On weakly étale morphisms
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Abstract

We show that the weakly étale morphisms, used to define the pro-étale site of a scheme, are characterized by a lifting property similar to the one which characterizes formally étale morphisms. In order to prove this, we prove a theorem called Henselian descent which is a “Henselized version” of the fact that a scheme defines a sheaf for the fpqc topology. Finally, we show that weakly étale algebras over regular rings arising in geometry are ind-étale and that weakly étale algebras do not always lift along surjective ring homomorphisms.

1 Introduction

Let $f : X \to Y$ be a morphism of schemes. We say $f$ has the Henselian lifting property if for every solid commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(A/I) & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
\text{Spec}(A) & \xrightarrow{\exists !} & Y.
\end{array}
$$

where $(A, I)$ is a Henselian pair, there exists a unique dashed arrow making the diagram commute.

**Theorem 1.** A morphism of schemes has the Henselian lifting property if and only if it is weakly étale.

Recall that a morphism of schemes is weakly étale if it is flat and its diagonal is flat. The schemes weakly étale over a scheme $S$ form the underlying category of the pro-étale site $S_{pro-\acute{e}t}$ as defined in [BS15]. Theorem 1 is the natural analogue of the characterization of étale morphisms as formally étale morphisms which are locally of finite presentation.

Let $(A, I)$ be a Henselian pair. Let $A \to B$ be a ring map which is weakly étale and faithfully flat. Denote $B^h$ and $(B \otimes_A B)^h$ the Henselizations of $B$ and $B \otimes_A B$ with respect to $IB$ and $I(B \otimes_A B)$.

**Theorem 2.** Let $X$ be a scheme. With notation as above, the diagram

$$
X(A) \to X(B^h) \Rightarrow X((B \otimes_A B)^h)
$$

is an equalizer.

Theorem 2 arises in the course of the proof of Theorem 1 but we think it is of independent interest. We call it Henselian descent as it resembles the fact that a scheme defines a sheaf for the fpqc topology. Note that Theorem 2 does not follow formally from this since one need not have $B^h \otimes_A B^h = (B \otimes_A B)^h$.

**Theorem 3.** Let $A$ be an excellent regular domain which contains a field. Let $A \to B$ be a weakly-étale ring map. Then $A \to B$ is ind-étale.

We use Theorem 3 to show that there exists a surjective ring map $A \to A/I$ and a weakly étale $A/I$-algebra $B$ which does not lift to a weakly étale $A$-algebra. On the other hand, Bhatt and Scholze proved in [BS15, Lemma 2.2.12] that every ind-étale $A/I$-algebra lifts to an ind-étale $A$-algebra.

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2 Reinterpretation of a result of Gabber

In this section we translate some of Gabber’s results about étale cohomology of Henselian pairs in [Gab94] to statements about pro-étale cohomology. The default topology under consideration in this section is the pro-étale topology and all cohomology groups and pullbacks are with respect to this topology unless otherwise stated. Recall that for a scheme $S$ there is a morphism of sites $\nu : S_{\text{pro-}\acute{e}t} \to S_{\acute{e}t}$ and pro-étale sheaves in the essential image of the pullback functor $\nu^{-1}$ are called classical, see [BS15, Definition 5.1.3].

Example 4. Let $X$ be a scheme locally of finite type over $\mathbb{Z}$. Then the functor $U \mapsto X(U) = \text{Mor}(U, X)$ defines a classical sheaf on $S_{\text{pro-}\acute{e}t}$. More generally, if $X \to S$ is a morphism locally of finite presentation, then the functor $U \mapsto \text{Mor}_S(U, X)$ defines a classical sheaf on $S_{\text{pro-}\acute{e}t}$. This follows from the characterization of classical sheaves [BS15, Lemma 5.1.2] and the functorial description of being locally of finite presentation [Sta22, Tag 01ZC].

