DERIVATION OF GEOSTROPHIC EQUATIONS AS A RIGOROUS LIMIT OF COMPRESSIBLE ROTATING AND HEAT CONDUCTING FLUIDS WITH THE GENERAL INITIAL DATA

YOUNG-SAM KWON
Department of Mathematics
Dong-A University, Busan, Korea

ANTONIN NOVOTNY*
University of Toulon, IMATH
BP 20139, 839 57 La Garde, France

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Abstract. We investigate a distinguished low Mach and Rossby - high Reynolds and Péclet number singular limit in the complete Navier-Stokes-Fourier system towards a strong solution of a geostrophic system of equations. The limit is effectuated in the context of weak solutions with ill prepared initial data. The main tool in the proof is based on the relative energy method.

1. Introduction. This paper deals with the Navier-Stokes-Fourier system describing evolution of a viscous compressible fluid in the regime of fast rotation. One of the distinguished singular limits of interest in this situation is small Mach and Rossby, and large Reynolds and Péclet number limit. Recently, Feireisl and the second author have studied these issues for the inviscid incompressible limit of the compressible Navier-Stokes equations for the rotating fluids in the barotropic regime, [18], [19]. The present paper may be viewed as an extension of the results of [18], [19] to the general case including the Navier-Stokes-Fourier equations. In contrast to [18], [19], we take into account the more realistic situation when the fluid is heat conducting and equation describing the evolution of temperature is not disregarded.

Let us consider viscous compressible gas with the density $\varrho = \varrho(t,x)$, the velocity $u = u(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$, $t \in (0,T)$, $x \in \Omega$, which fills the infinite straight layer

$$\Omega = \mathbb{R}^2 \times (0,1)$$

rotating around the $x_3$ axis

$$f = [0,0,1].$$

Its evolution is governed by the scaled complete Navier-Stokes-Fourier system:

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* Corresponding author: Anotnin Novotny.

395
\[
\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho u) &= 0, \\
\partial_t (\varrho u) + \text{div}_x (\varrho u \otimes u) + \frac{1}{\varepsilon} \varrho \mathbf{f} \times u + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) &= \varepsilon^a \text{div}_x \mathbb{S}(\vartheta, \nabla_x u) + \varrho \nabla_x G, \\
\partial_t (\varrho s(\varrho, \vartheta)) + \text{div}_x (\varrho s(\varrho, \vartheta)u) + \varepsilon^b \text{div}_x \left( \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) &= \frac{1}{\vartheta} \left( \varepsilon^2 \varrho \mathbb{S}(\vartheta, \nabla_x u) \cdot \nabla_x u - \varepsilon^b \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta \right).
\end{align*}
\]

In these equations, \( p = p(\varrho, \vartheta) \) is the pressure, \( s = s(\varrho, \vartheta) \) the specific entropy, the symbol \( \mathbb{S}(\vartheta, \nabla_x u) \) denotes the viscous stress satisfying \textit{Newton’s law}

\[
\mathbb{S}(\vartheta, \nabla_x u) = \mu(\vartheta) \left( \nabla_x u + \nabla_x^T u - \frac{2}{3} \text{div}_x u \mathbf{I} \right) + \eta(\vartheta) \text{div}_x u \mathbf{I},
\]

and \( \mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) \) is the heat flux determined by \textit{Fourier’s law}

\[
\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta,
\]

where the quantities \( \mu > 0, \eta \geq 0, \kappa > 0 \) are temperature dependent transport coefficients and \( \mathbf{I} \) is the identity \( 3 \times 3 \) matrix. The quantity \( \nabla_x G \) corresponds to the force of gravitation. We assume that

\[
\nabla_x G \in L^\infty \cap L^1(\Omega).
\]

The system is completed with the boundary conditions

\[
u \cdot \mathbf{n}\big|_{\partial \Omega} = 0, \quad [\mathbb{S}(\nabla_x u) \cdot \mathbf{n}] \times \mathbf{n}\big|_{\partial \Omega} = 0 \tag{1.8}
\]

imposed on the horizontal boundary, and the far field conditions

\[
\varrho \to \overline{\varrho}, \; \vartheta \to \overline{\vartheta}, \; u \to 0 \text{ as } |x| \to \infty, \tag{1.9}
\]

where \( \overline{\varrho} > 0 \) is the equilibrium density and \( \overline{\vartheta} > 0 \) is a positive equilibrium temperature. In (1.8) and in the sequel, \( \mathbf{n} \) denotes the outer normal to the boundary \( \partial \Omega \) of \( \Omega \).

The model (1.2 - 1.9) may be viewed as a rough approximation (\( f \)-plane model) of the Earth atmosphere in a plane tangent to the Earth at a certain latitude, see Vallis [50, Chapter 2, Section 2.3] and Durrant [7].

The momentum equation (1.3) contains a small parameter \( \varepsilon \) related to different nondimensional numbers resulting from the scale analysis: \textit{Rossby number} = \( \varepsilon \), \textit{Mach number} = \( \varepsilon \), \textit{Reynolds number} = \( \varepsilon^{-a} \), \textit{Péclet number} = \( \varepsilon^{-b} \), \( a, b > 0 \), see Klein [36].\footnote{In the main theorem, we will require also an upper bound for the exponent \( a \) characterizing the Reynolds number (namely \( a < 10/3 \)). This is a technical requirement dictated by the passage to the limit \( \varepsilon \to 0 \) in the convective term in the weak formulation (2.15) of the entropy balance (1.4), cf. Section 6.4.1.} We consider the singular limit problem for the \textit{ill-prepared initial data}

\[
\varrho(0, \cdot) = \varrho_{0, \varepsilon} := \overline{\varrho} + \varepsilon \varrho_{0, \varepsilon}^{(1)}, \quad \vartheta(0, \cdot) = \vartheta_{0, \varepsilon} := \overline{\vartheta} + \varepsilon \vartheta_{0, \varepsilon}^{(1)}, \quad u(0, \cdot) = u_{0, \varepsilon}, \tag{1.10}
\]

where \( \overline{\varrho} > 0, \; \overline{\vartheta} > 0 \) are the anticipated constant density and temperature enforced by the incompressible low Mach number limit.

Under this scaling, it is expected that the fast rotation will drive the motion to the horizontal plane flow regime while the gravitation acts against this process by driving the fluid to a vertical stratification which should be expressed by the buoyancy force. However, since the stratification is low (the Froude number in front of the gravity term is comparable to 1), the buoyancy force is expected to be
invisible in the limiting regime. Further, the low Mach number limit will drive the motion to an incompressible one, and high Reynolds number limit to a non-viscous one. A similar situation was considered in [37]: the authors were investigating situation when Mach number dominated the Rossby number and obtained in the limit a two-dimensional incompressible Euler-Boussinesq system coupled with the transport equation describing the evolution of temperature. In the present paper, we consider the situation when the Mach number is comparable with the Rossby number. This regime seems to be more appropriate from the point of view of the physics of atmosphere: the target system is a system of quasi-geostrophic Euler equations completed with the transport equation for temperature, cf. (1.12–1.14).

We denote
\[
\alpha = \frac{1}{\rho} \frac{\partial p(\rho, \vartheta)}{\partial \rho}, \quad \beta = \frac{1}{\rho} \frac{\partial p(\rho, \vartheta)}{\partial \vartheta}, \quad \delta = \frac{\partial s(\rho, \vartheta)}{\partial \vartheta}, \quad \omega = \frac{\rho}{\rho_s} \left( \alpha + \frac{\beta^2}{\delta} \right) .
\] (1.11)

Further, in the following, for a vector field \( \mathbf{b} = [b_1, b_2, b_3] \) we introduce the horizontal component \( b_h = [b_1, b_2] \) and write \( \mathbf{b} = [b_h, b_3] \). Similarly, the symbols \( \nabla_h, \text{div}_h, \Delta_h \) denote the differential operators acting on the horizontal variable \( x_h = (x_1, x_2) \), where \( x = (x_1, x_2, x_3) \).

Supposing we already know that, in some sense, \( \rho_\varepsilon - \rho_\varepsilon \rightarrow \rho, \quad \vartheta_\varepsilon - \vartheta_\varepsilon \rightarrow \vartheta, \quad u_\varepsilon \rightarrow v \) we may (formally) check (effectuating the asymptotic expansion of the quantities \( u_\varepsilon, \varrho_\varepsilon, \vartheta_\varepsilon \) in powers of \( \varepsilon \)) that \( q = q(t, x_h), \quad v = [v_h(t, x_h), 0] \) are interrelated through the diagnostic equation
\[
f \times v + \nabla q = 0 \quad \text{(1.12)}
\]
where the quantities
\[
q = \alpha \vartheta^{(1)} + \beta \vartheta^{(1)}, \quad \vartheta = \frac{\alpha \beta}{\alpha \delta + \beta^2} \vartheta^{(1)} + \frac{\alpha \delta}{\alpha \delta + \beta^2} \vartheta^{(1)}
\]
satisfy the quasigeostrophic Euler equation
\[
\partial_t \left( \Delta_h q - \frac{1}{\beta} \frac{\delta}{\beta^2 + \alpha \delta} q \right) + \nabla_h^* q \cdot \nabla_h \left( \Delta_h q - \frac{1}{\beta} \frac{\delta}{\beta^2 + \alpha \delta} q \right) = 0 \quad \text{(1.13)}
\]
and the transport equation
\[
\partial_t \vartheta + v \cdot \nabla \vartheta = 0 . \quad \text{(1.14)}
\]
Note that by virtue of (1.12),
\[
\nabla_h^* q = [-\partial_{x_2} q, \partial_{x_1} q] = v, \quad \Delta_h q = \text{curl}_h v_h ;
\]
whence \( q \) can be viewed as a kind of stream function, while the system (1.12), (1.13) possesses the same structure as the 2D Euler equations. In particular, we expect the solutions of (1.12), (1.13) to be as regular as the initial data and to exist globally in time.

Our intention is to translate these heuristic arguments to a rigorous mathematical statements. The exact statement of our results including the initial data for the target system (1.12 - 1.14) will be specified in Theorem 3.1 below. The main tool in the proof is the relative energy method developed for the Navier-Stokes-Fourier system in [14] in steps of Germain [27] and [20].

The phenomena discussed above have been investigated by many authors for the incompressible fluids or for the compressible fluids in the barotropic regime: The fact that highly rotating fluids become planar (two-dimensional), and, accordingly, fast
rotation has a regularizing effect, was observed by Babin, Mahalov, and Nicolaenko [1], [2], Bresch, Desjardins, and Gerard-Varet, [3], Chemin et al. [4], Gallagher, Saint-Raymond [25], [26], Fanelli [22], [23], [18], [19], among others. The inviscid limit is a well studied and partially still open challenging problem, see Clopeau, Mikelić, Robert [5], Kato [31], Masmoudi [40], [41], [42], Sammartino and Caflisch [44], [45], Swann [47], Temam and Wang [48], [49], Iftimie, Sueur [29], Gie, Kelliher [28], Masmoudi, Rousset [39], Wang, Xin, Zang [51], to name only a few. The low Mach number limits were investigated in the pioneering papers by Ebin [8], and Klainerman and Majda [34], and later reexamined in the context of weak solutions by Lions and Masmoudi [38], see also the survey by Danchin [6], Gallagher [24], and Schochet [46].

The low Mach and low Mach/high Reynolds, Péclet number regimes in the context of the compressible heat-conducting flows have been investigated recently in [15], [16], [17]. To the best of our knowledge, the effects of the fast rotation in the complete Navier-Stokes-Fourrier system have not yet been investigated in the mathematical literature (with exception of [37] which deals with a different distinguished limit). Taking into account the full thermodynamics of the primitive as well as the target systems is the first key feature of the present paper.

We address the problem with the ill prepared initial data in the framework of weak solutions for the Navier-Stokes-Fourier system (1.2 - 1.4), developed in [13], and later extended to problems on unbounded domains in [30]. This is the second key feature of the paper.

The two dimensional quasi-geostrophic Euler equations - which form a part of the target system in the investigated distinguished limit- is known to possess a unique global in time classical solution. Consequently, we obtain the convergence of the sequence of weak solutions to the solutions of the target equations on the (arbitrary) large time interval, where the weak solutions to the primitive equations exist. This is another key feature of the present paper.

Last but not least, an important issue is the behavior of the oscillatory part of the weak solutions to the scaled system. In contrast to [15] dealing with the heat conducting flows without rotation, these are described by a system with coefficients depending on the scaling parameter \( \varepsilon \), similarly as in [16]. In contrast to [18] dealing with the rotating barotropic fluids, the oscillating quantities (and the corresponding equations for the acoustic-Poincaré waves) are not straightforward to identify.

The scaling adopted in this paper leads to a weak stratification. There are no rigorous proofs of distinguished limits for the fast rotating compressible flows in the regimes leading to strong stratification (when the Mach and Froude numbers are comparable) neither in the barotropic nor in the heat-conducting case) for the ill prepared initial data. This is an interesting open problem. The only so far “treatable” problems in this situation are those with the well prepared initial data, cf. [21].

