HERMITE-HADAMARD AND SIMPSON-LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE HARMONICALLY
CONVEX FUNCTIONS

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Abstract. In this paper, a new identity for differentiable functions is derived. A consequence of
the identity is that the author establishes some new general inequalities containing all of the
Hermite-Hadamard and Simpson-like type for functions whose derivatives in absolute value at
certain power are harmonically convex. Some applications to special means of real numbers are also
given.

1. Introduction

Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers
and \( a, b \in I \) with \( a < b \). The following inequality

\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)\,dx \leq \frac{f(a) + f(b)}{2} \tag{1.1} \]

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality
for convex functions. Note that some of the classical inequalities for means can be derived from
(1.1) for appropriate particular selections of the mapping \( f \). Both inequalities hold in the reversed
direction if \( f \) is concave.

Following inequality is well known in the literature as Simpson inequality:

Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping
on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then the following inequality holds:

\[
\left| \frac{1}{3} \left[ f(a) + f(b) \right] + \frac{2}{b-a} \int_{a}^{b} f(x)\,dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^{4}.
\]

For some results which generalize, improve and extend the Hermite-Hadamard
and Simpson inequalities, we refer the reader to the recent papers (see [1, 2, 3, 4, 5, 6, 8]).

In [5], the author introduced the concept of harmonically convex functions and
established some results connected with the right-hand side of new inequalities
similar to the inequality (1.1) for these classes of functions. Some applications to
special means of positive real numbers are also given.

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**Definition 1.** Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically convex, if
\[
(1.2) \quad f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\]
for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (1.2) is reversed, then \( f \) is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 2.** Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold
\[
(1.3) \quad f \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.
\]

The above inequalities are sharp.

Some results connected with the right part of (1.3) was given in [5] as follows:

**Theorem 3.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q \geq 1 \), then
\[
(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \lambda_1^{-\frac{q}{p}} \left[ \lambda_2 |f'(a)| + \lambda_3 |f'(b)| \right]^{\frac{q}{p}},
\]
where
\[
\begin{align*}
\lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right), \\
\lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right), \\
\lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right) \\
&= \lambda_1 - \lambda_2.
\end{align*}
\]

**Theorem 4.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2x} \left( \frac{1}{p+1} \right)^{\frac{q}{p}} (\mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q)^{\frac{1}{q}},
\]
where
\[
\mu_1 = \frac{1}{b-a}, \quad \mu_2 = \frac{1}{a-b}.
\]
where

\[ \mu_1 = \frac{a^{2-2q} + b^{1-2q} [(b - a) (1 - 2q) - a]}{2 (b - a)^2 (1 - q) (1 - 2q)}, \]

\[ \mu_2 = \frac{b^{2-2q} - a^{1-2q} [(b - a) (1 - 2q) + b]}{2 (b - a)^2 (1 - q) (1 - 2q)}. \]

In this paper, we shall give some general integral inequalities connected with the left and right parts of (1.3), as a result of this, we shall obtained some new midpoint, trapezoid and Simpson like-type inequalities for differentiable harmonically convex functions.

2. Main results

In order to prove our main results we need the following lemma:

Lemma 1. Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a differentiable function on \( I^o \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a,b] \) then for \( \lambda \in [0,1] \) we have the equality

\[
(1 - \lambda) f\left(\frac{2ab}{a+b}\right) + \lambda \left(\frac{f(a) + f(b)}{2}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx
= \frac{ab(b-a)}{2} \left[ \int_0^{1/2} \frac{\lambda - 2t}{A_t^2} f'(\frac{ab}{A_t}) dt + \int_{1/2}^1 \frac{2 - \lambda - 2t}{A_t^2} f'(\frac{ab}{A_t}) dt \right],
\]

where \( A_t = tb + (1-t)a \).

