GLOBAL WELL-POSEDNESS FOR THE TWO-DIMENSIONAL MAXWELL-NAVIER-STOKES EQUATIONS

CHANGXING MIAO AND XIAOXIN ZHENG

Abstract. In this paper, we investigate Cauchy problem of the two-dimensional full Maxwell-Navier-Stokes system, and prove the global-in-time existence and uniqueness of solution in the borderline space which is very close to $L^2$-energy space by developing the new estimate of $\sup_{j \in \mathbb{Z}^2} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \frac{1}{\sqrt{\phi_{i,k}} u(\tau)} \|^2_{L^2_2(\mathbb{R}^2)} \, d\tau < \infty$. This solves the open problem in the framework of borderline space purposed by Masmoudi in [16].

1. Introduction

We consider a coupled system of equations consisting of the Navier-Stokes equations of fluid dynamics and Maxwells equations of electromagnetism. The coupling comes from the Lorentz force in the fluid equation and the electric current in the Maxwell equations which takes the following form

$$
\begin{align*}
\begin{cases}
    u_t + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi = j \times B & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
    E_t - \text{curl} B = -j, \\
    B_t + \text{curl} E = 0, \\
    \text{div} u = \text{div} B = 0, \\
    j = \sigma(E + u \times B).
\end{cases}
\end{align*}
$$

System (1.1) should be supplemented with an initial condition

$$
u(0, x) = u_0(x), \quad B(0, x) = B_0(x), \quad E(0, x) = E_0(x),$$

where $u_0(x)$ and $B_0(x)$ satisfy $\text{div} u_0 = \text{div} B_0 = 0$. Here, $u = (u^1, u^2, u^3)(t, x_1, x_2)$ stands for velocity of the fluid. $E = (E^1, E^2, E^3)(t, x_1, x_2)$ and $B = (B^1, B^2, B^3)(t, x_1, x_2)$ electric field and magnetic field, respectively. The scalar function $\pi$ is the pressure which can be recovered at least formally by $u$ and $j \times B$ via Calderón-Zygmund operators, that is,

$$
\pi = -\mathbb{P}(u \cdot \nabla) u - (j \times B),
$$

where $\mathbb{P}$ is the Leray projector. $j$ is the electric current which is given by Ohm’s law and $j \times B$ is the Lorentz force. In addition, $\nu$ is the viscosity and $\sigma$ is the electric conductivity. For simplicity, we will take $\nu = \sigma = 1$ in the following parts.

This system has strong physical background, the reader can refer to [3, 7] for more physical introduction concerning on magnetohydrodynamics. By the divergence-free condition and the following vanishing condition that

$$
\int_{\mathbb{R}^2} j \cdot (u \times B) \, dx + \int_{\mathbb{R}^2} (j \times B) \cdot u \, dx = 0,
$$

Date: May 1, 2018.

2010 Mathematics Subject Classification. 35Q30; 35B40; 76D05.

Key words and phrases. Maxwell-Navier–Stokes equations; weak solutions; uniqueness; Localization.
it is easy to show that for a smooth solution,

$$
\| (u, E, B) (t) \|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \left( \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 + \| j(\tau) \|_{L^2(\mathbb{R}^2)}^2 \right) \, d\tau = \| (u_0, E_0, B_0) \|_{L^2(\mathbb{R}^2)}^2.
$$

(1.3)

This natural energy equality is very similar to that for the bi-dimensional Navier-Stokes equations. As we know, with the help of the energy estimate, Leray [15] showed that the bi-dimensional Navier-Stokes system has a unique global-in-time weak solution. Inspired by this Leray theory, a natural question is that does system (1.1) exist a unique global-in-time weak solution enjoying the energy estimate (1.3). However, due to the hyperbolic nature of the Maxwell equation, it is difficult to get compactness of $B$ and hence passing to the limit in the product $j \times B$ seems to be a challenge problem. This leads to that it is difficult to get global-in-time existence of the $L^2$ energy weak solution. The essential reason is the lack of the control of $\int_0^\infty \| u(t) \|_{L^2}^2 \, dt$. In fact, the $L^2$ energy estimate just provides us the control of $\int_0^\infty \| u(t) \|_{\text{BMO}}^2 \, dt$ which is bounded by $\int_0^\infty \| \nabla u(t) \|_{L^2}^2 \, dt$. Hence, proving global existence of weak solutions to system (1.1) in the energy space $(L^2)^3$ or the borderline space $L^2 \times L^2_{\log} \times L^2_{\log}$ seems to be an open problem.

From above, we easily find that $\int_0^\infty \| u(t) \|_{L^\infty}^2 \, dt$ is a very important quantity to show the global regularity of weak solutions. Unfortunately, we don’t have the control of this quantity. But, it is very close to $\int_0^\infty \| u(t) \|_{\text{BMO}}^2 \, dt$. With this observation, Masmoudi [16] proved the existence and uniqueness of global strong solutions in the $H^s(\mathbb{R}^2)$ framework to problem (1.1) with $s > 0$. His proof highly relies on a time-space logarithmic inequality that enabled him to upper estimate the $L^\infty$-norm of the velocity field by the energy norm and higher Sobolev norms. Another line of research was pursued by Ibrahim and Keraani [13], they proved a local-in-time strong solution in the borderline space $\dot{B}^{0}_{2,1} \times (L^2_{\log})^2$ by using parabolic regularization arguments giving control of the $L^\infty$ norm of the velocity field of the solution. Based on this, a global-in-time result for small initial data and a local-in-time result for the large initial data in the borderline space $L^2 \times (L^2_{\log})^2$ were obtained in [12] by establishing an $L^2 \times L^\infty$ estimate on the velocity field. Very recently, Ibrahim, Masmoudi and Lemarié-Rieusset in [10] proved the existence of time-periodic small solutions and their asymptotic stability for the 3D Navier-Stokes-Maxwell problem in the presence of external time-periodic forces.

In our paper, our target is to show the global-in-time existence and uniqueness of solution for the large initial data in the borderline space $L^2 \times (L^2_{\log})^2$. Therefore, the main task is to bridge the gap between $\int_0^\infty \| u(t) \|_{\text{BMO}}^2 \, dt$ and $\int_0^\infty \| u(t) \|_{L^\infty}^2 \, dt$. But the previous methods for problem (1.1) including the argument used in [10] do not work. This requires us to develop a new method to overcome this difficulty. Now, we take the linear heat equation as an example to illustrate our main idea. Our strategy is to use micro-analysis in physical space to bootstrap the regularity of solution. Let $f$ be the smooth solution of the linear heat equation $\partial_t f - \Delta f = 0$. Multiplying this linear heat equation by $\varphi_{j,k} f$, we see that

$$
\frac{1}{2} \partial_t (\varphi_{j,k} f^2) - \varphi_{j,k} f \Delta f = 0,
$$
where \( \varphi_{j,k} \) is the solution of the eigenvalue problem, see Lemma 2.6 for details. Integrating the above equality in space variable over \( \mathbb{R}^2 \) and using Corollary 2.7, one has

\[
\| \sqrt{\varphi_{j,k}} f \|_{L^\infty_t L^2_x(\mathbb{R}^2)}^2 + \lambda_1 2^{2j} \| \sqrt{\varphi_{j,k}} f \|_{L^2_t L^2_x(\mathbb{R}^2)}^2 + 2 \| \sqrt{\varphi_{j,k}} \nabla f \|_{L^2_t L^2_x(\mathbb{R}^2)}^2
\]

\[
= \| \sqrt{\varphi_{j,k}} f_0 \|_{L^2_x(\mathbb{R}^2)}^2 - \int_0^t \int_{\partial B_{2^{-j}(k)}} f^2 \nabla \varphi_{j,k} \cdot n \, dS \, d\tau.
\]

By the trace theorem and the Hölder inequality, we finally get that

\[
\| f \|_{L^\infty_t L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla f(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau + \lambda_1 \sup_{j \in \mathbb{Z}} 2^{2j} \| \sqrt{\varphi_{j,k}} f \|_{L^2_t L^2(\mathbb{R}^2)}^2
\]

\[
\leq C \| f_0 \|_{L^2_x(\mathbb{R}^2)}^2 + C \int_0^t \| f(\tau) \|_{H^1(\mathbb{R}^2)}^2 \, d\tau.
\]

This together with the following natural \( L^2 \)-energy estimate

\[
\| f(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla f(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \leq \| f_0 \|_{L^2(\mathbb{R}^2)}^2
\]

allows us to infer that

\[
\sup_{j \in \mathbb{Z}} 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{j,k}} f(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau < \infty,
\]

which plays the key role in our proof. In virtue of the Morrey-Campanato type characterization of \( L^\infty(\mathbb{R}^2) \), we know that this quantity is very close to \( L^2_t L^\infty \). Thus, with this global-in-time bound, we further establish the global-in-time bound of solution in in the borderline space \( L^2 \times (L^2_{\text{log}})^2 \) in terms of techniques in harmonic analysis. As a result, we eventually get the control of \( \int_0^t \| u(t') \|_{L^\infty}^2 \, dt' \). This enables us to remove the small assumption for initial data in [12].

Now we state our main result as follows:

**Theorem 1.1.** Let \( u_0 \in L^2(\mathbb{R}^2) \) and \( (E_0, B_0) \in (L^2_{\text{log}}(\mathbb{R}^2))^2 \). Then system (1.1) admits a unique global-in-time solution \( (u(t), E(t), B(t)) \in C_b(\mathbb{R}^+; L^2(\mathbb{R}^2)) \times (C(\mathbb{R}^+; L^2_{\text{log}}(\mathbb{R}^2)))^2 \) such that (1.3) and

\[
\| u(t) \|_{L^2(\mathbb{R}^2)}^2 + \| (E, B) \|_{L^\infty_t L^2_{\text{log}}(\mathbb{R}^2)}^2 + \int_0^t \| j(\tau) \|_{L^2_{\text{log}}(\mathbb{R}^2)}^2 \, d\tau + \int_0^t \| u(\tau) \|_{L^\infty(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C \left( t, \| u_0 \|_{L^2(\mathbb{R}^2)}, \| (E_0, B_0) \|_{L^2_{\text{log}}(\mathbb{R}^2)} \right).
\]

**Remark 1.2.** Compared with result in [13, 12], we extend the local-in-time solution established in [13] to the global-in-time solution in theorem 1.1, while we removes the small assumption for initial data in [12].

**Remark 1.3.** Let us point out that in our paper, we develop the following new estimate

\[
\sup_{j \in \mathbb{Z}} 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{j,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau < \infty.
\]

In terms of the Morrey-Campanato type characterization of \( L^\infty(\mathbb{R}^2) \), we easily find that it is very close to \( L^2_t L^\infty \)-estimate for \( u \). In the other words, this type space can be viewed as the Chemin-Lerner space in the framework of localization.
Acknowledgments: This work was supported in part by the National Natural Science Foundation of China. C. Miao was also supported by Beijing Center for Mathematics and Information Interdisciplinary Sciences.

2. Preliminary

2.1. Littlewood-Paley Theory and the functional spaces. In this subsection, we first review the so-called Littlewood-Paley decomposition described, e.g., in [2, 4, 5, 17]. Next, we introduce some useful functional spaces such as Morrey-Campanato space and its properties. Let $(\chi, \psi)$ be a couple of smooth functions with values in $[0, 1]$ such that $\chi$ is supported in the ball $\{\xi \in \mathbb{R}^d | |\xi| \leq \frac{4}{3}\}$, $\varphi$ is supported in the ring $\{\xi \in \mathbb{R}^d | \frac{4}{3} \leq |\xi| \leq \frac{5}{3}\}$ and

$$\chi(\xi) + \sum_{j \in \mathbb{N}} \psi(2^{-j}\xi) = 1 \quad \text{for each } \xi \in \mathbb{R}^d.$$

For any $u \in S'(\mathbb{R}^d)$, one can define the dyadic blocks as

$$\Delta_{-1} u = \chi(D) u \quad \text{and} \quad \Delta_j u := \Psi(2^{-j}D) u \quad \text{for each } j \in \mathbb{N}.$$

We also define the following low-frequency cut-off:

$$S_j u := \chi(2^{-j}D) u.$$

According to the support in frequency space, it is easy to verify that

$$u = \sum_{j \geq -1} \Delta_j u, \quad \text{in } S'(\mathbb{R}^d),$$

and this is called the inhomogeneous Littlewood-Paley decomposition. It has nice properties of quasi-orthogonality:

$$\Delta_j \Delta_{j'} u \equiv 0 \quad \text{if } |j - j'| \geq 2.$$

$$\Delta_j (S_{j'-1} u \Delta_{j'} v) \equiv 0 \quad \text{if } |j - j'| \geq 5.$$

We shall also use the homogeneous Littlewood-Paley operators as follows:

$$\hat{S}_j u := \chi(2^{-j}D) u \quad \text{and} \quad \hat{\Delta}_j u := \Phi(2^{-j}D) u \quad \text{for each } j \in \mathbb{Z},$$

which enjoy the properties of quasi-orthogonality as above for inhomogeneous operator.

Definition 2.1. Let $S'_h(\mathbb{R}^d)$ be the space of tempered distributions $u$ such that

$$\lim_{q \to -\infty} \hat{S}_j u = 0, \quad \text{in } S'(\mathbb{R}^d).$$

Definition 2.2. For any $u, v \in S'_h(\mathbb{R}^d)$, the product $uv$ has the homogeneous Bony decomposition:

$$uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v),$$

where the paraproduct term

$$\hat{T}_u v = \sum_{j \leq k-2} \hat{\Delta}_j u \hat{\Delta}_k v = \sum_j \hat{S}_{j-1} u \hat{\Delta}_j v,$$

and the remainder term

$$\hat{R}(u, v) = \sum_j \hat{\Delta}_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j := \sum_{k=-1}^1 \hat{\Delta}_{j-k}.$$
In the similar way, we can define the inhomogeneous Bony decomposition:
\[ uv = T_u v + T_v u + R(u, v), \]
one can refer to [2] for the details. Now we introduce the Bernstein lemma which will be useful throughout this paper.

**Lemma 2.1.** Let \( 1 \leq a \leq b \leq \infty \) and \( f \in L^a(\mathbb{R}^d) \). Then there exists a positive constant \( C \) such that for \( q, k \in \mathbb{N} \),
\[
\sup_{|\alpha|=k} \| \partial^\alpha \hat{S}_t f \|_{L^b(\mathbb{R}^d)} \leq C^k 2^{q(\frac{k+d}{2}+1)} \| \hat{S}_t f \|_{L^a(\mathbb{R}^d)},
\]
\[
C^{-k} 2^{qk} \| \hat{\Delta}_t f \|_{L^b(\mathbb{R}^d)} \leq \sup_{|\alpha|=k} \| \partial^\alpha \hat{\Delta}_t f \|_{L^a(\mathbb{R}^d)} \leq C^k 2^{qk} \| \hat{\Delta}_t f \|_{L^a(\mathbb{R}^d)}.
\]

**Definition 2.3.** Let \( s \in \mathbb{R} \), \( (p, q) \in [1, \infty]^2 \) and \( u \in \mathcal{S}^\prime(\mathbb{R}^d) \). Then we define the inhomogeneous Besov spaces as
\[
B_{p,q}^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}^\prime(\mathbb{R}^d) \mid \| u \|_{B_{p,q}^s(\mathbb{R}^d)} < \infty \right\},
\]
where,
\[
\| u \|_{B_{p,q}^s(\mathbb{R}^d)} := \begin{cases} \left( \sum_{j \geq -1} 2^{jsq} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{j \geq -1} 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^d)} & \text{if } q = \infty. \end{cases}
\]

**Definition 2.4.** Let \( s \in \mathbb{R} \), \( (p, q) \in [1, \infty]^2 \) and \( u \in \mathcal{S}^\prime(\mathbb{R}^d) \). Then we define the inhomogeneous Fourier-Herz spaces as
\[
FB_{p,q}^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}^\prime(\mathbb{R}^d) \mid \| u \|_{FB_{p,q}^s(\mathbb{R}^d)} < \infty \right\},
\]
where,
\[
\| u \|_{FB_{p,q}^s(\mathbb{R}^d)} := \begin{cases} \left( \sum_{j \geq -1} 2^{jsq} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{j \geq -1} 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^d)} & \text{if } q = \infty. \end{cases}
\]

**Definition 2.5.** For \( s, \sigma \in \mathbb{R} \) and \( \alpha > 0 \), we define the space \( H_{\alpha}^{s,\sigma}(\mathbb{R}^2) \) by its norm
\[
\| u \|_{H_{\alpha}^{s,\sigma}(\mathbb{R}^2)} := \sum_{q \leq 0} 2^{2qs} \| \hat{\Delta}_q u \|_{L^2(\mathbb{R}^2)}^2 + \sum_{q > 0} q^\alpha 2^{2q}\sigma \| \hat{\Delta}_q u \|_{L^2(\mathbb{R}^2)}^2.
\]
Finally, we define \( \tilde{L}_t^s \tilde{H}_{\alpha}^{s,\sigma}(\mathbb{R}^2) \) by its norm
\[
\| u \|_{\tilde{L}_t^s \tilde{H}_{\alpha}^{s,\sigma}(\mathbb{R}^2)} := \sum_{q \leq 0} 2^{2qs} \| \hat{\Delta}_q u \|_{L_t^2 L^2(\mathbb{R}^2)}^2 + \sum_{q > 0} q^\alpha 2^{2q}\sigma \| \hat{\Delta}_q u \|_{L_t^2 L^2(\mathbb{R}^2)}^2.
\]

Through the whole paper, we denote \( \dot{H}_1^{0,0}(\mathbb{R}^2) \) by \( L_{log}^2(\mathbb{R}^2) \) for the sake of simplicity.

Next, we introduce localization in physical space. Firstly, we define partition of unity that we shall use through our paper.

**Proposition 2.2.** Let \( B_1(0) := \{ \xi \in \mathbb{R}^2 \mid |\xi| \leq 1 \} \). There exists radial function \( \phi \), valued in the interval \([0,1]\), belonging to \( \mathcal{D}(B_1(0)) \), and such that
\[
\sum_{k \in \mathbb{Z}^2} \phi(2^j x + k) = 1, \quad \forall x \in \mathbb{R}^2, \text{ and } j \in \mathbb{Z}, \tag{2.1}
\]
\[
\text{Supp} \phi_{j,i} \cap \text{Supp} \phi_{j,k} = \emptyset, \quad \text{if } |i-k| \geq 5, \tag{2.2}
\]
and
\[
\frac{1}{16} \leq \sum_{k \in \mathbb{Z}^2} \phi_{j,k}^2(x) \leq 1, \quad \text{for all } x \in \mathbb{R}^2 \text{ and } j \in \mathbb{Z}.
\] (2.3)

Here and what in follows, we denote \( \phi_{j,k} = \phi(2^j x - k) \).

**Proof.** Let us choose a radial smooth function \( \zeta \) satisfying
\[
\zeta(x) = \begin{cases} 
1 & |x| \leq \sqrt{2}^2; \\
0 & |x| \geq 1.
\end{cases}
\]

Thus, we have that if a couple \((i,k)\) satisfying \(|k - i| \geq 5\),
\[B_1(i) \cap B_1(k) = \emptyset.\] (2.4)

Now, we let
\[S(x) = \sum_{k \in \mathbb{Z}^2} \zeta(x + k).\]

It is obvious that \(S(x + k) = S(x)\) for all \(k \in \mathbb{Z}^2\). According to property (2.4), we know that the above summation \(S(x)\) is finite on \(\mathbb{R}^2\). Thus, the function \(S(x)\) is smooth on \(\mathbb{R}^2\).

On the other hand, we have
\[\bigcup_{k \in \mathbb{Z}^2} B_1(k) = \mathbb{R}^2.\]

Since the function \(\zeta\) is nonnegative and has value 1 near \(B_{\sqrt{2}^2}(0)\), it follows from the covering property that the function \(S\) is positive.

Now, we claim that the function \(\phi = \frac{\zeta}{S}\) is suitable. In fact, it is obvious that \(\phi\) belongs to \(\mathcal{D}(B_1(0))\) and
\[
\sum_{k \in \mathbb{Z}^2} \phi(x - k) = \sum_{k \in \mathbb{Z}^2} \frac{\zeta(x - k)}{S(x - k)} = \sum_{k \in \mathbb{Z}^2} \frac{\zeta(x - k)}{S(x)} = 1, \quad \forall \ x \in \mathbb{R}^2 \text{ and } j \in \mathbb{Z}.
\]

Now, it remains for us to prove (2.3). Let us denote that for \(m = 0, 1, 2, 3\),
\[I_m^j : = \sum_{k_1 = 4i + m} \phi^2(2^j x + k) = 1, \quad \forall \ x \in \mathbb{R}^2 \text{ and } j \in \mathbb{Z},\]

where \(k = (k_1, k_2)^T\) and \(i \in \mathbb{Z}^2\).

