Instanton Expansions for Mass Deformed 
\( N = 4 \) Super Yang-Mills Theories

J. A. Minahan, D. Nemeschansky and N. P. Warner

*Physics Department, U.S.C.*

*University Park, Los Angeles, CA 90089*

We derive modular anomaly equations from the Seiberg-Witten-Donagi curves for softly broken \( N = 4 \) \( SU(n) \) gauge theories. From these equations we can derive recursion relations for the pre-potential in powers of \( m^2 \), where \( m \) is the mass of the adjoint hypermultiplet. Given the perturbative contribution of the pre-potential and the presence of “gaps” we can easily generate the \( m^2 \) expansion in terms of polynomials of Eisenstein series, at least for relatively low rank groups. This enables us to determine efficiently the instanton expansion up to fairly high order for these gauge groups, *e. g.* eighth order for \( SU(3) \). We find that after taking a derivative, the instanton expansion of the pre-potential has integer coefficients. We also postulate the form of the modular anomaly equations, the recursion relations and the form of the instanton expansions for the \( SO(2n) \) and \( E_n \) gauge groups, even though the corresponding Seiberg-Witten-Donagi curves are unknown at this time.

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1. Introduction

One of the more remarkable consequences of string duality is that it sometimes has highly non-trivial implications for field theory limits of the string, and it has thus led to new insights for some purely field theoretic questions. An example of this is the use of heterotic/type II duality to re-derive \cite{1-3} the effective actions of Seiberg and Witten for \( N = 2 \) supersymmetric Yang-Mills theories \cite{4}. This re-derivation showed rather explicitly how the Yang-Mills theory could be viewed as a compactification of the (0,2) non-critical string in six dimensions, where the compactifying manifold is the Riemann surface of \cite{4}, and in which (a particular form of) the Seiberg-Witten differential represents the local string tension on the Riemann surface. With this insight, the study of the stable states in the BPS spectrum of the field theory is reduced to the study of classical geodesics on this Riemann surface\cite{3,4}.

Toroidal compactifications of the \( E_8 \), (0, 1) supersymmetric non-critical strings in six dimensions have also been extensively studied \cite{6-12}. In \cite{11} it was shown how such string compactifications could also be characterized by a generalized Seiberg-Witten differential on the compactification torus. The slightly unusual feature was that the analogue of the instanton expansion for the four dimensional effective action actually counted electric BPS excitations of the non-critical string. This could be partially understood via the compactification of the six dimensional theory on a circle to five dimensions, in which four dimensional instantons have the interpretation as five-dimensional BPS solitons \cite{13}.

In \cite{12} the effective action and differential were extensively used to generate characters of the \( E_n \) non-critical strings compactified on a torus, but with the string winding around only one of the circles. There were two natural moduli in these characters: the string tension, which indexed the string winding number, and the complex structure, \( \tau \), of the torus, which indexed the momentum states of the string for fixed winding number. In \cite{14} it was shown that the partition function, \( G_n(\tau) \), of all the string momentum states of winding number \( n \) is an almost modular form in \( \tau \). That is, \( G_n \) can be written as a polynomial in the Eisenstein functions \( E_2, E_4 \) and \( E_6 \). Moreover, a recursion relation for \( G_n \) was derived, and this combined with requirement that certain momentum states be absent, completely determined the \( G_n \), and in principle enables the computation of the \( G_n \) to arbitrary order. These results were, of course, consistent with the numbers generated in \cite{4} by counting rational curves in a del Pezzo surface.

Given the similarity of the effective actions for the non-critical strings and the effective actions of the more usual quantum field theories, it is natural to ask whether the instanton expansion of quantum field theories exhibits a similar structure. Specifically, the
softly broken $N = 4$ supersymmetric gauge theories have an $SL(2, \mathbb{Z})$ invariance when the hypermultiplet mass, $m$, is zero (and $N = 4$ supersymmetry is restored). At this point there are no instanton corrections to the effective action. Turning on the mass introduces non-trivial corrections to the effective action, but it also breaks the $S$-duality of the theory. If one now expands the effective action in $\frac{m}{M_j}$, where the $M_j$ are the masses of the charged vector bosons, one gets a series whose coefficients, $G_\alpha$, are functions of the original $N = 4$ coupling parameter, $\tau$. In this paper we show that these coefficient functions are, once again almost modular forms (they be written as polynomials in the Eisenstein functions $E_2$, $E_4$ and $E_6$). We also show that these coefficient functions satisfy a recursion relation. Once again this recursion relation can be used to completely determine the instanton expansion given two other physical pieces of input: (i) The (known) perturbative correction to the effective action, and (ii) the requirement that the $m \to \infty$ limit of the instanton series is finite.

Apart from being of theoretical interest in precisely cataloguing the breaking of $S$-duality, the foregoing proves of major computational significance in that it enables one to determine the multi-instanton corrections to the effective action rather efficiently. We used it to determine the $SU(2)$ effective action to instanton order 24, and the $SU(3)$ effective action to instanton order eight. We also use it to conjecture the forms of the instanton expansions for the softly broken $N = 4$ theories with $SO(2n)$ and $E_n$ gauge groups, in spite of the fact that the corresponding Seiberg-Witten actions are unknown.

Since gauge theories and non-critical strings are intimately related, one would like to find a stringy interpretation of our results, and the recursion relation in particular. This, as yet eludes us, but our results suggest that one should be able to relate the instanton expansion to a topological field theory, and a topological string theory in particular. First, we find if we take a $\tau$ derivative on the pre-potential, and expand in $\frac{m}{M_j}$ and $q = e^{2\pi i \tau}$, then all the coefficients are integers with uniform signs for each monomial in $\frac{m}{M_j}$. This of course suggests a topological amplitude, but whether it is a topological Yang-Mills theory or a topological string, or both is unclear from this observation. Rather more “stringy” behaviour is suggested by one of our results for the $SU(2)$ theory, which will presumably generalize to other gauge groups. If one computes the pre-potential in the limit as $m \to M_W$, where $M_W$ is the mass of the $W$-boson, and subtracts the divergent perturbative piece, one finds that the instanton series for the pre-potential is very reminiscent of string threshold corrections\[15\]:

$$\partial^2_{M_W} F_{\text{instanton}} = \frac{2\pi i}{24} \left( \log (\eta(2\tau)) - \log (\eta(\tau)) \right). \quad (1.1)$$
Thus we feel that our results will ultimately provide insight into the formulation of Yang-Mills theories in terms of non-critical string theories.

In section 2 we derive our recursion relation for the $SU(2)$ gauge theory, and then use it and the methods outlined above to derive the instanton expansion. This section is also a model for the subsequent sections: Section 3 generalizes the results to $SU(n)$, while section 4 discusses other simply laced groups. Section 5 contains some brief concluding remarks.

2. Instanton expansions in the $SU(2)$ theory

2.1. Derivation of the Recursion Relation

The result presented here is actually a special case of the recursion relation derived in the next section. However we present a derivation here for several reasons: (i) to make this section self-contained, (ii) to provide a more explicit guide to the computation in section 3, and (iii) to highlight the similarities with the derivation of the recursion relation for the non-critical string [14].

The mass deformed $SU(2)$ curve is given by

$$y^2 = \prod_{i=1}^{3} \left( x - e_i u - \frac{1}{4} e_i^2 m^2 \right), \quad (2.1)$$

where $e_i$ are the combinations of Jacobi theta functions given in [4]. The $SU(2)$ invariant $u$ is related to $\text{tr}\phi^2$ by the relation [16]

$$u = \text{tr}(\phi^2) + m^2 \sum_{n=0}^{\infty} \alpha_n e^{2\pi i n} \quad (2.2)$$

It is not necessary to know the $\alpha_n$ for what follows.

The quantities $e_i$ can be rewritten in terms of Eisenstein series, with

$$E_4 = \frac{3}{2} \sum_i e_i^2 \quad E_6 = -\frac{9}{2} \sum_{i \neq j} e_i e_j \quad (2.3)$$

If we define a new variable $\chi = m^2/(12u)$, then after a rescaling of $x$ and $y$ the mass deformed curve can be rewritten as

$$y^2 = 4x^3 - \frac{4\beta^4 u^2}{3}(E_4 + 2\chi E_6 + \chi^2 E_4^2)x$$

$$- \frac{8\beta^6 u^3}{27}(E_6 + 3E_4^2 \chi + 3E_4 E_6 \chi^2 + (2E_6^2 - E_4^3) \chi^3), \quad (2.4)$$

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where the scale $\beta$ will be fixed below.

We now compare the curve in (2.4) to the standard elliptic curve
\[ y^2 = 4x^3 - \frac{g_2(\tilde{\tau})}{\omega^4}x - \frac{g_3(\tilde{\tau})}{\omega^6} \]  
where $\omega$ is one of the periods and $\tilde{\tau}$ is the modulus for the curve, which depends on $\tau$ and $\chi$. Using $g_2(\tilde{\tau}) = \frac{4}{3}\pi^4 \tilde{E}_4$ and $g_3(\tilde{\tau}) = \frac{8}{27}\pi^6 \tilde{E}_6$ and comparing (2.4) with (2.5), we see that
\[ \omega = \frac{\pi}{\beta u^{1/2}} \left( \frac{\tilde{E}_4}{E_4 + 2\chi E_6 + \chi^2 E_4^2} \right)^{1/4}. \]  

The natural coordinate, $\phi$, on the moduli space of the gauge theory [4] is given by
\[ \phi = \frac{\pi}{\beta} \int \frac{du}{u^{1/2}} \left( \frac{\tilde{E}_4}{E_4 + 2\chi E_6 + \chi^2 E_4^2} \right)^{1/4}. \]  

In the limit that $m$ approaches zero, one has $\tilde{\tau} \to \tau$ and $\phi \to u^{1/2}$, thus from (2.7), $\beta = 2\pi$. We can expand the integral in powers of $u$, where in the large $u$ limit, we would find all half integer powers, except for an integration constant. This integration constant is set to zero so that the $Z_2$ invariance under $\phi \to -\phi$, $u \to u$ is preserved.

