Geometric construction of bases of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$

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Abstract

We present an efficient algorithm for the construction of a basis of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$ via the Poincaré-Lefschetz duality theorem. Denoting by $g$ the first Betti number of $\overline{\Omega}$ the idea is to find, first $g$ different 1-boundaries of $\overline{\Omega}$ with supports contained in $\partial\Omega$ whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$, and then to construct in $\Omega$ a homological Seifert surface of each one of these 1-boundaries. The Poincaré-Lefschetz duality theorem ensures that the relative homology classes of these homological Seifert surfaces in $\Omega$ modulo $\partial\Omega$ form a basis of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$. We devise a simply procedure for the construction of the required set of 1-boundaries of $\overline{\Omega}$ that, combined with a fast algorithm for the construction of homological Seifert surfaces, allows the efficient computation of a basis of $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$ via this very natural geometrical approach. Some numerical experiments show the efficiency of the method and its performance comparing with other algorithms.

1 Introduction

Consider a bounded domain $\Omega$ of $\mathbb{R}^3$ whose closure $\overline{\Omega}$ is polyhedral and whose boundary $\partial\Omega$ is sufficiently regular, like that used for finite element approximation of differential problems. Our aim is to develop a set of fast and robust algorithms for the automatic identification and construction of that homological structures that influence the solvability of differential problems defined on $\Omega$. Let us consider, for instance, the curl-div system

$$\begin{align*}
curl u &= F \quad \text{in} \ \Omega \\
\div u &= G \quad \text{in} \ \Omega \\
u \cdot n &= g \quad \text{on} \ \partial\Omega
\end{align*}$$

It is well-know that the solution of this problem is not unique if $g$, the first Betti number of $\Omega$, is greater than zero. Two different ways to fix a unique solution are to prescribe the circulation around a set of 1-cycles in $\Omega$ that are representatives of a basis of the first homology group $H_1(\overline{\Omega}; \mathbb{Z})$ of $\overline{\Omega}$ or to prescribe the flux through a set of surfaces that are representatives of a basis of the second relative homology group $H_2(\overline{\Omega}, \partial\Omega; \mathbb{Z})$ of $\overline{\Omega}$ modulo $\partial\Omega$.

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Let us consider a triangulation of \( \Omega \); namely, a tetrahedral mesh of \( \Omega \). The incidence matrices of such a triangulation, tetrahedra-to-faces, faces-to-edges and edges-to-vertices, are the integer matrix representations of the so-called boundary operators associated with the given triangulation. The standard procedure to compute the homology and cohomology groups of \( \Omega \) is based on the computation of the Smith normal form of these integer matrices, a computationally demanding algorithm even in the case of sparse matrices (see e.g. [21] and [13]). Thus, before the Smith normal form procedure is employed, the problem size is reduced using fast algorithms (usually algorithms that run in linear time) that remove homologically irrelevant parts of the triangulation (see e.g. [19], [20]). An implementation of these techniques have been integrated in the finite element mesh generator Gmesh by Pelikka et al. (see [22]). Other software that perform homology and cohomology computations, with less emphasis on finite element modeling, are CHomP [10], jPlex [21] and GAP homology [10].

A specific approach for the construction of a basis of \( H_2(\Omega; \partial \Omega; \mathbb{Z}) \) has been proposed by Kotiuga in [19], [17], [18] and [13]. There the aim is to construct the so-called “cuts” of \( \Omega \); namely, surfaces-with-boundary \( \{ S_i \}_{i=1}^g \) of \( \Omega \) with \( \partial S_i \subseteq \partial \Omega \) which permit to construct a single-valued magnetic scalar potential in \( \Omega \) \( \setminus \bigcup_{i=1}^g S_i \) of any given current density in \( \Omega \). These cuts are nonsingular polyhedral representatives of a basis of \( H_2(\Omega, \partial \Omega; \mathbb{Z}) \). The algorithm consists in two main steps. Starting with a basis of \( H_1(\Omega, \mathbb{Z}) \), in the first step, one constructs a basis \( \{ f_i \}_{i=1}^g \) of the cohomology group \( H^1(\Omega; \mathbb{Z}) \) approximating a differential problem with a finite element method. Then the second step is to construct the cuts of \( \Omega \) as level sets of the maps \( \{ f_i \}_{i=1}^g \). The representatives of the basis are regular surfaces and this justify the substantial complexity of the procedure.

In this paper we focus on the construction of a basis of \( H_2(\Omega; \partial \Omega; \mathbb{Z}) \) using a geometric approach based on the Poincaré-Lefschetz duality theorem. Here we are not interested in questions concerning regularity. Indeed the representatives of the basis that we construct are formal linear combinations (with integer coefficients) of oriented faces of the given triangulation that we call homological Seifert surfaces. This allows to gain in efficiency from the computational point of view.

Let us precise what we meant when we said that the boundary \( \partial \Omega \) of \( \Omega \) is sufficiently regular. In what follows we will assume that \( \partial \Omega \) is locally flat; that is, for every point \( x \in \partial \Omega \), there exist an open neighborhood \( U_x \) of \( x \in \mathbb{R}^3 \) and a homeomorphism \( \phi_x : U_x \rightarrow \mathbb{R}^3 \) such that \( \phi_x(U_x \cap \partial \Omega) = P \), where \( P \) is the coordinate plane \( \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \) (see [3, 4]). This kind of domains includes all Lipschitz polyhedral domains, but also domains like the crossed bricks (see, e.g., Fig. 3.1 in [13]). Let \( \mathcal{T} \) be a triangulation of \( \Omega \). A 1-cycle \( \gamma \) of \( \mathcal{T} \) is a formal linear combination (with integer coefficients) of oriented edges of \( \mathcal{T} \) with zero boundary. The 1-cycle \( \gamma \) is said to be a 1-boundary of \( \mathcal{T} \) if it is equal to the boundary of a formal linear combination \( S \) of oriented faces of \( \mathcal{T} \). If such a \( S \) exists, we call it homological Seifert surface of \( \gamma \) in \( \mathcal{T} \).

Given \( g \) different 1-boundaries \( \{ \sigma_n^1 \}_{n=1}^g \) of \( \mathcal{T} \) with supports contained in \( \partial \Omega \) and whose homology classes in \( \mathbb{R}^3 \setminus \Omega \) form a basis of \( H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z}) \), and given for each \( n = 1, \ldots, g \) a homological Seifert surface \( S_n \) of \( \sigma_n^1 \) in \( \mathcal{T} \), the Poincaré-Lefschetz duality theorem ensures that the relative homology classes of the surfaces \( \{ S_n \}_{n=1}^g \) in \( \Omega \) modulo \( \partial \Omega \) form a basis of
In $\mathbb{H}$ we propose and analyze a very efficient algorithm that, given a 1-boundary $\gamma$ of $\mathcal{T}$, computes a homological Seifert surfaces of $\gamma$ in $\mathcal{T}$. Hence this algorithm allows the construction of a basis of $H_2(\mathbb{H}, \partial \Omega; \mathbb{Z})$ once we know a set of 1-boundaries $\sigma'_1, \ldots, \sigma'_g$ of $\mathcal{T}$ with supports contained in $\partial \Omega$ and whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$.

If $\partial \Omega$ is connected, an algorithm for the construction of such a set of 1-boundaries have been analyzed in $\mathbb{H}$. The first step is to construct a set of $2g$ 1-cycles $\{\gamma_i\}_{i=1}^{2g}$ of $\partial \Omega$ that are representatives of a basis of $H_1(\partial \Omega; \mathbb{Z})$. The second step is to compute $g$ linear combinations $\{\tilde{\sigma}_n = \sum_{i=1}^{2g} B_{n,i} \gamma_i\}_{n=1}^g$ of these 2g 1-cycles $\gamma_i$, whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of the homology group $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$. If $\partial \Omega$ is connected, the 1-cycles $\tilde{\sigma}_n$ of $\partial \Omega$ turn out to be also 1-boundaries of $\mathbb{H}$ so we can take $\sigma'_n = \tilde{\sigma}_n$ for $n = 1, \ldots, g$.

The authors extend to the case of a non connected boundary $\partial \Omega$ the construction of representatives of a basis of $H_1(\partial \Omega; \mathbb{Z})$ and then the construction of $g$ independent linear combinations of these 1-cycles that are representatives of a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$. But, being $\partial \Omega$ not connected, the elements of this basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ are not necessarily 1-boundaries of $\mathbb{H}$.

For instance in Figure $\mathbb{H}$ a) the domain $\Omega$ is an open solid torus with a coaxial smaller closed solid torus removed and the homology class of the 1-cycle $\tilde{\sigma}_1$ of $\partial \Omega$, represented by a continuous line, is different from zero in $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ (indeed it is a representative of an element of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$, but $\tilde{\sigma}_2$ is not a 1-boundary of $\mathbb{H}$). To obtain a 1-boundary $\sigma'_1$ of $\mathbb{H}$ homologous to $\tilde{\sigma}_1$ in $\mathbb{H}$, we need to add a 1-cycle $\sigma_1$ of $\partial \Omega$, like the one represented by the dotted line: $\sigma'_1 := \tilde{\sigma}_1 + \sigma_1$. Now $\sigma'_1$ is the boundary of the homological Seifert surface $S_1$ of $\mathbb{H}$ represented in Figure $\mathbb{H}$ a). Analogously the homology class of the 1-cycle $\tilde{\sigma}_2$ of $\partial \Omega$, represented by a continuous line in Figure $\mathbb{H}$ b), is different from zero in $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$. Together with $\tilde{\sigma}_1$ they are representatives of a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$. However $\tilde{\sigma}_2$ is not a 1-boundary of $\mathbb{H}$ and to obtain a 1-boundary $\sigma'_2$ of $\mathbb{H}$ homologous to $\tilde{\sigma}_2$ in $\mathbb{H}$, we need to add a 1-cycle $\sigma_2$ of $\partial \Omega$, like the one represented by the dotted line: $\sigma'_2 := \tilde{\sigma}_2 + \sigma_2$ is the boundary of the homological Seifert surface $S_2$ of $\mathbb{H}$ represented in Figure $\mathbb{H}$ b).

