Limit theorems for the total scalar curvature

Shota Hamanaka

1*Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama-cho, Toyonaka, 560-0043, Osaka, Japan.

Corresponding author(s). E-mail(s): hamanaka1311558@gmail.com;

Abstract

We prove that the lower bound of the total scalar curvatures on a closed \(n\)-manifold is preserved under the \(W^{1,p}\) \((p > n)\) convergence of the Riemannian metrics provided that each scalar curvature is nonnegative. We also discuss certain weighted version of this type of theorem.

Keywords: Scalar curvature, Ricci flow, Heat flow, Weak notions of the scalar curvature lower bound

MSC Classification: 53C21, 53E20

1 Introduction

Gromov [8] proved the following “\(C^0\)-limit theorem”.

**Theorem 1.1** ([8, p.1118] and [2]). Let \(M\) be a (possibly open) smooth manifold and \(\kappa : M \to \mathbb{R}\) a continuous function. Consider a sequence of \(C^2\)-Riemannian metrics \(g_i\) on \(M\) that converges to a \(C^2\)-Riemannian metric \(g\) in the local \(C^0\)-sense. Assume that for all \(i = 1, 2, \cdots\) the scalar curvature \(R(g_i)\) of \(g_i\) satisfies \(R(g_i) \geq \kappa\) everywhere on \(M\). Then \(R(g) \geq \kappa\) everywhere on \(M\).

In contrast, let \(M\) be the same as in the above theorem, for given a continuous function \(\sigma : M \to \mathbb{R}\), the set \(\{g \in M | R(g) \leq \sigma\}\) is \(C^0\)-dense in the set \(M\) of all smooth Riemannian metrics on \(M\) ([19, Theorem B]). On the other hand, in our forthcoming paper [9], we will show that in a fixed conformal class, the upper bound of the total scalar curvature is preserved under \(C^0\)-convergence of metric tensors. Gromov [8] proved the above theorem by using a gluing technique and the resolution of Geroch’s conjecture. Later, Bamler [2] gave an alternative proof of this theorem using the Ricci–DeTurck flow. On the other hand, Lee–Topping [18] recently proved
that nonnegativity of scalar curvature is not preserved in dimension at least four under the topology of uniform convergence of Riemannian distance. For other studies on the behavior of the scalar curvature lower bound under various weak topologies, see, for example, [5, 10, 12].

As we will see below, we can observe that the point-wise version of Gromov’s theorem (Theorem 1.1) is false in general. That is, we can easily construct an example of $C^2$-Riemannian metrics $(g_i)$ on a smooth manifold $M$ which satisfies the following: $g_i$ converges to a $C^2$-Riemannian metric $g$ on $M$ in the $C^0$-sense as $i \to \infty$. And there is a point $p \in M$ such that for some $\kappa \in \mathbb{R}$,

$$R(g_i)(p) \geq \kappa \quad \text{for all } i \in \mathbb{N},$$

but

$$R(g)(p) < \kappa.$$ 

Indeed, Lohkamp gave an example in [4, Lecture Series 2, Counterexample 2.3.2].

Example 1.1 ([4, Lecture Series 2, Counterexample 2.3.2], cf. Example 5.2 and 5.3 in this paper). For each $i \in \mathbb{N}$, we define a smooth metric on $\mathbb{R}^n$ as

$$g_i := \begin{cases} \exp(2f_i) \cdot g_{\text{Eucl}} & \text{on } D_{\alpha,i} := \{ x \in \mathbb{R}^n \mid |x|_{g_{\text{Eucl}}}^2 \leq \alpha/i \}, \\ g_{\text{Eucl}} & \text{in } \mathbb{R}^n \setminus D_{2\alpha,i}. \end{cases}$$

Here, $g_{\text{Eucl}}$ denotes the Euclidean metric on $\mathbb{R}^n$, $|x|_{g_{\text{Eucl}}}$ is the Euclidean distance from $x$ to the origin $o$, the smooth function

$$f_i := \frac{\alpha}{i} - x_1^2 - x_2^2 - \cdots - x_n^2$$

whose support is contained in $D_{2\alpha,i}$, and $\alpha \in \mathbb{R}$ is a positive constant. Then, using the following fact, we have $R(g_i)(o) = C(n) > 0$. Moreover, $g_i$ converges to $g_{\text{Eucl}}$ uniformly in the $C^0$-sense on $\mathbb{R}^n$.

Fact 1.1 (Conformal change of the scalar curvature). For a Riemannian metric $g$ and a $C^2$-function $\phi$, set $\tilde{g} := e^{2\phi} g$, then

$$R(\tilde{g}) = e^{-2\phi} R(g) - 2(n-1) e^{-2\phi} \Delta_g \phi - (n-2)(n-1) e^{-2\phi} |d\phi|^2_{g}.$$ 

The proof of this fact is a straightforward calculation. Note that the scalar curvature lower bounds are guaranteed only at the origin $o$ in this example. But we can never take a metric whose scalar curvature is bounded from below by some positive constant on a small neighborhood of $o$ and nonnegative on the whole manifold $\mathbb{R}^n$, and which is equal to the Euclidean metric outside a compact subset of $\mathbb{R}^n$. Indeed, if such a metric exists, then we can construct a metric on the $n$-dimensional torus whose scalar curvature is nonnegative everywhere and positive somewhere. But, this is impossible by the resolution of Geroch’s conjecture (see [7, 20, 21]). Of course, we can apply Theorem 1.1 to this sequence $(g_i)$ and $M = \mathbb{R}^n$. But, we can only take the lower bound $\kappa$ such that $\sup_{\mathbb{R}^n} \kappa \leq 0$ in this case because the support of $f_i$ shrinks as $i \to \infty$. Hence, we can only obtain the trivial fact $R(g_{\text{Eucl}}) = 0 \geq \kappa$ even if we use Theorem 1.1.
In this paper, we investigate some total scalar curvature versions of Theorem 1.1. More precisely, we will consider the following problem: Let $M$ be a (possibly non-compact) smooth manifold, $\{g_i\}_{i \in \mathbb{N}}$ a sequence of $C^2$-Riemannian metrics and $g$ a $C^2$-Riemannian metric on $M$. If $g_i$ converges to $g$ in some sense (with respect to $g$) and

$$\int_M R(g_i) \, dv_{g_i} \geq \kappa \quad \text{for all } i$$

for some constant $\kappa \in \mathbb{R}$. Here $R_{g_i}, dv_{g_i}$ denote respectively the scalar curvature of $g_i$ and the Riemannian volume measure of $g_i$. Then, does

$$\int_M R(g) \, dv_g \geq \kappa$$

hold? Or, more generally, let us consider a sequence of $C^2$ metrics $g_i$ on $M$ that converge to a $C^2$ metric in some sense and a sequence of functions $f_i$ on $M$ that converges to a function $f$ in some sense that satisfies

$$\int_M R(g_i)e^{-f_i} \, dv_{g_i} \geq \kappa \quad \text{for all } i$$

for some constant $\kappa \in \mathbb{R}$. Then, we ask whether

$$\int_M R(g)e^{-f} \, dv_g \geq \kappa$$

holds or not.

We emphasize that we will only consider the situation in which metrics converge to some metric with respect to a certain topology which is weaker than $C^2$, each metric is at least $C^2$, and the underlying manifolds we consider are assumed to be smooth.

If $M^2$ is closed (i.e., compact without boundary) surface, from the Gauss-Bonnet theorem,

$$\int_{M^2} R(g) \, dv_g = 4\pi \chi(M)$$

for each Riemannian metric $g$ on $M$. Here $\chi(M)$ denotes the Euler characteristic of $M$. Hence it is sufficient to consider the above problem (unweighted case) in dimension $n \geq 3$ and, unless otherwise mentioned, we will assume below that the dimensions of manifolds are greater than or equal to three.

On the other hand, if $M^{2n}$ is a closed complex $n$ (real $2n$)-manifold ($n \geq 1$) and $g, g_i$ ($i = 1, 2, \cdots$) are Kähler metrics on $M$. Let $\omega$ and $\omega_i$ be the Kähler forms of $g$ and $g_i$ respectively. Assume that $g_i$ converges to $g$ (hence $\omega_i$ converges to $\omega$) in the $C^0$-sense as $i \to \infty$. Then

$$\int_M R(g_i) \, dv_{g_i} = \frac{4\pi}{(n-1)!} (c_1(M) \cdot [\omega_i^{n-1}](M))$$

$$\rightarrow \frac{4\pi}{(n-1)!} (c_1(M) \cdot [\omega^{n-1}](M)) = \int_M R(g) \, dv_g \quad (i \to \infty),$$
where $c_1(M)$ denotes the first Chern class of $M$. Note that we assumed here that the limiting metric $g$ is Kähler metric on $M$ as well, but, in our main theorem 3, we will not assume that the limiting metric $g$ is a Ricci soliton. Although it deviates a bit from our subject, the following interesting result about lower bounds of scalar curvature integrals is also known.

**Theorem 1.2** ([4, Lecture Series 1, Theorem 4.1]). Let $M$ be a compact $n$-dimensional manifold ($n \geq 3$) carrying a hyperbolic metric $g_0$. There is a neighborhood $U$ of $g_0$ in the space of all Riemannian metrics with the $C^2$-topology such that for any $g \in U$,

$$
\int_M (R(g) - \kappa) n/2 \, d\text{vol}_g \geq \int_M (R(g_0) - \kappa) n/2 \, d\text{vol}_{g_0},
$$

and equality if and only if $g$ is isometric to $g_0$. Here, $R(g) := \max\{-R(g), 0\}$.

Our first main result in this paper is the following.

**Main Theorem 1.** Let $p > n$. Let $M^n$ be a closed manifold of dimension $n \geq 3$ and $g$ a $C^2$ Riemannian metric on $M$. Assume that $(g_i)$ is a sequence of $C^2$ Riemannian metrics on $M$ such that $g_i$ converges to $g$ on $M$ in the $W^{1,p}$-sense as $i \to \infty$,

$$
\int_M R(g_i) \, d\text{vol}_{g_i} \geq \kappa \quad \text{for some constant } \kappa \in \mathbb{R},
$$

and $R(g_i) \geq 0$ on $M$ for each $i$. Then

$$
\int_M R(g) \, d\text{vol}_g \geq \kappa.
$$

**Remark 1.1.** From the assumption $R(g_i) \geq 0$, of course, the only meaningful case is $\kappa \geq 0$. For the case of $\kappa = 0$, this claim follows from Theorem 1.1. However, Main Theorem 1 states that this statement still holds for all $\kappa \geq 0$.

Here, we said that a sequence of metrics $(g_i)_i$ converges to a metric $g$ on $M$ in the $W^{1,p}$-sense if $g_i$ and all first weak derivatives of it respectively converge to those of $g$ with respect to the $L^p$-norm of $g$. (Since $M$ is compact, if $(g_i)$ converges in the $W^{1,p}$-sense with respect to $g$, then it also converges $W^{1,p}$-sense with respect to any fixed reference metric on $M$.) Note that from Morrey’s embedding, there is a continuous embedding: $W^{1,p} \hookrightarrow C^{0,1-\frac{d}{p}}$ if $p > n$. Therefore the same statement of Main Theorem 1 still holds even if one replace $W^{1,p}$ ($p > n$) with $C^{0,\alpha}$ for all $\alpha \in (0,1]$. On the other hand, in Main Theorem 1, if we weaken the assumption from $W^{1,p}$-convergence to $C^0$-convergence, then the same statement (without the assumption that each $g_i$ has nonnegative scalar curvature) no longer holds true in general. Indeed, we will give an example (Example 5.3) on every closed Riemannian $n$-manifold for $n \geq 3$.

As a corollary of Main Theorem 1 and Theorem 1.1, we obtain the following.

**Corollary 1.1.** Let $p > n$, and let $\mathcal{M}$ be the space of all $C^2$-Riemannian metrics on a closed manifold $M$. For any nonnegative continuous function $\sigma \in C^0(M, \mathbb{R}_{\geq 0})$ and
constant $\kappa \in \mathbb{R}$, the subspace

$$\left\{ g \in \mathcal{M} \mid \int_{\mathcal{M}} R(g) \, d\text{vol}_g \geq \kappa, \ R(g) \geq \sigma \text{ on } \mathcal{M} \right\} \subset \mathcal{M}$$

is closed in $\mathcal{M}$ with respect to the $W^{1,p}$-topology.

As our second main result in this paper, we will prove the following theorem in a more general setting.

**Main Theorem 2.** Let $p > n^2/2$. Suppose that $M^n$ is a closed $n$-manifold ($n \geq 2$), $g$ is a $C^2$ Riemannian metric on $M$, and $(g_i)$ is a sequence of $C^2$ Riemannian metrics on $M$ such that $g_i$ converges to $g$ on $M$ in the $W^{1,p}$-sense as $i \to \infty$. Let $(f_i)$ be a family of functions on $M$ and $f$ a function on $M$. Assume the following:

1. There is a positive constant $\Lambda > 0$ such that $f$ and $f_i$ ($i \geq 0$) are $\Lambda$-Lipschitz functions on $M$.
2. $f_i \overset{C^0}{\longrightarrow} f$ uniformly on $M$.
3. $R(g_i) \geq 0$ on $M$ for all $i$.
4. $\int_M R(g_i) e^{-f_i} d\text{vol}_{g_i} \geq \kappa$ ($\kappa \in \mathbb{R}$).

Then

$$\int_M R(g) e^{-f} d\text{vol}_g \geq \kappa.$$  

This is non-trivial even in the two-dimensional case because $f_i$ is non-constant in general, hence we cannot use the Gauss-Bonnet theorem. For example, (1) and (2) are automatically satisfied in case that $e^{-f_i} d\text{vol}_{g_i} = d\text{vol}_{g_0} = e^{-f} d\text{vol}_g$ for some $C^0$ metric $g_0$, and $g_i \overset{C^1}{\longrightarrow} g$. Indeed, since each $f_i$ can be locally written as $f_i = \log(\sqrt{\det g_i}) - \log(\sqrt{\det g_0})$ ($f$ is also represented in the same form) and $g_i \overset{C^1}{\longrightarrow} g$, so the norm of the first derivatives $|\nabla f_i|_{g_i}$ are uniformly bounded and $f_i \overset{C^1}{\longrightarrow} f$ uniformly on $M$. And, we speculate the assumptions $R(g_i) \geq 0$ ($i = 1, 2, \cdots$) in Main Theorem 1 and 2 may not be needed. As a corollary of Main Theorem 2, we can obtain the following and from it, we can also define a new weak notion of scalar curvature lower bounds.

