Family of Boundary Poisson Brackets

K. BERING *
Institute for Fundamental Theory
Department of Physics
University of Florida
Florida 32611, USA

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Abstract

We find a new \(d\)-parameter family of ultra-local boundary Poisson brackets that satisfy the Jacobi identity. The two already known cases [hep-th/9305133, hep-th/9806249 and hep-th/9901112] of ultra-local boundary Poisson brackets are included in this new continuous family as special cases.

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*Email address: bering@phys.ufl.edu, bering@nbi.dk
1 Introduction

We have seen an increasing number of theories during the last few years where boundaries or topological defects play a central role. Open strings ending on D-branes are one of the more recent examples and surface terms in gravity is another.

Typically a physical system has to fulfill extra constraints at a boundary. There can be sound physical motives for imposing these constraints (for instance local conservation of a quantity), but they can also appear for more ad hoc reasons.

The work of [1, 2] use a generalized notion of functional differentiability, that led to two new boundary Poisson brackets. They generalize the Poisson bracket of Lewis, Marsden, Montgomery and Ratiu [4]. It would be worthwhile to go back and re-examine various physical systems in this framework. It might lead to new ways of imposing (or not imposing!) boundary conditions and solving the system.

In this letter we shall stay in a general canonical formalism, and develop the Poisson brackets wrt. this extended notion of functional differentiability.

2 Review of Boundary Poisson Brackets

Consider a $d + 1$ dimensional space-time $\Sigma \times \mathbb{R}$, where space $\Sigma$ is a region of $\mathbb{R}^d$, with a spatial boundary $\partial \Sigma$. Consider a phase space of (bosonic) coordinate and momenta field variables $\phi^A(x, t)$, $A = 1, \ldots, 2N$. Time plays no role in the following, so we shall suppress $t$ in our formulae. We denote the non-degenerate symplectic structure by $\omega^{AB}$, which we for simplicity take to be ultra-local and constant.

Our building blocks for the boundary Poisson bracket [1, 2, 3] are the tower of higher Euler-Lagrange derivatives

$$\frac{\delta F}{\delta \phi^{(k)}(x)}$$

of a functional $F$. (See for instance Olver [3, p.365].) They have the property that

$$\delta F = \int_\Sigma d^d x \sum_{k=0}^{\infty} \partial_k \left[ \frac{\delta F}{\delta \phi^{(k)}(x)} \delta \phi^{A}(x) \right]$$

for arbitrary infinitesimal variations of the fields $\phi^{A}(x) \rightarrow \phi^{A}(x) + \delta \phi^{A}(x)$. The case $k = 0$ corresponds to the usual Euler-Lagrange derivative. Note, that the terms with $k \neq 0$, by the divergence theorem, can be recast into an integral over the boundary $\partial \Sigma$. Clearly our ability to probe the higher derivatives diminishes as we constrain the dynamical fields $\phi^{A}(x)$ with more boundary conditions. Here we want to investigate the maximal effect of the boundary terms, and hence we shall not impose any boundary conditions. (Needless to say that if boundary conditions at a later stage become necessary, for instance during quantization, this will cause no inconsistency, because it just restricts the number of field configurations.)
With this in mind, it is easy to see that the “bulk” Poisson bracket
\[
\{F, G\}_{(0)} \equiv \int_{\Sigma} d^{d}x \frac{\delta F}{\delta \phi^{A}(0)(x)} \omega^{AB} \frac{\delta G}{\delta \phi^{B}(0)(x)}
\] (2.3)
built out of the usual Euler-Lagrange derivatives does not generically satisfy the Jacobi identity: There can be a non-zero total derivative term left over. It is natural to ask if it is possible to modify the “bulk” Poisson bracket (2.3) with a boundary term such that the Jacobi identity is restored identically.

3 A $d$-Parameter Family of Boundary Brackets

We limit ourselves to the following ultra-local Ansatz for the full boundary Poisson bracket
\[
\{F, G\} = \sum_{k, \ell=0}^{\infty} c_{k\ell} \int_{\Sigma} d^{d}x \partial^{k+\ell} \left[ \frac{\delta F}{\delta \phi^{A}(k)(x)} \omega^{AB} \frac{\delta G}{\delta \phi^{B}(\ell)(x)} \right],
\] (3.1)
where $c_{k\ell}$ is a sequence of constant coefficients. The “bulk” coefficient $c_{00} \equiv 1$ by definition. Soloviev [1] found that
\[
\forall k, \ell : c_{k\ell} = 1
\] (3.2)
is a solution. Recently, we found another solution [2]
\[
c_{k\ell} = \delta_{\min(k,\ell),0} = \begin{cases} 1 & \text{if } k = 0 \text{ or } \ell = 0 \\ 0 & \text{otherwise} \end{cases}.
\] (3.3)
Our main new result is that
\[
c_{k\ell}(s) = \frac{(s)_{k}(s)_{\ell}}{(s)_{k+\ell}} = \frac{\Gamma(k+s)\Gamma(\ell+s)}{\Gamma(k+\ell+s)\Gamma(s)} = \frac{B(k+s,\ell+s)}{B(k+\ell+s,s)}
\] (3.4)
is a solution for arbitrary complex parameter $s \in ((\mathbb{C} \cup \{\infty\}) \setminus (-\mathbb{N}))^d$ on $d$ copies of the Riemann sphere except for the negative integers $s \in (-\mathbb{N})^d$, $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$, where some of the coefficients have poles. Here $(s)_{n} = \Gamma(s+n)/\Gamma(s)$ is the Pochhammer symbol in $d$ dimensions.

