NOTE ON BRENDLE-EICHMAIR’S PAPER
“ISOPERIMETRIC AND WEINGARTEN SURFACES IN THE SCHWARZSCHILD MANIFOLD”

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Abstract. In this short note, we show that the assumption “convex” in Theorem 7 of Brendle-Eichmair’s paper [4] is unnecessary.

1. Introduction

For \( n \geq 3 \), let \( \lambda : [0, \bar{r}) \to \mathbb{R} \) be a smooth positive function which satisfies the following conditions (see [4]):

(H1) \( \lambda'(0) = 0 \) and \( \lambda''(0) > 0 \).

(H2) \( \lambda'(r) > 0 \) for all \( r \in (0, \bar{r}) \).

(H3) The function
\[
2 \frac{\lambda''(r)}{\lambda(r)} - \frac{(n - 2) - \lambda'(r)^2}{\lambda(r)^2}
\]

is non-decreasing for \( r \in (0, \bar{r}) \).

(H4) \( \frac{\lambda''(r)}{\lambda(r)} + \frac{1 - \lambda'(r)^2}{\lambda(r)^2} > 0 \) for all \( r \in (0, \bar{r}) \).

Now we consider the manifold \( M = \mathbb{S}^{n-1} \times [0, \bar{r}) \) equipped with a Riemannian metric \( \bar{g} = dr \otimes dr + \lambda(r)^2 g_{\mathbb{S}^{n-1}} \). Let \( \Sigma \) be a closed embedded star-shaped hypersurface in \( (M, \bar{g}) \), where star-shaped means that the unit outward normal \( \nu \) satisfies \( \langle \partial_r, \nu \rangle \geq 0 \). Denote by \( \sigma_p \) the \( p \)-th elementary symmetric polynomial of the principal curvatures. In fact, for this manifold \( (M, \bar{g}) \) Brendle and Eichmair proved the following theorem

**Theorem 1** (Theorem 7 of [4]). Let \( \Sigma \) be a closed embedded hypersurface in the manifold \( (M, \bar{g}) \) that is star-shaped and convex. If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( \mathbb{S}^{n-1} \times \{ r \} \) for some \( r \in (0, \bar{r}) \).

In this note, we show that the assumption “convex” in Theorem [4] is unnecessary. That is we have

**Theorem 2.** Let \( \Sigma \) be a closed, embedded and star-shaped hypersurface in the manifold \( (M, \bar{g}) \). If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( \mathbb{S}^{n-1} \times \{ r \} \) for some \( r \in (0, \bar{r}) \).

\[\text{Date: May 3, 2014.}\]
\[\text{The research of the authors was supported by NSFC No. 11271214.}\]
Note that the conditions (H1)-(H4) are all satisfied on the de Sitter-Schwarzschild manifolds (see [4]). So we have the following Corollary

**Corollary 3.** Let $\Sigma$ be a closed, embedded and star-shaped hypersurface in the de Sitter-Schwarzschild manifold $(M, \bar{g})$. If $\sigma_p$ is constant, then $\Sigma$ is a slice $\mathbb{S}^{n-1} \times \{r\}$ for some $r \in (0, \bar{r})$.

2. **Proof of Theorem 2**

In this section, by observing the existence of an elliptic point on $\Sigma$ and some basic facts about the function $\sigma_p$, we can remove the assumption “convex”.

Let $X = \lambda(r) \partial_r$. It is easy to see that $X$ is a conformal vector field satisfying $\nabla X = \lambda' \bar{g}$. Following the argument as Lemma 5.3 in [1], we have

**Lemma 4.** Let $\psi : \Sigma \to (M, \bar{g})$ be a closed hypersurface. Then there exists an elliptic point $x$ on $\Sigma$, i.e., all the principal curvatures are positive at $x$.

**Proof.** Let $h = \pi_I \circ \psi : \Sigma \to I$ be the height function on $\Sigma$, where $\pi_I$ is the projection $\pi_I(r, \theta) = r$. At any point $x \in \Sigma$, we have

$$\nabla h = (\nabla r)^\top = (\partial_r)^\top.$$  \hspace{1cm} (1)

Let $\{e_1, \cdots, e_{n-1}\}$ be a local orthonormal frame on $\Sigma$, and assume that the second fundamental form $h_{ij} = \langle \nabla_{e_i} \nu, e_j \rangle$ is diagonal with eigenvalues $\kappa_1, \cdots, \kappa_{n-1}$. Then

$$\nabla_{e_i} \nabla h = \nabla_{e_i} (\frac{1}{\lambda(h)} \lambda(h) \partial_r)^\top$$

$$= -\frac{\lambda'}{\lambda} (\nabla_{e_i} h) \partial_r^\top + \frac{1}{\lambda} \nabla_{e_i} (\lambda \partial_r^\top).$$  \hspace{1cm} (2)

