SINGULAR LOCUS OF INSTANTON SHEAVES ON $\mathbb{P}^3$

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Abstract. We prove that the singular locus of rank 2 instanton sheaf $E$ on $\mathbb{P}^3$ which is not locally free has pure dimension 1. Moreover, we also show that the dual and double dual of $E$ are isomorphic locally free instanton sheaves, and that the sheaves $\text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3})$ and $E^{**}/E$ are rank 0 instantons. We also provide explicit examples of instanton sheaves of rank 3 and 4 illustrating that all of these claims are false for higher rank instanton sheaves.

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1. Introduction

Mathematical instanton bundles have been intensely studied by several authors since its introduction in the late 1970’s. They derive their nomenclature from Gauge Theory: mathematical instanton bundles on odd dimensional complex projective spaces $\mathbb{C}P^{2k+1}$ are precisely those holomorphic vector bundles that arise, via the twistor correspondence, in relation with quaternionic instantons on quaternionic projective spaces $\mathbb{H}P^k$, see [9] for details, and also [11].

The simplest case of such objects are rank 2 instanton bundles on $\mathbb{C}P^3$, and there is a vast literature about them. One outstanding problem that resisted solution for a couple of decades regards the irreducibility and smoothness of the moduli space $\mathcal{I}(c)$ of rank 2 instanton bundles on $\mathbb{C}P^3$ of charge $c$. It was known since 2003 that $\mathcal{I}(c)$ is smooth and irreducible for $c \leq 5$, see [1] and the references therein. Recently, Tikhomirov has proved in [14, 15] irreducibility for arbitrary $c$, while the second named author and Verbitsky established smoothness for every $c$, see [8].
The next step is to understand how instanton bundles degenerate, that is to study non locally free instanton sheaves. Maruyama and Trautmann were the first to consider limits of instantons on $\mathbb{P}^3$ in [10]; in [6] instanton sheaves on arbitrary projective spaces $\mathbb{P}^n$ are studied. In this paper we only consider instanton sheaves on $\mathbb{P}^3$; these are defined as torsion free sheaves $E$ on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_3(E) = 0$, and satisfying

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0,$$

compare with [10, 1.1, page 216] and [6, page 69]. The charge of $E$ is given by its second Chern class $c_2(E)$.

The goal of this paper is to study the singular locus of rank 2 instanton sheaves, showing that they are of pure dimension 1. In the process, the rank 0 instantons introduced by Hauzer and Langer in [4, Definition 6.1] also appear. A rank 0 instanton is a pure dimension 1 sheaf $Z$ on $\mathbb{P}^3$ such that $h^0(Z(-2)) = h^1(Z(-2)) = 0$; $d := h^0(Z(-1))$ is called the degree of $Z$; see Section 3.2 below for details. We say that two rank 0 instantons $Z_1$ and $Z_2$ are dual to each other if $\text{Ext}^2(Z_1, \mathcal{O}_{\mathbb{P}^3}) \cong Z_2$ and $\text{Ext}^2(Z_2, \mathcal{O}_{\mathbb{P}^3}) \cong Z_1$.

More precisely, with the previous definitions in mind, we prove the following.

**Main Theorem.** If $E$ is a rank 2 non locally free instanton sheaf on $\mathbb{P}^3$, then

1. its singular locus has pure dimension 1;
2. $E^\vee$ and $E^{\vee\vee}$ are isomorphic locally free instanton sheaves;
3. the sheaves $\text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3})$ and $E^{\vee\vee}/E$ are dual rank 0 instantons of degree $c_2(E) - c_2(E^{\vee\vee})$.

In addition, if $c_2(E) - c_2(E^{\vee\vee}) = 1$, then the singular locus of $E$ consists of a single line.

All of these claims are false for instanton sheaves of higher rank. Indeed, we provide explicit examples of instanton sheaves of rank 3 and 4 on $\mathbb{P}^3$ for which $E^\vee$ and $E^{\vee\vee}$ are not locally free and not instanton sheaves, respectively, see Section 5.3. In addition, we also describe rank 3 instanton sheaves whose singular loci are either of dimension 0 or not of pure dimension 1, see Section 5.2 below.

**Acknowledgements.** MG was supported by a PhD grant from CAPES, Brazil; most of the results present here were obtained as part of his thesis. MJ is partially supported by the CNPq grant number 302477/2010-1 and the FAPESP grant number 2011/01071-3.

2. **Sheaves and monads**

In this section, we fix the notation that will be used in this work and we recall some basic definitions and results. We work over a fixed algebraically closed field $\kappa$ of characteristic zero. By a projective variety $X$ we understand a nonsingular, projective, integral, separated noetherian scheme of finite type over $\kappa$. All sheaves on $X$ are coherent sheaves of $\mathcal{O}_X$-modules.

2.1. **Sheaves.** Let $E$ be a coherent sheaf on projective variety $X$; its **support** is the closed set

$$\text{Supp}(E) = \{x \in X \mid E_x \neq 0\},$$

where $E_x$ denotes the stalk of $E$ over the point $x \in X$. The dimension of $E$, denoted by $\dim(E)$, is defined to be the dimension $\text{Supp}(E)$ as an algebraic set. A coherent
sheaf $E$ is said to be of pure dimension $d$ if $\dim E = d$ and every nonzero subsheaf of $E$ also has dimension $d$.

