Double-extended Kerr–Schild form for 5D electrovacuum solutions

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Abstract
Five-dimensional Einstein–Maxwell–Chern–Simons equations are investigated in the framework of an extended Kerr–Schild strategy to search for black holes solutions. The fulfillment of Einstein equations constrains the Chern–Simons coupling constant to a value determined by the trace of the energy-momentum tensor of the electromagnetic configuration.

Keywords Higher dimensional gravity · Electrovacuum solutions · Black holes · Kerr–Newman black holes · Exact solutions · General relativity

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# 1 Introduction

The search for Kerr–Newman black holes in higher dimensions has been an active field of research in the last decades. The 5\(D\) extremal charged black hole with a sole angular momentum was obtained in Ref. [1]. This solution does not solve the 5\(D\) Einstein–Maxwell equations since it contains Chern-Simons contributions. Numeric [2, 3] and perturbative [4] methods have also been tried to build this kind of 5\(D\) black hole solutions, but only one angular momentum was considered in these attempts too.

A family of asymptotically flat rotating solutions to vacuum Einstein equations in \(d\) dimensions was built by Myers and Perry [5]. These solutions are characterized by \([(d−1)/2]\) independent angular momenta. Kerr-NUT-(A)dS black holes in higher dimensions have been studied in Ref. [6].

In five dimensions, the charged rotating black hole with two independent angular momenta was obtained by Chong, Cvetič, Lü, and Pope (CCLP) [7]. This geometry appeared in the framework of five-dimensional minimal gauged supergravity. It is not a solution to Einstein–Maxwell equations since it is subjected to a specific value of the Chern-Simons (CS) coupling constant. It is the charged version of the 5\(D\) Kerr-NUT-(A)dS black hole. The separability of Hamilton-Jacobi and Klein-Gordon in the CCLP chart was studied in Ref. [8]. The existence of a Killing-Yano tensor for this solution is established in Ref. [9].

Equal angular momenta solutions to Einstein–Maxwell–Chern–Simons (EMCS) equations with arbitrary values of the CS coupling constant have been investigated in Ref. [3, 4], although from a numerical or perturbative perspective. The analytical building of solutions to EMCS equations with arbitrary CS coupling constant and two independent angular momenta was studied in Ref. [10], by introducing a properly extended Plebański-Demiański Ansatz [11] in 5\(D\). In this work we will renew the search for this kind of solutions in the framework of a double-extended Kerr–Schild form for the metric tensor. We will describe the flat 5\(D\) seed geometry by means of a null geodesic tetrad in a \((2+2)D\) submanifold together with an orthogonal unitary vector. The chart to be used renders this basis in a simple form, what will prove to be very friendly for algebraic manipulations. We will show that the specific CS coupling constant of the CCLP solution seems to be inescapable, since it is determined by the trace of the energy-momentum tensor of the pointlike solution to Maxwell-Chern-Simons equation.

In Sect. 2 we introduce the basic properties of the chart describing the seed metric. In Sect. 3 we describe the Kerr-Schild strategy to search for black-hole solutions, and its different extensions. In Sect. 4 we show the pointlike solution to Maxwell-Chern-Simons equation in a double-extended Kerr-Schild background. In Sect. 5 we solve...
the EMCS equations for the pointlike charge. In Sect. 6 we establish the relationship between signature (3,2) and (1,4). In Sect. 7 we display the conclusions.

2 From flat spacetime to rotating black holes

Let be \( \{n_a\} \) a basis for the tangent space of a 5D spacetime \((a = 0, \ldots, 4)\), and \(\{n^a\} \) the respective dual basis of 1-forms,

\[
n^a(n_b) = \delta^a_b .
\]  

(1)

The metric tensor will be formulated as

\[
g = n^0 \otimes n^2 + n^2 \otimes n^0 + n^1 \otimes n^3 + n^3 \otimes n^1 + n^4 \otimes n^4 .
\]  

(2)

Due to the duality (1) the inverse metric results to be

\[
g^{-1} = n_2 \otimes n_0 + n_0 \otimes n_2 + n_1 \otimes n_3 + n_3 \otimes n_1 + n_4 \otimes n_4 .
\]  

(3)

The form of the metric implies that vectors \(\{n_0, n_1, n_2, n_3\} \) are null; they form a basis of a 2 + 2 spacetime,

\[
|n_a|^2 = n_a \cdot n_a = g(n_a, n_a) = 0 , \quad a = 0, 1, 2, 3 .
\]  

(4)

Vector \(n_4\) is unitary,

\[
|n_4|^2 = n_4 \cdot n_4 = g(n_4, n_4) = 1 .
\]  

(5)

The metric (2) and the inverse metric (3) can be written as

\[
g = \tilde{\eta}_{ab} n^a \otimes n^b , \quad g^{-1} = \tilde{\eta}^{ab} n_a \otimes n_b ,
\]  

(6)

where

\[
\tilde{\eta}^{ab} = \tilde{\eta}_{ab} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]  

(7)

is an orthogonal transformation of the \((2 + 3)D\) Minkowski symbol:

\[
\tilde{\eta}_{ab} = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2}
\end{pmatrix}
\]  

(8)
For some purposes it is useful to call
\[ m = n_4, \] (9)
and preserve the symbol \( n_a \) for the null tetrad \((a = 0, 1, 2, 3)\).

Other properties emerging from the form of the metric (2) are: \( m \) is orthogonal to the null tetrad,
\[ m \cdot n_a = g(m, n_a) = 0, \quad a = 0, 1, 2, 3, \] (10)
and vectors \( n_a \) fulfill the orthogonality relations
\[ n_0 \cdot n_1 = 0 = n_2 \cdot n_3, \quad n_1 \cdot n_2 = 0 = n_0 \cdot n_3. \] (11)

We remark the way to “lower the indices” of the null tetrad, since the metric (2) mixes the characters:
\[ g(n_2, \cdot) = n^0, \quad g(n_3, \cdot) = n^1, \quad \text{etc.} \] (12)

### 2.1 Flat spacetime

The former framework will be used to search for solutions to Einstein–Maxwell equations (electrovacuum solutions) in five dimensions. Let us first introduce the flat metric, which will play the role of a seed metric in following steps. The flat spacetime can be represented by choosing a chart \((t, \phi, \psi, r, p)\) such that
\[ n^2 = dr, \quad n^3 = dp. \] (13)

The basis of the cotangent space is completed with the 1-forms
\[ n^0 = dt - \frac{a^2 + p^2}{a} d\phi - \frac{b^2 + p^2}{b} d\psi, \] (14)
\[ n^1 = dt - \frac{a^2 + r^2}{a} d\phi - \frac{b^2 + r^2}{b} d\psi, \] (15)
\[ m = r \ p \ ( - \frac{b}{a^2} d\phi - \frac{a}{b^2} d\psi ). \] (16)

