Mesonic Spectrum of Two Dimensional Supersymmetric Theories.

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Abstract
We consider a bound state problem for a family of supersymmetric gauge theories with fundamental matter. These theories can be obtained by a dimensional reduction of supersymmetric QCD from three dimensions to 1+1 and subsequent truncation of some of the fields. We find that the models without adjoint scalar converge to well-defined continuum limits and calculate the resulting spectra of these theories. We also find the critical value of coupling at which an additional massless state is observed. By contrast, the models containing adjoint scalars, seem to have a continuous mass spectrum in the limit of infinite volume.
1 Introduction.

One of the most important problems in Quantum Field Theory is the study of the bound state spectra of non-abelian gauge theories. There are several approaches to this problem. For QCD like theories lattice gauge theory is probably the most popular approach since the approximation does not break the most important symmetry, gauge symmetry. Similarly for supersymmetric theories, supersymmetric DLCQ (SDLCQ) is probably the most powerful approach since the approximation does not break the most important symmetry, supersymmetry. In this paper will consider supersymmetric theories and follow this latter approach.

Long ago ’t Hooft [1] showed that two dimensional models can be a powerful laboratory for studying the bound state problem. These models remain popular to this day since they are easy to solve and share many properties with their four dimensional cousins, most notably stable bound states. Supersymmetric two dimensional models are particularly attractive since they are also super-renormalizable. Given that the dynamics of gauge field is responsible for strong interaction and the formation of bound states, it comes as no surprise that a great deal of effort has gone into investing bound states of pure glue in supersymmetric models [2, 3, 4]. While such theories capture the essential properties of the mass spectrum and some of them are relevant for the string theory [5], the wavefunctions are quite different from the ones for mesons and baryons. Extensive study of the meson spectrum of non–supersymmetric theories has been done (see [6] for a review), but the problem has not been addressed in a context of supersymmetric models. In this paper we will introduce seven new two dimensional supersymmetric models that have not been previously studied that are particularly useful for studying mesons within a two dimensional supersymmetric laboratory. Throughout this paper we use a word “meson” to indicate the group structure of the state. Namely we define a meson as a bound state whose wave-function can be written as a linear combination of parton chains, each chain starts and ending with a creation operator in fundamental representation. In supersymmetric theories the states, defined this way, can have either bosonic or fermionic statistics.

To simplify the calculation we will consider only the large $N$ limit [1], which has proven to be a powerful approximation for bound state calculations. While baryons can be constructed in this limit [7], they have an infinite number of partons and thus practical calculations for such states are compli-
cated. In this paper we concentrate our attention on the mesonic spectrum. Note that throughout this paper we completely ignore the zero mode problem \[8\], however it is clear that considerable progress on this issue could be made following our earlier work on the zero modes of the two dimensional supersymmetric model with only adjoint fields \[9\].

The paper has the following organization. In section 2 we consider the three dimensional SQCD and dimensionally reduce it to 1 + 1. We perform the light-cone quantization of the resulting theory by applying canonical commutation relations at fixed $x^+$ and choosing the light-cone gauge ($A^+ = 0$) for the vector field. After solving the constraint equations we end up with a model containing 4 dynamical fields. We construct the supercharge for the dimensionally reduced theory and observe that it can also be used to define models with less supersymmetry. In particular, in section 3 we study the mesonic spectrum of systems without dynamical quarks. We find that one of these systems (we call it $A\lambda$) has many light states in SDLCQ, which probably give rise to a continuous spectrum in the continuum limit. The other system, containing only dynamical gluinos, seems not to have a well-defined bound state problem: all masses are pushed to infinity in the continuum limit. In section 4 we study the systems which do not include the adjoint scalar. They all share the same properties: a well-defined continuum spectrum and the existing of a critical value of the coupling constant at which the lowest mass bound state becomes massless. Finally, in section 5 we study the remaining theories which include the adjoint boson and at least one of the fundamental fields. We find that all this models have a continuous spectrum, which seems to be a general property of two dimensional supersymmetric systems with adjoint scalars \[10\].

2 Supersymmetric Systems with fundamental matter.

We consider the family of supersymmetric models in two dimensions which can be obtained as the result of dimensional reduction of SQCD$_{2+1}$ and possible truncation of some fields in the resulting two dimensional theory. Our starting point is the three dimensional action:

\[
S = \int d^3x \text{tr} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{i}{2} \bar{\Lambda} \Gamma^\mu D_\mu \Lambda + D_\mu \xi^\dagger D^\mu \xi + i \bar{\Psi} D_\mu \Gamma^\mu \Psi - \right.
\]
This action describes the system of a gauge field $A_\mu$ and its superpartner $\Lambda$, both taking values in the adjoint representation of $SU(N)$, and two complex fields, a scalar $\xi$ and a Dirac fermion $\Psi$, transforming according to the fundamental representation of the same group. Thus in matrix notation the covariant derivatives are given by:

$$D_\mu \Lambda = \partial_\mu \Lambda + ig[A_\mu, \Lambda], \quad D_\mu \xi = \partial_\mu \xi + ig' A_\mu \xi, \quad D_\mu \Psi = \partial_\mu \Psi + ig' A_\mu \Psi. \quad (2)$$

