I. INTRODUCTION

Entanglement as a purely quantum mechanical property was already recognized around 1935 by Einstein, Podolsky and Rosen [1].

A pure state of a bipartite quantum system is entangled if and only if it can not be written as a product state. Consider the two qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

also called singlet state. One can show that this state can not be written in the form $|\psi\rangle = |a\rangle|b\rangle$ with single qubit states $|a\rangle$ and $|b\rangle$.

In this paper we consider the most general case, a multipartite mixed state $\rho$ on a Hilbert space $H = \otimes_{j=1}^n H_j$. It is in general called entangled if it can not be written in the form

$$\rho = \sum_i p_i \left( \otimes_{j=1}^n |\psi_i^{(j)}\rangle \langle \psi_i^{(j)}| \right),$$

with non-negative probabilities $p_i$, $\sum_i p_i = 1$, and $|\psi_i^{(j)}\rangle$ being states on $H_j$. Otherwise the state is called separable.

Entanglement of pure bipartite states $|\psi\rangle$ is usually quantified by the entanglement entropy

$$E(|\psi\rangle) = -Tr \left( \rho^A \log_2 \rho^A \right),$$

$$\rho^A = Tr_B \left| \psi \right| \left( \psi \right).$$

For mixed states different measures were proposed. In this paper we will consider the geometric measure of entanglement $E_{Ge}$ proposed in [4] and compare it to measures based on fidelity $[3,6,7,8,9]$. Those are measures of the form

$$E_f(\rho) = f \left( \max_{\sigma \in S} F(\rho, \sigma) \right),$$

where $F$ is the quantum fidelity, $f$ is a proper chosen function and $S$ is the set of separable states of the form $[1]$.

By construction, $E_{Ge}$ was supposed to be different from measures based on fidelity. In this paper we show that this is not the case. One of the main results of this paper will be, that $E_{Ge}$ is also a fidelity-based measure.

The structure of the paper is as follows. In Section II we give important definitions. In Section III we give main results for pure states, mixed states and two qubit states. A conclusion is given in Section IV.

Appendix A concentrates on bipartite pure states and in Appendix B we prove a proposition needed for our main result.

II. DEFINITIONS

First we restate the definition of fidelity $F(\rho, \sigma)$ between two quantum states $\rho$ and $\sigma$:

$$F(\rho, \sigma) = \left( Tr \left[ \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right] \right)^2. \quad (2)$$

It is important to notice that many authors define the fidelity as the square root of our definition, important example are the authors of [10].

In the following we always consider $n$-partite states on finite dimensional Hilbert space $H = \otimes_{i=1}^n H_i$.

For a mixed state $\rho$ we now define the fidelity of separability:

$$F_{sep}(\rho) = \max_{\sigma \in S} F(\rho, \sigma), \quad (3)$$

maximization is done over the set of $n$-partite separable states $S$ of the form $\sigma = \sum_i p_i \left( \otimes_{j=1}^n |\psi_i^{(j)}\rangle \langle \psi_i^{(j)}| \right)$. 

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For pure states geometric measure of entanglement (GME) is defined as \[8, 11\]

\[
E_{Ge} (|\psi\rangle) = 1 - \Lambda^2_{\text{max}} (|\psi\rangle),
\]
\[
\Lambda_{\text{max}} (|\psi\rangle) = \max_{|\phi\rangle \in S} |\langle \phi |\psi \rangle|,
\]

the maximization runs over all separable pure states \(|\phi\rangle = \otimes_{i=1}^{n} |\phi^{(i)}\rangle\). Extension to mixed states is made over the convex roof construction \[11\]:

\[
E_{Ge} (\rho) = \sum_{i} p_{i} E_{Ge} (|\psi_{i}\rangle),
\]

minimization is done over all pure state decompositions of \(\rho\).

In \[6\] authors defined the revised geometric measure of entanglement (RGME)

\[
E_{RGMe} (\rho) = 1 - F_{\text{sep}} (\rho).
\]

Groverian measure of entanglement for a mixed state \(\rho\) is defined as \[3, 7\]

\[
E_{Gr} (\rho) = \sqrt{1 - F_{\text{sep}} (\rho)}.
\]

Finally Bures measure of entanglement is defined as \[9\]

\[
E_{B} (\rho) = 2 \left( 1 - \sqrt{F_{\text{sep}} (\rho)} \right).
\]

As the measures \[7, 8\] and \[9\] are all simple functions of \(F_{\text{sep}}\), we will only give results for \(F_{\text{sep}}\) in the following sections.

