Perestroikas of Shocks and Singularities of Minimum Functions

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Abstract

The shock discontinuities, generically present in inviscid solutions of the forced Burgers equation, and their bifurcations happening in the course of time (perestroikas) are classified in two and three dimensions – the one-dimensional case is well known. This classification is a result of selecting among all the perestroikas occurring for minimum functions depending generically on time, the ones permitted by the convexity of the Hamiltonian of the Burgers dynamics. Topological restrictions on the admissible perestroikas of shocks are obtained. The resulting classification can be extended to the so-called viscosity solutions of a Hamilton–Jacobi equation, provided the Hamiltonian is convex.

Keywords: singularities, transitions, shocks, Burgers equation, viscosity solutions, Hamilton–Jacobi equation, minimum functions.

1 Introduction

The subject of this paper is singularities and perestroikas (= bifurcations, metamorphoses, or transitions) of shocks in plane and space (two and three dimensions). Shocks are discontinuities of limit potential solutions of the Burgers equation with vanishing viscosity and external potential force. The Burgers equation is just the Navier–Stokes equation without the pressure term – its theory is well described in the survey [1].

If we remove the viscosity term as well, we will get the equation describing irrotational flow of a medium consisting of non-interacting particles moving in an external potential force field. This is a model of formation of the large-structure of the Universe or the so-called “Zel'dovich approximation” [2], [3]. According to this model, after some time the density becomes infinite because the fastest particles overrun the slowest ones. The points with infinite density form caustics in space which evolve with time. Generic singularities and perestroikas of caustics are described in [4], [5].

Limit potential solutions of the Burgers equation describe motion when the particles cannot pass through each other – they adhere in some sense. The adhesion is a result of interaction between them described by the vanishing viscosity. The shocks of limit solutions, like the above caustics, are formed by the points with infinite density. But there is a big difference: a particle cannot leave a shock but can leave a caustic. In other words, the matter accumulates in the shocks. This is the so-called adhesion model of matter evolution in the Universe describing the formation of cellular structure of the matter described, for example, in [6].

Shocks appear even if the initial condition is smooth – here and further this term means infinite differentiability. We describe local singularities and perestroikas of shocks in physically interesting dimensions \( d \leq 3 \) with generic smooth initial conditions. It means that we are interested only in singularities and perestroikas which are stable with respect to any sufficiently small perturbations of the smooth initial condition. Other singularities and perestroikas can be killed by its arbitrarily small perturbation.

It turns out that at typical times such a generic shock has singularities from a finite list and at separate times experiences perestroikas. In order to describe them we consider the world shock lying in the space-time and formed by the points where our limit solution is not smooth – our shocks in space or instant shocks are just sections of the world shock with isochrones \( t = \text{const} \). The number of topologically different generic perestroikas proves to be finite as well.

An example of a world shock in the well known case \( d = 1 \) is shown in Figure [†]. The instant shocks are sets

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Figure 1. Singularities of world shocks in plane generate perestroikas of instant shocks on line of isolated points, the world shock is a curve in plane which can have triple points and end-points. Any triple point sticks a pair of points of an instant shock, any end-point generates its new point. These 2 perestroikas exhaust all generic ones – any other perestroika can be killed by an arbitrarily small perturbation of the initial condition.

Figure 2. Singularities of world shocks and perestroikas of instant shocks in plane of isolated points, the world shock is a curve in plane which can have triple points and end-points. Any triple point sticks a pair of points of an instant shock, any end-point generates its new point. These 2 perestroikas exhaust all generic ones – any other perestroika can be killed by an arbitrarily small perturbation of the initial condition.

Figures 3, 4, and 5 show the case $d = 3$. In this case it is impossible to draw singularities of world shocks because they lie in 4-dimensional space-time. An instant shock is a surface which can have singularities which occur on world shocks of the previous dimension $d = 2$. Again, at typical times there are no other singularities but at separate times they occur. Namely, an instant shock can experience perestroikas shown in the figures by black arrows. These 26 perestroikas exhaust all generic ones – any other perestroika can be killed by an arbitrarily small perturbation of the initial condition.

In more detail, we consider a material $d$-dimensional medium whose velocity is potential and described by the Burgers equation with the potential force term:

\[
\begin{align*}
\nabla \psi (x, 0) &= \varphi_0 (x) \\
\psi (x, t) &= \nabla \psi
\end{align*}
\]

where $x \in \mathbb{R}^d$ is a point of the medium, $v(x, t)$ is the velocity at the point $x$ at the time $t$, $\nu > 0$ is the viscosity of the medium, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$ is the usual $\nabla$-operator in $\mathbb{R}^d$, and $\Delta = \nabla \cdot \nabla$ is the Laplacian. The force potential $U(x, t)$ and the initial condition $\varphi_0 (x)$ are assumed to be smooth. ( Everywhere in the present paper it means infinite differentiability.) Let $\varphi$ be the limit solution as the viscosity vanishes. It has
the following minimum representation:

\[ \varphi(x,t) = \lim_{\nu \to 0} \psi(x,t) = \min_{u} F(u, x, t) \]  

(2)

where \( F \) is a smooth function which, in the mechanics terminology, is just the action defined by the Lagrangian

\[ L(x, x, t) = \frac{1}{2} |x|^2 - U(x, t) \]

along its extremal trajectory ending at the point \((x, t)\) with the velocity \(u\). This fact is well known in the unforced case \( U \equiv 0 \). If \( U \neq 0 \) then it can be proved using the so-called viscosity solutions of the Hamilton–Jacobi equation described, for example, in [7]. In the unforced case \( L = |x|^2/2 \)

\[ F(u, x, t) = \varphi_0(x - ut) + \frac{1}{2} |u|^2 t \]

and we can rewrite our minimum representation as

\[ \varphi(x, t) = \min_{a} W(a, x, t) \]

\[ W(a, x, t) = \varphi_0(a) + \frac{|a - x|^2}{2t} \]

(3)

where \( a = x - ut \) and \( x \) are the Lagrangian and Eulerian coordinates of a particle moving with the velocity \( u \). This formula can be got explicitly without the viscosity solution theory. We shall describe the both approaches in Section 4.

The action \( F \) is smooth but, nevertheless, the limit solution \( \varphi \) is continuous only and can be non-differentiable at some points. So, the velocity \( v \) can have discontinuities. For example, it can happen at a point \((x_*, t_*)\) such that the function \( F(u, x_*, t_*) \) of \( u \) attains its minimum for two different values \( u_1 \neq u_2 \):

\[ F(u_1, x_*, t_*) = F(u_2, x_*, t_*) \leq F(u, x_*, t_*) \]

for any \( u \).

![Figure 3. Perestroikas of instant shocks in space (beginning)](image)

Figure 3. Perestroikas of instant shocks in space (beginning)

### Table

| \( A_1^2 \) | \( A_1^4 \) | \( A_3^1 \) |
| --- | --- | --- |
| ![Image] | ![Image] | ![Image] |

![Figure 6. Competition of minima generates shocks and caustics](image)

Figure 6. Competition of minima generates shocks and caustics

The points where the limit solution is not smooth form the world shock lying in space-time and the instant shocks which are in space and vary in time. In other words, the instant shocks are the sections of the world shock with isochrones \( t = \text{const} \). Figure 6 shows how competition of minima of the functions \( F(u, x, t) \) generates a world shock in one-dimensional case \( d = 1 \). Namely, at the points \((x_*, t_*)\) of the world shock the
corresponding function $F(u, x, t)$ of $u$ has at least two global minima or its minimum is degenerate. So, the world shock is the Maxwell stratum of the family of functions of $u$ depending on $x$ and $t$. The caustics are shown by dots – at any their point the corresponding function has a degenerate critical value which is not minimal for a typical point of the caustic.

We describe local singularities and perestroikas of instant shocks in physically interesting cases of dimension $d \leq 3$ if the initial condition is generic. Formally, it means that our description works only for some open dense subset in the space of all smooth initial conditions. It is assumed that this space is provided with the so-called Whitney topology – its definition can be found, for example, in [5] (2.3, Remark 2). In other words, we are interested only in singularities and perestroikas which are stable with respect to any sufficiently small perturbations of the initial condition. Other singularities and perestroikas can be killed by its arbitrarily small perturbation.

It turns out that such a generic instant shock is a hypersurface which at typical times has singularities from a finite (up to smooth diffeomorphisms) list. At these times the instant shock does not locally change with respect to smooth diffeomorphisms. But at separate times it can experience local perestroikas – the instant shocks at the perestroika time and at close times are not diffeomorphic to each other. The instant shock at the perestroika time itself has a singularity which does not belong to the finite list.

We get a finite topological classification of generic perestroikas of instant shocks. Their smooth classifi-

| $A_1^3$ | $A_3$ | $A_5$ |
|---------|-------|-------|
| ![Diagram](image1.png) | ![Diagram](image2.png) | ![Diagram](image3.png) |

Figure 4. Perestroikas of instant shocks in space (continuation)

| $A_1A_3$ | $A_1^2A_3$ |
|----------|-------------|
| ![Diagram](image4.png) | ![Diagram](image5.png) |

Figure 5. Perestroikas of instant shocks in space (end)
cation proves to be infinite – continuous invariants or moduli appear. The finite topological classification described in detail in Section 3 is the first main result of the present paper (Theorem 4). The second main result is the shock determinator described in Section 4 (Theorems 5, 6, and 7) which allows to define what singularity or perestroika is occurring at a given point of space-time for a concrete initial condition of the unforced Burgers equation.

The general plan of the classification is the following.

1.1 Step 1: Minimum representations for limit solutions

Firstly, we describe in Section 4 how to get the minimum representations of limit solutions. The representation (3) can be obtained finding explicitly the potential solution of the unforced Burgers equation with the help of the so-called Hopf–Cole transformation which was published by Forsyth back in 1906 [8]. This is a well known way, but we describe it.

The representation (3) in the forced case is an automatic corollary of the theory of generalized or viscosity solutions of a Hamilton–Jacobi equation constructed [9], [10], [7], we brief this approach in Section 4 as well.

