Two-Grid based Adaptive Proper Orthogonal Decomposition Algorithm for Time Dependent Partial Differential Equations

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Abstract In this article, we propose a two-grid based adaptive proper orthogonal decomposition (POD) algorithm to solve the time dependent partial differential equations. Based on the error obtained in the coarse grid, we propose an error indicator for the numerical solution obtained in the fine grid. Our new algorithm is cheap and easy to implement. We implement our new method to the solution of time-dependent advection-diffusion equations with Kolmogorov flow and ABC flow. The numerical results show that our method is more efficient than the existing POD algorithms.

Keywords Proper orthogonal decomposition · Galerkin projection · Error indicator · Adaptive · Two grid

1 Introduction

Time dependent partial differential equations play an important role in scientific and engineering computing. Many physical phenomena are described by time dependent partial differential equations, for example, seawater intrusion [4], heat transfer [10], fluid equations [6,39]. The design and analysis of high efficiency numerical schemes for time dependent partial differential equations has always been an active research topic.

For the spatial discretization of the time dependent partial differential equations, some traditional methods, for example, the finite element method [8], the
finite difference method [22], the plane wave method [20], can be used. However, the dimension of the discretized systems from these traditional methods is usually very large when the spatial dimension is three. Therefore, if we always use these traditional methods to do the spatial discretization at each time interval, the computational cost will be very high [5,9,33].

The proper orthogonal decomposition (POD) approach is a dimensionality reduction algorithm, and is widely used in computational physics, engineering etc [3,7,18,24,28]. The basic idea of the POD algorithm is to start with an ensemble of data, called snapshots, collected from numerical solution obtained by one of the traditional methods over some interval \([0,T_0]\), then construct POD modes by performing SVD to these snapshots [19,23,28]. Usually, the number of the POD modes will be much less than the degree of freedom in the traditional methods. Therefore, it will be much cheaper to discretize the time dependent partial differential equations in the subspace spanned by these POD modes. However, if the POD modes are not well constructed, approximation error obtained by the POD algorithm will degrade the accuracy of solutions.

To reduce the approximation error of POD algorithm, some adaptive POD approach is then introduced [26,30,31,34]. Similar to the idea of adaptive finite element method [12,15], some error indicators are needed to determine whether the POD modes are reliable or not. If it is not reliable, then POD modes must be updated from time to time by some strategies. Therefore, it is important to find an efficient error indicator. To our knowledge, the existing adaptive POD methods mainly use the residual to construct an error indicator [14,30,34]. This type of error indicator is efficient, however, it is usually expensive to compute.

In this paper, we propose a two-grid based adaptive POD algorithm, where we use the error obtained in the coarse mesh to construct the error indicator, which is used to tell us if we need to update the POD subspace in the fine mesh or not. Since the degree of freedom of coarse mesh is much less than that of the fine mesh, it is cheap to calculate the error indicator. By our method, we can easily compute the error indicator, and then update the POD subspace when needed.

The rest of this paper is organized as follows. First, we give some preliminaries in Section 2, including basic introduction for the finite element method, the POD-Galerkin method, and the residual based adaptive POD method. Then, we propose our two-grid based adaptive POD algorithm in Section 3. Next, we apply our new method to the simulation of some typical time dependent partial differential equations, including advection-diffusion equation with three-space dimensional velocity field, such as Kolmogorov flow and ABC flow, and use these tests to show the efficiency and the advantage of our method to the existing methods in Section 4. Finally, some concluding remarks are given in Section 5.

2 Preliminaries

We consider the following general time dependent partial differential equation

\[
\begin{align*}
\begin{cases}
  u_t - D_0 \Delta u + B(x,y,z,t) \cdot \nabla u + c(x,y,z,t)u = f(x,y,z,t), & \text{in } \Omega \times (0,T) \\
  u(x,y,z,0) = h(x,y,z), \\
  u(x+l,y,z,t) = u(x,y,z,t+l) = u(x,y,z,t) = u(x,y,z,t),
\end{cases}
\end{align*}
\]
where $\Omega = [0, t]^3$, $f \in L^2(0, T; L^2(\Omega))$, $c \in L^\infty(\Omega)$, $B \in C(0, T; W^{1, \infty}(\Omega)^3)$ and $D_0$ is a constant.

Define a bilinear form

$$a(t; u, v) = D_0(\nabla u, \nabla v) - D_0 \int_{\partial \Omega} \frac{\partial u}{\partial n} v \, d\sigma + (B \cdot \nabla u, v) + (cu, v), \forall u, v \in H^1(\Omega),$$

where $(\cdot, \cdot)$ stands for the inner product in $L^2(\Omega)$, and the function space $V = \{v \in H^1(\Omega): v_{|\mathbf{x}=0} = v_{|\mathbf{x}=t} = v_{|\mathbf{y}=0} = v_{|\mathbf{y}=t} = v_{|\mathbf{z}=0} = v_{|\mathbf{z}=l}\}$.

Then the variational form of equation (1) can be written as follows: find $u \in V$ such that

$$(\frac{\partial u}{\partial t}, v) + a(t; u, v) = (f(x, y, z, t), v), \forall v \in V. \quad (2)$$

In order to solve (2) numerically, we choose the implicit Euler scheme \cite{1,36} for the temporal discretization. We partition the time interval into $N \in \mathbb{N}$ subintervals with equal length $\delta t = T/N$, and set $u^k(x, y, z) = u(x, y, z, t_k)$ where $t_k = k \ast \delta t$, for $k \in \{0, 1, \ldots, N\}$. Then the semi-discretization scheme of (2) can be written as:

$$\left(\frac{u^k(x, y, z) - u^{k-1}(x, y, z)}{\delta t}, v\right) + a(t_k; u^k(x, y, z), v) = (f(x, y, z, t_k), v), \forall v \in V. \quad (3)$$

2.1 Standard finite element method

In this subsection, we introduce the standard finite element discretization for the equation (3). For more general introduction on standard finite element method, please refer to e.g. \cite{32,35}.