Thanks to property (2.2), it is obvious that
\[1 = \left( \sum_{k \in \mathbb{Z}^2} \phi(2^j x + k) \right)^2 \leq 16 \sum_{m=0}^4 I_m^j.\]

This estimate yields (2.3) and we end the proof of Proposition 2.2. \(\square\)

**Lemma 2.3.** (i) Let \( \Phi \in \mathcal{S}(\mathbb{R}^2) \), then there holds
\[
\| \Phi * f \|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{k \in \mathbb{Z}^2} \| \phi_{0,k} f \|_{L^1(\mathbb{R}^2)},
\]

where \( C \) is a positive constant independent of \( f \).
(ii) Let \( i, j \in \mathbb{Z} \) and \( i \leq j \). Then, we have that for each \( q \in [1, \infty] \),
\[
\| \phi_{i,k}f \|_{L^q(\mathbb{R}^2)} \leq 2^{1+2(j-i)} \sup_{k \in \mathbb{Z}^2} \| \phi_{j,k}f \|_{L^q(\mathbb{R}^2)}.
\] (2.6)

**Proof.** Estimate (2.6) follows from the covering theorem directly. So we just show estimate for \( (2.5) \). In view of \( (2.1) \), one can write
\[
\Phi \ast f(x) = \int_{\mathbb{R}^2} \Phi(x-y)f(y) \, dy = \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \Phi(x-y)\phi_{0,k}(y)f(y) \, dy
\]
\[
= \sum_{|k-x|<5} \int_{\mathbb{R}^2} \Phi(x-y)\phi_{0,k}(y)f(y) \, dy + \sum_{|k-x|\geq 5} \int_{\mathbb{R}^2} \Phi(x-y)\phi_{0,k}(y)f(y) \, dy.
\]
On one hand, it is obvious from the Hölder inequality that
\[
\sum_{|k-x|<5} \int_{\mathbb{R}^2} \Phi(x-y)\phi_{0,k}(y)f(y) \, dy \leq \sum_{|k-x|<5} \| \Phi \|_{L^\infty(\mathbb{R}^2)} \| \phi_{0,k}f \|_{L^1(\mathbb{R}^2)}
\]
\[
\leq C \sup_{k \in \mathbb{Z}^2} \| \phi_{0,k}f \|_{L^1(\mathbb{R}^2)}.
\] (2.7)

On the other hand, by the Hölder inequality and the property of support of \( \phi \), we readily have
\[
\sum_{|k-x|\geq 5} \int_{\mathbb{R}^2} \Phi(x-y)\phi_{0,k}(y)f(y) \, dy
\]
\[
= \sum_{i\geq 0} \int_{2^{2i}\leq |x-y|\leq 2^{2i+1}} \Phi(x-y) \sum_{k \in \mathbb{Z}^2} \phi_{0,k}(y)f(y) \, dy
\]
\[
\leq \sum_{i\geq 0} 2^{-4i} \int_{2^{2i}\leq |x-y|\leq 2^{2i+1}} |x-y|^4 \Phi(x-y) \sum_{|k-x|\leq 2^{i+2}} \phi_{0,k}(y)f(y) \, dy
\]
\[
\leq \sum_{i\geq 0} 2^{-4i} \sup_{x \in \mathbb{R}^2} |x|^4 \Phi(x) \sum_{|k-x|\leq 2^{i+2}} \| \phi_{0,k}f \|_{L^1(\mathbb{R}^2)}
\]
\[
\leq C \sup_{k \in \mathbb{Z}^2} \| \phi_{0,k}f \|_{L^1(\mathbb{R}^2)} \sum_{i\geq 0} 2^{-2i} \leq C \sup_{k \in \mathbb{Z}^2} \| \phi_{0,k}f \|_{L^1(\mathbb{R}^2)}.
\] (2.8)

Collecting estimate (2.7) and estimate (2.8) yields the desired result (2.5). \qed

**Lemma 2.4.** Let \( j \in \mathbb{N} \) and \( 2^j \sup_{k \in \mathbb{Z}^2} \| \phi_{j,k}f \|_{L^2(\mathbb{R}^2)} < \infty \). Then there holds
\[
\| S_j f \|_{L^\infty(\mathbb{R}^2)} \leq C 2^j \sup_{k \in \mathbb{Z}^2} \| \phi_{j,k}f \|_{L^2(\mathbb{R}^2)};
\]
where \( C \) is a positive constant independent of \( j \) and \( f \).

**Proof.** By changing a variable, one can conclude that
\[
|S_j f(x)| = \left| 2^{2j} \int_{\mathbb{R}^2} \Phi(2^j y) f(x-y) \, dy \right| = \left| \int_{\mathbb{R}^2} \varphi(y) f\left( \frac{2^j x - y}{2^j} \right) \, dy \right|.
\]
Let \( f_j(x) := f(x/2^j) \), then we have
\[
|S_j f(x)| = \left| \int_{\mathbb{R}^2} \Phi(y) f_j(2^j x - y) \, dy \right| = |\hat{\Delta}_0 f_j(2^j x)|,
\]
By using the first estimate in Lemma 2.3 and the Hölder inequality, we know that
\[ \|S_0 f_j\|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{k \in \mathbb{Z}^2} \|\phi_{0,j} f_j\|_{L^1(\mathbb{R}^2)} \]
\[ = C \sup_{k \in \mathbb{Z}^2} \|\phi_{0,k} f(\cdot/2^j)\|_{L^1(\mathbb{R}^2)} \leq C \sup_{k \in \mathbb{Z}^2} \|\phi_{0,k} f(\cdot/2^j)\|_{L^2(\mathbb{R}^2)}. \]
This implies
\[ \|S_0 f_j\|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{k \in \mathbb{Z}^2} \|\phi_{0,k} f(\cdot/2^j)\|_{L^2(\mathbb{R}^2)} \]. \quad (2.9)

Clearly, we have by changing a variable that
\[ \left( \int_{\mathbb{R}^2} \left| \phi_{0,k}(y) f\left(\frac{y}{2^j}\right) \right|^2 \, dy \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^2} \left| \phi(y-k) f\left(\frac{y}{2^j}\right) \right|^2 \, dy \right)^{\frac{1}{2}} \]
\[ = \left( 2^{2j} \int_{\mathbb{R}^2} \left| \phi(2^j y) f(y) \right|^2 \, dy \right)^{\frac{1}{2}} \]
\[ \leq 2^j \sup_{k \in \mathbb{Z}^2} \|\phi_{j,k} f\|_{L^2(\mathbb{R}^2)}. \]

Inserting this estimate into (2.9) yields the desired result. \qed

2.2. The principal eigenvalue of Laplace operator. Next, we review some statements concerning on the principal eigenvalue of Laplace operator.

Proposition 2.5 (Chap 6.5 Theorem 2, [8]). Assume that $U$ is open and bounded, and $\partial U$ is smooth. There hold that

(i) We have
\[ \lambda_1 := \min \left\{ B[u, u] := \int_U |\nabla u(x)|^2 \, dx \mid u \in H^1_0(U), \|u\|_{L^2(U)} = 1 \right\}. \]
Furthermore, the above minimum is attained for a function $w_1$, positive in $U$, which solves
\[ \begin{cases} 
-\Delta w_1 = \lambda_1 w_1 & \text{in } U, \\
w_1 = 0 & \text{on } \partial U.
\end{cases} \]

(ii) Finally, if $u \in H^1_0(U)$ is any weak solution of
\[ \begin{cases} 
-\Delta u = \lambda_1 u & \text{in } U, \\
u = 0 & \text{on } \partial U,
\end{cases} \]
then $u$ is a multiple of $w_1$.

Next, we will introduce an important property of the solution of the eigenvalue problem, which is the main ingredient of our proof.

Lemma 2.6. Let $f \in S(\mathbb{R}^2)$ and $\varphi$ be the solution of the above eigenvalue problem
\[ \begin{cases} 
-\Delta \varphi = \lambda_1 \varphi & \text{in } B_2(0), \\
\varphi|_{\partial B_2(0)} = 0
\end{cases} \quad (2.10) \]
satisfying $\varphi(x) = 0$ for all $x \in \mathbb{R}^2 \setminus B_2(0)$. Then, we have that for $\varphi_r = \varphi(x/r)$,
\[ -\int_{\mathbb{R}^2} \Delta f \varphi_r \, dx = \frac{\lambda_1}{2r^2} \int_{\mathbb{R}^2} f^2 \varphi_r \, dx + \int_{\mathbb{R}^2} \varphi_r |\nabla f|^2 \, dx + \frac{1}{2} \int_{\partial B_2}(0) f^2 \varphi_r \cdot n \, dS. \]
Proof. Integration by parts yields
\[- \int_{\mathbb{R}^2} \Delta f f \varphi_r \, dx = - \frac{1}{2} \int_{\mathbb{R}^2} (\Delta f + \Delta ff) \varphi_r \, dx \]
\[= - \frac{1}{2} \int_{\mathbb{R}^2} \Delta f^2 \varphi_r \, dx + \int_{\mathbb{R}^2} |\nabla f|^2 \varphi_r \, dx. \tag{2.11} \]
Integrating by parts again and using that \(\varphi_r\) solves the following eigenvalue problem
\[\begin{cases} 
- \Delta \varphi_r = \frac{\lambda_1}{r^2} \varphi_r, & x \in \mathbb{R}^2, \\
\varphi_r|_{\partial B_{2r}(0)} = 0, \end{cases} \]
we easily find that
\[- \frac{1}{2} \int_{\mathbb{R}^2} \Delta f^2 \varphi_r \, dx = - \frac{1}{2} \int_{\mathbb{R}^2} f^2 \Delta \varphi_r \, dx + \frac{1}{2} \int_{\partial B_{2r}(0)} f^2 \nabla \varphi_r \cdot n \, dS \]
\[= \frac{\lambda_1}{2r^2} \int_{\mathbb{R}^2} f^2 \varphi_r \, dx + \frac{1}{2} \int_{\partial B_{2r}(0)} f^2 \nabla \varphi_r \cdot n \, dS. \]
Plugging this estimate into (2.11) yields the desired result and then we finish the proof of the lemma. \(\square\)

Corollary 2.7. Let \(\varphi_{j,k} := \varphi(2^j x - k)\) with \(\varphi\) defined in Lemma 2.6. Then, we have
\[(i)\text{ for } f \in \mathcal{S}(\mathbb{R}^2), \]
\[- \int_{\mathbb{R}^2} \Delta f f \varphi_{j,k} \, dx = \frac{\lambda_1 2^{2j}}{2} \int_{\mathbb{R}^2} f^2 \varphi_{j,k} \, dx + \int_{\mathbb{R}^2} \varphi_{j,k} |\nabla f|^2 \, dx \]
\[+ \frac{1}{2} \int_{\partial B_{2^{-j+1}(k)}} f^2 \nabla \varphi_{j,k} \cdot n \, dS. \tag{2.12} \]
\[(ii)\text{ there exists two positive constants } M_l \text{ and } M_u \text{ such that} \]
\[M_l \varphi_{j,k} \leq \varphi_{j,k} \varphi_{j,k} \leq M_u \varphi_{j,k}. \tag{2.13} \]
\[(iii) \text{ orthogonal property:} \]
\[\varphi_{j,i} \varphi_{j,k} = 0 \quad \text{for} \quad |i - k| \geq 5. \]

With this test function in hand, we will give a refined \(L^2\)-estimate for smooth solution of the linear heat equation.

Proposition 2.8. Let the scalar function \(f\) be a smooth solution of the following linear heat equation in the plane:
\[\partial_t f - \Delta f = 0, \quad f|_{t=0} = f_0. \tag{2.14} \]
Then, there hold that
\[\|f(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla f(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau = \|f_0\|_{L^2(\mathbb{R}^2)}^2, \]
and
\[\lambda_1 \sup_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\phi_{j,k}} f(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 \, d\tau \leq C(t, \|f_0\|_{L^2(\mathbb{R}^2)}). \tag{2.15} \]
Proof of Proposition 2.8. Firstly, the standard $L^2$-inner argument enables us to conclude that for all $t \geq 0$,

$$\|f(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \|\nabla f(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau = \|f_0\|_{L^2(\mathbb{R}^2)}^2. \quad (2.16)$$

Multiplying (2.14) by $\varphi_{j,k} f$, we see that

$$\frac{1}{2} \partial_t (\varphi_{j,k} f^2) - \varphi_{j,k} f \Delta f = 0.$$ 

Integrating the above equality in space variable over $\mathbb{R}^2$ and using equality (2.12), one has

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varphi_{j,k}} f(t)\|_{L^2(\mathbb{R}^2)}^2 + \frac{\lambda_1 2^j}{2} \int_{\mathbb{R}^2} f^2 \varphi_{j,k} \, dx + \int_{\mathbb{R}^2} \varphi_{j,k} |\nabla f|^2 \, dx = \frac{1}{2} \int_{\partial B_{2^{-j+1}(k)}} f^2 \nabla \varphi_{j,k} \cdot n \, dS.$$ 

By the Cauchy-Schwartz inequality, we can infer that

$$- \int_{\partial B_{2^{-j+1}(k)}} f^2 \nabla \varphi_{j,k} \cdot n \, dS \leq \|f\|^2_{L^4(\partial B_{2^{-j+1}(k)})} \|\nabla \varphi_{j,k}\|_{L^2(\partial B_{2^{-j+1}(k)})} \leq \|f\|^2_{L^4(\partial B_{2^{-j+1}(k)})} \|\nabla \varphi_{j,k}\|_{L^2(\partial B_{2^{-j+1}(k)})}. \quad (2.18)$$

By the trace theorem (see for example Theorem 5.36 in [1]) and the Hölder inequality, we easily find that

$$\|f\|_{L^4(\partial B_{2^{-j+1}(k)})} \leq C \|f\|_{W^{1,\frac{4}{3}}(B_{2^{-j+1}(k)})} \leq C 2^{-\frac{j}{4}} \|f\|_{H^1(B_{2^{-j+1}(k)})} \leq C 2^{-\frac{j}{4}} \sum_{|k'-k| \leq 2} \left( \|\sqrt{\phi_{j,k'} f}\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\phi_{j,k'} \nabla f}\|_{L^2(\mathbb{R}^2)} \right), \quad (2.19)$$

where $W^{1,\frac{4}{3}}(\Omega)$ is the general Sobolev space.

On the other hand, we find that

$$\|\nabla \varphi_{j,k}\|_{L^2(\partial B_{2^{-j+1}(k)})} = 2^j \left( \int_{\partial B_{2^{-j+1}(0)}} |\nabla \varphi|^2(2) \, dS \right)^{\frac{1}{2}} = C 2^{\frac{j}{2}}. \quad (2.20)$$

Inserting estimates (2.19) and (2.20) into (2.13) leads to

$$- \int_{\partial B_{2^{-j}(k)}} f^2 \nabla \varphi_{j,k} \cdot n \, dS \leq C \|f\|_{H^1(B_{2^{-j+1}(k)})}^2 \leq C \sum_{|k'-k| \leq 2} \left( \|\sqrt{\phi_{j,k'} f}\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\phi_{j,k'} \nabla f}\|_{L^2(\mathbb{R}^2)} \right). \quad (2.21)$$
With this estimate, summing equality (2.17) over $k \in \mathbb{Z}^2$ and integrating the resulting equality with respect to time $t$, we immediately have

$$\lambda_1 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k} f}(\tau) \right\|^2_{L^2(\mathbb{R}^2)} d\tau$$

$$\leq \sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k} f_0} \right\|^2_{L^2(\mathbb{R}^2)} + C \sum_{k \in \mathbb{Z}^2} \sum_{k' \leq 2} \int_0^t \left( \left\| \varphi_{j,k'} f(\tau) \right\|_{L^2(\mathbb{R}^2)} + \left\| \varphi_{j,k'} \nabla f(\tau) \right\|_{L^2(\mathbb{R}^2)} \right) d\tau.$$

By the discrete Young inequality and (2.1), one has

$$\sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k} f_0} \right\|^2_{L^2(\mathbb{R}^2)} = \sum_{k \in \mathbb{Z}^2} \sum_{|k-k'| < 5} \left\| \varphi_{j,k} \sqrt{\varphi_{j,k'} f_0} \right\|^2_{L^2(\mathbb{R}^2)}$$

$$\leq C \sum_{k \in \mathbb{Z}^2} \sum_{|k-k'| < 5} \left\| \varphi_{j,k'} f_0 \right\|^2_{L^2(\mathbb{R}^2)}$$

$$\leq C \sum_{k' \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k'} f_0} \right\|^2_{L^2(\mathbb{R}^2)} = C \left\| f_0 \right\|^2_{L^2(\mathbb{R}^2)}.$$ 

On the other hand, according to the property of (2.13), we easily find that

$$\left\| \sqrt{\varphi_{j,k} f} \right\|^2_{L^\infty_t L^2(\mathbb{R}^2)} \leq C \left\| \varphi_{j,k} \sqrt{\varphi_{j,k} f} \right\|^2_{L^\infty_t L^2(\mathbb{R}^2)},$$

which implies that

$$\sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k} f} \right\|^2_{L^\infty_t L^2(\mathbb{R}^2)} \leq C \sum_{k \in \mathbb{Z}^2} \left\| \sqrt{\varphi_{j,k} f} \right\|^2_{L^\infty_t L^2(\mathbb{R}^2)}.$$

In a similar fashion as above, it is easy to conclude that

$$\lambda_1 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \left\| \varphi_{j,k} f(\tau) \right\|^2_{L^2(\mathbb{R}^2)} d\tau \leq C \lambda_1 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \left\| \varphi_{j,k} f(\tau) \right\|^2_{L^2(\mathbb{R}^2)} d\tau.$$

Collecting these estimates above, we readily have

$$\lambda_1 2^{2j} \int_0^t \sum_{k \in \mathbb{Z}^2} \left\| \varphi_{j,k} f(\tau) \right\|^2_{L^2(\mathbb{R}^2)} d\tau \leq C \left\| f_0 \right\|^2_{L^2(\mathbb{R}^2)} + C \int_0^t \left\| f(\tau) \right\|^2_{H^1(\mathbb{R}^2)} d\tau.$$

Taking supremum the above inequality over $j \in \mathbb{Z}$ together with estimate (2.16) gives the required result. \qed

In the last part of this section, we are devoted to show a estimate for the tri-linear term which will be used in the proof.

**Lemma 2.10.** There holds that

$$\int_0^t \int_{\mathbb{R}^2} f g h \, dx \, dt \leq \left\| f \right\|_{L^1_t L^2(\mathbb{R}^2)} \left\| \nabla g \right\|_{L^2_t L^2(\mathbb{R}^2)} \left\| h \right\|_{L^\infty_t L^2(\mathbb{R}^2)}$$

$$+ 2N \sup_{k \in \mathbb{Z}} \left\| \hat{S}_k g \right\|_{L^2_t L^\infty(\mathbb{R}^2)} \left\| f \right\|_{L^2_t L^2(\mathbb{R}^2)} \left\| h \right\|_{L^\infty_t L^2(\mathbb{R}^2)}$$

$$+ \sup_{k \in \mathbb{Z}} \left\| \hat{S}_k g \right\|_{L^2_t L^\infty(\mathbb{R}^2)} \left\| h \right\|_{L^\infty_t B^0_{2,2}(\mathbb{R}^2)} \left( \sum_{|k| \geq N} \left\| \hat{\Delta}_k f \right\|^2_{L^1_t L^2(\mathbb{R}^2)} \right)^{1/2}.$$
Remark 2.11. Note that for any \( i \leq -1 \), we have from the support property that

\[
\dot{S}_i f = S_0 \dot{S}_i f.
\]

This implies that \( \| \dot{S}_i f \|_{L^\infty} \leq C \| S_0 f \|_{L^\infty} \) for \( i \leq -1 \), hence we have

\[
\sup_{i \in \mathbb{Z}} \| \dot{S}_i f \|_{L^\infty} \leq C \sup_{i \geq 0} \| S_i f \|_{L^\infty}.
\]

So, we often use \( \sup_{i \geq 0} \| S_i f \|_{L^\infty} \) instead of \( \sup_{i \in \mathbb{Z}} \| \dot{S}_i f \|_{L^\infty} \) in the following parts.

