The rank 1 real Wishart spiked model*

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Abstract

In this paper, we consider $N$-dimensional real Wishart matrices $Y$ in the class $W_R(\Sigma, M)$ in which all but one eigenvalues of $\Sigma$ is 1. Let the non-trivial eigenvalue of $\Sigma$ be $1 + \tau$, then as $N, M \to \infty$, with $M/N = \gamma^2$ finite and non-zero, the eigenvalue distribution of $Y$ will converge into the Marchenko-Pastur distribution inside a bulk region. When $\tau$ increases from zero, one starts to see a stray eigenvalue of $Y$ outside of the support of the Marchenko-Pastur density. As the this stray eigenvalue leaves the bulk region, a phase transition will occur in the largest eigenvalue distribution of the Wishart matrix. In this paper we will compute the asymptotics of the largest eigenvalue distribution when the phase transition occur. We will first establish the results that are valid for all $N$ and $M$ and will use them to carry out the asymptotic analysis. In particular, we have derived a contour integral formula for the Harish-Chandra Itzykson-Zuber integral $\int_{O(N)} e^{tr(XgYg^T)} g^Tdg$ when $X, Y$ are real symmetric and $Y$ is a rank 1 matrix. This allows us to write down a Fredholm determinant formula for the largest eigenvalue distribution and analyze it using orthogonal polynomial techniques. As a result, we obtain an integral formula for the largest eigenvalue distribution in the large $N$ limit characterized by Painlevé transcendents. The approach used in this paper is very different from a recent paper [23], in which the largest eigenvalue distribution was obtained using stochastic operator method. In particular, the Painlevé formula for the largest eigenvalue distribution obtained in this paper is new.

1 Introduction

Let $X$ be an $N \times M$ (throughout the paper, we will assume $M > N$ and $N$ is even) matrix such that each column of $X$ is an independent, identical $N$-variate random variable with normal distribution and zero mean. Let $\Sigma$ be its covariance matrix, i.e. $\Sigma_{ij} = E(X_{i1}X_{j1})$. Then $\Sigma$ is an $N \times N$ positive definite symmetric matrix and we will denote its eigenvalues by $1 + \tau_j$. The matrix $Y$ defined by $Y = \frac{1}{M}XX^T$ is a real Wishart matrix in the class $W_R(\Sigma, M)$ and $M$ is called the degree of freedom of the Wishart matrix. We can think of each column of $X$ as a draw from a $N$-variate random variable with the normal distribution.

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and zero mean, and $Y$ the the sample covariance matrix for the samples represented by $X$. Real Wishart matrices are good models of sample covariance matrices in many situations and have applications in many areas such as finance [49], [41], genetic studies and climate data. (See [48] for example.)

In many of these applications, one has to deal with data in which both $N$ and $M$ are large, while the ratio $M/N$ is finite and non-zero. In this case, the sample covariance matrix $Y$ becomes a poor approximation for the covariance matrix $\Sigma$. However, since it is often reasonable to approximate the data by $N$-variate gaussian random variables, a comparison between the eigenvalue distribution of the sample covariance matrix and Wishart matrices with a known covariance matrix will give a good estimate for the spectrum of the true covariance matrix $\Sigma$. In particular, in applications to principle component analysis, one would like to study the asymptotic behavior of the largest eigenvalue of $Y$ as $N, M \to \infty$ with $M/N \to \gamma^2 \geq 1$ fixed.

For many statistical data with large $N$ and $M$ and $M/N$ finite, it was noted in [48] that the eigenvalue distribution of the sample covariance matrix is well-approximated by the Marchenko-Pastur law [51] inside a bulk region (See [5], [6], [7], [8], [62].)

$$\rho(\lambda) = \frac{\gamma^2}{2\pi \lambda} \sqrt{(\lambda - b_-)(b_+ - \lambda)} \chi_{[b_- b_+]}$$

where $\chi_{[b_- b_+]}$ is the characteristic function for the interval $[b_-, b_+]$ and $b_\pm = (1 \pm \frac{1}{\gamma})^2$. However, outside of the bulk region, there are often a finite number of large eigenvalues at isolated locations. This behavior prompted the introduction of spiked models in [48], which are Wishart matrices with a covariance matrix $\Sigma$ such that only finitely many eigenvalues of $\Sigma$ are different from one. These non-trivial eigenvalues in $\Sigma$ will then be responsible for the spikes that appear in the eigenvalue distribution of the sample covariance matrix $\Sigma$. The number of these non-trivial eigenvalues in $\Sigma$ is called the rank of the spiked model.

Of particular interest is a phase transition that arises in the largest eigenvalue distributions when the first of these spikes starts leaving the bulk region. This phenomenon was first studied in [10] for the complex Wishart spiked model and then in [71] for the rank 1 quaternionic Wishart spiked model. Let $1 + \tau$ be the non-trivial eigenvalue in $\Sigma$, in both cases, it was shown that the phase transition in the largest eigenvalue distribution occurs when $\tau = \gamma^{-1}$. Their results are expressed in terms of the Hastings-McLeod solution of the Painlevé II equation, which is the unique solution to the Painlevé II equation

$$\phi_0''(\zeta) = \zeta \phi_0 + 2 \phi_0^3,$$

with the following asymptotic behavior

$$\phi_0 \sim Ai(\zeta), \quad \zeta \to +\infty,$$

$$\phi_0 = \sqrt{-\zeta} \left(1 + \frac{1}{8\zeta^3} + O(\zeta^{-6})\right), \quad \zeta \to -\infty.$$  

In particular, they have proved the following.
Theorem 1.1. (10 for complex and [72] for quarternions) Let \( W(\Sigma, M, 2) = W_C(\Sigma, M) \) and \( W(\Sigma, M, 4) = W_Q(\Sigma, M) \) be the \( N \)-dimensional complex and quaternionic Wishart matrices with \( M \) degrees of freedom respectively. Suppose all but one eigenvalues of \( \Sigma \) are 1 and the other eigenvalue is \( 1 + \tau \). Then as \( N, M \to \infty \) with \( M/N \to \gamma \geq 1 \) finite, we have

1. For \(-1 < \tau \leq \gamma^{-1}\), the largest eigenvalue distribution is given by

\[
\lim_{M \to \infty} P \left( \left( \lambda_{\text{max}} - (1 + \gamma^{-1})^2 \right) \frac{\gamma M^2}{(1 + \gamma)^3} \leq \zeta \right) = T_{W, \beta}(\zeta)
\]

where \( T_{W, \beta}(\zeta) \) is the Tracy-Widom distribution.

\[
T_{W, 2}(\zeta) = \exp \left( - \int_{\zeta}^{\infty} (y - \zeta) \phi_0(y) dy \right),
\]

\[
T_{W, 4}(\zeta) = \frac{1}{2} \sqrt{T_{W, 2}(\zeta)} \left( e^{-\frac{1}{2} \int_{\zeta}^{\infty} \phi_0(y) dy} + e^{\frac{1}{2} \int_{\zeta}^{\infty} \phi_0(y) dy} \right).
\]

2. For \( \tau = \gamma^{-1} \),

\[
\lim_{M \to \infty} P \left( \left( \lambda_{\text{max}} - (1 + \gamma^{-1})^2 \right) \frac{\gamma M^2}{(1 + \gamma)^3} \leq \zeta \right) = F_{\beta}(\zeta)
\]

where \( F_{\beta}(\zeta) \) is given by

\[
F_{2}(\zeta) = T_{W, 1}^2(\zeta), \quad F_{4}(\zeta) = T_{W, 1}(\zeta), \quad T_{W, 1}(\zeta) = \sqrt{T_{W, 2}(\zeta)} e^{-\frac{1}{2} \int_{\zeta}^{\infty} \phi_0(y) dy}.
\]

3. For \( \tau > \gamma^{-1} \),

\[
\lim_{M \to \infty} P \left( \left( \lambda_{\text{max}} - (\tau + 1) (1 + \gamma^{-2} \tau^{-1}) \right) \frac{\sqrt{\beta M^2}}{(1 + \tau) \sqrt{1 - \gamma^{-2} \tau^{-2}}} \leq \zeta \right) = \text{erf}(\zeta)
\]

where \( \text{erf}(\zeta) \) is the error function.

The functions \( T_{W, \beta} \) are called the Tracy-Widom distributions in the literature and they give the largest eigenvalue distributions for the real \( (\beta = 1) \), complex \( (\beta = 2) \) and quarternionic \( (\beta = 4) \) Wishart ensembles with \( \Sigma = I \), as well as the largest eigenvalue distributions for a large class of random matrix models.

Note that in [10], the phase transition was in fact computed for spiked models of any finite rank. These previous results naturally divide the range of the non-trivial eigenvalue \( 1 + \tau \) into 3 regimes, which are illustrated in Figure [11].
Figure 1: The 3 different regimes. Left: The subcritical regime where the perturbation in \( \Sigma \) is not strong enough to form a spike in the spectrum of \( Y \). Middle: The critical regime where the perturbation is just strong enough to form a spike at the edge of the spectrum. Right: The super-critical regime where a spike has left the spectrum.

1. The subcritical regime: When \(-1 < \tau < \gamma^{-1}\), the perturbation in \( \Sigma \) is not strong enough to form a spike in the eigenvalue distribution of \( Y \) and the largest eigenvalue distribution in the Wishart matrix is not affected by this non-trivial eigenvalue. For real Wishart ensembles, this case was studied in [39] and it was shown that the largest eigenvalue distribution remains the same as the case when \( \Sigma = I \). The result of [39] also applies to much more general sample covariance matrices that are not necessarily Gaussian.

2. The critical regime: When \( 1 - \gamma^{-1}\tau = O(M^{-\frac{2}{3}}) \), the perturbation is just strong enough to form a spike and a phase transition occurs in the largest eigenvalue distribution. The result in [10] for the complex phase transition in fact extends to the whole regime where the authors showed that the largest eigenvalue distribution is described by a function \( F_2(\zeta, w) \) which equals \( TW^2(\zeta) \) when \( w = 0 \). The critical regime for the real case is the subject of this paper. It was also studied recently in [23] using a completely different approach.

3. The super-critical regime: When \( \tau > 1 \), the perturbation in \( \Sigma \) is strong enough to form a spike and the largest eigenvalue is located at the spike instead of the edge of the bulk region. For rank 1 real Wishart spiked model, this was studied in [58]. The results in [58] has also been generalized to a large class of rank 1 spiked perturbed random matrix models in [72].

Despite having the most applications, the phase transition for real Wishart spiked model has not been solved until very recently [23]. The main goal of this paper is to obtain the asymptotic largest eigenvalue distribution for the rank 1 real Wishart spiked model in the critical regime. In a recent paper [23], the asymptotic largest eigenvalue distribution for the rank 1 real Wishart ensemble was obtained by using a completely different approach to ours. In [23], the authors first use the Housefolder algorithm to reduce a Wishart matrix into tridiagonal form. Such tridiagonal matrix is then treated as a discrete random Schrödinger operator and by taking an appropriate scaling limit, the authors obtained a
continuous random Schrödinger operator on the half-line. By doing so, the authors in [23] bypass the problem of determining the eigenvalue j.p.d.f. for the real Wishart ensemble and obtain the largest eigenvalue distribution in the asymptotic limit. In [23], the largest eigenvalue distribution is characterized in several different ways, one of which is the solution of a PDE.

**Theorem 1.2.** (Theorem 1.7 of [23]) Let \( w = \left( \frac{N}{(1+\gamma^{-1})^2} \right)^{\frac{1}{3}} (1 - \gamma \tau) \in (-\infty, \infty), \) then the following boundary value problem

\[
\frac{\partial F}{\partial \zeta} + \frac{2}{\beta} \frac{\partial^2 F}{\partial w^2} + (\zeta - w^2) \frac{\partial F}{\partial w} = 0,
\]

\( F(\zeta, w) \to 1, \quad \zeta, w \to +\infty \) together,

\( F(\zeta, w) \to 0, \quad w \to -\infty, \quad x < x_0 < \infty \)

has a unique solution \( F_\beta(\zeta, w) \) and

\[
\lim_{M \to \infty} P \left( \frac{\lambda_{\text{max}} - (1 + \gamma^{-1})^2}{(1 + \gamma)^\frac{2}{3}} \leq \frac{\gamma M^\frac{4}{3}}{(1 + \gamma)^\frac{2}{3}} \right) = F_\beta(\zeta, w),
\]

where \( \beta = 1, 2 \) and 4 for the real, complex and quarternionic Wishart ensembles respectively.

The distribution also has another characterization which requires more set up to explain. We will refer the readers to [23] for more details. The above result in [23] in fact also extends to the spiked perturbations of Gaussian ensembles in random matrix theory and to their general \( \beta \) analogues, which are defined in [23].

On the other hand, the approach in this paper uses orthogonal polynomial techniques that are closer to those in [10] and [71] and we obtain a characterization of the largest eigenvalue distribution in the critical regime in terms of Painlevé transcendent. It will be very interesting see how these two representations of the largest eigenvalue distribution can be converted into one another. We will now state our results and outline the method used in the paper.

## 2 Statement of result

We will now explain the approach used in this paper.

Let \( \lambda_j \) be the eigenvalues of the Wishart matrix, then the j.p.d.f. for the real Wishart ensemble is given by

\[
P(\lambda) = \frac{1}{Z_{M,N}} |\Delta(\lambda)| \prod_{j=1}^{N} \lambda_j^{N-N-1} \int_{O(N)} e^{-\frac{M}{2} tr(\Sigma^{-1}gYg^{-1})} g^T dg,
\]

(2.1)
where \( g^T dg \) is the Haar measure on \( O(N) \) and \( Z_{M,N} \) is a normalization constant. The j.p.d.f. for complex and quaternionic Wishart ensembles are similar

\[
P_2(\lambda) = \frac{1}{Z_{M,N}} |\Delta(\lambda)|^2 \prod_{j=1}^{N} \lambda_j^{M-N} \int_{U(N)} e^{-M\text{tr}(\Sigma^{-1}g'Yg^{-1})} g^1 dg,
\]

for complex Wishart and

\[
P_4(\lambda) = \frac{1}{Z_{M,N}} |\Delta(\lambda)|^4 \prod_{j=1}^{N} \lambda_j^{2(M-N)+1} \int_{Sp(N)} e^{-2M\text{Re}(\text{tr}(\Sigma^{-1}g'Yg^{-1}))} g^{-1} dg,
\]

for the quaternionic Wishart.

A major difficulty in the asymptotic analysis of the real Wishart ensembles is to find a simple expression for the j.p.d.f. in terms of its eigenvalues. For complex Wishart ensembles, the integral over \( U(N) \) in the expression of the j.p.d.f. can be evaluated using the Harish-Chandra [42] Itzykson Zuber [45] formula to obtain a compact determinantal formula for the j.p.d.f. in terms of the eigenvalues \( \lambda_j \). This has led to the results in [10] and results on the largest eigenvalue distributions for more general complex Wishart ensembles [38], [57]. In the quaternionic case, although the Harish-Chandra and Itzykson Zuber formula does not apply, the integral over \( Sp(N) \) in the j.p.d.f. can still be evaluated using Zonal polynomial expansion to obtain a determinantal formula for the j.p.d.f. in the rank 1 case [71]. In the real case, however, neither of these methods apply and so far there has not been any simple formula for the j.p.d.f. that allows the asymptotic analysis in the real case. Our first result is a contour integral formula that would allow the asymptotic analysis of the rank 1 real Wishart spiked model.

**Theorem 2.1.** Assuming \( N \) is even. Let the non-trivial eigenvalue in the covariance matrix \( \Sigma \) be \( 1 + \tau \) and suppose \( \tau \neq 0 \). Then the j.p.d.f. of the eigenvalues in the rank 1 real Wishart spiked model with covariance matrix \( \Sigma \) is given by

\[
P(\lambda) = \tilde{Z}_{M,N}^{-1} \int_{\Gamma} |\Delta(\lambda)| e^{\frac{M\tau}{2(N+1)}t} \prod_{j=1}^{N} e^{-\frac{N}{2} \lambda_j} \lambda_j^{\frac{M-N-1}{2}} (t - \lambda_j)^{-\frac{1}{2}} dt,
\]

where \( \Gamma \) is a contour that encloses all the points \( \lambda_1, \ldots, \lambda_N \) that is oriented in the counterclockwise direction and \( \tilde{Z}_{M,N} \) is the normalization constant. The branch cuts of the square root \((t - x)^{-\frac{1}{2}}\) is chosen to be the line \( \text{arg}(t - x) = \pi \).

We will present two different proofs of this in the paper. The first one is a geometric proof which involves choosing a suitable set of coordinates on \( O(N) \) and decompose the Haar measure into two parts so that the integral in (2.1) can be evaluated. This will be achieved in Sections 3.1 and 3.2. The second proof is an algebraic proof that uses the Zonal polynomial expansion to verify the formula in Theorem 2.1. This proof will be given in Appendix A where integral formulae of the form (2.2) for the complex and quaternionic Wishart ensembles will also be derived.
Remark 2.1. The integral formula derived here is very similar to a more general formula in [13], in which the matrix integral over $O(N)$ is given by

$$
\int_{O(N)} e^{-\text{tr}(XgYg^{-1})} g^T dg \propto \int e^{\text{tr}(S)} \prod_{j=1}^{N} \det(S - y_j X)\frac{dS}{\sqrt{-1}}
$$

where the integral of $S$ is over $\sqrt{-1}$ times the space of $N \times N$ real symmetric matrices and $y_j$ are the eigenvalues of $Y$. The measure $dS$ is the flat Lebesgue measure on this space. The proof in [13] is also similar to our geometric proof of (2.2) presented in the main text. The main difference between the derivation in [13] and our derivation is that in obtaining (2.2), we use the assumption that $Y$ is a rank 1 matrix to decompose the Haar measure to obtain the simpler formula (2.2) in the rank 1 case.

Remark 2.2. Shortly after the first part of this paper, which contains (2.2) appeared in the preprint server ArXiv, we learnt that D. Wang has also derived (2.2) independently and use it for the asymptotic analysis in the super-critical regime [72].

From the expression of the j.p.d.f., we see that the largest eigenvalue distribution is given by

$$\mathbb{P}(\lambda_{\text{max}} \leq z) = \int_{\lambda_1 \leq \ldots \leq \lambda_N \leq z} \cdots \int P(\lambda) d\lambda_1 \ldots d\lambda_N,$$

$$= \tilde{Z}_{M,N}^{-1} \int_{\Gamma} e^{\frac{M}{2} t} \int_{\lambda_1 \leq \ldots \leq \lambda_N \leq z} \cdots \int |\Delta(\lambda)| \prod_{j=1}^{N} w(\lambda_j) d\lambda_1 \ldots d\lambda_N dt$$

where $w(x)$ is

$$w(x) = e^{-\frac{M}{2} x} x^{\frac{M-N-1}{2}} (t - x)^{-\frac{1}{2}}$$

and $\Gamma$ is a close contour that encloses the interval $[0, z]$.

We can analyze the integrand as in [53], [68], [67] and [69]. By an identity of Brujin [27], we can express the multiple integral as a Pfaffian.

$$\int_{\lambda_1 \leq \ldots \leq \lambda_N \leq z} \cdots \int |\Delta(\lambda)| \prod_{j=1}^{N} w(\lambda_j) d\lambda_1 \ldots d\lambda_N$$

$$= Pf \left( \langle (1 - \chi_{[z,\infty)}) r_j(x), (1 - \chi_{[z,\infty)}) r_k(y) \rangle \right) \right)$$

where $r_j(x)$ is an arbitrary sequence of degree $j$ monic polynomials and $\langle f,g \rangle$ is the skew product

$$\langle f,g \rangle = \int_{0}^{\infty} \int_{0}^{\infty} \epsilon(x-y)f(x)g(y)w(x)w(y)dx dy.$$
upper or lower half plane near \( t \) such that the integral is well defined. As \( \Gamma \) will only intersect \((0, \infty)\) at a point \( x_0 > z \), the left hand side of (2.3), and hence the Pfaffian, will not be affected by such deformations. Then by following the method in \[68, 67\] and \[69\], we can write the Pfaffian as the square root of a Fredholm determinant. Let \( D \) be the moment matrix with entries \( \langle r_j, r_k \rangle \) and \( \det_2 \) be the regularized 2-determinant
\[
\det_2(I + A) = \det((I + A)e^{-A})e^{tr(A_{11} + A_{22})}
\]
for the \( 2 \times 2 \) matrix kernel \( A \) with entries \( A_{ij} \), then we have
\[
Pf \left( \langle (1 - \chi_{[z, \infty)}(x)r_j(x), (1 - \chi_{[z, \infty)}(y)r_k(y) \rangle_1 \right) = \sqrt{\det(D(t)) \det_2(I - \chi K \chi)}
\]
where \( \chi = \chi_{[z, \infty)} \) and \( K \) is the operator whose kernel is given by
\[
K(x, y) = \begin{pmatrix}
S_1(x, y) & -\frac{\partial}{\partial y} S_1(x, y) \\
IS_1(x, y) & S_1(y, x)
\end{pmatrix}
\] (2.7)
and \( S_1(x, y) \) and \( IS_1(x, y) \) are the kernels
\[
S_1(x, y) = -\sum_{j,k=0}^{N-1} r_j(x) w(x) \mu_{jk} \epsilon(r_k w)(y),
\] (2.8)
\[
IS_1(x, y) = -\sum_{j,k=0}^{N-1} \epsilon(r_j w)(x) \mu_{jk} \epsilon(r_k w)(y)
\]
and \( \mu_{jk} \) is the inverse of the matrix \( D \). As shown in \[73\], the kernel can now be expressed in terms of the Christoffel Darboux kernel of some suitable orthogonal polynomials, together with a correction term which gives rise to a finite rank perturbation to the Christoffel Darboux kernel. In this paper, we introduce a new proof of this using skew orthogonal polynomials and their representations as multi-orthogonal polynomials. By using ideas from \[2\] to write skew orthogonal polynomials in terms of orthogonal polynomials, we can express the skew orthogonal polynomials with respect to the weight \( w(x) \) in terms of a sum of Laguerre polynomials. Let \( \pi_{k,1} \) be the monic skew orthogonal polynomial of degree \( k \) with respect to the weight \( w(x) \).
\[
\langle \pi_{2k+1,1}, y^j \rangle_1 = \langle \pi_{2k,1}, y^j \rangle_1 = 0, \quad j = 0, \ldots, 2k - 1.
\] (2.9)
Then we can write these down in terms of Laguerre polynomials.

**Proposition 2.1.** Let \( L_k \) be the degree \( k \) monic Laguerre polynomial with respect to the weight \( w_0(x) \)
\[
\int_0^\infty L_k(x) L_j(x) w_0(x) dx = \delta_{jk} h_{j,0}, \quad w_0(x) = x^{M-N} e^{-Mx}.
\]
If \( (L_{2k-1}, L_{2k-2})_1 \neq 0 \), then the skew orthogonal polynomials \( \pi_{2k,1} \) and \( \pi_{2k+1,1} \) both exist and \( \pi_{2k,1} \) is unique while \( \pi_{2k+1,1} \) is unique up to an addition of a multiple of \( \pi_{2k,1} \). Moreover,
we have \( \langle L_{2k}, L_{2k-1} \rangle_1 = 0 \) and the skew orthogonal polynomials are given by

\[
\begin{align*}
\pi_{2k,1} &= L_{2k} - \frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1}, \\
\pi_{2k+1,1} &= L_{2k+1} - \frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1} + \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-2} + c_1 \pi_{2k,1},
\end{align*}
\]

where \( c \) is an arbitrary constant.

Next, by representing skew orthogonal polynomials as multi-orthogonal polynomials and write them in terms of the solution of a Riemann-Hilbert problem as in [63], we can apply the results of [28] and [9] to express the kernel \( S_1(x, y) \) as a finite rank perturbation of the Christoffel Darboux kernel of the Laguerre polynomials.

**Theorem 2.2.** Suppose \( \langle L_{N-1}, L_{N-2} \rangle_1 \langle L_{N-3}, L_{N-4} \rangle_1 \neq 0 \). Let \( S_1(x, y) \) be defined by (2.8) and choose the sequence of monic polynomials \( r_j(x) \) such that \( r_j(x) \) are arbitrary degree \( j \) monic polynomials that are independent on \( t \) and \( r_j(x) = \pi_{j,1}(x) \) for \( j = N-2, N-1 \). Then we have

\[
S_1(x, y) - K_2(x, y) = \epsilon \left( \pi_{N+1,1} w \right)(y) \left( \frac{0}{M} \frac{M}{2\lambda_{N-2,0}} \frac{M}{2\lambda_{N-1,0}} \right) \left( L_{N-2}(x) \right) w(x)
\]

where \( K_2(x, y) \) is the kernel of the Laguerre polynomials

\[
K_2(x, y) = \left( \frac{y(t-y)}{x(t-x)} \right)^{\frac{1}{2}} \frac{1}{w_0^\frac{1}{2}}(x) w_0^\frac{1}{2}(y) \frac{L_N(x)L_{N-1}(y) - L_N(y)L_{N-1}(x)}{h_{N-1,0}(x-y)}
\]

Note that the correction term on the right hand side of (2.10) is the kernel of a finite rank operator. Its asymptotics can be computed using the known asymptotics of the Laguerre polynomials and the method in [63], [31] and [33]. The actual asymptotic analysis of this correction term, however, is particularly tedious as one would need to compute the asymptotics of the skew orthogonal polynomials up to the third leading order term due to cancelations. To compute the contribution from the determinant \( \det \mathbb{D} \), we derive the following expression for the logarithmic derivative of \( \det \mathbb{D} \).

**Proposition 2.2.** Let \( \mathbb{D} \) be the moment matrix with entries \( \langle r_j, r_k \rangle_1 \), where the sequence of monic polynomials \( r_j(x) \) is chosen such that \( r_j(x) \) are arbitrary degree \( j \) monic polynomials that are independent on \( t \) and \( r_j(x) = \pi_{j,1}(x) \) for \( j = N-2, N-1 \). Then the logarithmic derivative of \( \det \mathbb{D} \) with respect to \( t \) is given by

\[
\frac{\partial}{\partial t} \log \det \mathbb{D} = - \int_{\mathbb{R}_+} \frac{S_1(x, x)}{t-x} \, dx.
\]
This then allows us to express the largest eigenvalue distribution $P(\lambda_{\text{max}} \leq z)$ as an integral of determinant.

**Theorem 2.3.** The largest eigenvalue distribution of the rank 1 real Wishart ensemble can be written in the following integral form.

$$P(\lambda_{\text{max}} \leq z) = C \int_{\Gamma} \exp \left( \frac{M \tau t}{2(1+\tau)} - \frac{1}{2} \int_{c_0}^{t} \int_{\mathbb{R}^+} \frac{S_1(x, x)}{s-x} \, dx \, ds \right) \sqrt{\det_2(I - \chi K \chi)} \, dt.$$ (2.13)

for some constant $c_0$ and $K$ is the operator with kernel given by (2.8). The integration contour $\Gamma$ is a close contour that encloses the interval $[0, z]$ in the anti-clockwise direction.

These results so far are valid for all $N$ and $M$ and are exact. As $N, M \to \infty$, the integral expression (2.13) can be used for the asymptotic analysis to obtain the largest eigenvalue distribution when the phase transition occurs. In this paper, we will demonstrate the asymptotic analysis for the case where $M/N \to \gamma^2 = 1$, but the analysis can also be applied to the case of any finite $\gamma$.

In the asymptotic limit, the $t$ integral in (2.13) can be computed using steepest descent analysis. In fact, we shall see that when $t \neq b_+$, the integrand in (2.13) is of order

$$\exp \left( \frac{M \tau t}{2(1+\tau)} - \frac{1}{2} \int_{c_0}^{t} \int_{\mathbb{R}^+} \frac{S_1(x, x)}{s-x} \, dx \, ds \right) \sqrt{\det_2(I - \chi K \chi)} \sim C e^{\frac{M \tau t}{2(1+\tau)} - \frac{1}{2} \int_{b_+}^{t} \rho(s) \log(t-s) \, ds},$$

where $\rho(s)$ is given in (1.1). The saddle point $t_{\text{saddle}}$ of this equation does not belong to the bulk region $[b_-, b_+]$ unless $\tau$ is at the critical value $\tau_c = \gamma^{-1}$. When $\tau < \tau_c$, the steepest descent contour $\Gamma$ can be deformed such that it does not intersect $[b_-, b_+]$. In this case, $t$ will always be of finite distance from $[b_-, b_+]$ and the factor $(t-x)^{-1/2}$ in $w(x)$ will have no effect on the asymptotics of the kernel $S_1$ at the edge point $b_+$. In this case, the kernel at the edge point will be given by the Airy kernel.

$$\lim_{N \to \infty} \frac{1 + \gamma}{\gamma M^{\frac{4}{3}}} S_1(x, y) = \frac{Ai(\xi_1)Ai'(\xi_2) - Ai(\xi_2)Ai'(\xi_1)}{\xi_1 - \xi_2} + \frac{1}{2} Ai(\xi_1) \int_{-\infty}^{\xi_2} Ai(s) \, ds$$

where $\xi_1 = (x - b_+) \frac{\gamma M^{\frac{4}{3}}}{(1+\gamma)^{\frac{4}{3}}}$ and $\xi_2 = (y - b_+) \frac{\gamma M^{\frac{4}{3}}}{(1+\gamma)^{\frac{4}{3}}}$ are finite.

In the critical case, however, the main contribution to the contour integral (2.13) comes from a small neighborhood of $t = b_+$ and the factor $(t-x)^{-1/2}$ in $w(x)$ will now significantly alter the behavior of the kernel $S_1$ and change it from the Airy kernel to into a more complicated kernel. This gives rise to the phase transition and a new distribution function. Our next result is the representation of this new distribution function in terms of the Hastings-McLeod solution of the Painlevé II equation. First let us define some functions that will appear in our formula.
Let $\psi(u, T) = (T - u)\frac{1}{2}$ and $H_j(u, T) = A \psi^{(j)}(u) \psi^{-1}(u, T)$ and let $S_{10}$, $S_{11}$ be

\[
S_{10}(T) = \int_{-\infty}^{\infty} H_1(u, s) du + \int_{-\infty}^{\infty} \left( \frac{\int_{-\infty}^{\infty} H_1(u, s) du}{\int_{-\infty}^{\infty} H_0(u, s) du} \right)^2 ds - \frac{2}{3} T^{\frac{3}{2}},
\]
\[
S_{11}(T) = 2 \int_{-\infty}^{0} \left( H_1 \int_{u}^{\infty} H_2 dv - \frac{(-u)^{\frac{1}{2}}}{2\pi(T - u)} \right) du
+ 2 \int_{0}^{\infty} H_1 \int_{u}^{\infty} H_2 dv du - \int_{-\infty}^{\infty} H_1 du \int_{-\infty}^{\infty} H_2 du.
\]

Define the function $S(T)$ to be

\[
S(T) = S_{10}(T) + \int_{0}^{T} S_{11}(s) ds - 1/2 \int_{0}^{\infty} (S_{11,+}(s) - S_{11,-}(s)) ds,
\tag{2.14}
\]

where the contour of integration in the first term remains in the upper half plane and $S_{11,\pm}$ are the boundary values of $S_{11}$ as it approaches the real axis in the upper/lower half plane.

Now let $U$ be the matrix

\[
U = \begin{pmatrix}
0 & 0 & 0 & -\psi \phi_0 \\
0 & 0 & \psi^{-1} \phi_0 & -\psi^{-1} \phi_0 \\
0 & -\sigma / \phi_0 & \frac{\partial}{\partial \xi} \log (\phi_0 / \psi) & 0 \\
-\phi_0 \psi^{-1} & 0 & \frac{1}{\psi \sigma} & 0
\end{pmatrix}
\]

and define $\vec{h}_j$ to be the vector $\vec{h}_j = \left( 0, 0, 0, \frac{\phi_j}{\psi} \right)^T$ for $j = 0, 1, 2$ and $\vec{h}_j = 0$ for $j = 3$ and $j = 4$, where $\phi_0$ is the Hastings-McLeod solution of Painlevé II (1.3) and

\[
\sigma(\xi) = \int_{\xi}^{\infty} \phi_0^2 d\xi, \quad \phi_1 = \phi'_0 + \sigma \phi'_0,
\]
\[
\phi_2 = \left( \xi + \int_{\xi}^{\infty} \phi_0(\xi) \phi_1(\xi) d\xi \right) \phi_0 - \sigma \phi_1.
\]

Let $\vec{v}_j$ be the vector that satisfies the linear system of ODEs with the following boundary condition

\[
\frac{\partial \vec{v}_j}{\partial \xi} = U(\xi) \vec{v}_j + \vec{h}_j, \quad \vec{v}_j \sim \left( 0, 0, 0, \int_{-\infty}^{\infty} H_j du \right), \quad \xi \to +\infty, \quad j = 0, 1, 2,
\]
\[
\vec{v}_3 \sim (0, 0, -1, 0), \quad \xi \to -\infty, \quad \vec{v}_4 \sim (0, 0, -1, 0), \quad \xi \to +\infty
\tag{2.15}
\]

Then the largest eigenvalue distribution at the phase transition is given by
Theorem 2.4. Suppose $N$ is even. Let $w = \left(\frac{N}{\pi}\right)^{\frac{1}{3}} (1 - \tau) \in (-\infty, \infty)$ and let $\zeta = (z - 4) (N/4)^{\frac{2}{3}}$, then as $N, M \to \infty$ such that $M/N \to \gamma^2 = 1$, the largest eigenvalue distribution at the phase transition is given by

$$\lim_{N \to \infty} \mathbb{P} \left( \lambda_{\max} - 4 \left(\frac{N}{4}\right)^{\frac{2}{3}} \leq \zeta \right) = C \sqrt{T} W_2(\zeta) \text{Im} \left( \int_{\Xi^+} e^{-\frac{w^2}{2} - \frac{1}{2} s} \left( \int_{-\infty}^{\infty} H_0 du \right)^{\frac{1}{2}} \times \left( \det (\delta_{jk} - (\alpha_j, \beta_k))_{1 \leq j, k \leq 3} \right)^{\frac{1}{2}} dT \right),$$

for some constant $C$, where $\Xi^+$ is a contour in the upper half plane that does not contain any zero of $\int_{-\infty}^{\infty} H_0 du$ and approaches $\infty$ in the sector $\pi/3 < \arg T < \pi$. It intersects $\mathbb{R}$ at the point $\zeta$. The entries in the $3 \times 3$ matrix are given by

$$\begin{align*}
(\alpha_1, \beta_1) &= \frac{v_{34} - v_{33} + 1}{2}, \quad (\alpha_1, \beta_2) = \frac{v_{32}}{2}, \quad (\alpha_1, \beta_1) = \psi^2 W(v_{33}, \phi_1 \psi^{-1})/2\sigma, \\
(\alpha_2, \beta_1) &= \frac{T}{2} q_0 - B_1 q_1 - q_2 - B_2 R_-, \quad (\alpha_3, \beta_1) = \frac{1}{2} q_1 + B_1 q_0 \\
(\alpha_2, \beta_j) &= \frac{T}{2} v_{j-2,2} - B_1 u_{-1,1-j} - u_{-1,2-j-2} - B_2 P_{-j-2}, \quad j = 2, 3, \\
(\alpha_3, \beta_j) &= \frac{1}{2} u_{-1,1-j} + B_1 v_{j-2,2},
\end{align*}$$

where $W(f, g)$ is the Wronskian $W(f, g) = fg' - gf'$ and $v_{jk}$ are the components of the vectors $v_j$ in (2.13). The functions $B_1, B_2, q_j, R_-, P_{-j}$ and $u_{-jk}$ are given by

$$\begin{align*}
q_j &= v_{j4} - v_{j3} - \int_{-\infty}^{\infty} H_j du, \quad u_{-jk} = \psi^2 W(v_{j3}, \phi_k \psi^{-1})/\sigma, \quad j = 1, 2, \\
R_- &= v_{44} - v_{43} + 1, \quad \tilde{P}_{-0} = v_{42}, \quad \tilde{P}_{-1} = \psi^2 W(v_{43}, \phi_1 \psi^{-1})/\sigma, \\
B_1 &= -\frac{1}{2} \int_{-\infty}^{\infty} H_1(u) du + \frac{1}{2} \int_{-\infty}^{\infty} H_0(u) du, \\
B_2 &= -\frac{B_1}{2} - \frac{T}{4} \int_{-\infty}^{\infty} H_0 du + \frac{B_1}{2} \int_{-\infty}^{\infty} H_1 du + \frac{1}{2} \int_{-\infty}^{\infty} H_2 du.
\end{align*}$$

The distribution in Theorem 2.4 is expressed in terms of integrals of the Airy function, together with the functions $v_{jk}$, which are solutions of linear ODEs with known boundary conditions. The coefficients of the ODEs satisfied by the $v_{jk}$ are given in terms of the Hastings-McLeod solutions and its derivatives and are therefore known functions.

Remark 2.3. The distribution in Theorem 2.4 can also be expressed in terms of an integral involving the solution of a Riemann-Hilbert problem. (see Appendix B) With the recent advancement in the numerical computation of Riemann-Hilbert problems [57], this representation may be useful for the numerical computation of the distribution in Theorem 2.4.
2.1 Other results

In obtaining the main result in Theorem 2.4, we have obtained some other results which may also be of interest to mathematicians and physicists working in random matrix theory.

2.1.1 Random matrix with external source

One application of the orthogonal polynomial approach developed in this paper is in the studies of random matrix with a rank 1 external source. Random matrix with external source are random matrix models on the space of real symmetric, Hermitian or Hermitian self-dual \( N \times N \) matrices with the following probability measure

\[
P(Y) dY = \frac{1}{Z} e^{-Mtr(V(Y) - AY)},
\]

for some real-valued function \( V(x) \) such that \( e^{-V(x)} \) decays fast enough as \( x \to \pm \infty \) and a real symmetric, Hermitian or Hermitian self-dual matrix \( A \). The function \( V(x) \) is usually called the potential while the matrix \( A \) is known as the external source. The real Wishart ensemble can be thought of as a special case when \( V(x) = \frac{x^2}{2} - \frac{M - N - 1}{2M} \log x \). Random matrices with external source are first studied by Brézin and Hikami [25], [26] and P. Zinn-Justin [74], [75] as a model of systems with both random and deterministic parts. For Hermitian random matrices, the external source model can be studied using multi-orthogonal polynomial and Riemann-Hilbert techniques [18] and there are many recent advancements in the asymptotic analysis of Hermitian external source models with non-Gaussian potential \( V(x) \) [12], [14], [15], [19], [20], [21], [22]. In [72], the contour integral formula (2.2) was derived independently and was used to study real symmetric random matrix with a rank 1 external source and a large class of potential \( V(x) \). By using the linear statistics results of Johansson [47] and a representation for the largest eigenvalue distribution that is different to (2.13), the largest eigenvalue distribution was obtained in the super-critical regime. The orthogonal polynomial approach developed in this paper can be used to extend the results in [72] to the critical regime. In fact, for real random matrix with a rank 1 external source, the expression (2.13) for the largest eigenvalue distribution remains valid, although the kernels \( S_1 \) and \( K \) will have to be modified according to the potential. For a polynomial potential \( V(x) \), the analogue of (2.10), which expresses the kernel \( S_1 \) in terms of orthogonal polynomials is well-known [73]. By using the asymptotics of orthogonal polynomials found in [36] and [37], such representation can be used to compute the asymptotics of the kernel \( S_1 \), which can then be used in (2.13) to obtain the largest eigenvalue distribution.

2.1.2 Orthogonal ensembles

A large part of this paper involves the analysis of an orthogonal ensemble with weight (2.4) and in doing so, we have obtained some new results for orthogonal ensembles.
The first of these results is the logarithmic derivative of the partition function \( Z_1 = \det D \) in Proposition 2.2. While the proposition is stated in the terms of the derivative of the parameter \( t \) in \( w(x) \), the proof can easily be generalized to obtain the logarithmic derivative of \( \det D \) for a general orthogonal ensemble with respect to any parameter.

\[
\frac{\partial}{\partial t} \log \det D = 2 \int_{\mathbb{R}_+} S_1(x, x) \partial_t \log w dx, \quad w(x) = e^{-NV(x)}.
\] (2.16)

For polynomial potential, the asymptotics of \( S_1(x, x) \) can be obtained through the asymptotics of orthogonal polynomials found in [36], [37]. This could then be used to compute the asymptotics of the partition function \( Z_1 = \det D \). (For unitary ensembles, this was done in [17] for the quartic potential \( V(x) = x^4 + tx^2 \) using a different differential identity.)

