Euclidean Algorithm for a Gravitational Lens in a Polynomial Equation

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The Euclidean algorithm in algebra is applied to a class of gravitational lenses for which the lens equation consists of any set of coupled polynomial equations in the image position. In general, this algorithm allows us to reduce an apparently coupled system to a single polynomial in one variable (say $x$ in Cartesian coordinates) without the other component (say $y$), which is expressed as a function of the first component. This reduction enables us to investigate the lensing properties in an algebraic manner: For instance, we can obtain an analytic expression of the caustics by computing the discriminant of the polynomial equation. To illustrate this Euclidean algorithm, we re-examine a binary gravitational lens and show that the lens equation is reduced to a single real fifth-order equation, in agreement with previous works. We apply this algorithm also to the linearized Kerr lens and find that the lens equation is reduced to a single real fifth-order one.

§1. Introduction

Gravitational lensing plays an important role in modern astronomy,1 in particular in the problems of determining the cosmological parameters, probing the mass profile of galaxies, and detecting a dark object such as MACHO. When the lens mass distribution is axially symmetric along the line of sight, the basic equation is reduced to one dimension. Except in this case, the system is coupled nonlinearly in general. It is thus believed that we need numerical treatments. Recently, however, the lens equation, which is apparently coupled, has been shown by chance to be reduced to a single real algebraic equation even for asymmetric cases, such as those of a two-point mass2 and an isothermal ellipsoid.3 It should be noted that the lens equation can be expressed as a single complex algebraic equation in the complex formalism.4 However, this equation is nonlinearly coupled in the real and imaginary parts, and therefore we must solve it numerically.

The reduction to a single real algebraic equation enables us in principle to study the lensing properties analytically or numerically in an efficient and accurate manner. For instance, we can obtain an analytic expression for the caustics,6 which is a curve in the source plane, and a criteria for a change in the number of images. The reason that this can be done is that this change corresponds to that in the number of real roots for the single real algebraic equation, which is determined by a discriminant in algebra.6

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It is worthwhile mentioning the resultant method, a part of the elimination theory in algebra. This method gives us a condition that two polynomial equations have a common root. For instance, let us take the Cartesian coordinates \((x, y)\) to denote these equations by \(P(x, y) = 0\) and \(Q(x, y) = 0\). We can apply the resultant method to our problem: First, these equations are considered only for \(y\). A common root \(y\) for them exists if the resultant vanishes, where the resultant is a function of \(x\), denoted by \(X(x)\). Similarly, a common root \(x\) exists if for the resultant \(Y(y)\), we have \(Y(y) = 0\), if we pay attention to only \(x\). However, all of the roots for these equations, \(X(x) = 0\) and \(Y(y) = 0\), do not necessarily satisfy the original equations \(P(x, y) = 0\) and \(Q(x, y) = 0\): An appropriate combination of a root \(x\) for \(X(x) = 0\) and \(y\) for \(Y(y) = 0\) is true, but the resultant method itself never tells about a way of choosing it.

In the examples mentioned above, the binary\(^2\) and the isothermal ellipsoid,\(^3\) we have a one-dimensional equation for one component of the coordinates and a linear equation for the other component of the coordinates, which is solved trivially and thus regarded as a function. In this sense, such a formalism, in which we have a one-dimensional equation and the associated function, is not known for a more general lens system. The main purpose of this paper is to demonstrate that the Euclidean algorithm is widely applicable to the lens equation for a pair of real polynomials. In section 2, we study a binary gravitational lens as an example. In the Appendix, a linearized Kerr lens is investigated as another example.

