Group Extensions of the Co-type of a Crossed Module and Strict Categorical Groups

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Abstract

Prolongations of a group extension can be studied in a more general situation that we call group extensions of the co-type of a crossed module. Cohomology classification of such extensions is obtained by applying the obstruction theory of monoidal functors.

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1 Introduction

A description of group extensions by means of factor sets leads to a close relationship between the extension problem of a type of algebras and the corresponding cohomology theory. This allows to study extension problems using cohomology as an effective method [6].

Let $A$ and $\Pi$ be two groups, $A$ abelian. An extension of $A$ by $\Pi$ is a short exact sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{\beta} \Pi \to 1. \quad (1)$$

A classical theorem in homological algebra asserts that the group of isomorphism classes of extensions of $A$ by $\Pi$ with a fixed operator $\varphi : \Pi \to \text{Aut} A$ is isomorphic to the second cohomology group $H^2_{\varphi}(\Pi, A)$, [9]. After that, the group $H^2$ was applied to the problem of classifying all group extensions in the different situations. This theorem has been made more precise by establishing a categorical equivalence between the category of extensions and a certain category whose objects are 2-cocycles [8]. With the notion of a categorical group (or a Gr-category [14]), many aspects of group extension problem are raised to a categorical level which help to obtain applications in algebra (see [4], [15]). This article belongs to this type.

The article is derived from the following classical problem. For a group extension $\Pi$ and a group homomorphism $\eta : \Pi' \to \Pi$, it follows from the existence of the pull-back of the pair $(\eta, p)$ that there is an extension $E\eta$ making the following diagram commute

$$E\eta : \quad 0 \to A \xrightarrow{j'} B' \xrightarrow{\beta'} \Pi' \to 1$$

$$E : \quad 0 \to A \xrightarrow{j} B \xrightarrow{\beta} \Pi \to 1$$
The problem is that with a given extension \( E' \) and a homomorphism \( \eta : \Pi' \to \Pi \), let us find all extensions \( E \) of \( A \) by \( \Pi \) such that \( E' = E\eta \). Then, the extension \( E \) is said to be a \( \eta \)-prolongation of \( E' \). A brief and general description of this problem was introduced in [16] (Proposition 5.1.1). In [13] we show the better descriptions in the case of the central extensions and \( \eta \) is an injection. Each prolongation induces a model which is “dual” to a group extension of the type of a crossed module (see Section 6). This leads to the notion of group extension of co-type of a crossed module studied in this paper.

The plan of this paper is, briefly, as follows. In Section 2 we recall reduced categorical groups, monoidal functors of type \((\varphi, f)\). In Section 3 we show the relation between the category of crossed modules and the category of strict Gr-categories, which is a useful tool in the next proofs. Next, we introduce the notion of a \( \zeta \)-extension of the co-type of a crossed module in Section 4, and we construct the obstruction theory of a \( \zeta \)-extension (Theorem 6). In Section 5 we present Schreier theory for \( \zeta \)-extensions of the co-type of a crossed module (Theorem 9). The last section is devoted to applying the results of previous sections to the problem of prolongations of a group extension in [13].

2 Preliminaries

For later use, we recall here some basic facts and results about categorical groups (see [12], [13]).

A categorical group is a monoidal category \((\mathcal{G}, \otimes, I, a, l, r)\) in which every object is invertible and the underlying category is a groupoid. If \((F, F') \) is a monoidal functor between categorical groups, the isomorphism \( F_\ast : I' \to FI \) can be deduced from \( F \) and \( \tilde{F} \). Thus, we will refer to \((F, \tilde{F})\) as a monoidal functor.

Two monoidal functors \((F, \tilde{F})\) and \((F', \tilde{F}')\) from \( \mathcal{G} \) to \( \mathcal{G}' \) are homotopic if there is a natural monoidal equivalence (or a homotopy) \( \alpha : (F, \tilde{F}, F_\ast) \to (F', \tilde{F}', F'_\ast) \), which is a natural equivalence such that \( F'_\ast = \alpha_I \circ F_\ast \).

Each categorical group \( \mathcal{G} \) determines three invariants, as follows:

1. The set \( \pi_0 \mathcal{G} \) of isomorphism classes of the objects in \( \mathcal{G} \) is a group where the operation is induced by the tensor product in \( \mathcal{G} \).
2. The set \( \pi_1 \mathcal{G} \) of automorphisms of the unit object \( I \) is a \( \pi_0 \mathcal{G} \)-module.
3. An element \([k] \in H^3(\pi_0 \mathcal{G}, \pi_1 \mathcal{G})\) is induced by the associativity constraint of \( \mathcal{G} \).

