On a $\tilde{C}_4$-ultrahomogeneous oriented graph

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Abstract

The notion of a $C$-ultrahomogeneous graph, due to Isaksen et al., is adapted for digraphs, and then a strongly connected $\tilde{C}_4$-ultrahomogeneous oriented graph on 168 vertices and 126 pairwise arc-disjoint 4-cycles is presented, with regular indegree and outdegree 3 and no circuits of lengths 2 and 3, by altering a definition of the Coxeter graph via pencils of ordered lines of the Fano plane in which pencils are replaced by ordered pencils.

Keywords: ultrahomogeneous oriented graph; Fano plane; ordered pencil

1 Introduction

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [12], [6], [11] and [7], (resp. [5], [10] and [2]). In [9], $C$-ultrahomogeneous graphs are defined and subsequently treated when $C =$ collection of either (a) complete graphs, or (b) disjoint unions of complete graphs, or (c) complements of those unions. In [3], a $(K_4, K_{2,2,2})$-ultrahomogeneous graph on 42 vertices, 42 copies of $K_4$ and 21 copies of $K_{2,2,2}$ is given that fastens objects of (a) and (c), namely $K_4$ and $K_{2,2,2}$, respectively, over copies of $K_2$.

In the present note and in [4], the notion of a $C$-ultrahomogeneous graph is extended as follows: Given a collection $C$ of (di)graphs closed under isomorphisms, a (di)graph $G$ is $C$-ultrahomogeneous (or $C$-UH) if every isomorphism between two $G$-induced members of $C$ extends to an automorphism of $G$. If $C = \{H\}$ is the isomorphism class of a (di)graph $H$, such a $G$ is said to be $\{H\}$-UH or $H$-UH.

In [4], the cubic distance-transitive graphs are shown to be $C_g$-UH graphs, where $C_g$ stands for cycle of minimum length, i.e. realizing the girth $g$; moreover, all these graphs but for the Petersen, Heawood and Foster graphs are shown to be $\tilde{C}_g$-UH digraphs, which allows the construction of novel $C$-UH graphs, in continuation to the work of [3], including a $(K_4, L(Q_3))$-UH graph on 102 vertices that fastens 102 copies of $K_4$ and 102 copies of the cuboctahedral graph $L(Q_3)$ over copies of $K_3$, obtained from the Biggs-Smith graph, $(\mathbb{P})$, by unzipping, powering and zipping back a collection of oriented $g$-cycles provided...
by the initial results. However, these graphs are undirected, so they are not properly digraphs.

In this note, a presentation of the Coxeter graph \( Cox \) is modified to provide a strongly connected \( \tilde{C}_4 \)-UH oriented graph \( D \) on 168 vertices, 126 pairwise arc-disjoint 4-cycles, with regular indegree and outdegree 3. In contrast, the construction of \[ \tilde{C}_4 \] used ordered pencils of unordered lines, instead.

We take the Fano plane \( F \) as having point set \( J_7 = \mathbb{Z}_7 \) (the cyclic group mod 7) and point-line correspondence \( \phi(j) = \{(j+1), (j+2), (j+4)\} \), for every \( j \in \mathbb{Z}_7 \) in order to color the vertices and edges of \( Cox \) as in Figure 1.

![Figure 1: Coloring the vertices and edges of Cox with elements of F](image)

This figure shows that each vertex \( v \) of \( Cox \) can be considered as an unordered pencil of ordered lines of \( F \), (brackets and commas avoided now):

\[
x b_{1c_1}, \; x b_{2c_2}, \; x b_{0c_0},
\]

(1)

corresponding to the three edges \( e_1, e_2, e_0 \) incident to \( v \), respectively, and denoted by \([x, b_1c_1, b_2c_2, b_0c_0]\), where \( x \) is the color of \( v \) in the figure, with \( b_i \) and \( c_i \) as the colors of the edge \( e_i \) and the endvertex of \( e_i \) other than \( v \), for \( i \in \{1, 2, 0\} \).

Moreover, two such vertices are adjacent in \( Cox \) if they can be written \([x, b_1c_1, b_2c_2, b_0c_0]\) and \([x', b'_1c'_1, b'_2c'_2, b'_0c'_0]\) (perhaps by means of a permutation of the entries \( b_i c_i \)) in such a way that \( \{b_i, c_i\} \cap \{b'_i, c'_i\} \) is constituted by just one element \( d_i \), for each \( i \in \{1, 2, 0\} \), and the resulting triple \( d_1d_2d_0 \) is a line of \( F \).