\section{Euclidean algorithm}

\subsection{Binary Gravitational Lens}

We consider a binary system consisting of two point masses, \(M_1\) and \(M_2\), and a separation vector \(L\) from object 1 to 2, which is located at a distance \(D_L\) from the observer. In units of the Einstein ring radius angle \(\theta_E\), the lens equation reads\(^4\)

\[
\beta = \theta - \left(\nu_1 \frac{\theta}{|\theta|^2} + \nu_2 \frac{\theta - \ell}{|\theta - \ell|^2}\right),
\]

where \(\beta = (a, b)\) and \(\theta = (x, y)\) denote the vectors representing the positions of the source and image, respectively, and we defined the mass ratios \(\nu_1\) and \(\nu_2\) and the angular separation vector \(\ell\) as

\[
\nu_1 = \frac{M_1}{M_1 + M_2},
\]

\[
\nu_2 = \frac{M_2}{M_1 + M_2},
\]

\[
\ell = \frac{L}{D_L \theta_E}.
\]

We have the identity \(\nu_1 + \nu_2 = 1\). For brevity, \(\nu_2\) is denoted by \(\nu\). Equation \((2.1)\) consists of a set of two coupled real quintic equations for \((x, y)\)

\[
|\theta|^2 |\theta - \ell|^2 (\theta - \beta) - (1 - \nu)|\theta - \ell|^2 \theta - \nu |\theta|^2 (\theta - \ell) = 0.
\]
Euclid Algorithm for a Gravitational Lens

For two point masses at different distances, the lens equation becomes more complicated. This has been previously investigated (e.g., Erdl and Schneider (1993)).

2.2. Euclidean algorithm

First, let us briefly review the Euclidean algorithm for integers. For example, we search for a common factor of 20 and 12. The larger integer, 20, is divided by the smaller one, 12, with the remainder 8, and 12 is divided by 8 with the remainder 4. Such a procedure is repeated until the remainder vanishes. Thus, we obtain

\[
\begin{align*}
20 &= 1 \times 12 + 8, \\
12 &= 1 \times 8 + 4, \\
8 &= 2 \times 4 + 0,
\end{align*}
\]

which shows that 4 is the greatest common divisor (G.C.D.) of 20 and 12. Actually, Eq. (2.6) can be rearranged as

\[
\begin{align*}
20 &= 5 \times 4 + 0, \\
12 &= 3 \times 4 + 0.
\end{align*}
\]

As another example, consider 9 and 4. In this case, we obtain

\[
\begin{align*}
9 &= 2 \times 4 + 1, \\
4 &= 4 \times 1 + 0.
\end{align*}
\]

Thus unity is the G.C.D. in this case. This procedure is called the Euclidean algorithm, which is applicable widely in algebra. Let us apply it to a pair of polynomials, given in Eq. (2.5), in which a common factor gives us a common root, namely an image position.

Equation (2.5) is coupled with respect to \(x\) and \(y\). For simplicity, the separation of the binary is assumed to be on the \(x\)-axis \(\ell = (\ell, 0)\). Concerning \(x\), Eq. (2.5) consists of a fifth-order equation and a fourth-order equation, respectively:

\[
\begin{align*}
f_5 &\equiv \sum_{i=0}^{5} c_5 x^i = 0, \\
f_4 &\equiv \sum_{i=0}^{4} d_4 x^i = 0.
\end{align*}
\]

Here, all of the coefficients are polynomials in \(y, a, b, \ell\) and \(\nu\):

\[
\begin{align*}
c_5 &= 1, \\
c_4 &= -(a + 2\ell), \\
c_3 &= 2y^2 + \ell^2 + 2a\ell - 1, \\
c_2 &= -2(a + \ell)y^2 + \ell(2 - a\ell - \nu), \\
c_1 &= y^4 + (\ell^2 + 2a\ell - 1)y^2 - \ell^2(1 - \nu),
\end{align*}
\]
We divide \( f_5 \) by \( f_4 \) with the quotient \( q_3 \) and the remainder \( f_3 \) as

\[
f_5 = q_3 f_4 + f_3,
\]

where \( f_3 \) is cubic in \( x \). This procedure can be repeated as

\[
f_4 = q_2 f_3 + f_2,
\]
\[
f_3 = q_1 f_2 + f_1,
\]
\[
f_2 = q_0 f_1 + f_0,
\]

where \( f_i \) is \( i \)-th order in \( x \) and \( q_i \) is linear in \( x \). We can eliminate \( f_3 \) and \( f_2 \) from these equations in order to show that both \( f_5 \) and \( f_4 \) have a common factor \( f_1 \) if \( f_0 \) vanishes. In short, Eqs. (2.9) and (2.10) are equivalent to the pair of equations \( f_0 = 0 \) and \( f_1 = 0 \). In the cases of integers considered above, Eqs. (2.6) and (2.7), \( f_1 \) and \( f_0 \) correspond to 4 and 0, respectively.

