Formal context for cryptographic models

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Abstract

To clarify what is involved in linking models to instruments, we adapt quantum mechanics to define models that display explicitly the points at which they can be linked to statistics of results of the use of instruments. Extending an earlier proof that linking models to instruments takes guesswork, we show: Any model of cryptographic instruments can be enveloped, nonuniquely, by another model that expresses conditions of instruments that must be met if the first model is to fit a set of measured outcomes. As a result, model $\alpha$ of key distribution can be enveloped in various ways to reveal alternative models that Eve can try to implement, in conflict with model $\alpha$ and its promise of security. A different enveloping model can help Alice and Bob by expressing necessities of synchronization that they manipulate to improve their detection of eavesdropping. Finally we show that models based on pre-quantum physics are also open to envelopment.

PACS numbers: 03.67.Dd, 03.65.Bz, 89.70.+c
I. INTRODUCTION

A designer Diana and the users Alice and Bob of cryptographic transmitting and receiving instruments, as well as the eavesdropper Eve, all employ various equations to model how results of the use of these instruments depend on what the participants do. Between a model as a set of equations and instruments made of glass and silicon there is a great divide. In choosing a model to analyze instruments or to be employed in a feedback loop where the model helps to operate instruments, one makes a link across this divide. While one can interpret measured results as refuting some candidate models, we recently proved that neither they nor logic can uniquely determine a quantum model: linking a model to instruments requires something beyond logic and measured data, something well named by the word guess [1].

Proofs of the security of quantum key distribution invoke inner products of quantum state vectors, and these depend on the model chosen. Here we prove that any given set of outcomes from a transmitter and receiver used to distribute a key can be fitted by many quantum-mechanical models which differ greatly among themselves in their inner products and hence in their implications for the security of a key. On one hand, this encourages Eve to invent snooping instruments even though she knows Alice and Bob have a proof of security, and, on the other hand, our findings encourage discovery and repair of “hidden security loopholes” [2] arising because their transmitting and receiving instruments “violate ... assumptions [that underlie their model] in ways not immediately apparent to Alice and Bob” [3].

To underpin an examination of the linking of models to instruments, in Section II we adapt quantum mechanics to define models that display explicitly the points at which they can be linked to statistics of results of the use of instruments. The models to be introduced express “what the participants do” in terms of commands sent to the instruments via Classical, digital Process-control Computers (CPC’s) that control them and that also record results from them; we call these CPC-oriented models. Section III extends an earlier
proof that linking CPC-oriented models to instruments takes guesswork: For any quantum-mechanical model of transmitting and receiving instruments there is another model (not unique) that expresses constraints in using the instruments that must be met if the first model is to fit a set of measured outcomes. We say the first model is *enveloped* by the second.

In Section IV we prove that for any quantum model $\alpha$ of key distribution, there exists an enveloping model $\beta$ that matches $\alpha$ with respect to measurements contemplated in $\alpha$ but that has smaller inner products and allows for other measurements, which, if Eve can implement them, allow undetected eavesdropping, in conflict with model $\alpha$ and its promise of security. For this reason, no proof can relieve Alice and Bob of the burden of making judgments about what models to link to their instruments, something implicit in [2,3], but here made vivid.

They can, however, put the same burden of judgment on Eve, for she too must use models. In Section V model $\alpha$ is enveloped by another model that expresses necessities of synchronization that Alice and Bob can manipulate to improve their detection of eavesdropping. In Section VI we indicate how models based on pre-quantum physics are also open to envelopment.

In summary, we find that instruments modeled are used in a context of circumstances and intentions which no model can fully describe. In creating an enveloping model, one formally expresses (rightly or wrongly) some hitherto unexpressed feature of this context. As will be proved, there is no end of opportunities to assert features of context, because any enveloping model can in turn be enveloped.

**II. LINKING INSTRUMENTS TO MODELS**

The central issue is the linking of uses of instruments to models. By *model* we mean a set of equations written in mathematical language, primarily quantum mechanics, with the intent of predicting statistics of the results of using instruments (such as transmitters and
receivers made of silicon and glass fibers *etc.*). Some of the equations of the set act as a set of assumptions from which the rest of the equations can be derived. Quantum mechanics provides a mathematical language in which to write down a wide variety of models, constrained by a grammar of logical constraints, so within a model conclusions can be proved to follow from assumptions. Because different sets of assumptions generate different quantum-mechanical models, quantum mechanics is a language, as distinct from a particular model written in that language; it has more room in it for diverse models that accord with any given set of experimental results than has been appreciated.

Although, as we shall see, instruments cannot be discussed independently of models, we separate them as best we can by the trick supposing the instruments are operated via digital computers. This will allow us to express “how the instruments are used” in terms of *commands* sent to the instruments by a Classical, digital Process-control Computer (CPC) that controls them and that also records results from them [1]. Instruments swallow commands and give back recordable results.

