A RESOLUTION (MINIMAL MODEL) OF THE PROP FOR
BIALGEBRAS

MARTIN MARKL

Abstract. This paper is concerned with a minimal resolution of the prop for bialgebras (Hopf algebras without unit, counit and antipode). We prove a theorem about the form of this resolution (Theorem 15) and give, in Section 5, a lot of explicit formulas for the differential.

1. Introduction and main results

A bialgebra is a vector space \( V \) with a multiplication \( \mu : V \otimes V \to V \) and a comultiplication (also called a diagonal) \( \Delta : V \to V \otimes V \). The multiplication is associative:

\[
\mu(\mu \otimes 1_{V}) = \mu(1_{V} \otimes \mu),
\]

where \( 1_{V} : V \to V \) denotes the identity map, the comultiplication is coassociative:

\[
(1_{V} \otimes \Delta)\Delta = (\Delta \otimes 1_{V})\Delta
\]

and the usual compatibility relation between \( \mu \) and \( \Delta \) is assumed:

\[
\Delta \circ \mu = (\mu \otimes \mu)T_{\sigma(2,2)}(\Delta \otimes \Delta),
\]

where \( T_{\sigma(2,2)} : V^{\otimes 4} \to V^{\otimes 4} \) is defined by

\[
T_{\sigma(2,2)}(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}) := v_{1} \otimes v_{3} \otimes v_{2} \otimes v_{4},
\]

for \( v_{1}, v_{2}, v_{3}, v_{4} \in V \) (the meaning of the notation \( \sigma(2,2) \) will be explained in Definition 17). We suppose that \( V \), as well as all other algebraic objects in this paper, are defined over a field \( k \) of characteristic zero.

Let \( B \) be the \( k \)-linear prop (see [9, 10] or Section 2 of this paper for the terminology) describing bialgebras. The goal of this paper is to describe a minimal model of \( B \), that is, a differential graded (dg) \( k \)-linear prop \( (M, \partial) \) together with a homology isomorphism

\[
(B, 0) \xrightarrow{\rho} (M, \partial)
\]

such that

(i) the prop \( M \) is free and

(ii) the image of \( \partial \) consists of decomposable elements of \( M \) (the minimality condition),

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see again Section 2 where free PROPs and decomposable elements are recalled.

The initial stages of this minimal model were constructed in [9, page 145] and [10, pages 215–216]. According to our general philosophy, it should contain all information about the deformation theory of bialgebras. In particular, the Gerstenhaber-Schack cohomology which is known to control deformations of bialgebras [3] can be read off from this model as follows.

Let $E_{\text{nd}}V$ denote the endomorphism PROP of $V$ and let a bialgebra structure $B = (V, \mu, \Delta)$ on $V$ be given by a homomorphism of PROPs $\beta : B \to E_{\text{nd}}V$. The composition $\beta \circ \rho : M \to E_{\text{nd}}V$ makes $E_{\text{nd}}V$ an $M$-module (in the sense of [10, page 203]), therefore one may consider the vector space of derivations $\text{Der}(M, E_{\text{nd}}V)$. For $\theta \in \text{Der}(M, E_{\text{nd}}V)$ define $\delta \theta := \theta \circ \partial$. It follows from the obvious fact that $\rho \circ \partial = 0$ that $\delta \theta$ is again a derivation, so $\delta$ is a well-defined endomorphism of the vector space $\text{Der}(M, E_{\text{nd}}V)$ which clearly satisfies $\delta^2 = 0$. Then

$$H_b(B; B) \cong H(\text{Der}(M, E_{\text{nd}}V), \delta),$$

where $H_b(B; B)$ denotes the Gerstenhaber-Schack cohomology of the bialgebra $B$ with coefficients in itself.

Algebras (in the sense recalled in Section 2) over $(M, \partial)$ have all rights to be called strongly homotopy bialgebras, that is, homotopy invariant versions of bialgebras, as follows from principles explained in the introduction of [12]. This would mean, among other things, that, given a structure of a dg-bialgebra on a chain complex $C_*$, then any chain complex $D_*$, chain homotopy equivalent to $C_*$, has, in a certain sense, a natural and unique structure of an algebra over our minimal model $(M, \partial)$.

For a discussion of PROPs for bialgebras from another perspective, see [16]. Constructions of various other (non-minimal) resolutions of the PROP for bialgebras, based mostly on a dg-version of the Boardman-Vogt $W$-construction, will be the subject of [6]. A completely different approach to bialgebras and resolutions of objects governing them can be found in a series of papers by Shoikhet [20, 21, 22], and also in a recent draft by Saneblidze and Umble [19]. A general theory of resolutions of PROPs is, besides [15], also the subject of Vallette’s thesis and its follow-up [24, 25].

Let us briefly sketch the strategy of the construction of our model. Consider objects $(V, \mu, \Delta)$, where $\mu : V \otimes V \to V$ is an associative multiplication as in (1), $\Delta : V \to V \otimes V$ is a coassociative comultiplication as in (2), but the compatibility relation (3) is replaced by

$$\Delta \circ \mu = 0.$$

**Definition 1.** A half-bialgebra or briefly $\frac{1}{2}$bialgebra is a vector space $V$ equipped with a multiplication $\mu$ and a comultiplication $\Delta$ satisfying (1), (3) and (4).

We chose this strange name because (1) is indeed, in a sense, one half of the compatibility relation (3). For a formal variable $\epsilon$, consider the axiom

$$\Delta \circ \mu = \epsilon \cdot (\mu \otimes \mu)T_{\sigma(2,2)}(\Delta \otimes \Delta).$$
At $\epsilon = 1$ we get the usual compatibility relation (3) between the multiplication and the diagonal, while $\epsilon = 0$ gives (4). Therefore (3) can be interpreted as a perturbation of (4) which may be informally expressed by saying that bialgebras are perturbations of $\frac{1}{2}$bialgebras. Experience with homological perturbation theory \cite{Koszul} leads us to formulate:

**Principle.** *The prop $B$ for bialgebras is a perturbation of the prop $\frac{1}{2}B$ for $\frac{1}{2}$bialgebras. Therefore there exists a minimal model of the prop $B$ that is a perturbation of a minimal model of the prop $\frac{1}{2}B$ for $\frac{1}{2}$bialgebras.*

We therefore need to know a minimal model for $\frac{1}{2}B$. In general, PROPs are extremely huge objects, difficult to work with, but $\frac{1}{2}$bialgebras exist over much smaller objects than PROPs. These smaller objects, which we call $\frac{1}{2}$PROPs, were introduced in an e-mail message from M. Kontsevich \cite{Kontsevich} who called them small PROPs. The concept of $\frac{1}{2}$PROPs makes the construction of a minimal model of $\frac{1}{2}B$ easy. We thus proceed in two steps.

**Step 1.** We construct a minimal model $(\Gamma(\Xi), \partial_0)$ of the prop $\frac{1}{2}B$ for $\frac{1}{2}$bialgebras. Here $\Gamma(\Xi)$ denotes the free PROP on the space of generators $\Xi$, see Theorem 13.

**Step 2.** Our minimal model $(M, \partial)$ of the prop $B$ for bialgebras will be then a perturbation of $(\Gamma(\Xi), \partial_0)$, that is,

$$(M, \partial) = (\Gamma(\Xi), \partial_0 + \partial_{\text{pert}}),$$

see Theorem 15.

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My particular thanks are due to M. Kontsevich whose e-mail \cite{Kontsevich} shed a new light on the present work and stimulated a cooperation with A.A. Voronov which resulted in \cite{Voronov}. Also the referee’s remarks were extremely helpful.
2. Structure of props and \( \frac{1}{2} \) props

Let us recall that a \( k \)-linear PROP \( A \) (called a theory in \([9, 10]\)) is a sequence of \( k \)-vector spaces \( \{ A(m, n) \}_{m,n \geq 1} \) with compatible left \( \Sigma_m \)-right \( \Sigma_n \)-actions and two types of equivariant compositions, vertical:

\[
\circ : A(m, u) \otimes \Sigma_u A(u, n) \to A(m, n), \quad m, n, u \geq 1,
\]

and horizontal:

\[
\circ : A(m_1, n_1) \otimes A(m_2, n_2) \to A(m_1 + m_2, n_1 + n_2), \quad m_1, m_2, n_1, n_2 \geq 1,
\]

together with an identity \( 1 \in A(1,1) \). PROPS should satisfy axioms which could be read off from the example of the endomorphism PROP \( \mathcal{E}nd_V \) of a vector space \( V \), with \( \mathcal{E}nd_V(m, n) \) the space of linear maps \( \text{Hom}_k(V^\otimes m, V^\otimes n) \), \( 1 \in \mathcal{E}nd_V(1,1) \) the identity map, horizontal composition given by the tensor product of linear maps, and vertical composition by the ordinary composition of maps. One can therefore imagine elements of \( A(m, n) \) as ‘abstract’ maps with \( n \) inputs and \( m \) outputs. See \([8, 10]\) for precise definitions.

We say that \( X \) has biarity \( (m,n) \) if \( X \in A(m,n) \). We will sometimes use the operadic notation: for \( X \in A(m,k) \), \( Y \in A(1,l) \) and \( 1 \leq i \leq k \), we write

\[
X \circ_i Y := X \circ (1^\otimes(i-1) \otimes Y \otimes 1^\otimes(k-i)) \in A(m,k+l-1)
\]

and, similarly, for \( U \in A(k,1) \), \( V \in A(l,n) \) and \( 1 \leq j \leq l \) we denote

\[
U_j \circ V := (1^\otimes(j-1) \otimes U \otimes 1^\otimes(l-j)) \circ V \in A(k+l-1,n).
\]

In \([10]\) we called a sequence \( E = \{ E(m,n) \}_{m,n \geq 1} \) of left \( \Sigma_m \)-right \( \Sigma_n \)-\( k \)-bimodules a core, but we prefer now to call such sequences \( \Sigma \)-bimodules. For any such a \( \Sigma \)-bimodule \( E \), there exists the free PROP \( \Gamma(E) \) generated by \( E \). It also makes sense to speak, in the category of PROPS, about ideals, presentations, modules, etc, see \([24]\) Chapter 2 for details.

Recall that an algebra over a PROP \( A \) is (given by) a PROP morphism \( \alpha : A \to \mathcal{E}nd_V \). A PROP \( A \) is augmented if there exist a homomorphism \( \epsilon : A \to 1 \) (the augmentation) to the trivial PROP \( 1 := \mathcal{E}nd_k \). Therefore an augmentation is the same as a structure of an \( A \)-algebra on the one-dimensional vector space \( k \).

Let \( A^+ := \text{Ker}(\epsilon) \) denote the augmentation ideal of an augmented PROP \( A \). The space \( D(A) := A^+ \circ A^+ \) is then called the space of decomposables and the quotient \( Q(A) := A^+ / D(A) \) the space of indecomposables of the augmented PROP \( A \). Observe that each free PROP \( \Gamma(E) \) is canonically augmented, with the augmentation defined by \( \epsilon(E) := 0 \).

