Loss networks

Stan Zachary and Ilze Ziedins
Heriot-Watt University and University of Auckland

March 3, 2009

We review the theory of loss networks, including recent results on their dynamical behaviour. We give also some new results.

AMS 2000 subject classification: Primary: 60K20; Secondary: 60K25.

1 Introduction

In a loss network calls, or customers, of various types are accepted for service provided that this can commence immediately; otherwise they are rejected. An accepted call remains in the network for some holding time, which is generally independent of the state of the network, and throughout this time requires capacity simultaneously from various network resources.

The loss model was first introduced by Erlang as a model for the behaviour of just a single telephone link (see Brockmeyer et al. (1948). The typical example remains that of a communications network, in which the resources correspond to the links in the network, and a call of any type requires, for the duration of its holding time, a fixed allocation of capacity from each link over which it is routed (Kelly, 1986). This is the case for a traditional circuit-switched telephone network, but the model is also appropriate to modern computer communications networks which support streaming applications with minimum bandwidth requirements (Kelly et al., 2000). There are also other examples: for instance, in a cellular mobile network similar capacity constraints arise from the need to avoid interference (Abdalla and Boucherie, 2002).

The mathematics of such networks has been widely studied, with interest in both equilibrium and, more recently, dynamical behaviour. Of particular importance are questions of call acceptance and capacity allocation (for example, routing), with the aim of ensuring good network performance which is additionally robust with respect to variations in network parameters. Call arrival rates, in particular, may fluctuate greatly. An excellent review of the state-of-the-art at the time of its publication is given by Kelly (1991)—see also the many papers cited therein, and the later survey by Ross (1995).

We take as our model the following. Let $\mathcal{R}$ denote the finite set of possible call, or customer, types. Calls of each type $r \in \mathcal{R}$ arrive at the network as a Poisson process with rate $\nu_r$, and each such call, if accepted by the network (see below), remains in it for a holding time which is exponentially distributed with mean $\mu_r^{-1}$. We shall discuss later the extent to which these assumptions, in particular the latter, are necessary. Calls which are rejected do not retry and are simply considered lost. All arrival processes and holding times are independent of one another. We denote the state of the network at time $t$ by $n(t) = (n_r(t), r \in \mathcal{R})$, where $n_r(t)$ is the number of calls of each type $r$ in progress at
that time. The process \( n(\cdot) \) is thus Markov. It takes values in some state space \( \mathcal{N} \subset \mathbb{Z}_+^{|\mathcal{R}|} \), where \( R = |\mathcal{R}| \). We assume \( \mathcal{N} \) to be defined by a number of resource constraints

\[
\sum_{r \in \mathcal{R}} A_{jr} n_r \leq C_j, \quad j \in \mathcal{J},
\]

(1)

indexed in a finite set \( \mathcal{J} \), where the \( A_{jr} \) and the \( C_j \) are nonnegative integers. Typically we think of a call of each type \( r \) as having a simultaneous requirement, for the duration of its holding time, for \( A_{jr} \) units of the capacity \( C_j \) of each resource \( j \); however, we show below that the resource constraints (1) can also arise in other ways. As noted above, in applications of this model to communications networks, the network resources usually correspond to the links in the network, and when discussing the model in that context we shall generally find it convenient to use this terminology. We shall also find it helpful to define, for each \( r \in \mathcal{R} \), the parameter \( \kappa_r = \nu_r / \mu_r \); many quantities of interest depend on \( \nu_r \) and \( \mu_r \) only through their ratio \( \kappa_r \).

We shall say that a network is uncontrolled whenever calls are accepted subject only to the condition that the resulting state of the network belongs to the set \( \mathcal{N} \). Uncontrolled networks are particularly amenable to mathematical analysis and are in certain senses very well-behaved. In addition, such a network has the important insensitivity property: the stationary distribution of the process \( n(\cdot) \) is unaffected by the relaxation of the assumption that the call holding time distributions are exponential, and depends on these holding time distributions only through their means. This is essentially a consequence of the detailed balance property considered in Section 2.1.

However, as we shall also see, the performance of uncontrolled networks may be far from optimal. A more general control strategy is given by requiring that a call of type \( r \), which arrives when the state of the network (immediately prior to its arrival) is \( n \), is accepted if and only if \( n \in A_r \) for some acceptance set \( A_r \). The sets \( A_r \) may be chosen so as to optimise, in some appropriate sense, the network’s performance. Such networks do not in general possess the insensitivity property described above.

Of interest in a loss network are both the stationary distribution \( \pi \) and the dynamics of the process \( n(\cdot) \). For the former it is usual to compute, for each \( r \), the stationary blocking probability \( B_r \), that a call of type \( r \) is rejected; here we shall find it slightly easier to work with the stationary acceptance (or passing) probability \( P_r = 1 - B_r \). We note immediately that, by Little’s Theorem, the stationary expected number of calls of each type \( r \) in the network is given by

\[
E_\pi n_r = \kappa_r P_r,
\]

(2)

where \( \kappa_r \) is as defined above. Thus acceptance probabilities may be regarded as one of the key performance measures in the stationary regime.

It will be convenient to refer to the above model of a loss network—in which arriving calls have fixed resource requirements and in which the only control in the network is the ability to reject calls—as the canonical model. When considering communications networks, it is natural to extend this model by allowing also the possibility of alternative routing, in which calls choose their route according to the current state of the network. Here the state space should properly be expanded to record the number of calls of each type on each route (but see below). We consider such models in Section 3.2.

In the case where we allow not only alternative routing, but also repacking of calls already in the network, the model simplifies again, and it is once more only necessary for the state
space to record the number of calls of each type in progress. Consider the simple example of a communications network consisting of three links with capacities $C_1', C_2', C_3'$, and three call types, in which calls of each type $r = 1, 2, 3$ require either one unit of capacity from the corresponding link $r$ or one unit of capacity from each of the other two links (in each case the distribution of the call holding time is assumed to be the same). If repacking is allowed, the state of the system may be given by $n = (n_1, n_2, n_3)$ as usual, and it is easy to check that a call of any type may be accepted if and only if the resulting state of the network satisfies the constraints

$$n_r + n_{r'} \leq C_r' + C_{r'}, \quad r \neq r', \quad r, r' \in \mathcal{R}.$$ 

This is therefore an instance of the uncontrolled network discussed above, in which the coefficients $A_{jr}$ and $C_j$ must be appropriately defined.

Exact calculations for large loss networks typically exceed the capabilities of even large computers, and we are thus led to consider approximations. Mathematical justification for these approximations is usually based on asymptotic results for one of two limiting schemes. In the first, which we shall refer to as the Kelly limiting scheme (see Kelly, 1986), the sets $\mathcal{R}$, $\mathcal{J}$, the matrix $A = (A_{jr})$, and the parameters $\mu_r$ are held fixed, while the arrival rates $\nu_r$ and the capacities $C_j$ are all allowed to increase in proportion to a scale parameter $N$ which tends to infinity. In the second, which is known as the diverse routing limit (see Whitt, 1985, and Ziedins and Kelly, 1989), the capacity of each resource is held constant, while the sets $\mathcal{R}$ and $\mathcal{J}$ (and correspondingly the size of the matrix $A$) are allowed to increase, and the arrival rates for call types requiring capacity at more than one resource to decrease, in such a way that the total traffic offered to each resource is also held constant (in particular, this requires that the arrival rate for any call type that requires capacity at more than one resource becomes negligible in the limit). Results for the latter scheme in particular are used to justify assumptions of independence in many approximations.

For each time $t$, define $m(t) = (m_j(t), j \in \mathcal{J})$, where $m_j(t)$ denotes the current occupancy, or usage, of each resource $j$ in a loss network. Define also $\pi'$ to be the stationary distribution of the process $m(\cdot)$. In particular, for the canonical model defined above, for each $t$,

$$m_j(t) = \sum_{r \in \mathcal{R}} A_{jr} n_r(t);$$

here the process $m(\cdot)$ takes values in the set

$$\mathcal{M} = \{m \in \mathbb{Z}_+^J : 0 \leq m_j \leq C_j, \ j \in \mathcal{J}\},$$

where we write $J = |\mathcal{J}|$, and the distribution $\pi'$ is given by

$$\pi'(m) = \sum_{n : A n = m} \pi(n), \quad m \in \mathcal{M}.$$ 

In general the process $m(\cdot)$ takes values in a space of significantly lower dimension than that of the process $n(\cdot)$. This is especially so in models of communications networks which incorporate alternative routing. It is a recurrent theme in the study of loss networks that, in general, at least approximately optimal control of a network is obtained by basing admission decisions and, in communications networks, routing decisions, solely on the state of the process $m(\cdot)$ at the arrival time of each call. Further, in this case, a knowledge of the distribution $\pi'$ is sufficient to determine call acceptance probabilities. We shall also see
that good estimates of $\pi'$ are generally given by assuming its (approximate) factorisation as

$$\pi'(m) = \prod_{j \in J} \pi'_j(m_j),$$

(6)

where each $\pi'_j$ is normalised to be a probability distribution. This is a further recurrent theme in the study of loss networks.

