Black hole perturbations in vector-tensor theories: the odd-mode analysis

Ryotaro Kase,\textsuperscript{a} Masato Minamitsuji,\textsuperscript{b} Shinji Tsujikawa\textsuperscript{a} and Ying-li Zhang\textsuperscript{a}

\textsuperscript{a}Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
\textsuperscript{b}Centro de Astrofísica e Gravitação – CENTRA, Departamento de Física, Instituto Superior Técnico – IST, Universidade de Lisboa – UL, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
E-mail: r.kase@rs.tus.ac.jp, masato.minamitsuji@tecnico.ulisboa.pt, shinji@rs.kagu.tus.ac.jp, yingli@rs.tus.ac.jp

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Abstract. In generalized Proca theories with vector-field derivative couplings, a bunch of hairy black hole solutions have been derived on a static and spherically symmetric background. In this paper, we formulate the odd-parity black hole perturbations in generalized Proca theories by expanding the corresponding action up to second order and investigate whether or not black holes with vector hair suffer ghost or Laplacian instabilities. We show that the models with cubic couplings $G_3(X)$, where $X = -A_\mu A^\mu/2$ with a vector field $A_\mu$, do not provide any additional stability condition as in General Relativity. On the other hand, the exact charged stealth Schwarzschild solution with a nonvanishing longitudinal vector component $A_1$, which originates from the coupling to the Einstein tensor $G^{\mu\nu}A_\mu A_\nu$ equivalent to the quartic coupling $G_4(X)$ containing a linear function of $X$, is unstable in the vicinity of the event horizon. The same instability problem also persists for hairy black holes arising from general quartic power-law couplings $G_4(X) \supset \beta_4 X^n$ with the nonvanishing $A_1$, while the other branch with $A_1 = 0$ can be consistent with conditions for the absence of ghost and Laplacian instabilities. We also discuss the case of other exact and numerical black hole solutions associated with intrinsic vector-field derivative couplings and show that there exists a wide range of parameter spaces in which the solutions suffer neither ghost nor Laplacian instabilities against odd-parity perturbations.

Keywords: modified gravity, astrophysical black holes, dark energy theory

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1 Introduction

The observational discovery of late-time cosmic acceleration in 1998 [1, 2] has led to the idea that extra degrees of freedom (DOFs) beyond those appearing in General Relativity (GR) and/or Standard Model of particle physics may be responsible for the acceleration [3–8]. One of such extra DOFs is a scalar field minimally coupled to gravity-dubbed quintessence [9–13]. If the scalar field has a direct coupling to the gravity sector, it is known that Horndeski theories [14] are most general scalar-tensor theories with second-order equations of motion [15–17]. Indeed, Horndeski theories have been applied to the late-time cosmic acceleration [18–21] as well as to the behavior of solutions in the Solar System [22–26].

The scalar field is not the unique possibility for realizing the cosmic acceleration, but a vector field $A_\mu$ can also play a similar role. If we respect the U(1)-gauge symmetry of a vector field, however, there are no nontrivial derivative interactions with gravity responsible for the cosmic acceleration [27, 28]. This situation is substantially modified in the presence of a vector field without the U(1)-gauge symmetry, i.e., a generalized Proca field. The action of generalized Proca theories with derivative couplings to gravity was derived in refs. [29–34] from the demand of keeping the equations of motion up to second order to avoid an Ostrogradski instability. Such new derivative interactions can give rise to the late-time cosmic acceleration [35–37], while recovering the behavior in GR around local objects with weak gravitational backgrounds [38, 39].

In modified gravity theories, the deviation from GR can manifest itself in strong gravitational backgrounds like black holes (BHs) and neutron stars (NSs) [40–45]. The recent direct detections of gravitational waves (GWs) by Advanced LIGO/Virgo from binary BH mergers [46–49] and a binary NS merger together with its optical counterpart [50–52] began to offer the possibility for probing GR and its possible deviation in the nonlinear regime of gravity. In the Einstein-Maxwell system of GR, there is a no-hair conjecture stating that the asymptotically flat and stationary BH solutions are described only by three parameters, i.e., mass, electric charge, and angular momentum [53–56]. The no-hair property is valid for a canonical scalar field minimally coupled to gravity [57, 58] and also for a scalar field which has a direct coupling to the Ricci scalar [59–61]. Bekenstein [62, 63] showed that, for a massive vector field theory with the broken U(1) gauge symmetry, the vector field must vanish due to the regularity on the horizon, and hence the resulting solution is given by the stationary BHs in GR without the vector hair.

In Horndeski theories, there exist BH solutions with scalar hair on the static and spherically symmetric background for a scalar field $\phi$ coupled to a Gauss-Bonnet term [64–68] or for a time-dependent scalar $\phi = q t + \psi(r)$ with nonminimal derivative couplings to the Einstein tensor [69–72] (see also refs. [73–76]). The latter corresponds to a stealth Schwarzschild BH solution with a nontrivial scalar hair, which was shown to be plagued by an instability problem against odd-parity perturbations in the vicinity of the BH event horizon [77].

In the Einstein-Proca theory, the static and spherically symmetric BH solution corresponds to the Schwarzschild spacetime without vector hair [62, 63]. On the other hand, in generalized Proca theories with derivative self-interactions and nonminimal derivative couplings to gravity, there exist a bunch of exact and numerical BH solutions with vector hair [79–87]. The cubic and quartic derivative interactions, which include the vector Galileon as specific cases, give rise to BH solutions with a primary Proca hair [82, 83]. For quintic power-law couplings the BH configuration regular throughout the horizon exterior does not
exist, but there are hairy solutions with secondary Proca hair in the presence of sixth-order and intrinsic vector-mode couplings. Unlike Horndeski theories, the existence of temporal vector component besides longitudinal mode allows one to realize many hairy BH solutions without tuning the models. We also note that the large temporal vector component with cubic and quartic derivative couplings allows the possibility for realizing the mass of NSs larger than that in GR [80, 88].

In this paper, we will formulate the odd-parity BH perturbations by extending the Regge-Wheeler formalism [89, 90] and investigate whether or not BH solutions derived in refs. [79, 82, 83] suffer ghost or Laplacian instabilities. On the static and spherically symmetric background, the perturbations can be decomposed into odd- and even-parity modes. In Horndeski theories, the stabilities of BH solutions against odd- and even-parity perturbations were investigated in refs. [77, 91, 92] (see also refs. [93–96]). The analysis of odd-parity modes shows that the exact stealth BH solution with a time-dependent scalar is plagued by instabilities in the vicinity of the event horizon [77]. In the present work, we generally derive conditions for the absence of ghost and Laplacian instabilities associated with odd-parity perturbations in generalized Proca theories.

We will show that the charged stealth BH solution with a nonvanishing longitudinal component $A_1$, found for the coupling to the Einstein tensor $G^{\mu \nu} A_\mu A_\nu$ [79] equivalent to the quartic coupling $G_4(X)$ containing a linear function of $X$, does not simultaneously satisfy conditions for the absence of ghost and Laplacian instabilities in the vicinity of the event horizon. Such an instability problem against odd-parity perturbations also persists for the other general quartic power-law couplings $G_4(X) \supset \beta_4 X^n$ with $A_1 \neq 0$. For the other branch with $A_1 = 0$, the BH solutions arising from quartic power-law interactions can be consistent with the conditions for the absence of ghosts and Laplacian instabilities throughout the horizon exterior.

We will also investigate whether ghost and Laplacian instabilities against odd-parity perturbations are present or not for hairy BHs arising from cubic, quintic, and intrinsic vector-mode couplings. The cubic derivative interactions do not give rise to any additional stability conditions as in GR. The quintic power-law couplings do not lead to background solutions regular throughout the horizon exterior, but the exact BH solution found in ref. [83] suffer neither ghost nor Laplacian instabilities. For the BH solutions associated with intrinsic vector-mode couplings, we will also show the existence of theoretically consistent parameter space.

Our paper is organized as follows. In section 2, we show the equations of motion in generalized Proca theories on the static and spherically symmetric background. In section 3, we derive the second-order action of odd-parity perturbations as well as conditions for the absence of ghost and Laplacian instabilities. In section 4, we study the stability of exact BH solutions against odd-parity perturbations, including the quartic coupling $G_4(X) \supset X/4$ as a special case. In section 5, we extend the analysis to general quartic power-law couplings $G_4(X) \supset \beta_4 X^n$ for the two branches characterized by $A_1 \neq 0$ and $A_1 = 0$. In sections 6 and 7, we will discuss the cases of hairy BH solutions realized by intrinsic vector-mode power-law couplings. Section 8 is devoted to conclusions.
2 Background equations

The action of generalized Proca theories with a vector field $A_{\mu}$ is given by [29–34]

$$S = \int d^{4}x \sqrt{-g} \left( F + \sum_{i=2}^{6} \mathcal{L}_{i} \right),$$

(2.1)

with

$$\mathcal{L}_{2} = G_{2}(X, F),$$
$$\mathcal{L}_{3} = G_{3}(X)\nabla_{\mu} A^{\mu},$$
$$\mathcal{L}_{4} = G_{4}(X)R + G_{4,X}(X) \left[ (\nabla_{\mu} A^{\mu})^{2} - \nabla_{\mu} A_{\nu} \nabla^{\nu} A^{\mu} \right],$$
$$\mathcal{L}_{5} = G_{5}(X)G_{\mu\nu} \nabla^{\mu} A^{\nu} - \frac{1}{6} G_{5,X}(X) \left[ (\nabla_{\mu} A^{\mu})^{3} - 3 \nabla_{\mu} A^{\mu} \nabla_{\rho} A_{\sigma} \nabla_{\sigma} A^{\rho} + 2 \nabla_{\mu} A_{\sigma} \nabla^{\mu} A^{\rho} \nabla_{\sigma} A_{\nu} \right]$$

$$- g_{5}(X) \tilde{F}^{\alpha\beta} \nabla_{\alpha} A_{\beta},$$
$$\mathcal{L}_{6} = G_{6}(X)\tilde{\mathcal{L}}^{\mu\nu\alpha\beta} \nabla_{\mu} A_{\nu} \nabla_{\alpha} A_{\beta} + \frac{1}{2} G_{6,X}(X) \tilde{F}^{\mu\nu} \nabla_{\alpha} A_{\mu} \nabla_{\beta} A_{\nu},$$

(2.2)

(2.3)

(2.4)

(2.5)

(2.6)

where $g_{\mu\nu}$ is the four-dimensional metric tensor, $g$ is the determinant of $g_{\mu\nu}$, and $\nabla_{\mu}$, $R$, $G_{\mu\nu}$ represent the covariant derivative, the Ricci scalar, the Einstein tensor associated with $Y$ respectively, and

$$X = -\frac{1}{2} A_{\mu} A^{\mu}, \quad F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}, \quad F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad \tilde{\mathcal{L}}^{\mu\nu\alpha\beta} = \frac{1}{4} \mathcal{E}^{\mu\nu\rho\sigma} \mathcal{E}^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}.$$  

(2.7)

Here, $\mathcal{E}^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor satisfying the normalization $\mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}_{\mu\nu\alpha\beta} = -4!$, and $R_{\rho\sigma\gamma\delta}$ is the Riemann tensor. While the function $G_{2}$ is dependent on the two quantities $X$ and $F$, the functions $G_{3,4,5,6}$ and $g_{5}$ depend on $X$ alone with the notation of partial derivatives $G_{i,X} \equiv \partial G_{i} / \partial X$.

In eq. (2.2), we can also take into account the dependence of another term $Y = A^{\mu} A^{\nu} F_{\mu} F_{\nu}$ respecting the parity invariance [37]. On the static and spherically symmetric spacetime this quantity is expressed as $Y = 4FX$ [83], so it is redundant to include the $Y$ dependence for discussing the background BH solutions. The dynamics of perturbations on the static and spherically symmetric background may be affected by the quantity $Y$, but we will not consider the explicit $Y$ dependence in $G_{2}$ throughout this paper. The Lagrangians containing the functions $G_{2} = -2g_{4}(X)F$, $g_{5}(X)$, and $G_{6}(X)$ correspond to intrinsic vector-modes [35–37]. The generalized Proca theories given by the action (2.1) lead to the second-order equations of motion with five propagating degrees of freedom.

We consider a static and spherically symmetric background given by the line element

$$ds^{2} = \bar{g}_{\mu\nu} dx^{\mu} dx^{\nu} = -f(r) dt^{2} + h^{-1}(r) dr^{2} + r^{2} \left( d\theta^{2} + \sin^{2} \theta d\varphi^{2} \right),$$

(2.8)

with the vector field in the form

$$\bar{A}_{\mu} = (A_{0}(r), A_{1}(r), 0, 0),$$

(2.9)

where $f$, $h$, $A_{0}$, and $A_{1}$ are functions of the radial coordinate $r$. On the background (2.8), the $\theta$ and $\varphi$ components of spatial vector component $A_{i}$ need to vanish.
In the following, we use the following notations

\[ X = X_0 + X_1, \quad X_0 = \frac{A_0^2}{2f}, \quad X_1 = -\frac{hA_0^2}{2}. \]  

(2.10)

The gravitational equations of motion arising from the action (2.1) are [83]

\[
\left( c_1 + \frac{c_2}{r} + \frac{c_3}{r^2} \right) h' + c_4 + \frac{c_5}{r} + \frac{c_6}{r^2} = 0, \tag{2.11}
\]

\[
-\frac{h}{f} \left( c_1 + \frac{c_2}{r} + \frac{c_3}{r^2} \right) f' + c_7 + \frac{c_8}{r} + \frac{c_9}{r^2} = 0, \tag{2.12}
\]

\[
\left( c_{10} + \frac{c_{11}}{r} \right) f''' + \left( c_{12} + \frac{c_{13}}{r} \right) f'' + \left( \frac{c_2}{2f} + \frac{c_{14}}{r} \right) f' h' + \left( c_{15} + \frac{c_{16}}{r} \right) f' + \left( -\frac{c_8}{2h} + \frac{c_{17}}{r} \right) h' + c_{18} + \frac{c_{19}}{r} = 0, \tag{2.13}
\]

(2.11)

(2.12)

