BOUNDARY VALUE PROBLEM WITH MEASURES FOR FRACTIONAL ELLIPTIC EQUATIONS INVOLVING SOURCE NONLINEARITIES

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Abstract. We are concerned with positive solutions of equation (E) \((-\Delta)^s u = f(u)\) in a domain \(\Omega \subset \mathbb{R}^N (N > 2s)\), where \(s \in (\frac{1}{2}, 1)\) and \(f \in C^\beta_{\text{loc}}(\mathbb{R})\), for some \(\beta \in (0, 1)\). We establish a universal a priori estimate for positive solutions of (E), as well as for their gradients. Then for \(C^2\) bounded domain \(\Omega\), we prove the existence of positive solutions of (E) with prescribed boundary value \(\rho \nu\), where \(\rho > 0\) and \(\nu\) is a positive Radon measure on \(\partial \Omega\) with total mass 1, and discuss regularity property of the solutions. When \(f(u) = u^p\), we demonstrate that there exists a critical exponent \(p_s := \frac{N + s}{N - s}\) in the following sense. If \(p \geq p_s\), the problem does not admit any positive solution with \(\nu\) being a Dirac mass. If \(p \in (1, p_s)\) there exits a threshold value \(\rho^* > 0\) such that for \(\rho \in (0, \rho^*)\), the problem admits a positive solution and for \(\rho > \rho^*\), no positive solution exists. We also show that, for \(\rho > 0\) small enough, the problem admits at least two positive solutions.

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1. INTRODUCTION

In this article we consider boundary value problem for the fractional elliptic equation with source nonlinearity

\[(1.1) \quad (-\Delta)^s u = f(u) \quad \text{in } \Omega,\]

where, unless otherwise stated, \(\Omega \neq \mathbb{R}^N (N > 2s)\) is a \(C^2\) bounded domain in \(\mathbb{R}^N\), \(s \in (\frac{1}{2}, 1)\), and \(f \in C^\beta_{\text{loc}}(\mathbb{R})\), for some \(\beta \in (0, 1)\). Here \((-\Delta)^s\) denotes the fractional Laplace operator

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defined as follows
\[ (-\Delta)^s u(x) = \lim_{\varepsilon \to 0} (-\Delta)^s_{\varepsilon} u(x), \]
where
\[ (-\Delta)^s_{\varepsilon} u(x) := a_{N,s} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \]
and \( a_{N,s} = \frac{2^{2s} \Gamma(N/2+s)}{\pi^{N/2} \Gamma(1-s)} \). When \( s = 1 \), \((-\Delta)^s\) coincides the classical laplacian \(-\Delta\) and the equation
\[ -\Delta u = f(u) \quad \text{in } \Omega \]
has been the research objective of many mathematicians in the literature. One of the first attempt in this direction was obtained in [7] for the case \( f(u) = u^p \) \((p > 1)\), showing the existence of a critical exponent \( \frac{N+1}{N-1} \) for the solvability of (1.3). More precisely, it was shown in [7] that if \( p \in \left(1, \frac{N+1}{N-1}\right) \) then, for any \( \mu \in \mathcal{M}^+ (\partial \Omega) \) (= the space of positive finite measures on \( \partial \Omega \)), there exists a solution of
\[ \begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = \mu & \text{on } \partial \Omega, \end{cases} \]
while if \( p \geq \frac{N+1}{N-1} \) there exists no solution of (1.4) with \( \mu \) being a Dirac measure concentrated at a point on \( \partial \Omega \). This type of problem was reconsidered by Bidaut-Veron and Yarur [8], in which they established sharp estimates of Green kernel and Poisson kernel and provided a necessary and sufficient condition for the existence of a solution of (1.4). When \( f \) satisfies a so-called subcriticality condition, an existence result for (1.3) was recently obtained by Chen et al. in [15] by using Schauder fixed point theorem, essentially based on estimates related to weighted Marcinkiewicz spaces. Recently, Bidaut-Véron et al. [6] provided new criteria, expressed in terms of appropriate capacities, for the solvability of problem (1.4). The approach employed in the above papers was then adapted to the setting in which the Laplace operator is shifted by a Hardy potential [30, 24].

It is worth noting that any solution of (1.4) is naturally bounded from below by the Poisson operator \( \mathbb{P}[\mu] \) which is the unique solution of the linear problem associated to (1.4). However, it is interesting to investigate an upper estimate for solutions of (1.3). In [32], Poláčik et al. developed a general method, based on rescaling arguments combined with a key doubling property, for derivation of universal, pointwise, a priori upper estimates of solutions to (1.3).

The aforementioned results are motivation for the present paper, the goal of which is twofold: (i) to establish a priori estimates for solutions of (1.1), as well as their gradient and (ii) to study the existence, nonexistence and multiplicity of solutions to the boundary value problem with measures for (1.1).

We set
\[ \omega(x) := \frac{1}{1 + |x|^{N+2s}} \quad \text{and} \quad L^1(\mathbb{R}^N, \omega) := \{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} |u| \omega dx < \infty \}. \]

Regarding the first aspect of our goal, we deal with viscosity solutions which are defined as follows:

**Definition 1.1.** (Viscosity solution) We say that a function \( u : \mathbb{R}^N \to \mathbb{R} \) which is continuous in \( \Omega \) and in \( L^1(\mathbb{R}^N, \omega) \) is a viscosity super-solution (sub-solution) of (1.1) if for every point
$x_0 \in \Omega$ and some neighborhood $V$ of $x_0$ with $V \subset \Omega$ and for every $\phi \in C^2(V)$ such that $u(x_0) = \phi(x_0)$ and
\[ u(x) \geq \phi(x) \quad \text{(resp. } u(x) \leq \phi(x)) \quad \text{for all } x \in V, \]
defining
\[ \tilde{u} := \begin{cases} \phi & \text{in } V, \\ u & \text{in } \mathbb{R}^N \setminus V, \end{cases} \]
we have
\[ (-\Delta)^s \tilde{u}(x_0) \geq f(\tilde{u}(x_0)) \quad \text{(resp. } (-\Delta)^s \tilde{u}(x_0) \leq f(\tilde{u}(x_0))). \]
We say that $u$ is a viscosity solution of (1.1) if it is a viscosity super-solution and also a viscosity sub-solution of (1.1).

Set $\delta(x) := \text{dist}(x, \partial \Omega)$ and define
\[ p_c := \frac{N}{N - 2s}. \]

Our first main result provides pointwise a priori estimates of viscosity solutions, as well as their gradient.

**Theorem 1.2.** Let $p \in (1, p_c)$ and $\Omega$ be an arbitrary domain in $\mathbb{R}^N$ (possibly unbounded). Assume $f \in C^\beta_{\text{loc}}(\mathbb{R})$ for some $\beta \in (0, 1)$ satisfies
\[ \lim_{t \to \infty} t^{-p} f(t) = L \in (0, \infty). \]
Then there exists a positive constant $C = C(N, s, f)$ such that for any nonnegative viscosity solution $u$ of (1.1), there holds
\[ u(x) + |\nabla u(x)|^{\frac{2s}{p+2s-1}} \leq C(1 + \delta(x)^{-\frac{2s}{p-1}}) \quad \forall x \in \Omega. \]

**Remark 1.3.** We would like to mention that in [3, Lemma 10], Barrios et al. have proved (1.8) for $C^2$ domain $\Omega$ assuming the solution $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$, whereas in our Theorem 1.2, estimate (1.8) is valid for any nonnegative viscosity solution (which may not be bounded) in any arbitrary domain and the constant $C$ does not depend on $\Omega$ or $u$. In particular, Theorem 1.2 includes solutions with singularities on the boundary. Moreover, in Proposition 3.3 we show that (1.8) holds for any nonnegative $C^\beta_{\text{loc}}(\Omega)$ distributional solution.

**Remark 1.4.** Since (1.8) deals with gradient estimate, assumption $s > 1/2$ is important as it ensures that gradient of any nonnegative solution of (1.1) exists (see proof of Theorem 3.1 for details). Furthermore, this assumption is needed for the wellposedness of the notion $s$-boundary trace in Definition 1.5 (see [31]) for more details).

**Throughout this paper we assume** $s \in (1/2, 1)$.

Our next interest lies on the existence of solutions to the boundary value problem with measures for (1.1). Before stating the main results, we introduce necessary notations.

For $\phi \geq 0$, denote by $\mathcal{M}(\Omega, \phi)$ the space of Radon measures $\tau$ on $\Omega$ satisfying $\int_{\Omega} \phi \, d|\tau| < \infty$ and by $\mathcal{M}(\partial \Omega)$ the space of bounded Radon measures on $\partial \Omega$ and by $\mathcal{M}^+(\partial \Omega)$ the space of bounded positive Radon measures on $\partial \Omega$. 

Let $G_s$ and $M_s$ be the Green kernel and the Martin kernel of $(-\Delta)^s$ in $\Omega$ respectively. We denote the associated Green operator $G_s$ and Martin operator $M_s$ as follows:

$$G_s[\tau] := \int_{\Omega} G_s(., y) d\tau(y), \quad \tau \in \mathcal{M}(\Omega, \delta^s),$$

$$M_s[\mu] := \int_{\partial \Omega} M_s(., z) d\mu(z), \quad \mu \in \mathcal{M}(\partial \Omega).$$

For more details, see Section 2.

For $\beta > 0$, we set

$$\Sigma_{\beta} := \{x \in \Omega : \delta(x) = \beta\}, \quad \Omega_{\beta} := \{x \in \Omega : \delta(x) < \beta\}, \quad D_{\beta} := \{x \in \Omega : \delta(x) > \beta\}.$$

In the nonlocal framework, the classical concept of boundary trace introduced by Marcus and Véron (see [27, Definition 1.3.6]) is not valid, hence one needs its nonlocal counterpart to tackle the boundary value problem with measure for (1.1). Recently, Nguyen and Véron [31] introduced a notion of normalized boundary trace which is defined as follows:

**Definition 1.5.** (s-boundary trace) Let $s > 1/2$. We say that a function $u \in L^1_{loc}(\Omega)$ possesses an $s$-boundary trace on $\partial \Omega$ if there exists a measure $\mu \in \mathcal{M}(\partial \Omega)$ such that

$$\lim_{\beta \to 0} \beta^{1-s} \int_{\Sigma_{\beta}} |u - M_s[\mu]| dS = 0.$$

The $s$-boundary trace of $u$ is denoted by $\text{tr}_s(u)$.

Note that the idea of the notion stems from the following two-sided estimate (see [31, Corollary 2.10])

$$C^{-1} \|\mu\|_{\mathcal{M}(\partial \Omega)} \leq \beta^{1-s} \int_{\Sigma_{\beta}} M_s[\mu] dS \leq C \|\mu\|_{\mathcal{M}(\partial \Omega)} \quad \forall \mu \in \mathcal{M}(\partial \Omega), \beta > 0 \text{ small}.$$ 

The notion is well-defined thanks to the fact that $s > \frac{1}{2}$ as explained in the remark following [31, Definition 2.13]. A remarkable feature of this notion is that it enables to examine $\text{tr}_s(G_s[\tau]) = 0$ for every $\tau \in \mathcal{M}(\Omega, \delta^s)$ and $\text{tr}_s(M_s[\mu]) = \mu$ for every $\mu \in \mathcal{M}(\partial \Omega)$ (see [31]), which is essential to investigate the problem

$$\begin{cases}
(-\Delta)^s u + f(u) = 0 & \text{in } \Omega, \\
\text{tr}_s(u) = \mu, \\
u = 0 & \text{in } \Omega^c = \mathbb{R}^N \setminus \Omega.
\end{cases}$$

In [31], Nguyen and Véron proved that

$$p_s := \frac{N + s}{N - s}$$

is a critical exponent for (1.10). More precisely, they showed the existence, uniqueness and stability result in the case $p \in (1, p_s)$ and removability result in the case $p \geq p_s$. For the study of boundary singularities of solutions to the equation in (1.10) in different setting, we refer to [2, 14, 13, 22].

In light of the above notion, the boundary value problem for (1.1) can be formulated in the following manner

$$\begin{cases}
(-\Delta)^s u = f(u) & \text{in } \Omega, \\
\text{tr}_s(u) = \mu, \\
u = 0 & \text{in } \Omega^c,
\end{cases}$$

where $\Omega$ is a $C^2$ bounded domain in $\mathbb{R}^N$.

