Reflected Discontinuous Backward Doubly Stochastic Differential Equation With Poisson Jumps.

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Abstract. In this paper we prove the existence of a solution for reflected backward doubly stochastic differential equations with poisson jumps (RBDSDEPs) with one continuous barrier where the generator is continuous and also we study the RBDSDEPs with a linear growth condition and left continuity in $y$ on the generator. By a comparison theorem established here for this type of equation we provide a minimal or a maximal solution to RBDSDEPs.

Keyword Reflected Backward doubly stochastic differential equations, random Poisson measure, minimal solution, comparison theorem, discontinuous generator.

1 Introduction.

A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [11] in 1994, which is a class of backward doubly stochastic differential equations (BDSDEs for short)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\widehat{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where $\xi$ is a random variable termed the terminal condition, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are two jointly measurable processes, $W$ and $B$ are two mutually independent standard Brownian motion, with values, respectively in $\mathbb{R}^d$ and $\mathbb{R}$. Several authors interested in weakening this assumption see Bahlali et al [3], Boufoussi et al. [5], Lin. Q [8] and [9], N’zi et al. [10], Shi et al. [12], Wu et al. [14], Zhu et al. [16]. A class of backward doubly stochastic differential equations with jumps was study by Sun el al. [13], Zhu et al. [15] They have proved the existence and uniqueness of solutions for this type of BDSDEs under uniformly Lipschitz conditions.

In addition, Bahlali et al [2] prove the existence and uniqueness of solutions to reflected backward doubly stochastic differential equations (RBDSDEs) with one continuous barrier and uniformly Lipschitz coefficients. The existence of a maximal and a minimal solution for RBDSDEs with continuous generator is also established.

In this paper, we study the now well-know reflected backward doubly stochastic differential equations with jumps (RBDSDEPs for short):

$$Y_t = \xi + \int_t^T f(s, \Lambda_s)ds + \int_t^T g(s, \Lambda_s)d\widehat{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T \int_E U_s (\epsilon) \tilde{\mu} (d\epsilon, ds), \quad 0 \leq t \leq T,$$

where $\Lambda_s = (Y_s, Z_s, U_s)$.

Motivated by the above results and by the result introduced by Fan. X, Ren. Y [6] and Zhu, Q., Shi, Y [15, 16], we establish firstly the existence of the solution of the reflected BDSDE with Poisson jumps (RBDSDE in short) under the continuous coefficient, also we prove the existence solution of a RBDSDEP where the coefficient $f$ satisfy a linear growth and left continuity in $y$ conditions on the generator of this type of equation.

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The organization of the paper is as follows. In Section 2, we give some preliminaires and we consider the spaces of processus also we define the Itô’s formula. In Section 3, we proof a comparison theorem, section 4 under a continuous conditions on \( f \) we obtain the existence of a minimal solution of RBDSDEP, and finally in section 5, we study RBDSDEP where the generator \( f \) satisfied a left continuity in \( y \) and linear growth conditions.

2 Notation, assumption and definition.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. For \( T > 0 \), We suppose that \((\mathcal{F}_t)_{t \geq 0}\) is generated by the following three mutually independent processes:

(i) Let \( \{W_t, 0 \leq t \leq T\} \) and \( \{B_t, 0 \leq t \leq T\} \) be two standard Brownian motion defined on \((\Omega, \mathcal{F}, P)\) with values in \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively.

(ii) Let random Poisson measure \( \mu \) on \( E \times \mathbb{R}_+ \) with compensator \( \nu (dt, dc) = \lambda (dc) dt \), where the space \( E = \mathbb{R} - \{0\} \) is equipped with its Borel field \( \mathcal{E} \) such that \( \{\bar{\mu}(\{0\} \times A) = (\mu - \nu)(\{0\} \times A)\} \) is a martingale for any \( A \in \mathcal{E} \) satisfying \( \lambda (A) < \infty \). \( \lambda \) is a \( \sigma \)-finite measure on \( \mathcal{E} \) and satisfies \( \int_E \left( 1 \wedge |e|^2 \right) \lambda (de) < \infty \).

Let \( \mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t) \), \( \mathcal{F}_t^\mu := \sigma(\mu_s; 0 \leq s \leq t) \) and \( \mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T) \), completed with \( P \)-null sets. We put, \( \mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^\mu \vee \mathcal{F}_{t,T}^B \). It should be noted that \((\mathcal{F}_t)\) is not an increasing family of \( \sigma \)-fields, and hence it is not a filtration.

For \( d \in \mathbb{N}^* \), \( |\cdot| \) stands for the Euclidian norm in \( \mathbb{R}^d \) \( \times \{0\} \).

We consider the following spaces of processus:

- We denote by \( S^2(0, T, \mathbb{R}^d) \), the set of continuous \( \mathcal{F}_t \)-measurable processes \( \{\varphi_t; t \in [0, T]\} \), which satisfy \( \mathbb{E} (\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty \).

- Let \( \mathcal{M}^2(0, T, \mathbb{R}^d) \) denote the set of \( d \)-dimensional, \( \mathcal{F}_t \)-measurable processes \( \{\varphi_t; t \in [0, T]\} \), such that \( \mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty \).

- \( \mathcal{A}^2 \) set of continuous, increasing, \( \mathcal{F}_t \)-measurable process \( K: [0, T] \times \Omega \rightarrow [0, +\infty) \) with \( K_0 = 0, \mathbb{E}(K_T)^2 < +\infty \).

- \( L^2 \) set of \( \mathcal{F}_T \)-measurable random variables \( \xi: \Omega \rightarrow \mathbb{R} \) with \( \mathbb{E} |\xi|^2 < +\infty \).

- We denote by \( \mathcal{L}^2(0, T, \bar{\mu}, \mathbb{R}^d) \), the space of mappings \( U: \Omega \times [0, T] \times E \rightarrow \mathbb{R}^d \) which are \( \mathcal{P} \otimes \mathcal{E} \) measurable such that

\[
\|U_t\|^2_{\mathcal{L}^2(0, T, \bar{\mu}, \mathbb{R}^d)} = \mathbb{E} \int_0^T \|U_t\|^2_{\mathcal{L}^2(E, \mathcal{F}, \lambda, \mathbb{R}^d)} dt < \infty,
\]

where \( \mathcal{P} \otimes \mathcal{E} \) denoted the \( \sigma \)-algebra of \( \mathcal{F}_t \)-predictable sets of \( \Omega \times [0, T] \) and

\[
\|U_t\|^2_{\mathcal{L}^2(E, \mathcal{F}, \lambda, \mathbb{R}^d)} = \int_E |U_t(e)|^2 \lambda (de).
\]

- Notice also the space \( \mathcal{D}^2(\mathbb{R}) = S^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{L}^2(0, T, \bar{\mu}, \mathbb{R}) \times \mathcal{A}^2 \) endowed with the norm

\[
||\langle Y, Z, U, K\rangle||_{\mathcal{D}^2} = ||Y||_{\mathcal{L}^2} + ||Z||_{\mathcal{M}^2} + ||U||_{\mathcal{L}^2} + ||K||_{\mathcal{A}^2}.
\]

is a Banach space.
Definition 2.1. A solution of a reflected BDSDEs is a quadruple of processes \((Y, Z, K, U)\) which satisfies

\[
\begin{align*}
&i) Y \in S^2 \left(0, T, \mathbb{R} \right), \ Z \in \mathcal{M}^2 \left(0, T, \mathbb{R}^d \right), K \in \mathcal{A}^2, U \in \mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R} \right), \\
&ii) Y_t = \xi + \int_t^T \mathcal{f}(s, Y_s, Z_s, U_s) ds + \int_t^T \mathcal{g}(s, Y_s, Z_s, U_s) d\mathcal{B}_s \\
&\quad + \int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \bar{\mu}(ds, de), \quad 0 \leq t \leq T, \\
&iii) S_t \leq Y_t, \quad 0 \leq t \leq T \quad \text{and} \quad \int_0^T (Y_t - S_t) dK_t = 0.
\end{align*}
\]

We give the following assumptions \((H)\) on the data \((\xi, \mathcal{f}, \mathcal{g}, S)\):

\((H.1)\) \(f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R} \right) \to \mathbb{R}; \ g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R} \right) \to \mathbb{R}\) be jointly measurable such that for any \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R} \right)\)

\[
\begin{align*}
\mathcal{f}(\cdot, \omega, y, z, u) &\in \mathcal{M}^2 \left(0, T, \mathbb{R} \right), \\
\mathcal{g}(\cdot, \omega, y, z, u) &\in \mathcal{M}^2 \left(0, T, \mathbb{R} \right).
\end{align*}
\]

\((H.2)\) There exist constant \(C > 0\) and a constant \(0 < \alpha < 1\) such that for every \((\omega, t) \in \Omega \times [0, T]\) and \((y, y') \in \mathbb{R}^2, \ (z, z') \in (\mathbb{R}^d)^2, \ (u, u') \in (\mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R} \right))^2\)

\[
\begin{align*}
(i) \quad &\left| \mathcal{f}(t, \omega, y, z, u) - \mathcal{f}(t, \omega, y', z', u') \right|^2 \leq C \left[ |y - y'|^2 + |z - z'|^2 + |u - u'|^2 \right], \\
(ii) \quad &\left| \mathcal{g}(t, \omega, y, z, u) - \mathcal{g}(t, \omega, y', z', u') \right|^2 \leq C \left[ |y - y'|^2 + \alpha \left( |z - z'|^2 + |u - u'|^2 \right) \right].
\end{align*}
\]

\((H.3)\) The terminal value \(\xi\) be a given random variable in \(L^2\).