As a matter of fact, our description of perestroikas works for the shocks of viscosity or generalized solutions of the Hamilton–Jacobi equation with a strictly convex (up to momenta) smooth Hamiltonian [13]. In this case there exists a minimum representation which is analogous to (3) and published, for example, in [12]. This representation shows that the viscosity solution in the convex case is the solution of the classical variational problem with a free left end:

$$\varphi(x, t) = \min_{\gamma(t) = x} \left\{ \varphi_0(\gamma(0)) + \int_0^t L(\dot{\gamma}(\tau), \gamma(\tau), \tau) \, d\tau \right\}$$

where $\gamma : [0, t] \rightarrow \mathbb{R}^d$ is a differentiable path in space and the Lagrangian is the Legendre transformation of the Hamiltonian:

$$L(x, \dot{x}, t) = \max_p \{ p \dot{x} - F(p, x, t) \}.$$

The Hamiltonian and the Lagrangian are both strictly convex down with respect to momenta. In other words, the matrices of their second derivatives must be positive definite:

$$\|H_{pp}(p, x, t)\| > 0, \quad \|L_{xx}(x, x, t)\| > 0.$$

Therefore, our classification describes singularities and perestroikas of solutions of the above variational problem. (The Lagrangian $L$ is assumed to be fixed but the initial condition $\varphi_0$ is generic.) In fact, everything is vice versa: at first we classify singularities and perestroikas in the variational problem with a convex Lagrangian, and after that we apply this classification to the limit solutions taking into account the minimum representations.

1.2 Step 2: Perestroikas of shocks for generic $F$

Secondly, we forget that $F$ is the action and consider it just as a generic smooth function. In Section 4 we describe, using methods of Singularity Theory, all singularities and perestroikas of shocks for $d \leq 3$ which are stable with respect to small perturbation of the function $F$ itself. Other singularities and perestroikas can be killed by an arbitrarily small perturbation of the function $F$.

Singularities of instant shocks at typical times and singularities of world shocks were described in [12]. Perestroikas of instant shocks were published for the first time in [14]. The both classification lists are proved to be finite for appropriate equivalence relations: smooth for singularities and topological for perestroikas.

But can the action $F$ be considered as generic? Indeed, $F$ is not an arbitrary function, it is defined by the initial condition and the force potential. For example, in the unforced case there is the formula (3) and the action does not allow any perturbation as a function of Lagrangian coordinates $a$, Eulerian coordinates $x$, and time $t$. Only the initial condition $\varphi_0$ is a generic smooth function of the Lagrangian coordinates $a$ — the other part of the formula (3) is a fixed function. Does this fact give us new singularities and perestroikas of shocks which are stable with respect to small perturbations of the initial condition only but not the action itself? Maybe, there exist singularities or perestroikas of shocks which cannot be killed by an arbitrarily small perturbation of the initial condition but vanish after such a perturbation of the action?

It turns out that the answers to these questions are negative — a general theorem implying this fact is published in [15]. Therefore, all singularities and perestroikas of instant shocks with generic initial conditions belong to our lists.

1.3 Step 3: Topological restrictions forbidding some perestroikas

On the other hand, do there exist singularities or perestroikas from the list of the step 2 which cannot be realized by shocks when $F$ is really the action? It turns out that such singularities do not exist but there are a few perestroikas forbidden for any force potential
U. Apparently, for the first time this fact has been observed by S.N. Gurbatov and A.I. Saichev. Firstly, an isochrone $t = \text{const}$ cannot be tangent to a world shock at its smooth point – this restriction forbids all perestroikas $A_2^{0j}$. Secondly, a few perestroikas are irreversible – they can occur in only one direction of time: for example, a triangle from shocks can vanish during the perestroika $A_2^{0j}$ but cannot appear. A universal topological restriction is the following:

A local shock after a perestroika is contractible (homotopically equivalent to a point). For example, the triangle from shocks is homotopically a circle but not a point, hence it cannot appear.

This topological restriction was explicitly checked in [14] only for the perestroikas from the list of the step 2. Namely, the forbidden perestroikas cannot be realized by any action in view of algebraic conditions described in Section 5 for the unforced case. In Section 5 we prove this restriction for an essentially wider class of perestroikas in all dimensions. As a matter of fact, it is almost equivalent to another topological restriction suggested and proved in [16] for all dimensions. As a matter of years?

### 1.4 Step 4: Realization of not forbidden perestroikas

At last we show that all perestroikas not forbidden by the topological restriction can be realized by instant shocks of the Burgers equation. Moreover, Theorems of Section 3 allow, knowing the initial condition, to define what singularity or perestroika from our list is realized by the instant shock at a given point of space-time. In particular, they give all genericity conditions for the initial condition which guarantee that the instant shock has only described singularities and experiences only described perestroikas. These theorems are proved in Section 6.

### 2 The Classification

#### 2.1 Simple minima

If in proper local smooth (curvilinear, in general!) coordinates a function on $\mathbb{R}^d$ has the form

$$u_1^2 + u_2^2 + \cdots + u_d^2 + \text{const}$$

then it attains a local minimum at the point $u = 0$. In Singularity Theory such minimum has the name $A_{2j-1}$. The index $\mu = 2j - 1$ is called the multiplicity of the minimum. The number $\kappa = \mu - 1 = 2j - 2$ is called the codimension of the minimum. Any non-degenerate minimum of a smooth function is $A_1$ – this is the well known Morse lemma. The minima $A_3$, $A_5$,... are degenerate, the ranks of their second differentials are equal to $d - 1$. So, it is easy to find a minimum which does not belong to the series $A_{2j-1}$ – the simplest one is:

$$X_9: u^4 + 2Cu^2v^2 + v^4 + \text{const}, \quad C > -1, \ C \neq 1,$$

where $C$ is a modulus or a continuous invariant up to smooth changes of the variables. (If $C = 1$ then 0 is a minimum too but its name is not $X_9$.)

Any smooth function with a minimum $A_3$ in $\mathbb{R}^d$ can be represented in some orthonormal coordinates $(a, b)$ with origin at the minimum point as

$$K a^4 + \omega(z) + O_{>4}(a, z) + \text{const},$$

$$\omega(z) = \omega_1 z_1^2 + \cdots + \omega_{d-1} z_{d-1}^2,$$

$$z = b - Ca^2$$

where $b, z, C \in \mathbb{R}^{d-1}$, $K, \omega_1, \ldots, \omega_{d-1} > 0$, and $O_{>4}$ denotes terms of order greater than 4 if $\deg a = 1$ and $\deg z = 2$. The straight line $b = 0$ is invariant and called kernel. If $C \neq 0$ then the points

$$(a, s C/|C|), \quad (a, s) \in \mathbb{R}^2$$

form the invariant kernel plane, and the equation

$$|C|(a^2 + s^2) - s = 0$$

defines the kernel circle which lies in the kernel plane and is tangent to the kernel line $s = 0$. The kernel circle and line divide the kernel plane into three domains: the disk $D$, the domain $U$, and the semiplane $P$ shown in Figure 7.

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![Figure 7. Kernel circle and line divide kernel plane of minimum $A_3$ into three domains](image-url)

Any smooth function with a minimum $A_5$ in $\mathbb{R}^d$ can be represented in some orthonormal coordinates $(a, b)$
with origin at the minimum point as
\[ K a^0 + \omega(z) + O_{>6}(a) + O_{>8}(a, z) + \text{const}, \]
\[ \omega(z) = \omega_1 z_1^2 + \cdots + \omega_d-1 z_{d-1}^2, \quad (5) \]
\[ z = b - Ca^2 - Da^3 - Ea^4 \]
where \( b, z, C, D, E \in \mathbb{R}^{d-1}, K, \omega_1, \ldots, \omega_d-1 > 0, \) and \( O_{>n} \) denotes terms of order greater than \( n \) if \( \deg a = 1 \) and \( \deg z = 4. \)

### 2.2 Classification of shock points

If a point \((x_*, t_*)\) lies on a shock then the function \( F(\cdot, x_*, t_*)\) attains its minimal value at a few points or at a degenerate one. The names of these minima can be written as one word like \( A_1^2 \) which means: “two (non-degenerate) minima \( A_1 \) and one (degenerate) minimum \( A_3^3 \).” Such word is the name of the point \((x_*, t_*)\).

For example, a typical point of a generic shock is \( A_1^3 \). In its neighborhood the shock is a smooth surface which acquires singularities at the points \( A_1^3, A_3, \ldots \). The points \( A_1 \) are out of the shock at all.

### 2.3 Shocks on line: \( d = 1 \)

If the action \( F \) is just an arbitrary generic smooth function, the world shock is a curve in plane which, besides smooth points \( A_1^3 \), can have triple points \( A_3^3 \) and end-points \( A_3 \). A generic curve with such prescribed singularities could generate 6 different perestroikas of its sections which are just sets of isolated points. All these possibilities – \( A_1^2[\pm] \) (a couple of points appears or vanishes), \( A_1^{++} \) (a point doubles or two points stick), \( A_3^3 \) (a point vanishes or appears) – are shown in Figures 3 and 8.

![Figure 8. Impossible perestroikas of instant shocks on line](image)

Which of these perestroikas are realized by “real” shocks, i.e. the shocks of limit potential solutions of the Burgers equation? According to the well known entropy condition, only two \( A_1^+ \) (two points stick) and \( A_1^- \) (a point appears) can happen – they are shown in Figure 4. The 4 forbidden perestroikas are shown in Figure 5. They can be explained by our topological restriction as well: the empty set and a couple of points are not contractile.

### 2.4 Shocks in plane: \( d = 2 \)

Let the action \( F \) be an arbitrary generic smooth function. Then the instant shock at a typical time is a curve which, besides smooth points \( A_1^3 \), can have triple points \( A_3^3 \) and end-points \( A_3 \) – the same singularities occur on world shocks of the previous dimension \( d = 1 \). The world shock is a surface which, besides smooth points \( A_1^3 \), can have the following singularities: \( A_1^3, A_3, A_1^3, A_1^3, A_3, A_1^3, A_1^3, A_3, A_1^3, A_3 \) shown in Figure 2. The singularities \( A_1^3, A_1^3, A_3, A_3 \) are new in comparison with the previous dimension and occur at isolated points of the world shock. The singularities \( A_1^3, A_3, A_3, A_3, A_3, A_3, A_3, A_3 \) form curves – triple lines and edges.

A generic surface with such prescribed singularities generates 18 different perestroikas of its sections with isochrones – all of them are shown in Figure 4 by white and black arrows. The perestroika \( A_1^3 \) occurs when an isochrone is tangent to the surface at a smooth point, the perestroikas \( A_1^3 \) and \( A_3 \) happen if an isochrone is tangent to a triple line or to an edge. But only 9 perestroikas are not forbidden by the topological restriction and can be realized by “real” shocks. They are shown by black arrows. (White ones show the forbidden perestroikas.)