(2.13)

where the coefficients \( c_{1,2,\ldots,19} \) are given in appendix A, and a prime represents the derivative with respect to \( r \). Varying the action (2.1) with respect to \( A_0 \) and \( A_1 \), it follows that

\[
rf \left[ 2fh(rA_0'' + 2A_1') + r(fh' - f'h)A_0' \right] (1 + G_{2,F}) + r^2 hA_0'' [2fhA_0'' - (f'h - f'h)A_0'] G_{2,FF}
-2r^2 f^2 A_0 G_{2,X} - 2r^2 f A_0' \left( fh^2 A_1 A_1' - hA_0 A_0' + f'h X_0 - f'h X_1 \right) G_{2,XF}
-rf A_0 \left[ 2rfh A_1' + (r'fh + rf'h + 4fh) A_1 \right] G_{3,X} + 4f^2 A_0 (rh' + h - 1) G_{4,X}
-8f A_0 [rfh^2 A_1 A_1' - (r'fh + rf'h + fh) X_1] G_{4,XX}
-f A_0 \left[ f(3h - 1)(h', A_1 + h(h - 1)(f' A_1 + 2f A_1')) \right] G_{5,X}
-2fh A_0 X_1 \left[ 2fh A_1' + (f'h + fh') A_1 \right] G_{5,XX} - 2f \left[ f(3h - 1) h' A_0' + h(h - 1)(2f A_0' - f' A_0) \right] G_{6}
-4fh A_0 X_1 (hA_0 A_0' - 2fh^2 A_1 A_1' - 2f'h X_0 + 2fh X_1) G_{6,XX}
-2f \left[ 4fh^2 X_1 A_0'' - 2h(h X - X_0) f' A_0' + f(6h - 1) h' X_1 A_0' + h(h - 1) A_0 A_0' \right] G_{6,XX}
-2fh^2 (3h - 1) A_0 A_0 A_1' G_{6,X} - 4fh \left[ 2rfh A_1 A_1'' \right. \left. - \{(r'fh - 3rf'h - 2fh) A_1 - 2rfh A_1' \} A_0' \right] g_5
-4rfh A_0' [hA_0 A_0' A_1 + 4fh X_1 A_1' - 2A_1 (f'h X_0 - f'h X_1)] G_{5,XX} = 0, \tag{2.14}
\]

(2.14)

and

\[
A_1 \left[ r^2 f G_{2,X} - 2(r f'h + fh - f) G_{4,X} + 4h(r A_0 A_0' - rf' X - f X_1) G_{4,XX} \right.
- h A_0^2 (3h - 1) G_{6,X} - 2h^2 X_1 A_0^2 G_{6,XX} \right] = r \left[ r(f' X - A_0 A_0') + 4f X_1 \right] G_{3,X} + 2fh X_1 G_{5,X}
+(A_0 A_0' - f' X) [(1 - h) G_{5,X} - 2h X_1 G_{5,XX}]
-2rh A_0^2 (g_5 + 2X_1 g_5, X). \tag{2.15}
\]

(2.15)

For the theories where only the functions \( G_i(X) \) with even indices \( i \) are present, the right hand side of eq. (2.15) vanishes. Then, there exists the branch satisfying \( A_1 = 0 \). For the theories containing \( G_i(X) \) with odd indices \( i \), the general solution to eq. (2.15) corresponds to a nonvanishing longitudinal component. There are a bunch of exact and numerical BH solutions with \( A_1 \neq 0 \) and \( A_1 = 0 \) for the above system.
3 Second-order action for odd-parity perturbations

On the static and spherically symmetric background (2.8), we consider small metric perturbations $h_{\mu\nu}$ with the vector-field perturbation $\delta A_\mu$. Then, the four-dimensional metric $g_{\mu\nu}$ and the vector field $A_\mu$ including perturbations are given, respectively, by
\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \delta A_\mu.
\] (3.1)

The perturbations on the background (2.8) can be decomposed into even- and odd-parity modes in the following manner. Under the rotation in two-dimensional sphere with the coordinates $\theta$ and $\varphi$, the metric perturbations $h_{tt}, h_{tr}, h_{rr}$ transform as scalars. Any scalar quantity $\Phi$ can be expressed in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$ as
\[
\Phi(t, r, \theta, \varphi) = \sum_{l,m} \Phi_{lm}(t, r)Y_{lm}(\theta, \varphi),
\] (3.2)
where $\Phi_{lm}$ is a function of $t$ and $r$. This scalar mode has the parity $(-)^l$ under the two-dimensional rotation. The perturbations with parities $(-)^l$ and $(-)^{l+1}$ are called even-mode and odd-mode, respectively [89, 90]. Under the two-dimensional rotation, the perturbations $h_{\theta\theta}, h_{r\theta}, h_{r\varphi}$ transform as vectors, while $h_{\theta\varphi}, h_{\varphi\varphi}$ transform as tensors. We can decompose any vector field $V_a$ and any symmetric tensor $T_{ab}$ into the following forms [94]:
\[
V_a(t, r, \theta, \varphi) = \nabla_a \Phi_1 + E_{ab} \nabla^b \Phi_2,
\] (3.3)
\[
T_{ab}(t, r, \theta, \varphi) = \nabla_a \nabla_b \Psi_1 + \gamma_{ab} \Psi_2 + \frac{1}{2} \left( E_{ac} \nabla_c \nabla_a \Psi_3 + E_{bc} \nabla_c \nabla_b \Psi_3 \right),
\] (3.4)
where the subscripts $a, b$ are either $\theta$ or $\varphi$, and $\Phi_1, \Phi_2, \Psi_1, \Psi_2, \Psi_3$ are scalar quantities. The tensor $E_{ab}$ is given by $E_{ab} = \sqrt{\gamma} \varepsilon_{ab}$, where $\gamma$ is the determinant of two dimensional metric $\gamma_{ab}$ on the sphere and $\varepsilon_{ab}$ is the anti-symmetric symbol with $\varepsilon_{\theta\varphi} = 1$. The terms containing the anti-symmetric tensor $E_{ab}$ in eqs. (3.3) and (3.4) correspond to the odd-parity modes, whereas the other terms correspond to the even-parity modes. On using eq. (3.2) in eqs. (3.3) and (3.4), the vector $V_a$ and the tensor $T_{ab}$ can be expressed in terms of the sum of spherical harmonics and their $\theta, \varphi$ derivatives multiplied by functions containing the $(t, r)$ dependence over $l$ and $m$.

In this paper, we focus on the odd-parity perturbations in generalized Proca theories. From the above argument, the metric perturbations corresponding to the odd modes can be expressed in the forms
\[
h_{tt} = h_{tr} = h_{rr} = 0, \quad (3.5)
\]
\[
h_{ta} = \sum_{l,m} Q_{lm}(t, r) E_{ab} \partial^b Y_{lm}(\theta, \varphi), \quad (3.6)
\]
\[
h_{ra} = \sum_{l,m} W_{lm}(t, r) E_{ab} \partial^b Y_{lm}(\theta, \varphi), \quad (3.7)
\]
\[
h_{ab} = \frac{1}{2} \sum_{lm} U_{lm}(t, r) \left[ E_{ac} \nabla_c \nabla_b Y_{lm}(\theta, \varphi) + E_{bc} \nabla_c \nabla_a Y_{lm}(\theta, \varphi) \right], \quad (3.8)
\]
where $Q_{lm}, W_{lm}, U_{lm}$ are functions of $t$ and $r$. The vector-field perturbation for the odd-parity sector is given by
\[
\delta A_t = \delta A_r = 0, \quad (3.9)
\]
\[
\delta A_a = \sum_{l,m} \delta A_{lm}(t, r) E_{ab} \partial^b Y_{lm}(\theta, \varphi), \quad (3.10)
\]
where \( \delta A_{lm} \) is a function of \( t \) and \( r \). All the perturbation variables are not necessarily physical because of the gauge degrees of freedom. Under the infinitesimal gauge transformation \( x_\mu \rightarrow x_\mu + \xi_\mu \) with

\[
\xi_t = \xi_r = 0, \quad \xi_a = \sum_{lm} \Lambda_{lm}(t,r) E_{ab} \partial^b Y_{lm}(\theta, \varphi),
\]

the perturbations \( Q_{lm}(t,r), W_{lm}(t,r), U_{lm}(t,r) \) in eqs. (3.6)–(3.8) transform, respectively, as

\[
Q_{lm} \rightarrow Q_{lm} + \dot{\Lambda}_{lm}, \quad W_{lm} \rightarrow W_{lm} + \Lambda'_{lm} - \frac{2}{r} \Lambda_{lm}, \quad U_{lm} \rightarrow U_{lm} + 2 \Lambda_{lm},
\]

where a dot represents the derivative with respect to \( t \). For \( l \geq 2 \), we will choose the Regge-Wheeler gauge \( U_{lm} = 0 \), after which we are left with two variables \( Q_{lm} \) and \( W_{lm} \). For the dipole mode \( l = 1 \), the perturbation \( h_{ab} \) vanishes identically, so we need to handle it separately. We also note that the odd-parity perturbations do not exist for the monopole mode \( l = 0 \).

We substitute eqs. (3.5)–(3.10) into the action (2.1) and expand it up to second order in perturbations. In the actual computation, we confirm that it is sufficient to set \( m = 0 \) and perform the integrals with respect to \( \theta \) and \( \varphi \). In appendix B, we show two integrals of spherical harmonics used for our computation. Performing the integrations with respect to \( t \) and \( r \) by parts, the resulting second-order action for odd-parity perturbations yields

\[
S_{\text{odd}} = \sum_{l,m} L \int dt dr \mathcal{L}_{\text{odd}},
\]

with

\[
L = l(l + 1),
\]

and

\[
\mathcal{L}_{\text{odd}} = r^2 \sqrt{\frac{f}{h}} \left[ C_1 \left( \dot{W}_{lm} - Q'_{lm} + \frac{2}{r} Q_{lm} \right)^2 + 2 \left( C_2 \dot{\delta} A_{lm} + C_3 \delta A'_{lm} + C_4 \delta A_{lm} \right) \right. \\
\left. \times \left( \dot{W}_{lm} - Q'_{lm} + \frac{2}{r} Q_{lm} \right) + C_5 \dot{\delta} A^2_{lm} + C_6 \dot{\delta} A_{lm} \delta A'_{lm} + C_7 \delta A^2_{lm} \right. \\
\left. + (L - 2) \left( C_8 W^2_{lm} + C_9 W_{lm} \delta A_{lm} + \frac{A_0}{f} C_9 W_{lm} Q_{lm} + C_{10} Q^2_{lm} + C_{11} Q_{lm} \delta A_{lm} \right) \\
\right. + (LC_{12} + C_{13}) \delta A^2_{lm} \right],
\]

where the background-dependent coefficients \( C_{1-13} \) are given in appendix C.

### 3.1 The modes of \( l \geq 2 \)

For the modes of \( l \geq 2 \), there are two dynamical fields \( W_{lm} \) and \( \delta A_{lm} \). The variable \( Q_{lm} \) in eq. (3.15) is non-dynamical, so one can derive a constraint equation for it by varying the action (3.13) with respect to \( Q_{lm} \). Due to the presence of the term \( Q^2_{lm} \) in the action, however, the corresponding constraint equation cannot be explicitly solved for \( Q_{lm} \). This problem can be tackled by using the method of a Lagrange multiplier (as already done in the
in refs. [91–93, 95]). We introduce a Lagrangian multiplier \( \chi(t, r) \) and rewrite eq. (3.15) as

\[
\mathcal{L}_{\text{odd}} = r^2 \sqrt{\frac{f}{h}} C_1 \left\{ 2 \chi \left( \dot{W}_{lm} - Q_{lm}' + \frac{2}{r} Q_{lm}' + \frac{C_2 \delta A_{lm} + C_3 \delta A_{lm}' + C_4 \delta A_{lm}'}{C_1} \right) - \chi^2 \right\} - (C_2 \delta A_{lm} + C_3 \delta A_{lm}' + C_4 \delta A_{lm})^2
\]

\[
+ C_5 \delta A_{lm}^2 + C_6 \delta A_{lm} \delta A_{lm}' + C_7 \delta A_{lm}^2
\]

\[
+(L - 2) \left( C_8 W_{lm}^2 + C_9 W_{lm} \delta A_{lm} + \frac{A_0}{f} C_9 W_{lm} Q_{lm} + C_{10} Q_{lm}^2 + C_{11} Q_{lm} \delta A_{lm} \right)
\]

\[
+ (LC_{12} + C_{13}) \delta A_{lm}^2 \right]. \quad (3.16)
\]

Varying the corresponding action with respect to \( \chi \) and substituting the equation of \( \chi \) into eq. (3.16), one recovers the original Lagrangian (3.15). Variations of the Lagrangian (3.16) with respect to \( W_{lm} \) and \( Q_{lm} \) lead, respectively, to

\[
C_1 \dot{\chi} - (L - 2) \left[ C_8 W_{lm} + \frac{C_9}{2} \left( \delta A_{lm} + \frac{A_0}{f} Q_{lm} \right) \right] = 0, \quad (3.17)
\]

\[
C_1 \chi' + \frac{2rf h C_1'}{2rf h} \left[ \delta A_{lm} + \delta A_{lm}' + (8fh + rfh - rf h') C_1 \right] \chi
\]

\[
+(L - 2) \left( \frac{A_0 C_9}{2f} W_{lm} + C_{10} Q_{lm} + \frac{C_{11}}{2} \delta A_{lm} \right) = 0. \quad (3.18)
\]

We solve eqs. (3.17) and (3.18) for \( W_{lm} \) and \( Q_{lm} \), respectively, and then substitute these solutions into the Lagrangian (3.16). After integrating by parts, the Lagrangian (3.16) can be written in the form

\[
(L - 2) \mathcal{L}_{\text{odd}} = r^2 \sqrt{\frac{f}{h}} \left( \ddot{\chi}^t \mathbf{K} \dot{\chi} + \dot{\chi}^t \mathbf{R} \dot{\chi} + \dot{\chi}^t \mathbf{G} \dot{\chi} + \dot{\chi}^t \mathbf{T} \chi + \dot{\chi}^t \mathbf{S} \chi + \dot{\chi}^t \mathbf{M} \chi \right), \quad (3.19)
\]

where \( \dot{\chi}^t = (\chi, \delta A_{lm}) \), and \( K, R, G, T, S, M \) are \( 2 \times 2 \) matrices. Setting \( \delta A_{lm} = 0, A_0 = q = \text{constant} \) and \( A_1 = \psi'(r) \), eq. (3.19) reduces to the second-order action for the odd-parity perturbations in shift-symmetric Horndeski theories with the linear time-dependent scalar field \( \phi = qt + \psi(r) \). We confirmed that, for the specific case \( G_3 = G_5 = 0 \) in shift-symmetric Horndeski theories, the second-order Lagrangian derived above coincides with that derived in refs. [77, 78].

