Analysis of the effect of bending rigidity on Gaussian Curvature for clamped and simply supported thin circular plates

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Abstract—Negative Gaussian curvature in deformed surface subjected to concentrated loads is a key weakening factor on the surface accuracy of umbrella antennas. The negative impacts can be mitigated by increasing the bending rigidity of the reflector surface. Thus, firstly, this paper is concerned with the effect of bending rigidity on Gaussian curvature based on the analysis of bending circular thin plate with large deflection. Exact expressions of Gaussian curvature for deformed circular thin plates are derived using the analytical solution of the von-Karman plate equation and the properties of positive and negative for Gaussian curvature is investigated. After solving the zeroes of Gaussian curvature using the Newton iteration method, Gaussian curvature distribution of the deformed surface is obtained.

1. Introduction

The analysis of thin plates has been an active research topic due to its relevance to mechanical, aerospace and civil engineering. The nonlinear bending of thin plates has attracted much attention. Static and dynamic analysis of plate [1] in large deformation has been one of the important topics in solid mechanics for a long time. In mesh antennas, when the mesh surface is formed under point loads, it can also be regarded as a thin plate with extremely low bending rigidity. The ideal reflective surface of the antenna is a parabolic reflective surface with positive Gaussian curvatures. However, due to the low bending rigidity, the mesh surface is also modelled as a membrane with no bending and compressive deformation [2-3]. The deformed surface subjected to a concentrated load usually exhibited negative Gaussian curvature, which greatly weakened the profile accuracy. In order to improve the negative Gaussian curvature, new reflective surface materials with higher bending rigidity than traditional mesh reflector are proposed [4], by which the reflector can well maintain the positive Gaussian curvature profile.

For simplification, the research in this paper commenced with the bending of thin circular plates. It is well known that the bending circular plate has a strong non-linearity, which manifests itself in geometry, material or a combination of both. Due to the complexity of mathematics, the nonlinear bending problem of thin plates is difficult to obtain an exact solution. Fortunately, Plaut [5] obtained the exact analytical solution of von Karman plate equations with large deflections after assuming $N_r \approx N = N$, based on which, the study in this paper will be carried out.

In this paper, the exact analytical solutions of Gaussian curvature of a circular thin plate were obtained by assuming $N_r \approx N_r \approx N$ based on von Karman equations. The effect of bending stiffness
on Gaussian curvature under different boundary conditions was analyzed by solving the zeroes of Gaussian curvature using Newton iteration method.

2. Governing equations and boundary conditions

Figure.1 Sketch of an Axisymmetric Flat Punch Onto a circular thin plate

The governing equations of axisymmetric large deflection bending of a thin circular plate with radius \( a \) subjected to concentrated load \( F \) take the forms of [5]:

\[
\frac{d}{dr} \left( r \frac{dw}{dr} \right) = \frac{1}{D} \left( \frac{F}{2\pi r} + N_r \left( \frac{dw}{dr} \right) \right)
\]

\[
\frac{d}{dr} \left( N_r + N_t \right) + \frac{Eh}{2r} \left( \frac{dw}{dr} \right) = 0
\]

where \( w, h, r, E \) are the deflection, thickness, radial coordinate of the plate and Young’s modulus respectively; \( N_r, N_t, D=Eh^3/12(1-v^2) \) are the radial, tangential membrane stresses and plate flexural stiffness, where \( v \) is Poisson’s ratio.

It is convenient to normalize Eq. 1 by the dimensionless quantities as follows for the plate shown asFig. 1:

\[
\xi = \frac{r}{a}, \quad W = \frac{w}{h}, \quad \zeta = \frac{c}{a}, \quad \beta = \sqrt{\frac{Na^2}{D}, \quad \varphi = \frac{Fa^2}{2\pi Dh}, \quad \theta = \frac{dw}{d\xi}}
\]

Therefore, Eq. 1 is recast as follows:

\[
\xi^2 \frac{d^2\theta}{d\xi^2} + \xi \frac{d\theta}{d\xi} - (1 + \beta^2 \xi^2) \theta = \varphi \theta
\]

The boundary conditions of this problem are

\[
\begin{align*}
\theta &= 0, \xi = 0 \\
\frac{d\theta}{d\xi} &= \frac{d^2w}{d\xi^2} = 0, \xi = 1
\end{align*}
\]

for simply supported boundary;

\[
\begin{align*}
\theta &= 0, \xi = 0 \\
\theta &= 0, \xi = 1
\end{align*}
\]

for clamped boundary.

The axisymmetric bending of a thin circular plate with large deflection can be solved by the boundary value problem as shown in Eq. 3-5.

