GROUP ACTION ON INSTANTON BUNDLES OVER $\mathbb{P}^3$

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Abstract: Denote by $MI(k)$ the moduli space of $k$-instanton bundles $E$ of rank 2 on $\mathbb{P}^3 = \mathbb{P}(V)$ and by $Z_k(E)$ the scheme of $k$-jumping lines. We prove that $[E] \in MI(k)$ is not stable for the action of $SL(V)$ if $Z_k(E) \neq \emptyset$. Moreover $\dim Sym(E) \geq 1$ if $\text{length} Z_k(E) \geq 2$. We prove also that $E$ is special if and only if $Z_k(E)$ is a smooth conic. The action of $SL(V)$ on the moduli of special instanton bundles is studied in detail.

1 Introduction

A $k$-instanton bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ is a stable 2-bundle $E$ such that $c_1 = 0$, $c_2 = k$ and $H^1(E(-2)) = 0$ (see [Har78] and [BH78]). The instanton bundles which are trivial on the fibers of the twistor map $\mathbb{P}^3 \to S^4$ correspond to self dual Yang Mills $Sp(1)$-connections on $S^4$, according to the ADHM-correspondence. The moduli space of $k$-instanton bundles on $\mathbb{P}^3$ will be denoted by $MI_{\mathbb{P}^3}(k)$.

Our aim is to study the natural action of $GL(V)$ over $MI_{\mathbb{P}^3}(k)$. In this sense this paper can be seen as the natural continuation of the study begun in [AO99] trying to answer a general problem raised by Simpson who asked about the stable points of the $GL(V)$-action on the moduli space of bundles over $\mathbb{P}(V)$. In [CO00] we proved that $MI_{\mathbb{P}^3}(k)$ is an affine variety and this fact simplifies a lot the geometry of the group action. An easy consequence is that all points of $MI_{\mathbb{P}^3}(k)$ are semistable according to Mumford’s geometric invariant theory. For $k \geq 3$ the generic $[E] \in MI_{\mathbb{P}^3}(k)$ has trivial stabilizer $Sym(E)$ and hence, for $k \geq 3$, among the nonstable points $[E]$ are all the points $[E]$ such that $Sym(E)$ has positive dimension. In this work, we will focus our attention on the connected component $Sym^0(E)$ containing the identity.

A classical tool for the study of vector bundles is the scheme of jumping lines (see [OSS80]). The restriction of a $k$-instanton bundle $E$ on a line $L$ is isomorphic to $\mathcal{O}_L(-p) \oplus \mathcal{O}_L(p)$ for some $p$ depending on $L$ and satisfying the inequalities $0 \leq p \leq k$.

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It is known that \( p = 0 \) is the general value and if \( p \geq 1 \) then \( L \) is called a jumping line of order \( p \). A useful criterion that we prove is the following (see Theorem 4.4, Example 4.6):

**Theorem A:** Let \( E \) be a \( k \)-instanton bundle. If \( E \) has a jumping line of maximal order \( k \), then \([E]\) is not stable and the converse is not true.

Moreover, we have found an example of a \( k \)-instanton bundle \( E \) with \( Z_k(E) \neq \emptyset \) and \( \text{Sym}^0(E) = 0 \) (see Example 4.7).

In [ST90] was observed that the action of \( SL(2) \simeq SL(U) \) over \( V \simeq U \otimes U \) is useful to get the cohomology and other geometrical properties of special instanton bundles. Unfortunately special instanton bundles are not invariant for such action. Instead, they are invariant for the action of \( SL(2) \simeq SL(U) \) over \( V \simeq U \otimes \mathbb{C}^2 \) where \( SL(U) \) acts trivially over \( \mathbb{C}^2 \). Our starting point was the remark that the correct action to be considered is \( SL(2) \times SL(2) \simeq SL(U) \times SL(U') \simeq \text{Spin}(V) \) acting over \( V \simeq U \otimes U' \) for a suitable isomorphism (see Remark 4.2), hence for a suitable symmetric form on \( V \). In fact the symmetry group of the kernel bundle considered in [ST90] is \( SL(U) \times SL(U') \) and for any special instanton bundle \( E \) we get that \( \text{Sym}(E) \) lies always in such a group (we believe that this fact is true for any instanton bundle). This fact conduces us to introduce the definition of fine action (see Definition 4.3). In particular a subgroup \( G \subseteq GL(V) \) has a fine action if there are 2-dimensional vector spaces \( U \) and \( U' \) such that \( G \subseteq GL(U) \times GL(U') \) and \( G \) acts according to fixed isomorphism \( V \simeq U \otimes U' \).

In Section 5 (see Theorem 5.1 and Theorem 5.3) we will prove

**Theorem B:** Let \( E \) be a \( k \)-instanton bundle. If length \( Z_k(E) \geq 2 \) then \( \text{Sym}^0(E) \) contains \( G \) of positive dimension having a fine action.

In addition, by means of Example 4.6, we will see that the converse of Theorem B is not true. On the other hand, we will see

**Theorem C:** The following are equivalent

(i) \( Z_k(E) \) is a conic.

(ii) \( E \) is a special \( k \)-instanton bundle.

(iii) \( \text{Sym}^0(E) \supseteq SL(U') \) having a fine action.
The equivalence between (i) and (ii) is proved in Corollary 2.9. (ii) implies (iii) is proved in (3) meanwhile (iii) implies (ii) is proved in Proposition 4.13.

After this result, we are led to conjecture that for any instanton bundle $E$, $\text{Sym}^{0}(E)$ has a fine action.

In Section 4 we study the group action over the subvariety $\text{MI}_{\mathbb{P}^{3}}(k)$ of special $k$-instanton bundles (see Definition 2.4) and our main results are Corollary 4.12 and Theorem 4.14. More precisely, it is known that every special $k$-instanton bundle is defined by $(k + 1)$ lines through the classical Serre correspondence and that there is a pencil of lines whose branch locus can be studied with the classical machinery of binary forms and which is a $GL(V)$-invariant. In Theorem 4.14 we show that in general there are exactly $\frac{1}{k+1} \binom{2k}{k}$ $GL(V)$-orbits of special $k$-instanton bundles with the same branch locus. We mainly apply results of Trautmann ([Tra88]) that we report in a form suitable for our purposes. Although Trautmann did not apply his results to instanton bundles, probably because the group action approach on moduli has only nowadays a stronger importance, some of these applications were certainly known to him.

We want to point out that we have chosen to present the actions in matrix form, due to the fact that this is suitable in view of computer implementing ([Anc96]).

2 Generalities about instantons over $\mathbb{P}^{3}$

The goal of this section is to prove some general results about $k$-instanton bundles on $\mathbb{P}^{3}$ that will be needed later. To this end, we start fixing some notation and recalling some known facts (See for instance [BT87]).

**Notation 2.1.** $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^{3}}(d)$ denotes the invertible sheaf of degree $d$ on $\mathbb{P}^{3} = \mathbb{P}(V)$. For any coherent sheaf $E$ on $\mathbb{P}^{3}$ we denote $E(d) = E \otimes \mathcal{O}_{\mathbb{P}^{3}}(d)$ and by $h^{i}E(d)$ the dimension of $H^{i}(\mathbb{P}^{3}, E(d)) = H^{i}E(d)$.