**Proof of Lemma 2.10.** According to the Bony para-product decomposition, one writes

\[
\int_0^t \int_{\mathbb{R}^2} f g h \, dx \, dt = \sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\tilde{T}_q g) \, dx \, dt + \sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\tilde{T}_h g) \, dx \, dt
\]

\[
+ \sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\tilde{R}(g, h)) \, dx \, dt.
\]

For the first term in the right side of the above equality, by the Hölder inequality and the support property of paraproduct, we have

\[
\sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\tilde{T}_q g) \, dx \, dt = \sum_{q \in \mathbb{Z}} \sum_{|k-q| \leq 5} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g \dot{\Delta}_k h) \, dx \, dt
\]

\[
= \sum_{q \in \mathbb{Z}} \sum_{|k-q| \leq 5} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^H \dot{\Delta}_k h) \, dx \, dt
\]

\[
+ \sum_{q \in \mathbb{Z}} \sum_{|k-q| \leq 5} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^L \dot{\Delta}_k h) \, dx \, dt.
\]

where \( g^L_N = \dot{S}_N g \) and \( g^H_N = (I_d - \dot{S}_N) g \).

For the para-product term, by the property of support and the Hölder inequality, we see that

\[
\sum_{q \in \mathbb{Z}} \sum_{|k-q| \leq 5} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^H_N \dot{\Delta}_k h) \, dx \, dt
\]

\[
= \sum_{q \geq N-5} \sum_{|k-q| \leq 5} \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^H_N \dot{\Delta}_k h) \, dx \, dt
\]

\[
\leq \sum_{q \geq N-5} \sum_{|k-q| \leq 5} \int_0^t \| \dot{\Delta}_q f (\tau) \|_{L^2(\mathbb{R}^2)} \| \dot{S}_{k-1} g(\tau) \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_k h(\tau) \|_{L^2(\mathbb{R}^2)} \, d\tau
\]

\[
\leq \sup_{k \in \mathbb{Z}} \| \dot{S}_{k-1} g \|_{L^2_t L^\infty(\mathbb{R}^2)} \sum_{q \geq N-5} \| \dot{\Delta}_q f \|_{L^2_t L^2(\mathbb{R}^2)} \sum_{|k-q| \leq 5} \| \dot{\Delta}_k h \|_{L^\infty_t L^2(\mathbb{R}^2)}
\]

\[
\leq \sup_{k \in \mathbb{Z}} \| \dot{S}_k g \|_{L^2_t L^\infty(\mathbb{R}^2)} \| h \|_{L^\infty_t B_{2,\infty}^0(\mathbb{R}^2)} \left( \sum_{q \geq N-5} \| \dot{\Delta}_q f \|_{L^2_t L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}.
\]

On the other hand, we find that

\[
\int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^L_N \dot{\Delta}_k h) \, dx \, dt = \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1} g^L_N \dot{\Delta}_k h) \, dx \, dt
\]
\[ + \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f^M_N (\dot{S}_{k-1}^N g^L_N \dot{\Delta}_k h) \, dx \, d\tau \]
\[ + \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q f^L_N (\dot{S}_{k-1}^N g^L_N \dot{\Delta}_k h) \, dx \, d\tau \]
\[ := I + II + III, \]

where \( f^M_N = \sum_{q=-N}^N \dot{\Delta}_q f \).

In a similar way as above, we can obtain

\[
\sum_{q \in \mathbb{Z}} \sum_{|k-q|\leq 5} I \leq \sup_{k \in \mathbb{Z}} \| \dot{S}_k g \|_{L^2([0,t]; L^\infty(\mathbb{R}^2))} \| f^L_N \|_{L^2([0,t]; L^2(\mathbb{R}^2))} \| h \|_{L^\infty([0,t]; B^0_{2,2}(\mathbb{R}^2))}
\]

and

\[
\sum_{q \in \mathbb{Z}} \sum_{|k-q|\leq 5} III \leq \sup_{k \in \mathbb{Z}} \| \dot{S}_k g \|_{L^2([0,t]; L^\infty(\mathbb{R}^2))} \| f^H_N \|_{L^2([0,t]; L^2(\mathbb{R}^2))} \| h \|_{L^\infty([0,t]; B^0_{2,2}(\mathbb{R}^2))}.
\]

Now we need to tackle with the term involving the middle frequency of \( f \). By the discrete Young inequality, we readily have

\[
\sum_{-N < q \leq N} \sum_{|k-q|\leq 5} ^t \int_0 \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1}^N g^L_N \dot{\Delta}_k h) \, dx \, d\tau
\]
\[
\leq \sum_{-N < q \leq N} \sum_{|k-q|\leq 5} \int_0 ^t \| \dot{\Delta}_q f (\tau) \|_{L^2(\mathbb{R}^2)} \| \dot{S}_{k-1}^N g(\tau) \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_k h(\tau) \|_{L^2(\mathbb{R}^2)} \, d\tau
\]
\[
\leq 2N \sup_{k \in \mathbb{Z}} \| \dot{S}_{k-1}^N g \|_{L^2([0,t]; L^\infty(\mathbb{R}^2))} \sup_{q \in \mathbb{Z}} \left( \| \dot{\Delta}_q f \|_{L^2([0,t]; L^2(\mathbb{R}^2))} \sum_{|k-q|\leq 5} \| \dot{\Delta}_k h \|_{L^\infty(\mathbb{R}^2)} \right)
\]
\[
\leq 2N \sup_{k \in \mathbb{Z}} \| \dot{S}_{k-1}^N g \|_{L^2([0,t]; L^\infty(\mathbb{R}^2))} \sup_{q \in \mathbb{Z}} \| \dot{\Delta}_q f \|_{L^2([0,t]; L^2(\mathbb{R}^2))} \sup_{q \in \mathbb{Z}} \| \dot{\Delta}_q h \|_{L^\infty(\mathbb{R}^2)}
\]
\[
\leq 2N \sup_{k \in \mathbb{Z}} \| \dot{S}_k g \|_{L^2([0,t]; L^\infty(\mathbb{R}^2))} \| f \|_{L^\infty([0,t]; L^2(\mathbb{R}^2))} \| h \|_{L^\infty([0,t]; L^2(\mathbb{R}^2))}.
\]

By the Hölder inequality, the second term can be bounded as follows:

\[
\sum_{q \in \mathbb{Z}} \int_0 ^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{T}_h g) \, dx \, d\tau
\]
\[
= \sum_{q \in \mathbb{Z}} \sum_{|k-q|\leq 5} \int_0 ^t \int_{\mathbb{R}^2} \dot{\Delta}_q f (\dot{S}_{k-1}^N h \dot{\Delta}_k g) \, dx \, d\tau
\]
\[
\leq \sum_{q \in \mathbb{Z}} \sum_{|k-q|\leq 5} \int_0 ^t \| \dot{\Delta}_q f (\tau) \|_{L^2(\mathbb{R}^2)} \| \dot{S}_{k-1}^N h(\tau) \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_k g(\tau) \|_{L^\infty(\mathbb{R}^2)} \, d\tau
\]
\[
\leq \| h \|_{L^\infty(\mathbb{R}^2)} \sum_{q \in \mathbb{Z}} \| \dot{\Delta}_q f \|_{L^2(\mathbb{R}^2)} \sum_{|k-q|\leq 5} \| \dot{\Delta}_k g \|_{L^2(\mathbb{R}^2)}
\]
\[
\leq \| f \|_{L^2(\mathbb{R}^2)} \| \nabla g \|_{L^2(\mathbb{R}^2)} \| h \|_{L^\infty(\mathbb{R}^2)}.
\]
As for the remainder term, by the support property of the remainder term and the Hölder inequality, we get
\[
\sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \tilde{\Delta}_q f(\tilde{R}(g, h)) \, dx \, d\tau = \sum_{q \in \mathbb{Z}} \sum_{k \geq q-5} \int_0^t \int_{\mathbb{R}^2} \tilde{\Delta}_q f(\tilde{\Delta}_k g \tilde{\Delta}_k h) \, dx \, d\tau
\leq \sum_{q \in \mathbb{Z}} \sum_{k \geq q-5} \int_0^t \|\tilde{\Delta}_q f\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k g \tilde{\Delta}_k h\|_{L^2(\mathbb{R}^2)} \, d\tau
\leq \sum_{q \in \mathbb{Z}} \sum_{k \geq q-5} 2^q \int_0^t \|\tilde{\Delta}_q f\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k g \tilde{\Delta}_k h\|_{L^1(\mathbb{R}^2)} \, d\tau.
\]

Furthermore, by the Hölder inequality, we obtain
\[
\int_0^t \|\tilde{\Delta}_q f\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k g \tilde{\Delta}_k h\|_{L^1(\mathbb{R}^2)} \, d\tau
\leq \int_0^t \|\tilde{\Delta}_q f(\tau)\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k g(\tau)\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k h(\tau)\|_{L^2(\mathbb{R}^2)} \, d\tau.
\]
Inserting this estimate into (2.22), we get from the discrete Young inequality and the Hölder inequality that
\[
\sum_{q \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} \tilde{\Delta}_q f(\tilde{R}(g, h)) \, dx \, d\tau
\leq \int_0^t \sum_{q \in \mathbb{Z}} \sum_{k \geq q-5} 2^q \|\tilde{\Delta}_q f(\tau)\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k g(\tau)\|_{L^2(\mathbb{R}^2)} \|\tilde{\Delta}_k h(\tau)\|_{L^2(\mathbb{R}^2)} \, d\tau
\leq C \int_0^t \|f(\tau)\|_{L^2(\mathbb{R}^2)} \|\nabla g(\tau)\|_{L^2(\mathbb{R}^2)} \|h(\tau)\|_{L^2(\mathbb{R}^2)} \, d\tau
\leq C \|f\|_{L^2(\mathbb{R}^2)} \|\nabla g\|_{L^2(\mathbb{R}^2)} \|h\|_{L^\infty(\mathbb{R}^2)} L^2(\mathbb{R}^2).
\]
Collecting all these estimates yields the desired result. \(\square\)

### 3. A priori estimates

This section is devoted to show some useful a priori estimates for the smooth solution of problem (1.1) which can be viewed as an preparation for proving our theorems. Let us begin by proving the \(L^2\)-energy estimate of solution \((u, B, E)\).

**Proposition 3.1.** Let \((u_0, E_0, B_0) \in (L^2(\mathbb{R}^2))^3\), and \((u, B, E)\) be a smooth solution of problem (1.1). Then we have that for all \(t \geq 0\),
\[
\| (u, E, B)(t) \|^2_{L^2(\mathbb{R}^2)} + 2 \int_0^t \| \nabla u(\tau) \|^2_{L^2(\mathbb{R}^2)} \, d\tau + 2 \int_0^t \| j(\tau) \|^2_{L^2(\mathbb{R}^2)} \, d\tau
= \| u_0 \|^2_{L^2(\mathbb{R}^2)} + \| E_0 \|^2_{L^2(\mathbb{R}^2)} + \| B_0 \|^2_{L^2(\mathbb{R}^2)}.
\]
where \(\| (u, E, B)(t) \|^2_{L^2(\mathbb{R}^2)} = \| u(t) \|^2_{L^2(\mathbb{R}^2)} + \| E(t) \|^2_{L^2(\mathbb{R}^2)} + \| B(t) \|^2_{L^2(\mathbb{R}^2)}\).
Proof. The proof of the theorem is standard, we also give the proof for completeness. Taking the $L^2$-inner product of $(u, E, B)$, we immediately have

$$
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|
abla u(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau = \int_{\mathbb{R}^2} (j \times B) \cdot u \, dx,
$$

$$
\frac{1}{2} \frac{d}{dt} \|E(t)\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \text{curl} \, B \cdot E \, dx - \int_{\mathbb{R}^2} j \cdot E \, dx,
$$

and

$$
\frac{1}{2} \frac{d}{dt} \|B(t)\|_{L^2(\mathbb{R}^2)}^2 = -\int_{\mathbb{R}^2} \text{curl} \, E \cdot B \, dx.
$$

Note that

$$
\int_{\mathbb{R}^2} \text{curl} \, B \cdot E \, dx - \int_{\mathbb{R}^2} \text{curl} \, E \cdot B \, dx = 0
$$

and from the relation $j = E + u \times B$ that

$$
\int_{\mathbb{R}^2} (j \times B) \cdot u \, dx - \int_{\mathbb{R}^2} j \cdot E \, dx
$$

$$
= -\int_{\mathbb{R}^2} |j|^2 \, dx + \int_{\mathbb{R}^2} j \cdot (u \times B) \, dx + \int_{\mathbb{R}^2} (j \times B) \cdot u \, dx
$$

$$
= -\int_{\mathbb{R}^2} |j|^2 \, dx.
$$

Collecting all these estimates, we readily have

$$
\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{L^2(\mathbb{R}^2)}^2 + \|E(t)\|_{L^2(\mathbb{R}^2)}^2 + \|B(t)\|_{L^2(\mathbb{R}^2)}^2 \right)
$$

$$
+ \int_0^t \|
abla u(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau + \int_0^t \|j(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau = 0.
$$

Integrating the above equality with respect to time $t$ yields the desired result (3.1). \qed

**Proposition 3.2.** Let $(u, B, E)$ be the smooth solution of problem (1.1). Then there exist a constant $C = C(t, \|(u_0, B_0, E_0)\|_{L^2(\mathbb{R}^2)}) > 0$ such that

$$
\|u\|_{L^\infty L^2(\mathbb{R}^2)} + \|(E, B)\|_{L^\infty B_{2,2}(\mathbb{R}^2)}^2 + \int_0^t \|
abla u(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau
$$

$$
+ \lambda_1 \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\varphi_{i,k} u(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau + \int_0^t \|j(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq C.
$$

**Proof.** Multiplying the first equations of system (1.1) by the cut-off function $\varphi_{i,k} u$, we have that

$$
\frac{1}{2} \partial_t (\varphi_{i,k} u^2) + \frac{1}{2} (u \cdot \nabla) (\varphi_{i,k} u^2) - \varphi_{i,k} u \Delta u = \varphi_{i,k} u (j \times B) + (u \cdot \nabla \varphi_{i,k}) u^2 - \varphi_{i,k} u \cdot \nabla \varphi.
$$
Integrating the above equality with respect to space variable $x$ over $\mathbb{R}^2$ yields

$$
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\varphi_{i,k} u(t)} \|_{L^2(\mathbb{R}^2)}^2 + \| \sqrt{\varphi_{i,k} \nabla u(t)} \|_{L^2(\mathbb{R}^2)}^2 + 2 \lambda_1 2^{|i|} \| \sqrt{\varphi_{i,k} u(t)} \|_{L^2(\mathbb{R}^2)}^2 \right)
= \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot (j \times B) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} (u \cdot \nabla \varphi_{i,k}) |u|^2 \, dx - \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \pi \, dx
- \frac{1}{2} \int_{\partial B_{2^{-i}}(k)} |u|^2 \nabla \varphi_{i,k} \cdot n \, dS.
$$

Next, summing the above equality over $k \in \mathbb{Z}^2$ and integrating the resulting inequality in time $t$ provides

$$
\frac{1}{2} \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k} u(t)} \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k} \nabla u(\tau)} \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
+ 2 \lambda_1 2^{|i|} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k} u(\tau)} \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
= \frac{1}{2} \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k} u_0} \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} u (j \times B) \, dx \, d\tau + \int_0^t \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} (u \cdot \nabla \varphi_{i,k}) u^2 \, dx \, d\tau
- \int_0^t \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \pi \, dx \, d\tau - \frac{1}{2} \int_0^t \sum_{k \in \mathbb{Z}^2} \int_{\partial B_{2^{-i}}(k)} |u|^2 \nabla \varphi_{i,k} \cdot n \, dS \, d\tau.
$$

By the Hölder inequality and (2.13), we easily find that

$$
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot (j \times B) \, dx = - \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} j \cdot (u \times B) \, dx
\leq C \int_{\mathbb{R}^2} |u(x)||j(x)||B(x)| \, dx.
$$

By Lemma 2.10, we see that

$$
\sum_{k \in \mathbb{Z}^2} \int_0^t \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot (j \times B) \, dx \, d\tau
\leq C \int_0^t \int_{\mathbb{R}^2} |u(t, x)||j(t, x)||B(t, x)| \, dx \, d\tau
\leq C \| j \|_{L^2_t L^2(\mathbb{R}^2)} \| \nabla u \|_{L^2_t L^2(\mathbb{R}^2)} \| B \|_{L^\infty_t L^2(\mathbb{R}^2)}
+ C N \sup_{k' \in \mathbb{Z}} \| S_{k'-1} u \|_{L^2_t L^\infty(\mathbb{R}^2)} \| j \|_{L^2_t L^2(\mathbb{R}^2)} \| B \|_{L^\infty_t L^2(\mathbb{R}^2)}
+ C \sup_{k' \in \mathbb{Z}} \| S_{k'-1} u \|_{L^2_t L^\infty(\mathbb{R}^2)} \| B \|_{L^\infty_t B^0_{2,2}(\mathbb{R}^2)} \left( \sum_{|i| \geq N} \| \hat{\Delta} j \|_{L^2_t L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}.
$$
For the second term, the Hölder inequality and the interpolation inequality allow us to conclude that
\[
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} (u \cdot \nabla \varphi_{i,k}) |u|^2 \, dx = - \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) u \cdot u \, dx \\
\leq C \int_{\mathbb{R}^2} |u(x)|^2 |\nabla u(x)| \, dx \\
\leq C \|u\|_{L^4(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)} \\
\leq C \|u\|_{L^2(\mathbb{R}^2)} \|\nabla u\|_{L^2(\mathbb{R}^2)}^2. 
\]

From this, it follows that
\[
\sum_{k \in \mathbb{Z}^2} \int_0^t \int_{\mathbb{R}^2} (u \cdot \nabla \varphi_{i,k}) |u|^2 \, dx \, dt \leq C \int_0^t \|u(\tau)\|_{L^2(\mathbb{R}^2)} \|\nabla u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau. \tag{3.4}
\]

Now, we turn to show the term involving the pressure. Since $\text{div} \, u = 0$, the pressure can be expressed by
\[
\pi = -\frac{\text{div}}{\Delta} ((u \cdot \nabla) u) + \left(\frac{\text{div}}{\Delta} \right)(j \times B) := \pi_1 + \pi_2.
\]

Therefore, we have
\[
\int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi \, dx = \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_1 \, dx + \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_2 \, dx.
\]

To bound the second integral in the right side of the above equality, we need to resort to the following lemma.