Note that $X = \lambda \partial_r$ is a conformal vector field, we have

$$\nabla_{e_i} (\lambda \partial_r^\top) = \nabla_{e_i} (\lambda \partial_r - \langle \lambda \partial_r, \nu \rangle \nu)$$

$$= (\nabla_{e_i} (\lambda \partial_r - \langle \lambda \partial_r, \nu \rangle \nu))^\top$$

$$= \lambda' e_i - \langle \lambda \partial_r, \nu \rangle \kappa_i e_i.$$  \hspace{1cm} (3)

Substituting (3) into (2) gives that

$$\nabla_{e_i} \nabla h = -\frac{\lambda'}{\lambda} (\nabla_{e_i} h) \partial_r^\top + \frac{1}{\lambda} (\lambda' - \langle \lambda \partial_r, \nu \rangle \kappa_i) e_i.$$  \hspace{1cm} (4)

Now we consider the maximum point $x$ of $h$. We have $\nabla h = 0, \nu = \partial_r$ and $\nabla^2 h \leq 0$ at $x$. Then from (4), we get

$$\kappa_i \geq \frac{\lambda'}{\lambda} > 0, \quad i = 1, \cdots, n - 1,$$

i.e., $x$ is an elliptic point of $\Sigma$. \hfill $\square$
Remark 5. If we assume that the closed embedded hypersurface $\Sigma$ in $M$ satisfies $\langle \partial_r, \nu \rangle > 0$, then $\Sigma$ can be parametrized by a graph on $\mathbb{S}^{n-1}$ (see [5]):

$$\Sigma = \{(r(\theta), \theta) : \theta \in \mathbb{S}^{n-1}\}.$$ 

Define a function $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}$ by $\varphi(\theta) = \Phi(r(\theta))$, where $\Phi(r)$ is a positive function satisfying $\Phi' = 1/\lambda$. Let $\varphi_i, \varphi_{ij}$ be covariant derivatives of $\varphi$ with respect to $g_{\mathbb{S}^{n-1}}$. Define $v = \sqrt{1 + |\nabla \varphi|^2_{g_{\mathbb{S}^{n-1}}}}$. Then the same calculation as in Proposition 5 in [5] gives that the second fundamental form of $\Sigma$ has the expression

$$h_{ij} = \frac{\lambda'}{v \lambda} g_{ij} - \frac{\lambda}{v} \varphi_{ij},$$

where $g_{ij}$ is the induced metric on $\Sigma$ from $(M, \bar{g})$. At the maximum point $x$ of $\varphi$, we have $\varphi_i = 0, \varphi_{ij} \leq 0$. Then we have $h_{ij} \geq \frac{\lambda'}{\lambda} g_{ij}$, i.e., $x$ is an elliptic point of $\Sigma$. Note that the maximum point $x$ of $\varphi$ is also a maximum point of $r$.

Recall that for $1 \leq k \leq n - 1$ the convex cone $\Gamma_k^+ \subset \mathbb{R}^{n-1}$ is defined by

$$\Gamma_k^+ = \{ \vec{\kappa} \in \mathbb{R}^{n-1} | |\sigma_j(\vec{\kappa})| > 0 \text{ for } j = 1, \ldots, k \},$$

or equivalently

$$\Gamma_k^+ = \text{component of } \{ \sigma_k > 0 \} \text{ containing the positive cone.}$$

It is clearly that $\Gamma_k^+$ is a cone with vertex at the origin and $\Gamma_k^+ \subset \Gamma_{j}^+$ for $j \leq k$.

We write $\sigma_0 = 1, \sigma_k = 0$ for $k > n - 1$, and denote $\sigma_{k;i}(\vec{\kappa}) = \sigma_k(\vec{\kappa})|_{\kappa_i=0}$, i.e., $\sigma_{k;i}(\vec{\kappa})$ is the $k$-th elementary symmetric polynomial of $(\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_{n-1})$. Then we have the following classical result (see, e.g., [10] Lemma 2.3, [6,9]).

Lemma 6. If $\vec{\kappa} \in \Gamma_k^+$, then $\sigma_{k-1;j}(\vec{\kappa}) > 0$ for each $1 \leq i \leq n - 1$ and

$$\sigma_{j-1} \geq \frac{j}{n-j} \left(\frac{n-1}{j}\right)^{1/j} \sigma_j^{(j-1)/j}, \text{ for } 1 \leq j \leq k. \quad (5)$$

The following Lemma shows that on connected closed hypersurface in $(M, \bar{g})$, the positiveness of $\sigma_p$ implies that the principal curvatures $\vec{\kappa} \in \Gamma_p^+$.

