A KACZMARZ ALGORITHM FOR SEQUENCES OF PROJECTIONS, INFINITE PRODUCTS, AND APPLICATIONS TO FRAMES IN IFS $L^2$ SPACES

PALLE JORGENSEN, MYUNG-SIN SONG, AND FENG TIAN

Abstract. We show that an idea, originating initially with a fundamental recursive iteration scheme (usually referred as “the” Kaczmarz algorithm), admits important applications in such infinite-dimensional, and non-commutative, settings as are central to spectral theory of operators in Hilbert space, to optimization, to large sparse systems, to iterated function systems (IFS), and to fractal harmonic analysis. We present a new recursive iteration scheme involving as input a prescribed sequence of selfadjoint projections. Applications include random Kaczmarz recursions, their limits, and their error-estimates.

Contents

1. Introduction 1
2. Slice-singular measures 2
3. Frames, projections, and Kaczmarz algorithms 4
   3.1. Algorithms, and products of projections 6
   3.2. The case of rank-1 projections in Hilbert space 10
   3.3. Random Kaczmarz constructions and sequences of projections 11
   3.4. Solutions to $Ax = y$ in finite, and in infinite, dimensional spaces 13
4. System of isometries 16
5. General iterated function system (IFS)-theory 18
6. Sierpinski and random power series 21
References 26

1. Introduction

In this paper, we consider certain infinite products of projections. Our framework is motivated by problems in approximation theory, in harmonic analysis, in frame theory, and the context of the classical Kaczmarz algorithm [Kac37]. Traditionally, the infinite-dimensional Kaczmarz algorithm is stated for sequences of vectors in a specified Hilbert space $\mathcal{H}$, (typically, $\mathcal{H}$ is an $L^2$-space.) We shall here formulate it

2000 Mathematics Subject Classification. Primary 47L60, 46N30, 46N50, 42C15, 65R10, 05C50, 05C75, 31C20, 60J20, 26E40, 41A65; Secondary 46N20, 22E70, 31A15, 58J65, 81S25, 68T05.

Key words and phrases. Hilbert space, Kaczmarz algorithm, randomized Kaczmarz algorithm, sequences of projections in Hilbert space, convergence, infinite products, frames, analysis/synthesis, interpolation, optimization, overdetermined linear systems, transform, feature space, iterated function system, fractal, Sierpinski gasket, harmonic analysis, approximation, infinite-dimensional analysis, integral decomposition, random variables, strong operator topology.
instead for sequences of projections. As a corollary, we get explicit and algorithmic criteria for convergence of certain infinite products of projections in $\mathcal{H}$.

**Organization and main results.**

Our first two sections outline a certain frame-harmonic analysis. This is the immediate focus of our present applications, but our main results, dealing with general projection valued Kaczmarz algorithms, we believe, are of independent interest. They include Theorem 3.5 (products of projections,) and its related results, Corollaries 3.8, 3.11, 3.15, and 3.16. The connection between infinite products of projections, on the one hand, and more classical Kaczmarz recursions (for frames), on the other, is spelled out in Corollaries 3.16 and 3.17. Our main result for random Kaczmarz algorithms is Theorem 3.20, combined with Corollary 3.21. In the remaining three sections, we return to applications, iterated function system, fractals, and random power series.

Our extension of the Kaczmarz algorithm to sequences of projections is highly nontrivial: While in general convergence questions for infinite products of projections (in Hilbert space) is difficult (see e.g., [Aro50, Rue82, Rue04, AJL18]), we show that our projection-valued formulation of Kaczmarz’ algorithm yields an answer to this convergence question; as well as a number of applications to stochastic analysis, and to frame-approximation questions in the Hilbert space $L^2(\mu)$, where $\mu$ is in a class of iterated function system (IFS) measures (see [Hut81, Hut95, DJ07, HJW16, JS18a]). The latter refers to a precise multivariable setting, and the class of measures $\mu$ we consider are fractal measures. (The notion of “fractal” is defined here relative to the rank $d$ of the ambient Euclidean space $\mathbb{R}^d$ for the particular IFS measure $\mu$ under consideration.) Indeed, our measures $\mu$ will be singular relative to the Lebesgue measure on $\mathbb{R}^d$. In addition to singularity questions for $\mu$ itself, one must also consider properties of the marginal measures for $\mu$, and the corresponding slice-direct integral decompositions. Our first two applications will be the IFS-measures for the Sierpinski gasket and the Sierpinski carpet, so $d = 2$.

In the next section, we introduce this family of measures $\mu$, called slice-singular measures. We then turn to our Kaczmarz algorithm for sequences of projections, and its applications.

### 2. Slice-singular measures

The purpose of the current paper is to perform a systematic analysis of fractal measures embedded in higher dimensions $d$, such as Sierpinski triangles ($d = 2$), and higher dimensional analogues, $d > 2$. The analysis for $d = 1$ begins with the following variant of the F&M Riesz theorem:

Consider a choice of period interval, $[0, 1]$, or $[-\pi, \pi]$, a positive finite measure $\mu$ with support in the chosen period interval; and the usual Fourier frequencies realized as complex exponentials $e_n$, $n \in \mathbb{Z}$. Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

**Theorem 2.1 (F&M Riesz).** The subset $\{e_n \mid n \in \mathbb{N}_0\}$ is total in $L^2(\mu)$ if and only if $\mu$ is singular with respect to Lebesgue measure.

The corresponding result is false when $d > 1$, and the question is: What is a natural extension of F&M Riesz’ theorem to higher dimensions, modeling the above formulation? One of the motivations for this is a certain construction of frame algorithms in $L^2(\mu)$; in the form started for $d = 1$ in [DJ07, HJW16, HJW18a, HJW18b]. For general frame theory, including projection valued frames, see e.g.,
Theorem 2.1 does not extend to 2D, or higher dimensions. In 1D, the standard F&M Riesz theorem is used at a crucial point; but there is not a direct extension of the theorem in one variable. To get a harmonic analysis of $L^2(\mu)$, with $\text{supp}(\mu) \subset \mathbb{R}^d$, $d \geq 2$, one must assume instead that $\mu$ is slice singular; see Definition 2.3. It is possible to view the result as an extension of F&M Riesz’ theorem to higher dimensions.

For the sake of stressing the idea, we shall consider the case $d = 2$ in most detail.

**Notation.** Let $(X, \mathcal{F})$ be a measurable space. $\mathcal{M}(X)$ denotes all Borel measures on $\mathcal{F}$. The set $\mathcal{M}^+(X)$ consists of all positive measures in $\mathcal{M}(X)$, and $\mathcal{M}^+_1(X)$ the subset of probability measures. We shall also use standard multi-index notations.

Let $(X \times Y, \mathcal{B}_X \times \mathcal{B}_Y, \mu)$ be a measure space, where $X, Y$ are equipped with $\sigma$-algebras $\mathcal{B}_X, \mathcal{B}_Y$ respectively, and $\mu$ is defined on the product $\sigma$-algebra.

**Lemma 2.2 (Disintegration).** Every positive measure $\mu$ on $X \times Y$ w.r.t. the product $\sigma$-algebra yields a unique representation as follows:

(i) $\xi := \mu \circ \pi^{-1}$ is a measure on $(X, \mathcal{B}_X)$;

(ii) There exists a conditional measure $\sigma^x(dy) := \sigma(x, dy)$ on $(Y, \mathcal{B}_Y)$, defined for a.a. $x \in X$, such that

$$d\mu = \int \sigma^x(dy) d\xi(x).$$

The precise meaning of (2.1) is as follows: For all measurable functions $F$ on $X \times Y$, we have

$$\iint_{X \times Y} F d\mu = \int_{X} \left( \int_{Y} F(x, y) \sigma^x(dy) \right) d\xi(x).$$

The decomposition (2.2) is often referred to as a *Rohlin disintegration formula*.

**Definition 2.3.** A Borel measure $\mu$ on $J^2 := [0, 1] \times [0, 1]$ is called slice singular iff (Def.)

(i) $\xi = \mu \circ \pi^{-1}$ is singular; and

(ii) for a.a. $x$ w.r.t. $\xi$, the measure $\sigma^x(\cdot)$ is singular.

“Singular” is defined relative to Lebesgue measure.

**Theorem 2.4.** If $\mu$ is slice singular on $J^2$, then $\{e_n\}_{n \in \mathbb{N}_0}$ has dense span in $L^2(\mu)$, where $e_n(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}$, for all $n = (n_1, n_2) \in \mathbb{N}_0^2$, and $x = (x_1, x_2) \in J^2$.

**Proof.** We shall show that, if $\langle F, e_n \rangle_{L^2(\mu)} = 0$, $\forall n \in \mathbb{N}_0^2$, then $F = 0$ $\mu$-a.e. But

$$\langle F, e_n \rangle_{L^2(\mu)} = \int_0^1 e_{n_1}(x) \left( \int_0^1 e_{n_2}(y) \overline{F(x, y)} \sigma^x(dy) \right) d\xi(x)$$

$$= 0, \forall n = (n_1, n_2) \in \mathbb{N}_0^2$$

$$\downarrow \text{ (since } \xi \text{ is singular)} \tag{2.3}$$

$$\int_0^1 e_{n_2}(y) \overline{F(x, y)} \sigma^x(dy) = 0, \text{ a.a. } x, \forall n_2 \in \mathbb{N}_0$$

$$\downarrow \text{ (since } \sigma^x(\cdot) \text{ is singular a.a. } x) \tag{2.4}$$
\[ F(x, y) = 0, \text{ a.a. } (x, y) \text{ w.r.t. } \mu. \]

This gives the desired conclusion that \( \{e_n\}_{n \in \mathbb{N}^2} \) is total in \( L^2(\mu) \). \( \square \)

**Example 2.5** \( (d = 2) \). \( \mu \in \mathcal{M}^+(\mathbb{T}^2) \), \( W = \) Sierpinski gasket/carpet (Figure 2.1).

Note that, for a.a. \( x \) w.r.t. \( \xi \), the measure \( \sigma^x \) on \( A(x) = \{ y \mid (x, y) \in W \} \) is a fractal measure with variable gap size; and by Kakutani’s theorem, for a.a. \( x \), \( \sigma^x(dy) \) is singular relative to the Lebesgue measure. Hence we can apply F&M Riesz as in (2.3), and (2.4).

The detailed properties of the fractals from Figure 2.1 (A) and (B) will be derived in Section 6 below.

![Figure 2.1. Examples of slice singular measures.](image)

While the Sierpinski constructions in Figure 2.1 are better known as self-affine planar sets, it is in fact the corresponding *measures* which are important for algorithms and for frame-harmonic analysis. As it turns out, the particular affine maps (see (6.1), (6.2), and Figure 6.2 below) going into the Sierpinski constructions are in fact special cases of a more general family of iterated function systems (IFS.) They are discussed in detail in sections 5 and 6 below. Brief preview: Given a system of contractive mappings, affine or conformal, there are then two associated fixed-point problems, one for compact sets, and the other for probability measures: The case of the sets \( W \) is discussed in (5.10), and the measures \( \mu \) in (5.8). For a fixed IFS, the set in question arises as the support of an associated IFS-measure \( \mu \). Probabilistic features of these constructions are outlined in sect 5, and their fractal properties, in sect 6, below. In particular, we show that these planar Sierpinski measures \( \mu \) are slice-singular.

