STABILITY AND INSTABILITY RESULTS OF THE KIRCHHOFF PLATE EQUATION WITH DELAY TERMS ON THE BOUNDARY CONTROL

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Abstract. In this paper, we consider the Kirchhoff plate equation with delay terms on the boundary control are added (see system (1.1) below). We give some instability examples of system (1.1) for some choices of delays. Finally, we prove its well-posedness, strong stability without any geometric condition and exponential stability under a multiplier geometric control condition.

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1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with boundary \( \Gamma \) of class \( C^4 \) consisting of a clamped part \( \Gamma_0 \neq \emptyset \) and a rimmed part \( \Gamma_1 \neq \emptyset \) such that \( \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \). We consider the following Kirchhoff plate equation with delay terms on the boundary controls:

\[
\begin{align*}
  u_{tt}(x,t) + \Delta^2 u(x,t) &= 0 \quad \text{in } \Omega \times (0,\infty), \\
  u(x,t) = \partial_\nu u(x,t) &= 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
  B_1 u(x,t) &= -\beta_1 \partial_\nu u_t(x,t) - \beta_2 \partial_\tau u_t(x,t - \tau_1) \quad \text{on } \Gamma_1 \times (0,\infty), \\
  B_2 u(x,t) &= \gamma_1 u_t(x,t) + \gamma_2 u_t(x,t - \tau_2) \quad \text{on } \Gamma_1 \times (0,\infty), \\
  u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\
  u_t(x,t) &= f_0(x,t) \quad \text{on } \Gamma_1 \times (-\tau_1,0), \\
  \partial_\nu u_t(x,t) &= g_0(x,t) \quad \text{on } \Gamma_1 \times (-\tau_2,0).
\end{align*}
\]

Here and below, \( \beta_1, \gamma_1, \tau_1 \) and \( \tau_2 \) are positive real numbers, \( \beta_2 \) and \( \gamma_2 \) are non-zero real numbers, \( \nu = (\nu_1, \nu_2) \) is the unit outward normal vector along \( \Gamma \), and \( \tau = (-\nu_2, \nu_1) \) is the unit tangent vector along \( \Gamma \). The constant

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0 < \mu < \frac{1}{2} is the Poisson coefficient and the boundary operators \( B_1 \) and \( B_2 \) are defined by
\[
B_1 f = \Delta f + (1 - \mu)C_1 f
\]
and
\[
B_2 f = \partial_\nu \Delta f + (1 - \mu)\partial_\nu C_2 f,
\]
where
\[
C_1 f = 2\nu_1 \nu_2 f_{x_1 x_2} - \nu_1^2 f_{x_2 x_2} - \nu_2^2 f_{x_1 x_1} \quad \text{and} \quad C_2 f = (\nu_1^2 - \nu_2^2) f_{x_1 x_2} - \nu_1 \nu_2 (f_{x_1 x_1} - f_{x_2 x_2}).
\]
Moreover, easy computations show that
\[
C_1 f = -\partial_\nu^2 f - \partial_\nu \nu_2 f_{x_1} + \partial_\nu \nu_1 f_{x_2} \quad \text{and} \quad C_2 f = \partial_\nu \nu_1 f_{x_1} - \partial_\nu \nu_2 f_{x_2}.
\]

Now, we reformulate system (1.1). For this aim, as in [14], we introduce the following auxiliary variables
\[
\begin{align*}
&z^1(x, \rho, t) := \partial_\nu u_t(x, t - \rho \tau_1), \quad x \in \Gamma_1, \rho \in (0, 1), t > 0, \\
&z^2(x, \rho, t) := u_t(x, t - \rho \tau_2), \quad x \in \Gamma_1, \rho \in (0, 1), t > 0.
\end{align*}
\]
Then, system (1.1) becomes
\[
\begin{align}
&u_{tt} + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
&u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \\
&B_1 u + \beta_1 \partial_\nu u_t + \beta_2 z^1(x, \rho, t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty), \\
&B_2 u - \tau_1 z^1(x, \rho, t) + \tau_2 z^2(x, \rho, t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, 1) \times (0, \infty), \\
&\tau_2 z^2(x, \rho, t) + z^2(x, \rho, t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, 1) \times (0, \infty),
\end{align}
\]
with the following initial conditions
\[
\begin{align*}
&u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot) \quad \text{in} \quad \Omega, \\
&z^1(\cdot, \rho, 0) = f_0(\cdot, -\rho \tau_1) \quad \text{on} \quad \Gamma_1 \times (0, 1), \\
&z^2(\cdot, \rho, 0) = g_0(\cdot, -\rho \tau_2) \quad \text{on} \quad \Gamma_1 \times (0, 1).
\end{align*}
\]

The energy of system (1.4)-(1.10) is given by
\[
E(t) = \frac{1}{2} \left\{ a(u, u) + \int_\Omega |u_t|^2 \, dx + \tau_1 |\beta_2| \int_{\Gamma_1} \int_0^1 |z^1(\cdot, \rho, t)|^2 \, d\rho d\Gamma + \tau_2 |\gamma_2| \int_{\Gamma_1} \int_0^1 |z^2(\cdot, \rho, t)|^2 \, d\rho d\Gamma \right\},
\]
where the sequilinear form \( a : H^2(\Omega) \times H^2(\Omega) \longrightarrow \mathbb{C} \) is defined by
\[
a(f, g) = \int_\Omega \left[ f_{x_1 x_1} \overline{g_{x_1 x_1}} + f_{x_2 x_2} \overline{g_{x_2 x_2}} + \mu \left( f_{x_1 x_1} \overline{g_{x_2 x_2}} + f_{x_2 x_2} \overline{g_{x_1 x_1}} \right) \right] + 2(1 - \mu) f_{x_1 x_2} \overline{g_{x_1 x_2}} \, dx.
\]

We first recall the following Green’s formula (see [11]):
\[
a(f, g) = \int_\Omega \Delta^2 \overline{g} \, dx + \int_{\Gamma} (B_1 f \partial_\nu \overline{g} - B_2 f \overline{g}) \, d\Gamma, \quad \forall f \in H^4(\Omega), \ g \in H^2(\Omega).
\]

For further purposes, we need a weaker version of it. Indeed as \( \mathcal{D}(\Omega) \) is dense in \( E(\Delta^2, L^2(\Omega)) := \{ f \in H^2(\Omega) \mid \Delta^2 f \in L^2(\Omega) \} \) equipped with its natural norm, we deduce that \( f \in E(\Delta^2, L^2(\Omega)) \) (see Theorem 5.6 in [13]) satisfies \( B_1 f \in H^{-\frac{1}{2}}(\Gamma) \) and \( B_2 f \in H^{-\frac{1}{2}}(\Gamma) \) with
\[
a(f, g) = \int_\Omega \Delta^2 \overline{g} \, dx + \langle B_1 f, \partial_\nu \overline{g} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \langle B_2 f, \overline{g} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}, \quad \forall g \in H^2(\Omega).
\]

Similar to [1], for any regular solution \( U = (u, u_t, z^1, z^2) \) of system (1.4)-(1.10), the energy \( E(t) \) satisfies the following estimation
\[
\frac{d}{dt} E(t) \leq - (\beta_1 - |\beta_2|) \int_{\Gamma_1} |\partial_\nu u_t|^2 \, d\Gamma - (\gamma_1 - |\gamma_2|) \int_{\Gamma_1} |u_t|^2 \, d\Gamma.
\]

Let us recall some previous literature.
In 1993, Rao in [17] studied the stabilization of the Kirchhoff plate equation with non-linear boundary controls (in the linear case, it corresponds to system (1.1) with $\beta_2 = \gamma_2 = 0$), under a multiplier geometric control condition he established an exponential energy decay rate.

Time delays appear in several applications such as in physics, chemistry, biology, thermal phenomena not only depending on the present state but also on some past occurrences (see [7, 10]). In the last years, the control of partial differential equations with time delays have become popular among scientists, since in many cases time delays induce some instabilities see [3, 4, 5, 6].

