Type II Topp-Leone Dagum distribution for modeling failure times data

K.M. Sakthivel 1* and K. Dhivakar 2

Abstract
In this paper, we introduce four parameter continuous probability distribution and named it as Type II Topp-Leone Dagum distribution which is generated using Type II Topp-Leone generated family of distributions. We observe different desirable properties of Type II Topp-Leone Dagum distribution. We present expressions for important statistical measures such as moments, moment generating function, cumulant generating function, inverted moments, probability weighted moments, reliability function, hazard rate function, reversed hazard function, cumulative hazard function, second failure rate function, mean waiting time, mean residual life, Bonferroni index, Lorenz curve and generalized entropy. For the proposed distribution, the parameters of the distribution are estimated by using maximum likelihood method. Finally, we used failure time of air conditioners data to study performance of the proposed distribution.

Keywords
Type II Topp-Leone generated family, Dagum distribution, Generalized entropy, Probability weighted moment, Maximum likelihood estimation.

AMS Subject Classification
60E05, 62F10, 62E10.

1. Introduction
Statistical modeling is the best and ultimate way of studying uncertainty of any phenomena and statistical analysis of lifetime data are random in applied sciences such as biological sciences, medical sciences, environment sciences, engineering, finance, and actuarial science, among others. Particularly,
2. Proposed Distribution

2.1 Type II Topp-Leone Generated Family

Type II Topp-Leone generated family is proposed by M. Elgarhy et al., [9]. The probability density function and cumulative distribution function are respectively given by

\[
f(x) = 2\tau g(x)G(x)[1 - G^2(x)]^{\tau - 1}, \tau > 0.
\] (2.1)

and

\[
F(x) = 1 - [1 - G^2(x)]^{\tau}, \tau > 0.
\] (2.2)

where, \( \tau \) is a shape parameter. \( G(x) \) and \( g(x) \) are cumulative distribution function and probability density function of the base distribution.

2.2 Type II Topp-Leone Dagum Distribution

A random variable \( X \) is said to have Type II Topp-Leone Dagum distribution with parameters \( \tau, \sigma, \theta \) and \( \beta \). If its probability density function is given by

\[
f(x; \tau, \sigma, \theta, \beta) = 2\tau \sigma \theta \beta x^{-\theta - 1} (1 + \sigma x^{-\theta})^{-\beta - 1} \times [1 - (1 + \sigma x^{-\theta})^{-2\beta}]^{\tau - 1}
\] (2.3)

where, \( x > 0, \tau > 0, \sigma > 0, \theta > 0 \) and \( \beta > 0 \).

Consider binomial series,

\[
(x - y)^r = \sum_{j=0}^{\infty} \binom{r}{j} (-1)^j x^{-j} y^j
\]

Hence,

\[
[1 - (1 + \sigma x^{-\theta})^{-2\beta}]^{\tau - 1} = \sum_{j=0}^{\infty} (-1)^j \binom{\tau - 1}{j} \times (1 + \sigma x^{-\theta})^{-2\beta j}
\] (2.4)

Using (2.3) in (2.4), we have the probability density function of \( X \) is

\[
f(x; \tau, \sigma, \theta, \beta) = \sum_{j=0}^{\infty} (-1)^j \binom{\tau - 1}{j} 2\tau \sigma \theta \beta x^{-\theta - 1} \times (1 + \sigma x^{-\theta})^{-2\beta(j+1)}
\] (2.5)

and the cumulative distribution function (cdf) is

\[
F(x; \tau, \sigma, \theta, \beta) = 1 - \left[1 - (1 + \sigma x^{-\theta})^{-2\beta}\right]^{\tau}
\] (2.6)

with, \( x > 0, \tau > 0, \sigma > 0, \theta > 0 \) and \( \beta > 0 \).

where, \( \sigma \) is a scale parameter, \( \theta \) and \( \beta \) are shape parameter, \( \tau \) is parameter of Type II Topp-Leone Generated family. Type II Topp-Leone Dagum distribution is symbolically denoted by TTHLD \((\tau, \sigma, \theta, \beta)\). Figures 1 to 8, depict the shape of Pdf, Cdf for various values of the parameters of Type II Topp-Leone Dagum distribution.
Figure 1. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 1$, $\sigma = 2$, $\theta = 4$ and different values of $\beta$.

Figure 2. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 4$, $\sigma = 2$, $\beta = 0.5$ and different values of $\theta$.