The paper is organized as follows. The necessary preliminary material including various concepts of weak solutions to the Navier-Stokes-Fourier system is collected in Section 2. Section 3 contains the main result on the asymptotic limit for \( \varepsilon \to 0 \), the proof of which is the main objective of the remaining part of the paper. In Section 4, the relative energy inequality is used to establish the necessary uniform bounds independent of \( \varepsilon \to 0 \). The problem of propagation and dispersion of the associated Poincaré-acoustic waves including the frequency cut-off technique
is discussed in Section 5. The proof of convergence towards the limit system is completed in Section 6.

Throughout the paper, we use the standard notation for Sobolev and Bochner spaces, see, e.g., the book of Evans [9].

2. Preliminaries, weak solutions to the Navier-Stokes-Fourier system.

2.1. Structural restrictions imposed on constitutive relations. The singular limit problem described in the previous section will be investigated under certain physically motivated assumptions on the constitutive laws and transport coefficients of the fluid. These assumptions guarantee, among others, existence of weak solutions. We refer the reader to [13, Chapter 3] for more information about the physical background.

The pressure $p = p(\rho, \vartheta)$ is given by the formula
\[ p(\rho, \vartheta) = \frac{\vartheta}{\rho^{3/2}} P \left( \frac{\vartheta}{\rho^{3/2}} \right) + \frac{a}{3} \vartheta^4, \quad a > 0; \tag{2.1} \]
the specific internal energy $e = e(\rho, \vartheta)$ and the specific entropy $s = s(\rho, \vartheta)$ read
\[ e(\rho, \vartheta) = \frac{3}{2} \vartheta \vartheta^{3/2} P \left( \frac{\rho}{\vartheta^{3/2}} \right) + a \vartheta^4, \tag{2.2} \]
\[ s(\rho, \vartheta) = S \left( \frac{\rho}{\vartheta^{3/2}} \right) + \frac{4a}{3} \vartheta^3 \rho, \tag{2.3} \]
where
\[ P \in C^1[0, \infty) \cap C^3(0, \infty), \quad P(0) = 0, \quad P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.4} \]
\[ \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0, \tag{2.5} \]
\[ 0 < \frac{3}{2} P(Z) - P'(Z)Z < c \text{ for all } Z > 0, \tag{2.6} \]
and
\[ S'(Z) = -\frac{3}{2} \frac{\vartheta P(Z) - P'(Z)Z}{Z^2}, \quad \lim_{Z \to \infty} S(Z) = 0. \tag{2.7} \]
The relation (2.6) translates positivity and uniform boundedness of the specific heat at constant volume.

Integrating (2.7) and employing the bound (2.6) We easily verify that $0 < S(Z) \leq c(1 + |\log Z|)$ (cf. (2.6), (2.7)). Therefore, by virtue of (2.3),
\[ \vartheta |s(\rho, \vartheta)| \leq c(\vartheta + \vartheta^3 + \vartheta |\log \rho| + \vartheta |\log \vartheta|^+) \tag{2.8} \]
This estimate will be needed later.

The transport coefficients $\mu, \eta, \kappa$ are effective functions of the temperature, $\mu, \eta \in C^1[0, \infty)$ are globally Lipschitz in $[0, \infty)$, $0 < \mu_1(1 + \vartheta) \leq \mu(\vartheta), \eta(\vartheta) \geq 0, \text{ for all } \vartheta > 0, \tag{2.9}$
\[ \kappa \in C^1[0, \infty), \quad 0 < \kappa(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \pi(1 + \vartheta^3) \text{ for all } \vartheta \geq 0. \tag{2.10} \]
2.2. Weak dissipative solutions. For $\varrho : (0, T) \times \Omega \mapsto (0, \infty)$, $r, \Theta, \vartheta : (0, T) \times \Omega \mapsto (0, \infty)$ and $\mathbf{u}, \mathbf{U} : (0, T) \times \Omega \mapsto \mathbb{R}^3$, we define the relative energy functional in the spirit of [14],

$$
E_{\varepsilon} \left( \varrho, \vartheta, \mathbf{u} | r, \Theta, \mathbf{U} \right) = \int_{\Omega} \left[ \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2} E(\varrho, \vartheta) \right] \, dx,
$$

(2.11)

where

$$
E(\varrho, \vartheta) = H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial r} (r - r) - H_\Theta(r, \Theta) \quad \text{and} \quad H_\Theta(\varrho, \vartheta) = \varrho (e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta))
$$

(2.12)

are the relative energy function and the Helmholtz function, respectively.

Now, we are ready to introduce the weak dissipative solutions. Following [13], [14], we say that a trio of functions $[\varrho, \vartheta, \mathbf{u}]$ represents a weak dissipative solution of the Navier-Stokes-Fourier system (1.2 - 1.10) in $(0, T) \times \Omega$ if:

1. $\varrho \geq 0, \vartheta > 0$ a.e. in $(0, T) \times \Omega$,

$$
(\varrho - \overline{\varrho}) \in L^\infty(0, T; L^2 + L^{5/3}(\Omega)), \quad (\vartheta - \overline{\vartheta}) \in L^\infty(0, T; L^2 + L^4(\Omega)), \quad \nabla_x \vartheta, \nabla_x \log(\vartheta) \in L^2(0, T; L^2(\Omega; \mathbb{R}^3)),
$$

$$
\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0,
$$

where $[\overline{\varrho}, \overline{\vartheta}]$ are anticipated constant density and temperature introduced in (1.10);

2. the density $\varrho \in C_{\text{weak}}([0, T]; L^1(K))$ for any compact $K \subset \overline{\Omega}$ and the equation of continuity (1.2) holds as a family of integral identities

$$
\int_{\Omega} \left[ \varrho(\tau, \cdot) \varphi(\tau, \cdot) - \varrho_{0, \tau} \varphi(0, \cdot) \right] \, dx = \int_{0}^{T} \int_{\mathbb{R}^3} \left( \varrho \vartheta \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) \, dx \, dt
$$

(2.13)

for any $\tau \in [0, T]$ and any test function $\varphi \in C_0^\infty([0, T] \times \overline{\Omega})$;

3. the linear momentum $\varrho \mathbf{u} \in C_{\text{weak}}([0, T]; L^1(K; \mathbb{R}^3))$ for any compact $K \subset \overline{\Omega}$ and the momentum equation (1.3) is satisfied in the sense of distributions,

$$
\int_{\Omega} \left[ \varrho \mathbf{u}(\tau, \cdot) \cdot \varphi(\tau, \cdot) - \varrho_{0, \tau} \mathbf{u}_{0, \tau} \varphi(0, \cdot) \right] \, dx
$$

(2.14)

$$
= \int_{0}^{T} \int_{\Omega} \left( \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \cdot \mathbf{u} : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi \right) \, dx \, dt
$$

for any $\tau \in [0, T]$, and any $\varphi \in C_0^\infty([0, T] \times \overline{\Omega}; \mathbb{R}^3), \varphi \cdot \mathbf{n}|_{\partial \Omega} = 0$;

4. the entropy production equation (1.4) is relaxed to the entropy inequality

$$
\int_{\Omega} \left[ \varrho_{0, \tau} s(\varrho_{0, \tau}, \varrho_{0, \tau}) \varphi(0, \cdot) - \varrho s(\varrho, \vartheta) \varphi(\tau, \cdot) \right] \, dx
$$

(2.15)

$$
+ \int_{0}^{T} \int_{\Omega} 1 \left( \varepsilon^{3+\alpha} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^5 \mathbb{G}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \right) \varphi \, dx \, dt
$$

$$
\leq - \int_{0}^{T} \int_{\Omega} \left( \varrho s(\varrho, \vartheta) \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \varepsilon^5 \mathbb{G}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \varphi \right) \, dx \, dt
$$

for a.a. $\tau \in [0, T]$ and any test function $\varphi \in C_0^\infty([0, T] \times \overline{\Omega}), \varphi \geq 0$.

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\[2\] The vector space $C_{\text{weak}}(\overline{T}; X)$ with $X$ a Banach space is a space of functions defined on $[0,T]$ belonging to the space $L^\infty(I; X)$ and continuous in $T$ with respect to the weak topology of $X$ (meaning that $f \in C_{\text{weak}}(\overline{T}; X)$ if $f \in L^\infty(I; X)$ and $t \mapsto F(f(t))$ belongs to $C(\overline{T})$ for any $F \in X^*$, where $X^*$ is the dual space to $X$).
5. The relative energy inequality

\[
\left[ \varepsilon \left( \rho, \vartheta, u | r, \Theta, U \right) \right]_{t=0}^{T} + \int_{0}^{T} \int_{\Omega} \left( \varepsilon^2 S(\vartheta, \nabla u) : \nabla u - \varepsilon b^{-2} q(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta \right) \, dx \, dt \\
\leq \int_{0}^{T} \int_{\Omega} \left( \varepsilon^2 S(\vartheta, \nabla u) : \nabla u - \varepsilon b^{-2} q(\vartheta, \nabla \vartheta) \cdot \nabla \vartheta \right) \, dx \, dt \\
+ \int_{0}^{T} \int_{\Omega} \left( \rho \left( \partial_t u + u \cdot \nabla u \right) \cdot (U - u) \right) \, dx \, dt - \int_{0}^{T} \int_{\Omega} \rho \nabla x G \cdot (U - u) \, dx dt \\
+ \frac{1}{\varepsilon^2} \int_{0}^{T} \int_{\Omega} \frac{1}{r} \left( \rho - \vartheta \right) \nabla x p(r, \Theta) \, dx dt + \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \rho \left( f \times u \right) \cdot (U - u) \, dx dt \\
+ \frac{1}{\varepsilon^2} \int_{0}^{T} \int_{\Omega} \rho \left( \nabla x p(r, \Theta) \right) \frac{1}{r} \cdot (U - u) \, dx dt \\
- \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \left[ \rho \left( s(\rho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta + \rho \left( \Theta s(\rho, \vartheta) - s(r, \Theta) \right) u \cdot \nabla x \Theta \right] \, dx dt
\]

holds for a.a. \( t \in (0, T) \) and for any trio of continuously differentiable test functions \( (r, \Theta, U) \) defined on \([0, T] \times \Omega\),

\[
r > 0, \ \Theta > 0, \ U \cdot n|_{\partial \Omega} = 0,
\]

\[
r - \vartheta, \Theta - \bar{\Theta} \in C([0, T]; W^{k,2}(\Omega; \mathbb{R}^3)), \quad \partial_t r, \partial_t \Theta \in C([0, T]; W^{k-1,2}(\Omega; \mathbb{R}^3)),
\]

\[
U \in C([0, T]; W^{k,2}(\Omega; \mathbb{R}^3)), \quad \partial_t U \in C([0, T]; W^{k-1,2}(\Omega; \mathbb{R}^3)), \quad k \geq 3.
\]

**Remark 2.1.**

1. Under the hypotheses \((2.1 - 2.10)\), the existence of weak dissipative solutions in \((0, T) \times \Omega\) to the Navier-Stokes-Fourier system \((1.2-1.10)\) emanating from the initial data \((1.10)\) (possessing, in particular, regularity \((3.2), (3.3), (3.4)\)) has been shown in [30], see also [13, Chapter 3] and [17].

2. Of course, the boundary conditions \((1.8)\) play an important role in the derivation of \((2.16)\). It is of interest to compare the results and approach of this paper with the investigations of high Reynolds number limits in the incompressible case, see notably, Iftimie, Sueur [29], Masmoudi, Rousset [39], Gie, Kelliher [28], Wang, Xin, Zang [51].

3. Any weak dissipative solution \([\rho, \vartheta, u]\) of problem \((1.2-1.10)\) always depends on \( \varepsilon \). In what follows, we shall therefore write \([\rho_\varepsilon, \vartheta_\varepsilon, u_\varepsilon]\) instead of \([\rho, \vartheta, u]\) whenever it is important to underline this dependence.