The action $(1)$ is invariant under supersymmetry transformations which are parameterized by a two–component Majorana fermion $\varepsilon$:

$$\delta A_\mu = i \frac{1}{2} \varepsilon \Gamma_\mu \Lambda, \quad \delta \Lambda = - \frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \varepsilon, \quad \delta \xi = i \frac{1}{2} \varepsilon \Psi, \quad \delta \Psi = - \frac{1}{2} \Gamma^\mu \varepsilon D_\mu \xi. \quad (3)$$

We introduced the commutator of the Dirac matrices:

$$\Gamma^{\mu\nu} = \frac{1}{2} [\Gamma^\mu \Gamma^{\nu}] .$$

Using standard techniques one can evaluate the Noether current corresponding to these transformations:

$$\varepsilon q^\mu = \frac{i}{4} \varepsilon \Gamma^{\alpha\beta} \Gamma^\mu \text{tr} (\Lambda F_{\alpha\beta}) + \frac{i}{2} D^\mu \xi \varepsilon \Psi + \frac{i}{2} \xi \varepsilon \Gamma^{\mu\nu} D_\nu \Psi - \frac{i}{2} \varepsilon \Psi D^\mu \xi + \frac{i}{2} D_\nu \Psi \Gamma^{\mu\nu} \varepsilon \xi. \quad (4)$$

We will consider the reduction of this system to two dimensions, which means that the field configurations are assumed to be independent of the space–like dimension $x^2$. In the resulting two dimensional system we will implement light–cone quantization, which means that initial conditions as well as canonical commutation relations will be imposed on a light–like surface $x^+ = \text{const}$. In particular we construct the supercharge by integrating the current $(4)$ over the light–like surface:

$$\varepsilon Q = \int dx^- dx^2 \left( \frac{i}{4} \varepsilon \Gamma^{\alpha\beta} \Gamma^+ \text{tr} (\Lambda F_{\alpha\beta}) + \frac{i}{2} D_\mu \xi \varepsilon \Psi + \frac{i}{2} \xi \varepsilon \Gamma^{+\nu} D_\nu \Psi - \frac{i}{2} \varepsilon \Psi D^+ \xi + \frac{i}{2} D_\nu \Psi \Gamma^{+\nu} \varepsilon \xi \right) . \quad (5)$$
Since all fields are assumed to be independent of $x^2$, the integration over this coordinate gives just a constant factor, which we absorb by a field redefinition.

If we consider a specific representation for the Dirac matrices in three dimensions:

$$
\Gamma^0 = \sigma_2, \quad \Gamma^1 = i\sigma_1, \quad \Gamma^2 = i\sigma_3,
$$

then the Majorana fermion $\Lambda$ can be chosen to be real, it is also convenient to write the fermions in the component form:

$$
\Lambda = (\lambda, \bar{\lambda})^T, \quad \Psi = (\psi, \bar{\psi})^T, \quad Q = (Q^+, Q^-)^T
$$

In terms of this decomposition the superalgebra has explicit $(1,1)$ form:

$$
\{Q^+, Q^+\} = 2\sqrt{2}P^+, \quad \{Q^-, Q^-\} = 2\sqrt{2}P^-, \quad \{Q^+, Q^-\} = 0.
$$

The traditional way of solving the bound state problem [6] is based on a simultaneous diagonalization of the momentum $P^+$ and the Hamiltonian $P^-$, but as one can see from the structure of (8), the same problem can be solved by diagonalizing $P^+$ and $Q^-$ instead [2].

In order to solve the bound state problem we impose the light cone gauge ($A^+ = 0$), then the supercharges are given by:

$$
Q^+ = 2 \int dx^- \left( \lambda \partial_- A^2 + \frac{i}{2} \partial_- \xi \bar{\psi} - \frac{i}{2} \psi \bar{\xi} + \frac{i}{2} \xi \bar{\psi} + \frac{i}{2} \partial_- \psi \xi \right),
$$

$$
Q^- = -2 \int dx^- \left( -\lambda \partial_- A^- + i\xi \bar{D}_2 \psi - iD_2 \psi \xi + \frac{i}{\sqrt{2}} \partial_- (\bar{\psi} \xi - \xi \bar{\psi}) \right).
$$

Note that apart from a total derivative these expressions involve only left-moving components of fermions ($\lambda$ and $\psi$). In fact in the light–cone formulation only these components are dynamical. To see this we consider the equations of motion that follow from the action (11), in the light cone gauge. Three of them serve as constraints rather than as dynamical statements:

$$
\partial_2 A^- = gJ, \quad J = i[A_2, \partial_- A^2] + \frac{1}{\sqrt{2}}\{\lambda, \lambda\} - \hbar \partial_- \xi \bar{\psi} + \hbar \xi \partial_- \bar{\psi} + \sqrt{2}h\psi \bar{\psi}^\dagger,
$$
Table 1: Interacting supersymmetric models with fundamental matter.
The numbers refer to the section in the paper where appropriate model is studied, and symbol — appears when the models which do not exist.