The singular locus of $E$ is defined as the following closed set

$$\text{Sing}(E) = \{ x \in X \mid E_x \text{ not free } O_{X,x} - \text{module} \}$$

$$= \bigcup_{p=1}^{\dim X} \text{Supp} \mathcal{E}xt^p(E, O_X).$$

Recall also that $E$ is torsion free if the canonical map into the double dual sheaf $E^{\vee\vee}$ is injective; if such map is an isomorphism, then $E$ is reflexive. One can show that if $E$ is torsion free, then $\text{codim Sing}(E) \geq 2$, and if $E$ is reflexive, then $\text{codim Sing}(E) \geq 3$.

Two sheaves derived from $E$ will play important parts later on,

$$S_E := \mathcal{E}xt^1(E, O_X) \text{ and } Q_E := E^{\vee\vee}/E.$$ Clearly, $\text{Supp}(S_E) \subseteq \text{Sing}(E)$; note also that $\text{Supp}(Q_E) \subseteq \text{Sing}(E)$. Indeed, if $x \notin \text{Sing}(E)$, then $E_x$ is free as an $O_{X,x}$-module, hence $E_x \simeq (E^{\vee\vee})_x$ and $x \notin \text{Supp}(Q_E)$. Moreover, it is not difficult to see that if $E^{\vee\vee}$ is locally free, then in fact $\text{Supp}(Q_E) = \text{Sing}(E)$.

Hartshorne proves in [3] the following results that will play key roles in the proof of our main results.

**Proposition 1.** ([3, Prop. 1.10]) If $F$ is a rank 2 reflexive sheaf on $\mathbb{P}^3$, then $F^{\vee} \cong F \otimes (\text{det } F)^{-1}$.

**Proposition 2.** ([3, Prop. 2.5]) If $F$ is a reflexive sheaf on $\mathbb{P}^3$, then there are isomorphisms

$$H^0(F^{\vee}(-4)) \simeq H^3(F)^{\vee}, \quad H^3(F^{\vee}(-4)) \simeq H^0(F)^{\vee}$$

and an exact sequence

$$0 \to H^1(F^{\vee}(-4)) \to H^2(F)^{\vee} \to H^0(S_F(-4)) \to H^2(F^{\vee}(-4)) \to H^1(F)^{\vee} \to 0.$$  

Another key ingredient is the following criterion due to Roggero.

**Proposition 3.** (cf. [13, Thm. 2.3]) If $F$ is a rank 2 reflexive sheaf with $c_1(F) = 0$ on $\mathbb{P}^3$, then $F$ is locally free if and only if $h^2(F(p)) = 0$ for some $p \leq -2$.

### 2.2. Monads.

Recall that a monad on $X$ is a complex $M^\bullet$ of locally free sheaves on $X$ of the following form:

$$M^\bullet : A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

which is exact on the first and last terms. The sheaf $E = \ker \beta / \text{Im } \alpha$ is called the cohomology of the monad $M^\bullet$.

The degeneration locus of the monad [4] consists of the following set

$$\Delta(M^\bullet) = \{ x \in X \mid \alpha(x) \text{ is not injective} \}.$$ 

**Lemma 4.** If $E$ is the cohomology sheaf of a monad $M^\bullet$ as in [4], then $\Delta(M^\bullet) = \text{Supp}(S_E) = \text{Sing}(E)$. In particular, $\text{Supp}(Q_E) \subseteq \text{Supp}(S_E)$.

**Proof.** One can break the monad [4] into the short exact sequences

$$0 \to K \to B \xrightarrow{\beta} C \to 0$$
where $K := \ker\beta$, and

(7) $0 \rightarrow A \xrightarrow{\alpha} K \rightarrow E \rightarrow 0$

Dualizing (7) we obtain

(8) $0 \rightarrow E^\vee \rightarrow K^\vee \xrightarrow{\alpha^\vee} A^\vee \rightarrow \text{Ext}^1(E, \mathcal{O}_P) \rightarrow 0$

and $\text{Ext}^p(E, \mathcal{O}_P) = 0$ for $p \geq 2$. It then follows immediately that $\text{Supp}(S_E) = \text{Sing}(E)$. To see that $\Delta(M) = \text{Supp}(S_E)$, note that $x \in \text{Supp}(S_E)$ if and only if the map of stalks $(\alpha^\vee)_x$ is not surjective; this happens if and only if the map of fibers $\alpha^\vee(x)$ is not surjective, which is equivalent to $x \in \Delta(M)$. □

In general, the dual of a monad may not be a monad, only a complex of the form

(9) $M^\vee : C^\vee \xrightarrow{\beta^\vee} B^\vee \xrightarrow{\alpha^\vee} A^\vee$

whose first map is injective. It may have two nontrivial cohomology sheaves: $H^0(M^\vee) := \ker(\alpha^\vee)/\text{Im}(\beta^\vee)$ and $H^1(M^\vee) := \text{coker}(\alpha^\vee)$.

**Lemma 5.** If $E$ is the cohomology sheaf of a monad $M^\bullet$ of the form (4), then $E^\vee \simeq H^0(M^\vee)$ and $S_E := H^1(M^\vee)$.

