On Certain Generalizations of Rogers-Ramanujan
Type Identities

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Abstract

We state and prove a number of unilateral and bilateral \(q\)-series identities and explore some of their consequences. Those include certain generalizations of the \(q\)-binomial sum which also generalize the \(q\)-Airy function introduced by Ramanujan, as well as certain identities with an interesting variable-parameter symmetry based on limiting cases of Heine’s transformation of basic hypergeometric functions.

MSC (2010): Primary 33D15 ; Secondary 33D70
Keywords: Basic Hypergeometric Functions, Heine Transform, Ramanujan’s \(q\)-Airy Function, Bilateral \(q\)-series

1 Introduction

The theory of \(q\)-series is well known for a number of fascinating identities with far reaching number theoretic consequences. A famous such example comes from the Rogers-Ramanujan identities

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}},
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}},
\]

where we follow the standard notations for \(q\)-shifted factorials and basic hypergeometric series as in the books [4], [6], [7]. References for the Rogers-Ramanujan identities, their origins and many of their applications are in [1], [2], and [3]. In particular we recall the partition theoretic interpretation of the first Rogers–Ramanujan identity as the partitions of an integer \(n\) into parts \(\equiv 1\) or \(4\) (mod 5) are equinumerous with the partitions

∗Research supported by a QNRF grant from Qatar Foundation NPRP No. : 7-1360-1-254
†Research supported by a QNRF grant from Qatar Foundation NPRP No. : 7-1360-1-254
of \( n \) into parts where any two parts differ by at least 2. The Roger-Ramanujan identities had many extensions and generalizations to different settings. One noteworthy generalization is to extend the identities in (1.2) to evaluate the sum
\[
\sum_{n=0}^{\infty} q^{n^2+mn} (q; q)_n,
\]
m = 0, \pm 1, \pm 2, \cdots , \text{see [5].}
One can view the Rogers-Ramanujan identities as evaluations of Ramanujan’s function defined by
\[
A_q(z) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} (-z)^n.
\]
Namely, (1.1) evaluates \( A_q(-1) \), (1.2) evaluates \( A_q(-q) \).

Another remarkable identity, which is even simpler, is the \( q \)-binomial identity [6, (II.3)]
\[
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.
\]
(1.3)

In recent work by Ismail and Zhang [8], the following function was considered
\[
A_q^{(\alpha)}(a; t) = \sum_{n=0}^{\infty} \frac{(a)_n q^{\alpha n^2} t^n}{(q; q)_n}.
\]
(1.4)
Clearly, \( A_q^{(\alpha)}(a; t) \) specializes to \( A_q(z) \) with \( a = 0, \alpha = 1 \) and \( t = -z \). Furthermore, it specializes to the left hand side of (1.3) when \( \alpha = 0 \). Naturally, one wouldn’t expect a simple closed form identity for such a general function, nonetheless we present in section 2 below some generalizations of identities on \( A_q^{(\alpha)} \) relating different values of the parameter \( \alpha \). We also address similar generalizations to a bilateral analogue of (1.4) related to the Ramanujan \( 1 \psi_1 \) sums [6, (II.29)]
\[
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad \left| \frac{b}{a} \right| < |z| < 1.
\]
(1.5)

In section 3, we consider the series
\[
F(a, c; z) := \sum_{k=0}^{\infty} \frac{(a; q)_k (-1)^k q^{k(k-1)/2} z^k}{(q, c; q)_k},
\]
which can be obtained as a limit of basic hypergeometric series. We then utilize hypergeometric transformations to obtain some infinite and terminating series identities.

2 Unilateral and Bilateral analogues of Rogers-Ramanujan Identities

We start by recalling the following results which were proved in [8].
Lemma 2.1. [8, Lemma 4.1] For nonnegative integer \( j, k, \ell, m, n \) and \( \rho = e^{2\pi i/3} \) we have

\[
\sum_{k=0}^{n} \frac{(a; q)_k (a; q)_{n-k} (-1)^k}{(q; q)_k (q; q)_{n-k}} = \begin{cases} 
0 & n = 2m + 1 \\
\frac{(a^2; q^2)}{(q^2; q^2)_m} & n = 2m 
\end{cases} 
\tag{2.1}
\]

and

\[
\sum_{j+k+\ell = n \atop j, k, \ell \geq 0} \frac{(a; q)_j (a; q)_k (a; q)_{\ell} \rho^{k+2\ell}}{(b; q)_j (b; q)_k (b; q)_\ell} = \begin{cases} 
0 & 3 \nmid n \\
\frac{(a^3; q^3)}{(q^3; q^3)_m} & n = 3m 
\end{cases} 
\tag{2.2}
\]

