INFINITELY MANY STRINGS IN DE SITTER SPACETIME:
EXPANDING AND OSCILLATING ELLIPTIC FUNCTION SOLUTIONS

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Abstract

The exact general evolution of circular strings in $2 + 1$ dimensional de Sitter spacetime is described closely and completely in terms of elliptic functions. The evolution depends on a constant parameter $b$, related to the string energy, and falls into three classes depending on whether $b < 1/4$ (oscillatory motion), $b = 1/4$ (degenerated, hyperbolic motion) or $b > 1/4$ (unbounded motion). The novel feature here is that one single world-sheet generically describes infinitely many (different and independent) strings. The world-sheet time $\tau$ is an infinite-valued function of the string physical time, each branch yields a different string. This has no analogue in flat spacetime. We compute the string energy $E$ as a function of the string proper size $S$, and analyze it for the expanding and oscillating strings. For expanding strings ($\dot{S} > 0$): $E \neq 0$ even at $S = 0$, $E$ decreases for small $S$ and increases $\propto S$ for large $S$. For an oscillating string ($0 \leq S \leq S_{\text{max}}$), the average energy $\langle E \rangle$ over one oscillation period is expressed as a function of $S_{\text{max}}$ as a complete elliptic integral of the third kind.

For each $b$, the two independent solutions $S_+$ and $S_-$ are analyzed. For $b < 1/4$, all the strings of the $S_-$ solution are unstable ($S_{\text{max}} = \infty$) and never collapse to a point ($S_{\text{min}} \neq 0$). $S_+$ describes one stable ($S_{\text{max}}$ is bounded) oscillating string and $\langle E \rangle$ is an increasing function of $b$ for $0 \leq b \leq 1/4$. For $b > 1/4$, all strings (for both $S_+$ and $S_-$) are unstable and have a collapse during their evolution. For $b = 1/4$, $S_-$ describes two strings (one stable and one unstable for large de Sitter radius), while $S_+$ describes one stable non-oscillating string.
1 Introduction and Results

The study of string dynamics in curved spacetime reveals new insights and new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetime) [1, 2, 3]. The results of this programme are relevant both for fundamental (quantum) strings and for cosmic strings, which behave essentially in a classical way.

Among the cosmological backgrounds of interest, de Sitter spacetime occupies a special place. It is on one hand relevant for inflation, and on the other hand, string propagation turns out to be specially interesting there. String-instability, in the sense that the string proper length grows indefinitely (proportional to the expansion factor of the universe) is particularly present in de Sitter spacetime [1, 4, 5].

Recently, several progresses in the understanding of string propagation in de Sitter spacetime have been performed [6-8]. The classical string equations of motion (plus the string constraints) were shown to be integrable in D-dimensional de Sitter spacetime [7]. They are equivalent to a non-linear sigma model on the grassmannian $SO(D, 1)/O(D)$ with periodic boundary conditions (for a closed string). In addition, the string constraints imply a zero world-sheet energy-momentum tensor, and these constraints are compatible with the integrability. Moreover, the exact string dynamics in de Sitter spacetime is equivalent to a generalized sh-Gordon model with a potential unbounded from below [7]. The sh-Gordon function $\alpha(\sigma, \tau)$ has here a clear physical meaning: $H^{-1}\exp[\alpha(\sigma, \tau)/2]$ determines the proper size of the string ($H$ is the Hubble constant). In $2 + 1$ dimensions, the string dynamics is exactly described by the standard sh-Gordon equation.

More recently, a novel feature for strings in de Sitter spacetime was found: Exact multi-string solutions [8]. Exact circular string solutions were found [9] describing two different strings. One string is stable (the proper size is bounded), and the other one is unstable (the proper size blows up) for large de Sitter radius. Soliton methods, (the so-called ”dressing method” in soliton theory) were implemented using the linear problem (Lax pair) of this system, in order to construct systematically exact string solutions [8]. The one-soliton string solution constructed in this way, generically describe five different and independent strings: one stable string and four unstable strings. These solutions (even the stable string) do not oscillate in time. Exact string solutions oscillatory in time in de Sitter universe were not found until now.
In this paper, we go further in the investigation of exact string solutions in de Sitter spacetime. We find exact string solutions describing \textit{infinitely many} different and independent strings. The novel feature here is that we have one single world-sheet but multiple (infinitely many) strings. The world-sheet time \( \tau \) turns out to be an infinite-valued function of the target string time \( X^0 \) (which can be the hyperboloid time \( q^0 \), the cosmic time \( T \) or the static coordinate time \( t \)). Each branch of \( \tau \) as a function of \( q^0 \) corresponds to a different string. In flat spacetime, multiple string solutions are necessarily described my multiple world-sheets. Here, a single world-sheet describes infinitely many different and simultaneous strings as a consequence of the coupling with the spacetime geometry. These strings do not interact among themselves; all the interaction is with the curved spacetime.

We apply the circular string \textit{Ansatz}:

\[
t = t(\tau), \quad \theta = \sigma, \quad r = \frac{1}{H} f(\tau)
\]

in 2 + 1 dimensional de Sitter spacetime, particularly convenient in terms of the static de Sitter coordinates \( (t, r, \theta) \) (we also describe the solutions in the hyperboloid and cosmic parametrizations). The string equations of motion and constraints can be solved directly and completely in terms of elliptic functions. They reduce to two decoupled first order differential equations for the time component \( t(\tau) \) and the string radius \( f(\tau) \):

\[
\dot{t} = \frac{\sqrt{b}}{H(1-f^2)}
\]

\[
\dot{f}^2 + V(f^2) = b; \quad V(f^2) = f^2 - f^4
\]

The \( f \)-equation is solved by: \( f^2(\tau) = \wp(\tau - \tau_o) + 1/3 \) where \( \wp \) is the Weierstrass elliptic function with discriminant \( \Delta = 16b^2(1 - 4b) \), and \( b \) and \( \tau_o \) are integration constants \( \tau_o \) is generally complex and must be chosen so that \( f(\tau) \) is real for real \( \tau \). The solutions depend on one constant parameter \( b \) related to the string energy, and fall into three classes, depending on whether \( b < 1/4 \) (\( \Delta > 0 \)), \( b = 1/4 \) (\( \Delta = 0 \)) or \( b > 1/4 \) (\( \Delta < 0 \)). As can be seen in the diagram \( (f^2, V(f^2)) \), Fig.1., in which the full string dynamics takes place, these cases correspond to oscillatory motion and to infinite (unbounded) motion.
The proper string size $S$ and energy $E$ of the circular strings are given for all $f$ by:

$$S = \frac{1}{H} f, \quad E = \frac{1}{\alpha' H} \left( f \dot{f} - \sqrt{b} \right) \frac{f^2 - 1}{f^2 - 1}.$$

We find for an expanding string ($\dot{f} > 0$), see section 5:

$$E(f \approx 0) = \frac{\sqrt{b}}{\alpha' H} (1 - f),$$
$$E(f = 1) = \frac{1}{\alpha' H} \frac{(1 + b)}{2 \sqrt{b}},$$
$$E(f >> 1) = \frac{1}{\alpha' H} f.$$

Notice also that the energy is non-zero, even at the collapse ($f = 0$), (except for the degenerate case $f = b = 0$, in which there is no string at all). It follows from these expressions that for a string expanding from zero radius, the energy first decreases, and then increases for large $f$, proportional to the invariant string size. For a string oscillating between $f = 0$ and $f_{\text{max}} = \sqrt{(1 - \sqrt{1 - 4b})/2}$ (in the $b < 1/4$-case, see Fig.1.), the average energy $<E>$ over a period $T$ is:

$$H \alpha' <E> = \frac{2\sqrt{b}}{T \sqrt{1 - f_{\text{max}}^2}} \Pi(f_{\text{max}}^2, \frac{f_{\text{max}}}{\sqrt{1 - f_{\text{max}}^2}}),$$

in terms of the complete elliptic integral of third kind $\Pi$.[9]

The string solutions in de Sitter spacetime enjoy conserved quantities associated with the $O(3, 1)$ rotations on the hyperboloid. For the circular solutions under consideration here, the only non-zero component is given by:

$$L_{01} = -L_{10} = 2\pi \sqrt{b}.$$

In the $b = 1/4$-case, the Weierstrass elliptic function degenerates into a hyperbolic function:

$$f^2(\tau) = \frac{1}{2} [1 + \sinh^{-2} (\frac{\tau - \tau_o}{\sqrt{2}})].$$
Two real independent solutions appear for the choices $\tau_o = i\pi/2$ and $\tau_o = 0$, respectively:

$$f_\pm^2(\tau) = \frac{1}{2} [\tanh(\frac{\tau}{\sqrt{2}})]^\pm 2.$$  

(They were previously found in ref. [3]) We have also the solution $f_0^2 = 1/2$, corresponding to a stable string with constant proper size $S_0 = 1/(\sqrt{2}H)$ (i.e., sh-Gordon function $\alpha = 0$). This solution was found in ref. [6] and we do not discuss it here.

