Zero temperature Glauber dynamics on complex networks

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Abstract. We study the Glauber dynamics at zero temperature of spins placed on the vertices of an uncorrelated network with a power-law degree distribution. Application of mean-field theory yields as main prediction that for symmetric disordered initial conditions the mean time to reach full order is finite or diverges as a logarithm of the system size \(N\), depending on the exponent of the degree distribution. Extensive numerical simulations contradict these results and clearly show that the mean-field assumption is not appropriate to describe this problem.

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1. Introduction

The zero temperature Glauber dynamics for the Ising model is one of the simplest ordering dynamics that can be devised for a set of spin-like variables. Its sole physical ingredient is the tendency of each spin to align with the majority of its neighbors, and for this reason Glauber dynamics has been long investigated as a model for the ordering of magnets as well as of other systems where the evolution is driven by the minimization of interfacial energy without conservation constraints (1). Despite its simplicity, the zero temperature Glauber dynamics exhibits nontrivial phenomena already on regular lattices. In $d = 1$ its ordering process can be mapped to an ensemble of coalescing random walkers and, as a consequence, the mean ordering time grows with the system size $N$ as $N^2$ (2). The same diffusive behavior occurs at $T > 0$ in higher dimensions, but for strictly vanishing temperature there is a probability that the dynamics remains trapped in states not fully ordered (2; 4). In $d = 2$ the lack of global ordering occurs about 3 times out of 10, in the limit of large systems, and it is associated to the formation of two or more ordered stripes spanning the system along one direction. In three (and presumably higher) dimensions the probability of reaching the fully ordered state goes to zero as the system size increases: domains of opposite magnetization coexist, with some “blinders” active at the boundaries (3; 4).

The simplicity of its local interactions makes Glauber dynamics a natural model for the evolution of some social systems, whose dynamics can be defined in terms of agents characterized by a binary variable (for instance their opinion with respect to some issue) that is updated in response to the pressure of their peers. While for physical applications it is appropriate to study spins on regular $d$-dimensional lattices, the modelization of social systems immediately raises the question of the behavior of Glauber dynamics when the interaction pattern is disordered, i.e., agents live on the vertices of a network (5). In this scenario, the nontrivial topological attributes of networks, such as the small-world property (6) and the scale-free property (7), are expected to play some relevant role. Steps in the direction of investigating these effects have been recently taken for other types of ordering dynamics, including the voter model (8; 9; 10; 11) and the Axelrod model (12).

Among the first investigations of Glauber dynamics on networks we can mention the work of Boyer and Miramontes (13), who found that the long-range connections of a Watts-Strogatz network (6) induce a pinning of the ordering process when domains reach a characteristic size, depending on the density of such connections. The Watts-Strogatz network is a rather special type of network, where, despite the small-world effect, the concept of a local $d$-dimensional regular neighborhood of a site maintains a significance, at least for a small density of long-range connections. Other kinds of networks, such as the random (Erdős-Rényi) network (14) are instead not a perturbation of a regular lattice, and also in this case it has been shown that the zero temperature Glauber dynamics does not lead to full ordering (15; 16). However, for networks of finite size $N$, it occurs that some realizations of the process actually lead to complete order, while
others get trapped in an infinitely long-lived (meta)stable state. The probability that a run ends up in a disordered state turns out to grow with \( N \): order is not reached for asymptotically large \( N \) \( \text{(16)} \).

Very recently, Zhou and Lipowsky \( \text{(17)} \) have presented a mean-field (MF) treatment of the zero temperature Glauber dynamics on generic uncorrelated complex networks whose degree distribution \( P(k) \), defined as the probability that a vertex is connected to other \( k \) vertices, decreases as a power-law, \( P(k) \sim k^{-\gamma} \), with a characteristic degree exponent \( \gamma \). Within this approach, they find that a dynamic transition takes place for the particular value \( \gamma_c = 5/2 \). For \( \gamma > 5/2 \), the system reaches the ordered state in a time that diverges logarithmically with the system size. For \( \gamma < 5/2 \), on the other hand, the ordering time is finite and independent from \( N \). These results are claimed to be confirmed by numerical simulations. The results reported in Ref. \( \text{(17)} \) are remarkable in several respects. In the first place, they seem to contradict the difficulty observed to reach complete order generally observed for Glauber dynamics on disordered interaction patterns. Secondly, the transition in the dynamical properties occurs for a value of the exponent \( \gamma \) that has not been associated to special behavior for other models on networks with a power-law degree distribution, in which transitions typically occur for \( \gamma = 3 \) \( \text{(18; 19; 20)} \), below which the second moment of the degree distribution diverges and the network becomes scale-free \( \text{(7)} \).

In this paper we present a refined study of the MF theory for the zero temperature Glauber dynamics presented in Ref. \( \text{(17)} \), recasting it more conveniently in a rate equation in the continuous time limit. In our analysis we take into account the finite size effects of the network in a more precise way, elucidating what happens at \( \gamma = 5/2 \). In particular, we present a discussion of the differences shown by the dynamics depending on the initial conditions, that is, depending on whether the initial state is completely random and uncorrelated with zero average magnetization (symmetric initial state), or it corresponds to a state with a given initial, small, magnetization (asymmetric initial state). In the former case, we show that a proper description of the ordering process needs to take into account the initial diffusive contribution of the dynamics, not explicitly present in the MF equations for infinite network size. Our results confirm the MF prediction of a different behavior of the mean ordering time depending on whether \( \gamma \) is larger or smaller that 5/2. In order to check the predictions of our MF analysis, we report the results of extensive numerical simulations. These simulations prove that, for symmetric initial conditions, the dynamics behaves in a way incompatible with the MF predictions (as already observed numerically in other nonequilibrium dynamics on complex networks \( \text{(21)} \)), showing no qualitative differences for \( \gamma \) above or below 5/2, a value that seems to emerge as an artifact of an invalid MF theory. Our findings reconcile the results for Glauber dynamics on scale-free networks with those obtained on other types of structures and calls for approaches, beyond mean-field, able to capture the observed phenomenology.
2. Mean-field theory

