Abstract. We study the bialgebra structures on quiver coalgebras and the monoidal structures on the categories of locally nilpotent and locally finite quiver representations. It is shown that the path coalgebra of an arbitrary quiver admits natural bialgebra structures. This endows the category of locally nilpotent and locally finite representations of an arbitrary quiver with natural monoidal structures from bialgebras. We also obtain theorems of Gabriel type for pointed bialgebras and hereditary finite pointed monoidal categories.

1. Introduction

This paper is devoted to the study of natural bialgebra structures on the path coalgebra of an arbitrary quiver and monoidal structures on the category of its locally nilpotent and locally finite representations. A further purpose is to establish a quiver setting for general pointed bialgebras and pointed monoidal categories.

Our original motivation is to extend the Hopf quiver theory \[4, 7, 8, 12, 13, 25, 31\] to the setting of generalized Hopf structures. As bialgebras are a fundamental generalization of Hopf algebras, we naturally initiate our study from this case. The basic problem is to determine what kind of quivers can give rise to bialgebra structures on their associated path algebras or coalgebras.

It turns out that the path coalgebra of an arbitrary quiver admits natural bialgebra structures, see Theorem 3.2. This seems a bit surprising at first sight by comparison with the Hopf case given in \[8\], where Cibils and Rosso showed that the path coalgebra of a quiver \(Q\) admits a Hopf algebra structure if and only if \(Q\) is a Hopf quiver which is very special. Bialgebra structures on general pointed coalgebras are also considered via quivers thanks to the Gabriel type theorem of coalgebras, see \[3, 5\]. Similar to the Hopf case obtained in \[31\], we give a Gabriel type theorem for general pointed bialgebras, see Proposition 3.3.

Our another motivation comes from finite monoidal categories which are natural generalization of finite tensor categories \[10\]. To the knowledge of the authors, not quite much is known for the construction and classification of finite monoidal categories which are not tensor categories, i.e., rigid monoidal categories \[9\]. By taking advantage of the well-developed quiver representation theory, the quiver presentation of a pointed bialgebra \(B\) can help us to investigate the monoidal category of right \(B\)-comodules. Accordingly, some classification results of pointed monoidal categories are obtained, see Proposition 4.1 and Corollary 4.2.

Bialgebra structures on the path coalgebra of a quiver \(Q\) induce monoidal structures on the category \(\text{Rep}^{nilf} Q\) of locally nilpotent and locally finite representations of \(Q\). These monoidal structures are also expected to be useful for the studying of the category \(\text{Rep}^{nilf} Q\) itself. For example, the tensor product of quiver representations naturally leads to the Clebsch-Gordan problem, i.e., the decomposition of the tensor product of any two representations into indecomposable summands, and the computation of the representation ring of \(\text{Rep}^{nilf} Q\), etc. Note that the tensor product given here is different from the vertex-wise and arrow-wise tensor product used in \[14, 15, 18, 19\], which in general is not from bialgebra, and therefore

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should provide different information for the categories of quiver representations. This interesting problem is the third motivation and will be treated in the future.

We remark that a standard dual process will give rise to natural bialgebra structures on the path algebra and monoidal structures on the category of representations of a finite quiver. We prefer the path coalgebraic approach as this is more convenient for exposition and, more importantly, allows infinite quivers.

Throughout, we work over a field \( k \). Vector spaces, algebras, coalgebras, bialgebras, linear mappings, and unadorned \( \otimes \) are over \( k \). The readers are referred to \([24, 30]\) for general knowledge of coalgebras and bialgebras, and to \([1, 28, 29]\) for that of quivers and their applications to (co)algebras and representation theory.

2. Quivers, Representations and Path Coalgebras

As preparation, in this section we recall some basic notions and facts about quivers, representations and path coalgebras.

A quiver is a directed graph. More precisely, a quiver is a quadruple \( Q = (Q_0, Q_1, s, t) \), where \( Q_0 \) is the set of vertices, \( Q_1 \) is the set of arrows, and \( s, t : Q_1 \rightarrow Q_0 \) are two maps assigning respectively the source and the target for each arrow. Note that, in this paper the sets \( Q_0 \) and \( Q_1 \) are allowed to be infinite. If \( Q_0 \) and \( Q_1 \) are finite, then we say \( Q \) is a finite quiver. For \( a \in Q_1 \), we write \( a : s(a) \rightarrow t(a) \). A vertex is, by convention, said to be a trivial path of length 0.