As discussed in [1], parsing a stream of data from instruments into a sequence of measurement occurrences, each with a quantum-mechanical *outcome*, cannot be determined from the data alone, but takes extra hypotheses, indeed a kind of stripped-down model, determined in part by guesswork, which we call a *parsing rule*. Parsing requires guesswork, both to assert the statistical independence of one segment of data from another, and to select criteria by which to weed out artifacts in the data attributed to instrumental imperfections, such as false and missed detections. Using a parsing rule, one parses a stream of data from the instruments into a sequence of measurement occurrences, assumed statistically independent, and one formats the data segment for each occurrence into (1) how the instruments were configured and (2) an outcome from the instruments. The parsing rule makes no statement about values of probabilities of outcomes, but does assert that such values exist; it provides a range of outcomes that are possible to record as well as set of possible command sequences. In this way it limits the models that can be tested by measured data that it parses.
To view the linking of instruments to models we postulate an analytic frame in which (a) instruments write via some parsing rule what we and other scientists interpret as numerical outcomes, (b) they write these numbers in memories of CPC’s, and (c) CPC’s send commands to some or all of the instruments. We view each set-up of instruments in terms of records in CPC memories of commands sent to the instruments by CPC’s and of outcomes from the instruments. Notice that we make no assumption that an instrument works as the manufacturer says it does, nor that it works the way any model compatible with the parsing rule says it does, nor that it functions statistically the same on Tuesday as it does on Monday. While such assumptions are made in the models to be discussed, the analytic frame provides room to consider cases in which the instruments write numbers that conflict with any or all models.

We define a CPC-oriented model of a set-up of instruments to be a model that expresses conditional probabilities of outcomes given commands to the instruments. For instance, a model $\alpha$ to be introduced in Section IV will express a conditional probability of outcome $j$ given a command $b_A$ from Alice’s CPC to her transmitter and a command $b_E$ from Eve’s CPC to her eavesdropping receiver, written $\Pr_\alpha(j|b_A, b_E)$. The subscript marks it as an assertion within model $\alpha$, leaving room to consider a different model $\beta$ that asserts a different numerical value $\Pr_\beta(j|b_A, b_E) \neq \Pr_\alpha(j|b_A, b_E)$.

The same CPC’s that control the instruments house in their memories CPC-oriented models and programs designed using them. These models and model-derived programs are used off-line to simulate the instruments; they are used on-line, not to simulate the instruments, but to help operate them, for example in a feedback loop of Bob’s receiver, as discussed in Section V. By considering both the instruments and the models as they are reflected in files of a CPC, we conceptually separate (as well as possible) these CPC-oriented

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1This extends our earlier analysis of instruments controlled by a single CPC programmed according to some model $\Box\Box\Box$ to deal with a setup of instruments controlled by several CPC’s.
models from the instruments modeled while allowing for interaction between models and instruments.

Like any set of equations, a CPC-oriented model can be copied, so copies of the same model can be used concurrently in different places for the same or different purposes. What can be done with a model or a program depends on where it is, for example on whether it is written in Alice’s CPC or in Eve’s. Because the models used in programming one CPC need not be the same as those used in programming another, several CPC’s controlling interacting instruments can work from different models concurrently. Where and when and how a CPC-oriented model is used is traceable in the execution sequences of the CPC’s in which copies of the model are housed, so the CPC frame allows analysis of various of CPC-oriented models and model-derived programs used to operate instruments that interact. Some or all of the instruments can be modeled by more than one model, and one model can conflict with another. Some models model other models: a component of Eve’s model can be her model α’ of Alice and Bob’s model α; this tells her (rightly or wrongly) how Alice and Bob, using their model α, will decide on their security, distinct from how Eve decides using her model β. Conversely, Alice and Bob’s model α contains as a component a model β’, their model of Eve’s model β. There is no necessary stopping place in modeling models.

If a model α invokes all the assumptions of a model β and possibly more, we say model α specializes model β, or that model β generalizes model α (meaning it has fewer assumptions). This is the first of several types of relations among models that will be used to express interactions between the invention and the modeling of transmitting and receiving instruments used in cryptographic key distribution.

III. MODELS OF COMMUNICATION

Ignoring eavesdropping for the moment, we focus on Alice communicating to Bob, as described quantum mechanically. Consider Alice transmitting $m$ quantum bits of raw data to Bob, with Alice using one CPC to control her transmitter and Bob using another to
control his receiver. They want to jointly implement a CPC-oriented model $\gamma$ of quantum communication, which says at each of a sequence of $m$ occurrences, Alice causes her CPC to command the preparation of a state vector, choosing $u$ for 0 or $v$ for 1 at random. Her CPC records her choices of 0 or 1. Bob’s receiver has, say, light detectors, one interpreted in model $\gamma$ as detecting $u$ to indicate Alice’s choice of 0 and another detecting $v$ to indicate Alice’s transmission of 1. Bob’s CPC records the decisions of his receiver, described as deciding on 0 or 1 or, if neither detector fires, ‘inconclusive’ \[11\].

