THE ATIYAH-HITCHIN BRACKET FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION.

I. GENERAL POTENTIALS.

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Abstract. This is the first in a series of papers on Poisson formalism for the cubic nonlinear Schrödinger equation with repulsive nonlinearity and its relation to complex geometry. In this paper we study general continuous potentials. We demonstrate that the Weyl functions of the corresponding auxiliary Dirac spectral problem carry a natural Poisson structure. We call it the Atiyah–Hitchin Poisson bracket. We show that the Poisson bracket on the phase space is the image of the Atiyah–Hitchin bracket on Weyl functions under the inverse spectral transform.

1. Introduction.

1.1. Statement of the problem. All 1+1 differential equations like the Korteweg-de-Vries, the modified Korteweg-de-Vries, the cubic nonlinear Schrödinger equation, the Toda lattice, the Camassa–Holm equation etc., which are analyzed by the inverse spectral transform, are Hamiltonian systems.

The cubic NLS with repulsive nonlinearity

\[ i\psi = -\psi'' + 2|\psi|^2\psi, \]

where \( \psi(x,t) \) is a complex function, will serve as our model example. We consider this problem on the entire line, i.e., \( x \in \mathbb{R} \). We assume that the phase space \( \mathcal{M} \) consists of functions \( \psi(x) \) and we do not impose any condition on \( \psi(x) \) except continuity. We restrict our attention to some specific function classes, e.g., periodic or rapidly decaying, in subsequent papers. To make apparent the algebraic nature of our considerations, we assume that all functionals \( A, B : \mathcal{M} \to \mathbb{C} \) are Frechet differentiable, all integrals converge, etc.

The cubic NLS equation is a Hamiltonian system

\[ \psi^* = \{\psi, \mathcal{H}\}, \]

with the classical bracket

\[ \{A, B\} = 2i \int \frac{\delta A}{\delta \psi(x)} \frac{\delta B}{\delta \psi(x)} dx, \]

(1.1)

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\(^1\)Prime ‘ signifies the derivative in the variable \( x \) and dot • the derivative with respect to time.
and Hamiltonian
\[ H = \frac{1}{2} \int |\psi'|^2 + |\psi|^4 \, dx. \]

The NLS equation arises as a compatibility condition for the commutator relation for some specially chosen differential operators. This leads to an auxiliary linear spectral problem for the Dirac operator
\[ \mathcal{D} f = \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i \partial_x + \begin{pmatrix} 0 & -i\psi \\ i\psi & 0 \end{pmatrix} \right] f = \frac{\lambda}{2} f \]
acting in a space of vector functions
\[ f(x, \lambda) = \begin{bmatrix} f_1(x, \lambda) \\ f_2(x, \lambda) \end{bmatrix}. \]

The goal of this and subsequent papers is to demonstrate that the Hamiltonian formalism (Poisson bracket) is built into the complex geometry of the Dirac spectral problem.

1.2. Description of results. We associate to a potential \( \psi(x) \) of the Dirac spectral problem a pair \((\Gamma, \mathcal{X})\), i.e., we have a map:
\[ \mathcal{M} \rightarrow (\Gamma, \mathcal{X}). \]

We call this map the direct spectral transform. In this pair \( \Gamma \) is a two-sheeted covering of the complex plane of the spectral parameter cut along the real line (see Figure 1). For any point \( Q \in \Gamma \), its projection to the plane of the spectral parameter \( \lambda \) is denoted by \( \lambda(Q) \). The holomorphic function \( \mathcal{X}(Q) \) defined on \( \Gamma \) is the so-called Weyl function of the Dirac spectral problem. We show that the pair

![Figure 1. The spectral cover.](image-url)
\( (\Gamma, \mathcal{X}) \) carries a natural Poisson structure. We call it the Atiyah-Hitchin bracket. Apart from insignificant constants the AH bracket is given by the formula
\[
\{ \mathcal{X}(Q), \mathcal{X}(P) \} = \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda(Q) - \lambda(P)}.
\]
(1.2)

This is the first theorem of the paper.

It turns out that the direct spectral transform can be inverted
\[(\Gamma, \mathcal{X}) \longrightarrow \mathcal{M}.\]
We call this map the inverse spectral transform. We show that the image of the AH bracket under the inverse spectral transform becomes the bracket 1.1 on phase space. This is our second theorem.

The pair \((\Gamma, \mathcal{X})\) introduced in this paper is a skeleton of an analogous construction for periodic and scattering potentials in subsequent papers [14] and [15]. In these cases there is an additional structure which allows one to identify the values of the meromorphic function \(\mathcal{X}\) on different banks of the cut.

The question of construction of canonical coordinates can not be resolved without assumptions on a potential. Canonical coordinates are given in terms of singularities of the Weyl functions on the real line, or in other words in terms of the spectrum of the auxiliary operator. At the moment the only equations which have been analyzed are the finite Toda lattice, [11], and the Camassa–Holm equation with rapidly decaying initial data of one sign, [12]. For both of these systems the spectrum of the corresponding auxiliary spectral problem is an isolated discrete set\(^2\). We refer to [11] and [12] for details.

1.3. **Historical remark.** Another approach which relates Hamiltonian theory and complex geometry was developed recently.

In a remarkable paper Krichever and Phong [6] proposed a construction of a symplectic formalism for integrable equations with periodic initial data. For the latest exposition of their results and connections with Seiberg-Witten theory see [2]. The Krichever-Phong approach is designed for 2+1 systems like the Kadomtsev-Petviashvili equation and the 2D-Toda lattice. It can also be extended to the scattering case, [13]. In the simplest cases the symplectic form can be inverted explicitly to obtain the Poisson bracket. For example the Gardner-Zakharov-Faddeev bracket for KdV or classical bracket 1.1 (both are constant on the phase space) can be obtained this way. In the case of a bracket with variable coefficients the situation is different. It is not possible to invert explicitly the corresponding higher (second) symplectic form even for the case of NLS with periodic boundary conditions, see [13]. These prompt us to find another approach to the Poisson formalism.

\(^2\)For the Camassa-Holm equation and the open Toda lattice the corresponding Poisson bracket on the Weyl function is also given by formula 1.2 or its’ reduction.
1.4. Content of the paper. In Section 2 we define the Atiyah-Hitchin bracket for rational maps of $\mathbb{CP}^1$ into itself. We state its basic properties and prove that the AH bracket is invariant under linear fractional transformations. The direct spectral transform and inverse spectral transform are defined in Section 3. We study the Poisson bracket in Section 4, which consists of two parts. In the first part we compute the image of the classical bracket for the Weyl function. In the second part we show that the bracket on phase space is an image of the AH bracket under the inverse spectral transform.

2. The Atiyah–Hitchin bracket.

Let $\mathcal{X}(\lambda) : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the rational function of degree $N$ such that

- $\mathcal{X}(\infty) = 0$,
- the preimage $\mathcal{X}^{-1}(\infty)$ consists of exactly $N$ points.

To any $N$–monopole solution $m_N(x)$, $x \in \mathbb{R}^3$, of the Bogomolny equation Atiyah and Hitchin, see [1], associate the scattering function $X(\lambda)$, with these properties. Thus we have the direct spectral transform

$$m_N(x) \rightarrow \mathcal{X}(\lambda).$$

This map is injective due to a theorem of S. Donaldson, [1].