Let \( \Gamma(\mathcal{A}, \mathcal{Y}) \) be the free PROP generated by one operation \( \mathcal{A} \) of biarity \( (1,2) \) and one operation \( \mathcal{Y} \) of biarity \( (2,1) \). More formally, \( \Gamma(\mathcal{A}, \mathcal{Y}) := \Gamma(E) \) with \( E \) the \( \Sigma \)-bimodule \( k \cdot \mathcal{A} \otimes k[\Sigma_2] \oplus k[\Sigma_2] \otimes k \cdot \mathcal{Y} \). As we explained in \([9, 10]\), the PROP \( B \) describing bialgebras has a presentation

\[
B = \Gamma(\mathcal{A}, \mathcal{Y}) / I_B,
\]
where \( I_B \) denotes the ideal generated by
\[
\mathcal{A} - \mathcal{A}, \mathcal{Y} - \mathcal{Y} \quad \text{and} \quad \mathcal{X} - \mathcal{X}.
\]
In the above display we denoted
\[
\mathcal{A} := \mathcal{A} (\mathcal{A} \otimes 1), \quad \mathcal{A} := \mathcal{A} (1 \otimes \mathcal{A}), \quad \mathcal{Y} := (\mathcal{Y} \otimes 1) \mathcal{Y}, \quad \mathcal{Y} := (1 \otimes \mathcal{Y}) \mathcal{Y},
\]
\[
\mathcal{X} := \mathcal{Y} \circ \mathcal{X} \quad \text{and} \quad \mathcal{X} := (\mathcal{A} \otimes \mathcal{A}) \circ (2, 2) \circ (\mathcal{Y} \otimes \mathcal{Y}),
\]
where \( \sigma(2, 2) \in \Sigma_4 \) is the permutation
\[
\sigma(2, 2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}
\]
or diagrammatically
\[
\sigma(2, 2) = \begin{array}{cccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]
\[
\bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc
\]
\[
\bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc
\]
We will use the similar notation for elements of free PROPs throughout the paper. All our ‘flow diagrams’ should be read from the bottom to the top.

**Remark 2.** Enriquez and Etingof described in [1] a basis of the \( k \)-linear space \( B(m, n) \) for arbitrary \( m, n \geq 1 \) as follows. Let \( \mathcal{A} \in B(1, 2) \) be the equivalence class, in \( B = \Gamma(\mathcal{A}, \mathcal{Y}) / I_B \), of the generator \( \mathcal{A} \in \Gamma(\mathcal{A}, \mathcal{Y})(1, 2) \) (we use the same symbol both for a generator and its equivalence class). Define \( \mathcal{A}^{[a]} := 1 \in B(1, 1) \) and, for \( a \geq 2 \), let
\[
\mathcal{A}^{[a]} := \mathcal{A} (\mathcal{A} \otimes 1) (\mathcal{A} \otimes 1)^{\otimes 2} \cdots (\mathcal{A} \otimes 1)^{\otimes (a-2)} \in B(1, a).
\]
Let \( \mathcal{Y}^{[b]} \in B(b, 1) \) has the obvious similar meaning. According to [1] Proposition 6.2, the elements
\[
(\mathcal{A}^{[a_1]} \otimes \cdots \otimes \mathcal{A}^{[a_m]}) \circ \sigma \circ (\mathcal{Y}^{[b_1]} \otimes \cdots \otimes \mathcal{Y}^{[b_n]}),
\]
where \( \sigma \in \Sigma_N \) for some \( N \geq 1 \), and \( a_1 + \cdots + a_m = b_1 + \cdots + b_m = N \), form a \( k \)-linear basis of \( B(m, n) \). This result can also be found in [6].

We have already observed that PROPs, and namely free ones, are extremely huge objects. For instance, the space \( \Gamma(\mathcal{A}, \mathcal{Y})(m, n) \) is infinite-dimensional for any \( m, n \), and even its quotient \( B(m, n) \) is infinite-dimensional, as follows from Proposition 6.2 of [1] recalled in Remark 2. Therefore it might come as a surprise that there are \emph{three} natural gradings of \( \Gamma(\mathcal{A}, \mathcal{Y})(m, n) \) by finite-dimensional pieces.

Since elements of free PROPS are represented by formal sums of graphs [15, Section 2], it makes sense to define the \emph{genus} \( \text{gen}(X) \) of a monomial \( X \) in a free PROP as the genus \( \dim H^1(G_X; \mathbb{Q}) \) of the graph \( G_X \) corresponding to \( X \). For example, \( \text{gen}(\mathcal{A}) = \text{gen}(\mathcal{X}) = 0 \), while
\[
\text{gen}(\mathcal{X}) = 1.
\]
There is another grading called the \emph{path grading} \( \text{pth}(X) \) implicitly present in [5], defined as the total number of directed paths connecting inputs with outputs of \( G_X \). Properties
of the genus and path gradings are discussed in [15, Section 5]. The following proposition follows immediately from the results of [15].

**Proposition 3.** For any fixed $d$, the subspaces

$$\text{Span}\{X \in \Gamma(\{,\}) (m, n); \ \text{gen}(X) = d\} \text{ and } \text{Span}\{X \in \Gamma(\{,\}) (m, n); \ \text{pth}(X) = d\}$$

are finite dimensional.

The following formula relating the path and genus gradings was also derived in [15]:

$$\text{pth}(X) \leq mn(\text{gen}(X) + 1) \text{ for } X \in \Gamma(\{,\}) (m, n).$$

There is, of course, also the obvious grading $\text{grd}(X)$ given by the number of vertices of the graph $G_X$. Using this grading, the decomposables of a free PROP can be described as

$$D(\Gamma(E)) = \text{Span}\{X \in \Gamma(E); \ \text{grd}(X) \geq 2\}.$$ 

Let us recall the following important definition [5, 15].

**Definition 4.** A $\frac{1}{2}$PROP is a collection $s = \{s(m, n)\}$ of dg $(\Sigma_m, \Sigma_n)$-bimodules $s(m, n)$ defined for all couples of natural numbers except $(m, n) = (1, 1)$, together with compositions

$$\circ_i : s(m_1, n_1) \otimes s(1, l) \to s(m_1, n_1 + l - 1), \ 1 \leq i \leq n_1,$$

and

$$j \circ : s(k, 1) \otimes s(m_2, n_2) \to s(m_2 + k - 1, n_2), \ 1 \leq j \leq m_2,$$

that satisfy the axioms satisfied by operations $\circ_i$ and $j \circ$, see [5], [6], in a general PROP.

**Remark 5.** Observe that $\frac{1}{2}$PROPS as introduced above cannot have a unit $1 \in s(1, 1)$. We choose this convention from the following reasons. There exist an obvious unital version of $\frac{1}{2}$PROPS, but for all examples of interest, including $\frac{1}{2}$bialgebras, the corresponding unital $\frac{1}{2}$PROP would satisfy $s(1, 1) \cong k$. Since there clearly exists a canonical one-to-one correspondence between unital $\frac{1}{2}$PROPS enjoying this property and non-unital $\frac{1}{2}$PROPS in the sense of the above definition, the unit would carry no information.

Moreover, working without units enables one to define the ‘obvious grading’ $\text{grd}(-)$ of free $\frac{1}{2}$PROPS in a very natural way, without using graphs. The same reason lead us in [11] to introduce pseudo-operads as non-unital versions of operads. The above considerations do not apply to PROPS because $P(1, 1)$ is typically an infinite-dimensional space.

Let us denote by $\Gamma_{\frac{1}{2}}(\{,\})$ the free $\frac{1}{2}$PROP generated by operations $\{,\}$ and $\{,\}$. The following proposition, which follows again from [15], gives a characterization of the subspaces

$$\Gamma_{\frac{1}{2}}(\{,\}) (m, n) \subset \Gamma(\{,\}) (m, n)$$

in terms of the genus and path gradings introduced above.
Proposition 6. The subspace $\Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(m, n)$ is, for $(m, n) \neq (1, 1)$, spanned by all monomials $X \in \Gamma(\mathcal{A}, \mathcal{Y})(m, n)$ such that (i) $\text{gen}(X) = 0$ and (ii) $\text{pth}(X) = mn$.

Equivalently, $\Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(m, n)$ is the span of elements of the form $U \circ V$ with some monomials $U \in \Gamma(\mathcal{A})(m, 1)$ and $V \in \Gamma(\mathcal{Y})(1, n)$.

Loosely speaking, elements of $\Gamma_{1/2}(\mathcal{A}, \mathcal{Y})$ are formal sums of graphs made of two trees grafted by their roots. Now it is completely obvious that $\Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(m, n)$ is finite-dimensional for any $m$ and $n$. The following example shows that both assumptions (i) and (ii) in Proposition 6 are necessary.

Example 7. It is clear that $\text{gen}(\mathcal{X}) = 0$, $\text{pth}(\mathcal{X}) = 3$, and it is indeed almost obvious that $\mathcal{X} \not\in \Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(2, 2)$. An example for which (ii) is satisfied but (i) is violated is provided by $\text{gen}(\mathcal{Y}) = 1$ and $\text{pth}(\mathcal{Y}) = 4$.

Proposition 6 then gives a rigorous proof of the more or less obvious fact that $(\mathcal{X}) \not\in \Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(2, 2)$.

On the other hand, $\text{gen}(\mathcal{X}) = 0$ and $\text{pth}(\mathcal{X}) = 4$, which corroborates that $\mathcal{X} \in \Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(2, 2)$.

Observation 8. Bialgebras cannot be defined over $\frac{1}{2}\text{PROP}$s, because the compatibility axiom [6] contains an element which does not belong to $\Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(2, 2)$, see [12].

In contrast, $\frac{1}{2}$bialgebras are algebras over the $\frac{1}{2}\text{PROP}$ $\frac{1}{2}b$ defined as

$$\frac{1}{2}b := \Gamma_{1/2}(\mathcal{A}, \mathcal{Y})/i_{1/2}b,$$

with the ideal $i_{1/2}b$ generated by $\mathcal{A} - \mathcal{A}$, $\mathcal{Y} - \mathcal{Y}$ and $\mathcal{X}$.

For a generator $\xi$ of biarity $(m, n)$, let $\text{Span}_{\Sigma, \Sigma}(\xi) := k[\Sigma_m] \otimes k \cdot \xi \otimes k[\Sigma_n]$, with the obvious mutually compatible left $\Sigma_m$-right $\Sigma_n$-actions.

The first step in pursuing the Principle formulated in Section 1 is to describe a minimal model of the $\frac{1}{2}\text{PROP}$ for $\frac{1}{2}$bialgebras in the category of $\frac{1}{2}\text{PROP}$s. This can be done as follows. Theorem 18 of [15] implies that $\frac{1}{2}b$ is a Koszul quadratic $\frac{1}{2}\text{PROP}$, therefore its minimal model is given by the cobar dual $\Omega_{\frac{1}{2}\text{PROP}}(\frac{1}{2}b)$ of the quadratic dual $\frac{1}{2}b^!$ of $\frac{1}{2}b$. This cobar dual is, by definition, a dg-$\frac{1}{2}\text{PROP}$ of the form $(\Omega_{\frac{1}{2}}(\Xi), \partial_0)$, with

$$\Xi := \Lambda \downarrow (\frac{1}{2}b^!)^*,$$

where $\Lambda$ denotes the sheared suspension [2], $\downarrow$ the usual desuspension of a graded vector space and $(-)^*$ the linear dual. Because, by [15] Example 16], $\frac{1}{2}b^!(m, n) \cong k$ for any $(m, n) \neq (1, 1)$, one immediately sees that $\Xi := \text{Span}_{\Sigma, \Sigma}(\xi^m_{m, n})_{m, n \in I}$ with

$$I := \{m, n \geq 1, (m, n) \neq (1, 1)\},$$

where the generator $\xi^m_{m, n}$ of biarity $(m, n)$ has degree $n + m - 3$. 
It remains to describe the differential $\partial_0$ which is, by definition, the unique derivation extending the linear dual of the structure operations of $\frac{1}{2}\mathcal{b}$. The result is given in the following theorem.