The solution $f_-$ describes two different strings, I and II, as it can be seen from the hyperboloid time $q_0^0(\tau)$, eq. (5.19), Fig.2a. Here $\tau$ is a two-valued function of $q_0^0$: String I corresponds to $-\infty < \tau < 0$ and string II to $0 < \tau < \infty$. The proper size and energy $S_-$ and $E_-$ for both strings are given by eq. (5.20). For $q_0^0 \to \infty$, string I is unstable, while string II is stable. Both $S_-(q_0^0 \to \infty)$ and $E_-(q_0^0 \to \infty)$ blow up for string I (for which $q_0^0 \to \infty$ corresponds to $\tau \to 0$), while they tend to constant values for string II (for which $q_0^0 \to \infty$ corresponds to $\tau \to \infty$). String I starts with minimal size $S_- = 1/(\sqrt{2}H)$ and $E_- = 1/(\alpha'H)$ at $\tau = -\infty$ and blows up at $\tau = 0$. String II starts with infinite size at $\tau = 0$ but approaches $S_- = 1/(\sqrt{2}H)$ and $E_- = 1/(\alpha'H)$ for $\tau \to \infty$.

The solution $f_+$ of this $b = 1/4$ case describes only one stable string-$q_0^+(\tau)$, eq. (5.26), is a monotonically increasing function of $\tau$ with proper size $S_+$ and energy $E_+$ given by eqs. (5.26). The string starts with $S_+ = 1/(\sqrt{2}H)$, $E_+ = 1/(\alpha'H)$ at $q_0^+ = -\infty$, it contracts until it collapses ($S_+ = 0$, $E_+ = 1/(2\alpha'H)$), then it expands until it reaches the original size and energy for $q_0^+ = \infty$. The average energy is $< E_+ >= 1/(\alpha'H)$ which is equal to the maximal energy. The string has minimal energy shortly after the collapse, as follows from the general expression (5.11). For $b = 1/4$ the evolution is always non-oscillatory. Even the stable string does not oscillate in time.

For $b < 1/4$ there exist two real independent solutions for the choices $\tau_o = 0$ and $\tau_o = \omega'$, where $\omega'$ is the imaginary semi-period, eq. (4.13), of the Weierstrass function:

$$f_-^2(\tau) = \wp(\tau) + 1/3,$$
$$f_+^2(\tau) = \wp(\tau + \omega') + 1/3.$$  

$f_-$ and $f_+$ are oscillating solutions as functions of $\tau$. The solution $f_-$ describes infinitely many strings; $f_-$ has infinitely many branches $[0, 2\omega], [2\omega, 4\omega], ...$.
each of which corresponds to a different string ($\omega$ is the real semi-period, eq. (4.15), of the Weierstrass function). This can be seen from the hyperboloid time $q^0_-(\tau)$, Fig.3a.: The world-sheet time $\tau$ is an infinite-valued function of $q^0_-$.

The hyperboloid time $q^0_-$ blows up at the boundaries of the branches $\tau = \pm 2N\omega$ ($N$ being an integer):

$$|q^0_-(\tau)| \sim \frac{1}{|2N\omega - \tau|}.$$  

Further insight is obtained by considering the cosmic time $T_-$ and the static coordinate time $t_-$. Closed expressions for them are given by eqs. (5.33)-(5.35) and (5.40), in terms of Weierstrass $\zeta$ and $\sigma$-functions, and also rewritten in terms of elliptic theta-functions. The cosmic time $T_-$ is singular at $\tau = 0$, $\tau = x/\mu$, $\tau = 2\omega$ and similarly in the other branches. ($x$, $\mu$ are two real constants. $x$ is expressed as an incomplete elliptic integral of first kind while $\mu = \sqrt{(1 + \sqrt{1 - 4b})/2}$). The static coordinate time $t_-$, on the other hand, is regular at the boundaries of the branches, but is singular at $\mu \tau = 2KN \pm x$:

$$t_-(\tau) \sim \frac{1}{2\pi} \log |\mu \tau - 2KN \mp x|,$$

where $K$ is a complete elliptic integral of the first kind. It must be noticed that although $f_-$ is periodic in $\tau$, the cosmic time is not, i.e. $T_-(\tau) \neq T_-(\tau + 2\omega)$, eq. (5.43). This implies that the infinitely many strings are different (the difference in their invariant proper size for a given cosmic time $T$ is given by eq. (5.48)), but they are all of the same type: unstable. For instance, in the branch $\tau \in [0, 2\omega]$, the string starts at $\tau = 0$ ($q^0_- = -\infty$) with infinite size, then contracts to the minimal size $HS_- = \sqrt{(1 + \sqrt{1 - 4b})/2}$ and eventually expands towards infinite size at $\tau = 2\omega$ ($q^0_- = \infty$). These solutions never collapse.

For the solution $f_+$ of the $b < 1/4$-case, the string dynamics takes place inside the horizon. $f_+$, being a regularly oscillating function of $\tau$, is then also a regularly oscillating function of the string times $q^0_+$, $T_+$ and $t_+$. The static coordinate time $t_+$, from which one easily deduces $q^0_+$ and $T_+$, is given in terms of theta-functions, eq. (5.51), and reexpressed in terms of the Jacobi zeta-function, eq. (5.52). The solution $f_+$ describes one stable string oscillating between its minimal size $S_+ = 0$ (collapse) and its maximal size $HS_+ =$
The string energy $E_+$ is given by eq. (5.56), and the average energy $<E_+>$ is a monotonically increasing function of $b$, for $b \in [0, 1/4]$. It must be noticed that the string oscillations here do not follow a pure harmonic motion as in flat Minkowski spacetime, but they are precise superpositions of all frequencies $(2n-1)\Omega$, $(\Omega = \pi\mu/(2K)$, $n = 1, 2, ..., \infty$); here the non-linearity of the string equations of motion fixes the relation between the mode coefficients, and the basic frequency $\Omega$ depends on the string energy.

For $b > 1/4$ two real independent solutions are obtained for $\tau_o = 0$ and $\tau_o = \omega'_2$, where $\omega'_2$ is the imaginary semi-period of the Weierstrass function:

$$f_+^2(\tau) = \varphi(\tau) + 1/3,$$
$$f_-^2(\tau) = \varphi(\tau + \omega'_2) + 1/3.$$

In this case $f_-$ again describes infinitely many strings, all of them are unstable. The difference with the $b < 1/4$-case, is that here the strings have a collapse during their evolution. For instance, in the branch $\tau \in [0, 2\omega_2]$, where $\omega_2$ is the real semi-period of the Weierstrass function, the string starts with infinite size at $\tau = 0$ ($q^0_\tau = -\infty$), it then contracts until it collapses to a point and then it expands towards infinite size again (at $\tau = 2\omega_2$ ($q^0_\tau = \infty$)).

In contrast to the $b < 1/4$-case, the solution $f_+$ is here just a time translated version of $f_-$:

$$f_+^2(\tau) = f_-^2(\tau + \omega_2),$$

and describes therefore essentially the same features as the solution $f_-$.  

