Fast Low-Rank Tensor Decomposition by Ridge Leverage Score Sampling

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Abstract

Low-rank tensor decomposition generalizes low-rank matrix approximation and is a powerful technique for discovering low-dimensional structure in high-dimensional data. In this paper, we study Tucker decompositions and use tools from randomized numerical linear algebra called ridge leverage scores to accelerate the core tensor update step in the widely-used alternating least squares (ALS) algorithm. Updating the core tensor, a severe bottleneck in ALS, is a highly-structured ridge regression problem where the design matrix is a Kronecker product of the factor matrices. We show how to use approximate ridge leverage scores to construct a sketched instance for any ridge regression problem such that the solution vector for the sketched problem is a \((1 + \varepsilon)\)-approximation to the original instance. Moreover, we show that classical leverage scores suffice as an approximation, which then allows us to exploit the Kronecker structure and update the core tensor in time that depends predominantly on the rank and the sketching parameters (i.e., sublinear in the size of the input tensor). We also give upper bounds for ridge leverage scores as rows are removed from the design matrix (e.g., if the tensor has missing entries), and we demonstrate the effectiveness of our approximate ridge regression algorithm for large, low-rank Tucker decompositions on both synthetic and real-world data.
1 Introduction

Tensor decomposition has a rich multidisciplinary history, but it has only recently become ubiquitous due to a surge of applications in data mining, machine learning, and signal processing Kolda and Bader (2009); Rabanser et al. (2017); Sidiropoulos et al. (2017). The most prevalent tensor decompositions are the CP and Tucker decompositions, which are often thought of as generalized singular value decompositions. Consequently, there are natural notions of low-rank tensor decomposition. Unlike matrix factorization, however, even computing the analogs of rank for a tensor is NP-hard Hillar and Lim (2013). Therefore, most low-rank tensor decomposition algorithms fix the rank structure in advance and then optimize the variables of the decomposition to fit the data. While conceptually simple, this technique can be extremely effective for many real-world applications since high-dimensional data is often inherently low-dimensional.

One of the cornerstones of low-rank tensor decomposition is the alternating least squares (ALS) algorithm. For CP and Tucker decompositions, ALS cyclically optimizes disjoint blocks of variables while keeping all others fixed. If no additional constraints or regularization are imposed, then each step of ALS is an ordinary least squares problem. For Tucker decompositions, the higher order singular value decomposition (HOSVD) and higher order orthogonal iteration (HOOI) algorithms Kolda and Bader (2009) are popular alternatives, but (1) do not scale as easily since they compute SVDs of matricizations of the data tensor in each step, and (2) do work well for data with missing entries. Therefore, we focus on ALS for Tucker decompositions and incorporate a new technique from randomized numerical linear algebra called ridge leverage score sampling to speed up its bottleneck step. Recently, Cheng et al. (2016) and Larsen and Kolda (2020) showed that (statistical) leverage score sampling is useful for accelerating ALS for CP decompositions. Leverage scores measure the importance of each observation in a least squares problem, and were recently generalized by Alaoui and Mahoney (2015) to account for Tikhonov regularization, hence the name ridge leverage scores. The CP decomposition algorithms in Cheng et al. (2016); Larsen and Kolda (2020), as well as the work of Diao et al. (2019) on Kronecker product regression, exploit the fact that the necessary leverage score distribution can be closely approximated by a related product distribution. This ultimately leads to efficient sampling subroutines. We follow a similar approach to give a fast sampling-based ALS algorithm for Tucker decompositions, and we show how to seamlessly extend leverage score sampling methods to account for L2 regularization (i.e., approximate ridge regression).

1.1 Our Contributions

This work gives several new results for ridge leverage score sampling and its applications in low-rank tensor decomposition. Below is a summary of our contributions:

1. Our first result is a method for augmenting approximate ridge leverage score distributions such that (1) we can sample from the augmented distribution in the same amount of time as the original distribution, and (2) if we sample rows according to this augmented distribution, then we can construct a sketched version for any ridge regression problem such that the solution vector for the sketch gives a \((1 + \varepsilon)-\)approximation to the original instance. We then show that the statistical leverage scores of the design matrix for any ridge regression problem are a useful overestimate of the \(\lambda\)-ridge leverage scores when augmented. Moreover, we quantify how the sample complexity of sketching algorithms decreases as the value of \(\lambda\) increases (i.e., as the effective dimensionality of the ridge regression problem shrinks).
2. Our second key result explores how ridge leverage score sampling can be used to compute low-rank Tucker decompositions of tensors. We consider the ubiquitous alternating least squares algorithm and speed up its core tensor update step—a notorious bottleneck for Tucker decomposition algorithms. We use our approximate ridge regression subroutine and exploit the fact that the design matrix in every core tensor update is a Kronecker product of the factor matrices. In particular, this means that the leverage score distribution of the design matrix is a product distribution of the leverage score distributions for the factor matrices, hence we can sample rows from the augmented distribution in time sublinear in the number of its rows. Our core tensor update is designed to be fast both in theory and in practice, since its time complexity is predominantly a function of the rank and sketching parameters.

3. Next, as a step towards better understanding alternating least squares for tensor completion, we derive upper bounds for the $\lambda$-ridge leverage scores when rows are removed from the design matrix (i.e., if the tensor has missing entries). While these bounds hold for general matrices and can be pessimistic if a large fraction of rows are removed, the proofs provide useful insight into exactly how $\lambda$-ridge leverage scores generalize classical leverage scores.

4. Lastly, we demonstrate how our approximate core tensor update based on fast (factored) leverage score sampling leads to massive improvements in the running time for low-rank Tucker decompositions while preserving the original solution quality of ALS. Specifically, we profile this algorithm using large, dense synthetic tensors and the movie data of Malik and Becker (2018), which explores sketching Tucker decompositions in a data stream model.

1.2 Related Works

Tensor Decomposition. The algorithms of Cheng et al. (2016) and Larsen and Kolda (2020) that use leverage score sampling with ALS to compute (unregularized) CP decompositions are most directly related. Avoiding degeneracies in a CP decomposition using ALS has carefully been studied in Comon et al. (2009). For first order methods, a step of gradient descent typically takes as long as an iteration of ALS since both involve computing the same quantities (Sidiropoulos et al., 2017, Remark 3). SGD-based methods, however, are known to be efficient for certain structured CP decompositions Ge et al. (2015). For Tucker decompositions, Frandsen and Ge (2020) recently showed that if the tensor being learned has an exact Tucker decomposition, then all local minima are globally optimal. Streaming algorithms for Tucker decompositions based on sketching and SGD have also recently been explored by Malik and Becker (2018) and Traore et al. (2019), respectively. The more general problem of low-rank tensor completion is a fundamental approach for estimating the values of missing data Acar et al. (2011); Jain et al. (2013); Jain and Oh (2014); Filipović and Jukić (2015), and has led to notable breakthroughs in computer vision Liu et al. (2012). Popular approaches for tensor completion are based on Riemannian optimization Kressner et al. (2014); Kasai and Mishra (2016); Nimishakavi et al. (2018) alternating least squares Zhou et al. (2013); Grasedyck et al. (2015); Liu and Moitra (2020), and projected gradient methods Yu and Liu (2016).

Ridge Leverage Scores. Alaoui and Mahoney (2015) recently extended the idea of statistical leverage scores to the setting of ridge regression, and used these scores to derive a sampling distribution that reduces the sketch size (i.e., the number of sampled rows) to the effective dimension of the problem. Since then, sampling from approximate ridge leverage score distributions has played a critical role in fundamental works for sparse low-rank matrix approximation Cohen et al. (2017),
an improved Nyström method via recursive sampling Musco and Musco (2017), bounding the statistical risk of ridge regression McCurdy (2018), a new sketching-based iterative method for ridge regression Chowdhury et al. (2018), and improved bounds for the number of random Fourier features needed for ridge regression as a function of the effective dimension Li et al. (2019). Closely related are fast recursive algorithms for computing approximate leverage scores Cohen et al. (2015) and for solving overconstrained least squares Li et al. (2013). Recent works have also explored sketching for sparse Kronecker product regression, which exploit a similar product distribution property of leverage scores Diao et al. (2018, 2019).

2 Preliminaries

Notation and Terminology. The order of a tensor is the number of its dimensions, also known as ways or modes. Scalars (zeroth-order tensors) are denoted by normal lowercase letters $x \in \mathbb{R}$, vectors (first-order tensors) by boldface lowercase letters $\mathbf{x} \in \mathbb{R}^n$, and matrices (second-order tensors) by boldface uppercase letters $\mathbf{X} \in \mathbb{R}^{m \times n}$. Higher-order tensors are denoted by boldface script letters $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$. For higher-order tensors, we use normal uppercase letters to denote the size of an index set (e.g., $[N] = \{1, 2, \ldots, N\}$). The $i$-th entry of a vector $\mathbf{x}$ is denoted by $x_i$, the $(i, j)$-th entry of a matrix $\mathbf{X}$ by $x_{ij}$, and the $(i, j, k)$-th entry of a third-order tensor $\mathbf{X}$ by $x_{ijk}$.

Linear Algebra. Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix and $\mathbf{0}_{m \times n}$ denote the $m \times n$ zero matrix. Denote the transpose of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ by $\mathbf{A}^\top$ and the Moore–Penrose inverse by $\mathbf{A}^+$. The singular value decomposition (SVD) of $\mathbf{A}$ is a factorization of the form $\mathbf{U} \Sigma \mathbf{V}^\top$, where $\mathbf{U} \in \mathbb{R}^{m \times m}$ and $\mathbf{V} \in \mathbb{R}^{n \times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix with non-negative real numbers on its diagonal. The entries $\sigma_i(\mathbf{A})$ of $\Sigma$ are the singular values of $\mathbf{A}$, and the number of non-zero singular values is equal to $\tau = \text{rank}(\mathbf{A})$. The compact SVD is a similar decomposition where $\Sigma \in \mathbb{R}^{r \times r}$ is a diagonal matrix containing only the non-zero singular values. Lastly, we denote the Kronecker product of two matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$ by $\mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{(mp) \times (nq)}$.

