EXISTENCE AND UNIQUENESS OF $S^1$-INVARIANT KÄHLER-RICCI SOLITONS

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ABSTRACT. We use the momentum construction for $S^1$-invariant Kähler metrics as developed by Hwang-Singer to construct new examples of steady Kähler-Ricci solitons. We also prove that these solitons are unique in their Kähler class, provided the vector field and the asymptotic behavior are fixed.

1. Introduction

A steady Kähler-Ricci soliton is a Kähler manifold $(M, g)$ whose Kähler form $\omega$ satisfies

$$\text{Ric}(\omega) = -\mathcal{L}_X \omega$$

for some vector field $X$ which is the real part of a holomorphic vector field. Solutions to (1) are natural generalizations of Ricci-flat metrics and arise as self-similar solutions to Ricci flow.

If the vector field $X$ is non-zero, the manifold must be non-compact [Ive93]. In general, there is no classification for steady Kähler-Ricci solitons available and only few examples are known. Even if a manifold admits a Kähler-Ricci soliton, it is not understood which subset of the Kähler cone contains further examples of Ricci solitons. It is also not clear, how many solitons there are in each Kähler class.

All known examples with $X \neq 0$ are divided into two classes. One class contains explicitly constructed solutions by using ODE methods ([Ham88], [Cao96], [CV96], [PTFV99], [FIK03], [Li10], [DW11], [FW11], [Yan12]), while the other examples are obtained by using PDE gluing methods ([BM17]). The explicit examples are constructed on Euclidean space or on holomorphic vector bundles over Kähler manifolds, while the gluing method produces solitons on certain crepant resolutions of orbifolds $\mathbb{C}^n/G$.

In this article, we use the momentum construction introduced by Hwang-Singer [HS02] to find new examples of steady Kähler-Ricci solitons. More precisely, we prove the following theorem.

**Theorem 1.1.** Let $\pi : K_M \to (M, g_M)$ be the canonical line bundle over a compact Kähler manifold. Assume that the Ricci form of $g_M$ is positive semi-definite and has constant eigenvalues with respect to $g_M$. Then $K_M$ admits a 1-parameter family of complete steady Kähler-Ricci solitons in the Kähler class $[\pi^* \omega_M]$. 


Theorem 1.1 generalises results obtained in [Cao96], [CV96], [PTFV99], [FIK03], [DW11] and [Yan12]. The main difference is that we do not assume \((M, g_M)\) to be a Kähler-Einstein Fano manifold, but only require that \(\text{Ric}(\omega_M)\) has constant eigenvalues.

Under the same assumption, Hwang-Singer [HS02] used Calabi’s ansatz to construct Kähler-Einstein metrics on line bundles. They observed that the constancy of eigenvalues is sufficient to reduce the Kähler-Einstein equation to a single ODE, which is linear after applying a certain transformation. We prove Theorem 1.1 by adapting their construction to the case of steady Kähler-Ricci solitons.

Theorem 1.1 produces new examples if the base \(M\) is a flag variety. More concretely, consider the canonical bundle over \(M = \mathbb{P}(T^*\mathbb{C}P^n)\), the projectivization of the cotangent bundle \(T^*\mathbb{C}P^n\). Previously, it was only known that compactly supported Kähler classes admit steady solitons ([PTFV99], [DW11], [Yan12]), whereas Theorem 1.1 shows they sweep out the entire Kähler cone.

Another interesting feature of Hwang-Singer’s construction is that it can also be applied to certain vector bundles of rank \(\geq 2\). Then we obtain a result analogue to Theorem 1.1.

**Theorem 1.2.** Let \(\pi : E \to D\) be a holomorphic vector bundle of rank \(m\) over a compact Kähler manifold \((D, \omega_D)\). Assume that \(E\) admits a Hermitian metric \(h\) such that the corresponding curvature form \(\gamma\) of the tautological bundle \((\mathcal{O}_{\mathbb{P}(E)}(-1), h)\) is negative semi-definite and has constant eigenvalues with respect to the Kähler metric \(\omega_M = p^*\omega_D - \gamma\), where \(p : M = \mathbb{P}(E) \to D\) is the natural projection. Additionally, suppose that

\[
\text{Ric}(\omega_M) = -m\gamma.
\]

Then \(E\) admits a 1-parameter family of complete steady Kähler-Ricci solitons in the class \([\pi^*\omega_D]\).

This can be applied to certain sums of line bundles and again, if the base is a flag variety, it constructs steady solitons in each Kähler class, generalising results in [Li10] and [DW11] (Theorem 4.20).

Given a Kähler-Ricci soliton, it is an interesting question whether or not it is unique in its Kähler class. It is natural to fix a vector field for this question because there can be families of solitons as in Theorem 1.1 and 1.2 for instance. In general, this question seems to be largely open.

In the special case of Ricci-flat Kähler metrics, the question of uniqueness is studied under additional assumptions on the asymptotic behaviour of the metric ([Joy00], [Got12], [CH13], [HHN15]). For example, asymptotically conical Ricci-flat metrics are unique in their Kähler class [CH13].

In the different setting of solitons with \(X \neq 0\), there are only few results such as [BM17]. Assuming that two steady solitons \(\omega_1, \omega_2\) with the
same vector field are related by $\omega_1 = \omega_2 + \sqrt{-1} \partial \bar{\partial} u$, [BM17][Proposition 1.2] shows that $\omega_1 = \omega_2$ provided $u$ and its derivatives tend to zero at infinity.

In this work, we extend the previous result for the metrics constructed in Theorem 1.1 and 1.2.

**Theorem 1.3.** Let $E \to D$ be a holomorphic vector bundle satisfying the assumptions in Theorem 1.1 or 1.2 and denote the steady Kähler-Ricci solitons constructed in Theorem 1.1 or 1.2 by $\omega_\phi$. Suppose that $\omega$ is a Kähler-Ricci soliton on $E$ with the same vector field as $\omega_\phi$ such that $[\omega] = [\omega_\phi] \in H^2(E)$. If moreover $\omega_\phi - \omega \in C^\infty(\Lambda^2 T^* E)$ for some $\delta > 2$, then $\omega_\phi = \omega$.

We reduce the proof of Theorem 1.3 to [BM17][Proposition 1.2] by proving a $\partial \bar{\partial}$-Lemma with controlled growth. Assuming that $\omega_\phi - \omega$ is asymptotic to zero, in a suitable sense, we show that there exists a smooth function $u$ such that $\omega_\phi - \omega = \sqrt{-1} \partial \bar{\partial} u$ and $u \in C^\infty_{\delta+2}(E)$, i.e. $u$ and all its derivatives tend to zero because $2 - \delta < 0$.

The strategy for finding such a function $u$ is analogue to [CH13][Section 3]. The main point is proving that all harmonic 1-forms of a certain growth behaviour are identically zero which requires non-negative Ricci curvature. We will see that this is indeed true for the metrics $\omega_\phi$ constructed in Theorem 1.1 and 1.2.

This article is structured as follows. In Section 2, we recall Hwang-Singer’s construction of Kähler metrics and prove Theorem 1.1. For proving Theorem 1.2, we have to make some adjustments which are explained in Section 3. The metrics are studied more closely in Section 4. Here we observe in particular that the curvature of these metrics is bounded and that the Ricci curvature is non-negative. Then, in Section 5, we prove Theorem 1.3 by studying the Laplace operator and harmonic 1-forms of the metrics constructed in Theorem 1.1 and 1.2.

After the first version of this paper was uploaded to the arXiv, Conlon and Deruelle [CD20] posted a preprint on the arXiv containing a new existence result for steady Kähler-Ricci solitons. There is some overlap between their main result [CD20][Theorem A] and our Theorems 1.1 and 1.2: compare Remarks 2.6 and 3.4 below.

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2. Calabi’s Ansatz for line bundles

Hwang-Singer’s construction combines Calabi’s ansatz with ideas from symplectic geometry ([HS02]). If \( \pi : (L, h) \to (M, \omega_M) \) denotes a Hermitian holomorphic line bundle over a Kähler manifold, then Calabi’s idea ([Cal79]) was to search for Kähler metrics of the form
\[
\pi^* \omega_M + \sqrt{-1} \partial \bar{\partial} f(t).
\]
Here, \( t \) denotes the logarithm of the fibre-wise norm function induced by \( h \) and \( f \) is a convex function of one variable. Instead of describing the metric (3) in terms of the potential \( f \), Hwang-Singer introduced a new variable \( \tau = \tau(t) \) and a function \( \phi = \phi(\tau) : (0, \infty) \to \mathbb{R}_+ \) which is related to the Legendre transformation \( F \) of \( f \) by \( \phi = 1/\phi'' \). In particular, \( \phi \) determines the metric (3) uniquely.

Assuming that the curvature form of \( h \) has constant eigenvalues, we will see in this section that the non-linear Kähler-Ricci soliton equation (1) is equivalent to a single, linear ODE in the function \( \phi \), which can be solved explicitly. This leads to a proof of Theorem 1.1. Additionally, we discuss the main examples to which Theorem 1.1 applies.

2.1. Notation and set-up. We begin by briefly recalling Calabi’s construction of Kähler metrics in the special case of the canonical bundle. We follow the presentation in [HS02] [Section 2].

Let \((M^n, \omega_M)\) be a Kähler manifold of complex dimension \( n \) and equip its canonical line bundle \( \pi : K_M \to M \) with the Hermitian metric \( h \) induced by \( \omega_M \). Let \( \gamma \) be the curvature form of \( h \) and assume that \(-\gamma \geq 0\), i.e. \( \gamma \) is negative semi-definite. Recall that \( \gamma \) is given by
\[
\gamma = -\sqrt{-1} \partial \bar{\partial} \log h(s, \bar{s}) = -\text{Ric}(\omega_M),
\]
where \( s : U \to K_M \) is a local holomorphic section of \( K_M \) and \( \text{Ric}(\omega_M) \) denotes the Ricci form of \( \omega_M \). We introduce the radial function \( r : K_M \to \mathbb{R}_{\geq 0} \) defined by \( r(v) = \sqrt{h(v, \bar{v})} \) and outside the zero section, we define a new function \( t : K_M \setminus M \to \mathbb{R} \) by \( t = 2 \log r \). The pullback \( \pi^* \gamma \) is a \( \partial \bar{\partial} \)-exact form on \( K_M \setminus M \) and satisfies
\[
\pi^* \gamma = -\sqrt{-1} \partial \bar{\partial} t.
\]

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a smooth function satisfying
\[
\lim_{t \to -\infty} f'(t) = 0 \quad \text{and} \quad f'' > 0.
\]
Then Calabi’s Ansatz searches for Kähler metrics \( \omega \) of the form
\[
\omega = \pi^* \omega_M + \sqrt{-1} \partial \bar{\partial} f(t) = \pi^* \omega_M - f'(t) \pi^* \gamma + f''(t) \sqrt{-1} \partial t \wedge \bar{\partial} t.
\]
Note that \( \omega \) is defined on \( K_M \setminus M \), the canonical bundle with the zero section removed, and it is positive since we assumed \(-\gamma \geq 0\) and (5). Depending on the behaviour of \( f(t) \) as \( t \to \pm \infty \), \( \omega \) can be extended to all of \( K_M \) and define a complete metric. When this can happen is explained in the next subsection.
We conclude this subsection by describing the Calabi metric $\omega$ in terms of the Legendre transformation of its potential $f$, which is well-defined since $f$ is convex by (5). We now briefly recall this transformation. Let $I = \text{Im} f' \subset \mathbb{R}_+$ be the image of $f'$ and define the new variable $\tau := f'(t) \in I$. We write $I = (0, \tau_2)$, which means that
\[
\lim_{s \to -\infty} \tau(s) = \lim_{t \to -\infty} f'(t) = 0, \quad \lim_{s \to +\infty} \tau(s) = \lim_{t \to +\infty} f'(t) = \tau_2.
\]
We point out that in general $\tau_2 \leq +\infty$, but in the case considered in subsequent sections, we have in fact that $\tau_2 = +\infty$. The Legendre transform $F : I \to \mathbb{R}$ is defined by the formula
\[
f(t) + F(\tau) = t\tau.
\]
One can check that $F$ is also strictly convex, so that we can define a new function $\varphi : I \to \mathbb{R}_+$ by
\[
\varphi(\tau) = \frac{1}{F''(\tau)}.
\]
Then we obtain the following relations
\[
\frac{d\tau}{dt} = f''(t) = \varphi(\tau), \quad f'''(t) = \frac{d\varphi}{dt} = \varphi'/(\tau)\varphi(\tau).
\]
In particular, (5) translates into
\[
\varphi > 0 \quad \text{on} \quad I = (0, \tau_2).
\]
We can then express the metric $\omega$ obtained from Calabi’s construction (6) as
\[
\omega = \pi^*\omega_M - \tau^*\gamma + \frac{1}{\varphi(\tau)} \sqrt{-1}\partial\bar{\partial} \tau
\]
by using equations (6) and (7).

The function $\varphi$ is called the momentum profile of $\omega$. We note that it is possible to reconstruct the Kähler potential $f$ of $\omega$ from its momentum profile by
\[
f(t) = \int_0^{\tau(t)} \frac{xdx}{\varphi(x)}.
\]
Hence, the Kähler metric given by Calabi’s Ansatz (6) is uniquely determined by its momentum profile. We emphasize this by writing $\omega = \omega_{\varphi}$.

**Completeness of $\omega_{\varphi}$.** The Kähler metric $\omega = \omega_{\varphi}$ given by (9) is a priori only defined on $K_M \setminus M$ and is in general not complete. Whether or not $\omega_{\varphi}$ extends across the zero section to a complete metric is determined by the behaviour of the momentum profile $\varphi$ toward the endpoints of $I = (0, \tau_2)$. This is well-understood and there is the following well-known proposition, whose proof can be found in [HS02] [Section 2] or [FW11] [Section 6], for example.
Proposition 2.1. Let $\omega_\varphi$ be given by (9). Suppose the profile $\varphi : I \to \mathbb{R}$ has a zero of integer order at each endpoint of $I = (0, \tau_2)$. Then $\omega_\varphi$ extends across the zero section if and only if $\varphi(0) = 0$ and $\varphi'(0) = 1$.