III. RESULTS

A. Pure states

In the following we will consider pure states \(|\psi\rangle \in \otimes_{i=1}^{n} \mathcal{H}_{i}\).

Proposition 1. For pure state \(|\psi\rangle \in \otimes_{i=1}^{n} \mathcal{H}_{i}\) holds:

\[
F_{\text{sep}} (|\psi\rangle) = \max_{|\phi\rangle \in S} |\langle \phi |\psi \rangle|^2 = \Lambda^2_{\text{max}} (|\psi\rangle),
\]

maximization is done over separable pure states \(|\phi\rangle = \otimes_{j=1}^{n} |\phi^{(j)}\rangle\).

Proof. To evaluate \(F_{\text{sep}}\) for a pure state \(|\psi\rangle\) we need to find a separable state \(\sigma\) that maximizes the fidelity among all separable states, such that

\[
F (|\psi\rangle, \sigma) = \max_{\rho_{\text{sep}} \in S} F (|\psi\rangle, \rho_{\text{sep}}).
\]

Then \(F_{\text{sep}} (|\psi\rangle) = F (|\psi\rangle, \sigma)\). Set now \(\sigma = \sum_{i} q_{i} |\phi_{i}\rangle \langle \phi_{i}|\) with separable states \(|\phi_{i}\rangle = \otimes_{j=1}^{n} |\phi_{i}^{(j)}\rangle\). Then we see:

\[
F (|\psi\rangle, \sigma) = \langle \psi |\sigma |\psi \rangle = \sum_{i} q_{i} |\langle \psi |\phi_{i}\rangle|^{2}.
\]

We define \(|\phi_{1}\rangle\) to have the largest overlap with \(|\psi\rangle\), that is \(|\langle \psi |\phi_{1}\rangle| \geq |\langle \psi |\phi_{2}\rangle|\). From \[12\] follows:

\[
F (|\psi\rangle, \sigma) \leq \sum_{i} q_{i} |\langle \psi |\phi_{i}\rangle|^{2} = |\langle \psi |\phi_{1}\rangle|^{2} = F (|\psi\rangle, |\phi_{1}\rangle).
\]

In maximization \[11\] we can restrict ourselves to pure states, such that \(\sigma\) can be chosen to be pure: \(\sigma = |\phi\rangle \langle \phi|\), \(|\psi\rangle = \otimes_{i=1}^{n} |\phi^{(i)}\rangle\). As \(F (|\psi\rangle, |\phi\rangle) = |\langle \phi |\psi \rangle|^{2}\) the proof is complete. \(\square\)

In \[8\] the author showed that for bipartite pure states \(|\psi\rangle\) with Schmidt decomposition \(|\psi\rangle = \sum \lambda_{i} |\phi_{i}^{(1)}\rangle \otimes |\phi_{i}^{(2)}\rangle\) the overlap \(\Lambda_{\text{max}} (|\psi\rangle)\) is given by the largest Schmidt coefficient

\[
\Lambda_{\text{max}} (|\psi\rangle) = \lambda = \max_{i} \lbrace \lambda_{i} \rbrace.
\]

An alternative proof can be found in Appendix \[A\] Thus we can state the following proposition.

Proposition 2. For bipartite pure state \(|\psi\rangle\) with largest Schmidt coefficient \(\lambda\) holds:

\[
F_{\text{sep}} (|\psi\rangle) = \lambda^{2}.
\]

B. Mixed states

Now we consider mixed states \(\rho\) on a finite dimensional Hilbert space \(\mathcal{H} = \otimes_{i=1}^{n} \mathcal{H}_{i}\). A purification of \(\rho\) is a pure state \(|\psi\rangle \in \mathcal{H}_{0} \otimes \mathcal{H}\) such that \(\rho = \text{Tr}_{0}[|\psi\rangle \langle \psi|]\).

Proposition 3. Let \(\rho\) be a mixed quantum state with particular decomposition \(\lbrace \rho_{i}, |\psi_{i}\rangle \rangle \rbrace\) such that \(\rho = \sum_{i} p_{i} |\psi_{i}\rangle \langle \psi_{i}|\). Every purification of \(\rho\) can be written in the form

\[
|\psi\rangle = \sum_{i} \sqrt{p_{i}} |\psi_{i}^{(0)}\rangle |\psi_{i}\rangle
\]

with \(|\psi_{i}^{(0)}\rangle \in \mathcal{H}_{0}, \langle \psi_{i}^{(0)} |\psi_{j}^{(0)}\rangle = \delta_{ij}\).