The non-vanishing metric components of \( K, R, G \) are given by

\[
K_{11} = q_1, \quad K_{22} = (L - 2) q_2, \quad \text{(3.20)}
\]

\[
R_{11} = \frac{A_0 C_9}{f C_{10}} K_{11}, \quad R_{22} = \frac{(L - 2)(C_1 C_6 - 2 C_2 C_5)}{C_1}, \quad \text{(3.21)}
\]

\[
G_{11} = \frac{C_8}{C_{10}} K_{11}, \quad G_{22} = \frac{(L - 2)(C_1 C_5 - C_3^2)}{C_1}, \quad \text{(3.22)}
\]

where

\[
q_1 = \frac{4 f^2 C_9^2 C_{10}}{A_0^2 C_9^2 - 4 f^2 C_8 C_{10}}, \quad \text{(3.23)}
\]

\[
q_2 = \frac{C_1 C_5 - C_3^2}{C_1}. \quad \text{(3.24)}
\]
From eq. (3.20), the conditions for the absence of ghosts are

\begin{align}
q_1 &> 0, \\
q_2 &> 0.
\end{align}

(3.25)  
(3.26)

Assuming that the solutions of \( \vec{X}^l \) are in the form \( e^{i(\omega t - kr)} \), the dispersion relation for large \( \omega \) and \( k \) yields

\[ \text{det}(\omega^2 K - \omega k R + k^2 G) = 0. \]

(3.27)

We derive the propagation speed \( c_r = \frac{dr_\gamma}{d\tau} \) along the radial direction in proper time outside the horizon \( (f > 0, h > 0) \), where \( d\tau = \sqrt{f} dt \) and \( dr_\gamma = dr / \sqrt{h} \). Since this is related to the propagation speed \( \tilde{c}_r \) in the coordinates \( t \) and \( r \) as \( \tilde{c}_r = \sqrt{f} h c_r \), we substitute the relation \( \omega = \tilde{c}_r k = \sqrt{f} h c_r \) into eq. (3.27) and solve it for \( c_r \). On using eqs. (3.20)–(3.22), this leads to the propagation speeds along the radial direction:

\begin{align}
c_r^1 &= A_0 C_3 \pm \sqrt{F_1}, \\
c_r^2 &= C_1 C_6 - 2 C_2 C_3 \pm \sqrt{F_2},
\end{align}

(3.28)  
(3.29)

where

\begin{align}
F_1 &\equiv A_0^2 C_9^2 - 4 f^2 C_6 C_{10}, \\
F_2 &\equiv C_1^2 (C_6^2 - 4 C_7 q_2) - 4 C_1 C_3 (C_2 C_6 - C_3 C_5).
\end{align}

(3.30)  
(3.31)

Both \( c_r^1 \) and \( c_r^2 \) can be either positive or negative, depending on the direction along which the odd-parity perturbations propagate. Since \( c_r^2 \) does not arise for \( \delta A_{lm} = 0 \), \( c_r^1 \) and \( c_r^2 \) correspond to the radial sound speeds arising from the gravity sector and the vector-field sector, respectively. The existence of real solutions to eqs. (3.28) and (3.29) requires the following two conditions:

\begin{align}
F_1 &\geq 0, \\
F_2 &\geq 0,
\end{align}

(3.32)  
(3.33)

under which the small-scale Laplacian instability can be avoided.

In the large \( \omega \) and \( L \) limit, the two matrices \( M \) and \( T \), besides \( K \), lead to contributions to the propagation speed \( c_\Omega \) along the angular direction. Picking up the dominant contributions to \( M \) and \( T \) for the large \( L \) limit, it follows that

\[ M_{11} = -LC_1, \quad M_{22} = L(L - 2)C_1, \quad T_{12} = -T_{21} = -(L - 2)D_2, \]

(3.34)

where

\[ D_1 \equiv C_{12} + \frac{f C_6 C_{11}^2 + C_9^2 (f C_{10} - A_0 C_{11})}{4 f C_1^2 C_{10}} q_1, \quad D_2 \equiv C_2 + \frac{C_9 (2 f C_{10} - A_0 C_{11})}{4 f C_1 C_{10}} q_1. \]

(3.35)

The matrix component \( M_{12} \) contains the term proportional to \( L - 2 \), but it does not affect \( c_\Omega \) derived below in the limit \( L \to \infty \). On using the solution of \( \vec{X}^3 \) in the form \( e^{i(\omega t - kr)} \), the dispersion relation is given by

\[ \text{det}(\omega^2 K - i\omega T + M) = 0. \]

(3.36)
The propagation speed along the angular direction in proper time is \( c_\Omega = r d\theta / d\tau = \dot{\Omega}_\Omega / \sqrt{J} \), where \( \dot{\Omega}_\Omega = r d\theta / dt \). We substitute the relation \( \omega^2 = \frac{c_\Omega^2 l^2}{r^2} = \frac{c_\Omega^2 f t^2}{r^2} \) into eq. (3.36) and solve it for \( c_\Omega^2 \). Taking the \( L \to \infty \) limit at the end, we obtain the two propagation speed squares as

\[
c_\Omega^2 = \frac{r^2}{2 f q_1 q_2} \left[ C_1 q_2 - D_1 q_1 + D_2^2 \pm \sqrt{(C_1 q_2 + D_1 q_1)^2 + D_2^2(2C_1 q_2 - 2D_1 q_1 + D_2^2)} \right].
\]  

(3.37)

To avoid the Laplacian instability for large \( L \), we require the conditions

\[
c_\Omega^2 \leq 0.
\] 

(3.38)

Thus, we have shown that the conditions (3.25), (3.26), (3.32), (3.33), and (3.38) need to hold for avoiding ghost and Laplacian instabilities.

### 3.2 The dipole mode \( l = 1 \)

The Regge-Wheeler gauge adopted for \( l \geq 2 \) corresponds to the gauge choice eliminating the perturbation \( h_{ab} \). For the dipole mode \( l = 1 \), however, the perturbation \( h_{ab} \) vanishes identically, so the gauge is not fixed in eq. (3.13). For \( l = 1 \), i.e., \( L = 2 \), the terms multiplied by the factor \((L - 2)\) in the Lagrangian (3.15) vanish.

We recall that, under the gauge transformation \( x_\mu \to x_\mu + \xi_\mu \) with eq. (3.11), the perturbations \( Q_{1m} \) and \( W_{1m} \) transform as eq. (3.12). For \( l = 1 \), we choose the gauge in which \( W_{1m} \) vanishes, such that

\[
\Lambda_{1m}(t, r) = -2 r \int d\tilde{r} \frac{W_{1m}(t, \tilde{r})}{\tilde{r}^2} + r^2 C(t),
\] 

(3.39)

where \( C(t) \) is an arbitrary function of \( t \) corresponding to a residual gauge degree of freedom. Varying the action (3.13) with respect to \( W_{1m} \) and \( Q_{1m} \), and setting \( \dot{W}_{1m} = 0 \) at the end, it follows that

\[
\dot{\mathcal{E}} = 0,
\] 

(3.40)

\[
(r^2 \mathcal{E})' = 0,
\] 

(3.41)

where

\[
\mathcal{E} \equiv r^2 \sqrt{\frac{J}{\hbar}} \left[ C_1 \left( Q_{1m} - \frac{2}{r} Q_{1m} \right) - \left( C_2 \delta A_{1m} + C_3 \delta A'_{1m} + C_4 \delta A_{1m} \right) \right].
\] 

(3.42)

The solutions to eqs. (3.40) and (3.41) are given by

\[
\mathcal{E} = \frac{C_1}{r^2},
\] 

(3.43)

where \( C_1 \) is an integration constant.\(^1\) Since the \( Q_{1m} \)-dependent terms in the Lagrangian (3.15) appear only in the form of \( Q_{1m}^2 - 2Q_{1m} \), one can eliminate those terms by using the solu-

\[^1\text{Integrating eq. (3.43) with eq. (3.42) to solve for } Q_{1m}, \text{ it follows that}
\]

\[
Q_{1m} = r^2 \int d\tilde{r} \frac{1}{C_1 r^2} \left( \frac{\hbar}{J} + C_2 \delta A_{1m} + C_3 \delta A'_{1m} + C_4 \delta A_{1m} \right) + r^2 C_2(t),
\] 

where \( C_2(t) \) is a time-dependent gauge mode. This gauge mode can be eliminated by setting the residual gauge degree of freedom \( \Lambda_{1m} = r^2 C(t) \) to be \( C(t) = \int dt C_2(t) \). In the case where \( \delta A_{1m} = 0, Q_{1m} \) does not depend on time, and the integration constant \( C_1 \) is related to the angular momentum of a slowly rotating BH [77, 91].
tion (3.43) with eq. (3.42). After the integration by parts, the Lagrangian (3.15) reduces to
\[
\mathcal{L}_{\text{odd}} = r^2 \sqrt{\frac{\mathcal{F}}{\mathcal{F}} \left[ \frac{C_1 C_5 - C_2^2}{C_1} \delta A_{1m}^2 + \left( C_6 - \frac{2 C_2 C_3}{C_1} \right) \delta A_{1m} \delta A_{1m}' + \left( C_7 - \frac{C_3^2}{C_1} \right) \delta A_{1m}'^2 \right. \\
- \frac{2 C_3 C_4}{C_1} \delta A_{1m} \delta A_{1m} + \left( 2 C_{12} + C_{13} - \frac{C_4^2}{C_1} \right) \delta A_{1m}'^2 + \frac{h C_2^2}{C_1 r^3 f} \right], \tag{3.44}
\]

This result shows that only the vector perturbation \( \delta A_{1m} \) propagates. The coefficient of \( \delta A_{1m}'^2 \) coincides with \( q_2 \) given by eq. (3.24). Moreover, from the first three terms in eq. (3.44), one can show that the radial propagation speed of \( \delta A_{1m} \) is the same as \( c_{v,2} \) given by eq. (3.29). Thus, the dipole perturbation does not give rise to any additional condition for the absence of ghost and Laplacian instabilities to those derived for \( t \geq 2 \).

### 3.3 GR and cubic couplings \( G_3(X) \)

In GR with an electromagnetic field described by the Lagrangian \( F \), the functions \( G_i(X) \) in the action (2.1) are
\[
G_4 = \frac{M_{\text{pl}}^2}{2}, \quad G_2 = G_3 = G_5 = G_6 = g_5 = 0, \tag{3.45}
\]
where \( M_{\text{pl}} \) is the reduced Planck mass. Integrating eqs. (2.11)–(2.15) with respect to \( r \) and using the boundary conditions \( f = h = 1 \) at \( r \to \infty \), we obtain the Reissner-Nordström (RN) solution:
\[
f = h = 1 - \frac{2 M}{r} + \frac{Q^2}{2 M_{\text{pl}}^2 r^2}, \quad A_0 = P + \frac{Q}{r}, \tag{3.46}
\]
with \( A_1 \) unfixed, where \( M \) and \( Q \) are mass and electric charge, respectively, and \( P \) is an arbitrary constant. The odd-parity perturbations about the RN solution were originally studied in ref. [97]. Substituting the functions (3.45) into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), it follows that
\[
q_1 = \frac{M_{\text{pl}}^2 h}{4 f^2}, \quad q_2 = \frac{1}{2 r^2 f}, \quad \mathcal{F}_1 = \frac{M_{\text{pl}}^4 f h}{4 r^5}, \quad \mathcal{F}_2 = \frac{M_{\text{pl}}^4 h^3}{16 r^8 f^3}, \quad c_{r_1}^2 = c_{r_2}^2 = c_{\Omega_1}^2 = c_{\Omega_2}^2 = 1, \tag{3.47}
\]
where \( f = h \). Since all these quantities are positive outside the event horizon, the RN solution (3.46) suffers neither ghost nor Laplacian instabilities.

Let us then consider the cubic coupling \( G_3(X) \) in the presence of the Einstein-Hilbert term, i.e.,
\[
G_3 = G_3(X), \quad G_4 = \frac{M_{\text{pl}}^2}{2}, \quad G_2 = G_5 = G_6 = g_5 = 0. \tag{3.48}
\]
The \( G_3 \)-dependent terms appear only via \( C_{13} \) in eq. (3.15), which does not affect eqs. (3.23), (3.24), (3.28), (3.29), and (3.37). Hence the quantities \( q_1, q_2, c_{r_1}, c_{r_2}, c_{\Omega_1}^2 \) are not modified relative to those in GR given by eq. (3.47). Thus, the hairy BH solutions arising from the cubic couplings [82, 83] satisfy the conditions for the absence of ghost and Laplacian instabilities against odd-parity perturbations.

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4 Exact solutions

In this section, we study whether or not the exact BH solutions obtained in refs. [79, 82, 83] are free from ghost and Laplacian instabilities against odd-parity perturbations. These exact solutions were derived by imposing the following two relations:

\[ f = h, \quad X = X_c, \]  

where \( X_c \) is a constant. On using eq. (2.10), the latter condition of eq. (4.1) translates to

\[ A_1 = \epsilon \sqrt{\frac{A_0^2 - 2fX_c}{f}}, \]  

where \( \epsilon = \pm 1 \).

The exact solution found in ref. [79] follows from the coupling \( G_4(X) = M_{pl}^2/2 + X/4 \), \( G_2 = G_3 = G_5 = G_6 = g_5 = 0 \). Imposing the two conditions (4.1), a family of exact BH solutions was obtained in refs. [82, 83] in the presence of derivative couplings other than \( G_4(X) \). In the following, we first study the stability of the exact BH solution present for the quartic coupling, and then proceed to the cases of other exact solutions.

4.1 Quartic coupling

For general quartic couplings \( G_4(X) \), eq. (2.15) gives

\[ A_1 \left[ f h r f' + (h-1)f \right] G_{4,X} - h (A_1^2 f h r + A_0^2 f^2 h - A_0^2 f r + 2A_0 A_0' f r) G_{4,XX} \right] = 0. \]  

There are two branches characterized by \( A_1 \neq 0 \) and \( A_1 = 0 \). The exact solution derived in ref. [79] belongs to the former one, so we first study this case and then proceed to the latter branch.