Proof. It suffices to note that

\[
I_1 = \frac{ab(b-a)}{2} \int_0^{1/2} \frac{\lambda - 2t}{A_t^2} f'(\frac{ab}{A_t}) dt
= (2t - \lambda) f\left(\frac{ab}{A_t}\right)|_{0}^{1/2} - 2 \int_0^{1/2} f\left(\frac{ab}{A_t}\right) dt
= (1 - \lambda) f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - 2 \int_0^{1/2} f\left(\frac{ab}{A_t}\right) dt.
\]

Setting \( x = \frac{ab}{A_t} \) and \( dx = -\frac{ab(b-a)}{A_t^2} dt \), which gives

\[
I_1 = (1 - \lambda) f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - \frac{2ab}{b-a} \int_{2ab/(a+b)}^b \frac{f(x)}{x^2} dx.
\]
Similarly, we can show that

\[ I_2 = ab(b-a) \int_{1/2}^{1} \frac{2 - \lambda - 2t}{A_t^2} f'' \left( \frac{ab}{A_t} \right) dt \]

\[ = \lambda f(a) + (1 - \lambda) f \left( \frac{2ab}{a+b} \right) - \frac{2ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx. \]

Thus,

\[ \frac{I_1 + I_2}{2} = (1 - \lambda) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \]

which is required. \( \square \)

**Theorem 5.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \([a, b]\) for \( q \geq 1 \) and then we have the following inequality for \( \lambda \in [0, 1] \)

\[
(1-\lambda) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{ab(b-a)}{2} \left\{ C_1^{1-\frac{1}{q}} (\lambda; a, b) \left[ C_2 (\lambda; a, b) |f'(a)|^q + C_3 (\lambda; a, b) |f'(b)|^q \right]^{\frac{1}{q}} 
+ C_1^{1-\frac{1}{q}} (\lambda; b, a) \left[ C_3 (\lambda; b, a) |f'(a)|^q + C_2 (\lambda; b, a) |f'(b)|^q \right]^{\frac{1}{q}} \right\},
\]

where

\[
C_1(\lambda; u, \vartheta) = \frac{1}{(\vartheta - u)^2}
\times \left[ -4 + \frac{\left[ \lambda (\vartheta - u) + 2u \right][3u + \vartheta]}{u(u + \vartheta)} + 2 \ln \left( \frac{2u(u + \vartheta)}{(2 + \lambda (\vartheta - u))^2} \right) \right],
\]

\[
C_2(\lambda; u, \vartheta) = \frac{1}{(\vartheta - u)^2}
\times \left\{ \left[ \lambda (\vartheta - u) + 4u \right] \ln \left( \frac{[\lambda (\vartheta - u) + 2u]^2}{2u(u + \vartheta)} \right) 
- \frac{[\lambda (\vartheta - u) + 2u][5u + 3\vartheta]}{u(3\vartheta + u)} + 7u + \vartheta \right\},
\]

and

\[
C_3(\lambda; u, \vartheta) = C_1(\lambda; u, \vartheta) - C_2(\lambda; u, \vartheta), \ u, \vartheta > 0.
\]
Proof. Let \( A_t = tb + (1-t)a \). From Lemma 1 and using the power mean inequality, we have

\[
\left| (1 - \lambda) f\left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \left[ \left( \frac{1}{0} \right) \left( \frac{1}{\lambda - 2t} \right) \right]^{1 - \frac{1}{q}} \left( \int_0^1 \left| \frac{f'(ab)}{A_t} \right|^q \, dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{1/2}^1 \left| \frac{2 - \lambda - 2t}{A_t^2} \right| \, dt \right)^{1 - \frac{1}{q}} \left( \int_{1/2}^1 \left| \frac{f'(ab)}{A_t} \right|^q \, dt \right)^{\frac{1}{q}}.
\]

Hence, by harmonically convexity of \(|f'|^q| on [a, b] , we have

\[
\left| (1 - \lambda) f\left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}
\]

\[
\times \left\{ \left( \int_0^{1/2} \left| \frac{\lambda - 2t}{A_t^2} \right| \, dt \right)^{1 - \frac{1}{q}} \left( \int_0^{1/2} \left| \frac{\lambda - 2t}{A_t^2} \right| \, dt \right)^{\frac{1}{q}} \left( \int_0^{1/2} \left| \frac{f'(ab)}{A_t} \right|^q \, dt \right)^{\frac{1}{q}} \right\}
\]

\[
\leq \frac{ab(b-a)}{2} C_1^{1 - \frac{1}{q}} (\lambda; a, b) \left[ C_2(\lambda; a, b) |f'(a)|^q + C_3(\lambda; a, b) |f'(b)|^q \right]^{\frac{1}{q}}
\]

\[
\leq \frac{ab(b-a)}{2} C_1^{1 - \frac{1}{q}} (\lambda; a, b) \left[ C_2(\lambda; a, b) |f'(a)|^q + C_3(\lambda; a, b) |f'(b)|^q \right]^{\frac{1}{q}}.
\]