### 2.5 Shocks in space: \( d = 3 \)

Again, let the action \( F \) be an arbitrary generic smooth function. Then the instant shock at a typical time is a surface which, besides smooth points \( A_1^3 \), can have triple lines \( A_3^3 \), edges \( A_3 \), and separate singularities \( A_1^3 \) and \( A_1^3 A_3 \) – the same singularities occur on world shocks of the previous dimension \( d = 2 \). A world shock, besides smooth points \( A_1^3 \), can have the following singularities: \( A_1^3, A_3, A_1^3, A_1^3, A_1^3, A_1^3, A_1^3 A_3, A_3, A_1^3, A_3 \), and \( A_5 \). The singularities \( A_1^3, A_2^3, A_3, A_3, A_5 \) are new in comparison with the previous dimension and occur at isolated points of the world shock. The singularities \( A_1^3, A_1^3 A_3, A_3, A_3, A_1^3 \) form curves and the singularities \( A_1^3 \) and \( A_3 \) form surfaces.

In this case there are 42 different perestroikas – all of them are shown in Figures 3, 4, 5, 8 by white and black arrows. But only 26 of them are not forbidden by the topological restriction and can be realized by “real” shocks. They are shown by black arrows. (White ones show the forbidden perestroikas.)
2.6 Normal forms

Formulas describing singularities of shocks are given in Table I. The functions from the second column of Table I are called normal forms of a limit solution in local smooth coordinates \((\alpha, \beta, \ldots)\) with respect to adding a smooth function. The set of points, where a normal form is not smooth, describes a world shock in a neighborhood of a point with the indicated name. Its coordinates are \(\alpha = \beta = \cdots = 0\). In general, the coordinates \((\alpha, \beta, \ldots)\) are curvelinear and mix space coordinates \(x\) and time \(t\). Besides, these normal forms describe limit solutions and instant shocks at typical times. In this case \((\alpha, \beta, \ldots)\) are local smooth coordinates in space.

Perestroikas of instant shocks are described by the normal forms of time from Table I. The world shock is described by the second column, the time coordinate is

\[
t(\alpha, \beta, \ldots) = t_* + \tau(\alpha, \beta, \ldots)
\]

where \(t_*\) is the perestroika’s instant and \(\tau\) is a normal form from the following three columns which correspond to the dimensions \(d = 1, 2, 3\) respectively. An empty cell of Table I means that the corresponding singularity of a world shock does not occur in the shown dimension.

The perestroikas for \(d = 1\) from Table I are shown in Figures 2 and 3. Plus or minus denoted by the sign “±” in the normal form of time from the table corresponds to the same sign in perestroikas’ names \(A^+_1\), \(A^+_2\), and \(A^+_3\) in the figures.

The perestroikas for \(d = 2, 3\) from Table I are shown in Figures 1 and 8 by arrows. Namely, each perestroika is shown as the three sections of the world shock with isochrones

\[
t(\alpha, \beta, \ldots) = t_-, t_+ > t_*.
\]

An arrow shows the time direction from \(t_-\) to \(t_+\). The \(t_+\)-sections have only generic singularities, but one point of the \(t_*\)-section is not generic – it is the point of the perestroika, its coordinates are \(\alpha = \beta = \cdots = 0\). The perestroikas which do not change after the time inversion are shown by up-down arrows \(\uparrow, \downarrow\) in the figures, their functions \(\tau\) from Table I do not contain ±. The sign “±” in Table I means that the perestroika does change after the time inversion, \(\uparrow\) in the figures means +, \(\downarrow\) means –.

Unfortunately, if we considered only smooth coordinates it would be impossible to get such simple normal forms of time – there would be quite a lot of continuous invariants. In other words, smooth classification of perestroikas of instant shocks is infinite. So, in order to describe perestroikas of shocks we allow the coordinates \((\alpha, \beta, \ldots)\) to be sectionally continuously differentiable but, nevertheless, they must be smooth outside the world shock. Therefore, the first derivatives \(\partial(\alpha, \beta, \ldots)/\partial(x, t)\) can have jumps on the world shock itself.

The first main result of the present paper is

**Theorem 1.** Let \(d \leq 3\), \(T > 0\), and the initial condition be a generic smooth function. Then any perestroika of the instant shock happening before the time \(T\) is described in some coordinates \((\alpha, \beta, \ldots)\) by a pair of normal forms from Table I – the limit solution and time. The coordinates \((\alpha, \beta, \ldots)\) are sectionally continuously differentiable and smooth outside the world shock of the limit solution.

**Remark.** The term “generic” means “from some open dense subset”. It is assumed that the space of all smooth initial conditions is provided with the so-called Whitney topology – its definition can be found, for example, in [5] (2.3, Remark 2). If \(T = \infty\) then the theorem is true only for some dense subset of the initial conditions which maybe is not open.

**Remark.** Another way to interpret our classification of perestroikas is to use smooth coordinates but another equivalence for limit solutions. Namely, instead of adding a smooth function we allow a smooth diffeomorphism of the graphs of the limit solution and its normal form at the same time. All details are discussed in 5.3.

2.7 Smoothness of \(F\)

In the present paper we require infinite smoothness of the action \(F\) which can be guaranteed by infinite smoothness of the initial condition \(\varphi_0\) and the force potential \(U\).

In the present paper we require infinite smoothness of the action \(F\) which can be guaranteed by infinite smoothness of the initial condition \(\varphi_0\) and the force potential \(U\). Probably, Theorem I is correct when we can guarantee \(F \in C^{d+3}\) with respect to all variables \((u, x, t)\). Then the coordinates \((\alpha, \beta, \ldots)\) from Theorem I will be sectionally continuously differentiable as before but only continuously differentiable outside the world shock of the limit solution. The crucial point in this question is Lemma – the change \(h\) must be continuously differentiable at least.
### 3 The Shock Determinator

The worth of Theorem 1 for applications is problematic. How can one find out whether or not a given initial condition is generic? What does really happen at some point of space-time if the initial condition is fixed and cannot be perturbed? Briefly speaking, how is it possible to apply Theorem 1 in a concrete situation?

The answer for the unforced Burgers equation is given by Theorems 2, 3, and 4. Theorem 2 recognizes the singularity of the instant shock at a given point of space-time. If none of its conditions is satisfied then we can try to apply Theorems 3 and 4 which recognize the perestroika of the instant shock. If they do not work too, it means that something non-generic occurs at our point. Namely, the singularity or perestroika is not stable with respect to small perturbations of the initial condition – it is not generic and, therefore, is not described by Theorem 1.

For the unforced Burgers equation there is the representation (3). Let \((x_*, t_*)\) be a fixed point of space-time and

\[ w(\cdot) = W(\cdot, x_*, t_*). \]

The information about global minima of the function \(w\) is written like \(w \in A_1^2 A_3\) – this example means that \(w\) has three global minima, two of them are \(A_1\) and the other one is \(A_3\).

**Theorem 2.** (Singularities) If \(d = 1, 2, 3\) and \(w \in A_1\), then \(x_*\) lies outside the instant shock at the time \(t_*\).

If \(d = 1, 2, 3\) and \(w \in A_1^2\), then \(x_*\) is a smooth point \(A_1^2\) of the instant shock at the time \(t_*\).

If \(d = 2, 3\), \(w \in A_1^3\), and the minimum points do not belong to the same straight line, then \(x_*\) is a singularity \(A_1^3\) of the instant shock at the time \(t_*\).

If \(d = 3\), \(w \in A_1^4\), and the minimum points do not belong to the same plane, then \(x_*\) is a singularity \(A_1^4\) of the instant shock at the time \(t_*\).

If \(d = 2, 3\), \(w \in A_3\) such that \(C \neq 0\), then \(x_*\) is a singularity \(A_3\) of the instant shock at the time \(t_*\).

If \(d = 3\), \(w \in A_1 A_3\) at the minimum \(A_2 C \neq 0\), and the kernel plane does not contain the minimum point \(A_1\), then \(x_*\) is a singularity \(A_1 A_3\) of the instant shock at the time \(t_*\).

In order to recognize perestroikas we use their extended names and signatures.

The *extended name* of a perestroika codes its name...
and the linear part of the normal form of time from Table 1. For example, the normal form \(-\alpha + \beta - \gamma\) in the case \(A_2^5A_3\) gives the extended name \(A_2^5\). Namely, the two first signs corresponding to the two minima \(A_1\) (see the normal form of limit solution) give \(A_1^-\) – the order is irrelevant, the third sign corresponding to the minimum \(A_3\) gives \(A_3^+\). The cases \(A_2^\pm\) are more complicated: the extended name shows not only the signs of the coefficients of the linear part but the opposite sign of their sum as well. For example, the normal form \(-\alpha - \beta - \gamma + \delta\) in the case \(A_2^5\) gives the extended name \(A_2^5\) – the signs themselves give \(-\ldots\) and \(+\) is the sign of the number \(-(-1-1-1+1)=2\), the order is irrelevant again. A general definition will be given in \(\mathbb{Z}^2\).

The signature of a perestroika is just the signature of the quadratic part of the normal form of time from Table 1 – the quantities of the positive and negative squares respectively. The signature is written as \([p,q]\), the trivial ones \([0,0]\) are omitted. For example, the normal form \(-\alpha - \beta^2 - \gamma^2 + \delta^2\) in the case \(A_2^5\) gives the signature \([1,2]\) – the order is irrelevant as usual. The corresponding perestroika is recognized as \(A_2^5\).

The extended names and signatures of the perestroikas from Figure 2 and Figures 3, 4, 5 are shown in Tables 2 and 3 respectively. In other words, Tables 2 and 3 can be considered as “maps” of the figures of perestroikas. Namely, the top line of each cell of the tables indicates the extended name and signature of the perestroika \(\uparrow\) from the corresponding cell of the figures, the bottom line corresponds to the perestroika \(\downarrow\).

**Theorem 3. (Perestroikas – Extended Names)** If \(d = 1, 2, 3\), \(w \in A_1^d\), and the points of the minima belong to the same straight line, then \((x_*, t_*)\) is a perestroika

\[ A_{1}^{d-}\] of the instant shock.

If \(d = 2, 3\), \(w \in A_1^d\), and the points of the minima belong to the same plane, but neither any three of them are on the same straight line nor all of them are on the same circle, then \((x_*, t_*)\) is a perestroika

\[ A_{1}^{d++}\] provided that the convex hull of our four points is a triangle;

\[ A_{1}^{d++}\] provided that the convex hull of our four points is a quadrangle.

