Conforming virtual element approximations of the two-dimensional Stokes problem

Gianmarco Manzini\textsuperscript{a} and Annamaria Mazzia\textsuperscript{b}

\textsuperscript{a} Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, via Ferrata 1, 27100 Pavia, Italy
\textsuperscript{b}Dipartimento di Ingegneria Civile, Edile e Ambientale - ICEA, Università di Padova, 35131 Padova, Italy

Abstract

The virtual element method (VEM) is a Galerkin approximation method that extends the finite element method to polytopal meshes. In this paper, we present two different conforming virtual element formulations for the numerical approximation of the Stokes problem that work on polygonal meshes. The velocity vector field is approximated in the virtual element spaces of the two formulations, while the pressure variable is approximated through discontinuous polynomials. Both formulations are inf-sup stable and convergent with optimal convergence rates in the $L^2$ and energy norm. We assess the effectiveness of these numerical approximations by investigating their behavior on a representative benchmark problem. The observed convergence rates are in accordance with the theoretical expectations and a weak form of the zero-divergence constraint is satisfied at the machine precision level.

Key words: Incompressible two-dimensional Stokes equation, virtual element method, enhanced formulation, error analysis.
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1 Introduction

Many physical phenomena in physics and engineering can be modeled by the Stokes flow [55]. Noteworthy applications are, for example, Stokes flows in porous media [6], design and development of efficient fibrous filters [63] and micro-fluid devices [75], dynamics of droplets [61], bio-suspensions and sedimentation [59]. A very successful approach for the numerical treatment of the Stokes equations in variational form is based on the finite element method (FEM) [37, 48, 58]. The FEM normally uses triangular and quadrilateral meshes in the two-dimensional (2D) case and tetrahedral and hexahedral meshes in the three-dimensional case (3-D). Furthermore, in the last two decades a great effort has been devoted in the design of numerical methods for partial differential equations (PDEs) suitable to polygonal and polyhedral meshes [16, 62, 78, 79]. To this end, it is worth mentioning the mimetic finite difference (MFD) method [16, 65] and its variational reformulation that led to the virtual element method (VEM) [7]. The MFD was designed to preserve several fundamental properties of PDEs, such as the maximum/minimum principle, the conservation of fundamental quantities in physics (mass, momentum, energy) and the solution symmetries. The MFD method was successfully applied to the numerical approximation on unstructured polygonal and polyhedral meshes of diffusion problems [34, 35], convection–diffusion problems [42], elasticity problems [66], gas dynamic problems [38], and electromagnetic problems [60]. On the other hand, the VEM is a finite element method that does not require the explicit knowledge of the basis functions and use of quadrature formulas to compute the bilinear forms of the Galerkin formulation. Indeed, the VEM can handle the construction of the bilinear forms on general polygonal
and polyhedral elements through special polynomial projections of the basis functions and their derivatives (gradients, curl, divergence). Such projections are computable from the degrees of freedom of the virtual element functions and ensure the polynomial consistency of the bilinear forms. The connection between the VEM and the FEM on polygonal/polyhedral meshes is thoroughly investigated in [43, 53, 68], between VEM and discontinuous skeletal gradient discretizations in [53], and between the VEM and the BEM-based FEM method in [41].

The VEM was originally formulated in [7] as a conforming FEM for the Poisson problem. Then, it was later extended to convection-reaction-diffusion problems with variable coefficients in [2, 10]. Meanwhile, the nonconforming formulation for diffusion problems was proposed in [5] as the finite element reformulation of [64]. Mixed VEM for elliptic problems were introduced in [33], and later extended to meshes with curved edges in [49]. Implementation of mixed methods is discussed in [50–52].

The connection with de Rham diagrams and Nedelec elements and the application to the electromagnetics has been explored in [9]. A practical application of these concepts can be found in [11, 71]. Other significant applications of the VEM on general meshes are found, for example, in [3, 4, 19–22, 24–30, 36, 39, 40, 44–46, 56, 70, 72–74, 80, 81].

In this work, we consider two possible numerical formulations of the VEM for the discretization of the two-dimensional (2D) Stokes equation. In both formulation, we approximate the two components of the velocity vector separately by using a variant of the conforming virtual element space originally proposed in [7] and already considered in [67]. In the first formulation we assume that the edge trace of each component of the velocity is a polynomial of degree $k + 1$, where $k$ is the maximum degree of the polynomials that are in the virtual element space. This definition of the scalar virtual element space is a special case of the generalized local virtual element space that is proposed in [23, Section 3]. In the second formulation, we assume that only the trace of the normal component of the velocity vector is a polynomial of degree $k + 1$, while the trace of the tangential component is a polynomial of degree $k$. For both formulations, we also consider the modified (“enhanced”) definition of the virtual element space [2], which allows us to construct the $L^2$ orthogonal projection onto the polynomials of degree $k$. In both formulations, the scalar unknown, e.g., the pressure, is approximated by discontinuous polynomials on the mesh elements. These two virtual element formulations satisfy the inf-sup stability condition, which is crucial to prove the well-posedness of the method, and can be proved to have an optimal convergence rate for the approximation errors in the $L^2$ norm and in the $H^1$-seminorm. A similar approach for the incompressible Stokes equations led to the low-order accurate MFD methods in [13, 14], that are equivalent to the formulations proposed in our work for $k = 1$.

All our numerical experiments confirm the expected optimal behavior of these two formulations, whose accuracy is comparable, although the second formulation requires less degrees of freedom than the first one. The zero divergence constraint is satisfied in a variational sense, i.e., the projection of the divergence on the subset of polynomials used in the scheme formulation is zero. It is worth mentioning that other virtual element approaches were recently proposed in the literature that approximate the Stokes velocity in such a way that its divergence is a polynomial that is set to zero in the scheme. This strategy provides an approximation of the Stokes velocity that satisfies the zero divergence constraint in a pointwise sense. We refer the interested reader to the works of References [12, 17, 18, 22, 47]. However, the polynomial projection of the velocity divergence in our VEM is zero up to the machine precision, so if we consider such projection as the virtual element approximation of the velocity divergence, this approximation is identically zero almost everywhere in the computational domain.

1.1 Structure of the paper

The outline of the paper is as follows. In Section 2, we introduce the Stokes problem. In Section 3, we discuss two different virtual element formulations for numerically solving this problem. In Section 4, we investigate the convergence of these formulations theoretically, and derive optimal convergence rates in the energy and $L^2$ norms for the velocity approximation and in the $L^2$ norm for the pressure approximation. In Section 5, we assess the accuracy of these virtual element approximations by investigating their behavior on a representative benchmark problem. In Section 6, we offer our final conclusions.

1.2 Notation and technicalities

We use the standard definition and notation of Sobolev spaces, norms and seminorms, cf. [1]. Let $k$ be a nonnegative integer number. The Sobolev space $H^k(\omega)$ consists of all square integrable functions with all square integrable weak
derivatives up to order $k$ that are defined on the open, bounded, connected subset $\omega$ of $\mathbb{R}^2$. As usual, if $k = 0$, we prefer the notation $L^2(\omega)$. We will also use the subspace of $L^2(\Omega)$ denoted by $L^2_0(\Omega)$ and defined on the computational domain $\Omega$ as
\[
L^2_0(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \right\}.
\]
Norm and seminorm in $H^k(\omega)$ are denoted by $\| \cdot \|_{k,\omega}$ and $| \cdot |_{k,\omega}$, respectively. We use the integral notation to denote the $L^2$-inner product between vector-valued fields, although for notation’s conciseness, we may prefer to use the notation \((\cdot, \cdot)\) in a few situations.

### 1.3 Mesh definition and regularity assumptions

For exposition’s sake, we consider an open, bounded, polygonal domain $\Omega$ and a family of mesh decompositions of $\Omega$ denoted by $\mathcal{T} = \{ \Omega_h \}_h$. Each mesh $\Omega_h$ is a set of non-overlapping, bounded (closed) elements $E$ such that $\Omega = \cup_{E \in \Omega_h} E$, where $\Omega$ is the closure of $\Omega$ in $\mathbb{R}^2$. The subindex $h$, which labels each mesh $\Omega_h$, is the maximum of the diameters $h_E = \sup_{x,y \in E} |x - y|$. Each element $E$ has a non-intersecting polygonal boundary $\partial E$ formed by $N^E_v$ straight edges $e$ connecting the $N^E_v$ polygonal vertices. The sequence of vertices forming $\partial E$ is oriented in the counter-clockwise direction and the vertex coordinates are denoted by $x_v = (x_v, y_v)$. We denote the measure of $E$ by $|E|$, its barycenter (center of gravity) by $x_E := (x_E, y_E)$, the unit normal vector to each edge $e \in \partial E$ and pointing out of $E$ by $n_{E,e}$, and the length of $e$ by $h_e$. Moreover, we assume that the orientation of the mesh edges in every mesh is fixed \emph{once and for all}, so that we can unambiguously introduce $n_e$, the unit normal vector to edge $e$. The orientation of this vector is independent of the element $E$ to which $e$ belongs, and may differ from $n_{E,e}$ only by the multiplicative factor $-1$.

**Mesh regularity assumptions.** In the definition of the admissible meshes, we first assume that the elemental boundaries are “polylines”, i.e., continuously connected portions of straight lines. Then, we need the following regularity assumptions on the family of mesh decompositions $\{ \Omega_h \}_h$ in order to use the interpolation and projection error estimates from the theory of polynomial approximation of functions in Sobolev spaces [32].

**Assumption 1.1 (Mesh regularity)**

- There exists a positive constant $\varrho$ independent of $h$ such that for every polygonal element $E$ it holds that
  
  (M1) $E$ is star-shaped with respect to a disk with radius $\geq \varrho h_E$;
  (M2) for every edge $e \in \partial E$ it holds that $h_e \geq \varrho h_E$.

**Remark 1.2** The star-shapedness property (M1) implies that all the mesh elements are simply connected subsets of $\mathbb{R}^2$. The scaling property (M2) implies that the number of edges in all the elemental boundaries is uniformly bounded from above over the whole mesh family $\{ \Omega_h \}_h$.

These mesh assumptions are quite general and, as observed from the very first publication on the VEM, see, for example, [7], allow the method a great flexibility in the geometric shape of the mesh elements. For example, we can consider elements with hanging nodes as in the adaptive mesh refinement (AMR) technique and elements with a non-convex shape. In this work we avoid elements with intersecting boundaries, elements with “holes”, and elements totally surrounding other elements. However, elements with such more challenging shapes have already been considered in the virtual element formulation to show the robustness of the method [73]. A recent review of the mesh regularity assumptions in the VEM literature and a thorough investigation of the VEM performance on mesh families with extreme characteristics can also be found in [76, 77].