Model $\gamma$ is composed of:

1. a set $A_\gamma$ of command strings that Alice’s CPC can send to generate states, here just the set $\{0, 1\}$; a set $B_\gamma$ of commands that Bob’s CPC can send to his receiver, which in this case is empty;

2. a Hilbert space $\mathcal{H}_\gamma = \mathbb{C}^2$ \textit{(i.e., the vector space of complex dimension 2)};

3. a function for states as functions of commands (here only Alice’s commands), $|v_\gamma\rangle : A_\gamma \to \mathcal{H}_\gamma$ such that $|v_\gamma(0)\rangle = u$ and $|v_\gamma(1)\rangle = v$;

4. a set of possible outcomes of Bob’s measurement, indexed by $j$ ranging over natural numbers or some subset of natural numbers, here 0 for $|v_\gamma(0)\rangle$, 1 for $|v_\gamma(1)\rangle$, and 2 for ‘inconclusive’;

5. a function from Bob’s commands to positive operator valued measures (POVM’s) on $\mathcal{H}_\gamma$, here simplified to the single POVM $M_\gamma$ consisting of a set of detection operators $M_\gamma(j)$ with

$$\sum_j M_\gamma(j) = 1,$$

$$(\forall j) M_\gamma(j) \geq 0 \text{ and } M_\gamma(j) = M_\gamma(j)^\dagger.$$ $\text{Model } \gamma \text{ asserts the probability of outcome } j \text{ given a command } b_\gamma \in A_\gamma \text{ for state preparation to be}$$

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\[
Pr_{\alpha}(j|b_A) = \langle v_{\alpha}(b_A)|M_{\gamma}(j)|v(b_A) \rangle. \tag{1}
\]

In relating model $\gamma$ to results in his CPC, Bob thinks of his CPC as recording detection results of Alice’s $m$-bit transmission in a sequence of $m$ memory segments, each of which can hold two bits, coded 00 for Alice’s ‘0’, 01 for Alice’s ‘1’, and 10 for ‘inconclusive’. We will refer to these two-bit memory segments in connection with timing, to which we now turn.

**A. Need for synchronization**

Model $\gamma$ is an armchair view of Bob’s receiver that lacks the detail necessary to design it. To design a receiver that works according to model $\gamma$, Diana must provide for synchronizing it to Alice’s transmitter within some allowed leeway. For this Diana envelops model $\gamma$ with a more detailed model $\delta$ that expresses the conditions of synchronization that must be maintained between Alice’s transmitter and Bob’s receiver if model $\gamma$ is to accord the records of Alice’s commands and Bob’s detections. Diana provides for Bob’s receiver to meet these conditions by adjusting the rate of Bob’s clock in response to measured results interpreted in model $\delta$. She designs this feedback loop by choosing a classical-control model $\epsilon$, to be discussed shortly. Without the gaps in synchronization defined by model $\delta$ and their containment within the allowed leeway in accord with model $\epsilon$, Bob’s CPC, driven by its clock, would mistime its routing of a detector signal to the $k$-th memory segment, resulting in an erroneous record.\(^2\)

To express the effect on reception of the drift of the clock of Bob’s CPC relative to Alice’s clock, Diana (having learned from Einstein) defines synchrony in terms of measurements

\(^2\)Although for a short transmission line of a fixed delay, Bob and Alice can use the same clock to drive their CPC’s synchronously, but for variable delay, *e.g.* if Bob is in motion, he needs his own clock, independently adjustable\(^3\). And even where the single-clock design works, Bob’s receiver must adjust its phase.
made of Alice’s signal arriving at Bob’s clock; however, in using a quantum model of the measurement of that signal, she must cope with quantum indeterminacy, which limits Bob’s receiver’s knowledge of arrival times to what it can deduce via Bayes rule from outcomes. To produce suitable outcomes, Diana invents a model \( \delta \), which has a Hilbert space \( \mathcal{H}_\delta \), of dimension higher than that of \( \mathcal{H}_\gamma \), along with states \( |v_\delta(b_A, s)\rangle \) that are functions not only of Alice’s commands in \( A_\delta = A_\gamma \), but also of a skew \( s \) of Bob’s clock relative to an imagined ideally synchronized clock. Diana designs Bob’s receiver to measure the \( k \)-th signal from Alice when the clock of Bob’s CPC reads \( t_k \). When we imagine Bob’s clock reads \( t_k \) as the ideal clock reads \( t_k - s_k \), we say Bob’s clock is fast by a skew \( s_k \). The state measured by Bob’s receiver when his clock reads \( t_k \) is then \( |v_\delta(b_A, s_k)\rangle = U_\delta(-s_k)|v_\delta(b_A, 0)\rangle \), where \( U_\delta \) is a unitary-operator-valued function of \( s_k \) by which Diana expresses skew.

In order to allow different possible outcomes for different values of the skew \( s_k \) at the reception of the \( k \)-th of the \( m \) signals from Alice, model \( \delta \) must assume more possible outcomes than the 0, 1, and ‘inconclusive’ of model \( \gamma \), so the POVM \( M_\delta \) has more than the three detection operators of \( M_\gamma \). When restricted to skews \( s_k \) of magnitude smaller than some allowed bound \( s_0 \), model \( \delta \) projects onto model \( \gamma \) as follows:

1. \( |v_\delta(b, s_k)\rangle \mapsto |v_\gamma(b)\rangle \), and
2. the detection operators of \( M_\delta \) partition into three sets, such that the sum of operators for each set maps to a single detection operator \( M_\gamma(j) \) with respect to probabilities of detection of \( j \).