**Theorem 9.** There is a minimal model of the $\mathcal{b}$ PROP $\frac{1}{2}\mathcal{b}$

\[(\frac{1}{2}\mathcal{b}, \partial = 0) \xleftarrow{\rho} (\Gamma_{\frac{1}{2}}(\Xi), \partial_0),\]

with the map $\rho_\frac{1}{2}$ defined by

\[\rho_\frac{1}{2}(\xi_1) := \mathbb{A}, \quad \rho_\frac{1}{2}(\xi_1^2) := \mathbb{Y},\]

while $\rho_\frac{1}{2}$ is trivial on all remaining generators. The differential $\partial_0$ is given by the formula

\[\partial_0(\xi_n^m) := (-1)^m \xi_1^m \circ \xi_1^1 + \sum_U (-1)^{(s+1)+m} \xi_u^m \circ_i \xi_1^1 \xi_s^1 + \sum_V (-1)^{(t+1)+1} \xi_1^t \circ \xi_v^n,\]

where we set $\xi_1^1 := 0$,

\[U := \{u, s \geq 1, u + s = n + 1, 1 \leq i \leq u\}\]

and

\[V = \{t, v \geq 1, t + v = m + 1, 1 \leq j \leq v\}\].

It follows from the remarks preceding Theorem 9 that a quadratic Koszul $\frac{1}{2}\text{PROP}$ admits a canonical functorial minimal model, given by the cobar dual of its quadratic dual. It can also be proved that minimal models of $\frac{1}{2}\text{PROPS}$ are unique up to isomorphism.

**Example 10.** If we denote $\xi_2 = \mathbb{A}$ and $\xi_1^2 = \mathbb{Y}$, then $\partial_0(\mathbb{A}) = \partial_0(\mathbb{Y}) = 0$. If $\xi_2 = \mathbb{X}$, then $\partial_0(\mathbb{X}) = \mathbb{X}$.

With the obvious, similar notation,

\[\begin{align*}
\partial_0(\mathbb{A}) &= \mathbb{A} - \mathbb{A}, \\
\partial_0(\mathbb{A}) &= \mathbb{A} - \mathbb{A} + \mathbb{A} - \mathbb{A} - \mathbb{A}, \\
\partial_0(\mathbb{Y}) &= \mathbb{Y} - \mathbb{Y}, \\
\partial_0(\mathbb{X}) &= \mathbb{X} - \mathbb{X} + \mathbb{X}, \\
\partial_0(\mathbb{X}) &= -\mathbb{X} + \mathbb{X} - \mathbb{X}, \\
\partial_0(\mathbb{X}) &= -\mathbb{X} + \mathbb{X} - \mathbb{X} + \mathbb{X} - \mathbb{X}, \\
\partial_0(\mathbb{X}) &= \mathbb{X} + \mathbb{X} + \mathbb{X} - \mathbb{X} + \mathbb{X} - \mathbb{X}, \text{ etc.}
\end{align*}\]

Observe that (14) for $m = 1$ gives

\[\partial_0(\xi_1^1) = \sum_U (-1)^{(s+1)+1} \xi_u^1 \circ_i \xi_s^1,\]

where $U$ is as in Theorem 9. Therefore the sub-$\frac{1}{2}\text{PROP}$ generated by $\xi_2, \xi_3, \xi_4, \ldots$ is in fact isomorphic to the minimal model $\mathcal{A}_\infty$ for the operad of associative algebras as described in [11].
It is well-known that $A_\infty$ is the operad of cellular chains of a cellular topological operad $K = \{K_n\}_{n \geq 2}$ such that each $K_n$ is an $(n - 2)$-dimensional convex polyhedron – the Stasheff associahedron (see [14, Section 1.6]). The formulas for the differential $\partial_0(\xi^1)$ then reflect the decomposition of the topological boundary of the top dimensional cell of $K_n$ into the union of codimension one faces. For example, the two terms in the right-hand side of (15) correspond to the two endpoints of the interval $K_3$, the five terms in the right-hand side of (16) to the five edges of the pentagon $K_4$, etc.

**Remark 11.** Just as there are non-$\Sigma$ operads as simplified versions of operads without the actions of symmetric groups [14, Definition II.1.14], there are obvious notions of non-$\Sigma$ props and non-$\Sigma$ $1/2$ props. Of a particular importance for us will be the free non-$\Sigma$ $1/2$ prop $\Gamma_{1/2}(\Xi)$ generated by $\Xi := \text{Span}(\{\xi^m_{n,I}\}_{m,n,I})$, where $\xi^m_n$ and $I$ are as in Theorem 9. There clearly exists, for any $m$ and $n$, a $\partial_0$-invariant factorization of $\Sigma_m$-$\Sigma_n$ spaces

$$\Gamma_{1/2}(\Xi)(m,n) \cong k[\Sigma_m] \otimes \Gamma_{1/2}(\Xi)(m,n) \otimes k[\Sigma_n].$$

Therefore, the acyclicity of $(\Gamma_{1/2}(\Xi), \partial_0)$ is equivalent to the acyclicity of $(\Gamma_{1/2}(\Xi), \partial_0)$. Observe that there is no analog of factorization (17) for props.

**Remark 12.** Another way to control the combinatorial explosion of props was suggested by W.L. Gan who introduced dioperads. Roughly speaking, a dioperad is a prop in which only compositions based on graphs of genus zero are allowed, see [2] for details.

Dioperads are slightly bigger than $1/2$ props. The piece $\Gamma_D(\mathcal{A}, \mathcal{Y})(m,n)$ of the free dioperad $\Gamma_D(\mathcal{A}, \mathcal{Y})$ is spanned by genus zero monomials of $\Gamma(\mathcal{A}, \mathcal{Y})(m,n)$, with no restriction on the path grading. Therefore, for instance,

$$\forall \gamma \in \Gamma_D(\mathcal{A}, \mathcal{Y})(2,2), \text{ while } \exists \gamma \not\in \Gamma_{1/2}(\mathcal{A}, \mathcal{Y})(2,2),$$

see Example 7. The relation between props, dioperads and $1/2$ props is analyzed in [15], where we also explain why $1/2$ props are better suited for our purposes than dioperads.

Let us finish Step 1 formulated in Section 1 by describing a minimal model of the prop $1/2 B$, following again [15]. Observe first that the prop $1/2 B$ is generated by the $1/2$ prop $1/2 b$. By this we mean that $1/2 B = L_{\square}(1/2 b)$, where $L_{\square} : 1/2 \text{PROP} \to \text{PROP}$ is the left adjoint to the forgetful functor $\square : \text{PROP} \to 1/2 \text{PROP}$. The functor $L_{\square}$ is, by [15, Theorem 4], exact. This surprisingly deep statement follows from the fact, observed by M. Kontsevich in [5], that $L_{\square}$ is a polynomial functor in the sense recalled in [6, Definition 1]. The last thing we need to realize is that $L_{\square}(\Gamma_{1/2}(\Xi), \partial_0) = (\Gamma(\Xi), \partial_0)$, where the differential $\partial_0$ is in both cases given by the same formula on the space of generators. We conclude that the application of the functor $L_{\square}$ to the minimal model of the $1/2$ prop $1/2 b$ described in Theorem 9 gives a minimal model of the prop $1/2 B$. We obtain

**Theorem 13.** The dg-prop

$$M_0 := (\Gamma(\Xi), \partial_0),$$

where the generators $\Xi$ are as in Theorem 9 and the differential $\partial_0$ is given by formula (14), is a minimal model of the prop $1/2 B$ for $1/2$ bialgebras.
Remark 14. For a $\frac{1}{2}\text{PROP} s$, let $P(s)$ be the augmented PROPs whose augmentation ideal equals $s$, whose compositions $o_i$ and $j\circ$ of (13) and (10) are those of $s$, and other compositions (that is, those not allowed for $\frac{1}{2}\text{PROP}s$) are set to be zero. Theorem 13 expresses the fact that the $\text{PROP} P(\frac{1}{2}b^1)$ is the quadratic dual of the $\frac{1}{2}\text{PROP}$ $B$ in the category of PROPs in the sense of B. Vallette [24, 25].

3. Main theorem and the proof - first attempt

Let us formulate the main theorem of the paper.

Theorem 15. There exists a minimal model $(M, \partial)$ of the $\text{PROP} B$ for bialgebras that is a perturbation of the minimal model $(M_0, \partial_0)$ of the $\frac{1}{2}\text{PROP} B$ for $\frac{1}{2}$bialgebras described in Theorem 13. By this we mean that

$$(M, \partial) = (\Gamma(\Xi), \partial_0 + \partial_{\text{pert}}),$$

where the generators $\Xi = \text{Span}_{\Sigma, \Sigma}(\{\xi^m_{n}\}_{m, n \in I})$ are as in Theorem 9 and $\partial_0$ is a derivation given by formula (14). The perturbation $\partial_{\text{pert}}$ raises the genus and preserves the path grading. More precisely, $\partial_{\text{pert}} = \partial_1 + \partial_2 + \partial_3 + \cdots$, where $\partial_g$ raises the genus by $g$, preserves the path grading and, moreover,

$$\partial_g(\xi^m_n) = 0 \text{ for } g > (m - 1)(n - 1).$$

Uniqueness of minimal models for PROPs is discussed in Section 8. Observe that (19) implies $\partial(\xi^1_n) = \partial_0(\xi^1_n)$ for all $n$. Therefore the sub-dg-operad generated in $(M, \partial)$ by $\xi^1_2, \xi^3_3, \xi^4_4, \ldots$ is isomorphic to the operad describing strongly homotopy associative algebras.

Formulas for the perturbed differential $\partial_{\text{pert}}(\xi^m_n)$ are, for some small $m$ and $n$, given in Section 5. Although Theorem 15 does not describe the perturbation $\partial_{\text{pert}}$ explicitly, it describes the space of generators $\Xi$ of the underlying free PROP. This itself seems to be very nontrivial information. It will also be clearer later that $\partial_0$ is in fact the quadratic part (with respect to the ‘obvious’ grading recalled in Section 3) of the perturbed differential $\partial$, therefore, using the terminology borrowed from rational homotopy theory, the unperturbed model $(M_0, \partial_0)$ describes the ‘homotopy Lie algebra’ of the $\text{PROP} B$.

Let us try to prove Theorem 15 by constructing naïvely a perturbation $\partial_{\text{pert}}$ as

$$\partial_{\text{pert}} = \partial_1 + \partial_2 + \partial_3 + \cdots,$$

where each $\partial_g$ is a derivation raising the genus by $g$. Observe that $\partial_g(\xi^m_n)$ must be a sum of decomposable elements, because the generators are of genus 0. It is, of course, enough to define $\partial_{\text{pert}}$ on the generators $\xi^m_n \in \Xi$ and extend it as a derivation.