In this case, the resulting metric on $K_M$ is complete if and only if at the upper endpoint $\tau_2$, one of the following conditions (i) and (ii) holds:

(i) The endpoint $\tau_2$ is finite and $\varphi$ vanishes at least to second order.
(ii) The endpoint $\tau_2$ is infinite and $\varphi$ grows at most quadratically.

Remark 2.2. Note that [HS02][Proposition 2.3] is identical with Proposition 2.1, except that Hwang and Singer require $\varphi'(0) = 2$ instead of $\varphi'(0) = 1$. This is due to the fact that our Kähler potential $f$ is twice the potential function used by Hwang and Singer; compare (6) with [HS02][(1.1)].

If the metric $\omega_\varphi$ extends to the total space of $K_M$, we would like to identify its de Rham cohomology class. Since we assumed (8), i.e. $I = (0, \tau_2)$, it follows immediately that $[\omega_\varphi] = [\pi^*\omega_M] \in H^2(K_M)$. We refer to the class $[\pi^*\omega_M]$ as the Kähler class of $\omega_\varphi$.

More generally, we define a Kähler class on $K_M$ simply to be a class in $H^2(K_M)$ containing positive $(1,1)$ forms and the Kähler cone is the set of all such Kähler classes. Using this definition, the projection map $\pi^* : H^2(M) \to H^2(K_M)$ identifies the Kähler cone of the compact base $M$ with the Kähler cone of $K_M$. Indeed, given a Kähler form on $K_M$, its restriction to $M$ clearly is a Kähler form on $M$. Conversely, given a Kähler form $\omega_M$ on $M$, Calabi’s Ansatz always produces a positive $(1,1)$ form in the class $[\pi^*\omega_M]$, for example consider $\omega_\varphi$ with $\varphi(\tau) = \tau$, which extends to $K_M$ by Proposition 2.1.

The Ricci form. In this paragraph, we provide a description of the Ricci-form of $\omega_\varphi$. The computations can be found, for example, in [HS02][Section 2.1].

Denote the Kähler metric of $\omega_M$ by $g_M$ and the curvature form of $(K_M, h)$ by $\gamma$. It gives rise to an endomorphism $B : T^{1,0}M \to T^{1,0}M$ of the holomorphic tangent bundle, which is locally defined by $B := g_M^{-1}\gamma = \bar{g}_M^\gamma g_M$. As in Theorem 1.1 we assume from now on that the eigenvalues of $B$ are constant over $M$. This condition is sufficient to reduce the soliton equation (11) to an ODE. These conditions guarantee that the function $Q : I \times M \to \mathbb{R}_+$ defined by

$$Q = \det (g_M^{-1}(\omega_M - \tau\gamma)) = \det (\operatorname{Id} - \tau B)$$

only depends on the parameter $\tau$, i.e. is constant over $M$. Also observe that $Q$ is a positive function because $-\gamma \geq 0$ and $\tau \geq 0$. $Q$ naturally appears in the computation of $\operatorname{Ric}(\omega_\varphi)$. Indeed, the Ricci form is given
2.2. Reduction to an ODE. We use the previously derived formula for the Ricci curvature to show that the Kähler-Ricci soliton equation is equivalent to an ODE in the function \( \varphi(\tau) \). Our presentation is similar to [FW11][Section 4].

By definition, the soliton vector field \( X \) must be the real part of a holomorphic vector field, i.e. \( L_X J = 0 \). On the line bundle \( K_M \), there is a natural choice for \( X \), which we now describe. \( K_M \) admits a holomorphic \( \mathbb{C}^* \)-action by fibre-wise multiplication and the corresponding holomorphic vector field \( Z \) is given by \( Z = z_0 \frac{\partial}{\partial z_0} \), where \( z_0 \) denotes the fibre coordinate of \( K_M \). In terms of the radial function \( t \) defined at the beginning of this section, we can write \( Z \) as

\[
Z = \text{Re} \, Z + \sqrt{-1} \text{Im} \, Z = \frac{\partial}{\partial t} - \sqrt{-1} J \frac{\partial}{\partial t}.
\]

So it is natural to set \( X := \mu \text{Re} \, Z = \mu \frac{\partial}{\partial t} \) for some constant \( 0 \neq \mu \in \mathbb{R} \). Before deriving the ODE, we need to calculate the following Lie-derivative:

\[
L_X \omega_{\varphi} = d(t_X \omega_{\varphi}) = \sqrt{-1} \partial \bar{\partial} (L_X f)(t) = \mu \sqrt{-1} \partial \bar{\partial} f'(t).
\]

Here, we used \( 2 \sqrt{-1} \partial \bar{\partial} = dJd \) and \( L_X J = 0 \) to obtain the second equality. We shall write out equation (13) in terms of fibre and base direction, as we did for the Ricci-form in (12):

\[
-L_X \omega_{\varphi} = \mu \varphi(\tau) \pi^* \gamma - \mu \frac{\varphi'}{\varphi}(\tau) \sqrt{-1} \partial \tau \wedge \bar{\partial} \tau.
\]

Now we are in position to see by comparing (12) and (15) that the soliton equation (1) for \( \omega_{\varphi} \) is equivalent to the following two equations

\[
\text{Ric}(\omega_M) + \frac{(\varphi'Q)'(\tau)}{Q}(\tau) \gamma = \mu \varphi(\tau) \gamma
\]

\[
\left( \frac{(\varphi'Q)'}{Q} \right)'(\tau) = \mu \varphi'(\tau).
\]

Since \( \text{Ric}(\omega_M) = -\gamma \), we see that differentiating (16) gives (17), so that we proved the following Lemma:

**Lemma 2.3.** Suppose that \( \omega_{\varphi} \) is a Kähler metric with momentum profile \( \varphi \). Then (1) with \( X = \mu \frac{\partial}{\partial t} \) is equivalent to the following equation:

\[
\varphi'(\tau) + \left( \frac{Q'}{Q}(\tau) - \mu \right) \varphi(\tau) = 1
\]
For the rest of this paragraph, we study the solution \( \varphi \) to Equation (18). This is a linear ODE of the form \( y' + p(x)y = q(x) \), which has an explicit one-parameter family of solutions given by

\[
(19) \quad y = \exp \left( -\int p(x)dx \right) \left( \int q(x) \exp \left( \int p(x)dx \right) dx + K \right).
\]

Applying (19) to (18), we have

\[
(20) \quad \varphi(\tau) = \frac{e^{\mu \tau}}{Q(\tau)} \left( \int_0^\tau e^{-\mu x}Q(x)dx + K \right),
\]

where \( K \geq 0 \) is determined by the initial value \( \lim_{\tau \to 0} \varphi(\tau) \). Justified by (ii) of Proposition \( \text{\ref{prop:initial}} \), we will assume that \( K = 0 \).

One can compute the integral (18) explicitly in terms of the coefficients \( b_j \geq 0 \) of the polynomial \( Q(\tau) = \det (\Id - \tau B) = b_k \tau^k + b_{k-1} \tau^{k-1} + \cdots + b_0 \). Note that the degree \( k \) of \( Q \) could be less than \( n \) since \( B \) is allowed to have zero eigenvalues. In fact, it is straightforward to see that

\[
(21) \quad \varphi(\tau) = \nu(0) \frac{e^{\mu \tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)},
\]

where \( \nu \) is given by

\[
(22) \quad \nu(\tau) = \sum_{j=0}^k \sum_{l=0}^j \frac{b_j j!}{\mu^j+1-l} \tau^l.
\]

We point out that the explicit expression for \( \nu \) is not relevant, but rather that it has the form

\[
(23) \quad \varphi(\tau) = \nu(0) \frac{e^{\mu \tau}}{Q(\tau)} + \frac{(-b_k/\mu)\tau^k + R_{k-1}(\tau)}{Q(\tau)}
\]

for a polynomial \( R_{k-1} \) of degree \( k - 1 \). Hence, we found an explicit solution for the soliton ODE (18). Also note that \( \varphi \) is defined on \( [0, +\infty) \) since \( Q(0) > 0 \). Moreover, \( \varphi \) is clearly positive on \( (0, +\infty) \).

With these observations, we can now finish the proof of Theorem \( \text{\ref{thm:main}} \).

2.3. Proof of Theorem \( \text{\ref{thm:main}} \) Let \( K_M \to (M, g_M) \) be the canonical bundle whose semi-negative curvature form \( \gamma = -\text{Ric}(\omega_M) \) has constant eigenvalues w.r.to \( g_M \). Suppose \( \varphi : (0, +\infty) \to \mathbb{R} \) is given by (20) with \( K = 0 \) and, as before, let \( \omega_\varphi \) be defined by (9). Since \( \varphi(\tau) > 0 \) for all \( \tau > 0 \), \( \omega_\varphi \) defines a Kähler metric and hence is a steady Kähler-Ricci soliton by Lemma \( \text{\ref{lem:steady}} \). We note that these metrics can only be complete if \( \mu < 0 \). This can be proven similarly to [FKK03][Lemma 5.1].
Hence we assume $\mu < 0$. From (23), we have the following asymptotic behaviour for large $\tau$:

$$\varphi(\tau) = -\frac{1}{\mu} + O(1/\tau).$$

(24)

Also recall that $\varphi$ and the potential $f$ are related by

$$\frac{df'(t)}{dt} = \varphi(f'(t)).$$

(25)

Using (25) together with (24), we conclude that the corresponding potential $f(t)$ is indeed defined for all $t \in \mathbb{R}$, i.e. $\omega_\varphi$ is defined on $K_M \setminus M$.

It remains to check that $\omega_\varphi$ extends across the zero section and defines a complete metric as $t \to +\infty$. By the first part of Proposition 2.1, $\omega_\varphi$ extends provided $\varphi(0) = 0$ and $\varphi'(0) = 1$. Since we assumed $K = 0$ in (20), we have $\varphi(0) = 0$. Plugging this into (18) gives $\varphi'(0) = 1$, as desired. The completeness as $t \to +\infty$ follows immediately from the asymptotic expansion (24) and (ii) of Proposition 2.1.

2.4. Examples. Theorem 1.1 immediately recovers all known examples of steady Kähler-Ricci solitons on the total space of line bundles ([Cao96], [CV96], [PTFV99], [DW11], [Yan12]). In these cases, the base is a product of Kähler-Einstein manifolds and the considered Kähler classes are represented by convex combinations of Kähler-Einstein metrics on each factor.

If the base manifold is a flag variety, Theorem 1.1 produces examples, which have not been mentioned before. In this case, steady solitons sweep out the entire Kähler cone.

Example 2.4 (Products). Let $(M_i, \omega_i), i = 1, \ldots, r$ be Kähler-Einstein manifolds with non-negative scalar curvature and denote their canonical bundles by $K_{M_i} \to M_i$. We consider the bundle

$$K_M = p_1^*K_{M_1} \otimes \cdots \otimes p_r^*K_{M_r} \to M := M_1 \times \cdots \times M_r,$$

where $p_i : M \to M_i$ is the projection. Then Theorem 1.1 applies and gives a complete steady soliton in each Kähler class of the form $\sum_{i=1}^r \alpha_i[p_i^*\omega_i] \in H^2(M)$ with $\alpha_i > 0$.

The case $r = 1$ was first considered in [Cao96] and [CV96] for $M = \mathbb{CP}^n$ and in [PTFV99] for a general Kähler-Einstein Fano manifold. For $r > 1$, these solitons are found in [DW11][Theorem 4.20].

Example 2.5 (Flag varieties). Let $G$ be a complex semisimple Lie group, $P \subset G$ a parabolic subgroup and $K \subset G$ a maximal compact subgroup. Then $K$ acts transitively on the flag manifold $M = G/P$. It is well-known that $M$ admits a $K$-invariant complex structure so that its anti canonical bundle is ample, compare [Bes87][Chapter 8] for example.
The previously mentioned results only produce solitons on the canonical bundle $K_M$ whose Kähler class is a multiple of $[\pi^*c_1(M)]$. In general, however, $H^2(M)$ is not spanned by $[c_1(M)]$.

We claim that every Kähler class admits a steady Kähler-Ricci soliton. Indeed, every Kähler class on $M$ admits a $K$-invariant Kähler form $\omega_K$ whose Ricci form $\text{Ric}(\omega_K)$ is also $K$-invariant. This means that the eigenfunctions of $\text{Ric}(\omega_K)$ w.r.t. $\omega_K$ must be $K$-invariant and hence constant since $K$ acts transitively on $M$. So Theorem 1.1 can be applied and proves the existence of a steady soliton in the class $[\pi^*\omega_K]$.

Remark 2.6. The new metrics in Example 2.5 can also be obtained from the recent result [CD20] [Theorem A], which was posted after the first version of this paper was uploaded to the arXiv.

3. Calabi metrics on vector bundles

Given a vector bundle $E \to D$, Hwang-Singer’s idea was to apply their construction to the tautological bundle $\mathcal{O}_{P(E)}(-1)$ over $\mathbb{P}(E)$, the projectivization of $E$ (HS02 [Section 3.2]). In this section, we explain the changes which are necessary to prove Theorem 1.2 and provide some examples.

The main difference is that one has to choose a new background metric on $\mathbb{P}(E)$, with respect to which the eigenvalues of the curvature form are computed. Then the discussion of the previous section can be applied and again, the soliton equation (11) reduces to a simple ODE. In this new setting, however, the function $Q$ defined by (11) will have zeros at $\tau = 0$, so there are some details which have to be checked.

3.1. Constructing a Kähler metric. As in [HS02] [Section 3.2], we explain how to adapt the machinery from the previous section to the tautological line bundle.