It is easily check that

\[
\int_0^{1/2} \frac{\lambda - 2t}{A_t^2} \, dt = C_1(\lambda; a, b) = \frac{1}{(b-a)^2}
\]

\[
\times \left[ -4 + \frac{\lambda (b-a) + 2a}{(a+b)} - 2a + 2 \ln \left( \frac{2a (a+b)}{(2a + \lambda (b-a))^2} \right) \right],
\]

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\[
\int_0^{1/2} \frac{1}{\lambda - 2t} t dt = C_2(\lambda; a, b) = \frac{1}{(b - a)^3}
\]
\[
\times \left\{ [\lambda (b - a) + 4a] \ln \left( \frac{\lambda (b - a) + 2a}{2a (a + b)} \right) - \frac{\lambda (b - a) + 2a (5a + 3b)}{a + b} + 7a + b \right\},
\]
\[
\int_0^{1/2} \frac{1}{\lambda - 2t} (1 - t) dt = C_3(\lambda; a, b) = C_1(\lambda; a, b) - C_2(\lambda; a, b),
\]
\[
\int_{1/2}^1 \frac{1}{\lambda - 2t} dt = C_1(\lambda; b, a), \quad \int_{1/2}^1 \frac{1}{\lambda - 2t} (1 - t) dt = C_2(\lambda; b, a),
\]
and
\[
\int_{1/2}^1 \frac{2 - \lambda - 2t}{A_t^2} dt = C_3(\lambda; b, a) = C_1(\lambda; b, a) - C_2(\lambda; b, a).
\]
This concludes the proof. \(\square\)

**Corollary 1.** Under the assumptions Theorem 1 with \(\lambda = 0\), we have
\[
\left| f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b f(x) dx \right| \leq \frac{ab (b - a)}{2}
\]
\[
\times \left\{ C_1^{1/3} (0; a, b) \left[ C_2(0; a, b) |f'(a)|^q + C_3(0; a, b) |f'(b)|^q \right]^{1/2} - C_1^{1/3} (0; b, a) \left[ C_3(0; b, a) |f'(a)|^q + C_2(0; b, a) |f'(b)|^q \right]^{1/2} \right\},
\]
where
\[
C_1(0; u, \vartheta) = \frac{2}{(\vartheta - u)^3} \left\{ \ln \left( \frac{u + \vartheta}{2u} \right) - \frac{\vartheta - u}{u + \vartheta} \right\},
\]
\[
C_2(0; u, \vartheta) = \frac{1}{(\vartheta - u)^3} \left\{ (3u + \vartheta) (\vartheta - u) \frac{u + \vartheta}{u + \vartheta} + 4u \ln \left( \frac{2u}{u + \vartheta} \right) \right\},
\]
\[
C_3(0; u, \vartheta) = \frac{1}{(\vartheta - u)^3} \left\{ \frac{2 (u + \vartheta)}{\vartheta - u} \ln \left( \frac{u + \vartheta}{2u} \right) - \frac{u + 3\vartheta}{u + \vartheta} \right\}, \quad u, \vartheta > 0.
\]

**Corollary 2.** Under the assumptions Theorem 1 with \(\lambda = 1\), we have
\[
\left| f(a) + f(b) \right| - \frac{ab}{b - a} \int_a^b f(x) dx \leq \frac{ab (b - a)}{2}
\]
\[ \times \left\{ C_1^{1 - \frac{q}{2}}(1; a, b) \left[ C_2(1; a, b) |f'(a)|^q + C_3(1; a, b) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \]

where

\[ C_1(1; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{\vartheta - u}{3u} + 2 \ln \left( \frac{2u}{u + \vartheta} \right) \right], \]

\[ C_2(1; u, \vartheta) = \frac{1}{(\vartheta - u)^3} \left[ (3u + \vartheta) \ln \left( \frac{u + \vartheta}{2u} \right) - 3(\vartheta - u) \right], \]

\[ C_3(1; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{u + \vartheta - u + 3\vartheta}{\vartheta - u} + 3(\vartheta - u) \ln \left( \frac{u + \vartheta}{2u} \right) \right], \]