We construct $\partial_{\text{pert}}(\xi^m_n)$ inductively. Let $N := m + n$. For $N = 3$, we must put

$$\partial_{\text{pert}}(\mathcal{A}) = \partial_{\text{pert}}(\mathcal{Y}) = 0.$$

Also for $N = 4$ the formula for the differential is dictated by the axioms of bialgebras:

$$\partial_{\text{pert}}(\mathcal{Y}) := \partial_{\text{pert}}(\mathcal{A}) := 0 \text{ and } \partial_{\text{pert}}(\mathcal{X}) := -\mathcal{C}.$$
For $N = 5$ we put
\[ \partial_{\text{pert}}(\mathbf{A}) = \partial_{\text{pert}}(\mathbf{Y}) := 0; \]
\( \partial_{\text{pert}}(\mathbf{X}) \) and \( \partial_{\text{pert}}(\mathbf{X}) \) are given by formulas
\begin{align*}
(20) \quad \partial_{\text{pert}}(\mathbf{X}) & := (\mathbf{X} \otimes \mathbf{X}) \circ (\mathbf{Y} \otimes \mathbf{X} - \mathbf{X} \otimes \mathbf{Y}) - (\mathbf{X} \otimes \mathbf{X} + \mathbf{X} \otimes \mathbf{X}) \circ (\mathbf{Y} \otimes \mathbf{Y} \otimes \mathbf{Y}), \\
(21) \quad \partial_{\text{pert}}(\mathbf{X}) & := (\mathbf{X} \otimes \mathbf{X} - \mathbf{X} \otimes \mathbf{X}) \circ (\mathbf{Y} \otimes \mathbf{Y}) + (\mathbf{X} \otimes \mathbf{X} \otimes \mathbf{X}) \circ (\mathbf{Y} \otimes \mathbf{Y} + \mathbf{Y} \otimes \mathbf{Y}).
\end{align*}
In the above displays, \( \sigma(2, 2) \) is the same as in (\ref{becker}),
\[
\sigma(3, 2) := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}
\]
with our usual convention that the ‘flow diagrams’ should be read from the bottom to the top, and \( \sigma(3, 2) := \sigma(2, 3)^{-1} \). Higher terms of the perturbed differential can be constructed by the standard homological perturbation theory as follows.

Suppose we have already constructed \( \partial_{\text{pert}}(\xi^u_v) \) for all \( u + v < N \) and fix some \( m \) and \( n \) such that \( m + n = N > 5 \). We are looking for \( \partial_{\text{pert}}(\xi^m_n) \) of the form
\[
(23) \quad \partial_{\text{pert}}(\xi^m_n) = \partial_1(\xi^m_n) + \partial_2(\xi^m_n) + \partial_3(\xi^m_n) + \cdots
\]
where \( \text{gen}(\partial_g(\xi^m_n)) = g \). Condition \( (\partial_g + \partial_{\text{pert}})^2(\xi^m_n) = 0 \) can be rewritten as
\[
\sum_{s + t = g} \partial_s \partial_t(\xi^m_n) = 0 \quad \text{for each } g \geq 1.
\]
We must therefore find inductively elements \( \partial_g(\xi^m_n), g \geq 1 \), solving the equation
\[
(24) \quad \partial_0 \partial_g(\xi^m_n) = - \sum_{s + t = g \atop t < g} \partial_s \partial_t(\xi^m_n).
\]
Observe that the right-hand side of (24) makes sense, because \( \partial_g(\xi^m_n) \) is a combination of \( \xi^u_v^m \)’s with \( u + v < N \), therefore \( \partial_g \partial_t(\xi^m_n) \) has already been defined. To verify that the right-hand side of (24) is a \( \partial_0 \)-cycle is also easy:
\[
\partial_0(- \sum_{s + t = g \atop t < g} \partial_s \partial_t(\xi^m_n)) = - \sum_{s + t = g \atop t < g} \partial_0 \partial_s \partial_t(\xi^m_n) = \sum_{s + t = g \atop t < g} \sum_{a + b = s \atop b < a} \partial_a \partial_b \partial_t(\xi^m_n)
\]
\[
= \sum_{1 \leq i \leq g} \partial_i(\sum_{k + l = g - i} \partial_k \partial_l(\xi^m_n)) = 0.
\]
The degree of the right-hand side of (24) is \( N - 5 \), which is a positive number, by our assumption \( N > 5 \). This implies that (24) has a solution, because \( (\Gamma(\Xi), \partial_0) \) is, by Theorem 13, \( \partial_0 \)-acyclic in positive dimensions. \( \square \)

There is however a serious flaw in the above proof: there is no reason to assume that the sum (24) is finite, that is, that the right-hand side of (24) is trivial for \( g \) sufficiently large!!! This convergence problem can be fixed by finding subspaces \( F(m, n) \subset \Gamma(\Xi)(m, n) \) satisfying the properties listed in the following definition.

**Definition 16.** The collection \( F \) of subspaces \( F(m, n) \subset \Gamma(\Xi)(m, n) \) is friendly if
(i) for each \( m \) and \( n \), there exists a constant \( C_{m,n} \) such that \( F(m,n) \) does not contain elements of genus \( > C_{m,n} \),

(ii) \( F \) is stable under all derivations (not necessary differentials) \( \omega \) satisfying \( \omega(\Xi) \subset F \),

(iii) \( \partial_0(\Xi) \subset F \), \( \mathcal{D} \in F(2,2) \), the right-hand side of (20) belongs to \( F(2,3) \) and the right-hand side of (21) belongs to \( F(3,2) \), and

(iv) \( F \) is \( \partial_0 \)-acyclic in positive degrees.

Observe that (ii) with (iii) imply that \( F \) is \( \partial_0 \)-stable, therefore (iv) makes sense. Observe also that we do not demand \( F(m,n) \) to be \( \Sigma_m-\Sigma_n \) invariant.

Suppose we are given such a friendly collection. We may then, in the above naïve proof, assume inductively that

\[
\partial_{\text{pert}}(\xi^m_n) \in F(m,n).
\]

Indeed, (25) is satisfied for \( m+n = 3, 4, 5 \), by (iii). Condition (ii) guarantees that the right-hand side of (24) belongs to \( F(m,n) \), while (iv) implies that (24) can be solved in \( F(m,n) \). Finally, (i) guarantees, in the obvious way, the convergence.

In this paper, we use the friendly collection \( S \subset \Gamma(\Xi) \) of special elements, introduced in Section 4. The collection \( S \) is generated by the free non-\( \Sigma \) \( \frac{1}{2} \text{PROP} \uplus \Gamma(\Xi) \), see Remark 11, by a suitably restricted class of compositions that naturally generalize those involved in \( \mathcal{D} \).

Another possible choice was proposed in [15], namely the friendly collection defined by

\[
F(m,n) := \{ f \in \Gamma(\Xi); \text{ pth}(f) = mn \}.
\]

This choice is substantially bigger than the collection of special elements and contains ‘strange’ elements, such as

\[
\mathcal{D} \in F(2,2)
\]

which we certainly do not want to consider. We believe that special elements are, in a suitable sense, the smallest possible friendly collection.

Properties of special elements are studied in Sections 6 and 7. Section 9 then contains a proof of Theorem 15.

4. Special elements

We introduce, in Definition 23 special elements in arbitrary free props. We need first the following:

**Definition 17.** For \( k,l \geq 1 \) and \( 1 \leq i \leq kl \), let \( \sigma(k,l) \in \Sigma_{kl} \) be the permutation given by

\[
\sigma(i) := k(i - 1 - (s-1)l) + s,
\]

where \( s \) is such that \( (s-1)l < i \leq sl \). We call permutations of this form special permutations.
To elucidate the nature of these permutations, suppose we have associative algebras $U_1, \ldots, U_k$. The above permutation is exactly the permutation used to define the induced associative algebra structure on the product 

\[(U_1 \otimes \cdots \otimes U_k) \otimes \cdots \otimes (U_1 \otimes \cdots \otimes U_k) / l\text{-times}/,\]

that is, the permutation which takes 

\[(u_1^1 \otimes u_2^1 \otimes \cdots \otimes u_k^1) \otimes (u_1^2 \otimes u_2^2 \otimes \cdots \otimes u_k^2) \otimes \cdots \otimes (u_1^l \otimes u_2^l \otimes \cdots \otimes u_k^l)\]

to

\[(u_1^1 \otimes u_2^2 \otimes \cdots \otimes u_k^l) \otimes (u_1^2 \otimes u_2^1 \otimes \cdots \otimes u_k^l) \otimes \cdots \otimes (u_1^l \otimes u_2^1 \otimes \cdots \otimes u_k^l).

**Example 18.** We have already seen examples of special permutations: the permutation $\sigma(2,2)$ in $\mathbb{S}$ and the permutation $\sigma(3,2)$ in $\mathbb{P}$. Observe that, for arbitrary $k, l \geq 1$, $\sigma(k, 1) = 1_{\Sigma_k}$, $\sigma(1, l) = 1_{\Sigma_l}$ and $\sigma(k, l) = \sigma(l, k)^{-1}$.

Special elements are defined using a special class of compositions defined as follows.

**Definition 19.** Let $P$ be an arbitrary PROP. Let $k, l \geq 1$, $a_1, \ldots, a_l \geq 1$, $b_1, \ldots, b_k \geq 1$, $A_1, \ldots, A_l \in P(a_i, k)$ and $B_1, \ldots, B_k \in P(l, b_j)$. Then define the $(k,l)$-fraction

\[
\frac{A_1 \cdots A_l}{B_1 \cdots B_k} := (A_1 \otimes \cdots \otimes A_l) \circ \sigma(k, l) \circ (B_1 \otimes \cdots \otimes B_k) \in P(a_1 + \cdots + a_l, b_1 + \cdots + b_k).
\]

**Example 20.** If $k = 1$ or $l = 1$, the $(k,l)$-fractions give the ‘operadic’ compositions:

\[
\frac{A_1 \otimes \cdots \otimes A_l}{B_1} = (A_1 \otimes \cdots \otimes A_l) \circ B_1 \quad \text{and} \quad \frac{A_1}{B_1 \otimes \cdots \otimes B_k} = A_1 \circ (B_1 \otimes \cdots \otimes B_k).
\]

We are going to use ‘dummy variables,’ that is, for instance, $A \in P(*, n)$ for a fixed $n \geq 1$ means that $A \in P(m, n)$ for some $m \geq 1$.

