Understanding the tsunami with a simple model

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Received 20 March 2006, in final form 12 April 2006
Published 16 May 2006
Online at stacks.iop.org/EJP/27/855

Abstract
In this paper, we use the approximation of shallow water waves (Margaritondo G 2005 Eur. J. Phys. 26 401) to understand the behaviour of a tsunami in a variable depth. We deduce the shallow water wave equation and the continuity equation that must be satisfied when a wave encounters a discontinuity in the sea depth. A short explanation about how the tsunami hit the west coast of India is given based on the refraction phenomenon. Our procedure also includes a simple numerical calculation suitable for undergraduate students in physics and engineering.

1. Introduction

Tsunamis are water waves with long wavelengths that can be triggered by submarine earthquakes, landslides, volcanic eruption and large asteroid impacts. These non-dispersive waves can travel for thousands of kilometres from the disturbance area where they have been created with a minimum loss of energy. As any wave, tsunamis can be reflected, transmitted, refracted and diffracted.

The physics of a tsunami can be very complex, especially if we consider its creation and behaviour next to the beach, where it can break. However, since tsunamis are composed of waves with very large wavelengths, sometimes greater than 100 km, they can be considered as shallow waves, even in oceans with depths of a few kilometres.

The shallow water approximation considerably simplifies the problem and still allows us to understand a lot of the physics of a tsunami. Using such an approximation, Margaritondo \cite{1} deduced the dispersion relation of tsunami waves extending a model developed by Behroozi and Podolefsky \cite{2}. Since energy losses due to viscosity and friction at the bottom \cite{3} can be neglected in the case of shallow waves, Margaritondo, considering energy conservation, explained the increase of the wave height when a tsunami approaches the coast, where the depth of the sea and the wave velocity are both reduced.

In this paper we use one of the results of \cite{1} in order to deduce the wave equation and include the variation of the seabed. Thus, we are able to explain the increase of the wave amplitude when passing from deeper to shallow water. Also, we discuss the refraction of
tsunami waves. This phenomenon allowed the tsunami of 26 December 2004, created at the Bay of Bengal, to hit the west coast of India (a detailed description is given in [4]). Both these inclusions—the seabed topography and the wave refraction—were pointed out by Chu [5] as necessary to understand some other phenomena observed in tsunamis.

This paper is organized as follows. The wave equation and the water flux conservation are used in section 2 in order to explain how and how much a shallow wave increases when passing from deeper to shallow water. In section 3, we extend the results obtained in section 2 to study how a wavepacket propagates in a water tank where the depth varies; in this section we use some numerical procedures that can be extended to the study of any wave propagating in a non-homogeneous medium. Also, the refraction of the 2004 tsunami in the south of India is discussed in section 3. The shallow wave and the continuity equations are deduced in appendix A.

2. Reflection and transmission of waves in one dimension

Consider a perturbation on the water surface in a rectangular tank with a constant depth. In the limit of large wavelengths and a small amplitude compared with the depth of the tank, the wave equation can be simplified to (see appendix A)

\[
\frac{\partial^2 y}{\partial t^2} = gh \frac{\partial^2 y}{\partial x^2}, \tag{1}
\]

where \(y(x, t)\) is the vertical displacement of the water surface at a time \(t\) propagating in the \(x\) direction, \(g\) is the gravity acceleration and \(h\) is the water depth.

Equation (1) is the most common one-dimensional wave equation. It is a second-order linear partial differential equation and, since \(g\) and \(h\) are constants, any function \(y = f(x \pm vt)\) is a solution (\(v = \sqrt{gh}\) is the wave velocity). An interesting aspect of equation (1) is that a propagating pulse does not present a dispersion due to the same velocity of all wavelengths and, thus, preserves its shape. Light in vacuum (and, in a very good approximation, in air) is non-dispersive. Also, sound waves in air are nearly non-dispersive. (If dispersion was important in the propagation of the sound in air, a sound would be heard differently in different positions, i.e. music and conversation would be impossible.)

However, the velocity of a shallow water wave varies with the depth. Thus, shallow water waves are dispersive in a non-uniform sea depth.

In order to study the evolution of a tsunami in a rectangular box with variable depth, which will be detailed in the next section, we approximate the irregular depth by successive steps. So, in the next paragraphs we will explain the treatment used when a wave encounters a discontinuity.