If \(d = 3\), \(w \in A_1^3\), and neither any four of the points of the minima belong to the same plane nor all five of them belong to the same sphere, then \((x_*, t_*)\) is a perestroika \(A_3^+\) of the instant shock. More precisely, let us consider the ball of the minimal radius containing all our five points. Four of them lie on the boundary of the ball and generate four planes which divide it into 11 parts: the 1 inner tetrahedron, the 4 domes adjoining the faces of the tetrahedron, and the 6 lobules adjoining its edges. We get:

\[ A_{1}^{d--}\] if the fifth point is inside of the tetrahedron;

\[ A_{1}^{d-}\] if the fifth point is inside of a dome;

\[ A_{1}^{d++}\] if the fifth point is inside of a lobule.

If \(d = 1, 2, 3\), \(w \in A_1\) such that \(C = 0\) when \(d = 2\) or 3, then \((x_*, t_*)\) is a perestroika

\[ A_{3}^{-}\] of the instant shock.

If \(d = 2, 3\), \(w \in A_1A_3\), at the minimum \(A_3\) \(C \neq 0\), the minimum point \(A_1\) belongs to the kernel plane of the minimum \(A_3\) but does not lie either on the kernel line or on the kernel circle, then \((x_*, t_*)\) is a perestroika

\[ A_{1}^{-}\] of the instant shock. More precisely, we get (see Figure 2.7):

\[ A_{1}^{d-}A_{3}^{-}\] if \(S\) lies in the disk \(\mathcal{D}\);

\[ A_{1}^{-}A_{3}^{-}\] if \(S\) lies in the domain \(\mathcal{U}\);

\[ A_{1}^{-}A_{3}^{-}\] if \(S\) lies in the semiplane \(\mathcal{P}\).

Otherwise, when the points \(A_1\) are on one side of the kernel plane:

\[ A_{1}^{-}A_{3}^{-}\] if \(S\) lies in the disk \(\mathcal{D}\) or in the semiplane \(\mathcal{P}\), or degenerates into the point \(O\) in the case of tangency of the sphere and the kernel plane;

\[ A_{1}^{-}A_{3}^{-}\] if \(S\) lies in the domain \(\mathcal{U}\).

If \(d = 3\), \(w \in A_3\) such that

\[
\begin{vmatrix}
C_1 & D_1 & E_1 \\
C_2 & D_2 & E_2 \\
1 & 0 & C_2^2 + C_3^2
\end{vmatrix} \neq 0,
\]

then \((x_*, t_*)\) is a perestroika

\[ A_{5}^{+}\] provided that the indicated determinants have different signs;

\[ A_{5}^{-}\] provided that the indicated determinants have the same sign.

As we saw before, the singularities with the same name can occur at isolated points of the world shocks.
– in this case in order to recognize the perestroika generated by each of them it is enough to know the extended name of the perestroika – in this case its signature is trivial [0, 0]. Otherwise, the singularities can form a submanifold $X$ of a positive dimension. In this case the perestroikas generated by them occur at the points where $X$ is tangent to isochrones. In order to recognize such a perestroika it is necessary to know its extended name and signature which is the signature of the critical point of the restriction $t|_X$ of time onto $X$. (This critical point is non-degenerate if the initial condition is generic.) The dimension of $X$ can be calculated through the multiplicities of the competing global minima:

$$\dim X = d + 1 - \kappa, \quad \kappa = \mu_1 + \ldots + \mu_m - 1$$

where $\kappa$ is, of course, just the codimension of $X$.

**Theorem 4. (Perestroikas – Signatures)** If $\dim X = 0$ then the signature of a perestroika is trivial $[0, 0]$. Otherwise, it is equal to the signature of a quadratic form $\Xi$ on $\mathbb{R}^{\dim X}$ which must not be degenerate in order to the perestroika to be generic. The form $\Xi$ is defined below.

### 3.1 Cases $A_1^3$ and $A_1^4$

Let $d + 2 > m \geq 3$ and the function $w$ have $m$ global minima $A_1$ at points $A_1, \ldots, A_m$ which lie in the same $(m-2)$-plane $\Lambda$ but do not belong to the same $(m-3)$-dimensional sphere or plane. Then the system of $d + 1$ linear equations and one inequality

$$\begin{cases}
  c_1 + \cdots + c_m = 0 \\
  A_1 c_1 + \cdots + A_m c_m = 0 \\
  |A_1|^2 c_1 + \cdots + |A_m|^2 c_m < 0
\end{cases}$$

has a solution $c$. Let $\Lambda^\perp$ denote the $(d + 2 - m)$-dimensional linear space of all (co)vectors being normal to $\Lambda$ and

$$\Xi = c_1 \Phi_1 |\Lambda^\perp + \cdots + c_m \Phi_m |\Lambda^\perp$$

where $c$ is a solution of our system and $\Phi_i$ is the quadratic form which is inverse to the matrix of the second derivatives of $w$ at the minimum point $A_i$:

$$\Phi_i = \Theta_i^{-1}, \quad \Theta_i = \|w_{aa}(A_i)\|.$$  

The signature of the quadratic form $\Xi : \Lambda^\perp \to \mathbb{R}$ is the signature of the perestroika.

### 3.2 Case $A_3$

Let the function $w$ have a minimum $A_3$, and $(a, b)$ be orthonormal coordinates giving its representation (6). We will need the following extra terms from there:

$$w(a, b) = Ka^4 + L_1 a^3 z_1 + \cdots + L_{d-1} a^3 z_{d-1} + \Omega_0(z) + a \Omega_1(z) + a^2 \Omega_2(z) + \ldots$$

where $z = b - C a^2$, $\Omega_0 = \omega, \Omega_1$, and $\Omega_2$ are quadratic forms on $\mathbb{R}^{d-1}$ or just symmetric $(d-1) \times (d-1)$-
matrices of real numbers. Let \( \Omega = \Omega_0 + a \Omega_1 + a^2 \Omega_2 \),
\[
\Theta = \begin{pmatrix}
4K & 3L_1 a^2 & \cdots & 3L_{d-1} a^2 \\
L_1 & \vdots & & \\
\vdots & \ddots & \ddots & \\
L_{d-1} & & & 2\Omega
\end{pmatrix}
\]
and \( \Psi_0, \Psi_1, \) and \( \Psi_2 \) be \((d - 1) \times (d - 1)\)-matrices of real numbers such that
\[
\Theta^{-1} = \begin{pmatrix}
\vdots & \cdots & \vdots \\
\vdots & \Psi & \vdots \\
\vdots & \cdots & \vdots
\end{pmatrix} + O_{>2}(a)
\]
where \( \Psi = \Psi_0 + a \Psi_1 + a^2 \Psi_2 \). It is easy to see that \( \Psi \) is symmetric because \( \Omega \) is symmetric.

Let \( d > 1 \) and the function \( w \) have a global minimum \( A_3 \) such that \( C = 0 \). The signature of the matrix
\[
\Xi = -\Psi_2
\]
is the signature of the perestroika.

### 3.3 Case \( A_1 A_3 \)

Let \( d = 3, w \in A_1 A_3 \), at the minimum \( A_2 \), \( C \neq 0 \), the minimum point \( A_1 \) lie in the kernel plane \( \Lambda \) of the minimum \( A_3 \) but do not belong to the kernel circle. If \( a = (a, b) \) are orthonormal coordinates giving for \( w \) the representation (3) and \((a_1, s_1 \ C/|C|)\) are the coordinates of the minimum point \( A_1 \) then the system of linear equations and one inequality
\[
\begin{cases}
c_1 + c_4 = 0 \\
a_1 c_1 + c_3 = 0 \\
s_1 c_1 + |C| c_2 = 0 \\
(a_1^2 + s_1^2) c_1 + c_2 < 0
\end{cases}
\]
has a solution \( c. \) Let \((0, n) = (0, C_2, -C_1)\) be a normal to \( \Lambda \) (co)vector and
\[
\Xi = c_1 \Phi(0, n) + c_2 \Psi_2(n) + c_3 \Psi_1(n) + c_4 \Psi_0(n)
\]
where \( c \) is a solution of our system, \( \Phi \) is the quadratic form defined in 3.3 for the minimum point \( A_1 \), and the matrices of the quadratic forms \( \Psi_{0,1,2} \) are defined in 3.2.

The sign of the number \( \Xi \) is the signature of the perestroika.

### 4 The Minimum Representations

#### 4.1 Unforced case

We can rewrite our equations (3) as one equation for \( \psi \):
\[
\psi_t + \frac{1}{2} \nabla \psi^2 = \nu \Delta \psi.
\]

The Hopf–Cole transformation (which was published by Forsyth back in 1906 [8]) \( \psi = -2\nu \ln \theta \) reduces this equation to the \( d \)-dimensional heat equation
\[
\psi_t = \nu \Delta \psi
\]
which can be solved explicitly if there are no boundaries. We will get
\[
\psi(x, t) = \nu \ln(4\pi \nu t) - \\
-2\nu \ln \int_{\mathbb{R}^d} \exp \left\{ -\frac{W(a, x, t)}{2\nu} \right\} da_1 \ldots da_d
\]
where \( W \) is defined in 3.3. This implies our minimum representation (3) for the limit solution as the viscosity vanishes \( \nu \to 0 \).

#### 4.2 Arbitrary potential

In this case we get \( h[\psi] = \nu \Delta \psi \) where
\[
\begin{align*}
 h[\psi](x, t) & = \psi_t(x, t) + \frac{1}{2} |\nabla \psi(x, t)|^2 + U(x, t). \\
\end{align*}
\]
A continuous function \( \varphi \) is called a viscosity solution of the Hamilton–Jacobi equation \( h[\varphi] = 0 \) if and only if for any point \((x_s, t_s)\):

- any smooth function \( \phi(x, t) \) such that \( \varphi - \phi \) has a local minimum at \((x_s, t_s)\) satisfies the inequality \( h[\phi](x_s, t_s) \geq 0 \), and
- any smooth function \( \phi(x, t) \) such that \( \varphi - \phi \) has a local maximum at \((x_s, t_s)\) satisfies the inequality \( h[\phi](x_s, t_s) \leq 0 \).