|          | no fundamentals | ψ  | ξ  | ψξ |
|----------|-----------------|----|----|----|
| λ        | —               | 3  | 3  | 3  |
| $A^2$    | —               | —  | —  | —  |
| $\lambda A^2$ | 3           | 3  | 3  | 3  |

\[
\partial_\tau \tilde{\lambda} = -\frac{ig}{\sqrt{2}} \left( [A^2, \lambda] + i h \xi \psi^\dagger - i h \psi \xi^\dagger \right), \quad (12)
\]

\[
\partial_\tau \tilde{\psi} = -\frac{ig'}{\sqrt{2}} A^2 \psi + \frac{g}{\sqrt{2}} \lambda \xi. \quad (13)
\]

We introduced the relative coupling for the fundamental matter: $h = g'/g$. Apart from the zero mode problem \[8\], one can invert the first constraint to write the auxiliary field $A^-$ in terms of physical degrees of freedom. Substituting the result into the expression for the supercharge and omitting the boundary term, we get:

\[
Q^- = -2 \int dx^- \left( g J \frac{1}{\partial_\tau} \lambda + g' \xi^\dagger A^2 \psi + g' \psi^\dagger A^2 \xi \right). \quad (14)
\]

This supercharge gives rise to a whole family of supersymmetric theories. Namely one can see that the expression (14) is meaningful even if we exclude some of the fields from the theory. As soon as we have at least one fermion and at least one field in the adjoint representation, (14) defines an interacting theory with supersymmetry. Some of these theories were studied before (namely the pure adjoint systems with \[2\] or without \[3\] bosons), but many of them are new. In this paper we will study the mesonic spectrum of all these models, their field content is summarized in the table 1.

In order to solve the bound state problem we apply the methods of Supersymmetric DLCQ. Namely we compactify the two dimensional theory on a light–like circle ($-L < x^- < L$), and impose periodic boundary conditions on all physical fields. This leads to the following mode expansions:

\[
A^2_{ij}(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left( a_{ij}(k)e^{-ik\pi x^-/L} + \bar{a}^t_{ji}(k)e^{ik\pi x^-/L} \right), \quad (15)
\]
\[ \lambda_{ij}(0, x^-) = \frac{1}{2^\frac{1}{4}} \sqrt{2L} \sum_{k=1}^{\infty} \left( b_{ij}(k) e^{-ik\pi x^-/L} + b_{ji}^\dagger(k) e^{ik\pi x^-/L} \right), \] (16)

\[ \xi_i(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left( c_i(k) e^{-ik\pi x^-/L} + \tilde{c}_i^\dagger(k) e^{ik\pi x^-/L} \right), \] (17)

\[ \psi_i(0, x^-) = \frac{1}{2^\frac{1}{4}} \sqrt{\frac{1}{L}} \sum_{k=1}^{\infty} \left( d_i(k) e^{-ik\pi x^-/L} + \tilde{d}_i^\dagger(k) e^{ik\pi x^-/L} \right). \] (18)

We drop the zero modes of the fields. Including them could lead to new and interesting effects (see [9], for example), but this is beyond the scope of this work. In the light–cone formalism one treats \( x^+ \) as a time direction, thus the commutation relations between fields and their momenta are imposed on the surface \( x^+ = 0 \). For the system under consideration this means:

\[ \left[ A_{ij}^2(0, x^-), \partial_- A_{kl}^2(0, y^-) \right] = i \left( \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \delta(x^- - y^-), \] (19)

\[ \left\{ \lambda_{ij}(0, x^-), \lambda_{kl}(0, y^-) \right\} = \sqrt{2} \left( \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \delta(x^- - y^-), \] (20)

\[ \left[ \xi_i(0, x^-), \partial_- \xi_j(0, y^-) \right] = i \delta_{ij} \delta(x^- - y^-), \] (21)

\[ \left\{ \psi_i(0, x^-), \psi_j(0, y^-) \right\} = \sqrt{2} \delta_{ij} \delta(x^- - y^-). \] (22)

These relations can be rewritten in terms of creation–annihilation operators:

\[ [a_{ij}, a_{kl}^\dagger] = \left( \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \quad \left\{ b_{ij}, b_{kl}^\dagger \right\} = \left( \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \] (23)

\[ [c_i, c_j^\dagger] = \delta_{ij}, \quad [\tilde{c}_i, \tilde{c}_j^\dagger] = \delta_{ij}, \quad [d_i, d_j^\dagger] = \delta_{ij}, \quad [\tilde{d}_i, \tilde{d}_j^\dagger] = \delta_{ij}. \] (24)

In this paper we will discuss numerical results obtained in the large \( N \) limit, i.e. we neglect \( 1/N \) terms in the above expressions. Although \( 1/N \) corrections may lead to interesting effects [11], they are beyond the scope of this work. In the presence of fundamental matter, however, one can also define different limits by making the number of flavors comparable with number of colors [12]. While this is an interesting direction for a future exploration, in the current paper we concentrate on models with one flavor and an infinite number of colors.