A summary picture of the main properties of the solutions of this paper is presented in Table I. The paper is organized as follows: In section 2 we formulate the string dynamics in 2 + 1 dimensional de Sitter spacetime and apply to it the circular string Ansatz. In section 3 we describe the problem in the static parametrization. In section 4 we find the closed and complete string solutions in terms of elliptic functions. Section 5 deals with the physical interpretation of these solutions and the concept of infinitely many strings.
2 String Dynamics in 2+1 de Sitter Space-time

It is well known that the 2 + 1 dimensional de Sitter spacetime can be considered as a 3 dimensional hyperboloid embedded in 4 dimensional flat Minkowski space:

\[ ds^2 = \frac{1}{H^2} \eta_{\mu\nu} dq^\mu dq^\nu, \]  

(2.1)

where \( \mu = (0, 1, 2, 3) \), \( \eta_{\mu\nu} = diag(-1, 1, 1, 1) \), \( H \) is the Hubble constant and we require:

\[ \eta_{\mu\nu} q^\mu q^\nu = 1. \]  

(2.2)

The equations of motion for the bosonic string in the conformal gauge takes the form [6]:

\[ q^\mu_+ - e^\alpha q^\mu = 0, \]  

(2.3)

where:

\[ e^{\alpha(\tau, \sigma)} \equiv -\eta_{\mu\nu} q^\mu_+ q^\nu_. \]  

(2.4)

and we have introduced the notation \( q^\mu_\pm = \frac{1}{2}(\partial_\sigma \pm \partial_\tau)q^\mu \), etc. The equations of motion are as usual supplemented by the constraints, that take the form:

\[ \eta_{\mu\nu} q^\mu_\pm q^\nu_\pm = 0. \]  

(2.5)

It was shown by de Vega and Sánchez [7] that the function \( \alpha(\tau, \sigma) \) fulfills the sh-Gordon equation:

\[ \ddot{\alpha} - \alpha'' - e^\alpha + e^{-\alpha} = 0. \]  

(2.6)

Therefore, one can first look for solutions \( \alpha(\tau, \sigma) \) to this equation and then work backwards trying to solve the linear (in \( q^\mu \)) equation (2.3) and finally impose the constraints (2.5). In the special case of circular string configurations under consideration here it will turn out, however, that the original equations of motion can be solved directly (and completely), so in this case it is not necessary to first solve equation (2.6). Before we come to the special solutions let us remark that \( \alpha(\tau, \sigma) \) generally determines the invariant string size. This follows from the observation that the line element (2.1) can be written as:

\[ ds^2 = \frac{1}{2H^2} e^{\alpha(\tau, \sigma)}(d\sigma^2 - d\tau^2), \]  

(2.7)
i.e., we can identify:

\[ S(\tau, \sigma) \equiv \frac{1}{\sqrt{2H}} e^{\alpha(\tau, \sigma)/2} \quad (2.8) \]

as the invariant string size.

The circular strings are obtained by the following Ansatz:

\[ q^\mu = (q^0(\tau), q^1(\tau), f(\tau) \cos \sigma, f(\tau) \sin \sigma) \quad (2.9) \]

In this case the normalization condition (2.2) reads:

\[ (q^0)^2 - (q^1)^2 - f^2 = -1. \quad (2.10) \]

The equations of motion (2.3) become:

\[ \ddot{q}^0 = e^\alpha q^0, \quad (2.11) \]
\[ \ddot{q}^1 = e^\alpha q^1 \quad (2.12) \]

as well as:

\[ \ddot{f} + f = e^\alpha f. \quad (2.13) \]

The definition (2.4) is:

\[ e^\alpha = (\dot{q}^0)^2 - (\dot{q}^1)^2 - \dot{f}^2 + f^2, \quad (2.14) \]

and the constraints (2.5) become:

\[ (q^0)^2 - (q^1)^2 - \dot{f}^2 - f^2 = 0. \quad (2.15) \]

The 6 equations (2.10)-(2.15) give an overconstrained system for \((q^0, q^1, f, \alpha)\). Due to the non-linear nature it looks quite complicated, but we will now see that the complete solution can be easily found.

Subtraction of (2.14) and (2.15) immediately yields:

\[ e^\alpha = 2\dot{f}^2 \quad (2.16) \]

and then eq. (2.13) is integrated to:

\[ \dot{f}^2 + f^2 - f^4 = \text{const} \equiv b. \quad (2.17) \]
This equation is generally solved in terms of a Weierstrass elliptic function, and we shall return to the explicit solution in section 4. Equation (2.10) is formally solved by:

\[ q^0 = \sqrt{1 - f^2} \sinh Ht, \]
\[ q^1 = \sqrt{1 - f^2} \cosh Ht, \]  
(2.18)

for \( 1 - f^2 \geq 0 \) and by:

\[ q^0 = \sqrt{f^2 - 1} \cosh Ht, \]
\[ q^1 = \sqrt{f^2 - 1} \sinh Ht, \]  
(2.19)

for \( 1 - f^2 \leq 0 \). Here \( t = t(\tau) \) and as usual we should make an extra copy of \((q^0, q^1)\) to cover all of the de Sitter geometry (since \( q^0 + q^1 \geq 0 \), using eqs. (2.18)-(2.19)). Now equation (2.15) is fulfilled provided:

\[ \dot{t} = \frac{\sqrt{b}}{H(1 - f^2)} \]  
(2.20)

and (2.11),(2.12) are trivially fulfilled! The original system of equations and constraints has now been reduced to the two separated first order equations (2.17) and (2.20).

Let us for a moment return to the function \( \alpha \) introduced in eq. (2.4). From eq. (2.16) we find that \( 4f \dot{f} = e^\alpha \dot{\alpha}, \ 4f \ddot{f} = (\ddot{\alpha} + \dot{\alpha}^2/2)e^\alpha \), so that eq. (2.17) leads to:

\[ \ddot{\alpha} - e^\alpha + 4be^{-\alpha} = 0. \]  
(2.21)

The redefinitions \( \tilde{\tau} = (4b)^{1/4} \tau, \ \alpha(\tau) = \frac{1}{2} \log 4b + \tilde{\alpha}(\tilde{\tau}) \) yield:

\[ \frac{d^2 \tilde{\alpha}}{d\tilde{\tau}^2} - e^{\tilde{\alpha}} + e^{-\tilde{\alpha}} = 0, \]  
(2.22)

that is, the sh-Gordon equation, as was proved more generally by de Vega and Sánchez [7]. Note that in the special case where \( b = 1/4 \) (corresponding to \( E = -2 \) in the notation of Ref.[6]) the redefinitions are trivial.

### 3 Static Parametrization and String Radius

In this section we will show that the results of the previous section can be obtained in an easier way by starting directly from the static parametrization.
In the static parametrization the line element of \(2+1\) dimensional de Sitter spacetime takes the form:

\[
ds^2 = -(1 - H^2 r^2)dt^2 + \frac{dr^2}{(1 - H^2 r^2)} + r^2 d\theta^2.
\]  

(3.1)

Writing \(x^\mu = (t, r, \theta)\) and \(g_{\mu\nu} = \text{diag}(-(1 - H^2 r^2), (1 - H^2 r^2)^{-1}, r^2)\) the equations of motion and constraints for the bosonic string in the conformal gauge read \((\mu = 0, 1, 2)\):

\[
\ddot{x}^\mu - x^{\nu\mu} + \Gamma^\mu_{\rho\sigma}(\dot{x}^{\rho} \dot{x}^{\sigma} - x^{\rho} x^{\nu}) = 0,
\]

\[
g_{\mu\nu}\dot{x}^\mu \dot{x}^{\nu} = g_{\mu\nu}(\dot{x}^\mu \dot{x}^{\nu} + x^{\mu} x^{\nu}) = 0,
\]

(3.2)

where the non-vanishing components of the Christoffel symbol are:

\[
\Gamma^r_{rr} = \frac{H^2 r}{1 - H^2 r^2}, \quad \Gamma^r_{tt} = -H^2 r(1 - H^2 r^2),
\]

\[
\Gamma^r_{\theta\theta} = -r(1 - H^2 r^2), \quad \Gamma^t_{rr} = \frac{-H^2 r}{1 - H^2 r^2}, \quad \Gamma^\theta_{\theta r} = \frac{1}{r}.
\]

The Ansatz for the circular string in the static coordinates is:

\[
t = t(\tau), \quad \theta = \sigma, \quad r = \frac{1}{H} f(\tau).
\]

(3.3)

In this case, equations (3.2) lead to:

\[
\ddot{t} - 2 \frac{t \dot{f}^2}{1 - t^2} = 0,
\]

\[
\ddot{f} + \frac{f \dot{f}^2}{1 - t^2} - f(1 - f^2) H^2 t^2 + f(1 - f^2) = 0,
\]

\[
-(1 - f^2) H^2 t^2 + \frac{\dot{f}^2}{1 - t^2} + f^2 = 0.
\]

(3.4)

The \(\ddot{t}\)-equation is immediately integrated and then all three equations are consistently solved by:

\[
\dot{f}^2 + f^2 - f^4 = b,
\]

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\[ i = \frac{\sqrt{b}}{H(1-f^2)}. \tag{3.5} \]

where \( b \) is an integration constant. This is in agreement with the results of section 2; compare with equations (2.17) and (2.20). Note that the constant \( b \) must be positive to ensure a real static coordinate time and that there is a horizon at \( f^2 = 1 \).