### 3. Frames, projections, and Kaczmarz algorithms

While earlier approaches to the Kaczmarz algorithm in Hilbert space have dealt with recursive constructions of vectors, as needed in optimization problems, or in harmonic analysis, we present here an extension of the algorithm to the context of countable systems of self-adjoint projections in a Hilbert space. As outlined in subsequent sections of our paper, the projection setting is motivated directly by applications; the *randomized* Kaczmarz algorithms, just one of them.

For the benefit of readers, and for later reference, we include below a brief review of fundamentals for the classical Kaczmarz algorithm, and its variants. This
also gives us a suitable framework for our present results: An operator theoretic extension of Kaczmarz, with applications to multivariable fractal measures.

**Literature guide:** In addition to Kaczmarz’ pioneering paper [Kac37], there are also the following more recent developments of relevance to our present discussion [EP01, Pop01, HS05, KM06, Szw07, Pop10, EN11, CT13, IZ13, LZ15, NSW16, Che18, Pop18, Zha19], as well as [HJW16, HJW18a, HJW18b].

The classical Kaczmarz algorithm is an iterative method for solving systems of linear equations, for example, \( Ax = b \), where \( A \) is an \( m \times n \) matrix.

Assume the system is consistent. Let \( x_0 \) be an arbitrary vector in \( \mathbb{R}^n \), and set

\[
x_k := \arg\min_{\langle a_j, x \rangle = b_j} \| x - x_{k-1} \|^2, \quad k \in \mathbb{N};
\]

(3.1)

where \( j = k \mod m \), and \( a_j \) denotes the \( j^{th} \) row of \( A \). At each iteration, the minimizer is given by

\[
x_k = x_{k-1} + \frac{b_j - \langle a_j, x_{k-1} \rangle}{\| a_j \|^2} a_j.
\]

(3.2)

That is, the algorithm recursively projects the current state onto the hyperplane determined by the next row vector of \( A \).

There is a stochastic version of (3.2), where the row vectors of \( A \) are selected randomly [SV09]. Also see Sections 3.3 and 3.4 below.

**Remark 3.1.** Following standard conventions in approximation theory, we use the notation \( \arg\min \) for denoting the vector which realizes a specified optimization; in this case (see Figure 3.1), we refer to the minimum problem on the right hand side in eq (3.1). So in the particular instance of the Kaczmarz algorithm (3.2), we are in finite dimensions, and there is then an easy, geometric, and explicit formula for the \( \arg\min \) vector occurring in each step of the algorithm, see Figure 3.1.

The Kaczmarz algorithm can be formulated in the Hilbert space setting as follows:

**Definition 3.2.** Let \( \{ e_j \}_{j \in \mathbb{N}_0} \) be a spanning set of unit vectors in a Hilbert space \( \mathcal{H} \), i.e., \( \text{span} \{ e_j \} \) is dense in \( \mathcal{H} \). For all \( x \in \mathcal{H} \), let \( x_0 = e_0 \), and set

\[
x_k := x_{k-1} + e_k \langle e_k, x - x_{k-1} \rangle.
\]

(3.3)

We say the sequence \( \{ e_j \}_{j \in \mathbb{N}_0} \) is effective if \( \| x_k - x \| \to 0 \) as \( k \to \infty \), for all \( x \in \mathcal{H} \).

**Remark 3.3.** A key motivation for our present analysis is an important result by Stanisław Kwapień and Jan Mycielski [KM06], giving a criterion for stationary sequences (referring to a suitable \( L^2(\mu) \)) to be effective.

**Observation.** Equation (3.3) yields, by forward induction:

\[
x - x_k = (1 - P_k) (x - x_{k-1})
\]

\[
= (1 - P_k) (1 - P_{k-1}) (x - x_{k-2})
\]

\[
\vdots
\]

\[
= (1 - P_k) (1 - P_{k-1}) \cdots (1 - P_0) x,
\]

where \( P_j \) is the orthogonal projection onto \( e_j \).
3.1. Algorithms, and products of projections

We now present an extension of the Kaczmarz algorithm; an extension to a setting of an infinite sequence of selfadjoint projections, as opposed to the classical case of sequences of vectors in Hilbert space. There are more general results on limits of iterated products of selfadjoint projections. See [AJL18] and also [Aro50, Rue82, Rue04]. For applications of infinite products of operators to central problems in mathematical physics, see e.g., papers by D. Ruelle et al [RT71, Rue79, Rue82].

Preliminaries

Let $\mathcal{H}$ be a Hilbert space. An operator $P : \mathcal{H} \to \mathcal{H}$ is said to be a selfadjoint projection iff (Def.) $P = P^* = P^2$. It is known that there is a bijective correspondence between:

(i) all closed subspaces $\mathcal{M} \subset \mathcal{H}$; and

(ii) the set of all selfadjoint projections $P$.

If $\mathcal{M}$ is as in (i), then $P$ may be obtained from the axioms for $\mathcal{H}$; and we have

$$P\mathcal{H} = \mathcal{M} = \{ x \in \mathcal{H} ; Px = x \} .$$

Conversely, if $P$ is given as in (ii), then $\mathcal{M}$ (see (3.4)) is a closed subspace in $\mathcal{H}$.

The ortho-complement

$$\mathcal{M}^\perp := \mathcal{H} \ominus \mathcal{M} = \{ x \in \mathcal{H} ; Px = 0 \}$$

is the closed subspace corresponding to the selfadjoint projection $P^\perp := 1 - P$. (Here, we denote the identity operator in $\mathcal{H}$ by 1, as it is the unit in the $C^*$-algebra $\mathcal{B}(\mathcal{H})$.)

Remark 3.4. For our present purpose, all projections will be assumed selfadjoint. On occasion, to save space, we shall simply say “projection” when selfadjointness is implicit. (We note that selfadjoint projections yield orthogonal sum-splittings, and are therefore often, equivalently, referred to as orthogonal projections.)
We shall further make use of the lattice operations corresponding to the correspondence (i)↔(ii) above:

If \( \mathcal{M}_i, i = 1, 2, \) are closed subspaces with corresponding projections \( P_i, i = 1, 2; \) then TFAE:

\[
\mathcal{M}_1 \subseteq \mathcal{M}_2, \quad \text{and} \quad P_1 = P_2 P_2. \tag{3.6}
\]

Moreover, for a pair of projections \( \{P_i\}_{i=1,2}, \) TFAE:

\[
P_1 = P_1 P_2 \quad \Downarrow \quad P_1 = P_2 P_1 \quad \Downarrow \quad \|P_1 x\| \leq \|P_2 x\|, \forall x \in \mathcal{H} \quad \Downarrow \quad \langle x, P_1 x \rangle \leq \langle x, P_2 x \rangle, \forall x \in \mathcal{H}. \tag{3.7}
\]

**Caution:** In general, the class of selfadjoint projections is *not* closed under products, under sums, or under differences.

**Theorem 3.5.** Let \( \{P_j\}_{j \in \mathbb{N}_0} \) be a system of selfadjoint projections in a Hilbert space \( \mathcal{H}. \) For all \( n \in \mathbb{N}_0, \) set

\[
T_n = (1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0), \quad \text{and} \quad Q_n = P_n (1 - P_{n-1}) \cdots (1 - P_0), \quad Q_0 = P_0. \tag{3.8}
\]

Then,

\[
1 - T_n^* T_n = \sum_{j=0}^n Q_j^* Q_j, \quad \text{and} \quad 1 - T_n = \sum_{j=0}^n Q_j. \tag{3.9}
\]

**Remark 3.6.** The operator products introduced in formulas (3.8) and (3.9) above will play an important role in our subsequent considerations. Hence, when we refer to \( Q_n \) and \( T_n, \) we shall mean the particular operator products in (3.8) and (3.9). The input in our algorithm will be a fixed system of selfadjoint projections, \( P_n. \)

Note that the factors making up the operator products in (3.8) and (3.9) are non-commuting. We stress that non-commutativity is an important (and subtle) feature of the theory of operator frames; see e.g., [JT17].

**Proof of Theorem 3.5.** One checks that \( T_n = T_{n-1} - Q_n, \) so that

\[
T_n^* T_n = (T_{n-1}^* - Q_{n-1}^*) (T_{n-1} - Q_n)
= T_{n-1}^* T_{n-1} - T_{n-1}^* Q_n - Q_{n-1}^* T_{n-1} + Q_{n-1}^* Q_n
= T_{n-1}^* T_{n-1} - Q_{n-1}^* Q_n - Q_{n-1}^* Q_n + Q_{n-1}^* Q_n
= T_{n-1}^* T_{n-1} - Q_{n-1}^* Q_n
= T_{n-2}^* T_{n-2} - Q_{n-2}^* Q_{n-1} - Q_{n-1}^* Q_n
\]

...
\[
= 1 - P_0 - \sum_{j=1}^{n} Q_j^* Q_j \\
= 1 - \sum_{j=0}^{n} Q_j^* Q_j.
\]

Since \( Q_n = T_{n-1} - T_n \), so
\[
\sum_{j=0}^{n} Q_j = Q_0 + (T_0 - T_1) + (T_1 - T_2) + \cdots + (T_{n-1} - T_n) \\
= P_0 + 1 - P_0 - T_n = 1 - T_n. \]

\[\square\]

Let \( \mathcal{H} \) be a Hilbert space, and let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of bounded operators in \( \mathcal{H} \), i.e., \( A_n \in \mathcal{B}(\mathcal{H}), \forall n \in \mathbb{N} \). We shall need the following two notions of convergence in \( \mathcal{B}(\mathcal{H}) \).

**Definition 3.7.**

(i) We say that \( A_n \to 0 \) in the strong operator topology (SOT) iff (Def.)
\[
\lim_{n \to \infty} \|A_n x\| = 0 \quad \text{for all vectors } x \in \mathcal{H}.
\]

(ii) We say that \( A_n \to 0 \) in the weak operator topology (WOT) iff (Def.)
\[
\lim_{n \to \infty} \langle x, A_n y \rangle = 0 \quad \text{for all pairs of vectors } x, y \in \mathcal{H}. \]

Here \( \langle \cdot, \cdot \rangle \) refers to the inner product in \( \mathcal{H} \).

**Corollary 3.8.** The following are equivalent:

(i) \( 1 = \sum_{j \in \mathbb{N}_0} Q_j^* Q_j \) in the weak operator topology.

(ii) \( 1 = \sum_{j \in \mathbb{N}_0} Q_j \) in the strong operator topology.

(iii) \( T_n \to 0 \) in the strong operator topology.

**Remark 3.9.** Under suitable conditions on \( Q_n \) one can show that the convergence in part (i) of the corollary also holds in the strong operator topology.

**Definition 3.10.** The system \( \{P_j\}_{j \in \mathbb{N}_0} \) is called effective if \( T_n \to 0 \) in the strong operator topology.