It turns out that, according to [8], there exists a unique viscosity solution of the Hamilton–Jacobi equation \( h[\varphi] = 0 \) with an initial condition \( \varphi_0 \). But the limit solution
\[
\varphi(x, t) = \lim_{\nu \to 0} \psi(x, t)
\]
satisfies the two above conditions and, therefore, is the viscosity one. (This fact is a motivation behind the above definition and term “viscosity solution”.) Let
\[
F(u, x, t) = \varphi_0(\gamma(0)) + \int_0^t L(\gamma(\tau), \gamma(\tau), \tau) \, d\tau
\]
where \( \gamma : [0, t] \to \mathbb{R}^d \) is the solution of the Cauchy problem:
\[
\begin{cases}
\dot{\gamma}(\tau) = -\nabla U(\gamma(\tau), \tau) \\
\gamma(t) = u \\
\gamma(0) = x
\end{cases}
\]
and $L(\dot{x}, x, t) = |\dot{x}|^2/2 - U(x, t)$ is the Lagrangian. According to \cite{12}, the function
\[
\varphi(x, t) = \min_u F(u, x, t)
\]
is a unique viscosity solution of the Hamilton–Jacobi $h[\varphi] = 0$ equation. Therefore it is the minimum representation for our limit solution.

4.3 Arbitrary convex Hamiltonian

Our minimum representation is true for the limit solution of any equation of the kind:
\[
\psi_t + H(\nabla \psi, x, t) = \nu \Delta \psi
\]
if the Hamiltonian $H$ is strictly convex down with respect to momenta. In other words, the matrix of its second derivatives must be positive definite:
\[
\|H_{pp}(p, x, t)\| > 0.
\]
In the previous case it is true because $H(p, x, t) = |p|^2/2 + U(x, t)$.

Let the Lagrangian be the Legendre transformation of the Hamiltonian:
\[
L(\dot{x}, x, t) = \max_p \{p \dot{x} - H(p, x, t)\},
\]
and the trajectory $\gamma$ is the solution of the analogous Cauchy problem for the Euler–Lagrange equation
\[
\frac{d}{dt} \frac{\partial}{\partial \gamma} L(\dot{\gamma}, \gamma, \tau) = \frac{\partial}{\partial \gamma} L(\dot{\gamma}, \gamma, \tau).
\]
Then everything works again. Our classification of perestroikas of shocks is applicable to this most general situation as well.

When the Hamiltonian is not convex only one-dimensional case is investigated \cite{17}, \cite{18}. The situation is more complicated because there is no a minimum representation like before. Instead a min-max representation exists \cite{12} but it does not help.

5 Minimum Functions

5.1 Families of functions

A generic smooth function can have only non-degenerate local minima $A_1$. All degenerate minima can be killed by an arbitrarily small perturbation of the original function. In other words, degenerate minima are not stable with respect to small perturbations of the original function. For example, $u^4$ – the simplest degenerate minimum $A_3$ of one variable – can be turned into a non-degenerate minima by the perturbation $u^4 + \varepsilon u^2$ where $\varepsilon > 0$. However, degenerate minima become stable with respect to small perturbations if we consider not functions but families of them. For example, the family
\[
F(u, \lambda) = u^4 + \lambda_1 u^2 + \lambda_2 u
\]
of functions of $u$ depending on the parameters $\lambda = (\lambda_1, \lambda_2)$ has the minimum $A_3$ if $\lambda = 0$. Any sufficiently small perturbation $F^*$ of $F$ will attain a degenerate minimum along $u$ at some point $u = u_*$ for some parameter values $\lambda = \lambda_*$. In order to find $u_*$ and $\lambda_*$ it is enough to solve the system of equations $F^*_{uu}(u_*, \lambda_*) = F^*_{uuu}(u_*, \lambda_*) = F^*_{uunn}(u_*, \lambda_*) = 0$ which has a unique solution according to the implicit function theorem (as well as the same system for $F$).

In general, the degenerate minimum $A_{2j+1}$ can be attained by function depending generically on $l \geq 2j$ parameters, and can be killed by an arbitrarily small perturbation of the original family for lesser $l$. The following Theorem describes locally all generic families containing minima $A_{2j+1}$ of one variable.

Lemma 1. Let a function $F(\cdot, \lambda_*)$ from a generic family $F$ of functions of one variable depending on $l \geq 2j$ parameters have a minimum $A_{2j+1}$ at a point $u_*$. Then the family $F$ can be reduced in a neighborhood of the point $(u_*, \lambda_*)$ to the following normal form:
\[
F(u, \lambda) = u^{2j+2} + \lambda_1 u^{2j} + \cdots + \lambda_{2j} u
\]
by a smooth change $\lambda \to h(\lambda)$ of the parameters, a smooth change $u \to g(u, \lambda)$ of the variable depending smoothly on the parameters, and adding $F(u, \lambda) \to F(u, \lambda) + c(\lambda)$ a smooth function of the parameters.

Indeed, after the change $u \mapsto u + u_*$ we can write
\[
F(u, \lambda) = k(u, \lambda) u^{2j+2} + b_0(\lambda) u^{2j+1} +
\]
\[
+ b_1(\lambda) u^{2j} + \cdots + b_{2j}(\lambda) u + c(\lambda)
\]
where $k(0, \lambda_*) = 1$ and $b_0(\lambda_*) = b_1(\lambda_*) = \cdots = b_{2j}(\lambda_*)$. The change $u \mapsto u/\sqrt[2j+2]{k(u, \lambda)}$ gives $k = 1$, the change $u \mapsto u - b_0(u, \lambda)/(2j+2)$ kills $b_0$. It remains to consider $b_1, \ldots, b_{2j}$ as new parameters in a neighborhood of $\lambda_*$ (taking into account that $F$ is generic) and kill $c$.

The minimum $X_0$ occurs in generic families of functions only if the number of parameters $l \geq 7$. Namely, the 3 second derivatives and the 4 third derivatives must be equal to 0 – we get exactly 7 extra conditions at the minimum point.
5.2 Singularities

Let $F(u, \lambda)$ be a family of functions of $u = (u_1, \ldots, u_d)$ depending smoothly on parameters $\lambda = (\lambda_1, \ldots, \lambda_l)$. Then the function

$$
\varphi(\lambda) = \min_u F(u, \lambda)
$$

is called the minimum function of the family $F$.

If the family $F$ is smooth and the function $F(\cdot, \lambda_*)$ has only the global minimum which is not degenerate then the minimum function $\varphi$ is smooth at a neighborhood of the point $\lambda_*$. If the function $F(\cdot, \lambda_*)$ attains its global minimal value at a few points or its global minimum is degenerate, then the minimum function $\varphi$ can have singularity at the point $\lambda_*$. In order to investigate singularities of the minimum functions of smooth generic families let us note that, while the number of parameters $l \leq 6$, it is enough to consider only functions of one variable. Indeed, if $l \leq 6$ and a function $F(\cdot, \lambda_*)$ attains its minimum at a point $u_*$ we get $\|F_{uu}(u_*, \lambda_*)\| \geq d - 1$ because the case $d - 2$ requires at least 7 parameters. Let, for example,

$$
\det \left| \frac{\partial^2 F}{\partial (u_2, \ldots, u_d)^2} \right| \neq 0
$$

then

$$
\varphi(\lambda) = \min_{\bar{u}} F(\bar{u}, \lambda), \quad \bar{F}(u_1, \lambda) = \min_{u_2, \ldots, u_d} F(u, \lambda)
$$

where $\bar{F}$ is a smooth family of smooth one-variable functions.

If $F$ is a generic family of smooth functions depending smoothly on $l \leq 4$ parameters $\lambda = (\lambda_1, \ldots, \lambda_l)$ then only the following combinations of minima on the same level are possible:

- $A_1$;
- $A_1^2$ if $l \geq 1$;
- $A_1^3, A_3$ if $l \geq 2$;
- $A_1^4, A_1 A_3$ if $l \geq 3$;
- $A_1^5, A_1^2 A_3, A_5$ if $l = 4$.

For example, the combination $A_1^2 A_3$ means that our function has two minima $A_1$, one minimum $A_3$, and the values of the function at all of them are the same.

Applying Theorem 5.3 in all these cases to the last minimum and considering the values at the other minima $A_1$ as new parameters we get:

**Theorem 5.3** If $F$ is a generic family of smooth functions depending smoothly on $l \leq 4$ parameters $\lambda = (\lambda_1, \ldots, \lambda_l)$, then its minimum function

$$
\varphi(\lambda) = \min_u F(u, \lambda)
$$

can be reduced in a neighborhood of any point $\lambda_*$ to one of the following normal forms such that $\kappa \leq l$:

- $A_1^{\kappa + 1}, \kappa \geq 0$: $\min\{\lambda_1, \ldots, \lambda_\kappa, 0\}$;
- $A_1^{\kappa - 2} A_3, \kappa \geq 2$:
  $$
  \min_{\kappa} \left\{ \lambda_1, \ldots, \lambda_{\kappa - 2}, \min_u (u^4 + \lambda_{\kappa - 1} u^2 + \lambda_{\kappa} u) \right\};
  $$
- $A_5, \kappa = 4$: $\min_u (u^6 + \lambda_1 u^4 + \lambda_2 u^3 + \lambda_3 u^2 + \lambda_4 u)$;

by a smooth change of the parameters and adding a smooth function of them.

These normal forms are given in Table I where $(\alpha, \beta, \gamma, \delta) = (\lambda_1, \ldots, \lambda_4)$. All of them except $A_1$ are not smooth at some points which form a shock in $\lambda$-space. The shock is smooth in the case $A_1^2(\lambda_1 = 0)$, has a boundary in the case $A_3(\lambda_1 \leq 0, \lambda_2 = 0)$, and has more complicated singularities in the other cases. Each of the shocks is a cylinder with $(l - \kappa)$-dimensional generators. The “most” singular points are defined by the equations $\lambda_1 = \cdots = \lambda_{\kappa} = 0$ and are said to be the corresponding type ($A_1^{\kappa + 1}, A_1^{\kappa - 2} A_3$, or $A_5$). Any other point has a type with lesser $\kappa$.

These shocks are interpreted as world shocks if $l = d + 1$ and instant shocks at typical times if $l = d$ provided that the action $F$ is a smooth generic function. But instant shocks can experience perestroikas at separate times which has not been described by us yet.

5.3 Perestroikas

Let $F(u, x, t)$ be a family of functions of $u$ depending smoothly on a point $x \in \mathbb{R}^d$ of space and time $t$,

$$
\varphi(x, t) = \min_u F(u, x, t),
$$

$$
S(\varphi) = \{(x, t) \in \mathbb{R}^{d + 1} | \varphi \notin C^\infty(x, t)\}
$$

be the minimum function and its world shock.