But outcomes in model \( \delta \) tell more than these projections. At each signal reception, Bob’s receiver records in his CPC not only a decision among 0, 1, and ‘inconclusive’ but also finer distinction from which his CPC estimates its clock skew (via Bayes rule and a prior probability distribution that Diana assumes for skew). In order to record the outcomes that help estimate skew and guide clock-rate adjustment, Bob’s receiver, designed using model \( \delta \), needs a memory segment for the \( k \)-th reception of more than two bits. Hence, the record
previously discussed in connection with model $\gamma$ is extracted from a larger record required by model $\delta$.3

To contain skews within the tolerable bound $s_0$, Diana chooses a classical model $\epsilon$ by which to design a program that, when executed by Bob’s CPC, responds to estimated skews by sending a command from the CPC to set a ‘faster-slower’ lever on the clock that drives that CPC; the command is a value of a control function $F_\epsilon$ that takes as its argument a computer file consisting of skews calculated from recently recorded detection results and recently issued commands to the clock itself. Although the quantum state to be controlled has a history that Bob’s CPC can only estimate via Bayes rule from outcomes, the design of a control function $F_\epsilon$ is within the discipline of classical feedback design. If model $\delta$ is implemented and if model $\epsilon$ succeeds in generating steering commands that are adequate, the skews are held within the bound $\pm s_0$ so that Bob’s detection results fulfill the intention of model $\gamma$ and, additionally, allow his CPC to make skew estimates necessary to guide clock adjustment.

Remark 1: Models, such as $\gamma$ and $\delta$, express desires and obstacles more flexibly than do inputs used for this purpose in control theory [8, 9]. Alice expresses what she wants by choosing model $\gamma$ altogether, not just by an ‘input’ of 0 or 1 to her transmitter. Because Diana wants Alice and Bob’s instruments to work in accord with Alice’s model $\gamma$, in spite of the obstacle of clock drift, she chooses models $\delta$ and its classical companion, model $\epsilon$.

Remark 2: The number of bits that arrive at Bob’s receiver is model-dependent: whether a detection result for a signal is seen as two bits (ignoring skew) or as more bits (allowing

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3Must there exist a quantum mechanical model that accords with experimental results of measurements of a skew-dependent state? Yes, because, any digital record can be interpreted (nonuniquely) as a record of quantum outcomes, and for any set of outcomes with their relative frequencies as functions of commands, many quantum mechanical models have probabilities that exactly fit [1].

4For discussion of Bayes rule in a non-quantum context of control, see [8].
for skew) depends on whether the record of the signal detection is interpreted using model \( \gamma \) or model \( \delta \).

Recall the freedom always present in quantum mechanical modeling to shift the boundary between the ‘system’ modeled and the measuring instrument, for instance by counting more of the measuring instrument as part of the system \([10]\). In view of this freedom, we conclude:

**Remark 3:** Every quantum mechanical model is contingent in the sense that it is projected onto by a restriction of an enveloping model that shows other possibilities.

**IV. MODELS OF VULNERABILITY TO EAVESDROPPING**

Widely discussed quantum-mechanical models of key distribution assert a nonzero inner product between quantum state vectors that Alice communicates to Bob, with the consequence that eavesdropping almost always leaves tracks in the form of errors that Bob and Alice can detect. If Alice’s transmitter, Bob’s receiver, and Eve’s snooping instruments can be counted on to work in accord with any of these models, then Alice can send Bob a key secure against undetected eavesdropping. The models can all be translated into CPC-oriented models to make visible the points at which they can be linked to results of the use of instruments, and it is to the credit of some of these models that relative frequencies of experimental results accord reasonably well with conditional probabilities of outcomes derived from the states and operators posited by the models. But we are sloppy if we forget that quantum states are terms in models, rather than model-independent features of instruments.

In linking a CPC-oriented model \( \alpha \) to instruments, one identifies commands in model \( \alpha \) with commands sent from CPC’s to the instruments, for example commands \( b_A, b_B, \) and \( b_E \) from CPC’s controlled by Alice, Bob, and Eve, respectively; one also parses results of the use of instruments in response to commands as quantum-mechanical outcomes, so that one can compare relative frequencies of these results to the conditional probabilities asserted by model \( \alpha \), e.g. \( \Pr_\alpha(j|b_A, b_B, b_E) \) as the conditional probability of a quantum outcome \( j \) given...
commands $b_A$, $b_B$, and $b_E$. (The outcome $j$ can be seen as several fragments, for example one for Bob and one for Eve, allowing for analysis of mutual information between Eve and Bob, etc.) It is to be noticed that this procedure sets up a divide that runs through the CPC between state vectors as terms in models, on one side, and on the other side the commands to and results from instruments. A large part of the story told here amounts to noticing this divide.

