THE COCENTER OF GRADED AFFINE HECKE ALGEBRA AND THE DENSITY THEOREM

DAN CIUBOTARU AND XUHUA HE

Abstract. We determine a basis of the (twisted) cocenter of graded affine Hecke algebras with arbitrary parameters. In this setting, we prove that the kernel of the (twisted) trace map is the commutator subspace (Density theorem) and that the image is the space of good forms (trace Paley-Wiener theorem).

1. Introduction

The affine Hecke algebras arise naturally in the theory of smooth representations of reductive $p$-adic groups. Motivated by the relation with abstract harmonic analysis for $p$-adic groups (such as the trace Paley-Wiener theorem and the Density theorem [BDK, Ka, Fl]), as well as the study of affine Deligne-Lusztig varieties (such as the “dimension=degree” theorem [He2, Theorem 6.1]), it is important to describe the cocenter of affine Hecke algebras, i.e., the quotient of the Hecke algebra by the vector subspace spanned by all commutators. In this paper, we solve the related problem for the graded affine Hecke algebras introduced by Lusztig [Lu1].

To describe the results, let $H$ be the graded Hecke algebra attached to a simple root system $\Phi$ and complex parameter function $k$, Definition 2.2.1. As a $C$-vector space, $H$ is isomorphic to $C[W] \otimes S(V)$, where $W$ is the Weyl group of $\Phi$, and $S(V)$ is the symmetric algebra of $V$, the underlying (complex) space of the root system.

Let $\delta$ be an automorphism of order $d$ of the Dynkin diagram of $\Phi$ which preserves the parameters $k$, and form the extended algebra $H' = H \rtimes (\delta)$. (The automorphism $\delta$ could of course be trivial.) The cocenter $\bar{H}' = H'/[H', H']$ of $H'$ and the $\delta$-twisted cocenter $\bar{H}_\delta = H'/[H, H]_\delta$ of $H$ are related in section 3.1.

In section 5.1 we construct a set of elements $\{w_C f_{J_C, i}\}$ of $\bar{H}$, where $C$ runs over the $\delta$-twisted conjugacy classes in $W$. To each class $C$, we attach a $\delta$-stable subset $J_C$ of the Dynkin diagram, and pick $w_C \in C \cap W_{J_C}$, where $W_{J_C}$ is the parabolic reflection subgroup of $W$ defined by $J_C$; the elements $f_{J_C, i}$ are chosen in $S(V)$, see 5.1 for the precise definitions. Our first result gives a basis for $\bar{H}_\delta$ (and hence a basis for $\bar{H}'$), which is independent of the parameter function.

Theorem A. The set $\{w_C f_{J_C, i}\}$ is a basis for the vector space $\bar{H}_\delta$.

The proof that the set $\{w_C f_{J_C, i}\}$ spans $\bar{H}_\delta$ relies of certain results about $\delta$-twisted conjugacy classes in the Weyl group, section 5 as well as the use of a filtration in $\bar{H}$ and its associated graded object, which allows us to reduce the proof.
to the case when the parameter function is identically 0. The case \( k \equiv 0 \) is proved directly in Proposition 6.1.1.

For the linear independence we use the representation theory of \( \mathbb{H} \) to produce modules whose traces “separate” the elements \( w_C f_{J, \delta} \). This is done in conjunction with a proof of the Density theorem and (twisted) trace Paley-Wiener theorem for graded Hecke algebras. More precisely, let \( R^\delta(\mathbb{H}) \) be the \( \mathbb{Z} \)-span of the \( \delta \)-stable irreducible \( \mathbb{H} \)-modules \( \text{Irr}_\delta \mathbb{H} \), and let \( R^*_\delta(\mathbb{H}) = \text{Hom}_\mathbb{C}(R^\delta(\mathbb{H}), \mathbb{C}) \) be the (complex) dual space. The twisted trace map is a linear map
\[
\text{tr}^\delta : \mathbb{H}_\delta \to R^*_\delta(\mathbb{H}),
\]
see section 4.1. If \( R^*_\delta(\mathbb{H})_{\text{good}} \) is the subspace of good forms (Definition 4.1.1), the image of the trace map is automatically in \( R^*_\delta(\mathbb{H})_{\text{good}} \).

**Theorem B.** The map \( \text{tr}^\delta : \mathbb{H}_\delta \to R^*_\delta(\mathbb{H})_{\text{good}} \) is a linear isomorphism.

This is a graded affine Hecke algebra analogue of results from \( p \)-adic groups, [BDK], [Ka], and [Fl]. However, our proof of injectivity (which uses the explicit spanning set of \( \mathbb{H}_\delta \)) is essentially different. Our approach also leads to the following result on the dimension of the space of \( \delta \)-elliptic representations \( R^\delta_0(\mathbb{H}) \) (4.2.2).

**Theorem C.** The dimension of the \( \delta \)-twisted elliptic representation space \( R^\delta_0(\mathbb{H}) \) is equal to the number of \( \delta \)-twisted elliptic conjugacy classes in \( W \).

When \( \delta = 1 \) and the parameter function \( k \) is positive, this result was previously known from [OS], where it was obtained by different methods. Using the explicit description of the cocenter \( \mathbb{H}_\delta \), we can argue that the dimension is at most the number of \( \delta \)-elliptic conjugacy classes. To show equality, we construct explicitly in section 8 via a case-by-case analysis, a set of linearly independent elements of \( R^\delta_0(\mathbb{H}) \) with the desired cardinality and other interesting properties, see Theorem 8.1.1.

Finally, using Clifford theory for \( \mathbb{H}' \) and the relation between \( \mathbb{H}_\delta \), \( R(\mathbb{H}') \) and \( \mathbb{H}_\delta \), \( R^\delta(\mathbb{H}) \) \( (i = 1, d) \) respectively, we obtain:

**Corollary D.** The trace map \( \text{tr} : \mathbb{H}' \to R(\mathbb{H}')^*_\text{good} \) is a linear isomorphism.

For Hecke algebras with real parameters, similar results were announced recently by Solleveld [So3], as part of his calculations of Hochschild homology. The proofs are based on deep results from [So2], where a version of the Aubert-Baum-Plymen conjecture, involving the extended quotient of the “first kind”, is proved. In particular, the paper uses a \( \mathbb{Q} \)-basis of \( R(\mathbb{H}')_\mathbb{Q} \) which depends analytically on the real parameter function.

Our method is different from loc. cit. and appears to be more elementary. For example, based on knowledge of minimal length elements, (twisted) elliptic elements of finite Weyl groups (section 8), and explicit basis of the elliptic space \( R^\delta_0(\mathbb{H}) \), constructed in section 8 we are able to handle arbitrary complex parameters. We also obtain explicit \( \mathbb{Q} \)-basis of \( R(\mathbb{H}')_\mathbb{Q} \) depending linearly on the complex parameter function. It is not clear to us if there is a connection between the basis in [So3] and the one constructed in the present paper. Finally, our approach seems to be related naturally to the Aubert-Baum-Plymen conjecture with the extended quotient of the “second kind” [ABP].
2. Preliminaries

2.1. Root system. Let $\Phi = (V_0, R, V_0^\vee, R^\vee)$ be a semisimple real root system:

(i) $V_0$ and $V_0^\vee$ are finite dimensional real vector spaces with a perfect pairing $( , ): V_0 \times V_0^\vee \to \mathbb{R}$;
(ii) $R \subset V \setminus \{0\}$ spans $V$;
(iii) there is a bijection $R \leftrightarrow R^\vee$, $\alpha \leftrightarrow \alpha^\vee$, such that $(\alpha, \alpha^\vee) = 2$;
(iv) the reflection $s_\alpha: V_0 \to V_0$, $s_\alpha(v) = v - (v, \alpha^\vee)\alpha$ (resp., $s_\alpha: V_0^\vee \to V_0^\vee$, $s_\alpha(v^\vee) = v^\vee - (\alpha, v^\vee)\alpha^\vee$) preserves $R$ (resp. $R^\vee$) for every $\alpha \in R$.

We assume, in addition, that the root system is reduced ($\alpha \in R$ implies $2\alpha \notin R$) and crystallographic ($(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$).

Let $W$ be the finite Weyl group, i.e., the subgroup of $GL(V)$ (identified to a subgroup of $GL(V^\vee)$ too) generated by $s_\alpha$, $\alpha \in R$. Fix a choice of positive roots $R^+ \subset R$ with corresponding positive coroots $R^{\vee,+} \subset R^\vee$. Denote $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ and $\rho^\vee = \frac{1}{2}\sum_{\alpha \in R^+} \alpha^\vee$. Let $I \subset R^+$ be a basis of $R^+$, the simple roots.

For every subset $J \subset I$, let $R_J$ be the subset of $R$ generated by $J$, $R_J^\vee$ the corresponding coroots, $R_J^+ = R^+ \cap R_J$, $R_J^{\vee,+} = R^{\vee,+} \cap R_J^\vee$, and $W_J$ the parabolic Weyl group generated by $\{s_\alpha : \alpha \in J\}$. Let $\Phi_J = (V_0, R_J, V_0^\vee, R_J^\vee)$ be the (nonsemisimple) subroot system. Denote

$$V_{0,J} = \mathbb{R}\text{-span of } R_J, \quad V_{0,J}^\vee = \mathbb{R}\text{-span of } R_J^\vee, \quad \Phi_J^\vee = (V_0,J, R_J, V_0^\vee, R_J^\vee).$$

Set $V = \mathbb{C} \otimes_\mathbb{R} V_0$ and $V^\vee = \mathbb{C} \otimes_\mathbb{R} V_0^\vee$, and similarly for $V_J, V_J^\vee$. Let $S(V)$ denote the symmetric algebra of $V$. Let $\mathbb{C}[W]$ denote the group algebra of $W$.

If $\delta$ is an automorphism of $I$ such that $(\alpha, \beta^\vee) = (\delta(\alpha), \delta(\beta^\vee))$, for all $\alpha, \beta \in I$, then $\delta$ induces an automorphism of $\Phi$; denote $W' = W \rtimes \langle \delta \rangle$, where $\langle \delta \rangle$ is the cyclic group generated by $\delta$.

2.2. Graded affine Hecke algebra. Let $k : R^+ \to \mathbb{C}$, $k(\alpha) = k_\alpha$, be a $W$-invariant function.

**Definition 2.2.1 ([Lus]).** The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Phi, k)$ attached to the root system $\Phi$ and parameter function $k$ is the unique associative complex algebra with identity generated by $w \in W$ and $S(V)$ such that:

(i) $\mathbb{H} \cong \mathbb{C}[W] \otimes_\mathbb{C} S(V)$ as $(\mathbb{C}[W], S(V))$-bimodules;
(ii) $(1 \otimes \omega)(s_\alpha \otimes 1) - (s_\alpha \otimes 1)(1 \otimes s_\alpha(\omega)) = k_\alpha(\omega, \alpha^\vee)$, for all $\alpha \in I, \omega \in V$.

In the sequel, we write $f$ for $1 \otimes f$, $f \in S(V)$, and $w$ for $w \otimes 1$, $w \in W$.

From Definition 2.2.1(ii), it is easy to deduce that

$$f \cdot s_\alpha - s_\alpha \cdot s_\alpha(f) = k_\alpha \Delta_\alpha(f), \quad \alpha \in I, \quad f \in S(V), \quad (2.2.1)$$

where $\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}$ is the difference operator. Moreover, by induction on $w \in W$, one can then verify that

$$f \cdot w = w \cdot w^{-1}(f) + \sum_{w' < w} w'f_{w'}, \quad w \in W, \quad f \in S(V), \quad (2.2.2)$$

for some $f_{w'} \in S(V)$, where $<$ denotes the Bruhat order in $W$. This relation will be used implicitly in the proofs below.

The center of $\mathbb{H}$ is $Z(\mathbb{H}) = S(V)^W$ ([Lus] Proposition 4.5]). Since $\mathbb{H}$ is finite over $Z(\mathbb{H})$, every simple $\mathbb{H}$-module is finite dimensional, and the center $Z(\mathbb{H})$ acts
by scalars (central character) in every irreducible module. The central characters are thus parameterized by $W$-orbits in $V^\vee$. Denote
\[ \Theta(\mathbb{H}) = W \backslash V^\vee \] and $\text{cc} : \text{Irr} \mathbb{H} \to \Theta(\mathbb{H}),$
the central character map, a finite-to-one map. We say that an irreducible $\mathbb{H}$-module $\pi$ has real central character if $\text{cc}(\pi) \in W \backslash V^\vee$.

Let $\delta$ be an automorphism of $I$ as in section 2.1 and suppose the parameter function $k$ satisfies $k_\alpha = k_{\delta(\alpha)}$ for all $\alpha \in R^+$. In this case, $\delta$ defines an automorphism of $\mathbb{H}$, and we may define the extended graded affine Hecke algebra $\mathbb{H}' = \mathbb{H} \rtimes (\delta)$. The center of $\mathbb{H}'$ is $Z(\mathbb{H}') = S(V)^{W'}$, and the central characters of $\mathbb{H}'$-modules are parameterized by $W'$-orbits in $V^\vee$. Denote $\Theta(\mathbb{H}') = W' \backslash V^\vee$.

Let $R(\mathbb{H}')$ denote the Grothendieck group of finite dimensional $\mathbb{H}'$-modules. For every ring $K \supset \mathbb{Z}$, set $R(\mathbb{H}')_K = K \otimes_{\mathbb{Z}} R(\mathbb{H}')$.

2.3. The elements $\bar{\omega}$. The algebra $\mathbb{H}'$ has a natural conjugate-linear anti-involution $\ast$ defined on generators ([BM1, section 5]) by
\[ w^* = w^{-1}, \ w \in W, \ \delta^* = \delta^{-1}, \ \omega^* = \overline{w_0 \cdot w_0(\omega) \cdot w_0}, \ \omega \in V, \] where $w_0$ is the long Weyl group element. This definition is motivated by the relation with Iwahori-Hecke algebras and $p$-adic groups (see [BM1]). A direct computation shows that
\[ \omega^* = -\overline{\omega} + 2p_\omega, \ \text{where} \ p_\omega = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(\omega, \alpha^\vee) s_\alpha \in \mathbb{C}[W], \ \omega \in V_0. \] (2.3.2)
Set
\[ \bar{\omega} = \omega - p_\omega, \ \omega \in V. \] (2.3.3)
In particular, $\bar{\omega}^* = -\bar{\omega}$. Notice also that $\delta(\bar{\omega}) = \bar{\delta}(\omega)$.

2.4. A filtration of $\mathbb{H}'$. Define a notion of degree in $\mathbb{H}'$ as follows. From Definition 2.2.1 one sees that every $h \in \mathbb{H}'$ can be uniquely written as $h = \sum_{w \in W'} w a_w$, where $a_w \in S(V)$. Define the degree of $h$ to be the maximum of degrees in $S(V)$ of all $a_w$. Set $\mathcal{F}^j \mathbb{H}'$ to be the set of elements of $\mathbb{H}'$ of degree less than or equal to $j$. This defines a filtration
\[ \mathbb{C}[W'] = \mathcal{F}^0 \mathbb{H}' \subset \mathcal{F}^1 \mathbb{H}' \subset \mathcal{F}^2 \mathbb{H}' \subset \ldots, \]
and let $\mathbb{H}'_0$ be the associated graded object. It is apparent from the commutation relation in Definition 2.2.1 that $\mathbb{H}'_0$ may be naturally identified with the (extended) graded affine Hecke algebra $\mathbb{H}_0'$ with parameter function $k = 0$.

