SIGNALIZERS AND BALANCE IN GROUPS
OF FINITE MORLEY RANK

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Abstract

There is a longstanding conjecture, due to Gregory Cherlin and Boris Zilber, that all simple groups of finite Morley rank are simple algebraic groups. The most successful approach to this conjecture has been Borovik’s program analyzing a minimal counterexample, or simple $K^*$-group. We show that a simple $K^*$-group of finite Morley rank and odd type is either algebraic or has Prüfer rank at most two. This result signifies a switch from the general methods used to handle algebraic groups, to the specialized methods which must be used to identify $PSL_2$, $PSL_3$, $PSp_4$, and $G_2$.

The Algebraicity Conjecture for simple groups of finite Morley rank, also known as the Cherlin-Zilber conjecture, states that simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields. In the last 15 years, the main line of attack on this problem has been the Borovik program of transferring methods from finite group theory. This program has led to considerable progress; however, the conjecture itself remains decidedly open. We divide groups of finite Morley rank into four types, odd, even, mixed, and degenerate, according to the structure of their Sylow 2-subgroups. For even and mixed type the Algebraicity Conjecture has been proven, and connected degenerate type groups are now known to have trivial Sylow 2-subgroups [13]. The present paper is part of the program to analyze a minimal counterexample to the conjecture in odd type, where the Sylow 2-subgroup is divisible-abelian-by-finite. It is the final paper in a sequence proving that such a minimal counterexample, or simple non-algebraic $K^*$-group, has Prüfer 2-rank at most two.

High Prüfer Rank Theorem 1. A simple $K^*$-group of finite Morley rank with Prüfer 2-rank at least three is algebraic.

This will be a consequence of the following so-called trichotomy, which is proved in the present paper. Here the traditional term “trichotomy” refers to the fact that there is also the Prüfer 2-rank $\leq 2$ case, which is largely unexplored at present.

Generic Trichotomy Theorem 2. Let $G$ be a simple $K^*$-group of finite Morley rank and odd type with Prüfer 2-rank $\geq 3$. Then either
1. $G$ has a proper 2-generated core, or
2. $G$ is an algebraic group over an algebraically closed field of characteristic not 2.

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The High Prüfer Rank Theorem then follows by applying the next two results.

**Strong Embedding Theorem 3 [14].** Let $G$ be a simple $K^*$-group of finite Morley rank and odd type, with normal 2-rank $\geq 3$ and Prüfer 2-rank $\geq 2$. Suppose that $G$ has a proper 2-generated core $M$. Then $G$ is a minimal connected simple group, and $M$ is strongly embedded.

**Minimal Simple Theorem 4 [19].** Let $G$ be a minimal connected simple group of finite Morley rank and of odd type. Suppose that $G$ contains a proper definable strongly embedded subgroup $M$. Then $G$ has Prüfer 2-rank one.

It may seem odd that the first of these results is appearing last. In fact, an earlier version of the trichotomy theorem began this sequence of developments. Namely, Borovik first proved the trichotomy theorem under a *tameness* assumption in [10], and the present author had explored eliminating tameness in [17]. In [10], Borovik produces the proper 2-generated core with a tame nilpotent signalizer functor theorem [15, Thm. B.30] (see also [17, Thm. 6.2]), an approach mirrored in the present paper. In [17], we show that the “most unipotent part” of a solvable signalizer functor is a nilpotent signalizer functor. This was believed to quickly eliminate tameness from [10]. However, more careful investigations revealed that obtaining a signalizer functor remained problematic.

In §2.1 of the present paper, we resolve this difficulty by constructing signalizer functors of a “sufficiently unipotent” reduced rank. The most serious obstacle is explained in Example 2.1. Our approach forces subsequent analysis to restrict itself to components of the centralizers of involutions which involve sufficiently large fields, a worrying but ultimately harmless restriction. Indeed, all complexities introduced by this approach are dispensed with in §2.1. This seems to be a different approach from that used by finite group theorists, who work with so-called *weakly balanced* signalizer functors [24, §29]; a similar method might work here as well. We call our approach partial balance.

The first section of this article covers necessary background material, including the definitions of a signalizer functor and the 2-generated core.

The second section contains the delicate definitions of partial balance, and of the associated family $\mathcal{E}_X$ of components from the centralizers of involutions. This section also contains a version of Asar’s theorem (Theorem 2.12) which states that $\mathcal{E}_X \neq \emptyset$, as well as a criterion for $\langle \mathcal{E}_X \rangle = G$ (Theorem 2.18). Borovik’s earlier unpublished work on the analysis of Lie rank two components [11] has heavily influenced this final result, although partial balance has given these results a more technical flavor.

The third section provides a suitable version of Berkman and Borovik’s Generic Identification Theorem [8]. It is the role of section two to verify the two hypotheses of this argument: reductivity for, and generation by, the centralizers of involutions. Our partial balance approach provides only a weak form of the reductivity hypothesis, which necessitates some alterations in the proof of the Generic Identification Theorem. All such critical changes are confined to §3.1, but there are important modifications throughout §3. The reader unfamiliar with the Generic Identification Theorem should consider exploring §3 before §1 or §2.

This is by no means the end of the story. The Prüfer 2-rank $\geq 3$ hypothesis used...
here is weaker than the normal 2-rank $\geq 3$ hypothesis originally used by Borovik [10]. As part of the ongoing program in odd type, [12] will show that Borovik’s original trichotomy holds, without the tameness hypothesis. We view [12] as a bridge between the “generic case” which is treated here, and the “quasi-thin” case (the identification of $\text{PSp}_4$, $G_2$, and $\text{PSL}_3$).

1. Preliminaries

This first section recalls various definitions and facts which are used throughout §2, and less pervasively in §3.

1.1. $K$-groups

We proceed, in this paper, by analyzing a so-called simple $K^*$-group of finite Morley rank. A $K^*$-group is a group whose proper definable simple sections are all algebraic. Similarly, a $K$-group is a group whose definable simple sections are all algebraic. So the proper subgroups of our $K^*$-group are clearly $K$-groups. One major $K$-group fact used throughout this article is the following generation principle.

**Fact 1.1** [10, Thm. 5.14]; see also [16, Thm. 3.25]. *Let $G$ be a connected $K$-group of finite Morley rank and odd type. Let $V$ be a four-subgroup acting definably on $G$. Then

$$G = \langle C_G^\circ(v) \mid v \in V^\# \rangle$$

Another major $K$-group fact worth recalling at the outset is the following “reductivity” criterion, which requires two definitions.

**Definition 1.2.** A quasisimple subnormal subgroup of a group $G$ is called a component of $G$ (see [15, p. 118 (2)]). We define $E(G)$ to be the connected part of the product of components of $G$, or equivalently the product of the components of $G^\circ$ (see [15, Lemma 7.10iv]). Such components are normal in $G^\circ$ by [15, Lemma 7.1iii], and indeed $E(G) \triangleleft G$.

**Definition 1.3.** The odd part $O(G)$ of a group $G$ of finite Morley rank is the maximal definable connected normal $2^\perp$-subgroup of $G$.

Clearly $O(G)$ is solvable if $G$ is a $K$-group.

**Fact 1.4** [10, Thm. 5.12]. *Let $H$ be a connected $K$-group of finite Morley rank and odd type with $O(H) = 1$. Then $H = F^\circ(H) \ast E(H)$ is isomorphic to a central product of quasisimple algebraic groups over algebraically closed fields of characteristic not 2 and of a definable normal divisible abelian group $F^\circ(H)$.*

This fact motivates signalizer functor theory, whose goal is to show that $O(H) = 1$, or something similar, when $H$ is the centralizer of an involution. The version we aim at here is Corollary 2.11 below.
1.2. Algebraic groups

A key tool in our program is the fact that a group of finite Morley rank acting faithfully as a group of automorphisms of an algebraic group must itself be algebraic.

Definition 1.5. Given an algebraic group $G$, a maximal torus $T$ of $G$, and a Borel subgroup $B$ of $G$ which contains $T$, we define the group $\Gamma$ of graph automorphisms associated to $T$ and $B$, to be the group of algebraic automorphisms of $G$ which normalize both $T$ and $B$.

Fact 1.6 [15, Thm. 8.4]. Let $G \rtimes H$ be a group of finite Morley rank where $G$ and $H$ are definable, $G$ an infinite quasi-simple algebraic group over an algebraically closed field, and $C^G_H(\langle \phi \rangle)$ is trivial. Then, viewing $H$ as a subgroup of $\text{Aut}(G)$, we have $H \leq \text{Inn}(G)\Gamma$, where $\text{Inn}(G)$ is the group of inner automorphisms of $G$ and $\Gamma$ is the group of graph automorphisms of $G$, relative to a fixed choice of Borel subgroup $B$ and maximal torus $T$ contained in $B$.

An algebraic group is said to be reductive if it has no unipotent radical. Such a group is a central product of semisimple algebraic groups and algebraic tori. The centralizer of an involution in a reductive algebraic group over a field of characteristic $\neq 2$ is itself reductive.

Fact 1.7 [35, Theorem 8.1]. Let $G$ be a quasisimple algebraic group over an algebraically closed field. Let $\phi$ be an algebraic automorphism of $G$ whose order is finite and relatively prime to the characteristic of the field. Then $C^G_\phi$ is nontrivial and reductive.

Proof. We know both that $C_H(\phi) \leq C_H(\phi \mod Q)$ as well as $[\phi, C_H(\phi \mod Q)] \leq Q$. There is a homomorphism $C_H(\phi \mod Q) \to Q$ given by $x \mapsto [\phi, x]$. As $Z(G)$ is finite, $t$ centralizes $C^G_\phi(\mod Q)$, as desired. So $C^G_{G/Z(G)}(\phi) = C^G_\phi(\mod Q)$. We may therefore assume that $G$ is its own universal central extension.

Since $\phi$ is algebraic and has finite order, $G \rtimes \langle \phi \rangle$ is an algebraic group which contains $\phi$ as an inner automorphism. Since the order of $\phi$ is finite and relatively prime to the characteristic, $\phi$ is a semisimple automorphism of $G$. So the result follows from Theorem 8.1 of [35].

Proof. We may assume that $G$ is its own universal central extension since $C^G_{G/Z(G)}(\phi) = C^G_\phi(\mod Q)$. Since $\phi$ is algebraic and has finite order, $G \rtimes \langle \phi \rangle$ is an algebraic group which contains $\phi$ as an inner automorphism. Since the order of $\phi$ is finite and relatively prime to the characteristic, $\phi$ is a semisimple automorphism of $G$. So the result follows from Theorem 8.1 of [35].

More specialized facts about algebraic groups will appear in §3.

1.3. Unipotent groups

While there is no intrinsic definition of unipotence in a group of finite Morley rank, there are various analogs of the “unipotent radical”: the Fitting subgroup, the $p$-unipotent operators $U_p$, for $p$ prime, and their “characteristic zero” analogs $U_{0,r}$ from [17, 16]. We recall their definitions below.
Definition 1.8. The Fitting subgroup $F(G)$ of a group $G$ of finite Morley rank is the subgroup generated by all its nilpotent normal subgroups.

The Fitting subgroup is itself nilpotent and definable [5, 30, 15, Thm. 7.3], and serves as a rough notion of unipotence in some contexts. However, the Fitting subgroup of a solvable group $H$ may not be contained in the Fitting subgroup of a solvable group containing $H$.

Definition 1.9. A connected definable $p$-subgroup of bounded exponent in a group $H$ of finite Morley rank is said to be $p$-unipotent. We write $U_p(H)$ for the subgroup generated by all $p$-unipotent subgroups of $H$.

Clearly $U_p(H)$ need not be solvable when $H$ is a non-solvable algebraic group in characteristic $p$; however, a $p$-unipotent $K$-group is solvable, and hence nilpotent by the following.

Fact 1.10 [22, Cor. 2.16]; [1, Fact 2.36]. Let $H$ be a connected solvable group of finite Morley rank. Then $U_p(H) \leq F^0(H)$ is itself $p$-unipotent, and hence nilpotent.

Thus the $p$-unipotent radical $U_p$ will automatically behave well, inside a solvable group. Its only weakness is that it may be trivial.

Fact 1.11 [15, Thm. 9.29 & §6.4]. Let $G$ be a connected solvable group of finite Morley rank. Then a Sylow $p$-subgroup $P$ of $G$ is connected, and $P = U_p(G) * T$ for a divisible abelian $p$-group $T$.