**Proof.** Dualizing (6) and breaking (8) into short exact sequences, we obtain the following three short exact sequences:

(10) $0 \rightarrow C^\vee \xrightarrow{\beta^\vee} B^\vee \rightarrow K^\vee \rightarrow 0,$

(11) $0 \rightarrow E^\vee \rightarrow K^\vee \xrightarrow{\alpha^\vee} T \rightarrow 0,$

where $T = \text{Im}(\alpha^\vee)$, and

(12) $0 \rightarrow T \rightarrow A^\vee \rightarrow S_E \rightarrow 0.$

On the other hand, a complex of the form (9) whose first map is injective can be broken down into three short exact sequences as follows

(13) $0 \rightarrow C^\vee \xrightarrow{\beta^\vee} B^\vee \rightarrow V \rightarrow 0,$

where $V := \text{coker}(\beta^\vee)$,

(14) $0 \rightarrow H^0(M^\vee) \rightarrow V \xrightarrow{\alpha^\vee} T \rightarrow 0,$

where $T = \text{Im}(\alpha^\vee)$, and

(15) $0 \rightarrow T \rightarrow A^\vee \rightarrow H^1(M^\vee) \rightarrow 0.$

The desired conclusion follows from the comparison of the two sets of sequences. □

We have one more observation regarding reflexive sheaves on 3-dimensional varieties.

**Lemma 6.** Let $M_\bullet$ be a monad as in equation (4) whose degeneration locus has codimension at least 3. Then its cohomology sheaf $E$ is reflexive.
Proof. Dualizing sequence (12) we conclude that $T^\vee \simeq A$ and $\mathcal{E}xt^1(T, \mathcal{O}_{P^3}) = \mathcal{E}xt^2(S_E, \mathcal{O}_{P^3}) = 0$, because $S_E$ is supported in codimension at least 3. Then dualizing sequence (11) we obtain, since $K$ is locally free:

$$0 \to A \xrightarrow{\alpha} K \to E^{\vee \vee} \to 0;$$

in other words, $E^{\vee \vee} \simeq \text{coker } \alpha^{\vee \vee} \simeq \text{coker } \alpha \simeq E$, as desired. \qed

3. Instantons on $\mathbb{P}^3$

We are finally in position to focus on our main object of study. In this Section, we review the definitions of instanton sheaves and rank 0 instantons on $\mathbb{P}^3$ and prove new basic results regarding their structure.

3.1. Instanton sheaves. Recall from [6, p. 69] that an instanton sheaf on $\mathbb{P}^3$ is a torsion free coherent sheaf $E$ with $c_1(E) = 0$ and

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$}

Moreover, the integer $c := h^1(E(-1))$ is called the charge of $E$. One can check that it coincides with $c_2(E)$.

As observed in the Introduction, locally free instanton sheaves of rank 2 are precisely (mathematical) instanton bundles. In fact, one can show that instanton sheaves are precisely those that can be obtained as the cohomology of a linear monad of the form

$$\mathcal{O}_{P^3}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{P^3}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{P^3}(1)^{\oplus c},$$

where $r$ is the rank of $E$.

If $E$ is a locally free instanton sheaf on $\mathbb{P}^3$, then it is easy to see that its dual $E^\vee$ and its double dual $E^{\vee \vee}$ are also instanton sheaves. The same is not true in general; in fact, we will see below that if $E$ is a reflexive instanton sheaf which is not locally free, then $E^{\vee}$ is not instanton. However, the sheaves $E^\vee$ and $E^{\vee \vee}$ still retain the following properties.

Lemma 7. If $E$ is an instanton sheaf on $\mathbb{P}^3$, then

(i) $h^0(E^\vee(-1)) = h^1(E^\vee(-2)) = 0$;
(ii) $h^2(E^\vee(-1)) = h^0(S_E(-2));$
(iii) $h^2(E^{\vee \vee}(-1)) = h^3(E^{\vee \vee}(-3)) = 0$;
(iv) $h^1(E^{\vee \vee}(-1)) = h^3(Q_E(-2))$.

If, in addition, $E$ is either $\mu$-semistable or of trivial splitting type, then $h^3(E^{\vee}(-3)) = h^0(E^{\vee \vee}(-1)) = 0$.

Proof. Dualizing the monad (17) we obtain the complex

$$\mathcal{O}_{P^3}(-1)^{\oplus c} \xrightarrow{\beta^\vee} \mathcal{O}_{P^3}^{\oplus r+2c} \xrightarrow{\alpha^\vee} \mathcal{O}_{P^3}(1)^{\oplus c}.$$}

We know from Lemma 5 that $E^\vee$ is the 0th cohomology of (18), while its first cohomology yields the sheaf $S_E$. Breaking (18) into short exact sequences as in the proof of Lemma 5 and passing to cohomology, we obtain that (i) and (ii).

Since $E$ is torsion free, we know that $\dim \text{Sing}(E) \leq 1$; since $\text{Supp}(Q_E) \subseteq \text{Sing}(E)$, we conclude that $\dim Q_E \leq 1$. Now we use the short exact sequence

$$0 \to E \to E^{\vee \vee} \to Q_E \to 0$$

to obtain (iii) and (iv), since $h^2(Q_E(k)) = h^3(Q_E(k)) = 0$ for every $k \in \mathbb{Z}$.
Finally, if \( E \) is either \( \mu \)-semistable or of trivial splitting type, then the double dual sheaf \( E^{\vee\vee} \) is, respectively, either \( \mu \)-semistable or of trivial splitting type; in either case, we have that \( h^0(E^{\vee\vee}(-1)) = 0 \). The vanishing of \( h^3(E^{\vee}(-3)) \) then follows from Lemma 2.

\[\square\]

The last result in this section describes the sheaves \( \mathcal{E}xt^p(S_E, \mathcal{O}_{\mathbb{P}^3}) \) when \( E \) is an instanton sheaf. It also provides, in particular, a relation between the sheaves \( S_E \) and \( Q_E \).