2.5. Parabolic subalgebras. Let $J \subset I$ be given. The parabolic subalgebra $\mathbb{H}_J$ of $\mathbb{H}$ is the subalgebra generated by $W_J$ and $S(V)$. Denote $\mathbb{H}_J^0$ the subalgebra of $\mathbb{H}_J$ spanned by $W_J$ and $S(V_J)$; this is the graded affine Hecke algebra for the semisimple root system $\Phi_J^0$ with parameter function $k_J = k|_{R_J^+}$. It is clear that
\[ V = V_J \oplus V^{W_J}, \ \mathbb{H}_J = \mathbb{H}_J^0 \otimes_{\mathbb{C}} S(V^{W_J}). \] (2.5.1)
For every $\mathbb{H}_J$ module $X$, define the parabolically induced $\mathbb{H}$-module
\[ \text{Ind}_{\mathbb{H}_J}^\mathbb{H}(X) = \mathbb{H} \otimes_{\mathbb{H}_J} X. \] (2.5.2)
In particular, if \( \sigma \) is an \( \mathbb{H}^\sigma \)-module, and \( \chi_\nu : S(V^{W_\nu}) \to \mathbb{C} \) is a character parameterized by \( \nu \in (V^{W_\nu})^{W_\nu} \), one can form the induced \( \mathbb{H} \)-module
\[
X(J, \sigma, \nu) = \text{Ind}_{\mathbb{H}_J}^{\mathbb{H}} (\sigma \otimes \chi_\nu).
\] (2.5.3)

3. The cocenter and Clifford theory

3.1. \( \delta \)-commutators. We retain the notation from the previous section. In particular, \( \delta \) is an automorphism of the Dynkin diagram of order \( d \) and \( \mathbb{H}_\delta = \mathbb{H} \rtimes \langle \delta \rangle \)
is the extended graded affine Hecke algebra.

**Definition 3.1.1.** Let \( h, h' \in \mathbb{H} \). Then \([h, h']_\delta = hh' - h'\delta(h)\) is called the \( \delta \)-commutator of \( h \) and \( h' \). Let \([\mathbb{H}, \mathbb{H}]_\delta \) be the submodule of \( \mathbb{H} \) generated by all \( \delta \)-commutators. Similarly, we may define \([h, h']_\delta \) and \([\mathbb{H}, \mathbb{H}]_\delta \), for all \( i \in \mathbb{N} \).

It is easy to see that \( \delta \) sends \([h, h']_\delta \) to \([\delta(h), \delta(h')]_\delta \) and thus sends \([\mathbb{H}, \mathbb{H}]_\delta \) to itself. Hence \((1 - \delta) : \mathbb{H} \to \mathbb{H}\) induces a map
\[
(1 - \delta) : \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta \to \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta.
\] (3.1.1)

We denote by \( \mathbb{H}^{[\delta]} \) the \( \delta \)-coinvariants of \( \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta \), i.e., the quotient of \( \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta \) by the image of \((1 - \delta)\).

We prove the following result.

**Proposition 3.1.1.** Set \( \mathbb{H}' = \mathbb{H}'/[\mathbb{H}', \mathbb{H}'] \), where \( \mathbb{H}' = \mathbb{H} \rtimes \langle \delta \rangle \). Then

1. \( \mathbb{H}' = \mathbb{H}'/[\mathbb{H}', \mathbb{H}'] \), where \( d \) is the order of \( \delta \).

2. The map \( h \mapsto h\delta \) induces an isomorphism of vector spaces from \( \mathbb{H}^{[\delta]} \) to \( \mathbb{H}'/(\mathbb{H}', \mathbb{H}') \).

**Proof.** We have the decompositions \( \mathbb{H}' = \mathbb{H}'/[\mathbb{H}', \mathbb{H}'] \) and
\[
[\mathbb{H}', \mathbb{H}'] = \sum_{j,k} [\mathbb{H}\delta^j, \mathbb{H}\delta^k] = \mathbb{H}/(\sum_i [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}]). \tag{3.1.2}
\]

Notice that \( \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \subset \mathbb{H}\delta^i \). Thus \([\mathbb{H}', \mathbb{H}'] \cap \mathbb{H}\delta^i = \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \) and \( \mathbb{H}' = \mathbb{H}'/[\mathbb{H}', \mathbb{H}'] \). Part (1) is proved.

Now we show that

(a) For any \( i \), \( \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \delta^{-i} = (1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta \).

By definition, \([\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] = [\mathbb{H}\delta^{i+j}, \delta^{-j}\mathbb{H}] \) is generated by
\[
[h\delta^{i+j}, \delta^{-j}h'] = h\delta^{i}h' - \delta^{-j}hh'\delta^{i+j} = (h\delta^{i}(h') - \delta^{-j}(h'h))\delta^i.
\]

Notice that
\[
h\delta^{i}(h') - \delta^{-j}(h'h) = -[h', h]_\delta + (1 - \delta)^{-j}(h'h) \in (1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta,
\]
since \((1 - \delta) \mathbb{H} \subset (1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta \). Thus \([\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \subset (1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta \).

On the other hand,
\[
h - \delta(h) = [h\delta^{-1}, \delta] \delta^{-i} \in \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \delta^{-i}.
\]

Hence \((1 - \delta) \mathbb{H} \subset \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \delta^{-i} \).

Moreover, as before, we have that
\[
[h', h]_\delta = -[h\delta^{i+j}, \delta^{-j}h'] \delta^{-i} + (1 - \delta) \mathbb{H} \subset \sum_j [\mathbb{H}\delta^{i+j}, \mathbb{H}\delta^{-j}] \delta^{-i}.
\]
Thus (a) is proved. Then
\[
\mathbb{H} \delta / (\mathbb{H}^r, \mathbb{H}^t \cap \mathbb{H} \delta) \cong \mathbb{H} / (\mathbb{H}^r, \mathbb{H} \delta^{-1} \cap \mathbb{H}) = \mathbb{H} / \sum_j [\mathbb{H} \delta^{-j}, \mathbb{H} \delta^{-j} \delta^{-i}]
\]
\[
= \mathbb{H} / ((1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta).
\]

It is easy to see that \((1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta \) is the image of \((1 - \delta) : \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta \to \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta\). Thus \(\mathbb{H} / ((1 - \delta) \mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta)\) is isomorphic to the quotient of \(\mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta\) by the image of \((1 - \delta)\).

Part (2) is proved. \(\square\)

### 3.2. Clifford theory for \(\mathbb{H}'\)

Let \(\Gamma = \langle \delta \rangle\). If \((\pi, X)\) is a finite dimensional \(\mathbb{H}\)-module, let \((\delta^{\pi, \delta} X)\) denote the \(\mathbb{H}\)-module with the action \(\delta^{\pi, \delta}(h)x = \pi(\delta^{-i}(h))x\), for all \(x \in X, h \in \mathbb{H}\). Suppose \(X\) is irreducible. Define the inertia group \(\Gamma_X = \{ \delta^i : X \cong \delta^i X \}\).

Fix a family of isomorphisms \(\phi_{\delta^i} : X \to \delta^{-i} X, \delta^i \in \Gamma_X\) (each one of these isomorphisms is unique up to scalar). In general, this defines factor set (2-cocycle)
\[
\beta : \Gamma_X \times \Gamma_X \to \mathbb{C}^\times, \quad \phi_{\delta^i} \phi_{\delta^j} = \beta(\delta^i, \delta^j) \phi_{\delta^{i+j}}.
\]

However, in our particular case, \(\Gamma_X\) is a cyclic subgroup, generated by say \(\delta^X\) and we can normalize the isomorphisms \(\phi_{\delta^i}\) such that \(\phi_{\delta^1 X} = \delta^X X\). This has the consequence that the factor set \(\beta\) can be chosen to be trivial.

If \(U\) is an irreducible \(\Gamma_X\)-module, there is an action of \(\mathbb{H} \times \Gamma_X\) on \(X \otimes U\):
\[
(h \delta^i)(x \otimes u) = h \phi_{\delta^i}(x) \otimes \delta^i u. \tag{3.2.1}
\]

One can form the induced \(\mathbb{H}'\)-module \(X \times U = \text{Ind}_{\mathbb{H} \times \Gamma_X}^{\mathbb{H}' \times \Gamma_X} (X \otimes U)\). The main results in Clifford theory are summarized next.

**Theorem 3.2.1** ([RR Appendix A]).

1. If \(X\) is an irreducible \(\mathbb{H}\)-module and \(U\) an irreducible \(\Gamma_X\)-module, the induced \(\mathbb{H}'\)-module \(X \times U\) is irreducible.

2. Every irreducible \(\mathbb{H}'\)-module is isomorphic to an \(X \times U\).

3. If \(X \times U \cong X' \times U'\), then \(X, X'\) are \(\langle \delta \rangle\)-conjugate \(\mathbb{H}\)-modules, and \(U \cong U'\) as \(\Gamma_X\)-modules.

We need a formula for the trace of an \(\mathbb{H}'\)-module. For every \(\delta' \in \Gamma\) and \((\pi, X) \in \text{lrr}^{\delta'} \mathbb{H}\), let \(\phi_{\delta'} \in \text{End}_\mathbb{C}(X)\) be the intertwiner as before. Define the twisted trace
\[
\text{tr}^{\delta'}(\pi) : \mathbb{H} \to \mathbb{C}, \quad \text{tr}^{\delta'}(\pi)(h) = \text{tr}(\pi(h) \circ \phi_{\delta'}).
\]
Let also \(\text{Tr}(\cdot, \cdot) : \mathbb{H}' \times R(\mathbb{H}') \to \mathbb{C}\) be the trace pairing, i.e., \(\text{Tr}(h, \pi) = \text{tr}(\pi(h)), h \in \mathbb{H}', \pi \in R(\mathbb{H}')\).

**Lemma 3.2.1.** Let \(X \times U\) be an irreducible \(\mathbb{H}'\)-module as in Theorem 3.2.1. For \(h \in \mathbb{H}, \delta' \in \Gamma\),
\[
\text{Tr}(h \delta', X \times U) = \begin{cases} 
\delta'(U) \sum_{\gamma \in \Gamma_X} \text{tr}^{\delta'}(X)(\gamma^{-1}(h)), & \text{if } \delta' \in \Gamma_X, \\
0, & \text{if } \delta' \notin \Gamma_X,
\end{cases} \tag{3.2.2}
\]
where \(\delta'(U)\) is the root of unity by which \(\delta'\) acts in \(U\).
Proof. As a vector space, $X \times U = \text{Ind}_{\mathbb{H} \rtimes \Gamma}^{\mathbb{H}} (X \otimes U) = \sum_{\gamma \in \Gamma_X} \gamma \otimes (X \otimes U)$. Then the action of $h\delta'$ is

$$(h\delta') \cdot \gamma \otimes (x \otimes u) = h\delta' \gamma \otimes (x \otimes u) = h\gamma' \delta'' \otimes (x \otimes u),$$

for some $\gamma' \in \Gamma / \Gamma_X$, $\delta'' \in \Gamma_X$.

$$\gamma' \cdot (\gamma')^{-1}(h)\delta'' \otimes (x \otimes u) = \gamma' \otimes ((\gamma')^{-1}(h)\delta''(x) \otimes \delta'' \cdot u).$$

For this to have a nonzero contribution to the trace, $\gamma' = \gamma$, which is equivalent, since $\Gamma$ is abelian, with $\delta' \in \Gamma_X$. Suppose now this is the case (so $\gamma' = \gamma$ and $\delta'' = \delta'$). Then $\delta' \cdot u = \delta'(U)u$ and the claim follows from the definition of the twisted trace.

\[ \square \]

Set $R(\mathbb{H})_C = R(\mathbb{H}) \otimes \mathbb{Z} \mathbb{C}$. We give a decomposition of $R(\mathbb{H})_C$ which is dual to the decomposition $\mathbb{H} = \oplus_{i=0}^{d-1} \mathbb{H}[i]$ from Proposition 3.1.1 in a sense to be made precise in the next section.

Let $\mathcal{O}$ be a $\Gamma$-orbit on $\text{Irr}(\mathbb{H})$. Set $\Gamma_{\mathcal{O}} = \Gamma_X$ for any $X \in \mathcal{O}$. This is well-defined since $\Gamma$ is cyclic. Then for any irreducible $\Gamma_{\mathcal{O}}$-module $U$ and $X \in \mathcal{O}$, $X \times U = \oplus_{Y \in \mathcal{O}} Y \otimes U$ is independent of the choice of $X$. We denote it by $\mathcal{O} \times U$.

By Theorem 3.2.1, $\text{Irr}(\mathbb{H}) = \{ \mathcal{O} \times U \}$, where $\mathcal{O}$ runs over $\Gamma$-orbits on $\text{Irr}(\mathbb{H})$ and $U$ runs over isomorphism classes of irreducible representations of $\Gamma_{\mathcal{O}}$.

Suppose that $\delta' \in \Gamma_{\mathcal{O}}$. Let $\text{U}_{\mathcal{O},i}$ be the virtual representation of $\Gamma_{\mathcal{O}}$ whose character is the characteristic function on $\delta'$. Then $\{ \mathcal{O} \times \text{U}_{\mathcal{O},i} \}$ is a basis of $R(\mathbb{H})_C$.

Let $R[i](\mathbb{H})_C$ be the subspace of $R(\mathbb{H})_C$ spanned by $\mathcal{O} \times \text{U}_{\mathcal{O},i}$, where $\mathcal{O}$ runs over $\Gamma$-orbits on $\text{Irr}(\mathbb{H})$ with $\delta' \in \Gamma_{\mathcal{O}}$. Then

$$R(\mathbb{H})_C = \oplus_{i=0}^{d-1} R[i](\mathbb{H})_C.$$  \hfill (3.2.3)

By definition, $R[i](\mathbb{H})_C$ is a vector space with basis $(\text{Irr}^{\delta'} \mathbb{H})_{\Gamma}$. Hence, the map $X \in \text{Irr}^{\delta'} \mathbb{H} \mapsto X \times \text{U}_{\Gamma_X,i}$ induces an isomorphism $R^{\delta'}(\mathbb{H})_{R_i \mathbb{C}} \rightarrow R^{[i]}(\mathbb{H})_C$. Here $R^{\delta'}(\mathbb{H})_{R_i \mathbb{C}}$ is the $\Gamma$-coinvariants of $R^{\delta'}(\mathbb{H})_C$.

By Lemma 3.2.1 for $0 \leq i, j < d$ with $i \neq j$, $\text{Tr}(\mathbb{H}\delta^i, \mathcal{O} \times \text{U}_{\mathcal{O},j}) = 0$.

4. (Twisted) Trace Paley-Wiener Theorem

In this section, we prove that trace Paley-Wiener theorem in the setting of graded affine Hecke algebra. The proof follows the general outline for the similar theorems for $p$-adic groups, \cite{BDK} and \cite{EF}, but for certain steps, e.g., Lemma 4.6.1 we give different arguments.