Tensor Products. The fibers of a tensor are vectors created by fixing all but one index (e.g., for a third-order tensor $\mathbf{X}$, the column, row, and tube fibers are denoted by $\mathbf{x}_{ijk}, \mathbf{x}_{i\cdot jk},$ and $\mathbf{x}_{ij \cdot},$ respectively). The mode-$n$ unfolding of a tensor $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the matrix $\mathbf{X}_n(\mathbf{X}) \in \mathbb{R}^{I_n \times (I_1 \cdot I_2 \cdot \cdots \cdot I_{n-1} I_{n+1} \cdot \cdots \cdot I_N)}$ that arranges the mode-$n$ fibers of $\mathbf{X}$ as the columns of $\mathbf{X}_n(\mathbf{X})$ ordered lexicographically by index. Going one step further, the vectorization of $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the vector $\text{vec}(\mathbf{X}) \in \mathbb{R}^{I_1 I_2 \cdots I_N}$ formed by vertically stacking the entries of $\mathbf{X}$ ordered lexicographically by index (e.g., this transforms matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$ into a tall vector vec($\mathbf{X}$) by stacking its columns). The $n$-mode product of a tensor $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ and matrix $\mathbf{A} \in \mathbb{R}^{J \times I_n}$ is denoted by $\mathbf{Y} = \mathbf{X} \times_n \mathbf{A}$ with $\mathbf{Y} \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$. Intuitively, this operation multiplies each mode-$n$ fiber of $\mathbf{X}$ by the matrix $\mathbf{A}$. Elementwise, this operation is expressed as follows:

$$
(\mathbf{X} \times_n \mathbf{A})_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \cdots i_{n-1} i_n} a_{j i_n}.
$$

The Frobenius norm $\|\mathbf{X}\|_F$ of a tensor $\mathbf{X}$ is the square root of the sum of the squares of all its entries.
Tucker Decomposition. The Tucker decomposition decomposes a tensor $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ into a core tensor $\mathbf{G} \in \mathbb{R}^{R_1 \times R_2 \times \cdots \times R_N}$ and multiple factor matrices $A^{(n)} \in \mathbb{R}^{I_n \times R_n}$. We can express the problem of finding a Tucker decomposition of $\mathbf{X}$ as minimizing the loss function

$$L\left(\mathbf{G}, A^{(1)}, \ldots, A^{(N)}\right) = \|\mathbf{X} - G_{1}A^{(1)} \times_2 \cdots \times_N A^{(N)}\|_F^2 + \lambda \left(\|\mathbf{G}\|_F^2 + \sum_{n=1}^{N} \|A^{(n)}\|_F^2\right),$$

where $\lambda$ is a regularization parameter. The elements of $\widehat{\mathbf{X}} = G_{1}A^{(1)} \times_2 \cdots \times_N A^{(N)}$ are

$$\hat{x}_{i_1i_2\ldots i_N} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} g_{r_1r_2\ldots r_N} a_{i_1r_1}^{(1)} a_{i_2r_2}^{(2)} \cdots a_{i_Nr_N}^{(N)}.$$  

Equation (1) shows that $\hat{\mathbf{X}}$ is the sum of $R_1R_2 \cdots R_N$ rank-1 tensors. The tuple $(R_1, R_2, \ldots, R_N)$ is the multilinear rank of the decomposition and is chosen to be much smaller than the dimensions of $\mathbf{X}$. Sometimes columnwise orthogonality constraints are enforced on the factor matrices, and hence the Tucker decomposition can be thought of as a higher-order SVD, but such constraints are not required.

Ridge Leverage Scores. The $\lambda$-ridge leverage score of the $i$-th row of a matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ is

$$\ell_i^\lambda(\mathbf{A}) \overset{\text{def}}{=} a_i: (\mathbf{A}^\top \mathbf{A} + \lambda \mathbf{I})^+ a_i^\top.$$

We also define the related cross $\lambda$-ridge leverage score as $\ell_{ij}^\lambda(\mathbf{A}) \overset{\text{def}}{=} a_i: (\mathbf{A}^\top \mathbf{A} + \lambda \mathbf{I})^+ a_j^\top$. The matrix of cross $\lambda$-ridge leverage scores is $\mathbf{A}(\mathbf{A}^\top \mathbf{A} + \lambda \mathbf{I})^+ \mathbf{A}^\top$, and we denote its diagonal by $\ell^\lambda(\mathbf{A})$ since this vector contains the $\lambda$-ridge leverage scores of $\mathbf{A}$. Ridge leverage scores generalize the statistical leverage scores of a matrix, in that setting $\lambda = 0$ recovers the leverage scores of $\mathbf{A}$, which we denote by the vector $\ell(\mathbf{A})$. If we let $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^\top$ be the compact SVD of $\mathbf{A}$, it can be shown that

$$\ell_{ij}^\lambda(\mathbf{A}) = \sum_{k=1}^{r} \frac{\sigma_k^2(\mathbf{A})}{\sigma_k^2(\mathbf{A}) + \lambda} u_{ik} u_{jk},$$

where $r = \text{rank}(\mathbf{A})$. Therefore, it follows that each $\ell_i^\lambda(\mathbf{A}) \leq 1$ since $\mathbf{U}$ is an orthogonal matrix. The effective dimension $d_{\text{eff}}$ of the ridge regression problem is the sum of $\lambda$-ridge leverage scores:

$$d_{\text{eff}} = \sum_{i=1}^{n} \ell_i^\lambda(\mathbf{A}) = \sum_{i=1}^{n} \sum_{k=1}^{r} \frac{\sigma_k^2(\mathbf{A})}{\sigma_k^2(\mathbf{A}) + \lambda} u_{ik}^2 = \sum_{k=1}^{r} \frac{\sigma_k^2(\mathbf{A})}{\sigma_k^2(\mathbf{A}) + \lambda} \leq r. \quad (4)$$

The regularization parameter $\lambda$ shrinks the dimensionality of the problem, so $d_{\text{eff}} = r$ if $\lambda = 0$. The $\lambda$-ridge leverage score of a row measures its importance when constructing the row space of $\mathbf{A}$ in the context of ridge regression. See Alaoui and Mahoney (2015); Cohen et al. (2015) for further details and intuition about $\lambda$-ridge leverage scores.

3 Approximate Ridge Regression by Leverage Score Sampling

We start by introducing a row sampling-based approach for approximately solving any ridge regression problem via $\lambda$-ridge leverage scores. Computing ridge leverage scores is often as expensive as solving
the ridge regression problem itself, and thus is not immediately useful for constructing a smaller, sketched instance to use as a proxy. However, in this section, we show how to use $\lambda$-ridge leverage scores overestimates to construct a feasible sampling distribution over the rows of an augmented design matrix, and then we use this new distribution to efficiently sketch an ordinary least squares problem whose solution vector is a $(1 + \varepsilon)$-approximation to the input ridge regression problem. Furthermore, we show that classical leverage scores are a sufficient overestimate of $\lambda$-ridge leverage scores, for any regularization strength $\lambda \geq 0$, and always result in efficient sketching subroutines.

It will be useful to first give some context into the derivation of $\lambda$-ridge leverage scores introduced by Alaoui and Mahoney (2015). Let us rewrite the ridge regression objective as an ordinary least squares problem in terms of an augmented design matrix and response vector:

$$x_{\text{opt}} = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2 + \lambda\|x\|_2^2 = \arg\min_{x \in \mathbb{R}^d} \left\| \begin{bmatrix} A & \sqrt{\lambda}I_d \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2 = \arg\min_{x \in \mathbb{R}^d} \|Ax - \bar{b}\|_2^2.$$  (5)

Lemma 3.1. Let $A \in \mathbb{R}^{n \times d}$ be any matrix and let $\bar{A} \in \mathbb{R}^{(n+d) \times d}$ be defined as in (5). For each row index $i \in [n]$, the $\lambda$-ridge leverage scores of $A$ are equal to the corresponding leverage scores of $\bar{A}$. Concretely, we have $\ell_\lambda^i(A) = \ell_i(\bar{A})$.

Proof. The lemma follows from $\bar{A}^T\bar{A} = A^T A + \lambda I$ and the definition of $\lambda$-ridge leverage scores. \qed

Lemma 3.1 gives an alternate explanation for why the effective dimension $d_{\text{eff}}$ shrinks as $\lambda$ increases, since the gap $d - d_{\text{eff}}$ is the sum of leverage scores for the $d$ augmented rows corresponding to the regularization terms.

Next we quantify approximate ridge leverage scores from a probability distribution point of view.

Definition 3.2. The vector $\hat{\ell}_\lambda(A) \in \mathbb{R}^n$ is a $\beta$-overestimate for the $\lambda$-ridge leverage score distribution of $A \in \mathbb{R}^{n \times d}$ if, for all $i \in [n]$, it satisfies

$$\frac{\hat{\ell}_\lambda^i(A)}{\|\hat{\ell}_\lambda(A)\|_1} \geq \beta \frac{\ell_\lambda^i(A)}{\|\ell_\lambda(A)\|_1} = \beta \frac{\ell_\lambda^i(A)}{d_{\text{eff}}}.$$  

Whenever $\beta$-overestimates are used in a leverage score-based sketching method, the sample complexity of the sketching algorithm increases by a factor of $O(1/\beta)$. Therefore, we want to construct approximate distributions that minimize the maximum relative decrease in probability. For ridge leverage score sampling (in contrast with classical leverage scores), it is possible to have $\beta > 1$ due to a decrease in effective dimension $d_{\text{eff}}$, which is beneficial since it means fewer samples are needed.

Now we define an augmented probability distribution, which will allow us to seamlessly apply sketching tools for ordinary least squares to ridge regression. The augmented distribution is constructed from a ridge leverage score $\beta$-overestimate, and one of its key properties is that we can sample from it in the same amount of time that we can sample from the $\beta$-overestimate distribution.
Definition 3.3. Let \( \mathcal{D}(\hat{\ell}^\lambda(A), d'_{\text{eff}}) \) denote the augmented distribution of the \( \beta \)-overestimate \( \hat{\ell}^\lambda(A) \), where \( d'_{\text{eff}} \geq 0 \) is a lower bound for the effective dimension of \( A \). The sample space of this distribution is the index set \([n + d]\), and its probability mass function is defined as

\[
\Pr(X = i) \propto \begin{cases} 
\hat{\ell}^\lambda(A) & \text{if } i \in [n], \\
\min\{1, d - d'_{\text{eff}}\} & \text{if } i \in [n + d] \setminus [n].
\end{cases}
\]

Note that sampling from this augmented distribution does not require any information about the leverage scores for the \( d \) additional rows in the augmented design matrix in Equation (5). To efficiently generate a sample from this distribution, we first flip a coin to branch on the two subsets of indices and then we sample from each conditional distribution accordingly.

Next we claim the augmented distribution of a \( \beta \)-overestimate for a ridge leverage score distribution is a \( \beta' \)-overestimate for the leverage score distribution of the augmented design matrix in Equation (5). The proofs for all of the remaining results in this section are deferred to Appendix A.

Lemma 3.4. If \( \hat{\ell}^\lambda(A) \) is a \( \beta \)-overestimate for the \( \lambda \)-ridge leverage score distribution of \( A \in \mathbb{R}^{n \times d} \), then the probability vector for the distribution \( \mathcal{D}(\hat{\ell}^\lambda(A), d'_{\text{eff}}) \) is a \( \beta' \)-overestimate for the leverage score distribution of \( \hat{A} \in \mathbb{R}^{(n + d) \times d} \), where

\[
\beta' = \min \left\{ \left( 1 + \frac{d \min\{1, d - d'_{\text{eff}}\}}{\|\hat{\ell}^\lambda(A)\|_1} \right)^{-1} \beta d, \frac{\|\hat{\ell}^\lambda(A)\|_1}{d} + \min\{1, d - d'_{\text{eff}}\} \right\}^{-1}.
\]

This lemma is a simple consequence of the definitions of a \( \beta \)-overestimate and the effective dimension. Although Equation (6) may initially seem unwieldy since \( d_{\text{eff}} \) could be difficult to compute, we can cleanly lower bound \( \beta' \) using the fact that \( d_{\text{eff}} \leq d \).

The next result follows immediately from the proof of Lemma 3.4. Observe that there is no explicit dependency on the effective dimension in this statement, as it gets cancelled out when using classical leverage scores as the overestimates.

Corollary 3.5. The leverage scores of \( A \in \mathbb{R}^{n \times d} \) are a \( (d_{\text{eff}}/\text{rank}(A)) \)-overestimate for the \( \lambda \)-ridge leverage scores of \( A \). Therefore, the distribution \( \mathcal{D}(\ell(A), d'_{\text{eff}}) \) is a \( (\text{rank}(A)/d + \min\{1, d - d'_{\text{eff}}\})^{-1} \)-overestimate for the leverage scores of \( \hat{A} \in \mathbb{R}^{(n + d) \times d} \).