Because \( f_1 \) is linear in \( x \), it takes the form

\[
f_1 = \left( D(y)x + E(y) \right) G(y),
\]

where \( D(y) \) and \( E(y) \) are polynomials in \( y \), \( a \), \( b \), \( \ell \) and \( \nu \), and \( G(y) \) is an algebraic function. Because \( G(y) \) plays no role in the following discussion, we write \( D(y) \) and \( E(y) \) as

\[
D(y) = b^3 \ell^2 - 2b^3 \ell^2 \nu + 2b^3 \ell^2 \nu^2 \\
\quad + (-2b^2 - 4ab^2\ell + 4b^3 \ell^2 + a^2 b^2 \ell^2 + b^4 \ell^2 + 8ab^2 \ell \nu \\
\quad - 10b^2 \ell^2 \nu - 2ab^2 \ell^3 \nu + b^2 \ell^4 \nu + 6b^2 \ell^2 \nu^2) y \\
\quad + (-2a^2 b - 6b^3 - 4a^3 b \ell - 4ab^3 \ell + 3a^2 b \ell^2 + 3b^3 \ell^2 + 4ab \ell \nu \\
\quad + 8a^3 b \ell \nu + 8ab^3 \ell \nu - 8b^2 \ell \nu - 12a^2 b \ell^2 \nu - 4b^3 \ell^2 \nu + 6ab^3 \nu \\
\quad - b\nu + 6b^2 \nu^2) y^2 \\
\quad + (2a^2 + 2b^2 - 4a^2 b^2 - 4b^4 + 4a^3 \ell + 4ab^2 \ell - 4a^2 \ell^2 - 4b^2 \ell^2 \\
\quad - 4a^2 \ell \nu - 8a^3 \ell \nu + 12a^2 \ell^2 \nu - 4a b^3 \nu + 2\ell^2 \nu^2) y^3 \\
\quad + (4a^2 b + 4b^3 - 8ab \ell \nu + 4b^2 \nu) y^4,
\]

\[
E(y) = -b^3 \ell^3 + 2b^3 \ell^3 \nu - b^3 \ell^3 \nu^2 \\
\quad + (b^2 \ell + 3ab^2 \ell^2 - 3b^2 \ell^3 - a^2 b^2 \ell^3 - b^4 \ell^3 - 4ab^2 \ell^2 \nu
\]

\[
= 0.
\]
Euclid Algorithm for a Gravitational Lens

\[ x = \frac{E(y)}{D(y)}, \quad (2.27) \]

if the component \( y \) is given. The \( y \) component is obtained by solving \( f_0 = 0 \), which does not depend on \( x \). After factorization, this is reduced to a fifth-order equation in \( y \),

\[ \sum_{i=0}^{5} e_i y^i = 0, \quad (2.28) \]

where

\[ e_5 = -4(a^2 + b^2)(a^2 + b^2 - 2a\ell + \ell^2), \quad (2.29) \]

\[ e_4 = 4b \left( -a^2 + a^4 - b^2 + 2a^2b^2 + b^4 - 2a^3\ell - 2ab^2\ell + a^2\ell^2 + b^2\ell^2 \right) + (2a\ell - \ell^2)\nu, \quad (2.30) \]

\[ e_3 = -a^2 - b^2 + 8a^2b^2 + 8b^4 - 2a^3\ell - 10ab^2\ell + 2a^2\ell^2 - 4a^4\ell^2 + 6b^2\ell^2 \]