In 2006, Nicaise and Pignotti in [14] studied the multidimensional wave equation with boundary feedback and a delay term at the boundary, by considering the following system:

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau) \quad \text{on } \Gamma_N \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \\
u_t(x, t) &= f_0(x, t) \quad \text{on } \Gamma_N \times (-\tau, 0),
\end{align*}
\]

where $\mu_1$ and $\mu_2$ are positive real numbers, and $\Omega$ is an open bounded domain of $\mathbb{R}^n$ with a boundary $\Gamma$ of class $C^2$ and $\Gamma = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$. Under the assumption $\mu_2 < \mu_1$, an exponential decay is achieved. If this assumption does not hold, they found a sequences of delays $\{\tau_k\}_k$, $\tau_k \to 0$, for which the corresponding solutions have increasing energy.

To the best of our knowledge, it seems that there is no result in the existing literature concerning the case of the Kirchhoff plate equation with boundary controls and time delay. The goal of the present paper is to fill this gap by studying both stability and instability of system (1.1).

The outline of this paper is as follows. In section 2, if $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$, we give some instability examples of system (1.1) for some particular choices of delays. In subsection 3.1, we prove the well-posedness of our system. The subsection 3.2 is devoted to establish the strong stability of our system by following a general criteria of Arendt and Batty. Finally, in the subsection 3.3, under the (MGC) condition, we show that system (1.1) is exponentially stable.

Let us finish this introduction with some notation used in the remainder of the paper: The usual norm and semi-norm of the Sobolev space $H^s(\Omega)$ ($s > 0$) are denoted by $\| \cdot \|_{H^s(\Omega)}$ and $| \cdot |_{H^s(\Omega)}$, respectively. By $A \lesssim B$, we mean that there exists a constant $C > 0$ independent of $A$, $B$ and a natural parameter $n$ such that $A \leq CB$.

2. Instability results

In this section, we will give some instability examples of system (1.1) in the cases $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$. This is achieved by distinguishing between the following cases:

(IS1) $|\beta_2| = \beta_1$ and $|\gamma_2| = \gamma_1$,

(IS2) $|\beta_2| \geq \beta_1$ and $|\gamma_2| \geq \gamma_1$ and $|\beta_2| - \beta_1 + |\gamma_2| - \gamma_1 > 0$.

**Theorem 2.1.** If (IS1) or (IS2) hold, then there exist sequences of delays and solutions of (1.1) corresponding to these delays such that their standard energy is constant.

**Proof.** We seek for a solution of system (1.1) in the form

\[
u(x, t) = e^{i\lambda t} \varphi(x), \quad \text{with } \lambda \neq 0.
\]
Inserting (2.1) in (1.1), we get
\[
\begin{cases}
-\lambda^2 \varphi + \Delta^2 \varphi = 0 \text{ in } \Omega, \\
\varphi = \partial_n \varphi = 0 \text{ on } \Gamma_0, \\
B_1 \varphi = -i\lambda (\beta_1 + \beta_2 e^{-i\lambda \tau_1}) \partial_n \varphi \text{ on } \Gamma_1, \\
B_2 \varphi = i\lambda (\gamma_1 + \gamma_2 e^{-i\lambda \tau_2}) \varphi \text{ on } \Gamma_1.
\end{cases}
\]
(2.2)

Let \( \theta \in H^2_{\Gamma_0}(\Omega) \). Multiplying the first equation in (2.2) by \( \theta \), then using Green’s formula, we get
\[
-\lambda^2 \int_{\Omega} \varphi \theta dx + a(\varphi, \theta) + i\lambda (\beta_1 + \beta_2 e^{-i\lambda \tau_1}) \int_{\Gamma_1} \partial_n \varphi \theta d\Gamma + i\lambda (\gamma_1 + \gamma_2 e^{-i\lambda \tau_2}) \int_{\Gamma_1} \varphi \theta d\Gamma = 0,
\]
for all \( \theta \in H^2_{\Gamma_0}(\Omega) \). Now, if \( |\beta_2| \geq |\beta_1| \) and \( |\gamma_2| \geq |\gamma_1| \), then we assume that
\[
\cos(\lambda \tau_1) = -\frac{\beta_1}{\beta_2} \quad \text{and} \quad \cos(\lambda \tau_2) = -\frac{\gamma_1}{\gamma_2}.
\]
(2.4)

Thus, we choose
\[
\beta_2 \sin(\lambda \tau_1) = \sqrt{\beta_2^2 - \beta_1^2} \quad \text{and} \quad \gamma_2 \sin(\lambda \tau_2) = \sqrt{\gamma_2^2 - \gamma_1^2}.
\]
Inserting (2.4) and (2.5) in (2.3), we obtain
\[
-\lambda^2 \int_{\Omega} \varphi \theta dx + a(\varphi, \theta) + \lambda \sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} |\partial_n \varphi|^2 d\Gamma + \lambda \sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} \varphi \theta d\Gamma = 0,
\]
for all \( \theta \in H^2_{\Gamma_0}(\Omega) \). Now, taking \( \theta = \varphi \) in (2.6), we obtain
\[
-\lambda^2 \int_{\Omega} |\varphi|^2 dx + a(\varphi, \varphi) + \lambda \sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} |\partial_n \varphi|^2 d\Gamma + \lambda \sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} |\varphi|^2 d\Gamma = 0.
\]
(2.7)

Without loss of generality, we can assume that
\[
\|\varphi\|_{L^2(\Omega)} = 1.
\]
(2.8)

Thus, from (2.7) and (2.8), we get
\[
\lambda^2 - a(\varphi, \varphi) - \lambda \sqrt{\beta_2^2 - \beta_1^2} q_{\nu}(\varphi) - \lambda \sqrt{\gamma_2^2 - \gamma_1^2} q(\varphi) = 0,
\]
where
\[
q(\varphi) = \int_{\Gamma_1} |\varphi|^2 d\Gamma \quad \text{and} \quad q_{\nu}(\varphi) = \int_{\Gamma_1} |\partial_n \varphi|^2 d\Gamma.
\]
(2.9)

We define
\[
W := \{ w \in H^2_{\Gamma_0}(\Omega) \mid \|w\|_{L^2(\Omega)} = 1 \}.
\]

Now, we distinguish two cases.

**Case 1:** If (IS\(_1\)) holds, then from (2.9), we have
\[
a(\varphi, \varphi) = \lambda^2.
\]
(2.11)

Let us define
\[
\lambda^2 := \min_{w \in W} a(w, w).
\]
(2.12)

Now, if \( \varphi \) verifies
\[
a(\varphi, \varphi) = \min_{w \in W} a(w, w),
\]
them it easy to see that \( \varphi \) is a solution of (2.3) and consequently (2.1) is a solution of (1.1). Moreover, from (2.1) and (1.1), we get
\[
E(t) = E(0) \geq a(\varphi, \varphi) + \lambda^2 \int_{\Omega} |\varphi|^2 dx = 2\lambda^2 > 0, \quad \forall t \geq 0.
\]

Thus, the energy of (1.1) is constant and positive. Further from our assumptions
\[
\cos(\lambda \tau_1) = -1, \quad \sin(\lambda \tau_1) = 0, \quad \cos(\lambda \tau_2) = -1, \quad \sin(\lambda \tau_2) = 0,
\]
system (2.2) becomes
\[
\begin{aligned}
-\lambda^2 \varphi + \Delta^2 \varphi &= 0 \quad \text{in } \Omega, \\
\varphi &= \partial_n \varphi = 0 \quad \text{on } \Gamma_0, \\
B_1 \varphi &= 0 \quad \text{on } \Gamma_1, \\
B_2 \varphi &= 0 \quad \text{on } \Gamma_1.
\end{aligned}
\]
(2.13)

So, we can take a sequence \((\lambda_n)\) of positive real numbers defined by
\[
\lambda_n^2 = \Lambda_{n}^2, \quad n \in \mathbb{N},
\]
where \(\Lambda_{n}^2\), \(n \in \mathbb{N}\), are the eigenvalues for the bi-Laplacian operator with the boundary conditions (2.13)_2-(2.13)_4.