We can always guarantee \( \beta' \geq 1/2 \) by using the statistical leverage scores as the \( \beta \)-overestimate for \( \lambda \)-ridge leverage scores since \( \text{rank}(A) \leq d \) and by letting \( d'_{\text{eff}} = 0 \). We choose to write the statements above, however, in terms of \( d'_{\text{eff}} \) since they perfectly recover leverage score overestimates for ordinary least squares when \( \lambda = 0 \) and the effective dimension \( d_{\text{eff}} = d \).

We are now ready to present our approximate ridge regression algorithm and its guarantees. This algorithm constructs an augmented distribution from the provided \( \beta \)-overestimate and uses Lemma 3.4 together with several well-known sketching ideas from randomized numerical linear algebra Drineas et al. (2006, 2011); Woodruff (2014) to bound the number of row samples needed from the augmented design matrix in Equation (5). We explain how all of the leverage score sampling building blocks interact in Appendix A. In particular, we show that the number of samples \( s \) defined on Line 3 of...
Algorithm 1 is sufficient for the sketch matrix $S$ to be a subspace embedding of the augmented least squares problem.

**Algorithm 1** Approximate ridge regression using a $\beta$-overestimate for $\lambda$-ridge leverage scores.

1: function `APPROXIMATERIDGEREGRESSION` ($A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, $\beta$-overestimate $\hat{\ell}(A)$ for the \(\lambda\)-ridge leverage scores of $A$, lower bound $d'_\text{eff}$ for the effective dimension of $A$, $\varepsilon$, $\delta$

2: Normalize $\hat{\ell}(A)$ so that $\|\hat{\ell}(A)\|_1 = d$ and set $\beta' \leftarrow \min\{\beta, 1/(1 + \min\{1, d - d'_\text{eff}\})\}$

3: Set number of samples $s \leftarrow \lceil 4d'/\beta' \max\{420\ln(4d/\delta), 1/(\delta\varepsilon)\} \rceil$

4: Set $\overline{A} \leftarrow [A ; \sqrt{\lambda}I_d]$ and $\overline{b} \leftarrow [b ; 0]$ as in Equation (5)

5: Initialize sketch matrix $S \leftarrow \overline{0}_{s \times (n+d)}$

6: for $i = 1$ to $s$ do

7: Sample $j \sim D(\hat{\ell}(A), d'_\text{eff})$ from the augmented distribution and set $s_{ij} \leftarrow 1/\sqrt{\Pr(j)}$

8: return $\tilde{x}_\text{opt} \leftarrow \arg\min_{x \in \mathbb{R}^d} \|S\overline{A}x - S\overline{b}\|_2$

**Theorem 3.6.** Algorithm 1 samples $s = O(d \max\{1/\beta, 1\} \max\{\ln(d/\delta), 1/(\delta\varepsilon)\})$ rows and returns a vector $\tilde{x}_\text{opt} \in \mathbb{R}^d$ such that, with probability at least $1 - \delta$, we have

$$\|Ax_\text{opt} - b\|_2^2 + \lambda \|x_\text{opt}\|_2^2 \leq (1 + \varepsilon) \left(\|Ax_\text{opt} - b\|_2^2 + \lambda \|x_\text{opt}\|_2^2\right).$$

Let $t$ denote the time complexity of sampling from $D(\hat{\ell}(A), d'_\text{eff})$ and let $T(n', d')$ be the time needed to solve a least squares problem of size $n' \times d'$. The running time of Algorithm 1 is $O(st + T(s, d))$.

One of the main takeaways from this result is that if we use leverage scores as the input $\beta$-overestimate, Corollary 3.5 ensures that only $O(d \max\{\log(d/\delta), 1/(\delta\varepsilon)\})$ row samples are needed since $\beta' \geq 1/2$.

This holds for any regularization strength $\lambda \geq 0$ since $\lambda = 0$ corresponds to full effective dimension and is in some sense the hardest type of ridge regression problem. Finally, we note that the running times in Theorem 3.6 are templated so that the result can easily be combined with fast sampling routines and modern least squares algorithms.

## 4 Fast Low-Rank Tucker Decomposition

Now we use the `APPROXIMATERIDGEREGRESSION` algorithm to accelerate the core tensor update steps in the alternating least squares (ALS) algorithm for Tucker decompositions. To achieve this, we exploit the Kronecker product structure of the design matrix in the core tensor subproblem and use the leverage scores of the design matrix as an overestimate for the true $\lambda$-ridge leverage scores. The leverage scores of a Kronecker product matrix factor cleanly into the product of leverage scores of its factor matrices, which allows us to sample from the augmented leverage score distribution in time that is sublinear in the number of rows of this design matrix. A similar technique was recently used in the context of ALS for tensor CP decompositions, where the leverage scores of a Kronecker product matrix were used as an overestimate for the leverage scores of a Khatri–Rao product design matrix Cheng et al. (2016); Larsen and Kolda (2020).

We start by presenting the ALS algorithm for Tucker decompositions and briefly analyze the amount of work in each step. To update factor matrix $A^{(n)}$, it solves $I_n$ ridge regression problems, all of which
share the same design matrix of size \((I_1 \cdots I_{n-1} I_{n+1} \cdots I_N) \times R_n\). For each core tensor update, we solve a ridge regression problem whose design matrix has dimensions \((I_1 I_2 \cdots I_N) \times (R_1 R_2 \cdots R_N)\).更新的核张量是迭代ALS中最昂贵的步骤，因此我们的动机是通过一个近似岭回归子程序来使其更快。

Algorithm 2 Alternating least squares (ALS) algorithm for regularized Tucker decomposition.

1: function ALS(tensor \(X \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\), multilinear rank \((R_1, R_2, \ldots, R_N)\), regularization \(\lambda\))
2: Initialize random core tensor \(\mathbf{S} \in \mathbb{R}^{R_1 \times R_2 \times \cdots \times R_N}\)
3: Initialize random factor matrix \(A^{(n)} \in \mathbb{R}^{I_n \times R_n}\) for \(n = 1\) to \(N\)
4: repeat
5: for \(n = 1\) to \(N\) do
6: Set \(K \leftarrow G_{(n)}(A^{(1)} \otimes \cdots \otimes A^{(n-1)} \otimes A^{(n+1)} \otimes \cdots \otimes A^{(N)})\) and \(B \leftarrow X^{(n)}\)
7: for \(i = 1\) to \(I_n\) do
8: Update factor matrix row \(a_i^{(n)} \leftarrow \arg\min_y \|yK - b_i\|_2^2 + \lambda\|y\|_2^2\)
9: Update \(S \leftarrow \arg\min_{y'} \|A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(N)} \cdot \text{vec}(S') - \text{vec}(X)\|_2^2 + \lambda\|\text{vec}(S')\|_2^2\)
10: until convergence
11: return \(S, A^{(1)}, A^{(2)}, \ldots, A^{(N)}\)

4.1 Approximate Core Tensor Update

Next we explore the structure of \(\lambda\)-ridge leverage scores for Kronecker product matrices and describe how to efficiently sample from the augmented leverage score distribution. The following result shows how \(\lambda\)-ridge leverage scores of a Kronecker product matrix decompose according to the SVDs of its factor matrices. In the special case of leverage scores (i.e., \(\lambda = 0\)), the expression completely factors, which we can exploit since it induces a product distribution. The proof of this result repeatedly uses the mixed-product property for Kronecker products with the pseudoinverse-based definition of \(\lambda\)-ridge leverage scores in Equation (2). We defer the proofs of all results in this section to Appendix B.

Lemma 4.1. Suppose \(K = A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(N)}\), where each factor matrix \(A^{(n)} \in \mathbb{R}^{I_n \times R_n}\), and let \((i_1, i_2, \ldots, i_N)\) denote the canonical row indexing of \(K\) according to its factors. If the SVD of \(A^{(n)} = U^{(n)} \Sigma^{(n)} V^{(n)^T}\), then for \(\lambda > 0\), the cross \(\lambda\)-ridge leverage scores of \(K\) are

\[
\ell_{(i_1, \ldots, i_N), (j_1, \ldots, j_N)}(K) = \sum_{t \in T} \frac{\prod_{n=1}^{N} \sigma_{i_n}^2(A^{(n)})}{\prod_{n=1}^{N} \sigma_{j_n}^2(A^{(n)})} + \lambda \left( \prod_{n=1}^{N} u_{i_n}^{(n)} \right) \left( \prod_{n=1}^{N} u_{j_n}^{(n)} \right),
\]

where the sum is over the row index set \(T = [I_1] \times [I_2] \times \cdots \times [I_N]\). Therefore, given the SVDs of the factor matrices, we can compute each cross \(\lambda\)-ridge leverage score of \(K\) in \(O(R_1 R_2 \cdots R_N \cdot N)\) time. Furthermore, for (statistical) cross leverage scores, we have

\[
\ell_{(i_1, \ldots, i_N), (j_1, \ldots, j_N)}(K) = \prod_{n=1}^{N} \ell_{i_n j_n}(A^{(n)}).
\]

Now we present our fast sampling-based core tensor update. The algorithm first computes the leverage scores for each factor matrix, and then it initializes a data structure to query entries in
the Kronecker product design matrix \( \mathbf{K} \) without explicitly constructing it (to avoid creating a memory bottleneck). Similarly, the algorithm then initializes a data structure to sample from the leverage score distribution of \( \mathbf{K} \) by independently sampling the index for each dimension from the corresponding factor matrix leverage score distribution. This ensures that the time complexity for sampling all row indices of \( \mathbf{K} \) is sublinear in the number of its rows. Finally, the algorithm calls the approximate ridge regression subroutine with these data structures as implicit inputs and updates the core tensor accordingly.

**Algorithm 3** Fast core tensor update using approximate ridge regression.

```
1: function FASTCORETENSORUPDATE(\( \mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \), factors \( \mathbf{A}^{(n)} \in \mathbb{R}^{I_n \times R_n} \), \( \lambda, \varepsilon, \delta \))
2: Compute factor matrix leverage scores \( \ell(\mathbf{A}^{(n)}) \) by Equation (2) for \( n = 1 \) to \( N \)
3: Initialize data structure to query entries of design matrix \( \mathbf{K} \leftarrow \mathbf{A}^{(1)} \otimes \mathbf{A}^{(2)} \otimes \cdots \otimes \mathbf{A}^{(N)} \)
4: Initialize data structure to sample from \( \ell(\mathbf{K}) \) using factored leverage scores
5: Set core tensor \( \text{vec}(\mathbf{g}) \leftarrow \text{APPROXIMATERIDGEREGRESSION}(\mathbf{K}, \text{vec}(\mathbf{X}), \ell(\mathbf{K}), \lambda, 0, \varepsilon, \delta) \)
```

**Theorem 4.2.** Let \( R = R_1 R_2 \cdots R_N \). Algorithm 3 gives a \((1 + \varepsilon)\)-approximation to the optimal core tensor weights with probability at least \( 1 - \delta \) in time \( O(R^2 \max\{\log(R/\delta), 1/(\delta \varepsilon)\} + \sum_{n=1}^{N} I_n R_n^2) \), where \( \omega < 2.373 \) is the matrix multiplication exponent, and uses \( O(R^2 + \sum_{n=1}^{N} I_n R_n) \) space.

The approximation and running time guarantees in Theorem 4.2 follow primarily from Corollary 3.5 and Theorem 3.6, which show that leverage scores are a sample-efficient overestimate for any ridge regression problem. One appealing consequence of an approximate core tensor update whose running time is predominantly a function of the multilinear rank and sketching parameters (as opposed to the size of the input tensor \( I_1 I_2 \cdots I_N \)) is that the max block improvement (MBI) algorithm Chen et al. (2012) becomes feasible for a much larger set of problems, since ALS is a special case of block coordinate descent.