**Lemma 8.** If \( E \) is an instanton sheaf on \( \mathbb{P}^3 \), then

(i) \( \mathcal{E}xt^1(S_E, \mathcal{O}_{\mathbb{P}^3}) = 0 \);

(ii) \( \mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq Q_E \);

(iii) \( \mathcal{E}xt^3(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq S_{E^{\vee}} \).

**Proof.** Dualizing the sequence

\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \overset{\alpha}{\to} K \to E \to 0 \]

and breaking into short exact sequences we obtain

\[ 0 \to E^{\vee} \to K^{\vee} \overset{\alpha^{\vee}}{\to} T \to 0, \]

where \( T = \text{Im}(\alpha^{\vee}) \), and

\[ 0 \to T \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c} \to S_E \to 0, \]

We can then gather (20) and the dual of (21) into the following diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \\
\downarrow & & \downarrow \\
0 & \to & T^{\vee} \\
\downarrow & & \downarrow \\
K^{\vee} & \overset{\alpha^{\vee}}{\to} & K \\
\downarrow & & \downarrow \\
E^{\vee\vee} & \to & S_T \\
\downarrow & & \downarrow \\
Q_E & \to & 0 \\
\downarrow & & \\
0 & & \\
\end{array}
\]

where we recall that \( S_T = \mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3}) \).

It follows that \( T^{\vee} \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \) and \( \mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3}) \simeq Q_E \). Note also that \( \mathcal{E}xt^2(T, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^1(E^{\vee}, \mathcal{O}_{\mathbb{P}^3}) \) and \( \mathcal{E}xt^3(T, \mathcal{O}_{\mathbb{P}^3}) = 0 \).

Dualizing (22), it follows that \( \mathcal{E}xt^1(S_E, \mathcal{O}_{\mathbb{P}^3}) = 0, \mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3}) \) and \( \mathcal{E}xt^3(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(T, \mathcal{O}_{\mathbb{P}^3}) \).

\[\square\]

### 3.2. Rank 0 instantons.

The notion of rank 0 instanton on \( \mathbb{P}^3 \) was introduced in [4 Defn. 6.1]. A coherent sheaf \( Z \) on \( \mathbb{P}^3 \) is called a rank 0 instanton if it has pure dimension 1 and \( h^0(Z(-2)) = h^1(Z(-2)) = 0 \). The integer \( d := h^0(Z(-1)) \) is called the degree of \( Z \).
One can show, see [3] Lemma 6.2, that $Z$ is a rank 0 instanton if and only if there is a complex of the form (a.k.a. a perverse instanton)

$$Z_\bullet : \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d}$$

such that $\mathcal{H}^{-1}(Z_\bullet) = \mathcal{H}^0(Z_\bullet) = 0$ and $\mathcal{H}^1(Z_\bullet) = Z$; here, $d$ is precisely the degree of $Z$.

In other words, $Z$ is a rank 0 instanton of degree $d$ if and only if it admits a resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \to Z \to 0. \tag{23}$$

It follows immediately that $\mathcal{E}xt^3(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$. Note also that $\mathcal{E}xt^1(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$, since $\text{codim} Z = 2$.

**Lemma 9.** If $Z$ is a rank 0 instanton, then so is $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$.

**Proof.** Break (23) into short exact sequences to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\sigma} I \to 0, \tag{24}$$

where $I = \text{Im} \tau = \text{coker} \sigma$, and

$$0 \to I \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \to Z \to 0. \tag{25}$$

Dualizing (23), we conclude that $I^\vee \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d}$, since $\mathcal{E}xt^1(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$, and $\mathcal{E}xt^1(I, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$.

Thus dualizing (24) we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\tau^\vee} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\sigma^\vee} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \to \mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3}) \to 0, \tag{26}$$

thus $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$ is a rank 0 instanton as well.

As seen on the proof above, $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$ is obtained essentially by dualizing the resolution that defines $Z$. For this reason, we say that $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$ is the dual of $Z$. Two rank 0 instantons $Z_1$ and $Z_2$ are dual to each other if $\mathcal{E}xt^2(Z_1, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_2$ and $\mathcal{E}xt^2(Z_2, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_1$.

Finally, we now analyze whether the sheaves $S_E$ and $Q_E$ are rank 0 instantons. Note that $\dim S_E \leq 1$ and $\dim Q_E \leq 1$, since $E$ is torsion free; however, both sheaves may have zero dimensional subsheaves. Our next two results provide sufficient conditions for $S_E$ and $Q_E$ to be rank 0 instantons.

**Lemma 10.** If $E$ is an instanton sheaf which is not locally free and such that $E^\vee$ is instanton, then $S_E$ is rank 0 instanton.

**Proof.** If $E^\vee$ is instanton, then $h^0(S_E(-2)) = h^0(E^\vee(-2)) = 0$, by Lemma 7 item (ii); in particular, $\text{Supp}(S_E)$ cannot have zero dimensional subsheaves, so it has pure dimension 1.

To see that $h^1(S_E(-2)) = 0$, note that the sequences (24) and (25) yields

$$h^1(S_E(-2)) = h^2(T(-2)) = h^3(E^\vee(-2)) = 0,$$

since $E^\vee$ is instanton.

**Lemma 11.** If $E$ is an instanton sheaf on $\mathbb{P}^3$ which is not reflexive and such that $E^{\vee\vee}$ is instanton, then $Q_E$ is rank 0 instanton. Moreover, there is a short exact sequence

$$0 \to S_{E^{\vee\vee}} \to S_E \to \mathcal{E}xt^2(Q_E, \mathcal{O}_{\mathbb{P}^3}) \to 0. \tag{27}$$
Proof. Consider the sequence
\[ 0 \to E \to E^{\vee \vee} \to Q_E \to 0. \]
If \(E\) and \(E^{\vee \vee}\) are both instantons, one immediately gets from the cohomology sequence that \(h^0(Q_E(-2)) = h^1(Q_E(-2)) = 0\), so \(Q_E\) is a rank 0 instanton.