Proof. Let \(|\phi\rangle\) be an arbitrary purification of \(\rho\) with Schmidt decomposition

\[
|\phi\rangle = \sum_{i} \sqrt{q_{i}} |\phi_{i}^{(0)}\rangle |\phi_{i}\rangle.
\]

\(|\phi_{i}\rangle\) are eigenstates and \(q_{i}\) are corresponding eigenvalues of \(\rho\). According to \[10\] Theorem 2.6 on page 103 there is a unitary matrix \(U\) such that

\[
\sqrt{q_{i}} |\phi_{i}\rangle = \sum_{j} u_{ij} \sqrt{p_{j}} |\psi_{j}\rangle.
\]
With (18) in (17) we get
\[ |\phi\rangle = \sum_j \sqrt{p_j} |\psi_j^{(0)}\rangle |\psi_j\rangle \]
with \( |\psi_j^{(0)}\rangle = \sum_i u_{ij} |\psi_i^{(0)}\rangle \), and thus \( \langle \psi_i^{(0)} | \psi_j^{(0)} \rangle = \delta_{ij} \).

This is exactly the form (10), this ends the proof.

A direct consequence of Proposition 3 is that every purification of a separable state of the form (11) can be written in the form
\[ |\psi\rangle = \sum_i \sqrt{p_i} (\otimes_{j=0}^n |\psi_i^{(j)}\rangle) \]
with \( \langle \psi_i^{(0)} | \psi_j^{(0)} \rangle = \delta_{ij} \), \( \sum_i p_i = 1 \), \( p_i \geq 0 \).

With this in mind we can prove the following theorem.

**Theorem 4.** For a multipartite mixed state \( \rho \) on a finite dimensional Hilbert space \( \mathcal{H} = \otimes_{j=1}^n \mathcal{H}_j \) holds:
\[ F_{\text{sep}} (\rho) = \max_{\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \text{ and } \sum_i p_i F_{\text{sep}} (|\psi_i\rangle)} \]
maximization is done over all pure state decompositions of \( \rho \).

**Proof.** Let \( \sigma \) be a separable state that maximizes the fidelity among all separable states, that is \( F_{\text{sep}} (\rho) = F (\rho, \sigma) \). According to (11), Equation 9.72 on page 411 holds:
\[ F (\rho, \sigma) = \max_{\sigma = \sum_i p_i |\psi_i\rangle \langle \psi_i|} |\langle \psi | \phi \rangle|^2 , \]
where \( |\psi\rangle \) is a purification of \( \rho \) and the maximization is done over all purifications of \( \sigma \). In the following \( |\phi\rangle \) will denote a particular purification that realizes the maximum: \( F (\rho, \sigma) = |\langle \psi | \phi \rangle|^2 \).

Using Proposition 3 we write
\[ |\phi\rangle = \sum_i \sqrt{q_i} \otimes_{j=0}^n |\phi_i^{(j)}\rangle \]
with \( \sum_i q_i = 1 \) and \( \langle \phi_i^{(0)} | \phi_j^{(0)} \rangle = \delta_{ij} \). We prove in Appendix (13) that
\[ |\langle \psi | \phi \rangle|^2 = \sum_i \max_{\langle \phi_i^{(0)} | \phi_j^{(0)} \rangle = \delta_{ij}} \left| \langle \psi \otimes_{j=0}^n \phi_i^{(j)} \rangle \right|^2 . \]

Using Proposition 3 we write: \( |\psi\rangle = \sum_i \sqrt{p_i} |\psi_i^{(0)}\rangle |\psi_i\rangle \).

Noting that there always is a unitary matrix \( u \) such that \( |\psi_i^{(0)}\rangle = \sum_j u_{ij} |\phi_j^{(0)}\rangle \) we rewrite \( |\psi\rangle \) as follows:
\[ |\psi\rangle = \sum_{i,j} \sqrt{p_i u_{ij}} |\phi_j^{(0)}\rangle |\psi_i\rangle = \sum_{j} \sqrt{p_j} |\phi_j^{(0)}\rangle |\psi_j\rangle , \]
where \( \sqrt{p_j} |\phi_j^{(0)}\rangle = \sum_i u_{ij} \sqrt{p_i} |\psi_i\rangle \) and \( \rho = \sum_j p_j |\phi_j^{(0)}\rangle \langle \phi_j^{(0)}| \).