### 2.3. Limit system

We turn our attention to the Cauchy problem for the limiting system \((1.12-1.13)\). We see that the problem \((1.13)\) enjoys strong similarity with the standard Euler system. In particular, we may use the abstract theory of Oliver [43, Theorem 3] to obtain the following result:

**Proposition 2.1.** Suppose that

\[
q_0 \in W^{k,2}(\mathbb{R}^2) \text{ for } k \geq 4.
\]

Then the problem \((1.15)\) with initial data

\[
q(0) = q_0
\]
admits a solution \( q \), unique in the class

\[
q \in C([0, T]; W^{k,2}(\mathbb{R}^2)) \cap C^1([0, T]; W^{k-1,2}(\mathbb{R}^2)).
\]  

(2.18)

and there is \( c > 0 \) such that

\[
\|q\|_{L^\infty(0, T; W^{k,2}(\Omega))} + \|\partial_t q\|_{L^\infty(0, T; W^{k-1,2}(\Omega))} \leq c\|q_0\|_{W^{k,2}(\Omega)}.
\]  

(2.19)

In particular, it is worth noting that, similarly to the 2D-Euler system, the solution \( q \) can be constructed globally in time. Finally, \( \theta \) satisfies the transport equation (1.14) with \( v = \nabla^\perp q \), where \( q \) is given in Proposition 2.1. We have, by the standard theory:

**Proposition 2.2.** Suppose that

\[
\theta_0 \in W^{1,2}(\mathbb{R}^2)
\]

and \( v = \nabla^\perp q := [-\partial_x q, \partial_y] \), where \( q \) is the solution of (1.13) constructed in Proposition 2.1 emanating from \( q_0 \in W^{k,2}(\mathbb{R}^2) \), \( k \geq 4 \). Then equation (1.14) with initial data

\[
\theta(0) = \theta_0
\]  

(2.20)

admits a unique solution \( \theta \) in class

\[
C([0, T]; W^{1,2}(\mathbb{R}^2)) \cap C^1([0, T]; L^2(\mathbb{R}^2))
\]  

(2.21)

and there exists \( c = c(\|q_0\|_{k,2}) > 0 \) such that

\[
\|\partial_t \theta\|_{L^\infty(0, T; W^{1,2}(\Omega))} + \|\partial_t \theta\|_{L^\infty(0, T; L^2(\Omega))} \leq c\|\theta_0\|_{W^{1,2}(\Omega)}.
\]  

(2.22)

2.4. **Poincaré waves and the kernel of the wave propagator.** Linearizing the equations (1.2–1.4) near the constant state \((\overline{\vartheta}, \overline{\varphi}, 0)\) and neglecting (formally) the (“small”) nonlinear terms we find that \( \alpha \frac{\vartheta - \overline{\varphi}}{\varepsilon} + \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \), and \( u_\varepsilon \) verify the equation for Poincaré waves

\[
\varepsilon \partial_t \left( \alpha \frac{\vartheta - \overline{\varphi}}{\varepsilon} + \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \right) + \omega \nabla_x u_\varepsilon \approx 0,
\]  

(2.23)

\[
\varepsilon \partial_t u_\varepsilon + f \times u_\varepsilon + \nabla_x \left( \alpha \frac{\vartheta - \overline{\varphi}}{\varepsilon} + \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \right) \approx 0
\]  

(2.24)

while the quantity \( \delta \frac{\vartheta - \overline{\varphi}}{\varepsilon} - \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \) verifies the transport equation

\[
\partial_t \left( \delta \frac{\vartheta - \overline{\varphi}}{\varepsilon} - \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \right) + u_\varepsilon \cdot \nabla \left( \delta \frac{\vartheta - \overline{\varphi}}{\varepsilon} - \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \right) + \left( \delta \frac{\vartheta - \overline{\varphi}}{\varepsilon} - \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \right) \text{div} u_\varepsilon \approx 0.
\]  

(2.25)

We may therefore hope to be able to control the time derivatives of the projection of the linear momentum \( \varrho_\varepsilon u_\varepsilon \) on the kernel \( K[\mathcal{B}(\omega)] \) of the wave propagator \( \mathcal{B}(\omega) \) of the wave equation (2.23–2.24), and the time derivatives of the entropy deviations \( \approx \delta \frac{\vartheta - \overline{\varphi}}{\varepsilon} - \beta \frac{\varphi - \overline{\varphi}}{\varepsilon} \) when \( \varepsilon \to 0 \). The time derivative of the deviations \( \frac{\varrho - \overline{\varphi}}{\varepsilon} \) and the time derivative of the projection of the linear momentum to \( [K[\mathcal{B}(\omega)]]_{+1} \) may explode as \( \varepsilon \to 0 \) for the initial data belonging to the orthogonal complement of the kernel of \( \mathcal{B}(\omega) \). However, as suggested in Lemma 2.1 later, the \( L^q(\Omega), q > 2 \) norms of these deviations may vanish as \( \varepsilon \to 0 \), under certain circumstances due to the dispersion. We will now put this heuristic argumentation on the rigorous grounds.
2.4.1. Kernel of the wave propagator. The wave propagator of the equation for the Poincaré waves is

\[ \mathcal{B}(\omega) : L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3) \rightarrow L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3), \]

\[ \mathcal{B}(\omega) \left[ \begin{array}{c} \xi \\ V \end{array} \right] \mapsto \left[ \begin{array}{c} \omega \text{div}_x V \\ f \times V + \nabla_x \xi \end{array} \right] \]

where \( \omega \) is positive number. Its domain of definition is

\[ \mathcal{D}[\mathcal{B}(\omega)] = \{ [r, V] \mid r \in W^{1,2}(\Omega), V \in L^2(\Omega; \mathbb{R}^3), \text{div}_x V \in L^2(\Omega), V \cdot n = V_3|_{\partial \Omega} = 0 \}. \]

It admits a non zero kernel

\[ \mathcal{K}(\mathcal{B}(\omega)) = \{ [q, v] \mid q = q(x_h), q \in W^{1,2}(\mathbb{R}^2), v = [v_h(x_h), 0], \text{div}_h v_h = 0, f \times v + \nabla_x q = 0 \}. \]

Let \( \mathcal{P}(\omega) \) denotes the projection

\[ \mathcal{P}(\omega) : L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3) \rightarrow \mathcal{K}(\mathcal{B}(\omega)). \]

Exactly as in \cite[Section 4.1.1]{19} we can show that

\[ \mathcal{P}(\omega)[r, U] = [q, v] \]

if and only if

\[ -\Delta_h q + q = \int_0^1 \text{curl}_h U_h \, dx_3 + \int_0^1 r \, dx_3 \text{ in } \mathbb{R}^2, v = [v_1, v_2, 0], v_1 = -\partial_{x_2} q, v_2 = \partial_{x_1} q. \]

(2.26)

2.4.2. Dispersive estimates for Poincaré waves. We shall consider the following equation for the Poincaré waves with small parameter \( \varepsilon \in (0, 1) \) that describes the oscillatory part of the weak solutions:

\[ \varepsilon \partial_t \xi + \omega \text{div}_x V = 0, \]

(2.27)

\[ \varepsilon \partial_t V + f \times V + \nabla_x \xi = 0, \quad V \cdot n|_{\partial \Omega} = 0, \quad \xi(0) = \xi_0, \quad V(0) = V_0. \]

(2.28)

(2.29)

The following lemma dealing with the dispersive estimates for solutions of problem (2.27–2.29) is proved in \cite[Section 5]{18}.

**Lemma 2.1.** Let \( \omega > 0, \varepsilon \in (0, 1) \), Suppose that the initial data

\[ \left[ \begin{array}{c} \xi_0 \\ V_0 \end{array} \right] \in [\mathcal{K}(\mathcal{B}(\omega))]^\perp \cap_{i=1}^{\infty} W^{1,1}(\Omega; \mathbb{R}^4) \]

and do not contain low and high frequencies in their Fourier modes, meaning that

\[ \mathcal{G} : \mathbb{R}^2 \times \mathbb{Z} \ni (\zeta, k) \mapsto \left[ \begin{array}{c} \xi_0 \\ V_0 \end{array} \right] \]

has compact support in \( (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \), where

\[ \widehat{g}(\zeta, k) = \int_0^1 \left[ \mathcal{F}_{x_h \rightarrow \zeta} [g(\cdot, x_3)](\zeta) e^{-2\pi i x_h k} \right] dx_3 \]

with \( \mathcal{F}_{x_h \rightarrow \zeta} \) the Fourier transform on \( L^1(\mathbb{R}^2) \),

\[ \mathcal{F}_{x_h \rightarrow \zeta}[f] = \int_{\mathbb{R}^2} f(x_h) e^{-2\pi i x_h \cdot \zeta} \, dx_h. \]
Then the problem (2.27–2.29) admits a unique solution
\[ [\xi_\varepsilon, \mathbf{V}_\varepsilon] \in C^1([0, T]; W^{l,2}(\Omega; \mathbb{R}^4)) \]
and there exists \( c = c(l, q, T) > 0 \) (independent of \( \varepsilon \)) such that
\[
\|\xi_\varepsilon\|_{L^\infty(0,T; W^{l,2}(\Omega))} \leq c, \quad \|\mathbf{V}_\varepsilon\|_{L^\infty(0,T; W^{l,2}(\Omega))} \leq c, \quad 2 \leq q \leq \infty, \quad l = 0, 1, \ldots ,
\]
\[
\|\xi_\varepsilon\|_{L^{l}(0,T; W^{l,2}(\Omega))} \rightarrow 0, \quad \|\mathbf{V}_\varepsilon\|_{L^{l}(0,T; W^{l,2}(\Omega))} \rightarrow 0, \quad 2 < q \leq \infty, \quad l = 0, 1, \ldots
\]
as \( \varepsilon \rightarrow 0 \).

3. Main result. In this section we introduce the main result.

**Theorem 3.1.** Let the thermodynamic functions \( p, e, s \) and the transport coefficients \( \mu, \eta, \kappa \) satisfy the hypotheses (2.1 - 2.7), (2.9), (2.10). Let the potential \( G \) be given by (1.7). Let the exponents \( a, b \), determining the Reynold and Péclet number scales, satisfy
\[
b > 0, \quad 0 < a < \frac{10}{3}.
\]

Let the initial data (1.10) be chosen in such a way that
\[
\{\theta_0^{(\varepsilon)}\}_{\varepsilon > 0}, \{\vartheta_0^{(\varepsilon)}\}_{\varepsilon > 0} \text{ are bounded in } L^2 \cap L^\infty(\Omega), \quad \theta_0^{(\varepsilon)} \rightarrow \theta_0, \quad \vartheta_0^{(\varepsilon)} \rightarrow \vartheta_0 \text{ in } L^2(\Omega),
\]
\[
\{u_0, \vartheta_0\}_{\varepsilon > 0} \text{ is bounded in } L^2(\Omega; \mathbb{R}^3), \quad u_0 \rightarrow u_0 \text{ in } L^2(\Omega; \mathbb{R}^3),
\]
where
\[
\theta_0^{(1)} \in W^{k-1/2} \cap W^{1,\infty}(\Omega), \quad \vartheta_0^{(1)} \in W^{1/2} \cap W^{1,\infty}(\Omega), \quad u_0 \in W^{k,2}(\Omega; \mathbb{R}^3) \text{ for a certain } k \geq 3.
\]

Let \( q_0 = q_0(x_h) \) be the unique solution of the elliptic problem
\[
- \Delta_h q_0 + q_0 = \int_0^1 \text{curl}_h [u]_h \, dx_3 + \int_0^1 \vartheta_0^{(1)} \, dx_3 \in W^{1,2}(\mathbb{R}^2).
\]

Finally, let \([\varphi_\varepsilon, \vartheta_\varepsilon, u_\varepsilon]\) be a weak dissipative solution of the Navier-Stokes-Fourier system (1.2 - 1.10) in \((0, T) \times \Omega\).

Then
\[
\operatorname{ess} \sup_{t \in (0,T)} \|\varphi_\varepsilon - \vartheta\|_{L^{5/3}_{\text{loc}}(\Omega)} \leq \varepsilon c,
\]
\[
\int_0^1 \frac{\vartheta_\varepsilon - \vartheta}{\varepsilon} \, dx \rightarrow \vartheta^{(1)} \text{ in } L^\infty(0,T; L^2_{\text{loc}} + L^{5/3}_{\text{loc}}(\Omega)) \text{ and weakly-* in } L^\infty(0,T; L^2 + L^{5/3}(\Omega)),
\]
\[
\sqrt{\varrho} u_\varepsilon \rightarrow \sqrt{\varrho} v \text{ in } L^\infty_{\text{loc}}((0,T]; L^2_{\text{loc}}(\Omega; \mathbb{R}^3)) \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)),
\]
and
\[
\int_0^1 \frac{\vartheta_\varepsilon - \vartheta}{\varepsilon} \, dx_3 \rightarrow \vartheta^{(1)} \text{ in } L^\infty_{\text{loc}}((0,T]; L^2_{\text{loc}}(\mathbb{R}^2)), \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega)),
\]
where
\[
[v, \varphi^{(1)}, \vartheta^{(1)}](x_h, x_3) = [v_h(x_h), 0, \varphi^{(1)}(x_h), \vartheta(x_h)], \quad x_h = (x_1, x_2), \quad v_h = (v_1, v_2) = \nabla_h q
\]
with
\[
q = q(t, x_h) = \alpha \varphi^{(1)}(t, x_h) + \beta \vartheta^{(1)}(t, x_h).
\]
Moreover, the couple
\[ \left( q, \theta = \frac{\alpha \delta}{\delta \alpha + \beta^2} q^{(1)} + \frac{\alpha \beta}{\delta \alpha + \beta^2} \right) \]
belongs to the class (2.18), (2.21) and solves (in the classical sense) the two-dimensional geostrophic Euler-Boussinesq system (1.12), (1.13), (1.14), with initial data (2.17), (2.20), where \( q_0 \) solves (3.5) and
\[ \theta_0 = \frac{2}{\alpha \delta + \beta^2} \int_0^1 q^{(1)} dx_3 - \frac{\alpha \beta}{\alpha \delta + \beta^2} \int_0^1 \theta^{(1)} dx_3. \] (3.10)

The rest of the paper is devoted to the proof of Theorem 3.1.

