CONTINUITY OF MONGE-AMPÈRE POTENTIALS IN BIG COHOMOLOGY CLASSES

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Abstract. Extending Di Nezza-Lu’s approach [16] to the setting of big cohomology classes, we prove that solutions of degenerate complex Monge-Ampère equations on compact Kähler manifolds are continuous on a Zariski open set. This allows us to show that singular Kähler-Einstein metrics on log canonical varieties of general type have continuous potentials on the ample locus outside of the non-klt part.

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1. Introduction

Finding canonical metrics on complex varieties is a fundamental problem of complex geometry. As evidenced by recent developments in Kähler geometry in connection with the Minimal Model Program, it is natural and necessary to allow the varieties $Y$ in question to be singular. Working on a desingularization $\pi : X \to Y$ is led to consider degenerate complex Monge-Ampère equations of the form

$$(\theta + dd^c \varphi)^n = e^{\lambda \varphi} f dV,$$

where $\theta$ is a smooth closed real $(1, 1)$-form representing a big cohomology class $\alpha$ on $X$, $\lambda \in \{0, \pm 1\}$, and $f$ is a density of the form $f = e^{\psi_+ - \psi_-}$, the functions $\psi_+, \psi_-$ being quasi-plurisubharmonic on $X$. The integrability properties of $f$ depend on the singularities of $Y$.

Finding a Kähler-Einstein metric on a stable variety $Y$ boils down to solving (1.1) on $X$ for $\lambda = 1$ and $f \in L^{1-\delta}(X, dV)$ for all $\delta \in (0, 1)$. Building on the variational approach developed in [5], Berman and Guenancia [6] have proved that the equation (1.1) admits a unique finite energy solution $\varphi$, in the sense of [20, 5]. When $\{\theta\}$ is additionally nef, they established the smoothness of $\varphi$ on a Zariski open set. As in the classical case of Yau [28], the main difficulty lies in establishing an $a \text{ priori}$ $C^0$-estimate. Unfortunately the (normalized) solution $\varphi$ to (1.1) is in general unbounded, so a natural idea is to try and bound such solution from below by a reference quasi-plurisubharmonic function.
This motivates the following general question: for which densities \( f \geq 0 \) is the solution \( \varphi \) locally bounded in some Zariski open subset of \( X \)? The main result of this paper is the following:

**Theorem 1.1.** Let \( X \) be a compact Kähler manifold of dimension \( n \) and fix a smooth closed real \((1,1)\)-form \( \theta \) which represents a big cohomology class. Let \( \varphi \in \mathcal{E}(X, \theta) \) be the unique normalized solution to

\[
(1.2) \quad (\theta + dd^c \varphi)^n = f dV, \quad \sup_X \varphi = 0.
\]

Assume that \( f \leq e^{-\varphi} \) for some quasi-plurisubharmonic function \( \varphi \) on \( X \). Then \( \varphi \) is continuous on \( \text{Amp}(\theta) \setminus E_1(\varphi) \), where \( \text{Amp}(\theta) \) is the ample locus of \( \theta \) and \( E_1(\varphi) = \{ x \in X : \nu(\varphi, x) \geq 1 \} \), with \( \nu(\varphi, x) \) being the Lelong number of \( \varphi \) at \( x \).

Let us recall that \( E_1(\varphi) \) which is called the Lelong super-level set of \( \varphi \), is an analytic subset of \( X \) by Siu’s result [24]. We refer the reader to Section 2.3 for the definition of the ample locus of a big cohomology class.

Since \( \mu = f dV \) is non-pluripolar, it is known [8, Section 3] that there exists a unique normalized solution \( \varphi \in \mathcal{E}(X, \theta) \), so the point is to study its regularity. The idea of the proof is that we first use Demailly’s equisingular approximation [12, 13] (see Theorem 2.1) to replace \( \varphi \) by a quasi-psh function \( \varphi_1 \) which has analytic singularities with polar locus \( Z \) contained in the set of points where the Lelong number of \( \varphi \) is greater than or equal to 1. We then adapt the approach of Di Nezza and Lu [16] (see Theorem 3.2) to prove the continuity of \( \varphi \) in the complement of \( E_1(\varphi) \) in the ample locus of \( \theta \). We also prove a slightly stronger version of Theorem 1.1 valid for more singular densities (see Theorem 3.1).

When the density \( f \) is smooth in a Zariski open set (i.e. outside an analytic subset), one expects the solution \( \varphi \) to be smooth in a Zariski open set (see [18, Question 21, 22]), but we are unable to prove this for the moment. When \( f \) belongs to \( L^p(X) \) for some \( p > 1 \), the Hölder continuity of \( \varphi \) on the ample locus was shown in [14]. The smoothness of \( \varphi \) when \( f \) is smooth is largely open.

Under the extra assumption that the class \( \{ \theta \} \) is nef, the regularity properties for the solutions for the degenerate Monge-Ampère equation (1.2) have been studied by many authors (see [8, 6, 16] and the references therein). The strategy in these papers is that one first establishes a relative uniform estimate which allows to adapt classical ideas of Yau [28] and Siu [25] to obtain locally uniform estimates for the Laplacian, and one finally uses Evans-Krylov’s general regularity theory to conclude. In the above case functions in \( \mathcal{E}(X, \theta) \) have zero Lelong numbers (see [20, Corollary 1.8], [10, Theorem 1.1]). Using this property Di Nezza and Lu [16] have generalized Kolodziej’s approach [22] to establish a relative uniform estimate. Let us mention that in the general case of a big class even the "least singular" potential \( V_0 \) may have positive Lelong numbers. To overcome this difficulty we exploit fine properties of quasi-plurisubharmonic envelopes inspired by [11, 23].

Our approach allows us to deal with non-nef data. As an application we prove that the unique singular Kähler-Einstein metric obtained in [6] is continuous on some Zariski open subset. More precisely, we have the following:

**Corollary 1.2.** Let \( (Y, \Delta) \) be a projective log canonical pair of general type, i.e. the canonical line bundle \( K_Y + \Delta \) is big. Then there is a unique singular Kähler-Einstein metric \( \omega \) on \( Y \) such that

\[
\text{Ric}(\omega) = -\omega + [\Delta]
\]

in the weak sense of currents, and such that \( \int_Y \omega^n = \text{vol}(K_Y + \Delta) \). Furthermore \( \omega \) has continuous potentials on \( \text{Amp}(K_Y + \Delta) \cap (Y, \Delta)_{\text{reg}} \setminus [\Delta] \). Here, \((Y, \Delta)_{\text{reg}}\)
denotes the locus of points \( p \in Y \) where the pair \((Y, \Delta)\) is log smooth at \( p \), i.e. \( Y \) is smooth and \( \Delta \) has simple normal crossing (snc) support on a neighborhood of \( p \), and \(|\Delta|\) denotes the integral part of \( \Delta \), i.e. if \( \Delta = \sum d_i \Delta_i \), then \(|\Delta| = \sum |d_i| \Delta_i \).

Recall that the ample locus of a big line bundle \( L \), denoted by \( \text{Amp}(L) \) may be defined as the ample locus of its first Chern class \( \alpha = c_1(L) \) (see Definition 2.2).