Let us consider an Ising model with spin variables $\sigma = \pm 1$ located at the vertices of a network, which is fully characterized by the adjacency matrix $A_{ij}$, taking the value $A_{ij} = 1$ if the vertices $i$ and $j$ are connected by an edge, while $A_{ij} = 0$ otherwise. The local field $h_i(t)$ experienced by vertex $i$, and due to the spins located at its nearest neighbors, is thus given by

$$h_i(t) = \sum_j A_{ij} \sigma_j(t).$$  \hfill (1)

The Glauber dynamics at zero temperature is defined as follows: At each time step $t$ we pick at random one of the vertices and we update the value of its spin according to the local field, that is,

$$\sigma_i = +1 \text{ if } h_i(t) > 0$$  \hfill (2)

$$\sigma_i = -1 \text{ if } h_i(t) < 0$$  \hfill (3)

$$\sigma_i = \pm 1 \text{ with probability } 1/2 \text{ if } h_i(t) = 0.$$  \hfill (4)

After each spin update, time is increased as $t \to t + 1/N$, where $N$ is the number of vertices in the network.

Starting from a disordered uncorrelated configuration, in which each spin is taken to be positive at random with probability $p$, Glauber dynamics tends to make neighboring spins equal in sign. Thus, the main questions raised by this dynamics are whether all spins become equal in the long run and, if this is the case, how much time this total ordering requires, and how it depends on the initial fraction $p$ of positive spins and on the system size $N$. Particularly interesting in this respect is the symmetric initial state in which $p = 1/2$, since in this case the dynamics must first break the symmetry of the initial state in order to reach full order.

To construct a MF theory for the zero temperature Glauber dynamics, we consider, following Ref. (17), the average dynamical variable $q_k(t)$, defined as the probability that a vertex of degree $k$ is in a $+1$ state. The evolution of the variable $q_k(t)$ depends on the average local field felt by the vertices of degree $k$, $h_k$. In terms of this variable, and according to the rules defining the Glauber dynamics, we can write a rate equation for $q_k(t)$ in the continuous time approximation (which is valid for infinitely large network sizes) that takes the form

$$\frac{dq_k(t)}{dt} = -q_k(t) \text{Prob}[h_k < 0] - q_k(t) \text{Prob}[h_k = 0] \frac{1}{2}$$

$$+ (1 - q_k(t)) \text{Prob}[h_k > 0] + (1 - q_k(t)) \text{Prob}[h_k = 0] \frac{1}{2},$$  \hfill (5)

where $\text{Prob}[h_k > 0]$, $\text{Prob}[h_k < 0]$, and $\text{Prob}[h_k = 0]$ are the normalized probabilities that the local field at the vertices of degree $k$ is positive, negative or zero, respectively. Rearranging the terms in Eq. (5) we can write

$$\frac{dq_k(t)}{dt} = -q_k(t) + \text{Prob}[h_k > 0] + \frac{1}{2} \text{Prob}[h_k = 0].$$  \hfill (6)
In order to estimate the probabilities of the local field $h_k$, the relevant quantity to consider is the probability $Q_k$ that an edge departing from a vertex of degree $k$ points to a $+1$ spin. For a generic network, statistically characterized by its degree distribution $P(k)$ and its degree correlations given by the conditional probability $P(k'|k)$ that a vertex of degree $k$ is connected to a vertex of degree $k'$ (22), the probability that an edge from a vertex of degree $k$ points to a $+1$ spin is given by

$$Q_k = \sum_{k'} P(k'|k)q_{k'},$$

that is, it is proportional to the probability that a vertex $k$ is connected to a vertex of degree $k'$ times the probability that this vertex is in a $+1$ state, averaged over all possible values of the degree $k'$. For random uncorrelated networks, the conditional probability takes the simplified form $P(k'|k) = kP(k)/\langle k \rangle$, and therefore the probability $Q_k$ is independent of $k$ and can be written as

$$Q = \frac{1}{\langle k \rangle} \sum_k kP(k)q_k.$$ 

(8)

Assuming that the probability of having a positive spin at the end of an edge departing from a $k$ vertex is independent of the values of the spins at the extremes of the other $k-1$ edges (which corresponds to a MF assumption), we have that the probability that $\ell$ edges from a $k$ vertex point to positive spins is given by a binomial distribution. The local field at a $k$ vertex can be zero only if $k$ is even and exactly half of its edges point to $+1$ spins. Thus

$$\text{Prob}[h_k = 0] = \binom{k}{k/2} Q^{k/2}(1 - Q)^{k/2},$$

(9)

for $k$ even. On the other hand, $h_k$ is positive when more than half of its edges point to $+1$ spins. Therefore, the probability of observing a positive local field is

$$\text{Prob}[h_k > 0] = \sum_{\ell = [k+1]/2}^k \binom{k}{\ell} Q^\ell(1 - Q)^{k - \ell},$$