The function $\mathcal{X}(\lambda)$ has the form

$$\mathcal{X}(\lambda) = \frac{\sum_{i=1}^{N} a_i \lambda_i}{\lambda^N + \sum_{j=0}^{N-1} b_j \lambda^j} = -\frac{q(\lambda)}{p(\lambda)}.$$

The monic polynomial $p(\lambda)$ is determined by its roots $\lambda_1, \ldots, \lambda_N$. The polynomial $q(\lambda)$ of degree $N-1$ can be determined from its values at the roots of denominator. Therefore,

$$\lambda_1, \ldots, \lambda_N, q(\lambda_1), \ldots, q(\lambda_N);$$

are global complex coordinates on this space of maps.

Let $\delta$ denote a variation of the parameters $\lambda_1, \ldots, \lambda_N, q(\lambda_1), \ldots, q(\lambda_N)$, while $d$ is a differential of the parameter $\lambda$. The Atiyah–Hitchin symplectic structure $\omega$ is defined by the formula

$$\omega = \sum_{k=1}^{N} \frac{\delta q(\lambda_k)}{q(\lambda_k)} \wedge \delta \lambda_k.$$

The corresponding Poisson bracket is specified by canonical relations:

$$\{q(\lambda_n), \lambda_k\} = \delta^n_k \frac{q(\lambda_n)}{4}.$$

All other brackets vanish

$$\{\lambda_n, \lambda_k\} = \{q(\lambda_n), q(\lambda_k)\} = 0.$$
The bracket turns the space of maps $X(\lambda) : \mathbb{C}\mathbb{P}^1 \to \mathbb{C}\mathbb{P}^1$ of degree $N$ with the above mentioned properties into a Poisson manifold.

Consider some function $X_0$, a point of the Poisson manifold. Fix some points $\lambda$ and $\mu$ on the sphere away from the poles of $X_0$. Then the points $\lambda$ and $\mu$ considered as an argument of $X$, where $X$ is from a small vicinity of $X_0$, are well defined functions in this vicinity. Their values at $X$ are functions of the coordinates $\lambda_1, \ldots, \lambda_N, q(\lambda_1), \ldots, q(\lambda_N)$. As it was demonstrated by Faybusovich and Gekhtman, [3], the bracket for $X(\lambda)$ and $X(\mu)$ is given by the formula

$$\{X(\lambda), X(\mu)\} = \frac{(X(\lambda) - X(\mu))^2}{\lambda - \mu}. \quad (2.3)$$

Here some miracle occurs. The Poisson bracket for two functions ($\lambda$ and $\mu$) on the Poisson manifold is given in terms of these functions and their argument ($X$). In [11] for rational functions we gave a direct proof that (2.3) implies 2.1–2.2. We list properties of the AH bracket.

Evidently, (2.3) is skew–symmetric with respect to $\lambda$ and $\mu$. It is also linear in its arguments

$$\{aX(\lambda) + bX(\mu), X(\nu)\} = a\{X(\lambda), X(\nu)\} + b\{X(\mu), X(\nu)\}, \quad (2.4)$$

where $a$ and $b$ are constants. The Leibnitz rule holds

$$\{X(\lambda)X(\mu), X(\nu)\} = X(\lambda)\{X(\mu), X(\nu)\} + X(\mu)\{X(\lambda), X(\nu)\}. \quad (2.5)$$

The bracket satisfies the Jacobi identity

$$\{X(\lambda), \{X(\mu), X(\nu)\}\} + \{X(\mu), \{X(\nu), X(\lambda)\}\} + \{X(\nu), \{X(\lambda), X(\mu)\}\} = 0.$$  

The following is particularly useful to us.

\[\text{Instead of } X(\lambda) \text{ and } X(\mu) \text{ one should write } \lambda(X) \text{ and } \mu(X).\]
Lemma 2.1. The bracket 2.3 is invariant under linear fractional transformations

\[ \mathcal{X} \rightarrow \mathcal{X}' = \frac{a \mathcal{X} + b}{c \mathcal{X} + d}, \quad (2.6) \]

where \( a, b, c, d \) are constants.

**Proof.** Consider a transformation of the form

\[ \mathcal{X} \rightarrow \mathcal{X}' = \frac{1}{c \mathcal{X} + d}. \quad (2.7) \]

Then

\[ \{\mathcal{X}'(\lambda), \mathcal{X}'(\mu)\} = \frac{1}{(c \mathcal{X}(\lambda) + d)^2} \frac{1}{(c \mathcal{X}(\mu) + d)^2} \{c \mathcal{X}(\lambda) + d, c \mathcal{X}(\mu) + d\} \]

\[ = \frac{1}{(c \mathcal{X}(\lambda) + d)^2} \frac{1}{(c \mathcal{X}(\mu) + d)^2} \frac{(c \mathcal{X}(\lambda) + d) - (c \mathcal{X}(\mu) + d)^2}{\lambda - \mu} \]

\[ = \frac{(\mathcal{X}'(\lambda) - \mathcal{X}'(\mu))^2}{\lambda - \mu}. \]

To finish the proof we note that two consecutive transformations of the form 2.7 produce the whole group 2.6. \( \square \)

The Atiyah–Hitchin bracket in coordinate free form 2.3 will appear for a much wider class than rational functions. Formula 2.3 itself can be a starting point for construction of the bracket on an infinite algebra of complex observables \( \mathcal{X}(\lambda), \lambda \in \mathbb{C} \mathbb{P}^1, \) meromorphically dependent on the parameter. Indeed, we start defining the bracket for two observables at different points by 2.3 and the extend it to all polynomials using 2.4 and 2.5. It can be verified in a long but simple calculation that 2.4 and 2.5 imply the Jacobi identity for the AH bracket. It is of interest to find all formulas similar to 2.3 with the properties of a Poisson bracket and compatible with the Cauchy theorem.

3. The spectral problem.

The NLS equation,

\[ i \psi^* = -\psi'' + 2|\psi|^2 \psi, \]

where \( \psi(x, t) \) is a smooth complex function, is a Hamiltonian system

\[ \psi^* = \{\psi, \mathcal{H}\}, \]

with Hamiltonian \( \mathcal{H} = \frac{1}{2} \int |\psi'|^2 + |\psi|^4 \, dx = \text{energy} \) and the bracket

\[ \{A, B\} = 2i \int \frac{\delta A}{\delta \bar{\psi}(x)} \frac{\delta B}{\delta \psi(x)} - \frac{\delta A}{\delta \psi(x)} \frac{\delta B}{\delta \bar{\psi}(x)} \, dx. \quad (3.1) \]

The NLS equation is a compatibility condition for the commutator

\[ [\partial_t - V_3, \partial_x - V_2] = 0, \]
with

\[ V_2 = \frac{i\lambda}{2} \sigma_3 + Y_0 = \begin{pmatrix} \frac{-i\lambda}{2} & 0 \\ 0 & \frac{i\lambda}{2} \end{pmatrix} + \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix} \]

and

\[ V_3 = \frac{\lambda^2}{2} i\sigma_3 - \lambda Y_0 + |\psi|^2 i\sigma_3 - i\sigma_3 Y_0'. \]

We often omit the lower index and write \( V = V_2 \).