**Corollary 3.11.** Suppose the system \( \{P_j\}_{j \in \mathbb{N}_0} \) is effective. Then, for all \( x \in \mathcal{H} \),
\[
x = \sum_{j \in \mathbb{N}_0} Q_j x. \tag{3.12}
\]

Moreover, for all \( x, y \in \mathcal{H} \),
\[
\langle x, y \rangle = \sum_{j \in \mathbb{N}_0} \langle Q_j x, Q_j y \rangle; \tag{3.13}
\]

and in particular,
\[
\|x\|^2 = \sum_{j \in \mathbb{N}_0} \|Q_j x\|^2. \tag{3.14}
\]

**Remark 3.12.** The system of operators \( \{Q_j\}_{j \in \mathbb{N}_0} \) in Corollary 3.11 has frame-like properties. Specifically, the mapping
\[
\mathcal{H} \ni x \mapsto (Q_j x) \in l^2(\mathbb{N}_0) \otimes \mathcal{H}
\]
plays the role of an analysis operator, and the synthesis operator $V^*$ is given by

$$I^2(N_0) \otimes \mathcal{H} \ni \xi \mapsto \sum_{j \in N_0} Q_j^* \xi_j.$$  

Note that $1 = V^* V$, by part (i) of Corollary 3.8; and eq. (3.14) is the generalized Parseval identity. Also see Proposition 3.13 below.

**Proposition 3.13.** Let $\{P_j\}_{j \in N_0}$ be an effective system. Then there exists a Hilbert space $\mathcal{H}$, an isometry $V : \mathcal{H} \to \mathcal{H}$, and self-adjoint projections $E_j$ in $\mathcal{H}$, such that $Q_j^* Q_j = V^* E_j V$, for all $j \in N_0$. Thus,

$$1 = \sum_{j \in N_0} Q_j^* Q_j = \sum_{j \in N_0} V^* E_j V. \quad (3.15)$$

**Proof.** Let $\mathcal{H} = l^2(N_0) \otimes \mathcal{H} = \oplus_{N_0} \mathcal{H}$, and set $V : \mathcal{H} \to \mathcal{H}$ by

$$V x = (Q_j x)_{j \in N_0}.$$  

Then, for all $x \in \mathcal{H}$ and $y = (y_j) \in \mathcal{H}$,

$$\langle V x, y \rangle_{\mathcal{H}} = \sum_j \langle Q_j x, y_j \rangle = \left( x, \sum_j Q_j^* y_j \right)_{\mathcal{H}}.$$  

Hence the adjoint operator $V^*$ is given by

$$V^* y = \sum_{j \in N_0} Q_j^* y_j.$$  

For all $j \in N_0$, let $E_j : \mathcal{H} \to \mathcal{H}$ be the projection,

$$E_j y = (0, \ldots, 0, y_j, 0, \ldots), \forall y = (y_j) \in \mathcal{H}.$$  

Then $Q_j^* Q_j = V^* E_j V$, and (3.15) follows from this. \hfill \square

Let $\mathcal{H}$ be a fixed Hilbert space. We shall have occasion to use Dirac’s notation for rank-one operators in $\mathcal{H}$: If $u, v \in \mathcal{H}$, we set $|u\rangle\langle v|$ the operator, which is defined by

$$|u\rangle\langle v| (x) = \langle v, x \rangle_{\mathcal{H}} u;$$

or in physics terminology,

$$|u\rangle\langle v| = |u\rangle \langle v|_{\mathcal{H}}.$$  

Note the following: For vectors $u_i, v_i, i = 1, 2$, we have:

$$\langle u_1 | v_1 \rangle \langle u_2 | v_2 \rangle = (u_1, u_2)_{\mathcal{H}} | u_1 \rangle \langle v_2 |.$$  

For the adjoint operators, we have:

$$|u\rangle\langle v|^* = |v\rangle\langle u|.$$  

If $B \in \mathcal{B}(\mathcal{H})$, we have

$$B |u\rangle\langle v| = |Bu\rangle\langle v|;$$

and

$$|u\rangle\langle v| B = |u\rangle \langle B^* v|.$$
3.2. The case of rank-1 projections in Hilbert space

Let \( \{P_j\}_{j \in \mathbb{N}_0} \) be a system of rank-1 projections, i.e., \( P_j = |e_j\rangle \langle e_j| \), where \( \{e_j\}_{j \in \mathbb{N}_0} \) is a set of unit vectors in \( \mathcal{H} \). When the system \( \{e_j\} \) is independent, then the corresponding family of projections \( P_j = |e_j\rangle \langle e_j| \) is non-commutative.

It follows from (3.9) that every \( Q_j \) is a rank-1 operator with range in \( \text{span} \{e_j\} \).

Thus there exists a unique \( g_j \in \mathcal{H} \) such that
\[
Q_j = |e_j\rangle \langle g_j|, \quad j \in \mathbb{N}_0.
\] (3.16)

Lemma 3.14. Given \( \{P_j\}_{j \in \mathbb{N}_0} \) a sequence of s.a. projections in \( \mathcal{H} \); set
\[
Q_n := P_n P_{n-1}^\perp \cdots P_1^\perp P_0^\perp,
\] (3.17)
where \( P_j^\perp := 1 - P_j \); then
\[
Q_n = P_n \left( 1 - \sum_{j=0}^{n-1} Q_j \right).
\] (3.18)

Proof. By definition, we have
\[
Q_n = P_n (1 - P_{n-1}) P_{n-2}^\perp \cdots P_0^\perp = P_n P_{n-2}^\perp \cdots P_0^\perp - P_n P_{n-1} P_{n-2}^\perp \cdots P_0^\perp = P_n P_{n-2}^\perp \cdots P_0^\perp - P_n Q_{n-1} = P_n P_{n-3}^\perp \cdots P_0^\perp - P_n Q_{n-2} - P_n Q_{n-1}
\]
\[
\vdots
\]
\[
= P_n P_0^\perp - P_n Q_1 - P_n Q_2 - \cdots - P_n Q_{n-1} = P_n - \sum_{j=0}^{n-1} P_n Q_j = P_n \left( 1 - \sum_{j=0}^{n-1} Q_j \right).
\]

\[\square\]

Corollary 3.15. The vectors \( \{g_j\} \) in (3.16) are determined recursively by
\[
g_0 = e_0
\] (3.19)
\[
g_n = e_n - \sum_{j=0}^{n-1} \langle e_j, e_n \rangle g_j.
\] (3.20)

Proof. For all \( x \in \mathcal{H} \), it follows from Lemma 3.14, that
\[
Q_n x = P_n x - \sum_{j=0}^{n-1} P_n Q_j x
\]
\[
\Downarrow
\]
\[
e_n \langle g_n, x \rangle = e_n \langle e_n, x \rangle - \sum_{j=0}^{n-1} e_n \langle e_n, e_j \rangle \langle g_j, x \rangle.
\]
That is, \( g_n = e_n - \sum_{j=0}^{n-1} \langle e_j, e_n \rangle g_j \). \[\square\]
Corollary 3.16. Assume \( \{|e_j\rangle\langle e_j|\}_{j \in \mathbb{N}_0} \) is effective, and let \( Q_j = |e_j\rangle\langle g_j| \) be as above. Then, for all \( x \in \mathcal{H} \), we have

\[
x = \sum_{j \in \mathbb{N}_0} \langle g_j, x \rangle e_j.
\]

(3.21)

In particular, for all \( A \in \mathcal{B}(\mathcal{H}) \), then

\[
Ax = \sum_{j \in \mathbb{N}_0} \langle A^* g_j, x \rangle e_j.
\]

(3.22)

Moreover, for all \( x, y \in \mathcal{H} \),

\[
\langle x, y \rangle = \sum_{j \in \mathbb{N}_0} \langle x, g_j \rangle \langle g_j, y \rangle,
\]

and

\[
||x||^2 = \sum_{j \in \mathbb{N}_0} ||g_j||^2.
\]

Proof. By assumption, \( Q_j^* Q_j = |g_j\rangle\langle g_j| \), hence

\[
\langle x, y \rangle = \sum_{j \in \mathbb{N}_0} \langle x, Q_j^* Q_j y \rangle = \sum_{j \in \mathbb{N}_0} \langle x, g_j \rangle \langle g_j, y \rangle.
\]

Corollary 3.17. The system \( \{|e_j\rangle\langle e_j|\}_{j \in \mathbb{N}_0} \) is effective iff \( \{g_j\}_{j \in \mathbb{N}_0} \) is a Parseval frame in \( \mathcal{H} \).

Remark 3.18. We note that when \( \mu \) is slice singular, then the Fourier frequencies \( \{e_n\}_{n \in \mathbb{N}_0} \) is effective in \( L^2(\mu) \), and every \( f \in L^2(\mu) \) has Fourier series expansion.

This conclusion is based on (3.20) and (3.21) from Corollaries 3.15 & 3.16. In more detail: Assume \( \mu \) is slice singular, and take \( \mathcal{H} = L^2(\mu) \). We may then think of Corollary 3.16 as a (generalized) Fourier expansion result since every \( f \) in the specified \( L^2(\mu) \) space admits a non-orthogonal Fourier expansion in terms of explicit coefficients and the standard Fourier functions \( e_n \). Indeed, the corresponding generalized Fourier coefficients are computed with the use of the functions \( g_n \) of the Kaczmarz algorithm, see eq. (3.21) and Corollary 3.15.

We stress that while the coefficients in the expansion for \( f \) are explicitly given in (3.21), this is nonetheless a non-orthogonal expansion; see also [HJW16, HJW18a].

3.3. Random Kaczmarz constructions and sequences of projections

In the discussion below, the word “random” will refer to a fixed probability space \( (\Omega, \mathcal{F}, P) \), where \( \Omega \) is a set (sample space), \( \mathcal{F} \) is a \( \sigma \)-algebra (specified events), and \( P \) is a probability measure defined on \( \mathcal{F} \). Random variables will then be measurable functions on \( (\Omega, \mathcal{F}) \). For example, if \( \xi : \Omega \to \mathcal{B}(\mathcal{H}) \) is an operator valued random variable, measurability will then refer to the \( \sigma \)-algebra of subsets in \( \mathcal{B}(\mathcal{H}) \) which are w.r.t. the usual operator topology.

Equivalently, \( \xi : \Omega \to \mathcal{B}(\mathcal{H}) \) is a random variable iff (Def.) for all pairs of vectors \( x, y \in \mathcal{H} \), then the functions

\[
\Omega \to \mathbb{C}, \quad \omega \mapsto \langle x, \xi(\omega) y \rangle_{\mathcal{H}}
\]

are measurable w.r.t. the standard Borel \( \sigma \)-algebra \( \mathcal{B}_{\mathbb{C}} \) of subsets of \( \mathbb{C} \).
Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we shall denote the corresponding expectation \(\mathbb{E}\), i.e.,

\[
\mathbb{E}(\cdots) \overset{\text{Def.}}{=} \int_{\Omega} (\cdots) \, d\mathbb{P}.
\] (3.24)

Theorem 3.20 below is a stochastic variant of the classical Kaczmarz algorithm; also see Theorem 3.5. For recent development and applications, we refer to [Fri05, Pop10, SEC14, LZ15, CESV15, LS15, NSW16, CC18, Che18].

Let \(\mathcal{H}\) be a Hilbert space. Given a family of selfadjoint projections \(\{P_j\}_{j \in \mathbb{N}_0}\) in \(\mathcal{H}\), let \(\xi : \Omega \to \mathcal{B}(\mathcal{H})\) be a random variable, such that

\[
\mathbb{P}(\xi = P_j) = p_j, \ j \in \mathbb{N}_0,
\] (3.25)

where \(p_j > 0\), and \(\sum_{j \in \mathbb{N}_0} p_j = 1\).

