A_k(x_p) A_{j_1}(x_1) \cdots A_{j_p}(x_p) \cdots A_{j_n}(x_n) \rangle \rangle.
\end{align*} \quad (100)$$

In the final term here, the first derivative operator in $f(\partial, \partial)$, defined in (30), acts only on the $z$ argument of the delta-function, and the second derivative operator denotes $\frac{\partial}{\partial x_p}$ acting only on the $A_k(x_p)$ field in the angle brackets. The $\mathcal{A}$ indicates that $A$ be taken out of the correlator. The function $\delta(z - x_p)$ denotes a two-dimensional delta function. The derivation of the second line in (100) from the first makes use of the identity

$$\langle \langle \frac{\delta \Gamma}{\delta A_i} \mathcal{O} \rangle \rangle = \langle \langle \frac{\delta \mathcal{O}}{\delta A_i} \rangle \rangle, \quad (101)$$

for arbitrary $\mathcal{O}$, which can be proved by using the definition (99), and performing a functional integration by parts.

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Using the identity
\[
\partial^{2i+3}(z-x_p)\quad (z-x_p) = \pi(2i+2)!\delta(z-x_p),
\tag{102}
\]
we may now integrate (100) to obtain
\[
\langle A_i(z) A_j(x_1) \cdots A_{jn}(x_n) \rangle =
\sum_{p=1}^{n} c_p (2j_p +2)! \delta^{j_p} \left( \frac{(z-x_p)^{2i+2}}{(z-x_p)^{2}} \right) \langle A_{j_1}(x_1) \cdots \mathcal{N}_{j_p}(x_p) \cdots A_{jn}(x_n) \rangle
\tag{103}
\]
\[
- \sum_{k=0}^{[i+k]/2} \sum_{\ell=0}^{k} \sum_{p=1}^{n} c_p (2j_p +2)! c_k (2k+2)! \delta_{i+k-2\ell,j_p} f_{2\ell}^{ik}(\partial_z, -\partial_A_k)
\times \left( \frac{(z-x_p)^{2i+2}}{(z-x_p)^{2}} \right) \langle A_k(x_p) A_{j_1}(x_1) \cdots \mathcal{N}_{j_p}(x_p) \cdots A_{jn}(x_n) \rangle.
\]
We have also, for convenience, rescaled the gauge fields \( A_i \) according to
\[
A_i \to \frac{1}{c_i (2i+2)!} A_i.
\tag{104}
\]
This recursion relation may be used to calculate arbitrary \( N \)-point correlation functions for the gauge fields. For example, the two-point function turns out to be
\[
\langle A_i(x, \bar{x}) A_j(y, \bar{y}) \rangle = c_i (2i+2)! \delta^{ij} \frac{(x-y)^{2i+2}}{(x-y)^{2}}.
\tag{105}
\]
Substituting this back into (103), we find that the three-point function is given by
\[
\langle A_i(x, \bar{x}) A_j(y, \bar{y}) A_k(z, \bar{z}) \rangle = c_j (2j+2)! f_{i+k-j}^{jk}(\partial_x, \partial_y) \frac{(y-z)^{2k+2}(x-y)^{2i+2}}{(y-z)^{2}(x-y)}
\tag{106}
\]
\[
+ c_k (2k+2)! f_{i+j-k}^{ij}(\partial_x, \partial_y) \frac{(z-y)^{2j+2}(x-z)^{2i+2}}{(z-y)^{2}(x-z)}.
\]
Using (14), (30) and (31), we find the following expression for the three-point function:
\[
\langle A_i(x, \bar{x}) A_j(y, \bar{y}) A_k(z, \bar{z}) \rangle = N_{ijk} \frac{(x-y)^{i+j-k+1}(y-z)^{j+k-i+1}(z-x)^{k+i-j+1}}{(x-y)(y-z)(z-x)},
\tag{107}
\]
where \( N_{ijk} \) is defined by
\[
N_{ijk} \equiv \frac{(2i+2)!(2j+2)!(2k+2)!}{(i+j-k+1)!(j+k-i+1)!(k+i-j+1)!} P_{ijk}.
\tag{108}
\]
Here, \( P_{ijk} \) is given by
\[
P_{ijk} = \frac{1}{2} c_k g_{i+j-k}^{ij}.
\tag{109}
\]

$P_{ijk}$ is manifestly symmetric in $i$ and $j$. Although it is not manifest, it is in fact totally symmetric in $i$, $j$ and $k$, by virtue of the identity

$$c_j \phi_{i+k-j}^{ij} = c_k \phi_{i+j-k}^{ij}. \quad (110)$$

Thus we may rewrite $P_{ijk}$ in the manifestly-symmetric form

$$P_{ijk} = \frac{1}{6} \left( c_k \phi_{i+j-k}^{ij} + c_j \phi_{i+k-j}^{ik} + c_i \phi_{j+k-1}^{jk} \right). \quad (111)$$

The three-point function in (107) is in agreement with the general structure of three-point function for conformal fields $A_i$, $A_j$ and $A_k$ with conformal dimensions $(-1-i, 1)$, $(-1-j, 1)$ and $(-1-k, 1)$ respectively [1,2,3].