Dualizing (28) one obtains (27) using the fact that \(\text{Ext}^1(Q_E, \mathcal{O}_{\mathbb{P}^3}) = 0\) (because \(Q_E\) is supported in codimension 2) and \(\text{Ext}^2(E^{\vee \vee}, \mathcal{O}_{\mathbb{P}^3}) = 0\) (because \(E^{\vee \vee}\) is instanton).

\[ \square \]

3.3. Two examples. We complete this section with two examples that highlight the necessity of the hypothesis used in Lemmata 10 and 11.

First, note that if \(E\) is a reflexive instanton sheaf that is not locally free, then \(\dim S_E = 0\) and \(S_E\) is not a rank 0 instanton. Moreover, \(E^{\vee}\) is not instanton.

Indeed, the first claim is clear, since the singular locus of reflexive sheaves must have codimension at least 3. Since \(\dim S_E = 0\), then \(h^0(S_E(-2)) \neq 0\), thus \(E^{\vee}\) is not instanton by Lemma 7, item (ii).

Here is a concrete example of a reflexive instanton sheaf which is not locally free, taken from [6, Example 5]. Consider the following instanton monad,

\[ (29) \quad \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \]

where \(\alpha\) and \(\beta\) are defined by

\[ \alpha = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \\ x_3 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & 0 \end{pmatrix}. \]

Its cohomology sheaf is a rank 3 instanton sheaf, here denoted \(E\), of charge 1. Its degeneracy locus, hence \(\text{Sing}(E)\), consists of a single point, namely \([0 : 0 : 0 : 1]\), hence \(E\) is reflexive, but not locally free. In particular, \(E^{\vee}\) is not instanton.

We shall further discuss rank 3 instantons of charge 1 in Section 5.1 below.

Next, we provide an example of an instanton sheaf which is not reflexive and such that \(E^{\vee \vee}\) is not an instanton, taken from [6, Example 3]. Consider first the following linear monad,

\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0; \]

according to [2, p. 5503], its cohomology sheaf is of the form \(\mathcal{I}_C(1)\), the twisted ideal of a space curve \(C \hookrightarrow \mathbb{P}^3\).

Floydad’s result [2, Main Theorem] also guarantees, for any \(c \geq 1\), the existence of a monad of the form:

\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2c+4} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c+1} \to 0, \]

whose cohomology is a rank 3 locally free sheaf \(F\) with \(c_1(F) = -1\).

The direct sum \(E := F \oplus \mathcal{I}_C(1)\) provides the desired example: \(E\) is a non reflexive instanton sheaf of rank 4 and charge \(c + 2\) (\(c \geq 1\)), being the cohomology of the linear monad

\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c+2} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2c+8} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c+2} \to 0. \]
However, $E^{\vee\vee} = F^{\vee\vee} \oplus O_{\mathbb{P}^3}(1)$ is not an instanton sheaf, since $H^0(E^{\vee\vee}(-1)) \simeq H^0(O_{\mathbb{P}^3}) \neq 0$. Note that $Q_E \simeq O_C(1)$, thus $Q_E$ may not be a rank 0 instanton. Furthermore, note also that $E^{\vee}$ is locally free, but not an instanton either.

4. Singular locus of rank 2 instanton sheaves

We are finally in position to establish the main result of this paper.

Let $E$ be a rank 2 instanton sheaf on $\mathbb{P}^3$. Applying Proposition 1 to its dual sheaf $F = E^{\vee}$, we conclude that $E^{\vee\vee} \simeq E^{\vee}$, since det$(E^{\vee}) = O_{\mathbb{P}^3}$. It then follows from Lemma 7 that both $E^{\vee}$ and $E^{\vee\vee}$ are instanton sheaves.

The fact that $S_E$ and $Q_E$ are rank 0 instantons follow from Lemma 10 and Lemma 11 respectively.

Furthermore, $E^{\vee}$ and $E^{\vee\vee}$ must be locally free by a direct application of Proposition 8 since $h^2(E^{\vee}(-2)) = h^2(E^{\vee\vee}(-2)) = 0$. In particular, $S_{E^{\vee\vee}} = 0$, thus $S_E \simeq \mathcal{E}xt^2(Q_E, O_{\mathbb{P}^3})$ by sequence 27. Since also $Q_E \simeq \mathcal{E}xt^2(S_E, O_{\mathbb{P}^3})$ by Lemma 8 item (ii), we have that $Q_E$ and $S_E$ are dual rank 0 instantons.

Finally, since Sing$(E) = \text{Supp}(S_E) = \text{Supp}(Q_E)$, it follows that Sing$(E)$ has pure dimension 1.

Note, in addition, that if $E$ has charge $c$ and $E^{\vee\vee}$ has charge $c'$, then $Q_E$ and $S_E$ are rank 0 instantons of degree $d := c - c'$. In fact, their Hilbert polynomials are given by $P_{S_E}(k) = P_{Q_E}(k) = dk + 2d$, so that $Q_E$ may be regarded as points in the quot scheme Quot$(dk+2d)(E^{\vee\vee})$.