4. Uniform bounds. In this section, we derive the uniform bounds on the family of dissipative weak solutions \( (\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon) \) independent of the scaling parameter \( \varepsilon \to 0 \).

4.1. Energy bounds. Taking \( r = \bar{\varrho}, \Theta = \bar{\vartheta}, \mathbf{U} = 0 \) as test functions in the relative energy inequality (2.16) we obtain
\[ \int_\Omega \left[ \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} E(\varrho_\varepsilon, \vartheta_\varepsilon, \bar{\varrho}, \bar{\vartheta}) \right] dx + \int_0^T \int_\Omega \frac{1}{\varrho_\varepsilon} \left( \varepsilon^a S(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon - \varepsilon^{b-2} q(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon \right) dx dt \leq \int_\Omega \left[ \frac{1}{2} q_0 \varrho_\varepsilon |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2} E(\varrho_0, \vartheta_0, \bar{\varrho}, \bar{\vartheta}) \right] dx + \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x G \cdot \mathbf{u}_\varepsilon \ dx \ dt \ dx \]
for a.a. \( \tau \in (0, T) \).

In accordance with the structural properties of the thermodynamic functions imposed through (2.1 - 2.7), the relative energy function enjoys the following properties: For any compact \( N \subset N' \subset (0, \infty)^2 \) there exists a strictly positive constant \( c = c(N, N') \), depending only on \( N, N' \) and the structural properties of \( P \), such that for any \( (r, \Theta) \in N' \),
\[ E(\varrho, \vartheta) r, \Theta) \geq c \left( |\varrho - r|^2 + |\vartheta - \Theta|^2 \right) \] if \( (\varrho, \vartheta) \in N' \),
\[ E(\varrho, \vartheta) r, \Theta) \geq c \left( 1 + \varrho^{5/3} + \vartheta^4 \right) \] if \( (\varrho, \vartheta) \in (0, \infty)^2 \setminus N' \). (4.3)

Similarly to [13, Chapter 4.7], we introduce a decomposition of a function \( h \):
\[ h = [h]_{\text{ess}} + [h]_{\text{res}} \]
for a measurable function \( h \),
where
\[ [h]_{\text{ess}} = \int 1_{\{\bar{\varrho}/2 < \varrho < \bar{\varrho}, \bar{\vartheta}/2 < \vartheta < 2\bar{\vartheta}\}}, [h]_{\text{res}} = h - h_{\text{ess}}. \]

Thanks to the hypotheses (3.2), (3.3), the first integral including the initial data on the right-hand side of (4.1) remains bounded uniformly for \( \varepsilon \to 0 \). We remark that
\[ \left| \int_\Omega \varrho_\varepsilon \nabla_x G \cdot \mathbf{u}_\varepsilon \ dx \right| \leq c \int_\Omega \left( \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \left( [\varrho_\varepsilon]_{\text{res}}^{5/3} + 1 \right) \sum \right) dx; \]
whence, the right time space integral in (4.1) can be “absorbed” by means of a Gronwall-type argument.

Combining (4.1), (4.2), (4.3) with the hypotheses (2.1 - 2.10) we deduce the following estimates:
\[ \text{ess sup}_{t \in (0, T)} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \leq c, \] (4.4)
Now, the bound (4.9) involving \( \vartheta \) follows immediately from (4.11) and (4.5), (4.6). To get (4.9) with \( \log \vartheta - \log \vartheta_{\text{res}} \), we take in (4.10), \( f = \log \vartheta - \log \vartheta_{\text{res}} \). 

\[ \frac{\vartheta}{\varepsilon} \rightarrow {0} \text{ in } L^{\infty}(0, T; L^{5/3}(\Omega)), \quad \frac{\vartheta - \bar{\vartheta}}{\varepsilon} \rightarrow {0} \text{ in } L^{\infty}(0, T; L^{4}(\Omega)). \]  

(4.12)

\[ \frac{\vartheta_{\text{res}}}{\varepsilon} \rightarrow {\vartheta^{(1)}} \text{ weakly-* in } L^{\infty}(0, T; L^{2}(\Omega)), \quad \frac{\vartheta - \bar{\vartheta}}{\varepsilon} \rightarrow {\vartheta} \text{ weakly-* in } L^{\infty}(0, T; L^{2}(\Omega)). \]  

(4.13)

Now, we use (4.4–4.6) to define \( \mathbf{u} \)

\( \vartheta \mathbf{u} \rightarrow \bar{\vartheta} \mathbf{u} \text{ weakly-* in } L^{\infty}(0, T; (L^{2} + L^{5/4})(\Omega; \mathbb{R}^{3})); \) 

(4.14)
Finally, multiplying the momentum balance (2.14) by \( \varepsilon \) and passing to the limit \( \varepsilon \to 0 \) we deduce
\[
\overline{\nabla} \cdot z = -f \times \mathbf{u}, \quad \mathbf{u} = [u_h(t, x_h), 0], \quad z = z(t, x_h), \quad z = \alpha \varphi^{(1)} + \beta \theta,
\]
where we have used also (4.12–4.14).

4.3. Preparation of the initial data. Following [18], we start by introducing suitable smoothing operators imposed on the initial data. Taking a family of smooth functions parameterized by \( \eta \in (0, 1) \)
\[
\psi_\eta \in C^{\infty}_c(0, \infty), 0 \leq \psi_\eta \leq 1, \quad \psi_\eta \to 1 \text{ as } \eta \to 0,
\]
and
\[
\phi_\eta = \phi_\eta(x_h) \in C^{\infty}_c(\mathbb{R}^2), \quad 0 \leq \phi_\eta \leq 1, \quad \phi_\eta \to 1 \text{ as } \eta \to 0,
\]
we introduce
\[
\left[ \phi_0^{(1)} \right]_\eta (x_h, x_3) = \sum_{|k| \leq 1/\eta} F_{\zeta \to x_h}^{-1} \left[ \psi_\eta(|\zeta|) \left( \hat{\phi}_0^{(1)} \right)(\zeta, k) \right] \exp(2\pi ikx_3),
\]
\[
\left[ \varphi_0^{(1)} \right]_\eta (x_h, x_3) = \sum_{|k| \leq 1/\eta} F_{\zeta \to x_h}^{-1} \left[ \psi_\eta(|\zeta|) \left( \hat{\varphi}_0^{(1)} \right)(\zeta, k) \right] \exp(2\pi ikx_3),
\]
and, similarly,
\[
[u_0 \mathbf{e}_{x}]_\eta (x_h, x_3) = \sum_{|k| \leq 1/\eta} F_{\zeta \to x_h}^{-1} \left[ \psi_\eta(|\zeta|) \left( \hat{u_0 \mathbf{e}_{x}} \right)(\zeta, k) \right] \exp(2\pi ikx_3).
\]

Now, we define \( \xi_{0,\varepsilon,\eta} \) and \( q_{0,\varepsilon,\eta} \) by setting
\[
\alpha \left[ \phi_0^{(1)} \right]_\eta + \beta \left[ \varphi_0^{(1)} \right]_\eta = \xi_{0,\varepsilon,\eta} - q_{0,\varepsilon,\eta}
\]
where \( q_{0,\varepsilon,\eta}(x) = q_{0,\varepsilon,\eta}(x_h) \) solves,
\[
-\Delta_h q_{0,\varepsilon,\eta} + q_{0,\varepsilon,\eta} = \int_1^1 \text{curl}_h [u_0 \mathbf{e}_{x}]_\eta \, dx_3 + \int_0^1 \left( \left[ \alpha \phi_0^{(1)} \right]_\eta + \beta \left[ \varphi_0^{(1)} \right]_\eta \right) \, dx_3
\]
and \( v_{0,\varepsilon,\eta}(x) = [v_h \mathbf{e}_{x,\eta} + v_{0,\varepsilon,\eta}] \), with \( [v_{0,\varepsilon,\eta}]_1 = -\partial_{x_2} q_{0,\varepsilon,\eta}, \quad [v_{0,\varepsilon,\eta}]_2 = \partial_{x_1} q_{0,\varepsilon,\eta} \).

Employing the elementary properties of the Fourier transform, the Marcinkiewicz theorem about multipliers, and Sobolev imbeddings, it is not difficult to see that the couple \( (\xi_{0,\varepsilon,\eta}, \mathbf{V}_{0,\varepsilon,\eta}) \) satisfies all assumptions of Lemma 2.1 imposed on the initial data, and that \( [q_{0,\varepsilon,\eta}, v_{0,\varepsilon,\eta}] \in \mathcal{K}^1(\mathbb{H}) \). Moreover, in view of (3.2–3.4),
\[
(1 + |x|^\gamma) \left[ \phi_0^{(1)} \right]_\eta \to (1 + |x|^\gamma) \left[ \phi_0^{(1)} \right]_\eta, \quad (1 + |x|^\gamma) \left[ \varphi_0^{(1)} \right]_\eta \to (1 + |x|^\gamma) \left[ \varphi_0^{(1)} \right]_\eta \text{ in } W^{l,q}(\Omega),
\]
\[
(1 + |x|^\gamma) \left[ u_0 \mathbf{e}_{x} \right]_\eta \to (1 + |x|^\gamma) \left[ u_0 \mathbf{e}_{x} \right]_\eta \text{ in } W^{l,q}(\Omega; \mathbb{R}^3), \quad 1 < q \leq \infty, \quad l, r, 0 = 1, \ldots
\]
as \( \varepsilon \to 0 \) when \( \eta \) is fixed.

Coming back to equation (4.19), we deduce by the same token as those of (4.21),
\[
(1 + |x|^\gamma) q_{0,\varepsilon,\eta} \to (1 + |x|^\gamma) q_0 \text{ in } W^{l,q}(\mathbb{R}^2), \quad 1 < q \leq \infty, \quad r, l = 0, 1, \ldots
\]

Before letting \( \eta \to 0 \), we recall the Parseval inequality for \( \hat{g} \),
\[
\| g \|_{L^2(\Omega)}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} |\hat{g}(\zeta, k)|^2 \, d\zeta
\]
and realize that
\[ \|\partial_r g\|_{L^2_r(\Omega)}^2 = \sum_{k \in \mathcal{Z}} \int_{\mathbb{R}^2} (2\pi k)^{2r} |\hat{g}(\zeta, k)|^2 d\zeta, \quad r = 1, 2, \ldots. \]

Employing these facts together with assumptions (3.2–3.4), we get
\[ \left[ \varrho_0^{(1)} \right]_\eta \to \varrho_0^{(1)}, \quad \left[ \varrho_0^{(1)} \right]_\eta \to \varrho_0^{(1)} \text{ in } W^{1,q}(\Omega), \quad 2 \leq q < \infty \quad (4.23) \]
\[ \left[ \varrho_0^{(1)} \right]_\eta \to \varrho_0^{(1)} \text{ in } W^{k-1,2}(\Omega), \quad \left[ q_0 \right]_\eta \to \varrho_0, \text{ in } W^{k,2}(\Omega), \quad \text{where } k \geq 4 \]
as \eta \to 0.

5. **Designing the test functions in the relative energy inequality.** Our aim is to employ the relative energy inequality (2.16) to prove the convergence of the sequence of weak dissipative solutions \([\varrho_\varepsilon, \vartheta_\varepsilon, u_\varepsilon]\) of the problem (1.2–1.10) to the corresponding strong solution of the target system (1.12–1.14) with the initial data (3.5), (3.10). To this end, we choose in (2.16) the test functions \([r, \Theta, U]\) in the following way:

\[ r = r_{\varepsilon, \eta} = \overline{r} + \varepsilon R_{\varepsilon, \eta}, \quad \Theta = \Theta_{\varepsilon, \eta} = \overline{\Theta} + \varepsilon T_{\varepsilon, \eta}, \quad U = U_{\varepsilon, \eta} = v_{\varepsilon, \eta} + V_{\varepsilon, \eta}; \quad (5.1) \]

where the trio \((R_{\varepsilon, \eta}, T_{\varepsilon, \eta}, U_{\varepsilon, \eta})\) is determined as follows.

- (i) First, we solve in \([0, T] \times \mathbb{R}^2\) the 2-D quasi-geostrophic Euler equation for streamlines

\[ \partial_t \left( \Delta_h q_{\varepsilon, \eta} - \frac{\delta}{\beta^2 + \alpha \delta} q_{\varepsilon, \eta} \right) + \nabla_h^\perp q_{\varepsilon, \eta} \cdot \nabla_h \left( \Delta_h q_{\varepsilon, \eta} - \frac{\delta}{\beta^2 + \alpha \delta} q_{\varepsilon, \eta} \right) = 0 \quad (5.2) \]

where the initial condition \(q_{0, \varepsilon, \eta}\) is defined in (4.19) and numbers \(\alpha, \beta, \delta\) in (1.11). We set

\[ v_{\varepsilon, \eta} = [\nabla_h^\perp q_{\varepsilon, \eta}, 0]. \quad (5.3) \]

- (ii) Second, we solve in \([0, T] \times \Omega\) the wave equation for the Poincaré waves

\[ \varepsilon \partial_t^2 \xi_{\varepsilon, \eta} + \omega \text{div} V_{\varepsilon, \eta} = 0, \quad (5.4) \]

\[ \varepsilon \partial_t^2 V_{\varepsilon, \eta} + f \times V_{\varepsilon, \eta} + \nabla \xi_{\varepsilon, \eta} = 0 \]

\[ \xi_{\varepsilon, \eta}(0, \cdot) = \xi_{0, \varepsilon, \eta}(\cdot), \quad V_{\varepsilon, \eta}(0, \cdot) = V_{0, \varepsilon, \eta}(\cdot), \]

where the positive number \(\omega\) is defined in (1.11) and the initial conditions \(\xi_{0, \varepsilon, \eta}, V_{0, \varepsilon, \eta}\) are defined in (4.18–4.20).