Suppose further that there exists a constant \(C, 0 < C < 1\), such that

\[
\mathbb{E}[\|\xi x\|^2] = \sum_{j \in \mathbb{N}_0} p_j \|P_j x\|^2 \geq C \|x\|^2, \ \forall x \in \mathcal{H}.
\] (3.26)

**Definition 3.19.** Let \(\xi, \eta : \Omega \to \mathcal{B}(\mathcal{H})\) be two operator-valued random variables. We say \(\xi\) and \(\eta\) are independent iff (Def.) for all \(x, y \in \mathcal{H}\), the scalar valued random variables \(\langle x, \xi y \rangle\), and \(\langle x, \eta y \rangle\) are independent.

We shall use the standard abbreviation i.i.d. for independent, identically distributed; also in the case of an indexed family of operator valued random variables. In the present case, the common distribution is specified by fixing the data in (3.25).

The key feature of our present randomized Kaczmarz algorithm is that it outputs a recursively generated sequence of operator valued random variables; see (3.27) and (3.28). Each output, in turn, will be a product of a specified i.i.d. system of projection valued random variables. The latter i.i.d. system serves as input into the algorithm.

**Theorem 3.20.** Let \(\{\xi_j\}_{j \in \mathbb{N}_0}\) be an i.i.d. realization of \(\xi\) from (3.25). Fix \(\xi_0 = P_0\), and set

\[
T_n = (1 - \xi_n) (1 - \xi_{n-1}) \cdots (1 - \xi_0), \ \text{and} \quad Q_n = \xi_n (1 - \xi_{n-1}) \cdots (1 - \xi_0), \ Q_0 = \xi_0.
\] (3.27)

Note that each product in (3.27) and (3.28) is an operator-valued random variable.

Then, for all \(x \in \mathcal{H}\), we have:

\[
\lim_{n \to \infty} \mathbb{E}[\|T_n x\|^2] = 0.
\] (3.29)

**Proof.** For all \(x \in \mathcal{H}\), we have

\[
T_n x = T_{n-1} x - \xi_n T_{n-1} x.
\]

But each \(\xi_n\) is a random variable taking values in the set of selfadjoint projections, as specified in (3.25), and so

\[
\|T_n x\|^2 = \|T_{n-1} x\|^2 - \|\xi_n T_{n-1} x\|^2.
\]

It follows from (3.26) that

\[
\mathbb{E}_{\xi_1, \cdots, \xi_{n-1}}[\|T_n x\|^2] = \mathbb{E}_{\xi_1, \cdots, \xi_{n-1}}[\|T_{n-1} x\|^2] - \mathbb{E}_{\xi_1, \cdots, \xi_{n-1}}[\|\xi_n T_{n-1} x\|^2] \\
\leq \|T_{n-1} x\|^2 (1 - C).
\]
Therefore, by taking expectation again, we get
\[
E \left[ \| T_n x \|^2 \right] \leq E \left[ \| T_{n-1} x \|^2 \right] (1 - C)
\]
\[\vdots\]
\[
\leq E \left[ \| T_0 x \|^2 \right] (1 - C)^n
\]
\[\leq \| x_0 - x \|^2 (1 - C)^n \rightarrow 0, \; n \rightarrow \infty. \]

Corollary 3.21. Let \( T_n \) and \( Q_n \) be as in (3.27)–(3.28), then the following hold.

(i) For all \( x \in \mathcal{H} \),
\[
\lim_{n \rightarrow \infty} E \left[ \left\| x - \sum_{j=0}^{n} Q_j x \right\|^2 \right] = 0. \tag{3.30}
\]

(ii) For all \( x, y \in \mathcal{H} \),
\[
\lim_{n \rightarrow \infty} E \left[ \left\| \langle x, y \rangle - \sum_{j=0}^{n} \langle x, Q_j^* Q_j y \rangle \right\|^2 \right] = 0. \tag{3.31}
\]

Proof. The assertion (3.30) follows from (3.29) and (3.11).

By (3.10), we have \( \| T_n x \|^2 = \langle x, T_n^* T_n x \rangle = \langle x, x \rangle - \sum_{j=0}^{n} \langle x, Q_j^* Q_j x \rangle \), and so
\[
E \left[ \langle x, T_n^* T_n x \rangle \right] \rightarrow 0, \; n \rightarrow \infty.
\]

Now (3.31) follows from this and the polarization identity. \( \square \)

Remark 3.22 (Fusion frames, and measure frames). Our present equation (3.26) may be viewed as an instance of what is now called fusion frames, and developed extensively by Casazza et al. [CK04, CKL08, CK08], and by others. In addition, we note that our present (3.31) is closely related to a formulation a certain notion of measure frames, see e.g., [FJKO05, EO13, Oko16, WO17], and its extensions in [JS18a].

3.4. Solutions to \( Ax = y \) in finite, and in infinite, dimensional spaces

A natural extension of the classical Kaczmarz algorithm is to solve the equation
\[ Ax = y, \]
when \( x, y \) are vectors in an infinite-dimensional Hilbert space \( \mathcal{H} \), and \( A, A^{-1} \) are both bounded operators in \( \mathcal{H} \); see Figure 3.1.

Equivalently, when \( \{ \varphi_j \}_{j \in \mathbb{N}} \) is an ONB (or a Parseval frame) in \( \mathcal{H} \), we shall consider the system of equations
\[
\langle \varphi_j, Ax \rangle = \langle \varphi_j, y \rangle
\]
\[\uparrow\]
\[
\langle A^* \varphi_j, x \rangle = \langle \varphi_j, y \rangle.
\]

Question 3.23. Given the complex numbers \( \langle A^* \varphi_j, x \rangle, \; j \in \mathbb{N} \), is it possible to recover \( x \) using the Kaczmarz method?

The closest analog to the finite-dimensional setting is the class of Hilbert-Schmidt operators, and we shall recall the basics below.
Definition 3.24. Assume $\mathcal{H}$ is separable. \( A : \mathcal{H} \to \mathcal{H} \) is Hilbert-Schmidt iff (Def.) $\exists$ an ONB \( \{e_i\}_{i \in \mathbb{N}_0} \), such that
\[
\sum_i \|Ae_i\|^2 < \infty. \tag{3.32}
\]
We denote the set of all Hilbert-Schmidt operators in $\mathcal{H}$ by $\text{HS}(\mathcal{H})$.

Note that $A \in \text{HS}(\mathcal{H})$ iff $A^* A$ is trace class, and for an ONB $\{e_i\}$, we have
\[
\sum_i \|Ae_i\|^2 = \sum_i \langle e_i, A^* A e_i \rangle = \text{tr} (A^* A). \tag{3.33}
\]

Lemma 3.25. $\text{HS}(\mathcal{H}) \cong \mathcal{H} \otimes \overline{\mathcal{H}}$, where $\overline{\mathcal{H}}$ denotes the conjugate Hilbert space.

Proof. If $\{e_i\}$ is an ONB, set $|e_i\rangle \langle e_j|$ w.r.t. the inner product
\[
(A, B) \mapsto \text{tr} (A^* B), \tag{3.34}
\]
for all $A, B \in \text{HS}(\mathcal{H})$. Hence,
\[
\langle A, B \rangle_{\text{HS}} = \sum_i \langle e_i, A^* B e_i \rangle_{\mathcal{H}} \text{ (=} \text{tr} (A^* B)). \tag{3.35}
\]

We shall show that
\[
\text{HS}(\mathcal{H}) \cap \{ |e_i\rangle \langle e_j| \}_{i,j \in \mathbb{N}_0} = 0, \tag{3.36}
\]
i.e., $\{A_{ij} := |e_i\rangle \langle e_j|\}$ is total in $\text{HS}(\mathcal{H})$. To see this, note that
\[
\text{tr} (|u\rangle \langle v|) = \langle v, u \rangle_{\mathcal{H}}, u, v \in \mathcal{H}.
\]
In fact, one checks that
\[
\text{tr} (|u\rangle \langle v|) = \sum_i \langle e_i, u \rangle \langle v, e_i \rangle = \sum \langle v, u \rangle, \text{ by Parseval.}
\]
Now, if $B \in \text{HS}(\mathcal{H})$, then
\[
(B, |e_i\rangle \langle e_j|)_{\text{HS}} = \text{tr} (B^* |e_i\rangle \langle e_j|) = \text{tr} (|B^* e_i\rangle \langle e_j|) = \text{tr} (|e_i\rangle \langle B e_j|) = \langle B e_j, e_i \rangle_{\mathcal{H}}, \text{ by (3.36)}.
\]
Therefore, if $(B, |e_i\rangle \langle e_j|)_{\text{HS}} = 0$, for all $i, j \in \mathbb{N}_0$, then $B = 0$; since
\[
Be_j = \sum_i \langle e_i, Be_j \rangle_{\mathcal{H}} e_i.
\]

Now, back to Question 3.23. From earlier discussion, the answer depends on whether the sequence $\{A^* \varphi_j\}$ is effective. In general, we do not get an effective sequence, even if $A$ is assumed Hilbert-Schmidt. However, under certain conditions (see (3.37)) the random Kaczmarz algorithm applies, and we get an approximate sequence that converges to $x$ in expectation. See details below.
Lemma 3.26. Suppose $A$ is a bounded operator in $\mathcal{H}$ with bounded inverse. Fix a Parseval frame $\{\varphi_j\}_{j \in \mathbb{N}}$ in $\mathcal{H}$, let $P_j$ be the projection onto $A^*\varphi_j$, $j \in \mathbb{N}$.
Assume further that
\[1/\|A^{-1}\|^2 < \sum_k \|A^*\varphi_k\|^2 < \infty.\] (3.37)

Then, there exists a probability distribution $\{p_j\}$ on $\{P_j\}$, given by
\[p_j = \|A^*\varphi_j\|^2 / \sum_k \|A^*\varphi_k\|^2,\] (3.38)
such that, for all $h \in \mathcal{H}$,
\[\sum_{j \in \mathbb{N}} p_j \|P_j h\|^2 \geq C \|h\|^2,\] (3.39)
where $C$ is a constant, $0 < C < 1$.

Proof. For all $h \in \mathcal{H}$, we have:
\[\|h\|^2 = \|A^{-1}Ah\|^2 \leq \|A^{-1}\|^2 \|Ah\|^2 = \|A^{-1}\|^2 \sum |\langle \varphi_j, Ah \rangle|^2 = \|A^{-1}\|^2 \sum |\langle A^*\varphi_j, h \rangle|^2 \]
\[= \|A^{-1}\|^2 \sum_k \|A^*\varphi_k\|^2 \sum_j \frac{\|A^*\varphi_j\|^2}{\sum_k \|A^*\varphi_k\|^2} \left(\|A^*\varphi_j\|^2 \langle A^*\varphi_j, h \rangle\right)^2 \]
\[= \|A^{-1}\|^2 \sum_k \|A^*\varphi_k\|^2 \sum_j p_j \|P_j h\|^2.\]
The desired conclusion follows from this. \qed

Corollary 3.27. Let the setting be as in Lemma 3.26. An approximate solution to $Ax = y$ is obtained recursively as follows:
Let $\xi : \Omega \to \mathcal{B}(\mathcal{H})$ be a random projection, s.t. $\mathbb{P}(\xi = P_j) = p_j$ (see (3.38)), and $\{\xi_j\}$ be an i.i.d. realization of $\xi$. Then, with $x_0 \neq 0$ fixed, and
\[x_j := x_{j-1} + \xi_j (x - x_{j-1}), \quad j \in \mathbb{N},\] (3.40)
we have:
\[\lim_{j \to \infty} \mathbb{E} \left[\|x_j - x\|^2\right] = 0.\] (3.41)

Note that, in (3.40) if $\xi = P_k$, then
\[\xi x = \frac{\langle A^*\varphi_k, x \rangle}{\|A^*\varphi_k\|^2} A^*\varphi_k = \frac{\langle \varphi_k, y \rangle}{\|A^*\varphi_k\|^2} A^*\varphi_k.\]

Proof. By Lemma 3.26, the estimate (3.39) holds with the probabilities specified in (3.38). See also condition (3.26). Moreover, it follows from (3.40) that
\[x - x_j = (1 - \xi_j) (1 - \xi_{j-1}) \cdots (1 - \xi_1) x_0.\]
Therefore, by Theorem 3.20, the assertion in (3.41) holds (with a suitable choice of index $j$). \qed
4. System of isometries

Below we discuss a particular aspect of our problem where the polydisk $\mathbb{D}^d$ will play an important role. As outlined below, the polydisk is a natural part of our harmonic analysis of frame-approximation questions in the Hilbert space $L^2(\mu)$, where $\mu$ is in a suitable class of IFS-measures, i.e., the multivariable setting for fractal measures.