If \( Q_0 \) is a finite quiver. For \( a \in Q_1 \), we write \( a : s(a) \rightarrow t(a) \). A vertex is, by convention, said to be a trivial path of length 0. A path \( g \) is a sequence of concatenated arrows of the form \( p = a_n \cdots a_1 \) with \( s(a_{i+1}) = t(a_i) \) for \( i = 1, \ldots, n-1 \). By \( Q_n \) we denote the set of the paths of length \( n \). A quiver is said to be acyclic if it has no cyclic paths, i.e. nontrivial paths with identical starting and ending vertices.

Let \( Q \) be a quiver and \( kQ \) the associated path space which is the \( k \)-span of its paths. There is a natural coalgebra structure on \( kQ \) with comultiplication as split of paths. Namely, for a trivial path \( g \), set \( \Delta(g) = g \otimes g \) and \( \varepsilon(g) = 1 \); for a non-trivial path \( p = a_n \cdots a_1 \), set

\[
\Delta(p) = t(a_n) \otimes p + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + p \otimes s(a_1)
\]

and \( \varepsilon(p) = 0 \). This is the so-called path coalgebra of the quiver \( Q \).

There exists on \( kQ \) an intuitive length gradation \( kQ = \bigoplus_{n \geq 0} kQ_n \), and with which the comultiplication \( \Delta \) complies perfectly. It is clear that the path coalgebra \( kQ \) is pointed, and the set of group-like elements \( G(kQ) \) is \( Q_0 \). Moreover, the coradical filtration of \( kQ \) is

\[
kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots ,
\]

therefore it is coradically graded in the sense of Chin-Musson \([5]\).

The path coalgebra is exactly the dual notion of the path algebra of a quiver, which is certainly more familiar. Dual to the freeness of path algebras, path coalgebras are cofree. Precisely, for an arbitrary quiver \( Q \), the vector space \( kQ_0 \) is a subcoalgebra of \( kQ \), and over the vector space \( kQ_1 \) there is an induced \( kQ_0 \)-bicomodule structure via

\[
\delta_L(a) = t(a) \otimes a, \quad \delta_R(a) = a \otimes s(a)
\]

for each \( a \in Q_1 \); the path coalgebra has another presentation as the so-called cotensor coalgebra (see, e.g. \([31, 32]\))

\[
\text{CoT}_{kQ_0}(kQ_1) = kQ_0 \oplus kQ_1 \oplus kQ_1 \square kQ_1 \oplus \cdots
\]

and hence enjoys the following

Universal Mapping Property: Let \( f : C \rightarrow kQ \) be a coalgebra map, \( \pi_n : kQ \rightarrow kQ_n \) the canonical projection and set \( f_n := \pi_n \circ f : C \rightarrow kQ_n \) for each

\[
\text{CoT}_{kQ_0}(kQ_1) = kQ_0 \oplus kQ_1 \oplus kQ_1 \square kQ_1 \oplus \cdots
\]
that the category of right modules equivalently, the set of locally nilpotent and locally finite representations of $\text{Rep}(Q)$ is said to be locally nilpotent if for all $g \in Q_0$ and all $x \in V_g$, there exist at most finitely many paths $p$ with source $g$ satisfying $V_p(x) \neq 0$. A representation is called locally finite if it is a directed union of finite-dimensional representations. We denote by $\text{Rep}^{\text{nilf}}(Q)$ the full subcategory of locally nilpotent and locally finite representations of $\text{Rep}(Q)$. It is well-known that the category of right $\mathbb{k}Q$-comodules is equivalent to $\text{Rep}^{\text{nilf}}(Q)$, see [20].

3. Quiver Bialgebras

In this section we show that the path coalgebra of an arbitrary quiver can be endowed with natural bialgebra structures. A Gabriel type theorem for pointed bialgebras is also given. Some examples are presented.

We start with the definition of bialgebra bimodules. Let $B$ be a bialgebra. A $B$-bialgebra bimodule is a vector space $M$ which is a $B$-bimodule and simultaneously a $B$-comodule such that the $B$-bicomodule structure maps are $B$-bimodule maps, or equivalently, the $B$-bimodule structure maps are $B$-bicomodule maps.

Lemma 3.1. Let $Q$ be a quiver. The associated path coalgebra $\mathbb{k}Q$ admits a bialgebra structure if and only if $Q_0$ has a monoid structure and $\mathbb{k}Q_1$ can be given a $\mathbb{k}Q_0$-bialgebra bimodule structure. Moreover, the set of graded bialgebra structures on the path coalgebra $\mathbb{k}Q$ is in one-to-one correspondence with the set of pairs $(S, M)$ in which $S$ is a monoid structure on $Q_0$ and $M$ is a $\mathbb{k}S$-bialgebra bimodule structure on $\mathbb{k}Q_1$.