In order to investigate perestroikas of the instant shocks let us consider the following equivalence of two world shocks:

$$
S(\varphi') \sim S(\varphi) \iff S(\varphi') = g(S(\varphi))
$$

where

$$
g: \mathbb{R}^{d + 1} \to \mathbb{R}^{d + 1}, (x, t) \mapsto (x, t + t_s)
$$
is a sectionally continuously differentiable homeomorphism of space-time shifting time and being a smooth diffeomorphism outside the world shocks themselves.

REMARK. Unfortunately, if we assume that $g$ is a smooth diffeomorphism everywhere we get an infinite local classification of perestroikas of shocks even if $F$ is generic and $d = 2$. Allowing smooth changes of time instead shifts does not help. For example, the instant shock at the time of the second perestroika $A_1^1$ in plane (see Figure 2) will have at least one continuous invariant – the cross-ratio of the four rays.

In order to obtain $g$ let us consider the graph of the minimum function $\varphi$

$$\Gamma(\varphi) = \{(y, x, t) \in \mathbb{R} \times \mathbb{R}^{d+1} \mid y = \varphi(x, t)\}$$

and define the equivalence of two graphs:

$$\Gamma(\varphi') \sim \Gamma(\varphi) \iff \Gamma(\varphi') = G(\Gamma(\varphi))$$

where

$$G : \mathbb{R} \times \mathbb{R}^{d+1} \to \mathbb{R} \times \mathbb{R}^{d+1}, (\cdot, \cdot, t) \mapsto (\cdot, \cdot, t + t_*)$$

is a smooth diffeomorphism shifting time. Then

$$g = \pi \circ G \circ \varphi,$$

$$\tilde{\varphi} : (x, t) \mapsto (\varphi(x, t), x, t), \quad \pi : (y, x, t) \mapsto (x, t).$$

THEOREM 6.14. If $F$ is a generic smooth family and $d \leq 3$ then all perestroikas of instant shocks are locally described by Table 4 with respect to the above equivalence.

Theorem 5 provides normal forms of world shocks if we set $l = d + 1$, but the reducing changes of variables mix space coordinate and time. In other words, $\lambda$-coordinates from Theorem 5 depend on both $x$ and $t$. After such mixing we can represent time as a generic smooth function of $\lambda$ without critical points. In the cases of Theorem 5 such function $t$ describes a perestroika of the instant shock if its restriction on the set of the “most” singular points $\lambda_1 = \cdots = \lambda_\kappa$ has a critical point or $\kappa = d + 1$. Shifting the coordinates $(\lambda_{\kappa+1}, \ldots, \lambda_{d+1})$ we get

$$t(\lambda) = t_* + \tau(\lambda),$$

$$\tau(0) = 0, \quad \tau_{\lambda_{\kappa+1}}(0) = \cdots = \tau_{\lambda_{d+1}}(0) = 0.$$

The graph of the minimum function $\varphi$ is a subset of the front $\Sigma(F)$ generated by the family $F$:

$$\Gamma(\varphi) \subset \Sigma(F) = \{(y, x, t) \in \mathbb{R} \times \mathbb{R}^{d+1} \mid \exists u : y = F(u, x, t), F_u(u, x, t) = 0\}.$$

If the function $F(\cdot, x_*, t_*)$ has $m$ global minima with the same minimal value $y_*$ then in a neighborhood of the point $(y_*, x_*, t_*)$ the front $\Sigma(F)$ consists of $m$ branches. In the cases of Theorem 5 we can, mixing $y$ and $\lambda$, define $\Sigma(F)$ in the following more symmetric way:

$$\Sigma(F) = \{(\xi, \eta) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^n \mid \exists u : P(u) = P'(u) = 0\}$$

where $n = d + 1 - \kappa$ and in the cases

1. $A_1^{\kappa+1}$: $m = \kappa + 1$ and $P = \xi_1 \cdots \xi_{\kappa+1}$;
2. $A_1^{\kappa-2}A_3$: $m = \kappa - 1$ and

$$P = \xi_1 \cdots \xi_{\kappa-2}(u^4 + \xi_{\kappa-1}u^2 + \xi_{\kappa}u + \xi_{\kappa+1});$$
3. $A_5$, $\kappa = 4$: $m = 1$ and

$$P = u^6 + \xi_1 u^4 + \xi_2 u^3 + \xi_3 u^2 + \xi_4 u + \xi_5.$$

Namely, $\xi_1 = \lambda_1 - y, \ldots, \xi_{m-1} = \lambda_m - y, \xi_m = \lambda_{m+1} - y, \ldots, \xi_{\kappa+1} = \lambda_{\kappa+1} - y$.

Let $T : \mathbb{R}^{\kappa+1} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function without critical points such that

$$T_{\xi_1}(0) \neq 0, \ldots, T_{\xi_{\kappa+1}}(0) \neq 0; \quad (7)$$

and if $n > 0$ then its restriction onto the plane $\xi = 0$ has a non-degenerate critical point at 0:

$$T_{\eta_1}(0) = \cdots = T_{\eta_n}(0) = 0, \quad \det \|T_{\eta\eta}(0)\| \neq 0. \quad (8)$$

The extended name of $T$ codes one of the normal forms $A_1^{\kappa+1}$, $A_1^{\kappa-2}A_3$, or $A_5$ for $\Sigma(F)$ and the signs of the first derivatives (1). For example: $A_1^{\kappa+1-\cdots}$ means $A_1^{\kappa}$ and indicates that among the 5 numbers (2) there are 2 positive and 3 negative ones (the order is not important – in extended names pluses are always followed by minuses): $A_1^{\kappa+1} - A_3^1$ means $A_1^{\kappa}A_3$ and indicates that one of the numbers $T_{\xi_1}(0)$ and $T_{\xi_{\kappa+1}}(0)$ is positive, the other is negative, and the number $T_{\xi_{\kappa+1}}(0)$ is positive; $A_3^{-}$ means $A_3$ and indicates that $T_{\xi_1}(0) < 0$.

If $n \neq 0$ the signature of $T$ is the signature of the second differential (3) and is denoted by $[p, q]$ where $p + q = n$.

Here are a few examples. In the case $A_1^2$, $n = 2$ the function $-\xi_1 + \xi_2 - \eta_1\eta_2$ has extended name $A_1^1$ and signature $[1, 1]$; in the case $A_3$, $n = 1$ the function $\xi_1 - \xi_3 - \eta_1^2$ has extended name $A_3^2$ and signature $[0, 1]$; in the case $A_1A_3$, $n = 1$ the function $-\xi_1 - \xi_2 + \eta_1^2$ has extended name $A_1^1A_3^1$ and signature $[1, 0]$; in the case $A_5$, $n = 0$ the function has extended name $A_5^{-}$.

Theorem 6 follows now from Theorem 5 proved in [19].
Let two smooth functions do not have critical points with the same critical value, satisfy (2) and (3), and have the same extended name and the same signature. Then they are translated to each other by a smooth diffeomorphism preserving the normal form of Σ(F).

Indeed, the translations from Theorem 5 are performed by vector fields being tangent to Σ(F) (see [12]). They are, of course, tangent to Π(F) as well. Let \( T = \tau \circ \pi \). The extended name of \( T \) can be arbitrary except \( A^{1,-+} \) and \( A^{1,-} \) because in the case \( A^{1} \)

\[
T_{\xi_{1}} + \cdots + T_{\xi_{m}} = 0 \quad (y = \xi_{1} + \cdots + \xi_{m} \text{ and } T \text{ does not depend on } y).
\]

But the normal forms \( \tau \) enumerated in Table 3 give for \( T \) all possible combinations of signatures and extended names with these exceptions.

5.4 Case where \( F \) is an action

If \( F \) is the action generated by a fixed convex smooth Lagrangian \( L \) then the front \( \Sigma(F) \) is the many-valued solution of the Hamilton–Jacobi equation with the initial condition \( \varphi_{0} \). The Hamiltonian is smooth because it is the Legendre transformation of the Lagrangian – see Section 5. Therefore, if the initial condition \( \varphi_{0} \) is generic then the conditions (2) and (3) are satisfied – this is proved in [13] for many-valued solutions of the Hamilton–Jacobi equation with any smooth Hamiltonian. Hence, Theorem 5 is true in this situation as well.

6. The Topological Restrictions

Let \((x_{*}, t_{*}) \in \mathbb{R}^{d+1}\) be an isolated perestroika of an instant shock. It means the point \((x_{*}, t_{*})\) has a neighborhood \( V \subset \mathbb{R}^{d+1}\) such that other perestroikas do not occur in \( V \).

**Theorem 8.** For a sufficiently small open ball \( B \subset \mathbb{R}^{d} \) containing \( x_{*} \) and for some \( \varepsilon > 0 \) the local instant shocks \( S_{t} \cap B \) are contractible (homotopically equivalent to a point) for all \( t \in [t_{*}, t_{*} + \varepsilon) \).

Originally Theorem 8 was discovered and proved in 1989, [14] only for the generic perestroikas when \( d \leq 3 \). But the proof was not topological. Namely, the restriction was algebraic and forbade some perestroikas from the lists of generic ones. It turned out that these perestroikas were exactly the same which were forbidden by the above topological restriction. In order to explain this fact topologically, Baryshnikov in 1990, [16], [20] suggested and proved another restriction which is true for all isolated (at least) perestroikas in all dimensions:

**Theorem 9.** For a sufficiently small open ball \( B \subset \mathbb{R}^{d} \) containing \( x_{*} \) and for some \( \varepsilon > 0 \) the local complements \( B \setminus S_{t} \) to the instant shocks are homotopically equivalent to each other for all \( t \in [t_{*}, t_{*} + \varepsilon) \).

It is easy to check that for the generic perestroikas if \( d \leq 3 \) these two restrictions are equivalent to each other. As a matter of fact they are equivalent for all isolated perestroikas. Indeed, roughly speaking the shocks themselves are homotopically equivalent if and only if their complements are homotopically equivalent. For example, for the homologies it directly follows from the Alexander duality between \((S_{t} \cap B) \cup \partial B \) and \( B \setminus S_{t} \) where \( \partial B \) is the ball’s boundary. But the local instant shock \( S_{t} \cap B \) at the perestroika’s time is contractible for a sufficiently small ball \( B \)!

Below we explain why the homology and homotopy groups of the complements of \( B \setminus S_{t} \) are the same for \( t \in [t_{*}, t_{*} + \varepsilon) \). These facts follow from Theorem 8 and are sufficient for all applications. As a matter of fact, we get weak homotopically equivalences which imply strong ones for CW-complexes. All details can be found in [14].

