SECRET KEY AGREEMENT ON WIRETAP CHANNELS WITH TRANSMITTER SIDE INFORMATION

Ashish Khisti

University of Toronto
Toronto, ON, Canada
phone: + (1) 416-978-7215, fax: + (1) 416-978-4425, email: akhisti@comm.utoronto.ca
web: http://www.comm.utoronto.ca/~akhisti

ABSTRACT

Secret-key agreement protocols over wiretap channels controlled by a state parameter are studied. The entire state sequence is known (non-causally) to the sender but not to the receiver and the eavesdropper. Upper and lower bounds on the secret-key capacity are established both with and without public discussion. The proposed coding scheme involves constructing a codebook to create common reconstruction of the state sequence at the sender and the receiver and another secret-key codebook constructed by random binning. For the special case of Gaussian channels, with no public discussion, the secret-key generation with dirty paper problem, the gap between our bounds is at-most 1/2 bit and the bounds coincide in the high signal-to-noise ratio and high interference-to-noise ratio regimes. In the presence of public discussion our bounds coincide, yielding the capacity, when then the channels of the receiver and the eavesdropper satisfy an independent noise condition.

1. INTRODUCTION

Many applications in cryptography require that the legitimate terminals have shared secret keys, not available to unauthorized parties. Information theoretic security encompasses the study of source and channel coding techniques to generate secret-keys between legitimate terminals. In the channel coding literature, an early work in this area is the wiretap channel model [1]. It consists of three terminals one sender, one receiver and one eavesdropper. The sender communicates to the receiver and the eavesdropper over a discrete-memoryless broadcast channel. A notion of equivocation-rate the normalized conditional entropy of the transmitted message given the observation at the eavesdropper, is introduced, and the tradeoff between information rate and equivocation rate is studied. Perfect secrecy capacity, defined as the maximum information rate under the constraint that the equivocation rate approaches the information rate asymptotically in the block length is of particular interest. Information transmitted at this rate can be naturally used as a shared secret-key between the sender and the receiver. In the source coding setup [2,3] the two terminals observe correlated source sequences and use a public discussion channel for communication. Any information sent over this channel is available to an eavesdropper. The terminals generate a common secret-key that is concealed from the eavesdropper in the same sense as the wiretap channel the equivocation.

In the present paper we consider a secret-key agreement problem when the sender and the receiver communicate over a channel controlled by a state parameter. The state parameter is known to the sender but not to the receiver or the eavesdropper. The problem of transmitting information on such channels, without secrecy constraints, is studied in [4]. A random binning strategy is proposed and shown to achieve the capacity. Costa [5] studies the problem of communicating over an additive noise Gaussian channel with an additive interference sequence known to the transmitter and establishes that there is no loss in capacity if the interference sequence is known only to the transmitter. In the present work, we study the problem of generating a common secret key between the sender and the receiver over such channels. Our proposed coding scheme is not based on the Gel’fand-Pinsker binning technique for sending an information message over such channels. Instead our codebook is designed to create a common reconstruction sequence at the sender and the receiver and distilling a secret-key based on this common sequence.

In related works, the problem of secret-message transmission over wiretap channels controlled by a state parameter is studied in [6,7]. In these works an achievable coding scheme is proposed that combines Gel’fand-Pinsker coding and coding for the wiretap channel. As discussed earlier, our coding scheme is based on a different approach and in general yields higher achievable rates. A related problem of common reconstruction of state sequences has been studied recently in [8,9]. The problem of secret-key agreement with symmetric channel state information at the sender and the legitimate receiver has been studied in [10]. However the coding scheme involved is based on the fact that the terminals have knowledge of the common state sequence to begin with. After this paper was submitted, we learnt about a recent work [11] where the problem of communicating over channels with non-causal CSI is used as a building block for characterizing the tradeoff between secret-key and secret-message transmission.

2. PROBLEM SETUP

As Fig. 1 illustrates, the channel model has three terminals — a sender, a receiver and an eavesdropper. The sender communicates with the other two terminals over a discrete-memoryless-channel with transition probability $p_{y_e|x,s}(\cdot)$ where $x$ denotes the channel input symbol, whereas $y_r$ and $y_e$ denote the channel output symbols at the receiver and the eavesdropper respectively. The symbol $s$ denotes a state variable that controls the channel transition probability. We assume that it is sampled i.i.d. from a distribution $p_s$ in each channel use. Further, the entire sequence $s^n$ is known to the sender before the communication begins.