If \( K_Y + \Delta \) is additionally nef, then the potential of \( \omega \) was already known to be smooth on \( \text{Amp}(K_Y + \Delta) \cap (Y, \Delta)_{\text{reg}} \setminus |\Delta| \), as follows from the combined arguments of Boucksom-Eyssidieux-Guedj-Zeriahi \[8\] (see also \[4\]) and Berman-Guenancia \[6\].

**Organization of the paper.** The paper is organized as follows. In Section 2 we recall basic pluripotential theory that will be needed later on. The proof of Theorem 1.1 is given in Section 3.1, while Corollary 1.2 is proved in Section 3.2.

**Notations.** In the whole article we fix

- \( X \) a \( n \)-dimensional compact Kähler manifold,
- \( dV \) a smooth volume form on \( X \),
- \( \alpha \in H^{1,1}(X, \mathbb{R}) \) a big cohomology class, and \( \theta \) a smooth representative of \( \alpha \)
- a Kähler form \( \omega \) so that \( \omega \geq \theta \).

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## 2. Preliminaries

The purpose of this section is to recall some essential materials in pluripotential theory which will be used later.

### 2.1. Quasi-psh functions.

Recall that an upper semi-continuous function \( \varphi : X \to \mathbb{R} \cup \{-\infty\} \) is called quasi-plurisubharmonic (quasi-psh for short) if it is locally the sum of a smooth and a plurisubharmonic (psh for short) function. We say that \( \varphi \) is \( \theta \)-plurisubharmonic (\( \theta \)-psh for short) if it is quasi-psh, and \( \theta + dd^c \varphi \geq 0 \) in the sense of currents, where \( d^c \) is normalized so that \( dd^c = \frac{1}{n} \partial \overline{\partial} \).

By the \( dd^c \)-lemma any closed positive \((1,1)\)-current \( T \) cohomologous to \( \theta \) can be written as \( T = \theta + dd^c \varphi \) for some \( \theta \)-psh function \( \varphi \) which is furthermore unique up to an additive constant.

We let \( \text{PSH}(X, \theta) \) denote the set of all \( \theta \)-psh functions which are not identically \( -\infty \). This set is endowed with the \( L^1(X) \)-topology. By Hartog’s lemma \( \varphi \mapsto \sup_X \varphi \) is continuous in this weak topology. Since the set of closed positive currents in a fixed cohomology class is compact (in the weak topology), it follows that the set of \( \varphi \in \text{PSH}(X, \theta) \), with \( \sup_X \varphi = 0 \) is compact.

Quasi-psh functions are in general singular, and a convenient way to measure their singularities is the Lelong numbers. Let \( x_0 \in X \). Fixing a holomorphic chart \( x_0 \in V_{x_0} \subset X \), the Lelong number \( \nu(\varphi, x_0) \) of a quasi-psh function \( \varphi \) at \( x_0 \in X \) is defined as follows:

\[
\nu(\varphi, x_0) := \sup\{\gamma \geq 0 : \varphi(z) \leq \gamma \log \|z - x_0\| + O(1) \text{ on } V_{x_0}\}.
\]

We remark here that this definition does not depend on the choice of local charts. In particular, if \( \varphi = \log |f| \) in a neighborhood \( V_{x_0} \) of \( x_0 \), for some holomorphic function \( f \), then \( \nu(\varphi, x_0) \) is equal to the vanishing order \( \text{ord}_{x_0}(f) := \sup\{k \in \mathbb{N} : D^\gamma f(x_0) = 0, \forall |\gamma| < k\} \). We can also define the Lelong super-level sets, for \( c > 0 \),

\[
E_c(\varphi) := \{x \in X : \nu(\varphi, x) \geq c\}.
\]
We also use the notation $E_c(T)$ for a closed positive $(1,1)$-current $T$. A well-known result of Siu [24] asserts that the Lelong super-level sets $E_c(\varphi)$ are analytic subsets of $X$. We refer the reader to [12, Remark 3.2] for a simple proof.

2.2. Demailly’s equisingular approximation. We next recall the basic result on the approximation of psh functions by psh functions with analytic singularities. For details about this, we refer the reader to [12, 13].

Following Demailly [12], a closed positive $(1,1)$-current $T = \theta + dd^c \varphi$ and its global potential $\varphi$ are said to have analytic singularities if there exists $c > 0$ such that

$$\varphi = c \log \left[ \sum_{j=1}^N |f_j|^2 \right] + v,$$

locally on $X$, where $v$ is a smooth function and the $f_j$’s are holomorphic functions.

Thanks to $dd^c$-Lemma, the problem of approximating a positive closed $(1,1)$-current is reduced to approximating a quasi-psh function. The following result of Demailly [12, 13] on the equisingular approximation of a quasi-psh function by quasi-psh functions with analytic singularities is crucial:

**Theorem 2.1** (Demailly’s equisingular approximation). Let $\varphi$ be a $\theta$-psh function on $X$. There exists a decreasing sequence of quasi-psh functions $(\varphi_m)$ such that

1. $(\varphi_m)$ converges pointwise and in $L^1(X)$ to $\varphi$ as $m \to +\infty$,
2. $\varphi_m$ has the same singularities as $1/2m$ times a logarithm of a sum of squares of holomorphic functions,
3. $\theta + dd^c \varphi_m \geq -\varepsilon_m \omega$, where $\varepsilon_m > 0$ decreases to 0 as $m \to +\infty$,
4. $\int_X e^{2m(\varphi_m - \varphi)} dV < +\infty$;
5. $\varphi_m$ is smooth outside the analytic subset $E_{1/m}(\varphi)$.

Proof. We refer the reader to [13, Theorem 1.6, Lemma 1.10] for a proof. \qed

2.3. Big cohomology classes. A cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is big if it contains a Kähler current, i.e. there is a positive closed current $T \in \alpha$ and $\varepsilon > 0$ such that $T \geq \varepsilon \omega$. Theorem 2.1 enables us in particular to approximate a Kähler current $T$ inside its cohomology class by Kähler currents $T_m$ with analytic singularities, with a very good control of the singularities. A big class therefore contains plenty of Kähler currents with analytic singularities.

**Definition 2.2.** We let $\text{Amp}(\alpha)$ denote the ample locus of $\alpha$, i.e. the Zariski open subset of all points $x \in X$ for which there exists a Kähler current $T_x \in \alpha$ with analytic singularities such that $T_x$ is smooth in a neighborhood of $x$.

It follows from the work of Boucksom [7, Theorem 3.17 (ii)] that one can find a single Kähler current $T_0 \in \alpha$ with analytic singularities such that

$$\text{Amp}(\alpha) = X\backslash \text{Sing}(T_0).$$

In particular $T_0$ is smooth in the ample locus $\text{Amp}(\alpha)$.

Given $\varphi, \psi \in \text{PSH}(X, \theta)$, we say that $\varphi$ is less singular than $\psi$, and denote by $\psi \preceq \varphi$, if there exists a constant $C$ such that $\psi \leq \varphi + C$ on $X$. We say that $\varphi, \psi$ have the same singularity type, and denote by $\varphi \simeq \psi$ if $\varphi \preceq \psi$ and $\psi \preceq \varphi$.