Asymptotic analysis of the partition function is of importance in extending the universality results of orthogonal and symplectic ensembles to general weights \( w \). At the moment, universality in the orthogonal and symplectic ensembles are proven for a large class of potentials \( V(x) \) [31], [33], [63], [60], [61]. However, except for the quartic case \( V(x) = x^4 + tx^2 \), all the available results are restricted to the case where the limiting eigenvalue distribution is supported on a single interval. The main obstacle in extending these results to more general potential is the computation of

\[
\det Q = \left( \frac{Z_{N,4}(w)Z_{2N,1}(w)}{2^N N!Z_{2N,2}(w)} \right)^2
\]

in the limit \( N \to \infty \), where \( Z_{n,\beta}(w) \) is the partition functions of the ensembles (See remark 2.4 of [63] and remark 1.5 of [31]).

In order to proof the universality in orthogonal and symplectic ensembles using the method in [31], [32] and [63], one needs to show that \( \lim_{N \to \infty} \det Q \neq 0 \). While the leading order asymptotics of these partition functions can be found using the estimates in [47], their combined contributions to \( \det Q \) cancel in the leading order and hence higher order terms in the asymptotics of \( Z_{n,\beta}(w) \) are needed to show that \( \lim_{N \to \infty} \det Q \neq 0 \). At the moment, asymptotics for the sub-leading order terms of partition functions are only available for \( \beta = 2 \). By using a differential identity for \( \log Z_{N,2} \), the authors in [17] computed the sub-leading order terms in \( Z_{N,2} \) for the potential \( V(x) = x^4 + tx^2 \) as \( N \to \infty \). A combination of the method in [17] and the differential identity (2.16) may provide a way to compute the sub-leading order terms in the partition function \( Z_{N,1} \) and help extend the universality results in orthogonal and symplectic ensembles to more general potentials.

Another interesting observation is Corollary 4.1, in which we showed that \( \langle L_k, L_{k-1} \rangle_1 = 0 \) whenever \( k \) is even. This turns out to be a very useful identity in the analysis of the phase transition when the double scaling limit \( t - 4 = T (4/N)^{\frac{1}{2}} \) for the ensemble with
weight (2.4) has to be considered. When analyzing this double scaling limit, the identity 
\[ \langle L_k, L_{k-1} \rangle_1 = 0 \] leads to cancelation in the leading order terms of the kernel \( S_1 \). This enables us to show that \( S_1 \) is of order \( N^\frac{2}{3} \) instead of \( N^\frac{1}{3} \), which is essential for the scaled limit of the determinant \( \det_2 (I - \chi K \chi) \) to exist. We believe this type of identity will also be useful in the analysis of other double scaling limits in orthogonal ensembles.

The paper is organized as follows. In Section 3 we will prove the contour integral formula (2.2) and in Section 4 the Christoffel-Darboux formula (2.10) for the kernel \( S_1 \) will be derived. In Section 5 we will prove the differential identity for the moment matrix \( D \) in Proposition 2.2. The results in these sections are all exact and apply to all \( N \) and \( M \).

We will start the asymptotic analysis in Section 6 in which the asymptotics for the kernel \( S_1 \), \( \det D \) and \( \det_2 (I - \chi K \chi) \) will be obtained. Finally, we will express the asymptotics of the determinant \( \det_2 (I - \chi K \chi) \) in terms of the Painlevé transcendents in Section 7.

Throughout the paper, we shall assume that \( N \) is even and that \( M - N > 0 \).

3 Contour integral formula for the j.p.d.f.

In this section we will prove the integral formula for the j.p.d.f. in Theorem 2.1.

3.1 Haar measure on \( SO(N) \)

In this section, we will find a convenient set of coordinate on \( O(N) \) to evaluate the integral

\[
\int_{O(N)} e^{-\frac{M}{2} tr(\Sigma^{-1} g Y g^{-1})} g^T dg
\]

that appears in the expression of the j.p.d.f. (2.1). As both \( \Sigma^{-1} \) and \( Y \) are symmetric matrices, they can be diagonalized by matrices in \( O(N) \). We can therefore replace both \( \Sigma^{-1} \) and \( Y \) by the diagonal matrices \( \Sigma_d^{-1} \) and \( \Lambda_d \).

\[
\Sigma_d^{-1} = \text{diag} \left( \frac{1}{1 + \tau_1}, \ldots, \frac{1}{1 + \tau_N} \right), \quad \Lambda_d = \text{diag} (\lambda_1, \ldots, \lambda_N)
\]

The group \( O(N) \) has two connected components, \( SO(N) \) and \( O_-(N) \) that consists of orthogonal matrices that have determinant 1 and -1 respectively. Let \( T \) be the matrix

\[
T = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & I_{N-2}
\end{pmatrix}
\]
then the left multiplication by $T$ defines an diffeomorphism from $O_-(N)$ to $SO(N)$. In particular, we can write the integral over $O(N)$ in (2.1) as

$$I(\Sigma, \Lambda) = \int_{O(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g,$$

$$= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g + \int_{O_-(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g,$$

$$= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g + \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} T g \Lambda d g^{-1} T^{-1})} g^T d g,$$

$$= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g + \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\tilde{\Sigma}_d^{-1} g \Lambda d g^{-1})} g^T d g,$$

where $\tilde{\Sigma}_d$ is the diagonal matrix with the first two entries of $\Sigma_d$ swapped.

$$\tilde{\Sigma}_d^{-1} = \text{diag} \left( \frac{1}{1 + \tau_2}, \frac{1}{1 + \tau_1}, \ldots, \frac{1}{1 + \tau_N} \right).$$

Note that $g^T d g$ is also the Haar measure on $SO(N)$.

As we are considering the rank 1 spiked model, we let $\tau_1 = \ldots = \tau_{N-1} = 0$ and $\tau_N = \tau$. Therefore $\tilde{\Sigma}_d = \Sigma_d$ and we have

$$I(\Sigma, \Lambda) = 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g,$$

(3.1)

Let $g_{ij}$ be the entries of $g \in SO(N)$. Then the integral $I$ can be written as

$$I(\Sigma, \Lambda) = 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g \Lambda d g^{-1})} g^T d g,$$

$$= 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} - I_N) g \Lambda d g^{-1})} e^{-\frac{M}{2} \text{tr}(g \Lambda d g^{-1})} g^T d g,$$

$$= 2 \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{SO(N)} e^{\tau M \sum_{j=1}^{N} \lambda_j g_{jN}^2} g^T d g,$$

We will now find an expression of the Haar measure and use it to compute the integral $I(\Sigma, \Lambda)$.

First let us define a set of coordinates on $SO(N)$ that is convenient for our purpose. We will then express the Haar measure on $SO(N)$ in terms of these coordinates.

An element $g \in SO(n)$ can be written in the following form

$$g = (\vec{g}_1, \ldots, \vec{g}_n), \quad |\vec{g}| = 1, \quad \vec{g}_i \cdot \vec{g}_j = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

This represents $SO(N)$ as the set of positively oriented orthonormal frames in $\mathbb{R}^N$ whose coordinate axis are given by the vectors $\vec{g}_i$. As the vector $\vec{g}_N$ is a unit vector, we can write
its components as

\[ g_{1N} = \cos \phi_1, \quad g_{jN} = \prod_{k=1}^{j-1} \sin \phi_k \cos \phi_j, \quad j = 2, \ldots, n - 1, \tag{3.2} \]

\[ g_{NN} = \prod_{k=1}^{N-1} \sin \phi_k \]

The remaining vectors \( \vec{g}_1, \ldots, \vec{g}_{N-1} \) form an orthonormal frame with positive orientation in a copy of \( \mathbb{R}^{N-1} \) that is orthogonal to \( \vec{g}_N \). Therefore the set of vectors \( \vec{g}_1, \ldots, \vec{g}_{N-1} \) can be identified with \( SO(N - 1) \). To be precise, let \( \vec{u} \) be a unit vector in \( \mathbb{R}^N \) and let \( G(\vec{u}) \in SO(N) \) be a matrix that maps \( \vec{u} \) to the vector \( (0, \ldots, 0, 1)^T \). As \( G \) is orthogonal, we have

\[ G(\vec{g}_N)\vec{g}_j = (v_{j1}, \ldots, v_{j,N-1}, 0)^T, \quad j < N \tag{3.3} \]

In particular, the matrix \( V \) with entries \( v_{ij} \) for \( 1 \leq i, j \leq N - 1 \) is in \( SO(N - 1) \). The following then gives a set of coordinates on \( SO(N) \).

\[ g = (\vec{g}_N, V). \tag{3.4} \]

In the above equation, \( \vec{g}_N \) is identified with the coordinates \( \phi_j \) in (3.2), while the matrix \( V \) is identified with the coordinates in \( SO(N - 1) \) that correspond to \( \vec{g}_N \). In terms of these coordinates, the left action of an element \( S \in SO(N) \) on \( g \) is given by the following.

\[ Sg = (SG(\vec{g}_1), \ldots, SG(\vec{g}_{N-1}), SG(\vec{g}_N))^T = (SG(\vec{g}_N)^{-1}v_1, \ldots, SG(\vec{g}_N)^{-1}v_{N-1}, SG(\vec{g}_N))^T \]

Then as in (3.3), we have

\[ G(S\vec{g}_N)SG(\vec{g}_N)^{-1}\vec{v}_j = (\tilde{v}_{j1}, \ldots, \tilde{v}_{j,N-1}, 0)^T. \]

The matrix \( \tilde{V} \) with entries \( \tilde{v}_{ij} \) are again in \( SO(N-1) \), therefore the matrix \( G(S\vec{g}_N)SG(\vec{g}_N)^{-1} \) is of the form

\[ G(S\vec{g}_N)SG(\vec{g}_N)^{-1} = \begin{pmatrix} \tilde{S}_{N-1} & \tilde{s} \\ 0 & s_N \end{pmatrix} \tag{3.5} \]

From the fact that \( G(S\vec{g}_N)SG(\vec{g}_N)^{-1} \) is an orthogonal matrix, it is easy to check that \( \tilde{s} = 0 \) and \( s_N = \pm 1 \). To determine \( s_N \), let us consider the action of \( G(S\vec{g}_N)SG(\vec{g}_N)^{-1} \) on \( (0, 0, \ldots, 1)^T \). We have

\[ G(S\vec{g}_N)SG(\vec{g}_N)^{-1}(0, 0, \ldots, 1)^T = G(S\vec{g}_N)S\vec{g}_N = (0, 0, \ldots, 1)^T \]

Therefore \( s_N = 1 \) and \( \tilde{S}_{N-1} \) is in \( SO(N-1) \). The action of \( S \) on \( g \) is therefore given by

\[ Sg = (S\vec{g}_N, \tilde{S}_{N-1}V). \tag{3.6} \]
We will now write the Haar measure on $SO(N)$ in terms of the coordinates (3.4). These coordinates give a local diffeomorphism between $SO(N)$ and $S^{N-1} \times SO(N-1)$ as $\tilde{g}_N \in S^{N-1}$ and $V \in SO(N-1)$. Let $dX$ be a measure on $S^{N-1}$ that is invariant under the action of $SO(N)$ and $VTdV$ be the Haar measure on $SO(N-1)$, then the following measure
\[ dH = dX \wedge VTdV, \]
is invariant under the left action of $SO(N)$. Let $S \in SO(N)$, then its action on the point $(\tilde{g}_N,V)$ is given by (3.6), where $\tilde{S}_{N-1}$ depends only on the coordinates $\phi_1, \ldots, \phi_{N-1}$.

Therefore under the action of $S$, the measure $dH$ becomes
\[ dH \rightarrow dX \wedge V^T \tilde{S}_{N-1}^T \tilde{S}_{N-1} \tilde{S}_{N-1} \tilde{S}_{N-1} dV = dX \wedge VTdV, \tag{3.7} \]
as $dX$ is invariant under the action of $S$. Therefore if we can find a measure on $S^{N-1}$ that is invariant under the action of $SO(N)$, then $dX \wedge VTdV$ will give us a left invariant measure on $SO(N)$. Since the left invariant measure on a compact group is also right invariant, this will give us the Haar measure on $SO(N)$. As the metric on $S^{N-1}$ is invariant under the action of $SO(N)$, it is clear that the volume form on $S^{N-1}$ is invariant under the action of $SO(N)$. Let $dX$ be the volume form on $S^{N-1}$, then from (3.7), we see that the measure $dX \wedge VTdV$ is invariant under the action of $SO(N)$.

**Proposition 3.1.** Let $dX$ be the volume form on $S^{N-1}$ given by
\[ dX = \sin^{N-2}(\phi_1) \sin^{N-1}(\phi_2) \ldots \sin(\phi_{N-2}) \wedge_{j=1}^{N-1} d\phi_j \]
in terms of the coordinates $\phi_1, \ldots, \phi_{N-1}$ in (3.2) and (3.4), then the Haar measure on $SO(N)$ is equal to a constant multiple of
\[ dH = dX \wedge VTdV, \]
where $VTdV$ is the Haar measure on $SO(N-1)$ in terms of the coordinates (3.4).

We can now compute the integral $I(\Sigma, \Lambda)$.

### 3.2 Integral formula

By using the expression of the Haar measure derived in the last section, we can now write the integral $I(\Sigma, \Lambda)$ as
\[ I(\Sigma, \Lambda) = 2 \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{SO(N)} e^{\frac{M}{2} \sum_{j=1}^{N} \lambda_j \tilde{g}_j} g^Tdg, \]
\[ = 2 \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{SO(N-1)} VTdV \int_{S^{N-1}} e^{\frac{M}{2} \sum_{j=1}^{N} \lambda_j \tilde{g}_j} dX, \]
\[ = 2C \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{S^{N-1}} e^{\frac{M}{2} \sum_{j=1}^{N} \lambda_j \tilde{g}_j} dX, \]
for some constant $C$, where the $N - 1$ sphere $S^{N-1}$ in the above formula is defined by
$
\sum_{j=1}^{N} g_{jN}^2 = 1$
and $dX$ is the volume form on it. If we let $g_{jN} = x_j$, then the above can be written as

$$
I(\Sigma, \Lambda) = 2C \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{\mathbb{R}^N} e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} \delta \left( \sum_{j=1}^{N} x_j^2 - 1 \right) dx_1 \ldots dx_N. \tag{3.8}
$$

This can be seen most easily by the use of polar coordinates in $\mathbb{R}^N$, which are given by

$$
x_1 = r \cos \phi_1, \quad x_j = r \prod_{k=1}^{j-1} \sin \phi_k \cos \phi_j, \quad j = 2, \ldots, N - 1,
$$

$$
x_N = r \prod_{k=1}^{N-1} \sin \phi_k.
$$

Then the volume form in $\mathbb{R}^N$ is given by

$$
dx_1 \ldots dx_N = r^{N-1} \sin^{N-2} \phi_1 \ldots \sin \phi_{N-2} dr d\phi_1 \ldots d\phi_{N-1}
$$

Therefore in terms of polar coordinates, we have

$$
\int_{\mathbb{R}^N} e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} \delta \left( \sum_{j=1}^{N} x_j^2 - 1 \right) dx_1 \ldots dx_N
$$

$$
= \int_0^\pi d\phi_1 \int_0^{2\pi} d\phi_2 \ldots \int_0^{2\pi} d\phi_{N-1} \int_0^\infty dr \delta \left( \sum_{j=1}^{N} r^2 - 1 \right) r^{N-1}
$$

$$
\times e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} \sin^{N-2} \phi_1 \ldots \sin \phi_{N-2}
$$

$$
= \int_{S^{N-1}} e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} dX.
$$

To compute the integral $I(\Sigma, \Lambda)$, we use a method in the studies of random pure quantum systems (see, e.g. [52]). The idea is to consider the Laplace transform of the function $I(\Sigma, \Lambda, t)$ defined by

$$
I(\Sigma, \Lambda, t) = 2C \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{\mathbb{R}^N} e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} \delta \left( \sum_{j=1}^{N} x_j^2 - t \right) dx_1 \ldots dx_N,
$$

then $I(\Sigma, \Lambda, 1) = I(\Sigma, \Lambda)$. The Laplace transform of $I(\Sigma, \Lambda, t)$ in the variable $t$ is given by

$$
\int_0^\infty e^{-st} I(\Sigma, \Lambda, t) dt = 2C \prod_{j=1}^{N} e^{-\frac{M}{2} \lambda_j} \int_{\mathbb{R}^N} e^{\frac{M}{2(1 + \tau)} \sum_{j=1}^{N} \lambda_j x_j^2} \delta \left( \sum_{j=1}^{N} x_j^2 - t \right) dx_1 \ldots dx_N
$$

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Then, provided \( \text{Re}(s) > \max_j (\lambda_j) \), the integral can be computed explicitly to obtain

\[
\int_0^\infty e^{-st} I(\Sigma, \Lambda, t) dt = 2C \prod_{j=1}^N e^{-\frac{\tau M}{2} \lambda_j} \left(s - \frac{\tau M}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}}
\]

Taking the inverse Laplace transform, we obtain an integral expression for \( I(\Sigma, \Lambda) \).

\[
I(\Sigma, \Lambda) = \frac{C}{\pi i} \int_{\Gamma} e^s \prod_{j=1}^N e^{-\frac{\tau M}{2} \lambda_j} \left(s - \frac{\tau M}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}} ds,
\]

where \( \Gamma \) is a contour that encloses all the points \( \frac{\tau M}{2(1+\tau)} \lambda_1, \ldots, \frac{\tau M}{2(1+\tau)} \lambda_N \) that is oriented in the counter-clockwise direction. Rescaling the variable \( s \) to \( s = Mt \), we obtain

\[
I(\Sigma, \Lambda) \propto \int_{\Gamma} e^{\frac{Mt}{2(1+\tau)}} \prod_{j=1}^N e^{-\frac{\tau M}{2} \lambda_j} (t - \lambda_j)^{-\frac{1}{2}} dt,
\]

This then give us an integral expression for the j.p.d.f. in Theorem 2.1.

For the purpose of computing the largest eigenvalue distribution \( P(\lambda_{\text{max}} \leq z) \), we can assume that the eigenvalues are all smaller than or equal to a constant \( z \).

### 4 Skew orthogonal polynomials and the kernel \( S_1 \)

In this section we will prove the representation of the kernel \( S_1 \) in Theorem 2.2. We will do so by using the multi-orthogonal polynomial representation of skew orthogonal polynomials in [70] and then apply the Christoffel-Darboux formula for multi-orthogonal polynomials in [28] to write the kernel \( S_1 \) as a finite sum of multi-orthogonal polynomials. We then simplify this sum further by using a result in [9]. This gives a new proof to a well-known result of Widom [73].

#### 4.1 Skew orthogonal polynomials

As explain in the introduction, we need to find the skew orthogonal polynomials with the weight (2.4). Let us consider the skew orthogonal polynomials with respect to the weight \( w \) in (2.4). We shall use the ideas in [2] to express the skew orthogonal polynomials in terms of a linear combinations of Laguerre polynomials.

Let \( Q_j(x) \) to be the degree \( j + 2 \) polynomial

\[
Q_j(x) = \frac{d}{dx} \left(x^{j+1}(t - x)w(x)\right) w^{-1}(x), \quad j \geq 0.
\]

Then as we assume \( M - N > 0 \), it is easy to see that

\[
\langle f(x), H_j(y) \rangle_1 = \langle f(x), x^j \rangle_2, \quad (4.1)
\]
for any \( f(x) \) such that \( \int_0^\infty f(x)w(x)dx \) is finite, where the product \( \langle \rangle_2 \) is defined by
\[
\langle f(x)g(x) \rangle_2 = \int_0^\infty f(x)g(x)w_0(x)dx, \quad w_0(x) = x^{M-N}e^{-Mx}.
\] (4.2)

Note that \( w_0(x) \) is not the square of \( w(x) \). The fact that \( w_0(x) \) is the weight for the Laguerre polynomials allows us to express the skew orthogonal polynomials for the weight \([2.4]\) in terms of Laguerre polynomials.

In particular, this implies that the conditions \([2.9]\) is equivalent to the following conditions
\[
\langle \pi_{2k,1}, y^j \rangle_1 = 0, \quad j = 0, 1,
\]
\[
\langle \pi_{2k,1}, y^j \rangle_2 = 0, \quad j = 0, \ldots, 2k - 3.
\] (4.3)

and the exactly same conditions for \( \pi_{2k+1,1}(x) \). In particular, the second condition implies the skew orthogonal polynomials can be written as
\[
\pi_{2k,1}(x) = L_{2k}(x) + \gamma_{2k,1}L_{2k-1}(x) + \gamma_{2k,2}L_{2k-2}(x),
\]
\[
\pi_{2k+1,1}(x) = L_{2k+1}(x) + \gamma_{2k+1,0}L_{2k}(x) + \gamma_{2k+1,1}L_{2k-1}(x) + \gamma_{2k+1,2}L_{2k-2}(x),
\]
where \( L_j(x) \) are the degree \( j \) monic Laguerre polynomials that are orthogonal with respect to the weight \( w_0(x) \).

\[
L_n(x) = \frac{(-1)^n e^{Mx}x^{-M-N}}{M^n} \frac{d^n}{dx^n} \left( e^{-Mx}x^{n-M-N} \right),
\]
\[
= x^n - \frac{(M-N+n)n}{M}x^{n-1} + O(x^{n-2}).
\] (4.4)

The constants \( \gamma_{k,j} \) are to be determined from the first condition in \([4.3]\). We will now show that if \( \langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0 \), then the skew orthogonal polynomials \( \pi_{2k,1} \) and \( \pi_{2k+1,1} \) exist and that \( \pi_{2k,1} \) is unique. First let us show that the first condition in \([4.3]\) is equivalent to
\[
\langle \pi_{2k,1}, L_{2k-j} \rangle_1 = 0, \quad j = 1, 2.
\]

To do this, we will first define a map \( \varrho_N \) from the span of \( L_{2k-1} \) and \( L_{2k-2} \) to the span of \( y \) and \( 1 \).

Let \( P(x) \) be a polynomial of degree \( m \). Then we can write the polynomial \( P(x) \) as
\[
P(x) = \frac{d}{dx} (q(x)x(t-x)w(x))w^{-1}(x) + R(x)
\] (4.5)

where \( q(x) \) is a polynomial of degree \( m - 2 \) and \( R(x) \) is a polynomial of degree less than or equal to 1. By writing down the system of linear equations satisfied by the coefficients of \( q(x) \) and \( R(x) \), we see that the polynomials \( q(x) \) and \( R(x) \) are uniquely defined for any given \( P(x) \). In particular, the map \( f : P(x) \mapsto R(x) \) is a well-defined linear map from the space of polynomial to the space of polynomials of degrees less than or equal to 1. Let \( \varrho_k \) be the following restriction of this map.
Definition 4.1. For any polynomial $P(x)$, let $f$ be the map that maps $P(x)$ to $R(x)$ in (4.5). Then the map $\varrho_k$ is the restriction of $f$ to the linear subspace spanned by the orthogonal polynomials $L_k, L_{k-1}$.

We then have the following.

**Lemma 4.1.** If $\langle L_k, L_{k-1} \rangle_1 \neq 0$, then the map $\varrho_k$ is invertible.

**Proof.** Suppose there exists non-zero constants $a_1$ and $a_2$ such that

$$a_1 L_k + a_2 L_{k-1} = \frac{d}{dx}(q(x)x(t-x)w)w^{-1}$$

for some polynomial $q(x)$ of degree $k - 2$, then by taking the skew product $\langle \rangle_1$ of this polynomial with $L_k$, we obtain

$$a_2 \langle L_{k-1}, L_k \rangle_1 = \langle a_1 L_k + a_2 L_{k-1}, L_k \rangle_1 = \langle q(x), L_k \rangle_2 = 0.$$  

As $q(x)$ is of degree $k - 2$. Since $\langle L_{k-1}, L_k \rangle_1 \neq 0$, this shows that $a_2 = 0$. By taking the skew product with $L_{k-1}$, we conclude that $a_1 = 0$ and hence the map $\varrho_k$ has a trivial kernel.

In particular, we have the following.

**Corollary 4.1.** If $k$ is even, then $\langle L_k, L_{k-1} \rangle_1 = 0$.

**Proof.** Let $q(x)$ be a polynomial of degree $k - 2$ that satisfies the following conditions

$$\int_{\mathbb{R}_+} \frac{d}{dx}(q(x)w_4(x))x^j w_4(x)dx = 0, \quad j = 0, \ldots, k - 2, \quad (4.6)$$

where $w_4(x) = x^{\frac{k-j-1}{2}}(t-x)^j e^{-\frac{M}{M+1}}$. A non trivial polynomial $q(x)$ of degree $k-2$ that satisfies these conditions exists if and only if the moment matrix with entries $\int_{\mathbb{R}_+} \frac{d}{dx}(x^j w_4(x))x^j w_4(x)dx$ has a vanishing determinant. For even $k$, the moment matrix is of odd dimension and antisymmetric and hence its determinant is always zero.

Assuming $k$ is even and let $q(x)$ be a polynomial that satisfies (4.6). By taking the inner product $\langle \rangle_2$ with $x^j$, we see that there exists non-zero constants $a_1$ and $a_2$ such that

$$a_1 L_k + a_2 L_{k-1} = \frac{d}{dx}(q(x)x(t-x)w)w^{-1},$$

Therefore by Lemma 4.1, we see that if $k$ is even, we will have $\langle L_k, L_{k-1} \rangle_1 = 0$.  

Lemma 4.1 shows that if $\langle L_i, L_{i-1} \rangle_1 \neq 0$, then there exist two independent polynomials $R_0(y)$ and $R_1(y)$ in the span of $y$ and 1 such that $R_j(y) = \varrho_i(L_{i-j})$. Then we have

$$R_j(y) = -\frac{d}{dy}(q_j(y)y(t-y)w)w^{-1} + L_{i-j}(y), \quad j = 0, 1.$$
In particular, the skew product of $R_j(y)$ with $L_{i-1}$, $l < 2$ is given by

$$\langle L_{i-1}(x), R_j(y) \rangle_1 = -\langle L_{i-1}, q_j \rangle_2 + \langle L_{i-1}, L_{i-j} \rangle_1.$$  

As $q_j$ is a polynomial of degree less than or equal to $i - 2$ and $l < 2$, the first term on the right hand side is zero. Therefore we have

$$\langle L_{i-1}(x), R_j(y) \rangle_1 = \langle L_{i-1}, L_{i-j} \rangle_1, \quad l < 2, \quad j = 0, 1. \quad (4.7)$$

We can now show that the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ exist if $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$.

**Proposition 4.1.** If $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$, then the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ both exist and $\pi_{2k,1}$ is unique while $\pi_{2k+1,1}$ is unique up to an addition of a multiple of $\pi_{2k,1}$. Moreover, we have $\langle L_{2k}, L_{2k-1} \rangle_1 = 0$ and the skew orthogonal polynomials are given by

$$\pi_{2k,1} = L_{2k} - \frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1},$$

$$\pi_{2k+1,1} = L_{2k+1} - \frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1} + \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-2} + c\pi_{2k,1}, \quad (4.8)$$

for $k \geq 2$, where $c$ is an arbitrary constant.

**Proof.** Let $\pi_{2k,1}$ and $\pi_{2k+1,1}$ be polynomials defined by

$$\pi_{2k,1}(x) = L_{2k}(x) + \gamma_{2k,1} L_{2k-1}(x) + \gamma_{2k,2} L_{2k-2}(x),$$

$$\pi_{2k+1,1}(x) = L_{2k+1}(x) + \gamma_{2k+1,1} L_{2k-1}(x) + \gamma_{2k+1,2} L_{2k-2}(x),$$

for some constants $\gamma_{j,k}$. If we can show that $\langle \pi_{2k-1,1}, y^j \rangle_1 = 0$ for $j = 0, 1$ and $l = -1, 0$, then $\pi_{2k-1,1}$ will be the skew orthogonal polynomial. Let $R_0$ and $R_1$ be the images of $L_{2k-1}$ and $L_{2k-2}$ under the map $\phi_{2k-1}$. Then by the assumption in the Proposition, they are independent in the span of $y$ and 1. Therefore the conditions $\langle \pi_{2k-1,1}, y^j \rangle_1 = 0$ are equivalent to $\langle \pi_{2k-1,1}, R_j(y) \rangle_1 = 0$. By taking $i = 2k - 1$ in (4.7), we see that this is equivalent to $\langle \pi_{2k-1,1}, L_{2k-1-j} \rangle_1 = 0$. This implies

$$\langle \pi_{2k,1}, L_{2k-1} \rangle_1 = \langle L_{2k}, L_{2k-1} \rangle_1 + \gamma_{2k,2} \langle L_{2k-2}, L_{2k-1} \rangle_1 = 0,$$

$$\langle \pi_{2k,1}, L_{2k-2} \rangle_1 = \langle L_{2k}, L_{2k-2} \rangle_1 + \gamma_{2k,1} \langle L_{2k-1}, L_{2k-2} \rangle_1 = 0.$$

Hence we have

$$\gamma_{2k,1} = -\frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}, \quad \gamma_{2k,2} = \frac{\langle L_{2k}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1},$$

which exist and are unique as $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$. This determines $\pi_{2k,1}$ uniquely. By Corollary 4.1, we have $\langle L_{2k}, L_{2k-1} \rangle_1 = 0$ and hence $\gamma_{2k,2} = 0$. Similarly, the coefficients for $\pi_{2k+1,1}$ are

$$\gamma_{2k+1,1} = -\frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}, \quad \gamma_{2k+1,2} = \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}.$$
Again, these coefficients exist and are unique. However, as \( \langle \pi_{2k,1}, \pi_{2k,1} \rangle_1 = 0 \) and \( \langle \pi_{2k,1}, y^j \rangle_1 = 0 \) for \( j = 0, \ldots, 2k - 1 \), adding any multiple of \( \pi_{2k,1} \) to \( \pi_{2k+1,1} \) will not change the orthogonality conditions \( \langle \pi_{2k+1,1}, y^j \rangle_1 = 0 \) that is satisfied by \( \pi_{2k+1,1} \) and hence \( \pi_{2k+1,1} \) is only determined up to the addition of a multiple of \( \pi_{2k,1} \).

4.2 The Christoffel Darboux formula for the kernel

In [59], skew orthogonal polynomials were interpreted as multi-orthogonal polynomials and represented as the solution of a Riemann-Hilbert problem. This representation allows us to use the results in [28] to derive a Christoffel-Darboux formula for the kernel (2.8) in terms of the Riemann-Hilbert problem.

Let us recall the definitions of multi-orthogonal polynomials. First let the weights \( w_0, w_1 \) and \( w_2 \) be

\[
w_0(x) = x^{M-N} e^{-Mx}, \quad w_l(x) = w(x) \int_{\mathbb{R}^+} e(x - y)L_{N-l-2}(y)w(y)dy, \quad l = 1, 2.
\]

Note that the weights \( w_l(x) \) are defined with the polynomials \( L_{N-l-3} \) and \( L_{N-l-4} \) instead of \( L_{N-l-1} \) and \( L_{N-l-2} \). This is because the construction below involves the polynomial \( \pi_{N-2,1} \) as well as \( \pi_{N,1} \). By taking \( i = N - 3 \) in (4.7), we see that the orthogonality conditions for \( \pi_{N,1} \) is also equivalent to

\[
\langle \pi_{N,1}, x^j \rangle_2 = 0, \quad j = 0, \ldots, N - 3, \quad \langle \pi_{N,1}, L_{N-j} \rangle_2 = 0, \quad j = 3, 4,
\]

provided \( \langle L_{N-3}, L_{N-4} \rangle_1 \) is also non-zero.

Then the multi-orthogonal polynomials of type II \( P_{N,l}^{II}(x) \), \[3, \[4, \[18, \[40] \] \] \] \] are polynomials of degree \( N - 1 \) such that

\[
\int_{\mathbb{R}^+} P_{N,l}^{II}(x)x^j w_0(x)dx = 0, \quad 0 \leq j \leq N - 3,
\]

\[
\int_{\mathbb{R}^+} P_{N,l}^{II}(x)w_m(x)dx = -2\pi i\delta_{lm}, \quad l, m = 1, 2. \quad (4.9)
\]

**Remark 4.1.** More accurately, these are in fact the multi-orthogonal polynomials with indices \((N - 2, \vec{e}_i)\), where \( \vec{e}_1 = (0, 1) \) and \( \vec{e}_2 = (1, 0) \).

We will now define the multi-orthogonal polynomials of type I. Let \( P_{N,l}^{I}(x) \) be a function of the following form

\[
P_{N,l}^{I}(x) = B_{N,l}(x)w_0 + \sum_{k=1}^2 (\delta_{kl}x + A_{N,k,l})w_k, \quad (4.10)
\]

where \( B_{N,l}(x) \) is a polynomial of degree \( N - 3 \) and \( A_{N,k,l} \) is independent on \( x \). Moreover, let \( P_{N,l}^{I}(x) \) satisfy

\[
\int_{\mathbb{R}^+} P_{N,l}^{I}(x)x^j dx = 0, \quad j = 0, \ldots, N - 1.
\]
Then the polynomials $B_{N,l}(x)$ and $\delta_{kl}x + A_{N,k,l}$ are multi-orthogonal polynomials of type I with indices $(N,2,1)$ for $l = 1$ and $(N,1,2)$ for $l = 2$. We will now show that $P_{N,1}^{II}$ and $P_{N,2}^{II}$ exist and are unique if both $\langle L_{N-1}, L_{N-2} \rangle_1$ and $\langle L_{N-3}, L_{N-4} \rangle_1$ are non-zero.

**Lemma 4.2.** Suppose both $\langle L_{N-1}, L_{N-2} \rangle_1$ and $\langle L_{N-3}, L_{N-4} \rangle_1$ are non-zero, then the polynomials $P_{N,1}^{II}$ and $P_{N,2}^{II}$ exist and are unique.

**Proof.** Let us write $L_{N-1}$ as

$$L_{N-1} = \frac{d}{dx} \left( q_l(x)(t - x)w \right) w^{-1} + R_l(x), \quad l = 1, \ldots, 4.$$ 

for some polynomials $q_l$ of degree $N - l - 2$ and $R_l$ of degree 1, then by Lemma 4.1, we see that both $q_{N-1}$ and $q_{N-3}$ are invertible and hence the composition $\varphi_{N-1}^{-1} \varphi_{N-3}$ is also invertible. In particular, there exists a $2 \times 2$ invertible matrix with entries $c_{l,k}$ and polynomials $\hat{q}_l$ of degree $N - 3$ such that

$$L_{N-1} = \frac{d}{dx} \left( \hat{q}_l(x)(t - x)w \right) w^{-1} + c_{l-1,1}L_{N-1} + c_{l-2,2}L_{N-2}, \quad l = 3, 4.$$ 

Then by the first condition in (4.9), we see that

$$\int_{R_+} P_{N,l}^{II}w_j(x)dx = \left\langle P_{N,l}^{II}, c_{l,1}L_{N-1} + c_{l,2}L_{N-2} \right\rangle_1 = -2\pi i \delta_{ij}, \quad l = 1, 2, \quad j = 1, 2. \quad (4.11)$$

As the matrix with entries $c_{ij}$ is invertible, we see that the linear equations (4.11) has a unique solution in the linear span of $L_{N-1}$ and $L_{N-2}$ if and only if $\langle L_{N-1}, L_{N-2} \rangle_1 \neq 0$. \quad \square

As we shall see, existence and uniqueness of $P_{N,l}^{II}$ would imply that the multi-orthogonal polynomials of type I also exist and are unique. As in [59], the multi-orthogonal polynomial together with the skew orthogonal polynomials form the solution of a Riemann-Hilbert problem. Let $Y(x)$ be the matrix

$$Y(x) = \left( \begin{array}{cccc}
\pi_{N,1}(x) & C(\pi_{N,1}w_0) & C(\pi_{N,1}w_1) & C(\pi_{N,1}w_2) \\
\kappa \pi_{N-2,1}(x) & \cdot & \cdot & \cdot \\
1_{N,1}(x) & \cdot & \cdot & \cdot \\
2_{N,2}(x) & \cdot & \cdot & \cdot
\end{array} \right), \quad (4.12)$$

where $\kappa$ is the constant

$$\kappa = -\frac{2\pi i}{\langle \pi_{N-2,1}, \pi_{N-3} \rangle_2} = -\frac{4\pi i}{M h_{N-1,1}}, \quad h_{2j-1,1} = \langle \pi_{2j-2,1}, \pi_{2j-1,1} \rangle_1. \quad (4.13)$$

and $C(f)$ is the Cauchy transform

$$C(f)(x) = \frac{1}{2\pi i} \int_{R_+} \frac{f(s)}{s - x}ds. \quad (4.14)$$
Then by using the orthogonality conditions of skew orthogonal polynomials and multi-orthogonal polynomials, together with the jump discontinuity of the Cauchy transform, one can check that $Y(x)$ satisfies the following Riemann-Hilbert problem.

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$,
2. $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_0 & w_1(z) & w_2(z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $z \in \mathbb{R}_+$,
3. $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^N \\ z^{-N+2} \\ z^{-1} \\ z^{-1} \end{pmatrix}$, $z \to \infty$.

Similarly, multi-orthogonal polynomials of type I can also be arranged to satisfy a Riemann-Hilbert problem. Let $\epsilon$ be the operator $\epsilon(f)(x) = \frac{1}{2} \int_0^\infty \epsilon(x-y)f(y)dy.$ (4.16)

First note that the functions $\psi_j(x) = \epsilon(\pi_{j,1} w)$ for $j \geq 2$ can be express in the form of (4.10). By Lemma 4.1, we can write $\pi_{j,1}(x)$ as

$$\pi_{j,1}(x) = \frac{d}{dx}(B_j(x)(t-x)w)w^{-1} + A_{j,1}L_{N-3} + A_{j,2}L_{N-4},$$ (4.17)

where $B_j(x)$ is a polynomial of degree $j - 2$.

Let $X(z)$ be the matrix value function defined by

$$X(z) = \begin{pmatrix} -\frac{\epsilon M}{2}C(\psi_{N-2}w) & \frac{\epsilon M}{2}B_{N-2} & \frac{\epsilon M}{2}A_{N-2,1} & \frac{\epsilon M}{2}A_{N-2,2} \\ -\frac{\epsilon M}{2}C(\psi_Nw) & \frac{\epsilon M}{2}B_N & \frac{\epsilon M}{2}A_{N,1} & \frac{\epsilon M}{2}A_{N,2} \\ -C(P_{N,1}^1) & \cdots & \cdots \\ -C(P_{N,2}^1) & \cdots & \cdots \end{pmatrix}. \tag{4.18}$$

Then by using the orthogonality and the the jump discontinuity of the Cauchy transform, it is easy to check that $X^{-T}(z)$ and $Y(z)$ satisfy the same Riemann-Hilbert problem and hence the multi-orthogonal polynomials of type I also exist and are unique.

We will now show that the kernel $S_1(x,y)$ given by (2.8) can be expressed in terms of the matrix $Y(z)$.

