In this paper, we employ the machinery first shown in Ahluwalia (Ahluwalia 2020. *Phys. Eng. Sci.* **476**, 20200249. doi:10.1098/rspa.2020.0249) and EPL **131**, 41001. (doi:10.1209/0295-5075/131/41001)), to obtain a new class of spin-half mass-dimension one fermions. Such spinors, after an appropriated dual structure examination, may serve as expansion coefficients of a local field.

1. Introduction

According to Lounesto's classification [1], spin-$\frac{1}{2}$ spinor fields can be split into two major classes, regular and singular, respectively, containing Dirac spinor fields and flag-dipole spinor fields [2,3]; the last class in particular can be subject to further splitting, giving rise to flagpoles (among which Majorana spinors) and dipoles (among which Weyl spinors). Dirac, Majorana and Weyl spinors are the pillars upon which we build quantum field theory and the standard model of particle physics, and they are all very well-known. However, from what we can understand from the method followed by Lounesto, the whole spinor classification is capable of assessing the physical properties of many more spinorial structures than those that are normally considered to be fundamental (and that is Dirac, Majorana and Weyl spinors). In order to avoid leaving fundamental information behind, it may, therefore, be wise to extend investigations so as to include all possible spinor structures.

Quite recently, in [4,5], a new mathematical tool has been investigated, the so-called square root of $4 \times 4$ identity matrix, which provides a mechanism to define the expansion coefficients of spin one-half fermionic fields [4]. The given method allows us to construct new classes of fermionic fields, based on eigenvectors of a linearly independent set of re-arrangements of the
Clifford matrices, which are, however, characterized for their being of mass-dimensional one. They are, therefore, a sort of *bosonic fermion*. They are essentially single-helicity spinors that do not necessarily satisfy the Dirac equations, called Elko [6]. By construction, they are naturally neutral, and thus potential candidates for dark matter [6]. An intrinsic feature of mass-dimensional one fermions is that their spin-sums are not Lorentz covariant [3,4,7]. Consequently, the associated propagators are non-local, but endowed with a *preferred* direction. By introducing the $\tau$-deformation for the adjoint structure dual [8,9], one can easily check that Lorentz invariance can be restored [3,10]. The Lorentz invariant formulation is obtained by removing the breaking term, namely $G$ in the spin-sum. From the physical point of view, the removal of $G$ via $\tau$-deformation is mathematically well-posed.

The algorithm described above, about the square root of unity, may then provide new candidates to populate the mass-dimensional one sector, but it can also give rise to yet another new entity, namely spin-half bosons [5]. They are, therefore, *fermionic bosons*.

In the present paper, however, we will leave fermionic bosons aside, focusing on the new type of bosonic fermions. However, we are not going to focus on Elko, but rather on a type of spinor that cannot be interpreted due to the fact that it does not hold real eigenvalues under the action of $C$ operator. In fact, one could allow some couplings that are not allowed for Elko fields owing to their being neutral. Hence, the spinors introduced here are new entities belonging to the class of flag pole spinors within the Lounesto classification [1].

The paper is organized as follows: In §§2 and 3, we revisit the linearly independent square root of the identity matrix and introduce the eigenvectors of the $\Gamma_{12}$ set, exploiting the appropriated adjoint structure. In §4, we define the quantum field operators, build the propagator and compute the Hamiltonian.

### 2. The background and the new spinors

The mechanism is based on the linearly independent square roots of identity [11], given by

\[
\begin{pmatrix}
I \\
i\gamma_1 & i\gamma_2 & i\gamma_3 & \gamma_0 \\
i\gamma_2\gamma_3 & i\gamma_1\gamma_3 & i\gamma_1\gamma_2 & \gamma_0 \\
i\gamma_3\gamma_0 & i\gamma_0\gamma_2 & i\gamma_0\gamma_1 & \gamma_2 \\
i\gamma_0\gamma_1 & i\gamma_1\gamma_0 & i\gamma_1\gamma_2 & \gamma_3
\end{pmatrix}
\]  

(2.1)

and

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

denoting the above set of gamma matrices by $\Gamma_{\ell}$, $\ell = 1, \ldots, 16$, in which $\Gamma_1 = I$ and $\Gamma_{16} = i\gamma_0\gamma_1\gamma_2\gamma_3$. The extra factor $i$ ensures that $\Gamma^2_{\ell} = +I$ providing real eigenvalues. The above representation is irreducible [11].