**Definition 2.2.** A $k$-instanton bundle on $\mathbb{P}^{3}$ is an algebraic vector bundle $E$ of rank 2 with Chern classes $c_{1}(E) = 0$ and $c_{2}(E) = k$, which is stable (i.e. $h^{0}E = 0$) and satisfies the vanishing $h^{1}E(-2) = 0$. We will denote by $\text{MI}_{\mathbb{P}^{3}}(k)$ the moduli space of $k$-instanton bundles on $\mathbb{P}^{3}$.

Let $E$ be a $k$-instanton bundle. It is well known (see [BH78]) that there is a
vector space $I$ of dimension $k$ and a symplectic vector space $(W, J)$ of dimension $2k + 2$ such that $E$ is the cohomology of a symplectic monad

\[(1) \quad I^* \otimes \mathcal{O}(-1) \xrightarrow{A^t} W \otimes \mathcal{O} \xrightarrow{A} I \otimes \mathcal{O}(1).\]

That is, $E \simeq \text{Ker} A/\text{Im} A^t$, where the transpose $A^t$ is computed with respect to the symplectic form $J$. We have isomorphisms $I \simeq H^1(E(-1))$, $W \simeq H^1(E \otimes \Omega^1)$.

The vector space $W \otimes I \otimes V = \text{Hom}(W \otimes \mathcal{O}, I \otimes \mathcal{O}(1))$ contains the subvariety $Q$ given by morphisms $A$ such that the sequence (1) is a complex, that is, such that $AJA^t = 0$. In that case, we shall say that $A$ represents $E$. $\text{GL}(I) \times \text{Sp}(W)$ acts on $Q$ by $(g, s) \cdot A = gAs$. Let $Q^0$ be the open subvariety of $Q \subseteq W \otimes I \otimes V$ which consists of morphisms $A \in Q$ that are surjective. By [BH78] (see also [CO00]) all points of $Q^0$ are stable for the action of $\text{GL}(I) \times \text{Sp}(W)$ and $\text{MI}_3^p(k)$ is isomorphic to the geometric quotient $Q^0/\text{GL}(I) \times \text{Sp}(W)$ according to Mumford’s GIT. In particular any morphism between $k$-instanton bundles lifts to a morphism between the corresponding monads.

**Definition 2.3.** Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$. A hyperplane $H \subseteq \mathbb{P}^3$ is an unstable hyperplane of $E$ if $h^0E|_H \neq 0$. The set of unstable hyperplanes of $E$ has a natural structure of scheme and it will be denoted by $W(E) \subseteq \mathbb{P}^3^*$.\n
The fact that, for any $k$-instanton bundle $E$ on $\mathbb{P}^3$, $h^0E(1) \leq 2$ ([BT87]) allows to pose the following definition

**Definition 2.4.** A $k$-instanton bundle $E$ on $\mathbb{P}^3$ is called special if $h^0E(1) = 2$.

We denote by $\text{MI}^s_3^p(k)$ the moduli space of special $k$-instanton bundles. By its definition, it is constructed as a closed subscheme of the affine variety $\text{MI}_3^p(k)$ ([CO00]). In particular, $\text{MI}^s_3^p(k)$ is also affine and it has dimension $2k + 9$ (see e.g. Proposition 2.12). Notice that any $E \in \text{MI}_3^p(2)$ is special ([BT87]).

For any $k$-instanton bundle $E$ we have $\text{dim} W(E) \leq 2$. By a theorem of Coanda ([Coa92]) a $k$-instanton bundle $E$ is special if and only if $\text{dim} W(E) = 2$ and in this case $W(E)$ is a smooth quadric.

Now we are going to give a similar result for the scheme of $k$-jumping lines.

**Definition 2.5.** Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$ and $1 \leq p \in \mathbb{Z}$. A line $l$ on $\mathbb{P}^3$ is called a $p$-jumping line of $E$ if $E|_l = \mathcal{O}_l(-p) \oplus \mathcal{O}_l(p)$ and we denote by

\[Z_j(E) = \{ \quad l \in G(\mathbb{P}^3, \mathbb{P}^3) \quad | \quad E|_l = \mathcal{O}_l(-i) \oplus \mathcal{O}_l(i) \quad j \leq i \quad \}\]
being $G(\mathbb{P}^1, \mathbb{P}^3)$ the Grassmannian of lines on $\mathbb{P}^3$.

$Z_j(E)$ has a natural structure of closed subscheme of $G(\mathbb{P}^1, \mathbb{P}^3)$. From the definition it follows that for any $k$-instanton bundle $E$ on $\mathbb{P}^3$ there is the following filtration

$$G(\mathbb{P}^1, \mathbb{P}^3) = Z_0(E) \supseteq Z_1(E) \supseteq Z_2(E)....$$

Moreover, by [Ski97]: Proposition 13 and [Rao97], $Z_{k+1}(E) = \emptyset$. Hence a line in $Z_k(E)$ is called a maximal order jumping line.

**Remark 2.6.** $Z_k(E)$ is obtained by cutting the Plücker quadric $Q_4 \subseteq \mathbb{P}^5$ with a linear space. In particular $\text{deg } Z_k(E) \leq 2$. Let $(x_0, \ldots, x_3)$ be homogeneous coordinates on $\mathbb{P}(V)$ and let $A = \sum_{i=0}^{3} A_i x_i \in W \otimes I \otimes V$ be a matrix representing $E$. Then it is well known that $Z_k(E)$ is obtained by cutting the Plücker quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$ with the linear space $\sum_{i<j} p_{ij} A_i J A_j^t = 0$.

In the following result we relate the maximal order jumping lines of $E$ with unstable hyperplanes of $E$.

**Lemma 2.7.** Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$ and $l \in Z_k(E)$. Then any $H \supseteq l$ is an unstable hyperplane of $E$, i.e., $H \in W(E)$.

**Proof.** Since $H \supseteq l$ we have the following exact sequence

$$0 \to E(-1)|_H \to E|_H \to E|_l \to 0.$$ 

Assume that $H$ is not an unstable plane. Taking cohomology to the above exact sequence we get the injection $H^0 E|_l \hookrightarrow H^1 E(-1)|_H$. Since by Serre’s duality $h^2 E(-1)|_H = h^0 E(-2)|_H = 0$, this gives us

$$k + 1 = h^0 E|_l \leq h^1 E(-1)|_H = -\chi(E(-1)|_H) = k$$

which is a contradiction. Hence $H$ is an unstable hyperplane of $E$. 

**Remark 2.8.** In particular, since for any $k$-instanton bundle $E$ on $\mathbb{P}^3$ we have $h^0 E|_H \leq 1$ ( [KO99]; Theorem 1.2), we get immediately from the above proof that $Z_j(E) = \emptyset$ for any $j \geq k + 1$.

**Corollary 2.9.** Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$. We have $\dim Z_k(E) \leq 1$. $E$ is special if and only if $\dim Z_k(E) = 1$ and in this case $Z_k(E)$ is a smooth conic in $G(\mathbb{P}^1, \mathbb{P}^3)$. 

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Proof. It follows from Lemma 2.7 that if $\dim Z_k(E) \geq 1$ then $\dim W(E) \geq 2$. Hence, by [Cox92], $E$ is a special instanton bundle. \qed

Remark 2.10. Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$. If $E$ has three different $k$-jumping lines then $E$ is a special $k$-instanton bundle.