Thus the metric exhibits a symmetry under the exchange \( r \leftrightarrow p, n^0 \leftrightarrow n^1 \). The components of \( g^\text{flat} \) are
\[
    g^\text{flat}_{\mu\nu} = \begin{pmatrix}
        0 & 0 & 0 & 1 & 1 \\
        0 & \frac{b^2}{a^2} & \frac{r^2}{a} & -a^2 + p^2 & -a^2 + r^2 \\
        0 & \frac{q^2}{a^2} & \frac{r^2}{b} & -b^2 + p^2 & -b^2 + r^2 \\
        1 - \frac{a^2 + p^2}{a} & -\frac{b^2}{a^2} & \frac{r^2}{b^2} & 0 & 0 \\
        1 - \frac{a^2 + r^2}{a} & -\frac{b^2}{a^2} & \frac{r^2}{b^2} & 0 & 0
    \end{pmatrix}, \] (17)
whose determinant is
\[
\det(g_{\mu\nu}^{\text{flat}}) = \frac{r^2 p^2}{a^4 b^4} (a^2 - b^2)^2 (r^2 - p^2)^2 .
\]

(18)

As can be seen, in this Ansatz the components of $g^{\text{flat}}$ look much simpler than they are in other Ansätze appearing in the literature, what is a great advantage for doing calculations. The parameters $a, b$ will be related to the independent angular momenta of the 5D black hole solutions. To avoid the vanishing of $\det(g_{\mu\nu}^{\text{flat}})$, the otherwise arbitrary parameters $a, b$ must be unequal; in fact, the second and third columns of $\det(g_{\mu\nu}^{\text{flat}})$ would be equal in case that $a = b$. The fact that the metric (17) describes a flat spacetime is straightforwardly verifiable by computing the Riemann tensor. The components of the metric tensor in the chart $(t, \phi, \psi, r, p)$ also show that the Killing vectors $\partial \phi, \partial \psi$ have positive squared norm, while $\partial_t$ is null.

Duality (1) implies that
\[
\begin{align*}
n_0 &= \frac{1}{r^2 - p^2} \left( (a^2 + b^2 + r^2) \frac{\partial}{\partial t} + \frac{a^3}{a^2 - b^2} \frac{\partial}{\partial \phi} + \frac{b^3}{b^2 - a^2} \frac{\partial}{\partial \psi} \right), \\
n_1 &= -\frac{1}{r^2 - p^2} \left( (a^2 + b^2 + p^2) \frac{\partial}{\partial t} + \frac{a^3}{a^2 - b^2} \frac{\partial}{\partial \phi} + \frac{b^3}{b^2 - a^2} \frac{\partial}{\partial \psi} \right), \\
m &= \frac{1}{r p} \left( ab \frac{\partial}{\partial t} + \frac{a^2 b}{a^2 - b^2} \frac{\partial}{\partial \phi} + \frac{b^2 a}{b^2 - a^2} \frac{\partial}{\partial \psi} \right), \\
n_2 &= \frac{\partial}{\partial r} \doteq k, \\
n_3 &= \frac{\partial}{\partial p} \doteq K.
\end{align*}
\]

(19)  (20)  (21)

According to Eq. (12), by lowering indices to the vectors $k, K$ one gets
\[
g(k, ) = n^0 \doteq k, \quad g(K, ) = n^1 \doteq K .
\]

(23)

### 2.2 Ansatz to search for black hole solutions

To search for black hole solutions we will introduce four functions $X, Y, X', Y'$ to be determined by solving the Einstein–Maxwell equations. These unknown functions enter the metric through the replacements
\[
\begin{align*}
dr &\rightarrow n^2 \doteq dr - \frac{Y(r)}{2 (r^2 - p^2)} k, \\
\frac{X(p)}{2 (r^2 - p^2)} K, \\
dp &\rightarrow n^3 \doteq dp + \frac{Y'(r) k + X'(p) K}{r^{-1} p^{-1} (r^2 - p^2)},
\end{align*}
\]

(24)

1 In five dimensions, equal angular momenta implies a static geometry [12].
where \( k, K \) are the (null) 1-forms (23). To keep the duality of bases we must also change

\[
\mathbf{n}_0 \rightarrow \mathbf{n}_0 + \frac{Y(r)}{2 \left( r^2 - p^2 \right)} \mathbf{k} + \frac{r \, p \, \gamma'(r)}{r^2 - p^2} \mathbf{m},
\]

\[
\mathbf{n}_1 \rightarrow \mathbf{n}_1 - \frac{X(p)}{2 \left( r^2 - p^2 \right)} \mathbf{K} - \frac{r \, p \, \lambda'(p)}{r^2 - p^2} \mathbf{m},
\]

(25)

where \( \mathbf{k}, \mathbf{K} \) are the null vectors (22). This strategy to introduce gravity starting from a seed flat metric makes contact with both the Kerr–Schild [13–15] and the Plebañski-Demiański [11, 16] Ansätze, which were originally designed to try with 4D solutions. Since the Ansatz (24, 25) does not alter the 5-form of volume \( n^0 \wedge n^1 \wedge n^2 \wedge n^3 \wedge n^4 = \mathbf{k} \wedge \mathbf{K} \wedge dr \wedge dp \wedge m \), the determinant (18) does not change.

### 2.3 Null geodesics

For a null vector \( k^\mu \) to be geodesic it must satisfy the equation \(^2\)

\[
k^\mu \left( \partial_\mu k_\nu - \partial_\nu k_\mu \right) = 0 \quad \text{or} \quad dk(\mathbf{k},) = 0.
\]

(26)

It is easy to verify that \( dk(\mathbf{k},) \) (i.e., \( dn^0 \left( \frac{\partial}{\partial r}, \right) \)) and \( d\mathbf{K}(\mathbf{K},) \) (i.e., \( dn^1 \left( \frac{\partial}{\partial p}, \right) \)) are zero for the original bases. On the other hand, the Ansatz (24, 25) does not modify the 1-forms \( n^0 = k \) and \( n^1 = K \), nor the null vectors \( n_2 = \mathbf{k} \) and \( n_3 = \mathbf{K} \). Thus, \( \mathbf{k}, \mathbf{K} \) remain null and geodesic whatever the unknown functions \( X, Y, X', Y' \) are. \(^3\)

For a later use we will mention two other equations that are independent of the functions \( X, Y, X', Y' \). Since vector \( \mathbf{m} \) is not modified by the Ansatz, it also remains valid that

\[
m^\mu \left( \partial_\mu k_\nu - \partial_\nu k_\mu \right) = 0, \quad m^\mu \left( \partial_\mu K_\nu - \partial_\nu K_\mu \right) = 0.
\]

(27)