Let $\mathcal{T}_h$ be a regular mesh over $\Omega$, that is, there exists a constant $\gamma^*$ such that \cite{11}

$$\frac{h_\tau}{\rho_\tau} < \gamma^*, \forall \tau \in \mathcal{T}_h$$

where $h_\tau$ is the diameter of $\tau$ for each $\tau \in \mathcal{T}_h$ and $\rho_\tau$ is the diameter of the biggest ball contained in $\tau \in \mathcal{T}_h$, $h = \max h_\tau, \tau \in \mathcal{T}_h$. Denote $\# \mathcal{T}_h$ the number of degree of freedom of mesh $\mathcal{T}_h$. Define the finite element space as

$$V_h = \{v_h : v_h|_e \in P_e, \forall e \in \mathcal{T}_h \text{ and } v_h \in C^0(\overline{\Omega})\} \cap V,$$

where $P_e$ is a set of polynomial function on element $e$.

Let $\{\phi_{h,i}\}_{i=1}^n$ be a basis for $V_h$, that is

$$V_h := \text{span}\{\phi_{h,1}, \phi_{h,2}, \ldots, \phi_{h,n}\}.$$

Then the numerical approximation of $u^k(x, y, z)$ can be expressed as

$$u_h^k(x, y, z) = \sum_{i=1}^n \beta_{h,i}^k \phi_{h,i}(x, y, z). \quad (4)$$
Fig. 1: Sketch of the POD method. $I_{FEM}, I_{POD}$ refers to the time intervals where (3) is discretized in $V_h$ and the subspace spanned by the POD modes, respectively.

Inserting (4) into (3), and setting $v = \phi_{h,j}, j = 1, 2, 3, \ldots, n$, respectively, we obtain

$$\sum_{i=1}^{n} \frac{\beta_{h,i}^k - \beta_{h,i}^{k-1}}{\delta t} (\phi_{h,i}, \phi_{h,j}) + \beta_{h,i}^k a(t_k; \phi_{h,i}, \phi_{h,j}) = (f(x, y, z, t_k), \phi_{h,j}). \quad (5)$$

We can rewrite (5) as

$$\sum_{i=1}^{n} \beta_{h,i}^k [(\phi_{h,i}, \phi_{h,j}) + \delta t a(t_k; \phi_{h,i}, \phi_{h,j})] = \delta t (f(x, y, z, t_k), \phi_{h,j}) + \sum_{i=1}^{n} \beta_{h,i}^{k-1} (\phi_{h,i}, \phi_{h,j}). \quad (6)$$

Define

$$A_{h,ij}^k = (\phi_{h,j}, \phi_{h,i}) + \delta t a(t_k; \phi_{h,j}, \phi_{h,i}), \quad u_h^k = (\beta_{h,1}^k, \beta_{h,2}^k, \beta_{h,3}^k, \ldots, \beta_{h,n}^k)^T,$$

$$b_h^k = \delta t * \left((f, \phi_{h,1}), \ldots, (f, \phi_{h,n})\right)^T, \quad C_{h,ij} = (\phi_{h,j}, \phi_{h,i}).$$

Then (6) can be written as the following algebraic form

$$A_{h}^k u_h^k = b_h^k + C_{h} u_h^{k-1}. \quad (7)$$

2.2 POD method

The POD method is a widely used dimensionality reduction algorithm[18,21,29]. It can capture the principal component of the numerical solution by using proper orthogonal decomposition. People then can construct POD modes based on the principal component. Usually, the number of POD modes will be much smaller than the degree of freedom for traditional methods. Therefore, the computational cost can usually be reduced largely by using the POD method. The sketch of the POD method is shown in Fig. 1, from which we can see the POD method for discretizing problem (3) include the following steps.

1. **Snapshots**

Discretize (3) in $V_h$ on the interval $[0, T_0]$, and collect the numerical solution per $\delta M$ steps($\delta M$ is a parameter to be specified in the numerical experiment).

Set $n_s = \left\lfloor \frac{T_0}{\delta t \cdot \delta M} \right\rfloor$, where $\lfloor * \rfloor$ mean round down, and denote

$$U_h = [u_h^0, u_h^{\delta M}, \ldots, u_h^{n_s \cdot \delta M}],$$
2. **POD modes**

Perform SVD to the snapshots $U_h \in \mathbb{R}^{n \times (n_a+1)}$, and obtain

$$U_h = RSV^T,$$

where $R = [R_1, R_2, \ldots, R_r] \in \mathbb{R}^{n \times r}$, $V = [V_1, \ldots, V_r] \in \mathbb{R}^{(n_a+1) \times r}$ are the left and right projection matrices, respectively, and $S = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. The rank of $U_h$ is $r$, and obviously $r \leq \min(n, n_a + 1)$.

Set $m = \min\{k| \sum_{i=1}^{k} s_{i,i} > \gamma \ast \text{Trace}(S)\}$ ($\gamma$ is a parameter to be specified in the numerical experiment), then the POD modes are constructed by

$$(\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m}) = (\phi_{h,1}, \phi_{h,2}, \ldots, \phi_{h,n})\tilde{R},$$

where $\tilde{R} = [R_1, R_2, \ldots, R_m]$.

For the convenience of the following discussion, we summarize the process of constructing POD modes as routine POD Mode($U_h, \gamma_1, \Phi_h, m, \Psi_h$), where $\Phi_h = (\phi_{h,1}, \phi_{h,2}, \ldots, \phi_{h,n})$ and $\Psi_h = (\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m})$, please see Algorithm 1 for the details.