We now explore the modular properties of these various terms. Under the transformation $\tau \to \frac{a\tau + b}{c\tau + d}$, the $E_k$ transform as $E_k \to (c\tau + d)^k E_k$. Hence the curve in (2.4) transforms nicely under these modular transformations if we also assume that $u \to (c\tau + d)^2 u$, $m \to m$. Hence $\chi$ has the transformation $\chi \to (c\tau + d)^{-2} \chi$.

By comparing (2.5) and (2.4) we can find another relation,
\[ \frac{\tilde{E}_4^3}{\tilde{E}_6^2} = \frac{(E_4 + 2\chi E_6 + \chi^2 E_4^2)^3}{(E_6 + 3E_4^2\chi + 3E_4E_6\chi^2 + (2E_6^2 - E_4^3)\chi^3)^2} \]  

From (2.8), and the fact that $\tilde{\tau} \to \tau$ as $m \to 0$, we immediately learn that the modular transformation on $\tau$ induces the transformation $\tilde{\tau} \to \frac{a\tau + b}{c\tau + d}$. It then follows that the difference $\tilde{\tau} - \tau$ transforms as:
\[ \tilde{\tau} - \tau \to (c\tilde{\tau} + d)^{-1}(c\tau + d)^{-1}(\tilde{\tau} - \tau). \]  

Next consider the dual coordinate $\phi_D$ which is the integral over $u$ of the dual period $\omega_D = \tilde{\tau} \omega$:
\[ \phi_D = \frac{1}{2} \int \frac{du}{u^{1/2}} \left( \frac{\tilde{E}_4}{E_4 + 2\chi E_6 + \chi^2 E_4^2} \right)^{1/4} \tilde{\tau}. \]  

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If we now consider $\phi_D - \tau \phi$, then from (2.10), (2.7), (2.9) and the action of the modular transformation on $u$, we have

$$\phi_D - \tau \phi \to (\phi_D - \tau \phi)(c\tau + d)^{-1}. \tag{2.11}$$

Hence we may infer that the $u$ expansion of $\phi_D - \tau \phi$ is

$$\phi_D - \tau \phi = \sum_{n=0} p_{2n}(\tau) \frac{m^{2n+2}}{u^{1/2+n}} \tag{2.12}$$

where the terms $p_{2n}(\tau)$ are modular forms of weight $2n$. Hence, they are composed of rational functions of $E_4$ and $E_6$. Indeed, for $\chi$ small, one can see explicitly from (2.10) that the $p_{2n}(\tau)$ are polynomials in $E_4$ and $E_6$.

However, in order to find the instanton expansion for the free energy and the coupling, we actually want the expansion of $\phi_D$ in powers of $\phi$, not $u$. Moreover, $u$ is a somewhat arbitrary parameter for an elliptic curve, while $2\phi$ is the physical mass of the gauge bosons. But, it is clear from (2.7) that an expansion of $\phi_D$ in terms of $\phi$ will not have coefficients that are modular forms; under the modular transformation, $\phi$ transforms to $c\phi_D + d$.

Luckily, this loss of modular invariance is actually quite mild and in fact can be calculated precisely. Furthermore, the breaking of the modular invariance is described by a modular anomaly equation which can be used to generate a recursion relation.

Let $H = \tilde{\tau} - \tau$ and define a new expression $h$,

$$h = \frac{H}{1 - \frac{2\pi i}{12} E_2 H}, \tag{2.13}$$

where $E_2$ is the regulated Eisenstein series of weight two and which is also given by $E_2 = \frac{24}{2\pi i} \partial_\tau \log(\eta(\tau))$. The function $E_2$ is not quite a modular form, under a modular transformation it generates an extra piece, with

$$E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \left( E_2(\tau) + \frac{12}{2\pi i} \frac{c}{c\tau + d} \right). \tag{2.14}$$

Using (2.14) and (2.11) we find that under a modular transformation, $h \to (c\tau + d)^{-2} h$. Therefore, the $u$ expansion of $h$ is comprised solely of modular forms and so has no $E_2$ terms in the expansion. Thus,

$$\frac{\partial H}{\partial E_2} = \frac{\partial}{\partial E_2} \left( \frac{h}{1 + \frac{2\pi i}{12} E_2 h} \right) = -\frac{2\pi i}{12} H^2 \tag{2.15}$$
Next consider the term
\[(\tilde{E}_4/E_4)^{1/4}(\tilde{\tau} - \tau),\] (2.16)
where \(\tilde{E}_4\) is given by
\[\tilde{E}_4 = E_4 + 2\chi E_6 + \chi^2 E_4^2.\] (2.17)
The expression in (2.10) is a modular function of weight \(-2\), therefore its \(u\) expansion is independent of \(E_2\). Using this fact and (2.15) we find that
\[\frac{\partial}{\partial E_2} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4} = \frac{2\pi i}{12} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4} (\tilde{\tau} - \tau).\] (2.18)

We now want to treat \(\tau\) and \(\phi\) as the independent variables. As a function of \(\tau\) and \(\phi\), \(u\) can have some \(E_2\) dependence. To this end let us act on (2.7) with \(\partial_{E_2}\), giving the equation
\[0 = \frac{1}{2} \frac{\partial u}{\partial E_2} \frac{1}{u^{1/2}} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4} + \frac{1}{2} \int \frac{du}{u^{1/2}} \frac{\partial}{\partial E_2} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4}.\] (2.19)
The derivative with respect to \(E_2\) inside the integral on the right-hand side refers to the explicit \(E_2\) dependence in the \(u\) expansion of \((\tilde{E}_4/E_4)^{1/4}\). Hence, using (2.18) and (2.10) in (2.19) one finds
\[\frac{\partial u}{\partial E_2} = -\frac{2\pi i}{12} \left( \frac{\tilde{E}_4}{E_4} \right)^{-1/4} (\phi_D - \tau \phi)\] (2.20)
We can also act on (2.7) with \(\partial_{\phi}\), which gives the equation
\[1 = \frac{1}{2} \frac{\partial u}{\partial \phi} \frac{1}{u^{1/2}} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4}.\] (2.21)
Therefore, from (2.20) we have
\[\frac{\partial u}{\partial E_2} = -\frac{2\pi i}{12} \frac{\partial u}{\partial \phi} (\phi_D - \tau \phi)\] (2.22)
Using (2.22) in the two relations
\[\partial_E (\phi_D - \tau \phi) = \frac{\partial u}{\partial E_2} \partial_u (\phi_D - \tau \phi)\]
\[\partial_{\phi} (\phi_D - \tau \phi) = \frac{\partial u}{\partial \phi} \partial_u (\phi_D - \tau \phi)\] (2.23)
we derive the equation
\[ \partial E_2(\phi_D - \tau \phi) = -\frac{2\pi i}{24} \partial_\phi (\phi_D - \tau \phi)^2. \] (2.24)

Since \( \phi_D = \partial_\phi F \) where \( F \) is the pre-potential, (2.24) can be integrated, giving
\[ \partial E_2 \tilde{F} = -\frac{2\pi i}{24} (\partial_\phi \tilde{F})^2 + C(\tau), \] (2.25)

where \( \tilde{F} = F - \frac{1}{2} \tau \phi^2 \) and \( C(\tau) \) is a \( \phi \) independent term.

The equation in (2.23) has exactly the same form as an equation in [14], where recursion relations were derived for the instanton expansion of the \( E_8 \) non-critical string. However, the expressions for \( \tilde{F} \) are completely different since the expansion parameters are different and the perturbative contribution for the noncritical string is trivial.

The key point is that \( \tilde{F} = 0 \) if \( m = 0 \). Therefore, the recursion relation we derive should relate the higher power terms in the \( m \) expansion of \( \tilde{F} \) to the lower power terms. Since only even powers of \( m \) appear, the series expansion for \( \tilde{F} \) has the form
\[ \tilde{F} = \frac{1}{8\pi i} f_1(\tau) m^2 \log(2\phi/m^2)^2 - \frac{1}{4\pi i} \sum_{n=2}^\infty \frac{f_n(\tau)}{(2n-2)} \frac{m^{2n}}{(2\phi)^{2n-2}}, \] (2.26)

up to a \( \phi \) independent piece that does not effect the physics. We have included a factor of 2 in front of \( \phi \) for later convenience. The coefficients \( f_n(\tau) \) are polynomials of \( E_2, E_4 \) and \( E_6 \), with weight \( 2n - 2 \), so that the total weight of \( \tilde{F} \) is zero. If we plug the expansion of \( \tilde{F} \) in (2.26) into (2.25), we find the recursion relation
\[ \partial E_2 f_n = \left( \frac{n-1}{6} \right) \sum_{m=1}^{n-1} f_m f_{n-m}. \] (2.27)

Hence, up to the \( E_2 \) independent terms, if we know \( f_1 \), then we can generate the whole instanton expansion.

