**Dissipative surface solitons in periodic structures**

Y. V. Kartashov¹, V. V. Konotop² (a) and V. A. Vysloukh³

¹ ICFO-Institut de Ciencies Fotòniques, and Universitat Politècnica de Catalunya, Mediterranean Technology Park 08860 Castelldefels (Barcelona), Spain, EU
² Centro de Física Teórica e Computacional and Departamento de Física, Faculdade de Ciências, Universidade de Lisboa - Avenida Professor Gama Pinto 2, Lisboa 1649-003, Portugal, EU
³ Departamento de Física y Matemáticas, Universidad de las Americas - Puebla, Santa Catarina Martir, 72820, Puebla, Mexico

*Received 18 May 2010; accepted in final form 31 July 2010. Published online 30 August 2010*

**PACS**

42.65.Tg – Optical solitons; nonlinear guided waves
42.65.Jx – Beam trapping, self-focusing and defocusing; self-phase modulation
42.65.Wi – Nonlinear waveguides

**Abstract** – We report dissipative surface solitons forming at the interface between a semi-infinite lattice and a homogeneous Kerr medium. The solitons exist due to the balance between amplification in the near-surface lattice channel and two-photon absorption. The stable dissipative surface solitons exist in both focusing and defocusing media, when propagation constants of corresponding states fall into a total semi-infinite or into one of total finite gaps of the spectrum (i.e. in a domain where propagation of linear waves is inhibited for the both media). In a general situation, the surface solitons form when the amplification coefficient exceeds the threshold value. When a soliton is formed in a total finite gap there exists also the upper limit for the linear gain.

Shallow periodic modulations of a refractive index may considerably affect the diffraction of low-power beams and properties of stationary nonlinear excitations [1]. Of particular interest are truncated periodic structures that support so-called surface lattice solitons propagating along the interfaces between periodic and uniform media. While the history of linear surface modes goes back to the seminal paper of Tamm [2], one-dimensional surface modes in nonlinear lattices were predicted in [3] and subsequently obtained in both focusing [4] and defocusing [5,6] materials, while their two-dimensional counterparts were generated in [7]. A characteristic feature of surface solitons is that they usually exist above the energy flow threshold. Surface lattice solitons inherit some properties of solitons in bulk lattices, in a sense that their shapes and transverse extent depend on the location of the propagation constant in a band-gap lattice spectrum. Moreover, there exists no sharp transition between modes strongly localized at the surface and modes localized at some distance from the surface or bulk modes: the localization domain moves smoothly from the boundary to some domain in the bulk as one follows the one-parametric branch of the solutions [8]. A surface can also support breather solutions, i.e. localized modes with periodically changing shapes [8]. Much richer behavior is observed when a defect is added into otherwise periodic guiding structure. In particular, when a defect is introduced into a surface lattice channel it may dramatically affect the conditions for surface soliton formation and result in the appearance of new types of surface modes [9].

The above-referred studies were dedicated to conservative systems. An interesting and still open problem is the impact of nonconservative surfaces on the existence and properties of the localized modes. A particularly interesting situation arises when a gain is applied at the interface between a homogeneous and a periodic medium. It is relevant to mention that the effect of a localized gain was analyzed for the existence and interaction of gap solitons in shallow fiber Bragg gratings [10], in media of two-level atoms [11], for the formation of spatial dissipative solitons in systems governed by the complex Ginzburg-Landau equation [12], and for the dynamic emission of moving lattice solitons in systems without dissipation [13].

In this letter we address the properties of truly stationary dissipative solitons forming at the edge of a semi-infinite lattice and existing due to the balance between

(a) E-mail: konotop@ciic.fc.ul.pt
localized gain in the near-surface lattice channel and strong two-photon absorption in a cubic medium. Dissipative surface solitons [14] were studied before only in the truncated discrete systems governed by the Ginzburg-Landau equation with uniform gain (the model introduced in [15]). Here we show that in an experimentally realistic setting dissipative surface solitons are pinned by the "defect" channel where gain is realized. Such modes are attractors and thus can be excited from a sufficiently large class of initial conditions, ranging from localized or extended regular patterns to noisy patterns. We argue that surface solitons form when the gain coefficient exceeds the threshold value, and if the propagation constant belongs to a finite gap, there exist also the upper bound for the allowed gain coefficient at which surface solitons can still be found.