The entire discussion that we have given above for $W_\infty$ gravity may be repeated for the case of $W_{1+\infty}$ gravity. $W_{1+\infty}$ is an algebra of similar type to $W_\infty$ with an additional spin-1 current. The details of its structure constants, and central terms are given in section 2. $W_{1+\infty}$ gravity can be obtained straightforwardly by gauging an additional spin-1 current to $W_\infty$ gravity. For the case of $W_{1+\infty}$ gravity, we shall again use the tilded notations introduced in section 2, and the formulae that we have derived for recursion relations and correlation functions for for $W_\infty$ gravity hold mutatis mutandis for $W_{1+\infty}$ gravity, except now the index $i$ is allowed to take -1.

The anomaly-freedom condition given in (82) allows us to expand the gauge fields $A_i(z, \bar{z})$ of $W_\infty$ gravity as follows:

$$A_i(z, \bar{z}) = \sum_{a=-i-1}^{i+1} (-1)^{i+1+a} \binom{2i+2}{i+1+a} J^i_a(\bar{z}) z^{i+1+a}, \quad (112)$$

where $J^i_a(\bar{z})$ are dynamic fields of arbitrary functions of $\bar{z}$. Substituting this into the two-point function (105) and three-point function (107) for the gauge fields $A$, we obtain the two-point and three-point functions for the “expansion coefficients” $J^i_a(\bar{z})$. For the two-point function, one finds

$$\langle J^i_a(\bar{x}), J^j_b(\bar{y}) \rangle = \frac{K_{ab}^{ij}}{(\bar{x} - \bar{y})^2}, \quad (113)$$

where the $K_{ab}^{ij}$ are given by

$$K_{ab}^{ij} = (-1)^{i+1+a} c_i (i+1+a)! (i+1-a)! \delta^i_j \delta_{a+b,0}. \quad (114)$$
After some algebra, one finds that the three-point function for \( J^i_m \) can be written as
\[
\langle \langle J^i_a(\bar{x})J^j_b(\bar{y})J^k_c(\bar{z}) \rangle \rangle = \frac{Q^{ijk}_{abc}}{(\bar{x}-\bar{y})(\bar{y}-\bar{z})(\bar{z}-\bar{x})},
\]
where the coefficients \( Q^{ijk}_{abc} \) are given by
\[
Q^{ijk}_{abc} = \delta_{a+b+c,0} \times \sum_{d=0}^{k+i-j+1} \frac{(i+1+a)!(i+1-a)!(j+1+b)!(j+1-b)!(k+1+c)!(k+1-c)!P_{ijk}(-)^{j+1-a+c+d}}{(j-k-a+d)!(i+1+a-d)!(j-i+c+d)!(k+1-c-d)!(k+i-j+1-d)!d!}
\]
(116)

As discussed in the previous section, we may repeat the above analysis for the case of \( W_{1+\infty} \) gravity; the expressions above will then be replaced by analogous tilded expressions.

Our goal now is to compare the two-point and three-point functions (105)-(107) for \( J^i_a \) with those dictated by an \( SL(\infty, R) \) Kac-Moody symmetry. To do this, we start by setting up our notations for \( SL(\infty, R) \). In the literature, a certain class of \( SL(\infty, R) \) has been discussed rather extensively [19,13], where \( SL(\infty, R) \) is viewed as the tensor algebras of \( SL(2, R) \). As discussed earlier, one of them is related to the \( W_{\infty} \) algebra in much the same way as \( SL(2, R) \) is a subalgebra of the Virasoro algebra. In this context, \( SL(\infty, R) \) was termed the “wedge” algebra of \( W_{\infty} \). Owing to this intimate connection between \( SL(\infty, R) \) and \( W_{\infty} \), many notations (e.g. index structure) for \( SL(\infty, R) \) appear to be \( W_{\infty} \)-like. So special care is necessary to tell them apart. Particularly it is important to emphasize that, just as the \( SL(2, R) \) subalgebra of Virasoro does not directly bear any relevance to the \( SL(2, R) \) Kac-Moody symmetry in the case of gravity, so the “wedge” algebra \( SL(\infty, R) \) of \( W_{\infty} \) does not per se imply the existence of an \( SL(\infty, R) \) Kac-Moody symmetry for \( W_{\infty} \) gravity. Nonetheless, as we shall see, such a remarkable symmetry does exist.