### 3 Mesons involving static (s)quarks.

We begin our consideration with the simplest models involving only adjoint matter. While the glueball spectrum of such models was studied before
it might also be interesting to look at the meson–like states. In the
string interpretation of these theories such states would correspond to
open strings with freely moving endpoints. In the QCD language the model
corresponds to a system of interacting gluons and gluinos which is bounded
by nondynamical (s)quark and anti-(s)quark. In the large $N$ limit we will
have to consider only a single (s)quark—anti-(s)quark pair, thus the Fock
space is constructed from the states of the following type:

$$\bar{f}_{i_1} a_{i_1i_2}^\dagger (k_1) \ldots b_{i_{n+1}i_n}^\dagger (k_n) \ldots f_{i_p} |0\rangle.$$  \hspace{1cm} (25)

Here $|0\rangle$ is a vacuum defined by annihilation operators $a_{ij}$ and $b_{ij}$ and $\bar{f}_i$ and $f_{ip}$ are sets of $c$–numbers.

The supercharges for such a system can be constructed by eliminating
fundamental matter from the expressions (9) and (14):

$$Q^+ = 2 \int dx^- \lambda \partial_- A^2,$$

$$Q^- = -2g \int dx^- \left( i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}} \{\lambda, \lambda\} \right) \frac{1}{\partial_-} \lambda.  \hspace{1cm} (26)$$

The supercharge $Q^-$ has an interesting property which can be seen from
its mode expansion. Acting on states (25), this operator changes the
numbers of bosons $a_{ij}^\dagger$ by an even number (0 or ±2), thus one can perform the
diagonalization of $P^-$ on two separate spaces: those containing either even
or odd number of bosons. The second supercharge $Q^+$ makes the situation
even more interesting: it maps one of these spaces into another and thus
leads to the same massive spectra in both sectors (see [14]). We have studied
the mesonic mass spectrum of $(Q^-)^2$ at low values of the harmonic resolution
$k$ and found no massless states, thus the spectra in two different sectors are
completely identical. Note that this property is not satisfied for the “glueball
spectrum”: there we found a lot of exact massless states [14, 10].

In figure 1a we present the results of the numerical diagonalization of the
Hamiltonian. Note that we considered only one of two equivalent sectors.
Although we have not seen any massless states at any finite value of resolution
(we considered $K = 4 \ldots 7$), it appears that at least the two lowest states
converge to $M = 0$ in the continuum limit. This conclusion is supported by
the quadratic fit to the data. The third lowest state is also well–defined and
an extrapolation of its mass gives a value of 1.83 for the continuum limit.
To reveal the structure of higher states we need some additional information
Figure 1: (a) Mesonic mass spectrum of $A\lambda$ model in units of $g_{YM}^2 N/\pi$ as a function of $1/k$. (b) Eigenvalues of Hamiltonian $P^- = M^2/k$ in units of $g_{YM}^2 N/\pi$ for the mesonic sector of $\lambda$ model.

about their wavefunctions, this will lead to a clear distinction between nearly degenerate states.

While this direction is definitely worth pursuing, we already can formulate some interesting properties of the mesonic mass spectrum. Unlike the glueball case, there are no massless mesons at any finite value of resolution. In the continuum limit, however, the massless mesons appear, but the number of such massless states is still an open question. We also see an example of a meson state converging to a finite mass, and the data presented in Figure 1a suggests that there are many such states in the continuum limit. We should mention that the convergence properties of this model are very good, which seems to be a general properties of supersymmetric DLCQ as opposed to the traditional DLCQ approach.

As we mentioned in the previous section, the supercharge $Q^-$ is well-defined as soon as we have at least one fermion. This suggests an interesting truncation of the supersymmetric adjoint model [3]: one can eliminate the bosonic field $A^2$ from the theory. This reduces the number of supersymmetries to $(1, 0)$ and the remaining supercharge is given by:

$$Q^- = -\frac{g}{\sqrt{2}} \int dx^- \{\lambda, \lambda\} \frac{1}{\partial^- \lambda}. \quad (27)$$

While the “closed string” sector of this theory has well-defined continuum bound states [3, 15], all mesonic masses appear to be pushed to infinity in the continuum limit. In fact if one looks at the spectrum of DLCQ Hamiltonian
$P^-$, rather than the mass operator $M^2 = 2KP^-$, a finite limit can be found. The result, presented in figure 3b, has a peculiar property: an extrapolation of the four lowest eigenvalues gives 3.08, 3.01, 3.09 and 3.06 accordingly. Thus it seems that the spectrum of the Hamiltonian has some kind of a threshold. This interesting observation, however, does not undermine the fact that this model does not have any sensible continuum limit.

4 Models without adjoint scalar.

As we already saw in the previous section, the simplest supersymmetric model can be constructed by truncating all fields in the supercharge (14), except for gluino $\lambda$. In this section we add the dynamics of fundamental matter to the model. There are three different ways of doing this, and they give rise to the systems which we call $\lambda \Psi$, $\lambda \xi$ and $\lambda \Psi \xi$. We will show that all three systems have a well-defined mass spectrum and their bound states exhibit similar behavior under a variation of the coupling constant $h$.