(10)

where $[x]$ is the smallest integer larger than or equal to $x$. In this way, we can write the rate equation for the probabilities $q_k$ as

$$\frac{dq_k(t)}{dt} = -q_k(t) + \Phi_k(Q),$$

(11)

where

$$\Phi_k(Q) = \sum_{\ell = [k/2]}^k \left[ 1 - \frac{1}{2} \Delta_{\ell,k/2} \right] \binom{k}{\ell} Q^\ell(1 - Q)^{k - \ell}.$$ 

(12)

By introducing Eq. (11) into the definition Eq. (8) we obtain a closed rate equation for the quantity $Q$, namely

$$\dot{Q} = -Q + \Psi(Q),$$

(13)

with

$$\Psi(Q) = \frac{1}{\langle k \rangle} \sum_k kP(k)\Phi_k(Q).$$

(14)
It is useful to write the dynamics in terms of the new variable \( y = Q - 1/2 \). In the initial state, where all spins point up or down with probability \( p \) and \( 1 - p \), respectively, we have \( Q(t = 0) = p \) and therefore \( y(t = 0) = p - 1/2 \). In the symmetric disordered state, with \( p = 1/2 \), we have \( y(t = 0) = 0 \), while the ordered states with all spins up or down correspond to \( y = 1/2 \) and \( y = -1/2 \), respectively. Thus, \( y \) plays the role of a convenient order parameter. The rate equation for the quantity \( y \) can be written as

\[
\dot{y} = -y - \frac{1}{2} + \psi(y),
\]

where we have defined the new function

\[
\psi(y) \equiv \Psi\left(\frac{1}{2} + y\right) = \sum_{k} k P(k) \Phi_k \left(\frac{1}{2} + y\right).
\]

The function \( \psi(y) \) is equal to \( 1/2 \) for \( y = 0 \) and goes symmetrically to \( 1 \) [0] for \( y = 1/2 \) \([y = -1/2]\). In terms of this variable, we can study the time that it takes for the system to become ordered by defining the ordering time \( t_{ord} \) such that \( |y(t_{ord})| = y_F \), where \( y_F \) is an arbitrary chosen positive value such that \( |p - 1/2| < y_F < 1/2 \).

For the explicit evaluation of \( \psi(y) \), presented in the Appendix, it turns out that, for an uncorrelated network with degree distribution \( P(k) \sim k^{-\gamma} \), there are different limiting regimes for different values of \( y \) and the degree exponent \( \gamma \). For simplicity we consider only the case \( y > 0 \); analogous considerations apply in the symmetric case \( y < 0 \).

2 < \( \gamma < 5/2 \): The behavior of \( \psi(y) \) can be approximated, depending on the value of \( y \) and of the minimum degree \( k_0 \) and degree cut-off \( k_c \) of the network in the following form:

\[
\psi(y) - 1/2 \approx c_2(\gamma) k_0^{\gamma-2} k_c^{5/2-\gamma} y, \quad y < y_c
\]

\[
\psi(y) - 1/2 \approx c_3(\gamma) k_0^{\gamma-2} y^{2(\gamma-2)}, \quad y_c < y < y_{NL}
\]

\[
\psi(y) - 1/2 \approx 1/2, \quad y > y_{NL}
\]

where \( y_c = k_c^{-1/2} \) and \( y_{NL} = k_0^{-1/2} \).

Inserting these forms into the dynamic equation Eq. (15), we obtain the temporal evolution of the order parameter \( y \). For \( y < y_c \), there is a regime with \( y \) growing exponentially with time, i.e.

\[
y(t) = y_T \exp\left(\left[c_2(\gamma) k_0^{\gamma-2} k_c^{5/2-\gamma} - 1\right](t - t_T)\right],
\]

with an exponential factor that depends on the network cut-off \( k_c \). Here we have denoted as \( t_T \) and \( y_T \) the values of \( t \) and \( y \) at the beginning of this regime. For reasons that will be clear below, these may or may not coincide with the initial values \( t = 0 \) and \( y(0) \). For \( y_c < y < y_{NL} \), the order parameter grows instead as a power-law,

\[
y(t) = \left[y_c^{1/2(\gamma-2)} + c_3(\gamma) k_0^{\gamma-2}(t - t_c)\right]^{1/[1-2(\gamma-2)]},
\]

where \( t_c \) is the time at which the order parameter reaches the value \( y_c \). Finally, for \( y > y_{NL} \), \( \psi(y) \) is approximately equal to 1 and the dynamic equation is \( \dot{y} = 1/2 - y \), so that

\[
y(t) = \frac{1}{2} + \left(y_{NL} - \frac{1}{2}\right) e^{-(t-t_{NL})},
\]
where \( t_{NL} \) is the time at which the value \( y_{NL} \) is reached, that is, \( y(t_{NL}) = y_{NL} \).

\( \gamma > 5/2 \): In this case \( y_c \) does not play any role; the function \( \psi(y) \) is independent of \( k_c \) and it can be approximated for \( y < y_{NL} \) as

\[
\psi(y) \sim 1/2 + c_1(\gamma)k_0^{1/2}y. \tag{21}
\]

The behavior \( y \) is thus again exponential, but now with a prefactor independent of \( k_c \), namely,

\[
y(t) = y_T \exp[(c_1(\gamma)k_0^{1/2} - 1)(t - t_T)], \tag{22}
\]

For \( y > y_{NL} \), \( \psi(y) \) the scenario is the same of the previous case, with an exponential approach to the asymptotic value \( y = 1/2 \).