Lemma 7. Let $\Sigma$ be a connected, closed hypersurface in $(M, \bar{g})$. If $\sigma_p > 0$ on $\Sigma$, then we have $\sigma_j > 0$ on $\Sigma$ for each $1 \leq j \leq p - 1$.

Proof. We believe that the proof of this Lemma can be found in literature, for example, see the proof of Proposition 3.2 in [3]. For convenience of the readers, we include the proof here. Lemma 4 implies that there exists an elliptic point $x$ on $\Sigma$. By continuity there exists an open neighborhood $\mathcal{U}$ around $x$ such that the principal curvatures are positive in $\mathcal{U}$. Hence $\sigma_k$ are positive in $\mathcal{U}$ for each $1 \leq k \leq n - 1$. Denote by $\mathcal{G}_j$ the connected component of the set $\{ x \in \Sigma : \sigma_j|_x > 0 \}$ containing $\mathcal{U}$.

Claim 8. For each $j$, we have $\mathcal{G}_{j+1} \subset \mathcal{G}_j$. 

Proof of the Claim. For each \( k \), define the open set
\[
\mathcal{V}_k = \bigcap_{j=1}^k G_j.
\]
It suffices to show that \( \mathcal{V}_k = G_k \). Since \( \sigma_j > 0 \) in \( \mathcal{V}_k \) for \( 1 \leq j \leq k \), Lemma 6 implies that at each point of this open set \( \mathcal{V}_k \) the inequalities (5) hold. By continuity (3) also hold at the boundary of \( \mathcal{V}_k \). If a point \( y \) of the boundary of \( \mathcal{V}_k \) belongs to \( G_k \), then (5) implies \( y \in G_j \) for each \( j \leq k \) and therefore belongs to \( \mathcal{V}_k \). This shows that the boundary of \( \mathcal{V}_k \) is contained in the boundary of \( G_k \). Since by definition \( \mathcal{V}_k \subset G_k \) and they are both open sets, \( G_k \) is connected, we have \( \mathcal{V}_k = G_k \). This completes the proof of the Claim. \( \square \)

Now we continue the proof of Lemma 7. We will show that \( G_{p-1} \) is closed. Pick a point \( y \) at the boundary of \( G_{p-1} \). By continuity \( \sigma_{p-1} \geq 0 \) at \( y \). Then Claim 8 implies that \( \sigma_j \geq 0 \) at \( y \) for each \( 1 \leq j \leq p-1 \). If \( \sigma_{p-1} = 0 \) at \( y \), by hypothesis \( \sigma_p > 0 \) and using Lemma 6, we have
\[
0 = \sigma_{p-1} \geq \frac{p}{n-p} \left( \frac{n-1}{p} \right)^{1/p} \sigma_{p-1}^{(p-1)/p} > 0,
\]
which is a contradiction. This implies \( \sigma_{p-1} \neq 0 \) at \( y \), and \( y \) belongs to the interior of \( G_{p-1} \). Therefore \( G_{p-1} \) is closed. Since it is also open, and then \( G_{p-1} = \Sigma \) by the connectedness of \( \Sigma \). Then Claim 8 shows that \( G_j = \Sigma \) for each \( 1 \leq j \leq p-1 \), this implies \( \sigma_j > 0 \) for \( 1 \leq j \leq p-1 \) on \( \Sigma \) and completes the proof of Lemma 7. \( \square \)

Now we can prove Theorem 2. As in [4], it suffices to prove the Heintze-Karcher-type inequality and Minkowski-type inequality.

If \( \sigma_p \) is a constant on \( \Sigma \), then Lemma 3 implies \( \sigma_p = \text{const} > 0 \). Denote by \( \vec{\kappa} = (\kappa_1, \ldots, \kappa_{n-1}) \) the principal curvatures of \( \Sigma \). Then Lemma 7 implies \( \vec{\kappa} \in \Gamma^+_1 \) on \( \Sigma \). Thus \( \vec{\kappa} \in \Gamma^+_1 \) and \( \Sigma \) is mean convex. So the Heintze-Karcher-type inequality
\[
(n-1) \int_\Sigma \frac{\lambda'}{H} \geq \int_\Sigma \langle X, \nu \rangle
\]
can be obtained as in [2].