![Figure 1: The boundaries.](image)

The main theoretical result of this paper is, starting from a set of $2g$ 1-cycles of $\partial \Omega$ representing a basis of $H_1(\partial \Omega; \mathbb{Z})$, to identify a set of $g$ 1-boundaries of $\mathbb{H}$ whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$. This is done in Section 2. In Section 3 we make precise some implementation issues concerning the effective construction of the mentioned 1-boundaries. Moreover, for the sake of completeness, we briefly describe the algorithm for the construction of homological Seifert surfaces studied in $\mathbb{H}$. The main tool in both cases is the closed block dual
barycentric complex of a triangulation. Combining these two procedures we obtain an algorithm for the construction of a basis of $H_2(\Omega, \partial \Omega; \mathbb{Z})$. Finally in Section 4 we present some numerical results illustrating the robustness and efficiency of this geometrical approach. We include also some comparisons with the results obtained using the cohomology solver integrated in Gmsh.

2 The construction of the 1-boundaries

Let $\mathcal{T} = (V, E, F, K)$ be a finite triangulation of $\overline{\Omega}$ where $V$ is the set of vertices, $E$ the set of edges, $F$ the set of faces and $K$ the set of tetrahedra of $\mathcal{T}$. Let $\mathcal{T}_0 = (V_0, E_0, F_0)$ be the triangulation of $\partial \Omega$ induced by $\mathcal{T}$; namely we have that $V_0 = V \cap \partial \Omega$, $E_0$ is the set of edges of $\mathcal{T}$ with both vertices in $V_0$ and $F_0$ is the set of faces with all vertices in $V_0$.

As indicated in the introduction, if $\partial \Omega$ is connected, then the desired 1-boundaries $\sigma_m^g$ are constructed in [14]. More precisely, under this connectedness condition the authors construct 1-cycles $\sigma_1, \ldots, \sigma_g, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_g$ of $\mathcal{T}_0$ in such a way that their homology classes in $\mathcal{T}_0$ form a basis of $H_1(\partial \Omega; \mathbb{Z})$ and it holds:

- $\sigma_1, \ldots, \sigma_g$ bounds in $\mathbb{R}^3 \setminus \Omega$ and their homology classes in $\Omega$ form a basis of $H_1(\Omega; \mathbb{Z})$,
- $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_g$ bounds in $\Omega$ and their homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$.

By defining $\sigma_m^g := \sigma_m$ for every $m \in \{1, \ldots, g\}$, we are done. We now consider the more complicated case in which $\partial \Omega$ is not connected.

Let us recall some results from Section 6 of [2]. As we have said, $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ denote the connected components of $\partial \Omega$. By the Jordan separation theorem, each open subset $\mathbb{R}^3 \setminus \Gamma_r$ of $\mathbb{R}^3$ has two connected components, both having $\Gamma_r$ as boundary. Denote by $D_r$ the bounded connected component of $\mathbb{R}^3 \setminus \Gamma_r$ and by $g_r$ the first Betti number of its closure $\overline{D_r}$ in $\mathbb{R}^3$.

Rearranging the indices $r \in \{0, 1, \ldots, p\}$ if necessary, we can suppose that $\Gamma_0$ is the “external” component of $\partial \Omega$; namely, it holds: $\overline{\Omega} = \overline{D_0} \cup \bigcup_{r=1}^p D_r$ and hence $\mathbb{R}^3 \setminus \Omega = (\mathbb{R}^3 \setminus D_0) \cup \bigcup_{r=1}^p \overline{D_r}$. The reader reminds that $H_1(\partial \Omega; \mathbb{Z})$ is isomorphic to $\bigoplus_{r=0}^p H_1(\Gamma_r; \mathbb{Z})$, so we have that $2g = \sum_{r=0}^p 2g_r$ or, equivalently, $g = \sum_{r=0}^p g_r$.

For convenience, if $c$ is a 1-cycle of $\mathbb{R}^3$ with support contained in a subset $Z$ of $\mathbb{R}^3$, then we denote by $[c]_Z$ the homology class of $c$ in $Z$.

For every $r \in \{0, 1, \ldots, p\}$, $\partial D_r = \Gamma_r$ is connected, so, as we said above, we can construct 1-cycles $\{\sigma_{r,s}^g\}_{s=1}^{g_r}$ of $\mathcal{T}_0$ with support contained in $\Gamma_r$ such that:

$\{[\sigma_{r,1}]_{D_r}, \ldots, [\sigma_{r,g_r}]_{D_r}\}$ is a basis of $H_1(\overline{D_r}; \mathbb{Z})$,

$[\sigma_{r,s}]_{\mathbb{R}^3 \setminus D_r} = 0$ for every $s \in \{1, \ldots, g_r\}$.

It follows that

$\{[\sigma_{0,s}]_\Omega\}_{s=1}^{g_0} \cup \{[\sigma_{1,s}]_\Omega\}_{s=1}^{g_1} \cup \ldots \cup \{[\sigma_{p,s}]_\Omega\}_{s=1}^{g_p}$ is a basis of $H_1(\overline{\Omega}; \mathbb{Z})$.
for every $r \in P$. Similarly, for every $r \in P$ and for every $s \in \{1, \ldots, g_r\}$, the support of $\sigma_{r,s}$ is contained in $\Gamma_r \subset \Omega_r$. In this way, thanks to \((\ref{eq:assertion6})\), there exist, and are unique, integers $\{a_{i,j}^{0,s}\}_{i,j}$ such that

$$[\sigma_{0,s}]_{\Omega_0} = \sum_{i \in P} \sum_{j=1}^{g_{i,s}} a_{i,j}^{0,s} [\sigma_{i,j}]_{\Omega_0}.$$  \hfill (11)

Similarly, for every $r \in P$ and for every $s \in \{1, \ldots, g_r\}$, the support of $\sigma_{r,s}$ is contained in $\Gamma_r \subset \Omega_r$. In this way, thanks to \((\ref{eq:assertion6})\), there exist, and are unique, integers $\{a_{i,j}^{0,s}\}_{i,j}$ such that

$$[\sigma_{r,s}]_{\Omega_r} = \sum_{j=1}^{g_{0,r}} a_{0,j}^{r,s} [\sigma_{0,j}]_{\Omega_r} + \sum_{i \in P} \sum_{j=1}^{g_{i,s}} a_{i,j}^{r,s} [\sigma_{i,j}]_{\Omega_r}.$$  \hfill (12)

Define the 1-cycles $\{\hat{\sigma}_{0,s}\}_{s=1}^{g_0} \cup \{\sigma_{1,s}'\}_{s=1}^{g_1} \cup \ldots \cup \{\sigma_{p,s}'\}_{s=1}^{g_p}$ of $\mathcal{T}_0$ by setting

$$\hat{\sigma}_{0,s} := \sum_{i \in P} \sum_{j=1}^{g_{i,s}} a_{i,j}^{0,s} \sigma_{i,j}.$$  \hfill (13)
for every \( s \in \{1, \ldots, g_r\} \), and

\[
\sigma'_{r,s} := \sigma_{r,s} - \sum_{j=1}^{g_0} a_{r,j} \sigma_{0,j} - \sum_{i \in P_r} \sum_{j=1}^{g_0} a_{i,j} \tilde{\sigma}_{i,j}
\]

(14)

for every \( r \in P \) and for every \( s \in \{1, \ldots, g_r\} \).

**Theorem 1.** The 1-cycles of \( T_0 \) defined in (13) and in (14) have the following properties:

1. They are 1-boundaries of \( T \); namely, their homology classes in \( \overline{\Omega} \) are null.

2. \([\tilde{\sigma}'_{0,s}]_{\mathbb{R}^3 \setminus \Omega} = [\tilde{\sigma}_{0,s}]_{\mathbb{R}^3 \setminus \Omega} \) for every \( s \in \{1, \ldots, g_0\} \) and \([\sigma'_{r,s}]_{\mathbb{R}^3 \setminus \Omega} = [\sigma_{r,s}]_{\mathbb{R}^3 \setminus \Omega} \) for every \( r \in \{1, \ldots, p\} \) and for every \( s \in \{1, \ldots, g_r\} \). In particular, the set

\[
\left\{ [\tilde{\sigma}'_{0,s}]_{\mathbb{R}^3 \setminus \Omega}\right\}_{s=1}^{g_0} \cup \left\{ [\tilde{\sigma}'_{1,s}]_{\mathbb{R}^3 \setminus \Omega}\right\}_{s=1}^{g_0} \cup \ldots \cup \left\{ [\tilde{\sigma}'_{p,s}]_{\mathbb{R}^3 \setminus \Omega}\right\}_{s=1}^{g_0}
\]

is a basis of \( H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z}) \).

3. Let \( S_{0,s} \) be a homological Seifert surface of \( \tilde{\sigma}'_{0,s} \) for every \( s \in \{1, \ldots, g_0\} \) and let \( S_{r,s} \) be a homological Seifert surface of \( \sigma'_{r,s} \) for every \( r \in \{1, \ldots, p\} \) and for every \( s \in \{1, \ldots, g_r\} \). Then the homology classes of such surfaces \( \{S_{r,s}\}_{r \in \{0,1,\ldots,p\}, s \in \{1,\ldots,g_r\}} \) in \( \overline{\Omega} \) modulo \( \partial \Omega \) form a basis of \( H_2(\overline{\Omega}, \partial \Omega; \mathbb{Z}) \).