**Definition 1.6.** (Weak solution) Let $\mu \in \mathcal{M}(\partial \Omega)$. A function $u$ is called a weak solution of (1.12) if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \delta^s)$ and
\[
(1.13) \quad \int_{\Omega} u(-\Delta)^s \xi dx = \int_{\Omega} f(u) \xi dx + \int_{\Omega} \mathcal{M}_s[\mu](-\Delta)^s \xi dx, \quad \forall \xi \in \mathcal{X}_s(\Omega),
\]
where $\mathcal{X}_s(\Omega) \subset C(\mathbb{R}^N)$ denotes the space of test functions $\xi$ satisfying

(i) $\text{supp}(\xi) \subset \Omega$,

(ii) $(-\Delta)^s \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^s \xi(x)| \leq C$ for some $C > 0$,

(iii) there exists $\varphi \in L^1(\Omega, \delta^s)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^s \xi| \leq \varphi$ a.e. in $\Omega$, for all $\epsilon \in (0, \epsilon_0]$.

We observe that, by [31, Proposition A], $u$ is a weak solution of (1.12) if and only if $u$ can be written in the form
\[
(1.14) \quad u = \mathcal{G}_s[f(u)] + \mathcal{M}_s[\mu].
\]

Our next result, which is proved by combining the bootstrap argument and regularity results (see [33, 34, 35]), depicts the relation between weak solutions and viscosity solutions.

**Theorem 1.7.** Let $\mu \in \mathcal{M}^+(\partial \Omega)$ and $p \in (1, p_s)$, where $p_s$ be as in (1.11). Assume $f \in C(\mathbb{R}^+)$ satisfies
\[
(1.15) \quad 0 \leq f(t) \leq at^p + b, \quad a, b > 0.
\]
If $u$ is a nonnegative weak solution of (1.12) then $u \in C^{2s+\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$. In particular, $u$ is a viscosity solution and satisfies (1.8).

The following theorem is devoted to an existence result.

**Theorem 1.8.** Let $\mu \in \mathcal{M}^+(\partial \Omega)$ and $p \in (1, p_s)$, where $p_s$ be as in (1.11). Assume $f \in C(\mathbb{R}^+)$ satisfies (1.15). There exist $\bar{b}$ and $\hat{\rho}$ such that if $b \in (0, \bar{b})$ and $\|\mu\|_{\mathcal{M}(\partial \Omega)} < \hat{\rho}$, then problem (1.12) admits a nonnegative weak solution $u \geq \mathcal{M}_s[\mu]$. Moreover, $u$ is a viscosity solution of (1.1) and satisfies (1.8).

Let us discuss the approach used in the proof of Theorem 1.8. As for the existence part, we translate (1.12) to an equivalent problem with zero boundary condition satisfied by $v = u - \mathcal{M}_s[\mu]$. In the spirit of [15], owing to the estimates of Green kernel and Martin kernel, together with Schauder fixed point theorem, we can construct a sequence of approximating solutions $\{v_n\}$ for the new problem provided that $\|\mu\|_{\mathcal{M}(\partial \Omega)}$ is small (see Lemma 4.3). Putting $u_n = v_n + \mathcal{M}_s[\mu]$ and using Vitali convergence theorem for the limit process, one can finally show that the sequence $\{u_n\}$ converges to a weak solution of (1.12). The rest of the theorem follows straightforward from Proposition 1.7.

When $f(u) = u^p$, the class of weak solutions of (1.12) can be much better described. For the convenience, we write (1.12) with $f(u) = u^p$ in the form

\[
(P_\rho) \quad \begin{cases} 
(-\Delta)^s u = u^p \quad \text{in } \Omega \\
\text{tr}_s(u) = \rho \nu \\
u = 0 \quad \text{in } \Omega^c,
\end{cases}
\]
where $\nu \in \mathcal{M}^+(\partial \Omega)$ such that $\|\nu\|_{\mathcal{M}(\partial \Omega)} = 1$ and $\rho$ is a positive parameter.
Theorem 1.9. Let $p > 1$, $\rho > 0$, $\nu \in \mathfrak{M}^+(\partial \Omega)$ such that $\|\nu\|_{\mathfrak{M}(\partial \Omega)} = 1$ and $p_s$ be as in (1.11).

Case I: $p \in (1, p_s)$. There exists a threshold value $\rho^* > 0$ for $(P_\rho)$ such that the following holds.

(i) If $\rho \leq \rho^*$ then problem $(P_\rho)$ admits a minimal positive weak solution $u_\rho$. Moreover $\{u_\rho\}_{\rho \leq \rho^*}$ is an increasing sequence which converges, as $\rho \to \rho^*$, to the minimal solution $u_{\rho^*}$ of $(P_{\rho^*})$ in $L^1(\Omega)$ and in $L^p(\Omega, \delta^s)$. (ii) If $\rho > \rho^*$ then problem $(P_\rho)$ does not admit any positive weak solution.

Case II: $p \geq p_s$. Then for every $\rho > 0$ and $z \in \partial \Omega$, problem $(P_\rho)$ with $\nu = \delta_z$ ($\delta_z$ denotes the Dirac measure concentrated at $z$) does not admit any positive weak solution.

The above theorem is a nonlocal analogue of [7, Theorem 1.3 and Corollary 2.3].

Remark 1.10. It is worthwhile to compare the absorption case with the source case. It was proved in [31] that when $f(u) = u^p$ with $p \in (1, p_s)$, then for any $\rho > 0$ and $z \in \partial \Omega$ problem (1.10) with $\mu = \rho \delta_z$ admits a unique solution $u_{\rho,z}$. Moreover $u_{\rho,z} = \lim_{\rho \to \infty} u_{\rho,z}$ is a solution of the equation in (1.10). However, this type of phenomenon does not occur in the case of source nonlinearity due to Theorem 1.9 Case I, (ii).

Now we assume that $0 \in \partial \Omega$. It is interesting that when $\nu = \delta_0$ ($\delta_0$ denotes the Dirac measure concentrated at 0) and $\rho > 0$ small, there are at least two weak solutions of $(P_\rho)$: the first one is the minimal solution $u_\rho$ given in Theorem 1.9 and the second one is constructed using Mountain Pass theorem. Further, the second solution is strictly greater than the minimal solution and this is reflected in the next theorem.

Theorem 1.11. Assume $p \in (1, p_s)$, where $p_s$ be as in (1.11), $0 \in \partial \Omega$ and $\nu = \delta_0$. Then there exists $\rho_0 \in (0, \rho^*)$ such that for any $\rho \in (0, \rho_0)$, $(P_\rho)$ admits at least two positive weak solutions $u$ and $u_\rho$ satisfying $u > u_\rho$. Here $u_\rho$ is the minimal solution given in Theorem 1.9.

Remark 1.12. The main reason that we have obtained the existence of second solution only in the range $(0, \rho_0) \subseteq (0, \rho^*)$ but not in entire $(0, \rho^*)$ is that the minimal solution $u_\rho$ is stable only in $(0, \rho_0)$ but may not stable in entire $(0, \rho^*)$ (see Definition 5.3 and Proposition 5.5). This is due to the fact that the eigenfunction $\varphi_1$, corresponding to the first eigenvalue of the weighted linearized eigenvalue problem

\begin{align}
(\Delta)^s \varphi = p \mu \varphi^{p-1} & \quad \text{in } \Omega \\
\varphi = 0 & \quad \text{in } \Omega^c,
\end{align}

which belongs to $H^s(\mathbb{R}^N)$, may not belong to $X_s(\Omega)$. If $\varphi_1 \in X_s(\Omega)$, then using [17, Lemma 2.2], it would hold $\int_\Omega u_\rho (\Delta)^s \varphi_1 dx = \int_\Omega \varphi_1 (\Delta)^s u_\rho dx$ and from this it can be shown that $u_\rho$ is stable for $\rho \in (0, \rho^*)$.

The rest of the paper is organized as follows. Section 2 is preliminaries, where we quote various important results from different papers which will be used in proving above theorems. In Section 3 we prove Theorem 1.2 and discuss the relation between different notions of solutions. Section 4 deals with the existence and regularity properties of positive solution of (1.12). In particular, we prove Theorem 1.7 and Theorem 1.8. In Section 5, we prove Theorem 1.9 and Theorem 1.11. Finally in Appendix, we consider equations of the type $(-\Delta)^s u = f(x, u, \nabla u)$ and we establish an a priori estimate for positive viscosity solutions of that equation and for their gradients.
We would like to remark that, in a forthcoming paper [5], we generalize the above a priori estimate and existence results to the case of systems.

**Notations:** Throughout this paper we denote by \( \delta(x) = \text{dist}(x, \partial \Omega) \), on the other hand by \( \delta_y \) we denote the Dirac mass concentrated at \( y \). By the notation \( u \in L^1(\mathbb{R}^N, \phi) \), we mean \( \int_{\mathbb{R}^N} |u(x)|\phi(x) \, dx < \infty \). Similarly we define \( L^1(\mathbb{R}^N, \delta^s) \). \( \Omega^c \) is defined as compliment of \( \Omega \). Throughout the present paper, we denote by \( c, c', c_1, c_2, C, \ldots \) positive constants that may vary from line to line. If necessary, the dependence of these constants will be made precise.

## 2. Preliminaries

In this section, we collect some results necessary for our analysis.

### 2.1. \( s \)-harmonic functions

Let us recall the definition of \( s \)-harmonic functions in the probabilistic sense from [9, page 55]. Let \( (X_t, P^x) \) be the standard symmetric \( 2s \)-stable Lévy process in \( \mathbb{R}^N \) (i.e. stationary with independent increments) with characteristic function

\[
E^0 e^{i\xi X_t} = e^{-t|\xi|^{2s}} \quad \xi \in \mathbb{R}^N, t \geq 0.
\]

Denote by \( E^x \) the expectation with respect to the distribution \( P^x \) of the process starting from \( x \in \mathbb{R}^N \). Assume without loss of generality that sample paths of \( X_t \) are right-continuous with finite left-hand limits a.s. It is known that \( (X_t) \) is a strong Markov process and its transition probabilities is defined by

\[
P_t(x, A) := P^x(X_t \in A) = \mu_t(A - x),
\]

where \( \mu_t \) is the one-dimensional distribution of \( X_t \) with respect to \( P^0 \). It is well known that \( -(\Delta)^s \) is the generator of the process \( (X_t, P^x) \).

If \( D \subset \mathbb{R}^N \) is a Borel subset, we define \( t_D := \inf\{t \geq 0 : X_t \not\in D\} \), i.e. \( t_D \) is the first exit time from \( D \). If \( D \) is bounded then \( t_D < \infty \) a.s. Denote

\[
E^x u(X_{t_D}) := E^x\{u(X_{t_D}) : t_D < \infty\}.
\]

**Definition 2.1.** Let \( u \) be a Borel measurable function in \( \mathbb{R}^N \). We say that \( u \) is \( s \)-harmonic in \( \Omega \) in probabilistic sense if for every bounded open set \( D \Subset \Omega \),

\[
u(x) = E^x u(X_{t_D}), \quad x \in D.
\]

We say that \( u \) is singular \( s \)-harmonic in \( \Omega \) in probabilistic sense if \( u \) is \( s \)-harmonic in probabilistic sense and \( u = 0 \) in \( \Omega^c \).

The following result follows from [9, Corollary 3.10 and Theorem 3.12].

**Proposition 2.2.** Let \( u \in L^1(\mathbb{R}^N, \omega) \). Then

(i) \( u \) is \( s \)-harmonic in \( \Omega \) in probabilistic sense if and only if \( -(\Delta)^s u = 0 \) in \( \Omega \) in the sense of distributions.

(ii) \( u \) is singular \( s \)-harmonic in \( \Omega \) in probabilistic sense if and only if \( u \) is \( s \)-harmonic in \( \Omega \) in the sense of distributions and \( u = 0 \) in \( \Omega^c \).

### 2.2. Green kernel and Martin kernel

We denote by \( G_s \) the Green kernel of \( -(\Delta)^s \) in \( \Omega \) respectively. More precisely, for every \( y \in \Omega \),

\[
\begin{cases}
(\Delta)^s G_s(., y) = \delta_y & \text{in } \Omega \\
G_s(., y) = 0 & \text{in } \Omega^c,
\end{cases}
\]
where \( \delta_y \) is the Dirac mass at \( y \). Fix any reference point \( x_0 \in \Omega \), the Martin kernel \( M_s \) of \(( -\Delta )^s \) in \( \Omega \) is defined by

\[
M_s(x, z) := \lim_{\Omega \ni y \to z} \frac{G_s(x, y)}{G_s(x_0, y)} \quad \forall x \in \mathbb{R}^N, \ z \in \partial \Omega.
\]

The Martin boundary is the set \( \Omega^* \setminus \Omega \), where \( \Omega^* \) is the smallest compact set for which \( M_s(x, z) \) is continuous in \( z \) in the extended sense. Martin boundary of \( \Omega \) can be identified with the Euclidean boundary \( \partial \Omega \) when \( \Omega \) is a Lipschitz bounded domain (see [19, Theorem 3.6]). It follows from [19] that the mapping \(( x, z) \mapsto M_s(x, z) \) is continuous on \( \Omega \times \partial \Omega \) and for any \( z \in \partial \Omega \), \( M_s(\cdot, z) \) is \( s \)-harmonic in \( \Omega \) with \( M_s(\cdot, z) = 0 \) in \( \Omega^c \) and \( M_s(x_0, z) = 1 \).