\((H.4)\) \((S_t)_{t \geq 0}\), is a continuous progressively measurable real valued process satisfying

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left( S^+_t \right)^2 \right) < +\infty, \quad \text{where} \quad S^+_t := \max(S_t, 0).
\]

\((H.5)\) \(S_T \leq \xi, \ \mathbb{P}\)-almost surely.

Theorem 2.1. \([6]\) Assume that \((H.1) - (H.5)\) holds. Then Eq (1.1) admits a unique solution \((Y, Z, U, K) \in \mathcal{D}^2 \left(\mathbb{R} \right)\).

The result depends on the following extension of the well-known Itô’s formula. Its proof follows the same way as lemma 1.3 of [11]

Lemma 2.1. Let \(\alpha \in S^2 \left(0, T, \mathbb{R}^k \right), \ (\beta, \gamma) \in \mathcal{M}^2 \left(\mathbb{R}^k \right)^2, \ \eta \in \mathcal{M}^2 \left(\mathbb{R}^{k \times d} \right)\) and \(\sigma \in \mathcal{L}^2 \left(0, T, \bar{\mu}, \mathbb{R}^k \right)\) such that:

\[
\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \eta_s dW_s + \int_0^t dK_s + \int_0^t \int_E \sigma_s(e) \bar{\mu}(ds, de),
\]

then \((i)\)

\[
|\alpha_t|^2 = |\alpha_0|^2 + 2 \int_0^t \langle \alpha_s, \beta_s \rangle ds + 2 \int_0^t \langle \alpha_s, \gamma_s \rangle dB_s + 2 \int_0^t \langle \alpha_s, \eta_s \rangle dW_s + 2 \int_0^t \langle \alpha_s, dK_s \rangle \\
+ 2 \int_0^t \int_E \langle \alpha_s, \sigma(s) \bar{\mu}(ds, de) \rangle - \int_0^t |\gamma_s|^2 ds + \int_0^t |\eta_s|^2 ds + \int_0^t \int_E |\sigma_s(e)|^2 \lambda(de) ds \\
+ \sum_{0 \leq t \leq T} (\Delta \alpha_s)^2.
\]

\((ii)\)

\[
\mathbb{E} |\alpha_t|^2 + \mathbb{E} \int_t^T |\eta_s|^2 ds + \mathbb{E} \int_t^T \int_E |\sigma_s(e)|^2 \lambda(de) ds \leq \mathbb{E} |\alpha_T|^2 + 2 \mathbb{E} \int_t^T \langle \alpha_s, \beta_s \rangle ds + 2 \mathbb{E} \int_t^T \langle \alpha_s, \eta_s \rangle dW_s + 2 \mathbb{E} \int_t^T \langle \alpha_s, dK_s \rangle + \mathbb{E} \int_t^T |\gamma_s|^2 ds.
\]
3 Comparison theorem.

Given two parameters \((\xi^1, f^1, g, T)\) and \((\xi^2, f^2, g, T)\), we consider the reflected BDSDEPs, \(i = 1, 2\)

\[
Y_t^i = \xi^i + \int_t^T f^1(s, Y_s^i, Z_s^i, U_s^i) ds + \int_t^T g(s, Y_s^i, Z_s^i, U_s^i) d\tilde{B}_s \\
+ \int_t^T dK_s^i - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s^i(\epsilon) \mu(\epsilon, ds) , \quad 0 \leq t \leq T. \tag{3.1}
\]

**Theorem 3.1.** Assume that the reflected BDSDEP associated with dates \((\xi^1, f^1, g, T), (resp (\xi^2, f^2, g, T))\)

has a solution \((Y_t^1, Z_t^1, K_t^1, U_t^1)_{t \in [0,T]}\), (resp \((Y_t^2, Z_t^2, K_t^2, U_t^2)_{t \in [0,T]}\)). Each one satisfying the assumption \(\text{(H)}\), assume moreover that:

\[
\begin{align*}
\xi^1 &\leq \xi^2, \\
\forall t \leq T, \ S_t^1 &\leq S_t^2, \\
f^1(t, Y_t, Z_t, U_t) &\leq f^2(t, Y_t, Z_t, U_t).
\end{align*}
\]

Then we have \(P \text{- a.s.}, \quad Y_t^1 \leq Y_t^2.\)

**Proof:** Let us show that \((Y_t^1 - Y_t^2)^+ = 0\), using the equation (3.1), we get

\[
\begin{align*}
\tilde{Y}_t = Y_t^1 - Y_t^2 = &\quad \tilde{\xi} + \int_t^T (f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)) ds + \int_t^T (g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)) d\tilde{B}_s \\
&+ \int_t^T (dK_s^1 - dK_s^2) - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(\epsilon) \lambda(\epsilon, ds) ,
\end{align*}
\]

where \(\tilde{\xi} = \xi^1 - \xi^2\), \(\tilde{Z} = Z^1 - Z^2\) and \(\tilde{U} = U^1 - U^2\).

Since \(\int_t^T \tilde{Y}_s^+ (g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)) d\tilde{B}_s\) and \(\int_t^T \tilde{Y}_s^+ \tilde{Z}_s dW_s\) are uniformly integrable martingale then taking expectation, we get by applying Lemma 2.1

\[
\begin{align*}
\mathbb{E} \left| \tilde{Y}_t \right|^2 &\leq \mathbb{E} \left| \tilde{\xi} \right|^2 + 2\mathbb{E} \int_t^T \tilde{Y}_s^+ (f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)) ds \\
&+ 2\mathbb{E} \int_t^T \tilde{Y}_s^+ (dK_s^1 - dK_s^2) + \mathbb{E} \int_t^T \int_E \left| U_s(\epsilon) \right|^2 \lambda(\epsilon, ds) ds.
\end{align*}
\]

Since

\[
\begin{align*}
\int_t^T (\tilde{Y}_s^+)^2 &\leq 0, \\
\int_t^T \tilde{Y}_s^+ (dK_s^1 - dK_s^2) &\leq 0,
\end{align*}
\]

we get

\[
\begin{align*}
\mathbb{E} \left| \tilde{Y}_t \right|^2 &\leq \mathbb{E} \int_t^T \left| \tilde{Z}_s \right|^2 ds + \int_t^T \int_E \left| U_s(\epsilon) \right|^2 \tilde{\mu}(\epsilon, ds) ds \\
&\leq 2\mathbb{E} \int_t^T \tilde{Y}_s^+ (f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)) ds \\
&+ \mathbb{E} \int_t^T \int_E \left| U_s(\epsilon) \right|^2 \left| g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2) \right|^2 ds,
\end{align*}
\]
we obtain, by hypothesis (H.2), and Young’s inequality the following inequality

\[ \begin{align*}
2E \int_0^T (\hat{Y}_s^+)^2 + (f^1(s,Y^1_s,Z^1_s,U^1_s) - f^2(s,Y^2_s,Z^2_s,U^2_s)) ds \\
\leq (2C + 2C^2 \epsilon)E \int_0^T |\hat{Y}_s^+|^2 ds + \epsilon^{-1}E \int_0^T \left( |\hat{Z}_s|^2 + \int_E |\bar{U}_s|^2 \lambda(de) \right) ds,
\end{align*} \]

also we applying the assumption (H.2) for g, we get

\[ ||g(s,Y^1_s,Z^1_s,U^1_s) - g(s,Y^2_s,Z^2_s,U^2_s)||^2 \leq C |\bar{Y}_s|^2 ds + \alpha \left\{ |\bar{Z}_s|^2 + |\bar{U}_s|^2 \right\}. \]

Then, we have the following inequality

\[ \begin{align*}
& \mathbb{E} \left\{ \left| (\hat{Y}_t^+)^2 + \int_t^T 1_{\{\bar{Y}_s > 0\}} |\hat{Z}_s|^2 ds + \int_t^T \int_E 1_{\{\bar{Y}_s > 0\}} |\bar{U}_s(\epsilon)|^2 \bar{\mu}(de) ds \right\} \\
& \leq (2C + 2C^2 \epsilon) \mathbb{E} \int_t^T |\hat{Y}_s^+|^2 ds + \epsilon^{-1} \mathbb{E} \int_t^T \left( |\hat{Z}_s|^2 + \int_E |\bar{U}_s|^2 \lambda(de) \right) ds \\
& \quad + C \mathbb{E} \int_t^T 1_{\{\bar{Y}_s > 0\}} |\bar{Y}_s|^2 ds + \alpha \mathbb{E} \int_t^T 1_{\{\bar{Y}_s > 0\}} |\bar{Z}_s|^2 + \int_E 1_{\{\bar{Y}_s > 0\}} |\bar{U}_s|^2 \lambda(de) ds,
\end{align*} \]

choosing \( \epsilon \) such that \( 0 < \epsilon^{-1} + \alpha \leq 1 \), we have

\[ \mathbb{E} \left| (\hat{Y}_t^+)^2 \right| \leq (3C + 2C^2 \epsilon) \mathbb{E} \int_t^T |\hat{Y}_s^+|^2 ds, \]

using Gronwall’s lemma implies that

\[ \mathbb{E} \left[ \left| (\hat{Y}_t^+)^2 \right| \right] = 0, \]

finally, we have

\[ Y^1_t \leq Y^2_t. \]

\section{Reflected BDSDEPs with continuous coefficient.}

In this section we are interested in weakening the conditions on \( f \). We assume that \( f \) and \( g \) satisfy the following assumptions:

\textbf{(H.6)} There exists \( 0 < \alpha < 1 \) and \( C > 0 \) s.t. for all \( (t, \omega, y,z,u) \in [0,T] \times \Omega \times \mathbb{R} \times L^2 (E, \mathcal{E}, \lambda, \mathbb{R}) \),

\[ \begin{align*}
& \left| f(t,\omega,y,z,u) \right| \leq C \left( 1 + |y| + |z| + |u| \right), \\
& \left| g(t,\omega,y,z,u) - g(t,\omega,y',z',u') \right|^2 \leq C \left| y - y' \right|^2 + \alpha \left\{ \left| z - z' \right|^2 + \left| u - u' \right|^2 \right\}.
\end{align*} \]

\textbf{(H.7)} For fixed \( \omega \) and \( t \), \( f(t,\omega,\cdot,\cdot,\cdot,\cdot) \) is continuous.