Specifically, we consider the propagation of laser radiation in a lattice with spatially localized linear gain and nonlinear losses that can be described by the nonlinear Schrödinger (NLS) equation for the dimensionless light field amplitude $q$:

$$i\eta_\xi = -\frac{1}{2}q_{\eta \eta} - [R(\eta) - i\gamma(\eta)]q - \sigma|q|^2q - i\alpha|q|^3q.$$  \hspace{1cm} (1)

Here $\eta = x/x_0$ and $\xi = z/L_{dif}$ are the transverse and longitudinal coordinates normalized to the characteristic beam width $x_0$ and to the diffraction length $L_{dif} = kx_0^2$, $k = 2\pi n_0/\lambda$, $\lambda$ is the wave number, $n_0$ is the unperurbed refractive index, $I_0$ is the characteristic intensity, $L_{nl} = n_0/kn_2I_0$ is the nonlinear self-action length, $p_i = L_{dif}/L_{gain}$ is the linear gain coefficient, $L_{gain}$ is the amplification length, $\alpha = L_{dif}/L_{loss}$ is the coefficient of nonlinear losses, $L_{loss} = 1/\alpha x_0 I_0$ characterizes the length of the two-photon absorption and $\sigma = 1$ ($\sigma = -1$) corresponds to the focusing (defocusing) nonlinearity.

Spatial solitons were successfully observed in nonlinear waveguide arrays made of an AlGaAs alloy below the half-band-gap, $\lambda \approx 1.53 \mu m$. Such arrays have a typical length of 6 mm, a waveguide separation of 4-7 $\mu m$ and an effective core area of about 20 $\mu m^2$ [16]. In this case (as well as in our model) the two-photon absorption is the dominating mechanism of optical losses: the linear absorption coefficient is around 0.1 cm$^{-1}$, while the typical value of the two-photon absorption coefficient is $\alpha_2 \approx 0.3$ cm$^{-1}$, and the representative peak soliton intensity is above $I_0 \approx 5$ GW/cm$^2$. Importantly, the same semiconductor material (AlGaAs) is widely used for the production of the wide-band (typical bandwidth of 60-70 nm) semiconductor optical amplifiers with rather high optical gain in the same spectral range [17]. In particular, for the above-mentioned structure, $n_0 = 3.34$ and the Kerr coefficient $n_2 = 1.6 \times 10^{-13}$ cm$^2$/W. Then, for a laser beam with $x_0 = 5 \mu m$ one obtains $L_{dif} \approx L_{nl} \approx 0.34$ mm and $L_{loss} \approx 6.67$ mm, that corresponds to $\sigma \approx 0.05$ (TE polarization is under consideration). Then, $p_i = 0.1$ is achieved for $L_{gain} \approx 0.34$ cm.

While qualitatively the results reported below are valid for general periodic modulations $R(\eta)$, the quantitative analysis is performed for the semi-infinite lattice of the form $R(\eta) = p_\tau \sin^2(2\eta)$, with $p_\tau$ being the depth of the lattice proportional to the refractive index modulation, which is placed at $\eta > 0$ (notice that the period of the structure can also be made $\pi/2$ by simple renormalization). At $\eta \leq 0$ we have a homogeneous medium where $R(\eta) \equiv 0$. We suppose that localized gain, whose profile is described by the function $\gamma(\eta)$, is realized in the vicinity of a near-surface lattice channel. In the simplest case we consider $\gamma(\eta)$ exactly coinciding with $R(\eta)$ in the first channel of the lattice, i.e. $\gamma(\eta) = p_\tau \sin^2(2\eta)$ ($p_\tau > 0$ is the linear gain coefficient) for $0 < \eta < \pi/2$, and $\gamma(\eta) \equiv 0$ otherwise, but the situation when the peak position of $\gamma(\eta)$ is shifted with respect to first maximum of $R(\eta)$ by a distance $\eta_0$ will be considered too.