Remark 5. If $\mathcal{F}$ is a sheaf on $S_{\acute{e}t}$ and $U \to S$ is an étale morphism of schemes then $\Gamma(U, \nu^{-1} \mathcal{F}) = \Gamma(U, \mathcal{F})$ and if $\mathcal{F}$ is an abelian sheaf then $H^p(U, \nu^{-1} \mathcal{F}) = H^p(U, \mathcal{F})$ for every $p$, where in both equations the right hand side is étale cohomology and the left is pro-étale cohomology. See [BS15, Lemma 5.1.2, Corollary 5.1.6].

Lemma 6. Let $(A, I)$ be a Henselian pair. Denote $i : \text{Spec}(A/I) \to \text{Spec}(A)$ the inclusion. Let $A \to B$ be a weakly étale ring map. Let $\mathcal{F}$ be a classical sheaf on $\text{Spec}(A)_{\text{pro-}\acute{e}t}$. Let $B^h$ denote the Henselization of $B$ with respect to $IB$. Then

$$\Gamma(\text{Spec}(B^h), \mathcal{F}) = \Gamma(\text{Spec}(B/IB), i^{-1} \mathcal{F}).$$

If in addition $\mathcal{F}$ is a torsion abelian sheaf, then

$$H^p(\text{Spec}(B^h), \mathcal{F}) = H^p(\text{Spec}(B/IB), i^{-1} \mathcal{F})$$

for every $p$.

Remark 7. The statement makes sense since $\text{Spec}(B^h)$ is an object of the pro-étale site of $\text{Spec}(A)$ whose pullback to $\text{Spec}(A/I)_{\text{pro-}\acute{e}t}$ is $\text{Spec}(B/IB)$. This is because $B \to B^h$ is ind-étale and $B/IB = B^h/IB^h$.

Proof. If $A = B$ then in view of Remark 3 this follows from Gabber’s results [Gab94, Theorem 1, Remark 2]. A small point is that if $\mathcal{F}$ is an abelian torsion sheaf which is classical then it is $\nu^{-1}$ of an abelian torsion sheaf on $S_{\acute{e}t}$ as follows from Remark 3. For general $A \to B$, let $f : \text{Spec}(B^h) \to \text{Spec}(A)$ and $j : \text{Spec}(B/IB) \to \text{Spec}(B^h)$ denote the obvious morphisms. Then $f^{-1} \mathcal{F}$ is a classical sheaf on $\text{Spec}(B^h)_{\text{pro-}\acute{e}t}$ by [BS15, Lemma 5.4.1] so we know by the first sentence of the proof that

$$\Gamma(\text{Spec}(B^h), f^{-1} \mathcal{F}) = \Gamma(\text{Spec}(B/IB), j^{-1} f^{-1} \mathcal{F}).$$

Since $\text{Spec}(B^h)$ is an object of the pro-étale site of $\text{Spec}(A)$ the left hand side is just $\Gamma(\text{Spec}(B^h), \mathcal{F})$ by [BS15, Lemma 4.2.7]. Similarly, if $\tilde{f}$ denotes the morphism $\text{Spec}(B/IB) \to \text{Spec}(A/I)$ then the right hand side is

$$\Gamma(\text{Spec}(B/IB), \tilde{f}^{-1} i^{-1} \mathcal{F}) = \Gamma(\text{Spec}(B/IB), i^{-1} \mathcal{F}),$$

since $\text{Spec}(B/IB)$ is an object of $\text{Spec}(A/I)_{\text{pro-}\acute{e}t}$. If $\mathcal{F}$ is an abelian torsion sheaf, the same argument verbatim with $\Gamma$ replaced by $H^p$ proves the second statement.

3 Henselian descent

In this section, $(A, I)$ always denotes a Henselian pair. For a weakly étale $A$-algebra $B$, we let $B^h$ denote the Henselization of $B$ with respect to $IB$. For such a $B$ we obtain a diagram

$$A \to B^h \rightrightarrows (B \otimes_A B)^h.$$

Our goal is to prove Theorem 2 which implies in particular (by taking $X = \mathbb{A}^1_{\mathbb{Z}}$) that if $A \to B$ is weakly étale and faithfully flat, then (3) is an equalizer.