**Proof.** (1) Let \( s \in \{1, \ldots, g_0\} \). We must prove that \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}} = 0 \). Observe that \( \mathbb{R}^3 = D_0 \cup \overline{\Omega}_0 \) and \( \overline{\Omega} = D_0 \cap \overline{\Omega}_0 \). In this way, the Mayer-Vietoris exact sequence associated with the splitting \( \mathbb{R}^3 = D_0 \cup \overline{\Omega}_0 \) implies that the following inclusion homomorphism is an isomorphism:

\[
i_*, j_* : H_1(\overline{\Omega}_0; \mathbb{Z}) \longrightarrow H_1(D_0; \mathbb{Z}) \oplus H_1(\overline{\Omega}_0; \mathbb{Z}),
\]

where \( i_* \) and \( j_* \) are the homomorphisms induced by the inclusions \( i : \overline{\Omega}_0 \hookrightarrow \mathbb{R}^3 \) and \( j : \overline{\Omega}_0 \hookrightarrow \mathbb{R}^3 \), respectively. It follows that \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}} = 0 \) if and only if \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}_0} = i_*(\tilde{\sigma}'_{0,s})_{\mathbb{R}^3} = 0 \) and \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}_0} = j_*(\tilde{\sigma}'_{0,s})_{\mathbb{R}^3} = 0 \). By (11) and (13), we have that \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}_0} = 0 \). Since \( D_r \subset D_0 \) for every \( r \in P \), equality (4) ensures that \([\tilde{\sigma}'_{i,j}]_{\overline{\Omega}} = 0 \) for every \( i \in P \) and for every \( j \in \{1, \ldots, g_i\} \). In this way, by (13), we infer that \([\tilde{\sigma}'_{i,j}]_{\overline{\Omega}} = 0 \). This proves that \([\tilde{\sigma}'_{0,s}]_{\overline{\Omega}} = 0 \), as desired.

For any given \( r \in P \) and \( s \in \{1, \ldots, g_r\} \), the proof of the fact that \([\tilde{\sigma}'_{r,s}]_{\overline{\Omega}} = 0 \) is similar. One must consider the Mayer-Vietoris sequence associated with splitting \( \mathbb{R}^3 = (\mathbb{R}^3 \setminus D_r) \cup \overline{\Omega}_0 \), points (12) and (14), and the inclusions \( \mathbb{R}^3 \setminus D_0 \subset \mathbb{R}^3 \setminus D_r \) and \( \overline{\Omega}_0 \subset \mathbb{R}^3 \setminus D_r \) for every \( i \in P_r \), together with equalities (2) and (4).

(2) Since \( \mathbb{R}^3 \setminus D_0 \subset \mathbb{R}^3 \setminus \Omega \) and \( \overline{\Omega}_0 \subset \mathbb{R}^3 \setminus \Omega \) for every \( i \in P \), equalities (2) and (4) imply that \([\sigma_{0,j}]_{\mathbb{R}^3 \setminus \Omega} = 0 \) for every \( j \in \{1, \ldots, g_0\} \) and \([\tilde{\sigma}_{i,j}]_{\mathbb{R}^3 \setminus \Omega} = 0 \) for every \( i \in P \) and for every \( j \in \{1, \ldots, g_i\} \). By (13) and (14), we have that \([\tilde{\sigma}'_{0,s}]_{\mathbb{R}^3 \setminus \Omega} = [\tilde{\sigma}_{0,s}]_{\mathbb{R}^3 \setminus \Omega} \) for every \( s \in \{1, \ldots, g_0\} \) and \([\tilde{\sigma}'_{r,s}]_{\mathbb{R}^3 \setminus \Omega} = [\tilde{\sigma}_{r,s}]_{\mathbb{R}^3 \setminus \Omega} \) for every \( r \in P \) and for every \( s \in \{1, \ldots, g_r\} \). This proves the first part of (2). The second part of (2) now follows immediately from (4).

(3) The existence of the homological Seifert surfaces \( S_{r,s} \) is equivalent to (1). Point (3) is a direct consequence of the second part of (2) and of the Poincaré-Lefschetz duality theorem. \( \square \)

We conclude this section by computing the coefficients \( a_{r,j}^{r',s} \). To do it we need to recall the notion of linking number and some properties that will be useful in the sequel. See, e.g., Rolfsen [24] pp. 132–136], Seifert and Threlfall [25] Sects. 70, 73, 77]. The linking number is
an integer that, given two 1-cycles $\gamma$ and $\eta$ of $\mathbb{R}^3$ with disjoint supports; namely, $|\gamma| \cap |\eta| = \emptyset$, represents the number of times that each curve winds around the other. A possible geometric way to give a rigorous definition is as follows. Choose a homological Seifert surface $S_\eta = \sum_{q=1}^{k} b_q f_q$ of $\eta$ in $\mathbb{R}^3$. It is well-known (and easy to see) that there exists a 1-cycle $\hat{\gamma} = \sum_{p=1}^{h} \hat{\omega}_p \hat{e}_p$ homologous to $\gamma$ in $\mathbb{R}^3 \setminus |\eta|$ (and “arbitrarily close to $\gamma$” if necessary), which is transverse to $S_\eta$ in the following sense: for every $p \in \{1, \ldots, h\}$ and for every $q \in \{1, \ldots, k\}$, the intersection $|\hat{\omega}_p| \cap |f_q|$ is either empty or consists of a single point, which does not belong to $|\partial \hat{\omega}_p| \cup |\partial f_q|$.

For every $p \in \{1, \ldots, h\}$ and for every $q \in \{1, \ldots, k\}$, define $L_{pq} := 0$ if $|\hat{\omega}_p| \cap |f_q| = \emptyset$ and $L_{pq} := \text{sign}(\tau(\hat{\omega}_p) \cdot \nu(f_q))$ otherwise. The linking number $\ell_\kappa(\gamma, \eta)$ between $\gamma$ and $\eta$ is the integer defined as follows:

$$\ell_\kappa(\gamma, \eta) := \sum_{p=1}^{h} \sum_{q=1}^{k} b_p b_q L_{pq}.$$  

This definition is well-posed: it depends only on $\gamma$ and $\eta$, not on the choice of $S_\eta$ and of $\hat{\gamma}$.

The linking number is symmetric $\ell_\kappa(\gamma, \eta) = \ell_\kappa(\eta, \gamma)$, and bilinear $\ell_\kappa(a\gamma, \eta) = a \ell_\kappa(\gamma, \eta)$ for every $a \in \mathbb{Z}$ and, if $\gamma^* \in Z_1(\mathbb{R}^3; \mathbb{Z})$ with $|\gamma^*| \cap |\eta| = \emptyset$, $\ell_\kappa(\gamma + \gamma^*, \eta) = \ell_\kappa(\gamma, \eta) + \ell_\kappa(\gamma^*, \eta)$.

The linking number is a homological invariant in the following sense: if a 1-cycle $\gamma^*$ of $\mathbb{R}^3$ is homologous to $\gamma$ in $\mathbb{R}^3 \setminus |\eta|$, then

$$\ell_\kappa(\gamma, \eta) = \ell_\kappa(\gamma^*, \eta).$$

In particular, we have:

$$\ell_\kappa(\gamma, \eta) = 0 \text{ if } \gamma \text{ bounds in } \mathbb{R}^3 \setminus |\eta|. \quad (17)$$

The linking number can be used to recognize 1-boundaries of $\mathcal{T}$ among 1-cycles of $\mathcal{T}$. This is possible by the Alexander duality theorem. Indeed, such a theorem ensures that $H_1(\mathbb{R}^3 \setminus \overline{\Omega}; \mathbb{Z})$ is isomorphic to $H_1(\overline{\Omega}; \mathbb{Z})$, and hence to $\mathbb{Z}^g$ if $g$ is the first Betti number of $\overline{\Omega}$. Furthermore, if $\sigma_1^*, \ldots, \sigma_g^*$ are 1-cycles of $\mathbb{R}^3$ with support in $\mathbb{R}^3 \setminus \overline{\Omega}$ whose homology classes in $\mathbb{R}^3 \setminus \overline{\Omega}$ form a basis of $H_1(\mathbb{R}^3 \setminus \overline{\Omega}; \mathbb{Z})$, then it holds:

a 1-cycle $\sigma$ of $\mathcal{T}$ is a 1-boundary of $\mathcal{T}$ if and only if $\ell_\kappa(\sigma, \sigma_i^*) = 0$ for every $i \in \{1, \ldots, g\}$.

For this topic, we refer the reader to [6] and to the references mentioned therein.

The linking number can be computed via a double integral:

$$\ell_\kappa(\gamma, \eta) = \frac{1}{4\pi} \oint_{\gamma} \left( \oint_{\eta} \frac{y - x}{|y - x|^3} \times ds(y) \right) \cdot ds(x). \quad (18)$$

For an efficient computation of the linking number see e.g. [4].