4.1.1 \( A_1 \neq 0 \)

If the second derivative \( G_{4,XX} \) obeys the relation \( G_{4,XX}(X_c) = 0 \) with (4.1), then eq. (4.3) reduces to \( rf' + f - 1 = 0 \). This is integrated to give \( f = h = 1 - 2M/r \), where \( M (> 0) \) is an integration constant. There exists an exact solution consistent with eqs. (2.11)–(2.14) for \( G_{4,X}(X_c) = 1/4 \). Then, for the quartic coupling given by

\[ G_4(X) = G_4(X_c) + \frac{1}{4}(X - X_c) + \sum_{n=3} b_n(X - X_c)^n, \]  

we have the exact solution [79, 82, 83]

\[ f = h = 1 - \frac{2M}{r}, \quad A_0 = P + \frac{Q}{r}, \quad A_1 = \epsilon \sqrt{\frac{2P(MP + Q)r + Q^2}{r - 2M}}, \]  

where \( P \) and \( Q \) are integration constants with \( X_c = P^2/2 \). In the following, we choose \( \epsilon = +1 \) without loss of generality. In spite of the existence of charge \( Q \) and nonvanishing longitudinal vector component \( A_1 \), the metric components are the same as those of the Schwarzschild metric. We call this the charged stealth Schwarzschild solution. The model \( G_4(X) = M_{pl}^2/2 + X/4 \), which is equivalent to the Lagrangian \( \mathcal{L} = M_{pl}^2 R/2 + F + (1/4)G^{\mu\nu} A_\mu A_\nu \) studied in ref. [79], corresponds to the special case of eq. (4.4) with the functions \( G_4(X_c) = M_{pl}^2/2 + X_c/4 \) and \( b_n = 0 \) for \( n \geq 3 \).
Let us study whether the solution (4.5) satisfies the stability conditions derived in section 3 outside the event horizon \((r > 2M)\). Due to the complexities of such conditions for general \(r\), we focus on those in the vicinity of the horizon. Substituting eqs. (4.4)–(4.5) into eqs. (3.23)–(3.24) and then expanding them around \(r = 2M\), the quantities \(q_1\) and \(q_2\) reduce, respectively, to

\[
q_1 = \frac{[P^2 - 4G_4(X_c)](2MP + Q)^2}{32G_4(X_c)(r - 2M)^2} + \mathcal{O}((r - 2M)^{-1}),
\]

(4.6)

\[
q_2 = \frac{(2MP + Q)^2}{32M^2[2P^2 - 4G_4(X_c)](r - 2M)^2} + \mathcal{O}((r - 2M)^{-1}).
\]

(4.7)

The two no-ghost conditions (3.25) and (3.26) are satisfied for

\[
P^2 > 4G_4(X_c) \quad \text{and} \quad G_4(X_c) > 0.
\]

(4.8)

The two quantities (3.32) and (3.33) associated with the radial propagation speeds are given by

\[
\mathcal{F}_1 = -\frac{[P^2 - 4G_4(X_c)]G_4(X_c)}{4096M^{10}}(r - 2M)^2 + \mathcal{O}((r - 2M)^3),
\]

(4.9)

\[
\mathcal{F}_2 = \frac{[P^2 - 4G_4(X_c)][3P^2 - 16G_4(X_c)]}{65536M^8} + \mathcal{O}(r - 2M).
\]

(4.10)

Under the conditions (4.8) for the absence of ghosts, we have \(\mathcal{F}_1 < 0\) and hence the propagation speed \(c_{r1}\) is imaginary. The other condition \(\mathcal{F}_2 \geq 0\) is satisfied for \(P^2 \geq 16G_4(X_c)/3\).

For general quartic couplings \(G_4(X)\), the term \(D_2\) defined in eq. (3.35) identically vanishes, so the two branches of eq. (3.37) reduce to \(c_{\Omega+}^2 = C_1r^2/(f_{q1})\) and \(c_{\Omega-}^2 = -D_1r^2/(f_{q2})\), respectively. Around the event horizon, it follows that

\[
c_{\Omega+}^2 = -\frac{8MG_4(X_c)}{(2MP + Q)^2}(r - 2M) + \mathcal{O}((r - 2M)^2),
\]

(4.11)

\[
c_{\Omega-}^2 = \frac{2M[3P^2 - 16G_4(X_c)]}{(2MP + Q)^2}(r - 2M) + \mathcal{O}((r - 2M)^2).
\]

(4.12)

While \(c_{\Omega-}^2\) is positive for \(P^2 \geq 16G_4(X_c)/3\), the other propagation speed squared \(c_{\Omega+}^2\) is always negative under the latter condition of eq. (4.8). In other words, from eqs. (4.6), (4.7), and (4.11), we obtain

\[
q_1q_2c_{\Omega+}^2 = -\frac{(2MP + Q)^2}{128M(r - 2M)^3} + \mathcal{O}((r - 2M)^{-2}),
\]

(4.13)

which is negative. Thus, under the no-ghost conditions \(q_1 > 0\) and \(q_2 > 0\), the exact BH solution (4.5) present for the quartic coupling (4.4) is plagued by the Laplacian instability \(c_{\Omega+}^2 < 0\) and by the problem of imaginary \(c_{r1}\) around the event horizon.

### 4.1.2 \( A_1 = 0 \)

We proceed to the other branch characterized by \(A_1 = 0\). Under the two conditions (4.1), we can exactly solve eqs. (2.11)–(2.14) for \(G_4(X_c) = 0\) and \(G_4(X_c) = X_c/2\). Thus, for the quartic coupling given by

\[
G_4(X) = \frac{X_c}{2} + \sum_{n=2} b_n (X - X_c)^n,
\]

(4.14)
there exists the following extremal RN BH solution [82, 83]:

\[ f = h = \left(1 - \frac{M}{r}\right)^2, \quad A_0 = P \left(1 - \frac{M}{r}\right), \quad A_1 = 0, \quad (4.15) \]

where \( P^2 = 2X_c \). Substituting eqs. (4.14)–(4.15) into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), we obtain

\[ q_1 = \frac{P^2}{8f}, \quad q_2 = \frac{1}{2r^2f}, \quad F_1 = \frac{P^4f^2}{16r^8}, \quad F_2 = \frac{P^4}{64r^8}, \quad c_r^2 = c_r^2 = c_\Omega = 1. \quad (4.16) \]

Since all these quantities are positive, the exact BH solution (4.15) suffers neither ghost nor Laplacian instabilities.

### 4.2 Quintic coupling

For the quintic coupling

\[ G_5(X) = G_5(X_c) + \sum_{n=2} b_n (X - X_c)^n \quad (4.17) \]

with \( X_c = M^2_{\text{pl}} \), the following exact RN solution with charge \( Q \) is present [82, 83]:

\[ f = h = 1 - \frac{2M}{r} + \frac{Q^2}{2M^2_{\text{pl}r^2}}, \quad A_0 = -\frac{2MM^2_{\text{pl}}}{Q} + \frac{Q}{r}, \quad A_1 = \epsilon \frac{2M^3_{\text{pl}} \sqrt{2(2M^2M^2_{\text{pl}} - Q^2)r^2}}{Q[2M^2_{\text{pl}}r(2M - r) - Q^2]} . \quad (4.18) \]

This has two branches: (i) \( A_1 \neq 0 \) (realized for \( 2M^2M^2_{\text{pl}} > Q^2 \)), and (ii) \( A_1 = 0 \) (realized for \( 2M^2M^2_{\text{pl}} = Q^2 \)).

Substituting eqs. (4.17)–(4.18) into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), we find that the quantities \( q_1, q_2, F_1, F_2, c_r^2, c_r^2, c_\Omega^2 \) are the same as those in GR given by eq. (3.47). Hence the exact solution (4.18) is plagued by neither ghost nor Laplacian instabilities.

### 4.3 Sixth-order coupling

For general sixth-order couplings \( G_6(X) \), eq. (2.15) reduces to

\[ A_1 A_0^2 \left[A_1^2 h^2 G_{6,XX} + (1 - 3h) G_{6,X} \right] = 0 . \quad (4.19) \]

There are two non-trivial branches characterized by (i) \( A_1 = 0 \) and (ii) \( A_0^2 = 0 \). In the following, we study the stability of solutions in each branch separately.

#### 4.3.1 \( A_1 = 0 \)

For the branch \( A_1 = 0 \), we have \( A_0^2 = 2fX_c \) from eq. (4.2). In this case, the model with the sixth-order coupling

\[ G_6(X) = \sum_{n=2} b_n (X - X_c)^n , \quad (4.20) \]

where \( X_c = M^2_{\text{pl}} \), gives rise to the following exact solution with the extremal RN metric [82, 83]:

\[ f = h = \left(1 - \frac{M}{r}\right)^2, \quad A_0 = \epsilon \sqrt{2}M^2_{\text{pl}} \left(1 - \frac{M}{r}\right), \quad A_1 = 0 . \quad (4.21) \]
Substituting eqs. (4.20)–(4.21) into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), it follows that the quantities \( q_1, q_2, F_1, F_2, c_{11}^2, c_{12}^2, c_{13}^2 \) are of the same forms as those in GR given by eq. (3.47) with \( f = h = (1 - M/r)^2 \). Thus, the exact solution (4.21) exhibits neither ghost nor Laplacian instabilities.

### 4.3.2 \( A_0' = 0 \)

The other branch corresponds to \( A_0 = P = \text{constant} \) with \( A_1 = \epsilon \sqrt{P^2 - 2 f X_c} / f \). Since all the \( G_6 \)-dependent terms in eqs. (2.11)–(2.15) are multiplied by \( A_0' \), the background solution simply reduces to the stealth Schwarzschild solution given by

\[
F = h = 1 - \frac{2M}{r}, \quad A_0 = P, \quad A_1 = \epsilon \sqrt{r(P^2 - 2 r X_c + 4 M X_c) / (r - 2 M)}, \tag{4.22}
\]

which arises for arbitrary couplings \( G_6(X) \). For the existence of \( A_1 \), we require that \( P^2 r - 2 r X_c + 4 M X_c > 0 \). Substituting the solution (4.22) into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), we find that \( q_1, F_1, c_{11}^2 \), and \( c_{13}^2 \) are of the same forms as eq. (3.47) with \( f = h = 1 - 2M/r \).

On the other hand, the quantities \( q_2, F_2, c_{12}^2 \), and \( c_{13}^2 \) are affected by the sixth-order coupling \( G_6(X) \). In the vicinity of the event horizon, we have

\[
q_2 = -\frac{P^2 G_{6,X}(X_c)}{8 M^2 (r - 2M)^2} + \mathcal{O}((r - 2M)^{-1}), \quad c_{13}^2 = -\frac{G_6(X_c) + 2 M^2}{M^2 G_{6,X}(X_c)} (r - 2M) + \mathcal{O}((r - 2M)^2), \tag{4.23}
\]

respectively. The conditions \( q_2 > 0 \) and \( c_{13}^2 \geq 0 \) translate, respectively, to

\[
G_{6,X}(X_c) < 0, \tag{4.24}
\]
\[
G_6(X_c) \geq -2 M^2. \tag{4.25}
\]

Around the event horizon, the condition \( F_2 \geq 0 \) corresponds to

\[
[G_6(X_c) - 4 M^2][G_6(X_c) - 4 M^2 + 2 X_c G_{6,X}(X_c)] \geq 0, \tag{4.26}
\]

under which \( c_{12}^2 = 1 + \mathcal{O}(r - 2M) \). Then, for \( X_c > 0 \), the Laplacian instabilities are absent under the conditions

\[
-2 M^2 \leq G_6(X_c) \leq 4 M^2 \quad \text{or} \quad G_6(X_c) \geq 4 M^2 - 2 X_c G_{6,X}(X_c). \tag{4.27}
\]

At spatial infinity \( (r \gg 2M) \), the quantities \( q_2, c_{12}^2 \), and \( c_{13}^2 \) reduce to those in GR with \( f \to 1 \) and \( h \to 1 \). In summary, the solution (4.22) is subject to neither ghost nor Laplacian instabilities under the conditions (4.24) and (4.27).

### 4.4 Other intrinsic vector-mode couplings

#### 4.4.1 Coupling \( g_4(X) \)

Let us consider the model given by the coupling

\[
G_2(X, F) = -2 g_4(X) F, \tag{4.28}
\]

which was originally introduced in ref. [29] as an intrinsic vector mode in the quartic Lagrangian \( L_4 \). In this case, eq. (2.15) reduces to

\[
g_{4,X} A_0'^2 A_1 = 0. \tag{4.29}
\]
For the model
\[ g_4(X) = g_4(X_c) + \sum_{n=2} b_n (X - X_c)^n, \]  
which satisfies the condition \( g_4, X_c = 0 \), there exists the RN-type exact solution
\[ f = h = 1 - \frac{2M}{r} + \frac{Q^2}{2M_\text{pl}^2 r^2} [1 - 2g_4(X_c)], \quad A_0 = P + \frac{Q}{r}, \quad A_1 = \epsilon \sqrt{A_0^2 - 2fX_c}. \]  
Plugging the solution (4.31) with (4.30) into eqs. (3.24) and (3.31), it follows that
\[ q_2 = \frac{1 - 2g_4(X_c)}{2r^2f}, \quad \mathcal{F}_2 = \frac{M_\text{pl}^4 [1 - 2g_4(X_c)]^2}{16r^8} \geq 0. \]  
The other quantities \( q_1, \mathcal{F}_1, c_{\Omega_1}^2, c_{\Omega_+}^2, c_{\Omega_-}^2 \) are of the same forms as eq. (3.47) with \( f \) and \( h \) given by eq. (4.31). Then, the solution (4.31) is plagued by neither ghost nor Laplacian instabilities for
\[ g_4(X_c) < \frac{1}{2}. \]  

From eq. (4.29), we also have the branch characterized by \( A_0' = 0 \). In this case, the \( g_4 \)-dependent terms in the background equations completely vanish, so the resulting solution is the same as eq. (4.22). On using this solution, we find that neither ghost nor Laplacian instabilities arise under the condition same as eq. (4.33).

We note that eq. (4.29) admits the other branch \( A_1 = 0 \). For the model (4.30) with \( g_4(X_c) = 1/2 \), there exists the other stealth Schwarzschild solution \( f = h = 1 - 2M/r, A_0 = \epsilon \sqrt{2(1 - 2M/r)X_c}, A_1 = 0 \). In this case the quantity \( q_2 \) exactly vanishes, so the strong-coupling problem is present.

### 4.4.2 Coupling \( g_5(X) \)

For the quintic intrinsic vector-mode coupling \( g_5(X) \), eq. (2.15) gives
\[ [f g_5 - (A_0^2 - 2fX_c) g_5, X] A_0^2 = 0. \]  

For the branch characterized by \( f g_5 = (A_0^2 - 2fX_c) g_5, X \), the model satisfying the conditions \( g_5(X_c) = 0 \) and \( g_5, X_c = 0 \), e.g.,
\[ g_5(X) = \sum_{n=2} b_n (X - X_c)^n, \]  
gives rise to the RN solution (3.46) with the longitudinal vector component (4.2). In this case, we find that all the quantities in eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37) are the same as those in GR given by eq. (3.47).