### 1.4 Polynomials

Hereafter, $\mathbb{P}_\ell(E)$ denotes the linear space of polynomials of degree up to $\ell$ defined on $E$, with the useful convention that $\mathbb{P}_{-1}(E) = \{ 0 \}$; $[\mathbb{P}_\ell(E)]^2$ denotes the space of two-dimensional vector-valued fields of polynomials of degree up to $\ell$ on $E$; $[\mathbb{P}_\ell(E)]^{2 \times 2}$ denotes the space of 2 × 2-sized tensor-valued fields of polynomials of degree up to $\ell$ on $E$. Similar definitions also hold for the space of univariate polynomials defined on all mesh edges $e$. Then, we define the
linear space of discontinuous scalar, vector and tensor polynomial fields by collecting together the local definitions, so that
\[ \mathbb{P}_\ell(\Omega_h) := \left\{ q \in L^2(\Omega) : q|_E \in \mathbb{P}_\ell(E) \forall E \in \Omega_h \right\}, \]
\[ [\mathbb{P}_\ell(\Omega_h)]^2 := \left\{ q \in [L^2(\Omega)]^2 : q|_E \in [\mathbb{P}_\ell(E)]^2 \forall E \in \Omega_h \right\}, \]
\[ [\mathbb{P}_\ell(\Omega_h)]^{2 \times 2} := \left\{ \kappa \in [L^2(\Omega)]^{2 \times 2} : \kappa|_E \in [\mathbb{P}_\ell(E)]^{2 \times 2} \forall E \in \Omega_h \right\}. \]
We will also use the norm and seminorm:
\[ ||v||_{1,h}^2 = ||v||_{0,\Omega}^2 + ||v||_{1,h}^2 \quad \text{with} \quad |v|_{1,h}^2 = \sum_{E \in \Omega_h} |v|_{1,E}^2 \quad (2) \]
for every function \( v \) defined in the broken Sobolev space
\[ [H^1(\Omega_h)]^2 = \left\{ v \in [L^2(\Omega)]^2 : v|_E \in [H^1(\Omega)]^2 \forall E \in \Omega_h \right\}, \]
which is the space of square integrable vector-valued functions whose restriction to every mesh element \( E \) is in \([H^1(\Omega)]^2\). Space \( \mathbb{P}_\ell(\Omega) \) is the span of the finite set of \textit{scaled monomials of degree up to} \( \ell \), that are given by
\[ \mathcal{M}_\ell(E) = \left\{ \left( \frac{x - x_E}{h_E} \right)^{\alpha} \text{ with } |\alpha| \leq \ell \right\}, \]
where
- \( x_E \) denotes the center of gravity of \( E \) and \( h_E \) its characteristic length, as, for instance, the edge length or the cell diameter;
- \( \alpha = (\alpha_1, \alpha_2) \) is the two-dimensional multi-index of nonnegative integers \( \alpha_i \) with degree \( |\alpha| = \alpha_1 + \alpha_2 \leq \ell \) and such that \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \) for any \( x \in \mathbb{R}^2 \) and \( \partial^{\alpha}|/\partial x^\alpha = \partial^{\alpha}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \).

The dimension of \( \mathbb{P}_\ell(\Omega) \) equals \( N_\ell = (\ell + 1)(\ell + 2)/2 \), the cardinality of the basis set \( \mathcal{M}_\ell(E) \).

Let \( v \) and \( \mathbf{v} = (v_x, v_y)^T \) denote a (smooth enough) scalar and vector-valued field. Then,
- the elliptic projection \( \Pi_\ell^\nabla \mathbf{v} \in \mathbb{P}_\ell(\Omega) \) is the solution of the variational problem
\[ \int_E \nabla (v - \Pi_\ell^\nabla \mathbf{v}) \cdot \nabla q \, dx = 0 \quad \forall q \in \mathbb{P}_\ell(\Omega), \quad (3) \]
\[ \int_{\partial E} (v - \Pi_\ell^\nabla \mathbf{v}) \cdot ds = 0; \quad (4) \]
- the orthogonal projection \( \Pi_\ell^0 \mathbf{v} \in \mathbb{P}_\ell(\Omega) \) is the solution of the variational problem
\[ \int_E (v - \Pi_\ell^0 \mathbf{v}) q \, dx = 0 \quad \forall q \in \mathbb{P}_\ell(\Omega); \quad (5) \]
- the orthogonal projection of a vector-valued field \( \mathbf{v} = (v_x, v_y)^T \) is the solution of the variational problem
\[ \int_E (\mathbf{v} - \Pi_\ell^0 \mathbf{v}) \cdot \mathbf{q} \, dx = 0 \quad \forall \mathbf{q} \in [\mathbb{P}_\ell(\Omega)]^2, \quad (6) \]
and can be computed componentwisely, i.e., \( \Pi_\ell^0 \mathbf{v} = (\Pi_\ell^0 v_x, \Pi_\ell^0 v_y)^T \in [\mathbb{P}_\ell(\Omega)]^2 \), where \( \Pi_\ell^0 v_x \) and \( \Pi_\ell^0 v_y \) are the scalar orthogonal projections defined above;
- the gradient of vector \( \mathbf{v} \) and its orthogonal projection \( \Pi_\ell^0 \nabla \mathbf{v} \in [\mathbb{P}_\ell(\Omega)]^{2 \times 2} \) onto the linear space of \( 2 \times 2 \)-sized matrix-valued polynomials of degree \( \ell \), which are defined componentwisely as follows:
\[ \nabla \mathbf{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix} \quad \text{and} \quad \Pi_\ell^0 \nabla \mathbf{v} = \begin{pmatrix} \Pi_\ell^0 \frac{\partial v_x}{\partial x} & \Pi_\ell^0 \frac{\partial v_x}{\partial y} \\ \Pi_\ell^0 \frac{\partial v_y}{\partial x} & \Pi_\ell^0 \frac{\partial v_y}{\partial y} \end{pmatrix}, \quad (7) \]
and this latter one is the solution of the variational problem:
\[
\int_E (\nabla v - \Pi E \nabla v) : \kappa \, dx = 0 \quad \forall \kappa \in [P_1(E)]^{2 \times 2}.
\] (8)

2 The Stokes problem and the virtual element discretization

The incompressible Stokes problem for the vector-valued field \( u \) and the scalar field \( p \) is governed by the system of equations:
\[
\begin{align*}
-\Delta u + \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \Gamma,
\end{align*}
\] (9-11)
on the computational domain \( \Omega \) with boundary \( \Gamma \). We refer to \( u \) and \( p \) as the Stokes velocity and the Stokes pressure. To ease the exposition, we consider only the case of homogeneous Dirichlet boundary conditions, see (11). However, the extension to nonhomogeneous Dirichlet boundary conditions is deemed straightforward and the general case is considered in the section of numerical experiments.

The variational formulation of (9)-(11) reads as: Find \( (u, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \) such that
\[
a(u, v) + b(v, p) = (f, v) \quad \forall v \in [H^1_0(\Omega)]^2, \\
b(u, q) = 0 \quad \forall q \in L^2_0(\Omega),
\] (12-13)
where the bilinear forms \( a(\cdot, \cdot) : [H^1(\Omega)]^2 \times [H^1(\Omega)]^2 \to \mathbb{R} \) and \( b(\cdot, \cdot) : [H^1(\Omega)]^2 \times L^2(\Omega) \to \mathbb{R} \) are
\[
a(v, w) := \int_\Omega \nabla v : \nabla w \, dx \quad \forall v, w \in H^1(\Omega), \\
b(v, q) := -\int_\Omega q \text{div } v \, dx \quad \forall v \in H^1(\Omega), q \in L^2(\Omega).
\] (14-15)

In the following section, it will be convenient to split these bilinear forms on the mesh elements by rewriting them in the following way:
\[
a(v, w) = \sum_{E \in \mathcal{O}_h} a^E(v, w) \quad \text{with} \quad a^E(v, w) = \int_E \nabla v : \nabla w \, dx, \\
b(v, q) = \sum_{E \in \mathcal{O}_h} b^E(v, q) \quad \text{with} \quad b^E(v, q) = -\int_E q \text{div } v \, dx.
\] (16-17)

The bilinear form \( a(\cdot, \cdot) \) is continuous and coercive. The bilinear form \( b(\cdot, \cdot) \) is continuous and satisfies the inf-sup condition:
\[
\inf_{q \in L^2_0(\Omega) \setminus \{0\}} \sup_{v \in [H^1_0(\Omega)]^2} \frac{b(v, q)}{||v||_{1, \Omega} ||q||_{0, \Omega}} \geq \beta,
\] (18)
for some real, strictly positive constant \( \beta \). These properties imply the existence and uniqueness of the solution pair \((u, p)\), and, so, the well-posedness of the variational formulation (12)-(13), and the stability inequality
\[
||u||_{1, \Omega} + ||p||_{0, \Omega} \leq C ||f||_{-1, \Omega},
\]
for a right-hand side forcing term \( f \in H^{-1}(\Omega) \), and a constant \( C \) that depends only on \( \Omega \), cf. [31, 57, 58].

Let \( k \geq 1 \) be a given integer number. Our virtual element discretizations have the general abstract form: Find \( (u_h, p_h) \in \bar{V}_h^{k} \times Q_{k-1}^h \)
\[
a_h(u_h, v_h) + b_h(v_h, p_h) = (f_h, v_h) \quad \forall v_h \in \bar{V}_h^{k}, \\
b_h(u_h, q_h) = 0 \quad \forall q_h \in Q_{k-1}^h.
\] (19-20)

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Here, $V^h_k$ is a finite-dimensional conforming subspace of $[H^1_0(\Omega)]^2$ and $Q^h_{k-1}$ a finite-dimensional discontinuous subspace of $L^2_0(\Omega)$. We use the integer $k$, which is a polynomial degree, to denote the accuracy of the method. The vector field $u_h$ and the scalar field $p_h$ are the virtual element approximation of $u$ and $p$, respectively. The bilinear forms $a_h(\cdot, \cdot) : V^h_k \times V^h_k \to \mathbb{R}$ and $b_h(\cdot, \cdot) : V^h_k \times Q^h_{k-1} \to \mathbb{R}$ are the virtual element approximations to the corresponding bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$. The linear functional $(\ell_h, \cdot)$ is the virtual element approximation of the right-hand side of (12). The definition of all these mathematical objects is discussed in the next section, where we present, analyze and investigate numerically two new virtual element formulations that are suitable to polygonal meshes.

3 Virtual element approximations of the Stokes problem

We present two different virtual element approximations of the 2-D Stokes problem in variational form. For both formulations, the Stokes pressure is approximated by a piecewise polynomial function that belongs to the space $P_k(\Omega)$, i.e., the subspace $[P_k(\Omega)]^2 \subset V^h_k(\Omega)$. This orthogonal projection is required in the formulation of the right-hand side of Eq. (19).

In the rest of this section, we first review the general construction of the virtual element approximation. Then, for each formulation

- (i) we explicitly define the local virtual element space and its degrees of freedom and discuss their unisolvence;
- (ii) we prove that the following polynomial projections of $\nabla v_h$, $\text{div} v_h$ and $v_h$ are computable for every virtual element vector-valued field $v_h$ using only the degrees of freedom of $v_h$: $\Pi_{k-1}^{0,E} \nabla v_h \in [P_{k-1}(\Omega)]^2$; $\Pi_{k-1}^{0,E} \text{div} v_h \in P_{k-1}(\Omega)$; $\Pi_{k}^{1,E} v_h \in [P_k(\Omega)]^2$; $\Pi_{k}^{0,E} v_h \in [P_k(\Omega)]^2$ where $\bar{k} = \text{max}(0, k - 2)$ for the regular space definition or $\bar{k} = k$ for the enhanced space definition; (we recall that the formal definitions of these operators are given in (4)-(8)).