Corollary 3. Under the assumptions Theorem 5 with \( \lambda = 1/3 \), we have

\[ \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{2ab}{a + b} \right) \right] - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b - a)}{2} \]

\[ \times \left\{ C_1^{1 - \frac{q}{2}}(1/3; a, b) \left[ C_2(1/3; a, b) |f'(a)|^q + C_3(1/3; a, b) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \]

where

\[ C_1(1/3; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{(\vartheta - u)(\vartheta - 3u)}{3u} + 2 \ln \left( \frac{18u(u + \vartheta)}{(5u + \vartheta)^2} \right) \right], \]

\[ C_2(1/3; u, \vartheta) = \frac{1}{(\vartheta - u)^3} \left[ \left( \frac{11u + \vartheta}{3} \right) \ln \left( \frac{18u + \vartheta}{(5u + \vartheta)^2} \right) + \frac{4u(\vartheta - u)}{3(u + \vartheta)} \right], \]

\[ C_3(1/3; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{\vartheta^2 - 4u\vartheta - u^2}{3u} + \frac{5u + 7\vartheta}{3(\vartheta - u)} \ln \left( \frac{18u + \vartheta}{(5u + \vartheta)^2} \right) \right], \]

Theorem 6. Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( f' \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q > 1 \) and then we have the following inequality for \( \lambda \in [0, 1] \)

\[ (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right) \leq \frac{ab(b - a)}{2} \]

\[ \times \left\{ C_1^{1 - \frac{q}{2}}(\lambda; p; a, b) \left( \left| \frac{f'(2ab)}{a + b} \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\} + \left( \frac{f'(a)}{4} \right)^{\frac{1}{q}} \]

where

\[ C_1(\lambda; p; u, \vartheta) = \frac{1}{\lambda^{1/2}} \int_0^{1/2} \frac{|\lambda - 2t|^p}{(tb + (1 - t)a)^p} dt, \quad u, \vartheta > 0. \]

and \( 1/p + 1/q = 1 \).
Proof. Let $A_t = tb + (1 - t)a$. Using Lemma 1 and Hölder’s integral inequality, we deduce

$$
(1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx
$$

\[
\leq \frac{ab(b - a)}{2} \left[ \int_0^{1/2} \frac{|\lambda - 2t|^{p'}}{A_t^{2p'}} \left( \int_0^{1/2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right) \right]^{\frac{1}{p'}} + \left( \int_0^{1/2} \frac{|2 - \lambda - 2t|^{p'}}{A_t^{2p'}} \left( \int_0^{1/2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right) \right)^{\frac{1}{p'}}.
\]

Using the harmonically convexity of $|f|^q$, we obtain the following inequalities from inequality (1.3):

$$
\frac{1}{2} \int_0^{1/2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \leq \frac{1}{2} \left( \frac{2ab}{b - a} \int_a^b \frac{|f'(x)|^q}{x^2} dx \right) \leq \frac{|f'(2ab/a + b)|^q + |f'(b)|^q}{4}
$$

(2.4)

and

$$
\frac{1}{1/2} \int_0^{1/2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \leq \frac{1}{2} \left( \frac{2ab}{b - a} \int_a^b \frac{|f'(x)|^q}{x^2} dx \right) \leq \frac{|f'(2ab/a + b)|^q + |f'(a)|^q}{4}.
$$

(2.5)

A combination of (2.3)- (2.5) gives the required inequality (2.2). \qed

Corollary 4. Under the assumptions Theorem 3 with $\lambda = 0$, we have

$$
\left| f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b - a)}{2}
$$

\[
\times \left\{ \int_0^{1/2} \left( \frac{f'(2ab/a + b)}{4} \right)^q \right\}^{\frac{1}{q}} + \left\{ C_{1/2} (0, p; a, b) \left( \frac{f'(2ab/a + b)}{4} + |f'(a)|^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]
Corollary 5. Under the assumptions Theorem 6 with $\lambda = 1$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}$$

$$\times \left\{ C^\frac{1}{q} \left[1, p; a, b\right] \left( \frac{|f'(\frac{2ab}{a+b})|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} + C^\frac{1}{q} \left[1, p; b, a\right] \left( \frac{|f'(\frac{2ab}{a+b})|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} \right\}.$$ 