Let $\pi : E \to (D, \omega_D)$ be a holomorphic vector bundle of rank $m \geq 2$ equipped with an Hermitian metric $h$ and assume that the Kähler manifold $D$ has complex dimension $d$. As in the case of line bundles, we define $r : E \to \mathbb{R}_{\geq 0}$ to be the radial function induced by $h$ and let $t = \log r^2$. Then Calabi’s Ansatz has the form

$$\omega = \pi^*\omega_D + \sqrt{-1}\partial\bar{\partial} f(t).$$

By construction, the projectivization of $E$ is naturally a fibre bundle $p : \mathbb{P}(E) \to D$, with fibre isomorphic to $\mathbb{CP}^{m-1}$. Recall that the natural map $\mathcal{O}_{P(E)}(-1) \subset p^*E \to E$ identifies $\mathcal{O}_{P(E)}(-1) \setminus \mathbb{P}(E) \cong E \setminus D$. By abuse of notation, we denote the bundle projection of $\mathcal{O}_{P(E)}(-1)$ also by $\pi$, so that we have a commuting diagram

$$L := \mathcal{O}_{P(E)}(-1) \longrightarrow E$$

$$\pi \downarrow \quad \quad \quad \quad \pi$$

$$M := \mathbb{P}(E) \longrightarrow D$$

(26)
In the notation from the previous section, let us denote the complex dimension of $M$ by $n$, i.e. $n = d + m - 1$. Via the natural identification $L \setminus M \cong E \setminus D$, $h$ induces a Hermitian metric on $L$, which we also simply denote by $h$. Hence, we can view $r$ as a function on $L$ and, if $\gamma$ is the curvature form of $(L, h)$, we have as before $\pi^*\gamma = -\sqrt{-1}\partial\bar{\partial}t$ with $t = \log r^2$. We again assume that $-\gamma \geq 0$. Then we are looking for metrics of the form

$$\omega = \pi^*\omega_D - f'(t)\pi^*\gamma + f''(t)\sqrt{-1}\partial\bar{\partial}t$$

where we require that $f : \mathbb{R} \to \mathbb{R}$ satisfies (3) to obtain a positive form. As before, we set $\tau := f'(t)$ and define $\varphi : (0, \tau_2) \to \mathbb{R}_+$ by (7), so that it also satisfies (8). Hence, $\omega_\varphi$ can also be expressed as in (3).

For the computation of Ricci curvature below, we need to choose a background Kähler metric $\omega_M$ on $M$. Define

$$\omega_M = p^*\omega_D - \gamma,$$

which is clearly positive in base direction of the fibration $p : \mathbb{P}(E) \to D$. To see that $\omega_M$ is positive in fibre direction, we note that $-\gamma$ restricts to the Fubini-Study metric on each fibre $\cong \mathbb{C}F^{m-1}$.

**The Ricci form.** The calculation is in principle the same as in the line bundle case, but the polynomial $Q$ does have zeros. Let $B = g_M^{-1}\gamma$ be the curvature endomorphism of $\gamma$, where $g_M$ is the metric with Kähler form given by (28) and assume that the eigenvalues of $B$ are constant over $M$. Then we define a function $Q$ by

$$Q = \det(g_M^{-1}(p^*\omega_D - \tau\gamma)),$$

which can be viewed as a function $Q : (0, \tau_2) \to \mathbb{R}_{\geq 0}$. Indeed, we can write

$$g_M^{-1}(p^*\omega_D - \tau\gamma) = g_M^{-1}(\omega_M - (\tau - 1)\gamma) = \text{Id} - (\tau - 1)B,$$

so that $Q$ is constant over $M$, i.e. it only depends on $\tau$. If $\beta_1, \ldots, \beta_n$ are the eigenvalues of $B$, we must have $\beta_{d+1} = \cdots = \beta_n = -1$ by the definition of $\omega_M$ and $\beta_1, \ldots, \beta_d \leq 0$ by assumption. From (30), we conclude that $Q$ is given by

$$Q(\tau) = \tau^{n-d} \prod_{j=1}^d (1 + \beta_j - \tau\beta_j) = \tau^{n-d}\hat{Q}(\tau),$$

for some polynomial $\hat{Q}$. Since $p^*\omega_D$ is positive in base direction, we conclude from (30) that $1 + \beta_j > 0$ for all $j = 1, \ldots d$. Hence, $\hat{Q}(0) > 0$ and $Q$ has a zero at $\tau = 0$ of order $n - d = m - 1$.

As in (12), one can find the following expression for the Ricci form:

$$\text{Ric}(\omega_\varphi) = \pi^*\text{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}\pi^*\gamma - \frac{1}{\varphi} \left(\frac{(\varphi Q)'}{Q}\right)' \partial\tau \wedge \bar{\partial}\tau.$$
3.2. The ODE. The natural $\mathbb{C}^*$-action on $E$ by biholomorphisms induces a holomorphic vector field $Z$. On $L \setminus M$, which is the tautological bundle with the zero section removed, the real part of $Z$ is given by $\Re Z = \partial/\partial t$, so we are looking for Ricci solitons with vector field $X = \mu \partial/\partial t$. Again, we find

\begin{equation}
-\mathcal{L}_X \omega_\varphi = -\mu \sqrt{-1} \partial \bar{\partial} f'(t) = \mu \varphi'(\tau) \pi^* \gamma - \mu \frac{\varphi'(\tau)}{\varphi(\tau)} \sqrt{-1} \partial \tau \wedge \bar{\partial} \tau.
\end{equation}

Combining (27) with (32) and (33), one can check that the soliton equation (1) is equivalent to

\begin{equation}
\text{Ric}(\omega_M) = c \gamma
\end{equation}

for some integration constant $c \in \mathbb{R}$. In fact, we must have $c = -m$ since the first Chern class of $M = P(E)$ is given by

\begin{equation}
c_1(M) = -mc_1(\mathcal{O}_P(E)(-1)) + p^* c_1(E) + p^* c_1(D).
\end{equation}

Equation (35) has the same form as (18), but with a different $Q$. Hence, the solution $\varphi$ is given by

\begin{equation}
\varphi(\tau) = \frac{e^{\mu \tau}}{Q(\tau)} \left( \int_0^\tau m e^{-\mu x} Q(x) dx \right),
\end{equation}

if we assume the integration constant to be zero.

We end this section by studying the solution $\varphi$. Let us write $Q(\tau) = b_{k+n-d} \tau^{k+n-d} + \cdots + b_{n-d} \tau^{n-d}$ with coefficients $b_j \geq 0$ for $j = 1 + n - d, \ldots, k + n - d$ and $b_{n-d} = Q(0) > 0$. Adapting (21) and (22) to this case, we obtain

\begin{equation}
\varphi(\tau) = \nu(0) \frac{e^{\mu \tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)}
\end{equation}

as well as

\begin{equation}
\nu(\tau) = m \sum_{j=n-d}^{k+n-d} \sum_{l=0}^j \frac{b_j}{l!} \frac{\tau^l}{\mu^{j+1-l}}.
\end{equation}

A priori, $\varphi$ given by (37) is defined on the interval $(0, +\infty)$ and because $Q(0) = 0$ one needs to check that $\varphi$ and its derivatives have a limit as $\tau \to 0$. To see that this is the case, note that we can rewrite (39) as

\begin{equation}
\nu(0) e^{\mu \tau} - \nu(\tau) = m \sum_{j=n-d}^{k+n-d} \frac{b_j j!}{\mu^{j+1}} \sum_{l=j+1}^\infty \frac{\mu^l}{l!} \tau^l,
\end{equation}

i.e. $\tau^{-(n-d)}(\nu(0) e^{\mu \tau} - \nu(\tau))$ tends to zero as $\tau \to 0$. Since $Q$ vanishes of order $n - d$ at $\tau = 0$, we then deduce from (38) that $\lim_{\tau \to 0} \varphi = 0$. Similarly, it follows that all derivatives of $\varphi$ have a limit as $\tau \to 0$. 

3.3. Proof of Theorem 1.2. The proof is now analogue to Section 2.3. The only part that might a priori be different is the extension of $\omega_\varphi$ to a complete metric on $E$. However, one can check that Proposition 2.1 also applies to the vector bundle case, see [HS02] [Lemma 3.7].

As before, one can check that $\varphi$ has the behaviour required by Proposition 2.1. Indeed, one can compute that $\varphi(0) = 0$ and $\varphi'(0) = 1$, as desired. Sending $\tau \to +\infty$, we conclude the following asymptotic expansion from (37) and (38)

$$\varphi(\tau) = -\frac{m}{\mu} + O(1/\tau),$$

and so we obtain a complete metric on the total space $E$.

3.4. Examples. We briefly discuss three different situations to which Theorem 1.2 applies. New examples of steady solitons are given in Example 3.2.

Example 3.1 (Complex plane). We let $D$ be a single point and $E \cong \mathbb{C}^n$ be the trivial bundle over $D$. Let $h$ be the Euclidean metric on $E$, so that $\omega_M = -\gamma$ is the Fubini-Study metric on $M = \mathbb{CP}^{m-1}$. This is the situation first studied in [Cao96].

Example 3.2 (Sum of line bundles). Let $(D, \omega_D)$ be a Kähler-Einstein Fano manifold of Fano index $m$. Define $L := K_D^{1/m}$ and consider the $m$-fold sum of $L$ with itself, i.e. $E = L \otimes \mathbb{C}^m$. Then we have $M = \mathbb{P}(E) = \mathbb{CP}^{m-1} \times D$ and

$$\mathcal{O}_{\mathbb{P}(E)}(-1) = p_1^* \mathcal{O}_{\mathbb{CP}^{m-1}}(-1) \otimes p_2^* L,$$

where $p_1, p_2$ denote the projections onto the first and second factor of $M$, respectively. Let $\omega_{FS}$ be the Fubini-Study metric on $\mathbb{CP}^{m-1}$, so that $\gamma = -p_1^* \omega_{FS} - 1/mp_2^* \text{Ric}(\omega_D)$ is the curvature form of $\mathcal{O}_{\mathbb{P}(E)}(-1)$, and define $\omega_M = p_2^* \omega_D - \gamma$. Then we clearly have

$$\text{Ric}(\omega_M) = mp_1^* \omega_{FS} + p_2^* \text{Ric}(\omega_D) = -m\gamma,$$

since $\omega_D$ is Kähler-Einstein. Moreover, the eigenvalues of $\text{Ric}(\omega_M)$ w.r.t. $\omega_M$ are constant, so that Theorem 1.2 can be applied. These examples of steady solitons are obtained in [Li10] [Theorem 2.1] and [DW11] [Theorem 4.20].

If the base $D = G/P$ is a flag manifold for $G$ a complex semisimple Lie group and $P \subset G$ a parabolic subgroup, one can find steady solitons in every Kähler class, similarly as in Example 2.5.

To see this, assume that $\omega_D$ represents a given Kähler class (not necessarily the first Chern class of $D$). We can pick $\omega_D$ to be $K$-invariant, where $K \subset G$ is a maximal compact subgroup. Since $\text{Ric}(\omega_D)$ is also $K$-invariant, the form $-\gamma = p_1^* \omega_{FS} + 1/mp_2^* \text{Ric}(\omega_D)$ is invariant under the diagonal action of $SU(m) \times K$ and also positive.

We claim that $\text{Ric}(-\gamma) = -m\gamma$. By [Bes87] [Theorem 8.2], we know that there exists a $SU(m) \times K$-invariant Kähler-Einstein metric $\omega_{KE} \in$
Also recall that the Ricci forms of all $SU(m) \times K$-invariant Kähler metrics agree, i.e. $\text{Ric}(-\gamma) = \text{Ric}(\omega_{KE})$. Since $-m\gamma$ and $\omega_{KE}$ are in the same Kähler class, we deduce from the uniqueness part of Calabi’s conjecture that $-m\gamma = \omega_{KE} = \text{Ric}(-\gamma)$.

As the form $\omega_M = p^*\omega_D - \gamma$ is also invariant under $SU(m) \times K$, we conclude

$$\text{Ric}(\omega_M) = \text{Ric}(-\gamma) = -m\gamma,$$

and hence the assumptions in Theorem 1.2 are satisfied.

**Example 3.3** (Cotangent bundle of $\mathbb{CP}^d$). Let $D = \mathbb{CP}^d$ be projective space equipped with the Fubini-Study metric and consider $E = T^*\mathbb{CP}^d$, the cotangent bundle of $\mathbb{CP}^d$. $E$ is naturally a $SU(d+1)$-homogeneous vector bundle, where the fibre action is given by the coadjoint action of $SU(d+1)$ on its Lie algebra. Since $\mathbb{CP}^d$ is a rank 1 symmetric space, the induced action of $SU(d+1)$ on $M = \mathbb{P}(E)$ is transitive. Verifying the assumptions of Theorem 1.2 is now similar to the previous Example. These steady solitons on $T^*\mathbb{CP}^d$ are of cohomogeneity one and are contained in [DW11][Section 5].

**Remark 3.4.** All the previous examples can also be constructed from [CD20][Theorem A].

### 4. Properties of $\omega_\varphi$

In this short section, we study curvature properties of the previously constructed metric $\omega_\varphi$. We show that $\omega_\varphi$ has bounded curvature and that its Ricci curvature is non-negative. Moreover, we obtain estimates on the growth of the function $f$ and its derivatives.

Recall that $f = f(t)$ is the Kähler potential of $\omega_\varphi$ as defined in (9) and $\varphi = \varphi(\tau)$ is its momentum profile, see (9). If $\omega_\varphi$ is a steady Kähler-Ricci soliton constructed in Theorem 1.1 or Theorem 1.2, then $\varphi$ satisfies (18) or (35), respectively. This ODE is in turn determined by the polynomial $Q = Q(\tau)$ defined by either (11) or (29). The statements in this section mainly reduce to understanding $Q$ and how it affects the asymptotic behaviour of $\varphi$, compare (21) or (38) depending on the rank of the underlying vector bundle.

We begin by considering the Ricci curvature of $\omega_\varphi$. More precisely, we prove the following theorem, which we need in the subsequent section. It generalises the observation made in [Yan12][Case 7].

**Theorem 4.1.** The complete steady Kähler-Ricci solitons constructed in Theorem 1.1 and 1.2 have non-negative Ricci curvature. Moreover, if the curvature form $-\gamma$ is positive definite, then the Ricci curvature is positive away from the zero section.