Then, setting
\[
\lambda_n \tau_1 = (2k + 1)\pi, \quad k \in \mathbb{N} \quad \text{and} \quad \lambda_n \tau_2 = (2l + 1)\pi, \quad l \in \mathbb{N},
\]
we get the following sequences of delays
\[
\tau_{1,n,k} = \frac{(2k + 1)\pi}{\lambda_n}, \quad k, n \in \mathbb{N} \quad \text{and} \quad \tau_{2,n,l} = \frac{(2l + 1)\pi}{\lambda_n}, \quad l, n \in \mathbb{N},
\]
which becomes arbitrarily small (or large) for suitable choices of the indices \(n,k,l \in \mathbb{N}\). Therefore, we have found sets of time delays for which system (1.1) is not asymptotically stable.

**Case 2:** If (IS2) holds, then from (2.9), we have
\[
\lambda = \frac{1}{2} \left[ \sqrt{\beta_2 - \beta_1^2} q_\nu(\varphi) + \sqrt{\gamma_2 - \gamma_1^2} q(\varphi) \pm \sqrt{\left( \sqrt{\beta_2 - \beta_1^2} q_\nu(\varphi) + \sqrt{\gamma_2 - \gamma_1^2} q(\varphi) \right)^2 + 4a(\varphi, \varphi)} \right].
\]
(2.14)

Let us define
\[
\lambda := \frac{1}{2} \min_{\nu \in W} \left\{ \sqrt{\beta_2 - \beta_1^2} q_\nu(w) + \sqrt{\gamma_2 - \gamma_1^2} q(w) \right. \\
\left. + \sqrt{\left( \sqrt{\beta_2 - \beta_1^2} q_\nu(w) + \sqrt{\gamma_2 - \gamma_1^2} q(w) \right)^2 + 4a(w, w)} \right\}.
\]
(2.15)

Let us prove that if the minimum in the right-hand side of (2.15) is attained at \(\varphi\), that is
\[
\min_{\nu \in W} \left\{ \sqrt{\beta_2 - \beta_1^2} q_\nu(w) + \sqrt{\gamma_2 - \gamma_1^2} q(w) + \sqrt{\left( \sqrt{\beta_2 - \beta_1^2} q_\nu(w) + \sqrt{\gamma_2 - \gamma_1^2} q(w) \right)^2 + 4a(w, w)} \right\},
\]
then \(\varphi\) is a solution of (2.6). For this aim, take for \(\varepsilon \in \mathbb{R}\)
\[
w = \varphi + \varepsilon \theta \quad \text{with } \theta \in H^2_{\Gamma_0}(\Omega) \quad \text{such that } \int_{\Omega} \varphi \overline{\theta} dx = 0.
\]
(2.17)

Thus, we have
\[
||w||^2_{L^2(\Omega)} = ||\varphi||^2_{L^2(\Omega)} + \varepsilon^2 ||\theta||^2_{L^2(\Omega)} = 1 + \varepsilon^2 ||\theta||^2_{L^2(\Omega)}.
\]
(2.18)

Now, if we define
\[
f(\varepsilon) := \frac{1}{1 + \varepsilon^2 ||\theta||_{L^2(\Omega)}} \left( \sqrt{\beta_2 - \beta_1^2} q_\nu(\varphi + \varepsilon \theta) + \sqrt{\gamma_2 - \gamma_1^2} q(\varphi + \varepsilon \theta) \right.
\]
\[
+ \sqrt{\left( \sqrt{\beta_2 - \beta_1^2} q_\nu(\varphi + \varepsilon \theta) + \sqrt{\gamma_2 - \gamma_1^2} q(\varphi + \varepsilon \theta) \right)^2 + 4a(\varphi + \varepsilon \theta, \varphi + \varepsilon \theta)},
\]
(2.19)
The Hilbert space $H$ which gives
\[
    a(\varphi, \varphi) = \sqrt{\beta_2^2 - \beta_1^2} q_\nu(\varphi) + \sqrt{\gamma_2^2 - \gamma_1^2} q(\varphi) + \sqrt{\left( \sqrt{\beta_2^2 - \beta_1^2} q_\nu(\varphi) + \sqrt{\gamma_2^2 - \gamma_1^2} q(\varphi) \right)^2 + 4a(\varphi, \varphi)},
\]
which gives
\[
    f'(0) = 0.
\]
Consequently, after an easy computation, we obtain
\[
    (2.20) \quad a(\varphi, \theta) + \lambda \sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_1} \partial_\nu \varphi \partial_\nu \theta d\Gamma + \lambda \sqrt{\gamma_2^2 - \gamma_1^2} \int_{\Gamma_1} \varphi \theta d\Gamma = 0.
\]
Since any function $\tilde{\varphi} \in H^2_{\Gamma_0}(\Omega)$ can be decomposed as
\[
    \tilde{\varphi} = \alpha \varphi + \theta \quad \text{with} \quad \alpha \in \mathbb{R} \quad \text{and} \quad \theta \in H^2_{\Gamma_0}(\Omega) \quad \text{such that} \quad \int_{\Omega} \varphi \tilde{\theta} dx = 0,
\]
from (2.20) and (2.7), we obtain that $\varphi$ satisfies (2.6). Thus, for such $\lambda > 0$
\[
    \lambda \tau_1 = \arccos \left( -\frac{\beta_1}{\beta_2} \right) + 2k\pi, \quad k \in \mathbb{N} \quad \text{and} \quad \lambda \tau_2 = \arccos \left( -\frac{\gamma_1}{\gamma_2} \right) + 2l\pi, \quad l \in \mathbb{N},
\]
define a sequences of time delays for which (1.1) is not asymptotically stable.

3. Stability results

In this section, we will prove the wellposedness, strong stability and exponential stability of system (1.4)-(1.10). For this aim, we make the following assumptions

\[(\text{H}) \quad \beta_1, \gamma_1 > 0, \quad \beta_2, \gamma_2 \in \mathbb{R}^*, \quad |\beta_2| < \beta_1 \quad \text{and} \quad |\gamma_2| < \gamma_1.\]

3.1. Wellposedness of the system. In this subsection, we will prove the wellposedness of system (1.4)-(1.10). Under the hypothesis (H) and from (1.15), system (1.4)-(1.10) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t) \leq 0$). Let us define the Hilbert space $\mathcal{H}$ by

\[
    \mathcal{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega) \times (L^2(\Gamma_1 \times (0,1)))^2,
\]
where

\[
    H^2_{\Gamma_0}(\Omega) = \{ f \in H^2(\Omega) \mid f = \partial_\nu f = 0 \text{ on } \Gamma_0 \}.
\]
The Hilbert space $\mathcal{H}$ is equipped with the following inner product

\[
    (U, U_1)_{\mathcal{H}} = a(u, u_1) + \int_{\Omega} \nabla U dx + \tau_1 |\beta_2| \int_{\Gamma_1} \int_0^1 z^1 z^2 d\rho d\Gamma + \tau_2 |\gamma_2| \int_{\Gamma_1} \int_0^1 z^2 d\rho d\Gamma,
\]
where $U = (u, v, z^1, z^2)^T$, $U_1 = (u_1, v_1, z_1^1, z_2^2)^T \in \mathcal{H}$.