For the computation of the coefficients $\alpha_i^r$ we will use also the fact that since $\partial \Omega$ has a collar in $\mathbb{R}^3 \setminus \Omega$, there exist 1-cycles $\{ \hat{\sigma}_{0,s} \}_{s=0}^{g_0} \cup \{ \hat{\sigma}_{1,s} \}_{s=1}^{g_1} \cup \ldots \cup \{ \hat{\sigma}_{r,s} \}_{s=1}^{g_r}$ of $\mathbb{R}^3$ with support contained in $\mathbb{R}^3 \setminus \overline{\Omega}$ (obtained by slightly retracting the 1-cycles $\{ \sigma_{0,s} \}_{s=0}^{g_0} \cup \{ \sigma_{1,s} \}_{s=1}^{g_1} \cup \ldots \cup \{ \sigma_{r,s} \}_{s=1}^{g_r}$ of $\mathcal{T}_\partial$ inside $\mathbb{R}^3 \setminus \overline{\Omega}$) such that $[\hat{\sigma}_{0,s}]_{\mathbb{R}^3 \setminus \Omega} = [\sigma_{0,s}]_{\mathbb{R}^3 \setminus \Omega}$ for every $s \in \{1, \ldots, g_0\}$ and $[\sigma_{r,s}]_{\mathbb{R}^3 \setminus \Omega} = [\sigma_{r,s}]_{\mathbb{R}^3 \setminus \Omega}$ for every $r \in P$ and for every $s \in \{1, \ldots, g_r\}$. In particular, thanks to [9] and [10], we infer that

$$\bigcup_{i \in P} \{ [\sigma_{i,s}]_{\mathbb{R}^3 \setminus \overline{\Omega}} \}_{s=1}^{g_i} \text{ is a basis of } H_1(\mathbb{R}^3 \setminus \overline{\Omega}; \mathbb{Z}) \quad (19)$$
\[ \{ \hat{\sigma}_{0,i} \} \in \mathbb{R}^3 \setminus \{ \Gamma_r \} \] is a basis of \( H_1(\mathbb{R}^3 \setminus \{ \Gamma_r \}; \mathbb{Z}) \) for every \( r \in P \).

For every \( k, i \in \{ 0, 1, \ldots, p \} \), define the \((g_k \times g_i)\)-matrix \( A_{k,i} \) as follows:

\[
\begin{align*}
A_{0,0} & := (\ell_k(\hat{\sigma}_{0,i}, \sigma_0,j))_{i,j} \in \mathbb{Z}^{g_k \times g_0}, \\
A_{0,i} & := (\ell_k(\hat{\sigma}_{0,i}, \sigma_0,j))_{i,j} \in \mathbb{Z}^{g_k \times g_i} \quad \text{if} \ i \in P, \\
A_{k,0} & := (\ell_k(\sigma_k,i, \sigma_0,j))_{i,j} \in \mathbb{Z}^{g_k \times g_0} \quad \text{if} \ k \in P, \\
A_{k,i} & := (\ell_k(\sigma_k,i, \sigma_0,j))_{i,j} \in \mathbb{Z}^{g_k \times g_i} \quad \text{if} \ k, i \in P \text{ and } k \neq i, \\
A_{k,k} & := (\ell_k(\sigma_k,i, \hat{\sigma}_{k,j}))_{i,j} \in \mathbb{Z}^{g_k \times g_k} \quad \text{if} \ k \in P.
\end{align*}
\]

**Lemma 2.** For every \( k, i \in \{ 1, \ldots, p \} \) the matrices \( A_{0,i} \) and \( A_{k,0} \) are equal to zero, and if \( k \neq i \) then also the matrix \( A_{k,i} \) is equal to zero.

**Proof.** First we notice that for any \( l \in \{ 1, \ldots, g_k \} \) the support of the 1-cycle \( \sigma_k,i \) is contained in \( \Gamma_k \) while for any \( j \in \{ 1, \ldots, g_i \} \) the support of the 1-cycle \( \hat{\sigma}_{i,j} \) is contained in \( \Gamma_i \). Hence, if \( k \neq l \) then the 1-cycles \( \sigma_k,i \) and \( \hat{\sigma}_{i,j} \) are disjoint and \( \ell_k(\sigma_k,i, \hat{\sigma}_{i,j}) = \ell_k(\sigma_k,i, \sigma_k,i) \) is well defined. Moreover \( \ell_k(\sigma_k,i, \sigma_0,j) = \ell_k(\sigma_k,i, \sigma_0,j) \) and \( \ell_k(\sigma_k,i, \hat{\sigma}_{i,j}) = \ell_k(\sigma_k,i, \hat{\sigma}_{i,j}) \).

Now it is not difficult to see that \( A_{0,i} = 0 \) if \( i \in P \); because for any \( j \in \{ 1, \ldots, g_i \} \) we have \( \hat{\sigma}_{i,j} = \partial_2 S_{i,j} \subset \{ \hat{\sigma}_{0,i} \} \subset \Gamma_k \) while for any \( l \in \{ 1, \ldots, g_k \} \), \( |\hat{\sigma}_{0,i}| \subset \Gamma_0 \). Since \( \Gamma_0 \cap \bigcup \{ \Gamma_i \} = \emptyset \) if \( i \in P \), then \( A_{0,i} = \ell_k(\hat{\sigma}_{0,i}, \sigma_0,j) = 0 \). Analogously \( A_{k,0} = 0 \) if \( k \in P \) because for any \( j \in \{ 1, \ldots, g_k \} \), \( |\sigma_k,i| \subset \Gamma_k \), and \( |\sigma_k,i| \subset \Gamma_k \). Again we have \( \Gamma_k \cap \mathbb{R}^3 \setminus D_0 = \emptyset \) if \( k \in P \) and then \( A_{k,0} = \ell_k(\sigma_k,i, \hat{\sigma}_{0,i}) = 0 \). Finally \( A_{k,i} = 0 \) if \( k, i \in P \) and \( k \neq i \) because for any \( j \in \{ 1, \ldots, g_k \} \), \( \hat{\sigma}_{i,j} = \partial_2 S_{i,j} \subset \{ \hat{\sigma}_{0,i} \} \), for any \( l \in \{ 1, \ldots, g_i \} \), \( |\sigma_k,i| \subset \Gamma_k \) and \( |\sigma_k,i| \subset \Gamma_k \).

**Computation of the coefficients** \((a_{k,i}^{0,s})_{i,j} \) **for** \( s \in \{ 1, \ldots, g_i \} \)

Let \( G_o := \sum_{i \in P} g_i = g - g_0 \) and let \( A_{(0)} \) be the diagonal block matrix with blocks \((A_{k,k})_{k \in P} \in \mathbb{Z}^{G_o \times G_o} \). It is important to observe that the entries of \( A_{(0)} \) are the linking numbers between the representatives of a basis of \( H_1(\mathbb{R}^3 \setminus \{ \Gamma_r \}; \mathbb{Z}) \) (see [13]) and the representatives of a basis of \( H_1(\{ \Gamma_r \}; \mathbb{Z}) \) (see [7]). In this way, the Alexander duality theorem applied to \( \hat{\Omega} \) ensures that

\[
\det (A_{(0)}) = 1.
\] (21)

Define the row vectors \( \alpha_i^{0,s} := (\alpha_{i,1}^{0,s}, \ldots, \alpha_{i,p}^{0,s}) \) and \( \beta_i^{0,s} := (\ell_k(\sigma_{i,1}, \hat{\sigma}_{0,s}), \ldots, \ell_k(\sigma_{i,p}, \hat{\sigma}_{0,s})) \) for every \( i \in P \), and the column vectors

\[
\alpha^{0,s} := (\alpha_1^{0,s}, \ldots, \alpha_p^{0,s})^T \in \mathbb{Z}^{G_o} \quad \text{and} \quad \beta^{0,s} := (\beta_1^{0,s}, \ldots, \beta_p^{0,s})^T \in \mathbb{Z}^{G_o},
\]

where the superscript \(^T\) denotes the transpose operation.

Bearing in mind the linearity of linking number and its homological invariance, equation (11) implies that

\[
\ell_k(\sigma_{k,h}, \hat{\sigma}_{0,s}) = \sum_{i \in P} \sum_{j=1}^{g_i} \alpha_{i,j}^{0,s} \ell_k(\sigma_{i,h}, \hat{\sigma}_{i,j}) \quad \text{if} \ k \in P \text{ and } h \in \{ 1, \ldots, g_k \}.
\] (22)
Linear system (22) in the unknowns \((\alpha_{i,j}^{r,s})_{i,j}\) can be rewritten in the following compact form:

\[
A_{(0)}\alpha^{0,s} = \beta^{0,s}, \tag{23}
\]

where \(\alpha^{0,s}\) is the unknown. Thanks to (21), equation (11) is equivalent to (23).

In this way, we conclude that the coefficients \((\alpha_{i,j}^{r,s})_{i,j}\) can be computed by solving linear system (23), namely solving \(p\) linear systems each one of dimension \(g_r, r = 1, \ldots, p\).