We know ([Hart78] for $k = 2$ and [BT87] in general) that if $E \in MI_{\mathbb{P}^3}(k)$ then $Z_k(E)$ is a smooth conic on $G(\mathbb{P}^1, \mathbb{P}^3)$ and that $W(E)$ is a smooth quadric. In addition, $Z_k(E)$ corresponds to one of the two rulings of $W(E)$ and the generic $s \in H^0 E(1)$ vanishes on $k$ disjoint lines of the other ruling. Moreover, it is well-known that each $E \in MI^s_{\mathbb{P}^3}(k)$ can be identified with a $g_{k+1}^1$ without base points on $Z_k(E) \simeq \mathbb{P}^1$ and that we can choose this line as the line of all hyperplanes containing the line $\{x_0 = x_1 = 0\}$. In fact, the dual of a smooth quadric is isomorphic to the quadric itself, and we can consider $Q_E \subseteq \mathbb{P}^3$ as the dual of $W(E)$ and then the tangent hyperplanes to $Q_E$ belong to $W(E)$. The $g_{k+1}^1$ is constructed by cutting with the zero locus of $s \in H^0 E(1)$ on a fixed line of the ruling in $W(E)$ corresponding to $Z_k(E)$. We will define the branch locus of this $g_{k+1}^1$ as the branch locus of the bundle. It is defined up to $SL(2)$-action.

Notation 2.11. We will denote by $G^k$ the open subset in $G(\mathbb{P}^1, \mathbb{P}^3)$ which consists of $g_{k+1}^1$ without fixed points and we will denote by $W \subseteq Gr(\mathbb{P}^2, \mathbb{P}^5)$ the open subset of planes which cut the Plücker quadric in a smooth conic.

The above discussion shows the following

Proposition 2.12. ([Huns82], [BT87]; Coroll. 4.7) The morphism $\pi_1 : MI_{\mathbb{P}^3}(k) \rightarrow W$ which takes $E$ to the plane spanned by the conic $Z_k(E)$ is a fibration such that all the fibers are isomorphic to $G^k$.

Let $Y \subseteq \mathbb{P}(S^2 V) \simeq \mathbb{P}^9$ be the open subvariety of smooth quadrics in $\mathbb{P}(V)$. There is another fibration $\pi_2 : MI^s_{\mathbb{P}^3}(k) \rightarrow Y$ which takes $E$ to the quadric $W(E)$. All the fibers of $\pi_2$ are isomorphic to the disjoint union of two copies of $G^k$ and we have a commutative diagram

\[
\begin{array}{ccc}
MI^s_{\mathbb{P}^3}(k) & \xrightarrow{\pi_2} & Y \\
\| & & \downarrow j \\
MI^s_{\mathbb{P}^3}(k) & \xrightarrow{\pi_1} & W
\end{array}
\]
where $j$ is a $2:1$ finite map. In addition, for any $Q \in \mathcal{W}$ the fiber $j^{-1}(Q)$ is given by the two rulings of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

In [AO98]: Proposition 1.1, special $k$-instanton bundles were characterized in terms of their monads. However that description, as the one given in [BT87], is not useful for our purposes because the monads considered there were not symplectic and the techniques that we develop in Section 4 do not apply. So we have to introduce an alternative description. With this aim we introduce some notations

**Definition 2.13.** A $(k+1) \times (k+1)$-Hankel matrix $H = (\alpha_{ij}) = (\alpha_{i+j-2})$ is one having equal elements along each diagonal line parallel to the secondary diagonal, i.e.

\[
H = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-1} & \alpha_k \\
\alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \alpha_{k+1} & \alpha_{k+2} & & \\
\alpha_{k-1} & \alpha_k & \cdots & \cdots & \cdots & \\
\alpha_k & \alpha_{k+1} & \cdots & \cdots & \alpha_{2k-1} & \alpha_{2k}
\end{pmatrix}.
\]

Let $X = [x^k, -kx^{k-1}y, (\frac{k}{2})x^{k-2}y^2, \ldots, (-1)^k y^k]$ and $H$ be a $(k+1) \times (k+1)$-Hankel matrix. Following [Tra88]: 2.4, we denote by $f_H$ the form of degree $2k$ defined by $X \cdot H \cdot X^t$.

**Definition 2.14.** Denote by $\Delta$ the invariant for $f_H$ defined by the determinant

\[
\begin{vmatrix}
\alpha_0 & \alpha_1 & \cdots & \alpha_k \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_k & \alpha_{k+1} & \cdots & \alpha_{2k}
\end{vmatrix}.
\]

Let $(x_0, \ldots, x_3)$ be homogeneous coordinates on $\mathbb{P}(V)$. We set

\[
I_k(x_0, x_1) := \begin{pmatrix}
x_0 & x_1 & \cdots & \cdots \\
\ldots & \cdots & \cdots & \cdots \\
x_0 & x_1
\end{pmatrix}
\quad \text{and} \quad
\tilde{I}_k(x_0, x_1) := \begin{pmatrix}
x_1 & x_1 & \cdots & \cdots \\
x_0 & x_1 & \cdots & \cdots \\
x_0 & x_1
\end{pmatrix}.
\]
The $k \times (k+1)$-matrix $I_k$ and the $(k+2) \times (k+1)$-matrix $\tilde{I}_k$ are both well known in the theory of vector bundles. In fact, the matrix $I_k$ defines a surjective morphism of vector bundles on $\mathbb{P}^1 = \mathbb{P}(U)$ given by the natural multiplication $S^k U \otimes \mathcal{O} \to S^{k-1} U \otimes \mathcal{O}(1)$, whose kernel is isomorphic to $\mathcal{O}(-k)$. More interesting is the fact that the converse holds. In fact, it is well known (see e.g. [AO99]) the following

**Proposition 2.15.** Every surjective morphism of vector bundles on $\mathbb{P}^1$

$$\mathcal{O}^{k+1}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)^k$$

is represented, in a suitable system of coordinates $(x_0, x_1)$, by the matrix $I_k$.

**Proof.** The proposition is equivalent to the fact that there is a unique $\text{SL}(2) \times \text{SL}(k) \times \text{SL}(k+1)$-orbit for nondegenerate matrices in $\mathbb{P} (\mathbb{C}^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^{k+1})$. \hfill $\square$

**Notation 2.16.** Given $H$ a nondegenerate $(k+1) \times (k+1)$-Hankel matrix, we denote

$$A := [I_k(x_0, x_1)|I_k(x_2, x_3) \cdot H].$$

**Lemma 2.17.** Let $J$ be the standard skew-symmetric nondegenerate matrix and let $A$ be as in (3). The following holds

(i) $AJA^t = 0$.

(ii) Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$ defined by $A$. Then $h^0 E(1) = 2$ and $E$ is special.

(iii) Conversely, every special $k$-instanton bundle is the cohomology bundle of a monad (3) for a suitable system of coordinates.