Figure 3. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 1$, $\theta = 3$, $\beta = 2$ and different values of $\sigma$.

Figure 4. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\sigma = 7$, $\theta = 2$, $\beta = 4$ and different values of $\tau$.

Figure 5. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 2$, $\sigma = 4$, $\theta = 2$ and different values of $\beta$.

Figure 6. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3$, $\sigma = 2$, $\beta = 1$ and different values of $\theta$. 
3. Reliability Measures

If \( X \sim \text{TITLDD}(\tau, \sigma, \theta, \beta) \) then the different reliability measures of random variable are given by

3.1 Reliability function

\[
R(x) = 1 - \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau
\]  
(3.1)

3.2 Hazard rate function

\[
h(x) = \frac{2\tau \sigma \theta \beta x^{-\theta-1} \left( 1 + \sigma x^{-\theta} \right)^{-2\beta-1}}{1 - \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau} \\
\times \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^{\tau-1}
\]  
(3.2)

3.3 Reversed hazard function

\[
r(x) = \frac{2\tau \sigma \theta \beta x^{-\theta-1} \left( 1 + \sigma x^{-\theta} \right)^{-2\beta-1}}{1 - \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau} \\
\times \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^{\tau-1}
\]  
(3.3)

3.4 Cumulative hazard function

\[
H(x) = -\log \left[ 1 - \left( 1 - (1 + \sigma x^{-\theta})^{-2\beta} \right)^\tau \right]
\]  
(3.4)

3.5 Second failure rate function

\[
h(x) = \log \left[ \frac{1 - \left( 1 - (1 + \sigma x^{-\theta})^{-2\beta} \right)^\tau}{\left( 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^\tau \right)} \right]
\]  
(3.5)

3.6 Mean waiting time

Mean waiting time is defined by

\[
\varphi(x) = x - \frac{1}{F(x)} \int_0^x x f(x) dx
\]

The mean waiting time of Type II Topp-Leone Dagum distribution is given by

\[
\varphi(x) = x - \frac{\sum_{j=0}^\infty (-1)^j \left( \frac{\tau-1}{j} \right) 2\tau \sigma \theta \beta x^{-\theta-1}}{1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \tau} \\
\times \left( 1 + \sigma x^{-\theta} \right)^{-2\beta(j+1)-1} dx
\]  
(3.6)

3.7 Mean residual life

Mean residual life is defined by

\[
\phi(x) = \frac{1}{s(x)} \int_x^\infty x f(x) dx - x
\]

\[
\phi(x) = \frac{1}{1 - \left( 1 - (1 + \sigma x^{-\theta})^{-2\beta} \right)^\tau} \\
\times \int_0^\infty x \sum_{j=0}^\infty (-1)^j \left( \frac{\tau-1}{j} \right) 2\tau \sigma \theta \\
\times \beta x^{-\theta-1} \left( 1 + \sigma x^{-\theta} \right)^{-2\beta(j+1)-1} dx - x
\]  
(3.8)
The mean residual life function of Type II Topp-Leone Dagum distribution is given by

\[ \phi(x) = \sum_{j=0}^{\infty} (-1)^j (\tau - 1)^{2j\beta \sigma \theta} B(1 - \frac{1}{\theta}, 2\beta(j + 1) + 1, \frac{1}{\theta}) \left(1 - \left(1 - (1 + \sigma x^{-\theta})^{-2\beta}\right)^{\tau - 1}\right) \]

(3.9)

Figures (9) to (16) depict the shape of the reliability function and hazard rate of the Type II Topp-Leone Dagum distribution.

Figure 9. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of \( \tau = 2.2, \sigma = 3.8, \theta = 2.3 \) and different values of \( \beta \).

Figure 10. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of \( \tau = 3.3, \sigma = 2.4, \beta = 1.4 \) and different values of \( \theta \).

Figure 11. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of \( \tau = 3.3, \theta = 2.3, \beta = 1.2 \) and different values of \( \sigma \).

Figure 12. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of \( \sigma = 5.8, \theta = 4.3, \beta = 10 \) and different values of \( \tau \).