4.1. Trace forms. Define the trace linear map

$$\text{tr} : \mathbb{H}^* \rightarrow R(\mathbb{H})^*, \quad h \mapsto (f_h : R(\mathbb{H}) \rightarrow \mathbb{C}, \ f_h(\pi) = \text{Tr} \pi(h)).$$

It clearly descends to a linear map

$$\text{tr} : \mathbb{H}^* \rightarrow R(\mathbb{H})^*. \hfill (4.1.2)$$

This is compatible to the decompositions from Proposition 3.1.1 and 3.2.3 as follows. To simplify notation, we write the details in the case of $\delta$, the same results hold for every $\delta$. Let $R_{\delta}(\mathbb{H}) = \text{Hom}_\mathbb{C}(R^{\delta}(\mathbb{H})_C, \mathbb{C})$ be the space of $\mathbb{C}$-valued linear forms on the vector space spanned by $\text{Irr}^{\delta}(\mathbb{H})$. The twisted trace map

$$\text{tr}_{\delta} : \mathbb{H} \rightarrow R_{\delta}(\mathbb{H}), \quad h \mapsto (f_h^{\delta} : R^{\delta}(\mathbb{H})_C \rightarrow \mathbb{C}, \ f_h^{\delta}(\pi) = \text{Tr}^{\delta}(\pi)). \hfill (4.1.3)$$
descends to a linear map
\[ \text{tr}^\delta : \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta \to R^\delta_0(\mathbb{H}). \]  
(4.1.4)

Call a form \( f \in R^\delta_0(\mathbb{H}) \) a trace form if \( f = f^\delta_h \) for some \( h \in \mathbb{H} \) (or better \( h \in \mathbb{H} \)) and denote the subspace of trace forms by \( R^\delta_0(\mathbb{H})_{\text{tr}} \). This of course is the image of \( \text{tr}^\delta \).

For every \( J \subset I \) such that \( \delta(J) = J \), denote by \( (V^\vee)^W_{J^\times} \delta \) the fixed points of \( \delta \) on \( (V^\vee)^W_J \).

**Definition 4.1.1.** A form \( f \in R^\delta_0(\mathbb{H}) \) is called good if for every \( J \subset I \) such that \( \delta(J) = J \), and every \( \sigma \in \text{Irr}^\delta(\mathbb{H}^\vee_0) \), the function \( \nu \mapsto f(X(J, \sigma, \nu)) \) is a regular function on the variety \((V^\vee)^W_{J^\times} \delta \). Denote the subspace of good forms by \( R^\delta_0(\mathbb{H})_{\text{good}} \).

It is clear that \( R^\delta_0(\mathbb{H})_{\text{tr}} \subset R^\delta_0(\mathbb{H})_{\text{good}} \). The content of the trace Paley-Wiener theorem is that in fact the two spaces are equal:

**Theorem 4.1.1.** \( R^\delta_0(\mathbb{H})_{\text{tr}} = R^\delta_0(\mathbb{H})_{\text{good}} \).

The proof is presented in the next subsections.

4.2. **A filtration of** \( R^\delta(\mathbb{H}) \). For every \( J \subset I \), recall the functor of parabolic induction \( i_J : R(\mathbb{H}_J) \to R(\mathbb{H}) \), \( i_J(X) = \text{Ind}^\mathbb{H}_J(X) \) from section 2.5. For every \( 0 \leq \ell \leq |I| \), define the abelian subgroup

\[ R^\delta_\ell(\mathbb{H}) = \sum_{J=\delta(J) \subset I; |J| \leq |I| - \ell} i_J(R^\delta(\mathbb{H}_J)). \]  
(4.2.1)

Then \( R^\delta_0(\mathbb{H}) = R^\delta(\mathbb{H}) \), \( R^\delta_1(\mathbb{H}) = R^\delta_{\text{Ind}}(\mathbb{H}) = \sum_{J=\delta(J) \subset I} i_J(R^\delta(\mathbb{H}_J)) \), the subgroup of twisted parabolically induced modules, and \( R^\delta_0 \mid_{|I|} = 0 \). These subgroups form a decreasing filtration

\[ R^\delta(\mathbb{H}) = R^\delta_0(\mathbb{H}) \supset R^\delta_1(\mathbb{H}) \supset R^\delta_2(\mathbb{H}) \supset \ldots. \]

Set

\[ \overline{R}^\delta_\ell(\mathbb{H}) = R^\delta_\ell(\mathbb{H})/R^\delta_{\ell+1}(\mathbb{H}), \quad 0 \leq \ell < |I|, \]  
(4.2.2)

and \( \overline{R}^\delta(\mathbb{H}) = \oplus_{0 \leq \ell < |I|} \overline{R}^\delta_\ell(\mathbb{H}) \), the associated graded group.

Of particular interest is \( \overline{R}^\delta_0(\mathbb{H}) = R^\delta(\mathbb{H})/R^\delta_{\text{Ind}}(\mathbb{H}) \), the space of virtual \( \delta \)-elliptic modules. A module \( \pi \in \text{Irr}^\delta_0(\mathbb{H}) \) is called elliptic if the image of \( \pi \) in \( \overline{R}^\delta_0(\mathbb{H}) \) is nonzero. Let \( \Theta^\ell(\mathbb{H}) \) denote the set of elliptic central characters, i.e., the subset of \( \Theta(\mathbb{H}) \) of all central characters of elliptic \( \pi \in \text{Irr}^\delta_0(\mathbb{H}) \).

4.3. **Langlands classification.** The parabolic induction part of the Langlands classification for \( \mathbb{H} \) is proved in [Ev], see also [KR, Theorem 2.4].

Let \( (\pi, X) \) be a finite dimensional \( \mathbb{H} \)-module. For every \( \lambda \in V^\vee \), set

\[ X_\lambda = \{ x \in X : (\pi(a) - \lambda(a))^nx = 0 \}, \quad \text{for some } n \in \mathbb{N}, \quad \text{and all } a \in S(V) \}, \]

the generalized \( \lambda \)-weight space of \( S(V) \). The set of \( S(V) \)-weights of \( X \) is \( \Psi(X) = \{ \lambda \in V^\vee : X_\lambda \neq 0 \} \). It is easy to see that \( \Psi(X) \subset W \cdot \nu \), where \( \nu \) is (a representative of) the central character of \( X \).

For every \( \nu \in V^\vee \), write \( \nu = \Re \nu + \sqrt{-1} \Im \nu \), where \( \Re \nu, \Im \nu \in V^0_\vee \).

**Definition 4.3.1.** An irreducible \( \mathbb{H} \)-module \( X \) is called tempered (resp., discrete series) if for every \( \lambda \in \Psi(X) \), \( (\omega, \Re \lambda) \leq 0 \) (resp. \( (\omega, \Re \lambda) < 0 \)) for all \( \omega \in V_0 \) such that \( \langle \omega, \alpha^\vee \rangle > 0 \) for all \( \alpha \in I \).
Theorem 4.3.1 ([EV Theorem 2.1]).

(1) Every standard induced module $X(J, \sigma, \nu)$, where $\sigma$ is a tempered $\mathbb{H}_{\sigma}^J$-module and $\langle \alpha, \mathbb{R}\nu \rangle > 0$ for all $\alpha \in I \setminus J$, has a unique irreducible quotient $L(J, \sigma, \nu)$.

(2) Every irreducible $\mathbb{H}$-module is isomorphic to a Langlands quotient $L(J, \sigma, \nu)$ as in (1).

(3) If $L(J, \sigma, \nu) \cong L(J', \sigma', \nu')$, then $J = J'$, $\sigma \cong \sigma'$, and $\nu = \nu'$.

Moreover, it is implicitly proved in [EV] that if $L(J', \sigma', \nu')$ is an irreducible constituent of the standard module $X(J, \sigma, \nu)$ different than $L(J, \sigma, \nu)$, then $\mathbb{R}\nu > \mathbb{R}\nu'$, where $>$ is the partial order relation on $V_0'$ defined by

$$a > b \text{ if } \langle \alpha, a \rangle > \langle \alpha, b \rangle, \text{ for all } \alpha \in I. \quad (4.3.1)$$

It is immediate that $\delta X(J, \sigma, \nu) \cong X(\delta(J), \delta(\sigma), \delta(\nu))$. Suppose that $\pi \in \text{Irr}^J \mathbb{H}$ is given such that $\pi \cong L(J, \sigma, \nu)$. Then $\pi \cong L(\delta(J), \delta(\sigma), \delta(\nu))$, and Theorem 4.3.1 implies that necessarily $\delta(J) = J$, $\sigma \in \text{Irr}^J \mathbb{H}$, and $\delta(\nu) = \delta$. In particular, all irreducible constituents of $X(J, \sigma, \nu)$ are in $\text{Irr}^J \mathbb{H}$.

Lemma 4.3.1. Suppose $\pi$ is an irreducible $\delta$-elliptic $\mathbb{H}$-module. Then there exists an irreducible $\delta$-elliptic tempered $\mathbb{H}$-module $\pi'$ such that $cc(\pi) = cc(\pi')$.

Proof. This is the Hecke algebra analogue of [Fl Lemma 1.2]. By Theorem 4.3.1 and the remarks following it, $\pi \cong L(J, \sigma, \nu)$ where $\delta(J) = J$, $\sigma \in \text{Irr}^J \mathbb{H}$, and $\nu = \delta(\nu)$. In $R^\sigma(\mathbb{H})$, we have $\pi = X(J, \sigma, \nu) - \sum_k \pi_k$, where $\pi_k = L(J_k, \sigma_k, \nu_k) \in \text{Irr}^J \mathbb{H}$ and $\nu_k < \nu$. Thus, by induction on the length of the $\nu$-parameter, it follows that $\pi$ is a linear $\mathbb{Z}$-combination of standard modules

$$\pi = \sum_i a_i X(J_i', \sigma_i', \nu_i'),$$

where for every $i$, $\delta(J_i') = J_i'$, $\delta(\sigma_i') = \sigma_i'$, $\delta(\nu_i') = \nu_i'$, and $ccX(J_i', \sigma_i', \nu_i') = cc(\pi)$.

If for every $i$, $J_i' \neq I$, then $\pi \cong 0 \mod \mathcal{R}^J_{\text{Ind}}(\mathbb{H})$, which is a contradiction. Thus there exists $i$'s such that $J_i' = I$, and the corresponding $X(J_i', \sigma_i', \nu_i') = \sigma_i'$ are all tempered $\delta$-elliptic $\mathbb{H}$-modules (and have the same central character as $\pi$). \hfill $\square$

4.4. Induction and restriction in $R(\mathbb{H})$. If $K \subset J(\subset I)$ are given, denote by $i_K^J : R(\mathbb{H}_K) \rightarrow R(\mathbb{H}_J)$ the functor of induction, and by $r_K^J : R(\mathbb{H}_J) \rightarrow R(\mathbb{H}_K)$ the functor of restriction. We also have the corresponding functors, denoted again by $i_K^J$ and $r_K^J$ between $R^J(\mathbb{H}_K)$ and $R^K(\mathbb{H}_J)$. The following lemma is the analogue of [BDK Lemma 5.4] and [Fl Lemma 2.1].

Lemma 4.4.1. (i) For $L \subset K \subset J(\subset I)$, $i_L^J = i_K^J \circ i_L^K$ and $r_L^J = r_K^J \circ r_L^K$.

(ii) For $J, K \subset I$, let $KW^J$ be a set of representatives of minimal length for $W_K \backslash W / W_J$. Then

$$r_K^J \circ i_L^J = \sum_{w \in KW^J} i_K^J_{w} \circ w \circ r_J^J,$$

where $K_w = K \cap wJw^{-1}$ and $J_w = J \cap w^{-1}Kw$.

(iii) If $K = wJw^{-1}$, then $i_K^J \circ w = i_{wJ}^J$.

(iv) If $K = \delta(K)$ and $J = \delta(J)$, let $KW^J(\delta) \subset KW^J$ denote the subset of $\delta$-fixed elements. Then

$$r_K^J \circ i_L^J = \sum_{w \in KW^J(\delta)} i_K^J_{w} \circ w \circ r_J^J,$$

(4.4.2)
as functors from $R^\delta(\mathbb{H}_J)$ to $R^\delta(\mathbb{H}_K)$.

Proof. Claim (i) is obvious in our setting.

We prove claim (ii). We need to prove that
\[
\mathbb{H} \otimes_{\mathbb{H}_J} \sigma = \bigoplus_{w \in K W^J} \mathbb{H}_K \otimes_{\mathbb{H}_K} w \circ \sigma|_{\mathbb{H}_w}, \quad \text{as left } \mathbb{H}_K\text{-modules,} \tag{4.4.3}
\]
where $K_w = K \cap wJw^{-1}$ and $J_w = w^{-1}K_ww$.

Let $\{w_1, w_2, \ldots, w_t\}$ be the set of elements in $K W^J$ ordered such that $\ell(w_i) \leq \ell(w_{i+1})$. As $(\mathbb{H}_K, \mathbb{H}_J)$-bimodules,
\[
\mathbb{H} = \sum_{i=1}^{t} \mathbb{H}_K w_i \mathbb{H}_J.
\]
Define a $(\mathbb{H}_K, \mathbb{H}_J)$-bimodule filtration of $\mathbb{H}$ by setting $\mathbb{H}_s = \bigoplus_{\ell_{\mathbb{H}_J} \leq s} \mathbb{H}_K w_i \mathbb{H}_J$. Then $E_s = \mathbb{H}_s \otimes_{\mathbb{H}_J} \sigma$ defines a left $\mathbb{H}_K$-module filtration of $\mathbb{H} \otimes_{\mathbb{H}_J} \sigma$. Set $E_s = E_s/E_s-1$, a left $\mathbb{H}_K$-module. Then
\[
\mathbb{H} \otimes_{\mathbb{H}_J} \sigma \cong \oplus\mathbb{E}_s,
\]
as left $\mathbb{H}_K$-modules, so one needs to prove that there exists an $\mathbb{H}_K$-module isomorphism
\[
B_s : E_s \to \mathbb{H}_K \otimes_{\mathbb{H}_K} w_s \circ (\sigma|_{\mathbb{H}_w}). \tag{4.4.5}
\]
Notice that $E_s$ is generated by $w_s \otimes_{\mathbb{H}_J} \sigma$ as a $\mathbb{H}_K$-module. Set
\[
B_s(hw_s \otimes v) = h \otimes_{\mathbb{H}_K w_s} \tau_{w_s}(v), \quad h \in \mathbb{H}_K, \quad v \in \sigma, \tag{4.4.6}
\]
where $\tau_{w_s}$ is the isomorphism $\sigma \to w_s \circ \sigma$. We need to check that $B_s$ is well-defined. Since $\mathbb{H}_K w_s$ is generated by $W_{K w_s}$ and $S(V)$, it is sufficient to check on these generators.

Firstly, if $w \in W_{K w_s}$, $ww_s = w_s w'$, where $w' \in W_{J w_s}$. We have $B_s(ww_s \otimes_{\mathbb{H}_J} v) = w \otimes_{\mathbb{H}_K w_s} \tau_{w_s}(v) = 1 \otimes_{\mathbb{H}_K w_s} \tau_{w_s}(\sigma(v)) = 1 \otimes_{\mathbb{H}_K w_s} \tau_{w_s}(\sigma(w')v)$. On the other hand,
\[
B_s(w_s w' \otimes_{\mathbb{H}_J} v) = B_s(w_s \otimes_{\mathbb{H}_J} \sigma|_{J w_s} \otimes_{\mathbb{H}_J} \sigma(v)) = 1 \otimes_{\mathbb{H}_K w_s} \tau_{w_s}(\sigma(w')v).
\]
Secondly, let $a \in S(V)$ be given. Then
\[
aw_s = w_s \cdot w_s^{-1}(a) + \sum_{u < w_s} ua_u,
\]
for some $u \in W$, $a_u \in S(V)$. This means that $aw_s \equiv w_s \cdot w_s^{-1}(a)$ in $E_s$, i.e., modulo $E_s-1$. In the same way as for $w \in W_{K w_s}$, it is then easy to see that $B_s(aw_s) = B_s(w_s \cdot w_s^{-1}(a))$.