3. Solution of large deflection bending problem and derivation of Gaussian curvatures

3.1. Clamped circular thin plate

The profile is obtained by solving Eq. 3 and Eq. 5 [5] with respect to \( \xi \) from \( \zeta \) to 1,
\[ W = \frac{\varphi}{\beta^2} \left[ C_1 \beta [I_0(\beta \xi) - I_0(\beta)] - C_2 \beta [K_0(\beta \xi) - K_0(\beta)] - \ln \xi \right] \]  
(6)

where \( I_i(x), K_i(x) \) are the \( i \)th order of the first and second kind modified Bessel functions, respectively, and

\[ C_1 = \frac{1}{\beta^2} \left[ \frac{K_1(\beta \xi) - K_1(\beta) / \xi}{I_1(\beta) K_1(\beta \xi) - I_1(\beta \xi) K_1(\beta)} \right] \]
\[ C_2 = \frac{1}{\beta^2} \left[ \frac{I_1(\beta \xi) - I_1(\beta) / \xi}{K_1(\beta) K_1(\beta \xi) - K_1(\beta \xi) I_1(\beta)} \right] \]  
(7)

In von-Karman plate equations, due to the assumption of \( (w)^2 \ll 1 \), the Gaussian curvature of deformed circular plates at each point can be expressed as

\[ K_0 = \frac{1}{\xi} \frac{dW}{d\xi} \frac{d^2W}{d\xi^2} \]  
(8)

In order to solve Gaussian curvature, the first-order and second-order derivatives of Eq. 6 are obtained separately,

\[ \frac{dw}{d\xi} = \frac{\varphi}{\beta^2} \left[ C_1 \beta^2 I_1(\beta \xi) + C_2 \beta^2 K_1(\beta \xi) - \frac{1}{\xi} \right] \]
\[ \frac{d^2w}{d\xi^2} = \frac{\varphi}{\beta^2} \left[ C_1 \beta^3 [I_0(\beta \xi) \beta \xi - I_1(\beta \xi)] + C_2 \beta^3 [K_0(\beta \xi) \beta \xi - K_1(\beta \xi)] - \frac{\beta \xi}{\xi^2} \right] \]  
(9)

In the absence of wrinkles, it is obvious that \( \frac{d^2W}{d\xi^2} \) is constantly negative. The positivity and negativity of the Gaussian curvature is determined by \( \frac{d^2W}{d\xi^2} \). Therefore, the positivity and negativity of \( \frac{d^2W}{d\xi^2} \) will be discussed here.

Substituting the first equation of Eq. 9 into the second equation yields

\[ \frac{d^2W}{d\xi^2} (-\beta \xi) = \frac{\varphi}{\beta^2} \left( -C_1 \beta^3 [I_0(\beta \xi) \beta \xi - I_1(\beta \xi)] + C_2 \beta^3 [K_0(\beta \xi) \beta \xi - K_1(\beta \xi)] - \frac{\beta \xi}{\xi^2} \right) \]
\[ = \frac{\varphi}{\beta^2} \left( -C_1 \beta^4 I_0(\beta \xi) \xi + C_2 \beta^4 K_0(\beta \xi) \xi + \beta \frac{\varphi}{\beta^2} \left( C_1 \beta^2 I_1(\beta \xi) + C_2 \beta^2 K_1(\beta \xi) - \frac{1}{\xi} \right) \right) \]
\[ = \beta^2 \varphi \left( -C_1 I_0(\beta \xi) + C_2 K_0(\beta \xi) \right) + \beta \frac{d\varphi}{d\xi} \]  
(10)

By combination of Eq. 5 and Eq. 10, we have

\[ -\beta \frac{d^2W}{d\xi^2} \xi_1 = \beta^2 \varphi (-C_1 I_0(\beta \xi) + C_2 K_0(\beta \xi)) \]  
(11)

Substituting Eq. 7 into Eq. 11 yields

\[ -\beta \frac{d^2W}{d\xi^2} = \varphi \left( \frac{K_1(\beta \xi) - K_1(\beta) / \xi}{I_1(\beta) K_1(\beta \xi) - I_1(\beta \xi) K_1(\beta)} I_0(\beta) + \frac{I_1(\beta \xi) - I_1(\beta) / \xi}{K_1(\beta) I_1(\beta \xi) - K_1(\beta \xi) I_1(\beta)} K_0(\beta) \right) \]
\[ = \frac{\varphi}{\xi [K_1(\beta \xi) - K_1(\beta) I_1(\beta \xi) + I_1(\beta \xi) K_1(\beta)]} \]  
(12)

Due to the monotonicity of modified Bessel functions, the following equation is constantly negative

\[ K_1(\beta \xi) - K_1(\beta) I_1(\beta \xi) + I_1(\beta \xi) K_1(\beta) \neq 0, 1 \]  
(13)