We start with the following simple fact. Let

\[
U_{1} \subset U_{2} \subset \ldots, \quad U_{\infty} = \bigcup_{i=1}^{\infty} U_{i}
\]

be an infinite sequence of open subsets in \( \mathbb{R}^{d} \) such that all embedding \( U_{t} \hookrightarrow U_{t+1} \) induce the isomorphisms

\[
H_{*}(U_{t}) \cong H_{*}(U_{t+1}), \quad \pi_{*}(U_{t}) \cong \pi_{*}(U_{t+1})
\]

between the homology and homotopy groups. But their definitions use only compact sets and our sets are open, so we get the isomorphisms

\[
H_{*}(U_{t}) \cong H_{*}(U_{\infty}), \quad \pi_{*}(U_{t}) \cong \pi_{*}(U_{\infty}).
\]

If the embeddings do not induce isomorphisms than we can write:

\[
\lim_{t \to \infty} H_{*}(U_{t}) = H_{*}(U_{\infty}), \quad \lim_{t \to \infty} \pi_{*}(U_{t}) = \pi_{*}(U_{\infty}).
\]

Let \( U_{t} \) denote the set of the Lagrangian coordinates of all particles which are outside the instant shock \( S_{t} \) at a time \( t \). This set is open, diffeomorphic to the complement \( \mathbb{R}^{d} \setminus S_{t} \) to the shock, and decreases:

\[
U_{t_{1}} \subset U_{t_{2}} \text{ if } t_{1} > t_{2}
\]

because a particle cannot leave the shock. Besides,

\[
U_{t_{*}} = \bigcup_{t > t_{*}} U_{t}.
\]

Now let us assume that there are no perestroikas at all for \( t \in (t_{*}, t_{*} + \varepsilon) \). It means that for the embeddings \( U_{t_{1}} \hookrightarrow U_{t_{2}} \) induce isomorphisms

\[
H_{*}(U_{t_{1}}) \cong H_{*}(U_{t_{2}}), \quad \pi_{*}(U_{t_{1}}) \cong \pi_{*}(U_{t_{2}})
\]
between the homologies and homotopies for \( t_1 + \varepsilon > t_1 > t_2 > t_* \). Analogously to the above fact for sequences, we get the isomorphisms:

\[
H_*(U_t) \cong H_*(U_{t_*}), \quad \pi_*(U_t) \cong \pi_*(U_{t_*})
\]

for \( t \in (t_*, t_* + \varepsilon) \).

For an isolated perestroika we can consider the time \( t_* \) as initial – it means that the Lagrangian coordinates of particles are defined at \( t_* \). Besides, we take the sets \( U_t \cap B \) instead of \( U_t \).

### 7 Shock Recognition

For the unforced Burgers equation there is the representation \( (9) \). Let the function \( w(a) = W(a, x_*, t_*) \) have a collection

\[
\mathcal{X} = \{ A_{\mu_1} \cdots A_{\mu_m} \}
\]

of \( m \) global minima, \( y_* \) be the minimal value, and \( \theta(a) = w(a) - y_* \). Let \( \mathcal{E} \) be the algebra of smooth functions on \( \mathbb{R}^d \) and

\[
\mathcal{K}(\mathcal{X}) \subset \mathcal{E}
\]

be the “submanifold” of all functions having the same collection \( \mathcal{X} \) of global minima such that the values at them are equal to 0. In particular, \( \theta \in \mathcal{K}(\mathcal{X}) \).

Let us fix the initial condition \( \varphi_0 \) and consider the set of all functions from the family \( W - y \):

\[
\mathcal{W} = \{ W(x, t) - y \in \mathcal{E} \mid x \in \mathbb{R}^d, y, t \in \mathbb{R}, t > 0 \}.
\]

It is a \((d+2)\)-dimensional semiplane:

\[
\mathcal{W} = \{ W(x, t; \theta(a)) \mid a \in \mathbb{R}^d, y, t \in \mathbb{R}, t > 0 \}
\]

where \( \mathbb{R}^+ \) is the set of positive real numbers. Moreover, if we fix \( t_* \) and consider the corresponding functions from \( W - y \):  

\[
\mathcal{W}_t = \{ W(x, t; \theta(a)) - y \in \mathcal{E} \mid x \in \mathbb{R}^d, y \in \mathbb{R} \}
\]

then we get a \((d+1)\)-dimensional plane

\[
\mathcal{W}_t = \{ W(x, t; \theta(a)) \mid a \in \mathbb{R}^d, t \in \mathbb{R} \}
\]

where \( \mathbb{R}^+ \) is the set of positive real numbers. Moreover, if we fix \( t_* \) and consider the corresponding functions from \( W - y \):

\[
\mathcal{W}_{t_*} = \{ W(x, t; \theta(a)) \mid a \in \mathbb{R}^d, y \in \mathbb{R} \}
\]

then we get a \((d+1)\)-dimensional plane

\[
\mathcal{W}_{t_*} = \{ W(x, t; \theta(a)) \mid a \in \mathbb{R}^d, t \in \mathbb{R} \}
\]

1) The plane \( \mathcal{W}_{t_*} \) is transversal to \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \) as well:

\[
\mathcal{T}_0 \mathcal{W}_{t_*} + \mathcal{T}_0 \mathcal{K}(\mathcal{X}) = \mathcal{E}
\]

\[
(1, a_1, \ldots, a_d, [a]^2)_{\mathbb{R}} + \mathcal{T}_0 \mathcal{K}(\mathcal{X}) = \mathcal{E}
\]

Then the instant shock at the time \( t_* \) has the singularity \( \mathcal{X} \) at the point \( \mathcal{X} \).

2) The plane \( \mathcal{W}_{t_*} \) is tangent to \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \). Then the point \( (x_*, t_*) \) is a perestroika \( \mathcal{X} \) of the instant shock.

In order to find its extended name and signature let us note that the differential of the time coordinate at the point \( (y_*, x_*, t_*) \):

\[
dt : \mathcal{T}_0 \mathcal{W} \to \mathbb{R}
\]

defines a unique linear form

\[
dt : \mathcal{O}_\theta = \mathcal{E}/\mathcal{T}_0 \mathcal{K}(\mathcal{X}) \to \mathbb{R}
\]

on the cotangent space to \( \mathcal{K}(\mathcal{X}) \) which is uniquely given by its properties:

\[
dt([1]) = dt([a_1]) = \cdots = dt([a_d]) = 0,
\]

\[
dt ([a^2]) = -2t_*^2 < 0
\]

where \( [h] \in \mathcal{O}_\theta \) denotes the conjugacy class of the function \( h \in \mathcal{E} \). The last property follows from the equation \( dt([W_1(a, x_*, t_*))] = 1 \). This linear form is responsible for the extended name of the perestroika – it is explained below.

The signature of the perestroika is, by definition, the signature of the critical point of the restriction

\[
t|_\mathcal{X}, \quad X = \mathcal{W} \cap \mathcal{K}(\mathcal{X});
\]

its second differential in our case is the quadratic form

\[
\Sigma = dt \circ \Pi|_{\mathcal{T}_0 \mathcal{X}}, \quad \mathcal{T}_0 \mathcal{X} = \mathcal{T}_0 \mathcal{W}_{t_*} \cap \mathcal{T}_0 \mathcal{K}(\mathcal{X})
\]

\[
\mathcal{T}_0 \mathcal{W}_{t_*} = \{ 1, a_1, \ldots, a_d \}_{\mathbb{R}}
\]

where \( \Pi : \mathcal{T}_0 \mathcal{K}(\mathcal{X}) \to \mathcal{O}_\theta \) is the second quadratic form of the “submanifold” \( \mathcal{K}(\mathcal{X}) \) which is defined below.

The “submanifold” \( \mathcal{K}(\mathcal{X}) \) can be obtained from \( \theta \):

\[
\mathcal{K}(\mathcal{X}) = \{ \theta(g(a)) + h(a) \theta(a) \}
\]

where \( g : \mathbb{R}^d \to \mathbb{R}^d \) is a smooth diffeomorphism and \( h \in \mathcal{E} \). If \( g_* \) and \( h_* \) depend smoothly on a parameter \( s \in \mathbb{R} \) we can consider the path

\[
\theta_s(a) = \theta(g_*(a)) + h_s(a) \theta(a) \in \mathcal{O}_\theta;
\]

we assume that \( g_0(a) = a \) and \( h_0 \equiv 1 \), therefore \( \theta_0 = \theta \).

The derivatives of \( g_* \) along \( s \) define the vector field

\[
v_s(g_*(a)) = g_*(a),
\]
and we can write
\[ \dot{\theta}_s(a) = (v_s \cdot \nabla \theta)(g_s(a)) + \dot{h}_s(a)\theta(a). \tag{11} \]
In particular,
\[ \dot{\theta}_0 = v_0 \cdot \nabla \theta + \dot{h}_0\theta \in \mathcal{I}_\theta \subset \mathcal{E} \]
where \( \mathcal{I}_\theta \) is the ideal generated by \( \theta \) and its all first partial derivatives. It is the tangent space to \( \mathcal{K}(\mathcal{X}) \):
\[ \mathcal{I}_\theta \mathcal{K}(\mathcal{X}) = \mathcal{I}_\theta \]
and the factor algebra
\[ \mathcal{O}_\theta = \mathcal{E}/\mathcal{I}_\theta \]
is the cotangent space. Its dimension is equal to the common multiplicity \( \mu = \mu_1 + \cdots + \mu_m \) of the collection \( \mathcal{X} \).

Let \( A_1, \ldots, A_m \in \mathbb{R}^d \) be the minimum points \( A_{\mu_1}, \ldots, A_{\mu_m} \) respectively and
\[ \mathcal{I}_{\theta,i} = \mathcal{I}_\theta + s_i, \quad \mathcal{O}_{\theta,i} = \mathcal{E}/\mathcal{I}_{\theta,i} \]
where \( s_i \in \mathcal{E} \) is the ideal consisting of all functions such that each of them is equal to 0 in some neighborhood (depending on the function) of the point \( A_i \). Then we get the decompositions
\[ \mathcal{I}_\theta = \mathcal{I}_{\theta,1} \cap \cdots \cap \mathcal{I}_{\theta,m}, \quad \mathcal{O}_\theta = \mathcal{O}_{\theta,1} \oplus \cdots \oplus \mathcal{O}_{\theta,m}. \tag{12} \]

In each algebra \( \mathcal{O}_{\theta,i} \) there is a unique maximal ideal \( \mathfrak{m}_{\theta,i} \). Its degree \( \deg \mathfrak{m}_{\theta,j}^{-1} \) defines the so-called Jacobian ideal which is a unique minimal ideal and a one-dimensional \( \mathbb{R} \)-linear subspace of the algebra \( \mathcal{O}_{\theta,i} \). This ideal is generated by the element
\[ j_{\theta,i} = |a - A_{|i-1}^{-1}| \in \mathcal{O}_{\theta,i} \subset \mathcal{O}_\theta \]
(\( \mu_i \) is odd) and the ray \( \langle j_{\theta,i} \rangle_{\mathbb{R}_+} \subset \mathcal{O}_\theta \) does not depend on coordinates (\( \mathbb{R}_+ \) is positive numbers).