We shall give the structure constants for the 1-parameter family of \( GL(\infty, R) \) with generators \( V^i_m \), for which the \( m \) index is restricted in the range given by
\[
-i-1 \leq m \leq i+1,
\]
(117)
and \( i \) taking values \( \geq -1 \). This is the 1-parameter family of tensor algebras discussed in Sec. 2. Thus the algebras have the following commutation relations [13]
\[
[V^i_m, V^j_n] = \sum_{\ell \geq 0} g^{ij}_{2\ell}(m, n; s) V^{i+j-2\ell}_{m+n},
\]
(118)
where
\[ g^{ij}_\ell(m, n; s) = \frac{\phi^{ij}_\ell(s)}{2(\ell+1)!} N^{ij}_\ell(m, n), \] (119)
and
\[ \phi^{ij}_\ell(s) = 4 F_3 \left[ \begin{array}{cccc}
-\frac{1}{2} - 2s, & \frac{3}{2} + 2s, & -\frac{1}{2} \ell - \frac{1}{2}, & -\frac{1}{2} \ell \\
-i - \frac{1}{2}, & -j - \frac{1}{2}, & i + j - \ell + \frac{5}{2}, & 1
\end{array} \right], \] (120)
and \( N^{ij}_\ell(m, n) \) is the same as that given in (11) (except the restriction on the index \( m \) (117)).

Note that these quantities are defined here for all integer values of the subscript argument, although only those with \textit{even} values occur in (118). Odd values for the subscript argument will play an important role presently. Note also that the quantities \( \phi^{ij}_2\ell \) and \( \tilde{\phi}^{ij}_2\ell \) introduced in (13) and (23) correspond to \( \phi^{ij}_2(s) \) with \( s = 0 \) and \( s = -\frac{1}{2} \) respectively, which implies that these two tensor algebras are the wedge algebras of \( W_\infty \) and \( W_{1+\infty} \) respectively. In fact, as discussed in [13], \( W_\infty \) and \( W_{1+\infty} \) can be viewed as the analytic extensions “beyond the wedge” of the \( s = 0 \) and \( s = -\frac{1}{2} \) \( GL(\infty, R) \) algebras. It is precisely these two tensor algebras, which we call \( SL(\infty, R) \) and \( GL(\infty, R) \) respectively, whose corresponding Kac-Moody algebras emerge in \( W_\infty \) gravity and \( W_{1+\infty} \) gravity.

We are now ready to discuss the correlation functions and recursion relations for this family of \( GL(\infty, R) \) Kac-Moody algebras, to show how they are related to our results for \( W \) gravity correlation functions. For reasons that will become clear shortly, it is convenient at this stage to discuss first the case for \( GL(\infty, R) \) with \( s = -\frac{1}{2} \), and its relation to \( W_{1+\infty} \) gravity.

For an arbitrary algebra \( G \), one can write down recursion relations for Kac-Moody currents \( j^A(\bar{z}) \) [43]:
\[
\langle \langle j^A(\bar{z}) j^{B_1}(\bar{x}_1) \cdots j^{B_n}(\bar{x}_n) \rangle \rangle = -\frac{K}{2} \sum_p \frac{\eta^{AB_p}}{(\bar{z} - \bar{x}_p)^2} \langle \langle j^{B_1}(\bar{x}_1) \cdots \kappa^{B_p}(\bar{x}_p) \cdots j^{B_n}(\bar{x}_n) \rangle \rangle
+ \sum_p \frac{\eta^{AB_p C_p}}{(\bar{z} - \bar{x}_p)} \langle \langle j^{C_p}(\bar{x}_p) j^{B_1}(\bar{x}_1) \cdots \kappa^{B_p}(\bar{x}_p) \cdots j^{B_n}(\bar{x}_n) \rangle \rangle. \]
(121)

Here, \( \eta^{AB} \) is the Cartan-Killing metric, \( f^{AB C} \) are the structure constants, and \( A, B, \ldots \) are adjoint indices for \( G \). Normally, for a finite-dimensional algebra, one defines the Cartan-Killing metric by means of the trace of generators \( T^A \), \textit{i.e.} as \( \eta^{AB} \equiv -2\text{Tr}(T^A T^B) \). For an infinite-dimensional algebra such as \( GL(\infty, R) \), this can be problematical. We can, however, define a Cartan-Killing metric, \textit{i.e.} a symmetric 2-index invariant tensor, in the following way.
It was shown in [13,14] that the $W_\infty$ and $W_{1+\infty}$ algebras, and hence in particular their wedge subalgebras, can be viewed as being derived from some corresponding associative-product algebras. This is in fact true for the whole family of the $GL(\infty, R)$ algebras under discussion [13,14]. In the case of $W_\infty$, these multiplications are called the “lone-star product.” For the family of the $GL(\infty, R)$ algebras, the operation is basically tensor product, modulo certain ideal. Explicitly they take the form for $GL(\infty, R)$

$$V_m^i \star V_n^j = \frac{1}{2} \sum_{\ell \geq -1} g_{ij}^{\ell}(m, n; s) V_{m+n}^{i+j-\ell}. \quad (122)$$