The two previously found solutions (3.2) and (3.3) correspond to $s = \infty$ and $s = 0$, respectively.

4 $x$-pointwise Poisson Bracket

We have assumed that all relevant functionals are of the local form
\[
F = \int_{\Sigma} d^{d}x \ f(x),
\] (4.1)
for some function $f(x) \equiv f\left(\partial^{k}\phi(x), x\right)$, that can depend on the dynamical fields $\phi^{A}(x)$ and on its spatial derivative $\partial^{k}\phi^{A}(x)$ up to a finite order $N$. 


The notion of higher functional derivatives, if defined merely from the descriptive property (2.2), is not unique. We emphasize that we use the canonical choice of the higher Euler-Lagrange derivatives:

$$\delta F \over \delta \phi^{A(k)}(x) = E_{A(k)} f(x) \equiv \sum_{m \geq k} \binom{m}{k} (-\partial)^{m-k} P_{A(m)} f(x)$$  \hspace{1cm} (4.2)

where $P_{A(m)} f(x)$ denotes the partial derivative of $f(x)$ wrt. $\partial^m \phi^A(x)$. It is easy to see that they obey property (2.2). The $x$-pointwise Poisson bracket reads

$$\{f, g\}(x) = \sum_{k, \ell=0}^{\infty} c_{k \ell} \partial^{k+\ell} \left[ E_{A(k)} f(x) \omega^{AB} E_{B(\ell)} g(x) \right]$$  \hspace{1cm} (4.3)

5 Fourier Transformed Bracket

It is convenient to resum the higher derivatives in a series,

$$P_A(q)f \equiv \sum_{k=0}^{\infty} q^k P_{A(k)}f$$  \hspace{1cm} (5.1)

$$E_A(q)f \equiv \sum_{k=0}^{\infty} q^k E_{A(k)}f = \exp \left[-\partial \frac{\partial}{\partial q}\right] P_A(q)f$$

and to introduce the Fourier transform

$$P_A(y)f \equiv \int d^d q \ e^{-qy} P_A(q)f$$  \hspace{1cm} (5.2)

$$E_A(y)f \equiv \int d^d q \ e^{-qy} E_A(q)f = e^{-\partial y} P_A(y)f$$

The Ansatz [3,4] for the boundary Poisson bracket becomes of the form

$$\{f, g\} = \int d^d y \ d^d y_A \ d^d y_B \ T(y, y_A, y_B) \ e^{\partial y} \left[ E_A(y_A)f \omega^{AB} E_B(y_B)g \right]$$  \hspace{1cm} (5.3)

for some kernel function $T(y, y_A, y_B)$. The $d$-parameter solution (3.3) can be written

$$T(y, y_A, y_B) = \int d^d q \ e^{-qy} \Phi_2(s, s; y_A, y_B)$$  \hspace{1cm} (5.4)

where $\Phi_2$ is a confluent hypergeometric function in two variables (in $d$ dimensions):

$$\Phi_2(\mu, \nu; \lambda \mid x, y) \equiv \sum_{k, \ell=0}^{\infty} \frac{(\mu)_k (\nu)_\ell x^k y^\ell}{(\lambda)_{k+\ell} \ k! \ \ell!}$$  \hspace{1cm} (5.5)

where

$$\forall i = 1, \ldots, d : \ \lambda_i \notin (-\mathcal{N}_0) \ \lor \ \mu_i = \nu_i = \lambda_i = 0$$  \hspace{1cm} (5.6)

and where $\mathcal{N}_0 \equiv \{0, 1, 2, 3, \ldots\}$. 

3
6 Sufficient Condition for the Jacobi Identity

To show that a bracket is a Poisson Bracket, the non-trivial step is to prove the Jacobi identity. It follows in straightforwardly, similar to the derivation given in Appendix B of \[3\], that

$$
\int d^d\tilde{y} \ T(y+y_B, y_A, y_B+\tilde{y}) \ T(\tilde{y}+y_C, y_C, y_D) - (A \leftrightarrow D, B \leftrightarrow C) = 0
$$

is a sufficient condition for the Jacobi identity of (6.3). In our case, (5.4) the condition (6.1) holds, because it is an identity for the \(\Phi_2\) function. After a \(y \rightarrow q\) Fourier transformation it reads:

$$
\int d^d\tilde{y} \ d^d\tilde{q} \ e^{-qy_B} \ \Phi_2 (s, s;|qy_A, q(y_B+\tilde{y})) \ e^{-\tilde{q}(\tilde{y}+y_C)} \ \Phi_2 (s, s;|\tilde{q}y_C, \tilde{q}y_D)
$$

$$
- (A \leftrightarrow D, B \leftrightarrow C) = 0, \quad s \notin (-\mathbb{N})^d.
$$