We start by exploiting the eigenvectors of $\Gamma_{12} = i\gamma_0\gamma_2\gamma_3$ set, from [5], given (in the rest frame referential) by

\[
\lambda_1(0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2(0) = \sqrt{m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]  

(2.2)

and

\[
\lambda_3(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_4(0) = \sqrt{m} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.
\]  

(2.3)
We have defined these spinors as rest spinors $\lambda_i(0)$. To obtain the spinors for an arbitrary momentum, $\lambda_i(p) = \kappa \lambda_i(0)$, we act with the $(1/2, 0) \oplus (0, 1/2)$ boost operator

$$\kappa = \sqrt{\frac{E + m}{2m}} \left( \begin{array}{cc} \frac{p \cdot p}{E + m} & \sigma \frac{p}{E + m} \\ \sigma \frac{p}{E + m} & -\frac{p \cdot p}{E + m} \end{array} \right),$$

(2.4)

in which $\sigma$ stands for the Pauli matrices.

Since we have defined the momentum-dependent spinors, we may next verify the $\lambda$ spinor dynamics and its behaviour under action of the $C$, $P$, and $T$ discrete symmetries.

We start inspecting the action of the Dirac operator, yielding the following relations

$$\gamma_\mu p^\mu \lambda_1(p) = m \lambda_3(p) \quad \text{and} \quad \gamma_\mu p^\mu \lambda_3(p) = m \lambda_1(p)$$

(2.5)

and

$$\gamma_\mu p^\mu \lambda_2(p) = -m \lambda_4(p) \quad \text{and} \quad \gamma_\mu p^\mu \lambda_4(p) = -m \lambda_2(p).$$

(2.6)

As we can see, the introduced spinors do not satisfy a first-order equation in momentum space. However, the action of $\gamma_\mu p^\mu$ in the above relations induces the following relation

$$(p_\mu p^\mu - m^2)\lambda_i(p) = 0,$$

(2.7)

thus, as one can see, $\lambda_i(p)$ spinors fulfill the Klein–Gordon equation. The above results can be displayed in the following fashion

$$P\lambda_1(p) = \lambda_3(p) \quad \text{and} \quad P\lambda_3(p) = \lambda_1(p),$$

(2.8)

and

$$P\lambda_2(p) = -\lambda_4(p) \quad \text{and} \quad P\lambda_4(p) = -\lambda_2(p),$$

(2.9)

in which, $P = m^{-1} \gamma_\mu p^\mu$, as noted in [12].

With respect to the charge-conjugation ($C = \gamma_2 K$) and time-reversal operator ($T = i\gamma_5 C$), we have

$$C\lambda_1(p) = -i\lambda_1(p) \quad \text{and} \quad C\lambda_2(p) = i\lambda_2(p),$$

(2.10)

$$C\lambda_3(p) = i\lambda_3(p) \quad \text{and} \quad C\lambda_4(p) = -i\lambda_4(p),$$

(2.11)

$$T\lambda_1(p) = -\lambda_2(p) \quad \text{and} \quad T\lambda_2(p) = \lambda_1(p),$$

(2.12)

and

$$T\lambda_3(p) = \lambda_4(p) \quad \text{and} \quad T\lambda_4(p) = -\lambda_3(p).$$

(2.13)

With the previous results at hand, we are able to compute $P^2 = +1$, $C^2 = +1$, $T^2 = -1$, $(CPT)^2 = +1$, holding similarity with other mass-dimension one fermions [4].

### 3. Defining the dual structure

At this point, we move into a more detailed analysis concerning the dual structure definition [13,14].

Under the Dirac dual structure, the spinors presented in (2.2) and (2.3) hold a null norm

$$\bar{\lambda}_j(p)\lambda_j(p) = 0, \quad \text{for all } j.$$

(3.1)

So, following the same algorithm in the recent literature [8], we define the dual structure in its most general form

$$\bar{\lambda}_j(p) = [\mathcal{O}\lambda_j(p)]^\dagger \gamma_0,$$

(3.2)

in which the $\mathcal{O}$ operator has the following structure

$$\mathcal{O} = \frac{1}{2m} [\lambda_1(p)\bar{\lambda}_1(p) - \lambda_2(p)\bar{\lambda}_2(p) + \lambda_3(p)\bar{\lambda}_3(p) - \lambda_4(p)\bar{\lambda}_4(p)].$$