### 4.2 Missing Data and Ridge Leverage Score Upper Bounds

Now we take a step towards the more general tensor completion problem, where the goal is to learn a tensor decomposition that fits the observed entries well and also generalizes to unseen data. For Tucker decompositions and the ALS algorithm, this corresponds to removing rows of the Kronecker product design matrix \( \mathbf{K} \) (i.e., observations in the input tensor) before updating the core. To apply sampling-based sketching techniques in this setting and accelerate the core tensor update, we need to understand how the \( \lambda \)-ridge leverage scores of a matrix change as its rows are removed. In particular, we need to give accurate upper bounds for the ridge leverage scores of the matrix with removed rows. Our first result shows how ridge leverage scores increase as the rows of the matrix are removed.

**Theorem 4.3.** Let \( \mathbf{L} = \mathbf{A}(\mathbf{A}^\top + \lambda \mathbf{I})^{-1} \mathbf{A}^\top \) be the cross \( \lambda \)-ridge leverage score matrix of \( \mathbf{A} \in \mathbb{R}^{n \times d} \). For any \( S \subseteq [n] \), let \( \tilde{\mathbf{A}} \) denote the matrix containing only the rows of \( \mathbf{A} \) indexed by \( S \). Let \( \overline{S} = [n] \setminus S \) denote the set of missing row indices, and let \( L_{\overline{S}, \overline{S}} \) be the principal submatrix of \( \mathbf{L} \) containing only the entries indexed by \( \overline{S} \). Then, for any \( \lambda > 0 \) and \( i \in S \), we have

\[
\ell_i^\lambda(\tilde{\mathbf{A}}) \leq \ell_i^\lambda(\mathbf{A}) + \frac{1}{1 - \lambda_{\max}(L_{\overline{S}, \overline{S}})} \sum_{j \in \overline{S}} \ell_{ij}^\lambda(\mathbf{A})^2,
\]
where \( \lambda_{\text{max}}(M) \) denotes the maximum eigenvalue of matrix \( M \).

The proof relies on the Woodbury matrix identity and Cauchy interlacing theorem and provides useful insight into the role that cross ridge leverage score \( \ell^\lambda_{ij}(A) \) plays when row \( j \) is removed but \( i \) remains. We also generalize a well-known property of cross leverage scores, which allows to sample from a product distribution in the tensor completion setting when combined with Theorem 4.3.

**Lemma 4.4.** For any \( A \in \mathbb{R}^{n \times d} \) and \( i \in [n] \), the sum of the squared cross \( \lambda \)-ridge leverage scores is at most the ridge \( \lambda \)-leverage score itself, with equality iff \( \lambda = 0 \). Concretely, \( \sum_{j=1}^n \ell^\lambda_{ij}(A)^2 \leq \ell^\lambda_i(A) \).

These results imply that the ridge leverage scores of \( A \) are a \( \beta \)-overestimate for ridge leverage scores of \( \tilde{A} \), where \( 1/\beta = 1 + 1/(1 - \lambda_{\text{max}}(\text{LS}, S)) \). We discuss how this affects the sample complexity for sketching in Appendix B (e.g., if the factor matrices for \( K \) are semi-orthogonal, then \( 1/\beta \leq 2 + 1/\lambda \)).

5 Experiments

We compare the performance of ALS with and without the FastCoreUpdate algorithm by computing low-rank Tucker decompositions of large, dense synthetic tensors and real-world movie data from the TensorSketch experiments of Malik and Becker (2018). Our experiments build on Tensorly Kossaifi et al. (2019) and were run on an Intel Xeon W-2135 processor (8.25 MB cache and 3.70 GHz) using 128GB of RAM. We refer to the ALS algorithm that uses ridge leverage score sampling for the core as ALS-RS.

**Synthetic Benchmarks.** First we stress test ALS and ALS-RS on large, dense tensors. We consider various low-rank Tucker decompositions and show that the running time to update the core in ALS-RS remains fixed as a function of the core size, without any loss in solution accuracy. These profiles further illustrate that the core tensor update is a severe bottleneck in an unmodified ALS algorithm.

Specifically, for each input shape we generate a random Tucker decomposition with an \((8, 8, 8)\) core tensor. All of the entries in the factor matrices and core tensor are i.i.d. uniform random variables from \([0, 1]\). We add Gaussian noise from \( \mathcal{N}(0, 1) \) to one percent of the entries in this Tucker tensor, and call the resulting tensor \( Y \). The noise gives a lower bound of 0.10 for the RMSE when learning \( Y \).

Next we initialize a random Tucker decomposition with a smaller core size and fit it to \( Y \) using ALS and ALS-RS, both starting from the same initial state. For each input shape and multilinear rank of the learned Tucker decomposition, we run ALS and ALS-RS for 10 iterations and report the mean running time of each step type in Table 1. In all the experiments we set \( \lambda = 0.001 \), and for ALS-RS we set \( \varepsilon = 0.1 \) and \( \delta = 0.1 \). The columns labeled F1, F2, and F3 correspond to factor matrix updates.

We draw several conclusions from Table 1. First, by using ridge leverage score sampling, ALS-RS guarantees fast, high-quality core tensor updates while solving a substantially smaller ridge regression problem in each iteration. Comparing the core columns of ALS and ALS-RS at fixed ranks as the input shape increases clearly demonstrates this. Second, ALS and ALS-RS converge to the same local optimum for this data, which is evident from the RMSE columns since both start from the same seed.
Table 1: Mean running times per step type for ALS and ALS-RS on dense synthetic tensors.

| Input Shape      | Rank   | F1 (s) | F2 (s) | F3 (s) | Core (s) | RMSE   | Core (s) | RMSE   |
|------------------|--------|--------|--------|--------|----------|--------|----------|--------|
| (512, 512, 512)  | (2, 2, 2) | 2.0    | 2.0    | 0.3    | 7.5      | 0.364  | 3.9      | 0.364  |
|                  | (4, 2, 2) | 2.1    | 2.0    | 0.3    | 13.3     | 0.363  | 9.4      | 0.363  |
|                  | (4, 4, 2) | 2.1    | 2.1    | 0.3    | 24.8     | 0.328  | 23.1     | 0.328  |
|                  | (4, 4, 4) | 2.1    | 2.1    | 0.4    | 48.1     | 0.284  | 54.7     | 0.284  |
| (1024, 512, 512) | (2, 2, 2) | 4.5    | 4.4    | 0.6    | 15.4     | 0.392  | 3.9      | 0.392  |
|                  | (4, 2, 2) | 4.6    | 4.4    | 0.7    | 26.9     | 0.387  | 9.3      | 0.387  |
|                  | (4, 4, 2) | 4.7    | 4.6    | 0.7    | 50.3     | 0.342  | 22.7     | 0.342  |
|                  | (4, 4, 4) | 4.7    | 4.6    | 0.9    | 96.8     | 0.300  | 53.9     | 0.300  |
| (1024, 1024, 512)| (2, 2, 2) | 13.3   | 13.2   | 1.3    | 35.0     | 0.425  | 3.9      | 0.425  |
|                  | (4, 2, 2) | 13.4   | 13.0   | 1.3    | 58.0     | 0.413  | 9.3      | 0.413  |
|                  | (4, 4, 2) | 13.7   | 13.7   | 1.4    | 104.0    | 0.382  | 22.7     | 0.382  |
|                  | (4, 4, 4) | 13.7   | 13.7   | 1.9    | 196.9    | 0.321  | 54.0     | 0.321  |
| (1024, 1024, 1024)| (2, 2, 2) | 27.7   | 27.6   | 2.7    | 67.9     | 0.406  | 3.9      | 0.406  |
|                  | (4, 2, 2) | 28.8   | 27.6   | 2.7    | 110.9    | 0.403  | 9.3      | 0.403  |
|                  | (4, 4, 2) | 28.9   | 28.8   | 2.8    | 196.3    | 0.355  | 22.8     | 0.356  |
|                  | (4, 4, 4) | 28.5   | 28.3   | 3.8    | 367.1    | 0.294  | 54.0     | 0.294  |

**Movie Experiments.** Now we use ALS and ALS-RS to compute low-rank Tucker decompositions of a video of a nature scene. Malik and Becker (2018) used their TensorSketch algorithm for Tucker decompositions on this data since most frames are very similar, except for a couple disturbances when a person walks by. Thus, the video is expected to be compressible. It consists of 2493 frames, each an image of size (1080, 1920, 3) corresponding to height, width, and the RGB channel. Similar to the experiments in Malik and Becker (2018), we construct tensors of shape (100, 1080, 1920, 3) from 100-frame slices of the video for our comparison. We again let $\lambda = 0.001$, and set $\varepsilon = 0.1$ and $\delta = 0.1$ for ALS-RS.

We use a Tucker decomposition with a core of shape (2, 4, 4, 2), and we present the mean running times for each update step type over 5 iterations, after which both algorithms have converged. The update steps of ALS took (8.8, 3.1, 8.8, 163.1, 9935.5) seconds, corresponding to F1, F2, F3, F4, and the core update. In contrast, ALS-RS took (9.2, 2.7, 9.3, 221.6, 85.8) seconds per step type. Both algorithms converge to 0.188 RMSE, and at each step of every iteration, the RMSEs of the two solutions differed in absolute value by at most 2.98e-5. Hence, they converged along the same paths. Using ALS-RS in this experiment sped up the core tensor update by more than a factor of 100 and shifted the performance bottleneck to the factor matrix update corresponding to the RGB channel.

6 Conclusion

This work accelerates ALS for regularized low-rank Tucker decompositions by using ridge leverage score sampling in the core tensor update. Specifically, it introduces a new approximate ridge regression
algorithm that augments classical leverage score distributions, and it begins to explore properties of ridge leverage scores in a dynamic setting. These results also immediately extend the leverage score sampling-based CP decomposition algorithms in Cheng et al. (2016); Larsen and Kolda (2020) to support L2 regularization. Lastly, our synthetic and real-world experiments demonstrate that this approximate core tensor update leads to substantial improvements in the running time of ALS for low-rank Tucker decompositions while preserving the original solution quality of ALS.

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A Missing Analysis from Section 3

Here we show how to use $\beta$-overestimates for the ridge leverage scores of a design matrix $A \in \mathbb{R}^{n \times d}$ to create a substantially smaller ordinary least squares problem whose solution vector gives a $(1 + \varepsilon)$-approximation to the original ridge regression problem. Our proof relies on several sketching and leverage score sampling results from randomized numerical linear algebra Drineas et al. (2006, 2011); Woodruff (2014). The leverage score sampling part of our argument was recently organized all in one place in (Larsen and Kolda, 2020, Appendix B), as these prerequisite results are well-understood but scattered across many references.

A.1 Fast Approximate Least Squares

We start by considering the overdetermined least squares problem defined by a matrix $A \in \mathbb{R}^{n \times d}$ and response vector $b \in \mathbb{R}^n$, where $n \geq d$ and rank($A$) = $d$. Define the optimal sum of squared residuals to be

$$\mathcal{R}^2 = \min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2.$$  \hspace{1cm} (7)

Assume for now that $A$ is full rank, and let the compact SVD of the design matrix be $A = U_A \Sigma_A V_A^T$. By definition, $U_A \in \mathbb{R}^{n \times d}$ is an orthonormal basis for the column space of $A$. Let $U_A^\perp \in \mathbb{R}^{n \times (n-d)}$ be an orthonormal basis for the $(n-d)$-dimensional subspace that is orthogonal to the column space of $A$. For notational simplicity, let $b^\perp = U_A^\perp U_A^T b$ denote the projection of $b$ onto the orthogonal subspace $U_A^\perp$. The vector $b^T$ is important because its norm is equal to the norm of the residual vector. To see this, observe that $x$ can be chosen so that $Ax$ perfectly matches the part of $b$ in the column space of $A$, but cannot (by definition) match anything in the range of $U_A^\perp$:

$$\mathcal{R}^2 = \min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2 = \left\| U_A^\perp U_A^T b \right\|_2^2 = \left\| b^\perp \right\|_2^2.$$  \hspace{1cm} \hspace{1cm} (8)

We denote the solution to the least squares problem by $x_{\text{opt}}$, hence we have $b = Ax_{\text{opt}} + b^\perp$.

