ON BENJAMINI–SCHRAMM LIMITS OF CONGRUENCE SUBGROUPS

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ABSTRACT. A sequence of orbifolds corresponding to pairwise non-conjugate congruence lattices in a higher rank semisimple group over zero characteristic local fields is Benjamini–Schramm convergent to the universal cover.

1. INTRODUCTION

A semisimple analytic group $G$ is defined as follows. Let $I$ be a finite index set. Assume that $k_i$ is a zero characteristic local field and $G_i$ is a connected simply-connected $k_i$-isotropic almost $k_i$-simple linear $k_i$-group for every $i \in I$. Denote $G_i = G_i(k_i)$ so that in particular $G_i$ is an almost simple non-compact linear group admitting a $k_i$-analytic structure. Let $G = \prod_{i \in I} G_i$.

Definition. A sequence of lattices $(\Gamma_n)_{n \in \mathbb{N}}$ in $G$ weakly central\(^1\) if for every compact subset $Q \subset G$ we have that

$$\Pr\left(\left\{g\Gamma_n \in G/\Gamma_n : g\Gamma_ng^{-1} \cap Q \subset Z(G)\right\}\right) \xrightarrow{n \to \infty} 1$$

This note is dedicated to establishing the following result.

Theorem 1. Assume that $|I| \geq 2$. Then every sequence of pairwise non-conjugate congruence lattices in $G$ is weakly central.

Recall that every irreducible lattice in $G$ is arithmetic whenever $|I| \geq 2$. A congruence lattice is a particular kind of an irreducible arithmetic lattice strictly containing a congruence subgroup. See §3 for a precise definition of this notion. In particular whenever lattices in $G$ are known to satisfy the congruence subgroup property a stronger formulation of Theorem 1 is possible.

We remark that if $|I| = 1$ and rank($G$) $\geq 2$ then every sequence of pairwise non-conjugate lattices is weakly central by [1, 12]. The recent works of Raimbault [20] and Fraczyk [9] establish closely related results for congruence lattices in the rank one groups $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{C})$. We also mention [1, §5] dealing with congruence subgroups in a fixed uniform arithmetic lattice.

Convergence of Plancherel measures. Let $\nu^G$ denote the Plancherel measure on the unitary dual $\hat{G}$ of $G$. For every uniform lattice $\Gamma$ in $G$, the quasi-regular representation $\rho_\Gamma$ of $G$ in $L^2(G/\Gamma)$ decomposes as a direct sum of irreducible representations. Every irreducible representation $\pi \in \hat{G}$ appears in $\rho_\Gamma$ with finite

\(^1\)Of course, this definition makes sense for any locally compact group. In the case that $G$ is totally disconnected a weakly central sequence is called a Farber sequence.
multiplicity \( m(\pi, \Gamma) \). The corresponding relative Plancherel measure is

\[
\nu_T = \frac{1}{\text{vol}(G/\Gamma)} \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \delta_\pi
\]

Combining Theorem 1 with \([1, 1.2]\) and \([12, 1.3]\) we obtain a generalization of one of the main results of \([1, 12]\) on convergence of relative Planche rel measures.

**Corollary 2.** Assume that \(|I| \geq 2\). Let \(\Gamma_n\) be any sequence of pairwise non-conjugate uniformly discrete torsion-free congruence lattices in \(G\). Then \(\nu_{T_n}(E) \xrightarrow{n \to \infty} \nu^G(E)\) for every relatively quasi-compact \(\nu^G\)-regular subset \(E \subset \hat{G}\).

The applications of Corollary 2 to limit multiplicities formulas and normalized Betti numbers \([1, 1.3, 1.4]\) carry over to our setting as well.

**Benjamini–Schramm convergence.** The semisimple analytic group \(G\) is acting by isometries on a contractible non-positively curved metric space \(X\), as follows. Let \(X_i\) be the symmetric space or Bruhat-Tits building associated to \(G_i\) for every \(i \in I\), depending on whether \(k_i\) is Archimedean or not. Take \(X = \prod_{i \in I} X_i\) equipped with the product metric.