2.2. The instanton expansion from the recursion relation

The first term \( f_1 \) in (2.26) can be found from the one-loop piece of \( \tilde{F} \). The perturbative piece also contributes to the higher \( f_n \), but no instanton term contributes to \( f_1 \). The perturbative pre-potential is given by
\[ \tilde{F}_{\text{pert}} = \frac{-1}{8\pi i} \left( (2\phi)^2 \log(2\phi)^2 - \frac{1}{2} \left( (2\phi + m)^2 \log(2\phi + m)^2 + (2\phi - m)^2 \log(2\phi - m)^2 \right) \right). \] (2.28)
The first few terms in its \( m \) expansion are

\[
\bar{\mathcal{F}}_{\text{pert}} = \frac{1}{8\pi i} \left( m^2 \log(2\phi)^2 - \frac{1}{6} \frac{m^4}{(2\phi)^2} - \frac{1}{30} \frac{m^6}{(2\phi)^4} \right) + O(m^8) \tag{2.29}
\]

Therefore, \( f_1 = 1 \), and so from (2.27), \( f_2 = \frac{1}{6} E_2 \). The expansions of the \( E_n \) are given by

\[
E_2 = 1 - 24 \sum_{n=1} \sigma_1(n)q^n \\
E_4 = 1 + 240 \sum_{n=1} \sigma_2(n)q^n \\
E_6 = 1 - 504 \sum_{n=1} \sigma_3(n)q^n,
\tag{2.30}
\]

where \( q = e^{2\pi i \tau} \) and \( \sigma_k(n) = \sum d|n d^k \). Comparing \( f_2 \) and (2.29), we see that the coefficient in front of \( E_2 \) is correctly normalized. The next term in the series is \( f_3 \), which satisfies \( f_3 = \frac{1}{18} E_2^2 + \alpha E_4 \). We can determine \( \alpha \) by comparing to the \( m^6 \) term in (2.28). Matching this to the \( q \) independent piece in \( f_3 \), we find that \( \alpha = 1/90 \). The next three terms can be found in a similar manner, and are given by

\[
f_4 = \frac{5}{216} E_2^2 + \frac{1}{90} E_2 E_4 + \frac{11}{7560} E_6 \\
f_5 = \frac{7}{648} E_2^4 + \frac{7}{810} E_2^2 E_4 + \frac{19}{22680} E_4^2 + \frac{11}{5670} E_2 E_6 \\
f_6 = \frac{7}{1296} E_2^5 + \frac{1}{162} E_2^3 E_4 + \frac{17}{11340} E_2 E_4^2 + \frac{11}{6048} E_2^2 E_6 + \frac{37}{142560} E_4 E_6.
\tag{2.31}
\]

The perturbative expansion does not give enough information to compute \( f_7 \) and beyond. This is because at \( f_7 \), there are two independent modular forms of weight 12, generated by \( E_4^3 \) and \( E_6^2 \). So we need another piece of information to set their coefficients. In order to proceed, let us look at the \( q \) expansions for the first six terms. The expansions are

\[
f_1 = 1 \\
f_2 = \frac{1}{6} - 4q - 12q^2 - 16q^3 + 28q^4 + O(q^5) \\
f_3 = \frac{1}{15} + 48q^2 + 256q^3 + 720q^4 + O(q^5) \\
f_4 = \frac{1}{28} - 30q^2 - 960q^3 - 6570q^4 + O(q^5) \\
f_5 = \frac{1}{45} + \frac{3584}{3} q^3 + 24864q^4 + O(q^5) \\
f_6 = \frac{1}{66} - 480q^3 - 43020q^4 + O(q^5)
\tag{2.32}
\]
An obvious feature is that after the constant term, the \( q \) expansion starts at \( q^2 \) for \( f_3 \) and \( f_4 \) and at \( q^3 \) for \( f_5 \) and \( f_6 \).

We can understand the presence of these “gaps” by considering the \( N = 2 \) pure gauge limit when the mass of the adjoint scalar is taken to infinity. At the same time we also must take \( \tau \) to imaginary infinity in order that the resulting cut off is finite. In the limit of large \( m \) and \( q \) the perturbative part of the pre-potential is to leading order

\[
\mathcal{F} \approx \frac{1}{2} \tau \phi^2 + \frac{1}{2\pi i} \phi^2 \log (m^2/\phi^2) = \frac{1}{2\pi i} \phi^2 \log \left( m^2 q^{1/2}/\phi^2 \right) \quad (2.33)
\]

Hence, in order to keep things finite, we should scale the cutoff as \( \Lambda^2 = m^2 q^{1/2} \). If we now look at the contributions to the instantons, we see that any term of the form \( (m^4 q)^s m \) will diverge in this limit. Thus, if we want to avoid such terms, we require that the \( q \) expansion of \( f_{2n} \) starts at \( q^n \) and that of \( f_{2n+1} \) starts at \( q^{n+1} \).

We can use the gaps to fix the coefficients of the modular forms. In fact, the presence of the gaps has made the system overdetermined and provides a nontrivial check on our formalism. The gaps have also obviated the need for the \( m \) expansion of \( \tilde{F}_{pert} \) beyond the leading order term. So, to find \( f_7 \) we can adjust the coefficients of \( E_4^3 \) and \( E_6^2 \) such that there is no \( q \) and \( q^2 \) term in the expansion. We can then check that everything is consistent by making sure that there is no \( q^3 \) term either.

Using Mathematica we were able to generate the first 48 terms. The next six terms in the expansion are

\[
f_7 = \frac{11}{3888} E_2^6 + \frac{11}{2592} E_2^4 E_4 + \frac{1199}{680400} E_2^2 E_4^2 + \frac{2281}{23351328} E_4^3 + \frac{121}{81648} E_2^3 E_6 \\
+ \frac{4127}{7484400} E_2 E_4 E_6 + \frac{145945800}{145945800} E_6^2 \\
f_8 = \frac{143}{93312} E_2^7 + \frac{1001}{349920} E_2^5 E_4 + \frac{377}{218700} E_2^3 E_4^2 + \frac{62459}{250192800} E_2 E_4^3 + \frac{1573}{1399680} E_2^4 E_6 \\
+ \frac{56797}{76982400} E_2^2 E_4 E_6 + \frac{7907}{159213600} E_4^2 E_6 + \frac{43151}{778377600} E_2 E_6^2
\]
\[ f_9 = \frac{715}{83908} E_2^8 + \frac{1001}{52480} E_2^6 E_4 + \frac{533}{349920} E_2^4 E_4^2 + \frac{146057}{37529200} E_2^2 E_4^3 + \frac{107803}{7695160704} E_4^4 + \frac{143}{174960} E_2^5 E_6 + \frac{1537}{1924560} E_2^3 E_4 E_6 + \frac{54928}{383107725} E_2 E_4^2 E_6 + \frac{202831}{2451889440} E_2^2 E_6^2 + \frac{2431}{5038848} E_2^9 + \frac{221}{174960} E_2 E_4 + \frac{533}{349920} E_2 E_4^2 + \frac{146057}{37529200} E_2 E_4^3 + \frac{107803}{7695160704} E_4^4 + \frac{143}{174960} E_2^5 E_6 + \frac{1537}{1924560} E_2^3 E_4 E_6 + \frac{54928}{383107725} E_2 E_4^2 E_6 + \frac{202831}{2451889440} E_2^2 E_6^2 + \frac{2431}{5038848} E_2^9 + \frac{221}{174960} E_2 E_4 + \frac{533}{349920} E_2 E_4^2 + \frac{146057}{37529200} E_2 E_4^3 + \frac{107803}{7695160704} E_4^4 + \frac{143}{174960} E_2^5 E_6 + \frac{1537}{1924560} E_2^3 E_4 E_6 + \frac{54928}{383107725} E_2 E_4^2 E_6 + \frac{202831}{2451889440} E_2^2 E_6^2 \]