In defining the secret-key capacity we separately consider the cases when a public discussion channel is and is
not present.

2.1 No Public Discussion

A length $n$ encoder is defined as follows. The sender samples a random variables $u$ from the conditional distribution $p_{u|x}(x_i^t)$. The encoding function produces a channel input sequence $x^t = f_n(u,s^n)$ and transmits it over $n$ uses of the channel. At time $i$ the symbol $x_i$ is transmitted and the legitimate receiver and the eavesdropper observe output symbols $y_{ri}$ and $y_{ei}$ respectively, sampled from the conditional distribution $p_{y_{ri},y_{ei}|x,s}(\cdot)$. The sender and receiver compute secret keys $\kappa = g_n(u,s^n)$ and $l = h_n(y^n)$. A rate $R$ is achievable if there exists a sequence of encoding functions such that for some sequence $\varepsilon_n$ that vanishes as $n \to \infty$, we have that $Pr(\kappa \neq l) \leq \varepsilon_n$ and $
frac{1}{n} I(\kappa; y^n_e) \leq \varepsilon_n$.

The largest achievable rate is the secret-key capacity.

2.2 Presence of Public Discussion

When a public discussion channel is present, the described protocol follows closely the interactive communication protocol in [3]. The sender transmits symbols $x_1, \ldots, x_n$ at times $0 < i_1 < i_2 < \cdots < i_n$ over the wiretap channel. At these times the receiver and the eavesdropper observe symbols $y_{ri_1}, \ldots, y_{ri_n}$ and $y_{ei_1}, \ldots, y_{ei_n}$ respectively. In the remaining times the sender and receiver exchange messages $\psi_i$ and $\phi_i$ where $1 \leq t \leq k$. For convenience we let $i_{n+1} = k + 1$. The eavesdropper observes both $\psi_i$ and $\phi_i$.

More specifically the sender and receiver sample random variables $u$ and $v$ from conditional distributions $p_{u|x}(x_i^t)$ and $p_{v}(\cdot)$ and observe that $v$ is independent of $(u,s^n)$.

- At times $0 < t < i_1$, the sender generates $\phi_i = \Phi_i(u,s^n,\psi^{i-1})$ and the receiver generates $\psi_i = \Psi_i(v,\phi^{i-1})$. These messages are exchanged over the public channel.
- At times $i_j$, $1 \leq j \leq n$, the sender generates $x_j = X_j(u,s^n,\psi^{i-1})$ and sends it over the channel. The receiver and eavesdropper observe $y_{ei}$ and $y_{ei}$ respectively. For these times we set $\psi_i = \phi_i = 0$.
- For times $i_j < t < i_{j+1}$, where $1 \leq j < n$, the sender and receiver compute $\phi_i = \Phi_i(u,s^n,\psi^{i-1})$ and $\psi_i = \Psi_i(v,y^n,\phi^{i-1})$ respectively and exchange them over the public channel.
- At time $k+1$, the sender and receiver compute $\kappa = g_n(u,s^n)$ and the receiver computes $l = h_n(v,y^n,\phi^k)$. We require that for some sequence $\varepsilon_n$ that vanishes as $n \to \infty$, $Pr(\kappa \neq l) \leq \varepsilon_n$ and

$$
\frac{1}{n} I(\kappa; y^n_e,\psi^k,\phi^k) \leq \varepsilon_n.
$$

The secret-key rate is defined as $\frac{1}{n} H(\kappa)$ and the largest achievable secret-key rate is the capacity.

3. MAIN RESULTS

Our main results are upper and lower bounds on the secret-key capacity, which coincide in some special cases. We again consider the cases of no public discussion and public discussion separately.

3.1 No Public Discussion

We first provide an achievable rate (lower bound) on the secret-key capacity.

Theorem 1 An achievable secret-key rate without public discussion is

$$
R^- = \max_{p_{u},p_{\phi}(\cdot)} I(u;y_{ri}) - I(u;y_{ei}),
$$

where the maximization is over all auxiliary random variables $u$ that satisfy the Markov condition $u \to (x,s) \to (y_{ri},y_{ei})$ and furthermore satisfy the constraint that

$$
I(u;y_{ri}) - I(u;s) \geq 0.
$$
The intuition behind the coding scheme is as follows. Upon observing $s^n$, the sender communicates the best possible reproduction $u^n$ of the state sequence to the receiver. Now both the sender and the receiver observe a common sequence $u^n$. The set of all codewords $u^n$ is binned into $2^nR^n$ bins and the bin-index is declared to be the secret key.