The next lemma is due to [20, Corollary 1.3] and [19, Theorem 3.9].

**Lemma 2.5.** There exists a constant \( c = c(N, s, \Omega) \) such that

\[
\|[\tau]_M \|_{\mathcal{M}_{k,s,\gamma}(\Omega, \delta_{\alpha})} \leq c \|[\tau]_M \|_{\mathcal{M}(\Omega, \delta_{\gamma})} \quad \forall \tau \in \mathcal{M}(\Omega, \delta_{\gamma}).
\]

**Lemma 2.6.** ([17, Proposition 2.2] and [31, Lemma 2.7]) (i) Let \( \alpha, \gamma \in [0, s] \) and \( k_{s,\gamma} \) be as in (2.4). There exists a constant \( c = c(N, s, \alpha, \gamma, \Omega) > 0 \) such that

\[
\|[G_s[\tau]]_M\|_{\mathcal{M}_{k_{s,\gamma}}(\Omega, \delta_{\alpha})} \leq c \|[\tau]_{\mathcal{M}(\Omega, \delta_{\gamma})} \quad \forall \tau \in \mathcal{M}(\Omega, \delta_{\gamma}).
\]
(ii) Let $\alpha > -s$. There exists a constant $c = c(N, s, \alpha, \Omega)$
\begin{equation}
\|M_k[\mu]\|_{M^{N+\alpha}_{N-\alpha}((\Omega, \delta^s))} \leq c \|\mu\|_{\mathcal{M}(\partial\Omega)} \quad \forall \mu \in \mathcal{M}(\partial\Omega).
\end{equation}

**Lemma 2.7.** [35, Proposition 1.4] (i) If $t > \frac{N}{2s}$ then there exists a constant $c = c(N, s, t, \Omega)$ such that
\begin{equation}
\|G_s[\tau]\|_{L^\infty(\Omega)} \leq c \|\tau\|_{L^t(\Omega)} \quad \forall \tau \in L^t(\Omega).
\end{equation}
(ii) If $1 < t < \frac{N}{2s}$, then there exists a constant $c = c(N, s, t)$ such that
\begin{equation}
\|G_s[\tau]\|_{L^{N-\frac{Nt}{2s}}(\Omega)} \leq c \|\tau\|_{L^t(\Omega)} \quad \forall \tau \in L^t(\Omega).
\end{equation}

The next result is due to Nguyen and Veron (see [31, Lemma 3.3]).

**Lemma 2.8.** Assume $z \in \partial\Omega$ and $1 < q < p_s$, where $p_s$ is as defined in (1.11). Then there exists a constant $c = c(N, s, q, \Omega)$ such that
\begin{equation}
G_s[M_s[\cdot, z]^q](x) \leq c|x-z|^{N+s-(N-s)q}M_s(x, z) \quad \forall x \in \Omega.
\end{equation}

*Proof.* Estimate (2.9) follows by combining Lemma 2.3 along with [31, Lemma 3.3].

**Lemma 2.9.** Assume $\mu \in \mathcal{M}^+(\partial\Omega)$ and $1 < q < p_s$, where $p_s$ is as defined in (1.11). Then there exists a constant $\tilde{C} = \tilde{C}(N, s, q, \Omega)$ such that
\begin{equation}
G_s[M_s[\mu]^q](x) \leq \tilde{C}\|\mu\|_{\mathcal{M}(\partial\Omega)}^{q-1}M_s[\mu] \quad \forall x \in \Omega.
\end{equation}

*Proof.* Combining Jensen's inequality with Lemma 2.8, we obtain (2.10) (also see [7, Theorem 1.1]).

### 3. A priori estimates

In this section, we adapt the method introduced by Poláčik et al [32], based on a topological argument, called the Doubling lemma (see [32, Lemma 5.1]), to establish a priori estimate of solutions, as well as their gradient. All the results in this section are valid for an arbitrary domain $\Omega$.

**Theorem 3.1.** Assume $f(u) = u^p$ with $1 < p < p_c$, where $p_c$ is defined as in (1.6) and $\Omega$ is an arbitrary domain in $\mathbb{R}^N$. Then there exists $C = C(N, p, s)$ such that for any nonnegative viscosity solution $u$ of (1.1), it holds
\begin{equation}
|u(x)| + |\nabla u(x)|^{\frac{2p}{p+2s-1}} \leq C\delta(x)^{-\frac{2s}{p+2s-1}} \quad \forall x \in \Omega.
\end{equation}

*Proof.* By definition of viscosity solution, we have $u, f(u) \in L^\infty_{loc}(\Omega)$ and therefore by [26, Lemma 4.2] it follows that $u \in C^\gamma_{loc}(\Omega)$ for some $\gamma \in (0, 1)$. Consequently, [13, Theorem 2.1] yields $u \in C^{2s+\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$ and thus $(-\Delta)^s u$ makes sense pointwise and $u \in C^1(\Omega)$ since $s > 1/2$. Now suppose (3.1) fails. Then there exist sequences $\Omega_k, u_k \in L^1(\mathbb{R}^N, \omega)$ where $\omega$ is given in (1.5), $y_k \in \Omega_k$ such that $u_k$ is a nonnegative solution of
\begin{equation}
(-\Delta)^s u = u^p \quad \text{in} \quad \Omega_k,
\end{equation}
and
\begin{equation}
M_k := u_k^{\frac{p-1}{2s}} + |\nabla u_k|^{\frac{p-1}{p+2s-1}}, \quad k = 1, 2, \ldots
\end{equation}
satisfy
\begin{equation}
M_k(y_k) > 2k\text{dist}^{-1}(y_k, \partial\Omega_k).
\end{equation}
By [32, Lemma 5.1 and Remark 5.2 (b)], it follows that there exits \( x_k \in \Omega_k \) such that
\[
M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k\text{dist}^{-1}(x_k, \partial \Omega_k)
\]
and
\[
M_k(z) \leq 2M_k(x_k) \quad \forall z \in B(x_k, kM_k(x_k)^{-1}).
\]
Now set
\[
\lambda_k := M_k(x_k)^{-1}
\]
and define
\[
v_k(y) := \lambda_k^{\frac{2s}{p-1}} u_k(x_k + \lambda_k y), \quad y \in \mathbb{R}^N.
\]
Note that, for \( y \in B(0, k) \), \( x_k + \lambda_k y \in B(x_k, k\lambda_k) = B(x_k, kM_k(x_k)^{-1}) \subset \Omega_k \) (see [32, Remark 5.2 (b)]). Therefore,
\[
(-\Delta)^s v_k(y) = \lambda_k^{\frac{2sp}{p-1}} (-\Delta)^s u_k(x_k + \lambda_k y) = v_k(y)^p, \quad y \in B(0, k).
\]
Moreover, from (3.6) and the definition of \( \lambda_k \), it follows that
\[
M_k(y) \leq 2 \quad \forall y \in B(0, k)
\]
and
\[
M_k(0) = 1.
\]
**Step 1:** We show that up to a subsequence, \( v_k \to v \) in \( C^\alpha_{loc}(\mathbb{R}^N) \), for some \( \alpha \in (0, 1) \).

For this, first we define \( \eta \in C^\infty_0(\mathbb{R}^N) \) such that
\[
\eta := \begin{cases} 
1 & \text{in } B(0, R_1) \\
0 & \text{in } B(0, R_2)^c,
\end{cases}
\]
where \( 0 < R_1 < R_2 \). Then define
\[
w_k(x) := a_{N,s} \int_{\mathbb{R}^N} \eta(y) \frac{1}{|x-y|^{N-2s}} v_k^p(y) dy,
\]
where \( a_{N,s} = \frac{2^{2s} \pi^{N/2}}{\Gamma(1-s)} \). Therefore, \( w_k \) satisfies
\[
(-\Delta)^s w_k = \eta v_k^p \quad \text{in } \mathbb{R}^N.
\]
We observe that, for \( k > R_2 \), \(|v_k(y)| \leq 2^{\frac{2s}{p-1}} \) in \( B(0, R_2) \), which can be easily checked using (3.10), (3.6)-(3.8) and (3.3). Therefore, for \( x, z \in B(0, R_1) \)
\[
|w_k(x) - w_k(z)| \leq a_{N,s} \int_{\mathbb{R}^N} \eta(y) \left| \frac{1}{|x-y|^{N-2s}} - \frac{1}{|z-y|^{N-2s}} \right| v_k^p(y) dy
\]
\[
\leq 2^{\frac{2s}{p-1}} a_{N,s} \int_{B(0, R_2)} \eta(y) \left| \frac{1}{|x-y|^{N-2s}} - \frac{1}{|z-y|^{N-2s}} \right| dy.
\]
Put \( D_1 = \{ y \in B(0, R_2) : |x-y| \geq |y-z| \} \) and \( D_2 = \{ y \in B(0, R_2) : |x-y| \leq |y-z| \} \). Observe that, thanks to mean value theorem we have
\[
|\rho^{2s-N} - t^{2s-N}| = (N - 2s)\theta^{2s-N-1} |\rho - t|,
\]
for some $\theta \in (\min(r,t), \max(r,t))$ and $r = |x-y|$ and $t = |y-z|$. Therefore using the above expression, we obtain
\begin{equation}
\int_{B(0,R_2)} |x-y|^{2s-N} - |y-z|^{2s-N} \, dy
\end{equation}
\begin{align*}
= & \int_{D_1} |x-y|^{2s-N} - |y-z|^{2s-N} \, dy + \int_{D_2} |x-y|^{2s-N} - |y-z|^{2s-N} \, dy \\
\leq & (N-2s)|x-z| \int_{D_1} |y-z|^{2s-N-1} \, dy + (N-2s)|x-z| \int_{D_2} |x-y|^{2s-N-1} \, dy \\
\leq & C(N,s,R_1,R_2)|x-z|.
\end{align*}
Here to obtain the last estimate, we have used $s > \frac{1}{2}$. Thus $w_k$ is uniformly Lipschitz continuous in $B(0,R_1)$. Consequently,
\begin{equation}
\|v_k - w_k\|_{L^\infty(B(0,R_1))} < C,
\end{equation}
where $C = C(p,N,s,R_1)$. Next, we define,
$$
\psi_k(x) := v_k(x) - w_k(x).
$$
Clearly, $\psi_k$ is $s$-harmonic in $B(0,R_1)$ in the viscosity sense. Also, it is easy to see that $\psi_k \in L^1(\mathbb{R}^N, \omega)$ for each $k$. Thus, by [11, Theorem 4.1], $\psi_k \in C^\alpha_{loc}(B(0,R_1))$ for some $\beta$. By a direct computation it can be shown that $\psi_k$ is $s$-harmonic in $B(0,R_1)$ in the sense of distribution sense. Hence by Proposition 2.2, it follows that $\psi_k$ is $s$-harmonic in $B(0,R_1)$ in the probabilistic sense. Next we define,
\begin{equation}
\tilde{\psi}_k := \begin{cases} 
\|\psi_k\|_{L^\infty(B(0,R_1))} - \psi_k & \text{in } B(0,R_1) \\
0 & \text{in } B(0,R_1)^c.
\end{cases}
\end{equation}
Thus $\tilde{\psi}_k$ is nonnegative in $\mathbb{R}^N$ and $s-$harmonic function in $B(0,R_1)$. Consequently applying [10, Lemma 3.2], we have for any $x \in B(0,R') \cap B(0,R_1)$,
$$
|\nabla \psi_k(x)| = |\nabla \tilde{\psi}_k(x)| \leq C \frac{\tilde{\psi}_k(x)}{|R_1 - R'|} \leq C(\|\psi_k\|_{L^\infty(B(0,R_1))} + \|w_k\|_{L^\infty(B(0,R_1))}) < C,
$$
where $C = C(p,N,s,R_1,R')$. Hence $\psi_k$ is uniformly Lipschitz in $B(0,R')$. This in turn implies $v_k = \psi_k + w_k$ is uniformly Lipschitz in $B(0,R')$. Therefore, applying Ascoli-Arzelà theorem, we obtain $v_k \to v$ in $C^\alpha(B(0,R'))$, for some $\alpha \in (0,1)$.