The following geometric notion is equivalent to saying that a sequence of lattices is weakly central, as explained in \([12, \S 3]\).

**Definition.** Let \((\Gamma_n)\) be a sequence of lattices in \(G\). The orbifolds \(\Gamma_n \backslash X\) Benjamini–Schramm converge to \(X\) if for every radius \(0 < R < \infty\) the probability that an \(R\)-ball with base point taken uniformly at random is contractible tends to one as \(n \to \infty\).

As an example, we provide a geometric application of Theorem 1 to arithmetic orbifolds, relying on the congruence subgroup property \([22]\).

**Corollary 3.** Let \(F\) be a number field with ring of integers \(O_F\). Assume that \(F\) has \(r\) real embeddings and \(2s\) complex embeddings with \(r + s \geq 2\). Consider the irreducible arithmetic lattice

\[
\text{SL}_2(O_F) \hookrightarrow \prod_{i=1}^r \text{SL}_2(\mathbb{R}) \times \prod_{i=1}^s \text{SL}_2(\mathbb{C})
\]

Let \(X = (\mathbb{H}^2)^r \times (\mathbb{H}^3)^s\) be a product of two and three dimensional hyperbolic spaces. Then the orbifolds corresponding to any sequence of distinct finite-index subgroups of \(\text{SL}_2(O_F)\) are Benjamini-Schramm convergent to \(X\).

**On properties \((T)\) and \((\tau)\).** Benjamini-Schramm convergence of lattices was first investigated in \([1]\), where it was shown that any sequence of pairwise non-conjugate irreducible lattices in a semisimple Lie group with high rank and property \((T)\) is weakly central. General local fields were dealt with in \([12]\). These proofs rely on property \((T)\), most crucially in order to invoke the Stuck–Zimmer theorem \([14, 23]\).

Our approach is to make use of property \((\tau)\) instead, avoiding the Stuck–Zimmer theorem which is presently unknown in the absence of property \((T)\). More precisely, we rely on property \((\tau)\) with respect to congruence lattices. This is sometimes called the Selberg property as it generalizes his famous theorem on congruence subgroups of the modular group. It is crucial that Selberg’s property is, perhaps surprisingly, uniform with respect to all the congruence lattices inside \(G\) — see Theorem A.

In addition, we rely on topological properties of the Chabauty space of semisimple analytic groups recently established by Gelander and the author \([12]\), thereby replacing yet another argument of \([1]\) which previously required property \((T)\).
Spectral gap and essentially free actions. Towards proving Theorem 1 we study a Borel $G$-space obtained by taking a certain limit with respect to a sequence of congruence lattices. Selberg’s property implies that this limiting $G$-space has spectral gap — see §2 for a discussion of this notion. The following theorem allows us to deduce that such an action is essentially free, provided that $|I| \geq 2$.

**Theorem 4.** Let $G$ be a product of at least two locally compact simple groups and $X$ a Borel $G$-space admitting an invariant probability measure $\mu$. Assume that $X$ is properly ergodic, irreducible and has spectral gap. Then $(X, \mu)$ is essentially free.

We do not claim any originality for Theorem 4 — it is a formal corollary of the well-known work of Bader–Shalom [2] and the fact that an action with spectral gap is not weakly amenable [8]. We state it here merely as an observation and in the hope that it may prove useful in other situations as well.

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2. Uniform spectral gap

Let $G$ be a compactly generated locally compact group and $(X, \mu)$ a Borel $G$-space with an invariant probability measure. Recall that the $G$-action on $X$ has spectral gap if the Koopman representation of $G$ in the space $L^2_0(X, \mu)$ of functions with zero integral does not almost admit invariant vectors.

