AFFINE PAVINGS OF HESSENBERG VARIETIES FOR SEMISIMPLE GROUPS

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Abstract. In this paper we consider certain closed subvarieties of the flag variety, known as Hessenberg varieties. We prove that Hessenberg varieties corresponding to nilpotent elements which are regular in a Levi factor are paved by affines. We provide a partial reduction from paving Hessenberg varieties for arbitrary elements to paving those corresponding to nilpotent elements. As a consequence, we generalize results of Tymoczko asserting that Hessenberg varieties for regular nilpotent and arbitrary elements of $\mathfrak{gl}_n(\mathbb{C})$ are paved by affines. For example, our results prove that any Hessenberg variety corresponding to a regular element is paved by affines. As a corollary, in all these cases the Hessenberg variety has no odd dimensional cohomology.

1. Introduction and Results

This paper investigates the topological structure of Hessenberg varieties, a family of subvarieties of the flag variety introduced in [5]. We prove that under certain conditions Hessenberg varieties over a complex, linear, reductive algebraic group $G$ have a paving by affines. This paving is given explicitly by intersecting these varieties with the Schubert cells corresponding to a particular Bruhat decomposition, which form a paving of the flag variety. This result generalizes results of J. Tymoczko in [9, 10, 11].

Let $G$ be a linear, reductive algebraic group over $\mathbb{C}$, $B$ a Borel subgroup, and let $\mathfrak{g}$, $\mathfrak{b}$ denote their respective Lie algebras. A Hessenberg space $H$ is a linear subspace of $\mathfrak{g}$ that contains $\mathfrak{b}$ and is closed under the Lie bracket with $\mathfrak{b}$. Fix an element $X \in \mathfrak{g}$ and a Hessenberg space $H$. The Hessenberg variety, $B(X, H)$, is the subvariety of the flag variety $G/B = B$ consisting of all $g \cdot \mathfrak{b}$ such that $g^{-1} \cdot X \in H$ where $g \cdot X$ denotes the adjoint action $Ad(g)(X)$.

We say that a nilpotent element $N$ of a reductive Lie algebra $\mathfrak{m}$ is a regular nilpotent element in $\mathfrak{m}$ if $N$ is in the dense adjoint orbit within the nilpotent elements of $\mathfrak{m}$. Suppose $N$ is a regular nilpotent element in a Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$. In this case, we prove that there is a torus action on $B(N, H)$ with a fixed point set consisting of a finite collection of points. This action yields a vector bundle over each fixed point, giving an affine paving of $B(N, H)$ by its intersection with the Schubert cells paving $B$. Our argument is inspired by the proof by C. De Concini, G. Lusztig and C. Procesi that Springer fibers are paved by affines [4]. The main result is as follows.

Theorem. Fix a Hessenberg space $H$ with respect to $\mathfrak{b}$. Let $N \in \mathfrak{g}$ be a nilpotent element such that $N$ is regular in some Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$. Then there is an affine paving of $B(N, H)$ given by the intersection of each Schubert cell in $B$ with $B(N, H)$.

Theorem [4.10] below gives the complete statement of this result. This generalizes Theorem 4.3 in [10] which states that the Hessenberg variety $B(N, H)$ is paved by affines when $N \in \mathfrak{g}$ is a regular nilpotent element. Moreover, we can extend this result to the Hessenberg variety $B(X, H)$ corresponding to the arbitrary element $X \in \mathfrak{g}$ when $X$ is semisimple or the nilpotent part of $X$ in its Jordan decomposition satisfies the conditions of the main theorem (see Theorem [5.4]). This implies that $B(X, H)$ is paved by affines for all regular elements $X$. We are therefore able to extend
Tymoczko’s result that the Hessenberg variety is paved by affine cells from all elements in \( \mathfrak{gl}(n, \mathbb{C}) \), given in [9], to certain elements of an arbitrary linear, reductive Lie algebra. Although our results are greatly influenced by results of Tymoczko, our proofs use a different approach.

The second section of this paper covers background information and facts used in the following sections. In the third, we prove a key lemma which states that in certain cases the intersection of the Hessenberg variety \( \mathcal{B}(X, H) \) with each Schubert cell is smooth. Section 4 consists primarily of the statement and proof of Theorem 4.10. Last, we consider the case in which \( X \in \mathfrak{g} \) is an arbitrary element with Jordan decomposition \( X = S + N \) in section 5. As a corollary of the results in this section, we compute the dimensions of the affine cells paving \( \mathcal{B}(X, H) \) when \( X \) is semisimple and when \( X \) is an arbitrary regular element of \( \mathfrak{g} \).

