Data-driven Reconstruction of Delay Differential Equation using Evolutionary Computation

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Abstract

This paper presents a variant of sparse representation modeling method, which has a promising performance of reconstruction of delay differential equation from measurement data. In the new method, a parameterized dictionary of candidate functions is constructed against the traditional expanded dictionary. The parameterized dictionary uses one function with variables to represent a class of functions. It has the ability to express functions in the continuous space so as to exponentially decrease the dimension of the dictionary. This property makes it possible to construct realizable dictionary associated with delay differential equation. Based on the parameterized dictionary, the reconstruction problem is rewritten and treated as mixed-integer nonlinear programming with both binary and continuous variables. Such optimization problem is hard to solve with the traditional mathematical methods while the emerging evolutionary computation provides competitive solutions. Experiments are carried out in 5 test systems including 3 well-known chaotic delay differential equations such as Mackey-Glass system. The results show the effectiveness of the new method in the reconstruction of delay differential equation.

Keywords: reconstruction, delay differential equation, mixed-integer nonlinear programming, evolutionary computation, parameterized dictionary

1. Introduction

Reconstruction of system dynamics from measurement data is a longstanding interest topic in physics[1]. Reconstruction problem, also known as the inverse engineering, exists in a wide range of physical systems. Time delay systems[2, 3, 4, 5], stochastic systems[6], partial differential equations systems[7, 8] and networks[9] are some of the most significant and challenging ones. Various types of information are distilled from the unknown system, such as fractal dimensions, Lyapunov exponents,
entropy and governing equations\cite{10, 11}, aiming at describing and understanding the system. Many data-driven reconstruction methods have been applied in this field such as embedding\cite{12}, symbolic regression\cite{13}, statistical inference\cite{14}, etc. Among all these methods, sparse representation modelling is considered the most promising one because it brings simplicity and interpretability\cite{15, 16, 17}. In sparse representation modelling, dictionary of possible system dynamic items is constructed with prior knowledge and regularization is adopted to introduce sparsity. To establish such model, Least absolute shrinkage and selection operator(LASSO\cite{18}), sparse Bayesian learning(SBL)\cite{19} and multi-objective evolutionary algorithms(MOEA\cite{20}) are applied and proved effective.

Although sparse representation modeling is the paradigm of the reconstruction problem, it fails in the reconstruction of delay differential equation(DDE) because there are two inevitable challenges as illustrated in the following:

- **The dictionary of candidate functions is hard to construct.** The dictionary must be designed complete and accurate enough to contain the true dynamics items, especially the delay. For example, the dictionary of a simple system with the formulation of $\dot{x}(t) = x(t - 1.5) + x^2$ is expected to be $\Theta(x) = [1, x, x(t - 0.1), \ldots, x(t - 1.5), x^2, x^2(t - 0.1), \ldots]$, or more complicated. Obviously, enhancing the degree of discrete delay exponentially increases the dimension of the dictionary. As is known, high dimensional dictionary matrix is expensive in both hardware storage and software computation. Hence, it is unrealistic to construct an appropriate dictionary.

- **The level of sparsity is hard to determine in regularization.** The right value of the sparsity controller, hyperparameter $\lambda$ could only be determined by brute-force search. If $\lambda$ is inappropriate, the exact reconstruction fails. To make matters worse, there may exists no right $\lambda$ if the right items is not included in the dictionary.

Above obstacles make the reconstruction of DDE an unfinished question. Hence, aiming at exact reconstruction of DDE, a variant of sparse representation modeling method is proposed in this paper. A parameterized dictionary is newly presented to express candidate functions in a low-dimensional space so as to overcome the curse of dimensionality. Reconstruction problem is then transformed into a mixed-integer nonlinear programming(MINLP) problem, which is finally efficiently solved by evolutionary computation(EC). To show the effectiveness, 5 reconstruction problems containing 3 well-known chaotic DDE are tested and the new method is analysed.
The remainder of this paper is organized as follows. Section 2 gives the formulation of reconstruction problem. Section 3 introduces the details of the proposed method. Section 4 presents the experiments and the results. Section 5 provides a discussion of the new method. Section 6 concludes this paper.