- (iii) Third, we solve in \([0, T] \times \Omega\) the transport equation

\[ \partial_t Z_{\varepsilon, \eta} + v_{\varepsilon, \eta} \cdot \nabla_x Z_{\varepsilon, \eta} = 0 \quad (5.5) \]

\[ Z_{\varepsilon, \eta}(0) = Z_{0, \varepsilon, \eta} = [\delta \varrho_{0, \varepsilon, \eta}^{(1)} - \beta \varrho_{0, \varepsilon, \eta}^{(1)}], \]

where \(\varrho_{0, \varepsilon, \eta}^{(1)}, \vartheta_{0, \varepsilon, \eta}^{(1)}\) are specified in (4.16) and

\[ v_{\varepsilon, \eta}(x_h) = \nabla^\perp q_{\varepsilon, \eta}(x_h) = [-\partial_{x_2} q_{\varepsilon, \eta}(x_h), \partial_{x_1} q_{\varepsilon, \eta}(x_h), 0]. \quad (5.6) \]

- (v) Finally \(R_{\varepsilon, \eta}, T_{\varepsilon, \eta}\) will be determined from the linear algebraic system

\[ \alpha R_{\varepsilon, \eta} + \beta T_{\varepsilon, \eta} - q_{\varepsilon, \eta} = \xi_{\varepsilon, \eta}, \quad \delta T_{\varepsilon, \eta} - \beta R_{\varepsilon, \eta} = Z_{\varepsilon, \eta}. \quad (5.7) \]
In what follows, in order to simplify notation, we write simply
\[ \xi_{0,\varepsilon}, V_{0,\varepsilon}, q_{0,\varepsilon}, \theta_{0,\varepsilon}, v_{0,\varepsilon} \] instead of \( \xi_{0,\varepsilon,\eta}, V_{0,\varepsilon,\eta}, q_{0,\varepsilon,\eta}, \theta_{0,\varepsilon,\eta}, v_{0,\varepsilon,\eta} \),
\[ \xi_{\varepsilon}, V_{\varepsilon}, q_{\varepsilon}, Z_{\varepsilon}, R_{\varepsilon}, T_{\varepsilon}, v_{\varepsilon} \] instead of \( \xi_{\varepsilon,\eta}, V_{\varepsilon,\eta}, q_{\varepsilon}, Z_{\varepsilon,\eta}, R_{\varepsilon,\eta}, T_{\varepsilon,\eta}, v_{\varepsilon,\eta} \),
where \( \eta > 0 \) is kept fixed.

5.1. **Properties of solutions of the wave equation.** We turn our attention to the wave equation (5.4). We observe by virtue of (4.20), (4.21) and (4.22) that its initial data satisfy all assumptions of Lemma 2.1. Employing this lemma we deduce regularity and dispersive properties of its solutions, namely
\[
\| \xi_{\varepsilon} \|_{L^\infty (0,T; W^{l,q}(\Omega))} \leq c(\eta), \quad \| V_{\varepsilon} \|_{L^\infty (0,T; W^{l,q}(\Omega))} \leq c(\eta), \quad 2 \leq q \leq \infty, \quad l = 0, 1, \ldots,
\]
\[
\| \xi_{\varepsilon} \|_{L^p (0,T; W^{l,q}(\Omega))} \rightarrow 0, \quad \| V_{\varepsilon} \|_{L^p (0,T; W^{l,q}(\Omega))} \rightarrow 0, \quad 2 < q \leq \infty, \quad 1 \leq p < \infty, \quad l = 0, 1, \ldots
\]
as \( \varepsilon \rightarrow 0 \), where we have used interpolation to obtain the last line. Recall finally the energy conservation for the wave equation (5.4) that reads
\[
\| \xi_{\varepsilon}(t) \|_{L^2(\Omega)}^2 + \| V_{\varepsilon}(t) \|_{L^2(\Omega)}^2 = \| \xi_{0,\varepsilon} \|_{L^2(\Omega)}^2 + \| V_{0,\varepsilon} \|_{L^2(\Omega)}^2
\]
for all \( t \in [0,T] \).

5.2. **Properties of the geostrophic approximation of the Euler equation.** We turn our attention to equation (5.2). Taking into account Proposition 2.1, we deduce the energy identity:
\[
\int_{\mathbb{R}^2} \left( \frac{1}{\theta} \frac{\delta}{\beta^2 + \alpha \delta} |q_{\varepsilon}(t, \cdot)|^2 + |\nabla_h q_{\varepsilon}(t, \cdot)|^2 \right) dx = \int_{\mathbb{R}^2} \left( \frac{1}{\theta} \frac{\delta}{\beta^2 + \alpha \delta} |q_{0,\varepsilon}|^2 + |\nabla_h q_{0,\varepsilon}|^2 \right) dx
\]
Further, from the estimate in Proposition 2.1 and equation (5.2), we deduce bounds,
\[
\| q_{\varepsilon}, \partial_t q_{\varepsilon} \|_{L^\infty (0,T; W^{l,q}(\mathbb{R}^2))} \leq c(\eta), \quad l = 0, 1, \ldots, \quad 1 < q \leq \infty.
\]
Consequently, we have the convergence
\[
q_{\varepsilon} \rightarrow q_\eta \text{ in } C^1([0,T], W^{l,q}(\mathbb{R}^2)), \quad l = 0, 1, \ldots, \quad 1 < q \leq \infty,
\]
where \( q_\eta \) solves the geostrophic Euler equation (1.13) with initial conditions \( [q_0]_\eta \). Moreover, \( q_\eta \) satisfies energy identity (5.11).
Since (4.23) holds, we deduce from (5.12)
\[
g_\eta \rightarrow q \text{ in } C([0,T]; W^{k-1,2}(\mathbb{R}^2) \text{ and } \ast\text{-weakly in } L^\infty(0,T; W^{k,2}(\mathbb{R}^2)),
\]
\[
\partial_0 q_\eta \rightarrow \partial_0 q \text{ in } C([0,T]; W^{k-2,2}(\mathbb{R}^2) \text{ and } \ast\text{-weakly in } L^\infty(0,T; W^{k-1,2}(\mathbb{R}^2)),
\]
as \( \eta \rightarrow 0^+ \), where \( q \) is a unique classical solution of the geostrophic approximation of the Euler system (1.13) in the class (2.21) with initial condition \( q_0 \) defined in (3.5).
5.3. Properties of the solutions to the transport equation. For fixed $\eta > 0$, the initial data for the transport equation (5.5) belong to $Z_{0,\varepsilon} \in W^{l,q}(\Omega)$, $l = 0, 1, \ldots, 1 \leq q \leq \infty$, and by virtue of (5.12) the transport vector field obeys estimates

$$\|v_x\|_{C([0,T];W^{l,q}(\Omega))} \leq c(\eta), \quad l = 0, 1, \ldots, 2 \leq q \leq \infty$$

uniformly with $\varepsilon$. The transport equation therefore yields

$$\|\partial_t Z_{\varepsilon}, Z_{\varepsilon}\|_{L^\infty([0,T];W^{l,q}(\Omega))} \leq c(\eta), \quad l = 0, 1, \ldots, 1 \leq q \leq \infty,$$

and the family

$$\{Z_{\varepsilon}\}_{\varepsilon > 0}$$

is precompact in $C([0,T];W^{l,q}(\Omega))$, $l = 0, 1, \ldots, 1 \leq q \leq \infty$. Recalling the definition of $R_{\varepsilon}$, $T_{\varepsilon}$ via $\xi_{\varepsilon}$, $q_{\varepsilon}$ and $Z_{\varepsilon}$, see (5.7), we deduce from the dispersive estimates (5.9) of $\xi_{\varepsilon}$, precompactness (5.16) of $Z_{\varepsilon}$ and convergence (5.13) of $q_{\varepsilon}$ that

$$Z_{\varepsilon} \to Z_{\eta}$$

strongly in $C([0,T];W^{l,q}([0,T];W^{l,q}(\Omega)))$, $l = 0, 1, \ldots, 1 \leq q \leq \infty$, (5.17)

$$(R_{\varepsilon}, T_{\varepsilon}) \to (R_{\eta}, T_{\eta})$$

in $L^p(0,T;W^{l,q}(\Omega;\mathbb{R}^2))$, $l = 0, 1, \ldots, 2 < q \leq \infty$, $1 \leq p < \infty$, (5.18)

where $Z_{\eta}$ is the unique solution of the transport equation

$$\partial_t Z_{\eta} + v_{\eta} \cdot \nabla_x Z_{\eta} = 0, \quad Z_{\eta}(0,\cdot) = [Z_0]_{\eta} = \delta[\vartheta^{(1)}_0]_{\eta} - \beta[\varrho^{(1)}_0]_{\eta},$$

and, due to (5.7), (5.9), (5.13), (5.17), (5.18),

$$\alpha R_{\eta} + \beta T_{\eta} - q_{\eta} = 0, \quad Z_{\eta} = \delta T_{\eta} - \beta R_{\eta}.$$ (5.20)

Employing (5.20), the transport equation (5.19) rewrites

$$\partial_t T_{\eta} + v_{\eta} \cdot \nabla_x T_{\eta} = \frac{\beta}{\alpha + \beta^2} \partial_t q_{\eta}, \quad \frac{\delta\alpha + \beta^2}{\alpha} T_{\eta}(0,\cdot) = \delta[\vartheta^{(1)}_0]_{\eta} - \beta[\varrho^{(1)}_0]_{\eta} + \frac{\beta}{\alpha} q_{\eta}.$$ (5.21)

We deduce from this equality and from (4.23) the bounds

$$\|\partial_t T_{\eta}\|_{L^\infty([0,T];L^q(\Omega))} \leq c, \quad \|T_{\eta}\|_{L^\infty([0,T];W^{l,q}(\Omega))} \leq c, \quad 2 \leq q < \infty$$

uniformly with respect to $\eta \in (0,1)$; whence

$$T_{\eta} \to T$$

in $C([0,T],L^2(\Omega))$ and $^*-$weakly in $L^\infty(0,T;W^{1,2}(\Omega))$ as $\eta \to 0$, (5.22)

where $T$ satisfies the transport equation

$$\partial_t T + v \cdot \nabla_x T = \frac{\beta}{\alpha + \beta^2} \partial_t q, \quad \frac{\delta\alpha + \beta^2}{\alpha} T(0,\cdot) = \delta[\vartheta^{(1)}_0] - \beta[\varrho^{(1)}_0] + \frac{\beta}{\alpha} q_0.$$ (5.23)

where $v_{\eta}(t,x) = [\nabla^T q_{\eta},0]$.

Finally, we set

$$\theta(t,x_h) = \int_0^1 T(t,x_h,x_3)dx_3 - \frac{\beta}{\delta\alpha + \beta^2} q(t,x_h)$$ (5.24)

and remark that

$$\partial_t \theta + v_h \cdot \nabla_x \theta = 0, \quad \theta(0) = \frac{\alpha}{\delta\alpha + \beta^2} \left( \delta \int_0^1 v^{(1)}_0 dx_3 - \beta \int_0^1 e^{(1)}_0 dx_3 \right),$$ (5.25)

where $v_h = \nabla^T q$ and by virtue of what was recalled in Propositions 2.1--2.2, $\theta$ belongs to the class $C^{(2.21)}$. 
6. **Convergence to the target system.** In this section, we use the test functions (5.1) in the relative energy inequality (2.16). Fixing \( \eta > 0 \) we perform the limit for \( \varepsilon \to 0 \). This will be carried over in several steps in the spirit of [15]. We omit the subscript \( \eta \) whenever no confusion arises.