**Definition.** A sequence $(X_n, \mu_n)$ of Borel $G$-spaces with invariant probability measures has uniform spectral gap if the natural representation of $G$ on $\bigoplus_n L^2_0(X_n, \mu_n)$ does not almost admit invariant vectors.

For example, if $G$ has property $(T)$ then any family of ergodic $G$-invariant probability measures has uniform spectral gap. More generally, such uniformity is useful when passing to weak-$*$ limits of probability measures on a given compact $G$-space.

**Proposition 5.** Let $X$ be a compact $G$-space and $\mu_n$ a sequence of invariant Borel probability measures on $X$ with uniform spectral gap. If $\mu$ is a weak-$*$ limit of the sequence $\mu_n$ then $\mu$ has spectral gap.

Let $\pi$ and $\pi_n$ denote the Koopman representations on the Hilbert spaces $L^2(X, \mu)$ and $L^2(X, \mu_n)$ for $n \in \mathbb{N}$. Let $\|\cdot\|$ and $\|\cdot\|_n$ denote the norms on these spaces.

**Proof of Proposition 5.** Let $\nu$ be probability measure on $G$ which is symmetric, absolutely continuous with respect to the Haar measure and such that $\text{supp}(\nu \ast \nu)$ is a generating set for $G$. Uniform spectral gap for the measures $\mu_n$ means that

$$\|\pi_n(\nu)|_{L^2_0(X, \mu_n)}\|_n < \beta$$

for some constant $0 < \beta < 1$ and all $n \in \mathbb{N}$. This fact is established in [3, G.4.2].

We claim that $\|\pi(\nu)|_{L^2_0(X, \mu)}\| < \beta$ as well. In estimating the norm of a continuous operator we may restrict attention to a dense subspace. Consider any non-zero
continuous function \( f \in C(X) \cap L^2_0(X, \mu) \). Note that \( \pi(\nu)f \in C(X) \cap L^2_0(X, \mu) \) as well. Denote
\[
a_n = \int f \, d\mu_n, \quad f = f'_n + a_n 1_X
\]
so that \( f'_n \in C(X) \cap L^2_0(X, \mu_n) \) for all \( n \in \mathbb{N} \).

Since \( \mu \) is a weak-* limit of the measures \( \mu_n \) we have that \( \lim_n a_n = \int f \, d\mu = 0 \).
To estimate the operator norm of \( \pi(\nu) \) on the space \( L^2_0(X, \mu) \) we calculate
\[
\|\pi(\nu)f\| = \lim_n \|\pi(\nu)f\|_n \leq \limsup_n \|\pi(\nu)f'_n\|_n + \lim_n |a_n| < \beta \limsup_n \|f'_n\|_n \leq \beta \left( \lim_n \|f\|_n + \lim_n |a_n| \right) = \beta \|f\|.
\]
Therefore the \( G \)-space \( (X, \mu) \) has spectral gap as well [3, G.4.2]. \( \square \)

Assume that \( G \) splits as a direct product \( G = G_1 \times \cdots \times G_n \) of \( n \) factors. It is natural to consider the restriction of the \( G \)-action to each factor \( G_i \) individually.

**Definition.** \( (X, \mu) \) has strong spectral gap if the restricted action of each factor has spectral gap. A sequence \( (X_n, \mu_n) \) of Borel probability \( G \)-spaces has strong uniform spectral gap if these restricted actions have uniform spectral gap.

Recall that a \( G \)-action is irreducible if each factor \( G_i \) is acting ergodically. As spectral gap clearly implies ergodicity, strong spectral gap implies irreducibility.

3. Congruence lattices and the Selberg property

Let \( G \) be semisimple analytic group, so that \( G \) is a direct product of \( G_i = G_i(k_i) \) where each \( k_i \) is a local field and \( i \) ranges over a finite index set \( I \).