We note that the lower bound can be easily extended to the case of two-sided CSI. If the receiver observes another state sequence $s_r$, correlated with $s$ according to a joint distribution $P_{s,s_r}(\cdot,\cdot)$ then the achievable rate expression (3) holds provided that we augment the received symbol by $(y_r,s_r)$.

Finally for the case of symmetric CSI i.e., when $s_r = s$, the constraint (4) is redundant as clearly $I(u;y_r,s) - I(u;y_c) \geq 0$ holds. Furthermore the resulting achievable rate,

$$R^- = \max_{P_u,P_x|s,u} I(u;y_r,s) - I(u;y_c)$$

is indeed the secret-key capacity as established in our earlier work [10].

Finally we note that the problem of secret-key agreement is different from the secret-message transmission problem considered in [6, 12, 7]. This is because the secret-key can be an arbitrary function of the state sequence (known only to the transmitter) whereas the secret-message needs to be independent function of the state sequence. For comparison, the best known lower bound on the secret-message transmission problem is stated below.

**Proposition 1** [6, 12, 7] An achievable secret message rate for wiretap channel with non-causal transmitter CSI is

$$R \leq \max_{P_u,P_x|s,u} I(u;y_r) - \max(I(u;s),I(u;y_c)).$$ (5)

We note that whenever the maximizing $u$ satisfies, $I(u;y_r) > I(u;s) > I(u;y_c)$, the secret-key rate (3) is strictly better than the secret-message rate (5).

The following theorem develops an upper bound on secret-key capacity that is amenable to numerical computation.

**Theorem 2** The secret-key capacity in absence of public discussion is upper bounded by $C \leq R^+$, where

$$R^+ = \min_{P_y,P_{y|x}} \max_{P_{x|y}} I(x,s;y_c|y_c),$$ (6)

where $\mathcal{P}$ denotes all the joint distributions $P_{y|x,s}^{*}$ that have the same marginal distribution as the original channel.

The intuition behind the upper bound is as follows. We create a degraded channel by revealing the output of the eavesdropper to the legitimate receiver. We further assume a channel with two inputs $(x^n, s^n)$ i.e., the state sequence $s^n$ is not arbitrary, but rather a part of the input codeword with distribution $P_s$. The secrecy capacity of the resulting wiretap channel is then given by $I(x,s;y_c|y_c).

Our proposed upper and lower bounds coincide, yielding capacity in some special cases. We present one such case in section 3.3.

### 3.2 With Public Discussion

In this section we provide lower and upper bounds on the secret-key capacity with public discussion. We first provide a lower bound below.

**Theorem 3** An achievable secret-key rate with public discussion is:

$$R^\text{disc} = \max_{P_{x|y}} \left( \max I(x,s;y_r) - I(y_c,y_r), R^- \right)$$ (7)

where $R^-$ is the lower bound attained without public discussion in Theorem 1.

The achievability scheme involves a natural modification of Maurer’s coding scheme [3, 2] to incorporate the presence of the state parameter and involves a single round of discussion. In particular, the sender generate a sequence $x^n$ according to the conditional distribution $P_{x|y}(x|s)$ and transmits over $n$ channel uses. At the end of the transmission, the receiver sends the bin index of $y^n$, so that the sender can recover this sequence given $(x^n, y^n)$. Next we provide an upper bound on the secret-key capacity under public discussion.

**Theorem 4** An upper bound on the secret-key capacity is

$$R^+ = \max_{P_{x|y}} I(x,s;y_c|y_c).$$ (8)

We note that the upper bound expression (8) is similar to the upper bound expression in (5) except that we cannot minimize over the joint-probability distribution in (8). This is because the public discussion channel provides a mechanism for feedback and hence the capacity does depend on the joint distribution (not just the marginal distributions). The proof for the upper bound expression in Theorem 4 also significantly more elaborate as it accounts for public discussion.

We note that if the channel additionally satisfies $y_r \rightarrow (x,s) \rightarrow y_c$ then the upper and lower bounds in Theorem 3 and 2 coincide. In particular if $P_{x|y}$ is the maximizing distribution in (8), we have that

$$R^\text{disc} \geq I(x,s;y_r) - I(y_c,y_r)$$

$$= I(y_c,x,s;y_r) - I(y_c,y_r) - I(y_c,y_r|x,s)$$

$$= I(y_c,x,s;y_r) - I(y_c,y_r)$$

$$= I(x,s;y_c|y_c) = R^-_{\text{disc}}.$$

Since $R^-_{\text{disc}} \leq R^-_{\text{disc}}$, it follows that the two expressions must be equal. This is summarized in the result below.