In Section 2 we consider the stationary behaviour of uncontrolled networks, reviewing both exact results and approximations for large networks. Our approach is based on the use of an elegant recursion due to Kaufman (1981) and to Dziong and Roberts (1987) which delivers all the classical results in regard to, for example, stationary acceptance probabilities, with a certain simplicity.

More general networks are studied in Section 3. In Section 3.1 we study the problem of optimal control in a single-resource network, where a reasonably tractable analysis of stationary behaviour is again possible, and where we show that either exactly or approximately optimal control may be obtained with the use of strategies based on reservation parameters. In Section 3.2 we consider multiple-resource networks, allowing in particular the possibility of alternative routing. We again derive approximations which are known to work extremely well in practice. In Section 4 we consider the dynamical behaviour of large loss networks. This is important for the study of the long-run, and hence also the equilibrium, behaviour of networks in the case where a direct equilibrium analysis is impossible. The study of network dynamics is also the key to understanding their stability. Finally, in Section 5 we mention some wider models and discuss some open problems.

2 Uncontrolled loss networks: stationary behaviour

We study here the stationary behaviour of the uncontrolled network introduced above, in which calls of any type are accepted subject only to the condition that the resulting state $n$ of the network belongs to the state space $\mathcal{N}$ defined by the capacity constraints (1). In particular we shall see, in Section 2.3 and subsequently, that most quantities of interest, in particular acceptance probabilities, may be calculated, exactly or approximately, without the need to calculate the full stationary distribution $\pi$ of the process $n(\cdot)$.

2.1 The stationary distribution

For each $r \in R$, let $\delta_r$ be the vector whose $r$th component is 1 and whose other components are 0. Recall that, under the assumptions introduced above, $n(\cdot)$ is a Markov process. For $n, n - \delta_r \in \mathcal{N}$ and $r \in R$, its transition rates between $n$ and $n - \delta_r$ are $n_r \mu_r$ and $\nu_r$. It thus follows that the stationary distribution $\pi$ of the process $n(\cdot)$ is given by the solution of the detailed balance equations

$$\pi(n)n_r \mu_r = \pi(n - \delta_r)\nu_r, \quad r \in R, \quad n \in \mathcal{N},$$

(7)

where, here and elsewhere, we make the obvious convention that $\pi(n - \delta_r) = 0$ whenever $n_r = 0$. That is,

$$\pi(n) = G^{-1} \prod_{r \in R} \frac{n_r^{\nu_r}}{n_r!}, \quad n \in \mathcal{N},$$

(8)

where the normalising constant $G^{-1}$ is determined by the requirement that $\sum_{n \in \mathcal{N}} \pi(n) = 1$. The simple product form of the stationary distribution (8) is a consequence of the fact
that the equations (7) do have a solution, that is, it is a consequence of the reversibility of the stationary version of the process \( n(\cdot) \). Note also that here the stationary distribution \( \pi \) depends on the parameters \( \nu_r \) and \( \mu_r \) only through their ratios \( \kappa_r = \frac{\nu_r}{\mu_r}, r \in \mathcal{R} \). This result is not in general true for networks with controls.

In the variation of our model in which calls of each type \( r \) have holding times which are no longer necessarily exponential (but with unchanged mean \( \mu_r^{-1} \)), it is well-known that the stationary distribution \( \pi \) of the process \( n(\cdot) \) continues to satisfy the detailed balance equations (7) and hence also (8). For a proof of this insensitivity property, see Burman et al. (1984).

The stationary probability that a call of type \( r \) is accepted, is given by

\[
P_r = \sum_{n \in \mathcal{N}_r} \pi(n),
\]

where \( \mathcal{N}_r = \{n \in \mathcal{N} : n + \delta_r \in \mathcal{N}\} \). In Section 2.3 we give a recursion which permits a reasonably efficient calculation of the probabilities \( P_r \) in networks of small to moderate size. However, the exact calculation of acceptance probabilities is usually difficult or impossible in large networks. We shall therefore also discuss various approximations.

### 2.2 The single resource case

Consider first the case \( \mathcal{R} = \{1\} \) of a single call type. For convenience we drop unnecessary subscripts denoting dependence on \( r \in \mathcal{R} \); in particular we write \( \kappa = \nu/\mu \). We then have \( \mathcal{N} = \{n : n \leq C\} \) for some positive integer \( C \). The stationary distribution \( \pi \) is a truncated Poisson distribution, and the stationary acceptance probability \( P \) is given by Erlang’s well-known formula, that is, by

\[
P = 1 - \pi(C) = 1 - E(\kappa, C),
\]

where

\[
E(\kappa, C) = \frac{\kappa^C / C!}{\sum_{n=0}^{\infty} \kappa^n / n!}.
\]

Note also that, from (2), the expected number of calls in progress under the stationary distribution \( \pi \) is given by \( \kappa P \).

While exact calculation of blocking probabilities via Erlang’s formula (10) is straightforward, it nevertheless provides insight to give approximations for networks in which \( C \) and \( \kappa \) are both large. Formally, we consider the Kelly limiting scheme in which \( C \) and \( \kappa \) are allowed to tend to infinity in proportion to a scale parameter \( N \) with \( p = C / \kappa \) held fixed.

The cases \( p > 1 \), \( p = 1 \) and \( p < 1 \) correspond to the network being, in an obvious sense, underloaded, critically loaded, and overloaded respectively. A relatively straightforward analysis of (10) shows that,

\[
P \rightarrow \min(1, p) \quad \text{as } N \rightarrow \infty.
\]

For \( p \geq 1 \) the error in the approximation \( P \approx 1 \) may be estimated by replacing the truncated Poisson distribution of \( n \) by a truncated normal distribution: for \( p > 1 \) it may be shown to decay at least exponentially fast in \( N \), while for the critically loaded case \( p = 1 \) it may be shown to be \( O(N^{-1/2}) \) as \( N \rightarrow \infty \). For the overloaded case \( p < 1 \) the approximation \( P \approx p \) may be refined as follows. Observe that in this case, and since \( \kappa \) and \( C \) are large, it follows from either (7) or (8) that the stationary distribution of free
capacity in the network is approximately geometric and so the stationary expected free capacity is given by the approximation

\[ C - E_p(n) \approx \frac{p}{1 - p}. \]  

(12)

Combining this with (2) leads to the very much more refined approximation for the stationary acceptance probability given by

\[ P \approx p - \frac{p}{\kappa(1 - p)}. \]  

(13)

The error in this approximation may be shown to be \( o(N^{-1}) \) as \( N \to \infty \), so also that in the original approximation \( P \approx p \) is \( O(N^{-1}) \).

### 2.3 The Kaufman-Dziong-Roberts (KDR) recursion

For the general model of an uncontrolled network, we now take the set \( \mathcal{N} \) to be given by a set of capacity constraints of the form (1). We give here an efficient recursion for the determination of stationary acceptance probabilities, due in the case \( J = \{1\} \) to Kaufman (1981) and in the general case to Dziong and Roberts (1987).

Recall that \( \pi' \) is the stationary distribution of the process \( m(\cdot) \) defined in the Introduction. Since a call of type \( r \) arriving at time \( t \) is accepted if and only if \( m_j(t-) + A_{jr} \leq C_j \) for all \( j \) such that \( A_{jr} \geq 1 \) (where \( m(t-) \) denotes the state of the process \( m(\cdot) \) immediately prior to the arrival of the call), it follows that a knowledge of \( \pi' \) is sufficient to determine stationary acceptance probabilities. Typically the size \( J \) of the set \( J \) is smaller than the size \( R \) of the set \( \mathcal{R} \), and so the dimension of the space \( \mathcal{M} \) defined by (4) is smaller than that of \( \mathcal{N} \). Thus a direct calculation of \( \pi' \), avoiding that of \( \pi \), is usually much more efficient for determining acceptance probabilities.