The present paper relies on the theory of “characteristic zero” unipotence introduced in [17]. We now turn our attention to this definition, as well as some facts from [17, 18, 16].

Definition 1.12. We say that a connected abelian group of finite Morley rank is indecomposable if it has a unique maximal proper definable connected subgroup, denoted $J(A)$ (see [17, Lemma 2.4]). We define the reduced rank $\bar{r}(A)$ of a definable indecomposable abelian group $A$ to be the Morley rank of the quotient $A/J(A)$, i.e. $\bar{r}(A) = \text{rk}(A/J(A))$. For a group $G$ of finite Morley rank, and any integer $r$, we define

$$U_{0,r}(G) = \left\{ A \leq G \mid \begin{array}{l} A \text{ is a definable indecomposable group,} \\
\bar{r}(A) = r, \text{ and } A/J(A) \text{ is torsion-free} \end{array} \right\}.$$ 

We say that $G$ is a $U_{0,r}$-group (alternatively $(0,r)$-unipotent group) if $U_{0,r}(G) = G$. We also set $\bar{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$.

We view the reduced rank parameter $r$ as a scale of unipotence, with larger values being more unipotent. By the following fact, analogous to Fact 1.10, the “most unipotent” groups, in this scale, are nilpotent.

Fact 1.13 [16, Thm. 2.21]; [17, Thm. 2.16]. Let $H$ be a connected solvable group of finite Morley rank. Then $U_{0,\bar{r}_0(H)}(H) \leq F(H)$. 

Fact 1.14 [18, Cor. 4.6]. Let $G = HT$ be a group of finite Morley rank, with $H$ and $T$ definable and nilpotent, and $H < G$. Suppose that $T$ is a $U_{0,r}$-group for some $r \geq \bar{r}_0(H)$. Then $G$ is nilpotent.

A good torus is a divisible abelian group of finite Morley rank whose definable connected subgroups are the definable hulls of their torsion. We arrive at a good torus when all our various notions of unipotence are trivial.

Fact 1.15 [16, Thm. 2.19]; [17, Thm. 2.15]. Let $H$ be a connected solvable group of finite Morley rank. Suppose $U_p(H) = 1$ for all $p$ prime, and $U_{0,\bar{r}_0(H)}(H) = 1$. Then $H$ is a good torus.

In a similar vein, the notion of $(0,r)$-unipotence provides a useful decomposition of a nilpotent group.

Fact 1.16 [18, Cor. 3.6]; [16, Thm. 2.31]. Let $G$ be a connected nilpotent group of finite Morley rank. Then $G = D*B$ is a central product of definable characteristic subgroups $D, B \leq G$ where $D$ is divisible and $B$ is connected of bounded exponent. Let $T$ be the torsion part of $D$. Then we have decompositions of $D$ and $B$ as follows.

$$D = d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots$$

$$B = U_2(G) \times U_3(G) \times U_5(G) \times \cdots$$

The next fact tells us when $q$-unipotence is preserved by taking centralizers, a fact used to produce a signalizer functor in Lemma 2.5 below.

Fact 1.17 [17, Fact 3.4]; [2]. Let $G$ be a connected solvable $p^\perp$-group of finite Morley rank, and let $P$ be a finite $p$-group of definable automorphisms of $G$. Then $C_G(P)$ is connected.

There is also a “characteristic zero” analog of the foregoing.

Fact 1.18 [17, Lemma 3.6]. Let $G$ be a nilpotent $(0,r)$-unipotent $p^\perp$-group of finite Morley rank, and let $P$ be finite $p$-group of definable automorphisms of $G$. Then $C_G(P)$ is $(0,r)$-unipotent.

In a similar vein, commutator subgroup of connected or $(0,r)$-unipotent groups tend to retain these properties.

Fact 1.19 [15, Cor. 5.29]. Let $H$ be a definable connected subgroup of a group $G$ of finite Morley rank and let $X \subset G$ be any subset of $G$. Then the group $[H,X]$ is definable and connected.

Fact 1.20 [18, Cor. 3.6]. Let $G$ be a solvable group of finite Morley rank, let $S \subset G$ be any subset, and let $H$ be a nilpotent $U_{0,r}$-group which is normal in $G$. Then $[H,S] \leq H$ is a $U_{0,r}$-group.
1.4. 2-Local structure

As the goal of our project is to constrain the 2-local structure, we need a few parameters to measure the complexity of a Sylow 2-subgroup. We define the 2-rank $m(G)$ of a group $G$ to be the maximum rank of its elementary abelian 2-subgroups. The Prüfer 2-rank $pr(G)$ is the maximum $k$ such that there is a Prüfer 2-subgroup $\mathbb{Z}(2^\infty)^k$ inside $G$, and the normal 2-rank $n(G)$ is the maximum 2-rank of a normal elementary abelian 2-subgroup of $G$. In an odd type group of finite Morley rank, these various ranks are all finite, and we have

$$m(G) \geq n(G) \geq pr(G).$$

These notions are well-defined because the Sylow 2-subgroups of a group of finite Morley rank are conjugate [9, 15, Thm. 10.11].

We use $\mathcal{E}_2(H)$ to denote the set of elementary abelian 2-subgroups $U \leq H$ with $m(U) \geq k$. We give $\mathcal{E}_2(H)$ a graph structure by placing an edge between $U, V \in \mathcal{E}_2(H)$ whenever $[U, V] = 1$. We say $H$ is 2-connected if the graph $\mathcal{E}_2(H)$ is connected, and we refer to the components of $\mathcal{E}_2(H)$ as 2-connected components otherwise.

**Fact 1.21** compare [4, 46.2]. Let $S$ be a locally finite 2-group. Then

1. If $m(S) > 2$ then the graph $\mathcal{E}_2(S)$ has a unique nonsingleton 2-connected component given by

$$\mathcal{E}^0_2(S) := \{X \in \mathcal{E}_2(S) : m(C_S(X)) > 2\},$$

and $\mathcal{E}^0_2(S)$ contains any $X \in \mathcal{E}_2(S)$ with $X \vartriangleleft S$.

2. If $n(S) > 2$ then $S$ is 2-connected.

**Proof.** Since $S$ is locally finite, this reduces to the finite case, found in [4, 46.2].

1.5. Proper 2-generated core

**Definition 1.22.** Consider a group $G$ of finite Morley rank and a 2-subgroup $S$ of $G$ with $m(S) \geq 3$. We define the 2-generated core $\Gamma_{S,2}(G)$ of $G$ (associated to $S$) to be the definable hull of the group generated by all normalizers of groups in $\mathcal{E}_2(S)$:

$$\Gamma_{S,2}(G) = d(\langle N_G(U) : U \in \mathcal{E}_2(S) \rangle)$$

We also define the weak 2-generated core $\Gamma^0_{S,2}(G)$ of $G$ (associated to $S$) to be the definable hull of all normalizers of groups in the nonsingleton 2-connected component $\mathcal{E}^0_2(S)$.

$$\Gamma^0_{S,2}(G) = d(\langle N_G(U) : U \in \mathcal{E}_2(S), m(C_S(U)) > 2\rangle).$$

We say that $G$ has a proper 2-generated core, or a proper weak 2-generated core, when, for a Sylow 2-subgroup $S$, $\Gamma_{S,2}(G) < G$ or $\Gamma^0_{S,2}(G) < G$, respectively.

Both notions of 2-generated core are well-defined, by the conjugacy of Sylow 2-subgroups. By Fact 1.21-2, the 2-generated core and the weak 2-generated core coincide when $n(G) \geq 3$, as is the case for much of the rest of this paper. When they differ, the weak 2-generated core is the more useful notion.
For an elementary abelian 2-group $V$ acting definably on $G$, we define $\Gamma_V(G)$ to be the group generated by the connected centralizers of involutions in $V$.

$$\Gamma_V(H) = \langle C_H^G(v) : v \in V^\# \rangle.$$  

**Proposition 1.23.** Let $G$ be a simple $K^*$-group of finite Morley rank and odd type, with $m(G) \geq 3$, and let $S$ be a Sylow 2-subgroup of $G$. Suppose that $\Gamma_E(G) < G$ for some $E \in \mathcal{E}_2^0(S)$. Then $G$ has a proper weak 2-generated core.

This depends on a lemma.

**Lemma 1.24.** Let $G$ be a simple $K^*$-group of finite Morley rank and odd type. Then $\Gamma_U(G) = \Gamma_V(G)$ for any $U, V$ in the same connected component of the graph $\mathcal{E}_2(G)$.

**Proof.** It is enough to prove the result for $U, V$ with $[U, V] = 1$. For any $v \in V^\#$, simplicity implies that $C_G^G(v)$ is a proper subgroup of $G$, and hence a $K$-group. Since $U$ normalizes $C_G^G(v)$, $C_G^G(v) = \Gamma_U(C_G^G(v))$ by Fact 1.1. So $\Gamma_V(G) \leq \Gamma_U(G)$, and the result follows by symmetry. \hfill $\Box$

**Proof of Proposition 1.23.** We may assume $E \leq S$ by conjugacy of Sylow 2-subgroups. Since involutions of $G$ have infinite centralizers by [15, Ex. 13 & 15 p. 79], the result will follow from the following claim, and simplicity.

$$\Gamma_{S,2}^0(G) \leq N_G(\Gamma_E(G)) \quad \text{for any } E \in \mathcal{E}_2^0(S)$$

By Lemma 1.24 and Fact 1.21-1, $\Gamma_E(G) = \Gamma_U(G)$ for any $U \in \mathcal{E}_2^0(S)$. For any $U \in \mathcal{E}_2^0(S)$,

$$N_G(U) \leq N_G(\Gamma_U(G)) = N_G(\Gamma_E(G)).$$

Thus $\Gamma_{S,2}^0(G) \leq N_G(\Gamma_E(G))$, as desired. \hfill $\Box$

We will encounter a variation of the preceding in the next section (see Lemma 2.15 and Proposition 2.17).

The following black hole principle for proper 2-generated cores reverses the roles of the subgroups $\Gamma_{S,2}^0(G)$ and $\Gamma_E(G)$ in Proposition 1.23.

**Lemma 1.25.** Let $G$ be an infinite simple $K^*$-group of finite Morley rank and odd type, and let $S$ be a 2-subgroup of $G$ satisfying $m(S) \geq 3$. Then $C_S^G(x) \leq \Gamma_{S,2}^0(G)$ for every $x \in I(S)$ with $m(C_S(x)) > 2$.

**Proof.** There is an $E \in \mathcal{E}_2^0(S)$ with $x \in E$ and $m(E) \geq 3$ by Fact 1.21-1. So there is an $E_1 \in \mathcal{E}_2^0(S)$ with $E_1 \leq E$ and $E_1 \cap \langle x \rangle = 1$. For any $y \in E_1^\#$, we have $C_{CG(x)}(y) \leq C_G(y, x)$ and $\langle y, x \rangle \in \mathcal{E}_2^0(S)$. By simplicity, Fact 1.1 yields

$$C_G^G(x) = \Gamma_{E_1}(C_G^G(x)) \leq \Gamma_{S,2}^0(G)$$

\hfill $\Box$
In particular, given a simple $K^*$-group $G$, Lemma 1.25 says
\[ \Gamma_E(G) \leq \Gamma_{E,2}(G) \] for any $E \in \mathcal{E}_3(G)$.

Now Proposition 1.23 and Lemma 1.25 yield the following.

**Proposition 1.26.** Let $G$ be a simple $K^*$-group of finite Morley rank and odd type, with $m(G) \geq 3$, and let $S$ be a Sylow 2-subgroup of $G$. If $\Gamma_{E,2}(G) < G$ for some $E \in \mathcal{E}_3(G)$, then $G$ has a proper weak 2-generated core, i.e. $\Gamma_{S,2}^0(G) < G$.

### 1.6. Signalizer functors

Signalizer functors are used in both the finite case and in the finite Morley rank case to produce a dichotomy between a proper 2-generated core, and a reductivity condition for centralizers of involutions.

**Definition 1.27.** Consider a group $G$ of finite Morley rank, and an elementary abelian 2-subgroup $E \leq G$. An $E$-signalizer functor on $G$ is a family $\{ \theta(s) \}_{s \in E^\#}$ of definable $E$-invariant 2-⊥-subgroups of $G$ satisfying:

- a. $\theta(s) \triangleleft C_G(s)$ for each $s \in E^\#$.
- b. $\theta(s) \cap C_G(t) \leq \theta(t)$ for any $s, t \in E^\#$.