**Lemma 4.1.** Fix $d > 1$, and let $\mathbb{D}^d$ be the polydisk. Let $H_2(\mathbb{D}^d)$ be the corresponding Hardy space. Let $\mu$ be a Borel probability measure on $\mathbb{T}^d \simeq [0,1]^d$. Then there is a bijective correspondence between:

(i) isometries $V : L^2(\mu) \to H_2(\mathbb{D}^d)$; and

(ii) Parseval frames $\{g_n\}$ in $L^2(\mu)$.

The correspondence is as follows:

(i) $\to$ (ii). Given $V$, isometric; set $g_n := V^*(z^n)$, where $n \in \mathbb{N}_0$.

(ii) $\to$ (i). Given $\{g_n\}$ a fixed Parseval frame in $L^2(\mu)$, set

$$
(Vf)(z) = \sum_{n \in \mathbb{N}_0} \langle g_n, f \rangle_{L^2(\mu)} z^n, \quad z \in \mathbb{D}^d.
$$

**Proof:** The fact that there is a correspondence between isometries and Parseval frames is general. Let $\mathcal{H}_1$ be a separable Hilbert space, then there is a bijective correspondence between the following two:

(i) A Parseval frame $\{(g_n)_{n \in \mathbb{N}}\}$ in $\mathcal{H}_1$ (with a suitable choice of index);

(ii) A pair $(\mathcal{H}_2, V)$, where $\mathcal{H}_2$ is a Hilbert space, and $V : \mathcal{H}_1 \to \mathcal{H}_2$ is isometric.

(Note that there is a similar result for Bessel frames as well.) The correspondence is as follows.

Given a Parseval frame $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{H}_1$, take $\mathcal{H}_2 := l^2(\mathbb{N})$, and set $Vf = \sum_{n} \langle g_n, f \rangle_{\mathcal{H}_1} \delta_n$, where $\{\delta_n\}_{n \in \mathbb{N}}$ is the standard ONB in $l^2(\mathbb{N})$.

Conversely, let $(\mathcal{H}_2, V)$ be such that $\mathcal{H}_1 \xrightarrow{V} \mathcal{H}_2$ is isometric. Choose an ONB $\{\beta_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}_2$, and set $g_n = V^* \beta_n$. Then $\{g_n\}$ is a Parseval frame in $\mathcal{H}_1$. Indeed, for all $h \in \mathcal{H}_1$, one checks that,

$$
\sum_{n} |\langle g_n, h \rangle_{\mathcal{H}_1}|^2 = \sum_{n} |\langle V^* \beta_n, h \rangle_{\mathcal{H}_2}|^2 = \sum_{n} |\langle \beta_n, Vh \rangle_{\mathcal{H}_2}|^2 = \|Vh\|_{\mathcal{H}_2}^2 = \|h\|^2_{\mathcal{H}_1}.
$$

The lemma follows by setting $\mathcal{H}_1 = L^2(\mu)$, and $\mathcal{H}_2 = H_2(\mathbb{D}^d)$. \qed

**Definition 4.2.** Fix $d > 1$. For all $x \in \mathbb{T}^d$, and all $z \in \mathbb{D}^d$, let

$$
K^* (z, x) = \prod_{j=1}^{d} \frac{1}{1 - z_j e(x_j)}.
$$

Let $\mu \in \mathcal{M}(\mathbb{T}^d)$, and set

$$
(C_\mu f)(z) = \int_{\mathbb{T}^d} f(x) K^* (z, x) d\mu(x)
$$

(4.1)

(4.2)
assumes a disintegration is isometric. It follows that given by

\[
\{ \text{has (non-orthogonal) Fourier expansion} \}
\]

where \( \hat{\mu}(n) = \int_{\mathbb{R}^d} e_n(x) \, d\mu(x), \) \( n \in \mathbb{N}_0^d. \)

Let \( L^2(\mu) \) be as above, where \( \mu \in \mathcal{M}^+ (\mathbb{T}^2), \) \( \xi = \mu \circ \pi_1^{-1}, \) and \( \mu \) assumes a disintegration \( d\mu = \int \sigma^x (dy) \, d\xi(x). \)

**Theorem 4.3** (see e.g., [Sar94, BS13]). Assume \( \mu \) is slice singular. There are then two associated isometries:

\[
L^2(\xi) \overset{V_\xi}{\longrightarrow} H_2(\mathbb{D}), \quad (V_\xi f)(z) = \frac{(C_\xi f)(z)}{(C_\xi 1)(z)},
\]

and

\[
L^2(\sigma^x) \overset{V_{\sigma^x}}{\longrightarrow} H_2(\mathbb{D}), \quad (V_{\sigma^x} f)(z) = \frac{(C_{\sigma^x} f)(z)}{(C_{\sigma^x} 1)(z)}.
\]

**Proof sketch.** Let \( \nu \) be a positive Borel measure on \([0, 1], \) and \( C_\mu \) be the Cauchy transform from (4.2). Assume \( \nu \) is singular.

Then, by F.M Riesz (see Theorem 2.1), the set \( \{ e_n \}_{n \in \mathbb{N}_0} \) is total in \( L^2(\nu) \). Moreover, it follows from [KM06], that \( \{ e_n \}_{n \in \mathbb{N}_0} \) is effective. Thus, every \( f \in L^2(\nu) \) has (non-orthogonal) Fourier expansion

\[
f = \sum_{n \in \mathbb{N}_0} \langle g_n, f \rangle_{L^2(\nu)} e_n,
\]

where \( \{ g_n \} \) is the Parseval frame in \( L^2(\nu) \) constructed from Kaczmarz’ algorithm. See also Remark 3.18. One may verify that

\[
(V_\nu f)(z) = \frac{C_\nu f}{C_\nu 1} = \sum_{n \in \mathbb{N}_0} \langle g_n, f \rangle_{L^2(\nu)} z^n,
\]

and so \( V_\nu : L^2(\nu) \to H_2(\mathbb{D}) \) is isometric.

The theorem follows from this, and the assumption that \( \mu \) is slice singular. \( \square \)

**Corollary 4.4.** The mapping

\[
V_\mu : L^2(\mu) \longrightarrow H_2(\mathbb{D}^2) (= H_2(\mathbb{D}) \otimes H_2(\mathbb{D}))
\]

given by

\[
(V_\mu f)(z_1, z_2) = V_\xi \left( (V_{\sigma^x(\cdot)} F(x, \cdot))(z_2) \right)(z_1)
\]

is isometric. It follows that \( \{ g_n := V_\mu^*(z^n) \}_{n \in \mathbb{N}_0^2} \) is a Parseval frame in \( L^2(\mu). \)

**Proof.** Follows from Theorem 4.3 and Lemma 4.1. \( \square \)

**Remark 4.5.** From the above discussion, we see that if \( V : L^2(\mu) \to H_2(\mathbb{D}^2) \) is an isometry, then \( \{ g_n := V^*(z^n) \}_{n \in \mathbb{N}_0^2} \) is a Parseval frame in \( L^2(\mu). \) This implication holds in general. Since there are “many” such isometries, it follows that there are “many” Parseval frames. For more details, see [HJW16, HJW18a, HJW18b] and the reference therein.
Lemma 4.6. Let $K$ be a kernel on $\mathbb{D}^d$, and $\mu$ be a measure on $\mathbb{T}^d$. Then for all $z \in \mathbb{D}^d$, we have $\lim_{r \to 1} K(z, re(x)) = K^*(z,x)$, a.a. $x \in \mathbb{T}^d$; and
$$V^*_\mu(K(\cdot, z)) = K^*(z,x),$$
a.a. $x$ w.r.t. $\mu$.

5. General iterated function system (IFS)-theory

In this section we turn to an analysis of the IFS measures (see e.g., [Hut81, Hut95, DJ07, BHS08, HJW16, JS18b]), as introduced in Sections 1 and 2 (see (5.3) below). The notion of iterated function systems (IFS) for the case of measures fits the following general idea of patterns with self-similarity across different scales. Also here, the IFS-measures are created by recursive repetition of a simple process in an ongoing feedback loop.

Recall that an IFS measure is obtained from a recursive algorithm involving successive iteration of a finite system of maps in a metric space. IFS systems are self-similar because the same fixed choice of “scaling” mappings is used in each step of the algorithm. (The simplest IFS measures arise from the standard Cantor construction applied to a finite interval. But the idea works much more generally.) Then the chosen finite index-set for the mappings is called an alphabet, denoted $B$. We shall analyze here the IFS measures with the aid of symbolic dynamics on a probability space $\Omega$, made up of infinite words in $B$. Then a fixed choice of probability weights on $B$ leads to an associated infinite product measure, called $P$, on $\Omega$, see (5.2). By Kakutani’s theorem, distinct weights yield mutually singular infinite product measures.

We shall construct a random variable $X$ on $\Omega$ such that the IFS then arises as the image under $X$, and the IFS measure $\mu$ becomes the distribution of $X$. Intuitively, $X$ is an infinite address map; see also eq (5.2) and Theorem 5.1. While the choice of such system of maps could be rather general, we shall restrict attention here to the case of a finite number of contractive affine mappings in $\mathbb{R}^d$, $d$ fixed; see e.g., (6.2) for the case of the standard Sierpinski gasket, where $d = 2$. In this case, the associated maximal entropy measure $\mu$ (see (6.5)) is a probability measure prescribed by the uniform distribution on $B$.

Let $(M, d)$ be a complete metric space. Fix an alphabet $B = \{b_1, \ldots, b_N\}$, $N \geq 2$, and let $\{\tau_b\}_{b \in B}$ be a contractive IFS with attractor $W \subset M$, i.e.,
$$W = \bigcup_b \tau_b(W).$$
(5.1)

In fact, $W$ is uniquely determined by (5.1).

Let $\{p_b\}_{b \in B}$, $p_b > 0$, $\sum_{b \in B} p_b = 1$, be fixed. Set $\Omega = B^\mathbb{N}$, equipped with the product topology. Let
$$P = \prod\limits_{\mathbb{N}} p = \underbrace{p \times p \times p \cdots}_{\text{$\mathbb{N}$ product measure}}$$
(5.2)
be the infinite-product measure on $\Omega$ (see [Kak43, Hid80]).

