Singular mean-field control games with applications to optimal harvesting and investment problems

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Abstract

This paper studies singular mean field control problems and singular mean field stochastic differential games. Both sufficient and necessary conditions for the optimal controls and for the Nash equilibrium are obtained. Under some assumptions the optimality conditions for singular mean-field control are reduced to a reflected Skorohod problem, whose solution is proved to exist uniquely. Some examples are given. In particular, a simple singular mean-field game is studied where the Nash equilibrium exists but is not unique.

1 Introduction

The irreversible investment problem is a classical problem in economics, with a long history. In short, the problem is the following. A factory is facing an increased demand for its product. Should it invest in more production capacity to meet the demand? The problem is

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that buying additional production capacity is an expensive, irreversible investment (usually production equipment cannot easily be sold after use) and the future demand for the product is uncertain. So the risk is that the factory ends up having paid for an additional capacity it does not need. On the other hand, if the factory does not increase the capacity, it might miss an opportunity for an increased sale.

This is a classical problem that has been studied by many authors in different contexts. See e.g. Pindyck (1988, 1991, 1991), Kobila (1993) and the references therein. Mathematically the problem can be formulated as a singular control problem. In this paper we study such singular control problems in the context of mean-field Itô processes and under model uncertainty. We interpret model uncertainty in the sense of Knight uncertainty, i.e. uncertainty about the underlying probability measure. Using the Girsanov theorem for Itô processes we can parametrize the family of densities of possible underlying probability measures by a stochastic process \( \theta(t) \). This leads to a stochastic differential game in which one of the players is the investor controlling the investment strategy and the other player controls the model by choosing the model parameter \( \theta \). Since the investment is irreversible, the control of the investor is a singular control, i.e. a non-decreasing non-negative stochastic process \( \xi(t) \). Thus we are dealing with a singular mean-field stochastic differential game.

Recently, there have been several papers dealing with mean-field control problems. See e.g. Meyer-Brandis et al (2012), Anderson & Djehiche (2011) and Hamadène (1998). A recent paper dealing with mean-field singular control is L. Zhang (2012). Our paper extends this paper in several directions: For example, we consider more general mean-field operators. And we allow the profit rate \( f \) (see below) to depend on the state \( X \), on the mean-field term \( Y \), and on the singular control \( \xi \). We also allow both the coefficient \( \lambda \) in the singular part of the state equation and the singular cost coefficient \( h \) in the performance functional to depend on the state \( X \). Moreover, we consider general games between two players of such singular control problems with asymmetric information.

Our paper is organized as follows: In Section 2 we present three motivating examples, In Section 3 we formulate the general mean-field singular stochastic control problem and prove a sufficient maximum principle and a necessary maximum principle. In Section 4 we reduce the maximum principle to a Skorohod problem and prove the existence and uniqueness of the solution. In Section 5 we prove a sufficient maximum principle for general singular mean-field stochastic games, and we obtain as a corollary a corresponding maximum principle for zero-sum games. In Section 6 we apply the results above to an optimal harvesting problem of a mean-field system and to model uncertainty singular control, in particular model uncertainty irreversible investment type problems.
2 Three motivating examples

2.1 Optimal harvesting from a mean-field system

Suppose we model the size \( X^0(t) \) of an unharvested population at time \( t \) by an equation of the form

\[
\begin{align*}
\frac{dX^0(t)}{dt} &= E[X^0(t)]b(t)dt + X^0(t)\sigma(t)dB(t); \quad t \in [0, T] \\
X^0(0) &= x > 0.
\end{align*}
\]

(2.1)

Here, and in the following, \( B(t) = B(t, \omega) \) is a Brownian motion on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}, P) \) satisfying the usual conditions. \( P \) is a reference probability measure. We assume that \( b(t) \) and \( \sigma(t) \) are given predictable processes. We may regard (2.1) as a limit as \( n \to \infty \) of a large population interacting system of the form

\[
\begin{align*}
\frac{dx^{i,n}(t)}{dt} &= \frac{1}{n} \sum_{j=1}^{n} x^{j,n}(t) b(t)dt + x^{i,n}(t)\sigma(t)dB^i(t), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

(2.2)

Thus the mean-field term \( E[X(t)] \) represents an approximation to the weighted average \( \frac{1}{n} \sum_{j=1}^{n} x^{j,n}(t) \) for large \( n \). Now suppose we introduce harvesting of the population. The size of the harvested population \( X(t) = X^\xi(t) \) at time \( t \) can then be modeled by a mean-field singular control stochastic differential equation of the form

\[
\begin{align*}
\frac{dX(t)}{dt} &= E[X(t)]b(t)dt + X(t)\sigma(t)dB(t) - \lambda_0(t)d\xi(t); \quad t \in [0, T] \\
X(0) &= x > 0
\end{align*}
\]

(2.3)

where \( \xi(t) \) is a non-decreasing predictable process with \( \xi(0^-) = 0 \), representing the harvesting effort, while \( \lambda_0(t) > 0 \) is a given harvesting efficiency coefficient.

The performance functional is assumed to be of the form

\[
J(\xi) = E\left[ \int_0^T h_0(t)X(t)d\xi(t) + KX(T) \right],
\]

(2.4)

where \( h_0(t) \) is a given adapted price process and \( K = K(\omega) \) is a given salvage price, assumed to be \( \mathcal{F}_T \)-measurable. The problem is to find \( \xi^* \) such that

\[
J(\xi^*) = \sup_{\xi} J(\xi).
\]

(2.5)

Such a process \( \xi^* \) is called an optimal singular control. This is an example of a mean-field singular control problem. We will return to this problem in Section 6.1.

2.2 Optimal irreversible investments under model uncertainty

Let \( \xi(t) \) denote the production rate capacity of a production plant and let \( D(t) \) denote the demand rate at time \( t \). At any time \( t \) the production capacity can be increased by \( d\xi(t) \)
at the price $\lambda_0(t, D(t))$ per capacity unit. The number of units sold per time unit is the minimum of the demand $D(t)$ and the capacity $\xi(t)$. The total expected net profit of the production is assumed to be

$$J(\xi, \theta) = E^{Q_0} \left[ \int_0^T a(t, E[\varphi(D(t))]) \min[D(t), \xi(t)] dt + g(D(T)) - \int_0^T \lambda_0(t, D(t)) d\xi(t) \right],$$

(2.6)

where $g(D(T))$ is some salvage value of the closed-down production plant, $\varphi$ is a given real function and $a(t, E[\varphi(D(t))])$ is the unit sales price of the production. Here $\{Q^\theta\}_{\theta \in \Theta}$ is a family of probability measures representing the model uncertainty. We let $A_G$ denote the set of right-continuous, non-decreasing $G$-adapted processes $\xi(\cdot)$ with $\xi(0^-) = 0$, where $\mathbb{G} := \{G_t\}_{t \geq 0}$ is a given subfiltration of $\mathbb{F}$, in the sense that $G_t \subseteq F_t$ for all $t$. Heuristically, $G_t$ represents the information available to the investor at time $t$. We assume that the demand $D(t)$ is given by a jump diffusion of the form

$$\begin{cases}
dD(t) = D(t^-) [\alpha(t, \omega) dt + \beta(t, \omega) dB(t)], 0 \leq t \leq T \\
D(0) > 0
\end{cases},$$

(2.7)

where $\alpha(t, \omega), \beta(t, \omega)$ are given $\mathbb{F}$-adapted processes. We want to maximize the expected total net profit under the worst possible scenario, i.e. find $(\xi^*, \theta^*) \in A_G \times \Theta$ such that

$$\sup_{\xi \in A_G} \left\{ \inf_{\theta \in \Theta} J(\xi, \theta) \right\} = \inf_{\theta \in \Theta} \left\{ \sup_{\xi \in A_G} J(\xi, \theta) \right\} = J(\xi^*, \theta^*),$$

(2.8)

This is an example of a (partial information) singular control game of a jump diffusion. Note that the system is non-Markovian, both because of the mean-field term and the partial information constraint. See Sections 6.2-6.4.

### 2.3 A mean field singular game

Suppose the demand $X(t)$ for a certain product at time $t$ is given by a mean field SDE of the form

$$\begin{cases}
dX(t) = E[X(t)] b(t) dt + X(t) \sigma(t) dB(t) \\
X(0) = x > 0
\end{cases}.$$

There are two competing companies producing this product, with production rate capacities represented by nondecreasing adapted processes $\xi_1, \xi_2$, respectively. The expected profit of the company $i$ is assumed to have to the form

$$J_i(\xi_1, \xi_2) = E \left[ \int_0^T \pi(t) \min(X(t), \xi_1(t) + \xi_2(t)) dt + \int_0^T h_i(t) d\xi_i(t) \right],$$
where $\pi(t) > 0$ is the price per unit sold and $h_i(t) < 0$ the production cost per unit for the factory $i, i = 1, 2$. We want to find a Nash equilibrium, i.e. a pair $(\xi_1^*, \xi_2^*) \in A_1 \times A_2$ such that

$$\sup_{\xi_1 \in A_1} J_1(\xi_1, \xi_2^*) = J_1(\xi_1^*, \xi_2^*)$$

and

$$\sup_{\xi_2 \in A_2} J_2(\xi_1^*, \xi_2) = J_2(\xi_1^*, \xi_2^*),$$

where $A_i$ is the family of admissible controls $\xi_i$ for company number $i; i = 1, 2$. We will return to this problem in Section 6.5.

### 3 Maximum principle for singular mean field control problems

#### 3.1 Problem statement

We first recall some basic concepts and results from Banach space theory. Let $V$ be an open subset of a Banach space $X$ with norm $\| \cdot \|$ and let $F : V \to \mathbb{R}$.

(i) We say that $F$ has a directional derivative (or Gâteaux derivative) at $x \in X$ in the direction $y \in X$ if

$$D_y F(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(x + \varepsilon y) - F(x))$$

exists.

(ii) We say that $F$ is Fréchet differentiable at $x \in V$ if there exists a linear map $L : X \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|} |F(x + h) - F(x) - L(h)| = 0.$$  

In this case we call $L$ the gradient (or Fréchet derivative) of $F$ at $x$ and we write

$$L = \nabla_x F.$$

(iii) If $F$ is Fréchet differentiable, then $F$ has a directional derivative in all directions $y \in X$ and

$$D_y F(x) = \nabla_x F(y) = \langle \nabla_x F, y \rangle.$$

In particular, if $X = L^2(P)$ the Fréchet derivative of $F$ at $X \in L^2(P)$, denoted by $\nabla_X F$, is a bounded linear functional on $L^2(P)$, which we can identify with a random variable in $L^2(P)$. For example, if $F(X) = E[\varphi(X)]; X \in L^2(P)$, where $\varphi$ is a real $C^1$- function such that $\varphi(X) \in L^2(P)$ and $\frac{\partial \varphi}{\partial x}(X) \in L^2(P)$, then $\nabla_X F = \frac{\partial \varphi}{\partial x}(X)$ and $\nabla_X F(Y) = E[\frac{\partial \varphi}{\partial x}(X)Y]$ for $Y \in L^2(P)$.  

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Consider a mixed regular and singular controlled system with state process \( X(t) = X(t) \) of the form
\[
dX(t) = b(t, X(t), Y(t), \xi(t), u(t), \omega)dt + \sigma(t, X(t), Y(t), \xi(t), u(t), \omega)dB(t)
+ \lambda(t, X(t), u(t), \omega)d\xi(t),
\]
\[ (3.1) \]
where
\[ Y(t) = F(X(t, \cdot)) \]
\[ (3.2) \]
and \( F \) is a Fréchet differentiable operator on \( L^2(P) \). We assume that all the coefficients \( b, \sigma, \lambda, f, g \) and \( h \) are Fréchet differentiable \((C^1)\) with respect to \( x, y, \xi, u \) with derivatives in \( L^2(m \times P) \), where \( m \) denoted Lebesgue measure on \([0, T]\). Note that we allow the coefficients to depend on both controls \( \xi \) and \( u \). This might be relevant, for example, in harvesting models. See \((4.26)\). The performance functional is assumed to be of the form
\[
J(\xi, u) = E \left[ \int_0^T f(t, X(t), Y(t), \xi(t), u(t), \omega)dt + g(X(T), Y(T), \omega)
+ \int_0^T h(t, X(t), u(t), \omega)\xi(dt) \right].
\]
\[ (3.3) \]
We may interpret the function \( f \) as a profit rate, \( g \) as a bequest or salvage value function and \( h \) as a cost rate for the use of the singular control \( \xi \).