Lemma 8. If $A \to B$ is weakly étale and faithfully flat, then so is $A \to B^h$. If $f : X \to \text{Spec}(A)$ is a flat morphism of schemes whose image contains $V(I)$, then $f$ is faithfully flat.
Proof. Since $A \to B \to B^h$ are all weakly étale (in particular, flat) and $A/I \to B^h/IB^h = B/IB$ is faithfully flat the first statement will follow if we prove the second. The second statement is true because $f(X)$ is closed under generalization and contains $V(I)$, and $I$ is contained in the Jacobson radical of $A$ so every point of $\text{Spec}(A)$ is the generalization of a point of $V(I)$.

Proof of Theorem 2. By Lemma \ref{lemma} and faithfully flat descent, it follows that $X(A) \to X(B^h)$ is injective, so the problem is to show that an element $g \in X(B^h)$ equalized by the two arrows to $X((B \otimes_A B)^h)$ comes from an element of $X(A)$.

Step 1. Reduction to the case $X$ is quasi-compact and quasi-separated.

We immediately reduce to the case $X$ is quasi-compact by replacing $X$ with a quasi-compact open containing the image of $g : \text{Spec}(B^h) \to X$. A quasi-compact scheme $X$ when viewed as an fpqc sheaf can be written as a filtered colimit $\text{colim}_X X_i$ with $X_i$ a quasi-compact and quasi-separated scheme and each $X_i \to X$ a local isomorphism. In particular, for every quasi-compact scheme $T$ one has $X(T) = \text{colim}_i X_i(T)$. See [Sta22, Tag 01ZA]. Then if we know (2) is exact when $X$ is replaced by $X_i$, taking the colimit over $i$ proves the result since filtered colimits are exact in the category of sets.

Step 2. Reduction to the case $X$ is of finite type over $Z$.

Now assume $X$ is a quasi-compact and quasi-separated scheme. Then by absolute Noetherian approximation [Sta22, Tag 01ZA], $X$ is a cofiltered limit with affine transition maps $\text{lim}_{i \in I} X_i$ where each $X_i$ is a scheme of finite type over $Z$. Then if we know (2) is exact when $X$ is replaced by $X_i$, taking the limit over $i$ proves the result for $X$.

Step 3. Proof when $X$ is of finite type over $Z$.

By Example \ref{example} $X$ represents a classical sheaf $\mathcal{F}$ on the pro-étale site of $\text{Spec}(A)$. Let $i : \text{Spec}(A/I) \to \text{Spec}(A)$ denote the inclusion. Then by Lemma \ref{lemma} the diagram \ref{equation} is identified with

$$
\Gamma(\text{Spec}(A/I), i^{-1}\mathcal{F}) \to \Gamma(\text{Spec}(B/IB), i^{-1}\mathcal{F}) \Rightarrow \Gamma(\text{Spec}(B/IB \otimes_{A/I} B/IB), i^{-1}\mathcal{F})
$$

which is an equalizer since $i^{-1}\mathcal{F}$ is a sheaf on the pro-étale site of $\text{Spec}(A/I)$.

In analogy with classical descent theory, it is natural to ask whether the complex

$$
\check{\mathcal{C}}(B/A)^h = B^h \to (B \otimes_A B)^h \to (B \otimes_A B \otimes_A B)^h \to \cdots
$$

is exact in positive degrees. This is the complex obtained from the Amitsur complex $\check{\mathcal{C}}(B/A)$ by Henselizing the terms $B \otimes_A \cdots \otimes_A B$ with respect to $I(B \otimes_A \cdots \otimes_A B)$. In fact, we will show now that this exactness can fail even for standard Zariski coverings $B = A_{f_1} \times \cdots \times A_{f_n}$ where $f_i \in A$ generate the unit ideal.