As a consequence of the previous expressions, the total ordering time has, in the most general case, four contributions, namely,

\[
t_{ord} = t_T + \Delta t_c + \Delta t_{NL} + \Delta t_{ord}, \tag{23}
\]

where \( \Delta t_c = t_c - t_T \), \( \Delta t_{NL} = t_{NL} - t_c \), and \( \Delta t_{ord} = t_{ord} - t_{NL} \).

The simplest term is the last one, \( \Delta t_{ord} \), that, from Eq. (20), is given by

\[
\Delta t_{ord} \approx \ln \left[ \frac{1 - 2y_{NL}}{1 - 2y_F} \right]. \tag{24}
\]

It is independent of \( N, p \), and, in practice, also of \( k_0 \). The other terms depend instead on the value of \( \gamma \).

For \( \gamma > 5/2 \), \( \Delta t_c \) and \( \Delta t_{NL} \) are merged in a single term, which is, from Eq. (22),

\[
\Delta t_c + \Delta t_{NL} \approx k_0^{-1/2} \ln \left[ \frac{y_{NL}}{y_T} \right]. \tag{25}
\]

For \( 2 < \gamma < 5/2 \), instead, we obtain from Eq. (19)

\[
\Delta t_{NL} \approx \frac{y_{NL}^{1-2(\gamma-2)} - y_c^{1-2(\gamma-2)}}{k_0^{3-2}}, \tag{26}
\]

while from Eq. (18) we have

\[
\Delta t_c \approx k_0^{2-\gamma}k_c^{-5/2} \ln \left[ \frac{y_c}{y_T} \right]. \tag{27}
\]

Let us now discuss the additional term \( t_T \). The deterministic equation of motion (15) holds strictly only in the limit of infinite network size. For finite \( N \) an additional random term appears, giving a diffusive contribution to the dynamics of \( y \) that competes with the deterministic drift described by Eq. (15). As it will be shown below, this contribution vanishes as \( N \) diverges, so that it can be generally neglected with respect to the drift, unless \( y(0) = 0 \), i.e., for symmetric initial conditions. In such a case also the drift vanishes, and the point \( y = 0 \) is an unstable equilibrium point, that can be characterized by a potential \( V(y) = -f''(\psi(y)) - y - 1/2)dy \) with a parabolic shape for very small \( y \). Under the deterministic MF dynamics a system placed at \( y = 0 \) will stay there forever, and the diffusive term is thus needed to provide the perturbation
necessary to take the system out of the initial equilibrium point. The interplay between drift and diffusion determines the time $t_T$ at which drift starts to dominate.

The origin of this diffusive term is easy to understand: As discussed at the beginning of this Section, the Glauber dynamics at zero temperature is defined by means of a sequential updating, in which at each time interval $\Delta t = 1/N$ a single spin is flipped, causing an average increment of the dynamic variable $\Delta Q = 1/N$. The initial condition $y = 0$, $Q = 1/2$, corresponds to a state in which every site is randomly assigned a $+1$ or a $-1$ spin with probability $p = 1/2$. Therefore, the initial evolution is dominated by a random flipping of spins, driven by the value of the local field $h_i$, that induces a diffusive motion of the variable $y$ until a nonzero value is reached, large enough to drive the dynamics out of the unstable equilibrium point. To be more concrete, when $y \simeq 0$, or $Q \simeq 1/2$, the drift term appearing in the equation of motion is very small. In particular, the deterministic drift velocity is given by

$$v = (1 - Q)\Psi(Q) - Q[1 - \Psi(Q)] = \psi(y) - 1/2 - y. \quad (28)$$

The diffusive component, on the other hand, has a diffusive constant that can be estimated as

$$D = (\Delta Q)^2/(2\Delta t) = 1/(2N). \quad (29)$$

For small values of $y$, the drift grows with $y$, while diffusion is independent of it. Starting with $y(0) \simeq 0$, for small $|y|$ diffusion prevails and the drift can be neglected. The diffusion leads to increasing fluctuations, and after some time a fluctuation leads to a threshold value $|y_T|$ sufficiently large for the bias to start prevailing, and drive the variable far from the equilibrium point. From this time on, bias dominates and the diffusive term in the equation of motion becomes negligible.

The value of $y_T$ can be determined as follows: When the system is in the position $y$, the time $t_D$ needed to go back diffusively to 0 is given by $|y| = \sqrt{Dt_D}$. During this time interval the bias will induce a drift $|v(y)t_D| = |v(y)|y^2/D$. If the drift is smaller than $|y|$ the system is able to go back to 0, and diffusion dominates. If the drift is larger than $|y|$ then bias dominates. The condition that defines $y_T$ is then

$$v(y_T)y_T^2/D = |y_T|. \quad (30)$$

Inserting Eq. (21) into Eq. (30) we obtain, for $\gamma > 5/2$,

$$|y_T| \approx N^{-1/2}k_0^{-1/4}, \quad (31)$$

while the time spent diffusing is

$$t_T \approx \frac{y_T^2}{D} \approx k_0^{-1/2}. \quad (32)$$

Notice that this time does not depend on $N$. On the other hand, for $2 < \gamma < 5/2$, we obtain, using the scaling of the cut-off with the network size, given by $k_c \sim N^{1/2}$ for uncorrelated scale-free networks (23),

$$|y_T| \approx k_0^{(2-\gamma)/2}N^{(\gamma-9/2)/4}. \quad (33)$$
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Figure 1. Sketch of the behavior of the evolution of $y(t)$ as a function of time for $2 < \gamma < 5/2$. Both axes are on logarithmic scale.

and

$$t_T \approx k_0^{(2-\gamma)} N^{(\gamma-5/2)/2},$$

(34)
a time that goes to zero when increasing $N$. We are now in the position to summarize the MF estimates for the ordering time.