**Corollary 1.2 (Corollary 4.3).** Let $p > n^2/2$ and $\kappa$ a constant. Suppose that $M$ is a closed manifold of dimension $n \geq 2$, $g$ is a $C^2$ Riemannian metric on $M$ and $(g_i)$ is a sequence of $C^2$ metric on $M$. Assume the following:

1. A sequence $(\phi_i)$ of nonnegative continuous functions on $M$ satisfying: for any positive constant $a > 0$ there is a positive constant $\Lambda > 0$ such that $\log(\phi_i + a)$ is $\Lambda$-Lipschitz on $M$ for all $i$,
2. $(\phi_i)$ converges to some nonnegative continuous function $\phi$ in the uniformly $C^0$-sense on $M$,
3. $R(g_i) \geq 0$ on $M$ for each $i$,
4. $\int_M R(g_i) \phi_i d\text{vol}_{g_i} \geq \kappa \int_M \phi_i d\text{vol}_{g_i}$,
5. $g_i$ converges to $g$ in the $W^{1,p}$-sense.
Then,
\[ \int_M R(g) \phi \, dvol_g \geq \kappa \int_M \phi \, dvol_g. \]

**Remark 1.2.** From (3), (5) and Remark 4.4, it is known that \( \int_M R(g) \psi \, dvol_g \geq 0 \) for all smooth nonnegative function \( \psi \). Hence it is reasonable to consider case that \( \kappa \geq 0 \) in this setting.

We give the necessary notions and prove this corollary in Section 4. Based on this type of limit theorem, we can define a new generalized notion of scalar curvature lower bound via the existence of certain types of approximate sequences as follows.

**Definition 1.1.** Let \( M^n \) be a smooth closed \( n \)-manifold and \( \kappa \) a constant. For any \( W^{1,p} (p > n^2/2) \) metric \( g \) on \( M \), \( g \) is of \( R \geq \kappa \) in the approximate distributional sense if for any nonnegative continuous function \( \phi \) there is an approximate sequence \( (\phi_i) \) satisfying (1), (2) in Corollary 1.2, and there exists a \( W^{1,p} (p > n^2/2) \)-approximate sequence of \( C^2 \)-metrics \( (g_i) \) satisfying (3)-(5) in Corollary 1.2.

Suppose now that a metric \( g \) in Definition 1.1 is actually \( C^2 \). Then, from Corollary 1.2, \( R(g) \geq \kappa \) in the approximate distributional sense on \( M \) implies that the same bound \( R(g) \geq \kappa \) holds in the distributional sense on \( M \). As a result, \( R(g) \geq \kappa \) in the conventional sense of Remark 4.4. Note that \( R(g) \geq 0 \) in the conventional sense in this case due to (3) of Corollary 1.2 and Theorem 1.1.

For example, if \( (M,g_i) \) is a complete gradient shrinking or steady Ricci soliton, then \( R(g_i) \geq 0 \) on \( M \) (see [6, Corollary 2.5], [26, Theorem 1.3]). Hence, from Main Theorem 1, if furthermore each total scalar curvature is bounded from below by some (nonnegative) constant, then such a lower bound is preserved under the \( W^{1,p} (p > n) \) convergence of metrics. On the other hand, if we assume that each metric is a Ricci soliton with an additional assumption, then we can obtain a similar statement under weaker \( C^0 \) convergence of metrics. Our third main theorem is the following.

**Main Theorem 3.** Let \( M^n \) be a closed \( n \)-manifold and \( g \) a \( C^2 \) Riemannian metric on \( M \). Let \( (g_i) \) be a sequence of Ricci solitons on \( M \) (i.e., \( -2 \text{Ric}(g_i) = \mathcal{L}_{Y_i} g_i - 2 \lambda_i g_i \) for some constant \( \lambda_i \in \mathbb{R} \) and a vector field \( Y_i \in \Gamma(TM) \)) such that \( g_i \) converges to \( g \) on \( M \) in the \( C^0 \)-sense as \( i \to \infty \). Assume

\[ \int_M R(g_i) \, dvol_{g_i} \geq \kappa \quad \text{for some constant } \kappa \in \mathbb{R}. \]

Moreover, assume that \( \lambda_i \leq C_+ \) for all \( i \) and some constant \( C_+ \in \mathbb{R} \) if \( \kappa \geq 0 \) (resp. \( \lambda_i \geq C_- \) for some \( C_- \in \mathbb{R} \) if \( \kappa < 0 \)). Then

\[ \int_M R(g) \, dvol_g \geq \kappa. \]

On a closed manifold, every Ricci soliton is a self-similar solution of the Ricci flow equation, and vice versa. This self-similarity is one of the reasons why the assumption of convergence in Main Theorem 3 can be weaker than \( W^{1,p} \).

The rest of the paper will be arranged as follows. In Section 2, we will prepare some lemmas to prove our main theorems in the next section. In Section 3, we will prove
our main theorems. In Section 4, we will prove some corollaries of our main theorems. In Section 5, we will give several examples that partially suggest that the assumptions of regularity or compactness of manifolds in Main Theorems 1 and 2 are optimal.

## 2 Preliminaries

First, we need the following stability result for the Ricci-DeTurck flow like what is proved in [2, Lemma 2]. More precisely, we need the following.

**Lemma 2.1** ([13, 23]). Let \((M, h)\) be a closed Riemannian manifold endowed with a \(C^2\)-Riemannian metric \(h\). Then, there are constants \(\tau, \varepsilon > 0\), \(C < \infty\) such that the following is true: Consider a \(C^2\)-Riemannian metric \(g\) that is \((1 + \varepsilon)\)-bilipschitz close to \(h\). Then there is a continuous family of Riemannian metrics \((g_t)_{t \in [0, \tau]}\) on \(M\) such that the following holds:

1. For all \(t \in [0, \tau]\), the metric \(g_t\) is \(1.1\)-bilipschitz to \(h\).
2. \((g_t)\) is smooth on \(M \times (0, \tau]\) and the map \([0, \tau] \rightarrow C^2(M, S^2 M), t \mapsto g_t\) is continuous. In particular, \(t \mapsto R_{g_t}\) is continuous on \([0, \tau]\).
3. \(g_0 = g\) and \((g_t)_{t \in [0, \tau]}\) is a solution to the Ricci DeTurck flow equation

\[
(RDE) \quad \frac{\partial}{\partial t} g_t = -2 \text{Ric}(g_t) - L_{X_{h(g_t)}} g_t,
\]

where \(L_{X_{h(g_t)}} g_t\) denotes the Lie derivative of \(g_t\) with respect to the time-dependent vector field \(X_{h(g_t)}\) defined in Remark 2.1 below.
4. For any \(t \in (0, \tau]\) and any \(m = 0, 1, 2\) we have

\[
|\nabla^m h g_t|_h < \frac{C}{t^{m/2}}
\]

where \(|\nabla^m h g_t|_h\) denotes the norm with respect to \(h\) of the covariant derivatives of \(g(t)\) by the Levi-Civita connection of \(h\).
5. If \((g_{t, t})_{t \in [0, \tau]}\) is a sequence of solutions to (RDE) that are continuous on \(M \times [0, \tau]\) and smooth on \(M \times (0, \tau]\) and if \(g_{t, 0}\) converges to some metric \(g_0\) in the \(C^0\)-sense, then there is a subsequence of \((g_{t, t})\) that converges to \((g_t)\) in the \(C^0\)-sense on \(M \times [0, \tau]\) and in the locally smooth sense on \(M \times (0, \tau]\) with respect to \(h\).

**Proof.** The items (a), (c) and (d) are the result of [23, Theorem 1.1]. (e) follows from the standard argument using the derivative estimates in [23, Theorem 1.1] and Arzela-Ascoli’s theorem. Differentiate the equation (66) in [22] twice with respect to \(\nabla h\), then it satisfies a parabolic equation treated in [13] (see also the last part of [2, Proof of Lemma 2]). Then, (b) follows from [13, Theorem 4.2].

Thanks to [12, Theorem 3.11], if we assume that \(g_t\) converges to \(g\) in the \(W^{1,p}(p > n)\)-sense, we also obtain the following.

**Lemma 2.2** (c.f. [23, Theorem 4.3]). Let \((M, g)\) be a closed Riemannian manifold endowed with a \(C^2\)-Riemannian metric \(g\). Then there are constants \(\varepsilon, \tau > 0\) and \(C < \infty\) such that the following is true: Consider a \(C^2\)-Riemannian metric \(g'\) on \(M\) that is \(\varepsilon\)-close to \(g\) in the \(W^{1,p}(p > n)\)-sense with respect to \(g\) (i.e., \(|g - g'|_{W^{1,p}(M, g)} < \varepsilon\), \(\tau < \infty\)).
where \( |g - g'|_{W^{1,p}(\varphi)} \) denotes the \( W^{1,p} \)-norm of the \((0,2)\)-tensor \( g - g' \) with respect to \( g \). Then there is a continuous family of Riemannian metrics \((\tilde{g}_t)_{t \in [0,\tau]} \) on \( M \) such that (a)-(c) and (e) in Lemma 2.1 hold. Moreover, the following (d') holds instead of (d) in Lemma 2.1.

(d') For any \( t \in (0,\tau] \) we have

\[
|\nabla g_\tau|_g < \frac{C}{t^{n/2p} - \varepsilon} \quad \text{and} \quad |\nabla^2 g_\tau|_g < \frac{C}{t^{n/2p} - \varepsilon}
\]

Proof. The statements except for the item (d') are the results of Lemma 2.1 (a)-(c), respectively. The item (d') is the result of [12, Theorem 3.11]. Indeed, since \( g' \) is \( W^{1,p} \)-close to \( g \), the \( W^{1,p} \)-bounded assumption in [12, Theorem 3.11] holds instead of \( \varepsilon \).

\[
\int_M |\nabla g'|_g^p \,	ext{dvol}_g \leq A = A(\varepsilon, g)
\]

is satisfied. Therefore, we obtain the assertion from [12, Theorem 3.11].

\[\Box\]

Remark 2.1. Let \((g_t)_{t \in [0,T]} \) \((0 < T)\) be a solution of the Ricci-DeTurck flow equation with \( g_0 = g \). Choose a background metric \( \bar{g} \) on \( M \) and define the Bianchi operator

\[
X_{\bar{g}}(h) = (\bar{g} + h)^{ij}(\bar{g} + h)^{pq} \left( -\nabla_{p} h_{ij} + \frac{1}{2} \nabla_{p} h_{ij} \right),
\]

(1)

which assigns a vector field to every symmetric 2-form \( h \) on \( M \). Let \((\Phi_t)_{t \in I} \) be the flow generated by the time-dependent family of vector fields \( X_{\bar{g}}(g_t) \), i.e.,

\[
\frac{\partial}{\partial t} \Phi_t = X_{\bar{g}}(g_t) \circ \Phi_t \quad \text{with} \quad \Phi_0 = \text{id}_M.
\]

(2)

We call this flow the \textit{Ricci-DeTurck diffeomorphism} associated to \( g_t \) below. Then \( \tilde{g}_t := \Phi_t^* g_t \) satisfies the \textit{Ricci flow equation}:

\[
\frac{\partial}{\partial t} \tilde{g}_t = -2 \text{Ric}(\tilde{g}_t) \quad \text{with} \quad \tilde{g}_0 = g_0 = g.
\]

(3)

For each \( g_t \), from Lemma 2.2, we have a Ricci-DeTurck flow \( g_t(t) \) defined on \([0,\tau]\) for some positive time \( \tau \), which is independent of \( i \). We also have the corresponding Ricci flow \( \tilde{g}_t(t) \) and the Ricci-DeTurck diffeomorphism \( \Phi_t \) both defined on the same interval \([0,\tau]\). Moreover, under the condition of Lemma 2.1 (e), we can see that for each \( t_0 \in (0,\tau) \), \( \{\tilde{g}_t(t_0)\}_{t \in [t_0,\tau]} \) smoothly subconverges to \( \{\tilde{g}_t(t_0)\}_{t \in [t_0,\tau]} \).

The next lemma will be used in the proof of Main Theorem 2. In particular, the assumption \( p > n^2/2 \) is needed for this lemma.

\textbf{Lemma 2.3} (Subconvergence of the Ricci-DeTurck diffeomorphisms). Let \( M \) be a smooth manifold and \( g \) a \( C^2 \)-Riemannian metric on \( M \). Assume a sequence of \( C^2 \)-Riemannian metrics \( (g_t) \) on \( M \) converges to \( g \) on \( M \) in the \( W^{1,p} \) \((p > n^2/2)\)-sense.
(Φ_{i,t})_{t∈[0,τ)} be the corresponding Ricci-DeTurck diffeomorphism of \( g_i \) with background metric \( g \) defined as in (2). Then, there is a subsequence of \( (Φ_{i,t})_{t∈[0,τ/2]} \) that converges to a time-dependent map \( (Φ_t)_{t∈[0,τ/2]} \) such that

- \( Φ_t \) is a homeomorphism for all \( t ∈ [0, τ/2] \),
- for all \( t ∈ [0, τ/2] \), \( Φ_t, Φ_t^{-1} \) are continuously differentiable and \( \text{d}Φ_t \) and \( \text{d}Φ_t^{-1} \) are mutually inverse,
- there is a subsequence \( (Φ_{ik,t})_{t∈[0,τ/2]} \) such that for all \( t ∈ [0, τ/2] \), \( Φ_{ik,t}, Φ_{ik,t}^{-1}, \text{d}Φ_{ik,t} \) and \( \text{d}Φ_{ik,t}^{-1} \) are converges to \( Φ_t, Φ_t^{-1}, \text{d}Φ_t \) and \( \text{d}Φ_t^{-1} \) respectively,
- \( Φ_0 = \text{id}_M \), where \( \text{id}_M : M → M \) denotes the identity map.

**Proof.** First note that there are a positive time \( τ = τ(M,g) > 0 \) and a positive constant \( C = C(M,g) < ∞ \) such that for sufficiently large \( i \), Lemma 2.2 holds for \( g \) and \( g_i \).