This identity is the special case \(s = t\), of a more general identity

$$
\int d^d\tilde{y} \ d^d\tilde{q} \ e^{-qy_B} \ \Phi_2 (s, t; 2t-s |qy_A, q(y_B+\tilde{y})) \ e^{-\tilde{q}(\tilde{y}+y_C)} \ \Phi_2 (s, s; t |\tilde{q}y_C, \tilde{q}y_D)
$$

$$
- (A \leftrightarrow D, B \leftrightarrow C) = 0, \quad (6.3)
$$

which is defined for pairs \((s, t)\) satisfying

$$
\forall i = 1, \ldots, d: \ t_i, 2t_i - s_i \notin (-\mathbb{N}_0) \quad \vee \quad s_i = t_i = 0.
$$

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A Proof of the \(\Phi_2\) Identity (6.3)

For completeness we provide a proof for the \(\Phi_2\) identity (6.3). Let us assume that we are given a pair \((s, t)\) satisfying (6.4). It is enough to give the proof for

$$
t-s \notin (-\mathbb{N}_0)^d,
$$

because once this case is proven, the remaining case would then follow from a continuity argument. Assuming \((A.1)\), we can rewrite the \(\Phi_2\) functions as

$$
\Phi_2 (s, t; 2t-s |qy_A, q(y_B+\tilde{y})) = \int d^d\tilde{y}_A \ d^d\tilde{q}_A \ e^{-\tilde{q}_A\tilde{y}_A} \ \Phi (s, t-s |\tilde{q}_A y_A) \ \beta (t-s, t |q\tilde{y}_A, q(y_B+\tilde{y})) ,
$$

$$
\Phi_2 (s, s; t |\tilde{q}y_C, \tilde{q}y_D) = \int d^d\tilde{y}_D \ d^d\tilde{q}_D \ e^{-\tilde{q}_D\tilde{y}_D} \ \beta (s, t-s |\tilde{q}_D y_D) \ \Phi (s, t-s |\tilde{q}_D y_D) \quad (A.2)
$$

Here \(\Phi\) is the usual confluent hypergeometric function in one variable (also known as \(1F_1\))

$$
\Phi (\mu; \nu | x) = \sum_{k=0}^{\infty} \frac{(\mu)_k x^k}{(\nu)_k k!}, \quad \nu \notin (-\mathbb{N}_0)^d. \quad (A.3)
$$

\(^1\)It is rather remarkable that the case \(s = t\), which is the case (6.3) that we ultimately are interested in, is excluded by this assumption!
We have introduced a convenient notation
\[
\beta(\mu, \nu | x, y) \equiv \Phi_2(\mu, \nu; \mu + \nu | x, y) = \sum_{k, \ell=0}^{\infty} \frac{B(\mu+k, \nu+\ell)}{B(\mu, \nu)} \frac{x^k y^\ell}{k! \ell!}, \quad \mu + \nu \notin (-\mathbb{N}_0)^d, \quad (A.4)
\]
for a special case of the confluent hypergeometric function \(\Phi_2\). We choose the name \(\beta\) because of its relationship with the Euler Beta function. Of crucial importance is the Kummer transformation
\[
e^x \beta(\mu, \nu | x, y) = \beta(\mu, \nu | x+z, y+z), \quad (A.5)
\]
which can easily be deduced from the integral representation:
\[
\beta(\mu, \nu | x, y) = \frac{1}{B(\mu, \nu)} \int_0^1 du u^{\mu-1}(1-u)^{\nu-1} e^{xu+y(1-u)}, \quad \text{Re}(\mu), \text{Re}(\nu) > 0. \quad (A.6)
\]
We insert the expressions (A.2) into equation (6.3), then we apply a suitable Kummer transformation on each of the two \(\beta\) functions and finally we do a translation of the integration variables
\[
y_A' = y_A - y_B, \quad y_D' = y_D - y_C. \quad (A.7)
\]
Then equation (6.3) becomes
\[
\int d^d y \ d^d q \ d^d y_A' \ d^d q_A \ d^d y_D' \ d^d q_D \ e^{-q_A(y_A'+y_B)} \ e^{-q_D(y_D'+y_C)} \ \Phi(s; t-s | 0, y_A y_D) = 0. \quad (A.8)
\]
This is true, because
\[
\int d^d y \ d^d q \ e^{-q} \ \beta(t-s, t | q_A, q) \ \beta(s, t-s | 0, q_D) \Phi(s; t-s | 0, q_D y_D) = 0, \quad (A.9)
\]
so that the \((A \leftrightarrow D, B \leftrightarrow C)\) symmetry becomes manifest.

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