Construction of the virtual element bilinear form $a_h$. Using these projection operators, we define the virtual element bilinear form $a_h(\cdot, \cdot)$ as the sum of local bilinear forms $a^E_h(\cdot, \cdot) : V^h_k(\Omega) \times V^h_k(\Omega) \to \mathbb{R}$ as follows:

$$a_h(v_h, w_h) = \sum_{E \in \Omega_h} a^E_h(v_h, w_h)$$

where

$$a^E_h(v_h, w_h) = \int_{E} \Pi_{k-1}^{0,E} \nabla v_h : \Pi_{k-1}^{0,E} \nabla w_h \, dx + S^E_h((1 - \Pi_{k}^{0,E})v_h, (1 - \Pi_{k}^{0,E})w_h).$$

Here, $S^E_h(\cdot, \cdot) : V^h_k(\Omega) \times V^h_k(\Omega) \to \mathbb{R}$ is the local bilinear form providing the stabilization term, and $\Pi_{k}^{0,E}$ denote either the $L^2$-orthogonal projection $\Pi_{k}^{0,E}$ (when computable) or the elliptic projection $\Pi_{k}^{0,E}$. The term $S^E_h(\cdot, \cdot)$ can be any symmetric, positive definite bilinear form for which there exist two real, positive constant $\sigma_*$ and $\sigma^*$ independent of $h$ (and E) such that
\[
\sigma a^E(v_h, v_h) \leq S_h^E(v_h, v_h) \leq \sigma^* a^E(v_h, v_h) \quad \forall v_h \in V_h^E(\Omega) \cap \ker(\Pi_k^E),
\]
where \(a^E(\cdot, \cdot)\) is defined in (16). Several possible stabilizations have been proposed over the last few years and are available from the technical literature, cf. [69]. The local bilinear form \(a_h^E(\cdot, \cdot)\) has two fundamental properties that are used in the analysis:

- **Polynomial consistency**: for every vector field \(v_h \in V_h^E(\Omega)\) and vector polynomial field \(q_h \in [P_k(\Omega)]^2\) it holds:
  \[
a_h^E(v_h, q_h) = a^E(v_h, q_h);
\]
  (25)

- **Stability**: there exist two real, positive constants \(\alpha_s\) and \(\alpha^*\) independent of \(h\) such that
  \[
  \alpha_s a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h) \quad \forall v_h \in V_h^E(\Omega).
\]
  (26)

Both constants \(\alpha_s\) and \(\alpha^*\) may depend on the polynomial degree \(k\) and the mesh regularity constant \(\rho\).

By adding all the elemental contributions, we find that \(a_h(\cdot, \cdot)\) is a coercive bilinear form on \(V_h^E \times V_h^E\):
\[
a_h(v_h, v_h) \geq \alpha_s |v_h|_{1, \Omega}^2.
\]
(27)

A second straightforward consequence of (26) and the symmetry of \(a_h^E(\cdot, \cdot)\) is that this bilinear form is an inner product on \(V_h^E(\Omega) \setminus \mathbb{R}\). Using the Cauchy-Schwarz inequality, it holds that:
\[
\alpha_s a^E(v_h, w_h) \leq \left[\alpha_s a^E(v_h, v_h)\right]^{\frac{1}{2}} \left[\alpha_s a^E(w_h, w_h)\right]^{\frac{1}{2}} \leq \alpha^* \left[\alpha_s a^E(v_h, v_h)\right]^{\frac{1}{2}} \left[\alpha_s a^E(w_h, w_h)\right]^{\frac{1}{2}} = \alpha_s |v_h|_{1, E} |w_h|_{1, E},
\]
(28)

which implies that the local bilinear form \(a_h^E(\cdot, \cdot)\) is continuous on \(V_h^E(\Omega) \times V_h^E(\Omega)\). The global continuity of \(a_h(\cdot, \cdot)\) follows on summing all the local terms and using again the Cauchy-Schwarz inequality:
\[
a_h(v_h, w_h) = \sum_{E \in \Omega_h} a_h^E(v_h, w_h) \leq \alpha^* \sum_{E \in \Omega_h} |v_h|_{1, E} |w_h|_{1, E} \leq \alpha^* \left(\sum_{E \in \Omega_h} |v_h|_{1, E}^2\right)^{\frac{1}{2}} \left(\sum_{E \in \Omega_h} |w_h|_{1, E}^2\right)^{\frac{1}{2}}.
\]
(29)

**Construction of the virtual element bilinear forms** \(b_h\). Similarly, we define the virtual element bilinear form \(b_h(\cdot, \cdot)\) as the sum of local bilinear forms \(b_h^E(\cdot, \cdot) : V_h^E(\Omega) \times P_{k-1}(\Omega) \to \mathbb{R}\) as follows:
\[
b_h(v_h, q_h) = \sum_{E \in \Omega_h} b_h^E(v_h, q_h) \quad \text{where} \quad b_h^E(v_h, q_h) = \int_E q_h \Pi_{k-1}^E \div v_h \, dx.
\]
(30)

From the definition of the orthogonal projection operator \(\Pi_{k-1}^E\), it immediately follows that
\[
b_h^E(v_h, q_h) = b^E(v_h, q_h) \quad \forall v_h \in V_h^E(\Omega), \; q_h \in P_{k-1}(\Omega).
\]
(31)

If we add this relation over all the mesh elements, we find that
\[
b_h(v_h, q_h) = b(v_h, q_h) \quad \forall v_h \in V_h^E(\Omega), \; q_h \in P_{k-1}(\Omega),
\]
(32)

which will be used in the analysis of the next section.

**Remark 3.1** Since \(\Pi_{k-1}^E(\div u_h)\) for all elements \(E\) is a polynomial of degree \(k - 1\), equation (20) is equivalent to require that \(\Pi_{k-1}^E(\div u_h) = 0\) in \(E\). This condition is the discrete analog in \(P_{k-1}(\Omega)\) of the incompressibility condition \(\div u = 0\).

**Construction of the virtual element right-hand side.** In every polygonal element \(E\), we approximate the right-hand side vector \(f\) with its polynomial projection \(f_h := \Pi_k^E f\) onto the local polynomial space \(P_k(E)\). We consider two possible choices of \(k\) given the integer \(k \geq 1\):

- \(\bar{k} = \max(k - 2, 0)\): this is the setting proposed in the original paper [7];
- \(\bar{k} = k\): this is the setting proposed in Ref. [2], which requires the enhanced definition of the virtual element space.

We discuss the enhanced definition of the virtual element space of both formulations in the next sections.
Finally, the right hand-side of equation (19) is given by
\[ \langle f_h, v_h \rangle = \sum_{E \in \Omega_h} \int_E \Pi_k f_h \cdot v_h \, dx = \sum_{E \in \Omega_h} \int_E f_h \cdot \Pi_k v_h \, dx, \] (33)

where the second equality follows on applying the definition of the orthogonal projector \( \Pi_k \).

We recall the following results pertaining these two possible approximations of the right-hand side, which follows on noting that \( 1 - \Pi_k \) is orthogonal to \( \Pi_0 \) in the \( L^2 \)-inner product. Assuming \( f \in [H^s(\Omega)]^2 \) with \( 1 \leq s \leq k \), we find that
\[ \|f - \Pi_k f\|_{0,E} \leq Ch^{s+1} \|f\|_{s,E} \|v_h\|_{1,E}. \] (34)

For \( k = 0 \) and assuming \( f \in [L^2(\Omega)]^2 \), we find that
\[ \|f - \Pi_0 f\|_{0,E} \leq Ch \|f\|_{0,E} \|v_h\|_{1,E}. \] (35)

### 3.1 Formulation F1

We set the virtual element space for the velocity vector-valued fields of the first formulation as
\[ V^{F1,h}_k(E) = \left[ V^{F1,h}_k(E) \right]^2, \]
where the corresponding scalar virtual element space is given by
\[ V^{F1,h}_k(E) := \left\{ v_h \in H^1(E) : v_h \big|_{\partial E} \in C^0(\partial E), \, v_h \big|_e \in \mathbb{P}_{k+1}(e) \forall e \in \partial E, \Delta v_h \in \mathbb{P}_{k-2}(E) \right\}. \] (36)

With a small abuse of notation, we denote the enhanced version of the local space with the same symbol \( V^{F1,h}_k(E) \), and we consider the following definition:
\[ V^{F1,h}_k(E) := \left\{ v_h \in H^1(E) : v_h \big|_{\partial E} \in C^0(\partial E), \, v_h \big|_e \in \mathbb{P}_{k+1}(e) \forall e \in \partial E, \Delta v_h \in \mathbb{P}_k(E), \right\}, \] (37)

where \( \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E) \) is the space of polynomials of degree exactly equal to \( k \) or \( k-1 \). This definition uses the elliptic projection operator \( \Pi_{k,E} \), which is computable from the degrees of freedom defined below, cf. Lemma 3.7.

**Remark 3.2** The virtual element space (36) and its modified version (37) differ from the spaces respectively defined in References [7] and [2] because all the edge traces of a virtual element function are polynomials of degree \( k+1 \) instead of \( k \). This definition is a special case of the generalized local virtual element space that is considered in [23, Section 3] for the discretization of the Poisson equation. In fact, the local scalar space (36) can be obtained by setting \( k_0 = k+1 \) in [23, Eq. (7)] (with the same meaning for the parameter \( k \)).

**Remark 3.3** Assuming that the trace on the edges of the elemental boundary is a polynomial of degree \( k+1 \) instead of \( k \) does not change the convergence rate of the method and implies that an additional degree of freedom is needed for each velocity components on every edge, thus increasing the complexity and the computational costs. However, it makes the proof of the inf-sup condition almost straightforward, which is crucial to prove the well-posedness and convergence of the method. So, this formulation allows us to build a stable numerical approximation to the Stokes problem that holds on any kind of polygonal meshes, including triangular and square meshes, for all orders of accuracy \( k \geq 1 \).
The degrees of freedom of this formulation for the spaces defined in (36) and (37) are given by:

- (F1-a) for $k \geq 1$, the vertex values $v_h(x_v), v \in \partial E$;
- (F1-b) for $k \geq 1$, the polynomial edge moments of $v_h$
  \[ \frac{1}{|e|} \int_{e} v_h(s) q_h(s) \, ds \quad \forall q_h \in \mathbb{P}_{k-1}(e) \]  
  for every edge $e \in \partial E$;
- (F1-c) for $k \geq 2$, the polynomial cell moments of $v_h$
  \[ \frac{1}{|E|} \int_{E} v_h(x) q_h(x) \, dx \quad \forall q_h \in \mathbb{P}_{k-2}(E). \]  

Figure 1 shows the degrees of freedom for each component of the velocity vector and the pressure for $k = 1, 2, 3$ on an hexagonal element.

**Lemma 3.4 (Unisolvency of the degrees of freedom)** The degrees of freedom (F1-a), (F1-b), and (F1-c) are unisolvency in the space $V^{F1,h}_k(E)$ for both the definitions given in (36) and (37).

**Proof.** The proof of the unisolvency of the degrees of freedom (F1-a)-(F1-c) for $V^{F1,h}_k(E)$ follows by adapting the arguments used in [7, Proposition 1] for the space defined in (36) and [2, Proposition 2] for the space defined in (37). We briefly sketch the proof of the unisolvency for the space defined in (36). For every virtual element function in $V^{F1,h}_k(E)$, we consider the integration by parts:

\[ \int_{E} |\nabla v_h|^2 \, dx = - \int_{E} v_h \cdot \Delta v_h + \sum_{e \in \partial E} \int_{e} v_h \, n_e \cdot \nabla v_h \, ds = (1) + (11). \]  

Now, assume that the degrees of freedom (F1-a), (F1-b), and (F1-c) are all zero. Then,

- for $k = 1$, it holds that $\Delta v_h = 0$; for $k \geq 2$, it holds that $\Delta v_h$ is a polynomial of degree $k - 2$ and (1) is a degree of freedom, hence it is zero by hypothesis;
- the trace of $v_h$ along each edge $e \in \partial E$ is a polynomial of degree $k + 1$ that can be recovered by the interpolation of the degrees of freedom (F1-a) and (F1-b). Since these degrees of freedom are zero by hypothesis, their trace interpolation is zero.