**Step 1 (C₀-convergence):** From Lemma 2.2 (d') and the definition (1), the norm (with respect to \( g \)) of the time-dependent vector field defined in (1) is bounded by some positive constant \( C = C(M,g) \). Thus, applying Gronwall’s lemma to (2), by the same argument in the proof of Lemma 2.1 in [5], one can obtain that

\[
d_g(Φ_{i,t}(p), Φ_{i,s}(p)) ≤ C|t^{1−\frac{1}{p}} s^{1−\frac{1}{p}}|, \quad t, s ∈ [0, τ/2], \quad p ∈ M.
\]  

Here, \( C \) depends only on \( M, g \) and \( τ \). Unlike that in [5, Lemma 2.1], one can get the above type of estimate (in particular, the right-hand side is not \( |t^{1−\frac{1}{p}} s^{1−\frac{1}{p}}| \) but \( |t^{1−\frac{1}{p}} s^{1−\frac{1}{p}}| \)). This follows from the first-derivative estimate (Lemma 2.2 (d')) and hence \( |X_ρ(g_{i,t})|_g \) is bounded by \( Ct^{1−\frac{1}{p}} \) where \( C = C(M,g) \) is a positive constant. On the other hand, taking the derivative of both sides of (2), and using the estimate of the second derivatives (Lemma 2.2 (d')) we obtain that

\[
\frac{∂}{∂t}|\text{d}Φ_{i,t}|_g \leq \frac{C}{t^{1−\frac{1}{p}}}|\text{d}Φ_{i,t}|_g, \quad t ∈ (0, τ/2],
\]

where \( |\text{d}Φ_{i,t}|_g \) denotes the maximum of the operator norm of \( \text{d}Φ_{i,t} : TM → TM \) with respect to \( g \) on \( M \). Hence, for all \( t ∈ (0, τ/2] \), we have

\[
\frac{d}{dt} \left( \frac{u(t)}{v(t)} \right) \leq 0,
\]

where \( u(t) = |\text{d}Φ_{i,t}|_g \) and \( v(t) = e^{Ct^{1−\frac{1}{p}}−\frac{t}{p}} \). Then, since \( u(t) \) is continuous on \( [0, τ/2] \), from the mean value theorem and this estimate of time-derivative, we have

\[
|\text{d}Φ_{i,t}|_g ≤ C e^{Ct^{1−\frac{1}{p}}−\frac{t}{p}}, \quad t ∈ [0, τ/2].
\]  

Since the pullback metric \( Φ^*_{i,t} g_{i,t} \) satisfies the Ricci flow equation:

\[
\frac{∂}{∂t}(Φ^*_{i,t} g_{i,t}) = -2 \text{Ric}(Φ^*_{i,t} g_{i,t}),
\]

where \( \text{Ric} \) is the Ricci curvature of \( g_{i,t} \) on \( M \), then it follows that

\[
|\text{d}Φ_{i,t}|_g \leq C e^{Ct^{1−\frac{1}{p}}−\frac{t}{p}}, \quad t ∈ [0, τ/2].
\]
by Lemma 2.2 (d’) and the above estimate, there is a constant $C = C(M, g, \tau)$ such that for all points $(p, t) \in M \times (0, \tau/2)$ and all vectors $v \in T_p M$,

$$\left| \frac{d}{dt} i^*|v|^2_{\mathfrak{g}_{t_1}, g_{t_1}} \right| \leq \frac{C}{t^{\frac{3}{4}} + \frac{4}{3}} |v|^2_{\mathfrak{g}_{t_1}, g_{t_1}}.$$ 

From this estimate, Lemma 2.2 (b) and the mean value theorem, we obtain

$$e^{-C t^2 \frac{1}{t} - \frac{3}{4} |v|_{g}^2} \leq |v|_{\mathfrak{g}_{t_1}, g_{t_1}}^2 \leq e^{C t^2 \frac{1}{t} - \frac{3}{4} |v|_{g}^2}.$$ 

Since $g_i$ converges to $g$ in the $C^0$-sense, for all sufficiently large $i$, we have

$$e^{-C t^2 \frac{1}{t} - \frac{3}{4} |v|_{g}^2} \leq |v|_{\mathfrak{g}_{t_1}, g_{t_1}}^2 \leq e^{C t^2 \frac{1}{t} - \frac{3}{4} |v|_{g}^2}$$

for some constant $\tilde{C} = \tilde{C}(M, g, \tau)$. Therefore, from Lemma 2.2 (a), the $C^0$-closedness of $g_i$ and $g$, and the previous estimate, there is a positive constant $C = C(M, g, \tau)$ such that for all sufficiently large $i$,

$$d_g(\Phi_{t_1}(p), \Phi_{t_1}(q)) \leq C d_{g_i}(\Phi_{t_1}(p), \Phi_{t_1}(q)) \leq C d_g(p, q). \quad (6)$$

Combining the inequalities (4) and (6), we have

$$d_g(\Phi_{t_1}(p), \Phi_{t_2}(q)) \leq C \left( |l^{1 - \frac{1}{t_1}} - s^{1 - \frac{1}{t_2}}| + d_g(p, q) \right), \quad t_1, t_2 \in [0, \tau/2], \ p, q \in M.$$ 

Then, by Arzela-Ascoli’s theorem, a subsequence $(\Phi_{i_k, t})_{t \in [0, \tau/2]}$ of $(\Phi_{t, t})_{t \in [0, \tau/2]}$ converges to a time-dependent map $\Phi_{t} : M \rightarrow M$ $(t \in [0, \tau/2])$ as $i_k \rightarrow \infty$. In exactly the same way, one can prove that there is a subsequence $(\Phi_{i_{k_l}, t})_{t \in [0, \tau/2]}$ of $(\Phi_{i_k, t})_{t \in [0, \tau/2]}$ that converges to some time-dependent map $\tilde{\Phi}_{t} : M \rightarrow M$ $(t \in [0, \tau/2])$ as $i_k \rightarrow \infty$. But, since $\Phi_{i_{k_l}, t}^{-1} \circ \Phi_{i_{k_l}, t} = id_M$, $\tilde{\Phi}_{t}^{-1}$ exists and $\tilde{\Phi}_{t}^{-1} = \Phi_{t}$ for each $t \in [0, \tau/2]$. To prevent complicated in expression, we will simply write this converging subsequence $(\Phi_{i_{k_l}, t})_{t \in [0, \tau/2]}$ as $(\Phi_{t, t})_{t \in [0, \tau/2]}$.

**Step 2 (C$^1$-convergence):** Next, we will show that the first derivatives of $\Phi_{t, t}$ subconverges as $i \rightarrow \infty$. Let $0 \leq a < 1$. Since $g_{g_i}$ satisfies a parabolic type PDE (see [22, Section 4, Equation (4)]), from the derivative estimate Lemma 2.2 we obtain that

$$\frac{\partial}{\partial t} (\nabla g_{g_i} \cdot t^a) - \Delta_g (\nabla g_{g_i} \cdot t^a) \leq a C t^{1 - \frac{1}{t} - \frac{3}{4} |v|_{g}^2} + C t^{a - \frac{1}{t} - \frac{3}{4} |v|_{g}^2} + C t^{a - \frac{1}{t} - \frac{3}{4} |v|_{g}^2}.$$ 

Then, by the parabolic $W_a^{2,1}$-estimate ([14, Ch. 4, Section 3, Theorem 7]), we have

$$||\nabla g_{g_i} \cdot t^a||_{W_a^{2,1}(M \times (0, \tau/2), g)} \leq C.$$
Similarly, there is a subsequence \((\Phi^i)\) for some 0 \(< t \leq 1\) so that there is a subsequence \((\Phi^i_k)\). Thus, from Morrey’s embedding theorem,

\[ ||\nabla g^i_t||_{C^\alpha(M,g)} \leq Ct^{-\alpha}, \quad t \in (0,\tau/2] \]

for all such \(a, q\) and \(t \in (0,\tau/2]\). Since, \(p > n^2/2 \geq n\), we can choose 0 < \(a < 1\) sufficiently close to 1 so that

\[ n < \frac{1}{1 + \frac{\alpha}{2p} - a}. \quad (7) \]

Hence, as noted above, the weaker relations:

\[ n < \frac{1}{\frac{\alpha}{2p} - a}. \quad (8) \]

are satisfied as well under the above condition (7). Thus, from Morrey’s embedding theorem,

\[ ||\nabla g^i_t||_{C^\alpha(M,g)} \leq Ct^{-\alpha}, \quad t \in (0,\tau/2] \]

for some 0 < \(\alpha < 1\). Therefore, by the same arguments that derived (5) above, we obtain that

\[ |d\Phi^i_t|_{C^\alpha(M,g)} \leq C \exp t^{1-\alpha}, \quad t \in [0,\tau/2]. \]

Then, by the inequalities (5) and (8), we can apply the Arzela-Ascoli’s theorem and obtain that there is a subsequence \((\Phi^i_{k_i,t})\) such that as \(i_k \to \infty\), \((\Phi^i_{k_i,t})\) converges to \((\Phi^i_t)\) for some time-dependent map \((\Phi^i_t)\). Moreover, \((\Phi^i_t)\) are differentiable on \(M\) and \((d\Phi^i_{k_i,t})\) converges to \((d\Phi^i_t)\) as \(i_k \to \infty\). Similarly, there is a subsequence \((\Phi^i_{k_i,t})\) of \((\Phi^i_{k_i,t})\) such that

\[ (\Phi^i_{k_i,t}) \to (\Phi^i_t) \quad \text{and} \quad (d\Phi^i_{k_i,t}) \to (d\Phi^i_t) \]

as \(i_k \to \infty\). Since \(\Phi^i_{k_i,t} \circ \Phi^i_{k_i,t} = \Phi^i_{k_i,t} \circ \Phi^i_{k_i,t} = \id_M\) and \(d\Phi^i_{k_i,t} \circ d\Phi^i_{k_i,t} = d\Phi^i_{k_i,t} \circ d\Phi^i_{k_i,t} = d\Phi^i_{k_i,t} \circ d\Phi^i_{k_i,t}\) are invertible and \(d\Phi^i_{k_i,t} = d\Phi^i_t\) for all \(t \in [0,\tau/2]\). Finally, \(\Phi^i_0 = \id_M\) easily follows from the definition (2) and the above construction of \((\Phi^i_t)\).

**Remark 2.2.** In Lemma 2.3, it is not known that the limit \((\Phi^i_t)\) is the Ricci-DeTurck diffeomorphism of the Ricci flow starting at \(g\) with background metric \(g\). Therefore, let \((g^i_t)\) be the Ricci-DeTurck flow starting at \(g\) with background metric \(g\), then we don’t know whether or not \((\Phi^i_t g^i_t)\) is the solution of the Ricci flow equation starting at \(g\).

Next, we need the following stability of the heat flow with the Ricci flow background, which is proved by Lee and Tam [17, Theorem 3.1].
Lemma 2.4 ([17, Theorem 3.1]). Let \((M^n, g_0)\) be a closed manifold of dimension \(n \geq 2\) and \(\{f_i\}\) a family of functions on \(M\) satisfying the assumptions (1) and (2) in Main Theorem 2. Suppose \(g(t) (t \in [0, \tau])\) be a solution of the Ricci flow equation starting at \(g_0\) such that \(|\text{Rm}(g(t))| \leq a \cdot t^{-1}\) for some \(a > 0\) on \([0, \tau]\). Then for all \(i_0 \in \mathbb{N}\), there are positive constants \(\tau_0 = \tau_0(n, \Lambda, i_0) > 0\) and \(C_0 = C_0(n, a, \Lambda, i_0) > 0\) such that the following holds. For all \(i \geq i_0\), there exists \(F_i(t) \in C^\infty(M) \quad (t \in (0, \min\{\tau, \tau_0\}))\) satisfies the heat flow equation:

\[
\frac{\partial}{\partial t} F_i(t) = \Delta_{g_i} F_i(t)
\]

such that

(A) \( (\Lambda^{-2} - 2(n - 1)t) (F_i)^* \eta_{\text{Euc}} \leq g(t) \),
(B) \( \sup_{x \in M} d_{\eta_{\text{Euc}}} (F_i(x, t), f_i(x)) \leq C_0 \sqrt{t} \).

Moreover, for any integer \(l \geq 0\), there is a constant \(C = C(n, l, a, \Lambda, i_0) > 0\) such that for all \(t \in (0, \min\{\tau, \tau_0\}]\),

\[
|\nabla^l dF_i| \leq C \cdot t^{-1/2}.
\]

Here, \(\eta_{\text{Euc}}\) denotes the Euclidean metric on \(\mathbb{R}\).

Proof. Since \(M\) is compact, the image of \(f\) is compact in the target space \(\mathbb{R}\). From the assumption (2) in Main Theorem 2, there is a compact neighborhood \(N(\subset \mathbb{R})\) of the image of \(f\) such that the image of \(f_i\) is contained in \(N\) for all \(i \geq i_0\). Then, from the assumption (1) of Main Theorem 2, one can apply the proof of Lemma 3.1 and Theorem 3.1 in [17] to \((M, g(t)), (N, h := g_{\text{Euc}}|_N)\) and \(f_i : M \to N \subset \mathbb{R}\) for all \(i \geq i_0\). Hence we obtain the desired assertions from Theorem 3.1 in [17].

The following lemma is a key to proving our main theorems. Note that we only need \(p > n\) for this lemma.

Lemma 2.5 (cf. [2, Lemma 4]). Let \(M^n\) be a closed \(n\)-manifold \((n \geq 2)\) and \(g\) a \(C^2\)-Riemannian metric on \(M\). Suppose \(f : M \to \mathbb{R}\) be a \(\Lambda\)-Lipschitz function for some \(\Lambda > 0\). Assume that a \(C^2\)-Riemannian metric \(g'\) on \(M\) is sufficiently close to \(g\) in the \(W^{1, p}(M, g)\)-sense so that Lemma 2.2 holds for \(g'\). Then for any given positive constant \(\delta > 0\), there is a constant \(\tau = \tau(M, g, |g - g'|_{W^{1, \infty}(M, g)}, \delta, \Lambda) > 0\) such that the following holds: Assume that \(\int_M R(g') e^{-f} \text{dvol}_{g'} \geq a\) for some \(a \in \mathbb{R}\) and \(R(g') \geq 0\) on \(M\). Then there is a solution \((g'_t)_{t \in [0, \tau]}\) to the Ricci flow equation (3) with the initial metric \(g'\) and a solution \((f'_t)_{t \in [0, \tau]}\) of the heat flow equation (9) with the Ricci flow background such that \(\int_M R(g'_t) e^{-f} \text{dvol}_{g'_t} \geq a - \delta\) for all \(t \in [0, \tau]\).