Thus $B_s$ is well-defined, and it is clearly a surjective $\mathbb{H}_K$-homomorphism. Thus $\dim E_s \geq \dim \mathbb{H}_K \otimes_{\mathbb{H}_K} w_s \circ (\sigma|_{\mathbb{H}_w}) = |W_K/W_{K w_s}| \dim \sigma$. Summing over $s$ and using (4.4.4), we find $|W/W_J| \dim \sigma = \sum \dim E_s \geq \sum |W_K/W_{K w_s}| \dim \sigma$. But since $|W/W_J| = \sum |W_K/W_{K w_s}|$, it follows that every $B_s$ must in fact be an isomorphism. Claim (ii) is proved.

Claim (iii) follows from (ii) and the parabolic induction part of Langlands classification (Theorem 4.3.1) identically with the proof of Lemma 5.4 of $\text{[BDK]}$. For (iv), one can adapt the proof of (ii) exactly as in $\text{[Fl]}$ Lemma 2.1(iv)].

For every $J = \delta(J) \subset I$, define the operator
\[
T_J : R^\delta(\mathbb{H}) \to R^\delta(\mathbb{H}), \quad T_J = i_J^t \circ r_J^t. \tag{4.4.7}
\]
Formal manipulations with the properties in Lemma 4.4.2 yield the following formulas (see [BDK, Corollary 5.4]).

**Lemma 4.4.2.**

(i) \( T_K \circ i_J^* = \sum_{w \in K W^J(\delta)} i_{J_w}^* \circ r_{J_w}^* \), where \( J_w = J \cap w^{-1}Kw \).

(ii) \( T_K \circ T_J = \sum_{w \in K W^J(\delta)} T_{J_w} \).

As a consequence, one sees that the operators \( T_K \) respect the filtration \( \{ R_0^\delta(x) \} \) from section 4.2. Moreover, if \( \ell = |K| \), then \( T_K \) acts on the quotient \( \overline{R}_\ell(x) \) by:

\[
T_K i_J^*(\sigma) = \begin{cases} 
|W_K^\delta| i_J^*(\sigma), & \text{if } J \sim K, \\
0, & \text{if } |J| = \ell \text{ and } J \not\sim K.
\end{cases}
\]

As in [BDK, section 5.5], define

\[
A_\ell = \prod_{K=\delta(\mathbb{K}), |K| = \ell} (T_K - |W_K^\delta|), \quad A = A_{|I|} \circ A_{|I|-1} \circ \cdots \circ A_0.
\]

Since every \( A_\ell \) preserves the filtration and kills \( \overline{R}_\ell(x) \), it follows that \( A \) kills \( R_0^\delta(x) \).

**Definition 4.4.1.** A linear form \( f \in R_0^\delta(x) \) is called \( \delta \)-elliptic if \( f(R_0^\delta(x)) = 0 \). Let \( R_0^\delta(x)_0 \) denote the space of \( \delta \)-elliptic linear forms.

Thus, the adjoint operator \( A^* : R_0^\delta(x) \to R_0^\delta(x) \) has the image in \( R_0^\delta(x)_0 \). On the other hand, from Lemma 4.4.2, it is apparent that \( A \) is of the form \( A = a + \sum_{J=\delta(\mathbb{K})} c'_J T_J \), for some integers \( a \neq 0 \) and \( c'_J \). But then \( A^* = a + \sum_{J=\delta(\mathbb{K})} c'_J T_J^* \), where \( T_J^* = r_J^* \circ i_J^* \). (Here for simplicity, we write \( r_J^* \) instead of \( (r_J^*)^* \) and similarly for \( i_J^* \).) Set \( \mathcal{A} = \frac{1}{a} A^* \).

### 4.5. Inclusion and restriction for \( \mathbb{H} \)

For every \( J \subset I \), let \( \tilde{i}_J : \mathbb{H}_J \to \mathbb{H} \) denote the inclusion. Define \( \tilde{T}_J : \mathbb{H} \to \mathbb{H}_J \) as follows. Given \( h \in \mathbb{H} \), let \( \psi_h : \mathbb{H} \to \mathbb{H} \) be the linear map given by left multiplication by \( h \). This can be viewed as a right \( \mathbb{H}_J \)-module morphism. Since \( \mathbb{H} \) is free of finite rank right \( \mathbb{H}_J \)-module, with basis \( W^J \), one can consider \( \text{tr} \psi_h \in \mathbb{H}_J \). Set \( \tilde{T}_J(h) = \text{tr} \psi_h \). Set \( \tilde{T}_J = \tilde{i}_J \circ \tilde{T}_J : \mathbb{H} \to \mathbb{H}_J \). As in section 4.3, for every \( K \subset J \), we may also define \( \tilde{i}_K^J \) and \( \tilde{T}_K^J \).

**Lemma 4.5.1.** The maps \( \tilde{i}_J \) and \( \tilde{T}_J \) are \( \text{Tr} \cdot \text{-adjoint to } r_J \) and \( i_J \), respectively, i.e.:

(a) \( \text{Tr} \tilde{i}_J(h, \pi) = \text{Tr}(h, r_J(\pi)) \), for \( h \in \mathbb{H}_J, \pi \in R(\mathbb{H}) \);

(b) \( \text{Tr}(h, i_J(\pi)) = \text{Tr}(\tilde{T}_J(h), \pi), \) for \( h \in \mathbb{H}, \pi \in R(\mathbb{H}_J) \).

Thus \( \tilde{T}_J \) is \( \text{Tr} \cdot \text{-adjoint to } T_J \) as well.

**Proof.** Claim (a) is obvious. For (b), let \( M \) be the space of the representation \( \pi \) and \( \{ v_1, \ldots, v_n \} \) an orthonormal basis of \( M \) with respect to an inner product \( \langle \cdot, \cdot \rangle_M \). A basis for \( i_J(\pi) \) is \( \{ x \otimes v_i : i = 1, n, x \in W^J \} \). Define \( \langle \cdot, \cdot \rangle \) on \( i_J(\pi) \) by declaring this basis orthonormal. For \( h \in \mathbb{H} \), view left multiplication of \( h \) as a right \( \mathbb{H}_J \)-module map \( \mathbb{H} \to \mathbb{H} \), and then

\[
h \cdot x = xh_x + \sum_{x' \in W^J \setminus \{ x \}} x'h_{x'}.
\]
for some $h_x, h_x^* \in \mathbb{H}_J$. Notice that $ar{T}_J(h) = \sum_{x \in W^J} h_x$. Then

$$\text{Tr}(h, i_J(\pi)) = \sum_{i=1}^{n} \sum_{x \in W^J} \langle h \cdot x \otimes v_i, v_i \rangle = \sum_{i=1}^{n} \sum_{x \in W^J} \langle xh_x \otimes v_i, v_i \rangle = \sum_{i=1}^{n} \sum_{x \in W^J} \langle x \otimes \pi(h_x)v_i, x \otimes v_i \rangle = \sum_{i=1}^{n} \sum_{x \in W^J} \langle \pi(h_x)v_i, v_i \rangle_M = \text{Tr}(\bar{T}_J(h), \pi).$$

\[\]

The analogous discussion with section 4.3 holds here and also the $\delta$-twisted version. In particular, define the filtration of $\mathbb{H}$:

$$\mathcal{E}_0^\delta \mathbb{H} \supset \mathcal{E}_1^\delta \mathbb{H} \supset \cdots \mathcal{E}_\ell^\delta \mathbb{H} \supset \cdots,$$

where $\mathcal{E}_\ell^\delta \mathbb{H} = \sum_{J \subset I, \ell(J) = |J|} \text{Tr}(\mathbb{H}, J) = \sum_{J \subset I, \ell(J) = |J|} \text{Tr}(\mathbb{H}, J).$ Set $\mathbb{H}_\ell = \mathcal{E}_\ell^\delta \mathbb{H} / \mathcal{E}_{\ell+1}^\delta \mathbb{H}$ and

$$\bar{A}_{\ell} = \prod_{\delta(K) = K, |K| = \ell} (\bar{T}_K - |W^K|), \quad \bar{A} = \bar{A}_{|J|} \circ A_{|J|} - 1 \circ \cdots.$$

As before, $\bar{A}_{\ell}$ preserves the filtration and kills $\mathcal{E}_{\ell+1}^\delta \mathbb{H}$. Thus

$$\bar{A}(\sum_{J \subset I, \ell(J) = |J|} \mathbb{H}, J) = 0.$$  \hspace{1cm} (4.5.1)

4.6. Proof of Theorem 4.1.1
Denote $\text{Irr}^\delta(\mathbb{H})_{\text{ell}} = cc^{-1}(\Theta^\delta(\mathbb{H})_0)$, and recall that $cc$ is finite-to-one. Set $R^\delta(\mathbb{H})_{\text{ell}}$ to be the linear span of $\text{Irr}^\delta(\mathbb{H})_{\text{ell}}$.

Lemma 4.6.1. The space $\overline{\text{Irr}}^\delta_0(\mathbb{H})$ is finite-dimensional, in particular, the set $\text{Irr}^\delta(\mathbb{H})_{\text{ell}}$ is finite.

Proof. Suppose $\{\pi_1, \pi_2, \ldots, \pi_k\}$ is a set in $\subset R^\delta(\mathbb{H})$ such that its image in $\overline{\text{Irr}}^\delta_0(\mathbb{H})$ is linearly independent. Applying the operator $A$ from section 4.3 one obtains a linearly independent set $\{A(\pi_1), A(\pi_2), \ldots, A(\pi_k)\}$ in $R^\delta(\mathbb{H})$. This is because $A(\pi) \equiv a\pi$ in $\overline{\text{Irr}}^\delta_0(\mathbb{H})$, for a nonzero integer $a$.

Since the characters of simple modules are linear independent, so are the characters of any linear independent set in $R^\delta(\mathbb{H})$. Thus there exist elements $h_1, h_2, \ldots, h_k$ of $\mathbb{H}$, such that the matrix $(\text{Tr}(h_i, A(\pi_j)))_{i,j}$ is invertible. By Lemma 4.5.1 the matrix $(\text{Tr}(A(h_i)A, \pi_j))_{i,j}$ is invertible. Since $\bar{A}$ vanishes on $\sum_{I, \ell(J) = |J|} \mathbb{H}_J$ by (4.5.1), it follows that

$$k \leq \dim \mathbb{H} / \left( \mathbb{H}, \mathbb{H}, \mathbb{H} \mathbb{H} + \mathcal{E}_1^\delta \mathbb{H} \right).$$

By Proposition 6.2.1 proved in section 6, the right hand side is bounded above by the number of $\delta$-elliptic conjugacy classes in $W$. This proves the first claim.

For the second claim, for every central character $\lambda \in \Theta^\delta(\mathbb{H})_0$, let $R^\delta(\mathbb{H})_{\lambda}$ be the span of $cc^{-1}(\lambda) \subset \text{Irr}^\delta\mathbb{H}$, and let $\overline{\text{Irr}}^\delta_0(\mathbb{H})_{\lambda}$ be the image of $R^\delta(\mathbb{H})_{\lambda}$ in $\overline{\text{Irr}}^\delta_0(\mathbb{H})$. Then

$$R^\delta(\mathbb{H})_{\text{ell}} = \bigoplus_{\lambda \in \Theta^\delta(\mathbb{H})_0} R^\delta(\mathbb{H})_{\lambda} \quad \text{and} \quad \overline{\text{Irr}}^\delta_0(\mathbb{H})_{\text{ell}} = \bigoplus_{\lambda \in \Theta^\delta(\mathbb{H})_0} \overline{\text{Irr}}^\delta_0(\mathbb{H})_{\lambda}.$$

This is because irreducible $\mathbb{H}$-modules with different central characters are necessarily independent, and the central character is the same for all constituents of a parabolically induced from a module with central character.
Since $R^\delta_0(\mathbb{H})$ is finite dimensional, then $\Theta^\delta(\mathbb{H})_0$ must be finite. Since $\mathbb{C}$ is a finite to one map, $\text{Im}^\delta(\mathbb{H})_{\text{ell}}$ is also finite.

\begin{remark}
The proof of Lemma 4.6.1 we presented is different than the argument from [BDK]. The classical proof (adapted to this setting under the assumption that $k$ is real valued) shows that the set of $\delta$-elliptic central characters $\Theta^\delta(\mathbb{H})_0$ is finite, as follows. Firstly, the set $\Theta^\delta(\mathbb{H})_0$ is a finite union of locally closed (in the Zarisky topology) subsets of $\Theta$, see [BDK], Proposition 1.1. Secondly, let $*: \Theta(\mathbb{H}) \to \Theta(\mathbb{H})$ be the anti-algebraic involution given by the hermitian dual. More precisely, if $\nu \in \Theta(\mathbb{H})$ is the central character of an irreducible module $\pi$, let $\nu^*$ be the central character of the hermitian dual of $\pi$ with respect to the operation $*$. Since $k$ is real, it follows from [Op], Proposition 2.35 that every tempered $\mathbb{H}$-module is $*$-unitary. In particular, using Lemma 4.3.1 $\nu = \nu^*$ for every $\nu \in \Theta^\delta(\mathbb{H})_0$. It follows that $\Theta^\delta(\mathbb{H})_0$ is finite.

Let $f \in R^\delta(\mathbb{H})_{\text{good}}$ be given. Since $\text{Im}^\delta(\mathbb{H})_{\text{ell}}$ is a finite set and the (twisted) characters of irreducible $\mathbb{H}$-modules are linearly independent, we can choose $f_1 \in R^\delta_0(\mathbb{H})_{\text{tr}}$ such that $f(\pi) = f_1(\pi)$ for all $\pi \in \text{Im}^\delta(\mathbb{H})_{\text{ell}}$. By replacing $f$ with $f - f_1$, we may therefore assume, without loss of generality, that $f(R^\delta(\mathbb{H})_{\text{ell}}) = 0$.

Apply to $f$ the operator $A$ defined in the previous section. Then $A(f) \in R^\delta(\mathbb{H})_0$, i.e., $A(f)(R^\delta_{\text{Ind}}(\mathbb{H})) = 0$. Recall that $A = 1 + \sum_{J = \delta(J) \leq I} c_J T_J^*$, for some $c_J \in \mathbb{Q}$. The operators $T_J$ preserve central characters, and therefore preserve $R^\delta(\mathbb{H})_{\text{ell}}$, hence $A(f)$ vanishes on $R^\delta(\mathbb{H})_{\text{ell}}$, because $f$ does. Since $R^\delta(\mathbb{H}) = R^\delta(\mathbb{H})_{\text{ell}} + R^\delta_{\text{Ind}}(\mathbb{H})$, it follows that $A(f) = 0$, and thus $f = -\sum_{J = \delta(J) \leq I} c_J T_J^*(f)$.