We define the linear unbounded operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ by:

\[
    D(A) = \left\{ U = (u, v, z^1, z^2)^T \in D_{\Gamma_0}(\Delta^2) \times H^2_{\Gamma_0}(\Omega) \times (L^2(\Gamma_1 \times H^1(0,1)))^2 \mid \begin{array}{l}
    B_1 u = -\beta_3 \partial_\nu v - \beta_2 z^1(\cdot, 1), \\
    B_2 u = \gamma_1 v + \gamma_2 z^2(\cdot, 1), \\
    v(z^1(\cdot, 0) = \partial_\nu v, \quad z^2(\cdot, 0) = v \text{ on } \Gamma_1
    \end{array} \right\}
\]
where

\[
    D_{\Gamma_0}(\Delta^2) = \{ f \in H^2_{\Gamma_0}(\Omega) \mid \Delta^2 f \in L^2(\Omega), \quad B_1 f \in L^2(\Gamma_1), \quad \text{and} \quad B_2 f \in L^2(\Gamma_1) \}\]

and

\[
    A \begin{pmatrix} u \\ v \\ z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} v \\ -\Delta_2 u \\ -\frac{1}{\tau_1} z^1 \\ -\frac{1}{\tau_2} z^2 \end{pmatrix}, \forall U = (u, v, z^1, z^2)^T \in D(A).
\]
Remark 3.1. From the fact that $2\Re(u_{x_1x_1}, u_{x_2x_2}) = |u_{x_1x_1} + u_{x_2x_2}|^2 - |u_{x_1x_1}|^2 - |u_{x_2x_2}|^2$, we remark that

$$
|u_{x_1x_1}|^2 + |u_{x_2x_2}|^2 + 2\mu \Re(u_{x_1x_1}, u_{x_2x_2}) + 2(1 - \mu)|u_{x_1x_1}|^2
$$

$$
= (1 - \mu)|u_{x_1x_1}|^2 + (1 - \mu)|u_{x_2x_2}|^2 + \mu|u_{x_1x_1} + u_{x_2x_2}|^2 + 2(1 - \mu)|u_{x_1x_1}|^2 \geq 0,
$$

consequently, from (1.12), we get

$$
a(u, u) \geq (1 - \mu)|u|_{H^2(\Omega)}.
$$

Hence the sesquilinear form $a$ is coercive on $H^2_{\Omega}(\Omega)$, since $\Gamma_0$ is non empty. On the other hand, from (1.14) (see also Lemma 3.1 and Remark 3.1 in [17]), we remark that

$$
a(f, g) = \int_{\Omega} \Delta^2 \bar{g} dx + \int_{\Gamma_1} (B_1 f \partial_\nu \bar{g} - B_2 f \bar{g}) d\Gamma,
$$

where $f \in D_{\Gamma_0}(\Delta^2), \ g \in H^2_{\Gamma_0}(\Omega)$. 

\[ \square \]

Now, if $U = (u, u, z^1, z^2)^T$ is solution of (1.4)-(1.10) and is sufficiently regular, then system (1.4)-(1.10) can be written as the following first order evolution equation

$$
U_t = AU, \quad U(0) = U_0,
$$

where $U_0 = (u_0, u_1, f_0(\cdot, -\rho_1), g_0(\cdot, -\rho_2))^T \in \mathcal{H}$. 

**Proposition 3.1.** Under the hypothesis (H), the unbounded linear operator $A$ is $m$-dissipative in the energy space $\mathcal{H}$.

**Proof.** For all $U = (u, v, z^1, z^2)^T \in D(A)$, from (3.1) and (3.2), we have

$$
\Re(\mathcal{A}U, U) = \Re\left\{a(v, u) - \int_{\Omega} \Delta^2 u dx - |\beta_2| \int_{\Gamma_1} \int_0^1 z_\rho^1 \bar{v} dx d\Gamma - |\gamma_2| \int_{\Gamma_1} \int_0^1 z_\rho^2 \bar{v} dx d\Gamma\right\}.
$$

Using (3.4) and the fact that $U \in D(A)$, we obtain

$$
\Re(\mathcal{A}U, U) = -\beta_1 \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma - \Re\left\{\beta_2 \int_{\Gamma_1} z^1(\cdot, 1) \partial_\nu \bar{v} d\Gamma\right\} - \gamma_1 \int_{\Gamma_1} |v|^2 d\Gamma - \Re\left\{\gamma_2 \int_{\Gamma_1} z^2(\cdot, 1) \bar{v} d\Gamma\right\}
$$

$$
- \frac{|\beta_2|}{2} \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma + \frac{|\beta_2|}{2} \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma - \frac{|\gamma_2|}{2} \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma + \frac{|\gamma_2|}{2} \int_{\Gamma_1} |\bar{v}|^2 d\Gamma.
$$

Now, by using Young’s inequality, we get

$$
\left\{\begin{array}{l}
- \Re\left\{\beta_2 \int_{\Gamma_1} z^1(\cdot, 1) \partial_\nu \bar{v} d\Gamma\right\} \leq \frac{|\beta_2|}{2} \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma + \frac{|\beta_2|}{2} \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma,

- \Re\left\{\gamma_2 \int_{\Gamma_1} z^2(\cdot, 1) \bar{v} d\Gamma\right\} \leq \frac{|\gamma_2|}{2} \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma + \frac{|\gamma_2|}{2} \int_{\Gamma_1} |\bar{v}|^2 d\Gamma.
\end{array}\right.
$$

Inserting (3.7) in (3.6) and using the hypothesis (H), we obtain

$$
\Re(\mathcal{A}U, U) \leq -(\beta_1 - |\beta_2|) \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma - (\gamma_1 - |\gamma_2|) \int_{\Gamma_1} |v|^2 d\Gamma \leq 0, \quad \forall U \in D(A)
$$

which implies that $A$ is dissipative. Now, let us prove that $A$ is maximal. For this aim, if $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we look for $U = (u, v, z^1, z^2)^T \in D(A)$ unique solution of

$$
-AU = F.
$$

Equivalently, we have the following system

$$
\begin{align*}
-v &= f_1, \\
\Delta^2 u &= f_2, \\
\frac{1}{\tau_1} z_\rho^1 &= f_3, \\
\frac{1}{\tau_2} z_\rho^2 &= f_4.
\end{align*}
$$
with the following boundary conditions
\begin{equation}
3.14
u = \partial_{\nu} u = 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad B_1 u = -\beta_1 \partial_{\nu} v - \beta_2 z^1(\cdot, 1), \quad B_2 u = \gamma_1 v + \gamma_2 z^2(\cdot, 1), \quad z^1(\cdot, 0) = \partial_{\nu} v, \quad z^2(\cdot, 0) = v \quad \text{on} \quad \Gamma_1.
\end{equation}

From (3.10) and the fact that \( F \in \mathcal{H} \), we get
\begin{equation}
3.15
v = -f_1 \in H^2_{\Gamma_0}(\Omega).
\end{equation}

From (3.12), (3.13), (3.14) and the fact that \( F \in \mathcal{H} \), we obtain
\begin{equation}
3.16
z^1_\rho \in L^2(\Gamma_1 \times (0, 1)) \quad \text{and} \quad z^1(\cdot, \rho) = \tau_1 \int_0^\rho f_3(\cdot, s)ds + \partial_{\nu} v
\end{equation}
and
\begin{equation}
3.17
z^2_\rho \in L^2(\Gamma_1 \times (0, 1)) \quad \text{and} \quad z^2(\cdot, \rho) = \tau_2 \int_0^\rho f_4(\cdot, s)ds + v.
\end{equation}

Consequently, from (3.15), (3.16), (3.17) and the fact that \( f_3, f_4 \in L^2(\Gamma_1 \times (0, 1)) \), we deduce that
\( z^1, z^2 \in L^2(\Gamma_1; H^1(0, 1)) \).

