Finite-Time Trajectory Tracking Control of Output-Constrained Uncertain Quadrotor

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ABSTRACT
In this article, a finite-time robust tracking control of output constrained multirotor unmanned aerial vehicle (UAV) is proposed. A finite-time sliding mode control (SMC) technique with barrier Lyapunov function (BLF) is used to assure robustness of the derived control laws while maintaining the output in specified constraints. A comparison of the proposed controller is carried out with conventional SMC to manifest the effectiveness of the output-constrained tracking control. Numerical simulations of quadrotor UAV with exogenous disturbances and time-invariant output constraints demonstrate the efficacy of the proposed controller regarding robustness, finite-time convergence, and chattering reduction.

INDEX TERMS
Barrier function, finite-time convergence, output constraint, quadrotor UAV, sliding mode control, trajectory tracking.

I. INTRODUCTION
Robust control design of multirotor unmanned air vehicles (MUAVs) is one of the most explored topics in recent times. This is because of their high maneuverability, superior agility, small size, vertical takeoff and landing ability, and the ability to handle the heavy payload. Owing to these advantages, MUAVs find many applications in today’s world. Some of the potential applications of MUAVs are target searching and tracking, air pollution monitoring, traffic surveillance, area mapping, disaster relief, reconnaissance, navigation, payload transportation, rescue operations, precision agriculture, environmental assessment, aerial photography, and site survey, etc. [1]–[7].

MUAVs are underactuated electro-mechanical systems having more degrees of freedom than control inputs [8]. Their dynamical model is highly nonlinear, complicated and states are strongly coupled. Because of these characteristics, robust control design for the altitude and attitude tracking is a challenging task. MUAVs suffer from different forms of perturbations and output constraints during flight missions.

Due to the strong disturbance and high maneuverability, MUAV can easily reach the boundary of the output constraint. The altitude of the MUAV for an indoor application must remain in an upper bound to avoid violation of output constraint. Similarly, the attitude system should follow the desired output constraint for operations such as transporting liquid. If the attitude angles violate output constraints, it may cause the liquid to spill. Thus, output-constrained flight control makes the flight operation safe.

There has been significant research on the control design for MUAVs. Being relatively simple in structure, most of the research is carried out on quadrotor unmanned air vehicle (UAV). Conventionally, nonlinear system models are simplified to linear equivalents by using Taylor’s approximation, gain-scheduling, feedback linearization, and dynamic inversion, etc. The controllers designed based on these methods are susceptible to modeling uncertainties and external disturbances [9]. Lyapunov based methods have, hence, gained more attention of researchers as these controllers guarantee robustness to matched and mismatched uncertainties [10].

Furthermore, recent trends of research on the control of quadrotor emphasize achieving robustness against unexpected perturbations. Some recent trends towards robust control of MUAV cover, for example, adaptive backstepping control [11], integral backstepping sliding mode control (SMC) [12], adaptive fuzzy gain-scheduling SMC [13], adaptive hierarchical SMC [14], disturbance observer-based robust tracking control [15]. A novel second-order SMC is proposed for a class of sliding dynamics under an upper-triangular structure and finite-time stability is achieved [16]. In this regard, in [17] the authors proposed two nonlinear controllers, the feedback linearizing control with the higher-order term derivative and an adaptive SMC for...
stabilization of quadrotor. The authors found that under modeling uncertainties and noisy conditions, SMC performs better than the feedback linearizing control. A controller design based on backstepping approach is presented by authors in [18]. In [19], a nonlinear H-infinity controller based on output feedback linearization for unmatched perturbation, and higher-order sliding mode estimator are used. Under external disturbances and modeling uncertainties, robust state estimation was considered. When the system tracks the desired attitude, the control strategy-based on disturbance rejection is considered. A nonlinear feedback controller with robust disturbance observer is proposed by authors in [20]. To compensate for the dynamic uncertainties, a robust disturbance observer was proposed whereas, for desired tracking performance, a nonlinear feedback controller was proposed. The output-constrained problem of the UAV, however, still needs attention.

The stability of systems is of paramount interest in the control system. For a closed-loop nonlinear system, it is the least requirement that a system is stable in the Lyapunov sense which means that the states of the system should never leave a certain known bound. The quadratic Lyapunov-based control methods generally ensure asymptotic stability in which state error approaches zero as the time approaches infinity. Furthermore, if states follow monotonic asymptotic convergence then the system is said to be exponential stable [21]. Recent developments in nonlinear control techniques guarantee the convergence of system states within finite-time depending upon the initial conditions of the system. This type of stability is named finite-time stability [22]. The upper bound on the time in which stability is achieved is obtained as a function of controller parameters as well as initial conditions and is named as settling time function. In this article, Lyapunov based controllers with the advancement of finite-time stability are compared and their corresponding settling time functions are evaluated.