By induction on $|J|$, we may assume that $R^\delta(\mathbb{H})_{\text{good}} = R^\delta(\mathbb{H})_{\text{tr}}$. Since $\mathbb{H}_J = \mathbb{H}_J \otimes S(V^{W_J})$, it is straightforward that also $R^\delta(\mathbb{H})_{\text{good}} = R^\delta(\mathbb{H})_{\text{tr}}$. It is also easy to see that \begin{equation}
i^*_J(R^\delta(\mathbb{H})_{\text{good}}) \subset R^\delta(\mathbb{H})_{\text{good}} \text{ and } r^*_J(R^\delta(\mathbb{H})_{\text{tr}}) \subset R^\delta(\mathbb{H})_{\text{tr}};\end{equation}

in the case of $p$-adic groups, the second inclusion requires an argument, see [BDK], section 5.3, but since for $\mathbb{H}$, $r_J$ is just restriction, it is immediate.

Thus $T_J^*(f) = r_J^*(i_J^*(f))$ is in $R^\delta(\mathbb{H})_{\text{tr}}$, and so is $f$, concluding the proof.

Now we define the good forms for $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$.

\begin{definition}
For any $J \subset I$ and $\sigma \in \text{Irr}(\mathbb{H}_J^0)$, we set 
$$\Gamma_{J, \sigma} = \{\delta^i; \delta^i(J) = J, \delta^i \sigma \cong \sigma\}. $$

A form $f \in R^*(\mathbb{H}')$ is called good if for every $J \subset I$, $\sigma \in \text{Irr}(\mathbb{H}_J^0)$ and irreducible representation $U$ of $\Gamma_{J, \sigma}$, the function $\nu \mapsto f(X(J, \sigma, \nu) \rtimes U)$ is a regular function on the variety $(V^\vee)^{W_J \times \Gamma_{J, \sigma}}$. Denote the subspace of good forms by $R^*(\mathbb{H}')_{\text{good}}$.

As a consequence of Theorem 1,1,1 and Clifford theory (section 3.2), we obtain the trace Paley-Wiener Theorem for $\mathbb{H}'$.

\begin{corollary}
$R^*(\mathbb{H}')_{\text{tr}} = R^*(\mathbb{H}')_{\text{good}}$.

\end{corollary}

\begin{proof}
It is obvious that $R^*(\mathbb{H}')_{\text{tr}} \subset R^*(\mathbb{H}')_{\text{good}}$. By 3.2.3, 
$$R(\mathbb{H}')_C = \bigoplus_{i=0}^{d-1} R^i(\mathbb{H}')_C \cong \bigoplus_{i=0}^{d-1} R^i(\mathbb{H})_C.$$
Hence
\[ R^* (\mathbb{H}^I) = \oplus_{i=0}^{d-1} \text{Hom}_C(R^d (\mathbb{H})_C, C) = \oplus_{i=0}^{d-1} \text{Hom}_C(R^d (\mathbb{H})_C, C)^\Gamma = \oplus_{i=0}^{d-1} R^d_i (\mathbb{H})^\Gamma. \]

By definition, \( R^* (\mathbb{H}^I)_\text{good} \subset \oplus_{i=0}^{d-1} R^d_i (\mathbb{H})^\Gamma \), where \( R^d_i (\mathbb{H})^\Gamma \equiv R^d_i (\mathbb{H}) \cap R^d_i (\mathbb{H})_\text{good} \). Notice that for \( 0 \leq i, j < d \), \( \text{tr}(\mathbb{H}) |_{R^d_i (\mathbb{H})}_C = 0 \) unless \( i = j \).

By Theorem 3.1.1, the image of the map \( \text{tr}_\delta : \mathbb{H} / [\mathbb{H}, \mathbb{H}]_\delta \to R^d (\mathbb{H})_\delta \) is \( R^d (\mathbb{H})_\text{good} \). Hence the image of \( \mathbb{H}^I \) is \( R^d (\mathbb{H})^\Gamma_\text{good} \). By Proposition 3.1.1
\[ \text{tr}(\mathbb{H}) = \oplus_{i=0}^{d-1} R^d_i (\mathbb{H})^\Gamma_\text{good} = R^* (\mathbb{H}^I)_\text{good}. \]

\[ \Box \]

5. Twisted elliptic conjugacy classes in the finite Weyl group

In this section, we discuss the (twisted) conjugacy classes of finite Coxeter groups. These results will be used in the rest of this paper. In this section, we fix a finite irreducible Coxeter group \( (W, I) \) and a group automorphism \( \delta : W \to W \) with \( \delta(I) = I \). Let \( d \) be the minimal positive integer such that \( \delta^d(i) = i \) for all \( i \in I \).

5.1. Twisted conjugacy classes. For \( w \in W \), set \( \text{supp}_\delta (w) = \cup_{n=0}^{d-1} \delta^n \text{supp}(w) \). Then \( \text{supp}_\delta (w) \) is a \( \delta \)-stable subset of \( I \).

We define the \( \delta \)-twisted conjugation action of \( W \) on itself by \( w \cdot _\delta w' = w w' \delta(w)^{-1} \). Any orbit is called a \( \delta \)-twisted conjugacy class of \( W \). A \( \delta \)-conjugacy class \( O \) of \( W \) is called elliptic if \( O \cap W_j = \emptyset \) for all proper \( \delta \)-stable subset \( J \) of \( I \), i.e., \( \text{supp}_\delta (w) = I \) for all \( w \in O \). An element \( w \in W \) is called \( \delta \)-elliptic if it is contained in an elliptic \( \delta \)-conjugacy class of \( W \).

Recall that \( V \) is the vector space spanned by \( \alpha_i \) (for \( i \in I \)). As before, we regard \( W \) as a subgroup of \( GL(V) \) and \( \delta \) as an element in \( GL(V) \) in the natural way. For \( w \in W \), set
\[ p_{w, \delta} (q) = \det(q \cdot \text{id}_V - w \delta). \]
Then it is easy to see that \( p_{w, \delta} (q) = p_{w', \delta} (q) \) if \( w \) is \( \delta \)-conjugate to \( w' \).

We have the following well-known result for elliptic conjugacy classes. We include the proof here for completeness.

Proposition 5.1.1. Let \( O \) be a \( \delta \)-twisted conjugacy class of \( W \). The following are equivalent:

1. \( O \) is elliptic;
2. \( p_{w, \delta} (1) \neq 0 \) for some (or equivalently, any) \( w \in O \);
3. For some (or equivalently, any) \( w \in O \), there is no nonzero point in \( V \) that is fixed by \( w \delta \).

Proof. (2) \( \Rightarrow \) (1) is proved in \([He1]\), Lemma 7.2.

(3) \( \Rightarrow \) (2) is obvious.

(1) \( \Rightarrow \) (3). Let \( w \in O \) and \( v \neq 0 \) with \( w \delta(v) = v \). Let \( x \in W \) with \( x(v) \) dominant. Set \( \bar{v} = x(v) \) and \( u' = x^{-1} w \delta(x) \). Then \( u' \in O \) and \( u' \delta(\bar{v}) = \bar{v} \). Since \( \bar{v} \) is dominant, \( \delta(\bar{v}) \) is also dominant. Hence \( \delta(\bar{v}) - \bar{v} = \delta(\bar{v}) - u' \delta(\bar{v}) \) is a linear combination of \( \alpha_i \) with nonnegative coefficients. As \( (\delta(\bar{v}), \rho') = (\bar{v}, \rho') \), we must have that \( \delta(\bar{v}) = \bar{v} \) and \( u' \delta(\bar{v}) = \delta(\bar{v}) \). Hence \( u' \) is generated by \( s_{\alpha} \), where \( \alpha \) runs over simple roots in \( V \) such that \( (\bar{v}, \alpha') = 0 \). In particular, \( u' \) is in a proper \( \delta \)-stable parabolic subgroup of \( W \). Thus \( O \) is not elliptic. \( \Box \)
5.2. Minimal length elements. We follow the notation in [GP] section 3.2.

Given \( w, w' \in W \) and \( i \in I \), we write \( w \lesssim w' \) if \( w' = s_i w \delta(s_i) \) and \( \ell(w') \leq \ell(w) \). If \( w = w_0, w_1, \ldots, w_n = w' \) is a sequence of elements in \( W \) such that for all \( k \), we have \( w_{k-1} \lesssim w_k \) for some \( j \in I \), then we write \( w \rightarrow_\delta w' \). If \( w \rightarrow_\delta w' \) and \( w' \rightarrow_\delta w \), then we say that \( w \) and \( w' \) are in the same \( \delta \)-cyclic shift class and write \( w \approx_\delta w' \).

For \( w \in W \) and \( i \in I \), define the length function \( \ell_i(w) \) as the number of generators in \( I \) conjugate to \( s_i \) occurring in a reduced expression of \( w \). By [GP] Exercise 1.15, it is independent of the choice of reduced expression of \( w \).

Set
\[
\ell_i(w) = \sum_{k=0}^{d-1} l_{i, \delta(k)}(w).
\]

Then it is easy to see that if \( w \approx_\delta w' \), then \( \ell_i(w) = \ell_i(w') \) for all \( i \in I \).

We have the following main result on elliptic conjugacy classes of \( W \).

**Theorem 5.2.1.** Let \( O \) be an elliptic \( \delta \)-twisted conjugacy class in \( W \) and \( O_{\min} \) be the set of minimal length elements in \( O \). Then

1. For each \( w \in O \), there exists \( w' \in O_{\min} \) such that \( w \rightarrow_\delta w' \).
2. Let \( w, w' \in O_{\min} \), then \( w \approx_\delta w' \). In particular, \( \ell_i(w) = \ell_i(w') \) for all \( i \in I \).
3. Let \( O' \) be an elliptic \( \delta \)-conjugacy classes of \( W \). Let \( w \in O_{\min} \) and \( w' \in O'_{\min} \). Then \( O = O' \) if and only if \( p_{w, \delta}(q) = p_{w', \delta}(q) \) and \( \ell_i(w) = \ell_i(w') \) for all \( i \in I \).

**Remark 5.2.1.** It was first proved via a case-by-case analysis for untwisted case by Geck and Pfeiffer in [GP] Theorem 3.2.7 and for twisted case by the second-named author in [He1] Theorem 7.5. A case-free proof for part (1) and (2) was found recently in [HN]. It would be interesting to find a case-free proof for part (3) and/or Theorem 5.2.2 below.

The following result can be checked easily from the list of Dynkin diagrams.

**Lemma 5.2.1.** Let \( J \subset I \) with \( \delta(J) = J \). Then we may write \( J = J_1 \cup J_2 \) with \( \delta(J_i) = J_i \) for \( i = 1, 2 \) and

1. \( J_1 \) is a union of connected components of type \( A \);
2. For any connected component \( K \) of \( J_1 \), either \( \delta |_K \) is identity or there exists another connected component \( K' \neq K \) of \( J_1 \) with \( \delta(K) = K' \);
3. Either (i) \( J_2 = \emptyset \) or (ii) \( J_2 \) is a connected component of \( J \) not of type \( A \) or (iii) \( J_2 \) is a connected component of \( J \) of type \( A \) and \( \delta |_{J_2} \) is nontrivial.

We have the following consequence that elliptic classes never fuse.

**Theorem 5.2.2.** Let \( J \subset I \) with \( \delta(J) = J \). Let \( C \) be a \( \delta \)-twisted conjugacy class of \( W \) such that \( C \cap W_J \) contains a \( \delta \)-elliptic element of \( W_J \). Then \( C \cap W_J \) is a single \( \delta \)-twisted conjugacy class of \( W_J \).

**Remark 5.2.2.** The untwisted case was due to Geck and Pfeiffer in [GP] Theorem 3.2.11. The general case can be proved in a similar way by using Theorem 5.2.1 (3) and Lemma 5.2.1. We omit the details.
In the rest of this section, we discuss some further properties on elliptic conjugacy classes of a parabolic subgroup of $W$.

**Proposition 5.2.1.** Let $C$ be a $\delta$-twisted conjugacy class of $W$. Let $J$ be a minimal $\delta$-stable subset of $I$ with $C \cap W_J \neq \emptyset$. Let $w \in C \cap W_J$ and $J' \subset I$ with $\delta(J') = J'$. Then $xw\delta(x)^{-1} \in W_{J'}$ if and only if $x = x'x_1$ for some $x' \in W_{J'}$ and $x_1 \in W^\delta \cap J'W^J$ such that $x_1(J) \subset J'$.

**Remark 5.2.3.** It is easy to see that $J$ is a minimal $\delta$-stable subset of $I$ with $C \cap W_J \neq \emptyset$ if and only if $C \cap W_J$ contains a $\delta$-elliptic element of $W_J$.

**Proof.** If $x = x'x_1$ for some $x' \in W_{J'}$ and $x_1 \in W^\delta \cap J'W^J$ such that $x(J) \subset J'$, then $xw\delta(x)^{-1} = x'(x_1wx_1^{-1})\delta(x)^{-1} \in W_{J'}$.

Suppose that $xw\delta(x)^{-1} \in W_{J'}$. We write $x$ as $x = y'x_1y$, where $y \in W_J$, $y' \in W_{J'}$ and $x_1 \in J'W^J$. Set $w' = yw\delta(y)^{-1} \in C \cap W_J$. Then $x_1w'\delta(x_1)^{-1} \in W_{J'}$. Hence $x_1w' \in x_1W_J \subset J'x_1W_J$ and $x_1w' \in W_{J'}\delta(x_1) \subset W_J\delta(x_1)W_J$. Since $x_1, \delta(x_1) \in J'W^J$, we must have that $x_1 = \delta(x_1) \in W^\delta$.

Now $w' \in W_J \cap x_1^{-1}W_{J'}x_1' = W_Jx_1^{-1}(J')$. By our assumption on $J$, $J \cap x_1^{-1}(J') = J$. So $x_1(J) \subset J'$. Hence $x \in W_Jx_1$.

**Proposition 5.2.2.** Let $J \subset I$ with $\delta(J) = J$ and $w$ be a $\delta$-elliptic element in $W_J$. Set $Z_{W,\delta}(w) = \{x \in W ; xw\delta(x)^{-1} = w\}$ and $N_{W,\delta}(W_J) = \{x \in W ; xW_J\delta(x)^{-1} = W_J\}$. Then

$$W_JZ_{W,\delta}(w) = N_{W,\delta}(W_J) = W_JZW_J,$$

where $Z = \{z \in J W^J ; z = \delta(z), J = z(J)\}$.

**Proof.** Let $C$ be the $\delta$-twisted conjugacy class of $W$ that contains $w$ and $C_J$ be the $\delta$-twisted conjugacy class of $W_J$ that contains $w$. Let $x \in N_{W,\delta}(W_J)$. By Theorem 5.2.2, $xw\delta(x)^{-1} \in C \cap W_J = C_J$. Therefore $xw\delta(x)^{-1} = yw\delta(y)^{-1}$ for some $y \in W_J$. Hence $y^{-1}x \in Z_{W,\delta}(w)$ and $x \in W_JZ_{W,\delta}(w)$.

On the other hand, if $x \in W_JZ_{W,\delta}(w)$, then $xw\delta(x)^{-1} \in C_J$. By Proposition 5.2.1, $x \in W_JZW_J \subset N_{W,\delta}(W_J)$.