For the other branch \( A_0' = 0 \) of eq. (4.34), we obtain the stealth Schwarzschild solution same as eq. (4.22) for arbitrary couplings \( g_5(X) \). Substituting this solution into eqs. (3.23)–(3.24), (3.28)–(3.31), and (3.37), the quantities \( q_1, \mathcal{F}_1, c_{\Omega_1}^2, \) and \( c_{\Omega_-}^2 \) are of the same forms as eq. (3.47) with \( f = h = 1 - 2M/r \). On the other hand, around the horizon, the leading-order terms in \( q_2 \) and \( c_{\Omega_-}^2 \) are given, respectively, by
\[ q_2 = \frac{|P| g_5(X_c)}{4M(r - 2M)^2} + \mathcal{O}((r - 2M)^{-1}), \quad c_{\Omega_-}^2 = \frac{M |P| + X_c g_5(X_c)}{MP^2 g_5(X_c)} (r - 2M) + \mathcal{O}((r - 2M)^2). \]  
\[ (4.36) \]
Then, the conditions $q_2 > 0$ and $c_{r2}^2 \geq 0$ translate to
\begin{align}
g_s(X_c) &> 0, \quad \text{(4.37)} \\
M|P| + X_c g_s(X_c) &\geq 0, \quad \text{(4.38)}
\end{align}
respectively. The condition $\mathcal{F}_2 \geq 0$ corresponds to
\begin{equation}
[M - |P|g_s(X_c)] \left[ M - P^2 \frac{X_c}{|P|} g_s(X_c) \right] \geq 0, \quad \text{(4.39)}
\end{equation}
under which $c_{r2}^2 = 1 + \mathcal{O}(r - 2M)$. There exists the parameter space in which all the conditions (4.37)–(4.39) are satisfied. At spatial infinity ($r \gg 2M$), the quantities $q_2, c_{r2}, c_{\Omega -}$ are the same as those in GR.

5 Quartic power-law couplings

In section 4.1, we studied whether or not the conditions for the absence of ghost and Laplacian instabilities are satisfied for some exact BH solutions by imposing the two conditions (4.1). Now, without imposing the conditions (4.1), we extend the analysis to more general quartic power-law couplings given by
\begin{equation}
G_4(X) = \frac{M_{\text{pl}}^2}{2} + \beta_4 M_{\text{pl}}^2 \left( \frac{X}{M_{\text{pl}}} \right)^n, \quad \text{(5.1)}
\end{equation}
where $\beta_4$ and $n \geq 1$ are constants. From eq. (4.3), there are several branches characterized by $A_1 \neq 0$ or $A_1 = 0$. For $n \geq 3$, there is a branch satisfying $A_1^2 = A_0^2/(f h)$ besides other branches discussed below. This corresponds to the RN solution obeying the specific relation $X = 0$ from eq. (2.10). In this case we have $G_4 = M_{\text{pl}}^2/2$ and $G_{4,X} = G_{4,XX} = 0$, so the quantities $q_1, q_2, \mathcal{F}_1, \mathcal{F}_2, c_{r1}^2, c_{r2}^2, c_{\Omega -}$ trivially reduce to those in GR given by eq. (3.47). In the following, we will focus on other branches of eq. (4.3).

5.1 Branch with $A_1 \neq 0$

Let us consider the quartic power-law model with $n \geq 2$. Solving eq. (4.3) for $A_1$, there is the branch satisfying
\begin{equation}
A_1 = \pm \sqrt{\frac{A_0^2 [f(h - 1) + (2n - 1)rf'h] - 4r A_0 A'_0(n - 1)fh}{fh[(2n - 1)rf'h - (1 + h - 2nh)f]}}. \quad \text{(5.2)}
\end{equation}
The effect of coupling $\beta_4$ on $f$ and $h$ appears as corrections to the RN metric expressed in the form
\begin{equation}
f_{\text{RN}} = h_{\text{RN}} = \left( 1 - \frac{r_h}{r} \right) \left( 1 - \mu \frac{r_h}{r} \right), \quad \text{(5.3)}
\end{equation}
where $\mu$ is a constant in the range $0 < \mu < 1$. In the vicinity of the event horizon characterized by the distance $r_h$, we expand $f, h, A_0$ as follows:
\begin{equation}
f = \sum_{i=1}^{\infty} f_i (r - r_h)^i, \quad h = \sum_{i=1}^{\infty} h_i (r - r_h)^i, \quad A_0 = a_0 + \sum_{i=1}^{\infty} a_i (r - r_h)^i, \quad \text{(5.4)}
\end{equation}
where \(f_i, h_i, a_0, a_i\) are constants. We assume that \(a_0 > 0\) without loss of generality. Substituting eq. (5.4) into eqs. (2.11)–(2.15) and solving them iteratively, the coefficients \(f_1, h_1, a_1\) are known to be

\[
f_1 = h_1 = \frac{1 - \mu}{r_h} > 0, \quad a_1 = \frac{\sqrt{2\mu M_{\text{pl}}}}{r_h} + \tilde{a}_1, \tag{5.5}\]

where \(\tilde{a}_1\) is a \(\beta_4\)-dependent constant that vanishes in the limit \(\beta_4 \to 0\) [82, 83]. The coupling \(\beta_4\) arises in metric components at the orders of \(i \geq 2\). To study the stability of hairy BH solutions against odd-mode perturbations for the model (5.1), we do not need the explicit expressions of coefficients in eq. (5.4). Plugging eq. (5.4) into eq. (5.2), the longitudinal component around the event horizon behaves as

\[
A_1 = \frac{a_0}{f_1(r - r_h)} + \frac{a_0[f_2 + h_2 - 2f_1^2(n - 1) - f_1(f_2 + h_2)(2n - 1)r_h + 2a_1f_1(f_1r_h - 1)]}{2f_1^2[(2n - 1)f_1r_h - 1]} + \mathcal{O}(r - r_h), \tag{5.6}\]

which diverges as \(r \to r_h\). This divergent property also persists for the exact solution discussed in section 4.1, but the regularity of the coordinate-independent scalar quantity \(\int A_\mu dx^\mu\) is ensured by introducing advanced and retarded null coordinates [82, 83]. We recall that, due to the property \(D_2 = 0\) for general quartic couplings \(G_4(X)\), the two solutions of eq. (3.37) reduce to \(\tilde{c}_{D+}^2 = C_1 r^2/(q_1 f)\) and \(\tilde{c}_{D-}^2 = -D_1 r^2/(q_2 f)\). Substituting eq. (5.4) into eqs. (3.23), (3.24), and \(\tilde{c}_{D+}^2\), and then picking up the leading-order contributions around the event horizon, the product \(q_1 q_2 \tilde{c}_{D+}^2\) yields

\[
q_1 q_2 \tilde{c}_{D+}^2 = -\left(\frac{(n - 1)a_0(a_0 + 2a_1 r_h)}{M_{\text{pl}}^2(2n - 1)f_1 r_h - 1}\right)^{2(n-1)} \frac{n^2 a_0^2 \beta_4^2}{4 f_1^2 r_h^2 (r - r_h)^2} + \mathcal{O}((r - r_h)^{-2}), \tag{5.7}\]

which is always negative for \(\beta_4 \neq 0\). We note that the charged stealth Schwarzschild solution discussed in section 4.1.1, i.e., \(n = 1\) and \(\beta_4 = 1/4\) in eq. (5.1), also has the same property. Setting \(n = 1\), \(\beta_4 = 1/4\), \(r_h = 2M\), \(f_1 = 1/(2M)\), and \(a_0 = P + Q/(2M)\) in eq. (5.7), we reproduce the result given by eq. (4.13).

From eq. (5.7), the conditions for the absence of ghosts \((q_1 > 0, q_2 > 0)\) and those for no Laplacian instability along the angular direction \((\tilde{c}_{D+}^2 \geq 0)\) cannot be simultaneously satisfied. Even if \(\beta_4\) is very close to 0, the first term on the right hand side of eq. (5.7) dominates over the other terms in the limit that \(r \to r_h\). This means that, no matter how small the coupling constant \(\beta_4\) is, the hairy BH solutions present for the quartic model (5.1) are unstable in the vicinity of the event horizon. This instability is mostly attributed to the existence of nonvanishing longitudinal component \(A_1\) which exhibits the divergence at \(r = r_h\).

### 5.2 Branch with \(A_1 = 0\)

Let us proceed to the other branch characterized by \(A_1 = 0\) for the power-law model (5.1) with \(n \geq 1\). For this branch, the coefficients \(C_2\) and \(C_5\) in eq. (3.24) simply reduce to \(C_2 = 0\) and \(C_5 = 1/(2fr^2)\), respectively, so that

\[
q_2 = \frac{1}{2fr^2}, \tag{5.8}\]

Then, the second no-ghost condition (3.26) is always satisfied throughout the horizon exterior.
The coupling $\beta_4$ works as corrections to the RN metric given by eq. (5.3). Plugging eq. (5.4) into eqs. (2.11)–(2.14), the resulting iterative solution around the event horizon has the leading-order terms:

$$
f_1 = h_1 = \frac{1 - \mu}{r_h} > 0, \quad a_0 = 0, \quad a_1 = \frac{\sqrt{2} \mu M_{pl}}{r_h}, \quad (5.9)
$$

so that the temporal component $A_0$ vanishes on the event horizon. The coupling $\beta_4$ appears in the coefficients of expansions of $f, h, A_0$ at the order of $(r - r_h)^{n+1}$ [82, 83]. Substituting the iterative solution (5.4) with (5.9) into eq. (3.23), we obtain

$$
q_1 = \frac{M_{pl}^2}{4f_1 (r - r_h)} + \mathcal{O} \left( (r - r_h)^0 \right), \quad (5.10)
$$

which means that the first no-ghost condition (3.25) is satisfied. Around the event horizon, the quantities (3.30) and (3.31) yield

$$
\mathcal{F}_1 = \frac{M_{pl}^4 f_1^2}{4r_h^5} (r - r_h)^2 + \mathcal{O} \left((r - r_h)^3\right), \quad \mathcal{F}_2 = \frac{M_{pl}^4}{16r_h^8} + \mathcal{O} \left(r - r_h\right), \quad (5.11)
$$

which are both positive, as required for the absence of Laplacian instabilities in the radial direction.

For general quartic couplings $G_4(X)$ with the branch $A_1 = 0$, the propagation speeds along the radial and angular directions have the following relations:

$$
c_{r_1}^2 = c_{\Omega_+}^2 = \frac{fG_4}{fG_4 - A_0^2 G_{4,XX}},
$$

$$
c_{r_2}^2 = c_{\Omega_-}^2 = \frac{fG_4 + A_0^2 G_{4,XX}(G_{4,X} - 1)}{fG_4 - A_0^2 G_{4,XX}}. \quad (5.13)
$$

On using the expanded solution (5.4) around the event horizon with eq. (5.9), eqs. (5.12) and (5.13) reduce to

$$
c_{r_1}^2 = c_{\Omega_+}^2 = 1 + \beta_4 \frac{n}{2^{n-2}} \left( \frac{2\mu}{1 - \mu} \right)^n \left( \frac{r}{r_h} - 1 \right)^n + \mathcal{O} \left( (r - r_h)^{n+1} \right), \quad (5.14)
$$

$$
c_{r_2}^2 = c_{\Omega_-}^2 = 1 + \beta_4^2 \frac{n^2}{2^{2n-3}} \left( \frac{2\mu}{1 - \mu} \right)^{2n-1} \left( \frac{r}{r_h} - 1 \right)^{2n-1} + \mathcal{O} \left( (r - r_h)^{2n} \right), \quad (5.15)
$$

which approach 1 in the limit that $r \to r_h$. Hence the Laplacian instabilities are absent around $r = r_h$. The small deviations of $c_{r_1}^2$ and $c_{r_2}^2$ from 1 arise from the coupling $\beta_4$ in the vicinity of the event horizon.

At spatial infinity ($r \gg r_h$), the solutions of $f, h, A_0$ expanded as a series of $1/r$ are given by

$$
f = 1 - \frac{2M}{r} \left[ 1 - \frac{2\beta_4 n(2MM_{pl} \hat{P} + \sqrt{2}Q) \hat{P}^{2n}}{MM_{pl} \hat{P} (2\beta_4 (2n-1) \hat{P}^{2n-1})} \right] + \frac{1}{r^2} \left( \frac{Q^2}{2M_{pl}^4} + \mu_1 \right) + \mathcal{O}(r^{-3}), \quad (5.16)
$$

$$
h = 1 - \frac{2M}{r} + \frac{1}{r^2} \left( \frac{Q^2}{2M_{pl}^2} + \mu_2 \right) + \mathcal{O}(r^{-3}), \quad (5.17)
$$

$$
A_0 = P + \frac{Q}{r} + \frac{\mu_3}{r^2} + \mathcal{O}(r^{-3}), \quad (5.18)
$$
where \( \bar{P} = P/(\sqrt{2}M_{\text{pl}}) \), and \( \mu_1, \mu_2, \mu_3 \) are \( r \)-independent constants containing the dependence of \( \beta_4 \) (which vanish in the limit that \( \beta_4 \to 0 \)). The terms \( \mu_1, \mu_2, \mu_3 \) do not appear for the dominant contributions to \( q_1, q_2, c_{r1}^2, c_{r2}^2, c_{t1}^2, c_{t2}^2 \). The leading-order term of \( q_1 \) is given by

\[
q_1 = \frac{M_{\text{pl}}^2}{4} \left[ 1 - 2(2n - 1)\beta_4 \bar{P}^{2n} \right]^2 \left[ 1 + 2\beta_4 \bar{P}^{2n} \right],
\]

so the ghost instabilities are absent for

\[
\beta_4 \bar{P}^{2n} > -\frac{1}{2}.
\]

The leading-order contributions to the quantities (3.30) and (3.31) are

\[
\mathcal{F}_1 = \frac{M_{\text{pl}}^4}{4 \pi^8} \left[ 1 + 2\beta_4 \bar{P}^{2n} \right] \left[ 1 - 2(2n - 1)\beta_4 \bar{P}^{2n} \right],
\]

\[
\mathcal{F}_2 = \frac{M_{\text{pl}}^4}{16 \pi^8} \left[ 1 - 2(2n - 1)\beta_4 \bar{P}^{2n} \right] \left[ 1 - 2(2n - 1)\beta_4 \bar{P}^{2n} + 4n^2 \beta_4^2 \bar{P}^{4n-2} \right].
\]

Under the condition (5.20), the two conditions \( \mathcal{F}_1 \geq 0 \) and \( \mathcal{F}_2 \geq 0 \) are satisfied for

\[
1 - 2(2n - 1)\beta_4 \bar{P}^{2n} \geq 0 \quad \text{and} \quad 1 - 2(2n - 1)\beta_4 \bar{P}^{2n} + 4n^2 \beta_4^2 \bar{P}^{4n-2} \geq 0.
\]

The leading-order sound speeds are given by

\[
c_{r1}^2 = c_{r1}^2 = 1 + \frac{4n\beta_4 \bar{P}^{2n}}{1 - 2(2n - 1)\beta_4 \bar{P}^{2n}}, \quad c_{r2}^2 = c_{t2}^2 = 1 + \frac{4n^2 \beta_4^2 \bar{P}^{4n-2}}{1 - 2(2n - 1)\beta_4 \bar{P}^{2n}},
\]

which are both positive under the inequalities (5.20) and (5.23). Hence there are neither ghost nor Laplacian instabilities under the conditions (5.20) and (5.23), which are well satisfied for \( |\beta_4| \) and \( |\bar{P}| \) smaller than the order of unity.