2. Problem Formulation

The proposed method is a variant of sparse representation modelling. Hence, sparse representation modelling is briefly reviewed in the beginning.

2.1. Sparse representation modelling

Consider a system governed by delay differential equations (DDEs) as:

$$\frac{d}{dt} x(t) = f[x(t), x(t-\tau)]$$

where $x \in \mathbb{R}^n$ is the system state, $f$ means the unknown system dynamics and $\tau$ stands for the time-delays. First, we collect the measurement data $x(t)$ at the sampling times $t_1$ to $t_m$ and approximate $\dot{x}(t)$ through numerical difference. Then, the dictionary $\Theta(x)$ is constructed which contains possible items of $f$ according to prior knowledge. For example, a dictionary may consist of constant, polynomial and time-delay items:

$$\Theta(x) = [1 \ x \ x^2 \ ... \ x(t-\tau_1) \ x(t-\tau_2) \ ...]$$

After that, sparse coefficients matrix $\Xi = [\xi_1 \ \xi_2 \ ... \ \xi_n]$ is defined in which $\xi_i$ is a sparse vector. Then the sparse regression problem is formulated as:

$$\dot{X} = \Theta(X)\Xi,$$

where $\dot{X}$ and $X$ are $m \times n$ matrix as:

$$\dot{X} = \begin{bmatrix} \dot{x}_1(t_1) & \cdots & \dot{x}_n(t_1) \\ \vdots & \ddots & \vdots \\ \dot{x}_1(t_m) & \cdots & \dot{x}_n(t_m) \end{bmatrix}, \ X = \begin{bmatrix} x_1(t_1) & \cdots & x_n(t_1) \\ \vdots & \ddots & \vdots \\ x_1(t_m) & \cdots & x_n(t_m) \end{bmatrix}.$$

This problem can be treated as $n$ optimization subproblems as:

$$\xi_i^* = \arg \min \| \Theta(X)\xi_i - \dot{X}_i \|_2 + \lambda \|\xi_i\|_0 \quad i = 1, 2, ..., n,$$

where $\xi_i$ and $X_i$ are the $i$th column of $\Xi$ and $X$. $\lambda$ is the regularization hyperparameter and the subscript 2 and 0 stands for $L_2$ and $L_0$ norm. The solution of Eq.(5) is the sparse coefficients matrix $\Xi^*$, thus we obtain the sparse representation of the system and finish the reconstruction.
2.2. Proposed method
As illustrated in the introduction, sparse representation modelling has limitation in the reconstruction of DDE. A new method is proposed based on sparse representation modelling, which has a different formulation of the reconstruction problem.

2.2.1. Parameterized dictionary
A parameterized dictionary is novelly presented. Its definition and analysis are introduced in this section.

**Definition 1.** \( p_i = [p_{i1}, p_{i2}, \ldots, p_{in_p}]^T \). \( n_p \) is the dimension of \( p_i \).

**Definition 2.** For \( x \in \mathbb{R}^n \), \( g(p_i) = x_1^{p_{i1}}(t - p_{i2})x_2^{p_{i3}}(t - p_{i4}) \cdots x_n^{p_{in_p} - 1}(t - p_{in_p}) \cdots \).

**Definition 3.** \( p = [p_1, p_2, \ldots, p^T_M] \). \( M \) is a given number which represents the max number of the reconstruction items. \( p \) is the parameters vector of the dictionary which satisfies \( p \in \mathbb{R}^{Mn_p} \).

The parameterized dictionary is constructed as:

\[
\Theta(x, p) = [g(p_1), g(p_2), \ldots, g(p_M)]^T.
\]  

(6)

It is apparent that the key to construct a parameterized dictionary is to construct \( g(p_i) \) and determine \( M \).