## 2 Presentation of a \( \tilde{C}_4 \)-UH digraph

Consider the oriented graph \( D \) whose vertices are the ordered pencils of ordered lines of \( F \), as in (1) above. Each such vertex will be denoted \((x, b_1c_1, b_2c_2, b_0c_0)\), where \( b_1b_2b_0 \) is a line of \( F \). An arc between two vertices of \( D \), say from \((x, b_1c_1, b_2c_2, b_0c_0)\) and \((x', b'_1c'_1, b'_2c'_2, b'_0c'_0)\), is established if and only if

\[
\begin{align*}
x &= c'_1, & b'_{i+1} &= c_{i+1}, & b'_{i-1} &= c_{i-1}, & b'_i &= b_i; \\
x' &= c_i, & c'_{i+1} &= b_{i-1}, & c'_{i-1} &= b_{i+1},
\end{align*}
\]
for some, \( i \in \{1, 2, 0\} \). This way, we obtain oriented 4-cycles in \( D \), such as

\[
((0, 26, 54, 31), (6, 20, 43, 15), (0, 26, 31, 54), (6, 20, 15, 43)).
\]

A simplified notation for the vertices \((x, yz, uv, pq)\) of \( D \) is \( yup_x \). With such a notation, the adjacency sub-list of \( D \) departing from the vertices of the form \( yup_0 \) is (with rows indicated \( a, b, c, d, e, f \), to be used below):

\[
\begin{align*}
1240 & : 1653, 3256, 3645; & 2350 & : 2146, 6341, 6156; & 3460 & : 3521, 1425, 1562; & 1560 & : 1423, 3524, 3462; \\
1420 & : 1563, 3465, 3526; & 2530 & : 2416, 6514, 6436; & 3640 & : 3251, 1652, 1245; & 1650 & : 1243, 3642, 3254; \\
2140 & : 2356, 6153, 6345; & 3250 & : 3641, 1246, 1651; & 4360 & : 4126, 5321, 5163; & 5160 & : 5324, 4124, 4362; \\
2410 & : 2536, 6435, 6513; & 3520 & : 3461, 1564, 1422; & 4630 & : 4216, 5612, 5235; & 5610 & : 5234, 4632, 4215; \\
4120 & : 4365, 5163, 5326; & 5230 & : 5614, 4216, 4634; & 6340 & : 6152, 2351, 2146; & 6150 & : 6342, 2143, 2354; \\
4210 & : 4635, 5236, 5613; & 5320 & : 5364, 4361, 4124; & 6430 & : 6512, 2413, 2531; & 6510 & : 6435, 2533, 2143.
\end{align*}
\]

From this sub-list, the adjacency list of \( D \), for its 168 = 24 \times 7 vertices, is obtained via translations mod 7. Let us represent each vertex \( yup_0 \) of \( D \) by means of a symbol \( j_i \), where \( i = a, b, c, d, e, f \) stands for the successive rows of the table above and \( j = \phi^{-1}(yup) \in \{0, 1, 2, 4\} \). These symbols \( j_i \) are assigned to the lines \( yup \) avoiding 0 \( \in \mathcal{F} \), and thus to the vertices \( yup_0 \), as follows:

\[
\begin{array}{c|cccc}
 i & j_0 & j_1 & j_2 & j_4 \\
 \hline
 a & 124 & 235 & 346 & 156 \\
 b & 142 & 253 & 364 & 165 \\
 c & 214 & 325 & 436 & 516 \\
 d & 241 & 352 & 463 & 561 \\
 e & 412 & 523 & 634 & 615 \\
 f & 421 & 532 & 643 & 651 \\
\end{array}
\]

![Figure 2: Split representation of the quotient graph \( D/\mathbb{Z}_7 \)](image)

With these symbols adopted, the quotient graph \( D/\mathbb{Z}_7 \) can be considered as a voltage graph with group \( \mathbb{Z}_7 \) and derived graph \( D \), ([S]), admitting a split representation into the three connected digraphs of Figure 2, whose vertices are indicated by the symbols \( j_i \) of their representatives \( yup_0 \), and in which: (1) the 18 oriented 4-cycles that are shown are interpreted all with counterclockwise orientation; (2) For each \( i \in \{a, \ldots, f\} \), the three vertices indicated by 0\( _i \) represent just one vertex of \( D/\mathbb{Z}_7 \), so they must be identified; (3) the leftmost arc in each one of the three connected digraphs must be identified with the corresponding rightmost arc by parallel translation; (4) if an arc \( e \) of \( D/\mathbb{Z}_7 \) has voltage \( \nu \in \mathbb{Z}_7 \), initial vertex \( j_i \) and terminal vertex \( j_i' \), then a representative
yup, of \( j_i \) initiates an arc in \( D \) that covers \( \bar{e} \) and has terminal vertex \( y'u'p'_{(\nu+\mu)} \), where \( yu_p \) and \( y'u'p' \) are represented respectively by \( j_i \) and \( j'_i \).

All the oriented 4-cycles of \( D \) are obtained by uniform translations mod 7 from these 18 oriented 4-cycles. Thus, there are just \( 126 = 7 \times 18 \) oriented 4-cycles of \( D \). Our construction of \( D \) shows that the following statement holds.

**Theorem 1** The oriented graph \( D \) is a strongly connected \( \tilde{C}_4 \)-UH digraph on 168 vertices, 126 pairwise disjoint oriented 4-cycles, with regular indegree and outdegree both equal to 3 and no circuits of lengths 2 and 3.

\[\square\]

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