(3.3)
It is readily seen that \( O^2 = 1 \) and \( O^{-1} = O \). The action of the \( O \) operator over the introduced spinors reads

\[
O\lambda_1(p) = \lambda_3(p) \quad \text{and} \quad O\lambda_2(p) = \lambda_4(p)
\]  

(3.4) 

and

\[
O\lambda_3(p) = \lambda_1(p) \quad \text{and} \quad O\lambda_4(p) = \lambda_2(p).
\]  

(3.5) 

A straightforward calculation, after the dual structure is settled, provides

\[
\tilde{\lambda}_1(p) = \lambda_3(p)\gamma_0 \quad \text{and} \quad \tilde{\lambda}_2(p) = \lambda_4(p)\gamma_0
\]  

(3.6) 

and

\[
\tilde{\lambda}_3(p) = \lambda_1(p)\gamma_0 \quad \text{and} \quad \tilde{\lambda}_4(p) = \lambda_2(p)\gamma_0,
\]  

(3.7) 

with orthonormal relations yielding a real and invariant norm

\[
\tilde{\lambda}_j(p)\lambda_j(p) = +2m \quad \text{for} \quad j = 1,3
\]  

(3.8) 

and

\[
\tilde{\lambda}_j(p)\lambda_j(p) = -2m \quad \text{for} \quad j = 2,4.
\]  

(3.9) 

With the appropriate dual structure at hand, one may compute the spin-sum, obtaining

\[
\sum_{j=1,3} \lambda_j \tilde{\lambda}_j (p) = m(\mathbb{I} + M(p,E))
\]  

(3.10) 

and

\[
\sum_{j=2,4} \lambda_j \tilde{\lambda}_j (p) = -m(\mathbb{I} - M(p,E)),
\]  

(3.11) 

in which \( M(p,E) \) reads

\[
M(p,E) = \begin{pmatrix}
0 & 0 & \frac{(E+m+p_z)p_x}{m(E+m)} & 1 + \frac{p_y(p_y-ip_x)}{m(E+m)} \\
0 & 0 & 1 + \frac{p_y(p_y+ip_x)}{m(E+m)} & \frac{(E+m+p_z)p_x}{m(E+m)} \\
-\frac{(E+m-p_z)p_x}{m(E+m)} & 1 + \frac{p_y(p_y-ip_x)}{m(E+m)} & 0 & 0 \\
1 + \frac{p_y(p_y+ip_x)}{m(E+m)} & -\frac{(E+m+p_z)p_x}{m(E+m)} & 0 & 0
\end{pmatrix}.  
\]  

(3.12) 

The following completeness relation

\[
\frac{1}{2m} \left[ \sum_{j=1,3} \lambda_j(p) \tilde{\lambda}_j(p) - \sum_{j=2,4} \lambda_j(p) \tilde{\lambda}_j(p) \right] = \mathbb{I}
\]  

(3.13) 