In this section, we construct a random variable $X : \Omega \to M$ with value in $M$ (a measure space $(M, \mathcal{B}_M)$), such that the distribution $\mu := P \circ X^{-1}$ is a Borel
probability measure supported on $W$, satisfying
\[ \mu = \sum_{b \in B} p_b \mu \circ \tau_b^{-1}. \] (5.3)
That is, $\mu$ is the IFS measure.

**Theorem 5.1.** For points $\omega = (b_{i_1}, b_{i_2}, b_{i_3}, \ldots) \in \Omega$ and $k \in \mathbb{N}$, set
\[ \omega_{|k} = (b_{i_1}, b_{i_2}, \ldots, b_{i_k}), \quad \text{and} \]
\[ \tau_{\omega_{|k}} = \tau_{b_{i_k}} \circ \cdots \circ \tau_{b_{i_2}} \circ \tau_{b_{i_1}}. \] (5.5)
Then $\bigcap_{k=1}^{\infty} \tau_{\omega_{|k}}(M)$ is a singleton, say $\{x(\omega)\}$. Set $X(\omega) = x(\omega)$, i.e.,
\[ \{X(\omega)\} = \bigcap_{k=1}^{\infty} \tau_{\omega_{|k}}(M); \] (5.6)
then:
(i) $X: \Omega \rightarrow M$ is an $(M,d)$-valued random variable.
(ii) The distribution of $X$, i.e., the measure
\[ \mu = P \circ X^{-1} \] (5.7)
is the unique Borel probability measure on $(M,d)$ satisfying:
\[ \mu = \sum_{b \in B} p_b \mu \circ \tau_b^{-1}; \] (5.8)
equivalently,
\[ \int_M f d\mu = \sum_{b \in B} p_b \int_M (f \circ \tau_b) d\mu, \] (5.9)
holds for all Borel functions $f$ on $M$.
(iii) The support $W_\mu = \text{supp}(\mu)$ is the minimal closed set (IFS), $\neq \emptyset$, satisfying
\[ W_\mu = \bigcup_{b \in B} \tau_b(W_\mu). \] (5.10)

**Proof.** We shall make use of standard facts from the theory of iterated function systems (IFS), and their measures; see e.g., [Hut81, Hut95, BHS08].

**Monotonicity:** When $\omega \in \Omega$ is fixed, then $\tau_{\omega_{|k}}(M)$ is a monotone family of compact subsets in $M$ s.t.
\[ \tau_{\omega_{|k+1}}(M) \subset \tau_{\omega_{|k}}(M). \] (5.11)
Since $\tau_b$ is strictly contractive for all $b \in B$, we get
\[ \lim_{k \to \infty} \text{diameter } (\tau_{\omega_{|k}}(M)) = 0, \] (5.12)
and so the intersection in (5.6) is a singleton depending only on $\omega$.

**The $\sigma$-algebras on $(\Omega, P)$ and $(X,d)$:** The $\sigma$-algebra of subsets of $\Omega$ is generated by cylinder sets. Specifically, if $f = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})$ is a finite word, the corresponding cylinder set is
\[ E(f) = \{ \omega \in \Omega \mid \omega_j = b_{i_j}, \ 1 \leq j \leq k \} \subset \Omega. \] (5.13)
The Borel $\sigma$-algebra on $M$ is determined from the fixed metric $d$ on $M$.

The measure $P (= P_p)$ is specified by its values on cylinder sets; i.e, set
\[ P(E(f)) = p_{b_{i_1}} p_{b_{i_2}} \cdots p_{b_{i_k}} =: p_f. \] (5.14)
Proof of (5.8). The argument is based on the following: On \( \Omega \), introduce the shifts \( \bar{\tau}_b (b_1, b_2, b_3, \cdots) = (b, b_1, b_2, b_3, \cdots), b \in B \). Let \( X \) be as in (5.6)-(5.7), then
\[
\tau_b \circ X = X \circ \bar{\tau}_b, \tag{5.15}
\]
which is immediate from (5.6).

We now show (5.9), equivalently (5.8). Let \( f \) be a Borel function on \( M \), then
\[
\int_M f \, d\mu = \int_\Omega (f \circ X) \, d\mathbb{P} \quad \text{(by (5.7))} \]
\[
= \sum_{b \in B} p_b \int_\Omega f \circ X \circ \bar{\tau}_b \, d\mathbb{P} \quad \text{(since \( \mathbb{P} \) is the product measure \( \times \mathbb{P} \), see (5.14))} \]
\[
= \sum_{b \in B} p_b \int_\Omega f \circ \tau_b \circ X \, d\mathbb{P} \quad \text{(by (5.15))} \]
\[
= \sum_{b \in B} p_b \int_M f \circ \tau_b \, d\mu \quad \text{(by (5.7))}
\]
which is the desired conclusion. \( \Box \)

In general, the random variable \( X : \Omega \to W \) (see (5.6)) is not 1-1, but it is always onto. It is 1-1 when the IFS is non-overlap; see Definition 5.2 below.

Definition 5.2. We say that \( (\tau_b, W) \) is "non-overlap" if for all \( b, b' \in B \), with \( b \neq b' \), we have \( \tau_b (W) \cap \tau_{b'} (W) = \emptyset \).

Corollary 5.3. Assume \( p \neq p' \), i.e., \( p_b \neq p'_b \), for some \( b \in B \). (Recall that \( \sum_{b \in B} p_b = \sum_{b \in B} p'_b = 1 \), \( p_b, p'_b > 0 \).) Let \( \mathbb{P} = \times \mathbb{P} \), and \( \mathbb{P}' = \times \mathbb{P}' \) be the corresponding infinite product measures; and let \( \mu = \mathbb{P} \circ X^{-1}, \mu' = \mathbb{P}' \circ X^{-1} \) be the respective distributions. Then \( \mu \) and \( \mu' \) are mutually singular.

Proof. This is an application of Kakutani’s theorem on infinite product measures. See [Kak43, Kak48]. \( \Box \)

Remark 5.4 (Affine IFSs). Let \( B = \{ b_1, \cdots, b_N \} \) be a subset of \( \mathbb{R}^d \), and fix a \( d \times d \) matrix \( M \). Assume \( M \) is expansive, i.e., \( |\lambda| > 1 \), for all eigenvalues \( \lambda \) of \( M \). Then the mapping \( \Omega = \{1, \cdots, N\}^\mathbb{N} \xrightarrow{X} W_B \) from (5.6) is given by
\[
\omega = (i_1, i_2, i_3 \cdots) \mapsto x := \sum_{j=1}^{\infty} M^{-j} b_{i_j}.
\]
Note that \( x \) has a random expansion, with the alphabets \( b_i \in B \), as a sequence of i.i.d. random variables with distribution \( p = (p_1, \cdots, p_N) \).
6. Sierpinski and random power series

Given a probability measure \( \mu \) on \( I^d \) where \( I = [0, 1] \), a key property that \( \mu \) may, or may not, have is that the Fourier frequencies \( \{ e_n \}_{n \in \mathbb{N}^d} \) are total in \( L^2 (\mu) \), i.e., that the closed span of \( \{ e_n \}_{n \in \mathbb{N}^d} \) is \( L^2 (\mu) \).

The result in \( d = 1 \), that, if \( \nu \) on \( I \) is singular, then the set \( \{ e_n \}_{n \in \mathbb{N}} \) is total in \( L^2 (\nu) \), fails for \( d = 2 \). There are examples when \( \mu \) on \( I^2 \) is positive, singular w.r.t. the 2D Lebesgue measure, but \( \{ e_n \}_{n \in \mathbb{N}^2} \) is not total in \( L^2 (\mu) \).

Example 6.1. Take \( \mu = \lambda_1 \times \nu \) (see Figure 6.1), where \( \lambda_1 \) is Lebesgue measure and \( \nu \) is a singular measure in \( I \), then \( \{ e_n \}_{n \in \mathbb{N}^2} \) is not total in \( L^2 (\mu) \).

![Figure 6.1. \( \lambda_1 = \text{Lebesgue}, \, \nu \perp \lambda_1 \)](image)

For the Sierpinski case (affine IFS), with the Sierpinski measure \( \mu \), total does hold in \( L^2 (\mu) \). See details below.

Let the alphabets be
\[
B = \{ b_0, b_1, b_2 \} := \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.
\]

Set
\[
M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad \tau_j (x) = M^{-1} (x + b_j).
\]

The Sierpinski gasket (Figure 6.2) is the IFS attractor \( W \) satisfying
\[
W = \bigcup_{j=0}^{2} \tau_j (W).
\]

We have the random variable \( B^\mathbb{N} \xrightarrow{X} W \), given by
\[
\omega = (b_{i_1}, b_{i_2}, b_{i_3}, \cdots) \mapsto x = \sum_{k=1}^{\infty} M^{-k} b_{i_k}.
\]

As a Cantor set, \( W \) (the Sierpinski gasket) is the boundary of the tree symbol representation; see Figure 6.3.

Recall that every \( \omega \in B^\mathbb{N} \) is an infinite word \( \omega = (b_{i_1}, b_{i_2}, b_{i_3}, \cdots) \), with \( i_k \in \{0, 1, 2\} \). Setting \( \omega|_{n} = (b_{i_1}, \cdots, b_{i_n}) \), a finite truncated word, and \( \tau_{\omega|_{n}} = \tau_{i_n} \circ \cdots \circ \tau_{i_1} \); then \( \bigcap_{n} \tau_{\omega|_{n}} (W) = \{ x \} \), i.e., the intersection is a singleton. And we set \( X (\omega) = x \).

Let \( p \) be the probability distribution on \( B \), where
\[
p = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).
\]
Let $P = \mathbb{X}^\infty p$, and $\mu = P \circ X^{-1}$ be the corresponding IFS measure, i.e., $\mu$ is the unique Borel probability measure on $W$, s.t.

$$d\mu = \frac{1}{3} \sum_{j=0}^{2} \mu \circ \tau_j^{-1}. \quad (6.5)$$

See Section 5 for details.

**Remark 6.2.**

(i) The Hausdorff dimension of $W$ is $\ln 3/\ln 2$, where $3 = \# \{B\}$ and $2$ = scaling number.

(ii) Let $O_j$ be the triangles removed from the $j^{th}$ iteration (Figure 6.2), and let $O = \bigcup_{j=1}^{\infty} O_j$. Then,

$$\lambda_2 (O) = \lambda_2 \left( \bigcup_{j=1}^{\infty} O_j \right) = \frac{1}{2} \left[ \frac{1}{4} + \frac{3}{4^2} + \frac{3^2}{4^3} + \cdots \right] = \frac{1}{2},$$

and so $\lambda_2 (W) = 0$, where $\lambda_2$ denotes the 2D Lebesgue measure. Note that $\mu (W) = 1$.

**Lemma 6.3.** Let $W$ be the Sierpinski gasket, and $\mu$ be the corresponding IFS measure. Let $\hat{\mu}$ be the Fourier transform of $\mu$, i.e., $\hat{\mu} (\lambda) := \int_W e^{i2\pi \lambda x} d\mu (x)$. Then

$$\hat{\mu} (\lambda) = \frac{1}{3} \left[ 1 + e^{i\pi\lambda_1} + e^{i\pi\lambda_2} \right] \hat{\mu} (\lambda/2), \quad (6.6)$$
where \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \).