The structure constants are antisymmetric under the interchange of $(im)$ with $(jn)$ when $\ell$ is even, and symmetric when $\ell$ is odd. Thus the commutator in (118) may be written as

$$[V_m^i, V_n^j] = V_m^i \star V_n^j - V_n^j \star V_m^i. \quad (123)$$

The lone-star product for the $W_{1+\infty}$ algebra contains the spin-1 generators $V_m^{-1}$. It turns out that the generator $V_0^{-1}$ commutes with all other generators in the algebra; thus it may be viewed as the identity operator in the algebra [14]. This enables us to define an invariant 2-index symmetric tensor, sidestepping the problem mentioned previously of how to attach a meaning to the operation of taking the trace of products of generators. In other words the lone-star algebra provides us with a rule for extracting the singlet part in the symmetric product of two generators; this is precisely the function played by the trace operation in the usual definition of a Cartan-Killing metric. Thus we define the Cartan-Killing metric as

$$\eta_{mn}^{ij}(s) \equiv g_{i+j+1}^{ij}(m, n; s). \quad (124)$$

For $W_{1+\infty}$, the parameter $s$ takes the value $-\frac{1}{2}$. In this case we write

$$\tilde{\eta}_{mn}^{ij} \equiv g_{i+j+1}^{ij}(m, n; -\frac{1}{2}), \quad (125)$$

in accordance with our previous notation. For $GL(\infty, R)$, $s$ can take generic values.

Having defined an invariant Cartan-Killing metric for $GL(\infty, R)$, we are now in a position to compare the correlation functions for the $GL(\infty, R)$ Kac-Moody algebra with those that we obtained from $W_{1+\infty}$ gravity. From (121), the two-point function for $GL(\infty, R)$ Kac-Moody currents $J_m^a(\bar{z})$ ($i \geq -1; -i-1 \leq a \leq i+1$) is given by

$$\langle \langle J_m^i(\bar{x}) J_n^j(\bar{y}) \rangle \rangle = -\frac{K}{2} \frac{\eta_{ab}^{ij}}{(\bar{x} - \bar{y})^2}. \quad (126)$$
This should be compared with our expression (113) obtained from $W_{1+\infty}$ gravity. Equivalence of the two expressions would require that $\tilde{\eta}_{ab}^{ij}$ and $\tilde{K}_{ab}^{ij}$ should be related by

$$-\frac{K}{2} \tilde{\eta}_{ab}^{ij} = \tilde{K}_{ab}^{ij},$$

(127)

for some value of the constant $K$. (Recall that $\tilde{K}_{ab}^{ij}$ is the analogue of (113) for the case of $W_{1+\infty}$ gravity; i.e. with $c_i$ replaced by $\tilde{c}_i$, given by (24). We are using $\tilde{\eta}_{ab}^{ij}$ for the Cartan-Killing metric, since $s = -\frac{1}{2}$ for the $GL(\infty, R)$ is the wedge subalgebra of $W_{1+\infty}$.) One can verify that (127) does indeed hold, with $K$ given by $K = -\frac{1}{4}$.

For the three-point function, one finds from (121) that the result for $GL(\infty, R)$ is given by

$$\langle \langle J_a^i(\bar{x})J_b^j(\bar{y})J_c^k(\bar{z}) \rangle \rangle = \tilde{f}_{abc}^{ijk}(\bar{x} - \bar{y})(\bar{y} - \bar{z})(\bar{z} - \bar{x}),$$

(128)

where $\tilde{f}_{abc}^{ijk}$ denote the structure constants of the $GL(\infty, R)$ wedge subalgebra of $W_{1+\infty}$ with all three indices “upstairs.” (We use the location of the “spin” indices $i, j, k$ to define the notion of upstairs and downstairs.) The “two up, one down” structure constants, which one reads off directly from the commutation relations (118), are defined in general by

$$[V_m^i, V_n^j] = f_{mnk}^{ijp} V_p^k$$

(129)

(summed over “spin” index $k$ and Fourier-mode index $p$), and so

$$\tilde{f}_{mnk}^{ijp} = g_{i+j-k}(m, n; -\frac{1}{2}) \delta_{m+n,p}.$$  

(130)