\[-2a^2b^2\ell - b^4\ell^2 + 2a^3\ell^3 + 2ab^2\ell^3 - a^2\ell^4 - b^2\ell^4 \]

\[ + (2a\ell + 4a^3\ell + 4ab^2\ell - 6a^2\ell^2 + 2b^2\ell^2 + 2a\ell^3)\nu - \ell^2\nu^2, \quad (2.31) \]

\[ e_2 = \left( a^2 + 5b^2 + 2a^3\ell + 2ab^2\ell - 2a^2\ell^2 + 4a^4\ell^2 - 2b^2\ell^2 + 2a^2b^2\ell^2 + b^4\ell^2 \right) - 2a^3\ell^3 - 2ab^2\ell^3 + a^2\ell^4 + b^2\ell^4 \]

\[ + (-2a\ell - 4a^3\ell - 4ab^2\ell + 4\ell^2 + 6a^2\ell^2 + 2b^2\ell^2 - 2a\ell^3)\nu \]

\[ - 3\ell^2\nu^2 \], \quad (2.32) \]

\[ e_1 = 1 + 2a\ell - 2\ell^2 + a^2\ell^2 + b^2\ell^2 - 2a\ell^3 + \ell^4 \]

\[ + (-4a\ell + 5\ell^2 + 2a^3\ell - \ell^4)\nu - 3\ell^2\nu^2, \quad (2.33) \]

\[ e_0 = -b^3\ell^2(-1 + \nu)\nu. \quad (2.34) \]

The common root of Eqs. (2.9) and (2.10) is thus obtained by (1) solving Eq. (2.28) for \( y \) and (2) substituting a root \( y \) into Eq. (2.27) to determine \( x \).
2.3. Particular cases

There can exist multiple roots in algebraic equations. A condition for a true multiple root is a discriminant in algebra, which corresponds to the caustics in the theory of gravitational lenses. An apparent multiple root is due to the adopted coordinate system: For instance, let us imagine two roots with the same value of $y$ but different $x$. These two images appear on a line parallel to the $x$-axis. A rotation of the coordinate system breaks such an apparent multiplicity. Here, we discuss this apparent multiplicity in our algorithm.

In the preceding subsection, we have assumed $f_i \neq 0$. However, $f_1 = 0$ may be satisfied automatically for a particular set of the lens and source parameters. Then, a common factor for Eqs. (2.9) and (2.10) becomes $f_2$, which is second order in $x$. Hence, apparent multiple roots are located at the same $y$ but different $x$. In this particular case, a practical strategy for solving the lens equation in our algorithm is (1) solving Eq. (2.28) for $y$ to find single roots and multiple roots, (2a) substituting a single root $y$ into Eq. (2.27) to determine $x$, and (2b) substituting a multiple root $y$ into $f_2 = 0$ and solving it as a square equation for $x$. If the multiplicity is three, the multiple root $y$ must be substituted into $f_3 = 0$, which becomes a cubic equation for $x$.

Let us consider the example $(a, b) = (1/2, 1/4)$ for $\ell = 1$ and $\nu = 1/2$. Then, the fifth-order equation is factorized as

$$(2 + 5y)^2(-1 - 12y - 4y^2 + 16y^3) = 0. \quad (2.35)$$

For its multiple root $y = -2/5$, $f_2$ becomes

$$f_2 = -\frac{123}{50000}(25x^2 - 25x - 16). \quad (2.36)$$

Hence $f_2 = 0$ is solved as $x = (5 \pm \sqrt{89})/10$.

§3. Conclusion

We demonstrated that the Euclidean algorithm works well in the case of a binary gravitational lens: The image position for Eqs. (2.9) and (2.10), which are a set of coupled polynomial equations, is obtained by (1) solving Eq. (2.28) for $y$ and (2) substituting a root $y$ into Eq. (2.27) to determine $x$. In the case of multiple roots, we must substitute the multiple root $y$ into $f_2 = 0$ and solve it as a square equation for $x$. If the multiplicity is three, the multiple root $y$ must be substituted into $f_3 = 0$, which becomes a cubic equation for $x$.