Consequently, $\nabla v_h = 0$, which implies that $v_h$ is constant on $E$, and this constant is zero since it coincides with the value of all its degrees of freedom, which we assume to be zero. The proof of the unisolvency for the space defined in (36) is completed by noting that the number of the degrees of freedom equals the dimension of space $V^{F1,h}_k(E)$. Similar modifications to the argument of [2, Proposition 2] make it possible to prove the unisolvency for the enhanced virtual element space defined in (37). \qed

**Lemma 3.5** Let $E$ be an element of mesh $\Omega_h$. For every virtual element function $v_h \in V^{F1,h}_k(E)$, the polynomial projection $\Pi^{0,E}_{k-1} \nabla v_h$ is computable using the degrees of freedom (F1-a), (F1-b), and (F1-c) of $v_h$.

**Proof.** To prove that $\Pi^{0,E}_{k-1} (\nabla v_h)$ is computable, we explicitly prove that $\Pi^{0,E}_{k-1} (\partial v_h/\partial x)$ is computable. Then, the same argument can be applied to prove that $\Pi^{0,E}_{k-1} (\partial v_h/\partial y)$ is also computable. To this end, we start from the definition of the orthogonal projection and integrate by parts:
\[
\int_{\Omega} q_h \Pi^0_E \frac{\partial v_h}{\partial x} \, dx = \int_{\Omega} q_h \frac{\partial v_h}{\partial x} \, dx = \int_{\Omega} v_h \frac{\partial q_h}{\partial x} \, dx + \sum_{e \in \partial \Omega} \int_{e} v_h q_h \, ds = (I) + (II),
\]
which holds for every \( q_h \in P_{k-1}(\Omega) \). Term (I) is computable since \( \partial q_h/\partial x \in P_{k-2}(\Omega) \) and this integral is determined by the degrees of freedom of \( v_h \) in (F1-c). Term (II) is computable since the polynomial \( q_h \) is known and \( v_h|_{\partial e} \in P_{k+1}(e) \) can be interpolated from the degrees of freedom of \( v_h \) given by (F1-a) and (F1-b) on every edge \( e \in \partial \Omega \).

**Remark 3.6** For all scalar virtual element functions \( v_h \in V^{F1,h}_k(\Omega) \), the polynomial projections \( \Pi^0_{k-1}(\partial v_h/\partial x) \) and \( \Pi^0_{k-1}(\partial v_h/\partial y) \) forming \( \Pi^0_{k-1} \nabla v_h \) are computable by using the degrees of freedom of \( v_h \). Consequently, the polynomial projections \( \Pi^0_{k-1} \nabla v_h \in [P_{k-1}(\Omega)]^2 \times 2 \) and \( \Pi^0_{k-1} \text{div} v_h \in P_{k-1}(\Omega) \) are computable for all virtual vector-valued fields \( v_h \in [V^{F1,h}_k(\Omega)]^2 \).

**Lemma 3.7** Let \( E \) be an element of mesh \( \Omega_h \). For all virtual element functions \( v_h \in V^{F1,h}_k(\Omega) \), the polynomial projection \( \Pi^0 E v_h \in P_k(\Omega) \) is computable from the degrees of freedom of \( v_h \).

**Proof.** The same argument of Lemma 3.5 is used here. We start from the definition of the elliptic projection and we integrate by parts:

\[
\int_{\Omega} \nabla \Pi^V_k \nabla v_h \cdot \nabla q_h \, dx = \int_{\Omega} \nabla v_h \cdot \nabla q_h \, dx = - \int_{\Omega} v_h \Delta q_h \, dx + \sum_{e \in \partial \Omega} \int_{e} v_h n_e \cdot \nabla q_h \, ds = (I) + (II).
\]

Since in (42) we take \( q_h \in P_k(\Omega) \) and \( \Delta q_h \in P_{k-2}(\Omega) \), term (I) is computable using the degrees of freedom (F1-c) of \( v_h \). Similarly, since \( v_h|_{\partial e} \in P_{k+1}(e) \) is computable from an interpolation of the degrees of freedom (F1-a) and (F1-b), term (II) is computable.

**Remark 3.8** \( \Pi^V_k v_h \) is computable componentwisely for every vector-valued virtual element field \( v_h \in V^{F1,h}_k(\Omega) \) and is used in the stabilization term of \( a^k(\cdot, \cdot) \), cf. (24).

### 3.2 Formulation F2

We denote the tangential and normal components of \( v_h \) along the edge \( e \in \partial \Omega \) by \( v_h|_e \cdot t_e \) and \( v_h|_e \cdot n_e \), where \( t_e \) and \( n_e \) are the unit tangential and orthogonal vector of \( e \). The virtual element space of the second formulation is defined as:

\[
V^{F2,h}_k(\Omega) := \left\{ v_h \in [H^1(\Omega)]^2 : v_h|_{\partial \Omega} \in [C^0(\partial \Omega)]^2, \right. \\
\left. v_h|_e \cdot t_e \in P_k(e), v_h|_e \cdot n_e \in P_{k+1}(e), \Delta v_h \in P_{k-2}(\Omega) \right\}.
\]

With a small abuse of notation we denote the “enhanced” version of this space with the same symbol “\( V^{F2,h}_k \)“:

\[
V^{F2,h}_k(\Omega) := \left\{ v_h \in [H^1(\Omega)]^2 : v_h|_{\partial \Omega} \in [C^0(\partial \Omega)]^2, \right. \\
\left. v_h|_e \cdot t_e \in P_k(e), v_h|_e \cdot n_e \in P_{k+1}(e), \Delta v_h \in P_{k}(\Omega) \right\},
\]

\[
\int_{\Omega} (v_h - \Pi^V_k v_h) \cdot q_h \, dx = 0 \quad \forall q_h \in [P_k(\Omega) \setminus P_{k-2}(\Omega)]^2,
\]

where \( P_k(\Omega) \setminus P_{k-2}(\Omega) \) is the space of polynomials of degree exactly equal to \( k \) and \( k - 1 \). This definition uses the elliptic projection operator \( \Pi^V_k \), which is computable from the degrees of freedom defined below, cf. Lemma 3.13.

Note that the normal component of \( v_h \) is a polynomial of degree \( k + 1 \) while the tangential component is a polynomial of degree \( k \). These conditions are reflected by the following degrees of freedom, which are the same for the virtual element functions defined in both (43) and (44):

- **(F2-a)** for \( k \geq 1 \), the vertex values \( v_h(x_e) \);
- **(F2-b)** for \( k \geq 1 \), the polynomial edge moments of \( v_h \cdot n_e \):

\[
\frac{1}{|e|} \int_e v_h \cdot n_e q_h \, ds \quad \forall q_h \in P_{k-1}(e)
\]
for every edge $e \in \partial E$:
- (F2-c) for $k \geq 2$, the polynomial edge moments of $v_h \cdot t_e$:
  \[
  \frac{1}{|e|} \int_{e} v_h \cdot t_e q_h \, ds \quad \forall q_h \in P_{k-2}(e) \tag{46}
  \]
for every edge $e \in \partial E$:
- (F2-d) for $k \geq 2$, the polynomial cell moments of $v_h$:
  \[
  \frac{1}{|E|} \int_{E} v_h \cdot q_h \, d\mathbf{x} \quad \forall q_h \in \left[ P_{k-2}(E) \right]^2. \tag{47}
  \]

Figure 2 shows the degrees of freedom of the velocity vector and the pressure for $k = 1, 2, 3$ on an hexagonal element.

**Remark 3.9** In this virtual element space, the normal component of $v_h$ has an increased polynomial degree. For example, for $k = 1$ the vector field $v_h \in V_{1}^{F2,h}(E)$ is such that $v_h \cdot n_e \in P_2(e)$ and $v_h \cdot t_e \in P_1(e)$ for every edge $e \in \partial E$. These degrees of freedom are the same used in the low-order MFD method of Reference [13] and our VEM is actually a reformulation of this mimetic scheme in the variational setting and a generalization to orders of accuracy that are higher than one. The analysis of the mimetic method and its extension to the three-dimensional case is presented in [15] and considers the additional edge degrees of freedom as associated with edge bubble functions.

**Remark 3.10** Using the degrees of freedom (F2-a) and (F2-b) the edge traces $v_h \cdot n_e \in P_{k+1}(e)$ and $v_h \cdot t_e \in P_k(e)$ are computable by solving a suitable interpolation problem. Consider the edge $e = (x', x''_e)$ defined by the vertices $x'$ and $x''_e$. Then,
- to interpolate $v_h \cdot n_e \in P_{k+1}(e)$ we need $k+2$ independent pieces of information, which are provided by $v_h(x') \cdot n_e$, $v_h(x''_e) \cdot n_e$ from the degrees of freedom (F2-a) and by the $k$ moments of $v_h \cdot n_e$ from the degrees of freedom (F2-b);
- to interpolate $v_h \cdot t_e \in P_k(e)$ we need $k+1$ independent pieces of information, which are provided by $v_h(x') \cdot t_e$, $v_h(x''_e) \cdot t_e$ from the degrees of freedom (F2-a) and by the $k-1$ moments of $v_h \cdot t_e$ from the degrees of freedom (F2-c).

**Lemma 3.11 (Unisolvence of the degrees of freedom)** The degrees of freedom (F2-a)-(F2-d) are unisolvent for both the regular and enhanced definition of $V_{k}^{F2,h}(E)$, respectively given in (43) and (44).

**Proof:** The argument that we use to prove the assertion of the lemma is similar to the one used to prove the unisolvency of the degrees of freedom of the first formulation. First, consider a vector field in the virtual element space defined in (43). An integration by parts yields:

\[
\int_{E} |\nabla v_h|^2 \, d\mathbf{x} = - \int_{E} v_h \cdot \Delta v_h \, d\mathbf{x} + \sum_{e \in \partial E} \int_{e} v_h \cdot \nabla v_h \cdot n_e \, ds = (I) + (II). \tag{48}
\]

Next, we assume that all the degrees of freedom in (F2-a), (F2-b), (F2-c), and (F2-d) are zero. Then,
- (I) is zero because $\Delta v_h \in \left[ P_{k-2}(E) \right]^2$, and, hence, it is a degree of freedom of type (F2-d) for $k \geq 2$ or zero for $k = 1$;
- to see that (II) is also zero, we use the orthogonal decomposition $v_h = (v_h \cdot n_e)n_e + (v_h \cdot t_e)t_e$ and note that $v_h \cdot n_e = 0$ and $v_h \cdot t_e = 0$ since these traces are computed by the interpolation of the degrees of freedom (F2-a),
(F2-b), and (F2-c), and these data are zero by hypothesis. Therefore, \( v_{h|e} = 0 \) on every edge \( e \in \partial E \) and all the edge integrals of \((I)\) must be zero.

It follows that \( \nabla v_h = 0 \), i.e., all the spatial derivatives of the components of \( v_h \) are zero. Therefore, the vector-valued field \( v_h \) is constant on \( E \) and since all its degrees of freedom are zero the constant must be zero. The assertion of the lemma is finally proved by noting that the number of the degrees of freedom is equal to the dimension of \( V_k^{F2,h}(E) \).

The unisolvence of the degrees of freedom (F2-a)-(F2-d) for the enhanced space defined in (44) follows by similarly adjusting the argument that is used in the proof of [2, Proposition 2].

\[ \square \]

**Lemma 3.12** Let \( E \) be an element of mesh \( \Omega_h \). For every virtual element function \( v_h \in V_k^{F2,h}(E) \), the polynomial projection \( \Pi_k^0 E \nabla v_h \) is computable from the degrees of freedom (F2-a), (F2-b), and (F2-c) of \( v \).

