Spatially heterogeneous dynamics in granular compaction

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We prove the emergence of spatially correlated dynamics in slowly compacting dense granular media by analyzing analytically and numerically multi-point correlation functions in a simple particle model characterized by slow non-equilibrium dynamics. We show that the logarithmically slow dynamics at large times is accompanied by spatially extended dynamic structures that resemble the ones observed in glass-forming liquids and dense colloidal suspensions. This suggests that dynamic heterogeneity is another key common feature present in very different jamming materials.

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When a pile of grains is gently shaken, its volume fraction increases so slowly that the process hardly becomes stationary on experimental timescales [1, 2]. This is reminiscent of the slow relaxation observed in glass-formers, as noted long ago [3]. From a more fundamental point of view, it is tempting to build upon analogies and suggest that granular media, glasses, and other jamming systems can be described by common theoretical approaches [4]. In recent years, several aspects of glasses and granular media have been studied with similar approaches. Static structures have been studied to understand the relevance of the network of force chains between grains or atoms to dynamics of the jammed state [5]. Also, since both grains and glasses undergo non-equilibrium ‘glassy’ dynamics, the idea that an effective thermodynamics can be used has received considerable attention [6, 7].

In this work we also transfer knowledge from one field to the other and show that the glassy dynamics of granular media is characterized by the appearance of spatio-temporal structures similar to the ones described as dynamic heterogeneity in glass-formers [8] and dense colloidal suspensions [9]. Dynamic heterogeneity is believed to play a crucial role in the glass formation, and forms the core of recent theoretical descriptions [10]. Physically, it stems from the existence of spatial correlations in the local dynamics that extend beyond the ones revealed by static pair correlations. To study dynamic heterogeneity, correlators that probe more than two points in space and time have to be considered [8–11]. These spatial fluctuations have never been studied in models or experiments on granular compaction, although caged particle dynamics was recently studied in a sheared system [12]. Here we prove the emergence of large dynamic lengthscales in a particle model which is a variant of the parking lot model, introduced and studied in detail in the context of granular compaction [13–15, 16, 17]. We take advantage of its relative simplicity to compute analytically multi-point correlations studied in glass-formers and confirm our results by numerical simulations.

We consider a variant of the parking lot model [13] introduced in Ref. [14]. The model is a one-dimensional process in which hard blocks of unit size are first irreversibly deposited at random positions on a line of linear size $L$ until no place is available for more depositions. Timescales are counted from the end of the deposition process which corresponds to $t_w = 0$. In a second step, particles are allowed to diffuse with a coefficient of diffusion $D$. The last dynamical rule defining the model consists of deposition events. When particles have diffused in such a way that a void of unit size opens, the void is instantaneously filled by a new particle and the density, $\rho(t_w) = N(t_w)/L$, increases by $1/L$, $N(t_w)$ being the total number of particles present at time $t_w$. These rules lead to slow dynamics, because the larger the density the longer it takes to open a void. A more general version of this model includes evaporation [13, 15]. It displays the generic features observed in granular compaction: logarithmic increase of the density [13, 14, 15], aging [15, 17], non-Gaussian density fluctuations [16], effective temperatures and link with Edwards entropy [15], hysteresis effects [15]. The time evolution of the density also describes well the experiments,

$$\rho(t_w) = \rho_\infty - \left[ a_0 + a_1 \ln \left( \frac{t_w - t_0}{\tau} \right) \right]^{-1}, \quad (1)$$

where $\rho_\infty$, $a_0$, $a_1$, $\tau$ and $t_0$ are fitting parameters. For the present model without evaporation Eq. (1) holds at large $t_w$ with $\rho_\infty = a_1 = \tau = 1$, $a_0 = t_0 = 0$. Let us remark here that such logarithmic compaction has also been obtained from different models with deposition, evaporation and diffusion, like those studied in Ref. [15]. Here we focus on local dynamic quantities and their spatial correlations through both analytic calculations and Monte Carlo simulations. The results of the simulations have been obtained by averaging over $2 \cdot 10^4$ independent histories with $L = 250$. Time is measured in units of the diffusion constant, $D$, which is set to unity.