We close this section by the following observation: The line element in the static coordinates (3.1) is now:

\[ ds^2 = \frac{f^2}{H^2}(d\sigma^2 - d\tau^2), \tag{3.6} \]

that should be compared with eqs. (2.7)-(2.8), that is:

\[ f^2 = \frac{e^\alpha}{2} \equiv H^2 S^2, \tag{3.7} \]

which is in agreement with eq. (2.16).

\section{4 Expanding and Oscillating Elliptic Function Solutions}

We now come to the explicit solution of equation (2.17) or equivalently of the first equation of (3.5). The \( \dot{t} \)-equation will be analysed in detail in the following section where we consider the physical interpretation of the various solutions. Equation (2.17) is solved by:

\[ f^2(\tau) = \wp(\tau - \tau_o) + 1/3, \tag{4.1} \]

where \( \wp \) is the Weierstrass elliptic \( \wp \)-function:

\[ \dot{\wp}^2 = 4\wp^3 - g_2\wp - g_3, \tag{4.2} \]

with invariants:

\[ g_2 = 4(\frac{1}{3} - b), \quad g_3 = \frac{4}{3}(\frac{2}{9} - b), \tag{4.3} \]

discriminant:

\[ \Delta \equiv g_2^3 - 27g_3^2 = 16b^2(1 - 4b), \tag{4.4} \]
and roots \((e_1, e_2, e_3)\):

\[
\frac{1}{6}(1 + 3\sqrt{1 - 4b}), \quad \frac{1}{6}(1 - 3\sqrt{1 - 4b}), \quad -\frac{1}{3}.
\]

Finally \(\tau_o\) is a complex integration constant that must be carefully chosen to obtain a real \(f(\tau)\) for real \(\tau\); recall that the Weierstrass function is only real on a lattice in the complex plane.

It is convenient to write equation (2.17) in the form:

\[
\dot{f}^2 + V(f^2) = b; \quad V(f^2) \equiv f^2 - f^4
\]

and to consider separately the 3 cases \(b = 1/4 (\Delta = 0)\), \(b < 1/4 (\Delta > 0)\) and \(b > 1/4 (\Delta < 0)\), see Fig.1.

\(b = 1/4\): In this case the Weierstrass function reduces to a hyperbolic function and eq. (4.1) becomes:

\[
f^2(\tau) = \frac{1}{2}(1 + \sinh^{-2}\left(\frac{\tau - \tau_o}{\sqrt{2}}\right)).
\]

Two real independent solutions are obtained by the choices \(\tau_o = 0\) and \(\tau_o = i\pi/\sqrt{2}\), respectively:

\[
\begin{align*}
    f_+^2(\tau) &= \frac{1}{2} \tanh^{-2}\frac{\tau}{\sqrt{2}}, \\
    f_-^2(\tau) &= \frac{1}{2} \tanh^2\frac{\tau}{\sqrt{2}}.
\end{align*}
\]

Note that:

\[
\begin{align*}
    f_+^2(-\infty) &= \frac{1}{2}, \quad f_-^2(0) = \infty, \quad f_-^2(\infty) = \frac{1}{2}, \\
    f_+^2(-\infty) &= \frac{1}{2}, \quad f_+^2(0) = 0, \quad f_+^2(\infty) = \frac{1}{2}.
\end{align*}
\]

These are the 2 solutions originally found by de Vega, Sánchez and Mikhailov [6], corresponding to \(\alpha > 0\) and \(\alpha < 0\) respectively. The interpretation of these solutions as a function of the world-sheet time \(\tau\) is clear from Fig.1: The solution (4.8) expands from \(f_+^2 = 1/2\) towards infinity and then contracts until it reaches its original size. The solution (4.9) contracts from \(f_-^2 = 1/2\) until it collapses. It then expands again until it reaches its original size. The physical interpretation, that is somewhat more involved, was described
in Ref. [3] and will be shortly reviewed in section 5. There is actually also a stationary solution for \( b = 1/4 \), corresponding to \( \alpha = 0 \), i.e. a string sitting on the top of the potential; see Fig.1. This solution with constant string size \( S = 1/(\sqrt{2}H) \) was discussed in Ref. [3] and will not be considered here.

**b < 1/4:** Here two real independent solutions are obtained by the choices \( \tau_o = 0 \) and \( \tau_o = \omega' \), respectively:

\[
\begin{align*}
    f^2_2(\tau) &= \wp(\tau) + 1/3, \\
    f^2_2(\tau + \omega') &= \wp(\tau + \omega') + 1/3,
\end{align*}
\]

(4.11) (4.12)

where \( \omega' \) is the imaginary semi-period of the Weierstrass function. It is explicitly given by [4]:

\[
\omega' = i \frac{\sqrt{2}K'(k)}{\sqrt{1 + \sqrt{1 - 4b}}} ; \quad k = \sqrt{\frac{1 - \sqrt{1 - 4b}}{1 + \sqrt{1 - 4b}}}
\]

(4.13)

Note that:

\[
\begin{align*}
    f^2_2(0) &= \infty, \quad f^2_2(\omega) = (1 + \sqrt{1 - 4b})/2, \quad f^2_2(2\omega) = \infty, \\
    f^2_+ (0) &= 0, \quad f^2_+ (\omega) = (1 - \sqrt{1 - 4b})/2, \quad f^2_+ (2\omega) = 0,
\end{align*}
\]

(4.14)

where \( \omega \) is the real semi-period of the Weierstrass function:

\[
\omega = \frac{\sqrt{2}K(k)}{\sqrt{1 + \sqrt{1 - 4b}}},
\]

(4.15)

and \( K' \) and \( K \) are the complete elliptic integrals of first kind. The interpretation of these solutions as a function of \( \tau \) is clear from Fig.1: The solution (4.11) oscillates between infinity and its minimal size \( f^-_2 = (1 + \sqrt{1 - 4b})/2 \) at the boundary of the potential, while the solution (4.12) oscillates between 0 and its maximal size \( f^+_2 = (1 - \sqrt{1 - 4b})/2 \). The physical interpretation will be considered in the following section.

**b > 1/4:** In this last case two real independent solutions are obtained by the choices \( \tau_o = 0 \) and \( \tau_o = \omega'_2 \), respectively:

\[
\begin{align*}
    f^2_-(\tau) &= \wp(\tau) + 1/3,
\end{align*}
\]

(4.16)
\[ f_2^2(\tau) = \wp(\tau + \omega_2') + 1/3, \quad (4.17) \]

where \( \omega_2' \) takes the explicit form:

\[ \omega_2' = i \frac{K'(\hat{k})}{b^{1/4}}; \quad \hat{k} = \sqrt{\frac{1}{2} + \frac{1}{4\sqrt{b}}}. \quad (4.18) \]

Note that:

\[
\begin{align*}
    f_2^2(0) &= \infty, & f_2^2(\omega_2) &= 0, & f_2^2(2\omega_2) &= \infty, & \ldots \\
    f_2^2(0) &= 0, & f_2^2(\omega_2) &= \infty, & f_2^2(2\omega_2) &= 0, & \ldots
\end{align*}
\quad (4.19)
\]

where:

\[ \omega_2 = \frac{K(\hat{k})}{b^{1/4}}. \quad (4.20) \]

It should be stressed that in this case the primitive semi-periods are \( \hat{\omega} = (\omega_2 - \omega_2')/2 \) and \( \hat{\omega}' = (\omega_2 + \omega_2')/2 \), i.e. \( (2\hat{\omega}, 2\hat{\omega}') \) spans a fundamental period parallelogram in the complex plane.

The interpretation of the solutions (4.16)-(4.17) as a function of \( \tau \) follows from Fig.1: Both of them oscillates between zero size (collapse) and infinite size (instability). The physical interpretations will follow in the next section.