**Proof.** We start from the definition of the orthogonal projection:

\[
\int_E \Pi_{k-1}^0 \nabla v_h : \tau_h \, dx = \int_E \nabla v_h : \tau_h \, dx \quad \forall \tau_h \in [P_{k-1}(E)]^{2 \times 2}.
\]

To prove that the right-hand side is computable from the degrees of freedom of \( v_h \), we integrate by parts:

\[
\int_E \nabla v_h : \tau_h \, dx = - \int_E v_h \cdot \text{div} \tau_h \, dx + \sum_{e \in \partial E} \int_{\partial E} v_h \cdot \tau_h \cdot n_e \, ds = (I) + (II).
\]

Since \( \text{div} \tau_h \in [P_{k-2}(E)]^2 \), term \((I)\) is computable using the values (F2-d) of \( v_h \). Then, we observe that the traces \( v_{h|e} \cdot n_e \in P_{k+1}(e) \) and \( v_{h|e} \cdot t_e \in P_k(e) \) are computable from the degrees of freedom (F2-a)-(F2-c). On using the decomposition \( v_h = (v_h \cdot n_e) n_e + (v_h \cdot t_e) t_e \), we conclude that the trace \( v_{h|e} \) is computable. Therefore, all edge integrals and ultimately \((II)\) are computable.

\[ \square \]

**Lemma 3.13** Let \( E \) be an element of mesh \( \Omega_h \). For every virtual element function \( v_h \in V_k^{F2,h}(E) \), the polynomial projection \( \Pi_k^V E v_h \in [P_{k-1}(E)]^{2 \times 2} \) is computable from the degrees of freedom of \( v_h \).

**Proof.** Consider the definition of the elliptic projection operator:

\[
\int_E \nabla \Pi_k^V E v_h : \nabla q_h \, dx = \int_E \nabla v_h : \nabla q_h \, dx \quad q_h \in [P_k(E)]^{2}.
\]

We integrate the right-hand side by parts:

\[
\int_E \nabla v_h : \nabla q_h \, dx = - \int_E v_h \cdot \Delta q_h \, dx + \sum_{e \in \partial E} \int_{\partial E} v_h \cdot \nabla q_h \cdot n_e \, ds = (I) + (II).
\]

Since we take \( q_h \in [P_k(E)]^2 \) and \( \Delta q_h \in [P_{k-2}(E)]^2 \), the first integral in term \( (I) \) is the moment of \( v_h \) against a vector polynomial of degree \( k - 2 \) and is, thus, computable using the degrees of freedom of \( v_h \) provided by (F2-d). Then, we observe that the traces \( v_{h|e} \cdot n_e \in P_{k+1}(e) \) and \( v_{h|e} \cdot t_e \in P_k(e) \) are computable from the degrees of freedom (F2-a)-(F2-c). On using the decomposition \( v_h = (v_h \cdot n_e) n_e + (v_h \cdot t_e) t_e \), also the trace \( v_{h|e} \) is computable, cf. Remark 3.10. Therefore, all edge integrals and ultimately \((II)\) are computable.

\[ \square \]

### 4 Wellposedness and convergence analysis

In this section, we first prove the wellposedness of the two virtual element formulations of Section 3. Then, we prove that these two formulations are convergent and we derive error estimates in the energy norm and the \( L^2 \) norm for the velocity field and the \( L^2 \) norm for the pressure field. The analysis is the same for both formulations \( F1 \) and \( F2 \), regardless of using the non-enhanced or the enhanced definition of the virtual element space. For this reason, we use the generic symbol \( V_k^h(E) \) to refer to the two virtual element spaces introduced in Section 3, i.e., \( V_k^{F1,h}(E) \) and \( V_k^{F2,h}(E) \).

Hereafter, we use the capitol letter "\( C \)" to denote a generic constant that is independent of \( h \) but may depend on the other parameters of the discretization, e.g., the polynomial degree \( k \), the mesh regularity constant \( \rho \), the stability constants \( \alpha_\ast \) and \( \alpha_+ \), etc. The constant \( C \) may take a different value at any occurrence.
In some mathematical proofs, we may find it convenient to write “\(A^{(X)}\)” to mean that “\(A = B\) follows from equation \((X)\),” i.e., to stack the equation reference number on the symbols “\(=\), “\(\leq\), “\(\geq\)” etc.

4.1 Wellposedness of the virtual element approximation

To prove the wellposedness of our formulations, we must verify that the virtual element space \(V^k_h\) and the discontinuous polynomial space \(Q^{k-1}_h\) are such that: (i) the bilinear form \(a_h(\cdot, \cdot)\) is bounded and coercive; (ii) the bilinear form \(b_h(\cdot, \cdot)\) is bounded and satisfies the inf-sup condition. Properties (i) are the immediate consequence of the stability property (26) and the Cauchy-Schwarz inequality, which imply (27) and (29). We rewrite these two inequalities here for the reader’s convenience:

\[
|a_h(v_h, w_h)| \leq \alpha^w |v_h|_{1, \Omega} |w_h|_{1, \Omega} \quad \forall v_h, w_h \in V^k_h, \quad (53)
\]

\[
\alpha_s |v_h|^2_{1, \Omega} \leq a_h(v_h, v_h) \quad \forall v_h \in V^k_h. \quad (54)
\]

Similarly, we can readily prove the boundedness of the bilinear form \(b_h(\cdot, \cdot)\) by using the Cauchy-Schwarz inequality, so that

\[
|b_h(v_h, q_h)| \leq \sqrt{2}|v_h|_{1, \Omega} ||q_h||_{0, \Omega} \quad \forall v_h \in V^k_h, q_h \in P_{k-1}(\Omega_h).
\]

Instead, the discrete inf-sup condition is proved in the following lemma, which relies on the construction of a suitable Fortin operator, see [31]. The construction of this operator is the same for both the regular and the enhanced versions of formulations \(F1\) and \(F2\).

**Lemma 4.1 (Inf-sup condition)** The bilinear form \(b_h(\cdot, \cdot)\) is inf-sup stable on \(V^k_h \times Q^{k-1}_h\) for the formulations \(F1\) and \(F2\) and for any given polynomial degree \(k \geq 1\).

**Proof.** The proof is essentially based on the construction of a Fortin operator \(\pi_F : [H^1(\Omega)]^2 \rightarrow V^k_h\) such that

\[
b(v, q_h) = b_h(\pi_F v, q_h) \quad \forall q_h \in P_{k-1}(\Omega_h), \quad (55)
\]

\[
||\pi_F v||_{1, \Omega} \leq |v|_{1, \Omega}, \quad (56)
\]

for all \(v \in [H^1(\Omega)]^2\), cf., e.g., [31]. As the proof is based on rather standard arguments, see e.g., [17, Proposition 3.1], we only briefly mention its three main steps.

In the first step, reasoning as in [70, Proposition 4.2] for the non-enhanced virtual element space and [39, Theorem 5 (case \(d = 2\))] for the enhanced virtual element space, we can prove the existence of a quasi-interpolation operator \(\pi^E : [H^{s+1}(E)]^2 \rightarrow V^k_h(E), 0 \leq s \leq k\) for all elements \(E \in \Omega_h\) such that

\[
|v - \pi^E F v|_{0,E} + h_E |v - \pi^E F v|_{1,E} \leq Ch_E^{s+1}|v|_{s+1,E}.
\]

Adding all elemental contributions, it is easy to see that

\[
||v - \pi^E v||_{1, \Omega} \leq C||v||_{1, \Omega},
\]

where \(\pi_1 : [H^{s+1}(\Omega)]^2 \rightarrow V^k_h\) is the global quasi-interpolation operator such that \((\pi_1 v)|_E = \pi^E_1(v)|_E\) for all \(E \in \Omega_h\).

In the second step, for any \(v \in [H^1(\Omega)]^2\) we consider a vector-valued virtual element function \(v_h\) such that

(i) for \(k \geq 1\), for all mesh edges \(e\), it holds that

\[
\int_E q_h v \cdot n_e ds = \int_E q_h v \cdot n_e ds \quad \forall q_h \in P_{k-1}(e), \quad (57)
\]

where we recall that \(n_e\) is the unit normal vector to the edge \(e\), whose orientation is fixed once and for all;

(ii) for \(k \geq 2\) and for all \(E \in \Omega_h\), it holds that

\[
\int_E v_h \cdot q_h dx = \int_E v \cdot q_h dx \quad \forall q_h \in [P_{k-2}(E)]^2. \quad (58)
\]
The vector-valued field $v_h$ is easily determined in $V_h^b$ by properly setting the degrees of freedom of the formulations $F1$ and $F2$. In particular, if $v_h \cdot n_e = v_{h,x} n_{e,x} + v_{h,y} n_{e,y}$ for $n_e = (n_{e,x}, n_{e,y})^T$ and $v_h = (v_{h,x}, v_{h,y})^T$, then it holds that

- condition (i) is verified by setting accordingly the degrees of freedom ($F1$-b) of formulation $F1$ and ($F2$-b) of formulation $F2$;
- condition (ii) is verified by setting accordingly the degrees of freedom ($F1$-c) of formulation $F1$ and ($F2$-d) of formulation $F2$.

All the remaining degrees of freedom are set to zero. The unisolveny property ensures that such $v_h$ exists and is unique in $V_h^b$. We denote the correspondence between $v$ and $v_h$ by introducing the elemental operator $\pi_h^E : [H^1(E)]^2 \rightarrow V_h^b(E)$, which is such that $\pi_h^E v = v_h$, and the global operator $(\pi_h^E v)|_\Omega = \pi_h^E (v|_\Omega)$ for all $E \in \Omega_h$.

In the third and last step, we define the Fortin operator as $\pi_F v = \pi_1 v + \pi_2 (1 - \pi_1) v$. This operator satisfies (55) and (56). The discrete inf-sup condition then follows immediately from the Fortin argument by using these relations and the continuous inf-sup condition (18).

\[ \square \]

The properties of coercivity and boundedness of $a_h(\cdot, \cdot)$ and inf-sup stability (cf. Lemma 4.1) and boundedness of $b_h(\cdot, \cdot)$ implies the wellposedness of the two virtual element formulations considered in this work. We formally state this result in the next theorem.

**Theorem 4.2 (Well-posedness)** The virtual element formulations $F1$ and $F2$ for any given polynomial degree $k \geq 1$ have one and only one solution pair $(u_h, p_h) \in V_h^b \times Q_h^{k-1}$, which is such that

\[ \|u_h\|_{1,\Omega} + |p_h|_{0,\Omega} \leq C \|f\|_{0,\Omega}. \]  

(59)

The proof is omitted as this is a standard result in the numerical approximation of saddle-point problems, cf. [31].

4.2 Preliminary results

To derive the error estimates in the energy norm and the $L^2$ norm, we need three technical lemmas that are preliminary reported here. The first two lemmas are reported without the proof as they are well-known results from the approximation theory, see [32, 54]. In particular, the first lemma provides an estimate of the projection error and is the vector version of the analogous result reported in [7] for the scalar case.

**Lemma 4.3 (Projection error)** Under Assumptions (M1)-(M2), for every vector-valued field $v \in [H^{s+1}(E)]^2$ with $1 \leq s \leq \ell$ for some given integer number $\ell$, there exists a vector polynomial $v_\pi \in [P_\ell(E)]^2$ such that

\[ \|v - v_\pi\|_{0,E} + h_E |v - v_\pi|_{1,E} \leq C h_E^{s+1} |v|_{s+1,E}. \]  

(60)

where $C$ is some positive constant that is independent of $h_E$ but may depend on the polynomial degree $\ell$ and the mesh regularity constant $\varphi$.