**Computation of the coefficients \((\alpha_{i,j}^{r,s})_{i,j}\) for \(r \in P\) and \(s \in \{1, \ldots, g_r\}\).**

Given \(k \in P\), we define the integer \(k_r \in \{0, 1, \ldots, p\} \setminus \{r\}\) by setting \(k_r := k - 1\) if \(k \leq r\) and \(k_r := k\) if \(k > r\). Let \(G_r := \sum_{i \in P_r} g_i = g - g_r\) and let \(A_{(r)}\) be the diagonal block matrix 
\[
(A_{k_r,i_r})_{k,i \in P} \in \mathbb{Z}^{G_r \times G_r}.
\]

By applying the Alexander duality theorem to \(\Pi_r\) (see (20) and (8)), we obtain:

\[
\det (A_{(r)}) = 1. \tag{24}
\]

Define the row vectors \(\alpha_0^{r,s} := (\alpha_{0,1}^{r,s}, \ldots, \alpha_{0,g_0}^{r,s})\), \(\beta_0^{r,s} := (\ell_k(\sigma_{0,1}, \sigma_{r,s}), \ldots, \ell_k(\sigma_{0,g_0}, \sigma_{r,s}))\) and, for every \(i \in P_r\), \(\alpha_i^{r,s} := (\alpha_{i,1}^{r,s}, \ldots, \alpha_{i,g_i}^{r,s})\) and \(\beta_i^{r,s} := (\ell_k(\sigma_{i,1}, \sigma_{r,s}), \ldots, \ell_k(\sigma_{i,g_i}, \sigma_{r,s}))\). Define also the column vectors

\[
\alpha^{r,s} := (\alpha_0^{r,s}, \alpha_1^{r,s}, \ldots, \alpha_{r-1}^{r,s}, \alpha_r^{r,s}, \alpha_{r+1}^{r,s}, \ldots, \alpha_p^{r,s})^T \in \mathbb{Z}^{G_r}
\]

and

\[
\beta^{r,s} := (\beta_0^{r,s}, \beta_1^{r,s}, \ldots, \beta_{r-1}^{r,s}, \beta_r^{r,s}, \beta_{r+1}^{r,s}, \ldots, \beta_p^{r,s})^T \in \mathbb{Z}^{G_r}
\]

By using equation (12) and the linking number, we infer that

\[
\ell_k(\sigma_{0,h}, \sigma_{r,s}) = \sum_{j=1}^{g_0} \alpha_{0,j}^{r,s} \ell_k(\sigma_{0,j}, \sigma_{r,s}) + \sum_{i \in P_r} \sum_{j=1}^{g_i} \alpha_{i,j}^{r,s} \ell_k(\sigma_{i,j}, \sigma_{r,s}) \tag{25}
\]

if \(h \in \{1, \ldots, g_0\}\) and

\[
\ell_k(\sigma_{k,h}, \sigma_{r,s}) = \sum_{j=1}^{g_0} \alpha_{0,j}^{r,s} \ell_k(\sigma_{k,h}, \sigma_{0,j}) + \sum_{i \in P_r} \sum_{j=1}^{g_i} \alpha_{i,j}^{r,s} \ell_k(\sigma_{k,h}, \sigma_{i,j}) \tag{26}
\]

if \(k \in P_r\) and \(h \in \{1, \ldots, g_k\}\). Equations (25) and (26) can be rewritten as follows:

\[
A_{(r)}\alpha^{r,s} = \beta^{r,s}. \tag{27}
\]

Also in this case, for each \(r \in P\) matrix \(A_{(r)}\) is block diagonal. Thanks to (24), equation (12) and linear system (27) are equivalent. Once again, we conclude that the coefficients \((\alpha_{i,j}^{r,s})_{i,j}\) can be computed by resolving linear system (27).

### 3 Homological issues for implementation

Given two different points \(a, b\) in \(\mathbb{R}^3\), we denote by \([a, b]\) the oriented segment of \(\mathbb{R}^3\) from \(a\) to \(b\). The segment of \(\mathbb{R}^3\) of vertices \(a, b\) is called support of \([a, b]\) and it is denoted by \(|[a, b]|\). The unit tangent vector \(\tau([a, b])\) of the oriented segment \([a, b]\) is given by \(\tau([a, b]) := \frac{b - a}{|b - a|}\). The
barycenter of \( e = [a, b] \) is the point of \( \mathbb{R}^3 \), \( B(e) = (a + b)/2 \). A (piecewise linear) 1-chain of \( \mathbb{R}^3 \) is a finite formal linear combination \( \sum_{i=1}^m \alpha_i e_i \) of oriented segments \( e_i = [a_i, b_i] \) of \( \mathbb{R}^3 \) with integer coefficients \( \alpha_i \). We denote by \( C_1(\mathbb{R}^3; \mathbb{Z}) \) the abelian group of 1-chains in \( \mathbb{R}^3 \).

Analogously, if \( a, b, c \) are three different not aligned points in \( \mathbb{R}^3 \), we denote by \([a, b, c] \) the oriented triangle of \( \mathbb{R}^3 \). The triangle of \( \mathbb{R}^3 \) of vertices \( a, b, c \) is called support of \([a, b, c] \) and it is denoted by \([a, b, c] \). The unit normal vector \( \nu([a, b, c]) \) of the oriented triangle \([a, b, c] \) is obtained by the right hand rule: \( \nu([a, b, c]) := \frac{(b-a) \times (c-a)}{|(b-a) \times (c-a)|} \). The barycenter of \( f = [a, b, c] \) is the point of \( \mathbb{R}^3 \), \( B(f) = (a + b + c)/3 \). A (piecewise linear) 2-chain of \( \mathbb{R}^3 \) is a finite formal linear combination \( \sum_{i=1}^n \beta_i f_i \) of oriented triangles \( f_i = [a_i, b_i, c_i] \) of \( \mathbb{R}^3 \) with integer coefficients \( \beta_i \). We denote by \( C_2(\mathbb{R}^3; \mathbb{Z}) \) the abelian group of 2-chains in \( \mathbb{R}^3 \).

Finally, if \( a, b, c, d \) are four different not coplanar points in \( \mathbb{R}^3 \), we denote by \([a, b, c, d] \) the oriented tetrahedron of \( \mathbb{R}^3 \). The tetrahedron of \( \mathbb{R}^3 \) of vertices \( a, b, c, d \) is called support of the oriented tetrahedron \([a, b, c, d] \) and it is denoted by \([a, b, c, d] \). The barycenter of \( t = [a, b, c, d] \) is the point of \( \mathbb{R}^3 \), \( B(t) = (a + b + c + d)/4 \). A (piecewise linear) 3-chain of \( \mathbb{R}^3 \) is a finite formal linear combination \( \sum_{i=1}^{n} \delta_i t_i \) of oriented tetrahedra \( t_i = [a_i, b_i, c_i, d_i] \) of \( \mathbb{R}^3 \) with integer coefficients \( \delta_i \). We denote by \( C_3(\mathbb{R}^3; \mathbb{Z}) \) the abelian group of 3-chains in \( \mathbb{R}^3 \).

We indicate by \( E, F \) and \( K \) the sets of oriented edges, oriented faces and oriented tetrahedra of \( T \), respectively.

Let us recall the definitions of dual vertices, dual edges and dual faces of \( T \). We equip the dual edges and the dual faces with the natural orientation induced by the right hand rule.

- For every tetrahedron \( t \in K \), the dual vertex \( D(t) \) of \( T \) associated with \( t \) is defined as the barycenter of \( t \): \( D(t) := B(t) \).

We denote by \( V' \) the set \( \{D(t) \in \mathbb{R}^3 \mid t \in K \} \) of all dual vertices of \( T \).

- For every oriented face \( f = [v, w, y] \in F \), the oriented dual edge \( D(f) \) of \( T \) associated with \( f \) is the element of \( C_1(\mathbb{R}^3; \mathbb{Z}) \) defined as follows: if \( K(f) \) denotes the set \( \{t \in K \mid [v, w, y] \subseteq t \} \); namely, the set of tetrahedra of \( T \) incident on \( f \), we set

\[
D(f) := \sum_{t \in K(f)} \text{sign}(\nu(f) \cdot \tau([B(f), B(t)])) \cdot [B(f), B(t)],
\]

where \( \text{sign}: \mathbb{R} \setminus \{0\} \rightarrow \{-1, 1\} \) denotes the function given by \( \text{sign}(s) := -1 \) if \( s < 0 \) and \( \text{sign}(s) := 1 \) otherwise.

\( D(f) \) can be described as follows. If the (oriented) face \( f \) is internal, then \( f \) is the common face of two tetrahedra \( t_1 \) and \( t_2 \) of \( T \), and the support of \( D(f) \) is the union of the segment joining \( B(f) \) with \( B(t_1) \) and of the segment joining \( B(f) \) and \( B(t_2) \). If \( f \) is a boundary face, then \( f \) is face of just one tetrahedron \( t \), and the support of \( D(f) \) is the segment joining \( B(f) \) with \( B(t) \). In both cases, \( D(f) \) is endowed with the orientation induced by \( f \) via the right hand rule.

We denote by \( E' \) the set \( \{D(f) \in C_1(\mathbb{R}^3; \mathbb{Z}) \mid f \in F \} \) of all oriented dual edges of \( T \).

- For every oriented edge \( e = [v, w] \in E \), the oriented dual face \( D(e) \) of \( T \) associated with \( e \) is the element of \( C_2(\mathbb{R}^3; \mathbb{Z}) \) defined as follows: if \( F(e) \) denotes the set \( \{f \in F \mid [v, w] \subseteq f \} \); namely, the set of faces of \( T \) incident on \( e \), then we set

\[
D(e) := \sum_{f \in F(e)} \sum_{t \in K(f)} \text{sign}(\tau(e) \cdot \nu([B(e), B(f), B(t)])) \cdot [B(e), B(f), B(t)].
\]
The reader observes that the support of $D(e)$ is the union of triangles of $\mathbb{R}^3$ with vertices $B(e), B(f)$, and $B(t)$, where $f$ varies in $F(e)$ and $t$ in $K(f)$. Such triangles are oriented by $e$ via the right hand rule.

We denote by $F'$ the set of oriented dual faces of $T$. The preceding three definitions determine the bijection $D : K \cup F \cup E \rightarrow V' \cup E' \cup F'$ such that $D(K) = V'$, $D(F) = E'$, and $D(E) = F'$.

We need also to describe the closed block dual barycentric complex of the triangulation $T_\partial$ of $\partial \Omega$ induced by $T$. Recall that $V_\partial$, $E_\partial$, and $F_\partial$ denote the sets of vertices, oriented edges and oriented faces of $T_\partial$, respectively.

- For every oriented face $f \in F_\partial$, the dual vertex $D_\partial(f)$ of $T_\partial$ associated with $f$ is defined as the barycenter of $f$: $D_\partial(f) := B(f)$.