For simplicity we write \( p_i \) instead of \( p_j \) and \( |\psi_i\rangle \) instead of \( |\phi_j^{(0)}\rangle \).

With (23) in (22) we get:
\[ |\langle \psi | \phi \rangle|^2 = \max_{\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|} \left( \sum_i p_i A_{\text{max}}^2 (|\psi_i\rangle) \right) . \]

This ends the proof.

Using Theorem 4 we can prove the following proposition.

**Proposition 5.** Geometric and revised geometric measure of entanglement are equal:
\[ E_{\text{Ge}} (\rho) = 1 - F_{\text{sep}} (\rho) . \]

**Proof.** Using definition (4), (6) of \( E_{\text{Ge}} \) and (11) we write
\[ E_{\text{Ge}} (\rho) = \min_{\sum_i p_i F_{\text{sep}} (|\psi_i\rangle)} \]
minimization is done over all pure state decompositions of \( \rho \). Using \( \sum_i p_i = 1 \) this becomes
\[ E_{\text{Ge}} (\rho) = 1 - \max_{\sum_i p_i F_{\text{sep}} (|\psi_i\rangle)} \]
Using Theorem 4 the proof is complete.

**C. Two qubits**

In (4) geometric measure for two qubit states was derived:
\[ E_{\text{Ge}} (\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - C (\rho)^2} \right) \]
with concurrence \( C (\rho) \). With (25) we can compute \( F_{\text{sep}} \) for two qubit states:
\[ F_{\text{sep}} (\rho) = \frac{1}{2} \left( 1 + \sqrt{1 - C (\rho)^2} \right) . \]

Using (26) all entanglement measures based on fidelity can be computed for two qubit states. For Bures measure of entanglement we get
\[ E_B (\rho) = 2 - 2 \sqrt{\frac{1 + \sqrt{1 - C (\rho)^2}}{2}} , \]
we already presented this result in (13).
IV. CONCLUDING REMARKS

In this paper we established a simple connection between the geometric measure of entanglement and entanglement measures based on fidelity. Using it, all results obtained for geometric measure can also be used for other measures and vice versa.

One of our main results is Theorem 4. In words it can be expressed as follows: the fidelity of separability is an upside down version of a convex roof measure of entanglement. This result underlines the importance of fidelity for quantum information theory, especially for construction of entanglement measures. We believe that it will be useful for further research in this direction.

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Appendix A: BIPARTITE PURE STATES

Let $|\psi\rangle$ be a bipartite pure state with Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |i^{(1)}\rangle \otimes |i^{(2)}\rangle.$$  

Further we define $\lambda = \max \{\lambda_i\}$. We will now show that $\lambda = \max_{|\phi\rangle \in S} |\langle \psi | \phi \rangle|$, $|\phi\rangle = |\phi^{(1)}\rangle \otimes |\phi^{(2)}\rangle$. Another proof was given in [8].

First we rewrite the states as follows:

$$|\phi^{(1)}\rangle = \sum_i a_i^* |i^{(1)}\rangle, \quad |\phi^{(2)}\rangle = \sum_i b_i |i^{(2)}\rangle.$$  

Now note that

$$|\langle \psi | \phi \rangle| = \left| \sum_i \lambda_i a_i^* b_i \right| = |\langle a | Y | b \rangle| \tag{A1}$$

with $Y$ being diagonal matrix with entries $\lambda_i$, $|a\rangle$ and $|b\rangle$ are normalized vectors with entries $a_i$ and $b_i$.

We have to maximize (A1) over all normalized vectors $|a\rangle$ and $|b\rangle$. For this we will prove the following theorem:

Theorem 6. For a Hermitian matrix $H$ with eigenvalues $\lambda_i$ and two normalized vectors $|a\rangle$ and $|b\rangle$ holds:

$$|\langle a | H | b \rangle| \leq \max_i |\lambda_i|.$$  

Proof. We will maximize $|\langle a | H | b \rangle|^2$:

$$|\langle a | H | b \rangle|^2 = |\langle a | Z | a \rangle|,$$

where $Z = H |b\rangle \langle b|H$ is a Hermitian matrix with rank 1, thus the only nonzero eigenvalue of $Z$ is $|b |H^2 |b\rangle$. Further from [14, Theorem 4.2.2 on page 176] follows that $|\langle a | Z | a \rangle| \leq (b |H^2 |b\rangle \leq \max_i \lambda_i^2$. This ends the proof.