Based on the data: a group \( \Pi \), a \( \Pi \)-module \( A \) and \( k \in Z^3(\Pi, A) \), we construct a categorical group, denoted by \( \text{Red}(\Pi, A, k) \) whose objects are elements \( x \in \Pi \) and the morphisms are automorphisms \((x, a) : x \to x\), where \( x \in \Pi, a \in A \). The composition of two morphisms is induced by the
addition in $A$

$$(x, a) \circ (x, b) = (x, a + b).$$

The tensor products are given by

$$x \otimes y = x.y, \quad x, y \in \Pi,$$

$$\quad (x, a) \otimes (y, b) = (xy, a + xb), \quad a, b \in A.$$  

The unit constraints of the categorical group $\text{Red}(\Pi, A, k)$ are strict, and its
associativity constraint is $\mathbf{a}_{x,y,z} = (xyz, k(x,y,z)).$

In the case where $\Pi, A, [k]$ are three invariants of a categorical group $G$
then $\text{Red}(\Pi, A, k)$ is monoidally equivalent to $G$ and it is called a reduction
of $G$, hence denoted by $G(k)$.

A functor $F : \text{Red}(\Pi, A, k) \to \text{Red}(\Pi', A', k')$ is of type $(\varphi, f)$ if

$$F(x) = \varphi(x), \quad F(x, a) = (\varphi(x), f(a)),$$

where $\varphi : \Pi \to \Pi'$, $f : A \to A'$ are group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$, for $x \in \Pi, a \in A$. Note that if $\Pi'$-module $A'$ is considered as a $\Pi$-module under the action $xa' = \varphi(x).a'$, then $f : A \to A'$ is a homomorphism of $\Pi$-modules. In this case, we call $(\varphi, f)$ a pair of homomorphisms and call

$$\xi = \varphi^*k' - f_*k \in Z^3(\Pi, A')$$

an obstruction of the functor $F$, where $\varphi^*, f_*$ are canonical homomorphisms

$$Z^3(\Pi, A) \xrightarrow{f_*} Z^3(\Pi, A') \xrightarrow{\varphi^*} Z^3(\Pi', A').$$

The results on monoidal functors of type $(\varphi, f)$ stated in [12] are summarized
in the following proposition.

**Proposition 1.** Let $G$ and $G'$ be two categorical groups, $G(k)$ and $G'(k')$ be
their reductions, respectively.

i) Every monoidal functor $(F, \tilde{F}) : G \to G'$ induces one $G(k) \to G'(k')$
of type $(\varphi, f)$.

ii) Every monoidal functor $G(k) \to G'(k')$ is a functor of type $(\varphi, f)$.

iii) A functor $F : G(k) \to G'(k')$ of type $(\varphi, f)$ is realizable, that is, it
induces a monoidal functor, if and only if its obstruction $[\xi]$ vanishes in
$H^3_F(\Pi, A')$. Then, there is a bijection

$$\text{Hom}_{(\varphi, f)}[G(k), G'(k')] \leftrightarrow H^3_F(\Pi, A'),$$

where $\text{Hom}_{(\varphi, f)}[G(k), G'(k')]$ is the set of all homotopy classes of monoidal
functors of type $(\varphi, f)$ from $G(k)$ to $G'(k')$. 

3
3 Categorical groups associated to a crossed module

A categorical group is strict, according to Joyal and Street [7], if all of its constraints are strict and every object has a strict inverse \((x \otimes y = 1 = y \otimes x)\). Brown and Spencer [3] called it a \(G\)-groupoid. The authors of [3] showed that there is a categorical equivalence between the category of crossed modules and that of \(G\)-groupoids, and hence crossed modules can be studied by means of category theory. The Brown-Spencer equivalence has recently developed for the category of (braided) crossed bimodules (see [10], Theorems 4.3, 4.4).

**Definition.** A crossed module is a quadruple \((B, D, d, \theta)\) where \(d : B \to D\), \(\theta : D \to \text{Aut}B\) are group homomorphisms such that the following relations hold

\[ C_1. \quad \theta d = \mu, \]
\[ C_2. \quad d(\theta_x(b)) = \mu_x(d(b)), \quad x \in D, b \in B, \]

where \(\mu_x\) is an inner automorphism given by conjugation of \(x\).

**Definition.** A homomorphism \((f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')\) of crossed modules consists of group homomorphisms \(f_1 : B \to B', f_0 : D \to D'\) satisfying

\[ H_1. \quad f_0 d = d' f_1, \]
\[ H_2. \quad f_1(\theta_x(b)) = \theta'_{f_0(x)} f_1(b), \]

for all \(x \in D, b \in B\).

In the present paper, the crossed module \((B, D, d, \theta)\) is sometimes denoted by \(B \xrightarrow{d} D\). For convenience, we denote by the addition for the operation in \(B\) and by the multiplication for that in \(D\).

The following properties follow from the definition of a crossed module.