The second lemma reports an estimate of the approximation errors for the interpolants $v_I$ and $q_I$. According to [7], we define the local interpolation $v_I \in V_I^b(E)$ of (a smooth enough) field $v$ as the virtual element field that has the same degrees of freedom. Similarly, we define the local interpolation $q_I \in Q_I^{k-1}$ of (a smooth enough) scalar function $q$ as the polynomial function that has the same degrees of freedom. Therefore, $(q_I)|_\Omega \in P_{k-1}(E)$ for all elements $E \in \Omega_h$, and

\[ \int_\Omega q_I(x) \, dx = 0, \]  

(61)

since according to (21) it also holds that $q_I \in L^2_0(\Omega)$.

**Lemma 4.4 (Interpolation error)** Under Assumptions (M1)-(M2), for every vector-valued field $v \in [H^{s+1}(E)]^2$ and scalar function $q \in H^s(E)$ with $1 \leq s \leq \ell$, for some given integer number $\ell$, there exist a vector-valued field $v_I \in V_I^b(E)$ and a scalar field $q_I \in P_{\ell-1}(E)$ such that
\[ ||v - v_I||_{0,E} + h_E|v - v_I|_{1,E} \leq C h_E^{s+1} |v|_{s+1,E}, \]  

\[ ||q - q_I||_{0,E} + h_E|q - q_I|_{1,E} \leq C h_E^s |q|_{s,E}. \]  

for some positive constant \( C \) that is independent of \( h_E \) but may depend on the polynomial degree \( \ell \) and the mesh regularity constant \( \varrho \).

In the last lemma of this section we prove a relation between \( u_h, u_I, p_h \) and \( p_I \) that will be used in the convergence analysis of the next sections.

**Lemma 4.5** Let \((u, p) \in [H^{s+1}(\Omega)]^2 \times L_0^2(\Omega), s \geq 1, \) be the exact solution of the variational formulation of the Stokes problem given in (12)-(13) and \((u_I, p_I) \in V_k^h \times Q_k -1\) the corresponding virtual element interpolation. Let \((u_h, p_h) \in V_k^h \times Q_k -1\) be the virtual element approximation to \((u, p)\) solving (19)-(20). Then, it holds that

\[ b(u_h - u_I, p_h - p_I) = 0. \]  

**Proof.** Let \( E \) be an element of mesh \( \Omega_h \) and \( k \geq 1 \) an integer number. Consider the function \( v \in [H^{s+1}(\Omega)]^2, s \geq 1, \) and its virtual element interpolant \( v_I \in V_k^h(\Omega). \) Integrating by parts twice and using the definition of the interpolant \( v_I \), we find that:

\[-b^E(v, q_h) = \int_E q_h \text{div} v \, dx = - \int_E v_h \cdot \nabla v \, dx + \sum_{e \in \partial E} \int_{e} q_h n_e \cdot v \, ds \]

\[ = - \int_E \nabla q_h \cdot v_I \, dx + \sum_{e \in \partial E} \int_{e} q_h n_e \cdot v_I \, ds = \int_E q_h \text{div} v_I \, dx = -b^E(v_I, q_h), \]  

which holds for all \( q_h \in \mathbb{P}_{k-1}(E). \) The identity chain (65) implies that \( b^E(v, q_h) = b^E(v_I, q_h), \) and, adding this relation over all elements \( E \) yields \( b(v, q_h) = b(v_I, q_h). \) By taking \( v = u, \) equation (13) implies that \( b(u, q_h) = b(u_I, q_h) = 0. \) Likewise, by taking \( v_I = u_h, \) equations (32) and (20) imply that \( b(u_h, q_h) = b_h(u_h, q_h) = 0. \) Taking the difference of the left-most left-hand side of the two previous identities yields \( b(u_h - u_I, q_h) = 0, \) which holds for all \( q_h \in Q_k -1. \) The assertion of the lemma readily follows by taking \( q_h = p_h - p_I. \) \( \square \)

### 4.3 Error estimate in the energy norm

**Theorem 4.6** Let \( u \in [H^{s+1}(\Omega) \cap H_0^1(\Omega)]^2 \) and \( p \in H^s(\Omega) \cap L_0^2(\Omega), 1 \leq s \leq k, \) be the solution of the variational formulation of the Stokes problem given in (12)-(13). Let \((u_h, p_h) \in V_k^h \times Q_k -1\) be the solution of the virtual element variational formulation (19)-(20) under the mesh regularity assumptions (M1) – (M2) and for any polynomial degree \( k \geq 1. \) Then, there exists a real, strictly positive constant \( C \) independent of \( h \) such that the following abstract estimate holds:

\[ ||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} \leq C \left( ||u - u_I||_{1,\Omega} + ||u - u_e||_{1,h} + ||p - p_I||_{0,\Omega} + \sup_{v_h \in V_k^h \setminus \{0\}} \frac{|\langle f_h, v_h \rangle - \langle f, v_h \rangle|}{||v_h||_{1,\Omega}} \right) \]

(66)

where \( u_I \in V_k^h \) and \( p_I \in Q_k -1 \) are the interpolants of \( u \) and \( p \) from Lemma 4.4, and \( u_e \in [\mathbb{P}_k(\Omega_h)]^2 \) is any polynomial approximation of \( u \) that is defined in accordance with Lemma 4.3. Moreover, if \( f \in [H^t(\Omega)]^2, t \geq 0, \) it holds that

\[ ||u - u_h||_{1,\Omega} + ||p - p_h||_{0,\Omega} \leq C \left( h^s(||u||_{s+1,\Omega} + ||p||_{s,\Omega}) + h^{\min(\tilde{k},s-1)} ||f||_{t,\Omega} \right), \]

(67)

where \( \tilde{k} \) is defined as in (33).

**Proof.** We add and subtract \( u_I \) and \( p_I \) in the two terms of the left-hand side of (66) and use the triangle inequality:

\[ ||u - u_h||_{1,\Omega} \leq ||u - u_I||_{1,\Omega} + ||u_I - u_h||_{1,\Omega}. \]

(68)

\[ ||p - p_h||_{0,\Omega} \leq ||p - p_I||_{0,\Omega} + ||p_I - p_h||_{1,\Omega}. \]

(69)
The two terms $|u - u_h|_{\Omega}$ and $|p - p_h|_{\partial \Omega}$ are in the right-hand side of (66). We can estimate them by applying Lemma 4.4 to obtain (67). Instead, to estimate the second term of the right-hand side of (68) and (69), we proceed as follows. Let $\delta_h = u_h - u_f \in V_h$. Starting from the coercivity inequality (54), we find that:

$$\alpha_h |\delta_h|^2_{1,\Omega} \leq a_h(\delta_h, \delta_h)$$

We derive an upper bound of term $\langle R_1 \rangle$ as follows:

$$\langle R_1 \rangle = \sup_{\psi_h \in V_h^k \setminus \{0\}} \frac{|(f_h, \delta_h) - (f, \delta_h)|}{|\psi_h|_{1,\Omega}} |\delta_h|_{1,\Omega}.$$  

We derive an upper bound of term $(R_2)$ by using the Cauchy-Schwarz inequality:

$$|\langle R_2 \rangle| = |b(\delta_h, p - p_f)| \leq ||\Delta \delta_h||_{0,\Omega} ||p - p_f||_{0,\Omega} \leq C |\delta_h|_{1,\Omega} ||p - p_f||_{0,\Omega}.$$  

To derive an upper bound of term $(R_3)$, we use the continuity of $a_h(\cdot, \cdot)$, cf. (53), and $a(\cdot, \cdot)$, we add and subtract $u$ in the first summation argument, and, in the last step, we use definition (2) of the broken seminorm $| \cdot |_{1,h}$ to find that

$$|\langle R_3 \rangle| = \sum_{E \in \Omega_h} \left| a_h^E(u_f - u_{\pi}, \delta_h) + a_h^E(u_{\pi} - u, \delta_h) \right| \leq \sum_{E \in \Omega_h} \left( \alpha^* |u_f - u_{\pi}|_{1,E} + |u_{\pi} - u|_{1,E} \right) |\delta_h|_{1,E}$$

$$\leq \sum_{E \in \Omega_h} \left( \alpha^* |u_f - u|_{1,E} + (1 + \alpha^*) |u - u_{\pi}|_{1,E} \right) |\delta_h|_{1,E} \leq \left( \alpha^* |u - u_f|_{1,E} + (1 + \alpha^*) |u - u_{\pi}|_{1} \right) |\delta_h|_{1,\Omega}.$$  

Let $\sigma_h = p_h - p_f \in Q_h$. In view of the discrete inf-sup condition, cf. Lemma 4.1, there exists a real, strictly positive constant $\beta$ and a virtual element vector-valued field $v_h$ such that

$$\beta ||\sigma_h||_{0,\Omega} |v_h|_{1,\Omega} \leq b_h(v_h, \sigma_h) \quad \text{[split } \sigma_h = p_h - p_f\text{]}$$

$$= b_h(v_h, p_h) - b_h(v_h, p_f) \quad \text{[use (19)]}$$

$$= a_h(u_h, v_h) + (f_h, v_h) - b_h(v_h, p_f) \quad \text{[add (12)]}$$

$$= a_h(u_h, v_h) + [a(u, v_h) + h(v_h, p) - (f, v_h)] + (f_h, v_h) - b_h(v_h, p_f) \quad \text{[use (16) and (23)]}$$

$$= (f_h, v_h) - (f, v_h) + b(v_h, p) - b_h(v_h, p_f) + \sum_{E \in \Omega_h} \left( a_h^E(u, v_h) - a_h^E(u_h, v_h) \right) \quad \text{[use (25) with } q_h = u_{\pi}\text{]}$$

$$= [\langle R_4 \rangle] + [(R_5)] + [(R_6)].$$

We derive an upper bound of term $\langle R_4 \rangle$ using the same steps as for the bound of term $\langle R_1 \rangle$ with $v_h$ instead of $\delta_h$:  

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\[ |(R_4)| = |\langle f_h, v_h \rangle - (f, v_h)\rangle \leq \sup_{v_h \in V_h} \frac{|\langle f_h, v_h \rangle - (f, v_h)\rangle}{|v_h|_{1,\Omega}} |v_h|_{1,\Omega}. \]

We derive an upper bound of term \((R_5)\) using the same steps as for the bound of term \((R_2)\) with \(v_h\) instead of \(\delta_h\):

\[ |(R_5)| \leq |v_h|_{1,\Omega} ||p_I - p||_{0,\Omega}. \]

We derive an upper bound of term \((R_6)\) using the same steps as for the bound of term \((R_3)\) with \(v_h\) instead of \(\delta_h\)
and \(u_h\) instead of \(u\):

\[ |(R_6)| \leq \left( \alpha^* |u - u_h|_{1,\Omega} + (1 + \alpha^*)|u - u_{\pi}|_{1,\Omega} \right) |v_h|_{1,\Omega}. \]

Finally, we use the bound of terms \((R_1) - (R_3)\) to control \(|u_I - u_h|_{1,\Omega}\) in (68). Then, we use the bound of terms \((R_4) - (R_4)\) and \(|u - u_h|_{1,\Omega}\) to control \(|p_I - p_{\pi}|_{1,\Omega}\) in (69). The first assertion of the theorem follows on using the resulting inequalities to control the left-hand side of (66). The estimate (67) follows from a straightforward application of Lemmas 4.3 and 4.4, and estimates (34)-(35) in the right-hand side of (66). \(\square\)

### 4.4 Error estimate in the \(L^2\) norm for the velocity field

**Theorem 4.7** Let \(u \in [H^{s+1}(\Omega) \cap H_0^1(\Omega)]^2\) and \(p \in H^s(\Omega) \cap L^2_0(\Omega), 1 \leq s \leq k,\) be the exact solution of the variational formulation of the Stokes problem given in (12)-(13) with \(f \in [H^t(\Omega)]^2, 0 \leq t.\) Let \((u_h, p_h) \in V_h \times Q_{k-1}^h\) be the solution of the virtual element variational formulation (19)-(20) under the mesh regularity assumptions (M1) – (M2). Then, it holds:

\[ ||u - u_h||_{0,\Omega} \leq C \left( h^{s+1} \left( ||u||_{s+1,\Omega} + ||p||_{s,\Omega} \right) + \delta^{\min(t,k)+1} ||f||_{t,\Omega} \right) \]

for some real, strictly positive constant \(C\) independent of \(h\) and \(k\) is defined as in (33).