**Step 2:** From (3.10) and (3.11) it follows $v$ is bounded in $\mathbb{R}^N$ and $v$ is nontrivial. Moreover, $v_k \geq 0$ implies $v \geq 0$. Let $\tilde{v}_k$ be the function obtained by extending $v_k$ to be zero outside $B(0,k)$. Then it is easy to see that $(-\Delta)^s \tilde{v}_k \geq \tilde{v}_k$ in $B(0,k)$. Passing to the limit, by using [12, Lemma 5] (see also [13, Lemma 2.4]), we obtain $(-\Delta)^s v \geq v^p$ in $\mathbb{R}^N$, which is a contradiction due to [21, Theorem 1.3] since $p < \frac{N}{N-2s}$. Hence the theorem follows.

**Remark 3.2.** It is necessary to emphasize that $u$ is not assumed to be bounded in $\mathbb{R}^N$, therefore $v_k$ may not be bounded in $\mathbb{R}^N$. This yields a difficulty in proving the convergence of the sequence $\{v_k\}$ since the local Schauder estimate in [33] cannot be applied. However, we overcome this issue by employing an estimate on the gradient of nonnegative $s$-harmonic function.
Proof of Theorem 1.2. The proof is similar to that of Theorem 3.1. We point out here
the main differences. Suppose the assertion of this theorem does not hold. Then there exist
sequences $\Omega_k, u_k \in L^1(\mathbb{R}^N, \omega), y_k \in \Omega_k$ such that $u_k$ satisfies
\begin{equation}
(-\Delta)^s u_k = f(u_k) \quad \text{in} \quad \Omega_k,
\end{equation}
and let $M_k$ be defined by (3.3). Then $M_k$ satisfies
\begin{equation}
M_k(y_k) > 2k \left(1 + \text{dist}^{-1}(y_k, \partial \Omega_k)\right).
\end{equation}
Moreover, we have
\begin{equation}
(-\Delta)^s (\nu - (-\Delta)^{1-s} \nu) = 0 \quad \text{in} \quad B(0, \frac{1}{2}),
\end{equation}
which is a distributional solution of
\begin{equation}
\frac{2sp}{p-1} \leq f_k(v_k(y)) \leq C_f' \quad \forall y \in B(0,k).
\end{equation}
Note that as $M_k(x_k) \geq M_k(y_k) > 2k$, we have $\lambda_k \to 0$ as $k \to \infty$. Therefore, by an easy
computation it follows
\begin{equation}
-C_f < f(t) \leq C_f(1 + t^p) \quad \forall t \geq 0.
\end{equation}
Next we show that other types of solutions satisfy (1.8) too. We say that a function
$u : \mathbb{R}^N \to \mathbb{R}$ is a distributional solution of (1.1) if $u \in L^1(\mathbb{R}^N, \omega)$ and $u$
\text{ satisfies (1.1) in the sense of distribution.}

Proposition 3.3. Assume $p \in (1, p_c)$, where $p_c$ is as defined in (1.6), $f$ is as in Theorem 1.2
and $\Omega$ is an arbitrary domain in $\mathbb{R}^N$. Let $u \in C_\text{loc}^\gamma(\Omega)$, for some $\gamma \in (0, 1)$, be a nonnegative
\text{distributional solution of (1.1). Then } u \text{ is a viscosity solution of (1.1) and estimate (1.8)
\text{holds.}

Proof. Suppose $u \in C_\text{loc}^\gamma(\Omega)$ is a nonnegative distributional solution of (1.1). Since $f \in C_\text{loc}^{\beta}(\mathbb{R})$, we obtain $f(u) \in C_\text{loc}^{\tilde{\beta}}(\Omega)$, for some $\tilde{\beta} \in (0, \beta)$.

Step 1: We show that $u \in C_\text{loc}^{2s+\alpha}(\Omega)$, for some $\alpha \in (0, 1)$.

To prove this step we use an idea from [13]. Without loss of generality, we assume $B(0, 1) \subset \Omega$ and $f(u) \in C^{\tilde{\beta}}(B(0,1))$. Let $\eta \in C_0^\infty(B(0,1))$ such that $\eta \equiv 1$ in $B(0, \frac{1}{2})$ and $0 \leq \eta \leq 1$. Now let us consider the equation
\begin{equation}
-\Delta w = \eta f(u) \quad \text{in} \quad \mathbb{R}^N.
\end{equation}
Using Schauder estimate of Laplacian, we have $w \in C^{2, \tilde{\beta}}$ and $(-\Delta)^{1-s} w \in C^{2s+\tilde{\beta}}$, see [37] or [23, Theorem 3.1]. Moreover, we have
\begin{equation}
(-\Delta)^s (u - (-\Delta)^{1-s} w) = 0 \quad \text{in} \quad B(0, \frac{1}{2}),
\end{equation}
i.e., \( u - (-\Delta)^{1-s}w \) is \( s \)-harmonic in the sense of distribution. It is easy to note that \( u - (-\Delta)^{1-s}w \in L^1(\mathbb{R}^N, \omega) \). Therefore, by Proposition 2.2, \( u - (-\Delta)^{1-s}w \) is \( s \)-harmonic in the probabilistic sense. Moreover, from the proof of [9, Theorem 3.12], it also follows that \( u - (-\Delta)^{1-s}w \in C^2(B(0, 1/2)) \). Further, using the definition of viscosity solution, it is easy to see that (3.21) is satisfied in the viscosity sense as well. Consequently, we can use [11, Theorem 1.1] and [11, Remark 9.4] (see also Theorem 4.1 there), to obtain that there exist \( \alpha \) such that \( u - (-\Delta)^{1-s}w \in C^{2s+\alpha}(B(0, 1/2)) \). Hence \( u \in C^{2s+\alpha}(B(0, 1/2)) \), for some \( \alpha > 0 \) and this completes the proof of step 1.

**Step 2:** By Step 1, \( (-\Delta)^s u(x) \) is well defined for all \( x \in \Omega \) and thus (1.1) is satisfied in pointwise sense as well. Therefore, again using the definition of viscosity solution, it is easy to see that \( u \) is a viscosity solution of (1.1). Hence estimate (1.8) follows from Theorem 1.2. \( \square \)

A function \( u : \mathbb{R}^N \to \mathbb{R} \) is called a classical solution of (1.1) if \( u \in C(\Omega) \cap L^1(\mathbb{R}^N, \omega) \), \((-\Delta)^s u(x)\) is well-defined for all \( x \in \Omega \) and \( u \) satisfies (1.1) in pointwise sense.

**Theorem 3.4.** Assume \( p \in (1, p_c) \), where \( p_c \) is as defined in (1.6), \( f \) is as in Theorem 1.2 and \( \Omega \) is an arbitrary domain in \( \mathbb{R}^N \). Let \( u \) be a nonnegative classical solution of (1.1). Then estimate (1.8) holds.

**Proof.** Using the definition of viscosity solution, it is not difficult to see that \( u \) is a viscosity solution of (1.1) and hence estimate (1.8) follows from Theorem 1.2. \( \square \)

## 4. Existence and regularity

This section is devoted to the regularity and existence of weak solutions of (1.12). We begin with the proof of the regularity property.

**Proof of Theorem 1.7.** We will use the bootstrap argument. Assume that \( u \) is a nonnegative weak solution of (1.12). Then \( f(u) \in L^1(\Omega, \delta^s) \) and \( u = G_s[f(u)] + M_s[\mu] \).

Let \( x_0 \in \Omega \) and \( r > 0 \) such that \( B(x_0, 2r) \subset \subset \Omega \). For any \( j \in \mathbb{N} \), set \( B_j := B(x_0, 2^{-j}r) \). For any \( j \in \mathbb{N} \), we can write

\[
(4.1) \quad u = G_s[\chi_{\Omega \setminus B_j} f(u)] + G_s[\chi_{B_j} f(u)] + M_s[\mu].
\]

Observe that, for \( x \in B_{j+1} \), by (2.2),

\[
G_s[\chi_{\Omega \setminus B_j} f(u)] = \int_{\Omega \setminus B_j} f(u(y)) G_s(x, y) dy \\
\leq C\delta(x)^s \int_{\Omega \setminus B_j} f(u(y)) \delta(y)^s |x - y|^{-N} dy \\
\leq C 2^N r^{-N} \| f(u) \|_{L^1(\Omega, \delta^s)} < \infty.
\]

Therefore,

\[
(4.2) \quad G_s[\chi_{\Omega \setminus B_j} f(u)] \in L^\infty(B_{j+1}) \quad \forall j \in \mathbb{N}.
\]

Next, by employing Lemma 2.5 and Lemma 2.6, we obtain for \( q \in (1, p_s) \),

\[
\| u \|_{L^q(\Omega, \delta^s)} \leq \| G_s[f(u)] \|_{L^q(\Omega, \delta^s)} + \| M_s[\mu] \|_{L^q(\Omega, \delta^s)} \\
\leq C(\| G_s[f(u)] \|_{M^{p_s}(\Omega, \delta^s)} + \| M_s[\mu] \|_{M^{p_s}(\Omega, \delta^s)}) \\
\leq C(\| f(u) \|_{\mathcal{M}(\Omega, \delta^s)} + \| \mu \|_{\mathcal{M}(\partial \Omega)}) < C'.
\]
That is, \( u \in L^q(\Omega, \delta^s) \) for every \( q \in (1, p_s) \). In particular, since \( p \in (1, p_s) \), it follows that \( u \in L^p(\Omega, \delta^s) \) and consequently \( \chi_{B_0} u \in L^p(B_0) \). By applying Lemma 2.6 (i) with \( \alpha = \gamma = 0 \), we deduce that \( \mathcal{G}_s[\chi_{B_0} u^p] \in M^p(B_0) \). Furthermore, Lemma 2.5 yields \( M^p(B_0) \subset L^q(B_0) \) for every \( 1 < q < p_c \). Thus \( \mathcal{G}_s[\chi_{B_0} u^p] \in L^q(B_0) \) for every \( 1 < q < p_c \). Since \( f(u) \leq C(1 + u^p) \), we have \( \mathcal{G}_s[\chi_{B_0} f(u)] \in L^q(B_0) \) for every \( 1 < q < p_c \). This and (4.1) – (4.2) yield \( u \in L^q(B_3) \) for every \( 1 < q < p_c \). Put 
\[
t_0 := \frac{1}{2}(1 + \frac{p_s}{p}) > 1.
\]
Then \( 1 < pt_0 < p_s < p_c \) and hence \( u \in L^{pt_0}(B_3) \). By the assumption, \( f(u) \in L^{t_0}(B_3) \). Without loss of generality, we assume that \( t_0 \neq \frac{N}{2s} \). If \( t_0 > \frac{N}{2s} \) then by Lemma 2.7 (i), \( \mathcal{G}_s[\chi_{B_3} f(u)] \in L^\infty(B_3) \). This and (4.1) – (4.2) imply \( u \in L^\infty(B_0) \). If \( t_0 < \frac{N}{2s} \) then by Lemma 2.7 (ii) we obtain \( \mathcal{G}_s[\chi_{B_3} f(u)] \in L^{pt_1}(B_3) \) where 
\[
t_1 := \frac{1}{p N - 2t_0 s}.
\]
Then from (4.1) – (4.2), \( u \in L^{pt_1}(B_6) \). By the assumption, \( f(u) \in L^{t_1}(B_6) \). We have 
\[
\frac{t_1}{t_0} = \frac{N}{p N - 2t_0 s} > \frac{N}{p N - 2s} > t_0.
\]
This implies that \( t_1 > t_0 > t_0 > 1 \).

Again, we may assume that \( t_1 \neq \frac{N}{2s} \). If \( t_1 > \frac{N}{2s} \) then by Lemma 2.7 (i), \( \mathcal{G}_s[\chi_{B_6} f(u)] \in L^\infty(B_6) \). Hence \( u \in L^\infty(B_9) \). If \( t_1 < \frac{N}{2s} \), by Lemma 2.7 (ii), \( \mathcal{G}_s[\chi_{B_6} f(u)] \in L^{pt_2}(B_6) \) where 
\[
t_2 := \frac{1}{p N - 2t_1 s}.
\]
Then by (4.1) – (4.2), \( u \in L^{pt_2}(B_9) \) and by the assumption \( f(u) \in L^{t_2}(B_9) \). We have 
\[
\frac{t_2}{t_1} = \frac{t_1}{t_0} \frac{2t_0 s}{2t_1 s} > \frac{t_1}{t_0} > t_0.
\]
This implies that \( t_2 > t_1 t_0 > t_0 \).