6.1. **Viscous and heat conducting terms.** We show by direct calculation, splitting the terms in their essential and residual parts and using assumptions (2.9–2.10), uniform bounds (4.5–4.11), regularity (2.18), estimates (5.9), (5.10), (5.15), and definition of \([R_e, \Theta_e, U_e]\), cf. (5.1), (5.3), (5.7), that the dissipative terms related to the viscosity and to the heat conductivity on the right-hand side of (2.16) become negligible as \( \varepsilon \to 0 \). More precisely:

\[
\varepsilon^\alpha S(\partial \varepsilon, \nabla_x u_e) : \nabla_x U_e \to 0 \text{ in } L^2((0, T) \times \Omega) + L^2(0, T; [L^1(\Omega)]^3) \text{ as } \varepsilon \to 0,
\]
and

\[
\varepsilon^{b-2} q(\partial \varepsilon, \nabla_x \Theta_e) \to 0 \text{ in } L^2((0, T) \times \Omega) + L^1((0, T) \times \Omega) \text{ as } \varepsilon \to 0.
\]

Consequently, the relative energy inequality (2.16) reduces to

\[
\left[ E_\varepsilon \left( \varrho \varepsilon, \varrho \varepsilon, u_e \right| r_e, \Theta_e, U_e \right]_{t=0}^T \leq \int_0^T \int_\Omega \left( \varrho \varepsilon \partial_t U_e + \varrho \varepsilon u_e \cdot \nabla_x U_e \right) \cdot (U_e - u_e) \, dx \, dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \frac{r_e - \varrho \varepsilon}{r_e} U_e \cdot \nabla_x p(r_e, \Theta_e) \, dx \, dt + \int_0^T \int_\Omega \varrho \varepsilon \left( f \cdot u_e \right) \cdot (U_e - u_e) \, dx \, dt
\]

\[
- \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \varrho \varepsilon \nabla_x p(r_e, \Theta_e) \cdot (U_e - u_e) \, dx \, dt - \int_0^T \int_\Omega \varrho \varepsilon \nabla_x (U_e - u_e) \, dx \, dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \frac{r_e - \varrho \varepsilon}{r_e} \varrho \varepsilon \partial_t p(r_e, \Theta_e) \, dx \, dt + \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left( p(r_e, \Theta_e) - \frac{p(\varrho \varepsilon, \varrho \varepsilon)}{\varrho \varepsilon} \right) \div U_e \, dx \, dt
\]

\[
- \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left[ \varrho \varepsilon \left( s(\varrho \varepsilon, \varrho \varepsilon) - s(r_e, \Theta_e) \right) \partial_t \Theta_e + \varrho \varepsilon \left( s(\varrho \varepsilon, \varrho \varepsilon) - s(r_e, \Theta_e) \right) u_e \cdot \nabla_x \Theta_e \right] \, dx \, dt + \chi(\varepsilon, \eta),
\]

where \( \chi \) denotes a generic function satisfying

\[
\lim_{\eta \to 0} \left( \lim_{\varepsilon \to 0} \chi(\varepsilon, \eta) \right) = 0.
\]

6.2. **Velocity convection.** In this section, we shall handle the expression

\[
\int_0^T \int_\Omega \left[ \varrho \varepsilon (U_e - u_e) \cdot \partial_t U_e + \varrho \varepsilon u_e \cdot \nabla_x U_e \cdot (U_e - u_e) \right] \, dx \, dt
\]

\[
= \int_0^T \int_\Omega \varrho \varepsilon (U_e - U_e) \cdot \nabla_x U_e \cdot (U_e - u_e) \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \left[ \varrho \varepsilon (U_e - u_e) \cdot \partial_t U_e + \varrho \varepsilon U_e \cdot \nabla_x U_e \cdot (U_e - u_e) \right] \, dx \, dt
\]

\[
\leq c \int_0^T \mathcal{E}(\varrho \varepsilon, \varrho \varepsilon, u_e \big| r_e, \Theta_e, U_e) \, dt
\]

\[
+ \int_0^T \int_\Omega \varrho \varepsilon \partial_t U_e \cdot (U_e - u_e) \, dx \, dt + \int_0^T \int_\Omega \varrho \varepsilon U_e \cdot \nabla_x U_e \cdot (U_e - u_e) \, dx \, dt.
\]

Using Lemma 2.1 and relations (4.5), (4.6), (4.12), (5.9), we verify that:
YOUNG-SAM KWON AND ANTONIN NOVOTNY

By virtue of (2.19), (2.22) (4.5), (4.12), (5.1), (5.7), (5.9), (5.10), (5.12), (5.15) we find

\[ \int_0^T \int_\Omega \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon} \nabla_x p(r_\varepsilon, \Theta_\varepsilon) \cdot \nabla_x U_\varepsilon \, dx \, dt = \left( \frac{r_\varepsilon - \varrho_\varepsilon}{\varepsilon} \right) \frac{1}{\varrho + \varepsilon R_\varepsilon} U_\varepsilon \cdot \nabla_x q_\varepsilon = V_\varepsilon \cdot \nabla_x q_\varepsilon. \]

Consequently, by virtue of (2.19), (2.22) (4.5), (4.12), (5.1), (5.7), (5.9), (5.10), (5.12), (5.15) we find

\[ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon} U_\varepsilon \cdot \nabla_x p(r_\varepsilon, \Theta_\varepsilon) \, dx \, dt = \chi(\varepsilon, \eta). \]

**6.3. Force terms.**

6.3.1. Vanishing pressure term. In this section we consider the term

\[ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon} U_\varepsilon \cdot \nabla_x p(r_\varepsilon, \Theta_\varepsilon) \, dx \, dt, \]

where

\[ \nabla_x p(r_\varepsilon, \Theta_\varepsilon) = \varepsilon \left[ \left( \partial_{\varrho} p(r_\varepsilon, \Theta_\varepsilon) - \partial_{\varrho} p(\varrho, \varrho) \right) \nabla_x R_\varepsilon + \left( \partial_{\varrho} p(r_\varepsilon, \Theta_\varepsilon) - \partial_{\varrho} p(\varrho, \varrho) \right) \nabla_x T_\varepsilon \right] + \varepsilon \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon - q_\varepsilon) + \varrho \nabla_x q_\varepsilon. \]

We realize that

\[ \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon} = \varepsilon \left( R_\varepsilon - \frac{\varrho_\varepsilon - \varrho}{\varepsilon} \right) \frac{1}{\varrho + \varepsilon R_\varepsilon} U_\varepsilon \cdot \nabla_x q_\varepsilon = V_\varepsilon \cdot \nabla_x q_\varepsilon. \]
6.3.2. Balance of pressure, gravity and Coriolis forces. According to (4.12), (4.14), (5.3), (5.9) and (5.13),
\[
\int_0^\tau \int_\Omega \varrho_e \nabla_x G \cdot (U_\varepsilon - u_\varepsilon) \, dx \, dt \to \int_0^\tau \int_\Omega \varrho \nabla_x G \cdot (v_\eta - u) \, dx \, dt = 0, \quad v_\eta = [\nabla_h q_\varepsilon, 0].
\]
By the same token as in Section 6.3.1, we therefore transform the pressure/gravity term as follows:
\[
\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho_e \nabla_x p(r_\varepsilon, \Theta_\varepsilon) \cdot (U_\varepsilon - u_\varepsilon) \, dx \, dt - \int_0^\tau \int_\Omega \varrho_e \nabla_x G \cdot (U_\varepsilon - u_\varepsilon) \, dx \, dt \quad (6.8)
\]
\[
= \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho_e \frac{\nabla_x p(r_\varepsilon, \Theta_\varepsilon)}{r_\varepsilon} \, dx \, dt + \chi(\varepsilon, \eta)
\]
\[
= \int_0^\tau \int_\Omega \bar{\varrho}(-\partial_t V_\varepsilon - \frac{1}{\varepsilon}(f \times V_\varepsilon) - \frac{1}{\varepsilon}(f \times v_\varepsilon)) \cdot (U_\varepsilon - u_\varepsilon) \, dx \, dt + \chi(\varepsilon, \eta)
\]
\[
= \int_0^\tau \int_\Omega \bar{\varrho}(-\partial_t V_\varepsilon \cdot V_\varepsilon - \partial_t V_\varepsilon + \partial_t \varrho \cdot V_\varepsilon) \cdot u_\varepsilon \, dx \, dt \quad (6.9)
\]
where we have used (5.4), (5.6), (5.9) to deduce the last two lines.

Finally, the Coriolis force in the relative energy inequality (6.6) reads
\[
\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho_e (f \times u_\varepsilon) \cdot (U_\varepsilon - u_\varepsilon) \, dx \, dt = -\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \bar{\varrho} (f \times U_\varepsilon) \cdot u_\varepsilon \, dx \, dt + \chi(\varepsilon, \eta). \quad (6.9)
\]
Taking into account the results (6.3–6.5) and (6.7–6.9) we can rewrite inequality (6.6) in the form
\[
\left[\begin{array}{c} \varepsilon \left( \varrho_e, \varrho_e, u_\varepsilon \right) \end{array}\right]_{t=0} ^\tau \leq \int_0^\tau \int_\Omega \bar{\varrho} \left( \partial_t v_\varepsilon + \nabla_x U_\varepsilon \right) \cdot (v_\varepsilon - u_\varepsilon) \, dx \, dt
\]
\[
+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \partial_t r_\varepsilon \partial_t p(r_\varepsilon, \Theta_\varepsilon) \, dx \, dt
\]
\[
+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \partial_t \left( p(r_\varepsilon, \Theta_\varepsilon) - \partial_t p(\varrho_e, \varrho_e) \right) \text{div} U_\varepsilon \, dx \, dt
\]
\[
- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left( \varrho_e \left( s(\varrho_e, \varrho_e) - s(r_\varepsilon, \Theta_\varepsilon) \right) \right) \partial_t \Theta_\varepsilon + \varrho_e \left( \varrho_e - s(r_\varepsilon, \Theta_\varepsilon) \right) U_\varepsilon \cdot \nabla_x \Theta_\varepsilon \, dx \, dt + \chi(\varepsilon, \eta).
\]

6.4. Balance of entropy and pressure.

6.4.1. Replacing velocity in the entropy convective term. Our intention in this section is to “replace” u_\varepsilon by U_\varepsilon in the remaining (last) convective term in (6.10). To this end, we write
\[
\int_0^\tau \int_\Omega \varrho_e \left( s(\varrho_e, \varrho_e) - s(r_\varepsilon, \Theta_\varepsilon) \right) \frac{U_\varepsilon \cdot \nabla_x \Theta_\varepsilon}{\varepsilon^2} \, dx \, dt
\]
\[
= \int_0^\tau \int_\Omega \varrho_e \left( s(\varrho_e, \varrho_e) - s(r_\varepsilon, \Theta_\varepsilon) \right) U_\varepsilon \cdot \nabla_x T_\varepsilon \, dx \, dt + \int_0^\tau \int_\Omega \varrho_e \left( s(\varrho_e, \varrho_e) - s(r_\varepsilon, \Theta_\varepsilon) \right) \left( U_\varepsilon - U_\varepsilon \right) \cdot \nabla_x T_\varepsilon \, dx \, dt,
\]
where
\[
\left| \int_0^\tau \int_\Omega \varrho_e \left( s(\varrho_e, \varrho_e) - s(r_\varepsilon, \Theta_\varepsilon) \right) \frac{U_\varepsilon - U_\varepsilon \cdot \nabla_x T_\varepsilon}{\varepsilon} \, dx \, dt \right|
\]
\[ \leq C \int_0^T \| \nabla_x T_\epsilon \|_{L^\infty(\Omega)} \int_\Omega \left( g_\epsilon [u_\epsilon - U_\epsilon]^2 + \left[ \frac{\theta_\epsilon - T_\epsilon}{\epsilon} \right]_{ess}^2 + \left[ \frac{\varrho_\epsilon - \Theta_\epsilon}{\epsilon} \right]_{ess}^2 \right) \, dx \, dt \]

\[ \leq C \int_0^T \| \nabla_x T_\epsilon \|_{L^\infty(\Omega)} \mathcal{E}_\epsilon \left( \theta_\epsilon, \bar{\varrho}_\epsilon, u_\epsilon \bigg| r_\epsilon, \Theta_\epsilon, U_\epsilon \right) \, dt \]

and

\[ \int_0^T \int_\Omega \frac{s(\varrho_\epsilon, \theta_\epsilon) - s(r_\epsilon, \Theta_\epsilon)}{\epsilon} \left( u_\epsilon - U_\epsilon \right) \cdot \nabla_x T_\epsilon \, dx \, dt = \chi(\epsilon, \eta) \text{ provided } 0 < a < 10/3. \]

When estimating the essential part we have used relation (4.2). In order to estimate the residual part we have employed relations (2.8), (4.6), definition (5.7) and estimates (4.6–4.9) to treat \( \theta_\epsilon, \varrho_\epsilon \), estimates (5.9), (5.10), (5.11), (5.12), (5.15) to treat \( Z_\epsilon, R_\epsilon, T_\epsilon, \zeta_\epsilon, \varrho_\epsilon \) \( V_\epsilon \) and \( \varphi_\epsilon \) together with (4.8) and Sobolev imbedding to treat \( u_\epsilon \).