is also provided. Note that \( M(p,E) \) explicitly breaks the Lorentz covariant structure. Thus, we are forced to re-examine the dual structure in such a way that the spin-sums as computed above become Lorentz invariant. It turns out that it happens only if one multiplies the Lorentz-violating piece in the spin-sums by a parameter involving the inverse of the spin-sums themselves. The new adjoint structure (henceforth indicated as \( \tilde{\lambda} \)) must be written in terms of the inverse of the
matrix \((I \pm \mathcal{M}(p, E))\). Having established this much, we define the new adjoint as
\[
\tilde{\lambda}_j(p) = \lambda_j(p)(I + \mathcal{M}(p, E))^{-1} \quad \text{for } j = 1, 3
\]
and
\[
\tilde{\lambda}_j(p) = \lambda_j(p)(I - \mathcal{M}(p, E))^{-1} \quad \text{for } j = 2, 4,
\]
yielding the following spin-sums
\[
\sum_{j=1,3} \lambda_j \tilde{\lambda}_j(p) = m(I + \mathcal{M}(p, E))(I + \mathcal{M}(p, E))^{-1}
\]
and
\[
\sum_{j=2,4} \lambda_j \tilde{\lambda}_j(p) = -m(I - \mathcal{M}(p, E))(I - \mathcal{M}(p, E))^{-1},
\]
and they are Lorentz invariant. A quick inspection reveals that the right-hand side of equations (3.10) and (3.11) do not admit inverse, that is \(\det(I \pm \mathcal{M}(p, E)) = 0\). Bearing in mind that \(\mathcal{M}^2(p, E) = I\) and that the eigenvalues of \(I\) and \(\mathcal{M}(p, E)\) are equal to \(\pm 1\), we are able to compute the inverse by applying the \(\tau\)-deformation algorithm presented in [9], furnishing
\[
(I + \mathcal{M}(p, E))^{-1} = \frac{I - \tau \mathcal{M}(p, E)}{1 - \tau^2}
\]
and
\[
(I - \mathcal{M}(p, E))^{-1} = \frac{I + \tau \mathcal{M}(p, E)}{1 - \tau^2}.
\]
Combining equations (3.6) and (3.7) with (3.14) and (3.15), we are able to write the dual structure in the following fashion
\[
\tilde{\lambda}_1(p) = 2 \tilde{\lambda}_1(p) \left( \frac{I - \tau \mathcal{M}(p, E)}{1 - \tau^2} \right),
\]
\[
\tilde{\lambda}_2(p) = 2 \tilde{\lambda}_2(p) \left( \frac{I + \tau \mathcal{M}(p, E)}{1 - \tau^2} \right),
\]
\[
\tilde{\lambda}_3(p) = 2 \tilde{\lambda}_3(p) \left( \frac{I - \tau \mathcal{M}(p, E)}{1 - \tau^2} \right),
\]
\[
\tilde{\lambda}_4(p) = 2 \tilde{\lambda}_4(p) \left( \frac{I + \tau \mathcal{M}(p, E)}{1 - \tau^2} \right),
\]
in which the constant multiplicative factor 2 is necessary to keep the relations (3.8) and (3.9) unchanged. After such a redefinition, the spin-sums finally read
\[
\sum_{j=1,3} \lambda_j(p) \tilde{\lambda}_j(p) = 2mI
\]
and
\[
\sum_{j=2,4} \lambda_j(p) \tilde{\lambda}_j(p) = -2mI.
\]
Clearly, they are Lorentz-invariant quantities. As can be seen in the current literature, the mechanism described above is the price to pay to establish the invariance of spin-sums, through the redefinition of the dual structure. A very important point regarding the construction performed in this work is the direct observation of the structure of the eigenvectors of \(\Gamma_\ell\) and its classification within Lounesto classification. As far as we know, see [15,16, and references therein], only a specific subclass of Lounesto’s class 2 is composed of spinors that lead to a theory of local quantum fields. Thus, if any of the eigenvectors of \(\Gamma_\ell\) belong to this subclass, as well as [5], we are automatically working in the scope of a local theory, otherwise, as shown in this work and also in [4,6], a careful analysis of the dual structure must be done in order to ensure locality. Thus, the roadmap for the presented construction (and also for future constructions) should be based
on the right appreciation of the dual structure—leading to a Lorentz invariant and non-vanishing norm and also a Lorentz invariant spin sums.

4. The propagator and Fermi statistics

Having settled the algebraic structure, we now move to investigate the quantum structure of the field in its general form. With \( \lambda_j(p) \) spinors as expansion coefficients we have

\[
\begin{align*}
    b(x) &\equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\nu E(p)}} \left[ \sum_{j=1,3} a_j(p)\lambda_j(p) e^{-ip\cdot x} + \sum_{j=2,4} b_j^\dagger(p)\lambda_j(p) e^{ip\cdot x} \right], \\
    \tilde{b}(x) &\equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\nu E(p)}} \left[ \sum_{j=1,3} a_j^\dagger(p)\tilde{\lambda}_j(p) e^{ip\cdot x} + \sum_{j=2,4} b_j(p)\tilde{\lambda}_j(p) e^{-ip\cdot x} \right],
\end{align*}
\]

as its adjoint. The \( \nu \) parameter is left free to be fixed later, at our convenience. Such a factor is related to the type of the field we are considering: if we are dealing with the field which obeys only a second-order equation (as mass-dimension one fermions do), then \( \nu = m \); otherwise, if one is handling a field which fulfils the first-order equation (as for the Dirac spinors) one sets \( \nu = 1 \). For now, we are not going to impose any relationship between the particle creation/annihilation operators. We are going to look for such relationships during the construction of the amplitude of propagation.