## 5 Physical Interpretation. Infinitely many Strings

We now discuss the physical interpretation of the results obtained in section 4. For that purpose it is convenient to also describe the solutions in terms of hyperboloid and/or comoving (cosmic) coordinates. The hyperboloid coordinates were already introduced in section 2, and the relation to the comoving coordinates \((T, X^1, X^2)\) is given by:

\[
\begin{align*}
    q^0 &= \sinh HT + \frac{H^2}{2} e^{HT} [(X^1)^2 + (X^2)^2], \\
    q^1 &= \cosh HT - \frac{H^2}{2} e^{HT} [(X^1)^2 + (X^2)^2], \\
    q^2 &= H e^{HT} X^1, & q^3 &= H e^{HT} X^2,
\end{align*}
\quad (5.1-5.3)\]
for $-\infty < T, X^1, X^2 < +\infty$. That is:

$$T = \frac{1}{H} \log(q^0 + q^1), \quad X^1 = \frac{1}{H(q^0 + q^1)} q^2, \quad X^2 = \frac{1}{H(q^0 + q^1)} q^3. \quad (5.4)$$

The relation to the static coordinates of sections 3 and 4 follows from eqs. (2.9) and (2.18)-(2.19). The relation of the comoving coordinates to the static parametrization (3.1) is given by:

$$T = \log \frac{|1 - H^2 r^2|}{2H} + t, \quad (5.5)$$

$$X^1 = \frac{r \cos \theta}{|1 - H^2 r^2|} e^{-Ht}, \quad (5.6)$$

$$X^2 = \frac{r \sin \theta}{|1 - H^2 r^2|} e^{-Ht}, \quad (5.7)$$

and the line element becomes:

$$ds^2 = -dT^2 + e^{2HT}[(dX^1)^2 + (dX^2)^2], \quad (5.8)$$

The string solution is then expressed in terms of the comoving coordinates through equations (3.3),(4.1) and (2.20) for $r, \theta = \sigma$ and $t$.

In describing the physical properties of our string solutions we will also need the string energy that is computed from the spacetime string energy-momentum tensor $(X^A = T, X^1, X^2)$:

$$\sqrt{-G}T^{AB}(X) = \frac{1}{2\pi \alpha'} \int d\sigma d\tau (\dot{X}^A \dot{X}^B - X'^A X'^B) \delta^{(3)}(X - X(\tau, \sigma)). \quad (5.9)$$

In the cases under consideration here, the cosmic time $X^0 = T \equiv T(\tau)$ is a function of $\tau$ only, and the string energy becomes:

$$E(T) = \int d^2 X \sqrt{-G} T^{00}(X) = \frac{1}{\alpha'} \frac{dT}{d\tau}. \quad (5.10)$$

From this expression we can actually get a lot of information about the energy without using the explicit time evolution of the strings found in section 4. From eqs. (5.5) and (3.5) it follows:

$$H\alpha'E = \frac{f \dot{f} - \sqrt{b}}{f^2 - 1}; \quad \dot{f}^2 = b + f^2(f^2 - 1), \quad (5.11)$$
giving the energy as a function of the invariant string size. We immediately see that the energy is non-zero except for the degenerate case \( f = b = 0 \), where there is no string at all. At the horizon, the energy for an expanding string \((\dot{f} > 0)\) is:

\[
H\alpha' E = \frac{b + 1}{2\sqrt{b}}; \quad f = 1
\]

(5.12)

while at \( f = 0 \):

\[
H\alpha' E = \sqrt{b}; \quad f = 0
\]

(5.13)

Considering a string expanding from \( f = 0 \) we find for small \( f \):

\[
H\alpha' E \approx \sqrt{b}(1 - f), \quad f \approx 0
\]

(5.14)

while for large \( f \):

\[
H\alpha' E \sim f; \quad f >> 1
\]

(5.15)

i.e. the string energy first decreases but eventually increases proportionally to the invariant string size. Considering instead a string configuration oscillating between 0 and a maximal radius \( f_{\text{max}} \) \((\dot{f}_{\text{max}} = 0)\) we can calculate the average energy by integrating over a period (say) \( T \). The first term of eq. (5.11) does not contribute since it is a total derivative of a periodic function and from the second term we find:

\[
H\alpha' < E > = \frac{\sqrt{b}}{T} \int_0^T \frac{d\tau}{1 - f^2}
\]

\[
= \frac{2\sqrt{b}}{T} \int_0^{f_{\text{max}}} \frac{df}{|\dot{f}|(1 - f^2)}
\]

\[
= \frac{2\sqrt{b}}{T} \int_0^{f_{\text{max}}} \frac{df}{(1 - f^2)(f^2 - f_{\text{max}}^2)(f^2 - (1 - f_{\text{max}}^2))}
\]

\[
= \frac{2\sqrt{b}}{T\sqrt{1 - f_{\text{max}}^2}} \Pi(f_{\text{max}}^2, f_{\text{max}}/\sqrt{1 - f_{\text{max}}^2}), \quad (5.16)
\]

where \( \Pi \) is the complete elliptic integral of third kind \([9]\) and \( f_{\text{max}} \) is the smallest root of \( \dot{f} = 0 \):

\[
f_{\text{max}} = \sqrt{(1 - \sqrt{1 - 4b})/2}.
\]

(5.17)
Notice that this result has been obtained without using any information about the detailed time evolution of the string.

Let us finally remark that the string solutions in 2+1 dimensional de Sitter spacetime enjoy as conserved quantities those associated with the $O(3,1)$ rotations on the hyperboloid (2.2):

$$L_{\mu\nu} = -L_{\nu\mu} = \int_0^{2\pi} d\sigma (q_\mu \dot{q}_\nu - q_\nu \dot{q}_\mu).$$  \hspace{1cm} (5.18)

After these introductory remarks let us now reanalyse our string solutions in the three cases $b = 1/4$, $b < 1/4$ and $b > 1/4$.

5.1 The hyperbolic $b = 1/4$ solutions

This case was analysed in Ref.[6], but let us restate and reinterpret some of the results here. This is justified by the fact that many of the important features of this case generalize to the $b \neq 1/4$ solutions (the elliptic solutions).

We first consider the $f_-$-solution (4.8). The hyperboloid time $q^0(\tau)$ is obtained from equations (2.18), (2.19) and their $(q^0 + q^1 \leq 0)$-counterparts, and by integrating eq. (2.20). The result is:

$$q^0(\tau) = \sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \coth \frac{\tau}{\sqrt{2}}.$$  \hspace{1cm} (5.19)

When we plot this function (Fig.2a.) we see that the string solution actually describes 2 strings (I and II) [6], since $\tau$ is a two-valued function of $q^0$. For both strings the invariant size and energy are given by eqs. (3.7) and (5.10), respectively:

$$S_- = \frac{1}{\sqrt{2}H} \coth | \frac{\tau}{\sqrt{2}} |,$$

$$E_- = \frac{1}{\alpha' H} | 1 + \frac{1}{\cosh \sqrt{2} \tau - \frac{1}{\sqrt{2}} \sinh \sqrt{2} \tau - 1} |.$$  \hspace{1cm} (5.20)

but string I corresponds to $\tau \in (-\infty, 0]$ and string II to $\tau \in [0, \infty]$. Therefore, $q^0_0 \rightarrow \infty$ corresponds to $\tau \rightarrow 0_-$ for string I, but to $\tau \rightarrow \infty$
for string II. More generally, when $q_0^{-} \to \infty$ both the invariant size and
the energy grow infinitely for string I, while they approach constant values
for string II. We conclude that string I is an unstable string for $q_0^{-} \to \infty$,
while string II is a stable string. More details about the connection between
hyperboloid time and world-sheet time for these solutions can be found in
Ref.[6].

Let us consider now the cosmic time, used to calculate the energy (5.20),
of the solution $f_-^-$ in a little more detail:

$$T^-(\tau) = \frac{1}{H} (\tau + \log \left| \frac{1}{\sqrt{2}} \coth \frac{\tau}{\sqrt{2}} - 1 \right|).$$

(5.21)

When we plot this function (Fig.2b.) we find that $\tau$ is a three-valued function
of $T^-$. What happens is that the time interval $\tau \in \] 0, \infty [$ for string II splits
into two parts. These features are easily understood when returning to the
effective potential Fig.1.: String I starts at $f_-^- = 1/2$ for $\tau = HT^- = -\infty$, it
then expands through the horizon $f_-^- = 1$ at:

$$\tau = -\sqrt{2} \log(1 + \sqrt{2}), \quad HT^- = \log 2 - \sqrt{2} \log(1 + \sqrt{2})$$

(5.22)

and continues towards infinity for $\tau \to 0_-, \ HT^- \to \infty$. String II starts at
infinity for $\tau = 0_+, \ HT^- = \infty$ and contracts through the horizon at:

$$\tau = \sqrt{2} \log(1 + \sqrt{2}), \quad HT^- = -\infty.$$  

(5.23)

This behaviour, approaching the horizon from the outside, corresponds to
the going backwards in cosmic time -part of Fig.2b. String II then continues
contracting from $f_-^- = 1$ at:

$$\tau = \sqrt{2} \log(1 + \sqrt{2}), \quad HT^- = -\infty,$$

(5.24)

until it collapses at $\tau = HT^- = \infty$.