Proof. From Lemma 2.2 and 2.4, there is a sufficiently small \(1 > \tau' > 0\) such that there is a Ricci flow \((g''(t))_{t \in [0, \tau']}\) starting at \(g''\) and there is a heat flow \((f'')_{t \in [0, \tau']}\) with
$f_t \to f$ as $t \to 0$ uniformly. Along these $(g'_t)$ and $(f_t)$, for all $t \in (0, \tau']$,

$$
\frac{d}{dt} \left( \int_M R(g'_t) e^{-f_t} d\text{vol}_{g'_t} \right)
$$

\begin{align*}
\overset{\text{(1)}}{=} & \int_M \left( \Delta_{g'(t)} R(g'_t) + 2|\text{Ric}(g'_t)|_{g'_t}^2 - R(g'_t)^2 \right) e^{-f_t} d\text{vol}_{g'_t} + \int_M \left( \frac{\partial}{\partial t} e^{-f_t} \right) R(g'_t) d\text{vol}_{g'_t} \\
\overset{\text{(2)}}{=} & \int_M \left( 2|\text{Ric}(g'_t)|_{g'_t}^2 - R(g'_t)^2 \right) e^{-f_t} d\text{vol}_{g'_t} - 2 \int_M \left( \Delta_{g'_t} f_t - |\nabla f_t|^2 \right) R(g'_t) e^{-f_t} d\text{vol}_{g'_t} \\
\overset{\text{(3)}}{=} & \int_M \left( 2 R(g'_t)^2 - R(g'_t)^2 \right) e^{-f_t} d\text{vol}_{g'_t} - 2 \int_M \left( \Delta_{g'_t} f_t - |\nabla f_t|^2 \right) R(g'_t) e^{-f_t} d\text{vol}_{g'_t} \\
= & \left( \frac{2}{n} - 1 \right) \int_M R(g'_t)^2 e^{-f_t} d\text{vol}_{g'_t} - 2 \int_M \left( \Delta_{g'_t} f_t - |\nabla f_t|^2 \right) R(g'_t) e^{-f_t} d\text{vol}_{g'_t} \\
\overset{\text{(4)}}{\geq} & -C t^{-\frac{n}{4} - \frac{3}{4}} \int_M R(g'_t) e^{-f_t} d\text{vol}_{g'_t}.
\end{align*}

Here, $C$ is a positive constant depending on $M, \Lambda$ and $|g - g'|_{W^{1,\infty}(M,g)}$, and

- (1) follows from the evolution of the scalar curvature and the volume form under the Ricci flow, and $\partial_t f_t = \Delta_{g'_t} f_t$ $(t \in (0, \tau'])$.
- (2) follows from applying the divergence formula to the term

$$\int_M \Delta_{g'(t)} R(g'(t)) e^{-f_t} d\text{vol}_{g'(t)},$$

- (3) follows from the Cauchy-Schwarz inequality $u|\text{Ric}(g'_t)|_{g'_t}^2 \geq R(g'_t)^2$,
- (4) have been obtained as follows. From the assumption $R(g') \geq 0$, by the maximum principle under the Ricci flow, we have $R(g'_t) \geq 0$. Since the scalar curvature is invariant under the pullback action by a diffeomorphism, from the $C^1$-closedness assumption, one can apply the same derivative estimate $(d')$ in Lemma 2.2 to one $R(g'_t)$ in the integrand of the left-hand side of the inequality. Moreover, applying the derivative estimate for the heat flow as in Lemma 2.4 to the second integrand of the left-hand side of the last inequality, we obtain the desired estimate.

Set $u(t) := \int_M R(g_t) e^{-f_t} d\text{vol}_{g_t}$ and $v(t) := \exp \left( C \left( 1 - \frac{n}{4} - \frac{3}{4} \right) t^{-\frac{n}{4} - \frac{3}{4}} \right)$. Then, by the above calculation, we have

$$
\frac{d}{dt} \left( \frac{u(t)}{v(t)} \right) = \frac{u'(t)v(t) - u(t)v'(t)}{v^2(t)} \geq \frac{-C t^{-\frac{n}{4} - \frac{3}{4}} u(t)v(t) - u(t) \left( -C t^{-\frac{n}{4} - \frac{3}{4}} \right) v(t)}{v^2(t)} = 0
$$

for all $t \in (0, \tau']$. Since $u(t)$ is continuous on $[0, \tau']$ from Lemma 2.1 (b) and Lemma 2.4 (B), by the mean value theorem and the previous derivative estimate, we finally
obtain that
\[ \int_M R(g'(t)) e^{-f(t)} d\text{vol}_{g'(t)} \geq v(t) \cdot \int_M R(g') e^{-f} d\text{vol}_{g'}, \quad t \in [0, \tau']. \]

Therefore, if we take sufficiently small \(0 < \tau' < \tau\), the desired assertion holds for such a constant \(\tau > 0\).

3 Proof of Main Theorems

In this section, we prove the main theorems. First, we prove Main Theorem 2 because we will use almost the same method in the proof of Main theorems 1 and 3. The idea of proof here is the same as that of Bamler [2], by contradiction. To do this, we suppose that the weighted total scalar curvature of the limiting metric is less than or equal to \(\kappa - \delta\) for some \(\delta > 0\). On the other hand, combining Lemma 2.2, 2.3, 2.4 and 2.5, we can deduce that the weighted total scalar curvature of the limiting metric is greater than or equal to \(\kappa - \delta/2\). This contradicts our supposition. We prescribe these arguments with more precision below.

**Proof of Main Theorem 2.** We show the assertion by contradiction. Suppose that \(\int_M R(g) e^{-f} d\text{vol}_g < \kappa\).

Then there is a positive constant \(\delta > 0\) such that \(\int_M R(g) e^{-f} d\text{vol}_g \leq \kappa - \delta < \kappa\).

On the other hand, since \(\int_M R(g_i) e^{-f_i} d\text{vol}_{g_i} \geq \kappa\), from Lemma 2.2, Lemma 2.5 and Lemma 2.4, there are a \(\tau = \tau(\delta)\), a Ricci flow \((\tilde{g}_{i,t})_{t \in [0, \tau]}\) and a heat flow \((f_{i,t})_{t \in (0, \tau]}\) such that \(\int_M R(\tilde{g}_{i,t}) e^{-f_{i,t}} d\text{vol}_{\tilde{g}_{i,t}} \geq \kappa - \frac{1}{2} \delta\)

and there also exists a heat flow \((f_{i})_{t \in [0, \tau]}\) with \(f_{i,t} \to f\) as \(t \to 0\) uniformly on \(M\). By Lemma 2.3, as \(i \to \infty\), we have a solution of the Ricci–DeTurck flow equation \((\tilde{g}_{i})_{t \in [0, \tau]}\) (retake a sufficiently small \(\tau\) if necessary) starting at \(g\) and a time-dependent \(C^1\)-diffeomorphism \((\Phi_t)_{t \in [0, \tau]}\) with \(\Phi_0 = \text{id}_M\) such that

\[ \int_M R(\Phi_t^* g_t) e^{-f_{t}} d\text{vol}_{\Phi_t^* g_t} = \int_M R(g_t) e^{-f_t \circ \Phi_t^{-1}} d\text{vol}_{g_t} \geq \kappa - \frac{1}{2} \delta \]

for all \(t \in [0, \tau]\). On the other hand, by Lemma 2.2 (b), \(\Phi_0 = \text{id}_M\) and \(f_t \to f\) \(t \to 0\), we have

\[ \int_M R(g) e^{-f} d\text{vol}_g = \lim_{t \to 0} \int_M R(g_t) e^{-f_t \circ \Phi_t^{-1}} d\text{vol}_{g_t} \geq \kappa - \frac{1}{2} \delta > \kappa - \delta. \]
This contradicts our supposition
\[ \int_M R(g) e^{-f} \, d\text{vol}_g \leq \kappa - \delta \]
and concludes the proof. \hfill \Box

Next, we give a proof of Main Theorem 1. This is simpler than the proof of Main Theorem 2 because the unweighted total scalar curvature \( \int_M R(g) \, d\text{vol}_g \) is diffeomorphism invariant, hence it does not need to use Lemma 2.3.

**Proof of Main Theorem 1.** We first prove the first half of the theorem. Then we can prove this part in the same way as in the proof of Main Theorem 2 since we can use Lemma 2.5 with \( f \equiv 0 \) (hence \( f_t \equiv 0 \)). Note that since the total scalar curvature is diffeomorphism invariant, from Remark 2.1, we have
\[ \int_M R(\tilde{g}_{i,t}) \, d\text{vol}_{\tilde{g}_{i,t}} = \int_M R(g_{i,t}) \, d\text{vol}_{g_{i,t}}, \]
where \( \tilde{g}_{i,t}, g_{i,t} \) are the Ricci-DeTurck flow and the Ricci flow starting at \( g_i \) respectively. Note also that the condition \( p > n \) is sufficient to apply Lemma 2.5. Therefore, under the assumption of Main Theorem 1, arguing in the same way in the proof of Main Theorem 2 (and using the same notation), we obtain
\[ \int_M R(g_i) \, d\text{vol}_{g_i} \geq \kappa - \frac{1}{2} \delta \]
fors all \( t \in (0, \tau] \), where \( g_i \) is the Ricci-DeTurck flow starting at \( g \). Then we can deduce a contradiction as \( t \to 0 \) as in the same way in the proof of Main Theorem 2.

Next, we prove the theorem for \( n = 3 \) without assuming \( R(g_i) \geq 0 \). This statement follows from the following more general claim.

**Claim 3.1** (cf. [4, Lecture Series 2, Proposition 2.3.1]). Let \((M^n, g)\) be a closed Riemannian manifold of dimension \( n \geq 2 \). Suppose that a sequence of \( C^2 \)-Riemannian metrics \((g_i)\) on \( M\) converges to \( g \) in the \( W^{1,2}\)-sense. Suppose also that \( M\) is parallelizable i.e., the tangent bundle of \( M\) is trivial. Then
\[ \int_M R(g_i) \, d\text{vol}_{g_i} \to \int_M R(g) \, d\text{vol}_g \quad \text{as} \quad i \to \infty. \]

**Proof of Claim 3.1.** We will use the similar technique in the proof of [4, Lecture Series 2, Proposition 2.3.1]. Since \( M\) is parallelizable, we can take a global section \( \{v_1(x), \cdots, v_n(x)\}_{x \in M}\) of the orthonomal frame bundle of \( M\). Define the 1-form \( \omega_k(\cdot) := g(v_k, \cdot) \) and the vector field \( v_k^i \) \((i = 1, 2, \cdots, k = 1, 2, \cdots, n)\) be the dual of \( \omega_k \) with respect to \( g_i \). Next, we will use the Bochner identity for 1-forms:
\[ \Delta_g = \nabla^*_g \nabla_g + \text{Ric}(g) \]
Here $\Delta_g = D^*_g D_g$ (resp. $\Delta_{g_i} = D^*_{g_i} D_{g_i}$) for the Dirac operator $D_g$ (resp. $D_{g_i}$) related to the deRham complex. Thus, we have for $g_i$ and $v^i_k$ using this Bochner identity and the integration by parts,

$$\int_M (||D_{g_i}\omega_k||^2_{g_i} - ||\nabla_{g_i}\omega_k||^2_{g_i})\,d\text{vol}_{g_i} = \int_M \text{Ric}(g_i)(v^i_k, v^i_k)\,d\text{vol}_{g_i}.$$ 

Since $g_i \to g$ in the $W^{1,2}$-sense on $M$, the left-hand side of the above equality converges to the following quantity

$$\int_M (||D_g\omega_k||^2_g - ||\nabla_g\omega_k||^2_g)\,d\text{vol}_{g} = \int_M \text{Ric}(g)(v_k, v_k)\,d\text{vol}_{g}.$$ 

Taking the sum from 1 to $n$ for $k$, we obtain that

$$\int_M R(g_i)\,d\text{vol}_{g_i} \to \int_M R(g)\,d\text{vol}_{g} \quad \text{as} \quad i \to \infty.$$ 

This completes the proof of the claim. \qed

Since every oriented closed three manifold is parallelizable, using this claim we have

$$\int_{M^3} R(g_i)\,d\text{vol}_{g_i} \to \int_{M^3} R(g)\,d\text{vol}_{g} \quad \text{as} \quad i \to \infty.$$ 

(Of course, if necessary, we discuss in a similar way after taking its orientable double cover.) In particular, if $\int_{M^3} R(g_i)\,d\text{vol}_{g_i} \geq \kappa$ for all $i$, then $\int_{M^3} R(g)\,d\text{vol}_{g} \geq \kappa$. This completes the proof of Main Theorem 1. \qed

**Example 3.1.** For example, the following manifolds are known to be parallelizable:

1. Every orientable closed three manifold.
2. $n$-dimensional sphere $S^n$ where $n = 0, 1, 3$ or 7.
3. Every Lie group.
4. The product of parallelizable manifolds.

**Question 3.1.**

- In Claim 3.1, is the parallelizability of $M$ necessary?
- What is the relation between parallelizability of a closed manifold and $W^{1,2}$-convergence of metric tensors on it?

Next, we give a proof of Main Theorem 3.

**Proof of Main Theorem 3.** The proof is similar to the one of Main Theorem 2. However, since each metric is Ricci soliton, the Ricci flow starting from such a metric is homothetic. Using this fact, we can prove a similar statement to Theorem 2 under the
assumption of lower $C^0$ regularity. Indeed, the solution $(\tilde{g}_{i,t})_{t \in [0,\tau)}$ of the Ricci flow equation starting from $g_i$ constructed in Lemma 2.1 is

$$\tilde{g}_{i,t} = (1 - 2\lambda_i t)^{-1} \Phi_t^* (g_i) \quad t \in [0,\tau),$$

where $\Phi_{i,t}$ is the family of diffeomorphisms generated by the time-dependent vector field $X_i(t) := (1 - 2\lambda_i t)^{-1} Y_i$ with the initial condition $\Phi_{i,0} = \text{id}_M$, and $\tau > 0$ is the uniform existence time of the flows guaranteed in Lemma 2.1. Thus, we have

$$\int_M R(\tilde{g}_{i,t}) \, d\text{vol}_{\tilde{g}_{i,t}} = (1 - 2\lambda_i t)^{n/2 - 1} \int_M R(g_i) \, d\text{vol}_{g_i}. \tag{10}$$

for all $t \in [0,\tau)$. We will first consider the case of $\kappa \geq 0$. From (10), for any $\delta > 0$, there is a positive time $\tau = \tau(\delta, C_+) > 0$ such that for any diffeomorphism $\Phi : M \to M$ and $t \in [0,\tau]$,

$$\int_M R(\Phi^* \tilde{g}_{i,t}) \, d\text{vol}_{\Phi^* \tilde{g}_{i,t}} = \int_M R(\tilde{g}_{i,t}) \, d\text{vol}_{\tilde{g}_{i,t}} \geq \kappa - \delta.$$

Similarly, when $\kappa < 0$, we see that for any $\delta > 0$, there is a positive time $\tau = \tau(\delta, C_-) > 0$ such that for any diffeomorphism $\Phi : M \to M$ and $t \in [0,\tau]$,

$$\int_M R(\Phi^* \tilde{g}_{i,t}) \, d\text{vol}_{\Phi^* \tilde{g}_{i,t}} = \int_M R(\tilde{g}_{i,t}) \, d\text{vol}_{\tilde{g}_{i,t}} \geq \kappa - \delta.$$

In particular, we take the inverse of the Ricci-DeTurck diffeomorphism (see Remark 2.1) of $\tilde{g}_{i,t}$ as $\Phi$ here, then we have

$$\int_M R(g_{i,t}) \, d\text{vol}_{g_{i,t}} \geq \kappa - \delta \quad \text{for all } t \in [0,\tau],$$

where $g_{i,t}$ is the Ricci-DeTurck flow starting at $g_i$. Hence, we can use this claim instead of Lemma 2.5 in the proof of Main Theorem 2. Therefore, using Lemma 2.1 instead of Lemma 2.2, we can prove the assertion in the same way as in the proof of Main Theorem 2.