We begin with the pure fermionic system $\lambda \Psi$. This system has $(1,0)$ supersymmetry and the supercharge has the form:

$$Q^- = -\frac{g}{\sqrt{2}} \int dx^- \left( \{\lambda, \lambda\} + 2h\psi\psi^\dagger \right) \frac{1}{\partial^-} \lambda.$$  \hspace{1cm} (28)

After substituting the expansions (16), (18) one gets the mode decomposition of the supercharge:

$$Q^- = \frac{i2^{-1/4}g\sqrt{L}}{\pi} \sum_{k_1=1}^\infty \sum_{k_2=1}^\infty \left\{ \frac{h}{k_1} \left[ \tilde{d}_j^i(k_2)b^\dagger_{ij}(k_1)d_j(k_1+k_2) + \tilde{d}_j^i(k_1+k_2)d_j(k_1) + b^\dagger_{ij}(k_1)d_j(k_1+k_2) + d^i(k_1+k_2)b_{ij}(k_1)d_j(k_2) \right] + \left( \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_1+k_2} \right) \left[ b^\dagger_{ik}(k_1)b^\dagger_{kj}(k_2)b_{ij}(k_1+k_2) + b^\dagger_{ij}(k_1+k_2)b_{ik}(k_1)b_{kj}(k_2) \right] \right\}.$$  \hspace{1cm} (29)

The mass spectrum of this theory is presented in figure 4a (we truncated it at $M^2 = 25$ and chose $h = 1$). There is one massless state and all other masses converge to finite continuum limits. We illustrate this convergence for the five lowest states in the Table 2. Using the structure of the supercharge (29), one can easily calculate the wavefunction of the massless state for any finite resolution as well as its continuum limit. This state appears to have
only two partons in it and the wavefunction is equal to a constant:
\[
|M = 0, K\rangle = C \sum_{n=1}^{K-1} \hat{d}_i^\dagger(n)\hat{d}_i^\dagger(K-n)|0\rangle \rightarrow \int_0^{P^+} dp \tilde{D}_i^\dagger(p)D_i^\dagger(P^+ - p)|0\rangle.
\] (30)

Here we have introduced the continuum modes \(D_i^\dagger\) and \(\tilde{D}_i^\dagger\). One can see that the state with the wavefunction (30) stays massless for an arbitrary values of coupling \(h\). It interesting to note that a massless state with constant wavefunction was also observed in a non–supersymmetric adjoint QCD in two dimensions [16].

It is also interesting to look at other masses as functions of the coupling constant. Figure 2b shows this dependence for the resolution \(K = 4\). We should note that this behavior is typical for all values of resolution and for all of the systems we are considering in this section. In particular one can see that the lowest states stays near \(M^2 = 0\) for a wide range of negative couplings. A closer look at this state at resolutions 4 and 5 is presented in figure 3. Looking at higher resolutions, we observe, that this state become massless for some value of coupling at all resolutions except \(K = 5\), thus the graph 3a is typical, while 3b is an artifact of the DLCQ. Interestingly, such odd behavior at \(k = 5\) is observed for all three systems we are studying here. The values of the critical coupling and the extrapolation to the continuum limit is presented in Table 3. The wavefunction of this massless state is concentrated in the two–parton sector: at resolutions 4, 6, 7 and 8 the two–particle sector contains 82%, 92%, 93% and 95% of wavefunction, however we believe that in continuum limit the wavefunction has small, but nonzero contributions from sectors with an arbitrarily large number of partons. This property is common for massless states in other supersymmetric theories [14].

While we were not able to solve for the continuum wavefunction in two–parton sector, the SDLCQ results presented in figure 4 points to a linear behavior in the continuum limit. Note that for the state we are looking at:
\[
|M = 0, K, h\rangle \approx \sum_{n=1}^{K-1} f(n, K-n)\hat{d}_i^\dagger(n)\hat{d}_i^\dagger(K-n)|0\rangle,
\] (31)

the wavefunction is antisymmetric: \(f(p, q) = -f(q, p)\), so in figure 4 we present only the region \(p \leq q\).

Let us now discuss two other theories without adjoint scalars. As we already mentioned, their properties are similar to the ones of the \(\lambda\psi\) system,
Figure 2: Pure fermionic model. (a) Mass spectrum in units of $g^2N/\pi$ at $h = 1$ as function of $1/k$. (b) Mass eigenvalues at $k = 4$ as functions of coupling $h$.

Figure 3: Pure fermionic model: lowest nonzero mass as function of coupling $h$ at resolutions 4 (a) and 5 (b).