We now consider briefly the $f_+^-$-solution (4.9). In this case the hyperboloid
time is given by:

$$q_0^+ (\tau) = \sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \tanh \frac{\tau}{\sqrt{2}},$$

(5.25)
which is a monotonically increasing function of $\tau$. The $f_+$-solution therefore describes only one string. The proper size and energy are given by:

\[ S_+ = \frac{1}{\sqrt{2}H} \tanh \left| \frac{\tau}{\sqrt{2}} \right|, \]

\[ E_+ = \frac{1}{\alpha'H}(1 - \cosh \frac{\sqrt{2}\tau}{2} - \frac{1}{\sqrt{2}} \sinh \frac{\sqrt{2}\tau}{2} + 1). \]

(5.26)

We can also express this solution in terms of the cosmic time:

\[ T_+ (\tau) = \frac{1}{H} \left[ \tau + \log(1 - \frac{1}{\sqrt{2}} \tanh \frac{\tau}{\sqrt{2}}) \right], \]

(5.27)

but since everything now takes place well inside the horizon this will not really give us more insight. The string starts with $f_+^2 = 1/2$ for $\tau = HT_+ = -\infty$ with the energy $E_+ = 1/(\alpha'H)$. It then contracts until it collapses for $\tau = HT_+ = 0$ where the energy is reduced to $E_+ = 1/(2\alpha'H)$. It now expands again and eventually reaches $f_+^2 = 1/2$ for $\tau = HT_+ = \infty$, where it has regained its original energy. Note that the string has the minimal energy for $\tau = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$, i.e. shortly after expanding from $f = 0$, thus confirming the general result obtained from equation (5.11). We also remark that in this case the average energy is actually equal to the maximal energy $< E_+ > = 1/(\alpha'H)$.

Let us close this analysis of the $b = 1/4$ case by mentioning that for both the $f_+$ and the $f_-$-solutions there is only one non-vanishing component of $L_{\mu\nu}$ introduced in (5.18). It is explicitly given by:

\[ L_{10} = \pi. \]

(5.28)

5.2 The elliptic $b < 1/4$ solutions

The calculations are here going to be somewhat more complicated since we deal with elliptic functions as compared to hyperbolic functions. It is therefore convenient to first introduce a more compact notation. Defining:

\[ \mu \equiv \sqrt{\frac{1 + \sqrt{1 - 4b}}{2}}, \quad \nu \equiv \sqrt{\frac{1 - \sqrt{1 - 4b}}{2}}, \]

(5.29)
so that $\mu^2 + \nu^2 = 1$, $0 \leq \nu \leq 1/\sqrt{2} \leq \mu \leq 1$, $\mu\nu = \sqrt{b}$, we find from eqs. (4.13),(4.15):

$$k = \frac{\nu}{\mu}, \quad \omega = \frac{K(k)}{\mu}, \quad \omega' = i \frac{K'(k)}{\mu}. \quad (5.30)$$

The solutions (4.11) and (4.12) can be written as:

$$f^2_-(\tau) = \frac{\mu^2}{\operatorname{sn}^2[\mu\tau | k]}, \quad (5.31)$$

$$f^2_+(\tau) = \nu^2 \operatorname{sn}^2[\mu\tau | k], \quad (5.32)$$

respectively.

Consider first the $f_-$-solution (5.31). It is clear from eq. (4.14) and the periodicity in general that we have infinitely many branches $[0, 2\omega]$, $[2\omega, 4\omega]$, .... We will see in a moment that each of these branches actually corresponds to one string, that is, the $f_-$-solution describes infinitely many strings. For that purpose we will need the hyperboloid time and the cosmic time as a function of $\tau$. Both of them are expressed in terms of the static coordinate time $t$, that is obtained by integrating (2.20):

$$Ht_-(\tau) = \zeta(x/\mu)\tau + \frac{1}{2} \log \left| \frac{\sigma(\tau - x/\mu)}{\sigma(\tau + x/\mu)} \right|, \quad (5.33)$$

where $\zeta$ and $\sigma$ are the Weierstrass $\zeta$ and $\sigma$-functions and $x$ is a real constant obeying $\operatorname{sn}[x | k] = \mu$, i.e. $x$ is expressed as an incomplete elliptic integral of the first kind. The expression (5.33) can be further rewritten in terms of theta-functions:

$$Ht_-(\tau) = \frac{\mu\tau \pi}{2K} \frac{\vartheta'_{1}(\pi x/2K)}{\vartheta_{1}(\pi/2K)} + \frac{1}{2} \log \left| \frac{\vartheta_{1}(\pi(\mu\tau-x)/2K)}{\vartheta_{1}(\pi(\mu\tau+x)/2K)} \right|, \quad (5.34)$$

and finally as:

$$Ht_-(\tau) = \frac{1}{2} \log \left| \frac{\sin[\pi \mu\tau/2K]}{\sin[\pi(\mu\tau+x)/2K]} \right| + \frac{\mu\tau \pi}{2K} \frac{\vartheta'_{1}(\pi x/2K)}{\vartheta_{1}(\pi/2K)}$$

$$- 2 \sum_{n=1}^{\infty} \frac{\varphi_{n}^{2n}}{n(1-q^{2n})} \frac{\sin[n\pi\mu\tau/K]}{\sin[n\pi/K]}, \quad (5.35)$$

22
where \( q = e^{-\pi K'/K} \). In the latter expression we have isolated all the real singularities in the first term. To be more specific we see that the static coordinate time is singular for \( \mu \tau \to 2KN \mp x \), where \( N \) is an integer, with the asymptotic behaviour:

\[
t_-(\tau) \to \pm \frac{1}{2H} \log |\mu \tau - 2KN \mp x|; \quad \tau \to \frac{2K}{\mu} N \pm \frac{x}{\mu}.
\]  

(5.36)

On the other hand \( t_-(\tau) \) is completely regular at the boundaries of the branches, i.e. for \( \tau = 0, \pm 2\omega, \pm 4\omega, \ldots \). These results can be easily translated to the hyperboloid time and cosmic time obtained from eqs. (2.18),(2.19) and (5.5), respectively. We find the explicit form of the hyperboloid time \( q_0^-(\tau) \) from eqs. (2.18)-(2.19),(5.31) and (5.34):

\[
q_0^-(\tau) = -\frac{\Omega \vartheta_1'(0)}{2\pi} e^{\frac{\Omega \vartheta_1'(\Omega \tau)}{\vartheta_1(\Omega \tau)}} \frac{\vartheta_1(\Omega(y - \tau)) + e^{-\frac{\Omega \vartheta_1'(\Omega \tau)}{\vartheta_1(\Omega \tau)}} \vartheta_1(\Omega(y + \tau))}{\vartheta_1(\Omega\tau) \vartheta_1(\Omega y)},
\]

(5.37)

where:

\[
\Omega \equiv \frac{\pi \mu}{2K}, \quad y \equiv \frac{x}{\mu}.
\]

(5.38)

Notice that the singularities (5.36) that originated by the zeroes of \( \vartheta_1(\Omega(y \pm \tau)) \) cancel in \( q_0^-(\tau) \) so that \( q_0^-(2\omega N \pm y) \) is finite. \( q_0^-(\tau) \) blows up for \( \tau = 0, \pm 2\omega, \pm 4\omega, \ldots \), like:

\[
|q_0^-| \propto \left| \frac{1}{2N\omega - \tau} \right|,
\]