There are two advantages of the parameterized dictionary.

- It avoids the expansion of the detailed candidate functions through compressing them into the parameters. As a consequence, the expressed space of the dictionary can be roughly large but still keep low-dimensional. In the meantime, the accuracy problem of the dictionary disappears because the parameters can be continuous.

- Simplicity and interpretability of the reconstruction system is obtained without the introduction of sparsity because \( M \) is an artificially given number and can be small or big as user’s wish. Since \( M \) is the max number of the reconstruction items, it doesn’t need tuning once it is given.

**Remark 1.** For better explanation of the parameterized dictionary, a system with the formulation of \( \dot{x}(t) = x(t - 1.5) + x^2 \) is analysed as the example. Define \( g(p_i) = x^{p_{i1}}(t - p_{i2}) \). Its parameterized dictionary is expressed as \( \Theta(x, p) = [g(p_1), g(p_2), \ldots, g(p_M)]^T \), or expanded as \( \Theta(x, p) = [x^{p_{M1}}(t - p_{M2}), \ldots, x^{p_{M1}}(t - p_{M2})]^T \).
2.2.2. MINLP formulation

Based on the parameterized dictionary, reconstruction problem can be formulated as:

\[
\{\xi^*_i, P^*_i\} = \arg \min \| \Theta(X, P_i)\xi_i - \dot{X}_i \|_2 \quad i = 1, 2, ..., n,
\]

where \( P_i \) is the \( i \)th column of the parameter variables matrix \( P \) and \( P \in \mathbb{R}^{Mnp \times n} \). Note that regularization is not used in Eq.(7), so the hyperparameter tuning problem no more exists.

Eq.(7) is a non-convex optimization problem. A feasible idea is to reformulate it as mixed-integer nonlinear programming. Without loss of generality, consider an \( n \)-dimensional system, in which \( \xi_i \) is expanded as \( [\xi_{i1}, \xi_{i2}, 0, ..., \xi_{il}, 0, ..., \xi_{iM}]^T \).

**Definition 4.** The simplified coefficients vector \( d_i = [1, 1, 0, ..., 1, 0, ..., 1]^T \) which satisfies \( \xi_i = [\xi_{i1}, \xi_{i2}, 0, ..., \xi_{il}, 0, ..., \xi_{iM}]^T \circ [1, 1, 0, ..., 1, 0, ..., 1]^T \), where the operator ”\( \circ \)” is the Hadamard product of matrix.

0 and 1 in \( d_i \) stands for the zero and non-zero items in \( \xi_i \) and the simplified coefficients Matrix \( D \) is defined as the combination of all \( d_i \). Hence, the reconstruction problem is transformed into mixed-integer nonlinear programming which is formulated as:

\[
\{d^*_i, P^*_i, \xi^*_i\} = \arg \min \| \Theta(X, P_i)(d_i \circ \xi_i) - \dot{X}_i \|_2 \quad i = 1, 2, ..., n.
\]

The generation of an MINLP solution has three steps. Firstly, determine \( d_i \) which represents the trade-off of the dictionary items. Secondly, determine \( P_i \). Thirdly, perform least square method to obtain \( \xi_i \). Thus, the MINLP problem can be treated in a bi-level optimization framework. In detail, the outside optimization aims to find the optimal \( d_i \) while the inside optimization searches for the optimal \( P_i \) and its relating \( \xi_i \). It is clear that the outside optimizes the binary variables and the inside optimizes the continuous variables.