**Lemma 3.3.** For each $\varepsilon > 0$, there exist an absolute constant $C > 0$ such that
\[
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k}(u \cdot \nabla) \pi_2 \, dx \\
\leq C \|j\|_{L^2(\mathbb{R}^2)}^2 \|B\|_{L^2(\mathbb{R}^2)} + \varepsilon \left(\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \lambda_1 2^{2i} \sum_{k \in \mathbb{Z}^2} \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)}^2 \right). \tag{3.5}
\]

**Proof of Lemma 3.3.** Firstly, we split the integral into the following two parts:
\[
\int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_2 \, dx = \sum_{|k| \leq 5} \int_{\mathbb{R}^2} \nabla \left(\frac{\text{div}}{\Delta} \right)(\phi_{i,k}(j \times B)) \cdot (\varphi_{i,k} u) \, dx \\
+ \sum_{|k| > 5} \int_{\mathbb{R}^2} \nabla \left(\frac{\text{div}}{\Delta} \right)(\phi_{i,k}(j \times B)) \cdot (\varphi_{i,k} u) \, dx.
\]
For the first term in the above equality, by integrating by parts and using the Hölder inequality, we have

\[
\int_{\mathbb{R}^2} \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (j \times B) \right) \cdot (\varphi_{i,k} u) \, dx \\
= -2 \sum_{|k'| \leq 5} \int_{\mathbb{R}^2} \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (j \times B) \right) \phi_{i,k'} (u \cdot \nabla) \varphi_{i,k} \, dx \\
\leq 2 \sum_{|k'| \leq 5} \left\| \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (j \times B) \right) \right\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^2)} \left\| \phi_{i,k'} u \cdot \nabla \varphi_{i,k} \right\|_{\dot{B}^0_{2,1}(\mathbb{R}^2)}.
\]

By the Hölder inequality, we easily find that

\[
\left\| \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (j \times B) \right) \right\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^2)} \leq C \left\| \phi_{i,k} (j \times B) \right\|_{\dot{B}^{-1}_{2,\infty}(\mathbb{R}^2)} \\
\leq C \left\| \phi_{i,k} (j \times B) \right\|_{L^1(\mathbb{R}^2)} \\
\leq C \left\| \sqrt{\phi_{i,k}} \right\|_{L^2(\mathbb{R}^2)} \left\| \sqrt{\phi_{i,k}} B \right\|_{L^2(\mathbb{R}^2)}.
\]

Thanks to the Bony paraproduct decomposition, we have

\[
(\phi_{i,k'} u) \cdot \nabla \varphi_{i,k} = \sum_{\ell=1}^2 \hat{T}_{\partial_k \varphi_{i,k}} (\phi_{i,k'} u^\ell) + \sum_{\ell=1}^2 \hat{T}_{\phi_{i,k'} u^\ell} (\partial_k \varphi_{i,k}) \\
+ \sum_{\ell=1}^2 \hat{R} (\partial_k \varphi_{i,k}, \phi_{i,k'} u^\ell).
\]

A simple calculation allows us to conclude that

\[
\left\| \hat{T}_{\partial_k \varphi_{i,k}} (\phi_{i,k'} u^\ell) \right\|_{\dot{B}^0_{2,1}(\mathbb{R}^2)} \leq C \sum_{\ell'=1}^{2} \left\| \hat{S}_{\ell'-1} \nabla \varphi_{i,k} \right\|_{L^\infty(\mathbb{R}^2)} \left\| \hat{\Delta}_{\ell'} (\phi_{i,k'} u) \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \sum_{\ell'=1}^{2} 2^{-\ell'} \left\| \hat{S}_{\ell'-1} \nabla \varphi_{i,k} \right\|_{L^\infty(\mathbb{R}^2)} 2^{\ell'} \left\| \hat{\Delta}_{\ell'} (\phi_{i,k'} u) \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \left\| \nabla \varphi_{i,k} \right\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)} \left\| \nabla (\phi_{i,k'} u) \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \left\| \nabla \varphi_{i,k} \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla (\phi_{i,k'} u) \right\|_{L^2(\mathbb{R}^2)}.
\]

In the similar fashion as above, \( \hat{T}_{\phi_{i,k'} u^\ell} \partial_k \varphi_{i,k} \) can be bounded as follows:

\[
\left\| \hat{T}_{\phi_{i,k'} u^\ell} \partial_k \varphi_{i,k} \right\|_{\dot{B}^0_{2,1}(\mathbb{R}^2)} \\
\leq C \sum_{\ell'=1}^{2} \left\| \hat{S}_{\ell'-1} (\phi_{i,k'} u) \right\|_{L^\infty(\mathbb{R}^2)} \left\| \hat{\Delta}_{\ell'} (\nabla \varphi_{i,k}) \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \sum_{\ell'=1}^{2} 2^{-\ell'} \left\| \hat{S}_{\ell'-1} (\phi_{i,k'} u) \right\|_{L^\infty(\mathbb{R}^2)} 2^{\ell'} \left\| \hat{\Delta}_{\ell'} (\nabla \varphi_{i,k}) \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \left\| \phi_{i,k'} u \right\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)} \left\| \Delta \varphi_{i,k} \right\|_{L^2(\mathbb{R}^2)} \leq C \left\| \Delta \varphi_{i,k} \right\|_{L^2(\mathbb{R}^2)} \left\| \phi_{i,k'} u \right\|_{L^2(\mathbb{R}^2)}.
\]
We turn to show the remainder term \( \dot{R}(\partial_t \varphi_{i,k}, \phi_{i,k'} u^t) \). We observe that
\[
\| \dot{R}(\partial_t \varphi_{i,k}, \phi_{i,k'} u^t) \|_{B^0_{2,1}(\mathbb{R}^2)} \leq C \| \dot{R}(\nabla \varphi_{i,k}, \phi_{i,k'} u) \|_{B^1_{2,1}(\mathbb{R}^2)} \\
\leq C \sum_{k' \in \mathbb{Z}} 2^{k'} \| \Delta' \nabla \varphi_{i,k} \|_{L^2(\mathbb{R}^2)} \| \Delta' (\phi_{i,k'} u) \|_{L^2(\mathbb{R}^2)} \\
\leq C \| \nabla \varphi_{i,k} \|_{L^2(\mathbb{R}^2)} \| \phi_{i,k'} u \|_{B^1_{2,2}(\mathbb{R}^2)} \\
\leq C \| \nabla \varphi_{i,k} \|_{L^2(\mathbb{R}^2)} \| \nabla (\phi_{i,k'} u) \|_{L^2(\mathbb{R}^2)}.
\]
Hence, we have
\[
\int_{\mathbb{R}^2} \nabla \left( \frac{\nabla}{-\Delta} \right) (\phi_{i,k} (j \times B)) \cdot (\varphi_{i,k} u) \, dx \\
\leq C \sum_{|k' - k| \leq 5} \| \sqrt{\phi_{i,k}} B \|_{L^2(\mathbb{R}^2)} \| \nabla (\varphi_{i,k} u) \|_{L^2(\mathbb{R}^2)} \\
\leq C \| \phi_{i,k} B \|_{L^2(\mathbb{R}^2)}^2 + \varepsilon \sum_{|k' - k| \leq 5} \| \nabla (\phi_{i,k'} u) \|_{L^2(\mathbb{R}^2)}^2.
\]
(3.6)

Now, we turn to bound the integral term
\[
\sum_{|\tilde{k} - k| > 5} \int_{\mathbb{R}^2} \nabla \left( \frac{\nabla}{-\Delta} \right) (\phi_{i,k} (j \times B)) \cdot (\varphi_{i,k} u) \, dx.
\]

The term \( \nabla \left( \frac{\nabla}{-\Delta} \right) (\phi_{i,k} (j \times B)) \) can be rewritten as
\[
\nabla \left( \frac{\nabla}{-\Delta} \right) (\phi_{i,k} (j \times B)) = \int_{\mathbb{R}^2} K(x - z) \phi_{i,k} (z) (j \times B)(z) \, dz,
\]
where the kernel \( K(x) \) satisfies \( |K| \leq c \frac{1}{|x|^2} \).

Hence, the above equality allows us to write
\[
\int_{\mathbb{R}^2} \sqrt{\varphi_{i,k}} \left( \frac{\nabla}{-\Delta} \right) (\phi_{i,k} (j \times B)) \cdot (\sqrt{\varphi_{i,k}} u) \, dx \\
= \int_{\mathbb{R}^2} \sqrt{\varphi_{i,k}} K * (\phi_{i,k} (j \times B)) \cdot (\sqrt{\varphi_{i,k}} u) \, dx.
\]

Since \( |\tilde{k} - k| > 5 \), by the Hölder inequality, we have
\[
\int_{\mathbb{R}^2} \sqrt{\varphi_{i,k}} K * \left( \phi_{i,k} (j \times B) \right) \| \sqrt{\varphi_{i,k}} u \| \, dx \\
= \int_{\mathbb{R}^2} \sqrt{\varphi_{i,k}} K_{\tilde{k} k} * \left( \phi_{i,k} (j \times B) \right) \| \sqrt{\varphi_{i,k}} u \| \, dx \\
\leq \| \sqrt{\varphi_{i,k}} (K_{\tilde{k} k} * \left( \phi_{i,k} (j \times B) \right)) \|_{L^1(\mathbb{R}^2)} \| \sqrt{\varphi_{i,k}} u \|_{L^1(\mathbb{R}^2)} \\
\leq C 2^{-k} \| \sqrt{\varphi_{i,k}} (K_{\tilde{k} k} * \left( \phi_{i,k} (j \times B) \right)) \|_{L^1(\mathbb{R}^2)} \| \sqrt{\varphi_{i,k}} u \|_{L^1(\mathbb{R}^2)}.
\]

where \( K_{\tilde{k} k} = c_{\frac{1}{|x|^2}} \chi_{B_{2^{-1/2}}(0)}(0) \) deduced from the support property of \( \varphi_{i,k} \) and \( \phi_{i,\tilde{k}} \).
We get from the Young inequality that
\[
\left\| \sqrt{\psi_{i,k}} \left( K_{k,k} \ast \phi_{i,k} (j \times B) \right) \right\|_{L^\infty (\mathbb{R}^2)} \leq \left\| K_{k,k} \right\|_{L^\infty (\mathbb{R}^2)} \left\| \phi_{i,k} (j \times B) \right\|_{L^1 (\mathbb{R}^2)} \\
\leq \frac{C 2^{2i}}{|k-k'|^2} \left\| \sqrt{\psi_{i,k}} j \right\|_{L^2 (\mathbb{R}^2)} \left\| \sqrt{\psi_{i,k}} B \right\|_{L^2 (\mathbb{R}^2)}.
\]
Therefore, in virtue of the discrete Young inequality and Cauchy-Schwarz inequality, we obtain
\[
\sum_{k \in \mathbb{Z}^2} \sum_{|k' - k| > 5} \int_{\mathbb{R}^2} \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (j \times B) \right) \cdot (\varphi_{i,k} u) \, dx \\
\leq C \sum_{k \in \mathbb{Z}^2} 2^i \sum_{|k' - k| > 5} \frac{1}{|k-k'|^2} \left\| \sqrt{\psi_{i,k}} j \right\|_{L^2 (\mathbb{R}^2)} \left\| \sqrt{\psi_{i,k}} B \right\|_{L^2 (\mathbb{R}^2)} \left\| \sqrt{\varphi_{i,k}} u \right\|_{L^2 (\mathbb{R}^2)} \\
\leq C \left\| j \right\|_{L^2 (\mathbb{R}^2)}^2 \left\| B \right\|_{L^2 (\mathbb{R}^2)}^2 + \varepsilon \lambda_1 \sum_{k \in \mathbb{Z}^2} 2^{2i} \left\| \sqrt{\varphi_{i,k}} u \right\|_{L^2 (\mathbb{R}^2)}^2.
\]
This estimate together with (3.6) yields
\[
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( j \times B \right) \cdot (\varphi_{i,k} u) \, dx \\
\leq C \left\| j \right\|_{L^2 (\mathbb{R}^2)}^2 \left\| B \right\|_{L^2 (\mathbb{R}^2)}^2 + \varepsilon \sum_{k \in \mathbb{Z}^2} \left( \lambda_1 2^{2i} \sum_{|k' - k| \leq 5} \left\| \phi_{i,k} u \right\|_{L^2 (\mathbb{R}^2)}^2 + \left\| \sqrt{\varphi_{i,k} \nabla u} \right\|_{L^2 (\mathbb{R}^2)}^2 \right) \quad (3.7) \\
\leq C \left\| j \right\|_{L^2 (\mathbb{R}^2)}^2 \left\| B \right\|_{L^2 (\mathbb{R}^2)}^2 + \varepsilon \sum_{k \in \mathbb{Z}^2} \left( \lambda_1 2^{2i} \left\| \phi_{i,k} u \right\|_{L^2 (\mathbb{R}^2)}^2 + \left\| \sqrt{\varphi_{i,k} \nabla u} \right\|_{L^2 (\mathbb{R}^2)}^2 \right).
\]
Hence, we end the proof of Lemma 3.3 \qed

Now, we need to bound the integral \( \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_1 \, dx \) which is contained in the lemma below.

**Lemma 3.4.** For each \( \varepsilon > 0 \), there exist a absolute constant \( C > 0 \) such that
\[
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_1 \, dx \leq C \left\| u \right\|_{L^2 (\mathbb{R}^2)}^2 \left\| \nabla u \right\|_{L^2 (\mathbb{R}^2)}^2 + \varepsilon \lambda_1 \sum_{k \in \mathbb{Z}^2} 2^{2i} \left\| \sqrt{\phi_{i,k} u} \right\|_{L^2 (\mathbb{R}^2)}^2. \quad (3.8)
\]

**Proof of Lemma 3.4.** We see that
\[
\int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi_1 \, dx = \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( (u \cdot \nabla) u \right) \, dx.
\]
One can write
\[
\int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( (u \cdot \nabla) u \right) \, dx = \sum_{|k' - k| \leq 5} \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (u \otimes u) \right) \, dx \\
+ \sum_{|k' - k| > 5} \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \left( \frac{\text{div}}{-\Delta} \right) \left( \phi_{i,k} (u \otimes u) \right) \, dx.
\]
Integration by parts leads to
\[
\int_{\mathbb{R}^2} \varphi_{i,k} u \cdot \nabla \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \, dx \\
= -2 \int_{\mathbb{R}^2} \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) u \cdot \nabla \varphi_{i,k} \, dx \\
= -2 \sum_{|k-k'| \leq 5} \int_{\mathbb{R}^2} \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \phi_{i,k} u \cdot \nabla \varphi_{i,k} \, dx.
\]

Moreover, by the Hölder inequality, we immediately have
\[
\int_{\mathbb{R}^2} \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \phi_{i,k} u \cdot \nabla \varphi_{i,k} \, dx \\
\leq \left\| \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla \varphi_{i,k} \right\|_{L^\infty(\mathbb{R}^2)} \left\| \varphi_{i,k} u \right\|_{L^2(\mathbb{R}^2)}.
\]
(3.9)

The Hölder inequality and the interpolation theorem give
\[
\sum_{k \in \mathbb{Z}^2} \left\| \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \right\|_{L^2(\mathbb{R}^2)}^2 \leq C \sum_{k \in \mathbb{Z}^2} \left\| \phi_{i,k'} (u \otimes u) \right\|_{L^2(\mathbb{R}^2)}^2 \\
\leq C \|u\|_{L^4(\mathbb{R}^2)}^4 \leq C \|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2.
\]

Plugging this estimate in (3.9) provides
\[
\sum_{k \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) u \cdot \nabla \varphi_{i,k} \, dx \\
\leq C \sum_{k \in \mathbb{Z}^2} 2^2 \left\| \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \right\|_{L^2(\mathbb{R}^2)} \sum_{|k-k'| \leq 5} \left\| \phi_{i,k'} u \right\|_{L^2(\mathbb{R}^2)} \\
\leq C \sum_{k \in \mathbb{Z}^2} \|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon \lambda_1 \sum_{k \in \mathbb{Z}^2} 2^{2i} \left\| \sqrt{\phi_{i,k}} u \right\|_{L^2(\mathbb{R}^2)}^2.
\]
(3.10)

On the other hand, we observe that
\[
\nabla \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) = G \ast (\phi_{i,k'} (u \otimes u)),
\]
where $G(x)$ satisfies $|G(x)| \leq \frac{c}{|x|^4}$.

It follows that
\[
\int_{\mathbb{R}^2} \nabla \left( \frac{\text{div div}}{-\Delta} \right) (\phi_{i,k'} (u \otimes u)) \cdot (\varphi_{i,k} u) \, dx = \int_{\mathbb{R}^2} \sqrt{\varphi_{i,k}} G \ast (\phi_{i,k'} (u \otimes u)) \cdot (\sqrt{\varphi_{i,k}} u) \, dx.
\]

Since $|k' - k| > 5$, we find that the supports of $\varphi_{i,k}$ and $\phi_{i,k'}$ are disjoint and thus we have
\[
\sqrt{\varphi_{i,k}} G \ast (\phi_{i,k'} (u \otimes u)) = \sqrt{\varphi_{i,k}} (G \chi_{[0,2^{-(1+i)}|k'-k|]}) \ast (\phi_{i,k'} (u \otimes u)).
\]
Letting $G_{kk'} := G\chi_{[0,|k-k'|2^{-(1+i)}]}$ and using the Hölder inequality, we immediately get

$$
\int_{\mathbb{R}^2} \nabla\left(\frac{\div \div}{-\Delta}\right)(\phi_{i,k'}(u \otimes u)) \cdot (\phi_{i,k} u) \, dx
= \int_{\mathbb{R}^2} \sqrt{\varphi_{i,k} G_{ik'}} (\phi_{i,k'}(u \otimes u)) \cdot (\sqrt{\varphi_{i,k}} u) \, dx
\leq \|\sqrt{\varphi_{i,k}}\|_{L^\infty} \|G_{kk'} (\phi_{i,k'}(u \otimes u))\|_{L^\infty(\mathbb{R}^2)} \|\sqrt{\varphi_{i,k}} u\|_{L^1(\mathbb{R}^2)}
\leq C^{-1} \|G_{kk'} (\phi_{i,k'}(u \otimes u))\|_{L^\infty(\mathbb{R}^2)} \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)}.
$$

By the discrete Young inequality, we have

$$
\|G_{kk'} (\phi_{i,k'}(u \otimes u))\|_{L^\infty(\mathbb{R}^2)} \leq \|G_{kk'}\|_{L^2(\mathbb{R}^2)} \|\phi_{i,k'}(u \otimes u)\|_{L^2(\mathbb{R}^2)}
\leq C \frac{2^{2i}}{|k-k'|^3} \|\sqrt{\varphi_{i,k'}} u\|_{L^2(\mathbb{R}^2)}.
$$

This estimate enables us to infer that

$$
\sum_{k \in \mathbb{Z}^2} \sum_{|k'-k| > 5} \int_{\mathbb{R}^2} \nabla\left(\frac{\div \div}{-\Delta}\right)(\phi_{i,k'}(u \otimes u)) \cdot (\phi_{i,k} u) \, dx
\leq C \sum_{k \in \mathbb{Z}^2} 2^i \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)} \sum_{|k'-k| > 5} \frac{1}{|k'-k|^3} \|\sqrt{\varphi_{i,k'}}(u \otimes u)\|_{L^2(\mathbb{R}^2)}
\leq C 2^i \left(\sum_{k \in \mathbb{Z}^2} \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)}^2\right)^{\frac{1}{2}} \|u \otimes u\|_{L^2(\mathbb{R}^2)}
\leq C 2^i \left(\sum_{k \in \mathbb{Z}^2} \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)}^2\right)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^2)} \|\nabla u\|_{L^2(\mathbb{R}^2)}
\leq C \|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon \lambda_1 \sum_{k \in \mathbb{Z}^2} 2^{2i} \|\sqrt{\varphi_{i,k}} u\|_{L^2(\mathbb{R}^2)}^2.
$$

This estimate together with estimate (3.11) gives the desired result in Lemma 3.4. ∎

From Lemma 3.3 and Lemma 3.4, we know that

$$
\sum_{k \in \mathbb{Z}^2} \int_0^t \int_{\mathbb{R}^2} \varphi_{i,k} (u \cdot \nabla) \pi \, dx \, dt
\leq C \int_0^t \|j(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\nabla B(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau + C \int_0^t \|u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
+ \varepsilon \sum_{k \in \mathbb{Z}^2} \int_0^t \left(\lambda_1 2^{2i} \sum_{|k'-k| \leq 5} \|\sqrt{\varphi_{i,k'}} u(\tau)\|_{L^2(\mathbb{R}^2)}^2 + \|\sqrt{\varphi_{i,k}} \nabla u(\tau)\|_{L^2(\mathbb{R}^2)}^2\right) \, d\tau.
$$

Repeating the same argument as used in (2.21), we can show that

$$
-\frac{1}{2} \int_{\partial B_{2^{-i}}(k)} |u|^2 \nabla \varphi_{i,k} \cdot n \, dS \leq C \sum_{|k'-k| \leq 2} \left(\|\sqrt{\varphi_{i,k'}} u\|_{L^2(\mathbb{R}^2)}^2 + \|\sqrt{\varphi_{i,k}} \nabla u\|_{L^2(\mathbb{R}^2)}^2\right), \quad (3.12)
$$
Combining (3.4), (3.7), (3.11) and (3.12) gives

\[
\sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k}} u(t) \|_{L^2(\mathbb{R}^2)}^2 + 2 \sum_{k \in \mathbb{Z}^2} \int_0^t \| \sqrt{\varphi_{i,k}} (\nabla u)(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\
+ 2\lambda_1 2^{2i} \sum_{k \in \mathbb{Z}^2} \int_0^t \| \sqrt{\varphi_{i,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k}} u_0 \|_{L^2(\mathbb{R}^2)}^2 + \sum_{k \in \mathbb{Z}^2} \int_0^t \int_{\mathbb{R}^2} \varphi_{i,k} u \cdot (j \times B) \, dx \, d\tau
\]

\[
+ C \int_0^t \| u(\tau) \|_{L^2(\mathbb{R}^2)} \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau + C \sum_{k \in \mathbb{Z}^2} \sum_{|k' - k| \leq 2} \int_0^t \| \sqrt{\varphi_{i,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
+ C \int_0^t \| j(\tau) \|_{L^2(\mathbb{R}^2)}^2 \| B(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau + C \int_0^t \| u(\tau) \|_{L^2(\mathbb{R}^2)} \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
+ \varepsilon \sum_{k \in \mathbb{Z}^2} \int_0^t \left( \lambda_1 2^{2i} \sum_{|k' - k| \leq 5} \| \sqrt{\varphi_{i,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 + \| \sqrt{\varphi_{i,k}} \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \right) \, d\tau
\]

\[
+ C \sum_{k \in \mathbb{Z}^2} \sum_{|k' - k| \leq 2} \int_0^t \| \sqrt{\varphi_{i,k}} \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau.
\]