To complete the proof of the Main Theorem, we assume that $d = 1$. Since $Q_E$ is a rank 0 instanton of degree 1, $Q_E(-1)$ admits a resolution of the form

$$0 \to O_{\mathbb{P}^3}(-2) \to O_{\mathbb{P}^3}(-1)^{\oplus 2} \to O_{\mathbb{P}^3} \to Q_E(-1) \to 0.$$ 

Therefore, in fact, $Q_E(-1) \simeq \iota_*O_\ell$ for some line $\iota : \ell \hookrightarrow \mathbb{P}^3$. In other words, the singular locus of $E$ consists of a single line.

In particular, we also obtain the following claim.

**Corollary 12.** Every rank 2 non locally free instanton sheaf $E$ of charge 1 is of the form

$$0 \to E \to O_{\mathbb{P}^3}^{\oplus 2} \to \iota_*O_\ell(1) \to 0,$$

for some line $\ell \in \mathbb{P}^3$.

A complete classification of possible singular loci for rank 2 instanton sheaves of charge $c$ seems to be a hard problem, since it requires an understanding of the quot schemes of rank 2 locally free instanton sheaves. A procedure to construct instanton sheaves with a prescribed singular locus is given in [7, Section 3]. To be precise, let $E$ be an instanton sheaf of charge $c$, and consider triples $(\Sigma, L, \varphi)$ for $E$ consisting of the following:

(i) an embedding $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ of a reduced, locally complete intersection curve of arithmetic genus $g$ and degree $d$;

(ii) an invertible sheaf $L \in \text{Pic}^{g-1}(\Sigma)$ such that $h^0(\iota_*L) = h^1(\iota_*L) = 0$;

(iii) a surjective morphism $\varphi : E \to \iota_*L(2)$.

It follows that $F := \ker \varphi$ is an instanton sheaf of the same rank as $E$ and charge $c+d$, if $E$ is locally free, then $Q_F = \iota_*L(2)$, so that the singular locus of $F$ is precisely $\Sigma$. The difficulty, of course, is proving the existence of the surjective morphism $\varphi$. In [7], the cases of rational curves and elliptic quartic curves is considered.
5. Singular locus of rank 3 instanton sheaves

In this section we show that instanton sheaf of rank larger than 2 may have 0-dimensional singularities, as well as singular loci which are not of pure dimension. These phenomena first occur for instanton sheaves of rank 3 and charge 1 and 2, respectively.

5.1. Rank 3 instantons of charge 1. We will now consider linear monads of the form

\[(30) \quad \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^\oplus 5 \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1).\]

Note that any surjective map \(\mathcal{O}_{\mathbb{P}^3}^\oplus 5 \to \mathcal{O}_{\mathbb{P}^3}(1)\) may, after a linear change of homogeneous coordinates and a change of basis on the free sheaf \(\mathcal{O}_{\mathbb{P}^3}^\oplus 5\), be written in the following form:

\[\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & 0 \end{pmatrix}.\]

It follows that the map \(\alpha\) is given by

\[(31) \quad \alpha = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \phi \end{pmatrix},\]

where \(\sigma_j \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))\) must satisfy the monad equation

\[(32) \quad \Sigma_j x_j \sigma_j = 0.\]

Moreover, the injectivity of \(\alpha\) is equivalent to at least one of the sections \(\sigma_j, \phi\) being non-trivial.

However, \(\phi \equiv 0\) if and only if the monad (30) decomposes as a sum of two monads

\[(\mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^\oplus 4 \to \mathcal{O}_{\mathbb{P}^3}(1)) \bigoplus (0 \to \mathcal{O}_{\mathbb{P}^3} \to 0)\]

which in turns is equivalent to the cohomology of the (30) splitting as a direct sum \(E \oplus \mathcal{O}_{\mathbb{P}^3}\), for some instanton sheaf \(E\) of rank 2 and charge 1.

We assume therefore, from now on, that \(\phi \neq 0\). Let \(\Gamma\) be the subspace of \(H^0(\mathcal{O}_{\mathbb{P}^3}(1))\) spanned by the sections \(\sigma_j\) and \(\phi\); in particular, \(\dim \Gamma \geq 1\).

We observe that \(\dim \Gamma = 1\) if and only if the cohomology of the monad (30) splits as a sum \(\Omega^1_{\mathbb{P}^3}(1) \oplus \mathcal{O}_\varphi\), where \(\varphi\) is the hyperplane defined by the equation \(\{ \phi = 0 \}\).

Indeed, \(\dim \Gamma = 1\) if and only if \(\sigma_j = \lambda_j \phi\) for each \(j = 1, 2, 3, 4\); thus one can find a basis for the free sheaf \(\mathcal{O}_{\mathbb{P}^3}^\oplus 5\) in which the map \(\alpha\) is of the form

\[\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \phi \end{pmatrix} .\]

It follows that the monad (30) decomposes as a sum of two linear monads

\[\left(0 \to \mathcal{O}_{\mathbb{P}^3}^\oplus 4 \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)\right) \bigoplus \left(\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^3} \to 0\right)\]

which in turns is equivalent to the cohomology of the (30) splitting as a direct sum \(\Omega^1_{\mathbb{P}^3}(1) \oplus \mathcal{O}_\varphi\).
**Proposition 13.** There exists a 1-1 correspondence between indecomposable instanton sheaves of rank 3 and charge 1 on \( \mathbb{P}^3 \), and non-trivial extensions of \( \mathcal{O}_\wp \) by \( \Omega^1_{\mathbb{P}^3}(1) \), for some hyperplane \( \wp \subset \mathbb{P}^3 \).