Proof. Assume first that the path coalgebra $\mathbb{k}Q$ admits a bialgebra structure. By considering its graded version induced by the coradical filtration (see, e.g. [24, 26]), we can assume further that the bialgebra structure on $\mathbb{k}Q$ is coradically graded. Note that the identity $1$ is a group-like, so $1$ lies in $Q_0$ which is the set of group-like elements of $\mathbb{k}Q$. For any two elements $g, h \in Q_0$, we have $\Delta(gh) = \Delta(g)\Delta(h) = gh \otimes gh$ and $\varepsilon(gh) = \varepsilon(g)\varepsilon(h) = 1$, therefore $gh \in Q_0$. Hence the restriction of the multiplication of $\mathbb{k}Q$ onto $Q_0$ gives rise to a monoid structure. The $\mathbb{k}Q_0$-bicomodule structure on $\mathbb{k}Q_1$ is given as in Subsection 2.3. The multiplication of $\mathbb{k}Q_0$ provides a bimodule structure on $\mathbb{k}Q_1$. Finally note that the axioms for bialgebras guarantee that the so-defined $\mathbb{k}Q_1$ is a $\mathbb{k}Q_0$-bialgebra bimodule.

Conversely, assume that $Q_0$ can be endowed a monoid structure and the vector space $\mathbb{k}Q_1$ has a $\mathbb{k}Q_0$-bialgebra bimodule structure. By the method of Nichols [26], these data can be used to construct a graded bialgebra structure on the path coalgebra $\mathbb{k}Q$ by the universal mapping property. Nichols’ construction was applied to quiver setting for Hopf algebras in [7, 8, 12, 13]. For completeness we include the construction in below.
The cotensor coalgebra $\text{CoT}_{kQ_0}(kQ_1)$ is exactly the path coalgebra $kQ$. The $kQ_0$-bimodule structure on $kQ_1$ can be extended to a multiplication on $kQ$ via the universal mapping property of $kQ$. Let $M_0 : kQ \otimes kQ \rightarrow kQ_0$ be the composition of the canonical projection $\pi_0 \otimes \pi_0 : kQ \otimes kQ \rightarrow kQ_0 \otimes kQ_0$ and the multiplication of the monoid algebra $kQ_0$, and $M_1 : kQ \otimes kQ \rightarrow kQ_1$ the composition of the canonical projection $\pi_0 \otimes \pi_1 \bigoplus \pi_1 \otimes \pi_0 : kQ \otimes kQ \rightarrow kQ_0 \otimes kQ_1 \bigoplus kQ_1 \otimes kQ_0$

and the sum of left and right module actions. Then it is clear that $M_0$ is a coalgebra map and $M_1$ is a $kQ_0$-bicomodule map. Let $M_n = \Delta_2^{(n-1)} \circ M_1^\otimes n$, where $\Delta_2$ is the coproduct of the tensor product coalgebra $kQ \otimes kQ$. For any path $p$ of length $n$, it is easy to see that $M_1(p) = 0$ if $l \neq n$. Therefore $M = \sum_{n \geq 0} M_n$ is a well-defined coalgebra map and moreover respects the length gradation. The associativity for the map $M$ can be deduced from the associativity of the bimodule action without difficulty by a standard application of the universal mapping property as before. The unit map is obvious. Hence we have defined an associative algebra structure and we obtain a graded bialgebra structure on $kQ$.

The latter one-to-one correspondence is obvious. □

Now we state our first main result.

**Theorem 3.2.** Let $Q$ be a quiver. The associated path coalgebra $kQ$ always admits bialgebra structures.

**Proof.** By Lemma 3.1, it is enough to provide a monoid structure on $Q_0$ and a $kQ_0$-bialgebra bimodule on $kQ_1$. In the first place, we fix the $kQ_0$-bicomodule structure on $kQ_1$ as in Subsection 2.3. If $Q_0$ has only one element, then let it be the unit group and take the trivial $kQ_0$-bimodule structure on $kQ_1$. Obviously, this defines a necessary bialgebra bimodule. Now we assume $Q_0$ contains at least 2 elements. Take an arbitrary element, say $e \in Q_0$, and set it to be the identity, i.e., let $eg = g = ge$ for all $g \in Q_0$. Take any element $z \in Q_0$ other than $e$, and make it to be a “zero” element. That is, let $gz = z = zg$ for any $g \in Q_0$. For any two elements $g, h \in Q_0 - \{e, z\}$, set $gh = z$. Here, $g = h$ is allowed. One can verify without difficulty that this endows $Q_0$ with a monoid structure. For the $kQ_0$-bimodule structure on $kQ_1$, define $e.a = a = a.e$, $f.a = 0 = a.f$