Example 9. According to the blog post [4J18] there is a sheaf of rings $\mathcal{O}^h$ on the Zariski site of $\text{Spec}(A/I)$ characterized by $\Gamma(D(f) \cap \text{Spec}(A/I), \mathcal{O}^h) = A_f^h$ for $f \in A$, where $A_f^h$ is the Henselization of $A_f$ with respect to $IA_f$. In loc. cit., an example is constructed of a Henselian pair $(A, I)$ with $H^1_{\text{zar}}(\text{Spec}(A/I), \mathcal{O}^h) \neq 0$. Since Čech cohomology and cohomology always agree in degree 1, we have

$$
0 \neq H^1_{\text{zar}}(\text{Spec}(A/I), \mathcal{O}^h)) = \text{colim}_{A \to B} H^1(\check{\mathcal{C}}(B/A)^h)
$$

where the colimit is taken over standard Zariski coverings $A \to B$. Since filtered colimits are exact, this implies that there exists a standard Zariski covering $A \to B$ for which the complex $\check{\mathcal{C}}(B/A)^h$ is not exact in degree 1.

In positive characteristic, such an example is not possible essentially by Gabber’s affine analog of proper base change.

Lemma 10. Assume $A$ is a $\mathbb{Z}/N$-algebra for some integer $N > 1$. Let $M$ be an $A$-module. Then the complex of $A$-modules

$$
M \otimes_A \check{\mathcal{C}}(B/A)^h = M \otimes_A B^h \to M \otimes_A (B \otimes_A B)^h \to M \otimes_A (B \otimes_A B \otimes_A B)^h \to \cdots
$$

is quasi-isomorphic to $M$ placed in degree zero.

Proof. Let $\mathcal{F}$ be the sheaf on $\text{Spec}(A)_{\text{pro-\acute{e}t}}$ characterized by the formula $\mathcal{F}(U) = M \otimes_A B$ for $U = \text{Spec}(B)$ an affine object of $\text{Spec}(A)_{\text{pro-\acute{e}t}}$. It follows from [Sta22, Tag 023M] that $\mathcal{F}$ is indeed a sheaf and that for $U$ an affine object of $\text{Spec}(A)_{\text{pro-\acute{e}t}}$ we have $H^p(U, \mathcal{F}) = 0$ for $p > 0$. See [Sta22, Tag 03F9] for the vanishing. One
sees using the description of classical sheaves that $F$ is classical. Moreover, $F$ is torsion since $N = 0$ in $A$. Therefore Gabber’s result, Lemma 6, implies that for $A \to B$ weakly étale,

$$H^p(\text{Spec}(B/IB), i^{-1}F) = \begin{cases} M \otimes_A B^h & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(5)

Since this vanishing also holds when $B$ is replaced by any tensor product $B \otimes_A \cdots \otimes_A B$, a classical result on Čech cohomology (Sta22 Tag 03F7) shows that the Čech complex $\mathcal{C}(\text{Spec}(B/IB) \to \text{Spec}(A/I), i^{-1}F)$ is quasi-isomorphic to $R\Gamma(\text{Spec}(A/I), i^{-1}F)$, which is $M[0]$ by (5) applied to $A = B$. On the other hand, the $p = 0$ case of (5) applied with $B$ replaced by $B \otimes_A \cdots \otimes_A B$ shows that this Čech complex coincides with the complex (4).

4 Lifting for weakly étale morphisms

We now turn to the proof of Theorem 1 starting with some formal consequences of the Henselian lifting property.

Lemma 11. 1. An open immersion has the Henselian lifting property.

2. A composition of morphisms having the Henselian lifting property has the Henselian lifting property.

3. If $f$ and $g$ are composable morphisms of schemes such that $f$ and $f \circ g$ have the Henselian lifting property, then so has $g$.