Localized dissipative solitons are searched in the form $q(\eta, \xi) = w(\eta) \exp(ib\xi + ib\theta(\eta))$ with $w(\eta)$ and $\theta(\eta)$ being the amplitude and the real stationary phase of the field, and $b$ being the propagation constant. Exponentially localized modes in periodic media [18] and conservative surface modes [3–5] emerge when the propagation constant $b$ falls into one of the gaps of the spectrum of the periodic guiding structure. It turns out that this is also true in the case of dissipative surface solitons. To prove this, we rewrite (1) in terms of the real functions $w(\eta)$ and $v(\eta) \equiv \eta_\theta$:

$$w_{\eta \eta} - 2bw - v^2w + 2\sigma w^3 + 2Rw = 0,$$  \hspace{1cm} (2a)

$$v(\eta)w_{\eta} - 2v^2w + 2\sigma w^4 = 0,$$  \hspace{1cm} (2b)

and consider first the limit $\eta \to -\infty$. Since in this limit $R \equiv 0$, one readily finds the explicit asymptotics

$$w = A_e^{-\sqrt{2b\eta}} - \sigma A^3 e^{3\sqrt{2b\eta}}/(8b) + O(e^{5\sqrt{2b\eta}}),$$  \hspace{1cm} (3a)

$$v = -\alpha A^3 e^{2\sqrt{2b\eta}}/(2\sqrt{2\eta}) + O(e^{\sqrt{2b\eta}}),$$  \hspace{1cm} (3b)

where $A_e$ is a real constant, depending on the total energy flow $U = \int_{-\infty}^{\infty} w^2d\eta$. The first (trivial) consequence of the obtained asymptotics is that localized solutions exist only in domains outside the linear spectrum, i.e. at $b > 0$. The second important conclusion is that in the asymptotic region $w^2w \sim \exp(5\sqrt{2b\eta})$, i.e. it decays faster than $w^2$. Thus, at $\eta \to -\infty$ the shape of the mode is described by the conservative properties of the medium, i.e. $q(\eta, \xi)$ exponentially approaches the standard stationary NLS soliton.

A similar, but more sophisticated, analysis can be performed for the limit $\eta \to \infty$, where the linear lattice is present. One still can prove that $v^2w$ decays faster than $w^2$, and thus the leading order for the field amplitude is given by the Floquet theorem: $w = A_e \exp(-\mu \eta)F_n(\eta)$, where $F_n(\eta) = \pi/2$ or $\pi$ periodic function, $\mu$ is the Floquet exponent which is determined by the detuning of the propagation constant from the band-edge towards the $n$-th stop gap (the lower and upper boundary of $n$-th stop gap).
Dissipative surface solitons in periodic structures

Fig. 1: (Colour on-line) (a) Band-gap spectrum of the infinite lattice. (b) $b$ vs. $p_i$ in focusing (curve 1) and defocusing (curve 2) media at $\alpha = 1$. The horizontal lines show the limits of the total gaps. $U$ vs. $p_i$ in focusing (c) and defocusing (d) media. Panels (b)–(d) correspond to $b_i = 0$. (e) $U$ vs. $\eta_s$ at $\alpha = 0.5$. (f) Maximal positive ($\eta_s^+$) and negative ($\eta_s^-$) shifts of the amplifying domain with respect to $R(\eta)$ at which the surface soliton still exists vs. $p_i$ at $\alpha = 0.5$. The circles in (b) and (e) correspond to the solitons shown in figs. 2(a) and (b), the circles in (c), (d) correspond to the solitons shown in the right column of fig. 2.

gap will be designated by $b_n^-$ and $b_n^+$, respectively, the number of the first finite gap is set to be $n = 1$, while the semi-infinite gap is denoted by $n = 0$, and $A_+$ is the normalization constant (see, e.g., [18] for more details). We illustrate the band-gap spectrum in fig. 1(a).