### 2.4 Kerr-NUT-(A)dS geometry

Remarkably, the metric \( g \) is flat not only when functions \( X, Y, X', Y' \) are zero. It remains flat if

\[
X_{\text{flat}}(p) = \gamma + \alpha \, p^2 - \beta^2 \, p^{-2}, \quad Y_{\text{flat}} (r) = \gamma + \alpha \, r^2 - \beta^2 \, r^{-2},
\]

\[
X_{\text{flat}}(p) = \delta + \varepsilon \, p^2 + \beta \, p^{-2}, \quad Y_{\text{flat}} (r) = \delta + \varepsilon \, r^2 + \beta \, r^{-2},
\]

(28)

what means that the free coefficients \( \alpha, \beta, \gamma, \delta, \varepsilon \) entail changes of coordinates. The Ansatz (25) allows to easily introduce the cosmological constant \( \Lambda \), the mass \( m \) and

\(^2\) A geodesic null vector \( \mathbf{k} \) fulfills \( k_\mu k^\mu = 0 \) and \( k_\mu k_{\nu;\mu} = 0 \); then it is \( 0 = k^\mu (k_\nu;\mu - k_{\mu;\nu}) = k^\mu (k_{\nu;\mu} - k_{\mu;\nu}) \).

\(^3\) On the contrary, the null vectors \( \mathbf{n}_0, \mathbf{n}_1 \) are geodesic only in the seed flat spacetime we began with.
the NUT charge \( n \); the 5D Kerr-NUT-(A)dS geometry is given by
\[
X(p) = X_{\text{flat}}(p) + 2n - \frac{\Lambda}{6} p^4, \quad Y(r) = Y_{\text{flat}}(r) - 2m - \frac{\Lambda}{6} r^4,
\]
(29)
as can be verified by solving the vacuum Einstein equations with cosmological constant [17]. At this level, the solution acquires two independent angular momenta characterized by the values of \( a, b \). Notice that the case \( m = -n \) is not a black hole but is de Sitter spacetime, since \( m, n \) would be absorbed into a (single) constant \( \gamma \) in \( X_{\text{flat}}, Y_{\text{flat}} \). As seen, the 5D mass and NUT charge are somehow degenerate since one of them can be absorbed into the other one by shifting the constant \( \gamma \) [18]. It could be said that the 5D mass and NUT charge are introduced by replacing the single constant \( \gamma \) in Eq. (28) with two different constants. Analogously, the electric charge will be introduced in Sect. 4 by replacing the single constant \( \beta \) in \( X_{\text{flat}}, Y_{\text{flat}} \) with two different constants.

3 The double-extended Kerr–Schild form

By replacing the Ansatz (25) in the inverse metric (3) we obtain a structure of type double-extended Kerr–Schild (dxKS),
\[
g^{-1} = h^{-1} + \mathcal{H}_0 \, k \otimes k + \mathcal{H}_1 \, K \otimes K + J_0 \, (k \otimes m + m \otimes k) \\
+ J_1 \, (K \otimes m + m \otimes K),
\]
(30)
where \( h \) is the seed metric, \( k \) and \( K \) are mutually orthogonal null geodesic vector fields with respect to both \( g \) and \( h \), and \( m \) is a unitary vector orthogonal to both \( k \) and \( K \). In the Ansatz (25) the functions \( \mathcal{H}, J \) are
\[
\mathcal{H}_0 = \frac{Y(r)}{r^2 - p^2}, \quad \mathcal{H}_1 = -\frac{X(p)}{r^2 - p^2}, \quad J_0 = \frac{r \, p \, Y(r)}{r^2 - p^2}, \quad J_1 = -\frac{r \, p \, X(p)}{r^2 - p^2},
\]
(31)
and the seed metric \( h \) is the one in Eq. (2). Of course, we can get rid of functions \( J \) in the 5D Kerr-NUT-(A)dS solution, since the free constants \( \delta, \varepsilon, \beta \) can be chosen to be zero. Even the cosmological term in Eq. (29) could be hidden in \( h^{-1} \) to cast the Kerr-(A)dS solution into the simpler KS form [13–15] where \( \mathcal{H}_1, J_0, J_1 \) are zero. The KS form is connected to the Newman-Janis algorithm to get the rotating charged black hole in four dimensions [19].

The case of the extended Kerr–Schild form (xKS)
\[
g^{-1} = h^{-1} + \mathcal{H} \, k \otimes k + J \, (k \otimes m + m \otimes k),
\]
(32)
has been studied as a way of looking for solutions in higher dimensions [20], in particular as a form of containing the CCLP solution [21, 22]. However, the \( dxKS \)
form provide us with a wider menu of possibilities to succeed in searching for new electrovacuum solutions.

The metric $g$ related to $g^{-1}$ in Eq. (30) is

$$g = h + \left( \mathcal{J}_0^2 - \mathcal{H}_0 \right) k \otimes k + \left( \mathcal{J}_1^2 - \mathcal{H}_1 \right) K \otimes K + \mathcal{J}_0 \mathcal{J}_1 \left( k \otimes K + K \otimes k \right) - \mathcal{J}_0 \left( k \otimes m + m \otimes k \right) - \mathcal{J}_1 \left( K \otimes m + m \otimes K \right). \quad (33)$$

For $g^{-1}$ in Eq. (30) to be the inverse of $g$, the 1-forms $k$, $K$, $m$ in Eq. (33) must be defined as

$$k_\mu = h_{\mu \nu} k^\nu, \quad K_\mu = h_{\mu \nu} K^\nu, \quad m_\mu = h_{\mu \nu} m^\nu \quad (34)$$

Thus, the proof is achieved by using that $k_\mu k^\mu = K_\mu K^\mu = k_\mu m^\mu = K_\mu m^\mu = 0$, and $h_{\mu \nu} m^\mu m^\nu = 1$. Notice that $m_\mu$ does not coincide with $g_{\mu \nu} m^\nu$:

$$g_{\mu \nu} m^\nu = h_{\mu \nu} m^\nu - \mathcal{J}_0 k_\mu - \mathcal{J}_1 K_\mu = m_\mu - \mathcal{J}_0 k_\mu - \mathcal{J}_1 K_\mu. \quad (35)$$

This is not an obstacle for vector $m$ to be unitary in both metrics $h$ and $g$, since $m$ is orthogonal to $k$ and $K$.