**Definition 2.3.** A $\theta$-psh function is said to have minimal singularities if it is less singular than any $\theta$-psh function.

Such a function is not unique in general, only its class of singularities is. Following Demailly, one defines the extremal function

$$V_\theta := \sup\{ \varphi \in \text{PSH}(X, \theta) : \varphi \leq 0 \}.$$
It is a $\theta$-psh function with minimal singularities. By the analysis above $V_\theta$ is locally bounded on the ample locus $\text{Amp}(\alpha)$. Of course we have $V_\theta \equiv 0$ if $\theta$ is semi-positive.

### 2.4. Non-pluripolar Monge-Ampère operator.

Let $\varphi_1, \cdots, \varphi_n \in \text{PSH}(X, \theta)$ with minimal singularities. Then they are locally bounded on the ample locus $\text{Amp}(\alpha)$. Following the construction of Bedford-Taylor [1, 2] in the local setting, it has been shown in [8, Section 1.2] that the product

$$(\theta + dd^c \varphi_1) \wedge \cdots \wedge (\theta + dd^c \varphi_n)$$

is well-defined as a positive Radon measure on $\text{Amp}(\alpha)$ and it has finite total mass. One can then extend it trivially on the whole $X$.

In particular, if $\varphi_1 = \cdots = \varphi_n = \varphi$ then this procedure defines the (non-pluripolar) Monge-Ampère measure of a function $\varphi \in \text{PSH}(X, \theta)$ with minimal singularities. For a general $\varphi \in \text{PSH}(X, \theta)$, its canonical approximants $\varphi^j := \max(\varphi, V_\theta - j)$, $j > 0$ have minimal singularities. One can show that the sequence of Borel positive measures $1_{\{\varphi > V_\theta - j\}}(\theta + dd^c \varphi^j)$ is increasing in $j$. Its (strong) limit

$$\text{MA}_\theta(\varphi) = (\theta + dd^c \varphi)^n := \lim_{j \to +\infty} 1_{\{\varphi > V_\theta - j\}}(\theta + dd^c \varphi^j)^n$$

is the non-pluripolar Monge-Ampère measure of $\varphi$. The volume of a big class $\alpha = \{\theta\}$ is given by the total mass of the non-pluripolar Monge-Ampère measure of $V_\theta$, i.e.

$$\text{Vol}(\alpha) := \int_X \text{MA}_\theta(V_\theta).$$

We say that $\varphi \in \text{PSH}(X, \theta)$ has full Monge-Ampère mass if $\int_X \text{MA}_\theta(\varphi) = \text{Vol}(\alpha)$. We let

$$\mathcal{E}(X, \theta) := \left\{ \varphi \in \text{PSH}(X, \theta) : \int_X \text{MA}_\theta(\varphi) = \text{Vol}(\alpha) \right\}$$

denote the set of $\theta$-psh functions with full Monge-Ampère mass. Note that $\theta$-psh functions with minimal singularities have full Monge-Ampère mass (see [8, Theorem 1.16] for more details), but the converse is not true.

We recall here the plurifine locality of the non-pluripolar product, which will be used several times in this paper.

**Lemma 2.4.** Assume that $\varphi, \psi$ are $\theta$-psh function such that $\varphi = \psi$ on an open set $U$ in the plurifine topology. Then

$$1_U \text{MA}_\theta(\varphi) = 1_U \text{MA}_\theta(\psi).$$

We stress in particular that sets of the form $\{u < v\}$, where $u, v$ are quasi-psh functions, are open in the plurifine topology.

**Proof.** The proof for locally bounded functions can be found in [3, Corollary 4.3] or [8, Section 1.2]. For the general case we write $\varphi$ (resp. $\psi$) as the decreasing limits of its canonical approximants $\varphi^t := \max(\varphi, V_\theta - t)$ (resp. $\psi^t := \max(\psi, V_\theta - t)$). We observe that $\varphi^t$ (resp. $\psi^t$) is locally bounded on the ample locus $\text{Amp}(\theta)$. By the result for locally bounded functions we have

$$1_{U \cap \{\varphi > V_\theta - t\}} \text{MA}_\theta(\varphi^t) = 1_{U \cap \{\psi > V_\theta - t\}} \text{MA}_\theta(\psi^t).$$

Letting $t \to +\infty$, we conclude the proof.

### 2.5. Capacities.
2.5.1. The Monge-Ampère capacity. For the convenience of the reader we recall here a few facts contained in [21]. Let $K$ be a Borel subset of $X$. The Monge-Ampère capacity is

$$\text{Cap}_\omega(K) := \sup \left\{ \int_K \omega + dd^c u^n : u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}.$$

Lemma 2.5. Let $\nu = gdV$ be a Radon positive measure with $0 \leq g \in L^p(dV)$ for some $p > 1$. Then there exists $B > 0$ depending on $n, p, \omega, dV$ and $\|g\|_{L^p(dV)}$ such that, for all Borel subsets $K$ of $X$,

$$\nu(K) \leq B \cdot \text{Cap}(K)^4.$$

Proof. This result was proved by Kolodziej; see [22, Section 2.5]. We refer the readers to [19, Proposition 3.1] for an alternative proof. □

2.5.2. The generalized capacity. We present here a generalization of this notion introduced by Di Nezza and Lu [16, 15] (see also [9, Section 4.1]).

Definition 2.6. Let $\psi \in \text{PSH}(X, \theta)$. We define the $\psi$-relative capacity of a Borel subset $K \subset X$ by

$$\text{Cap}_{\theta, \psi}(K) := \sup \left\{ \int_K MA_{\theta}(u) : u \in \text{PSH}(X, \theta), \psi - 1 \leq u \leq \psi \right\}.$$

Note that when $\theta$ is Kähler, a related notion of capacity has been studied in [15, 16]. The (generalized) Monge-Ampère capacity plays a vital role in establishing uniform estimates for complex Monge-Ampère equation (see e.g. [19, 8, 15, 16] and the references therein). We shall use the $\psi$-capacity $\text{Cap}_{\theta, \psi}$ in the proof of Theorem 3.2.

The following results are important for the sequel.

Lemma 2.7. Fix $\varphi \in \mathcal{E}(X, \theta)$ and $\psi \in \text{PSH}(X, \theta)$. Then the function

$$H(t) := \text{Cap}_{\theta, \psi}(\{\varphi < \psi - t\}), \quad t \in \mathbb{R},$$

is right-continuous and $H(t) \to 0$ as $t \to +\infty$.