**Proposition 2.** Let \((B, D, d, \theta)\) be a crossed module.

i) \(\text{Ker}d \subset Z(B)\).

ii) \(\text{Im}d\) is a normal subgroup in \(D\).

iii) The homomorphism \(\theta\) induces a homomorphism \(\varphi : D \to \text{Aut}(\text{Ker}d)\) given by

\[ \varphi_x = \theta_x|\text{Ker}d. \]

iv) \(\text{Ker}d\) is a left \(\text{Coker}d\)-module with the action

\[ sa = \varphi_x(a), \quad a \in \text{Ker}d, \quad x \in s \in \text{Coker}d. \]

As mentioned above, a categorical group can be seen as a crossed module [3], [7]. To help motivate the reader, we present this fact in detail.

For each crossed module \((B, D, d, \theta)\), one can construct a strict categorical group \(\mathcal{G}_{B \to D} = \mathcal{G}\), called the categorical group associated to the crossed module \(B \to D\), as follows.

\[ \text{Ob}\mathcal{G} = D, \quad \text{Hom}(x, y) = \{b \in B \mid x = d(b)y\}, \]
where $x, y$ are objects of $G$. The composition of two morphisms is given by

$$(x \overset{b}{\to} y \overset{c}{\to} z) = (x \overset{b+c}{\to} z).$$

The tensor functor is given by $x \otimes y = xy$ and

$$(x \overset{b}{\to} y) \otimes (x' \overset{b'}{\to} y') = (xx' \overset{b+b'+\theta y b'}{\to} yy'). \quad (3)$$

Conversely, for a strict categorical group $(G, \otimes)$, we define a crossed module $C_G = (B, D, d, \theta)$ as follows. Set

$D = \text{Ob}G, \quad B = \{ x \overset{b}{\to} 1 | x \in D \}.$

The operations on $D$ and on $B$ are given by

$xy = x \otimes y, \quad b + c = b \otimes c,$

respectively. Then, the set $D$ becomes a group in which the unit is 1, the inverse of $x$ is $x^{-1}$ ($x \otimes x^{-1} = 1$). The set $B$ is a group in which the unit is the morphism $(1 \overset{id}{\to} 1)$ and the inverse of $(x \overset{b}{\to} 1)$ is the morphism $(x^{-1} \overset{\bar{b}}{\to} 1)(b \otimes \bar{b} = id_1)$.

The homomorphisms $d : B \to D$ and $\theta : D \to \text{Aut} B$ are respectively given by

$$d(x \overset{b}{\to} 1) = x,$$

$$\theta_y(x \overset{b}{\to} 1) = (xy^{-1} \overset{id_b+b+id_{y^{-1}}}{\to} 1).$$

The following result shows the relationship between homomorphisms of crossed modules and monoidal functors of associated categorical groups.

**Proposition 3** ([1]). Let $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ be a homomorphism of crossed module.

i) There is a functor $F : G_B \to G_{B'}$ given by

$$F(x) = f_0(x), \quad F(b) = f_1(b),$$

where $x \in \text{Ob}G, \ b \in \text{Mor}G$.

ii) Natural isomorphisms $\tilde{F}_{x,y}$ together with $F$ is a monoidal functor if and only if $\tilde{F}_{x,y} = \varphi(\overline{x}, \overline{y})$, where $\varphi \in Z^2(\text{Coker}d, \text{Ker}d')$.

Note. In the category of $G$-groupoids in [3], the morphisms $(\tilde{F}, \tilde{F})$ satisfy $\tilde{F} = id.$
4 Group extensions of the co-type of a crossed module

In this section we introduce a concept which is “dual” to the concept of group extension of type $B \xrightarrow{d} D$ in [1, 2]. As will be showed later, it is also regarded as a generalization of the prolongation problem of group extensions [13].

**Definition.** Let $d : B \to D$ be a crossed module. A group *extension* of $A$ of co-type $B \xrightarrow{d} D$ is a diagram of group homomorphisms

$$
\begin{array}{cccccc}
B & \xrightarrow{d} & D \\
\downarrow{\beta} & & \\
0 & \xrightarrow{j} & E & \xrightarrow{p} & D & \xrightarrow{1},
\end{array}
$$

where the bottom row is exact, $j(A) \subset Z(E)$, the pair $(\beta, \text{id}_D)$ is a morphism of crossed modules.

Since the bottom row is exact and since $p \circ \beta \circ i = d \circ i = 0$, where $i : \text{Ker} d \to B$ is an inclusion, there exists a unique homomorphism $\zeta : \text{Ker} d \to A$ such that the left hand side square commutes

$$
\begin{array}{cccccc}
\text{Ker} d & \xrightarrow{i} & B & \xrightarrow{d} & D \\
\downarrow{\zeta} & & \downarrow{\beta} & & \\
0 & \xrightarrow{j} & E & \xrightarrow{p} & D & \xrightarrow{1}.
\end{array}
$$

This homomorphism is defined by

$$j(\zeta(c)) = \beta(ic), \ c \in \text{Ker} d. \quad (5)$$

Moreover, $\zeta$ depends only on the equivalence class of the extension $E$.