### Algorithm 1 POD Mode($U_h, \gamma_1, \Phi_h, m, \Psi_h$)

**Input:** $U_h, \gamma_1, \Phi_h = (\phi_{h,1}, \phi_{h,2}, \ldots, \phi_{h,n})$.

**Output:** $m$ and POD modes $\{\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m}\}$.

**Step1:** Perform SVD to $U_h$, and obtain $U_h = RSV^T$, where $S = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

**Step2:** Set $m = \min\{k| \sum_{i=1}^{k} s_{i,i} > \gamma \ast \text{Trace}(S)\}$.

**Step3:** $$(\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m}) = \Phi_h\tilde{R}[\cdot, 1:m].$$

3. **Galerkin projection**

For $t > T_0$, we discretize the equations (3) in the space span$\{\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m}\}$, which is sometimes called the POD subspace. That is, the solution of (3) is approximated by

$$u_h^k \text{POD}(x, y, z) = \sum_{i=1}^{m} \beta_{h,i}^k \psi_{h,i}(x, y, z).$$

Inserting (10) into (3), and setting $v = \psi_{h,j}$, $j = 1, 2, 3, \ldots, m$, respectively, we then obtain the following discretized problem:

$$\sum_{i=1}^{m} \beta_{h,i}^k [(\psi_{h,i}, \psi_{h,j}) + \delta t a(t_k; \psi_{h,i}, \psi_{h,j})] = \delta t (f(x, y, z, t_k), \psi_{h,j})$$

$$+ \sum_{i=1}^{m} \beta_{h,i}^{k-1} (\psi_{h,i}, \psi_{h,j}).$$

(11)

We define

$$\tilde{A}_{h,i,j} = (\psi_{h,j}, \psi_{h,i}) + \delta t a(t_k; \psi_{h,j}, \psi_{h,i}),$$

$$\tilde{b}_{h,i} = \delta t \ast ((f, \psi_{h,1}), \ldots, (f, \psi_{h,m}))^T, \quad \tilde{C}_{h,i,j} = (\psi_{h,j}, \psi_{h,i}),$$

$$u_{h, \text{POD}} = (\beta_{h,1}^k, \beta_{h,2}^k, \beta_{h,3}^k, \ldots, \beta_{h,m}^k)^T.$$
Then (11) can be written as following algebraic form

$$\tilde{\mathbf{A}}_h^k \mathbf{u}_{h,\text{POD}}^k = \tilde{\mathbf{b}}_h^k + \tilde{\mathbf{C}}_h \mathbf{u}_{h,\text{POD}}^{k-1}. \quad (12)$$

By some simple calculation, we have that

$$\tilde{\mathbf{A}}_h^k = \tilde{\mathbf{R}}^T \mathbf{A}_h^k \tilde{\mathbf{R}}, \quad \tilde{\mathbf{b}}_h^k = \tilde{\mathbf{R}}^T \mathbf{b}_h^k, \quad \tilde{\mathbf{C}}_h = \tilde{\mathbf{R}}^T \mathbf{C}_h \tilde{\mathbf{R}}.$$

Summarizing the above discussion, we then get the standard POD method for discretizing (3), which is shown as Algorithm 2.

**Algorithm 2** POD method

1: Give $\delta t, \gamma_1, T_0, \delta M$, and the mesh $T_h$. Set $n_s = \lfloor T_0 / \delta t \rfloor$.
2: Discretize (3) in $V_h$ on interval $[0, T_0]$, and obtain $\mathbf{u}_h^k, k = 0, \ldots, \lfloor T_0 / \delta t \rfloor$.
3: Take snapshots $\mathbf{U}_h$ at $t_0, t_\delta M, \ldots, t_{n_s} \cdot \delta M$, respectively, that is $\mathbf{U}_h = [\mathbf{u}_h^0, \mathbf{u}_h^{\delta M}, \ldots, \mathbf{u}_h^{n_s \cdot \delta M}]$.
4: Construct POD modes $\Psi_h$ by PODMode($\mathbf{U}_h, \gamma_1, \Phi_h, m, \Psi_h$).
5: $t = T_0$.
6: while $t \leq T$ do
7: \hspace{1em} $t = t + \delta t, k = k + 1$.
8: \hspace{1em} Discretize (3) in the subspace span($\psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m}$), and obtain $\mathbf{u}_h^k,\text{POD}$.
9: end while

2.3 Adaptive POD method

When the numerical solution changes greatly as time increase, the original POD modes cannot catch the behavior of the solution efficiently, then the POD algorithm will result in wrong results. To overcome this problem, some adaptive POD algorithms are proposed[2,31]. Similar to the adaptive finite element method[15], the adaptive POD method consists of the loop of the form

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Update}. \quad (13)$$

Here, we introduce this loop briefly. Suppose we have the POD modes $\{\psi_{h,1}, \ldots, \psi_{h,m}\}$, which can be written as

$$\begin{pmatrix} \psi_{h,1}, \psi_{h,2}, \ldots, \psi_{h,m} \end{pmatrix} = \begin{pmatrix} \phi_{h,1}, \phi_{h,2}, \ldots, \phi_{h,m} \end{pmatrix} \tilde{\mathbf{R}},$$

where $\tilde{\mathbf{R}} = [R_1, R_2, \ldots, R_m]$.

First, we discretize the equation (3) in the space spanned by these POD modes $\{\psi_{h,1}, \ldots, \psi_{h,m}\}$, and obtain the POD approximation. Then, we construct an error indicator to Estimate the error of the POD approximation. Next, we Mark the time where the error indicator is too large. Finally, we Update the POD modes at the marked time, and obtain the new POD modes. We repeat this loop in the new time interval until the terminal point.

The design of error indicator is an essential part in the Estimate step. For the error indicator, there are usually two requirements, one is that it can estimate the
error of the approximated solution very well, the other is that it should be very cheap to compute. In fact, the main difference between different adaptive POD methods lies in the construction of the error indicator.