**Proof.** As seen above, every indecomposable rank 3 instanton sheaf \( E \) of charge 1 on \( \mathbb{P}^3 \) is the cohomology of a monad of the form (30) with \( \phi \neq 0 \) and \( \dim \Gamma \geq 2 \).

The key observation here is that, in this case, the monad (30) can be written as a (non-trivial) extension of two simpler linear monads

\[
\begin{array}{c}
0 \\
\mathcal{O}_{\mathbb{P}^3}^\oplus 4 \\
\mathcal{O}_{\mathbb{P}^3}(-1) \\
\mathcal{O}_{\mathbb{P}^3}(-1) \\
0
\end{array}
\begin{array}{c}
\omega \\
\alpha \\
\beta \\
\phi \\
0
\end{array}
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^3}(1) \\
\mathcal{O}_{\mathbb{P}^3}^\oplus 5 \\
\mathcal{O}_{\mathbb{P}^3}(1) \\
\mathcal{O}_{\mathbb{P}^3} \\
0
\end{array}
\begin{array}{c}
\approx \\
\approx \\
\approx \\
\rightarrow \\
0
\end{array}
\]

where the map \( \omega \) is given by
\[
\omega = \left( \begin{array}{cccc}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{array} \right),
\]
and \( \alpha \) is given by (31).

Passing to cohomology, this short exact sequence of complexes yields precisely the short exact sequence

\[
(34) \quad 0 \to \Omega^1_{\mathbb{P}^3}(1) \to E \to \mathcal{O}_\wp \to 0
\]

between their middle cohomologies, where \( \wp \) is the hyperplane defined by the equation \( \phi = 0 \).

Conversely, if \( E \) is a non-trivial extension of \( \mathcal{O}_\wp \) by \( \Omega^1_{\mathbb{P}^3}(1) \), for some hyperplane \( \wp \subset \mathbb{P}^3 \), then one can lift the short exact sequence (34) to a short exact sequence of complexes as in (33). From the considerations above, we know that the linear monad thus obtained in the middle row is such that \( \phi \neq 0 \) and \( \dim \Gamma \geq 2 \), thus its cohomology sheaf is an indecomposable rank 3 instanton sheaf \( E \) of charge 1. \( \square \)

The previous Proposition allows us to provide a neat description of the moduli space of indecomposable rank 3 instanton sheaves of charge 1, which we will denote here by \( \mathcal{T}^{\ellf}(3,1) \).

If \( E \) is such an object, let \( \wp_E \) be the corresponding hyperplane in \( \mathbb{P}^3 \), obtained via the sequence (34); this yields a map
\[
\varpi : \mathcal{T}^{\ellf}(3,1) \to (\mathbb{P}^3)^\vee.
\]

Next, note that \( \varpi \) is surjective. Indeed, consider first the sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_\wp \to 0,
\]
i.e. $\varphi$ is the hyperplane within $\mathbb{P}^3$ given by the equation $\phi = 0$. Applying the functor $\text{Hom}(\cdot, \Omega^1_{\mathbb{P}^3}(1))$, we conclude that

$$\text{Ext}^1(\mathcal{O}_\varphi, \Omega^1_{\mathbb{P}^3}(1)) \simeq H^0(\Omega^1_{\mathbb{P}^3}(2)).$$

In particular, we have $\dim \text{Ext}^1(\mathcal{O}_\varphi, \Omega^1_{\mathbb{P}^3}(1)) = 6$, independently of the hyperplane $\varphi$. Thus for every hyperplane $\varphi \in (\mathbb{P}^3)^\vee$ one has non-trivial extensions by $\Omega^1_{\mathbb{P}^3}(1)$, and every such extension defines an element $[E] \in \mathcal{I}^{1}(3, 1)$ such that $\varpi([E]) = \varphi$.

Furthermore, we also concluded that the fibres of $\varpi$ are precisely the projectivization of $\text{Ext}^1(\mathcal{O}_\varphi, \Omega^1_{\mathbb{P}^3}(1))$.

Summarizing the above discussion, we have the following result.

**Proposition 14.** The moduli space $\mathcal{I}^{1}(3, 1)$ of indecomposable rank 3 instanton sheaves of charge 1 on $\mathbb{P}^3$ is a projective variety of dimension 8, given by the total space of a $\mathbb{P}^5$-bundle over $(\mathbb{P}^3)^\vee$.

We are finally able to characterize the singular loci of rank 3 instanton sheaves of charge 1.

**Proposition 15.** The singular locus of a rank 3 instanton sheaf $E$ of charge 1 which is not locally free is either a point, if $E$ is reflexive, or a line, if $E$ is not reflexive.

**Proof.** First, let $E$ be a reflexive instanton sheaf of rank 3 and charge 1. Then $\dim \text{Sing}(E) = 0$, and we must show that $\text{Sing}(E)$ is a point.

Indeed, recall that $\text{Sing}(E)$ coincides with the degeneration locus of the monad (30), which is given by the common zeros of the sections $\sigma_j$ and $\phi$; such set has dimension zero if and only if it consists of a single point.

Note, in addition, that $E$ is reflexive if and only if $\dim \Gamma = 3$. We have already proved the only if part; conversely, if $\dim \Gamma = 3$, then $\text{Sing}(E)$ consists of a single point, hence $E$ must be reflexive by Lemma [6].

Now if $E$ is indecomposable and not reflexive, then $\dim \Gamma = 2$, which means that the degeneration locus of the monad (30), and hence $\text{Sing}(E)$, is a line.