We want to find \((\xi^*, u^*) ) \in \mathcal{A}\) such that
\[
J(\xi^*, u^*) = \sup_{(\xi, u) \in \mathcal{A}} J(\xi, u).
\]
\[ (3.4) \]
Here \( \mathcal{A} \) is a given family of \( \mathcal{G} \)-predictable processes such that the corresponding state equation has a unique solution \( X \) such that \( \omega \to X(t, \omega) \in L^2(P) \) for all \( t \). We let \( \mathcal{A} \) denote the set of possible values of \( u(t); t \in [0, T] \) when \( (\xi, u) \in \mathcal{A} \).

\(3.2\) A sufficient maximum principle for singular mean field control

In this subsection we prove a sufficient maximum principle for the singular control games described above. To this end, define the Hamiltonians \( H \) as follows:
\[
H(t, x, y, \xi, u, p, q)(dt, \xi(dt)) = H_0(t, x, y, \xi, u, p, q)dt + \{\lambda(t, x, u)p + h(t, x, u)\}\xi(dt),
\]
\[ (3.5) \]
where
\[
H_0(t, x, y, w) = f(t, x, y, \xi, u) + b(t, x, y, \xi, u)p + \sigma(t, x, y, \xi, u)q.
\]
\[ (3.6) \]
The associated mean-field BSDE for the adjoint processes is

\[
\begin{align*}
dp(t) &= -\frac{\partial H_0}{\partial x}(t, X(t), Y(t), \xi(t), u(t), p(t), q(t))dt \\
&\quad - \frac{\partial H_0}{\partial y}(t, X(t), Y(t), \xi(t), u(t), p(t), q(t))\nabla_{X(t)} F)dt + q(t)dB(t) \\
p(T) &= \frac{\partial g}{\partial x}(X(T), Y(T)) + E[\frac{\partial g}{\partial y}(X(T), Y(T))]\nabla_{X(T)} F.
\end{align*}
\] (3.7)

The sufficient maximum principle for this singular mean field control is stated as follows.

**Theorem 3.1 (Sufficient maximum principle for mean-field singular control)** Let \( \hat{\xi}, \hat{u} \in \mathcal{A} \), with corresponding solutions \( \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{\xi}(t), \hat{\xi}(t), \hat{\xi}(t) \) of (3.1) and (3.7). Suppose the following conditions hold

- The function \( X, \xi, u \rightarrow H(t, X, F(X), \xi, u, \hat{\xi}(t), \hat{\xi}(t)) \) (3.8) is concave with respect to \( (X, \xi, u) \in L^2(P) \times \mathbb{R} \times \mathbb{R} \) for all \( t \in [0, T] \).

- (The conditional maximum property)
  \[
  \underset{v \in \mathcal{A}}{\text{ess sup}} \ E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), v, \hat{\xi}(t), \hat{q}(t)) | \mathcal{G}_t] \\
  = E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{u}(t), \hat{\xi}(t), \hat{q}(t)) | \mathcal{G}_t].
  \] (3.9)

- (Variational inequality)
  \[
  \underset{\xi}{\text{ess sup}} \ E[H(t, \hat{X}(t), \hat{Y}(t), \xi, \hat{u}(t), \hat{\xi}(t), \hat{q}(t)) | \mathcal{G}_t] \\
  = E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{u}(t), \hat{\xi}(t), \hat{q}(t)) | \mathcal{G}_t]
  \] (3.10)

Then \( (\hat{\xi}(t), \hat{u}(t)) \) is an optimal control for \( J(\xi, u) \).

**Proof.** This theorem is a straightforward consequence of Theorem 5.1 below. We refer to the proof there. \( \square \)

### 3.3 A necessary maximum principle for singular mean field control

In the previous section we gave a verification theorem, stating that if a given control \( (\hat{\xi}, \hat{u}) \) satisfies (3.8)-(3.10), then it is indeed optimal for the singular mean field control problem. We now establish a partial converse, implying that if a control \( (\hat{\xi}, \hat{u}) \) is optimal for the singular mean field control problem, then it is a conditional critical point for the Hamiltonian.
To achieve this, we start with the setup of [12] as follows. For \( \xi \in \mathcal{A} \), let \( \mathcal{V}(\xi) \) denote the set of \( \mathcal{G} \)-adapted processes \( \eta \) of finite variation such that there exists \( \delta = \delta(\xi) > 0 \) satisfying

\[
\xi + a\eta \in \mathcal{A} \text{ for all } a \in [0, \delta].
\]  

(3.11)

Then for \( \xi \in \mathcal{A} \) and \( \eta \in \mathcal{V}(\xi) \) we have, by our smoothness assumptions on the coefficients,

\[
\lim_{a \to 0^+} \frac{1}{a} (J(\xi + a\eta) - J(\xi)) = E \left[ \int_0^T \left\{ \frac{\partial f}{\partial x}(t, X(t), Y(t), \xi(t), u(t)) Z(t) \right. \right.
\]

\[
+ \frac{\partial f}{\partial y}(t, X(t)Y(t), \xi(t), u(t)) \langle \nabla_{X(t)} F, Z(t) \rangle \left. \right\} dt
\]

\[
+ \int_0^T \frac{\partial f}{\partial \xi}(t, X(t), Y(t), \xi(t), u(t)) \eta(dt) \right]
\]

\[
+ E \left[ \frac{\partial g}{\partial x}(X(T), Y(T)) Z(T) + \frac{\partial g}{\partial y}(X(T), Y(T)) \langle \nabla_{X(T)} F, Z(T) \rangle \right]
\]

\[
+ E \left[ \int_0^T \left\{ \frac{\partial h}{\partial x}(t, X(t), \xi(t), u(t)) Z(t) \xi(dt) + \int_0^T h(t, X(t), \xi(t), u(t)) \eta(dt) \right\} \right]
\]

(3.12)

where

\[
Z(t) := \lim_{a \to 0^+} \frac{1}{a} \left( X^{(\xi + a\eta)}(t) - X^{(\xi)}(t) \right).
\]  

(3.13)

Note that by the chain rule we have

\[
\lim_{a \to 0^+} \frac{1}{a} \left( Y^{(\xi + a\eta)}(t) - Y^{(\xi)}(t) \right) = \lim_{a \to 0^+} \frac{1}{a} \left( F\left( (X^{(\xi + a\eta)}(t) - F(X^{(\xi)}(t)) \right) = \langle \nabla_{X(t)} F, Z(t) \rangle.
\]  

(3.14)

Moreover,

\[
dZ(t) = \left( \frac{\partial h}{\partial x}(t) Z(t) + \frac{\partial h}{\partial y}(t) \langle \nabla_{X(t)} F, Z(t) \rangle \right) dt
\]

\[
+ \left( \frac{\partial \sigma}{\partial x}(t) Z(t) + \frac{\partial \sigma}{\partial y}(t) \nabla_{X(t)} F, Z(t) \right) dB(t)
\]

\[
+ \frac{\partial \lambda}{\partial x}(t) Z(t) \xi(dt) + \lambda(t) \eta(dt) \right); \ Z(0) = 0.
\]  

(3.15)

**Theorem 3.2** (Necessary maximum principle for singular mean-field control problem) Suppose \((\hat{x}, \hat{u}) \in \mathcal{A}\) is optimal, i.e. satisfies (3.4). Then

\[
E \left[ \frac{\partial H_0}{\partial u}(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), u, \hat{p}(t), \hat{q}(t)) \bigg| \mathcal{G}_t \right] = 0.
\]  

(3.16)
Moreover, the following variational inequalities hold.

\[
\begin{align*}
E \left[ \frac{\partial f}{\partial \xi} (t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{u}(t)) + \lambda(t, \hat{X}(t), \hat{u}(t)) \hat{p}(t) + h(t, \hat{X}(t), \hat{u}(t)) | \mathcal{G}_t \right] & \leq 0 \\
for all \ t \in [0, T] \quad \text{and} \quad \\
E \left[ \frac{\partial f}{\partial \xi} (t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{u}(t)) + \lambda(t, \hat{X}(t), \hat{u}(t)) \hat{p}(t) + h(t, \hat{X}(t), \hat{u}(t)) | \mathcal{G}_t \right] d\hat{\xi}(t) & = 0
\end{align*}
\]

(3.17)

Proof. To simplify the notation denote

\[
\hat{f}(t) := f(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t)) \\
\hat{\lambda}(t) := \lambda(t, \hat{X}(t)), \quad \hat{h}(t) := h(t, \hat{X}(t)).
\]

We need to prove that if \( \hat{\xi} \in \mathcal{A}_G \) is optimal, i.e. if

\[
\sup_{\xi \in \mathcal{A}_G} J(\xi) = J(\hat{\xi}) \quad (3.18)
\]

then \( \hat{\xi} \) satisfies the following variational inequalities:

\[
E \left[ \frac{\partial \hat{f}}{\partial \xi} (t) + \hat{\lambda}(t) \hat{p}(t) + \hat{h}(t) | \mathcal{G}_t \right] \leq 0 \quad \text{for all} \ t \in [0, T] \quad (3.19)
\]

and

\[
E \left[ \frac{\partial \hat{f}}{\partial \xi} (t) + \hat{\lambda}(t) \hat{p}(t) + \hat{h}(t) | \mathcal{G}_t \right] \hat{\xi}(dt) = 0 \quad \text{for all} \ t \in [0, T]. \quad (3.20)
\]

To this end, choose \( \xi \in \mathcal{A}_G \) and \( \eta \in \mathcal{V}(\xi) \) and compute

\[
\frac{d}{da} J(\xi + a\eta) |_{a=0} = A_1 + A_2 + A_3 + A_4, \quad (3.21)
\]

where

\[
\begin{align*}
A_1 &= E \left[ \int_0^T \left\{ \frac{\partial f}{\partial x}(t)Z(t) + \frac{\partial f}{\partial y}(t)\langle \nabla_{X(t)} F, Z(t) \rangle \right\} dt \right] \\
A_2 &= E \left[ \int_0^T \frac{\partial f}{\partial \xi}(t)\eta(dt) \right] \\
A_3 &= E \left[ \frac{\partial g}{\partial x}(X(T), Y(T))Z(T) + \frac{\partial g}{\partial y}(X(T), Y(T))\langle \nabla_{X(T)} F, Z(T) \rangle \right] \\
A_4 &= E \left[ \int_0^T \frac{\partial h}{\partial x}(t)Z(t)\xi(dt) + h(t)\eta(dt) \right].
\end{align*}
\]

(3.22)
By the definition of $H_0$ we have
\[
A_1 = E \left[ \int_0^T Z(t) \left\{ \frac{\partial H_0}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) \right\} dt \right. \\
+ \left. \int_0^T \langle \nabla_{X(t)} F, Z(t) \rangle \left( \frac{\partial H_0}{\partial y}(t) - \frac{\partial b}{\partial y}(t)p(t) - \frac{\partial \sigma}{\partial y}(t)q(t) \right) dt \right].
\] (3.23)

By the terminal condition of $p(T)$ (see (3.7)) we have
\[
A_3 = E[(p(T), Z(T))]
= E \left[ \int_0^T p(t)dZ(t) + \int_0^T Z(t)dp(t) + \int_0^T q(t) \left( \frac{\partial \sigma}{\partial x}(t)Z(t) + \frac{\partial \sigma}{\partial y}(t)\langle \nabla_{X(t)} F, Z(t) \rangle \right) dt \right]
= E \left[ \int_0^T p(t) \left\{ \frac{\partial b}{\partial x}(t)Z(t) + \frac{\partial b}{\partial y}(t)\langle \nabla_{X(t)} F, Z(t) \rangle \right\} dt \right. \\
+ \left. \int_0^T p(t)\frac{\partial \lambda}{\partial x}(t)Z(t)\xi(dt) + \int_0^T p(t)\lambda(t)\eta(dt) \\
- \int_0^T \left( \frac{\partial H_0}{\partial x}(t)Z(t) + \frac{\partial H_0}{\partial y}(t)\langle \nabla_{X(t)} F, Z(t) \rangle \right) dt \\
+ \int_0^T q(t) \left( \frac{\partial \sigma}{\partial x}(t)Z(t) + \frac{\partial \sigma}{\partial y}(t)\langle \nabla_{X(t)} F, Z(t) \rangle \right) dt \right].
\] (3.24)

Combining (3.21)-(3.24) we get
\[
\left. \frac{d}{da}J(\xi + a\eta) \right|_{a=0} = E \left[ \int_0^T \left\{ \frac{\partial f}{\partial \xi}(s) + \lambda(s)p(s) + h(s) \right\} \eta(ds) \right].
\]

In particular, if we apply this to an optimal $\xi = \hat{\xi}$ for $J$ we get, for all $\eta \in V(\hat{\xi})$,
\[
E \left[ \int_0^T \left\{ \frac{\partial \hat{f}}{\partial \xi}(s) + \hat{\lambda}(t)\hat{p}(s) + \hat{h}(s) \right\} \eta(ds) \right] = \frac{d}{da}J(\hat{\xi} + a\eta)_{a=0} \leq 0.
\] (3.25)

If we choose $\eta$ to be a pure jump process of the form
\[
\eta(s) = \sum_{0< t_i \leq s} \alpha(t)
\]
where $\alpha(t) > 0$ is $\mathcal{G}_t$-measurable, (3.25) gives
\[
E \left[ \left( \frac{\partial \hat{f}}{\partial \xi}(t) + \hat{\lambda}(t)\hat{p}(t) + \hat{h}(t) \right) \alpha \right] \leq 0 \text{ for all } t.
\]

Since this holds for all such $\eta$, we conclude that
\[
E \left[ \left( \frac{\partial \hat{f}}{\partial \xi}(t) + \hat{\lambda}(t)\hat{p}(t) + \hat{h}(t) \right) \mid \mathcal{G}_t \right] \leq 0 \text{ for all } t \in [0, T].
\] (3.26)
Finally, applying (3.26) to
\[ \eta(dt) = \hat{\xi}(dt) \in \mathcal{V}(\hat{\xi}) \]
and then to
\[ \eta(dt) = -\hat{\xi}(dt) \in \mathcal{V}(\hat{\xi}) \]
we get, for all \( t \in [0,T] \),
\[ E \left[ \frac{\partial \hat{f}}{\partial \xi}(t) + \hat{\lambda}(t)\hat{p}(t) + \hat{\eta}(t) \mid \mathcal{G}_t \right] \hat{\xi}(dt) = 0 \text{ for all } t \in [0,T]. \quad (3.27) \]
With (3.26) and (3.27) the proof is complete.