**Remark 3.1.** If $g_i$ is a shrinking Ricci soliton (i.e., $\lambda_i > 0$ in the above situation), then the maximal existence time of the corresponding Ricci flow is $(2\lambda_i)^{-1}$. But, since we assume $g_i$ converges to $g$ in the $C^0$-sense, Lemma 2.1 implicitly prevents $\lambda_i \to \infty$ as $i \to \infty$.

### 4 Some corollaries

In the assumption in Main Theorem 1, the nonnegativity of each scalar curvature can be replaced with general lower bound of each scalar curvature provided that a suitable condition about the volumes is additionally assumed.

**Corollary 4.1** (for Main Theorem 1). Let $M^n$ be a closed $n$-manifold and $g$ a $C^2$-Riemannian metric on $M$. Let $(g_i)$ be a sequence of $C^2$-Riemannian metrics on $M$
that converges to $g$ on $M$ in the $W^{1,p}$-sense ($p > n$). Assume that there are a constant $\kappa \in \mathbb{R}$ and a continuous function $\sigma \in C^0(M)$ such that for all $i$,

- $\int_M R(g_i) \, d\text{vol}_{g_i} \geq \kappa$,
- $R(g_i) \geq \sigma$ on $M$, and
- $\text{Vol}(M,g_i) \geq \text{Vol}(M,g)$.

Here $\text{Vol}(M,g)$ is the volume of $M$ with respect to $g$. Then

$$\int_M R(g) \, d\text{vol}_g \geq \kappa.$$ 

**Remark 4.1.** If the limiting metric $g$ here is a hyperbolic metric (i.e., its sectional curvature is constant $-1$) and each metric $g_i$ has its volume entropy $h(g_i) \leq n - 1$ ($n \geq 3$), then $\text{Vol}(M,g_i) \geq \text{Vol}(M,g)$ for all $i$. This is the result of [3]. The volume entropy $h(G)$ of a closed Riemannian manifold $(M,G)$ is given by the limit

$$h(G) = \lim_{r \to \infty} \frac{\log \left( \text{Vol} \left( B_G(p_0, r), \tilde{G} \right) \right)}{r},$$

where $\text{Vol} \left( B_G(p_0, r), \tilde{G} \right)$ is the volume of a ball of radius $r$ in the universal cover $(\tilde{M}, \tilde{G})$. (In particular, Bishop-Gromov’s volume comparison implies that a manifold with Ricci curvature lower bound $\text{Ric}_G \geq -(n-1)G$ satisfies $h(G) \leq n - 1$.) Moreover, as a deep corollary of Perelman’s resolution of the Geometrization conjecture of W. Thurston, a very strong generalization of this result [3] in dimension 3 holds (see [1]):

- if $(M,g_0)$ is a closed hyperbolic 3-manifold, for any metric $g$ on $M$,
  - if $R(g) \geq -6$ then $\text{Vol}(M,g) \geq \text{Vol}(M,g_0)$.

**Proof of Corollary 4.1.** There is a sufficient large natural number $n \in \mathbb{N}$ such that the Riemannian product manifold $(N := M \times S^n, g_N := g_i \times g_{\text{std}})$ satisfies $R(g_N) \geq 0$, where $(S^n,g_{\text{std}})$ is the standard Riemannian $n$-sphere of constant scalar curvature $n(n-1)$. Indeed, since

$$R(g_N) = R(g_i) + R(g_{\text{std}}) = R(g_i) + n(n-1) \geq \min_M \sigma + n(n-1),$$

we see that $R(g_N) \geq 0$ for sufficiently large $n$. Therefore we have

$$\int_N R(g_i) \, d\text{vol}_{g_N} = \int_{M \times S^n} (R(g_i) + R(g_{\text{std}})) \, d\text{vol}_{g_i \times g_{\text{std}}}$$

$$= \text{Vol}(S^n,g_{\text{std}}) \int_M R(g_i) \, d\text{vol}_{g_i}$$

$$+ \text{Vol}(M,g_i) \text{Vol}(S^n,g_{\text{std}}) n(n-1).$$
On the other hand, from our assumption, \( g_i^N = g_i \times g_{\text{std}} \) converges to \( g \times g_{\text{std}} \) in the \( W^{1,p} \)-sense and
\[
\int_N R(g_i^N) \, d\nu_{g_i^N} \geq \text{Vol}(S^n, g_{\text{std}}) \kappa + \text{Vol}(M, g) \text{Vol}(S^n, g_{\text{std}}) \, n(n-1).
\]
Therefore, from Main Theorem 1,
\[
\text{Vol}(S^n, g_{\text{std}}) \int_M R(g) \, d\nu_{g} + \text{Vol}(M, g) \text{Vol}(S^n, g_{\text{std}}) \, n(n-1)
= \int_N R(g \times g_{\text{std}}) \, d\nu_{g \times g_{\text{std}}}
\geq \text{Vol}(S^n, g_{\text{std}}) \kappa + \text{Vol}(M, g) \text{Vol}(S^n, g_{\text{std}}) \, n(n-1).
\]
Then, subtracting \( \text{Vol}(M, g) \text{Vol}(S^n, g_{\text{std}}) \, n(n-1) \) from both sides and dividing both sides by \( \text{Vol}(S^n, g_{\text{std}}) > 0 \), we obtain \( \int_M R(g) \, d\nu_g \geq \kappa. \)

We also present another corollary of Corollary 4.1 here. In order to do this, we need to recall the definition of the Yamabe constant:

**Definition 4.1 (Yamabe constant).** The Yamabe constant \( Y(M, g) \) of a closed Riemannian manifold \((M, g)\) is defined as
\[
Y(M, g) := \inf \left\{ \int_M R(\tilde{g}) \, d\nu_{\tilde{g}} \middle| \tilde{g} \in [g] \text{ and } \text{Vol}(M, \tilde{g}) = 1 \right\},
\]
where \([g] := \left\{ \tilde{g} = u^{-\frac{n}{n-2}} g \mid u \in C^\infty(M), u > 0 \text{ on } M \right\}\) is the conformal class of the metric \( g \). By the definition, \( Y(M, g) \) depends only on the conformal class \([g]\) of \( g \). A Riemannian metric \( \tilde{g} \in [g] \) with \( \text{Vol}(M, \tilde{g}) = 1 \) is called *Yamabe metric* of \([g]\) if
\[
Y(M, g) = \int_M R(\tilde{g}) \, d\nu_{\tilde{g}}.
\]

**Corollary 4.2 (for Corollary 4.1).** Let \( M^n \) be a closed \( n \)-manifold and \( g \) a \( C^2 \)-Riemannian metric on \( M \). Let \( (g_i) \) be a sequence of \( C^2 \)-Riemannian metrics on \( M \) that converges to \( g \) on \( M \) in the \( W^{1,p} \)-sense \( (p > n) \), and \( \text{Vol}(M, g_i) = 1 \). Assume that \( g \) is a Yamabe metric of \([g]\), and there are a constant \( \kappa \in \mathbb{R} \) and a continuous function \( \sigma \in C^0(M) \) such that for all \( i \), \( Y(M, g_i) \geq \kappa \) and \( R(g_i) \geq \sigma \) on \( M \). Then \( Y(M, g) \geq \kappa \).

**Proof.** From the definition of \( Y(M, g_i) \) and \( \text{Vol}(M, g_i) = 1 \), we have
\[
\int_M R(g_i) \, d\nu_{g_i} \geq \kappa.
\]
Since $g_i \to g$ in the $W^{1,p}$-sense ($p > n$) and $\text{Vol}(M, g_i) = 1$, we also have $\text{Vol}(M, g) = 1$. Hence, from Corollary 4.1, we have

$$\int_M R(g) \, d\text{vol}_g \geq \kappa.$$ 

Therefore, since $g$ is a Yamabe metric of $[g]$ and $\text{Vol}(M, g) = 1$, we obtain

$$Y(M, g) = \int_M R(g) \, d\text{vol}_g \geq \kappa.$$

**Remark 4.2.** If $\sigma \geq \kappa$, this corollary directly follows from Gromov’s theorem (Theorem 1.1 in this paper). Indeed, since $g_i \to g$ in the $W^{1,p}$-sense for $p > n$ (in particular, in the $C^0$-sense), from Theorem 1.1, we have $R(g) \geq \sigma$. On the other hand, since $g$ is a Yamabe metric of $[g]$ (hence, its scalar curvature $R(g)$ is constant) and $\text{Vol}(M, g) = 1$, we have

$$Y(M, g) = \int_M R(g) \, d\text{vol}_g \geq \sigma \geq \kappa.$$ 

However, if $\sigma \leq \kappa$, this corollary does not follow from Gromov’s theorem.

Up to this point, we have only dealt with case where the underlying manifold is closed. In a special situation, we can prove a similar statement as Claim 3.1 for a non-compact manifold.

**Proposition 4.1** (Conformal deformations on an open manifold). Let $(M^n, g)$ be a non-compact Riemannian $n$-manifold ($n \geq 3$) with $\int_M R(g) \, d\text{vol}_g < +\infty$ and $u_i : M \to \mathbb{R}$ a sequence of positive $C^2$-functions. Assume that each $u_i$ is equal to 1 outside a compact set and $u_i \to 1$ uniformly $C^1$ sense on $M$.

Set $g_i := u_i^{4/(n-2)} g$. Then, it holds that

$$\int_M R(g_i) \, d\text{vol}_{g_i} \xrightarrow{i \to \infty} \int_M R(g) \, d\text{vol}_g.$$ 

**Proof.** From the formula for the scalar curvature and the volume form under this conformal change:

$$R(g_i) = -\frac{4}{n-2} \Delta_g u_i + u_i^{-\frac{4}{n-2}} R(g), \quad \text{vol}_{g_i} = u_i^{\frac{4}{n-2}} \text{vol}_g,$$ 

20
we have
\[
\int_M R(g_i) \, dvol_{g_i} = -\frac{4n-1}{n-2} \int_M u_i \frac{2n-4}{2n-2} \Delta_g u_i \left( u_i \frac{2n}{2n-2} \, dvol_g \right) + \int_M u_i^{\frac{2n}{n-2}} R(g) \, dvol_g \\
= -\frac{4n-1}{n-2} \int_M u_i \Delta_g u_i \, dvol_g + \int_M u_i^{\frac{2n}{n-2}} R(g) \, dvol_g \\
= \frac{4n-1}{n-2} \int_M |\nabla_g u_i|^2 \, dvol_g + \int_M u_i^{\frac{2n}{n-2}} R(g) \, dvol_g.
\]

We have used the divergence formula and the fact that \( u_i \) is equal to 1 outside a compact set in the third equality. Since \( u_i \xrightarrow{1 \to \infty} 1 \) \( C^1 \)-uniformly on \( M \), we have
\[
\int_M |\nabla_g u_i|^2 \, dvol_g \to 0 \quad \text{as} \quad i \to \infty
\]
and
\[
\int_M \frac{2n}{n-2} R(g) \, dvol_g \to \int_M R(g) \, dvol_g \quad \text{as} \quad i \to \infty
\]
Therefore we obtain the desired assertion from the above equality. \( \square \)

**Remark 4.3.** The proof of Proposition 4.1 also provides the following: In Main Theorem 1, if \( p = \infty \), and if for each \( i \), \( g_i = u_i \cdot g \) for some positive \( C^2 \)-functions \( u_i : M \to \mathbb{R}_+ \), then the assumption \( R(g_i) \geq 0 \) \( i = 1, 2, \ldots \) is not needed.

Lee and LeFloch \[15\] defined a notion of distributional scalar curvature for smooth manifolds that have a metric tensor that has only certain lower regularity.  

**Definition 4.2** (Distributional scalar curvature ([15, Definition 2.1], [12, Section 2])). Let \( M \) be a smooth manifold endowed with a smooth background metric \( h \). Given any Riemannian metric \( g \) with \( L^\infty_{loc}(M) \cap W^{1,2}_{loc}(M) \) regularity and locally bounded inverse \( g^{-1} \in L^\infty_{loc}(M) \), the scalar curvature distribution \( R_g \) is defined for every compactly supported smooth test function \( u : M \to \mathbb{R} \) by
\[
\langle R_g, u \rangle := \int_M \left( -V \cdot \nabla \left( \frac{dvol_g}{dvol_h} \right) + F \frac{dvol_g}{dvol_h} \right) \, dvol_h,
\]
where \( V = (V^k) \in \Gamma(M) \) is given by \( V^k := g^{ij} \Gamma^k_{ij} - g^{ik} \Gamma^j_{ji} \), \( F \) is a function as
\[
F := R_h - \nabla_k g^{ij} \Gamma^k_{ij} + \nabla_k g^{ik} \Gamma^j_{ji} + g^{ij} \left( \Gamma^k_{ij} \Gamma^l_{kl} - \Gamma^k_{jk} \Gamma^l_{il} \right)
\]
and \( \Gamma^k_{ij} := \frac{1}{2} g^{kl} \left( \nabla_i g_{jl} + \nabla_j g_{il} - \nabla_l g_{ij} \right) \). Here, \( \nabla \) denotes the Levi-Civita connection of \( h \).

Let \( \kappa \) be a continuous function on \( M \). We say that \( R_g \geq \kappa \) in the distributional sense if \( \langle R_g, u \rangle - \int_M \kappa u \, dvol_g \geq 0 \) for any nonnegative compactly supported test function \( u \in C^\infty(M) \cap C^\infty_0(M) \).