**Proof.** (i) is a straightforward verification. Let $T_k = \begin{pmatrix} & & & 1 \\ & & \vdots & \\ & 1 & & \end{pmatrix}$ and let

$$\tilde{A} = \left[ I_k(x_0, x_1) \cdot T_{k+1} \cdot H^{-1} | I_k(x_2, x_3) \cdot T_{k+1} \right].$$
Then $AJ\tilde{A}^t = 0$. Since $\tilde{A}$ has $(k + 2)$ rows it follows $h^0 E(1) = 2$ which proves (ii). When $H$ moves among the nondegenerate Hankel matrices, the moduli space of bundles given by (2) is isomorphic to $\mathbb{P}(S^{2k}U) \setminus \{ \Delta = 0 \}$ (see Corollary 3.2). In Theorem 3.5 we quote a result of Trautmann which shows that $\mathbb{P}(S^{2k}U) \setminus \{ \Delta = 0 \}$ is isomorphic to $G^k$. Since this is exactly a fiber of the map $\pi_1$ of Proposition 2.12, we obtain (iii).

**Remark 2.18.** With the notations of the above proof set

$$\tilde{H} = \begin{pmatrix}
\alpha_0 & \alpha_1 & \cdots & \alpha_k & \alpha_{k+1} \\
\alpha_1 & \cdots & \alpha_{k+1} & \alpha_{k+2} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{k-1} & \alpha_k & \cdots & \alpha_{2k}
\end{pmatrix}.$$

Then, $\tilde{H} \cdot T_{k+2} \cdot \tilde{A} = A$.

We will end with the following result which is a straightforward verification and we left it to the reader.

**Lemma 2.19.** Let $E$ be a special $k$-instanton bundle defined by the monad (3). Then $W(E)$ is given by all the hyperplanes which are tangent to the quadric $x_0x_3 - x_1x_2$ and $Z_k(E)$ is the conic obtained by cutting the Plücker quadric with the plane $\{ p_{02} = p_{13} = p_{03} + p_{12} = 0 \}$.

## 3 Useful facts about $SL(2)$-actions

For any $g \in \mathfrak{sl}(U)$, we denote by $s^k g \in \mathfrak{sl}(S^k U)$ the image of $g$ through the Lie algebra representation of $\mathfrak{sl}(U)$ given by the $k$-symmetric power. Notice that $\mathfrak{sl}(U)$ acts on $S^2(S^k U)$ by

$$\mathfrak{sl}(U) \times (S^2(S^k U)) \rightarrow (S^2(S^k U))$$

$$(g, A) \mapsto (s^k g)^t \cdot A + A \cdot s^k g.$$

**Lemma 3.1.** Let $H$ be a $(k + 1) \times (k + 1)$-Hankel matrix determined by $\alpha = (\alpha_0, \ldots, \alpha_{2k})$ and consider $g \in \mathfrak{sl}(U)$. Then the following holds

(i) $H' = (s^k g)^t \cdot H + H \cdot s^k g$ is a Hankel matrix.
(ii) If \( \alpha' = (\alpha'_0, ..., \alpha'_{2k}) \) are the coefficients of \( H' \) then \((s^{2k}_g)^t \cdot (\alpha)^t = (\alpha')^t\).

(iii) The natural action of \( sl(U) \) on forms of degree \( 2k \) takes \( f_H \) to \( f_{H'} \).

**Proof.** It follows from direct computation. \( \square \)

**Corollary 3.2.** The \((k + 1) \times (k + 1)\)-Hankel matrices forms an invariant subspace of \((S^2(S^kU))\) isomorphic to \( S^{2k}U \).

**Proof.** It is an immediate consequence of Lemma 3.1. \( \square \)

There is an analog description for the Lie group \( SL(U) \). If \( g \in SL(U) \), we denote by \( S^k g \in SL(S^kU) \) the image of \( g \) through the group representation of \( SL(U) \) given by the \( k \)-symmetric power.

As a consequence of the fact that \( S^k(\exp g) = \exp (s^k g) \), from Lemma 3.1 we obtain

**Lemma 3.3.** Let \( H \) be a \((k + 1) \times (k + 1)\)-Hankel matrix determined by \( \alpha = (\alpha_0, ..., \alpha_{2k}) \) and consider \( g \in SL(U) \). Then the following holds

(i) \( H'' = (s^k g)^t \cdot H \cdot S^k g \) is a Hankel matrix.

(ii) If \( \alpha'' = (\alpha''_0, ..., \alpha''_{2k}) \) are the coefficients of \( H'' \) then \((s^{2k}_g)^t \cdot (\alpha)^t = (\alpha'')^t\).

(iii) The natural action of \( SL(U) \) on forms of degree \( 2k \) takes \( f_H \) to \( f_{H'} \). \( \square \)

The following is a known result whose generalization can be found in [AO99].

**Proposition 3.4.** Let \( g \in SL(2) \) be such that \( g \cdot \left( \begin{array}{c} x'_0 \\ x'_1 \end{array} \right) = \left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) \). Then

\[
(S^k g)^t \cdot I_k(x_0, x_1) \cdot (S^k g)^{-1} = I_k(x'_0, x'_1).
\]

**Theorem 3.5.** ([Tru88]; Prop. 2.2) The morphism \( \mathbb{P}(S^{2k}U) \setminus \{\Delta = 0\} \longrightarrow G^k \subseteq G(\mathbb{P}^1, \mathbb{P}(S^{k+1}U)) \) defined by

\[
\alpha \mapsto L_\alpha = \{(f_0, \ldots, f_{k+1}) \in S^{k+1}U|(f_0, \ldots, f_{k+1}) \cdot \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k+1} & \alpha_{k+2} & \cdots & \alpha_{2k} \end{pmatrix} = 0\}
\]

being \( \alpha = (\alpha_0, \ldots, \alpha_{2k}) \), is a \( SL(U) \)-equivariant isomorphism. \( \square \)
We have a filtration of \( SL(U) \)-invariant closed subvarieties \( H_{2k} \subseteq \ldots \subseteq H_2 \subseteq H_1 = \mathbb{P}(S^{2k}(U)) \) where \( H_j = \{ f \in S^{2k}(U) | f \) has a root of multiplicity \( \geq j \} \). In particular, \( H_{2k} \) is the rational normal curve of degree \( 2k \) and \( H_2 \) is the discriminant hypersurface. The very basic example of Mumford’s GIT (already known to Hilbert) shows that \( H_{k+1} \) consists of the locus where all the \( SL(U) \)-invariant vanish (i.e. not semistable points) and the \( SL(U) \)-quotient map is defined on \( \mathcal{P}^0 = \mathbb{P}(S^{2k}(U)) \setminus H_{k+1} \). We note that \( H_{k+1} \subseteq \{ \Delta = 0 \} \). It is well known that the orbit of \( f \in \mathcal{P}^0 \) is not closed in \( \mathcal{P}^0 \) if and only if \( f \in H_k \) (i.e. not stable points). In fact, the orbits of points in \( H_k \) all contain in their closure the orbit of \( x^k y^k \in H_k \), which is the only orbit which is two-dimensional (all the other orbits are three-dimensional). This particular \( SL(U) \)-orbit corresponds to a particular \( SL(V) \)-orbit of special instanton bundles, that we will define in a while.

Now consider the analogous filtration \( W_{k+1} \subseteq \ldots \subseteq W_2 \subseteq W_1 = \mathbb{P}(S^{k+1}(U)) \) where \( W_j = \{ f \in S^{k+1}(U) | f \) has a root of multiplicity \( \geq j \} \). In particular, \( W_{k+1} \) is the rational normal curve of degree \( k + 1 \) and \( W_2 \) is the discriminant hypersurface of degree \( 2k \).