Using (A2) in (A1) and noting that $Y \geq 0$ we immediately get

$$|\langle \psi | \phi \rangle| \leq \lambda$$

for all separable states $|\phi\rangle$ and equality is attained if $|\phi\rangle = |1^{(1)}\rangle |1^{(2)}\rangle$, where $\lambda_1 = \lambda$. This proves that

$$\max_{|\phi\rangle \in S} |\langle \psi | \phi \rangle| = \lambda,$$

as stated above.

Appendix B: OPTIMAL PURIFICATIONS

Let $\sigma$ be a separable state on $n$-partite Hilbert space $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$. Then it can be written as

$$\sigma = \sum_i q_i \left( \otimes_{j=1}^n |\phi_i^{(j)}\rangle \langle \phi_i^{(j)}| \right). \tag{B1}$$

According to Proposition 3 every purification of $\sigma$ can be written as a state $|\phi\rangle \in \mathcal{H}_0 \otimes \mathcal{H}$ of the form

$$|\phi\rangle = \sum_i \sqrt{q_i} \otimes_{j=1}^n |\phi_i^{(j)}\rangle \tag{B2}$$

with $|\phi_i^{(0)}\rangle |\phi_i^{(0)}\rangle = \delta_{ij}$, $\sum_i q_i = 1$.

For a given pure state $|\psi\rangle \in \mathcal{H}_0 \otimes \mathcal{H}$ we now want to maximize $|\langle \psi | \phi \rangle|^2$ among all states $|\phi\rangle$ of the form (B2). Noting that

$$|\langle \psi | \phi \rangle|^2 = \left| \sum_i \sqrt{q_i} (|\psi\rangle \otimes_{j=1}^n |\phi_i^{(j)}\rangle) \right|^2 \tag{B3}$$

we will now show that $q_i$ can be eliminated.

Proposition 7. For any pure state $|\psi\rangle \in \mathcal{H}_0 \otimes \mathcal{H}$ holds:

$$\max \mathcal{H}_{tr}(\rho |\rho\rangle) \max_{|\phi\rangle_{i=1}^n} |\langle \psi | \phi \rangle|^2 = \sum_i \max_{|\phi_i^{(0)}\rangle_{i=1}^n} |\langle \psi | \otimes_{j=1}^n |\phi_i^{(j)}\rangle|^2.$$  

(B4)

Proof. Note that following inequality holds:

$$|\langle \psi | \phi \rangle| = \left| \sum_i \sqrt{q_i} (|\psi\rangle \otimes_{j=1}^n |\phi_i^{(j)}\rangle) \right| \leq \sum_i \sqrt{q_i} \left| \langle \psi | \otimes_{j=1}^n |\phi_i^{(j)}\rangle \right| \tag{B5}$$
In maximizing $|\langle \psi | \phi \rangle|$ we are free to choose the phases of $|\phi^{(j)}_{i}\rangle$, this can always be done such that on rhs of (B5) equality holds, that is

$$\max_{\text{Tr}_0[|\phi\rangle\langle \phi|]} |\langle \psi | \phi \rangle| = \max_i \sum_i \sqrt{q_i} \left| \langle \psi | \otimes_{j=0}^n \phi^{(j)}_{i}\rangle \right|.$$  \hspace{1cm} (B6)

Maximization on rhs is done over all $|\phi^{(j)}_{i}\rangle$ with the only restriction $\langle \phi^{(0)}_{i}| \phi^{(0)}_{j}\rangle = \delta_{ij}$, and over all $q_i$ restricted by $\sum_i q_i = 1$. Maximization over $q_i$ can be evaluated using Lagrange multipliers with the result

$$\sqrt{q_i} = \frac{\left| \langle \psi | \otimes_{j=0}^n \phi^{(j)}_{i}\rangle \right|}{\sqrt{\sum_i \left| \langle \psi | \otimes_{j=0}^n \phi^{(j)}_{i}\rangle \right|^2}}.$$  \hspace{1cm} (B7)

Using (B7) in (B6) we get (B4). This ends the proof. \[\square\]