Inserting estimate (3.3) into the above estimate, we readily have

\[
\| u(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau + \lambda_1 \sup_{i \in \mathbb{Z}^2} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + C (1 + \| u \|_{L^2_t L^\infty_x(\mathbb{R}^2)}) \| \nabla u \|_{L^2_t L^2_x(\mathbb{R}^2)}^2 + C \| B \|_{L^2_t L^2_x(\mathbb{R}^2)} \| j \|_{L^2_t L^2_x(\mathbb{R}^2)}^2
\]

\[
+ C \sup_p \| S_{p-1} j \|_{L^2_t L^\infty_x(\mathbb{R}^2)} \left( \sum_{|q| \geq N} \| \hat{j}_q \|_{L^1_t L^2_x(\mathbb{R}^2)}^2 \right)^{1/2} \| B \|_{L^t L^2_x(\mathbb{R}^2)}
\]

\[
+ C N \sup_p \| S_{p-1} j \|_{L^2_t L^\infty_x(\mathbb{R}^2)} \| j \|_{L^2_t L^2_x(\mathbb{R}^2)} \| B \|_{L^\infty_t L^2_x(\mathbb{R}^2)}
\]

\[
+ \varepsilon \lambda_1 2^{2i} \sum_{k \in \mathbb{Z}^2} \sum_{|k' - k| \leq 3} \int_0^t \| \sqrt{\varphi_{i,k'}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau.
\]

Moreover, by estimate (3.11), we have

\[
\| u(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau + \lambda_1 \sup_{i \in \mathbb{Z}^2} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \| \sqrt{\varphi_{i,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C t + C \sup_p \| S_{p-1} j \|_{L^2_t L^\infty_x(\mathbb{R}^2)} \left( \sum_{|q| \geq N} \| \hat{j}_q \|_{L^1_t L^2_x(\mathbb{R}^2)}^2 \right)^{1/2} \| B \|_{L^t L^2_x(\mathbb{R}^2)}
\]

\[
+ C N \sup_p \| S_{p-1} j \|_{L^2_t L^\infty_x(\mathbb{R}^2)} \cdot
\]

where the positive constant $C$ only depends on the initial data, independent of $t$. 
Next, applying $\dot{\Delta}_q$ to the second equation and the third equation, respectively, and taking $L^2$-norm of the resulting equations, we have
\begin{align}
\sup_{\tau \in [0,t]} \left\| \dot{\Delta}_q E(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 + \sup_{\tau \in [0,t]} \left\| \dot{\Delta}_q B(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left\| \dot{\Delta}_q j(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\
\leq \left\| \dot{\Delta}_q E_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \dot{\Delta}_q B_0 \right\|_{L^2(\mathbb{R}^2)}^2 - \int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q (u \times B) \cdot \dot{\Delta}_q j \, dx \, d\tau.
\end{align}

Lemma 3.5. There holds that
\begin{align}
\int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q (u \times B) \cdot \dot{\Delta}_q j \, dx \, d\tau \\
\leq \sup_{p \in \mathbb{Z}} \left\| \dot{S}_p u \right\|_{L^2(\mathbb{R}^2)} \left\| \dot{\Delta}_q j \right\|_{L^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \left\| \dot{\Delta}_p B \right\|_{L^3(\mathbb{R}^2)} + C \int_0^t C_q \left\| \dot{\Delta}_q j(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| B(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla u(\tau) \right\|_{L^2(\mathbb{R}^2)} \, d\tau,
\end{align}
where $c_q \in \ell^2$.

Proof of Lemma 3.5. Thanks to the Bony decomposition, we decompose $u^\ell B^m$ into three parts:
\[ u^\ell B^m = \dot{T}_u^\ell B^m + \dot{T}_{B^m} u^\ell + \dot{R}(u^\ell, B^m). \]

By the Hölder inequality, we have
\begin{align}
\int_{\mathbb{R}^2} \dot{\Delta}_q \left( \dot{T}_{B^m} u^\ell \right) \dot{\Delta}_q j^i \, dx &= \int_{\mathbb{R}^2} \dot{\Delta}_q \left( \sum_{|p-q| \leq 5} \dot{S}_{p-1} B^m \dot{\Delta}_p u^\ell \right) \dot{\Delta}_q j^i \, dx \\
&\leq \sum_{|p-q| \leq 5} \left\| \dot{S}_{p-1} B \right\|_{L^\infty(\mathbb{R}^2)} \left\| \dot{\Delta}_p u \right\|_{L^2(\mathbb{R}^2)} \left\| \dot{\Delta}_q j \right\|_{L^2(\mathbb{R}^2)} \\
&= \sum_{|p-q| \leq 5} 2^{-p} \left\| \dot{S}_{p-1} B \right\|_{L^\infty(\mathbb{R}^2)} 2^p \left\| \dot{\Delta}_p u \right\|_{L^2(\mathbb{R}^2)} \left\| \dot{\Delta}_q j \right\|_{L^2(\mathbb{R}^2)} \\
&\leq C c_q \left\| \dot{\Delta}_q j(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| B(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla u(\tau) \right\|_{L^2(\mathbb{R}^2)},
\end{align}
where $c_q \in \ell^2$.

This estimate means
\begin{align}
\int_0^t \int_{\mathbb{R}^2} \dot{\Delta}_q \left( \dot{T}_{B^m} u^\ell \right) \dot{\Delta}_q j^i \, dx \, d\tau \\
&\leq C \int_0^t C_q \left\| \dot{\Delta}_q j(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| B(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla u(\tau) \right\|_{L^2(\mathbb{R}^2)} \, d\tau.
\end{align}

By the Hölder inequality again and the Bernstein inequality, the remainder term can be bounded as follows:
\begin{align}
\int_{\mathbb{R}^2} \dot{\Delta}_q \left( \dot{R}(B^m, u^\ell) \right) \dot{\Delta}_q j^i \, dx &= \int_{\mathbb{R}^2} \dot{\Delta}_q \left( \sum_{p \geq q-5} \dot{\Delta}_p B^m \dot{\Delta}_p u^\ell \right) \dot{\Delta}_q j^i \, dx \\
&\leq \sum_{p \geq q-5} \left\| \dot{\Delta}_q \left( \dot{\Delta}_p B^m \dot{\Delta}_p u \right) \right\|_{L^2(\mathbb{R}^2)} \left\| \dot{\Delta}_q j \right\|_{L^2(\mathbb{R}^2)}
Moreover, by the Young inequality, we obtain
\[
\int_{\mathbb{R}^2} \Delta_q \left( \tilde{R}(B^m, u^\ell) \right) \Delta_q j^i \, dx \leq \sum_{p \geq q - 5} 2^q \left\| \Delta_q (\tilde{\Delta}_p B \tilde{\Delta}_p u) \right\|_{L^1(\mathbb{R}^2)} \left\| \Delta_q j \right\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq \sum_{p \geq q - 5} 2^q \left\| \tilde{\Delta}_p B \right\|_{L^2(\mathbb{R}^2)} \left\| \tilde{\Delta}_p u \right\|_{L^2(\mathbb{R}^2)} \left\| \Delta_q j \right\|_{L^2(\mathbb{R}^2)}.
\]
where \( c_q \in \ell^2 \).

Thus we have
\[
\int_0^t \int_{\mathbb{R}^2} \Delta_q \left( \tilde{R}(B^m, u^\ell) \right) \Delta_q j^i \, dx \, d\tau \leq C \int_0^t c_q \| \Delta_q j(\tau) \|_{L^2(\mathbb{R}^2)} \| B(\tau) \|_{L^2(\mathbb{R}^2)} \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)} \, d\tau.
\]  \( (3.19) \)

At last, we deal with the para-product term involving the low frequency of \( u \). Note that
\[
\int_0^t \int_{\mathbb{R}^2} \Delta_q (\tilde{T}_u B^m) \Delta_q j^i \, dx \, d\tau = \int_0^t \int_{\mathbb{R}^2} \Delta_q \left( \sum_{|p-q| \leq 5} \tilde{S}_{p-1} u^\ell \tilde{\Delta}_p B^m \right) \Delta_q j^i \, dx \, d\tau
\]
\[
\leq \sum_{|p-q| \leq 5} \left\| \tilde{S}_{p-1} u \right\|_{L^2 L^\infty(\mathbb{R}^2)} \left\| \tilde{\Delta}_p B \right\|_{L^2 L^2(\mathbb{R}^2)} \left\| \Delta_q j \right\|_{L^2 L^2(\mathbb{R}^2)}
\]
\[
\leq \sup_{p \in \mathbb{Z}} \left\| \tilde{S}_p u \right\|_{L^2 L^\infty(\mathbb{R}^2)} \left\| \Delta_q j \right\|_{L^2 L^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \left\| \tilde{\Delta}_p B \right\|_{L^2 L^2(\mathbb{R}^2)}.
\]
Combining this estimate with estimate (3.18) and estimate (3.19) implies the required result. \( \square \)

Now, we come back to the proof of Proposition 3.2. Inserting estimate (3.17) into (3.16) leads to
\[
\sup_{\tau \in [0,t]} \left\| \Delta_q E(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 + \sup_{\tau \in [0,t]} \left\| \Delta_q B(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left\| \Delta_q j(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]
\[
\leq \left\| \Delta_q E_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \Delta_q B_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \sup_{p \in \mathbb{Z}} \left\| \tilde{S}_p u \right\|_{L^2 L^\infty(\mathbb{R}^2)} \left\| \Delta_q j \right\|_{L^2 L^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \left\| \tilde{\Delta}_p B \right\|_{L^2 L^2(\mathbb{R}^2)}
\]
\[
+ C \int_0^t c_q \left\| \Delta_q j(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| B(\tau) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla u(\tau) \right\|_{L^2(\mathbb{R}^2)} \, d\tau.
\]
Summing the above inequality over $q \in \mathbb{Z}$ provides us

\[
\|E\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}^2 + \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}^2 + \int_0^t \|\dot{j}(\tau)\|_{L_t^2(\mathbb{R}^2)}^2 \, d\tau \\
\leq \|E_0\|_{L_t^2(\mathbb{R}^2)}^2 + \|B_0\|_{L_t^2(\mathbb{R}^2)}^2 + C \int_0^t \sum_{q \in \mathbb{Z}} c_q \|\Delta_q \dot{j}\|_{L_t^2(\mathbb{R}^2)} \|B\|_{L_t^2(\mathbb{R}^2)} \|\nabla u\|_{L_t^2(\mathbb{R}^2)} \, d\tau \\
+ \sup_{p \in \mathbb{Z}} \|\dot{S}_p u\|_{L_t^2 L_t^\infty(\mathbb{R}^2)} \sum_{q \in \mathbb{Z}} \left( \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \|\Delta_p B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \right) .
\]  

(3.20)

On one hand, by the Hölder inequality, one has

\[
\int_0^t \sum_{q \in \mathbb{Z}} c_q \|\Delta_q \dot{j}(\tau)\|_{L_t^2(\mathbb{R}^2)} \|B(\tau)\|_{L_t^2(\mathbb{R}^2)} \|\nabla u(\tau)\|_{L_t^2(\mathbb{R}^2)} \, d\tau \\
\leq C \int_0^t \|\dot{j}(\tau)\|_{L_t^2(\mathbb{R}^2)} \|B(\tau)\|_{L_t^2(\mathbb{R}^2)} \|\nabla u(\tau)\|_{L_t^2(\mathbb{R}^2)} \, d\tau \\
\leq C \|\dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \|B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \|\nabla u\|_{L_t^2 L_t^2(\mathbb{R}^2)} .
\]  

(3.21)

On the other hand, the high-low frequency technique enables us to infer that

\[
\sum_{q \in \mathbb{Z}} \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \|\Delta_p B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \\
= \sum_{|q| > N} \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \|\Delta_p B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \\
+ \sum_{-N \leq q \leq N} \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \sum_{|p-q| \leq 5} \|\Delta_p B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \\
\leq \left( \sum_{|q| \geq N} \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)} + CN \|B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \|\dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} ,
\]  

(3.22)

where the positive integer $N$ to be fixed later.

Plugging both estimates (3.21), (3.22) in (3.20) yields

\[
\|E\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}^2 + \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}^2 + \int_0^t \|\dot{j}(\tau)\|_{L_t^2(\mathbb{R}^2)}^2 \, d\tau \\
\leq \|E_0\|_{L_t^2(\mathbb{R}^2)}^2 + \|B_0\|_{L_t^2(\mathbb{R}^2)}^2 + \sup_p \|\dot{S}_p u\|_{L_t^2 L_t^\infty(\mathbb{R}^2)} \left( \sum_{|q| \geq N} \|\Delta_q \dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)} \\
+ CN \sup_p \|\dot{S}_p u\|_{L_t^2 L_t^\infty(\mathbb{R}^2)} \|B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \|\dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \\
+ C \|\dot{j}\|_{L_t^2 L_t^2(\mathbb{R}^2)} \|B\|_{L_t^\infty L_t^2(\mathbb{R}^2)} \|\nabla u\|_{L_t^2 L_t^2(\mathbb{R}^2)} .
\]  

(3.23)
This together with estimate (3.15) entails
\[
\|(E, B)\|^2_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)} + \|u(t)\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|j(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau
\]
\[
+ \int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau + \lambda_1 \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{i,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau
\]
\[
\leq C t + C \sup_{p \geq -1} \|S_p |u|\|_{L_t^p L^\infty(\mathbb{R}^2)} \left( \sum_{|q| \geq N} \|\hat{\Delta}_q j\|^2_{L_t^2 L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}
\]
\[
+ C N \sup_{p \geq -1} \|S_p |u|\|_{L_t^2 L^\infty(\mathbb{R}^2)}.
\]
By resorting to Lemma 2.3, we readily have
\[
\sup_{p \geq -1} \|S_p |u|\|_{L_t^p L^\infty(\mathbb{R}^2)} \leq C \sup_{p \geq -1} \left( 2^{2p} \int_0^t \left( \sup_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{p,k}} u(\tau)\|_{L^2(\mathbb{R}^2)} \right)^2 \ d\tau \right)^{\frac{1}{2}}
\]
\[
\leq C \sup_{p \geq -1} \left( 2^{2p} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{p,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \sup_{p \geq -1} 2^{2p} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{p,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau \right)^{\frac{1}{2}}.
\]
Inserting this estimate into (3.24) and using the Cauchy-Schwarz inequality, we immediately have
\[
\|(E, B)\|^2_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)} + \|B\|^2_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)} + \|u(t)\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|j(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau
\]
\[
+ \int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau + \lambda_1 \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{i,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau
\]
\[
\leq C + \tilde{C} \left( \sum_{|q| \geq N} \|\hat{\Delta}_q j\|^2_{L_t^2 L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \left( \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{i,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau \right)^{\frac{1}{2}} \|B\|_{L_t^\infty \dot{B}_{2,2}^0(\mathbb{R}^2)}
\]
\[
+ C N^2 + \frac{\lambda_1}{8} \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{i,k}} u(\tau)\|^2_{L^2(\mathbb{R}^2)} \ d\tau.
\]
(3.25)
Since
\[
\sum_{q \in \mathbb{Z}} \int_0^\infty \|\hat{\Delta}_q j(t)\|^2_{L^2(\mathbb{R}^2)} \ dt \leq C \int_0^\infty \|j(t)\|^2_{L^2(\mathbb{R}^2)} \ dt
\]
\[
\leq C \left( \|u_0\|^2_{L^2(\mathbb{R}^2)} + \|B_0\|^2_{L^2(\mathbb{R}^2)} + \|E_0\|^2_{L^2(\mathbb{R}^2)} \right),
\]
we can choose the integer $N$ sufficiently large such that
\[
\tilde{C} \left( \sum_{|q| \geq N} \|\hat{\Delta}_q j\|^2_{L_t^2 L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \leq \frac{\lambda_1}{8}.
\]
From this, estimate (3.25) reduces to
\[
\|E\|_{L^2_t L^2_{x,t}(\mathbb{R}^2)}^2 + \|B\|_{L^2_t B^0_{2,2}(\mathbb{R}^2)}^2 + \lambda_1 \sup_{i \in \mathbb{Z}} 2^{2i} \int_0^t \sum_{k \in \mathbb{Z}^2} \|\sqrt{\phi_{i,k}} u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau \leq C. \quad (3.26)
\]
So, we complete the proof of this proposition. \qed

Based on the estimates for \( u \) established in Proposition 3.2, we further show the globally-in-time a priori estimates for smooth solutions in the borderline space.

**Proposition 3.6.** Let \( u_0 \in L^2(\mathbb{R}^2) \), \( B_0 \in L^2_{\text{log}}(\mathbb{R}^2) \) and \( E_0 \in L^2_{\text{log}}(\mathbb{R}^2) \). Then, for any smooth solution \((u, E, B)\), there exists a positive constant \( C \) such that
\[
\|E\|_{L^\infty_t L^2_{\text{log}}(\mathbb{R}^2)}^2 + \|B\|_{L^\infty_t L^2_{\text{log}}(\mathbb{R}^2)}^2 + \int_0^t \|\tilde{j}(\tau)\|_{L^2_{\text{log}}(\mathbb{R}^2)}^2 \, d\tau \leq C(t, \|u_0\|_{L^2(\mathbb{R}^2)}, \|(E_0, B_0)\|_{L^2_{\text{log}}(\mathbb{R}^2)}) .
\]

**Proof.** First of all, the same argument as in proving (3.16) provides
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q E(t)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q B(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta_q E(t)\|_{L^2(\mathbb{R}^2)}^2 = -\int_{\mathbb{R}^2} \Delta_q (u \times B) \cdot \Delta_q E \, dx .
\]
Thanks to the Bony-paraproduct decomposition, the integral in the right side of the above equality can be written as
\[
\int_{\mathbb{R}^2} \Delta_q (u^m B^t) \Delta_q E^i \, dx = \int_{\mathbb{R}^2} \Delta_q (\hat{T}_u B^m) \Delta_q E^i \, dx + \int_{\mathbb{R}^2} \Delta_q (\hat{T}_{B^m} u^t) \Delta_q E^i \, dx + \int_{\mathbb{R}^2} \Delta_q \hat{R}(u^t, B^m) \Delta_q E^i \, dx .
\]
By the H"older inequality, we find that for \( q > 5 \),
\[
\int_{\mathbb{R}^2} \Delta_q (\hat{T}_{B^m} u^t) \Delta_q E^i \, dx \leq \sum_{|q' - q| \leq 5} \|\hat{S}_{q'-1} B\|_{L^1(\mathbb{R}^2)} \|\Delta_q u\|_{L^2(\mathbb{R}^2)} \|\Delta_q E\|_{L^2(\mathbb{R}^2)} \leq C c_q \sup_{q \geq 1} \sqrt{q^2 - \frac{q}{q'}} \|\hat{S}_{q'-1} B\|_{L^1(\mathbb{R}^2)} \|u\|_{B^s_{2,2}(\mathbb{R}^2)} \|\Delta_q E\|_{L^2(\mathbb{R}^2)} , \quad (3.27)
\]
where \( c_q \in \ell^2 \).
Note that
\[
\sup_{q \geq 1} \sqrt{q^2 - \frac{q}{q'}} \|\hat{S}_{q'-1} B\|_{L^1(\mathbb{R}^2)} \leq \sup_{q \geq 1} \sqrt{q^2 - \frac{q}{q'}} \sum_{1 \leq i \leq q - 2} \|\Delta_i B\|_{L^4(\mathbb{R}^2)} + \sup_{q \geq 1} \sqrt{q^2 - \frac{q}{q'}} \|\hat{S}_0 B\|_{L^4(\mathbb{R}^2)} \leq \sum_{1 \leq i \leq k-2} \sqrt{k - i} \times 2^{-\frac{k-i}{2}} \sqrt{2} \frac{1}{2} \|\Delta_i B\|_{L^4(\mathbb{R}^2)} + C \|B\|_{L^2(\mathbb{R}^2)} \leq \sum_{1 \leq i \leq k-2} \frac{k - i}{2} \sqrt{2} \frac{1}{2} \|\Delta_i B\|_{L^4(\mathbb{R}^2)} + C \|B\|_{L^2(\mathbb{R}^2)} + \sum_{1 \leq i \leq k-2} \frac{k - i}{2} \sqrt{2} \frac{1}{2} \|\Delta_i B\|_{L^4(\mathbb{R}^2)} + C \|B\|_{L^2(\mathbb{R}^2)} .
\[ \leq \sup_{i \geq 1} \sqrt{i} 2^{-\frac{i}{2}} \| \Delta_i B \|_{L^4(\mathbb{R}^2)} + C \| B \|_{L^2(\mathbb{R}^2)} \leq C \| B \|_{L^2_{\log}(\mathbb{R}^2)}. \]

Plugging this estimate in (3.27) and applying the Cauchy-Schwarz inequality to the resulting estimate, we get

\[
\int_{\mathbb{R}^2} \hat{\Delta_q} (T_{B^m} u^t) \hat{\Delta_q} E^i \, dx \leq \frac{C_{c_q}}{\sqrt{q}} \| B \|_{L^2_{\log}(\mathbb{R}^2)} \| u \|_{L^4_{\tau,1}(\mathbb{R}^2)} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)}
\leq \frac{C_{c_q}}{\sqrt{q}} \| B \|_{L^2_{\log}(\mathbb{R}^2)} \| \nabla u \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)}
\leq \frac{C_{c_q}^2}{q} \| \nabla u \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)}^2. \tag{3.28}
\]

As for the remainder term, it can be bounded as follows:

\[
\int_{\mathbb{R}^2} \hat{\Delta_q} \hat{R}(u^t, B^m) \hat{\Delta_q} E^i \, dx = \sum_{q' \geq q-5} \int_{\mathbb{R}^2} \hat{\Delta_q} \left( \hat{\Delta_q} u^t \hat{\Delta_q} B^m \right) \hat{\Delta_q} E^i \, dx
\leq C \sum_{q' \geq q-5} 2^q \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)} \tag{3.29}
\]

A simple calculation yields that for \( q > 5 \),

\[
\sum_{q' \geq q-5} 2^q \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)}
= \frac{1}{\sqrt{q}} \sum_{q' \geq q-5} \frac{\sqrt{q}}{\sqrt{q'}} 2^q - q' \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \sqrt{q} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)}
\leq \frac{1}{\sqrt{q}} \sum_{q' \geq q-5} \sqrt{q - q'} 2^q - q' \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \sqrt{q} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)}
+ \frac{1}{\sqrt{q}} \sum_{q' \geq q-5} 2^q - q' \sqrt{q} \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \sqrt{q} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)}.
\]

Moreover, by the discrete Young inequality, one has that for \( q > 5 \),

\[
\sum_{q' \geq q-5} 2^q \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)} = \sum_{q' \geq q-5} 2^q - q' \| \hat{\Delta_q} u \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta_q} B \|_{L^2(\mathbb{R}^2)}
\leq \frac{C_{c_q}}{\sqrt{q}} \| \nabla u \|_{L^2(\mathbb{R}^2)} \| B \|_{L^2_{\log}(\mathbb{R}^2)},
\]

where \( c_q \in l^2 \).