The downstairs index can then be raised using the Cartan-Killing metric defined by (124) (with $s = -\frac{1}{2}$), to give

$$\tilde{f}_{mnp}^{ijk} = \eta_{pq}^{k\ell} \tilde{f}_{mn\ell}^{ijp} = \sum_{\ell} g_{k+\ell+1}(p, m+n; -\frac{1}{2}) g_{i+j-\ell}(m, n; -\frac{1}{2}).$$  

(130)

Comparing with our expression (115) for $W_{1+\infty}$ gravity (with $Q_{abc}^{ijk}$ replaced by the appropriate tilded version, as described in the previous section), we find that indeed

$$\tilde{f}_{abc}^{ijk} = 8Q_{abc}^{ijk}.$$  

(131)

Thus the three-point functions derived on the one hand from $W_{1+\infty}$ gravity, and on the other hand from $GL(\infty, R)$ Kac-Moody currents (with the parameter $s$ chosen to have the
value $s = -\frac{1}{2}$ appropriate to the $GL(\infty, R)$ wedge subalgebra of $W_{1+\infty}$ are in agreement. As we shall discuss at the end of this section, one can also establish the equivalence of the general recursion relations for correlation functions for $W_{1+\infty}$ gravity and $GL(\infty, R)$ Kac-Moody currents. Thus we have established that $W_{1+\infty}$ gravity has a hidden $GL(\infty, R)$ Kac-Moody symmetry, generalizing the $SL(2, R)$ symmetry of two-dimensional gravity found by Polyakov.

The situation is a little more subtle for the case of $W_\infty$ gravity. The reason for this is that for the hidden Kac-Moody algebra in this case, the corresponding tensor algebra $SL(\infty, R)$ turns out to be the wedge subalgebra of $W_\infty$ generated by $V_i^m$ with $i \geq 0$, which does not contain the spin-1 generator at the apex. Since there is no spin-1 current in the algebra, our procedure for defining an invariant Cartan-Killing metric breaks down in this case. (The expression (124) vanishes inside the wedge if $s$ is chosen to have the special value $s = 0$.) However, since the expression (124) is non-degenerate for all other values of $s$, we may approach $s = 0$ via a limiting procedure, in which we first rescale the generators, $V_i^m \to sV_i^m$, before sending $s$ to zero. Although other structure constants in the lone-star product will now diverge, the relevant ones relating $V_i^m$ and $V_j^n$ to $V_{-1}^0$ will now be finite, and this is sufficient for the purpose of obtaining an $SL(\infty, R)$-invariant symmetric 2-index tensor. Thus we may define the Cartan-Killing metric for $SL(\infty, R)$ as

$$\eta_{mn}^{ij} \equiv \left. \frac{d}{ds} \theta_{i+j+1}^{ij}(m, n; s) \right|_{s=0}. \quad (133)$$

This can be recast in the form

$$\eta_{mn}^{ij} = \frac{\Psi_{i+j+1}^{ij}}{2(i+j+2)!} N_{i+j+1}^{ij}(m, n), \quad (134)$$

where

$$\Psi_{\ell}^{ij} = -\sum_{k=0}^{[(\ell+1)/2]} \frac{4k}{(4k^2-1)} \frac{(-\frac{1}{2})_k(\frac{3}{2})_k(-\frac{1}{2}\ell-\frac{1}{2})_k(-\frac{1}{2}\ell)_k}{k!(i+j+\frac{5}{2})_k(i+j-\frac{1}{2})_k(i+j-\frac{1}{2})_k(i+j-\frac{1}{2})_k}. \quad (135)$$

One can now verify that the two-point function (113) derived from $W_\infty$ gravity, and the two-point function (126) for $SL(\infty, R)$ Kac-Moody currents, with $\eta_{ab}^{ij}$ given by (134) and (135), in fact coincide; specifically, we find that $\eta_{ab}^{ij} = 8K_{ab}^{ij}$. Similarly, the three-point functions coincide, where the downstairs index on the $SL(\infty, R)$ structure constants is raised using our definition (133) for the Cartan-Killing metric; we find that $f_{abc}^{ijk} = 8Q_{abc}^{ijk}$. Again, as we shall discuss presently, the equivalence of the $W_\infty$ gravity and $SL(\infty, R)$ Kac-Moody
recursion relations can also be established in general. Thus we see that $W_\infty$ gravity has an underlying $SL(\infty, R)$ Kac-Moody symmetry.