It is well known (and easy to check) that a pencil \( L \in Gr(\mathbb{P}^1, \mathbb{P}(S^{k+1}(U))) \) has a fixed point (that is it does not belong to \( G^k \)) if and only if \( L \) is contained in some osculating space to the rational normal curve \( W_{k+1} \). Moreover, the branch locus of \( L \) can be identified with \( B_L \in \mathbb{P}(S^{2k}(U)) \) being \( B_L \) isomorphic to \( L \cap W_2 \). It follows the basic fact that

\[
L \cap W_{s+1} \neq \emptyset \iff B_L \in H_s \quad \text{for} \quad 1 \leq s \leq k
\]

and we also remark that

\[
L_\alpha \cap W_{k+1} \neq \emptyset \iff \alpha \in H_k
\]

(compare it with [Tra88]:Prop. 2.6). The map \( G^k \to \mathbb{P}(S^{2k}(U)) \) which takes \( L \) to \( B_L \) extends to a map

\[
R: Gr(\mathbb{P}^1, \mathbb{P}(S^{k+1}(U))) \to \mathbb{P}(S^{2k}(U))
\]

which is finite and equivariant (see [Tra88]:Remark 2.5, and also our proof of Theorem [4.14]). In particular, a pencil \( L \in G^k \) is not stable if and only if \( L \cap W_{k+1} \neq \emptyset \). Notice once more that all the orbits of \( L \) such that \( L \cap W_{k+1} \neq \emptyset \) contain in the closure the unique orbit given by the chordal variety to \( W_{k+1} \).

\footnote{The only correction to be done in the diagram at page 41 of [Tra88] is that the arrow \( H \) must be dotted.}
We resume the above discussion in the following

Corollary 3.6. All points of $G^k$ are semistable for the action of $SL(U)$ and the only not stable ones are the pencils with a point of multiplicity $(k + 1)$. All the not stable orbits contain in the closure the unique orbit of pencils with two points of multiplicity $(k + 1)$. $\Box$

4 The action on $MI_{\mathbb{P}^3}(k)$ and $MI^s_{\mathbb{P}^3}(k)$

Along this section, we will keep the notations introduced before. Since a morphism between $k$-instanton bundles lifts to a morphism between the corresponding monads, we get that the action of $GL(V)$ over $MI_{\mathbb{P}^3}(k)$ lifts to the action of $GL(I) \times Sp(W) \times GL(V)$ over $Q^0$. This means that if $E$ and $E'$ are $k$-instanton bundles in the same $GL(V)$-orbit then any two representatives $A, A' \in Q^0$ are in the same $GL(I) \times Sp(W) \times GL(V)$-orbit. The (connected components containing the identity of the) stabilizers and the stable points of the two actions correspond to each other (this technique was used in \cite{AO99}).

As a consequence of the main theorem in \cite{CO00} we get

Theorem 4.1. All points in $MI_{\mathbb{P}^3}(k)$ are semistable for the $GL(V)$-action.

Proof. Since $Q^0$ is the complement in $Q$ of an invariant hypersurface (see \cite{CO00}) all points in $Q^0$ are semistable. In addition, the two GIT-quotient $MI_{\mathbb{P}^3}(k)/GL(V)$ and $Q^0/GL(I) \times Sp(W) \times GL(V)$ are isomorphic. $\Box$

Remark 4.2. Let $U$ and $U'$ be vector spaces of dimension 2. Fix three isomorphisms $V \simeq U \otimes U'$, $W \simeq S^k U \otimes U'$ and $I \simeq S^{k-1}U$. Then, there is an induced natural multiplication map $A \in Hom(W \otimes O, I \otimes O(1))$ which is $GL(U) \times GL(U')$-invariant. Indeed it can be shown that ker $A$ is a rank $k + 2$ bundle on $\mathbb{P}(V)$ with symmetry group in $GL(V)$ isomorphic to $GL(U) \times GL(U')/\mathbb{C}^*$. In the case $U = U'$, this construction was considered in \cite{ST90}, where ker $A$ was called the kernel bundle. We could work in $SL(V)$, which is the universal cover of $Aut(\mathbb{P}(V))$, and in this case the symmetry group of ker $A$ is $Spin(V) \simeq SL(U) \times SL(U')$, but we prefer to consider the $GL$ groups, which is a not essential variation.

After this remark, we are led to pose the following definition, which will play an important role in the sequel.
Definition 4.3. A subgroup $G \subseteq \text{GL}(V)$ is said to have a fine action over $\text{MI}_{\mathbb{P}^3}(k)$ if there are vector spaces $U$ and $U'$ of dimension 2 such that $G \subseteq \text{GL}(U) \times \text{GL}(U')$ and there exists a lifted action of $G$ over $W \otimes I \otimes V$ which is determined by some fixed isomorphisms $V \simeq U \otimes U'$, $W \simeq S^k U \otimes U'$, $I \simeq S^{k-1} U$.

Theorem 4.4. Let $E$ be a $k$-instanton bundle such that $Z_k(E) \neq \emptyset$. Then $[E]$ is a not stable point in $\text{MI}_{\mathbb{P}^3}(k)$ for the $\text{GL}(V)$-action.

Proof. By [Ski97], $E$ is represented by

$$A = [I(x_0, x_1)|H(x_0, x_1, x_2, x_3)]$$

where $H$ is a $k \times (k + 1)$ matrix of linear forms. Let $\lambda \simeq \mathbb{C}^* \subseteq \text{SL}(U')$ acting in such a way that the splitting $W = U'^{k+1}$ defines two eigenspaces in $W$ with positive and negative weights that correspond to the two blocks in which are divided the columns of $A$. Moreover, the positive eigenspace on $V = U' \oplus U'$ is generated by $x_0, x_1$ and the negative one by $x_2, x_3$. The action on $I$ is trivial. $\mathbb{C}^*$ has a fine action according to Definition 4.3 and we get

$$\lim_{t \to 0} \lambda(t) \cdot A = [I(x_0, x_1)|H(0, 0, x_2, x_3)]$$

which still represents a $k$-instanton. $\square$

Remark 4.5. By [Rao97] we know that if $Z_k(E) \neq \emptyset$ then $[E]$ is a smooth point in $\text{MI}_{\mathbb{P}^3}(k)$. It should be interesting to know if the unstable points for the $\text{GL}(V)$-action over $\text{MI}_{\mathbb{P}^3}(k)$ are smooth points.

Let us see, by means of an example, that the converse of Theorem 4.4 is no longer true.

Example 4.6. Let

$$A = \begin{pmatrix} 3x_0 & x_1 & 4x_3 & 5x_3 & x_2 \\ 5x_0 & x_1 & 3x_3 & 4x_0 \\ 5x_0 & x_1 & 5x_3 & x_2 \\ 5x_0 & x_1 & 3x_3 & 4x_0 \end{pmatrix}.$$ 

It is easy to check that $A^t A^t = 0$ and that the associated morphism is surjective (the rank is 3 on every $(x_0, \ldots, x_3) \in \mathbb{P}^3$) so that $A$ represents a 3-instanton bundle $E$. Moreover $Z_3(E) = \emptyset$ and $\mathbb{C}^* \subseteq \text{Sym}(E)$ having a fine action. The weights are as follows:
• on $I : -2, 0, 2$
• on $W : 5, 3, 1, -1, -5, -3, -1, 1$
• on $V : x_0(-3), x_1(-1), x_2(1), x_3(3)$

which correspond to weights $-1$ and $1$ on $U$ and $-2$ and $2$ on $U'$ with the isomorphisms $I \simeq S^2U$, $W \simeq S^3U \otimes U'$, $V \simeq U \otimes U'$. In particular $[E]$ is a not stable point for the $GL(V)$-action over $MI_{P^3}(k)$ and this shows that the converse of Theorem 4.4 is not true.