**Remark 4.4.** If a metric \( g \) is \( C^2 \), then the scalar curvature distribution \( \langle R_g, u \rangle \) coincides with \( \int_M R(g) u \, dvol_g \).
For more details about the distributional scalar curvature and related results, see [12, 15, 24, 25]. From Gromov’s $C^0$-limit theorem (Theorem 1.1), there has already been a definition of scalar curvature lower bounds for $C^0$ metrics (see [17, Definition 1.2] for example). Namely, a $C^0$ metric $g$ on a smooth manifold $M$ is of $R(g) \geq \kappa$ on $M$ in the Gromov’s sense if and only if there exists a sequence of $C^2$ metrics $(g_i)$ such that $g_i$ converge $C^0$-locally to $g$ and satisfy $R(g_i) \geq \kappa$ on $M$. Note that Burkhardt-Guim [5] pointed out that her definition (via the Ricci–DeTurck flow) and this Gromov’s definition are actually equivalent on a closed manifold. For example, on tori, there is no metric $g$ which is of $R(g) \geq \kappa > 0$ in the Gromov’s sense (or equivalently in the sense of [5, Definition 1.2]) from the resolution of Geroch’s conjecture [7, 20, 21]. In contrast, a metric $g$ that is of $R(g) \geq \kappa > 0$ in the sense of the definition 4.2 might exist on a torus. At least on a manifold whose Yamabe invariant is nonpositive, the question of how different these definitions are is related to Schoen’s conjecture (cf. [12, 16]). As a corollary of Main Theorem 2, we can obtain the following. This is the same as Corollary 1.2 in Section 1.

**Corollary 4.3.** Let $p > n^2/2$ and $\kappa$ a constant. Suppose that $M$ is a closed manifold of dimension $n \geq 2$, $g$ is a $C^2$ Riemannian metric on $M$ and $(g_i)$ is a sequence of $C^2$ metric on $M$. Assume the following:

1. a sequence $(\phi_i)$ of nonnegative smooth functions on $M$ satisfying: for any positive constant $a > 0$ there is a positive constant $\Lambda > 0$ such that $\log(\phi_i + a)$ is $\Lambda$-Lipschitz on $M$ for all $i$,
2. $(\phi_i)$ converges to some nonnegative continuous function $\phi$ in the uniformly $C^0$-sense on $M$,
3. $R(g_i) \geq 0$ on $M$ for each $i$,
4. $\int_M R(g_i) \phi_i \ dv_{g_i} \geq \kappa \int_M \phi_i \ dv_{g_i}$,
5. $g_i$ converges to $g$ in the $W^{1,p}$-sense.

Then

$$\int_M R(g) \phi \ dv_{g} \geq \kappa \int_M \phi \ dv_{g}.$$

**Proof.** From (3) and (4), for any (small) positive constant $a > 0$,

$$\int_M R(g_i)(\phi_i + a) \ dv_{g_i} \geq \kappa \int_M \phi_i \ dv_{g_i}$$

$$= \kappa \int_M \phi \ dv_{g} + \kappa \left( \int_M \phi_i \ dv_{g_i} - \int_M \phi \ dv_{g} \right).$$

Then, applying Main Theorem 2 to $f_i = - \log(\phi_i + a)$ and $f = - \log(\phi + a)$, we obtain that

$$\int_M R(g)(\phi + a) \ dv_{g} \geq \kappa \int_M \phi \ dv_{g} + \kappa \delta$$

for all $0 < \delta << 1$. Here, $a, \delta > 0$ can be taken arbitrarily small, and we obtain the desired inequality:

$$\int_M R(g)\phi \ dv_{g} \geq \kappa \int_M \phi \ dv_{g}.$$ 

$\square$
Question 4.1. Can we replace the condition (3) and (4) with “\( R(g_i) \geq \kappa \) in the distributional sense for some nonnegative constant \( \kappa \geq 0 \)?

If we make the regularity of convergence much stronger \( W^{1,p} (p > n^2/2) \) in [12, Theorem 3.2 (1)], then it can be proven that the above question is positively true.

5 Examples

In this section, we construct some examples of Riemannian metrics \( (g_i) \) on a smooth manifold \( M \) such that \( g_i \) converges to a metric \( g \) in some sense and

\[
\int_M R(g_i) \, d\text{vol}_{g_i} \geq \kappa \text{ for some constant } \kappa \in \mathbb{R}, \text{ but } \int_M R(g) \, d\text{vol}_g < \kappa.
\]

- In Example 5.1, \( M = \mathbb{R}^n \setminus \{0\} \) \((n \geq 3)\) and \( g_i \to g_{\text{Eucl}} \) in the locally uniformly smooth sense in \( \mathbb{R}^n \setminus \{0\} \), but not in the \( C^0 \)-sense. Note that the limiting metric \( g_{\text{Eucl}} \) in \( \mathbb{R}^n \setminus \{0\} \) is incomplete.
- In Example 5.2, \( M = \mathbb{R}^n \) \((n \geq 3)\) and \( g_i \to g_{\text{Eucl}} \) in the uniformly \( C^0 \)-sense, but not in the \( C^1 \)-sense.
- In Example 5.3, \((M,g)\) is an arbitrary closed Riemannian \( n \)-manifold with \( n \geq 3 \) and \( g_i \to g \) in the \( C^0 \)-sense, but not in the \( C^1 \)-sense.
- In Example 5.4, \( M = \mathbb{R}^n \) \((n \geq 2)\) and \( g_i \to g_{\text{Eucl}} \) in the uniformly \( C^1 \)-sense, but not in the \( C^2 \)-sense.

If \( M \) is a compact smooth manifold and \( C^2 \)-Riemannian metrics \( \{g_i\} \) converges to a \( C^2 \)-Riemannian metric \( g \) on \( M \) in the \( C^2 \) sense as \( i \to \infty \), then \( \max_M R(g_i) \to \max_M R(g) \). So, by Lebesgue’s dominated convergence theorem, we have

\[
\int_M R(g_i) \, d\text{vol}_{g_i} \to \int_M R(g) \, d\text{vol}_g \text{ as } i \to \infty.
\]

However, if \( M \) is non-compact, it is not known whether or not there is a Lebesgue integrable function \( f : M \to \mathbb{R} \) such that \( |R(g_i)| \leq f \) a.e. on \( M \) for all \( i \). Hence, in this situation, Lebesgue’s dominated convergence theorem cannot be applied in general. Indeed, the following example implies that Main Theorem 1 (without the assumption that each \( g_i \) has nonnegative scalar curvature) does not hold in general if \( M \) is non-compact.

Example 5.1 (\( C^\infty \) locally uniformly convergence and incomplete limiting metric)

The limiting metric of the first example constructed below is incomplete on \( \mathbb{R}^n \) \((n \geq 3)\).

Consider the smooth positive function \( u : \mathbb{R}^n \to \mathbb{R} \) defined as

\[
u = \phi \left( i^{-1} + e^{-ir^2} \right) + 1
\]

and \( (\mathbb{R}^n, g_i := u_i^{-1} \cdot g_{\text{Eucl}}) \) \((n \geq 3)\). Here \( \phi : \mathbb{R}^n \to [0,1] \) is a smooth cut-off function such that \( \phi \equiv 1 \) on the closed ball \( B_{r_0} := \{ x \in \mathbb{R}^n \mid r(x) \leq r_0 \} \) and \( \phi \equiv 0 \) outside of the \( \varepsilon /2 \)-neighbourhood \( (B_{r_0})_{\varepsilon /2} \) of \( B_{r_0} \) where \( r_0 > 0 \) is an arbitrarily fixed.
positive constant. Here, \(g_{\text{Eucl}}\) denotes the Euclidean metric on \(\mathbb{R}^n\), \(r : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) is the Euclidean distance function from the origin \(o \in \mathbb{R}^n\), and

\[
l := \frac{n + 2}{4} \quad \text{when} \quad \begin{cases} n = 2m + 1 \ (m \geq 1), \\ n = 2m \ (m \geq 2). \end{cases}
\]

Then for each \(i\), \((\mathbb{R}^n, g_i)\) is a non-compact smooth Riemannian manifold with

\[
R(g_i) = u_i^{-\frac{n+2}{n-2}} \left( -\frac{n-1}{n-2} \Delta_{g_{\text{Eucl}}} u_i + R(g_{\text{Eucl}})u_i \right) \\
= u_i^{-\frac{n+2}{n-2}} \left( -\frac{n-1}{n-2} \Delta_{g_{\text{Eucl}}} u_i \right). \tag{11}
\]

Moreover, \(g_i\) converges to \(g_{\text{Eucl}}\) in the locally \(C^\infty\)-sense in \(\mathbb{R}^n \setminus \{o\}\) but not in the \(C^0\)-sense on \(\mathbb{R}^n\). Note that \((\mathbb{R}^n \setminus \{o\}, g_{\text{Eucl}})\) is incomplete. On \(\overline{B_{r_0}}\),

\[
|\nabla u_i|^2 = \sum_{j=1}^n \left| \frac{\partial}{\partial x^j} i^{-1+i e^{-ir^2}} \right|^2 = \sum_{j=1}^n \left| \frac{\partial}{\partial x^j} \frac{\partial}{\partial r} i^{-1+i e^{-ir^2}} \right|^2 \\
= \sum_{j=1}^n \left| \frac{x^j}{r} (-2)^i r e^{-ir^2} \right|^2 \\
= 4i^2 r^2 e^{-2ir^2}.
\]

When \(i \to \infty\),

\[
R(g_i) \to \begin{cases} 0 & \text{on } \overline{B_{r_0}} \setminus \{o\}, \\ \infty & \text{at } o. \end{cases}
\]

Indeed, we can observe such a behavior from the form of the scalar curvature on \(\overline{B_{r_0}}\) as follows.

\[
R(g_i) = -\frac{4(n-1)}{n-2} u_i^{-\frac{n+2}{n-2}} (-2i^2 n + 4i^2 r^2) e^{-ir^2} \tag{12}
\]

From (11) and the divergence formula, we have

\[
\int_{\mathbb{R}^n} R(g_i) \, d\text{vol}_{g_i} = -\frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} u_i^{-\frac{n+2}{n-2}} \Delta_{g_{\text{Eucl}}} u_i \left( u_i \frac{2\pi}{n-2} \right) \, d\text{vol}_{g_{\text{Eucl}}} \\
= -\frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} u_i \Delta_{g_{\text{Eucl}}} u_i \, d\text{vol}_{g_{\text{Eucl}}} \\
= \frac{4(n-1)}{n-2} \int_{\overline{B_{r_0}}} |\nabla u_i|^2 \, d\text{vol}_{g_{\text{Eucl}}} \\
\geq \frac{16(n-1)}{n-2} \int_{\overline{B_{r_0}}} i^2 r^2 e^{-2ir^2} \, d\text{vol}_{g_{\text{Eucl}}} \\
= \frac{16(n-1)}{n-2} \text{Vol}(S^{n-1}) \int_{0}^{r_0} i^2 r^2 e^{-2ir^2} \, r^{n-1} \, dr, \tag{13}
\]

24
where $\text{Vol}(S^{n-1})$ denotes the volume of $(n-1)$-sphere with respect to the standard metric. Here,

\[
\int_0^{r_0} r^{n+1} e^{-2ir^2} dr = \left[ -\frac{1}{2i} r^n e^{-2ir^2} \right]_0^{r_0} + \frac{n}{2i} \int_0^{r_0} r^{n-1} e^{-2ir^2} dr
\]

\[
= -\frac{1}{2i} r_0^n e^{-2ir_0^2} + \frac{n}{2i} \int_0^{r_0} r^{n-1} e^{-2ir^2} dr
\]

Set the left hand side of this equation as $I_{n+1} := \int_0^{r_0} r^{n+1} e^{-2ir^2} dr$. Then we have

\[
\begin{align*}
I_{n+1} &= -\frac{e^{-2ir_0^2}}{2n} \sum_{k=0}^{m} \prod_{s=0}^{k} \left( \frac{n-2s}{2} \right) + \prod_{s=0}^{m} \left( \frac{2s+1}{n} \right) I_0 & \text{if } n = 2m + 1 \ (m \geq 1), \\
I_{n+1} &= -\frac{e^{-2ir_0^2}}{2n} \sum_{k=0}^{m} \prod_{s=0}^{k} \left( \frac{n-2s}{2} \right) + \prod_{s=1}^{m} \left( \frac{2s}{n} \right) I_1 & \text{if } n = 2m \ (m \geq 2).
\end{align*}
\]

Moreover,

\[
I_0 = \int_0^{r_0} e^{-2ir^2} dr \geq \left( \int_0^{r_0} \int_0^{\pi} e^{-2ir^2} r dr d\theta \right)^{1/2} = \sqrt{\frac{\pi}{2} \left( \frac{1}{2i} - e^{-2ir_0^2} \right)},
\]

and

\[
I_1 = \int_0^{r_0} r e^{-2ir^2} dr = \left[ -\frac{1}{2i} e^{-2ir^2} \right]_0^{r_0} = \frac{1}{2i} - \frac{1}{2i} e^{-2ir_0^2}.
\]

Combining these, as $i \to \infty$, we can see that the rightmost term of (13) converges to

\[
\begin{align*}
16 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \sqrt{\pi} \prod_{s=0}^{m} \left( \frac{2s+1}{2} \right) & > 0 & \text{if } n = 2m + 1 \ (m \geq 1), \\
8 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \prod_{s=1}^{m} s & > 0 & \text{if } n = 2m \ (m \geq 2).
\end{align*}
\]

Hence, for all sufficiently large $i$,

\[
\int_{\mathbb{R}^n} R(g_i) \, dvol_{g_i} \geq \begin{cases} 
8 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \sqrt{\pi} \prod_{s=0}^{m} \left( \frac{2s+1}{2} \right) & > 0 & \text{if } n = 2m + 1 \ (m \geq 1), \\
4 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \prod_{s=1}^{m} s & > 0 & \text{if } n = 2m \ (m \geq 2).
\end{cases}
\]

Note that $R(g_i)$ cannot be nonnegative on $\mathbb{R}^n$ by the positive mass theorem or the resolution of Geroch’s conjecture on tori [7, 20, 21]. Indeed, from (12),

\[
R(g_i)(o) = 8 \frac{n(n-1)}{n-2} i^l \left( i^{-1+l} + 1 \right)^{-\frac{n+2}{n-2}} > 0,
\]

25
and

\[ R(g_i)(x) = -i^{n-1} \left( i^{-1}e^{-\frac{t^2}{u^2}} + 1 \right) \left( -2n + ir_0^2 \right) e^{-\frac{t^2}{u^2}} < 0 \]

for any point \( x \in \{ x \in \mathbb{R}^n \mid r(x) = \frac{2 \pi}{n} \} \) and sufficiently large \( i \).

Next, we will construct a counterexample (to Main Theorem 1 without the assumption that each \( g_i \) has nonnegative scalar curvature), in which each \( g_i \) is complete.