In Fig. 1 we illustrate the results reported in this paper, namely the emergence of spatio-temporal correlations in dynamic trajectories of compacting granular systems. In this figure, black (white) denote particles that are more (less) mobile than the average on a timescale of the order...
of the structural relaxation time. The time extension of the trajectory is $10^5$, the spatial extension $L = 500$, and the final density $\rho(10^5) \approx 0.91$. The emergence of large-scale correlations in the local dynamics is evident, since domains that extend spatially over a hundred particles can be observed, despite the absence of any such large static correlations between particles. Spatial clustering of fast and slow regions is the hallmark of dynamic heterogeneity \cite{8, 9, 10, 11}. A closer look at the left part of the figure shows that the dynamic lengthscale is visibly smaller at shorter times, suggesting that spatial correlations between particles. Spatial clustering is reduced, for clarity. By definition, the structure factor is built from two-time two-point quantities which is the minimum requirement to probe the spatio-temporal patterns of Fig. 1. The small $q$ limit in Eq. (4) defines a dynamic “four-point” susceptibility,\

$$\chi_k(t, t_w) = S_k(q = 0, t, t_w),$$

which can be rewritten as the variance of the fluctuations of the spatially averaged two-time dynamics. Physically, dynamic fluctuations increase when the number of independently relaxing objects decreases \cite{11}. Normalizations ensure that $\chi_k$ is finite in the thermodynamic limit, except at a dynamic critical point \cite{12}.

The quantities introduced above can be estimated analytically as follows. Between two deposition events, the total free volume is $L(1 - \rho)$ and thus the slow dynamics can be shown \cite{20} to be equivalent to a continuum version of a symmetric exclusion process at density $\tilde{\rho} = \rho/(1 - \rho)$, i.e. involving caged dynamics. This process is then mapped to a fluctuating interface model \cite{21, 22}. Labeling particles in the order of their position from the origin we introduce an interface position $y(x, t)$ for the $i$th particle encountered on the lattice, such that $y(x, t) = r_i(t)$, $x$ replacing $i$ as a label for the particle. We then write $y(x, t) = h(x, t) + x/\rho$, were $h(x, t)$ is the deviation from the position of the particle from an uniform configuration. The connection to symmetric exclusion process allows one to show \cite{21, 22} that $h(x, t)$ follows an Edwards-Wilkinson dynamics \cite{23},

$$\frac{\partial h}{\partial t} = \alpha \nabla^2 h + \eta,$$

where $\alpha = \tilde{\rho}^2 \propto \ln(t_w)$. The equilibrium measure reads

$$P_{eq}[h(x)] = \exp \left(-\frac{1}{2\alpha} \int_{-\infty}^{\infty} dx \left| \nabla h(x) \right|^2 \right),$$
which shows that deviations from the averaged position of a particle arise with an elastic penalty in the interface representation. This elastic behavior makes the following calculation similar to the evaluation of the elastic contribution to the dynamical susceptibility [14] in supercooled liquids performed in Ref. [24].

Equation (6) is first solved in the Fourier space,

$$\hat{h}(q,t) = e^{-aq^2\tau} \hat{h}(q,t_w) + \int_{t_w}^{t} dt' \eta(q,t') e^{-\alpha(t-t')} , \quad (8)$$

where we have defined the time difference $\tau = t - t_w$. A crucial approximation is made here since $\alpha$ in Eq. (8) has in fact a logarithmic time dependence. This amounts to neglecting the effect of further deposition events on the particles already present at $t_w$. Since deposition is such a rare event at large times this approximation should capture the evolution of the local dynamics, as our numerical simulations shall confirm. Using (8), one easily gets

$$F_s(k,t,t_w) = \exp \left( -k^2 \frac{2\tau}{\alpha(t_w)} g(0) \right) , \quad (9)$$