For \( j, k, m, \ell, n \in F \), we have

\[
\sum_{j+k=n} \frac{(a; q)_j (a; q)_k (-1)^k}{(b; q)_j (b; q)_k} = \begin{cases} 
0 & n = 2m + 1 \\
\frac{(q, b/a; q)_\infty (a^2; q^2)_m}{(-q, b/a, q/a; q)_\infty (b^2; q^2)_m} & n = 2m 
\end{cases} 
\tag{2.3}
\]

and

\[
\sum_{j+k+\ell = n} \frac{(a; q)_j (a; q)_k (a; q)_{\ell} \rho^{k+2\ell}}{(b; q)_j (b; q)_k (b; q)_\ell} = 0
\tag{2.4}
\]

for \( 3 \nmid n \),

\[
\sum_{j+k+\ell = 3m} \frac{(a; q)_j (a; q)_k (a; q)_{\ell} \rho^{k+2\ell}}{(b; q)_j (b; q)_k (b; q)_\ell} = \frac{(q, b/a; q)_\infty (b^2, q^3 a^{-3}; q^3)_\infty (a^3; q^3)_m}{(b, q/a; q)_\infty (q^3, b^3 a^{-3}; q^3)_m (b^3; q^3)_m}.
\tag{2.5}
\]

We start by proving an extension of these results to general primitive roots of unity. In order to make our notation compact; we define for \( r \geq 2 \) and \( n \geq 0 \) sets

\[
C_r(n) := \left\{ (k_1, \ldots, k_r) : k_i \in F \text{ and } \sum_{i=1}^{r} k_i = n \right\}. \tag{2.6}
\]

Also, let \( C^+_r(n) \) be the subset of \( C_r(n) \) whose entries are all nonnegative. Furthermore, we let \( \zeta_r \) denote the primitive \( r \)th root of unity \( e^{2\pi i/r} \).

Lemma 2.2. The following identities hold for \( r \geq 2 \)

\[
\sum_{(k_1, \ldots, k_r) \in C^+_r(n)} \frac{(a; q)_k_1 (a; q)_k_2 \cdots (a; q)_k_r \zeta_r^{\sum_{i=1}^{r} i k_i}}{(q; q)_k_1 (q; q)_k_2 \cdots (q; q)_k_r} = \begin{cases} 
0 & r \nmid n \\
\frac{(a^r; q^r)_m}{(q^r; q^r)_m} & n = rm 
\end{cases} 
\tag{2.7}
\]

and

\[
\sum_{(k_1, \ldots, k_r) \in C_r(n)} \frac{(a; q)_k_1 \cdots (a; q)_k_r (\zeta_r)^{\sum_{i=1}^{r} i k_i}}{(b; q)_k_1 \cdots (b; q)_k_r} = \begin{cases} 
0 & r \nmid n \\
\frac{(q, b/a; q)_\infty (b^r, q^r a^{-r}; q^r)_\infty (a^r; q^r)_m}{(b, q/a; q)_\infty (q^r, b^r a^{-r}; q^r)_m (b^r; q^r)_m} & n = rm 
\end{cases} 
\tag{2.8}
\]
Proof. We start by noting that, for $|t| < 1$ we have
\[
\frac{(at; q)_\infty}{(t; q)_\infty} \frac{(a \zeta_1 t; q)_\infty}{(\zeta_1 t; q)_\infty} \cdots \frac{(a r^{r-1} t; q)_\infty}{(\zeta_1^{r-1} t; q)_\infty} = \frac{(a^r t^r; q^r)}{(t^r; q^r)}, \quad |t| < 1.
\]
Employing (1.3) we see that
\[
\prod_{i=0}^{r-1} \left( \sum_{k_i=0}^{\infty} \frac{(a; q)_{k_i}}{(q; q)_{k_i}} (\zeta_i^r x)^{k_i} \right) = \frac{(a^r x^r; q^r)}{(x^r; q^r)} = \sum_{m=0}^{\infty} \frac{(a^r; q^r)_m}{(q^r; q^r)_m} x^{rm},
\]
and (2.7) follows by comparing the coefficients of $x^n$ in (2.9).