For each \( r \in \mathcal{R} \), define the vector \( A_r = (A_{jr}, j \in J) \). For each \( m \in \mathcal{M} \) and \( r \in \mathcal{R} \), summing the detailed balance equations (7) over \( n \) such that \( An = m \) and using also (5) yields

\[ \kappa_r \pi'(m - A_r) = E(n_r | m) \pi'(m), \quad r \in \mathcal{R}, \quad m \in \mathcal{M}, \]  

(14)

where

\[ E(n_r | m) = \frac{\sum_{n: An = m} n_r \pi(n)}{\sum_{n: An = m} \pi(n)} \]

is the stationary expected value of \( n_r \) given \( An = m \). Since, for each \( m \) and each \( j \), we have \( \sum_{r \in \mathcal{R}} A_{jr} E(n_r | m) = m_j \), it follows from (14) that

\[ \sum_{r \in \mathcal{R}} A_{jr} \kappa_r \pi'(m - A_r) = m_j \pi'(m), \quad m \in \mathcal{M}, \quad j \in J. \]  

(15)

This is the Kaufman-Dziong-Roberts (KDR) recursion on the set \( \mathcal{M} \), enabling the direct determination of successive values of \( \pi'(m) \) as multiples of \( \pi'(0) \). The entire distribution \( \pi' \) is then determined uniquely by the requirement that \( \sum_{m \in \mathcal{M}} \pi'(m) = 1 \).

### 2.4 Approximations for large networks

We now suppose that \( \kappa_r, r \in \mathcal{R} \), and \( C_j, j \in J \), are sufficiently large that the exact calculation of the stationary distributions \( \pi \) or \( \pi' \) is impracticable. We seek good approximations for the latter and for acceptance probabilities.
A simple approximation  We give first a simple approximation, due to Kelly (1986), which generalises the approximation \( P \approx \min(1, p) \) of Section 2.2 for the single-resource case. To provide asymptotic justification we again consider the Kelly limiting scheme, in which the parameters \( \kappa_r \) and \( C_j \) are allowed to increase in proportion to a scale parameter \( N \), the sets \( \mathcal{R} \), \( \mathcal{J} \) and the matrix \( A \) being held fixed. We assume (this is largely for simplicity) that the matrix \( A \) is such that for each \( m \in \mathcal{M} \) there is at least one \( n \in \mathbb{Z} \) such that \( An = m \). (This implies in particular that the matrix \( A \) is of full rank.) We outline an argument based on the equations (14) and the KDR recursion (15).

Suppose that \( \pi'(m) \) is maximised at \( m^* \in \mathcal{M} \). The distribution (8) of \( \pi \) is a truncation of a product of independent Poisson distributions each of which has a standard deviation which is \( O(N^{1/2}) \) as the scale parameter \( N \) increases. From this and from the mapping of \( \pi \) to \( \pi' \), it follows that all but an arbitrarily small fraction of the distribution of \( \pi' \) is concentrated within a region \( \mathcal{M}^* \subseteq \mathcal{M} \) such that the components of \( m \in \mathcal{M}^* \) differ from those of \( m^* \) by an amount which is again \( O(N^{1/2}) \) as \( N \) increases. Further, it is not too difficult to show from the above condition on the matrix \( A \) that, for each \( r \), \( E(n_r | m) \) varies smoothly with \( m \), and that within \( \mathcal{M}^* \) we may make the approximation

\[
E(n_r | m) \approx E(n_r | m^*)
\]

(16)

The error yet again being \( O(N^{1/2}) \) as \( N \) increases. It now follows from (14) that within \( \mathcal{M}^* \) we have

\[
\pi'(m) \approx \pi'(m^*) \prod_{j \in \mathcal{J}} p^{m_j^*-m_j},
\]

where necessarily, since \( m^* \) maximises \( \pi'(m) \),

\[
0 \leq p_j \leq 1, \quad j \in \mathcal{J},
\]

(17)

\[
p_j = 1, \quad \text{for } j \text{ such that } m_j^* < C_j.
\]

Further, from (15),

\[
\sum_{r \in \mathcal{R}} A_{jr} \kappa_r \prod_{k \in \mathcal{J}} p^{A_{kr}} = m_j^* \leq C_j, \quad j \in \mathcal{J}.
\]

(19)

Thus, from (16), within \( \mathcal{M}^* \) the stationary distribution \( \pi' \) of \( m(\cdot) \) does indeed have the approximate factorisation (6), where each of the component distributions \( \pi'_j \) is here geometric (and where in the case \( p_j = 1 \) the geometric distribution becomes uniform). Further, for each \( r \) and each \( j \), we have

\[
\pi'_j(\{m_j : m_j \leq C_j - A_{jr}\}) \approx p^{A_{jr}}.
\]

Thus the stationary acceptance probabilities \( P_r \) are given by the approximation

\[
P_r \approx \prod_{j \in \mathcal{J}} p^{A_{jr}}, \quad r \in \mathcal{R}.
\]

(20)

Kelly (1986) considered an optimisation problem from which it follows that the equations (17)–(19) determine the vectors \( m^* \) and \( p = (p_j, j \in \mathcal{J}) \) uniquely. He further showed, in an approach based on consideration of the stationary distribution \( \pi \), that the approximation (20) becomes exact as the scale parameter \( N \) tends to infinity.
A refined approximation The (multiservice) reduced load or knapsack approximation (Dziong and Roberts, 1987, see also Ross, 1995) is a more refined approximation than that defined above. It is given by retaining the approximate factorisation (6) of the stationary distribution \( \pi' \) of \( m(\cdot) \). However, subject to this assumed factorisation, the estimation of the component distributions \( \pi'_j \) is refined.

For each \( j \in J \) and \( r \in \mathcal{R} \), define

\[
p_{jr} = \frac{C_j - A_{jr}}{\sum_{m_j=0} \pi'_j(m_j)};
\]

note that \( p_{jr} = 1 \) if \( A_{jr} = 0 \). For fixed \( j \), substitution of (6) into the KDR recursion (15) and summation over all \( m_k \) for all \( k \neq j \) yields

\[
\sum_{r \in R} A_{jr} \left( \kappa_r \prod_{k \neq j} p_{kr} \right) \pi'_j(m_j - A_{jr}) = m_j \pi'_j(m_j), \quad 1 \leq m_j \leq C_j, \quad j \in J
\]

(22)

(where, as usual, we make the convention \( \pi'_j(m_j) = 0 \) for \( m_j < 0 \)). This is the one-dimensional KDR recursion associated with a single resource constraint \( j \), and is readily solved to determine \( \pi'_j \) and hence the probabilities \( p_{jr}, r \in \mathcal{R} \), in terms of the probabilities \( p_{kr}, r \in \mathcal{R} \), for all \( k \neq j \). We are thus led to a set of fixed point equations in the probabilities \( p_{jr} \), for which the existence—but not always the uniqueness, see Chung and Ross (1993)—of a solution is guaranteed. From (6), the probability that a call of type \( r \) is accepted is then given by

\[
P_r = \prod_{j \in J} p_{jr}.
\]

We remark that the recursion (22) corresponds to a modified network in which there is a single resource constraint \( j \) and each arrival rate \( \kappa_r \) is reduced to \( \kappa_r \prod_{k \neq j} p_{kr} \). This reduced load approximation is of course exact in the case of a single-resource network.

In the case where each \( A_{jr} \) can only take the values 0 or 1 we may set \( p_j = p_{jr} \) for \( r \) such that \( A_{jr} = 1 \). The fixed point equations (21) and (22) then reduce to

\[
p_j = 1 - E \left( \sum_{r \in \mathcal{R}} \kappa_r \prod_{k \neq j} p_{kr}^{A_{jr}} \right)
\]

(24)

where \( E \) is the Erlang function (10). This case is the well-known Erlang fixed point approximation (EFPA) and has a unique solution, see Kelly (1986), and also Ross (1995). It yields acceptance probabilities which are known to be asymptotically exact in the Kelly limiting scheme discussed above, and also, under appropriate conditions, in the diverse routing limit discussed in the Introduction—see Whitt (1985), and Ziedins and Kelly (1989). The EFPA also has an extension to the case of general \( A_{jr} \), which may be regarded as a simplified version of the reduced load approximation. As with the latter approximation the EFPA may here have multiple solutions.

3 Controlled loss networks: stationary behaviour

We now study the more general version of a loss network, in which calls are subject to acceptance controls, and the issues are those of achieving optimal performance.
3.1 Single resource networks

We consider a simple model which illustrates some ideas of optimal control—in particular those of robustness of the control strategy with respect to variations in arrival rates (which may in practice be unknown, or vary over time).