Inserting this estimate into (3.29) leads to

\[
\int_{\mathbb{R}^2} \hat{\Delta_q} \hat{R}(u^t, B^m) \hat{\Delta_q} E^i \, dx \leq \frac{C_{c_q}}{\sqrt{q}} \| \nabla u \|_{L^2(\mathbb{R}^2)} \| B \|_{L^2_{\log}(\mathbb{R}^2)} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)}
\leq \frac{C_{c_q}^2}{q} \| \nabla u \|_{L^2(\mathbb{R}^2)}^2 \| B \|_{L^2_{\log}(\mathbb{R}^2)}^2 + \frac{1}{4} \| \hat{\Delta_q} E \|_{L^2(\mathbb{R}^2)}^2. \tag{3.30}
\]
Lastly, we tackle with the para-product term $\int_{\mathbb{R}^2} \hat{\Delta}_q (\hat{T}_u^t B^m) \hat{\Delta}_q E^i \, dx$. We see that for $q > 5$,

$$
\int_{\mathbb{R}^2} \hat{\Delta}_q (\hat{T}_u^t B^m) \hat{\Delta}_q E^i \, dx = \sum_{|k-q| \leq 5} \int_{\mathbb{R}^2} \hat{\Delta}_q (\hat{S}_{k-1} u^f \hat{\Delta}_k B^m) \hat{\Delta}_q E^i \, dx \\
\leq \sum_{|k-q| \leq 5} \| \hat{S}_{k-1} u \|_{L^\infty(\mathbb{R}^2)} \| \hat{\Delta}_k B \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta}_q E \|_{L^2(\mathbb{R}^2)} \\
\leq C \| \hat{S}_{q+5} u \|_{L^\infty(\mathbb{R}^2)} \sum_{|k-q| \leq 5} \| \hat{\Delta}_k B \|_{L^2(\mathbb{R}^2)} \| \hat{\Delta}_q E \|_{L^2(\mathbb{R}^2)} \\
\leq \frac{CC_q}{\sqrt{q}} \| \hat{S}_{q+5} u \|_{L^\infty(\mathbb{R}^2)} \| B \|_{L^2_{\log}(\mathbb{R}^2)} \| \hat{\Delta}_q E \|_{L^2(\mathbb{R}^2)}.
$$

By the Cauchy-Schwarz inequality, we readily have that for $q > 5$,

$$
\int_{\mathbb{R}^2} \hat{\Delta}_q (u \times B) \cdot \hat{\Delta}_q E \, dx \leq \| \hat{S}_{q+5} u \|_{L^\infty(\mathbb{R}^2)}^2 \| \hat{\Delta}_q E \|_{L^2(\mathbb{R}^2)}^2 + \frac{CC_q^2}{q} \| B \|_{L^2_{\log}(\mathbb{R}^2)}^2.
$$

Collecting all these estimates (3.28), (3.30), (3.31) yields that for $q > 5$,

$$
\frac{d}{dt} \| \hat{\Delta}_q E(t) \|_{L^2(\mathbb{R}^2)}^2 + \frac{d}{dt} \| \hat{\Delta}_q B(t) \|_{L^2(\mathbb{R}^2)}^2 + \| \hat{\Delta}_q E(t) \|_{L^2(\mathbb{R}^2)}^2 \\
\leq \left( \| \hat{\Delta}_q E(0, B_0) \|_{L^2(\mathbb{R}^2)}^2 + C \int_0^t \| B(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 \, d\tau \right)
$$

In view of the Gronwall inequality, we immediately have that for $q > 5$,

$$
\frac{d}{dt} \| \hat{\Delta}_q E(t) \|_{L^2(\mathbb{R}^2)}^2 + \| \hat{\Delta}_q B(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \hat{\Delta}_q E(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\
\leq e^{Ct} \| \hat{\Delta}_q E(0, B_0) \|_{L^2(\mathbb{R}^2)}^2 + C \int_0^t \| B(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 \, d\tau
$$

Note that

$$
\int_0^t \| \hat{S}_{q+5} u(\tau) \|_{L^\infty(\mathbb{R}^2)}^2 \, d\tau \leq C2^{2q} \int_0^t \sup_{k \in \mathbb{Z}^2} \| \sqrt{\phi_{q,k}} u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau
$$

Multiplying (3.32) by $q$ and summing the resulting inequality over $q > 5$, we get by using the $L^2$-estimate (3.11) that

$$
\sum_{q > 5} q \| \hat{\Delta}_q E \|_{L^\infty L^2(\mathbb{R}^2)}^2 + \sum_{q > 5} q \| \hat{\Delta}_q B \|_{L^\infty L^2(\mathbb{R}^2)}^2 + \sum_{q > 5} q \int_0^t \| \hat{\Delta}_q E(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\
+ \| u(t) \|_{L^2(\mathbb{R}^2)}^2 + \| E(t) \|_{L^2(\mathbb{R}^2)}^2 + \| B(t) \|_{L^2(\mathbb{R}^2)}^2 + \| E(t) \|_{L^2(\mathbb{R}^2)}^2
$$

$$
\leq C \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + C \| (E_0, B_0) \|_{L^2_{\log}(\mathbb{R}^2)}^2 + C \int_0^t \| B(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 \, d\tau
$$
\[ + C \int_0^t \| \nabla u(\tau) \|^2_{L^2(\mathbb{R}^2)} \| B(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau. \]

Since
\[
\| E \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \| B \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \int_0^t \| E(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau
\]
\[
\leq \sum_{q > 5} q \| \hat{\Delta}_q E \|^2_{L^\infty_t L^2(\mathbb{R}^2)} + \sum_{q > 5} q \| \hat{\Delta}_q B \|^2_{L^\infty_t L^2(\mathbb{R}^2)} + \sum_{q > 5} q \int_0^t \| \hat{\Delta}_q E(\tau) \|^2_{L^2(\mathbb{R}^2)} \, d\tau
\]
\[
+ C \| u(t) \|^2_{L^2(\mathbb{R}^2)} + C \| E(t) \|^2_{L^2(\mathbb{R}^2)} + C \| B(t) \|^2_{L^2(\mathbb{R}^2)} + C \| E(t) \|^2_{L^2(\mathbb{R}^2)},
\]
we have
\[
\| E \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \| B \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \int_0^t \| E(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau
\]
\[
\leq C + C \int_0^t \| B(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau + C \int_0^t \| \nabla u(\tau) \|^2_{L^2(\mathbb{R}^2)} \| B(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau.
\]
By the Gronwall inequality again, we eventually get that
\[
\| E \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \| B \|^2_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} + \int_0^t \| E(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau
\]
\[
\leq C e^{C t} + C \int_0^t \| \nabla u(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau.
\]
Based on this regularity, we turn to show that \( \int_0^t \| j(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau < \infty \). Since \( j = E + u \times B \) and \( \int_0^t \| E(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau < \infty \), we just need to show that \( \int_0^t \| (u \times B)(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau < \infty \). Thanks to the Bony paraproduct decomposition, one writes
\[ u^t B^i = \hat{T}_{u^t} B^i + \hat{T}_{B^i} u^t + \hat{R}(u^t, B^i). \]
According to the definition of \( L^2_{\log}(\mathbb{R}^2) \), we have
\[
\int_0^t \| \hat{T}_{u^t} B^i(\tau) \|^2_{L^2_{\log}(\mathbb{R}^2)} \, d\tau
\]
\[
\leq C \int_0^t \sum_{q \leq 5} \| \hat{\Delta}_q (\hat{T}_{u^t} B^i) \|^2_{L^2(\mathbb{R}^2)} \, d\tau + C \int_0^t \sum_{q > 5} q \| \hat{\Delta}_q (\hat{T}_{u^t} B^i) \|^2_{L^2(\mathbb{R}^2)} \, d\tau
\]
\[
:= I + II.
\]
By the H"older inequality, we immediately have
\[
I \leq C \sum_{q \leq 5} \sum_{|p-q| \leq 5} \int_0^t \| \hat{S}_{p-q} u(\tau) \|^2_{L^\infty(\mathbb{R}^2)} \| \hat{\Delta}_p B(\tau) \|^2_{L^2(\mathbb{R}^2)} \, d\tau
\]
\[
\leq C \sup_{p \in \mathbb{Z}} \int_0^t \| \hat{S}_{p-q} u(\tau) \|^2_{L^\infty(\mathbb{R}^2)} \, d\tau \sum_{p \in \mathbb{Z}} \| \hat{\Delta}_p B \|^2_{L^\infty L^2(\mathbb{R}^2)}
\]
\[
\leq C \sup_{i \in \mathbb{Z}} 2^{2i} \sup_{k \in \mathbb{Z}^2} \| \sqrt{\phi_{i,k}} u(\tau) \|^2_{L^2(\mathbb{R}^2)} \, d\tau \| B \|^2_{L^\infty L^2(\mathbb{R}^2)}.
\]
Similarly, we have

$$II \leq C \sum_{q > 5} \sum_{|p-q| \leq 5} q \int_0^t \left\| \hat{S}_{p-1}B(\tau) \right\|_{L^\infty(\mathbb{R}^2)}^2 \left\| \hat{A}_p B(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 \, d\tau$$

$$\leq C \sup_{p \in \mathbb{Z}} \int_0^t \left\| \hat{S}_{p-1}u(\tau) \right\|_{L^\infty(\mathbb{R}^2)}^2 \, d\tau \sum_{p \geq 1} p \left\| \hat{A}_p B \right\|_{L^\infty L^2(\mathbb{R}^2)}^2$$

$$\leq C \sup_{\ell \in \mathbb{Z}} 2^{2\ell} \int_0^t \sup_{k \in \mathbb{Z}^2} \left\| \sqrt{\phi_{i,k}} u(\tau) \right\|_{L^2(\mathbb{R}^2)}^2 \, d\tau \left\| B \right\|_{L^\infty L^2(\mathbb{R}^2)}^2.$$
By the Hölder inequality and the discrete Young inequality, one has

\[
K_1 \leq C \sum_{q \leq 5} \sum_{p \geq q-5} \int_0^t 2^{2q}\|\tilde{\Delta}_p u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\tilde{\Delta}_p B(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C \sum_{q \leq 5} \sum_{p \geq q-5} \int_0^t 2^{2(q-p)}2^{2p}\|\tilde{\Delta}_q u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\tilde{\Delta}_q B(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C\|B\|_{L^p_t L^2(\mathbb{R}^2)}^2 \sum_{p \geq 1} 2^{2p}\|\tilde{\Delta}_p u\|_{L^2_t L^2(\mathbb{R}^2)}^2 \leq C\|\nabla u\|_{L^p_t L^2(\mathbb{R}^2)}^2 \|B\|_{L^p_t L^2(\mathbb{R}^2)}^2.
\]

In the similar fashion, we can obtain

\[
K_2 \leq C \sum_{q > 5} \sum_{p \geq q-5} \int_0^t 2^{2q}\|\tilde{\Delta}_p u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\tilde{\Delta}_p B(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C \sum_{q > 5} \sum_{p \geq q-5} \int_0^t 2^{2(q-p)}2^{2p}\|\tilde{\Delta}_q u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \|\tilde{\Delta}_q B(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C\|B\|_{L^p_t L^2(\log \mathbb{R}^2)}^2 \sum_{p \geq 1} 2^{2p}\|\tilde{\Delta}_p u\|_{L^2_t L^2(\mathbb{R}^2)}^2 \leq C\|\nabla u\|_{L^p_t L^2(\mathbb{R}^2)}^2 \|B\|_{L^p_t L^2(\log \mathbb{R}^2)}^2.
\]

Therefore, we finally get

\[
\int_0^t \|(u \times B)(\tau)\|_{L^2_t(\mathbb{R}^2)}^2 \, d\tau
\]

\[
\leq C \left( \sup_{i \in \mathbb{Z}} 2^{2i} \sup_{k \in \mathbb{Z}^2} \sqrt{\phi_{i,k}} \|\tilde{\Delta}_p u(\tau)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u\|_{L^p_t L^2(\mathbb{R}^2)} \right) \|B\|_{L^p_t L^2(\log \mathbb{R}^2)}^2.
\]

By Proposition 3.2 and estimate 3.33, we know that \(\int_0^t \|(u \times B)(\tau)\|_{L^2_t(\mathbb{R}^2)}^2 \, d\tau < \infty\). So, we finish the proof of the proposition. \(\square\)

Based on this regularity in the borderline space, we can show the global-in-time bound for \(\int_0^t \|u(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau\), which plays an important role in the proof of some known results such as [12].

**Proposition 3.7.** Let \(u_0 \in L^2(\mathbb{R}^2)\), \(B_0 \in L^2(\mathbb{R}^2)\) and \(E_0 \in L^2(\mathbb{R}^2)\). Then, for any smooth solution \((u, E, B)\), there holds that

\[
\int_0^t \|\tilde{u}(\tau)\|_{L^2(\mathbb{R}^2)}^2 \, d\tau \leq C(t, \|u_0\|_{L^2(\mathbb{R}^2)}, \|E_0, B_0\|_{L^2(\mathbb{R}^2)}).
\]

**Proof.** By Duhamel formula, one writes the solution \(u\) in the following form

\[
u(t, x) = u_{2d} + u_2,
\]

where

\[
u_2(t, x) = \int_0^t e^{(t-\tau)\Delta}P(j \times B)(\tau) \, d\tau
\]

and

\[
u_{2d}(t, x) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}P(u \otimes u)(\tau) \, d\tau
\]

where
which is a solution of the following equations governed by
\[
\begin{aligned}
\partial_t v + (u \cdot \nabla)u - \Delta v + \nabla \pi &= 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\text{div} v &= 0, \\
v|_{t=0} &= u_0.
\end{aligned}
\] (3.35)

First of all, we are going to show
\[
\int_0^t \| \hat{v}(\tau) \|^2_{L^1(\mathbb{R}^2)} \, d\tau < \infty,
\]
which is the direct consequence of the following proposition.

**Proposition 3.8.** Let \(v\) be a solution of the nonlinear equations (3.35). Then, we have
\[
\int_0^t \| \hat{v}(\tau) \|^2_{L^1(\mathbb{R}^2)} \, d\tau \leq C \left( \| u_0 \|_{L^2(\mathbb{R}^2)} \right).
\]

**Remark 3.9.** Let us point out that in this proposition we give a new method to show that the Leray solution of the two-dimensional Navier-Stokes equations satisfies
\[
\int_0^t \| u_{2d}(\tau) \|^2_{L^\infty(\mathbb{R}^2)} \, d\tau < \infty,
\]
which was shown in [6]. More importantly, we also prove that \(\int_0^t \| u_{2d}(\tau) \|^2_{L^1(\mathbb{R}^2)} \, d\tau < \infty\).

**Proof of Proposition 3.8.** By Duhamel formula, we have that
\[
v(x, t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \text{div}(u \otimes u) \, d\tau.
\]

Taking Fourier transform yields
\[
\hat{v}(\xi, t) = e^{-t|\xi|^2} \hat{u}_0 + \int_0^t e^{-(t-s)|\xi|^2} \left( I_d - \frac{\xi \xi_j}{|\xi|^2} \right) i\xi \cdot (\hat{u} \otimes \hat{u}) \, d\tau.
\]

For the linear part, Proposition 3.2 allows us to get
\[
\left\| e^{-t|\xi|^2} \hat{u}_0 \right\|_{L^2(\mathbb{R}^+; L^1(\mathbb{R}^2)))} = \left\| \frac{1}{t} \right\| e^{-t|\xi|^2} \hat{u}_0 \right\|_{L^1(\mathbb{R}^2)} \left\| L^2(\mathbb{R}^+; \mathbb{R}^2)) \sim \| u_0 \|_{FB^{1,2}}.
\]

For the nonlinear part, we see that
\[
\left\| e^{-(t-s)|\xi|^2} \left( I_d - \frac{\xi \xi_j}{|\xi|^2} \right) i\xi \cdot (\hat{u} \otimes \hat{u}) \, d\tau \right\|_{L^2(\mathbb{R}^+; L^1(\mathbb{R}^2)))} \leq \left\| \int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-(t-s)|\xi|^2} \right\|_{L^2(\mathbb{R}^2)} \left\| \xi \right\|^{-\frac{1}{2}} \left\| (\hat{u} \cdot \nabla) u \right\|_{L^2(\mathbb{R}^n)} \, d\tau \right\|_{L^2(\mathbb{R}^+)} \leq \left\| \int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-(t-s)|\xi|^2} \right\|_{L^2(\mathbb{R}^2)} \left\| \xi \right\|^{-\frac{1}{2}} \left\| (\hat{u} \cdot \nabla) u \right\|_{L^2(\mathbb{R}^2)} \, d\tau \right\|_{L^2(\mathbb{R}^2)}.
\]

Thanks to the Bony paraproduct decomposition and \(\text{div} u = 0\), we have the following estimate for the bilinear term
\[
\left\| |\xi|^{-\frac{1}{2}} \left( \hat{u} \cdot \nabla \right) u \right\|_{L^2(\mathbb{R}^2)} \leq C \left\| \Lambda \frac{1}{2} u(t) \right\|_{L^2(\mathbb{R}^2)} \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^2)}.
\]
Therefore, we have
\[
\left\| \int_0^t e^{-(t-\tau)|\xi|^2} \left( \mathbb{I}_d - \frac{\xi \cdot \xi}{|\xi|^2} \right) i\xi \cdot \left( \hat{u} \otimes \hat{u} \right) d\tau \right\|_{L^2(\mathbb{R}^+; L^1(\mathbb{R}))} \\
\leq C \left\| \int_0^t (t-\tau)^{-\frac{s}{2}} \| \Lambda^\frac{1}{2} u(t) \|_{L^2(\mathbb{R}^2)} \| \nabla u(t) \|_{L^2(\mathbb{R}^2)} d\tau \right\|_{L^2(\mathbb{R}^+)} \\
\leq C \left( \int_0^t \left( \int_0^x \right) \frac{1}{(x-\tau)^{\frac{s}{2}}} \| \Lambda^\frac{1}{2} u(t) \|_{L^2(\mathbb{R}^2)} \| \nabla u(t) \|_{L^2(\mathbb{R}^2)} d\tau \right) \\
\leq C \| \Lambda^\frac{1}{2} u(t) \|_{L^4(\mathbb{R}^+; L^2(\mathbb{R}^2))} \| \nabla u \|_{L^2(\mathbb{R}^+; L^2(\mathbb{R}^2))},
\]
where we have used the following lemma.