The signs of the numbers
\[ c_1 = d t(j_{\theta,1}), \ldots, c_m = d t(j_{\theta,m}) \]
form the extended name of perestroika.

If \( \mathcal{X} = A_1^m \) then \( \mu = m \),
\[ \mathcal{I}_{\theta,i} = (a_1 - A_{1i}, \ldots, a_d - A_{id}) \]
is a maximal ideal where \( (A_{1i}, \ldots, A_{id}) \) are the coordinates of the minimum point \( A_i \).
\[ \mathcal{O}_{\theta,i} \cong \mathbb{R}, \quad j_{\theta,i} = [1], \quad \mathcal{O}_\theta \cong \mathbb{R}^m, \]
and the decomposition of \([1], [a_1], \ldots, [a_d], [a]^2 \) can be written as a \((d + 2) \times \mu\)-matrix:
\[
M = \begin{pmatrix}
1 & \cdots & 1 & 0 & 0 & 1 \\
[a] & \cdots & [a] & 0 & 1 & 0 \\
[a]^2 & \cdots & [a]^2 & 0 & 0 & 0 \\
A_1 & \cdots & A_1 & \mathbf{C} & 0 & 0 \\
|A_1|^2 & \cdots & |A_1|^2 & 0 & 1 & 0 \\
|A|^2 & \cdots & |A|^2 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Let \( \mathcal{X} = A_1^l A_3, l = m - 1 \) and \((a, b)\) be orthonormal coordinates giving the representation \([4]\) for the minimum \( A_3 \) of \( \theta \). Then \( \mu = m + 2 \) and using the coordinates \((a, z)\) we get
\[ \mathcal{I}_{\theta,m} = (a^3 + O_{>3}, z_1 + O_{>2}, z_{d-1} + O_{>2}); \]
the Nakayama lemma implies that
\[ \mathcal{I}_{\theta,m} = (a^3, z_1, \ldots, z_{d-1}). \]
Therefore, \( \mathcal{O}_{\theta,m} \cong \mathbb{R}[a]/(a^3), j_{\theta,m} = [a]^2, \)
\[ \mathcal{O}_\theta \cong \mathbb{R}^{m-1} \oplus \mathbb{R}[a]/(a^3). \]
Using the basis \([a]^2, [a], [1] \) in \( \mathbb{R}[a]/(a^3) \) the decomposition of \([1], [a], [b_1], \ldots, [b_{d-1}], [a]^2 \) can be written as a \((d + 2) \times \mu\)-matrix
\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
E & D & C & 0 & 0 & 0 \\
|C|^2 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
where \( C, D, E \) comes from the representation \([3]\): \( [b] = C[a]^2 + D[a^3] + E[a^4] \) (mod \( \mathcal{I}_\theta \)).
Let, in these all cases,
\[ dt = (c_1, \ldots, c_\mu) \]
be the coordinates of the linear form \( dt \) written as a \( \mu \)-column \( c \) and the \((d + 1) \times \mu\)-matrix \( M' \) be the matrix \( M \) without the last row \( M_{d+2} \):
\[ M = \begin{bmatrix} M' \\ M_{d+2} \end{bmatrix}. \]

So, if \( \text{rank } M = \mu \) then the semiplane \( \mathcal{W} \) is transversal to \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \) and the world shock has the singularity \( \mathcal{X} \) at the point \( (x_*, t_*) \). There are two possible cases:

1) If \( \text{rank } M' = \mu \) too then the plane \( \mathcal{W}_t \) is transversal to \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \) as well and the instant shock at the time \( t_0 \) has the singularity \( \mathcal{X} \) at the point \( (x_*, t_0) \). This is an algebraic form of Theorem 3.

2) If \( \text{rank } M' = \mu - 1 \) then the plane \( \mathcal{W}_t \) is tangent to \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \) and the point \( (x_*, t_0) \) be the coordinates of the linear form \( d \) \( \mathcal{K}(\mathcal{X}) \) at the point \( \theta \). Besides, the system of \( d + 1 \) linear equations and one inequality coming from (3):
\[
\begin{align*}
M'c &= 0 \\
M_{d+2}c &< 0
\end{align*}
\]
have a solution and the signs of its components \( c_1, \ldots, c_\mu \) give the perestroika's extended name. Neither of them must equal to \( 0 \) otherwise the perestroika is not generic and does not belong to our lists. This is an algebraic form of Theorem 4.

In order to calculate the perestroika's signature let us define the second quadratic form of the "submanifold" \( \mathcal{K}(\mathcal{X}) \subset \mathcal{E} \) at the point \( \theta \):
\[ \Pi_{\theta}(x, y) = [u \cdot \nabla u] = [v \cdot \nabla v] \mod \mathcal{I}_\theta, \]
\[ x = u \cdot \nabla \theta + h \theta \in \mathcal{I}_\theta, \]
\[ y = v \cdot \nabla \theta + k \theta \in \mathcal{I}_\theta. \]

These formulas show that \( \Pi \) is correctly defined (depends only on \( r, \eta \) but not on \( u, v, k, h \)), symmetric, and linear over \( \mathcal{E} \). Besides,
\[ \mathcal{J}_\theta = \mathcal{I}_\theta^2 + (\theta) \subset \ker \Pi_{\theta}. \]

As usual, the geometrical sense of the second quadratic form \( \Pi \) is the curvature of \( \mathcal{K}(\mathcal{X}) \):
\[ \Pi_{\theta}(\hat{\theta}_0, \hat{\theta}_0) = \hat{\theta}_0 \mod \mathcal{I}_\theta \]

because differentiating (11) we get
\[ \hat{\theta}_0 = v_0 \cdot \nabla (v_0 \cdot \nabla \theta) + \dot{v}_0 \cdot \nabla \theta + \ddot{v}_0 \theta. \]

We can decompose the second quadratic form using the decomposition (12):
\[ \Pi_{\theta} = \Pi_{\theta,1} + \cdots + \Pi_{\theta,m}, \quad \Pi_{\theta,i} : \mathcal{I}_{\theta,i} \to \mathcal{O}_{\theta,i}, \]
\[ \mathcal{J}_{\theta,i} = \mathcal{I}_{\theta}^2 + (\theta) + s_i \subset \ker \Pi_{\theta,i}. \]

Let \( \mathcal{X} = A_{n}^\theta \), then \( \mathcal{J}_{\theta,i} = \mathcal{I}_{\theta,i}^2 \). How to calculate the matrix \( \Phi_{\theta} \)? We know that
\[ \partial_{uv} \theta = \sum_q (\theta_i)_{pq} (a_q - A_{iq}) \mod \mathcal{J}_{\theta,i}, \]
then the equality
\[ \Pi_{\theta,i}(\partial_{uv} \theta, a_r - A_{ir}) = \partial_{uv} a_r = \delta_{pr} \]
gives \( \Phi_{\theta} = \Theta_{\theta}^{-1} \mod \mathcal{I}_{\theta,i} \). But, according to (11),
\[ \mathcal{T}_\theta X = \langle a_1, \ldots, a_d \rangle_{\mathbb{R}} \cap \mathcal{I}_\theta = \Lambda_{\perp} \]
and \( \Xi = c_1 \Phi_{\theta} |_{\Lambda_{\perp}} + \cdots + c_n \Phi_{\theta} |_{\Lambda_{\perp}} \).

Let \( \mathcal{X} = A_{3}^\theta \), then \( \mathcal{I}_\theta = (a^3, z_1, \ldots, z_{d-1}) \), \( \mathcal{J}_\theta = \mathcal{I}_\theta^2 + (a^3) \). How to calculate the matrix \( \Phi_{\theta} = \Pi_{\theta}(r_1, r_2) \) where \( (r_1, \ldots, r_d) = (a^3, z_1, \ldots, z_{d-1}) \)? If the coordinates \( u = (a, z) \) we get
\[ \partial_{uv} \theta = \sum_q (\theta_i)_{pq} (a_q - A_{iq}) \mod \mathcal{I}_\theta, \]
then the equality \( \Pi_{\theta}(\partial_{uv} \theta, r_r) = \partial_{uv} r_r \) gives
\[ \Psi' = \Theta_{\theta}^{-1} \left[ \begin{array}{cc}
3a^2 & 0 \\
0 & \text{id} \end{array} \right] \mod \mathcal{I}_\theta \]
where \( \text{id} \) is the identity \((d - 1) \times (d - 1)\)-matrix and the last multiplier is diagonal.

If \( C = 0 \) then \( b = z \) see (1) and
\[ \mathcal{T}_\theta X = \langle 1, a, b_1, \ldots, b_{d-1} \rangle_{\mathbb{R}} \cap \mathcal{I}_\theta = \langle z_1, \ldots, z_{d-1} \rangle_{\mathbb{R}}. \]

But \( \Psi = \Psi'_{\langle z_1, \ldots, z_{d-1} \rangle} \) and, according to (10),
\[ \Xi = c_1 \Psi_2 + c_2 \Psi_1 + c_3 \Psi_0 = -\Psi_2 \]
because \( c = (-1, 0, 0) \).

If \( \mathcal{X} = A_{1}A_{3}, d = 3 \), \( C \neq 0 \), and \( A_1 \) belongs to the kernel plane \( \Lambda \). Then
\[ \mathcal{T}_\theta X = \langle 1, a, b_1, b_2 \rangle_{\mathbb{R}} \cap \mathcal{I}_\theta = \langle C_2 z_1 - C_1 z_2 \rangle_{\mathbb{R}} = \Lambda_{\perp} \]
and, according to (13),
\[ \Xi = c_1 \Phi_{0}(n) + c_2 \Psi_2(n) + c_3 \Psi_1(n) + c_4 \Psi_0(n) \]
where \( n = (C_2, -C_1) \).
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