As is well known, the fermionic statistics are written as

\[
\{a_i(p), a_j^\dagger(p')\} = (2\pi)^3 \delta^3(p - p')\delta_{ij} \quad \text{and} \quad \{a_i(p), a_j(p')\} = 0 = \{a_i^\dagger(p), a_j^\dagger(p')\},
\]

whereas the bosonic statistics are

\[
[a_i(p), a_j^\dagger(p')] = (2\pi)^3 \delta^3(p - p')\delta_{ij} \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')].
\]

Similar relations are expected for \( b_i(p) \) and \( b_i^\dagger(p) \) operators.

To determine the statistics for the \( b(x) \) and \( \tilde{b}(x) \) we follow a similar programme as scrutinized in [17], where the authors looked for the main aspects of causality and Fermi statistics for mass-dimension one fermions. We consider two events, \( x \) and \( x' \), and note that the amplitude to propagate from \( x \) to \( x' \) is usually written in the following form

\[
i\mathcal{D}(x - x') = \xi \left( \frac{\langle b(x')\tilde{b}(x) \rangle \theta(t' - t) \pm \langle \tilde{b}(x)b(x') \rangle \theta(t - t')}{\langle \mathcal{T} \tilde{b}(x')b(x) \rangle} \right).
\]

One must keep in mind the plus sign holding for bosons and the minus sign for fermions [18] and that \( \mathcal{T} \) is the time ordering operator. At this stage, it is worth re-writing the two Heaviside step functions of equation (4.5) in their integral form

\[
\theta(t' - t) = -\frac{1}{2\pi i} \int d\omega \frac{e^{i\omega(t' - t)}}{\omega - i\epsilon},
\]

and

\[
\theta(t - t') = -\frac{1}{2\pi i} \int d\omega \frac{e^{i\omega(t - t')}}{\omega - i\epsilon},
\]
in which $\epsilon, \omega \in \mathbb{R}$. Now, we are able to write the amplitude of propagation

$$iD(x - x') = -\xi \lim_{\epsilon \to 0^+} \left[ \frac{d^4p}{(2\pi)^4} \frac{1}{2mE(p)} \right] \frac{d\omega}{2\pi i} \times \left[ \sum_{j=1,3} \frac{\lambda_j(p)\bar{\lambda}(p)}{\omega - i\epsilon} e^{i(\omega - E(p))(t' - t)} e^{ip(x' - x)} \right.$$  

$$\pm \left. \sum_{j=2,4} \frac{\lambda_j(p)\bar{\lambda}(p)}{\omega - i\epsilon} e^{-i(\omega - E(p))(t' - t)} e^{ip(x' - x)} \right].$$  

(4.8)

By employing these results, substituting $\omega \to p_0 = -\omega + E(p)$ in the first term and $\omega \to p_0 = \omega - E(p)$ in the second term and using the spin-sums defined above, equation (4.8) can be written as

$$iD(x - x') = i\xi \lim_{\epsilon \to 0^+} \left[ \frac{d^4p}{(2\pi)^4} \frac{1}{2mE(p)} e^{-ip_\mu(x^\mu - x'^\mu)} \right] \times \left[ \sum_{j=1,3} \frac{\lambda_j(p)\bar{\lambda}(p)}{E(p) + p_0 - i\epsilon} \pm \sum_{j=2,4} \frac{\lambda_j(p)\bar{\lambda}(p)}{E(p) - p_0 - i\epsilon} \right].$$  

(4.9)

hence furnishing

$$D(x - x') = \xi \left[ \int \frac{d^4p}{(2\pi)^4} \frac{1}{2mE(p)} e^{-ip_\mu x^\mu} \sum_{j=1,3} \lambda_j(p)\bar{\lambda}(p) \right.$$  

$$+ \beta \left. \int \frac{d^4p}{(2\pi)^4} \frac{1}{2mE(p)} e^{-ip_\mu x^\mu} \sum_{j=2,4} \lambda_j(p)\bar{\lambda}(p) \right].$$  

(4.10)

The introduced parameter $\beta$ stands for a real constant being $\beta = +1$ for bosons and $\beta = -1$ for fermions. After a straightforward calculation, we have

$$D(x - x') = \xi \int \frac{d^4p}{(2\pi)^4} \frac{1}{2mE(p)} e^{-ip_\mu x^\mu} \times \left[ \frac{2m\|}{E(p) - \sqrt{p^2 + m^2 - i\epsilon}} + \frac{(-2m\|\beta}{E(p) + \sqrt{p^2 + m^2 - i\epsilon}} \right],$$  