4. The base change of a morphism with the Henselian lifting property has the Henselian lifting property.

Proof. We prove 1; 2-4 are formal. It suffices to show that if we are given a diagram \[ \begin{array}{ccc} \text{Spec}(A) & \to & \text{Spec}(B) \\ \downarrow & & \downarrow \\ \text{Spec}(C) & \to & \text{Spec}(D) \end{array} \]

with $f$ an open immersion, then the image of $\text{Spec}(A) \to Y$ is contained in the open $f(X) \subset Y$. This is because the closed $V(I)$ maps into $f(X)$ and every point of $\text{Spec}(A)$ is a generalization of a point of $V(I)$ since $I$ is contained in the Jacobson radical of $A$.

Proof of $\implies$ direction of Theorem 1. If $f$ has the Henselian lifting property then so does its diagonal by Lemma 11. Hence it suffices to show $f$ is flat. If $U \subset X$ is an affine open mapping into an affine open $V \subset Y$ then it follows from Lemma 11 that $U \to V$ has the Henselian lifting property. Thus it suffices to show that a ring map $R \to S$ with the Henselian lifting property is flat. Choose a surjection $P \to S$ of $R$-algebras with $P$ a possibly infinite polynomial algebra over $R$. Let $P^h$ denote the Henselization of $P$ with respect to $\text{Ker}(P \to S)$. Then form the solid diagram

$$\begin{array}{ccc} S & \xleftarrow{h} & S \\ \uparrow & & \uparrow \\ P^h & \xleftarrow{\kappa} & R \end{array}$$

and fill in the dashed arrow. Now $S$ is a direct summand of $P^h$ as an $R$-module, and $P^h$ is flat over $R$, completing the proof.

Flat immersions of schemes which are not open immersions are somewhat rare, but they do occur: The diagonal of a weakly étale morphism is often an example. We point out here that they share the following property in common with the open immersions:

Lemma 12. Let $j : Z \to X$ be a flat immersion of schemes. Let $f : T \to X$ be a morphism. Then $f$ factors through $Z$ (necessarily uniquely) if and only if $f(T) \subset j(Z)$ set-theoretically.

Proof. The only if direction is clear. The if direction is local so we may assume given a flat surjection of rings $A \to A/I$ and a ring map $A \to B$ such that $IB \subset p$ for every $p \in \text{Spec}(A)$. We have to show $A \to B$ factors through $A/I$. By Sta22 Tag 04PS, a flat surjection $A \to A/I$ of rings satisfies $A/I \cong (1 + I)^{-1}A$ as $A$-algebras. By the universal property of localization, it suffices now to show every element $1 + f$ with $f \in I$ is mapped to a unit of $B$. This is because $1 + f$ is not contained in any prime of $B$ by the assumption that they all contain $IB \ni f$.

Corollary 13. Let $f : X \to Y$ be a morphism of schemes.
1. If $f$ is a flat immersion then $f$ has the Henselian lifting property.

2. If the diagonal of $f$ is flat and $(A, I)$ is a henselian pair, then any dashed arrow in a diagram (1), if it exists, is unique.

Proof. 1. Suppose given a solid diagram (1) with $(A, I)$ a Henselian pair. We have to show there exists a unique dashed arrow. By Lemma 12 it suffices to show $\text{Spec}(A)$ maps into the subset $f(X) \subset Y$. But if $f$ is flat then $f(X)$ is closed under generalization and contains the image of $V(I) \subset \text{Spec}(A)$, so we conclude as in the proof of Lemma 11.

2. If the diagonal of $f$ is flat then it is a flat immersion so by 1 it has the Henselian lifting property. By a formal argument, this implies uniqueness of dashed arrows for all diagrams of the form (1).

The idea for proving the more difficult direction of Theorem 1 is to use the uniqueness just proven together with Henselian descent to reduce to constructing a dashed arrow locally.