Figure 13. Hazard rate function of Type II Topp-Leone Dagum distribution for fixed value of \( \tau = 3, \sigma = 2, \theta = 4 \) and different values of \( \beta \).
4. Statistical Properties

4.1 Moments

The moment about the origin is defined by

\[
\mu'_r = \int_{-\infty}^{\infty} x^r f(x) \, dx, \quad \text{when } X \text{ is continuous.}
\]

\[
\mu'_r = \int_{-\infty}^{\infty} x^r \left[ \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \sigma \theta \beta x^{-\theta - 1} \right] \, dx
\]

\[
(1 + \sigma x^{-\theta})^{-2\beta(j+1)-1} \, dx
\]

\[
= \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}} \int_{0}^{\infty} \frac{u^{-\beta}}{(1+u)^{2\beta(j+1)+1}} \, du
\]

Hence, the \( r \)-th moment of Type II Topp-Leone Dagum distribution is given by

\[
\mu'_r = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}} B(1 - \frac{r}{\theta}, 2\beta(j+1) + \frac{r}{\theta})
\]

In particular,

\[
E(X) = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}}
\]

\[
\times B(1 - \frac{1}{\theta}, 2\beta(j+1) + \frac{1}{\theta})
\]

\[
E(X^2) = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}}
\]

\[
\times B(1 - \frac{2}{\theta}, 2\beta(j+1) + \frac{2}{\theta})
\]

\[
E(X^3) = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}}
\]

\[
\times B(1 - \frac{3}{\theta}, 2\beta(j+1) + \frac{3}{\theta})
\]

\[
E(X^4) = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}}
\]

\[
\times B(1 - \frac{4}{\theta}, 2\beta(j+1) + \frac{4}{\theta})
\]

Variance,

\[
V(X) = \left[ \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}} \right]
\]

\[
\times B(1 - \frac{2}{\theta}, 2\beta(j+1) + \frac{2}{\theta})
\]

\[
- \left[ \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^{\frac{\beta}{\theta}} \right]^2
\]

\[
\times B(1 - \frac{1}{\theta}, 2\beta(j+1) + \frac{1}{\theta})^2
\]
4.2 Moment generating function

The moment generating function of the random variable $X$ is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx,$$

where $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x) \, dx$$

The moment generating function of Type II Topp-Leone Dagum distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.4)

4.3 Characteristic function

The characteristic function of the random variable $X$ is defined by

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x) \, dx,$$

where $e^{itx} = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!}$

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_{0}^{\infty} x^r \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.5)

The characteristic function of Type II Topp-Leone Dagum distribution is given by

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.6)

4.4 Cumulant generating function

Cumulant generating function is defined by

$$K_X(t) = \log M_X(t)$$

The cumulant generating function of Type II Topp-Leone Dagum distribution is given by

$$K_X(t) = \log \left[ \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right) \right]$$

(4.7)

4.5 Inverted moments

The $r^{th}$ inverted moment is defined by

$$\mu_r^+ = \int_{-\infty}^{\infty} x^{-r} f(x) \, dx$$

$$\mu_r^+ = \int_{0}^{\infty} x^{-r} \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.8)

The $r^{th}$ inverted moment of Type II Topp-Leone Dagum distribution is given by

$$\mu_r^+ = \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) - \frac{r}{\theta}\right)$$

(4.9)

4.6 Central moments

The $r^{th}$ central moment is defined by

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu_1)^r f(x) \, dx$$

$$\mu_r = \sum_{m=0}^{\infty} \binom{r}{m} (-1)^m (\mu_1)^m (\mu_r - m)$$

The $r^{th}$ central moment of Type II Topp-Leone Dagum distribution is given by

$$\mu_r = \sum_{m=0}^{\infty} \binom{r}{m} (-1)^m \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.11)

4.7 Probability weighted moments

The probability weighted moment of the random variable $X$ is defined by

$$\tau_{r,h} = \mathbb{E} \left[ X^r F(x)^h \right] = \int_{-\infty}^{\infty} x^r F(x)^h f(x) \, dx$$

$$\tau_{r,h} = \int_{0}^{\infty} x^r \left[ \sum_{j=0}^{\infty} \left( -1 \right)^j \left( \frac{\tau - 1}{j} \right) 2\beta \sigma \right] \times B\left(1 - \frac{r}{\theta}, 2\beta (j + 1) + \frac{r}{\theta}\right)$$

(4.12)

Here we use binomial series expansion

$$(x - y)^r = \sum_{j=0}^{\infty} \binom{r}{j} (-1)^j x^{r-j} y^j$$
It is to be noted that,

\[
[1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^{-1}]^h \sum_{l=0}^{m} \binom{h}{j} \left( \begin{array}{c} l \\ m \\ j \end{array} \right) (1 + \sigma x^{-\theta})^{-2\beta m}
\]