In practical applications, it is often desired that the output of the system should be constrained. For example, in indoor operations, it is desired that the UAV never leaves the closed-quarter, or its attitude angles are subjected to certain limitations. Similarly, in electrical applications, the current through a branch might be constrained based on appliance specifications. These constraints are more crucial during transients. As a solution, the barrier Lyapunov function (BLF) is introduced in [23] which ensures that if initially, the system fulfills certain conditions on output, then for all future time, it is guaranteed to satisfy the output constraints. Some further developments for the discrete nonlinear systems with event triggered constraints and tracking control are given in [24], [25].

Fu et al. in [26] applied the BLF-based robust backstepping control on MUAVs to counter external disturbances. Finite-time stabilization with the use of BLF is achieved in [27] in presence of parametric uncertainties. The integral BLF is used in conjunction with backstepping and vision-based control in [28] to restrict drastic attitude motion. The BLF based backstepping is implemented on quadrotor and performance is compared with PID and SMC in [29]. Some recent developments have been made on the design of SMC for output constrained systems. A second-order SMC for an output constrained system under external disturbances is developed in [30]. This approach is based on BLF and finite-time convergence is attained while maintaining the output of the system in the desired constraints. There is no significant work on the application of SMC in conjunction with BLF on MUAVs. Though this methodology is developed in [31], [32] and has been used on applications like robotic manipulators [31], fuel cell [32], distributed crowd dynamics [33], and hypersonic flight vehicles [34], etc.

In this article, the finite-time tracking for altitude and attitude systems of a quadrotor UAV is achieved under output constraints. The overall novelty of this article is two-fold: 1) a finite time SMC based tracking controller in conjunction with barrier Lyapunov function is devised for second-order systems. 2) The proposed control topology is implemented on quadrotor dynamics and it is shown by comparison that this technique can be very useful especially in indoor operations as compared to a finite-time conventional SMC. Numerical simulations of quadrotor UAV with exogenous disturbances and time-invariant output constraints demonstrate the efficacy of the proposed controller regarding robustness, finite-time convergence, and chattering reduction.

The article is organized as follows: Section II introduces the problem statement and definitions and lemmas which are used in the design process and convergence analysis of the proposed controllers. Section III presents the finite-time conventional SMC without considering any output constraints. The output-constrained finite-time SMC is designed in Section IV. Control laws devised in Section III are implemented on a quadrotor UAV for tracking control problem in Section IV. Section V presents and discusses the results of numerical simulation and Section VI presents the conclusion.

II. PROBLEM STATEMENT AND PRELIMINARIES
A. PROBLEM FORMULATION

The model is devised using the Euler-Lagrange approach. In Fig. 1, the configuration of the quadrotor according to the earth-fixed inertial frame and the body-fixed frame is shown. The fixed inertial frame is considered as a reference frame, whereas the body frame positioned at the center of gravity is fixed with respect to the UAV platform. The mathematical model is derived based on the assumption that the origin of the body frame is located at the center of gravity of the UAV. The structure of the quadrotor UAV as well as the propellers are assumed to be rigid.

The model of the quadrotor is [35]:
\[
\dot{\phi} = \dot{\theta} \dot{\psi} a_1 + \dot{\theta} a_2 \Omega_\phi + b_1 U_2
\]  
\[
\dot{\theta} = \dot{\phi} \dot{\psi} a_3 + \dot{\phi} a_4 \Omega_\theta + b_2 U_3
\]  
\[
\dot{\psi} = \dot{\theta} \dot{\phi} a_5 + b_3 U_4
\]
where the Euler angle vector $[\phi, \theta, \psi]^T$ denotes the attitude of UAV in the body frame and the states represent roll, pitch, and yaw, the vector $[x, y, z]$ describe the position in the inertial frame, $m$ is the total mass of the quadrotor, $g$ is the gravity, $l$ is the arm length, and $\Omega_i$ is the residual angular speed of the propeller. The moment of inertia along $x, y, z$ axes are denoted by $I_x, I_y, I_z$ respectively whereas $a_1, \ldots, a_5$ and $b_1, \ldots, b_3$ are simplified versions of constants written as $a_1 = (I_y - I_z) / I_x, a_2 = (I_z - I_y) / I_x, a_3 = (I_x - I_y) / I_z, a_4 = (I_y - I_z) / I_x$, and $b_1 = l / I_x, b_2 = l / I_y, b_3 = l / I_z$ where $J_r$ is the inertia of the rotor. The control inputs are denoted by $U_1, \ldots, U_4$, given by:

$$\begin{align*}
U_1 &= b \left( \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2 \right) \\
U_2 &= b \left( -\Omega_2^2 + \Omega_4^2 \right) \\
U_3 &= b \left( \Omega_1^2 - \Omega_3^2 \right) \\
U_4 &= d \left( -\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2 \right)
\end{align*}$$

where $d$ represents the drag coefficient, $b$ is the lift coefficient of each propeller, and $\Omega_1 \ldots \Omega_4$ is the rotational speed of each propeller. For simplification, the states and rate of change of states are refined to formulate the first-order differential equations: let $x_1 = \dot{\phi}, x_2 = \dot{\theta}, x_3 = \theta, x_4 = \dot{\theta}, x_5 = \dot{\psi}, x_6 = \dot{\psi}, x_7 = \dot{z}, x_8 = \dot{x}, x_{10} = \dot{x}, x_{11} = \dot{y}, x_{12} = \dot{y}$, then the standard first-order representation of the system is:

$$\dot{X} = f(X, U)$$

where

$$f(X, U) = \begin{bmatrix}
x_2 \\
x_4 x_6 a_1 + x_4 a_2 \Omega_r + b_1 U_2 + \delta_\phi(x, t) \\
x_2 x_6 a_3 - x_2 a_4 \Omega_r + b_2 U_3 + \delta_\theta(x, t) \\
x_6 x_2 a_4 + b_3 U_4 + \delta_\psi(x, t) \\
x_8 g - (\cos x_5 \cos x_1) U_1 / m + \delta_h(x, t) \\
x_{10} (\sin x_1 \sin x_5 + \cos x_1 \sin x_3 \cos x_5) U_1 / m \\
x_{12} (-\sin x_1 \cos x_5 + \cos x_1 \sin x_3 \sin x_5) U_1 / m
\end{bmatrix}$$

and $\delta_\phi, \delta_\theta, \delta_\psi, \delta_h$ represent the uncertain terms due to exogenous disturbance.

Most of the controllers designed for the quadrotor assume that the angles $\phi$ and $\theta$ lie within $[-\pi/2, \pi/2]$ interval. Then the control is designed by saturating these states within this interval. However, in real problems, it may not always be true due to disturbance and unmodeled dynamics. Robust controllers play a vital role in mitigating the effect of the uncertainties in the system by compromising the robustness. A more practical solution to the problem is that the control law should be devised such that the output should remain within a prescribed bound. We have proposed a finite-time output-constrained robust tracking control technique that ensures that the magnitude of the output never crosses a pre-specified bound in presence of exogenous disturbances.

### B. PRELIMINARIES

Consider the second-order nonlinear system in Brunovsky normal form:

$$\begin{cases}
\dot{\xi}_1 = \xi_2 \\
\dot{\xi}_2 = f(\xi) + g(\xi)u + \delta(\xi, t) \\
y - \xi_1
\end{cases}$$

where $\xi_1, \xi_2 \in \mathbb{R}$ are the states, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ represents the output, $f$ and $g$ are continuous functions and $\delta(\xi, t)$ is the uncertain term due to external disturbance. The initial values of states are denoted by $\xi(0) = [\xi_0 \xi_2]$. The output $y(t)$ is required to converge to the desired output $y_d(t)$, fulfilling the output constraint such that

$$|y(t)| \leq h_c$$

$\forall t \geq 0$, where $h_c > 0$.

**Assumption 1**: The uncertain term $\delta(\xi, t)$ in (13) is bounded such that $|\delta(\xi, t)| \leq L_d$. Where $L_d$ is a positive constant.

**Remark 1**: We have considered the bounded uncertainties only. Thus the only knowledge about the uncertain term is its upper bound. This assumption is practical because for many systems we have an estimate of the upper bound of the uncertain term.
Definition 1 ([36]): Consider the system
\[
\dot{\xi} = f(\xi) , \xi(0) = \xi_0
\] (15)
where \( f: \mathbb{R}^{n+} \times \mathcal{D} \to \mathbb{R}^n \) is defined continuously on open neighborhood \( \mathcal{D} \) of the origin and \( f(0, 0) = 0 \). The origin is said to be globally finite-time stable if it is globally asymptotically stable and any solution \( \xi(0, \xi_0) \) reaches origin at some finite moment, thus \( \xi(t, \xi_0) = 0, \forall t \geq T(\xi_0) \) and \( \forall \xi_0 \in \mathbb{R}^n \). The function \( T(\xi_0) \) is the settling time function.

Definition 2 ([37]): Consider the system (15). A barrier Lyapunov function \( V(\xi) \) is a scalar function, defined on an open region \( \mathcal{D} \) containing the origin, that is continuous, positive-definite, has continuous first-order partial derivatives at every point of \( \mathcal{D} \), has the property \( V(\xi) \to \infty \) as \( \xi \to \partial \mathcal{D} \), and satisfies \( V(\xi) \leq b, \forall t \geq 0 \) along the solution of (15) for \( \xi_0 \in \mathcal{D} \), where \( b \) is a positive constant.