**Example 21.** For $\begin{array}{c} a \end{array}, \begin{array}{c} b \end{array} \in P(*, 2)$ and $\begin{array}{c} c \end{array}, \begin{array}{c} d \end{array} \in P(2, *)$,

\[
\begin{array}{c} a \end{array} \begin{array}{c} b \end{array} \otimes \begin{array}{c} c \end{array} \begin{array}{c} d \end{array} = (\begin{array}{c} a \end{array} \otimes \begin{array}{c} b \end{array}) \circ \sigma(2,2) \circ (\begin{array}{c} c \end{array} \otimes \begin{array}{c} d \end{array}) = \begin{array}{c} a \end{array} \begin{array}{c} b \end{array} \otimes \begin{array}{c} c \end{array} \begin{array}{c} d \end{array}.
\]

Similarly, for $\begin{array}{c} x \end{array}, \begin{array}{c} y \end{array} \in P(*, 3)$ and $\begin{array}{c} z \end{array}, \begin{array}{c} u \end{array}, \begin{array}{c} v \end{array} \in P(2, *)$,

\[
\begin{array}{c} x \end{array} \begin{array}{c} y \end{array} \otimes \begin{array}{c} z \end{array} \begin{array}{c} u \end{array} \begin{array}{c} v \end{array} = (\begin{array}{c} x \end{array} \otimes \begin{array}{c} y \end{array}) \circ \sigma(3,2) \circ (\begin{array}{c} z \end{array} \otimes \begin{array}{c} u \end{array} \otimes \begin{array}{c} v \end{array}) = \begin{array}{c} x \end{array} \begin{array}{c} y \end{array} \otimes \begin{array}{c} z \end{array} \begin{array}{c} u \end{array} \begin{array}{c} v \end{array}.
\]

If $P$ is a dg-PROP with differential $\partial$, then it easily follows from Definition 19 that

\[
\partial \left( \frac{A_1 \cdots A_l}{B_1 \cdots B_k} \right) = \sum_{1 \leq i \leq l} (-1)^{\deg(A_1) + \cdots + \deg(A_{i-1})} \frac{A_1 \cdots \partial A_i \cdots A_l}{B_1 \cdots B_k} + \sum_{1 \leq j \leq k} (-1)^{\deg(A_1) + \cdots + \deg(A_l) + \deg(B_1) + \cdots + \deg(B_{j-1})} \frac{A_1 \cdots \partial B_j \cdots B_k}{B_1 \cdots B_k}.
\]
Suppose that the PROP $\mathbb{P}$ is free, therefore the genus of monomials of $\mathbb{P}$ is defined. It is clear that, under the notation of Example 21,
\[
\text{gen} \left( \begin{array}{cccc}
\grave{a} & \grave{b} \\
\grave{c} & \grave{d}
\end{array} \right) = 1 + \text{gen}(\grave{a}) + \text{gen}(\grave{b}) + \text{gen}(\grave{c}) + \text{gen}(\grave{d})
\]
and also that
\[
\text{gen} \left( \begin{array}{cccc}
x & y \\
z & u \\
u & v
\end{array} \right) = 2 + \text{gen}(x) + \text{gen}(y) + \text{gen}(z) + \text{gen}(u) + \text{gen}(v).
\]
The following lemma generalizes the above formulas.

**Lemma 22.** Let $\mathbb{P}$ be a free PROP. Then the genus of the $(k, l)$-fraction is given by
\[
\text{gen} \left( \frac{A_1 \cdots A_l}{B_1 \cdots B_k} \right) = (k - 1)(l - 1) + \sum_{i=1}^{l} \text{gen}(A_i) + \sum_{j=1}^{k} \text{gen}(B_j).
\]

**Proof.** A straightforward and easy verification. \qed

**Definition 23.** Let us define the collection $S \subset \Gamma(\Xi)$ of special elements to be the smallest collection of linear subspaces $S(m, n) \subset \Gamma(\Xi)(m, n)$ such that:

(i) $1 \in S(1, 1)$ and all generators $\xi^m \in \Xi$ belongs to $S$, and
(ii) if $k, l \geq 1$ and $A_1, \ldots, A_l, B_1, \ldots, B_k \in S$, then
\[
\frac{A_1 \cdots A_l}{B_1 \cdots B_k} \in S.
\]

**Remark 24.** One may introduce special PROPs as objects similar to PROPs, but for which only compositions used in the definition of special elements (i.e. the ‘fractions’) are allowed. The collection $S \subset \Gamma(\Xi)$ would then be the free special PROP generated by $\Xi$.

**Example 25.** Let the boxes denote arbitrary special elements. Then the elements
\[
\begin{array}{ccc}
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\end{array}
\text{ and } \begin{array}{ccc}
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\end{array}
\]
are also special, while the elements
\[
\begin{array}{ccc}
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\end{array}
\text{ and } \begin{array}{ccc}
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\end{array}
\]
are not special. As an exercise, we recommend calculating the genera of these composed elements in terms of the genera of individual boxes. Other examples of special elements can be found in Section 5.

The following lemma states that the path grading of special elements from $S(m, n)$ equals $mn.$
Lemma 26. Let \( m, n \geq 1 \), let \( X \in S(m, n) \) be a monomial and let \( 1 \leq i \leq m, 1 \leq j \leq n \). Then there exists, in the graph \( G_X \), exactly one directed path connecting the \( i \)-th output with the \( j \)-th input. In particular, \( \text{pth}(X) = mn \) for any \( X \in S(m, n) \).

**Proof.** The statement is certainly true for generators \( \xi_m^m \). Suppose we have proved it for some \( A_1, \ldots, A_l, B_1, \ldots, B_k \in S \) and consider

\[
X := \frac{A_1 \cdots A_l}{B_1 \cdots B_k}.
\]

There clearly exist unique \( 1 \leq s \leq l \) and \( 1 \leq t \leq k \) such that the \( i \)-th output of \( X \) is an output of \( A_s \) and the \( j \)-th input of \( X \) is an input of \( B_t \).

It follows from the definition of \( \sigma(k, l) \) that the \( t \)-th input of \( A_s \) is connected to the \( s \)-th output of \( B_t \) and that the \( t \)-th input of \( A_s \) is the only input of \( A_s \) which is connected to some output of \( B_t \). These considerations obviously imply that there is, in \( G_X \), a unique directed path connecting the \( i \)-th output with the \( j \)-th input. \( \square \)

In the following lemma we give an upper bound for the genus of special elements.

Lemma 27. Let \( X \in S(m, n) \) be a monomial. Then \( \text{gen}(X) \leq (m-1)(n-1) \).

**Proof.** A straightforward induction on the ‘obvious’ grading. If \( \text{grd}(X) = 1 \), then \( X \) is a generator and Lemma 27 holds trivially. Each \( X \in S(m, n) \) with \( \text{grd}(X) > 1 \) can be decomposed as

\[
X = \frac{A_1 \cdots A_u}{B_1 \cdots B_u},
\]

with some \( 1 \leq v \leq m, 1 \leq u \leq n, A_i \in S(a_i, u), B_j \in S(v, b_j) \), \( a_i \geq 1, b_j \geq 1, 1 \leq i \leq v, 1 \leq j \leq u \), \( \sum_1^v a_i = m, \sum_1^u b_j = n \), such that \( \text{grd}(A_i), \text{grd}(B_j) < \text{grd}(X) \). By Lemma 22 and the induction assumption

\[
\text{gen}(X) = (u-1)(v-1) + \sum_1^v \text{gen}(A_i) + \sum_1^u \text{gen}(B_j)
\]

(by induction)

\[
\leq (u-1)(v-1) + \sum_1^v (a_i - 1)(u - 1) + \sum_1^u (v - 1)(b_j - 1)
= (u - 1)(v - 1) + (m - v)(u - 1) + (v - 1)(n - u)
= (m - 1)(n - 1) - (m - v)(n - u) \leq (m - 1)(n - 1). \quad \square
\]

**Remark 28.** Observe that the subspaces \( S(m, n) \subset \Gamma(\Xi)(m, n) \) are not \( \Sigma_m - \Sigma_n \) invariant. It easily follows from Proposition 10 and Lemma 26 that the subspace \( S_0 \) of \( S \) spanned by genus zero monomials coincides with the free non-\( \Sigma \frac{1}{2} \text{PROP} \Gamma_{\frac{1}{2}}(\Xi) \).

**Theorem 29.** Special elements form a friendly collection.

**Proof.** Condition (i) of Definition 16 with \( C_{m,n} = (m-1)(n-1) \), follows from Lemma 27. Condition (ii) follows from the fact, observed in Remark 24, that \( S \) is the free special PROP while (iii) is completely clear. In contrast, acyclicity (iv) is a very deep statement which we formulate as:
Proposition 30. The vector spaces $S(m, n)$ of special elements are $\partial_0$-acyclic in positive degrees, for each $m, n \geq 1$.

Proposition 30 is proved in Section 7. □

5. Explicit calculations

In this section we give a couple of formulas for the perturbed differential (the formulas for the unperturbed differential $\partial_0$ were given in Example 10). The first nontrivial one expresses the compatibility axiom, the second two are (20) and (21):

\[
\begin{align*}
\partial(\mathbf{x}) &= \partial_0(\mathbf{x}) - \frac{\mathbf{A}}{\mathbf{Y}} \\
\partial(\mathbf{x}) &= \partial_0(\mathbf{x}) + \frac{\mathbf{A}}{\mathbf{Y}} - \frac{\mathbf{A}}{\mathbf{Y}} - \frac{\mathbf{A}}{\mathbf{Y}} - \frac{\mathbf{A}}{\mathbf{Y}} \\
\partial(\mathbf{x}) &= \partial_0(\mathbf{x}) - \frac{\mathbf{A}}{\mathbf{Y}} + \frac{\mathbf{A}}{\mathbf{Y}} + \frac{\mathbf{A}}{\mathbf{Y}} + \frac{\mathbf{A}}{\mathbf{Y}}
\end{align*}
\]

Let us pause a little and formulate the following conjecture.

Conjecture 31. There exists a series of convex $(m+n-3)$-dimensional polyhedra $B_n^m$ such that the differential $\partial(s_n^m)$ is the sum of the codimension-one faces of these polyhedra.

These polyhedra should generalize the case of $A_\infty$-algebras discussed in Example 10 in the sense that $B_1^1 = B_1^n = K_n$ for $n \geq 2$. Clearly $B_2^2$ is the interval, while $B_3^2 = B_3^2$ is the heptagon depicted in Figure 1. Before we proceed, we need to simplify our notation by an almost obvious ‘linear extension’ of $(k, l)$-fractions.

Notation 32. Let $k, l, s, t \geq 1$, $A_i^1, \ldots, A_i^s \in S(*, k)$ and $B_i^1, \ldots, B_i^t \in S(l, *)$. Then define

\[
\sum_{s,t} A_i^1 \cdots A_i^s B_i^1 \cdots B_i^t \ := \ \sum_{s,t} \frac{A_i^s \cdots A_i^s}{B_i^1 \cdots B_i^t}.
\]

For example, with this notation the formula for $\partial(\mathbf{x})$ can be simplified to

\[
\begin{align*}
\partial(\mathbf{x}) &= \partial_0(\mathbf{x}) + \frac{\Delta(\mathbf{A})}{\mathbf{Y}} - \frac{\Delta(\mathbf{A})}{\mathbf{Y}} \\
&= \partial_0(\mathbf{x}) + \frac{\Delta(\mathbf{A})}{\mathbf{Y}} - \frac{\Delta(\mathbf{A})}{\mathbf{Y}}
\end{align*}
\]

where $\Delta$ is the Saneblidze-Umble diagonal in the associahedron [18].
The next term is

$$\partial(\mathcal{X}) = \partial_0(\mathcal{X}) - \frac{\mathcal{X}}{\mathcal{X}} - \frac{\mathcal{X}}{\mathcal{Y}} + \frac{\mathcal{X}}{\mathcal{X}} + \frac{\mathcal{X}}{\mathcal{Y}} + \frac{\mathcal{X}}{\mathcal{Y}} - \frac{\mathcal{X}}{\mathcal{Y}} - \frac{\mathcal{X}}{\mathcal{Y}} - \frac{\mathcal{X}}{\mathcal{Y}} - \frac{\mathcal{X}}{\mathcal{Y}}$$

$$+ \frac{\mathcal{Y}}{\mathcal{X} \mathcal{Y} - \mathcal{Y} \mathcal{X} - \mathcal{Y} \mathcal{X} - \mathcal{Y} \mathcal{X} - \mathcal{Y} \mathcal{X}} - \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y}}$$

$$- \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X} + \mathcal{Y} \mathcal{Y} \mathcal{X} - \mathcal{Y} \mathcal{Y} \mathcal{X} - \mathcal{Y} \mathcal{Y} \mathcal{X}} + \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}} + \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}} + \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}} - \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}} - \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}} - \frac{\mathcal{Y}}{\mathcal{Y} \mathcal{Y} \mathcal{X}}.$$

Observe that the last term of the above equation is

$$- \frac{\Delta^{(3)}(\mathcal{X})}{\Delta^{(3)}(\mathcal{Y})},$$

where $\Delta^{(3)}(-) := (\Delta \otimes \mathbb{1})\Delta(-)$ denotes the iteration of the Saneblidze-Umble diagonal which is coassociative on $\mathcal{X}$ and $\mathcal{Y}$ (see [13]). The corresponding 3-dimensional polyhedron $B_3^3$ is shown in Figure 2.
The plane projection of 3-dimensional polyhedron $B_3^3$ from one of its square faces. Polyhedron $B_3^3$ has 30 2-dimensional faces (8 heptagons and 22 squares), 72 edges and 44 vertices.