Suppose that $\mathcal{R} = \{1, 2\}$ and that as usual calls of each type $r$ arrive at rate $\nu_r$ and have holding times which are exponentially distributed with mean $\mu_r^{-1}$. Suppose further that there is a single resource of capacity $C$ and that a call of either type requires one unit of this capacity, so that the constraints (1) here reduce to $n_1 + n_2 \leq C$. We assume that calls of type 1 have greater value per unit time than those of type 2, so that it is desirable to choose the acceptance regions $A_r$, $r = 1, 2$, so as to maximise the linear function

$$
\phi(P_1, P_2) := a_1 \kappa_1 P_1 + a_2 \kappa_2 P_2,
$$

for some $a_1 > a_2 > 0$ (where, again as usual, for each call type $r$, $\kappa_r = \nu_r / \mu_r$ and $P_r$ is the stationary acceptance probability.) An upper bound for the expression in (25) is given by the solution of the linear programming problem, in the variables $P_1, P_2$,

$$
\text{maximise } \phi(P_1, P_2), \quad \text{subject to } P_r \in [0, 1] \text{ for } r = 1, 2, \quad \kappa_1 P_1 + \kappa_2 P_2 \leq C
$$

(26)

(where the latter constraint follows from (2)). It is easy to see that the solution of this problem is characterised uniquely by the conditions

$$
P_1 = P_2 = 1, \quad \text{whenever } \kappa_1 P_1 + \kappa_2 P_2 < C, \quad \text{(27)}
P_2 = 0, \quad \text{whenever } P_1 < 1. \quad \text{(28)}
$$

It is clearly not possible to choose the acceptance regions $A_1, A_2$ so that the corresponding values of $P_1, P_2$ solve exactly the problem (26). However, we show below that this solution may be achieved asymptotically as the size of the system is allowed to increase, and further that there is an asymptotically optimal control that is both simple and robust with respect to variations in the parameters $\kappa_1, \kappa_2$.

We consider first the form of the optimal control in the special case $\mu_1 = \mu_2$. Here since, at the arrival time $t$ of any call, those calls already within the system are indistinguishable with respect to type, it is clear that the optimal decision on call admission is a function only of the arriving call type and of the total volume $m(t-) = n_1(t-) + n_2(t-)$ of calls already in the system. A formal proof is a straightforward exercise in Markov decision theory. Further, simple coupling arguments show that, for an incoming call of either type arriving at time $t$ and any $0 < m < C$, if it is advantageous to accept the call when $m(t-) = m$, then it is also advantageous to accept the call when $m(t-) = m - 1$. It follows that the optimal acceptance regions are of the form

$$
A_1 = \{n: n_1 + n_2 < C\} \quad \text{(29)}
$$

$$
A_2 = \{n: n_1 + n_2 < C - k\} \quad \text{(30)}
$$

for some reservation parameter $k$, whose optimal value depends on $C, \kappa_1$ and $\kappa_2$.

Consider now the general case where we do not necessarily have $\mu_1 = \mu_2$, and suppose that $C, \nu_1$ and $\nu_2$ are large. More formally we again have in mind the Kelly limiting scheme in which these parameters are allowed to increase in proportion to some scale parameter $N$ which tends to infinity (while $\mu_1, \mu_2$ are held fixed). We further suppose
that the acceptance regions are again as given by (29) and (30), where the reservation parameter $k$ increases slowly with $N$, i.e. in such a way that

$$k \to \infty, \quad k/C \to 0, \quad \text{as } N \to \infty.$$  

(31)

It is convenient to let $P_1, P_2$ denote the limiting acceptance probabilities. In the case $\kappa_1 + \kappa_2 \leq C$, it is not difficult to see that, since $k/C \to 0$ as $N \to \infty$, we have $P_1 = P_2 = 1$, so that $P_1, P_2$ solve the optimisation problem (26). Consider now the case $\kappa_1 + \kappa_2 > C$. Here, again since $k/C \to 0$ as $N \to \infty$, it follows that, in the limit, the capacity of the network is fully utilised. Further, if $\kappa_1$ is sufficiently large that $P_1 < 1$ (informally, even for large $N$, calls of type 1 are being rejected in significant numbers), then the effect of the increasing reservation parameter $k$ is such that, again in the limit, the network remains sufficiently close to capacity to ensure that no calls of type 2 are accepted, and hence $P_2 = 0$. It now follows that when $\kappa_1 + \kappa_2 > C$, the limiting acceptance probabilities $P_1, P_2$ satisfy the conditions (27) and (28) and so again solve the optimisation problem (26).

The above analysis demonstrates the asymptotic optimality of any strategy based on the use of a reservation parameter $k$, provided only that, in the limiting regime, $k$ increases in accordance with (31). In practice, in a large network (here for large $C$), only a small value of $k$ is required in order to achieve optimal performance. We also observe that the performance of a reservation parameter strategy is indeed robust with respect to variations in $\kappa_1, \kappa_2$.

This analysis also extends easily to the case where there are more than two call types, and also, with a little more difficulty, to that where the capacity constraint is of the form $\sum_{r \in R} A_r n_r \leq C$ for general positive integers $A_r$ (see Bean et al., 1995). Here a different reservation parameter may be used for each call type, and, in the Kelly limiting scheme, a complete prioritisation and optimal control are again achieved asymptotically by allowing the differences between the reservation parameters to increase slowly.

3.2 Multiple resource models

Consider now the general case of the canonical model in which there is a set of resources $J$ and in which state $n$ of the network is subject to the constraints (11). Suppose that it is again desirable to choose admission controls so as to maximise the linear function $\phi(P) := \sum_{r=1}^{R} a_r \kappa_r P_r$ of the stationary acceptance probabilities $P_r$, for given constants $a_r$, $r \in R$. As in Section 3.1 we may consider the linear programming problem

$$\text{maximise } \phi(P), \quad \text{subject to } P_r \in [0, 1] \text{ for } r \in R, \quad \sum_{r=1}^{R} A_{jr} \kappa_r P_r \leq C_j \text{ for } j \in J, \quad (32)$$

which provides an upper bound on the achievable values of the objective function $\phi$. It is easy to see that this value may be asymptotically achieved within the Kelly limiting regime by reserving capacity $A_{jr} \kappa_r P_r$ at each resource $j$ solely for calls of each type $r$, where here $P$ is the solution of the problem (32). However this strategy is neither optimal in networks of finite capacity, nor is it robust with respect to variations in the parameters $\kappa_r$. At the opposite end of the spectrum from this complete partitioning policy is that of complete sharing. The latter can lead to unfairness if there are asymmetric traffic patterns, with the potential for some call types to receive better service than others. In practice it is expected that good strategies will be based on the sharing of resources and the use of reservation parameters—as was shown to be optimal for single resource networks in Section 3.1.
In the case of communications networks it is natural to allow also alternative routing, as described in the Introduction. An upper bound for the achievable performance is given by supposing that repacking is possible, i.e. that calls in progress may be rerouted as necessary. In this case, our model for the network reduces to an instance of the canonical model (as defined in the Introduction) with appropriately redefined set $\mathcal{J}$, matrix $A = (A_{jr})$ and capacities $C_j$. The upper bound on $\phi(P)$ given by the linear programming problem (32) is then also an upper bound in the more usual case in which repacking is not allowed. In the latter case practical control strategies are again based on the use of appropriate reservation parameters, and there is some hope that performance close to the upper bound above may be achieved in networks with sufficiently large capacities or sufficient diversity of routing, even without repacking. In applications reservation parameters are generally used to prioritise different traffic streams. In networks with alternative routing they also prevent the occurrence of network instabilities, where, for fixed parameter values, the network may have two or more relatively stable operating regimes—one in which most calls are directly routed, and others in which many calls are alternatively routed, with a resulting severe degradation of performance (see Gibbens et al., 1990, Kelly, 1991). By giving priority to directly routed traffic, the use of reservation parameters prevents the network from slipping into an inefficient operating state.

There have been numerous investigations of control strategies for communications networks that employ either fixed or, particularly, alternative routing. Such strategies are often studied in the context of fully connected networks. Two of the most commonly studied are least busy alternative (LBA) routing and dynamic alternative routing (DAR). LBA routing seeks to route calls directly if possible, and otherwise routes them via that path which minimises the maximum occupancy on any of its links. Directly routed calls are usually “protected” with some form of reservation parameter (Kelly, 1991, Marbukh, 1993). Hunt and Laws (1993) showed that, for fully connected networks which permit only two-link alternative routes, LBA routing is asymptotically optimal in the diverse routing limit (see Section 4.5). This policy is robust to changes in traffic patterns, but has the difficulty that it requires information on the current states of all possible alternative paths before an alternative routing decision is made.

A much simpler routing scheme is DAR (Gibbens et al., 1989, Gibbens and Kelly, 1990). In this scheme, for each pair of nodes, a record is maintained of the current preferred alternative route, and this is the one that is used if a call cannot be routed directly. If neither the direct route nor the current preferred alternative route are available, then the call is rejected, and a new preferred alternative route is chosen at random from those available. Directly routed traffic is again usually protected by a reservation parameter. This policy is easy to implement. It does not require information about the current state of the system to be held at any node, just a record of the current preferred alternative route to other nodes. It is also robust to changes in traffic patterns—alternative routes on which the load increases will be discarded and replaced by routes on which the load is lower. Neither LBA routing nor DAR require traffic rates to be known or estimated (except approximately, in order to set the appropriate level of the reservation parameters).