### 3.1. The direct spectral transform.

The commutator relation produces an auxiliary linear problem

\[ f'(x, \lambda) = V(x, \lambda) f(x, \lambda), \quad f(x, \lambda) = \begin{bmatrix} f_1(x, \lambda) \\ f_2(x, \lambda) \end{bmatrix}. \tag{3.2} \]

This can be written as an eigenvalue problem for the Dirac operator (see the introduction). Let \( f^T \) denote the transposition of the vector \( f \) and let \( f^* \) denote the adjoint of the vector \( f \). Let \( L^2(a, b) \) be a space of vector functions with the property

\[ \int_a^b f^*(x, \lambda) f(x, \lambda) \, dx < \infty. \]

The Weyl solution, \cite{18},

\[ e(x, y, \lambda) = \begin{bmatrix} e_1(x, y, \lambda) \\ e_2(x, y, \lambda) \end{bmatrix} \]

is the solution of 3.2 which belongs to \( L^2[y, +\infty) \) or \( L^2(-\infty, y] \), where \( y \) is an arbitrary point on the line. Due to the theorem of Levitan and Martunov, \cite{7} section 8.6, for a continuous potential \( \psi(x) \) the Dirac operator is always in the limit point case. It means that for \( \lambda \) with \( \Im \lambda \neq 0 \) there exists one solution from \( L^2[y, +\infty) \) and another solution from \( L^2(-\infty, y] \). Evidently the Weyl solutions are determined up to a multiplicative constant.

Pick some \( \alpha \in [0, \pi) \). Consider the fundamental system of solutions \( p_\alpha(x, y, \lambda) \) and \( u_\alpha(x, y, \lambda) \) of 3.2 normalized by

\[ p_\alpha(x, y, \lambda)|_{x=y} = \begin{bmatrix} i e^{i\alpha} \\ -i e^{-i\alpha} \end{bmatrix}, \quad u_\alpha(x, y, \lambda)|_{x=y} = \begin{bmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{bmatrix}. \tag{3.3} \]

The Weyl solution \( e(x, y, \lambda) \) from \( L^2[y, +\infty) \) or \( L^2(-\infty, y] \) is proportional to a linear combination of \( p_\alpha \) and \( u_\alpha \):

\[ e \sim a p_\alpha + b u_\alpha, \]

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\(^4\)Here and below \( \sigma \) denotes the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
but the Weyl function $X^\pm(\alpha, \lambda) = a/b$ is defined uniquely. Evidently, the solution $X^\pm p_\alpha + u_\alpha$ belongs to $L^2([y, +\infty])$ for $\lambda$ with $\Re \lambda \neq 0$.

**Example 3.1.** The trivial potential $\psi(x) \equiv 0$.

$$p_\alpha(x, y, \lambda) = \begin{bmatrix} ie^{ia-\frac{ia}{2}(x-y)} \\ -ie^{-ia+\frac{ia}{2}(x-y)} \end{bmatrix}, \quad u_\alpha(x, y, \lambda) = \begin{bmatrix} e^{ia-\frac{ia}{2}(x-y)} \\ e^{-ia+\frac{ia}{2}(x-y)} \end{bmatrix}. $$

The Weyl function $X^y(\alpha, \lambda) = i$ if $\Re \lambda > 0$ and $-i$ for $\Re \lambda < 0$. The Weyl function $X^\alpha(\alpha, \lambda) = i$ if $\Re \lambda > 0$ and $i$ for $\Re \lambda < 0$. The only potential which has such Weyl functions vanishes identically.

**Lemma 3.2.** Any two functions $X^\pm(\alpha, \lambda)$ and $X^\pm(\beta, \lambda)$ are related by the equation

$$X^\pm(\alpha, \lambda) = X^\pm(\beta, \lambda) \frac{\cos(\alpha - \beta) - \sin(\alpha - \beta)}{\sin(\alpha - \beta) + \cos(\alpha - \beta)}. \quad (3.4)$$

**Proof.** First we will obtain the expression for the function $X^\pm(\alpha, \lambda)$ in terms of the square integrable solution $e(x, y, \lambda)$. This solution is proportional to $X^\pm p_\alpha + u_\alpha$ with some constant $c$. Thus $X^\pm p_\alpha + u_\alpha = ce$ and normalization conditions 3.3 imply the system

$$X^\pm(\alpha, \lambda) ie^{i\alpha} + e^{i\alpha} = ce_1(y, y, \lambda),$$

$$-X^\pm(\alpha, \lambda) ie^{-i\alpha} + e^{-i\alpha} = ce_2(y, y, \lambda).$$

Solving for $X^\pm(\alpha, \lambda)$,

$$X^\pm(\alpha, \lambda) = \frac{e_1 e^{-i\alpha} - e_2 e^{i\alpha}}{e_1 e^{-i\alpha} + e_2 e^{i\alpha}}. \quad (3.5)$$

This identity can be written as

$$X^\pm(\alpha, \lambda) = \frac{(e_1 e^{-i\beta} - e_2 e^{i\beta}) \cos(\alpha - \beta) - (e_1 ie^{-i\beta} + e_2 ie^{i\beta}) \sin(\alpha - \beta)}{(e_1 e^{-i\beta} + e_2 e^{i\beta}) \cos(\alpha - \beta) + (e_1 ie^{-i\beta} - e_2 ie^{i\beta}) \sin(\alpha - \beta)}.$$ 

Dividing the numerator and denominator by $e_1 ie^{-i\beta} + e_2 ie^{i\beta}$ we obtain 3.4. Therefore, if $X^\pm(\alpha, \lambda)$ is known for some value of the parameter $\alpha$, then it is known for all other values of $\alpha$.

If $f(x, \lambda)$ is a solution of the auxiliary problem 3.2 corresponding to $\lambda$, then $\hat{f} = \sigma_1 f$ is a solution corresponding to $\bar{\lambda}$. Formula 3.5 implies, for $\lambda$ with $\Re \lambda \neq 0$, 

$$X^\pm(\bar{\lambda}) = \frac{e_1(\bar{\lambda}) e^{-i\alpha} - e_2(\bar{\lambda}) e^{i\alpha}}{e_1(\bar{\lambda}) e^{-i\alpha} + e_2(\bar{\lambda}) e^{i\alpha}} = \frac{\bar{e}_2(\lambda) e^{-i\alpha} - \bar{e}_1(\lambda) e^{i\alpha}}{\bar{e}_2(\lambda) e^{-i\alpha} + \bar{e}_1(\lambda) e^{i\alpha}} = X^\pm(\lambda).$$

The function $X^\pm(\alpha, \lambda)$ for fixed $y$ maps the upper/lower half-plane into the upper/lower half-plane.
For a nonzero potential the Weyl function $X_\alpha^+(y, \lambda)$ takes values in the upper half-plane when $\lambda$ is in the upper half-plane. It is represented by the integral, [5],

$$X_\alpha^+(y, \lambda) = b\lambda + a + \int_{-\infty}^{+\infty} \left[ \frac{1}{t-\lambda} - \frac{t}{t^2+1} \right] d\sigma_\alpha(t),$$

with

$$b \geq 0, \quad a \in \mathbb{R}, \quad \int_{-\infty}^{+\infty} \frac{d\sigma_\alpha(t)}{t^2+1} < \infty.$$

The spectral measure $d\sigma_\alpha$ corresponds to a self-adjoint extension of the operator $\mathcal{D}$ on $L^2[y, +\infty)$ specified by the boundary condition

$$f_1(y)e^{-i\alpha} = f_2(y)e^{i\alpha}.$$

The function $X_\alpha^-(y, \lambda)$ has a similar representation.