By induction, we can construct a sequence \( \{t_k\} \) such that \( t_k \neq \frac{N}{2s} \), 
\[
t_k := \frac{1}{p N - 2t_{k-1} s},
\]
\( t_k > t_0^{k+1} \) and \( \mathcal{G}_s[\chi_{B_{3k}} f(u)] \in L^{pt_k}(B_{3k}) \) and \( u \in L^{pt_k}(B_{3(k+1)}) \). Since \( t_0 > 1 \), there exists \( k \) large enough such that \( t_k > \frac{N}{2s} \). Then, by employing again Lemma 2.7 (ii), we deduce that \( u \in L^\infty(B_{3(k+1)}) \). Thus \( u \in L^\infty_{loc}(\Omega) \). By regularity results [34], we deduce that \( u \in C^{2s+c}(\Omega) \). This implies that \( u \) is a viscosity solution and hence (3.1) holds. \( \square \)

**Lemma 4.1.** Assume \( f(u) = u^p \) with \( p > 1 \) and \( \mu \in \mathcal{W}^+(\partial \Omega) \). If \( u \) is a solution of (1.12) then there is a constant \( c = c(N, s, p, \Omega) \) such that 
\[
\|u\|_{L^1(\Omega)} + \|u\|_{L^p(\Omega, \delta^s)} \leq c(1 + \|\mu\|_{\mathcal{W}(\partial \Omega)}).
\]

**Proof.** We prove this lemma in the spirit of [7]. Let \( (\lambda_1, \varphi_1) \) be the first eigenvalue and corresponding positive eigenfunction of \( (-\Delta)^s \) in \( X_0 \) (see the definition of \( X_0 \) in (5.12)). By [17, Lemma 2.1(ii)], \( \varphi_1 \in X_0(\Omega) \). Thus by taking \( \zeta = \varphi_1 \) in (1.13), we obtain 
\[
\lambda_1 \int_\Omega u \varphi_1 dx = \int_\Omega u^p \varphi_1 dx + \lambda_1 \int_\Omega M_s[\mu] \varphi_1 dx.
\]
We recall the Young’s inequality
\[ ab \leq \varepsilon a^p + C(\varepsilon)b^{\frac{1}{p-1}}, \quad a, b > 0, \varepsilon > 0, \]
where \( C(\varepsilon) = \frac{p-1}{p}(\varepsilon p)^{1/(p-1)} \). Since \( p > 1 \), using the above Young’s inequality with \( \varepsilon = (2\lambda_1)^{-1} \), \( a = u \varphi_1^p \), and \( b = \varphi_1^{1-p} \), we obtain
\[
\int_{\Omega} u \varphi_1 dx \leq (2\lambda_1)^{-1} \int_{\Omega} u^p \varphi_1 dx + (2\lambda_1)^{1-\frac{1}{p}} \int_{\Omega} \varphi_1 dx.
\]
Substituting (4.5) into (4.4) yields
\[
\int_{\Omega} u^\varphi_1 dx \leq 2\lambda_1 \int_{\Omega} M_s[u] \varphi_1 dx \leq (2\lambda_1)^{\frac{p}{p-1}} \int_{\Omega} \varphi_1 dx.
\]
Since the second term on the left hand-side of (4.6) is nonnegative, taking into account that \( c^{-1}\delta^s \leq \varphi_1 \leq c\delta^s \) for some constant \( c > 0 \), we have
\[
\|u\|_{L^p(\Omega, \delta^s)}^p \leq c(2\lambda_1)^{\frac{p}{p-1}} \int_{\Omega} \delta^s dx \leq c'.
\]
Next, combining (1.14) along with Lemma 2.5 yields
\[
\|u\|_{L^1(\Omega)} \leq \|G_s[u\varphi]\|_{L^1(\Omega)} + \|M_s[\mu]\|_{L^1(\Omega)} \leq C(\|G_s[u\varphi]\|_{L^{p_1}(\Omega)} + \|M_s[\mu]\|_{L^\infty(\Omega)}).
\]
Further, using [17, Proposition 2.2] (with \( \alpha = s = \beta, \gamma = 0 \)) and Lemma 2.6 (with \( \alpha = 0 \)) in the RHS of the above expression, we obtain
\[
\|u\|_{L^1(\Omega)} \leq C(\|u\|_{L^p(\Omega, \delta^s)} + \|\mu\|_{\mathcal{M}(\partial\Omega)}).
\]
Hence (4.3) holds by combining (4.7) and (4.8).

**Lemma 4.2.** Assume \( f(u) = u^p \), \( p \in (1, p_s) \), where \( p_s \) is defined as in (1.11) and \( \mu \in \mathcal{M}^+(\partial\Omega) \). Assume in addition that there exists a function \( U \in L^p(\Omega, \delta^s) \) such that \( U \geq G_s[U\varphi] + M_s[\mu] \). Then there exists positive minimal weak solution \( u_\mu \) of (1.12) satisfying
\[
M_s[\mu] \leq u_\mu \leq U.
\]
**Proof.** Put \( u_0 := M_s[\mu] \) and
\[
u_n := G_s[u_n^{p-1}] + M_s[\mu], \quad n \geq 1.
\]
Clearly \( u_0 \leq U \) and hence
\[
u_1 = G_s[u_0^p] + M_s[\mu] \leq G_s[U\varphi] + M_s[\mu] \leq U.
\]
By induction, we can show that \( u_n \leq U \) for every \( n \geq 1 \). Moreover, it is easy to see that \( \{u_n\} \) is an increasing sequence. Hence \( u_n \uparrow u_\mu \leq U \in L^p(\Omega, \delta^s) \). Therefore \( G_s[u_n^p] \uparrow G_s[u_\mu^p] \) a.e. in \( \Omega \). Letting \( n \to \infty \) in (4.10), we deduce that
\[
u_\mu = G_s[u_\mu^p] + M_s[\mu].
\]
This means that \( u_\mu \) is a weak solution of (1.12).

Next we show that \( u_\mu \) is the minimal solution of (1.12), that is, for any positive weak solution \( u \) of (1.12), we have \( u_\mu \leq u \). This follows as we have
\[
u = G_s[u\varphi] + M_s[\mu] \geq u_0,
\]
and this in turn implies
\[
u \geq G_s[u_0^p] + M_s[\mu] \geq u_1.
\]
By induction it follows that \( u \geq u_n \), for all \( n \geq 1 \). Hence \( u \geq u_\mu \).
Next we are concerned with solutions to the problem

\[
\begin{cases}
(\Delta)^s u = f(u) & \text{in } \Omega \\
\text{tr}_s(u) = \rho u & \text{in } \Omega^c, \\
u = 0 & \text{in } \partial \Omega,
\end{cases}
\]

(4.11)

where \( \nu \in \mathcal{M}^+(\partial \Omega) \) such that \( \|\nu\|_{\mathcal{M}(\partial \Omega)} = 1 \). Let \( \{f_n\} \) be a sequence of \( C^1 \) nonnegative functions defined on \( \mathbb{R}^+ \) such that

\[
f_n(0) = f(0), \quad f_n \leq f_{n+1} \leq f, \quad \sup_{\mathbb{R}^+} f_n = n \quad \text{and} \quad \lim_{n \to \infty} \| f_n - f \|_{L^\infty(\mathbb{R}^+)} = 0.
\]

(4.12)

Lemma 4.3. Assume \( f \) satisfies (1.15) and \( \{f_n\} \subset C^1(\mathbb{R}^+) \) is a sequence satisfying (4.12). Then there exist \( \hat{\Lambda} > 0 \) depending on \( N, s, p \) such that for every \( b \in (0, \hat{b}) \) and \( \rho \in (0, \hat{\rho}) \) the following problem

\[
\begin{cases}
(\Delta)^s v = f_n(v + \rho M_s[\mu]) & \text{in } \Omega \\
\text{tr}_s(v) = 0 & \text{in } \Omega^c, \\
v = 0 & \text{in } \partial \Omega,
\end{cases}
\]

(4.13)

admits a nonnegative solution \( v_n \) satisfying

\[
\| v_n \|_{L^p(\Omega, \delta^s)} \leq \hat{\Lambda}.
\]

(4.14)

Proof. We aim to use Schauder fixed point theorem in order to prove the existence of positive solutions of (4.13). For \( n \in \mathbb{N} \), define the operator \( S_n \) by

\[
S_n(v) := G_s[f_n(v + \rho M_s[\mu])] \quad \forall v \in L^1(\Omega), \quad v \geq 0.
\]

(4.15)

Fix \( q \in (p, p_s) \) and set

\[
Q(v) := \| v \|_{L^q(\Omega, \delta^s)} \quad \forall v \in L^q(\Omega, \delta^s).
\]

(4.16)

Step 1: Since \( q < p_s = k_{s,s} \) where \( p_s \) is given in (1.11) and \( k_{s,s} \) is given in (2.4), applying Lemma 2.5 we have

\[
Q(S_n(v)) = \| G_s[f_n(v + \rho M_s[\mu])] \|_{L^q(\Omega, \delta^s)} \leq C \| G_s[f_n(v + \rho M_s[\mu])] \|_{M^{k_{s,s}}(\Omega, \delta^s)}.
\]

Consequently, choosing \( \alpha = s \) in (2.5) and using (1.15), for any \( v \in L^q(\Omega, \delta^s) \cap L^1(\Omega) \) we obtain from the above inequality that

\[
Q(S_n(v)) \leq C \| f_n(v + \rho M_s[\mu]) \|_{L^1(\Omega, \delta^s)} \\
\leq C \| a(v + \rho M_s[\mu]) + \rho b \|_{L^1(\Omega, \delta^s)} \\
\leq C \left( a \int_\Omega v^q \delta^s dx + a \rho^p \int_\Omega M_s[\nu]^p \delta^s dx + b \int_\Omega \delta^s dx \right)
\]

(4.17)

where \( C = C(N, s, q, \Omega) \). By Hölder inequality,

\[
\int_\Omega v^p \delta^s dx \leq \left( \int_\Omega v^{q \delta^s} dx \right)^{\frac{p}{q}} \left( \int_\Omega \delta^{\frac{q}{p} - 1} dx \right)^{\frac{q}{q - p}} \leq C \left( \int_\Omega v^q \delta^s dx \right)^{\frac{p}{q}} = C Q(v)^p.
\]

(4.18)

Combining (4.17), (4.18) and (2.6), we obtain

\[
Q(S_n(v)) \leq C(aQ(v)^p + a\rho^p + b).
\]

(4.19)

Therefore if \( Q(v) \leq \Lambda \) then

\[
Q(S_n(v)) \leq C(a\Lambda^p + a\rho^p + b).
\]
Since \( p > 1 \), there exist \( \hat{\rho}, \hat{b} > 0 \) such that for any \( \rho \in (0, \hat{\rho}) \) and \( b \in (0, \hat{b}) \) the algebraic equation

\[
C(a\Lambda^p + a\rho^p + b) = \Lambda
\]

admits a largest root \( \hat{\Lambda} > 0 \). Therefore,

\[
(4.20) \quad Q(v) \leq \hat{\Lambda} \implies Q(S_n(v)) \leq \hat{\Lambda}.
\]

**Step 2:** We apply Schauder fixed point theorem to our setting. Set

\[
\mathcal{O} := \{ \phi \in L^1(\Omega) : \phi \geq 0, \ Q(\phi) \leq \hat{\Lambda} \}.
\]

Clearly, \( \mathcal{O} \) is a convex, closed subset of \( L^1(\Omega) \).

In light of (4.20), \( S_n \) is well-defined in \( \mathcal{O} \) and \( S_n(\mathcal{O}) \subset \mathcal{O} \). Now, suppose \( \phi_m \to \phi \) in \( L^1(\Omega) \) as \( m \to \infty \). Since \( f_n(\phi_m + \rho M_s[\nu]) \leq n \) for every \( m \) and the fact that \( G_s : L^1(\Omega, \delta^s) \to L^1(\Omega) \) is compact (see [17, Proposition 2.6]), we have \( S_n(\phi_m) \to S_n(\phi) \) in \( L^1(\Omega) \) as \( m \to \infty \). Therefore \( S_n \) is continuous.