Proof. The proof is almost the same as the one of [16, Lemma 2.6] in the Kähler case, i.e. $\theta = \omega$ is Kähler. For the reader’s convenience, we give the proof here. The right-continuity is straightforward. For the second statement, we first assume that $\psi \leq V_0$. Fix $u \in \text{PSH}(X, \theta)$ such that $\psi - 1 \leq u \leq \psi$. The generalized comparison principle ([8, Corollary 2.3]) yields

$$\int_{\{\varphi < \psi - t\}} MA_{\theta}(u) \leq \int_{\{\varphi < u - t + 1\}} MA_{\theta}(u) \leq \int_{\{\varphi < u - t + 1\}} MA_{\theta}(\varphi) \leq \int_{\{\varphi < V_0 - t + 1\}} MA_{\theta}(\varphi).$$

The last term goes to 0 as $t \to +\infty$ as $\varphi \in \mathcal{E}(X, \theta)$. This finishes the proof. □

Lemma 2.8. Let $\psi$ be a quasi-psh function such that $\theta + dd^c \psi \geq \delta \omega$ for some $\delta \in (0, 1)$. Then for any Borel set $K \subset X$,

$$\text{Cap}_\omega(K) \leq \frac{1}{\delta^n} \text{Cap}_{\theta, \psi}(E).$$

Proof. Let $u$ be a $\omega$-psh function such that $-1 \leq u \leq 0$. We then have that $\varphi := \psi + \delta u$ is a candidate defining $\text{Cap}_{\theta, \psi}$. It follows that

$$\delta^n \int_K (\omega + dd^c u) \leq \int_K (\theta + dd^c \psi + dd^c(\delta u)) \leq \text{Cap}_{\theta, \psi}(K),$$

and taking the supremum over all $u$ we get the desired estimate. □
Generalizing [22, Ineq. (2.3.2)], we have the following result which is a simple consequence of the generalized comparison principle ([8, Corollary 2.3]).

**Lemma 2.9.** Let $\psi \in \text{PSH}(X, \theta)$ and $\varphi \in \mathcal{E}(X, \theta)$. Then for all $t > 0$ and $0 < s \leq 1$ we have

$$s^n \text{Cap}_{\theta, \psi}(\{\varphi < \psi - t - s\}) \leq \int_{\{\varphi < \psi - t\}} \text{MA}_{\theta}(\varphi).$$

**Proof.** Let $u$ be a $\theta$-psh function such that $\psi - 1 \leq u \leq \psi$. We then have

$$\{\varphi < \psi - t - s\} \subset \{\varphi < u + (1-s)\psi - t\} \subset \{\varphi < \psi - t\}.$$

Since $s^n \text{MA}_{\theta}(u) \leq \text{MA}_{\theta}(su + (1-s)\psi)$ and $\varphi$ has full Monge-Ampère mass, it follows from the generalized comparison principle ([8, Corollary 2.3]) that

$$s^n \int_{\{\varphi < \psi - t - s\}} \text{MA}_{\theta}(u) \leq \int_{\{\varphi < \psi - t\}} \text{MA}_{\theta}(su + (1-s)\psi)$$

$$\leq \int_{\{\varphi < su + (1-s)\psi - t\}} \text{MA}_{\theta}(su + (1-s)\psi)$$

$$\leq \int_{\{\varphi < su + (1-s)\psi - t\}} \text{MA}_{\theta}(\varphi) \leq \int_{\{\varphi < \psi - t\}} \text{MA}_{\theta}(\varphi).$$

Since $u$ was taken arbitrarily as a candidate in the definition of $\text{Cap}_{\theta, \psi}$, the proof therefore finishes. \qed

### 2.6. Quasi-psh envelopes

For a Borel function $h$, we let $P_{\theta}(h)$ denote the largest $\theta$-psh function lying below $h$:

$$P_{\theta}(h) := (\sup\{\varphi \in \text{PSH}(X, \theta) : \varphi \leq h \text{ on } X\})^*.$$

**Proposition 2.10.** Fix $\varphi \in \mathcal{E}(X, \theta)$. Then for any $b > 0$, $P_{\omega}(b \varphi - b V_\theta)$ is a $\omega$-psh function with full Monge-Ampère mass.

The proof given below is inspired by [11, Lemma 4.3].

**Proof.** We first show that the function $P_{\omega}(b \varphi - b V_\theta) \not\equiv -\infty$ for all $b > 0$.

For each $j \in \mathbb{N}$ we set $\varphi_j := \max(\varphi, V_\theta - j)$ and $\psi_j := P_{\omega}(b \varphi_j - b V_\theta)$. We observe that $(\psi_j)$ is a decreasing sequence of $\omega$-psh functions, and $\psi_j \geq -jb$ for each $j$. Therefore the proof would follow if we could show that $\lim_j \psi_j$ is not identically $-\infty$. We let for each $j$, $D_j := \{\psi_j = b \varphi_j - b V_\theta\}$ denote the contact set. Observe that the sets $D_j$ are non-empty for $j$ large enough. Fix $t > 0$. We see that

$$\{\psi_j \leq -t\} \cap D_j = \{\varphi_j \leq V_\theta - t/b\} \subset \{\varphi \leq V_\theta - t/b\}.$$

Set $\tilde{\omega} := (\frac{1}{b} + 1) \omega$. By Lemma 2.11 below and plurifine locality we have for $j > t/b$,

$$\int_{\{\psi_j \leq -t\}} (\omega + dd^c \psi_j)^n \leq \int_{\{\psi_j \leq -t\}} 1_{D_j}(\omega + dd^c \psi_j)^n$$

$$\leq b^n \int_{\{\psi_j \leq -t\}} 1_{D_j}(\tilde{\omega} + dd^c \psi_j)^n$$

$$\leq b^n \int_{\{\varphi \leq V_\theta - t/b\}} (\tilde{\omega} + dd^c \varphi)^n$$

$$= b^n \left( \int_X (\tilde{\omega} + dd^c \varphi)^n - \int_{\{\varphi > V_\theta - t/b\}} (\tilde{\omega} + dd^c \varphi)^n \right),$$

since $\varphi_j = \varphi$ on $\{\varphi > V_\theta - t/b\}$ for $j > t/b$. Suppose by contradiction that $\sup_X \psi_j \to -\infty$ as $j \to +\infty$. It thus follows that $\{\psi_j \leq -t\} = X$ for $j$ large.
enough, \( t \) being fixed. Hence, for \( j > 0 \) large enough, (2.1) becomes
\[
\int_X \omega^n \leq b^n \left( \int_X (\tilde{\omega} + dd^c \varphi_j)^n - \int_{\{\varphi > \nu_0 - t/b\}} (\tilde{\omega} + dd^c \varphi)^n \right).
\]
Letting \( j \to +\infty \), we obtain
\[
(2.2) \quad \int_X \omega^n \leq b^n \left( \int_X (\tilde{\omega} + dd^c \varphi)^n - \int_{\{\varphi > \nu_0 - t/b\}} (\tilde{\omega} + dd^c \varphi)^n \right),
\]
where we have used that
\[
(\tilde{\omega} + dd^c \varphi)^n = \sum_{k=0}^n \binom{n}{k} (\tilde{\omega} - \theta)^k \wedge (\theta + dd^c \varphi_j)^{n-k} \to (\tilde{\omega} + dd^c \varphi)^n
\]
in the weak sense of measures on \( X \), thanks to [9, Theorem 2.3, Remark 2.5]. Finally, letting \( t \to +\infty \), in (2.2) we obtain a contradiction. Consequently, \( \psi_j \) decreases to a \( \omega \)-psh function, we infer that \( P_\omega(b\varphi - b\nu_0) \) is a \( \omega \)-psh function for any \( b > 0 \).