We observe that the second condition is equivalent to the "balance" condition
\[ \theta(s) \cap C_G(t) = \theta(t) \cap C_G(s) \] for any $s, t \in E^\#$.

In practice, we will only be interested in signalizer functors satisfying the following stronger invariance condition, which is used to produce a proper 2-generated core.

\[ \theta(s)^g = \theta(s^g) \] for all $s \in E^\#$ and all $g \in G$ for which $s^g \in E$.  \((\dagger)\)

As one would expect, we say $\theta$ is a connected or nilpotent signalizer functor if the groups $\theta(s)$ are connected or nilpotent, respectively, for all $s \in E^\#$.

We now show that signalizer functors yield a proper weak 2-generated core.

**Theorem 1.28.** Let $G$ be a simple $K^*$-group of finite Morley rank and odd type, and let $S$ be a Sylow 2-subgroup of $G$. Suppose that, for some $E \in \mathcal{E}_3(S)$, $G$ admits a nontrivial connected nilpotent $E$-signalizer functor $\theta$ satisfying $(\dagger)$. Then $G$ has a proper weak 2-generated core.

The key fact underlying this result is the Nilpotent Signalizer Functor Theorem.

**Definition 1.29.** We say that an $E$-signalizer functor on a group $G$ of finite Morley rank is complete if:

- a. $\theta(E) = \langle \theta(s) : s \in E^\# \rangle$ is a solvable $p^\perp$-group, and
- b. $\theta(s) = C_{\theta(E)}(s)$ for any $s \in E^\#$.

### Nilpotent Signalizer Functor Theorem 5 [10, 17, 15, Theorem B.30].

Let $G$ be a group of finite Morley rank, and let $E \leq G$ be a finite elementary abelian 2-group of rank at least 3. Let $\theta$ be a connected nilpotent $E$-signalizer functor. Then $\theta$ is complete and $\theta(E)$ is nilpotent.
We shall work with the proper group $\theta(E)$ in the same manner as we did with $\Gamma_E(G)$ in Proposition 1.23.

We also recall a variation on Fact 1.1.

**Fact 1.30** [17, Fact 3.7]. Let $H$ be a solvable $p^\perp$-group of finite Morley rank. Let $E$ be a finite elementary abelian $p$-group acting definably on $H$. Then

$$H = \langle C_H(E_0) \mid E_0 \leq E, [E : E_0] = p \rangle$$

**Proof of Theorem 1.28.** It suffices to show that $\Gamma_{E,2}(G) < G$ by Proposition 1.26. The Nilpotent Signalizer Functor Theorem says that $\theta$ is complete and $\theta(E)$ is nilpotent. Since $G$ is simple, our result will follow from $\Gamma_{E,2}(G) \leq N_G(\theta(E))$.

For any $U, V \in \mathcal{E}_2(E)$, we have that $\theta(U) = \theta(V)$ because

$$\theta(u) \leq \langle C_{\theta(u)}(v) \mid v \in V^\# \rangle \leq \theta(V) \quad \text{for any } u \in U.$$

by Fact 1.30 and the signalizer functor property. Thus $\theta(U) = \theta(E)$. For any $U \in \mathcal{E}_2(E)$ and any $g \in N_G(U)$, our hypothesis (†) yields

$$\theta(E)^g = \theta(U)^g = \theta(U^g) = \theta(U) = \theta(E).$$

Thus $\Gamma_{E,2}(G) \leq N_G(\theta(E)) < G$, as desired.

2. **Balance and components**

In the tame setting of [10], Borovik states that $O(C_G(i))$ is a nilpotent signalizer functor. In view of Theorem 1.28, it then follows that either $G$ has a proper 2-generated core, if $O(C_G(i)) \neq 1$, or else $C_G(i)$ is “reductive” in the sense of Fact 1.4.

It also follows, from Proposition 1.23, that either $G$ has a proper 2-generated core, or else $\Gamma_{\Omega_1(S^+)}(G) = G$. These two facts constitute the reductivity and generation conditions of the Generic Trichotomy Theorem [8], so a tame version of the Generic Trichotomy Theorem then follows. In fact, [8] uses these two conditions merely to establish that $G$ is generated by the quasisimple components of the centralizers of toral involutions, and hence by their root $\text{SL}_2$-subgroups. The remainder of the argument focuses on these root $\text{SL}_2$-subgroups, treating them as an abstract family of root $\text{SL}_2$-subgroups for $G$, and eventually applying the Curtis-Tits Theorem.

In this section, we turn our attention towards “unbalanced groups” where $O(C_G(i))$ is not necessarily a signalizer functor, in order to eliminate the hypothesis of tameness. Instead, we use the “most unipotent” parts of $O(C_G(i))$ as signalizer functors. In Theorem 2.9, these signalizer functors are used to prove a dichotomy between a proper 2-generated core, and our $\tilde{B}$-property (see §2.1 below). Corollary 2.11 then provides a limited form of the reductivity proved in Fact 1.4. However, this weaker form of reductivity does not admit such a quick proof of generation by components. So our version of this result, Theorem 2.18 below, requires a considerably more delicate argument.

2.1. **Partial balance**

We require an example to explain the failure of balance.
Example 2.1. Consider a field \((k,T,+,\cdot)\) of finite Morley rank, with \(T < k^*\) torsion-free, and \(G := SO_4(k)\) \((D_4)\). By Table 4.3.1 on p. 145 of [26], there are involutions \(i,j\) in \(G\), lying in a common torus, such that

\[
C_G(i) \cong SL_4(k) \ast k^* \quad \text{and} \quad C_G(j) \cong SL_3(k) \ast SL_3(k)
\]

So \(O(C_G(i)) = O(k^*) = T \neq 1\) and \(O(C_G(j)) = 1\). However, every inner involutive automorphism of \(SL_m\) is a central product with one copy of \(k^*\). So \(O(C_G(\cdot))\) is not a signalizer functor. In fact, the reductivity hypothesis of Fact 1.4 fails too, although its conclusions still holds since the centralizer is still reductive.

Our solution to this is to choose a reduced rank \(\tilde{r}^* (\cdot)\) which is the largest possible problematic reduced rank in \(k^*\), and work above it by using the fact that \(rk(k^*) > \tilde{r}_0(k^*)\).

Definition 2.2. Consider a simple \(K^*\)-group \(G\) of finite Morley rank and let \(X\) be a subgroup of \(G\) with \(m(X) \geq 3\). We write

\[
I^0(X) := \{i \in I(X) : m(C_X(i)) \geq 3\}
\]

for the set of involutions from eight-groups in \(E_4(X)\). We define

\[
\tilde{r}^O(X) := \sup\{\tilde{r}_0(O(C_G(i))) : i \in I^0(X)\}
\]

as the supremum of the reduced ranks of the odd parts of the centralizers in \(G\) of the involutions in \(I^0(X)\). We also define \(\tilde{r}^* (X)\) to be the supremum of \(\tilde{r}_0(k^*)\) as \(k\) ranges over the base fields of the quasisimple components of the quotients \(C_G^0(i)/O(C_G(i))\) associated to involutions \(i \in I^0(X)\).

One can easily check that

\[
\tilde{r}^O(G) = \max_{E \in E_4(G)} \tilde{r}^O(E) \quad \text{and} \quad \tilde{r}^* (G) = \max_{E \in E_4(G)} \tilde{r}^* (E).
\]

We recall that, for a nonsolvable group \(L\) of finite Morley rank, \(U_{0,r}(L)\) and \(U_p(L)\) need not be solvable, as quasisimple algebraic groups are generated by the unipotent radicals of their Borel subgroups. Proposition 2.13 below will shed further light on the definition of \(\tilde{r}^* (\cdot)\) by providing a converse to this observation.

Definition 2.3. We continue in the notation of Definition 2.2. For a definable subgroup \(H\) of \(G\), we define \(\hat{U}_X(H)\) to be the subgroup of \(H\) generated by \(U_p(H)\) for \(p\) prime as well as by \(U_{0,r}(H)\) for \(r > \tilde{r}^* (X)\). As an abbreviation, we use \(\hat{F}_X(H)\) to denote \(F^\infty(\hat{U}_X(H))\), and \(\hat{E}_X(H)\) to denote \(E(\hat{U}_X(H))\). We use \(\hat{E}_X^\infty\) to denote the set of components of \(E_X(C_G(i)) = E(\hat{U}_X(C_G(i)))\) for \(i \in I^0(Y)\) with \(Y \leq X\), and we set \(\hat{E}_X = \hat{E}_X^\infty\).

\(\hat{U}_X(H)\) is the subgroup of \(H\) which is generated by its unmistakably unipotent subgroups. These definitions are all sensitive to the choice of \(X\), which is usually a fixed eight-group.

Definition 2.4. We say that a simple \(K^*\)-group \(G\) with \(m(G) \geq 3\) satisfies the \(\bar{B}\)-property if, for every \(2\)-subgroup \(X\) \(\leq G\) with \(m(X) \geq 3\) and every \(t \in I^0(X)\), the group \(\hat{U}_X(O(C_G^0(t)))\) is trivial. This is equivalent to
(B.-1) \( U_p(O(C_G(t))) = 1 \) for all \( t \in I^0(G) \) and every prime \( p \).
(B.-2) \( \bar{\rho}^O(X) \leq \bar{\rho}^*(X) \) for every 2-subgroup \( X \leq G \) with \( m(X) \geq 3 \), and

The \( \bar{B} \)-property is an unbalanced alternative to Borovik’s \( B \)-conjecture: that \( O(C_G(i)) = 1 \) for all \( i \in I(G) \). Although the \( \bar{B} \)-property is significantly more delicate than the strong \( B \)-property, the next two subsections will establish results about the components in \( \bar{E}_X \) which are similar to Borovik’s.

Our goal in this subsection is to verify that the failure of the \( \bar{B} \)-property leads to a proper weak 2-generated core. For this, we need two appropriate signalizer functors.

**Lemma 2.5.** Let \( G \) be a simple \( K \)-group of finite Morley rank and odd type with \( m(G) \geq 3 \), and let \( E \in E_3(G) \). Then \( \{ U_p(O(C_G(t))) \mid t \in E^\# \} \) is a connected nilpotent \( E \)-signalizer functor satisfying

\[
\theta(s)^s = \theta(s^s) \quad \text{for all} \quad s \in E^\# \quad \text{and} \quad g \in G \quad \text{for which} \quad s^g \in E.
\]

We need the following two facts.

**Fact 2.6** [15, Ex. 11 p. 93 & Ex. 13c p. 72]. Let \( G \) be a group of finite Morley rank and let \( H \triangleleft G \) be a definable subgroup. If \( x \in G \) is an element such that \( \bar{x} \in G/H \) is a \( p \)-element, then \( xH \) contains a \( p \)-element. Furthermore, if \( H \) and \( G/H \) are \( p^+ \)-groups, then \( G \) is a \( p^+ \)-group.

**Fact 2.7** [2, 17, Fact 3.2]. Let the group \( G = H \ltimes T \) be a semidirect product of finite Morley rank. Suppose \( T \) is a solvable \( \pi \)-group of bounded exponent and \( Q \triangleleft H \) is a definable solvable \( \pi^+ \)-subgroup. Then

\[
C_H(T)Q/Q = C_{H/Q}(T).
\]

**Proof of Lemma 2.5.** Let \( \theta(t) := U_p(O(C_G(t))) \). We observe that \( \theta(s)^s = \theta(s^g) \) for every involution \( s \in I(G) \) and every \( g \in G \). By Fact 1.10, \( \theta(s) \) is nilpotent. \( \theta(s) \) is connected by definition.

Let \( s, t \in E^\#; \) in particular \( [s, t] = 1 \). Also let \( K_s = O(C_G(s)) \). Since \( C_G(s) \) is a \( K \)-group, Fact 1.4 says \( C_G^\circ(s)/K_s = G_1 \times \cdots \times G_n \rtimes F \) is the central product of finitely many quasisimple algebraic groups \( G_1, \ldots, G_n \) and of a definable divisible abelian group \( F \). Since \( F \) is abelian, \( O(C_G^\circ(t)) \) is \( 1 \) by Fact 2.6. We now consider the action of \( t \) on the components. For any component \( G_k \), either \( t \) normalizes \( G_k \), or else \( t \) swaps \( G_k \) with another component \( G_i = G_i^k \). In the second case, the centralizer of \( t \) is some diagonal subgroup of \( G_k^i \rtimes G_k \), i.e. \( C_{G_k^i}(t) \cong \{(g, \sigma(g)) \mid g \in G_k \} \cong G_k \) for some automorphism \( \sigma \) of \( G_k \). So we may assume that \( t \) normalizes each \( G_k \) with \( k \leq m \), and \( C_{G_{m+1} \times \cdots \times G_n}(t) \cong G_{n+m} \rtimes \cdots \rtimes G_n \).