We next show that \( S_n \) is a compact operator. Let \( \{ \phi_m \} \subset \mathcal{O} \) be a bounded sequence in \( L^1(\Omega) \). For each fixed \( n \) put

\[
\psi_m := S_n(\phi_m) = G_s[f_n(\phi_m + \rho M_s[\nu])].
\]

Since, the mapping \( G_s : L^1(\Omega, \delta^s) \to L^1(\Omega) \) is compact, using dominated convergence theorem, there exist a subsequence, still denoted by \( \{ \psi_m \} \), and a function \( \psi \) such that \( \psi_m \to \psi \) in \( L^1(\Omega) \). Thus \( S_n \) is compact.

Hence, by Schauder fixed point theorem there is a function \( 0 \leq v_n \in L^1(\Omega) \) such that \( S_n(v_n) = v_n \) and \( Q(v_n) \leq \hat{\Lambda} \) where \( \hat{\Lambda} \) is independent of \( n \). Therefore \( v_n \) is a nonnegative weak solution of (4.13), i.e.

\[
(4.21) \quad \int_{\Omega} v_n(-\Delta)^s \xi dx = \int_{\Omega} f_n(v_n + \rho M_s[\nu]) \xi dx \quad \forall \xi \in X_s(\Omega).
\]

Further, \( p < q \) and \( Q(v_n) \leq \hat{\Lambda} \) implies \( \|v_n\|_{L^p(\Omega, \delta^s)} \leq \bar{\Lambda} \), where \( \bar{\Lambda} = C\hat{\Lambda} \) and \( C = C(p, q, s, \Omega) \). \( \square \)

**Proof of Theorem 1.8.** Let \( b \in (0, \hat{b}) \) and \( \rho \in (0, \hat{\rho}) \), where \( \hat{b} \) and \( \hat{\rho} \) be as in Lemma 4.3. For each \( n \), set \( u_n := v_n + \rho M_s[\nu] \) where \( v_n \) is the solution constructed in Lemma 4.3. Then \( \text{tr}_s(u_n) = \rho \nu \) and

\[
(4.22) \quad \int_{\Omega} u_n(-\Delta)^s \xi dx = \int_{\Omega} f_n(u_n) \xi dx + \rho \int_{\Omega} M_s[\nu](-\Delta)^s \xi dx \quad \forall \xi \in X_s(\Omega).
\]

Since \( \{v_n\} \subset \mathcal{O} \), \( \{v_n^p\} \) is uniformly bounded in \( L^1(\Omega, \delta^s) \). Since \( f_n \leq f \) and by assumption (1.15), \( \{f_n(v_n + \rho M_s[\nu])\} \) is uniformly bounded in \( L^1(\Omega, \delta^s) \). By [17, Proposition 2.6], the mapping \( G_s : L^1(\Omega, \delta^s) \to L^1(\Omega) \) is compact, hence, up to a subsequence, \( \{v_n\} \) is convergent in \( L^1(\Omega) \).

Therefore there exists a function \( u \) such that \( u_n \to u \) in \( L^1(\Omega) \) and a.e. in \( \Omega \). Consequently \( f_n(u_n) \to f(u) \) a.e. in \( \Omega \).

As \( \{v_n\} \) is uniformly bounded in \( L^q(\Omega, \delta^s) \), so is \( \{u_n\} \). By Hölder inequality, we deduce that \( \{u_n^p\} \) is equi-integrable with respect to \( \delta^s dx \) in \( \Omega \). Then we use assumption (1.15) to obtain that \( \{f_n(u_n)\} \) is equi-integrable with respect to \( \delta^s dx \) in \( \Omega \). Thus Vitali convergence theorem guarantees that \( f_n(u_n) \to f(u) \) in \( L^1(\Omega, \delta^s) \). Therefore, letting \( n \to \infty \) in (4.22) and using [17, Lemma 2.1(i)] yields

\[
(4.23) \quad \int_{\Omega} u(-\Delta)^s \xi dx = \int_{\Omega} f(u) \xi dx + \rho \int_{\Omega} M_s[\nu](-\Delta)^s \xi dx \quad \forall \xi \in X_s(\Omega).
\]
This means $u$ is a weak solution of (4.11).

\[ \square \]

5. Power source

5.1. Minimal solution.

**Theorem 5.1.** Assume $p \in (1, p_s)$ where $p_s$ is defined as in (1.11). Then there exists a positive constant $\tilde{\rho}$ such that for any $p \in (0, \tilde{\rho})$ problem $(P_\rho)$ admits the minimal positive weak solution $u_\rho$.

**Proof.** We aim to use Lemma 4.2 to prove this theorem. To this end, we construct a super solution. For $\theta > 0$, put

\[ U := \rho M_s[\nu] + \theta \rho^p G_s[M_s[\nu]^p]. \]  

Then

\[ (-\Delta)^s U = \theta \rho^p M_s[\nu]^p. \]  

Using (2.10), (5.1) and the fact that $\|\nu\|_{\mathbb{H}^s(\partial \Omega)} = 1$ (as stated in $(P_\rho)$), we obtain

\[ U^p \leq (\rho + \tilde{C} \theta \rho^p)^p M_s[\nu]^p. \]  

Therefore, if

\[ (\rho + \tilde{C} \theta \rho^p)^p \leq \theta \rho^p, \]  

then it holds

\[ U \geq \rho M_s[\nu] + G_s[U^p]. \]  

We see that (5.4) is equivalent to

\[ (1 + \tilde{C} \theta \rho^{p-1})^p \leq \theta. \]  

Note that the function $h(\theta) := (1 + \tilde{C} \theta \rho^{p-1})^p$ can intersect the line $g(\theta) = \theta$ if

\[ \tilde{C} \rho^{p-1} \leq \frac{1}{p} \left( \frac{p-1}{p} \right)^{p-1}. \]  

Define

\[ \tilde{\rho} := \left( \frac{1}{\tilde{C} p} \right)^{\frac{1}{p-1}} \left( \frac{p-1}{p} \right). \]  

Therefore, if $\rho < \tilde{\rho}$ then $h(\tilde{\theta}) \leq \tilde{\theta}$ for $\tilde{\theta} = \left( \frac{p}{p-1} \right)^p$. Hence, for $\tilde{\theta}$ we have chosen, $U$ satisfies (5.5). Consequently, by Lemma 4.2 there exists a minimal solution $u_\rho$ of $(P_\rho)$ satisfying

\[ \rho M_s[\nu] \leq u_\rho \leq U. \]  

\[ \square \]

**Proof of Theorem 1.9.** We consider two cases.

**Case 1:** $p \in (1, p_s)$. Put

\[ A := \{ \rho > 0 : (P_\rho) \text{ admits a positive solution} \} \quad \text{and} \quad \rho^* := \sup A. \]

By Theorem 5.1, $(P_\rho)$ admits a positive solution for $\rho > 0$ small, therefore $A \neq \emptyset$.

**Claim 1:** $\rho^*$ is finite.
To see that, let \( \rho \in \mathcal{A} \) and \( u_\rho \) be the minimal positive weak solution of \((P_\rho)\). Using (4.6) with \( \mu = \rho\nu \), we obtain

\[
2\lambda_1 \rho \int_\Omega M_s[\nu] \varphi_1 dx \leq (2\lambda_1)^{\frac{1}{p^*}} \int_\Omega \varphi_1 dx.
\]

This yields

\[
\rho \leq (2\lambda_1)^{\frac{1}{p^*}} \frac{\int_\Omega \varphi_1 dx}{\int_\Omega M_s[\nu] \varphi_1 dx}.
\]

Hence

\[
\rho^* \leq (2\lambda_1)^{\frac{1}{p^*}} \frac{\int_\Omega \varphi_1 dx}{\int_\Omega M_s[\nu] \varphi_1 dx} < \infty.
\]

**Claim 2:** \( (0, \rho^*) \subseteq \mathcal{A} \).

Note that to see the claim, it is enough to prove that if \( \mathcal{A} \ni \rho' < \rho^* \) and \( 0 < \rho < \rho' \) then \( \rho \in \mathcal{A} \). Since \( \rho' \in \mathcal{A} \), due to Theorem 5.1, there exists a minimal positive solution \( u_{\rho'} \) of \((P_{\rho'})\) which is greater than \( \rho M_s[\nu] \). By a similar argument as in the proof of Theorem 5.1, we can show that \((P_\rho)\) admits a minimal weak solution \( u_\rho < u_{\rho'} \), i.e. \( \rho \in \mathcal{A} \).

**Claim 3:** \( \rho^* \in \mathcal{A} \).

Observe that, the claim is equivalent to proving that problem \((P_{\rho^*})\) admits a positive solution. Let \( \{\rho_n\} \subset \mathcal{A} \) be a nondecreasing sequence converging to \( \rho^* \). For each \( n \), let \( u_{\rho_n} \) be the minimal positive weak solution of \((P_{\rho_n})\). Then \( u_{\rho_n} \in L^1(\Omega) \cap L^p(\Omega, \delta^s) \) and it satisfies

\[
(5.7) \quad \int_\Omega u_{\rho_n} (-\Delta)^s \xi dx = \int_\Omega u_{\rho_n}^p \xi dx + \rho_n \int_\Omega M_s[\nu] (-\Delta)^s \xi dx \quad \forall \xi \in X_s(\Omega).
\]

It follows from Lemma 4.1 that the sequence \( \{u_{\rho_n}\} \) is uniformly bounded in \( L^1(\Omega) \) and in \( L^p(\Omega, \delta^s) \). By the formulation

\[
(5.8) \quad u_{\rho_n} = G_s[u_{\rho_n}^p] + \rho_n M_s[\nu],
\]

and the fact that \( G_s : L^1(\Omega, \delta^s) \to L^1(\Omega) \) is compact (see [17, Proposition 2.6]), we derive that there exist a function \( u_{\rho^*} \) and a subsequence, still denoted by the same notation, such that \( \{u_{\rho_n}\} \) converges, as \( n \to \rho^* \), to \( u_{\rho^*} \) in \( L^1(\Omega) \) and a.e in \( \Omega \).

Further, thanks to Lemma 2.5, for \( q \in (p, p_s) \) we have

\[
\|u_{\rho_n}\|_{L^q(\Omega, \delta^s)} \leq C\|u_{\rho_n}\|_{M^{p_s}(\Omega, \delta^s)} \leq C(\|G_s[u_{\rho_n}^p]\|_{M^{p_s}(\Omega, \delta^s)} + \rho_n\|M_s[\nu]\|_{M^{p_s}(\Omega, \delta^s)}).
\]

Consequently, applying (2.5) (with \( \gamma = s = \alpha \)) and (2.6) (with \( \alpha = s \)) to the right-hand side of the above inequality, we obtain

\[
\|u_{\rho_n}\|_{L^q(\Omega, \delta^s)} \leq C(\|u_{\rho_n}\|_{L^p(\Omega, \delta^s)}^p + \rho^*\|\nu\|_{M(\partial\Omega)}) \leq C(1 + \rho^*)
\]

Thus \( \{u_{\rho_n}\} \) is uniformly bounded in \( L^q(\Omega, \delta^s) \). We invoke Holder inequality to infer that \( \{u_{\rho_n}^p\} \) are equi-integrable in \( L^1(\Omega, \delta^s) \). By Vitali’s convergence theorem, up to a subsequence, \( u_{\rho_n}^p \to u_{\rho^*}^p \) in \( L^1(\Omega, \delta^s) \). Therefore, letting \( n \to \infty \) in (5.7) yields

\[
(5.9) \quad \int_\Omega u_{\rho^*} (-\Delta)^s \xi dx = \int_\Omega u_{\rho^*}^p \xi dx + \rho^* \int_\Omega M_s[\nu] (-\Delta)^s \xi dx \quad \forall \xi \in X_s(\Omega).
\]

This means \( u_{\rho^*} \) is a solution of \((P_{\rho^*})\).