**Theorem 5** The secret-key capacity with public discussion for a DMC channel that satisfies $y_r \rightarrow (x,s) \rightarrow y_c$ is given by

$$C_{\text{disc}} = \max_{P_{x|y}} I(x,s;y_c|y_c).$$ (9)

### 3.3 Gaussian Case

We now study the Gaussian special case under an average power constraint. The channel to the legitimate receiver and the eavesdropper is expressed as:

$$y_r = x + s + z_r$$

$$y_c = x + s + z_c,$$ (10)
where \( z_t \sim \mathcal{N}(0, 1) \) and \( z_e \sim \mathcal{N}(0, 1 + \Delta) \) denote the additive white Gaussian noise and are assumed to be sampled independently. The state parameter \( s \sim \mathcal{N}(0, Q) \) is also sampled i.i.d. at each time instance and is independent of both \( z_t \) and \( z_e \). Furthermore, the channel input satisfies an average power constraint \( E[|x|^2] \leq P \). As the title indicates, we call this setup, secret sharing with dirty paper.

Thus the parameter \( P \) denotes the signal-to-noise ratio, the parameter \( Q \) denotes the interference-to-noise ratio, whereas \( \Delta \) denotes the degradation level of the eavesdropper. We now provide lower and upper bounds on the secret-key capacity with and without public discussion. For simplicity in exposition we limit our analysis to the case when \( P \geq 1 \).

**Proposition 2** Assuming that \( P \geq 1 \), a lower bound on the secret-key agreement capacity is given by,

\[
R^- = \frac{1}{2} \log \left( 1 + \frac{\Delta (P + Q + 2\sqrt{PQ})}{P + Q + 1 + \Delta + 2\sqrt{PQ}} \right),
\]

where \( p < 1 \) is the largest value that satisfies

\[
P(1 - p^2) \geq 1 - \frac{1}{P + Q + 1}.
\]

**Proposition 3** In absence of public discussion, an upper bound on the secret-key capacity is given by,

\[
R^+ = \frac{1}{2} \log \left( 1 + \frac{\Delta (P + Q + 2\sqrt{PQ})}{P + Q + 1 + \Delta + 2\sqrt{PQ}} \right)
\]

It can be readily verified that the upper and lower bounds coincide in several asymptotic regimes.

**Proposition 4** The upper and lower bounds on secret-capacity without public discussion satisfying the following

\[
\forall P \geq 0, R_+ - R_- \leq \frac{1}{2}
\]

\[
\lim_{P \to \infty} R_+ - R_- = 0
\]

\[
\lim_{Q \to \infty} R_+ - R_- = 0
\]

**Proposition 5** In the presence of public discussion, the secret-key capacity is given by the following expression,

\[
R^+ = \frac{1}{2} \log \left( 1 + \frac{(1 + \Delta)(P + Q + 2\sqrt{PQ})}{P + Q + 1 + \Delta + 2\sqrt{PQ}} \right)
\]

4. WITHOUT PUBLIC DISCUSSION

In this section we provide the coding scheme and the upper bound for the case when there is no public discussion.

4.1 Proof of Theorem

A sequence of length \( n \) code is described as follows.

4.1.1 Codebook Generation

- Generate a total of \( 2^{n(I(u; y_e) - \varepsilon_n)} \) sequences. Each sequence is sampled i.i.d. from a distribution \( p_u(.) \).
- Select a rate \( R = I(u; y_e) - I(u; y_e) - \varepsilon_n \) and randomly partition the set sequences in the previous step into \( 2^nR \) bins so that there are \( 2^{n(I(u; y_e) - \varepsilon_n)} \) sequences in each bin.

4.1.2 Encoding

- Given a state sequence \( s^n \) the encoder selects a sequence \( u^n \) randomly from the list of all possible sequences that are jointly typical with \( s^n \).
- At time \( i = 1, 2, \ldots, n \) the encoder transmits symbol \( x_i \) generated by sampling the distribution \( p_{x_i|u,s}(\cdot|u_i, s_i) \).

4.1.3 Secret-key generation

- The decoder upon observing \( y^n_e \) finds a sequence \( u^n \) jointly typical with \( y^n_e \).
- Both encoder and the decoder declare the bin-index of \( u^n \) to be the secret-key.