The relation with the Saneblidze-Umble diagonal $\Delta$ is even more manifest in the formula

$$
\partial(X) = \partial_0(X) + \frac{\Delta(\mathbf{A})}{\mathbf{XY} + \mathbf{XX} + \mathbf{YY}} + \frac{\Delta(\mathbf{A})}{\mathbf{XYY} - \mathbf{YY} + \mathbf{YYX}} + \frac{\Delta(\mathbf{A})}{\mathbf{YYYY}}.
$$

The corresponding $B_4^2$ is shown in Figure 3.

6. Calculus of special elements

This section provides a preparatory material for the proof of the $\partial_0$-acyclicity of the space $S(m, n)$ given in Section 7. As in the proof of Lemma 27, each monomial $X \in S(m, n)$ is represented as

$$
X = \frac{A_1 \cdots A_v}{B_1 \cdots B_u},
$$

for $1 \leq v \leq m$, $1 \leq u \leq n$, $A_i \in S(a_i, u)$, $B_j \in S(v, b_j)$, with $\sum_v a_i = m$ and $\sum_u b_j = n$. Very crucially, representation (26) is not unique, as illustrated in the following example.
Example 33. It is easy to verify that

\[(27) \quad \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}.\]

Therefore, the element \(X \in S(2, 3)\) above can be either represented as

\[X = \frac{A_1 A_2}{B_1 B_2},\]

with \(A_1 = A_2 = \bigstar \in S(1, 2),\)

\[B_1 = \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array} \in S(2, 2)\]

and \(B_2 = \bigstar \in S(2, 1)\), or as

\[X = \frac{A_1' A_2'}{B_1' B_2' B_3'},\]

with

\[A_1' = A_2' = \bigstar \in S(1, 3)\]

and \(B_1' = B_2' = B_3' = \bigstar \in S(2, 1)\). Of a bit different nature is the relation

\[(28) \quad \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}.\]
or a similar one

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\otimes \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
= \begin{array}{c}
\begin{array}{c}
\otimes \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\end{equation}

where \( \mathcal{O} \) is an arbitrary element of \( S(2,2) \).

It follows from the above remarks that

\begin{equation}
S(m, n) = \text{Span}(\xi^m) \oplus \bigoplus_M S(a_1, u) \otimes \cdots \otimes S(a_v, u) \otimes S(v, b_1) \otimes \cdots \otimes S(v, b_u) / R(m, n)
\end{equation}

where

\[ M = \{1 \leq v \leq m, 1 \leq u \leq n, (v,u) \neq (1,n), (m,1) \text{ and } \sum_1^v a_i = m, \sum_1^u b_j = n\} \]

and \( R(m, n) \) accounts for the non-uniqueness of presentation (26). Observe that if \( R(m, n) \) were trivial, then the \( \partial_0 \)-acyclicity of \( S(m,m) \) would follow immediately from the Künneth formula and induction.

**Example 34.** We have

\[ S(2,2) \cong \text{Span}(\xi^2) \oplus [S(2,1) \otimes S(1,2)] \oplus [S(1,2) \otimes S(1,2) \otimes S(2,1) \otimes S(2,1)] \]

\[ \cong \text{Span}(X) \oplus \text{Span}(X) \oplus \text{Span} \left( \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array} \right), \]

the relations \( R(2,2) \) are trivial. On the other hand, the left-hand side of (28) represents an element of \( S(3,3) \) by

\[ \begin{array}{c}
\begin{array}{c}
\otimes \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\end{array} \in S(2,3) \otimes S(1,3) \otimes S(2,1) \otimes S(2,1) \otimes S(2,1), \]

while the right-hand side of (28) represents the same element by

\[ \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\otimes \begin{array}{c}
\begin{array}{c}
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge \\
\wedge
\end{array}
\end{array}
\end{array} \in S(1,2) \otimes S(1,2) \otimes S(1,2) \otimes S(3,2) \otimes S(3,1). \]

Therefore \( R(3,3) \) must contain a relation that identifies these two elements.

Let us describe the space of relations \( R \). Suppose that \( s,t \geq 1, c_1, \ldots, c_s, d_1, \ldots, d_t \geq 1 \) are natural numbers and let \((c; d)\) denote the array \((c_1, \ldots, c_s; d_1, \ldots, d_t)\). A crucial rôle in the following definition will be played by a matrix

\begin{equation}
C = (C_{ij})_{1 \leq i \leq s, 1 \leq j \leq t}
\end{equation}

with entries \( C_{ij} \in S(d_i, c_j) \). Finally, let

\[ X = \frac{A_1 \cdots A_v}{B_1 \cdots B_u} \]

be a monomial as in (26).
Definition 35. Element $X$ is called $(c;d)$-up-reducible if $u = c_1 + \cdots + c_s, \ v = t$ and
\begin{equation}
A_i = \frac{A_{i1} \cdots A_{id_i}}{C_{i1} \cdots C_{is}}
\end{equation}
for some $A_{ib} \in S(\ast, s), \ 1 \leq i \leq t, \ 1 \leq b \leq d_i$, where $C_{ij}$ are entries of a matrix as in (31). Dually, $X$ is called $(c;d)$-down-reducible if $v = d_1 + \cdots + d_t, \ u = s$ and
\begin{equation}
B_j = \frac{C_{ij} \cdots C_{jt}}{B_{1j} \cdots B_{cj}}
\end{equation}
for some $B_{aj} \in S(t, \ast), \ 1 \leq j \leq s, \ 1 \leq a \leq c_j$, again with $C = (C_{ij})$ as in (31). We denote by $Up(c;d)$ (resp. $Dw(c;d)$) the subspace spanned by all $(c;d)$-(resp. down) -reducible monomials.

Proposition 36. The spaces $Up(c;d)$ and $Dw(c;d)$ are isomorphic. The isomorphism is given by the identification of the up-reducible element
\begin{equation}
\begin{array}{ccc}
A_{11} \cdots A_{1d_1} & A_{21} \cdots A_{2d_2} & \cdots \\
C_{11} \cdots C_{1s} & C_{21} \cdots C_{2s} & \cdots \\
B_{11} \cdots B_{c_1} & B_{12} \cdots B_{c_2} & \cdots \\
\end{array}
\end{equation}
with the down-reducible element
\begin{equation}
\begin{array}{ccc}
A_{11} \cdots A_{1d_1} & A_{21} \cdots A_{2d_2} & \cdots \\
C_{11} \cdots C_{1s} & C_{12} \cdots C_{1s} & \cdots \\
B_{11} \cdots B_{c_1} & B_{12} \cdots B_{c_2} & \cdots \\
\end{array}
\end{equation}
Relations $R$ in (30) are generated by the above identifications.

Proof. The proof follows from analyzing the underlying graphs.

We call the relations described in Proposition 36 the $(c;d)$-relations. These relations are clearly compatible with the differential $\partial_0$ and do not change the genus. They are trivial if $d_j = c_i = 1$, for all $i, j$.

Example 37. Equation (27) of Example 33 is an equality of two $(2,1;1,1)$-reducible elements with $A_{11} = A_{21} = \ast, \ B_{11} = B_{21} = B_{12} = \varepsilon$ and the matrix (31) given by
\begin{equation}
C = \begin{pmatrix}
\ast & \varepsilon \\
\ast & \varepsilon
\end{pmatrix}
\end{equation}
Equation (28) of Example 33 is an equality between two $(2,1;2,1)$-reducible elements with $A_{11} = A_{12} = A_{21} = \ast, \ B_{11} = B_{21} = B_{12} = \varepsilon$ and
\begin{equation}
C = \begin{pmatrix}
\ast & \varepsilon & \varepsilon \\
\ast & \varepsilon & \varepsilon
\end{pmatrix}
\end{equation}
We leave it as an exercise to interpret also (29) in terms of $(c;d)$-relations.
Example 38. Let us write presentation (30) for $S(2, 3)$. Of course, $S(2, 3) = S_0(2, 3) \oplus S_1(2, 3) \oplus S_2(2, 3)$, where the subscript denotes the genus. Then $S_0(2, 3)$ is represented as the quotient of

$$Span(\zeta^2_2) \oplus [S_0(2, 2) \otimes S(1, 1) \otimes S(1, 2)] \oplus [S(2, 1) \otimes S(1, 3)] \oplus [S_0(2, 2) \otimes S(1, 2) \otimes S(1, 1)] \cong$$

$$\cong Span(\mathbf{X}) \oplus Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{Z}}{\mathbf{I}} \right),$$

where $\mathbf{X}$ is an arbitrary element of $S_0(1, 3)$ and $\mathbf{Y}$ is an element of $S_0(2, 2)$, modulo relations $R(2, 3)$ that identify the up-(2; 1)-reducible element

$$\frac{\mathbf{X}}{\mathbf{I}} \in Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right)$$

with the down-(2; 1)-reducible element

$$\frac{\mathbf{Y}}{\mathbf{I}} \in Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right)$$

and identify the up-(2; 1)-reducible element

$$\frac{\mathbf{X}}{\mathbf{I}} \in Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right)$$

with the down-(2; 1)-reducible element

$$\frac{\mathbf{Y}}{\mathbf{I}} \in Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right).$$

With the obvious similar notation, $S_1(2, 3)$ is the quotient of

$$Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{Z}}{\mathbf{I}} \right),$$

where again $\mathbf{X} \in S_0(2, 2)$ is an arbitrary element, modulo relations $R(2, 3)$ that identify the up-(1, 1; 2)-reducible generator of the second summand with

$$\frac{\mathbf{X}}{\mathbf{I}} \in Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right)$$

and the up-(1, 1; 2)-reducible generator of the fourth summand with

$$\frac{\mathbf{Y}}{\mathbf{I}} \in Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right).$$

Finally, $S_2(2, 3)$ is the quotient of

$$Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right) \oplus Span \left( \frac{\mathbf{Y}}{\mathbf{I}} \right)$$

modulo $R(2, 3)$ identifying the up-(2, 1; 1)-reducible element

$$\frac{\mathbf{X}}{\mathbf{I}} \in Span \left( \frac{\mathbf{X}}{\mathbf{I}} \right)$$
with the generator of the first summand, and the up-(1, 2; 1, 1)-reducible element
\[ \begin{array}{c}
\text{reducible element}
\end{array} \in \text{Span} \left( \begin{array}{c}
\text{generator of last summand}
\end{array} \right) \]
with the generator of the last summand.