We have already seen that the term \( f_{2n} \) starts its series expansion at \( q^n \) and \( f_{2n+1} \) starts its series expansion at \( q^{n+1} \). Therefore, in order to compute the entire \( n \) instanton contribution to \( \mathcal{F} \), we need to compute the series expansions up to the \( q^n \) term of \( f_i \) for \( i \) from 1 to 2n. Hence, with 48 terms we can compute the first 24 terms in the instanton
expansion. The first eight are

\[ \mathcal{F}_1 = \frac{1}{2\pi i} \frac{m^4}{(2\phi)^2} q \]

\[ \mathcal{F}_2 = \frac{1}{2\pi i} \left( 3 \frac{m^4}{(2\phi)^2} - 6 \frac{m^6}{(2\phi)^4} + 5 \frac{m^8}{2 (2\phi)^6} \right) q^2 \]

\[ \mathcal{F}_3 = \frac{1}{2\pi i} \left( 4 \frac{m^4}{(2\phi)^2} - 32 \frac{m^6}{(2\phi)^4} + 80 \frac{m^8}{(2\phi)^6} - \frac{224}{3} \frac{m^{10}}{(2\phi)^8} + 24 \frac{m^{12}}{(2\phi)^{10}} \right) q^3 \]

\[ \mathcal{F}_4 = \frac{1}{2\pi i} \left( 7 \frac{m^4}{(2\phi)^2} - 90 \frac{m^6}{(2\phi)^4} + \frac{1095}{2} \frac{m^8}{(2\phi)^6} - 1554 \frac{m^{10}}{(2\phi)^8} + 2151 \frac{m^{12}}{(2\phi)^{10}} \right. \]
\[ \left. - 1430 \frac{m^{14}}{(2\phi)^{12}} + \frac{1469}{4} \frac{m^{16}}{(2\phi)^{14}} \right) q^4 \]

\[ \mathcal{F}_5 = \frac{1}{2\pi i} \left( 6 \frac{m^4}{(2\phi)^2} - 192 \frac{m^6}{(2\phi)^4} + 2144 \frac{m^8}{(2\phi)^6} - 12096 \frac{m^{10}}{(2\phi)^8} + \frac{187056}{5} \frac{m^{12}}{(2\phi)^{10}} \right. \]
\[ \left. - 65472 \frac{m^{14}}{(2\phi)^{12}} + \frac{323232}{5} \frac{m^{16}}{(2\phi)^{14}} - 33600 \frac{m^{18}}{(2\phi)^{16}} + \frac{35768}{5} \frac{m^{20}}{(2\phi)^{18}} \right) q^5 \]

\[ \mathcal{F}_6 = \frac{1}{2\pi i} \left( 12 \frac{m^4}{(2\phi)^2} - 360 \frac{m^6}{(2\phi)^4} + 6210 \frac{m^8}{(2\phi)^6} - 58016 \frac{m^{10}}{(2\phi)^8} + 314016 \frac{m^{12}}{(2\phi)^{10}} \right. \]
\[ \left. - 1033120 \frac{m^{14}}{(2\phi)^{12}} + 2114840 \frac{m^{16}}{(2\phi)^{14}} - 2698080 \frac{m^{18}}{(2\phi)^{16}} + \frac{6249064}{3} \frac{m^{20}}{(2\phi)^{18}} \right. \]
\[ \left. - 890112 \frac{m^{22}}{(2\phi)^{20}} + 161588 \frac{m^{24}}{(2\phi)^{22}} \right) q^6 \]

\[ \mathcal{F}_7 = \frac{1}{2\pi i} \left( 8 \frac{m^4}{(2\phi)^2} - 128 \frac{m^6}{(2\phi)^4} + 14880 \frac{m^8}{(2\phi)^6} \right. \]
\[ \left. - 207168 \frac{m^{10}}{(2\phi)^8} + 1727856 \frac{m^{12}}{(2\phi)^{10}} \right. \]
\[ \left. - 9109760 \frac{m^{14}}{(2\phi)^{12}} + \frac{219699584}{7} \frac{m^{16}}{(2\phi)^{14}} - \frac{71919360}{18} \frac{m^{18}}{(2\phi)^{16}} + 109991904 \frac{m^{20}}{(2\phi)^{18}} \right. \]
\[ \left. - \frac{774893568}{7} \frac{m^{22}}{(2\phi)^{20}} + 70299264 \frac{m^{24}}{(2\phi)^{22}} - 25518592 \frac{m^{26}}{(2\phi)^{24}} + \frac{28244800}{7} \frac{m^{28}}{(2\phi)^{26}} \right) q^7 \]

\[ \mathcal{F}_8 = \frac{1}{2\pi i} \left( 15 \frac{m^4}{(2\phi)^2} - 930 \frac{m^6}{(2\phi)^4} + 62635 \frac{m^8}{2 (2\phi)^6} \right. \]
\[ \left. - 605934 \frac{m^{10}}{(2\phi)^8} + 7192017 \frac{m^{12}}{(2\phi)^{10}} \right. \]
\[ \left. - 55338690 \frac{m^{14}}{(2\phi)^{12}} + \frac{1146161679}{4} \frac{m^{16}}{(2\phi)^{14}} - \frac{1022595770}{18} \frac{m^{18}}{(2\phi)^{16}} + 2552909057 \frac{m^{20}}{(2\phi)^{18}} \right. \]
\[ \left. - 4471880166 \frac{m^{22}}{(2\phi)^{20}} + \frac{1091921219}{2} \frac{m^{24}}{(2\phi)^{22}} - 4541726970 \frac{m^{26}}{(2\phi)^{24}} \right. \]
\[ \left. + 2451618975 \frac{m^{28}}{(2\phi)^{26}} - 773708598 \frac{m^{30}}{(2\phi)^{28}} + \frac{866589165}{8} \frac{m^{32}}{(2\phi)^{30}} \right) q^8 \]

The two instanton expression matches the result of D'Hoker and Phong for $SU(2)$ [7].

We can directly find the instanton expansion for $N = 2$ $SU(2)$ Super Yang-Mills by
taking the limit \( m \to \infty \), \( m^4q = \Lambda^4 \). The surviving terms in the instanton expansion are those with the highest power of \( m \). When can readily check that these terms match those found by Matone [18].

A striking feature of the \( F_n \) terms in (2.35) is that after taking a \( \tau \) derivative, the coefficients of the \( m^2 \) expansions are integers. This suggests that the instanton expansion is computing topological invariants, and the natural guess is that they are Euler numbers for instanton moduli spaces.

2.3. The coupling near the \( U(1) \) singularity

The instanton expansion also has an interesting behavior as \( m \) approaches \( 2\phi \). At this point a charged BPS state is massless, hence the perturbative piece has a log singularity, Let us consider the expansion of the coupling as a series in \( (1 - m^2/(2\phi)^2) \). The full perturbative contribution of this effective \( U(1) \) theory is

\[
\tau_{eff} = \frac{1}{2\pi i} \log \left( 1 - \frac{m^2}{(2\phi)^2} \right)^2 + \frac{1}{2\pi i} \log(C(\tau)),
\]

where \( C(\tau) \) is a function to be determined. The nonperturbative pieces of the effective \( U(1) \) theory appear as higher powers of \( (1 - m^2/(2\phi)^2) \). To find \( C(\tau) \), consider the expression \( \frac{\tilde{E}_3 - \tilde{E}_6}{E_4^3} \). Using (2.8), (2.3) and (2.17), we find that

\[
\frac{\tilde{E}_3 - \tilde{E}_6}{E_4^3} = \frac{E_4^3 - E_6^2}{E_4^3} \prod_{i=1}^{3} (1 - 3\chi e_i)^2.
\]

The hypermultiplet is massless if \( \chi = 1/(3e_1) \), which corresponds to \( u = m^2e_1/4 \). At this singularity

\[
\tilde{q} = 0, \quad e_1^2 \tilde{E}_4 = (e_1 - e_2)^2(e_1 - e_3)^2,
\]

therefore, near the singularity we can approximate (2.37) as

\[
1728\tilde{q} = \frac{1728\eta^{24}(\tau)e_1^2}{(e_1 - e_2)^4(e_1 - e_3)^4} \left( 1 - \frac{m^2e_1}{4u} \right)^2 + O(\tilde{q}^2).
\]

However, we want to express (2.39) in terms of \( \phi \) and not \( u \). Since the singularity occurs at \( \phi^2 = m^2/4 \), we can approximate \( \phi^2 \) as

\[
\left( 1 - \frac{m^2}{4\phi^2} \right) = \alpha \left( 1 - \frac{m^2 e_1}{4u} \right) + O \left( \left( 1 - \frac{m^2 e_1}{4u} \right)^2 \right).
\]
In order to determine $\alpha$ in (2.40), consider the expression for $\phi$ in (2.7), with $\beta = 2\pi$. Taking a derivative with respect to $u$, we have
\[
\left. \frac{\partial \phi}{\partial u} \right|_{u = m^2 e_1/4} = \frac{1}{2u^{1/2}} \left( \frac{\tilde{E}_4}{E_4} \right)^{1/4} \left|_{u = m^2 e_1/4} = \frac{1}{m \sqrt{(e_1 - e_2)(e_1 - e_3)}} \right.
\]
(2.41)

Taking a $u$ derivative on (2.40), we have
\[
\frac{m^2}{2\phi^3} \frac{\partial \phi}{\partial u} \bigg|_{u = m^2 e_1/4} = \frac{m^2 e_1}{4u^2} \bigg|_{u = m^2 e_1/4} = \frac{4\alpha}{m^2 e_1}
\]
(2.42)

Thus, we have
\[
\alpha = \frac{e_1}{\sqrt{(e_1 - e_2)(e_1 - e_3)}},
\]
(2.43)

and therefore, $C(\tau)$ is
\[
C(\tau) = \frac{\eta^{24}(\tau)}{(e_1 - e_2)^3(e_1 - e_3)^3} = \frac{\eta^{24}(2\tau)}{\eta^{24}(\tau)},
\]
(2.44)

where we have used the fact that $e_1 - e_2 = \vartheta_3^4$ and $e_1 - e_3 = \vartheta_4^4$. We can now explicitly check that the instanton expansions in (2.35) are consistent with (2.44).