**Note on terminologies.** Since the homomorphism $\theta'$ of the crossed module $E \xrightarrow{p} D$ is the conjugation and since $j(A) \subset Z(E)$, $\theta'_x$ acts on $A$ as an identity. Thus, the group $A$ can be seen as a $D$-module with the trivial action. Then,

$$\zeta(sc) = \zeta(c), \ s \in \text{Coker} \ d, c \in \text{Ker} d. \quad (6)$$

Indeed, By Proposition [2] $\theta_x(c) \in \text{Ker} d$, so one has

$$j\zeta(\theta_x(c)) = \beta(\theta_x c) = \beta \circ i(\theta_x c) \quad (5)$$

$$\Rightarrow \quad \theta'_x(\beta c) = \theta'_x(j\zeta(c)) = j\zeta(c).$$

Since $j$ is injective, we obtain [10]. Thus, it defines a trivial Coker $d$-module structure on $\text{Im} \zeta$.  

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The homomorphism $\zeta : \text{Ker } d \to A$ satisfying the condition (6) is called an abstract $\zeta$-kernel of the crossed module $B \xrightarrow{d} D$. An extension of $A$ of co-type $B \xrightarrow{d} D$ inducing $\zeta : \text{Ker } d \to A$ is said to be an extension of the abstract $\zeta$-kernel, or a $\zeta$-extension of co-type $B \xrightarrow{d} D$.

• The obstruction theory: the case $\zeta$ is surjective

From now on, assume that $\zeta : \text{Ker } d \to A$ is an onto homomorphism. We use the obstruction theory of monoidal functors to deal with the existence of $\zeta$-extensions.

Let $G = \mathbb{G}_{B \to D}$ be the categorical group associated to crossed module $B \to D$. Since $\pi_0 G = \text{Coker } d$ and $\pi_1 G = \text{Ker } d$, the reduced categorical group $G(k)$ is of form $G(k) = \text{Red}(\text{Coker } d, \text{Ker } d, k)$, $[k] \in H^3(\text{Coker } d, A)$, where the associativity constraint $k$ is defined as follows. Choose a set of representatives $\{x_s \mid s \in \text{Coker } d\}$ in $D$. For each $x \in s$ choose an element $b_x \in B$ satisfying $x_s = d(b_x)x$, $b_{xs} = 0$. According to [14], the family $(x_s, b_x)$ is called a stick. It defines a monoidal functor $(H, \tilde{H}) : G(k) \to G$ by

$$H(s) = x_s, \quad H(s, a) = a, \quad \tilde{H}_{r,s} = -b_{x_rx_s}.$$  

Then, $k$ is determined by the following commutative diagram

$$
\begin{align*}
\begin{array}{ccc}
  x_s(x_{rt}) & \xrightarrow{x_s \otimes \tilde{H}_{r,t}} & x_s x_{rt} \\
  \| & & \downarrow k(s, r, t) \\
(\tilde{H}_{s,r} \otimes x_t) x_{rt} & \xrightarrow{x_{sr} x_t} & \tilde{H}_{s,rt} \otimes x_{rt}.
\end{array}
\end{align*}
$$

By the relation (3), this diagram implies

$$\theta_{x_s}(\tilde{H}_{r,t}) + \tilde{H}_{s,rt} + k(s, r, t) = \tilde{H}_{s,r} + \tilde{H}_{sr,t}.$$  

We write $k = \delta(\tilde{H})$ even though the function $\tilde{H}$ takes values in $B$. The cohomology class $\text{Obs}(\zeta) = [\zeta, k] \in H^3(\text{Coker } d, A)$ is called the obstruction of the abstract $\zeta$-kernel.

The onto homomorphism $\zeta : \text{Ker } d \to A$ induces a quotient category $G/\text{Ker } \zeta$ with the same objects of $G (= D)$, but morphisms are homotopy classes of morphisms in $G$, i.e., elements of the group $\mathbb{B} = B/\text{Ker } \zeta$. The category $G/\text{Ker } \zeta$ is just the categorical group associated to the crossed module $(\mathbb{B}, \mathbb{A}, \mathbb{F}, D)$ induced by the crossed module $(B, d, \theta, D)$.

**Lemma 4.** If the obstruction $\text{Obs}(\zeta)$ vanishes in $H^3(\text{Coker } d, A)$, there exists a monoidal functor $\text{Red}(D, A, 0) \to G/\text{Ker } \zeta$.  

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Proof. If \( \text{Obs}(\zeta) \) vanishes in \( H^3(\text{Coker} d, A) \), then \( \zeta_* k = \delta g \), where \( g : (\text{Coker} d)^2 \to A \). Consider a functor

\[
F : \text{Red}(D, A, 0) \to \text{Red}(\text{Coker} d, A, \delta g),
\]

for \( F = (q, \text{id}) \), where \( q \) is the natural projection. The obstruction of \( F \) is

\[
q^*(\delta g) = \delta (q^* g).
\]

Thus, \( F \) together with \( \bar{F} = q^* g \) is a monoidal functor. It follows the existence of a monoidal functor from \( \text{Red}(D, A, 0) \) to \( \mathbb{G}/\text{Ker} \zeta \).