It follows from (3.11), (3.14), (3.16) and (3.17) that
\begin{equation}
3.18
\begin{cases}
\Delta^2 u = f_2 \quad \text{in} \quad \Omega, \\
u = \partial_{\nu} u = 0 \quad \text{on} \quad \Gamma_0, \\
B_1 u = (\beta_1 + \beta_2)\partial_{\nu} f_1 - \tau_1 \beta_2 \int_0^1 f_3(\cdot, s)ds \quad \text{on} \quad \Gamma_1, \\
B_2 u = -(\gamma_1 + \gamma_2) f_1 + \tau_2 \gamma_2 \int_0^1 f_4(\cdot, s)ds \quad \text{on} \quad \Gamma_1.
\end{cases}
\end{equation}

Let \( \varphi \in H^2_{\Gamma_0}(\Omega) \). Multiplying the first equation in (3.18) by \( \varphi \) and integrating over \( \Omega \), then using Green’s formula, we obtain
\begin{equation}
3.19
a(u, \varphi) = l(\varphi), \quad \forall \varphi \in H^2_{\Gamma_0}(\Omega),
\end{equation}
where
\begin{align*}
al(\varphi) &= \int_\Omega f_2 \varphi dx + \int_{\Gamma_1} \left((\beta_1 + \beta_2)\partial_{\nu} f_1 - \tau_1 \beta_2 \int_0^1 f_3(\cdot, s)ds\right) \partial_{\nu} \varphi d\Gamma \\
&\quad + \int_{\Gamma_1} \left((\gamma_1 + \gamma_2) f_1 + \tau_2 \gamma_2 \int_0^1 f_4(\cdot, s)ds\right) \varphi d\Gamma.
\end{align*}

It is easy to see that, \( a \) is a sesquilinear, continuous and coercive form on \( H^2_{\Gamma_0}(\Omega) \times H^2_{\Gamma_0}(\Omega) \) and \( l \) is an antilinear and continuous form on \( H^2_{\Gamma_0}(\Omega) \). Then, it follows by Lax-Milgram theorem that (3.19) admits a unique solution \( u \in H^2_{\Gamma_0}(\Omega) \). By taking the test function \( \varphi \in \mathcal{D}(\Omega) \), we see that the first identity of (3.18) holds in the distributional sense, hence \( \Delta^2 u \in L^2(\Omega) \). Coming back to (3.19), and again applying Green’s formula (1.14), we find that
\begin{equation}
3.20
B_1 u = (\beta_1 + \beta_2) \partial_{\nu} f_1 - \tau_1 \beta_2 \int_0^1 f_3(\cdot, s)ds \quad \text{on} \quad \Gamma_1
\end{equation}
and
\begin{equation}
3.21
B_2 u = -(\gamma_1 + \gamma_2) f_1 + \tau_2 \gamma_2 \int_0^1 f_4(\cdot, s)ds \quad \text{on} \quad \Gamma_1.
\end{equation}

Further since \( F \in \mathcal{H} \), we deduce that \( u \in \mathcal{D}_{\Gamma_0}(\Delta^2) \). Consequently, if we define \( U = (u, v, z^1, z^2)^T \) with \( u \in H^2_{\Gamma_0}(\Omega) \) the unique solution of (3.19), \( v = -f_1 \), and \( z^1 \) (resp. \( z^2 \)) defined by (3.16) (resp. (3.17)), \( U \) belongs to \( D(A) \) is the unique solution of (3.9). Then, \( A \) is an isomorphism and since \( \rho(A) \) is open set of \( \mathbb{C} \) (see Theorem 6.7 (Chapter III) in [9]), we easily get \( R(\lambda I - A) = \mathcal{H} \) for a sufficiently small \( \lambda > 0 \). This, together with the dissipativeness of \( A \), imply that \( D(A) \) is dense in \( \mathcal{H} \) and that \( A \) is m-dissipative in \( \mathcal{H} \) (see Theorems 4.5, 4.6 in [15]). The proof is thus complete.
According to Lumer-Phillips theorem (see [15]), Proposition 3.1 implies that the operator $A$ generates a $C_0$-semigroup of contractions $e^{tA}$ in $H$ which gives the well-posedness of (3.5). Then, we have the following result:

**Theorem 3.1.** For all $U_0 \in H$, system (3.5) admits a unique weak solution $U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, H)$. Moreover, if $U_0 \in D(A)$, then the system (3.5) admits a unique strong solution $U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$.

3.2. **Strong Stability.** In this subsection, we will prove the strong stability of system (1.4)-(1.10). The main result of this section is the following theorem.

**Theorem 3.2.** Under the hypotheses (H) and (GC), the $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ is strongly stable in $H$; i.e., for all $U_0 \in H$, the solution of (3.5) satisfies

$$
\lim_{t \to +\infty} \|e^{tA}U_0\|_H = 0.
$$

According to Arendt-Batty [2], to prove Theorem 3.2, we need to prove that the operator $A$ has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable. The proof of these results is not reduced to the analysis of the point spectrum of $A$ on the imaginary axis since its resolvent is not compact. Hence the proof of Theorem 3.2 has been divided into the following two Lemmas.

**Lemma 3.1.** For all $\lambda \in \mathbb{R}$, $i\lambda I - A$ is injective i.e.,

$$
\ker(i\lambda I - A) = \{0\}.
$$

**Proof.** From Proposition 3.1, we have $0 \in \rho(A)$. We still need to show the result for $\lambda \in \mathbb{R}^+$. For this aim, suppose that $\lambda \neq 0$ and let $U = (u, v, z^1, z^2) \in D(A)$ be such that

$$
AU = i\lambda U.
$$

Equivalently, we have the following system

\begin{align}
(3.21) & \quad v = i\lambda u, \\
(3.22) & \quad -\Delta^2 u = i\lambda v, \\
(3.23) & \quad -\frac{1}{\tau_1} z^1 = i\lambda z^1, \\
(3.24) & \quad -\frac{1}{\tau_2} z^2 = i\lambda z^2.
\end{align}

From (3.8), (3.20) and (H), we get

$$
0 = \Re \langle i\lambda \|U\|_H^2 \rangle = \Re \langle AU, U \rangle_H \leq -(\beta_1 - |\beta_2|) \int_{\Gamma_1} |\partial_\nu v|^2 d\Gamma - (\gamma_1 - |\gamma_2|) \int_{\Gamma_1} |v|^2 d\Gamma \leq 0.
$$

Thus, we have

$$
\partial_\nu v = v = 0 \quad \text{on} \quad \Gamma_1,
$$

which gives, from (3.21) and the fact that $\lambda \neq 0$, that

$$
u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma_1.
$$

Using (3.23), (3.24), (3.25) and the fact that $z^1(\cdot, 0) = \partial_\nu v$, $z^2(\cdot, 0) = v$ on $\Gamma_1$, we obtain

\begin{align}
(3.27) & \quad z^1(\cdot, \rho) = \partial_\nu ve^{-i\lambda \tau_1 \rho} = 0 \quad \text{on} \quad \Gamma_1 \times (0, 1), \\
(3.28) & \quad z^2(\cdot, \rho) = ve^{-i\lambda \tau_2 \rho} = 0 \quad \text{on} \quad \Gamma_1 \times (0, 1).
\end{align}

Now, from (3.25), (3.27), (3.28) and the fact that $U \in D(A)$, we get

$$
B_1 u = \Delta u + (1 - \mu)C_1 u = 0 \quad \text{on} \quad \Gamma_1,
$$

$$
B_2 u = \partial_\nu \Delta u + (1 - \mu)\partial_\nu C_2 u = 0 \quad \text{on} \quad \Gamma_1.
$$

Using (3.26) and the fact that $\nabla u = \partial_\tau u + \partial_\nu u \nu$ on $\Gamma_1$, we obtain

$$
\nu_{z_1} = \nu_{z_2} = 0 \quad \text{on} \quad \Gamma_1.
$$

\[9\]
Now, from (1.2), (3.26) and (3.31), we get
\begin{equation}
C_1 u = C_2 u = 0 \text{ on } \Gamma_1,
\end{equation}
consequently, from (3.29) and (3.30), we get
\begin{equation}
\Delta u = \partial_\nu \Delta u = 0 \text{ on } \Gamma_1.
\end{equation}
Inserting (3.21) in (3.22), we obtain
\begin{equation}
\begin{cases}
\lambda^2 u - \Delta^2 u = 0 \text{ in } \Omega, \\
u = \partial_\nu u = 0 \text{ on } \Gamma_0, \\
u = \partial_\nu u = \Delta u = \partial_\nu \Delta u = 0 \text{ on } \Gamma_1.
\end{cases}
\end{equation}
Holmgren uniqueness theorem (see [12]) yields
\begin{equation}
u = 0 \text{ in } \Omega.
\end{equation}
Finally, from (3.21), (3.27), (3.28), and (3.35), we get
\begin{equation}
U = 0.
\end{equation}
\hfill\Box

**Lemma 3.2.** Under the hypothesis (H), for all \( \lambda \in \mathbb{R} \), we have
\begin{equation}
R(i\lambda I - A) = \mathcal{H}.
\end{equation}