The Euclidean algorithm is applicable to any pair of polynomials. The linearized Kerr lens is investigated in the Appendix. It would be interesting to investigate $N$ point masses or some analytic models for a galaxy if the lens equation is a polynomial. These interesting applications will be considered in the future.
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Appendix A

— A Linearized Kerr Lens —

We consider the linearized Kerr spacetime with mass \( m \) and specific angular momentum vector \( \mathbf{q} \). In units of the Einstein ring angular radius \( \theta_E \), the lens equation for this spacetime is expressed as \([3], [5]\)

\[
\beta = \theta - \frac{\theta}{|\theta|^2} + \left( \frac{s \times \mathbf{\mu}}{|\theta|^2} - \frac{2(s \times \mathbf{\mu} \cdot \theta)}{|\theta|^4} \right),
\]

(A.1)

where \( \mathbf{\mu} \) denotes the unit vector along the line of sight, and we have defined

\[
s = \frac{D_S \theta_E \mathbf{q} - (\mathbf{q} \cdot \mathbf{\mu}) \mathbf{\mu}}{4 D_{LS} m}.
\]

(A.2)

Here, \( D_S \) and \( D_{LS} \) denote the distance between the observer and the source and the distance between the lens and the source, respectively. Although the orientation of the spin vector \( \mathbf{q} \) is arbitrary, only the projection of \( \mathbf{q} \) onto the lens plane makes contribution. On the lens plane, we adopt the Cartesian coordinates such that \( s = (0, s) \), \( \beta = (a, b) \) and \( \theta = (x, y) \). The lens equation becomes fifth-order and fourth-order in \( x \), respectively as

\[
g_5 \equiv (x - a)(x^2 + y^2)^2 + (s - x)(x^2 + y^2)^2 - 2sx^2 = 0,
\]

(A.3)

\[
g_4 \equiv (y - b)(x^2 + y^2)^2 - y(x^2 + y^2)^2 - 2sxy = 0,
\]

(A.4)

which are coupled nonlinearly.

By applying the Euclidean algorithm, the pair of polynomial lens equations are reduced to

\[
x = -\frac{H(y)}{I(y)},
\]

(A.5)

\[
\sum_{i=0}^{5} k_i y^i = 0,
\]

(A.6)

where we have defined

\[
H(y) = 2b^3 s y^2 + 2b^2 y^3 (a + s + a^2 s + b^2 s) + 2ab y^4 (a^2 + b^2 - 2as) + 2y^5 (-a^3 - ab^2 + a^2 s - b^2 s),
\]

(A.7)

\[
I(y) = b^3 s y^2 + b^2 y (-1 + 4as + 3s^2) + b y^2 (a^2 - 3b^2 + 2as + 4a^3 s + 4ab^2 s + 3s^2) + y^3 (a^2 + b^2 - 2a^2 b^2 - 2b^4 - 2as - 4a^3 s + s^2) + 2by^4 (a^2 + b^2 - 2as),
\]

(A.8)
and

\[ k_5 = 4(a^2 + b^2)^2, \quad (A\cdot9) \]
\[ k_4 = -4b(-a^2 + a^4 - b^2 + 2a^2b^2 + b^4 + 2as), \quad (A\cdot10) \]
\[ k_3 = a^2 + b^2 - 8a^2b^2 - 8b^4 - 2as(1 + 2a^2 + 2b^2) + s^2, \quad (A\cdot11) \]
\[ k_2 = b[-a^2 - 5b^2 + 2as(1 + 2a^2 + 2b^2) + 3s^2], \quad (A\cdot12) \]
\[ k_1 = b^2(-1 + 4as + 3s^2), \quad (A\cdot13) \]
\[ k_0 = b^3s^2. \quad (A\cdot14) \]

Thus the lens equation is reduced to a single, real fifth-order one.

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