**Proof.** Immediate from (6.5). More specifically, we have

\[
\hat{\mu}(\lambda) = \frac{1}{3} \sum_{j=0}^{2} e^{i2\pi \lambda \cdot \tau_j(x)} d\mu(x) \\
= \frac{1}{3} \left( \int_W e^{i2\pi \lambda / 2(x+(0,1))} d\mu(x) + \int_W e^{i2\pi \lambda / 2(x+(0,1))} d\mu(x) \right) \\
= \frac{1}{3} \left( 1 + e^{i\pi \lambda_1} + e^{i\pi \lambda_2} \right) \hat{\mu}(\lambda / 2),
\]

which is the assertion (6.6). \(\square\)

By general theory (see Section 2), the IFS measure \(\mu\) as in (6.5) has a disintegration

\[
d\mu = \int_0^1 \sigma^x (dy) d\xi(x), \tag{6.7}
\]

where

\[
\xi = \mu \circ \pi_1^{-1} \tag{6.8}
\]

with \(\text{supp}(\xi) \subset [0,1]\). Note, if \(S \subset [0,1]\) is a measurable subset, then

\[
\xi(S) = \mu(\{(x,y) \mid x \in S\}). \tag{6.9}
\]

**Lemma 6.4.** Let \(W\) be the Sierpinski gasket. Then points in \(W\) are represented as random power series

\[
\left[ \begin{array}{c} x \\ y \end{array} \right] \in W \iff \left\{ \begin{array}{c} x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \\ y = \sum_{k=1}^{\infty} \eta_k 2^{-k} \end{array} \right. \tag{6.10}
\]

where \((\varepsilon_k), (\eta_k)\) are defined on \(\Omega = \{0,1\}^\mathbb{N}\), i.e., the binary probability space.

Moreover, \(\varepsilon_k\) is i.i.d. on \(\{0,1\}, k \in \mathbb{N}\), with distribution \((2/3, 1/3)\). That is, \(\text{Prob}(\varepsilon_k = 0) = 2/3\), and \(\text{Prob}(\varepsilon_k = 1) = 1/3\). The same conclusion holds for \(\eta_k\) as well.

**Proof.** This follows from (6.3) and (6.4).

In detail, let \(X : B^\mathbb{N} \to W\) be the random variable from (6.3), \(X(\omega) = \sum_{k=1}^{\infty} M^{-k} b_k\), for all \(\omega \in B^\mathbb{N}\); then

\[
W \ni \left[ \begin{array}{c} x \\ y \end{array} \right] = X(\omega) = \sum_{k=1}^{\infty} \left[ \begin{array}{c} 2^{-k} \\ 0 \\ 2^{-k} \end{array} \right] b_k \\
= \sum_{k=1}^{\infty} \left[ \begin{array}{c} 2^{-k} \\ 0 \\ 2^{-k} \end{array} \right] \left\{ \begin{array}{c} [0] \\ [0] \\ [1] \end{array} \right\} \\
= \left[ \begin{array}{c} \sum_{k=1}^{\infty} 2^{-k} \varepsilon_k (x) \\ \sum_{k=1}^{\infty} 2^{-k} \eta_k (x) \end{array} \right],
\]

where

\[
\text{Pr}(\varepsilon_k = 0) = \text{Pr}(\eta_k = 0) = 2/3, \\
\text{Pr}(\varepsilon_k = 1) = \text{Pr}(\eta_k = 1) = 1/3.
\]
Also we have the following conditional probabilities:

\[ \Pr(\eta_k = 0 | \varepsilon_k = 0) = \frac{1}{2}, \]
\[ \Pr(\eta_k = 1 | \varepsilon_k = 0) = \frac{1}{2}, \]
\[ \Pr(\eta_k = 0 | \varepsilon_k = 1) = 1. \]

One checks that

\[ \Pr(\eta_k = 0) = \Pr(\eta_k = 0 | \varepsilon_k = 0) \Pr(\varepsilon_k = 0) + \Pr(\eta_k = 0 | \varepsilon_k = 1) \Pr(\varepsilon_k = 1) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{2}{3}, \]
\[ (6.11) \]
\[ \Pr(\eta_k = 1) = \Pr(\eta_k = 1 | \varepsilon_k = 0) \Pr(\varepsilon_k = 0) + \Pr(\eta_k = 1 | \varepsilon_k = 1) \Pr(\varepsilon_k = 1) = \frac{1}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}. \]
\[ (6.12) \]

See the diagram in Figure 6.4.

Lemma 6.5. Let \( \mu \) be the IFS measure of the Sierpinski gasket as above, and \( \xi = \mu \circ \pi_1^{-1} \) be as in (6.7)–(6.9), so that \( \mu \) has the disintegration in (6.7).

(i) Then the measure \( \xi \) is singular and non-atomic. More precisely, \( \xi \) is the product measure \( \chi_1^{\infty} \{2/3, 1/3\} \) defined on \( \{0, 1\}^\mathbb{N} \).

(ii) For a.a. \( x \) w.r.t \( \xi \), the measure \( \sigma^x(dy) \) (in the \( y \)-variable) is singular. Hence \( \mu \) is slice singular (see Definition 2.3), and \( \{e_n\}_{n \in \mathbb{N}^2} \) is total in \( L^2(\mu) \).

Proof. For all points \( (x, y) \in W \), let \( x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}, y = \sum_{k=1}^{\infty} \eta_k 2^{-k} \) be as in (6.10). Then (6.11) & (6.12) hold for all \( x \in I \).

Therefore, we get the product measure \( \xi = \chi_1^{\infty} \{2/3, 1/3\} \) on the space \( \Omega = \chi_1^{\infty} \{0, 1\} \); see Figure 6.5. By contrast, \( \lambda = \chi_1^{\infty} \{1/2, 1/2\} \) is Lebesgue measure; hence \( \xi \) and \( \lambda \) are mutually singular by Kakutani’s theorem. (See Corollary 5.3 above.)

Note that, for a.a. \( x \), the measure \( \sigma^x(dy) \) is the middle interval gap supported on \( A(x) = \{y \mid (x, y) \in W\} \), and we conclude that \( \sigma^x(dy) \) is singular w.r.t. Lebesgue measure for a.a. \( x \). By Theorem 2.4, it follows that \( \{e_n\}_{n \in \mathbb{N}^2} \) is total in \( L^2(\mu) \). □

Remark 6.6. There is a Markov chain associated with the transition probabilities (see Figure 6.4). Note that

\[
\begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix},
\]
The measure $\xi$, or $d\xi(x)$ as an infinite product measure.

so the conditional expectation can be expressed as a Perron-Frobenius problem with the row vector $\left[\frac{2}{3} \ 1/3\right]$ as a left Perron-Frobenius vector.

As another example, consider the fractal Eiffel Tower $W_{Ei}$ (see Figure 6.6). In this case, we have

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\},$$

and $p = (1/4, 1/4, 1/4, 1/4)$. It follows that each coordinate of points in $W_{Ei}$ has representation $\sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $\{\varepsilon_k\}$ is i.i.d. with $Pr(\varepsilon_k = 0) = 3/4$, and $Pr(\varepsilon_k = 1) = 1/4$. The transition probabilities are given by the diagram below.

One checks that

$$\begin{bmatrix} 3/4 & 1/4 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix}. $$

Conjecture 6.7. Given an affine contractive IFS measure $\mu$ supported in $[0,1]^d$, let $T = (T_{ij})$ be the corresponding Markov transition matrix. Then the following are equivalent:

(i) The Fourier frequencies $\{e_n\}_{n \in \mathbb{N}^d}$ are total in $L^2(\mu)$.

(ii) The Perron-Frobenius vector $v$ ($vT = v$, or $\sum_j v_j T_{ji} = v_i$) is non-constant, i.e., not proportional to $(1,1,\cdots,1)$.

Remark 6.8 (The Sierpinski carpet). In the above, we carried out all the detailed computation justifying our conclusions for the case of the Sierpinski gasket, Figure 2.1 (A). Recall that Figure 2.1 (B) represents the Sierpinski carpet, a close cousin; and the reader will be able to fill in details from inside the section, spelling out the changes from (A) to (B). In case (B), naturally, the particular affine transformations
\( \xi = \mu \circ \pi_1^{-1} = \chi_1 \{ 3/4, 1/4 \} \)

\[ Pr(\varepsilon_k = 0) = 3/4, \quad Pr(\varepsilon_k = 1) = 1/4 \]

Figure 6.6. Construction of the fractal Eiffel Tower.

(6.1)–(6.2) are a bit different (i.e., for case (B)), but they are of the same nature. In particular, it follows that the maximal entropy (IFS) measure for the Sierpinski carpet is also slice-singular. Moreover, the other conclusions from Lemmas 6.4, and 6.5, and Remark 6.6, carry over from case (A) to case (B), \textit{mutatis mutandis}. As the underlying ideas and methods involved are the same, interested readers will be able to fill in details.

Moreover the above remarks, regarding extension of the conclusions for case (A) to that of (B), also apply \textit{mutatis mutandis}, to the case of Figure 6.6, the fractal Eiffel Tower. There again, we conclude that the associated maximal entropy (IFS) measure is also slice-singular.

\textit{Acknowledgement.} The co-authors thank the following colleagues for helpful and enlightening discussions: Professors Daniel Alpay, Sergii Bezuglyi, Ilwoo Cho, Wayne Polyzou, Eric S. Weber, and members in the Math Physics seminar at The University of Iowa.

\textbf{References}

[AJL18] Daniel Alpay, Palle Jorgensen, and Izchak Lewkowicz, \textit{W-Markov measures, transfer operators, wavelets and multiresolutions}, Frames and harmonic analysis, Contemp. Math., vol. 706, Amer. Math. Soc., Providence, RI, 2018, pp. 293–343. MR 3796644

[Aro50] N. Aronszajn, \textit{Theory of reproducing kernels}, Trans. Amer. Math. Soc. \textbf{68} (1950), 337–404. MR 0051437

[BCKL17] Travis Bemrose, Peter G. Casazza, Victor Kaftal, and Richard G. Lynch, \textit{The unconditional constants for Hilbert space frame expansions}, Linear Algebra Appl. \textbf{521} (2017), 1–18. MR 3611473

[BHS08] Michael F. Barnsley, John E. Hutchinson, and Örjan Stenflo, \textit{V-variable fractals: fractals with partial self similarity}, Adv. Math. \textbf{218} (2008), no. 6, 2051–2088. MR 2431670

[BS13] Anton Baranov and Donald Sarason, \textit{Quotient representations of inner functions}, Recent trends in analysis, Theta Ser. Adv. Math., vol. 16, Theta, Bucharest, 2013, pp. 35–46. MR 3411042

[CC18] Yuxin Chen and Emmanuel J. Candès, \textit{The projected power method: an efficient algorithm for joint alignment from pairwise differences}, Comm. Pure Appl. Math. \textbf{71} (2018), no. 8, 1648–1714. MR 3847751

[CESV15] Emmanuel J. Candès, Yonina C. Eldar, Thomas Strohmer, and Vladislav Voroninski, \textit{Phase retrieval via matrix completion [reprint of MR3032952]}, SIAM Rev. \textbf{57} (2015), no. 2, 225–251. MR 3345342