Acceptance probabilities for controlled loss networks are usually estimated using a generalised version of the reduced load or knapsack approximation of Section 2.4. As there, we make the approximation (6) for the stationary distribution $\pi'$ of the resource occupancy process $m(\cdot)$. Each of the marginal distributions $\pi'_j$ is estimated as the stationary distribution of a Markov process on $\{0, \ldots, C_j\}$ which approximates the behaviour of the
resource \( j \) considered in isolation. Let \( p_{jr} \) be the probability under this distribution that a call of type \( r \) is accepted, subject to the controls of the model, with \( p_{jr} = 1 \) if \( A_{jr} = 0 \). In the case of the canonical model, in which no alternative resource usage is allowed, calls of each type \( r \) are assumed to arrive at resource \( j \) at a rate \( \nu_r \prod_{k \neq j} p_{kr} \)—this is the “reduced load” for calls of type \( r \) at this resource; further, calls of this type arriving at this resource are subject to the acceptance controls of the model and, if accepted, depart at rate \( \mu_r \) as usual. The estimated stationary distribution \( \pi'_j \) then determines the acceptance probabilities \( p_{jr} \) at the resource \( j \). Thus we are again led to a set of fixed point equations which determine—not always uniquely—the acceptance probabilities \( p_{jr} \) for all \( r \in \mathcal{R} \) and \( j \in \mathcal{J} \). Finally the stationary network acceptance probability \( P_r \) for calls of each type \( r \) is again given by \( P_r = \prod_{j \in \mathcal{J}} p_{jr} \).

In the case of a communications network where the canonical model is extended by allowing the possibility of alternative routing, it is necessary to modify the above approximation. Suppose, for example, that a link (resource) \( j \) forms part of the second choice route for calls of type \( r \). Then, in the one-dimensional process associated with link \( j \), the arrival rate for calls of type \( r \) is taken to be the product of the arrival rate \( \nu_r \) at the network, the probability that a call of this type is rejected on its first-choice route, and (as before) the probabilities that the call can be accepted at each of the remaining resources on the alternative route (see e.g. Gibbens and Kelly, 1990).

The basis of the reduced load approximation is the approximate factorisation of the distribution \( \pi'_j \) above. In the case of controlled networks, this approximation fails to become exact under the Kelly limiting regime in which capacities and arrival rates increase in proportion. It may, however, be expected to hold under sufficiently diverse routing. It is known to be remarkably accurate in most applications.

4 Dynamical behaviour and stability

4.1 Fluid limits for large capacity networks

We now consider the dynamical behaviour of large networks. As well as such behaviour being of interest in its own right—for example in networks in which input rates change suddenly, fixed points of network dynamics correspond to equilibrium, or quasi-equilibrium, states of the network (see below). The identification of such points is often the key to understanding long-term behaviour, in particular to resolving stability questions and determining stationary distributions where (as is usual) the latter may not be directly calculated. However, we note that it is characteristic of loss networks that, from any initial state, equilibrium is effectively achieved within a very few call holding times, so that transient performance is of less significance than is the case for networks which permit queueing.

We describe a theory first suggested by Kelly (1991). We yet again assume the Kelly limiting scheme described in the Introduction, in which the network topology is held fixed and arrival rates and capacities are allowed to increase in proportion. More explicitly, we consider a sequence of networks satisfying our usual Markov assumptions (though this is not strictly necessary) and indexed by a scale parameter \( N \). All members of the sequence are identical in respect of the (finite) sets \( \mathcal{R}, \mathcal{J} \), the matrix \( A = (A_{jr}, j \in \mathcal{J}, r \in \mathcal{R}) \), and the departure rates \( \mu_r \), \( r \in \mathcal{R} \). For the \( N \)th member of the sequence, calls of each type \( r \) arrive at rate \( N \nu_r \) for some vector of parameters \( \nu \), and the capacity of each resource \( j \).
is $NC_j$ for some vector of parameters $C$, where, for simplicity, we take each $C_j$ to be integer-valued. As always, it is convenient to define $\kappa_r = \nu_r/\mu_r$ for each $r \in \mathcal{R}$.

We now describe the rules whereby calls are accepted. For each $N$, let $n^N(t) = (n^N_r(t), r \in \mathcal{R})$, where $n^N_r(t)$ is the number of calls of type $r$ in progress at time $t$. Define also the free capacity process $\bar{m}^N(\cdot) = (\bar{m}^N_j(\cdot), j \in \mathcal{J})$ where each $\bar{m}^N_j(t) = NC_j - \sum_{r \in \mathcal{R}} A_{jr} n^N_r(t)$ is the free capacity of resource $j$ at time $t$. A call of type $r$ arriving at time $t$ is accepted if and only if the free capacity $\bar{m}^N(t^-)$ of the system, immediately prior to its arrival, belongs to some acceptance region $\mathcal{A}_r \subset \mathbb{Z}_+^{\mathcal{J}}$. We take the acceptance regions $\mathcal{A}_r$, $r \in \mathcal{R}$, to be independent of $N$, although, in a refinement of the theory, some dependence may be allowed. Note that, in a change from our earlier conventions, the acceptance regions $\mathcal{A}_r$ are defined in terms of the free capacity of each system.

While the above description defines instances of the canonical model of the Introduction, more sophisticated controls, such as those involving the use of alternative routing in communications networks, may be modelled by the suitable redefinition of input streams and acceptance sets (see Hunt and Kurtz, 1994).

For each $N$, define the normalised process $x^N(\cdot) = n^N(\cdot)/N$, which takes values in the space

$$X = \{x \in \mathbb{R}_+^J : \sum_{r \in \mathcal{R}} A_{jr} x_r \leq C_j \text{ for all } j \in \mathcal{J}\}. \quad (33)$$

Assume that, as $N \to \infty$, the initial state $x^N(0)$ converges in distribution to some $x(0) \in X$, which, for simplicity, we take to be deterministic. Then we might expect that the process $x^N(\cdot)$ should similarly converge in distribution to a fluid limit process $x(\cdot)$ taking values in the space $X$, with dynamics given by

$$x_r(t) = x_r(0) + \int_0^t (\nu_r \bar{P}_r(u) - \mu_r x_r(u))du, \quad r \in \mathcal{R}, \quad (34)$$

where, for each $t$, $\bar{P}_r(t)$ corresponds to the limiting rate at which calls of each type $r$ are being accepted at time $t$.

A rigorous convergence result is given by Hunt and Kurtz (1994). A somewhat technical condition (always likely to be satisfied in applications) is required on the acceptance sets $\mathcal{A}_r$. However, the main difficulty is that in some, usually rather pathological, cases the limiting acceptance rates $\bar{P}_r(t)$ may fail to be unique.

In many cases, though, it is possible to show that, for each $r$, there does exist a unique function $P_r$ on $X$ such that, for each $t$, we have $\bar{P}_r(t) = P_r(x(t))$. In general, the trajectories of the limit process $x(\cdot)$ are then deterministic functions of their initial positions $x(0)$. The fixed points $\hat{x}$ of the limit process $x(\cdot)$ are given by the solutions of

$$\nu_r P_r(\hat{x}) = \mu_r \hat{x}_r, \quad r \in \mathcal{R}. \quad (35)$$

In the case of a single fixed point $\hat{x}$, to which all trajectories of $x(\cdot)$ converge, it may be shown that the stationary distribution of the original normalised process $x^N(\cdot)$ converges to that concentrated on the single point $\hat{x}$. Then in particular, for each $r$, $P_r(\hat{x})$ is the limiting stationary acceptance probability for calls of type $r$. In the case of multiple fixed points, those which are locally stable correspond to “quasi-stationary” distributions of the process $x^N(\cdot)$, i.e. regimes which are maintained over periods of time which are lengthy but finite.
4.2 Single resource networks

As the simplest non-trivial application of the above theory, we consider the case \( J = 1 \) of a single resource, for which equilibrium behaviour was described in Section 3.1. It is again convenient to write \( A_r \) for \( A_{1r} \) for each \( r \), and similarly \( C \) for \( C_1 \). The technical condition referred to above on the acceptance sets \( \bar{A}_r \subseteq \mathbb{Z}_+ \), here reduces to the requirement that, for each \( r \), either \( m \in \bar{A}_r \) for all sufficiently large \( m \in \mathbb{Z}_+ \)—we let \( \mathcal{R}^* \) denote the set of such \( r \)—or \( m \notin \bar{A}_r \) for all sufficiently large \( m \in \mathbb{Z}_+ \).