**Lemma 3.10 (9 18).**

- Let \( 1 < p, q, r < \infty, 0 < s_1, s_2 \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \) and \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{r}. \) Then there holds
  \[
  \| f \ast g \|_{L^{r,s}(\mathbb{R}^d)} \leq C(p, q, s_1, s_2) \| f \|_{L^{p,s_1}(\mathbb{R}^d)} \| g \|_{L^{q,s_2}(\mathbb{R}^d)}.
  \]
- Let \( 0 < p, q, r \leq \infty, 0 < s_1, s_2 \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \) and \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}. \) Then we have the Hölder inequality for Lorentz spaces
  \[
  \| fg \|_{L^{r,s}(\mathbb{R}^d)} \leq C(p, q, s_1, s_2) \| f \|_{L^{p,s_1}(\mathbb{R}^d)} \| g \|_{L^{q,s_2}(\mathbb{R}^d)}.
  \]

Collecting these estimates, we immediately get
\[
\left( \int_0^\infty \| \hat{u}(\tau) \|_{L^1(\mathbb{R}^2)}^2 d\tau \right)^{\frac{1}{2}} \leq C \| u_0 \|_{L^2(\mathbb{R}^2)}^2 + C \| \Lambda^\frac{1}{2} u \|_{L^4(\mathbb{R}^+; L^2(\mathbb{R}^2))} \| \nabla u \|_{L^2(\mathbb{R}^+; L^2(\mathbb{R}^2))}.
\]
This together with the energy estimate
\[
\| u(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla u(\tau) \|_{L^2(\mathbb{R}^2)}^2 d\tau \leq \| u_0 \|_{L^2(\mathbb{R}^2)}^2
\]
and the Hausdorff-Young inequality entails the desired result. \( \square \)

Next, we just need to bound the following the quantity including \( u_2. \) Taking the Fourier transform and taking \( L^1 \)-norm, we readily have
\[
\left( \int_0^t \| \hat{u}_2(s) \|_{L^1(\mathbb{R}^2)}^2 ds \right)^{\frac{1}{2}} \leq \left( \int_0^t \left( \int_0^s e^{-(s-\tau)|\xi|^2} \mathcal{F}(\hat{u}_2(B)(\tau)) d\tau \right)^2 \right)^{\frac{1}{2}} ds \\
\leq C \left( \int_0^t \left( \int_0^s e^{-(s-\tau)|\xi|^2} \mathcal{F}(\hat{u}_2(B)(\tau)) d\tau \right)^2 \right)^{\frac{1}{2}} ds.
\]
The inhomogeneous Bony paraproduct decomposition allows us to write
\[
\left( \int_0^t \left( \int_0^s e^{-(s-\tau)|\xi|^2} \mathcal{F}(\hat{u}_2(B)(\tau)) d\tau \right)^2 \right)^{\frac{1}{2}} ds \\
= \left( \int_0^t \left( \int_0^s e^{-(s-\tau)|\xi|^2} \mathcal{F}(\hat{T}_j B^m)(\tau) d\tau \right)^2 \right)^{\frac{1}{2}} ds.
\]
where $1 \leq j, m \leq 3$.

By the Minkowski inequality, the Hölder inequality and the Young inequality, we have

\[
I \leq \left\| \left( \int_0^t \int_0^s e^{-(s-r)|x|^2} (\widehat{T^m_{j}} B^m)(\tau) \, d\tau \right)^2 \right\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}}
\]

\[
\leq C \left\| \frac{1}{|x|^2} \left\| \widehat{T^m_{j}} B^m \right\|_{L^1(\mathbb{R}^2)} \right\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}}
\]

\[
\leq C \sum_{q=-1}^{\infty} 2^{-2q} \left\| S_q^{-1} \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \left( \sum_{q=-1}^{\infty} \left\| \Delta_q \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2} \right)^{\frac{1}{2}}.
\]

Note that

\[
\sum_{q=-1}^{\infty} 2^{-2q} \left\| S_q^{-1} \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2} \leq C \sum_{q=-1}^{\infty} 2^{-2q} \left\| \Delta_q \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2}
\]

\[
\leq C \sum_{q=-1}^{\infty} \left\| \Delta_q \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2}.
\]

Plugging this estimate in (3.36) and using the Plancherel theorem, we see that

\[
I \leq C \left( \sum_{q=-1}^{\infty} \left\| \Delta_q \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2} \right)^{\frac{1}{2}} \left( \sum_{q=-1}^{\infty} \left\| \Delta_q \hat{B} \right\|_{L^2(\mathbb{R}^2)}^{2} \right)^{\frac{1}{2}}
\]

\[
\leq C \| \hat{B} \|_{L^2(\mathbb{R}^2)} \| \hat{B} \|_{L^2(\mathbb{R}^2)} \left\| B \right\|_{L^\infty(\mathbb{R}^2)} \leq C \| \hat{B} \|_{L^2(\mathbb{R}^2)} \left\| B \right\|_{L^\infty(\mathbb{R}^2)}.
\]

In the similar way, one has

\[
II \leq C \| \hat{B} \|_{L^2(\mathbb{R}^2)} \left\| B \right\|_{L^\infty(\mathbb{R}^2)}.
\]
It remains for us to bound the term $III$. We bound it as follows:

$$III \leq C \left( \int_0^t \left\| \int_0^s e^{-(s-r)|\xi|^2} R(j^\ell, \hat{B}^m)(\tau) \, d\tau \right\|_{L^1(\mathbb{R}^2)}^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq C \sum_{p=1}^{\infty} \left( \int_0^t \left\| \int_0^s \psi_p e^{-(s-r)|\xi|^2} R(j^\ell, \hat{B}^m)(\tau) \, d\tau \right\|_{L^1(\mathbb{R}^2)}^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq C \sum_{p=1}^{\infty} \left( \int_0^t \left\| \int_0^s 2^{2p} e^{-c2^2p(s-r)} R(j^\ell, \hat{B}^m)(\tau) \, d\tau \right\|_{L^{\infty}(\mathbb{R}^2)}^2 \, ds \right)^{\frac{1}{2}}. \quad (3.37)$$

By the Young inequality and the Hölder inequality, one has

$$\sum_{p=1}^{\infty} \left( \int_0^t \left\| \int_0^s 2^{2p} e^{-c2^2p(s-r)} \| \psi_p R(j^\ell, \hat{B}^m) \|_{L^{\infty}(\mathbb{R}^2)}(\tau) \, d\tau \right\|_{L^2(\mathbb{R}^2)}^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq \sum_{p=1}^{\infty} \left( \int_0^t \| \psi_p R(j^\ell, \hat{B}^m) \|_{L^{\infty}(\mathbb{R}^2)}(\tau) \, d\tau \right)^{\frac{1}{2}}$$

$$\leq \sum_{p=1}^{\infty} \sum_{k\geq p-5} \| \hat{\Delta_k} j \|_{L^2_t L^2(\mathbb{R}^2)} \| \hat{\Delta_k} B \|_{L^\infty_t L^2(\mathbb{R}^2)}$$

Inserting this estimate into $(3.37)$ and using Fubini theorem, we readily have

$$III \leq C \sum_{p=1}^{\infty} \sum_{k\geq p-5} \| \hat{\Delta_k} j \|_{L^2_t L^2(\mathbb{R}^2)} \| \hat{\Delta_k} B \|_{L^\infty_t L^2(\mathbb{R}^2)}$$

$$= C \sum_{k=1}^{\infty} \sum_{1 \leq p \leq k+5} \| \hat{\Delta_k} j \|_{L^2_t L^2(\mathbb{R}^2)} \| \hat{\Delta_k} B \|_{L^\infty_t L^2(\mathbb{R}^2)}$$

$$= C \sum_{k=1}^{\infty} (k + 1) \| \hat{\Delta_k} j \|_{L^2_t L^2(\mathbb{R}^2)} \| \hat{\Delta_k} B \|_{L^\infty_t L^2(\mathbb{R}^2)}$$

$$\leq C \| j \|_{L^2_t L^2_{\log}(\mathbb{R}^2)} \| B \|_{L^\infty_t L^2_{\log}(\mathbb{R}^2)}.$$ 

Collecting estimates for $I, II, III$, we end the proof of this proposition. \qed

4. Proof of main results

In this section, we are going to show the main theorems. Let us begin with the uniqueness of solution.

4.1. Uniqueness. This subsection is devoted to prove the uniqueness of solutions established in our theorems. To do this, it suffices to show the following proposition.
Proposition 4.1. Let $E, B, \tilde{E}, \tilde{B} \in C_b([0, T]; L^2_{\log}(\mathbb{R}^2))$, $j, \tilde{j} \in L^2([0, T]; L^2_{\log}(\mathbb{R}^2))$ and $u, \tilde{u} \in C_b([0, T]; L^2(\mathbb{R}^2)) \cap L^2([0, T]; H^1(\mathbb{R}^2))$ satisfying

$$
\int_0^T \| (u, \tilde{u})(\tau) \|_{L^\infty(\mathbb{R}^2)}^2 \, d\tau < \infty.
$$

Assume that $(u, E, B, p)$ and $(\tilde{u}, \tilde{E}, \tilde{B}, \tilde{p})$ be two solutions of system (1.1) associated with the same initial data. Then $(u, E, B, p) \equiv (\tilde{u}, \tilde{E}, \tilde{B}, \tilde{p})$ on interval $[0, T]$.

Proof. Letting $(\delta u, \delta E, \delta B, \delta p) := (u - \tilde{u}, E - \tilde{E}, B - \tilde{B}, \pi - \tilde{\pi})$, then we easily find that the difference $(\delta u, \delta E, \delta B, \delta p)$ satisfies

$$
\begin{aligned}
&\partial_t \delta u + (u \cdot \nabla) \delta u - \Delta \delta u + \nabla \delta \pi = j \times \delta B + \delta j \times \tilde{B} - (\delta u \cdot \nabla)\tilde{u} \\
&\partial_t \delta E - \text{curl} \delta B = -\delta j, \\
&\partial_t \delta B + \text{curl} \delta E = 0 \\
&\text{div } \delta u = \text{div } \delta B = 0,
\end{aligned}
$$

(4.1)

where $\delta j = \delta E + \delta u \times B + \tilde{u} \times \delta B$. It corresponds to the following initial condition

$$(\delta u, \delta E, \delta B)|_{t=0} = (0, 0, 0).$$

Taking the standard $L^2$-estimate of $\delta u$ yields

$$
\frac{1}{2} \frac{d}{dt} \| \delta u(t) \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla \delta u(t) \|_{L^2(\mathbb{R}^2)}^2
$$

$$
= \int_{\mathbb{R}^2} (j \times B) \cdot \delta u \, dx + \int_{\mathbb{R}^2} (j \times \delta B) \cdot \delta u \, dx - \int_{\mathbb{R}^2} (\delta u \cdot \nabla)\tilde{u} \cdot \delta u \, dx.
$$

By the Hölder inequality and the interpolation theorem, we see that

$$
- \int_{\mathbb{R}^2} (\delta u \cdot \nabla)\tilde{u} \cdot \delta u \, dx \leq \int_{\mathbb{R}^2} |\delta u|^2 |\nabla \tilde{u}| \, dx
$$

$$
\leq \| \nabla \tilde{u} \|_{L^2(\mathbb{R}^2)} \| \delta u \|_{L^4(\mathbb{R}^2)}^2
$$

$$
\leq C \| \nabla \tilde{u} \|_{L^2(\mathbb{R}^2)} \| \delta u \|_{L^2(\mathbb{R}^2)} \| \nabla \delta u \|_{L^2(\mathbb{R}^2)}
$$

$$
\leq C \| \nabla \tilde{u} \|_{L^2(\mathbb{R}^2)}^2 \| \delta u \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \| \nabla \delta u \|_{L^2(\mathbb{R}^2)}^2.
$$

By the same argument as in the proof of Proposition 3.6, we have

$$
\int_{\mathbb{R}^2} (j \times B) \cdot \delta u \, dx + \int_{\mathbb{R}^2} (j \times \delta B) \cdot \delta u \, dx
$$

$$
\leq C \left( \| (B, j)(t) \|_{L^2_{\log}(\mathbb{R}^2)}^2 + \| u(t) \|_{L^\infty(\mathbb{R}^2)}^2 \right) \left( \| \delta B(t) \|_{L^2_{\log}(\mathbb{R}^2)}^2 + \| \delta E(t) \|_{L^2_{\log}(\mathbb{R}^2)}^2 \right)
$$

$$
+ \frac{1}{4} \| \nabla \delta u \|_{L^2(\mathbb{R}^2)}^2.
$$

Collecting the above estimates, we readily have

$$
\| \delta u(t) \|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \| \nabla \delta u(s) \|_{L^2(\mathbb{R}^2)}^2 \, ds
$$

$$
\leq \int_0^t C \left( \| (B, j)(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 + \| u(\tau) \|_{L^\infty(\mathbb{R}^2)}^2 \right) \left( \| \delta B(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 + \| \delta E(\tau) \|_{L^2_{\log}(\mathbb{R}^2)}^2 \right) \, d\tau
$$

$$
+ C \int_0^t \| \nabla \tilde{u}(\tau) \|_{L^2(\mathbb{R}^2)}^2 \| \delta u(\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau.
$$

(4.2)
Taking $L^2_{\log}$-norm of $(\delta E, \delta B)$ and integrating the resulting equality with respect to time $t$, we obtain
\[
\|\delta E(t)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \|\delta B(t)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \int_0^t \|\delta E(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau
\]
\[
\leq \sum_{q=-\infty}^0 \int_0^t \int_{\mathbb{R}^2} |\Delta_q(\delta u \times B)\Delta_q\delta E| \, dx \, d\tau + \sum_{q=-\infty}^0 \int_0^t \int_{\mathbb{R}^2} |\Delta_q(u \times \delta B)\Delta_q\delta E| \, dx \, d\tau
\]
\[
+ \sum_{q=1}^\infty q \int_0^t \int_{\mathbb{R}^2} |\Delta_q(\delta u \times B)\Delta_q\delta E| \, dx \, d\tau + \sum_{q=1}^\infty q \int_0^t \int_{\mathbb{R}^2} |\Delta_q(u \times \delta B)\Delta_q\delta E| \, dx \, d\tau.
\]
By the same argument as in the proof of Proposition 3.6, we have
\[
\sum_{q=-\infty}^0 \int_0^t \int_{\mathbb{R}^2} |\Delta_q(\delta u \times B)\Delta_q\delta E| \, dx \, d\tau + \sum_{q=-\infty}^0 \int_0^t \int_{\mathbb{R}^2} |\Delta_q(u \times \delta B)\Delta_q\delta E| \, dx \, d\tau
\]
\[
+ \sum_{q=1}^\infty q \int_0^t \int_{\mathbb{R}^2} |\Delta_q(\delta u \times B)\Delta_q\delta E| \, dx \, d\tau + \sum_{q=1}^\infty q \int_0^t \int_{\mathbb{R}^2} |\Delta_q(u \times \delta B)\Delta_q\delta E| \, dx \, d\tau
\]
\[
\leq C \int_0^t \|B(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)}\|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau + C \int_0^t \|\tilde{u}(\tau)\|^2_{L^\infty(\mathbb{R}^2)}\|\delta B(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau
\]
\[
+ \frac{1}{4} \int_0^t \|\delta E(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau.
\]
Collecting all estimates of $(\delta E, \delta B)$ gives
\[
\|\delta E(t)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \|\delta B(t)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \int_0^t \|\delta E(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau
\]
\[
\leq C \int_0^t \|B(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)}\|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau + C \int_0^t \|\tilde{u}(\tau)\|^2_{L^\infty(\mathbb{R}^2)}\|\delta B(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau
\]
This estimate together with (4.2) enables us to conclude that
\[
\|\delta u(t)\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau + \|\delta E(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \int_0^t \|\delta E(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau
\]
\[
\leq \int_0^t C\|(B, j)(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \|u(\tau)\|^2_{L^\infty(\mathbb{R}^2)}\|(\delta B(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} + \|\delta j(\tau)\|^2_{L^2_{\log}(\mathbb{R}^2)} d\tau.
\]
Since $(\delta u, \delta E, \delta B)|_{t=0} = (0, 0, 0)$, there exists a time $t_0 \in [0, T]$ such that
\[
\|(\delta u, \delta E, \delta B)(t)\|_{L^2} \equiv 0 \text{ on } t \in [0, t_0] \quad \text{and} \quad \|(\delta u, \delta E, \delta B)(t)\|_{L^2} > 0 \text{ on } (t_0, T).
\]
If $t_0 = T$ then the uniqueness follows. Therefore, we assume $t_0 < T$. By the Gronwall inequality, it follows that $(\delta u, \delta E, \delta B) \equiv 0$ on $[t_0, T]$. So, we eventually get the uniqueness of solution. ☐
4.2. **Existence.** In this subsection, we focus on the existence statement of Theorem 4.1. To do this, we will adopt the following approximate scheme:

\[
\begin{cases}
\partial_t u^N + (u^N \cdot \nabla) u^N - \Delta u^N + \nabla p^N = j^N \times B^N, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t E^N - \text{curl } B^N = j^N, \\
\partial_t B^N + \text{curl } E^N = 0,
\end{cases}
\]

(4.3)

where the constant \( C \) is determined by the solution \((u^N, E^N, B^N)\) of (4.3).

Since \((u_0, B_0, E_0) \in \left( L^2(\mathbb{R}^3) \right)^3\), we have \((u_0, B_0, E_0) \in \cap_{s>0} \left( H^s(\mathbb{R}^3) \right)^3\). From the main theorem proved in [16], we know that the approximate system (4.3) exists a unique global solution \((u^N, E^N, B^N) \in \left( C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \right)^3\) for all \( s > 0 \). Thanks to some a priori estimates established in Section 3, it follows from the Fatou lemma that

\[
\| u^N(t) \|^2_{L^2(\mathbb{R}^2)} + \| (E^N, B^N) \|^2_{L^2_{\tau} L^2_{\phi}(\mathbb{R}^2)} + \int_0^t \| \nabla u^N(\tau) \|^2_{L^2(\mathbb{R}^2)} d\tau + \lambda_1 \sup_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \| \sqrt{\phi_{i,k}} u^N(\tau) \|^2_{L^2(\mathbb{R}^2)} d\tau + \int_0^t \| j^N(\tau) \|^2_{L^2_{\phi}(\mathbb{R}^2)} d\tau \leq C,
\]

(4.4)

where the constant \( C \) does not depend on parameter \( N \).