**Proof.** From Proposition 3.1, we have \( \lambda \in \rho(A) \). We still need to show the result for \( \lambda \in \mathbb{R}^* \). For this aim, for \( F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H} \), we look for \( U = (u, v, z^1, z^2)^T \in D(A) \) solution of
\begin{equation}
(i\lambda I - A)U = F.
\end{equation}
Equivalently, we have the following system
\begin{align}
i\lambda u - v &= f_1, \\
i\lambda v + \Delta^2 u &= f_2, \\
i\lambda z^1 + \frac{1}{\tau_1} z^1 &= f_3, \\
i\lambda z^2 + \frac{1}{\tau_2} z^2 &= f_4,
\end{align}
with the following boundary conditions
\begin{equation}
u = \partial_\nu u = 0 \text{ on } \Gamma_0 \text{ and } B_1 u = -\beta_1 \partial_\nu v - \beta_2 \partial_\nu z^1 = 0, \\
B_2 u = \gamma_1 v + \gamma_2 z^2 = 0, \\
z^1(\cdot, 0) = \partial_\nu v, \ z^2(\cdot, 0) = v \text{ on } \Gamma_1.
\end{equation}
From (3.39), (3.40) and (3.41), we deduce that
\begin{equation}
z^1(\cdot, \rho) = \partial_\nu v e^{-i\lambda\tau_1\rho} + \tau_1 \int_0^\rho f_3(x, s) e^{i\lambda\tau_1(s-\rho)} ds \text{ on } \Gamma_1 \times (0, 1),
\end{equation}
\begin{equation}
z^2(\cdot, \rho) = v e^{-i\lambda\tau_2\rho} + \tau_2 \int_0^\rho f_4(x, s) e^{i\lambda\tau_2(s-\rho)} ds \text{ on } \Gamma_1 \times (0, 1).
\end{equation}
It follows from (3.37), (3.38), (3.41), (3.42) and (3.43) that
\begin{equation}
\begin{cases}
-\lambda^2 u + \Delta^2 u = i\lambda f_1 + f_2 \text{ in } \Omega, \\
u = \partial_\nu u = 0 \text{ on } \Gamma_0, \\
B_1 u = -C_{1\lambda} (\partial_\nu u + \frac{\rho}{\tau_1} \partial_\nu f_1) - F_{1\lambda} \text{ on } \Gamma_1, \\
B_2 u = D_{1\lambda} (u + \frac{\rho}{\tau_1} f_1) + G_{1\lambda} \text{ on } \Gamma_1,
\end{cases}
\end{equation}
where
\begin{align}
C_{1\lambda} = i\lambda (\beta_1 + \beta_2 e^{-i\lambda\tau_1}), \\
F_{1\lambda} = \beta_2 \tau_1 \int_0^1 f_3(x, s) e^{i\lambda\tau_1(s-1)} ds,
\end{align}
and
\[ D_{i\lambda} = i\lambda(\gamma_1 + \gamma_2 e^{-i\lambda\tau_2}), \quad G_{i\lambda} = \gamma_2\tau_2 \int_0^1 f_i(x, s)e^{i\lambda\tau_2(s-1)}ds. \]

Let \( \varphi \in H^2_{\Gamma_0}(\Omega) \). Multiplying the first equation in (3.44) by \( \varphi \), integrating over \( \Omega \), then using Green’s formula, we obtain
\[ b(u, \varphi) = l(\varphi), \quad \forall \varphi \in \mathbb{V} := H^2_{\Gamma_0}(\Omega), \]
where
\[ b(u, \varphi) = b_1(u, \varphi) + b_2(u, \varphi), \]
with
\[
\begin{align*}
  b_1(u, \varphi) &= a(u, \varphi), \\
  b_2(u, \varphi) &= -\lambda^2 \int_{\Omega} u\varphi dx + C_{i\lambda} \int_{\Gamma_1} \partial_{\nu}u\partial_{\nu}\varphi d\Gamma + D_{i\lambda} \int_{\Gamma_1} u\varphi d\Gamma.
\end{align*}
\]
and
\[ l(\varphi) = \int_{\Omega} (i\lambda f_1 + f_2)\varphi dx - \int_{\Gamma_1} \left( \frac{i}{\lambda} C_{i\lambda} \partial_{\nu}f_1 + F_{i\lambda} \right) \partial_{\nu}\varphi d\Gamma - \int_{\Gamma_1} \left( \frac{i}{\lambda} D_{i\lambda} + G_{i\lambda} \right) \varphi d\Gamma. \]

Let \( \mathbb{V}' \) be the dual space of \( \mathbb{V} \). Let us define the following operators
\[ \mathbb{B} : \mathbb{V} \rightarrow \mathbb{V}' \quad \text{and} \quad \mathbb{B}_i : \mathbb{V} \rightarrow \mathbb{V}' \quad i \in \{1, 2\}, \]
such that
\[
\begin{align*}
  \mathbb{B}(\varphi) &= b(u, \varphi), \quad \forall \varphi \in \mathbb{V}, \\
  \mathbb{B}_i(\varphi) &= b_i(u, \varphi), \quad \forall \varphi \in \mathbb{V}, \quad i \in \{1, 2\}.
\end{align*}
\]
We need to prove that the operator \( \mathbb{B} \) is an isomorphism. For this aim, we divide the proof into two steps:

**Step 1.** In this step, we prove that the operator \( \mathbb{B}_2 \) is compact. For this aim, let us define the following Hilbert space
\[ H^2_{\Gamma_0}(\Omega) := \{ \varphi \in H^s(\Omega) \mid \varphi = \partial_{\nu}\varphi = 0 \text{ on } \Gamma_0 \} \quad \text{with } s \in \left( \frac{3}{2}, 2 \right). \]
Now, from (3.46) and a trace theorem, we get
\[
|b_2(u, \varphi)| \lesssim \|u\|_{L^2(\Omega)}\|\varphi\|_{H^2(\Omega)} + \|\partial_{\nu}u\|_{L^2(\Gamma_1)}\|\partial_{\nu}\varphi\|_{L^2(\Gamma_1)} + \|u\|_{L^2(\Gamma_1)}\|\varphi\|_{L^2(\Gamma_1)}
\]
\[
\lesssim \|u\|_{H^s(\Omega)}\|\varphi\|_{H^2(\Omega)},
\]
for all \( s \in \left( \frac{3}{2}, 2 \right) \). As \( \mathbb{V} \) is compactly embedded into \( H^s_{\Gamma_0}(\Omega) \) for any \( s \in \left( \frac{3}{2}, 2 \right) \), \( \mathbb{B}_2 \) is indeed a compact operator.

This compactness property and the fact that \( \mathbb{B}_1 \) is an isomorphism imply that the operator \( \mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2 \) is a Fredholm operator of index zero. Now, following Fredholm alternative, we simply need to prove that the operator \( \mathbb{B} \) is injective to obtain that it is an isomorphism.

**Step 2.** In this step, we prove that the operator \( \mathbb{B} \) is injective (i.e. \( \ker(\mathbb{B}) = \{0\} \)). For this aim, let \( u \in \ker(\mathbb{B}) \) which gives
\[ b(u, \varphi) = 0, \quad \forall \varphi \in \mathbb{V}. \]
Equivalently, we have
\[ a(u, \varphi) - \lambda^2 \int_{\Omega} u\varphi dx + C_{i\lambda} \int_{\Gamma_1} \partial_{\nu}u\partial_{\nu}\varphi d\Gamma + D_{i\lambda} \int_{\Gamma_1} u\varphi d\Gamma = 0, \quad \forall \varphi \in \mathbb{V}. \]
Thus, we find that
\begin{align*}
-\lambda^2 u + \Delta^2 u &= 0 \text{ in } \mathcal{D}'(\Omega), \\
u = \partial_n u &= 0 \text{ on } \Gamma_0, \\
B_1 u &= -C_{i\lambda} \partial_n u \text{ on } \Gamma_1, \\
B_2 u &= D_{i\lambda^2}\nu \text{ on } \Gamma_1.
\end{align*}

Therefore, the vector \( U \) defined by

\[
U = (u, i\lambda u, i\lambda e^{-i\lambda^2 \tau}, \partial_n u, i\lambda e^{-i\lambda^2 \tau} \nu)^\top
\]

belongs to \( D(A) \) and satisfies

\[
i\lambda U - AU = 0,
\]

and consequently \( U \in \ker(i\lambda I - A) \). Hence Lemma 3.1 yields \( U = 0 \) and consequently \( u = 0 \) and \( \ker(\mathbb{B}) = \{0\} \).