Lemma 5. Each monoidal functor \( \text{Red}(D, A, 0) \to \mathbb{G}/\text{Ker} \zeta \) defines a \( \zeta \)-extension of co-type \( B \overset{d}{\to} D \).

Proof. Construction of the crossed product from a monoidal functor \((\Gamma, \bar{\Gamma}) : \text{Red}(D, A, 0) \to \mathbb{G}/\text{Ker} \zeta \).

The morphism \( \bar{\Gamma} \) defines an associated function \( g : D^2 \to A \) by \( \bar{\Gamma}_{s,r} = (1, g(s, r)) \). Now, we set \( \varphi : \text{Coker} d \to \text{Aut} \overline{B} \) by

\[
\varphi_s(\overline{b}) = \overline{x_s b} \overline{(= \overline{b}_{x_s}(b))}.
\]  

Since \( x_r x_s = d(\bar{\Gamma}_{r,s}) x_{rs} \), the functions \( \varphi, g \) satisfy the rule

\[
\varphi_s \varphi_r = \mu_g(s, r) \varphi_{sr}.
\]

Since \( \delta g = 0 \), according to Lemma 8.1 \([9]\) one can defines a crossed product \( E_g = \overline{B} \times \text{Coker} d \). Namely, \( E_g = \overline{B} \times \text{Coker} d \) and the operation on \( E_g \) is

\[
(\overline{b}, s) + (\overline{c}, r) = (\overline{b} + \varphi_s(\overline{c}) + g(s, r), sr).
\]

In this group \((0, 1)\) is the zero, while the negative of the element \((\overline{b}, s)\) is \((\overline{b}, s^{-1})\), where \( \varphi_s(\overline{b}) = -\overline{b} - g(s, s^{-1}) \). One obtains an exact sequence

\[
E_g : 0 \to A \overset{j_g}{\to} E_g \overset{p_g}{\to} D \to 1,
\]

where \( j_g(\zeta(c)) = (\overline{c}, 1) \), \( p_g(\overline{b}, s) = db.x_s \). Indeed,

\[
p_g j_g(\zeta(c)) = p_g(\overline{c}, 1) = dc.x_s = 1,
\]

and for \((\overline{b}, s) \in \text{Ker}(p_g)\), then \( p_g(\overline{b}, s) = db.x_s = 1 \). By the uniqueness of the representation in \( D \), we have \( db = 1 \) and \( x_s = 1 \), it follows that \( b \in \text{Ker} d \) and \( s = 1 \), or \((\overline{b}, s) \in \text{Im}(j_g)\).

We prove that \( j_g(A) \subset Z(E_g) \). For \( b, c \in B \), one has

\[
\mu_{(\overline{b}, s)}(\overline{c}, 1) = (\mu_{\overline{b}} \varphi_s(\overline{c}), 1)
\]

(10)
If \( c \in \text{Ker} \, d \), then by (6), \( \varphi_s(\overline{c}) = \overline{c} \). Hence,
\[
\mu(\overline{b}, s)(\overline{c}, 1) = (\mu_{\overline{c}}(\overline{c}), 1) = (\overline{b} + c - \overline{b}, 1) = (\overline{c}, 1).
\]

Since \( j_g(A) \subset Z(E_g) \) and \( p_g \) is a surjection, \( E_g \xrightarrow{p_g} D \) is a crossed module in which the homomorphism \( \theta' : D \to \text{Aut} \, E_g \) is the conjugation. To define the morphism \( (\beta, \text{id}_D) \) of crossed modules, one set
\[
\beta : B \to E_g, \quad \beta(b) = (\overline{b}, 1).
\]

This correspondence is a homomorphism thanks to the relation (9). Clearly, \( p_g \circ \beta = d \). Moreover, for all \( c \in B \) and \( x = db.x_s \in D \), we have
\[
\beta(\theta_x(c)) = \beta(\theta_{db}(\theta_x(c))) = (\mu_b \theta_x(c), 1) = (\mu_{\overline{b}} \varphi_s(\overline{c}), 1).
\]

Since \( \theta_x' = \mu(\overline{b}, s) \),
\[
\theta_x' \beta(c) = \mu(\overline{b}, s)(\overline{c}, 1) = (\mu_{\overline{b}} \varphi_s(\overline{c}), 1)
\]

Thus, the relation \( H_2 \) holds, and \( E_g \) is a \( \zeta \)-extension of co-type \( B \xrightarrow{d} D \). \( \Box \)

We state one of the paper’s main results.

**Theorem 6.** Let \( \zeta : \text{Ker} \, d \to A \) be the abstract \( \zeta \)-kernel of the crossed module \( B \xrightarrow{d} D \). Then, the vanishing of the obstruction \( \text{Obs}(\zeta) \) in \( H^3(\text{Coker} \, d, A) \) is necessary and sufficient for there to exist a \( \zeta \)-extension of co-type \( B \xrightarrow{d} D \).