(4.11)

the only relevant physical result comes from the choice $\beta = -1$. This last observation is equivalent to the choice (4.3). After some mathematical manipulations in equation (4.11), we obtain

$$D(x - x') = 2\xi \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} \frac{\|}{p_\mu p^\mu - m^2 + i\epsilon}.$$  

(4.12)

Now, to fix the factor $\xi$, we normalize the integral of $D(x - x')$ over all possible $(x - x')$ space–time (which corresponds to the amplitude for the particle to be found anywhere in the Universe), so that

$$2\xi \int \frac{d^4p}{(2\pi)^4} (2\pi)^4 \delta(p^\mu) e^{-ip_\mu x^\mu} \frac{\|}{p_\mu p^\mu - m^2 + i\epsilon} = 1.$$  

(4.13)

Then the above equation translates into

$$2\xi \frac{1}{-m^2 + i\epsilon} = 1,$$  

(4.14)

and thus the normalization of the $\xi$ factor (taking the limit $\epsilon \to 0$) is

$$\xi = -\frac{m^2}{2}.$$  

(4.15)

This gives

$$D(x - x') = -m^2 \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} \frac{\|}{p_\mu p^\mu - m^2 + i\epsilon}. $$  

(4.16)
so that
\[(\partial_\mu \partial_\mu' + m^2) S_{\text{FD}}(x' - x) = -\delta^4(x' - x).\] (4.17)

The Feynman–Dyson propagator is finally
\[S_{\text{FD}}(x' - x) \overset{\text{def}}{=} -\frac{1}{m^2} \delta(x - x')\]
\[= \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu(x'' - x')} \frac{\mathbb{I}}{p_\mu p_\mu' - m^2 + i\epsilon}.\] (4.18)

This is identical to that of a scalar Klein–Gordon field. Thus, such results allow us to write the Lagrangian density
\[\mathcal{L}(x) = \partial_\mu \bar{b}(x) \partial^\mu b(x) - m^2 \bar{b}(x)b(x).\] (4.19)

Following the arguments presented in Weinberg [19, p. 500 and 502], Dirac, Weyl or Majorana spinors obey a first-order derivative field equation. Such a feature implies a propagator that for large momentum is proportional to \(p^{-1}\). This asymptotic behaviour of the associated propagator results in the fact that mass-dimension must be 3/2. However, note the unique kinematic operator that is satisfied by \(\lambda\) spinor is the Klein–Gordon equation, which is a second-order derivative equation. For this specific case, for a large momentum, the propagator is proportional to \(p^{-2}\), contrasting the previous cases. Thus, we are led to conclude that the mass-dimension of the \(b\) field is 1, rather than 3/2. Such a result is in agreement with other mass-dimension one fermions, as reported in [4,6].

We have derived, rather than assumed, the Fermi statistics for the introduced fields. Once the field equation was verified, one can easily determinate the momentum conjugate to \(b(x)\)
\[\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{b}(x)} = \frac{\partial \bar{b}(x)}{\partial t}.\] (4.20)

Using the previous results, after some algebra, one can determine the locality structure of the new fermionic field. We start by evaluating the \(b(t, x) - \pi(t, x')\) anti-commutator
\[\{b(t, x), \pi(t, x')\} = \frac{i}{4m} \left( \sum_{i=1,3} \lambda_i(p) \lambda_i^- (p) - \sum_{i=2,4} \lambda_i(-p) \lambda_i^- (-p) \right),\] (4.21)
so that, bearing in mind relations (3.8) and (3.9), one finds
\[\frac{1}{4m} \left( \sum_{i=1,3} \lambda_i(p) \lambda_i^- (p) - \sum_{i=2,4} \lambda_i(-p) \lambda_i^- (-p) \right) = \mathbb{I},\] (4.22)
yielding
\[\{b(t, x), \pi(t, x')\} = i\delta^3(x - x')\mathbb{I}.\] (4.23)
Consequently, the remaining two equal time anti-commutators vanish
\[\{b(t, x), b(t, x')\} = \{\pi(t, x), \pi(t, x')\} = 0,\] (4.24)
thus, the field \(b\) is local.

It is worth mentioning that, given the similarities with the Elko spinors [6], and also with the recent spinors defined in [4], the spinors introduced here have essentially the same Hamiltonian and zero-point energy.