Hence,

\[
\tau_{rh} = \sum_{j,l,m=0}^{\infty} (-1)^{i+l+m} \left( \begin{array}{c} \tau - l \\ j \end{array} \right) \left( \begin{array}{c} l \\ m \end{array} \right) 2\tau \sigma \theta \beta \\
\times \int_{0}^{\infty} x^{-\theta-1}(1 + \sigma x^{-\theta})^{-2\beta(j+1)-1}dx
\]

\[
= \sum_{j,l,m=0}^{\infty} (-1)^{i+l+m} \left( \begin{array}{c} \tau - l \\ j \end{array} \right) \left( \begin{array}{c} l \\ m \end{array} \right) 2\tau \sigma \theta \beta
\times \int_{0}^{\infty} \frac{u^{-\tau}}{(1+u)^{2\beta(j+m+1)+1}} du
\]

Thus, the probability weighted moment of Type II Topp-Leone Dagum distribution is given by

\[
\tau_{rh} = \sum_{j,l,m=0}^{\infty} (-1)^{i+l+m} \left( \begin{array}{c} \tau - l \\ j \end{array} \right) \left( \begin{array}{c} l \\ m \end{array} \right) 2\tau \sigma \theta \beta
\times B(1 - \frac{r}{\theta} 2\beta(j+m+1) + \frac{r}{\theta})
\]

\[\text{(4.13)}\]

\[\text{(4.14)}\]

5. Order Statistics

Let \(Y_i\) be the \(i^{th}\) order statistics of the random sample \(X_1, X_2,..., X_n\). The pdf of the \(j^{th}\) order statistics for Type II Topp-Leone Dagum distribution is given by

\[
f_j(x) = \frac{n!}{(j-1)(n-j)!} \left[ 2\tau \sigma \theta \beta x^{-\theta-1} \times (1 + \sigma x^{-\theta})^{-2\beta} \right]^{j-1}
\]

\[
\times \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^{-1} - \left( \begin{array}{c} l \\ m \end{array} \right) \right]^{n-j}
\]

\[\text{(5.1)}\]

The pdf of the smallest order statistics \(X_1\) is given by

\[
f_{(1,n,x)} = n \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^{-1} \right]^{n-1}
\]

\[
\times \left[ 2\tau \sigma \theta \beta x^{-\theta-1} (1 + \sigma x^{-\theta})^{-2\beta} \right]^{n-1}
\]

\[\text{(5.2)}\]

The pdf of the largest order statistics \(X_n\) is given by

\[
f_{(n,n),x} = n \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^{-1} \right]^{n-1}
\]

\[
\times \left[ 2\tau \sigma \theta \beta x^{-\theta-1} (1 + \sigma x^{-\theta})^{-2\beta} \right]^{n-1}
\]

\[\text{(5.3)}\]

And the pdf of the median order statistics is given by

\[
f_{m+1,n}(x) = \frac{(2m+1)}{m!m!} \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^{-1} \right]^m
\times \left[ 1 - (1 - ((1 + \sigma x^{-\theta})^{-2\beta}))^{m} \right]^m
\times 2\tau \sigma \theta \beta x^{-\theta-1} (1 + \sigma x^{-\theta})^{-2\beta-1}
\times \left( 1 - (1 + \sigma x^{-\theta})^{-2\beta} )^{-1} \right]
\]

\[\text{(5.4)}\]

The joint distribution of the \(i^{th}\) and \(j^{th}\) order statistics for \(1 \leq i < j \leq n\) is given by

\[
f_{i,j}(x_i, x_j) = C [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1}
\times [1 - F(x_j)]^{n-j} f(x_i) f(x_j)
\]

where,

\[
C = \frac{n!}{(i-1)! (j-i-1)! (n-j)!}
\]

\[\text{(5.5)}\]

Special case if \(i = 1\) and \(j = n\). We get the joint distribution of minimum and maximum of order statistics

\[
f_{1,n,x}(x_1, x_n) = n(n-1) [F(x_n) - F(x_1)]^{n-2} f(x_1) f(x_n)
\]

\[
f_{1,n,x}(x_1, x_n) = n(n-1)
\]

\[\text{(5.6)}\]

where,

\[
q_1 = (1 + \sigma x_i^{-\theta}), q_n = (1 + \sigma x_n^{-\theta}).
\]
5.1 Generalized entropy