**Proof.** First, we consider the solitons constructed on line bundles in Theorem 1.1. Let $\omega_\varphi$ be the Kähler metric given by (9) with $\varphi$ satisfying
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Recall that the Ricci curvature is given by
\[ \text{Ric}(\omega_\varphi) = -\mathcal{L}_X(\omega_\varphi) = \mu \varphi(\tau) \pi^* \gamma - \mu \frac{\varphi'}{\varphi} \sqrt{-1} \partial \bar{\partial} \tau. \]

Since \( \varphi(0) = 0, \varphi > 0 \) on \((0, \infty)\), and \( \mu \gamma \geq 0 \), we only need to show that \( \varphi' > 0 \). To see that this is the case, we define a function
\[ H(\tau) := \frac{Q^2}{Q' - \mu Q} e^{-\mu \tau} - \int_0^\tau e^{-\mu x} Q(x) dx. \]

Using the ODE \((18)\), it is straightforward to prove that \( \varphi' \geq 0 \) iff \( H \geq 0 \). As \( H(0) > 0 \) for \( Q \) given by \((11)\), we are done if we can show that \( H' \geq 0 \). From the definition of \( H \), we compute
\[ H'(\tau) = e^{-\mu \tau} \frac{Q}{(Q' - \mu Q)^2} ((Q')^2 - QQ'') \]
so that \( H' \geq 0 \) if and only if \( (Q')^2 - QQ'' \geq 0 \). The later condition can be checked easily starting from the explicit expression for \( Q \). Indeed, let \( \beta_1, \ldots, \beta_n \) be the eigenvalues of the endomorphism \( B = g_M^1 \gamma : T^{1,0}M \to T^{1,0}M \), and write
\[ Q(\tau) = \det (\text{Id} - \tau B) = \prod_{j=1}^n (1 - \beta_j \tau). \]

Then we have
\[ \frac{(Q')^2 - QQ''}{Q^2} = \sum_{j=1}^n \frac{\beta_j^2}{(1 - \beta_j \tau)^2} \geq 0, \]
as required. For the second statement, it suffices to observe that \( \varphi'(\tau) > 0 \) if and only if \( (Q')^2 - QQ'' > 0 \), which is certainly true if \( \gamma < 0 \). This proves Theorem 4.1 for line bundles.

The arguments for the metrics in Theorem 1.2 are analogous. It also reduces to showing that \( (Q')^2 - QQ'' \geq 0 \), where \( Q \) is this time given by \((31)\).

Note that the non-negativity of Ricci curvature can also be expressed in terms of the potential function \( f \). In particular, we have the following

**Corollary 4.2.** Let \( \omega_\varphi \) be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2 and let \( f = f(t) \) be defined by \((6)\) or \((27)\), respectively. Then \( f'' \) is monotone increasing.

**Proof.** Recall from \((14)\) or \((33)\) that we have
\[ -\mathcal{L}_X \omega_\varphi = -\mu \sqrt{-1} \partial \bar{\partial} f'(t) = \mu f''(t) \pi^* \gamma - \mu f''(t) \sqrt{-1} \partial t \land \bar{\partial} t, \]
and so \( \text{Ric}(\omega_\varphi) = -\mathcal{L}_X \omega_\varphi \) can only be non-negative if \( f''(t) \geq 0 \) since \( \mu < 0 \). Thus, Theorem 1.1 implies that \( f'' \) is increasing. \( \square \)
We end this section by pointing out some growth properties of the potential function $f$.

Lemma 4.3. Let $\omega_\varphi$ be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2 and let $f = f(t)$ be related to $\varphi$ by (10). Then there is a constant $C > 0$ such that for all $t \geq C$, we have

\begin{equation}
C^{-1} \leq f''(t) \leq C \quad \text{and} \quad C^{-1}t \leq f'(t) \leq Ct.
\end{equation}

Moreover, for all $j \in \mathbb{N}_0$ and $t \geq C$

\begin{equation}
C^{-1}(1 + f'(t))^{-j} \leq |f^{(2+j)}(t)| \leq C(1 + f'(t))^{-j}.
\end{equation}

Proof. First note that the bound on $f''(t)$ in (46) implies the bound on $f'(t)$ after integrating the parameter $t$, so we only need to find $C > 0$ such that

\begin{equation}
C^{-1} \leq f''(t) \leq C
\end{equation}

for all $t \geq C$. Translating the problem into bounding $\varphi(\tau)$, we recall from (7) that

\begin{equation}
\tau = \tau(t) = f'(t) \quad \text{and} \quad \varphi(\tau(t)) = f''(t).
\end{equation}

Since $f'(t)$ is positive and increasing, we can choose a $C \geq 1$ such that the following estimate

\begin{equation}
\tau(t) = f'(t) \geq C^{-1}
\end{equation}

holds for all $t \geq C$. Then we recall the asymptotic expansion (10)

\begin{equation}
\varphi(\tau(t)) = -\frac{m}{\mu} + O(1/\tau(t))
\end{equation}

with $\mu < 0$ implying that $\varphi(\tau(t))$ is uniformly bounded from above because of (50). Together with (49), this proves the upper bound for $f''(t)$ in (48). For the lower bound, note that $f''(t) > 0$ is increasing and thus is bounded from below by some positive constant if $t \geq C$. Inequality (48) now follows, and so does (46).

Next, consider the case $j > 0$, i.e. we estimate $f^{(2+j)}(t)$. Differentiating (49) and using the chain rule, we see that

\begin{equation}
\varphi'(\tau(t)) \frac{d\tau}{dt}(t) = \varphi'(\tau(t)) \cdot f''(t).
\end{equation}

Taking further derivatives of this equation, we conclude that $f^{(2+j)}$ can be written as

\begin{equation}
f^{(2+j)} = \sum_{\alpha} c_\alpha \cdot \varphi^{(\alpha_1)} \cdot \ldots \cdot \varphi^{(\alpha_i)} \cdot (f'')^j,
\end{equation}

where the sum is over all multi-indices $\alpha$ with $\alpha_1 + \ldots + \alpha_i = j$ and $c_\alpha$ are constants only depending on the multi-index $\alpha$. Since $f''(t)$ satisfies (18), it is sufficient to estimate derivatives of $\varphi$. In fact, we have for all $\beta \in \mathbb{N}$ that

\begin{equation}
C^{-1} \tau^{-\beta} \leq |\varphi^{(\beta)}(\tau)| \leq C \tau^{-\beta},
\end{equation}

and thus

\begin{equation}
C^{-1} \tau^{-j} \leq |f^{(2+j)}(t)| \leq C \tau^{-j}.
\end{equation}
because $\varphi(\tau)$ behaves asymptotically like a rational function of the form $P/Q$ with polynomials $P(\tau), Q(\tau)$ having the same degree, see (23). Substituting $\tau(t) = f'(t)$ in (52) and combining the resulting estimate with (51), we finally obtain (47) as desired. □

The important point about Lemma 4.3 is estimate (46), i.e. that $f''(t)$ behaves like a constant and $f'(t)$ grows roughly like the function $t$ in the limit $t \to \infty$. This will be crucial in the next section because we want to understand the asymptotic geometry of $\omega_\varphi$.

Another interesting consequence of Lemma 4.3 is that the metrics $\omega_\varphi$ have bounded curvature and positive injectivity radius.

**Lemma 4.4.** The curvature tensor of the steady solitons constructed in Theorem 1.1 and 1.2 is uniformly bounded and each of these metrics has positive injectivity radius.

**Proof.** It is straightforward to see that the first claim reduces to bounding $f''(t), f'''(t)$ and $f^{(4)}(t)$, where $\varphi$ and $f$ are related by (10), so we focus on the second one.

According to [CGTS82] [Theorem 4.7], the lower bound on the injectivity radius follows if we can bound the volume of all unit balls uniformly from below. For this, recall that the function $t$ identifies $E \setminus D \cong \mathbb{R} \times S$, where $S$ is the $S^1$-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see (26).

Under this identification, the metric $g_\varphi$ admits the following decomposition on $\mathbb{R} \times S$

\begin{equation}
(53) \quad g_\varphi = f''(t) \left( dt^2 + (Jdt)^2 \right) + f'(t)\pi^*\hat{g} + \pi^*g_D,
\end{equation}

where $J$ denotes the complex structure on $E$, and $\hat{g}, g_D$ are the (2,0) tensors associated to $-\gamma, \omega_D$, respectively. By compactness of $D$, we only have to consider the set $\{t \gg 1\}$, on which $g_\varphi$ is uniformly equivalent to the metric

\begin{equation}
(54) \quad g_t := dt^2 + (Jdt)^2 + t\pi^*\hat{g} + \pi^*g_D,
\end{equation}

compare Lemma 4.3. Let us further denote $g_{S^1} := (Jdt)^2$ and rescale $g_t$ by some fixed constant so that the diameter $\text{diam}(S, g_{S^1})$ of each $S^1$-fibre satisfies $\text{diam}(S, g_{S^1}) = 1/4$. It then suffices to bound the volume of unit balls w.r.t. $g_t$ on the set $\{t \gg 1\}$ uniformly from below.

Let $x \in E$ with $t(x) \gg 1$ and denote the unit ball of $g_t$ around $x$ by $B_{g_t}(x, 1)$. We introduce families of metrics on $M$ and $S$ by declaring

\begin{align*}
g_{M, \tau} &:= \tau\hat{g} + p^*g_D \\
g_{S, \tau} &:= g_{S^1} + \pi^*g_{M, \tau}
\end{align*}

for each $\tau \geq 1$, where $p : M \to D$ is the projection as in (26) and $\pi : S \to M$. In particular, the projection $\pi$ becomes a Riemannian submersion $\pi : (S, g_{S, \tau}) \to (M, g_{M, \tau})$. Using this notation and writing $x = (t(x), y)$, we obtain the following inclusion

\begin{equation}
B := [t(x) - 1/2, t(x)] \times B_{g_{S, t(x)}}(y, 1/2) \subset B_{g_t}(x, 1).
\end{equation}
This is an immediate consequence of the decomposition (54) together with the fact that for all $p \in B$ we have $t(p) \leq t(x)$ and $g_{S,\tau_0} \leq g_{S,\tau_1}$ for all $\tau_0 \leq \tau_1$. Before estimating the $g_t$-volume $\text{Vol}_{g_t}(B_{g_t}(x,1))$ of the unit ball $B_{g_t}(x,1)$, we observe that

$$\tag{55} g_{S,t(x)-1/2} = g_{S,t(x)} - \frac{1}{2}\pi^* \hat{g} \geq g_{S,t(x)} - \frac{1}{2}(t(x) - 1)\pi^* \hat{g} \geq \frac{1}{2}g_{S,t(x)}$$

provided $t(x) \geq 2$. Using the inclusion $B \subset B_{g_t}(x,1)$ then implies that

$$\text{Vol}_{g_t}(B_{g_t}(x,1)) \geq \text{Vol}_{g_t}(B) \geq \frac{1}{2} \cdot \text{Vol}_{g_{S,t(x)-1/2}}(B_{g_{S,t(x)}}(y,1/2)) \geq 2 \cdot \frac{\dim S}{2} \cdot \text{Vol}_{g_{S,t(x)}}(B_{g_{S,t(x)}}(y,1/2)),$$

where we applied Fubini’s theorem in the second line, and the last inequality follows from (55). Thus, it remains to bound the $g_{S,t(x)}$-volume of $B_{g_{S,t(x)}}(y,1/2)$ uniformly from below.

We further reduce this volume bound to an integration on $M$ by observing that the projection $\pi : S \to M$ satisfies

$$\tag{56} \pi^{-1}(B_{g_{M,t(x)}}(\pi(y),1/4)) \subset B_{g_{S,t(x)}}(y,1/2)$$

Indeed, given a $b \in B_{g_{M,t(x)}}(\pi(y),1/4)$ and a length-minimizing curve $q : [0,1] \to M$ from $\pi(y)$ to $b$, we may lift $q$ to a horizontal curve $\tilde{q}$ in $S$ from $\tilde{q}(0) = y$ to some point $\tilde{q}(1) \in \pi^{-1}(b)$. For any $a \in \pi^{-1}(b)$, the triangle inequality for the distance function $\text{dist}_{g_{S,t(x)}}$ then yields

$$\text{dist}_{g_{S,t(x)}}(y,a) \leq \text{dist}_{g_{S,t(x)}}(y,\tilde{q}(1)) + \text{dist}_{g_{S,t(x)}}(\tilde{q}(1),a) \leq \text{dist}_{g_{M,t(x)}}(\pi(y),b) + \frac{1}{4} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

where the second inequality holds since we normalised each fibre $\pi^{-1}(b)$ to be of diameter 1/4 and the third one follows since $\pi : S \to M$ is a Riemannian submersion. Hence, we conclude that $a \in B_{g_{S,t(x)}}(y,1/2)$ as claimed.

Inclusion (56) yields an estimate on the $g_{S,t(x)}$-volume as follows. We write $\omega_{t(x)}$ for the Kähler form of $g_{M,t(x)}$ and $\chi$ for the characteristic function of the ball $B_{g_{S,t(x)}}(y,1/2)$, and then observe that

$$\int_{B_{g_{S,t(x)}}(y,1/2)} (Jdt) \wedge \pi^* \omega_{t(x)}^{\dim_{CM}} = \int_{B_{g_{M,t(x)}}(\pi(y),1/2)} \pi_* (\pi^* \chi Jdt) \cdot \omega_{t(x)}^{\dim_{CM}} \geq \int_{B_{g_{M,t(x)}}(\pi(y),1/4)} \pi_* (Jdt) \cdot \omega_{t(x)}^{\dim_{CM}} = \text{Vol}_{g_{S,t(x)}}(S^1) \cdot \int_{B_{g_{M,t(x)}}(\pi(y),1/2)} \omega_{t(x)}^{\dim_{CM}}.$$
Here, \( \pi_*(\chi Jdt) \) denotes the function on \( M \) obtained by integrating \( \chi Jdt \) over fibres, i.e. \( \pi_*(\chi Jdt)(b) = \int_{\pi^{-1}(b)} \chi Jdt \), so that the first equality follows from Fubini’s theorem. In the second line, we used that \( \chi \equiv 1 \) on the set \( \pi^{-1}(B_{g_{M,t}(x)}(\pi(y), 1/4)) \) by (56) and the final equation holds because the volume of each \( S^1 \)-fibre is the same by (54). Thus, it remains to find a constant \( C > 0 \), independent of \( x = (t(x), y) \), such that

\[
\text{Vol}_{g_{M,t}(x)}(B_{g_{M,t}(x)}(\pi(y), 1/4)) \geq C^{-1} > 0. \tag{57}
\]

To prove this, let us first assume that \( \hat{g} \) is positive definite. Then we can find a constant \( C_0 > 0 \), only depending on the eigenvalues of \( \hat{g} \) w.r.t. \( g_M \), such that

\[
C_0^{-1} \tau g_M \leq g_{M,\tau} \leq C_0 \tau g_M,
\]

for all \( \tau \geq 1 \) and with \( g_M := g_{M,1} \). Since we can rescale by a fixed constant, it suffices to bound the \( \tau g_M \)-volume of \( B_{\tau g_M}(z, 1) \) from below by a constant independent of both \( z \in M \) and \( \tau \gg 1 \). We note that

\[
B_{\tau g_M}(z, 1) = B_{g_M}(z, \tau^{-\frac{1}{2}})
\]

and by compactness, the \( g_M \)-volume \( \text{Vol}_{g_M}(B_{g_M}(z, \tau^{-\frac{1}{2}})) \) is, up to some uniform constant, bounded from below by \( \tau^{-\dim g_M} \) for \( \tau \) sufficiently large. This shows that \( \text{Vol}_{\tau g_M}(B_{\tau g_M}(z, 1)) = \tau^{-\dim g_M} \text{Vol}_{g_M}(B_{g_M}(z, \tau^{-\frac{1}{2}})) \) is indeed uniformly bounded from below and (57) then follows.