It remains to show that \( P_\omega(b\varphi - b\nu_0) \) has full Monge-Ampère mass. Observe that \( P_\omega(b\varphi - b\nu_0) \geq \frac{1}{b} P_\omega(A\varphi - A\nu_0) \) for \( A > b \). Using monotonicity of mass (see e.g. [27, Theorem 1.2]), we obtain
\[
\int_X (\omega + dd^c P_\omega(b\varphi - b\nu_0))^n \geq \left( 1 - \frac{b}{A} \right)^n \int_X \omega^n + \left( \frac{b}{A} \right)^n \int_X (\omega + dd^c P_\omega(A\varphi - A\nu_0))^n.
\]
Letting \( A \to +\infty \) we thus finish the proof.

**Lemma 2.11.** Fix \( b > 0 \), \( \varphi \) and \( P_\omega(b\varphi - b\nu_0) \in \text{PSH}(X, \omega) \). Then the measure \( (\omega + dd^c P_\omega(b\varphi - b\nu_0))^n \) is supported on the contact set \( D := \{ P_\omega(b\varphi - b\nu_0) = b\varphi - b\nu_0 \} \), and
\[
1_D(\omega + dd^c P_\omega(b\varphi - b\nu_0))^n \leq b^n 1_D \left( \left( 1 + \frac{1}{b} \right) \omega + dd^c \varphi \right)^n.
\]

**Proof.** We refer the readers to [11, Lemma 4.4] for a proof of the first statement.

For the second one, set \( u = \frac{1}{b} P_\omega(b\varphi - b\nu_0) + \nu_0 \). Then \( u \) is a \( \tilde{\omega} := \left( \frac{1}{b} + 1 \right) \omega \)-psh function, and \( u \leq \varphi \). It follows from [21, Corollary 10.8] that
\[
(2.3) \quad 1_{\{u = \varphi\}}(\tilde{\omega} + dd^c u)^n \leq 1_{\{u = \varphi\}}(\tilde{\omega} + dd^c \varphi)^n.
\]
Furthermore, the measure \( (\omega + dd^c P_\omega(b\varphi - b\nu_0))^n \) is supported on the contact set \( D = \{ b\varphi - b\nu_0 = P_\omega(b\varphi - b\nu_0) \} = \{ u = \varphi \} \), hence
\[
(2.4) \quad \frac{1}{b^n}(\omega + dd^c P_\omega(b\varphi - b\nu_0))^n = 1_{\{u = \varphi\}} \frac{1}{b^n}(\omega + dd^c P_\omega(b\varphi - b\nu_0))^n \leq 1_{\{u = \varphi\}}(\tilde{\omega} + dd^c u)^n.
\]
Combining (2.3) and (2.4) we obtain the desired estimate. \( \square \)

3. Regularity of solutions

3.1. **Proof of the Main Theorem.** In this section we prove Theorem 1.1. The key ingredient is an adaptation of Di Nezza-Lu’s approach [16] (see also [15]).

Given a non-negative Radon measure \( \mu \) whose total mass is \( \text{Vol}(\alpha) \), we consider the Monge-Ampère equation
\[
(3.1) \quad \text{MA}_\theta(\varphi) = \mu.
\]
The systematic study of such equations in big cohomology classes has been initiated in [8]. It has been shown there that (3.1) admits a unique normalized solution \( \varphi \in \mathcal{E}(X, \theta) \) if and only if \( \mu \) is a *non pluripolar* measure on \( X \).

Our goal is to prove the following:
Theorem 3.1. Assume \( \nu = gdV \) to be a Radon measure, with \( 0 \leq g \in L^p(dV) \) for some \( p > 1 \). Let \( \mu \) be a non-pluripolar measure such that \( \mu(X) = \text{Vol}(\alpha) \). Assume that \( \mu = fdV \), with \( f \leq e^{-\varphi} \) for some quasi-psh function \( \varphi \) on \( X \). Let \( \varphi \in \mathcal{E}(X, \theta) \) be the unique normalized solution to (3.1). Then \( \varphi \) is continuous on \( \text{Amp}(\alpha) \setminus E_{\frac{q}{16}}(\phi) \), where \( q \) denotes the conjugate exponent of \( p \).

Note that Theorem 1.1 in the introduction is a particular case of Theorem 3.1. We first establish this result under an extra assumption. More precisely, we have the following theorem, which is closely similar to [16, Theorem 3.1] in the case that \( \theta \) is Kähler.

Theorem 3.2. Let \( \varphi \in \mathcal{E}(X, \theta) \) be normalized by \( \sup_X \varphi = 0 \). Assume that \( \text{MA}_0(\varphi) \leq e^{-\rho}gdV \), for some quasi-psh function \( \rho \) on \( X \), and \( 0 \leq g \in L^p(dV) \), with \( p > 1 \). Assume that \( \rho \) is locally bounded on an open set \( U \subset \text{Amp}(\alpha) \). Then \( \varphi \) is continuous on \( U \).

Proof of Theorem 3.2. We fix a \( \theta \)-psh function \( \rho_0 \) on \( X \) such that

\[
\theta + dd^c \rho_0 \geq 2\delta_0 \omega,
\]

for some small constant \( \delta_0 > 0 \). Replacing \( \rho_0 \) by \( \rho_0 - \sup_X \rho_0 \), we can always assume that \( \rho_0 \leq V_\theta \). Moreover, we can choose \( \rho_0 \) such that it is smooth in the ample locus, with analytic singularities thanks to [7, Theorem 3.17 (ii)].

We will divide the proof in three steps.

Step 1. We prove that \( \varphi \) is locally bounded on \( U \).

We pick \( a > 0 \) so small that \( a\phi \) belongs to \( \text{PSH}(X, \delta_0 \omega) \). Set \( \psi := \rho_0 + a\phi \). We thus have \( \theta + dd^c \psi \geq \delta_0 \omega \), and \( \psi \leq V_\theta + a\phi \). We claim that

\[
\varphi \geq \psi - A,
\]

for \( A > 0 \) depending only \( \delta_0, p, dV, \|\|_{L^p(dV)} \), and \( \int_X e^{-2P_\alpha(a^{-1}\varphi - a^{-1}V_\theta)}g dV \).

We remark that by Proposition 2.10, for any \( b > 0 \), \( P_\alpha(b\varphi - bV_\theta) \) is a \( \omega \)-psh function with full Monge-Ampère mass, hence it has zero Lelong numbers [see 20, Corollary 1.8]. Therefore, Skoda’s integrability theorem [26] ensures that \( e^{-P_\alpha(b\varphi - bV_\theta)} \) belongs to \( L^q(dV) \) for all \( q < +\infty \). In particular, \( \int_X e^{-2P_\alpha(b\varphi - bV_\theta)}g dV \) is finite for any \( b > 0 \).