The three next Lemmas will be useful in the sequel.

We recall the following classical lemma. It can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [1].

\textbf{Lemma 4.1.} Let \( f : \Omega \times [0,T] \times \mathbb{R} \times L^2 (E, \mathcal{E}, \lambda, \mathbb{R}) \rightarrow \mathbb{R} \) be a measurable function such that:

1. For a.s. every \( (t,\omega) \in [0,T] \times \Omega \), \( f(t,\omega,y,z,u) \) is a continuous.
2. There exists a constant $C > 0$ such that for every $(t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$, 
\[ |f(t, \omega, y, z, u)| \leq C(1 + |y| + |z| + |u|). \]

Then exists the sequence of function $f_n$
\[
f_n(t, \omega, y, z, u) = \inf_{(y', z', u') \in \mathcal{B}(\mathbb{R})} \left[ f \left(t, \omega, y', z', u' \right) + n \left( |y - y'| + |z - z'| + |u - u'| \right) \right],
\]
is well defined for each $n \geq C$, and it satisfies, $d\mathbb{P} \times dt - a.s.$

(i) Linear growth: \( \forall n \geq 1, (y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times L^2, \ |f_n(t, \omega, y, z, u)| \leq C(1 + |y| + |z| + |u|). \)

(ii) Monotonicity in $n$: \( \forall y, z, u, f_n(t, \omega, y, z, u) \) is increases in $n$.

(iii) Convergence: \( \forall (t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathcal{B}^2(\mathbb{R}), \) if \( (t, \omega, y_n, z_n, u_n) \rightarrow (t, \omega, y, z, u) \), then \( f_n(t, \omega, y_n, z_n, u_n) \rightarrow f(t, \omega, y, z, u). \)

(iv) Lipschitz condition: \( \forall n \geq 1, (t, \omega) \in [0, T] \times \Omega, \forall (y, z, u) \in \mathcal{B}^2(\mathbb{R}) \) and \( \left(y', z', u' \right) \in \mathcal{B}^2(\mathbb{R}), \) we have \( \left| f_n(t, \omega, y, z, u) - f_n(t, \omega, y', z', u') \right| \leq n \left( |y - y'| + |z - z'| + |u - u'| \right). \)

Now given $\xi \in L^2$, $n \in \mathbb{N}$, we consider $(Y^n, Z^n, K^n, U^n)$ and (resp $(V, N, K, M)$) be solutions of the following reflected BDSDEPs:
\[
\begin{aligned}
Y_t^n &= \xi + \int_t^T f_n(s, Y_s^n, Z_s^n, U_s^n) ds + \int_t^T g(s, Y_s^n, Z_s^n, U_s^n) dB_s \\
&+ \int_t^T dK^n_s - \int_t^T Z_s^n dW_s - \int_t^T E U^n_s(e) \, \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \\
S_t &\leq Y_t^n, \quad 0 \leq t \leq T, \quad \text{and} \quad \int_0^T (Y^n_t - S_t) dK^n_t = 0.
\end{aligned}
\]
respectivealy
\[
\begin{aligned}
V_t &= \xi + \int_t^T H(s, V_s, N_s, M_s) ds + \int_t^T g(s, V_s, N_s, M_s) dB_s \\
&+ \int_t^T dK_s - \int_t^T N_s dW_s - \int_t^T E M_s(e) \, \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \\
S_t &\leq V_t, \quad 0 \leq t \leq T, \quad \text{and} \quad \int_0^T (V_t - S_t) dK_t = 0,
\end{aligned}
\]
where $H(s, \omega, V, N, M) = C(1 + |V| + |N| + |M|)$.  

**Lemma 4.2.** (i) $a.s.$ for all $t$ and $\forall n \leq m$, $Y^n_t \leq Y^m_t \leq V_t$. 

(ii) Assume that (H.1), (H.3) - (H.7) is in force. Then there exists a constant $A > 0$ depending only on $C, \alpha, \xi$ and $T$ such that:
\[
||U^n||_{L^2(0, T, \tilde{\mu}, \mathbb{R})} \leq A, \quad ||Z^n||_{M^2(0, T, \mathbb{R}^d)} \leq A.
\]

**Proof:** The prove of the (i) follow from comparison theorem. It remains to prove (ii), by lemma 2.1, we have
\[
\mathbb{E} |Y^n_t|^2 + \mathbb{E} \int_t^T |Z_s^n|^2 ds + \mathbb{E} \int_t^T \int_E |U^n_s(e)|^2 \lambda(de) ds \\
\leq \mathbb{E} |\xi|^2 + 2\mathbb{E} \int_t^T Y^n_s f_n(s, Y^n_s, Z^n_s, U^n_s) ds + 2\mathbb{E} \int_t^T Y^n_s dK^n_s + \mathbb{E} \int_t^T |g(s, Y^n_s, Z^n_s, U^n_s)|^2 ds.
\]
By (i) in lemma 4.1, we have
\[
2\mathbb{E} \int_t^T Y_s^n f_n(s, Y_s^n, Z_s^n, U_s^n) ds \leq 2C \mathbb{E} \int_t^T Y_s^n (1 + |Y_s^n| + |Z_s^n| + |U_s^n|) ds
\]
\[
\leq \mathbb{E} \int_t^T |Y_s^n|^2 ds + TC^2 + 2C \mathbb{E} \int_t^T |Y_s^n|^2 ds + \frac{C^2}{\gamma_1} \mathbb{E} \int_t^T |Y_s^n|^2 ds
\]
\[
+ \gamma_1 \mathbb{E} \int_t^T |Z_s^n|^2 ds + \frac{C^2}{\gamma_2} \mathbb{E} \int_t^T |Y_s^n|^2 ds + \gamma_2 \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds,
\]
\[
\leq \left( 1 + 2C + \frac{C^2}{\gamma_1} + \frac{C^2}{\gamma_2} \right) \mathbb{E} \int_t^T |Y_s^n|^2 ds + TC^2
\]
\[
+ \gamma_1 \mathbb{E} \int_t^T |Z_s^n|^2 ds + \gamma_2 \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds,
\]
also by the hypothesis associated with \( g \), we get
\[
||g(s, Y_s^n, Z_s^n, U_s^n)||^2 \leq (1 + \epsilon) ||g(s, Y_s^n, Z_s^n, U_s^n) - g(s, 0, 0, 0)||^2 + \frac{1 + \epsilon}{\epsilon} ||g(s, 0, 0, 0)||^2,
\]
\[
\leq (1 + \epsilon) C |Y_s^n|^2 + (1 + \epsilon) \alpha \{ |Z_s^n|^2 + |U_s^n|^2 \} + \frac{1 + \epsilon}{\epsilon} ||g(s, 0, 0, 0)||^2.
\]
Choosing \( \gamma_1 = \gamma_2 = \frac{\epsilon^2}{T} \). Then, we obtain the following inequality
\[
\mathbb{E} \left( |Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \right)
\]
\[
\leq \mathbb{E} \left[ \xi^2 + TC^2 + \left( 1 + 2C + \frac{4C^2}{\epsilon^2} + (1 + \epsilon) C \right) \mathbb{E} \int_t^T |Y_s^n|^2 ds + 2 \int_t^T Y_s^n dK_t^n \right.
\]
\[
+ \left. \frac{\epsilon^2}{T} + (1 + \epsilon) \alpha \right] \left\{ \mathbb{E} \int_t^T |Z_s^n|^2 ds + \mathbb{E} \int_0^T \int_E |U_s^n(e)|^2 \lambda(de) ds \right\} + \frac{1 + \epsilon}{\epsilon} \mathbb{E} \int_0^T ||g(s, 0, 0, 0)||^2 ds.
\]
Consequently, we have
\[
\mathbb{E} \int_t^T \left( |Z_s^n|^2 + \int_E |U_s^n(e)|^2 \lambda(de) ds \right)
\]
\[
\leq \left( \frac{\epsilon^2}{2} + (1 + \epsilon) \alpha \right) \mathbb{E} \left\{ \int_t^T |Z_s^n|^2 ds + \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \right\} + \Lambda + \theta \mathbb{E} |K_t^n - K_t^n|^2,
\]
where
\[
\Lambda = \mathbb{E} \left[ \xi^2 + TC^2 + \frac{4C^2}{\epsilon^2} \mathbb{E} \int_t^T ||g(s, 0, 0, 0)||^2 ds + \frac{1}{T} \mathbb{E} \left( \sup_{0 \leq s \leq T} (S_s)^2 \right) + \left( 1 + 2C + \frac{4C^2}{\epsilon^2} + (1 + \epsilon) C \right) \mathbb{E} \left( \sup_{t \leq T} |Y_t^n|^2 \right) \right].
\]
Now choosing \( \epsilon \) and \( \alpha \) such that \( 0 \leq \frac{\epsilon^2}{T} + (1 + \epsilon) \alpha < 1 \), we obtain
\[
\mathbb{E} \int_t^T |Z_s^n|^2 ds + \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \leq \Lambda + \theta \mathbb{E} |K_t^n - K_t^n|^2. \tag{4.2}
\]
On the other hand, we have from Eq.(4.1)
\[
K_t^n - K_t^n = Y_t^n - \xi - \int_t^T f_n(s, Y_s^n, Z_s^n, U_s^n) ds - \int_t^T g(s, Y_s^n, Z_s^n, U_s^n) dB_s
\]
\[
+ \int_t^T Z_s^n dW_s + \int_t^T \int_E U_s^n(e) \tilde{\mu}(ds, de).
\]
Using the Hölder’s inequality and assumption (H.6), we have
\[
\mathbb{E} |K_t^n - K_t^n|^2 \leq C_1 + C_2 \left( \mathbb{E} \int_t^T |Z_s^n|^2 ds + \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \right),
\]
7
From inequality (4.2), we get
\[ \mathbb{E} \int_0^T \left( |Z^n_s|^2 + \int_E |U^n_s(e)|^2 \lambda(de) \right) ds \leq \Lambda + \theta C_1 + \theta C_2 \mathbb{E} \int_t^T \left( |Z^n_s|^2 + \int_E |U^n_t|^2 \lambda(de) \right) ds, \]
Finally choosing \( \theta \) such that \( 0 \leq \theta C_2 \leq 1 \), we obtain
\[ \mathbb{E} \int_t^T |Z^n_s|^2 ds + \mathbb{E} \int_t^T \int_E |U^n_s(e)|^2 \lambda(de) ds \leq \Lambda + \theta C_1 < \infty. \]
The prove of lemma 4.2 is complet. \( \square \)