**Proposition 4.2.** Suppose $\langle L_{N-3}, L_{N-4} \rangle_1 \langle L_{N-1}, L_{N-2} \rangle_1 \neq 0$ and let the kernel $S_1(x,y)$ be

$$S_1(x,y) = -\sum_{j,k=0}^{N-1} r_j(x)w(x)\mu_{jk}\epsilon(r_k w)(y), \tag{4.19}$$
where \( r_j(x) \) is an arbitrary degree \( j \) monic polynomial for \( j < N - 2 \) and \( r_j(x) = \pi_j(x) \) for \( j \geq N - 2 \). The matrix \( \mu \) with entries \( \mu_{jk} \) is the inverse of the matrix \( \mathbb{D} \) whose entries are given by

\[
(\mathbb{D})_{jk} = \langle r_j, r_k \rangle_1, \quad j, k = 0, \ldots, N - 1. \quad (4.20)
\]

Then the kernel \( S_1(x,y) \) exists and is equal to

\[
S_1(x,y) = \frac{w(x)w^{-1}(y)}{2\pi i(x-y)} \begin{pmatrix} 0 & w_0(y) & w_1(y) & w_2(y) \end{pmatrix} Y_{+1}^{-1}(y) Y_+^{-1}(x) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \quad (4.21)
\]

**Proof.** First note that, since \( \langle L_{N-1}, L_{N-2} \rangle_1 \neq 0 \) and \( \langle L_{N-3}, L_{N-4} \rangle_1 \neq 0 \), the skew orthogonal polynomials \( \pi_{N-l,1} \) exist for \( l = -1, \ldots, 2 \). In particular, the moment matrix \( \hat{\mathbb{D}} \) with entries

\[
(\hat{\mathbb{D}})_{jk} = \langle x^j, y^k \rangle_1, \quad j, k = 0, \ldots, N - 1
\]

is invertible. Since the polynomials \( \pi_{N-j,1} \) exist for \( j = -1, \ldots, 2 \), the sequence \( r_k(x) \) and \( x^k \) are related by an invertible transformation. Therefore the matrix \( \mathbb{D} \) in (4.20) is also invertible. As the matrix \( \mathbb{D} \) is of the form

\[
\mathbb{D} = \begin{pmatrix} \mathbb{D}_0 & 0 \\ 0 & h_{N-1,1} \mathcal{J} \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

where \( \mathbb{D}_0 \) has entries \( \langle r_j, r_k \rangle_1 \) for \( j, k \) from 0 to \( N - 3 \). From this, we see that the matrix \( \mu_{jk} \) is of the form

\[
\mu = \begin{pmatrix} \mathbb{D}_0^{-1} & 0 \\ 0 & -h_{N-1,1}^{-1} \mathcal{J} \end{pmatrix}.
\quad (4.22)
\]

As in [28], let us now expand the functions \( x r_j(x) \) and \( x \epsilon(r_j w) \).

\[
x r_j(x) = \sum_{k=0}^{N-1} c_{jk} r_k(x) + \delta_{N-1,j} \pi_{N,1}(x),
\]

\[
x \epsilon(r_j w)(x) = \sum_{k=0}^{N-1} d_{jk} \epsilon(r_k w) + d_{j,N} \psi_N + d_{j,N+1} \frac{P^I_{N,1}(x)}{w(x)} + d_{j,N+2} \frac{P^I_{N,2}(x)}{w(x)}.
\quad (4.23)
\]

Then the coefficients \( c_{jk} \) and \( d_{jk} \) for \( j, k = 0, \ldots, N - 1 \) are given by

\[
c_{jk} = \sum_{l=0}^{N-1} \langle x r_j, r_l \rangle_1 \mu_{lk}, \quad d_{jk} = \sum_{l=0}^{N-1} \mu_{kl} \langle x r_l, r_k \rangle_1.
\]

Therefore if we let \( \mathcal{C} \) be the matrix with entries \( c_{jk}, j, k = 0, \ldots N - 1 \) and \( \mathbb{D} \) be the matrix with entries \( d_{jk} \) for \( j, k = 0, \ldots N - 1 \), then we have

\[
\mathcal{C} = \mathbb{D}_1 \mu, \quad \mathbb{D} = (\mu \mathbb{D}_1)^T,
\quad (4.24)
\]

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where $\mathbb{D}_1$ is the matrix with entries $(xr_j, r_k)_1$ for $j, k = 0, \ldots, N - 1$. From (4.23), we obtain

$$(y - x)S_1(x, y) = \sum_{j=0}^{N-1} \pi_{N,1}(x)w(x)\mu_{N-1,j} \epsilon (r_jw)(y)$$

$$- \sum_{j,k=0}^{N-1} r_j(x)w(x)\mu_{jk} \left( d_{k,N}\psi_N(y) + d_{k,N+1} \frac{P_{N,1}^I(y)}{w(y)} + d_{k,N+2} \frac{P_{N,2}^I(y)}{w(y)} \right)$$

$$+ r^T(x)w(x) \left( C^T \mu - \mu D \right) \epsilon (rw)(y),$$

where $r(x)$ is the column vector with components $r_k(x)$. Now by (4.24), we see that

$$C^T \mu = \mu^T \mathbb{D}_1^T \mu, \quad \mu D = \mu \mathbb{D}_1^T \mu^T,$$

which are equal as $\mu^T = -\mu$. Now from the form of $\mu$ in (4.22), we see that $\mu_{N-1,j} = \delta_{j,N-2}h_{N-1,1}^{-1}$. Hence we have

$$\sum_{j=0}^{N-1} \pi_{N,1}(x)w(x)\mu_{N-1,j} \epsilon (r_jw)(y) = h_{N-1,1}^{-1} \pi_{N,1}(x)w(x)\psi_{N-2}(y). \quad (4.26)$$

Let us now consider the second term in (4.25). As in (4.17) we can write $\epsilon(r_kw)w$ as

$$\epsilon(r_kw)w = q_k(x)w_0 + D_{k,1}w_1 + D_{k,2}w_2,$$

where $q_k(x)$ is of degree $k - 2$. Then from the form of $P_{N,j}^I$ in (4.10) and the orthogonality condition (4.9), we see that the coefficients $d_{k,N+l}, l = 1, 2$ are given by

$$d_{k,N+l} = D_{k,l} = -\frac{1}{2\pi i} \int_{\mathbb{R}^+} P_{N,l}^I(x)\epsilon(r_kw)wdx.$$

For $k \neq N - 1$, the polynomial $q_k$ is of degree less than or equal to $N - 4$, while for $k = N - 1$, $B_N$ in (4.17) is a polynomial of degree $N - 2$ and hence the coefficient $d_{k,N}$ is zero unless $k = N - 1$. For $k = N - 1$, it is given by the leading coefficient of $q_{N-1}$ divided by the leading coefficient of $B_N$. Since

$$\pi_{N-1,1}(x) = \frac{d}{dx} (q_{N-1}(x)x(t - x)) w^{-1} + D_{k,1}L_{N-3} + D_{k,2}L_{N-4},$$

we see that both the leading coefficient of $q_{N-1}(x)$ and $B_N$ is $\frac{2}{\pi}$. Hence $d_{k,N}$ is $\delta_{k,N-1}$. This gives us

$$\sum_{j=0}^{N-1} r_j(x)w(x)\mu_{jk}d_{k,N}\psi_N(y) = -h_{N-1,1}^{-1} \pi_{N-2,1}(x)w(x)\psi_N(y).$$
To express the second term in (4.25) in terms of the multi-orthogonal polynomials, let us now express $P_{N,l}^{II}$ in terms of the polynomials $r_k$. Let us write $P_{N,l}^{II} = \sum_{j=0}^{N-1} a_j r_j(x)$. Then we have

$$\int_{\mathbb{R}^+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx = \sum_{j=0}^{N-1} a_j (\mathcal{D}_1)_{jk}, \quad \sum_{k=0}^{N-1} \int_{\mathbb{R}^+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx \mu_{kj} = a_j.$$ 

Hence $P_{N,l}^{II}(x)$ can be written as

$$P_{N,l}^{II}(x) = \left( \int_{\mathbb{R}^+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx \mu_{kj} \right) r_j(x) = -2\pi i \sum_{k,j=0}^{N-1} d_{k,N} \mu_{kj} r_j(x).$$

Therefore the second term in (4.25) is given by

$$\sum_{j,k=0}^{N-1} r_j(x) w(x) \mu_{jk} \left( d_{k,N} \psi_N(y) + d_{k,N+1} \frac{P_{N,1}^{I}(y)}{w(y)} + d_{k,N+2} \frac{P_{N,2}^{I}(y)}{w(y)} \right)$$

$$= -h_{N-1,1}^{-1} \pi_{N-2,1}(x) \psi_N(y) + \frac{1}{2\pi i} \sum_{l=1}^{2} P_{N,l}^{II}(x) w(x) w^{-1}(y) P_{N,l}^{I}(y).$$

From this and (4.26), we obtain

$$(y - x) S_1(x, y) = h_{N-1,1}^{-1} \pi_{N,1}(x) w(x) \psi_N(y) + h_{N-1,1}^{-1} \pi_{N-2,1}(x) w(x) \psi_N(y)$$

$$- \frac{1}{2\pi i} \sum_{l=1}^{2} P_{N,l}^{II}(x) w(x) w^{-1}(y) P_{N,l}^{I}(y).$$

By using the fact that $Y^{-1}(y) = X^T(y)$ and the expressions of the matrix $Y$ (4.12) and $X$ (4.18), together with (4.13), we see that this is the same as (4.21).  

4.3 The kernel in terms of Laguerre polynomials

We will now use a result in [9] to further simplify the expression of the kernel $S_1(x, y)$ so that its asymptotics can be computed using the asymptotics of Laguerre polynomials. Let us recall the set up in [9]. First let $Y(x)$ be a matrix satisfying the Riemann-Hilbert
problem

1. \( Y(z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R}_+ \),

\[
Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_0(z) & w_1(z) & \cdots & w_r(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+
\]

2. \( Y_+(z) = Y_-(z) \begin{pmatrix} z^n & z^{-n+r} \\ z^{-n} & z^{-1} & \cdots & z^{-1} \end{pmatrix}, \quad z \to \infty. \)

and let \( \mathcal{K}_1(x, y) \) be the kernel given by

\[
\mathcal{K}_1(x, y) = \frac{w_0(x)w_0^{-1}(y)}{2\pi i(x-y)} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} Y_+^{-1}(y)Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (4.27)
\]

Let \( \pi_{j,2}(x) \) be the monic orthogonal polynomials with respect to the weight \( w_0(x) \). Let \( \mathcal{K}_0(x, y) \) be the following kernel.

\[
\mathcal{K}_0(x, y) = w_0(x) \pi_{2,n}(x)\pi_{2,n-1}(y) - \pi_{2,n}(y)\pi_{2,n-1}(x) h_{n-1,2}(x-y)
\]

where \( h_{j,2} = \int_0^\infty \pi_{j,2}^2 w_0 dx \). Let \( \pi(z) \), \( v(z) \) and \( u(z) \) be the following vectors

\[
\pi(z) = (\pi_{n-2}, \ldots, \pi_{n-1})^T, \quad v(z) = (w_1, \ldots, w_r)^T w_0^{-1}, \quad u(z) = (I - \mathcal{K}_0^T) v(z)
\]

and let \( \mathcal{W} \) be the matrix \( \mathcal{W} = \int_{\mathbb{R}_+} \pi(z)v^T(z)w_0(z)dz \). Then the kernel \( \mathcal{K}_1(x, y) \) can be express as [9]

**Proposition 4.3.** Suppose \( \int_{\mathbb{R}_+} p(x)w_1(x)dx \) converges for any polynomial \( p(x) \). Then the kernel \( \mathcal{K}_1(x, y) \) defined by (4.27) is given by

\[
\mathcal{K}_1(x, y) - \mathcal{K}_0(x, y) = w_0(x)u^T(y)\mathcal{W}^{-1}\pi(x). \quad (4.30)
\]

**Remark 4.2.** Although in [9], the theorem is stated with the jump of \( Y(x) \) on \( \mathbb{R} \) instead of \( \mathbb{R}_+ \), while the weights are of the special form \( w_0 = e^{-NV(x)} \), \( w_j(x) = e^{a_j x} \), where \( V(x) \) is an even degree polynomial, the proof in [9] in fact remain valid as long as integrals of the form \( \int_{\mathbb{R}_+} p(x)w_1(x)dx \) converges for any polynomial \( p(x) \). This is true in our case.
We can now apply Proposition 4.3 to our case. In our case, the vectors \( \pi(z) \) and \( v(z) \) are given by

\[
\pi(z) = (L_{N-2} \quad L_{N-1})^T, \quad v(z) = (w_1 \quad w_2)^T w_0^{-1},
\]

while the matrix \( \mathcal{W} \) is given by

\[
\mathcal{W} = \begin{pmatrix}
\langle L_{N-2}, L_{N-3} \rangle_1 & \langle L_{N-2}, L_{N-4} \rangle_1 \\
\langle L_{N-1}, L_{N-3} \rangle_1 & \langle L_{N-1}, L_{N-4} \rangle_1
\end{pmatrix}
\]

By corollary 4.1, we see that \( \langle L_{N-2}, L_{N-3} \rangle_1 = 0 \) and hence the determinant of \( \mathcal{W} \) is

\[
\det \mathcal{W} = \langle L_{N-2}, L_{N-4} \rangle_1 \langle L_{N-1}, L_{N-3} \rangle_1.
\]

Suppose \( \det \mathcal{W} \neq 0 \) and that the multi-orthogonal polynomials \( P_{l_N}^l \) exist. Let us now consider the vector \( u(z) \). By the Christoffel-Darboux formula, we have

\[
u(z) = (I - \mathcal{K}_0^T) v(z) = v(z) - \sum_{j=0}^{N-1} \frac{L_j(z)}{h_{j,0}} \left( \langle L_j, L_{N-3} \rangle_1 \quad \langle L_j, L_{N-4} \rangle_1 \right),
\]

where \( h_{j,0} = \langle L_j, L_j \rangle_2 \). We shall show that \( L_{N-3} \) and \( L_{N-4} \) can be written in the following form

\[
L_{N-l} = \frac{d}{dx} \left( q_l(x)x(t - x)w \right) w^{-1} + Q_{l-2,1} \pi_{N+1,1} + Q_{l-2,2} \pi_{N,1}, \quad l = 3, 4,
\]

for some polynomial \( q_l(x) \) of degree \( N - 1 \). By Lemma 4.1, we see that if \( \langle L_{N-3}, L_{N-4} \rangle_1 \neq 0 \), then the map \( g_{N-3} \) in Definition 4.1 is invertible. Therefore if the restriction of the map \( f \) in Definition 4.1 is also invertible on the span of \( \pi_{N+1,1} \) and \( \pi_{N,1} \), we will be able to write \( L_{N-3} \) and \( L_{N-4} \) in the form of (4.31).

**Lemma 4.3.** Let \( f \) be the map in Definition 4.1 and let \( g_\pi \) be its restriction to the span of \( \pi_{N+1,1} \) and \( \pi_{N,1} \). Then \( g_\pi \) is invertible.

**Proof.** Suppose there exist \( a_1 \) and \( a_2 \) such that

\[
a_1 \pi_{N+1,1} + a_2 \pi_{N,1} = \frac{d}{dx} \left( q(x)x(t - x)w \right) w^{-1}
\]

for some polynomial \( q(x) \) of degree at most \( N - 1 \). Then we have

\[
\langle x^j, a_1 \pi_{N+1,1} + a_2 \pi_{N,1} \rangle_1 = \langle x^j q(x) \rangle_2 = 0, \quad j = 0, \ldots, N - 1.
\]

As the degree of \( q(x) \) is at most \( N - 1 \), this is only possible if \( q(x) = 0 \). \( \square \)
The composition of \( q_{N-3} \) and \( q_{-1} \) will therefore give us a representation of \( L_{N-3} \) and \( L_{N-4} \) in the form of (4.31). By using this representation and the fact that \( q_l(x) \) is of degree at most \( N - 1 \), we see that

\[
K_0^T (w_1 w_0^{-1}) = \sum_{j=0}^{N-1} \frac{L_j(x)}{h_{j,0}} (L_j, L_{N-l-2})_1
\]

\[
= \sum_{j=0}^{N-1} \frac{L_j(x)}{h_{j,0}} \left( (L_j, q_l)_2 + (L_j, Q_{l-1,1} \pi_{N+1,1} + Q_{l-2,2} \pi_{N,1})_1 \right),
\]

\[
= q_l(x).
\]

Therefore the vector \( u(x) \) is given by

\[
u(x) = w(x)w_0^{-1}(x)Q \epsilon (\pi_{N+1,1} w \pi_{N,1} w)^T
\]

where \( Q \) is the matrix with entries \( Q_{i,j} \). We will now determine the constants \( Q_{i,j} \).

**Lemma 4.4.** Let \( \pi_{N,1} \) and \( \pi_{N+1,1} \) be the monic skew orthogonal polynomial with respect to the weight \( w(x) \) and choose \( \pi_{N+1,1} \) so that the constant \( c \) in (4.31) is zero. Then the vector \( u(y) \) in (4.30) is given by

\[
u(x) = w(x)w_0^{-1}(x)Q \epsilon (\pi_{N+1,1} w \pi_{N,1} w)^T,
\]

where \( Q \) is the matrix whose entries \( Q_{i,j} \) are given by

\[
Q_{i,1} = -\frac{M \langle L_{N-1}, L_{N-i-2} \rangle_1}{2h_{N-1,0}},
\]

\[
Q_{i,2} = (Mt - (N + M)) \frac{\langle L_{N-1}, L_{N-i-2} \rangle_1}{2h_{N-1,0}} - M \frac{\langle L_{N-2}, L_{N-i-2} \rangle_1}{2h_{N-2,0}}.
\]

**Proof.** First let us compute the leading order coefficients of the polynomial \( q_l(x) \) in (4.31). Let \( q_l(x) = q_{l,N-1} x^{N-1} + q_{l,N-2} x^{N-2} + O(x^{N-3}) \), then we have

\[
\frac{d}{dx} (q_l(x)x(t-x)w^{-1} = \frac{M}{2} q_{l,N-1} x^{N+1},
\]

\[
+ \left( -\frac{1}{2}(N + M) q_{l,N-1} - \frac{Mt}{2} q_{l,N-1} + \frac{M}{2} q_{l,N-2} \right) x^N + O(x^{N-1})
\]

From (4.31), we see that

\[
q_{l,N-1} = -\frac{2}{M} Q_{l-2,1}.
\]

On the other hand, by orthogonality, we have

\[
\langle L_{N-1}, L_{N-l} \rangle_1 = \langle L_{N-1}, q_l \rangle_2 = q_{l,N-1} h_{N-1,0}.
\]
Therefore $Q_{l-2,1}$ is given by

$$Q_{l-2,1} = -\frac{M \langle L_{N-1}, L_{N-l} \rangle_1}{2h_{N-1,0}}$$

Let us now compute $Q_{i,2}$. By taking the skew product, we have

$$\langle L_{N-2}, L_{N-l} \rangle_1 = q_{i, N-1} \langle L_{N-2}, x^{N-1} \rangle_2 + q_{i, N-2} h_{N-2,0}$$

Now from (4.34), we obtain

$$\langle x^{j+1}, L_j \rangle_2 = \left( L_{j+1} + \frac{(M - N + j + 1)(j + 1)}{M} x^j, L_j \right)_2 = \frac{(M - N + j + 1)(j + 1)}{M} h_{j,0}.$$  \hspace{1cm} (4.35)

Hence $q_{l, N-2}$ is equal to

$$q_{l, N-2} = \frac{\langle L_{N-2}, L_{N-l} \rangle_1}{h_{N-2,0}} - q_{i, N-1} \frac{(M - 1)(N - 1)}{M}$$  \hspace{1cm} (4.36)

By using the expansion (4.8) of the skew orthogonal polynomials in terms of $L_{N-k}$, we have

$$\langle L_{N-1}, L_N \rangle_2 = \frac{M}{2} q_{i, N-1} \langle x^{N+1}, L_N \rangle_2 + Q_{l-2,2} h_{N,0}$$

$$+ \left( \frac{1}{2} (-N - M) q_{i, N-1} - \frac{M t}{2} q_{i, N-1} + \frac{M}{2} q_{i, N-2} \right) h_{N,0}$$

By substituting (4.36) into this, we obtain

$$Q_{l-2,2} = \left( \frac{1}{2} (M + N) + \frac{M t}{2} \right) \langle L_{N-1}, L_{N-l} \rangle_1 \frac{1}{h_{N-1,0}} - \frac{M}{2} \langle L_{N-2}, L_{N-l} \rangle_1 \frac{1}{h_{N-2,0}}$$

This proves the lemma.  \hspace{1cm} \square

Note that the matrix $\det(QW^{-1}) = M^2/(4h_{N-1,0}h_{N-2,0})$ and hence $QW^{-1}$ is invertible. From Lemma 4.4 and (4.30), we obtain Theorem 2.2.

We will now further simplify the expression for the correlation kernel $S_1(x, y)$. First we make the following observation. From the definition (4.19) of the kernel $S_1$, it is easy to check that the following identity is true.

$$- \frac{\partial}{\partial y} S_1(x, y) = \frac{\partial}{\partial x} S_1(y, x).$$  \hspace{1cm} (4.37)
We can use this simple observation in the expression (2.10) that we obtained in the last section. First let us write the Christoffel Darboux kernel terms of the solution of a Riemann-Hilbert problem, we have

\[ K_2(x, y) = \left(\frac{y(t - y)}{x(t - x)}\right)^{\frac{1}{2}} \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z_+^{-1}(y)Z_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.38} \]

where \( Z(x) \) is the following matrix

\[ Z(x) = \begin{pmatrix} L_N(x) \\ \kappa_{0, N-1}L_{N-1}(x) \end{pmatrix} C(L_Nw_0) \begin{pmatrix} \kappa_0, N-1 \kappa_0, N-1 \end{pmatrix} (L_{N-1}w_0) \frac{\sigma_3}{w_0^2}, \tag{4.39} \]

where \( \sigma_3 \) is the Pauli matrix \( \sigma_3 = \text{diag}(1, -1) \), \( \kappa_{0, n} = -\frac{2\pi i}{\kappa_{0, n}} \) and \( C(f) \) is the Cauchy transform.

It is well-known that if the logarithmic derivative of \( w_0 \) is rational, then the matrix \( Z(x) \) in (4.39) satisfies a linear system of ODE

\[ \frac{\partial}{\partial x} Z(x) = A(x)Z(x) \tag{4.40} \]

for some rational function \( A(x) \). By using (4.4) and the recurrence relation of the \( L_n \)

\[ L_n = -\frac{n}{M} \left( 2 + \frac{M - N - 1 - Mx}{n} \right) L_{n-1} - \frac{n(n-1)}{M^2} \left( 1 + \frac{M - N - 1}{n} \right) L_{n-2}, \tag{4.41} \]

we obtain the matrix \( A(x) \) as

\[ A(x) = \begin{pmatrix} -\frac{M}{2} + \frac{M+N}{2x} & \frac{N}{\kappa_{0, N-1}x} \\ -\frac{M\kappa_{0, N-1}}{x} & \frac{\kappa_{0, N-1}}{2} - \frac{M+N}{2x} \end{pmatrix} \tag{4.42} \]

Let us now consider the derivative \( \frac{\partial K_2(x, y)}{\partial y} \). We have, from (4.38)

\[ \frac{\partial K_2(x, y)}{\partial y} = \frac{t - 2y}{(x(t - x)(t - y))^\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{Z_+^{-1}(y)Z_+(x)}{4\pi i(x - y)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{y(t - y)}{x(t - x)}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{Z_+^{-1}(y)Z_+(x)}{2\pi i(x - y)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.43} \]

Similarly, \( \frac{\partial K_2(y, x)}{\partial x} \) is given by

\[ \frac{\partial K_2(y, x)}{\partial x} = \frac{t - 2x}{(x(t - x)(t - y))^\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{Z_+^{-1}(x)Z_+(y)}{4\pi i(y - x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{x(t - x)}{y(t - y)}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{Z_+^{-1}(x)Z_+(y)}{2\pi i(y - x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.44} \]

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The sum of these two derivatives is therefore given by

\[
\frac{\partial K_2(y, x)}{\partial x} + \frac{\partial K_2(x, y)}{\partial y} = h_{0,N-1}^{-1} w(x)w(y) \left( - L_N(x)L_N(y) \right) \\
+ (L_{N-1}(y)L_N(x) + L_{N-1}(x)L_N(y)) \left( \frac{M}{2}(x + y) - \frac{Mt + (M + N)}{2} \right) \\
- L_{N-1}(x)L_{N-1}(y)N.
\]

By using the recurrence relations of the Laguerre polynomials, we obtain

\[
\frac{\partial K_2(y, x)}{\partial x} + \frac{\partial K_2(x, y)}{\partial y} = h_{0,N-1}^{-1} w(x)w(y) \left( \frac{(N - 1)(M - 1)}{2M} L_{N-2,N}(x, y) \\
+ \frac{M}{2} L_{N+1,N-1}(x, y) + \frac{(M + N) - Mt}{2} L_{N-1,N}(x, y) \right)
\]

where \( L_{n,m}(x, y) \) is given by

\[
L_{n,m}(x, y) = L_n(x)L_m(y) + L_m(x)L_n(y).
\]

Let \( K_1(x, y) \) be the correction kernel in (2.10).

\[
K_1(x, y) = \epsilon \left( \pi_{N+1,1}w \pi_{N,1}w \right)(y) \left( \frac{0}{2h_{0,N-2}} \frac{-M}{2h_{0,N-2}} \right) \pi(x)w(x)
\]

Then we have

\[
\frac{\partial K_1(y, x)}{\partial x} + \frac{\partial K_1(x, y)}{\partial y} = w(x)w(y) \left( - \frac{M}{2h_{0,N-2}} L_{N,N-2}(x, y) \\
- \frac{M}{2h_{0,N-2}} L_{N+1,N-1}(x, y) + \frac{Mt - (N + M)}{2h_{0,N-1}} L_{N-1,N}(x, y) \\
+ \left( \frac{M}{2h_{0,N-2}} \frac{\langle L_{N-1}, L_{N-2} \rangle_1}{\langle L_{N-1}, L_{N-2} \rangle_1} \right) L_{N-1,N-2}(x, y) \\
+ \left( \frac{M}{h_{0,N-1}} \frac{\langle L_{N+1}, L_{N-2} \rangle_1}{\langle L_{N-1}, L_{N-2} \rangle_1} \right) L_{N-1,N-1}(x, y) \right)
\]

Then by using \( \frac{\partial S_1(y, x)}{\partial x} + \frac{\partial S_1(x, y)}{\partial y} = 0 \) and the formula for \( h_{0,n} \),

\[
h_{n,0} = \frac{n!(n + M - N)!}{M^{2n + M - N + 1}},
\]
we obtain the following.

\[
\langle L_{N+1}, L_{N-1} \rangle_1 = \frac{(N-1)(M-1)}{M^2} \langle L_N, L_{N-2} \rangle_1,
\]

\[
\langle L_{N+1}, L_{N-2} \rangle_1 = \frac{Mt - (N + M)}{M} \langle L_N, L_{N-2} \rangle_1
\]

In particular, the correction term \( K_1(x, y) \) in the kernel can be written as

\[
K_1(x, y) = -\frac{M}{2h_{0,N-2}} \left( \epsilon(L_Nw)(y)L_{N-2}w(x) - \frac{\langle L_N, L_{N-2} \rangle_1}{\langle L_{N-1}, L_{N-2} \rangle_1} \left( \epsilon(L_{N-1}w)(y)L_{N-2}w(x) - \epsilon(L_{N-2}w)(y)L_{N-1}w(x) \right) \right)
\]

\[
- \frac{M}{2h_{0,N-1}} \epsilon(L_{N+1}w)(y)L_{N-1}w(x) + \frac{Mt - (N + M)}{2h_{0,N-1}} \epsilon(L_{N}w)(y)L_{N-1}w(x).
\]

5 Derivative of the partition function

In this section we will derive a formula for the derivative of determinant of the matrix \( \mathbb{D} \) given in (4.20). We have the following.

**Proposition 5.1.** Let \( \mathbb{D} \) be the matrix given by (4.20), where the sequence of monic polynomials \( r_j(x) \) in (4.20) is chosen such that \( r_j(x) \) are arbitrary degree \( j \) monic polynomials that are independent on \( t \) and \( r_j(x) = \pi_{j,1}(x) \) for \( j = N-2, N-1 \). Then the logarithmic derivative of \( \det \mathbb{D} \) with respect to \( t \) is given by

\[
\frac{\partial}{\partial t} \log \det \mathbb{D} = -\int_{\mathbb{R}_+} S_1(x, x) \frac{t-x}{t} \, dx,
\]

where \( S_1(x, y) \) is the kernel given in (4.19).

**Proof.** First let us differentiate the determinant \( \det \mathbb{D} \) with respect to \( t \). We have

\[
\frac{\partial}{\partial t} \det \mathbb{D} = \det \left( \begin{array}{cccc}
\partial_tD_{00} & D_{01} & \cdots & D_{0,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
\partial_tD_{2n-1,0} & D_{2n-1,1} & \cdots & D_{N-1,N-1} \\
D_{00} & \partial_tD_{01} & \cdots & \partial_tD_{0,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
D_{2n-1,0} & \partial_tD_{2n-1,1} & \cdots & \partial_tD_{N-1,N-1} \\
\end{array} \right) + \cdots + \det \left( \begin{array}{cccc}
D_{00} & D_{01} & \cdots & \partial_tD_{0,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
D_{N-1,0} & D_{N-1,1} & \cdots & \partial_tD_{N-1,N-1} \\
\end{array} \right).
\]
Computing the individual determinants using the Laplace formula, we obtain
\[
\frac{\partial}{\partial t} \det \mathbb{D} = \det \mathbb{D} \sum_{i,j=0}^{N-1} \partial_t D_{ij} \mu_{ij}.
\] (5.2)

As \( r_j(x) \) are independent on \( t \) for \( j < N - 2 \), the derivative \( \partial_t D_{ij} \) is given by
\[
\partial_t D_{ij} = -\frac{1}{2} \left( \left\langle \frac{r_i}{t-x}, r_j \right\rangle + \left\langle r_i, \frac{r_j}{t-y} \right\rangle \right) = -\frac{1}{2} \left( \left\langle \frac{r_i}{t-x}, r_j \right\rangle - \left\langle \frac{r_j}{t-x}, r_i \right\rangle \right)
\]
for \( i, j < N - 2 \). For either \( i \) or \( j \) equal to \( N - 2 \) or \( N - 1 \), we have
\[
\partial_t D_{i,N-1} = \delta_{N-2,i} \left( -\frac{1}{2} \left( \left\langle \frac{\pi_{N-1,1}}{t-x}, \pi_{N-1,1} \right\rangle - \left\langle \frac{\pi_{N-2,1}}{t-x}, \pi_{N-2,1} \right\rangle \right) \right)
\]
\[
\partial_t D_{i,N-2} = \delta_{N-1,i} \left( \frac{1}{2} \left( \left\langle \frac{\pi_{N-1,1}}{t-x}, \pi_{N-1,1} \right\rangle - \left\langle \frac{\pi_{N-2,1}}{t-x}, \pi_{N-2,1} \right\rangle \right) \right).
\]

As \( \mathbb{D} \) is anti-symmetric, the derivatives \( \partial_t D_{N-1,i} \) and \( \partial_t D_{N-2,i} \) are given by \( \partial_t D_{N-1,i} = -\partial_t D_{i,N-1} \) and \( \partial_t D_{N-2,i} = -\partial_t D_{i,N-2} \). From these and (5.2), we obtain
\[
\frac{\partial}{\partial t} \det \mathbb{D} = \det \mathbb{D} \sum_{i,j=0}^{N-1} \left( \sum_{i=0}^{N-3} \left\langle \frac{r_i}{t-x}, r_j \right\rangle \right) \mu_{ij}
\]
\[
+ 2 \left( \left\langle \frac{\pi_{N-1,1}}{t-x}, \pi_{N-1,1} \right\rangle - \left\langle \frac{\pi_{N-2,1}}{t-x}, \pi_{N-2,1} \right\rangle \right) \left( \delta_{N-2,N-1} \right) \mu_{N-2,N-1},
\] (5.3)

where we have used the anti-symmetry of \( \mathbb{D} \) and \( \mu \) to obtain the last term. From the structure of the matrix \( \mu \) in (4.12), we see that the last term in (5.3) can be written as
\[
2 \left( \left\langle \frac{\pi_{N-2,1}}{t-x}, \pi_{N-1,1} \right\rangle - \left\langle \frac{\pi_{N-1,1}}{t-x}, \pi_{N-2,1} \right\rangle \right) \mu_{N-2,N-1}
\]
\[
= \sum_{i=0}^{N-1} \sum_{j=N-2}^{N-1} \left\langle \frac{r_i}{t-x}, r_j \right\rangle \mu_{ij} + \sum_{i=N-2}^{N-1} \sum_{j=0}^{N-1} \left\langle \frac{r_i}{t-x}, r_j \right\rangle \mu_{ij}
\]

From this, (5.3) and the expression of the kernel in (4.19), we obtain (5.1).
6 Asymptotic analysis

We will now use the asymptotics of the Laguerre polynomials to obtain asymptotic expression for the kernel $S_1(x,y)$. We will demonstrate the asymptotic analysis with the assumption that $M/N \to \gamma = 1$. The same analysis also be applied for general $\gamma$.

Let us recall the asymptotics of the Laguerre polynomials in different regions of $\mathbb{R}_+$. These asymptotics can be found in [70]. When $\gamma = 1$, the end points $b_-$ and $b_+$ of the support of $\rho$ in (1.1) are 0 and 4 respectively.

Let $n = N - j$, where $j = O(1)$. Let us define the function $\varphi(x)$ to be

$$\varphi(x) = \int_4^x \frac{1}{2} \sqrt{\frac{s - 4}{s}} ds,$$  

(6.1)

where the contour of integration is chosen such that it does not intersect the interval $(-\infty, 4)$. We can now use $\varphi(x)$ to define two maps $\tilde{f}_n$ and $f_n$ in the neighborhoods of 0 and 4.

**Definition 6.1.** Let $\varphi_+(x)$ be the branch of $\varphi(x)$ in the upper half plane and let $\delta > 0$ be sufficiently small, then the maps $\tilde{f}_n$ is defined in a disc of radius $\delta$ center at $b_-$ as follows.

$$\tilde{f}_n = \left(\frac{n}{2} (\varphi_+(x) - \varphi_+(0))\right)^2, \quad \gamma = 1.$$  

(6.2)

In the above expressions, the map $\tilde{f}_n$ is defined to be the analytic continuation of the expression in the right hand side to the whole disc.

Similarly, the map $f_n$ in a neighborhood of 4 is defined to be

$$f_n = \left(\frac{3}{2} n \varphi_+(x)\right)^{\frac{4}{3}}.$$  

(6.3)

As in [70], these maps are conformal inside the small discs around $b_{\pm,n}$, provided $\delta$ is sufficiently small. In particular, they behave as follows as near $b_{\pm,n}$. (Recall that $b_{-,n} = 0$ and $b_{+,n} = 4$ when $\gamma = 1$.)

$$\tilde{f}_n = -n^2 x \tilde{f}_n(x), \quad f_n = \left(\frac{n}{4}\right)^{\frac{4}{3}} (x - 4) \tilde{f}_n(x),$$  

(6.4)

where the maps $\tilde{f}_n(x)$ and $\tilde{f}_n(x)$ are of the form

$$\tilde{f}_n(x) = 1 + O(x), \quad \tilde{f}_n(x) = 1 + O(x - 4).$$

Note that $\tilde{f}_n(x)$ and $\tilde{f}_n(x)$ are independent on $n$.

We can now state the asymptotics of the Laguerre polynomials. These can be found, for example, in [70]. When $M - N = \alpha = O(1)$, the Laguerre polynomials have the following asymptotic expressions in different regions on $\mathbb{R}_+$ (See [70]).
Proposition 6.1. Suppose $M-N = \alpha = O(1)$, then for $n = N-j$ and $j = O(1)$, there exists sufficiently small $\delta > 0$ and the Laguerre polynomials $L_n(x)$ in (4.4) have the following asymptotic behavior on $\mathbb{R}_+$ as $n, N, M \to \infty$.

1. Uniformly for $x \in (0, \delta]$,
   \[
   L_n \left( \frac{n}{N} x \right) \left( w_0 \left( \frac{n}{N} x \right) \right) \frac{1}{2} = \frac{2\sqrt{\pi}(-1)^n(-\tilde{f}_n)^{\frac{1}{2}}(i\kappa_{n-1})^{-\frac{1}{2}}}{x^{\frac{1}{4}}(4-x)^{\frac{1}{4}}} \left( \sin \zeta J_\alpha \left( 2(-\tilde{f}_n)^{\frac{1}{2}} \right) \right.
   \]
   \[
   \times \left. (1 + O(N^{-1})) + \cos \zeta J_\alpha' \left( 2(-\tilde{f}_n)^{\frac{1}{2}} \right) (1 + O(N^{-1})) \right),
   \]
   where $J_\alpha$ is the Bessel function and $\zeta$ is the following
   \[
   \zeta = \frac{1}{2}(\alpha + 1) \arccos \left( \frac{x}{2} - 1 \right) - \frac{\pi\alpha}{4}. \quad (6.6)
   \]
   The branch cut of $\arccos(x)$ in the above is chosen to be $(-\infty, -1] \cup [1, \infty)$. The constant $\kappa_n$ is given by $\kappa_n = -\frac{2\pi i}{\kappa_{n,0}}$.

2. Uniformly for $x \in [\delta, 1 - \delta]$,
   \[
   L_n \left( \frac{n}{N} x \right) \left( w_0 \left( \frac{n}{N} x \right) \right) \frac{1}{2} = \frac{2(i\kappa_{n-1})^{-\frac{1}{2}}}{x^{\frac{1}{4}}(4-x)^{\frac{1}{4}}} \left( \cos \left( \eta_+ + i\eta_+ + \frac{\pi}{4} \right) \right.
   \]
   \[
   \times \left. (1 + O(N^{-1})) + \cos \left( \eta_- + i\eta_- + \frac{\pi}{4} \right) O(N^{-1}) \right),
   \]
   where $\eta_{\pm}$ are given by
   \[
   \eta_{\pm} = \frac{1}{2}(\alpha \pm 1) \arccos \left( \frac{x}{2} - 1 \right) \quad (6.8)
   \]

3. Uniformly for $x \in [4 - \delta, 4 + \delta]$,
   \[
   L_n \left( \frac{n}{N} x \right) \left( w_0 \left( \frac{n}{N} x \right) \right) \frac{1}{2} = \frac{2\sqrt{\pi}(i\kappa_{n-1})^{-\frac{1}{2}}}{x^{\frac{1}{4}}|x - 4|^{\frac{1}{4}}} \left( |f_n|^{\frac{1}{4}} \cos \eta_+(x)Ai(f_n) \right.
   \]
   \[
   \times \left. (1 + O(N^{-1})) - |f_n|^{\frac{1}{4}} \sin \eta_+(x)Ai'(f_n)(1 + O(N^{-1})) \right),
   \]

Remark 6.1. Reader may notice that the asymptotic formula presented in Proposition 6.1 is different from the ones in [70]. This is because the weight in [70] that is relevant to us is $w_0(x) = x^{\alpha}e^{-4nx}$ and a rescaling of the variable from $x$ to $y = \frac{4n}{N}x$ is needed to obtain the formula in Proposition 6.1 from the results in [70].

We will now use the asymptotic formulae in Proposition 6.1 to compute the skew products of the form $\langle L_{N-j}, L_{N-k} \rangle_1$. We shall follow the ideas in [31], [33] and [63].
6.1 Asymptotics for the skew products

Throughout the analysis, we will assume that \( t \) is of finite distance from \([0, 4)\). We will also need to consider the case where \(|t - 4|\) is of order \( N^{-\frac{3}{2}}\). We shall organize the results according to the different regions on \( \mathbb{R}_+ \). We will divide \( \mathbb{R}_+ \) into four regions \((0, N^{-\frac{1}{2}}], [N^{-\frac{1}{2}}, 4 - N^{-\varepsilon}], [4 - N^{-\varepsilon}, 4 + N^{-\varepsilon}]\) and \([4 + N^{-\varepsilon}, \infty)\) for some \( \varepsilon > 0 \). These regions are called the Bessel region, the bulk region, the Airy region and the exponential region.