For asymmetric initial conditions ($p \neq 1/2$, $y(0) > 0$) we can set $t_T = 0$ and $y_T = y(0)$. In this case, we find at leading order for $\gamma > 5/2$

$$t_{ord} \approx \text{const} + k_0^{-1/2} \ln y(0),$$

(35)

while for $2 < \gamma < 5/2$ we obtain

$$t_{ord} \approx \text{const} + k_0^{-1/2} N \frac{(\gamma-5/2)}{2} \ln \left[ \frac{N}{y(0)} \right],$$

(36)

As we can observe, for finite $y(0)$ the ordering time is finite in the infinite network size limit. For finite networks, it exhibits a decreasing size correction in the case $2 < \gamma < 5/2$.

For symmetric initial conditions, $y(0) = 0$, and $\gamma > 5/2$ we have, at leading order,

$$t_{ord} \approx \text{const} + k_0^{-1/2} \ln N,$$

(37)

that is, the ordering time diverges logarithmically with the network size $N$. For $2 < \gamma < 5/2$, on the other hand, we obtain

$$t_{ord} \approx \text{const} + k_0^{-1/2} N^{(\gamma-5/2)/2} \ln N.$$

(38)

The ordering time is now a decreasing function of $N$ that tends to a constant value in the limit $N \to \infty$. For illustration purposes a sketch of the different temporal scales and the related behaviors of $y$ is depicted in Fig. 1.

3. Numerical results

In this Section we compare the theoretical results derived above within the MF formalism with extensive numerical simulations of the Glauber dynamics at zero temperature,
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Figure 2. Plot of the probability that a run starting from the symmetric initial state $y(0) = 0$ does not lead to complete order as a function of $N$, for networks with various values of $\gamma$ and $k_0$.

performed on uncorrelated networks with a power-law degree distribution. The networks used have been generated using the uncorrelated configuration model (UCM) defined in Ref. (24). The network is built as follows: We start by assigning to each vertex $i$ in a set of $N$ initially disconnected vertices a random degree $k_i$, extracted from the probability distribution $P(k)$, subject to the constraint $k_0 \leq k_i \leq k_c = N^{1/2}$ and $\sum_i k_i$ even. Afterward, the network is constructed by randomly connecting the vertices with $\sum_i k_i/2$ edges, respecting the preassigned degrees and avoiding multiple and self-connections between vertices. Using this algorithm, it is possible to create random networks with any preassigned degree distribution. For the case of networks with a degree distribution given by a power law, $P(k) \sim k^{-\gamma}$, the restriction $k \leq N^{1/2}$ for any degree exponent guarantees that the networks generated in this way are completely uncorrelated (23). Notice that if such a restriction is not enforced, correlations unavoidably arise for $\gamma < 3$ (23).

We start considering the case $\gamma = 4$ as a representative of networks with $\gamma > 3$. In this case the connectivity pattern does not show strong degree fluctuations ($\langle k^2 \rangle < \infty$ for $N \to \infty$) and one does not expect to observe the special effects induced by the presence of hubs, relevant for other dynamical processes on networks. MF theory predicts the ordering time to be given by Eq. (35) for $y(0) > 0$ and by Eq. (37) for $y(0) = 0$. However, even in this simple case, we find that the observed phenomenology is in striking disagreement with the expected theoretical results.

The first, fundamental, discrepancy lies in the very existence of order in the system for large times. At the MF level, all instances of the dynamics order sooner or later. From the numerical evidence, however, we conclude that networks starting from a symmetric initial state do not get ordered in the large $N$ limit. To reach this conclusion, we have performed a large number of runs for several values of $N$, starting from $y(0) = 0$. We observe that some of them do not lead to a fully ordered state, remaining instead trapped in some (meta)stable states that last forever. Fig. 2 reports the fraction of these runs that do not lead to complete ordering. After a minimum for a network size
that apparently depends on the value of $k_0$, this fraction grows with $N$. Extrapolating for large $N$, we conclude that an infinite network never gets ordered. This absence of ordering of the Glauber dynamics at $T = 0$ on complex networks had already been pointed out in Ref. (16). Fig. 2 demonstrates also that the lack of ordering occurs for all values of $\gamma$, either larger or smaller than $5/2$. From this figure we can also observe that the fraction of nonordering runs decreases with increasing $k_0$, in such a way that for very large values of $k_0$ and moderate values of $N$, this fraction is apparently zero (i.e. all realizations get ordered). This fact, however, is a trivial consequence of the nonordering probability becoming too small for the number of runs that can be performed. In Ref. (17), where large values of $k_0$ were used, this absence of ordering was not noticed.

In view of this result, the very notion of an ordering time becomes somewhat fuzzy, in the sense that an unrestricted average over all runs will yield an infinite ordering time. For practical purposes, nevertheless, we can define an operational ordering time by performing an average over the restricted ensemble of those runs that actually order. Thus we have computed the average ordering time, defined as the average time $t_{ord}$ needed for the ordering parameter $y$ to reach the arbitrary value $|y_F| = 1/4$ (17). In Fig. 3 we report the values of $t_{ord}$ obtained for networks with $\gamma = 4$, starting from nonsymmetric initial conditions $y(0) > 0$ and different values of $k_0$, as a function of the network size $N$. As predicted from the MF result Eq. (35), we observe that $t_{ord}$ becomes independent of $N$, for sufficiently large network sizes. In the inset of Fig. 3 we plot the asymptotic value of $t_{ord}$ for the largest network size considered $N = 10^6$, as a function of $y(0)$. Here we can observe that the MF prediction $t_{ord} \sim -\ln y(0)$, Eq. (35), seems to be fulfilled, at least for sufficiently large values of $k_0$ and $y(0)$. For small values of $y(0)$ and $k_0$ the convergence of $t_{ord}$ with $N$ is rather slow, a fact that hides the asymptotic logarithmic behavior.