Let us now assume that \( \gamma \) has at least one zero eigenvalue w.r.t. \( g_M \). Since these eigenvalues are assumed to be constant over \( M \), its Kernel \( \text{Ker} \gamma \) defines a proper subbundle of \( T^{1,0}M \). Moreover, \( \gamma \) is closed, so that \( \text{Ker} \gamma \) is integrable according to Frobenius’ theorem. Thus, if \( n \) is the complex dimension of \( M \) and \( k \) the number of positive eigenvalues of \( \gamma \), we find a chart around each point defined on some neighborhood of the Euclidean unit ball \( B(1) \subset \mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k} \) around the origin such that each slice \( \{z_0\} \times \mathbb{C}^{n-k} \) in \( B(1) \) is an integral manifold for \( \text{Ker} \gamma \), i.e.

\[
\gamma(v, v) > 0 \quad \text{and} \quad \gamma(v, w) = 0 \quad \text{for all} \quad v \in T\mathbb{C}^k, w \in T\mathbb{C}^{n-k}.
\]

By compactness, we can cover \( M \) by finitely many of such Euclidean balls \( B_j(1) \) for \( j = 1, \ldots, N \) and also find a constant \( C_0 > 0 \) such that

\[
C_0^{-1} g_{\mathbb{C}^n} \leq g_M \leq C_0 g_{\mathbb{C}^n} \quad \text{on each} \quad B_j(1),
\]

where \( g_{\mathbb{C}^n} \) is the Euclidean metric on \( B_j(1) \) and the constant \( C_0 \) is independent of the ball \( B_j(1) \).

For \( \tau \geq 1 \), we also consider the following product metric

\[
g_{\mathbb{C}^n, \tau} := (1 + \tau) g_{\mathbb{C}^k} + g_{\mathbb{C}^{n-k}} \quad \text{on} \quad \mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}
\]
Then there exists a uniform constant $C > 0$, which only depends on $C_0$ and the $g_M$-eigenvalues of $\gamma$, such that
\begin{equation}
C^{-1}g_{\mathbb{C}^n,\tau} \leq g_{M,\tau} \leq C g_{\mathbb{C}^n,\tau} \quad \text{on each } B_j(1).
\end{equation}
Let $\varepsilon > 0$ be the Lebesgue number associated to the cover $\{B_j(1)\}_{j=1}^N$ of the manifold $(M, g_M)$, i.e. the ball $B_{g_M}(z, \varepsilon)$ is contained in $B_j(1)$ for some $j$. Note that since $g_M \leq g_{M,\tau}$, we also have $B_{g_M}(z, 1) \subset B_{g_M}(z, 1) \subset B_j(1)$. Additionally, we may assume that $\varepsilon < 1$, so that it suffices to bound the $g_{M,\tau}$-volume of the smaller ball $B_{g_M}(z, \varepsilon)$ from below because the constant $\varepsilon > 0$ is independent of both $z \in M$ and $\tau \geq 1$.

This, in turn, can be reduced to bounding the $g_{\mathbb{C}^n,\tau}$-volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Indeed, this is a direct consequence of (58), which implies that the volume forms of $g_{\mathbb{C}^n,\tau}$ and $g_{M,\tau}$ are uniformly equivalent on $B_{g_{M,\tau}}(z, \varepsilon)$ and also that we have the inclusion
$$B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon) \subset B_{g_{M,\tau}}(z, \varepsilon).$$

For the remaining lower volume bound, observe that the following product of Euclidean balls
\begin{equation}
B_{g_{\mathbb{C}^n}} \left( z, \frac{1}{2}C^{-\frac{1}{2}}\varepsilon(1 + \tau)^{-\frac{1}{2}} \right) \times B_{g_{\mathbb{C}^n}} \left( z, \frac{1}{2}C^{-\frac{1}{2}}\varepsilon \right)
\end{equation}
is contained in $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Applying Fubinis’ theorem to the product (59) and using the fact that the volume form of $g_{\mathbb{C}^n,\tau}$ is equal to $(1 + \tau)^k$-times the volume form of $g_{\mathbb{C}^n}$ then yields the required lower bound on the $g_{\mathbb{C}^n,\tau}$-volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$, which is independent of both $z \in M$ and $\tau \geq 1$. This finishes the proof.

\begin{flushright}
\Box
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5. Uniqueness in a Kähler class

The purpose of this section is to prove Theorem 1.3. We begin by briefly recalling notation from Sections 2 and 3 and then define the function spaces appearing in Theorem 1.3. We also explain how to reduce the proof to a $\partial \bar{\partial}$-Lemma, which is stated below (Theorem 5.4).

5.1. A $\partial \bar{\partial}$-Lemma. Throughout this section, let $\pi : E \to D$ be a rank $m$ holomorphic vector bundle over a compact Kähler manifold $(D, \omega_D)$. The complex dimension of $E$ (as a manifold) is denoted by $m+d$, where $d$ is the complex dimension of $D$. If $m=1$, we assume that it satisfies the conditions in Theorem 1.1 and if $m \geq 2$, we assume $E$ is given as in Theorem 1.2. Also recall that we defined a radial function $r : E \to \mathbb{R}_{\geq 0}$ by $r(v) = \sqrt{h(v, v)}$, which vanishes along the zero section of $E$ and we set $t := 2 \log r$. Note that we can use the function $t$ to identify $E$, with its zero section removed, as the product $\mathbb{R} \times S$, where $S$ is the $S^1$-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see Diagram (26). Under
this identification, the function \( t \) on \( E \setminus D \) corresponds to the projection onto the first factor of \( \mathbb{R} \times S \).

Let \( \omega_\varphi \) be the Kähler Ricci soliton constructed in Theorem \ref{thm:existence_kahler_ricci_soliton} or \ref{thm:existence_kahler_ricci_soliton_2}, i.e. \( \omega_\varphi \) is defined by \((9)\) with \( \varphi \) satisfying \((18)\) if \( E \) is a line bundle or by \((27)\) and \((35)\) if \( m \geq 2 \). We denote the corresponding Riemannian metric by \( g_\varphi \).

On the manifold \( \mathbb{R} \times S \), we can write the metric \( g_\varphi \) as follows. If \( J \) denotes the complex structure on \( E \) and \( g_D \) and \( \hat{g} \) are the \((2,0)\) tensors associated to \( \omega_D \) and \( -\gamma \), respectively, then
\[
g_\varphi = f''(t) \left( dt^2 + (Jdt)^2 \right) + f'(t) \pi^* \hat{g} + \pi^* g_D,
\]
where \( f \) can be reconstructed from \( \varphi \) via \((10)\). We would also like to point out that we allowed \( -\gamma \) to have zero-eigenvalues, i.e. \( \hat{g} \) is only semidefinite. As a consequence, the volume growth of \( g_\varphi \) will be determined by the zero-eigenvalues of \( -\gamma \).

Before stating the main theorem of this section, we require a definition of weighted function spaces. As a weight function, we choose \( w: E \to \mathbb{R}^+ \) to be defined by
\[
w(t) := 1 + f'(t).
\]
This choice is inspired by the work of Hein \cite{Hei11}. Indeed, the following lemma shows that \( w \) has the same properties as the function \( \rho \) in \cite{Hei11}[Theorem 1.6].

**Lemma 5.1.** Fix \( x_0 \in E \) and denote the distance function of \( g_\varphi \) by \( \rho(x) \). Then there exists a constant \( C > 0 \) such that
\[
C^{-1} w(t(x)) \leq (1 + \rho(x)) \leq C w(t(x))
\]
for all \( x \in E \) with \( w(t(x)) \geq C \). Moreover, \( w \) satisfies
\[
|\nabla w| + w |\Delta w| \leq C,
\]
where \( |\cdot|, \nabla \) and \( \Delta \) are associated with \( g_\varphi \).

**Proof.** We identify \( E \setminus D \cong \mathbb{R} \times S \) and without loss of generality, we can assume \( x_0 = (t_0, y_0) \in \mathbb{R} \times S \). Let \( (t, y) \in \mathbb{R} \times S \) with \( t_0 \leq t \) and consider a shortest path \( q_{t,y} = (q_t, q_y) : [0, 1] \to \mathbb{R} \times S \) from \( (t_0, y_0) \) to \( (t, y) \). Its length \( L(q_{t,y}) \) is given by
\[
L(q_{t,y}) = \int_0^1 \sqrt{g_\varphi (\dot{q}_{t,y}(\sigma), \dot{q}_{t,y}(\sigma))} d\sigma.
\]
Then \((62)\) reduces to finding a constant \( C > 0 \) such that
\[
C^{-1} w(t) \leq L(q_{t,y}) \leq C w(t)
\]
for all \( y \in S \) and all \( t \geq C \). In fact, it is sufficient to show inequality \((64)\) with \( w(t) \) replaced by \( t \) since there is a \( C > 0 \) such that
\[
C^{-1} t \leq w(t) \leq C t
\]
for \( t \geq C \), compare (46). Thus, we begin by choosing \( C > 0 \) such that (65) holds, and we increase \( C > 0 \) as we go along, if necessary.

For proving the lower bound in (64), we estimate

\[
L(q_t,y) \geq \int_0^1 \sqrt{f''(q_t,y)\dot{q}_t(\sigma)} d\sigma \geq \sqrt{f''(t_0)}(t-t_0),
\]

as required. Before showing the upper bound, we conclude from (46) that

\[
g_\varphi \leq C \left(dt^2 + tg_S\right),
\]

where we define \( g_S := (Jdt)^2 + \pi^*\hat{g} + \pi^*g_D \) with \( \hat{g} \) and \( g_D \) as in (60). Also observe that we can now assume \( q_t \) to be the linear path in the \( \mathbb{R} \)-factor, i.e. \( q_t(\sigma) = \sigma(t-t_0) + t_0 \). Then we obtain

\[
L(q_t,y) \leq C \int_0^1 \dot{q}_t(\sigma)d\sigma + C \int_0^1 \sqrt{\dot{q}_t(\sigma)} \cdot \sqrt{g_S(\dot{q}_y(\sigma), \dot{q}_y(\sigma))} d\sigma
\]

\[
\leq Ct + C \text{diam}(S,g_S)\sqrt{t}
\]

(66)

\[
\leq Ct,
\]

for all \( t \) sufficiently large and with \( \text{diam}(S,g_S) \) denoting the diameter of the compact manifold \( (S,g_S) \). Now (5.1) follows immediately.

For the second claim, observe from (27) that we have

\[
|\nabla w| = f''
\]

which is uniformly bounded according to Lemma 4.3. For bounding the Laplace operator \( \Delta w = \Delta f' \), recall that on a Kähler manifold, the Laplace operator \( \Delta \) satisfies \( \Delta = 2 \text{tr}_{\omega_\varphi}(-1\partial \bar{\partial}) \), where \( \text{tr}_{\omega_\varphi} \) denotes the trace computed w.r.t. \( g_{\varphi} \). Then we apply (33) to obtain

\[
\Delta f' = 2 \text{tr}_{\omega_\varphi} \left(\sqrt{-1\partial \bar{\partial}} f'\right)
\]

\[
= \frac{2}{\mu} \text{tr}_{\omega_\varphi} \left(L_X \omega_\varphi\right)
\]

\[
= 2 \left(\varphi' Q' - \varphi'\right),
\]

where the last equality holds since there is the following formula

\[
\text{tr}_{\omega_\varphi}(-\pi^*\gamma) = \frac{Q'}{Q},
\]

see [HS02]((2.22)). Using the soliton ODE (35), we continue

\[
w \Delta w = (1 + f') \Delta f'
\]

\[
= 2(1 + \tau) \left(\varphi' Q' + \varphi'\right)
\]

\[
= 2(1 + \tau) \left(m + \mu \varphi\right),
\]

which is also uniformly bounded because of the asymptotic expansion (10). This, together with the uniform bound on \( |\nabla w| \), implies (63). \( \square \)
Lemma 5.1 ensures that our definition of weighted function spaces below coincides with the one used in [Hei11]. These spaces are well-adapted to study the Laplace operator on a wide class of complete manifolds.

**Definition 5.2.** Let \( \Lambda^* T^* E \) be the exterior algebra of \( T^* E \) and consider \( \delta \in \mathbb{R} \) and \( k \in \mathbb{N}_0 \). We define \( C^k_\delta (\Lambda^* T^* E) \) to be the space of \( k \)-times continuously differentiable sections \( \eta \) of \( \Lambda^* T^* E \) such that the norm

\[
||\eta||_{C^k_\delta} := \sum_{j=0}^{k} \sup_E |w^j \nabla^j \eta|
\]

is finite, where \( w \) is given by \( (61) \) and \( \nabla, |\cdot| \) are associated to \( g_\varphi \). We also set

\[
C^\infty_\delta (\Lambda^* T^* E) := \bigcap_{k \in \mathbb{N}_0} C^k_\delta (\Lambda^* T^* E).
\]

In other words, elements in \( C^\infty_\delta (\Lambda^* T^* E) \) grow at most like \( w^\delta \) and their \( l \)-th derivatives at most like \( w^{\delta-l} \). Having introduced the necessary notation, we can now state the main result of this section.