6.1.1 The Bessel Region

Let us define \( \hat{L}_n(x) \) to be

\[
\hat{L}_n(x) = L_n \left( \frac{n}{N} x \right) w \left( \frac{n}{N} x \right) \left( \frac{(i\kappa_{n-1})}{2} \right) .
\]

From (6.6), we see that as \( x \to 0 \), \( \sin \zeta \) and \( \cos \zeta \) have the following behavior.

\[
\sin \zeta = 1 + O(x), \quad \cos \zeta = \frac{\alpha + 1}{2} \sqrt{x}(1 + O(x)).
\] (6.11)

Let us now express the integrals in the Bessel region in terms of the variable \( \tilde{f}_n \). From (6.2), we have

\[
x = -\frac{\tilde{f}_n}{n^2} \left( 1 - \frac{1}{12} \tilde{f}_n^2 + O \left( \frac{\tilde{f}_n^2}{n^4} \right) \right),
\]

\[
\left( \frac{n}{N} \left( t - \frac{n x}{N} \right) \right)^{\frac{1}{2}} = \left( \frac{-\tilde{f}_n}{tn^2} \right)^{\frac{1}{2}} \left( 1 + O \left( \frac{\tilde{f}_n}{n^2} \right) + O(n^{-1}) \right)
\] (6.12)

As \( x < N^{-\frac{1}{2}} \) in the Bessel region, we see that the term \( \tilde{f}_n^2 / n^4 \) is of order \( O(n^{-1}) \) at most. Then from the asymptotic formula (6.5), we have the following asymptotic formula for \( \hat{L}_n(x) \) in the Bessel region.

\[
\hat{L}_n(x) = \frac{(-1)^n \sqrt{2 \pi n^2}}{t^{\frac{1}{2}}} \left( \frac{J_\alpha \left( 2 \tilde{f}_n^2 \right)}{(\tilde{f}_n)^2} \left( 1 + \frac{k_{1,0}}{n} + \sum_{j=1}^{\infty} \frac{k_{1,j} \tilde{f}_n^j}{n^{2j}} \right) + \frac{\alpha + 1}{2n} \right) \left( 1 + \frac{k_{2,0}}{n} + \sum_{j=1}^{\infty} \frac{k_{2,j} \tilde{f}_n^j}{n^{2j}} \right)
\] (6.13)

for some constants \( k_{1,j} \) and \( k_{2,j} \) that are bounded in \( n \). By using the asymptotic formula for the Bessel function in [II], we obtain the following.
Lemma 6.1. Let \( k \geq 0 \), \( s > v \geq 0 \), and \( \alpha > 0 \), then as \( s \to \infty \), we have
\[
\int_0^s J_\alpha(u)u^k du = O(1) + O\left(s^{k-\frac{1}{2}}\right), \quad \int_0^s J'_\alpha(u)u^k du = O(1) + O\left(s^{k-\frac{1}{2}}\right).
\] (6.14)

Let \( s_1 = O(s) \) as \( s \to \infty \), and let \( B_1 = J_\alpha \), \( B_2 = J'_\alpha \), then as \( s \to \infty \), we have the following for the double integrals.
\[
\int_0^s B_i(u)u^k \int_u^{s_1} B_j(v)v^l dv du = O\left(s^{k+l}\right), \quad i, j = 1, 2, \quad k + l > 0.
\] (6.15)

Proof. Integrating the first formula by parts, we obtain
\[
\int_0^s J_\alpha(u)u^k du = \left(- \int_0^s J_\alpha(v)dv u^k\right)_v^s + k \int_0^s \left(\int_u^s J_\alpha(v)dv\right) u^{k-1} du
\]
By (9.2.1) and (11.4.17) of [1], we see that \( \int_0^s J_\alpha(v)dv = O\left(s^{-\frac{1}{4}}\right) \) as \( u \to s \) and \( s \to \infty \). The function \( \int_0^s J_\alpha(v)dv \) is therefore bounded and behaves as \( O\left(s^{-\frac{1}{4}}\right) \) as \( u \to s \). Therefore we have
\[
\int_0^s J_\alpha(u)u^k du = k \int_0^s \left(\int_0^s J_\alpha(v)dv\right) u^{k-1} du + O(1) + O\left(s^{k-\frac{1}{2}}\right) = O(1) + O\left(s^{k-\frac{1}{2}}\right).
\]
This proves the first equation in (6.14). Integrating the second equation by parts and using \( J_\alpha(0) = 0 \) for \( \alpha > 0 \), we obtain
\[
\int_0^s J'_\alpha(u)u^k du = J_\alpha(s) s^k - J_\alpha(v)v^k - k \int_0^s J_\alpha(u)u^{k-1} du.
\]
By using the following estimates for any \( C > 0 \),
\[
\sup_{y \in [0,C]} |y^{-\alpha} J_\alpha(y)| < \infty, \quad \sup_{y \in [C,\infty)} |\sqrt{y} J_\alpha(y)| < \infty,
\]
\[
\sup_{y \in [0,C)} |y^{-\alpha+1} J'_\alpha(y)| < \infty, \quad \sup_{y \in [C,\infty)} |\sqrt{y} J'_\alpha(y)| < \infty,
\] (6.16)
we obtain the second equation in (6.14). The estimate (6.15) now follows immediately from (6.14) and (6.16).

We can now compute the integrals in the skew products in the Bessel region.

Proposition 6.2. The single integral involving \( L_n(x)w(x) \) in the Bessel region is given by
\[
(i \kappa_{n-1})^{\frac{1}{2}} \int_0^{N^{-\frac{1}{2}}} L_n(x)w(x) dx = I_{1,n} = \frac{(-1)^n \sqrt{2\pi}}{\sqrt{nt}} + O\left(n^{-\frac{7}{4}}\right).
\] (6.17)

The double integral in the Bessel region is given by
\[
i \sqrt{\kappa_{n-1}\kappa_{m-1}} \int_0^{N^{-\frac{1}{2}}} L_n(x)w(x) \int_x^{N^{-\frac{1}{2}}} L_m(y)w(y) dy dx dy = J_1 = \frac{(-1)^{m+n+2\pi}}{nt} + O\left(n^{-\frac{5}{4}}\right).
\] (6.18)
Proof. Let us first prove the single integral. We have, by (6.10),
\[
(i\kappa_{n-1})^{1/2} \int_0^{N^{-1/2}} L_n(x)w(x)dx = \int_0^{N^{-1/2}} \hat{L}_n\left(\frac{N}{n} x\right) dx = \frac{n}{N} \int_0^{\frac{\sqrt{N}}{n}} \hat{L}_n(x)dx.
\]
From (6.13), we have
\[
\int_0^{\frac{\sqrt{N}}{n}} \hat{L}_n(x)dx = \frac{(-1)^n \sqrt{2\pi}}{\sqrt{lnn}} \int_0^{u_+} \left(J_\alpha(u) \left(1 + \frac{k_{10}}{n} + \sum_{j=1}^{\infty} \frac{(-1)^j k_{1j} u^{2j}}{(2n)^{2j}}\right) + \frac{\alpha + 1}{4} J'_\alpha(u) \left(\frac{u}{n} + \frac{k_{20} u}{n^2} + \sum_{j=1}^{\infty} \frac{(-1)^j k_{2j} u^{2j+1}}{2j n^{2j+1}}\right)\right) du,
\]
where we have changed the integration variable from \(x\) to \(u = 2(-\tilde{f}_n)^{1/2}\) and \(u_+\) is the corresponding upper limit \(u_+ = 2\sqrt{-\tilde{f}_n} \left(\frac{\sqrt{N}}{n}\right) = O\left(n^{\frac{3}{4}}\right)\). By Lemma 6.1, we have
\[
\int_0^{\frac{\sqrt{N}}{n}} \hat{L}_n(x)dx = \frac{(-1)^n \sqrt{2\pi}}{\sqrt{lnn}} \left(\int_0^{u_+} J_\alpha(u) du + O\left(n^{-\frac{5}{8}}\right)\right)
\]
By using the asymptotic formula for the Bessel function (\(\Pi\), (9.2.5), (9.2.9) and (9.2.10)), we have obtained the following estimate
\[
\int_{u_+}^{\infty} J_\alpha(u) du = O\left(u_+^{-\frac{1}{2}}\right) \quad (6.19)
\]
By using this and the fact that \(\int_0^{\infty} J_\alpha(u) du = 1\), we arrive at (6.17).

Let us now compute the double integral. Let \(n = N - j\) and \(m = N - k\), where \(j\) and \(k\) are finite. As in the single integrals, we will change the integration variables into \(u = 2\sqrt{-\tilde{f}_n}\) and \(v = 2\sqrt{-\tilde{f}_m}\). Let \(v_+ = 2\sqrt{-\tilde{f}_m} \left(\sqrt{N}/n\right) = O\left(m^{\frac{3}{4}}\right)\) be the upper limit in the variable \(v\) and \(\nu(u)\) be the value of \(v\) when \(y = \frac{n}{m} x\). Then by using (6.12), we have
\[
\nu(u) = \sqrt{\frac{m}{n}} u \left(1 + \frac{u^2}{24m} \frac{n - m}{n^2} + O\left(\frac{u^3}{n^4}\right)\right) \quad (6.20)
\]
Now the double integral is given by
\[
i^{\sqrt{\kappa_{n-1} \kappa_{m-1}}} \int_0^{N^{-1/2}} L_n(x)w(x) \int_x^{N^{-1/2}} L_m(y)w(y) dy dx = \frac{mn}{N^2} \int_0^{\frac{\sqrt{N}}{m}} \hat{L}_n(x) \int_{\frac{\nu(u)}{m}}^{\frac{\sqrt{N}}{m}} \hat{L}_m(y) dy dx.
\]
By using Lemma 6.1 and (6.16). We see that it is of the order
\[
\int_{0}^{\frac{\sqrt{N}}{n}} \hat{L}_n(x) \int_{\frac{\nu(u)}{m}}^{\frac{\sqrt{N}}{m}} \hat{L}_m(y) dy dx = \frac{(-1)^{m+n} 2\pi}{\sqrt{nm}} \int_0^{u_+} J_\alpha(u) \int_{\nu(u)}^{v_+} J_\alpha(v) dv du + O\left(N^{-\frac{5}{4}}\right).
\]
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To compute the first term, note that, since 
\[ \nu'(u)J_\alpha(\nu(u)) = -\frac{d}{du} \left( \int_{\nu(u)}^{u+} J_\alpha(v)dv \right), \]
we have, by (6.20) and mean value theorem, the following
\[ (1 + O(n^{-1})) \left( J_\alpha(u) + J'_\alpha(\xi_u) \left( \frac{(m-n)u}{2n} + O \left( n^{-\frac{3}{4}} \right) \right) \right) = -\frac{d}{du} \left( \int_{\nu(u)}^{u+} J_\alpha(v)dv \right), \]
where \( \xi_u \) is between \( u \) and \( \nu(u) \). As \( J_\alpha(u) \) is bounded, we have
\[ \int_{u+}^{\nu(u)} J_\alpha(u)J_\alpha(v)dvdv = \frac{1}{2} \left( \left( \int_{0}^{u+} J_\alpha(v)dv \right)^2 - \left( \int_{\nu(u)+}^{u+} J_\alpha(v)dv \right)^2 \right), \]
\[ -\int_{0}^{u+} \left( J'_\alpha(\xi_u) \left( \frac{(m-n)u}{2n} + O \left( n^{-\frac{3}{4}} \right) \right) + O (n^{-1}) \right) \int_{\nu(u)}^{u+} J_\alpha(v)dvdv. \]
By using Lemma 6.1 and (6.19), we obtain
\[ \int_{\nu(u)+}^{u+} J_\alpha(v)dv = O \left( n^{-\frac{3}{4}} \right), \quad \int_{0}^{u+} J_\alpha(v)dv = 1 + O \left( n^{-\frac{3}{4}} \right), \]
\[ \int_{0}^{u+} \left( J'_\alpha(\xi_u) \left( \frac{(m-n)u}{2n} + O \left( n^{-\frac{3}{4}} \right) \right) + O (n^{-1}) \right) \int_{\nu(u)}^{u+} J_\alpha(v)dvdv = O \left( n^{-\frac{3}{4}} \right). \]
This gives us
\[ \int_{0}^{u+} J_\alpha(u) \int_{\nu(u)}^{u+} J_\alpha(v)dvdv = \frac{1}{2} + O \left( n^{-\frac{3}{4}} \right). \]
This proves the proposition.

This concludes the analysis in the Bessel region, we will now consider the integrals in the bulk region.

### 6.1.2 The Bulk region

First recall the following matching formula in the region \( \left[ \sqrt{\frac{\pi}{2n}}, \delta \right] \) from [33].

**Lemma 6.2.** (Proposition 4.4 of [33])

1. Uniformly for \( x \in \left[ \sqrt{\frac{\pi}{2n}}, \delta \right] \), as \( n \to \infty \),
\[
(-\tilde{f}_n)^{\frac{1}{2}} \left( \sin \zeta J_\alpha \left( 2 \left( -\tilde{f}_n \right)^{\frac{1}{2}} \right) + \cos \zeta J'_\alpha \left( 2 \left( -\tilde{f}_n \right)^{\frac{1}{2}} \right) \right) = \frac{(-1)^n}{\sqrt{\pi}} \left( \cos F_n(x) + \iota_n \sin F_n(x) \right) + O(n^{-1}),
\]
where \( \iota_n = \frac{4\alpha^2-1}{16(-\tilde{f}_n)^{\frac{1}{2}}} \) and \( F_n = \eta_+ + in\varphi_+ - \frac{\pi}{4} \).
2. Uniformly for \( x \in [4 - \varepsilon, \frac{N}{n} c_-] \), as \( n \to \infty \), we have

\[
\cos \eta_+ |f_n|^{\frac{1}{4}} Ai(f_n) - \sin \eta_+ |f_n|^{-\frac{1}{4}} Ai'(f_n) = \frac{1}{\sqrt{\pi}} \cos F_n + O \left( \frac{1}{n(4 - x)^{\frac{3}{2}}} \right). \tag{6.22}
\]

This shows that throughout the bulk region, the function \( \hat{L}_n(x) \) in (6.10) has the following asymptotic behavior.

\[
\hat{L}_n(x) = \frac{2 \sqrt{N}}{\sqrt{nx^{\frac{3}{4}}(4 - x)^{\frac{3}{4}}}} (t - \frac{n}{x})^{\frac{1}{2}} (\cos F_n(x) + E) \tag{6.23}
\]

uniformly for \( x \in \left[ \frac{\sqrt{N}}{n}, \frac{N}{n} c_- \right] \), where \( E \) is an error term of order \( O \left( \frac{1}{n^{1 - \frac{1}{2}} x} \right) \) + \( O \left( \frac{1}{n^{1 - \frac{1}{2}} x} \right) \). From this we can now compute the single integral.

**Proposition 6.3.** Let \( c_- = 4 - N^{-\varepsilon} \) such that \( \varepsilon \leq \frac{1}{20} \). Suppose the distance from the point \( t \) to the interval \( \left[ \frac{\sqrt{N}}{n}, \frac{N}{n} c_- \right] \) is greater than \( CN^{-\varepsilon} \) for some \( C > 0 \) and the distance from \( t \) to 0 is finite. Then the single integral in the bulk region is of the following order.

\[
(i \kappa_{n-1})^{\frac{1}{2}} \int_{N^{-\frac{1}{2}}}^{c_-} L_n(x) w(x) dx = I_{2,n} = O \left( n^{-\frac{7}{8}} \right) \tag{6.24}
\]

**Proof.** By (6.10), we have

\[
(i \kappa_{n-1})^{\frac{1}{2}} \int_{N^{-\frac{1}{2}}}^{c_-} L_n(x) w(x) dx = \frac{n}{N} \int_{N^{-\frac{1}{2}}}^{c_-} \hat{L}_n(x) dx.
\]

From this and (6.23), we obtain

\[
\int_{\frac{\sqrt{N}}{n}}^{c_-} \hat{L}_n(x) dx = 2 \sqrt{\frac{n}{N}} \int_{\frac{\sqrt{N}}{n}}^{c_-} \frac{\cos F_n(x) + E}{x^{\frac{3}{4}}(4 - x)^{\frac{3}{4}} (t - \frac{n}{x})^{\frac{1}{2}}} dx. \tag{6.25}
\]

Since \( F_n = \eta_+ + in\varphi_- - \frac{x}{4} \), we see that its derivative is given by

\[
F'_n = -\frac{(\alpha + 1)}{2\sqrt{x(4 - x)}} + \frac{in}{2} \left( \sqrt{\frac{x - 4}{x}} \right)_+ = -\frac{\alpha + 1 + n(4 - x)}{2\sqrt{x(4 - x)}} \tag{6.26}
\]

where the plus subscript indicates the boundary value as \( \sqrt{x - 4}/\sqrt{x} \) approaches the real axis in the upper half plane.

This gives us the following estimate of the order of \( 1/F'_n \).

\[
\frac{1}{F'_n} = -\frac{2\sqrt{x}}{n\sqrt{4 - x}} \left( 1 + O \left( \frac{1}{n(4 - x)} \right) \right) \tag{6.27}
\]
We can now integrate the first term in (6.25) by parts to obtain
\[
\int \frac{\cos F_n(x)}{x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \, dx = \frac{\sin F_n}{F_n x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \bigg|_{\frac{Nc}{n} - \infty}^{\frac{\infty}{n}}
\]
\[- \int \frac{\sin F_n(x)}{x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \frac{d}{dx} \left(\frac{1}{F_n x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}}\right) \, dx
\]
As the distance from \(t\) to the bulk region is at least of order \(O(N^{-\varepsilon})\), the first term in the above equation is of order \(O(n^{-\frac{5}{4} \varepsilon})\), which is of order \(O\left(n^{-\frac{7}{8}}\right)\) as \(\varepsilon \leq 1/20\).

By repeating the integration by parts procedure to the second term, one can verify that it is of order at most \(O(n^{11/4\varepsilon - 2})\). This gives
\[
\int \frac{\cos F_n(x)}{x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \, dx = O\left(n^{-\frac{7}{8}}\right)
\]
Since the error term \(E\) is of order \(O\left(n^{-1}(4 - x)^{-\frac{1}{2}}\right) + O\left(n^{-1}x^{-\frac{1}{2}}\right)\), we obtain the following estimate for contribution from \(E\).
\[
2\sqrt{n \over N} \int \frac{E}{x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \, dx = O\left(n^{\frac{3}{4} \varepsilon - 1}\right)
\]
This concludes the proof of the proposition. \(\square\)

In particular, from the proof of Proposition 6.3, we see that
\[
\int \frac{\cos F_n(x)}{x^{\frac{3}{2}}(4 - x)^{\frac{1}{2}} \left(t - \frac{n}{N}x\right)^{\frac{3}{2}}} \, dx = O\left(n^{-\frac{7}{8}}\right), \quad y \in \left[\sqrt{\frac{N}{n}}, \frac{N}{n}c_-\right] \quad (6.28)
\]
Let us now proceed to compute the double integrals. To begin with, we have the following lemma that will help us to simplify the results. (Proposition 5.12 in [33])

**Lemma 6.3.** The function \(\theta(x)\) defined by
\[
\theta(x) = \frac{1}{2} \int_4^x \sqrt{\frac{4 - s}{s}} \, ds = -i \varphi_+, \quad x \in [0, 4]
\] satisfies the following
\[
\theta(x) - x\theta'(x) = -\arccos\left(\frac{x}{2} - 1\right) \quad (6.30)
\]
Proof. By differentiating $\theta(x)$ twice we see that

$$x\theta''(x) = -\frac{1}{\sqrt{x(4-x)}}.$$ 

As $\arccos(x)$ is given by

$$\arccos(x) = \int_x^1 \frac{1}{\sqrt{1-s^2}}ds,$$

we see that

$$\frac{d}{dx} (\theta(x) - x\theta'(x)) = -\frac{d}{dx} \arccos \left( \frac{x}{2} - 1 \right).$$

Integrating the above, we obtain

$$\theta(x) - x\theta'(x) = -\arccos \left( \frac{x}{2} - 1 \right) + C.$$ 

For some integration constant $C$. By evaluating the above equation at $x = 4$, we see that $C = 0$. This proves the lemma. 

The lemma implies the following. (Proposition 5.13, [33])

Lemma 6.4. Let $n = N - j$, $m = N - k$ where $j$ and $k$ are finite integers, then for $x \in \left[ \sqrt{N/(n-2)} \right.$,$\left. N/(n-2) \right]$, we have

$$F_m \left( \frac{x}{m} \right) - F_n(x) = (m - n) \arccos \left( \frac{x}{2} - 1 \right) + O \left( \frac{1}{m^{2k-1}} \right).$$ (6.32)

Proof. As in [33], let us write the left hand side of (6.32) as

$$F_m \left( \frac{x}{m} \right) - F_n(x) = \left( F_m \left( \frac{x}{m} \right) - F_m(x) \right) + (F_m(x) - F_n(x))$$

By repeat application of the mean value theorem, we see that the first term on the right hand side is given by

$$F_m \left( \frac{x}{m} \right) - F_m(x) = \frac{x(n-m)}{m} F_m'(x) + \frac{x^2(n-m)^2}{2m^2} F_m''(\xi)$$

for some $\xi \in [x, xn/m]$. From (6.26), we see that for $x \in \left[ \sqrt{N/(n-2)} \right.$,$\left. N/(n-2) \right]$, we have

$$\frac{x(n-m)}{m} F_m'(x) = (m - n)x\theta'(x) + O \left( \frac{1}{m\sqrt{4-x}} \right),$$

$$\frac{x^2(n-m)^2}{2m^2} F_m''(\xi) = O \left( \frac{1}{m^2(4-x)^{3/2}} \right) + O \left( \frac{1}{m\sqrt{4-x}} \right).$$
As $4 - x = N^{-\varepsilon}$ and $\varepsilon \leq 1/20$, we have
\[
F_m \left( x \frac{n}{m} \right) - F_n(x) = (m-n)x\theta'(x) + (F_m(x) - F_n(x)) + O (m^{\frac{7}{2}})
\]
\[
= (m-n)x\theta'(x) + (n-m)\theta(x) + O (m^{\frac{7}{2}})
\]
\[
= (m-n) \arccos \left( \frac{x}{2} - 1 \right) + O (m^{\frac{7}{2}}),
\]
where the last equality follows from lemma [6.3]. □

We can now compute the double integrals in the bulk region.

Proposition 6.4. The double integral inside the bulk region is of the following order.
\[
\int_{N^{-\varepsilon}} L_n(x)w(x) \int_{x}^{c_-} L_m(y)w(y)dydx = J_2
\]
\[
= \frac{4}{N} \int_{\frac{N}{m}}^{\frac{N}{m}} \sin \left( (n-m) \arccos \left( \frac{x}{2} - 1 \right) \right) \frac{dx}{x(4-x)(t-x)} + O \left( n^{-\frac{7}{2}} \right)
\]
where $c_- = 4 - N^{-\varepsilon}$.

Proof. The proof is similar to the computation in [33] and [31]. We have
\[
J_2 = \frac{mn}{N^2} \int_{\frac{N}{m}}^{\frac{N}{m}} \hat{L}_n(x) \int_{\frac{N}{m}}^{\frac{N}{m}} \hat{L}_m(y)dydx,
\]
Then by using (6.23), we see that the double integral is given by the following.
\[
J_2 = \frac{4\sqrt{mn}}{N} \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{\cos F_n(x) + E(x)}{x^{\frac{3}{4}}(4-x)^{\frac{1}{4}}(t - \frac{n}{N}x)^{\frac{1}{2}}} \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{\cos F_m(y) + E(y)}{y^{\frac{3}{4}}(4-y)^{\frac{1}{4}}(t - \frac{n}{N}y)^{\frac{1}{2}}} dydx
\]
First let us compute the error terms. By changing the order of the integration, we have
\[
\int_{\frac{N}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{x^{\frac{3}{4}}(4-x)^{\frac{1}{4}}(t - \frac{n}{N}x)^{\frac{1}{2}}} \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{E(y)}{y^{\frac{3}{4}}(4-y)^{\frac{1}{4}}(t - \frac{n}{N}y)^{\frac{1}{2}}} dydx
\]
\[
= \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{E(y)}{y^{\frac{3}{4}}(4-y)^{\frac{1}{4}}(t - \frac{n}{N}y)^{\frac{1}{2}}} \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{x^{\frac{3}{4}}(4-x)^{\frac{1}{4}}(t - \frac{n}{N}x)^{\frac{1}{2}}} dx dy
\]
\[
= \int_{\frac{N}{m}}^{\frac{N}{m}} \frac{E(y)}{y^{\frac{3}{4}}(4-y)^{\frac{1}{4}}(t - \frac{n}{N}y)^{\frac{1}{2}}} O \left( n^{-\frac{7}{2}} \right) dy = O \left( n^{-\frac{7}{2}} \right),
\]
where the last equality follows from (6.23). The order of the other error term can be estimated similarly. Let us now consider the leading order term. Integrating by parts, we
Hence by interchanging the order of integration, we obtain

\[ J_{20} = \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{x^\frac{3}{4}(4 - x)^\frac{1}{2}} \frac{\cos F_m(y)}{y^\frac{3}{4}(4 - y)^\frac{1}{2}} \, dx \, dy \]

\[ = \left( \frac{m}{n} \right)^{\frac{3}{4}} \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{F_m' \left( \frac{n}{m} x \right) x^\frac{3}{4}(4 - x)^\frac{1}{2}} \sin F_m \left( \frac{n}{m} x \right) \, dx \]

\[ - \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{x^\frac{3}{4}(4 - x)^\frac{1}{2}} \frac{\sin F_m(y)}{\frac{n}{m} x} \, dy \, dy \]

\[ + O \left( n^{-\frac{7}{4}} \right) \]

To evaluate the second term, note that for \( y \) in the interval \( \left[ \frac{\sqrt{N}}{m}, \frac{N}{m} \right] \), we have

\[ \sin F_m(y) \frac{d}{dy} \left( \frac{1}{F_m y^\frac{3}{4}(4 - y)^\frac{1}{2} \left( t - \frac{m}{N} y \right)^\frac{1}{2}} \right) = O \left( \frac{1}{m^{1-\epsilon}(4 - y)^\frac{3}{2} y^\frac{3}{2}} \right) \]

Hence by interchanging the order of integration, we obtain

\[ \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n(x)}{x^\frac{3}{4}(4 - x)^\frac{1}{2}} \frac{\cos F_m(y)}{x^\frac{3}{4}(4 - x)^\frac{1}{2}} \, dx \, dy \]

\[ = O \left( m^{-\frac{7}{4}} \right) + O \left( m^{-\frac{7}{4} + \frac{1}{2}} \right) = O \left( m^{-\frac{7}{4}} \right) \]

To compute the first term of \( J_{20} \), let us change the integration variable from \( x \) to \( \frac{n}{m} x \). Then we obtain

\[ J_{20} = \left( \frac{m}{n} \right)^{\frac{1}{4}} \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n \left( \frac{n}{m} x \right)}{F_m' \left( \frac{n}{m} x \right) x^\frac{3}{4}(4 - x)^\frac{1}{2}} \sin F_m \left( \frac{n}{m} x \right) \, dx + O(m^{-\frac{7}{4}}) \]

Now by (6.27) and the fact that both \( 4 - x \) and \( t - \frac{n}{N} x \) are of order at least \( n^{-\epsilon} \), we obtain the following estimate

\[ J_{20} = -2 \left( \frac{m}{n} \right)^{\frac{1}{4}} \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\cos F_n \left( \frac{n}{m} x \right) \sin F_m \left( \frac{n}{m} x \right)}{m x(4 - x) (t - x)} \, dx + O(m^{-2+3\epsilon}) + O \left( m^{-\frac{7}{4}} \right) \]

To evaluate the integral, we use the angle addition formula for sine to obtain

\[ J_{20} = - \left( \frac{m}{n} \right)^{\frac{1}{4}} \left( \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\sin (F_n(x) - F_n \left( \frac{n}{m} x \right))}{m x(4 - x) (t - x)} \, dx + \int_{\frac{n}{m}}^{\frac{N}{m}} \frac{\sin \left( \frac{n}{m} x \right) + F_m \left( \frac{n}{m} x \right)}{m x(4 - x) (t - x)} \, dx \right) \]

\[ + O(m^{-\frac{7}{4}}). \]
The first term can be simplified using \((6.32)\) while integration by parts shows that the second term is of order \(O \left( m^{-\frac{7}{4}} \right) \). This gives

\[
J_{20} = \int_{\frac{\sqrt{N}}{m}}^{\sqrt{N}} \frac{\sin \left( (n - m) \arccos \left( \frac{x}{2} - 1 \right) \right)}{mx(4 - x)(t - x)} \, dx + O(m^{-\frac{7}{4}}).
\]

As \(\frac{N}{m} c_+ - \frac{N}{m} c_-\) is of order \(m^{-1}\), we see that

\[
\int_{\frac{N}{m} c_+}^{\frac{N}{m} c_-} \frac{\sin \left( (m - n) \arccos \left( \frac{x}{2} - 1 \right) \right)}{mx(4 - x)(t - x)} \, dx = O \left( m^{-2+2\epsilon} \right).
\]

Similarly, we can change the lower limit to \(\sqrt{N} n c_-\) and add an error term of order \(O \left( m^{-2} \right)\).

This proves the proposition.

As we are only going to consider the values of \(m\) and \(n\) with \(m - n = \pm 1\) or \(\pm 2\), we can simplify the double integrals further. First let us consider the case when \(m - n = \pm 1\). A simple calculation shows that

\[
\sin \left( \arccos \left( \frac{x}{2} - 1 \right) \right) = \frac{\sqrt{x(4 - x)}}{2}.
\]

(6.34)

From this we obtain

\[
\sin \left( 2 \arccos \left( \frac{x}{2} - 1 \right) \right) = \frac{\sqrt{x(4 - x)(x - 2)}}{2}.
\]

(6.35)

We can now use these and residue calculation to compute the double integrals.

**Proposition 6.5.** Let \(F_l\) be

\[
F_l = \int_{\frac{\sqrt{N}}{m} c_-}^{\sqrt{N}} \frac{4 \sin \left( l \arccos \left( \frac{x}{2} - 1 \right) \right)}{Nx(4 - x)(t - x)} \, dx \quad (6.36)
\]

Then the asymptotics of the double integrals in the bulk region are given by the followings.

\[
J_2 = \pm \frac{2\pi}{N\sqrt{t(t-4)}} \mp F_1 + O \left( N^{-\frac{7}{4}} \right), \quad n - m = \pm 1,
\]

\[
J_2 = \pm \frac{2\pi}{N} \left( \frac{t - 2}{\sqrt{t(t-4)}} - 1 \right) \mp F_2 + O \left( N^{-\frac{7}{4}} \right), \quad n - m = \pm 2.
\]

(6.37)

**Proof.** We shall compute the integrals using Cauchy theorem. Since \((\sqrt{x-4})_+ = i\sqrt{4-x}\) on \([0, 4]\), we have, by using \((6.34)\), the following

\[
\int_0^4 \frac{\sin \left( \arccos \left( \frac{x}{2} - 1 \right) \right)}{x(4 - x)(t - x)} \, dx = i \int_0^4 \frac{1}{2\sqrt{x(4 - x)(t - x)}} \, dx,
\]

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and similar relations for \( n - m = 2 \). The right hand side can be computed using Cauchy’s theorem and we obtain

\[
\int_0^4 \frac{\sin\left((n - m) \arccos \left(\frac{x}{2} - 1\right)\right)}{x(4 - x)(t - x)} \, dx = \begin{cases} 
\frac{\pi}{2 \sqrt{t(t-4)}} & \text{if } n - m = 1; \\
\frac{\pi}{2} \left(\frac{t-2}{\sqrt{t(t-4)}} - 1\right) & \text{if } n - m = 2; 
\end{cases}
\]

By (6.34) and (6.35), we see that

\[
\int_0^{\frac{\sqrt{N}}{4}} \frac{4 \sin \left(l \arccos \left(\frac{x}{2} - 1\right)\right)}{N x(4 - x)(t - x)} \, dx = O \left(N^{-\frac{5}{4}}\right).
\]

From this and (6.33), (6.34) and (6.35), we have

\[
\mathcal{J}_2 = \int_0^4 \frac{\sin\left((n - m) \arccos \left(\frac{x}{2} - 1\right)\right)}{x(4 - x)(t - x)} \, dx - F_{n-m} + O \left(N^{-\frac{5}{4}}\right).
\]

This completes the proof of the Proposition.

This completes the analysis in the bulk region. We will now move onto the Airy region.

### 6.1.3 The Airy region

The analysis in the Airy region is more difficult compared to the other regions as we will need to consider the case where the point \( t \) is inside the Airy region. First let us compute the asymptotics of the function \( \hat{L}_n(x) \) in the Airy region.

As the asymptotics contain the functions \( Ai(f_n) \) and \( Ai'(f_n) \), we shall make a change of variable and write the asymptotics in terms of the function \( f_n \). First, from the expression of \( f_n \) in Definition 6.1, we see that the map \( f_n \) is of the following order inside the Airy region.

\[
f_n = \left(\frac{n}{4}\right)^{\frac{3}{2}} \left(x - 4\right) \left(1 - \frac{x - 4}{20} + O((x - 4)^2)\right), \tag{6.38}
\]

and hence \( x - 4 \) is of order

\[
x - 4 = \left(\frac{4}{n}\right)^{\frac{3}{2}} f_n \left(1 + \frac{1}{20} f_n \left(\frac{4}{n}\right)^{\frac{3}{2}} + O \left(\frac{f_n^2}{n^{\frac{3}{2}}}\right)\right). \tag{6.39}
\]

Let us introduce the scaled variable \( T \) to be

\[
T = \left(\frac{N}{4}\right)^{\frac{3}{2}} (t - 4). \tag{6.40}
\]

If \( T \) is close to the real axis, say \( T = T_R + i T_I, \) \( T_R, T_I \in \mathbb{R} \) and \( |T_I| < \delta \), then we will deform the integration contour \( \mathbb{R} \) in \( \langle L_n, L_m \rangle \) inside a small neighborhood of \( T_R \) into the
upper/lower half plane depending on the sign of $T_I$, so that the distance between $T$ and the integration contour is always greater than some finite $\delta > 0$. If $T \in \mathbb{R}$, then for definiteness, we will deform the contour into the lower half plane and also deform the branch cut of $(T - f_n)^{\frac{1}{2}}$ into the upper half plane so that the integrals are well-defined. As pointed out in the introduction, such deformations will not affect our final result. For simplicity of the notations, we will denote the integration contour in the Airy region by $\mathcal{R}$ in all cases, but with the understand that appropriate deformations of the contours are performed when $T$ is close to $\mathbb{R}$.

As $T$ is of finite distance from the integration contour, we have

\[
(t - \frac{n}{N}x)^{-\frac{1}{2}} = \left(\frac{4}{N}\right)^{-\frac{1}{2}} (T - f_n)^{-\frac{1}{2}}
\]

\[
\times \left(1 + \frac{4\frac{1}{2}(N - n)}{N^\frac{1}{2}(T - u)} + \frac{1}{T - u} G_1 \left(\frac{u(n - N)}{N}\right) + \frac{n}{4\frac{1}{2} N^\frac{1}{2}(T - u)} G_2 \left(\frac{u}{n^\frac{1}{2}}\right)\right)^{-\frac{1}{2}}
\]

\[
= \left(\frac{4}{N}\right)^{-\frac{1}{2}} (T - f_n)^{-\frac{1}{2}} \mathcal{G}_n^{-\frac{1}{2}}(f_n),
\]

where $G_1$ and $G_2$ are power series expansions in their arguments with coefficients independent on $n$ and $N$. Note that the series expansion $B$ starts at power 1 while the expansion $C$ starts at power 2.

Let us write $f_n = u$, then by using (6.41) and (6.39), we can write $\hat{L}_n(x)/u'$ as a series in terms of $u$.

\[
\hat{L}_n(x)/u' = \frac{N^\frac{1}{2} \sqrt{\pi}}{\sqrt{2n^\frac{1}{2}(T - u)^\frac{1}{2}}} \left(\frac{n}{4}\right)^{-\frac{1}{2}} A\left(\frac{u}{n^{\frac{1}{2}}}(T - u)^{\frac{1}{2}}\right) \left(a_0 + \sum_{k=1}^{\infty} a_k u^k\right)
\]

\[
+ \frac{\alpha + 1}{2} \left(\frac{n}{4}\right)^{-\frac{1}{2}} A\left(\frac{u}{n^{\frac{1}{2}}}(T - u)^{\frac{1}{2}}\right) b_0 + \sum_{k=1}^{\infty} \frac{b_k u^k}{n^{\frac{1}{2}}}\right) \mathcal{G}_n(u)
\]

For some constants $a_0$, $b_0$, $a_k$ and $b_k$ of the form

\[
a_0 = 1 + O(n^{-1}), \quad b_0 = 1 + O(n^{-1}), \quad a_k = a_{k,0} + O(n^{-1}), \quad b_k = b_{k,0} + O(n^{-1}),
\]

where $a_{k,0}$ and $b_{k,0}$ are independent on $n$.

As the functions $\hat{L}_n$ are close to each other inside the Airy region, we will introduce the following function $\mathcal{A}(U)$.

\[
\mathcal{A}(U) = \hat{L}_N(x(U))/u'(x(U)),
\]

where $x(U)$ is the inverse function of $U = f_N(x)$. The function $\mathcal{A}(U)$ is regarded as a function in the variable $U$. We shall express the double integrals in terms of the function $\mathcal{A}$.
As in (6.39) we can write \(x(U)\) as a power series in \(U\) (but with \(n\) replaced by \(N\)). In particular, we have

\[
\mathcal{A}(U) = \frac{\sqrt{\pi}}{\sqrt{2(T - U)^{\frac{1}{2}}}} \left( \frac{N}{4} \right)^{-\frac{1}{6}} Ai(U) \left( \tilde{a}_0 + \sum_{k=1}^{\infty} \frac{\tilde{a}_k U^k}{N^{\frac{2k}{3}}} \right) + \frac{\alpha + 1}{2} \left( \frac{N}{4} \right)^{-\frac{1}{6}} Ai'(U) \left( \tilde{b}_0 + \sum_{k=1}^{\infty} \frac{\tilde{b}_k U^k}{N^{\frac{2k}{3}}} \right) G_N(U)
\]

where \(\tilde{a}_k = a_k + O\left(n^{-1}\right)\) and \(\tilde{b}_k = b_k + O\left(n^{-1}\right)\). If we now let \(x\) be the inverse function of \(u = f_n(x)\) and use (6.41) and (6.42) to represent \(\tilde{L}_n(x(u))/u'(x(u))\) as a function in \(u\), then we see that, as a function of \(u\), we have

\[
\frac{\tilde{L}_n(x(u))}{u'(x(u))} = \left( \mathcal{A}(u) + \frac{Ai(u)}{(T - u)^{\frac{1}{2}}} \left( \frac{N}{4} \right)^{-\frac{1}{6}} C_{1,n}(u) + \frac{Ai'(u)}{(T - u)^{\frac{1}{2}}} \frac{Ai(u)}{N^{\frac{1}{2}}} C_{2,n}(u) \right) \times (\mathcal{E}_n(u) + \mathcal{D}_n(u)),
\]

where \(C_{i,n}(u)\) and \(\mathcal{D}_n(u)\) are power series of the form

\[
C_{i,n}(u) = \sum_{k,j} c_{i,k,j,n} \frac{u^k}{N^{2(k+j+1)}} (T - u)^j, \quad i = 1, 2,
\]

\[
\mathcal{D}_n(u) = \sum_{k,j} d_{k,j,l,n} \frac{u^k}{N^{2(k+j+1)}} (T - u)^j
\]

where \(d_{k,j,l,n}\) and \(c_{i,k,j,n}\) are bounded and \(k \geq 2j\). In the series \(C_{i,n}(u)\), both indices \(k\) and \(j\) start from 0, while in \(\mathcal{D}_n(u)\), \(k\) and \(l\) start from 1 and \(j\) from 0. The term \(\mathcal{E}_n(u)\) is of the form

\[
\mathcal{E}_n(u) = 1 - \frac{1}{2} v_{0,n} + \frac{3}{8} v_{0,n}^2 + O\left(N^{-1}\right),
\]

where \(v_{0,n}\) is given by

\[
v_{0,n} = \frac{4\pi(N - n)}{N^{\frac{1}{2}}(T - u)}
\]

We are now ready to compute the single integrals. First let us show the following

**Lemma 6.5.** Let \(k, j \geq 0\), then as \(s_{\pm} \to +\infty\) and \(|s_-|/|s_+| = O(1)\), we have

\[
\int_{s_-}^{s_+} \frac{u^k Ai(u)}{(T - u)^{\frac{1}{2}}} du = O\left(|s_-|^{k+\frac{1}{2}+\frac{3}{4}}\right) + O\left(1\right),
\]

\[
\int_{s_-}^{s_+} \frac{u^k Ai'(u)}{(T - u)^{\frac{1}{2}}} du = O\left(|s_-|^{k+\frac{1}{2}+\frac{1}{4}}\right) + O\left(1\right),
\]
Let \( \nu(u) \) be a function of \( u \) such that \( \nu(u) = O(u) \) as \( u \to \infty \) and let \( v_+ = O(s_+) \), then we have
\[
\int_{s_-}^{s_+} \frac{u^{k_1} Ai^{(i_1)}(u)}{(T - u)^{\frac{k_1}{2}}} \frac{u^{k_2} Ai^{(i_2)}(u)}{(T - u)^{\frac{k_2}{2}}} \, dv \, du = O\left(|s_+|^{k_1 + k_2 + \frac{i_1 + i_2 - i_1 - i_2}{2}}\right) + O(1), \tag{6.49}
\]
where \( Ai(0) = Ai \) and \( Ai^{(1)} = Ai' \).