For each point of the complex plane with nonzero imaginary part there are two Weyl solutions. To make the Weyl solution a single valued function of a point we introduce $\Gamma = \Gamma_+ \cup \Gamma_-$, a two sheeted covering of the complex plane (Figure 1). Each sheet $\Gamma_+$ or $\Gamma_-$ is a copy of the complex plane without the real line. Each point of the cover $\Gamma$ is a pair $Q = (\lambda, \pm)$ where $\lambda$ is a point of the complex plane and the sign $\pm$ specifies the sheet. We denote by $P_+$ and $P_-$ the infinity corresponding to the sheet $\Gamma_+$ or $\Gamma_-$ (Figure 2).

Let us introduce two components of the spectral cover

$$\Gamma_R = \{Q \in \Gamma_+, \Im \lambda(Q) > 0\} \cup \{Q \in \Gamma_-, \Im \lambda(Q) < 0\}$$

and

$$\Gamma_L = \{Q \in \Gamma_+, \Im \lambda(Q) < 0\} \cup \{Q \in \Gamma_-, \Im \lambda(Q) > 0\}.$$
Evidently $\Gamma = \Gamma_R \cup \Gamma_L$. On $\Gamma$ we define an involution $\epsilon_a$ by the rule

$$\epsilon_a : (\lambda, \pm) \rightarrow (\overline{\lambda}, \mp).$$

The involution permutes infinities $\epsilon_a : P_+ \rightarrow P_-$. Evidently $\Gamma_R$ or $\Gamma_L$ are invariant under the action of $\epsilon_a$.

We define the Weyl solution $e(x, y, Q)$, $x, y \in \mathbb{R}^1$, $Q \in \Gamma$, which is the vector function

$$e(x, y, Q) = \begin{bmatrix} e_1(x, y, Q) \\ e_2(x, y, Q) \end{bmatrix},$$

as a solution of 3.2 with $\lambda = \lambda(Q)$ and the following properties

- $e(x, y, Q)$ belongs to $L^2[y, +\infty)$ if $Q \in \Gamma_R$;
- $e(x, y, Q)$ belongs to $L^2(-\infty, y]$ if $Q \in \Gamma_L$;

normalized by the condition

$$e_1(x, y, Q)|_{x=y} = 1. \quad (3.6)$$

The standard transition matrix $M(x, y, \lambda) = (M^{(1)}, M^{(2)})$ is $2 \times 2$ matrix solution of the auxiliary linear problem 3.2 which satisfies the boundary condition $M(x, y, \lambda)|_{x=y} = I$. The symmetry of the matrix $V(x, \lambda)$

$$\sigma_1 V(x, \lambda) \sigma_1 = V(x, \bar{\lambda})$$

produces the same relation for the transition matrix

$$\sigma_1 M(x, y, \lambda) \sigma_1 = M(x, y, \bar{\lambda}). \quad (3.7)$$

This implies for the columns

$$M^{(1)}(x, y, \overline{\lambda}) = \sigma_1 \overline{M}^{(2)}(x, y, \lambda), \quad M^{(2)}(x, y, \overline{\lambda}) = \sigma_1 \overline{M}^{(1)}(x, y, \lambda). \quad (3.8)$$

Evidently the Weyl solution $e(x, y, Q)$ is a linear combination of the columns of the transition matrix. The normalization condition 3.6 implies

$$e(x, y, Q) = M^{(1)}(x, y, \lambda) + \mathcal{X}(y, Q)M^{(2)}(x, y, \lambda), \quad (3.9)$$

where the function $\mathcal{X}(y, Q)$ is called the Weyl function.

Formulas 3.8, 3.9 and the uniqueness of the Weyl function imply

$$\mathcal{X}(y, \epsilon_a Q) = \frac{1}{\mathcal{X}(y, Q)}. \quad (3.10)$$

**Example 3.3. The trivial potential** $\psi(x) \equiv 0$.

For the columns of the transition matrix we have

$$M^{(1)}(x, y, \lambda) = e^{-i\frac{\lambda}{2}(x-y)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M^{(2)}(x, y, \lambda) = e^{i\frac{\lambda}{2}(x-y)} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
Arguing as in the proof of Lemma 3.2, for $Q \in \Gamma_R$ we have

$$\mathcal{X}(y, Q) = \frac{i + \mathcal{X}_+^\alpha(y, \lambda)}{i - \mathcal{X}_-^\alpha(y, \lambda)} e^{-i2\alpha}, \quad \lambda = \lambda(Q).$$

(3.11)

Note that the transformation

$$z \mapsto z' = \frac{i + z}{i - z} e^{-i2\alpha}$$

establishes a 1:1 correspondence between the upper/lower half–plane and the exterior/interior of the unit circle. This implies, in particular, that the function $\mathcal{X}(y, Q)$ for $Q \in \Gamma_R$ with $\Im Q > 0$ or $\Im Q < 0$ takes values in the exterior or interior of the unit circle. The shift $\alpha \mapsto \beta$ described on the $z$–plane by formula 3.4 corresponds to rotation via the angle $2(\alpha - \beta)$ on the $z'$–plane.

For $Q \in \Gamma_L$ we also have

$$\mathcal{X}(y, Q) = \frac{i + \mathcal{X}_-^\alpha(y, \lambda)}{i - \mathcal{X}_+^\alpha(y, \lambda)} e^{-i2\alpha}, \quad \lambda = \lambda(Q).$$

(3.12)

The function $\mathcal{X}(\lambda, Q)$ for $Q \in \Gamma_L$ with $\Im Q < 0$ or $\Im Q > 0$ takes values in the exterior or interior of the unit circle.

We associated to a potential $\psi(x)$ defined on the entire line a pair $(\Gamma, \mathcal{X})$. Thus we defined a map

$$\mathcal{M} \quad \rightarrow \quad (\Gamma, \mathcal{X}),$$

(3.13)

which we call the direct spectral transform. This map is injective. If the function $\mathcal{X}(y, Q)$ is known for some fixed value of $y$, formulas 3.11-3.12 allow one to reconstruct $\mathcal{X}_-^\alpha(y, \lambda)$ and $\mathcal{X}_+^\alpha(y, \lambda)$ (or equivalently the spectral measure). By the theorem of Marchenko, [8], a potential is determined on the corresponding half line by its spectral measure.

Due to its injective character the direct spectral transform can be inverted

$$(\Gamma, \mathcal{X}) \quad \rightarrow \quad \mathcal{M}.$$ 

(3.14)

This map we call the inverse spectral transform. An effective procedure for the inverse spectral transform was constructed by Gelfand and Levitan, [4].