### 4 Electrodynamics in the dxKS Ansatz

Our aim is to obtain 5D charged rotating black holes, so we will now turn the attention to Maxwell equations in the $dxKS$ framework. The “pointlike” charge potential

$$A = \frac{Q}{2 \left( r^2 - p^2 \right)} \left( n^0 + n^1 \right) \quad (36)$$

yields a solution to Maxwell equations

$$d \ast F = 0 \quad (37)$$

not only in the seed flat metric but for arbitrary functions $X$, $Y$, as will be shown. In fact, the corresponding electromagnetic field is

$$F = dA = \frac{2Q r}{(r^2 - p^2)^2} n^0 \wedge n^2 - \frac{2Q p}{(r^2 - p^2)^2} n^1 \wedge n^3, \quad (38)$$

$$\ast F = \frac{2Q r}{(r^2 - p^2)^2} n^1 \wedge n^3 \wedge n^4 - \frac{2Q p}{(r^2 - p^2)^2} n^0 \wedge n^2 \wedge n^4. \quad (39)$$

No trace of $X$, $Y$ is left in $F$ and $\ast F$. However $\ast F$ is affected by the functions $\mathcal{X}$, $\mathcal{Y}$, which enter $\ast F$ through $n^4$. The field (38) no longer fulfills Maxwell equations (37) in

---

4 Both terms are gauge equivalents since $(r^2 - p^2)^{-1} k$ and $(r^2 - p^2)^{-1} K$ differs in an exact 1-form.

5 According to Eq. (24) it is $n^0 \wedge n^2 = n^0 \wedge n^2, n^1 \wedge n^3 = n^1 \wedge n^3$ (remember that $k = n^0$ and $K = n^1$).
a general $d x K S$ Ansatz because $d \ast F$ will depend on the functions $\mathcal{X}, \mathcal{Y}$ (or $\mathcal{J}_0, \mathcal{J}_1$) and their derivatives. Certain choices of these functions could eventually restore a proper dynamical behavior for the potential (36).

$$d \ast F = d \ast \hat{F} + d \left( \frac{2Q}{(r^2 - p^2)^2} n^1 \wedge n^3 \wedge -\mathcal{Y}(r) n^0 + \mathcal{X}(p) n^1 \right) - \frac{2Q p}{(r^2 - p^2)^2} n^0 \wedge n^2 \wedge -\mathcal{Y}(r) n^0 + \mathcal{X}(p) n^1 \right) \right),$$

where $\ast \hat{F}$ stands for the dual field in the seed metric, whose exterior derivative vanishes for the potential (36). Then the former expression reduces to

$$d \ast F = d \left( -\frac{2Q p r^2 \mathcal{Y}(r)}{(r^2 - p^2)^3} n^0 \wedge n^1 \wedge n^3 + \frac{2Q p^2 \mathcal{X}(p)}{(r^2 - p^2)^3} n^0 \wedge n^1 \wedge n^2 \right) \right),$$

Instead of a direct computation of the exact 4-form in the second term, we can resort to the relations

$$dn^a = \frac{1}{2} c^a_{cb} n^b \wedge n^c$$

where $c^a_{cb}$ are the structure functions defined by

$$[n_c, n_b] = c^a_{cb} n_a.$$  

(see for instance equation (6.176) in Ref. [23]). In the basis $\{\bar{n}_0, \bar{n}_1, n_2, n_3, n_4\}$ the structure functions are such that

$$c^a_{4b} = 0 = c^a_{b4}, \quad a \neq 4.$$  

Equations (41, 43) imply that the exact 4-form in the second term of Eq. (40) is necessarily proportional to $n^0 \wedge n^1 \wedge n^2 \wedge n^3$. So, from the form of $F$ in Eq. (38), it can be said that the exact 4-form is proportional to $F \wedge F$. If the proportionality factor is a constant then the Eq. (40) will become the Maxwell-Chern-Simons (MCS) dynamical equations, or Maxwell equation in case the proportionality factor be zero. As a conclusion, the functions $\mathcal{X}, \mathcal{Y}$ in Eq. (40) should be constrained to guarantee a proper dynamics for the electromagnetic field (38). In particular the flat functions $\mathcal{X}_{\text{flat}}, \mathcal{Y}_{\text{flat}}$ in Eq. (28) allows the field (38) to satisfy Maxwell equations (whatever $X, Y$ are). Moreover, by splitting $\beta$ in $\mathcal{X}_{\text{flat}}, \mathcal{Y}_{\text{flat}}$ into two different values,

$$\mathcal{X}(p) = \delta + \epsilon p^2 + (\beta + \mu_{\mathcal{X}} Q) p^{-2},$$

$$\mathcal{Y}(r) = \delta + \epsilon r^2 + (\beta + \mu_{\mathcal{Y}} Q) r^{-2},$$

\[\square]\ Springer
then Eq. (40) for the field (38) becomes a MCS equation:

$$d \star F = -(\mu X - \mu Y) F \wedge F.$$  \hfill (46)

### 5 Electrovacuum gravity

Our aim now is to use the pointlike-charge electromagnetic field (38) as a source in Einstein equations. The goal will be to solve them by finding suitable functions $X(p)$, $Y(r)$. The solution must become the 5D Kerr-NUT-(A)dS geometry (29) when the charge $Q$ is zero.

Let us examine the form of Einstein tensor in the $dxKS$ Ansatz. Again we can resort to the structure functions $c_{bc}^{a}$. By defining $\tilde{c}_{abc} = \eta^{ad} c_{bc}^{d}$, the connection takes the form

$$\Gamma^a_{bc} = \frac{1}{2} \tilde{\eta}^{ad} [c_{cdb} + c_{bdc} - c_{dbc}],$$ \hfill (47)

and the components of the Riemann tensor are

$$R^a_{bcd} = n_c (\Gamma^a_{bd}) - n_d (\Gamma^a_{bc}) + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{be} \Gamma^a_{cd} - c^e_{cd} \Gamma^a_{be},$$ \hfill (48)

where the vectors $n_c$ act as operators on the functions $\Gamma^a_{bd}$.

In a general $dxKS$ Ansatz the mixed Einstein tensor turns out to be diagonal only in the block corresponding to the null tetrad. However, if the functions $X(p)$ and $Y(p)$ are restricted to be the ones of Eqs. (44) and (45), then the mixed Einstein tensor $G$ is entirely diagonal in the basis $\{\tilde{n}_0, \tilde{n}_1, n_2, n_3, n_4\}$ of the Ansatz (24, 25); moreover, $G$ is linear in the unknown functions $X(p)$, $Y(r)$ [24, 25],

$$G = f(r, p) \left( \tilde{n}_0 \otimes n^0 + n_2 \otimes n^2 \right) + g(r, p) \left( \tilde{n}_1 \otimes n^1 + n_3 \otimes n^3 \right) + h(r, p) n_4 \otimes n^4,$$ \hfill (49)

where

$$f(r, p) = \frac{-p r (r^2 - p^2) X''(p) - 2 r^3 X'(p) - (p^3 - 3 p r^2) Y'(r)}{2 p r (r^2 - p^2)^2} + f_0(r, p)$$ \hfill (50)

$$g(r, p) = \frac{p r (r^2 - p^2) Y''(r) - 2 p^3 Y'(r) - (r^3 - 3 r p^2) X'(p)}{2 p r (r^2 - p^2)^2} + g_0(r, p)$$ \hfill (51)

$$h(r, p) = -\frac{X''(p) - Y''(r)}{2 (r^2 - p^2)} + h_0(r, p)$$ \hfill (52)

with

$$f_0(r, p) = -g_0(r, p)$$
\[ h_0(r, p) = -\frac{6}{p^4} \frac{Q (\mu_Y - \mu_X) (r^2 (\beta + \mu_X Q) - p^2 (\beta + \mu_Y Q))}{(r^2 - p^2)^3} \]

\[ h_0(r, p) = -\frac{3p^2 (2r^2 - p^2)(\beta + \mu_Y Q)^2}{r^4 (r^2 - p^2)^3} + \frac{3r^2 (2p^2 - r^2)(\beta + \mu_X Q)^2}{p^4 (r^2 - p^2)^3}. \] (54)