**Lemma 4.3.** Assume that (H.1), (H.3) – (H.7) is in force. Then the sequence \((Z^n, U^n)\) converges a.s. in \( \mathcal{M}^2(0, T, \mathbb{R}^d) \times L^2(0, T, \tilde{\mu}, \mathbb{R}) \).

**Proof:** Let \( n_0 \geq C \). From Eq.(4.1), we deduce that there exists a process \( Y \in \mathcal{S}^2(0, T, \mathbb{R}) \) such that \( Y^n \to Y \) a.s., as \( n \to \infty \). Applying Lemma 2.1 to \(|Y^n_t - Y_m|^2|, \) for \( n, m \geq n_0 \)
\[ \mathbb{E} \left( |Y^n_t - Y^m_t|^2 \right) + \int_t^T \left( |Z^n_s - Z^m_s|^2 ds + \int_E |U^n_s(e) - U^m_s(e)|^2 \lambda(de) ds \right) \]
\[ \leq 2 \mathbb{E} \left( \int_t^T |Y^n_s - Y^m_s| \left( f_n(s, Y^n_s, Z^n_s, U^n_s) - f_m(s, Y^m_s, Z^m_s, U^m_s) \right) ds + |g(s, Y^n_s, Z^n_s, U^n_s) - g(s, Y^m_s, Z^m_s, U^m_s)|^2 ds. \]
Since \( \int_t^T (Y^n_s - Y^m_s)(dK^n_s - dK^m_s) \leq 0 \), we deduce that
\[ \mathbb{E} \int_t^T |Z^n_t - Z^m_t|^2 ds + \mathbb{E} \int_t^T \int_E |U^n_s(e) - U^m_s(e)|^2 \lambda(de) ds \]
\[ \leq 2 \mathbb{E} \left( \int_t^T |f_n(s, Y^n_s, Z^n_s, U^n_s) - f_m(s, Y^m_s, Z^m_s, U^m_s)|^2 ds + |g(s, Y^n_s, Z^n_s, U^n_s) - g(s, Y^m_s, Z^m_s, U^m_s)|^2 ds. \]
Using Hölder’s inequality and assumption (H.6) for \( g \), we deduce that
\[ (1 - \alpha) \mathbb{E} \left( \int_t^T |Z^n_t - Z^m_t|^2 ds + \int_t^T \int_E |U^n_s(e) - U^m_s(e)|^2 \lambda(de) ds \right) \]
\[ \leq 2 \mathbb{E} \left( \int_t^T |f_n(s, Y^n_s, Z^n_s, U^n_s) - f_m(s, Y^m_s, Z^m_s, U^m_s)|^2 ds \right)^{\frac{1}{2}} + C \mathbb{E} \int_t^T |Y^n_s - Y^m_s|^2 ds. \]
Applying assumption (H.6) for \( f \) and the boundedness of the sequence \((Y^n, Z^n, U^n)\), we deduce that
\[ (1 - \alpha) \left( \mathbb{E} \int_t^T |Z^n_t - Z^m_t|^2 ds + \mathbb{E} \int_t^T \int_E |U^n_s(e) - U^m_s(e)|^2 \lambda(de) ds \right) \leq C^{\alpha\xi} \mathbb{E} \int_t^T |Y^n_s - Y^m_s|^2 ds, \]
where the constant \( C^{\alpha\xi} > 0 \) depend only \( \xi, C, \alpha \) and \( T \).
Which yields that \((Z^n)_{n \geq 0}\) respectively \((U^n)_{n \geq 0}\) is a Cauchy sequence in \( \mathcal{M}^2(0, T, \mathbb{R}^d) \), respectively in \( L^2(0, T, \tilde{\mu}, \mathbb{R}) \). Then there exists \((Z, U) \in \mathcal{M}^2(0, T, \mathbb{R}^d) \times L^2(0, T, \tilde{\mu}, \mathbb{R}) \) such that
\[ \mathbb{E} \int_0^T |Z^n_t - Z_s|^2 ds + \mathbb{E} \int_0^T \int_E |U^n_s(e) - U_s(e)|^2 \lambda(de) ds \to 0, \quad \text{as } n \to \infty. \]
Theorem 4.1. Assume that $(H.1), (H.3)-(H.7)$ holds. Then Eq (1.1) admits a solution $(Y, Z, U, K) \in D^2(\mathbb{R})$. Moreover there is a minimal solution $(Y^*, Z^*, U^*)$ of RBDSDEP (1.1) in the sense that for any other solution $(Y, Z, U)$ of Eq. (1.1), we have $Y^* \leq Y$.

Proof:
From Eq (4.1), it’s readily seen that $(Y^n)$ converges in $\mathcal{S}^2(0, T, \mathbb{R})$, $dt \otimes d\mathbb{P} - a.s.$ to $Y \in \mathcal{S}^2(0, T, \mathbb{R})$. Otherwise thanks to Lemma 4.3 there exists two subsequences still noted as the whole sequence $(Z^n)_{n \geq 0}$ respectively $(U^n)_{n \geq 0}$ such that

$$
\mathbb{E} \int_0^T |Z^n_t - Z_s|^2 ds \to 0 \text{ as } n \to \infty, \quad \text{and} \quad \mathbb{E} \int_0^T \int_E |U^n_s(e) - U_s(e)|^2 \lambda(de) ds \to 0, \text{ as } n \to \infty.
$$

Applying Lemma 4.1, we have $f_n(t, Y^n_s, Z^n_s, U^n_s) \to f(t, Y, Z, U)$ and the linear growth of $f_n$ implies

$$
|f_n(t, Y^n, Z^n, U^n)| \leq C \left(1 + \sup_n (|Y^n| + |Z^n| + |U^n|)\right) \in \mathbb{L}^1 ([0, T]; dt).
$$

Thus by Lebesgue’s dominated convergence theorem, we deduce that for almost all $\omega$ and uniformly in $t$, we have

$$
\mathbb{E} \int_t^T f_n(s, Y^n_s, Z^n_s, U^n_s) ds \to \mathbb{E} \int_t^T f(s, Y_s, Z_s, U_s) ds.
$$

We have by (H.6) the following estimation

$$
\mathbb{E} \int_t^T ||g(s, Y^n_s, Z^n_s, U^n_s) - g(s, Y_s, Z_s, U_s)||^2 ds 
\leq C \mathbb{E} \int_t^T |Y^n_s - Y_s|^2 ds + \alpha \mathbb{E} \int_t^T |Z^n_s - Z_s|^2 ds + \alpha \mathbb{E} \int_t^T \int_E |U^n_s(e) - U_s(e)|^2 \lambda(de) ds \to 0, \text{ as } n \to \infty,
$$

using Burkholder-Davis-Gundy inequality, we have

\[
\begin{cases}
\mathbb{E} \sup_{0\leq t \leq T} \left\| \int_t^T Z^n_s dW_s - \int_t^T Z_s dW_s \right\|^2 \to 0, \\
\mathbb{E} \sup_{0\leq t \leq T} \left\| \int_t^T E(U^n_s(e)) - \int_t^T E(U_s(e)) \right\|^2 \to 0, \\
\mathbb{E} \sup_{0\leq t \leq T} \left\| \int_t^T g(s, Y^n_s, Z^n_s, U^n_s) d\tilde{B}_s - \int_t^T g(s, Y_s, Z_s, U_s) d\tilde{B}_s \right\|^2 \to 0, \text{ in probability as, } n \to \infty.
\end{cases}
\]

Let the following reflected BDSDEPs with data $(\xi, f, g, S)$

\[
\begin{cases}
\hat{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + \int_t^T g(s, Y_s, Z_s, U_s) d\tilde{B}_s + \int_t^T dK_s \\
- \int_t^T Z_s dW_s - \int_t^T E(U_s(e)) d\tilde{B}_s, \\
S_t \leq \hat{Y}_t, \quad 0 \leq t \leq T \quad \text{and} \quad \int_0^T \left( \hat{Y}_t - S_t \right) dK_t = 0.
\end{cases}
\]