**Congruence lattices.** Let \( F \) be an algebraic number field and \( \mathbb{H} \) an absolutely simple linear \( F \)-group. Let \( R \subset V_F \) denote these infinite valuations such that \( \mathbb{H}(F_v) \) is non-compact for \( v \in R \). Assume that there is a finite set of valuations \( S \) with \( R \subset S \subset V_F \) and a bijection \( i : I \to S \) so that \( k_i \cong F_i(a) \) and \( G_i \) is \( k_i \)-isomorphic to \( \mathbb{H} \) for all \( i \in I \). In particular we may identify \( G \) with \( \prod_{v \in S} \mathbb{H}(F_v) \).

The group \( \mathbb{H}(F(S)) \) is an irreducible lattice in \( G \). Given a non-zero ideal \( a \) in the ring \( F(S) \) let \( \mathbb{H}(a) \) denote the kernel of the natural map \( \mathbb{H}(F(S)) \to \mathbb{H}(F(S)/a) \).

**Definition.** A congruence lattice is any lattice in \( G \) containing some \( \mathbb{H}(a) \) as above.

The following is essentially a reformulation of the well-known Selberg’s property.

**Theorem A** (Selberg’s property). Let \( G \) be a semisimple analytic group. Then the family of \( G \)-spaces \( G/\Gamma \) with normalized probability measures and \( \Gamma \) ranging over the congruence lattices in \( G \) has strong uniform spectral gap.

We begin our discussion of Theorem A with a few preliminary remarks.

- The two \( G \)-representations \( L^2(X, \mu) \) and \( L^2(X, \alpha \mu) \) are equivalent for every \( \alpha > 0 \), so that renormalizing a finite measure on a Borel \( G \)-space has no effect on spectral gap.
- Similarly, the two \( G \)-representations \( L^2(G/\Gamma) \) and \( L^2(G/\Gamma') \) are equivalent for every \( g \in G \).
- For a pair of lattices \( \Gamma, \Gamma' \) with \( \Gamma \leq \Gamma' \) the \( G \)-representation \( L^2(G/\Gamma') \) is contained in \( L^2(G/\Gamma) \).
In light of these remarks we may restrict our attention to the situation of the lattice $\mathbb{H}(\mathfrak{a})$ inside $\prod_{v \in S} \mathbb{H}(F_v)$ while making sure that the resulting strong spectral gap is independent of the field $F$, the group $\mathbb{H}$ and the ideal $\mathfrak{a}$.

**On Selberg’s property.** The existence of spectral gap for congruence subgroups of $\text{SL}_2(\mathbb{Z})$ regarded as lattices in $\text{SL}_2(\mathbb{R})$ is essentially Selberg’s classical theorem $[21]$. The Archimedean case, where $\mathbb{H}$ is still $\text{SL}_2$, $F$ is any number field and $S$ consists of infinite places is treated in $[24]$. The remaining case of $\mathbb{H} = \text{SL}_2(F)$ and $S$ an arbitrary set of places follows from the work of Gelbart–Jacquet $[13]$.

The Burger–Sarnak method $[5]$ allows to go beyond $\text{SL}_2$. This method was extended by Clozel–Ullmo $[7]$, in particular covering the $p$-adic case. We refer the reader to the useful discussion on $[16, \S 4.2]$.

**Theorem B** (Burger–Sarnak, Clozel–Ullmo). Let $\mathbb{H}_1$ be a semi-simple $F$-subgroup of $\mathbb{H}$. Then for every valuation $v \in S$ the restriction of $L^2(G/\mathbb{H}(\mathfrak{a}))$ to $\mathbb{H}_1(F_v)$ is weakly contained in

$$\bigoplus_{\sigma' < F(S)} L^2 \left( \prod_{v \in S} \mathbb{H}_1(F_v)/\mathbb{H}_1(\mathfrak{a}') \right)$$

where the direct sum is taken over all non-zero ideals in $F(S)$.