### 5.1 Vacuum solutions

Einstein vacuum equations with cosmological constant \( \Lambda \) are

\[ G + \Lambda \equiv 0. \] (55)

In the \( dxKS \) framework they amount to

\[ f(r, p) = g(r, p) = h(r, p) = -\Lambda. \] (56)

It is easy to verify that the 5D Kerr-NUT-(A)dS geometry (29) solves these equations (\( Q \) must be taken as zero in Eqs. (53, 54)). Anyway we will recover this result as a particular case of the charged geometry to be solved below.

### 5.2 Electrovacuum solutions

Einstein electrovacuum equations are

\[ G + \Lambda \equiv 8\pi T, \] (57)

where \( T \) is the mixed electromagnetic energy-momentum tensor,

\[ T^\mu_v = \frac{1}{4\pi} \left( F^{\mu\beta} F_{\beta v} - \frac{1}{4} \delta^\mu_v F^{\alpha\beta} F_{\alpha\beta} \right). \] (58)

This tensor is valid even in the MCS case because the Chern–Simons term in the Lagrangian, the 5-form \( A \wedge F \wedge F \), does not contain the metric; thus it does not contribute to the energy-momentum tensor. The expression of \( T \) for the pointlike-charge field (38) is

\[ T = -\frac{Q^2}{2\pi (r^2 - p^2)^3} \left( \bar{n}_0 \otimes n^0 - \bar{n}_1 \otimes n^1 + n_2 \otimes n^2 - n_3 \otimes n^3 \right) \]

\[ -\frac{Q^2 (r^2 + p^2)}{2\pi (r^2 - p^2)^4} \bar{n}_4 \otimes n^4. \] (59)
The $(2+2)D$ sector in $T$ is traceless. The additional dimension provides $T$ with a non-null trace. These trace will be responsible for preventing the fulfillment of Maxwell equations, as we are going to show.

By comparing Eqs. (49) and (59) one realizes that Eq. (57) implies three equations:

$$f(r, p) + \Lambda - \frac{4 Q^2}{(r^2 - p^2)^3} = 0 , \quad (60)$$

$$g(r, p) + \Lambda + \frac{4 Q^2}{(r^2 - p^2)^3} = 0 , \quad (61)$$

$$h(r, p) + \Lambda + \frac{4 Q^2 (r^2 + p^2)}{(r^2 - p^2)^4} = 0 . \quad (62)$$

Let us focus on the last one. Combining it with Eq. (52) and multiplying by $2(p^2 - r^2)$ we obtain

$$X''(p) - Y''(r) - 2 (r^2 - p^2) \left( h_0(r, p) + \Lambda + \frac{4 Q^2 (r^2 + p^2)}{(r^2 - p^2)^4} \right) = 0 . \quad (63)$$

Then, the last term must split into a sum of a function of $r$ and a function of $p$; so the second derivative $\partial^2 / \partial r \partial p$ of this term must be zero:

$$\frac{\partial^2}{\partial r \partial p} \left[ 2 (r^2 - p^2) \left( h_0(r, p) + \Lambda + \frac{4 Q^2 (r^2 + p^2)}{(r^2 - p^2)^4} \right) \right] = 96 Q^2 p r \left( 3(\mu_\mathcal{Y} - \mu_\mathcal{X})^2 - 4 \right) \frac{(r^2 + p^2)}{(r^2 - p^2)^5} = 0 , \quad (64)$$

which implies the necessary condition

$$(\mu_\mathcal{Y} - \mu_\mathcal{X})^2 = \frac{4}{3} . \quad (65)$$

Thus, the pointlike-charge solution only works if associated with the specific Chern-Simons coupling constant $|\mu_\mathcal{Y} - \mu_\mathcal{X}| = 2/\sqrt{3}$; at least this is the result in the $dxKS$ framework. Remarkably, the term $4$ in the parenthesis of Eq. (64) comes from the $n_4 \otimes n^4$ term in Eq. (59) (i.e., it comes from the trace of $T$); on the other hand, the term $3(\mu_\mathcal{Y} - \mu_\mathcal{X})^2$ comes from $h_0(r, p)$. Thus, the value of the CS coupling constant is dictated by the the additional dimensions that provide $T$ with a trace.\(^6\) Presumably, each odd dimension will correspond to a different CS coupling constant.

Coming back to Eq. (63), now we have

$$X''(p) - Y''(r) + \frac{6 (\beta + \mu_\mathcal{X} Q)^2}{p^4} + 2 \Lambda p^2 - \frac{6 (\beta + \mu_\mathcal{Y} Q)^2}{r^4} - 2 \Lambda r^2 = 0 . \quad (66)$$

\(^6\) Nevertheless, solutions to Maxwell equations do exist in five dimensions for those cases where the energy-momentum tensor results to be traceless. See, for instance, the pure radiation solution in Sect. VI of Ref. [10].
By using the separation constant $2\alpha$ it yields

\begin{align}
X''(p) + 2 \Lambda \ p^2 + 6 \left( \beta + \mu_X \ Q \right)^2 \ p^{-4} - 2 \alpha &= 0, \\
Y''(r) + 2 \Lambda \ r^2 + 6 \left( \beta + \mu_Y \ Q \right)^2 \ r^{-4} - 2 \alpha &= 0,
\end{align}

which have solutions

\begin{align}
X(p) &= d_2 \ p + d_1 - \frac{\Lambda}{6} \ p^4 - \left( \beta + \mu_X \ Q \right)^2 \ p^{-2} + \alpha \ p^2, \\
Y(r) &= c_2 \ r + c_1 - \frac{\Lambda}{6} \ r^4 - \left( \beta + \mu_Y \ Q \right)^2 \ r^{-2} + \alpha \ r^2.
\end{align}