Given a CPC-oriented model $\alpha$ of quantum key distribution that shows Alice and Bob to be secure against eavesdropping, one can envelop model $\alpha$ in a model $\beta$ that introduces a range of conditions; under some conditions model $\beta$ projects to model $\alpha$, agreeing with it, while under other conditions model $\beta$ leads to drastically different conclusions in conflict with those of model $\alpha$. Among these are conditions under which Eve can learn the key without leaving tracks that Alice and Bob can detect. This envelopment is possible because model $\beta$ can invoke states and their inner products that differ from those of model $\alpha$ while still agreeing with model $\alpha$ with respect to probabilities of outcomes for commands considered in model $\alpha$.

For example, we envelop model $\alpha$ with a model $\beta$ expressing conditions in which Alice’s transmitter leaks light into a channel accessible to Eve, but that is unknown to Alice and Bob (and is not expressed in model $\alpha$). There are two cases to consider, corresponding to two types of models. Deferring models of Eve’s use of a probe, we start with the simpler case of a model that segments the transmission of signals from Alice to Bob into (1) Alice’s transmission to Eve, followed by (2) Eve’s transmission to Bob. For such segmented transmission, suppose model $\alpha$ assumes that (1) Alice chooses commands from a set $A_\alpha = \{0, 1\}$, with command $b_A$ generating a state vector $|\nu_\alpha(b_A)\rangle \in \mathcal{H}_\alpha$, and (2) Eve commands her listening instruments with a command $b_E \in E_\alpha$ to make a measurement expressed by a POVM $M_\alpha(b_E)$ which has a detection operator $M_\alpha(b_E; j_E)$ acting on $\mathcal{H}_\alpha$, associated with outcome $j_E$. Model $\alpha$ implies that the conditional probability of Eve obtaining the outcome $j_E$ given her command $b_E$ and Alice’s command $b_A$ is
\[ \Pr_\alpha(j_E|b_A, b_E) = \langle v_\alpha(b_A)|M_\alpha(b_E; j_E)|v_\alpha(b_A)\rangle. \] (2)

**Proposition 1** Given any such (segmented) model \(\alpha\) with inner product \(\langle v_\alpha(0)|v_\alpha(1)\rangle\) and given any \(0 \leq r < 1\), there is a model \(\beta\) that gives the same conditional probabilities of Eve’s outcomes for all her commands belonging to \(E_\alpha\), so

\[ (\forall b_A \in A_\alpha, b_E \in E_\alpha)\Pr_\beta(j_E|b_A, b_E) = \Pr_\alpha(j_E|b_A, b_E) \] (3)

while

\[ |\langle v_\beta(0)|v_\beta(1)\rangle| = r|\langle v_\alpha(0)|v_\alpha(1)\rangle|. \] (4)

**Proof:** Motivated by the idea that, unknown to Alice, her transmitter signal might generate an additional “leakage” into an unintended spurious channel that Eve reads, we construct the following enveloping model \(\beta\) which assumes:

1. the same set of commands for Alice, so \(A_\beta = A_\alpha\),

2. a larger Hilbert space \(\mathcal{H}_\beta = \mathcal{H}_{\text{leak}} \otimes \mathcal{H}_\alpha\) in which Alice produces vectors \(|v_\beta(b_A)\rangle = |w_\beta(b_A)\rangle \otimes |v_\alpha(b_A)\rangle\), with \(|w_\beta(b_A)\rangle \in \mathcal{H}_{\text{leak}}\);

3. a larger set of commands for Eve, \(E_\beta = E_\alpha \sqcup E_{\text{extra}}\) (disjoint union);

4. a POVM-valued function of Eve’s commands to her measuring instruments, with detection operators

\[
M_\beta(b_E; j_E) = \begin{cases} 
1_{\text{leak}} \otimes M_\alpha(b_E; j_E) & \text{for all } b_E \in E_\alpha, \\
\text{Eve’s choice of POVM to distinguish } |v_\beta(0)\rangle & \text{from } |v_\beta(1)\rangle \text{ if } b_E \in E_{\text{extra}}.
\end{cases}
\] (5)

According to model \(\beta\), if Eve chooses any measurement command of \(E_\alpha\), Eq. (2) holds. But model \(\beta\) speaks not of the vectors \(|v_\alpha(b_A)\rangle\) but of other vectors having an inner product of magnitude
\[ |\langle v_\beta(0)|v_\beta(1)\rangle| = |\langle w(0)|w(1)\rangle||\langle v_\alpha(0)|v_\alpha(1)\rangle| \text{.} \tag{6} \]

The unit vectors \( |w(0)\rangle \) and \( |w(1)\rangle \) can be specified at will, so that the factor \( r \overset{\text{def}}{=} |\langle w(0)|w(1)\rangle| \) can be chosen to be as small as one pleases. \( \Box \)

If she can find and gain access to a channel carrying leakage state \( s \), Eve implements a model \( \beta \) with a value of \( r < 1 \), in which case she uses an optimal POVM to distinguish Alice’s 1’s and 0’s, with fewer ‘inconclusives’ than Alice and Bob think possible, and hence with less impact on Bob’s error rate. If Eve can do this, she has more information about the key for a given rate of Bob’s errors than Alice and Bob found possible when they bet on model \( \alpha \), thus vitiating Alice and Bob’s attempt to distribute a key secure against undetected eavesdropping.