3. Proposed Algorithm

Above bi-level optimization problem is an NP-hard problem with high nonlinearity. It cannot be efficiently solved by traditional mathematical methods. However, an emerging optimization method named evolutionary computation has the potential to obtain solutions with high quality and acceptable computation cost. Therefore, EC is adopted in both outside and inside optimization and it is introduced in the beginning as preliminary of the proposed algorithm.
3.1. Evolutionary computation

Evolutionary computation\cite{21, 22, 23} represents a class of nature-inspired optimization algorithms. It aims at global optimization and works in the absence of explicit problem formulation and gradient information. As a consequence, it has a broad application in many scientific and engineering problems\cite{24, 25} where traditional mathematical methods fail. To emphasis, combinational optimization\cite{26, 27} and multi-modal optimization\cite{28, 29} are some of the most important applications in EC field. A lot of evolutionary algorithms(EAs) have been presented and well studied. Among various algorithms, particle swarm optimization(PSO)\cite{30, 31} gains special attention as it has strong global optimization ability and is easily realised. Hence, PSO is introduced here to explain the mechanism of EA.

In PSO, particles (or individuals in other EAs) are the basic units of optimization. Each particle has two characteristics, which are position \( x_i \) and velocity \( v_i \). \( x_i \) represents the solution in optimization and \( v_i \) represents the search direction and
step size. Firstly, $N$ particles are randomly initialized with $x_i^k$ and velocity $v_i^k$, in which $k$ means the iteration number. Then, the objective value of each particle is evaluated. $pbest_i$ and $gbest$ are defined which stand for the best position a particle find in its own search history and the best position the particle swarm find in the whole search history. Thus, $pbest_i$ and $gbest$ in the present iteration can be obtained after evaluation. Next, particles are updated with velocities and positions according to the rule as:

$$
\begin{align*}
    v_i^{k+1} &= \omega v_i^k + c_1 r_1 \circ (pbest_i - x_i^k) + c_2 r_2 \circ (gbest - x_i^k) \\
    x_i^{k+1} &= x_i^k + v_i^{k+1}
\end{align*}
$$

(9)

where $\omega$, $c_1$ and $c_2$ are the hyperparameters which are often defaults and $r_1$ and $r_2$ are random vectors uniformly distributed in $[0,1]$. The operator ”$\circ$” is the Hadamard product of matrix. At this point, an iteration is over and the loop continues until the iteration comes to the max iteration as set. $gbest$ in the last iteration is the best solution of the problem. Note that $gbest$ is not equivalent to the global optimum although it is always a competitive solution. The flowchart of PSO is shown in Fig.1. An intuitive description of the optimization process is shown in Fig.2.

![Figure 2](image)

Figure 2: (a) is the visualization of a 2-dimensional Rastrigin function with the formulation of $f(x) = \sum_{i=1}^{2}(x_i^2 - 10\cos(2\pi x_i) + 20)$ and the global optimum is $[0, 0]^T$. (b)-(d) are the visualization of the optimization process. The function is plotted in contours and the black dots are the particles. (b) is the $k = 1$ snap shot(randomly initialization). (c) is the $k = 25$ snap shot. (d) is the $k = 100$ snap shot. It shows the particles converge to the global optimum after a certain number of iterations.

It is worth noting that there are binary versions of PSO[32, 33, 34] and many other EAs[35, 36, 37] for the optimization problem defined in binary space. Through adding a transfer function mapping the continuous and binary solution space[34], a continuous EA can be transferred into the binary version.
3.2. Algorithm

Due to the property of the bi-level optimization, the binary EA and the continuous EA are respectively used in the outside and the inside optimization, which are denoted as $EA_{out}$ and $EA_{in}$. The selection of $EA_{out}$ and $EA_{in}$ is open. They can be the binary and the continuous version of the same EA or the combination of two different EAs. The whole algorithm is described as follow:

Algorithm 1 Proposed algorithm
1: Set $M$ and construct $\Theta(x, p)$;
2: Set the hyperparameters of $EA_{out}$ and $EA_{in}$ including $N_{out}$ and $N_{in}$, $I_{out}^{max}$ and $I_{in}^{max}$, etc.;
3: for each $i = 1, 2, \ldots, n$ do
4: Initialize the population $d_{i}^{N_{out}}$;
5: while iteration $\leq I_{out}^{max}$ do
6: for each individual of $d_{i}^{N_{out}}$ do
7: Initialize the population $P_{i}^{N_{in}}$;
8: while iteration $\leq I_{in}^{max}$ do
9: for each individual of $P_{i}^{N_{in}}$ do
10: Evaluate Eq.(8);
11: Update $P_{i}$;
12: end for
13: end while
14: Obtain $P_{i}^{*}$ and $\xi_{i}^{*}$ of $P_{i}^{N_{in}}$ for $d_{i}$;
15: Update $d_{i}$;
16: end for
17: end while
18: Obtain $d_{i}^{*}, P_{i}^{*}$ and $\xi_{i}^{*}$;
19: end for
20: Combine $d_{i}^{*}, P_{i}^{*}$ and $\xi_{i}^{*}$ thus obtain $D^{*}, P^{*}$ and $\Xi^{*}$.

4. Experiments and Results

4.1. Experiment settings

To show the effectiveness, the proposed method is executed in 5 reconstruction problems. The characteristics of the test systems are listed in Table I. System 1 and 2 are governed by ordinary differential equations(ODEs). They can be treated as the simple version of DDEs with 0 delays. System 3-5 are well-known chaotic
| ID | Name         | Formulation                                                                 | Dictionary                                                                 | Solution space |
|----|--------------|-----------------------------------------------------------------------------|----------------------------------------------------------------------------|----------------|
| 1  | Linear system| $\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.1 & 2 & 0 \\ -2 & -0.1 & 0 \\ 0 & 0 & -0.3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ | $g(p_i) = x^{p_{i,1}} y^{p_{i,2}} z^{p_{i,3}}, M = 5$ | $p_{ij} \in [0,5], p_{ij} \in \mathbb{Z}$ |
| 2  | Lorenz       | $\begin{cases} \dot{x} = \sigma(y - x) & \sigma = 10 \\ \dot{y} = x(p - z) - y & \rho = 28 \\ \dot{z} = xy - \beta z & \beta = 8/3 \end{cases}$ | $g(p_i) = x^{p_{i,1}} y^{p_{i,2}} z^{p_{i,3}}, M = 5$ | $p_{ij} \in [0,5], p_{ij} \in \mathbb{Z}$ |
| 3  | Delayed Rössler | $\begin{cases} \dot{x} = -y - z + \alpha_1 x(t - \tau_1) + \alpha_2 x(t - \tau_2) \\ \dot{y} = x + \beta_1 y \\ \dot{z} = \beta_2 + z(x - \gamma) \end{cases}$ | $\alpha_1 = 0.2, \alpha_2 = 0.5, \beta_1 = \beta_2 = 0.2, \gamma = 5.7, \tau_1 = 1, \tau_2 = 2$ | $p_{ij} \in [0,5], p_{ij} \in \mathbb{Z}$ |
| 4  | Ikeda        | $x(t) = -x(t) + \alpha \sin(x(t - \tau))$ | $\alpha = 6, \tau = 1.59, \gamma = 5.7, \tau_1 = 1, \tau_2 = 2$ | $p_{ij} \in [0,5], p_{ij} \in \mathbb{Z}$ |
| 5  | Mackey-Glass | $\begin{cases} \dot{x}(t) = -b x(t) + \frac{\alpha_1 (x(t) - \tau)}{x(t - \tau)} \\ \dot{y}(t) = \frac{\alpha_2 (y(t) - \tau)}{y(t - \tau)} \end{cases}$ | $\alpha_1 = 0.2, \beta = 0.1, c = 10, \tau = 20$ | $p_{ij} \in [0,10], p_{ij} \in \mathbb{Z}$ |
DDEs. Specifically, system 3 has two different delays, system 4 has a delay with high accuracy and system 5 has a fraction item.

Data is collected in the simulation system with the sampling interval of 0.01s and is intercepted for reconstruction with the length of 20s in system 1-4 and 80s in system 5. The approximation of $\dot{x}(t)$ is calculated by center difference. The construction of parameterized dictionary with the solution space of each system is shown in Table I. Note that the time series of a single time-delay item like $x(t - \tau)$ is obtained from the entire sampling data, as a consequence of which, the real number field in the solution space of system 4 is actually realised with an accuracy of 0.01s.