4.1.4 Secrecy Analysis

We need to show that for the proposed encoder and decoder, the equivocation at the eavesdropper satisfies

\[
\frac{1}{n} H(\kappa|y^n_e) = I(u; y_e) - I(u; y_e) + o_n(1),
\]

where \( o_n(1) \) is a term that goes to zero as \( n \to \infty \).

Accordingly note that

\[
\frac{1}{n} H(\kappa|y^n_e) = \frac{1}{n} H(\kappa, u^n|y^n_e) - \frac{1}{n} H(u^n|y^n_e, \kappa)
\]

\[
= \frac{1}{n} H(u^n|y^n_e) - \frac{1}{n} H(u^n|y^n_e, \kappa)
\]

\[
= \frac{1}{n} H(u^n|y^n_e) - \varepsilon_n
\]

where the last step follows from the fact that there are at-most \( 2^{n(I(u; y_e) - \varepsilon_n)} \) sequences in each bin and hence the eavesdropper can decode the codeword \( u^n \) given the key \( \kappa \). It remains to lower-bound the first conditional entropy term.

\[
\frac{1}{n} H(u^n|y^n_e) = \frac{1}{n} H(u^n) + \frac{1}{n} H(y^n_e|u^n) - \frac{1}{n} H(y^n_e)
\]

\[
= \frac{1}{n} H(u^n) + \frac{1}{n} H(y^n_e|u^n, s^n) - \frac{1}{n} H(y^n_e) + \frac{1}{n} I(s^n; y^n_e|u^n)
\]

We now appropriately bound each term in (20). First note that since the sequence \( u^n \) is uniformly distributed among the set of all possible codeword sequences, it follows that

\[
\frac{1}{n} H(u^n) = \frac{1}{n} \log_2 \|\mathcal{U}\| - \varepsilon_n
\]

\[
= I(u; y_e) - \varepsilon_n
\]

Next, given \((u^n, s^n)\), as verified below, the channel to the
The legitimate receiver can decode $u^n$ (with high probability) and map it to the secret key. The eavesdropper’s noise-uncertainty sphere includes possible key values. Note that unlike the traditional dirty-paper code, the transmitter signal $x^n$ has a component along $s^n$. The achievable rate, does depend on the interference power and hence it is beneficial to amplify it using part of the transmit power. Also note that unlike a dirty-paper code we do not scale down $s^n$ before quantizing but use $\alpha = 1$.

Finally, in order to lower bound the term $I(s^n; y^n_e | u^n)$ we let $J$ to be a random variable which equals 1 if $(s^n, u^n)$ are jointly typical. Note that $\Pr(J = 1) = 1 - o_n(1)$.

\[
\frac{1}{n} I(s^n; y^n_e | u^n) = \frac{1}{n} H(s^n | u^n) - \frac{1}{n} H(s^n | u^n, y^n_e) \\
\geq \frac{1}{n} H(s^n | u^n, J = 1) - \frac{1}{n} H(s^n | u^n, y^n_e) - o_n(1) \\
\geq H(s | u) - \frac{1}{n} H(s^n | u^n, y^n_e) - o_n(1) \quad (25)
\]

where (25) follows from the fact that $s^n$ is an i.i.d. sequence and hence conditioned on the fact that $(s^n, u^n)$ is a pair of typical sequence there are $2^{nH(s | u) - m_n(1)}$ possible sequences $s^n$.

Substituting (21), (23), (24) and (25) in the lower bound (20) and using the fact that as $n \to \infty$, the summation
converges to the mean values,
\[
\frac{1}{n} H(\kappa | y^n_c) = I(u; y_i) + H(y_c | u, s) - H(y_c) + H(s | u) - H(s | u, y_c) - o_n(1)
\]
\[
= I(u; y_i) - I(y_c; u) - I(y_c; s | u) + I(y_c; s) - o_n(1)
\]
as required.