The proof of the claim above follows the approach of Di Nezza and Lu [16, 15]. To see this, fix \( s \in [0, 1], t > 0 \). Set \( d\nu = gdV, b = a^{-1} \). Using Lemma 2.9 and the assumption on \( \text{MA}_0(\varphi) \) we have

\[
s^n \text{Cap}_{\omega, \psi}(\{\varphi < \psi - t - s\}) \leq \int_{\{\varphi < \psi - t\}} \text{MA}_0(\varphi) \leq e^{b(\psi - \varphi)}e^{-\phi}d\nu
\]

\[
\leq \int_{\{\varphi < \psi - t\}} e^{-(b\varphi - bV_\theta)}d\nu
\]

\[
\leq \int_{\{\varphi < \psi - t\}} e^{-P_\alpha(b\varphi - bV_\theta)}d\nu,
\]

where we have used that \( b\psi \leq bV_\theta + \phi \) in the third inequality. Using Hölder’s inequality we have

\[
\int_{\{\varphi < \psi - t\}} e^{-P_\alpha(b\varphi - bV_\theta)}d\nu \leq (\nu(\{\varphi < \psi - t\}))^{1/2} \left( \int e^{-2P_\alpha(b\varphi - bV_\theta)}d\nu \right)^{1/2}.
\]
By Lemma 2.5, one can find a constant $B > 0$ depending on $n$, $p$, $\omega$, $dV$, and $\|g\|_{L^p(dV)}$ such that

$$\nu()^{1/2} \leq B(\text{Cap}_X(\cdot))^2.$$  

Since $\theta + dd^c \psi \geq \delta_0 \omega$ it follows from Lemma 2.8 that $\text{Cap}_X \leq \delta_0^{-n} \text{Cap}_{\theta, \psi}$, hence

$$\nu()^{1/2} \leq B\delta_0^{-2n} \text{Cap}_{\theta, \psi}(\cdot)^2.$$  

By (3.3), (3.4) and (3.5) we thus get

$$s^n \text{Cap}_{\theta, \psi}(\{\varphi < \psi - s - t\}) \leq C \text{Cap}_{\theta, \psi}(\{\varphi < \psi - t\})^2,$$

where $C$ depends on $\omega$, $\delta_0$, $n$, $p$, $\|g\|_{L^p(dV)}$, and $\int_X e^{-2P_c(b\varphi - b\psi)} d\nu$. Set

$$H(t) := (\text{Cap}_{\theta, \psi}(\{\varphi < \psi - t\}))^{1/n}, \quad t > 0.$$  

By the estimate (3.6) we get

$$sH(t+s) \leq C^{1/n} H(t)^2.$$  

It follows from Lemma 2.7 that the function $H$ is right-continuous and $H(\pm \infty) = 0$. We can thus apply [19, Lemma 2.4] which yields $H(t_0 + 2) = 0$, where $t_0 > 0$ is such that

$$H(t_0) < \frac{1}{2C^{1/n}}.$$  

Therefore, for $A = t_0 + 2$ we have $\varphi \geq \psi - A$ on $X \setminus P$ for some Borel subset $P$ such that $\text{Cap}_{\theta, \psi}(P) = 0$. By Lemma 2.8 we have $\text{Cap}_X(P) = 0$ so $P$ is a pluripolar set. Hence $\varphi \geq \psi - A$ everywhere.

Using Hölder’s inequality it follows from (3.3) (take $s = 1$) that

$$H(t)^n \leq \left( \int_X e^{-2P_c(b\varphi - b\psi)} d\nu \right)^{1/2} \left( \int_{\{\varphi < \psi - t+1\}} d\nu \right)^{1/2} \leq \left( \int_X e^{-2P_c(b\varphi - b\psi)} d\nu \right)^{1/2} \left( \frac{1}{t - 1} \int_X |\psi - \varphi| d\nu \right)^{1/2}.$$  

The last integral is bounded by a uniform constant: using Hölder’s inequality again we have $\int_X |\psi - \varphi| d\nu \leq \|g\|_{L^p(dV)} (\|\psi\|_{L^q(dV)} + \|\varphi\|_{L^q(dV)})$ with $q = p/(p - 1)$.

Since $\varphi$ belongs to the compact set of $\theta$-psh functions normalized by $\sup_X \varphi = 0$, its $L^p$ norm is bounded by an absolute constant only depending on $\theta$, $d\nu$ and $p$. Consequently, we can choose $t_0 > 0$ to be only dependent on $d\nu$, $p$, $\|g\|_{L^p(dV)}$, and an upper bound for $\int_X e^{-2P_c(b\varphi - b\psi)} d\nu$.

**Step 2.** There exists a sequence of functions $\varphi_j \in \text{PSH}(X, \theta) \cap \mathcal{O}(\text{Amp}(\alpha))$ which decreases towards $\varphi$.

For convenience, we normalize $\varphi$ so that $\sup_X \varphi = -1$. Let $0 \geq h_j$ be a sequence of smooth functions decreasing to $\varphi$. Then the sequence of $\theta$-psh functions $\varphi_j := P_\theta(h_j)$ decreases to $\varphi$ as $j \to +\infty$. Indeed, since the operator $P_\theta$ is monotone, the sequence $\varphi_j$ is decreasing to a $\theta$-psh function $u$ and since $\varphi_j \geq \varphi$ for all $j$ we have $u \geq \varphi$. Moreover, $u(x) \leq \varphi_j(x) \leq h_j(x), \forall x \in X$, for all $j$, hence $u(x) \leq \varphi(x)$, as claimed. Furthermore, we have that $\varphi_j$ is continuous in $\text{Amp}(\alpha)$ for each $j$. In fact, [4, Ineq. (1.2)] gives an upper bound on the Monge-Ampère measure of $\varphi_j$:

$$\text{MA}_\theta(\varphi_j) \leq 1_D \text{MA}_\theta(h_j), \quad D = \{\varphi_j = h_j\}.$$  

The equality also holds following the work of Di Nezza and Trapani [17]. In particular $\text{MA}_\theta(\varphi_j)$ has an $L^\infty$-density, hence it follows from [14, Theorem D] that $\varphi_j$ is Hölder continuous in $\text{Amp}(\alpha)$, as claimed.
Step 3. We show that the decreasing convergence \( \varphi_j \to \varphi \) is locally uniform on \( U \) and finish the proof.