By Facts 1.6 and 1.7, \( C_{G_k^i}(t) \) is reductive for \( k \leq m \). So \( O(C_{G_k^i}(t)) \) is a subgroup of an algebraic torus, and hence divisible abelian. Hence \( U_p(O(C_{G_{m+1} \times \cdots \times G_n}(t))) = 1 \).

So \( U_p(O(C_{G_k^i}(t)/K_k(t))) = 1 \). Since \( C_{G_k^i}(t)K_k/K_k = C_{G_k^i}(s)/K_k(t) \) by Fact 2.7, \( U_p(O(C_{G_k^i}(t)/K_k(t))) = U_p(O(C_{G_k^i}(t))) \leq K_k \).

For any \( t \in E^\# \), the group \( \bar{\theta}(t) \) is \( p \)-unipotent, and so \( 2^+ \) and nilpotent. By Fact 1.17, \( C_{\theta(t)}(s) \) is a connected. So \( C_{\theta(t)}(s) \leq U_p(O(C_{\theta(t)}(s))) \leq U_p(O(C_{C_G(t)}(s))) \leq U_p(K_s) = \theta(s) \).
Lemma 2.8. Let $G$ be a simple $K^*$-group of finite Morley rank and odd type with $m(G)\geq 3$. Let $E \in \mathcal{E}_3(G)$ and set $r := \bar{r}^O(E)$. If $r > \bar{r}^*(E)$ then $U_{0,r}(O(C_G(t)))$ is a connected nilpotent $E$-signalizer functor again satisfying \( (\dagger) \).

Proof. Let $\theta(t) := U_{0,r}(O(C_G(t)))$. We observe that $\theta(s)^g = \theta(s^g)$ for every involution $s \in I(G)$ and every $g \in G$. $\theta(s)$ is clearly connected and solvable. So $\theta(t)$ is nilpotent by Theorem 1.13.

Let $s, t \in E^\#$; in particular $|s, t| = 1$. Also let $K_s = O(C_G(s))$. Since $C_G(s)$ is a $K$-group, Fact 1.4 says $C_G(s)/K_s = G_1 \cdots G_n * F$ is the central product of finitely many quasisimple algebraic groups $G_1, \ldots, G_n$ and of a definable divisible abelian group $F$. Since $F$ is abelian, $O(C_G^s(t)) = 1$. We next show that $U_{0,r}(O(C^s_{G_1,\ldots,G_n}(t))) = 1$. For any component $G_k$, either $t$ normalizes $G_k$, or else $t$ swaps $G_k$ with another component $G_l = G_k$. In the second case, the centralizer of $t$ is some diagonal subgroup of $G_k \times G_l$, i.e. $C(t) \cong \{ (g, \sigma(g)) \mid g \in G_k \} \cong G_k$ for some automorphism $\sigma$ of $G_k$. So we may assume that $t$ normalizes each $G_k$ with $k \leq m$, and $C_{G_{m+1},\ldots,G_n}(t) \cong G_{m+1} \cdots G_n$. Consider a connected definable indecomposable abelian subgroup $A$ of $O(C_{G_1,\ldots,G_n}(t))$ with $A/J(A)$ torsion-free. Then there is an $k \leq m$ with a nontrivial projection map $\pi : A \to O(C_{G_k/Z(G_k)}(t))$. By [17, Lemma 2.9], the image $\pi(A)$ is also indecomposable abelian, and $\bar{r}_0(\bar{A}) = \bar{r}_0(\pi(A))$. By Facts 1.6 and 1.7, $C_{G_k/Z(G_k)}(t)$ is reductive, and hence $O(C_{G_k/Z(G_k)}(t))$ is an algebraic torus. It follows that

$$\bar{r}_0(A) = \bar{r}_0(\pi(A)) \leq \bar{r}_0(O(C_{G_k/Z(G_k)}(t))) \leq \bar{r}^*(E).$$

Since $r > \bar{r}^*(E)$, we have $U_{0,r}(O(C_{G_1,\ldots,G_n}(t))) = 1$. Since $C_{G_1,\ldots,G_n}(t)K_s/K_s = C_{C_{G_1,\ldots,G_n}(t)}(t)$ by Fact 2.7, $U_{0,r}(O(C_{C^s_{G_1,\ldots,G_n}(t)}(t))) = U_{0,r}(O(C_{C_{G_1,\ldots,G_n}(t)}(t))) \leq K_s$.

For any $t \in E^\#$, the centralizer $C_{\theta(t)}(s)$ is a connected $(0, r)$-nilpotent $2$-group by Fact 1.18. Thus

$$C_{\theta(t)}(s) \leq U_{0,r}(O(C_{C^s_{G_1,\ldots,G_n}(t)}(t))) \leq U_{0,r}(K_s) = \theta(s),$$

and $\theta$ is a signalizer functor.

We can now verify the $\tilde{B}$-property, in the absence of a proper $2$-generated core.

Theorem 2.9. Let $G$ be a simple $K^*$-group of finite Morley rank and odd type with $m(G) \geq 3$. Then either

1. $G$ has a proper weak $2$-generated core, or else

2. $G$ satisfies the $\tilde{B}$-property, i.e. $\bar{U}_X(O(C_G(t))) = 1$ for every $2$-subgroup $X \leq G$ with $m(X) \geq 3$ and every $t \in I^0(X)$.

Proof. We first suppose that $(\tilde{B}-1)$ fails, i.e. $U_q(O(C_G(t))) \neq 1$ for some involution $i \in I^0(G)$. There is an $E \in \mathcal{E}_3(G)$ containing $i$. By Lemma 2.5, $\theta(t) := U_p(O(C_G(t)))$ is a connected nilpotent $E$-signalizer functor satisfying \( (\dagger) \). So $G$ has a proper weak $2$-generated core by Theorem 1.28.

We next suppose that $(\tilde{B}-2)$ fails, i.e. $\bar{r}^O(X) > \bar{r}^*(X)$ for some $2$-subgroup $X \leq G$ with $m(X) \geq 3$. There is an $E \in \mathcal{E}_3(X)$ such that $\bar{r}^O(E) = \bar{r}^O(X)$. Let $\theta(t) := U_{0,r}(O(C_G(t)))$ where $r := \bar{r}^O(E)$ is the largest reduced rank appearing inside $O(C_G(t))$ for involutions $i \in E^\#$. Now $\theta(t) := U_{0,r}(O(C_G(t)))$ is a connected nilpotent $E$-signalizer functor satisfying \( (\dagger) \) by Lemma 2.8. By the choice of $r$, $\theta(t)$
is nontrivial for some involution \( i \in E^{\#} \). So \( G \) again has a weak proper 2-generated core by Theorem 1.28.

\[ \square \]

2.2. **Existence of components in \( \hat{E}_X \)**

In this subsection, we will use the \( \hat{B} \)-property to show that \( G \) is a group “of component type” in the sense that

\[ \hat{E}_X \neq \emptyset \] for every 2-subgroup \( X \leq G \) with \( m(X) \geq 3 \).

We employ the \( p \)-unipotent and 0-unipotent signalizer functors found in §2.1, via Theorem 2.9. Our major tool will be the following analog of Fact 1.4 which allows us to exploit the \( \hat{B} \)-property.

**Lemma 2.10.** Let \( H \) be a \( K \)-group of finite Morley rank and odd type, and let \( \hat{H} \) be the subgroup of \( H \) generated by

a. \( U_{0,r}(H) \) for \( r > \bar{r}_0(O(H)) \), as well as

b. \( U_p(H) \) for any prime \( p \) satisfying \( U_p(O(H)) = 1 \).

Then \( \hat{H} = E(\hat{H}) \ast F^\circ(\hat{H}) \) and \( F^\circ(\hat{H}) \) is abelian.

**Proof.** We first show that \([F(O(H)), \hat{H}] = 1\). Let \( A \) be a definable connected nilpotent subgroup of \( H \) with either

a. \( U_{0,r}(A) = A \) for some \( r > \bar{r}_0(O(H)) \), or

b. \( U_p(A) = A \) for some prime \( p \) for which \( U_p(O(H)) = 1 \).

Then \( A \cdot F(O(H)) \) is nilpotent by Fact 1.14. Since \( r > \bar{r}_0(O(H)) \), we have \([A, F(O(H))] = 1\) by Fact 1.16. So \([F(O(H)), \hat{H}] = 1\).

Since \( F(O(\hat{H})) \) is definably characteristic in \( \hat{H} \), \( F(O(\hat{H})) \leq F(O(H)) \), so \( F(O(\hat{H})) \leq Z(\hat{H}) \). Hence \( O(\hat{H}) \) is nilpotent, and

\[ O(\hat{H}) = F(O(\hat{H})) \leq Z(\hat{H}). \] (\( \ast \))

Consider \( K := \hat{H}/O(\hat{H}) \). By Fact 1.4, \( K = E(K) \ast F(K) \) and \( F(K) \) is abelian. By (\( \ast \)), the inverse image of \( F(K) \) in \( \hat{H} \) is nilpotent, and thus equals with \( F^\circ(\hat{H}) \). As any quasi-simple component \( L \) of \( E(K) \) is perfect, such a component \( L \) admits only finite central extensions by \([3]\). By (\( \ast \)), the inverse image \( \hat{L} \) of \( L \) in \( \hat{H} \) is isomorphic to a central product \( L \ast O(\hat{H}) \), which thus contains a component of \( \hat{H} \). Now \( \hat{H} = E(\hat{H}) \ast F^\circ(\hat{H}) \), as desired.

Therefore \( F^\circ(\hat{H}) \) satisfies our generation hypotheses. Clearly \( U_p(F^\circ(\hat{H})) \leq U_p(O(H)) = 1 \). By Fact 1.16, \( F^\circ(\hat{H}) \) is a central product of the \( U_{0,r}(F^\circ(\hat{H})) \) with \( r > \bar{r}_0(O(H)) \) \( \geq \bar{r}_0(O(H)) \). It follows that \( F^\circ(\hat{H}) \) is abelian since \( F(K) \) was abelian.

\[ \square \]

**Corollary 2.11.** Let \( G \) be a simple \( K^* \)-group of finite Morley rank and odd type with \( m(G) \geq 3 \) which satisfies the \( \hat{B} \)-property. Then, for every 2-subgroup \( X \leq G \) with \( m(X) \geq 3 \) and every \( i \in I^0(X) \), we have \( \hat{U}_X(C_G(i)) = \hat{E}_X(C_G(i)) \ast \hat{F}_X(C_G(i)) \) and \( \hat{F}_X(C_G(i)) \) is abelian.

We can now verify that \( \hat{E}_X \) is nonempty.
THEOREM 2.12. Let $G$ be a simple $K^*$-group of finite Morley rank and odd type with $m(G) \geq 3$. Then either
1. $G$ has a proper weak 2-generated core, or else
2. $\hat{E}_X \neq \emptyset$ for every 2-subgroup $X \leq G$ with $m(X) \geq 3$.

Proof. By Theorem 2.9, we may assume that $G$ satisfies the $\hat{B}$-property. Consider a 2-subgroup $X \leq G$ with $m(X) \geq 3$. There is an $E \in \mathcal{E}_3(X)$ with $\hat{E}(E) = E$ maximal. So $\hat{E}_X(X) \subseteq \hat{E}_X(E)$.

We first consider the case where $C^0_\gamma(i)$ is solvable for all $i \in E^\#$. In particular, $\hat{E}(E) = E$. For all $i \in E^\#$, $U_p(C_\gamma(i)) = U_p(O(C_\gamma(i))) = 1$ and $\hat{E}(E) \leq \hat{E}(E) = E$, since $G$ satisfies the $\hat{B}$-property. By Fact 1.15, $O(C^0_\gamma(i))$ is a good torus, and hence central in $C^0_\gamma(i)$ by [15, Theorem 6.16]. Since $C^0_\gamma(i)/O(C^0_\gamma(i))$ is abelian by Fact 1.4, $C^0_\gamma(i)$ is nilpotent, and divisible. Now $C^0_\gamma(i)'$ is torsion-free by [15, Theorem 6.9]. As $\hat{E}(E) = E$, $C^0_\gamma(i)$ is in fact abelian. By Fact 1.30, $C^0_\gamma(i) = \langle C^0_\gamma(i)(E_0) : E_0 \leq E, [E : E_0] = 2 \rangle$. As $C^0_\gamma(i) \neq 1$ by [15, Ex. 13 & 15 p. 79], there is some four-group $E_1 \leq E$ with $H := C^0_\gamma(E_1) \neq 1$. Since each $C^0_\gamma(i)$ is abelian, $H < C^0_\gamma(i)$ for all $i \in E^\#$, and thus $H < \Gamma(E_1(G))$. Since $G$ is simple, $\Gamma(E_1(G)) \leq N_G(H) < G$, and $G$ has a proper weak 2-generated core $\Gamma_\gamma(G) < G$ by Proposition 1.23. So we may assume that $C^0_\gamma(i)$ is nonsolvable for some $i \in E^\#$.