By Itô’s formula, we derive that

\[
\begin{align*}
\mathbb{E} \left| Y^n_t - \hat{Y}_t \right|^2 &\leq 2 \mathbb{E} \int_t^T \left( Y^n_s - \hat{Y}_s \right) \left( f_n(s, Y^n_s, Z^n_s, U^n_s) - f(s, Y_s, Z_s, U_s) \right) ds \\
&+ 2 \mathbb{E} \int_t^T \left( Y^n_t - \hat{Y}_t \right) \left( dK^n_s - dK_s \right) + \mathbb{E} \int_t^T \left| g(s, Y^n_s, Z^n_s, U^n_s) - g(s, Y_s, Z_s, U_s) \right|^2 ds \\
&- \mathbb{E} \int_t^T \int_E \left( U^n_s(e) - \hat{U}_s(e) \right) \left( d\tilde{B}_s \right) \lambda(de) ds - \mathbb{E} \int_t^T \left| Z^n_s - \hat{Z}_s \right|^2 ds.
\end{align*}
\]
Using the fact that $\mathbb{E} \int_t^T \left( Y^n_s - \bar{Y}_s \right) (dK^n_s - dK_s) \leq 0$, we get
\[
\mathbb{E} \left| Y^n_t - \bar{Y}_t \right|^2 + \mathbb{E} \int_t^T \int_{\mathcal{E}} \left| U^n_s (e) - \bar{U}_s (e) \right|^2 \lambda (de) \, ds + \mathbb{E} \int_t^T \left| Z^n_s - \bar{Z}_s \right|^2 \, ds
\leq 2\mathbb{E} \int_t^T \left( Y^n_s - \bar{Y}_s \right) (f_n (s, Y^n_s, Z^n_s, U^n_s) - f (s, Y_s, Z_s, U_s)) \, ds + \mathbb{E} \int_t^T \left| g (s, Y^n_s, Z^n_s, U^n_s) - g (s, Y_s, Z_s, U_s) \right|^2 \, ds,
\]
letting $n \to \infty$, we have $Y_t = \bar{Y}_t$, $U_t = \bar{U}_t$ and $Z_t = \bar{Z}_t$, $d\mathbb{P} \times dt$-a.e.

Let $(Y^*, Z^*, U^*, K^*)$ be a solution of (1.1). Then by Theorem 3.1, we have for any $n \in \mathbb{N}^*$, $Y^n \leq Y^*$. Therefore, $Y$ is a minimal solution of (1.1).

\section{RBDSDEPs with discontinuous coefficient.}

In this section we are interested in weakening the conditions on $f$. We assume that $f$ satisfy the following assumptions:

\begin{itemize}
  \item[(H.8)] There exists a nonnegative process $f_t \in \mathcal{M}^2 (0, T, \mathbb{R})$ and constant $C > 0$, such that
  \[\forall (t, y, z, u) \in [0, T] \times \mathcal{B}^2 (\mathbb{R}), \quad |f (t, y, z, u)| \leq f_t (\omega) + C (|y| + |z| + |u|).\]
  \item[(H.9)] $f (t, \cdot, z, u) : \mathbb{R} \to \mathbb{R}$ is a left continuous and $f (t, y, \cdot, \cdot)$ is a continuous.
  \item[(H.10)] There exists a continuous fonction $\pi : [0, T] \times \mathcal{B}^2 (\mathbb{R})$ satisfying for $y \geq y', (z, u) \in \mathbb{R}^d \times L^2 (E, \mathcal{E}, \lambda, \mathbb{R})$
  \[
  \begin{cases}
    |\pi (t, y, z, u)| \leq C (|y| + |z| + |u|), \\
    f (t, \omega, y, z, u) - f (t, \omega, y', z', u') \geq \pi (t, y - y', z - z', u - u').
  \end{cases}
  \]
  \item[(H.11)] $g$ satisfies (H.2, (ii)).
\end{itemize}

\subsection{Existence result.}

The two next Lemmas will be useful in the sequel.

\begin{lemma}
Assume that $\pi$ satisfies (H.10), $g$ satisfies (H.11) and $h$ belongs in $\mathcal{M}^2 (0, T, \mathbb{R})$. For a continuous processes of finite variation $A$ belong in $\mathcal{A}^2$, we consider the processes $(\bar{Y}, \bar{Z}, \bar{U}) \in \mathcal{S}^2 (0, T, \mathbb{E}) \times \mathcal{M}^2 (0, T, \mathbb{R}^d) \times L^2 (0, T, \bar{\mu}, \mathbb{R})$ such that:
\[
\begin{alignat}{2}
(i) \quad & \bar{Y}_t = \xi + \int_t^T (\pi (s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) + h (s)) \, ds + \int_t^T g (s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) d\bar{B}_s \\
& + \int_t^T dA_s - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_{\mathcal{E}} \bar{U}_s (e) \, \bar{\mu} (ds, de), & \quad 0 \leq t \leq T,
\end{alignat}
\]
\[
(ii) \quad \int_0^T \bar{Y}_t^- \, dA_s \geq 0.
\]

Then we have,

1. The RBDSDEPs (5.1) admits a minimal solution $(Y_t, Z_t, A_t, U_t) \in \mathcal{D}^2 (\mathbb{R})$.
2. If $h (t) \geq 0$ and $\xi \geq 0$, we have $\bar{Y}_t \geq 0$, $d\mathbb{P} \times dt$-a.s.

\textbf{Proof:} (1) Obtained from a previous part.

(2) Applying lemma 2.1 to $|Y_t^-|^2$, we have
\[
\mathbb{E} \left( \int_t^T \left| Y^n_t - \bar{Y}_t \right|^2 + \int_t^T \int_{\mathcal{E}} \left| U^n_s (e) - \bar{U}_s (e) \right|^2 \lambda (de) \, ds + \int_t^T \int_{\mathcal{E}} \left| \bar{U}_s (e) \right|^2 \lambda (de) \, ds \right)
\leq \mathbb{E} \left( \left| \xi \right|^2 - 2 \int_t^T \bar{Y}_s^- (\pi (s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) + h (s)) \, ds - 2 \int_t^T \bar{Y}_s^- \, dA_s + \int_t^T \int_{\mathcal{E}} \left| g (s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) \right|^2 \, ds \right).
\]
Since $h(t) \geq 0$, $\xi \geq 0$ and using the fact that $\int_0^T \tilde{Y}_t^- \, dA_s \geq 0$, we obtain
\[
\mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{Z}_s^- \right|^2 \, ds + \mathbb{E} \int_t^T \int_E 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds \\
\leq -2\mathbb{E} \int_t^T \tilde{Y}_s^- \pi_s (s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \, ds + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| g(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \right|^2 \, ds.
\]
According to assumptions (H.11), we get
\[
\mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{Z}_s^- \right|^2 \, ds + \mathbb{E} \int_t^T \int_E 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds \\
\leq -2\mathbb{E} \int_t^T \tilde{Y}_s^- \pi_s (s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \, ds + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| g(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \right|^2 \, ds \\
+ \alpha \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds,
\]
applying assumption (H.10) and using Young’s inequality, we have
\[
-2\mathbb{E} \int_t^T \tilde{Y}_s^- \pi_s (s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) \, ds \leq 2C \mathbb{E} \int_t^T \left| \tilde{Z}_s^- \right|^2 \, ds + \frac{1}{2c} \mathbb{E} \int_t^T \left| \tilde{Y}_s^- \right|^2 \, ds + 2C^2 \mathbb{E} \int_t^T \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds \\
+ \frac{1}{2c} \mathbb{E} \int_t^T \left| \tilde{Y}_s^- \right|^2 \, ds + 2C^2 \mathbb{E} \int_t^T \int_E 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds.
\]
Then
\[
\mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{Z}_s^- \right|^2 \, ds + \mathbb{E} \int_t^T \int_E 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \, ds \\
\leq (3C + c^{-1}) \mathbb{E} \int_t^T \left| \tilde{Y}_s^- \right|^2 \, ds + (\alpha + 2C^2) \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s^- < 0\}} \left( \left| \tilde{Z}_s^- \right|^2 + \int_E 1_{\{\tilde{Y}_s^- < 0\}} \left| \tilde{U}_s^- (e) \right|^2 \lambda(\text{de}) \right) \, ds
\]
Therefore, choosing $\epsilon$, $\alpha$ and $C$ such that $0 < \alpha + 2C^2 < 1$ and using Gronwall’s inequality, we have
\[
\mathbb{E} |\tilde{Y}_t^-|^2 = 0,
\]
$P - a.s.$ for all $t \in [0, T]$. Finally implies that $\tilde{Y}_t \geq 0$, $P - a.s.$ for all $t \in [0, T]$.