Corollary 6. Under the assumptions Theorem 8 with $\lambda = 1/3$, we have

$$\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{2ab}{a+b} \right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}$$

$$\times \left\{ C^\frac{1}{q} \left[\frac{1}{3}, p; a, b\right] \left( \frac{|f'(\frac{2ab}{a+b})|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} + C^\frac{1}{q} \left[\frac{1}{3}, p; b, a\right] \left( \frac{|f'(\frac{2ab}{a+b})|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} \right\}.$$ 

Theorem 7. Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$ and then we have the following inequality for $\lambda \in [0, 1]$

$$\left(1 - \lambda\right) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{ab(b-a)}{4}$$

$$\times \frac{C^\frac{1}{q} \left[\lambda, p \right](\lambda, p)}{(1 - q)(1 - 2q)(b-a)^2 \left( C_5(q; a, b) |f'(a)|^q + C_6(q; a, b) |f'(b)|^q \right)^\frac{1}{q}}$$

$$+ \left( C_5(q; a, b) |f'(a)|^q + C_6(q; b, a) |f'(b)|^q \right)^\frac{1}{q},$$

where

$$C_4(\lambda, p) = \frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p+1},$$

$$C_5(q; u, \vartheta) = \left[ \left( \frac{u + \vartheta}{2} \right)^{1-2q} \left[ \frac{\vartheta - 3u}{2} - q(\vartheta - u) \right] + u^{2-2q} \right],$$

$$C_6(q; u, \vartheta) = \left[ \left( \frac{u + \vartheta}{2} \right)^{1-2q} \left[ \frac{3\vartheta - u}{2} - q(\vartheta - u) \right] + u^{1-2q} [u - 2\vartheta + 2q(\vartheta - u)] \right], \quad u, \vartheta > 0$$

and $1/p + 1/q = 1$. 

HARMONICALLY CONVEX FUNCTIONS
Proof. Let $A_t = tb + (1 - t)a$. Using Lemma 1 and Hölder’s integral inequality, we deduce

$$
(1 - \lambda) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_0^b \frac{f(x)}{x^2} dx
$$

$$
\leq \frac{ab(b-a)}{2} \left[ \int_0^{1/2} \left| \frac{\lambda - 2t}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right) dt + \int_0^{1/2} \left| \frac{2 - \lambda - 2t}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right) dt \right]
$$

(2.7)

$$
\leq \frac{ab(b-a)}{2} \left\{ \left( \int_0^{1/2} |\lambda - 2t|^p dt \right) \left( \int_0^{1/2} \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^{1/2} |2 - \lambda - 2t|^p dt \right) \left( \int_0^{1/2} \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{\frac{1}{q}} \right\}
$$

Using the harmonically convexity of $|f'|^q$, we obtain

$$
\int_0^{1/2} \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \leq \frac{1}{2 (1 - q) (1 - 2q) (b-a)^2}
$$

(2.8) × \left\{ \left[ \left( \frac{a+b}{2} \right)^{1-2q} \left( \frac{b-3a}{2} - q(b-a) \right) + a^{2-2q} \right] |f'(a)|^q

+ \left[ \left( \frac{a+b}{2} \right)^{1-2q} \left( \frac{3b-a}{2} - q(b-a) \right) + a^{1-2q} [a - 2b + 2q(b-a)] \right] |f'(b)|^q \right\}

and

$$
\int_{1/2}^1 \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \leq \frac{1}{2 (1 - q) (1 - 2q) (b-a)^2}
$$

(2.9) × \left\{ \left[ b^{1-2q} [b - 2a - 2q(b-a)] + \left( \frac{a+b}{2} \right)^{1-2q} \left[ \frac{3a-b}{2} + q(b-a) \right] \right] |f'(a)|^q

+ \left[ \left( \frac{a+b}{2} \right)^{1-2q} \left[ \frac{a-3b}{2} + q(b-a) \right] + b^{2-2q} \right] |f'(b)|^q \right\}

Further, we have

$$
\int_0^{1/2} |\lambda - 2t|^p dt = \frac{1}{1/2} \int_0^{1/2} |2 - \lambda - 2t|^p dt = \frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{2 (p+1)}
$$

(2.10)