Proof of $\Leftarrow$ direction of Theorem 1. It remains to prove that if $X \to Y$ is weakly étale, then it has the Henselian lifting property. To prove this we have:

Claim. It is enough to show that if $(A, I)$ is a Henselian pair and $f : X \to \text{Spec}(A)$ is a weakly étale, faithfully flat morphism with $X$ quasi-compact and quasi-separated, then for any solid diagram

$$
\begin{array}{ccc}
\text{Spec}(A/I) & \xrightarrow{t} & X \\
\downarrow & & \downarrow f \\
\text{Spec}(A) & = & \text{Spec}(A)
\end{array}
$$

there exists a dashed arrow.

Proof of Claim. Assume given a diagram (1) with $X \to Y$ weakly étale. We only have to show a dashed arrow exists since uniqueness follows from Corollary 13.2. Producing a dashed arrow in (1) is equivalent to producing a dashed arrow in the diagram (6) with $X$ replaced by $X \times_Y \text{Spec}(A)$. Thus we see it suffices to find a dashed arrow in the diagram (6) for $f : X \to \text{Spec}(A)$ weakly étale. To do this, we may assume $X$ is quasi-compact by replacing $X$ by a quasi-compact open containing the image of $\text{Spec}(A/I)$. Then write $X = \text{colim}_i X_i$ as in Step 1 of the proof of Theorem 2 and replace $X$ by any $X_i$ through which $\text{Spec}(A/I) \to X$ factors to assume $X$ is quasi-compact and quasi-separated, and still weakly étale over $A$ since $X_i \to X$ is a local isomorphism. It is automatic from Lemma 8 that $f$ is faithfully flat. This proves the claim.

Now suppose given a diagram (6) as in the claim. Choose a surjective étale morphism $g : \text{Spec}(B) \to X$. Then $A \to B$ is faithfully flat and weakly étale. It may not be true that $g$ and $t : \text{Spec}(A/I) \to X$ give rise to the same morphism $\text{Spec}(B/IB) \to X$, and we will now remedy this. The cartesian diagram

$$
\begin{array}{ccc}
\text{Spec}(B) \times_X \text{Spec}(A/I) & \to & \text{Spec}(B) \times_A \text{Spec}(A/I) - \text{Spec}(B/IB) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta_f} & X \times_A X
\end{array}
$$

shows that the top horizontal arrow is weakly étale and quasi-compact since $\Delta_f$ is by assumption. Choose surjective étale morphism $\text{Spec}(D) \to \text{Spec}(B) \times_X \text{Spec}(A/I)$. Then $B/IB \to D$ is weakly étale and there is a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(D) & \to & \text{Spec}(B) \\
\downarrow & & \downarrow g \\
\text{Spec}(A/I) & \xrightarrow{t} & X
\end{array}
$$

In fact, by [BS15, Theorem 2.3.4], after replacing $D$ with a faithfully flat, ind-étale $D$-algebra, we may even assume $B/IB \to D$ is ind-étale. In this case, by [BS15, Lemma 2.2.12], there is an ind-étale $B$-algebra $B'$ with an isomorphism $B'/IB' = D$. Replace $B$ with $B'$. Then $g : \text{Spec}(B) \to X$ and $t : \text{Spec}(A/I) \to X$ indeed give rise to the same morphism $\text{Spec}(B/IB) \to X$ and furthermore, $A \to B$ is weakly étale by construction and faithfully flat by Lemma 8.
We are looking for an element of \( \text{Mor}_A(\text{Spec}(A), X) \) mapping to the given element \( t \in \text{Mor}_A(\text{Spec}(A/I), X) \). Set \( C = B \otimes_A B \) and let \( B^h, C^h \) denote the henselizations of \( B, C \) with respect to \( IB, IC \). Consider the diagram

\[
\begin{array}{ccc}
\text{Mor}_A(\text{Spec}(A), X) & \longrightarrow & \text{Mor}_A(\text{Spec}(B^h), X) \\
\downarrow & & \downarrow \\
\text{Mor}_A(\text{Spec}(A/I), X) & \longrightarrow & \text{Mor}_A(\text{Spec}(B/IB), X)
\end{array}
\]