Now by theorem (4.1), we consider the processes $(\tilde{Y}_t^0, \tilde{Z}_t^0, \tilde{K}_t^0, \tilde{U}_t^0)$, $(Y_t^0, Z_t^0, K_t^0, U_t^0)$ and sequence of processes $(\tilde{Y}_n^0, \tilde{Z}_n^0, \tilde{K}_n^0, \tilde{U}_n^0)_{n \geq 0}$ respectively minimal solution of the following RBDSDEPs for all $t \in [0, T]$

(i) $\tilde{Y}_t^0 = \xi + \int_t^T \left[ -C \left( \tilde{Y}_s^0 + \tilde{Z}_s^0 + \tilde{U}_s^0 \right) - f_s \right] \, ds + \int_t^T g(s, \tilde{Y}_s^0, \tilde{Z}_s^0, \tilde{U}_s^0) \, d\tilde{B}_s$ \\
$+ \int_t^T d\tilde{K}_s^0 - \int_t^T \tilde{Z}_s^0 \, dW_s - \int_t^T \int_E \tilde{U}_s^0 (e) \, \bar{\mu}(ds, de), 0 \leq t \leq T,$

(ii) $\tilde{Y}_t^0 \geq S_t$, 

(iii) $\int_0^T (\tilde{Y}_s^0 - S_s) \, d\tilde{K}_s^0 = 0$,

(i) $Y_t^0 = \xi + \int_t^T \left[ C \left( Y_s^0 + Z_s^0 + U_s^0 \right) + f_s \right] \, ds + \int_t^T g(s, Y_s^0, Z_s^0, U_s^0) \, d\tilde{B}_s$ \\
$+ \int_t^T dK_s^0 - \int_t^T Z_s^0 \, dW_s - \int_t^T \int_E U_s^0 (e) \, \bar{\mu}(ds, de), 0 \leq t \leq T,$

(ii) $Y_t^0 \geq S_t$, 

(iii) $\int_0^T (Y_s^0 - S_s) \, dK_s^0 = 0$,
and
\[
(i) \quad \bar{Y}_t^n = \xi + \int_t^T \left[ f(s, \bar{Y}_s^{n-1}, \bar{Z}_s^{n-1}, \bar{U}_s^{n-1})ds + \pi \left( s, \bar{Y}_t^n - \bar{Y}_t^{n-1}, \bar{Z}_t^n - \bar{Z}_t^{n-1}, \bar{U}_t^n - \bar{U}_t^{n-1} \right) \right] ds \\
\quad \quad + \int_t^T g(s, \bar{Y}_s^n, \bar{Z}_s^n, \bar{U}_s^n) dB_s + \int_t^T d\bar{K}_s^n - \int_t^T \bar{Z}_s^n dW_s - \int_t^T \int_E \bar{U}_s^n (e) \tilde{\mu} (ds, de), \quad 0 \leq t \leq T, \\
(ii) \quad \bar{Y}_t^n \geq S_t, \\
(iii) \quad \int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n = 0. 
\]

(5.4)

**Lemma 5.2.** Under the assumptions (H.1), (H.3), (H.5) and (H.8) – (H.11), we have for any \( n \geq 1 \) and \( t \in [0, T] \)
\[
\bar{Y}_t^n \leq \bar{Y}_t^{n+1} \leq \bar{Y}_t^0. 
\]

**Proof:** For any \( n \geq 0 \), we set
\[
\begin{align*}
\delta \rho_t^{n+1} &= \rho_t^{n+1} - \rho_t^n, \\
\Delta \psi^{n+1} (s, \delta \bar{Y}_s^{n+1}, \delta \bar{Z}_s^{n+1}, \delta \bar{U}_s^{n+1}) &= \psi (s, \delta \bar{Y}_s^n + \bar{Y}_s^n, \delta \bar{Z}_s^n + \bar{Z}_s^n, \delta \bar{U}_s^n + \bar{U}_s^n) - \psi (s, \bar{Y}_s^n, \bar{Z}_s^n, \bar{U}_s^n).
\end{align*}
\]

Using Eq.(5.4), we have
\[
\delta \bar{Y}_t^{n+1} = \int_t^T \left[ \pi \left( s, \delta \bar{Y}_s^{n+1}, \delta \bar{Z}_s^{n+1}, \delta \bar{U}_s^{n+1} \right) + \theta^n_s \right] ds + \int_t^T \Delta g^{n+1} (s, \delta \bar{Y}_s^{n+1}, \delta \bar{Z}_s^{n+1}, \delta \bar{U}_s^{n+1}) dB_s \\
\quad + \int_t^T d \left( \delta \bar{K}_s^{n+1} \right) - \int_t^T \delta \bar{Z}_s^{n+1} dW_s - \int_t^T \int_E \delta \bar{U}_s^{n+1} (e) \tilde{\mu} (ds, de),
\]
where
\[
\begin{align*}
\theta^n_s &= \Delta f^n (s, \delta \bar{Y}_s^n, \delta \bar{Z}_s^n, \delta \bar{U}_s^n) - \pi \left( s, \delta \bar{Y}_s^n, \delta \bar{Z}_s^n, \delta \bar{U}_s^n \right), \\
\text{and} \\
\theta^0 = f (s, \bar{Y}_s^0, \bar{Z}_s^0, \bar{U}_s^0) + C \left( \bar{Y}_s^0 + \bar{Z}_s^0 + \bar{U}_s^0 \right) + f_s, \forall n \geq 0.
\end{align*}
\]

According to its definition, on can show that \( \theta^n_s \) and \( \Delta g^n, \forall n \geq 0 \) satisfy all assumption of lemma 5.1. Moreover, since \( \bar{K}_t^n \) is a continuous and increasing process, for all \( n \geq 0, \Delta \bar{K}_t^{n+1} \) is a continuous process of finite variation. Using the same argument as in first part. On can show that
\[
\int_0^T \left( \bar{Y}_t^{n+1} - \bar{Y}_t^n \right) d \bar{K}_t^n = \int_0^T \left( \bar{Y}_t^{n+1} - \bar{Y}_t^n \right) \bar{K}_t^{n+1} - \int_0^T \left( \bar{Y}_t^{n+1} - \bar{Y}_t^n \right) d \bar{K}_t^n \geq 0.
\]

Applying lemma 5.1 we deduce that \( \delta \bar{Y}_t^{n+1} \geq 0 \), i.e. \( \bar{Y}_t^{n+1} \geq \bar{Y}_t^n \), for all \( t \in [0, T] \), we have \( \bar{Y}_t^{n+1} \geq \bar{Y}_t^n \geq \bar{Y}_t^0 \).

Now we show prove that \( \bar{Y}_t^{n+1} \leq \bar{Y}_t^0 \), by definition, we obtain
\[
\bar{Y}_t^0 - \bar{Y}_t^n = \int_t^T \left( -C \left( \left| Y_s^0 - \bar{Y}_s^{n+1} \right| + \left| Z_s^0 - \bar{Z}_s^{n+1} \right| + \left| U_s^0 - \bar{U}_s^{n+1} \right| \right) + \Lambda_s^{n+1} \right) \right) ds \\
\quad + \int_t^T \left( g (s, Y_s^0, Z_s^0, U_s^0) - g (s, \bar{Y}_s^{n+1}, \bar{Z}_s^{n+1}, \bar{U}_s^{n+1}) \right) dB_s \\
\quad + \int_t^T \left( \delta \bar{K}_s^0 - \delta \bar{K}_s^{n+1} \right) + \int_t^T \left( Z_s^0 - \bar{Z}_s^{n+1} \right) dW_s - \int_t^T \int_E \left( U_s^0 (e) - \bar{U}_s^{n+1} (e) \right) \tilde{\mu} (ds, de),
\]
where
Applying Lemma 2.1, we obtain

$$
\Lambda_s^{n+1} = C \left( \left| Y_s^0 - \tilde{Y}_s^{n+1} \right| + \left| Z_s^0 - \tilde{Z}_s^{n+1} \right| + \left| U_s^0 - \tilde{U}_s^{n+1} \right| + \left| Y_s^0 \right| + \left| Z_s^0 \right| + \left| U_s^0 \right| \right) + f_s - f(s, \bar{Y}_s^n, \bar{Z}_s^n, \bar{U}_s^n) - \pi \left( s, \delta \bar{Y}_s^{n+1}, \delta \bar{Z}_s^{n+1}, \delta \bar{U}_s^{n+1} \right).
$$

Also using Lemma 5.1 we deduce that $Y_t^0 - \tilde{Y}_t^{n+1} \geq 0$, i.e. $Y_t^0 \geq \tilde{Y}_t^{n+1}$, for all $t \in [0, T]$. Thus, we have for all $n \geq 0$

$$
Y_t^0 \geq \tilde{Y}_t^{n+1} \geq \tilde{Y}_t^0, \quad \text{dP} \times dt - \text{a.s., } \forall t \in [0, T].
$$

Now, our main result

**Theorem 5.1.** Under assumptions (H.1), (H.3), (H.5) and (H.8) - (H.11), the RBDSDEPs (1.1) has a minimal solution $(Y_t, Z_t, K_t, U_t)_{0 \leq t \leq T} \in \mathcal{D}^2(\mathbb{R})$.

**Proof:** Since $|\tilde{Y}_t^0| \leq \max(\tilde{Y}_t^0, Y_t^0) \leq |Y_t^0|$ for all $t \in [0, T]$, we have

$$
\sup_n \mathbb{E} \left( \sup_{0 \leq t \leq T} |\tilde{Y}_t^{n+1}|^2 \right) \leq \mathbb{E} \left( \sup_{0 \leq t \leq T} |\tilde{Y}_t^0|^2 \right) + \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^0|^2 \right) < \infty.
$$

Therefore, we deduce from the Lebesgue’s dominated convergence theorem that $(\tilde{Y}_t^n)_{n \geq 0}$ converges in $\mathcal{S}^2 (0, T, \mathbb{R})$ to a limit $Y$.