Now we build on a structural result of Drineas et al. (2011, Lemma 1) that establishes sufficient conditions on any sketching matrix $S \in \mathbb{R}^{s \times n}$ such that the solution $\tilde{x}_{\text{opt}}$ to the approximate least squares problem

$$\tilde{x}_{\text{opt}} = \arg\min_{x \in \mathbb{R}^d} \|S(Ax - b)\|_2^2$$  \hspace{1cm} \hspace{1cm} (9)

gives a relative-error approximation to the original least squares problem. The two conditions that we require of matrix $S$ are:

$$\sigma_{\text{min}}^2(SU_A) \geq 1/\sqrt{2}, \quad \text{and} \quad \left\| U_A^\perp S^T b^\perp \right\|_2^2 \leq \varepsilon \mathcal{R}^2/2,$$  \hspace{1cm} \hspace{1cm} (10)

for some $\varepsilon \in (0, 1)$. Note that while the APPROXIMATERIDGEREGRESSION algorithm is randomized, the following lemma is a deterministic statement. Failure probabilities will enter our analysis later when we show that this algorithm satisfies conditions (9) and (10) with sufficiently high probability.
Lemma A.1 (Drineas et al. (2011)). Consider the overconstrained least squares approximation problem in (7), and let the matrix \( U_A \in \mathbb{R}^{n \times d} \) contain the top \( d \) left singular vectors of \( A \). Assume the matrix \( S \) satisfies conditions (9) and (10) above, for some \( \varepsilon \in (0,1) \). Then, the solution \( \tilde{x}_{\text{opt}} \) to the approximate least squares problem (8) satisfies:

\[
\| A\tilde{x}_{\text{opt}} - b \|_2^2 \leq (1 + \varepsilon) R^2, \quad \text{and} \\
\| \tilde{x}_{\text{opt}} - x_{\text{opt}} \|_2^2 \leq \frac{1}{\sigma_{\min}^2(A)} \varepsilon R^2. 
\]

Proof. Let us first rewrite the sketched least squares problem induced by \( S \) as

\[
\begin{align*}
\min_{x \in \mathbb{R}^d} \| SAx - Sb \|_2^2 & = \min_{y \in \mathbb{R}^d} \| SA(x_{\text{opt}} + y) - S(Ax_{\text{opt}} + b^\perp) \|_2^2 \\
& = \min_{y \in \mathbb{R}^d} \| SAy - Sb^\perp \|_2^2 \\
& = \min_{z \in \mathbb{R}^d} \| SU_Az - Sb^\perp \|_2^2. 
\end{align*}
\]

Equation (13) follows since \( b = Ax_{\text{opt}} + b^\perp \), and (14) follows because the columns of \( A \) span the same subspace as the columns of \( U_A \). Now, let \( z_{\text{opt}} \in \mathbb{R}^d \) be such that \( U_Az_{\text{opt}} = A(\tilde{x}_{\text{opt}} - x_{\text{opt}}) \), and note that \( z_{\text{opt}} \) minimizes (14). This fact follows from

\[
\begin{align*}
\| SAx_{\text{opt}} - Sb^\perp \|_2^2 & = \| S\tilde{x}_{\text{opt}} - S(b - b^\perp) - Sb^\perp \|_2^2 = \| SA\tilde{x}_{\text{opt}} - Sb \|_2^2. 
\end{align*}
\]

Thus, by the normal equations, we have

\[(SU_A)^\top SU_Az_{\text{opt}} = (SU_A)^\top Sb^\perp.
\]

Taking the norm of both sides and observing that under condition (9) we have \( \sigma_i((SU_A)^\top SU_A) = \sigma_i^2(SU_A) \geq 1/\sqrt{2} \), for all \( i \), it follows that

\[
\| z_{\text{opt}} \|_2^2/2 \leq \| (SU_A)^\top SU_Az_{\text{opt}} \|_2^2 = \| (SU_A)^\top Sb^\perp \|_2^2. 
\]

Using condition (10), we observe that

\[
\| z_{\text{opt}} \|_2^2 \leq \varepsilon R^2. 
\]

To establish the first claim of the lemma, let us rewrite the squared norm of the residual vector as

\[
\begin{align*}
\| A\tilde{x}_{\text{opt}} - b \|_2^2 & = \| A\tilde{x}_{\text{opt}} - Ax_{\text{opt}} + Ax_{\text{opt}} - b \|_2^2 \\
& = \| A\tilde{x}_{\text{opt}} - Ax_{\text{opt}} \|_2^2 + \| Ax_{\text{opt}} - b \|_2^2 \\
& = \| U_Az_{\text{opt}} \|_2^2 + R^2 \\
& \leq (1 + \varepsilon) R^2, 
\end{align*}
\]

where (17) follows from the Pythagorean theorem since \( b - Ax_{\text{opt}} = b^\perp \), which is orthogonal to \( A \), and consequently \( A(x_{\text{opt}} - \tilde{x}_{\text{opt}}) \); (18) follows from the definition of \( z_{\text{opt}} \) and \( R^2 \); and (19) follows from (16) and the orthogonality of \( U_A \).
To establish the second claim of the lemma, recall that $A(x_{\text{opt}} - \tilde{x}_{\text{opt}}) = U_A z_{\text{opt}}$. Taking the norm of both sides of this expression, we have that

$$\|x_{\text{opt}} - \tilde{x}_{\text{opt}}\|^2_2 \leq \frac{\|U_A z_{\text{opt}}\|^2_{\sigma_{\text{min}}}(A)}{\sigma_{\text{min}}(A)} \leq \frac{\varepsilon R^2}{\sigma_{\text{min}}(A)}{\delta},$$

(20)

where (20) follows since $\sigma_{\text{min}}(A)$ is the smallest singular value of $A$ and $\text{rank}(A) = d$; and (21) follows from (16) and the orthogonality of $U_A$.

### A.2 Tools for Satisfying the Structural Conditions

Next we present two results that will be useful for proving that the sketching matrix $S$ satisfies the structural conditions in Equations (9) and (10). The first result is (Woodruff, 2014, Theorem 17), which states that $SU_A$ is a subspace embedding for the column space of $U_A$. This result can be thought of as approximate isometry and is noticeably stronger than the desired condition $\sigma^2_{\text{min}}(SU_A) \geq 1/\sqrt{2}$.

**Theorem A.2** (Woodruff (2014)). Consider $A \in \mathbb{R}^{n \times d}$ and its compact SVD $A = U_A \Sigma_A V_A^\top$. For any $\beta > 0$, let $\hat{\ell}(A)$ be a $\beta$-overestimate for the leverage score distribution of $A$. Let $s > 144d \ln(2d/\delta)/(\beta \varepsilon^2)$. Construct a sampling matrix $\Omega \in \mathbb{R}^{n \times s}$ and rescaling matrix $D \in \mathbb{R}^{s \times s}$ as follows. Initially, let $\Omega = 0^{n \times s}$ and $D = 0^{s \times s}$. For each column $j$ of $\Omega$ and $D$, independently, and with replacement, choose a row index $i \in [n]$ with probability $q_i = \hat{\ell}_i(A)/\|\hat{\ell}(A)\|_1$, and set $\omega_{ij} = 1$ and $d_{jj} = 1/\sqrt{q_i s}$. Then with probability at least $1 - \delta$, simultaneously for all $i$, we have

$$1 - \varepsilon \leq \sigma^2_i(D^\top \Omega^\top U_A) \leq 1 + \varepsilon.$$

To prove the second structural condition, we use the following result about squared-distance sampling for approximate matrix multiplication in (Drineas et al., 2006, Lemma 8). In our analysis for leverage score sampling-based ridge regression, it is possible (and beneficial) that $\beta > 1$ in the statement below. Therefore, we modify the original theorem statement and provide a proof to show that the result is unaffected.

**Theorem A.3** (Drineas et al. (2006)). Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times p}$, and $s$ denote the number of samples. Let $p \in \mathbb{R}^n$ be a probability vector such that, for all $i \in [n]$, we have

$$p_i \geq \beta \frac{\|a_i\|^2_2}{\|A\|^2_F},$$

for some constant $\beta > 0$. Sample $s$ row indices $(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(s)})$ from $p$, independently and with replacement, and form the approximate product

$$\frac{1}{s} \sum_{t=1}^s \frac{1}{p_{\xi^{(t)}}} a_{\xi^{(t)}}^\top b_{\xi^{(t)}} = (SA)^\top SB,$$
where $S \in \mathbb{R}^{s \times n}$ is the sampling and rescaling matrix whose $t$-th row is defined by the entries

$$s_{tk} = \begin{cases} \frac{1}{\sqrt{sp_k}} & \text{if } k = \xi_t, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\mathbb{E} \left[ \|A^\top B - (SA)^\top SB\|_F^2 \right] \leq \frac{1}{\beta s} \|A\|_F^2 \|B\|_F^2.$$

**Proof.** First, we analyze the entry of $(SA)^\top SB$ at index $(i, j)$. By viewing the approximate product as a sum of outer products, we can write this entry in terms of scalar random variables $X_t$, for $t \in [s]$, as follows:

$$X_t = \frac{a_{\xi(t)} b_{\xi(t)} j}{sp_{\xi(t)}} \implies [(SA)^\top SB]_{ij} = \sum_{t=1}^{s} X_t.$$

The expected values of $X_t$ and $X_t^2$ for all values of $t$ are

$$\mathbb{E}[X_t] = \sum_{k=1}^{n} p_k \frac{a_{ki} b_{kj}}{sp_k} = \frac{1}{s} (A^\top B)_{ij}, \text{ and}$$

$$\mathbb{E}[X_t^2] = \sum_{k=1}^{n} p_k \left( \frac{a_{ki} b_{kj}}{sp_k} \right)^2 = \frac{1}{s^2} \sum_{k=1}^{n} \frac{(a_{ki} b_{kj})^2}{p_k}.$$