3.2. The formal series for the Weyl functions. For the function $\mathcal{X}(y, Q)$, $Q \in \Gamma_+, \ \Im Q > 0$, and $z < y$ we have

$$\mathcal{X}(y, Q) = \frac{e_2(y, y, Q)}{e_1(y, y, Q)} = \frac{e_2(y, z, Q)}{e_1(y, z, Q)}.$$

This together with 3.2 implies the Ricatti type equation, $\lambda = \lambda(Q)$:

$$\frac{d\mathcal{X}(y, Q)}{dy} = i\lambda \mathcal{X}(y, Q) + \psi - \overline{\psi} \mathcal{X}^2(y, Q).$$

(3.15)

In fact, the function $\mathcal{X}(y, Q)$ satisfies the same equation in the variable $y$ for any $Q$. 

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For a generic potential the Weyl function $\mathcal{X}(y, Q)$ has a pole at infinity for $Q \in \Gamma_+$ (compare Example 3.3). The formal expansion has the form

$$\mathcal{X}(y, Q) = a_{-1}(y)\lambda + a_0(y) + \frac{a_1(y)}{\lambda} + \ldots, \quad Q \in (P_+);$$

with

$$a_{-1}(y) = \frac{1}{-i\psi(y)}, \quad a_0(y) = \frac{-i\psi(y)}{\psi^2(y)}, \quad \text{etc.}$$

The function $\mathcal{X}(y, Q)$ has a zero at infinity for $Q \in \Gamma_-$. It has the power series expansion

$$\mathcal{X}(y, Q) = \frac{b_1(y)}{\lambda} + \frac{b_2(y)}{\lambda^2} + \frac{b_3(y)}{\lambda^3} + \ldots, \quad Q \in (P_-);$$

where

$$b_1(y) = i\psi(y), \quad b_2(y) = \psi'(y), \quad \text{etc.}$$

The coefficients of the two expansions are obtained by substituting the series into the differential equation 3.15 and then by matching terms with the same power of $\lambda$. The expansions are connected in accordance with formula 3.10.

V. A. Marchenko, [9], studied the question of when the formal series have an asymptotic character. If the potential has some degree of differentiability then the formal series have an asymptotic character. According to [9] a converse statement is also true. If the formal series converge for large values of $\lambda$, then the corresponding potential is an infinitely differentiable function. In fact, for our purposes it is enough to have only the first term of the asymptotic expansion.

The Weyl functions $\mathcal{X}_+^\alpha(y, \lambda)$ and $\mathcal{X}_-^\alpha(y, \lambda)$ for fixed $\lambda$ and $\alpha$ are the functions of $y$ and the potential. Formula 3.5 and the spectral problem 3.2 imply the Ricatti–type equation for $\mathcal{X}_+^\alpha(y, \lambda)$:

$$\frac{d\mathcal{X}_+^\alpha(y, \lambda)}{dy} = -\frac{\lambda}{2}(\lambda^2(y+, \lambda)+1) - \mathcal{X}_+^\alpha(y, \lambda)(\overline{\psi}_\alpha + \psi_\alpha) + \frac{i}{2}(\mathcal{X}_+^\alpha(y, \lambda)+1)(\overline{\psi}_\alpha - \psi_\alpha),$$

where $\psi_\alpha = \psi e^{2i\alpha}$. In general, the function $\mathcal{X}(y, Q)$ is simpler to work with than $\mathcal{X}_\alpha^\pm(y, \lambda)$.

4. The Poisson Bracket.

4.1. The Atiyah–Hitchin bracket for the Weyl function. The main result of this section is Theorem 4.1. This theorem describes the image of the Poisson bracket 3.1 under the direct spectral transform 3.13.

**Theorem 4.1.** Let $y$ be some fixed point of the line and $\mathcal{X}(Q) = \mathcal{X}(y, Q)$ and $\mathcal{X}(P) = \mathcal{X}(y, P)$. 

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i. If $Q$ and $P$ belong to $\Gamma_R$, then
\[
\{\mathcal{X}(Q), \mathcal{X}(P)\} = 2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda(Q) - \lambda(P)}.
\]

ii. If $Q$ and $P$ belong to $\Gamma_L$, then
\[
\{\mathcal{X}(Q), \mathcal{X}(P)\} = -2 \times \frac{(\mathcal{X}(Q) - \mathcal{X}(P))^2}{\lambda(Q) - \lambda(P)}.
\]

iii. If $Q$ belongs to $\Gamma_R$, $P$ belongs to $\Gamma_L$, then
\[
\{\mathcal{X}(Q), \mathcal{X}(P)\} = 0.
\]

Remark 4.2. In the formulation of Theorem we have to consider various parts of the spectral cover separately because it is not simply-connected.

Formulas 3.11 and 3.12 and the invariance of the AH bracket under linear fractional transformations (Lemma 2.1) produce the bracket for $\mathcal{X}_\alpha^\pm(y, \lambda)$. We precede the proof with two auxiliary results.

**Lemma 4.3.** Let the vectors $f^\triangledown(x, \lambda)$, $f^\blacktriangle(x, \lambda)$ satisfy
\[
f^\triangledown'(x, \lambda) = V(x, \lambda)f^\triangledown(x, \lambda), \quad f^\blacktriangle'(x, \lambda) = V(x, \lambda)f^\blacktriangle(x, \lambda),
\]
and the vectors $g^\triangledown(x, \mu)$, $g^\blacktriangle(x, \mu)$ satisfy
\[
g^\triangledown'(x, \mu) = V(x, \mu)g^\triangledown(x, \mu), \quad g^\blacktriangle'(x, \mu) = V(x, \mu)g^\blacktriangle(x, \mu).
\]
The following identity holds:
\[
f_1^\triangledown f_2^\blacktriangle g_1^\triangledown g_2^\blacktriangle - f_2^\triangledown f_1^\blacktriangle g_1^\triangledown g_2^\blacktriangle = \frac{1}{i(\mu - \lambda)} \times \frac{d}{dx}\left[ (f^\triangledown^T J g^\triangledown) (f^\blacktriangle^T J g^\blacktriangle) \right].
\]

**Proof.** The identity can be verified by differentiation. \(\square\)

**Lemma 4.4.** If $Q \in \Gamma_R$, then the following identities hold:
\[
\frac{\delta \mathcal{X}(z, Q)}{\delta \psi(y)} = -[e_1(y, z, Q)]^2, \quad y \geq z; \quad (4.1)
\]
\[
\frac{\delta \mathcal{X}(z, Q)}{\delta \psi(y)} = [e_2(y, z, Q)]^2, \quad y \geq z. \quad (4.2)
\]
The derivatives vanish for $y < z$.

If $Q \in \Gamma_L$, then the following identities hold:
\[
\frac{\delta \mathcal{X}(z, Q)}{\delta \psi(y)} = [e_1(y, z, Q)]^2, \quad y \leq z; \quad (4.3)
\]
\[
\frac{\delta \mathcal{X}(z, Q)}{\delta \psi(y)} = -[e_2(y, z, Q)]^2, \quad y \leq z. \quad (4.4)
\]
The derivatives vanish for $y > z$.\(\)
The lemma was proved first in [16]. Here we present a simplified proof of formulas 4.1 and 4.2. The proof of other formulas 4.3 and 4.4 is the same. We split the proof into small steps.