The generalized entropy proposed by Cowell [2] and it is defined by

\[ GE(w, \delta) = \frac{1}{\delta(\delta - 1)\mu} \left[ \int_0^\infty x^\delta f(x)dx \right] - 1 \]

where, \( \delta > 0 \).

\[ GE(w, \delta) = C \cdot \frac{D}{D - 1} \quad (5.7) \]

where,

\[ C = \int_0^\infty x^\delta \left[ \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau\sigma\beta x^{-\theta - 1} \right] dx \]

\[ D = \delta(\delta - 1) \left[ \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 \right] \]

\[ \times B\left(1 - \frac{\delta}{\theta}, 2\beta(j + 1) + \frac{\delta}{\theta}\right) \]

Note that,

\[ \int_0^\infty x^\delta f(x)dx = \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 \]

\[ \times B\left(1 - \frac{\delta}{\theta}, 2\beta(j + 1) + \frac{\delta}{\theta}\right) \]

The Generalized entropy of Type II Topp-Leone Dagum distribution is given by

\[ GE(w, \delta) = \frac{A}{B} - 1 \quad (5.8) \]

where

\[ A = \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 \]

\[ \times B\left(1 - \frac{\delta}{\theta}, 2\beta(j + 1) + \frac{\delta}{\theta}\right) \]

\[ B = \delta(\delta - 1) \left[ \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 \right] \]

\[ \times B\left(1 - \frac{\delta}{\theta}, 2\beta(j + 1) + \frac{\delta}{\theta}\right) \]

5.2 Lorenz curve

Lorenz [15] curve is defined by

\[ L(x) = \frac{1}{\mu} \int_0^x xf(x)dx \]

\[ L(x) = \frac{1}{\mu} \int_0^x \left[ \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma \theta \right] \]

\[ \times \beta x^{-\theta - 1} \left( 1 + \sigma x^{-\theta} \right)^{-2\beta(j+1)-1} dx \]

5.3 Bonferroni index

The Bonferroni index is introduced by Bonferroni [1]. The Bonferroni index is defined by

\[ B(x) = \frac{L(x)}{F(x)} \]

The Bonferroni index of Type II Topp-Leone Dagum distribution is given by

\[ B(x) = \frac{E}{F} \quad (5.10) \]

where,

\[ E = \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 \]

\[ \times B\left(1 - \frac{1}{\theta}, 2\beta(j + 1) + \frac{1}{\theta}\right) \]

\[ F = \sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 B\left(1 - \frac{r}{\theta}, 2\beta(j + 1) + \frac{r}{\theta}\right) \]

\[ \times \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right] \]

5.4 Zenga Index

Zenga [20] index is defined by

\[ Z = 1 - \frac{\bar{\mu}_x}{\mu_v^z} \]

where,

\[ \bar{\mu}_x = \frac{1}{F(x)} \int_0^\infty xf(x)dx \]

\[ \mu_v^z = \frac{1}{1 - F(x)} \int_0^\infty xf(x)dx \]

\[ \bar{\mu}_x = \frac{\sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 B\left(1 - \frac{1}{\theta}, 2\beta(j + 1) + \frac{1}{\theta}\right) \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau}{1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau} \]

and

\[ \mu_v^z = \frac{\sum_{j=0}^\infty (-1)^j \left( \frac{\tau - 1}{j} \right) 2\beta\sigma^2 B\left(1 - \frac{r}{\theta}, 2\beta(j + 1) + \frac{r}{\theta}\right) \left[ 1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau}{1 - \left( 1 + \sigma x^{-\theta} \right)^{-2\beta} \right]^\tau} \]
The Zenga index of Type II Topp-Leone Dagum distribution is given by
\[ Z = 1 - \frac{G}{H} \]  
(5.11)
where
\[ G = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^2 B(1 - \frac{1}{\theta}, 2\beta(j + 1) + \frac{1}{\theta}; \nu) \times \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^\tau \right] \]
\[ H = \sum_{j=0}^{\infty} (-1)^j \left( \frac{\tau - 1}{j} \right) 2\tau \beta \sigma^2 B(1 - \frac{1}{\theta}, 2\beta(j + 1) + \frac{1}{\theta}) \times \left[ 1 - (1 - (1 + \sigma x^{-\theta})^{-2\beta})^\tau \right] \]