We now fix an $i \in E^\#$ and a component $L$ of $C^0_\gamma(i)/O(C^0_\gamma(i))$ so that $\hat{E}(E) = \hat{E}(E)$ where $k$ is the base field of $L$. Since $k$ is algebraically closed, $k^*$ contains torsion, and hence

$$\text{rk}(k^*) = \text{rk}(k^*) > \text{rk}(k^*) = \hat{E}(E).$$

Suppose toward a contradiction that $\hat{E}(E)(C_\gamma(i)) = 1$. By Corollary 2.11, we have $U_E(C_\gamma(i)) = \hat{E}(E)(C_\gamma(i)) \ast \hat{E}(E)(C_\gamma(i))$ and $\hat{E}(E)(C_\gamma(i))$ is abelian for every $i \in E^\#$. So $\hat{E}(E)(C_\gamma(i)) = \hat{E}(E)(C_\gamma(i))$ is abelian. If $\text{char}(k) > 0$ then $U_{\text{char}(k)}(C_\gamma(i)) \leq \hat{E}(E)(C_\gamma(i))$ is abelian. If $\text{char}(k) = 0$ then $U_{\text{char}(k)}(C_\gamma(i)) \leq \hat{E}(E)(C_\gamma(i))$ is abelian. Either case contradicts the existence of $L$.

We also observe that the definition of $\hat{E}_X$ restricts the fields involved as follows.

PROPOSITION 2.13. Let $H$ be a group of finite Morley rank which is isomorphic to a linear algebraic group over an algebraically closed field $k$. Then
1. If $U_{\text{char}(k)}(H) \neq 1$ for some $r > \text{rk}(k^*)$ then $\text{char}(k) = 0$ and $\text{rk}(k) = r$.
2. If $U_{\text{char}(k)}(H) \neq 1$ then $\text{char}(k) = p$.

If $H$ is quasisimple, these conditions imply $U_p(H) = H$ and $U_{\text{char}(k)}(H) = H$, respectively.

Proof. If $\text{char}(k) \neq p$, then $H$ has bounded $p$-rank, so the second point follows.

We now consider the first point. Let $A$ be a nontrivial $(0, r)$-unipotent abelian group, and let $A$ be the Zariski closure of $A$. Then $A = S \times U$ where $S$ is semisimple, and $U$ is the unipotent radical of $A$. If $A$ has nonunipotent elements, then $\hat{A} := AU/U$ is a nontrivial subgroup of the semisimple group $A/U$. As $\hat{A}/U$ is linear, $\hat{A}/U \hookrightarrow (k^*)^n$ for some $n$. But $U_{\text{char}(k)}(A) = \hat{A}$ by [17, Lemma 2.11], contradicting $r > \text{rk}(k^*)$. So $A$ consists of unipotent elements, i.e. $A \leq U (= \hat{A})$. Hence $\text{char}(k) = 0$. As $U$ is linear, $U \hookrightarrow (k^*_+)^n$ for some $n$. By [31, Cor. 3.3], there are no definable subgroups of $k^*_+$. So $U_{\text{char}(k)}(U) = 1$ unless $r = \text{rk}(k)$. 


The last remark follows from the fact that quasisimple algebraic groups are generated by the unipotent radicals of their Borel subgroups, or indeed by any conjugacy class.

2.3. Generation by components in $\hat{E}_X$

We next show that components in $\hat{E}_X$ generate $G$, i.e.

$$\langle \hat{E}_{\Omega_1(S^+)} \rangle = G$$

when $\text{pr}(S) \geq 3$.

However, these results will be proven in a form usable also when $\text{pr}(S) < 3$.

For any group $H$ of finite Morley rank, any 2-subgroup $X$ acting definably on $H$, and any $V \in \mathcal{E}_2(X)$, we define

$$\hat{\Gamma}_{X,V}(H) = \langle \hat{U}_X(C_H(v)) : v \in V^\# \rangle.$$

**Lemma 2.14.** Let $G$ be a $K^*$-group of finite Morley rank and odd type. Let $X$ be a 2-subgroup of $G$ with $\text{m}(X) \geq 3$. Suppose that $G$ satisfies the $\bar{B}$-property and that there is a four-group $E \in \mathcal{E}_3^0(X)$ which centralizes a Sylow 2-subgroup $T$ of $G$. Let $H := \langle \hat{E}_X(C_G(z)) : z \in E^\# \rangle$. Then the following hold.

1. For any $x, y \in E^\#$, we have $[\hat{F}_X(C_G(x)), \hat{F}_X(C_G(y))] = 1$ and the group $\hat{F}_X(C_G(x))$ normalizes $\hat{E}_X(C_G(y))$.
2. $\hat{U}_X(O(H)) = 1$.
3. $\hat{\Gamma}_{X,E}(G) \leq N^0_{\mathcal{G}}(\hat{E}_X(H))$.

**Proof.** As a notational convenience, we let $F_x := \hat{F}_X(C_G(x))$ and let $H_x := \hat{E}_X(C_G(x))$ for $x \in E^\#$. Since $G$ satisfies the $\bar{B}$-property, Corollary 2.11 says $\hat{U}_X(C_G(x)) = H_x \ast F_x$ and $F_x$ is abelian. Since $y \in E^\#$ normalizes $F_x$ for any $x \in E^\#$, there is a homomorphism $f_y : F_x \rightarrow F_x$ given by $u \mapsto [y, u]$. Since $E$ centralizes $T$ and $F_x \cap T$ is the Sylow 2-subgroup of $F_x$, there is no 2-torsion in $F_x \setminus \ker(f_y)$. So $[F_x, E] \leq O(F_x) \leq O(C_G^0(x))$ by Fact 2.6 and Fact 1.19. By Fact 1.20, $[U_{0,r}(F_x), E]$ is a $U_{0,r}$-group. Clearly $[U_p(F_x), E]$ is $p$-unipotent. Since $\hat{U}_X(C_G^0(x)) = H_x \ast F_x$, we have $F_x = \hat{U}_X(F_x)$. So $[F_x, E] \leq \hat{U}_X(O(C_G^0(x))) = 1$ by the $\bar{B}$-property. Now $F_x \leq \hat{U}_X(C_G^0(y))$ for every $x, y \in E^\#$. Since $F_y$ is central in $\hat{U}_X(C_G^0(y))$, we have $[F_x, F_y] = 1$. Also $F_x$ normalizes $H_y$, as $H_y \cap \hat{U}_X(C_G^0(y))$. Thus $F := \langle F_z : z \in E^\# \rangle$ is an abelian group of automorphisms of $H = (H_x : z \in E^\#)$.

Since $H_x$ is characteristic in $C_G(x)$ for all $x \in E^\#$, $E$ normalizes $H$. Suppose towards a contradiction that $U_X(O(H)) \neq 1$. So either

1. $K := U_{0,r}(O(H)) \neq 1$ for some $r > \bar{r}(X)$, or else
2. $K := U_p(O(H)) \neq 1$ for some prime $p$.

In either case, $K = \Gamma_K(K)$ by Fact 1.1. So a $K_x := C_K(x)$ is nontrivial. In case (1), we may choose $r$ maximal, so $K$ is nilpotent by Theorem 1.13. Hence $K_x$ is $(0, r)$-unipotent by Lemma 1.18. In case (2), $K$ is nilpotent by Fact 1.10. Hence $K_x$ is $p$-unipotent by Fact 1.17. In either case, $K_x$ is nilpotent and normalized by $H_x$. Since $K_x \leq \hat{U}_X(C_G(x))$, and $H_x$ is semisimple, we have $K_x \leq F_x$, in contradiction to the $\bar{B}$-property. Thus $U_X(O(H)) = 1$, as desired.

For the last part, we may assume $H < G$ is a $K$-group. So $H = \hat{E}_X(H) * \hat{F}_X(H)$ by Corollary 2.11. Since $\hat{E}_X(H)$ is characteristic in $H$, $F$ normalizes $\hat{E}_X(H)$. So $\hat{\Gamma}_{X,E}(G) = F \cdot H \leq N^0_{\mathcal{G}}(\hat{E}_X(H))$. □

We now provide the promised variant of Lemma 1.24 above.
Lemma 2.15. Let $G$ be a $K^*$-group of finite Morley rank and odd type with $m(G) \geq 3$, and which satisfies the $\mathcal{B}$-property. Let $T$ be a Sylow $2$-subgroup of $G$, and let $X \leq C_G(T)$ be a $2$-subgroup which centralizes $T$ and has $m(X) \geq 3$. Then $\tilde{\Gamma}_{X,U}(G) = \tilde{\Gamma}_{X,V}(G)$ for any two four-groups $U, V \in E_2^0(X)$.

We need the following algebraic fact.

Fact 2.16. Let $G$ be a quasisimple algebraic group over an algebraically closed field of characteristic not $2$ which is not of type $(P)\text{SL}_2$, and let $V$ be a four-group of algebraic automorphisms of $G$ which centralizes a maximal $2$-torus of $G$. Then

$$G = \langle E(C_G(x)) : x \in V^\# \rangle.$$  

Proof. Let $T$ be a maximal $2$-torus of $G$ centralized by $V$. For $i \in V^\#$, Fact 1.7 says $C_G^p(i)$ is reductive. Hence $C_G^p(i) = F_i \circ H_i$ where $H_i := E(C_G^p(i))$ and $F_i := F^0(C_G^p(i))$ is an algebraic torus. So $F_i$ is the Zariski closure of $F_i \cap T$, and hence is centralized by $V$. Now $F := \langle F_i \mid i \in V^\# \rangle$ is an algebraic torus normalizing $H = \langle H_i \mid i \in V^\# \rangle$.

We now show that $H$ is reductive. Suppose that $H$ has a nontrivial unipotent radical $U$. By Fact 1.1, $U_j := C_U(j) \neq 1$ for some $j \in V^\#$. But $U_j \lhd C_G^p(j)$, contradicting the reductivity of $C_G^p(j)$. So $H$ must be reductive.

Now $\Gamma_V(G) = FH \leq N_G^p(E(H))$. We observe that $H_j \neq 1$ for $j \in V^\#$, since $G \not\cong (P)\text{SL}_2$. So $H$ is not solvable, and $E(H) \neq 1$. Since $\Gamma_V(G) = G$ by Fact 1.1, it follows that $E(H) = G$ because $G$ is quasisimple, and hence $H = G$. 

Proof of Lemma 2.15. We may assume that $[U, V] = 1$ since $U$ and $V$ lie in the same $2$-connected component. It is enough to show that $\tilde{U}_X(C_G(u)) \leq \tilde{\Gamma}_{X,V}(\tilde{U}_X(C_G(u)))$ for any $u \in U^\#$. Since $G$ satisfies the $\mathcal{B}$-property, we have $\tilde{U}_X(C_G(u)) = \tilde{E}_X(C_G(u)) \circ \tilde{F}_X(C_G(u))$ by Corollary 2.11. By Fact 1.16, the abelian group $F_u := \tilde{F}_X(C_G(u))$ may be written as a product of various $U_p(F_u)$, with $p$ prime, and $U_0(r(F_u)$, with $r > r^*(X)$. By Fact 1.18, $C_{U_0(r(F_u)}(v)$ is $(0, r)$-unipotent. By Fact 1.17, $C_{U_0(F_u)}(v)$ is $p$-unipotent. So $C_{F_u}(v) \leq \tilde{U}_X(C_G(u))$. Fact 1.1 yields

$$\tilde{F}_X(C_G(u)) \leq \tilde{\Gamma}_{X,V}(\tilde{U}_X(C_G(u))).$$

Now consider a component $L \leq \tilde{E}_X(C_G(u))$. It suffices to show that $L = \tilde{\Gamma}_{X,V}(L)$.