**Proof.** Without loss of generality we assume that \( z = 0 \).

**Step 1.** This step prepares gradients. Let \( M^\bullet = \delta M \) be a variation of \( M(x, 0) \) in response to the variation of \( \psi(y) \) and \( \bar{\psi}(y) \), \( 0 \leq y \leq x \). Then \( M^\bullet = VM^\bullet + V^\bullet M \). The solution of this nonhomogenous equation is

\[
M^\bullet(x) = M(x) \int_0^x M^{-1}(\xi)V^\bullet(\xi)M(\xi) \, d\xi.
\]

Therefore,

\[
\frac{\delta M(x, 0)}{\delta \psi(y)} = M(x, y) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M(y, 0),
\]

\[
\frac{\delta M(x, 0)}{\delta \bar{\psi}(y)} = M(x, y) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M(y, 0).
\]

**Step 2.** The purpose of this step is to prove the formula

\[
\frac{\delta \mathcal{X}(0, Q)}{\delta \psi(y)} = \mathcal{X}(0, Q) \left[ \frac{m_{11}(y, 0)}{A} + \frac{m_{12}(y, 0)}{B} \right],
\]

where

\[
A = m_{21}(y, 0) - \mathcal{X}(y, Q)m_{11}(y, 0),
\]

\[
B = -m_{22}(y, 0) + \mathcal{X}(y, Q)m_{12}(y, 0).
\]

Consider the auxiliary spectral problem 3.2 on the finite interval \([a, b]\). The solution

\[
f(x, \lambda) = M^{(1)}(x, a, \lambda) + \mathcal{X}(b, a, Q)M^{(2)}(x, a, \lambda),
\]

with some \( \mathcal{X}(b, a, Q) \), satisfies the boundary condition \( f_1(b, \lambda) = f_2(b, \lambda) \) if

\[
\mathcal{X}(b, a, Q) = \frac{m_{11} - m_{21}}{m_{22} - m_{12}}(b, a, \lambda).
\]

The limit

\[
\mathcal{X}(0, Q) = \lim_{b \to +\infty} \mathcal{X}(b, 0, Q)
\]

exists because the spectral problem is in the limit–point case. Therefore,\(^5\)

\[
\nabla \mathcal{X}(0, Q) = \lim_{b \to +\infty} \nabla \mathcal{X}(b, 0, Q).
\]

\(^5\nabla = \delta/\delta \psi(y) \) or \( \delta/\delta \bar{\psi}(y) \).
To compute the derivative
\[
\nabla X(b, 0, \lambda) = \frac{m_{11} - m_{21}}{m_{22} - m_{12}} \nabla m_{11} - \nabla m_{21}(b, 0, \lambda) - \frac{m_{11} - m_{21}}{m_{22} - m_{12}} \nabla m_{22} - \nabla m_{12}(b, 0, \lambda).
\]

we use the formulas of Step 1:
\[
\frac{\delta X(b, 0, Q)}{\delta \psi(y)} = X(b, 0, Q) \frac{[m_{12} - m_{22}] (b, y) m_{11}(y, 0)}{[m_{11} - m_{21}] (b, 0)} - X(b, 0, Q) \frac{[m_{22} - m_{12}] (b, y) m_{12}(y, 0)}{[m_{22} - m_{12}] (b, 0)} = X(b, 0, Q) [m_{12} - m_{22}] (b, y)
\]
\[
\times \left[ \frac{m_{11}(y, 0)}{[m_{11} - m_{21}] (b, 0)} + \frac{m_{12}(y, 0)}{[m_{22} - m_{12}] (b, 0)} \right].
\]

Using the identity \( M(b, 0) = M(b, y) M(y, 0) \) and simple algebra, one finds
\[
X(b, 0, Q) \times \left[ \frac{m_{11}(y, 0)}{m_{21}(y, 0) - X(b, y, Q) m_{11}(y, 0)} + \frac{m_{12}(y, 0)}{-m_{22}(y, 0) + X(b, y, Q) m_{12}(y, 0)} \right].
\]

Now pass to the limit when \( b \to \infty \).

**Step 3.** The purpose of this step is to prove the formula
\[
\frac{\delta X(0, Q)}{\delta \psi(y)} = -X(0, Q) X(y, Q) \frac{m_{21}(y, 0) + m_{22}(y, 0)}{m_{21}(y, 0) - X(y, Q) m_{11}(y, 0)}. \tag{4.7}
\]

To compute the derivative \( \nabla X(b, 0, \lambda) \) given by 4.6 we use the formulas of Step 1
\[
\frac{\delta X(b, 0, Q)}{\delta \psi(y)} = X(b, 0, Q) \frac{[m_{11} - m_{21}] (b, y) m_{21}(y, 0)}{[m_{11} - m_{21}] (b, 0)} - X(b, 0, Q) \frac{[m_{21} - m_{11}] (b, y) m_{22}(y, 0)}{[m_{22} - m_{12}] (b, 0)} = X(b, 0, Q) [m_{11} - m_{21}] (b, y)
\]
\[
\times \left[ \frac{m_{21}(y, 0)}{[m_{11} - m_{21}] (b, 0)} + \frac{m_{22}(y, 0)}{m_{22} - m_{12} (b, 0)} \right].
\]

Using the identity \( M(b, 0) = M(b, y) M(y, 0) \), and simple algebra, one finds
\[
+ X(b, 0, \lambda) X(b, y, \lambda)
\]
\[
\times \left[ \frac{m_{21}(y, 0)}{-m_{21}(y, 0) + X(b, y, \lambda) m_{11}(y, 0)} + \frac{m_{22}(y, 0)}{m_{22} - X(b, y, \lambda) m_{12}(y, 0)} \right].
\]

Now pass to the limit when \( b \to \infty \).
Step 4. Consider $e(x, y, Q)$ proportional to $e(x, 0, Q)$. Then

$$
\mathcal{X}(y, Q) = \frac{e_2(y, y, Q)}{e_1(y, y, Q)} = \frac{e_2(y, 0, Q)}{e_1(y, 0, Q)}.
$$

(4.8)

Therefore,

$$
\mathcal{X}(y, Q)[m_{11}(y, 0) + \mathcal{X}(0, Q)m_{12}(y, 0)] = m_{21}(y, 0) + \mathcal{X}(0, Q)m_{22}(y, 0).
$$

After simple algebra,

$$
\frac{\mathcal{X}(0, Q)}{A} = \frac{1}{B}.
$$

(4.9)

Therefore, using 4.5, 4.8, and 4.9, we obtain

$$
\frac{\delta \mathcal{X}(0, Q)}{\delta \psi(y)} = \mathcal{X}(0, Q) \left[ \frac{m_{11}(y, 0) + \mathcal{X}(0, Q)m_{12}(y, 0)}{A} \right]
$$

$$
= \mathcal{X}(0, Q) \frac{e_1(y, 0, \lambda)}{m_{21}(y, 0) - \frac{e_2(y, 0, Q)}{e_1(y, 0, Q)} m_{11}(y, 0)}
$$

$$
= \mathcal{X}(0, Q) \frac{[e_1(y, 0, Q)]^2}{m_{21}(y, 0)e_1(y, 0, Q) - m_{11}(y, 0)e_2(y, 0, Q)}.
$$

The denominator does not depend on $y$ and can be computed for $y = 0$, where it is equal to $-\mathcal{X}(0, Q)$. Formula 4.1 is proved.