where $g(x) = \int_{-\infty}^{\infty} dq e^{iqx} (1 - e^{-q^2/2})/q^2$, so that $g(0) = (2\pi)^{1/2}$. Equation (9) is a classic result for the ordering process of one-dimensional random walkers [25, 26]. It shows that $F_s(k,t,t_w)$ displays aging behavior and scales with $\tau/\tau_g(t_w)$, with a logarithmically increasing relaxation timescale $\tau_g(t_w) \sim \alpha(t_w)/k^4$. Particles are therefore sub-diffusing, $\langle \delta r^2(t,t_w) \rangle \sim \sqrt{\tau/\alpha(t_w)}$, and the van Hove function [2] is a Gaussian. Gaussianity is a consequence of the diffusive nature of the particle motion at short times, which explains the discrepancy with the non-Gaussian distributions of displacements that have recently been measured experimentally in a bidimensional geometry [12].

The dynamic susceptibility [5] can also be computed,

$$\chi_k(t,t_w) = \int dx \frac{\cos \left( \frac{2k^2}{\rho(t_w)} \sqrt{2x} g \left( \frac{x}{\rho(t_w) \sqrt{2x}} \right) \right) - 1}{\cos \left( \frac{2k^2}{\rho(t_w)} \sqrt{2x} g(0) \right) - 1}$$

$$= \frac{\alpha(t_w)}{k^2} \mathcal{F} \left( \frac{\tau}{\tau_g(t_w)} \right) , \quad (10)$$

where $\mathcal{F}(x)$ is a scaling function defined from the first line of Eq. (10). Careful analysis of Eq. (10) shows that $\chi_k(t,t_w)$ goes from zero at $\tau = 0$ to the asymptotic value $\chi_k(t \to \infty, t_w) = \alpha(t_w)/(2k^2 g(0))$, via a maximum $\chi_k^\star \sim \alpha(t_w)/k^2$ when $\tau \sim \tau_g(t_w)$.

In Fig. 2 we show the dynamic susceptibility as a function of time separation $\tau$ for various $t_w$ at time separations corresponding to the maximum of the dynamic susceptibility, obtained in numerical simulations. We also show the analytical results. As for the dynamic susceptibility the agreement between analytical and numerical results is excellent.

At fixed $t_w$ the structure factor is characterized by a plateau at small $q$ whose height is given by $\chi_k^\star(t_w)$. When $q$ increases $S_k(q,t,t_w)$ leaves the plateau and decreases to 0 at large $q$. When $t_w$ is increased the plateau becomes higher and it ends at a smaller wavevector but its overall shape is unchanged. This implies a dynamic scale invari-
ance such as found in glass-formers [10, 19]: rescaling times by $\tau_k(t_w)$ and space by $\xi_k(t_w) \sim \alpha_\tau(t_w)/k^2$ makes trajectories statistically equivalent. Formally this means that the following scaling law is obeyed,

$$S_k(q, \tau_k(t_w), t_w) = \chi_k^*(t_w) \chi_k(q\xi_k(t_w)).$$

(11)

Again we note that data in Fig. 3 strikingly resemble dynamic structure factors measured both in realistic supercooled liquids [3] and coarse-grained models for the glass transition [14].

A major result of the above analysis is the existence of a dynamic length scale, $\xi_k(t_w)$, which grows logarithmically with time when compaction proceeds, and therefore diverges when the systems jams. A diverging lengthscale provides support to the temporal renormalization group argument developed in Ref. [14], but we see no obvious connection between the dynamic criticality described here and the various power law scalings observed in static systems approaching jamming from above [27]. Physically, collective rearrangements of particles are needed to create a void of unit size, the denser the system the more cooperative the dynamics. A naive determination of $\xi$ would rely on a free volume argument [28]. The mean free volume available to particle is $1/\rho$, so that the number of particles required to have a fluctuation of the free volume equal to unity is $N' \sim \rho^2 / \xi(t_w)$. This simple physical argument underlies the cooperative nature of the dynamics which is more formally captured by four-point correlation functions, Eqs. (4) and (5).

The idea that the size of collective motions increases when dynamics becomes slow is certainly not new. Multipoint correlation functions have now been measured in very different materials with similar qualitative results. That large dynamic length scales control glassy dynamics suggests the possibility that few universality classes underly and possibly unify the dynamical behaviour of a much wider diversity of jamming materials.

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