**Proof.** Necessary condition. Let \( E \) be a \( \zeta \)-extension of co-type \( B \to D \) satisfying the diagram (4). Then, the reduced categorical group of the categorical group \( G' \) associated to the crossed module \( E \xrightarrow{d} D \) is \( \text{Red}(1, A, 0) \). By Proposition 3, the pair \( (\beta, \text{id}_D) \) determines a monoidal functor \( (F, \tilde{F}) : G \to G' \).

By Proposition 1, \( (F, \tilde{F}) \) induces a monoidal functor of type \( (0, \zeta) \) from \( \text{Red}(\text{Coker} \, d, \text{Ker} \, d, k) \) to \( \text{Red}(1, A, 0) \). Also by Proposition 1, the obstruction \( [\zeta, k] \) of the pair \( (0, \zeta) \) vanishes in \( H^3(\text{Coker} \, d, A) \).

Sufficient condition. It follows directly from Lemma 4 and Lemma 5. \( \Box \)

### 5 Classification theorem

**Definition.** Two \( \zeta \)-extensions of co-type \( (B, D, d, \theta) \),
\[
0 \to A \xrightarrow{j} E \xrightarrow{p} D \to 1, \quad B \xrightarrow{\beta} E
\]
\[
0 \to A \xrightarrow{j'} E' \xrightarrow{p'} D \to 1, \quad B \xrightarrow{\beta'} E'
\]
are equivalent if there is an isomorphism \( \omega : E \to E' \) such that \( \omega j = j' \), \( p' \omega = p \) and \( \omega \beta = \beta' \).
We denote by \( \text{Ext}_B^D(D, A, \zeta) \)
the set of all equivalence classes of \( \zeta \)-extensions of co-type \( B \to D \) inducing \( \zeta \). We describe this set by means of the set

\[
\text{Hom}_{(0, \zeta)}(\text{Red}(D, A, 0), \frac{G}{\text{Ker} \zeta})
\]

of homotopy classes of monoidal functors of type \((0, \zeta)\) from \( \text{Red}(D, A, 0) \) to \( \frac{G}{\text{Ker} \zeta} \). First, let \( q : B \to \overline{B} = B/\text{Ker} \zeta \) and \( \sigma : D \to \text{Coker} d \) be the natural projections, one states the following lemma.

**Lemma 7.** If \( \zeta \) is surjective, then the commutative diagram (11) induces a short exact sequence

\[
0 \to \overline{B} \xrightarrow{\varepsilon} E \xrightarrow{\sigma p} \text{Coker} d \to 1,
\]

where \( \varepsilon(b + \text{Ker} \zeta) = \beta(b) \).

*Proof.* Obviously, \( \sigma p \) is surjective. It is easy to see that \( \text{Ker} \beta = \text{Ker} \zeta \), so \( \varepsilon \) is injective. The diagram (11) implies \( \sigma \varepsilon(\overline{b}) = \sigma p \beta(b) = \sigma d(b) = 1 \), this means that the above sequence is semi-exact. For \( e \in \text{Ker}(\sigma p) \), \( p(e) \in \text{Ker} \sigma = \text{Im} d \), and hence \( p(e) = d(\overline{b}) = p \beta(b) = p \varepsilon(\overline{b}) \). Then, \( e = \varepsilon(\overline{b}) + ja \). Since \( ja = j \zeta(c) = \beta(c) = \varepsilon(\overline{c}) \), \( e = \varepsilon(\overline{b + c}) \in \text{Im} \varepsilon \). Thus, the sequence (11) is exact.

**Lemma 8.** Each \( \zeta \)-extension of co-type \( B \to D \) is equivalent to a crossed product extension which is constructed from a monoidal functor of type \((0, \zeta)\), \((\Gamma, \tilde{\Gamma}) : \text{Red}(D, A, 0) \to \frac{G}{\text{Ker} \zeta} \).

*Proof.* Let \( E \) be a \( \zeta \)-extension of co-type \( B \xrightarrow{d} D \). By the proof of Theorem 6 there is a monoidal functor \((\Gamma, \tilde{\Gamma}) : \text{Red}(D, A, 0) \to \frac{G}{\text{Ker} \zeta} \). By Lemma 4 the crossed product \( E_g \), where \( g \) is the function associated with \( \tilde{\Gamma} \), is a \( \zeta \)-extension of co-type \( B \xrightarrow{d} D \) in which

\[
\beta_g : B \to E_g, \ b \mapsto (\overline{b}, 1).
\]

Thanks to the exact sequence (11) in Lemma 7 each element of \( E \) can be represented uniquely as \( \varepsilon \overline{b} + e_s \), where \( \{e_s, s \in \text{Coker} d\} \) is a set of representatives of Coker \( d \) in \( E \). It is easy to check that the correspondence

\[
\omega : E \to E_g, \ \varepsilon \overline{b} + e_s \mapsto (\overline{b}, s)
\]

is a group isomorphism. Moreover, \( \omega \) makes two extensions \( E \) and \( E_g \) equivalent.

\[ \square \]
Theorem 9 (Schreier theory for extensions of co-type of a crossed module).