**Proof.** In the derivation of the \(L^2\) error for the virtual element approximation of the velocity vector \(u\), we make use of the solution \((\Psi, \varphi) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times H^1(\Omega) \cap L^2_0(\Omega)]\) of the dual problem:

\[ -\Delta \Psi - \nabla \varphi = u - u_h \quad \text{in } \Omega, \]
\[ \text{div } \Psi = 0 \quad \text{in } \Omega. \]

Since \(\Psi \in [H^2(\Omega)]^2\) and \(\varphi \in H^1(\Omega)\), the application of Lemmas 4.3 and 4.4 yields

\[ |(\Psi - \Psi_I)|_{1,\Omega} + |(\Psi - \Psi_\pi)|_{1,h} \leq Ch||\Psi||_{2,\Omega}, \]

\[ ||\varphi - \varphi_I||_{0,\Omega} \leq Ch\varphi|_{1,\Omega}, \]

where \(\Psi_I\) and \(\varphi_I\) are the virtual element interpolant of \(\Psi\) and \(\varphi\) in \(V_h^k\) and \(Q_{k-1}^h\), respectively, \(\Psi_\pi\) is the polynomial approximation of \(\Psi\) according to Lemma 4.3, and \(|| \cdot ||_{1,\Omega}\) in (73) is the “broken” norm defined in Eq. (2). Under the assumption that the domain \(\Omega\) is convex, the solution pair \((\Psi, \varphi)\) has the following regularity property:

\[ ||\Psi||_{2,\Omega} + ||\varphi||_{1,\Omega} \leq C||u - u_h||_{0,\Omega}. \]

Then, we use the definition of the \(L^2\) norm, and note that the boundary integral on \(\partial \Omega\) of \(n \cdot (u - u_h)\), which is originated by an integration by parts, is zero since \(u = u_h = 0\) on \(\partial \Omega\), and we find that
We estimate separately each term and (75):

To derive an upper bound for term (75), we find that

\[ b \]  

where

\[ R \]

is suitable polynomial approximations of \( \Psi \) and inequalities (73) and (75), and we find that

\[ \| u - u_h \|_{0, \Omega} \leq C h \| u - u_h \|_{1, \Omega} \]  

(76)

We split term (R2) into three subterms by using (12), adding (19) and rearranging the terms:

\[ (R_2) = a(\Psi_f, u - u_h) - a(u_h, \Psi_f) \]

\[ = (f, \Psi_f) - b(\Psi_f, p) - a(u_h, \Psi_f) + (a_h(u_h, \Psi_f) + b_h(\Psi_f, p_h) - f_h, \Psi_f) \]

\[ = [(f, \Psi_f) - (f_h, \Psi_f)] + [b_h(\Psi_f, p_h) - b(\Psi_f, p)] + [a_h(u_h, \Psi_f) - a(u_h, \Psi_f)] \]

\[ = (R_{21}) + (R_{22}) + (R_{23}). \]  

(77)

To bound term (R21), we use inequalities (34) and (35), the boundedness of the interpolation operator, and inequality (75), and we find that

\[ |(R_{21})| \leq C h \| u - u_h \|_{0, \Omega} \| u - u_h \|_{1, \Omega}. \]  

(78)

To derive an upper bound for term (R22), we first note that \( b_h(\Psi_f, p_h) = b(\Psi_f, p_h) \) from (32) and that we can subtract \( b(\Psi_f, p_h - p) = 0 \), which is zero since \( \text{div} \Psi = 0 \), cf. (72). Then, we use the Cauchy-Schwarz inequality, inequalities (73) and (75), and we find that

\[ |(R_{22})| = |b(\Psi_f, p_h - p)| = |b(\Psi_f, \Psi_f, p_h - p)| \leq |\text{div}(\Psi_f - \Psi)|_{0, \Omega} |p_h - p|_{0, \Omega} \leq C h |u - u_h|_{0, \Omega} |p_h - p|_{0, \Omega}. \]  

(79)

To estimate (R23), we first note that the local consistency property of the bilinear form \( a_h(\cdot, \cdot) \) implies that

\[ a_h^E(u_h, \Psi_f) - a^F(u_h, \Psi_f) = a_h^E(u_h - u_e, \Psi_f - \Psi_e) - a^F(u_h - u_e, \Psi_f - \Psi_e), \]  

(80)

where \( u_e \) and \( \Psi_e \) are suitable polynomial approximations of \( u \) and \( \Psi \) satisfying the assumptions of Lemma 4.3. Then, we use this identity, Lemmas 4.3 and 4.4 and inequality (75) to obtain the bound on (R23) as follows:

\[ |(R_{23})| = |a_h(u_h, \Psi_f) - a(u_h, \Psi_f)| = \sum_{E \in \Omega_h} \left| a_h^E(u_h - u_e, \Psi_f - \Psi_e) - a^E(u_h - u_e, \Psi_f - \Psi_e) \right| \]

\[ \leq (1 + \alpha^*) \sum_{E \in \Omega_h} |u_h - u_e|_{1, E} |\Psi_f - \Psi_e|_{1, E} \leq (1 + \alpha^*) \left( \sum_{E \in \Omega_h} |u_h - u_e|_{1, E}^2 \right)^{1/2} \left( \sum_{E \in \Omega_h} |\Psi_f - \Psi_e|_{1, E}^2 \right)^{1/2}. \]  

(81)
We add and subtract $u$ and $\Psi$, and use the triangular inequality to find that
\begin{equation}
|u_h - u_{\pi}|^2_{1,E} = (|u_h - u|_{1,E} + |u - u_{\pi}|_{1,E})^2 \leq 2|u_h - u|^2_{1,E} + 2|u - u_{\pi}|^2_{1,E}, \tag{82}
\end{equation}
\begin{equation}
|\Psi_I - \Psi_{\pi}|^2_{1,E} = (|\Psi_I - \Psi|_{1,E} + |\Psi - \Psi_{\pi}|_{1,E})^2 \leq 2|\Psi_I - \Psi|^2_{1,E} + 2|\Psi - \Psi_{\pi}|^2_{1,E}. \tag{83}
\end{equation}

Using inequalities (82), (83), (73), and (75), we find that
\begin{equation}
\left|\mathbf{R}_{23}\right|^2 \leq C\left(|u_h - u|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) \left(|\Psi_I - \Psi|_{1,\Omega} + |\Psi - \Psi_{\pi}|_{1,h}\right) \tag{73}
\end{equation}
\begin{equation}
\leq C\left(|u_h - u|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) h|\Psi|_{2,\Omega} \leq Ch\left(|u_h - u|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) ||u - u_h||_{0,\Omega}. \tag{84}
\end{equation}

We derive an upper bound for term $\mathbf{R}_3$ by using the Cauchy-Schwarz inequality, and the inequalities (74) and (75):
\begin{equation}
\left|\mathbf{R}_3\right| \leq Ch\left(|u_{h} - u_{\pi}|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) h|\Phi|_{2,\Omega} \leq Ch\left(|u_{h} - u_{\pi}|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) ||u - u_h||_{0,\Omega}. \tag{74}
\end{equation}
\begin{equation}
\leq C\left(|u_{h} - u_{\pi}|_{1,\Omega} + |u - u_{\pi}|_{1,h}\right) ||u - u_h||_{0,\Omega}. \tag{75}
\end{equation}

Finally, we note that term $\mathbf{R}_4$ is zero by using (13) and (20) (set $q = q_h = \varphi_I$):
\begin{equation}
\mathbf{R}_4 = b(u - u_h, \varphi_I) = b(u, \varphi_I) - b_h(u_h, \varphi_I) = 0. \tag{86}
\end{equation}

The assertion of the theorem follows by using the bounds of terms $\mathbf{R}_i$, for $i = 1, 2, 3$ and $\mathbf{R}_4 = 0$ to estimate the left-hand side of (70). Theorem 4.6 to bound the resulting term $u_h - u|_{1,\Omega} + ||p - p_h||_{0,\Omega}$ and Lemma 4.3 to bound $|u - u_{\pi}|_{1,h}$.

5 Numerical experiments

We assess the convergence property of the two virtual element method formulations considered in this paper by numerically solving problem (12)-(13) on the computational domain $\Omega = [0, 1] \times [0, 1]$. The Dirichlet boundary conditions and the source term are set accordingly to the manufactured solution $u = (u_x, u_y)^T$ and $p$ given by
\begin{align*}
u_x(x, y) &= \cos(2\pi x) \sin(2\pi y), \\
u_y(x, y) &= -\sin(2\pi x) \cos(2\pi y), \\
p(x, y) &= e^{x+y} - (e - 1)^2.
\end{align*}

Our implementation of the virtual element method uses the basis of orthogonal polynomials in all mesh elements, which is well-known to control the ill-conditioning of the final linear system very efficiently.