Replacing these expressions for $X$ and $Y$, Eqs. (60) and (61) turn out to be

\begin{align}
2 \ c_2 \ r^3 + d_2 \left( p^3 - 3 \ p \ r^2 \right) &= 0, \\
2 \ d_2 \ p^3 + c_2 \left( r^3 - 3 \ r \ p^2 \right) &= 0,
\end{align}

which hold true only if $c_2 = d_2 = 0$. As in the vacuum case, $c_1 = -2m$ is interpreted as the mass, and $d_1 = 2n$ as the NUT charge; but only their difference is physically meaningful since they can be added with a constant $\gamma$ without affecting the solution. In sum, the charged rotating black hole solution to Einstein–Maxwell–Chern–Simons equations (57, 46) in the $dxKS$ Ansatz (30, 31) is characterized by the functions

\begin{align}
X(p) &= 2n + \gamma + \alpha \ p^2 - \frac{\Lambda}{6} \ p^4 - \left( \beta + \mu_X \ Q \right)^2 \ p^{-2}, \\
\chi(p) &= \delta + \varepsilon \ p^2 + \left( \beta + \mu_X \ Q \right) p^{-2}, \\
Y(r) &= -2m + \gamma + \alpha \ r^2 - \frac{\Lambda}{6} \ r^4 - \left( \beta + \mu_Y \ Q \right)^2 \ r^{-2}, \\
\gamma(r) &= \delta + \varepsilon \ r^2 + \left( \beta + \mu_Y \ Q \right) r^{-2},
\end{align}

where $|\mu_Y - \mu_X| = 2/\sqrt{3}$.\footnote{Equations (46, 53, 54) show that changing the signs of $\mu_X$, $\mu_Y$ is equivalent to changing the sign of $Q$; so it is enough to consider that $\mu_Y - \mu_X = 2/\sqrt{3}$ when replacing in the equations.}

As mentioned in Sect. 2.D, the free constants $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ represent choices of charts in the maximally symmetric case ($m + n = 0 = Q$). Clearly, $\gamma$ and $\beta$ in the solution (73, 74) are absorbed in a shift of $m, n, \mu_X, \mu_Y$,

\begin{align}
m \longrightarrow m' &= m - \frac{\gamma}{2}, \quad n \longrightarrow n' = n + \frac{\gamma}{2}, \\
\mu_X, \mu_Y \longrightarrow \mu'_{X,Y} &= \mu_X, \mu_Y + \beta \ Q^{-1},
\end{align}

the magnitudes $m + n$ and $\mu_X - \mu_Y$ being unaffected by the shift. The CCLP solution as originally made known [7, 8], was built with $\mu_X' = 0$ and $\mu_Y' = 2/\sqrt{3}$. By choosing
\[ \delta = 0 = \epsilon \] one concludes that the CCLP solution fits the \( x KS \) form with the (A)dS metric playing the role of \( h \). \[8\]

The shift (75) could suggest that the only physically meaningful magnitudes are \( m + n \) and \( \mu_Y - \mu_X \), so extending the degeneration between \( m \) and \( n \) typical of the uncharged solution. However, as shown in the Appendix A, the form of the Kretschmann scalar for the charged solution indicates that such degeneration is actually broken by the charge.

### 6 Signature \((1,4)\)

Up to now we have worked with a metric of signature \((3,2)\), which greatly simplifies the algebraic aspects of the problem by allowing the metric to take the \( dx KS \) form. Even though the signature \((3,2)\) can be relevant on its own in different contexts (two-time physics [26], AdS/dS spacetimes [27, 28], Kleinian spaces [29]), an important feature of our approach is that we can retrieve the Lorentzian signature by means of a proper Wick rotation.

Let us consider the coordinate transformation

\[
dt' = dt - \frac{a^2 + b^2 + p^2 + ab X(p)}{X(p)} \, dp - \frac{a^2 + b^2 + r^2 + ab Y(r)}{Y(r)} \, dr , \tag{76}
\]

\[
d\phi' = d\phi - \frac{a + b X(p)}{(a^2 - b^2) \, X(p)} a^2 dp - \frac{a + b Y(r)}{(a^2 - b^2) \, Y(r)} a^2 dr , \tag{77}
\]

\[
d\psi' = d\psi + \frac{b + a X(p)}{(a^2 - b^2) \, X(p)} b^2 dp + \frac{b + a Y(r)}{(a^2 - b^2) \, Y(r)} b^2 dr , \tag{78}
\]

The components of the metric (33) in the chart \((t', \phi', \psi', r, p)\) are

\[
g_{\mu\nu} = \begin{pmatrix}
M & 0 & 0 \\
0 & -\frac{p^2+r^2}{X(p)} & 0 \\
0 & 0 & -\frac{p^2+r^2}{X(p)}
\end{pmatrix} . \tag{79}
\]

So the block \( r, p \) has been diagonalized; instead, \( M \) remains as a non-diagonal \( 3 \times 3 \) matrix. In particular the vectors \( \partial/\partial r \) and \( \partial/\partial p \) are no longer null in this chart. \[9\]

The coordinate transformation (76)–(78) preserves the volume,

\[
\det(g_{\mu\nu}) = \frac{r^2 p^2}{a^4 b^4} \left( a^2 - b^2 \right)^2 \left( -p^2 + r^2 \right)^2 . \tag{80}
\]

By analyzing the form of the components \( g_{\mu\nu} \) in Eq. (79), one realizes that the metric tensor can be displayed in the extended Plebański-Demiański Ansatz of Ref. \[8\]

---

8 A suitable coordinate transformation is needed to compare the components of the metric with those of Refs. [7, 10] (see Appendix B).

9 Vectors \( \partial/\partial r, \partial/\partial p \) has changed, since they now differentiate along curves of constant \( t', \phi', \psi' \).
\[ g = \left. \begin{array}{c}
-\frac{Y(r)}{r^2 + p^2} \omega^0 \otimes \omega^0 + \frac{X(p)}{r^2 + p^2} \omega^1 \otimes \omega^1 + \frac{r^2 p^2}{a^2 b^2} \omega^2 \otimes \omega^2 \\
+ \frac{p^2 + r^2}{Y(r)} \omega^3 \otimes \omega^3 - \frac{p^2 + r^2}{X(p)} \omega^4 \otimes \omega^4,
\end{array} \right. \] (81)

where the 1-forms \( \omega^a, \omega^2 \) in the coordinate basis \( \{dt', d\phi', d\psi', dr, dp\} \) read (after dropping the primes)

\[
\begin{align*}
\omega^0 &= dt - a^{-1}(a^2 + p^2) d\phi - b^{-1}(b^2 + p^2) d\psi, \\
\omega^1 &= dt - a^{-1}(a^2 + r^2) d\phi - b^{-1}(b^2 + r^2) d\psi, \\
\omega^2 &= \frac{b^2}{a} d\phi + \frac{a^2}{b} d\psi, \\
\omega^3 &= dr, \\
\omega^4 &= dp.
\end{align*}
\] (82-85)

As can be seen, the chart we are using here is simpler than the one used in Ref. [10].