Letting \( u^{M,N} = u^M - u^N, E^{M,N} = E^M - E^N \) and \( B^{M,N} = B^M - B^N \), we see that the triple \((u^{M,N}, E^{M,N}, B^{M,N})\) solves the following system in \(\mathbb{R}^+ \times \mathbb{R}^2\):

\[
\begin{cases}
\partial_t u^{M,N} + (u^M \cdot \nabla) u^{M,N} - \Delta u^{M,N} + \nabla p^{M,N} = j^M \times B^{M,N} + j^{M,N} \times B^N + (u^{M,N} \cdot \nabla) u^N, \\
\partial_t E^{M,N} - \text{curl } B^{M,N} = -j^{M,N}, \\
\partial_t B^{M,N} + \text{curl } E^{M,N} = 0,
\end{cases}
\]

(4.5)

By the same argument in proving the uniqueness, we can infer that

\[
\| u^{M,N}(t) \|^2_{L^2} + \int_0^t \| \nabla u^{M,N}(\tau) \|^2_{L^2} d\tau + \sup_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \| \phi_{i,k} u^{M,N}(\tau) \|^2_{L^2} d\tau
\]

\[
+ \| E^{M,N} \|^2_{L^2_{\tau} B^0_{1,2}} + \| B^{M,N} \|^2_{L^2_{\tau} B^0_{1,2}} + \int_0^t \| E^{M,N}(\tau) \|^2_{L^2} d\tau 
\]

\[
\leq C \int_0^t \| \nabla u^N(\tau) \|^2_{L^2} \| B^{M,N}(\tau) \|^2_{L^2} d\tau + C \int_0^t \| j^N(\tau) \|^2_{L^2} + \| B^N(\tau) \|^2_{L^2} \| B^{M,N}(\tau) \|^2_{L^2} d\tau
\]

\[
+ C \sup_{q \in \mathbb{Z}} \int_0^t \| \dot{S}_q u^N(\tau) \|^2_{L^\infty} d\tau \int_0^t \| E^{M,N}(\tau) \|^2_{L^2} d\tau + C \int_0^t \| \nabla u^N(\tau) \|^2_{L^2} \| u^{M,N}(\tau) \|^2_{L^2} d\tau
\]

\[
+ \| (u_0^{M,N}, E_0^{M,N}, B_0^{M,N}) \|^2_{L^2}.
\]
Performing the Gronwall inequality and using the uniform estimate (4.4), we get
\[
\|u^{M,N}(t)\|_{L_2}^2 + \int_0^t \|\nabla u^{M,N}(\tau)\|_{L_2}^2 d\tau + \sup_i 2^{2i} \int_0^t \sum_k \|\varphi_{i,k} u^{M,N}(\tau)\|_{L_2}^2 d\tau 
\]
\[
+ \|E^{M,N}\|_{L_\infty_t B^0_{2,2}}^2 + \|B^{M,N}\|_{L_\infty_t B^0_{2,2}}^2 + \int_0^t \|E^{M,N}(\tau)\|_{L_2}^2 d\tau 
\leq \|(u_0^{M,N}, E_0^{M,N}, B_0^{M,N})\|_{L_2}^2 e^{Ct}.
\]
This implies that \(\{(u^N, B^N, E^N)\}_{N=1}^\infty\) is a Cauchy sequence in the Banach space
\[
X := (L_\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1)) \times L_\infty([0,T]; L^2) \times L_\infty([0,T]; L^2).
\]
Therefore, there exists a strong limit \((u, E, B)\) such that
\[
u^N \to u \in L_\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1) \quad \text{as} \quad N \to \infty; \quad (4.7)
\]
\[
E^N \to E \in L_\infty([0,T]; L^2) \quad \text{as} \quad N \to \infty; \quad (4.8)
\]
\[
B^N \to B \in L_\infty([0,T]; L^2) \quad \text{as} \quad N \to \infty. \quad (4.9)
\]
Next, we want to show that
\[
j^N \to j \in L^2([0,T]; L^2) \quad \text{as} \quad N \to \infty. \quad (4.10)
\]
Note that \(j^N = \sigma(E^N + u^N \times B^N)\). Since (4.8) holds, we just need to show that
\[
u^N \times B^N \to u \times B \in L^2([0,T]; L^2) \quad \text{as} \quad N \to \infty.
\]
One writes
\[
u^N \times B^N - u \times B = (u^N - u) \times B^N + u \times (B^N - B).
\]
With the help of the Bony para-product decomposition and the Hölder inequality, we can show that
\[
\|(u^N - u) \times B^N\|_{L^2([0,T]; L^2)} 
\leq C \|B^N\|_{L_\infty([0,T]; L^2)} \left( \|(u^N - u)\|_{L^2([0,T]; \dot{H}^1)} + \sup_r \frac{1}{r^2} \int_0^t \sum_k \|\varphi_{r,k}(u^N - u)(\tau)\|_{L_2}^2 d\tau \right).
\]
This together with the uniform estimate (4.4), (4.7) and estimate (4.6) entails
\[
\|(u^N - u) \times B^N\|_{L^2([0,T]; L^2)} \to 0 \quad \text{as} \quad N \to \infty.
\]
In the same way, we have
\[
\|u \times (B^N - B)\|_{L^2([0,T]; L^2)} \to 0 \quad \text{as} \quad N \to \infty.
\]
Hence, we have the required convergence (4.10). The main task is now to show that \((u, E, B)\) is a solution of system (1.1) in the sense of distribution. Let the vector \(\omega \in \mathcal{S} (\mathbb{R}^2)\) satisfying \(\text{div } \omega = 0\), and \(\vartheta(t) \in \mathcal{D} ([0, T])\). Then, we have
\[
\langle u^N(0), \omega \rangle \vartheta(0) + \int_0^T \langle u^N(t), \omega \rangle \vartheta(t) dt 
\]
\[
+ \int_0^T \langle u^N, (u^N \cdot \nabla) \rangle \vartheta(t) dt + \int_0^T \langle j^N \times B^N, \omega \rangle \vartheta(t) dt = 0;
\]
\[
\langle E^N(0), \omega \rangle \vartheta(0) + \int_0^T \langle B^N(t), \text{curl } \omega \rangle \vartheta(t) dt = \int_0^T \langle j^N, \omega \rangle \vartheta(t) dt;
\]
By the Hölder inequality, one has
\[ \langle B^N(0), \omega \rangle \vartheta(0) - \int_0^T \langle E^N(t), \text{curl} \omega \rangle \vartheta(t) \, dt = 0, \]
where \( \langle, \rangle \) denotes the standard \( L^2 \)-inner product.

For the linear term, it is easy to show that, as \( N \to \infty \),
\[ \langle u^N(0), \omega \rangle \vartheta(0) + \int_0^T \langle u^N(t), \omega \rangle \vartheta(t) \, dt \to \langle u(0), \omega \rangle \vartheta(0) + \int_0^T \langle u(t), \omega \rangle \vartheta(t) \, dt; \]
\[ \langle E^N(0), \omega \rangle \vartheta(0) + \int_0^T \langle B^N(t), \text{curl} \omega \rangle \vartheta(t) \, dt \to \langle E(0), \omega \rangle \vartheta(0) + \int_0^T \langle B(t), \text{curl} \omega \rangle \vartheta(t) \, dt; \]
\[ \langle B^N(0), \omega \rangle \vartheta(0) - \int_0^T \langle E^N(t), \text{curl} \omega \rangle \vartheta(t) \, dt \to \langle B(0), \omega \rangle \vartheta(0) - \int_0^T \langle E(t), \text{curl} \omega \rangle \vartheta(t) \, dt \]
and
\[ \int_0^T \langle j^N, \omega \rangle \vartheta(t) \, dt \to \int_0^T \langle j, \omega \rangle \vartheta(t) \, dt. \]

So, we need to show that as \( N \to \infty \),
\[ \int_0^T \langle u^N, (u^N \cdot \nabla) \omega \rangle \vartheta(t) \, dt + \int_0^T \langle j^N \times B^N, \omega \rangle \vartheta(t) \, dt \]
\[ \to \int_0^T \langle u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt + \int_0^T \langle j \times B, \omega \rangle \vartheta(t) \, dt. \]

A simple calculation yields
\[ \int_0^T \langle u^N - u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt \]
\[ = \int_0^T \langle u^N - u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt + \int_0^T \langle u, ((u - u^N) \cdot \nabla) \omega \rangle \vartheta(t) \, dt. \]

By the Hölder inequality, one has
\[ \int_0^T \langle u^N - u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt \]
\[ \leq \| \vartheta \|_L^1(\mathbb{R}^+) \| \nabla \omega \|_{L^\infty(\mathbb{R}^2)} \| u^N - u \|_{L^\infty([0,T]; L^2(\mathbb{R}^2))} \| u \|_{L^\infty([0,T]; L^2(\mathbb{R}^2))}. \]

This combined with the uniform estimate (1.4) and (1.7) leads to
\[ \int_0^T \langle u^N - u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt \to 0 \quad \text{as} \quad N \to \infty. \]

Performing the same argument, we can obtain
\[ \int_0^T \langle u, ((u - u^N) \cdot \nabla) \omega \rangle \vartheta(t) \, dt \to 0 \quad \text{as} \quad N \to \infty. \]

Thus, we have
\[ \int_0^T \langle u^N, (u^N \cdot \nabla) \omega \rangle \vartheta(t) \, dt \to \int_0^T \langle u, (u \cdot \nabla) \omega \rangle \vartheta(t) \, dt \quad \text{as} \quad N \to \infty. \]

Similarly, we have
\[ \int_0^T \langle j^N \times B^N, \omega \rangle \vartheta(t) \, dt \to \int_0^T \langle j \times B, \omega \rangle \vartheta(t) \, dt \quad \text{as} \quad N \to \infty. \]
From above, we show that \((u, E, B)\) is a distributional solution of system (1.1).

Now, we begin to show the time continuity of solution. Since \(u \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}^0_{2,2}(\mathbb{R}^2))\), there exists a positive integer \(N\) such that
\[
\sum_{k \geq N} \|\Delta_k u\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))}^2 < \frac{\varepsilon}{2}.
\] (4.11)

For all \(t_1, t_2 \in \mathbb{R}^+\), we assume \(t_2 > t_1\) without lose of generality. By computations, one has
\[
\|S_N(u(t_2) - u(t_1))\|_{L^2} \leq \int_{t_1}^{t_2} \|S_N \partial_t u(\tau)\|_{L^2} d\tau.
\] (4.12)

Recall that
\[
\partial_t u = \Delta u - \mathbb{P}(u \cdot \nabla) + j \times B.
\]
It follows form the Bernstein inequality that
\[
\|S_N \partial_t u\|_{L^2(\mathbb{R}^+; L^2)} \leq \|S_N \Delta u\|_{L^2(\mathbb{R}^+; L^2)} + \|\mathbb{P} S_N ((u \cdot \nabla) + j \times B)\|_{L^2(\mathbb{R}^+; L^2)}
\leq C2^N \|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} + C \|S_N ((u \cdot \nabla) + j \times B)\|_{L^2(\mathbb{R}^+; L^2)}
\leq C2^N \|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} + C^2 \|S_N ((u \cdot \nabla) + j \times B)\|_{L^2(\mathbb{R}^+; L^2)}\]
\[
\leq C2^N \|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} + C\|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} \|u\|_{L^\infty(\mathbb{R}^+; L^2)}
\leq C2^N \|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} + C\|\nabla u\|_{L^2(\mathbb{R}^+; L^2)} < \infty.
\]

Inserting this estimate into (4.12) leads to
\[
\|S_N(u(t_2) - u(t_1))\|_{L^2} \leq C \|S_N \partial_t u\|_{L^2(\mathbb{R}^+; L^2)}(t_2 - t_1)^{\frac{1}{2}}.
\]
According to the low-high decomposition technique and (4.11), we obtain
\[
\|u(t_2) - u(t_1)\|_{L^2} \leq \|S_N(u(t_2) - u(t_1))\|_{L^2} + 2 \sum_{k \geq N} \|\Delta_k u\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))}^2
\leq C(t_2 - t_1)^{\frac{1}{2}} + \varepsilon.
\]
This implies \(u(t) \in C(\mathbb{R}^+; L^2(\mathbb{R}^2))\). In the same way as used for \(u\), we can obtain that \(E(t) \in C(\mathbb{R}^+; L^2_{\log}(\mathbb{R}^2))\) and \(B(t) \in C(\mathbb{R}^+; L^2_{\log}(\mathbb{R}^2))\).

Now, we begin to show the existence statement in Theorem 1.1. By the compact argument, we know that system (1.1) admits a unique global-in-time solution \((u, B, E)\). By Proposition 3.6 and Corollary 3.7, we get from Fatou’s lemma that \(\|(E, B)\|_{L^\infty_t L^2_{\log}(\mathbb{R}^2)} \leq C(t)\) and \(u \in L^2_{\text{loc}}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))\). Thus, we finish the proof of our theorems.

Appendix

In this appendix, we will give a lemma and a proposition which have been used in Section 3.

Lemma E.1 ([2], Lemma 2.35). For any positive \(s\), there holds that
\[
\sup_{t > 0} \sum_{j \in \mathbb{Z}} t^s 2^{2js} e^{-c t 2^{2j}} < \infty.
\]
Proposition E.2. Let $s$ be a positive real number and $(p, r) \in [1, \infty]^2$. Then there exists a constant $C > 0$ such that

$$C^{-1} \|u\|_{F_{B^p_{r,s}}^s} \leq \left\| t^s |\xi|^2 \hat{u} \right\|_{L_p^{r, (r + \frac{d}{p})}} \leq C \|u\|_{F_{B^p_{r,s}}^s} \quad \text{for all } u \in S'_h.$$

Proof. According to the support property of $\varphi_j$, we see that

$$\left\| t^s \varphi_j e^{-t|\xi|^2} \hat{u} \right\|_{L_p^r} \leq C t^s 2^{js} e^{-ct2^j} 2^{-2js} \left\| \Delta_j u \right\|_{L_p^r}.$$ 

Using the fact that $u \in S'_h$ and the definition of the homogeneous Fourier-Herz spaces, we have

$$t^s |\xi|^2 \hat{u} \left\|_{L_p^r} \leq \sum_{j \in \mathbb{Z}} t^s \varphi_j e^{-t|\xi|^2} \hat{u} \left\|_{L_p^r} \leq C \|u\|_{F_{B^p_{r,s}}^s} \sum_{j \in \mathbb{Z}} t^s 2^{js} e^{-ct2^j} c_{r,j},$$

where $\|c_{r,j}\|_r = 1$.

If $r = \infty$, then the inequality readily follows from Lemma E.1.

If $r < \infty$, then using the Hölder inequality and Lemma E.1, we obtain

$$\int_0^\infty t^s \|e^{-t|\xi|^2} \hat{u}\|_{L_p^r} \frac{dt}{t} \leq C \|u\|_{F_{B^p_{r,s}}^s} \int_0^\infty \left( \sum_{j \in \mathbb{Z}} t^s 2^{js} e^{-ct2^j} c_{r,j} \right)^{r-1} \left( \sum_{j \in \mathbb{Z}} t^s 2^{js} e^{-ct2^j} c_{r,j} \right)^{-1} \frac{dt}{t} \leq C \|u\|_{F_{B^p_{r,s}}^s} \int_0^\infty \sum_{j \in \mathbb{Z}} t^s 2^{js} e^{-ct2^j} c_{r,j} \frac{dt}{t}.$$ 

Using Fubini’s theorem, one infers that

$$\int_0^\infty t^s \|e^{-t|\xi|^2} \hat{u}\|_{L_p^r} \frac{dt}{t} \leq C \|u\|_{F_{B^p_{r,s}}^s} \sum_{j \in \mathbb{Z}} c_{r,j} \int_0^\infty t^s 2^{js} e^{-ct2^j} \frac{dt}{t} \leq C T(s) \|u\|_{F_{B^p_{r,s}}^s} \quad \text{with} \quad T(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$ 

To prove the other inequality, we use the following identity

$$\widehat{\Delta_j u} = \frac{1}{\Gamma(s + 1)} \int_0^\infty t^s |\xi|^{s+1} e^{-t|\xi|^2} \Delta_j \hat{u} \frac{dt}{t}.$$ 

Since $e^{-t|\xi|^2} = e^{-\frac{t}{2} |\xi|^2} e^{-\frac{t}{2} |\xi|^2}$, we have

$$\left\| \Delta_j u \right\|_{L_p^r} \leq C \int_0^\infty t^s 2^{j(s+1)} e^{-ct2^j} \left\| \Delta_j e^{-\frac{t}{2} \Delta} \hat{u} \right\|_{L_p^r} \frac{dt}{t} \leq C \int_0^\infty t^s 2^{j(s+1)} e^{-ct2^j} \left\| e^{-t \Delta} \hat{u} \right\|_{L_p^r} \frac{dt}{t}.$$
If $r = \infty$, then we have
\[
\|\hat{\Delta}_j u\|_{L^p} \leq C \left( \sup_{t > 0} t^s \|e^{-t\Delta} u\|_{L^p} \right) \int_0^\infty 2^{2j(s+1)} e^{-ct2^j} \, dt \\
\leq C 2^{2js} \left( \sup_{t > 0} t^s \|e^{-t\Delta} u\|_{L^p} \right).
\]
If $r < \infty$, we write
\[
\sum_{j \in \mathbb{Z}} 2^{-2j r} \|\hat{\Delta}_j u\|_{L^p}^r \leq C \sum_{j \in \mathbb{Z}} 2^{2j r} \left( \int_0^\infty t^s e^{-ct2^j} \|e^{-t\Delta} u\|_{L^p} \, dt \right)^{r/2}.
\]
The Hölder inequality with the weight $e^{-ct2^j}$ implies that
\[
\left( \int_0^\infty t^s e^{-ct2^j} \|e^{-t\Delta} u\|_{L^p} \, dt \right)^r \leq \left( \int_0^\infty e^{-ct2^j} \, dt \right)^{r-1} \int_0^\infty t^s e^{-ct2^j} \|e^{-t\Delta} u\|_{L^p} \, dt \\
\leq C 2^{-2j(r-1)} \int_0^\infty t^r s e^{-ct2^j} \|e^{-t\Delta} u\|_{L^p} \, dt.
\]
By resorting to Lemma E.1 and the Fubini theorem, we readily get
\[
\sum_{j \in \mathbb{Z}} 2^{-2j r} \|\hat{\Delta}_j u\|_{L^p}^r \leq C \sum_{j \in \mathbb{Z}} 2^{-2j} \int_0^\infty t^r s e^{-ct2^j} \|e^{-t\Delta} u\|_{L^p} \, dt \\
\leq C \int_0^\infty \left( \sum_{j \in \mathbb{Z}} 2^{-2j} e^{-ct2^j} \right) t^r \|e^{-t\Delta} u\|_{L^p} \, dt \\
\leq C \int_0^\infty t^r \|e^{-t\Delta} u\|_{L^p} \, dt.
\]
The proposition is thus proved. \hfill \Box

REFERENCES

[1] R. A. ADAMS and J. J.F. FOURNIER, Sobolev spaces, Second Edition, Academic Press, Amsterdam, 2003.
[2] H. BAHOURI, J.-Y. CHEMIN and R. DANCHIN, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der mathematischen Wissenschaften 343, Springer-Verlag, 2011.
[3] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge Monographs on Plasma Physics, vol. 1, Cambridge University Press, Cambridge, 1993.
[4] M. Cannone, Ondelettes, paraproducts et Navier-Stokes. Nouveaux essais, Diderot éditeurs, Paris, 1995.
[5] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations. Handbook of Mathematical fluid Dynamics, Vol. III, North-Holland, Amsterdam, 2004.
[6] J-Y. CHEMIN and I. GALLAGHER, On the global wellposedness of the 3-D Navier-Stokes equations with large initial data, Ann. Sci. cole Norm. Sup. (3) 39 (2006), no. 4, 679–698.
[7] P.A. Davidson, An Introduction to Magnetohydrodynamics, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
[8] L C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Volume 19, American Mathematical Society Providence, Rhode Island, 1998.
[9] L. GRAFAKOS, Classical Fourier Analysis, Second edition, Graduate Texts in Mathematics 249, Springer, New York, 2008.
[10] S.Ibrahim, N. Masmoudi and P.G. Lemarié-Rieusset, Time-periodic forcing and asymptotic stability for the Navier-Stokes-Maxwell equations, arXiv:1601.06458.
[11] P. G. Lemarié-Rieusset, Recent Developments in the Navier–Stokes Problem, Chapman & Hall/CRC Press, Boca Raton, 2002.
[12] P. Germain, S. Ibrahim and N. Masmoudi, Well-posedness of the Navier-Stokes-Maxwell equations, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 144 (2014), 71–86.
[13] S. Ibrahim and S. Keraani, Global small solution of the Maxwell-Navier-Stokes equations, SIAM J. Math. Anal. 43 (2011), 2275–2295.
[14] A. Kufner, O. John and S. Fučík, Functional spaces, Prague, Academia, 1977.
[15] J. Leray, Essai sur le mouvement dun liquide visqueux emplissant lespace, Acta Mathematica, 63, (1933), 193–248.
[16] N. Masmoudi, Global well posedness for the Maxwell-Navier-Stokes system in 2D, J. Math. Pures. Appl. 93 (2010), 559–571.
[17] C. Miao, J. Wu and Z. Zhang, Littlewood-Paley Theory and Applications to Fluid Dynamics Equations, Monographs on Modern pure mathematics, No.142. Science Press, Beijing, 2012.
[18] R. O’Neil, Convolution operators and $L(p,q)$ spaces, Duke Math. J. 30 (1963), 129–142.

(C. Miao) Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, P.R. China.
E-mail address: miao_changxing@iapcm.ac.cn

(X. Zheng) School of Mathematics and Systems Science, Beihang University, Beijing 100191, P.R. China
E-mail address: xiaoxinzheng@buaa.edu.cn