**Proof.** First note that, as the Airy function decays exponentially as \( u \to \infty \),
\[
Ai(u) = \frac{1}{2 \sqrt{\pi}} u^{-\frac{1}{4}} e^{-\frac{2}{3} u^{\frac{3}{2}}} \left(1 + O\left(u^{-\frac{3}{4}}\right)\right), \quad u \to +\infty,
\]
\[
Ai(-u) = \sin\left(\frac{2}{3} u^{\frac{3}{2}} + \frac{\pi}{4}\right) \sqrt{\frac{\pi}{u^{\frac{1}{4}}} \left(1 + O\left(u^{\frac{3}{4}}\right)\right)}, \quad u \to -\infty.
\]
From this we have
\[
\int_{s_-}^{s_+} u^{k} Ai(u) \frac{1}{(T - u)^{\frac{k}{2}}} \, dv \, du = O(1),
\]
as \( s_+ \to +\infty \). Therefore let us consider the integral in the negative real axis. Integrating by parts, we obtain
\[
\int_{s_-}^{0} u^{k} Ai(u) \frac{1}{(T - u)^{\frac{k}{2}}} \, dv \, du = \left(\frac{u^{k} \int_{-\infty}^{0} Ai(v) \, dv}{(T - u)^{\frac{k}{2}}}\right)_{s_-}^{0} - k \int_{s_-}^{0} u^{k-1} \int_{-\infty}^{u} Ai(v) \, dv \, dv \, du
\]
\[
- \frac{j}{2} \int_{s_-}^{0} u^{k} \int_{-\infty}^{u} \frac{Ai(v) \, dv}{(T - u)^{\frac{k+1}{2}}} \, dv \, du
\]
Now by (6.50), we see that \( \int_{-\infty}^{u} Ai(v) \, dv = O\left(u^{-\frac{3}{4}}\right) \) as \( u \to -\infty \) and is bounded for \( u \in \mathbb{R} \). This gives us the following estimate
\[
\int_{s_-}^{0} u^{k} Ai(u) \frac{1}{(T - u)^{\frac{k}{2}}} \, dv \, du = O\left(|s_-|^{k-\frac{3}{2} - \frac{3}{4}}\right) + O(1)
\]
For integrals involving the derivative \( Ai'(u) \), we again note that \( Ai'(u) \) is decaying exponentially as \( u \to +\infty \) and we again have
\[
\int_{0}^{s_+} u^{k} Ai'(u) \frac{1}{(T - u)^{\frac{k}{2}}} \, dv \, du = O(1).
\]
For the integral on the negative real axis, we perform integration by parts again and use the estimate
\[
|Ai(s)| \leq C \left(1 + |s|\right)^{-\frac{3}{4}}, \quad s < 0 \tag{6.52}
\]
for some constant $C$. This shows that the integral on the negative real axis is of order
\[
\int_{s_-}^{0} u^k A_i'(u) \frac{du}{(T-u)^{\frac{1}{2}}} = O \left( |s_-|^{k-\frac{1}{2}-\frac{1}{4}} \right) + O(1).
\]
This proves (6.48). The estimate (6.49) now follows immediately from (6.48) and (6.50).

Let us now transform the limits in the Airy region into the variable $u$. As in the previous cases, we are interested in the integral
\[
(i\kappa_{n-1})^{\frac{1}{2}} \int_{c_-}^{c_+} L_n(x) w(x) dx = \mathcal{I}_{\Lambda,n} = \frac{n}{N} \int_{c_-}^{c_+} \mathcal{L}_n(x) dx,
\]
The following is an immediate consequence of (6.44) and the estimates (6.48).

**Proposition 6.6.** Let $T$ be defined as in (6.40), then the single integral in the Airy region is given by
\[
\frac{N}{n} (i\kappa_{n-1})^{\frac{1}{2}} \int_{c_-}^{c_+} L_n(x) w(x) dx = \int_{u_-}^{u_+} \mathcal{A} \mathcal{E}_n du + O \left( N^{-\frac{5}{6}} \right) \quad (6.53)
\]
This completes the analysis of the single integral in the Airy region. We will now analyze the double integrals.

By (6.10), we have
\[
i\sqrt{\kappa_{n-1}\kappa_{m-1}} \int_{c_-}^{c_+} L_n(x) w(x) \int_{x}^{c_+} L_m(y) w(y) dy dx = \mathcal{J}_3 = \frac{mn}{N^2} \int_{c_-}^{c_+} \mathcal{L}_n(x) \int_{x}^{c_+} \mathcal{L}_m(y) dy dx, \quad (6.54)
\]
We will change the integration variables to $u = f_n(x)$ and $v = f_m(y)$. We will denote the limits of the outer integral by $u_\pm$ and the upper limit of the inner integral by $v_+$. Let us now compute the lower limit of the inner integral. As both $u = f_n(x)$ and $v = f_m(y)$ are conformal inside the Airy region, they can be written as a series of each other. Then by (6.39) and the analogue for $f_m$, together with the fact that at the lower bound, $y = \frac{n}{m}x$, we obtain
\[
4 \left( \frac{n}{m} - 1 \right) + \frac{n}{m} \left( \left( \frac{4}{n} \right)^{\frac{1}{2}} u + \frac{1}{20} \left( \frac{4}{n} \right)^{\frac{3}{2}} u^2 + O \left( \frac{u^3}{n^2} \right) \right) = \left( \frac{4}{m} \right)^{\frac{1}{2}} v + \frac{1}{20} \left( \frac{4}{m} \right)^{\frac{3}{2}} v^2 + O \left( \frac{v^3}{m^2} \right)
\]
The following can easily be seen by writing $v$ as a series expansion in $u$. 

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Lemma 6.6. Let $u = f_n(x)$ and let $\nu(u)$ be the value of $v = f_m$ at $y = \frac{n}{m} x$, then as $n, m \to \infty$ with $m - n$ finite, we have

$$\nu(u) = v_0 + (1 + v_1) u + \sum_{l=2}^{\infty} v_l u^l, \quad v_0 = (n - m) \left( \frac{4}{m} \right)^{\frac{1}{3}} + O \left( m^{-\frac{1}{3}} \right), \quad (6.55)$$

$$\delta v_1 = \frac{1}{15} \frac{m - n}{m} + O \left( m^{-2} \right), \quad v_l = O \left( m^{-\frac{2l+1}{3}} \right), \quad l \geq 2.$$  

By (6.49) and (6.44), we see that the double integral is given by

$$i \sqrt{\kappa_{n-1}\kappa_{m-1}} \int_{c_-}^{c_+} L_n(x) w(x) \int_{c_-}^{c_+} L_m(y) w(y) dx dy = J_3$$

$$= \frac{mn}{N^2} \left( \int_{u_-}^{u_+} \varepsilon_n A(u) \int_{v(u)}^{v_+} \varepsilon_m A(v) dv du \right) + O \left( N^{-\frac{1}{2}} \right). \quad (6.56)$$

Let us consider the terms in $\int_{u_-}^{u_+} \varepsilon_n A(u) \int_{v(u)}^{v_+} \varepsilon_m A(v) dv du$ in (6.56). This term can be written as

$$\int_{u_-}^{u_+} \varepsilon_n A(u) \int_{v(u)}^{v_+} \varepsilon_m A(v) dv du$$

$$= \sum_{j,k=0}^{2} \frac{h_{kj}}{N^{\frac{1}{2}}} ((N-n)^j(N-m)^k A(j,k) + (N-n)^k(N-m)^j A(k,j)) \quad (6.57)$$

for some constants $h_{kj}$ that are independent on $N, n$ and $m$, where $A(j,k)$ are defined as follows.

$$A(j,k) = \int_{u_-}^{u_+} \frac{A(u)}{(T-u)^j} \int_{v(u)}^{v_+} \frac{A(v)}{(T-v)^k} dv du. \quad (6.58)$$

First let us consider the leading order term in (6.57), which is given by $A(0,0)$.

Let $L(u)$ be the following function in $u$.

$$L(u) = \frac{dx_N}{du} \frac{2^{\frac{1}{2}}(-u)^{\frac{1}{2}}(T-u)^{\frac{1}{2}}}{x_N(u)^{\frac{1}{2}}(4 - x_N(u))^\frac{1}{2}} \quad (6.59)$$

where $x_N(u)$ is defined as the inverse function to $u = f_N(x_N)$.

Note that $L(u)$ is of order

$$L(u) = \left( \frac{N}{4} \right)^{-\frac{1}{2}} \left( 1 + O \left( \frac{u}{N^{\frac{1}{2}}} \right) \right) \quad (6.60)$$

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where the error term is uniform in $T$ as $T \to N^{\frac{2}{3}}$. Then $A(u)$ can be written as

$$A(u) = \sqrt{\frac{\pi}{2}} u^\frac{1}{2} \cos \eta_+(x_N)Ai(u) \left(1 + O(N^{-1})\right) - u^\frac{1}{4} \sin \eta_+(x_N)Ai'(u) \left(1 + O(N^{-1})\right) \mathcal{L}(u)$$

The following can be derived using (6.22).

Lemma 6.7. The derivatives of $A(u)$ behave as follows as $u \to -CN^{\frac{2}{3} - \varepsilon}$ for some $C > 0$.

$$A^{(2k-1)}(u) = \frac{1}{\sqrt{2}} \left( \frac{(-1)^{k-1}(-u)^{\frac{2k-1}{4}} \sin F_N(x_N(u))}{(-u)^{\frac{1}{2}}(T-u)^{\frac{3}{4}}} \mathcal{L}(u) + O\left(\frac{u^{k-\frac{3}{4}}}{(T-u)^{\frac{3}{4}}}\right) \right),$$

$$A^{(2k)}(u) = \frac{1}{\sqrt{2}} \left( \frac{u^k \cos F_N(x_N(u))}{(-u)^{\frac{1}{2}}(T-u)^{\frac{3}{4}}} \mathcal{L}(u) + \left(\frac{u^{k-\frac{3}{4}}}{(T-u)^{\frac{3}{4}}}\right) \right),$$

(6.61)

where $k \geq 0$ and $x_N(u)$ is the inverse of $u = f_N(x)$.

Changing the upper limit $u_+$ into $u_-$, we can write the term $A(0,0)$ as

$$A(0,0) = \frac{1}{2} \left( \int_{u_-}^{u_+} A(u)du \right)^2 - \sum_{j=0}^{\infty} \int_{u_-}^{u_+} A(u)A^{(j)}(u)(\nu - u)^{j+1}du + O \left( e^{-\frac{1}{4}u_+^{\frac{3}{2}}} \right)$$

(6.62)

where $\nu$ is treated as a function of $u$ by using Lemma 6.6.

Let us estimate the order of these terms.

Lemma 6.8. Let $u_-/N^{\frac{2}{3}} = O(N^{-\varepsilon})$. If $j$ is odd, then we have

$$\int_{u_-}^{u_+} A(u)A^{(j)}(u)(\nu - u)^{j+1}du = \left\{ O \left( m^{-2+\frac{3}{2}} \right) \right\}$$

(6.63)

If $j$ is even, then we have

$$\int_{u_-}^{u_+} A(u)A^{(j)}(u)(\nu - u)^{j+1}du = \frac{1}{4} \int_{u_-}^{0} u^{\frac{3}{2}}L^2(u)(\nu - u)^{j+1}du$$

$$+ \left\{ O \left( N^{-\frac{3}{2}} \right), \quad j > 0; \right\}$$

$$O \left( N^{-\frac{3}{2}} \right), \quad j = 0.$$  

(6.64)

Proof. The first equation can be proven using integration by parts. By (6.55), (6.61) and (6.60), we have

$$\int_{u_-}^{u_+} A(u)A^{(j)}(u)(\nu - u)^{j+1}du = -A(u_-)A^{(j-1)}(u_-)(\nu - u_-)^{j+1}$$

$$- \int_{u_-}^{u_+} A(u)A^{(j-1)}(u)(\nu - u)^{j+1}du + O \left( \left( \frac{u^{\frac{3}{2}}}{m^{\frac{3}{2}}} \right) m^{-2} \right)$$

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From (6.61), (6.55) and (6.60), the first term is of order \( O \left( \left( \frac{u}{m} \right)^{\frac{i+4}{2}} m^{-2} \right) \). Repeating the integration by parts, we obtain (6.63).

To prove (6.64), we have, by (6.61), the following

\[
A(u)A^{(j)}(u) - \frac{1}{4} \frac{u^\frac{i}{2} \mathcal{L}^2(u)}{(-u)^{\frac{i}{2}}(T-u)} - \frac{1}{4} \frac{u^\frac{i}{2} \cos 2F_N(x_N)}{(-u)^{\frac{i}{2}}(T-u)} \mathcal{L}^2(u) = O \left( u^{\frac{i+6}{2}} m^{-\frac{1}{3}} \right)
\]  

(6.65)
as \( u \to u_- \). After multiplying by \( (\nu(u) - u)^{j+1} \) and integrate, the error term \( O \left( u^{\frac{i+6}{2}} m^{-\frac{1}{3}} \right) \) gives a contribution of order \( O \left( N^{-\frac{4}{3}} \right) \) for \( j > 0 \) and \( O \left( N^{-\frac{2}{3}} \right) \) for \( j = 0 \).

The term with the \( \cos 2F_N \) factor can be integrated by parts using

\[F_N = \eta_+ + iN\varphi_+ - \frac{\pi}{4} = \frac{2}{3} (-u)^{\frac{i}{2}} + O \left( \frac{u^{\frac{i}{2}}}{N^{\frac{1}{2}}} \right) - \frac{\pi}{4} \]

\[\frac{dF_N}{du} = (-u)^{\frac{i}{2}} + O \left( \frac{1}{u^{\frac{i}{2}}N^{\frac{1}{2}}} \right)\]

Then by splitting the interval \([u_-, 0]\) into \([u_-, u_0]\) and \([u_0, 0]\) for some \( u_0 \) between \( u_- \) and 0, and integrating the integral on \([u_-, u_0]\) by parts, we have

\[
\int_{u_-}^{0} \left( A(u)A^{(j)}(u) - \frac{1}{4} \frac{u^\frac{i}{2}}{(-u)^{\frac{i}{2}}(T-u)} \mathcal{L}^2(u) \right) (\nu(u) - u)^{j+1} \, du
\]

\[= O \left( \frac{u_{\max(0,j-4)}}{m^{\frac{i+4}{2}}} m^{-2} \right) + O \left( m^{-\frac{i+1}{2}} \right)\]

Hence we have

\[
\int_{u_-}^{0} \left( A(u)A^{(j)}(u)(\nu - u)^{j+1} - \frac{1}{4} \frac{u^\frac{i}{2}(\nu - u)^{j+1}}{(-u)^{\frac{i}{2}}(T-u)} \mathcal{L}^2(u) \right) \, du
\]

\[= \begin{cases} O \left( N^{-\frac{4}{3}} \right), & j > 0; \\ O \left( N^{-\frac{2}{3}} \right), & j = 0. \end{cases}\]  

(6.66)

As \( A(u)A^{(j)}(u)(\nu - u)^{j+1} \) is integrable in \([0, \infty)\) and have exponential decay at \(+\infty\), (6.64) now follows from (6.66).

We would now like to combine the terms in (6.62) with the end point terms \( \mathcal{F}_i \) in (6.36) from the double integral in the bulk region.
Lemma 6.9. Let $F_{n-m}$ be given by (6.36), then we have

$$F_{n-m} + \frac{1}{4} \sum_{k=0}^{\infty} \int_{u_{-}}^{u_{+}} \frac{u^{k} (\nu(u) - u)^{2k+1}}{(-u)^{2}(T-u)(2k+1)!} \mathcal{L}^2(u)du =$$

$$4 \int_{\frac{n}{m}x}^{4} \frac{(n-m)\phi}{Nx(4-x)(t-x)}dx + \frac{1}{4} \int_{u_{-}}^{u_{+}} \frac{(-u)^{2}}{\nu(u) - u} \mathcal{L}^2(u)du + O\left(N^{-\frac{3}{4}}\right). \quad (6.67)$$

Proof. From (6.36), we can write $F_{n-m}$ in the following form. (After renaming the integration variable from $x$ to $x_{N}$)

$$F_{n-m} = 4 \int_{\frac{n}{m}x}^{4} \frac{\sin ((n-m)\phi(x_{N}))}{Nx_{N}(4-x_{N})(t-x_{N})}dx_{N}$$

$$= 4 \sum_{k=0}^{\infty} \int_{\frac{n}{m}x}^{4} \frac{(-1)^{k}((n-m)\phi(x_{N}))^{2k+1}}{Nx_{N}(4-x_{N})(t-x_{N})(2k+1)!}dx_{N} \quad (6.68)$$

where $\phi(x_{N}) = \arccos \left(\frac{N}{x_{N}} - 1\right)$. Let $u = f_{N}(x_{N})$. Then from the proof of Lemma 6.4, we can express $\phi(x_{N})$ as follows when $u \to u_{-}$.

$$(n-m)\phi(x_{N}) = F_{n}(x_{N}) - F_{m}(x_{N} \frac{n}{m}) + O\left(\frac{1}{N}\left(\frac{u}{N^{\frac{3}{4}}}\right)^{-\frac{1}{2}}\right)$$

Now $F_{n}(x_{N})$ is given by

$$F_{n}(x_{N}) = \sin(\phi_{x_{N}}) + \eta_{+}(x_{N}) = \frac{\pi}{4} = \frac{n}{N} \frac{2}{3}(-u)^{\frac{3}{2}} + \eta_{+}(x_{N}) - \frac{\pi}{4}.$$  

Similarly, we have $F_{m}(x_{N}) = \frac{2}{3}(-f_{m})^{\frac{3}{2}} + \eta_{+}(x_{N}) - \frac{\pi}{4}$ and $f_{m}(x_{N}n/m) = \nu(f_{n})$. Then by Lemma 6.6 and $f_{n}(x_{N}) = \left(\frac{n}{m}\right)^{\frac{3}{2}} F_{N}(x_{N}) = \left(\frac{n}{m}\right)^{\frac{3}{2}}u$, we obtain

$$F_{m} \left(\frac{n}{m}x_{N}\right) = \frac{2}{3} \left((-\nu(u))^{\frac{3}{2}} + \frac{n-N}{N}u(-\nu(u))^{\frac{3}{2}}\right)$$

$$+ \eta_{+} \left(\frac{n}{m}x_{N}\right) - \frac{\pi}{4} + O\left(\frac{1}{N} \left(\frac{u}{N^{\frac{3}{2}}}\right)^{-\frac{1}{2}}\right)$$

From (6.8), we obtain

$$F_{n}(x_{N}) - F_{m} \left(\frac{n}{m}x_{N}\right) = (-u)^{\frac{3}{2}}(\nu(u) - u) + O\left(\frac{1}{N} \left(\frac{u}{N^{\frac{3}{2}}}\right)^{-\frac{1}{2}}\right)$$

as $u \to u_{-}$. From (6.59), we see that $\mathcal{L}^2(u)$ is given by

$$\mathcal{L}^2(u) = \frac{8(-u)^{\frac{3}{2}}(T-u)}{x_{N}^{\frac{3}{2}}(4-x_{N})^{\frac{1}{2}}(t-x_{N}) \left(\frac{dx_{N}}{du}\right)^{2}}.$$
By using $u = f_N(x_N)$, (6.3) and (6.1), we obtain $d_u x_N = \frac{2\phi^2}{N_x} \sqrt{\frac{x_N}{x_N} - 4}$. Therefore $L^2(u)$ is can be written as

$$L^2(u) = \frac{16u(T - u)}{N x_N (4 - x_N) (t - x_N)} \, dx_N.$$ 

This gives us

$$\frac{4(-1)^k ((n - m)\phi(x_N))^{2k+1}}{N x_N (4 - x_N) (t - x_N)} = -\frac{1}{4} \frac{u^k (\nu(u) - u + O(N^{-\frac{2}{3}} u^{-1}))^{2k+1}}{(-u)^{\frac{3}{2}}(T - u)} \, L^2(u) \, du \, dx_N$$

as $u \to u_-$. As $\nu - u = O\left(N^{-\frac{1}{2}}\right)$ and $u_- / N^{\frac{3}{2}} = O\left(N^{-\varepsilon}\right)$, we see that for $k > 1$,

$$\frac{4(-1)^k ((n - m)\phi(x_N))^{2k+1}}{N x_N (4 - x_N) (t - x_N)} + \frac{1}{4} \frac{u^k (\nu(u) - u)^{2k+1}}{(-u)^{\frac{3}{2}}(T - u)} L^2(u) \, du \, dx_N = O\left(\frac{u^k - \frac{3}{2}}{N^{2(2k+1)}}\right) \tag{6.69}$$

as $u \to u_-$. Since $\phi(x_N) = \arccos \left(\frac{x}{4} - 1\right)$ it behaves as $\sqrt{4 - x_N}$ as $x \to 4$. Therefore the function on the left hand side of (6.69) is integrable for $u \in (u_-, 0)$. Therefore by using $\varepsilon \leq 1/20$, we have

$$\int_{u_-}^4 \frac{4(-1)^k ((n - m)\phi(x))^{2k+1}}{N x(4 - x)(t - x)} \, dx = -\frac{1}{4} \int_{u_-}^0 \frac{u^k (\nu(u) - u)^{2k+1}}{(-u)^{\frac{3}{2}}(T - u)} L^2(u) \, du + O\left(N^{-\frac{3}{2}}\right)$$

for $k > 0$ and of order $O\left(N^{-\frac{3}{2}}\right)$ for $k = 0$. This, together with (6.68), implies the Lemma.

By (6.62), (6.68) and Lemma 6.8 6.9, we see that $A(0, 0)$ can be written as

$$A(0, 0) - F_{n-m} = \frac{1}{2} \left( \int_{u_-}^{u_+} A(u) \, du \right)^2 - \int_{u_-}^{u_+} A^2(u)(\nu - u) \, du$$

$$- 4 \int_{u_-}^{4} \frac{((n - m)\phi)}{N x(4 - x)(t - x)} \, dx + O\left(N^{-\frac{3}{2}}\right) \tag{6.70}$$

By (6.55), we see that the terms in the sum involving $A$ are of the form

$$\int_{u_-}^{u_+} A^2(u)(\nu - u) \, du = \frac{4^4(n - m)}{N^\tau} \int_{u_-}^{u_+} A^2(u) \, du + O\left(N^{-\frac{3}{2} + \frac{\tau}{2}}\right) \tag{6.71}.$$ 

Let us now compute the other terms in (6.57). These terms are of the form

$$\frac{h_{kj}}{N^{2k}} ((N - n)^j (N - m)^k A(j, k) + (N - n)^k (N - m)^j A(k, j))$$

for some for some constants $h_{kj}$ that are independent on $N$, $n$ and $m$.

To compute these terms, let us first prove the following.
Lemma 6.10. Let $F$ and $G$ be integrable functions on $[u_-, u_+]$ such that $F$ and $G$ are of order $e^{-ku^2}$ as $u \to +\infty$ for some $k > 0$. Then we have

$$
\int_{u_-}^{u_+} F(u) \int_{\nu(u)}^{\nu(u)+} G(v) dv du + \int_{u_-}^{u_+} \int_{\nu(u)}^{\nu(u)+} F(v) dv du = \int_{u_-}^{u_+} F(u) du \int_{\nu(u)}^{\nu(u)+} G(v) dv du + \int_{u_-}^{u_+} F(u) \int_{\nu(u)}^{\nu(u)+} G(v) dv du \tag{6.72}
$$

Integrating the first term by parts, we have

$$
\int_{u_-}^{u_+} F(u) \int_{\nu(u)}^{\nu(u)+} G(v) dv du = \int_{u_-}^{u_+} F(u) du \int_{\nu(u)}^{\nu(u)+} G(u) dv du - \int_{u_-}^{u_+} G(u) \int_{\nu(u)}^{\nu(u)+} F(v) dv du + O \left( e^{-ku^2_+} \right)
$$

as $F$ is of order $e^{-ku^2}$ when $u \to +\infty$. This implies

$$
\int_{u_-}^{u_+} F(u) \int_{\nu(u)}^{\nu(u)+} G(v) dv du = \int_{u_-}^{u_+} F(u) du \int_{\nu(u)}^{\nu(u)+} G(u) dv du - \int_{u_-}^{u_+} G(u) \int_{\nu(u)}^{\nu(u)+} F(v) dv du + O \left( e^{-ku^2_+} \right)
$$

This, together with (6.73), implies the lemma. \qed

By taking $F = A/(T-u)^j$ and $G = A/(T-u)^k$ for $A(j,k)$, we have the following.

Corollary 6.1. The terms in (6.71) are given by

$$(N-m)^{k-j} A(j,k) + (N-n)^{k-j} A(k,j) = AH(j,k) \left( (N-m)^{k-j} - (N-n)^{k-j} \right) + SH(j,k) \left( (N-m)^{k-j} + (N-n)^{k-j} \right) + \frac{1}{2} \left( (N-m)^{k-j} + (N-n)^{k-j} \right) \int_{u_-}^{u_+} A_j du \int_{u_-}^{u_+} A_k dv du + O \left( e^{-ku^2_+} \right), \tag{6.74}
$$

where $AH(j,k)$ and $SH(j,k)$ and $A_j$ are given by

$$
A_j(u) = \frac{A(u)}{(T-u)^j}, \quad AH(j,k) = \frac{1}{2} \left( A(j,k) - A(k,j) \right),
$$

$$
SH(j,k) = \frac{1}{2} \left( \int_{u_-}^{u_+} A_j(u) \int_{\nu(u)}^{\nu(u)+} A_k(v) dv du + \int_{u_-}^{u_+} A_k(u) \int_{\nu(u)}^{\nu(u)+} A_j(v) dv du \right)
$$

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We can now consider the terms in (6.57). We shall first consider the terms where \( j \) and \( k \) are not both zero. From (6.49), we see that if \( j + k > 2 \), then the term will be of order \( O\left( N^{-\frac{4}{3}} \right) \). Therefore we shall only consider the cases where \( j + k \leq 2 \). Let us now compute the terms \( A(j, k) \).

**Lemma 6.11.** Let \( A_0(j, k) \) be

\[
A_0(j, k) = \int_{u_-}^{u_+} A_j(u) \int_{u}^{v_+} A_k(v) dv du,
\]

(6.75)

Then for \( j + k \geq 1 \), we have

\[
A(j, k) = A_0(j, k) + \frac{4\pi(m-n)}{N^{\frac{4}{3}}} \int_{u_-}^{u_+} A_j(u)A_k(u) du + O\left( N^{-1} \right).
\]

(6.76)

**Proof.** This can be proved by using the mean value theorem. By repeat use of mean value theorem, we have

\[
\int_{\nu(u)}^{u} A_k(v) dv = A_k(u)(u - \nu(u)) - \frac{1}{2} A'(\xi_u)(u - \nu(u))^2
\]

for some \( \xi_u \) between \( u \) and \( \nu(u) \). Therefore we have

\[
A(j, k) = A_0(j, k) + \int_{u_-}^{u_+} A_j(u)A_k(u)(u - \nu(u)) du - \frac{1}{2} \int_{u_-}^{u_+} A_j(u)A'_{k}(\xi_u)(u - \nu(u))^2 du
\]

Note that \( A_j A_k \) is of order \( N^{-\frac{4}{3}} u^{-\frac{3}{2}j - k} \) as \( u \to -\infty \) and decays exponentially as \( u \to +\infty \). Then by Lemma [6.6], we see that

\[
\int_{u_-}^{u_+} A_j(u)A_k(u)(u - \nu(u))^2 du = \frac{4\pi(m-n)^i}{N^{\frac{4}{3}}^{i}} \int_{u_-}^{u_+} A_j A_k du + O\left( N^{-\frac{4}{3}} \right),
\]

where \( i = 1, 2 \). Similarly, since \( j + k \geq 1 \), the function \( A_j(u)A'_k(\xi_u) \) is of order \( N^{-\frac{4}{3}} u^{-2} \) as \( u \to -\infty \) and decays exponentially as \( u \to +\infty \). Hence the integral involving \( A_j(u)A'_k(\xi_u) \) is of order \( N^{-1} \). This proves the lemma.

From this, (6.77) and (6.55), we can compute the terms \( AH(j, k) \) and \( SH(j, k) \).

**Corollary 6.2.** Let \( AH_0(i, j) \) be

\[
AH_0(i, j) = \frac{1}{2} (A_0(i, j) - A_0(j, i))
\]

Then the terms \( AH(j, k) \) and \( SH(j, k) \) are of order

\[
AH(j, k) = AH_0(j, k) + O\left( N^{-1} \right), \quad SH(j, k) = \frac{4\pi(m-n)}{N^{\frac{4}{3}}} \int_{u_-}^{u_+} A_j A_k du + O\left( N^{-1} \right).
\]
By using this, (6.49) and (6.55), together with the formula for $A(0,0)$ (6.70) and (6.71), we arrive at the following.

**Lemma 6.12.** Let $n - m = k$. Then we have

$$J_3 - \frac{1}{2} I_{n,3} I_{m,3} - \mathcal{F}_{n-m} = k J_{31} + k(2(N-n) + k) J_{32}$$

$$+ \frac{1}{2} I_{n,3} \int_{u_m}^{v_m} A(u) E_m du + O \left( N^{-\frac{4}{3} + \frac{2}{3}} \right)$$

(6.78)

where the terms $J_{3j}$ are given by

$$J_{31} = -\frac{4^3}{2N^2} AH_0(0,1) - \frac{4}{N^2} \int_{N_1}^{4} \int_{N_1}^{4} N x (4-x)(t-x) dx - \frac{4^3}{N^2} \int_{u_m}^{u_m} A^2(u) du,$$

$$J_{32} = -\frac{4^4}{2N^2} \left( \int_{u_m}^{u_+} A_0 A_1 du + \frac{3}{4} A H_0(0,2) \right)$$

(6.79)

As we shall see, the terms $J_{31}$ and $J_{32}$ will in fact not enter into the expression of the kernel $K_1$. What is important is the structure of equation (6.78).

**Remark 6.2.** Throughout this section, all the dependence on $T$ are through factors of $(T-u)^{-\frac{1}{2}}$ for $j > 0$, with the exception of the function $L(u)$. From the definition of $L(u)$ in (6.59), we see that it can be written in the form (6.60)

$$L(u) = \left( \frac{N}{4} \right)^{-\frac{1}{2}} \left( 1 + O \left( \frac{u}{N^{\frac{1}{4}}} \right) \right)$$

with an error term that is uniform in $T$ as long as $T$ is not on the integration contour of $u$ and $v$. Therefore all error terms in this section is uniform in $T$ as $|T| \to N^{\frac{4}{3}}$ on the contour $\Xi_+$ in Theorem 2.2.

### 6.1.4 The exponential region

Inside the exponential region $[4 + N^{-\epsilon}, \infty)$, we have the following matching formula for the polynomials (See Lemma 4.8 of [33]).

**Lemma 6.13.** For any $d > 0$ there exists a constant $k > 0$ such that, uniformly for $x \in [4 + N^{-d}, \infty)$, we have

$$\hat{L}_n(x) = O \left( e^{-k(x-4)n^{\frac{4}{3}}} \right).$$

From this, it is clear that the contribution from the exponential region is of order $O \left( e^{-kN^{\frac{4}{3} - d}} \right)$ for some $k > 0.$
6.2 Asymptotics of the skew inner product

We can now compute the asymptotics of the skew inner products \( \langle L_n, L_m \rangle \). We have

\[
\langle L_n, L_m \rangle = \frac{1}{2} \int_{-\infty}^{\infty} L_n(x)w(x)dx \int_{-\infty}^{\infty} L_m(y)w(y)dy - \int_{-\infty}^{\infty} L_n(x) \int_{x}^{\infty} L_m(y)w(y)dydx.
\] (6.80)

By breaking the range of these integrals into different regions, we obtain

\[
\frac{N^2i\sqrt{\kappa_{n-1}\kappa_{m-1}}}{nm} \langle L_n, L_m \rangle = \sum_{j=1}^{3} \left( \frac{1}{2} \mathcal{I}_{n,j} \mathcal{I}_{m,j} - \mathcal{J}_j \right) + \frac{1}{2} (\mathcal{I}_{n,3} (\mathcal{I}_{m,2} + \mathcal{I}_{m,1}) - (\mathcal{I}_{n,1} + \mathcal{I}_{n,2}) \mathcal{I}_{m,3} \mathcal{I}_{n,3}) \int_{u_-}^{v_-} \mathcal{A}(u)\mathcal{E}_m du \]

\[- \mathcal{I}_{n,3} (\mathcal{I}_{m,2} + \mathcal{I}_{m,1}) - (\mathcal{I}_{n,1} + \mathcal{I}_{n,2}) \mathcal{I}_{m,3} \mathcal{I}_{n,3}) \int_{u_-}^{v_-} \mathcal{A}(u)\mathcal{E}_m du \]

\[
= \sqrt{\frac{2\pi}{Nt}} ((-1)^n \mathcal{I}_{n,3} - (-1)^n \mathcal{I}_{m,3}) + O \left( N^{-\frac{11}{2}} \right).
\] (6.81)

for some \( k > 0 \). Then, as \( \mathcal{I}_{n,2} \) and \( \mathcal{I}_{m,2} \) are of order \( O \left( N^{-\frac{7}{2}} \right) \) (see (6.24)), we see that the last term in the above sum is of order \( O \left( N^{-\frac{11}{2}} \right) \). By using mean value theorem and the asymptotic formulae for \( \mathcal{A}(u) \), we see that \( \int_{u_-}^{v_-} \mathcal{A}(u)\mathcal{E}_m du = O \left( N^{-\frac{7}{2}} \right) \). From these, (6.17), (6.24) and (6.53), we obtain

\[
\mathcal{I}_{n,3} (\mathcal{I}_{m,2} + \mathcal{I}_{m,1}) - (\mathcal{I}_{n,1} + \mathcal{I}_{n,2}) \mathcal{I}_{m,3} \mathcal{I}_{n,3}) \int_{u_-}^{v_-} \mathcal{A}(u)\mathcal{E}_m du \]

\[
= \sqrt{\frac{2\pi}{Nt}} ((-1)^n \mathcal{I}_{n,3} - (-1)^n \mathcal{I}_{m,3}) + O \left( N^{-\frac{11}{2}} \right).
\] (6.81)

From this and (6.17), (6.18), (6.37) and Lemma 6.12 we arrive at the following.

**Proposition 6.7.** The product \( \langle L_n, L_m \rangle \) is given by

\[
\frac{N^2i\sqrt{\kappa_{n-1}\kappa_{m-1}}}{nm} \langle L_n, L_m \rangle = - \left( k\mathcal{J}_{31} + k(2(N - n) + k)\mathcal{J}_{32} \right) + k\mathcal{J}_{B1} + \mathcal{J}_{B2} + \mathcal{J}_{B3,k} + \left\{ \begin{array}{l l}
(-1)^n \left( I_0 + (2(n - N) - k) I_1 \right), & \text{k odd;} \\
(-1)^{n+1} k I_1, & \text{k even.}
\end{array} \right. + O \left( N^{-\frac{11}{2}} \right)
\] (6.82)

where \( \mathcal{J}_{3j} \) are given in (6.79) and \( \mathcal{J}_{B1}, \mathcal{J}_{B2} \) and \( I_j \) are given by

\[
I_0 = -\sqrt{\frac{2\pi}{Nt}} \int_{u_-}^{u_+} \mathcal{A}(u) du, \quad I_1 = -\sqrt{\frac{2\pi}{4\sqrt{Nt}}} \left( \frac{4}{N} \right)^{\frac{1}{2}} \int_{u_-}^{u_+} \mathcal{A}(u) du,
\]

\[
\mathcal{J}_{B1} = \frac{2\pi}{N+4\pi\sqrt{T}} - \frac{2\pi}{N}, \quad \mathcal{J}_{B2} = \frac{2\pi}{N},
\] (6.83)

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while $J_{B3,k}$ is given by

$$J_{B3,1} = 0, \quad J_{B3,2} = \frac{2\pi 4^{\frac{1}{2}}}{N^{\frac{3}{2}}} \sqrt{\frac{T}{t}}. \quad (6.84)$$

We can simplify the expression further by the observation that $\langle L_n, L_m \rangle_1 = 0$ whenever $n - m = 1$ and $n$ is even. (See Corollary 4.1. Taking $k = 1$ and $n$ even in (6.82), we have

$$- \left( J_{31} + (2(N - n) + 1)J_{32} \right) + J_{B1} + J_{B2} + I_0 + (2(n - N) - 1) I_1 = O \left( N^{-\frac{25}{24}} \right)$$

As this holds for any finite $N - n$ as long as $n$ is even, we obtain the following relations.

$$J_{32} = -I_1 + O \left( N^{-\frac{25}{24}} \right), \quad -J_{31} + J_{B1} + I_0 = -J_{B2} + O \left( N^{-\frac{25}{24}} \right). \quad (6.85)$$

Note that although in the above equation, integration limits $u_{\pm}$ depend on $n$, the effect of changing these limits will only result in terms of order $O \left( N^{-\frac{25}{24}} \right)$ and hence we can consider them as fixed under the change of $n$.

Let us now compute the factor $\kappa_{n-1}\kappa_{m-1}$. We have $\kappa_n = -\frac{2\pi i}{h_{n,0}}$. By (4.49), we see that $h_{n,0}/h_{m,0} = 1 + O \left( N^{-1} \right)$. From this, Proposition (5.7) and (6.85), we see that the skew inner products that we need have the following asymptotics.

$$\frac{2\pi}{h_{n,0}} \langle L_{N-1}, L_{N-2} \rangle_1 = -2I_0 + 6I_1 + O \left( N^{-\frac{25}{24}} \right),$$

$$\frac{2\pi}{h_{n,0}} \langle L_n, L_{N-2} \rangle_1 = -2I_0 + 2I_1 - J_{B2} + J_{B3,2} + O \left( N^{-\frac{25}{24}} \right). \quad (6.86)$$

Remark 6.3. From (6.86), we see that for large enough $N$, the products $\langle L_{N-1}, L_{N-2} \rangle_1$ and $\langle L_{N-3}, L_{N-4} \rangle_1$ will be non-zero if $\int_{-\infty}^{\infty} \frac{A_t}{(T-u)^{\frac{3}{2}}} du \neq 0$. Since this is an analytic function with jump on $\mathbb{R}$ that is not identically zero, we can choose the contour $\Xi_+$ in Theorem 2.4 such that $\int_{-\infty}^{\infty} \frac{A_t}{(T-u)^{\frac{3}{2}}} du \neq 0$ on $\Xi_+$. We can therefore assume that $\int_{-\infty}^{\infty} \frac{A_t}{(T-u)^{\frac{3}{2}}} du \neq 0$.

Note that, by Remark 6.2, these error terms are uniform in $T$ as $|T| \to N^{\frac{3}{2}}$. Let us look at the behavior of the skew products when $|T| \to N^{\frac{3}{2}}$ and when $t$ remain finite.

First by (6.83), we see that as $T \to \infty$, the integrals $I_j$ are of the following orders as $T \to \infty$.

$$I_j = O \left( T^{-\frac{2}{3} + \frac{1}{2}} N^{-\frac{2}{3} + \frac{1}{4}} \right). \quad (6.87)$$

(The statement is clear if $|T| > u_-$. If $|T| < u_-$, then by breaking the range of the integral into $(u_-, -|T|^{\frac{1}{2}})$ and $(-|T|^{\frac{1}{2}}, u_+)$ and integrate by parts the integral over $(u_-, -|T|^{\frac{1}{2}})$ using (6.61), one can check that (6.87) is correct.)
By using \( t = 4 + \left( \frac{1}{N} \right)^{2/3} T \) in (6.84), we see that \( J_{B3,2} \) is of the following order.

\[
J_{B3,2} = O \left( N^{-1} \left( \frac{T}{N^{2/3}} \right)^{2/3} \right), \quad |T| \to N^{2/3} \text{ and } t \text{ finite.} \tag{6.88}
\]

From these we obtain the behavior of the skew products as \( T \to \infty \).

**Lemma 6.14.** The skew products \( \langle L_n, L_m \rangle_1 \) in (6.86) are of the following orders as \( T \to \infty \).

\[
\frac{2\pi}{h_{N,0}} \langle L_n, L_m \rangle_1 = O \left( T^{-2/3} N^{-2/3} \right), \quad |T| \to N^{2/3} \text{ and } t \text{ finite.} \tag{6.89}
\]

We can now compute the asymptotic of the kernel.