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2. Preliminaries

We state results and definitions from the literature which will be used in later sections. All algebraic groups in this paper are assumed to be complex and linear. Let \( G, \mathfrak{g}, \) and \( \mathcal{B} \) be as in the section above.

2.1. Notation. In each section, we fix a standard Borel subgroup and call it \( B \). Let \( T \subset B \) be a fixed maximal torus with Lie algebra \( \mathfrak{t} \) and denote by \( W \) the Weyl group associated to \( T \). Fix a representative \( w \in N_G(T) \) for each Weyl group element \( w \in W = N_G(T)/T \). Let \( \Phi^+, \Phi^- \) and \( \Delta \) denote the positive, negative, and simple roots associated to the previous data. Let \( \mathfrak{g}_\gamma \) denote the root space corresponding to \( \gamma \in \Phi \) and fix a generating root vector \( E_\gamma \in \mathfrak{g}_\gamma \). Write \( U \) for the maximal unipotent subgroup of \( B, U^- \) for its opposite subgroup, and \( u \) and \( u^- \) for their respective Lie algebras.

Given a standard parabolic subgroup \( Q \) of \( G \) with Levi decomposition \( MU_Q \), we denote the Lie algebras of \( Q, M \) and \( U_Q \) by \( \mathfrak{q}, \mathfrak{m} \) and \( \mathfrak{u}_Q \) respectively. Then \( B_M := B \cap M \) is a standard Borel subgroup of \( M \) with Lie algebra \( \mathfrak{b}_M := \mathfrak{b} \cap \mathfrak{m} \). Since \( Q \) is standard, \( M \) corresponds to a subset \( \Delta_M \) of simple roots. Denote by \( \Phi(u_Q) \) and \( \Phi_M \) the subsets of roots so that

\[
\mathfrak{u}_Q = \bigoplus_{\gamma \in \Phi(u_Q)} \mathfrak{g}_\gamma \quad \text{and} \quad \mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\gamma \in \Phi_M} \mathfrak{g}_\gamma.
\]

In particular, \( \mathfrak{m} \) has triangular decomposition \( \mathfrak{m} = \mathfrak{u}_M^- \oplus \mathfrak{t} \oplus \mathfrak{u}_M^+ \) where

\[
\mathfrak{u}_M^+ = \bigoplus_{\gamma \in \Phi_M^+} \mathfrak{g}_\gamma \quad \text{and} \quad \mathfrak{u}_M^- = \bigoplus_{\gamma \in \Phi_M^-} \mathfrak{g}_\gamma,
\]

with \( \Phi_M^+ = \Phi_M \cap \Phi^+ \). Let \( U_M \) denote the unipotent subgroup of \( G \) with Lie algebra \( \mathfrak{u}_M \). Then \( U_M \) is the maximal unipotent subgroup of \( B_M \), and \( u = \mathfrak{u}_M \oplus \mathfrak{u}_Q \).

2.2. Hessenberg Varieties. We give the precise definition of a Hessenberg variety.

Definition 2.1. A subspace \( H \subset \mathfrak{g} \) is a Hessenberg space with respect to \( \mathfrak{b} \) if \( \mathfrak{b} \subset H \) and \( H \) is a \( \mathfrak{b} \)-submodule.

Denote by \( \Phi_H \subset \Phi \) the subset of roots such that \( H = \mathfrak{t} \oplus \bigoplus_{\gamma \in \Phi_H} \mathfrak{g}_\gamma \). Then the conditions that \( H \) is a Hessenberg space are equivalent to requiring that \( \Phi^+ \subset \Phi_H \) and \( \Phi_H \) be closed with respect to addition with roots from \( \Phi^+ \). Let \( X \in \mathfrak{g} \) and \( H \) be some fixed Hessenberg space. Set

\[
G(X, H) = \{ g \in G : g^{-1} \cdot X \in H \}.
\]
where $g \cdot X$ denotes $Ad(g)(X)$. Then $G(X, H)$ is a subvariety of $G$ which is invariant under right multiplication by $B$. Let

$$B(X, H) = \{ g \cdot b \in B : g \in G(X, H) \}$$

denote its image in the flag variety $B$. This is the Hessenberg variety associated to $X$ and $H$. Note that when $H = b$, $B(X, H)$ is the variety of Borel subalgebras containing $X$, denoted $B^X$, and called the Springer variety of $X$. In the other extreme, when $H = g$, the Hessenberg variety is the full flag variety $B$.