Step 5. Using 4.7, 4.8 and 4.9 we obtain

$$
\frac{\delta \mathcal{X}(0, Q)}{\delta \psi(y)} = -\mathcal{X}(0, Q) \mathcal{X}(y, Q) \left[ \frac{m_{21}(y, 0) + \mathcal{X}(0, Q)m_{22}(y, 0)}{A} \right]
$$

$$
= -\mathcal{X}(0, Q) \mathcal{X}(y, Q) \frac{e_2(y, 0, Q)}{m_{21}(y, 0) - \mathcal{X}(y, Q)m_{11}(y, 0)}
$$

$$
= -\mathcal{X}(0, Q) \frac{[e_2(y, 0, Q)]^2}{m_{21}(y, 0)e_1(y, 0, Q) - m_{11}(y, 0)e_2(y, 0, Q)}.
$$

The denominator does not depend on $y$ and can be computed for $y = 0$, where it is equal to $-\mathcal{X}(0, Q)$. Formula 4.2 is proved.

Now we are ready to prove the theorem. □
Proof. We will prove the first formula. All others can be treated the same way. By Lemma 4.4,
\[ \{\chi(Q), \chi(P)\} = \]
\[ = 2i \int_{y}^{+\infty} \frac{\delta \chi(y, Q)}{\delta \psi(\xi)} \frac{\delta \chi(y, P)}{\delta \psi(\xi)} - \frac{\delta \chi(y, Q)}{\delta \psi(\xi)} \frac{\delta \chi(y, P)}{\delta \psi(\xi)} \ d\xi \]
\[ = 2i \int_{y}^{+\infty} -e_2^2(\xi, y, Q)e_1^2(\xi, y, P) + e_1^2(\xi, y, Q)e_2^2(\xi, y, P) \ d\xi. \]
Using the identity of Lemma 4.3,
\[ = \frac{2i}{i(\lambda(P) - \lambda(Q))} [e(\xi, y, Q)^T J e(\xi, y, P)]^2 |_{y}^{+\infty} \]
\[ = 2 \times \frac{\left(\chi(Q) - \chi(P)\right)^2}{\lambda(Q) - \lambda(P)}. \]
The formula is proved. \( \square \)

4.2. Computation of the Poisson bracket for field variables. The inverse spectral transform 3.14 maps the AH bracket on Weyl functions given by Theorem 4.1 to the phase space. The main result of this section is the following

**Theorem 4.5.** The AH bracket for the field variables \( \psi(x) \) and \( \overline{\psi}(x) \) is given by the formulas\(^6:\)
\[\{\psi(z), \psi(y)\} = 0, \quad (4.10)\]
\[\{\overline{\psi}(z), \overline{\psi}(y)\} = 0, \quad (4.11)\]
\[\{\psi(z), \psi(y)\} = 2i \delta(z - y). \quad (4.12)\]
These identities are an equivalent form of the Poisson bracket 3.1.

The inverse spectral transform 3.14 is very implicit. However, if we know \( \chi(y, Q) \) for all values of the variable \( y \), then the series presented in 3.2 imply
\[ \lim_{Q \to P_+} \frac{\lambda(Q)}{\chi(y, Q)} = -i \overline{\psi}(y) \quad (4.13)\]
and
\[ \lim_{Q \to P_+} \lambda(Q) \chi(y, Q) = i \psi(y). \quad (4.14)\]

\(^6\)The identities are understood in the sense of generalized functions: \( u(x) = v(x) \) if for any \( f(x) \in C_0^\infty \) we have \( \int u(x)f(x)dx = \int v(x)f(x)dx \).
The limits are complex conjugate of each other due to 3.10 and
\[
\lim_{Q \to P_-} \lambda(Q) \mathcal{X}(x, Q) = \lim_{Q \to P_-} \lambda(\epsilon_a \epsilon_a Q) \mathcal{X}(x, \epsilon_a \epsilon_a Q) = \lim_{Q \to P_-} \frac{\lambda(\epsilon_a Q)}{\mathcal{X}(x, \epsilon_a Q)}.
\]

To prove theorem 4.5 using these formulas one needs to compute the bracket \(\{\mathcal{X}(x, Q), \mathcal{X}(y, P)\}\) when \(x \neq y\). Fortunately one needs this formula only asymptotically when one of the points \(Q\) or \(P\) tends to infinity.

**Lemma 4.6.** Suppose \(Q \to P_\pm\) from the imaginary direction. Then
\[
\{\mathcal{X}(y, Q), \mathcal{X}(x, P)\} \sim e^{-i\lambda(Q)(x-y)} \{\mathcal{X}(x, Q), \mathcal{X}(x, P)\}.
\]

**Proof.** The identity
\[
e(y, z, Q) = M(y, x, \lambda)e(x, z, Q), \quad \lambda = \lambda(Q),
\]
implies
\[
\mathcal{X}(y, Q) = \frac{m_{22}(y, x, \lambda)\mathcal{X}(x, Q) + m_{21}(y, x, \lambda)}{m_{12}(y, x, \lambda)\mathcal{X}(x, Q) + m_{11}(y, x, \lambda)}.
\]
If \(\lambda = i\tau, \tau \to \pm\infty\), then
\[
M(y, x, \lambda) \sim e^{-i\lambda/2(y-x)\sigma_3},
\]
\(i.e.,\) the transition matrix near infinity behaves like the solution of the free \((Y_0 = 0)\) equation. To see this write the integral equation for the transition matrix
\[
M(y, x, \lambda) = e^{-i\lambda/2(y-x)\sigma_3} + \int_x^y d\xi e^{-i\lambda/2(y-\xi)\sigma_3}Y_0(\xi)M(\xi, x, \lambda).
\]
In symbolic form \(M = R + AM\), where \(R\) is the solution of the free equation, and \(A\) is the integral operator. Now expand the solution into the Neumann series
\[
M = R + AR + A^2R + \ldots.
\]
and take the first term of the expansion. This produces the stated asymptotics. \(\Box\)

**Remark 4.7.** We cut the tail of the Neumann series. An estimate for the tail for infinitely smooth potentials is given in [17] and for general square integrable potentials in [10].

The next lemma establishes that the Poisson tensor is real.

**Lemma 4.8.** The Poisson brackets for the field variables \(\psi(x)\) and \(\overline{\psi}(x)\) are real
\[
\{\psi(y), \psi(z)\} = \{\overline{\psi}(y), \overline{\psi}(z)\},
\]
\[
\{\psi(y), \overline{\psi}(z)\} = \{\psi(y), \overline{\psi}(z)\}.
\]
Proof. We will prove the first identity. The second identity can be proved along the same lines.