Here the functions \( P_r \) defined above always exist (see Hunt and Kurtz, 1994). To identify them, define, for each \( x \in X \), the Markov process \( \bar{m}_x(\cdot) \) on \( \mathbb{Z}_+ \) with transition rates given by

\[
\bar{m} \rightarrow \begin{cases} 
\bar{m} - A_r & \text{at rate } \nu_r I_{\{m \in \bar{A}_r\}} \\
\bar{m} + A_r & \text{at rate } \mu_r x_r, 
\end{cases}
\]  

(36)

Let \( \pi_x \) be the stationary distribution of this process where it exists. Define \( \bar{X} \subseteq X \) by

\[
\bar{X} = \{ x \in X : \sum_{r \in \mathcal{R}} A_r x_r = C \text{ and } \pi_x \text{ exists} \}. 
\] (37)

(The set \( \bar{X} \) may be thought of as consisting of those points in \( X \) for which the limiting dynamics are “blocking”.) Then, for \( x \in \bar{X} \), we have \( P_r(x) = \pi_x(\bar{A}_r) \) for all \( r \); for \( x \in X \setminus \bar{X} \), we have \( P_r(x) = 1 \) for \( r \in \mathcal{R}^* \) and \( P_r(x) = 0 \) for \( r \notin \mathcal{R}^* \). The fixed points \( \hat{x} \) of the limiting dynamics (in general there may be more than one such) are then given by the solutions of (35).

Consider now the case of reservation-type controls, and suppose that the call types are arranged in order of decreasing priority. The acceptance regions are thus given by \( \bar{A}_r = \{ m : \bar{m} \geq k_r + A_r \} \) for some \( 0 = k_1 \leq k_2 \leq \cdots \leq k_R \) and we have \( \mathcal{R}^* = \mathcal{R} \). It is easy to see that, in the light traffic case given by \( \sum_{r \in \mathcal{R}} A_r \kappa_r \leq C \), the single fixed point \( \hat{x} \) of the limiting dynamics is given by \( \hat{x}_r = \kappa_r \) for all \( r \), and that all trajectories of these dynamics converge to \( \hat{x} \). In the heavy traffic case given by \( \sum_{r \in \mathcal{R}} A_r \kappa_r > C \), define \( \bar{X} \subseteq X \) by

\[
\bar{X} = \{ x \in X : \sum_{r \in \mathcal{R}} A_r x_r = C \text{ and } x_r < \kappa_r \text{ for all } r \in \mathcal{R} \}. 
\]

Then it is straightforward to show that \( \bar{X} \subseteq \bar{X} \) and that all fixed points of the limiting dynamics lie within \( \bar{X} \) (see Bean et al., 1995). In the case where \( A_r = 1 \) for all \( r \), it is also straightforward to show that there is a unique fixed point. It is unclear whether it is possible, for more general \( A_r \), to have more than one fixed point.

Now define \( r_0 \geq 0 \) to be the maximum value of \( r \in \mathcal{R} \) such that \( \sum_{r \leq r_0} A_r \kappa_r \leq C \). Suppose that the reservation parameters \( k_1, \ldots, k_r \) are allowed to increase. Further consideration of the processes \( \pi_x \) shows that, in the limit (formally as these reservation parameters tend to infinity), the fixed point \( \bar{x} \) is necessarily unique and is such that \( P_r(\bar{x}) = 1 \) for all \( r \leq r_0 \), with, in the heavy traffic case, \( 0 \leq P_{r_0+1}(\bar{x}) \leq 1 \) and \( P_r(\bar{x}) = 0 \) for all \( r \geq r_0 + 2 \). Since the stationary distributions associated with our sequence of networks converge to that concentrated on the unique fixed point \( \bar{x} \), it follows that the reservation strategy does indeed approximate, and in the limit achieve, the complete prioritisation of call types discussed in Section 3.1. As mentioned there, and as easily verified from the above analysis, quite small values of the reservation parameters \( k_1, \ldots, k_r \) are sufficient to achieve a very good approximation to this prioritisation.
Even in the present single-resource case it is possible to achieve nonuniqueness of the fixed points of the limit process $x(\cdot)$ by the use of more general, and sufficiently perverse, controls, in particular with the use of acceptance sets of the form $\bar{A}_r = \{\bar{m}: A_r \leq \bar{m} \leq k_r + A_r\}$ for some $k_r \geq 0$ (see Bean et al., 1997). Thus we may construct networks which have several (very different) regimes which are quasi-stationary in the sense discussed above.

4.3 Multi-resource networks: the uncontrolled case

We now consider multi-resource networks, and again study the behaviour of the fluid limit process $x(\cdot)$ associated with the Kelly limiting scheme. Here in general a rich variety of behaviour is possible. However, in the case of the uncontrolled networks of Section 2, in which calls of all types are accepted subject only to the availability of sufficient capacity, the process $x(\cdot)$ is rather well-behaved. Note that here, in terms of the available free capacity, the acceptance sets are given by, for each $r \in \mathcal{R}$,

$$\bar{A}_r = \{\bar{m}: \bar{m}_j \geq A_j r \text{ for all } j\}. \quad (38)$$

Recall also that $X$ is as given by (33). Define the (real-valued) concave function $f$ on $X$ by

$$f(x) = \sum_{r \in \mathcal{R}} (x_r \log \nu_r - x_r \log \mu_r x_r + x_r) \quad (39)$$

and let $\hat{x}$ be the value of $x$ which maximises $f(x)$ in $X$. Kelly (1986) shows that, as $N \to \infty$, the stationary distribution of the process $x^N(\cdot)$ converges to that concentrated on the single point $\hat{x}$. (Indeed this is the basis of his original derivation of the the limiting acceptance probabilities considered in Section 2.4.)

Assume for the moment the unique existence of the functions $P_r$ on $X$ introduced above. Then, for the fluid limit process $x(\cdot)$, it follows from (34) and (39) that $df(x(t))/dt = g(x(t))$ where the function $g$ on $X$ is given by

$$g(x) = \sum_{r \in \mathcal{R}} \frac{\partial f(x)}{\partial x_r} \left(\nu_r P_r(x) - \mu_r x_r\right)$$

$$= \sum_{r \in \mathcal{R}} \left(\log \nu_r - \log \mu_r x_r\right) \left(\nu_r P_r(x) - \mu_r x_r\right).$$

Analogously to the preceding section, for each $x \in X$, the limiting acceptance probabilities $P_r(x)$ are given by consideration of the stationary distribution of a “free capacity” Markov process whose transition rates depend on $x$. Some simple analysis of the equilibrium equations which define this stationary distribution (see Zachary, 2000) now shows that $g(x) \geq 0$ for all $x \in X$ with equality if and only if $x = \hat{x}$.

Thus the dynamics of the limit process $x(\cdot)$ are such that, away from the point $\hat{x}$, the function $f(x(\cdot))$ is always strictly increasing. It thus acts as a Lyapunov function, ensuring that all trajectories of the process $x(\cdot)$ converge to the single fixed point $\hat{x}$. Indeed a rigorous application of the fluid limit theory of Hunt and Kurtz (1994) (again see Zachary, 2000, for details) shows this result continues to hold even if the functions $P_r$ on $X$ are not uniquely defined (whether this can ever happen in the case of uncontrolled networks remains an open problem). The result therefore establishes an important stability property of uncontrolled networks, and guarantees that the stationary distribution describes the typical behaviour of the network.
For general multi-resource networks, the fluid limit process $\mathbf{x}(\cdot)$ associated with the Kelly limiting scheme may fail to be unique, and may in particular exhibit multiple fixed points. We describe in some detail an elementary example, which is a simplification of one due to Hunt (1995b). Suppose that $R = 3$, $J = 2$, and that the matrix $A$ is given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. $$

Thus in particular calls of types 1 and 2 each require capacity from a single resource, while calls of type 3 require capacity from both resources in the network. Suppose further that the (free capacity) acceptance sets are given by, for some $k_1, k_2 \geq 1$,

$$\mathcal{A}_1 = \{ \mathbf{m} : 1 \leq \bar{m}_1 \leq k_1 \}, \quad \mathcal{A}_2 = \{ \mathbf{m} : 1 \leq \bar{m}_2 \leq k_2 \}, \quad \mathcal{A}_3 = \{ \mathbf{m} : \bar{m}_1 \geq 1, \bar{m}_2 \geq 1 \}. $$

(As Hunt remarks, this is not entirely unrealistic: in more complex networks, operating under some form of alternative routing, certain resources may have calls of certain types routed over them precisely when the network is in general very busy.) Finally suppose that $\mu_r = 1$ for all $r$ and that the vectors $\mathbf{v}$ and $\mathbf{C}$ defined in Section 4.1 (each to be scaled by $N$ for the $N$th member of the sequence of networks) are given by $\mathbf{v} = (\nu_1, \nu_2, \nu_3)$ and $\mathbf{C} = (C, C)$.