Assumption 2: For any \( h \), \( h > 0 \), there exist positive constants \( Y_0, A_0, Y_1, Y_2, \ldots, Y_n \) satisfying \( Y_0 \leq A_0 < h \), such that the desired trajectory \( y_d(t) \), as well as its successive time derivatives, satisfy \( |y_d(t)| < Y_0, |y_d(t)| \leq Y_1, \ldots, |y_d(n)(t)| \leq Y_n \) for all \( t \geq 0 \).

Remark 2: The desired trajectory is time-varying thus Assumption 2 makes sure that not only the desired trajectory is bounded by some constant which is less than the required constraint \( h \), but also its successive derivatives are bounded by some constant values.

Lemma 1 ([38]): Consider the system (15) and a continuous function \( V: \mathbb{R}^n \to \mathbb{R} \geq 0 \), continuously differentiable on \( \mathbb{R}^n \), class \( K_\infty \) functions \( \alpha, \bar{\alpha} \) and a positive constant \( \kappa \) satisfying
\[
\alpha(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \quad (16)
\]
\[
\dot{V}(x) = \kappa - \kappa \sqrt{V(x)} \quad (17)
\]
for all \( x \neq 0 \), then the origin is globally finite-time stable.

Lemma 2 ([23]): Consider the system (13). Then for any positive constant \( h_b \), let \( Z_1 := \{ \xi_1 \in \mathbb{R} : -h_b < \xi_1 < h_b \} \subset \mathbb{R} \). Suppose that there exist continuously differentiable and positive definite functions \( V_1 \) and \( V_2 \), such that \( V_1(\xi_1) \to \infty \) as \( \xi_1 \to h_b \) or \( \xi_1 \to -h_b \) and
\[
\beta(|\xi_2|) \leq V_2(\xi_2) \leq \bar{\beta}(|\xi_2|) \quad (18)
\]
where \( \beta, \bar{\beta} \) are class \( K_\infty \) functions. Consider \( V_1 \) and \( V_2 \) such that \( V(\xi) = V_1(\xi_1) + V_2(\xi_2) \) and \( \xi_1 \in (h_b, h_b) \). If,
\[
\dot{V} = \frac{\partial V}{\partial \xi_1} \dot{\xi}_1 \leq 0
\] (19)
then \( \dot{\xi}_1(t) \) remains in the open set \( \{ \xi_1 \in (-h_b, h_b) \} \) \( \forall t \geq 0 \).

Lemma 3 ([39]): For \( a > 0 \), the following inequality holds \( \forall x \) such that \( |x| < a \):
\[
\log \left( \frac{a^2}{a^2 - x^2} \right) \leq \frac{x^2}{a^2 - x^2}
\] (20)

III. FINITE-TIME TRAJECTORY TRACKING WITH CONVENTIONAL SMC
We have designed a finite-time trajectory tracking controller using the conventional SMC technique without output constraint in this section. We consider a general second-order system given by (13). For tracking performance, the output should match the desired output trajectory \( y_d \). Let \( \eta_1 = \xi_1 - y_d \) be the error of actual and desired output and \( \eta_2 = \dot{\eta}_1 = \dot{\xi}_2 - \dot{y}_d \). Then the dynamical system in transformed coordinates can be represented as
\[
\dot{\eta}_1 = \eta_2
\]
\[
\dot{\eta}_2 = f(\xi) + g(\xi) u + \sigma(\xi, t) - \dot{y}_d
\] (21)

The goal is to devise an input \( u \) such that after some finite-time, \( \eta_1 = 0 \). We consider the sliding manifold
\[
\sigma = c_1 \eta_1 + \eta_2
\] (23)

Theorem 1: Consider the nonlinear system (13), and the sliding manifold (23), under Assumption 1, for the control input
\[
u = \frac{1}{g(\xi)} (-c_1 \eta_2 - f(\xi) + \dot{y}_d - \kappa \sigma(\xi))
\] (24)
with \( c_1 > 0 \) and \( \kappa = L_a + \kappa_a/\sqrt{2} \), where \( \kappa_a > 0 \), \( \eta_1 = \xi_1 - y_d \), and \( \eta_2 = \dot{\xi}_2 - \dot{y}_d \). The output tracks the desired trajectory \( y_d \), and after some finite-time \( \eta_1(t, \xi) = 0 \). Moreover, there exists a time \( T(\eta_0) : \mathbb{R}^2 \to \mathbb{R}^+ \)
\[
T(\eta_0) \leq \frac{\sqrt{2}}{\kappa_a} |c_1 \eta_{10} + \eta_{20}|
\] (25)
such that \( \sigma = 0 \) for all \( t > T(\eta_0) \), where \( \eta_{10} \) and \( \eta_{20} \) are the values of \( \eta_1 \) and \( \eta_2 \) at time \( t = 0 \).