**Theorem 5.3.** Let \( \omega_\varphi \) be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2. Assume that \( \omega \) is a Kähler-Ricci soliton on \( E \) with the same vector field as \( \omega_\varphi \) such that \( [\omega] = [\omega_\varphi] \in H^2(\mathbb{E}) \). If moreover \( \omega_\varphi - \omega \in C^\infty_\delta (\Lambda^2 T^* E) \) for some \( \delta > 2 \), then \( \omega_\varphi = \omega \).

The main part of proving Theorem 5.3 will be a \( \partial \bar{\partial} \)-Lemma, with controlled growth. In fact, we will prove

**Theorem 5.4.** Let \( \delta > 2 \) and \( \eta \in C^\infty_\delta (\Lambda^* T^* E) \) be a real \((1,1)\) form. If \( \eta \) is \( d \)-exact, then \( \eta = \sqrt{-1} \partial \bar{\partial} u \) for some \( u \in C^\infty_\delta (\mathbb{E}) \).

Assuming this result, Theorem 5.3 follows immediately.

**Proof of Theorem 5.3.** By Theorem 5.4 there exists a \( u \in C^\infty_\delta (\mathbb{E}) \) such that \( \omega_\varphi - \omega = \sqrt{-1} \partial \bar{\partial} u \). Since \( 2 - \delta < 0 \), \( u \) and all its derivatives tend to zero at infinity, so we can apply the maximum principle [BM17][Proposition 1.2] and conclude that \( \omega_\varphi = \omega \). \( \square \)

The remainder of this section is devoted to proving Theorem 5.4. We follow the ideas for asymptotically conical metrics given in [CH13][Section 3], which rely on two main ingredients. Firstly, we need to understand solutions to Poisson’s equation \( \Delta u = h \) and their growth behaviour (Section 5.2). Secondly, we need to show that harmonic \((1,0)\) forms of certain growth behaviour are identically zero (Section 5.3). The proof of Theorem 5.4 will then be finished in Section 5.4.
5.2. The Laplace Operator. We start by considering the Laplace operator $\Delta$ of the metric $g_\phi$ acting on suitably weighted Hölder spaces, which we now define.

**Definition 5.5.** Let $\text{dist}(x, y)$ be the distance between $x, y \in E$ measured w.r.t. $g_\phi$ and denote the injectivity radius of $g_\phi$ by $i_0$. (Note that $i_0 > 0$ by Lemma 4.4). For $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, we define a seminorm on the space of all tensor fields $T$ on $E$ by

$$[T]_{C^{0,\alpha}_\delta} := \sup_{x \neq y \in E, \text{dist}(x,y) < \frac{i_0}{2}} \left( \min(w(x), w(y))^{-\delta} \frac{|T_x - T_y|}{\text{dist}(x,y)^{\alpha}} \right),$$

where the norm $| \cdot |$ is induced by $g_\phi$ and the difference $T_x - T_y$ is defined by using parallel transport along the minimal geodesic from $x$ to $y$.

The weighted Hölder space $C^{k,\alpha}_\delta(E)$ is then defined to be the subset of all $u \in C^k_\delta(E)$ for which the norm

$$||u||_{C^{k,\alpha}_\delta} := ||u||_{C^k_\delta} + [\nabla^k u]_{C^{0,\alpha}_{\delta-k-\alpha}}$$

is finite.

The Laplace operator $\Delta$ acts as

$$\Delta : C^{2,\alpha}_{2+\delta}(E) \to C^{0,\alpha}_\delta(E),$$

for any $\delta \in \mathbb{R}$ and we are interested in the surjectivity of this operator. A partial answer to this question is provided in [Hei11].

Given $h \in C^{0,\alpha}_\delta(E)$ with $\delta < -2$, we can essentially always solve Poisson’s equation $\Delta u = h$, but it is not clear how the solution $u$ will behave as $t \to \infty$. This depends on the volume growth of $g_\phi$, which is related to the degree $k$ of the polynomial $Q$ defined in (11) for $m = 1$ or (31) for $m \geq 2$. Alternatively, it is evident from the definition of $Q$ that $k$ is equal to $m + d - 1$ minus the number of zero-eigenvalues of $\gamma$. (Recall that $m + d - 1$ is the complex dimension of $\mathbb{P}(E)$.)

More precisely, we will prove the following important proposition about the existence of solutions to $\Delta u = h$.

**Proposition 5.6.** Let $\delta > 2$ and suppose $h \in C^{0,\alpha}_{-\delta}(E)$.

(i) If $k \leq 1$, assume $\int h \omega_\phi^{m+d} = 0$ additionally. Then there exists a $u \in C^{2,\alpha}(E)$ such that $\Delta u = f$ and the integral $\int |\nabla u|^2 \omega_\phi^{m+d}$ is finite.

(ii) If $k > 1$ and $2 < \delta < k + 1$, then there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\epsilon})$ as well as $|\nabla u| = O(w^{2-\delta+\epsilon})$ for all $\epsilon > 0$.

Before proceeding with its proof, we first of all need to check that we can indeed apply Hein’s work [Hei11][Theorem 1.5, 1.6], i.e. we have to verify that the metric $(E, g_\phi)$ satisfies Hein’s condition $\text{SOB}(\beta)$. For the sake of completeness, we recall [Hei11][Definition 1.1] here.
**Definition 5.7** ([Hei11][Definition 1.1]). A Riemannian manifold \((M, g)\) is called SOB(\(\beta\)) if there exists a \(x_0 \in M\) and a constant \(C \geq 1\) satisfying the following:

(i) The set \(B(x_0, s_1) \setminus \overline{B}(x_0, s_0)\) is connected for all \(s_1 > s_0 \geq C\),
(ii) \(\text{Vol}(B(x_0, s)) \leq C s^3\) holds for all \(s \geq C\),
(iii) \(\text{Vol}(B(x, (1 - C^{-1})\rho(x))) \geq C^{-1} \rho(x)^3\) holds for all \(x \in M\) with \(\rho(x) \geq C\),
(iv) \(\text{Ric}_x \geq -C \rho(x)^{-2}\) holds if \(\rho(x) \geq C\).

Here \(B(x_0, s)\) denotes the geodesic ball around \(x_0\), \(\text{Vol}(B(x_0, s))\) its volume and \(\rho(x)\) denotes the distance from \(x\) to \(x_0\).

As the next lemma shows, the soliton metrics \((E, g_\varphi)\) constructed in Theorem 1.1 and 1.2 are SOB\((k + 1)\).

**Lemma 5.8.** The metric \((E, g_\varphi)\) is SOB\((k + 1)\), where \(k\) is equal to \(m + d - 1\) minus the number of zero-eigenvalues of the curvature form \(\gamma\) on \(\mathbb{P}(E)\).

**Proof.** We fix \(x_0 \in D \subset E\) to be a point on the zero-section of \(E\). Thanks to Theorem 4.1, Condition (iv) in Definition 5.7 is clearly satisfied, so we focus on (i), (ii), (iii).

Beginning with the volume estimates (ii) and (iii), we consider the volume form of \(g_\varphi\), which is given by

\[
\frac{\omega_\varphi^{m+d}}{(m+d)!} = \frac{\sqrt{-1}}{(m+d)!} f^{m} Q(f') \partial t \wedge \bar{\partial} t \wedge (\pi^* \omega_D - \pi^* \gamma)^{m+d-1},
\]

where the polynomial \(Q\) is defined by (11) if \(m = 1\) or (29) if \(m \geq 2\). Recall from above, that the degree of \(Q\) is equal to \(k\) as defined in Lemma 5.8. If we then choose \(C \geq 1\) such that (46) is satisfied, we obtain for large \(t \geq C:\)

\[
C^{-1} t^k \leq f''(t) Q(f'(t)) \leq C t^k.
\]

Moreover, Lemma 5.1 implies that

\[
C^{-1} t(x) \leq \rho(x) \leq C t(x)
\]

if \(\rho(x) \geq C\). In the estimates that follow, we increase \(C > 0\) if necessary but it still denotes a uniform constant which only depends on the geometry of \((E, g_\varphi)\) and the choice of base point \(x_0\).

For verifying (ii), let \(s \geq C\) and observe that (69) implies

\[
B(x_0, s) \subset B(x_0, C) \cup \{ y \in E \mid 0 \leq t(y) \leq Cs \}.
\]
Integrating over these sets and using (67), we obtain
\[
\text{Vol}(B(x_0, s)) \leq C + C \int_0^s \int_S f^n(t)Q(f'(t))dt \wedge \tilde{\omega} \wedge (\pi^* (\omega_D - \gamma))^{m+d-1}
\]
\[
\leq C + C \int_0^s t^k dt
\]
\[
\leq Cs^{k+1},
\]
where we used (68) in the second line. This proves (ii) of Definition 5.1 with \( \beta = k + 1 \).

For showing (iii), the goal is to choose a new \( C_0 \geq 1 \) such that for all \( x \in E \) with \( \rho(x) \gg 1 \) sufficient large, we have an inclusion of the form (70)
\[
B(x, (1 - C_0^{-1}) \rho(x)) \supset \left\{ t(y) \in \left[ t(x) + 1, t(x) + C_0^{-1} \rho(x) - \sqrt{\rho(x)} \right] \right\}.
\]
Indeed, if (70) holds, we can integrate and use (68) to estimate
\[
\text{Vol}(B(x, (1 - C_0^{-1}) \rho(x))) \geq C_0^{-1} \int_{t(x)+1}^{t(x)+C_0^{-1} \rho(x) - \sqrt{\rho(x)}} \sigma^k d\sigma
\]
\[
\geq C_0^{-1} \rho(x)^{k+1},
\]
which is (iii) with \( \beta = k + 1 \) as required. Hence it remains to check inclusion (70). To see that this is true, we again introduce the metric \( g_S := (Jdt)^2 + \pi^* \tilde{\gamma} + \pi^* g_D \) on the cross-section \( S \) as in the proof of Lemma 5.1 so that
\[
g_S \leq C (dt^2 + tg_S) =: g_t.
\]
To estimate the distance function of \( g_t \) from above, we proceed as in (66). Given \( x, y \in E \) with \( C \leq t(x) \) and \( t(x) \leq t(y) \), we consider a path \( q : [0, 1] \to E \) from \( q(0) = x \) to \( q(1) = y \), which we write as \( q = (q_1, q_2) \) under the identification \( E \setminus D \cong \mathbb{R} \times S \). Furthermore, we assume that \( q_1(\sigma) = \sigma(t(y) - t(x)) + t(x) \) is the linear path from \( t(x) \) to \( t(y) \), so that we estimate using (66)
\[
\text{dist}_{g_t}(x, y) \leq C \int_0^1 \dot{q}_1(\sigma) d\sigma + C \int_0^1 \sqrt{q_1(\sigma)} \sqrt{g_S(\dot{q}_2(\sigma), \dot{q}_2(\sigma))} d\sigma
\]
\[
\leq C(t(t(y) - t(x)) + \text{diam}(S, g_S) \left( \sqrt{t(y) - t(x)} + \sqrt{t(x)} \right)
\]
Together with (69) and (71), this implies
\[
\text{dist}_{g_S}(x, y) \leq C \left( t(y) - t(x) + \sqrt{t(y) - t(x)} + C \sqrt{\rho(x)} \right)
\]
from which we can deduce inclusion (70). Indeed, let \( C > 0 \) satisfy (72) and define a new constant \( C_0 > 0 \) by \( C_0^{-1} = C^{-1}(1 - C^{-1}) \). If we then
assume that \( \rho(x) \gg 1 \) is large enough so that \( C_0^{-1}\rho(x) - \sqrt{\rho(x)} > 1 \), we estimate for all \( y \in E \) with \( t(x) + 1 \leq t(y) \leq t(x) + C_0^{-1}\rho(x) - \sqrt{\rho(x)} \):

\[
\text{dist}_{g_\varphi}(x, y) \leq C(t(y) - t(x)) + C\sqrt{\rho(x)} \\
\leq CC_0^{-1}\rho(x) - C\sqrt{\rho(x)} + C\sqrt{\rho(x)} \\
= (1 - C^{-1})\rho(x).
\]

Here we obtained the first inequality by applying \( t(y) - t(x) \geq 1 \) to (72) and the second inequality makes use of the upper bound on \( t(y) \). This shows inclusion (70) and thus (iii).

It remains to verify Condition (i). By compactness of \( D \), we can choose \( C > 1 \) such that for all \( s \geq C \) the ball \( B(x_0, s) \) contains a tubular neighborhood of the zero section. Given \( x \in E \setminus D \), we denote the complex line thorough \( x \) by \( L_x \cong \mathbb{C} \). We need to understand the shape of the intersection of \( L_x \) with the set \( B(s_0, s_1)(x_0) := B(x_0, s_1) \setminus B(x_0, s_0) \) for all \( s_1 > s_0 \geq C \).

First, we claim that for each \( x \in B(x_0, s) \), the radial path \( g_{rad} \) in \( L_x \) from \( x \) to 0 belongs to \( L_x \). In order to prove this, use the identification \( E \setminus D \cong \mathbb{R} \times S \) and write \( x = (a_1, b) \). Let \( q : [0, 1] \to E \) be a shortest geodesic from \( q(0) = x_0 \) to \( q(1) = (a_1, b) \). On \( E \setminus D \), we decompose \( q = (q_1, q_2) \) and let us assume for the moment that \( q_1(\sigma) \) is increasing in \( \sigma \in [0, 1] \). Given \( a_0 < a_1 \), we then choose a \( \sigma_0 \in (0, 1) \) with \( q_1(\sigma_0) = a_0 \) and reparameterize the path \( q \) by declaring \( q_{\sigma_0}(\sigma) := (q_1(\sigma_0\sigma), q_2(\sigma)) \), so that \( q_{\sigma_0} \) is a path from \( x_0 \) to \((a_0, b) \). It follows from (60) that we have

\[
g_{g_\varphi}(q_{\sigma_0}(\sigma), q_{\sigma_0}(\sigma)) \leq g_{\varphi}(q(\sigma), q(\sigma))
\]

for all \( \sigma \in [0, 1] \), since \( f'' \) and \( f' \) are both increasing and we assumed that \( q_1(\sigma_0\sigma) \leq q_1(\sigma) \). Then we conclude \( L(q_{\sigma_0}) \leq L(q) \) and thus \( \rho(a_0, b) \leq \rho(a_1, b) \) for all \( a_0 < a_1 \) and \( b \in S \), as we claimed.