Example 4.7. The generic 3-instanton bundle $E$ with $Z_3$ given by one point has $Sym^0(E) = 0$. This can be checked with the help of Macaulay system ([BS]) by using Skiti monad ([Ski97]) with a generic Hankel matrix. Moreover, it seems likely that the generic $k$-instanton bundle $E$ with $Z_k \neq \emptyset$ has $Sym^0(E) = 0$. On the other hand, there are examples of $k$-instanton bundles such that $dim Sym(E) = 2$.

Keeping the notations introduced in Section 2, we recall a result from [CO00].

Theorem 4.8. There is a $Sp(W) \times SL(I) \times SL(V)$-invariant homogeneous polynomial $D$ over $W \otimes I \otimes V$ of degree $2k(k+1)$ such that $A \in Q^0$ if and only if $A \in Q$ and $D(A) \neq 0$.

Corollary 4.9. Let $A \in Q^0$ representing a $k$-instanton bundle $E$ and $\lambda: \mathbb{C}^* \rightarrow Sp(W) \times SL(I) \times SL(V)$ be a morphism such that its image is contained in $Sym(E)$. Then, if \( \lim_{t \to 0} \lambda(t) \cdot A \) exists, it belongs to $Q^0$.

Proof. It follows from the fact that $D(\lambda(t) \cdot A)$ is constant with respect to $t$.

Remark 4.10. After Corollary 4.9, a classification of all $\mathbb{C}^*$-invariant $k$-instanton bundles should give the classification of unstable points in $MI_{P^3}(k)$, but we postpone this study.

For special $k$-instanton bundles the description of the group action is quite precise. By the above geometric description, if $g^*E = E$, then $g$ leaves the smooth quadric $W(E)$ fixed and does not exchange the two rulings. Hence, $g \in SL(U) \times SL(U')$, for some complex vector spaces $U$ and $U'$ of dimension two, acting over $\mathbb{P}^3 = \mathbb{P}(U \otimes U')$ and one has to check how $SL(U) \times SL(U')$ acts on the space $G^k$. Notice that the first $SL(U')$ does not change anything so, for any $E \in MI_{P^3}^a(k)$

\[
(3) \quad SL(U') \subseteq Sym(E)
\]
and it follows that

\[(4) \quad MI_{P^3}(k)/SL(V) \cong G^k/SL(U)\]

(compare it with Corollary 3.6). This description was performed for \(k = 2\) by Hartshorne ([Har78]) and in the general case by Spindler and Trautmann ([ST90]), although they considered \(U = U'\).

Let us now see that under this isomorphism, the isomorphism class of the bundle is uniquely determined by the \(SL(2)\)-class of its associated Hankel matrix. Indeed we have

**Lemma 4.11.** (i) For every \(g \in SL(2)\)

\[
\begin{pmatrix}
(S^k g^{-1})^t & 0 \\
0 & S^k g
\end{pmatrix} \in Sp(\mathbb{C}^{2k+2})
\]

with respect to the standard \(J\).

(ii) For any \((k+1) \times (k+1)\)-Hankel matrix \(H\)

\[
(S^{k-1}g)^t \cdot [I_k(x_0, x_1)|I_k(x_2, x_3) \cdot H] \cdot \begin{pmatrix}
(S^k g^{-1})^t & 0 \\
0 & S^k g
\end{pmatrix} = [I_k(x'_0, x'_1)|I_k(x'_2, x'_3) \cdot H']
\]

being \(H' = (S^k g)^t \cdot H \cdot S^k g\) a Hankel matrix.

(iii) The isomorphism \(MI_{P^3}(k)/SL(V) \cong G^k/SL(U)\) takes the \(SL(V)\) class of the bundle to the \(SL(2)\)-class of its associated Hankel matrix given in (4).

**Proof.** We will prove (iii) since (i) and (ii) are easy to check. Given \(E\) a special \(k\)-instanton bundle choose coordinates such that \(\{x_0 x_3 - x_1 x_2 = 0\} \in W(E)\). Take two \(k\)-jumping lines \(L_1\) and \(L_2\) such that \(H^0(E_{|L_1})\) and \(H^0(E_{|L_2})\) are orthogonal spaces into \(W\) with respect to \(J\) (here \(H^0(E_{|L_i})\) are considered into \(W\) by the monad (3)). Moreover we can assume that \(L_1 = \{x_0 = x_1 = 0\}\) and \(L_2 = \{x_2 = x_3 = 0\}\). With this choice of coordinates there exists a matrix \(A\) representing \(E\) as in (3) containing a Hankel matrix \(H\). \(SL(2)\) acts on these choices, the \(SL(2)\)-class of \(H\) is uniquely determined and characterizes the \(SL(V)\)-orbit of \(E\).

As a consequence we obtain the following nice description
Corollary 4.12. Let $MI^s_{\mathbb{P}^3}(k)$ be the moduli space of special $k$-instanton bundles on $\mathbb{P}^3$. Then all points of $MI^s_{\mathbb{P}^3}(k)$ are semistable for the action of $SL(V)$ and the only not stable ones have a section vanishing on a line with multiplicity $(k+1)$. All the not stable orbits contain in the closure the unique orbit of bundles having two distinct sections each one vanishing on a (different) line with multiplicity $(k+1)$.

Proof. It follows from the isomorphism ([]) and Lemma 4.11.

Proposition 4.13. Let $E$ be a $k$-instanton bundle such that $SL(U') \subseteq Sym(E)$ having a fine action. Then $E$ is special.

Proof. By the assumption there is a $C^* \subseteq SL(U')$ with a two-dimensional eigenspace in $V$ of positive weight. Then there is a matrix representing $E$ such that in the first $k \times (k+1)$ submatrix only the coordinates of this eigenspace appear. It follows that the line spanned by this coordinates is a $k$-jumping line. Since $Z_k(E)$ is $SL(U')$-invariant and not empty, it is one-dimensional and the result follows from Corollary 2.9.

Let us now briefly describe different group actions of $G \subset Sym(E)$, being $E$ a special $k$-instanton bundle.