The proof of (2.8) is similar. We start by noting that for $|ba^{-1}| < |x| < 1$, we have
\[
\prod_{i=0}^{r-1} \left( \sum_{k_i=0}^{\infty} \frac{(q, b/a, a \zeta_1^i z, q \zeta_1^{-i}/az; q)_\infty}{(b, q/a, \zeta_1^i z, b \zeta_1^{-i}/az; q)_\infty} \right) = \frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty} \frac{(a^r z^r, q^r a^{-r} z^{-r}; q^r)_\infty}{(z^r, b^{-r} a^{-r} z^{-r}; q^r)_\infty}
\]
applying the Ramanujan \( \psi \) sum (1.5) to that identity establishes (2.8).

Lemma (2.2) enables us to prove the following result generalizing Theorem 4.2 of [8]

**Theorem 2.3.** For $\alpha \geq 0$ and integer $r \geq 2$ we have
\[
A_q^{(\alpha)} (a^r; t^r) = \sum_{k_1, \ldots, k_{r-1}=0} (a; q)_{k_1} \cdots (a; q)_{k_{r-1}} \frac{\zeta_{r-1} \cdots \zeta_1 q^{\alpha (\zeta_{r-1}^i + \cdots + \zeta_1^i)}/t^{\zeta_{r-1}^i + \cdots + \zeta_1^i}}{(q; q)_{k_1} \cdots (q; q)_{k_{r-1}}} A_q^{(\alpha)} \left( a; q^{2\alpha (\zeta_{r-1}^i + \cdots + \zeta_1^i)}/t \right).
\]
(2.9)

\[
A_q^{(\alpha)} (a^r; t^r) = \sum_{k_2, \ldots, k_r=0} (a; q)_{k_2} \cdots (a; q)_{k_r} \frac{\zeta_{r-2} \cdots \zeta_1 q^{\alpha (\zeta_{r-2}^i + \cdots + \zeta_1^i)}/t^{\zeta_{r-2}^i + \cdots + \zeta_1^i}}{(q; q)_{k_2} \cdots (q; q)_{k_r}} A_q^{(\alpha)} \left( a; \zeta_1 q^{2\alpha (\zeta_{r-2}^i + \cdots + \zeta_1^i)}/t \right).
\]
(2.10)

Proof. The proof of (2.9) follows from the following series identities
\[
A_q^{(\alpha)} (a^r; t^r) = \sum_{m=0}^{\infty} \frac{(a^r; q^r)_m}{(q^r; q^r)_m} t^m
\]
\[
= \sum_{n=0}^{\infty} \sum_{\mathbb{C}^+ \cap \langle n \rangle} (a; q)_{k_1} \cdots (a; q)_{k_r} \frac{\zeta_{r-1} \cdots \zeta_1 q^{\alpha \zeta_{r-1}^i + \cdots + \zeta_1^i} t^n}{(q; q)_{k_1} \cdots (q; q)_{k_r}} A_q^{(\alpha)} \left( a; q^{2\alpha (\zeta_{r-1}^i + \cdots + \zeta_1^i)}/t \right).
\]
(2.11)

The proof of (2.10) is almost identical, except that in the last step the inner sum is performed over $k_1$ rather than $k_r$. \(\square\)
Following [8] we also consider the following generalization of the $1\psi_1$ function. For $\alpha \geq 0$, define $B_q^{(\alpha)}$ by

$$B_q^{(\alpha)}(a, b; x) = \sum_{n=-\infty}^{\infty} \frac{(a; q)^n q^{\alpha n^2} x^n}{(b; q)^n}. \quad (2.12)$$

Note that $B_q^{(\alpha)}$ is a bilateral series analogue of $A_q^{(\alpha)}$ (with one additional parameter in fact as a result of the generality afforded by the $1\psi_1$ formula). Again using Lemma (2.2) we obtain the following result.