5. Concluding remarks and outlooks

In this communication, we have shown the possibility of constructing an entirely new class of mass-dimension one fermionic field, in which the eigenvectors of the \(\Gamma_\ell\) play the role of expansion coefficient after a judicious examination of the dual structure. As we may see, the constructed field is local. As first noted in [4,5], such a mechanism may also serve as a general method to explore
other classes of the $\Gamma_\ell$ set. Such a possibility provides us with new fermions of mass-dimension one, as well as new fields altogether [5].

Interestingly enough, another important consequence concerning the results derived above lies in the fact that the dual of the expansion coefficients defined is not the Dirac dual, and since the propagator essentially depends on the expansion coefficients’ functions and their dual structures, a crevice opens, and one can evade the Weinberg no go theorem, as first reported in [20]. By exploring a freedom in the dual structure, as computed in (3.14) and (3.15), requiring the Lorentz invariance of the norm and also the Lorentz invariance of the spin-sums, such a mechanism shows the results are covariant only under a subgroup of Lorentz [21], not being in conflict with Weinberg works.

These new types of mass-dimension one fermions, as well as other fields in general, still have an unknown physics, which deserves to be carefully explored in detail from a dynamical perspective, and in various scenarios (such as cosmology and phenomenology). The only restriction for the possible interactions is driven by the argument of power counting re-normalizability and gauge symmetry.

Data accessibility. This article has no additional data.

Authors’ contributions. R.J.B.R.: conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, supervision, validation, visualization, writing—original draft, writing—review and editing; L.F.: conceptualization, data curation, formal analysis, investigation, methodology, resources, validation, visualization, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. No funding has been received for this article.

Appendix A. Further investigations on $\Gamma_\ell$

As a parenthetic remark, it is possible to verify that momentum-dependent and dual spinors also correspond to eigenvalues of $\Gamma_{12}$. The eigenvectors of $\Gamma_{12}$, in equations (2.2) and (2.3), are defined in the rest frame referential. If one wishes to obtain the ‘momentum dependent’ eigenvectors of $\Gamma$, one needs to analyse the following

$$\Gamma_i \lambda_i(0) = \pm \lambda_i(0), \quad (A 1)$$

where ‘±’ stands for the +1 and −1 eigenvalues. Thus, multiplying both sides by Lorentz boost operator, $\kappa$, and inserting the identity ($\kappa^{-1}\kappa$) between $\Gamma_{12}$ and $\lambda_i(0)$, we have

$$\kappa \Gamma_{12} (\kappa^{-1}\kappa) \lambda_i(0) = \pm \kappa \lambda_i(0), \quad (A 2)$$

remembering $\lambda(p) = \kappa \lambda_i(0)$, such an equation can be written in the following fashion

$$\kappa \Gamma_{12} \kappa^{-1} \lambda(p) = \pm \lambda(p). \quad (A 3)$$

Note the eigenvectors of $\kappa \Gamma_{12} \kappa^{-1}$ stand for the momentum-dependent spinors

$$\kappa \Gamma_{12} \kappa^{-1} \lambda(p) = \pm \lambda(p). \quad (A 4)$$

Now, for the dual structure relations presented along the text, we have

$$\lambda_j(p) \gamma_0 \gamma_0^\dagger \Gamma_{12} \kappa^{-1} \Gamma^\dagger \gamma_0 = \pm \lambda_j(p). \quad (A 5)$$

The dual stands eigenvector of the $\gamma_0 \Gamma^\dagger \Gamma_{12} \kappa^{-1} \Gamma^\dagger \gamma_0$ operator.
Finally, looking towards developing the same procedure for the dual structure including r-deformation, we have for $j = 1, 3$

$$\tilde{\lambda}_j(p)(\mathbb{1} + M(p, E))\gamma_0\sigma^j [\gamma I_{12} \kappa^{-1}]^j \sigma^j \gamma_0 (\mathbb{1} + M(p, E))^{-1} = \tilde{\lambda}_j(p),$$

(A 6)

and for $j = 2, 4$

$$\tilde{\lambda}_j(p)(\mathbb{1} - M(p, E))\gamma_0\sigma^j [\gamma I_{12} \kappa^{-1}]^j \sigma^j \gamma_0 (\mathbb{1} - M(p, E))^{-1} = -\tilde{\lambda}_j(p).$$

(A 7)

Note that the dual stands for eigenvectors of the $\mathbb{1} + M(p, E))\gamma_0\sigma^j [\gamma I_{12} \kappa^{-1}]^j \sigma^j \gamma_0 (\mathbb{1} + M(p, E))^{-1}$ and $\mathbb{1} - M(p, E))\gamma_0\sigma^j [\gamma I_{12} \kappa^{-1}]^j \sigma^j \gamma_0 (\mathbb{1} - M(p, E))^{-1}$ operators, once spinors $\lambda_1$ and $\lambda_3$ hold a similar (but not the same) dual structure when compared with $\lambda_2$ and $\lambda_4$; thus, the relations (A 6) and (A 7) are a direct consequence of equations (3.14) and (3.15).