[CH18] Peter G. Casazza and John I. Haas, IV, \textit{On Grassmannian frames with spectral constraints}, Sampl. Theory Signal Image Process. \textbf{17} (2018), no. 1, 17–28. MR 3817340
[Che18] Xuemei Chen, The Kaczmarz algorithm, row action methods, and statistical learning algorithms, Frames and harmonic analysis, Contemp. Math., vol. 706, Amer. Math. Soc., Providence, RI, 2018, pp. 115–127. MR 3796634

[CK04] Peter G. Casazza and Gitta Kutyniok, Frames of subspaces, Wavelets, frames and operator theory, Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113. MR 2066823

[CK08] ________, Robustness of fusion frames under erasures of subspaces and of local frame vectors, Radon transforms, geometry, and wavelets, Contemp. Math., vol. 464, Amer. Math. Soc., Providence, RI, 2008, pp. 149–160. MR 2440135

[CKL08] Peter G. Casazza, Gitta Kutyniok, and Shidong Li, Fusion frames and distributed processing, Appl. Comput. Harmon. Anal. 25 (2008), no. 1, 114–132. MR 2419707

[CT13] Wojciech Czaja and James H. Tanis, Kaczmarz algorithm and frames, Int. J. Wavelets Multiresolut. Inf. Process. 11 (2013), no. 5, 1350036, 13. MR 3117886

[DJ07] Dorin Ervin Dutkay and Palle E. T. Jorgensen, Fourier frequencies in affine iterated function systems, J. Funct. Anal. 247 (2007), no. 1, 110–137. MR 2319756

[EN11] Yonina C. Eldar and Deanna Needell, Acceleration of randomized Kaczmarz method via the Johnson-Lindenstrauss lemma, Numer. Algorithms 58 (2011), no. 2, 163–177. MR 2835851

[EO13] Martin Ehler and Kasso A. Okoudjou, Probabilistic frames: an overview, Finite frames, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2013, pp. 415–436. MR 2964017

[EP01] David J. Evans and Constantin Popa, Projections and preconditioning for inconsistent least-squares problems, Int. J. Comput. Math. 78 (2001), no. 4, 599–616. MR 1898684

[FJKO05] Matthew Fickus, Brody D. Johnson, Keri Kornelson, and Kasso A. Okoudjou, Convolutional frames and the frame potential, Appl. Comput. Harmon. Anal. 19 (2005), no. 1, 77–91. MR 2147063

[FL19] Hartmut Führ and Jakob Lemvig, System bandwidth and the existence of generalized shift-invariant frames, J. Funct. Anal. 276 (2019), no. 2, 563–601. MR 3906284

[Fri05] Benjamin Friedlander, A subspace method for space time adaptive processing, IEEE Trans. Signal Process. 53 (2005), no. 1, 74–82. MR 2113344

[HH19] Jie Huang and Ting-Zhu Huang, A nonstationary accelerating alternating direction method for frame-based Poissonian image deblurring, J. Comput. Appl. Math. 352 (2019), 181–193. MR 3894265

[Hid80] Takeyuki Hida, Brownian motion, Applications of Mathematics, vol. 11, Springer-Verlag, New York-Berlin, 1980, Translated from the Japanese by the author and T. P. Speed. MR 562914

[HJW16] John E. Herr, Palle E. T. Jorgensen, and Eric S. Weber, Positive Matrices in the Hardy Space with Prescribed Boundary Representations via the Kaczmarz Algorithm, ArXiv e-prints (2016), arXiv:1603.08852.

[HJW18a] John E. Herr, Palle E. T. Jorgensen, and Eric S. Weber, A matrix characterization of boundary representations of positive matrices in the Hardy space, Frames and harmonic analysis, Contemp. Math., vol. 706, Amer. Math. Soc., Providence, RI, 2018, pp. 255–270. MR 3796641

[HJW18b] John E. Herr, Palle E.T. Jorgensen, and Eric S. Weber, A characterization of boundary representations of positive matrices in the hardy space via the abel product, Linear Algebra and its Applications (2018).

[HKLW07] Deguang Han, Keri Kornelson, David Larson, and Eric Weber, Frames for undergraduates, Student Mathematical Library, vol. 40, American Mathematical Society, Providence, RI, 2007. MR 2367342

[LLL18] Deguang Han, David R. Larson, and Rui Liu, Dilations of operator-valued measures with bounded p-variations and framings on Banach spaces, J. Funct. Anal. 274 (2018), no. 5, 1466–1490. MR 3778680

[HLS15] Ryan Hotovy, David R. Larson, and Sam Scholze, Binary frames, Houston J. Math. 41 (2015), no. 3, 875–899. MR 3423689

[HS05] Rainis Haller and Ryszard Szwarc, Kaczmarz algorithm in Hilbert space, Studia Math. 169 (2005), no. 2, 123–132. MR 2140451

[Hut81] John E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981), no. 5, 713–747. MR 625600
[Hut95], Fractals: a mathematical framework, Complex. Int. 2 (1995), 14 HTML documents. MR 1656855

[IZ13] Andrey Aleksandrovich Ivanov and Aleksandr Ivanovich Zhdanov, Kaczmarz algorithm for Tikhonov regularization problem, Appl. Math. E-Notes 13 (2013), 270–276. MR 3159297

[JS07] Palle E. T. Jorgensen and Myung-Sin Song, Entropy encoding, Hilbert space, and Karhunen-Loève transforms, J. Math. Phys. 48 (2007), no. 10, 103503, 22. MR 2362796

[JS18a] Palle Jorgensen and Feng Tian, Infinite-dimensional measure spaces and frame analysis, Acta Appl. Math. 155 (2018), 41–56. MR 3800275

[JS18b] Palle Jorgensen and Feng Tian, Markov chains and generalized wavelet multiresolutions, J. Anal. 26 (2018), no. 2, 259–283. MR 3882025

[JT17] Palle Jorgensen and Feng Tian, Non-commutative analysis, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, With a foreword by Wayne Polyzou. MR 3642406

[KA19] M. M. Khader and M. Adel, Introducing the windowed Fourier frames technique for obtaining the approximate solution of the coupled system of differential equations, J. Pseudo-Differ. Oper. Appl. 10 (2019), no. 1, 241–256. MR 3910931

[Kac37] S. Kaczmarz, Angenäherte Auflösung von Systemen linearer Gleichungen, Bulletin International de l’Académie Polonaise des Sciences et des Lettres 35 (1937), 355–357.

[Kak43] Shizuo Kakutani, Notes on infinite product measure spaces. II, Proc. Imp. Acad. Tokyo 19 (1943), 184–188. MR 0014404

[Kak48] Shizuo Kakutani, On equivalence of infinite product measures, Ann. of Math. (2) 49 (1948), 214–224. MR 0023331

[KL19] Victor Kaftal and David R. Larson, Admissible sequences of positive operators, Trans. Amer. Math. Soc. 371 (2019), no. 5, 3721–3742. MR 3896128

[KM06] Stanisław Kwapien and Jan Mycielski, Erratum to the paper: “On the Kaczmarz algorithm of approximation in infinite-dimensional spaces” [Studia Math. 148 (2001), no. 1, 75–86; mr1881441], Studia Math. 176 (2006), no. 1, 93. MR 2263965

[Kol83] A. N. Kolmogorov, On logical foundations of probability theory, Probability theory and mathematical statistics (Tbilisi, 1982), Lecture Notes in Math., vol. 1021, Springer, Berlin, 1983, pp. 1–5. MR 735967

[LS15] Shuyang Ling and Thomas Strohmer, Self-calibration and biconvex compressive sensing, Inverse Problems 31 (2015), no. 11, 115002, 31. MR 3424852

[LZ15] Junhong Lin and Ding-Xuan Zhou, Learning theory of randomized Kaczmarz algorithm, J. Mach. Learn. Res. 16 (2015), 3341–3365. MR 3450541

[NSW16] Deanna Needell, Nathan Srebro, and Rachel Ward, Stochastic gradient descent, weighted sampling, and the randomized Kaczmarz algorithm, Math. Program. 155 (2016), no. 1–2, Ser. A, 549–573. MR 3439812

[Oko16] Kasso O. Okoudjou, Preconditioning techniques in frame theory and probabilistic frames, Finite frame theory, Proc. Sympos. Appl. Math., vol. 73, Amer. Math. Soc., Providence, RI, 2016, pp. 105–142. MR 3526434

[Pop01] C. Popa, Oblique projections as preconditioner in Kaczmarz-like algorithms, Proceedings of the Ninth Symposium of Mathematics and its Applications, Rom. Acad., Timișoara, 2001, pp. 118–122. MR 2208766

[Pop10] Constantin Popa, A hybrid Kaczmarz-conjugate gradient algorithm for image reconstruction, Math. Comput. Simulation 80 (2010), no. 12, 2272–2285. MR 2721177

[Pop18] C. Popa, Convergence rates for Kaczmarz-type algorithms, Numer. Algorithms 79 (2018), no. 1, 1–17. MR 3846956

[RT71] David Ruelle and Floris Takens, On the nature of turbulence, Comm. Math. Phys. 20 (1971), 167–192. MR 0284067

[Rue79] D. Ruelle, Analyticity properties of the characteristic exponents of random matrix products, Adv. in Math. 32 (1979), no. 1, 68–80. MR 534172

[Rue82] David Ruelle, Characteristic exponents and invariant manifolds in Hilbert space, Ann. of Math. (2) 115 (1982), no. 2, 243–290. MR 647807

[Rue04] D. Ruelle, Thermodynamic formalism, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004, The mathematical structures of equilibrium statistical mechanics. MR 2129258
[Sar94] Donald Sarason, Sub-Hardy Hilbert spaces in the unit disk, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, John Wiley & Sons, Inc., New York, 1994, A Wiley-Interscience Publication. MR 1289670

[SEC14] Mahdi Soltanolkotabi, Ehsan Elhamifar, and Emmanuel J. Candès, Robust subspace clustering, Ann. Statist. 42 (2014), no. 2, 669–699. MR 3210983

[SV09] Thomas Strohmer and Roman Vershynin, A randomized Kaczmarz algorithm with exponential convergence, J. Fourier Anal. Appl. 15 (2009), no. 2, 262–278. MR 250924

[Szw07] Ryszard Szwarc, Kaczmarz algorithm in Hilbert space and tight frames, Appl. Comput. Harmon. Anal. 22 (2007), no. 3, 382–385. MR 2311862

[WO17] Clare Wickman and Kasso Okoudjou, Duality and geodesics for probabilistic frames, Linear Algebra Appl. 532 (2017), 198–221. MR 3688637

[Zha19] Jian-Jun Zhang, A new greedy Kaczmarz algorithm for the solution of very large linear systems, Appl. Math. Lett. 91 (2019), 207–212. MR 3896982

(Palle E.T. Jorgensen) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, U.S.A.
E-mail address: palle-jorgensen@uiowa.edu
URL: http://www.math.uiowa.edu/~jorgen/

(Myung-Sin Song) Department of Mathematics, Southern Illinois University Edwardsville, Edwardsville, IL 62026, U.S.A.
E-mail address: msong@siue.edu
URL: http://www.siue.edu/~msong/

(Feng Tian) Department of Mathematics, Hampton University, Hampton, VA 23668, U.S.A.
E-mail address: feng.tian@hamptonu.edu