In order to demonstrate that the recursion relations for correlation functions of the $J_i^k(\bar{z})$ that follow by substituting (112) into the $W_\infty$ gravity recursion relations are the same as the Kac-Moody recursion relations (121), only a little more algebra than we have already carried out is required. The following identity,

$$f_{2\ell}^{ik}(\partial_{\bar{z}}, -\partial_{A_k})(z-x_p)^{2i+2}A_k(x_p) =
\sum_{c=-k-1}^{k+1} \sum_{d=-1}^{i+1} (-)^{i+k+c+d+1} \binom{2i+2}{i+1+d} \binom{2k+2}{k+1+c} J_c^k(\bar{x}_p)x_p^{i+k+c+d+1-2\ell}z^{i+1-d}g_{2\ell}^{ik}(d; c; s)$$

(136)

can easily be established, where the $f_{2\ell}^{ik}$ quantities on the left-hand side are untilded when $s$ is chosen to be 0 (i.e. for $W_\infty$ gravity), and tilded when $s$ is chosen to be $-\frac{1}{2}$ (i.e. for $W_{1+\infty}$ gravity). Using this result in (136), the proof of the equivalence of the Kac-Moody and $W_\infty$ gravity recursion relations follows after some straightforward combinatoric manipulations.

To recapitulate, we have shown that there exists an $SL(\infty, R)$ Kac-Moody symmetry in quantum $W_\infty$ gravity, in close parallel to the existence of $SL(2, R)$ Kac-Moody symmetry in two-dimensional quantum gravity. However, there is one subtle point in the case of quantum $W_\infty$ gravity that two-dimensional quantum gravity does not share. A priori, there are many inequivalent $SL(\infty, R)$ algebras parametrized by $s$, which can be associated with quantum $W_\infty$ gravity, while there is a unique $SL(2, R)$ Kac-Moody algebra. It is natural as well as remarkable that the underlying Kac-Moody symmetry for quantum $W_\infty$ picks a specific $SL(\infty, R)$ algebra that turns out to be exactly the wedge algebra of $W_\infty$ itself. This point begs for some deeper understanding on the interplay between conformal algebras and current algebras in two-dimensional gauge theories, which has been discussed generously in literature.

To close we note that, since central extensions are not allowed for the generators of $w_\infty$ except the Virasoro sector, one can quickly deduce from the recursion relations of $w_\infty$ gravity that all correlators vanish. This indicates that the dynamics of $w_\infty$ is very simple if not trivial. After all, $w_\infty$ tends to be inconsistent at quantum level and become $W_\infty$ gravity upon renormalization, as illustrated in the previous section.

8. SYMMETRY IN THE $c = 1$ STRING MODEL

There has been a considerable amount of activity in the study of the lower-dimensional
string theories in the past two years, which was pioneered in [41,42]. The initial breakthrough was the discovery of non-perturbative solutions to two-dimensional quantum gravity coupled to some matter system [44] obtained by applying the techniques of matrix models. This success has led to solutions of various matrix models, which often have physical interpretations as two-dimensional gravity coupled to certain matter system. More importantly, the non-perturbative information extracted by this somewhat indirect means has stimulated a whole range of approaches to formulating and solving the problem of two-dimensional gravity coupled to matter. They include topological field theory [45,46], continuum Liouville field theory [36,40,47] and effective field theory [48].

Soon after the initial breakthrough it was realized within the framework of matrix models that much of the non-perturbative information is encoded in some generalized KdV hierarchy [49]. On the other hand, topological field theory re-interprets the solutions of matrix models and supplies the mathematical foundation for the solvability of these models [45]. This has led to the discovery of the so-called Virasoro constraints and $W$ constraints that dictate the solutions by giving rise to a set of recursion relations for the physical correlators [46], which suggest some underlying symmetry structure for two-dimensional gravity coupled to matter with $c \leq 1$.

Since the matrix-model approach to two-dimensional quantum gravity coupled to matter with the central charge $c$ is limited to be powerful only when $c \leq 1$, it is then especially interesting to understand the model with $c = 1$ in order to probe the region where $c > 1$. Since the models with $c < 1$ are shown to be described by $W_N$ constraints, it is natural to expect that, as $c \to 1$, $c = 1$ string theory is dictated by $W_{\infty}$ constraints. There is abundant but confusing literature on this point. One interesting success is that various finite-$N$ $W_N$ constraints can be embedded in $W_{1+\infty}$ constraints in the context of fermion Grassmannian [28].

In the meantime, continuum Liouville field theory, which was formulated for quantum gravity in two dimensional space-time in [36], has successfully reproduced some of the results of the matrix models and topological field theory. For example, some correlation functions are calculated in the context of using the Liouville field to describe two-dimensional gravity coupled some conformal matter [47], which reproduce those given by the solutions of matrix models and topological field theory. The advantage of this approach is that it offers an intuitively more physical picture of two-dimensional gravity so that many conventional techniques can be applied. One example is the successful application of BRST analysis to
the physical states of two-dimensional gravity coupled to conformal matter with \( c \leq 1 \) \( [7] \), which showed that there exist many new states with non-vanishing ghost number. In fact, indications for the existence of these new states first arose in the calculation of correlation functions, by both the continuum Liouville method \([50]\) and matrix-model analysis \([51]\), where they appear as poles in the correlation functions. The existence of these novel states is indicative of some large underlying symmetry.