Clozel made the final contribution towards Selberg’s property by dealing with arbitrary absolutely simple groups. In fact Theorem $[A]$ is essentially equivalent to $[6, \text{Thm. 3.1}]$. Clozel’s proof for a general $F$-group $\mathbb{H}$ depends on whether it is $F$-isotropic or not. If $\text{rank}_F(\mathbb{H}) \geq 1$ then $\mathbb{H}$ is known to admit $\text{SL}_2$ as a $F$-subgroup $[17, \text{I.1.6.3}]$. In that case one may rely on $[13]$ and Theorem $[B]$.

The main effort of $[6]$ is in dealing with the anisotropic case, as follows. If $\mathbb{H}$ is $F$-anisotropic then it admits a $F$-subgroup $\mathbb{H}_1$ with $\text{rank}_{F_v}(\mathbb{H}_1) = 1$ and such that

$$\mathbb{H}_1 \cong \text{SL}_1(D) \quad \text{or} \quad \mathbb{H}_1 \cong \text{SU}(D, *)$$

where $D$ is a division algebra of degree $p^2$ over $F$ or over a quadratic extension of $F$ in the first and second cases respectively, and $p$ is a prime $[6, \text{1.1}]$. A careful analysis $[6, \S 3.2]$ reveals that the case of $\text{SL}_1(D)$ can be reduced to $\text{SL}_2$. Similarly, the case of $\text{SU}(D, *)$ is reduced either to $\text{SL}_2$, $\text{SU}(3, F_v)$ with $v$ finite or $\text{SU}(n, 1)$ with $v$ infinite. The parameter $n$ is clearly bounded as $\mathbb{H}_1(F_v)$ embeds in $G$. Clozel then establishes spectral gap directly for congruence lattices in these last two families of rank one groups.

**Uniformity of spectral gap.** Note that while $[6, \text{Theorem 3.1}]$ is stated with respect to a fixed algebraic number field $F$ and group $\mathbb{H}$, the resulting spectral gap for the subgroup $\mathbb{H}_1(F_v)$ is independent of any such choices.

To conclude the discussion, observe that there are only finitely many possibilities for the group $\mathbb{H}_1(F_v)$ and that the validity of Theorem $[A]$ for these implies the same for $G$. We need to take into account the fact that $\mathbb{H}_1(F_v)$ regarded as a subgroup of $G$ depends on the chosen $F$-structure. The argument relies on the preliminary remarks made above and on Lemma $[6]$.

**Lemma 6.** Let $\mathbb{H}_1$ be an $F$-group and $v \in S$ a valuation such that $\text{rank}_{F_v}(\mathbb{H}_1) = 1$. Denote $H_1 = \mathbb{H}_1(F_v)$ and let $Q \subset H_1$ be a compact subset. Then there is a compact

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Note that this case already suffices for the purpose of our Corollary $[6]$.
subset \( Q \subset G_i(v) \) such that every \( F_v \)-homomorphism \( \varphi : H_1 \to G_i(v) \) satisfies 
\( \varphi(Q_1)^g \subset Q \) for some \( g \in G_i(v) \).

Proof. Denote \( i = i(v) \) to simplify notation. Let \( S \) be a maximal \( F_v \)-split torus of \( G_i \) and denote \( S = S(F_v) \). The group \( H_1 \) admits \( SL_2 \) as a \( F \)-subgroup by the Jacobson–Morozov theorem. Let \( T \) denote the diagonal subgroup of \( SL_2 \) and \( T = T(F_v) \). By the assumptions \( T \) is a maximal \( F_v \)-split torus of \( H_1 \).

Consider the following Cartan decompositions [19 3.2-3]

\[
SL_2(F_v) = K_0TK_0, \quad H_1 = K_1TK_1, \quad G_i = KSK
\]

with the corresponding maximal compact subgroups \( K_0 \leq SL_2(F_v) \), \( K_1 \leq H_1 \) and \( K \leq G_i \). These three decompositions can be made compatible in the sense that

\[
K_0 = K \cap SL_2(F_v), \quad K_1 = K \cap H_1 \quad \text{and} \quad T = S \cap SL_2(F_v) = S \cap H_1
\]

up to conjugating by an element of \( G_i \). This is possible in the Archimedean case by a theorem of Mostow [18], and in the non-Archimedean case due to the strong transitivity property of affine buildings [10].