As was mentioned before, the log of the right-hand side of (2.44) looks like the holomorphic part of a string threshold correction, which can also be expressed in terms of $F_1$, the genus one partition function of a topological field theory coupled to topological gravity\[15\]. If the target space is a two torus with Kahler modulus $t$, then $F_1$ satisfies
\[
\frac{\partial_t}{2 \pi i} F_1 = \frac{2}{2 \pi i} \partial_t \log \eta (\exp(2 \pi i t))
\]
(2.45)

The right-hand side is the generator for the number of inequivalent maps of a world-sheet torus to a target space torus. Hence, the $\tau$ derivative of $\frac{1}{12} \tau_{eff}$ in (2.36) seems to count the number of maps from a torus into a torus with modulus $2\tau$ minus the number of maps into a torus with modulus $\tau$. This is made even more suggestive if one considers the behavior of the Donagi-Witten curve at the singular point $m^2 = 4\phi^2$, where a genus two surface degenerates into a torus with Teichmuller parameter $2\tau$.

3. Recursion relations for $SU(n)$

It is relatively straightforward to generalize our results to other gauge groups, and to $SU(n)$ in particular. To do this we first recall some of the essential features of the quantum effective action of softly broken $N = 4$ supersymmetric $SU(n)$ gauge theories \[19\].

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3.1. The SU(n) quantum effective action

The relevant Riemann surface is defined as an $n$-sheeted foliation over a torus. That is, one introduces the standard Weierstrass torus, with modulus $\tau$:

$$y^2 = 4 \left(x - e_1(\tau)\right) \left(x - e_2(\tau)\right) \left(x - e_3(\tau)\right) ;$$

and defines the Riemann surface over it via:

$$F_n(t, x, y; u_j) \equiv \sum_{\ell=0}^{n-1} u_{n-\ell} P_{\ell}(t, x, y) = 0 ,$$

where $u_0 = 1$. The $P_{\ell}$ are quasi-homogeneous polynomials of weight $\ell$, (where $t, x, y$ are assigned weights 1, 2, 3), and are uniquely defined by requiring $P_{\ell} \sim t^\ell + \ldots$ as $t \to \infty$, and by specifying their factorization properties in the limit $x, y, t \to \infty$ [19]. We have deviated slightly from [19] in that we have taken $u_1 \neq 0$. We have done this for later mathematical convenience, and it may be thought of as as adding an extra $U(1)$ factor, converting the action to that of a $U(n)$ gauge theory. It should be noted that the surface (3.2) has genus $n$, while $SU(n)$ has rank $n - 1$. The reason for promoting the gauge group to $U(n)$ is that we want the genus of the surface to equal the rank of the gauge group.

The Seiberg-Witten differential for these models is given by [19–21]:

$$\lambda_{SW} = t \frac{dx}{y} = t \, d\xi ,$$

where $\xi$ is the standard “flat” coordinate on the base torus. The periods of this differential are thus

$$\phi_i = \oint_{a_i} \lambda_{SW} , \quad \phi_{D,i} = \oint_{b_i} \lambda_{SW} , \quad i = 1, \ldots, n .$$

There are thus $n$ of each of the $\phi_i$ and the $\phi_{D,i}$. However, let $t_j, j = 1, \ldots, n$ denote the roots of (3.2). Since $P_n$ contains no term $t^{n-1}$ term, it follows that $u_1 = \sum t_j$, and hence

$$\sum_{i=1}^{n} \phi_i = u_1 \oint_{a} \frac{dx}{y} , \quad \sum_{i=1}^{n} \phi_{D,i} = u_1 \oint_{b} \frac{dx}{y} , \quad i = 1, \ldots, n ,$$

where $a$ and $b$ are the cycles of the base torus. Thus the sum of the $\phi_i$ and $\phi_{D,i}$ are simply the standard periods of the base torus multiplied by $u_1$. This sum gives the effective action of the $U(1)$ factor in $U(n)$, and the fact that it depends trivially on the $u_j$ and the base torus reflects the triviality of the effective action of a pure $U(1)$ theory (with no coupling to charged matter). Setting $u_1 = 0$ as in [19] means that the $\phi_j$ and $\phi_{D,j}$ are not all
independent (and indeed this constraint projects one onto the rank \( n - 1 \) Prym variety of interest). We have really introduced \( u_1 \) to avoid having to deal with this constraint and thereby avoid littering the discussion below with projection operators.

The dependence on the hypermultiplet mass, \( m \), is implicit in the discussion above: The \( u_j \) are the invariants of the Higgs vevs divided by \( m^j \). To restore the mass dependence explicitly, one rescales \( t \to t/m \), replaces \( u_j \to u_j/m^j \), and multiplies \( F_n \) by \( m^n \). The net effect is the same as replacing \( x \to m^2 x, y \to m^3 y \) in \( F_n \). The \( m = 0 \) limit is now clear: One has \( P_\ell = t_\ell \), and the foliation over over the base is trivial. The Riemann surface is \( n \) disconnected copies of the base torus (3.1). The Seiberg-Witten differential on the \( j^{th} \) copy is \( t_j d\xi \), where \( t_j \) is the \( j^{th} \) root of (3.2). Note that \( t_j \) is constant over the base, and so the periods of \( \lambda_{SW} \) are simple multiples of the periods of the base torus.

3.2. The \( SU(n) \) recursion relation

As in section 2, we imagine expanding the periods of the Seiberg-Witten differential about \( m = 0 \), in a series in \( m \) and inverse powers of \( \phi_i \), or more precisely \( \phi_i - \phi_j \). Our goal is to generalize (2.24) and (2.25), and the key to doing it is to establish the analog of (2.15) and establish and use the modular properties of appropriate generalization of \( \phi_D - \tau \phi \).

Let \( \zeta_j \) be a canonically normalized basis of holomorphic differentials on the Riemann surface (3.2) at a general value of \( m \), i.e.

\[
\oint_{a_i} \zeta_j = \delta_{ij}, \quad \oint_{b_i} \zeta_j = \tilde{\Omega}_{ij}, \quad i, j = 1, \ldots, n. \tag{3.6}
\]

The period matrix is thus \( (\mathcal{I}, \tilde{\Omega}) \) where \( \mathcal{I} \) is the \( n \times n \) identity matrix. Let \( (\mathcal{I}, \Omega) \) be the the corresponding period matrix at \( m = 0 \). Since the surface at \( m = 0 \) is \( n \) disconnected copies of the the base, one has:

\[
\Omega = \tau \mathcal{I}. \tag{3.7}
\]

An element \( \mathcal{M} \) of the group \( Sp(2g, \mathbb{Z}) \) acts upon the cycles, and hence on the periods, by left multiplication:

\[
\begin{pmatrix} b_j \\ a_i \end{pmatrix} \to \mathcal{M} \begin{pmatrix} b_j \\ a_i \end{pmatrix}, \quad \begin{pmatrix} \phi_{D,j} \\ \phi_i \end{pmatrix} \to \mathcal{M} \begin{pmatrix} \phi_{D,j} \\ \phi_i \end{pmatrix}. \tag{3.8}
\]

If one writes \( \mathcal{M} \) in terms of its \( n \times n \) blocks then its action on \( \tilde{\Omega} \) may be written:

\[
\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \tilde{\Omega} \to (A \tilde{\Omega} + B)(C \tilde{\Omega} + D)^{-1}. \tag{3.9}
\]
The submatrices of $\mathcal{M}$ must satisfy:

$$A^T C = C^T A; \quad B^T D = D^T B; \quad A^T D - C^T B = D^T A - B^T C = I. \quad (3.10)$$

As in section 2, introduce a matrix $H = \tilde{\Omega} - \Omega$. Under the modular transformation, $\mathcal{M}$, one has:

$$H \rightarrow (\Omega C^T + D^T)^{-1} H (C \tilde{\Omega} + D)^{-1},$$

$$= (\Omega C^T + D^T)^{-1} H (C H + C \Omega + D)^{-1}. \quad (3.11)$$

Define a matrix, $h$, via

$$h = H (1 - \frac{2\pi i}{12} E_2(\tau) H)^{-1}. \quad (3.12)$$

This does not behave well under a general $Sp(2g, \mathbb{Z})$ transformation, but we now specialize to the $SL(2, \mathbb{Z})$ subgroup of $Sp(2g, \mathbb{Z})$ induced via modular transformations of the base torus, $i.e.$ we take $(A, B, C, D) = (aI, bI, cI, dI)$. One then finds that all the entries of the matrix $h$ are $(SL(2, \mathbb{Z}))$ modular forms of weight $-2$, $i.e.$ one has $h \rightarrow (c\tau + d)^{-2} h$. As before one can write the entries of $h$ in terms of the modular forms $E_4$ and $E_6$, and hence $H$ may be written in terms of $E_2, E_4$ and $E_6$, and once again (2.15) is satisfied, where $H$ is now a matrix.

From (3.8) and (3.10) it is easily seen that the column vectors $\phi_D - \Omega \phi$ transform under $Sp(2g, \mathbb{Z})$ as

$$\phi_D - \Omega \phi \rightarrow (\Omega C^T + D^T)^{-1} (\phi_D - \Omega \phi). \quad (3.13)$$

It follows that under the $SL(2, \mathbb{Z})$ subgroup, each element of $\phi_D - \Omega \phi$ transforms as a modular form of weight $-1$, exactly as in (2.11).