6. **Spanning set of (twisted) cocenter**

6.1. **Spanning set of $\mathbb{H}_0/\mathbb{H}_{0,0}\delta$**. Let $\mathcal{I}^\delta = \{J \subset I ; J = \delta(J)\}$. For $J, J' \in \mathcal{I}^\delta$, we write $J \sim \delta J'$ if there exists $w \in W^\delta \cap J'W^J$ such that $w(J) = J'$. For each $\sim \delta$-equivalence class in $\mathcal{I}^\delta$, we choose a representative. Such representatives form a subset of $\mathcal{I}^\delta$, which we denote by $\mathcal{I}_0^\delta$. Then there is a natural bijection between $\mathcal{I}_0^\delta$ and $\mathcal{I}^\delta/\sim \delta$.

For any $J \subset \mathcal{I}_0^\delta$, we choose a basis $\{f_{J,k}\}$ of $S(V^{W_J \times \delta})N_{W,\delta}(W_J)$.

For each $\delta$-twisted conjugacy class $C$ of $W$, we fix a minimal element $J_C \in \mathcal{I}_0^\delta$ such that $C \cap W_{J_C} \neq \emptyset$. Such $J_C$ is uniquely determined by $C$. We fix an element $w_C$ in $C \cap W_{J_C}$. By definition, $w_C$ is a $\delta$-elliptic element in $W_{J_C}$.

**Proposition 6.1.1.** We keep the notations as in 6.1. Then $\{w_C f_{J_C,\delta}\}$ spans $\mathbb{H}_0/\mathbb{H}_{0,0}\delta$ as a vector space.

**Proof.** Notice that for any $x, y \in W$ and $f \in S(V)$,

$$xy\delta(x)^{-1}f \equiv y\delta(x)^{-1}f \delta(x) = y\delta(x)^{-1}(f) \bmod [\mathbb{H}_0, \mathbb{H}_{0,0}\delta].$$
Thus $H_0$ is spanned by $w_C S(V)$, where $C$ runs over $\delta$-twisted conjugacy classes of $W$.

Now we fix a $\delta$-twisted conjugacy class $C$. Let $J = J_C$. We have that $V = V^{W_J} \oplus U$, where $U$ is the subspace of $V$ spanned by simple roots in $J$. This is a decomposition of $V$ as $W_J \times \delta$ submodules. Since $w$ is $\delta$-elliptic in $W_J$, $(1 - w_C \delta)$ is invertible on $U$. Also $V^{W_J} = V^{W_J} \times \delta \oplus (1 - \delta)V^{W_J}$. As $(1 - \delta)V^{W_J}$ is a direct sum of eigenspaces of $\delta$ with eigenvalues not equal to 1, $(1 - w_C \delta)$ is also invertible on $(1 - \delta)V^{W_J}$. Set $U' = U \oplus (1 - \delta)V^{W_J}$. Then $(1 - w_C \delta)$ is invertible on $U'$ and $V^{w_C \delta} = V^{W_J} \times \delta$.

Let $f \in S(V)$ and $u \in U'$. There exists $v \in U'$ such that $v - w_C \delta(v) = u$. Now

$$
\begin{align*}
uw_C f &= vw_C f - (w_C \delta(v)) w_C f = vw_C f - w_C f \delta(v) \\
&= [v, w_C f]_\delta \in [H_0, H_0]_\delta.
\end{align*}
$$

Hence $U' w_C S(V) \subseteq [H_0, H_0]_\delta$.

Since $V = V^{W_J} \times \delta \oplus U'$, we have that

$$
\begin{align*}
w_C S(V) &= w_C S(U') S(V^{W_J} \times \delta) = S(U') w_C S(V^{W_J} \times \delta) \\
&\subseteq w_C S(V^{W_J} \times \delta) + [H_0, H_0]_\delta.
\end{align*}
$$

Let $f \in S(V^{W_J} \times \delta)$. Then for any $x \in Z_{W, \delta}(w_C)$,

$$
w_C f \equiv x w_C f \delta(x)^{-1} = (x w_C \delta(x)^{-1}) \delta(x)(f) = w_C \delta(x)(f) \mod [H_0, H_0]_\delta.
$$

Hence $w_C f = \frac{1}{|Z_{W, \delta}(w_C)|} \sum_{x \in Z_{W, \delta}(w_C)} w_C \delta(x)(f) \in w_C S(V^{W_J} \times \delta) Z_{W, \delta}(w_C)) \mod [H_0, H_0]_\delta$.

Here the last inclusion follows from the fact that $\delta(Z_{W, \delta}(w_C)) = Z_{W, \delta}(\delta(w_C))$.

Since $w_C$ is $\delta$-elliptic in $W_J$, $\delta(w_C)$ is also $\delta$-elliptic in $W_J$. By Proposition 5.2.2,

$W_J Z_{W, \delta}(\delta(w_C)) = N_{W, \delta}(W_J)$ and

$S(V^{W_J} \times \delta) Z_{W, \delta}(w_C)) S(V^{W_J} \times \delta) W_J Z_{W, \delta}(w_C)) = S(V^{W_J} \times \delta)_{W_J Z_{W, \delta}(w_C))}.$

Therefore $\{w_C f \lambda(i)\}$ spans the image of $w_C S(V)$ in $H_0/[H_0, H_0]_\delta$. \qed

6.2. Spanning set of $H_0$. Let $\bar{H}$ be the graded algebra associated to the filtration of $H$ given by the degree of $S(V)$ defined in section 2.4. Recall that $\bar{H} \cong H_0$. By induction on degree one shows that if $\mathcal{L}$ is a spanning set of $\bar{H}/[\bar{H}, \bar{H}]_\delta$, then $\mathcal{L}$ is also a spanning set of $H/[H, H]_\delta$. To see this, first notice that (using the relations in $H$) the $\delta$-commutators preserve the filtration, i.e.,

$$[F^i H, F^j H] \subseteq F^{i+j} H.$$

More precisely, if $a_1$ and $a_2$ are elements of $S(V)$ homogeneous of degrees $l$ and $j$, respectively, then

$$[w_1 \cdot a_1, w_2 \cdot a_2]_\delta \in (w_1 w_2 \cdot w_2^{-1}(a_1)a_2 - w_2 \delta^l(w_1) \cdot \delta^l(a_1) \delta^l(w_1^{-1})a_2) + F^{i+j-1} H.$$

Let $\bar{h} \in H/[H, H]_\delta$ be given and assume that $h \in F^i H$. Since $h$ can be written as $h = \sum_{a} w_a a$ for some $a \in S(V)$, set $h_0 = \sum_{a} w_a a \in H_0$. Then $h = \sum_{\bar{x} \in \mathcal{L}} c_{\bar{x}} x_{0} \in [H_0, H_0]_\delta$, where only finitely many $c_{\bar{x}}$ are nonzero, $x_{0}$ is a representative of $\bar{x}$ in $H_0$. Moreover, we may choose the $x_{0}$’s that contribute to the sum to have (maximal)
We have that $x$ is homogeneous and $[y_{0,j}, y'_{0,j}] \in \mathcal{F}^i \mathbb{H}_0$.

Let $y_j, y'_j \in \mathbb{H}$ be the corresponding elements for $y_{0,j}, y'_{0,j}$, respectively. By (6.2.1), $[y_j, y'_j]_\delta$ differs from $[y_{0,j}, y'_{0,j}]_\delta$ by an element in $\mathcal{F}^{i-1} \mathbb{H}$. Then $h - \sum \bar{x} c_x x - \sum \bar{y} [y_j, y'_j]_\delta \in \mathcal{F}^{i-1} \mathbb{H}$, and the claim follows by induction. (Here $x$ denotes a representative of $\bar{x}$ in $\mathbb{H}$ corresponding to $x_0$.)

Now we have the following result.

**Proposition 6.2.1.** We keep the notations as in [6.1]. Then $\{w_C f_{J, i}\}$ spans $\tilde{\mathbb{H}}_\delta$ as a vector space.

Now we give the trace formula of the element from the spanning set on the induced representations.

**Proposition 6.2.2.** Let $J, J'$ be $\delta$-stable subsets of $I$. Let $w$ be an $\delta$-elliptic element in $W_J$ and $C$ be the $\delta$-twisted conjugacy class of $W$ that contains $w$. Let $M$ be an $\mathbb{H}'_{J'}$-module. Then for any $f, h \in S(V^{W_J \times (\delta)})$

$$
\text{Tr}(w f \delta, \text{Ind}_{\mathbb{H}'_{J'}}^\mathbb{H}', M) = \begin{cases} 
|N_{W, \delta}(W_J)/W_J| \text{Tr}(w f \delta, M), & \text{if } J = J'; \\
0, & \text{if } C \cap W_{J'} = \emptyset.
\end{cases}
$$

**Proof.** The space $\text{Ind}_{\mathbb{H}'_{J'}}^\mathbb{H}', M$ is the direct sum of $xM, x \in W_{J'}$.

Let $x \in W_{J'}$ and $m \in M$. We write $x$ as $x = x_1 x_2$, where $x_1 \in W_J$ and $x_2 \in J' W_{J'}$. Then

$$
wf \delta(xm) = wf \delta(x_1) \delta(x_2)m = w \delta(x_1) f \delta(x_2)m.
$$

Notice that $f \delta(x_2) = \delta(x_2) \delta(x_2)^{-1}(f) + \sum_{y < \delta(x_2)} y f_y = \delta(x_2) \delta(x)^{-1}(f) + \sum_{y < \delta(x_2)} y f_y$

for some $f_y \in S(V)$. We have that

$$
wf \delta(xm) = w \delta(x) \delta(x)^{-1}(f)m + \sum_{y < \delta(x_2)} w \delta(x_1) y f_y m.
$$

We show that

(a) For any $y < \delta(x_2), w \delta(x_1) y f_y m \in \sum_{x' \in J' W_{J'}, x' \neq x} x'M$.

If (a) is not true, then $w \delta(x_1) y \in xW_{J'}$. Hence $y \in (w \delta(x_1))^{-1} x W_{J'} \subset W_{J_2} W_{J'}$. Since $x_2 \in J' W_{J'}$, $y \geq x_2$. Thus $\ell(y) \geq \ell(x_2) = \ell(\delta(x_2)) > \ell(y)$. That is a contradiction. Hence (a) is proved.

It is easy to see that $w \delta(x) |\delta(x)^{-1}(f)|m \in \sum_{x' \in W_{J'}, x' \neq x} x'M$ if and only if $w \delta(x) \in xW_{J'}$, i.e., $x^{-1} w \delta(x) \in W_{J'}$.

In particular, if $C \cap W_{J'} = \emptyset$, then $x^{-1} w \delta(x) \notin W_{J'}$ for any $x$ and thus $\text{Tr}(wf \delta, \text{Ind}_{\mathbb{H}'_{J'}}^\mathbb{H}', M) = 0$.

If $J' = J$, then by Proposition 5.2.2, for any $x \in W_J$, $x^{-1} w \delta(x) \in W_J$ if and only if $x \in J W_{J'} \cap W^\delta$ and $x(J) = J$. For such $x$, $\delta(x)^{-1}(f) = f$. The number of elements $x$ satisfy such condition equals $|N_{W, \delta}(W_J)/W_J|$.

Now

$$
\text{Tr}(wf \delta, \text{Ind}_{\mathbb{H}'_{J'}}^\mathbb{H}', M) = \sum_{x \in J W_{J'} \cap W^J, x(J) = J} \text{Tr}(x^{-1} w f \delta x M).
$$

We have that $x^{-1} w f \delta x = x^{-1} w f x \delta$. By Theorem 5.2.2 there exists $x' \in W_J$ such that $x^{-1} w x = x' w \delta(x')^{-1}$. Then $x^{-1} w f \delta x = x' w \delta(x')^{-1} f \delta = x'(w f \delta)(x')^{-1}$. So
Let $M$ be a module of $H'$, and the minimality of $J$, it follows that

$$\text{Tr}(x^{-1}w f \delta, M) = \text{Tr}(x' w f \delta(x')^{-1}, M) = \text{Tr}(w f \delta, M).$$

Thus

$$\text{Tr}(w f \delta, \text{Ind}_{H'_j}^{H'} M) = |N_{W_j}(W_j)/W_j| \text{Tr}(w f \delta, M).$$

$\square$

7. Density theorem and basis theorem

7.1. Tempered modules. Let $\text{res}_{W'} : R(H') \to R(W')$ denote the linear map given by the restriction of $H'$-modules to $\mathbb{C}[W']$. Define

$$T_{\text{re}} = \{ \pi : \pi \text{ tempered } H'\text{-module with real central character} \}. \quad (7.1.1)$$

**Theorem 7.1.1.** Suppose the parameter function $k$ of $H$ is real-valued.

(a) The set $\text{res}_{W'} T_{\text{re}}$ is a $\mathbb{Q}$-basis of $R(W')_q$.

(b) If $H'$ has parameter function of geometric type in the sense of [Lu2], then $\text{res}_{W'} T_{\text{re}}$ is a $\mathbb{Z}$-basis of $R(W')$ and the change of bases matrix between $\text{res}_{W'} T_{\text{re}}$ and $\text{Irr} W'$ is upper uni-triangular in an appropriate ordering.

Part (a) of Theorem [7.1.1] is proved by homological algebra in [So1, Theorem 6.5]. Part (b) follows from [Lu2] together with Clifford theory, see [BC, 3.4-3.6]. In fact, (b) is now known to hold for all real parameters $k$ and all root systems except some cases in type $F_4$ (where it is again expected to be true).

7.2. Basis Theorem. We retain the notation from section 6.1. In particular, let $\{w_C f_{J_C, i}\}$ be the spanning set of $H_3'$ from Proposition 6.2.1, indexed by conjugacy classes $C$ in $W$ and for every $i$, a basis $\{f_{J_C, i}\}$ of $S(V^{W_C} \times \langle \delta \rangle)^{N_{W_j} W_{J_C}}$.

**Theorem 7.2.1** (Basis Theorem). The set $\{w_C f_{J_C, i}\}$ forms a basis of $H_3'$.

**Proof.** In light of Proposition 6.2.1, we need to prove that the set is linearly independent. To see this, we proceed by induction and use the formula for the trace of induced modules in Proposition 6.2.2 to separate the subsets for various $J'$. Let

$$\sum_{C,i} a_{C,i} w_C f_{J_C, i} = 0 \quad (7.2.1)$$

be a linear combination.

The base case is $J' = \emptyset$. For every character $\chi_\nu : H_\emptyset = S(V) \times \langle \delta \rangle \to \mathbb{C}$, parameterized by $\nu \in (V')^\delta$, consider the induced module $X(\nu) := \text{Ind}_{H_\emptyset}^{H'}(\chi_\nu)$. By Proposition 6.2.2, $\text{Tr}(w_C f_{C,i} \delta, X(\nu)) = 0$ for all $C \neq \emptyset$ and all $i$. Thus applying $\text{Tr}(-, X(\nu))$ to (7.2.1), we get $|W| \sum_i a_{1,i} f_{\emptyset, i} = 0$ for every $\nu \in W \setminus (V')^\delta$. This means that the polynomial function $\sum_i a_{1,i} f_{\emptyset, i}$ vanishes on its natural domain, thus $\sum_i a_{1,i} f_{\emptyset, i} = 0$ in $S(V)^W \times \langle \delta \rangle$. Since by construction, $\{f_{\emptyset, i}\}$ are linear independent in $S(V)^W \times \langle \delta \rangle$, it follows that $a_{1,i} = 0$ for all $i$.