Steps 1 and 2 guarantee that the operator \( \mathbb{B} \) is isomorphism. Furthermore it is easy to see that the operator \( l \) is an antilinear and continuous form on \( \mathcal{V} \). Consequently, (3.45) admits a unique solution \( u \in \mathcal{V} \). In (3.45), by taking test functions \( \varphi \in \mathcal{D}(\Omega) \), we see that the first identity of (3.44) holds in the distributional sense, hence \( \Delta^2 u \in L^2(\Omega) \). Coming back to (3.45), and again applying Green’s formula (1.14), we find that

\[
B_1 u = -C_{i\lambda}(\partial_n u + \frac{i}{\lambda}\partial_n f_1) - F_{i\lambda} \text{ on } \Gamma_1,
\]

and

\[
B_2 u = D_{i\lambda}(u + \frac{i}{\lambda} f_1) + G_{i\lambda} \text{ on } \Gamma_1.
\]

Further since \( u, \partial_n u, f_1, \partial_n f_1, F_{i\lambda} \) and \( G_{i\lambda} \) belong to \( L^2(\Gamma_1) \), we deduce that \( u \in \mathcal{D}(\Omega) \). Consequently, if \( u \in \mathcal{V} \) is the unique solution of (3.45) and if we define \( Z^1 \) (resp. \( Z^2 \)) by (3.42) (resp. (3.43)), we deduce that

\[
U = (u, i\lambda u - f_1, z^1, z^2)^\top
\]

belongs to \( D(A) \) and is the unique solution of (3.36). \( \square \)

**Proof of Theorem 3.2.** From Lemma 3.1, the operator \( A \) has no pure imaginary eigenvalues (i.e. \( \sigma_p(A) \cap i\mathbb{R} = \emptyset \)). Moreover, from Lemma 3.1 and Lemma 3.2, \( i\lambda I - A \) is bijective for all \( \lambda \in \mathbb{R} \) and since \( A \) is closed, we conclude, with the help of the closed graph theorem, that \( i\lambda I - A \) is an isomorphism for all \( \lambda \in \mathbb{R} \), hence that \( \sigma(A) \cap i\mathbb{R} = \emptyset \). \( \square \)

### 3.3. Exponential stability

In this subsection, we will prove the strong stability of system (1.4)-(1.10). Let us start up this subsection with the definition of our multiplier geometric control condition.

**Definition 3.1.** We say that the partition \((\Gamma_0, \Gamma_1)\) of the boundary \( \Gamma \) satisfies the multiplier geometric control condition \( \text{MGC} \) if there exists a point \( x_0 \in \mathbb{R}^2 \) and a positive constant \( \delta \) such that

\[
(GC) \quad h \cdot \nu \geq \delta^{-1} \text{ on } \Gamma_1 \quad \text{and} \quad h \cdot \nu \leq 0 \text{ on } \Gamma_0,
\]

where \( h(x) = x - x_0 \). \( \square \)

**Theorem 3.1.** Under the hypotheses (H) and (GC), the \( C_0 \)-semigroup \( e^{tA} \) is exponentially stable; i.e. there exists constants \( M \geq 1 \) and \( \epsilon \geq 0 \) independent of \( U_0 \in \mathcal{H} \) such that

\[
\|e^{tA}U_0\|_\mathcal{H} \leq Me^{-\epsilon t}\|U_0\|_\mathcal{H}, \forall t \geq 0.
\]

**Proof.** Since \( i\mathbb{R} \subset \rho(A) \) (see the previous subsection), according to [8] and [16], to prove Theorem 3.1, it remains to prove that

\[
\limsup_{\lambda \in \mathbb{R}, |\lambda| \to \infty} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty.
\]  

We will prove condition (3.50) by a contradiction argument. For this purpose, suppose that (3.50) is false, then there exists \( \{ (\lambda_n, U_n := (u_n, v_n, z^1_n, z^2_n)^\top) \} \) \( n \geq 1 \subset \mathbb{R}^* \times D(A) \) with

\[
|\lambda_n| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \|U_n\|_\mathcal{H} = 1, \forall n \geq 1,
\]
such that

\[(3.52)\quad (i\lambda_n I - A)U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^T \to 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as} \ n \to \infty.\]

For simplicity, we drop the index \(n\). Equivalently, from (3.52), we have

\[(3.53)\quad i\lambda u - v = f_1 \to 0 \quad \text{in} \quad H^2_{\text{loc}}(\Omega),\]

\[(3.54)\quad i\lambda v + \Delta^2 u = f_2 \to 0 \quad \text{in} \quad L^2(\Omega),\]

\[(3.55)\quad i\lambda z^1 + \frac{1}{\tau_1}z^1 = f_3 \to 0 \quad \text{in} \quad L^2(\Gamma_1 \times (0,1)),\]

\[(3.56)\quad i\lambda z^2 + \frac{1}{\tau_2}z^2 = f_4 \to 0 \quad \text{in} \quad L^2(\Gamma_1 \times (0,1)).\]

Taking the inner product of (3.52) with \(U\) in \(\mathcal{H}\) and using (3.8), we get

\[(\beta_1 - |\beta_2|) \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + (\gamma_1 - |\gamma_2|) \int_{\Gamma_1} |v|^2 d\Gamma \leq -\Re(\langle AU, U \rangle_{\mathcal{H}}) = \Re(\langle F, U \rangle_{\mathcal{H}}) \leq \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}},\]

From the above estimation, (H) and the fact that \(\|F\|_{\mathcal{H}} = o(1)\) and \(\|U\|_{\mathcal{H}} = 1\), we obtain

\[(3.57)\quad \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |v|^2 d\Gamma = o(1),\]

**Lemma 3.3.** Under the hypothesis (H), the solution \(U = (u, v, z^1, z^2)^T \in D(A)\) of (3.53)-(3.56) satisfies the following estimations

\[(3.58)\quad \int_{\Gamma_1} \int_0^1 |z^1|^2 d\rho d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma = o(1),\]

\[(3.59)\quad \int_{\Gamma_1} \int_0^1 |z^2|^2 d\rho d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma = o(1).\]

**Proof.** From (3.42), Cauchy-Schwarz inequality and the fact that \(\rho \in (0, 1)\), we get

\[
\int_{\Gamma_1} \int_0^1 |z^1|^2 d\rho d\Gamma \leq 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + 2\tau_1^2 \int_{\Gamma_1} \int_0^1 \left( \int_0^\rho |f_3(\cdot, s)| ds \right)^2 d\rho d\Gamma
\]

\[
\leq 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + 2\tau_1^2 \int_{\Gamma_1} \int_0^1 \rho \int_0^\rho |f_3(\cdot, s)|^2 dsd\rho d\Gamma
\]

\[
\leq 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + 2\tau_1^2 \left( \int_0^1 \rho d\rho \right) \int_{\Gamma_1} \int_0^1 |f_3(\cdot, s)|^2 dsd\Gamma
\]

\[
= 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + \tau_1^2 \int_{\Gamma_1} \int_0^1 |f_3(\cdot, s)|^2 dsd\Gamma.
\]