Finally, due to (5.18), (5.21)

\[ \| \nabla_x T_\epsilon \|_{L^\infty(\Omega)} \rightarrow \| \nabla_x T_\eta \|_{L^\infty(\Omega)} \text{ in } L^p(0, T), \quad 1 \leq p < \infty, \quad \text{as } \epsilon \to 0, \]

where

\[ \| \nabla_x T_\eta \|_{L^\infty(0, T; \Omega)} \leq c \text{ uniformly in } \eta \in (0, 1). \]

Consequently, we can rewrite the relative energy inequality inequality (6.10) in the form

\[ 0 \leq \int_0^T \int_\Omega \mathcal{E}_\epsilon \left( \theta_\epsilon, \bar{\varrho}_\epsilon, u_\epsilon \bigg| r_\epsilon, \Theta_\epsilon, U_\epsilon \right) \, dt \]

\[ \leq \int_0^T \int_\Omega \nabla(\varphi_\epsilon \nabla + v_\epsilon \cdot \nabla v_\epsilon) \cdot (v_\epsilon - u_\epsilon) \, dx \, dt \]

\[ + \frac{1}{\epsilon^2} \int_0^T \int_\Omega \frac{r_\epsilon - \theta_\epsilon}{r_\epsilon} \partial_t p(\epsilon, \Theta_\epsilon) \, dx \, dt \]

\[ + \frac{1}{\epsilon^2} \int_0^T \int_\Omega \left( p(r_\epsilon, \Theta_\epsilon) - p(\theta_\epsilon, \theta_\epsilon) \right) \text{div} U_\epsilon \, dx \, dt \]

\[ - \frac{1}{\epsilon^2} \int_0^T \int_\Omega \left[ s(\varrho_\epsilon, \theta_\epsilon) - s(r_\epsilon, \Theta_\epsilon) \right] \partial_t \Theta_\epsilon + \frac{1}{\epsilon} \left[ s(\varrho_\epsilon, \theta_\epsilon) - s(r_\epsilon, \Theta_\epsilon) \right] U_\epsilon \cdot \nabla \Theta_\epsilon \, dx \, dt + \chi(\epsilon, \eta) \]

\[ + c \int_0^T \mathcal{E}_\epsilon \left( \theta_\epsilon, \bar{\varrho}_\epsilon, u_\epsilon \bigg| r_\epsilon, \Theta_\epsilon, U_\epsilon \right) \, dt + \chi(\epsilon, \eta). \]

Indeed, in the residual component, the most difficult term in the passage to the limit is the term containing \([g_\epsilon s(\varrho_\epsilon, \theta_\epsilon)]\)\( \text{res} u_\epsilon \cdot \nabla_x T_\epsilon \). Recalling (2.8), its treatment amounts to estimate the terms \([\varrho_\epsilon^2]_{\text{res} u_\epsilon} \), \([g_\epsilon \log \varrho_\epsilon]_{\text{res} u_\epsilon} \), \([g_\epsilon [\log \varrho_\epsilon]^+]_{\text{res} u_\epsilon} \) in \( L^1((0, T) \times \Omega) \)-norm. To this end, we have, by virtue of the estimates (4.6), (4.8),

\[ \| [g_\epsilon s(\varrho_\epsilon, \theta_\epsilon)]_{\text{res} u_\epsilon} \|_{L^1((0, T) \times \Omega)} \leq \epsilon^{-a/2} \left( \| \varrho_\epsilon^2 \|_{L^\infty((0, T); L^{6/5}(\Omega))} + \| [g_\epsilon \log \varrho_\epsilon]_{\text{res} u_\epsilon} \|_{L^\infty((0, T); L^{6/5}(\Omega))} \right) \]

\[ \times \| \epsilon^{a/2} u_\epsilon \|_{L^2((0, T); W^{1, 2}(\Omega))} \leq c \epsilon^{(\frac{a}{2} - \frac{3}{2})} \| \epsilon^{a/2} u_\epsilon \|_{L^2((0, T); W^{1, 2}(\Omega))} ightarrow 0 \text{ whenever } 0 < a < \frac{10}{3}. \]

This is the only point in the whole proof, where we need the upper bound for the exponent \( a \).
6.4.2. **Handling the residual component of the right hand side of (6.11).** To begin, we observe that the residual components of all integrals at the right hand side on the second, third and fourth line of inequality (6.11) are negligible. To this end, we employ estimates (5.9), (5.15), (5.12) together with the system of equations (5.4) and formulas (5.7), to deduce

$$
\sup_{t \in (0, T)} \varepsilon \| \partial_t R_\varepsilon(t, \cdot) \|_{L^\infty(\Omega)}, \sup_{t \in (0, T)} \varepsilon \| \partial_t T_\varepsilon(t, \cdot) \|_{L^\infty(\Omega)} \leq c(\eta),
$$

(6.12)

$$
\varepsilon \| \partial_t R_\varepsilon \|_{L^\infty(\Omega)} \to 0, \varepsilon \| \partial_t T_\varepsilon \|_{L^\infty(\Omega)} \to 0 \text{ in } L^p(0, T), 1 \leq p < \infty.
$$

(6.13)

We also know that $\text{div}_x v_\varepsilon = 0$, cf. (5.6), and $\|p(\varrho_\varepsilon, \vartheta_\varepsilon)\|_{\text{res}} \leq c \varepsilon^2$, $\|\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\|_{\text{res}} \leq c \varepsilon^2$, cf. (2.4-2.5), (2.8) and (4.6). Now, we use these relations in combination with the uniform estimates (4.6), (2.19) and (5.9–5.10) employing the same arguments as in Section 6.4.1; after a long calculation, similar as in [15, Section 7.5], we finally get the desired result, namely

$$
- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[ \varrho_\varepsilon \left( s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r_\varepsilon, \Theta_\varepsilon) \right) \partial_t \Theta_\varepsilon + \varrho_\varepsilon \left( s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(r_\varepsilon, \Theta_\varepsilon) \right) U_\varepsilon \cdot \nabla \vartheta_\varepsilon \right]_{\text{res}} \, dx \, dt
$$

(6.14)

$$
- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[ \frac{\varrho_\varepsilon - r_\varepsilon}{r_\varepsilon} \partial_t p(r_\varepsilon, \Theta_\varepsilon) \right]_{\text{res}} \, dx \, dt
$$

$$
- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[ p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(r_\varepsilon, \Theta_\varepsilon) \right] \text{div} U_\varepsilon \right]_{\text{res}} \, dx \, dt
$$

$$
= \chi(\varepsilon, \eta).
$$

6.4.3. **Handling the essential component of the right hand side of (6.11).** In view of the preceding Section, we have to handle solely the essential part of the integrals at the first and second line of formula (6.11) whose integrands can be, roughly speaking, replaced by their linearization at $\overline{\varrho}, \overline{\vartheta}$. Since we already know that the corresponding residual components are negligible, we may omit the symbol $[\cdot]_{\text{res}}$ in all integrands.

We start with the following observations that can be obtained by using suitably several times Taylor formula, relation (5.7) and definition (5.1) - to express $R_\varepsilon$ and $T_\varepsilon$ as linear combination of $\alpha R_\varepsilon + \beta T_\varepsilon$ and $Z_\varepsilon = \delta T_\varepsilon - \beta R_\varepsilon$, together with estimates (4.5), (5.9–5.10), (5.12), (5.15) and equation (5.4).

1. First,

$$
\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon} \partial_t p(r_\varepsilon, \Theta_\varepsilon) \, dx \, dt
$$

$$
= \int_0^\tau \int_\Omega \left( R_\varepsilon - \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) \, dx \, dt + \chi(\varepsilon, \eta)
$$

$$
= \delta \frac{\varepsilon}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega (\alpha R_\varepsilon + \beta T_\varepsilon) \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) \, dx \, dt
$$

(6.15)

$$
- \frac{\beta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega (\delta T_\varepsilon - \beta R_\varepsilon) \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) \, dx \, dt
$$

$$
- \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) \, dx \, dt + \chi(\varepsilon, \eta).
$$
2. Second,

\[
\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left(p(r_x, \Theta_x) - p(q_x, \vartheta_x)\right) \text{div} U_x \, dx \, dt \\
= \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \left(\alpha R_x + \beta T_x - \frac{\alpha q_x - \overline{q}}{\varepsilon} - \frac{\beta \vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \text{div} V_x \, dt + \chi(\varepsilon, \eta) \\
= -\frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega (\alpha R_x + \beta T_x) \partial_t (\alpha R_x + \beta T_x - q_x) \, dx \, dt \\
\quad + \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left(\alpha \frac{q_x - \overline{q}}{\varepsilon} + \beta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \partial_t (\alpha R_x + \beta T_x - q_x) \, dx \, dt + \chi(\varepsilon, \eta) \\
= -\frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega (\alpha R_x + \beta T_x) \partial_t (\alpha R_x + \beta T_x) \, dx \, dt \\
\quad + \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left(\alpha \frac{q_x - \overline{q}}{\varepsilon} + \beta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \partial_t (\alpha R_x + \beta T_x) \, dx \, dt \\
\quad + \frac{1}{2} \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega |\partial_t q_x|^2 \, dx \, dt - \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left(\alpha \frac{q_x - \overline{q}}{\varepsilon} + \beta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \partial_t q_x \, dx \, dt + \chi(\varepsilon, \eta) \tag{6.16}
\]

3. Third,

\[
\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho_x \left(s(r_x, \Theta_x) - s(q_x, \vartheta_x)\right) \left(\partial_t \Theta_x + U_x \cdot \nabla_s \Theta_x\right) \, dx \, dt \\
= \int_0^\tau \int_\Omega \left(\delta T_x - \beta R_x + \beta \frac{q_x - \overline{q}}{\varepsilon} - \delta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \left(\partial_t T_x + U_x \cdot \nabla_s T_x\right) \, dx \, dt + \chi(\varepsilon, \eta) \\
= \frac{\beta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left(\delta T_x - \beta R_x + \beta \frac{q_x - \overline{q}}{\varepsilon} - \delta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) \left(\partial_t (\alpha R_x + \beta T_x) + U_x \cdot \nabla_x (\alpha R_x + \beta T_x - q_x)\right) \, dx \, dt \\
\quad + \frac{\beta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left(\delta T_x - \beta R_x + \beta \frac{q_x - \overline{q}}{\varepsilon} - \delta \frac{\vartheta_x - \overline{\vartheta}}{\varepsilon}\right) U_x \cdot \nabla_x q_x \, dx \, dt
Further we write

\[ q = \frac{1}{2} \frac{1}{\beta^2 + \alpha \delta} \int_0^\tau \left( \delta T_\epsilon - \beta R_\epsilon + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} - \delta \frac{\vartheta_\epsilon - \overline{\vartheta}}{\epsilon} \right) \left( \partial_t (\delta T_\epsilon - \beta R_\epsilon) \right) \] 

\[ + \mathbf{U}_\epsilon \cdot \nabla_x (\delta T_\epsilon - \beta R_\epsilon) \right) \, dx \, dt + \chi (\epsilon, \eta) \]

(6.17)

where we have used the fact that \( Z_\epsilon = \delta T_\epsilon - \beta R_\epsilon \) verifies (5.5) together with the bounds (5.12), (5.15) and dispersive estimates (5.9) to obtain the last line.

Summing together formulas (6.15–6.17), we observe cancellation of all non vanishing integrals and get

\[ \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \frac{r_\epsilon - \varrho_\epsilon}{r_\epsilon} \partial_t p(r_\epsilon, \Theta_\epsilon) \, dx \, dt + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \left( p(r_\epsilon, \Theta_\epsilon) - p(\varrho_\epsilon, \vartheta_\epsilon) \right) \left( \partial_t \mathbf{U}_\epsilon \right) \, dx \, dt \]

\[ + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \varrho_\epsilon \left( s(r_\epsilon, \Theta_\epsilon) - s(\varrho_\epsilon, \vartheta_\epsilon) \right) \left( \partial_t \mathbf{U}_\epsilon \right) \, dx \, dt \]

\[ + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt + \chi (\epsilon, \eta) \]

The relative energy inequality (6.10) now rewrites

\[ \mathcal{E}_\epsilon \left( \varrho_\epsilon, \vartheta_\epsilon, \mathbf{u}_\epsilon \right) \bigg|_{t=0}^{t=\tau} \]

\[ \leq \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

+ \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt + \chi (\epsilon, \eta) \]

Using (4.21), (4.23), (4.20) we obtain that

\[ \mathcal{E}_\epsilon \left( \varrho_{0,\epsilon}, \vartheta_{0,\epsilon}, \mathbf{u}_{0,\epsilon} \right) \bigg|_{t=0}^{t=\tau} = \chi (\epsilon, \eta); \]

Due to (4.13), (4.12), (5.3), (5.13)

\[ \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

\[ = \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

\[ = \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

where we have used (4.15), integration by parts and equation (1.13) with \( q_\eta \) on place of \( q \), cf. (5.2). Moreover,

\[ \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

\[ = \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]

\[ = \int_0^\tau \int_\Omega \left( \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \left( \frac{\varrho_\epsilon - \overline{\varrho}}{\epsilon} + \beta \frac{\Theta_\epsilon - \overline{\Theta}}{\epsilon} \right) \partial_t q_\epsilon \right) \, dx \, dt \]
Putting together the above computations we get
\[= \int_0^\tau \int_\Omega \nabla^\perp \partial_t q_\eta \cdot \nabla \Delta_h q_\eta \ dx \ dt,\]
where we have used integration by parts and expressed \(v_\eta\) via \(q_\eta\). Finally we recall that
\[\int_0^\tau \int_\Omega \vec{\rho} \partial_t v_\eta \cdot v_\eta \ dx \ dt + \int_0^\tau \int_\Omega \vec{\rho} v_\eta \cdot \nabla v_\eta \cdot v_\eta \ dx \ dt = \frac{1}{2} \int_0^\tau \int_\Omega \vec{\rho} \partial_t |\nabla v_\eta|^2 \ dx \ dt.\]