It follows from faithfully flat descent that the bottom row is an equalizer and from Henselian descent, Theorem \ref{faithfully-flat-desc} that the top row is also. It follows from Corollary \ref{henselian-desc} that the vertical arrows are injective. The morphism \( g : \text{Spec}(B) \to X \) induces an element \( g' \in \text{Mor}_A(\text{Spec}(B^h), X) \) which maps to the same element of \( \text{Mor}_A(\text{Spec}(B/IB), X) \) as \( t \in \text{Mor}_A(\text{Spec}(A/I), X) \). It is now a diagram chase to produce the desired element of \( \text{Mor}_A(\text{Spec}(A), X) \).

\[ \square \]

5 Weakly étale over regular

**Proposition 14.** Let \( Y \) be a Noetherian integral scheme which is regular and Nagata. Let \( f : X \to Y \) be an integral, weakly étale morphism of schemes. Then \( f \) is a co-filtered limit of finite-étale morphisms.

**Proof.** Let \( K \) denote the function field of \( Y \). Then the generic fibre \( X_K = \text{Spec}(A) \) where \( A \) is a weakly étale \( K \)-algebra. Thus \( A \) is the filtered colimit of its finite-type \( K \)-subalgebras \( A' \) which are all étale algebras over \( K \) by \[ \text{Sta22} \text{ Tag 0CRK}. \] On the other hand, \( f \) is the integral closure of \( Y \) in the \( K \)-algebra \( A \) since \( f \) is integral and \( X \) is integrally closed in \( A \) by \[ \text{Sta22} \text{ Tag 0RWW}. \] Hence \( f \) is the co-filtered limit with affine transition maps of the integral closures \( X' \to Y \) of \( Y \) in \( A' \). It suffices to show each \( X' \to Y \) is finite étale. Finiteness follows from the assumption that \( Y \) is Nagata, see \[ \text{Sta22} \text{ Tag 0GHE}. \] For étaleness, since \( X' \) is normal we may use purity of the branch locus \[ \text{Sta22} \text{ Tag 0BM}. \] It suffices to show that for each codimension one point \( x' \in X' \) with image \( y \) the extension \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X',x'} \) of DVRs is unramified with separable residue field extension. Now \( X \to X' \) is surjective: Its image is dense since \( A' \subset A \) and closed since \( f \) was assumed integral. Thus we may find \( x \in X \) mapping to \( x' \) and the composition \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \) is a weakly étale ring map by \[ \text{Sta22} \text{ Tag 0RQ}. \] Thus \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) induces an isomorphism on strict Henselizations by Olivier’s Theorem \[ \text{Sta22} \text{ Tag 0RZ}, \] so if \( \pi \in \mathcal{O}_{Y,y} \) is a uniformizer, then the image of \( \pi \) in \( \mathcal{O}_{X,x} \) must be in \( m_x \setminus m_x^2 \). But then the same must be true for the image of \( \pi \) in \( \mathcal{O}_{X',x'} \) so that \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X',x'} \) is unramified. For the residue field extension, we have a tower \( \kappa(x)/\kappa(x')/\kappa(y) \) and \( \kappa(x)/\kappa(y) \) is separable algebraic by \[ \text{Sta22} \text{ Tag 0RJ} \] hence so is \( \kappa(x')/\kappa(y) \).

By slightly modifying this argument we will prove Theorem \ref{ind-étale}. In place of classical Zariski–Nagata purity \[ \text{Sta22} \text{ Tag 0BM}, \] we will use the following refinement.

**Theorem 15 (\[ \text{Sta22} \text{ Tag 0EC} \].)** Let \( Y \) be an excellent regular scheme over a field. Let \( f : X \to Y \) be a finite type morphism of schemes with \( X \) normal. Let \( V \subset X \) be the maximal open subscheme where \( f \) is étale. Then the inclusion morphism \( V \to X \) is affine.

Next, we will improve the ad-hoc argument from the proof of Proposition \ref{étale} that \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X',x'} \) is unramified with separable residue field extension.