Notice that the coordinate transformation (76)–(78) involves the functions \( X, Y, \lambda', \lambda \) carrying the information on the gravitational field. So the components \( g_{\mu\nu} \) in the new chart can no longer be split into the sum of the components of a flat seed metric plus gravitational terms. Namely, the choice \( X = 0 = Y \) as a way to recover the flat metric is no longer allowed.

In Sect. 2 we have started with a flat seed metric of signature \( (3, 2) \); equation (81) shows that this signature is kept for every choice of \( X, Y \) and every sign of \( r^2 - p^2 \). Let us perform the Wick rotation of \( p \),

\[ p \rightarrow i \ p, \] (86)

therefore

\[ \omega^4 \otimes \omega^4 \rightarrow -\omega^4 \otimes \omega^4. \] (87)

If we further assume that that the functions \( X(ip), \lambda'(ip) \) are real—which is indeed the case for the solutions in Eqs. (73, 74)–, it follows that the metric signature depends only on the sign of \( X(ip) \): if \( X(ip) < 0 \), then the (real) metric (81) gets a Lorentzian signature \( (1, 4) \). The Wick rotation, besides, puts on an equal footing the coordinates \( r, p \), since the ubiquitous expression \((-p^2 + r^2)\) becomes \((p^2 + r^2)\). The zenith angle \( \theta \) of 5D ellipsoidal coordinates is introduced through the relation \( p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \).

It is important to note that although not every metric in the \( dxKS \) ansatz is transformed to the signature \( (1, 4) \) by the proposed coordinate transformation (if \( X(ip) > 0 \) the Wick rotation does not change the signature), every Lorentzian metric that can be expressed in the Plebański-Demiański Ansatz can also be expressed in the \( dxKS \) form

\[ \text{[10],} \]  

\[ Y(r) = -\frac{Y(r)}{r^2 + p^2} \omega^0 \otimes \omega^0 + \frac{X(p)}{r^2 + p^2} \omega^1 \otimes \omega^1 + \frac{r^2 p^2}{a^2 b^2} \omega^2 \otimes \omega^2 \\
+ \frac{p^2 + r^2}{Y(r)} \omega^3 \otimes \omega^3 - \frac{p^2 + r^2}{X(p)} \omega^4 \otimes \omega^4, \] (81)

\[ \text{where the 1-forms } \omega^a, \omega^2 \text{ in the coordinate basis } \{dt', d\phi', d\psi', dr, dp\} \text{ read (after dropping the primes)} \]
by performing the inverse transformation of (76)–(78), together with the corresponding Wick rotation.

7 Conclusions

Equations (73, 74) display the “pointlike” rotating electrovacuum solution in the $5D$ double extended Kerr–Schild ($dxKS$) framework (30, 31). The physical parameters in Eqs. (73, 74) are the mass $m$, the NUT charge $n$, $\mu_x Q$, and $\mu_y Q$. The difference $(\mu_y - \mu_x)Q$, which connects with the Chern-Simons coupling constant in Eq. (46), is $2Q/\sqrt{3}$, as dictated by the trace of the Einstein equations. The degeneracy between $m$ and $n$ exhibited by the uncharged solution is broken by the presence of the electric charge $Q$, as evidenced by the Kretschmann scalar. Since the EMCS equations depend on $\mu_x$ and $\mu_y$ only through their difference, one might expect a degeneracy between these parameters. However, by inspecting the Kretschmann scalar one concludes this is not the case. As the Ricci tensor depends only on $\mu_x - \mu_y$, the degeneracy breaking is located in the Weyl tensor.

Let us review the basic elements of the $5D$ $dxKS$ Ansatz. We start from a flat $(2+3)D$ seed spacetime, whose tangent and cotangent spaces are spanned by the dual bases $\{n_a\}$ and $\{n^a\}$ respectively (see Eq. (3)). The (by definition) null tetrad $\{n_0, n_1, n_2 = k, n_3 = K\}$ is constituted by four independent geodesic congruences. The way the Ansatz (24, 25) introduces gravity implies the deformation of $n_0, n_1$ by following some rules that are essential for the pointlike electromagnetic potential (36) to work:

(i) $n^0 \wedge n^2$ and $n^1 \wedge n^3$ do not change.
(ii) the spacetime volume is preserved (the determinant of the metric is not affected).
(iii) $k, K$ continue to be null and geodesic vectors in the deformed metric.
(iv) Since the Ansatz does not modify the vectors $k, K$, they are Lie-dragged by the independent Killing vectors $\partial_r, \partial_\phi, \partial_\psi$; in fact they are still the coordinate vectors $k = \partial/\partial r, K = \partial/\partial p$. In particular, the $5D$ $dxKS$ Ansatz keeps the two independent “rotational” symmetries of the seed metric.
(v) The Ansatz protects the property (43). This property means that $dn^a(m) = 0$ if $a \neq 4$; Eq. (27) corresponds to the cases $a = 0, 1$.

One might consider a further extension of Ansatz (30) by adding a term proportional to $k \otimes K + K \otimes k$. It turns out that, although such metric preserves most of the properties posed by (30), by solving the EMCS equations under this assumption we observe that they can only be fulfilled when this term is absent; at least for the electromagnetic source in Eq. (36).

Although we develop the $5D$ $dxKS$ Ansatz in the context of metric signature $(3,2)$, in Sect. 6 we have displayed a change of coordinate followed by a Wick rotation that throws the metric into the signature $(1,4)$, and leaves it in the Plebański-Demiański Ansatz of Ref. [10].

The computational simplicity is an important issue in this type of work. The charts we introduced in Sects. 2 and 6, which make the bases look as shown in Eqs. (13–16)
and (82–85), greatly facilitate the computational procedures, so providing a new way to advance in the search for higher-dimensional electrovacuum solutions.