Let $E$ be a special $k$-instanton bundle represented by a matrix $A$ (see Notation 2.16). In general we have $SL(2) \subseteq Sym(E)$ acting in the following way. If

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2)$$

then the action on $W$ is determined by

$$\begin{bmatrix} \alpha \cdot Id & \beta \cdot H \\ \gamma \cdot H^{-1} & \delta \cdot Id \end{bmatrix} \in Sp(W)$$

while the action on $I$ is trivial and the action on $V$ is given by

$$\begin{align*}
x_0 & \mapsto \alpha x_0 + \gamma x_2 \\
x_1 & \mapsto \alpha x_1 + \gamma x_3 \\
x_2 & \mapsto \beta x_0 + \delta x_2 \\
x_3 & \mapsto \beta x_1 + \delta x_3.
\end{align*}$$

Let us now describe some particular case. Consider the Hankel matrix $H$ given by

$$\alpha_i = \delta_{i,k}.$$
It corresponds to the form $x^ky^k$ which, for $k \geq 2$, is the only form such that $\Delta \neq 0$ and such that the stabilizer has dimension 4. In this case, $\mathbb{C}^* \cdot SL(2) \subseteq Sym(E)$ where $SL(2)$ acts as above and $\mathbb{C}^*$ acts in the following way. For $t \in \mathbb{C}^*$ we have

$$\begin{bmatrix} t^{k-1} \\ \vdots \\ t^{-k+1} \end{bmatrix} \in SL(I), \quad \begin{bmatrix} t^k \\ \vdots \\ t^k \\ \vdots \\ t^{-k} \end{bmatrix} \in Sp(W)$$

and the action on $V$ is determined by

$$x_0 \mapsto tx_0, \quad x_1 \mapsto t^{-1}x_1, \quad x_2 \mapsto tx_2, \quad x_3 \mapsto t^{-1}x_3.$$ 

A matrix description of the instanton bundle $E$ with $Sym(E) = \mathbb{C}^* \cdot SL(2)$ is

$$\begin{pmatrix} x_0 & x_1 & x_3 & x_2 \\ & \ddots & & \vdots \\ x_0 & x_1 & x_3 & x_2 \end{pmatrix}.$$ 

Finally, let us mention that bundles $E$ having two distinct sections each one vanishing on a (different) line with multiplicity $(k+1)$ are quite interesting, because they are characterized by the property $Sym(E) \simeq SL(U) \cdot \mathbb{C}^*$ with a fine action.

We will end this section given a more precise description of the correspondence, introduced in Section 2, between special $k$-instanton bundles and linear systems $\mathcal{g}_{k+1}^1$ without base points. More precisely we will prove
Theorem 4.14. Let $MI_{\mathbb{P}^3}^s(k)$ be the moduli space of special $k$-instanton bundles on $\mathbb{P}^3$. There is a natural morphism
\[ \phi : MI_{\mathbb{P}^3}^s(k) \to \mathbb{P}^{2k}/SL(U) \]
which takes $E$ to the branch locus defined by its pencil of sections and which factors through
\[ \phi' : MI_{\mathbb{P}^3}^s(k)/SL(V) \to \mathbb{P}^{2k}/SL(U). \]
$\phi'$ is finite of degree equal to $\deg(G(\mathbb{P}^1, \mathbb{P}(S^{k+1}U))) = \tfrac{1}{k+1}{2k \choose k} = c_{k+1}$, where $c_{k+1}$ is the $(k+1)$-th Catalan number (see e.g. [GKZ94]; pag. 239).

The above theorem appears in [New81] for $k = 2$ and in this case $MI_{\mathbb{P}^3}(2)/SL(V)$ is isomorphic to $\mathbb{A}^1$.

This result answers the question posed by R. Hartshorne in [Har78b] concerning the relation between the cross-ratio of the four branch points of a $g_3$ and the orbits for the action of $SL(U)$. Indeed, Hartshorne asked if the cross-ratio determined uniquely the orbit and P. Newstead showed that for each value of the cross-ratio there are exactly two orbits in $G^2$ with one exception (See [New81] for more details).

Proof of Theorem 4.14.

By the isomorphism (4), $MI_{\mathbb{P}^3}^s(k)/SL(V) \cong G^k/SL(U)$. Moreover, by Lemma 4.11, this isomorphism takes the $SL(V)$-class of the bundle to the $SL(2)$-class of its associated Hankel matrix given in (2). Hence, we will prove that there exists a finite morphism
\[ \psi : G^k/SL(U) \to \mathbb{P}^{2k}/SL(U) \]
of degree equal to $c_{k+1}$. To this end, let $G = Gr(\mathbb{P}^1, \mathbb{P}(S^{k+1}U))$ and denote by $p_{i,j}$ the Plücker coordinates on $G$. Define
\[ R : (p_{0,1}, \ldots, p_{k,k+1}) \to (q_0 : \cdots : q_{2k+1}) \]
where, for any $1 \leq m \leq 2k+1$, $q_m = \sum_{\mu+\nu=m, \mu<\nu} (\nu - \mu)p_{\mu,\nu}$ (compare it with [Tra88]; Pag. 41-42).

Claim: $R$ is induced by the projection from the linear subspace defined by
\[ q_0 = \cdots = q_{2k+1} = 0 \]
which is disjoint from $G$.

Proof of the Claim: First of all recall that, for any $\{i_0, i_1, i_2, i_3\} \subset \{1, \ldots, k\}$, the Plücker coordinates $p_{i,j}$ verify the following Plücker relations

$$p_{i_0, i_1} p_{i_2, i_3} - p_{i_0, i_2} p_{i_1, i_3} + p_{i_0, i_3} p_{i_1, i_2} = 0.$$  

We will prove by induction on $m = \mu + \nu$, that if $q_1 = \cdots = q_{2k+1} = 0$ then $p_{i,j} = 0$ for any pair $(i, j)$. If $m = 1$, the assumption $q_1 = 0$ implies $p_{0,1} = 0$. Let us assume $p_{\mu,\nu} = 0$ for $\mu < \nu$ such that $\mu + \nu = m$ and we will see that $p_{\mu,\nu} = 0$ for any $\mu < \nu$ such that $\mu + \nu = m + 1$. Considering the Plücker relations for a suitable sets of indices and by induction hypothesis we get

$$p_{i,j} p_{k,l} = 0$$

for any $i < j$, $k < l$ with $i + j = k + l = m + 1$. Hence, from the assumption

$$0 = q_{m+1} = \sum_{\mu + \nu = m+1, \mu < \nu} (\nu - \mu) p_{\mu,\nu}$$

we deduce that $p_{\mu,\nu} = 0$ for any $\mu < \nu$ with $\mu + \nu = m + 1$, which proves what we want.

It follows from the claim that $R$ is a finite morphism. Moreover, since $G^k \subset G$, by [Tra88]: Pag. 41, $R$ induces a finite morphism

$$\psi : G^k / SL(U) \longrightarrow \mathbb{P}^{2k} / SL(U)$$

of degree equal to $c_{k+1}$ which, by Theorem 3, sends the $SL(U)$-class of the linear system $g_{k+1}^1$ associated to the $SL(V)$-class of a bundle $E$, to the branch locus defined by its pencil of sections. 

Remark 4.15. A general special instanton bundle with $c_2 = 2$ has symmetry group $SL(2) \cdot G_8$ where $G_8$ is binary dihedral of order 8. There is a central extension

$$0 \rightarrow \mathbb{Z}_2 \rightarrow G_8 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow 0.$$  

In fact $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong D_4$. If the quartic form corresponding to $(\alpha_0, \ldots, \alpha_4)$ satisfies $I = 0$ then $\text{Sym}(E) \cong SL(2) \cdot G_{24}$ where $G_{24}$ is binary tetrahedral of order 24. There is a central extension

$$0 \rightarrow \mathbb{Z}_2 \rightarrow G_{24} \rightarrow A_4 \rightarrow 0.$$
If the bundle has only one section vanishing on a line with multiplicity 4 then the quartic form has a double root and $\text{Sym}(E) \cong \text{SL}(2) \cdot G_{16}$ where $G_{16}$ is binary dihedral of order 16. The bundle with two sections each one vanishing on a different line with multiplicity 4 has $\text{Sym}(E) \cong \text{SL}(2) \cdot \mathbb{C}^*$ and it is characterized by the property $\dim \text{Sym}(E) \geq 4$.