The process $\mathbf{x}(\cdot)$ takes values in the space $X = \{ \mathbf{x} \in \mathbb{R}_+: x_1 + x_3 \leq C, \ x_2 + x_3 \leq C \}$. Its dynamics may be determined through the fluid limit theory outlined above. For $\mathbf{x} \in X_0 := \{ \mathbf{x} \in X : x_1 + x_3 < C, \ x_2 + x_3 < C \}$ (corresponding to limit points of the dynamics well away from the capacity constraints) the limiting acceptance probabilities are well-defined and given by

$$P_1(\mathbf{x}) = P_2(\mathbf{x}) = 0, \quad P_3(\mathbf{x}) = 1. \quad (40)$$

For $\mathbf{x} \in X_1 := \{ \mathbf{x} \in X : x_1 + x_3 = C, \ x_2 + x_3 < C \}$ and for $\mathbf{x} \in X_2 := \{ \mathbf{x} \in X : x_1 + x_3 < C, \ x_2 + x_3 = C \}$ (corresponding in both cases to limit points of the dynamics such that only one capacity constraint is relevant) the limiting acceptance probabilities are again well-defined and given by consideration of a Markov process on $\mathbb{Z}_+$ as in the single resource case considered in Section 4.2 (For $\mathbf{x} \in X_1$, for example, it follows from the definition of $\mathcal{A}_2$ that the transition rates of this Markov process are as if $\nu_2 = 0$.) For $\mathbf{x} \in X_{12} := \{ \mathbf{x} \in X : x_1 + x_3 = C, \ x_2 + x_3 = C \}$ it is necessary to consider also a “free capacity” Markov process on $\mathbb{Z}_+^2$.

In the case $\nu_3 \leq C$, these Markov processes all fail to possess stationary distributions and the limiting acceptance probabilities are given by (40) for all $\mathbf{x} \in X$. Thus the limit process $\mathbf{x}(\cdot)$ is as if $\nu_1 = \nu_2 = 0$ and all trajectories of this process are deterministic functions of their initial values and tend to the single fixed point $\hat{\mathbf{x}} = (0, 0, \nu_3)$.

The case $\nu_3 > C$ is more interesting. Here it is readily verified that the limit process $\mathbf{x}(\cdot)$ possesses no fixed points in $X_0$. Within $X_1$ consideration of the stationary distribution of the Markov process defined in Section 4.2 shows that there is a single fixed point $\mathbf{x}^{(1)} = (a_1, 0, C - a_1)$ for some $a_1$ which is independent of $\nu_2$. Similarly within $X_2$ there is a single fixed point $\mathbf{x}^{(2)} = (0, a_2, C - a_2)$ for some $a_2$ which is independent of $\nu_1$. However, within $X_{12}$ the dynamics of the limit process $\mathbf{x}(\cdot)$ are not deterministic. It is further not difficult to show that all trajectories of $\mathbf{x}(\cdot)$ which avoid the set $X_{12}$ tend deterministically to one of the two fixed points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ above (depending on whether the set $X_1$ or the
set $X_2$ is hit first). Those trajectories of $x(\cdot)$ which do hit $X_{12}$ may, in an appropriate probabilistic sense, tend to either $x^{(1)}$ or $x^{(2)}$.

The interpretation of the above behaviour is the following. Suppose that $N$ is large and that, for example, resource 1 fills to capacity first. Then this resource remains full and blocks sufficient of the type 3 calls to ensure that resource 2 remains only partially utilised, with no calls of type 2 ever being accepted. This corresponds to a “quasi-stationary” state whose limit, as $N \to \infty$, is concentrated on the fixed point $x^{(1)}$. Alternatively, if resource 2 fills to capacity first, the network settles, for an extended period of time, to a quasi-stationary state whose limit is concentrated on the fixed point $x^{(2)}$. While, for finite $N$, transitions between these two quasi-stationary states will eventually occur, the time taken to do so can be shown to increase exponentially in $N$.

The behaviour in the above example is typical of that which may occur in more general networks—in particular those using alternative routing strategies—which are poorly controlled. Fluid limits may be used to study behaviour in networks with high capacities and correspondingly high arrival rates, and to choose values of, for example, reservation parameters so as to ensure that the network does not spend extended periods of time in states in which it is operating inefficiently. A realistic example here is the fully-connected network with alternative routing considered in Section 3.2.

As noted above, fluid limits may also be used to study equilibrium behaviour, especially in the case where all trajectories of the limit process $x(\cdot)$ tend to a unique fixed point $\hat{x}$. In particular we may show that, for the Kelly limiting regime considered here, the limiting stationary distribution of the free capacity processes $m_N(\cdot)$ in general only has a product form in the case of uncontrolled networks. This product-form assumption is the basis of the commonly used approximations considered in Section 3.2. Its justification owes more to the results for the diverse routing limit also considered there and in Section 4.5.

### 4.5 The diverse routing limit

In this section we consider the fluid limit obtained under the diverse routing regime discussed in the Introduction. Although a high degree of symmetry is required in order to obtain formal limits, the results obtained lend support to the commonly made assumptions of independence of resource blocking which are used, for example, in the construction of the approximations discussed in Section 3.2.

As outlined earlier, the diverse routing regime holds when the numbers of resources and possible “routes” in the network increase, while the total capacity and arrival rate at each resource remains constant. For this limit to exist we require a high degree of symmetry in the network. There are two canonical examples (with variants) that have been extensively studied. We describe both here using the terminology of communications networks.

The first is the so-called star network (see, for instance, Whitt, 1985, Ziedins and Kelly, 1989; Hunt, 1995a). Here there are $K$ links, each with capacity $C$. The scale parameter of the regime is then taken to be $K$. Assume that calls of any size $r \geq 1$ require unit capacity at each of $r$ resources and have holding times with unit mean. Then in a symmetric network there are $\binom{K}{r}$ possible choices of the set of links for such a call. Let the arrival rate for each such choice be $\nu^K_r = \nu_r / (\binom{K-1}{r-1})$, so that the total arrival rate at each resource for calls of size $r$ is exactly $\lambda_r$. For example, we may assume that the $K$ links are distributed around a central hub, through which all communications must pass. Many variants of this model are possible—multiple call sizes can coexist in the network, as can multiple capacities,
provided only that the proportion of links with any given capacity remains constant as \( K \) increases. The network is assumed to have fixed routing and the only permissible controls are those on admission.

Let \( x^K(t) = (x^K_j(t), j \in J) \) where \( x^K_j(t) \) is the proportion of links in which \( j \) units of capacity are in use at time \( t \). For the network without admission controls, Whitt (1985) showed that, given the initial point \( x(0) \), the process \( x^K(t) \) converges weakly to a deterministic limit process \( x(\cdot) \), which satisfies a set of first-order differential equations with a unique fixed point \( \bar{x} \), such that \( x(t) \to \bar{x} \) as \( t \to \infty \) for all initial \( x(0) \). The limit \( \bar{x} \) coincides exactly with that given by the Erlang fixed point approximation. Recall that the latter is obtained from the assumption that the stationary free capacity distributions on the various links of the network are independent of each other. For the case where all calls are of size two, Hunt (1995a) obtained a functional central limit theorem for the process \( x^K(\cdot) \), with the limit an Ornstein-Uhlenbeck diffusion process (as previously conjectured by Whitt), which was then extended to more general sizes and initial conditions by Graham and Meleard (1995). In the case of networks with admission controls very little has been proved. MacPhee and Ziedins (1996) studied such networks and gave a weak convergence result for the process \( x^K(\cdot) \). However, there remain many open questions about the behaviour of this process.

The second canonical example of the diverse routing regime is that of the fully connected network with alternative routing (Hunt and Laws, 1993). Here both admission and routing controls are possible. The network has \( N \) nodes; between each pair of these there is a link with capacity \( C \), so that the total number of links is \( K = \binom{N}{2} \). Here again \( K \) is the scale parameter. Calls arrive at each link at rate \( \nu \); each call has a unit capacity requirement and holding time of mean 1. There are three possible actions on the arrival of a call: (i) accept the call at that link, (ii) select a pair of links that form an alternative route between that pair of nodes and route the call along this, or (iii) reject the call. Hunt and Laws showed that an asymptotically optimal policy, in the sense of minimising the average number of lost calls in equilibrium, is to route a call directly if possible and otherwise to route it via an alternative route, provided that the remaining free capacity on each link of the alternative route is at least some reservation parameter \( k \), where the optimal choice of \( k \) is determined by the parameters \( K \) and \( \nu \). The optimal choice of alternative route is given by choosing that which is least busy, i.e. which maximises the minimum of the free capacities on the two links. The analysis of Hunt and Laws largely dispenses with the graph structure inherent in the choice of alternative routes, an assumption justified by analogy with earlier results of Crametz and Hunt (1991) in relation to the simpler model without reservation.

As in the example of the star network, of interest here is the process \( x^K(\cdot) \), defined as earlier. Hunt and Laws showed weak convergence of this process to a deterministic limit process. They showed that this limit process satisfies differential equations which yield the constraints for a linear programming problem, the solution to which gives an upper bound on the acceptance probabilities. (These constraints correspond to the detailed balance equations that in equilibrium govern the changes in occupancy of a single link.) They further showed that their policy achieves this upper bound.
5 Further developments and open questions

Our discussion has of necessity omitted many topics of interest, some of which we mention briefly here, as well as discussing some remaining open questions.