Proof: Consider the quadratic Lyapunov function \( V = 0.5a^2 \). Then
\[
\dot{V} = \sigma \dot{\sigma} = \sigma (c_1 \dot{\eta}_1 + \dot{\eta}_2)
\] (26)
\[
\dot{V} = \sigma (c_1 \eta_2 + f(\xi) + g(\xi) u + \sigma(\xi, t) - \dot{y}_d)
\] (27)

Substituting (24) into (27), we have
\[
\dot{V} \leq -|\sigma| (\kappa - L_a) = -\kappa_a \sqrt{V}
\] (28)
where \( \kappa = L_a + \kappa_a/\sqrt{2} \) and \( \kappa_a > 0 \). Thus, by the virtue of Lemma 1, the states \( \eta_1 \) and \( \eta_2 \) converge to the sliding manifold in finite-time.

To evaluate the settling time function, consider the worst-case i.e. \( \dot{V} = -\kappa_a \sqrt{V} \), the differential equation is rewritten in the following form:
\[
dV

\sqrt{\dot{V}(\eta(t))} = -\kappa_a dt
\] (29)

using the fact that \( V(\eta(t)) = 0 \), and consequently \( \sigma = 0 \forall t \geq T(\eta_0) \), the solution of (29) is given by:
\[
T(\eta_0) \leq \frac{2}{\kappa_a} \sqrt{V_0} = \frac{\sqrt{2}}{\kappa_a} |c_1 \eta_{10} + \eta_{20}|
\] (30)

where \( \eta_{10} = \eta_1(0) \) and \( \eta_{20} = \eta_2(0) \).

We have established the fact that conventional SMC achieves finite-time convergence even when exposed to exogenous disturbances and the finite-time in which the states of the systems reach the sliding surface is upper bounded by (30).
IV. FINITE-TIME TRAJECTORY TRACKING SMC WITH OUTPUT-CONSTRAINT

A trajectory tracking control for the output-constrained nonlinear system is developed using SMC with BLF (SMC-BLF). We present the following theorem to prove the robustness and find-time convergence of the proposed controller while satisfying the output constraints.

Theorem 2: Consider the nonlinear system (13), output constraint (14), the sliding manifold (23), under Assumption 1-2. For the control input

\[ u = \frac{1}{g(\xi)} [-c_1\eta_2 - f(\xi) + \dot{y}_d - L_\delta \sigma \]

\[ - \left( \frac{\eta_1 \sigma}{\hat{h}_b^2 - \eta_1^2} + \kappa \sqrt{|2\sigma|} + \frac{\eta_1}{\sqrt{\hat{h}_b^2 - \eta_1^2}} \right) \text{sgn} (\sigma) \]

(31)

with \( c_1 > 0, \kappa > 0, \eta_1 = \xi_1 - y_d, \eta_2 = \xi_2 - y_d, \) and \( |\eta_1(0)| < \hat{h}_b, \) the origin is globally finite-time stable. Therefore, the states \( \eta_1 \) and \( \eta_2 \) converge to the sliding manifold in finite-time. Moreover, if the initial conditions are defined to be \( \eta_0 = (\eta_{10}, \eta_{20}), \) the settling time \( T(\eta_0) : R^2 \to R^+ \) is upper bounded by:

\[ T(\eta_0) \leq \frac{\sqrt{c_1} \eta_{10} + \eta_{20} + \frac{1}{2} \log \left( \frac{\hat{h}_b^2}{\hat{h}_b^2 - \eta_{10}^2} \right)}{\kappa} \]

(32)

Proof: Consider the Lyapunov function as

\[ V(\eta) = V_1(\eta) + V_2(\eta) \]

where \( V_1(\eta) \) is defined by

\[ V_1(\eta) = |\sigma| \]

(34)

and the barrier Lyapunov function

\[ V_2(\eta) = \frac{1}{2} \log \left( \frac{\hat{h}_b^2}{\hat{h}_b^2 - \eta_1^2} \right) \]

(35)

The time derivative of \( V(\eta) \) is:

\[ \dot{V}(\eta) = \dot{\sigma} \text{sgn}(\sigma) + \frac{\eta_1}{\hat{h}_b^2 - \eta_1^2} \dot{\eta}_1 \]

(36)

\[ \dot{V}(\eta) = (c_1\eta_2 + f(\xi) + g(\xi) u + \delta(\xi, t) - \dot{y}_d) \text{sgn}(\sigma) \]

\[ + \frac{\eta_1}{\hat{h}_b^2 - \eta_1^2} \sigma - \frac{c_1 \eta_1^2}{\hat{h}_b^2 - \eta_1^2} \]