6. Parameter estimation

The principle of maximum likelihood essentially assumes that the sample is representative of the population and chooses as the estimator that value of the parameter which maximizes the probability density function. If \( X_1, X_2, \ldots, X_n \) are iid random variables with probability density function \( f_\theta(x_i) \) and the likelihood function is \( L(\theta/x_i) \).

\[ L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) \]

Hence, the likelihood function of Type II Topp-Leone Dagum distribution is given by
\[ L(\theta) = \prod_{i=1}^{n} \left[ 2\tau \sigma^2 \beta \xi_{i}^{-\theta} (1 + \sigma \xi_{i}^{-\theta})^{-2\beta - 1} \times (1 - (1 + \sigma \xi_{i}^{-\theta})^{-2\beta})^{\tau - 1} \right] \]  
(6.1)
and the log likelihood is
\[ logL(\theta) = nlog2 + nlog\tau + nlog\sigma + nlog\theta + nlog\beta \]
\[ + m + (-\theta - 1) \sum_{i=1}^{n} logx_i + (-2\beta - 1) \]
\[ \times (1 - (1 + \sigma \xi_{i}^{-\theta})^{-2\beta})^{\tau - 1} \times (1 - (1 + \sigma \xi_{i}^{-\theta})^{-2\beta}) \]

The partial derivatives first order of log \( L(\theta) \) with respect to parameter \( \tau, \sigma, \theta \) and \( \beta \) is given by
\[ \frac{\partial logL}{\partial \tau} = 0, \quad \frac{\partial logL}{\partial \sigma} = 0, \quad \frac{\partial logL}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial logL}{\partial \beta} = 0. \]
\[ \frac{\partial logL}{\partial \tau} = \frac{n}{\tau} + \sum_{i=1}^{n} \frac{(\tau - 1)}{(1 - (1 + \sigma \xi_{i}^{-\theta})^{-2\beta})} = 0 \]  
(6.2)

\[ \frac{\partial logL}{\partial \sigma} = \frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(-2\beta - 1)x_i}{(1 + \sigma x_i^{-\theta})^\theta} \]
\[ + \sum_{i=1}^{n} \frac{(\tau - 1)2\beta (1 + \sigma x_i^{-\theta})^{-2\beta - 1} x_i^{-\theta}}{(1 - (1 + \sigma x_i^{-\theta})^{-2\beta})} = 0 \]  
(6.3)
\[ \frac{\partial logL}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} logx_i + \sum_{i=1}^{n} \frac{(-\beta - 1)\sigma x_i^{-\theta} log(-\theta)}{(1 + \sigma x_i^{-\theta})} = 0 \]  
(6.4)
and
\[ \frac{\partial logL}{\partial \beta} = \frac{\beta}{n} - 2 \sum_{i=1}^{n} log(1 + \sigma x_i^{-\theta}) \]
\[ - \sum_{i=1}^{n} \frac{(\tau - 1)(1 + \sigma x_i^{-\theta})^{-2\beta} log(-2\beta)}{(1 - (1 + \sigma x_i^{-\theta})^{-2\beta})} = 0 \]  
(6.5)

The above mentioned four nonlinear equations can not solve analytically. We can solve these four nonlinear equations through numerical methods like Quasi–Newton method. However, we can also solve the above-mentioned nonlinear equations using the R software.

7. Application

We consider the data set given in Proschan [16]. The data set is about the duration of time between successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes. This data set studied by different authors including Huang and Oluyede [12] and Elbatal and Aryal [8]. The descriptive statistics is provided in Table 7.1 for Type II Topp-Leone Dagum distribution. We estimated the parameters for type II Topp-Leone Dagum distribution using method of estimation of maximum likelihood function. The estimated parameters values are given in table 7.2.

| n   | Mean | Median | Min | Max  | Q1   | Q3  |
|-----|------|--------|-----|------|------|------|
| 188 | 92.07| 54.00  | 1.00| 603  | 20.75| 603.00|

| Model | \( \sigma \) | \( \theta \) | \( \beta \) | \( \lambda \) | \( \tau \) | \( \phi \) |
|-------|--------------|--------------|-------------|-------------|-----------|---------|
| TITLDD| 0.0100.998   | 1.928        | 0.246       | -           | 4.592     | -       |
| KD    | 0.005.035    | 4.385        | 0.376       | -           | -         | 21.704  |
| TDD   | 1574.602     | 1.659        | 0.674       | 0.167       | -         | -       |
| DD    | 0.094.153    | 1.263        | 1.239       | -           | -         | -       |
And we obtained expression for reliability function, hazard function, second failure rate function, mean waiting time function, second failure rate function, mean waiting time function, and mean residual life function. The proposed distribution provided the better fit compared to other three models.