### 6.3 Asymptotics of the kernel \( S_1(x, y) \)

First let us used the results in the previous sections to compute the correction term to the kernel. Let us define the variables \( \xi_1 \) and \( \xi_2 \) to be

\[
\xi_1 = \left( \frac{N}{4} \right)^{2/3} (x - 4), \quad \xi_2 = \left( \frac{N}{4} \right)^{2/3} (y - 4). \tag{6.90}
\]

We will assume that \( \xi_1 \) and \( \xi_2 \) are bounded from below. First we need to compute the integrals \( \epsilon(\pi_n w) \), which is a linear combinations of the integrals of the \( L_n w \). We have

\[
\epsilon(\pi_n w)(y) = \int_y^\infty L_n(s)w(s)ds + \frac{1}{2} \int_{-\infty}^\infty L_n(s)w(s)ds.
\]

To compute the first term, let us first assume \( \xi_2 < N^{1/3} \). Then we have, by the asymptotic formula of the \( L_n(x) \) inside the Airy region, and the estimates [1],

\[
|Ai(t)| \leq Ce^{-(2/3)t^{3/2}}, \quad |Ai'(t)| < C(1 + t^{1/4})e^{-2/3t^{3/2}}, \quad t > 0, \tag{6.91}
\]

the following.

\[
\sqrt{\frac{\kappa n_{-1}N}{n}} \int_y^\infty L_n(s)w(s)ds = \int_{u\left(\frac{N}{n}y\right)}^\infty A\mathcal{E}_n du + O \left( e^{-k\xi_2 N^{-2/3}} \right) \tag{6.92}
\]

for some \( k > 0 \), where \( u = f_n(y) \). Note that by (6.91), we in fact have an exponential decay of order \( e^{-k\xi_2^{3/2}} \) in the error term above. However, the decay \( e^{-k\xi_2} \) will be sufficient for our purpose. The lower limit \( u \left( \frac{N}{n}y \right) \) can be expressed in terms of \( \xi_2 \) as follows.

\[
u \left( \frac{N}{n}y \right) = \xi_2 + \left( \frac{4}{N} \right)^{1/3} (N - n) - \left( \frac{4}{N} \right)^{2/3} \frac{\xi_2^2}{20} + O \left( \left( \frac{\xi_2}{N^{1/3}} \right)^j N^{-4/3} \right).
\]
where \( j > 0 \). By using mean value theorem and (6.91), we can write the integral as

\[
\int_{u(N/n)}^{\infty} \mathcal{AE}_n\,du = L_0(\xi_2) + \left(\frac{4}{N}\right)^{\frac{1}{4}} (N-n)L_1(\xi_2) + \frac{1}{2} \left(\frac{4}{N}\right)^{\frac{3}{4}} (N-n)^2L_2(\xi_2) + O\left(e^{-k\xi_2/N^{\frac{3}{2}}}\right),
\]

for some \( k > 0 \). The \( L_j \) in (6.93) are given by

\[
L_0 = \int_{\xi_2}^{\infty} A\,du + \frac{42\xi_2^2}{20N^2} A(\xi_2), \quad L_1 = -A(\xi_2) - \int_{\xi_2}^{\infty} \frac{A(u)}{2(T-u)}\,du, \quad L_2 = -A'(\xi_2) + \frac{A(\xi_2)}{T-\xi_2} + \frac{3}{4} \int_{\xi_2}^{\infty} \frac{A(u)}{(T-u)^2}\,du.
\]

On the other hand, if \( \xi_2 > N^{\frac{1}{8}} \), then from Lemma 6.13 and the asymptotic formula for the Airy function, we see that

\[
\sqrt{i\kappa_{n-1}N/n} \int_{u_-}^{\infty} L_n(s)w(s)\,ds = O\left(e^{-k\xi_2}\right), \quad \int_{u(N/n)}^{\infty} \mathcal{AE}_n\,du = O\left(e^{-k\xi_2}\right)
\]

for some constant \( k > 0 \). Therefore if we choose the constant \( k \) in (6.93) to be small enough, then (6.93) remains valid for all \( \xi_2 \) bounded below.

Similarly, the second term in (6.92) is given by

\[
\sqrt{i\kappa_{n-1}} \int_{-\infty}^{\infty} L_n(s)w(s)\,ds = \left(-\frac{1}{2}\right) \frac{n \sqrt{\pi}}{\sqrt{2Nt}} + \frac{1}{2} \int_{u_-}^{u_+} \mathcal{AE}_n(u)\,du + O\left(N^{-\frac{7}{8}}\right).
\]

Let \( u_- = f_N(c_-) \), then changing the lower limit in the above integral into \( u_-,N \) will only result in an error term of order \( O\left(N^{-1+\frac{7}{8}}\right) = O\left(N^{-\frac{1}{8}}\right) \). Similarly, changing the upper limit to \( +\infty \) will only result in an exponentially small error term. We can therefore change the lower limit in the integration to \( u_-,N \) and the upper limit to \( +\infty \). Let \( \xi_{2,N} \) be the value of \( \xi_2 \) at \( u = u_-,N \), then we have

\[
\sqrt{i\kappa_{n-1}} (L_n w)(y) = \Psi_0(\xi_2) + \left(\frac{4}{N}\right)^{\frac{1}{4}} (N-n)\Psi_1(\xi_2) + \frac{1}{2} \left(\frac{4}{N}\right)^{\frac{3}{4}} (N-n)^2\Psi_2(\xi_2) + \frac{(-1)^n \sqrt{\pi}}{\sqrt{2Nt}} + O\left(N^{-\frac{7}{8}}\right),
\]

(6.95)

where \( \Psi_j(\xi_2) \) is given by

\[
\Psi_j(\xi_2) = \frac{1}{2} L_j(\xi_{2,N}) - L_j(\xi_2), \quad j = 0, 1, 2.
\]
Similarly, the orthogonal polynomials $L_n(x)$ are given by

$$L_n(x)w(x) = \frac{\sqrt{h_{N,0}}}{2(T - \xi_1)^{\frac{3}{2}}} \left( \left( \frac{N}{4} \right)^{\frac{3}{2}} Ai(\xi_1) - \left( \frac{N}{4} \right)^{\frac{1}{2}} \left( \frac{\alpha + 1}{2} + (n - N) \right) Ai'(\xi_1) \right) + \left( \frac{N}{4} \right)^{-\frac{1}{2}} \left( \frac{(N - n)^{2}}{2} Ai''(\xi_1) + f_1(\xi_1) + (N - n)f_2(\xi_2) \right) + O \left( \frac{e^{-k\xi_1}}{N^{\frac{1}{2}}} \right) \tag{6.96}$$

where $f_1$ and $f_2$ are functions that are independent on $N$, $n$ and $T$. Then by using (4.51), we obtain the asymptotics for the correction kernel $K_1(x, y)$.

$$K_1(x, y) = \frac{Ai(\xi_1)\Psi_0(\xi_2)}{(T - \xi_1)^{\frac{3}{2}}} Q_{00} + \frac{Ai'(\xi_1)\Psi_0(\xi_2)}{(T - \xi_1)^{\frac{1}{2}}} Q_{10} + \frac{Ai(\xi_1)\Psi_1(\xi_2)}{(T - \xi_1)^{\frac{3}{2}}} Q_{01} + \frac{Ai'(\xi_1)\Psi_1(\xi_2)}{(T - \xi_1)^{\frac{1}{2}}} Q_{11} + \frac{Ai(\xi_1)\Psi_2(\xi_2)}{(T - \xi_1)^{\frac{3}{2}}} Q_{02} + \frac{Ai'(\xi_1)\Psi_2(\xi_2)}{(T - \xi_1)^{\frac{1}{2}}} Q_{12} + \frac{Ai(\xi_1)}{(T - \xi_1)^{\frac{3}{2}}} Q_{03} + \frac{Ai'(\xi_1)}{(T - \xi_1)^{\frac{1}{2}}} Q_{13} + O \left( \frac{e^{-k\xi_1}N^{\frac{1}{2}}}{1 + |T|} \right) + O \left( \frac{e^{-k\xi_1}N^{\frac{1}{2}}}{T} \right) \tag{6.97}$$

The coefficients $Q_{jk}$ are given by

$$Q_{00} = \left( \frac{N}{4} \right)^{\frac{3}{2}} \frac{NT}{4\sqrt{2\pi}}, \quad Q_{10} = \left( \frac{N}{4} \right)^{\frac{3}{2}} \frac{N(4I_1 + J_{B2} - J_{B3,2})}{8\sqrt{2\pi}I_0} + \left( \frac{4}{N} \right)^{\frac{1}{2}} \frac{NT}{4\sqrt{2\pi}},$$

$$Q_{01} = -\left( \frac{N}{4} \right)^{\frac{3}{2}} \frac{N(4I_1 + J_{B2} - J_{B3,2})}{8\sqrt{2\pi}I_0}, \quad Q_{11} = \left( \frac{N}{4} \right)^{-\frac{1}{2}} \frac{N}{4\sqrt{2\pi}},$$

$$Q_{02} = -\left( \frac{N}{4} \right)^{-\frac{1}{2}} \left( \frac{N}{2\sqrt{2\pi}} + \frac{3N(4I_1 + J_{B2} - J_{B3,2})}{16\sqrt{2\pi}I_0} \right),$$

$$Q_{20} = \left( \frac{4}{N} \right)^{\frac{3}{2}} \frac{NT}{8\sqrt{2\pi}} + \left( \frac{N}{4} \right)^{-\frac{1}{2}} \frac{3N(4I_1 + J_{B2} - J_{B3,2})}{16I_0\sqrt{2\pi}},$$

$$Q_{03} = \frac{N(4I_1 + J_{B2} - J_{B3,2})}{16\sqrt{I_0}} + \left( \frac{4}{N} \right)^{\frac{3}{2}} \frac{TN}{16\sqrt{t}},$$

$$Q_{13} = \left( \frac{4}{N} \right)^{\frac{3}{2}} \frac{N}{8\sqrt{t}} \left( -1 + \frac{T}{2} \right) \left( \frac{4}{N} \right)^{\frac{3}{2}} - \frac{3N(4I_1 + J_{B2} - J_{B3,2})}{4I_0}. \tag{6.98}$$

From (6.87) and (6.88), we see that as $T \to \infty$, the kernel $K_1$ has the following behavior

$$K_1(\xi_1, \xi_2) = O \left( N^{\frac{3}{2}} \right), \quad \text{uniformly as } |T| \to N^{\frac{3}{2}} \text{ and } t \text{ finite.}$$

In fact, by using (6.86), (6.89) and (4.51), we obtain the following.
Lemma 6.15. As $T \to \infty$ and $t$ remains bounded, the kernel $K_2$ and $K_1$ become the Airy kernels in the large $N$ limit.

\[
\left( \frac{4}{N} \right)^{\frac{3}{2}} K_2(\xi_1, \xi_2) = \frac{Ai(\xi_1)Ai'(\xi_2) - Ai(\xi_2)Ai'(\xi_1)}{\xi_1 - \xi_2} + O\left( \frac{e^{-k(\xi_1+\xi_2)}}{TN^{\frac{3}{2}}} \right),
\]

\[
\left( \frac{4}{N} \right)^{\frac{3}{2}} K_1(\xi_1, \xi_2) = \frac{1}{2} Ai(\xi_1) \int_{-\infty}^{\xi_2} Ai(u)du + o(1)e^{-k\xi_1}.
\]

for some $k > 0$.

Proof. The statement for the kernel $K_2$ follows immediately from the representation (2.11) and the Airy asymptotics of the Laguerre polynomials. To prove the statement for the correction term $K_1$, first note that, by (6.86), we see that

\[
\frac{2\pi}{hN_0} \langle L_N, L_{N-2} \rangle_1 = \frac{2\pi(t-2)}{N \sqrt{t(t-4)}} - \frac{\pi}{N} + O\left( N^{-\frac{25}{24}} \right),
\]

\[
\frac{2\pi}{hN_0} \langle L_{N-1}, L_{N-2} \rangle_1 = \frac{4\pi}{N \sqrt{t(t-4)}} + O\left( N^{-\frac{25}{24}} \right),
\]

(6.100)

uniformly for $|T| < cN^{\frac{3}{2}}$. By dividing the range of integration in $\int_{u_-}^{u_+} \frac{Ai}{(T-u)^{\frac{3}{2}}} du$ into $[u_-, -|T|^{\frac{1}{2}}]$ and $[-|T|^{\frac{1}{2}}, u_+]$, we see that

\[
\int_{u_-}^{u_+} \frac{Ai}{(T-u)^{\frac{3}{2}}} du = \frac{1}{T^{\frac{3}{2}}} \left( \int_{-\infty}^{\infty} Ai(u)du + O\left( T^{-\frac{3}{2}} \right) \right) = \frac{1}{T^{\frac{3}{2}}} \left( 1 + O\left( T^{-\frac{3}{2}} \right) \right),
\]

From this, (6.100), (6.95), (6.96) and (4.51), we see that in this limit, $K_1$ is given by

\[
\left( \frac{4}{N} \right)^{\frac{3}{2}} K_1(\xi_1, \xi_2) = \frac{1}{2} Ai(\xi_1) \left( \frac{1}{2} - \int_{\xi_2}^{\infty} Ai(u)du + O\left( T^{-\frac{3}{2}} \right) + O\left( N^{-\frac{1}{4}} \right) \right)
\]

\[
+ \left( \frac{1}{4} + O\left( N^{-\frac{29}{24}}T^{\frac{1}{2}} \right) + O\left( T^{-\frac{3}{2}} \right) \right) Ai(\xi_1) \left( 1 + O\left( N^{-\frac{1}{4}} \right) \right)
\]

\[
= \frac{1}{2} Ai(\xi_1) \left( \int_{-\infty}^{\xi_2} Ai(u)du + O\left( T^{-\frac{3}{2}} \right) + O\left( N^{-\frac{29}{24}}T^{\frac{1}{2}} \right) + O\left( N^{-\frac{1}{4}} \right) \right)
\]

This proves the lemma.

By using the explicit expressions for $A$ and $\Psi_j$, we obtain the asymptotic formula for the kernel when $T$ is finite.
Proposition 6.8. Let $T$ be of order $N^{\frac{1}{3} - c}$ for some positive $0 < c \leq \frac{1}{3}$, then for $\xi_1$ and $\xi_2$ bounded from below, the kernel $K_1(\xi_1, \xi_2)$ is given by

$$\left(\frac{4}{N}\right)^{\frac{3}{2}} K_1(\xi_1, \xi_2) = K_{1,\infty}(\xi_1, \xi_2) + O\left(N^{-\frac{1}{3}} e^{-k\xi_1}\right),$$

$$K_{1,\infty}(\xi_1, \xi_2) = \left(\frac{T}{2}H_0(\xi_1) \int_{-\infty}^{\xi_2} H_0(u) du + \frac{1}{2} H_1(\xi_1) \int_{-\infty}^{\xi_2} H_1(u) du \right)$$

(6.101)

$$B_1 \left( H_1(\xi_1) \int_{-\infty}^{\xi_2} H_0(u) du - H_0(\xi_1) \int_{-\infty}^{\xi_2} H_1(u) du \right)$$

$$- H_0(\xi_1) \int_{-\infty}^{\xi_2} H_2(u) du + B_2 H_0(\xi_1).$$

for some $k > 0$, where $H_j$, $B_1$ and $B_2$ are given by

$$H_j(u) = \frac{Ai^{(j)}(u)}{(T - u)^{\frac{1}{2}}} ,$$

$$B_1 = -\frac{1}{2} \int_{-\infty}^{\xi_2} H_1(u) du + 1,$$

$$B_2 = -\frac{B_1}{2} - \frac{T}{4} \int_{-\infty}^{\xi_2} H_0(u) du + \frac{B_1}{2} \int_{-\infty}^{\xi_2} H_1(u) du + \frac{1}{2} \int_{-\infty}^{\xi_2} H_2 du,$$

(6.102)

As pointed out in [31], [32] and [33], the eigenvalue statistics will not be affected by the rescaling

$$K \mapsto \left(\frac{N}{4}\right)^{-\frac{2}{3}} K\left(\frac{N}{4}\right)^{\frac{2}{3}}$$

(6.103)

of the matrix kernel. By rescaling the kernel in this way, all the entries will have the same order in the large $N$ limit. From now on, we shall use this rescaled kernel and denote it also by $K$.

From (6.101), we obtain the following estimate for the rescaled matrix kernel $K$.

Proposition 6.9. Let $K_\infty$ be the $2 \times 2$ matrix whose entries are given by

$$K_{\infty,11}(\xi_1, \xi_2) = K_{\infty,22}(\xi_2, \xi_1) = K_{1,\infty}(\xi_1, \xi_2) + K_{2,\infty}(\xi_1, \xi_2),$$

$$K_{\infty,12}(\xi_1, \xi_2) = -\frac{\partial K_{11,\infty}}{\partial \xi_2}, \quad K_{\infty,21}(\xi_1, \xi_2) = -\int_{\xi_1}^{\xi_2} K_{11,\infty}(u, \xi_2) du,$$

(6.104)

where $K_{2,\infty}(\xi_1, \xi_2)$ is given by

$$K_{2,\infty}(\xi_1, \xi_2) = \left(\frac{T - \xi_2}{T - \xi_1}\right)^{\frac{1}{2}} \frac{Ai(\xi_1)Ai'(\xi_2) - Ai(\xi_2)Ai'(\xi_1)}{\xi_1 - \xi_2}.$$

Suppose $T$ is of order $N^{\frac{1}{3} - c}$ for some positive $0 < c \leq \frac{1}{3}$ and that $\xi_1$ and $\xi_2$ are bounded from below. Let $K(\xi_1, \xi_2)$ be the rescaled matrix kernel in (6.103), then there exists $k > 0$
such that

\[
\left( \frac{4}{N} \right)^{\frac{2}{3}} K(\xi_1, \xi_2) = K_\infty(\xi_1, \xi_2) + \begin{pmatrix} O\left( N^{-\frac{1}{3}} e^{-k\xi_1} \right) & O\left( N^{-\frac{1}{3}} e^{-k(\xi_1+\xi_2)} \right) \\ O\left( N^{-\frac{1}{3}} e^{-k\xi_2} \right) & O\left( N^{-\frac{1}{3}} e^{-k\xi_2} \right) \end{pmatrix} + \begin{pmatrix} o(1) e^{-k\xi_1} & o(1) e^{-k(\xi_1+\xi_2)} \\ o(1) e^{-k\xi_2} & o(1) e^{-k\xi_2} \end{pmatrix} \frac{1}{2} \int_{-\infty}^{\xi_2} A_i(u)du,
\]

Proof. The statement for the 11th and 22nd entries follows immediately from (6.101), the representation (2.11) and the asymptotics of the Laguerre polynomials inside the Airy region. The statement for the 12th entry follows by replacing \( \epsilon(L_n w) \) in (6.95) by the asymptotics of the polynomials in (6.96) in the derivation of (6.97). The computation is the same as the derivation of (6.97) and we shall not carry out the details here. To obtain the results for the 21st entry, we use the fact that \( \epsilon(S_1(x, y)) \) is skew symmetric to obtain

\[
\epsilon(S_1(x, y)) = \epsilon(S_1(x, y)) - \epsilon(S_1(y, y)) = -\int_x^y S_1(t, y)dt.
\]

The statement for the 21st entry then follows from integrating (6.101) and the asymptotic formula for \( K_2 \).

A similar statement can be obtained for the Airy kernels when \( T \to \infty \).

Lemma 6.16. Let \( K_{airy}(\xi_1, \xi_2) \) be the following matrix kernel

\[
K_{airy,11}(\xi_1, \xi_2) = K_{airy,22}(\xi_2, \xi_1) = \frac{A_i(\xi_1)A_i'(\xi_2) - A_i(\xi_2)A_i'(\xi_1)}{\xi_1 - \xi_2} \quad \text{and} \quad \frac{1}{2} A_i(\xi_1) \int_{-\infty}^{\xi_2} A_i(u)du,
\]

\[
K_{airy,12}(\xi_1, \xi_2) = -\frac{\partial}{\partial \xi_2} K_{airy,11}(\xi_2, \xi_1), \quad K_{airy,21}(\xi_1, \xi_2) = -\int_{\xi_1}^{\xi_2} K_{airy,11}(u, \xi_2)du.
\]

Then for \( T \to \infty \) with \( t \) finite and \( \xi_1 \) and \( \xi_2 \) bounded from below, there exists \( k > 0 \) such that the rescaled matrix kernel \( K(\xi_1, \xi_2) \) in (6.103) is of the following order as \( N \to \infty \).

\[
\left( \frac{4}{N} \right)^{\frac{2}{3}} K(\xi_1, \xi_2) = K_{airy}(\xi_1, \xi_2) + \begin{pmatrix} o(1) e^{-k\xi_1} & o(1) e^{-k(\xi_1+\xi_2)} \\ o(1) e^{-k\xi_2} & o(1) e^{-k\xi_2} \end{pmatrix} \frac{1}{2} \int_{-\infty}^{\xi_2} A_i(u)du.
\]

In order to show that the convergence of the Fredholm determinant, we need the following bounds on the derivatives of the kernel \( K_2 \).

Lemma 6.17. For \( T \) of order \( N^{\frac{2}{3}-\epsilon} \) for some \( 0 < \epsilon \leq \frac{4}{3} \), we have

\[
\frac{\partial^l}{\partial \xi_1^l \partial \xi_2^j} \left( \left( \frac{4}{N} \right)^{\frac{2}{3}} K_2(\xi_1, \xi_2) - K_{2,\infty}(\xi_1, \xi_2) \right) = O\left( N^{-\frac{2}{3}} e^{-k(\xi_1+\xi_2)} \right)
\]

for some \( k > 0 \) and \( l, j = 0, 1 \).
Proof. The lemma is an immediate consequence of the following results in [32] (3.8)
\[
\left| \frac{\partial^i}{\partial \xi^i_1} \frac{\partial^j}{\partial \xi^j_2} \left( \left( \frac{4}{N} \right)^{\frac{3}{2}} K_{\text{lag}}(\xi_1, \xi_2) - K_{2,\text{airy}}(\xi_1, \xi_2) \right) \right| = O \left( N^{-\frac{3}{4}} e^{-k(\xi_1 + \xi_2)} \right)
\] (6.105)
where \( K_{2,\text{airy}} \) is the Airy kernel
\[
K_{2,\text{airy}}(\xi_1, \xi_2) = \frac{Ai(\xi_1)Ai'(\xi_2) - Ai(\xi_2)Ai'(\xi_1)}{\xi_1 - \xi_2}
\] (6.106)
and \( K_{\text{lag}} \) is the Christoffel Darboux kernel of the Laguerre polynomials
\[
K_{\text{lag}}(x, y) = w_0^2(x)w_0^2(y)\kappa_{N-1}^2 \frac{L_N(x)L_{N-1}(y) - L_N(y)L_{N-1}(x)}{x - y}
\]
and \( x = 4 + \xi_1 (4/N)^{\frac{3}{2}}, y = 4 + \xi_2 (4/N)^{\frac{3}{2}} \). As \( K_2 \) is the conjugate to \( K_{\text{lag}} \) and \( K_{2,\infty} \) is the conjugate to \( K_{2,\text{airy}} \),
\[
K_2 = \left( \frac{y(t - y)}{x(t - x)} \right)^{\frac{1}{2}} K_{\text{lag}}, \quad K_{2,\infty} = \left( \frac{T - \xi_2}{T - \xi_1} \right)^{\frac{1}{2}} K_{2,\text{airy}},
\]
the lemma follows from (6.105). \( \square \)

The corresponding statement when \( T \to \infty \) follows from the same argument but with \( K_{2,\infty} \) replaced by \( K_{2,\text{airy}} \).

With the estimates in Proposition 6.9 and Lemma 6.17, we can obtain the following asymptotic result for the determinant \( \det_2 (I - \chi K \chi) \).

Proposition 6.10. Let \( \zeta = (z - 4)(N/4)^{\frac{3}{2}}, \xi_1 = (x - 4)(N/4)^{\frac{3}{2}}, \xi_2 = (y - 4)(N/4)^{\frac{3}{2}}, \)
\( g(\xi) = \sqrt{1 + \xi^2} \) and \( G = \text{diag}(g, g^{-1}) \), then as \( N \to \infty \) and \( T \) of order up to \( o \left( N^{\frac{1}{2}} \right) \), we have
\[
\sqrt{\det_2 (I - \chi z G(\xi_1)K G^{-1}(\xi_2) \chi z)} = \sqrt{\det_2 (I - \chi z G(\xi_1)K_\infty G^{-1}(\xi_2) \chi z)} + o(1),
\]
and for \( T \to \infty \) while \( t \) remain finite, we have
\[
\sqrt{\det_2 (I - \chi z G(\xi_1)K G^{-1}(\xi_2) \chi z)} = \sqrt{\det_2 (I - \chi z G(\xi_1)K_{\text{airy}} G^{-1}(\xi_2) \chi z)} + o(1),
\]
where \( K \) is the rescaled kernel in (6.103) and \( \det_2 \) is the regularized 2-determinant \( \det_2 (I + A) = \det((I + A)e^{-A})e^{\text{tr}(A_{11} + A_{22})} \) for the \( 2 \times 2 \) matrix kernel \( A \) with entries \( A_{ij} \). The characteristic functions \( \chi_z \) and \( \chi_\zeta \) are \( \chi_z = \chi_{(z, \infty)} \) and \( \chi_\zeta = \chi_{(\zeta, \infty)} \) respectively.

The proof of this proposition is exactly the same as the proof of Corollary 1.4 in [32]. We shall not repeat the details of the proof here. As in [32], the function \( g(\xi) = \sqrt{1 + \xi^2} \) is to ensure that the 2-determinant exists and there is a great freedom in the choice of the \( g(\xi) \).

We will now analyze the asymptotics of the derivative \( \frac{\partial \log \det D}{\partial t} \) in (5.1).
6.4 Asymptotics of the derivative $\frac{\partial \log \det \mathbb{D}}{\partial t}$

We will now compute the asymptotics of the derivative $\frac{\partial \log \det \mathbb{D}}{\partial t}$ in (5.1). Let us write the integral in (5.1) in the following form

$$\frac{\partial}{\partial t} \log \det \mathbb{D} = - \int_{\mathbb{R}^+} \frac{K_2(x,x)}{t-x} \, dx - \int_{\mathbb{R}^+} \frac{K_1(x,x)}{t-x} \, dx,$$

(6.107)

where $K_2(x,y)$ is the kernel given by the Laguerre polynomials (2.11) and $K_1(x,y)$ is the correction term on the right hand side of (2.10).

To compute the contribution from the kernel $K_2$, we will use the following differential identity [16]. (See also [54] Lemma 2.1) Let $\hat{Z}$ be the matrix related to the matrix $Z$ in (4.39) by $\hat{Z} = Z w_0^{-\frac{\pi}{2}}$. Then we have

**Lemma 6.18.** Let $\hat{Z} = Z w_0^{-\frac{\pi}{2}}$, where $Z$ is the matrix in (4.39). Then we have

$$\int_{\mathbb{R}} \frac{K_2(x,x)}{t-x} \, dx = \frac{1}{2} \text{tr} \left( \hat{Z}^{-1} (t) \, \hat{Z}' (t) \, \sigma_3 \right)$$

(6.108)

The lemma can be proven in exactly the same way as Lemma 2.1 of [54]. The key is to use (4.38), the jump condition of $\hat{Z}$ on $\mathbb{R}^+$ and L’Hopital Rule to write $K_2(x,x)$ as

$$K_2(x,x) = \frac{1}{4\pi i} \left( \text{tr} \left( \hat{Z}_-^{-1} \hat{Z}_-' \sigma_3 \right) - \text{tr} \left( \hat{Z}_+^{-1} \hat{Z}_+' \sigma_3 \right) \right), \quad x \in \mathbb{R}^+$$

and then deform the contour of integration to obtain (6.108).

The asymptotics of the matrix $\hat{Z}$ can be found in [70]. For $x \not\in [0,4]$, the asymptotics of $\hat{Z}$ is given by

$$\hat{Z}(x) = (i \kappa_{N-1})^{-\frac{2\pi}{N}} R(x) P_\infty(x) e^{N \left( \frac{x}{2} - \phi(x) \right) \sigma_3},$$

(6.109)

where $R(x)$ is of the form $I + O (N^{-1})$ and $P_\infty(x)$ is a matrix bounded in $x$ for $x \not\in [0,4]$. Near the point 4, both the matrix $P_\infty$ and $P_\infty^{-1}$ have a forth root singularity. The derivative of $R(x)$ is of order $O (N^{-1})$ and the derivative of $P_\infty(x)$ remains bounded. From this, we have

$$\text{tr} \left( \hat{Z}^{-1} (t) \, \hat{Z}' (t) \, \sigma_3 \right) = N \left( 1 - \sqrt{\frac{t-4}{t}} \right) + O (1).$$

(6.110)

As we will see, this is the part that determines where the saddle point is. Once we have done the saddle point analysis later on in this section, we will see that at the phase transition, the saddle point will be inside the Airy region. We therefore also need the asymptotics of the matrix $\hat{Z}(x)$ inside the Airy region. This again, can be found in [70].
Lemma 6.19. ([70], Section 5.3) The asymptotics of the matrix $\dot{Z}(x)$ inside a small disc $U_\delta$ of radius $\delta$ around 4 is given by

$$\dot{Z}(x) = \frac{\sqrt{2}e^{\frac{i}{4}(i\kappa_{N-1})^{-\frac{4}{3}}}R(x)}{x^{\frac{1}{4}}(x-4)^{\frac{1}{4}}} \left( \begin{array}{cc} \cos \eta_+ & -i \sin \eta_+ \\ -i \cos \eta_- & -\sin \eta_- \end{array} \right) f_N(x)_{-\frac{4}{3}} A(f_N(x))w_0_{-\frac{4}{3}},$$

where $\eta_0$ is given by (6.8) and $A(\xi)$ is the matrix

$$A(\xi) = \sqrt{2}e^{\frac{i}{4} \frac{2\pi}{3}} \left( \begin{array}{cc} Ai(\xi) & Ai(\omega^2 \xi) \\ Ai'(\xi) & Ai'(\omega^2 \xi) \end{array} \right) e^{-i\frac{2\pi}{3} \sigma_3}, \quad \xi \in I,$n

$$A(\xi) = \sqrt{2}e^{\frac{i}{4} \frac{2\pi}{3}} \left( \begin{array}{cc} Ai(\xi) & Ai(\omega^2 \xi) \\ Ai'(\xi) & Ai'(\omega^2 \xi) \end{array} \right) e^{-i\frac{2\pi}{3} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right), \quad \xi \in II,$n

$$A(\xi) = \sqrt{2}e^{\frac{i}{4} \frac{2\pi}{3}} \left( \begin{array}{cc} Ai(\xi) & -\omega^2 Ai(\omega^2 \xi) \\ Ai'(\xi) & -Ai'(\omega^2 \xi) \end{array} \right) e^{-i\frac{2\pi}{3} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \quad \xi \in III,$n

$$A(\xi) = \sqrt{2}e^{\frac{i}{4} \frac{2\pi}{3}} \left( \begin{array}{cc} Ai(\xi) & -\omega^2 Ai(\omega^2 \xi) \\ Ai'(\xi) & -Ai'(\omega^2 \xi) \end{array} \right) e^{-i\frac{2\pi}{3} \sigma_3}, \quad \xi \in IV.$n

where $\omega = e^{\frac{2\pi i}{3}}$ and the regions $I$, $II$, $III$ and $IV$ are given by

$$I = \{\xi \mid 0 < \arg(\xi) < 2\pi/3\}, \quad II = \{\xi \mid 2\pi/3 < \arg(\xi) < \pi\}, \quad III = \overline{II}, \quad IV = \overline{I},$$

where the overline indicates complex conjugation. The matrix $R(x)$ is again of the form $I + O(N^{-1})$. Its derivative is of order $R'(x) = O(N^{-1})$.

From the behavior of the functions $\eta_0$, we see that the asymptotics of $\dot{Z}(x)$ inside the Airy region is of the form

$$\dot{Z}(x) = (i\kappa_{N-1})^{-\frac{2}{3}} R(x) P_a(x) N^{-\frac{2}{3}} A(f_N(x))w_0_{-\frac{2}{3}}, \quad (6.111)$$

where both $P_a(x)$ and $P_a^(-1)(x)$ are bounded and analytic inside $U_\delta$ (See [70]). Moreover, as $f_N(x) \to \infty$, there is a matching condition between the formula for $\dot{Z}(x)$ in the Airy region and its formula in the outside region (6.109).

$$P_a(x) N^{-\frac{2}{3}} A(f_N(x))w_0_{-\frac{2}{3}} = P_\infty(x) \left( I + O \left( f_N^{-\frac{2}{3}} \right) \right) e^{N \left( \frac{3}{2} - \varphi(x) \right) \sigma_3}. \quad (6.112)$$

From this and the fact that $P_\infty$ is of order $O \left( (x-4)^{-\frac{1}{2}} \right)$ as $x \to 4$, we see that (6.110) remains valid if $T = (t-4)(N/4)^{\frac{2}{3}}$ is large, but with some modifications in the error term.

$$\text{tr} \left( \dot{Z}^{-1}(t) \dot{Z}'(t) \sigma_3 \right) = N \left( 1 - \sqrt{\frac{t-4}{t}} \right) + O \left( N^T^{-\frac{2}{3}} \right) + O \left( N^{-\frac{2}{3}} \right). \quad (6.113)$$

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We shall divide the range of $T$ into the regimes $0 < |T| \leq N^{1/5}$ and $|T| \geq N^{1/5}$ and use (6.113) to for $|T| \geq N^{1/5}$. Let us now assume $|T| \leq N^{1/5}$. Then from (6.111) we obtain

$$
\text{tr} \left( \hat{Z}^{-1} (t) \hat{Z}' (t) \sigma_3 \right) = \text{tr} \left( A^{-1} (f_N (t)) A' (f_N (t)) f_N' \sigma_3 \right) + \text{tr} \left( P_a^{-1} P_a' N^{-\frac{2\delta}{4}} A \sigma_3 A^{-1} N^{-\frac{2\delta}{4}} \right) + M - \frac{M - N}{t} + O \left( N^{-\frac{\delta}{2}} \right),
$$

(6.114)

The error term is uniform in $T$ throughout $U_\delta$. By using the asymptotic formula for the Airy functions, we see that the matrix $A(t)$ has the following behavior as $t \to \infty$.

$$
A(t) = \frac{1}{\sqrt{2}} \xi^{-\frac{\alpha}{4}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\xi^2}{2}} \left( I + O(\xi^{-\frac{1}{2}}) \right) e^{-\frac{\xi^2}{2}} \sigma_3, \quad \xi \to \infty
$$

(6.115)

Therefore uniformly in $U_\delta$, the second term in (6.114) is of order

$$
\text{tr} \left( P_a^{-1} P_a' N^{-\frac{2\delta}{4}} A \sigma_3 A^{-1} N^{-\frac{2\delta}{4}} \right) = O \left( (N/T)^{\frac{\delta}{2}} \right)
$$

By (6.115), the first term in (6.114) has the following behavior as $f_N \to \infty$.

$$
\text{tr} \left( A^{-1} (f_N (t)) A' (f_N (t)) f_N' \sigma_3 \right) = -2f_N^{\frac{5}{2}} + O \left( f_N^{-1} \right), \quad f_N \to \infty.
$$

Since $f_N (t) = T \left( 1 + O \left( T/N^{\frac{\delta}{2}} \right) \right)$, we have, by mean value theorem,

$$
\text{tr} \left( A^{-1} (f_N (t)) A' (f_N (t)) f_N' \sigma_3 \right) = \text{tr} \left( A^{-1} (T) A' (T) f_N' \sigma_3 \right) + O \left( T^{\frac{\delta}{2}}/N^{\frac{\delta}{2}} \right)
$$

$$
= \left( \frac{N}{4} \right)^{\frac{\delta}{2}} \text{tr} \left( A^{-1} (T) A' (T) \sigma_3 \right) + O \left( T^{\frac{\delta}{2}} \right)
$$

Therefore after integration, we have

$$
\int_{t_0}^t \int_0^T K_2(x, x') dx' \frac{K_2(x, t-x)}{t-x} \, dx = \frac{1}{2} \int_{t_0}^T \text{tr} \left( A^{-1} (T) A' (T) \sigma_3 \right) dT + \frac{M(t-t_0)}{2} + O \left( N^{-\frac{\delta}{2}} \right),
$$

where $|T| \leq N^{\frac{1}{2}}$ and $|T_0| = N^{\frac{1}{2}}$. For $|T| \geq N^{\frac{1}{2}}$ and $|T_0| = N^{\frac{1}{2}}$, we have, by (6.113), the following

$$
\int_{t_0}^t \text{tr} \left( \hat{Z}^{-1} (t) \hat{Z}' (t) \sigma_3 \right) = N \int_{t_0}^t \left( 1 - \sqrt{\frac{t-4}{t}} \right) \, dt + O \left( N^{\frac{\delta}{2}} \right).
$$

Since the first term is of order $O \left( T^{\frac{\delta}{2}} \right) + O \left( N^{\frac{\delta}{2}} T \right)$, we see that for $|T| \geq N^{\frac{1}{2}}$, the first term will dominate over the error term. Summarizing, we have the following.

**Proposition 6.11.** Let $T = (t-4) (N/4)^{\frac{\delta}{2}}$. Then the contribution of the kernel $K_2$ to the logarithmic derivative of $\text{det} \, D$ is given by
1. Uniformly for $|T| \leq N^{1/5}$, we have
\[
\int_{t_0}^{t} \frac{K_2(x,x)}{t-x} \, dx = \frac{1}{2} \int_{t_0}^{T} \text{tr} \left( A^{-1}(T) A'(T) \sigma_3 \right) \, dT + \frac{M}{2} (t-t_0) + O \left( N^{-\frac{2}{15}} \right),
\] (6.116)
for any $|T_0| = N^{1/5}$, where the integration contour does not cross the jump contours of the matrix $A$.

2. Uniformly for $|T| > N^{1/5}$, we have
\[
\int_{t_0}^{t} \frac{K_2(x,x)}{t-x} \, dx = \frac{N}{2} \int_{t_0}^{t} \left( 1 - \sqrt{\frac{t-4}{t}} \right) \, dt + O \left( N^{-\frac{2}{15}} \right).
\] (6.117)
for any $|T_0| = N^{1/5}$.

where the integration contour does not cross $(-\infty, 4]$.

Let us now compute the contribution from the correction term $K_1(x,y)$. Before we compute the asymptotics, let us make a further simplification using (4.51). First by using the recurrence relation of the Laguerre polynomials (4.41), we see that
\[
L_N = \frac{tL_N}{t-x} - \frac{xL_N}{t-x} - \frac{L_N}{M(t-x)} (Mt - (M + N + 1)) - \frac{L_{N+1}}{t-x} - \frac{N}{M} \frac{L_{N-1}}{t-x}.
\]
Substituting this back into (4.51), we obtain
\[
\int_{R_+} \frac{K_1(x,y)}{t-x} \, dx = -\frac{M}{2h_{0,N-2}} \left( \frac{L_{N-2}}{t-x} \right)_{1} - 2 \frac{\langle L_N, L_{N-2} \rangle_1}{\langle L_{N-1}, L_{N-2} \rangle_1} \frac{\partial}{\partial t} \langle L_{N-1}, L_{N-2} \rangle_1 \right) \right.

- \frac{M}{2h_{0,N-1}} \frac{\partial}{\partial t} \langle L_{N+1}, L_{N-1} \rangle_1 - \frac{1}{2h_{0,N-1}} \left( \frac{L_N}{t-x} \right)_{1} \frac{\partial}{\partial t} \langle L_{N-1}, L_{N-1} \rangle_1 + \frac{N}{2h_{0,N-1}} \left( \frac{L_{N-1}}{t-x} \right)_{1}.
\] (6.118)

where we have used the fact that $\langle L_N, L_{N-1} \rangle_1 = 0$.

The main task is to compute the products $\langle L_n/(t-x), L_m \rangle_1$. The analysis is very similar to those in Section 6. First note that, outside of the Airy region, the analysis in Section 6 remains the same and we have

**Lemma 6.20.** The single and double integrals in $\langle L_n/(t-x), L_m \rangle_1$ have the following contributions in the Bessel region.

\[
(i^{k_n-1}) \int_0^{\infty} \frac{L_n(x)w(x)}{t-x} \, dx = \frac{(-1)^n \sqrt{2\pi}}{\sqrt{nt^{1/2}}} + O \left( n^{-\frac{2}{5}} \right).
\] (6.119)

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The double integral in the Bessel region is given by
\[
\begin{align*}
  i \sqrt{\kappa_n - 1} \int_0^{N^{-\frac{1}{2}}} L_n(x)w(x) \frac{t-x}{t} \int_x^{N^{-\frac{1}{2}}} L_m(y)w(y)dy & = \tilde{J}_1 = \left(\frac{(-1)^m n + 2\pi}{nt^2}\right) + O\left(n^{-\frac{5}{2}}\right).
\end{align*}
\] (6.120)

The single integral in the bulk region is given by
\[
\begin{align*}
  (i \kappa_n - 1)^{\frac{1}{2}} \int_{N^{-\frac{1}{2}}}^{c^-} \frac{L_n(x)w(x)}{t-x} dx & = \tilde{I}_2, n = \mathcal{O}\left(n^{-\frac{7}{8}}\right),
\end{align*}
\] (6.121)

Let \(n - m = k\), then the asymptotics of the double integrals in the bulk region is given by
\[
\begin{align*}
  \tilde{J}_2 & = \mathcal{O}\left(N^{-\frac{5}{4}}\right), k = 0, \\
  \tilde{J}_2 & = -\frac{\partial J_2}{\partial t} + \mathcal{O}\left(N^{-\frac{5}{4}}\right), k = 1, 2,
\end{align*}
\] (6.122)

where \(\tilde{J}_2\) is given by
\[
\begin{align*}
  \tilde{J}_2 & = i \sqrt{\kappa_n - 1} \int_{N^{-\frac{1}{2}}}^{c^-} L_n(x)w(x) \frac{t-x}{t} \int_x^{c^-} L_m(y)w(y)dy.
\end{align*}
\]

and \(J_2\) is given in (6.37).