(37)

\[ \dot{V}(\eta) = (c_1\eta_2 + f(\xi) + g(\xi) u + \delta(\xi, t) - \dot{y}_d) \text{sgn}(\sigma) \]

\[ + \frac{\eta_1}{\hat{h}_b^2 - \eta_1^2} \sigma \]

(38)

Substituting (31) into (38), we have

\[ \dot{V}(\eta) \leq -\kappa \sqrt{|2\sigma|} - \kappa \left( \frac{\eta_1^2}{\hat{h}_b^2 - \eta_1^2} \right)^{1/2} \]

(39)

using Lemma 3

\[ \dot{V}(\eta) \leq -\kappa \sqrt{2\sigma} - \kappa \left[ \log \left( \frac{\hat{h}_b^2}{\hat{h}_b^2 - \eta_1^2} \right) \right]^{1/2} \]

(40)

Defining \( \kappa_a = \kappa \sqrt{2}, \)

\[ \dot{V}(\eta) \leq -\kappa_a \sqrt{V(\eta)} \]

(41)

\[ \dot{V}(\eta) \leq -\kappa_a \sqrt{V(\eta)} \]

(42)

Thus, by the virtue of Lemma 1, the system under consideration is finite-time stable. Furthermore, by Lemma 2, we have \( \eta_1 < \hat{h}_b \) if \( \eta_1(0) < \hat{h}_b. \) Since \( y(t) = \eta_1 + y_d, \) and \( y_d \leq A_0, \) thus we have \( y(t) < \hat{h}_b + A_0 = \hat{h}_c. \) Therefore the output is always bounded by (14).

To evaluate the settling time function, consider the worst-case i.e. \( \dot{V}(\eta) = -\kappa_a \sqrt{V(\eta)}, \) which can be written as:

\[ \frac{dV}{\sqrt{V(\eta)(t)}} = -\kappa_a dt \]

(44)

using the fact that \( V(\eta(t)) = 0 \forall t \geq T(\eta_0), \) the solution of (44) is given by:

\[ T(\eta_0) \leq \frac{2}{\kappa_a} \sqrt{V_0} \]

(45)

where

\[ V_0 = |c_1\eta_{10} + \eta_{20}| + \frac{1}{2} \log \left( \frac{\hat{h}_b^2}{\hat{h}_b^2 - \eta_{10}^2} \right) \]

(46)

and \( \eta_{10} = \eta_1(0) \) and \( \eta_{20} = \eta_2(0). \)

From (31), a major concern of \( u \) becoming unbounded whenever \( |\eta_1| = \hat{h}_b. \) However, we have established that for the closed-loop system, \( \eta_1 \) will never reach \( \hat{h}_b \) for all time \( t \geq 0. \) Consequently, \( u \) will never become unbounded. Moreover, \( |\eta_1(0)| < \hat{h}_b \) is the required condition on the initial output value which is somehow a strict condition.

Remark 3: In (31), the two design parameters \( c_1 \) and \( \kappa \) are utilized that play a key role in the transient response of the system response. The parameters \( c_1 \) and \( \kappa \) can be chosen any positive number, however, a larger value of these parameters will result in a faster transient response in terms of settling time. Generally, these parameters can be chosen by trial and error.

V. FINITE-TIME SLIDING MODE CONTROL FOR QUADROTOR UAV

The controllers developed in Section III and Section IV are employed to designed trajectory tracking control for the quadrotor UAV. The block diagram of the complete system is shown in Fig. 2. The conventional SMC and SMC-BLF are designed in Subsection A and Subsection B respectively.

A. TRAJECTORY TRACKING USING FINITE-TIME SMC WITHOUT OUTPUT-CONSTRAINT

1) ROLL ANGLE TRACKING

A finite-time SMC controller is designed without output constraint for the roll angle tracking. For the second-order roll angle model, consider the sliding surface \( \sigma_0 = c_1 \phi \eta_{1\phi} + \eta_{2\phi} \), where \( \eta_{1\phi} = x_1 - x_{1d}, \) and \( \eta_{2\phi} = \dot{\eta}_{1\phi}, \) where \( x_{1d} \) is the desired trajectory for the roll angle. Taking the Lyapunov function
A finite-time SMC controller is designed without output constraint for the altitude system. For the second-order altitude system model, consider the sliding surface \( \sigma_h = c_{1h}\eta_{1h} + \eta_{2h} \), where \( \eta_{1h} = x_7 - x_7d \), and \( \eta_{2h} = \eta_{1h} \), where \( x_7d \) is the desired trajectory for the altitude. Taking the Lyapunov function \( V_h = 0.5a_h^2 \) and using the following control law for altitude tracking

\[
U_1 = \frac{m}{\cos x_5 \cos x_1} (c_{1h}\eta_{2h} + g - \ddot{x}_7d + \kappa_h \text{sgn} (\sigma_h))
\]

where \( \kappa_h = L_{\delta_h} + \kappa_{ah}/\sqrt{2} \) for some positive constant \( \kappa_{ah} \), and \( L_{\delta_h} \) represents the upper bound of matched uncertainty appearing in the altitude channel; results in \( \dot{V}_h \leq -\kappa_{ah}\sqrt{V_h} \).