To analyze expansions about $m = 0$, we need to consistently assign formal modular weights under $SL(2, \mathbb{Z})$. First, because $x$ is a Weierstrass function on the torus, it follows that $x$ has weight 2, and $y$ has weight 3. This means that $t$ must be assigned a formal weight of 1, and the $u_j$ are to be given a formal weight $j$ for the defining equation of the Riemann surface (3.2) to be modular invariant. Let $v_j$ denote the roots of $F_n = 0$ at $m = 0$, $i.e.$ the $v_j$ are the roots of $t^n + \sum_{j=1}^n u_j t^{n-j} = 0$. The $u_j$ are thus invariant polynomials in the $v_j$. Moreover, since $SL(2, \mathbb{Z})$ acts on each leaf of the foliation separately, it does not permute the $v_j$, and so the $v_j$ thus have a formal modular weight of 1. Suppose that one expands $\phi_{D,i} - \tau \phi_i$ into a Laurent series of the form:

$$\phi_{D,i} - \tau \phi_i = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} G_{\alpha,k}(\tau) \frac{m_k^{k+1}}{Q_{\alpha,k}(v_\ell)}, \quad (3.14)$$
where the $Q_{\alpha,k}(v_\ell)$ are some appropriately chosen set of homogeneous polynomials of degree $k$ in the $v_\ell$. It then follows that $G_{\alpha,k}$ is a function of modular weight $k - 1$, and therefore can be written in terms of $E_4$ and $E_6$ alone.

The final ingredient is to invert $\phi_i(v_j, \tau)$, to obtain $v_j(\phi_i, \tau)$, and consider its modular properties. The fact that such an inversion is possible follows from the implicit function theorem and the fact that at $m = 0$ one has $\phi_j = \omega_1 v_j$, where $\omega_1$ is one of the periods of the base torus. We now need to show that $\phi(v_j, \tau)$ and hence $v_j(\phi_i, \tau)$ can be written in terms of $E_2, E_4$ and $E_6$.

By definition, $\partial_{u_k}\phi_j = \alpha_{jk}$ and $\partial_{u_k}\phi_{D,j} = \alpha_{D,jk}$, where $(\alpha_{ij}, \alpha_{D,ij})$ is the period matrix of (3.2). Hence:

$$\frac{\partial}{\partial u_k} (\phi_{D,j} - \tau \phi_j) = (\alpha_{D,jk} - \tau \alpha_{jk}) = (\tilde{\Omega}_{ji} - \Omega_{ji}) \alpha_{ik} .$$ \hspace{1cm} (3.15)

We have just seen that the left-hand-side has an expansion in terms $E_4$ and $E_6$, and (3.12) along with the modular properties of $h$ show that $(\tilde{\Omega}_{ik} - \Omega_{ik})$ has an expansion in terms of $E_2, E_4$ and $E_6$. Therefore $\alpha_{ik}$ has such an expansion, which in turn implies that $\phi_j(v_k, \tau)$ has such an expansion.

The conclusion is that when we substitute $v_j(\phi_i, \tau)$ into a series of the form (3.14), one gets a series of the form:

$$\phi_{D,i} - \tau \phi_i = \sum_{k=0}^{\infty} \sum_{\alpha} G_{\alpha,k}(\tau) \frac{m^{k+1}}{Q_{\alpha,k}(\phi_\ell)} ,$$ \hspace{1cm} (3.16)

where $G_{\alpha,k}$ can be written in terms of $E_2, E_4$ and $E_6$, and has a modular weight of $k - 1$ (if one ignores the anomalous modular behaviour of $E_2$). Moreover, the $E_2$ dependence here is inherited implicitly via the $v_j$.

The steps of the proof now go much as in section 2. First, from the definition of $\phi_{D,i}$ and $\phi_i$, one has

$$\phi_i = \int_{u_k}^{u_k} \alpha_{ij} \, du_j , \quad \phi_{D,i} - \tau \phi_i = \int_{u_k}^{u_k} (\alpha_{D,ij} - \tau \alpha_{ij}) \, du_j .$$ \hspace{1cm} (3.17)

Since the $E_2$ dependence comes from the implicit dependence in $u_k$, one has

$$\frac{\partial (\phi_{D,i} - \tau \phi_i)}{\partial E_2} = (\alpha_{D,ij} - \tau \alpha_{ij}) \frac{\partial u_j}{\partial E_2} = (\tilde{\Omega}_{ik} - \tau \delta_{ik}) \alpha_{kj} \frac{\partial u_j}{\partial E_2}$$

$$= \frac{\partial (\phi_{D,i} - \tau \phi_i)}{\partial \phi_k} \alpha_{kj} \frac{\partial u_j}{\partial E_2} .$$ \hspace{1cm} (3.18)
Differentiating the first equation in (3.17) with respect to \( E_2 \) yields:

\[
0 = \alpha_{ij} \frac{\partial u_j}{\partial E_2} + \int u_k \left( \frac{\partial \alpha_{ij}}{\partial E_2} \right)_u \, du_j , \tag{3.19}
\]

where the differentiation of \( \alpha_{ij} \) is done with respect to the explicit \( E_2 \) dependence, rather that the implicit dependence via \( u_j \). Here and below we adopt the standard thermodynamic notation in which \( (\partial_x f)_y \) denotes the derivative of \( f \) with respect to \( x \) holding \( y \) constant. From (3.13) and the fact that \( \phi_D - \Omega \phi \) has modular weight \(-1\), it follows that \( (\partial_{E_2}(H \alpha))_u = 0 \). Moreover, inverting (3.12) one has \( (\partial_{E_2}H)_u = -\frac{2\pi i}{12}H^2 \), exactly as in (2.15). Combining these two facts one sees that:

\[
H \left( \frac{\partial \alpha}{\partial E_2} \right)_u = \frac{2\pi i}{12} H^2 \alpha . \tag{3.20}
\]

Using this, (3.19) and (3.17) in (3.18), one finally arrives at

\[
\frac{\partial}{\partial E_2}(\phi_{D,i} - \tau \phi_i) = -\frac{2\pi i}{24} \frac{\partial(\phi_{D,i} - \tau \phi_k)}{\partial \phi_k} (\phi_{D,k} - \tau \phi_k) \tag{3.21}
\]

\[
= -\frac{2\pi i}{24} \frac{\partial}{\partial \phi_i} \left( (\phi_{D,k} - \tau \phi_k)(\phi_{D,k} - \tau \phi_k) \right) .
\]

from which the one also obtains a recursion relation like (2.23) for the pre-potential,

\[
\partial_{E_2} \tilde{F} = -\frac{2\pi i}{24} \left( \frac{\partial_{\phi_i}\tilde{F}}{\partial_{\phi_i}\tilde{F}} \right) \left( \frac{\partial_{\phi_i}\tilde{F}}{\partial_{\phi_i}\tilde{F}} \right) . \tag{3.22}
\]

Note that in going to the second equality of (3.21), and hence to derive the recursion relation for the the pre-potential, we crucially used the integrability of the system: that is, we used the symmetry of \( \frac{\partial \phi_{D,i}}{\partial \phi_{k}} \) in \( i \) and \( k \).

### 3.3. The SU(3) instanton expansion

With the recursion relation in (3.22), we can now start computing the instanton expansion. The method is similar to that in section (2.2). The perturbative contribution to the pre-potential is

\[
\tilde{F}_{pert} = -\frac{1}{4\pi i} \sum_{i<j} \left( (\phi_{ij})^2 \log(\phi_{ij})^2 - \frac{1}{2} (\phi_{ij} + m)^2 \log(\phi_{ij} + m)^2 \right.
\]

\[
\left. + (\phi_{ij} - m)^2 \log(\phi_{ij} - m)^2 \right) , \tag{3.23}
\]

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where $\phi_{ij} = \phi_i - \phi_j$. As an expansion in $m^2$, we have

$$\tilde{F}_{\text{pert}} = \frac{1}{4\pi i} \sum_{i<j} \left( (\log(\phi_{ij})^2 + 3) m^2 + \frac{m^4}{6(\phi_{ij})^2} + \frac{m^6}{30(\phi_{ij})^4} + \frac{m^8}{84(\phi_{ij})^6} + O(m^8) \right) \tag{3.24}$$

The full pre-potential, as an expansion in $m$ has the form

$$\tilde{F} = \frac{1}{4\pi i} f_1(\tau, \phi_i) m^2 - \frac{1}{2\pi i} \sum_{n=2} f_n(\tau, \phi_i) m^{2n}. \tag{3.25}$$

The recursion relation for the $f_n$ is then

$$\partial_{E_2} f_n = \frac{n-1}{12} \sum_{m=1}^{n-1} \frac{(\partial_{\phi_i} f_m)(\partial_{\phi_i} f_{n-m})}{(2m-2)(2n-2m-2)}. \tag{3.26}$$

Unlike the perturbative piece, which only has terms of the form $(\phi_{ij})^{-2l}$, the instanton contributions will have terms that combine the different $\phi_{ij}$. The recursion relation will explicitly generate such terms. One then needs to adjust the $E_2$ independent pieces of the $f_n$ so that the $q$ expansion has no leading order terms that have this mixed form.