**Example 5.2** (Not \( C^1 \) but \( C^0 \)). Consider \( (\mathbb{R}^n, g_i := u_i^{-\frac{4}{n-2}} \cdot g_{Eucl}) \) (\( n \geq 3, \ i = 2, 3, \cdots \)). Here the smooth positive function \( u_i : \mathbb{R}^n \to \mathbb{R} \) has been defined as

\[ u_i = \phi \left( i^{-1} \sin(i r^2) \right) + 1. \]

Here, \( \phi : \mathbb{R}^n \to [0, 1] \) is a smooth cut-off function such that \( \phi \equiv 1 \) on \( \overline{B_{r_0}} := \{ x \in \mathbb{R}^n \mid r(x) \leq r_0 \} \) and \( \phi \equiv 0 \) outside of the \( \varepsilon/2 \)-neighborhood \( \overline{B_{r_0}} \), where \( r_0 > 0 \) is an arbitrarily fixed positive constant. Here, \( r : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is the Euclidean distance function from the origin \( o \in \mathbb{R}^n \). Then, for each \( i \), \( (\mathbb{R}^n, g_i) \) is a non-compact smooth Riemannian manifold with

\[
R(g_i) = u_i^{\frac{n-2}{n+2}} \left( -\frac{n-1}{n-2} \Delta_{g_{Eucl}} u_i + R(g_{Eucl}) u_i \right) \\
= u_i^{\frac{n-2}{n+2}} \left( -\frac{n-1}{n-2} \Delta_{g_{Eucl}} u_i \right),
\]

and \( g_i \to g_{Eucl} \) on \( \mathbb{R}^n \) in the uniformly \( C^0 \) but not in the \( C^1 \)-sense. On \( \overline{B_{r_0}} \):

\[
|\nabla u_i|^2 = \sum_{j=1}^n \left| \frac{\partial}{\partial x^j} i^{-1} \sin(i r^2) \right|^2 = \sum_{j=1}^n \left| \frac{\partial r}{\partial x^j} \frac{\partial}{\partial r} i^{-1} \sin(i r^2) \right|^2 \\
= \sum_{j=1}^n \left| \frac{x^j}{r} (2r) \cos(i r^2) \right|^2 \\
= 4r^2 \cos^2(i r^2) \\
= 2r^2(1 + \cos(2i r^2)).
\]

Note that \( R(g_i) \) is not nonnegative on \( \mathbb{R}^n \). Indeed, for sufficiently large \( i \) and \( k \in \mathbb{Z} \) such that \( \overline{B_{r_0}} \cap \left\{ x \in \mathbb{R}^n \mid r(x) = \sqrt{\frac{(2k-1)}{2n}} \right\} \neq \emptyset \), we can take a point \( x_i \in \overline{B_{r_0}} \cap \left\{ x \in \mathbb{R}^n \mid r(x) = \sqrt{\frac{(2k-1)}{2n}} \right\} \). Then

\[
R(g_i)(x_i) \to \begin{cases} 
\frac{8(n-1)(2k-1)\pi}{n} & \text{if } k \text{ is odd,} \\
-\frac{8(n-1)(2k-1)\pi}{n-2} & \text{if } k \text{ is even.}
\end{cases}
\]

\[ 26 \]
This is checked as follows. For sufficiently large \( i \), such a point \( x_i \) is contained in \( B_{r_0} \). Hence, from the above formula and the choice of the point \( x_i \), we have

\[
R(g_i)(x_i) = u_i^{-\frac{n+2}{n-2}} \left( -4 \frac{n-1}{n-2} \Delta_{\text{Eucl}} u_i \right)
\]

\[
= -4 \frac{n-1}{n-2} \left( i^{-1} + 1 \right) \frac{n+2}{n-2} \left( 2n \cos(ir(x_i)^2) - 4ir^2 \sin(ir(x_i)^2) \right)
\]

\[
= (-1)^{k+1} \frac{8(n-1)(2k-1)\pi}{n-2} \left( i^{-1} + 1 \right) \frac{n+2}{n-2}.
\]

Since \( (i^{-1} + 1) \frac{n+2}{n-2} \to 1 (i \to \infty) \), we can observe the desired behavior of the scalar curvature as above. Moreover, from (14) and the divergence formula, we have

\[
\int_{\mathbb{R}^n} R(g_i) \, d\text{vol}_{g_i} = -4 \frac{n-1}{n-2} \int_{\mathbb{R}^n} u_i \frac{n+2}{n-2} \Delta_{\text{Eucl}} u_i \left( u_i^{-\frac{n+2}{n-2}} \, d\text{vol}_{\text{Eucl}} \right)
\]

\[
= -4 \frac{n-1}{n-2} \int_{\mathbb{R}^n} u_i \Delta_{\text{Eucl}} u_i \, d\text{vol}_{\text{Eucl}}
\]

\[
= 4 \frac{n-1}{n-2} \int_{B_{r_0}} |\nabla u_i|^2 \, d\text{vol}_{\text{Eucl}}
\]

\[
\geq 8 \frac{n-1}{n-2} \int_{B_{r_0}} r^2 (1 + \cos(2ir^2)) \, d\text{vol}_{\text{Eucl}}
\]

\[
= 8 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \int_0^{r_0} r^2 (1 + \cos(2ir^2)) r^{n-1} \, dr.
\]

Here,

\[
\int_0^{r_0} r^2 (1 + \cos(2ir^2)) r^{n-1} dr = \left[ \frac{1}{n+2} r^{n+2} \right]_0^{r_0} + \int_0^{r_0} r^{n+1} \cos(2ir^2) \, dr
\]

\[
= \frac{1}{n+2} r_0^{n+2} + \left[ \frac{r^{n+1}}{n+2} \sin(2ir^2) \right]_0^{r_0} - \frac{n(n-2)i^{-2}}{16} \int_0^{r_0} r^{n-3} \cos(2ir^2) \, dr
\]

\[
\geq \frac{1}{n+2} r_0^{n+2} + \left[ \frac{r^{n+1}}{n+2} \sin(2ir^2) \right]_0^{r_0} - \frac{n(n-2)i^{-1}}{16} \int_0^{r_0} r^{n-3} \, dr
\]

\[
= \frac{1}{n+2} r_0^{n+2} + \frac{n^{-1} i^{-1}}{4} \sin(2ir_0^2) + \frac{n^{-2} i^{-2}}{16} r_0^{-2} \cos(2ir_0^2) - \frac{n^{-2} i^{-2}}{16} r_0^{-2}.
\]

Since

\[
\frac{n^{-1} i^{-1}}{4} \sin(2ir_0^2) + \frac{n^{-2} i^{-2}}{16} r_0^{-2} \cos(2ir_0^2) = \frac{n^{-2} i^{-2}}{16} r_0^{-2} \to 0 \text{ as } i \to \infty,
\]

27
there is a sufficiently large $i_0 = i_0(n, r_0)$ such that for all $i \geq i_0$,

$$\frac{r_0^{n-1}}{4}\sin(2ir_0^2) + \frac{ni^{n-2}}{16}r_0^{n-2}\cos(2ir_0^2) - \frac{ni^{n-2}}{16}r_0^{n-2} > -\frac{1}{2(n+2)}r_0^{n+2}.$$ 

Hence, for all $i \geq i_0$,

$$\int_{\mathbb{R}^n} R(g_i) \, d\text{vol}_{g_i} > \frac{n-1}{(n+2)(n-2)}\text{Vol}(S^{n-1})r_0^{n+2} > 0.$$

From the Morrey embedding, we have

$$C^1 \hookrightarrow W^{1,p} \hookrightarrow C^{0,\frac{p}{n}} \hookrightarrow C^0 \quad \text{if } p > n.$$ 

Therefore the same statement of Main Theorem 1 still holds even though one replace $W^{1,p}$ ($p > n$) with $C^{0,\alpha}$ for all $\alpha \in (0, 1]$. On the other hand, in Main Theorem 1, if we weaken the assumption from $W^{1,p}$ to $C^0$, then the same statement (without the assumption $R(g_i) \geq 0$) does not hold in general. Indeed, using the same local construction as in the previous example in dimension $\geq 3$, we can also construct a counterexample on a closed manifold to Main Theorem 1 (without the assumption that each $g_i$ has nonnegative scalar curvature) as follows. Note that all metrics $g_i$ in each such example has sign-changing scalar curvature, i.e., for each $i$, there are some points $x_i, y_i \in M$ s.t. $R(g_i)(x_i) < 0 < R(g_i)(y_i)$.

**Example 5.3 (On every closed manifold).** Consider $\left(\mathbb{M}^n, g_i := u^{\frac{4}{n-2}} \cdot g_0\right) \quad (n \geq 3, \ i = 2, 3, \cdots)$, where $\mathbb{M}^n$ is a closed $n$-manifold and $g_0$ is a Riemannian metric on $M$. Here the smooth positive function $u_i : M \to \mathbb{R}$ has been defined as

$$u_i = \phi \left( i^{-1} \sin(ih^2) \right) + 1.$$ 

Here, $\phi : M \to [0, 1]$ is a smooth cut-off function such that $\phi \equiv 1$ on $B_{r_0}(p) := \{ x \in M \mid d_{g_0}(p, x) \leq r_0 \}$ and $\phi \equiv 0$ outside of the $\varepsilon/2$-neighbourhood $B_{r_0}$ for some point $p \in M$ where $0 < r_0 < \text{inj}(M, g_0)$ is a sufficiently small positive constant. Here, $h := d_{g_0}(\cdot, p) : M \to \mathbb{R}_{\geq 0}$ is the distance function of $g_0$ from the point $p$ and $\text{inj}(M, g_0)$ is the injectivity radius of $(M, g_0)$. Then, for each $i$, $(M, g_i)$ is a smooth Riemannian manifold with

$$R(g_i) = u_i^{-\frac{4}{n-2}} \left( -\frac{n-1}{n-2} \Delta_{g_0} u_i + R(g_0)u_i \right).$$
and $g_i$ converges to $g_0$ on $M$ in the $C^0$ but not in the $C^1$-sense. In the same calculation as in the previous example, we have

$$\int_M R(g_i) \, dvol_{g_i}$$

$$= -\frac{n-1}{n-2} \int_M u_i^{\frac{n+2}{n-2}} (\Delta g_0 u_i + R_{g_0} u_i) \, dvol_{g_0}$$

$$= -\frac{n-1}{n-2} \int_M (u_i \Delta g_0 u_i + R_{g_0} u_i^2) \, dvol_{g_0}$$

$$= \frac{n-1}{n-2} \left[ \int_{B_{r_0}} |\nabla u_i|^2_{g_0} \, dvol_{g_0} + \int_M R(g_0) u_i^2 \, dvol_{g_0} \right]$$

$$\geq \frac{8}{n-2} \int_{B_{r_0}} h^2 (1 + \cos(2i h^2)) \, dvol_{g_0} + \int_M R(g_0) u_i^2 \, dvol_{g_0}$$

$$\geq \frac{8}{n-2} \tilde{C} \int_{r_0}^r h^2 (1 + \cos(2i h^2)) h^{n-1} \, dh + \int_M R(g_0) u_i^2 \, dvol_{g_0}. $$

Here, the constant $\tilde{C}$ depends only on $n$ and $g_0$. Thus, from the observation as in the previous example, there is $i_0 \in \mathbb{N}$ and a positive constant $C = C(n, g_0, r_0) > 0$ such that for all $i \geq i_0$,

$$8 \frac{n-1}{n-2} \int_{r_0}^r h^2 (1 + \cos(2i h^2)) h^{n-1} \, dh \geq C > 0.$$

Therefore, for all $i \geq i_0$,

$$\int_M R(g_i) \, dvol_{g_i} \geq C + \int_M R(g_0) u_i^2 \, dvol_{g_0}. $$

Moreover, by the definition of $u_i$,

$$\left| \int_M R(g_0) u_i^2 \, dvol_{g_0} - \int_M R(g_0) \, dvol_{g_0} \right| \leq \int_M |R(g_0)| (2i^{-1} + i^{-2}) \, dvol_{g_0}. $$

Hence, there is a sufficiently large $i_1$ such that for all $i \geq i_1$,

$$\left| \int_M R(g_0) u_i^2 \, dvol_{g_0} - \int_M R(g_0) \, dvol_{g_0} \right| \leq \frac{C}{2}. $$

Thus, for all $i \geq \max\{i_0, i_1\}$, we have

$$\int_M R(g_i) \, dvol_{g_i} \geq C + \int_M R(g_0) u_i^2 \, dvol_{g_0} > \int_M R(g_0) \, dvol_{g_0}. $$

Here, we have a question about the regularity of convergence in the assumption of Main Theorem 2.
Question 5.1. Are there any $C^2$-metrics $(g_i)$ and $\Lambda$-Lipschitz ($\Lambda > 0$) functions $(f_i)$ on a closed $n$-manifold $M^n$ ($n \geq 3$) satisfying the followings?

- $g_i$ converges to a $C^2$-metric $g$ in the $W^{1,\frac{2}{n}}$-sense,
- $f_i$ converges to a $\Lambda$-Lipschitz function $f$ in the uniformly $C^0$-sense,
- there is a constant $\kappa$ such that $\int_M R(g_i) e^{-f_i} \text{dvol}_{g_i} \geq \kappa > \int_M R(g) e^{-f} \text{dvol}_g$.

Or, additionally,

- there is a point $p_i \in M$ for each $i$ such that $R(g_i)(p_i) \to -\infty$ as $i \to \infty$.

In the following Example 5.4, we give another counterexample which is similar to the one in Example 5.2 ($n \geq 3$). However, in the following example of dimension $\geq 3$, the support of $u_i - 1$ is contained in a fixed compact subset. Hence, unfortunately, it is not possible to localize this construction directly and construct such a counterexample on a closed manifold as in Example 5.3. On the other hand, in the two-dimensional example of Example 5.4, the support of $e^{n_i} - 1$ is not compact for each $i$ (see the last half of Example 5.4).

Example 5.4 (Not $C^2$ but $C^1$). We will construct an example similar to the one in Example 5.2. However, in this example, the topology of the convergence of the metrics is different.