Thus, we have $\mathbb{E}[(SA)^\top SB]_{ij} = \sum_{t=1}^{s} \mathbb{E}[X_t] = (A^\top B)_{ij}$, which means the estimator is unbiased. Furthermore, since the estimated matrix entry is the sum of $s$ i.i.d. random variables, its variance is

$$\text{Var} \left( [(SA)^\top SB]_{ij} \right) = \sum_{t=1}^{s} \text{Var}(X_t)$$

$$= \sum_{t=1}^{s} \left( \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 \right)$$

$$= \sum_{t=1}^{s} \frac{1}{s^2} \sum_{k=1}^{n} \left( \frac{(a_{ki} b_{kj})^2}{p_k} - (A^\top B)_{ij}^2 \right)$$

$$= \frac{1}{s} \sum_{k=1}^{n} \left( \frac{(a_{ki} b_{kj})^2}{p_k} - (A^\top B)_{ij}^2 \right).$$
Now we apply this result to the expectation we want to bound:

\[
\mathbb{E}\left[\|A^\top B - (SA)^\top SB\|_F^2\right] = \sum_{i=1}^m \sum_{j=1}^p \mathbb{E}\left[\left(\|\left((SA)^\top SB\right)_{ij} - (A^\top B)_{ij}\|_2^2\right)\right]
\]

\[
= \sum_{i=1}^m \sum_{j=1}^p \mathbb{E}\left[\left(\|\left((SA)^\top SB\right)_{ij} - \mathbb{E}\left[\left((SA)^\top SB\right)_{ij}\right]\right|_2^2\right)\right]
\]

\[
= \sum_{i=1}^m \sum_{j=1}^p \text{Var}\left(\left((SA)^\top SB\right)_{ij}\right)
\]

\[
= \frac{1}{s} \sum_{i=1}^m \sum_{j=1}^p \sum_{k=1}^n \frac{(a_{ki}b_{kj})^2}{p_k} - \left(\|A^\top B\|_F^2\right)
\]

\[
= \frac{1}{s} \sum_{k=1}^n \|a_k\|_2^2 \|b_k\|_2^2 - \frac{n}{s} \|A^\top B\|_F^2
\]

\[
\leq \frac{1}{s} \sum_{k=1}^n \|a_k\|_2^2 \|b_k\|_2^2,
\]

where the last inequality uses the fact that the Frobenius norm of any matrix is nonnegative. Finally, by using the \(\beta\)-overestimate assumption on the sampling probabilities, we have

\[
\mathbb{E}\left[\|A^\top B - (SA)^\top SB\|_F^2\right] \leq \frac{1}{s} \sum_{k=1}^n \frac{\|a_k\|_2^2 \|b_k\|_2^2}{p_k}
\]

\[
\leq \frac{1}{s} \sum_{k=1}^n \left(\|A\|_F^2 \frac{\|a_k\|_2^2 \|b_k\|_2^2}{\beta \|a_k\|_2^2}\right)
\]

\[
= \frac{1}{s \beta} \|A\|_F^2 \sum_{k=1}^n \|b_k\|_2^2
\]

\[
= \frac{1}{s \beta} \|A\|_F^2 \|B\|_F^2,
\]

which is the desired upper bound. \(\square\)

### A.3 Generalizing to Ridge Regression

Now that all of our tools are in place, we show how to design an augmented distribution that allows us to efficiently use \(\beta\)-overestimates for the \(\lambda\)-ridge leverage score distribution of \(A\) to construct a sketch of the ridge regression problem whose solution gives a \((1 + \varepsilon)\)-approximation for the original loss function. Moreover, our analysis carefully tracks how \(\beta'\) (the overestimate factor for the augmented distribution) changes as a function of \(\beta\) and the effective dimension \(d_{\text{eff}}\) of the original problem.

**Lemma 3.4.** If \(\hat{\ell}(A)\) is a \(\beta\)-overestimate for the \(\lambda\)-ridge leverage score distribution of \(A \in \mathbb{R}^{n \times d}\), then the probability vector for the distribution \(D(\hat{\ell}(A), d_{\text{eff}})\) is a \(\beta'\)-overestimate for the leverage
score distribution of $\mathbf{A} \in \mathbb{R}^{(n+d) \times d}$, where

$$\beta' = \min \left\{ \left( 1 + \frac{d \min\{1, d - d'_{\text{eff}}\}}{\|\hat{\ell}^{\lambda}_{\text{eff}}\|_1} \right)^{-1} \beta \frac{d}{d_{\text{eff}}} \left( \frac{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1}{d} + \min\{1, d - d'_{\text{eff}}\} \right)^{-1} \right\}. \quad (6)$$

**Proof.** We start by analyzing the leverage score distribution for $\mathbf{A}$. For row indices $i \in [n]$, we have

$$\Pr(X = i) = \frac{\ell_{i}(\mathbf{A})}{d}.$$  

It follows from Lemma 3.1 and the definition of effective dimension that $\sum_{i=1}^{n} \ell_{i}(\mathbf{A}) = d_{\text{eff}}$. Hence, for any augmented row index $j \in \{n + 1, n + 2, \ldots, n + d\}$, we have

$$\Pr(X = j) \leq \frac{\min\{1, d - d'_{\text{eff}}\}}{d}. \quad (22)$$

Next, we analyze the distribution $\mathcal{D}((\hat{\ell}^{\lambda}(\mathbf{A}), d'_{\text{eff}}))$. Each row index $i \in [n]$ is sampled with probability

$$\Pr(Y = i) = \frac{\hat{\ell}_{i}^{\lambda}(\mathbf{A})}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1 + d \min\{1, d - d'_{\text{eff}}\}} \quad (23)$$

$$= \frac{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1 + d \min\{1, d - d'_{\text{eff}}\}}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1} \cdot \frac{\hat{\ell}_{i}^{\lambda}(\mathbf{A})}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1} \cdot \frac{\bf{\ell}_{i}(\mathbf{A})}{\|\bf{\ell}^{\lambda}(\mathbf{A})\|_1} \quad (24)$$

$$= \left( 1 + \frac{d \min\{1, d - d'_{\text{eff}}\}}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1} \right)^{-1} \beta \frac{\ell_{i}(\mathbf{A})}{d_{\text{eff}}} \Pr(X = i). \quad (25)$$

Equation (23) follows from the definition of $\mathcal{D}((\hat{\ell}^{\lambda}(\mathbf{A}), d'_{\text{eff}}))$; Equation (24) follows since $\hat{\ell}^{\lambda}(\mathbf{A})$ is a $\beta$-overestimate for the $\lambda$-ridge leverage score distribution of $\mathbf{A}$; and Equation (25) follows from the alternate characterization of $\lambda$-ridge leverage scores in Lemma 3.1.

Further, each row index $j \in \{n + 1, n + 2, \ldots, n + d\}$ is sampled with probability

$$\Pr(Y = j) = \frac{\min\{1, d - d'_{\text{eff}}\}}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1 + d \min\{1, d - d'_{\text{eff}}\}} \quad (26)$$

$$= \frac{d}{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1 + d \min\{1, d - d'_{\text{eff}}\}} \cdot \frac{\min\{1, d - d'_{\text{eff}}\}}{d} \quad (27)$$

$$\geq \left( \frac{\|\hat{\ell}^{\lambda}(\mathbf{A})\|_1}{d} + \min\{1, d - d'_{\text{eff}}\} \right)^{-1} \Pr(X = j). \quad (27)$$
Equation (26) follows from the definition of $\mathcal{D}(\hat{\mathbf{\ell}}^\lambda(\mathbf{A}), d'_{\text{eff}})$, and Equation (27) uses the upper bound for $\Pr(X = j)$ in Equation (22). Therefore, we have

$$\beta' = \min \left\{ \left( 1 + \frac{d \min\{1, d - d'_{\text{eff}}\}}{\|\hat{\mathbf{\ell}}^\lambda(\mathbf{A})\|_1} \right)^{-1} \frac{\beta d}{d_{\text{eff}}} \left( \frac{\|\hat{\mathbf{\ell}}^\lambda(\mathbf{A})\|_1}{d} + \min\{1, d - d'_{\text{eff}}\} \right)^{-1} \right\},$$

(28)

as desired.

\[\square\]

**Corollary 3.5.** The leverage scores of $\mathbf{A} \in \mathbb{R}^{n \times d}$ are a $(d_{\text{eff}}/\text{rank}(\mathbf{A}))$-overestimate for the $\lambda$-ridge leverage scores of $\mathbf{A}$. Therefore, the distribution $\mathcal{D}(\mathbf{\ell}(\mathbf{A}), d'_{\text{eff}})$ is a $(\text{rank}(\mathbf{A})/d + \min\{1, d - d'_{\text{eff}}\})^{-1}$-overestimate for the leverage scores of $\mathbf{A} \in \mathbb{R}^{(n+d) \times d}$.

**Proof.** The first part of the claim follows from the inequality

$$\frac{\ell_i(\mathbf{A})}{\|\mathbf{\ell}(\mathbf{A})\|_1} = \frac{\ell_i(\mathbf{A})}{\text{rank}(\mathbf{A})} \geq \frac{\ell_i^\lambda(\mathbf{A})}{\text{rank}(\mathbf{A})} = \frac{d_{\text{eff}}}{\text{rank}(\mathbf{A})} \cdot \frac{\ell_i^\lambda(\mathbf{A})}{d_{\text{eff}}}.$$ 

The second part is then a consequence of Lemma 3.4. Letting $\hat{\ell}^\lambda(\mathbf{A}) = \mathbf{\ell}(\mathbf{A})$ and $\beta = d_{\text{eff}}/\text{rank}(\mathbf{A})$, we have

$$\beta' = \left( \frac{d_{\text{eff}}}{d} \cdot \frac{\text{rank}(\mathbf{A})}{d_{\text{eff}}} + \min\{1, d - d_{\text{eff}}\} \right)^{-1},$$

which completes the proof. \[\square\]

Now we prove our main theorem in Section 3, which (1) shows that the solution vector to the sketched regression problem is a good approximation to the initial ridge regression problem, and (2) quantifies how many row samples are needed for a $(1 + \varepsilon)$-approximation as a function of the input overestimate factor $\beta$.

**Theorem 3.6.** Algorithm 1 samples $s = O(d \max\{1/\beta, 1\} \max\{\ln(d/\delta), 1/(\delta \varepsilon)\})$ rows and returns a vector $\tilde{\mathbf{x}}_{\text{opt}} \in \mathbb{R}^d$ such that, with probability at least $1 - \delta$, we have

$$\|\mathbf{A}\tilde{\mathbf{x}}_{\text{opt}} - \mathbf{b}\|^2_2 + \lambda\|\tilde{\mathbf{x}}_{\text{opt}}\|^2_2 \leq (1 + \varepsilon) \left( \|\mathbf{A}\mathbf{x}_{\text{opt}} - \mathbf{b}\|^2_2 + \lambda\|\mathbf{x}_{\text{opt}}\|^2_2 \right).$$

Let $t$ denote the time complexity of sampling from $\mathcal{D}(\hat{\mathbf{\ell}}^\lambda(\mathbf{A}), d'_{\text{eff}})$ and let $T(n', d')$ be the time needed to solve a least squares problem of size $n' \times d'$. The running time of Algorithm 1 is $O(st + T(s, d))$.

**Proof.** It suffices to find a $(1 + \varepsilon)$-approximate solution to the equivalent least squares problem

$$\arg \min_{\mathbf{x} \in \mathbb{R}^d} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

We know that the augmented distribution $\mathcal{D}(\hat{\mathbf{\ell}}^\lambda(\mathbf{A}), d'_{\text{eff}})$ is a $\beta'$-overestimate for the leverage score distribution of $\mathbf{A}$ by Lemma 3.4. This enables us to use standard leverage score-based sampling arguments to show that the sketching matrix $\mathbf{S} \in \mathbb{R}^{s \times n}$ satisfies the two desired structural conditions
Once we prove that Equations (9) and (10) both hold with probability at least $1 - \delta$, we apply Lemma A.1 to conclude that the solution vector to the sketched least squares problem

$$\tilde{x}_{opt} = \arg\min_{x \in \mathbb{R}^d} \| S\tilde{A}x - S\tilde{b} \|_2^2$$

is a $(1 + \varepsilon)$-approximation to the original ridge regression problem. Therefore, it remains to show that the structural conditions hold with sufficient probability.

**Structural Condition 1.** We show that Equation (9) holds for $\tilde{A}$ with probability at least $1 - \delta/2$ by using Theorem A.2. First, observe that the sampling matrix $\Omega$ and rescaling matrix $D$ in this theorem statement are a decomposition of the sketching matrix in Algorithm 1 (i.e., $D^\top\Omega^\top = S$). Second, observe that it suffices to set $\varepsilon = 1 - 1/\sqrt{2}$ in the statement of Theorem A.2 in order to satisfy (9).