4.2 Proof of Theorem 2
A sequence of length-\(n\) code satisfies:
\[
\frac{1}{n} H(\kappa | y^n_c) \leq \varepsilon_n
\]
(27)
\[
\frac{1}{n} H(\kappa | y^n_e) \geq \frac{1}{n} H(\kappa) - \varepsilon_n
\]
(28)
where (27) follows from the Fano’s Lemma since the receiver is able to recover the secret-key \(\kappa\) given \(y^n_c\) and (28) is a consequence of the secrecy constraint. Furthermore, note that \(\kappa \rightarrow (x^n, s^n) \rightarrow (y^n_c, y^n_e)\) holds as the encoder generates the secret key \(\kappa\). Thus we can bound the rate \(R = \frac{1}{n} H(\kappa)\) as below:
\[
nR \leq I(\kappa; y^n_c) + 2n\varepsilon_n
\]
\[
\leq I(\kappa; x^n, s^n, y^n_c | y^n_c) + 2n\varepsilon_n
\]
\[
= I(s^n, x^n; y^n_c | y^n_c) + 2n\varepsilon_n
\]
\[
= \sum_{i=1}^n I(s_i; x_i; y^n_c | y^n_c) + 2n\varepsilon_n
\]
\[
\leq nI(s, x; y^n_c | y^n_e) + 2n\varepsilon_n
\]
where the last step follows from the concavity of the conditional entropy term \(I(s, x; y^n_c | y^n_e)\) in the input distribution \(p_{x,s}\) (see e.g., [17])

Finally since the secret-key capacity only depends on the marginal distribution of the channel and not on the joint distribution we can minimize over all joint distributions with fixed marginal distributions.

5. WITH PUBLIC DISCUSSION

In this section we provide the proofs of the coding theorem and the converse for the case when there is a public discussion channel allowed.

5.1 Proof of Theorem 3
Our coding scheme is closely related to the coding theorem for the channel model in [1,2] and emulates the generation of correlated source sequences. It consists of the following steps:

- Fix a distribution \(p_{x|s}\). This induces a joint distribution \(p_{x,y_i,y_c,s}\). Let \(R = I(y_i; x, s) - I(y_i; y_c) - \varepsilon_n\)
- Partition the set of all typical sequences \(y^n_c\) into \(2^{n(H(y_c | x, s) - o_n(1))}\) bins. Furthermore partition the collection of \(2^{n(I(y_i; x, s))}\) sequences in each bin into further \(2^n\) sequences so that there are \(2^{n(I(y_i; x, s) - \varepsilon_n)}\) sequences in each sub-bin.
- Given symbol \(s_i\) at time \(i\), sample a symbol \(x_i\) from the conditional distribution \(p_{x|s}\) and transmit it over the channel.
- The receiver upon observing \(y^n_c\) transmits the bin index of this sequence over the channel. Using the bin index and the knowledge of \((x^n, s^n)\) the sender reproduces \(y^n_e\).
- Both the sender and the receiver declare the sub-bin index of \(y^n_e\) as the secret-key.

Following the secrecy analysis in [3,2] it can be shown that this construction satisfies the secrecy constraint [2] and furthermore attains a rate of
\[R = I(y_i; x, s) - I(y_i; y_c) + o_n(1)\].

5.2 Proof of Theorem 4
We now establish a corresponding upper bound on the secret-key capacity.

First, using the fact that the receiver is able to recover the secret-key and the eavesdropper is subjected to a secrecy
To establish constraint (2), we have that
\[
\frac{1}{n} H(\kappa; y^n, \nu, \Phi^k) \leq \varepsilon_n
\] (29)
and
\[
\frac{1}{n} I(\kappa; y^n, \psi^k, \Phi^k) \leq \varepsilon_n
\] (30)

Using the above relations and the fact that \( R = \frac{1}{n} H(\kappa) \), we note that
\[
nR \leq H(\kappa)
\]
\[
\leq I(\kappa; y^n, \nu, \Phi^k) - I(\kappa; y^n, \Phi^k) + 2n\varepsilon_n
\] (31)
\[
\leq I(\kappa; y^n, \nu, y^n, \Phi^k) - I(\kappa; y^n, \Phi^k) + 2n\varepsilon_n
\] (32)
\[
\leq I(\kappa; y^n, \nu, y^n, \Phi^k) + 2n\varepsilon_n
\] (33)

where we have introduced
\[
F_{r,j} = I(\kappa; s^n; y^n, y^n) (34)
\]
\[
G_{r,j} = I(\kappa; s^n; |\phi^n|, y^n, y^n) (35)
\]
\[
F_{e,j} = I(\kappa; s^n; |\phi^n|, y^n, y^n) (36)
\]
\[
G_{e,j} = I(\kappa; s^n; |\phi^n|, y^n, y^n) (37)
\]