Fix \( \lambda \in (0,1) \). For any \( j \in \mathbb{N} \), set

\[
\psi_j := \lambda \psi + (1 - \lambda) \varphi_j - (A + 2) \lambda,
\]

where \( A > 0 \) is the uniform constant so that \( \varphi \geq \psi - A \) (see (3.2)). If we pick \( \varepsilon_j \leq \frac{1}{2(A+1)} \delta_0 \) for \( j > 0 \) big enough, then \( \theta + dd^c \psi_j \geq \frac{1}{2} \delta_0 \omega \). We observe by definitions that \( \varphi_j \leq \psi_0 \), hence \( \psi_j \leq \lambda \psi + (1 - \lambda) \psi_0 \leq \psi_0 + \lambda \alpha \). Set

\[
H_j(t) := \left[ \text{Cap}_{\theta, \psi_j} \{ \varphi < \psi_j - t \} \right]^{1/n}, \quad t > 0.
\]

For any \( s \in [0,1] \), \( t > 0 \), we can argue as above to obtain

\[
sH_j(t+s) \leq C^{1/n} H_j(t)^2,
\]

for \( C > 0 \) only depending on \( p \), \( dV \), \( \|g\|_{L^p(dV)} \), \( \delta_0 \), \( \lambda \), and \( \int_X e^{-2P_{\theta=\varepsilon_0} (\varphi - c \psi) i dV} \), with \( c = (\lambda a)^{-1} \). Let \( \chi \) be an increasing convex weight such that \( \chi(0) = 0 \), \( \chi(-\infty) = -\infty \), and \( \varphi \) has finite \( \chi \)-energy (see [8, Proposition 2.11]). Since \( \varphi_j \geq \varphi \geq \psi - A \) we have \( \psi_j \leq \varphi_j - 2\lambda \). It follows from Lemma 2.9 (take \( s = \lambda \), \( \psi = \psi_j + \lambda \)) that

\[
\lambda^n \text{Cap}_{\theta, \psi_j} \{ \varphi < \psi_j \} \leq \int_{\{ \varphi < \psi_j + \lambda \}} \text{MA}_\theta(\varphi) \leq \int_{\{ \varphi < \psi_j - \lambda \}} \text{MA}_\theta(\varphi) \leq \frac{1}{\lambda} \int_X (\varphi - \psi_j - \lambda) \text{MA}_\theta(\varphi).
\]

The latter converges to 0 as \( j \to +\infty \) since \( \varphi_j \) decreases to \( \varphi \), namely \( H_j(0) \) goes to 0 as \( j \to +\infty \). We thus take \( j > 0 \) so big that \( H_j(0) \leq 1/(2C^{1/n}) \). It thus follows from [19, Remark 2.5] that \( H_j(t) = 0 \) if \( t \geq t_0 \) where \( t_0 \leq 4C^{1/n} H_j(0) \). We then have

\[
\varphi \geq \lambda \psi + (1 - \lambda) \varphi_j - (A + 2) \lambda - 4C^{1/n} H_j(0).
\]

We have \( H_j(0) \to 0 \) as \( j \to \infty \). Letting \( j \to +\infty \), we thus obtain

\[
\lim_{j \to +\infty} \inf \frac{\varphi - \varphi_j}{K} \geq -\lambda (\sup_K |\psi| + A + 2),
\]

for any compact subset \( K \) of \( U \). We observe by definition that \( \psi = \rho_0 + a \phi \) is bounded on \( K \). Finally, letting \( \lambda \to 0 \) in (3.8) we obtain that the convergence \( \varphi_j \to \varphi \) is locally uniform on \( U \), hence \( \varphi \) is continuous on \( U \). □

Proof of Theorem 3.1. By rescaling, we may assume without loss of generality that \( \phi \) is a \( \omega \)-psh function. By Theorem 2.1, we can find a quasi-psh function \( \phi_m \) with analytic singularities such that \( \omega + dd^c \phi_m \geq -\varepsilon_m \omega \) for \( \varepsilon_m > 0 \) decreasing to 0 as \( m \to +\infty \), and

\[
\int_X e^{2m(\phi_m - \phi)} dV < +\infty.
\]

Note that \( \phi_m \) is smooth outside the analytic set \( E_{1/m}(\phi) \subset X \). We see that

\[
\text{MA}_{\theta}(\varphi) \leq e^{-\phi_m} e^{(\phi_m - \phi)} i dV.
\]

Set now \( g' = e^{(\phi_m - \phi)} g \). We choose \( m = \lfloor q \rfloor \), where \( \lfloor q \rfloor \) denotes the integer part of \( q \). We have \( 2m > q \), hence there is a constant \( p' > 1 \) such that \( \frac{1}{p'} = \frac{1}{p} + \frac{1}{2m} \). It follows from Hölder’s inequality that \( g' \in L^{p'}(dV) \). Observe that \( \phi_m \) is smooth in the complement \( X \setminus E_{1/m}(\phi) \) of the analytic set \( E_{1/m}(\phi) \subset X \), in particular it is locally bounded on \( \text{Amp}(\alpha) \setminus E_{1/m}(\phi) \supset \text{Amp}(\alpha) \setminus E_{1/q}(\phi) \). We can thus apply Theorem 3.2 to complete the proof.

In particular if \( p = \infty \) then we can choose \( m = 1 \) to conclude the proof of Theorem 1.1. □
3.2. Kähler-Einstein metrics on log canonical pairs of general type.

3.2.1. Log canonical singularities. A pair \((Y, \Delta)\) is by definition a connected complex normal projective variety \(Y\) and an effective \(\mathbb{Q}\)-divisor \(\Delta\). We will say that the pair \((Y, \Delta)\) has log canonical singularities if \(K_Y + \Delta\) is \(\mathbb{Q}\)-Cartier, and if for some (or equivalently any) log resolution \(\pi : X \to Y\) of \((Y, \Delta)\), we have

\[ K_X = \pi^*(K_Y + \Delta) + \sum_i a_i E_i, \]

where \(E_i\) are either exceptional divisors or components of the strict transform of \(\Delta\), and the coefficients \(a_i \in \mathbb{Q}\) satisfy the inequality \(a_i \geq -1\) for all \(j\). The divisor \(\sum_i E_i\) has simple normal crossing support. We denote the singular set of \(Y\) by \(Y_{\text{sing}}\) and let \(Y_{\text{reg}} := Y \setminus Y_{\text{sing}}\).

Let \(m\) be a positive integer such that \(m(K_Y + \Delta)\) is Cartier. If we choose \(\sigma\) a local generator of \(m(K_Y + \Delta)\) defined on an open subset \(U\) of \(Y\), then \((i^{mn^2}\sigma \wedge \bar{\sigma})^{1/m}\) defines a smooth volume form on \(U \cap (Y_{\text{reg}} \setminus \text{Supp}(\Delta))\). If \(f_i\) is a local equation of \(E_i\) around a point \(\pi^{-1}(U)\), then we can see that

\[ \pi^* \left( i^{mn^2}\sigma \wedge \bar{\sigma} \right)^{1/m} = \prod_i |f_i|^{2n^2} dV \]

locally on \(\pi^{-1}(U)\) for some local volume form \(dV\). The previous construction leads to the following adapted measure which is introduced in [19, Sect. 6]:

**Definition 3.3.** Let \((Y, \Delta)\) be a pair and let \(h\) be a smooth hermitian metric on the \(\mathbb{Q}\)-line bundle \(\mathcal{O}_Y(K_Y + \Delta)\). The corresponding adapted measure \(\mu_{Y,h}\) on \(Y_{\text{reg}}\) is locally defined by choosing a nowhere zero section \(\sigma\) of \(\mathcal{O}_Y(m(K_Y + \Delta))\) over a small open set \(U\) and setting

\[ \mu_{Y,h} := \frac{(i^{mn^2}\sigma \wedge \bar{\sigma})^{1/m}}{|\sigma|^{2/m}_{h^m}}. \]

The point of the definition is that the measure \(\mu_{Y,h}\) does not depend on the choice of \(\sigma\). This measure can be extended by zero across \(Y_{\text{sing}} \cup \text{Supp}(\Delta)\). Remark that the restriction \(\sigma|_{Y_{\text{reg}}}\) can be viewed as a meromorphic form with a pole of order \(md_i\) on \(\Delta\), where \(\Delta = \sum_i d_i \Delta_i\) is the decomposition of \(\Delta\) into prime divisors (see [19, Sect. 6.3]). The Lelong-Poincaré formula yields

\[ -dd^c \log \mu_{Y,h} = [\Delta] - i\Theta_h(K_Y + \Delta) \]

on \(Y_{\text{reg}}\), where \([\Delta]\) is the integration current on \(\Delta\).