A combination of (2.7)–(2.10) gives the required inequality (2.6). \qed
Corollary 7. Under the assumptions Theorem 7 with $\lambda = 0$, we have
\[
\left| f \left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}}
\]
\[
\times \frac{1}{\left[ (1-q)(1-2q) (b-a)^2 \right]^{1/q}} \left\{ (C_5(q; a, b) |f'(a)|^q + C_6(q; a, b) |f'(b)|^q)^{\frac{1}{q}} \right\}
\]
\[+ (C_6(q; b, a) |f'(a)|^q + C_5(q; b, a) |f'(b)|^q)^{\frac{1}{q}} \right\}.
\]

Corollary 8. Under the assumptions Theorem 7 with $\lambda = 1$, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}}
\]
\[
\times \frac{1}{\left[ (1-q)(1-2q) (b-a)^2 \right]^{1/q}} \left\{ (C_5(q; a, b) |f'(a)|^q + C_6(q; a, b) |f'(b)|^q)^{\frac{1}{q}} \right\}
\]
\[+ (C_6(q; b, a) |f'(a)|^q + C_5(q; b, a) |f'(b)|^q)^{\frac{1}{q}} \right\}.
\]

Corollary 9. Under the assumptions Theorem 7 with $\lambda = 1/3$, we have
\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{2ab}{a+b} \right) \right] \right. \\
\left. - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{4(3^{p+1}(p+1))^{1/p}}
\]
\[
\times \frac{1 + 2^{p+1}}{\left[ (1-q)(1-2q)(b-a)^2 \right]^{1/q}} \left\{ (C_5(q; a, b) |f'(a)|^q + C_6(q; a, b) |f'(b)|^q)^{\frac{1}{q}} \right\}
\]
\[+ (C_6(q; b, a) |f'(a)|^q + C_5(q; b, a) |f'(b)|^q)^{\frac{1}{q}} \right\}.
\]

3. Some applications for special means

Let us recall the following special means of two nonnegative number $a, b$ with $b > a$:

1. The arithmetic mean
\[ A = A(a, b) := \frac{a+b}{2}. \]

2. The geometric mean
\[ G = G(a, b) := \sqrt{ab}. \]

3. The harmonic mean
\[ H = H(a, b) := \frac{2ab}{a+b}. \]

4. The Logarithmic mean
\[ L = L(a, b) := \frac{b-a}{\ln b - \ln a}. \]
Proposition 2. Let
\[ \text{Proposition 3.} \]
It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

Proposition 1. Let \( 0 < a < b \) and \( \lambda \in [0, 1] \). Then we have the following inequality
\[
\left| (1 - \lambda)H + \lambda A - \frac{G^2}{L} \right| \leq \frac{ab(b-a)}{2 \lambda} \left\{ C_1(\lambda; a, b) + C_1(\lambda; b, a) \right\},
\]
where \( C_1 \) is defined as in Theorem 5.

Proof. The assertion follows from the inequality (2.1) in Theorem 5 for \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x \).

Proposition 2. Let \( 0 < a < b \) and \( \lambda \in [0, 1] \). Then we have the following inequality
\[
\left| (1 - \lambda)H + \lambda A - \frac{G^2}{L} \right| \leq \frac{ab(b-a)}{2^{1+1/q}} \left\{ C_1^p(\lambda; p; a, b) + C_1^p(\lambda; p; b, a) \right\},
\]
where \( q > 1, 1/p + 1/q = 1 \) and \( C_1 \) is defined as in Theorem 6.

Proof. The assertion follows from the inequality (2.2) in Theorem 6 for \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x \).

Proposition 3. Let \( 0 < a < b \) and \( \lambda \in [0, 1] \). Then we have the following inequality
\[
\left| (1 - \lambda)H + \lambda A - \frac{G^2}{L} \right| \leq \frac{ab(b-a)}{4^{1/q} \left(1 - q \right) \left(1 - 2q \right) \left(b-a\right)^2} \left\{ \frac{C_5}{2} (q; a, b) + C_6(q; a, b) \right\},
\]
where \( q > 1, 1/p + 1/q = 1 \) and \( C_4, C_5 \) and \( C_6 \) are defined as in Theorem 7.

Proof. The assertion follows from the inequality (2.3) in Theorem 7 for \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x \).