Example 38 shows that presentation (30) is not economical. Moreover, we do not need to delve into the structure of \( S_0(m, n) \) because we already know that this piece of \( S(m, n) \), isomorphic to the free non-\( \Sigma\frac{1}{2} \text{PROP} \Gamma_1(\Xi) \), is \( \partial_0 \)-acyclic, see Remarks 11, 28 and Theorem 9. So we will work with the reduced form of presentation (30):

\[ S(m, n) = S_0(m, n) \oplus \bigoplus N S(a_1, u) \otimes \cdots \otimes S(a_v, u) \otimes S(v, b_1) \otimes \cdots \otimes S(v, b_u) / Q(m, n) \]

where

\[ N := M \cap \{ v \geq 2, u \geq 2 \} = \{ 2 \leq v \leq m, 2 \leq u \leq n, \text{ and } \sum v a_i = m, \sum u b_j = n \} \]

and \( Q(m, n) \subset R(m, n) \) is the span of \((c_1, \ldots, c_s; d_1, \ldots, d_t)\)-relations with \( s, t \geq 2 \).

Example 39. We have the following reduced representations:

\[ S(2, 2) = S_0(2, 2) \oplus \text{Span} \left( \begin{array}{c}
\text{generator of 2, 2}
\end{array} \right) \quad \text{and} \]

\[ S_1(2, 3) = \text{Span} \left( \begin{array}{c}
\text{generator of 2, 3}
\end{array} \right) \oplus \text{Span} \left( \begin{array}{c}
\text{generator of 1, 3}
\end{array} \right), \quad \text{where} \ \Box \in S_0(2, 2). \]

The reduced presentation of \( S_2(2, 3) \) is the same as the unreduced one given in Example 38.

We conclude that

\[ S(2, 3) \cong S_0(2, 3) \oplus S_0(2, 2) \oplus S_0(2, 2) \oplus S_0(1, 3)^{\otimes 2}. \]

This, by the way, immediately implies the \( \partial_0 \)-acyclicity of \( S(2, 3) \).

7. Acyclicity of the space of special elements

The proof of the \( \partial_0 \)-acyclicity, in positive dimensions, of \( S(m, n) \) is given by induction on \( K := m \cdot n \). The acyclicity is trivial for \( K \leq 2 \). Indeed, there are only three spaces to consider, namely \( S(1, 1) = \text{Span}(\mathbb{1}), S(1, 2) = \text{Span}(\mathbb{\Lambda}) \) and \( S(2, 1) = \text{Span}(Y) \). All these spaces are concentrated in degree zero and have trivial differential.

The acyclicity is, in fact, obvious also for \( K = 3 \) because \( S(1, 3) = S_0(1, 3) \) and \( S(3, 1) = S_0(3, 1) \) coincide with their tree parts. For \( K = 4 \) we have two ‘easy’ cases, \( S(4, 1) = S_0(4, 1) \) and \( S(1, 4) = S_0(1, 4) \), while the acyclicity of \( S(2, 2) \) follows from presentation (36).

Suppose we have proved the \( \partial_0 \)-acyclicity of all \( S(k, l) \) with \( k \cdot l < K \). Let us express the reduced presentation (34) as the short exact sequence

\[ 0 \to Q(m, n) \to L(m, n) \oplus S_0(m, n) \to S(m, n) \to 0, \]
where we denoted

\[ L(m, n) := \bigoplus_N S(a_1, u) \otimes \cdots \otimes S(a_v, u) \otimes S(v, b_1) \otimes \cdots \otimes S(v, b_u) \]

with \( N \) defined in (35). It follows from the Künneth formula and induction that \( L(m, n) \) is \( \partial_0 \)-acyclic while the acyclicity of \( S_0(m, n) \cong \bigoplus Q \) was established in Remark 11. Short exact sequence (37) then implies that it is in fact enough to prove that the space of relations \( Q(m, n) \) is \( \partial_0 \)-acyclic for any \( m, n \geq 1 \). This would clearly follow from the following claim.

**Claim 40.** For any \( w \in L(m, n) \) such that \( \partial_0(w) \in Q(m, n) \), there exists \( z \in Q(m, n) \) such that \( \partial_0(z) = \partial_0(w) \).

**Proof.** It follows from the nature of relations in the reduced presentation (32) that

\[ \partial_0(w) = \sum_{(c, d)} u_{↑(c,d)}^{(c,d)} - u_{↓(c,d)}^{(c,d)}, \]

where the summation runs over all \((c; d) = (c_1, \ldots, c_s; d_1, \ldots, d_t)\) with \( s, t \geq 2 \), and \( u_{↑(c,d)}^{(c,d)} \) (resp. \( u_{↓(c,d)}^{(c,d)} \)) is an \((c;d)\)-up (resp. down) reducible element such that \( u_{↑(c,d)}^{(c,d)} - u_{↓(c,d)}^{(c,d)} \in Q(m, n) \).

The idea of the proof is to show that there exists, for each \((c;d)\), some \((c;d)\)-up-reducible \( z_{↑(c,d)}^{(c,d)} \) and some \((c;d)\)-down-reducible \( z_{↓(c,d)}^{(c,d)} \) such that \( z^{(c,d)} := z_{↑(c,d)}^{(c,d)} - z_{↓(c,d)}^{(c,d)} \) belongs to \( Q(m, n) \) and

\[ u_{↑(c,d)}^{(c,d)} - u_{↓(c,d)}^{(c,d)} = \partial_0(z^{(c,d)}). \]

Then \( z := \sum_{(c,d)} z^{(c,d)} \) will certainly fulfill \( \partial_0(z) = \partial_0(w) \). We will distinguish five types of \((c; d)\)'s. The first four types are easy to handle; the last type is more intricate.

**Type 1:** All \( d_1, \ldots, d_t \) are \( \geq 2 \) and all \( c_1, \ldots, c_s \) are arbitrary. In this case \( u_{↑(c,d)}^{(c,d)} \) is of the form

\[ \frac{A_1 \cdots A_t}{B_1 \cdots B_u} \]

with

\[ A_i = \frac{A_{i1} \cdots A_{id_i}}{C_{i1} \cdots C_{is}} \]

as in (32). It follows from the definition that \( \partial_0 \) cannot create \((k, l)\)-fractions with \( k, l \geq 2 \). Therefore a monomial as in (40) may occur among monomials forming \( \partial_0(y) \) for some monomial \( y \) if and only if \( y \) itself is of the above form. Let \( z_{↑(c,d)}^{(c,d)} \) be the sum of all monomials in \( w \) whose \( \partial_0 \)-boundary nontrivially contributes to \( u_{↑(c,d)}^{(c,d)} \). Let \( z_{↓(c,d)}^{(c,d)} \) be the corresponding \((c;d)\)-down-reducible element. Then clearly \( u_{↑(c,d)}^{(c,d)} = \partial_0(z_{↓(c,d)}^{(c,d)}) \) and (39) is satisfied with \( z^{(c,d)} := z_{↑(c,d)}^{(c,d)} - z_{↓(c,d)}^{(c,d)} \) constructed above. In this way, we may eliminate all \((c;d)\)'s of Type 1 from (38).

**Type 2:** All \( c_1, \ldots, c_t = 1 \) and all \( d_1, \ldots, d_s \) are arbitrary. In this case \( u_{↑(c,d)}^{(c,d)} \) is of the form

\[ \frac{A_1 \cdots A_t}{B_1 \cdots B_u} \]
with

\[(42) \quad A_i = \frac{A_{i1} \cdots A_{id_i}}{C_{i1} \cdots C_{is}},\]

where \(C_{ij} \in S(d_i, 1)\), for \(1 \leq i \leq t, 1 \leq j \leq s\). When \(d_i = 1\),

\[A_i = A_{i1} \circ (C_{i1} \otimes \cdots \otimes C_{is}),\]

where \(C_{ij} \in S(1, 1)\) must be a scalar multiple of \(1\). We observe that element \(\text{(41)}\) is \((c; d)\)-up-reducible if and only if \(A_i\) is as in \(\text{(42)}\) where \(d_i \geq 2\); if \(d_i = 1\) then \(A_i\) may be arbitrary. We conclude, as in the previous case, that a monomial of the above form may occur in \(\partial_b(y)\) if and only if \(y\) itself is of the above form. Therefore, by the same argument, we may eliminate \((c; d)\)'s of Type 2 from \(\text{(38)}\).

**Type 3:** All \(c_1, \ldots, c_t \geq 2\) and \(d_1, \ldots, d_s\) are arbitrary. This case is dual to Type 1.

**Type 4:** All \(d_1, \ldots, d_t = 1\) and \(c_1, \ldots, c_s\) are arbitrary. This case is dual to Type 2.

**Type 5:** The remaining case. This means that \(1 \in \{c_1, \ldots, c_s\}\) but there exist some \(c_j \geq 2\), and \(1 \in \{d_1, \ldots, d_t\}\) but there exists some \(d_i \geq 2\). This is the most intricate case, because it may happen that, for some monomial \(y\), \(\partial_b(y)\) contains a \((c; d)\)- (up- or down-) reducible piece although \(y\) itself is not \((c; d)\)-reducible. For instance, let

\[y := \text{Diagram of \(y\)}\]

Then \(\partial_b(y)\) contains a \((2, 1; 2, 1)\)-up-reducible piece

\[\text{Diagram of \(\partial_b(y)\)}\]

though \(y\) itself is not \((2, 1; 2, 1)\)-up-reducible. Nevertheless, we see that \(\partial_b(y)\) contains also

\[\text{Diagram of \(\partial_b(y)\)}\]

which is not reducible. This is in fact a general phenomenon, that is, if \(\partial_b(y)\) contains a \((c; d)\)-up-reducible piece and if \(y\) is not \((c; d)\)-up-reducible, then \(\partial_b(y)\) contains also an irreducible piece. Therefore such \(y\) cannot occur among monomials forming up \(w\) in Claim \([40]\).

We conclude that \(y\) must also be \((c; d)\)-up-reducible and eliminate it from \(\text{(38)}\) as in the previous cases. Down-reducible pieces can be handled similarly. This finishes our proof of Claim \([40]\).
8. SOME GENERALITIES ON MINIMAL MODELS

In this section we discuss properties of minimal models of PROPs. We will see that minimal models of PROPs do not behave as nicely as for example minimal models of simply connected commutative associative algebras. We will start with an example of a PROP that does not admit a minimal model. Even when a minimal model of a given PROP exists, we are not able to prove that it is unique up to isomorphism, although we will show that it is still unique in a weaker sense. These pathologies of minimal models for PROPs are related to the absence of a suitable filtration required by various inductive procedures used in the “standard” theory of minimal models.

In this section we focus on minimal models of PROPs that are concentrated in (homological) degree 0. This generality would be enough for the purposes of this paper. Observe that even these very special PROPs need not have minimal models. An example is provided by the PROP

\[ X := \Gamma(u, v, w)/(u \circ v = w, v \circ w = u, w \circ u = v), \]

where \( u, v \) and \( w \) are degree 0 generators of biarity \((2, 2)\). Before we show that \( X \) indeed does not admit a minimal model, observe that a (non-negatively graded) minimal model of an arbitrary PROP concentrated in degree 0 is always of the form

\[ M = (\Gamma(E), \partial), \]

where \( E = \bigoplus_{i \geq 0} E_i \) with \( E_i := \{e \in E; \deg(e) = i\} \), and the differential \( \partial \) satisfying, for any \( n \geq 0 \),

\[ \partial(E_n) \subset \Gamma(E_{<n}), \quad E_{<n} := \bigoplus_{i<n} E_i. \]  

This means that \( M \) is special cofibrant in the sense of [12, Definition 17].

Free PROPs \( \Gamma(E) \) are canonically augmented, with the augmentation defined by \( \epsilon(E) = 0 \). This augmentation induces an augmentation of the homology of minimal dg-PROPs, therefore all PROPs with trivial differential which admit a minimal model are augmented. The contrary is not true, as shown by the example of the PROP \( X \) above with the augmentation given by \( \epsilon(u) = \epsilon(v) = \epsilon(w) := 0 \).