4 The optimality conditions

Since there have already been studies (see e.g. [9] and references therein) on the usual (nonsingular) mean field control problems, let us consider only the singular control \( \xi \), i.e. with no regular control \( u \). The system that we shall deal with, is described by
\[ dX(t) = b(t,X(t),Y(t),\xi(t))dt + \sigma(t,X(t),Y(t),\xi(t))dB(t) + \lambda(t,X(t),Y(t))d\xi(t). \quad (4.1) \]
The performance functional is
\[ J(\xi) = E \left[ \int_0^T f(t,X(t),Y(t),\xi(t))dt + g(X(T),Y(T)) + \int_0^T h(t,X(t),Y(t))d\xi(t) \right]. \quad (4.2) \]
The auxiliary backward stochastic differential equation with mean field is
\[ dp(t) = -\left[ f_x(t,X(t),Y(t),\xi(t)) + b_x(t,X(t),Y(t),\xi(t))p(t) \right. \\
\left. + \sigma_x(t,X(t),Y(t),\xi(t))q(t) \right] dt - \left[ f_y(t,X(t),Y(t),\xi(t)) \\
+ b_y(t,X(t),Y(t),\xi(t))p(t) + \sigma_y(t,X(t),Y(t),\xi(t))q(t) \right] \nabla_{X(t)} F dt \\
+ q(t)dB(t), \quad (4.3) \]
\[ p(T) = g_x(X(T),Y(T)) + E[g_y(X(T),Y(T))] \nabla_{X(T)} F. \quad (4.4) \]
To solve the above BSDE, we denote
\[ \begin{align*}
\alpha(t) &= -b_x(t,X(t),Y(t),\xi(t)) - b_y(t,X(t),Y(t),\xi(t))\nabla_{X(t)} F \\
\beta(t) &= -\sigma_x(t,X(t),Y(t),\xi(t)) - \sigma_y(t,X(t),Y(t),\xi(t))\nabla_{X(t)} F \\
\phi(t) &= -f_x(t,X(t),Y(t),\xi(t)) - f_y(t,X(t),Y(t),\xi(t))\nabla_{X(t)} F \\
\Theta &= g_x(X(T),Y(T)) + E[g_y(X(T),Y(T))] \nabla_{X(T)} F.
\end{align*} \quad (4.5) \]
We also denote
\[
\rho_{t,T} := \exp \left\{ \int_t^T \beta(s)dW_s + \int_t^T \left[ \alpha(s)ds - \frac{1}{2} \beta^2(s) \right] ds \right\}.
\]

Then application of Equation (2.11) of [7] to the above backward SDE yields
\[
p(t) = \mathbb{E} \left( \Theta_{t,T} + \int_t^T \rho_{t,r} \phi(r) dr \big| \mathcal{F}_t \right).
\] (4.6)

Substituting this into the maximum principle equations (3.17), we see that the maximum principle consists of the following equations:
\[
\frac{\partial f}{\partial \xi}(t, X(t), Y(t), \xi(t)) + \lambda(t, X(t), Y(t))p(t) + h(t, X(t), Y(t)) \leq 0 \tag{4.7}
\]
and
\[
\left[ \frac{\partial \phi}{\partial \xi}(t, X(t), Y(t), \xi(t)) + \lambda(t, X(t), Y(t))p(t) + h(t, X(t), Y(t)) \right] d\xi(t) = 0. \tag{4.8}
\]

Thus here is the strategy to solve the maximum principle equations (4.1), (4.7) and (4.8), where the \( p(t) \) in (4.7) and (4.8) is given by (4.6): First, assuming that \( X \) is known, we solve the backward stochastic differential equation (4.3) with the terminal condition (4.4) to obtain (4.6). Then substitute it into (4.7). This way we obtain an equation which will describe the domain \( D \) in which the process \( X(t) \) must be all the time. We control the process \( X(t) \) in such a way that when the process is in the interior of the domain \( D \), we don’t do anything. When the process reaches the boundary of \( D \), we exercise the minimal push to keep the process inside the domain \( D \). Here are some detailed explanation of the above strategy:

The equations (4.7) and (4.8) are essentially an equation for the “domain” of the state \( X(t) \) and the condition for the singular control \( \xi \) to satisfy. The equation (4.7) can be complicated since the solution \( p(t) \) may depend on the paths of \( X, Y \) and the path of control \( \xi \) itself up to time \( t \). Denote \( X_t = (X(s), 0 \leq s \leq t) \) the trajectory of \( X \) up to time \( t \). Then \( p(t) \) can be represented in general as \( p(t) = p(X(t), Y(t), \xi(t)) \).

We now consider a slightly more general situation, where the singular control may be any finite variation process, not necessarily increasing. The increasing case corresponds to \( r(t, X_t, Y_t, \xi_t) = \infty \) below.

Suppose that there are two functionals \( l, r : [0, T] \times C([0, T], \mathbb{R})^3 \to \mathbb{R} \) with \( l \leq r \) such that the equations (4.7) and (4.8) can be written as
\[
\begin{cases}
    l(t, X_t, Y_t, \xi_t) \leq X(t) \leq r(t, X_t, Y_t, \xi_t) \\
    \int_0^T [X(t) - l(t, X_t, Y_t, \xi_t)] d\xi(t) = 0 \\
    \int_0^T [r(t, X_t, Y_t, \xi_t) - X(t)] d\xi(t) = 0.
\end{cases} \tag{4.9}
\]
Then we are led to the problem of finding a finite variation (not necessarily increasing) control $\xi$ for the system

$$
\frac{dX(t)}{dt} = b(t, X(t), Y(t), \xi(t))dt + \sigma(t, X(t), Y(t), \xi(t))dB(t) + \lambda(t, X(t), Y(t))d\xi(t)
$$

(4.10)

satisfying (4.9). This is a Skorohod type problem. For simplicity, we restrict ourselves to the case when

$$
\lambda(t, x, y) = 1.
$$

**Theorem 4.1** Suppose that the following hold

1. $b$ and $\sigma$ are uniformly Lipschitz continuous. Namely, there is a positive constant $L$ such that

$$
|b(t, x_2, y_2, \xi_2) - b(t, x_1, y_1, \xi_1)| \leq L(|x_2 - x_1| + |y_2 - y_1| + |\xi_2 - \xi_1|).
$$

(4.11)

The same inequality holds for $\sigma$.

2. $l$ and $r$ are uniformly Lipschitz continuous, i.e.

$$
|r(t, X_t^2, Y_t^2, \xi_t^2) - r(t, X_t^1, Y_t^1, \xi_t^1)| \leq \kappa \sup_{0 \leq s \leq t} \left[ |X^2(s) - X^1(s)| + |\xi^2(s) - \xi^1(s)| \right] + L \int_0^t \sup_{0 \leq s \leq r} \left[ |X^2(s) - X^1(s) + |Y^2(s) - Y^1(s)| + |\xi^2(s) - \xi^1(s)| \right] dr
$$

(4.12)

for some $\kappa < 1/4$. The same inequality holds for $l$.

3. For any $t \in [0, T]$, $X$, $Y$ and $\xi$ in $C([0, T], \mathbb{R})$,

$$
l(t, X_t, Y_t, \xi_t) < r(t, X_t, Y_t, \xi_t).
$$

(4.13)

Then, Equations (4.9)-(4.10) have a unique solution.

Proof. We shall apply the Banach fixed point theorem to prove the theorem. Let us denote by $\mathbb{B}$ the Banach space of all continuous adapted processes $(X(t), \xi(t))$ which are square integrable. More precisely,

$$
\mathbb{B} = \left\{ (X, \xi) : X \text{ and } \xi \text{ are continuous and adapted and} \right. \|X(t)\|_{\mathbb{B}} := \left\{ E \sup_{0 \leq t \leq T} \left( |X(t)|^2 + |\xi(t)|^2 \right) \right\}^{1/2} < \infty \right\}.
$$
From (4.9) and (4.10), we define the following mapping on \( \mathbb{B} \):

\[
F(X, \xi) = (Z, \eta) : \quad (4.14)
\]

where \((Z, \eta)\) satisfies the inequalities:

\[
\begin{cases}
    l(t, X_t, Y_t, \xi_t) \leq Z(t) \leq r(t, X_t, Y_t, \xi_t) \\
    \int_0^T [X(t) - l(t, X_t, Y_t, \xi_t)] \, d\eta(t) = 0 \quad (4.15) \\
    \int_0^T [r(t, X_t, Y_t, \xi_t) - X(t)] \, d\eta(t) = 0,
\end{cases}
\]

and, in addition,

\[
dZ(t) = b(t, X(t), Y(t), \xi(t)) \, dt + \sigma(t, X(t), Y(t), \xi(t)) \, dB(t) + d\eta(t) \quad (4.16)
\]

For every given continuous pair \((X(t), \xi(t))\) in \( \mathbb{B} \), by the condition (3), Theorem 2.6 and Corollary 2.4 of \( \mathbb{B} \) the above Skorohod problem has a unique solution \((Z(t), \eta(t))\) and the solution pair \((Z(t), \eta(t))\) can be represented as

\[
\begin{align*}
\eta(t) &= \Xi(l, r, \psi)(t) \quad (4.17) \\
Z(t) &= \psi(t) - \eta(t), \quad (4.18)
\end{align*}
\]

where

\[
\psi(t) := \int_0^t b(s, X(s), Y(s), \xi(s)) \, ds + \int_0^t \sigma(s, X(s), Y(s), \xi(s)) \, dB(s)
\]

\[
l(u) := l(u, X_u, Y_u, \xi_u) \\
r(u) := r(u, X_u, Y_u, \xi_u)
\]

\[
\Xi(l, r, \psi)(t) := \max \left\{ \left( \psi(0) - r(0) \right)^+ \wedge \inf_{u \in [0,t]} \left( \psi(u) - l(u) \right) \right\},
\]

\[
\sup_{s \in [0,t]} \left( \psi(s) - r(s) \right) \wedge \inf_{u \in [s,t]} \left( \psi(u) - l(u) \right) \right\}.
\]

It is elementary that

\[
|\max\{b_1, b_2\} - \max\{a_1, a_2\}| \leq \max\{b_1 - a_1, b_2 - a_2\}
\]

\[
\left| \sup_{0 \leq t \leq T} g(t) - \sup_{0 \leq t \leq T} f(t) \right| \leq \sup_{0 \leq t \leq T} |g(t) - f(t)|
\]

\[
\left| \inf_{0 \leq t \leq T} g(t) - \inf_{0 \leq t \leq T} f(t) \right| \leq \sup_{0 \leq t \leq T} |g(t) - f(t)|.
\]