**Lemma 16.** Let \( A \to B \to C \) be local homomorphisms of local rings. Assume \( A \to B \) is essentially of finite presentation, \( B \) is a geometrically unibranch domain (for example if \( B \) is normal), \( A \to C \) is weakly étale, and the generic point of \( \text{Spec}(B) \) lies in the image of \( \text{Spec}(C) \to \text{Spec}(B) \). Then \( A \to B \) is the localization of an étale ring map.

**Proof.** This result is known if instead of assuming \( A \to C \) is weakly étale we assume it is the localization of an étale ring map, see \[ \text{Sta22} \text{ Tag 0GSA}. \] We will reduce to this case. First note that the hypotheses do not change if we replace \( C \) with its strict Henselization \( C^{sh} \). This is because \( C \to C^{sh} \) is ind-étale so that the composition \( A \to C^{sh} \) remains weakly étale, and \( C \to C^{sh} \) is faithfully flat so that the generic point of \( \text{Spec}(B) \) is still hit. Combining this with Olivier’s Theorem \[ \text{Sta22} \text{ Tag 0RZ} \] we see that we may assume \( C = A^{sh} \) is the strict henselization of \( A \). Then as \( A \)-algebras, \( C = \text{colim}_i C_i \) is a filtered colimit of local \( A \)-algebras which are localizations of étale \( A \)-algebras, and \( B \) is essentially of finite presentation, hence the map \( B \to C \) factors through some \( C_i \). Replace \( C \) with \( C_i \) and we are done.

\[ \square \]
Proof of Theorem \[3\]. As $B$ is the filtered colimit of its finite type $A$-subalgebras $C$, it suffices to show each inclusion $C \to B$ factors through an étale $A$-algebra. Let $K$ denote the fraction field of $A$. We have a commutative diagram

\[
\begin{array}{ccc}
B & \longrightarrow & B \otimes_A K \\
\uparrow & & \uparrow \\
C & \longrightarrow & C \otimes_A K.
\end{array}
\]

The top horizontal arrow is injective since $B$ is flat over $A$ and the right vertical arrow is injective since $K$ is flat over $A$. We conclude from the diagram that $C \to C \otimes_A K$ is also injective. Let $C'$ denote the integral closure of $C$ inside $C \otimes_A K$. Then $C' \subset B$ since $B$ is integrally closed in $B \otimes_A K$ by \cite{Sta22, Tag 092W}. Also $C \to C'$ is finite since $A$ is excellent and $C \otimes_A K$ is reduced, see \cite{Sta22, Tag 0CKR} and \cite{Sta22, Tag 03GH}. Replace $C$ by $C'$ to assume $C$ is a normal ring, still of finite type over $A$. Let $V \subset \text{Spec}(C)$ denote the étale locus of the ring map $A \to C$. Then $V = \text{Spec}(D)$ is affine by Theorem 15 and we will be done if we show the inclusion $C \to B$ factors through $D$. This is equivalent to showing that the image of $\text{Spec}(B) \to \text{Spec}(C)$ is contained in $V$. If $q \in \text{Spec}(C)$ is in the image then there is $p \in \text{Spec}(B)$ mapping to $q$ and then applying Lemma 16 to the local ring homomorphisms $A_{q,A} \to C_q \to B_p$ shows that $A \to C$ is étale at $q$ as needed.

Corollary 17. There exists a surjective ring homomorphism $A \to A/I$ and a weakly étale $A/I$-algebra which does not lift to a weakly étale $A$-algebra.

Proof. We take $A = C[x,y]$ and $I = ((y - x^2 + 1)(y + x^2 - 1))$. In \cite{Sta22, Tag 09AP} we find an example of a weakly étale $A/I$-algebra $B$ which is not ind-étale. If there were a lift to a weakly étale $A$-algebra $B$ then $A \to B$ would necessarily be ind-étale by Theorem 3. But then $A/I \to B/IB = B$ would be ind-étale also, a contradiction.

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