for all $a \in Q_1$ and all $f \in Q_0 - \{e\}$. Clearly, the bicomodule structure maps are bimodule maps and hence we have obtained a $kQ_0$-bialgebra bimodule structure on $kQ_1$. □

Note that, in the proof of the previous theorem we provide only a very “trivial” example of graded bialgebra structures for the path coalgebra. The classification of all the graded bialgebra structures on $kQ$, or equivalently the classification of all the suitable monoid structures on $Q_0$ and $kQ_0$-bialgebra bimodule structures on $kQ_1$, is still not clear and is a very interesting problem of quiver combinatorics.

However, if $Q_0$ can be afforded a group structure and further $kQ_1$ can be afforded a corresponding bialgebra bimodule structure over $kQ_0$, then the situation is much clearer. Indeed, in this case $kQ_0$ becomes a Hopf algebra and $kQ_1$ becomes a $kQ_0$-Hopf bimodule [26]. Now the the fundamental theorem of Hopf modules of Sweedler [30, Theorem 4.1.1] can be applied, and the category of $kQ_0$-Hopf bimodules is proved to be equivalent to the direct product of the representation categories of a class of subgroups of $Q_0$ by Cibils and Rosso [7]. Note that, a quiver $Q$ with $Q_0$ having a group structure and $kQ_1$ having a $kQ_0$-Hopf bimodule structure is far from arbitrary. Such quivers are called covering quivers in [13] and Hopf quivers in [8]. It would be of interest to generalize the classification of Hopf bimodules over groups to that of bialgebra bimodules over monoids.
By a theorem of Heyneman and Radford, see e.g. [24, Theorem 5.3.1], the coalgebra map $\Theta$ is injective since its restriction to the first term of the coradical filtration is injective. Again, by the universal mapping property, one can show that the map $\Theta$ is also an algebra map. Therefore, $\Theta$ is actually an embedding of bialgebras. The last condition $kQ_0 \oplus kQ_1 \subseteq gr B$ guarantees that the quiver $Q$ is unique. \qed
In the following we give some examples of bialgebras on the path coalgebras of quivers.

**Example 3.4.** Let $\mathcal{K}_n$ be the $n$-Kronecker quiver, i.e. a quiver of the form

\[
\bullet \to \bullet \to \bullet 
\]

Denote the arrows as $a_1, \ldots, a_n$. Let the source vertex be $e$ and the target vertex be $z$ as in Theorem 3.2. Then there is a bimodule structure on the space spanned by $\{a_i\}_{i=1}^n$ over the monoid $\{e, z\}$ defined by

\[ e.a_i = a_i = a_i.e, \quad z.a_i = 0 = a_i.z \]

for all $i$. We have the following multiplication formulas for the quiver bialgebra $k\mathcal{K}_n$:

\[ a_i \cdot a_j = 0, \quad \forall \ 1 \leq i, j \leq n. \]

**Example 3.5.** Let $\mathcal{S}_n$ be the $n$-subspace quiver, i.e.

\[
\bullet \\
\rightarrow \\
\cdot \cdot \cdot \\
\rightarrow \\
\bullet
\]

Denote the target vertex by $e$, the source vertices by $f_1, \ldots, f_n$, and the corresponding arrows by $a_1, \ldots, a_n$. Assign $e$ to be the identity, $f_1$ to be the “zero” element, we get a monoid structure on the set of vertices as in Theorem 3.2. The bimodule structure is defined similarly

\[ e.a_i = a_i = a_i.e, \quad f_i.a_j = 0 = a_j.f_i \]

for all $1 \leq i, j \leq n$. The multiplication of the quiver bialgebra $k\mathcal{S}_n$ is similar: $a_i \cdot a_j = 0, \forall \ i, j$.