From the expression of \( \Xi \), we easily see that

\[
\sup_{0 \leq s \leq t} |\Xi(l_2, r_2, \psi_2)(s) - \Xi(l_1, r_1, \psi_1)(s)| \leq 2 \sup_{0 \leq s \leq t} \left[ |l_2(s) - l_1(s)| + |r_2(s) - r_1(s)| \right]
\]

\[
+ 4 \sup_{0 \leq s \leq t} \left[ |\psi_2(s) - \psi_1(s)| \right]. \quad (4.19)
\]
Now we want to show that $\mathbb{B} \ni (X, \xi) \to F(X, \xi) = (Z, \eta)$ is a contraction on $\mathbb{B}$. Assume that $(X^1, \xi^1)$ and $(X^2, \xi^2)$ be two elements in $\mathbb{B}$ and let $(Z^1, \eta^1)$ and $(Z^2, \eta^2)$ be the corresponding solutions to \((4.15)-(4.16)\). Then for $i = 1, 2$, we have

$$
\eta_i(t) = \Xi(l_i, r_i, \psi_i)(t) \quad (4.20)
$$

$$
Z_i(t) = \psi_i(t) - \eta_i(t) \quad (4.21)
$$

where

$$
\psi_i(t) := \int_0^t b(s, X^i(s), Y^i(s), \xi^i(s))ds + \int_0^t \sigma_i(s, X^i(s), Y^i(s), \xi^i(s))dB(s)
$$

$$
l_i(u) := l(u, X^i_u, Y^i_u, \xi^i_u)
$$

$$
r_i(u) := r(u, X^i_u, Y^i_u, \xi^i_u)
$$

$$
\eta_i := \Xi(l_i, r_i, \psi_i)(t) = \max \left\{ (\psi_i(0) - r_i(0))^+ \wedge \inf_{u \in [0,t]} (\psi_i(u) - l_i(u)) \right\},
$$

$$
\sup_{s \in [0,t]} \left[ (\psi_i(s) - r_i(s)) \wedge \inf_{u \in [s,t]} (\psi_i(u) - l_i(u)) \right].
$$

From \((4.19)\) and then from the assumptions on $l$ and $r$, we see that

$$
E \sup_{0 \leq r \leq t} |\eta_2(r) - \eta_1(r)|^2 \
\leq 8E \sup_{0 \leq s \leq t} \left[ |l_2(s) - l_1(s)|^2 + |r_2(s) - r_1(s)|^2 \right] + 32E \sup_{0 \leq s \leq t} \left[ |\psi_2(s) - \psi_1(s)|^2 \right]
$$

$$
\leq 8\kappa^2 E \sup_{0 \leq s \leq t} \left[ |X^2(s) - X^1(s)|^2 + |\xi^2(s) - \xi^1(s)|^2 \right]
$$

$$
+ 32E \sup_{0 \leq s \leq t} |\psi_2(s) - \psi_1(s)|^2
$$

$$
+ C \int_0^t E \sup_{0 \leq r \leq t} \left[ |X^2(s) - X^1(s)|^2 + |\xi^2(s) - \xi^1(s)|^2 \right] dr.
$$

By standard argument from stochastic analysis, we have

$$
E \sup_{0 \leq s \leq t} |\psi_2(s) - \psi_1(s)|^2 \leq C \int_0^t E \sup_{0 \leq s \leq r} \left[ |X^2(s) - X^1(s)|^2 + |\xi^2(s) - \xi^1(s)|^2 \right] dr.
$$

Thus we have

$$
E \sup_{0 \leq r \leq t} |\eta_2(r) - \eta_1(r)|^2 \leq 8E \sup_{0 \leq s \leq t} \left[ |l_2(s) - l_1(s)|^2 + |r_2(s) - r_1(s)|^2 \right]
$$

$$
+ C \int_0^t E \sup_{0 \leq s \leq r} \left[ |X^2(s) - X^1(s)|^2 + |\xi^2(s) - \xi^1(s)|^2 \right] dr.
$$

\((4.22)\)
From (4.21) we have

\[
E \sup_{0 \leq r \leq t} |Z_2(r) - Z_1(r)|^2 \leq 2E \sup_{0 \leq r \leq t} |\eta_2(r) - \eta_1(r)|^2 + 2E \sup_{0 \leq r \leq t} |\psi_2(r) - \psi_1(r)|^2
\]
\[
\leq 16\kappa^2 E \sup_{0 \leq r \leq t} [(X^2(r) - X^1(r))^2 + (\xi^2(r) - \xi^1(r))^2] + 64E \sup_{0 \leq s \leq t} |\psi_2(s) - \psi_1(s)|^2
\]
\[
+ C \int_0^t E \sup_{0 \leq s \leq r} [(X^2(s) - X^1(s))^2 + (\xi^2(s) - \xi^1(s))^2] \, dr
\]
\[
\leq 16\kappa^2 E \sup_{0 \leq r \leq t} [(X^2(r) - X^1(r))^2 + (\xi^2(r) - \xi^1(r))^2] + C \int_0^t E \sup_{0 \leq s \leq r} [(X^2(s) - X^1(s))^2 + (\xi^2(s) - \xi^1(s))^2] \, dr .
\]

Combining the above inequality with (4.22), we have

\[
E \sup_{0 \leq r \leq t} |Z_2(r) - Z_1(r)|^2 + |\eta_2(r) - \eta_1(r)|^2
\]
\[
\leq 16\kappa^2 E \sup_{0 \leq r \leq t} [(X^2(r) - X^1(r))^2 + (\xi^2(r) - \xi^1(r))^2] + C \int_0^t E \sup_{0 \leq s \leq r} [(X^2(s) - X^1(s))^2 + (\xi^2(s) - \xi^1(s))^2] \, dr
\]
\[
\leq (16\kappa^2 + Ct) E \sup_{0 \leq r \leq t} [(X^2(r) - X^1(r))^2 + (\xi^2(r) - \xi^1(r))^2] .
\]

If \( \kappa < 1/4 \), then we can choose \( t_0 \) such that \( 16\kappa^2 + Ct < 1 \) for all \( t \leq t_0 \). Thus from (4.23), we conclude that \( F \) is a contraction mapping from \( \mathcal{B} \) to \( \mathcal{B} \). Following a routine argument, we see that the solution \( \xi(t), X(t) \) for the equations (4.9) and (4.10) up to time \( t_0 \). Since the constant \( C \) in \( 16\kappa^2 + Ct \) does not depends on the initial condition, we repeat this procedure to solve the equations (4.9) and (4.10) for on the interval \([0,T] \). \( \square \)

**Remark 4.2** From the proof of the theorem, we see that if we define the Picard iteration for \( n = 0, 1, 2, \cdots \),

\[
\begin{align*}
&l(t, X_t^{(n)}, Y_t^{(n)}, \xi_t^{(n)}) \leq X^{(n+1)}(t) \leq r(t, X_t^{(n)}, Y_t^{(n)}, \xi_t^{(n)}) \\
&\int_0^T \left[ X^{(n)}(t) - l(t, X_t^{(n)}, Y_t^{(n)}, \xi_t^{(n)}) \right] d\xi^{(n+1)}(t) = 0 \tag{4.24}
\end{align*}
\]

and

\[
dX^{(n+1)}(t) = b(t, X^{(n)}(t), Y^{(n)}(t), \xi^{(n)}(t)) dt + \sigma(t, X^{(n)}(t), Y^{(n)}(t), \xi^{(n)}(t)) dB(t) + d\xi^{(n+1)}(t) , \tag{4.25}
\]
where \( X_i^{(0)} = X(0), \xi_i^{(0)} = 0 \), then \( \langle X^{(n)}(t), \xi^{(n)}(t) \rangle \) will converge to the true solution \( \langle X(t), \xi(t) \rangle \) in \( \mathbb{B} \). This may be used to construct the numerical solutions.

**Example 4.3** Let us consider an optimal harvesting problem where the population density \( X(t) \) at time \( t \) is described by the linear controlled system

\[
\begin{align*}
\frac{dX(t)}{dt} &= \left[ b_0(t, \xi) + b_1(t)X(t) + b_2(t)EX(t) \right] dt \\
&\quad + \sigma(t, X(t), Y(t), \xi(t))dB(t) - d\xi(t). \tag{4.26}
\end{align*}
\]

We allow the coefficients \( b_0 \) and \( \sigma_0 \) to depend on the harvested amount \( \xi \) to model the situation where the harvesting has influence on the environment and hence on the population growth. We want to find \( \hat{\xi} \) such that

\[
\sup_{\xi \in A} J(\xi) = J(\hat{\xi}), \tag{4.27}
\]

where

\[
J(\xi) = E \left[ \int_0^T f(t, X(t), Y(t), \xi(t)) dt + g(X(T), Y(T)) + \int_0^T h(t, X(t)) \xi(dt) \right],
\]

with

\[
f(t, x, y, \xi) = f_1(t)x + f_2(t)y + f_3(t, \xi) \tag{4.28}
\]

and

\[
g(x, y) = Kx \tag{4.29}
\]

with \( K > 0 \). Then from (4.5) we get

\[
\begin{align*}
\alpha(t) &= -b_1(t) - b_2(t) \\
\beta(t) &= -\sigma_x(t, X(t), Y(t), \xi(t)) - \sigma_y(t, X(t), Y(t), \xi(t))\nabla X(t)F \\
\phi(t) &= -f_1(t) - f_2(t) \\
\Theta &= K.
\end{align*}
\]

Denote

\[
\rho_{t,T} = \exp \left\{ \int_t^T \beta(s)dW_s + \int_t^T \left[ \alpha(s)ds - \frac{1}{2} \beta^2(s) \right] ds \right\}. \tag{4.30}
\]

Since \( \alpha \) is deterministic, we have for all \( t \leq r \leq T \),

\[
\begin{align*}
\alpha(t, r) &= \mathbb{E} \left( \rho_{t,r} | \mathcal{F}_t \right) \\
&= \exp \left\{ \int_t^r \alpha(s)ds \right\} \mathbb{E} \left( \exp \left\{ \int_t^r \beta(s)dW_s - \frac{1}{2} \int_t^r \beta^2(s)ds \right\} | \mathcal{F}_t \right) \\
&= \exp \left\{ \int_t^r \alpha(s)ds \right\}.
\end{align*}
\]
Note that $\alpha(t,r)$ is a deterministic function. It is easy to see from (4.6) that
\begin{equation}
p(t) = K\alpha(t,T) + \int_t^T \alpha(t,r)\phi(r)dr \tag{4.30}
\end{equation}
is a deterministic function. Thus we have
\begin{equation}
\frac{\partial}{\partial \xi}f_3(t,\xi) + \lambda(t, X(t), Y(t))p(t) + h(t, X(t), Y(t)) \leq 0.
\end{equation}
If furthermore we assume
\begin{equation}
\phi(t) \geq 0, \quad \frac{\partial}{\partial \xi}f_3(t,\xi) = 0, \quad \text{and} \quad h(t, x, y) = h_0(t)x^\kappa, \tag{4.31}
\end{equation}
where $h_0(t)$ is positive and $\kappa$ is a constant, we get (noting that $\lambda(t) = -1$)
\begin{equation}
-p(t) + h_0(t)X^\kappa(t) \leq 0,
\end{equation}
or
\begin{equation}
X(t) \begin{cases} 
\leq \left( \frac{p(t)}{h_0(t)} \right)^{\frac{1}{\kappa}} & \text{if } \kappa > 0 \\
\geq \left( \frac{h_0(t)}{p(t)} \right)^{-\frac{1}{\kappa}} & \text{if } \kappa < 0.
\end{cases}
\end{equation}
In this case, we can take
\begin{equation}
\begin{cases}
 l(t, x, y, \xi) = 0 \quad \text{and} \quad r(t, x, y, \xi) = \left( \frac{p(t)}{h_0(t)} \right)^{\frac{1}{\kappa}} \quad \text{if } \kappa > 0 \\
l(t, x, y, \xi) = \left( \frac{h_0(t)}{p(t)} \right)^{-\frac{1}{\kappa}} \quad \text{and} \quad r(t, x, y, \xi) = \infty \quad \text{if } \kappa < 0.
\end{cases}
\tag{4.32}
\end{equation}
Note that $\kappa < 0$ means that unit price goes up when the population goes down (which becomes more precious). In this case, we want keep the population above a threshold $\hat{h}(t) = \left( \frac{h_0(t)}{p(t)} \right)^{-\frac{1}{\kappa}}$. It is interesting to note that when $h_0(t)$ is larger, this threshold $\hat{h}_0(t)$ is also larger. We have proved

**Theorem 4.4** Under the assumptions (4.28), (4.29) and (4.31), the solution $\hat{\xi}$ of the mean field singular control problem (4.27) is given by the solution $(\hat{X}, \hat{Y}, \hat{\xi})$ of the Skorohod reflection problem (4.9) and (4.26), with the boundaries $l$ and $r$ given by (4.32).
Next, we continue the above example but with $h$ being given by

$$h(t, x, y) := h_0(t)x^2 + h_1(t)x. \quad (4.33)$$

Namely, we continue to assume (4.26)-(4.30). But we replace (4.31) by

$$\phi(t) \geq 0, \quad \frac{\partial}{\partial \xi} f_3(t, \xi) = 0, \quad \text{and} \quad h(t, x, y) = h_0(t)x^2 + h_1(t)x. \quad (4.34)$$

where $h_0(t)$ is positive. Then, the inequalities (4.7) - (4.8) become

$$-p(t) + h_0(t)X^2(t) + h_1(t)X(t) \leq 0. \quad (4.35)$$

where

$$\begin{align*}
l(t) &:= \frac{h_1(t) - \sqrt{h_1^2(t) + 4h_0(t)p(t)}}{2h_0(t)} \quad (4.36) \\
r(t) &:= \frac{h_1(t) + \sqrt{h_1^2(t) + 4h_0(t)p(t)}}{2h_0(t)} \quad (4.37)
\end{align*}$$

Similar to Theorem 4.4, we have

**Theorem 4.5** Under the assumptions (4.28), (4.29) and (4.34), the solution $\hat{\xi}$ of the mean field singular control problem (4.27) is given by the solution $(\hat{X}, \hat{Y}, \hat{\xi})$ of the Skorohod reflection problem (4.9) and (4.26), with the boundaries $l$ and $r$ given by (4.35)-(4.37).