The above inequality, (3.57) and the fact that \(f_3 \to 0\) in \(L^2(\Gamma_1 \times (0,1))\) lead to the first estimation in (3.58). Now, from (3.42), we deduce that

\[z^1(\cdot, 1) = \partial_{\nu} ve^{-i\lambda_1} + \tau_1 \int_0^1 f_3(\cdot, s)e^{i\lambda_1(s-1)} ds \quad \text{on} \quad \Gamma_1,\]

consequently, by using Cauchy-Schwarz inequality, we get

\[
\int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma \leq 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + 2\tau_1^2 \int_{\Gamma_1} \left( \int_0^1 |f_3(\cdot, s)| ds \right)^2 d\Gamma
\]

\[
\leq 2 \int_{\Gamma_1} |\partial_{\nu} v|^2 d\Gamma + 2\tau_1^2 \int_{\Gamma_1} \int_0^1 |f_3(\cdot, s)|^2 dsd\Gamma.
\]

Therefore, from the above inequality, (3.57) and the fact that \(f_3 \to 0\) in \(L^2(\Gamma_1 \times (0,1))\), we get the second estimation in (3.58). The same argument as before yielding (3.59), the proof is complete.\(\square\)
Next, from the above estimations, (3.57) and the fact that \( U \in D(A) \), we get

\[
\int_{\Gamma_1} |B_1 u|^2 d\Gamma = o(1) \quad \text{and} \quad \int_{\Gamma_1} |B_2 u|^2 d\Gamma = o(1).
\]

**Lemma 3.4.** Under the hypothesis \((H)\), the solution \( U = (u, v, z^1, z^2) \top \in D(A) \) of (3.53)-(3.56) satisfies the following estimations

\[
\int_{\Gamma_1} |\partial_{\nu} u|^2 d\Gamma = o(\lambda^{-2}) \quad \text{and} \quad \int_{\Gamma_1} |u|^2 d\Gamma = o(\lambda^{-2}).
\]

**Proof.** Since \( U \in D(A) \), we have \( B_1 u = -\beta_1 \partial_{\nu} v - \beta_2 z^1(\cdot, 1) \) and \( B_2 u = \gamma_1 v + \gamma_2 z^2(\cdot, 1) \) on \( \Gamma_1 \).

Inserting (3.53) in the above equations, we get

\[
i\lambda \partial_{\nu} u = -\frac{1}{\beta_1} B_1 u + \partial_{\nu} f_1 - \frac{\beta_2}{\beta_1} z^1(\cdot, 1) \quad \text{on} \quad \Gamma_1,
\]

and

\[
i\lambda u = \frac{1}{\gamma_1} B_2 u + f_1 - \frac{\gamma_2}{\gamma_1} z^2(\cdot, 1) \quad \text{on} \quad \Gamma_1.
\]

From the above equations, we deduce that

\[
\int_{\Gamma_1} |\lambda \partial_{\nu} u|^2 d\Gamma \lesssim \frac{1}{\beta_1^2} \int_{\Gamma_1} |B_1 u|^2 d\Gamma + \int_{\Gamma_1} |\partial_{\nu} f_1|^2 d\Gamma + \frac{\beta_2^2}{\beta_1^2} \int_{\Gamma_1} |z^1(\cdot, 1)|^2 d\Gamma,
\]

and

\[
\int_{\Gamma_1} |\lambda u|^2 d\Gamma \lesssim \frac{1}{\gamma_1^2} \int_{\Gamma_1} |B_2 u|^2 d\Gamma + \int_{\Gamma_1} |f_1|^2 d\Gamma + \frac{\gamma_2^2}{\gamma_1^2} \int_{\Gamma_1} |z^2(\cdot, 1)|^2 d\Gamma.
\]

Using a trace theorem and the fact that \( a(f_1, f_1) = o(1) \), we get

\[
\int_{\Gamma_1} |\partial_{\nu} f_1|^2 d\Gamma \lesssim \|f_1\|_{H^2(\Omega)}^2 \lesssim a(f_1, f_1) = o(1),
\]

and

\[
\int_{\Gamma_1} |f_1|^2 d\Gamma \lesssim \|f_1\|_{H^2(\Omega)}^2 \lesssim a(f_1, f_1) = o(1).
\]

Inserting these estimations in (3.62) and (3.63), then using Lemma 3.3 and (3.60), we get the desired result. \( \square \)

**Lemma 3.5.** Under the hypotheses \((H)\) and \((\text{GC})\), the solution \( U = (u, v, z^1, z^2) \top \in D(A) \) of (3.53)-(3.56) satisfies the following estimations

\[
\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad a(u, u) = o(1).
\]

**Proof.** Inserting (3.53) in (3.54), we get

\[
-\lambda^2 u + \Delta^2 u = i\lambda f_1 + f_2 \quad \text{in} \quad \Omega.
\]

Multiplying the above equation by \((h \cdot \nabla)\), integrating over \( \Omega \), then taking the real part, we obtain

\[
\Re \left\{ -\lambda^2 \int_{\Omega} u(h \cdot \nabla) dx + \int_{\Omega} \Delta^2 u(h \cdot \nabla) dx \right\} = \Re \left\{ i\lambda \int_{\Omega} f_1(h \cdot \nabla) dx + \int_{\Omega} f_2(h \cdot \nabla) dx \right\}
\]

Now, by using Green’s formula and the fact that \( u = 0 \) on \( \Gamma_0 \), then using (3.61), we get

\[
\Re \left\{ -\lambda^2 \int_{\Omega} u(h \cdot \nabla) dx \right\} = \frac{1}{2} \int_{\Omega} |\lambda u|^2 dx - \frac{1}{2} \int_{\Omega} (h \cdot \nu) |\lambda u|^2 d\Gamma = \frac{1}{2} \int_{\Gamma_1} |\lambda u|^2 d\Gamma + o(1).
\]

Using the fact that \( \lambda^2 a(u, u) = O(1) \) and \( a(f_1, f_1) = o(1) \), we obtain

\[
\begin{align*}
\|\lambda \nabla u\|_{L^2(\Omega)} & \leq \|\lambda\|_{H^2(\Omega)} \lesssim \|\lambda\| a(u, u) = O(1), \\
\|f_1\|_{L^2(\Omega)} & \leq \|f_1\|_{H^2(\Omega)} \lesssim \sqrt{a(f_1, f_1)} = o(1).
\end{align*}
\]
Thus, from the above estimations and the fact that $f_2 \to 0$ in $L^2(\Omega)$, we obtain
\[
\Re \left\{ \lambda \int_\Omega f_1 (h \cdot \nabla u) \, dx + \int_\Omega f_2 (h \cdot \nabla u) \, dx \right\} = o(1).
\]
Inserting (3.66) in (3.65) and using (3.67), we obtain
\[
\frac{1}{2} \int_\Omega |\lambda u|^2 \, dx = -\Re \left\{ \int_\Omega \Delta^2 u (h \cdot \nabla u) \right\} + o(1).
\]
According to Lemma 5.4 in [1], for all $u \in D_{r_o}(\Delta^2)$, we have
\[
-\Re \left\{ \int_\Omega \Delta^2 u (h \cdot \nabla u) \right\} \leq -\frac{1}{2} a(u, u) + \frac{\epsilon_1 R^2}{2} \int_{\Gamma_1} |B_2 u|^2 \, d\Gamma + \left( \int_{\Gamma_1} |B_1 u|^2 \, d\Gamma \right)^{\frac{1}{2}} \left( \int_{\Gamma_1} |\partial_n u|^2 \, d\Gamma \right)^{\frac{1}{2}} + \frac{R^2 \epsilon_2}{2} \int_{\Gamma_1} |B_1 u|^2 \, d\Gamma,
\]
where $R = \| h \|_{L^\infty(\Omega)}$ and $\epsilon_1, \epsilon_2$ are positive constants. Consequently, using (3.61) and (3.60), we obtain
\[
-\Re \left\{ \int_\Omega \Delta^2 u (h \cdot \nabla u) \right\} \leq -\frac{1}{2} a(u, u) + o(1).
\]
Finally, inserting (3.70) in (3.68), we get
\[
\frac{1}{2} \int_\Omega |\lambda u|^2 \, dx + \frac{1}{2} a(u, u) = o(1).
\]
The proof is thus complete. \qed

**Proof of Theorem 3.1:** From Lemmas 3.3 and 3.5, we deduce that
\[
\| U \|_{\mathcal{H}} = o(1),
\]
which contradicts (3.51). \qed

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