In the case that $L \not\cong (P)\text{SL}_2$, Fact 2.16 yields

$$L = \langle E(C_L(x)) : x \in V^\# \rangle.$$ 

By Proposition 2.13, we have $E(C_L(x)) = \tilde{E}_X(C_L(x))$, and hence $L = \tilde{\Gamma}_{X,V}(L)$.

In the case that $L \cong (P)\text{SL}_2$, $(P)\text{SL}_2$ has no graph automorphisms, so every $x \in V^\#$ acts by some inner automorphism by Fact 1.6. We observe that $T \leq C_G(u)$, and thus $T$ is a Sylow $2$-subgroup of $C_G(u)$. So $L \cap T$ is a Sylow $2$-subgroup of $L$. Since $(P)\text{SL}_2$ contains no four-group centralizing a torus, there is now some $x \in V^\#$ which centralizes $L$, and the claim follows.

We now prove a version of Proposition 1.23.

Proposition 2.17. Let $G$ be an infinite simple $K^*$-group of finite Morley rank
and odd type with \( m(G) \geq 3 \). Let \( T \) be a Sylow\(^2\) 2-subgroup of \( G \), and let \( X \leq C_G(T) \) be a 2-subgroup which centralizes \( T \) with \( m(X) \geq 3 \). Suppose that \( \Gamma_{X,E}(G) < G \) for some \( E \in \mathcal{E}_{2}^1(X) \). Then \( G \) has a proper weak 2-generated core.

**Proof.** By Proposition 1.23, it is enough to show that \( \Gamma_E(G) < G \). Let \( A \in \mathcal{E}_3(X) \) be an eight-subgroup of \( X \) containing \( E \). By Lemma 1.25, \( \Gamma_E(G) \leq \Gamma_{A,2}(G) \). So the result will follow from the following claim and simplicity.

\[
\Gamma_{A,2}(G) \leq N_G(\tilde{\Gamma}_{X,E}(G))
\]

By Lemma 2.15 and Fact 1.21-1, \( \tilde{\Gamma}_{X,E}(G) = \tilde{\Gamma}_{X,U}(G) \) for any \( U \leq A \). For any four-group \( U \leq A \),

\[
N_G(U) \leq N_G(\tilde{\Gamma}_{X,U}(G)) = N_G(\tilde{\Gamma}_{X,E}(G)).
\]

Thus \( \Gamma_{A,2}(G) \leq N_G(\tilde{\Gamma}_{X,E}(G)) \), as desired. \( \Box \)

We now prove that our components generate \( G \).

**Theorem 2.18.** Let \( G \) be a \( K^* \)-group of finite Morley rank and odd type with \( m(G) \geq 3 \). Suppose that there is a four-group \( E \in \mathcal{E}_{2}^1(G) \) which centralizes a Sylow\(^2\) 2-subgroup \( T \) of \( G \), and that there is an eight-group \( X \in \mathcal{E}_3(C_G(T)) \) containing \( E \). Then either

1. \( G \) has a proper weak 2-generated core, or
2. \( \langle \mathcal{E}^X_E \rangle = \langle \tilde{E}_X(C_G(x)) : x \in E^\# \rangle = G \).

We need the following fact about involutive automorphisms of algebraic groups, which follows immediately from Table 4.3.1 on p. 145 of [26] and Fact 1.6.

**Fact 2.19.** Let \( G \) be a quasisimple algebraic group over an algebraically closed field of characteristic not 2 and let \( \alpha \) be a definable involutive automorphism of \( G \). If \( G \not\cong (P)\text{SL}_2 \), then \( E(C_G(\alpha)) \neq 1 \).

**Proof of Theorem 2.18.** We may assume that \( G \) satisfies the \( \tilde{B} \)-property by Theorem 2.9.

We first show that \( \tilde{E}_X(C_G^2(z)) \neq 1 \) for some \( z \in E^\# \) (note \( m(E) \geq 2 \)). We may assume that \( \tilde{E}_X \neq 1 \) by Theorem 2.12, so there is a component \( L \leq \tilde{E}_X(C_G(x)) \neq 1 \) for some \( x \in X^\# \). We observe that \( E \) normalizes \( L \) since \( X \) is an eight-group containing \( E \). So any \( z \in E^\# \) acts on \( L \) via an algebraic automorphism, by Fact 1.6, and hence \( C_L(z) \) is reductive by Fact 1.7. By Proposition 2.13, \( \tilde{U}_X(U) = U \) for any unipotent group in \( L \). As quasi-simple groups are generated by their unipotent subgroups, \( \tilde{U}_X(C_G(z)) \geq \tilde{E}_X(C_L(z)) = E(C_L^\circ(z)) \). If \( L \not\cong (P)\text{SL}_2 \), then \( E(C_L^\circ(z)) \neq 1 \) by Fact 2.19, and hence \( \tilde{E}_X(C_G(z)) \neq 1 \) by Corollary 2.11. So we may assume \( L \cong (P)\text{SL}_2 \). Since \( (P)\text{SL}_2 \) has no graph automorphisms, any \( z \in E^\# \) acts via inner automorphism, by Fact 1.6. Since \( (P)\text{SL}_2 \) contains no four-group centralizing a torus, there is now some \( z \in E^\# \) which centralizes \( L \), and \( \tilde{U}_X(C_G(z)) \) is non-abelian. By Corollary 2.11, \( \tilde{U}_X(C_G(z)) = \tilde{E}_X(C_G(z)) * \tilde{F}_X(C_G(z)) \) and \( \tilde{F}_X(C_G(z)) \) is abelian. So \( \tilde{E}_X(C_G^2(z)) \neq 1 \) for some \( z \in E^\# \).

Let \( H := \langle \tilde{E}_X(C_G(x)) : x \in E^\# \rangle \). Since \( \tilde{U}_X(O(H)) = 1 \) by Lemma 2.14, \( H = \tilde{E}_X(H) * \tilde{F}_X(H) \) and \( \tilde{F}_X(H) \) is abelian by Corollary 2.11. Since \( \tilde{E}_X(C_G^2(x)) \neq 1 \), we have \( \tilde{E}_X(H) \neq 1 \).
Since \( E \) centralizes \( T \), Lemma 2.14 says \( \tilde{\Gamma}_{X,E}(G) \leq N_G^0(\tilde{E}_X(H)) \). Hence \( \tilde{\Gamma}_{X,E}(G) \leq N_G^0(\tilde{E}_X(H)) < G \), and the theorem follows from Proposition 2.17.

3. The Generic Trichotomy Theorem

We now turn our attention toward proving the following, our main result.

**Generic Trichotomy Theorem 6.** Let \( G \) be a simple \( K^* \)-group of finite Morley rank and odd type with \( \text{pr}(G) \geq 3 \). Then either

1. \( G \) has a proper 2-generated core, i.e. \( \Gamma_{S,2}(G) < G \), or else
2. \( G \) is an algebraic group over an algebraically closed field of characteristic not 2.

Our strategy is to replicate the proof by Berkman and Borovik of the Generic Identification Theorem [8], being careful to use only “safe” components, under the assumption that (1) does not occur. So we adopt the following standing hypotheses and notation.

**Hypothesis 3.1.** We consider a simple \( K^* \)-group \( G \) of finite Morley rank and odd type with \( \text{pr}(G) \geq 3 \), and fix a Sylow 2-subgroup \( S \) of \( G \). We also suppose that the 2-generated core of \( G \) is not proper, i.e. \( \Gamma_{S,2}(G) = G \).

As \( n(2) \geq \text{pr}(G) \geq 3 \), we have \( \Gamma_{S,2}(G)^0 = \Gamma_{S,2}(G) = G \) by Fact 1.21-2. So, by Theorem 2.9,

\[ G \text{ satisfies the } \check{B}\text{-property.} \tag{3.1} \]

3.1. Root \( SL_2 \) subgroups

The first stage in our analysis is to select, and establish the properties of a family of abstract “root \( SL_2 \)-subgroups” of \( G \). The root \( SL_2 \)-subgroups of an algebraic group associated to a maximal torus \( T \) may be defined as those Zariski closed subgroups of \( G \) which are normalized by \( T \) and are isomorphic to \( (P)SL_2 \), or alternatively in terms of groups generated by opposite root groups. We employ several facts about root \( SL_2 \)-subgroups of algebraic groups.

**Fact 3.2** cf. [8, Fact 2.1]. Let \( G \) be a quasisimple algebraic group over an algebraically closed field. Let \( T \) be a maximal torus in \( G \) and let \( K, L \) be Zariski closed subgroups of \( G \) that are isomorphic to \( SL_2 \) or \( PSL_2 \) and are normalized by \( T \). Then

1. Either \( K \) and \( L \) commute or \( \langle K, L \rangle \) is a quasisimple algebraic group of type \( A_2 \), \( C_2 \), or \( G_2 \).
2. The subgroups \( K \) and \( L \) are root \( SL_2 \)-subgroups of \( \langle K, L \rangle \).
3. If \( \langle K, L \rangle \) is of type \( G_2 \), then \( G = \langle K, L \rangle \).

More generally, a semisimple subgroup of a simple algebraic group \( G \) which is normalized by a maximal torus \( T \) is called subsystem subgroup of \( G \), associated to \( T \). Berkman and Borovik refer to the full classification of semisimple subsystem
subgroups [33, 2.5] (see also §3.1 of [32]) for the proof of this fact. The elementary argument here is based on the following fact.

**Fact 3.3** [32, Prop. 3.1]; [33, 2.5]. Let $G$ be a simple algebraic group, let $T$ be a maximal torus of $G$, and let $X$ be a closed connected subgroup of $G$ which contains $T$. Then $X = DZU$ where $D$ is a subsystem subgroup normalized by $T$, $Z$ is a torus, and $U$ is the unipotent radical of $X$.

**Proof of Fact 3.2.** By Fact 3.3, $(K, L) = DZU$ where $D$ is a subsystem subgroup, $Z$ is a torus, and $U$ is the unipotent radical of $(K, L)$. There is an automorphism $\phi$ of the root system for $G$ which sends any root $\alpha \in I$ to its negative $-\alpha$, and $\phi$ translates to an automorphism $\phi$ of the group $G$ such that $\phi$ normalizes $T$ and $\phi(X_\alpha) = X_{-\alpha}$ [20]. Since $K$ and $L$ each contain one positive and one negative root from $I$, we find that $K$, $L$, and $(K, L)$ are all normalized by $\phi$. If $U$ is nontrivial, it contains a root group $X_\alpha$. Since $U$ is characteristic in $(K, L)$, the nilpotent group $U$ must also contain $X_{-\alpha}$, and hence contains a copy of $(P)SL_2$ [20]. So $U = 1$, and $(K, L) = D$. Since $D$ is semisimple, $D \cong A_1 \ast A_2$, $A_2$, or $G_2$, as desired.  

**Fact 3.4** see [21, p. 19]. Let $G$ be a semisimple algebraic group over an algebraically closed field, and fix a maximal algebraic torus $T$ of $G$. Then the following hold.

1. $G$ is generated by its root $SL_2$-subgroups associated with $T$.
2. The intersection $T \cap K$ of $T$ and a root $SL_2$-subgroup $K$ associated to $T$ is a maximal algebraic torus of $K$.

**Proof.** The fact that $G$ is generated by those root $SL_2$-subgroups which are normalized by $T$ can be found on p. 19 of [21]. For the second part, we observe that the maximal algebraic tori of $N_G(K)$, one of which is $T$ and one of which extends a maximal algebraic torus of $K$, are conjugate in $N_G(K)$. 

We also need to know that a root $SL_2$-subgroup is “cut out” by the centralizer of a 2-torus in the associated maximal torus, We remark that this is an essential point if one hopes to apply Fact 3.2, but it remains somewhat obscure in [8].

**Fact 3.5.** Let $G$ be a quasisimple algebraic group over an algebraically closed field of characteristic not 2. Let $T$ be a maximal algebraic torus of $G$, and let $L$ be a root $SL_2$-subgroup of $G$ normalized by $T$. Then $L = E(C_G(C_{S^\circ}(L)))$ where $S^\circ$ is the Sylow 2-subgroup of $T$.