**B. TRAJECTORY TRACKING USING FINITE-TIME SMC WITH OUTPUT CONSTRAINT**

1) ROLL ANGLE TRACKING

The procedure developed in Section IV is followed to develop a robust controller for the roll channel with output constraints. The roll angle output trajectory is required to be within a bond such that \(|x_1(t)| < R_{\phi0} \) for all time. Consider the Lyapunov function

\[
V_\phi = |\sigma_\phi| + \frac{1}{2} \log \left( \frac{h_{b\phi}^2}{h_{b\phi}^2 - \eta_{1\phi}^2} \right)
\]

and taking the control input for roll angle trajectory tracking as:

\[
U_2 = \frac{1}{b_1} \left[ -c_{1\phi}\eta_{2\phi} - x_4x_6a_1 + x_4a_2\Omega_r + \ddot{x}_1d - L_{3\phi}\sigma_\phi - \left( \eta_{1\phi}\frac{\eta_{1\phi}}{h_{b\phi}^2 - \eta_{1\phi}^2} + \sqrt{2}\eta_{1\phi} + \kappa_{\phi} \right) \text{sgn} (\sigma_\phi) \right]
\]

ensures the finite-time tracking while fulfilling the output constraint.

2) PITCH ANGLE TRACKING

The pitch angle output trajectory is required to be within a bond such that \(|x_3(t)| < R_{\theta0} \) for all time. Consider the Lyapunov function

\[
V_\theta = |\sigma_\theta| + \frac{1}{2} \log \left( \frac{h_{b\theta}^2}{h_{b\theta}^2 - \eta_{1\theta}^2} \right)
\]

And taking the control input for pitch channel trajectory tracking as:

\[
U_3 = \frac{1}{b_2} \left[ -c_{1\theta}\eta_{2\theta} - x_4x_6a_1 + x_4a_2\Omega_r + \ddot{x}_3d - L_{3\theta}\sigma_\theta - \left( \eta_{1\theta}\frac{\eta_{1\theta}}{h_{b\theta}^2 - \eta_{1\theta}^2} + \sqrt{2}\eta_{1\theta} + \kappa_{\theta} \right) \text{sgn} (\sigma_\theta) \right]
\]

ensures the finite-time tracking while fulfilling the output constraint.
TABLE 1. System’s parameters.

| Symbol | Description | Value   |
|--------|-------------|---------|
| m      | Mass of quadrotor | 1.1kg   |
| g      | Gravitational acceleration | 9.81m/s² |
| Ω_p   | Residual angular speed of propeller | 1 rad/s |
| L     | Arm length | 0.25m   |
| I_x   | Moment of inertia along x-axis | 7.5 × 10⁻³kgm² |
| I_y   | Moment of inertia along y-axis | 7.5 × 10⁻³kgm² |
| I_z   | Moment of inertia along z-axis | 1.3 × 10⁻²kgm² |
| J_r   | Rotor inertia | 2.8 × 10⁻³kgm² |

3) YAW ANGLE TRACKING

The yaw angle output trajectory is required to be within a bound such that \(|x_5(t)| < h_{c\psi}\) for all time. The procedure developed in the previous section is employed to formulate the control law for the pitch channel. Consider the Lyapunov function

\[
V_\psi = |\sigma_\psi| + \frac{1}{2} \log \left( \frac{\bar{h}_{b\psi}^2}{\bar{h}_{b\psi}^2 - \eta_{1\psi}^2} \right) \tag{55}
\]

and taking the control input for yaw channel trajectory tracking as follows:

\[
U_4 = \frac{1}{b_3} \left[ -c_1\psi \eta_2\psi - x_2x_4a_5 + \dot{x}_5d - L_\delta \sigma_\psi \right.
\]

\[
-\left( \frac{\eta_1\psi \sigma_\psi}{\bar{h}_{b\psi}^2 - \eta_{1\psi}^2} + \kappa_\psi \sqrt{2|\sigma_\psi| + \frac{\eta_1\psi}{\bar{h}_{b\psi}^2 - \eta_{1\psi}^2}} \text{sgn} (\sigma_\psi) \right) \tag{56}
\]

ensures the finite-time tracking while fulfilling the output constraint.