Hence, the claim holds if we show that \( q_1 \) is increasing. Recall that by definition, \( q_1 = t(q) \) and clearly \( q_1 \) increases if and only if \( r^2(q) = e^{t(q)} \) does, where \( r : E \to \mathbb{R}_{\geq 0} \) is defined at the beginning of Section 3. Since \( x_0 \) lies on the zero section of \( E \), we have \( r(q(0)) = 0 \) and consequently there is a \( \hat{\sigma} \in [0, 1] \) such that

\[
r(q(\sigma)) = 0 \text{ for all } \sigma \in [0, \hat{\sigma}] \text{ and } \frac{d}{d\sigma}r^2(q(\sigma)) > 0 \text{ on } (\hat{\sigma}, \hat{\sigma} + \varepsilon)
\]

for some small \( \varepsilon > 0 \). In particular, \( \lim_{\sigma \to \hat{\sigma}^+} q_1(\sigma) \geq 0 \) and we only have to rule out the existence of two points \( \sigma_1, \sigma_2 \in [\hat{\sigma}, 1] \) with \( \sigma_1 < \sigma_2 \) such that

\[
q_1(\sigma_1) = q_2(\sigma_2) \text{ and } q_1(\sigma_1) < q_1(\sigma) \text{ for all } \sigma \in (\sigma_1, \sigma_2).
\]
However, if this was the case, then the path $q$ cannot be length-
minimizing. Indeed, suppose that there are such numbers $\sigma_1, \sigma_2$ satis-
ifying (73). Then we define a new path $\tilde{q}$ from $x_0$ to $x$ by

$$\tilde{q}(\sigma) = \begin{cases} q(\sigma) & \text{if } \sigma \in [0, 1] \setminus (\sigma_1, \sigma_2) \\ (q_1(\sigma_1), q_2(\sigma)) & \text{if } \sigma \in (\sigma_1, \sigma_2). \end{cases}$$

Using the decomposition (60) and the fact that $f''$ is increasing, we see
that $L(\tilde{q}) < L(q)$, contradicting the minimality of $q$. It follows that $q_1$ must be increasing.

Now we can verify Condition (i), so consider any $s_1 > s_0 \geq C$.
As shown in the previous paragraph, both $L_x \cap B(x_0, s_j)$ with $j = 1, 2$ are star-shaped regions with center $0 \in L_x$, so the complement $L_x \cap B((s_0, s_1))(x_0)$ is diffeomorphic to a genuine open annulus in $C$. From this, we deduce that $B((s_0, s_1))(x_0)$ is a fibre bundle over $\mathbb{P}(E)$ with annuli in $C$ as fibres. In particular, $B((s_0, s_1))(x_0)$ is connected because $D$ is,
finishing the proof.

Before proving Proposition 5.6, we study the spaces $C^{k,\alpha}_\delta(E)$ fur-
ther. In fact, Lemma 5.1 allows us to obtain the expected embedding
theorems and also Schauder estimates for $\Delta$.

**Lemma 5.9** (Embeddings). Let $k, l \in \mathbb{N}$, $0 < \alpha_0, \alpha_1 < 1$ and $\delta_0 \leq \delta_1$. Then there are the following continuous embeddings:

(i) $C^{k}_{\delta_0}(E) \subset C^{l}_{\delta_1}(E)$ if $l \leq k$,

(ii) $C^{k,\alpha_0}_{\delta_0}(E) \subset C^{k,\alpha_1}_{\delta_1}(E)$ if $l \leq k$ and $\alpha_1 \leq \alpha_0$,

(iii) $C^{k+1}_{\delta}(E) \subset C^{k,1}_{\delta}(E)$. In particular, $C^{\infty}_{\delta}(E) = \bigcap_{k \in \mathbb{N}_0} C^{k,\alpha}_{\delta}(E)$.

The proof of this lemma is analogue to [CSCB79][Lemma 2], so we omit it here.

Now we are in a position to show Proposition 5.6

**Proof of Proposition 5.6**. Part (i) is a direct consequence of Theorem 1.5 in [Hei11]. Indeed, $(E, g_\varphi)$ is SOB$(k + 1)$ by Lemma 5.8 and, because of Lemma 5.1 we have that $|h| = O(\rho^{-\delta})$ with $\delta > 2$, where $\rho(x)$ denotes the distance to some fixed point $x_0$. Then (i) is precisely [Hei11][Theorem 1.5].

For (ii), we note that the function $w$ satisfies the assumption of The-
orem 1.6 in [Hei11], see Lemma 5.1. Consequently, [Hei11][Theorem 1.6] gives a $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$.

Then it only remains to verify the decay rate of $|\nabla u|$, which is a consequence of standard Schauder theory. Indeed, since the curvature
of $g_\varphi$ is bounded by Lemma 4.4, we can find $s > 0$ and $Q > 0$ such
that for all $x \in E$, there is a chart $\Phi_x$ from to the Euclidean ball
Moreover, the weight function $x$-norm. For simplicity, we suppress the chart $\Phi$ and all $\Phi^*$ estimates (\cite{GT01}[Theorem 6.2]) to the balls $B_x(s)$, we obtain that

\begin{equation}
|\nabla u|_{g_\varphi}(x) \leq Q|du|_{g_{\text{euc}}}(x)
\end{equation}

for some uniform constant $C_0 > 0$ depending only on $\alpha$, $s$ and $Q$. Moreover, the weight function $w$ is chosen so that there is a uniform constant $C_1 > 0$ such that for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$, we have $\frac{1}{w(y)} \leq w(x) \leq C_1 w(y)$. This follows directly from Lemma \ref{lemma:estimates} and the fact that $g_\varphi$ and $g_{\text{euc}}$ are uniformly equivalent on $B_x(s)$. Therefore, we continue to estimate for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$:

\[
u(y) \leq C w(y)^{2-\delta + \varepsilon} \leq C C_1^{2-\delta + \varepsilon} w(x)^{2-\delta + \varepsilon},
\]

i.e. $||u||_{C^{0,\alpha}(B_x(s))} = O(w(x)^{2-\delta + \varepsilon})$. Similarly, we conclude that

\[||h||_{C^{0,\alpha}(B_x(s))} = O(w(x)^{-\delta})\]

because $s$ is chosen strictly smaller than the injectivity radius and $h \in C^{0,\alpha}(E)$. In combination with (74) we consequently arrive at

|\nabla u|_{g_\varphi} = O(w^{2-\delta + \varepsilon}),

as claimed.

The issue with (ii) of Proposition \ref{proposition:convergence} is that one can in general not conclude $u = O(w^{2-\delta})$ if $u = O(w^{2-\delta + \varepsilon})$ for all $\varepsilon > 0$. For proving Theorem \ref{theorem:existence}, however, we would like to conclude that indeed $u = O(w^{2-\delta})$. The following proposition gives a criterion, when this conclusion is true.

**Lemma 5.10.** Let $\delta > 0$ and suppose that $\xi \in C_{1-\delta}(T^*E)$. If $\xi = du$ for some $u \in C^1(E)$, then there exists a constant function $u_c$ such that $u - u_c \in C_{-\delta}(E)$. If additionally $u \to 0$ as $t \to \infty$, then $u_c \equiv 0$.

**Proof.** This lemma is proven analogously to the corresponding statement for conical metrics \cite{Mar02}[Lemma 5.10]. First observe that we only need to find a constant $u_c$ such that $u - u_c \in C_{-\delta}(E)$ because $\nabla (u - u_c) = du \in C_{0}(T^*E)$ by assumption.

We work on $E \setminus D \cong \mathbb{R} \times S$ and fix a point $(t_0, y_0) \in \mathbb{R} \times S$. Viewing $S$ as the slice $\{0\} \times S$, we endow $S$ with a metric $g_S$ by restricting $g_\varphi$ to $S$. For a different point $(t, y)$, we let $q_{t_0,t}$ be the straight line path from $(t_0, y_0)$ to $(t, y)$ and $q_{y_0,y}$ be a path joining the points $(t, y_0)$ and $(t, y)$,
so that its projection onto $S$ is a length minimizing geodesic. Then we have by Stoke’s theorem

\[ u(t, y) - u(t_0, y_0) = \int_{q_{t_0, t}} \xi + \int_{q_{t_0, y}} \xi. \]  

(75)

As in the proof of [Mar02] [Lemma 5.10 (c)], the key is to notice that the integral of $\xi$ along the path $q_{t_0, \infty}$ is finite, where $q_{t_0, \infty}$ is the linear path from $(t_0, y_0)$ to $(+\infty, y_0)$. Indeed, since $\xi \in C_{-1-\delta}(T^*E)$ and $\delta > 0$, we can estimate

\[ \int_{q_{t_0, \infty}} \xi \leq \int_{t_0}^{\infty} |\xi(q_{t_0, \infty})| \, ds \]

\[ \leq ||\xi||_{C_{-1-\delta}} \int_{t_0}^{\infty} f'' w^{-1-\delta} \, ds \]

\[ \leq ||\xi||_{C_{-1-\delta}} w^{-\delta}(t_0) < +\infty, \]

(76)

Splitting the integral $\int_{q_{t_0, \infty}} \xi$ into two parts, we can rewrite (75) as follows:

\[ u(t, y) - u(t_0, y_0) - \int_{q_{t_0, \infty}} \xi = -\int_{q_{t, \infty}} \xi + \int_{q_{t_0, y}} \xi. \]

(77)

As in (76), it is easy to see that the right hand side of (77) is bounded by $w^{-\delta}(t)$. In fact, we have

\[ \int_{q_{t_0, y}} \xi \leq \int_{a}^{b} |\xi| \, \hat{q}_{y_0, y} \, \hat{\varphi} \, ds \]

\[ \leq C ||\xi||_{C_{-1-\delta}} w^{-1-\delta}(t) \sqrt{f(t)} \int_{a}^{b} |\hat{q}_{y_0, y}| \, ds \]

\[ \leq C ||\xi||_{C_{-1-\delta}} w^{-\delta}(t) \text{diam}(S, g_S), \]

(78)

where $q_{y_0, y}$ is defined on the interval $[a, b]$ and $C > 0$ is some constant independent of $t$. Combining with (76), we obtain

\[ |u(t, y) - u(t_0, y_0) - \int_{q_{t_0, \infty}} \xi| \leq ||\xi||_{C_{-1-\delta}} \left( (\delta^{-1} + C \text{diam}(S, g_S)) \right) w^{-\delta}(t), \]

i.e. $u - u_c \in C_{0-\delta}(E)$ where we set

\[ u_c = u(t_0, y_0) + \int_{q_{t_0, \infty}} \xi \]

\[ = u(t_0, y_0) + \lim_{t \to \infty} (u(t, y_0) - u(t_0, y_0)) \]

\[ = \lim_{t \to \infty} u(t, y_0). \]
Thus, it remains to show that \( u_c \) is indeed constant. Let \( q_{y_0, y_1} \) be a path in the slice \( \{ t \} \times S \) connecting two points \( (t, y_0) \) and \( (t, y_1) \). Then we have
\[
  u(t, y_1) - u(t, y_0) = \int_{q_{y_0, y_1}} \xi,
\]
and by (78), the right hand side of (79) goes to 0 as \( t \to \infty \). Hence
\[
  \lim_{t \to \infty} u(t, y_0) = \lim_{t \to \infty} u(t, y_1)
\]
for any \( y_0, y_1 \in S \), proving the lemma.

5.3. Vanishing of harmonic forms. We aim at proving a vanishing theorem for harmonic \((1,0)\)-forms on the manifold \((E, g_\phi)\). This will be needed for the \( \partial \bar{\partial} \)-Lemma. We start with a basic observation which is immediate from the standard Bochner formula.

**Lemma 5.11.** Any harmonic 1-form \( \beta \) on \((E, g_\phi)\) such that \( |\beta| \to 0 \) as \( t \to \infty \) must vanish identically.

**Proof.** Since \( \text{Ric}(\omega_\phi) \) is non-negative by Theorem 4.1, the Bochner formula reads
\[
  \Delta |\beta|^2 \geq 0,
\]
and the claim then follows from the Maximum principle.

It becomes more interesting if we replace the asymptotic condition of \( \beta \) in the previous lemma by requiring that \( \beta \) be square-integrable instead. If \( \beta \) is moreover of type \((1,0)\), it is also holomorphic and it must be zero by the following Theorem.

**Theorem 5.12.** Any \( L^2 \)-holomorphic \((1,0)\)-form on \((E, g_\phi)\) is identically zero.

**Proof.** We adapt the idea behind [MW15][Theorem 7]. Let \( \beta \) be a holomorphic \((1,0)\)-form, which is square integrable w.r.t. the metric \( g_\phi \). Then \( \bar{\partial} \beta = \bar{\partial}^* \beta = 0 \), and by the Kähler identities \( \Delta_d \beta = 0 \), i.e. \( \beta \) is harmonic. Since every \( L^2 \)-harmonic form on a complete manifold is closed and coclosed, we conclude \( d\beta = d^* \beta = 0 \). Observe that \( \beta \) and \( \pi^* j^* \beta \) are in the same de-Rham cohomology class, where \( \pi : E \to D \) is the projection and \( j : D \to E \) is the inclusion of \( D \) as the zero section. Hence \( \beta = \pi^* j^* \beta + \partial h \) for some function \( h \). It follows immediately that \( \partial \bar{\partial} h = 0 \).