**Theorem 2.4.** We have

$$B_q^{(\alpha)}(r^r, b^r; x^r) = \frac{(q, b/a; q)^r}{(q, b/a; q)^r} \frac{(b^r, q^r a^{-r}; q^r)_{\infty}}{\left( b^r, q^r a^{-r}; q^r \right)_{\infty}} \sum_{m=-\infty}^{\infty} \frac{(a; q)^m q^{\alpha m^2} x^m}{(b; q)^m} \cdot \quad (2.13)$$

**Proof.** The proof follows from the following series identities

$$\frac{(q, b/a; q)^r}{(q, b/a; q)^r} \frac{(b^r, q^r a^{-r}; q^r)_{\infty}}{\left( b^r, q^r a^{-r}; q^r \right)_{\infty}} \sum_{m=-\infty}^{\infty} \frac{(a; q)^m q^{\alpha m^2} x^m}{(b; q)^m} \cdot \quad (2.14)$$

**Corollary 2.5.** The following family of bilateral Rogers-Ramanujan type identities hold

$$\sum_{n=-\infty}^{\infty} \frac{q^{\alpha n^2} x^n}{1 - a^r q^{an}} = \frac{(q^r; q^r)_{\infty}}{(q; q)_{\infty}} \frac{(a, q/a; q)^r}{(a^r, q^r; q^r; q^r)_{\infty}} \sum_{m=-\infty}^{\infty} \frac{\zeta^{ik_1} q^{(\sum k_1)^2} x^{\sum k_1}}{(1 - a q^{k_1}) \cdot (1 - a q^{k_r})}. \quad (2.15)$$
Proof. Setting $\alpha = 1$ and $b = aq$ and using \((2.13)\) we see that
\[
B_{q^r}(a^r, a^{−r}; x^r) = \frac{(q, q; q)_\infty^r}{(qa, q/a; q)_\infty^r} \frac{(q^r a^r, q^r a^{−r}; q^r)_\infty}{(q^r, q^r; q^r)_\infty} (1 - a^r) \sum_{m=0}^{\infty} \frac{q^{m2} x^m}{(1 - a^r q^m)}
\]
(2.16)
and the result follows on simplification.

\[\square\]

3 Heine Transforms and related identities

We consider the basic hypergeometric function defined by
\[
2\phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| q, z \right) := \sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(c; q)_k (q; q)_k} z^k.
(3.1)
\]

We shall study the function
\[
F(a, c; z) := \sum_{k=0}^{\infty} \frac{(a; q)_k (-1)^k q^{k(k-1)/2} z^k}{(q; q)_k (c; q)_k} = \lim_{b \to \infty} 2\phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| q, \frac{z}{b} \right).
(3.2)
\]

where we have utilized the limit
\[
\lim_{b \to \infty} \frac{(b; q)_k}{b^k} = (-1)^k q^{k(k-1)/2}.
(3.3)
\]

We have the following alternative representation of $F(a, c; z)$

**Theorem 3.1.** For $c, z \neq q^{-m}$ (m nonnegative integer) we have
\[
F(a, c, z) = (z; q)_\infty \sum_{k=0}^{\infty} \frac{(az/c; q)_k (-1)^k q^{k(k-1)/2} c^k}{(q; q)_k (z; q)_k}.
(3.4)
\]

Proof. Recall the Heine transformation [3] (III.2)]
\[
2\phi_1 \left( \begin{array}{c} A, B \\ C \end{array} \right| q, z \right) = \frac{(C/B, Bz; q)_\infty}{(C; z; q)_\infty} 2\phi_1 \left( \begin{array}{c} A Bz/C, Bz \\ z, B \end{array} \right| q, C \right).
(3.5)
\]

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k (-1)^k q^{k(k-1)/2} z^k}{(q; q)_k} = \lim_{b \to \infty} 2\phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| q, \frac{z}{b} \right)
(3.6)
\]

\[
= \lim_{b \to \infty} \frac{(c/b; q)_\infty (z/b; q)_\infty}{(c; q)_\infty (z/b; q)_\infty} 2\phi_1 \left( \begin{array}{c} az/c, b \\ z \end{array} \right| q, \frac{c}{b} \right)
(3.7)
\]

\[
= \frac{(z; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(az/c; q)_k (-1)^k q^{k(k-1)/2} c^k}{(q; q)_k (z; q)_k}.
(3.8)
\]

For the transformation formula to be valid we need neither $c$ nor $z$ to be of the form $q^{-m}$. This proves \((3.3)\).

\[\square\]
Remark. Note that for \( c = z \), both sides of \((3.3)\) are identical. Otherwise the result above provides a way of interchanging the roles of \( c \) and \( z \) in which both sums are very similar except that the right hand side acquires the factor \( \frac{(z;q)_\infty}{(c;q)_\infty} \) and the expression \((a; q)_k\) is replaced by \((az/c; q)_k\).