One such topic is the application of large deviations techniques to loss networks in order to estimate, for example, blocking probabilities in cases where it is important to keep these very small. For an excellent introduction to this see, for instance, Shwartz and Weiss (1994); later papers include those by Simonian et al. (1997) and by Graham and O’Connell (2000).

In some models of communications networks, particularly those whose graph structure is tree-like, the network topology may be such as to lend itself to more accurate calculations of acceptance probabilities, involving recursions that do not make the link independence assumption that is such an essential feature of the approximations presented above (see Zachary and Ziedins, 1999).

Extensions of loss network models include recent work by Antunes et al. (2005) which studies a variant of the model where customers may obtain service sequentially at a number of resources, each of which is a loss system. The aim here is to model a cellular wireless system where a call in progress may move from base station to base station. Several authors have also considered explicitly systems with time-varying arrival rates and/or retries (see, for example, Jennings and Massey, 1997, and Abdalla and Boucherie, 2002).

A large number of interesting and important open problems remain. The approach to most of these seems to lie in a better understanding of network dynamics. There has been no systematic investigation of how to achieve asymptotically optimal control in a general network (for example in the sense of Section 3.2), using controls which are simple, decentralised, and robust with respect to variations in network parameters, although, for communications networks, there is a belief that this will usually combine some form of alternative routing with the use of reservation parameters to guarantee stability.

A further major problem is that of the identification of instability, where the state of a network may remain over extended periods of time in each of a number of “quasi-equilibrium” distributions, some of which may correspond to highly inefficient performance. Instability is further closely linked to problems of phase transition in the probabilistic models of statistical physics, and to the study of how phenomena such as congestion propagate through a network. At present results only exist for some very regular network topologies (see, for instance, Ramanan et al., 2002 and Luen et al., 2006).

Questions related to those above concern the identification of fluid limits, and in particular the problem of the uniqueness of their trajectories given initial conditions. It is notable that the uniqueness question has not yet been resolved even in the case of a general uncontrolled loss network, although it is known that here all trajectories do tend to the same fixed point, thus guaranteeing network stability. Further, while fixed points of fluid limits identify quasi-equilibrium states of a network, detailed behaviour within such states, and the estimation of the time taken to pass between them, requires a more delicate analysis based on the study of diffusion limits. Here relatively little work has been done (see Fricker et al., 2003).

Finally we mention that loss networks may be seen as a subclass of a more general class of stochastic models, with state space $\mathbb{Z}_+^R$ for some $R$ and fairly regular transition rates between neighbouring states. Notably their analysis has much in common with that of
processor-sharing networks, in which calls again have a simultaneous resource requirement. A unified treatment is still awaited.
References

[1] Abdalla, N. and Boucherie, R.J. (2002) Blocking probabilities in mobile communications networks with time-varying rates and redialing subscribers. Ann. Oper. Res. 112, 15–34.

[2] Antunes, N., Fricker, C., Robert, P. and Tibi, D. (2005) Stochastic networks with multiple stable points. Preprint.

[3] Bean, N.G., Gibbens, R.J. and Zachary, S. (1995) Asymptotic analysis of large single resource loss systems under heavy traffic, with applications to integrated networks. Adv. Appl. Probab. 27, 273–292.

[4] Bean, N.G., Gibbens, R.J. and Zachary, S. (1997) Dynamic and equilibrium behaviour of controlled loss networks. Ann. Appl. Probab. 7, 873–885.

[5] Brockmeyer, E., Halstrom, H.L. and Jensen, A. The Life and Works of A. K. Erlang. Academy of Technical Sciences, Copenhagen.

[6] Burman, D.Y., Lehoczky, J.P. and Lim, Y. (1984) Insensitivity of blocking probabilities in a circuit switching network. J. Appl. Probab. 21, 850–859.

[7] Chung, S.-P. and Ross, K.W. (1993) Reduced load approximations for multirate loss networks. IEEE T. Commun. 41, 1222–1231.

[8] Crametz, J.-P. and Hunt, P.J. (1991) A limit result respecting graph structure for a fully connected loss network with alternative routing. Ann. Appl. Probab. 1, 436–444.

[9] Dziong, Z. and Roberts, J.W. (1987) Congestion probabilities in a circuit-switched integrated services network. Perform. Evaluation 7, 267–284.

[10] Fricker, C., Robert, P. and Tibi, D. (2003) A degenerate central limit theorem for single resource loss systems. Ann. Appl. Probab. 13, 561-575.

[11] Gibbens, R.J., Kelly, F.P. and Key, P.B. (1989) Dynamic alternative routing—modelling and behaviour. Proc. 12th Int. Teletraffic Congress, Turin (ed. M. Bonatti). Elsevier.

[12] Gibbens, R.J. and Kelly, F.P. (1990) Dynamic routing in fully connected networks. IMA J. Math. Control and Inf. 7, 77–111.

[13] Gibbens, R.J., Hunt, P.J. and Kelly, F.P. (1990) Bistability in communication networks. In Disorder in Physical Systems: a Volume in Honour of John M. Hammersley (Eds. Grimmett, G. and Welsh, D.), Oxford University Press, 1990, 113–127.

[14] Graham, C. and Meleard, S. (1995) Dynamic asymptotic results for a generalized star-shaped loss network. Ann. Appl. Probab. 5, 666–680.

[15] Graham, C. and O’Connell, N. (2000) Large deviations at equilibrium for a large star-shaped loss network. Ann. Appl. Probab. 10, 104–122.

[16] Hunt, P.J. (1995a) Loss networks under diverse routing: the symmetric star network. Adv. Appl. Probab. 25, 255–272.

[17] Hunt, P.J. (1995b) Pathological behaviour in loss networks. J. Appl. Probab. 32, 519–533.

[18] Hunt, P.J. and Kurtz, T.G. (1994) Large loss networks. Stoch. Proc. Appl. 53, 363–378.
[19] Hunt, P.J. and Laws, C.N. (1993) Asymptotically optimal loss network control. _Math. Oper. Res._ **18**, 880–900.

[20] Jennings, O.B. and Massey, W.A. (1997) A modified offered load approximation for nonstationary circuit switched networks. _Telecommunication Systems_ **7**, 229–251.

[21] Kaufman, J.S. (1981) Blocking in a shared resource environment. _IEEE T. Commun._ **29**, 1474–1481.

[22] Kelly, F.P. (1986) Blocking probabilities in large circuit-switched networks. _Adv. Appl. Probab._ **18**, 473–505.

[23] Kelly, F.P. (1991) Loss networks. _Ann. Appl. Probab._ **1**, 319–378.

[24] Kelly, F.P., Key, P. and Zachary. S. (2000) Distributed admission control. _IEEE J. Sel. Areas Commun._ **18**, 2617–2628.

[25] Luen, B., Ramanan, K. and Ziedins, I. (2006) Nonmonotonicity of phase transitions in a loss network with controls. To appear in: _Ann. Appl. Probab._

[26] MacPhee, I.M. and Ziedins, I. (1996) Admission controls for loss networks with diverse routing. In _Stochastic Networks: Theory and Applications_, Number 4 in Royal Statistical Society Lecture Note Series, (Eds. Kelly, F.P., Zachary, S. and Ziedins, I.), Oxford University Press. pp. 205–214.

[27] Marbukh, V. (1993) Loss circuit switched communication network–performance analysis and dynamic routing. _Queueing Systems_ **13**, 111–141.

[28] Ramanan, K., Sengupta, A., Ziedins, I. and Mitra, P. (2002) Markov random field models of multicasting in tree networks. _Adv. Appl. Probab._ **34**, 58–84.

[29] Ross, K.W. (1995) _Multiservice loss models for broadband telecommunication networks_. Springer.

[30] Shwartz, A. and Weiss, A. (1994) _Large deviations for performance analysis_. Chapman and Hall.

[31] Simonian, A., Roberts, J. W., Theberge, F. and Mazumdar, R (1997) Asymptotic estimates for blocking probabilities in a large multi-rate loss network. _Adv. Appl. Probab._ **29**, 806–829.

[32] Whitt, W. (1985) Blocking when service is required from several facilities simultaneously. _AT&T Tech. J._ **64**, 1807–185.

[33] Zachary, S. (2000) Dynamics of large uncontrolled loss networks. _J. Appl. Probab._ **37** (3).

[34] Zachary, S. and Ziedins, I. (1999) Loss networks and Markov random fields. _J. Appl. Probab._ **36**, 403–414.

[35] Ziedins, I.B. and Kelly, F.P. (1989) Limit theorems for loss networks with diverse routing. _Adv. Appl. Probab._ **21**, 804–830.