We denote by $V'_\partial$ the set of all dual vertices of $T_\partial$.

- For every oriented edge $e \in E_\partial$, the oriented dual edge $D_\partial(e)$ of $T_\partial$ associated with $e$ is the element of $C_1(\mathbb{R}^3; \mathbb{Z})$ defined as follows. Let $f_1$ and $f_2$ be the oriented faces in $F_\partial$ incident on $e$, and let $n(f_1)$ and $n(f_2)$ be the outward unit normals of $\partial \Omega$ at $B(f_1)$ and at $B(f_2)$, respectively. Then we set

$$
D_\partial(e) := \sum_{i=1}^{2} \text{sign}(\tau(e) \cdot (n(f_i) \times \tau([B(e), B(f_i)]))) [B(e), B(f_i)].
$$

$D_\partial(e)$ can be described as follows. By interchanging $f_1$ with $f_2$ if necessary, we can suppose that $f_1$ is on the left of $e$ and $f_2$ on the right of $e$ with respect to the orientation of $\partial \Omega$ induced by its outward unit vector field. Then we have:

$$
D_\partial(e) = [B(f_1), B(e)] + [B(e), B(f_2)].
$$

We denote by $E'_\partial$ the set of all oriented dual edges of $T_\partial$.

- For every $v \in V_\partial$, the oriented dual face $D_\partial(v)$ of $T_\partial$ associated with $v$ is the element of $C_2(\mathbb{R}^3; \mathbb{Z})$ defined as follows. If $E_\partial(v)$ denotes the set of edges in $E_\partial$ incident on $v$ and, for any edge $e \in E_\partial$, $F_\partial(e)$ denotes the set of oriented faces in $\partial \Omega$ incident in $e$ then

$$
D_\partial(v) = \sum_{e \in E_\partial(v)} \sum_{f \in F_\partial(e)} \text{sign}(n(f) \cdot \nu([v, B(e), B(f)])) [v, B(e), B(f)],
$$

being $n(f)$ the outward unit normal vector of $\partial \Omega$ at $B(f)$.

We denote by $F'_\partial$ the set of all dual faces of $T_\partial$.

### 3.1 Construction of the retraction

From the computational point of view, in order to construct $g$ 1-boundaries with supports contained in $\partial \Omega$ whose homology classes in $\mathbb{R}^3 \setminus \Omega$ form a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ is more convenient
to construct 1-cycles \( \{ \sigma_0^+ \}_{j=0}^g \cup \{ \sigma_0^- \}_{j=0}^g \cup \cdots \cup \{ \sigma_p^+ \}_{j=0}^g \) of \( \mathbb{R}^3 \) with support contained in \( \Omega \) that are a retraction of the cycles \( \{ \sigma_0^+ \}_{j=0}^g \cup \{ \sigma_0^- \}_{j=0}^g \cup \cdots \cup \{ \sigma_p^+ \}_{j=0}^g \) of \( \partial \Omega \) such that 

\[
\beta_k(\sigma_0^+, \sigma_0^-) = \beta_k(\sigma_0^+, \sigma_0^-) \quad \text{and} \quad \beta_k(\sigma_k^+, \sigma_k^-) = \beta_k(\sigma_k^+, \sigma_k^-).
\]

Let us see how to compute such a retraction. We can consider just the case of a simple loop \( \eta \) with \( |\eta| \subset \partial \Omega \). Then for each \( v \in |\eta| \) there exist exactly two oriented edges \( e_p = [v_p, v] \) and \( e_s = [v, v_s] \) such that the coefficients of \( e_p \) and \( e_s \) in \( \eta \) are both equal one.

For each vertex \( v \in V_0 \), \( F_0(v) \) denotes the set of oriented faces in \( \partial \Omega \) incident in \( v \). Then if \( v \in |\eta| \) we denote \( \text{left}(v, \eta) \) the faces \( f \in F_0(v) \) that are on the left with respect to \( \eta \). More precisely, denoting by \( V(v) = \{ w \in V_0 \mid [w, v] \in E_0 \} \) we sort the vertices in \( V(v) \) in the following way: we set \( w_0 = v_p \) and for \( m > 0 \), \( w_m \) is the unique element of \( V(v) \) such that \( \nu([w_{m-1}, v, w_m]) \) coincides with the outward unit normal of \( \partial \Omega \) at these face. Clearly there exists \( m^* \geq 1 \) such that \( w_{m^*} = v_s \). We define

\[
\text{left}(v, \eta) := \{ f \in F_0 \mid |f| = [w_{m-1}, v, w_m] \} \text{ for some } m \in \{1, \ldots, m^*\}.
\]

Then we define \( \text{fun}(v, \eta) \) the 2-chain

\[
\text{fun}(v, \eta) = \sum_{e \in E_0(v)} \sum_{f \in F_0(v) \cap \text{left}(v, \eta)} \text{sign}(n(f)) \cdot \nu([v, B(e), B(f)])[v, B(e), B(f)],
\]

namely, the subchain of \( D_0(v) \) with support on the left of \( \eta \).

First we replace \( \eta \) with \( \tilde{\eta} = \eta - \partial_2 \left( \sum_{v \in |\eta|} \text{fun}(v) \right) \). Notice that since \( \partial \Omega \) is orientable (?????) then \( \tilde{\eta} \) is a formal linear combination of oriented boundary dual edges: \( \tilde{\eta} = \sum_{e \in E_0} c_e D_0(e) \).

Then we define the interior retraction \( \eta^+ \) in the following way: \( \eta^+ = \tilde{\eta} - \sum_{e \in E_0} c_e \partial_2 B(e) \). \( \eta^+ \) is a linear combination of oriented interior dual edges.

### 3.2 Construction of homological Seifert surfaces

Given an orientation of the edges and of the faces of the triangulation \( \mathcal{T} \) of \( \overline{\Omega} \), the problem of computing homological Seifert surfaces can be formulated as a linear system with as many unknowns as faces and as many equations as edges of \( \mathcal{T} \).

Let \( \gamma = \sum_{e \in E} a_e e \) be a given 1-boundary of \( \mathcal{T} \). A 2-chain \( S = \sum_{f \in \mathcal{F}} b_f f \) of \( \mathcal{T} \) is a homological Seifert surface of \( \gamma \) in \( \mathcal{T} \) if its coefficients \( \{ b_f \}_{f \in \mathcal{F}} \) satisfy the following equation:

\[
\sum_{f \in \mathcal{F}} b_f \partial_2 f = \sum_{e \in \mathcal{E}} a_e e,
\]

Equation (28)

We can write this equation more explicitly as a linear system. Given \( e \in \mathcal{E} \), let \( F(e) \) be the set \( \{ f \in \mathcal{F} \mid |e| \subset |f| \} \) of oriented faces in \( \mathcal{F} \) incident on \( e \) and let \( \phi_e : F(e) \to \{-1, 1\} \) be the function sending \( f \in F(e) \) into the coefficient of \( e \) in the expression of \( \partial_2 f \) as a formal linear combination of oriented edges in \( \mathcal{E} \). Equation (28) is equivalent to the linear system

\[
\sum_{f \in \mathcal{F}(e)} \phi_e(f) b_f = a_e \quad \forall e \in \mathcal{E},
\]

where the unknowns \( \{ b_f \}_{f \in \mathcal{F}} \) are integers.

The matrix of this linear system is the incidence matrix between faces and edges of \( \mathcal{T} \). Its entries take values in the set \( \{-1, 0, 1\} \). This matrix is very sparse because it has just three
nonzero entries per columns and the number of nonzero entries on each row is equal to the number of faces incident on the edge corresponding to the row. This kind of problems are usually solved using the Smith normal form, a computationally demanding algorithm even in the case of sparse matrices (see e.g. [21], [11]).

A first difficulty to devise a general and efficient algorithm to compute a homological Seifert surface $S$ of a given 1-boundary $\gamma$ of $T$ is that the problem has not a unique solution. If $t$ is the number of tetrahedra of $T$ and $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ are the connected components of $\partial \Omega$, then the kernel of the incidence matrix is a free abelian group of rank $t + p$; namely, it is isomorphic to $\mathbb{Z}^{t+p}$. One of its basis is given by the boundaries of tetrahedra of $T$ and by the 2-chains $\gamma_1, \ldots, \gamma_p$ associated with the triangulations of $\Gamma_1, \ldots, \Gamma_p$ induced by $T$.

A natural strategy to obtain a unique solution $S$ is to add $t + p$ equations, by setting equal to zero the unknowns corresponding to suitable faces $f_1, \ldots, f_{t+p}$ of $T$. From the geometric point of view, this is equivalent to impose that the homological Seifert surface $S$ of $\gamma$ does not contain the faces $f_1, \ldots, f_{t+p}$. From the computational point of view, it is equivalent to eliminate some unknowns of the problem to obtain an equivalent solvable linear system with a unique solution.

We will use graph techniques to describe which coefficients set equal zero. More precisely, we introduce the complete dual graph of $T$ denoted by $A'$.

To do that we need to recall some notions of homology theory (see e.g. [21]).

**Definition 3.** We call $A' := (V' \cup V'_0, E' \cup E'_0)$ complete dual graph of $T$. A 1-chain of $A'$ is a formal linear combination of oriented dual edges in $E' \cup E'_0$ with integer coefficients. A 1-chain $\gamma$ of $A'$ is called 1-cycle of $A'$ if $\partial_1 \gamma = 0$.