We can also consider the case that $h$ is given by (4.33) but with $h_0(t) < 0$. In this case, the domain (4.7) will be either

$$X(t) \leq \underline{h}(t) \quad \text{or} \quad X(t) \geq \overline{h}(t).$$

The interested readers may write down similar result for this case as well.

## 5 General singular mean-field games

### 5.1 Statement of the problem

In this section we consider the stochastic game of two players, each of them is to maximize his/her singular mean-field performance.

Denote $\xi = (\xi_1, \xi_2), u = (u_1, u_2), w = (w_1, w_2), \lambda = (\lambda_1, \lambda_2), h = (h_1, h_2)$ with $h_i = (h_{i,1}, h_{i,2})$, and let the pair $w_i = (\xi_i, u_i)$ represent the control of player $i; i = 1, 2$. 

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Suppose the process \(X(t) = X^{\xi, u}(t)\) under control of the two players satisfy the following stochastic differential equation with jumps.

\[
dX(t) = b(t, X(t), Y(t), \xi(t), u(t), \omega)dt + \sigma(t, X(t), Y(t), \xi(t), u(t), \omega)dB(t) + \lambda(t, X(t), u(t), \omega)d\xi(t),
\]

where

\[
Y(t) = F(X(t, \cdot)),
\]

and \(F\) is a Fréchet differentiable operator on \(L^2(P)\).

We put \(G^i = \{G^i_t\}_{t \geq 0}\) where \(G^i_t \subseteq F_t\) is the information available to player \(i\) at time \(t\). The \textit{performance functional} for player \(i\) is assumed to be on the form

\[
J_i(\xi, u) = E \left[ \int_0^T f_i(t, X(t), Y(t), w(t), \omega)dt + g_i(X(T), Y(T), \omega) + \int_0^T h_i(t, X(t), u(t), \omega)\xi(dt) \right]; \quad i = 1, 2.
\]

We want to find a \textit{Nash equilibrium} for this game, i.e. find \((\xi^*_1, u^*_1) \in \mathcal{A}^{(1)}\) and \((\xi^*_2, u^*_2) \in \mathcal{A}^{(2)}\) such that

\[
\sup_{(\xi^*_1, u^*_1) \in \mathcal{A}^{(1)}} J_1(\xi^*_1, u^*_1, \xi^*_2, u^*_2) = J_1(\xi^*_1, u^*_1, \xi^*_2, u^*_2)
\]

and

\[
\sup_{(\xi^*_2, u^*_2) \in \mathcal{A}^{(2)}} J_2(\xi^*_1, u^*_1, \xi^*_2, u^*_2) = J_2(\xi^*_1, u^*_1, \xi^*_2, u^*_2)
\]

Here \(\mathcal{A}^{(i)}\) is a given family of \(\mathcal{G}^{(i)}\)-predictable processes such that the corresponding state equation has a unique solution \(X\) such that \(\omega \rightarrow X(t, \omega) \in L^2(P)\) for all \(t\). We let \(\mathcal{A}^{(i)}\) denote the set of possible values of \(u_i(t); t \in [0, T]\) when \((\xi_i, u_i) \in \mathcal{A}^{(i)}; i = 1, 2\).

### 5.2 A sufficient maximum principle for the general non-zero sum case

Define two \textit{Hamiltonians} \(H_i; i = 1, 2\), as follows:

\[
H_i(t, x, y, \xi_1, u_1, \xi_2, u_2, p_i, q_i)(dt, \xi_1(dt), \xi_2(dt)) = H_{i,0}(t, x, y, \xi_1, u_1, \xi_2, u_2, p_i, q_i)dt + \sum_{j=1}^2 \{\lambda_j(t, x, u)p_i + h_{i,j}(t, x, u)\}\xi_j(dt)
\]

where

\[
H_{i,0}(t, x, y, w, p_i, q_i) := f_i(t, x, y, w) + b(t, x, y, \xi, u)p_i + \sigma(t, x, y, \xi, u)q_i.
\]

We assume that for \(i = 1, 2\), \(H = H_i\) is Fréchet differentiable \((C^1)\) in the variables \(x, y, \xi, u\).
The BSDE for the adjoint processes $p_i, q_i$ is
\[
\begin{cases}
    dp_i(t) = -\frac{\partial H_{i,0}}{\partial x}(t, X(t), Y(t), w(t), p_i(t), q_i(t)) dt \\
    \quad - \frac{\partial H_{i,0}}{\partial y}(t, X(t), Y(t), w(t), p_i(t), q_i(t)) \nabla_{X(t)} F dt \\
    + q_i(t) dB(t)
\end{cases}
\]  
\[p_i(T) = \frac{\partial g_i}{\partial x}(X(T), Y(T)) + \frac{\partial g_i}{\partial y}(X(T), Y(T)) \nabla_{X(T)} F; \quad i = 1, 2. \tag{5.8}\]

Note that (5.8) is an operator valued BSDE for each $\omega$.

**Theorem 5.1 (Sufficient maximum principle)** Let $(\xi_1, \hat{u}_1) \in \mathcal{A}^{(1)}$, $(\xi_2, \hat{u}_2) \in \mathcal{A}^{(2)}$ with corresponding solutions $\hat{X}, \hat{p}_i, \hat{q}_i, \hat{r}_i$ of (2.3) and (2.8). Assume the following:

- The maps
  \[X, w_1 \to H_1(t, X, F(X), w_1, \hat{w}_2(t), \hat{p}_1(t), \hat{q}_1(t)), \tag{5.9}\]
  and
  \[X, w_2 \to H_2(t, X, F(X), \hat{w}_1, w_2(t), \hat{p}_2(t), \hat{q}_2(t)), \tag{5.10}\]
  and
  \[X \to g_i(X, F(X)) \tag{5.11}\]
  are concave for all $t; i = 1, 2$.  

- (The conditional maximum properties)
  \[
es\sup_{u_1 \in A_1} [H_1(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), u_1, \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t)) \mid \mathcal{G}_t^{(1)}] \\
  = E[H_1(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t)) \mid \mathcal{G}_t^{(1)}] \tag{5.12}\]
  and
  \[
es\sup_{u_2 \in A_2} [H_2(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), u_2, \hat{p}_2(t), \hat{q}_2(t)) \mid \mathcal{G}_t^{(2)}] \\
  = E[H_2(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_2(t), \hat{q}_2(t)) \mid \mathcal{G}_t^{(2)}], \tag{5.13}\]

- (Variational inequalities)
  \[
es\sup_{\xi_1} [H_1(t, \hat{X}(t), \hat{Y}(t), \xi_1, \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t)) \mid \mathcal{G}_t^{(1)}] \\
  = E[H_1(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t)) \mid \mathcal{G}_t^{(1)}] \tag{5.14}\]
  and
  \[
es\sup_{\xi_2} [H_2(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \xi_2, \hat{u}_2(t), \hat{p}_2(t), \hat{q}_2(t)) \mid \mathcal{G}_t^{(2)}] \\
  = E[H_2(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_2(t), \hat{q}_2(t)) \mid \mathcal{G}_t^{(2)}], \tag{5.15}\]

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Then \((\xi_1, \hat{\xi}_1), (\xi_2, \hat{\xi}_2)\) is a Nash equilibrium, in the sense that (5.1) and (5.5) hold with 
\[ \xi^*_i := \hat{\xi}_i, u^*_i := \hat{u}_i; \ i = 1, 2. \]

Proof. By introducing a suitable increasing sequence of stopping times converging to \(T\),
we see that we may assume that all local martingales appearing in the proof below are martingales.
We refer to [15] for details. We first study the stochastic control problem (5.4).

Consider \(J_1(\xi_1, u_1, \xi_2, u_2) - J_1(\xi_1, \hat{\xi}_1, \hat{\xi}_2, \hat{u}_2) = I_1 + I_2 + I_3 + I_4\), where

\[
I_1 := E \left[ \int_0^T \{ f_1(t, X(t), Y(t), w_1(t), \hat{\omega}_2(t)) - f_1(t, \bar{X}(t), \bar{Y}(t), \hat{\omega}_1(t), \hat{\omega}_2(t)) \} dt \right]
\]

\[
I_2 := E[ g_1(X(T), Y(T)) - g_1(\bar{X}(T), \bar{Y}(T))] \]

\[
I_3 := E \left[ \int_0^T \{ h_1(t, X(t), u_1(t), \hat{\omega}_2(t)) - h_1(t, \bar{X}(t), \hat{\omega}(t)) \} dt \right]
\]

\[
I_4 := E \left[ \int_0^T \{ h_2(t, X(t), u_1(t), \hat{\omega}_2(t)) - h_2(t, \bar{X}(t), \hat{\omega}(t)) \} dt \right]
\]

By the definition of \(H_1\) we have

\[
I_1 = E \left[ \int_0^T \{ H_{1,0}(t, X_t, Y_t, u_1(t), \hat{\omega}_2(t), \hat{p}_1(t), \hat{q}_1(t)) - H_{1,0}(t, \bar{X}_t, \bar{Y}_t, \bar{p}_1(t), \bar{q}_1(t)) - (b - \hat{b})\hat{p}_1 - (\sigma - \hat{\sigma})\hat{q}_1 \} dt \right]
\]

(5.16)

By concavity of \(g_1\) and the Itô formula we have

\[
I_2 \leq E \left[ \frac{\partial g_1}{\partial x}(\bar{X}(T), \bar{Y}(T)) (X(T) - \bar{X}(T)) \right.
\]

\[
+ \frac{\partial g_1}{\partial y}(\bar{X}(T), \bar{Y}(T)) (\nabla_{X(T)} F, X(T) - \bar{X}(T)) \right]
\]

\[
= \langle \hat{p}_1(T), X(T) - \bar{X}(T) \rangle
\]

\[
= E \left[ \int_0^T \hat{p}_1(t) d\bar{X}(t) + \int_0^T \bar{X}(t) d\hat{p}_1(t) + \int_0^T \hat{q}_1(t) \bar{\sigma}(t) dt \right].
\]

(5.17)

where we have put

\[
\bar{X}(t) := X(t) - \hat{X}(t), \ \bar{\sigma}(t) := \sigma(t) - \hat{\sigma}(t).
\]

(5.18)

Note that

\[
E \left[ \int_0^T \hat{p}_1(t) d\bar{X}(t) \right] = E \left[ \int_0^T \hat{p}_1(t) (b - \hat{b}) dt \right.
\]

\[
+ \int_0^T \hat{p}_1(t) (\lambda_1 d\xi_1(t) - \lambda_1 d\hat{\xi}_1(t) + \lambda_2 d\xi_2(t) - \lambda_2 d\hat{\xi}_2(t)) \right].
\]

(5.19)
and that
\[
E \left[ \int_0^T \tilde{X}(t) d\tilde{p}_1(t) \right] = E \left[ \int_0^T \tilde{X}(t) \{ -\frac{\partial H_{1,0}}{\partial x}(t, X(t), Y(t), w(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t)) \right.
\]
\[
- \frac{\partial H_{1,0}}{\partial y}(t, X(t), Y(t), w(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t)) \nabla X(t) F \} dt \right] \tag{5.20}
\]

Combining (5.20) with \( I_3 \) and \( I_4 \) we get
\[
J_1(\xi_1, u_1, \hat{\xi}_2, \hat{u}_2) - J_1(\hat{\xi}_1, \hat{u}_1, \hat{\xi}_2, \hat{u}_2)
\leq E \left[ \int_0^T \{ H_1(t) - \hat{H}_1(t) - \frac{\partial \hat{H}_1}{\partial x}(X - \hat{X}) - \frac{\partial \hat{H}_1}{\partial y} \nabla F(X - \hat{X}) \} dt \right]
\]
\[
= E \left[ \int_0^T E[H(t) - \hat{H}_1(t) - \frac{\partial \hat{H}_1}{\partial x}(X - \hat{X}) - \frac{\partial \hat{H}_1}{\partial y} \nabla F(X - \hat{X}) | G^{(1)} \} dt \right] \tag{5.21}
\]

where \( \hat{H}_1(t) \) means that \( H_1 \) is evaluated at \( (t, \hat{X}(t), \hat{Y}(t), \hat{\xi}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t)) \), while \( H_1(t) \) means that \( H_1 \) is evaluated at \( (t, X(t), Y(t), \xi_1(t), u_1(t), \xi_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t)) \).