### 8. Conclusion

In this research paper, we introduced the Type II Topp-Leone Dagum distribution. This proposed distribution is generated from Type II Topp-Leone generating family distribution studied by M. Elgarhy et al., [9]. For the proposed distribution, we have study graphical behavior of probability density function, cumulative distribution function, hazard rate function and reliability function for the different parameters values. We have obtained some mathematical and statistical properties like moment, moment generating function, characteristic function, cumulant generating function, inverted moment, central moment, probability weighted moment, order statistics, Generalized entropy, Lorenz curve, Bonferroni index, Zenga index. And we obtained expression for reliability function, hazard rate function, reversed hazard rate function, cumulant hazard function, second failure rate function, mean waiting time and mean residual life function. The proposed distribution parameters are estimated by method of maximum likelihood. Finally, a real data set used to fit the proposed distribution, and it provided greater flexibility compared to other suitable probability models.

### References

[1] C. Bonferroni, Elmenti di statistica generale. Libreria Seber, Firenze,(1930).

[2] F.A. Cowell, On the structure of additive inequality measures, The Rev. of Eco.Stud., 47(1980), 521-531.

[3] G. M. Cordeiro, M. De Castro, A new family of generalized distributions, Jour. of Stat. and Comp. Simu., 81(2009), 883-893.

[4] C. Dagum, A new model of personal income distribution: specification and estimation, Eco. Appli., 30(1977), 413-437.

[5] C. Dagum, The generation and distribution of income, the Lorenz curve and the Gini ratio, Eco. Appli., 33(1980), 327-367.

[6] F. Domma, Landamento della Hazard function nel modello di Dagum a tre parametri, Quaderni di Statistica., 4(2002), 103-114.

[7] F. Domma, S. Giordano and M. Zenga, Maximum Likelihood Estimation in Dagum distribution from Censored Samples, J.of Appl. Stat., 38(2011), 2971-2985.

[8] I. Elbatal, G. Aryal, Transmuted Dagum Distribution, Chile. J. of Statist., 2(2015), 31-45.

[9] M. Elgarhy, M. Arslan Nasir, Farrukh Jamal, and Ganzhe Ozel, The type II Topp-Leone generated family of distributions: Properties and application, J. of stat. and mana. syst., (2018), 1529-1551.

[10] R. C. Gupta, P. L. Gupta, and R. D. Gupta, Modeling failure data by Lehmann alternatives, Com. in Stat. Theory and Methods., 27(1998), 887-904.

[11] A.S. Hassan, M. Elgarhy, A new family of exponentiated Weibull-generated distributions, Inter J. of Mat. and its Appl., 4(2016), 135–148.

[12] S. Huang and B.O. Oluyede, Exponentiated Kumaraswamy-Dagum distribution with applications to income and lifetime data, J.of Stat. Distr. and Appli., (2014), 1-20.

[13] C. Kleiber and Kotz, Statistical Size Distribution in Economics and Actuarial Sciences, new York: John Wiley and Sons Inc,(2003).

[14] C. Kleiber, A guide to the Dagum distribution: Duangkaamon, C.Modeling Income Distributions and Lorenz Curves Series: Economic Studies in Inequality, Social Exclusion and Well-Being, 5(2008), Springer, New York.

[15] M.O. Lorenz, Methods of measuring the concentration of wealth, Amer. Stat. Assoc., 9(1905), 209-219.

[16] F. Proschan, Theoretical explanation of observed decreasing failure rate, Technometrics., 5(1963), 375-383.

[17] G. Pietra, Della relazioni fra indici di variabilità, not I e II. Atti del Reale Istituto Veneto di Scienze, it Lettereed Arti., 74(1915), 775-804.

[18] W. Shaw, I. Buckley, The alchemy of probability distributions: beyond Gram–Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, arXiv:(2009), 0901-0434.

[19] C. W. Topp, F. C. Leone, A family of J-shaped frequency functions, J. of the Amer. Stat. Assoc., 50(1955), 209-219.

[20] M. Zenga, La curtosi (Kurtosis), Statistica., 56(1996), 87-101