In Fig. 4 we show the results obtained for the ordering time in networks with $\gamma = 4$ and symmetric initial conditions $y(0) = 0$, for different values of the minimum degree
Figure 4. Average value of the ordering time $t_{ord}$, for $\gamma = 4$ as a function of the network size, for symmetric initial conditions $y(0) = 0$ and several values of the minimum degree $k_0$. 

$k_0$. As we can see from this plot, the average ordering time $t_{ord}$ seems to grow with $N$ as a power-law, with effective exponents depending on the particular value of $k_0$. For large values of $k_0$, the exponent is so small that the behavior could be confused with a logarithmic one. However, for small $k_0$, the power-law growth is quite evident, in strong disagreement with the MF prediction, Eq. (37).

We have performed this same kind of analysis for two other values of the degree exponent, namely $\gamma = 2.75$ and $\gamma = 2.25$, in order to check the possible effects of a strongly inhomogeneous degree distribution, with diverging second moment, and also to assess whether the values of $\gamma$ smaller than $5/2$ affect the behavior of the ordering time, as predicted by MF theory. Results for asymmetric initial conditions $y(0) > 0$ for $\gamma = 2.25$ are reported in Fig. 5. In this case, we find that the convergence to the asymptotic value of the ordering time for large $N$ is extremely slow. As a consequence, it is not possible to make a definite statement about the validity of MF based on the present numerical evidence. For the largest $N$ that could be reached, the behavior of $t_{ord}$ for small $y(0)$ does not seem to be logarithmic, inset in Fig. 5, in apparent contradiction with Eq. (36). However, it is possible that this discrepancy could disappear if larger values of $N$ could be considered.

Much clearer is the situation for the symmetric initial condition $y(0) = 0$: the behavior is qualitatively the same (and different from MF) independently of the value of $\gamma$. We have already seen in Fig. 2 that the absence of ordering for large $N$ holds for any value of $\gamma$. But the similarity is stronger when the ordering time $t_{ord}$ is measured as a function of $N$. As displayed in Figs. 6 and 7, $t_{ord}$ diverges at large $N$ as a power-law, as opposed to the logarithmic growth and the constant behavior predicted by MF for $\gamma = 2.75$ and $\gamma = 2.25$, respectively.

The conclusion of this numerical analysis is that MF theory is unable to account for the behavior of the ordering time of the Glauber dynamics at zero temperature with symmetric initial conditions. The physical reason for this failure can be understood by
considering in more in detail the dynamical evolution of the system.

The symmetric initial state is created by randomly placing +1 and −1 spins on the network with probability \( p = 1/2 \). This procedure is akin to a percolation process \(^{25}\) of both the positive and negative spins with the same probability \( p \). In random networks, the percolation threshold is given by \(^{20}\)

\[
p_c = \frac{1}{\langle k \rangle} - 1. \tag{39}
\]

In scale-free networks with \( \gamma < 3 \), the degree fluctuations diverge, \( \langle k^2 \rangle \rightarrow \infty \), and \( p_c \rightarrow 0 \) in the limit of large network sizes. In the case of homogeneous networks, with finite \( \langle k^2 \rangle \), we observe that \( p_c \sim 1/\langle k \rangle \), and thus it becomes small for large average degree or \( k_0 \). Hence for reasonable values of \( k_0 \), all networks exhibit a very small percolation threshold. Therefore, symmetric initial conditions give rise to two giant components,
Figure 7. Average value of the ordering time $t_{ord}$, for $\gamma = 2.25$, as a function of the network size, for symmetric initial conditions $y(0) = 0$ and three values of the minimum degree $k_0$.

Figure 8. Temporal evolution of the size of the giant components ($G_+$ and $G_-), the size of the boundary $B$ and the order parameter $y$ for a run on a network with $N = 10^5$, $\gamma = 4$, $k_0 = 5$ with symmetric initial conditions.

$G_+$ and $G_-$, with approximate size $N/2$, each one corresponding to a connected set of vertices occupied by $+1$ or $-1$ spins, respectively. In Fig. 8 we plot, as a function of time and in a single run, the relative size of the giant components of $+1$ spins, $G_+(t)$, and $-1$ spins, $G_-(t)$, and the relative size of the boundary $B(t)$, defined as the fraction of edges with spins of different sign at the ends. For illustration, we also plot the value of the order parameter $y(t)$ corresponding to that particular ordering run. From this Figure we can observe that the dynamics is governed by the competition of the giant components of different signs, the $G_+$ growing at the expense of the $G_-$. It is noteworthy that during most of the evolution the boundary occupies a constant and sizeable fraction of all the edges. In the case shown in the Figure the competition ends abruptly with the positive giant component $G_+$ invading the whole system. In some cases instead, the competition ends up in a stalemate: the system reaches a stationary configuration
with the two coexisting giant components basically frozen with only some spins of even degree freely flipping back and forth at the boundary.