If $\zeta$-extensions of co-type $B \xrightarrow{d} D$ exist, then there is a bijection

$$\Omega : \text{Ext}_{B \rightarrow D}(D, A, \zeta) \rightarrow \text{Hom}_{(0,\zeta)}[\text{Red}(D, A, 0), G / \text{Ker } \zeta].$$

Proof. The correspondence $E \mapsto (\Gamma, \tilde{\Gamma})$ in Lemma defines a correspondence $[E] \mapsto [(\Gamma, \tilde{\Gamma})]$. The fact that $\Omega$ is injective implies by following steps.

Step 1: If monoidal functors $(\Gamma, \tilde{\Gamma})$ and $(\Gamma', \tilde{\Gamma}')$ are homotopic, then two extensions $E_g$ and $E_{g'}$ are equivalent.

Let $\Gamma, \Gamma' : \text{Red}(D, A, 0) \rightarrow G / \text{Ker } \zeta$ be two monoidal functors and $\alpha : \Gamma \rightarrow \Gamma'$ be a homotopic. Then, the following diagram commutes

$$\begin{array}{ccc}
\Gamma s \Gamma r & \xrightarrow{\Gamma s} & \Gamma s r \\
\downarrow \alpha_s \otimes \alpha_r & & \downarrow \alpha_{sr} \\
\Gamma' s \Gamma' r & \xrightarrow{\Gamma' s} & \Gamma' s r.
\end{array}$$

Since the morphisms $\alpha_s$ are of forms $(1, a_s)$, it follows from the above diagram that

$$g(s, r) - g'(s, r) = a_s + a_r - a_{sr} = (\delta a)(s, r). \quad (12)$$

Since $\zeta$ is surjective, $a_s = \zeta(z_s)$, where $z : \text{Coker } d \rightarrow \text{Ker } d$ is a normalized function.

Then, by (12), $\alpha$ determines a map $\omega : E_g \rightarrow E_{g'}$ by

$$(b, s) \mapsto [b + z_s, s]. \quad (13)$$

By the relation (13) and by the definition of operations in $E_g, E_{g'}$, the map $\omega$ is a group homomorphism. Further, it makes two extensions $E$ and $E_g$ equivalent.

Step 2: If two extensions $E_g$ and $E_{g'}$ are equivalent, then $(\Gamma, \tilde{\Gamma})$ and $(\Gamma', \tilde{\Gamma}')$ are homotopic.

Let $E_g$ and $E_{g'}$ be equivalent via the isomorphism $\omega : E_g \rightarrow E_{g'}$. From $p_g = p_{g'} : E_g \rightarrow E_{g'} \rightarrow D$, it follows that $\omega$ is of the form (13), where $z : \text{Coker } d \rightarrow \text{Ker } d$ is a normalized function. Since $\omega$ is a homomorphism, $\alpha = \zeta k$ is a homotopy between $\Gamma$ and $\Gamma'$.

It follows from Lemma 5 that $\Omega$ is surjective.

It follows from Proposition 1 and Theorem 9 that

Corollary 10. If $\zeta$-extensions of co-type $B \xrightarrow{d} D$ exist, then there is a bijection

$$\text{Ext}_{B \rightarrow D}(D, A, \zeta) \leftrightarrow H^2(\text{Coker } d, A).$$
6 Prolongations of a group extension

In this section we show an application of \( \zeta \)-extensions of co-type of a crossed module in order to obtain the results on prolongations of a group extension in the sense of [13]. Given a commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
\mathcal{B} : & 0 & \longrightarrow \text{Ker } \pi \overset{i}{\longrightarrow} B \overset{\pi}{\longrightarrow} \Pi \overset{\eta}{\longrightarrow} 1 \\
\mathcal{E} : & 0 & \longrightarrow A \overset{j}{\longrightarrow} E \overset{p}{\longrightarrow} D \overset{1}{\longrightarrow} \\
\end{array}
\]

where the rows are exact, \( \text{Ker } \pi \subset ZB, \eta \) is a normal monomorphism (in the sense that \( \eta \Pi \) is a normal subgroup of \( D \)) and \( \zeta \) is an epimorphism. Then, \( \mathcal{E} \) is said to be a \((\zeta, \eta)\)-prolongation of \( \mathcal{B} \).