**Definition 2.2.** We say $X \in g$ is in standard position with respect to $(b, t)$ if $X = S + N$ with $S \in t$ and $N \in u$.

**Remark 2.3.** For any $X \in g$, there exists $g \in G$ so that $g \cdot X$ in standard position with respect to $(b, t)$. Since the map $l_g : B \to B$, given by $l_g(a \cdot b) = ga \cdot b$ induces an isomorphism $l_g : B(X, H) \to B(g \cdot X, H)$, we may always assume $X$ is in standard position with respect to $(b, t)$.

### 2.3. Pavings

In what follows we show that for certain elements $X \in g$, $B(X, H)$ is paved by affines.

**Definition 2.4.** A paving of an algebraic variety $Y$ is a filtration by closed subvarieties

$$Y_0 \subset Y_1 \subset \cdots \subset Y_i \subset \cdots \subset Y_d = Y.$$

A paving is affine if $Y_i - Y_{i-1}$ is a finite, disjoint union of affine spaces.

There is a well known affine paving of the flag variety given by the Bruhat decomposition. Indeed, $B = \bigsqcup_{w \in W} X_w$ where each $X_w = BwB/B$ denotes the Schubert cell indexed by $w \in W$. Each $X_w$ has the following explicit description.

**Lemma 2.5.** Fix $w \in W$. The following are isomorphic varieties:

1. the Schubert cell $X_w = B\dot{w}B/B$;
2. the subgroup $U^w = \{ u \in U : \dot{w}^{-1}u\dot{w} \in U^- \}$; and
3. the Lie subalgebra, $u^w := Lie(U^w) = \bigoplus_{\alpha \in \Phi_w} g_{\alpha}$ where $\Phi_w = \{ \gamma \in \Phi^+ : w^{-1}(\gamma) \in \Phi^- \}$. In particular, $\dim U^w = |\Phi_w|$.

Additionally, $X_w = \bigsqcup_{w \leq u \leq w} X_u$ where $\leq$ denotes the Bruhat order on the Weyl group (see [1]). Set $B_i = \bigsqcup_{w \in W : |\Phi_w| = i} X_w$. Then the $B_i$ are closed subvarieties of $B$ which give an affine paving of $B$ since

$$B_i - B_{i-1} = \bigsqcup_{w \in W : |\Phi_w| = i} X_w \cong \bigsqcup_{w \in W : |\Phi_w| = i} u^w \cong \bigsqcup_{w \in W : |\Phi_w| = i} C^d.$$

The Hessenberg variety $B(X, H)$ is a closed subvariety of $B$, so the intersections $B_i \cap B(X, H) = \bigsqcup_{w \in W : |\Phi_w| = i} X_w \cap B(X, H)$ are closed. They form a paving of $B(X, H)$ where

$$B_i \cap B(X, H) - B_{i-1} \cap B(X, H) = \bigsqcup_{w \in W : |\Phi_w| = i} X_w \cap B(X, H).$$

To show this paving is affine, we will show that each $X_w \cap B(X, H)$ is homeomorphic to some affine space $C^d$. In summary,

**Remark 2.6.** $B(X, H)$ is paved by the intersections $B_i \cap B(X, H)$ and therefore paved by affines if $X_w \cap B(X, H) \cong C^d$ for all $w \in W$ and some $d \in \mathbb{Z}$.

Using the identification $X_w \cong U^w$, we can write the intersection explicitly as

$$X_w \cap B(X, H) = \{ u\dot{w} \cdot b : u \in U^w, u^{-1} \cdot X \in \dot{w} \cdot H \}.$$

A paving by affine cells computes the Betti numbers of an algebraic variety $Y$. 

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**Affine Pavings of Hessenberg Varieties**

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Lemma 2.7. Let $Y$ be an algebraic variety with an affine paving, $Y_0 \subset Y_1 \subset \cdots \subset Y_i \subset \cdots \subset Y_d = Y$. Then the nonzero compactly supported cohomology groups of $Y$ are given by $H_c^{2k}(Y) = \mathbb{Z}^{n_k}$, where $n_k$ denotes the number of affine components of dimension $k$.