The first four $f_n$ for the general $SU(n)$ theory are

$$f_1 = \sum_{i<j} \log(\phi_{ij})^2$$

$$f_2 = \frac{E_2}{6} \sum_{i<j} \frac{1}{\phi_{ij}^2}$$

$$f_3 = \left( \frac{E_2^2}{18} + \frac{E_4}{90} \right) \sum_{i<j} \frac{1}{\phi_{ij}^4} - \left( \frac{E_2^2}{144} - \frac{E_4}{144} \right) \sum_{i\neq j \neq k \neq i} \frac{1}{\phi_{ij}^2 \phi_{ik}^2}$$

$$f_4 = \left( \frac{5E_2^3}{216} + \frac{E_2 E_4}{90} + \frac{11E_6}{7560} \right) \sum_{i<j} \frac{1}{\phi_{ij}^6} - \left( \frac{E_2^3}{144} - \frac{E_2 E_4}{240} - \frac{E_6}{360} \right) \sum_{i\neq j \neq k \neq i} \frac{1}{\phi_{ij}^4 \phi_{ik}^2}$$

$$+ \left( \frac{E_2^3}{288} - \frac{E_2 E_4}{480} - \frac{E_6}{720} \right) \sum_{i<j<k} \frac{1}{\phi_{ij}^2 \phi_{ik}^2 \phi_{jk}^2}$$

$$+ \left( \frac{E_2^3}{432} - \frac{E_2 E_4}{144} + \frac{E_6}{216} \right) \sum_{i \neq j \neq k \neq l} \frac{1}{\phi_{ij}^2 \phi_{ik}^2 \phi_{il}^2} \tag{3.27}$$

Because of symmetry, the sums in (3.27) come with overall integer factors. Taking this into account, the $q$ expansions of these $f_n$ have coefficients that are integer multiples of $2n-2$. This leads us to conjecture that the expansion of $\partial_{\tau} \tilde{F}/m^2$ in $m^2/M_i^2$ and $q$, where the $M_i$ are the masses of the charged vector bosons, has only integer coefficients.
The number of different types of terms in $f_m$ increases rapidly as $m$ is increased, thus for relatively high rank gauge groups, the expansions become unwieldy. There is another problem with the computation of higher rank gauge groups which involves the gaps. For general $SU(n)$, when flowing from the $N = 4$ theory to the $N = 2$ SYM, the scaling is $m^{2n}q = \Lambda^{2n}$, with $\Lambda$ finite as $m \to \infty$. Therefore, $f_{nm+1}$ has one more gap than $f_{nm}$ and so the number of gaps jumps by one for every $n$ terms in the expansion in (3.25). This presents a problem for $SU(7)$ and higher. For these groups, the $f_7$ term does not have a gap. But we need more information than what is given by the perturbative expansion in order to find $f_7$. This is because the $E_2$ independent modular form that is contained in $f_7$ has weight 12 and the space of such forms has dimension 2. Therefore, we have to know at least some of the one instanton contribution in order to compute the coefficients in front of the $E_4^3$ and $E_6^2$ terms. Likewise, when we get to $f_{6m+1}$ we will need to know some of the contributions of the first $m$ instanton terms in order to find the coefficients of the $m + 1$ terms that span the modular forms of weight $12m$.

On the other hand, if the rank is low then the recursion relation is very useful and one can use Mathematica to generate the instanton expansion to high order. Let us consider the case of $SU(3)$. Instead of writing the expansion in terms of $\phi_{ij}$, it is more convenient to use the invariants $U$ and $\Delta$, where

$$U = \phi_{12}^2 + \phi_{13}^2 + \phi_{23}^2, \quad \Delta = \phi_{12}^2 \phi_{13}^2 \phi_{23}^2. \quad (3.28)$$

We were able to compute the first 24 terms in the expansion of (3.25), which is enough information to completely determine the first eight instantons. The contribution of the first four are

\begin{align*}
\mathcal{F}_1 &= \left( m^4 \frac{U^2}{2\Delta} - m^6 \frac{U}{\Delta} \right) q \\
\mathcal{F}_2 &= \left( m^4 \frac{3U^2}{2\Delta} + m^6 \left( \frac{18U}{\Delta} - \frac{3U^4}{4\Delta^2} \right) + m^8 \left( \frac{5U^6}{64\Delta^3} - \frac{3U^3}{4\Delta^2} - \frac{27}{\Delta} \right) \right. \\
&\quad \left. + m^{10} \left( -\frac{5U^5}{16\Delta^3} + \frac{9U^2}{\Delta^2} \right) + m^{12} \left( \frac{5U^4}{16\Delta^3} - \frac{21U^2}{2\Delta^2} \right) \right) q^2 \\
\mathcal{F}_3 &= \left( m^4 \frac{2U^2}{\Delta} + m^6 \left( \frac{116U}{\Delta} - \frac{4U^4}{\Delta^2} \right) + m^8 \left( \frac{5U^6}{2\Delta^3} - \frac{96U^3}{\Delta^2} + \frac{144}{\Delta} \right) \right. \\
&\quad \left. + m^{10} \left( -\frac{7U^8}{12\Delta^4} + \frac{67U^5}{3\Delta^3} + \frac{26U^2}{\Delta^2} \right) + m^{12} \left( \frac{3U^{10}}{64\Delta^5} - \frac{U^7}{4\Delta^4} - \frac{161U^4}{2\Delta^3} + \frac{436U^2}{\Delta^2} \right) \right. \\
&\quad \left. + m^{14} \left( -\frac{9U^9}{32\Delta^5} + \frac{207U^6}{16\Delta^4} - \frac{27U^3}{\Delta^3} - \frac{396}{\Delta^2} \right) + m^{16} \left( \frac{9U^8}{16\Delta^5} - \frac{125U^5}{4\Delta^4} + \frac{278U^2}{\Delta^3} \right) \right) q^3
\end{align*}
\[ \mathcal{F}_4 = \left( m^4 \frac{7U^2}{2\Delta} + m^6 \frac{332U}{\Delta} - \frac{45U^4}{4\Delta^2} \right) + m^8 \frac{1095U^6}{64\Delta^3} - \frac{2925U^3}{4\Delta^2} + \frac{2025}{\Delta} \]
\[ + m^{10} \left( -\frac{777U^8}{64\Delta^4} + \frac{512U^5}{8\Delta^3} - \frac{4941U^2}{\Delta^2} \right) + m^{12} \left( \frac{2151U^{10}}{512\Delta^5} - \frac{7647U^7}{32\Delta^4} + \frac{2118U^4}{\Delta^3} \right) \]
\[ + \frac{4905U}{\Delta^2} + m^{14} \left( -\frac{715U^{12}}{1024\Delta^6} + \frac{2139U^9}{64\Delta^5} + \frac{585U^6}{4\Delta^4} - \frac{18297U^3}{2\Delta^3} + \frac{3240}{\Delta^2} \right) \]
\[ + m^{16} \left( \frac{1469U^{14}}{32768\Delta^7} + \frac{1157U^{11}}{2048\Delta^6} - \frac{101781U^8}{512\Delta^5} + \frac{108279U^5}{32\Delta^4} - \frac{8901}{\Delta^3} \right) \]
\[ + m^{18} \left( -\frac{1469U^{14}}{4096\Delta^7} + \frac{2587U^{10}}{128\Delta^6} - \frac{387U^7}{32\Delta^5} - \frac{13323}{2\Delta^4} + \frac{14805U}{3\Delta^3} \right) \]
\[ + m^{20} \left( \frac{4407U^{12}}{4096\Delta^7} - \frac{41145U^9}{128\Delta^6} - \frac{22203U^6}{16\Delta^5} + \frac{1773U^3}{2\Delta^4} - \frac{9639}{\Delta^3} \right) \]
\[ + m^{22} \left( -\frac{1469U^{11}}{1024\Delta^7} + \frac{14937U^8}{128\Delta^6} - \frac{2484U^5}{\Delta^5} + \frac{10407U^2}{\Delta^4} \right) \]
\[ + m^{24} \left( \frac{1469U^{10}}{2048\Delta^7} - \frac{3861U^7}{64\Delta^6} + \frac{21777U^4}{16\Delta^5} - \frac{26229U}{4\Delta^4} \right) \bigg) q^4. \]

We can easily obtain the \( N = 2 \) SU(3) SYM instanton expansion from the mass deformed \( N = 4 \) expansion by taking the limit \( m \to \infty \) and \( q \to 0 \), and keeping finite \( \Lambda^6 = m^6q \). The first eight terms in this instanton expansion are

\[ F_1 = - \left( \frac{\Lambda^6}{\Delta} \right) U \]
\[ F_2 = - \left( \frac{\Lambda^6}{\Delta} \right)^2 U \left( \frac{21}{2} - \frac{5}{16} \psi \right) \]
\[ F_3 = - \left( \frac{\Lambda^6}{\Delta} \right)^3 U \left( \frac{682}{3} - \frac{265}{12} \psi + \frac{3}{8} \psi^2 \right) \]
\[ F_4 = - \left( \frac{\Lambda^6}{\Delta} \right)^4 U \left( \frac{26229}{4} - \frac{21777}{16} \psi + \frac{3861}{64} \psi^2 - \frac{1469}{2048} \psi^3 \right) \]
\[ F_5 = - \left( \frac{\Lambda^6}{\Delta} \right)^5 U \left( \frac{220878}{80709} - \frac{53499}{8} \psi^2 - \frac{244347}{1280} \psi^3 + \frac{4471}{2560} \psi^4 \right) \]
\[ F_6 = - \left( \frac{\Lambda^6}{\Delta} \right)^6 U \left( \frac{8201045}{2} - \frac{9420803}{32} \psi^2 + \frac{20093193}{1536} \psi^3 - \frac{48449263}{8192} \psi^4 \right) \]
\[ + \frac{1358263}{2048} \psi^4 - \frac{40397}{8192} \psi^5 \]
\[ F_7 = - \left( \frac{\Lambda^6}{\Delta} \right)^7 U \left( \frac{2278827252}{7} - 272724552\psi + \frac{749523903}{14}\psi^2 - \frac{3702963825}{896}\psi^3 \\ + \frac{37821921}{256}\psi^4 - \frac{35234427}{14336}\psi^5 + \frac{441325}{28672}\psi^6 \right) \]

\[ F_8 = - \left( \frac{\Lambda^6}{\Delta} \right)^8 U \left( \frac{108545170581}{8} - \frac{251496872289}{16}\psi + \frac{274083715485}{64}\psi^2 \\ - \frac{961670300877}{2048}\psi^3 + \frac{205509716343}{8192}\psi^4 - \frac{18914175245}{262144}\psi^5 \\ + \frac{10012215681}{1048576}\psi^6 - \frac{866589165}{16777216}\psi^7 \right) \] (3.30)

where \( \psi = U^3 / \Delta. \)