More recently it has been elegantly shown that there exists a so-called ground ring in the space of special physical states of the \( c = 1 \) string theory \([10]\), on which there act some large symmetry groups. It turns out that these symmetry groups are, roughly, some area-preserving diffeomorphisms and volume-preserving diffeomorphisms \([10]\). Shortly afterwards it was shown in the continuum Liouville theory \([8]\), by a different means from that of ref.\([10]\), that indeed there exist symmetry algebras with spin-1 vertex operators as their generators, whose structure constants are identical to that of the area-preserving diffeomorphism of a two-dimensional surface.

For the \( c = 1 \) model described by the two-dimensional critical string with coordinates \( X^\mu = (\phi, X) \) and a linear dilation background

\[
S = \frac{1}{4\pi\alpha'} \int \left( \partial_a X \partial^a X + \partial_a \phi \partial^a \phi - \sqrt{\alpha'} \phi R(2) \right),
\]

one has the following vertex operators characterized by labels \( J \) and \( m \):

\[
\Psi_{J \, m}(z) = \psi_{J \, m} e^{(J-1)\phi},
\]

where the \( \psi_{J \, m} \) are primary fields that form \( SU(2) \) multiplets with \( J \) either integer or half-integer, and \( m = (-J, -J+1, \ldots, J-1, J) \); they are constructed by hitting \( :e^{iJX} : \) repeatedly with the \( SU(2) \) lowering operator \( H_-(z) \) as follows.

\[
\psi_{J \, m}(z) \sim [H_-(z)]^{J-m} : e^{iJX(z)} :.
\]

Here we have introduced the \( SU(2) \) generators given by

\[
H_\pm(z) = \oint \frac{du}{2\pi i} : e^{\pm iJX(u+z)} :,
\]

\[
H_3(z) = \oint \frac{du}{4\pi} \partial X(u+z).
\]

Thus the following algebra for the vertex operators has been obtained \([8]\):

\[
\Psi_{J_1 \, m_1}(z) \Psi_{J_2 \, m_2}(w) \sim \frac{J_2 m_1 - J_1 m_2}{z-w} \Psi_{J_1 + J_2 - 1, m_1 + m_2}(w).
\]
Since the vertex operators are gravitationally dressed to be spin-1 fields, they form a current algebra. The group structure constants are extremely simple, and in fact identical to that of the $w_\infty$ algebra (21) after a proper shift in the index $J$. However there is a crucial difference in that in Eq.(141) the index $m$ is restricted to be with the wedge, so to speak, to form multiplets of $SU(2)$, while in Eq.(21) the Fourier index of $w_\infty$ generators runs from $-\infty$ to $+\infty$, which is essential to making the connection between $w_\infty$ and an area-preserving diffeomorphism, as shown in Sec. 2. Thus the vertex operators in Eq.(138) fill up exactly the wedge algebra of $w_\infty$, but not the whole $w_\infty$. Since $w_\infty$ is a contraction of $W_\infty$, its wedge subalgebra is also a contraction of the wedge subalgebras of $W_\infty$ or an $SL(\infty)$ algebra, which we shall call $SL_c(\infty)$. It is important to realize that $SL_c(\infty)$ as a subalgebra of $w_\infty$ does not have the interpretation of an area-preserving diffeomorphism in the sense discussed in Sec. 2.

The upshot is that the internal group of the current algebra in Eq.(141) is not $w_\infty$, but rather $SL_c(\infty)$. In other words the operator algebra is an $SL_c(\infty)$ Kac-Moody algebra. The authors of ref.[8] went on to conjecture that when a non-zero cosmological constant term $\lambda \int e^{-\phi}$ is introduced in the Lagrangian (137), there will be additional terms on the right-hand side of Eq.(141) so that the internal group for the operator algebra will be the wedge algebra of $W_\infty$ or, more precisely, an $SL(\infty)$ algebra. Since there are a family of $SL(\infty)$ parametrized by $s$ given in Sec. 2, and all of which contract to the same $SL_c(\infty)$, it is not possible to see which $SL(\infty)$ algebra corresponds to the internal group of the vertex operator algebra with non-vanishing cosmological constant. So far it remains a formidable task to carry out a direct calculation for the structure constants of this algebra. Some efforts have been made in this direction [58].