**Example 3.6.** Let $\mathcal{A}_\infty$ be the quiver

\[
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \ldots
\]

Index the vertices $g_i$, from left to right, by the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and consider the additive monoid structure, i.e. $g_i g_j = g_{i+j}$. Denote the arrow $g_i \rightarrow g_{i+1}$ by $a_i$. Define a bialgebra bimodule structure on the space of arrows by

\[ g_i a_j = a_{i+j}, \quad a_j g_i = q^j a_{i+j} \]

for all $i, j \in \mathbb{N}$, where $q \in \mathbb{k} - \{0\}$ is a parameter. Assume further that $q$ is not a root of unity. By a routine verification one can show that the axioms of bialgebra bimodules are satisfied. Let $p_{i}^{j}$ denote the path $a_{i+l-1} \cdots a_{i+1} a_i$ if $l \geq 1$, or $g_i$ if $l = 0$. Apparently, $\{p_i^j\}_{i,j \geq 0}$ is a basis of $k\mathcal{A}_\infty$. Then using Subsection 3.4 as well as induction we have the following multiplication formula

\[ p_i^j \cdot p_j^m = q^j \binom{l+m}{m} p_{i+j}^{l+m} \]

for all $i, j, l, m \in \mathbb{N}$. Here we use the quantum binomial coefficients as in [17]. For completeness, we recall their definition. For any $q \in \mathbb{k}$, integers $l, m \geq 0$, define

\[ l_q = 1 + q + \cdots + q^{l-1}, \quad l_q! = 1_q \cdots l_q, \quad \text{and} \quad \binom{l+m}{l}_q = \frac{(l+m)!_q}{l_q! m!_q}. \]

Clearly the quiver bialgebra $k\mathcal{A}_\infty$ is generated as an algebra by $g_i$ and $a_0$ with relation $a_0 g_1 = q g_1 a_0$. Therefore, $k\mathcal{A}_\infty$ is exactly the quantum plane of Manin [23] and the bialgebra structure given here is identical to that in [17, P118].
We remark that the quivers in the previous examples admit no Hopf algebra structures, as they are certainly not Hopf quivers. For simple quivers as in Examples 3.4 and 3.5, it is not difficult to classify all the possible graded bialgebra structures on the path coalgebras. The bialgebra structure given in Example 3.6 is different from the “trivial” one as given in the proof of Theorem 3.2.

Though theoretically any graded pointed bialgebra can be obtained as a large sub-bialgebra of a quiver bialgebra, there seems no chance to present a general method for such construction. However, in the following we will show that a large class of pointed bialgebras whose coradical filtration has length 2 can be constructed systematically.

**Proposition 3.7.** Let $Q$ be a quiver. Assume that in $Q_0$ there is a sink (i.e., admitting only incoming arrows) or a source (i.e., admitting only outgoing arrows). Then there exists on $kQ$ a bialgebra structure such that $kQ_0 \oplus kQ_1$ becomes its sub-bialgebra.

**Proof.** By assumption, there is a sink or a source in $Q_0$. Now fix such a vertex to be the identity $e$ of the monoid on $Q_0$ and take the graded bialgebra structure on $kQ$ as given in the proof of Theorem 3.2. Let $B = kQ_0 \oplus kQ_1$. We claim that $B$ is a sub-bialgebra of $kQ$. In fact, we only need to verify that the multiplication of arrows is closed in $B$. Given an arbitrary pair of arrows $a : g \rightarrow h$ and $b : u \rightarrow v$, by Subsection 3.4 the multiplication is

$$a \cdot b = [a.v][g,b] + [h.b][a.u].$$

Clearly the term $[a.v][g,b]$ survives, namely not $=0$, only if $v = g = e$, but this contradicts with the assumption that $e$ is a sink or a source. Similarly we have $[h.b][a.u] = 0$. This shows that the multiplication of $kQ$ is indeed closed in $B$. □

**Remark 3.8.** Coalgebras with coradical filtration having length 2 are studied by Kosakowska and Simson in [20], where a reduction to hereditary coalgebras is presented and the Gabriel quiver is discussed in terms of irreducible morphisms. The dual of such a coalgebra is an algebra of radical square zero. The class of radical square zero algebras is very important in the representation theory of Artin algebras (see e.g. [1, 2]). The dual of the previous proposition asserts that every elementary radical square zero algebra with condition $\text{Ext}^1(S, -) = 0$, or $\text{Ext}^1(-, S) = 0$ for some simple module $S$ has a bialgebra structure, therefore its module category has natural tensor product.

4. Monoidal Structures over Quiver Representations

In this section we consider the natural monoidal structures on the categories of locally nilpotent representations of quivers arising from bialgebra structures.