Note that by concavity of \( H_1 \) we have
\[
H_1(t, X, F(X), \xi_1, u_1, \hat{\xi}_2, \hat{u}_2, \hat{p}_1, \hat{q}_1) - H_1(t, \hat{X}, F(\hat{X}), \hat{\xi}_1, \hat{u}_1, \hat{\xi}_2, \hat{u}_2, \hat{p}_1, \hat{q}_1)
\leq \frac{\partial \hat{H}_1}{\partial x}(\hat{X})(X - \hat{X}) + \frac{\partial \hat{H}_1}{\partial y}(\hat{X}) \nabla \hat{X} F(X - \hat{X}) + \nabla \hat{X} \hat{H}_1(\hat{\xi})(\xi_1 - \hat{\xi}_1) + \frac{\partial \hat{H}_1}{\partial u_1}(\hat{u})(u_1 - \hat{u}_1) \tag{5.22}
\]

Therefore, to obtain that \( J_1 - \hat{J}_1 \leq 0 \), it suffices that
\[
E[\nabla \xi_1 \hat{H}_1(\hat{\xi}) | G^{(1)}](\xi_1 - \hat{\xi}_1) \leq 0 \tag{5.23}
\]
for all \( \xi_1 \), and that
\[
E[\frac{\partial \hat{H}_1}{\partial u_1}(\hat{u}) | G^{(1)}](u_1 - \hat{u}_1) \leq 0 \tag{5.24}
\]
for all \( u_1 \). The inequality (5.23) holds by our assumption (5.12), and the inequality (5.23) holds by our assumption (5.14). The difference
\[
J_2(\hat{\xi}_1, \hat{u}_1, \xi_2, u_2) - J_2(\hat{\xi}_1, \hat{u}_1, \hat{\xi}_2, \hat{u}_2)
\]
is handled similarly. \qed
5.3 The zero-sum game case

In the zero-sum case we have

\[ J_1(w_1, w_2) + J_2(w_1, w_2) = 0. \] (5.25)

Then the Nash equilibrium \((\hat{w}_1, \hat{w}_2) \in A_1 \times A_2\) satisfying (5.4)-(5.5) becomes a saddle point for

\[ J(w_1, w_2) := J_1(w_1, w_2). \] (5.26)

To see this, note that (5.4)-(5.5) imply that

\[ J_1(w_1, \hat{w}_2) \leq J_1(\hat{w}_1, \hat{w}_2) = -J_2(\hat{w}_1, \hat{w}_2) \leq -J_2(\hat{w}_1, w_2) \]

and hence

\[ J(w_1, \hat{w}_2) \leq J(\hat{w}_1, \hat{w}_2) \leq J(\hat{w}_1, w_2) \text{ for all } w_1, w_2. \]

From this we deduce that

\[ \inf_{w_2 \in A_2} \sup_{w_1 \in A_1} J(w_1, w_2) \leq \sup_{w_1 \in A_1} J(\hat{w}_1, \hat{w}_2) \leq \inf_{w_2 \in A_2} \sup_{w_1 \in A_1} J(w_1, w_2). \] (5.27)

Since we always have \(\inf \sup \geq \sup \inf\), we conclude that

\[ \inf_{w_2 \in A_2} \sup_{w_1 \in A_1} J(w_1, w_2) = \sup_{w_1 \in A_1} J(\hat{w}_1, \hat{w}_2) = \inf_{w_1 \in A_1} \sup_{w_2 \in A_2} J(w_1, w_2). \] (5.28)

i.e. \((\hat{w}_1, \hat{w}_2) \in A_1 \times A_2\) is a saddle point for \(J(w_1, w_2)\).

Hence we want to find \((\xi^*, \theta^*) \in A_\xi \times \Theta\) such that

\[ \sup_{\xi \in A_\xi} \left\{ \inf_{\theta \in \Theta} J(\xi, \theta) \right\} = \inf_{\theta \in \Theta} \left\{ \sup_{\xi \in A_\xi} J(\xi, \theta) \right\} = J(\xi^*, \theta^*), \] (5.29)

where

\[ J(\xi, u) = E \left[ \int_0^T f(t, X(t), Y(t), u(t), \omega) dt + g(X(T), Y(T), \omega) \right. \]
\[ + \left. \int_0^T h(t, X(t), u(t), \omega) \xi(dt) \right]. \] (5.30)

As shown in [13], in this case only one Hamiltonian \(H\) is needed, namely

\[ H(t, x, y, \xi_1, u_1, \xi_2, u_2, p, q)(dt, \xi_1(dt), \xi_2(dt)) \]
\[ = H_0(t, x, y, \xi_1, u_1, \xi_2, u_2, p, q)dt + \sum_{j=1}^{2} \{ \lambda_j(t, x, u)p + h_j(t, x, u) \} \xi_j(dt) \] (5.31)
where
\[ H_0 := H_{1,0}(t, x, y, w, p, q) = f(t, x, y, w) + b(t, x, y, u)p + \sigma(t, x, y, u)q \] (5.32)
and we have put \( g_i = g, h_i = h_{1,i} ; i = 1, 2 \) and \( f_1 = f = -f_2 \).
Moreover, there is only one couple \((p, q)\) of adjoint processes, given by the BSDE
\[
\begin{align*}
    dp(t) &= -\frac{\partial H_0}{\partial x}(t, X(t), Y(t), w(t), p(t), q(t))dt \\
    &\quad - \frac{\partial H_0}{\partial y}(t, X(t), Y(t), w(t), p(t), q(t))\nabla X(t)F)dt \\
    &\quad + q(t)dB(t) \\
    p(T) &= \frac{\partial g}{\partial x}(X(T), Y(T)) + E[\frac{\partial g}{\partial y}(X(T), Y(T))]\nabla X(T)F.
\end{align*}
\] (5.33)

We can now state the corresponding sufficient maximum principle for the zero-sum game:

Theorem 5.2 (Sufficient maximum principle for zero-sum singular mean-field games)
Let \((\hat{w}_1, \hat{w}_2) \in A_1 \times A_2\), with corresponding solutions \(\hat{X}(t), \hat{Y}(t), \hat{p}(t), \hat{q}(t)\). Suppose the following holds

- The function
  \[ X, w_1 \rightarrow H(t, X, F(X), w_1, \hat{w}_2(t), \hat{p}(t), \hat{q}(t)) \] (5.34)
is concave for all \( t \), the function
\[ X, w_2 \rightarrow H(t, X, F(X), \hat{w}_1(t), w_2, \hat{p}(t), \hat{q}(t)) \] (5.35)
is convex for all \( t \), and the function
\[ X \rightarrow g(X, F(X)) \] (5.36)
is affine.

- (The conditional maximum property)
\[
\begin{align*}
    \text{ess sup}_{v_1 \in A_1} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), v_1, \hat{\xi}_2(t), \hat{w}_2(t), \hat{p}(t), \hat{q}(t)) \mid G_t^{(1)}] \\
    &= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{w}_1(t), \hat{\xi}_2(t), \hat{w}_2(t), \hat{p}(t), \hat{q}(t)) \mid G_t^{(1)}] \quad (5.37)
\end{align*}
\]
and
\[
\begin{align*}
    \text{ess inf}_{v_2 \in A_2} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), v_2, \hat{\xi}_2(t), \hat{w}_2(t), \hat{p}(t), \hat{q}(t)) \mid G_t^{(2)}] \\
    &= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{w}_1(t), \hat{\xi}_2(t), \hat{w}_2(t), \hat{p}(t), \hat{q}(t)) \mid G_t^{(2)}]. \quad (5.38)
\end{align*}
\]
Moreover, the following variational inequalities hold:

\[
\begin{align*}
\text{(Variational inequalities)} \\
\text{ess sup} \ E[H(t, \hat{X}(t), \hat{Y}(t), \xi_1, \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}(t), \hat{q}(t)) | \mathcal{G}_t^{(1)}] &= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}(t), \hat{q}(t)) | \mathcal{G}_t^{(1)}] \quad (5.39) \\
\text{ess inf} \ E[H(t, \hat{X}(t), \hat{Y}(t), \xi_2, \hat{u}_2(t), \hat{p}(t), \hat{q}(t)) | \mathcal{G}_t^{(2)}] &= E[H(t, \hat{X}(t), \hat{Y}(t), \xi_1(t), \xi_2, \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}(t), \hat{q}(t)) | \mathcal{G}_t^{(2)}]. \quad (5.40)
\end{align*}
\]

Then \( \hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t)) \) is a saddle point for \( J(u_1, u_2) \).

\[ \square \]

5.4 A necessary maximum principle for the general case

In Section 6.2 we proved a verification theorem, stating that if a given control \((\hat{\xi}, \hat{u})\) satisfies certain conditions, then it is indeed optimal for the singular control game. We now establish a partial converse, implying that if a control \((\hat{\xi}, \hat{u})\) is optimal for the singular control game, then it is a conditional saddle point for the Hamiltonian.

\textbf{Theorem 5.3} (Necessary maximum principle for singular mean-field games)

Suppose \( \tilde{w}_1 = (\tilde{\xi}_1, \tilde{u}_1) \in \mathcal{A}^{(1)} \) and \( \tilde{w}_2 = (\tilde{\xi}_2, \tilde{u}_2) \in \mathcal{A}^{(2)} \) constitute a Nash equilibrium for the game, i.e. satisfies (5.41) and (5.42). Then

\[
E \left[ \frac{\partial H_{1,0}}{\partial u_1}(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1, \hat{\xi}_2(t), \hat{u}_2(t), \hat{p}_1(t), \hat{q}_1(t))_{u_1 = \hat{u}_1(t)} | \mathcal{G}_t^{(1)} \right] = 0 \quad (5.41)
\]
and

\[
E \left[ \frac{\partial H_{2,0}}{\partial u_2}(t, \hat{X}(t), \hat{Y}(t), \hat{\xi}_1(t), \hat{u}_1(t), \hat{\xi}_2(t), \hat{u}_2, \hat{p}_2(t), \hat{q}_2(t))_{u_2 = \hat{u}_2(t)} | \mathcal{G}_t^{(2)} \right] = 0. \quad (5.42)
\]

Moreover, the following variational inequalities hold:

\[
\begin{align*}
\left\{ \begin{array}{l}
E \left[ \frac{\partial f_i}{\partial \xi_i}(t, \hat{X}(t), \hat{Y}(t), \tilde{w}(t)) + \lambda_i(t, \hat{X}(t), \hat{u}(t))\hat{p}_i(t) + h_{ii}(t, \hat{X}(t), \hat{u}(t)) \right] \mathcal{G}_t^{(i)} \leq 0 \\
\text{for all } t, \ i = 1, 2
\end{array} \right.
\]
and

\[
\left\{ \begin{array}{l}
E \left[ \frac{\partial f_i}{\partial \xi_i}(t, \hat{X}(t), \hat{Y}(t), \tilde{w}(t)) + \lambda_i(t, \hat{X}(t), \hat{u}(t))\hat{p}_i(t) + h_{ii}(t, \hat{X}(t), \hat{u}(t)) \right] \mathcal{G}_t^{(i)} \ d\hat{\xi}_i(t) = 0 \\
\text{for all } t, \ i = 1, 2.
\end{array} \right. \quad (5.43)
\]
Proof. This theorem can be proved in a way similar to the proof of Theorem 3.2 with an adjustment to the stochastic game case. The adjustment is similar to one in the proof of Theorem 5.1. □

6 Applications

6.1 Return to the optimal harvesting problem

To illustrate our results, we apply it to the optimal harvesting problem (2.3), (2.4) in Section 2.1: Here the Hamiltonian (3.12) gets the form

$$H(t, x, y, \xi, p, q)(dt, d\xi) = [yb(t)p + x\sigma(t)q]dt + \{-\lambda_0(t)p + x h_0(t)\}d\xi(t)$$ (6.1)

By Theorem 5.1 the corresponding BSDE reduces to

$$\begin{cases}
dp(t) = -[b(t)p(t) + \sigma(t)q(t)]dt + q(t)dB(t) \\
p(T) = K
\end{cases}$$ (6.2)

with associated variational inequalities

$$\begin{cases}
-\lambda_0(t)p(t) + h_0(t)X(t) \leq 0; & t \in [0, T] \\
[-\lambda_0(t)p(t) + h_0(t)X(t)]d\hat{\xi}(t) = 0; & t \in [0, T]
\end{cases}$$ (6.3)

The system consisting of (2.3), (2.4) combined with (6.2), (6.3) represents a mean-field forward-backward reflected SDE. Combining the above with the result of Section 4 we get:

**Theorem 6.1** Assume that $\hat{X}(t), \hat{p}(t), \hat{\xi}(t)$ is a solution of the system (2.3), (2.4) $\&$ (6.2), (6.3). Then $\hat{\xi}(t)$ is an optimal harvesting strategy for the problem (2.5). Heuristically the optimal harvesting strategy can be described as follows:

- If
  $$\hat{p}(t) > \frac{h_0(t)\hat{X}(t)}{\lambda_0(t)},$$ i.e. $\hat{X}(t) < \frac{\lambda_0(t)\hat{p}(t)}{h_0(t)}$, (6.4)
  then do nothing (choose $\hat{\xi}(t) = 0$).