To confirm the above analytic estimations, we numerically compute the radial dependence of \( q_1, q_2 \) as well as \( c_{r1}^2, c_{r2}^2 \) for hairy BH solutions present in the model (5.1). For the branch \( A_1 = 0 \), we recall that \( c_{r1}^2 \) and \( c_{t1}^2 \) are equivalent to \( c_{r1}^2 \) and \( c_{r2}^2 \), respectively. We numerically integrate eqs. (2.11)-(2.14) for \( n = 2 \) by employing the boundary conditions of \( f, h, A_0 \) around the event horizon (see equations (6.17)-(6.19) in ref. [1]) and substitute the background solutions into (3.23)-(3.24) and (3.28)-(3.29). In figure 1, we plot \( q_1, q_2 \) and \( c_{r1}^2 - 1, c_{r2}^2 - 1 \) versus \( r/r_h \) for \( \beta_4 = 1 \) and \( \bar{P} = 0.5 \). In this case, the conditions (5.20) and (5.23) are consistently satisfied at spatial infinity. In figure 1, we can confirm that there are neither ghost nor Laplacian instabilities throughout the horizon exterior. The asymptotic values of \( c_{r1}^2 \) and \( c_{r2}^2 \) on the event horizon are equivalent to 1, while at spatial infinity, \( c_{r1}^2 \) and \( c_{r2}^2 \) approach constants different from 1. This difference is induced by the non-zero coupling \( \beta_4 \).

\[\text{For the model (5.1) with } A_1 \neq 0, \text{ the quantities } q_1, q_2 \text{ and } c_{r1}^2, c_{r2}^2, c_{t1}^2, c_{t2}^2 \text{ at spatial infinity also reduce to eqs. (5.19) and (5.24) at leading order, respectively, by reflecting the fact that } A_1 \text{ vanishes in the limit that } r \to \infty \text{ [82, 83].}\]
Figure 1. Numerical plots of $q_1$ and $q_2$ normalized by $M_{pl}^2/4$ and $r_h^{-2}$ respectively (left), and the deviations of $c^2_{r1}, c^2_{r2}$ from 1 (right) for the quartic power-law model (5.1) with $n = 2$ and $A_1 = 0$. We choose the parameters as $\beta_4 = 1$, $\tilde{P} = 0.5$, and $\mu = 0.2$.

6 Sixth-order power-law couplings

In this section, we consider the sixth-order power-law couplings given by

$$G_6(X) = \frac{\beta_6}{M_{pl}^2} \left( \frac{X}{M_{pl}^2} \right)^n,$$

with $G_4 = M_{pl}^2/2$, where $\beta_6$ and $n \geq 0$ are constants. In this case, the longitudinal mode obeys

$$\beta_6 A_0 A_1 (A_0^2 - f_h A_1^2)^{n-2} \left[ A_1^2 f_h ((2n+1)h - 1) - A_0^2 (3h - 1) \right] = 0.$$  

(6.2)

From eq. (6.2), there are four possible branches characterized by (i) $A'_0 = 0$, (ii) $A_1 = 0$, (iii) $A_0^2 = f_h A_1^2$ (present for $n \geq 3$), and (iv) $A_1^2 = A_0^2 (3h - 1)/[f_h ((2n+1)h - 1)]$ [82, 83]. The branches (i) and (iii) give rise to the stealth Schwarzschild solution with $A_1$ undetermined and the trivial RN solution, respectively. The branch (iv) does not exist throughout the horizon exterior ($0 < h < 1$). Then, we focus on the branch (ii), i.e.,

$$A_1 = 0.$$  

(6.3)

The theory with $n = 0$ corresponds to the U(1)-gauge invariant vector-field interaction advocated by Horndeski [98], in which case the hairy BH solution with $A_1 = 0$ was found in ref. [99]. In this case, we have $G_{6,X} = 0$, $C_9 = 0$ and $C_8 = -f_h C_{10}$, so the quantity (3.30) reduces to $F_1 = 4f^3 h C_{10}^2 \geq 0$. From eq. (3.28), it follows that

$$c^2_{r1} = 1 \quad \text{(for } n = 0).$$  

(6.4)

For the power-law models with $n \geq 0$, let us study whether the theoretical consistent conditions derived in section 3 are satisfied by using the iterative solutions (5.4) in the vicinity
of the event horizon. The coupling $\tilde{\beta}_6$ appears as corrections to the RN solution (5.3). For $n = 0$, the coefficients of eq. (5.4) consistent with the background eqs. (2.11)–(2.14) are

$$f_1 = h_1 = \frac{1 - \mu}{r_h}, \quad a_1 = \frac{M_{pl}}{r_h} \sqrt{\frac{2\mu}{1 + 2\tilde{\beta}_6}},$$

$$f_2 = -\frac{1 - 2\mu + (2 - 5\mu + 3\mu^2)\tilde{\beta}_6}{(1 + 2\tilde{\beta}_6) r_h^2}, \quad h_2 = -\frac{1 - 2\mu + (2 - \mu - \mu^2)\tilde{\beta}_6}{(1 + 2\tilde{\beta}_6) r_h^2},$$

$$a_2 = -\frac{M_{pl}}{r_h} \sqrt{\frac{2\mu}{1 + 2\tilde{\beta}_6}} \left(1 + \frac{1 - \tilde{\beta}_6}{2\tilde{\beta}_6}ight),$$

(6.5)

where $\tilde{\beta}_6 \equiv \beta_6/(r_h^2 M_{pl}^2)$, $\mu$ is a constant appearing in the RN metric (5.3), and $a_0 \neq 0$. Substituting these iterative solutions into eqs. (3.23) and (3.24), we obtain

$$q_1 = \frac{M_{pl}^2(1 + 2\tilde{\beta}_6) r_h}{4(1 - \mu)[1 + 2(1 - \mu)\tilde{\beta}_6](r - r_h)} + \mathcal{O}((r - r_h)^0),$$

$$q_2 = \frac{1 - (1 - \mu)\tilde{\beta}_6}{2(1 - \mu) r_h (r - r_h)} + \mathcal{O}((r - r_h)^0).$$

(6.6)

The leading-order terms of $c_{r2}^2, c_{\Omega+}^2, c_{\Omega-}^2$ in the vicinity of the event horizon yield

$$c_{r2}^2 = 1, \quad c_{\Omega+}^2 = \frac{1 + 2(1 - \mu)\tilde{\beta}_6}{1 + 2\tilde{\beta}_6}, \quad c_{\Omega-}^2 = \frac{[1 + 2(1 - \mu)\tilde{\beta}_6]^2}{(1 + 2\tilde{\beta}_6)[1 + (1 - \mu)\tilde{\beta}_6]}.$$  

(6.7)

If the coupling $\tilde{\beta}_6$ is in the range

$$-\frac{1}{2} < \tilde{\beta}_6 < \frac{1}{1 - \mu} \quad \text{(for } n = 0),$$

(6.8)

the conditions $q_1 > 0, q_2 > 0, c_{\Omega+}^2 \geq 0, c_{\Omega-}^2 \geq 0$ are satisfied. Since $\mathcal{F}_2 = M_{pl}^4[1 - (1 - \mu)\tilde{\beta}_6^2/(16r_h^8)]$, the condition (3.33) is automatically satisfied.

For the models with $n \geq 1$, the coefficients in the expansions (5.4) up to the order of $r - r_h$ are given by

$$f_1 = h_1 = \frac{1 - \mu}{r_h}, \quad a_0 = 0, \quad a_1 = \frac{\sqrt{2\mu} M_{pl}}{r_h}.$$  

(6.9)

The coupling $\beta_6$ appears for the terms whose orders are higher than $r - r_h$. Then, the quantities $q_1$ and $q_2$ reduce, respectively, to

$$q_1 \simeq \frac{M_{pl}^2}{4 f_1 (r - r_h)} + \mathcal{O}((r - r_h)^0), \quad q_2 \simeq \frac{1}{2 f_1 r_h^2 (r - r_h)} + \mathcal{O}((r - r_h)^0),$$

(6.10)

so that the two conditions (3.25) and (3.26) are satisfied. The quantities $\mathcal{F}_1$ and $\mathcal{F}_2$ are the same as those given in eq. (5.11), so the conditions (3.32) and (3.33) are also satisfied. For $n = 1$, the propagation speeds are given by

$$c_{r1}^2 = 1 + \mathcal{O}(r - r_h), \quad c_{r2}^2 = 1 + \mathcal{O}((r - r_h)^2),$$

$$c_{\Omega+}^2 = 1 + \mathcal{O}(r - r_h), \quad c_{\Omega-}^2 = 1 - \mu \tilde{\beta}_6 + \mathcal{O}(r - r_h).$$

(6.11)
Then, there are neither ghost nor Laplacian instabilities under the condition

$$\tilde{\beta}_6 \leq \frac{1}{\mu} \quad \text{(for } n = 1\!) .$$

(6.12)

If \( n \geq 2 \), then the propagation speeds yield

$$c^2_{r_1} = 1 + O((r - r_h)^n) , \quad c^2_{r_2} = 1 + O((r - r_h)^{2n}) , \quad c^2_{\Omega^+} = 1 + O((r - r_h)^n) , \quad c^2_{\Omega^-} = 1 + O((r - r_h)^{n-1}) ,$$

(6.13)

where we used the property \( f_i = h_i \) for \( i \leq n \) in the expansions of eq. (5.4) [83]. The coupling \( \tilde{\beta}_6 \) gives rise to the deviations of \( c^2_{r_1}, c^2_{r_2}, c^2_{\Omega^+}, c^2_{\Omega^-} \) from 1. From eq. (6.13), the Laplacian instabilities are absent in the vicinity of the event horizon.

At spatial infinity \((r \gg r_h)\), the solutions of \( f, h, A_0 \) expanded as a series of \( 1/r \) are given [82, 83] by

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{2M^2_{\text{pl}}r^2} - \frac{\tilde{\beta}_6 P^2n Q^2}{2^{1+n}M_{\text{pl}}^{4+2n}r^4} - \frac{2^{-n}\beta_6 P^2n-1 Q^2 [MP(6n - 5) + 8Qn]}{10M_{\text{pl}}^{1+2n}r^5} + O(r^{-6}) ,$$

(6.14)

$$h = 1 - \frac{2M}{r} + \frac{Q^2}{2M^2_{\text{pl}}r^2} + \frac{\beta_6 M P^2n Q^2(2n - 1)}{2^{1+n}M_{\text{pl}}^{4+2n}r^5} + O(r^{-6}) ,$$

(6.15)

$$A_0 = P + \frac{Q}{r} - \frac{2^{-n}\beta_6 M P^2n Q}{M_{\text{pl}}^{2+2n}r^4} - \frac{2^{-n}\beta_6 P^2n-1 Q (32M^2 M_{\text{pl}}^4 Pn + 28MM_{\text{pl}}^2 Qn - 3PQ^2)}{20M_{\text{pl}}^{4+2n}r^5} + O(r^{-6}) ,$$

(6.16)

which means that the coupling \( \tilde{\beta}_6 \) works as corrections to the RN solution (3.46). On using eqs. (6.14)–(6.16), it follows that \( q_1, q_2, F_1, F_2 \) approach the GR values (3.47) with \( f = h = 1 \). The propagation speed squares \( c^2_{r_1}, c^2_{r_2}, c^2_{\Omega^+}, c^2_{\Omega^-} \) also approach the asymptotic value 1. For instance, the deviation of \( c^2_{r_1} \) from 1 far outside the event horizon behaves as

$$c^2_{r_1} - 1 \simeq 2n\tilde{\beta}_6 \tilde{P}^2n \tilde{Q}^2 \left(\frac{r_h}{r}\right)^4 ,$$

(6.17)

where \( \tilde{P} = P/(\sqrt{2}M_{\text{pl}}) \) and \( \tilde{Q} = Q/(r_h M_{\text{pl}}) \). Then, \( c^2_{r_1} - 1 \) decreases rapidly for increasing \( r \).

By integrating eqs. (2.11)–(2.14) with respect to \( r \) with the boundary conditions of \( f, h, A_0 \) around the event horizon, we can numerically obtain hairy BH solutions induced by the coupling \( \tilde{\beta}_6 \). For such background solutions, we compute the quantities associated with the absence of ghost and Laplacian instabilities. In figure 2, we plot \( q_1, q_2 \) as well as the deviations of \( c^2_{r_1}, c^2_{r_2}, c^2_{\Omega^+}, c^2_{\Omega^-} \) from 1 as functions of \( r/r_h \) for the model parameters \( n = 1, \tilde{\beta}_6 = 0.7, \) and \( \mu = 0.2 \). In this case, the condition (6.12) is satisfied around the event horizon. As we see in the left panel of figure 2, both \( q_1 \) and \( q_2 \) are positive throughout the horizon exterior. In the right panel, we observe that \( c^2_{r_1}, c^2_{r_2}, c^2_{\Omega^+} \) approach 1 as \( r \to r_h \), while \( c^2_{\Omega^-} \to 1 - \mu \tilde{\beta}_6 \) in the same limit. The propagation speed squares show some deviations from 1 slightly outside the event horizon, but they rapidly approach the asymptotic value 1 for the distance \( r \gg r_h \). Thus, there are neither ghost nor Laplacian instabilities throughout the horizon exterior. Under the condition (6.8), this property also holds for the BH solution arising from the U(1) gauge-invariant interaction \((n = 0)\).
Figure 2. Numerical plots of $q_1$ and $q_2$ normalized by $M_{\text{pl}}^2/4$ and $r_h^{-2}$ respectively (left), and the deviations of $c_{r1}^2$, $c_{r2}^2$, $c_{\Omega+}^2$, $c_{\Omega-}^2$ from 1 (right) for the sixth-order power-law model (6.1) with $n = 1$. The model parameters are chosen to be $\beta_6 = 0.7$ and $\mu = 0.2$.