On the other hand from (5.4), we deduce that

$$
\tilde{Y}_0^{n+1} = \tilde{Y}_0^{n+1} + \int_0^T \left[ f(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) ds + \pi \left( s, \delta \tilde{Y}_s^{n+1}, \delta \tilde{Z}_s^{n+1}, \delta \tilde{U}_s^{n+1} \right) \right] ds
$$

$$
+ \int_0^T g(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) d\tilde{B}_s + \int_0^T d\tilde{K}_s^{n+1} - \int_0^T \tilde{Z}_s^{n+1} dW_s - \int_0^T \tilde{U}_s^{n+1} (e) \tilde{\mu} (ds, dc),
$$

applying Lemma 2.1, we obtain

$$
\mathbb{E} \left| \tilde{Y}_0^{n+1} \right|^2 + \mathbb{E} \int_0^T \left| \tilde{Z}_s^{n+1} \right|^2 ds + \mathbb{E} \int_0^T \left| \tilde{U}_s^{n+1} (e) \right|^2 \lambda (de) ds
$$

$$
\leq \mathbb{E} \left| \xi \right|^2 + 2\mathbb{E} \int_0^T \tilde{Y}_s^{n+1} \left[ f(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) + \pi \left( s, \delta \tilde{Y}_s^{n+1}, \delta \tilde{Z}_s^{n+1}, \delta \tilde{U}_s^{n+1} \right) \right] ds
$$

$$
+ 2\mathbb{E} \int_0^T \tilde{Y}_s^{n+1} d\tilde{K}_s^n + \mathbb{E} \int_0^T \left| g(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) \right|^2 ds.
$$

From (H.8) and (H.10), we get

$$
\tilde{Y}_t^{n+1} \left( f(t, \tilde{Y}_t^n, \tilde{Z}_t^n, \tilde{U}_t^n) + \pi \left( t, \delta \tilde{Y}_t^{n+1}, \delta \tilde{Z}_t^{n+1}, \delta \tilde{U}_t^{n+1} \right) \right)
$$

$$\leq \tilde{Y}_t^{n+1} \left( f_t (\omega) + 2C \left( \tilde{Y}_t^n + \tilde{Z}_t^n + \tilde{U}_t^n \right) + C \left( \tilde{Y}_t^{n+1} + \tilde{Z}_t^{n+1} + \tilde{U}_t^{n+1} \right) \right)
$$

$$\leq \frac{\tilde{Y}_t^{n+1}}{2} + \frac{f_t (\omega)}{2} + C^2 \left( \tilde{Y}_t^n + \tilde{Z}_t^n + \tilde{U}_t^n \right)^2 + \frac{C^2}{\epsilon_1} \left( \tilde{Y}_t^{n+1} \right)^2 + \frac{\epsilon_1}{2} \tilde{Z}_t^{n+1} + \frac{2C^2}{\epsilon_2} \left( \tilde{Y}_t^{n+1} \right)^2 + \frac{\epsilon_2}{2} \tilde{U}_t^{n+1}
$$

$$+ C \tilde{Y}_t^{n+1} + \tilde{Z}_t^{n+1} + \tilde{U}_t^{n+1} + \tilde{Y}_t^n + \tilde{Z}_t^n + \tilde{U}_t^n + \tilde{Y}_t^{n+1} + \tilde{Z}_t^{n+1} + \tilde{U}_t^{n+1} + \tilde{Y}_t^n + \tilde{Z}_t^n + \tilde{U}_t^n.
$$

Also applying (H.11), we obtain the following inequality

$$
\left| g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1}) \right|^2 \leq \left| g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1}) - g(s, 0, 0, 0) \right|^2 + \left| g(s, 0, 0, 0) \right|^2,
$$

$$\leq C \left( \tilde{Y}_s^{n+1} \right)^2 + \alpha \left( \tilde{Z}_s^{n+1} \right)^2 + \left( \tilde{U}_s^{n+1} \right)^2 + \left| g(s, 0, 0, 0) \right|^2.
$$

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Using Young's inequality, we get
\[ 2\mathbb{E} \int_0^T \tilde{Y}_s^{n+1} d\tilde{K}_s^{n+1} \leq 2\mathbb{E} \int_0^T S_s d\tilde{K}_s^{n+1} \leq \frac{1}{\theta} \mathbb{E} \left( \sup_{0 \leq t \leq T} |S_t|^2 \right) + \theta \mathbb{E} \left( \tilde{K}_T^{n+1} \right)^2. \]

Therefore, there exists a constant $C$ independent of $n$ such that for any $\epsilon_i$, where $i = 1 : 4$, we derive
\[
\mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds \\
\leq C + (\epsilon_3 + \alpha) \mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + (\epsilon_4 + \alpha) \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}|^2 \lambda(de) ds \\
+ \epsilon_1 \mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + \epsilon_2 \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}|^2 \lambda(de) ds + \theta \mathbb{E} \left( \tilde{K}_T^{n+1} \right)^2. \tag{5.5}
\]

Moreover, since
\[
\tilde{K}_T^{n+1} = \tilde{Y}_0^{n+1} - \xi - \int_0^T \left[ f(s, \tilde{Y}_s^{n}, \tilde{Z}_s^{n}, \tilde{U}_s^{n}) + \pi \left( s, \delta \tilde{Y}_s^{n+1}, \delta \tilde{Z}_s^{n+1}, \delta \tilde{U}_s^{n+1} \right) \right] ds \\
- \int_0^T g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1}) d\tilde{B}_s + \int_0^T \tilde{Z}_s^{n+1} dW_s + \int_0^T \int_E \tilde{U}_s^{n+1}(e) \tilde{\mu}(ds, de),
\]

Using Hölder’s inequality and assumption $(H.8), (H.10)$, there exists two constants $C_1$ and $C_2$ depend of $\xi, \alpha, \epsilon_i, i = 1, \ldots, 4$, and we have that
\[
\mathbb{E} \left| \tilde{K}_T^{n+1} \right|^2 \leq C_1 + C_2 \left( \mathbb{E} \int_0^T \left( |\tilde{Z}_s^{n+1}|^2 + |\tilde{U}_s^{n+1}|^2 \right) ds + \mathbb{E} \int_0^T \int_E \left( |\tilde{U}_s^{n}|^2 + |\tilde{U}_s^{n+1}|^2 \right) \lambda(de) ds \right),
\]

we come back to inequality (5.5), we obtain
\[
\mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds \\
\leq (C + \theta C_1) + (\epsilon_1 + \theta C_2) \mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + (\epsilon_2 + \theta C_2) \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n}|^2 \lambda(de) ds \\
+ (\epsilon_3 + \alpha + \theta C_2) \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds,
\]

we taking $\epsilon_1 = \epsilon_2 = \epsilon_0$ and $\epsilon_3 = \epsilon_4 = \bar{\epsilon}$, we have
\[
\mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds \\
\leq (C + \theta C_1) + (\epsilon_0 + \theta C_2) \left( \mathbb{E} \int_0^T |\tilde{Z}_s^{n}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n}|^2 \lambda(de) ds \right) \\
+ (\bar{\epsilon} + \theta C_2 + \alpha) \mathbb{E} \int_0^T \left( |\tilde{Z}_s^{n+1}|^2 + \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds \right),
\]

we choosing $\epsilon$, $\theta$ and $\alpha$ such that $0 \leq (\bar{\epsilon} + \theta C_2 + \alpha) < 1$, we get
\[
\mathbb{E} \int_0^T |\tilde{Z}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n+1}(e)|^2 \lambda(de) ds \\
\leq (C + \theta C_1) + (\epsilon_0 + \theta C_2) \left( \mathbb{E} \int_0^T |\tilde{Z}_s^{n}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{n}|^2 \lambda(de) ds \right) \\
\leq (C + \theta C_1) \sum_{i=0}^{i=n-1} (\epsilon_0 + \theta C_2)^i + (\epsilon_0 + \theta C_2)^n \left( \mathbb{E} \int_0^T |\tilde{Z}_s^{0}|^2 ds + \mathbb{E} \int_0^T \int_E |\tilde{U}_s^{0}|^2 \lambda(de) ds \right).
Now choosing \( \epsilon_0, \theta \) and \( C_2 \) such that \( \epsilon_0 + \theta C_2 < 1 \) and noting \( \mathbb{E} \int_0^T \left( |\tilde{Z}^n_s|^2 + \int_E |\tilde{U}_s|^2 \lambda(\mu) \right) \, ds < \infty \),

we obtain

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T |\tilde{Z}^{n+1}_s|^2 \, ds < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T \int_E |\tilde{U}^{n+1}_s(\epsilon)|^2 \lambda(\mu) \, ds < \infty,
\]

consequently, we deduce that

\[
\mathbb{E} \left| \tilde{K}^{n+1}_T \right|^2 < \infty.
\]