Thus, for existence of a localized surface mode one has to require that the propagation constant lies beyond the allowed band of the linear spectrum of the uniform medium, on the one hand, and belongs to one of the gaps of the periodic structure on the other hand. The respective domains will be termed total gaps (to distinguish them from the own gaps of the lattice: as it is clear total gaps represent a subset of the lattice gaps). For example, for the case of $p_r = 5$, studied below in details, there are only two total gaps: the semi-infinite total gap $b \in (2.875, \infty)$ and the total finite gap $(0, 1.840)$ while the lattice gaps, shown in fig. 1(a), are given by $(2.875, \infty)$, $(-0.645, 1.840)$, etc. (i.e. $b_0^- = 2.875$, $b_1^+ = 1.840$, $b_1^- = -0.645$ etc.)

The established constraints on $b$ naturally impose limitations on the possibility of the excitation of the surface modes. The mismatch between the boundaries of the gaps in the left- and right-hand structures implies the existence of a threshold value, $U_{\text{cut}}$, of the energy flow. In order to show this, let us consider a mode whose propagation constant $b$ belongs to the total semi-infinite gap and approaches $b_0^-$, i.e. $0 < b - b_0^- \ll 1$. Considering $\eta > 0$, we observe that when $b \rightarrow b_0^-$ the amplitude $w$ tends to zero: $w_{\text{max}} \rightarrow 0$ (see, e.g., [19] and references therein). On the other hand, considering (2) at $\eta = 0$ as an ODE defining the shape of the soliton, the smallness of $w$ means that the terms $w^2w$ in (2a) can be neglected and the asymptotic behavior is described by the conservative NLS equation, i.e. by (3a). Since, $b$ does not go to zero (due to finite value of $b_0^-$), also the nonlinear term $w^2$ can be neglected and the field behavior is described by the linear equation $w_{\eta\eta} = 2bw$. Thus, the function $w$ must be exponentially decaying, which for the linear ODE at hand is only possible if $w_\eta(0) = \sqrt{2bw}(0)$. As is clear, this is an extra condition, in addition to the continuity of $q$ and $q_\eta$, which must be satisfied at the boundary, i.e. at $\eta = 0$. In a general situation this is impossible with only two available constants $A_+$ and $A_-$, which are determined by the total energy $U$ and by the properties of the linear lattice at $\eta > 0$. In other words, by assuming that the amplitude of the mode can go to zero we have arrived at a contradiction. Thus, there exists a minimal threshold value of $w_{\text{max}}$ above which surface modes can exist. Taking now into account the Sobolev inequality $|w_{\text{max}}|^2 \leq 2U \int |w|^2d\eta$ we conclude that there exists a threshold for the energy flow. Moreover, since the limit $b \rightarrow b_0^-$ would imply $w_{\text{max}} \rightarrow 0$ we finally conclude that there exists also a cut-off value $b_{\text{cut}} > b_0^-$ such that only for $b > b_{\text{cut}}$ one can find dissipative surface modes. Below we illustrate these properties in numerical examples. We notice that the above arguments are also valid for the pure conservative case, and thus explain the threshold values for the energy flow $U$ observed in earlier studies [3–5].

Dissipative surface solitons exist not only due to the balance between diffraction, refraction, and nonlinearity, but also due to the balance between localized gain and nonlinear losses, expressed by the condition $\alpha \int_{-\infty}^{\infty} w^4d\eta = \int_{-\infty}^{\infty} \gamma(\eta)w^2d\eta$. Hence the propagation constant $b$ and the energy flow $U$ are determined by the gain $p_i$ and by nonlinear losses $\alpha$. Typical dependences $b(p_i)$ for surface solitons in the focusing medium are shown in fig. 1(b). The propagation constant $b$ in the focusing medium falls into the total semi-infinite gap, i.e. $b > b_0^-$, and monotonically grows with $p_i$. Such surface solitons exist above some threshold value of $b$, $b_{\text{cut}} > b_0^-$, respectively above the minimal value of the gain coefficient,
conservative lattice for various shifts of the amplifying domain with respect to the conservative lattice at $\alpha = 0.5$, $\eta_s = 0$. Panel (b) shows solitons residing in the first lattice channel for various shifts of the amplifying domain with respect to the conservative lattice at $p_i = 1$, $\alpha = 0.5$. Panel (c) shows solitons residing in the second lattice channel at $\alpha = 0.4$, $\eta_s = \pi/2$. Gray regions indicate guiding lattice channels, while cyan regions show amplifying domains. In (b) we show only the amplifying domain corresponding to $\eta_s = 0.7$.