Consider $\left(\mathbb{R}^n, g_i := u_i^{\frac{4}{n-2}} \cdot g_{\text{Eucl}}\right)$ ($n \geq 3$, $i = 2, 3, \cdots$). Here the smooth positive function $u_i : \mathbb{R}^n \to \mathbb{R}$ has been defined as

$$u_i = \phi_i \left( i^{-2} \sin(i r_i^2) \right) + 1.$$  

Here, $\phi_i : \mathbb{R}^n \to [0, 1]$ is a smooth cut-off function such that $\phi_i \equiv 1$ on $B_{r_i} := \{ x \in \mathbb{R}^n \mid r(x) \leq r_i \}$ and $\phi_i \equiv 0$ outside of the $\varepsilon/2$-neighborhood $B_{r_i}$, where $r_i := i \frac{\varepsilon}{\sqrt{n}}$. Note that $r_i \to \infty$ as $i \to \infty$. Here, $r : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the Euclidean distance function from the origin $o$. Then, for each $i$, $(\mathbb{R}^n, g_i)$ is a non-compact smooth Riemannian manifold with

$$R(g_i) = u_i^{-\frac{n+2}{n-2}} \left( -\frac{n-1}{n-2} \Delta_{g_{\text{Eucl}}} u_i + R(g_{\text{Eucl}}) u_i \right)$$

$$= u_i^{-\frac{n+2}{n-2}} \left( -\frac{n-1}{n-2} \Delta_{g_{\text{Eucl}}} u_i \right).$$
Moreover, \( g_i \) converges to \( g_{\text{Eucl}} \) on \( \mathbb{R}^n \) in the uniformly \( C^1 \) but not in the \( C^2 \)-sense.

On \( B_{r_i} \),

\[
|\nabla u_i|^2 = \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} i^{-2} \sin(ir^2) \right|^2 = \sum_{j=1}^{n} \left| \frac{\partial r}{\partial x_j} \frac{\partial}{\partial r} i^{-2} \sin(ir^2) \right|^2
= \sum_{j=1}^{n} \frac{x_j^2}{r^2} (2i^{-2}r) \cos(ir^2)
= 4i^{-2}r^2 \cos^2(ir^2)
= 2i^{-2}r^2(1 + \cos(2ir^2)).
\]

Note that \( R(g_i) \) is not nonnegative on \( \mathbb{R}^n \). Indeed, when \( i \to \infty \), \( R(x) \) oscillates for each \( x \in \{ x \in \mathbb{R}^n | r(x) \neq 0 \} \). Indeed, for sufficiently large \( i \), such a point \( x \) is contained in \( B_{r_i} \). Hence, from the above formula,

\[
R(g_i)(x) = u_i \rightarrow \frac{n+2}{n-2} \left( -\frac{n-1}{n-2} \Delta_{g_{\text{Eucl}}} u_i \right)
= -4 \left( \frac{n-1}{n-2} \right) \frac{2ni^{-1} \cos(ir(x)^2) - 4r(x)^2 \sin(ir(x)^2)}{(i^{-2} \sin(ir(x)^2) + 1)^{\frac{n-2}{2}}},
\]

Since \( (i^{-1} + 1)^{\frac{n+2}{n-2}} \rightarrow 1 \) and \( 2ni^{-1} \cos(ir(x)^2) \rightarrow 0 \) as \( i \to \infty \), we can easily observe the desired behavior of the scalar curvature. Moreover, by the divergence formula,

\[
\int_{\mathbb{R}^n} R(g_i) \, d\text{vol}_{g_i} \geq -\frac{4n-1}{n-2} \int_{\mathbb{R}^n} u_i \frac{\Delta_{g_{\text{Eucl}}} u_i}{u_i} \, d\text{vol}_{g_{\text{Eucl}}}
= -\frac{4n-1}{n-2} \int_{\mathbb{R}^n} u_i \Delta_{g_{\text{Eucl}}} u_i \, d\text{vol}_{g_{\text{Eucl}}}
= \frac{4n-1}{n-2} \int_{\mathcal{B}_{r_i}} |\nabla u_i|^2 \, d\text{vol}_{g_{\text{Eucl}}}
\geq 8 \frac{n-1}{n-2} \int_{\mathcal{B}_{r_i}} r^2 i^{-2} (1 + \cos(2ir^2)) \, d\text{vol}_{g_{\text{Eucl}}}
= 8 \frac{n-1}{n-2} \text{Vol}(S^{n-1}) \int_{0}^{r_i} r^2 i^{-2} (1 + \cos(2ir^2)) r^{n-1} \, dr.
\]
Here,

\[
\int_0^{r_i} i^{-2} r^2 (1 + \cos(2ir^2)) r^{n-1} dr = \left[ \frac{i^{-2} r^{n+2}}{n + 2} \right]_0^{r_i} + \int_0^{r_i} i^{-2} r^{n+1} \cos(2ir^2) dr
\]

\[= \frac{1}{n + 2} + \left[ \frac{r^{n-3}}{4} \sin(2ir^2) \right]_0^{r_i} + \frac{n_i^{-4}}{16} \left[ r^{n-2} \cos(2ir^2) \right]_0^{r_i}
\]

\[- \frac{n(n-2)i^{-4}}{16} \int_0^{r_i} r^{n-3} \cos(2ir^2) dr
\]

\[\geq \frac{1}{n + 2} + \frac{r^{n-3}}{4} \sin(2ir^2) + \frac{n_i^{-4}}{16} \left[ r^{n-2} \cos(2ir^2) \right]_0^{r_i}
\]

\[- \frac{n(n-2)i^{-4}}{16} \int_0^{r_i} r^{n-3} dr
\]

\[\geq \frac{1}{n + 2} \frac{\pi^{\frac{n}{2} - 1}}{4} \sin(2i^{\frac{1}{n+2}}) + \frac{n_i^{-4}}{16} \left( \frac{2i^{\frac{n-2}{n+2}}}{4} \cos(2i^{\frac{1}{n+2}}) - \frac{n_i^{-4}}{16} \left( \frac{2i^{\frac{n-2}{n+2}}}{4} \right) \right) \to 0 \text{ as } i \to \infty,
\]

there is a sufficiently large \(i_0 = i_0(n)\) such that for all \(i \geq i_0\),

\[\frac{\pi^{\frac{n}{2} - 1}}{4} \sin(2i^{\frac{1}{n+2}}) + \frac{n_i^{-4}}{16} \left( \frac{2i^{\frac{n-2}{n+2}}}{4} \cos(2i^{\frac{1}{n+2}}) - \frac{n_i^{-4}}{16} \left( \frac{2i^{\frac{n-2}{n+2}}}{4} \right) \right) > -\frac{1}{2(n+2)}.
\]

Hence for all \(i \geq i_0\),

\[\int_{\mathbb{R}^2} R(g_i) dvol > \frac{n - 1}{(n + 2)(n - 2)} \text{Vol}(S^{n-1}) > 0.
\]

Next, we will construct a two-dimensional example. Consider the smooth function \(u_i\) on \(\mathbb{R}^2\) defined by

\[u_i := e^{-ir^2} \sin \left( \frac{i}{2} r^2 \right) \quad (i = 1, 2, \ldots),
\]

where \(r(.) := |o - \cdot|\) denotes the Euclidean distance function from the origin \(o \in \mathbb{R}^2\). Then \(u_i\) uniformly converges to the constant function 0 in the \(C^1\) topology in \(\mathbb{R}^2\), but \(u_i\) does not converge to 0 in the \(C^2\) topology in \(\mathbb{R}^2\). Hence the sequence of complete metrics \((g_i := e^{u_i}g_{Euc})\) on \(\mathbb{R}^2\) uniformly converges to \(g_{Euc}\) in the \(C^1\) sense on \(\mathbb{R}^2\),
but \( g_i \) does not converge to \( g_{Eucl} \) in the \( C^2 \) sense on \( \mathbb{R}^2 \). Set \( a := -i, b := \frac{1}{2}a = -\frac{i}{2} \). Then we can check that

\[
\Delta u_i = (4a + 4a^2 r^2)e^{ar^2} \sin(br^2) + (4b \cos r^2 - 4b^2 r^2 \sin br^2)e^{ar^2} + 8abr^2 e^{ar^2} \cos br^2,
\]

and

\[
\int_{\mathbb{R}^2} R(g_i) \, \text{dvol}_{g_i} = -\int_{\mathbb{R}^2} \Delta_{g_{Eucl}} u_i \, \text{dvol}_{g_{Eucl}} = -\int_0^{2\pi} \int_0^\infty r \Delta_{g_{Eucl}} u_i \, dr \, d\theta.
\]

Moreover,

- \( I := \int_0^\infty r e^{ar^2} \cos br^2 \, dr = -\frac{b}{2(a^2 + b^2)} \),
- \( J := \int_0^\infty r e^{ar^2} \sin br^2 \, dr = -\frac{aJ - bI}{a^2 + b^2} \),
- \( K := \int_0^\infty r^3 e^{ar^2} \cos br^2 \, dr = \frac{aJ - bI}{2(a^2 + b^2)} \),
- \( L := \int_0^\infty r^3 e^{ar^2} \sin br^2 \, dr = -\frac{aJ - bI}{a^2 + b^2} \).

Combining these, we obtain that

\[
\int_{\mathbb{R}^2} R(g_i) \, \text{dvol}_{g_i} = -2\pi \left( 4bI + 4aJ - 4(a^2 - b^2) \frac{aJ - bI}{a^2 + b^2} - 8ab \frac{bJ - aI}{a^2 + b^2} \right)
\]

\[
= 2\pi \frac{8a^3 b}{(a^2 + b^2)^2}
\]

\[
= \frac{128\pi}{25} > 0 = \int_{\mathbb{R}^2} R(g_{Eucl}) \, \text{dvol}_{g_{Eucl}}.
\]

Note that we have used \( b = \frac{1}{2}a \) in the third equality.

**Question 5.2.** As we have seen in the above examples, in Main Theorem 1 (without the assumption that the scalar curvature of each \( g_i \) is nonnegative), we cannot weaken the assumptions that the manifold is closed and the convergence is in the sense of \( W^{1,p} \) (\( p > n \)) to that the manifold is open and the convergence is in the sense of \( C^0 \) respectively. Then, can we weaken the assumptions in Main Theorems in any sense?

**Remark 5.1.** In the above examples, we have constructed these counterexamples by deforming the Euclidean metric locally in a conformal direction. Then, due to the factors from changes of the volume forms, the total scalar curvatures are uniformly bounded from below by a positive constant. On the other hand, if we try to investigate similar examples for the weighted total scalar curvature \( \int_M R(g) e^{-f} \text{dvol}_g \) (i.e., counterexamples to Main Theorem 2), we cannot use the same method in the above examples since there is no contribution from the factors associated with the conformal changes of the volume forms in this situation.

**Acknowledgements** The author thanks Prof. Boris Botvinnik for suggesting the problem related to Main Theorem 2. The author also thanks Prof. Kazuo Akutagawa.
for giving him the opportunity to visit the University of Oregon from September 25 to October 10, 2022.

**Author Contributions** SH has written the manuscript.

**Funding** The author was supported by JSPS KAKENHI Grant Number 24KJ0153.

**Data availability** Not applicable.

**Declarations**

**Conflict of interest** The author declares that there is no conflict of interest.

**Ethics approval and consent to participate** Not applicable.

**Consent for publication** The author declares the consent for publication.

**References**

[1] M. T. Anderson, Canonical metrics on 3-manifolds and 4-manifolds, Asian J. Math. **10**, 127–163 (2006).

[2] R. H. Bamler, A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature, Math. Res. Letters **23**, 325–337 (2016).

[3] G. Besson, G. Courtois and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, GAFA **5**, 731–799 (1995).

[4] G. Besson, J. Lohkamp, P. Pansu and P. Petersen, Riemannian geometry, Fields institute monographs **4**, American Mathematical Society (1996).

[5] P. Burkhardt-Guim, Pointwise lower scalar curvature bounds for C⁰ metrics via regularizing Ricci flow, Geom. Funct. Anal. **29**, 1703–1772 (2019).

[6] B.-L. Chen, Strong uniqueness of the Ricci flow, J. Differ Geom. **82**, 363–382 (2009).

[7] M. Gromov and H. Blaine Lawson Jr., Spin and scalar curvature in the presence of a fundamental group. I, Ann. of Math. **111**, 209–230 (1980).

[8] M. Gromov, Dirac and Plateau billiards in domains with corners, Cent. Eur. J. Math. **12**, 1109–1156 (2014).

[9] S. Hamanaka, Upper bound preservation of the total scalar curvature in a conformal class, arXiv preprint arXiv:2301.05444v6 (2023).

[10] Y. Huang and M.-C. Lee, Scalar curvature lower bound under integral convergence, Math. Z. **303**(1), 2 (2023).

[11] S. Huang and L.-F. Tam, Short-Time Existence for Harmonic Map Heat Flow with Time-Dependent Metrics, J. Geom. Anal. **32**, 1–32 (2022).
[12] W. Jiang, W. Sheng and H. Zhang, Weak scalar curvature lower bounds along Ricci flow, Science China Mathematics 66(6), 1141–1160 (2023).

[13] H. Koch and T. Lamm, Parabolic equations with rough data, Math. Bohem. 140 (2015), 457–477.

[14] N. V. Krylov, Lectures on elliptic and parabolic equations in Sobolev spaces, Graduate Studies in Mathematics 96, American Mathematical Society (2008).

[15] D. A. Lee and P. G. LeFloch, The positive mass theorem for manifolds with distributional curvature, Comm. Math. Phys. 339, 99–120 (2015).

[16] M.-C. Lee and L.-F. Tam, Continuous metrics and a conjecture of Schoen, Trans. Am. Math. Soc. 378, 1531–1550 (2025).

[17] M.-C. Lee and L.-F. Tam, Rigidity of Lipschitz map using harmonic map heat flow, arXiv preprint arXiv:2207.11017 (2022), to appear in Amer. J. Math.

[18] M.-C. Lee and P. M. Topping, Metric limits of manifolds with positive scalar curvature, arXiv preprint arXiv:2203.01223v3 (2022).

[19] J. Lohkamp, Curvature h-principles, Ann. of Math. 142, 457–498 (1995).

[20] R. Schoen and S.-T. Yau, Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature, Ann. of Math. 110, 127–142 (1979).

[21] R. Schoen and S.-T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28, 159–183 (1979).

[22] W.-X. Shi, Deforming the metric on complete Riemannian manifolds, J. Differential Geom. 30, 223–301 (1989).

[23] M. Simon, Deformation of $C^0$ Riemannian metrics in the direction of their Ricci curvature, Commun. Anal. Geom. 10, 1033–1074 (2002).

[24] C. Sormani, W. Tian and C. Wang, An extreme limit with nonnegative scalar curvature, Nonlinear Anal. 239, 113427 (2024).

[25] W. Tian and C. Wang, Compactness of sequences of warped product circles over spheres with nonnegative scalar curvature, Math. Ann. 390, 1–57 (2024).

[26] Z. H. Zhang, On the completeness of gradient Ricci solitons, Proc. Am. Math. Soc. 137, 2755–2759 (2009).