Therefore, the sign of \( \frac{d^2W}{d\xi^2} \xi_1 \) is determined by the following equation:

\[ g(\xi) = [\xi K_1(\beta \xi) - K_1(\beta) I_0(\beta)] + [\xi I_1(\beta \xi) - I_1(\beta) K_0(\beta)] \]

By solving the derivative of \( g(\xi) \), we find that \( g'(\xi) < 0, \xi \in (0, 1) \); therefore,

\[ g(\xi) > g(1) = 0, \xi \in (0, 1) \Rightarrow \frac{d^2W}{d\xi^2} \xi_1 > 0 \]  
(14)
Thus, by substituting Eq. 14 into Eq. 8, it is found that the Gaussian curvature of the deformed clamped circular thin plate is negative around $\xi = 1$.

Similarly, by combination of Eq. 5 and Eq. 10, we have

$$-\beta \frac{d^2 \varphi}{d \xi^2} = \beta^2 \varphi \{-C_1 I_0 (\beta \xi) + C_2 K_0 (\beta \xi)\}$$

(15)

By combination of Eq. 7 and Eq. 15,

$$-\beta \frac{d^2 W}{d \xi^2} \bigg|_{\xi = \xi} = \frac{\varphi}{\xi[K_1(\beta \xi) - K_2(\beta \xi)]I_0 (\beta \xi) + [\xi I_1 (\beta \xi) - I_1 (\beta)]K_0 (\beta \xi)} \{[\xi K_2 (\beta \xi) - K_1 (\beta)]I_0 (\beta \xi) + [\xi I_2 (\beta \xi) - I_2 (\beta)]K_0 (\beta \xi)\}$$

(16)

Considering Eq. 13, the sign of $\frac{d^2 W}{d \xi^2}$ is determined by the following equation:

$$h(\xi) = \{[\xi K_2 (\beta \xi) - K_1 (\beta)]I_0 (\beta \xi) + [\xi I_2 (\beta \xi) - I_2 (\beta)]K_0 (\beta \xi)\}$$

(17)

By solving the derivative of $h(\xi)$, we find that $h'(\xi) > 0, \xi \in (0, 1)$; therefore,

$$h(\xi) < h(1) = 0, \xi \in (0, 1) \Rightarrow \frac{d^2 W}{d \xi^2} \bigg|_{\xi = \xi} < 0$$

(18)

By substituting Eq. 18 into Eq. 8, we can see that the Gaussian curvature of the deformed plate is positive around $\xi = \zeta$.

Due to the continuity of Gaussian curvature and its different signs at the two ends of the circular thin plate,

$$\begin{cases} K_G(\xi) > 0, \xi \in (0, 1) \\ K_G(1) < 0 \end{cases}$$

(19)

The following equation can be obtained,

$$\exists \lambda \in (\zeta, 1), K_G(\lambda) = 0$$

(20)

Similarly, by combination of Eq. 8 and Eq. 20,

$$\frac{d^2 W}{d \xi^2} \bigg|_{\xi = \lambda} = 0$$

(21)

That is to say, by solving Eq. 21 to get $\lambda$, the distribution Gaussian curvature for deformed circular thin plate can be obtained.

3.2. Simply supported circular thin plate

The exact solution of the linear modified Bessel differential equations shown as Eq. 3 is found to be

$$\theta = \varphi \left[ C_1 I_1 (\beta \xi) + C_2 K_1 (\beta \xi) - \frac{1}{\beta \xi} \right]$$

(22)

In order to obtain the exact solution of simply supported circular thin plate, the boundary conditions in Eq. 4 is substituted into Eq. 22 to obtained $C_1$ and $C_2$.

$$C_1 = \frac{1}{\beta^2 \xi K_1 (\beta \xi) [\beta I_0 (\beta \xi) - I_1 (\beta \xi)] + [\beta K_0 (\beta \xi) + K_1 (\beta \xi)] [\xi I_1 (\beta \xi) - I_1 (\beta)]}$$

$$C_2 = \frac{1}{\beta^2 \xi K_1 (\beta \xi) [\beta I_0 (\beta \xi) - I_1 (\beta \xi)] + [\beta K_0 (\beta \xi) + K_1 (\beta \xi)] [\xi I_1 (\beta \xi) - I_1 (\beta)]}$$

(23)