3.2.2. Kähler-Einstein metrics. Let \((Y, \Delta)\) be a log canonical pair, with \(n = \dim_{\mathbb{C}} Y\). Assume that \(K_Y + \Delta\) is a big \(\mathbb{Q}\)-line bundle.

We next recall the notion of (negatively curved) Kähler-Einstein metric attached to a pair \((Y, \Delta)\). There are various equivalent definitions for such an object (e.g. in [19, 8, 6]), we choose here the following definition in the sense of [6, Sect. 3].

**Definition 3.4.** We say that a closed positive current \(\omega_{KE} \in c_1(K_Y + \Delta)\) on \(Y\) is a (singular) Kähler-Einstein metric (KE for short) with negative curvature for \((Y, \Delta)\) if

1. The non-pluripolar product \(\omega_{KE}^n\) defines a (locally) absolutely continuous measure on \(Y_{\text{reg}}\) with respect to \(dz \wedge d\bar{z}\) and \(\log(\omega_{KE}^n/dz \wedge d\bar{z}) \in L^1_{\text{loc}}(Y_{\text{reg}})\), where \(z = (z_i)\) are local holomorphic coordinates;
2. \(\text{Ric}(\omega_{KE}) = -\omega_{KE} + [\Delta]\) on \(Y_{\text{reg}}\);
3. \(\int_{Y_{\text{reg}}} \omega_{KE}^n = \text{vol}(K_Y + \Delta)\).
The condition (1) allows us to define (on $Y_{\text{reg}}$) the Ricci curvature of $\omega_{KE}$ by setting $\text{Ric}(\omega_{KE}) := -\log(\omega_{KE}^n)$. Another way of thinking of this is to interpret the positive measure $\omega_{KE}^n|_{Y_{\text{reg}}}$ as a singular metric on $-K_{Y_{\text{reg}}}$ whose curvature is then $\text{Ric}(\omega_{KE})$ by definition.

Let $h$ be a smooth hermitian metric on $K_Y + \Delta$ with curvature $\eta$. Finding a singular Kähler-Einstein metric is equivalent to solving the following Monge-Ampère equation for an $\eta$-psh function with full Monge-Ampère mass

\begin{equation}
(\eta + dd^c \phi)^n = e^{\phi + \epsilon \mu_{Y,h}},
\end{equation}

for some $c \in \mathbb{R}$. Indeed, we set $\omega := \eta + dd^c \phi$. Since $\phi$ is locally integrable $\omega$ satisfies the condition (1) (see Eq. (3.9)). We also have

$\text{Ric}(\omega) = -dd^c \log \omega^n = -dd^c \phi - dd^c \log \mu_{Y,h} = -dd^c \phi - \eta + [\Delta]$ on $Y_{\text{reg}}$. Condition (3) is clearly satisfied.

We now prove Corollary 1.2 in the introduction. Assume the initial pair $(Y, \Delta)$ is lc of general type, i.e. the canonical bundle $K_Y + \Delta$ is big. We consider a log resolution $\pi : (X, D) \to (Y, \Delta)$ of the pair, with $K_X + D = \pi^*(K_Y + \Delta)$. Here, $D = \sum a_i D_i$ is a $\mathbb{R}$-divisor with simple normal crossing support (sncc for short) on $X$, consisting of $\pi$-exceptional divisors with coefficients in $(-\infty, 1]$, and of the strict transforms of the components of $\Delta$ with coefficients in $(0, 1]$. The (singular) Kähler-Einstein metric $\omega_{KE}$ for $(X, D)$, or equivalently the pull-back of the (singular) KE metric for $(Y, \Delta)$ by $\pi$ can be written as $\omega_{KE} = \theta + dd^c \varphi$ where $\theta = \pi^* \eta$ is a smooth representative of $c_1(K_X + D)$ and $\varphi$ is a $\theta$-psh function (with full Monge-Ampère mass) solving the following Monge–Ampère equation

\begin{equation}
\text{MA}_\theta(\varphi) = \frac{e^{\varphi} dV}{\prod_i |s_i|^{2\alpha_i}}.
\end{equation}

Here $s_i$ are sections cutting out $D_i$ above, $|\cdot|_i$ are smooth hermitian metrics on associated line bundles $\mathcal{O}_X(D_i)$, and $dV$ is the smooth volume form with prescribed Ricci curvature. We let $D_{\text{nklt}}$ denote the non-klt part of $D$, i.e. $D_{\text{nklt}} := \cup_{a_i = 1} D_i$. We remark that $\varphi$ goes to $-\infty$ near $D_{\text{nklt}}$ as $|s|^2$ is not integrable.

As a consequence of Theorem 3.2 we have the following

**Corollary 3.5.** There is a unique solution $\varphi \in \mathcal{E}(X, \theta)$ to the equation (3.11) which is continuous on $\text{Amp}(\theta) \setminus D_{\text{nklt}}$.

**Proof.** The existence of a unique solution $\varphi \in \mathcal{E}(X, \theta)$ to the Monge–Ampère equation (3.11) follows from [6, Theorem 4.2]. It remains to prove the continuity of $\varphi$. It is convenient to differentiate the "klt part" of $D$ from its "non-klt part", so we set $g = \prod_{a_i < 1} |s_i|^{-2\alpha_i} \in L^p$ for some $p > 1$, and $\phi = \sum_{a_i = 1} 2 \log |s_i|$. We see that $\phi$ is smooth outside of the non-klt locus $D_{\text{nklt}}$, in particular it is locally bounded on $\text{Amp}(\theta) \setminus D_{\text{nklt}}$. Since $\varphi$ is bounded from above, we can therefore apply Theorem 3.2 to complete the proof. \hfill $\square$

With the notations above, since $\{\theta\}$ is a pull-back by $\pi$ of a big class, we have

$\text{Amp}(\theta) = \pi^{-1}(\text{Amp}(K_Y + \Delta)) \setminus \text{Exc}(\pi)$

which projects onto $\text{Amp}(K_Y + \Delta) \cap (Y, \Delta)_{\text{reg}}$. We next observe that the projection of the non-klt locus $D_{\text{nklt}}$ is contained in $(Y, \Delta)_{\text{sing}} \cup [\Delta]$, with $[\Delta] = \sum_{a_i = 1} \Delta_i$.

The proof of Corollary 1.2 thus follows.

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