In the following corollary, we examine a case in which the above Theorem leads to transforming infinite sums into finite ones.

**Corollary 3.2.** Let \( \alpha, \gamma \) be real numbers such that none of \( \alpha + \gamma \) and \( \gamma - n \) are negative integers for any nonnegative integer \( n \) (in particular, \( \gamma \) itself can not be a negative integer). Then the following identity holds

\[
\sum_{k=0}^{\infty} \frac{(q^\alpha; q)_k (-1)^k q^{k(k-1)/2} q^{k(\gamma-n)}}{(q, q^{\alpha + \gamma}; q)_k} = \frac{(q^{\gamma-n}; q)_\infty}{(q^{\alpha + \gamma}; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (-1)^k q^{k(k-1)/2} q^{k(\gamma+n)}}{(q; q)_k (q^{\gamma-n}; q)_k}, \tag{3.5}
\]

**Proof.** This follows directly from \((3.3)\) by setting \( a = q^\alpha \) and \( c = q^\gamma \) to obtain

\[
F(q^\alpha, q^{\alpha + \gamma}; q^{\gamma-n}) = \frac{(q^{\gamma-n}; q)_\infty}{(q^{\alpha + \gamma}; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (-1)^k q^{k(k-1)/2} q^{k(\gamma+n)}}{(q; q)_k (q^{\gamma-n}; q)_k}. \tag{3.6}
\]

A well-known consequence of the \( q \)-binomial theorem \((1.3)\) for \(|x| < 1\) (see p. 490 of [4] for instance)

\[
1 = \frac{1}{(x; q)_n} = \sum_{k=0}^{\infty} \frac{(q; q)_{n+k-1}}{(q; q)_k (q; q)_{n-1}} x^{k}. \tag{3.7}
\]

Combining \((3.7)\) with our results above yields the following evaluation.

**Corollary 3.3.** For \(|x| < 1\) and integer \( n \geq 0 \) we have

\[
\sum_{k=0}^{\infty} \frac{(q; q)_{n+k-1}}{(q; q)_k (q; q)_{n-1}} x^{k} = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (-1)^k q^{k(k-1)/2} q^{k\gamma}}{(q; q)_k (xq^{-n}; q)_k} x^{k}. \tag{3.8}
\]

**Proof.** By setting \( \alpha = 0 \) in \((3.5)\) the left hand side simplifies to 1 since \((1; q)_0 = 1\) and \((1; q)_k = 0\) for \( k \geq 1 \). We thus obtain

\[
1 = \frac{(q^{\gamma-n}; q)_\infty}{(q^{\gamma}; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (-1)^k q^{k(k-1)/2} q^{k\gamma}}{(q; q)_k (q^{\gamma-n}; q)_k}. \tag{3.9}
\]

Note that

\[
\frac{(x; q)_\infty}{(xq^{-n}; q)_\infty} = \frac{1}{(xq^{-n}; q)_n}. \tag{3.10}
\]

Thus, setting \( x = q^{\gamma} \) in \((3.9)\) and applying \((3.7)\) we obtain \((3.8)\). \(\square\)
In fact, we can generalize the argument above to obtain the following family of formulas which exhibits a remarkable symmetry between two finite sums of different lengths.

**Theorem 3.4.** Let \( m, n \) be nonnegative integers, and assume \( x \neq q^l \) for any integer \( l \), then the following holds

\[
\sum_{k=0}^{m} \frac{(q^{-m};q)_k(-1)^kq^{k(k-2m-1)/2}x^k}{(q;x/q^m;q)_k} = \frac{(xq^{-n};q)_{\infty}}{(xq^{-m};q)_{\infty}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(-1)^kq^{k(k-2m-1)/2}x^k}{(q;x/q^n;q)_k}.
\]

**Proof.** This follows by setting \( \alpha = -m \) in (3.5) and noting that \( (q^{-m};q)_k = 0 \) for \( k \geq m + 1 \). It is well known that the expression \( (y;q)_\infty \) with \( |q| < 1 \) converges, and since both sums are now finite there are no convergence concerns, except that we need to guarantee that \( (x/q^m;q)_k \) (as well as \( (x/q^n;q)_k \) and \( (x/q^m;q)_\infty \) are nonzero), which leads to the restrictions placed on \( x \).

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