5 Link between $\text{Sym}$ and $Z_k$

The goal of this section is to study the symmetry group $\text{Sym}(E)$ of a $k$-instanton bundle $E$ such that $Z_k(E) \neq \emptyset$.

**Theorem 5.1.** Let $E$ be a $k$-instanton bundle on $\mathbb{P}^3$ such that $Z_k(E)$ contains a double point. Then there exists $C \subseteq \text{Sym}^0(E)$, where $C$ has a fine action.

**Proof.** By assumption and [Sk97] there is monad representing $E$ with

$$A = \begin{pmatrix} x_0 & x_1 & \cdots & x_0 \\ \vdots & \ddots & \ddots & \vdots \\ x_0 & x_1 \end{pmatrix} \begin{pmatrix} H(x_0, x_1, x_2, x_3) \end{pmatrix}$$

and the line $\{x_0 = x_1 = 0\} \in Z_k(E)$ with coordinates $(p_{01}, \ldots, p_{13}, p_{23}) = (0, \ldots, 0, 1)$ corresponds to a double point. By Remark 2.6, $Z_k(E)$ is given by $\sum_{i<j} p_{ij} A_i J A_j^t = 0$. The crucial point is that

$$A' = \begin{pmatrix} x_0 & x_1 & \cdots & x_0 \\ \vdots & \ddots & \ddots & \vdots \\ x_0 & x_1 \end{pmatrix} \begin{pmatrix} H(0, 0, x_2, x_3) \end{pmatrix}$$

represents another instanton $E'$. Moreover

$$A_0 J A_2^t = A_0' J A_2^t, \quad A_0 J A_3^t = A_0' J A_3^t, \quad A_1 J A_2^t = A_1' J A_2^t, \quad A_1 J A_3^t = A_1' J A_3^t,$$

$$A_2 J A_3^t = A_2' J A_3^t = 0$$

and $A_0' J A_1^t = 0$ while, in general, $A_0 J A_1^t \neq 0$. By assumption the variety $\sum_{i<j} p_{ij} A_i J A_j^t = 0$ contains a line tangent to the Plücker quadric $p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12} = 0$ in the point $(0, \ldots, 0, 1)$. Hence the system

$$p_{02} A_0 J A_2^t + p_{03} A_0 J A_3^t + p_{12} A_1 J A_2^t + p_{13} A_1 J A_3^t = 0$$

20
has a nonzero solution \((\tilde{p}_02, \tilde{p}_03, \tilde{p}_12, \tilde{p}_13)\) which gives the line \((0, s\tilde{p}_02, s\tilde{p}_03, s\tilde{p}_12, s\tilde{p}_13, t)\) where \((s, t) \in \mathbb{P}^1\). Notice that this system is the same which gives solutions in the unknowns \((p_01, \ldots p_{13}, p_{23})\) for \(Z_k(E')\) which has now the solutions \((u, s\tilde{p}_02, s\tilde{p}_03, s\tilde{p}_12, s\tilde{p}_13, t)\) where \((u, s, t) \in \mathbb{P}^2\). Therefore, \(Z_k(E')\) is obtained by cutting the Plücker quadric with a plane which, in particular, means that it is a conic. Hence, by Corollary 2.9, \(E'\) is special and by the Lemma \((2.17)\) there exists a nondegenerate \((k + 1) \times (k + 1)\)-Hankel matrix \(K\) such that \(I_k(x_2, x_3) \cdot K = H(0, 0, x_2, x_3)\). Define the morphism

\[
\mathbb{C} \rightarrow Sp(W), \\
t \mapsto S_t
\]

where \(S_t\) is represented by the matrix

\[
S_t = \begin{bmatrix}
Id & tK \\
0 & Id
\end{bmatrix}.
\]

We have

\[
A' \cdot S_t = \begin{pmatrix}
x_0 & x_1 & \cdots & H(0, 0, tx_0 + x_2, tx_1 + x_3) \\
\vdots & \ddots & \ddots & \vdots \\
x_0 & x_1
\end{pmatrix}
\]

Hence

\[
A \cdot S_t = \begin{pmatrix}
x_0 & x_1 & \cdots & H(x_0, x_1, tx_0 + x_2, tx_1 + x_3) \\
\vdots & \ddots & \ddots & \vdots \\
x_0 & x_1
\end{pmatrix}
\]

and \(\mathbb{C} \subseteq Sym^0(E)\) as we wanted.

\[\square\]

**Remark 5.2.** With the notations of the above proof, there is a subgroup \(\lambda\) isomorphic to \(\mathbb{C}^*\) given by \(x_0 \mapsto tx_0, x_1 \mapsto tx_1, x_2 \mapsto t^{-1}x_2, x_3 \mapsto t^{-1}x_3\) such that, by Corollary \((4.9)\) \(\lim_{t \to 0} \lambda(t) \cdot E\) is still a \(k\)-instanton bundle with four-dimensional symmetry group as in \((5)\).

**Theorem 5.3.** Let \(E\) be a \(k\)-instanton bundle on \(\mathbb{P}^3\) such that \(Z_k(E)\) contains two distinct points. Then there exists \(\mathbb{C}^* \subseteq Sym^0(E)\), where \(\mathbb{C}^*\) has a fine action.
Proof. By assumption and [Ski97], there is monad representing $E$ with

$$A = \begin{pmatrix}
  x_0 & x_1 & \cdots & \cdots & H(x_0, x_1, x_2, x_3) \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  x_0 & x_1 & \cdots & \cdots & H(x_0, x_1, x_2, x_3)
\end{pmatrix}$$

and the line $L_1 = \{x_0 = x_1 = 0\} \in Z_k(E)$. Since $Z_k(E)$ contains two distinct points, we can assume that $L_2 = \{x_2 = x_3 = 0\} \in Z_k(E)$. Moreover, from the fact that $h^0(E|L_2) = k+1$ we deduce that there exists a nondegenerate $(k+1) \times (k+1)$-Hankel matrix $M$ such that

$$H(x_0, x_1, 0, 0) = I_k(x_0, x_1) \cdot M.$$ 

Defining

$$S = \begin{bmatrix}
  \text{Id} & -M \\
  0 & \text{Id}
\end{bmatrix}$$

we get

$$A \cdot S = [I_k(x_0, x_1)] - M \cdot I_k(x_0, x_1) + H(x_0, x_1, x_2, x_3)] = [I_k(x_0, x_1)|\tilde{H}(x_2, x_3)]$$

where $\tilde{H}(x_2, x_3)$ is a matrix of linear forms only in $x_2, x_3$. Therefore, there exists $\mathbb{C}^*$ acting on $U$ with weights $-1, 0$ and on $U'$ with weights $0, 0$ such that $\mathbb{C}^* \subset \text{Sym}(E)$ having a fine action, which proves what we want.

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