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**Data Availability** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

**Appendix A: Kretschmann scalar**

The Kretschmann scalar for the solution (73, 74) is

$$K = \frac{10 \Lambda^2}{9} + 96 (m+n)^2 \frac{3p^4 + 10p^2r^2 + 3r^4}{(p^2 - r^2)^6} - \frac{768Q^2(\mu\mathbf{X} - \mu\mathbf{Y})}{(p^2 - r^2)^6}$$

$$+ (m+n)(\mu'\mathbf{X} + \mu'\mathbf{Y}) (p^2 + r^2)$$

$$+ \frac{4\Lambda Q^2(\mu\mathbf{X} - \mu\mathbf{Y})^2}{3(p^2 - r^2)^6} \left(p^{6} + 23p^{4}r^{2} + 23p^{2}r^{4} + r^{6}\right)$$

$$+ \frac{48Q^4(\mu\mathbf{X} - \mu\mathbf{Y})^2}{(p^2 - r^2)^7} \left((m+n)(p^{4} + 26p^{2}r^{2} + r^{4}) - 8(m' p^{4} + n' r^{4})\right)$$

$$- \frac{4Q^4(\mu\mathbf{X} - \mu\mathbf{Y})^2}{(-p^2 + r^2)^8} \left((\mu\mathbf{X} - \mu\mathbf{Y})^2 \left(28p^{2}r^{2} + 65(p^{4} + r^{4})\right)\right)$$

$$- 192(\mu'\mathbf{X} p^{2} - \mu'\mathbf{Y} r^{2})^2\right).$$

If the Kretschmann scalar (A1) admits the decomposition

$$K = R_{abcd}R^{abcd} = C_{abcd}C^{abcd} + \frac{4}{d-2} R_{ab}R^{ab} - \frac{2}{(d-1)(d-2)}R^2,$$

then the contributions breaking the degeneration are located in the Weyl tensor $C_{abcd}$.

The physical and geometric interpretation of the parameters $\gamma$ and $\beta$ in the charged case is yet to be understood. A possibility one must consider is that these constants represent mere coordinate choices. In such case one would be able to absorb these parameters present in the Kretschmann scalar in a redefinition of the coordinates $r$ and $p$. Note, however, that $K$ is linear in $\gamma$ and quadratic in $\beta$, while the coordinates $r$ and $p$ appear in a variety of powers ranging from $-12$ to 0. Furthermore, once the coordinate change is performed the scalars $R$ and $R_{ab}R^{ab}$ should continue to be independent of $\gamma$ and $\beta$. Because of these reasons we can see that such a change of coordinates would be hard to obtain if not impossible.
If the parameters $\gamma$ and $\beta$ cannot be absorbed in a coordinate change then different values of these parameters would represent different spacetimes. The fact that these constants appear on the Weyl tensor seems to indicate that they might play a relevant role in the algebraic classification of these spacetimes [30].

Appendix B: Relationship with the Plebański-Demiański Ansatz in Ref. [10]

Equation (81) in Sect. 6 displays an extended Plebański-Demiański Ansatz that works for signature $(3, 2)$; it uses a much simpler set of 1-forms $\{\omega^a, \omega^2\}$ than the one used in Ref. [10]. For the sake of completeness, we will show the way to obtain the metric as displayed in Ref. [10]. Let us perform the complex coordinate transformation

\[
dt' = dt - \frac{a^2 + b^2 + p^2 + abX(p)}{X(p)} dp - \frac{a^2 + b^2 + r^2 + abY(r)}{Y(r)} dr , \quad (B1)
\]

\[
d\phi' = a \lambda \, dt + \frac{(1 - a^2 \lambda)(a^2 - b^2)}{a^2} d\phi - \frac{a(1 + \lambda(b^2 + p^2)) + bX(p)}{X(p)} a^2 dp
\]

\[- \frac{a(1 + \lambda(b^2 + r^2)) + bY(r)}{Y(r)} a^2 dr , \quad (B2)
\]

\[
d\psi' = b \lambda \, dt - \frac{(a^2 - b^2)(1 - b^2 \lambda)}{b^2} d\psi + \frac{b(1 + \lambda(a^2 + p^2)) + aX(p)}{X(p)} b^2 dp
\]

\[+ \frac{b(1 + \lambda(a^2 + r^2)) + aY(r)}{Y(r)} b^2 dr , \quad (B3)
\]

\[
dr' = dr , \quad dp' = -i \, dp . \quad (B4)
\]

where $\lambda = \Lambda/6$, then we can see that the metric tensor (33) can be expressed as (we drop the primes)

\[
g = -\frac{Y(r)}{p^2 + r^2} \omega^0 \otimes \omega^0 + \frac{X(ip)}{p^2 + r^2} \omega^1 \otimes \omega^1 - \frac{a^2 b^2}{r^2 p^2} \omega^2 \otimes \omega^2
\]

\[+ \frac{p^2 + r^2}{Y(ip)} \, dr \otimes dr + \frac{p^2 + r^2}{X(ip)} \, dp \otimes dp , \quad (B5)
\]

where the 1-forms $\omega^a, \omega^2$ read

\[
\omega^0 \equiv \frac{(1 - p^2 \lambda)}{(1 - a^2 \lambda)(1 - b^2 \lambda)} \, dt - \frac{a(a^2 - p^2)}{(a^2 - b^2)(1 - a^2 \lambda)} \, d\phi' - \frac{b(b^2 - p^2)}{(b^2 - a^2)(1 - b^2 \lambda)} \, d\psi , \quad (B6)
\]

\[
\omega^1 \equiv \frac{(1 + r^2 \lambda)}{(1 - a^2 \lambda)(1 - b^2 \lambda)} \, dt - \frac{a(a^2 + r^2)}{(a^2 - b^2)(1 - a^2 \lambda)} \, d\phi - \frac{b(b^2 + r^2)}{(b^2 - a^2)(1 - b^2 \lambda)} \, d\psi , \quad (B7)
\]

\[
\omega^2 \equiv - \frac{(1 + r^2 \lambda)(1 - p^2 \lambda)}{(1 - a^2 \lambda)(1 - b^2 \lambda)} \, dt + \frac{a(a^2 + r^2)(a^2 - p^2)}{a(a^2 - b^2)(1 - a^2 \lambda)} \, d\phi + \frac{b(b^2 + r^2)(b^2 - p^2)}{b(b^2 - a^2)(1 - b^2 \lambda)} \, d\psi . \quad (B8)
\]
\[ \omega^2 \equiv \omega_0^2 - \frac{p^2 r^2}{2ab (r^2 + p^2)} \left[ \left( \mathcal{Y}(r) - \frac{ab}{r^2} \right) \omega^0 - \left( \mathcal{X}(ip) + \frac{ab}{p^2} \right) \omega^1 \right]. \] (B9)

The metric in Ref. [10] is retrieved by redefining \(-X(ip) \rightarrow X(p), -Y(r) \rightarrow Y(r), \mathcal{X}(ip) + abp^{-2} \rightarrow \mathcal{X}(p), \mathcal{Y}(r) - abr^{-2} \rightarrow \mathcal{Y}(r)\) together with a global change of sign (Ref. [10] uses signature \((4,1)\) instead of the signature \((1,4)\) used in this work). As mentioned in Sect. 6, the metric signature of this ansatz depends solely on the sign of the function \(X\). If \(X(ip) > 0\), then the metric (B5) has signature \((3,2)\). If \(X(ip) < 0\), the signature is \((1,4)\) and the complex coordinate transformation (B1)–(B3) renders the metric (B5) Lorentzian.

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