To complete the proof, it suffices to show that the following relations in (33) hold
\[
I(\kappa; s^n; |\phi^n|, y^n) = 0 (38)
\]
\[
F_{r,j} - F_{e,j} \leq I(\kappa; s^n; y^n) (39)
\]
\[
G_{r,j} - G_{e,j} \leq 0 (40)
\]

To establish (33) note that for \( 0 \leq k \leq i_1 - 1 \) we have that
\[
\Phi_k = \Phi_k(\kappa, s^n, \psi^{i_1 - 1})\] and likewise \( \Psi_k = \Psi_k(\kappa, \psi^{i_1 - 1})\). Using which
\[
I(\kappa; s^n; |\phi^n|, y^n) = 0
\]
\[
I(\kappa; s^n; y^n, y^n) = 0
\]
\[
I(\kappa; s^n; |\phi^n|, y^n) = 0
\]
Continuing this process we have that
\[
I(\kappa; s^n; |\phi^n|, y^n) = 0
\]
where the last relation follows from the fact that \( \nu \) is independent of \( s^n \).

In order to establish (39), we use (34) and (36) to get,
\[
F_{r,j} - F_{e,j} = I(\kappa; s^n; y^n, y^n) (\phi^n, y^n, y^n) (34)
\]
\[
H(\kappa; s^n; y^n, y^n) (\phi^n, y^n, y^n) (35)
\]
\[
F_{e,j} = I(\kappa; s^n; y^n, y^n) (\phi^n, y^n, y^n) (36)
\]
\[
G_{r,j} = I(\kappa; s^n; y^n, y^n) (\phi^n, y^n, y^n) (37)
\]
\[
G_{e,j} = I(\kappa; s^n; y^n, y^n) (\phi^n, y^n, y^n) (38)
\]
\[
F_{r,j} - F_{e,j} \leq I(\kappa; s^n; y^n) (39)
\]
\[
G_{r,j} - G_{e,j} \leq 0 (40)
\]

In this section we develop the corresponding results for the Gaussian case.
6.1 Proof of Prop. 2

The lower bound expression follows from Theorem 1 by choosing \( x \sim \mathcal{N}(0, P) \) to be a Gaussian random variable independent of \( s \) and by choosing \( u = x + \alpha s \). In this case, \( R = I(u; y_e) - I(u; y_c) = h(u|y_c) - h(u|y_e) \). Further evaluating each of the terms above with \( u = x + \alpha s \), note that
\[
h(u|y_c) = \frac{1}{2} \log \left( P + \alpha^2 Q + 2 \alpha \rho \sqrt{PQ} - \frac{(P + \alpha Q + (1 + \alpha) \rho \sqrt{PQ})^2}{P + Q + 1 + \Delta + 2 \rho \sqrt{PQ}} \right)
\]
and
\[
h(u|y_e) = \frac{1}{2} \log \left( P + \alpha^2 Q + 2 \alpha \rho \sqrt{PQ} - \frac{(P + \alpha Q + \rho (1 + \alpha) \sqrt{PQ})^2}{P + Q + 1 + 2 \rho \sqrt{PQ}} \right).
\]
This yields that
\[
R = \frac{1}{2} \log \left( 1 + \frac{\Delta}{1 + \frac{PQ(\alpha - 1)^2 (1 - \rho^2) + (P + \alpha^2 Q + 2 \rho \alpha \sqrt{PQ})}{P + Q + 1 + 2 \rho \sqrt{PQ}} \right)
\]
\[
+ \frac{1}{2} \log \left( \frac{P + Q + 1 + 2 \rho \sqrt{PQ}}{P + Q + 1 + \Delta + 2 \rho \sqrt{PQ}} \right).
\]
(44)

Note that the first term in the expression above is maximized when \( \alpha = 1 \). As we show below, this choice is indeed feasible when \( P \geq 1 \). In particular, the constraint (3) requires that
\[
h(u|s) \geq h(u|y_c)
\]
\[
\Rightarrow \frac{1}{2} \log P(1 - \rho^2) \geq \frac{1}{2} \log \left( \frac{PQ(\alpha - 1)^2 (1 - \rho^2) + (P + \alpha^2 Q) + 2 \rho \alpha \sqrt{PQ}}{P + Q + 1 + 2 \rho \sqrt{PQ}} \right),
\]
Substituting \( \alpha = 1 \) above we have that
\[
P(1 - \rho^2) \geq 1 - \frac{1}{P + Q + 1 + 2 \rho \sqrt{PQ}} \geq 1 - \frac{1}{P + Q + 1}
\]
(45)
as required.