For some \( \varepsilon > 0 \), consider the tube \( D_\varepsilon = \{ z \in E \mid r(z) \leq \varepsilon \} \) around the zero section. Then by Stoke’s theorem, there is the following formula
\[
  \int_{D_\varepsilon} |\partial h|^2 = - \int_{D_\varepsilon} \langle h, \partial^* \partial h \rangle + \int_{\partial D_\varepsilon} h \nu(\partial h).
\]
Here, \( \nu := \frac{X}{|X|} \) denotes the outward pointing unit normal vector to \( \partial D_\varepsilon \). As \( X \) is a real holomorphic vector field, the function \( \iota_X(\partial h) \) is also
holomorphic and we claim that it is in $L^2$. Indeed, using $\iota_X(\pi^*j^* \beta) = 0$, we observe that

$$|\iota_X(\partial h)| = |\iota_X(\beta)| \leq |X| \cdot |\beta|$$

so that $\iota_X(\partial h)$ is square-integrable since $X$ is bounded and $\beta$ is $L^2$. Hence, $\iota_X(\partial h)$ is an $L^2$-holomorphic function and must consequently be zero. Moreover, $2\iota_X(\partial h) = \Delta h = 0$ because $h$ is pluriharmonic. Thus, $\partial h$ vanishes identically on $D_x$ by (80). So $\partial h$ must be zero everywhere since it is a holomorphic $(1,0)$-form.

We conclude that $\beta = \pi^*j^* \beta$. However, a form pulled back from the base can never be in $L^2$, unless it vanishes identically. Indeed, let $\alpha$ be a 1-form on $D$ which is non-zero at some point $p$. Keeping the expression (60) in mind, we can always estimate in a neighborhood around $p$

$$\langle \pi^*\alpha, \pi^*\alpha \rangle \geq C w^{-1} > 0$$

for some $C > 0$ independent of $t$. It follows that $\int_E |\pi^*\alpha|^2 = +\infty$ since $w^{-1}$ is not integrable. This finishes the proof. □

5.4. The $\partial\bar{\partial}$-Lemma. In this paragraph, we prove Theorem 5.3 on the manifold $E$ analogue to [CH13][Theorem 3.11].

The first step is to find a primitive of $\eta$, with controlled growth. In fact, one can write down an explicit primitive for $\eta$ on the product $E \setminus D \cong \mathbb{R} \times S$ and then read off its growth behaviour. This is the idea behind the next proposition.

**Proposition 5.13.** Let $\delta > 2$ and $\eta \in C^\infty(\Lambda^2 T^*E)$ be a $d$-exact 2-form. Then $\eta = d\theta$ for some $\theta \in C^\infty_{\delta+1}(T^*E)$.

**Proof.** As in [CH13][Theorem 3.11], we first reduce the problem to finding a primitive for $\eta$ on the product $\mathbb{R} \times S$. By assumption, there exists a 1-form $\xi$ such that $\eta = d\xi$. Let $t_1 < t_2$ and define two compact sets $K_j$ with $j = 1, 2$

$$K_j = \{ z \in E \mid t(z) \leq t_j \},$$

where we view the zero section of $E$ to be the set $\{ z \in E \mid t(z) = -\infty \}$. We pick a cut-off function $\chi$ so that $\chi \equiv 0$ on $K_1$ and $\chi \equiv 1$ on the complement of $K_2$. Then we put $\hat{\xi} := \chi \xi$ and $\hat{\eta} := d\hat{\xi}$. Note that if $\hat{\eta} = d\hat{\theta}$ for some $\hat{\theta}$, then $\theta := \xi - \hat{\xi} + \hat{\theta}$ satisfies

$$d\theta = d\xi - d(\chi \xi) + \hat{\eta} = \eta.$$

Since $\theta = \hat{\theta}$ outside $K_2$, it suffices to find $\hat{\theta} \in C^\infty_{\delta-4}(\Lambda^*E)$ with $\hat{\eta} = d\hat{\theta}$ and $\hat{\theta} \equiv 0$ on $K_1$. The following construction of $\hat{\theta}$ can be found in the proof of [Mar02][Proposition 5.8].

For each $t \in \mathbb{R}$, there is an inclusion $i_t :\{t\} \times S \to \mathbb{R} \times S$ given by $i_t(y) = (t, y)$. Write $\hat{\eta} = dt \wedge \hat{\eta}_1 + \hat{\eta}_2$, where $\hat{\eta}_j$ are 1-parameter families
We have to prove that \( \hat{\theta} \) is well-defined, i.e. that the integral (82) exists. We start by looking at \( |\hat{\eta}_1| \). As \( dt \) and \( \hat{\eta}_1 \) are orthogonal to each other, we have that

\[
|dt \wedge \hat{\eta}_1| = |dt||\hat{\eta}_1| = \frac{1}{\sqrt{f''(t)}}|\hat{\eta}_1|.
\]

Using that \( dt \wedge \hat{\eta}_1 \) is orthogonal to \( \hat{\eta}_2 \), we can estimate

\[
|\hat{\eta}_1| = \sqrt{f''(t)}|dt \wedge \hat{\eta}_1| \leq \sqrt{f''(t)}|dt \wedge \hat{\eta}_1 + \hat{\eta}_2| = O(w^{-\delta}),
\]

since \( f'' \) is bounded and \( |\hat{\eta}_1| = O(w^{-\delta}) \) by assumption. To compute the integral (82), we work in coordinates. Let \((y_0 = t, y_1, \ldots, y_{2(m+d)-1})\) be real coordinates of \( \mathbb{R} \times S \) and write \( \hat{\eta}_1 = \sum_{j \geq 1} \hat{\eta}_{1,j} dy_j \). Then (82) becomes

\[
\hat{\theta}_t = -\sum_{j \geq 1} \left( \int_t^\infty i_s^* \hat{\eta}_{1,j} ds \right) dy_j.
\]

Note that the norms \( |dy_j| \) may not have the same asymptotic behaviour for different values of \( j = 1, \ldots, m + d - 1 \). In fact, it follows from (60) that we have

\[
|dy_j| = \begin{cases} O(w^{-\frac{j}{2}}) & \text{if } \pi^* \hat{g}_{jj} > 0, \\ O(1) & \text{if } \pi^* \hat{g}_{jj} = 0, \end{cases}
\]

As \( |\hat{\eta}_1| = O(w^{-\delta}) \), we conclude that either \( |\hat{\eta}_{1,j}| = O(w^{-\delta + \frac{j}{2}}) \) or \( |\hat{\eta}_{1,j}| = O(w^{-\delta}) \) and hence, the integrals in (85) are all finite because we chose \( -\delta + 1 < -1 \).

We also observe from (51) that \( \hat{\theta}_t = \hat{\theta}_s \) for all \( s, t \leq t_1 \), so \( \hat{\theta} \) extends to a smooth 1-form on \( E \). Moreover, we can read off (55) that \( |\hat{\theta}| = O(w^{-\delta + l}) \), i.e. \( \hat{\theta} \in C^0_{-\delta + l}(T^* E) \). It is possible to obtain estimates on derivatives of \( \hat{\theta} \) and to show that \( \hat{\theta} \in C^\infty_{-\delta + l}(T^* E) \). However, this is a long calculation which relies only on two main observations. First, we deduce from Lemma 4.3 that \( |\nabla^l dy_j| \) behaves asymptotically like \( |dy_j| w^{-l} \) for all \( l \geq 0 \) and \( j = 0, \ldots, 2(m + d) - 1 \). Secondly, we can conclude from \( \eta \in C^\infty_{-\delta}(\Lambda^* T^* E) \) that also \( |\nabla^l \hat{\eta}_1| = O(w^{-\delta - l}) \). Using formula (55), it is then straightforward to verify \( |\nabla^l \hat{\theta}| = O(w^{-\delta - l + 1}) \),
as claimed. We leave the details to the reader, but remark that the required estimate is similar to bounding $|\hat{\theta}|$.

It remains to show that $\hat{\eta} = \hat{d}\hat{\theta}$ by considering its components. In fact, it is an easy calculation ([Mar02][p.80]) to prove that
\[
\frac{\partial}{\partial t} \left( i^*_s(\hat{\eta} - d\hat{\theta}) \right) = 0,
\]
\[i.e. \ i^*_s(\hat{\eta} - d\hat{\theta}) = i^*_s(\hat{\eta} - d\hat{\theta}) \text{ for all } s, t \in \mathbb{R}. \]
Since $\hat{\eta}, \hat{\theta} \to 0$ as $t \to \infty$, we conclude that $i^*_s(\hat{\eta} - d\hat{\theta}) = 0$ for any $t \in \mathbb{R}$. Moreover, it is shown in [Mar02][p.80] that
\[
i^*_\rho \hat{\eta} = i^*_\rho d\hat{\theta},
\]
and hence $\hat{\eta} = \hat{d}\hat{\theta}$ as we claimed.

\[\square\]

**Proof of Theorem 5.4.** The strategy is the same as for the proof of [CH13][Theorem 3.11]. We start with some basic observations. By Proposition 5.13, there exists a $\theta \in C_{-\delta}(\Lambda^*E)$ such that $d\theta = \eta$. Since $\eta$ is real, $\theta$ will also be a real form, i.e. $\theta^{1,0} = \bar{\theta}^{0,1}$ if $\theta = \theta^{1,0} + \theta^{0,1}$ is the decomposition into types. Moreover, $\eta$ is of type $(1,1)$, so we must have that $\partial\theta^{1,0} = \bar{\partial}\theta^{0,1} = 0$.

If $\partial^*$ denotes the formal dual of $\partial$ (w.r.t. the $L^2$-metric induced by $g_\varphi$), then $\partial^*\theta^{1,0} \in C^\infty(E)$. We would like to find a solution $u$ to the equation $\Delta u = \partial^*\theta^{1,0}$, whose growth we can control. There are two cases to consider, corresponding to part (i) and (ii) of Proposition 5.6.

First, we consider the case where the degree of the polynomial $Q$ is greater or equal to 2. By (ii) of Proposition 5.6, there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^*\theta^{1,0}$ and $|u| + |\nabla u| = O(u^{2-\delta+\varepsilon})$. It follows that $\Delta u = \partial^*\theta^{1,0}$. Since $\theta^{1,0}$ is harmonic by the Kähler identities, $\partial^*\theta^{1,0} = \partial(\partial u - \theta^{1,0}) = 0$, and hence the 1-form $\partial u - \theta^{1,0}$ is harmonic by the Kähler identities.

Choosing $\varepsilon > 0$ small enough, we can assume that $2 - \delta + \varepsilon < 0$ and hence we see from $|\nabla u| = O(u^{2-\delta+\varepsilon})$ and $\theta \in C^\infty_{1-\delta}(\Lambda^*E)$ that
\[
|\partial u - \theta^{1,0}| \leq |du| + |\theta| \to 0,
\]
as $t \to \infty$. Then Lemma 5.11 implies $\partial u - \theta^{1,0} = 0$ and consequently,
\[
\eta = d\theta = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0} = \bar{\partial}\bar{\partial}u + \bar{\partial}u = -2\sqrt{-1}\partial\bar{\partial}\text{Im }u,
\]
where $\text{Im }u$ is the imaginary part of $u$.

It remains to show that $\text{Im }u \in C^\infty_{2-\delta}(E)$ as opposed to only $\text{Im }u \in C^{2,\alpha}(E)$ and $\text{Im }u = O(u^{2-\delta+\varepsilon})$. As we can choose $\varepsilon > 0$ so that $2 - \delta + \varepsilon < 0$, this improvement of the decay rate, however, follows immediately from Proposition 5.10 if we can show $d\text{Im }u \in C^\infty_{1-\delta}(\Lambda^*E)$. This last condition is clearly satisfied since $\theta^{1,0}, \theta^{0,1} \in C^\infty_{1-\delta}(\Lambda^*E)$ and $\theta^{1,0} - \theta^{0,1} = \partial u - \bar{\partial}u = d\text{Re }u + \sqrt{-1}d\text{Im }u$. This settles the first case.
For the second case, assume that $k \leq 1$. We want to use (i) of Proposition 3.6 to solve $\Delta u = \partial^* \theta^{1,0}$. This time, however, we only know that the solution $u$ satisfies $\int |\nabla u|^2 \omega_{\varphi}^{m+d} < +\infty$, and not necessarily that $u$ decays towards infinity. So the idea is to use the vanishing Theorems 5.12 instead.

Before applying Proposition 3.6 (i), we need to verify that $\int \partial^* \theta^{1,0} \omega_{\varphi}^{m+d}$ is zero. For any $t_0 \in \mathbb{R}$, define $K_{t_0} = \{z \in E \mid t(z) \leq t_0\}$ and consider the integral

$$\int_{K_{t_0}} \partial^* \theta^{1,0} \omega_{\varphi}^{m+d} = \int_{K_{t_0}} d^* \theta^{1,0} = \int_{\{t_0\} \times S} \ast \theta^{1,0},$$

where we used Stoke’s for the last equality. If we equip the slice $\{t_0\} \times S$ with the restriction of $g_{\varphi}$ and denote the corresponding volume by $\text{Vol}(\{t_0\} \times S)$, then we can estimate

$$\left| \int_{\{t_0\} \times S} \ast \theta^{1,0} \right| \leq \text{Vol}(\{t_0\} \times S) \sup_{\{t_0\} \times S} |\theta| = O(w^{k+1-\delta}(t_0)),$$

since $|\theta| = O(w^{1-\delta})$ and $\text{Vol}(\{t_0\} \times S) = O(w^k)$. It follows that the right hand side of (87) goes to zero as $t_0 \to +\infty$, as we assumed $k \leq 1$ and $\delta > 2$. Hence $\int \partial^* \theta^{1,0} \omega_{\varphi}^{m+d} = 0$, as claimed.

So we find a $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^* \theta^{1,0}$ and $\int |\nabla u|^2 \omega_{\varphi}^{m+d}$ is finite. In particular, the 1-form $\beta = \theta^{1,0} - \partial u$ is harmonic. Also note that

$$|\theta| \omega_{\varphi}^{m+d} = O(w^{2-2\delta+k})$$

with $2 - 2\delta + k < -1$, so that $\theta$ is in $L^2$, and thus $\beta$ is $L^2$ as well.

It follows that $d\beta = d^* \beta = 0$, and in particular, $\beta$ is an $L^2$-holomorphic $(1,0)$-form. Hence it must be identically zero by Theorem 5.12, i.e. $\theta^{1,0} = \partial u$. The rest of the proof is now analogous to the first case.

\[\square\]

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