For the Update of the POD modes, we can do as follows. Give a threshold $\eta_0$, when the error indicator $\eta_k \geq \eta_0$, which means that the error of POD approximation is too large, then we trace back to the previous time layer of the current time (which is labeled as the $p$-th time layer). Starting from the $p$-th time layer, we discretize equation (3) in the finite element space $V_h$ and obtain $u_h^p, u_h^{p+1}, \ldots, u_h^{p+\lceil \frac{\delta T}{\delta M} \rceil}$. Set $n_{s1} = \lfloor \frac{\delta T}{\delta M} \rfloor$ and denote $W_{h,1} = [u_h^p, u_h^{p+\delta M}, \ldots, u_h^{p+n_{s1}\delta M}]$.

Performing SVD to $W_{h,1}$, we get

$$W_{h,1} = R_1 S_1 V_1.$$  

Here $R_1 \in \mathbb{R}^{n \times r_1}$, $V_1 \in \mathbb{R}^{(n_{s1}+1) \times r_1}$ are the left and right projection matrices, respectively, and $S_1 = \text{diag}\{\sigma_{1,1}, \sigma_{1,2}, \ldots, \sigma_{1,r_1}\}$ with $\sigma_{1,1} \geq \sigma_{1,2} \geq \cdots \geq \sigma_{1,r_1} > 0$. The rank of $W_{h,1}$ is $r_1 \leq \min(n, n_{s1} + 1)$.

Set $m_1 = \min\{k : \sum_{i=1}^{k} S_{1,ii} \geq \gamma_2 \text{Trace}(S_1)\}$ ($\gamma_2$ is a parameter to be specified in the numerical experiment), then we combine the first $m_1$ column of $R_1$ and the old (previously used) POD modes, and get new matrix $W_{h,2}$. That is

$$W_{h,2} = [R_1[:,1:m_1], \tilde{R}].$$

We then perform SVD to $W_{h,2}$, and obtain

$$W_{h,2} = R_2 S_2 V_2.$$  

(14)

Here $R_2 \in \mathbb{R}^{n \times r_2}$, $V_2 \in \mathbb{R}^{(m_1+m) \times r_2}$ are the left and right projection matrices, respectively, and $S_2 = \text{diag}\{\sigma_{2,1}, \sigma_{2,2}, \ldots, \sigma_{2,r_2}\}$ with $\sigma_{2,1} \geq \sigma_{2,2} \geq \cdots \geq \sigma_{2,r_2} > 0$. The rank of $W_{h,2}$ is $r_2 \leq \min(n, m_1 + m)$.

Set $m_2 = \min\{k : \sum_{i=1}^{k} S_{2,ii} \geq \gamma_3 \text{Trace}(S_2)\}$ ($\gamma_3$ is a parameter to be specified in the numerical experiment), then the new POD modes are

$$\tilde{\psi}_{h,i}^{\text{new}} = \sum_{j=1}^{n} R_{2,j} \phi_{h,j}, \quad i = 1, 2, \ldots, m_2.$$  

(15)

For the convenience of the following discussion, we summarize the process of updating the POD modes as routine Update POD Mode($W_{h,1}, \gamma_2, \gamma_3, \Phi_h, m, \Psi_h$), which is shown as Algorithm 3.
Then we obtain the following framework of adaptive POD method for discretizing problem (3), see Algorithm 4 for the details.

Algorithm 4 Adaptive POD method

1: Give $\delta t, T_0, \delta T, \tau_1, \tau_2, \tau_3, \delta M$ and the mesh $T_h$. Set $n_s = \lfloor \frac{T_0}{\tau_3 \delta T} \rfloor$
2: Discretize (3) in $V_h$ on interval $[0, T_0]$ and obtain $u_{h,k}^n, \forall k \in [0, \lfloor T_0/\delta t \rfloor]$, then take snapshots $U_h$ at different times $t_0, t_\delta M, \ldots, t_n, s M$, that is $U_h = [u_{h,1}^0, u_{h,1}^{s M}, \ldots, u_{h,n}^{s M}]$.
3: Construct POD modes $\Psi_h$ by POD_Mode($U_h, \tau_1, \tau_2, m, \psi_h$).
4: $t = t_0$.
5: while $t \leq T$ do
6: $t = t + \delta t, k = k + 1$.
7: Discretize (3) in the space span{$\psi_{h,1}, \ldots, \psi_{h,m}$}, and obtain $u_{h,POD}^k$ then compute error indicator $\eta_k$ by some strategy.
8: if $\eta_k > \eta_0$ then
9: $t = t - \delta t, k = k - 1$.
10: Discretize (3) in $V_h$ on interval $[t, t + \delta T]$ to get $u_{h,k+i}^0, i = 1, \ldots, \frac{\delta T}{\delta t}$, from which to get snapshots $W_{h,1}^1$, then update POD models $\Psi_h$ by $\text{Update\_POD\_Mode}(W_{h,1}, \tau_1, \tau_2, \tau_3, m, \psi_h), k = k + \frac{\delta T}{\delta t}$.
11: end if
12: end while

2.4 Residual based adaptive POD method

The main difference between different adaptive POD methods is the way to construct the error indicator. Among the existing adaptive POD methods, the residual based adaptive POD method is the most widely used [14,30]. The residual based adaptive POD algorithm uses the residual to construct the error indicator. Based on the residual corresponding to the POD approximation, people construct the error indicator $\eta_k$ as:

$$
\eta_k = \frac{\|b_h^k + C_h \bar{R}u_{h,POD}^k - A_h^k \bar{R}u_{h,POD}^k\|_2}{\delta t \|\bar{R}u_{h,POD}^k\|_2}.
$$

(16)
3 Two-grid based adaptive POD method

For the residual based adaptive POD algorithm, we can see from (16) that in order to calculate the error indicator, we need to go back to the finite element space \( V_h \) to calculate the residual, which is too expensive. Here, we propose a two-grid based adaptive POD algorithm (TG-APOD). The main idea is to construct two finite element spaces, coarse finite element space and fine finite element space, then construct the POD subspace in the fine finite element space, and use the coarse finite element space to construct the error indicator to tell us when the POD modes need to be updated.