Binary PSO[32] and couple-based PSO[41] are selected as $EA_{out}$ and $EA_{in}$ in the experiment, which are respectively denoted as BPSO and CPSO. BPSO is the first version and the most widely used version of binary PSO. CPSO is a modified continuous PSO designed for multi-modal optimization problem. Their hyperparameter settings are listed in Table II, in which $M$ and $n_p$ are obtained directly from Table I. The hyperparameters are selected in a general way without tuning according to [32] and [41]. Due to the stochastic nature of EAs, each test case is run 100 times independently. All experiments are realised with Matlab code.

| Table 2: Hyperparameter settings of BPSO and CPSO. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| **BPSO**        | $N_{out}$       | $I_{out}^{max}$ | $V_{lim}^{out}$ | $\omega$       | $c_1$           | $c_2$           |
| $M$             | $N_{out}^2$     | 4               | 0.6             | 2               | 2               |
| **CPSO**        | $N_{in}$        | $I_{in}^{max}$  | $V_{lim}^{in}$  | $\omega_a$     | $\omega_b$     | $c_{1a}$        | $c_{1b}$        | $c_{2a}$        | $c_{2b}$        |
| $Mn_p$          | $N_{in}^2$      | 0.6             | 0.2             | 0.3             | 0.9             | 0.3             | 1.5             | 1.5             |

4.2. Analysis of the results

The reconstructed system formulation and the success ratio of each case are exhibited in Table III. The meaning of success is the exact reconstruction, which means any little deviation such as the time delay 1.58 in system 4 is considered fail. The optimal objective values in experiment 4 and 5 of 30 running times are shown in Table IV.

From Table III, it is shown that all systems are able to be exactly reconstructed. ODE systems have the success ratio of 1 while DDE systems have a lower success ratio. Combining Table II and Table III, it is shown that the success ratio decreases with increase of the complexity of the dictionary and the solution space. In theoretical aspect, the nonlinearity of the optimization problem Eq.(8) becomes higher as the dictionary and the solution space become more complex, which enhances the difficulty of optimization. This means EAs may trap into local optima when the
Table 3: Reconstruction results of 5 systems.

| ID | Reconstructed system | Success ratio |
|----|----------------------|---------------|
| 1  | \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.1000 & 1.9999 & 0 \\ -1.9999 & -0.1000 & 0 \\ 0 & 0 & -0.3000 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) | 100/100 |
| 2  | \( \begin{align*} \dot{x} &= 10.0301y - 10.0545x \\ \dot{y} &= 28.1237x - 1.0042xz - 1.0537y \\ \dot{z} &= 1.0006xy - 2.6657z \end{align*} \) | 100/100 |
| 3  | \( \begin{align*} \dot{x} &= -0.9998y - 0.9996z + \\ & \quad 0.1996x(t-1) + 0.5000x(t-2) \\ \dot{y} &= 1.0000x + 0.2000y \\ \dot{z} &= 0.1931 + 0.9995xz - 5.6958z \end{align*} \) | 85/100 |
| 4  | \( \dot{x}(t) = -0.9999x(t) + 5.9983 \sin(x(t-1.59)) \) | 55/100 |
| 5  | \( \dot{x}(t) = -0.9999x(t) + 0.1999x(t-20) \) | 8/100 |

Table 4: The optimal objective values in experiment 4 and 5 from 30 times running. The bold number mean it is a successful case.