7 Other intrinsic vector-mode couplings

Finally, we study whether or not hairy BH solutions arising from intrinsic vector modes $g_4(X)$ and $g_5(X)$ satisfy the conditions for the absence of ghosts and Laplacian instabilities. We will mostly discuss the power-law models $g_4(X) \propto X^n$ and $g_5(X) \propto X^n$, but as we see below, it is possible to derive several conditions in a general way without restricting their functional forms.

7.1 Coupling $g_4(X)$

Let us consider the coupling $G_2(X, F) = -2g_4(X)F$ with $G_4 = M_{\text{pl}}^2/2$. In this case, the longitudinal mode $A_1$ satisfies the relation (4.29). Independent of the branches arising from eq. (4.29), the quantities $q_1, q_2, c_{r1}^2, c_{r2}^2, c_{\Omega+}^2, c_{\Omega-}^2$ are given, respectively, by

$$q_1 = \frac{M_{\text{pl}}^2 h}{4f^2}, \quad q_2 = \frac{1 - 2g_4}{2r^2 f}, \quad c_{r1}^2 = c_{r2}^2 = c_{\Omega+}^2 = c_{\Omega-}^2 = 1,$$

with $\mathcal{F}_1 = M_{\text{pl}}^4 f h/(4r^8) \geq 0$ and $\mathcal{F}_2 = M_{\text{pl}}^4 (1 - 2g_4^2h^3/(16r^8f^3) \geq 0$. Hence there are neither ghost nor Laplacian instabilities under the condition

$$g_4 < \frac{1}{2}. \tag{7.2}$$

Let us consider the power-law coupling model given by

$$g_4(X) = \gamma_4 \left( \frac{X}{M_{\text{pl}}^2} \right)^n,$$  \tag{7.3}
where $\gamma_4$ and $n \geq 1$ are constants. From eq. (4.29), there are three branches characterized by (i) $A_0^2 = 0$, (ii) $A_1 = 0$, and (iii) $A_0^2 = A_0^2/(f h)$ (present for $n \geq 2$). The branches (i) and (iii) correspond to the stealth Schwarzschild solution and the RN solution, respectively. For the branch (ii), there exists a hairy BH solution where the coupling $\gamma_4$ appears in $f, h, A_0$ as corrections to the RN solution [82, 83].

For the branch $A_1 = 0$, the coupling (7.3) reduces to

$$g_4 = \gamma_4 \left( \frac{A_0^2}{2 f M_{pl}^2} \right)^n. \quad (7.4)$$

Around the event horizon, the leading-order solutions to $f, h, A_0$ are given by $f \simeq h \simeq (1-\mu)(r-r_h)/r_h$ and $A_0 \simeq \sqrt{2} \mu M_{pl}(r-r_h)/r_h$, respectively, with the corrections of coupling $\gamma_4$ appearing at the order of $(r-r_h)^{n+1}$ [82, 83]. Then, the term (7.4) vanishes in the limit $r \to r_h$, so the ghost instability is absent around the event horizon. At spatial infinity ($r \gg r_h$), the solutions behave as $f, h \to 1$ and $A_0 \to P = \text{constant}$, so the coupling (7.4) reduces to $g_4 \simeq \gamma_4 \tilde{P}^2n$, where $\tilde{P} = P/(\sqrt{2} M_{pl})$. In this regime, the condition (7.2) translates to $\gamma_4 \tilde{P}^{2n} < 1/2$. For $n \geq 1$, the temporal vector component squared $\gamma_4 \tilde{P}^{2n}$ monotonically increases from 0 ($r \simeq r_h$) to the asymptotic value $P^2$ ($r \to \infty$). Hence, under the condition $\gamma_4 \tilde{P}^{2n} < 1/2$, there is no ghost instability throughout the horizon exterior.

### 7.2 Coupling $g_5(X)$

We proceed to study the quintic intrinsic vector-mode coupling $g_5(X)$ with $G_4 = M_{pl}^2/2$. Independent of the functional form of $g_5$, we have the following relations:

$$q_1 = \frac{h M_{pl}^2}{4 f^2}, \quad c_{r_1}^2 = 1. \quad (7.5)$$

On the other hand, the quantities $q_2, c_{r_2}^2, c_{\Omega_+}^2, c_{\Omega_-}^2$ depend on the choice of $g_5(X)$.

In what follows, we focus on the power-law coupling model given by

$$g_5(X) = \frac{\gamma_5}{M_{pl}^2} \left( \frac{X}{M_{pl}^2} \right)^n, \quad (7.6)$$

where $\gamma_5$ and $n \geq 1$ are constants. From eq. (4.34), there are three branches characterized by $A_0^2 = 0$, $A_1 = \epsilon \sqrt{A_0^2/(f h)}$ (present for $n \geq 2$), and

$$A_1 = \epsilon \sqrt{\frac{A_0^2}{(1+2n)fh}}. \quad (7.7)$$

The first two branches correspond to the stealth Schwarzschild solution and the RN solution, respectively [83]. For the branch (7.7), there exists a hairy BH solution where the coupling $\gamma_5$ works as corrections to the RN metric, so we investigate its stability in the following.

For $n \geq 1$, the solutions expanded around $r = r_h$ are given by eq. (5.4) with the coefficients:

$$f_1 = h_1 = \frac{1-\mu}{r_h}, \quad f_2 = h_2 = \frac{2\mu - 1}{r_h^2}, \quad a_0 = 0, \quad a_1 = \frac{\sqrt{2} \mu M_{pl}}{r_h}, \quad a_2 = -\frac{\sqrt{2} \mu M_{pl}}{r_h^2}, \quad (7.8)$$
up to the order of \((r - r_h)^2\). The coupling \(\gamma_5\) appears at the order of \((r - r_h)^{n+2}\) in the expansions of \(f, h, A_0\) \cite{83}. On using this iterative solution, it follows that

\[
q_2 = \frac{1}{2 f_1 r_h^2 (r - r_h^2)} + \mathcal{O}((r - r_h)^0), \quad F_2 = \frac{M_{pl}^4}{16 r_h^8} + \mathcal{O}(r - r_h), \quad (7.9)
\]

so the conditions (3.26) and (3.33) are satisfied around \(r = r_h\). For \(\gamma_5 > 0\), the propagation speed squares yield

\[
c_{r_2}^2 = 1 + \mathcal{O}((r - r_h)^{n+1}), \quad c_{\Omega^+}^2 = 1 + \mathcal{O}((r - r_h)^{n+1}), \quad c_{\Omega^-}^2 = 1 + \mathcal{O}((r - r_h)^n), \quad (7.10)
\]

whereas, for \(\gamma_5 < 0\), the power-law dependence between \(c_{\Omega^+}^2 - 1\) and \(c_{\Omega^-}^2 - 1\) is exchanged. The coupling \(\gamma_5\) induces the deviations of \(c_{r_2}^2, c_{\Omega^+}^2, c_{\Omega^-}^2\) from 1. Thus, the hairy BH solution is free from the Laplacian instability in the vicinity of the event horizon.

For the distance \(r \gg r_h\), the coupling \(\gamma_5\) gives rise to corrections to the RN solution (3.46) at the order of \(1/r^2\) in \(f, h\) and at the order of \(1/r^2\) in \(A_0\) \cite{83}. From eq. (7.7), the longitudinal mode approaches a constant \(A_1 \rightarrow P/\sqrt{1 + 2n}\) as \(r \rightarrow \infty\). On using the iterative solution at spatial infinity, we find that \(q_2 \simeq 1/(2 r^2) > 0\) and \(F_2 = M_{pl}^4/(16 r_h^8) > 0\). Moreover, the propagation speed squares \(c_{r_2}^2, c_{\Omega^+}^2, c_{\Omega^-}^2\) approach 1 as \(r \rightarrow \infty\), so the Laplacian instability is absent at spatial infinity. In the intermediate regime between \(r = r_h\) and \(r \gg r_h\), we numerically confirmed that the hairy BH solutions arising from the power-law coupling with \(n \geq 1\) are plagued by neither ghost nor Laplacian instabilities. The coupling \(\gamma_5\) gives rise to the values of \(c_{r_2}^2, c_{\Omega^+}^2, c_{\Omega^-}^2\) different from 1 in the intermediate region, but they rapidly approach 1 for increasing \(r\). For example, the quantity \(c_{r_2}^2 - 1\) has the dependence proportional to \(1/r^2\) at spatial infinity. The qualitative behavior of \(c_{r_2}^2, c_{\Omega^+}^2, c_{\Omega^-}^2\) is similar to that shown in the right panel of figure 2.

8 Conclusions

In this paper, we formulated the odd-parity perturbations about the static and spherically symmetric BH solutions in generalized Proca theories by expanding the action up to the second order in perturbations. We derived the conditions under which hairy BH solutions in these theories suffer neither ghost nor Laplacian instabilities. The existence of a temporal vector component \(A_0\) besides a longitudinal mode gives rise to a bunch of exact and numerical BH solutions with vector hairs.

For odd-parity perturbations, there are two propagating degrees of freedom arising from the gravity sector and the vector field. For \(l \geq 2\), where \(l\) is an integer associated with the expansion in terms of spherical harmonics \(Y_l^m\), the conditions for avoiding ghost instabilities correspond to \(q_1 > 0\) and \(q_2 > 0\), where \(q_1, q_2\) are given by eqs. (3.23)–(3.24). We derived the propagation speeds in the radial direction in the forms (3.28)–(3.29), where the quantities \(F_1\) and \(F_2\) are required to be positive for the existence of real solutions of \(c_{r_1}\) and \(c_{r_2}\). In the angular direction, there are also two propagation speed squares \(c_{\Omega^\pm}^2\) derived in eq. (3.37), which must be positive to avoid Laplacian instabilities for large \(l\). We also showed that the analysis of dipole perturbations \((l = 1)\) does not give rise to additional conditions to those obtained for \(l \geq 2\). For the cubic couplings \(G_{3l}(X)\), there are no ghost and Laplacian instabilities against odd-parity perturbations as in GR.

The charged stealth Schwarzschild BH solution (4.5) with nonvanishing \(A_1\) present for the quartic coupling (4.4) satisfies neither \(F_1 \geq 0\) nor \(c_{\Omega^+}^2 \geq 0\) in the vicinity of the event
horizon under the no-ghost conditions (4.8). Then, this solution, which was firstly found in ref. [79] for the model $G_4(X) = M_{pl}^2/2 + X/4$, is prone to the Laplacian instability. For models with the quartic coupling (4.14) and with the quintic coupling (4.17), there exist the extremal RN solution (4.15) with $A_1 = 0$ and the RN solution with $A_1 \neq 0$, respectively. These two exact BH solutions are plagued by neither ghost nor Laplacian instabilities. Models with intrinsic vector modes, e.g., (4.20), (4.30), and (4.35), also give rise to exact BH solutions satisfying the relations (4.1). In such cases, we showed the existence of parameter spaces consistent with conditions for the absence of ghost and Laplacian instabilities.

For the quartic power-law models (5.1), there are hairy BH solutions where the coupling $\beta_4$ appears as corrections to the RN solution (3.46). For the branch $A_1 \neq 0$, we can iteratively derive the solutions to $f, h, A_0$ in the forms (5.4) around the event horizon. On using this expansion, we showed that the product $q_1 q_2 c_{\Omega}^2$ is given by eq. (5.7), which is negative for $\beta_4 \neq 0$. Even if $\beta_4$ is very small, the first term on the right hand side of eq. (5.7) dominates over the other terms as $r$ approaches $r_h$. Then, the hairy BH solutions present for the power-law model (5.1) with $A_1 \neq 0$ are prone to the instability problem in the vicinity of the event horizon for $\beta_4 \neq 0$ and $n \geq 1$. For the model (5.1) with the branch $A_1 = 0$, the BH solution suffers neither ghost nor Laplacian instabilities throughout the horizon exterior under the conditions (5.20) and (5.23). This suggests that the nonvanishing longitudinal mode $A_1$ with peculiar behavior (5.6) around $r = r_h$ is the main reason for the instability mentioned above.

We also investigated the case of hairy BH solutions with the branch $A_1 = 0$ arising from the sixth-order power-law coupling (6.1). The model with $n = 0$, which corresponds to the U(1) gauge-invariant derivative interaction, has the iterative BH solution (5.4) with the coefficients given by (6.5) and $a_0 \neq 0$. In this case, we showed that ghost and Laplacian instabilities are absent under the condition (6.8). For $n = 1$, the coupling $\beta_6$ leads to a nontrivial deviation of $c_{\Omega_-}^2$ from 1 around $r = r_h$, so that the Laplacian instability can be avoided under the condition (6.12). For $n \geq 2$, there are no particular bounds on the coupling $\beta_6$. In this case, $c_{\Omega_1}^2, c_{\Omega_2}^2, c_{\Omega_+}^2, c_{\Omega_-}^2$ approach 1 in both limits $r \to r_h$ and $r \to \infty$, with small deviations from 1 in the intermediate regime outside the event horizon.

For general intrinsic vector-mode couplings $g_4(X)$, there are neither ghost nor Laplacian instabilities for $g_4 < 1/2$. In case of the power-law couplings (7.3), the hairy BH solution with the branch $A_1 = 0$ has the property $g_4 = 0$ on the event horizon, so it is sufficient to satisfy the condition $g_4 < 1/2$ at spatial infinity. For the quintic intrinsic vector-mode power-law couplings (7.6) with $n \geq 1$, the conditions for the absence of ghost and Laplacian instabilities are satisfied throughout the horizon exterior without a particular bound on the coupling $\gamma_5$.

In summary, we showed that, apart from the quartic power-law models (5.1) with the branch $A_1 \neq 0$, there are hairy BH solutions which are free from ghost and Laplacian instabilities against the odd-parity perturbations. In tables 1 and 2 in appendix D, we summarize the stability of BH solutions found in the analysis of sections 4–6. We should emphasize, however, that the analysis of odd-parity perturbations alone is not sufficient to guarantee the complete stability of BHs in general. We need to ensure whether the BH solutions satisfy the same type of stability conditions against even-parity perturbations. Moreover, even if the absence of ghost and Laplacian instabilities for both odd- and even-parity sectors is proven for some BH solutions, we still need to ensure the absence of tachyonic eigenmodes for the radial perturbation equations in both sectors (See ref. [96] for related arguments in Horndeski theories). Finally, it also has to be checked that all the solutions to the linear and nonlinear perturbation equations obtained from regular initial data remain bounded throughout the horizon exterior [100–102].
The other important question is whether or not the BH solutions free from ghost and Laplacian instabilities can be further constrained from the observational points of view. The almost simultaneous detection of GWs from a NS merger and its short gamma-ray burst counterpart has significantly constrained the deviation of propagation speed of GWs from the speed of light to be less than order $10^{-15}$ [50–52]. For example, if we naively apply this bound to the large-distance modification of the propagation speeds of perturbations (5.24) derived from the $A_1 = 0$ branch of the quartic power-law coupling models (5.1), we would obtain a bound $|\beta_1 P^{2n}| \lesssim 10^{-15}$, assuming $|\beta_1 P^{2n}| \ll 1$ and $n = O(1)$. However, we have to be more careful when we relate the propagation speeds derived in this paper to the observed speed of GWs from local sources (BHs and NSs).