- If
  $$\hat{p}(t) = \frac{h_0(t)\hat{X}(t)}{\lambda_0(t)},$$ i.e. $\hat{X}(t) = \frac{\lambda_0(t)\hat{p}(t)}{h_0(t)}$, (6.5)
  then we harvest immediately from $\hat{X}(t)$ at a rate $d\hat{\xi}(t)$ which is exactly enough to prevent $\hat{X}(t)$ from going above $\frac{\lambda_0(t)\hat{p}(t)}{h_0(t)}$. 

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• If

\[ \hat{p}(0^-) < \frac{h_0(0)\hat{X}(0^-)}{\lambda_0(0)}, \text{i.e. } \hat{X}(0^-) > \frac{\lambda_0(0)\hat{p}(0^-)}{h_0(0)}, \]  

then we harvest immediately what is necessary to bring \( \hat{X}(0^-) \) down to the level of \( \frac{\lambda_0(0)\hat{p}(0^-)}{h_0(0)}. \)

6.2 Application to model uncertainty singular control

We represent model uncertainty by a family of probability measures \( Q = Q^\theta \) equivalent to \( P \), with the Radon-Nikodym derivative on \( \mathcal{F}_t \) given by

\[ \frac{d(Q \mid \mathcal{F}_t)}{d(P \mid \mathcal{F}_t)} = G^\theta(t) \]  

where, for \( 0 \leq t \leq T \), \( G^\theta(t) \) is an exponential martingale of the form

\[ dG^\theta(t) = G^\theta(t-)^{\theta(t)} dB(t); \quad G^\theta(0) = 1. \]  

Here \( \theta \) may be regarded as a scenario control. Let \( \mathcal{A}_1 := \mathcal{A}_\xi \) denote a given family of admissible singular controls \( \xi \) and let \( \mathcal{A}_2 := \Theta \) denote a given set of admissible scenario controls \( \theta \) such that

\[ E[\int_0^T \theta^2(t)dt] < \infty. \]  

Now assume that \( X_1(t) = X^\xi(t) \) is a singularly controlled mean-field Itô process of the form

\[ dX_1(t) = b_1(t, X_1(t), Y_1(t), \omega)dt + \sigma_1(t, X_1(t), Y_1(t), \omega)dB(t) \]
\[ + \lambda_1(t, X_1(t), \omega)d\xi(t), \]  

where

\[ Y_1(t) = F(X_1(t, \cdot)), \]  

and \( F \) is a Fréchet differentiable operator on \( L^2(P) \).

As before let \( \mathcal{G}^{(1)} = \{\mathcal{G}^{(1)}_t\}_{0 \leq t \leq T} \) and \( \mathcal{G}^{(2)} = \{\mathcal{G}^{(2)}_t\}_{0 \leq t \leq T} \) be given subfiltrations of \( \mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T} \), representing the information available to the controllers at time \( t \). It is required that \( \xi \in \mathcal{A}_1 \) be \( \mathcal{G}^{(1)} \)-predictable, and \( \theta \in \mathcal{A}_2 \) be \( \mathcal{G}^{(2)} \)-predictable. We set \( w = (\xi, \theta) \) and consider the stochastic differential game to find \( (\hat{\xi}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2 \) such that

\[ \sup_{\xi \in \mathcal{A}_1} \inf_{\theta \in \mathcal{A}_2} E_{Q^\theta}[j(\xi, \theta)] = \inf_{\theta \in \mathcal{A}_2} \sup_{\xi \in \mathcal{A}_1} E_{Q^\theta}[j(\xi, \theta)], \]  

where

\[ j(\xi, \theta) = \int_0^T \{f_1(t, X(t), Y(t), \xi(t), \omega) + \rho(\theta(t))\}dt \]
\[ + g_1(X(T), Y(T), \omega) + \int_0^T h_1(t, X(t), \omega)\xi(dt). \]  

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The term $E_{Q^\theta}[^{T}\int_{0}^{T} \rho(\theta(t))dt]$ can be seen as a penalty term, penalizing the difference between $Q^\theta$ and the original probability measure $P$.

Note that since $G^\theta(t)$ is a martingale we have

$$E_{Q^\theta}[j(\xi, \theta)] = E \left[ G^\theta(T)g_1(X(T), Y(T)) + \int_{0}^{T} G^\theta(t)\{f_1(t, X(t), Y(t), \xi(t)) + \rho(\theta(t))\}dt \right. \left. + \int_{0}^{T} G^\theta(t)h_1(t, X(t), Y(t))\xi(dt) \right] =: J(\xi, \theta). \quad (6.14)$$

We see that this is a mean-field singular control stochastic differential game of the type discussed in Section 4, with a two-dimensional state space

$$X(t) := (X_1(t), X_2(t)) := (X^\xi(t), G^\theta(t)) \quad (6.15)$$

and with

$$f(t, X(t), Y(t), \xi, \theta) := G^\theta(t)\{f_1(t, X_1(t), Y_1(t), \xi(t)) + \rho(\theta(t))\}$$

$$= X_2(t)\{f_1(t, X_1(t), Y_1(t), \xi(t)) + \rho(\theta(t))\}, \quad (6.16)$$

$$g(X(T), Y(T)) := G^\theta(T)g_1(X_1(T), Y_1(T)) = X_2(T)g_1(X_1(T), Y_1(T)), \quad (6.17)$$

and

$$h(t, X(t), Y(t)) := G^\theta(t)h_1(t, X_1(t), Y_1(t)) = X_2(t)h_1(t, X_1(t), Y_1(t)). \quad (6.18)$$

Using the result from Section 4, we get the following Hamiltonian for the game (6.12):

$$H(t, x_1, x_2, y_1, \xi, \theta, p, q, r)(dt, \xi(dt))$$

$$= H_0(t, x_1, x_2, y_1, \xi, \theta, p, q)dt + \{\lambda_1(t, x)p_1 + x_2h_1(t, x)\}\xi(dt) \quad (6.19)$$

where

$$H_0(t, x_1, x_2, y_1, \xi, \theta, p, q) = x_2\{f_1(t, x_1, y_1, \xi) + \rho(\theta)\}$$

$$+ b_1(t, x_1, y_1)p_1 + \sigma(t, x_1, y_1)q_1 + x_2\theta q_2. \quad (6.20)$$

The corresponding mean-field BSDEs for the adjoint processes become

$$\begin{cases}
    dp_1(t) = -\frac{\partial H_0}{\partial x_1}(t, X(t), Y(t), w(t), p(t), q(t))dt \\
    \quad - \frac{\partial H_0}{\partial y_1}(t, X(t), Y(t), w(t), p(t), q(t))\nabla_{X_1(t)}F dt + q_1(t) dB(t)
\end{cases} \quad (6.21)$$

$$p_1(T) = X_2(T)\frac{\partial g_1}{\partial x_1}(X_1(T), Y_1(T)) + E\frac{\partial g_1}{\partial y_1}(X_1(T), Y_1(T))\nabla_{X_1(T)}F$$

and

$$\begin{cases}
    dp_2(t) = -\{f_1(t, X_1(t), Y_1(t), \xi) + \rho(\theta(t)) + \theta(t)q_2(t)\}dt + q_2(t) dB(t) \\
    p_2(T) = g_1(X_1(T), Y_1(T)).
\end{cases} \quad (6.22)$$
Minimizing the Hamiltonian with respect to $\theta$ gives the following first order condition:

$$\frac{\partial \rho}{\partial \theta(t)} = -E[ q_2(t) \mid \mathcal{G}_t^{(2)}].$$

(6.23)

The variational inequalities (5.14) - (5.15) reduce to

$$\begin{cases}
E[\nabla \xi f_1(t, \hat{X}(t), \hat{Y}(t), \hat{\omega}(t)) + \lambda_1(t, \hat{X}_1(t)) \hat{p}(t) + h_1(t, \hat{X}_1(t)) \mid \mathcal{E}_t^{(1)}] \leq 0; \\
E[\nabla \xi f_1(t, \hat{X}(t), \hat{Y}(t), \hat{\omega}(t)) + \lambda_1(t, \hat{X}(t)) \hat{p}(t) + h_1(t, \hat{X}(t)) \mid \mathcal{E}_t^{(1)}]d\xi(t) = 0;
\end{cases}$$

(6.24)

### 6.3 A special case

For simplicity, consider the special case with

$$\mathcal{G}_t^{(i)} = \mathcal{F}_t,$$

and

$$\lambda_1(t, x) = \lambda_1(t), \quad h_1(t, x) = h_1(t)$$

(6.26)

i.e., $\lambda_1$ and $h_1$ do not depend on $x$.

Then, writing $X_1(t) = X^\xi(t)$, $Y_1(t) = Y^\xi(t)$ and $X_2(t) = G^\theta(t)$ and $f_1 = f$, $g_1 = g$, $b_1 = b$, $\sigma_1 = \sigma$, $\lambda_1 = \lambda$, the controlled system gets the form $(X^\xi, G^\theta)$, where $G^\theta$ is given by (6.8) and

$$dX^\xi(t) = b(t, X^\xi(t), Y^\xi(t))dt + \sigma(t, X^\xi(t), Y^\xi(t))dB(t) + \lambda(t)d\xi(t); \quad X^\xi(0) = x$$

(6.27)

The performance functional becomes

$$E_{Q^\theta} [j(\xi, \theta)] = E \left[ G^\theta(T) g(X^\xi(T), Y^\xi(T)) \right] + \int_0^T G^\theta(t) \{ f(t, X^\xi(t), Y^\xi(t), \xi(t)) + \rho(\theta(t)) \} dt =: J(\xi, \theta).$$

(6.28)

and the Hamiltonian becomes

$$H(t, x, g, y, \xi, \theta, p, q)(dt, \xi(dt)) = H_0(t, x, g, y, \xi, \theta, p, q)dt + \{ \lambda(t)p_1 + g h_1(t) \} \xi(dt),$$

(6.29)

where

$$H_0(t, x, g, y, \xi, \theta, p, q) = g \{ f_1(t, x, y, \xi) + \rho(\theta) \} + b_1(t, x, y)p_1 + \sigma(t, x, y)q_1 + g \theta q_2.$$

The corresponding mean-field BSDEs for the adjoint processes become

$$\begin{cases}
p_1(t) = -\frac{\partial H_0}{\partial x}(t, X^\xi(t), Y^\xi(t), \xi(t), p(t), q(t))dt \\
- \frac{\partial H_0}{\partial y}(t, X^\xi(t), Y^\xi(t), \xi(t), p(t), q(t))\nabla_{X^\xi(t)} F^t dt + q_1(t)dB(t) \\
p_1(T) = G^\theta(T) \frac{\partial g}{\partial x}(X^\xi(T), Y^\xi(T)) + E[ \frac{\partial g}{\partial y}(X^\xi(T), Y^\xi(T))] \nabla_{X^\xi(T)} F^T
\end{cases}$$

(6.30)
and
\[
\begin{align*}
dp_2(t) &= -\{f(t, X^\xi(t), Y^\xi(t), \xi) + \rho(\theta(t)) + \theta(t)q_2(t)\} dt + q_2(t)dB(t) \\
p_2(T) &= g_1(X^\xi(T), Y^\xi(T)).
\end{align*}
\]

(6.31)

Then the first order condition for a minimum of the Hamiltonian with respect to \(\theta_0\) reduces to
\[\rho'(\theta)(t) = -q_2(t).\]
\[(6.32)\]

The variational inequalities (6.19) become
\[
\begin{align*}
\nabla_\xi f_1(t, X(t), Y(t), \xi(t)) + \lambda_1(t)p_1(t) + h_1(t) &\leq 0; \\
[\nabla_\xi f_1(t, X_1(t), Y(t), \xi(t)) + \lambda_1(t)p_1(t) + h_1(t)]d\xi(t) &= 0;
\end{align*}
\]

(6.33)