(5.39)

where \( N \) is again an integer. This demonstrates that the world-sheet time \( \tau \) is actually an infinite valued function of \( q_0^- \), and that the solution \( f_- \) therefore describes infinitely many strings (see Fig.3a). This should be compared with the \( b = 1/4 \) case where we found a solution describing two strings. In that case the two strings were of completely different type and had completely different physical interpretations. In the present case we find infinitely many strings but they are all of the same type. In the branch \( \tau \in [0, 2\omega] \) (say) the string starts with infinite string size at \( \tau = 0, \quad q_0^- = -\infty \). It then contracts to its minimal size \( f_0^2 = (1 + \sqrt{1 - 4b})/2 \) and reexpands towards infinity at \( \tau = 2\omega, \quad q_0^- = \infty \). This solution, and the infinitely many others of the same type, are unstable strings.
The cosmic time of the $f_-$-solution is obtained by combining eqs. (5.5), (5.31) and (5.35). Expanding also the log-term of eq. (5.5) in terms of $q$ we find:

$$HT_-(\tau) = \Omega \tau \frac{\vartheta'}{\vartheta_1}(\Omega y) + \log | \frac{\Omega \vartheta_1((\Omega(\tau-y))\vartheta'_1(0))}{\pi \vartheta_1(\Omega \tau)\vartheta_1(\Omega y)} |$$

$$= \log | \frac{\Omega \sin(\Omega(\tau-y))}{\sin(\Omega \tau)} | - \log | \frac{\pi \vartheta_1(\Omega y)}{\vartheta_1'(0)} | + \Omega \tau \frac{\vartheta'}{\vartheta_1}(\Omega y)$$

$$- 4 \sum_{m=1}^{\infty} \frac{q^m}{m(1-q^m)} \sin(m\Omega y) \sin(m(2\tau - y)). \quad (5.40)$$

It can be shown that $0 \leq \Omega \leq 1$ and $1.246.. \leq y \leq \pi/2$. It follows that:

$$0 < y < \omega < 2\omega - y < 2\omega. \quad (5.41)$$

The cosmic time (5.40) is singular at $\tau = 0$, $y$, $2\omega$ but regular at $\tau = 2\omega - y$ and similarly in the other branches. The singularity of the static coordinate time (5.35) at $\tau = 2\omega - y$ has been canceled by adding the log-term of eq. (5.5), see Fig.3b. Therefore, the interpretation of the string solution in the branch $\tau \in [0, 2\omega]$ (say), as seen in comoving coordinates, is as follows: The string starts with infinite size and energy at $\tau = 0$, $HT_- = \infty$. It then contracts and passes the horizon from the outside at $\tau = y$, $HT_- = -\infty$. The string now continues contracting from the inside of the horizon at $\tau = y$, $HT_- = -\infty$ until it reaches the minimal size at:

$$\tau = \omega, \quad HT_- = \frac{\pi}{2} \frac{\vartheta'}{\vartheta_1}(\Omega y) + \frac{1}{2} \log \frac{1 - \sqrt{1 - 4b}}{2}. \quad (5.42)$$

The energy is here given by $H\alpha'E_- (\tau = \omega) = 2\sqrt{b}/(1 - \sqrt{1 - 4b})$. From now on the string expands again. It passes the horizon from the inside after finite cosmic time and continues towards infinity for $HT_- \to \infty$.

It is an interesting observation that the cosmic time is not periodic in $\tau$, i.e. $T_-(\tau) \neq T_-(\tau + 2\omega)$, although the string size is. Explicitly we find:

$$T_-(\tau + 2\omega) - T_-(\tau) = \frac{\pi}{H} \frac{\vartheta'}{\vartheta_1}(\Omega y). \quad (5.43)$$

This means that $f_-(\tau)$ really describes infinitely many strings with different invariant size at a given cosmic time. To be more specific let us consider a fixed cosmic time $T_-$ and the corresponding string times:

$$T_- \equiv T_-(\tau_1) = T_-(\tau_2) = ..., \quad (5.44)$$
where $\tau_1 \in [0, 2\omega[, \; \tau_2 \in [2\omega, 4\omega[\ldots$. Taking for simplicity a cosmic time $HT_\sim \gg 1$ we have (see Fig.3b.):

$$\tau_n = \frac{n\pi}{\Omega} + \epsilon_n, \quad \epsilon_n << 1.$$  \hspace{1cm} (5.45)

To the lowest orders we find from eq. (5.40):

$$HT_\sim = -\log \epsilon_n + n\pi \frac{\vartheta'_1}{\vartheta_1}(\Omega y) + O(\epsilon_n),$$  \hspace{1cm} (5.46)

so that:

$$\epsilon_n = \exp[-HT_\sim + n\pi \frac{\vartheta'_1}{\vartheta_1}(\Omega y) + \ldots]$$  \hspace{1cm} (5.47)

The invariant string sizes are then:

$$HS_\sim(\tau_n) = f_\sim(\tau_n) \approx \frac{1}{\epsilon_n} = \exp[HT_\sim - n\pi \frac{\vartheta'_1}{\vartheta_1}(\Omega y) + \ldots]$$  \hspace{1cm} (5.48)

i.e. they are separated by a multiplicative factor. This expression of course is only valid as long as $\epsilon_n << 1$, so $n$ should not be too large.

We now consider the $f_+$-solution (5.32). In this case the dynamics takes place well inside the horizon. The possible singularities of the hyperboloid time $q_+$ and the cosmic time $T_+$ therefore coincide with the singularities of the static coordinate time $t_+$. The static coordinate time is again obtained from eq. (2.20) which we first rewrite as:

$$H\dot{t}_+(\tau) = \frac{\sqrt{b}}{2/3 - \varphi(\tau + \omega')} = \sqrt{b} \left[ 1 + \frac{b}{\varphi(\tau) - (b - 1/3)} \right].$$  \hspace{1cm} (5.49)

Integration leads to:

$$Ht_+(\tau) = \tau(\sqrt{b} + \zeta(a)) + \frac{1}{2} \log | \frac{\sigma(\tau - a)}{\sigma(\tau + a)} |,$$  \hspace{1cm} (5.50)

where $a$ is a complex constant obeying $\varphi(a) = b - 1/3$, i.e. $sn[a\mu | k] = 1/\nu$. It follows that $a\mu = iK + x$ where $x$ is real and $sn[x | k] = \mu$. Again we can express the static coordinate time in terms of theta-functions:

$$Ht_+(\tau) = \tau[\sqrt{b} + \frac{\mu\pi}{2K} \frac{\vartheta'_4}{\vartheta_4}(\frac{\pi x}{2K})] + \frac{1}{2} \log | \frac{\vartheta_4(\frac{\pi(\mu\gamma - x)}{2K})}{\vartheta_4(\frac{\pi(\mu\gamma + x)}{2K})} |,$$  \hspace{1cm} (5.51)
or in terms of the Jacobi zeta-function \( zn \):

\[
Ht_+ (\tau) = \tau (\sqrt{b} + \mu \ zn(x, k)) - 2 \sum_{n=1}^{\infty} \frac{q^n}{n(1 - q^{2n})} \sin(\frac{n\pi \mu \tau}{K}) \sin(\frac{n\pi x}{K}), \quad (5.52)
\]

where \( q = e^{-\pi K'/K} \). In this form we see explicitly that \( t_+ \) consists of a linear term plus oscillating terms. The cosmic time takes the form (see eqs. (3.13) and (5.5)):

\[
HT_+ (\tau) = \frac{1}{2} \log(1 - \nu^2 \sin^2(\mu \tau \ | \ k)) + Ht_+ (\tau)
\]

\[
= \tau [\sqrt{b} + \Omega \frac{\vartheta_4(\Omega y)}{\vartheta_4(\Omega \tau)}] + \log | \frac{\Omega \frac{\vartheta_4(\Omega (\tau - y))}{\vartheta_4(\Omega \tau)}}{\pi \vartheta_1(\Omega y)} | . \quad (5.53)
\]

Notice that the argument of the log has no real zeroes:

\[
HT_+ (\tau) = \tau [\sqrt{b} + \Omega \frac{\vartheta_4(\Omega y)}{\vartheta_4(\Omega \tau)}] + \log | \frac{\Omega \frac{\vartheta_4(\Omega (\tau - y))}{\vartheta_4(\Omega \tau)}}{\pi \vartheta_1(\Omega y)} |
\]

\[-4 \sum_{n=1}^{\infty} \frac{q^n}{n(1 - q^{2n})} \sin(n\pi \mu \Omega) \sin(n\pi \Omega(2\tau - y)). \quad (5.54)
\]

The static coordinate time and the cosmic time are therefore completely regular functions of \( \tau \), and it follows that the string solution \( f_+ \), which is oscillating regularly as a function of world-sheet time \( \tau \), is also oscillating regularly when expressed in terms of hyperboloid time or cosmic time. This solution represents one stable string.