Indeed, assume that \( X \) has a minimal model \( \rho : (\Gamma(E), \partial) \to (X, 0) \). The map \( \rho \) induces the isomorphism

\[ H_0(\rho) : H_0(\Gamma(E), \partial) = \Gamma(E_0)/(\partial(E_1)) \xrightarrow{\cong} X. \]

Since \( X = k \oplus X(2, 2) \), \( E_0 = E_0(2, 2) \) and the above map is obviously an isomorphism of augmented PROPs. Therefore \( H_0(\rho) \) induces an isomorphism of the spaces of indecomposables. While it follows from the minimality of \( \partial \) that \( Q(\Gamma(E_0)/(\partial(E_1))) \cong E_0 \), clearly \( Q(X) = 0 \), from which we conclude that \( E_0 = 0 \), which is impossible.

Although we are not able to prove that minimal models are unique up to isomorphism, the following theorem shows that they are still well-defined objects of a certain derived category. Namely, let \( \text{ho-dgPROP} \) be the localization of the category \( \text{dgPROP} \) of differential non-negatively graded PROPs by homology isomorphisms.
Proposition 41. Let \( A \) be a PROP concentrated in degree 0. Then its minimal model (if exists), considered as an object of the localized category \( \text{ho-dgPROP} \), is unique up to isomorphism.

Proof. The proposition would clearly be implied by the following statement. Let \( \alpha : \mathcal{M}' \to A \) and \( \beta : \mathcal{M}'' \to A \) be two minimal models of \( A \). Then there exists a homomorphism \( h : \mathcal{M}' \to \mathcal{M}'' \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{h} & \mathcal{M}'' \\
\alpha \downarrow & & \beta \downarrow \\
A & & A
\end{array}
\]

commutes. Such a map \( h \) can be constructed by induction. Assume that \( \mathcal{M}' = (\Gamma(E), \partial') \), \( \mathcal{M}'' = (\Gamma(F), \partial'') \) and let \( h_0 : \Gamma(E_0) \to \mathcal{M}'' \) be an arbitrary lift in the diagram

\[
\begin{array}{ccc}
\Gamma(E_0) & \xrightarrow{h_0} & \mathcal{M}'' \\
\alpha|_{\Gamma(E_0)} \downarrow & & \beta \downarrow \\
A & & A
\end{array}
\]

Suppose we have already constructed, for some \( n \geq 1 \), a homomorphism

\[ h_{n-1} : \Gamma(E_{<n}) \to \mathcal{M}'', \]

such that \( \beta \circ h_{n-1} = \alpha|_{\Gamma(E_{<n})} \), where \( E_{<n} \) is as in (43). Let us show that \( h_{n-1} \) can be extended into \( h_n : \Gamma(E_{<n+1}) \to \mathcal{M}'' \) with the similar property. To this end, fix a \( k \)-linear basis \( B_n \) of \( E_n \) and observe that for each \( e \in B_n \) there exists a solution \( \omega_e \in \mathcal{M}'' \) of the equation

\[ \partial''(\omega_e) = h_{n-1}(\partial' e). \]

This follows from the fact that \( \partial'' h_{n-1}(\partial' e) = h_{n-1}(\partial' \partial' e) = 0 \) which means that the right-hand side of (44) is a \( \partial'' \)-cycle, therefore \( \omega_e \) exists by the acyclicity of \( \mathcal{M}'' \) in degree \( n-1 \). Define a linear map \( r_n : E_n \to \mathcal{M}'' \) by \( r_n(e) := \omega_e \), for \( e \in B_n \). Finally, define a linear equivariant map \( r_n : E_n \to \mathcal{M}'' \) by

\[ r_n(f) := \sum_{\tau, \sigma} \frac{1}{k! l!} \sigma^{-1} L_n(\sigma f \tau) \tau^{-1}, \]

where \( f \in E_n \) is of biarity \((k, l)\) and the summation runs over all \( \sigma \in \Sigma_k \) and \( \tau \in \Sigma_l \). It is easy to verify that the homomorphism \( h_n : \Gamma(E_n) \to \mathcal{M}'' \) determined by \( h_n(f) := r_n(f) \) for \( f \in E_n \), extends \( h_{n-1} \), and the induction goes on.

By modifying the proof of [12, Lemma 20], one may generalize Proposition 41 to an arbitrary non-negatively graded PROP with trivial differential.

Remark 42. One usually proves that two minimal models connected by a (co)homology isomorphisms are actually isomorphic. This is for instance true for minimal models of 1-connected commutative associative dg-algebras [7, Theorem 11.6(iv)], minimal models of connected dg-Lie algebras [23, Theorem II.4(9)] as well as for minimal models of augmented
However, ‘classical’ isomorphism theorems can still be proved if one imposes some additional assumptions on the type of minimal models involved, as illustrated by Theorem 43 below. Let us call a minimal model \((\Gamma(\Xi), \partial)\) of the bialgebra PROP \(B\) *special* if the differential \(\partial\) preserves the subspace of special elements and if it is of the form \(\partial = \partial_0 + \partial_{\text{pert}}\), where \(\partial_0\) is as in (14) and \(\partial_{\text{pert}}\) raises the genus.

**Theorem 43.** Let \(M' = (\Gamma(\Xi), \partial')\) and \(M'' = (\Gamma(\Xi), \partial'')\) be two special minimal models for the bialgebra PROP \(B\). Then there exists an isomorphism \(\phi : M' \to M''\) preserving the space of special elements.

**Proof.** Let us construct inductively a map \(\phi : (\Gamma(\Xi), \partial') \to (\Gamma(\Xi), \partial'')\) of augmented PROPS which preserves the space of special elements and which is the ‘identity modulo elements of higher genus.’ By this we mean that \(\phi|_{\Xi} = 1_{\Xi} + \eta\), where the image of the linear map \(\eta : \Xi \to \Gamma(\Xi)\) consists of special elements of positive genera.

The first step of the inductive construction is easy: we define \(\phi_0 : \Gamma(\Xi_0) \to \Gamma(\Xi)\) by \(\phi_0|_{\Xi_0} := 1_{\Xi_0}\). Suppose we have already constructed \(\phi_{n-1} : \Gamma(\Xi_{<n}) \to \Gamma(\Xi)\). As in the proof of Proposition 41 one must solve, for each element \(e\) of a basis of \(\Xi_n\), the equation

\[
\partial''\omega_e = \phi_{n-1}(\partial'e).
\]

The right-hand side is a \(\partial''\)-cycle, therefore a solution \(\omega_e\) exist by the acyclicity of \(M''\) in degree \(n-1\). But not every solution is good for our purposes. Observe that the right-hand side of (45) is of the form

\[
\phi_{n-1}(\partial_0 e + \partial_{\text{pert}} e) = \partial_0(e) + \vartheta_e,
\]

where \(\vartheta_e\) is a sum of special elements of positive genera. We leave as an exercise to prove that the \(\partial_0\)-acyclicity of the space of special elements implies, similarly as in the ‘naïve’ proof of Theorem 15 given in Section 3, that in fact there exists \(\omega_e\) of the form \(\omega_e = e + \eta_e\), where \(\eta_e\) is a sum of special elements of genus \(> 0\). Therefore

\[
\phi_n(e) := \omega_e = e + \eta_e, \quad e \in \Xi_n,
\]

defines an extension of \(\phi_{n-1}\) which preserves special elements and which is the identity modulo elements of higher genera.

The proof is concluded by showing that every endomorphism \(\phi : \Gamma(\Xi) \to \Gamma(\Xi)\) whose linear part is the identity and which preserves the space of special elements is invertible. We leave this statement as another exercise to the reader. \(\square\)
9. Proof of the main theorem and final reflections

Proof of Theorem 15. We already know from Theorem 29 that the collection $S$ of special elements is friendly, therefore the inductive construction described in Section 3 gives a perturbation $\partial_{\text{pert}} = \partial_1 + \partial_2 + \partial_3 + \cdots$ such that

$$\partial_g(\xi_n^m) \in S_g(m, n), \text{ for } g \geq 0.$$  

Equation (19) then immediately follows from Lemma 27 while the fact that $\partial_{\text{pert}}$ preserves the path grading follows from Lemma 26.

It remains to prove that $(M, \partial) = (\Gamma(\Xi), \partial_0 + \partial_{\text{pert}})$ really forms a minimal model of $B$, that is, to construct a homology isomorphism from $(M, \partial)$ to $(B, \partial = 0)$. To this end, consider the homomorphism

$$\rho : (\Gamma(\Xi), \partial_0 + \partial_{\text{pert}}) \to (B, \partial = 0)$$

defined, in presentation (7), by

$$\rho(\xi_1^1) := \lambda, \quad \rho(\xi_1^1) := \gamma,$$

while $\rho$ is trivial on all remaining generators. It is clear that $\rho$ is a well-defined map of dg-PROP-s. The fact that $\rho$ is a homology isomorphism follows from rather deep Corollary 27 of [15]. An important assumption of this Corollary is that $\partial_{\text{pert}}$ preserves the path grading. This assumption guarantees that the first spectral sequence of [15, Theorem 24] converges because of the inequalities given in [15, Exercise 21] and recalled here in (9). The proof of Theorem 15 is finished. □

Final reflections and problems. We observed that it is extremely difficult to work with free PROP-s. Fortunately, it turns out that most of classical structures are defined over simpler objects – operads, 1/2PROP-s or dioperads. In Remark 24 we indicated a definition of special PROP-s for which only compositions given by ‘fractions’ are allowed.

Let us denote by $sB$ the special PROP for bialgebras. It clearly fulfills $sB(m, n) = k$ for all $m, n \geq 1$ which means that bialgebras are the easiest objects defined over special PROP-s in the same sense in which associative algebras are the easiest objects defined over non-$\Sigma$-operads (recall that the non-$\Sigma$-operad $Ass$ for associative algebras fulfills $Ass(n) = k$ for all $n \geq 1$) and associative commutative algebras are the easiest objects defined over ($\Sigma$-)operads (operad $Com$ fulfills $Com(n) = k$ for all $n \geq 1$).

Let us close this paper by summarizing some open problems.

(1) Does there exist a sequence of convex polyhedra $B_{n}^{m}$ with the properties stated in Conjecture 31?  

(2) What can be said about the minimal model for the PROP for “honest” Hopf algebras with an antipode?  

(3) Explain why the Saneblidze-Umble diagonal occurs in our formulas for $\partial$. 
(4) Describe the generating function
\[ f(s, t) := \sum_{m, n \geq 1} \dim S(m, n) s^m t^n \]
for the space of special elements.

(5) Give a closed formula for the differential \(\partial\) of the minimal model.

(6) Develop a theory of homotopy invariant versions of algebraic objects over PROPs, parallel to that of [12] for algebras over operads. We expect that all main results of [12]
remain true also for PROPs, though there might be surprises and unexpected difficulties
related to the combinatorial explosion of PROPs.

(7) What can be said about the uniqueness of the minimal model? Is the minimal model
of an augmented PROP concentrated in degree 0 unique up to isomorphism? If not, is at
least a suitable completion of the minimal model unique?

There is a preprint [17] which might contain answers to Problems (1) and (5).

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E-mail address: markl@math.cas.cz

Mathematical Institute of the Academy, Žitná 25, 115 67 Prague 1, The Czech Republic