**Proof.** We may assume that $G$ is not isomorphic to $(P)SL_2$ because otherwise $G = L$. Let $S$ be a Sylow 2-subgroup of $G$ such that $S^\circ \leq T$. Any connected definable group of automorphisms of $G$ must be inner by Fact 1.6. Since $L$ is normalized by $S^\circ$, we have $pr(C_{S^\circ}(L)) = pr(G) - 1$. By Fact 1.7 (see also [21, Thm. 3.5.4]), $C_G(C_{S^\circ}(L))$ is reductive. So $pr(K) \leq 1$ where $K := E(C_G(C_{S^\circ}(L)))$. Since $L \leq K$ and $L$ and $K$ are both algebraic subgroups, we have $L = K$. 

We now proceed with the analysis of groups satisfying Hypothesis 3.1.
Lemma 3.6. For any $i \in \Omega_1(S^\circ)^\#$ and any definable connected quasisimple algebraic $L \leq E(C_G^0(i))$ which is normalized by $S^\circ$, we have
1. $S^\circ \cap L$ is a Sylow $2$-subgroup of $L$.
2. $T_L := C_L(S^\circ \cap L)$ is a maximal algebraic torus of $L$.
3. $S^\circ = C_{S^\circ}(L)(S^\circ \cap L)$, and $\text{pr}(G) = \text{pr}(S^\circ) = \text{pr}(C_{S^\circ}(L)) + (S^\circ \cap L)$.

For this, we need the following fact about algebraic groups.

Fact 3.7. Let $G$ be a quasisimple algebraic group over an algebraically closed field of characteristic not $2$, and let $T$ be a Sylow $2$-subgroup of $G$. Then $C_G(T)$ is a maximal algebraic torus of $G$.

Proof. In an algebraic group over an algebraically closed field, the maximal algebraic torus is the Zariski closure of a maximal algebraic torus of $G$ field of characteristic not $2$. By Fact 3.7, $T_L$ is a maximal algebraic torus of $L$. By Fact 1.6, the connected definable group $d(S^\circ)$ acts by inner automorphisms on $L$, so the third condition follows.

Proof of Lemma 3.6. Since $S^\circ$ is a Sylow $2$-subgroup of $N^0_L(L)$, the group $S^\circ \cap L$ is a Sylow $2$-subgroup of $L$. By Fact 3.7, $T_L$ is a maximal algebraic torus of $L$. By Fact 1.6, the connected definable group $d(S^\circ)$ acts by inner automorphisms on $L$, so the third condition follows.

Lemma 3.8. For any component $M \in \tilde{E}_{S^\circ}$, $M$ is normalized by $S^\circ$.

Proof. Clearly $M \triangleleft C_G^0(i)$ for some $i \in \Omega_1(S^\circ)$ (see [15, Lemma 7.1iii]). Since $S^\circ \leq C_G^0(i)$, $M$ is normalized by $S^\circ$.

Definition 3.9. Let $\Sigma$ be the set of all root $\text{SL}_2$-subgroups of components $K \in \tilde{E}_{S^\circ}$ which are associated to $T_K$, i.e. $\Sigma$ is the set of all Zariski closed subgroups of the components $K \in \tilde{E}_{S^\circ}$ which are normalized by $T_K$ and are isomorphic to $(\text{P})\text{SL}_2$.

Since the root $\text{SL}_2$-subgroups of $K$ generate $K$ by Fact 3.4-1, Theorem 2.18 yields the following.

$$\langle \Sigma \rangle = \langle \tilde{E}_{S^\circ} \rangle = G$$

\((*)\)

We view the subgroups in $\Sigma$ as abstract root $\text{SL}_2$-subgroups for $G$.

Lemma 3.10. For any $L \in \Sigma$, we have
1. $L$ is normalized by $S^\circ$,
2. $L = E(C_G(C_{S^\circ}(L)))$, and
3. $L$ is a Zariski closed subgroup of any definable quasisimple $K < G$ which contains $L$ and which is normalized by $S^\circ$.

Proof. Let $R_L := C_{S^\circ}(L)$ and let $M \in \tilde{E}_{S^\circ}$ be a component containing $L$ as a root $\text{SL}_2$-subgroup associated with $T_M$. By Lemma 3.6-3, $S^\circ = C_{S^\circ}(M)(M \cap S^\circ)$. Since $L$ is normalized by $M \cap S^\circ \leq T_M$, $L$ is normalized by $S^\circ$. By Fact 3.5, $L = E(C_M(M \cap R_L))$. Fix $i \in \Omega_1(S^\circ)^\#$ with $M \triangleleft C_G^0(i)$. Clearly $E(C_G(R_L)) \leq M$ because any other component of $C_G^0(i)$ meets $R_L$ in an infinite 2-torus. As $i \in R_L$, $L = E(C_G(R_L))$. 


Now for any definable quasisimple $K < G$ which contains $L$ and which is normalized by $S^\circ$, the group $R_L$ acts on $K$ by inner automorphisms by Fact 1.6, so $L = E(C_K(R_L))$ is Zariski closed.

**Lemma 3.11** cf. [8, Lemma 3.1]. For any distinct $K, L \in \Sigma$,
1. $C_{S^\circ}(K) \cap C_{S^\circ}(L) \neq 1$ and $M := \langle K, L \rangle$ is a $K$-group.
2. Either $K$ and $L$ commute or $M$ is an algebraic group of type $A_2$, $B_2 = C_2$, or $G_2$.
3. $S^\circ \cap M = \langle S^\circ \cap K \rangle \times \langle S^\circ \cap L \rangle$ is a Sylow $2$-subgroup of $M$.
4. $K$ and $L$ are root $\text{SL}_2$-subgroups of $M$ normalized by $T_M$.
5. $[T_K, T_L] = 1$.

**Proof.** Let $R_L := C^\circ_{S^\circ}(L)$. Since $S^\circ$ normalizes $K$ and $L$ by Lemma 3.10-1, and $K, L \cong \text{SL}_2$, we know that $\text{pr}(R_K), \text{pr}(R_L) = \text{pr}(G) - 1$ and $S^\circ = R_K R_L$ by Lemma 3.6-3. So
\[
\text{pr}(G) = \text{pr}(R_K) + \text{pr}(R_L) - \text{pr}(R_K \cap R_L) \]
and
\[
\text{pr}(R_K \cap R_L) = \text{pr}(G) - 2 \geq 1.
\]

Thus $C_{S^\circ}(K) \cap C_{S^\circ}(L) \neq 1$ and $M$ has a nontrivial center. Since $G$ is simple, $M < G$ is a $K$-group.

Let $i \in I(C_{S^\circ}(K) \cap C_{S^\circ}(L))$. By Corollary 2.11 (and the $\tilde{B}$-property), $K, L \leq \tilde{E}_{S^\circ}(C_G(i))$. If the groups belong to different components of $C^\circ_{S^\circ}(i)$, then they commute. If they both belong to the same component $H \in \tilde{E}_{S^\circ}$, then $H$ is a quasisimple algebraic group normalized by $S^\circ$ by Lemma 3.8. By Lemma 3.10-3, $K$ and $L$ are Zariski closed in $H$. By Fact 3.2-1, $M = \langle K, L \rangle$ is an algebraic group of type $A_2$, $B_2 = C_2$, or $G_2$. In either case, $S^\circ$ normalizes $M$, so $S^\circ \cap M$ is a Sylow $2$-subgroup of $M$ and $T_M$ is a maximal "algebraic" torus of $M$ by Lemma 3.6.

For (4) and (5), we may assume that $[L, K] \neq 1$ and $M$ is a quasisimple algebraic group. By Fact 3.2-2, $K$ and $L$ are root $\text{SL}_2$-subgroups of $M$. By Fact 3.4-2, $T_M = T_K * T_L$, so $[T_K, T_L] = 1$.

We give $\Sigma$ a graph structure by placing an edge between $L, K \in \Sigma$ when $[L, K] \neq 1$. Since $G$ is simple and $\langle \Sigma \rangle = G$, the graph $\Sigma$ is connected. By Lemma 3.11-2, any adjacent $L, K \in \Sigma$ algebraic groups over the same algebraically closed field. So all the elements of $\Sigma$ are algebraic groups over a common algebraically closed field $F$. Since $G$ has odd type, $\text{char}(F) \neq 2$. In particular, $\text{rk}(K) = \text{rk}(L)$ for all $K, L \in \Sigma$.

From this point on, our argument reduces to that given by Berkman and Borovik in [8], following the presentation of [7]. Indeed, we may now lighten our standing hypotheses to the following.

**Hypothesis 3.12.** We consider a simple group $G$ of finite Morley rank and odd type with $\text{pr}(G) \geq 3$, and fix a Sylow $2$-subgroup $S$ of $G$. Also choose some family $\tilde{E}$ of algebraic components from the centralizers of involutions in $S^\circ$, Let $\Sigma$ be the set of root $\text{SL}_2$-subgroups, from components in $\tilde{E}$, which are associated to $S^\circ$, in the sense of Definition 3.9. Suppose that Lemmas 3.6, 3.10, and 3.11 are satisfied, and
also that

\[ \langle \Sigma \rangle = G. \]  

\((\ast)\)

We give the analysis in full below.

3.2. Weyl group

We now turn our attention to the Weyl group of \( G \), continuing under Hypothesis 3.12.

**Lemma 3.13.** The natural torus \( T := \langle T_L : L \in \Sigma \rangle \) is divisible abelian. So \( T_L = T \cap L \).

**Proof.** By Lemma 3.11-5, the algebraic tori \( T_K \) for \( K \in \Sigma \) all commute, so the result follows.

**Definition 3.14.** For any \( L \in \Sigma \), \( W(L) := N_L(T_L)/T_L \) is the Weyl group of \( L \) and has order 2. We may identify \( W(L) \cong N_L(T)/C_L(T) \) with its image in \( W := N_G(T)/C_G(T) \), by Lemma 3.13. Now let \( r_L \) denote the single involution inside \( W(L) \), and define \( W_0 := \langle r_L \in W : L \in \Sigma \rangle \).

**Lemma 3.15** cf. [8, Lemma 3.5]. For any \( L, K \in \Sigma \), \( [K, L] = 1 \) if and only if \( [r_K, r_L] = 1 \).

**Proof.** It suffices to check this in \( (K, L) \). So the result follows from Fact 3.2-2 and Fact 3.4.

We will analyze \( W_0 \) by examining its action on \( S^0 \) and \( T \).

**Lemma 3.16** cf. [8, Lemmas 3.6 & 3.7]. \( S^0 \) is the Sylow 2-subgroup of \( T \) and \( C_G(S^0) = C_G(T) \). In particular, \( W_0 \) acts faithfully on \( S^0 \).

**Proof.** By Lemma 3.13, \( T \) is divisible abelian, so its Sylow 2-subgroup is connected by [15, Theorem 9.29].

Let \( D := S^0 \cap T \). Suppose toward a contradiction that \( D < S^0 \). For all \( K \in \Sigma \), \([S^0, r_K] = S^0 \cap K \), so \( r_K \) acts trivially on \( S^0 / D \). Let \( b \in S^0 \) satisfy \( |b/D| \geq 4 \) and let \( a \in S^0 \) satisfy \( a|_{W_0} = b \). Then \( \epsilon \in \Pi_{w \in W_0} a^w \) satisfies \( b/D = c/D \) and \( |c| \geq 4 \). Since \( S^0 = C_{S^0}(K) * (S^0 \cap K) \) and \( |C_{S^0}(K) \cap (S^0 \cap K)| \leq |Z(K)| = 2 \), we have \( |C_{S^0}(r_K) : C_{S^0}(K)| \leq 2 \). Since \( c \in C_{S^0}(r_K) \), \( c^2 \in C_{S^0}(K) \) for all \( K \in \Sigma \), and \( c^2 \neq 1 \). So \( c^2 \in C_G(\langle \Sigma \rangle) = Z(G) \), contradicting the simplicity of \( G \). Thus \( S^0 \) is the Sylow 2-subgroup of \( T \), and \( C_G(S^0) \geq C_G(T) \).

For the reverse direction, consider \( x \in C_G(S^0) \). Then, for every \( L \in \Sigma \), \( x \) centralizes \( C_{S^0}(L) \). So \( x \) normalizes \( L = E(C_G(C_{S^0}(L))) \) by Lemma 3.10-2. Since \( x \) centralizes the maximal 2-torus \( S^0 \cap L \), \( x \) must act on \( L \) as an element of \( T_L \) by Fact 1.6. Thus \( x \in C_G(T) \) and \( C_G(S^0) \leq C_G(T) \).