We first construct a coarse partition \( \mathcal{T}_H \) for the space domain \( \Omega \) with mesh size \( H \) which is much larger than \( h \) and a coarse partition for the time domain with time step \( \Delta t \) which is much bigger than \( \delta t \). The finite element space corresponding to the partition \( \mathcal{T}_H \) is denoted as \( V_H \). In our following discussion, coarse mesh means coarse spacial mesh size \( H \) together with coarse time step \( \Delta t \), and fine mesh means fine spacial mesh size \( h \) together with fine time step \( \delta t \). For simplicity, we require that there exist some integers \( M_1 \gg 1 \) and \( M_2 \gg 1 \), such that \( \Delta t = M_1 \delta t \) and \( H = M_2 h \).

We first discretize the partial differential equation (2) by the same time discretized scheme as that used for obtaining (3), that is, the implicit Euler scheme, with coarse time step \( \Delta t \), to obtained the following equation, which is similar as (3) but with a coarse time step \( \Delta t \),

\[
\frac{u^k(x,y,z) - u^{k-1}(x,y,z)}{\Delta t} + a(t_k; u^k(x,y,z), v) = (f(x,y,z,t_k), v), \forall v \in V. \tag{17}
\]

Then, we discretize the above equation in \( V_H \) and obtain the finite element approximation \( u^k_H \). We then discretize (17) by the adaptive POD method to obtain its adaptive POD approximation \( u^k_H \) with the error indicator \( \eta_k \) being defined as

\[
\eta_k = \frac{\|u^k_H - u^k_{H,\text{POD}}\|_2}{\|u^k_H\|_2}. \tag{18}
\]

Now, we introduce each step of the loop (13) one by one for our TG-APOD algorithm.

1. **Solve.** Discretize the problem (3) in the subspace spanned by the POD modes \( \{\psi_{h,1}, \cdots, \psi_{h,m}\} \).
2. **Estimate.** For \( t = k\Delta t \), we calculate the error indicator defined in (18), from which we decide if we need to update the POD modes.
3. **Mark.** Giving a threshold \( \eta_0 \), we set

\[
\text{flag}_k = \begin{cases} 
1, & \text{if } \eta_k > \eta_0, \\
0, & \text{otherwise}. 
\end{cases} \tag{19}
\]

4. **Update.** If \( \text{flag}_k = 1 \), then we move to the \( k-1 \)-th time layer, and discretize the equation (3) in the fine finite element space \( V_h \) on time interval \([t_{k-1}, t_{k-1} + \delta T] \), to get \( u^{k+1}_h, i = 1, \ldots, \frac{\delta T}{\delta t} \), from which to get snapshots \( W_{h,1} \), then update POD models \( \Psi_h \) by \( \text{Update}_{\text{POD Mode}}(W_{h,1}, \gamma_2, \gamma_3, \Phi_h, m, \Psi_h) \).
From the discussion above, we can see that the steps **Estimate** and **Mark** are only dependent on the coarse mesh. Therefore, in practice, we can first gather all the marked time layers on the coarse mesh, then share the marked time layers with the fine mesh, from which we can easily know when we should **Update** the POD modes on the fine mesh.

We summarize the discussion above and obtain our two-grid based adaptive POD method, and state it as Algorithm 5.

### Algorithm 5 Two-grid based adaptive POD method

1: Give coarse mesh $T_H$ with coarse time step $\Delta t$ and fine mesh $T_h$ with fine time step $\delta t$,
2: Discretize (3) in $V_H$ on interval $[0, T_0]$, and take the snapshots $U_H$.
3: Construct POD modes $\Psi_H$ by POD Mode($U_H, \gamma_1, \Phi_H, m, \Psi_H$).
4: $t = T_0$
5: while $t \leq T$
6: $t = t + \Delta t$
7: Discretize (3) in $V_H$ to get $u_H^k$, and discretize (3) in the space span{$\psi_{H,1}, \ldots, \psi_{H,m}$} to get $u_H^{k,POD}$,
8: if $\eta(t) > \eta_0$ then
9: $t = t - \Delta t$
10: $S = S \cup \{t\}$
11: Discretize (3) in $V_H$ on interval $[t, t + \delta T]$, and take snapshots $W_H, 1$, then update POD modes $\Psi_H$ by Update POD Mode($W_H, 1, \gamma_2, \gamma_3, \Phi_H, m, \Psi_H$).
12: end if
13: end while
14: Discretize (3) in $V_h$ on interval $[0, T_0]$, and get the snapshots $U_h$.
15: Construct POD modes $\Psi_h$ by POD Mode($U_h, \gamma_1, \Phi_h, m, \Psi_h$).
16: $t = T_0$
17: while $t \leq T$ do
18: $t = t + \delta t$
19: Discretize (3) in the space span{$\psi_{h,1}, \ldots, \psi_{h,m}$}, and obtain $u_h^{k,POD}$,
20: if $t \in S$ then
21: Discretize (3) in $V_h$ on interval $[t, t + \delta T]$, and take snapshots $W_h, 1$, then update POD mode $\Psi_h$ by Update POD Mode($W_h, 1, \gamma_2, \gamma_3, \Phi_h, m, \Psi_h$).
22: end if
23: end while

**Remark 1** For our two-grid based adaptive POD method, the steps **Estimate** and **Mark** are all carried out in the coarse mesh. Since $\Delta t \gg \delta t$, the number of calculation for the error indicator is much smaller than that carried in the fine time interval. Besides, $H \gg h$ means $\# T_H \ll \# T_h$, which implies that the cost for calculating the error indicator $\eta_k$ is cheap. These two facts make our two-grid based adaptive POD method much cheaper than the existing adaptive POD methods.