By induction, suppose we are left with a (smaller) linear combination as in (7.2.1). Let $J'$ be a minimal element in $T_0$ that appears in this combination. Suppose $C$ is a conjugacy class that occurs and $C \cap W_{J'} = \emptyset$. By the construction of the spanning set (see section 6.1) and the minimality of $J'$, we must have $J' = J_C$. Let $M$ be a module of $H_{J'}$. Applying $\text{Tr}(-, \text{Ind}_{H_{J'}}^{H'} M)$ to the linear combination, it follows that

$$|N_{W_j}(W_{J'})/W_{J'}| \sum_{C : J_C = J', i} a_{C,i} \text{Tr}(w_C f_{J', i} \delta, M) = 0, \text{ for all } M.$$
Specialize $M$ to $M = \sigma \otimes \chi_{\nu}$, where $\sigma$ is an irreducible tempered module with real central character of $(\mathbb{H}_{\nu})^{\mathbb{Z}_{\nu}}$, and $\chi_{\nu} : S(V^{W_{\nu} \times (\delta)}) \to \mathbb{C}$ is a character indexed by $\nu \in N_{W_{\nu}}(W_{\nu}) \setminus (V^{\nu})^{W_{\nu} \times (\delta)}$. By Theorem 7.3.1 when the parameter function $k$ takes real values, the set of such $\sigma$ separates the $w_{C}$’s. For arbitrary parameters $k$, we can specialize $\sigma$ to the representations explicitly constructed in Theorem 8.1.1(1) below. By the same discussion as for $J' = \emptyset$, the unramified characters $\chi_{\nu}$ separate the $f_{J',i}$’s. In conclusion, $a_{C,i} = 0$ for all $C$ such that $J_{C} = J'$.

**Remark 7.2.1.** Theorem 7.2.1 implies a description of the cocenter for the extended graded Hecke algebra $\mathbb{H}' = \mathbb{H} \rtimes \langle \delta \rangle$, $\delta^{d} = 1$. From Proposition 7.1.1 we have

$$\mathbb{H}' = \bigoplus_{i=0}^{d-1} \mathbb{H}^{[i]}, \text{ where } \mathbb{H}^{[i]} \text{ is the space of } \delta \text{-coinvariants in } \mathbb{H}_{\delta}. \quad (7.2.2)$$

Now Theorem 7.2.1 gives an explicit basis for each $\mathbb{H}^{[i]}$.

The following corollary is stated here for convenience. It was already proved in Lemma 4.6.1 and it is a consequence of the explicit description of the basis of $\mathbb{H}^{[i]}$. The case $\delta = 1$ was proved in the setting of affine Hecke algebras in [OS, Proposition 3.9] via different methods.

**Corollary 7.2.1.** The dimension of the $\delta$-elliptic space $\mathbb{P}_{0}^{\delta}(\mathbb{H})$ is at most the number of $\delta$-twisted elliptic conjugacy classes in $W$.

### 7.3. Density Theorem

In the notation of section 4.1 let $\text{tr}^{\delta} : \mathbb{H}_{\delta} \to R_{\delta}^{\sharp}(\mathbb{H})$ be the trace map.

**Theorem 7.3.1** (Density Theorem). The trace map $\text{tr}^{\delta} : \mathbb{H}_{\delta} \to R_{\delta}^{\sharp}(\mathbb{H})$ is injective. Equivalently, if $h \in \mathbb{H}$ is such that $\text{Tr}(h_{\delta}, \pi) = 0$ for all $\pi \in \text{Irr}^{\delta}(\mathbb{H})$, then $h \in [\mathbb{H}, \mathbb{H}]_{\delta}$.

**Proof.** Suppose $x = \sum_{C,i} a_{C,i} w_{C} f_{J,i} \in \mathbb{H}_{\delta}$ is in $\ker \text{tr}^{\delta}$. This means that for every $\pi \in R^{\delta}(\mathbb{H})_{\text{ss}}$, $\text{Tr}(x, \pi) = 0$. The inductive argument in the proof of Theorem 7.2.1 then implies that $a_{C,i} = 0$ for all $C, i$, and so $x = 0$.

In the setting of $\mathbb{H}' = \mathbb{H} \rtimes \langle \delta \rangle$, we defined the trace map $\text{tr} : \mathbb{H}' \to R(\mathbb{H}')^{\ast}$ in section 4.1. As a consequence of Theorem 7.3.1 and Clifford theory (section 3.2), we obtain a density theorem for $\mathbb{H}'$.

**Corollary 7.3.1.** The trace map $\text{tr} : \mathbb{H}' \to R(\mathbb{H}')^{\ast}$ is injective.

**Proof.** By Proposition 3.1.1 $\mathbb{H}' = \bigoplus_{i=0}^{d-1} \mathbb{H}^{[i]} / ([\mathbb{H}', \mathbb{H}'] \cap \mathbb{H}^{[i]})$. By (3.2.3), $R(\mathbb{H}')_{\mathbb{C}} = \bigoplus_{i=0}^{d-1} R_{\mathbb{C}}^{\delta}(\mathbb{H}')_{\mathbb{C}}$. Moreover, for $0 \leq i, j < d$, $\text{tr}(\mathbb{H}_{\delta}^{[i]})_{\mathbb{C}} = 0$ unless $i = j$. Thus $\ker \text{tr} = \bigoplus_{i=0}^{d-1} \ker \text{tr} \cap \mathbb{H}^{[i]}$.

Let $h_{\delta}' \in \mathbb{H}'$, $\delta' = \delta$, be such that $\text{tr}(h_{\delta}') = 0$. By Lemma 3.2.4 this is equivalent with the condition that for every $X \in \text{Irr}^{\delta}(\mathbb{H})$,

$$\sum_{\gamma \in \Gamma / \Gamma_{X}} \text{tr}^{\delta}(X)(\gamma^{-1}(h)) = 0.$$

Since for every $\gamma_{1} \in \Gamma_{X}$, $\text{tr}(X)(h) = \text{tr}(\gamma_{1} X)(h) = \text{tr}(X)(\gamma_{1}^{-1}(h))$, it follows that the equivalent condition is

$$\sum_{\gamma \in \Gamma} \text{tr}^{\delta}(X)(\gamma^{-1}(h)) = 0.$$
By Theorem 7.3.1 we have then that $\sum_{\gamma \in \Gamma} \gamma(h) \in [\mathbb{H}, \mathbb{H}]_\delta$. Writing
\[ dh = \sum_{\gamma} \gamma(h) + \sum_{\gamma \neq 1} (1 - \gamma)(h), \]
we see that $h \in (1 - \delta)\mathbb{H} + [\mathbb{H}, \mathbb{H}]_\delta$. By claim (a) in the proof of Proposition 3.1.1 this is the same as $h\delta' \in \mathbb{H}\delta' \cap [\mathbb{H}', \mathbb{H}']$, which is what we wanted to prove.

8. Bases of $R(\mathbb{H})$

In this section, we exhibit linearly independent sets in $\overline{R}_0(\mathbb{H})$ of cardinality equal to the number of $\delta$-elliptic conjugacy classes in $W$. This implies in particular that equality holds in Corollary 7.2.1 and that our sets are bases. Moreover, our construction is such that the $W$-structure of these bases elements does not change for various values of the parameter function $k$ of $\mathbb{H}$, and in addition the action of the Hecke algebra elements $\omega$ (equivalently $\overline{\omega}$) depends linearly in $k$.

8.1. The main result of the section follows.

**Theorem 8.1.1.** Let $\mathbb{H}_k = \mathbb{H}(\Psi, k)$ be a graded Hecke algebra associated to a simple root system $\Psi$ and parameter function $k : R^+ \to \mathbb{C}$ as in Definition 2.2.1. Let $\delta$ be an automorphism of the Dynkin diagram of $\Psi$ of order less than or equal to 3.

1. There exists a $\mathbb{Z}$-basis of $\overline{R}_0(\mathbb{H}_k)$ represented by a set of genuine representations $\{\pi_1, \ldots, \pi_m\} \subset R^\delta(\mathbb{H}_k)$, where $m$ is the number of $\delta$-elliptic conjugacy classes in $W$ such that
   - (P1) the restriction $\text{res}_W \pi_j$ is independent of $k$ for every $j = 1, m$;
   - (P2) the actions of $\pi_j(\omega)$, $j = 1, m$, $\omega \in V_{\mathbb{C}}$, depend linearly in the parameter function $k$.

2. There exists a $\mathbb{Q}$-basis of $R^\delta(\mathbb{H}_k)_\mathbb{Q}$ satisfying properties (P1) and (P2) above.

The proof of (1) will occupy the rest of the section; we exhibit an explicit basis $\{\pi_1, \ldots, \pi_m\}$ for each pair $(\Psi, \delta)$. We remark that except for simply-laced systems and parameter $k \equiv 0$, or certain special values of $k$ in type $F_4$, the bases we give consist of elements of $\text{irr}(\mathbb{H})_{\text{ell}}$ (for any parameter function). In type $F_4$, for almost all values of the parameter $k$, the same is true; however for a few special values of $k$ some of the $\pi_j$ may become reducible.

**Proof of (2).** Let $\overline{R}_0(\mathbb{H})$ be as in 7.2.2 and $\overline{R}_\delta(\mathbb{H}) = \oplus_i \overline{R}_0(\mathbb{H})$ be the associated graded object. As $\mathbb{Q}$-vector spaces $R^\delta(\mathbb{H})_\mathbb{Q} \cong \overline{R}_\delta(\mathbb{H})_\mathbb{Q}$, so it is sufficient to construct a basis of $\overline{R}_\delta(\mathbb{H})_\mathbb{Q}$ with the desired properties.

Recall the set $\mathcal{I}_0^\delta$ from section 6.1. For every $J \in \mathcal{I}_0^\delta$, the parabolic subalgebra $\mathbb{H}_J$ decomposes as $\mathbb{H}_J = \mathbb{H}_J^{ss} \otimes S(V^{W_J})$. Then we have the following decomposition:
\[ \overline{R}_\delta(\mathbb{H})_\mathbb{Q} = \bigoplus_{J \in \mathcal{I}_0^\delta} i_J(\overline{R}_0(\mathbb{H}_J^{ss}))_\mathbb{Q} \otimes (S(V^{W_J} \times (\delta))^{N_{W,\delta}(W_J)})^*, \]  
(8.1.1)

where the subgroup of unramified characters $(S(V^{W_J} \times (\delta))^{N_{W,\delta}(W_J)})^*$ can be canonically identified with $N_{W,\delta}(W_J) \backslash (V^{(\delta)})^{W_J}$. Let $\mathcal{B}_{J,0} := \mathcal{B}(R_0^\delta(\mathbb{H}_J^{ss}))$ be the basis
of $R_0^\Delta(H^n)$ given by (1). Then the desired basis for $\overline{R}_0^J(H^n)$ (and so of $R^\Delta(H^n)$) is
\[ \left\{ J(\pi \otimes \chi_\nu) : J \in \mathcal{I}_0, \pi \in \mathcal{B}_{J,0}, \nu \in N_{W,\delta}(W_J) \setminus (V^+)W_{\delta(\delta)} \right\}. \] (8.1.2)

8.2. $A_{n-1}$. Let $\mathbb{H}_{n,k}^A$ be the graded Hecke algebra of $GL(n)$ with (constant) parameter $k$ (for simplicity of formulas we consider $GL(n)$ rather than $SL(n)$) generated by $w \in S_n$ and $\{\varepsilon_1, \ldots, \varepsilon_n\}$. Let $s_{i,j}$ denote the reflection in the root $\varepsilon_i - \varepsilon_j$. As it is well-known, there is a surjective algebra morphism
\[ \phi_k : \mathbb{H}_{n,k}^A \to \mathbb{C}[S_n], \]
\[ \phi_k(w) = w, \ w \in S_n; \]
\[ \phi_k(\varepsilon_i) = k(s_{i,i+1} + s_{i,i+2} + \cdots + s_{i,n}), \ 1 \leq i < n; \]
\[ \phi_k(\varepsilon_n) = 0. \] (8.2.1)

Using the adjoint map $\phi_k^* : R(S_n) \to R(\mathbb{H}_{n,k}^A)$, we can lift every irreducible $S_n$-representation to an irreducible (in general non-hermitian) $\mathbb{H}_{n,k}^A$-module. If $\sigma$ is a partition of $n$ parameterizing an irreducible $S_n$-module, denote by $\pi_{n,k}^A(\sigma)$ the resulting simple $\mathbb{H}_{n,k}^A$-module.

There is a single elliptic conjugacy class in $S_n$, the class of $n$-cycles. The space $\overline{R}_0^J(\mathbb{H}_{n,k}^A)$ is one dimensional spanned by the class of the trivial $\mathbb{H}_{n,k}^A$-module $\pi_{n,k}^A((n))$.

8.3. $2A_{n-1}$. Let $\delta$ be the automorphism of order 2 of the Dynkin diagram of type $A_{n-1}$. The elliptic $\delta$-twisted conjugacy classes in $S_n$ are in one-to-one correspondence with partitions of $n$ where every part is odd, see [He1, §7.14]. Every irreducible $S_n$-representation is $\delta$-stable, i.e., $\text{Irr}(S_n) = \text{Irr}S_n$. The representations $\pi_{n,k}(\sigma)$ constructed in section 8.2 may seem therefore like good candidates for constructing a basis in $\overline{R}_0^J(\mathbb{H}_{n,k}^A)$, but the problem is that they are not typically $\delta$-invariant. This is because $\delta$ maps an irreducible $H$-module to its contragredient, and the modules $\pi_{n,k}(\sigma)$ are not self-contragredient in general. A basis of $\overline{R}_0^J(\mathbb{H}_{n,k}^A)$ is instead given by the unitary induced modules
\[ \{ \text{Ind}_{\mathbb{H}_{n,k}^A}^{\mathbb{H}_{n,k}^A}(\text{triv}) : \sigma \text{ partition of } n \text{ into distinct parts} \}, \] (8.3.1)
where $J(\sigma)$ is the subset of $I$ corresponding to the partition $\sigma$; more precisely, if $\sigma = (n_1, n_2, \ldots, n_\ell)$, then $\mathbb{H}_{n,k}^A(\sigma) = \prod_{i=1}^\ell \mathbb{H}_{n_i,k}$.

8.4. $B_n/C_n$. The set of elliptic conjugacy classes in $W(B_n)$ is in one-to-one correspondence with partitions of $n$, [Ca2]. For every partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$, let $w_\lambda$ be a representative of the corresponding elliptic conjugacy class, explicitly, $w_\lambda$ is a Coxeter element for $W(B_{\lambda_1}) \times \cdots \times W(B_{\lambda_k})$.

From Definition 2.2.1 one sees that there is an isomorphism between the graded Hecke algebra of type $B_n$ with parameters $k_1$ on the long roots and $k_2$ on the short roots, and the graded Hecke algebra of type $C_n$ with parameters $k_1$ on the short roots and $2k_2$ on the long roots. Because of this isomorphism, we consider only the graded Hecke algebra of type $B_n$ (with arbitrary parameters).