Proposition 4. Let \( 0 < a < b \), \( \lambda \in [0, 1] \) and \( q \geq 1 \). Then we have the following inequality
\[
\left| (1 - \lambda)H^2 + \lambda A(a^2, b^2) - G^2 \right| \leq \frac{ab(b-a)}{\left\{ C_1^1 \left( \frac{1}{2} \left( \lambda; a, b \right) \left[ C_2(\lambda; a, b)a^q + C_3(\lambda; a, b)b^q \right] \right) + \left( C_1^1 \left( \frac{1}{2} \left( \lambda; b, a \right) \left[ C_2(\lambda; b, a)a^q + C_3(\lambda; b, a)b^q \right] \right) \right\}},
\]
where $C_1$, $C_2$ and $C_3$ are defined as in Theorem 3.

**Proof.** The assertion follows from the inequality (2.1) in Theorem 5 for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 5.** Let $0 < a < b$ and $\lambda \in [0, 1]$. Then we have the following inequality

$$\left| (1-\lambda) H^2 + \lambda A(a^2, b^2) - G^2 \right| \leq \frac{ab(b-a)C_4^{1/p}(\lambda, p)}{2^{1/q}} \times \left\{ C_5^\frac{1}{q}(\lambda, p; a, b) A^\frac{1}{q}(H^q, b^q) + C_6^\frac{1}{q}(\lambda, p; b, a) A^\frac{1}{q}(a^q, H^q) \right\},$$

where $q > 1$ and $1/p + 1/q = 1$.

**Proof.** The assertion follows from the inequality (2.2) in Theorem 6 for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 6.** Let $0 < a < b$ and $\lambda \in [0, 1]$. Then we have the following inequality

$$\left| (1-\lambda) H^2 + \lambda A(a^2, b^2) - G^2 \right| \leq \frac{ab(b-a)C_4^{1/p}(\lambda, p)}{2^{1/q}} \times \left\{ (C_5(q; a, b)a^q + C_6(q; a, b)b^q) \frac{1}{q} + (C_5(q; b, a)a^q + C_6(q; b, a)b^q) \frac{1}{q} \right\},$$

where $q > 1$, $1/p + 1/q = 1$ and $C_4, C_5$ and $C_6$ are defined as in Theorem 4.

**Proof.** The assertion follows from the inequality (2.0) in Theorem 7 for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$. □

**Proposition 7.** Let $0 < a < b$, $n \in (-1, \infty) \setminus \{0\}$, $\lambda \in [0, 1]$ and $q \geq 1$. Then we have the following inequality

$$\left| (1-\lambda) H^{n+2} + \lambda A(a^{n+2}, b^{n+2}) - G^2 L_n^2 \right| \leq \frac{ab(b-a)(n+2)}{2} \times \left\{ C_1^\frac{1}{q}(\lambda; a, b) \left[ C_2(\lambda; a, b) a^{(n+1)q} + C_3(\lambda; a, b) b^{(n+1)q} \right] \right\}^{-\frac{1}{q}} + C_1^\frac{1}{q}(\lambda; b, a) \left[ C_3(\lambda; b, a) a^{(n+1)q} + C_2(\lambda; b, a) b^{(n+1)q} \right]^{-\frac{1}{q}},$$

where $C_1, C_2$ and $C_3$ are defined as in Theorem 3.

**Proof.** The assertion follows from the inequality (2.1) in Theorem 5 for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □

**Proposition 8.** Let $0 < a < b$ and $n \in (-1, \infty) \setminus \{0\}$. Then we have the following inequality

$$\left| (1-\lambda) H^{n+2} + \lambda A(a^{n+2}, b^{n+2}) - G^2 L_n^2 \right| \leq \frac{ab(b-a)(n+2)}{2^{1+1/q}} \times \left\{ C_1^\frac{1}{q}(\lambda; a, b) A^{\frac{1}{q}}(H^{(n+1)q}, b^{(n+1)q}) + C_1^\frac{1}{q}(\lambda; b, a) A^{\frac{1}{q}}(a^{(n+1)q}, H^{(n+1)q}) \right\},$$

where $q > 1$, $1/p + 1/q = 1$ and $C_1$ is defined as in Theorem 7.