### 4 Numerical examples

In this section, we apply our new method to two types of fluid equations, kolmogorov flow and ABC flow, which will show the efficiency of our two-grid based
adaptive POD algorithm. For these two types of equations, we compare our new algorithm with the POD algorithm and the residual based adaptive POD algorithm. In our test, we use the standard finite element approximation corresponding to the fine mesh as the reference solution, and the relative error of approximation obtained by the POD algorithm, or the residual based adaptive POD algorithm, or our new two-grid based adaptive POD algorithm is calculated as

$$\text{Error} = \frac{\| u^k_h - u^k_{h,*} \|_2}{\| u^k_h \|_2}, \quad (20)$$

where $u^k_h$ and $u^k_{h,*}$ represent finite element approximations and different kinds of POD approximations at the $k$-th time layer.

Our numerical experiments are carried out on LSSC-IV in the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, and our code is written based on the 3D adaptive finite element software platform PHG [27].

4.1 Kolmogorov flow

First, we consider the following advection-dominated 3D Kolmogorov flow equation with the cosines term [13, 17, 25].

\[
\begin{aligned}
&u_t - \epsilon \Delta u + B(x, y, z, t) \cdot \nabla u = f(x, y, z, t), \quad (x, y, z) \in \Omega, t \in [0, T], \\
&u(x, y, z, 0) = 0, \\
&u(x + 2\pi, y, z, t) = u(x, y + 2\pi, z, t) = u(x, y, z + 2\pi, t) = u(x, y, z, t), \quad (21)
\end{aligned}
\]

where

\[
\begin{aligned}
B(x, y, z, t) &= (\cos(y), \cos(z), \cos(x)) + (\sin(z), \sin(x), \sin(y)) \cos(t), \\
f(x, y, z, t) &= -\cos(y) - \sin(z) \ast \cos(t), \\
\Omega &= [0, 2\pi]^3, \quad T = 100.
\end{aligned}
\]

For this example, we have tested 4 different cases with $\epsilon = 0.5, 0.1, 0.05,$ and 0.01, respectively. We divide the $[0, 2\pi]^3$ into tetrahedrons to get the initial mesh containing 6 elements, then refine the initial mesh 22 times uniformly using bisection to get our computational mesh. We set $\delta t = 0.005$, and the type of the finite element basis is piecewise linear function ($P_1$). For the three cases with $\epsilon = 0.5, 0.1,$ and 0.05, the parameters are chosen as $T_0 = 1.5, \delta T = 1, \delta M = 5$; for case of $\epsilon = 0.01$, the parameters are chosen as $T_0 = 5, \delta T = 3, \delta M = 20$. In the two-grid based adaptive POD algorithm, we refine the initial mesh 16 times uniformly using bisection to obtain the coarsen mesh, and the time step of the coarse mesh is set to 0.09. We use 36 processors for the simulation. The detailed information of numerical results are listed in Table 1.

In Table 1, ‘DOFs’ means the degree of freedom, ‘Time’ is the wall time for the simulation, ‘Average Error’ is computed by averaging the errors of numerical solution for each time layer.

From Table 1, we can see that the number of those POD modes for each POD type method is much smaller than standard-FEM, which means that those POD
Table 1: The results of Kolmogorov flow with different $\epsilon$ obtained by Standard-FEM, POD, Residual-APOD, and TG-APOD, respectively.

| $\epsilon$ | Methods    | $\eta_0$ | DOFs | Average Error | Time(s) |
|------------|------------|----------|------|---------------|---------|
| 0.5        | Standard-FEM | –        | 4194304 | 0.193343 | 13786.49 |
|            | POD        | –        | 13    | 0.001019 | 547.75  |
|            | Residual-APOD | 4.0 $\times$ 10$^{-6}$ | 47 | 0.000280 | 6761.29 |
|            | TG-APOD    | 0.002    | 58    |           | 1843.86 |
| 0.1        | Standard-FEM | –        | 4194304 | 0.559466 | 11950.82 |
|            | POD        | –        | 22    | 0.005903 | 535.47  |
|            | Residual-APOD | 1.0 $\times$ 10$^{-5}$ | 107 | 0.003628 | 8222.21 |
|            | TG-APOD    | 0.005    | 121   |           | 3771.56 |
| 0.05       | Standard-FEM | –        | 4194304 | 0.805803 | 11379.64 |
|            | POD        | –        | 16    | 0.019034 | 546.51  |
|            | Residual-APOD | 3.0 $\times$ 10$^{-5}$ | 140 | 0.012306 | 9503.81 |
|            | TG-APOD    | 0.01     | 179   |           | 6204.12 |
| 0.01       | Standard-FEM | –        | 4194304 | 0.104584 | 12955.44 |
|            | POD        | –        | 43    | 0.100980 | 1265.95 |
|            | Residual-APOD | 2.0 $\times$ 10$^{-4}$ | 133 | 0.104584 | 1265.95 |
|            | TG-APOD    | 0.01     | 193   |           | 7197.82 |

algorithms have good performance in dimensional reduction. For this example, the error of the numerical solution obtained by the POD algorithm is too large. However, both the residual based adaptive POD method and our two-grid based adaptive POD method can obtain numerical solution with higher accuracy, which validates the effectiveness of the adaptive POD methods. We then compare our two-grid adaptive POD method with the residual based adaptive POD method. We can see that our method not only obtains numerical solutions with higher accuracy but also takes less cpu time than the residual-based adaptive POD method. This shows that our two-grid adaptive POD method is more effective.

To see more clearly, we compare the error of the numerical solutions obtained by the different methods in Fig. 2.