| Count | System 4 | System 5 | Count | System 4 | System 5 |
|-------|----------|----------|-------|----------|----------|
| 1     | 0.0840   | 0.0262   | 16    | 0.0829   | 0.0237   |
| 2     | 0.2398   | 0.0318   | 17    | 0.2501   | 0.0201   |
| 3     | 0.0972   | 0.0274   | 18    | 0.2368   | 0.0278   |
| 4     | 0.1013   | 0.0044   | 19    | 0.2476   | 0.0288   |
| 5     | 0.2468   | 0.0220   | 20    | 0.2383   | 0.0323   |
| 6     | 0.2434   | 0.0257   | 21    | 0.1003   | 0.0294   |
| 7     | 0.0558   | 0.0384   | 22    | 0.2390   | 0.0272   |
| 8     | 0.2589   | 0.0046   | 23    | 0.2376   | 0.0047   |
| 9     | 0.2500   | 0.0254   | 24    | 0.2576   | 0.0193   |
| 10    | 0.2440   | 0.0190   | 25    | 0.2369   | 0.0249   |
| 11    | 0.0300   | 0.0110   | 26    | 0.2445   | 0.0287   |
| 12    | 0.0103   | 0.0309   | 27    | 0.2488   | 0.0218   |
| 13    | 0.2487   | 0.0276   | 28    | 0.1038   | 0.0292   |
| 14    | 0.2424   | 0.0219   | 29    | 0.2438   | 0.0183   |
| 15    | 0.2377   | 0.0199   | 30    | 0.2508   | 0.0314   |
Eq. (8) becomes extremely multimodal. This problem could be moderated through enhancing the global search ability of the algorithm by tuning hyperparameters of $EA_{out}$ and $EA_{in}$ or selecting other EAs.

From Table IV, it is indicated that both successful and unsuccessful results have a good fitting accuracy. Due to the simplicity of the parameterized dictionary, the reconstructed governing equations must have a good generalization ability. Besides, an abnormality appears in Table IV. It is shown that the unsuccessful cases in system 4 even have a smaller optimal objective value. It means the algorithm finds the global optimum but the global optimum is not related to the true system dynamics. This is caused by the deviation of the center difference. If adopting an approximation method with higher accuracy or improving the accuracy of sampling, the approximation of $\dot{x}(t)$ could be more accurate and further decrease the abnormality.

A visualisation of experiment 1-4 with the comparison of the original and the reconstructed system is shown in Fig. 3.

5. Discussion

A brief discussion of the proposed method is given in the following.

- The total computational times of evaluation is $N_{out} \cdot I_{out_{\text{max}}} \cdot N_{in} \cdot I_{in_{\text{max}}}$. When they are set as in Table II, the computational times becomes $M \cdot M^2 \cdot M_{n_p} \cdot (M_{n_p})^2$. Hence, the computational complexity of the proposed algorithm is written as $O(n_{p}^3M^6)$. $M$ is a given constant. Therefore, the computational complexity can be rewritten as $O(n_{p}^3)$. $n_p$ is influenced by the construction of the parameterized dictionary which barely increases as the complexity of the dictionary grows.

- The selection of $EA_{out}$ and $EA_{in}$ is open. Although only PSO class of EAs is tested in the experiments, other EAs also work in theory aspect.

- The hyperparameters of $EA_{out}$ and $EA_{in}$ need tuning in applications in order to obtain stronger global optimization ability.

- Unsuccessful reconstruction is inevitable even when $N$ and $I_{\text{max}}$ are set big enough because the mechanism of EA is intrinsically stochastic search.

6. Conclusion

In summary, this paper proposes a new data-driven method for reconstructing the system governing equation. The details of the method is illustrated. 5 systems are
tested including 3 chaotic DDEs. The results show the effectiveness of the proposed method to reconstruct DDE dynamics. Last but not least, this method is a generic method that can be applied to find other governing equations like ODE and fractional differential equation (FDE).

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Figure 3: (a)-(d) are the exhibitions of the original system and the reconstructed system in experiments 1-4. In each subfigure, the blue dots stand for the original system dynamic and the yellow solid line represents the reconstructed. (a)-(c) are shown in the space trajectories while (d) is in the phase portrait. It is shown that the successfully reconstructed systems reproduce the dynamics of the original systems.