Due to the same boundary $\theta \bigg|_{\xi = \zeta} = 0$ with the clamped circular thin plate at $\xi = \zeta$, the sign of $\frac{d^2 W}{d \xi^2} \bigg|_{\xi = \zeta}$ is determined by the following equation:

$$h(\xi) = C_1 I_0 (\beta \xi) - C_2 K_0 (\beta \xi)$$

(24)
Substituting Eq. 23 into Eq. 24, we can find the following equation has the same positive and negative signs with
\[ g(\zeta) = -K_0(\beta\xi)(\zeta I_1(\beta\xi) + \beta I_0(\beta) - I_1(\beta)) + I_0(\beta\xi)(-\zeta K_1(\beta\xi) + [\beta K_0(\beta) + K_1(\beta)]) \] (25)

By solving the derivative of \( g(\zeta) \), \( g'(\zeta) > 0, \zeta \in (0,1) \); therefore,
\[ h(\zeta) < h(1) = 0, \zeta \in (0,1) \Rightarrow \frac{d^2W}{d\xi^2}|_{\xi=\zeta} < 0 \] (26)

that is to say the Gaussian curvature of the deformed plate is positive around \( \xi = \zeta \).

Due to the boundary condition \( \frac{d^2W}{d\xi^2} \) need to be resolved to know the sign of \( \frac{d^2W}{d\xi^2} \) around \( \xi = 1 \),
\[ \frac{d^2W}{d\xi^2} = \beta^2 \varphi(C_1 I_1(\beta\xi) + C_2 K_1(\beta\xi)) + \frac{1}{\xi^2} \frac{dW}{\xi} - \frac{1}{\xi} \frac{d^2W}{\xi^2} \] (27)

Substituting Eq. 9 and Eq. 23 into Eq. 27 yields that,
\[ \frac{d^3W}{d\xi^3}|_{\xi=1} = \varphi \left[ (C_1 I_1(\beta) + C_2 K_1(\beta))(1 + \beta^2) - \frac{1}{\beta} \right] \] (28)

The sign of \( \frac{d^3W}{d\xi^3} \) is related to \( \beta \) from Eq. 28, for example, when \( \zeta = 0.001 \),
\[ \frac{d^3W}{d\xi^3}|_{\xi=1} (\beta = 1.5) = 0.0586, \frac{d^3W}{d\xi^3}|_{\xi=1} (\beta = 2) = -0.0776 \] (29)

that is to say, the sign of \( \frac{d^2W}{d\xi^2} \) around \( \xi = 1 \) can be positive or negative with \( \beta \).

4. Results and Discussion
In this section, Newton’s iterative method was used to solve \( \lambda \) in Eq. 21 for clamped and simply supported circular thin plates to illustrate the distribution of Gaussian curvature in circular plates. Furthermore, a finite element model of umbrella antennas was developed to demonstrate the guidance of the above analysis to engineering.

In order to obtain \( \lambda \) in Eq. 21, \( \zeta \) need to be set at first. Here, \( \zeta \) is taken as 0.001 during calculation and the results are shown in Fig. 2.
From the analysis in the previous section, it is known that around $\xi = \zeta$, the Gaussian curvature of the deformed surface is positive, and the value of $\lambda$ determines the size of the region with positive Gaussian curvature. It can be seen from Fig. 2 that the region with positive Gaussian curvature decrease dramatically with $\beta$. Back to Eq. 2, we know that $\beta$ denotes the ratio of stretching stress to bending rigidity such that (1) when $\beta \to 0$, the plate is platelike and allows bending only, and (2) when $\beta \to \infty$, the plate is membrane like and allows stretching only. It should be noted that the Gaussian curvature of entire deformed simply supported plate is positive until $\beta$ increase to 1.7 and it has same region of positive Gaussian curvature with clamped plates when $\beta$ is more than 3. In contrast, the $\lambda$ has an upper limit about 0.37 for clamped plates. Therefore, in order to increase the ratio of the region with positive Gaussian curvature in reflectors of antennas, the simply supported boundary should be applied, instead of the clamped boundary, while, the reflective surface bending rigidity should be increased properly.

Revisiting Eq. 6, it can be seen that the deformation of surface can be determined by $\beta$ only, naturally including the central deflection $W_0$. Therefore, in combination with Fig. 2, we can see the effect of $W_0$ on $\lambda$, shown as Fig. 3.

Fig. 3 indicates that for a thin circular plate, only the dimensionless deflection of load point affects the deformed surface, while the effects from other material or geometric parameters can be regarded as the measures to adjust $W_0$. 
5. Conclusion
In this paper, thin circular plate with large deformation was studied based on von Kármán thin plate theory and the Gaussian curvature of deformed surfaces was analytically analyzed. The effect of boundary conditions (clamped and simply supported) on the Gaussian curvature in deformed surface was investigated and Newton's iterative method was used to obtain the distribution of Gaussian curvature in thin circular plate. The results show that the area of Gaussian curvature region of the deformed thin plate with simply supported boundary is larger than that with clamped boundary when the dimensionless deflection of load point is less than 3.

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