denoted below as $p_i^{\text{low}}$, and above the threshold energy flow $U_{\text{cut}}$. The energy flow monotonically increases with $p_i$ everywhere except for a very narrow region close to $p_i^{\text{low}}$ (fig. 1(c)). Typical profiles of dissipative surface solitons in a focusing medium are shown in fig. 2(a) when gain is realized in the near-surface lattice channel. For low values of $p_i$ the surface solitons expand considerably into the lattice region and acquire a shape reminiscent to shape of the Bloch state bordering the respective gap edge. Due to energy flow in the transverse direction and despite the presence of nonlinear losses in the entire medium solitons may extend far beyond the amplifying region. With the increase of the gain $p_i$ the light gradually concentrates in the near-surface channel. It should be stressed that solitons may form in a near-surface lattice channel even when the gain is displaced by a distance $\eta_s$ with respect to the first maximum of the lattice. The typical profile of a surface soliton supported by such shifted-gain landscape is shown in fig. 2(b) — here the maximum of the field remains in the surface channel despite the fact that gain is realized almost between the first and the second channels. The energy flow of surface solitons first decreases with $\eta_s$, and then increases when the shift approaches the maximal value beyond which the surface soliton cannot form in the first lattice channel (fig. 1(e)). The maximal possible shift of the gain landscape quickly increases with $p_i$ and saturates already at $p_i = 3$ (fig. 1(f)). Remarkably, surface solitons may form not only when the gain profile is shifted into the depth of the lattice (positive $\eta_s$), but also when the gain is shifted into a uniform medium (negative $\eta_s$). When the shift $\eta_s$ becomes sufficiently large solitons may form in the second, third, etc., channels of the lattice. Representative examples of profiles of dissipative surface solitons in the second lattice channel are shown in fig. 2(c). Such solitons feature smaller thresholds (both in terms of $p_i$ and $U$) for their existence than their counterparts in the first lattice channel.

In the case of a defocusing medium localized gain can support dissipative gap solitons featuring characteristic oscillating tails (inside the lattice) at the surface of a semi-infinite lattice. The propagation constant of such solitons falls into the finite total gap, $b \in (0, b_1^\text{up})$ (see fig. 1(b) where there is only one total gap) and decreases with $p_i$. Like their counterparts in a focusing medium, now the solitons emerging from the finite total gap exist above the minimal value of the gain coefficient $p_i^{\text{low}}$ (for this value of linear gain the propagation constant approaches a cut-off value that is close to the upper edge of the total finite gap). However, now, due to finiteness of the gap, there exists also the upper limit for the linear gain $p_i^{\text{up}}$ at which $b$ reaches zero value, i.e. dissipative gap solitons can be found for $p_i^{\text{low}} < p \leq p_i^{\text{up}}$. Respectively, the energy flow takes on the values from the finite interval, where energy flow is the increasing function of $p_i$ (fig. 1(d)). When $p_i \rightarrow p_i^{\text{low}}$ gap surface solitons expand dramatically into the lattice region, but remain well localized inside the uniform medium (fig. 3(a)), in accordance with the asymptotics (3). The best overall localization is achieved for intermediate $p_i$ values (fig. 2(e)). When $b$ approaches zero (respectively, $p_i$ approaches $p_i^{\text{up}}$) the gap surface solitons again become poorly localized due to appearance of long tails in the uniform medium (fig. 2(f)). The domains of existence of dissipative surface solitons in both focusing and defocusing media on the $(\alpha, p_i)$-plane are shown in fig. 3. The minimal gain $p_i^{\text{low}}$ required for the existence of solitons in a focusing medium as well as the width $p_i^{\text{up}} - p_i^{\text{low}}$ of the band of gain coefficients where solitons exist in defocusing medium, increase with nonlinear losses $\alpha$.