Our idea is to consider a suitable spanning tree $B'$ of $A'$ and to set equal to zero the unknowns corresponding to faces of $T$ whose dual edge belongs to $B'$. The total number of arcs in the spanning tree $B'$ is equal to the number of tetrahedra of $T$ plus the number of faces of $T$ contained in $\partial \Omega$ minus one, but, clearly, not all the arcs of $B'$ correspond to faces of $T$ since there are also arcs corresponding to edges of $T$ contained in $\partial \Omega$. The choice of $B'$ is promising if and only if the number of faces of $T$ whose dual edge belongs to $B'$; namely, the number of arcs of $B'$ not contained in $\partial \Omega$ is equal to $t + p$ but not all the spanning trees of $A'$ satisfy this equality. It is not difficult to see that for all spanning tree $B'$ of $A'$, $N_{B'} \geq t + p$. The equality holds true if and only if for each $i \in \{0, 1, \ldots, p\}$ the graph $B'_i$ induced by $B'$ on $\Gamma_i$ is a spanning tree of $A'_i$, the graph induced by $A'$ on $\Gamma_i$. If the spanning tree $B'$ of $A'$ has the latter property, then we call it Seifert dual spanning tree of $T$.

Let $B' = (V' \cup V'_0, N')$ be a Seifert dual spanning tree of $T$ and let $N'$ be its set of oriented dual edges. In [1] we proved that the following linear system

$$\begin{cases} \sum_{f \in F(e)} \phi_e(f) b_f = a_e & \text{if } e \in E \\ b_f = 0 & \text{if } e' \in N' \end{cases}$$

has a unique solution. In [1] we give also an explicit formula for the coefficients of the solution of (29). Roughly speaking the coefficient in $S$ of any face $f$ with $D(f) \in N'$ is equal to the linking number between $\gamma$ and the unique 1-cycle, $\sigma_{\sigma'}(D(f))$ of $A'$ with all the edges except $D(f)$ contained in $B'$. But this two cycles could intersect on $\partial \Omega$ and in this case is necessary, in order to define the linking number, to “retract” $\gamma$ inside $\Omega$. More precisely we prove that

$$b_f = \ell_\gamma(R_+ (\gamma), \sigma_{\sigma'}(D(f)))$$
for every \( f \in F \). The cycle \( R_+(\gamma) \) is defined in the following way. For every oriented edge \( e = [v, w] \) in \( \mathcal{E}_\partial \), choose a tetrahedron \( t_e \in K \) incident on \( e \) (namely, \( \{v, w\} \subseteq t_e \)), denote by \( d_e \) the barycenter of the triangle of \( \mathbb{R}^3 \) of vertices \( v, w, B(t_e) \), and define the 1-chain \( r_+(e) \) of \( \mathbb{R}^3 \) by setting

\[
r_+(e) := [v, d_e] + [d_e, w].
\]

Given \( \xi = \sum_{e \in E} \alpha_ee(e) \), we define:

\[
R_+(\xi) := \sum_{e \in E \setminus \mathcal{E}_\partial} \alpha_ee + \sum_{e \in \mathcal{E}_\partial} \alpha_ee_+(e).
\]

To compute the solution of (29) is convenient to adopt an elimination procedure and to use the explicit formula if it is necessary to restart the elimination procedure.

Let us set \( \mathcal{G} = \{ f \in F \mid D(f) \in N' \} \).

**Algorithm 1.**

1. \( \mathcal{R} := \mathcal{G}, \mathcal{D} := \mathcal{E} \).
2. while \( \mathcal{R} \neq \mathcal{F} \)
   
   (a) \( n_\mathcal{R} := \text{card}(\mathcal{R}) \)
   
   (b) for every \( e \in \mathcal{D} \)
   
   i. if every oriented face of \( F(e) \) belong to \( \mathcal{R} \)
      
      A. \( \mathcal{D} = \mathcal{D} \setminus \{ e \} \)
      
      ii. if exactly one oriented face \( f^* \in F(e) \) does not belong to \( \mathcal{R} \)
      
      A. compute \( b_f \) via (29)
      
      B. \( \mathcal{R} = \mathcal{R} \cup \{ f \} \)
      
      C. \( \mathcal{D} = \mathcal{D} \setminus \{ e \} \)
   
   (c) if \( \text{card}(\mathcal{R}) = n_\mathcal{R} \)
   
   i. pick \( f \notin \mathcal{R} \) and compute \( b_f = \ell_\alpha(R_+(\gamma), \sigma_{\alpha_\mathcal{R}}(D(f))) \)
   
   ii. \( \mathcal{R} = \mathcal{R} \cup \{ f \} \)

We have shown in [1] that very often, (2.c) never occurs and the homological Seifert surface can be computed by a very fast elimination procedure. In the examples that we tried the elimination procedure fails just when considering a non trivial computational domain that is a cube with a knotted cavity, and a boundary that embrace twice the cavity. In this case it was enough to use once the explicit formula to restart the elimination procedure.

Concerning the existence and the construction of internal homological Seifert surfaces of \( \gamma \); namely, homological Seifert surfaces of \( \gamma \) formed only by internal faces of \( T \) we proved in [1] that a necessary condition for the solution of an internal homological Seifert surface is that the boundary \( \gamma \) must be corner free, namely, no edge of \( \gamma \) belongs to two faces on \( \partial \Omega \) of the same tetrahedra. Clearly if the mesh is such that no tetrahedra has two faces on \( \partial \Omega \) then each boundary is corner free. Moreover in [1] we identify a family of Seifert dual spanning trees of \( T \), the so called strongly-Seifert dual spanning trees, such that if the boundary \( \gamma \) is corner free then the computed homological Seifert surface using such a Seifert dual spanning tree and Algorithm [1] is internal.
Let us denote by plug (the support of) the dual edge of a boundary face. A maximal plug set is a set of disjoint plugs that is not subset of any other set of disjoint plugs. If the mesh is such that no tetrahedra has two faces on $\partial \Omega$ then the set of all plugs is the unique maximal plug set. Notice that if a tetrahedra has two faces on $\partial \Omega$ then the dual edges of these two faces are not disjoint because the barycenter of the tetrahedra is a common point.

Given a Seifert dual spanning tree $B'$ of $T$, we say that $B'$ is a strongly-Seifert dual spanning tree of $T$ if it contains a maximal plug-set of $T$.

4 Some numerical experiments

We have implemented the algorithm proposed in this paper in C++. All computations have been run on an Intel Core i7-3720QM @ 2.60GHz laptop with 16Gb of RAM.

The first elementary example is a solid torus with a concentric toric cavity. The boundary of the domain has two connected components and none of them is homologically trivial. The generators of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ are the two cycles $\hat{\sigma}_1, \hat{\sigma}_2$ represented in Figure 1 as continuous lines. Clearly none of them is the boundary of a 2-chain contained in $\Omega$. Therefore, the first step is to complete each one with a cycle trivial in $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ in order to obtain a 1-boundary in the same homology class.

In Figure 2 we show the two representatives of $H_2(\overline{\Omega}, \partial \Omega; \mathbb{Z})$ on the left the one corresponding to the cycle $\hat{\sigma}_1$ and on the right the one corresponding to the cycle $\hat{\sigma}_2$.

![Figure 2: The torus with a toric cavity. Representatives of a basis of $H_2(\overline{\Omega}, \partial \Omega; \mathbb{Z})$ for the finest mesh are shown.](image-url)

Table 1 contains the details on the number of edges and faces in the complex for four different meshes and the corresponding computational time divided into four contributions: Mesh pre-processing represent the time spent for loading the mesh from hard disk and computing all incidences between the elements of the complex. Hiptmair–Ostrowski is the time spent for computing the bases of $H_1(\Omega; \mathbb{Z})$ and $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ with the algorithm introduced in [14]. We remark that each one of the $g$ elements of the constructed bases is supported in a single connected component of the boundary. Boundary retrieval is the time employed to find the 1-boundaries from the homology basis, which is the main contribution of this paper. Finally, elimination algorithm represents the time needed for the construction of the homological Seifert surfaces with the iterative elimination algorithm introduced in [1].

In Table 1 (and in the next tables) we include also the time spent by a state-of-the-art imple-
Table 1: The torus with a toric cavity: the number of geometric elements of the triangulation and the computational time.

| Mesh 1  | Mesh 2  | Mesh 3  | Mesh 4  |
|--------|--------|--------|--------|
| Edges  | 51521  | 145963 | 1321902| 10238231|
| Faces  | 76330  | 227314 | 2177158| 17210016|
| Mesh pre-processing [s] | 0.607  | 1.800  | 17.76  | 141.2 |
| Hiptmair–Ostrowski [s] | 0.084  | 0.216  | 0.863  | 3.909 |
| Boundary retrieval [s] | 0.012  | 0.034  | 0.122  | 0.513 |
| Elimination algorithm [s] | 0.061  | 0.193  | 2.720  | 24.52 |
| Total Time (this paper) [s] | 0.764  | 2.243  | 21.46  | > 2 hours |
| Total Time (GMSH [12]) [s] | 1.544  | 5.538  | 86.28  | > 2 hours |
| Speedup | 2.0    | 2.5    | 4.0    | —    |

In the algorithm proposed in this paper the more expensive computation is the one concerning the linking number that, in the worst case, has a computational cost proportional to the square of the number of edges in the boundary of Ω. The total number of linking numbers to be computed is ∑_{r=0}^{P}(2g_r)^2 for the automatic construction of generators of a basis of $H_1(\mathbb{R}^3 \setminus \Omega; \mathbb{Z})$ using the algorithm by Hiptmair–Ostrowski, plus the computation of the coefficient $\beta_{r,s}$, $r = 0, \ldots, P$ that are $\sum_{s=0}^{P} g_s \sum_{r\neq s}^{P} g_r = \sum_{r=0}^{P} g_r (g-r) = g^2 - \sum_{r=0}^{P} g_r$. So the total number is $g^2 + 3 \sum_{r=0}^{P} g_r^2$ that, for this first example means, 10 linking number to be computed.