We use the action of \( W_0 \) on \( S^0 \) to obtain a complex representation.
Lemma 3.17 cf. [8, §3.3]. \( W_0 \) has a faithful irreducible representation \( R \) over \( \mathbb{C} \) of dimension \( \text{pr}(G) \geq 3 \) in which the \( r_L \) act as reflections for \( L \in \Sigma \).

For this, we employ a Tate module over the \( 2 \)-adics.

Fact 3.18 [6, 8, §3.3]. Let \( T \) be a \( p \)-torus of Prüfer \( p \)-rank \( n \) in a group of finite Morley rank. Then \( \text{End}(T) \) can be faithfully represented as the ring of \( n \times n \) matrices over the \( p \)-adic integers \( \mathbb{Z}_p \cong \text{End}((\mathbb{Z}/p^\infty)) \).

Proof of Lemma 3.17. For every \( L \in \Sigma \) and every \( 2 \)-torus \( X \leq S^0 \) disjoint from \( L \), \( X \) must act on \( L \) as elements of \( S^0 \cap L \) by Fact 1.6, so there is some \( 2 \)-torus \( Y \leq C_{S^0}(L) \) with \( X \leq Y(S^0 \cap L) \). So \( S^0 = C_{S^0}(r_L) \ast (S^0 \cap L) \). By Lemmas 3.10-1 and 3.6, \( r_L \) inverts \( S^0 \cap L = [S^0, r_L] \). Thus \( r_L \) acts as a “reflection” on \( S^0 \).

By Lemma 3.16, \( W_0 \) acts faithfully on \( S^0 \). By Fact 3.18, \( W_0 \) has a faithful representation over the ring of \( 2 \)-adic integers \( \mathbb{Z}_2 \) which has dimension \( \text{pr}(S^0) \geq 3 \). By tensoring with \( \mathbb{C} \), \( W_0 \) has a faithful representation \( R \) over \( \mathbb{C} \) which has dimension \( \text{pr}(S^0) \geq 3 \). The \( r_L \)'s continue to act as reflections in this representation.

Now suppose towards a contradiction that \( W_0 \) acts reducibly on \( R \). Since the representation \( R \) is completely reducible, \( R = R_1 \oplus R_2 \) where \( R_1 \) and \( R_2 \) are proper \( W_0 \)-invariant subspaces.

Suppose that \( W_0 \) acts trivially on \( R_1 \). Then there is a \( 2 \)-torus \( \hat{R}_i \) centralized by all \( r_L \), and \( G \leq C_G(\hat{R}_i) \), a contradiction. So we may assume that \( W_0 \) acts non-trivially on both \( R_1 \) and \( R_2 \).

For \( L \in \Sigma \), the \(-1\)-eigenspace \([R, r_L]\) of \( r_L \) belongs to one of the two subspaces, either \( R_1 \) or \( R_2 \). \( r_L \) acts as a reflection on that subspace and centralizes the other. Let \( \Sigma_i := \{ L \in \Sigma : [R, r_L] \leq R_i \} \) for \( i = 1, 2 \). For \( L \in \Sigma_1 \) and \( K \in \Sigma_2 \), we have \([r_L, r_K] = 1\), and thus \([L, K] = 1\) by Lemma 3.15, in contradiction with the fact that \( \Sigma \) is connected.

To further constrain \( W_0 \), we next obtain representations of \( W_0 \) over almost all finite fields.

Lemma 3.19 cf. [8, §3.4]. For primes \( q > \max(|W_0|, 3) \) with \( q \neq \text{char}(L) \) for any \( L \in \Sigma \), \( W_0 \) has a faithful irreducible representation over \( \mathbb{Z}/q\mathbb{Z} \), where, for any \( L \in \Sigma \), the involution \( r_L \) acts by reflection.

Proof. Consider the elementary abelian \( q \)-group \( E_q \) generated by all elements of order \( q \) in \( T \). \( W_0 \) clearly acts on \( E_q \). Let \( N = N_G(T) \). Since \( C_G(T) \leq C_N(E_q) \), we may show that \( W_0 \) acts faithfully by showing that \( C_N(E_q) \leq C_G(T) \).

For any \( x \in C_N(E_q) \leq N \), \( x \) acts on \( \Sigma \) by conjugation. For any \( L \in \Sigma \), if \( L^x \neq L \) then \( L \) and \( L^x \) either commute or generate a quasisimple group as root \( \text{SL}_2 \)-subgroups by Lemma 3.11-2. In either case, \(|L \cap L^x| \leq 2\), in contradiction to the fact that \( L \cap E_q = L^x \cap E_q \). So \( x \) normalizes \( L \), and the element \( x \) acts on \( T \cap L \) as an element of \( N_L(T \cap L) \) by Fact 1.6. Since the Weyl group of \( \text{SL}_2 \) inverts the torus, any element of \( N_L(T \cap L) \setminus C_L(T \cap L) \) inverts some element of \( E_q \). So \( x \) centralizes \( T \cap L \) for all \( L \in \Sigma \). Now \( x \) centralizes \( T = (T \cap L)L \in \Sigma \), and \( W_0 \) acts faithfully on \( E_q \).

We also observe that \( W_0 \) acts by reflections on \( E_q \) because, for every \( L \in \Sigma \), \([E_q, r_L]\) has order \( q \) and is inverted by \( r_L \), i.e. \(|E_q \cap L| = q\).
Now suppose toward a contradiction that $W_0$ acts reducibly on $E_q$. Since $q > |W_0|$, the representation is completely reducible, and $E_q = R_1 \oplus R_2$ where $R_1$ and $R_2$ are proper $W_0$-invariant subspaces of $E_q$.

Suppose that $W_0$ acts trivially on $R_i$. For any $L \in \Sigma$, $R_i$ acts by inner automorphisms on $L$ by Fact 1.6. We recall that nontrivial Weyl group elements in $W(L)$ invert the torus $T_L$. Since $R_i$ centralizes $T_L$, we know that $R_i$ acts via conjugation by elements of $T_L$. Since $W_0$ centralizes $R_i$, we find that $R_i$ centralizes $L$. So $R_i \leq Z(\langle \Sigma \rangle) = Z(G)$, a contradiction. So we may assume that $W_0$ acts nontrivially on both $R_1$ and $R_2$.

For $L \in \Sigma$, the eigenspace $[E_q, r_L]$ of $r_L$ belongs to one of the two subspaces, either $R_1$ or $R_2$. So $r_L$ acts as a reflection on that subspace and centralizes the other. Let $\Sigma_i := \{ L \in \Sigma : [E_q, r_L] \leq R_i \}$ for $i = 1, 2$. Then $\Sigma = \Sigma_1 \cup \Sigma_2$. For $L \in \Sigma_1$ and $K \in \Sigma_2$, we have $[r_L, r_K] = 1$, and thus $[L, K] = 1$ by Lemma 3.15, in contradiction with the fact that $\Sigma$ is connected.

The two preceding lemmas provide sufficient information to identify the Weyl group $W_0$.

**Lemma 3.20** cf. [8, Lemma 3.11]. There exists an irreducible root system $I$ of type $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, or $F_4$ on which $W_0$ acts as a crystallographic reflection group.

This lemma follows from the following major fact, which depends on a detailed analysis of the irreducible complex reflection groups [34, 23].

**Fact 3.21** [7, Thm. 2.3]. Let $W$ be a finite group, $I \subseteq W$ a subset, and $n$ an integer, satisfying the following conditions.

(i) The set $I$ generates $W$, consists of involutions, and is closed under conjugation in $W$;

(ii) The graph $\Delta_I$ with vertices $I$ and edges $(i, j)$ for noncommuting pairs $i, j \in I$ is connected;

(iii) For all sufficiently large prime numbers $\ell$, $W$ has a faithful representation $V_\ell$ over the finite field $\mathbb{F}_\ell$ in which the elements of $I$ operate as complex reflections, with no common fixed vectors.

Then one of the following occurs.

(a) $W$ is a dihedral group acting in dimension $n = 2$, or cyclic of order two.

(b) $W$ is isomorphic to an irreducible crystallographic Coxeter group, that is, $A_n, B_n, C_n, D_n$ ($n \geq 3$), $E_n$ ($n = 6, 7, 8$), or $F_n$ ($n = 4$).

(c) $W$ is a semidirect product of a quaternion group of order 8 with the symmetric group $S_{n_1}$, acting naturally, represented in dimension 2.

If, in addition, over some field, $W$ has an irreducible representation of dimension at least 3, in which the elements of $I$ act as reflections, then case (b) applies.

**Proof of Lemma 3.20.** We observe that $\{ r_L : L \in \Sigma \}$ is a normal subset of $W_0$ which generates $W_0$. The noncommuting graph on this set is connected by Lemma 3.15. So Lemmas 3.19 and 3.17, complete the verification of the hypotheses of Fact 3.21.
We also show that all reflections in $W_0$ come from our root SL$_2$-subgroups.

**Lemma 3.22** cf. [8, Lemma 3.12]. Every $r \in W_0$ which is a reflection in the representation $R$ over $\mathbb{C}$ has the form $r_K$ for some $K \in \Sigma$.

Recall that the reflections of a Coxeter group correspond to roots in the associated root system (see [29, Lemma 5.7]), and hence there are at most two conjugacy classes of reflections.

**Fact 3.23** [28, 10.4 Lemma C]. A finite irreducible reflection group of type $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$ has only one conjugacy class of reflections. A finite irreducible reflection group of type $B_n$, $C_n$, $F_4$, and $G_2$ has two conjugacy classes of reflections, corresponding to the short and long roots.

Since the roots of only one length are closed under the action of the Coxeter group, they form the root system for a proper subgroup.

**Fact 3.24.** The subgroup of $B_n$, $C_n$, $F_4$, or $G_2$ generated by the reflections associated to roots of only one length is a proper subgroup.

**Proof of Lemma 3.22.** By Fact 3.23, there are at most two conjugacy classes of reflections in $I$, corresponding to the short and long roots. So we may assume that $I$ has more than one root length, i.e. $W_0 \cong B_n$, $C_n$, or $F_4$, and that the set $S := \{r_L : L \in \Sigma\}$ consists of only one of these conjugacy classes. By Fact 3.24, $\langle S \rangle < W_0$, a contradiction. \(\square\)

3.3. **Identification**

We continue the analysis of the preceding subsections, loosely following [8, §3.6]. We will invoke the Curtis-Tits theorem which may be expressed as follows: a simply connected quasisimple algebraic group is the free amalgam of the system of subgroups and inclusion maps corresponding to all root SL$_2$-subgroups and subgroups generated by pairs of such subgroups, taken relative to a fixed maximal torus [25]. The Generic Identification Theorem of Berkman and Borovik proceeds by passing from the full system of groups and subgroups to the collection of subsystems corresponding to pairs of roots, which are now known. A flexible form of this result is based on a result of Timmesfeld [36].

**Fact 3.25** [7, Prop. 2.3]. Let $\Phi$ be an irreducible root system (of spherical type) and rank at least 3, and let $\Pi$ be a system of fundamental roots for $\Phi$. Let $X$ a group generated by subgroups $X_r$ for $r \in \Pi$. Set $X_\Pi = \langle X_r, X_s \rangle$. Suppose that $X_\Pi$ is a group of Lie type $\Phi_\Pi$ over an infinite field, with $X_r$ and $X_s$ corresponding root SL$_2$-subgroups with respect to some maximal torus of $X_\Pi$. Then $X/Z(X)$ is isomorphic to a group of Lie type via a map carrying the subgroups $X_r$ to root SL$_2$-subgroups.

We now conclude the proof of the Generic Trichotomy Theorem, working, as usual, under Hypothesis 3.12. By Lemma 3.20, $I$ is the desired irreducible root system of spherical type and rank at least 3. For every vertex $i \in I$, there is an $r_i \in W_0$ which is a reflection in the representation $R$ over $\mathbb{C}$. There is also a reflection
signalizers and balance

For every $L \in \Sigma$, By Lemma 3.22, there is an $L_i \in \Sigma$ such that $r_i = r_{L_i}$, and

$$<L_i| i \in I> = <\Sigma> = G.$$ 

For $i, j \in I$, the group $M := <L_i, L_j>$ is of Lie type by Lemma 3.11-2 when $[L_i, L_j] \neq 1$. If $[L_i, L_j] = 1$ then $M = L_i * L_j$ which has Lie type because $L_i$ and $L_j$ are algebraic over the same field. By Lemma 3.11-4, $L_i$ and $L_j$ are root $SL_2$-subgroups corresponding to a maximal torus $T_M$ of $M$. Now $G$ is a Chevalley group by Fact 3.25, as desired.

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