Let $\mathbb{H}_{n,k_1,k_2}^B$ be the graded Hecke algebra of type $B_n$ with parameter $k_1$ on the long roots and parameter $k_2$ on the short roots. Let $W(B_n)$ be the Weyl group of type $B_n$ generated by simple reflections $s_{i,i+1}$ for the roots $\varepsilon_i - \varepsilon_{i+1}$ and $s_n$ for the
root $\epsilon_n$. In [CK], for every partition $\sigma$ of $n$, one constructed an irreducible unitary $\mathbb{H}_{n,k_1,k_2}$-module $\pi_{n,k_1,k_2}(\sigma)$ via the assignment

$$\pi_{n,k_1,k_2}(\sigma): \begin{cases} s_{i,i+1} \mapsto \pi_{n,k_1}^A(\sigma)(s_{i,i+1}), & 1 \leq i \leq n-1; \\
 s_n \mapsto \text{Id}; \\
 \epsilon_i \mapsto k_2\text{Id} + \pi_{n,k_1}^A(\sigma)(\epsilon_i), & 1 \leq i \leq n. \end{cases}$$

(8.4.1)

By definition $\pi_{n,k_1,k_2}(\sigma)|_{W(B_n)} = \sigma \times 0$, in the bipartition notation for irreducible $W(B_n)$-representations ($\text{Ca}1$).

An equivalent way to define $\pi_{n,k_1,k_2}(\sigma)$ is by the assignment

$$w \mapsto (\sigma \times 0)(w), \quad w \in W(B_n);$$

$$\tilde{\omega} \mapsto 0, \quad \omega \in V = \mathbb{C}(\epsilon_1, \ldots, \epsilon_n);$$

(8.4.2)

here $\tilde{\omega}$ is defined as in (2.5.3) and it depends linearly in $k_1, k_2$.

From the explicit description of representatives $w_\lambda$, it is obvious that the matrix $(\text{Tr}(w_\lambda, \pi_{n,k_1,k_2}(\sigma)))_{\lambda,\sigma}$ equals the character table of $S_n$, $(\text{Tr}((\lambda, \sigma))_{\lambda,\sigma}$, and thus it is invertible.

8.5. $D_n$ and $2D_n$. Let $\mathbb{H}_{n,k}$ be the graded Hecke algebra of type $D_n$ with constant parameter $k$. Then $\mathbb{H}_{n,k,0} \cong \mathbb{H}_{n,k} \times \langle \delta \rangle$, where $\delta$ is the automorphism of order 2 of $D_n$ interchanging the simple roots $\epsilon_{n-1} - \epsilon_n$ and $\epsilon_{n-1} + \epsilon_n$.

In particular, $\mathcal{R}_0(\mathbb{H}_{n,k,0}) \cong \mathcal{R}_0(\mathbb{H}_{n,k}) \oplus \mathcal{R}_0(\mathbb{H}_{n,k})$, and therefore this case follows from the general type $\mathcal{B}_n$ case using that the number of elliptic conjugacy classes in $W(D_n)$ plus the number of $\delta$-twisted elliptic conjugacy classes in $W(D_n)$ equals the number of elliptic conjugacy classes in $W(B_n)$ (see [He1] §7.20).

Denote the restriction of the module $\pi_{n,k,0}(\sigma)$ constructed before to $\mathbb{H}_{n,k}$ by $\pi_{n,k}(\sigma)$. Since $(\sigma \times 0)|_{W(D_n)}$ is irreducible, it follows that $\pi_{n,k}(\sigma)$ is a unitary irreducible $\mathbb{H}_{n,k}$-module.

8.6. $3D_4$. Let $\delta$ be the automorphism of order 3 of the Dynkin diagram of type $D_4$. From [He1] Lemma 7.21, there are four conjugacy classes of $\delta$-twisted elliptic elements in $W(D_4)$.

A set of four linearly independent $\delta$-fixed $\mathbb{H}(D_4)$-modules in $\mathcal{R}_0(\mathbb{H})$ is formed of $\pi_{4,k}(\sigma)$, where $\sigma$ is any of the partitions $(4)$, $(31)$, $(22)$, and $\pi'$, (8.6.1)

where

$$\text{Ind}_{\mathbb{H}(D_4)}^{\mathbb{H}(D_3)} \pi_{3,k}^A((3)) = \pi' \oplus \pi_{4,k}^D((22)), \text{ as } \mathbb{H}(D_4)-\text{modules.}$$

(8.6.2)

Notice that this induced module is zero in $\mathcal{R}_0(\mathbb{H}(D_4))$, but since $A_2 \subset D_4$ is not fixed under $\delta$, it contributes to $\mathcal{R}_0(\mathbb{H}(D_4))$.

8.7. $G_2$. For exceptional groups, we use the labeling of irreducible $W$-representations from [Ca1].

There are three elliptic conjugacy classes in $W(G_2)$. The irreducible $W(G_2)$-representations $\phi_{1,0}$, $\phi_{1,6}$, $\phi'_{1,3}$, $\phi''_{1,3}$ and $\phi_{2,2}$ can be extended to unitary $\mathbb{H}$-modules by letting $\tilde{\omega}$ act by 0 for all $\omega$, see [BM2]. In $\mathcal{R}_0(\mathbb{H})$, $\phi_{1,0}$ and $\phi_{1,6}$ are in the same class, and so are $\phi'_{1,3}$ and $\phi''_{1,3}$. The set of $\mathbb{H}$-modules supported on

$$\phi_{1,0}, \phi'_{1,3}, \phi_{2,2}$$

(8.7.1)
is linearly independent in $\mathcal{T}_0(\mathcal{F})$. The table containing the modules with their central characters is Table 1. The central characters are the residual central characters from $\mathcal{O}_p$. The $\omega_i$ are the fundamental weights and the parameters $k, k'$ are on long and short roots, respectively.

**Table 1. $G_2$**

| central character | $W$-structure |
|------------------|--------------|
| $k\omega_1 + k'\omega_2$ | $\phi_{1,0}$ |
| $k\omega_1 + (-k + k')\omega_2$ | $\phi_{1,3}$ |
| $k\omega_1 + \frac{1}{2}(-k + k')$ | $\phi_{2,2}$ |

8.8. $F_4$. There are nine elliptic conjugacy classes in $W(F_4)$. There are nine irreducible $W(F_4)$-representations that can be lifted to $\mathbb{H}$-modules by letting $\tilde{\omega}$ act by zero. They form only five independent classes in $\mathcal{F}_0(\mathbb{H})$, for which a set of representatives is

$$\phi_{1,0}, \phi_{1,12}', \phi_{2,4}', \phi_{2,4}''', \phi_{4,8}. \quad (8.8.1)$$

There remain four modules to be specified. If an $\mathbb{H}$-module is not supported on a single $W$-type, the next simplest construction is when the module is supported on a $W$-representation $\sigma' + \sigma$, where $\sigma$ and $\sigma'$ are irreducible $W(F_4)$-representations and $\text{Hom}_W[\sigma', \sigma \otimes \text{refl}] \neq 0$. We are able to construct the remaining four modules in this way: $\phi_{4,1} + \phi_{1,0}, \phi_{4,7}' + \phi_{1,12}'$ and $\phi_{8,3}' + \phi_{2,4}'$ and $\phi_{6,6}' + \phi_{4,2}$. Notice that the first of them, supported on $\phi_{4,1} + \phi_{1,0}$ ("the affine reflection representation") also falls into the framework of the $W$-graph construction of [Lu3] in the setting of affine Hecke algebras. In the last module, we remark that $\phi_{6,6}'$ is the second exterior power of the reflection representation $\phi_{4,2}$.

These results are tabulated in Table 2. The central characters are the residual ones from $\mathcal{O}_p$. The convention is that $\omega_i$ are the fundamental weights and $k$ is the parameter on the long roots, while $k'$ is the parameter on the short roots. In each of the nine families of representations constructed above, the actions of $\omega_i$ (equivalently $\tilde{\omega}_i$) are linear in $k$ and $k'$.

**Table 2. $F_4$**

| central character | $W$-structure |
|------------------|--------------|
| $k\omega_1 + k\omega_2 + k'\omega_1 + k'\omega_4$ | $\phi_{1,0}$ |
| $k\omega_1 + k\omega_2 + (-k + k')\omega_3 + k'\omega_4$ | $\phi_{2,4}'$ |
| $k\omega_1 + k\omega_2 + (-k + k')\omega_3 + k\omega_4$ | $\phi_{4,1} + \phi_{1,0}$ |
| $k\omega_1 + k\omega_2 + (-2k + k')\omega_3 + k'\omega_4$ | $\phi_{1,12}'$ |
| $k\omega_1 + k\omega_2 + (-2k + k')\omega_3 + 2k\omega_4$ | $\phi_{8,3}' + \phi_{2,4}'$ |
| $k\omega_1 + k\omega_2 + (-2k + k')\omega_3 + k\omega_4$ | $\phi_{4,7} + \phi_{1,12}'$ |
| $k\omega_1 + k\omega_2 + (-2k + k')\omega_3 + (3k - k')\omega_4$ | $\phi_{2,4}'$ |
| $k\omega_2 + (-k + k')\omega_4$ | $\phi_{4,8}$ |
| $k\omega_2 + (-k + k')\omega_4$ | $\phi_{6,6} + \phi_{4,1}$ |
8.9. $E$. This case (as well as every equal parameter untwisted $H$) is covered by the results of [Re] in terms of the geometric classification of $H$-modules [Lu2]. More precisely, a set of representatives for a basis of $\mathcal{R}_0(H)$ is given by (tempered) modules $\pi(e,\phi)$ parameterized geometrically by pairs

$$ (e, \phi), \quad (8.9.1) $$

where

(i) $e$ ranges over a set of representatives for quasi-distinguished ([Re, (3.2.2)]) nilpotent orbits in the Lie algebra corresponding to the root system;

(ii) $\phi$ is an irreducible representation of $A(e)$, the group of components of the centralizer of $e$ in the associated adjoint Lie group, such that

(iii) $\phi$ is a representative for its class in $\mathcal{R}_0(A(e))$, the elliptic representation space for $A(e)$ defined in [Re, §3.2].

Concretely, for type $E$, for every distinguished $e$ (in the sense of Bala-Carter [Ca1]), one allows every $\phi$ of “Springer type”, while for $e$ quasi-distinguished, but not distinguished, one allows only $\phi = 1$. For example, for $E_6$, a basis is labeled by the five pairs

$$(E_6, 1), (E_6(a_1), 1), (E_6(a_3), 1), (E_6(a_3), \text{sgn}), (D_4(a_1), 1). \quad (8.9.2)$$

8.10. $E_6$. Let $\delta$ be the automorphism of order 2 of the Dynkin diagram of type $E_6$. According to [GKP] Table III, there are nine $\delta$-twisted elliptic conjugacy classes in $W(E_6)$. A set of nine linear independent (in $\mathcal{R}_0(H(E_6))$) $\delta$-fixed irreducible modules is formed of the tempered modules $\pi(e, \phi)$ (see section 8.9), for the pairs

$$(E_6, 1), (E_6(a_1), 1), (E_6(a_3), 1), (E_6(a_3), \text{sgn}), (D_5, 1), \quad (8.10.1)$$

$$(D_5(a_1), 1), (A_4 + A_1, 1), (D_4(a_1), 1), (D_4(a_1), \text{refl}).$$

Notice that the subsets of type $D_5$ and $A_4 + A_1$ are not $\delta$-stable. The component group for $D_4(a_1)$ is $A(e) = S_3$.

Acknowledgements. After a first version of this paper was posted at arxiv.org, M. Solleveld pointed out to us that he proved similar results for real parameter in [So3, Proposition 2.5 & Theorem 3.1].

The first named author thanks the Department of Mathematics at Hong Kong University of Science and Technology for its hospitality and support during his visit, when part of the research for this paper was completed. D. C. was supported in part by nsf-dms 0968065 and X. H. was supported in part by HKRGC grant 602011.

References

[ABP] A.-M. Aubert, P. Baum, R. Plymen, Kazhdan-Lusztig parameters and extended quotients, arxiv:1102.4172.
[BM1] D. Barbasch, A. Moy, Reduction to real infinitesimal character in affine Hecke algebras, J. Amer. Math. Soc. 6 (1993), no. 3, 611–635.
[BM2] D. Barbasch, A. Moy, Classification of one $K$-type representations, Trans. Amer. Math. Soc. 351 (1999), no. 10, 4245–4261.
[BC] D. Barbasch, D. Ciubotaru, Unitary equivalences for reductive $p$-adic groups, to appear in Amer. J. Math.
[BDK] J. Bernstein, P. Deligne, D. Kazhdan, Trace Paley-Wiener theorem for reductive $p$-adic groups, J. d’Analyse Math. 47 (1986), 180–192.
[Ca1] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1993. xii+544 pp.

[Ca2] R. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. 25, no. 1 (1972), 1–59.

[CK] D. Ciubotaru, S. Kato, *Tempered modules in exotic Deligne-Langlands classification*, Adv. Math. 226 (2011), 1538–1590.

[Ev] S. Evens, *The Langlands classification for graded Hecke algebras*, Proc. Amer. Math. Soc. 124 (1996), no. 4, 1285–1290.

[Fl] Y. Flicker, *Bernstein’s isomorphism and good forms*, K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras, 1992 Summer Research Institute; Proc. Symp. Pure Math. 58 II (1995), 171–196, AMS, Providence RI.

[GP] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press Oxford University Press, New York, 2000.

[GKP] M. Geck, S. Kim, G. Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Weyl groups*, J. Algebra 229 (2000), 570–600.

[He1] X. He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. 215 (2007), no. 2, 469–503.

[He2] X. He *Geometric and homological properties of affine Deligne-Lusztig varieties*, arXiv:1201.4901.

[HN] X. He and S. Nie, *Minimal length elements of finite Coxeter groups*, arXiv:1108.0282v1, to appear in Duke Math. J.

[Ka] D. Kazhdan, *Representations groups over close local fields*, Journal dAnalyse Mathematique, 47 (1986), 175–179.

[KR] C. Kriloff, A. Ram, *Representations of graded Hecke algebras*, Represent. Theory 6 (2002), 31–69.

[Lu1] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. 2 (1989), 599–635.

[Lu2] G. Lusztig, *Cuspidal local systems and graded algebras II*, Representations of groups (Banff, AB, 1994), Amer. Math. Soc., Providence, 1995, 217–275.

[Lu3] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. 277 (1983), no. 2, 623–653.

[Op] E. Opdam, *On the spectral decomposition of affine Hecke algebras*, J. Inst. Math. Jussieu 3 (2004), no. 4, 531–648.

[OS] E. Opdam, M. Solleveld, *Homological algebra of affine Hecke algebras*, Adv. Math. 220 (2009), no. 5, 1549–1601.

[RR] A. Ram, J. Ramagge, *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, Birkhäuser, Trends in Mathematics (2003), 428–466.

[Re] M. Reeder, *Euler-Poincaré pairings and elliptic representations of Weyl groups and p-adic groups*, Compositio Math. 129 (2001), 149–181.

[Sol1] M. Solleveld, *Homology of graded Hecke algebras*, J. Algebra 323 (2010), 1622–1648.

[Sol2] M. Solleveld, *On the classification of irreducible representations of affine Hecke algebras with unequal parameters*, Represent. Theory 16 (2012), 1–87.

[Sol3] M. Solleveld, *Hochschild homology of affine Hecke algebras*, arXiv:1108.5286.

(D. Ciubotaru) Dept. of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: ciubo@math.utah.edu

(X. He) Dept. of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
E-mail address: maxhe@ust.hk