If this conjecture turns out to be true, there are a few implications. First of all, since the $SL(\infty)$ algebras do have an interpretation as area-preserving diffeomorphisms a la Hoppe [18,19], the internal group of the vertex operator algebra would thus regain a geometrical flavor. Secondly if the internal $SL(\infty)$ turns out to be the one that correspond exactly to the wedge subalgebra of $W_\infty$, these vertex operators can be plugged into Eq.(112), giving rise to the gauge fields of $W_\infty$ gauge theory. Thus for each multiplet of $SL(2)$ vertex operators, one has a $W_\infty$ gauge field, which carries higher spin on the world-sheet. This may suggest that the theory of $W_\infty$ gravity discussed earlier is of relevance to the $c = 1$ string model.

Unfortunately the vertex operators given in Eq.(138) and (139) are not the whole story; there are many more BRST invariant operators with ghost number zero as well as that with
non-vanishing ghost numbers [7,52]. It seems an insurmountable to carry out an explicit evaluation on the algebra for the full set of physical operators.

9. SUMMARY

In this paper we have attempted to give a overview of $W_\infty$ theory at both the classical and quantum levels. We started with realizations of $W_\infty$ and proceeded to build $W_\infty$ gravity and a $W_\infty$ string model. We have included two specific models, the first of which is $w_\infty$ gravity coupled to a real scalar, while the second is $W_\infty$ gravity coupled to a complex fermion. Quantum mechanically the first model that was discussed is not consistent and suffers from matter-dependent anomalies, the removal of which forces the theory to become a model of $W_\infty$ gravity coupled to a scalar. Now viewed from a different standpoint where the scalar is fermionized, the renormalized model corresponds to the second model we discussed for $W_\infty$ gravity coupled to a complex fermion that is completely consistent at quantum level, free from both matter-dependent and universal anomalies.

In quantizing the gauge fields as well as matter fields, we arrived at the anomalous $W_\infty$ Ward identities. We next reviewed the BRST analysis of $W_\infty$ and showed that the anomaly-freedom condition for the $W_\infty$ string is that the central charge of $W_\infty$ matter must be $-2$. We then showed that there exists an underlying $SL(\infty, R)$ Kac-Moody symmetry in $W_\infty$ gravity.

Finally we made a short excursion into the recent investigation on the lower-dimensional strings. In particular we looked into the $c = 1$ model, for which a vertex operator algebra was worked out in ref.[8]. This current algebra turns out to be a Kac-Moody algebra with its internal group being a contracted $SL(\infty, R)$ algebra. Its structure constants are identical to that of $w_\infty$.

Undoubtedly we have left out a great deal of topics in the theory of $W_\infty$. Many of them are not only interesting mathematically but also potentially relevant to string theory. We shall mention a few of them here to conclude our discussion.

Extensions of $W_\infty$ to supersymmetric $W_\infty$ are certainly very interesting algebraically. It was shown in ref.[26] that there exists an $N = 2$ supersymmetric $W_\infty$ algebra. From the vantage point of field theory, this implies that one can build supersymmetric $W_\infty$ gravity and a $W_\infty$ string [33]. The quantization of these theories should follow a similar line to our analysis outlined above. There are also other extensions of $W_\infty$ [25,53], the gauge theories of which would also be interesting.
Since the concept of topological field theory was introduced \[54\], it has flourished in its application to two-dimensional models \[45,46\]. Analogous to ordinary two-dimensional gravity, the so-called two-dimensional topological gravity can also be generalized to topological \(W\) gravity \[55\]. However, due to the lack of solid mathematical foundation for these topological \(W\) gravities, their utility still remains to be seen.

There is also a considerable amount of interest in understanding finite-\(N\) \(W_N\) gravity and the \(W_N\) string. The non-linearity in the symmetry of \(W_N\) introduces as much novelty as the difficulty it causes. An overview of this field can be found in other papers such as ref.\[56\].

Finally the concept of universal \(W\)-algebra that encompasses all finite-\(N\) \(W_N\) algebras as its truncations has been pursued extensively. Although it is not completely clear what the conclusion is, there have been some notable developments. In the context of field theory, it was shown in \[32\] that classical \(W_N\) gravity can be obtained as a truncation from the classical \(w_\infty\) gravity. It has also been shown that field theoretic realization of \(W_\infty\) at \(c = -2\) contains realizations of \(W_N\) at \(c = -2\) and gives rise to complete consistent \(W_N\) structure constants \[57\]. However, these encouraging signs are far from a proof that the linear \(W_\infty\) algebra is the universal \(W\)-algebra; there remains a good possibility that some non-linear generalization of \(W_\infty\), as advocated in \[11\], may prove to be the true universal \(W\)-algebra.

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