In Fig. 2, the x-axis is time, the y-axis is relative error of numerical solution. The results obtained by POD algorithm, residual based adaptive POD algorithm, and our two-grid based adaptive POD algorithm are reported in line with color darkslategray, blue, and red, respectively.

From the left figure of Fig. 2, we can see that the error curve obtained by the POD algorithm is above both those obtained by the two adaptive POD algorithms, which means that the adaptive POD algorithms are more efficient than the POD algorithm. From the right figure of Fig.2, we can see that our two-grid based adaptive POD algorithm is more efficient than the residual based adaptive POD algorithm.

4.2 ABC flow

We then consider the ABC flow. ABC flow was introduced by Arnold, Beltrami, and Childress [13] to study chaotic advection, enhanced transport and dynamo
Fig. 2: The change of error for solution of (21) with different $\epsilon$ obtained by POD algorithm, Residual-APOD algorithm, and TG-APOD algorithm, respectively.
effect, see [6, 16, 37, 38] for the details.

\[
\begin{aligned}
&\left\{ \\
&\quad u_t - \epsilon \Delta u + B(x, y, z, t) \cdot \nabla u = f(x, y, z, t), \quad (x, y, z) \in \Omega, t \in [0, T], \\
&\quad u(x, y, z, 0) = 0, \\
&\quad u(x + 2\pi, y, z, t) = u(x, y, z, t) = u(x, y, z + 2\pi, t) = u(x, y, z, t),
\end{aligned}
\]

where

\[
B(x, y, z, t) = \sin(z + \sin wt) + \cos(y + \sin wt), \\
\sin(x + \sin wt) + \cos(z + \sin wt), \\
\sin(y + \sin wt) + \cos(x + \sin wt)), \\
f(x, y, z, t) = -\sin(z + \sin wt) - \cos(y + \sin wt), \\
\Omega = [0, 2\pi]^3, T = 100.
\]

For this example, we also test 4 different cases with \( \epsilon = 0.5, 0.1, 0.05, \) and \( 0.01, \) respectively. We divide the domain \([0, 2\pi]^3\) into tetrahedrons to get the initial grid containing 6 elements, then refine the initial mesh 22 times uniformly using bisection to get our computational mesh. We set \( w = 1.0, \Delta t = 0.005, \) and choose the finite element basis to be piecewise linear function (\( P_1 \)). For the two cases with \( \epsilon = 0.5, 0.1, \) the parameters are chosen as \( T_0 = 1.5, \delta T = 1, \delta M = 5; \) for other cases with \( \epsilon = 0.05, 0.01, \) the parameters are chosen as \( T_0 = 5, \delta T = 4, \delta M = 20. \)

In the two-grid based adaptive POD algorithm, we refine the initial mesh 16 times uniformly using bisection to obtain the coarsen mesh, and the time step of the coarse mesh is set to 0.09. We use 36 processors for the simulation. We list the detailed information in Table 2. The notation in Table 2 has the same meaning as in Table 1.

Table 2: The results of ABC flow with different \( \epsilon \) obtained by Standard-FEM, POD, Residual-APOD, and TG-APOD algorithm, respectively.

| \( \epsilon \) | Methods | \( n_0 \) | DOFs | Average Error | Times (s) |
|---|---|---|---|---|---|
| 0.5 | FEM | – | 4194304 | – | 16690.55 |
| | POD | – | 12 | 0.303585 | 534.96 |
| | Res-APOD | 1.0 \times 10^{-5} | 52 | 0.010679 | 9656.49 |
| | TG-APOD | 0.02 | 53 | 0.007453 | 1875.04 |
| 0.1 | FEM | – | 4194304 | – | 15238.87 |
| | POD | – | 14 | 0.658382 | 536.73 |
| | Res-APOD | 1.0 \times 10^{-5} | 148 | 0.010298 | 13924.65 |
| | TG-APOD | 0.005 | 181 | 0.005460 | 8043.11 |
| 0.05 | FEM | – | 4194304 | – | 14457.24 |
| | POD | – | 42 | 0.504838 | 1485.62 |
| | Res-APOD | 1.0 \times 10^{-5} | 172 | 0.009505 | 14671.97 |
| | TG-APOD | 0.005 | 217 | 0.005441 | 8736.96 |
| 0.01 | FEM | – | 4194304 | – | 16074.96 |
| | POD | – | 61 | 0.676701 | 1878.01 |
| | Res-APOD | 1.0 \times 10^{-4} | 164 | 0.078336 | 14882.99 |
| | TG-APOD | 0.01 | 244 | 0.070448 | 11401.42 |

Similar to the first example, we can see from Table 2 that those POD type methods can indeed reduce the number of basis a lot. For this example, the accuracy for the approximation obtained by the POD method is so large that they
are meaningless. While both the residual based adaptive POD method and our two-grid based adaptive POD method can obtain numerical solution with high accuracy. Among the two adaptive POD methods, our two-grid adaptive POD method can get higher accuracy numerical solution than the residual based adaptive POD method. When it comes to time, our two-grid adaptive method takes less cpu time than the residual based adaptive POD method. These shows that our two-grid adaptive POD method is more effective.

Similarly, we show the error of the numerical solutions obtained by the different methods in Fig. 3.

In Fig. 3, the x-axis is time, the y-axis is the relative error of POD approximations. The results obtained by POD algorithm, residual based adaptive POD algorithm, and our two-grid based adaptive POD algorithm are reported in line with color darkslategray, blue, and red, respectively.

From Fig. 3, we can see more clearly that both the two-grid adaptive POD method and the residual based adaptive POD method behave much better than the POD method, and our two-grid based adaptive POD method behaves a little better than the residual based adaptive POD method.

We have some more tests for setting different parameters $w$. The results for cases with $w = 1, 1.5, 2, 2.5$ are shown in Table 3.