References

1. Louesto P. 2001 Clifford algebras and spinors, vol. 286. Cambridge, UK: Cambridge University Press.
2. Bueno Rogerio RJ, Coronado Villalobos CH. 2019 Some remarks on dual helicity flag-dipole spinors. Phys. Lett. A 383, 125873. (doi:10.1016/j.physleta.2019.125873)
3. Lee C-Y. 2021 Fermionic degeneracy and non-local contributions in flag-dipole spinors and mass dimension one fermions. Eur. Phys. J. C 81, 1–8. (doi:10.1140/epjc/s10052-020-08759-1)
4. Ahluwalia DV. 2020 A new class of mass dimension one fermions. Proc. R. Soc. A 476, 20200249. (doi:10.1098/rspa.2020.0249)
5. Ahluwalia DV. 2020 Spin-half bosons with mass dimension three-half: towards a resolution of the cosmological constant problem. EPL 131, 41001. (doi:10.1209/0295-5075/131/41001)
6. Ahluwalia DV. 2019 Mass dimension one fermions. Cambridge, UK: Cambridge University Press.
7. Ahluwalia-Khalilova DV, Grumiller D. 2005 Spin-half fermions with mass dimension one: theory, phenomenology, and dark matter. J. Cosmol. Astropart. Phys. 2005, 012. (doi:10.1088/1475-7516/2005/07/012)
8. Ahluwalia DV. 2017 The theory of local mass dimension one fermions of spin one half. Adv. Appl. Clifford Algebras 27, 2247–2285. (doi:10.1007/s00006-017-0775-1)
9. Bueno Rogerio RJ, Hoff da Silva JM. 2017 The local vicinity of spin sums for certain mass-dimension-one spinors. EPL 118, 10003. (doi:10.1209/0295-5075/118/10003)
10. Bueno Rogerio RJ. 2021 Singular spinors and their connection. Mod. Phys. Lett. A 36, 2150093. (doi:10.1142/S0217732321500930)
11. Schweber SS. 2011 An introduction to relativistic quantum field theory. New York, NY: Courier Corporation.
12. Sperança LD. 2014 An identification of the Dirac operator with the parity operator. Int. J. Mod. Phys. D 23, 1444003. (doi:10.1142/S0218271814440039)
13. Hoff da Silva JM, Cavalcanti RT. 2019 Further investigation of mass dimension one fermionic duals. Phys. Lett. A 383, 1683–1688. (doi:10.1016/j.physleta.2019.02.041)
14. Bueno Rogerio RJ, Coronado Villalobos CH. 2018 Non-standard Dirac adjoint spinor: the emergence of a new dual. EPL 121, 21001. (doi:10.1209/0295-5075/121/21001)
15. Bueno Rogerio RJ, Hoff da Silva JM, Coronado Villalobos CH. 2020 Regular spinors and fermionic fields. Preprint (https://arxiv.org/abs/2010.08597).
16. Bueno Rogerio RJ. 2019 Constraints on mapping the Louesto’s classes. Eur. Phys. J. C 79, 929. (doi:10.1140/epjc/s10052-019-7461-5)
17. Ahluwalia DV, Nayak AC. 2014 Elko and mass dimension one field of spin one-half: causality and fermi statistics. Int. J. Mod. Phys. D 23, 1430026. (doi:10.1142/S0218271814300262)
18. Greiner W. 1990 Relativistic quantum mechanics, vol. 3. Berlin, Germany: Springer.
19. Weinberg S. 1995 The quantum theory of fields, vol. 1. Cambridge, UK: Cambridge University Press.
20. Ahluwalia DV. 2017 Evading Weinberg’s no-go theorem to construct mass dimension one fermions: constructing darkness. EPL 118, 60001. (doi:10.1209/0295-5075/118/60001)
21. Ahluwalia DV, Horvath SP. 2010 Very special relativity as relativity of dark matter: the Elko connection. J. High Energy Phys. 2010, 78. (doi:10.1007/JHEP11(2010)078)