Recall that a monoidal category is a sextuple $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor, $1$ an object, $\alpha : \otimes \circ (\otimes \times \text{Id}) \rightarrow \otimes \circ (\text{Id} \times \otimes)$, $\lambda : 1 \otimes - \rightarrow \text{Id}$, $\rho : - \otimes 1 \rightarrow \text{Id}$ are natural isomorphisms such that the associativity and unitarity constraints hold, or equivalently the pentagon and the triangle diagrams are commutative, see e.g. [21] for detail.

Natural examples of monoidal structures are the categories of $B$-modules and $B$-comodules with $B$ a bialgebra, see e.g. [17, 24]. Recall that, if $U$ and $V$ are right $B$-comodules and $U \otimes V$ the usual tensor product of $k$-spaces, then the comodule structure of $U \otimes V$ is given by $u \otimes v \mapsto u_0 \otimes v_0 \otimes u_1 v_1$, where we use the Sweedler notation $u \mapsto u_0 \otimes u_1$ for comodule structure maps. The unit object is the trivial comodule $k$ with comodule structure map $k \mapsto k \otimes 1$. On the other hand, by the reconstruction formalism, monoidal categories with fiber functors are coming in this manner, see e.g. [9, 22].

The monoidal categories arising from quiver bialgebras (as their right comodule categories) share a common property, i.e., their simple objects all have $k$-dimension 1 and consist of a monoid. Stimulated by this and the notion of pointed tensor
From now on, the field \( k \) is assumed to be algebraically closed and the monoidal categories under consideration are \( k \)-linear abelian. A monoidal category is said to be finite, if the underlying category is equivalent to the category of finite-dimensional comodules over a finite-dimensional coalgebra. This is a natural generalization of the notion of finite tensor categories of Etingof and Ostrik [10].

Classification of finite monoidal categories is a fundamental problem. Our results in Section 3 indicate that even finite pointed monoidal categories are “over” pervasive, it is necessary to impose proper condition before considering the classification problem. In the following, by taking advantage of the theory of quivers and their representations, we give some classification results for some classes of finite pointed monoidal categories.

An abelian category is said to be hereditary, if the extension bifunctor \( \text{Ext}^n \) vanishes at each degree \( n \geq 2 \). Next we consider hereditary finite pointed monoidal categories. Though the following results are direct consequences from some well-known theorems of quivers and representations, we feel it is of interest to include them here.

**Proposition 4.1.** A hereditary finite pointed monoidal category with a fiber functor is equivalent to \( (\text{Rep} Q, F) \) with \( Q \) a finite acyclic quiver and \( F \) the forgetful functor from \( \text{Rep} Q \) to the category \( \text{Vec}_k \) of vector spaces.

**Proof.** By the standard reconstruction process, see e.g. [9, 22], a finite monoidal category with fiber functor is equivalent to \( (B-\text{comod}, F) \) in which \( B-\text{comod} \) is the category of finite-dimensional right comodules over a finite-dimensional bialgebra \( B \) and \( F \) is the forgetful functor. Note that, the category of right comodules over a finite-dimensional bialgebra \( B \) is pointed if and only if \( B \) is pointed. Now by the Gabriel type theorem for pointed coalgebras [5], there exists a unique quiver \( Q \) such that \( B \) is isomorphic to a large subcoalgebra, i.e. includes the space spanned by the set of vertices and arrows, of the path coalgebra \( kQ \). The hereditary condition of the category \( B-\text{comod} \) forces \( B \) to be hereditary as a coalgebra. This indicates that \( B \) is isomorphic to \( kQ \) as a coalgebra. Since \( B \cong kQ \) is finite-dimensional, the quiver \( Q \) must be finite and acyclic. Obviously, any representation of \( Q \) is automatically locally nilpotent, hence \( B-\text{comod} \) is equivalent to \( \text{Rep} Q \). \( \square \)

Now we can use Gabriel’s famous classification theorem [11] on quivers of finite-representation type, i.e. admitting only finitely many indecomposable representations up to isomorphism, to describe hereditary pointed monoidal categories in which there are only finitely many iso-classes of indecomposable objects. Follow the terminology of [16], a finite monoidal category is said to be of finite type if it has only finitely many iso-classes of indecomposable objects.

**Corollary 4.2.** A hereditary pointed monoidal category of finite type with a fiber functor is of the form \( \text{Rep} Q \) where \( Q \) is a finite disjoint union of quivers of ADE type.

Finally we give two examples of quiver monoidal categories. We also work out their Clebsch-Gordan formula and representation ring respectively.