Whether Eve can implement a measurement of leakage as called for in model \( \beta \) with \( r < 1 \) is unanswerable by modeling; it is a question that requires work on “the other side of the divide.” The point to be stressed is that the agreement between model \( \alpha \) and a set of measured results, no matter what results, is no logical guarantee against Eve implementing model \( \beta \) with a value of \( r \) less than 1, or even a value of 0 which would give her the whole key while causing no errors for Alice and Bob to detect.

**A. Models involving a defense function**

When noise in communications channels is recognized, privacy amplification is necessary to distill a secure key [11]. Arguments for the security of quantum key distribution with noisy channels, summarized and refined in Refs. [3,12,13], center on a defense function. The existence of a defense function depends on a proof (within some model) of a relation between Eve’s maximum Renyi information on whatever bits she directly or indirectly interrogates and a positive contribution to Bob’s error rate in receiving bits.

Defense functions have been analyzed for models of Eve’s use of a probe [15] and without restricting Alice’s transmission to a choice of only two state vectors. In such a model \( \alpha \), Alice chooses one of several state vectors in one Hilbert space \( \mathcal{H}_{\text{sig},\alpha} \) while Eve generates a
fixed vector in a different Hilbert space $\mathcal{H}_{\text{probe},\alpha}$, and the tensor product of Alice’s choice of state vector and Eve’s fixed probe vector evolves unitarily in an interaction, after which Eve and Bob make measurements, Eve confined to the probe sector and Bob to the signal sector. Like segmented models, probe models relate Eve’s information to Bob’s error rate in such a way that Bob’s error rate depends on inner products ascribed to the state vectors among which Alice chooses; in particular if the inner products for distinct signal vectors are all zero, Eve can learn everything without causing any effect that Alice and Bob can detect.

The Appendix displays consequences of leakage of Alice’s transmission for models involving Eve’s use of a probe: just as for models that segment the transmission, the state vectors used to model Alice’s transmission are model-dependent, and so are their inner products. To see the consequence for defense functions, suppose that Alice and Bob use model $\alpha$ which assumes that Alice chooses between state vectors $|v_\alpha(0)\rangle$ and $|v_\alpha(1)\rangle$ with inner product having a magnitude $S_\alpha = |\langle v_\alpha(1)|v_\alpha(0)\rangle|$. Assuming model $\alpha$, Alice and Bob determine a defense function $t(n, e_T)$, as discussed in [12]; in order to mark its dependence on model $\alpha$ and especially its dependence on the inner product of $S_\alpha$, we write this as $t_\alpha(n, e_T, S_\alpha)$. For any such model $\alpha$ and whatever the measured results with which it accords, we can show a model $\beta$ that agrees with model $\alpha$ insofar as these results are concerned, but disagrees with it about predictions of the detectability of Eve’s eavesdropping, because in place of the inner product(s) of model $\alpha$, model $\beta$ has inner product(s) smaller by our choice of $r$, for any $0 \leq r < 1$.

**Proposition 2** If a model $\alpha$ asserts that Alice and Bob can distill a key that is secure against measurements commanded by Eve from a set of commands $E_\alpha$, then there exists another model $\beta$ that matches the predictions of model $\alpha$ for the commands in $E_\alpha$ but that makes additional commands available to Eve that make the key insecure.

To prove this, one uses Proposition 3 of the Appendix that envelops any model $\alpha$ with a model $\beta$ in which $S_\beta = rS_\alpha$ with $r$ as small as one pleases. The effect of making $S_\beta$ smaller than $S_\alpha$ is visible for the case of B92 models [3] in Figure 4 of [12], where $S_\beta$ is denoted (in
notation with which our notation regrettably clashes) by \(\sin 2\alpha\). One sees that as \(S_\beta\) gets smaller, \(t_\beta\) gets bigger, so that at any fixed error rate, one can determine an \(r\) for which model \(\beta\) allows no distilled secure key. For the BB84 model as discussed in \([12]\), the effect of \(r \ll 1\) in an enveloping model \(\beta\) is to conflict with the BB84 model in such a way as to increase \(t\) and allow undetected eavesdropping.

Thus, just as for segmented eavesdropping discussed above, Eve can try to implement a model \(\beta\) which drastically increases what she can learn for a given error rate. Again, whether she can succeed in implementing such a model is another question, on the other side of the divide that runs through the CPC’s between models and instruments. No matter what measured results they stand on, Alice and Bob always face a choice between a model \(\alpha\) and an enveloping model \(\beta\) that challenges the security asserted by model \(\alpha\). Because both models make identical predictions about probabilities that connect with the measured data, Alice and Bob face a choice that no combination of logic and their fixed set of measured results can decide. They must make a judgment, or, to put it baldly, they must make a *guess* and act on it \([1]\).