Table 2: Pure fermionic model: lowest massive states.

| state | K=4  | K=5  | K=6  | K=7  | K=8  | K=9  | K=10 | K=∞  |
|-------|------|------|------|------|------|------|------|------|
| 1     | 5.82 | 5.71 | 5.63 | 5.57 | 5.52 | 5.48 | 5.46 | 5.21 |
| 2     | 12   | 12.25| 12.37| 12.44| 12.48| 12.50| 12.51| 12.90|
| 3     | 17.18| 17.37| 17.44| 17.46| 17.46| 17.46| 17.45| 17.67|
| 4     | —    | 17.56| 18.29| 18.75| 19.06| 19.28| 19.45| 21.39|
| 5     | —    | 22.54| 22.85| 23.84| 24.16| 24.38| 24.54| 26.80|
Table 3: Pure fermionic model: critical coupling as function of resolution.

| K   | K=4          | K=5          | K=6          | K=7          | K=8          | K=9          | K=∞          |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| h   | -1.50        | —            | -1.48        | -1.59        | -1.35        | -1.27        | -1.25        |

Figure 4: Pure fermionic model: DLCQ wavefunction of antisymmetric massless state appearing at the critical value of coupling. Only half of the wavefunction ($p \leq q$) is presented at resolutions 4 (a), 6 (b), 7 (c), 8 (d).

so we will consider both $\lambda \xi$ and $\lambda \psi \xi$ models only briefly. The $\lambda \xi$ system has $(1,0)$ supersymmetry and its supercharge reads:

$$Q^- = -\frac{g}{\sqrt{2}} \int dx^- \left( \{\lambda, \lambda\} + i\sqrt{2}h\xi \partial_- \xi \dagger - i\sqrt{2}h\partial_- \xi \xi \dagger \right) \frac{1}{\partial_-} \lambda. \quad (32)$$

The mass spectrum of this model is presented in figure 5a (again, we put $h = 1$) and the extrapolation of the masses to the continuum limit is given in Table 4. The masses exhibit the same coupling dependence as in the pure fermionic model (the only exception is the absence of the coupling-independent massless state), in particular there exists a critical coupling at which one of the states becomes massless. The values of this critical coupling are plotted in figure 5b and for the highest resolution we have $h_{cr} = -1.2$.

Finally we analyze the system which includes everything except for the adjoint scalar ($\lambda \psi \xi$ in our notation). This model has two supercharges, but we look only at one of them:

$$Q^- = -\frac{g}{\sqrt{2}} \int dx^- \left( \{\lambda, \lambda\} + i\sqrt{2}h\xi \partial_- \xi \dagger - i\sqrt{2}h\partial_- \xi \xi \dagger + 2h\psi \psi \dagger \right) \frac{1}{\partial_-} \lambda. \quad (33)$$

The second supercharge can be formally constructed:

$$Q^+ = 2 \int dx^- \left( \frac{i}{2} \partial_- \xi \dagger \psi - \frac{i}{2} \psi \dagger \partial_- \xi - \frac{i}{2} \xi \dagger \partial_- \psi + \frac{i}{2} \partial_- \psi \dagger \xi \right), \quad (34)$$
Table 4: $\lambda\xi$ model: lowest massive states.

| state | K=4 | K=5 | K=6 | K=7 | K=8 | K=9 | K=10 | K=∞ |
|-------|-----|-----|-----|-----|-----|-----|------|-----|
| 1     | 6.70| 6.68| 6.68| 6.69| 6.70| 6.70| 6.72 | 6.71|
| 2     | 13.16| 13.60| 13.87| 14.07| 14.23| 14.35| 14.46| 15.30|
| 3     | 18.13| 18.42| 18.58| 18.69| 18.77| 18.83| 18.88| 19.20|
| 4     | —   | 19.02| 19.93| 20.55| 21.00| 21.35| 21.63| 23.96|
| 5     | —   | 23.80| 24.55| 25.34| 25.77| 26.09| 26.35| 29.00|

but its square does not give the canonical $P^+$ but rather:

$$\{Q^+, Q^+\} = 2\sqrt{2}P^+ - 2\sqrt{2}\int dx^- \lambda \partial_- \lambda.$$  \hspace{1cm} (35)

Moreover, from this expression one concludes that

$$\left[\{Q^+, Q^+\}, Q^-\right] \neq 0,$$  \hspace{1cm} (36)

thus the two supercharges do not anticommute and they cannot be diagonalized simultaneously. Our formulation of the bound state problem is based on diagonalization of $P^+$ and $(Q^-)^2$, which still commute, and the $Q^+$ operator must be abandoned.

In the large $N$ calculations there are four different sectors to consider. One can start from either one of the four types of states:

$$\tilde{d}^i b^j \ldots b^j d^i |0\rangle, \quad \tilde{c}^i b^j \ldots b^j c^i |0\rangle, \quad \tilde{c}^i b^j \ldots b^j d^i |0\rangle, \quad \tilde{d}^i b^j \ldots b^j c^i |0\rangle,$$  \hspace{1cm} (37)

then $Q^-$ acts only inside the corresponding subspace. The first and second sectors reproduce the results we just obtained for $\lambda\psi$ and $\lambda\xi$ models. Two remaining models are mapped into each other under the Z2 transformation:

$$b_{ij} \leftrightarrow b_{ji}, \quad \tilde{c}_i \leftrightarrow c_i, \quad \tilde{d}_i \leftrightarrow d_i,$$  \hspace{1cm} (38)

which is a symmetry of $Q^-$. The spectrum for one of these sectors is presented in figure $6a$ and in the table $5$ and the critical coupling is plotted in figure $6b$. Note that we don’t have a coupling-independent massless state in this sector. The critical coupling for resolutions $K > 6$ is relatively constant at $h_{cr} = -2.0$. 