Putting together the above computations we get
\[
\lim_{\epsilon \to 0} \int_0^\tau \int_\Omega \vec{\rho} \left( \frac{\partial^e v_\epsilon + v_\epsilon \cdot \nabla v_\epsilon}{\epsilon} \right) \cdot (v_\epsilon - u_\epsilon) \ dx \ dt
= \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \partial_t q_\eta^2 \ dx dt + \frac{1}{2} \frac{1}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \vec{\rho} \partial_t |\nabla v_\eta|^2 \ dx \ dt.
\]

On the other hand, by (4.12–4.15) and (5.13),
\[
\frac{1}{2} \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \partial_t |q_\eta|^2 \ dx dt - \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \left( \frac{\partial e - \vec{\eta}}{\epsilon} + \frac{\beta \partial e - \vec{\eta}}{\epsilon} \right) \partial_t q_\eta \ dx dt \to
\frac{1}{2} \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \partial_t |q_\eta|^2 \ dx dt - \frac{\delta}{\beta^2 + \alpha \delta} \int_0^\tau \int_\Omega \partial_t q_\eta \ dx dt \text{ as } \epsilon \to 0. \quad (6.21)
\]

Consequently, seeing (6.20–6.21) and (5.11), the relative energy inequality (6.19) rewrites
\[\mathcal{E}_\epsilon \left( q_\epsilon, \partial e, u_\epsilon \big| r_\epsilon, \Theta, U_\epsilon \right) (\tau) \leq \chi(\epsilon, \eta) + c \int_0^\tau \mathcal{E}_\epsilon \left( q_\epsilon, \partial e, u_\epsilon \big| r_\epsilon, \Theta, U_\epsilon \right) \ dx \ dt. \quad (6.22)
\]

6.5. **Gronwall type argument.** Gronwall lemma states that inequality
\[\mathcal{E}(\tau) \leq A(\tau) + \int_0^\tau B(s) \mathcal{E}(s) ds \text{ a.e. in } (0, T)\]
implies
\[\mathcal{E}(\tau) \leq A(\tau) + \int_0^\tau A(s) B(s) e^{\int_0^s B(z) \ dx} ds \text{ a.e. in } (0, T) \quad (6.23)\]
under assumptions \(0 \leq B \in L^1(0, T), A, \mathcal{E}\) measurable functions on \((0, T)\) such that the integrals \(\int_0^\tau |\mathcal{E}'(s)| B(s) \ dx ds\) and \(\int_0^\tau |A(s) B(s)\ dx ds\) are finite.

Now, we apply (6.23) to (6.19) in order to get
\[\mathcal{E}_\epsilon \left( q_\epsilon, \partial e, u_\epsilon \big| r_\epsilon, \Theta, U_\epsilon \right) (\tau) \leq \chi(\epsilon, \eta)(1 + cTe^{c\tau}) \text{ for a.a. } \tau \in (0, T). \quad (6.24)\]

Due to (4.5–4.6),
\[q_\epsilon \to \vec{\eta} \text{ strongly in } L^\infty(0, T; L^2 + L^{5/3}(K)) \text{ for any compact } K \subset \Omega, \]
Employing these facts and the convergence of sequences \(v_\epsilon, R_\epsilon, T_\epsilon\) established in (5.13) and (5.18), together with relations (5.1), (2.11–2.12) we deduce from (6.24)
\[\limsup_{\epsilon \to 0} \int_K \left| \sqrt{\rho} \ u_\epsilon - \sqrt{\rho} v_\eta \right|^2 \ dx \quad (6.25)\]

\[\leq (1 + cTe^{c\tau}) \lim_{\epsilon \to 0} \chi(\epsilon, \eta), \text{ a.e. in } (0, T), \]
where we have used the lower bounds (4.2), (4.3) for the relative energy function.
Now, in view of relation (6.2) and convergence established in (5.14) and (5.22) we deduce relations (3.7–3.8) by letting $\eta \to 0$ in (6.25). By virtue of (5.24) the limit $\theta$ in (3.8) verifies transport equation (1.14) with the initial data (3.10). Theorem 3.1 is proved.

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**REFERENCES**

[1] A. Babin, A. Mahalov and B. Nicolaenko, Global regularity of 3D rotating Navier-Stokes equations for resonant domains, *Indiana Univ. Math. J.*, 48 (1999), 1133–1176.

[2] A. Babin, A. Mahalov and B. Nicolaenko, 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity, *Indiana Univ. Math. J.*, 50 (2001), 1–35.

[3] D. Bresch, B. Desjardins and D. Gerard-Varet, Rotating fluids in a cylinder, *Disc. Cont. Dyn. Syst.*, 11 (2004), 47–82.

[4] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical Geophysics*, volume 32 of Oxford Lecture Series in Mathematics and Applications, The Clarendon Press Oxford University Press, Oxford, 2006.

[5] T. Clopeau, A. Mikelić and B. Robert, On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction boundary conditions, *Nonlinearity*, 11 (1998), 1625–1636.

[6] R. Danchin, Low Mach number for viscous compressible flows, *M2AN Math. Model Numer. Anal.*, 39 (2005), 459–475.

[7] D. R. Durran, Is the Coriolis force really responsible for the inertial oscillation?, *Bull. Amer. Meteorological Soc.*, 74 (1993), 2179–2184.

[8] D. B. Ebin, The motion of slightly compressible fluids viewed as a motion with strong constraint force, *Ann. Math.*, 105 (1977), 141–200.

[9] L. C. Evans, *Partial Differential Equations*, Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[10] E. Feireisl, Low Mach number limits of compressible rotating fluids, *J. Math. Fluid Mechanics*, 14 (2012), 61–78.

[11] E. Feireisl and Y. Lu and A. Novotný, Rotating compressible fluids under strong stratification, *Nonlinear Analysis: Real World Applications*, 19 (2014), 11–18.

[12] E. Feireisl, I. Gallagher, D. Gerard-Varet and A. Novotný, Multi-scale Analysis of Compressible Viscous and Rotating Fluids, *Comm. Math. Phys.*, 314 (2012), 641–670.

[13] E. Feireisl and A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhäuser-Verlag, Basel, 2009.

[14] E. Feireisl and A. Novotný, Weak-strong uniqueness property for the full Navier-Stokes-Fourier system, *Arch. Rational Mech. Anal.*, 204 (2012), 683–706.

[15] E. Feireisl and A. Novotný, Inviscid incompressible limits of the full Navier-Stokes-Fourier system, *Commun. Math. Phys.*, 321 (2013), 605–628.

[16] E. Feireisl and A. Novotný, Inviscid incompressible limits under mild stratification: A rigorous derivation of the Euler–Boussinesq system, *Applied Mathematics and Optimization*, 70 (2014), 279–307.

[17] E. Feireisl and A. Novotný, The low mach number limit for the full Navier-Stokes-Fourier system, *Arch. Rational Mech. Anal.*, 186 (2007), 77–107.

[18] E. Feireisl and A. Novotný, Multiple scales and singular limits for compressible rotating fluids with general initial data, *Comm. Part. Diff. Eq.*, 39 (2014), 1104–1127.

[19] E. Feireisl and A. Novotný, Scale interactions in compressible rotating fluids, *Annali di Matematica Pura ed Applicata*, 193 (2014), 1703–1725.

[20] E. Feireisl, B. J. Jin and A. Novotný, Relative Entropies, Suitable Weak Solutions, and Weak-Strong Uniqueness for the Compressible Navier–Stokes System, *Journal of Mathematical Fluid Mechanics*, 14 (2012), 717–730.

[21] E. Feireisl, Y. Lu and A. Novotný, Rotating compressible fluids under strong stratification, *Nonlinear Analysis: Real World Applications*, 19 (2014), 11–18.

[22] F. Fanelli, Highly rotating viscous compressible fluids in presence of capillarity effects, *Math. Ann.*, 366 (2016), 981–1033.
[23] F. Fanelli, A singular limit problem for rotating capillary fluids with variable rotation axis, J. Math. Fluid Mech., 18 (2016), 625–658.

[24] I. Gallagher, Résultats récents sur la limite incompressible, Astérisque, Séminaire Bourbaki, 2003/2004 (2005), 29–57.

[25] I. Gallagher and L. Saint-Raymond, Weak convergence results for inhomogeneous rotating fluid equations, J. Anal. Math., 99 (2006), 1–34.

[26] I. Gallagher and L. Saint-Raymond, Mathematical study of the betaplane model: Equatorial waves and convergence results, Mem. Soc. Math. Fr., 107 (2006), v+116 pp.

[27] P. Germain, Weak-strong uniqueness for the isentropic compressible Navier-Stokes system, J. Math. Fluid Mech., 13 (2011), 137–146.

[28] G.-M. Gie and J. P. Kelliher, Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions, J. Diff. Equations, 235 (2012), 1862–1892.

[29] D. Iftimie and F. Sueur, Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions, Arch. Rat. Mech. Anal., 199 (2011), 145–175.

[30] D. Jesslé, B. J. Jin and A. Novotný, Navier-Stokes-Fourier system on unbounded domains: Weak solutions, relative entropies, weak-strong uniqueness, SIAM J. Math. Anal., 45 (2013), 1907–1951.

[31] T. Kato, Remarks on the zero viscosity limit for nonstationary Navier–Stokes flows with boundary, Seminar on Nonlinear Partial Differential Equations (Berkeley, Calif., 1983), 85–98, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984.

[32] T. Kato and G. Ponce, Well-Posedness of the Euler and Navier-Stokes Equations in the Lebesgue Spaces \( L^p \) (\( R^2 \)), Revista Matematica Iberoamericana, 2 (1986), 73–88.

[33] T. Kato and C. Y. Lai, Nonlinear evolution equations and the Euler flow, J. Funct. Anal., 56 (1984), 15–28.

[34] S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Comm. Pure Appl. Math., 34 (1981), 481–524.

[35] R. Klein, Asymptotic analyses for atmospheric flows and the construction of asymptotically adaptive numerical methods, Z. Angw. Math. Mech., 80 (2000), 765–777.

[36] R. Klein, Scale-dependent models for atmospheric flows, Annual Review of Fluid Mechanics. Vol. 42, Annu. Rev. Fluid. Mech., Annual Reviews, Palo Alto, CA, 42 (2010), 249–274.

[37] Y. Kwon, D. Maltese and A. Novotny, Multiscale analysis in the Compressible Rotating and Heat Conducting Fluid, J. Math. Fluid. Mech., 20 (2018), 421–444.

[38] P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid, J.Math.Pures Appl., 77 (1998), 585–627.

[39] N. Masmoudi and F. Rousset, Uniform regularity for the Navier-Stokes equations with Navier boundary conditions, Arch. Rational Mech. Anal., 203 (2012), 529–575.

[40] N. Masmoudi, The Euler limit of the Navier-Stokes equations and rotating fluid with boundary, Arch. Rational Mech. Anal., 142 (1998), 375–394.

[41] N. Masmoudi, Incompressible inviscid limit of the compressible Navier–Stokes system, Ann. Inst. H. Poincaré, Anal. non linéaire, 18 (2001), 199–224.

[42] N. Masmoudi, Examples of singular limits in hydrodynamics, Handbook of Differential Equations: Evolutionary Equations, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 3 (2007), 195–275.

[43] M. Oliver, Classical sofor a generalized Euler equation in two dimensions, J. Math. Anal. Appl. (9), 215 (1997), 471–484.

[44] M. Sammartino and R. Caffiisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space I, Existence for Euler and Prandtl equations, Comm. Math. Phys., 192 (1998), 433–461.

[45] M. Sammartino and R. Caffiisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space II, Construction of Navier-Stokes equations, Comm. Math. Phys., 192 (1998), 463–491.

[46] S. Schochet, The mathematical theory of low Mach number flows, M2AN Math. Model Numer. anal., 39 (2005), 441–458.

[47] H. S. G. Swann, The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in \( R^3 \), Trans. Amer. Math. Soc., 157 (1971), 373–397.

[48] R. Temam and X. Wang, On the behaviour of the solutions of the Navier-Stokes equations at vanishing viscosity, Annali Scuola Normale Pisa, 25 (1997), 807–828.
[49] R. Temam and X. Wang, Boundary layer associated with incompressible Navier-Stokes equations: The noncharacteristic boundary case, J. Differential Equations, 179 (2002), 647–686.

[50] G. K. Vallis, Atmospheric and Ocean Fluid Dynamics, Cambridge University Press, Cambridge, 2006.

[51] L. Wang, Z. Xin and A. Zang, Vanishing viscous limits for 3D Navier-Stokes equations with a Navier-slip boundary conditions, J. Math. Fluid. Mech., 14 (2012), 791–825.

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E-mail address: ykwon@dau.ac.kr
E-mail address: novotny@univ-tln.fr