Finally, we analyzed the stability of the obtained soliton solutions. We have performed both, the linear
stability analysis and the direct propagation method in the presence of input perturbations. Both of them showed that the dissipative surface solitons are exceptionally robust and can withstand even strong shape deformations almost in the entire existence domain (we considered the lowest branches only).

To conclude, we have demonstrated that an interface with a gain between a periodic and a homogeneous medium can support a diversity of the surface solitons. The propagation constants of the modes belong to one of the total gaps of the structure. Such modes are characterized by the presence of threshold values of the energy flow and cut-off values of the propagation constant and, respectively, of the linear gain. The dissipative surface solitons are attractors and therefore can be easily excited from a wide range of initial conditions. Our analysis clearly indicates that similar modes can be obtained in a more general situation where an interface with gain separates two different periodic media. In this last case one can expect larger diversity of the surface modes, especially when the composite structure is characterized by more than one total finite gap. Finally, the reported surface dissipative solitons seem to be very promising objects for spectroscopy, sensors, excitations of nano-particles, etc. since their technological manufacturing is already available.

REFERENCES

[1] Lederer F. et al., *Phys. Rep.*, 463 (2008) 1; Kartashov Y. V., Vysloukh V. A. and Torner L., *Prog. Opt.*, 52 (2009) 63.
[2] Tamm I. E., *Z. Phys.*, 76 (1932) 849.
[3] Makris K. G. et al., *Opt. Lett.*, 30 (2005) 2466.
[4] Suntsov S. et al., *Phys. Rev. Lett.*, 96 (2006) 063901.
[5] Kartashov Y. V., Vysloukh V. A. and Torner L., *Phys. Rev. Lett.*, 96 (2006) 073901.
[6] Rosenberg C. R. et al., *Phys. Rev. Lett.*, 97 (2006) 083901; Smirnov E. et al., *Opt. Lett.*, 31 (2006) 2338.
[7] Wang X. et al., *Phys. Rev. Lett.*, 98 (2007) 123903; Szameit A. et al., *Phys. Rev. Lett.*, 98 (2007) 173903.
[8] Bludov Yu. V. and Konotop V. V., *Phys. Rev. E*, 76 (2007) 046604.
[9] Molina M. I., Garanovich I. L., Sukhorukov A. A. and Kivshar Y. S., *Opt. Lett.*, 31 (2006) 2332; Chen W. H., He Y. J. and Wang H. Z., *Opt. Express*, 14 (2006) 11271; Szameit A. et al., *Opt. Lett.*, 34 (2009) 797; Malkova N. et al., *Opt. Lett.*, 34 (2009) 1633.
[10] Mak W. C., Malomed B. A. and Chu P. L., *Phys. Rev. E*, 67 (2003) 026608.
[11] Melnikov I. V. and Aitchison J. S., *Appl. Phys. Lett.*, 87 (2005) 201111.
[12] Lam C.-K., Malomed B. A., Chow K. W. and Wai P. K. A., *Eur. Phys. J. ST*, 173 (2009) 233.
[13] Kartashov Y. V., Vysloukh V. A. and Torner L., *Opt. Lett.*, 32 (2007) 2061.
[14] Mihalache D., Mazilu D., Lederer F. and Kivshar Y. S., *Phys. Rev. A*, 77 (2008) 043828.
[15] Efremidis N. K. and Christodoulides D. N., *Phys. Rev. E*, 67 (2003) 026606; Efremidis N. K., Christodoulides D. N. and Hizanidis K., *Phys. Rev. A*, 76 (2007) 043839.
[16] Eisenberg H. S., Silberberg Y., Morandotti R., Boyd A. R. and Aitchison J. S., *Phys. Rev. Lett.*, 81 (1998) 3383.
[17] Connelly M. J., *Semiconductor Optical Amplifiers* (Springer) 2002.
[18] Alfimov G. L., Konotop V. V. and Salerno M., *Europhys. Lett.*, 58 (2002) 7.
[19] Cruz H. A., Brazhnyi V. A., Konotop V. V. and Salerno M., *Physica D*, 238 (2009) 1372.