Series expansion analogous to eq. (5.54) hold for the string radius:

\[
f_+ (\tau) = \nu \ \text{sn}[\mu \tau \ | \ k] = \frac{2\pi}{K(k)\sqrt{1 + k^2}} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1 - q^{2n-1}} \sin((2n - 1)\Omega \tau). \quad (5.55)
\]

We see from eq. (5.54)-(5.55) that the string oscillations do not follow a pure harmonic motion as in flat Minkowski spacetime, but they are precise superpositions of all frequencies \((2n - 1)\Omega \) \((n = 1, 2, ..., \infty)\) with uniquely defined coefficients. The non-linearity of the string equations in de Sitter spacetime fixes the relation between the mode coefficients. In the present case the basic frequency \( \Omega \) depends on the string energy, while in Minkowski spacetime the frequencies are merely \( n \).
Using eqs. (5.5) and (5.32) the energy (5.11) of this solution is given by:

\[
E_+ = \frac{1}{\alpha'} \frac{d}{d \tau} \left( \frac{1}{2H} \log(1 - \nu^2 \text{sn}^2[\mu \tau \mid k]) + t_+(\tau) \right)
\]

\[
= \frac{1}{H\alpha'} \left[ \sqrt{b} + \mu \text{zn}(x,k) \right] + "\text{oscillating terms"},
\] (5.56)

and by averaging over a period \(2K/\mu\), the average energy \(<E_+>\) is just the square bracket term. Let us consider this energy in a little more detail. In the limit \(b \to 0\), where the amplitude of the string size oscillation goes to 0 (see Fig.1.), we find:

\[
k \to 0, \quad \mu \to 1, \quad x \to \pi/2,
\] (5.57)

and then \(<E_+> \to 0\). This is not surprising since in the limit \(b \to 0\) the \(f_+\)-solution ceases to exist. The other interesting limit is \(b \to 1/4\) where the elliptic solutions turn into hyperbolic solutions. We find:

\[
k \to 1, \quad \mu \to 1/\sqrt{2}, \quad \tanh x \to 1/\sqrt{2},
\] (5.58)

and then \(<E_+> \to 1/(H\alpha')\), in agreement with eq. (5.26). More generally a numerical analysis shows that \(<E_+>\) is a monotonically increasing function of \(b\) for \(b \in [0,1/4]\). The energy (5.56) should be compared with the average energy computed directly from eq. (5.16). In the special case here eq. (5.16) gives:

\[
H\alpha' <E_+> = \sqrt{b} \frac{\Pi(\nu^2,k)}{K(k)},
\] (5.59)

which is indeed equivalent to the square bracket term of (5.56) \[10\].

Finally we remark that the only non-vanishing component of \(L_{\mu \nu}\) introduced in (5.18) is again \(L_{10}\). This holds for both the \(f_-\) and the \(f_+\)-solutions:

\[
L_{10} = 2\pi \sqrt{b}.
\] (5.60)

This expression is actually general for all the solutions under consideration here since it is obtained directly from the equations of motion (3.5). The result (5.28) is therefore just the special case \(b = 1/4\), and (5.60) is also valid for \(b > 1/4\), that will be considered in the following subsection.
5.3 The elliptic $b > 1/4$ solutions

Before analysing the solutions let us also in this case introduce an alternative notation for the elliptic functions. We first write:

$$
\omega_2 = \frac{K(\hat{k})}{\alpha}, \quad \omega'_2 = \frac{iK'(\hat{k})}{\alpha}; \quad \alpha \equiv b^{1/4},
$$

(5.61)

and $\hat{k}$ is given by (4.18). The solutions (4.16) and (4.17) are then expressed in terms of Jacobi elliptic functions:

$$
f_-^2(\tau) = \alpha^2 \frac{1 + \text{cn}[2\alpha\tau | \hat{k}]}{1 - \text{cn}[2\alpha\tau | \hat{k}]},
$$

(5.62)

$$
f_+^2(\tau) = \alpha^2 \frac{1 - \text{cn}[2\alpha\tau | \hat{k}]}{1 + \text{cn}[2\alpha\tau | \hat{k}]},
$$

(5.63)

respectively.

Consider first the $f_-$-solution (5.62). The static coordinate time, obtained by integrating (2.20), is:

$$
Ht_-(\tau) = \zeta(x/\alpha)\tau + \frac{1}{2} \log \left| \frac{\sigma(\tau - x/\alpha)}{\sigma(\tau + x/\alpha)} \right|.
$$

(5.64)

This expression is formally identical to (5.33), but remember that the parameters of the Weierstrass functions are different now. The real constant $x$ is here obeying:

$$
\text{cn}[2x | \hat{k}] = \frac{1 - \alpha^2}{1 + \alpha^2}.
$$

(5.65)

In terms of theta-functions, following the same steps as in subsection 5.2, $t_-(\tau)$ can be written in the same form as (5.35), but with $q^2 = -e^{-\pi K'(\hat{k})/K(\hat{k})}$. The analysis of this function and of the corresponding hyperboloid time and cosmic time is now similar to the analysis in the $b < 1/4$-case after equation (5.35), so we shall skip the details. The result is that the $f_-$-solution describes infinitely many strings of the same type, in the same sense as in subsection 5.2. The only difference is that in the present case the strings have a collapse during their evolution: Considering the branch $\tau \in [0, 2\omega_2]$ (say) the string starts with infinite string size at $\tau = 0$, $q_0^- = -\infty$. It then contracts until
it collapses, after which it expands again and reaches infinite string size at \( \tau = 2\omega_2, q_0^\tau = \infty \). This is again an unstable string.

The physical interpretation of the \( f_+ \)-solution easily follows from the fact that:

\[
f_+^2(\tau) = f_-^2(\tau + \omega_2),
\]

i.e. it is just a time translated version of \( f_- \).

6 Conclusion

We have found the exact general evolution of circular strings in 2 + 1 dimensional de Sitter spacetime. We have expressed it closely and completely in terms of elliptic functions. The solution generically describes infinitely many (different and independent) strings, and depends on one constant parameter \( b \) related to the string energy. A summary of the main features and conclusions of our results is presented in Table I.
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Table I

Circular string evolution in de Sitter spacetime. For each $b$, there exists two independent solutions $f_-$ and $f_+$:

| $b$          | $f_-$                                      | $f_+$                                      |
|--------------|--------------------------------------------|--------------------------------------------|
| $b < 1/4$    | Infinitely many different strings. All are unstable ($f_-^{\text{max}} = \infty$), and never collapse to a point ($f_-^{\text{min}} = \sqrt{(1 + \sqrt{1 - 4b})/2} > 0$) | One stable oscillating string ($0 \leq f_+ \leq f_+^{\text{max}}$, where $f_+^{\text{max}} = \sqrt{(1 - \sqrt{1 - 4b})/2}$). $\langle E_+ \rangle$ is a bounded monotonically increasing function of $b \in [0, 1/4]$ |
| $b > 1/4$    | Infinitely many strings. All of them are unstable ($f_-^{\text{max}} = \infty$) and they collapse to a point ($f_-^{\text{min}} = 0$) | Infinitely many strings similar to $f_-$. In this case $f_+$ is just a time-translation of $f_-$: $f_+(\tau) = f_-(\tau + \omega_2)$ |
| $b = 1/4$    | Two different and non-oscillating strings $f_-^{(I)}$ and $f_-^{(II)}$. $f_-^{(I)}$ is unstable and $f_-^{(II)}$ is stable for large de Sitter radius. | One non-oscillating and stable string. $f_+^{\text{min}} = 0$, $E_+^{\text{min}} = \frac{1}{2H\alpha'}$, $f_+^{\text{max}} = \frac{1}{\sqrt{2H}}$, $E_+^{\text{max}} = \frac{1}{H\alpha'}$ |
Figure Captions

Fig.1. The potential $V(f^2) = f^2 - f^4$ defined in equation (4.6). For $b \leq 1/4$ it acts effectively as a barrier. The horizon corresponds to $f^2 = 1$.

Fig.2a. The hyperboloid time $q_0^0$ in the $b = 1/4$ case, given by equation (5.19), as a function of $\tau$. Notice that $\tau$ is a two-valued function of $q_0^0$.

Fig.2b. The cosmic time $T_-$ in the $b = 1/4$ case, given by equation (5.21), as a function of $\tau$. $T_-$ is singular for $\tau = 0$ and $\tau = \sqrt{2} \log(1 + \sqrt{2})$, and thus $\tau$ is a three-valued function of $T_-$. 

Fig.3a. The hyperboloid time $q_0^0$ as a function of $\tau$ in the elliptic case $b < 1/4$. Each of the infinitely many branches corresponds to one string.

Fig.3b. The cosmic time $T_-$ in the $b < 1/4$ case, given by equation (5.40), as a function of $\tau$. Notice that $T_-$ is not periodic, as explained in subsection 5.2.
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