Note that the choice of \(\varepsilon \leq 1/20\) ensures that the error terms are of the same order as before.

In the Airy region, the analysis is more different. Let us now compute these integrals inside the Airy region. As in Section 6.1.3, we have
\[
\begin{align*}
  \frac{N}{m} (i \kappa_{m-1})^{\frac{1}{2}} \int_{\frac{m^2}{N}}^{c^+} \hat{L}_m(y)dy & = \int_{v(u)}^{v^+} A E_m dv + \mathcal{O}\left(N^{-\frac{7}{4}}\right),
  \frac{N}{n} (i \kappa_{n-1})^{\frac{1}{2}} \int_{c^-}^{c^+} \frac{\hat{L}_n(x)}{t - \frac{x}{N}} dx & = \left(\frac{N}{4}\right)^{\frac{3}{2}} \int_{u_0}^{u_+} A_1 \hat{E}_n du + \mathcal{O}\left(N^{-\frac{7}{4}}\right),
\end{align*}
\] (6.123)

where \(A_1 = A/(T - u)\) and \(\hat{E}_n\) is of the form
\[
\hat{E}_n = 1 - \frac{3}{2} v_{0,n} + \frac{15}{8} v_{0,n}^2 + \mathcal{O}\left(N^{-1}\right),
\] (6.124)

where \(v_{0,n}\) is given in (6.47). By using mean value theorem, (6.55), we obtain the following estimate.
\[
\begin{align*}
  \int_{v(u)}^{v^+} A E_m dv & = \int_{u}^{v^+} A E_m dv + \left(\frac{4}{N}\right)^{\frac{1}{2}} (m-n)A(u)\hat{E}_m(u) \\
  & \quad - \left(\frac{4}{N}\right)^{\frac{3}{2}} \frac{(m-n)^2}{2} A'(u)\hat{E}_m(u) \left(1 + \mathcal{O}\left(\frac{u}{N}\right)\right) + \frac{1}{3} A''(\xi_u) (u - \nu)^3 \hat{E}_m(\xi_u)
\end{align*}
\] (6.125)
for some $\xi_u$ between $u$ and $\nu(u)$. As
\[
\left| \int_{u_-}^{u_+} A_1(u) A''(\xi_u)(\nu - u)^3 \mathcal{E}_m(\xi_u) \, du \right| < \frac{1}{N^{\frac{\delta}{2}}} \left| \int_{u_-}^{\frac{1}{2} |T|} \frac{C}{u^{\frac{\delta}{2}}(|T| - u)} \, du \right| + O \left( \frac{1}{(1 + |T|^2)N^{\frac{\delta}{2}}} \right),
\]
we see that this term is of order $O \left( N^{-\frac{\delta}{4}T^{-\frac{1}{2}}} \right)$ as $T \to \infty$ and $|T| < |u_-|$. Applying similar
argument to the other error terms in (6.125), we see that
\[
\int_{u_-}^{u_+} A_1 \mathcal{E}_n \int_{\nu(u)}^{v_+} A \mathcal{E}_m \, dv \, du
= \int_{u_-}^{u_+} A_1 \mathcal{E}_n \int_{u}^{v_+} A \mathcal{E}_m \, dv \, du + \left( \frac{4}{N} \right)^{\frac{\delta}{2}} (m - n) \int_{u_-}^{u_+} A_1 A \mathcal{E}_n \mathcal{E}_m \, du
- \left( \frac{4}{N} \right)^{\frac{\delta}{2}} (m - n)^2 \int_{u_-}^{u_+} A_1 \mathcal{A}' \, du + O \left( N^{-\frac{\delta}{4}T^{-\frac{1}{2}}} \right).
\]
Let us now determine the behavior of these terms as $|T| \to N^{\frac{\delta}{2}}$. We have

**Lemma 6.21.** As $|T| \to |u_-|$, the terms $\mathcal{A}_0(i, j)$ is of order $O \left( N^{-\frac{\delta}{4}T^{-i-j-1}} \right)$ in $T$.

**Proof.** Let us write the integral in the following form
\[
\mathcal{A}_0(i, j) = \int_{u_-}^{\frac{1}{2} |T|} A_i \int_{u}^{\frac{1}{2} |T|} A_j \, dv \, du + \int_{u_-}^{u_+} A_i \mathcal{A}_j \, dv \, du + \int_{u_-}^{u_+} A_i \int_{u}^{v_+} A_j \, dv \, du
\]
Then it is clear that the third term is of order $O \left( N^{-\frac{\delta}{4}T^{-i-j-1}} \right)$ in $N$ and $T$ as $|T| \to |u_-|$. By using the asymptotic formula (6.61) to integrate the second term by parts, we see that this term is also of order $O \left( N^{-\frac{\delta}{4}T^{-i-j-1}} \right)$. For the first term, we have, from (6.61), the following
\[
\int_{u_-}^{\frac{1}{2} |T|} A_i \int_{u}^{\frac{1}{2} |T|} A_j \, dv \, du < \int_{u_-}^{\frac{1}{2} |T|} \frac{C}{N^{\frac{\delta}{2}}u(|T| - u)^{i+j+1}} \, du
\]
for some constant $C > 0$ independent on $T$ and $N$. Integrating this gives us the result. \(\square\)

From Lemma 6.21 and the asymptotic behavior of $A$, we see that the various terms in
the above expansion are of the following behavior as $|T| \to N^{\frac{\delta}{2}}$.
\[
\int_{u_-}^{u_+} A_1 \mathcal{A}_1 \, du = O \left( N^{-\frac{\delta}{4}T^{-\frac{3}{2}}} \right), \quad \int_{u_-}^{u_+} A_1 \mathcal{A}' \, du = O \left( N^{-\frac{\delta}{4}T^{-\frac{1}{2}}} \right), \quad \int_{u_-}^{u_+} A_1^2 \, du = O \left( N^{-\frac{\delta}{4}T^{-\frac{3}{2}}} \right), \quad \mathcal{A}_0(i, j) = O \left( N^{-\frac{\delta}{4}T^{-i-j-1}} \right).
\]
From (6.126), we obtain the double integral inside of the Airy region.

Let us now compute the order of the following term in (6.118) inside the Airy region.

\[ D_1 = -\frac{M}{2h_{0,N-2}} \left< \frac{L_{N-2}}{t-x}, L_N \right>_1 + \frac{N}{2h_{0,N-1}} \left< \frac{L_{N-1}}{t-x}, L_{N-1} \right>_1. \] (6.128)

First by differentiating the identity \( \langle L_N, L_{N-1} \rangle_1 = 0 \), we can establish the following.

**Lemma 6.22.** Let \( PH(i, j) \) be

\[ PH(i, j) = \frac{1}{2} \int_{u_-}^{u_+} A_i du \int_{u_-}^{u_+} A_j du - A_0(i, j), \] (6.129)

and let \( \tilde{J}_{3k} \) be the followings.

\[ \tilde{J}_{31} = \left( \frac{N}{4} \right)^{1/4} \int_{u_-}^{u_+} A_1 du, \quad \tilde{J}_{32} = -\left( \frac{N}{4} \right)^{1/4} \frac{3}{2} PH(2, 0), \]

\[ \tilde{J}_{33} = \frac{3}{8} PH(1, 2) - \frac{1}{2} \left( \int_{u_-}^{u_+} A_1^2 du - \int_{u_-}^{u_+} A_1 A'du \right), \] (6.130)

\[ \tilde{J}_{34} = \frac{3}{8} (5PH(3, 0) - PH(1, 2)), \quad \tilde{J}_{35} = -2 \int_{u_-}^{u_+} A_1^2 du, \]

and let \( \tilde{I}_j \) be

\[ \tilde{I}_1 = \left( \frac{N}{4} \right)^{1/4} \frac{\sqrt{2\pi}}{4\sqrt{t}} \int_{u_-}^{u_+} A_1 du, \quad \tilde{I}_2 = \left( \frac{N}{4} \right)^{1/4} \frac{3\sqrt{2\pi}}{4\sqrt{t}} \int_{u_-}^{u_+} A_2 du. \] (6.131)

Let \( k = n - m \), then \( \langle \frac{L_n}{t-x}, L_m \rangle_1 \) is given by

\[
\frac{2\pi}{h_{N,0}} \left< \frac{L_n}{t-x}, L_m \right>_1 = -PH(1, 0) + k\tilde{J}_{31} + (N-n)\tilde{J}_{32} + k^2 \tilde{J}_{33} + (N-n)^2 \tilde{J}_{34} \\
+ k(N-n)\tilde{J}_{35} + (-1)^n \left( (-1)^k \tilde{I}_1 + (-1)^k (N-n) \tilde{I}_2 \right) + \frac{\partial}{\partial t} \tilde{J}_2 \\
+ O \left( N^{-\frac{3}{4}} / \left( 1 + |T|^{\frac{3}{2}} \right) \right) + O \left( N^{-\frac{5}{4}} / \left( 1 + |\sqrt{T}| \right) \right) \] (6.132)

The lemma follows from a straightforward calculation using (6.126) and Lemma 6.20.

As in the computation of the skew products \( \langle L_{2k}, L_{2k-1} \rangle_1 = 0 \) can be used to simplify the expressions \( \langle \frac{L_n}{t-x}, L_m \rangle_1 \). In this case, we differentiate the identity \( \langle L_{2k}, L_{2k-1} \rangle_1 = 0 \) with respect to \( t \). Then we obtain

\[ -2 \frac{\partial}{\partial t} \left< \frac{L_n}{t-x}, L_{n-1} \right>_1 = \left< \frac{L_n}{t-x}, L_{n-1} \right>_1 - \left< \frac{L_{n-1}}{t-x}, L_n \right>_1 = 0. \]
By using Lemma 6.22 we obtain the following identities.

\[ 2\tilde{J}_{31} - \tilde{J}_{32} - 2\tilde{I}_1 + 2\frac{\partial \tilde{J}_2}{\partial t} = E_1, \quad -\tilde{J}_{34} + \tilde{J}_{35} - \tilde{I}_2 = E_2, \tag{6.133} \]

where \( E_1 \) and \( E_2 \) are error terms that behave as the error terms in (6.132) and \( J_2 \) in the above is taken to be the expression in (6.37) with \( n - m = 1 \).

We can now compute the term \( D_1 \) in (6.128). By using Lemma 6.22 we obtain the following

\[ \frac{4\pi D_1}{M} = \tilde{J}_{35} - 4\tilde{J}_{33} + 2\frac{\pi}{N} \frac{\partial}{\partial t} \sqrt{\frac{t - 4}{t}} - \frac{\partial}{\partial t} (F_2 - 2F_1) + E. \tag{6.134} \]

where the error term \( E \) has the same order as the ones in (6.132).

Let us now show that \( F_2 - 2F_1 \) is of order \( O\left(N^{-1/2}\right) \).

From (6.33), we see that \( F_2 - 2F_1 \) is given by

\[ F_2 - 2F_1 = \int_0^4 \frac{\sin (2\phi) - 2 \sin \phi}{Nx(4 - x)} \frac{(t - x)}{dx} = \int_0^4 \frac{8 \sin \phi (\cos \phi - 1)}{Nx(4 - x)} (t - x) dx \]

where again, \( \phi = \arccos \left(\frac{t}{2} - 1\right) \). As \( \phi = \sqrt{4 - x (1 + O (x - 4))} \), we have

\[ F_2 - 2F_1 = - \int_0^4 \frac{4\sqrt{4 - x (1 + O (x - 4))} \frac{(t - x)}{dx}}{Nx}. \]

By the change of variable \( x = 4 + \left(\frac{1}{N}\right)^{1/2} \xi \) and \( t = 4 + \left(\frac{1}{N}\right)^{1/2} T \), it is easy to see that the above integral is of order \( O\left(N^{-1/2}\right) \). Hence after integration, we obtain

\[ \int_0^T D_1 dt = \frac{1}{2} \sqrt{\frac{t - 4}{t}} - \frac{1}{4} \int_{-\infty}^{\infty} \frac{Ai^2(u)}{(T - u)^2} du \]

\[ - \frac{3}{4} \int_0^T \left( \frac{Ai}{(T - u)^{3/2}}, \frac{Ai}{(T - u)^{3/2}} \right) dT + \int_0^T O(E) dT \tag{6.135} \]

where \( (f, g)_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \epsilon(x - y)f(x)g(y)dydx \) and the error \( E \) is of the order indicated in (6.132). In particular, the error term is of order \( o(1) \) uniformly in \( T = o \left( N^{1/4} \right) \). For \( T = O \left( N^{1/4} \right) \), the error term is of order \( O(1) \).

Let us now consider the other terms in (6.118). Let \( D_2 \) be

\[ D_2 = \frac{M}{h_{0, N-2}} \frac{(L_N, L_{N-2})}{(L_{N-1}, L_{N-2})} \frac{\partial}{\partial t} (L_{N-1}, L_{N-2}) - \frac{M}{h_{0, N-1}} \frac{\partial}{\partial t} (L_{N+1}, L_{N-1}) \]

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By using (6.86) and (4.50), we see that

$$D_2$$ is given by

$$\frac{2\pi D_2}{M} = \left( \frac{4I_1 + J_{B2} - J_{B3,2}}{2I_0} + O \left( N^{-\frac{2}{3T}} \right) \right) \frac{\partial}{\partial t} \left( -2I_0 + 6I_1 + O \left( N^{-\frac{2}{3T}} \right) \right)$$

$$+ \frac{\partial}{\partial t} \left( 4I_1 - J_{B3,2} + O \left( N^{-\frac{2}{3T}} \right) \right).$$

Note that the error term $$O \left( N^{-\frac{2}{3T}} \right)$$ is uniform in $$T$$ as $$|T| \to N^\frac{3}{4}$$ and its derivative is of the same order and is also uniform in $$T$$. From this, we obtain

$$\frac{2\pi D_2}{M} = \left( \frac{4I_1 + J_{B2} - J_{B3,2}}{2I_0} + O \left( N^{-\frac{2}{3T}} \right) \right) \frac{\partial}{\partial t} \left( -2I_0 + O \left( N^{-\frac{1}{T}} \right) \right)$$

$$+ \frac{\partial}{\partial t} \left( 4I_1 - J_{B3,2} + O \left( N^{-\frac{2}{3T}} \right) \right).$$

By using the definitions of $$I_0, I_1$$ and $$J_{B2}, J_{B3,2}$$ in Proposition 6.7, we obtain the following for $$D_2$$.

$$\int_t^{t'} D_2 dt = \frac{2}{\sqrt{t}} \int_{-\infty}^{\infty} H_1 du - \sqrt{\frac{t-4}{t}} \log \left( t^{\frac{1}{2}} \int_{-\infty}^{\infty} H_0 du \right)$$

$$- 2 \int_t^{T} \left( \frac{\int_{-\infty}^{\infty} H_1 du}{\sqrt{t} \int_{-\infty}^{\infty} H_0 du} \right)^2 dT + O \left( N^{-\frac{1}{4T}} \log T \right) + O \left( N^{-\frac{1}{3T}} T^{1/2} \right).$$

(6.136)

The orders of $$T$$ in the error terms indicates their behavior as $$|T| \to N^\frac{3}{4}$$. Summarizing, we obtain the contribution to the derivative from the correction kernel $$K_1(x, y)$$. 

Proposition 6.12. Let $$|T| < N^\frac{3}{4}$$. Then the contribution of $$K_1(x, y)$$ to the logarithmic derivative is given by

$$\int_t^{t'} \int_R \frac{K_1(x, x)}{t-x} dx dt = -\frac{1}{4} \int_{-\infty}^{\infty} \frac{Ai^2}{(T-u)^2} du + \int_{-\infty}^{\infty} H_1 du - \log \left( \int_{-\infty}^{\infty} H_0 du \right)$$

$$- \frac{3}{4} \int_t^{T} \left( \frac{Ai}{(T-u)^{3/2}} \right) \frac{Ai}{(T-u)^{3/2}} dT - \int_t^{T} \left( \frac{\int_{-\infty}^{\infty} H_1 du}{\int_{-\infty}^{\infty} H_0 du} \right)^2 dT + o(1).$$

(6.137)

where the error term is uniform in $$T$$. As $$T \to \infty$$ but $$t$$ remains finite, the function $$\int_t^{T} \int_R \frac{K_1(x, x)}{t-x} dx dt$$ will be of order $$O(1)$$ in $$N$$.

The fact that the integral remains bounded for $$t$$ finite follows from the estimates (6.127) and the expression (6.137).
6.5 Steepest descent analysis

Before we apply steepest descent analysis to the contour Γ, let us take a closer look at the integration constants in Proposition 6.11 and 6.12. Since the integration constants must be the same for all T while the functions involved have jump discontinuities on \( \mathbb{R} \), care must be taken when choosing these integration constants.

First note that the expressions in Proposition 6.11 and 6.12 can be simplified further through integration by parts. By repeat use of integration by parts, we can write the following terms in (6.135) as

\[
-\frac{3}{4} \left( \frac{A_i}{(T-u)^{3/2}} - \frac{A_i'}{2(T-u)^{3/2}} \right) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{A_i'^2}{(T-u)^3} \, du = 2 \int_{-\infty}^{0} \left( H_1 \int_{-\infty}^{u} H_2 \, dv - \frac{(-u)^{1/2}}{2\pi(T-u)} \right) \, du + 2 \int_{0}^{\infty} H_1 \int_{-\infty}^{u} H_2 \, dv \, du - \int_{-\infty}^{\infty} H_1 \, du \int_{-\infty}^{\infty} H_2 \, dv
\]

\[
- \int_{-\infty}^{0} \left( \frac{(A_i')^2 - A_i A_i''}{T-u} \right) \, du - \int_{0}^{\infty} \left( \frac{A_i'^2 - A_i A_i''}{T-u} \right) \, du.
\]

The term \( \frac{(-u)^{1/2}}{\pi(T-u)} \) is there to ensure the convergence of the respective integrals. From the jump discontinuities and the behavior at \( T \to \infty \), one can check that

\[
\int_{-\infty}^{0} \left( \frac{(A_i')^2 - A_i A_i''}{T-u} \right) \, du + \int_{0}^{\infty} \left( \frac{(A_i')^2 - A_i A_i''}{T-u} \right) \, du = \frac{1}{2} \text{tr} \left( A^{-1} (T) A'(T) \sigma_3 \right) + T^{1/2}
\]

Therefore we have, for \( |T| < N^{1/2} \),

\[
\int_{t}^{T} \int_{\mathbb{R}} \frac{S_1(x,x)}{t-x} \, dx \, dt = 2 \int_{-\infty}^{T} \int_{-\infty}^{0} \left( H_1 \int_{-\infty}^{u} H_2 \, dv - \frac{(-u)^{1/2}}{2\pi(T-u)} \right) \, dv \, du \, dT + 2 \int_{0}^{T} \int_{-\infty}^{u} H_1 \int_{-\infty}^{\infty} H_2 \, dv \, dudT - \int_{-\infty}^{T} \int_{-\infty}^{\infty} H_1 \, du \int_{-\infty}^{\infty} H_2 \, dv \, dT
\]

\[
+ \int_{-\infty}^{\infty} H_1 \, du - \log \left( \int_{-\infty}^{\infty} H_0 \, du \right) - \int_{-\infty}^{T} \left( \frac{\int_{-\infty}^{\infty} H_1 \, du}{\int_{-\infty}^{\infty} H_0 \, du} \right)^2 \, dT + \frac{Mt}{2} - \frac{2}{3} T^{3/2} + C + o(1),
\]

where \( C \) is an integration constant that has no jump on \( \mathbb{R} \). By Lemma 6.21, we see that the term \( \left( \frac{\int_{-\infty}^{\infty} H_1 \, du}{\int_{-\infty}^{\infty} H_0 \, du} \right)^2 \) is of order \( T^{-\frac{3}{2}} \) as \( |T| > N^{1/2} \). Hence the we can choose the base point of this integral to be \( +\infty \). For the other terms, integration by parts shows
that
\[ \mathcal{S}_{11} = 2 \int_{-\infty}^{0} \left( H_1 \int_{-\infty}^{u} H_2 dv - \frac{(-u)^{3/2}}{2\pi(T-u)} \right) du + 2 \int_{0}^{\infty} H_1 \int_{-\infty}^{u} H_2 dv du \]
\[ - \int_{-\infty}^{\infty} H_1 du \int_{-\infty}^{\infty} H_2 dv du = -\frac{1}{\pi} \int_{-\infty}^{0} \frac{2\pi(A')^2 - (-u)^{3/2}}{(T-u)} du \]
\[ - 2 \int_{0}^{\infty} \frac{(A')^2}{(T-u)} du + O(T^{-2}) \]
\[ (6.138) \]
as \( T \to \infty \). Therefore if we let \( 0_{\pm} \) be the origin on the positive and negative sides of \( \mathbb{R} \), then \( \int_{0_+}^{0_-} \mathcal{S}_{11} dT \in \mathbb{i} \mathbb{R} \) is bounded, where the integration contour goes from \( 0_+ \) to \( \infty \) along the positive side of \( \mathbb{R}_+ \) and then from \( \infty \) back to \( 0_- \) along the negative side of \( \mathbb{R}_- \). Therefore we can fix the integration constant by defining the integral of \( \mathcal{S}_{11} \) to be
\[ \int_{0_+}^{0_-} \mathcal{S}_{11} dT + 1/2 \int_{0_-}^{0_+} \mathcal{S}_{11} dT, \]
which is the same as \( \int_{0_-}^{0_+} \mathcal{S}_{11} dT - 1/2 \int_{0_-}^{0_+} \mathcal{S}_{11} dT \). We shall use these as definitions of the function \( \int_{T}^{T} \mathcal{S}_{11} dT \) in the upper and lower half planes respectively. Summarizing, we obtain

**Proposition 6.13.** The logarithm of \( \det \mathbb{D} \) is given by the following

1. **Uniformly for** \( |T| < N^{\frac{3}{4}} \), **we have**

\[ \det \mathbb{D} = C_0 \exp \left( -\mathcal{S}_{10} - \int_{0_+}^{T} \mathcal{S}_{11} dT \mp 1/2 \int_{0_-}^{0_+} \mathcal{S}_{11} dT - \frac{M}{2} t \right) \]
\[ \times \int_{-\infty}^{\infty} H_0 du \left( 1 + o(1) \right), \quad \pm \text{Im}(T) > 0 \]
\[ (6.139) \]

where \( C_0 \) is an integration constant and \( \mathcal{S}_{11} \) is given by \( (6.138) \), while \( \mathcal{S}_{10} \) is

\[ \mathcal{S}_{10} = \int_{-\infty}^{\infty} H_1 du + \int_{T}^{\infty} \left( \int_{-\infty}^{\infty} H_1 dv du \right)^2 dT - \frac{2}{3} T^{\frac{3}{2}}. \]

2. **Uniformly for** \( |T| > N^{\frac{3}{4}} \), **det** \( \mathbb{D} \) **is given by**

\[ \det \mathbb{D} = C_0 \exp \left( -\frac{N}{2} \int_{0}^{t} \left( 1 - \sqrt{\frac{t-4}{t}} \right) dt + O \left( N^{\frac{7}{8}} \right) \right). \]
\[ (6.140) \]

### 6.6 Saddle point analysis

We can now carry out the steepest descent analysis for the contour integral in \( t \). As we have seen in the last section, the regularized determinant \( \det_2 (I - \chi G K G^{-1} \chi) \) remains
bounded in $N$ for all values of $t$. Then from (6.140), we see that for $|T| > N^{1/2}$, the saddle point is the solution of the following equation.

$$\frac{2\tau}{(1 + \tau)} - \left(1 - \sqrt{\frac{t - 4}{t}}\right) = 0.$$ 

The saddle point is then given by

$$t_{saddle} = \frac{(1 + \tau)^2}{\tau}.$$ 

For $\tau \in (-1, 0)$, we have $t_{saddle} \in (-\infty, 0)$ and for $0 < \tau < 1/4$, we have and $t_{saddle} \in [4, \infty)$. Hence the saddle point $t_{saddle}$ will not intersect the bulk region $(0, 4)$ for any value of $\tau \neq \pm 1$.

In this case, we can deform the contour $\Gamma$ such that it does not intersect the interval $[0, 4]$ and by Proposition 6.10 we see that the kernels $K_1$ and $K_2$ become the respective Airy kernels for all $t \in \Gamma$. The largest eigenvalue distribution in this case becomes

$$\mathbb{P}\left(\left(\lambda_{\text{max}} - 4\right)\left(N/4\right)^{1/2} \leq \zeta\right) = C_0 \sqrt{\det_2 \left(I - \chi_{[\zeta, \infty)} G K_{\text{airy}} G^{-1} \chi_{[\zeta, \infty)}\right)} \int_{\Gamma} (f_1(t) + f_2(t, z)) \, dt$$

for some function $f_1(t)$ independent on $z$ and $f_2(t, z)$ of order $o(1)$. This gives us the Tracy Widom distribution for the largest eigenvalue. This is a known result in [39].

However, when $\tau = 1$, saddle point coincides with the right edge point. In this case, the main contribution of the integral in $t$ will come from a neighborhood of $t = 4$ and the kernels will become significantly different from the Airy kernel. This is the case when the phase transition happens. Let us now find the steepest descent contour in this case.

**Lemma 6.23.** There exists $\delta > 0$ such that

$$\text{Re}\left(\int_1^x \sqrt{\frac{s - 4}{s}} \, ds\right) < 0, \quad |\text{Im}(x)| < \delta, \quad -\delta < \text{Re}(x) < 4$$

(6.141)

**Proof.** The statement for $0 < \text{Re}(x) < 4$ is a standard property that follows from the Cauchy-Riemann equation. (See, e.g. [70].) Since

$$\frac{\partial}{\partial x} \text{Im}\left(\int_1^x \left(\sqrt{\frac{s - 4}{s}}\right) \pm \, ds\right) = \pm \sqrt{\frac{4 - x}{x}},$$

By the Cauchy-Riemann equations, we see that $\text{Re}\left(\int_1^x \left(\sqrt{\frac{s - 4}{s}}\right) \pm \, ds\right)$ is decreasing as we move away from the real axis. As this real part is zero on $[0, 4]$, we obtain the inequality (6.141) on for $0 < \text{Re}(x) < 4$. As the function $\int_1^x \sqrt{\frac{4 - x}{s}} \, ds$ behaves as $c_0 - 2\sqrt{2}(-x)^{1/2} + O(x)$ when $x \to 0$ for some $c_0 \in \mathbb{R}$, we see that in a sufficiently small disk $D$ center at $x = 0$, we have

$$\text{Re}\left(\int_1^x \left(\sqrt{\frac{s - 4}{s}}\right) \pm \, ds\right) < 0, \quad x \in D \setminus \mathbb{R}_+.$$ 

This completes the proof of the lemma. $\square$
We can therefore choose our integration contour as in Figure 2 to obtain

**Proposition 6.14.** Let \( w = \left( \frac{N}{4} \right)^{\frac{3}{4}} (1 - \tau) \in (-\infty, \infty) \) and let \( \zeta = (z - 4) \left( \frac{N}{4} \right)^{\frac{3}{4}} \), then as \( N \to \infty \), the largest eigenvalue distribution is given by

\[
\lim_{N \to \infty} \mathbb{P} \left( \lambda_{\text{max}} - 4 \left( \frac{N}{4} \right)^{\frac{3}{4}} \leq \zeta \right) = C \int_{\Xi} e^{-\frac{w}{2} - \frac{1}{2} S \left( \int_{-\infty}^{\infty} H_0 du \right)^{\frac{3}{4}}} \times \left( \det_2 \left( I - \chi_\zeta G(\xi_1) K_\infty G^{-1}(\xi_2) \chi_\zeta \right) \right)^{\frac{1}{2}} dT,
\]

for some constant \( C \), where \( \Xi \) is a symmetric contour that does not contain any zero of \( \int_{-\infty}^{\infty} H_0 du \) and approaches \( \infty \) in the sector \( \pi/3 < \arg T < 4\pi/3 \), \( \arg T \neq \pi \). It intersects the \( \mathbb{R} \) at a point \( T_0 > \zeta \). The function \( S \) is given by

\[
S = S_{10} + \int_{0_T}^{T} S_{11} dT \pm 1/2 \int_{0_T}^{0_T} S_{11} dT, \quad \pm \text{Im}(T) > 0.
\]

**Remark 6.4.** In the above formula, the integration should be understood as a sum of integrations performed over the contours \( \Xi_+ \) and \( \Xi_- \), where \( \Xi_\pm \) are the intersections of \( \Xi \) with the upper/lower half plane. Near the intersection point \( T_0 \), the boundary values of the integrand in the upper/lower half planes are to be taken when performing these integrals.

### 7 Fredholm determinant

In this section we will carry out the final part of the analysis and express the determinant \( \det_2 \left( I - \chi_{[z, \infty)} K \chi_{[z, \infty)} \right) \) in terms of the Hastings-McLeod solution of the Painlevé II equation. We will follow the approach in [67]. While the following operations are formal and did not take into account the fact that \( I - G_\chi K_\infty G^{-1}_\chi \) is not of trace class, the procedure and the resulting formula (7.3) can be justified rigorously as in [67].

Let \( S_{1,\infty} \) be the operator with the kernel \( S_{1,\infty} = K_{1,\infty} + K_{2,\infty} \), \( D \) the differential operator, \( \epsilon \) the operator with kernel \( \epsilon(\xi_1 - \xi_2) \) and \( \chi \) the multiplication by \( \chi_{[z, \infty)} \), where \( \zeta = (z - 4) \left( \frac{N}{4} \right)^{\frac{3}{4}} \). Then by Proposition 6.10, we would like to consider the determinant...
of the following operator.

\[ I - G \chi \left( \begin{array}{cc} S_{1,\infty} & \epsilon \frac{S_{1,\infty} D}{S_{1,\infty}^T} \end{array} \right) \chi G^{-1}, \tag{7.1} \]

where \( S_{1,\infty}^T \) is the transpose of \( S_{1,\infty} \) and \( \hat{\epsilon} f = \int_{\xi_2}^{\xi} f(t)dt \). Then by (6.101) and (6.104), we see that

\[
- \frac{\partial}{\partial \xi_2} K_{1,\infty}(\xi, \xi_2) - \frac{\partial}{\partial \xi_1} K_{1,\infty}(\xi_2, \xi) = \frac{\partial}{\partial \xi_2} K_{2,\infty}(\xi, \xi_2) + \frac{\partial}{\partial \xi_1} K_{2,\infty}(\xi_2, \xi_1) \\
= -TH_0(\xi_1)H_0(\xi_2) - H_1(\xi_1)H_1(\xi_2) + H_0(\xi_1)H_2(\xi_2) + H_2(\xi_1)H_0(\xi_1).
\]

This implies \( S_{1,\infty} D = DS_{1,\infty}^T \) and as \( D \hat{\epsilon} = I \), we can follow [67] to write the operator as

\[ I - AB, \quad A = G \left( \begin{array}{cc} \chi D & 0 \\ 0 & \chi \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) g, \quad B = g^{-1} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) \left( \begin{array}{cc} \hat{\epsilon}(S_{1,\infty}) & S_{1,\infty}^T \chi \\ \hat{\epsilon}(S_{1,\infty}) - \epsilon \chi D & S_{1,\infty}^T \chi \end{array} \right) \chi G^{-1} \]

then by using \( \det(I - AB) = \det(I - BA) \), we obtain

\[ I - g^{-1} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) \left( \begin{array}{cc} \hat{\epsilon}(S_{1,\infty}) \chi D & S_{1,\infty}^T \chi \\ \hat{\epsilon}(S_{1,\infty}) - \epsilon \chi D & S_{1,\infty}^T \chi \end{array} \right) \chi G^{-1} g; \]

from this, we see that the determinant can be written as the determinant of the scalar operator

\[ \det_2 \left( I - \chi GK_{\infty} G^{-1} \chi \right) = \det_2 \left( I - g^{-1} \left( \hat{\epsilon}(S_{1,\infty}) \chi D + S_{1,\infty}^T \chi - S_{1,\infty}^T \epsilon \chi D \right) g \right). \]

Let us now show that \( \hat{\epsilon} S_{1,\infty} = S_{1,\infty}^T \epsilon \). First by writing \( \hat{\epsilon} S_{1,\infty} \) as

\[ \hat{\epsilon} S_{1,\infty} = \epsilon(S_{1,\infty})(\xi_1, \xi_2) - \epsilon(S_{1,\infty})(\xi_2, \xi_2), \]

we see that the first term is equal to \( S_{1,\infty} \epsilon \) because \( S_{1,\infty} D = DS_{1,\infty}^T \), and hence \( \epsilon S_{1,\infty} = S_{1,\infty}^T \epsilon \). Let us compute the second term. Let \( f \) be an \( L^2 \) function, then

\[ \epsilon(S_{1,\infty})(\xi_2, \xi_2) f = -\int_{-\infty}^{\infty} \int_{\xi_2}^{\xi} S_{1,\infty}(t, \xi_2) f(\xi_2)d\xi_2 dt + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{1,\infty}(t, \xi_2) f(\xi_2)d\xi_2 dt. \tag{7.2} \]

These terms can be computed using integration by parts. The first term becomes

\[
\int_{-\infty}^{\infty} \int_{\xi_2}^{\xi} S_{1,\infty}(t, \xi_2) f(\xi_2)d\xi_2 dt = \int_{-\infty}^{\infty} S_{1,\infty}(\xi_2, \xi_2) \int_{-\infty}^{\xi_2} f(u)du d\xi_2 - \int_{-\infty}^{\infty} \int_{\xi_2}^{\infty} \frac{\partial}{\partial \xi_2} S_{1,\infty}(t, \xi_2) \int_{-\infty}^{\xi_2} f(u)du d\xi_2.
\]
As \( \frac{\partial}{\partial \xi_2} S_{1,\infty}(t, \xi_2) = \frac{\partial}{\partial t} S_{1,\infty}(\xi_2, t) \), we obtain
\[
\int_{-\infty}^{\infty} \int_{\xi_2}^{\infty} S_{1,\infty}(t, \xi_2) \int_{-\infty}^{\xi_2} f(u) du \, dt \, d\xi_2 = \int_{-\infty}^{\infty} S_{1,\infty}(\xi_2, t) \int_{-\infty}^{\xi_2} f(u) du \, dt \, d\xi_2 - \int_{-\infty}^{\infty} S_{1,\infty}(\xi_2, \xi_2) \int_{-\infty}^{\xi_2} f(u) du \, dt \, d\xi_2 = 0.
\]
The second term in (7.2) can be computed similarly and we obtain \( \tilde{\epsilon} S_{1,\infty} = S_{1,\infty}^T \epsilon \). Therefore we have
\[
\det_2 \left( I - \chi G K_{\infty} G^{-1} \chi \right) = \det_2 \left( I - g^{-1} \left( S_1^T \epsilon \chi D + S_1^T \chi - S_1^T \chi \epsilon \chi D \right) g \right).
\] (7.3)

As mentioned before, the procedure in obtaining (7.3) can be rigorously justified as in [67]. We shall therefore treat (7.3) as a rigorous formula and refer the readers to [67].

7.1 Differential equations for the Fredholm determinant

Before we compute the asymptotics of this determinant, let us first recall some facts about Fredholm determinants and their relations to Painlevé equations.

Let us now recall some basic facts about operators of the form (6.106). The Airy kernel (6.106) is an integrable kernel, that is, it has the form
\[
K(x, y) = \sum_{j=1}^{k} f_{1j}(x) f_{2j}(y) \frac{x - y}{x - y}.
\] (7.4)

In this case, we have \( k = 2 \) and \( f_{11} = -f_{22} = \text{Ai} \), while \( f_{21} = f_{12} = \text{Ai}' \). Operators of these form appear often in random matrix theory and have been studied extensively in the literature. These operators were first singled out as a distinguished class in [44] in which their properties were also studied. In random matrix theory, they were used to obtain the celebrated Tracy-Widom distribution.

Let us remind ourselves the following well-known facts about integrable operators. (See e.g. [30], [44], [65]) Let the kernel of an integrable operator on a contour \( \Sigma \) be given by (7.4). Suppose \( I - K \) is invertible, then the resolvent \( R \) given by
\[
R = (I - K)^{-1} K = (I - K)^{-1} - I,
\] (7.5)
is also an integrable operator with kernel given by
\[
R(x, y) = \frac{\sum_{j=1}^{k} F_{1j}(x) F_{2j}(y)}{x - y},
\] (7.6)

where \( F_{ij} \) are given by \( F_{ij} = (I - K)^{-1} f_{ij} \). Let us now specialize to the resolvent of the operator with the kernel \( K_{2,\text{airy}} \chi \) acting on \( \mathbb{R} \). Then the resolvent of this kernel is given by \( (I - K_{2,\text{airy}} \chi)^{-1} = I + R_0 \chi \), where \( R_0 \) has the kernel
\[
R_0(\xi_1, \xi_2) = \frac{\Phi_0(\xi_1) \Phi_1(\xi_2) - \Phi_0(\xi_2) \Phi_1(\xi_1)}{\xi_1 - \xi_2}.
\] (7.7)
where $\Phi_0$ and $\Phi_1$ are

$$
\Phi_0 = (I - K_{2, \text{airy}} \chi)^{-1} A_i, \quad \Phi_1 = (I - K_{2, \text{airy}} \chi)^{-1} A_i'.
$$

(7.8)

For our analysis, we will also need $\Phi_2$ defined by $\Phi_2 = (I - K_{2, \text{airy}} \chi)^{-1} A_i''$. The resolvent $R_0 \chi$ is closely related to the Hastings-McLeod solution of the Painlevé II equation. In fact, the determinant $\det (I + R_0 \chi)$ is well-known in the random matrix literature and is given by the Tracy-Widom distribution for the GUE \[66\].

$$
\det(I + R_0 \chi) = TW_2(\zeta).
$$

(7.9)

The operator $S_{1, \infty}$ has kernel $S_{1, \infty} = K_{1, \infty} + \psi^{-1} K_{2, \text{airy}} \psi$, where $\psi$ is the multiplication of the function $\psi = (T - \xi)^{1/2}$. We will also use the same notation to denote this square root function itself.