**Proof.** The assertion follows from the inequality (2.2) in Theorem 6 for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □
Proposition 9. Let \(0 < a < b\), \(\lambda \in [0, 1]\), and \(n \in (-1, \infty) \setminus \{0\}\). Then we have the following inequality

\[
\left| (1 - \lambda) H^{n+2} + \lambda A(a^{n+2}, b^{n+2}) - G^2 L_n^a \right| 
\leq \frac{ab(b - a)(n + 2)C_4^{1/p}(\lambda, p)}{4[(1 - q)(1 - 2q)(b - a)^2]^{1/q}} \left\{ \left( C_5(q; a, b)a^{(n+1)q} + C_6(q; a, b)b^{(n+1)q} \right)^{\frac{1}{q}} + \left( C_6(q; b, a)a^{(n+1)q} + C_5(q; b, a)b^{(n+1)q} \right)^{\frac{1}{q}} \right\},
\]

where \(q > 1\), \(1/p + 1/q = 1\) and \(C_5\) and \(C_6\) are defined as in Theorem 5.

Proof. The assertion follows from the inequality (2.6) in Theorem 7, for \(f : (0, \infty) \rightarrow \mathbb{R}, \ f(x) = x^{n+2}, \ n \in (-1, \infty) \cup \{0\}\).

Proposition 10. Let \(0 < a < b\), \(\lambda \in [0, 1]\) and \(q \geq 1\). Then we have the following inequality

\[
\left| (1 - \lambda) H^{2} \ln H + \lambda A(a^2 \ln a, b^2 \ln b) - G^2 \ln I \right| \leq ab(b - a)
\times \left\{ C_1^{\frac{1}{q-1}}(\lambda; a, b) \left[ C_2(\lambda; a, b)G^{2q}(a, A(1, \ln a)) + C_3(\lambda; a, b)G^{2q}(b, A(1, \ln b)) \right] \right\}^{\frac{1}{q}}
+ C_1^{\frac{1}{q-1}}(\lambda; b, a) \left[ C_3(\lambda; b, a)G^{2q}(a, A(1, \ln a)) + C_2(\lambda; b, a)G^{2q}(b, A(1, \ln b)) \right]^{\frac{1}{q}},
\]

where \(C_1, C_2\) and \(C_3\) are defined as in Theorem 5.

Proof. The assertion follows from the inequality (2.1) in Theorem 5, for \(f : (0, \infty) \rightarrow \mathbb{R}, \ f(x) = x^2 \ln x\).

Proposition 11. Let \(0 < a < b\), and \(\lambda \in [0, 1]\). Then we have the following inequality

\[
\left| (1 - \lambda) H^{2} \ln H + \lambda A(a^2 \ln a, b^2 \ln b) - G^2 \ln I \right| \leq \frac{ab(b - a)}{2^{1/q}}
\times \left\{ C_1^{\frac{1}{q-1}}(0, p; a, b)A \left( G^{2q}(H, A(1, \ln H)), G^{2q}(b, A(1, \ln b)) \right) \right\}^{\frac{1}{q}}
+ C_1^{\frac{1}{q-1}}(0, p; b, a)A \left( G^{2q}(H, A(1, \ln H)), G^{2q}(a, A(1, \ln a)) \right) \right\},
\]

where \(q > 1\), \(1/p + 1/q = 1\) and \(C_1\) is defined as in Theorem 6.

Proof. The assertion follows from the inequality (2.2) in Theorem 6, for \(f : (0, \infty) \rightarrow \mathbb{R}, \ f(x) = x^2 \ln x\).

Proposition 12. Let \(0 < a < b\), and \(\lambda \in [0, 1]\). Then we have the following inequality

\[
\left| (1 - \lambda) H^{2} \ln H + \lambda A(a^2 \ln a, b^2 \ln b) - G^2 \ln I \right| \leq \frac{ab(b - a)}{2}
\]
$$\frac{C_4^{1/p}(\lambda, p)}{\left(1 - q \right) \left(1 - 2q \right) (b - a)^{2}} \left\{ \left( C_5(q; a, b) G^{2q}(a, A(1, \ln a)) + C_6(q; a, b) G^{2q}(b, A(1, \ln b)) \right)^{\frac{1}{q}} 
 + \left( C_6(q; b, a) G^{2q}(a, A(1, \ln a)) + C_5(q; b, a) G^{2q}(b, A(1, \ln b)) \right)^{\frac{1}{q}} \right\},$$

where $q > 1$, $1/p + 1/q = 1$ and $C_5$ and $C_6$ are defined as in Theorem 7.

Proof. The assertion follows from the inequality (2.6) in Theorem 7 for $f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) = x^2 \ln x$. \hfill $\square$

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