6.2 Proof of Prop. 3

We evaluate the upper bound in Theorem 2 for the choice \( z_e = z_1 + z_2 \), where \( z_2 \sim \mathcal{N}(0, \Delta) \) is independent of \( z_1 \).
\[
I(x; s; y_c | y_e) = h(y_c | y_e) - h(y_c | y_e, x, s)
\]
\[
= h(y_c | y_e) - h(z_1 | z_2)
\]
\[
\leq \frac{1}{2} \log \left( \frac{P + Q + 1 + 2 \sqrt{PQ}}{P + Q + 1 + 2 \rho \sqrt{PQ}} - \frac{(P + Q + 1 + 2 \rho \sqrt{PQ})^2}{P + Q + 1 + \Delta + 2 \rho \sqrt{PQ}} \right) - \frac{1}{2} \log \left( 1 - \frac{1}{1 + \Delta} \right)
\]
where we have used the fact that the conditional entropy \( h(y_c | y_c) \) is maximized by a Gaussian distribution. The above expression gives (13).

6.3 Proof of Proposition 5

Since the Gaussian model satisfies the condition in Theorem 5 it suffices to evaluate \( C = I(x, s; y_1 | y_2) \).
\[
I(x, s; y_1 | y_2) = h(y_1 | y_2) - h(y_1 | y_2, x, s)
\]
\[
= h(y_1 | y_2) - h(z_1 | z_2)
\]
\[
= \frac{1}{2} \log 2 \pi e \left( \frac{P + Q + 1 + 2 \rho \sqrt{PQ}}{P + Q + 1 + 2 \rho \sqrt{PQ}} - \frac{(P + Q + 2 \rho \sqrt{PQ})^2}{P + Q + 1 + 2 \rho \sqrt{PQ} + \Delta} \right) - \frac{1}{2} \log 2 \pi e
\]
(48)
which upon simplifying yields the desired expression.

REFERENCES

[1] A. D. Wyner, “The wiretap channel,” Bell Syst. Tech. J., vol. 54, pp. 1355–87, 1975.
[2] U. M. Maurer, “Secret key agreement by public discussion from common information,” IEEE Trans. Inform. Theory, vol. 39, pp. 733–742, Mar. 1993.
[3] R. Ahlswede and I. Csiszar, “Common randomness in information theory and cryptography – Part I: Secret sharing,” IEEE Trans. Inform. Theory, vol. 39, pp. 1121–1132, Jul. 1993.
[4] S. I. Gel’fand and M. S. Pinsker, “Coding for channels with random parameters,” Problems of Control and Information Theory, vol. 9, pp. 19–31, 1980.
[5] M. H. Costa, “Writing on dirty paper,” IEEE Trans. Inform. Theory, vol. 29, pp. 439–441, May 1983.
[6] C. Mitrapant, H. Vinck, and Y. Luo, “An achievable region for the gaussian wiretap channel with side information,” IEEE Trans. Inform. Theory, vol. 52, pp. 2181–2190, May 2006.
[7] W. Liu and B. Chen, “Wiretap channel with two-sided state information,” in Proc. 41st Asilomar Conf. on Signals, Systems and Comp., Nov. 2007.
[8] Y. Steinberg, “Simultaneous transmission of data and state with common knowledge,” in Proc. Int. Symp. Inform. Theory, Toronto, Canada, Jul. 2008, pp. 935–939.
[9] ——, “Coding and common reconstruction,” IEEE Trans. Inform. Theory, vol. 55, pp. 4995–5010, Nov. 2009.
[10] A. Khisti, S. N. Diggavi, and G. W. Wornell, “Secret-key agreement using asymmetric in channel state information,” in Proc. Int. Symp. Inform. Theory, 2009.
[11] V. Prabhakaran, K. Esparza, and K. Ramchandran, “Secrecy via sources and channels,” IEEE Trans. Inform. Theory, submitted, Nov. 2009. [Online]. Available: http://www.ifp.illinois.edu/~vinodmp/publications/Secrecy09.pdf
[12] Y. Chen and H. Vinck, “Wiretap channel with side information,” in Proc. Int. Symp. Inform. Theory, Jun. 2006.
[13] A. Khisti, A. Tchamkerten, and G. W. Wornell, “Secure Broadcasting over Fading Channels,” IEEE Trans. Inform. Theory, Special Issue on Information Theoretic Security, pp. 2453–2469, 2008.