All observations related to this simple benchmark still hold true for other numerical experiments. In our second example the domain is the complement of the Borromean rings ($g = 3$) with respect to a box. The number of connected components of the boundary is 4 and the first Betti number of the domain is equal to 3. The number of linking numbers to be computed is $9 + 3(1 + 1 + 1) = 18$. In Figure 3 we show three representatives of a basis of $H_2(\overline{\Omega}, \partial \Omega, \mathbb{Z})$ for two different meshes.

In Example 3 the boundary of the domain has 2 connected components and its first Betti

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The number of linking number to be computed is $9 + 3(4 + 1) = 24$. In Figure 4 we show the three generators of $H_2(\Omega, \partial \Omega; \mathbb{Z})$ for the trefoil benchmark and again, in Table 3, we give the dimension of the four different meshes considered, the computational time and the speed up with respect to GMSH with results similar to the previous examples. In Example 4 the the domain is the complement of a Hopf link with respect to a two torus, as illustrated in Figure 5 where we show four surfaces that are representatives of a basis of $H_2(\Omega, \partial \Omega; \mathbb{Z})$. In this case the number of connected components of the boundary of the domain is 3, the first Betti number of the domain is 4, and the total number of linking numbers computed is $4^2 + 3 \times (4 + 1 + 1) = 34$. In Table 3 we report the information about the meshes considered and the computational time. The speed up with respect to GMSH is similar to previous examples.

As expected, for these four benchmark problems the algorithm proposed in this paper has
Figure 4: The trefoil knot benchmark. Representatives of a basis of $H_2(\overline{\Omega}, \partial \Omega; \mathbb{Z})$ generators for the finest mesh are shown.

| Benchmark trefoil knot | Mesh 1 | Mesh 2 | Mesh 3 | Mesh 4 |
|------------------------|--------|--------|--------|--------|
| Edges                  | 45018  | 176123 | 1260407| 10264628|
| Faces                  | 72305  | 283758 | 2086618| 17305967|
| Pre-processing time [s]| 0.554  | 2.103  | 16.72  | 153.6  |
| Hiptmair–Ostrowski [s] | 0.046  | 0.163  | 0.767  | 3.099  |
| Boundary retrieval [s] | 0.017  | 0.056  | 0.113  | 0.736  |
| Elimination algorithm [s]| 0.052  | 0.256  | 2.595  | 27.98  |
| Total Time (this paper) [s]| 0.669  | 2.578  | 20.20  | 185.4  |
| Total Time (GMSH [12]) [s]| 1.638  | 8.814  | 94287  | > 2 hours |
| Speedup                | 2.5    | 3.4    | 4.7    | –      |

Table 3: Benchmark trefoil knot: the number of geometric elements of the triangulation and the computational time.

Figure 5: The Hopf link benchmark. Representatives of a basis of $H_2(\overline{\Omega}, \partial \Omega; \mathbb{Z})$ generators for the finest mesh are shown.
Table 4: Benchmark Hopf link: the number of geometric elements of the triangulation and the computational time.

| Benchmark Hopf link | Mesh 1       | Mesh 2       | Mesh 3       | Mesh 4       |
|---------------------|--------------|--------------|--------------|--------------|
| Edges               | 39692        | 263041       | 2255753      | 10152372     |
| Faces               | 64007        | 434513       | 3794183      | 17148224     |
| Mesh pre-processing [s] | 0.857          | 3.183       | 30.98        | 153.1        |
| Hiptmair–Ostrowski [s] | 0.029          | 0.131       | 0.657        | 3.031        |
| Boundary retrieval [s] | 0.008          | 0.034       | 0.134        | 0.498        |
| Elimination algorithm [s] | 0.044          | 0.415       | 5.118        | 27.82        |
| Total Time (this paper) [s] | 0.938          | 3.763       | 36.89        | 184.5        |
| Total Time (GMSH [12]) [s] | 1.576          | 16.04       | 201.7        | > 2 hours    |
| Speedup             | 1.7          | 4.3          | 5.5          | –            |

Table 5: Benchmark plate with holes: the number of geometric elements of the triangulation and the computational time.

| Benchmark plate with holes | Mesh 1       | Mesh 2       | Mesh 3       | Mesh 4       |
|----------------------------|--------------|--------------|--------------|--------------|
| Edges                      | 45596        | 334526       | 1164992      | 7740566      |
| Faces                      | 65396        | 523825       | 1908897      | 12956479     |
| Pre-processing time [s]    | 0.493        | 4.102        | 15.79        | 118.6        |
| Hiptmair–Ostrowski [s]     | 3.251        | 23.14        | 17.60        | 50.66        |
| Boundary retrieval [s]     | 1.458        | 19.14        | 19.97        | 39.11        |
| Elimination algorithm [s]  | 0.198        | 1.789        | 6.931        | 55.95        |
| Total Time (this paper) [s] | 5.400          | 48.17       | 60.30        | 264.3        |
| Total Time (GMSH [12]) [s] | 2.044          | 27.86       | 138.1        | > 4 hours    |
| Speedup                    | 0.38         | 0.58         | 2.3          | –            |

In Figure 8 we can see that in this benchmark problem the time on small examples is dominated by the linking number computations so it is not strictly linear.
Figure 6: Time [s] vs mesh density [number of faces] for the GMSH code and the implementation of the algorithm proposed in this paper.

Figure 7: The plate with holes benchmark.

References

[1] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, and R. Specogna, Efficient construction of homological seifert surfaces. arXiv:1409.5487, 2015.
Figure 8: Time [s] vs mesh density [on the left number of faces, on the right number of faces on the boundary] for the GMSH code and the implementation of the algorithm proposed in this paper.

[2] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, and A. Valli, Construction of a finite element basis of the first de Rham cohomology group and numerical solution of 3d magnetostatic problems, SIAM J. Numer. Anal., 51 (2013), pp. 2380–2402.

[3] Z. Arai, A rigorous numerical algorithm for computing the linking number of links, Nonlinear Theory and Its Applications, 4 (2013), pp. 104–110.

[4] R. Benedetti, R. Frigerio, and R. Ghiloni, The topology of Helmholtz domains, Expo. Math., 30 (2012), pp. 319–375.

[5] M. Brown, Locally flat imbeddings of topological manifolds, Ann. of Math. (2), 75 (1962), pp. 331–341.

[6] J. Cantarella, D. DeTurck, and H. Gluck, Vector calculus and the topology of domains in 3-space, Amer. Math. Monthly, 109 (2002), pp. 409–442.

[7] CHomP. http://chomp.rutgers.edu/software, 2012.

[8] T. Dey and S. Guha, Computing homology groups of simplicial complexes in $\mathbb{R}^3$, J. ACM, 45 (1998), pp. 266–287.

[9] P. Dlotko and R. Specogna, Efficient cohomology computation for electromagnetic modeling, CMES, 60 (2010), pp. 247–277.

[10] J.-G. Dumas, F. Heckenbach, B. Saunders, and V. Welker, GAP Homology. http://www.eccis.udel.edu/dumas/Homology, 2011.

[11] J.-G. Dumas, B. D. Saunders, and G. Villard, On efficient sparse integer matrix Smith normal form computations, J. Symbolic Comput., 32 (2001), pp. 71–99. Computer algebra and mechanized reasoning (St. Andrews, 2000).
[12] C. Geuzaine and J.-F. Remacle, *Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities.*, International Journal for Numerical Methods in Engineering, 79 (2009), pp. 1309–1331.

[13] P. W. Gross and P. R. Kotiuga, *Electromagnetic Theory and Computation: a Topological Approach*, Cambridge University Press, New York, 2004.

[14] R. Hiptmair and J. Ostrowski, *Generators of \( H_1(\Gamma_\mathbb{H}, \mathbb{Z}) \) for triangulated surfaces: construction and classification*, SIAM J. Comput., 31 (2002), pp. 1405–1423.

[15] C. S. Iliopoulos, *Worst-case complexity bounds on algorithms for computing the canonical structure of finite abelian groups and the hermite and smith normal forms of an integer matrix*, SIAM J. Comput., 18 (1989), pp. 658–669.

[16] P. R. Kotiuga, *On making cuts for magnetic scalar potentials in multiply connected regions*, J. Appl. Phys., 61 (1987), pp. 3916–3918.

[17] —— , *Toward an algorithm to make cuts for magnetic scalar potentials in finite element meshes*, J. Appl. Phys., 63 (1988), pp. 3357–3359. Erratum: *J. Appl. Phys.*, 64 (1988), 4257.

[18] —— , *An algorithm to make cuts for scalar potentials in tetrahedral meshes based on the finite element method*, IEEE Trans. Magn., 25 (1989), pp. 4129–4131.

[19] P. Monk, *Finite Element Methods for Maxwell’s Equations*, Oxford University Press, Oxford, 2003.

[20] M. Mrozek and B. Batko, *Coreduction homology algorithm*, Discrete Comput. Geom., 41 (2009), pp. 96–118.

[21] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, 1984.

[22] M. Pellikka, S. Suuriniemi, L. Kettunen, and C. Geuzaine, *Homology and cohomology computation in finite element modeling*, SIAM J. Sci. Comput., 35 (2013), pp. B1195–B1214.

[23] P. Pilarczyk and P. Real, *Computation of cubical homology, cohomology, and (co)homological operations via chain contraction*, Adv. Comput. Math., 41 (2015), pp. 253–275.

[24] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, 1976.

[25] H. Seifert and W. Threlfall, *A Textbook of Topology*, Academic Press, New York, 1980.

[26] H. Sexton and M. Vejdemo-Johansson, *jPlex*, December 2008. http://comptop.stanford.edu/programs/jplex/.