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Figure 5: $\lambda\xi$ model: (a) Mass spectrum in units of $g^2N/\pi$ at $h = 1$ as function of $1/k$. (b) Critical coupling $h_{cr}$ as function of $1/k$.

Figure 6: $\lambda\psi\xi$ model: (a) Mass spectrum in units of $g^2N/\pi$ at $h = 1$ as function of $1/k$. (b) Critical coupling $h_{cr}$ as function of $1/k$.

Table 5: $\lambda\psi\xi$ model: lowest massive states.

| state | K=4  | K=5  | K=6  | K=7  | K=8  | K=9  | K=10 | K=∞  |
|-------|------|------|------|------|------|------|------|------|
| 1     | 6.26 | 6.20 | 6.16 | 6.13 | 6.11 | 6.10 | 6.11 | 5.98 |
| 2     | 12.58| 12.92| 13.12| 13.25| 13.35| 13.42| 13.48| 14.09|
| 3     | 17.66| 17.89| 18.01| 18.07| 18.11| 18.14| 18.16| 18.50|
| 4     | —    | 18.29| 19.11| 19.65| 20.03| 20.31| 20.53| 22.80|
| 5     | —    | —    | 23.70| 24.58| 24.96| 25.23| 25.44| 31.30|
5 Models containing adjoint bosons.

Let us now discuss the models with an adjoint scalar $A^2$. We will see that the general property of such theories is the presence of a large number of low energy bound states which complicates the extrapolation to the continuum limit. We begin with the system which does not have a gluino as a dynamical field. Although this model has two supercharges:

$$Q^- = -2g' \int dx^- \left( \xi^\dagger A^2 \psi + \psi^\dagger A^2 \psi \right),$$

$$Q^+ = 2 \int dx^- \left( \frac{i}{2} \partial_- \xi^\dagger \psi - \frac{i}{2} \psi^\dagger \partial_- \xi - \frac{i}{2} \xi^\dagger \partial_- \psi + \frac{i}{2} \partial_- \psi^\dagger \xi \right),$$

they do not commute, thus we will ignore the supercharge $Q^+$ in our consideration (this situation is analogous to the case of $\lambda \xi \psi$ model). We constructed mesonic spectrum of this model and the lowest masses are presented in figure $\text{fig}$. One can see that the number of light states grows with resolution, in fact we will argue that in the continuum limit this model has a continuous mass spectrum. But first let us look at the heaviest bound state one can construct at a given value of resolution. The masses of such states are plotted in figure $\text{fig}b$, and one can see that this graph has a good linear approximation:

$$M_{max}^2(K) = \left( \frac{g'}{\pi} \right)^2 \frac{2N}{\pi} \left( 1.08K - 1.86 \right),$$

where $K$ is a value of resolution. The negative constant in the above expression is not important, since one cannot consider $K < 4$ and the interesting limit is $K \to \infty$. For the total number of bosonic states at a particular resolution, simple combinatorics gives:

$$N_{total}(K) = 2^K.$$  

In order to analyze the continuum limit of the spectrum we will study the density of states $dN/N_{total}(K)$ as a function of the reduced mass: $dM^2/M_{max}^2(K)$. In particular we will plot the distribution function defined as

$$F(M^2/M_{max}^2(K)) = \frac{N(\text{Mass} < M)}{N_{total}}.$$   

Such plots for different resolutions are presented in color in figure $\text{fig}a$, which gives a convincing argument for the convergence of the function $F(x)$ in the
Figure 7: $A\psi\xi$ model: (a) Mass spectrum in units of $(g')^2N/\pi$ as function of $1/k$. (b) Maximum mass in units of $(g')^2N/\pi$ as function of resolution.

Figure 8: $A\psi\xi$ model: (a) Color plot of function $F(x)$ at resolutions $k = 6 \ldots 10$, (b) The same function at $K = 10$ as a best available approximation to the continuum limit.
continuum limit. In figure 8b we present this function for resolution 10, which is a good approximation to the continuum limit.

Let us now discuss the models that include both fields in the adjoint representation. First we consider the $A\lambda\psi$ system. It has two supercharges which don’t commute, so we look only at one of them:

$$Q^- = -2g \int dx^- \left( i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}} \{\lambda, \lambda\} + \sqrt{2} h \psi \psi^\dagger \right) \frac{1}{\partial_-} \lambda. \quad (44)$$

The mass spectrum, obtained as the result of the diagonalization of Hamiltonian $P^- = (Q^-)^2$, is presented in the figure 9a. As usual we put $h = 1$ and truncated the spectrum at some value of mass (in this case $M^2 = 10$). The states can be easily traced from one resolution to another, and a new low state appears at every even resolution and a new massless state appears at every odd resolution (thus there is one massless state at $K = 4$, two massless states at $K = 5$ and $K = 6$, 3 massless states at $K = 7, 8$ and so on). Such behavior was observed in other supersymmetric systems with adjoint scalars [10] and it points to a continuum spectrum in the limit $K = \infty$. One can also look at the coupling dependence of the states, in particular the lowest state with nonzero mass at $K = 4$ becomes massless at the coupling $h = -1.25$ (see figure 9b). But since this state ultimately becomes a part of a continuous spectrum, its properties are not as interesting as its counterpart in the $\lambda\psi$ model.