**Example 4.3.** Let \( A_n \) be the quiver

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

with \( n \geq 2 \) vertices \( v_1, \ldots, v_n \) and \( n-1 \) arrows \( a_1, \ldots, a_{n-1} \) where \( a_i : v_i \to v_{i+1} \).

Set \( v_1 \) to be the identity, \( v_2 \) to be the zero element, take the monoid structure on \( \{v_i|1 \leq i \leq n\} \) as in Theorem 3.2 and consider the corresponding bialgebra structure with multiplication given by

\[
v_1 \cdot a_i = a_i = a_i \cdot v_1, \quad v_j \cdot a_i = 0 = a_i \cdot v_j \quad (j \geq 2), \quad a_i \cdot a_j = 0.
\]
For a pair of integers $i, j$ satisfying $1 \leq i \leq j \leq n$, define a representation $V(i, j)$ of $\mathcal{A}_n$ by

$$V(i, j)_{V_k} = \begin{cases} k, & i \leq k \leq j \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad V(i, j)_{v_k} = \begin{cases} 1, & i \leq k \leq j - 1 \\ 0, & \text{otherwise} \end{cases}.$$  

It is well-known that the set $\{V(i, j) \mid 1 \leq i \leq j \leq n\}$ is a complete list of indecomposable representations of $\mathcal{A}_n$. For the Clebsch-Gordan problem of the category $\text{Rep} \mathcal{A}_n$, it is enough to consider the decomposition rule of $V(i, j) \otimes V(k, l)$ thanks to the Krull-Schmidt theorem. Given a representation $V(i, j)$, let $e_s(i \leq s \leq j)$ denote a basis element of the vector space $k$ assigned to the $s$-th vertex. Recall that the associated comodule structure map for $V(i, j)$ is given by

$$\delta(e_s) = \sum_{x=s}^{j} e_x \otimes p_x^{s-x},$$

where by $p_x^y$ we denote the path of length $y$ starting at the vertex $x$. Now for $e_s \otimes e_t \in V(i, j) \otimes V(k, l)$, we have

$$\delta(e_s \otimes e_t) = \begin{cases} \sum_{s=x}^{j} e_x \otimes e_t \otimes p_x^{s-x} + \sum_{y=t}^{l} e_s \otimes e_y \otimes p_t^{y-t}, & s = t = 1; \\ \sum_{y=t}^{l} e_s \otimes e_y \otimes p_t^{y-t}, & s = 1, t \geq 2; \\ \sum_{s=x}^{j} e_x \otimes e_t \otimes p_x^{s-x}, & s \geq 2, t = 1; \\ e_s \otimes e_t \otimes p_x^{s-x}, & s \geq 2, t \geq 2. \end{cases}$$

Therefore, it is clear that

$$V(i, j) \otimes V(k, l) = \begin{cases} V(i, j) \otimes V(k, l) \oplus V(2, 2)^{(j-i)(l-k)+1}, & i = k = 1; \\ V(k, l) \oplus V(2, 2)^{(j-i)(l-k)+1}, & i = 1, k \geq 2; \\ V(i, j) \oplus V(2, 2)^{(j-i)(l-k)+1}, & i \geq 2, k = 1; \\ V(2, 2)^{(j-i)(l-k)+1}, & i \geq 2, k \geq 2. \end{cases}$$

**Example 4.4.** Consider the infinite quiver $\mathcal{A}_\infty$. Take the bialgebra structure on $k \mathcal{A}_\infty$ as in Example 3.6 and keep the notations therein. For any pair of integers $i, j$ with $0 \leq i \leq j$, define a representation $V(i, j)$ of $\mathcal{A}_\infty$ as the previous example. Clearly, $\{V(i, j) \mid 0 \leq i \leq j\}$ is a complete set of locally nilpotent and locally finite indecomposable representations of $\mathcal{A}_\infty$. As in Example 4.3, take a basis element $e_s$ for the vector space $k$ in $V(i, j)$ attached to the $s$-th vertex $g_s$. The corresponding comodule structure map of $V(i, j)$ is

$$\delta(e_s) = \sum_{x=s}^{j} e_x \otimes p_x^{s-x}.$$  

Consider the tensor product $V(i, j) \otimes V(k, l)$. For $e_s \otimes e_t \in V(i, j) \otimes V(k, l)$, we have

$$\delta(e_s \otimes e_t) = \sum_{x=s}^{j} \sum_{y=t}^{l} \binom{x-s+y-t}{y-t} e_x \otimes e_y \otimes p_s^{x-s+y-t}.$$  

With this, it is not hard to see that

$$V(0, 1) \otimes V(0, n) = V(0, n+1) \oplus V(1, n)$$

and

$$V(i, j) \otimes V(1, 1) = V(i+1, j+1) = V(1, 1) \otimes V(i, j).$$

This implies that the representation ring of $\text{Rep}^{lnf} \mathcal{A}_\infty$ is generated by $V(0, 1)$ and $V(1, 1)$ and is isomorphic to the polynomial ring in two variables $\mathbb{Z}[X, Y]$.