In general it seems to be a formidable mathematical challenge to solve such a coupled system of forward-backward singular SDEs. However, in some cases a possible solution procedure could be described, as in the next example:

### 6.4 Optimal harvesting under uncertainty

Now we consider a model uncertainty version of the optimal harvesting problem in Section 6.1. For simplicity we put \(K = 1\). Thus we have the following mean-field forward system \((X, \xi, G^\theta)\), where \(G^\theta\) is given by (6.8) and
\[
\begin{align*}
\begin{cases}
\quad dX(t) &= E[X^\xi(t)]b(t)dt + X^\xi(t)\sigma(t)dB(t) - \lambda_0(t)d\xi(t) \\
\quad X^\xi(0^-) &= x > 0,
\end{cases}
\end{align*}
\]

(6.34)

with performance functional
\[
J(\xi, \theta) = E\left[\int_0^T G^\theta(t)\rho(\theta(t))dt + \int_0^T G^\theta(t)h_0(t)X^\xi(t)d\xi(t) + G^\theta(T)X^\xi(T)\right].
\]

(6.35)

The model uncertainty harvesting problem is to find \((\xi^*, \theta^*) \in \mathcal{A}_F \times \Theta\) such that
\[
\sup_{\xi \in \mathcal{A}_F} \left\{\inf_{\theta \in \Theta} J(\xi, \theta)\right\} = \inf_{\theta \in \Theta} \left\{\sup_{\xi \in \mathcal{A}_F} J(\xi, \theta)\right\} = J(\xi^*, \theta^*),
\]

(6.36)

Here the Hamiltonian is
\[
H(t, x, g, y, \xi, \theta, p, q)(dt, \xi(dt)) = \{g\rho(\theta) + yb(t)p_1 + x\sigma(t)q_1 + gh_0(t)\}dt + \{-\lambda_0(t)p_1 + xgh_0(t)\}\xi(dt).
\]

(6.37)

Minimizing the Hamiltonian with respect to \(\theta\) gives the first order equation
\[\rho'(\theta)(t) = -q_2(t).\]
\[(6.38)\]
The corresponding reflected backward system is

\[
\begin{cases}
p_1(t) = -[b(t)p_1(t) + \sigma(t)q_1(t)]dt - h_0(t)G^\theta(t)d\xi(t) + q_1(t)dB(t) \\
p_1(T) = G^\theta(T)
\end{cases}
\]  
(6.39)

and

\[
\begin{cases}
p_2(t) = -[\rho(\theta(t)) + \theta(t)q_2(t)]dt - h_0(t)X(t)d\xi(t) + q_2(t)dB(t) \\
p_2(T) = X(T),
\end{cases}
\]  
(6.40)

with variational inequalities

\[
\begin{cases}
-\lambda_0(t)p_1(t) + h_1(t) \leq 0; \\
[-\lambda_0(t)p_1(t) + h_1(t)]d\xi(t) = 0
\end{cases}
\]  
(6.41)

Then we get the following result:

**Theorem 6.2** Suppose there exists a solution \( \hat{X}(t) := X^\hat{\xi}(t) \), \( \hat{G}(t) := G^\hat{\theta}(t) \), \( \hat{p}_1(t) \), \( \hat{p}_2(t) \), \( \hat{q}_1(t) \), \( \hat{q}_2(t) \), \( \hat{\xi}(t) \), \( \hat{\theta}(t) \) of the coupled system of mean-field forward-backward singular stochastic differential equations consisting of the forward equations (6.34) and the reflected backward equations (6.39), (6.40), and satisfying the constraint (6.41). Then \( \hat{\xi}(t) \) is the optimal harvesting strategy and \( \hat{\theta}(t) \) is the optimal scenario parameter for the model uncertainty harvesting problem (6.36).

### 6.5 A mean field singular game

We now return to the mean field singular game described in Section 2.3.

In this case, we get from (5.6)

\[
H_i(t, x, y, \xi_1, \xi_2, p_i, q_i)(dt, \xi_1(dy), \xi_2(dt)) = \pi \min(x, \xi_1 + \xi_2) + yb(t)p_i + x\sigma(t)q_i + h_1\xi_1(dt) + h_2\xi_2(dt)
\]

and the adjoint equations (5.8) becomes

\[
\begin{cases}
p_i(t) = -[\chi_{[0,\xi_1+\xi_2]}(x)\pi(t) + \sigma(t)q_i(t) + b(t)p_i(t)] dt + q_i(t)dB(t) \\
p_i(T) = 0.
\end{cases}
\]

The variational inequalities (5.43) get the form

\[
\begin{cases}
\pi(t)\chi_{[0,X(t)]}(\xi_1(t) + \xi_2(t)) + h_i(t) \leq 0 \\
\pi(t)\chi_{[0,X(t)]}(\xi_1(t) + \xi_2(t)) + h_i(t) \leq 0;
\end{cases}
\]

and

\[
\left[\pi(t)\chi_{[0,X(t)]}(\xi_1(t) + \xi_2(t)) + h_i(t)\right] \xi_i(dt) = 0; \quad i = 1, 2.
\]

Optimal strategy for factory 1:

i) If \( \pi(t) + h_1(t) < 0 \), do nothing.
In other words, \((\xi, X)\) solves the reflected Skorohod problem

\[
\begin{aligned}
\xi_1(t) &\geq (X(t) - \xi_1^* (t)) \chi_{[0,\infty)}(\pi(t) + h_1(t)) \\
\xi_1(t) - (X(t) - \xi_1^*(t)) \chi_{[0,\infty)}(\pi(t) + h_1(t)) \xi_1(dt) &= 0. \\
\end{aligned}
\tag{6.42}
\]

So for given \(\xi_1^*\) we choose \(\xi_1 := R_1(\xi_1^*)\) solution of the reflected Skorohod problem \((6.42)\).

Similarly, given \(\xi_1^*\) we choose \(\xi_2 := R_2(\xi_1^*)\) as the solution of the reflected Skorohod problem

\[
\begin{aligned}
\xi_2(t) &\geq (X(t) - \xi_2^*(t)) \chi_{[0,\infty)}(\pi(t) + h_2(t)) \\
\xi_2(t) - (X(t) - \xi_1^*(t)) \chi_{[0,\infty)}(\pi(t) + h_2(t)) \xi_2(dt) &= 0. \\
\end{aligned}
\tag{6.43}
\]

Thus, to find the Nash equilibrium we need to solve the following coupled reflected Skorohod problem:

\[
\begin{aligned}
\xi_1(t) &\geq (X(t) - \xi_2(t)) \chi_{[0,\infty)}(\pi(t) + h_1(t)) \\
\xi_1(t) - (X(t) - \xi_2(t)) \chi_{[0,\infty)}(\pi(t) + h_1(t)) \xi_1(dt) &= 0; \\
\xi_2(t) &\geq (X(t) - \xi_1(t)) \chi_{[0,\infty)}(\pi(t) + h_2(t)) \\
\xi_2(t) - (X(t) - \xi_1(t)) \chi_{[0,\infty)}(\pi(t) + h_2(t)) \xi_2(dt) &= 0. \\
\end{aligned}
\tag{6.44}
\]

The above system of reflected Skorohod problem can be solved in the following way:

We divide the interval \([0,T]\) into \(0 = t_0 < t_1 < \cdots < t_n = T\) such that on each interval \([t_k, t_{k+1}]\) the signs of \(\pi(t) + h_1(t)\) and \(\pi(t) + h_2(t)\) remains unchanged.

On each interval \([t_k, t_{k+1}]\), we use the following control principles. If both of the inequalities \(\pi(t) + h_1(t) < 0\) and \(\pi(t) + h_2(t) < 0\) hold, then do nothing. If \(\pi(t) + h_1(t) < 0\) and \(\pi(t) + h_2(t) \geq 0\), then the first factory does not do anything. The second condition in \((6.44)\) becomes

\[
\begin{aligned}
\xi_2(t) &\geq (X(t) - \xi_1(t)) \\
[\xi_2(t) - (X(t) - \xi_1(t))] \xi_2(dt) &= 0.
\end{aligned}
\]

By Remark 2.7 (namely, Equation (2.8)) of \([3]\),

\[
\xi_2(t) = \sup_{t_k \leq s \leq t} (X(s) - \xi_1(t_k))^+, \quad t_k \leq t \leq t_{k+1}.
\]

Thus, we keep \(\xi_1(t) = \xi_1(t_k)\) unchanged and in the same time increase \(\xi_2(t)\) to \(X(t) - \xi_1(t)\). Similar result holds if \(\pi(t) + h_1(t) \geq 0\) and \(\pi(t) + h_2(t) < 0\).

If both \(\pi(t) + h_1(t) \geq 0\) and \(\pi(t) + h_2(t) \geq 0\), then \((6.44)\) becomes

\[
\begin{aligned}
\xi_1(t) &\geq (X(t) - \xi_2(t)) \\
[\xi_1(t) - (X(t) - \xi_2(t))] \xi_1(dt) &= 0; \\
\xi_2(t) &\geq (X(t) - \xi_1(t)) \\
[\xi_2(t) - (X(t) - \xi_1(t))] \xi_2(dt) &= 0.
\end{aligned}
\tag{6.45}
\]
Let \( \xi(t) = \xi_1(t) + \xi_2(t) \) and then (6.45) is equivalent to
\[
\begin{cases}
\xi(t) \geq X(t) \\
[\xi(t) - X(t)] \xi(dt) = 0.
\end{cases}
\]
Again by Remark 2.7 (namely, Equation (2.8)) of [3], we have
\[
\xi(t) = \sup_{t_k \leq s \leq t} X(s)^+, \quad t_k \leq t \leq t_{k+1}.
\]
Now we show that any decomposition of \( \xi(t) \) into the sum of two nondecreasing processes \( \xi_1(t) \) and \( \xi_2(t) \) will solve (6.45). In fact, assume \( \xi(t) = \xi_1(t) + \xi_2(t) \), where \( \xi_1 \) and \( \xi_2 \) are two nondecreasing processes. Since \( \xi(t) \geq X(t) \) and \( \xi_1 \) and \( \xi_2 \) are nondecreasing, we have
\[
\begin{cases}
[\xi(t) - X(t)] \xi_1(dt) \geq 0 \\
[\xi(t) - X(t)] \xi_2(dt) \geq 0.
\end{cases}
\]
Add them we have
\[
[\xi(t) - X(t)] \xi_1(dt) + [\xi(t) - X(t)] \xi_2(dt) = [\xi(t) - X(t)] \xi(dt) = 0.
\]
This implies
\[
\begin{cases}
[\xi(t) - X(t)] \xi_1(dt) = 0 \\
[\xi(t) - X(t)] \xi_2(dt) = 0.
\end{cases}
\]
Thus, \( \xi_1 \) and \( \xi_2 \) satisfies (6.45). Summarizing we have

**Theorem 6.3** Assume that we can divide the interval \([0, T]\) into \(0 = t_0 < t_1 < \cdots < t_n = T\) such that on each interval \([t_k, t_{k+1})\) the signs of \(\pi(t) + h_1(t)\) and \(\pi(t) + h_2(t)\) remain unchanged. Then we can recursively find the solution \(\xi_1\) and \(\xi_2\) on each interval \([t_k, t_{k+1})\) for \(k = 0, 1, \ldots, n - 1\). On the interval \([t_k, t_{k+1}]\), we have

(i) If both of the inequalities \(\pi(t) + h_1(t) < 0\) and \(\pi(t) + h_2(t) < 0\) hold, then do nothing.

(ii) If \(\pi(t) + h_1(t) < 0\) but \(\pi(t) + h_2(t) \geq 0\), then
\[
\xi_1(t) = \xi_1(t_k), \quad t_k \leq t \leq t_{k+1}
\]
and
\[
\xi_2(t) = \sup_{t_k \leq s \leq t} (X(s) - \xi_1(t_k))^+, \quad t_k \leq t \leq t_{k+1}.
\]
If \(\pi(t) + h_1(t) \geq 0\) but \(\pi(t) + h_2(t) < 0\), then
\[
\xi_2(t) = \xi_2(t_k), \quad t_k \leq t \leq t_{k+1}
\]
and
\[
\xi_1(t) = \sup_{t_k \leq s \leq t} (X(s) - \xi_2(t_k))^+, \quad t_k \leq t \leq t_{k+1}.
\]

(iii) If both of the inequalities \(\pi(t) + h_1(t) \geq 0\) and \(\pi(t) + h_2(t) \geq 0\) hold, then \(\xi_1\) and \(\xi_2\) can be any nondecreasing processes such that
\[
\xi_1(t) + \xi_2(t) = \sup_{t_k \leq s \leq t} X(s)^+, \quad t_k \leq t \leq t_{k+1}.
\]

Note in particular that in this case the Nash equilibrium is not unique.
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