First, the observed polarized GWs $h_+$ and $h_\times$ are the combination of even- and odd-parity perturbations in general [103], so we need to consider the propagation of even modes for deriving the speed of GWs appropriately. Second, the GWs recently detected by LIGO and Virgo from a NS merger travelled over the cosmological distance [50–52], so strictly speaking, it is required to consider the propagation of GWs on the time-dependent background. It will be of interest to derive the GW propagation speed from local sources by taking into account even-parity perturbations as well as the effect of time-dependent cosmological background. These issues will be left for future work.

A Coefficients in the background equations

In eqs. (2.11)–(2.13) the coefficients $c_{1,2,\ldots,19}$ are given by

$$c_1 = - A_1 X G_{3,X},$$
$$c_2 = - 2 G_4 + 4 (X_0 + 2 X_1) G_{4,X} + 8 X_1 X G_{4,XX},$$
$$c_3 = - A_1 (3 h X_0 + 5 h X_1 - X) G_{5,X} - 2 h A_1 X_1 X G_{5,XX},$$
$$c_4 = G_2 - 2 X_0 G_{2,X} - \frac{h}{f} (A_0 A_1 A'_0 + 2 f X A'_1) G_{3,X} - \frac{h A^2_0 (1 + 2 G_{2,F})}{2 f},$$
$$c_5 = - 4 h A_1 X_0 G_{3,X} - 4 h^2 A_1 A'_1 G_{4,X} + \frac{8 h}{f} (A_0 X_1 A'_0 - f h A_1 X A'_1) G_{4,XX}$$
$$+ \frac{2 h^2}{f} A_1 A^2_0 (g_5 + 2 X_0 g_{5,X}),$$
$$c_6 = 2 (1 - h) G_4 + 4 (h X - X_0) G_{4,X} + 8 h X_0 X_1 G_{4,XX}$$
$$- \frac{h}{f} [(h - 1) A_0 A_1 A'_0 + 2 f (3 h X_1 + h X_0 - X) A'_1] G_{5,X}$$
$$- \frac{2 h^2 X_1}{f} (A_0 A_1 A'_0 + 2 f X A'_1) G_{5,XX}$$
$$+ \frac{h A^2_0}{f} [(h - 1) G_6 + 2 (h X - X_0) G_{6,X} + 4 h X_0 X_1 G_{6,XX}],$$
$$c_7 = - G_2 + 2 X_1 G_{2,X} - \frac{h}{f} A_0 A_1 A'_0 G_{3,X} + \frac{h A^2_0 (1 + 2 G_{2,F})}{2 f},$$
$$c_8 = 4 h A_1 X_1 G_{3,X} + \frac{4 h}{f} A_0 A'_0 (G_{4,X} + 2 X_1 G_{4,XX}) - \frac{2 h^2}{f} A_1 A^2_0 (3 g_5 + 2 X_1 g_{5,X}),$$
\[c_9 = 2(h - 1)G_4 - 4(2h - 1)X_1G_{4,X} - 8hX_1^2G_{4,XX}
- \frac{h}{f}A_0A_1A'_0[(3h - 1)G_{5,X} + 2hX_1G_{5,XX}]
- \frac{h}{f}A_0^2[(3h - 1)G_6 + 2(6h - 1)X_1G_{6,X} + 4hX_1^2G_{6,XX}],\]

\[c_{10} = -\frac{2h}{f}(G_4 - 2XG_{4,X}),\]

\[c_{11} = -\frac{2h^2}{f}A_1XG_{5,X},\]

\[c_{12} = \frac{h}{f^2}[G_4 - 2(2X_0 + X_1)G_{4,X} - 4X_0XG_{4,XX}],\]

\[c_{13} = \frac{h^2}{f^2}A_1[(3X_0 + X_1)G_{5,X} + 2X_0XG_{5,XX}],\]

\[c_{14} = -\frac{h}{f}A_1[(3X_0 + 5X_1)G_{5,X} + 2X_1XG_{5,XX}],\]

\[c_{15} = \frac{h}{f^2}[2fA_1X_0G_{3,X} + 2(2A_0A'_0 - fhA_1A'_1)G_{4,X}
+ 4\{A_0(2X_0 + X_1)A'_0 - fhA_1A'_1\}G_{4,XX}
- hA_1A'_0(g_5 + 2X_0g_5,X)],\]

\[c_{16} = -\frac{h}{f^2}\left[2f(G_4 - 2X_0X_1G_{4,XX}) + h\left\{3A_0A_1A'_0 + 2f(X_0 + 3X_1)A'_1\right\}G_{5,X}
+ 2h\left\{A_0A_1(X_1 + 2X_0)A'_0 + 2fX_1A'_1\right\}G_{5,XX}
+ hA_0^2(G_6 + 2XG_{6,X} + 4X_0X_1G_{6,XX})\right],\]

\[c_{17} = -2G_4 + 8X_1(G_{4,X} + X_1G_{4,XX})
+ h\frac{A'_0}{f}\left[A_0A_1(3G_{5,X} + 2X_1G_{5,XX}) + A'_0\{3G_6 + 4X_1(3G_{6,X} + X_1G_{6,XX})\}\right],\]

\[c_{18} = 2G_2 - \frac{2h}{f}\left[(A_0A_1A'_0 + 2fX_1A'_1)G_{3,X} + 2(A_0A'_0 + A_0^2)G_{4,X}
+ 2A'_0(2X_0A'_0 - hA_0A_1A'_1)G_{4,XX}\right]
+ \frac{2h^2A'_0}{f^2}\left[f(2A_1A'_0 + A_0A'_1)g_5 + A'_0(A_0A_1A'_0 + 2fX_1A'_1)g_5,X\right] + \frac{h}{f}A_0^2,\]

\[c_{19} = \frac{2h}{f}\left[-2(A_0A'_0 + fhA_1A'_1)G_{4,X} + 4X_1(A_0A'_0 - fhA_1A'_1)G_{4,XX}
+ h(A_1A'_0 + A_0A_0A'_1 + A_0A_1A'_0)G_{5,X}
+ 2hA'_0(A_0X_1A'_1 + A_1X_0A'_0)G_{5,XX} + 2hA_0^2G_6
+ \frac{h}{f}\left\{(A_0A'_0^2 + 4fX_1A'_0^2 - 3fhA_1A'_0A'_1)G_{6,XX}
+ 2A'_0X_1(A_0A'_0 - fhA_1A'_1)G_{6,XX}\right\}\right].\]
\[ \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \left| \frac{\partial}{\partial \theta} Y_{l0}(\theta, \varphi) \right|^2 \sin \theta = L, \quad (B.1) \]

\[ \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \left[ \frac{1}{\sin \theta} \left| \frac{\partial}{\partial \theta} Y_{l0}(\theta, \varphi) \right|^2 + \sin \theta \left| \frac{\partial^2}{\partial \theta^2} Y_{l0}(\theta, \varphi) \right|^2 \right] = L^2, \quad (B.2) \]

where \( L \) is defined by eq. (3.14).

**C  Coefficients in the second-order action of odd-parity perturbations**

The coefficients \( C_{1-13} \) in eq. (3.15) are given by

\[ C_4 = \frac{h(rG_4 - 2 r XG_{4, X} + hA_1 XG_{5, X})}{2r^3 f}, \]

\[ C_2 = -\frac{h[2rfA_1 G_{4, X} - fhA_1^2 G_{5, X} + A_0 A'_0 (2hA_1 G_{6, X} - 2r g_5)]}{4r^3 f^2}, \]

\[ C_3 = \frac{h}{2r^3 f} \left[ r A_0 G_{4, X} - \frac{1}{2} h A_0 A_1 G_{5, X} - A'_0 (hG_6 - h^2 A_1^2 G_{6, X} + rh A_1 g_5) \right], \]

\[ C_4 = -\frac{h}{2r^3 f} \left[ A'_0 \left\{ r^2 (1 + G_{2, F} - G_{4, X}) - h \left( 2G_6 - \frac{1}{2} r A_1 G_{5, X} + 4r A_1 g_5 \right) + 2h^2 A_1^2 G_{6, X} \right\} \right. \\
\left. + 2A_0 \left( r G_{4, X} - \frac{1}{2} h A_1 G_{5, X} \right) \right], \]

\[ C_5 = \frac{1}{2r^3 f} \left[ r (1 + G_{2, F}) - h' (G_6 - h A_1^2 G_{6, X}) + 2h^2 A_1 A'_1 G_{6, X} \right. \\
\left. - (2h A_1 + rh' A_1 + 2rh A'_1) g_5 \right], \]

\[ C_6 = \frac{h (f A_0' - f' A_0) (rg_5 - h A_1 G_{6, X})}{r^3 f^2}, \]

\[ C_7 = -\frac{h}{2r^3 f} \left[ r f (1 + G_{2, F}) - f' h (G_6 - h A_1^2 G_{6, X}) - h (r f' A_1 + 2f A_1) g_5 \right], \]

\[ C_8 = -\frac{h}{4r^4 f} \left[ 2f G_4 + 2fh A_1^2 G_{4, X} + h A_1 (f' X - A_0 A'_0) G_{5, X} - h A_0^2 (G_6 - h A_1^2 G_{6, X}) \right], \]

\[ C_9 = \frac{h}{2r^4 f} \left[ 2f A_1 G_{4, X} + (f' X - A_0 A'_0) G_{5, X} + h A_0^2 A_1 G_{6, X} \right], \]

\[ C_{10} = \frac{1}{4r^4 f^3} \left[ 2f^2 G_4 - 2fh A_0^2 G_{4, X} + f \{ h A_0 A'_0 A_1 + f X (h' A_1 + 2h A'_1) \} G_{5, X} \right. \\
\left. - h A_0^2 (f G_6 + h A_1^2 G_{6, X}) \right], \]

\[ C_{11} = -\frac{1}{4r^4 f^3} \left[ 2f^2 A_0 G_{4, X} - f A_0 (f' h A_1 + fh A'_1) G_{5, X} \\
+ 2f (f' h A_0' - f h' A'_0 - 2fh A'_0) G_{6} + 2h A'_0 (f' A'_0 - f A_0 A'_0 + f^2 h A'_1 + 2f^2 h A_1 A'_1) G_{6, X} \right], \]
\[ C_{12} = -\frac{1}{4r^4f^3} \left[ 2f^3(1 + G_{2,F}) - f(2ff''h - ff'h + f'f'h')G_6 \\
+ f'h(f' A_0^2 - 2fA_0 A_1' + f'^2 A_1^2 + 2f'^2 h A_1 A_1') G_{6,X} \\
- 2f^2(f'h A_1 + f'h A_1 + 2f A_1)g_5 \right], \]
\[ C_{13} = -\frac{1}{8r^4f^3} \left[ 4r^2f^3 G_{2,X} + 2rf^2 \left( 4fh A_1 + rf'h A_1 + rf'h A_1 + 2rf h A_1' \right) G_{3,X} \\
- 2rf \{ 2rf f''h - f'h(rf' - 2f) + f'h(rf' + 2f) \} G_{4,X} \\
+ 2h \{ rf^2 A_1(rf' + 2f)(h' A_1 + 2h A_1') - r A_0(rf' - 2f)(2f A_0' - f' A_0) \\
+ 4f^2h(rf' + f)A_2^2 \} G_{4,XX} + h \{ r(2ff'' - ff'^2)h A_1 \\
+ f'f'(2h A_1 + 3rh' A_1 + 2rh A_1') \} G_{5,X} \\
- h^2 A_1 \{ rf'^2 A_0^2 + rf'^2 f'h A_2^2 + (4f^2 A_0 - 2rf f' A_0) A_0' + 2f^2 f'(r A_1 A_1' - 2X) \} G_{5,XX} \\
- 4f^2 h^2 A_0^2 (G_{6,X} - h A_1^2 G_{6,XX} + 2r A_1 g_{5,X}) \right]. \]

D Summary of the main results

In order to make it convenient for the readers to look for the main results of calculations in this paper, we summarize them into two tables below. The word “stable” means that the corresponding BH solution satisfies the conditions for the absence of ghosts and Laplacian instabilities associated with odd-parity perturbations. We caution that this does not necessarily guarantee the complete stability of BHs. In case of the cubic couplings \( G_3(X) \), as discussed in section 3.3, the quantities \( q_1, q_2, c_1, c_2, c_{11}^2 \) are simply the same as those in GR, so we do not list this case in tables.

| \( G_4(X) \) | \( G_5(X) \) | \( G_6(X) \) | \( g_4(X) \) | \( g_5(X) \) |
|-------------|-------------|-------------|-------------|-------------|
| Forms of couplings | (4.4) | (4.14) | (4.17) | Arbitrary | (4.30) | (4.35) | Arbitrary |
| Branches | \( A_1 \neq 0 \) | \( A_1 = 0 \) | \( A_1 \neq 0, A_1 = 0 \) | \( A_1 = 0 \) | \( A_1 \neq 0 \) | \( A_1 \neq 0 \) | \( A_1 = 0 \) |
| BH solutions | (4.5) | (4.15) | (4.18) | (4.21) | (4.22) | (4.31) | (3.46) | (4.22) |
| Stability conditions | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable | stable | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable |

Table 1. Summary of the stabilities of exact BH solutions discussed in section 4.

| \( G_4(X) = M_0^2/2 + \beta_4 M_{pl}^2 \left( X/M_{pl}^2 \right)^n \) | \( G_6(X) = (\beta_6/M_{pl}^2) \left( X/M_{pl}^2 \right)^n \) |
|-------------|-------------|
| Branches | \( A_1 \neq 0, r \sim r_h \) | \( A_1 = 0, r \sim r_h \) | \( A_1 = 0, r \gg r_h \) |
| Subcases | None | \( n = 0 \) | \( n = 1 \) | \( n \geq 2 \) | None |
| BH solutions | (5.4)–(5.6) | (5.4), (5.9) | (5.16)–(5.18) | (5.4), (6.5) | (5.4), (6.9) | (6.14)–(6.16) |
| Stability conditions | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable | \( q_1 q_2 c_{11}^2 < 0, \) unstable | stable |

Table 2. Summary of the stabilities of numerical BH solutions discussed from section 5 to 6.
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