**Claim 4:** \( u_{\rho^*} \) is the minimal positive weak solution of \((P_{\rho^*})\).

To see this, let \( u \) be any weak solution of \((P_{\rho^*})\) then we see that \( u \geq u_{\rho_n} \). Therefore \( u \geq u_{\rho^*} \).
**Case 2: \( p \geq p_s \).** Suppose by contradiction that for some \( \rho > 0 \) and \( z \in \partial \Omega \) there exists a positive weak solution \( u \) of \((P_\rho)\) with \( \nu = \delta_z \). Then \( u \in L^p(\Omega, \delta^s) \) and \( u \geq \rho M_s(\cdot, z) \). This, along with (2.3), implies
\[
\int_{\Omega} u(x)^p \delta(x)^s dx \geq \rho^p \int_{\Omega} M_s(x, z)^p \delta(x)^s dx
\]
\[
\geq C \int_{\Omega} |x-z|^{-NP} \delta(x)^{(p+1)} s dx
\]
\[
\geq C \int_{\{x \in \Omega: \delta(x) \geq \frac{1}{2} |x-z|\}} |x-z|^{-NP} \delta(x)^{(p+1)} s dx.
\]
Fix \( r_0 > 0 \) such that
\[
\mathcal{C} := \left\{ x \in \Omega : |x-z| \leq r_0, \, \delta(x) \geq \frac{1}{2} |x-z| \right\} \subseteq \left\{ x \in \Omega : \delta(x) \geq \frac{1}{2} |x-z| \right\}.
\]
Then
\[
(5.10) \quad \int_{\Omega} u(x)^p \delta(x)^s dx \geq c \int_{\mathcal{C}} |x-z|^{s(p+1)-NP} dx.
\]
Since \( p \geq p_s \), the integral on the right hand-side of (5.10) is divergent, which in turn implies that \( u \notin L^p(\Omega, \delta^s) \). Thus we get a contradiction. \( \square \)

5.2. **Mountain Pass type solution.** In this subsection we assume \( p \in (1, p_s) \) and we construct a second weak solution of \((P_\rho)\) when \( \rho \in (0, \rho_0) \), for certain \( \rho_0 \) which will be specified later. Towards that end, first we would like to apply mountain pass theorem to find a variational solution of
\[
(5.11) \quad \begin{cases} 
(\Delta)^s u = (u_\rho + u^+)^p - u^p_\rho & \text{in } \Omega, \\
 u = 0 & \text{in } \Omega^c,
\end{cases}
\]
where \( u^+ := \max(u, 0) \) and \( u_\rho \) is the minimal positive weak solution of \((P_\rho)\). For this, we define
\[
(5.12) \quad X_0 := \{ v \in H^s(\mathbb{R}^N) : v = 0 \quad \text{in } \Omega^c \},
\]
where \( H^s(\mathbb{R}^N) \) is the standard fractional Sobolev space on \( \mathbb{R}^N \). It is well-known that
\[
(5.13) \quad \|v\|_{X_0} := \left( \int_Q \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}},
\]
where \( Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c) \), is a norm on \( X_0 \) and \((X_0, \|\cdot\|_{X_0})\) is a Hilbert space, with the inner product
\[
\langle u, v \rangle_{X_0} := \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy.
\]
Put
\[
2^* := \frac{2N}{N-2s}.
\]
It is also well known that the embedding \( X_0 \hookrightarrow L^r(\mathbb{R}^N) \) is compact, for any \( r \in [1, 2^*] \) and \( X_0 \hookrightarrow L^{2^*}(\mathbb{R}^N) \) is continuous.

**Definition 5.2.** We say that \( u \in X_0 \) is a variational solution of (5.11) if
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y)) (\phi(x) - \phi(y))}{|x-y|^{N+2s}} dxdy = \int_{\Omega} \left[(u_\rho + u^+)^p - u^p_\rho \right] \phi dx \quad \forall \phi \in X_0.
\]
**Definition 5.3.** We say that a solution \( u \) of \((P_\rho)\) is stable (resp. semistable) if
\[
\|\phi\|^2_{X_0} > p \int_\Omega u^{p-1} \phi^2 \, dx, \quad (\text{resp. } \geq 0) \quad \forall \phi \in X_0 \setminus \{0\}.
\]

**Lemma 5.4.** [16, Proposition 2.3] Let \( 1 < p < p_* \). Then the embedding \( X_0 \hookrightarrow L^2(\Omega, \frac{dx}{|x|(N-s)(p-1)}) \) is continuous and compact.

**Proposition 5.5.** Assume \( 0 \in \partial \Omega, \ p \in (1,p_*), \ \rho < (0, \rho^*) \) and \( u_\rho \) is the minimal positive solution of problem \((P_\rho)\) with \( \nu = \delta_0 \), obtained in Theorem 1.9. Then there exits \( \rho_0 \in (0, \rho^*) \) such that \( u_\rho \) is stable for \( \rho \in (0, \rho_0) \). Moreover, there exists a positive constant \( C = C(N,s,p,\rho,\rho_0) \) such that
\[
\|\phi\|^2_{X_0} > p \int_\Omega u_\rho^{p-1} \phi^2 \, dx \geq C\|\phi\|^2_{X_0} \quad \forall \phi \in X_0 \setminus \{0\}.
\]

**Proof.**

**Step 1:** \( u_\rho \) is stable for \( \rho > 0 \) small.

Indeed, from the construction of \( u_\rho \), in the proof of Theorem 5.1, we have
\[
u_\rho \leq U \leq C\rho M_\nu(x,0) \leq C\rho \delta(x_s) |x|^{-N} \leq C\rho |x|^{-(N-s)}.
\]

Consequently, for any \( \phi \in X_0 \setminus \{0\} \), applying Lemma 5.4 we have
\[
\int \nu_\rho^{p-1} \phi^2 \, dx \leq C\rho^{p-1} \int \frac{\phi^2}{|x|^{(N-s)(p-1)}} \, dx \leq \frac{1}{p} \|\phi\|^2_{X_0},
\]
if we choose \( \rho > 0 \) small enough. This completes Step 1.

Define
\[
\mathcal{R} := \{ \rho > 0 : u_\rho \text{ is stable} \} \quad \text{and} \quad \rho_0 := \sup \mathcal{R}.
\]

**Step 2:** Either \( \mathcal{R} = (0, \rho_0) \) or \( \mathcal{R} = (0, \rho_0) \).

Clearly \( \rho_0 \leq \rho_* \). We claim that if \( \rho' \in \mathcal{R} \) then \( (0, \rho') \subseteq \mathcal{R} \). Indeed, if \( \rho' \in \mathcal{R} \) and \( \rho \in (0, \rho') \), then by Theorem 1.9, \( u_\rho < u_{\rho'} \). Consequently, for any \( \phi \in X_0 \setminus \{0\} \),
\[
\|\phi\|^2_{X_0} > p \int \nu_\rho^{p-1} \phi^2 \, dx > p \int \nu_{\rho'}^{p-1} \phi^2 \, dx.
\]

This implies that \( u_\rho \) is stable.

Now since \( \rho_0 = \sup \mathcal{R} \), for every \( n \in \mathbb{N} \), there exists \( \rho_n \in \mathcal{R} \) such that
\[
\rho_n \leq \rho_0 < \rho_n + \frac{1}{n}.
\]
If there exists \( n_0 \) such that \( \rho_{n_0} = \rho_0 \) then by the above observation we deduce that \( (0, \rho_0) \subseteq \mathcal{R} \) and hence \( (0, \rho_0) = \mathcal{R} \). Otherwise, if \( \rho_n < \rho_0 \) for every \( n \) then we can assume that \( \{\rho_n\} \) is an increasing sequence converging to \( \rho_0 \). This and the above observation imply that \( (0, \rho_0) = \mathcal{R} \).

**Step 3:** (5.15) holds for every \( \rho \in (0, \rho_0) \).

Towards this, let \( \rho \in (0, \rho_0) \) and put \( \rho' = \frac{\rho+\rho_0}{2} \). Set \( \alpha := \left( \frac{\rho}{\rho'} \right)^\frac{1}{2} < 1 \). Let \( \nu_{\rho'} \) and \( u_\rho \) be the minimal positive weak solutions of \((P_{\rho'})\) and \((P_\rho)\) respectively with \( \nu = \delta_0 \). Then
\[
(-\Delta)\nu^{\alpha u_{\rho'}} \geq (\alpha u_{\rho'})^p \quad \text{in} \quad \Omega.
\]

It is easy to see that \( \alpha \rho' > \rho \). Therefore,
\[
\alpha u_{\rho'} = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \quad \text{tr}_s(\alpha u_{\rho'}) = \alpha \rho' \delta_0 > \rho \delta_0.
\]
Thus, $\alpha u_\rho'$ is a super solution to $(P_\rho)$ with $\nu = \delta_0$. Consequently, Lemma 4.2 yields $\alpha u_\rho' \geq u_\rho$. Furthermore, as $\rho' < \rho_0$, $u_\rho'$ is stable. Therefore,

$$0 < \|\phi\|^2_{X_0} - p \int_\Omega \frac{\|\phi\|^2_{X_0} - p\alpha^{-1}}{1} \int_\Omega \frac{\|\phi\|^2_{X_0} - p\alpha^{-1}}{1} dx$$

$$= \alpha^{-1}(\alpha^{-1}\|\phi\|^2_{X_0} - p \int_\Omega \frac{\|\phi\|^2_{X_0} - p\alpha^{-1}}{1} dx).$$

(5.18)

Hence,

$$\|\phi\|^2_{X_0} - p \int_\Omega \frac{\|\phi\|^2_{X_0} - p\alpha^{-1}}{1} dx = (1 - \alpha^{-1})\|\phi\|^2_{X_0} + \alpha^{-1}\|\phi\|^2_{X_0} - p \int_\Omega \frac{\|\phi\|^2_{X_0} - p\alpha^{-1}}{1} dx$$

$$> (1 - \alpha^{-1})\|\phi\|^2_{X_0} = C\|\phi\|^2_{X_0},$$

where $C = (1 - \alpha^{-1})$. Hence (5.15) holds for every $\rho \in (0, \rho_0)$.

The energy functional associated to (5.11) is

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy - \int_\Omega H(u, u^+)dx \quad \forall u \in X_0,$$

where

$$H(r, t) := \frac{1}{p+1} \left[ (r + t^+)^{p+1} - r^{p+1} - (p+1)rt^+ \right].$$

We also observe that ([28, Lemma C.2(iii)]) for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$, such that

(5.19)

$$H(r, t) - \frac{p}{2}r^{p-1}t^2 \leq \varepsilon t^{p-1}t^2 + c_\varepsilon t^{p+1}, \quad r, t \geq 0.$$

Furthermore, by ([28, Lemma C.2(ii)])

(5.20)

$$H(r, t) \geq \frac{1}{p+1} t^{p+1}, \quad r, t \geq 0.$$

In particular, $H(r, t) > \frac{1}{p+1} t^{p+1}$, $r, t > 0$.

**Theorem 5.6.** Suppose $0 \in \partial \Omega$, $p \in (1, p_s)$ and $\rho \in (0, \rho_0)$, where $\rho_0$ is as defined in (5.17). Then problem (5.11) admits a nontrivial nonnegative variational solution.

**Proof.** First we prove that $I$ has the mountain pass geometry. Clearly, $I(0) = 0$. Using (5.15) and (5.19), we have

$$I(u) = \frac{1}{2} \left[ \|u\|^2_{X_0} - p \int_\Omega \frac{\|u\|^2_{X_0}}{1} dx \right] - \int_\Omega H(u, u^+)dx - \frac{p}{2} \int_\Omega \frac{u^{p-1}}{1} dx$$

$$\geq C \|u\|^2_{X_0} - \varepsilon \int_\Omega \frac{\|u\|^2_{X_0}}{1} dx - C\|u\|^{p+1}_{L^{p+1}(\Omega)}$$

$$\geq \left( \frac{C}{p} - \frac{\varepsilon}{p^2} \right) \|u\|^2_{X_0} - C\|u\|^{p+1}_{L^{p+1}(\Omega)},$$

where in the last line we have used the Sobolev inequality and the fact that $u_\rho$ is stable. Therefore as $p > 1$, there exists $r, b > 0$, such that $\inf_{\|u\|=r} I(u) = b > 0$. Next, let $u_0 \in X_0$ with $\|u_0\|_{X_0} = 1$. Then using (5.20) we obtain

$$I(tu_0) < \frac{t^2}{2} \|u_0\|^2_{X_0} - \frac{t^{p+1}}{p+1} \|u_0\|^{p+1}_{L^{p+1}(\Omega)}.$$

Thus there exists $\bar{u} \in X_0$ such that $\|\bar{u}\|_{X_0} > r$ and $I(\bar{u}) < 0$. 


Next, we show that $I$ satisfies Palais-Smale condition, i.e., let $\{v_n\} \subset X_0$ such that $I(v_n) \to c$ and $I'(v_n) \to 0$ in $(X_0)'$, the dual of $X_0$, we need to show that, up to a subsequence, $\{v_n\}$ converges to some $v$ in $X_0$. By a similar argument as in [16, Proposition 4.2] we see that $\{v_n\}$ is bounded in $X_0$. Therefore, there exists $v$ in $X_0$ such that up to a subsequence $v_n \rightharpoonup v$ in $X_0$. Thus $v_n \to v$ in $L^{p+1}(\Omega)$.