If, on the other hand, $E$ is decomposable (i.e. if $\dim \Gamma = 1$), then it decomposes as a sum $E' \oplus \mathcal{O}_{\mathbb{P}^3}$ with $E'$ being a non locally free rank 2 instanton of charge 1. It then follows from Proposition [12] that $\text{Sing}(E')$, which of course coincides with $\text{Sing}(E)$, is a line.

**Remark 16.** It is easy to see that every reflexive instanton sheaf of rank 3 on $\mathbb{P}^3$ is $\mu$-semistable. Indeed, every instanton sheaf $E$ on $\mathbb{P}^3$ satisfies $H^0(E(-1)) = H^0(E^\vee(-1)) = 0$, thus $\mu$-semistability follows from the criterion in [12] Remark 1.2.6 b, page 167.

On the other hand, by [6] Prop. 16], there are no $\mu$-stable instanton sheaves of rank 3 and charge 1. Furthermore, one can check from [33] that $h^0(E) = 1$, and it follows that $E$ is not (Gieseker) semistable either.

### 5.2. Rank 3 instanton sheaves of charge 2.

In the last part of this paper, we present two interesting examples of rank 3 instanton sheaves of charge 2.

We begin by showing how to construct a rank 3 instanton sheaf of charge 2 whose singular locus is the disjoint union of a line and a point.
The starting point is an indecomposable reflexive instanton sheaf $F$ of rank 3 and charge 1 which is not locally free. For instance, take the one obtained as cohomology of the monad (24): its singular locus is just the point $P = [0 : 0 : 0 : 1]$. One easily checks that it has trivial splitting type, that is, its restriction to a generic line is trivial.

Let $ι : ℓ → \mathbb{P}^3$ be a line in $\mathbb{P}^3$ that does not contain the point $[0 : 0 : 0 : 1]$ and to which the restriction of $F$ is trivial. As we have checked above, the sheaf $ι_*\mathcal{O}_ℓ(1)$ is a rank 0 instanton of degree 1. Moreover, since $F|_ℓ \simeq \mathcal{O}_ℓ^{⊕ 2}$, there are surjective maps $ϕ : F → ι_*\mathcal{O}_ℓ(1)$; let $E := \ker ϕ$.

From the short exact sequence

\[(35) \quad 0 → E → F → ι_*\mathcal{O}_ℓ(1) → 0\]

one easily checks that $E$ is a rank 3 instanton sheaf of charge 2. Indeed, since $F$ is an instanton sheaf, it is easy to see that $E$ is torsion free and that $c_1(E) = c_3(E) = 0$ and $c_2(E) = 2$. One also checks immediately from the cohomology sequence that

$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0$.

**Remark 17.** The construction of the two previous paragraphs is a particular case of an elementary transformation of an instanton sheaf, as outlined in [7, Section 3].

Note also that $E^⊥ \simeq F^⊥$, thus in particular $E^{\vee \vee} \simeq F$, so $E^{\vee \vee}$ is also an instanton sheaf, and the sequence $0 → E → E^{\vee \vee} → Q_E → 0$ matches the sequence (35), thus $Q_E = ι_*\mathcal{O}_ℓ(1)$. In addition, sequence (24) reduces to, in this case,

\[0 → j_*\mathcal{O}_P → S_E → ι_*\mathcal{O}_ℓ(1),\]

where $j$ denotes the inclusion of the point $P = [0 : 0 : 0 : 1]$ within $\mathbb{P}^3$. It follows immediately that $\text{Sing}(E) = P ∪ ℓ$, which is not of pure dimension (neither 0 or 1).

We conclude with an explicit example of a monad whose cohomology is a reflexive instanton sheaf of rank 3 and charge 2 whose singular locus consists of two distinct points. We take

\[(36) \quad \mathcal{O}_{\mathbb{P}^3}(-1)^{⊕ 2} α_{mn} → \mathcal{O}_{\mathbb{P}^3}^{⊕ 7} β_{mn} → \mathcal{O}_{\mathbb{P}^3}(1)^{⊕ 2}\]

with

\[α_{mn} = \left(\begin{array}{cccccc}
x_2 & x_4 & x_1 & m^2 x_4 & -x_3 & x_3 - m x_4 \\
-m^2 x_4 & -x_2 & -x_1 & x_3 & x_1 & \frac{1}{m} x_3 - x_4 \\
x_1 & x_3 & -x_2 & m^2 x_4 & -x_4 & \frac{1}{m} x_3 + x_4 \\
m x_3 & x_1 & x_2 & -x_1 & -x_3 & \frac{1}{m^2} x_3 \\
x_3 - m x_4 & x_1 & x_2 & \frac{1}{m} x_3 & -x_4 & -x_3 \\
x_3 - m x_4 & \frac{1}{m} x_3 - x_4 & \frac{1}{m} x_3 + x_4 & 0 & x_3 & x_3 \end{array}\right)\]

and

\[β_{mn} = \left(\begin{array}{cccccc}
-x_1 & x_3 & -x_2 & x_4 & x_3 + x_4 & 0 & x_3 \\
x_3 & -x_1 & m^2 x_4 & -x_2 & x_4 & x_3 & \frac{1}{m} x_3 \end{array}\right)\]

where $n$ is a root of the equation $n^3 + 2n^2 + n + 1 = 0$ and $m = n + 1/n$ (this guarantees that $β_0 = 0$). One checks that $β$ is surjective everywhere (since $m ≠ ±1$) and that $α$ fails to be injective only at the points $[n : 0 : 1 : 0]$ and $[0 : m : 0 : 1]$. 

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