For \(Q, P \in \Gamma_R\), using 4.14, we have

\[
\{\psi(y), \psi(z)\} = \{i\psi(y), i\psi(z)\} = -\lim_{Q, P \to P_\perp} \{\lambda(Q)\mathcal{X}(y, Q), \lambda(P)\mathcal{X}(z, P)\} = -\lim_{Q, P \to P_\perp} \lambda(Q)\bar{\lambda}(P)\{\mathcal{X}(y, Q), \mathcal{X}(z, P)\}.
\]

We assume that the limit is taken along the imaginary direction. Using Lemma 4.6,

\[
\therefore = -\lim_{Q, P \to P_\perp} \lambda(Q)\bar{\lambda}(P)\times \frac{2(\mathcal{X}(z, Q) - \mathcal{X}(z, P))^2}{\lambda(Q) - \bar{\lambda}(P)}.
\]

By Theorem 4.1,

\[
\therefore = -\lim_{Q, P \to P_\perp} \lambda(Q)\bar{\lambda}(P)e^{i\lambda(Q)(z-y)}\times \frac{2(\mathcal{X}(z, Q) - \mathcal{X}(z, P))^2}{\lambda(Q) - \bar{\lambda}(P)}.
\]

Now we use \(Q = \epsilon_a\epsilon_a Q, P = \epsilon_a\epsilon_a P\) and invariance of \(\Gamma_R\) under the action of \(\epsilon_a\).

From formula 3.10, after simple algebra

\[
\therefore = -\lim_{Q, P \to P_\perp} \lambda(Q)\lambda(P)e^{i\lambda(Q)(z-y)}\times \frac{2(\mathcal{X}(z, Q) - \mathcal{X}(z, P))^2}{\lambda(Q) - \bar{\lambda}(P)}
\]

\[
= -\lim_{Q, P \to P_\perp} \lambda(Q)\lambda(P)e^{i\lambda(Q)(z-y)}\{\frac{1}{\mathcal{X}(y, Q)}, \frac{1}{\mathcal{X}(z, P)}\}
\]

\[
= -\lim_{Q, P \to P_\perp} \lambda(Q)\lambda(P)\{\frac{1}{\mathcal{X}(y, Q)}, \frac{1}{\mathcal{X}(z, P)}\} = -\{i\psi(y), i\psi(z)\}.
\]

The last line follows from 4.13. \(\Box\)

Now we are ready to prove the main result.

Proof of Theorem 4.5. Due to Lemma 4.8, identities 4.10 and 4.11 are equivalent. We compute the bracket 4.10. Let \(Q, P \in \Gamma_R\) and \(Q \to P_\perp, P \to P_\perp\) along the imaginary direction. Using formula 4.14 and Lemma 4.6 for \(f(x) \in C^\infty_0\) and \(y \leq z\).
we have:

\[
\int_{-\infty}^{z} dy f(y) \{ \psi(z), \psi(y) \} = - \int_{-\infty}^{z} dy f(y) \{ i\psi(z), i\psi(y) \}
\]

\[
= - \lim_{Q,P \to P_-} \int_{-\infty}^{z} dy f(y) \{ \lambda(Q) \mathcal{X}(z,Q), \lambda(P) \mathcal{X}(y,P) \}
\]

\[
= - \lim_{Q,P \to P_-} \lambda(Q)\lambda(P) \int_{-\infty}^{z} dy f(y) \{ \mathcal{X}(z,Q), \mathcal{X}(y,P) \} \int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)}.
\]

Let \(Q, P\) be such that \(\lambda(Q) = -i\tau, \lambda(P) = -2i\tau\). Since \(\mathcal{X}\) has a zero at \(P_-\) we have, using Theorem 4.1 when \(\tau \to +\infty\),

\[
\lim_{Q,P \to P_-} \lambda(Q)\lambda(P) \{ \mathcal{X}(z,Q), \mathcal{X}(z,P) \} \sim \lambda(Q)\lambda(P) \frac{(\mathcal{X}(z,Q) - \mathcal{X}(z,P))^2}{\lambda(Q) - \lambda(P)} = O(\tau^{-1}).
\]

For the integral we have

\[
\int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)} = O(\tau^{-1}).
\]

Therefore,

\[
\{ \psi(z), \psi(y) \} = 0, \quad y \leq z.
\]

Using skew symmetry of the bracket and interchanging \(y\) and \(z\) we have

\[
\{ \psi(z), \psi(y) \} = 0, \quad y \geq z.
\]

Taking the sum of these two formulas, we obtain 4.10.

Now we compute the bracket 4.12. Let \(Q, P \in \Gamma_R\) and \(Q \to P_+, P \to P_-\) along the imaginary direction. Then using formulas 4.13, 4.14 and Lemma 4.6 for
If \( f(x) \in C_0^\infty \) and \( y \leq z \) we have:

\[
\int_{-\infty}^{z} dy f(y) \{ \bar{\psi}(z), \psi(y) \} = \int_{-\infty}^{z} dy f(y) \{-i\bar{\psi}(z), i\psi(y)\}
\]

\[
= \lim_{-\infty} \int_{-\infty}^{z} dy f(y) \left\{ \frac{\lambda(Q)}{X(z, Q)} \lambda(P) X(y, P) \right\}
\]

\[
= \lim_{-\infty} \int_{-\infty}^{z} dy f(y) \left\{ X(z, Q), X(y, P) \right\}
\]

\[
= \lim_{-\infty} \int_{-\infty}^{z} dy f(y) e^{-i\lambda(P)(z-y)}.
\]

Using Theorem 4.1,

\[
\{X(z, Q), X(z, P)\} = 2 \times \left( \frac{X(z, Q) - X(z, P)}{\lambda(Q) - \lambda(P)} \right)^2.
\]

Since \( X \) has a pole at \( P_+ \) and a zero at \( P_- \), we have asymptotically

\[
\sim -\frac{\lambda(Q)\lambda(P)}{X^2(z, Q)} \{X(z, Q), X(z, P)\} \sim -\frac{\lambda(Q)\lambda(P)}{X^2(z, Q)} \times 2 \times \frac{X^2(z, Q)}{\lambda(Q) - \lambda(P)}.
\]

Let \( P = \epsilon_a Q \) and \( \lambda(Q) = i\tau, \tau \to +\infty \); then

\[
\sim -\frac{\lambda(Q)\lambda(P)}{X^2(z, Q)} \{X(z, Q), X(z, P)\} \sim -\frac{2\tau^2}{i\tau + i\tau}.
\]

Using steepest descent we have

\[
\therefore = \lim_{\tau \to +\infty} -\frac{2\tau^2}{i\tau + i\tau} \int_{-\infty}^{z} dy f(y) e^{-\tau(z-y)} = if(z).
\]

Therefore,

\[
\{\bar{\psi}(z), \psi(y)\} = i\delta(z - y), \quad y \leq z. \tag{4.16}
\]

Since the bracket is real, by Lemma 4.8 we have

\[
\{\psi(z), \bar{\psi}(y)\} = -i\delta(z - y), \quad y \leq z.
\]

By the skew symmetry of the bracket and interchanging \( z \) and \( y \),

\[
\{\bar{\psi}(z), \psi(y)\} = i\delta(z - y), \quad z \leq y. \tag{4.17}
\]
Taking the sum of 4.16 and 4.17, we obtain 4.12.

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