Every time the tsunami encounters a step, part is transmitted and part is reflected. Then, consider a wave with an amplitude given by \(y = \cos(kx - \omega t)\), where \(k\) and \(\omega\) are, respectively, the wave number and the frequency incoming in a region where the depth of the water and also the wave velocity have a discontinuity as represented in figure 1. On the left side of the discontinuity, the perturbation is given by

\[
y_1(x, t) = \cos(kx - \omega t) + R \cos(kx + \omega t + \varphi_1), \tag{2}
\]

where \(R \cos(kx + \omega t + \varphi_1)\) corresponds to the reflected wave and \(\varphi_1\) is a phase to be determined by the boundary conditions. On the right side of the discontinuity, the wave amplitude is given by

\[
y_2(x, t) = T \cos(k'x - \omega t + \varphi_2), \tag{3}
\]
corresponding to the transmitted wave part. The wave numbers for $x < 0$ and $x > 0$ are, respectively,

$$k = \frac{\omega}{v}$$

(4)

and

$$k' = \frac{\omega}{v'}$$

(5)

where $v$ and $v'$ are the velocities of the wavepacket on the left and right sides of the discontinuity.

In order to determine $R$ and $T$, we must impose the boundary conditions at $x = 0$. For any instant, the wave should be continuous at $x = 0$: $\cos \omega t + R \cos(\omega t + \varphi_1) = T \cos(-\omega t + \varphi_2)$. The same should happen with the flux, $f(x, t)$, given by (see equation (A.13))

$$f(x, t) = h \frac{\partial z(x, t)}{\partial t}$$

(6)

where $z(x, t)$ is the horizontal displacement of a transversal section of water (see equation (A.3) for the relation between $z$ and $y$).

Imposing the boundary conditions $y_1(0, t) = y_2(0, t)$ and $f_1(0, t) = f_2(0, t)$, we can deduce $\sin \varphi_1 = \sin \varphi_2 = 0$. Then, choosing $\varphi_1 = \varphi_2 = 0$ we obtain

$$R = \frac{k' - k}{k + k'} = \frac{v - v'}{v + v'}$$

(7)

and

$$T = \frac{2k'}{k + k'} = \frac{2v}{v + v'}$$

(8)

It is worthwhile to mention here that other choices of $\varphi_1$ and $\varphi_2$ will change the signs of $R$ and $T$. However, in this case, it will also change the phases of the reflected and transmitted waves. Both modifications will compensate themselves and the shape of the wave will remain unchanged in relation to the choice $\varphi_1 = \varphi_2 = 0$. 

Figure 1. Evolution of a pulse propagating from a deep sea to a shallow sea (solid line). The dashed line is the wave velocity in units of $1/200$ m s$^{-1}$. The upper frame (A) shows the pulse before the velocity discontinuity, the middle frame (B) at the velocity discontinuity and the lower frame (C) after the discontinuity.
Reflection and transmission are very important effects in wave propagation; every time a travelling wave (light, water waves, pulses in strings, etc) encounters a discontinuity in the medium where it propagates, reflection and transmission occur. Since there are no energy losses, energy flux is conserved. The energy of a wave is proportional to the square of its amplitude [1]. Thus, the energy flux is proportional to the squared amplitude times the wave velocity. The energy flux of the incident wave at $x = 0$ is given by

$$\phi_{\text{inc}} = v$$

(the amplitude of the incident wave was chosen as 1).

The reflected and transmitted energy flux are given by

$$\phi_{\text{refl}} = R^2 v$$

and

$$\phi_{\text{trans}} = T^2 v'$$

respectively.

Using equations (7) and (8), it can immediately be shown that

$$\phi_{\text{inc}} = \phi_{\text{refl}} + \phi_{\text{trans}}.$$  (12)

It is worthwhile to mention here that equations (7), (8) and (12) are classical textbook results on waves at interfaces.

Figure 1 shows the evolution of a wavepacket in three distinct situations: before finding a step, passing the step and after passing the step. On the left side of $x = 0$ the depth is 2000 m and on the right side 10 m. We can note a growth of the wavepacket amplitude after passing a step due to the velocity variation; in this case $k'/k = 14$, and, then $T = 1.9$.

Using energy conservation, Margaritondo deduced that the wave amplitude, when going from a sea depth $h_1$ to $h_2$, increases by the factor $(h_1/h_2)^{1/2}$ [1]. According to this result, the amplitude in our example should grow by a factor of 3.8. However, the growth observed was 1.9. The difference is due to the fact that when a wavepacket encounters a discrete step of velocity, part of the energy is reflected (the reader can verify that equation (12) is satisfied). As will be shown in the following section, when the sea depth varies smoothly, the reflected wave can be neglected, and our results become equal to Margaritondo’s result.

3. Waves in a variable depth

In order to study the evolution of a tsunami when it propagates in a rectangular box where the depth varies, we initially made a wavepacket propagating into a crescent $x$ direction. The variable depth was approximated by a succession of steps taken as narrow as we wished.

The evolution of the wavepacket was calculated as follows.