In the initial state $B(t = 0) = 2p(1 - p)$. After a short transient $B(t)$ reaches the plateau. During this short time interval strong correlations build up in the system, in the sense that $+1$ spins become surrounded with high probability with a majority of positive spins, while most neighbors of a $-1$ spin are almost certainly negative. This fact can be quantitatively measured by computing the mean probability that a vertex with a given spin $\sigma$ has a positive local field $h$. This kind of measurement is reported in Fig. 9 for homogeneous networks with degree exponent $\gamma = 4$, where we display the probability that the local field of a site of degree $k_0$ is positive. Within the mean-field treatment, this quantity is given by the function $\Phi(k_0(1/2 + y))$, Eq. (12), plotted as a dashed line. The solid line in the middle is the numerical evaluation of the same quantity. The other two lines are instead numerical evaluations of the same probability, but averaged only over all sites with $s_i = 1$ (top) and $s_i = -1$ (bottom). The middle solid line (that is the average of the top and bottom lines) is quite similar to the analytical result, but the other two are completely different. The mean-field treatment assumes that the local field is completely uncorrelated from the value of the spin, and depends at most on the degree of the vertex in which the spin is located. In the real system, however, during virtually all the dynamics, a positive spin has almost surely a positive local field, while a negative spin has a negative field. These strong correlations, completely missed by the MF approach, are at the root of the nontrivial behavior shown by the zero temperature Glauber dynamics.

4. Conclusions

In this paper we have investigated the ordering process of the zero temperature Glauber dynamics when Ising spins are placed on the vertices of an uncorrelated network with
degree distribution decreasing as a power-law of exponent $\gamma$. We have first developed a mean-field theory for the problem, expanding and improving on the earlier approach presented in Ref. [17]. Within the MF theory we identify a suitable order parameter and compute in detail the time needed for the system to order. While for asymmetric initial conditions the ordering time is finite, for symmetric initial conditions there is a transition: $t_{\text{ord}}$ is finite for $\gamma < 5/2$, while it diverges logarithmically with the system size $N$ for $\gamma > 5/2$.

The validity of these analytical results has been checked by means of extensive numerical simulations. While for asymmetric initial conditions we find an apparently reasonable agreement with MF theory, when the probability of a positive initial spin is $p = 1/2$, the behavior is at odds with the MF results and highly nontrivial. In particular, it turns out that there is a finite probability for a run to get trapped forever in a stationary disordered state. This probability grows with $N$ so that complete ordering is not reached for asymptotically large $N$. This conclusion, in striking disagreement with the MF predictions, holds for all values of $\gamma$. If we compute the ordering time restricted only to those runs that actually lead to full order, other unexpected results arise. For large $N$ and independently of the degree exponent, $t_{\text{ord}}$ appears to diverge as a power-law of the system size, the effective exponent being a function of $\gamma$ and the smallest degree $k_0$. One of the main conclusions of our work is, therefore, that mean-field theory does not provide a correct description of the ordering dynamics of the Glauber model at zero temperature. In this sense, the peculiar exponent $\gamma_c = 5/2$ turns out to be an artifact produced by a MF theory that is not capable to describe the system, and it does not bear apparently any relevance for the true behavior of the Glauber dynamics.

The failure of the MF theory for symmetric initial conditions is reminiscent of similar observations made on other dynamics on scale-free networks [21]. In the present case, this failure can be traced back to the breakdown of the assumption that the local field of a spin is independent of its value. This breakdown is explicitly shown in the formation of two giant components of positive and negative spins, competing through an extensive boundary, which induces the presence of strong correlations.

It is important to remark that the nontrivial phenomenology uncovered in this paper is not a consequence of the scale-free property of the network, since it appears already for $\gamma = 4$. The nontriviality of this problem is thus totally unrelated from the heterogeneous or homogeneous nature of the substrate. Somewhat surprisingly, the behavior is qualitatively the same for any $\gamma > 2$: In this case a scale-free topology seems not to have any relevant effects. The key to a deeper understanding of this problem lies likely in a detailed analysis of the properties of the two giant clusters competing in the system and of the boundary separating them. Along with an investigation of the effect of temperature in the problem, this remains as an interesting open issue for future work.
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Appendix A.

In this Appendix we derive analytically the asymptotic behavior of the function

\[ \psi(y) = \sum_k k P(k) \Phi_k \left( \frac{1}{2} + y \right) \]  

for finite networks, in which the degree \( k \) is restricted to the range \( k \in [k_0, k_c] \), \( k_0 \) being the smallest degree in the network and \( k_c \) the degree cut-off or maximum degree present in the network. In the continuous degree approximation, the degree distribution takes in this case the form

\[ P(k) = \frac{\gamma - 1}{k^{\gamma - 1}} - \frac{1}{(k_c/k_0)^{\gamma - 1} k^{-\gamma}}. \]  

The function \( \Phi_k(1/2 + y) \) has the generic form, neglecting the Kronecker symbol,

\[ \Phi_k(1/2 + y) = \sum_{l=k/2}^k p_{k,l}(y), \]  

where

\[ p_{k,l}(y) = \binom{k}{l} (1/2 + y)^l (1/2 - y)^{k-l}. \]  

For large values of \( k \), the binomial distribution Eq. (A.4) can be approximated by a normal distribution,

\[ p_{k,l}(y) \approx \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -(l-m)^2/2\sigma^2 \right), \]  

with

\[ m = \sum_l l p_{k,l}(y) = k \left( 1/2 + y \right), \]  

\[ \sigma^2 = \sum_l l^2 p_{k,l}(y) - m^2 = k(1-4y^2)/4. \]  