Now we shall prove that \( (\tilde{Z}^n, \tilde{K}^n, \tilde{U}^n) \) is a Cauchy sequence in \( \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2 \times \mathcal{L}^2(0, T, \tilde{\mu}, \mathbb{R}) \),

set \( \Gamma^n_s = f(s, \tilde{Y}^n_{s-1}, \tilde{Z}^n_{s-1}, \tilde{U}^n_{s-1}) + \pi \left( s, \delta \tilde{Y}^n_s, \delta \tilde{Z}^n_s, \delta \tilde{U}^n_s \right) \),

we have

\[
\tilde{Y}^n_t - \tilde{Y}^m_t = \int_t^T (\Gamma^n_s - \Gamma^m_s) \, ds + \int_t^T (g(s, \tilde{Y}^n_s, \tilde{Z}^n_s, \tilde{U}^n_s) - g(s, \tilde{Y}^m_s, \tilde{Z}^m_s, \tilde{U}^m_s)) \, dB_s
\]

\[
+ \int_t^T (d\tilde{K}^n_s - d\tilde{K}^m_s) - \int_t^T (\tilde{Z}^n_s - \tilde{Z}^m_s) \, dW_s - \int_t^T \int_E (\tilde{U}^n_s(\epsilon) - \tilde{U}^m_s(\epsilon)) \, \tilde{\mu}(ds, d\epsilon),
\]

applying Lemma 2.1 to \( |\delta \tilde{Y}^{n,m}|^2 = |\tilde{Y}^n_s - \tilde{Y}^m_s|^2 \), we have

\[
\mathbb{E} \left| \tilde{Y}^n_t - \tilde{Y}^m_t \right|^2 + \mathbb{E} \int_t^T |\tilde{Z}^n_s - \tilde{Z}^m_s|^2 \, ds + \mathbb{E} \int_t^T \int_E |\tilde{U}^n_s - \tilde{U}^m_s|^2 \lambda(\mu) \, ds
\]

\[
\leq 2\mathbb{E} \int_t^T \left( \tilde{Y}^n_s - \tilde{Y}^m_s \right)^2 (\Gamma^n_s - \Gamma^m_s) \, ds + 2\mathbb{E} \int_t^T \left( \tilde{Y}^{n+1}_s - \tilde{Y}^n_s \right) \left( d\tilde{K}^n_s - d\tilde{K}^m_s \right)
\]

\[
+ \mathbb{E} \int_t^T \left( g(s, \tilde{Y}^n_s, \tilde{Z}^n_s, \tilde{U}^n_s) - g(s, \tilde{Y}^m_s, \tilde{Z}^m_s, \tilde{U}^m_s) \right)^2 \, ds,
\]

since \( \int_t^T \left( \tilde{Y}^{n+1}_s - \tilde{Y}^n_s \right) \left( d\tilde{K}^n_s - d\tilde{K}^m_s \right) \leq 0 \), we obtain

\[
\mathbb{E} \int_0^T |\tilde{Z}^n_s - \tilde{Z}^m_s|^2 \, ds + \mathbb{E} \int_0^T \int_E |\tilde{U}^n_s - \tilde{U}^m_s|^2 \lambda(\mu) \, ds
\]

\[
\leq 2\mathbb{E} \int_t^T \left( \tilde{Y}^n_s - \tilde{Y}^m_s \right)^2 (\Gamma^n_s - \Gamma^m_s) \, ds + \mathbb{E} \int_t^T \left( g(s, \tilde{Y}^n_s, \tilde{Z}^n_s, \tilde{U}^n_s) - g(s, \tilde{Y}^m_s, \tilde{Z}^m_s, \tilde{U}^m_s) \right)^2 \, ds.
\]

Applying Hölder’s inequality and assumption \( (H.11) \), we deduce that

\[
(1 - \alpha) \left\{ \mathbb{E} \int_t^T \left| \tilde{Z}^n_s - \tilde{Z}^m_s \right|^2 \, ds + \mathbb{E} \int_t^T \int_E \left| \tilde{U}^n_s - \tilde{U}^m_s \right|^2 \lambda(\mu) \, ds \right\}
\]

\[
\leq 2\mathbb{E} \left( \int_t^T \left| \tilde{Y}^n_s - \tilde{Y}^m_s \right|^2 \, ds \right)^{\frac{1}{2}} \left( \int_t^T |\Gamma^n_s - \Gamma^m_s|^2 \, ds \right)^{\frac{1}{2}} + C \mathbb{E} \int_t^T \left| \tilde{Y}^n_s - \tilde{Y}^m_s \right|^2 \, ds.
\]

The boundedness of the sequence \( (\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n, \tilde{U}^n) \), we deduce that \( \Lambda = \sup_{n \in \mathbb{N}} \left( \mathbb{E} \int_0^T |\Gamma^n_s|^2 \, ds \right) < \infty \),

this yields that

\[
(1 - \alpha) \mathbb{E} \int_t^T \left| \tilde{Z}^n_s - \tilde{Z}^m_s \right|^2 \, ds + \mathbb{E} \int_t^T \int_E \left| \tilde{U}^n_s - \tilde{U}^m_s \right|^2 \lambda(\mu) \, ds
\]

\[
\leq 4\Lambda \mathbb{E} \left( \int_t^T \left| \tilde{Y}^n_s - \tilde{Y}^m_s \right|^2 \, ds \right)^{\frac{1}{2}} + C \mathbb{E} \int_t^T \left| \tilde{Y}^n_s - \tilde{Y}^m_s \right|^2 \, ds.
\]
Which yields that $\left(\tilde{Z}^n\right)_{n \geq 0}$ respectively $\left(\tilde{U}^n\right)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{M}^2(0, T, \mathbb{R}^d)$ respectively in $L^2(0, T, \tilde{\mu}, \mathbb{R})$. Then there exists $(Z, U) \in \mathcal{M}^2(0, T, \mathbb{R}^d) \times L^2(0, T, \tilde{\mu}, \mathbb{R})$ such that,

$$E \int_t^T |\tilde{Z}^n_s - Z_s|^2 \, ds + E \int_t^T \int_E |\tilde{U}^n_s - U_s|^2 \, \lambda(de) \to 0, \quad \text{as } n \to \infty. \quad (5.6)$$

On the other hand, applying Burkholder-Davis-Gundy inequality and (5.6), we obtain

$$E \sup_{0 \leq t \leq T} \left| \int_t^T \tilde{Z}^n_s dW_s - \int_t^T Z_s dW_s \right|^2 \leq E \int_t^T \left| \tilde{Z}^n_s - Z_s \right|^2 \, ds \to 0, \quad \text{as } n \to \infty, \quad (5.7)$$

Therefore, from the properties of $(f, \pi)$, we have

$$\Gamma^n_s = f(s, \tilde{Y}^{n-1}_s, \tilde{Z}^{n-1}_s, \tilde{U}^{n-1}_s) + \pi \left(s, \delta \tilde{Y}^{n}_s, \delta \tilde{Z}^{n}_s, \delta \tilde{U}^{n}_s\right) \to f(s, Y_s, Z_s, U_s),$$

$P - a.s.$, for all $t \in [0, T]$ as $n \to \infty$. Then follows by dominated convergence theorem that

$$E \int_0^T |\Gamma^n_s - f(s, Y_s, Z_s, U_s)|^2 \, ds \to 0.$$

Since $\left(\tilde{Y}^n_s, \tilde{Z}^n_s, \tilde{U}^n_s, \Gamma^n_s\right)$ converges in $\mathcal{B}^2(\mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R})$ and

$$E \left(\sup_{0 \leq t \leq T} \left| \tilde{K}^n_t - \tilde{K}^m_t \right|^2 \right) \leq E \left(\tilde{Y}^n_0 - \tilde{Y}^m_0 \right)^2 + E \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}^n_t - \tilde{Y}^m_t \right|^2 \right) + E \int_0^T \left| \Gamma^n_s - \Gamma^m_s \right|^2 \, ds$$

$$+ E \sup_{0 \leq t \leq T} \left| \int_t^T g(s, \tilde{Y}^n_s, \tilde{Z}^n_s, \tilde{U}^n_s) - g(s, \tilde{Y}^m_s, \tilde{Z}^m_s, \tilde{U}^m_s) \right| \, ds$$

$$+ E \sup_{0 \leq t \leq T} \left| \int_t^T \left( \tilde{Z}^n_s - \tilde{Z}^m_s \right) \, dW_s \right|^2 + E \sup_{0 \leq t \leq T} \left| \int_t^T \tilde{U}^n_s(e) - \tilde{U}^m_s(e) \right| \, \bar{\mu}(ds, de) \right|^2,$$

for any $n, m \geq 0$, we deduce that

$$E \left(\sup_{0 \leq t \leq T} \left| \tilde{K}^n_t - \tilde{K}^m_t \right|^2 \right) \to 0,$$

as $n, m \to \infty$. Consequently, there exists a $\mathcal{F}_t$-measurable process $K$ with value in $\mathbb{R}$ such that

$$E \left(\sup_{0 \leq t \leq T} \left| K^n_t - K_t \right|^2 \right) \to 0, \quad \text{as } n \to \infty. \quad (5.7)$$

Finally, we have

$$E \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}^n_t - Y_t \right|^2 + \int_t^T \left| \tilde{Z}^n_s - Z_s \right|^2 \, ds + \int_t^T \int_E \left| \tilde{U}^n_s - U_s \right|^2 \, \lambda(de) \, ds + \sup_{0 \leq t \leq T} \left| K^n_t - K_t \right|^2 \right) \to 0, \quad \text{as } n \to \infty.$$
On the other hand, from the result of Saisho, we have
\[
\int_0^T \left( \tilde{Y}_s^n - S_s \right) d\tilde{K}_s^n \to \int_0^T \left( Y_s - S_s \right) dK_s, \quad \text{P-a.s., as } n \to \infty.
\]
Using the identity \( \int_0^T \left( \tilde{Y}_s^n - S_s \right) d\tilde{K}_s^n = 0 \) for all \( n \geq 0 \), we obtain \( \int_0^T \left( Y_s - S_s \right) dK_s = 0 \). Letting \( n \to +\infty \) in equation (1.1), we prove that \( (Y_t, Z_t, K_t, U_t)_{t \in [0,T]} \) is the solution to (1.1).

Let \( (Y^*, Z^*, U^*, K^*) \) be a solution of (1.1). Then by Theorem 3.1, we have for any \( n \in \mathbb{N}^* \), \( Y^n \leq Y^* \).

Therefore, \( Y \) is a minimal solution of (1.1).

**Remark 5.1.** Using the same arguments and the following approximating sequence
\[
f_n \left( t, \omega, y, z, u \right) = \sup_{\{y', z', u'\} \in B^2(\mathbb{R})} \left[ f \left( t, \omega, y', z', u' \right) - n \left( |y - y'| + |z - z'| + |u - u'| \right) \right],
\]
one can prove that the RBDSDE (1.1) has a maximal solution.

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