As we saw in our study of the models without gauge fields, the theories which differ only by replacing $\psi$ by $\xi$, behave in the same fashion. The same is true for the models with adjoint scalar. The spectrum of the $A\lambda\xi$ model is presented in figure 10, it also converges to a continuous spectrum and has a critical coupling $h$ at any $K \neq 5$. The figure 10b illustrates that it is not only low mass states that appear at high resolution, but all values of mass seem to be filled in the continuum limit.

Finally we consider the system without truncation, i.e. we study the $A\lambda\psi\xi$ model. Unlike all other systems we studied in this section, it has a complete $(1,1)$ supersymmetry. Thus the two supercharges given by (9) and (14) anticommute and can be diagonalized simultaneously. Thus, apart from the massless sector, the spectrum is four-fold degenerate and we can look only at a quarter of the theory, while diagonalizing the mass operator. In particular, we consider only bosonic states with an even number of creation operators $a^\dagger$. The combined action of $Q^+$ and $Q^-$ gives a boson with an odd number of creation operators $a^\dagger$, while the action of either one of the
Figure 9: $A\lambda\psi$ model: (a) Mass spectrum at $h = 1$ in units of $(g')^2N/\pi$ as function of $1/k$. (b) Lowest nonzero mass at $K = 4$ as function of coupling $h$.

Figure 10: $A\lambda\xi$ model: Mass spectrum at $h = 1$ in units of $(g')^2N/\pi$ as function of $1/k$. 
supercharges leads to the fermionic sectors. The low energy spectrum in a single sector at $h = 1$ is presented in figure [11], there are two massless states for every even value of resolution. One can see that the presence of these states is not the only difference between the odd and even values of $K$, it seems that we are dealing with the SDLCQ of two different theories. Of course, in the large $K$ limit they should converge to the same result, and, as figure [11] demonstrates, the resulting theory has many light states and the possibility of a continuum spectrum is not ruled out. Note that the lowest states appear in almost degenerate pairs. The explanation of this doubling poses an interesting question, which might be answered by performing a careful study of the wavefunctions. We also looked at the lowest masses as functions of a coupling $h$, and the result for resolution 4 is presented in figure [12]. This shows two peculiar properties of this model. First, there is a smooth interchange between almost degenerate states, as opposed to the level crossing we observed in the other theories. In addition to this, the model exhibits two critical couplings at resolution 4 and it is interesting to see whether this property persists at higher resolutions. Unfortunately this model is the hardest one to study since it has a maximal number of fields. Also additional calculational effort is require to handle the fact that the $A\lambda\psi\xi$ model (and only this model) has a complex supercharge. We hope to overcome both difficulties in the future work. In spite of these difficulties we can already conclude that the $A\lambda\psi\xi$ has many light states in the continuum limit and possibly it converges to a continuous spectrum as the other models with adjoint scalars.

6 Discussion.

In this work we studied the mesonic mass spectrum of various supersymmetric models. The calculations were performed in the framework of supersymmetric DLCQ, namely we compactified the light–like coordinate $x^-$ on a finite circle and performed a numerical diagonalization of supercharge $Q^-$. We found that the systems with adjoint scalars tend to give a continuous spectrum in the decompactification limit. For one of these systems ($A\psi\xi$ model) we found a limiting form of the distribution function for the mass. By contrast, the models without adjoint bosons have well defined spectra of bound states in the continuum limit. The only exception from this rule is the system containing only the gaugino field, which seems not to have any
Figure 11: Complete two dimensional model: Mass spectrum at $h = 1$ in units of $g^2 N/\pi$ as function of $1/k$.

Figure 12: Complete two dimensional model: (a) Lowest masses at resolution 4 as functions of coupling $h$, (b) Lowest mass at resolution 4 near critical couplings.
finite mass mesons in the decompactification limit. For the well defined systems we found the masses of the lightest mesons and demonstrated the fast convergence of SDLCQ approximation.

We also looked at the mass spectrum at different values of coupling constant. The nontrivial phase diagram is an essential property which distinguish the models we considered here from the two dimensional systems studied previously [3, 2]. The coupling constant of a pure gauge theory in two dimensions has a dimension of mass, thus all the bound state masses scale like $g$, leaving no space for nontrivial coupling constant dependence. One can avoid this problem by introducing the masses for a gauge field or its superpartner, but such terms usually lead to breaking of either a gauge invariance or supersymmetry. Another way of introducing a free parameter in the theory is to add a new supermultiplet with a different charge $g'$. Of course, this parameter is not completely free: the quantization of charge requires the value $g'/g$ to be a rational number, but formally we can study the bound states as functions of $h = g'/g$. We found an interesting property of the lowest nonzero mass: it vanishes at a particular negative value of $h$. The nature of this property is still unclear. One can also introduce the true free parameter by considering the massive matter supermultiplet (now there is no obstacle coming from gauge invariance) and we leave this possibility for a future investigation.

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