Assume without loss of generality that \( Q_1 = K_1T_1K_1 \) for some compact subset \( T_1 \subset T \). We claim that \( Q = KS_1K \) for some sufficiently large compact subset \( S_1 \subset S \) as required. Indeed, consider the two Lie algebras \( \mathfrak{h} = \mathcal{L}(H_1) \) and \( \mathfrak{g} = \mathcal{L}(G_i) \) and let \( d\varphi : \mathfrak{h} \to \mathfrak{g} \) denote the differential of \( \varphi \). Therefore \( d\varphi(\mathfrak{sl}_2) \) is a split \( \mathfrak{sl}_2 \)-triplet of \( \mathfrak{g} \) in the sense of [4 VIII.§11.1]. In this situation the restriction of \( d\varphi \) to the Lie algebra \( \mathcal{L}(T) \) of the torus \( T \) is bounded [4 VIII.§11.3]. \( \square \)

4. Weakly amenable actions and Theorem [4]

Let \( G \) be a locally compact group and \((X, \mu)\) a Borel \( G \)-space with an invariant probability measure. We use spectral gap to deduce essential freeness for such an action and establish Theorem [1].

Recall that a \( G \)-space \((X, \mu)\) is weakly amenable if the orbital equivalence relation generated by the action is amenable; see e.g. [25 Section 4.3] and [23] for details.

Theorem C (Stuck–Zimmer, Bader–Shalom). Let \( G \) be a direct product of at least two simple groups. Assume that the \( G \)-space \((X, \mu)\) is properly ergodic, irreducible and not weakly amenable. Then it is \( \mu \)-essentially free.

Proof. See [23] for the classical case of semisimple Lie groups, [14] for semisimple linear groups over local fields and [2] for general locally compact groups. \( \square \)

We conclude that Theorem [4] follows at once by combining Theorem [C] with the following observation, due to Creutz [8].

Proposition D. Let \( G \) be a second countable locally compact group. If the \( G \)-space \((X, \mu)\) is properly ergodic and weakly amenable then it has no spectral gap.

Roughly speaking, the proof of Proposition [D] is as follows. A weakly amenable action is orbit equivalent to an action of either \( \mathbb{Z} \) or \( \mathbb{R} \), depending on whether \( G \) is countable or not. The action of these amenable groups admits more than a single invariant mean on \( L^\infty(X, \mu) \). This fact is invariant under orbit equivalence, so that the action of \( G \) admits more than a single invariant mean as well. Therefore the action of \( G \) is not strongly ergodic and in particular there can be no spectral gap. See [8 7.3.1] for complete details and references concerning this argument.

It is interesting to note the similarity of the above theme with the role played by Selberg’s theorem in the solution of the Banach-Ruziewicz problem [15].
5. INVARIANT RANDOM SUBGROUPS AND A PROOF OF THEOREM 1

We now prove Theorem 1 relying on strong uniform spectral gap for congruence lattices. Invariant random subgroups are a main tool used in the proof.

Associated to any second countable locally compact group \( G \), the Chabauty space of closed subgroups denoted \( \text{Sub}(G) \). This is a compact \( G \)-space with the Chabauty topology and the conjugation action. An invariant random subgroup of \( G \) is a \( G \)-invariant probability measure on \( \text{Sub}(G) \). Let \( \text{IRS}(G) \) denote the compact convex space of all invariant random subgroups of \( G \) with the weak-* topology.