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References

[1] I. Assem, D. Simson, A. Skowronski, Elements of the Representation Theory of Associative Algebras. Vol. 1. Techniques of Representation Theory. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] X. Chen, H.-L. Huang, P. Zhang, Dual Gabriel theorem with applications, Sci. China Ser. A 49 (2006), no. 1, 9-26.

[4] X. Chen, P. Zhang, Comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver methods, J. Pure Appl. Algebra 211 (2007), no. 3, 862-876.

[5] W. Chin, S. Montgomery, Basic coalgebras, Modular interfaces (Riverside, CA, 1995), 41-47, AMS/IP Stud. Adv. Math. 4, Amer. Math. Soc., Providence, RI, 1997.

[6] W. Chin, I.M. Musson, The coradical filtration for quantized enveloping algebras, J. London Math. soc. (2) 53 (1996) 50-62.

[7] C. Cibils, M. Rosso, Algèbres des chemins quantiques, Adv. Math. 125 (1997) 171-199.

[8] C. Cibils, M. Rosso, Hopf quivers, J. Algebra 254 (2002) 241-251.

[9] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, lecture note for the MIT course 18.769, 2009. available at: www-math.mit.edu/ etingof/tenscat.pdf

[10] P. Etingof, V. Ostrik, Finite tensor categories, Moscow Math. J. 4 (2004) 627-654.

[11] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71-103.

[12] E.L. Green, Constructing quantum groups and Hopf algebras from coverings, J. Algebra 176 (1995) 12-33.

[13] E.L. Green, O. Solberg, Basic Hopf algebras and quantum groups, Math. Z. 229 (1998) 45-76.

[14] M. Herschend, Tensor products on quiver representations, J. Pure Appl. Algebra 212 (2008) 452-469.

[15] M. Herschend, On the representation ring of a quiver, Algebr. Represent. Theory, 12 (2009) 513-541.

[16] H.-L. Huang, G. Liu, Y. Ye, Quivers, quasi-quantum groups and finite tensor categories, Comm. Math. Phys. 303 (2011) 595-612.

[17] C. Kassel, Quantum Group, Graduate Texts in Math. 155, Springer-Verlag, New York, 1995.

[18] R. Kinser, The rank of a quiver representation, J. Algebra 320(6) (2008) 2363-2387.

[19] R. Kinser, Rank functions on rooted tree quivers, Duke Math. J. 152(1) (2010) 27-92.

[20] J. Kosakowska, D. Simson, Bipartite coalgebras and a reduction functor for coradical square complete coalgebras, Colloq. Math. 112 (2008), no. 1, 89-129.

[21] S. Mac Lane, Categories for the working mathematicians, 3rd Edition, Graduate Texts in Math. 5, Springer-Verlag, New York, 1998.

[22] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.

[23] Yu.I. Manin, Some remarks on Koszul algebras and quantum groups, Ann. Inst. Fourier. (Grenoble) 37(4) (1987) 191-205.

[24] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Regional Conf. Series in Math. 82, Amer. Math. Soc., Providence, RI, 1993.

[25] S. Montgomery, Indecomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. of the Amer. Math. Soc. 123 (1995) 2343-2351.

[26] W.D. Nichols, Bialgebras of type one, Communications in Algebra 6(15) (1978) 1521-1552.

[27] M. Rosso, Quantum groups and quantum shuffles, Invent. Math. 133 (1998) 399-416.

[28] D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, Colloq. Math. 90 (2001) 101-150.

[29] D. Simson, Coalgebras of tame comodule type, comodule categories, and a tame-wild dichotomy problem, in: Representations of algebras and related topics, 561-660, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.

[30] M. Sweedler, Hopf Algebras, W. A. Benjamin, Inc., New York, 1969.

[31] F. Van Oystaeyen, P. Zhang, Quiver Hopf algebras, J. Algebra 280(2) (2004) 577-589.

[32] D. Woodcock, Some categorical remarks on the representation theory of coalgebras, Comm. Algebra 25 (1997) 2775-2794.

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