We run our virtual element solver on three mesh families respectively composed by random quadrilateral meshes ($\mathcal{M}1$), general polygonal meshes ($\mathcal{M}2$), and concave element meshes ($\mathcal{M}3$). The construction of these mesh families is rather standard in the literature of the VEM and its description can easily be found, for example, in [28]. For every mesh family, we consider five refinements. The base mesh and the first refined mesh of each family are shown in Figure 3; mesh data are reported in Tables 1 and 2.

| Level | $\mathcal{M}1$ | $\mathcal{M}2$ | $\mathcal{M}3$ |
|-------|---------------|---------------|---------------|
| 1     | $3.72 \cdot 10^{-1}$ | $4.26 \cdot 10^{-1}$ | $3.81 \cdot 10^{-1}$ |
| 2     | $1.99 \cdot 10^{-1}$ | $2.50 \cdot 10^{-1}$ | $1.91 \cdot 10^{-1}$ |
| 3     | $1.01 \cdot 10^{-1}$ | $1.25 \cdot 10^{-1}$ | $9.54 \cdot 10^{-2}$ |
| 4     | $5.17 \cdot 10^{-2}$ | $6.21 \cdot 10^{-2}$ | $4.77 \cdot 10^{-2}$ |
| 5     | $2.61 \cdot 10^{-2}$ | $3.41 \cdot 10^{-2}$ | $2.38 \cdot 10^{-2}$ |

Table 1
Diameter $h$ of meshes $\mathcal{M}1$, $\mathcal{M}2$, and $\mathcal{M}3$. On any set of refined meshes, we measure the $H^1$ relative error for the velocity vector field by applying the formula
\begin{equation}
\text{error}_{H^1}(\Omega)(u) = \frac{|u - \Pi_h^0 u_h|_{1,h}}{|u|_{1,\Omega}} \approx \frac{|u - u_h|_{1,\Omega}}{|u|_{1,\Omega}}, \tag{87}
\end{equation}
Fig. 3. Base meshes (top row) and first refinement meshes (bottom row) of the three mesh families used in this section: (\(\mathcal{M}_1\)) random quadrilateral meshes; (\(\mathcal{M}_2\)) general polygonal meshes; (\(\mathcal{M}_3\)) concave element meshes.

| Level | \(N_{el}\) | \(N_{el}\) | \(N_{el}\) | \(N_{el}\) |
|-------|---------|---------|---------|---------|
| 1     | 16      | 25      | 16      | 73      |
| 2     | 64      | 81      | 84      | 171     |
| 3     | 256     | 289     | 312     | 628     |
| 4     | 1024    | 1089    | 1202    | 2406    |
| 5     | 4096    | 4225    | 4772    | 9547    |

Table 2
Number of elements \(N_{el}\) and vertices \(N\) of meshes \(\mathcal{M}_1\), \(\mathcal{M}_2\), and \(\mathcal{M}_3\).

and the \(L^2\) relative error by applying the formula

\[
\text{error}_{L^2(\Omega)}(u) = \frac{||u - \Pi_k^0 u_h||_{0,\Omega}}{||u||_{0,\Omega}} \approx \frac{||u - u_h||_{0,\Omega}}{||u||_{0,\Omega}}.
\]  

(88)

For the pressure scalar field we measure the \(L^2(\Omega)\) relative error by applying the formula

\[
\text{error}_{L^2(\Omega)}(p) = \frac{||p - p_h||_{0,\Omega}}{||p||_{0,\Omega}}.
\]  

(89)

In our implementations, the use of the enhancement spaces only changes the calculation of the right-hand side of Eq. (19). In fact, in the implementations of \(F1\) and \(F2\) using the non-enhanced space definitions, we approximate the right-hand side through the projection operator \(\Pi_k^0\) with \(k = max(0, k-2)\), while in the ones using the enhanced space definitions, we approximate the right-hand side through the projection operator \(\Pi_k^0\). However, since the nonenhanced and the enhanced versions have the same degrees of freedom, we can always compute the projection operator \(\Pi_k^0\), and use it to evaluate the approximation error as in (87) and (88) above. In the non-enhanced case, this is equivalent to a sort of post-processing of \(u_h\), which is known only through its degrees of freedom, to derive a polynomial approximation of \(u\) that is defined on the whole computational domain.

**Convergence results.** In Figures 9, 10, 11, and 12, we compare the approximation errors (87), (88), and (89) that are obtained when using the non-enhanced and the enhanced definitions of the virtual element space for the velocity approximation. In particular, we recall that formulation \(F1\) uses the space definitions (36) (non-enhanced) and (37)
Fig. 4. Error curves versus $h$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers show the errors for the first formulation using space (36); solid (black) lines with triangular markers show the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_{\hat{k}}^0$ with $\hat{k} = \max(0, k - 2)$. The mesh families used in each calculation are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 5. Error curves versus $h$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers show the errors for the first formulation using space (36); solid (black) lines with triangular markers show the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_0^k$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 6. Error curves versus $N_{dof}$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers shows the errors for the first formulation using space (36); solid (black) lines with triangular markers shows the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_k^0$ with $k = \max(0, k - 2)$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 7. Error curves versus $N_{dof}$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers shows the errors for the first formulation using space (36); solid (black) lines with triangular markers shows the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi^0$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 8. $L^2$-norm of the divergence of the velocity field using the non-enhanced virtual element space (36) (top panels) and the enhanced virtual element space space (43) (bottom panels). The right-hand side (33) is approximated by using the projection operator $\Pi_k^0$. Solid (red and black) lines with square markers refer to $\Pi_k^{0,E}(\text{div } u_h)$; dotted (blue) lines with circle markers refer to $\Pi_k^{0,E}(\text{div } u_h)$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.

(enlarged); formulation $F2$ uses the space definitions (43) (non-enhanced) and (44) (enhanced). All error curves in Figures 9 and 10, for $k = 1, \ldots, 6$ are shown in a log-log plot versus the mesh size parameter $h$. All error curves in Figures 11 and 12, for $k = 1, \ldots, 6$ are shown in a log-log plot versus the total number of degrees of freedom $N^{\text{dof}}$. Solid (red) lines with square markers show the errors for the formulation $F1$; solid (black) lines with triangular markers show the errors for the formulation $F2$. The mesh family is shown in the bottom-left corner and the slopes of the error curves reflect the numerical order of convergence of each scheme.

When the error on the velocity approximation is measured using the energy norm, both formulations $F1$ and $F2$ provide the optimal convergence rate, which scales as $O(h^k)$ as expected from Theorem 4.6, regardless of using the non-enhanced or the enhanced versions of the method. An optimal convergence rate, this time scaling like $O(h^{k+1})$, is also visible for all the error curves of both formulations in the $L^2$-norm as expected from Theorem 4.7 when using the enhanced definition of the virtual element spaces and the projection operator $\Pi_k^0$ in the right-hand side of the VEM. Optimal convergence rates are also visible for both formulations $F1$ and $F2$ if $k \neq 2$ when using the non-enhanced versions of the virtual element spaces and the projection operator $\Pi_k^0$ with $k = \max(0, k-2)$. Instead, when $k = 2$ the non-enhanced formulations $F1$ and $F2$ lose one order of convergence. This fact is in agreement with the behavior previously noted in [8], where the optimal convergence rate for $k = 2$ was obtained by changing (in some sense, “enhancing”) the construction of the right-hand side. We also note that there is not a significant difference when we compare the accuracy of the two formulation with respect to the number of degrees of freedom, although we expect that formulation $F2$ can be more convenient than formulation $F1$ as it has a smaller number of degrees of freedom.

**Free-divergence condition.** Regarding the approximation of the zero-divergence constraint, the polynomial projection $\Pi_{k-1}^{0,E}(\text{div } u_h)$ is close to the machine precision in all elements $E \in \Omega_h$ for all the formulations and meshes here considered. Although we do not have a direct control on the divergence of the virtual element approximation, a straightforward calculation using the free-divergence condition for the ground truth, i.e., $\text{div } u = 0$, and an application of Theorem 4.6 yield

$$||\text{div } u_h||_{0,\Omega} = ||\text{div } (u_h - u)||_{0,\Omega} \leq C||u_h - u||_{1,\Omega} \approx O(h^k).$$
Fig. 9. Error curves versus $h$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers show the errors for the first formulation using space (36); solid (black) lines with triangular markers show the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi^0_k$ with $\bar{k} = \max(0, k - 2)$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 10. Error curves versus $h$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers show the errors for the first formulation using space (36); solid (black) lines with triangular markers show the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_0^k$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 11. Error curves versus $N_{dof}$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers shows the errors for the first formulation using space (36); solid (black) lines with triangular markers shows the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_k^0$ with $\bar{k} = \max(0, k - 2)$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
Fig. 12. Error curves versus $N_{dof}$ for the velocity approximation using the energy norm (87) (top panels) and the $L^2$-norm (88) (mid panels), and for the pressure approximation using the $L^2$-norm (89) (bottom panels). Solid (red) lines with square markers shows the errors for the first formulation using space (36); solid (black) lines with triangular markers shows the errors for the second formulation using space (43). The right-hand side is approximated by using the projection operator $\Pi_k^0$. The mesh families used in each calculations are shown in the left corner of each panel and the expected convergence rates are reflected by the slopes of the triangles and corresponding numeric labels.
So, we expect that the “true” divergence of the numerical approximation \( u_h \) scales like \( O(h^k) \) for \( h \to 0 \).

Furthermore, we note that for both formulations \( F1 \) and \( F2 \) the projections \( \Pi_k^\ell(\text{div} \, u_h) \), \( \ell = k, k + 1 \), are computable from the degrees of freedom of \( u_h \) when using the enhanced version of the two spaces. This fact allows us to post-process \( \text{div} \, u_h \) and obtain the polynomial projections \( \Pi_k^{0,E}(\text{div} \, u_h) \) and \( \Pi_{k+1}^{0,E}(\text{div} \, u_h) \) in every element \( E \in \Omega_h \), which, in principle, could be better approximations than \( \Pi_{k-1}^{0,E}(\text{div} \, u_h) \). However, it is worth noting that \( \Pi_{k-1}^{0,E}(\text{div} \, u_h) \) is expected to be zero (not considering rounding effects and the ill-conditioning of the discretization) and a straightforward calculation using the boundedness of \( \Pi_0^\ell \) and again the result of Theorem 4.6 shows that

\[
\|\Pi_k^\ell \text{div} \, u_h\|_{0,\Omega} = \|\Pi_k^\ell \text{div} \, (u_h - u)\|_{0,\Omega} \leq C\|\text{div} \, (u_h - u)\|_{0,\Omega} \leq C\|u_h - u\|_{1,\Omega} \approx O(h^k),
\]

where \( C \approx \|\Pi_k^\ell\| \). So, we cannot expect a real gain by pursuing this route although this estimate concerns with the worst case scenario and a convergence rate to zero faster than \( O(h^k) \) is still possible. This effect is illustrated by the different error curves that are obtained using the three mesh families \( M1, M2, \) and \( M3 \) and are shown in the log-log plots of Figure 13. In this figure, the three top panels are related to formulation \( F1 \); the solid (red) curves show the behavior of the \( L^2 \)-norm of \( \Pi_k^0(\text{div} \, u_h) \); the dotted (blue) curves show the behavior of the \( L^2 \)-norm of \( \Pi_{k+1}^0(\text{div} \, u_h) \). Here, the deviation from zero looks decreasing like \( O(h^k) \) in agreement with (90). The three bottom panels are related to formulation \( F2 \); the solid (black) curves show the behavior of the \( L^2 \)-norm of \( \Pi_k^1(\text{div} \, u_h) \); the dotted (blue) curves show the behavior of the \( L^2 \)-norm of \( \Pi_{k+1}^1(\text{div} \, u_h) \). Here, the deviation from zero looks decreasing at a rate that is closer to \( O(h^{k+1}) \) for \( k \neq 2 \) especially on mesh families \( M1 \) and \( M3 \), and intermediate between \( h^2 \) and \( h^3 \) for \( k = 2 \) when using mesh family \( M2 \).

6 Conclusions

We studied two conforming virtual element formulations for the numerical approximation of the Stokes problem to unstructured meshes that work at any order of accuracy. The components of the vector-valued unknown are approxi-
imated by using variants of the conforming regular or enhanced virtual element spaces that were originally introduced for the discretization of the Poisson equation. The scalar unknown is approximated by using discontinuous polynomials. The stiffness bilinear form is approximated by using the orthogonal polynomial projection of the gradients onto vector polynomials of degree \( k - 1 \) and adding a suitable stabilization term. The zero divergence constraint is taken into account by projecting the divergence equation onto the space of polynomials of degree \( k - 1 \). Our convergence analysis proves that the method is well-posed and convergent and optimal convergence rates are obtained through error estimates in the energy norm and in the \( L^2 \)-norm. Such optimal convergence rates are confirmed by numerical results on a set of three different representative families of meshes. These methods work well also in the lowest-order case (e.g., for the polynomial order \( k = 1 \)) on triangular and square meshes, which are well-known to be potentially unstable. Moreover, our numerical experiments show that the divergence constraint is satisfied at the machine precision level by the orthogonal polynomial projection of the divergence of the approximate velocity vector.

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