• At time $t$, the wavepacket amplitude $y(x, t)$ was divided into $n$ small discrete transversal sections of length $\Delta x$ (in our case $\Delta x = 50000$ m).

• Every small part of the wavepacket $y(x, t)$ was investigated.

  (i) If in a time interval $\Delta t$ it stays in the same velocity step, then the wavepacket at $t + \Delta t$ was simply increased by $y(x, t)$ at the position $x + v(x) \Delta t$.

  (ii) If in a time interval $\Delta t$, we choose $\Delta t$ as 30 s, it encounters a velocity step, part is reflected and part is transmitted. The reflected and transmitted parts were calculated from equations (7) and (8) and added to the wavepacket at $t + \Delta t$ propagating to the left or to the right, respectively. The step width and the time interval $\Delta t$ were chosen such that the reflected or transmitted parts never encounter a second step.
Figure 2. The upper frame (A) shows an initial wavepacket travelling to the right. The frames (B) and (C) show, respectively, the tsunami in an intermediary position and near the coast. The tsunami velocity is given by the dashed line. Note that the wavepacket extension diminishes in the same proportion as the velocity. (Only the progressive part of the wavepacket is shown.)

Figure 3. Change of orientation of the wave crests (solid lines) in a variable sea depth. The dashed lines are the trajectories of the front waves. The sea depth varies with the $q$ coordinate.

Figure 2 shows three positions of the right-propagating wavepacket (the left propagating was omitted). The initial wavepacket, figure 2(A), has its centre at a depth of about 3930 m ($v \sim 200$ m s$^{-1}$); then the centre of the wavepacket goes to a position where the depth is about 165 m ($v \sim 40$ m s$^{-1}$), figure 2(C). The growth of the amplitude is about 1.7. The difference between our result and the one expected by Margaritondo [1] ($(3903/165)^{0.25} = 2.2$) is due to the fact that we approximate the continuous depth variation by discrete steps and, as a consequence, the left-propagating wavepacket was not negligible.

In the last paragraphs of this section, we present a short discussion of the refraction phenomenon.

Consider, for instance, a water wave propagating in a medium where the sea depth varies with $q$, as shown in figure 3 (for instance, $q$ can be the distance from the coast). The wave crest at $q_1$ has a velocity $v_1$ and at $q_2$ velocity $v_2$. It is a matter of geometry to show that a
wavefront will change orientation, making a curve with radius

\[ R = \frac{v}{|\frac{dv}{dq}|}. \]  

(13)

For instance, consider what happened in the south of India. The sea depth varies from about 2000 m at \( q_1 = 500 \) km far from the coast to about 100 m near the coast, \( q_2 \sim 0 \). Thus, \(|\frac{dv}{dq}| \approx 2.2 \times 10^{-4} \text{ s}^{-1} \). As a consequence, the radius formed by a wave crest varies from about 640 km far from the coast to about 140 km near the coast. This is about what we can see from figure 4. In figure 3, a tsunami wavefront propagation, part in deep water and part in shallow water near the coast, refracts and, in consequence, changes orientation. Figure 4 shows a refraction map for the 26 December 2004 tsunami. The dashed curves are the frontwaves at 100, 150, 200 and 300 min after the earthquake [4].

4. Discussion

The tsunami of 26 December 2004, obviously, did not propagate in a rectangular box, but, approximately, in a circular tank. As the tsunami propagates, its extension increases and, in consequence, its amplitude diminishes. However, when it approaches shallow waters, near the coast, its amplitude grows again as shown by the simplified model developed in this paper.

The developed model depends on two approximations: the wave amplitude is small and the length of the wavepacket is large when compared with the sea depth. Since the first approximation is not valid near the beach, we stopped the evolution when the front part of the tsunami wavepacket attained a depth of about 50 m, shown in figure 2(C).

In summary, with the model developed in this paper we showed how to use the approximation of shallow water waves to a variable depth. This simple model allows us to understand what occurs when a tsunami goes from deeper to shallow waters; the velocity of the rear part of the wavepacket is larger than the velocity of its front part, causing water to pile up. Also, refraction effects that are not present in a sea of constant depth can be observed near the coast.

Acknowledgment

MTY thanks the Brazilian agency FAPESP (Fundação de Amparo a Pesquisa do Estado de São Paulo) for financial support.
Appendix. Deduction of the wave equation for waves with wavelengths much greater than the water depth

We will deduce the wave equation in a simplified situation. We will make the following approximations (all of them can be applied to tsunamis located far from the beach): the part of the restoration force that depends on the surface tension can be neglected in the case of waves with large wavelengths, the wavelength or the extension of the wavepacket will be considered much longer than the depth of the water (in the case of tsunamis the wavelengths and the ocean depth can have, approximately, hundreds of kilometres and a few kilometres, respectively), the wave amplitude will be considered much smaller than the ocean depth. Another simplification is the tank where the wave propagates; we will consider a wave propagating in a rectangular box with vertical walls and constant depth. In this approximation of shallow water waves, all the droplets in the same transversal portion have the same oscillatory horizontal motion along the $x$ direction. Finally, friction at the bottom will be neglected [3].