4) ALTITUDE TRACKING

The altitude output trajectory is required to be within a bound such that \(|x_7(t)| < h_{ch}| for all time. The procedure developed in the previous section is used to design a controller for the pitch channel. Consider the Lyapunov function

\[
V_h = |\sigma_h| + \frac{1}{2} \log \left( \frac{\bar{h}_{bh}^2}{\bar{h}_{bh}^2 - \eta_{1h}^2} \right) \tag{57}
\]

and taking the control input for altitude channel trajectory tracking as follows:

\[
U_1 = \frac{m}{\cos x_5 \cos x_1} \left[ c_{1h}\eta_2 h + g - \dot{x}_7d + L_\delta \sigma_h + \left( \frac{\eta_1 \sigma_h}{\bar{h}_{bh}^2 - \eta_{1h}^2} \right) \right.
\]

\[
+ \frac{\kappa_h \sqrt{2|\sigma_h| + \eta_{1h}}}{\sqrt{\bar{h}_{bh}^2 - \eta_{1h}^2}} \text{sgn} (\sigma_h) \right] \tag{58}
\]

ensures the finite-time tracking while fulfilling the output constraints.

Remark 4: The finite-time control laws for altitude and attitude systems are designed where the x and y positions are taken as free-states. Since the attitude and altitude are required to follow the desired trajectory, thus x and y states remain stable and bounded.

VI. SIMULATION RESULTS

Simulation results of the devised controllers in the previous section are presented in this section. To show the effectiveness of the designed controllers, different values of initial conditions and output constraints are considered for each channel. Table 1 shows the physical parameters of UAV employed in simulations.

To illustrate the results of the designed controllers, the desired trajectory for the roll angle is \(x_{1d} = 0.2 + 0.3 \sin t\) with \(x_1(0) = 0.2\) rad and \(x_2(0) = 2\) rad/s. The output constraint for the roll angle is taken as \(h_{c\phi} = 0.55\). Since
−0.1 ≤ x_{1d} ≤ 0.5, hence the value of \( h_{bh} = 0.55 - 0.5 = 0.05 \). For pitch angle tracking, the desired trajectory is \( x_{3d} = 0.3 + 0.3 \sin t \) with \( x_3(0) = 0.3 \) rad and \( x_4(0) = 2 \) rad/s. The output constraint is defined as \( h_c = 0.65 \) with the objective that \( |x_3| < h_c \). Since \( 0 \leq x_{3d} \leq 0.6 \), hence the value of \( h_{ph} = 0.65 - 0.6 = 0.05 \). For yaw angle tracking, the desired trajectory is \( x_{5d} = 0.4 + 0.3 \sin t \) with \( x_5(0) = 0.4 \) rad and \( x_6(0) = 2 \) rad/s. The output constraint is \( h_{c\psi} = 0.75 \) with the objective that \( |x_5| < h_{c\psi} \). Since \( 0.1 \leq x_{5d} \leq 0.7 \), hence the value of \( h_{bh} = 0.75 - 0.7 = 0.05 \). For altitude tracking, the desired trajectory is \( x_{7d} = 4 + \sin t \) with initial condition \( x_7(0) = 4 \) rad and \( x_8(0) = 5 \) rad/s. The output constraint is \( h_{ch} = 5.5 \) with the objective that \( |x_7| < h_{ch} \). Since \( 3 \leq x_{7d} \leq 5 \), hence the value of \( h_{bh} = 5.5 - 5 = 0.5 \).

Both controllers are simulated with the same gains and time-varying sinusoidal disturbance of amplitude one.

Fig. 3, Fig. 4 to Fig. 6 show the trajectory tracking of both SMC and SMC-BLF controllers for roll, pitch, yaw angles, and altitude. Clearly, SMC violates the constraints while the SMC-BLF maintains the output constraints. Albeit, the settling time for both controllers is the same, because of the same gains and initial conditions. However, the SMC-BLF outperforms the conventional SMC not only in the output constraints maintain but also in the transient response.

Fig. 7 shows the control inputs for the conventional SMC. As expected, SMC exhibits chattering in the control input. Fig. 8 shows the control input for the SMC-BLF control inputs. The controller eliminates chattering and the control inputs are smoother as compared to the conventional SMC inputs.

VII. CONCLUSION
This article presents a robust nonlinear control technique for the output-constrained uncertain quadrotor UAV based on the use of BLF. A finite-time trajectory tracking control is developed and a comparison of the proposed control technique is presented with the conventional SMC. The results show that the SMC-BLF achieves finite-time convergence, robust to external disturbances, and reduces the chattering from the control input to a significant level. This study can be further extended by considering time-varying output constraints and designing SMC based on barrier Lyapunov function for an uncertain quadrotor.
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