By substituting the asymptotic kernels into (7.3), we obtain

$$
\det_2 (I - G \chi K_{\infty} \chi G^{-1}) = \det(I + g^{-1} R^T g \chi) \det \left( I - g^{-1} \tilde{K} g \right),
$$

(7.10)

where $\tilde{K}$ and $R$ are the operators with the following kernels

$$
\begin{align*}
R(\xi_1, \xi_2) &= \psi^{-1}(\xi_1) R_0(\xi_1, \xi_2) \psi(\xi_2), \\
\tilde{K} &= (I + R^T \chi) \left( K_{1, \infty}^T + K_{2, \text{airy}}^T \right) (1 - \chi) \epsilon \chi D + (I + R^T \chi) K_{1, \infty}^T \chi.
\end{align*}
$$

(7.11)

As the resolvent operator $R$ is conjugate to the resolvent $R_0$, we have $\det(I + g^{-1} R^T g \chi) = \det(I + R_0 \chi)$, which is the Tracy-Widom distribution for the GUE by (7.9). Let us now consider the determinant of $I - g^{-1} \tilde{K} g$. We will follow the steps in \[67\] to show that the operator $g^{-1} \tilde{K} g$ is of finite rank, that is, it is of the form

$$
\tilde{K} = \sum_{j=1}^{k} \alpha_j \otimes \beta_j,
$$

(7.12)

where $f \otimes h$ is the operator with kernel $f(x)h(y)$. As in \[67\], the operator $(1 - \chi) \epsilon \chi D$ is given by (See (16) in \[67\])

$$
(1 - \chi) \epsilon \chi D = (1 - \chi) \left( -\epsilon_\zeta \otimes \delta_\zeta + \epsilon_\infty \otimes \delta_\infty \right),
$$

(7.13)

where $\epsilon_\zeta$, $\epsilon_\infty$, $\delta_\zeta$ and $\delta_\infty$ are given by

$$
\epsilon_\zeta(\xi) = \epsilon(\xi - \zeta), \quad \epsilon_\infty(\xi) = -\frac{1}{2}, \quad \delta_\zeta = \delta(\xi - \zeta), \quad \delta_\infty = \delta(\xi - \infty).
$$

From these definitions, it is easy to see that

$$
\epsilon_\zeta(1 - \chi) = \epsilon_\infty(1 - \chi) = -\frac{1}{2}(1 - \chi).
$$

(7.14)
Now from (6.101), we see that the kernel $K_{1,\infty}$ is of rank 2.

$$K_{1,\infty} = H_0 \otimes G_0 + H_1 \otimes G_1,$$

$$G_0 = \left( \frac{T}{2} I(H_0) - B_1 I(H_1) - I(H_2) + B_2 \right), \quad G_1 = \left( \frac{1}{2} I(H_1) + B_1 I(H_0) \right)$$

(7.15)

where $I$ is the integration $I(f) = \int_{\xi}^{\infty} f(x)dx$. By using this, (7.13) and (7.14) in (7.11), we see that $\tilde{K}$ can be written as

$$\tilde{K} = \frac{1}{2} (I + R^T \chi) \left( \psi K_{2,airy} \psi^{-1}(1 - \chi) + (H_0, (1 - \chi)) G_0 \right)$$

$$+ (H_1, (1 - \chi)) G_1 \right) \otimes (\delta_{\zeta} - \delta_{\infty}) + (I + R^T \chi) \sum_{j=0}^{1} G_j \otimes H_j \chi,$$

where $(f, h) = \int_{\mathbb{R}} f(x)h(x)dx$ and we have used $(a \otimes b)(c \otimes d) = (b, c)a \otimes d$ to obtain the above equation. From this, we see that $\tilde{K}$ is indeed of the form (7.12) with $k = 3$ and $\alpha_j, \beta_j$ given by

$$\alpha_1 = \frac{1}{2} (I + R^T \chi) \left( \psi K_{2,airy} \psi^{-1}(1 - \chi) + (H_0, (1 - \chi)) G_0 + (H_1, (1 - \chi)) G_1 \right),$$

$$\beta_1 = \delta_{\zeta} - \delta_{\infty}, \quad \alpha_j = (I + R^T \chi)G_{j-2}, \quad \beta_j = H_{j-2} \chi, \quad j = 2, 3,$$

(7.16)

Then the determinant $\det \left( I - g^{-1} \tilde{K} g \right)$ is given by the determinant of the $3 \times 3$ matrix with entries $\delta_{ij} - (\alpha_i, \beta_j)$. We will now derive a close system of ODEs in the variable $\zeta$ for these matrix entries. From the expression of $G_j$ in (7.15), we see that the matrix entries involve the following quantities

$$Q_{-, j} = \psi(I + R_0 \chi) \psi^{-1} I(H_j), \quad u_{-, jk} = (Q_{-, j}, H_k \chi),$$

$$V_k = \left( \psi(I + R_0 \chi) K_{2,airy} \psi^{-1}(1 - \chi), H_k \chi \right), \quad k = 0, 1, \quad j = 0, 1, 2.$$  

(7.17)

We will also introduce some auxiliary functions

$$Q_{+, j} = \psi^{-1}(I + R_0 \chi) \psi I(H_j), \quad u_{+, jk} = \left( \psi^2 Q_{+, j}, H_k \chi \right),$$

$$R_{\pm} = \psi^{\pm 1} \int_{-\infty}^{\zeta} R_0(\zeta, \xi) \psi^{\pm 1}(\xi)d\xi, \quad \tilde{R}_{\pm} = -\psi^{\mp 1} \int_{\zeta}^{\infty} R_0(\zeta, \xi) \psi^{\pm 1}(\xi)d\xi,$$

$$\mathcal{P}_{+, 0} = \int_{-\infty}^{\zeta} \Phi_0 \psi d\xi, \quad \mathcal{P}_{-, j} = \int_{-\infty}^{\zeta} \Phi_j \psi^{-1} d\xi,$$

$$\tilde{\mathcal{P}}_{+, 0} = -\int_{\zeta}^{\infty} \Phi_0 \psi d\xi, \quad \tilde{\mathcal{P}}_{-, j} = -\int_{\zeta}^{\infty} \Phi_j \psi^{-1} d\xi, \quad k = 0, 1, \quad j = 0, 1, 2.$$  

(7.18)

which will be more convenient for the purpose of deriving the ODEs. Note that, as in (67) and (65), the $u_{\pm, jk}$ can be written as

$$u_{\pm, jk} = \left( \psi^{\pm 1} I(H_j), \psi H_k \chi \right) = \left( \psi^{\pm 1} I(H_j), (I + R_0 \chi) \psi H_k \chi \right)$$

$$= \left( \Phi_k, \psi^{\pm 1} I(H_j) \chi \right),$$

(7.19)
where we have used (7.8) in the above. By using the definition of the resolvent (7.5) to write $R_0 \chi$ as $R_0 \chi = K_{2, \text{airy}} \chi (I + K_{2, \text{airy}} \chi)^{-1}$, we see that

$$R_0(\xi_1, \xi_2) = O \left( e^{-\frac{2}{\epsilon} \xi_1^2} \right), \quad \xi_1 \to \infty, \quad Q_{\pm,j}(\infty) = \int_{-\infty}^{\infty} H_j d\xi.$$  (7.20)

We have the following relations between these variables.

**Lemma 7.1.** The functions $V_k$ and $Q_{+,k}$ in (7.14) can be written as

$$V_k = \mathcal{P}_{-,k} - (H_k, (1 - \chi)), \quad Q_{+,j} - Q_{-,j} = \frac{\Phi_0}{\psi} u_{-j1} - \frac{\Phi_1}{\psi} u_{-j0},$$  (7.21)

$$\mathcal{R}_+ - \mathcal{R}_- = \frac{\Phi_0(\zeta)}{\psi(\zeta)} \mathcal{P}_{-,1} - \frac{\Phi_1(\zeta)}{\psi(\zeta)} \mathcal{P}_{-,0}.$$  

**Proof.** By using the definition of the resolvent (7.5), we see that $(I + R_0 \chi) K_{2, \text{airy}} = R_0$ and hence $V_k$ can be written as

$$V_k = (R_0 \psi^{-1} (1 - \chi), \psi H_k \chi) = (\psi^{-1} (1 - \chi), R_0 \psi H_k \chi) = ((1 - \chi), \Phi_k \psi^{-1}) - (H_k, (1 - \chi)),$$

where we have used the property of integrable operators (7.8). The first equation then follows immediately from this.

By using $\psi = (T - \epsilon)^{\frac{1}{2}}$, one can easily verify the following.

$$\psi^{-1} R_0 \psi - \psi R_0 \psi^{-1} = \frac{\Phi_0}{\psi} \otimes \frac{\Phi_1}{\psi} - \frac{\Phi_1}{\psi} \otimes \frac{\Phi_0}{\psi}.$$  

The equation relating $Q_{\pm,j}$ and $\mathcal{R}_\pm$ then follows from this and (7.19). \qed

By subtracting $1/2(H_0, (1 - \chi))$ times the second column, and $1/2(H_1, (1 - \chi))$ times the third column from the first column of the determinant det $(I - (\alpha_j, \beta_k))$, we see that the determinant is the same as

$$\det (I - (\alpha_j, \beta_k)) = \det \begin{pmatrix} 1 & -\alpha_2, \beta_1 \\ -\frac{1}{2} \mathcal{P}_{-,0} & 1 - (\alpha_2, \beta_2) \\ -\frac{1}{2} \mathcal{P}_{-,1} & 1 - (\alpha_3, \beta_3) \end{pmatrix}.  \quad (7.22)$$

By using (7.20), (7.15), (7.17) and (7.18), we see that the other entries $(\alpha_i, \beta_j)$ can be written as

$$\begin{align*}
(\alpha_2, \beta_1) &= \frac{T}{2} \left( q_{-,0} - q_{-,0}^{(\infty)} \right) - B_1 \left( q_{-,1} - q_{-,1}^{(\infty)} \right) - \left( q_{-,2} - q_{-,2}^{(\infty)} \right) - B_2 \tilde{R}_-,
(\alpha_2, \beta_j) &= \frac{T}{2} \left( u_{-,0j-2} - B_1 u_{-,1j-1} - u_{-,2j-2} - B_2 \tilde{P}_{-,j-2} \right),
(\alpha_3, \beta_1) &= \frac{1}{2} \left( q_{-,1} - q_{-,1}^{(\infty)} \right) + B_1 \left( q_{-,0} - q_{-,0}^{(\infty)} \right),
(\alpha_3, \beta_j) &= \frac{1}{2} \left( u_{-,1j-2} + B_1 u_{-,0j-2} \right), \quad j = 2, 3.
\end{align*}  \quad (7.23)$$

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where \( q_{-,j} \) are the values of \( Q_{-,j} \) at \( \zeta \) and \( q_{-,j}^{(\infty)} \) are the values of \( Q_{-,j} \) at \( \infty \).

Let us now derive ODEs for these functions in the variable \( \zeta \). First recall the following formula that holds for arbitrary operator \( K \) that depends smoothly on a parameter \( \zeta \).

\[
\frac{\partial}{\partial \zeta} (I - K)^{-1} = (I - K)^{-1} \frac{\partial K}{\partial \zeta} (I - K)^{-1}.
\] (7.24)

Applying this to the operator \( K_{2,\text{airy}} \chi \), we obtain, as in [65], the followings.

\[
\frac{\partial}{\partial \zeta} K_{2,\text{airy}} \chi = -K_{2,\text{airy}}(\xi_1, \zeta)\delta(\zeta - \xi_2), \quad \frac{\partial}{\partial \zeta} (I - K_{2,\text{airy}} \chi)^{-1} = -R_0(\xi_1, \zeta)\rho(\xi, \xi_2) \quad (7.25)
\]

where \( \rho(\xi_1, \xi_2) \) is the kernel of \( I + R_0 \chi \), that is, \( \rho(\xi_1, \xi_2) = \delta(\xi_1 - \xi_2) + R_0(\xi_1, \xi_2) \chi \). Now from (6.106), we obtain the following for the derivative of \( \rho(\xi_1, \xi_2) \).

\[
\left( \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) K_{2,\text{airy}}(\xi_1, \xi_2) \chi = -Ai(\xi_1)Ai(\xi_2) + K_{2,\text{airy}}(\xi_1, \xi_2)\delta(\xi_2 - \zeta).
\]

This then implies the following for the derivative of \( \rho(\xi_1, \xi_2) \).

\[
\left( \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) \rho(\xi_1, \xi_2) = (I + R_0 \chi) \left( \left( \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) K_{2,\text{airy}}(\xi_1, \xi_2) \chi \right) (I + R_0 \chi)
\]

\[= -\Phi_0(\xi_1)\Phi_0(\xi_2) \chi + R_0(\xi_1, \zeta)\rho(\xi, \xi_2),
\]

where we have used the fact that the kernel of the operator \( (f \otimes g)A \) is the same as the kernel \( (f \otimes A^T g) \) and that \( \chi(I + R_0 \chi) \) is the same as its transpose, together with \( (I + R_0 \chi)K_{2,\text{airy}} \chi = R_0 \chi \).

From this and (7.25), we obtain the following for \( \rho(\xi_1, \xi_2) \) as in [67].

\[
\left( \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) \rho(\xi_1, \xi_2) = -\Phi_0(\xi_1)\Phi_0(\xi_2) \chi.
\] (7.26)

From (7.26) and \( \rho(\xi_1, \xi_2) = \delta(\xi_1 - \xi_2) + R_0(\xi_1, \xi_2) \chi \), we obtain the following for \( \xi_2 \in (\zeta, \infty) \).

\[
\left( \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) R_0(\xi_1, \xi_2) = -\Phi_0(\xi_1)\Phi_0(\xi_2), \quad \xi_2 \in (\zeta, \infty).
\] (7.27)

We can now derive a system of ODEs satisfied by the functions in (7.17) and (7.18).

**Lemma 7.2.** Let \( q_{\pm,j} \) and \( \phi_j \) be the values of \( Q_{\pm,j} \) and \( \Phi_j \) at \( \xi = \zeta \) respectively and let the function \( Q_j \) be \( Q_j = q_{+,j} - q_{-,j} \). Then functions \( u_{\pm,j,k} \), \( q_{\pm,j} \) and \( Q_j \) satisfy the following differential equations.

\[
\frac{\partial q_{\pm,j}}{\partial \zeta} = -\frac{\phi_0}{\psi(\zeta)} u_{+,j,0} + \frac{\phi_j}{\psi(\zeta)} + \frac{1}{2\psi^2(\zeta)} Q_j,
\]

\[
\frac{\partial Q_j}{\partial \zeta} = \frac{\partial}{\partial \zeta} \left( \frac{\partial}{\partial \zeta} \phi_0 \psi^{-1} \right) Q_j - \frac{\psi}{\phi_0} W \left( \frac{\phi_0}{\psi}, \frac{\phi_j}{\psi} \right) u_{-,j,0},
\]

\[
u_{-,j,k} = W \left( Q_j, \phi_k \psi^{-1} \right) / W \left( \phi_0 \psi^{-1}, \phi_j \psi^{-1} \right), \quad \frac{\partial u_{\pm,j,k}}{\partial \zeta} = -\psi^{\pm 1} q_{\pm,j,k} \phi_k,
\]

where \( W(f, g) \) is the Wronskian \( W(f, g) = fg' - gf' \).
Proof. First by using (7.25) and (7.19), one can show as in [65], that, the derivatives of the $u_{\pm,jk}$ are given by

$$\frac{\partial u_{\pm,jk}}{\partial \zeta} = -\left(R(\xi_1, \zeta), \psi^\pm I(H_j)\chi\right) \phi_k - \left(\psi^\pm I(H_j)\right)(\zeta)\phi_k,$$

(7.29)

$$= -\left(\rho(\xi_1, \zeta), \psi^\pm I(H_j)\chi\right) \phi_k = -\psi^\pm q_{\pm,j} \phi_k.$$ From this and (7.21), we see that

$$Q_j = \frac{\phi_0}{\psi(\zeta)} u_{-j1} - \frac{\phi_1}{\psi(\zeta)} u_{-j0},$$

$$\frac{\partial Q_j}{\partial \zeta} = \left(\frac{\phi_0}{\psi(\zeta)}\right)' u_{-j1} - \left(\frac{\phi_1}{\psi(\zeta)}\right)' u_{-j0},$$

(7.30)

where the prime denotes derivative of $\zeta$. By eliminating $u_{-j1}$ from the second equation, we obtain the differential equation for $Q_j$ in (7.28). From (7.30), we see that $u_{-jk}$ are given by

$$u_{-jk} = W\left(Q_j, \phi_k \psi^{-1}\right) / W\left(\phi_0 \psi^{-1}, \phi_1 \psi^{-1}\right).$$

Now by using the definition of $Q_{+j}$, we see that

$$\frac{\partial q_{+j}}{\partial \zeta} = \frac{1}{2\psi^2(\zeta)}q_{+j} + \psi^{-1}(\zeta) \int \frac{\partial p(\zeta, \xi_2)}{\partial \zeta} \psi(\xi_2) I(H_j)(\xi_2) d\xi_2.$$

By applying (7.26) to this, we obtain

$$\int \frac{\partial p(\zeta, \xi_2)}{\partial \zeta} \psi(\xi_2) I(H_j)(\xi_2) d\xi_2 = -\int \frac{\partial p(\zeta, \xi_2)}{\partial \xi_2} \psi(\xi_2) I(H_j)(\xi_2) d\xi_2 - \phi_0 u_{+,j0}$$

$$= \int \rho(\zeta, \xi_2) \frac{\partial}{\partial \xi_2} \left(\psi(\xi_2) I(H_j)(\xi_2)\right) d\xi_2 - \phi_0 u_{+,j0}$$

$$= -\frac{1}{2\psi(\zeta)}q_{-j} + \phi_j - \phi_0 u_{+,j0}.$$ This, together with (7.21), gives us the differential equation for $q_{+j}$. \qed

This gives us the first set of differential equations. Let us now derive the second set of differential equations for the functions $\mathcal{R}_{\pm}$ and $\mathcal{P}_j$.

**Lemma 7.3.** Let $\mathcal{R}_0$ be $\mathcal{R}_+ - \mathcal{R}_-$. Then the functions $\mathcal{R}_0$, $\mathcal{R}_+$, $\mathcal{P}_{-j}$ and $\mathcal{P}_{0,j}$ satisfy the following set of differential equations.

$$\frac{\partial \mathcal{R}_0}{\partial \zeta} = \frac{\partial \log (\phi_0 \psi^{-1})}{\partial \zeta} \mathcal{R}_0 - \frac{\psi(\zeta)}{\phi_0} W\left(\frac{\phi_0}{\psi}, \frac{\phi_1}{\psi}\right) \mathcal{P}_{0,j},$$

$$\frac{\partial \mathcal{R}_+}{\partial \zeta} = \frac{1}{2\psi^2} \mathcal{R}_0 - \frac{\phi_0}{\psi(\zeta)} \mathcal{P}_{+,j},$$

$$\frac{\partial \mathcal{P}_{-j}}{\partial \zeta} = \frac{\partial \mathcal{P}_{0,j}}{\partial \zeta} = \phi_0 \psi^{\pm 1}(\zeta) \left(1 - \mathcal{R}_\pm\right),$$

(7.31)

$$\mathcal{P}_{-j} = W\left(\mathcal{R}_0, \phi_j \psi^{-1}\right) / W\left(\phi_0 \psi^{-1}, \phi_1 \psi^{-1}\right).$$

The functions $\mathcal{R}_0$, $\mathcal{R}_+$, $\mathcal{P}_{-j}$ and $\mathcal{P}_{0,j}$ also satisfy the same system of linear ODE.
Proof. The proof is similar to (7.28). First, by (7.25), we can obtain the derivatives of \( \Phi_k(\xi) \) with respect to \( \zeta \).

\[
\frac{\partial}{\partial \zeta} \Phi_k(\xi) = -R_0(\xi, \zeta) \phi_k,
\]

From this, we obtain the derivatives of \( P_{\pm,0} \).

\[
\frac{\partial P_{\pm,j}}{\partial \zeta} = \phi_j \psi^{\pm 1} + \int_{-\infty}^{\xi} \frac{\partial \Phi_j(\xi)}{\partial \zeta} \psi^{\pm 1}(\xi) d\xi = \phi_j \psi^{\pm 1} - \phi_j \int_{-\infty}^{\xi} R_0(\xi, \zeta) \psi^{\pm 1}(\xi) d\xi.
\]

The differential equation for \( P_{\pm,0} \) now follows from the fact that \( R_0(\xi_1, \xi_2) \) is symmetric with respect to the interchange of \( \xi_1 \) and \( \xi_2 \).

From this and (7.21), we again have the following equations for \( R_0 \).

\[
R_0 = \frac{\phi_0}{\psi(\zeta)} P_{-1} - \frac{\phi_1}{\psi(\zeta)} P_{-0},
\]

\[
\frac{\partial R_0}{\partial \zeta} = \left( \frac{\phi_0}{\psi(\zeta)} \right)' P_{-1} - \left( \frac{\phi_1}{\psi(\zeta)} \right)' P_{-0}.
\]

This then gives us the Wronskian equations for \( P_{-j} \). Let us now derive the differential equation for \( R_+ \). First by using the fact that \( R_0 \) is symmetric with respect to the interchange of its arguments, we can write \( R_+ \) as \( R_+ = \psi^{-1} \int_{-\infty}^{\xi} R_0(\xi, \zeta) \psi^{-1}(\xi) d\xi \). Then we have

\[
\frac{\partial R_+}{\partial \zeta} = \frac{1}{2 \psi^2(\zeta)} R_+ + \psi^{-1}(\zeta) R_0(\zeta, \zeta) \psi(\zeta) + \psi^{-1}(\zeta) \int_{-\infty}^{\xi} \frac{\partial R_0(\xi, \zeta)}{\partial \zeta} \psi(\xi) d\xi.
\]

By using (7.21), the above becomes

\[
\frac{\partial R_+}{\partial \zeta} = \frac{1}{2 \psi^2(\zeta)} R_+ - \psi^{-1}(\zeta) \int_{-\infty}^{\xi} \frac{\partial R_0(\xi, \zeta)}{\partial \zeta} \psi(\xi) d\xi - \frac{\phi_0}{\psi(\zeta)} P_{+0}
\]

\[
= \frac{1}{2 \psi^2(\zeta)} R_+ - \frac{\phi_0}{\psi(\zeta)} P_{+0}.
\]

The same argument can be applied to the functions The functions \( \tilde{R}_0, \tilde{R}_+, \tilde{P}_{-j} \) and \( \tilde{P}_{0,j} \) to obtain the same system of linear ODEs.

The ODEs satisfied by the various functions can be further simplified. First, the function \( \phi_0 = (I - K_{airy}) \chi(\zeta) \) is known to be the Hastings-McLeod solution of the Painlevé II equation (1.3) (See [66]).

The Wronskian of \( \phi_0/\psi \) and \( \phi_1/\psi \) can be written as

\[
W(\phi_0/\psi, \phi_1/\psi) = \psi^{-2}(\phi_0 \phi_1' - \phi_1 \phi_0') = \psi^{-2} R_0(\zeta, \zeta).
\]

The function \( R_0(\zeta, \zeta) \), again is known to be the logarithmic derivative of the Tracy-Widom distribution for the GUE [66]. Therefore we have

\[
R_0(\zeta, \zeta) = \int_{\zeta}^{\infty} \phi_0^2(\xi) d\xi = \sigma(\zeta).
\]
To obtain $\phi_1$, let us take the derivative of $\phi_0$ and use (7.26), then we have
\[
\frac{\partial \phi_0}{\partial \zeta} = \int_{\mathbb{R}} \frac{\partial \rho(\zeta, \xi)}{\partial \zeta} Ai(\xi) d\xi = -\int_{\mathbb{R}} \frac{\partial \rho(\zeta, \xi)}{\partial \xi} Ai(\xi) d\xi - \phi_0(\Phi_0, Ai\chi),
\]
\[= \phi_1 - \phi_0(\Phi_0, Ai\chi).\]
As in the derivations of (7.29), we see that
\[\frac{\partial}{\partial \zeta}(\Phi_0, Ai\chi) = \phi_2, \quad \frac{\partial}{\partial \zeta}(\Phi_1, Ai\chi) = \phi_0\phi_1.\]
Since $(\Phi_0, Ai\chi) = (\Phi_1, Ai\chi) = 0$ at $\zeta = \infty$, we obtain
\[\phi_1 = \phi_0 + \phi_0 \int_{\zeta}^{\infty} \phi_0^2(\xi) d\xi = \phi_0^\prime + \sigma \phi_0 \quad (7.34)\]
To obtain $\phi_2$, we use $Ai''(\xi) = \xi Ai(\xi)$ to obtain
\[\Phi_2(\xi) = \xi (I + R_0\chi) Ai + \Phi_1(\xi)(\Phi_0, Ai\chi) - \Phi_0(\xi)(\Phi_1, Ai\chi).\]
Therefore we have
\[\phi_2 = \left( \zeta + \int_{\zeta}^{\infty} \phi_0(\xi)\phi_1(\xi) d\xi \right) \phi_0 - \sigma \phi_1. \quad (7.35)\]
Summarizing, we have the following.

**Proposition 7.1.** Let $U$ be the matrix
\[
U = \begin{pmatrix}
0 & 0 & 0 & -\psi^0 \\
0 & 0 & \psi^{-1}\phi_0 & -\psi^{-1}\phi_0 \\
0 & -\frac{\partial}{\partial \zeta} \log (\phi_0/\psi) & 0 & 0 \\
-\phi_0\psi^{-1} & 0 & 0 & 0
\end{pmatrix}
\]
and let $\vec{h}_j$ be the vector $\vec{h}_j = \left(0, 0, 0, \int_{\zeta}^{\infty} H_j d\xi\right)^T$ for $j = 0, 1, 2$ and $\vec{h}_j = 0$ for $j = 3$ and $j = 4$. Then the functions in (7.22) and (7.23) are given by
\[
q_{-j} - q_{-j}^{(\infty)} = v_{j4} - v_{j3} - \int_{-\infty}^{\infty} H_j d\xi, \quad u_{-j0} = v_{j2}, \quad u_{-jk} = \frac{\psi^2 W(v_{j3}, \phi_k \psi^{-1})}{\sigma},
\]
\[
R_{-} = v_{34} - v_{33} + 1, \quad P_{-0} = v_{32}, \quad P_{-1} = \frac{\psi^2 W(v_{33}, \phi_1 \psi^{-1})}{\sigma},
\]
\[
\bar{R}_{-} = v_{44} - v_{43} + 1, \quad \bar{P}_{-0} = v_{42}, \quad \bar{P}_{-1} = \frac{\psi^2 W(v_{43}, \phi_1 \psi^{-1})}{\sigma},
\]
where $\vec{v}_j$ is the vector that satisfies the linear system of ODEs
\[
\frac{\partial \vec{v}_j}{\partial \zeta} = U(\zeta)\vec{v}_j + \vec{h}_j, \quad \vec{v}_j \sim \left(0, 0, 0, \int_{-\infty}^{\infty} H_j d\xi\right)^T, \quad \zeta \rightarrow +\infty, \quad j = 0, 1, 2,
\]
\[
\vec{v}_3 \sim (0, 0, -1, 0)^T, \quad \zeta \rightarrow -\infty, \quad \vec{v}_4 \sim (0, 0, -1, 0)^T, \quad \zeta \rightarrow +\infty
\]
These functions can also be characterized using the connection between Fredholm determinants and Riemann-Hilbert problems. We will outline this connection in Appendix B.

Appendix: A proof of the j.p.d.f. formula using Zonal polynomials

We present here a simpler algebraic proof of Theorem 2.1 using Zonal polynomials. Zonal polynomials are introduced by James [46] and Hua [43] independently. They are polynomials with matrix argument that depend on an index \( k \) which is a partition of an integer \( k \). The real Zonal polynomials \( Z_p(X) \) take arguments in symmetric matrices and are homogeneous polynomials in the eigenvalues of its matrix argument \( X \). We shall not go into the details of their definitions, but only state the important properties of these polynomials that is relevant to our proof. Readers who are interested can refer to the excellent references of [55], [50] and [64].

Let \( p \) be a partition of an integer \( k \) and let \( l(p) \) be the length of the partition. We will use \( p \vdash k \) to indicate that \( p \) is a partition of \( k \). Let \( X \) and \( Y \) be \( N \times N \) symmetric matrices and \( x_i, y_i \) their eigenvalues. Given a partition \( p = (p_1, \ldots, p_{l(p)}) \) of the integer \( k \), we will order the parts \( p_i \) such that if \( i < j \), then \( p_i > p_j \). If we have 2 partitions \( p \) and \( p' = (p'_1, \ldots, p'_{l(p')}) \), then we say that \( p < p' \) if there exists an index \( j \) such that \( p_i = p'_i \) for \( i < j \) and \( p_j < p'_j \). Let the monomial \( x^p \) be \( x^{p_1} \cdots x^{p_{l(p)}} \), we say that \( x^{p'} \) is of a higher weight than \( x^p \) if \( p' > p \). Then the Zonal polynomial \( Z_p(X) \) is a homogenous polynomial of degree \( k \) in the eigenvalues \( x_j \) with the highest weight term being \( x^p \). It has the following properties.

\[
(\text{tr}(X))^k = \sum_{p \vdash k} Z_p(X),
\]

\[
\int_{O(N)} e^{-M t(XgYg^T)} \, dg = \sum_{k=0}^{\infty} \frac{M^k}{k!} \sum_{p \vdash k} \frac{Z_p(X)Z_p(Y)}{Z_p(I_N)}
\]

(A.1)

These properties can be found in the references [55], [50] and [64]. Another important property is the following generating function formula for the Zonal polynomials, which can be found in [50] and [64].

\[
\prod_{i,j=1}^{N} (1 - 2\theta x_i y_j)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p \vdash k} \frac{Z_p(X)Z_p(Y)}{d_p}
\]

(A.2)

where \( d_p \) are constants. In particular, if \( (k) \) is the partition of \( k \) with length 1, that is, \( (k) = (k, 0, \ldots, 0) \), then the constant \( d_{(k)} \) is given by

\[
d_{(k)} = \frac{1}{(2k - 1)!!}.
\]

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For the rank 1 spiked model, let us consider the case where all but one $y_j$ is zero and denote
the non-zero eigenvalue by $y$. Then from the fact that the highest weight term in $Z_p(Y)$
is $y_1^{p_1} \ldots y_{p(r)}^{p(r)}$, we see that the only non-zero $Z_p(Y)$ is $Z_{(k)}(Y)$, which by the first equation
in (A.1), is simply $y^k$. Therefore the formulae in (A.1) and (A.2) are greatly simplified in this case.

\[
\int_{O(N)} e^{-\text{tr}(X g Y g^T)} dg = \sum_{k=0}^{\infty} \frac{M^k Z_{(k)}(X) y^k}{k! Z_{(k)}(I_N)^k},
\]
\[
\prod_{i=1}^{N} (1 - 2 \theta x_i y)^{-\frac{1}{N}} = \sum_{k=0}^{\infty} \theta^k \frac{(2k - 1)!! Z_{(k)}(X) y^k}{k!}
\]
By using the generating function formula, we see that $Z_{(k)}(I_N)$ is given by
\[
Z_{(k)}(I_N) = \frac{(N/2 + k - 1)!! 2^k}{(N/2 - 1)!(2k - 1)!!}.
\]
By taking $\theta = \frac{1}{2t}$ in the second equation of (A.3), we see that
\[
\prod_{i=1}^{N} (t - x_i y)^{-\frac{1}{N}} = t^{-\frac{N}{2}} \sum_{k=0}^{\infty} (2t)^{-k} \frac{(2k - 1)!! Z_{(k)}(X) y^k}{k!}
\]
We can now compute the integral
\[
S(t) = \int_\Gamma e^{Mt} \prod_{i=1}^{N} (t - x_i y)^{-\frac{1}{N}} dt
\]
by taking residue at $\infty$, which is the $t^{-1}$ coefficient in the following expansion
\[
e^{Mt} \prod_{i=1}^{N} (t - x_i y)^{-\frac{1}{N}} = \sum_{k,j=0}^{\infty} \frac{M^j t^{-\frac{N}{2} + j} (2k - 1)!! Z_{(k)}(X) y^k}{2^k j! k!}
\]
This coefficient is given by
\[
S(t) = M^{t^{-1}} \sum_{k=0}^{\infty} \frac{Z_{(k)}(X) (2k - 1)!! y^k M^k}{2^k (N/2 + k - 1)! k!}
\]
\[
= \frac{M^{t^{-1}}}{(N/2 - 1)!} \sum_{k=0}^{\infty} \frac{Z_{(k)}(X) y^k M^k}{Z_{(k)}(I_N) k!} = \frac{M^{t^{-1}}}{(N/2 - 1)!} \int_{O(N)} e^{-\text{tr}(X g Y g^T)} dg.
\]
By taking $y = \frac{r}{2(1+\tau)}$, this proves Theorem 2.1. There also exist complex and quaternionic
Zonal polynomials $C_p(X)$ and $Q_p(X)$ which satisfy the followings instead.
\[
\int_{U(N)} e^{-\text{tr}(X g Y^*)} g^* dg = \sum_{k=0}^{\infty} \frac{M^k}{k!} \sum_{p^k} \frac{C_p(X) C_p(Y)}{C_p(I_N)},
\]
\[
\int_{Sp(N)} e^{-\text{Re}(\text{tr}(X g Y g^{-1}^*))} g^{-1} dg = \sum_{k=0}^{\infty} \frac{M^k}{k!} \sum_{p^k} \frac{Q_p(X) Q_p(Y)}{Q_p(I_N)}.
\]
Their generating functions are given by
\[
\prod_{i,j=1}^{N} (1 - 2\theta x_i y_j)^{-1} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p\vdash k} \frac{C_p(X)C_p(Y)}{c_p},
\]
\[
\prod_{i,j=1}^{N} (1 - 2\theta x_i y_j)^{-2} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p\vdash k} \frac{Q_p(X)Q_p(Y)}{q_p},
\]
where \(c(k)\) and \(q(k)\) are
\[
c(k) = \frac{1}{2k!}, \quad q(k) = \frac{1}{(k+1)!2^k}
\]

Then by following the same argument as in the real case, we can write down the following integral formulae for rank one perturbations of the complex and quaternionic cases.

\[
\int_{U(N)} e^{-Mt(\text{tr}(XgYg^*))} g^\dagger dg = (N-1)! \int_{\Gamma} e^{Mt} \prod_{i=1}^{N} (t-x_i y_i)^{-1} dt,
\]
\[
\int_{Sp(N)} e^{-M\text{Re}(\text{tr}(XgYg^{-1}))} g^{-1} dg = (2N-1)! \int_{\Gamma} e^{Mt} \prod_{i=1}^{N} (t-x_i y_i)^{-2} dt.
\]

**Appendix B: Connection to Riemann-Hilbert problem**

The functions \(\Phi_j\) involved in (7.18) can also be found by solving a Riemann-Hilbert problem (See e.g. [24], [29], [30], [34], [35]). Let \(K\) be an integrable operator acting on \(\mathbb{R}\) of the form (7.4) and \(f_i\) be the column vector with entries \(f_{ij}\). Then operator \(K\chi\) is also integrable. Its resolvent can be found by solving the following Riemann-Hilbert problem.

1. \(Y(\xi)\) is analytic in \(\mathbb{C} \setminus [\zeta, \infty)\),
2. \(Y_+(\xi) = Y_-(\xi) \left( I - 2\pi i f_1(\xi) f_2^T(\xi) \right), \quad \xi \in (\zeta, \infty)\),
3. \(Y(\xi) = I + O(\xi^{-1}), \quad \xi \to \infty\),
4. \(Y(\xi) = O(\log |\xi - \zeta|), \quad \xi \to \zeta\).

Then the vectors \(F_i\) with entries \(F_{ij}\) in (7.6) are given by \(Y_+(\xi)f_{ij}\).

This Riemann-Hilbert problem can be connected to the Riemann-Hilbert problem of the Painlevé II equation. We will now outline this connection. For more details, please see [29]. Let \(A(\xi)\) be the matrix in Lemma 6.19. Multiplying the solution \(Y\) of (A.1) by \(A\) on the right and then deform the regions suitably will transform the Riemann-Hilbert
problem (A.1) into the following Riemann-Hilbert problem.

1. \( X(\xi) \) is analytic in \( \mathbb{C} \setminus \Sigma \),
2. \( X_+(\xi) = X_-(\xi)J_X, \quad \xi \in \Sigma \),
3. \( X(\xi) = \frac{1}{\sqrt{2}} \xi^{-\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} e^{-\frac{\pi}{4}\sigma_3} \left( I + O(\xi^{-\frac{3}{2}}) \right) e^{-\frac{3\pi}{4}\sigma_3}, \quad \xi \to \infty \), (A.2)
4. \( X(\xi) = O(\log |\xi - \zeta|), \quad \xi \to \zeta \).

where the contour \( \Sigma \) is the union of

\( \Sigma_1 = (-\infty, \zeta], \quad \Sigma_2 = \zeta + e^{\frac{2\pi}{3}} \mathbb{R}_+, \quad \Sigma_3 = \zeta + e^{\frac{4\pi}{3}} \mathbb{R}_+ \).

The contours in \( \Sigma \) are all pointing towards \( \zeta \). The jump matrix \( J_X \) is given by

\[
J_X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \xi \in \Sigma_2 \cup \Sigma_3, \quad J_X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi \in \Sigma_1.
\]

Since \( X = YA \), we have the following.

\[
X_{11} = \sqrt{2\pi} e^{-\frac{\pi}{4}} (I - K_{2,\text{airy}}\lambda)^{-1} Ai, \quad X_{21} = \sqrt{2\pi} e^{\frac{\pi}{4}} (I - K_{2,\text{airy}}\lambda)^{-1} Ai'.
\] (A.3)

Note the difference between our Riemann-Hilbert problem (A.2) and the one given in (2.11-15) of [29]. In (A.2), as \( \xi \to \infty \), the next to the leading order term is of order \( \xi^{-1} \) smaller than the leading order term. This is due to the asymptotic behavior of the Airy functions in the matrix \( A \). As a result, our Riemann-Hilbert problem (A.2) will be uniquely solvable.

This is important for us to keep track of the functions \( \Phi_0 \) and \( \Phi_1 \) that appears in the resolvent. In [29], it was shown that the Riemann-Hilbert problem (A.2) can be solved by using the monodromy problem associating with the Painlevé II equation. Let \( \Sigma_\Psi \) be the union of the contours

\[
\Sigma_{\Psi,1} = e^{\frac{2\pi}{3}} \mathbb{R}_+, \quad \Sigma_{\Psi,2} = e^{\frac{4\pi}{3}} \mathbb{R}_+, \quad \Sigma_{\Psi,3} = e^{\frac{7\pi}{6}} \mathbb{R}_+, \quad \Sigma_{\Psi,4} = e^{-\frac{\pi}{3}} \mathbb{R}_+,
\]

and let \( S_\pm, S_1 \) and \( S_2 \) be the regions

\[
S_+ = \left\{ \xi \mid -\frac{\pi}{6} < \arg \xi < \frac{\pi}{6} \right\}, \quad S_- = -S_+, \quad S_1 = \left\{ \xi \mid \frac{\pi}{6} < \arg \xi < \frac{5\pi}{6} \right\}, \quad S_2 = -S_1.
\]

Then \( \Psi(\xi, v) \) is the solution to the following monodromy problem of the Painlevé II equation.

1. \( \Psi(\xi, v) \) is analytic in \( \mathbb{C} \setminus \Sigma_\Psi \),
2. \( \Psi_+ (\xi, v) = \Psi_- (\xi, v)J_\Psi, \quad \xi \in \Sigma_\Psi \),
3. \( \Psi(\xi, v) = \left( I + \frac{\Psi_\infty}{\xi} + O(\xi^{-2}) \right) e^{-i(\frac{3\pi}{4}\xi^3 + v\xi)}\sigma_3, \quad \xi \to \infty \),
4. \( \Psi(\xi, v) = \begin{cases} O \left( |\xi|^{\frac{2\pi}{3}} \right) \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}, & \xi \to 0 \text{ in } S_\pm, \\
O \left( |\xi|^{\frac{2\pi}{3}} \right), & \xi \to 0 \text{ in } S_1; \\
O \left( |\xi|^{\frac{2\pi}{3}} \right), & \xi \to 0 \text{ in } S_2. \end{cases} \) (A.4)

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where \( J_\Psi \) is given by

\[
J_\Psi = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad \xi \in \Sigma_{\Psi,1} \cup \Sigma_{\Psi,2}, \quad J_\Psi = \begin{pmatrix} 0 & -i \\ 1 & 1 \end{pmatrix}, \quad \xi \in \Sigma_{\Psi,3} \cup \Sigma_{\Psi,4},
\]

where the contours are all pointing towards infinity. The matrix \( \Psi \) also satisfies the Lax equation for the Painlevé II equation.

\[
\frac{\partial}{\partial v} \Psi = \begin{pmatrix} -i \xi & q \\ q & i\xi \end{pmatrix} \Psi,
\]

\[
\xi \frac{\partial}{\partial \xi} \Psi = \begin{pmatrix} -4i\xi^3 - (2i q + v)\xi & 4q\xi^2 + 2iq_v\xi - \frac{1}{2} \\ 4q\xi^2 - 2iq_v\xi - \frac{1}{2} & 4i\xi^3 + (2i q + v)\xi \end{pmatrix} \Psi,
\]

where \( q(v) \) is related to the Hastings-McLeod solution of Painlevé II by (see (1.47) of [29])

\[
2^{-\frac{4}{7}}(v + 2q^2(v) + 2q_v(v)) = \phi_0^2 \left( 2^{\frac{4}{7}}v \right).
\]

By using the Lax equations (A.5) and the behavior of \( \Psi \) at \( \xi \to \infty \), one can check that the next to the leading term \( \Psi_\infty \) in the expansion in (A.4) is given by

\[
\Psi_\infty = \begin{pmatrix} -i \xi & q \\ q & i \xi \end{pmatrix} \]

(A.7)

The authors in [29] then showed that the solution to the Riemann-Hilbert problem (A.2) is given by

\[
X(\xi, \zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} (\xi - \zeta)^{-\frac{1}{2}} \sigma_3 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Psi \left( 4^{-\frac{1}{7}}i (\xi - \zeta)^{\frac{1}{7}}, -2^{\frac{4}{7}} \zeta \right) e^{-\frac{1}{7}i \sigma_3}
\]

for some \( h \) independent on \( \xi \). The function \( h(\zeta) \) is determined by making sure that the next to the leading order term in \( X \) is of order indicated by (A.2). By using (A.7), one can check that the appropriate choice of \( h \) in this case is given by \( h = -q \). The properties of the Riemann-Hilbert problem in (A.4) is studied very thoroughly in [29], and these properties will hopefully be useful in the characterization of the functions in (7.22) and (7.23).

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