Let the compact SVD of $\tilde{A}$ be $\tilde{A} = U_{\tilde{A}}\Sigma_{\tilde{A}}V_{\tilde{A}}^\top$. Since $D(\hat{\ell}^\lambda(A), d'_{\text{eff}})$ is a $\beta'$-overestimate for the leverage score distribution of $\tilde{A}$, it follows from Theorem A.2 that with probability at least $1 - \delta/2$,

$$\sigma_{\text{min}}^2(SU_{\tilde{A}}) \geq 1/\sqrt{2},$$

as long as the number of samples $s$ is at least

$$s > \frac{144d\ln(4d/\delta)}{\beta'(1 - 1/\sqrt{2})^2} > \frac{1680d\ln(4d/\delta)}{\beta'}. \tag{29}$$

The algorithm rescales $\hat{\ell}^\lambda(A)$ on Line 2 so that $\| \hat{\ell}^\lambda(A)\|_1 = d$. Therefore, it follows from Lemma 3.4 and the fact that $d_{\text{eff}} \leq d$ that

$$\beta' = \min \left\{ \left( 1 + \frac{d \min\{1, d - d'_{\text{eff}}\}}{\| \hat{\ell}^\lambda(A)\|_1} \right)^{-1} \beta d_{\text{eff}}, \left( \frac{\| \hat{\ell}^\lambda(A)\|_1}{d} + \min\{1, d - d'_{\text{eff}}\} \right)^{-1} \right\}$$

$$\geq \min\{\beta, 1\}/(1 + \min\{1, d - d'_{\text{eff}}\}). \tag{29}$$

Thus, the number of samples is sufficient by our choice of $\beta'$ and $s$ on Lines 2–3 of Algorithm 1, so Equation (9) holds for $\tilde{A}$ with probability at least $1 - \delta/2$.

**Structural Condition 2.** We show that Equation (10) holds for $\tilde{A}$ with probability at least $1 - \delta/2$ by using Theorem A.3 and Markov's inequality. First observe that

$$U_{\tilde{A}}^\top B_{\tilde{A}}^\perp = U_{\tilde{A}}^\top \left( U_{\tilde{A}}^\perp U_{\tilde{A}}^\top B_{\tilde{A}} \right) = 0_{\text{rank}(\tilde{A})},$$

where $B_{\tilde{A}}^\perp$ is defined analogously to $b^\perp$ in Appendix A.1. Thus, the second structural condition can be seen as bounding how closely the sampled product approximates the zero vector. To do this, we can apply Theorem A.3 since the leverage scores of a matrix $M$ are the squared distances of the row
vectors of its compact left-singular matrix $U_M$. Combining these facts, it follows that
\[
E \left[ \| U_A^T S^T S \tilde{B}^T \|_2^2 \right] = E \left[ \| 0_{\text{rank}(A)} - U_A^T S^T S \tilde{B}^T \|_2^2 \right]
\]
\[
= E \left[ \| U_A^T \tilde{B}^\perp - U_A^T S^T S \tilde{B}^T \|_2^2 \right]
\]
\[
\leq \frac{1}{\beta^2 s} \| U_A^T \|_F \| \tilde{B}^\perp \|_2^2
\]
\[
= \frac{d}{\beta^2 s} \| \tilde{B}^\perp \|_2^2.
\]

Since the norm of the residual vector equals the norm of $\tilde{B}^\perp$, applying Markov’s inequality gives us
\[
\Pr \left( \| U_A^T S^T S \tilde{B}^T \|_2^2 \geq \frac{\varepsilon \| \tilde{B}^\perp \|_2^2}{2} \right) \leq \frac{2E \left[ \| U_A^T S^T S \tilde{B}^T \|_2^2 \right]}{\varepsilon \| \tilde{B}^\perp \|_2^2} \leq \frac{2d}{\varepsilon \beta^2 s}. \tag{30}
\]

Bounding the rightmost expression of (30) by the failure probability $\delta/2$ implies that the number of samples must be at least
\[
s \geq \frac{4d}{\beta^2 \delta \varepsilon}.
\]

The number of samples $s$ in APPROXIMATE RIDGE REGRESSION is large enough by (29) and Line 3 of Algorithm 1. Therefore, Equation (10) holds for $\bar{A}$ with probability at least $1 - \delta/2$.

Taking a union bound over the failure probabilities for the two structural conditions ensures that both are satisfied with probability at least $1 - (\delta/2 + \delta/2) = 1 - \delta$. Furthermore, since the running times of the two subroutines are parameterized by $t$ and $T(n', d')$, the result immediately follows. \(\square\)

## B Missing Analysis from Section 4

Now we prove several technical results about $\lambda$-ridge leverage scores and their upper bounds as rows are removed from the design matrix. First, we present an explicit formula for the $\lambda$-ridge leverage scores of a Kronecker matrix in terms of the singular value decompositions of its factor matrices.

**Lemma 4.1.** Suppose $K = A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(N)}$, where each factor matrix $A^{(n)} \in \mathbb{R}^{I_n \times R_n}$, and let $(i_1, i_2, \ldots, i_N)$ denote the canonical row indexing of $K$ according to its factors. If the SVD of $A^{(n)} = U^{(n)} \Sigma^{(n)} V^{(n)^T}$, then for $\lambda > 0$, the cross $\lambda$-ridge leverage scores of $K$ are
\[
\ell_{(i_1, \ldots, i_N), (j_1, \ldots, j_N)}^{(\lambda)}(K) = \sum_{t \in T} \frac{\prod_{n=1}^N \sigma_{i_n}^2(A^{(n)})}{\prod_{n=1}^N \sigma_n^2(A^{(n)})} + \lambda \left( \prod_{n=1}^N u_{i_n t_n}^{(n)} \right) \left( \prod_{n=1}^N u_{j_n t_n}^{(n)} \right),
\]
where the sum is over the row index set $T = [I_1] \times [I_2] \times \cdots \times [I_N]$. Therefore, given the SVDs of the factor matrices, we can compute each cross $\lambda$-ridge leverage score of $K$ in $O(R_1 R_2 \cdots R_N \cdot N)$ time. Furthermore, for (statistical) cross leverage scores, we have
\[
\ell_{(i_1, \ldots, i_N), (j_1, \ldots, j_N)}(K) = \prod_{n=1}^N \ell_{i_n j_n}(A^{(n)}).
\]
Putting everything together, the ridge regression algorithm using leverage scores as an overestimate for the mixed-product property of Kronecker products implies that

\[ K^T K = (A^T A) \otimes (B^T B) \otimes (C^T C). \]

Let \( A = U_A \Sigma_A V_A^T \) denote the SVD of \( A \) such that \( U_A \in \mathbb{R}^{I_1 \times I_1} \) and \( V_A \in \mathbb{R}^{R_1 \times R_1} \). It follows from the orthogonality of \( U_A \) that

\[ A^T A = V_A \Sigma_A^2 V_A^T, \]

where \( \Sigma_A^2 \) denotes \( \Sigma_A \Sigma_A^T \). Similarly, let \( B = U_B \Sigma_B V_B^T \) and \( C = U_C \Sigma_C V_C^T \). It follows from the mixed-product property that

\[
K^T K + \lambda I = (V_A \Sigma_A^2 V_A^T) \otimes (V_B \Sigma_B^2 V_B^T) \otimes (V_C \Sigma_C^2 V_C^T) + \lambda I
\]

\[
= (V_A \otimes V_B \otimes V_C) (\Sigma_A^2 \otimes \Sigma_B^2 \otimes \Sigma_C^2 + \lambda I) (V_A^T \otimes V_B^T \otimes V_C^T)
\]

Since \((XY)^+ = Y^+X^+\) if \( X \) or \( Y \) is orthogonal, we have

\[
(K^T K + \lambda I)^+ = ((V_A \otimes V_B \otimes V_C) (\Sigma_A^2 \otimes \Sigma_B^2 \otimes \Sigma_C^2 + \lambda I) (V_A^T \otimes V_B^T \otimes V_C^T))^+
\]

\[
= (V_A^T \otimes V_B^T \otimes V_C^T)^+ (\Sigma_A^2 \otimes \Sigma_B^2 \otimes \Sigma_C^2 + \lambda I)^+ (V_A \otimes V_B \otimes V_C)^+
\]

\[
= (V_A \otimes V_B \otimes V_C) (\Sigma_A^2 \otimes \Sigma_B^2 \otimes \Sigma_C^2 + \lambda I)^+ (V_A^T \otimes V_B^T \otimes V_C^T)
\]

Next, observe that

\[
K = (U_A \Sigma_A V_A^T) \otimes (U_B \Sigma_B V_B^T) \otimes (U_C \Sigma_C V_C^T)
\]

\[
= (U_A \otimes U_B \otimes U_C) (\Sigma_A \otimes \Sigma_B \otimes \Sigma_C) (V_A^T \otimes V_B^T \otimes V_C^T).
\]

Putting everything together, the \( \lambda \)-ridge cross leverage scores can be expressed as

\[
K(K^T K + \lambda I)^+ K^T = (U_A \otimes U_B \otimes U_C) \Lambda (U_A \otimes U_B \otimes U_C)^T,
\]

where

\[
\Lambda = (\Sigma_A \otimes \Sigma_B \otimes \Sigma_C) (\Sigma_A^2 \otimes \Sigma_B^2 \otimes \Sigma_C^2 + \lambda I)^+ (\Sigma_A \otimes \Sigma_B \otimes \Sigma_C).
\]

Equation (31) is the eigendecomposition of \( K(K^T K + \lambda I)^+ K^T \). In particular, \( \Lambda \in \mathbb{R}^{I_2 I_3 \times I_1 I_2 I_3} \) is a diagonal matrix of eigenvalues, where the \((i_1, i_2, i_3)\)-th eigenvalue is

\[
\lambda_{(i_1, i_2, i_3)} = \frac{\sigma_{i_1}^2(A) \sigma_{i_2}^2(B) \sigma_{i_3}^2(C)}{\sigma_{i_1}^2(A) \sigma_{i_2}^2(B) \sigma_{i_3}^2(C) + \lambda}.
\]

The claim about the values of \( \ell_{(i_1, i_2, i_3), (j_1, j_2, j_3)}^\lambda(K) \) then follows from the definition of cross \( \lambda \)-ridge leverage scores in Equation (3). Furthermore, the statistical leverage score property holds because setting \( \lambda = 0 \) gives an expression that is the product of the leverage scores of the factor matrices.

Next we analyze the time and space complexity of the fast core tensor update via our approximate ridge regression algorithm using leverage scores as an overestimate for the \( \lambda \)-ridge leverage scores.
Theorem 4.2. Let \( R = R_1 R_2 \cdots R_N \). Algorithm 3 gives a \((1 + \varepsilon)\)-approximation to the optimal core tensor weights with probability at least \( 1 - \delta \) in time \( O(R^\omega \max\{\log(R/\delta), 1/(\delta \varepsilon)\} + \sum_{n=1}^{N} I_n R_n^2) \), where \( \omega < 2.373 \) is the matrix multiplication exponent, and uses \( O(R^2 + \sum_{n=1}^{N} I_n R_n) \) space.