**V. MODULATION OF CLOCK RATE TO IMPROVE SECURITY**

While Alice and Bob may view their need for guesswork and judgment as bad news, they can put this need to good use if their system designer Diana recognizes that Eve is in the same boat: she too must act on guesswork. Recognizing this, Diana can design a key distribution system with features that make it harder for Eve to snoop.

As discussed in Section III, to accord with model \(\alpha\), any receiver whether Bob’s or Eve’s, must maintain close synchrony with Alice’s transmission in order to function. In both the segmented and the probe cases discussed above, the models \(\alpha\) and \(\beta\) can accord with measured results only if Bob’s and Eve’s receivers work in accord with enveloping models similar to model \(\delta\) that expresses clock skew contained within an allowed leeway. Recall that model \(\delta\) describes a receiver as parsing its results for each of Alice’s bits into two parts, one
indicating ‘0’, ‘1’, or ‘inconclusive’, the other indicating skew to be contained by adjusting the faster-slower lever of Bob’s clock, and, like Bob’s receiver, Eve’s must do this to make eavesdropping measurements at times that work.\footnote{In the segmented case, Bob, unaware, synchronizes his receiver to Eve’s re-transmission, even though he supposes he is synchronizing with Alice’s transmission.}

We suggest that Diana try to design Alice’s transmitter and Bob’s receiver to make the parsing by Eve’s receiver impossible without use of prior information that Alice has also encoded, and that Bob has better access to than does Eve. The idea is for Alice’s transmitter to be timed by a clock whose rate is intentionally randomly varied rapidly and over a wide range, and for Alice to encrypt indications of coming rate variations in her transmission to Bob. The eavesdropping problem is different (and harder) for these rate variations than for the key because they are more perishable. Quantum-mechanical models assert that the operation of the faster-slower lever on Eve’s receiver cannot be corrected \textit{ex post}; that is, if she intercepts Alice’s signal and records it using a receiver clock unsynchronized to Alice’s transmission, there is no way to reconstruct from her record what she would have received with a synchronized clock.

\textbf{VI. GENERALIZATION}

Extending the proof in \cite{1} that guesswork is necessary to the linking of quantum models to results of instruments, we have introduced the concept of enveloping models to prove that for any quantum-mechanical model $\alpha$ of key distribution there exists an enveloping model $\beta$ that agrees with $\alpha$ for commands dealt with by $\alpha$, but encompasses other possibilities, and leads to conclusions about security that conflict with those implied by model $\alpha$. By drawing on the quantum-mechanical separation of states and outcomes, this proof used more than was really necessary. All that is necessary is the separation made in quantum mechanics between what happens at an occurrence of a measure, an outcome, and what might have
happened, expressed as the set of possible outcomes. This separation is found not only in quantum mechanics, but in any statistical theory, and in particular in the usual electrical engineering of “non-quantum” systems, based on Maxwell’s electromagnetics to which one adjoins ideas of noise or the generation of random signals. In this non-quantum framework, the statistical outlook alone allows one to introduce CPC’s as a medium in which to see a divide between models and instruments within the CPC’s that manage both. Doing this, one puts the statements of a model in the form \( \Pr(\alpha | b_A, b_E) \). Then Propositions 1 and 2 can be proved without resort to quantum mechanics, so again the issues considered above arise: (a) what else might Eve measure that a model used by Alice and Bob has failed to account for, and (b) how might clock pumping help Alice and Bob? Thus the uncloseable possibility of enveloping any model \( \alpha \) by another model \( \beta \) that expresses extra conditions of the use of instruments is no peculiarity of using quantum rather than non-quantum models; it is endemic to any cryptographic modeling that invokes probabilities.

**ACKNOWLEDGMENTS**

We thank Howard E. Brandt for reading an early draft and giving us an astute critique, indispensable to this paper. The situations described in which Diana, Alice, Bob, and Eve take part are what Wittgenstein called language games \([16]\), with the language being the quantum mechanics of CPC-oriented models.
APPENDIX A: LEAKAGE CHANNEL IN CASE EVE USES A PROBE

This appendix proves for the case of Eve using a probe the analog of Proposition 1: if one model with its inner products predicts a set of probabilities for outcomes, so does another model having smaller inner products. (Thus, as in the segmented case, inner products depend on a choice of model undetermined by measured data.) The proof here makes no requirement that Alice choose among only two state vectors; she can choose from a set of any size. Transposing to the CPC-context a model expressing Eve’s use of a probe \[3\], one obtains a model \(\alpha\) that assumes:

1. a set of Alice’s commands \(A_\alpha\);
2. a Hilbert space \(\mathcal{H}_{\text{sig,}\alpha}\) for Alice’s signals and a disjoint Hilbert space \(\mathcal{H}_{\text{probe,}\alpha}\) for Eve’s probe;
3. a function assigning Alice’s states to commands (here only Alice’s commands), \(|v_\alpha\rangle : A_\alpha \to \mathcal{H}_{\text{sig,}\alpha}\);
4. a fixed starting state for Eve’s probe of \(|e_\alpha\rangle \in \mathcal{H}_{\text{probe,}\alpha}\);
5. a set \(E_\alpha\) of Eve’s possible commands to her measuring instruments;
6. unitary operators \(U_\alpha(b_E)\) for \(b_E \in E_\alpha\) acting on the product Hilbert space \(\mathcal{H}_{\text{probe,}\alpha} \otimes \mathcal{H}_{\text{sig,}\alpha}\) for the interaction of Eve’s probe with Alice’s signal;
7. a set \(O_E\) of possible outcomes of Eve’s measurements, indexed by \(j_E\);
8. for each of Eve’s commands \(b_E \in E_\alpha\), a POVM \(M_{E,\alpha}(b_E)\) with detection operators \(M_{E,\alpha}(b_E; j_E)\) that act on \(\mathcal{H}_{\text{probe,}\alpha}\);
9. a POVM for Bob’s receiver acting on \(\mathcal{H}_{\text{sig,}\alpha}\), with possible outcomes indexed by \(j_B\) and detection operators \(M_B(j_B)\).
This produces a quantum mechanical model $\alpha$ in which

$$\Pr_\alpha(j_B,j_E|b_A,b_E) = \langle v_\alpha(b_A)|e_\alpha|U_\alpha^\dagger(b_E)[M_B(j_B) \otimes M_E(b_E;j_E)]U_\alpha(b_E)|e_\alpha\rangle|v_\alpha(b_A)\rangle. \quad (A1)$$

**Proposition 3** Given any such (probe) model $\alpha$ with inner products $\langle v_\alpha(b_A)|v_\alpha(b'_A)\rangle$ and any $0 \leq r < 1$, there is a model $\beta$ that gives the same conditional probabilities of Eve’s and Bob’s outcomes for each command $b_E \in E_\alpha$ and for all of Bob’s commands, so

$$\forall b_A \in A_\alpha, b_E \in E_\alpha \Pr_\beta(j_E|b_A, b_E) = \Pr_\alpha(j_E|b_A, b_E) \quad (A2)$$

while

$$\forall b_A \neq b'_A |\langle v_\beta(b_A)|v_\beta(b'_A)\rangle| = r|\langle v_\alpha(b_A)|v_\alpha(b'_A)\rangle|. \quad (A3)$$

**Proof:** The proof extends the construction used in the proof of Proposition 1, with a model $\beta$ defined by

1. the same command set for Alice: $A_\beta = A_\alpha$;

2. signals expressed by a vector intended by Alice, as in model $\alpha$, tensored in to an unintended vector in an additional Hilbert space $\mathcal{H}_{\text{leak}}$, so Alice produces vectors $|v_\beta(b_A)\rangle = |w_\beta(b_A)\rangle \otimes |v_\alpha(b_A)\rangle$, with $|w_\beta(b_A)\rangle \in \mathcal{H}_{\text{leak}}$;

3. a fixed starting state for Eve’s probe of $|e_\alpha\rangle \in \mathcal{H}_{\text{probe},\alpha}$;

4. a larger set of commands for Eve, $E_\beta = E_\alpha \sqcup E_{\text{extra}}$ (disjoint union);

5. unitary operators $U_\beta(b_E)$ acting on $\mathcal{H}_{\text{leak}} \otimes \mathcal{H}_{\text{probe},\alpha} \otimes \mathcal{H}_{\text{sig},\alpha}$ for the interaction of Eve’s probe with Alice’s signal, defined so that

$$U_\beta(b_E) = \begin{cases} 1_{\text{leak}} \otimes U_\alpha(b_E) & \text{for all } b_E \in E_\alpha, \\ \text{Eve’s choice of unitary } U_\beta(b_E) & \text{if } b_E \in E_{\text{extra}}; \end{cases} \quad (A4)$$
6. a POVM-valued function of Eve’s commands to her measuring instruments, with detection operators defined so

\[
M_\beta(b_E; j_E) = \begin{cases} 
1 \otimes M_\alpha(b_E; j_E) & \text{for all } b_E \in E_\alpha, \\
\mathcal{H}_{\text{leak}} \otimes \mathcal{H}_{\text{probe},\alpha} & \text{if } b_E \in E_{\text{extra}}.
\end{cases}
\]  

(A5)

According to model $\beta$, if Eve chooses any measurement command of $E_\alpha$, Eq. (A2) holds. But model $\beta$ speaks not of the vectors $|v_\alpha(b_A)\rangle$ but of other vectors having an inner product (relevant to the security of quantum key distribution) of

\[
|\langle v_\beta(b_A)|v_\beta(b'_A)\rangle| = |\langle w(b_A)|w(b'_A)\rangle| |\langle v_\alpha(b_A)|v_\alpha(b'_A)\rangle|.
\]  

(A6)

The unit vectors $|w(b_A)\rangle$ can be specified so that $|w(b_A)\rangle = r^{1/2}|u(b_A)\rangle + (1 - r)|u_0\rangle$, with $\langle u(b_A)|u(b'_A)\rangle = 0$ for all $b_A \neq b'_A$ and $\langle u(b_A)|u_0\rangle = 0$ for all $b_A \in A_\beta$. With this specification Eq. (2) holds, and furthermore $r$ can be chosen as small as one wishes. $\square$
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