4. Recursion relations for other simply laced groups

Even though the Donagi-Seiberg-Witten curve is not known, we can still make a reasonable guess for a recursion relation in the instanton expansion for the \( SO(2n) \) and \( E_n \) groups. Recall that the recursion relation for \( SU(4) \) is

\[ \partial_{E^2} \tilde{F} = -\frac{2\pi i}{24} \sum_{i=1}^{4} \left( \partial_{\phi_i} \tilde{F} \right) \left( \partial_{\phi_i} \tilde{F} \right). \] (4.1)

The perturbative contribution to the pre-potential is

\[ \tilde{F}_{pert} = -\frac{1}{8\pi i} \sum_{i<j}^{4} \left( (\phi_i - \phi_j)^2 \log(\phi_i - \phi_j)^2 - \frac{1}{2} (\phi_i - \phi_j + m)^2 \log(\phi_i - \phi_j + m)^2 \right. \]

\[ + (\phi_i - \phi_j - m)^2 \log(\phi_i - \phi_j - m)^2 \left. \right) \] (4.2)

Since \( SU(4) \cong SO(6) \), we should be able to express the pre-potential and the recursion relation in terms of three \( SO(6) \) variables \( a_i \). To this end let

\[ a_1 = \frac{1}{2} (\phi_1 - \phi_2 - \phi_3 + \phi_4) \]

\[ a_2 = \frac{1}{2} (-\phi_1 + \phi_2 - \phi_3 + \phi_4) \]

\[ a_3 = \frac{1}{2} (-\phi_1 - \phi_2 + \phi_3 + \phi_4) \]

\[ a_4 = \frac{1}{2} (\phi_1 + \phi_2 + \phi_3 + \phi_4). \] (4.3)
Clearly the recursion relation becomes

\[ \partial E_2 \tilde{F} = -\frac{2\pi i}{24} \sum_{i=1}^{3} \partial a_i \tilde{F} \partial a_i \tilde{F}. \tag{4.4} \]

The perturbative piece has no \( a_4 \) dependence, and as a result \( \partial a_4 \tilde{F} = 0 \). Hence, we can restrict the sum in (4.4) from 1 to 3. The perturbative pre-potential in \( SO(6) \) coordinates is

\[
\tilde{F}_{\text{pert}} = \frac{-1}{8\pi i} \sum_{i<j}^{3} \left( (a_i - a_j)^2 \log(a_i - a_j)^2 + (a_i + a_j)^2 \log(a_i + a_j)^2 \\
- \frac{1}{2} \left( (a_i - a_j + m)^2 \log(a_i - a_j + m)^2 + (a_i - a_j - m)^2 \log(a_i - a_j - m)^2 \\
+ (a_i + a_j + m)^2 \log(a_i + a_j + m)^2 + (a_i + a_j - m)^2 \log(a_i + a_j - m)^2 \right) \right). 
\tag{4.5}\]

Given the form of the \( SO(6) \) recursion relation, it does not take a great leap of faith to postulate that the recursion relation for general \( SO(2n) \) is

\[ \partial E_2 \tilde{F} = -\frac{2\pi i}{24} \sum_{i=1}^{n} \partial a_i \tilde{F} \partial a_i \tilde{F}. \tag{4.6} \]

The perturbative pre-potential is the obvious generalization of (4.5). We can then construct an \( m^2 \) expansion as in (3.25) for \( \tilde{F} \). For general \( SO(2n) \), the first three terms in the expansion are

\[
f_1 = \sum_{i<j} \log \left( \phi_i^2 - \phi_j^2 \right)^2 \\
f_2 = \frac{E_2}{6} \sum_{i<j} \left( \frac{1}{(\phi_i - \phi_j)^2} + \frac{1}{(\phi_i + \phi_j)^2} \right) \\
f_3 = \left( \frac{E_2^2}{18} + \frac{E_4}{90} \right) \sum_{i<j} \left( \frac{1}{(\phi_i - \phi_j)^4} + \frac{1}{(\phi_i + \phi_j)^2} \right) - \left( \frac{E_2^2}{144} - \frac{E_4}{144} \right) \times \\
\times \sum_{i \neq j \neq k \neq i} \left( \frac{1}{(\phi_i - \phi_j)^2(\phi_i - \phi_k)^2} + \frac{1}{(\phi_i - \phi_j)^2(\phi_i + \phi_k)^2} + \frac{1}{(\phi_i + \phi_j)^2(\phi_i + \phi_k)^2} \right). \tag{4.7}\]

As for the \( E_n \) gauge groups, \( E_8 \) has the same recursion relation as \( SO(16) \). However, the perturbative contribution to the pre-potential is modified to include contributions from
the $SO(16)$ spinor. This change in the perturbative pre-potential propagates through the entire instanton expansion through the recursion relation.

The $E_7$ recursion relation has a sum of an $SO(12)$ and an $SU(2)$ piece on the right hand side. The perturbative pre-potential has contributions from the adjoints of each of these groups, plus a contribution from the $(32, 2)$ that fills out the $E_7$ adjoint. $E_6$ has a sum of an $SO(10)$ and a $U(1)$ on the right hand side of the recursion relation. The perturbative piece of the pre-potential has the contribution of the $SO(10)$ adjoint, plus a contribution from both $SO(10)$ spinors with opposite $U(1)$ charges.

5. Discussion

The recursion relation in (3.22) has a form that is reminiscent of a renormalization group equation derived in [17]. These authors find that

$$ \partial_\tau \mathcal{F} = \frac{1}{4\pi i} \sum_{j=1}^{n} \oint_{a_j} t^2 d\xi , \quad (5.1) $$

where $t$ and $\xi$ are Riemann surface coordinates defined in (3.2) and (3.3), and the integrals are around the $a_j$ periods of the surface (3.2). Expanding the right-hand side of (5.1) in powers of $q$ leads to the instanton expansion. The authors explicitly computed the entire two instanton contribution for any $SU(n)$. The two instanton contribution has powers of $m$ up to $m^{4n}$. So to some extent, the approach of [17] and our approach are complementary; for a given instanton number they can find contributions for arbitrarily high powers of $m$, while for a given power of $m$ we know the contributions of arbitrarily high powers of $q$.

There are many possible avenues for further development of the ideas presented here. One open problem is how to go beyond the limit $h = 6$, where $h$ is the dual Coxeter number. Recall that if $h \leq 6$, then one can use the perturbative expansion and the gaps to completely determine, order by order, the $m$ expansion of $\mathcal{F}$. For $h > 6$ the presence of a gap, and knowledge of all the lower order terms, is insufficient to fix all the terms in the expansion. However, our recursion relation does generate an overdetermined system of equations for the coefficients in the instanton expansion, particularly at higher orders. It is therefore conceivable that gaps and consistency at higher orders in the instanton expansion could resolve indeterminacy in the lower order terms. Thus, while technically more complex, it is possible that our recursion relation and the gaps could still determine the series for $h > 6$. It would also be valuable to see if one could combine our recursion relation with (5.1) and get deeper insight into the structure of the expansion.
It is hoped that the recursion relation in (4.6), or the instanton expansion derived from it, might lead to the Donagi-Witten curve for the $SO(2n)$ or $E_n$ theories. This is not completely farfetched. In [12] the instanton expansion for the $E_8$ string in the $\tau \to i\infty$ limit was used to compute the corresponding Seiberg-Witten curve. In that case, it was known that the expansion is made up of $E_8$ characters. This put restrictions on the form of the curve, and by going to high enough instanton number, was enough to determine all coefficients in the curve. In the $SO(2n)$ case and $E_n$ case, we in principle know the instantons, so it is conceivable that we can go back and find the curve.

Another interesting question is whether the recursion relations found here for the $N = 4$ theories can be generalized to $N = 2$ theories with vanishing $\beta$-functions. For example, the $SU(2)$ gauge theory with $N_f = 4$, with $m_1 = m_2 = m; m_3 = m_4 = 0$ has an identical effective action to the one discussed here. For general values of the mass parameters, if a recursion relation exists, we would expect it to involve Jacobi $\theta$-functions. Another example is the non-critical string with Wilson loops that break the $E_8$ global symmetry to a smaller subgroup. This recursion relation, if it exists, should also involve Jacobi $\theta$-functions.

Finally, there is the issue of the topological interpretation of the pre-potential, and in particular, of the integers that appear in the $\tau$ derivatives of $F_n$. Unlike the $E_8$ non-critical string, the expansions do not have the usual Gromov-Witten form. However, there are two natural ways in which one might hope to find such an explanation: either in terms of the topology of instanton moduli space or in terms of topological amplitudes of non-critical strings.

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