On the other hand, we can prove

**Proposition 9** (Minkowski-type inequality). For \( 1 \leq p \leq n-1 \), suppose that \( \Sigma \) is star-shaped and \( \sigma_p > 0 \). Then
\[
p \int_\Sigma \langle X, \nu \rangle \sigma_p \geq (n-p) \int_\Sigma \lambda' \sigma_{p-1}
\]

*Proof.* Let \( \xi = X - \langle X, \nu \rangle \nu \) and \( T^{(p)}_{ij} = \frac{\partial \sigma_p}{\partial h_{ij}} \). Then
\[
\nabla_i \xi_j = \nabla_i X_j - \langle X, \nu \rangle h_{ij} = \lambda' \delta_{ij} - \langle X, \nu \rangle h_{ij}
\]
Therefore
\[
\sum_{i,j=1}^{n-1} \nabla_i (\xi_j T_{ij}^{(p)}) = \lambda' \sum_{i=1}^{n-1} T_{ii}^{(p)} - \sum_{i,j=1}^{n-1} T_{ij}^{(p)} (X, \nu) h_{ij} + \sum_{i,j=1}^{n-1} \xi_j \nabla_i T_{ij}^{(p)} - n \sum_{i=1}^{n-1} T_{ii}^{(p)} - n \sum_{i,j=1}^{n-1} T_{ij}^{(p)} (X, \nu) + \sum_{i,j=1}^{n-1} \xi_j \nabla_i T_{ij}^{(p)}
\]

(8)

Next as the proof of Proposition 8 in [4], we can get
\[
\sum_{i,j=1}^{n-1} \xi_j \nabla_i T_{ij}^{(p)} = -n - p \sum_{j=1}^{n-1} \sigma_{p-2,j} (\bar{k}) \xi_j Ric(e_j, \nu)
\]

By direct calculation, we have
\[
Ric(e_j, \nu) = - (n-2) \left( \frac{\lambda''(r)}{\lambda(r)} + \frac{1 - \lambda'(r)^2}{\lambda(r)^2} \right) \xi_j \langle \partial_r, \nu \rangle.
\]

Thus, using the assumption “star-shaped” \( \langle \partial_r, \nu \rangle \geq 0 \) and the condition (H4), we have \( \xi_j Ric(e_j, \nu) \leq 0 \) for each \( 1 \leq j \leq n - 1 \). On the other hand, from Lemma 7 and Lemma 6, \( \bar{k} \in \Gamma^+_{p-1} \) on \( \Sigma \) and \( \sigma_{p-2,j} (\bar{k}) > 0 \) for each \( 1 \leq j \leq n - 1 \). Therefore we have
\[
\sum_{i,j=1}^{n-1} \xi_j \nabla_i T_{ij}^{(p)} \geq 0.
\]

(9)

Putting (9) into (8) and integrating on \( \Sigma \), we get the Proposition 8. □

Once obtaining the Heintze-Karcher-type inequality (6) and the Minkowski-type inequality (7), we can go through the remaining proof as in [4], which completes the proof of Theorem 2.

APPENDIX A. FURTHER REMARK

Finally we give a remark about the generalization of Theorem 2. For \( n \geq 3 \), let \( (N, g_N) \) be a compact Einstein manifold of dimension \( n - 1 \) satisfying \( Ric_N = (n-2)Bg_N \) for some constant \( B \). Moreover, let \( \lambda : [0, \bar{r}) \to \mathbb{R} \) be a smooth positive function which satisfies the following conditions:

(H1)’ \( \lambda'(0) = 0 \) and \( \lambda''(0) > 0 \).
(H2)’ \( \lambda'(r) > 0 \) for all \( r \in (0, \bar{r}) \).
(H3)’ The function
\[
2 \frac{\lambda''(r)}{\lambda(r)} - (n-2) \frac{B - \lambda'(r)^2}{\lambda(r)^2}
\]

is non-decreasing for \( r \in (0, \bar{r}) \).
(H4)’ \( \frac{\lambda''(r)}{\lambda(r)} + \frac{B - \lambda'(r)^2}{\lambda(r)^2} > 0 \) for all \( r \in (0, \bar{r}) \).
Let manifold \( M = N \times [0, \bar{r}) \) with a Riemannian metric \( \bar{g} = dr \otimes dr + \lambda(r)^2 g_N \). By use of the similar arguments as proof of Theorem 2, we can obtain the following generalization of Theorem 2

**Theorem 10.** Let \( \Sigma \) be a closed, embedded and star-shaped hypersurface in the manifold \((M, \bar{g})\). If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( N \times \{r\} \) for some \( r \in (0, \bar{r}) \).

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