Figure A1 illustrates the situation considered. The wave direction of propagation is $x$; $h_0$ is the unperturbed height of the water and $L$ is the box width.

Figure A2 shows the same box of figure A1 in a side view showing a perturbation in the water. A lamellar slice with width $\Delta x$ in $x$ and height $h$ at a time $t$ will have a height $h + y$ and a width $\Delta x + \Delta z$ when it occupies the position $x + z$ at an instant $t + \Delta t$. $z = z(x, t)$ is the horizontal displacement—along the $x$ direction—of a vertical lamellar slice with an equilibrium position at $x$. When the wave propagates, this part of the water oscillates to left and right. Equalling the volume of water in $\Delta x$ and $\Delta x + \Delta z$, we have

$$Lh \Delta x = L(h + y)(\Delta x + \Delta z) = L(h \Delta x + h \Delta z + y \Delta x + y \Delta z).$$  (A.1)
If we consider \( y \ll h \) and \( \Delta z \ll \Delta x \), then equation (A.1) becomes
\[
h\Delta z + y\Delta x = 0, \quad (A.2)
\]
or
\[
y = -h \frac{\partial z}{\partial x}. \quad (A.3)
\]
This last equation is the mass conservation equation of the fluid and relates the vertical displacement of the water surface, \( y \), with the horizontal displacement of a vertical slice of water.

To apply Newton’s second law to a small portion of the lamellar slice, \( \Delta h \), of water (see figure A2), we should calculate the total force, \( \Delta F \), acting on it. This force depends on the pressure difference between the opposite sides of the slice:
\[
\Delta F = \Delta h L \left( P(x) - P(x + \Delta x) \right) \approx -\Delta h L \frac{\partial P}{\partial x} \Delta x. \quad (A.4)
\]
Then, \( F = ma \) leads to
\[
-\Delta h L \frac{\partial P}{\partial x} \Delta x = \rho L \Delta h \Delta x \frac{\partial^2 z}{\partial t^2}, \quad (A.5)
\]
where \( m \) is the mass of the water slice, \( a \) is the acceleration of the transversal section of water given by the second partial derivative of \( z \) with respect to \( t \) and \( \rho \) is the water density.

Since
\[
\frac{\partial P}{\partial x} = \rho g \frac{\partial y}{\partial x}, \quad (A.6)
\]
where \( g \) is the gravity acceleration, we can write equation (A.5) as
\[
\frac{\partial^2 z}{\partial t^2} = -g \frac{\partial y}{\partial x}. \quad (A.7)
\]
Taking derivatives of both sides of equation (A.3) with respect to \( x \), we have
\[
\frac{\partial y}{\partial x} = -h \frac{\partial^2 z}{\partial x \partial t^2}. \quad (A.8)
\]
Finally, using equation (A.8) in equation (A.7) we obtain the wave equation [3, 6]
\[
\frac{\partial^2 z}{\partial t^2} = gh \frac{\partial^2 y}{\partial x^2}. \quad (A.9)
\]
Using equation (A.3) we can show that the vertical displacement of the water surface, \( y \), obeys an equivalent wave equation:
\[
\frac{\partial^2 y}{\partial t^2} = gh \frac{\partial^2 y}{\partial x^2}. \quad (A.10)
\]
The solutions of equations (A.9) and (A.10) are any function of \( x - vt \) or \( x + vt \), where the wave velocity is given by
\[
v = \sqrt{gh}. \quad (A.11)
\]
Equation (A.11) is a particular case of the general expression for the dispersion relation for waves in a water surface [6],
\[
v = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}} \tanh \frac{2\pi h}{\lambda}, \quad (A.12)
\]
where \( \lambda \) is the wavelength, \( \rho \) is the water density and \( \sigma \) is the water surface tension. In the case of a long wavelength and neglecting surface tension, equation (A.12) reduces to equation (A.11).

Equation (A.11) gives two useful conclusions for waves presenting the same characteristics as a tsunami: the wave velocity does not depend on the wavelength and the wavepacket does not disperse when propagating in a region of constant depth.
Since $z = z(x, t)$ is the horizontal displacement of a lamellar slice of water, the water flux, $f$, is given by
\[
f = Lh \frac{\partial z}{\partial t}.
\] (A.13)

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