Associated to any normal subgroup \( N \triangleleft G \) is the point mass \( \delta_N \in \text{IRS}(G) \). More interestingly, to any lattice \( \Gamma \) in \( G \) we associate \( \mu_\Gamma \in \text{IRS}(G) \) obtained by pushing forward the \( G \)-invariant probability on \( G/\Gamma \) to \( \text{Sub}(G) \) via the \( G \)-equivariant map \( G/\Gamma \to \text{Sub}(G), \ g\Gamma \mapsto g\Gamma g^{-1} \).

The following two propositions are to be compared with [12, 1.1] and [17, II.4.4].

**Proposition 7.** Let \( \mu \in \text{IRS}(G) \) be irreducible and essentially transitive. If \( \mu \) is an accumulation point of \( \{ \mu_\Gamma \} \) where \( \Gamma \) are lattices in \( G \) then \( \mu = \delta_M \) with \( M \leq Z(G) \).

**Proof.** Since \( \text{Sub}(G), \mu \) is essentially transitive there is a closed subgroup \( H \leq G \) such that \( \mu \) is supported on the conjugacy class of \( H \) and \( \text{Sub}(G), \mu \) is isomorphic to the homogeneous \( G \)-space \( G/N_G(H) \). The factors \( G_i \) of \( G \) are non-compact and the fields \( k_i \) have zero characteristic so that the density theorem of Borel applies [17, II.4.4]. Combined with the irreducibility of the action it implies either that \( N_G(H) \) is an irreducible lattice or that \( N_G(H) = G \) and \( \mu \) is equal to \( \delta_M \) for some normal subgroup \( M \triangleleft G \).

We deal with these two possibilities separately, relying on results from [12]. If \( N_G(H) \) is a lattice in \( G \) then \( H \) must be an irreducible lattice as well by the normal subgroup theorem of Margulis [17, IV]. Every irreducible lattice of \( G \) admits a Chabauty open neighborhood in \( \text{Sub}(G) \) consisting of conjugates [12, 1.9], so that the corresponding point \( \mu_H \) is isolated in the space of extreme points of \( \text{IRS}(G) \).

A non-discrete normal subgroup of \( G \) does not belong to the closure of the Chabauty subspace of discrete subgroups [12, 6.7]. Therefore the second case where \( \mu = \delta_M \) is impossible unless \( M \) is central, as required.

**Proposition 8.** Every sequence of distinct \( \mu_{\Gamma_n} \in \text{IRS}(G) \) associated to congruence lattices \( \Gamma_n \) in \( G \) is weak-* convergent to \( \delta_M \in \text{IRS}(G) \) for some \( M \leq Z(G) \).

**Proof.** Let \( \mu \in \text{IRS}(G) \) be any accumulation point of the sequence \( \mu_{\Gamma_n} \). We claim that \( \mu \) is equal to \( \delta_M \) for some \( M \leq Z(G) \), and in particular that \( \mu_{\Gamma_n} \) is convergent.

Theorem 4 implies that the sequence \( \mu_{\Gamma_n} \) of \( G \)-invariant Borel probability measures on the compact \( G \)-space \( \text{Sub}(G) \) has strong uniform spectral gap. By Proposition 7 the \( G \)-space \( \text{Sub}(G), \mu \) has strong spectral gap as well. In particular \( \mu \) is both irreducible and has spectral gap.

Making use of the argument on [23, p. 729] we may assume that \( G \) has trivial center to begin with. Since the \( G \)-action on \( \text{Sub}(G), \mu \) is certainly not essentially free, Theorem 4 implies that it must be essentially transitive. We conclude from Proposition 7.

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3See Gelander’s lecture notes [11] on the Chabauty topology, invariant random subgroups and Benjamini-Schramm spaces.
The following three modes of convergence are all equivalent — weak-$^*$ convergence of $\mu_{\Gamma_n}$ to a central subgroup in IRS $(G)$, Benjamini–Schramm convergence of $\Gamma_n \backslash X$ to $X$ and the fact that the sequence $\Gamma_n$ is weakly central.

The proof of Theorem 1 is now complete.

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