From Table 3, we can obtain the same conclusion as those from Table 1 and Table 2, that is, adaptive POD methods outperform the POD method a lot, and our two-grid based adaptive method outperforms the residual based adaptive method.

5 Concluding remarks

In this paper, we proposed a two-grid based adaptive POD method to solve the time dependent partial differential equations. We apply our method to some typical 3D advection-diffusion equations, with Kolmogorov flow and ABC flow. Numerical results show that our two-grid based adaptive POD algorithm is more effective than the residual based adaptive POD algorithm, especially than the original POD algorithm. Here, we simply use the relative error of the POD solution on the coarse spacial and temporal meshes to construct the error indicator. In our future work, we plan to construct some other error indicator based on our two-grid approach or some other approach, and study nonlinear or other types of time dependent partial differential equations.

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Fig. 3: The change of error for solution of (22) with different $\epsilon$ obtained by POD algorithm, Residual-APOD algorithm, and TG-APOD algorithm, respectively.
Table 3: Compare the result of (22) with different $w$ and $\epsilon$ obtained by standard-FEM, POD, Residual-APOD and TG-APOD algorithm, respectively

| $\epsilon$ | $w$ | Methods          | $\eta_0$ | DOFs | Average Error | Time (s)  |
|-----------|-----|------------------|---------|------|---------------|-----------|
| 0.5       | 1.5 | FEM              | –       | 4194304 | 0.294394     | 16745.85  |
|           |     | POD              | –       | 14    | 0.005888     | 9627.10   |
|           |     | Res-APOD 1.0 x 10^{-5} | 57 | 0.002513 | 2361.77   |
|           |     | TG-APOD 0.01    | 59      | 0.270138 | 730.91    |
| 0.5       | 2.0 | FEM              | –       | 4194304 | 0.270138     | 17258.89  |
|           |     | POD              | –       | 15    | 0.017753     | 561.23    |
|           |     | Res-APOD 1.0 x 10^{-5} | 44 | 0.006577 | 1931.23   |
|           |     | TG-APOD 0.02    | 59      | 0.005998 | 8890.87   |
| 0.5       | 2.5 | FEM              | –       | 4194304 | 0.216109     | 17161.84  |
|           |     | POD              | –       | 17    | 0.014760     | 9627.10   |
|           |     | Res-APOD 1.0 x 10^{-5} | 48 | 0.007357 | 1550.19   |
|           |     | TG-APOD 0.02    | 49      | 0.006577 | 1931.23   |
| 0.1       | 1.5 | FEM              | –       | 4194304 | 0.729666     | 15322.78  |
|           |     | POD              | –       | 15    | 0.010995     | 531.42    |
|           |     | Res-APOD 8.0 x 10^{-6} | 154 | 0.008529 | 6298.07   |
|           |     | TG-APOD 0.01    | 159     | 0.009955 | 14431.22  |
| 0.1       | 2.0 | FEM              | –       | 4194304 | 0.751065     | 15324.38  |
|           |     | POD              | –       | 16    | 0.011761     | 536.24    |
|           |     | Res-APOD 8.0 x 10^{-6} | 127 | 0.006577 | 730.91    |
|           |     | TG-APOD 0.01    | 143     | 0.006577 | 1931.23   |
| 0.1       | 2.5 | FEM              | –       | 4194304 | 0.657750     | 15104.35  |
|           |     | POD              | –       | 18    | 0.022907     | 589.66    |
|           |     | Res-APOD 8.0 x 10^{-6} | 113 | 0.021733 | 3574.45   |
|           |     | TG-APOD 0.02    | 114     | 0.021733 | 3574.45   |
| 0.05      | 1.5 | FEM              | –       | 4194304 | 0.445832     | 14141.53  |
|           |     | POD              | –       | 45    | 0.037319     | 12322.02  |
|           |     | Res-APOD 2.0 x 10^{-5} | 125 | 0.018763 | 6094.93   |
|           |     | TG-APOD 0.008   | 165     | 0.018763 | 12322.02  |
| 0.05      | 2.0 | FEM              | –       | 4194304 | 0.325020     | 14332.82  |
|           |     | POD              | –       | 47    | 0.018739     | 12347.84  |
|           |     | Res-APOD 1.0 x 10^{-5} | 127 | 0.006625 | 7628.33   |
|           |     | TG-APOD 0.008   | 224     | 0.006625 | 7628.33   |
| 0.05      | 2.5 | FEM              | –       | 4194304 | 0.245578     | 14355.11  |
|           |     | POD              | –       | 49    | 0.017663     | 1528.92   |
|           |     | Res-APOD 1.6 x 10^{-5} | 129 | 0.007226 | 6301.64   |
|           |     | TG-APOD 0.006   | 169     | 0.007226 | 6301.64   |
| 0.01      | 1.5 | FEM              | –       | 4194304 | 0.670596     | 16223.32  |
|           |     | POD              | –       | 46    | 0.141915     | 12703.29  |
|           |     | Res-APOD 1.8 x 10^{-4} | 126 | 0.089605 | 11494.74  |
|           |     | TG-APOD 0.008   | 246     | 0.089605 | 11494.74  |
| 0.01      | 2.0 | FEM              | –       | 4194304 | 0.557215     | 16376.39  |
|           |     | POD              | –       | 48    | 0.091943     | 12917.72  |
|           |     | Res-APOD 1.4 x 10^{-4} | 244 | 0.050542 | 8789.69   |
|           |     | TG-APOD 0.008   | 208     | 0.050542 | 8789.69   |
| 0.01      | 2.5 | FEM              | –       | 4194303 | 0.582593     | 16226.39  |
|           |     | POD              | –       | 49    | 0.072278     | 12573.98  |
|           |     | Res-APOD 6 x 10^{-5} | 129 | 0.041294 | 10908.90  |
|           |     | TG-APOD 0.01    | 209     | 0.041294 | 10908.90  |
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