**Claim 1:** $v_n \to v$ in $L^2(\Omega, u_\rho^{p-1}dx)$.

To see this note that as $u_\rho$ is the minimal solution of $(P_\rho)$, (5.16) holds. Consequently,

$$u_\rho(x)^{p-1} \leq C|x|^{-(N-s)(p-1)}.$$  

Hence by Lemma 5.4, the claim follows.

Define

$$h(r, t) := (r + t^+)^p - r^p.$$  

**Claim 2:** $\lim_{n \to \infty} \int_0^r h(u_\rho, v_n)(v_n - v) \, dx = 0$.

To see this first note that, by elementary computation it can be easily deduced that

$$h(r, t) \leq C(r^{p-1}|t| + |t|^p).$$  

Therefore,

$$\int_\Omega h(u_\rho, v_n)(v_n - v) \, dx \leq C \int_\Omega \left(\int_{\Omega} |v_n - v|^2 u_\rho^{p-1} \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_n|^2 u_\rho^{p-1} \, dx\right)^{\frac{1}{2}}$$

$$+ C \left(\int_\Omega |v_n|^{p+1} \, dx\right)^{\frac{p}{p+1}} \left(\int_\Omega |v_n - v|^{p+1} \, dx\right)^{\frac{1}{p+1}} \to 0.$$  

Thus the claim follows.

As a result, as $n \to \infty$,

$$o(1)\|v_n - v\|_{X_0} = \langle I'(v_n), v_n - v \rangle$$

$$= \langle v_n, v_n - v \rangle - \int_\Omega h(u_\rho, v_n)(v_n - v) \, dx$$

$$= \|v_n\|_{X_0}^2 - \langle v_n, v \rangle - \int_\Omega h(u_\rho, v_n)(v_n - v) \, dx.$$  

As $v_n \rightharpoonup v$ in $X_0$, taking the limit $n \to \infty$ and applying Claim 2, we obtain

$$\lim_{n \to \infty} \|v_n\|_{X_0}^2 = \|v\|_{X_0}^2.$$  

As a result, $I$ satisfies Palais-Smale condition.

Therefore, applying mountain pass theorem we get $i^*$ is a critical value of $I$, where

$$i^* = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) \geq b > 0,$$

and $\Gamma := \{ \gamma \in C([0, 1], X_0) : \gamma(0) = 0, \gamma(1) = \bar{u} \}$. This gives the existence of $u \in X_0$ such that $I(u) = i^*$ and $I'(u) = 0$. Note that $i^* > 0$ implies $u$ is nontrivial nonnegative solution of (5.11).
Proof of Theorem 1.11. Let \( 1 < p < p_s \) and \( u_\rho \) be the minimal positive solution of \((P_\rho)\), when \( \nu = \delta_0 \). Further, let \( \rho \in (0, \rho_0) \), where \( \rho_0 \) is defined in (5.17). Then, from Theorem 5.6, there is a nontrivial nonnegative variational solution \( v_\rho \in X_0 \) of (5.11). Namely,
\[
(5.22) \quad \int_{\Omega} (-\Delta)^{\frac{s}{2}} v_\rho (-\Delta)^{\frac{s}{2}} \phi \, dx = \int_{\Omega} \left[ (u_\rho + v_\rho)^p - u_\rho^p \right] \phi \, dx \quad \forall \phi \in X_0.
\]

Set
\[
T(\Omega) := \{ \phi \in C^\infty(\Omega) : \text{there exists } \psi \in C^\infty_0(\Omega) \text{ such that } \phi = G_s[\psi] \}.
\]
This is the space of test functions defined in [1, Page 41]. By [1, Lemma 5.6], \( T(\Omega) \subset X_0 \).

Therefore, we deduce from (5.22) that
\[
(5.23) \quad \int_{\Omega} v_\rho (-\Delta)^{s} \phi \, dx = \int_{\Omega} (-\Delta)^{\frac{s}{2}} v_\rho (-\Delta)^{\frac{s}{2}} \phi \, dx = \int_{\Omega} \left[ (u_\rho + v_\rho)^p - u_\rho^p \right] \phi \, dx \quad \forall \phi \in T(\Omega).
\]

Then [1, Lemma 5.12 and Lemma 5.13] ensures that \( T(\Omega) \subset X_s(\Omega) \) where \( X_s(\Omega) \) is given in Definition 1.6 and
\[
(5.24) \quad \int_{\Omega} v_\rho (-\Delta)^{s} \phi \, dx = \int_{\Omega} \left[ (u_\rho + v_\rho)^p - u_\rho^p \right] \phi \, dx \quad \forall \phi \in X_s(\Omega).
\]

This means that \( v_\rho \) is a weak solution of
\[
(5.25) \quad \begin{cases}
(-\Delta)^{s} v = (u_\rho + v_\rho)^p - u_\rho^p & \text{in } \Omega \\
\text{tr}_s(v) = 0 & \text{in } \Omega^c, \\
v = 0 & \text{in } \Omega^c,
\end{cases}
\]

Set \( u := u_\rho + v_\rho \). Clearly \( u \) is a weak solution of \((P_\rho)\) and hence by Theorem 1.7, \( u \in C^{2s+\alpha}_{loc}(\Omega) \) for some \( \alpha \in (0, 1) \). Similarly, Theorem 1.7 also implies \( u_\rho \in C^{2s+\alpha}_{loc}(\Omega) \). Therefore, \( v_\rho \in C^{2s+\alpha}_{loc}(\Omega) \). By the strong maximum principle (see [36, Proposition 2.17]), \( v_\rho > 0 \) in \( \Omega \), which yields \( u > u_\rho \). The proof is complete.

\[\square\]

APPENDIX A. EQUATIONS WITH NONLINEARITY IN THE GRADIENT

In this section, we establish a global priori estimate for the positive solutions (and their gradients) of the following type of equations
\[
(\text{A.1}) \quad (-\Delta)^{s} u = f(x, u, \nabla u) \quad \text{in } \Omega.
\]

Put
\[
(\text{A.2}) \quad q = \frac{2sp}{p + 2s - 1}.
\]

Assume \( f : \Omega \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory function, and assume that there exist \( p_1 \in (0, p) \), \( q_1 \in (0, q) \) and \( \tilde{C} \) such that
\[
(\text{A.3}) \quad -\tilde{C}(1 + t^{p_1} + |\xi|^{q_1}) \leq f(x, t, \xi) \leq \tilde{C}(1 + t^p + |\xi|^q), \quad x \in \Omega, \ t \geq 0, \ \xi \in \mathbb{R}^N.
\]

Theorem A.1. Suppose \( p \in (1, p_c) \), where \( p_c \) is as defined in (1.6), \( q \) is defined by (A.2) and \( f \) satisfies (A.3). Furthermore, assume that for every \( x \in \Omega \),
\[
(\text{A.4}) \quad \lim_{t \rightarrow \infty, \Omega \ni z \rightarrow x} t^{-p} f(z, t, t^{\frac{p+1}{2}-\frac{1}{2}} \xi) = L(x) \in (0, \infty)
\]
uniformly for \( \xi \) bounded. Moreover, if \( \Omega \) is unbounded then we assume that (A.4) also hold for \( x = \infty \). Then

...
(i) There exists a constant $C = C(N, s, p)$ such that, for any positive viscosity solution $u \in C^1_{\text{loc}}(\Omega)$ of (A.1), estimate (1.8) holds.

(ii) If $f \in C^\beta_{\text{loc}}(\Omega, \mathbb{R}, \mathbb{R}^N)$ (for some $\beta \in (0, 1)$) in each variable and $u \in C^{1+\gamma}_{\text{loc}}(\Omega)$, for some $\gamma \in (0, 1)$, is a positive distributional solution of (A.1), then $u$ is a viscosity solution of (A.1) and estimate (1.8) holds.

Proof. (i) We first prove the assertion for viscosity solution. Since the proof is similar to that of Theorem 1.2, we only point out the differences here. Suppose (1.8) does not hold, then there exist sequences $\Omega_k$, $u_k \in C^1_{\text{loc}}(\Omega)$, $y_k \in \Omega_k$ such that $u_k$ satisfies

\begin{equation}
(\Delta)^s u_k = f(y_k, u_k, \nabla u_k) \quad \text{in} \quad \Omega_k,
\end{equation}

in viscosity sense. Let $M_k$ be defined by (3.3). Then $M_k$ satisfies (3.18) and (3.5)-(3.6). We define $\lambda_k$ and $v_k$ as in (3.7) and (3.8) respectively. Then $v_k$ satisfies

\begin{equation}
(\Delta)^s v_k(y) = f_k(v_k(y)) := \lambda_k^{2sp} f(x_k + \lambda_k y, \lambda_k v_k(y), \lambda_k^{2+sp} \nabla v_k(y))
\end{equation}

for $y \in B(0, k)$. Furthermore, (3.10)-(3.11) hold. From (3.3), we deduce that there exists a constant $C_1$, $C_2 > 0$ such that

\begin{equation}
-C_1 \lambda_k < f_k(v_k(y)) \leq C_2, \quad \forall y \in B(0, k),
\end{equation}

for some $\varepsilon > 0$ and all $k$ large. Proceeding as in Step 1 in the proof of Theorem 3.1 we deduce that, up to a subsequence, $\{v_k\}$ converges to some function $v$ in $C^\alpha_{\text{loc}}(\mathbb{R}^N)$, for some $\alpha \in (0, 1)$. Doing the similar analysis as in Step 2 in the proof of Theorem 3.1, we obtain $v$ is nonnegative, nontrivial and bounded in $\mathbb{R}^N$. Let $\tilde{v}_k$ be the function obtained by extending $v_k$ to be zero outside $B(0, k)$. Then using (A.7), it is not difficult to check that $(-\Delta)^s \tilde{v}_k \geq 0$ in $B(0, k)$ for large $k$. Passing to the limit, by [12, Lemma 5], we obtain $(-\Delta)^s v \geq 0$ in $\mathbb{R}^N$. Consequently, [36, Proposition 2.17] yields $v > 0$ in $\mathbb{R}^N$.

Fixing $y \in \mathbb{R}^N$, we denote $\mu_k = \lambda_k^{\frac{2sp}{p+2s}} v_k(y), \xi_k = v_k^{\frac{p+2s-1}{2}}(y) \nabla v_k(y)$. This reduces

\begin{equation}
f_k(v_k(y)) = v_k^p(y) \mu_k^{sp} f(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+2s-1}{2s}} \xi_k).
\end{equation}

As $y$ is fixed, thanks to (3.10), and the fact that $v > 0$ and $\lambda_k \to 0$, it follows that $\mu_k \to \infty$ and $\xi_k$ remains bounded. If $\{x_k\}$ is bounded, then up to a subsequence $x_k \to x_0 \in \Omega$. Therefore, by (A.4)

\begin{equation}
f_k(v_k(y)) \to L(x_0)v^p(y), \quad \text{as} \quad k \to \infty.
\end{equation}

If $\Omega$ is unbounded and $x_k \to \infty$, then the additional assumption on $f$ implies that (A.8) still holds with $x = \infty$.

Claim: $f_k(v_k) \to L(x_0)v^p$ locally uniformly in $\mathbb{R}^N$.

To see the claim, we observe that as $v$ is continuous and strictly positive, there exists $m > 0$ such that $v(y) > m$ for $\overline{B}(0, R)$ and moreover as $v_k > 0$, for each $k$, $v_k^{-1} \to v^{-1}$ in $B(0, R)$. Therefore for large $k$, $v_k^{-\frac{2s}{p+2s-1}}$ is uniformly bounded in $B(0, R)$. Hence $\xi_k$ is uniformly bounded in $B(0, R)$. Consequently, By (A.4) we have $f_k(v_k) \to L(x_0)v^p$ uniformly in $B(0, R)$. Hence the claim follows.

Let $\tilde{v}_k$ be as before. Then $(-\Delta)^s \tilde{v}_k \geq f_k(\tilde{v}_k)$ in $B(0, k)$. Passing the limit, by [12, Lemma 5], we obtain $(-\Delta)^s v \geq L(x_0)v^p$ in $\mathbb{R}^N$. As $p < p_c < \frac{N}{N-2s}$, we get a contradiction to the Liouville type theorem [21, Theorem 1.3] as before. Hence the theorem follows.
(ii) Since $f \in C^\beta_{\text{loc}}$ in each variable and $u \in C^{1+\gamma}_{\text{loc}}(\Omega)$, we obtain $f(x, u, \nabla u) \in C^{\tilde{\beta}}_{\text{loc}}(\Omega)$, for some $\tilde{\beta} \in (0, \min(\beta, \gamma))$. Therefore, we can follow the similar arguments as in the proof of Proposition 3.3 to conclude the result. We omit the details. □

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