If we insert this form into the definition of \( \Phi_k(1/2 + y) \), and substitute the sum over \( l \) by an integral, we obtain

\[ \Phi_k(1/2 + y) = \frac{1}{2} \text{erf} \left( \frac{y \sqrt{2k}}{(1-4y^2)^{1/2}} \right) + \frac{1}{2} \text{erf} \left( \frac{\sqrt{2k}(1/2 - y)}{(1-4y^2)^{1/2}} \right), \]  

where \( \text{erf}(x) \) is the error function. For constant \( y \sqrt{k} \) and large \( k \), the argument of the second term in Eq. (A.8) becomes large, and we can approximate this term by...
unity. In the same limit, the \( y \) contribution in the denominator of the argument in the first term of this equation is negligible, so that

\[
\Phi_k(1/2 + y) = \frac{1}{2} + \frac{1}{2} \text{erf}\left(y \sqrt{2k}\right).
\]

(Eq. A.9)

In Fig. A1 we check the validity of this analytical expression by plotting it against the result of the numerical evaluation of Eq. (A.3) as a function of the rescaled variable \( y \sqrt{2k} \).

Inserting the functional form Eq. (A.9) into the definition of \( \psi(y) \), we obtain, within the continuous degree approximation, and neglecting terms of order \((k_c/k_0)^{1-\gamma}\),

\[
\psi(y) \simeq \frac{1}{2} + \frac{(\gamma - 2)k_0^{-2}}{2} \int_{k_0}^{k_c} k^{-\gamma+1} \text{erf}\left(y \sqrt{2k}\right) dk.
\]

(Eq. A.10)

The behavior of this function has different asymptotic regimes, depending on the value of \( y \). In the first place, if \( \sqrt{2k_c y} \) is small, then the argument in the error function is small for all values of \( k \), and we can substitute it by its linear approximation, \( \text{erf}(x) \simeq 2x/\sqrt{\pi} \).

| \( y \ll y_c \) | \( 2 < \gamma < 5/2 \) | \( \gamma > 5/2 \) |
|-----------------|-----------------|-----------------|
| \( y_c \ll y \ll y_{NL} \) | \( c_2(\gamma)k_0^{-2}k_c^{5/2-\gamma}y \) | \( c_1(\gamma)k_0^{1/2}y \) |
| \( y \gg y_{NL} \) | \( c_3(\gamma)k_0^{-2}y^{2(\gamma-2)} \) | \( c_1(\gamma)k_0^{1/2}y \) |

Table A1. Functional forms of \( \psi(y) - 1/2 \) as a function of \( y \) for the different ranges of \( \gamma \).
Then, for $y < y_c = k_c^{-1/2}$, we have

$$
\psi(y) \simeq \frac{1}{2} + \frac{\sqrt{2}(\gamma - 2)k_0^{\gamma - 2}}{\sqrt{\pi}} y \int_{k_0}^{k_c} k^{-\gamma + 3/2} dk. 
$$

(A.11)

This integral can be expressed analytically in terms of the incomplete Gamma function (26). Performing an expansion of the resulting expression for $y \ll y_{NL} = k_0^{-1/2}$, we obtain again two limiting regimes, depending on the degree exponent. For $\gamma > 5/2$, the leading term is again $\psi(y) \sim 1/2 + c_1(\gamma)k_0^{1/2} y$, that is, we obtain the same linear behavior as in the case $y < y_c$. For $2 < \gamma < 5/2$ instead, the integration over $k$ brings a singular behavior, so that we obtain $\psi(y) \sim 1/2 + c_2(\gamma)k_0^{3-2\gamma/2} k_c^{5/2-\gamma} y$. Here $c_1(\gamma)$ and $c_2(\gamma)$ are constants that depend only on the degree exponent $\gamma$.

When $y > y_c$, we must take into account the full functional form of the error function in order to evaluate Eq. (A.10). In this case we have that the argument of the error function is large at the upper limit of the integral. Given that the error function saturates to one for large argument, we can simplify the integral extending its upper limit to infinity. Therefore, we have that, in this range of $y$ values,

$$
\psi(y) \simeq \frac{1}{2} + \frac{\sqrt{2}(\gamma - 2)k_0^{\gamma - 2}}{\sqrt{\pi}} y \int_{k_0}^{\infty} k^{-\gamma + 1} \text{erf} \left( y\sqrt{2k} \right) dk.
$$

(A.13)

This last integral can be expressed analytically in terms of the incomplete Gamma function (26). Performing an expansion of the resulting expression for $y \ll y_{NL} = k_0^{-1/2}$, we obtain again two limiting regimes, depending on the degree exponent. For $\gamma > 5/2$, the leading term is again $\psi(y) \sim 1/2 + c_1(\gamma)k_0^{1/2} y$, that is, we obtain the same linear behavior as in the case $y < y_c$. For $2 < \gamma < 5/2$ instead, the integration over $k$ brings a singular behavior, so that we obtain $\psi(y) \sim 1/2 + c_3(\gamma)k_0^{3-2\gamma} y^{2(\gamma-2)}$. Here $c_3(\gamma)$ is again a constant that depends only on the degree exponent $\gamma$. Finally, for $y \gg y_{NL}$ the argument in the error function in Eq. (A.13) is large for all $k$, and we can approximate $\text{erf}(x) \simeq 1$. In this case, $\psi(y)$ can be taken to be equal to 1 for any value of $\gamma$. The behavior of the function $\psi(y)$ is summarized in Table A1.

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