Proof. The leverage scores of the factor matrix \( A^{(n)} \) can be computed in \( O(I_n R_n^2) \) time using the pseudoinverse expression in Equation (2). Next, compute cumulative density functions for each of the factor distributions in \( O(R_1 + R_2 + \cdots + R_N) \) time. This enables us to sample a row index of from the leverage score distribution of \( K \) in time \( O(\log(R_1) + \log(R_2) + \cdots + \log(R_N)) \) using \( N \) binary searches and the factorization property in Lemma 4.1. The number of samples in the approximate ridge regression subroutine is \( s = O(R \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \) since \( \beta' \geq 1/2 \) if leverage scores are used as the \( \lambda \)-ridge leverage score overestimate by Corollary 3.5. Therefore, we can sample all of the row indices of the ridge-augmented version of matrix \( K \) and construct the sketch matrix \( S \) in time \( O(R \log(R) \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \).

We solve the least squares problem with design matrix \( A = SK \) and response vector \( b = S \text{vec}(X) \) by computing \( A^\top A \) and \( A^\top b \) and using the normal equations. The matrix \( A^\top A \) is the outer product of \( s = O(R \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \) row vectors of length \( R \), and can therefore be constructed in \( O(sR^2) = O(R^3 \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \) time using the data structure to implicitly query entries of \( K \) from the factor matrices. Note that we first compute a row of \( K \) using this data structure, and then we compute the outer product in \( O(R^2) \) time. We can compute \( (A^\top A)^+ \) in \( O(R^\omega) \) time via Alman and Williams (2021), where \( \omega < 2.373 \), since \( A^\top A \) is an \( R \times R \) matrix.

We construct \( A^\top b \) similarly in \( O(R \cdot R \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \) time. Putting everything together and using the normal equations, we can compute \( \tilde{\mathbf{x}}_{\text{opt}} = (A^\top A)^+ A^\top b \) in \( O(R^2) \) time. Therefore, the total running time of Algorithm 3 is \( O(R^\omega \max\{\log(R/\delta), 1/(\delta \varepsilon)\}) \).

The approximation guarantee follows directly from Corollary 3.5 and Theorem 3.6 since the augmented leverage scores of \( K \) are a \((1/2)\)-overestimate of the true \( \lambda \)-ridge leverage scores. We achieve the claimed space complexity by using implicit representations of \( K \) and its leverage scores. \( \square \)

Now we prove two elucidatory upper bounds for \( \lambda \)-ridge leverage scores as rows are removed from the design matrix. The first upper bound crucially relies on the regularization strength \( \lambda \) being positive, but an analogous statement holds if we use the pseudoinverse all the way throughout the proof. This pseudoinverse-based result is useful for accurately bounding statistical leverage scores (i.e., \( \lambda = 0 \)) when removing a set of rows lowers the rank of the matrix. The second upper bound holds for any set of removed rows, and thus is not particularly strong in general. That said, if the factor matrices are well-structured (e.g., columnwise orthogonality constraints), then it provides nontrivial guarantees.

Theorem 4.3. Let \( L = A(A^\top A + \lambda I)^+ A^\top \) be the cross \( \lambda \)-ridge leverage score matrix of \( A \in \mathbb{R}^{n \times d} \). For any \( S \subseteq [n] \), let \( \bar{A} \) denote the matrix containing only the rows of \( A \) indexed by \( S \). Let \( \bar{S} = [n] \setminus S \) denote the set of missing row indices, and let \( L_{\bar{S}, \bar{S}} \) be the principal submatrix of \( L \) containing only the entries indexed by \( \bar{S} \). Then, for any \( \lambda > 0 \) and \( i \in S \), we have

\[
\ell_i^\lambda(\bar{A}) \leq \ell_i^\lambda(A) + \frac{1}{1 - \lambda_{\max}(L_{\bar{S}, \bar{S}})} \sum_{j \in \bar{S}} \ell_j^\lambda(A)^2,
\]
where $\lambda_{\text{max}}(M)$ denotes the maximum eigenvalue of matrix $M$.

**Proof.** Let $W$ be the 0-1 diagonal matrix such that $w_{ii} = 1$ if $i \in S$ and $w_{ii} = 0$ otherwise. Then

$$\tilde{A} \tilde{A} = A \tilde{A} - ((I - W)A)(I - W)A.$$ 

Let $B = (I - W)A$. For any $i \in S$, it follows that

$$\ell_i^\lambda(\tilde{A}) = a_i (A \tilde{A} + \lambda I)^a_i = a_i (A \tilde{A} - B \tilde{B} + \lambda I)^a_i.$$ 

The matrix $A \tilde{A} - B \tilde{B} + \lambda I$ is positive definite because $\lambda > 0$, and hence invertible. Therefore, by the Woodbury matrix identity, we have

$$(A \tilde{A} - B \tilde{B} + \lambda I)^{-1} = (A \tilde{A} + \lambda I)^{-1} + (A \tilde{A} + \lambda I)^{-1}B(I - (A \tilde{A} + \lambda I)^{-1}B)^{-1}(A \tilde{A} + \lambda I)^{-1}.$$ 

For each $i \in S$, let $v_i \in \mathbb{R}^{1 \times n}$ be the row vector whose $j$-th entry is equal to $\ell_{ij}^\lambda(A)$ if $j \in S$ and zero otherwise. Then

$$v_i = a_i (A \tilde{A} + \lambda I)^{-1}B.$$ 

Hence, we have

$$a_i (A \tilde{A} - B \tilde{B} + \lambda I)^{-1}a_i^\top = v_i [a_i (A \tilde{A} + \lambda I)^{-1} + (A \tilde{A} + \lambda I)^{-1}B(I - (A \tilde{A} + \lambda I)^{-1}B)^{-1}(A \tilde{A} + \lambda I)^{-1}] a_i^\top = \ell_i^\lambda(A) + v_i (I - (A \tilde{A} + \lambda I)^{-1}B)^{-1}v_i^\top.$$ 

In terms of principal submatrices, this is equivalent to

$$\ell_i^\lambda(\tilde{A}) = \ell_i^\lambda(A) + \ell_S^\lambda(A)^\top (I - L_{\overline{S}, S})^{-1} \ell_S^\lambda(A).$$ (33)

Therefore, it follows from the Rayleigh–Ritz theorem that

$$\ell_i^\lambda(\tilde{A}) \leq \lambda_{\text{max}}\left((I - L_{\overline{S}, S})^{-1}\right) \|\ell_S^\lambda(A)\|_2^2 = \lambda_{\text{max}}\left((I - L_{\overline{S}, S})^{-1}\right) \sum_{j \in S} \ell_{ij}^\lambda(A)^2.$$ 

We know from (32) that all of the eigenvalues of $L$ are nonnegative and strictly less than one since

$$\lambda_i(L) = \frac{\sigma_i^2(A)}{\sigma_i^2(A) + \lambda}.$$ 

Therefore, since $L$ is symmetric and $L_{\overline{S}, S}$ is a principal submatrix of $L$, Cauchy’s interlacing theorem implies that

$$0 \leq \lambda_i(L_{\overline{S}, S}) \leq \lambda_i(L) < 1.$$ (34)
Finally, it follows from a standard eigenvalue argument that
\[
\lambda_{\text{max}}\left( (I - L_{\mathcal{S},\mathcal{S}})^{-1} \right) = \frac{1}{1 - \lambda_{\text{max}}(L_{\mathcal{S},\mathcal{S}})}.
\]
which completes the proof.

**Corollary B.1.** If \( K = A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(N)} \) where \( A^{(n)} \in \mathbb{R}^{I_n \times R_n} \), \( L = K(K^tK + \lambda I)^+K^t \) is the matrix of cross \( \lambda \)-ridge leverage scores of \( K \), then for any \( \lambda > 0 \) and \( S \subseteq [n] \), we have
\[
\frac{1}{1 - \lambda_{\text{max}}(L_{\mathcal{S},\mathcal{S}})} \leq 1 + \frac{\prod_{n=1}^{N} \| A^{(n)} \|_2^2}{\lambda}.
\]

**Proof.** The claim is a direct consequence of the formula for the eigenvalues of \( L \) in Equation (32) and the application of Cauchy’s interlacing theorem in Equation (34). Specifically, we know that
\[
\lambda_{\text{max}}(L_{\mathcal{S},\mathcal{S}}) \leq \lambda_{\text{max}}(L) = \frac{\prod_{n=1}^{N} \sigma_{\text{max}}^2(A^{(n)})}{\prod_{n=1}^{N} \sigma_{\text{max}}^2(A^{(n)}) + \lambda} = \frac{\prod_{n=1}^{N} \| A^{(n)} \|_2^2}{\lambda}.
\]
Therefore, it follows that
\[
\frac{1}{1 - \lambda_{\text{max}}(L_{\mathcal{S},\mathcal{S}})} \leq 1 + \frac{\prod_{n=1}^{N} \| A^{(n)} \|_2^2 + \lambda}{\lambda} = 1 + \frac{\prod_{n=1}^{N} \| A^{(n)} \|_2^2}{\lambda}.
\]

Last, we prove a property of ridge leverage scores that is closely related to the fact that hat matrices in linear regression are idempotent. This inequality can be combined with the upper bounds above to give multiplicative bounds for \( \lambda \)-ridge leverage scores of a matrix after some of its rows are removed, and the coefficient in these multiplicative bounds is approximately equal to \( \beta^{-1} \) in the \( \beta \)-overestimate.

**Lemma 4.4.** For any \( A \in \mathbb{R}^{n \times d} \) and \( i \in [n] \), the sum of the squared cross \( \lambda \)-ridge leverage scores is at most the ridge \( \lambda \)-leverage score itself, with equality iff \( \lambda = 0 \). Concretely, \( \sum_{j=1}^{n} \ell_{ij}^\lambda(A)^2 \leq \ell_{i1}^\lambda(A) \).

**Proof.** First, observe that
\[
\sum_{j=1}^{n} \ell_{ij}^\lambda(A)^2 = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \frac{\sigma_{k1}^2(A)}{\sigma_k^2(A) + \lambda} u_{ik} u_{jk} \right)^2
\]
\[
= \sum_{j=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left( \frac{\sigma_{k1}^2(A)}{\sigma_k^2(A) + \lambda} \right) \left( \frac{\sigma_{k2}^2(A)}{\sigma_k^2(A) + \lambda} \right) u_{ik_1} u_{jk_1} u_{ik_2} u_{jk_2}
\]
\[
= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left( \frac{\sigma_{k1}^2(A)}{\sigma_k^2(A) + \lambda} \right) \left( \frac{\sigma_{k2}^2(A)}{\sigma_k^2(A) + \lambda} \right) u_{ik_1} u_{ik_2} \sum_{j=1}^{n} u_{jk_1} u_{jk_2}.
\]
Since $\mathbf{U}$ is an orthogonal matrix, we have

$$
\sum_{j=1}^{n} u_{jk_1} u_{jk_2} = \begin{cases} 
1 & \text{if } k_1 = k_2, \\
0 & \text{if } k_2 \neq k_2.
\end{cases}
$$

Therefore, it follows that

$$
\sum_{j=1}^{n} \hat{\ell}_{ij}^\lambda (\mathbf{A})^2 = \sum_{k_1=1}^{n} \left( \frac{\sigma_{k_1}^2 (\mathbf{A})}{\sigma_{k_1}^2 (\mathbf{A}) + \lambda} \right)^2 u_{ik}^2 \leq \sum_{k_1=1}^{n} \frac{\sigma_{k_1}^2 (\mathbf{A})}{\sigma_{k_1}^2 (\mathbf{A}) + \lambda} u_{ik}^2 = \hat{\ell}_{i}^\lambda (\mathbf{A}).
$$

Moreover, the inequality above implies that the two expressions are equal if and only if $\lambda = 0$. □