For the quotient group \( \overline{\mathcal{B}} = B / \text{Ker } \zeta \), the homomorphisms \( i, \eta \pi, \zeta, \beta \) in the commutative diagram (14) induce the homomorphisms \( \iota, \delta, \zeta, \beta \), respectively, such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Ker } \delta & \longrightarrow & \overline{\mathcal{B}} \overset{d}{\longrightarrow} D \\
\iota & \downarrow & \beta \\
0 & \longrightarrow & A \overset{j}{\longrightarrow} E \overset{p}{\longrightarrow} D \overset{1}{\longrightarrow}
\end{array}
\]

Besides, according to Theorem 2 [13], \( \mathcal{E} \) induces a homomorphism \( \theta : D \rightarrow \text{Aut}(\overline{\mathcal{B}}) \) such that the quadruple \( (\overline{\mathcal{B}}, D, d, \theta) \) is a crossed module.

**Theorem 11.** \( \mathcal{E} \) is a \( \zeta \)-extension of co-type \( (\overline{\mathcal{B}}, D, d, \theta) \).

**Proof.** In the diagram (15), since the bottom row is exact and \( jA \subset ZE \) (Theorem 10 [13]), the epimorphism \( p : E \rightarrow D \) together with the conjugation in \( E \) is a crossed module. It is easy to see that the pair \((\iota, id_{D})\) is a homomorphism of crossed modules, so \( \mathcal{E} \) is a \( \zeta \)-extension of co-type \( (\overline{\mathcal{B}}, D, d, \theta) \).

- The problem of prolongations of a group extension.

Given a diagram of group homomorphisms

\[
\begin{array}{ccc}
\mathcal{E} : & 0 & \longrightarrow \text{Ker } \pi \overset{i}{\longrightarrow} B \overset{\pi}{\longrightarrow} \Pi \overset{\eta}{\longrightarrow} 1 \\
\end{array}
\]

where the row is exact, \( i \) is an inclusion map, \( \text{Ker } \pi \subset ZB, \eta \) is a normal monomorphism, \( \zeta \) is surjective, and a group homomorphism \( \theta : D \rightarrow \text{Aut}(\overline{\mathcal{B}}) \) such that the quadruple \( (\overline{\mathcal{B}}, D, d, \theta) \) is a crossed module (where the notations \( \overline{\mathcal{B}}, d \) are defined as above). These data are denoted by the triple \( (\zeta, \eta, \theta) \),
called a pre-prolongation of $E$. A $(\zeta, \eta)$-prolongation of $E$ inducing $\theta$ is also called a covering of the pre-prolongation $(\zeta, \eta, \theta)$.

The “prolongation problem” is that of finding whether there is any covering of the pre-prolongation $(\zeta, \eta, \theta)$ of $E$ and, if so, how many.

According to [13], each pre-prolongation $(\zeta, \eta, \theta)$ of $E$ induces an obstruction $k$. This obstruction is just the obstruction of an abstract $\zeta$-kernel of the crossed module $\overline{B} \overset{d}{\rightarrow} D$. Thus, from the results on crossed modules in previous sections, one obtains the solution of the problem of prolongations of a group extension (Theorem 8 and Theorem 15 in [13]).

**Theorem 12.** Let $(\zeta, \eta, \theta)$ be a pre-prolongation.

i) The vanishing of the obstruction $[\zeta, k]$ in $H^3(\text{Coker } d, A)$ is necessary and sufficient for there to exist a covering of $(\zeta, \eta, \theta)$.

ii) If $[\zeta, k]$ vanishes, there is a bijection

$$\text{Ext}(\zeta, \eta)(D, A) \leftrightarrow H^2(\text{Coker } d, A),$$

where $\text{Ext}(\zeta, \eta)(D, A)$ is the set of equivalence classes of $(\zeta, \eta)$-prolongations of the extension $B$ inducing $\theta$.

**Proof.** i) According to Theorem[13] the vanishing of $[\zeta, k]$ in $H^3(\text{Coker } d, A)$ is necessary and sufficient for there to exist a $\overline{\zeta}$-extension $E$ of co-type $\overline{B} \overset{d}{\rightarrow} D$. Thanks to the following diagram, this is equivalent to the fact that $E$ is a covering of the pre-prolongation $(\zeta, \eta, \theta)$,

\[
\begin{array}{ccccccccc}
B : & 0 & \rightarrow & \text{Ker} \pi & i & B & \rightarrow & \Pi & \rightarrow & 1 \\
\downarrow \zeta & & \downarrow \pi_0 & & \downarrow \eta & & \downarrow \eta & & \downarrow 1 \\
E : & 0 & \rightarrow & A & j & E & \rightarrow & D & \rightarrow & 1.
\end{array}
\]

ii) It is clear that two coverings of the pre-prolongation $(\zeta, \eta, \theta)$ are equivalent if and only if they are two equivalent $\overline{\zeta}$-extensions of co-type $\overline{B} \overset{d}{\rightarrow} D$, that is, there is a bijection

$$\text{Ext}(\zeta, \eta)(D, A) \leftrightarrow \text{Ext}_{\overline{B} \overset{d}{\rightarrow} D}(D, A, \overline{\zeta}).$$

Now, by Corollary[10] we have the bijection

$$\text{Ext}(\zeta, \eta)(D, A) \leftrightarrow H^2(\text{Coker } d, A).$$

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