2.4. Associated Parabolic. Let $N \in \mathfrak{g}$ be a nonzero nilpotent element. By the Jacobson-Morozov theorem ([3], Theorem 3.7.4) there exists a homomorphism of algebraic groups $\phi : SL_2(\mathbb{C}) \to G$ such that

$$d\phi \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) = N.$$ 

Define a 1-parameter subgroup $\lambda : \mathbb{C}^* \to G$ so that $\lambda(z) = \phi \left( \begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right)$ for all $z \in \mathbb{C}^*$, and consider the $\lambda$-weight spaces of $\mathfrak{g}$:

$$\mathfrak{g}_i(\lambda) = \{ X \in \mathfrak{g} : \lambda(z) \cdot X = z^i X \ \forall z \in \mathbb{C}^* \}.$$ 

When there is no ambiguity we write $\mathfrak{g}_i$ instead of $\mathfrak{g}_i(\lambda)$. Now, $N \in \mathfrak{g}_2$ and we can decompose $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$. Let $L$ and respectively $P$ denote the connected algebraic subgroups of $G$ whose Lie algebras are $l := \mathfrak{g}_0$ and $p := \bigoplus_{i \geq 0} \mathfrak{g}_i$. It is known that

1. $P$ is a parabolic subgroup depending only on $N$ (not on the choice of $\phi$).
2. $P = LU_P$ is a Levi decomposition and its unipotent radical $U_P$ has Lie algebra $u_P = \bigoplus_{i \geq 0} \mathfrak{g}_i$.

Lemma 2.8. The maps

$$ad_N : u_P \to \bigoplus_{i \geq 3} \mathfrak{g}_i \quad \text{and} \quad ad_N : l \to \mathfrak{g}_2$$

are onto.

Generally, a 1-parameter subgroup $\lambda : \mathbb{C}^* \to T$ is dominant with respect to $\Phi^+$ if $\langle \gamma, \lambda \rangle \geq 0$ for all $\gamma \in \Phi^+$. Here $\langle \cdot, \cdot \rangle$ is the natural pairing between the character and cocharacter groups of $G$ defined by $\lambda(z) \cdot E_\gamma = z^{\langle \gamma, \lambda \rangle} E_\gamma$. If $\lambda$ is the 1-parameter subgroup associated to nilpotent element $N$ as above, then $\lambda$ is dominant if and only if $P$ is a standard parabolic subgroup.

2.5. A key lemma. There is a result yielding a vector bundle structure which we will use in the following sections. It is a special case of Theorem 9.1 in [2].

Lemma 2.9. Let $\pi : E \to Y$ be a vector bundle over a smooth variety $Y$ with a fiber preserving a linear $\mathbb{C}^*$-action on $E$ with strictly positive weights. Let $E_0 \subset E$ be a $\mathbb{C}^*$-stable, smooth, closed subvariety. Then the restriction $\pi : E_0 \to \pi(E_0)$ is a vector sub-bundle of $\pi : E \to Y$.

3. Fixed Point Reduction

Let $Q$ be a standard parabolic subgroup of $G$ with Levi decomposition $Q = MU_Q$. The Levi subgroup $M$ is a connected, reductive algebraic group. Thus its connected centralizer $Z := Z_G(M)^0 \subset T$ is a torus. Consider the action of $Z$ on the flag variety, $B$. We can explicitly calculate the fixed point set $B^Z$ using the following.

Proposition 3.1. ([3], Proposition 8.8.7) Each connected component of $B^Z$ is isomorphic to the flag variety of $M$, $B(M)$. In particular, the connected component containing $b_0 \in B^Z$ is $M \cdot b_0 \cong M/(M \cap B_0)$ where $B_0$ is the Borel subgroup of $G$ such that $\text{Lie}(B_0) = b_0$ and $M \cap B_0$ is a Borel subgroup of $M$. 
Remark 3.5. The fiber of the vector bundle $M$ corresponding to $\mu$ induced by where the base space $m$ subgroup $so$ without loss of generality we may assume $m = g_0(\mu)$ and $u_Q = \bigoplus_{i > 0} g_i(\mu)$.

Recall that given a standard Levi subgroup $M$ of $G$, we write $W_M$ to denote the subgroup of the Weyl group associated to $M$. Let

$$W^M = \{ v \in W : \Phi_v \subseteq \Phi(u_Q) \},$$

where $\Phi_v = \{ \gamma \in \Phi^+: w^{-1}(\gamma) \in \Phi^- \}$. The elements of $W^M$ form a set of minimal representatives for $W_M \backslash W$ in the following sense.

Lemma 3.2. ([8], Proposition 5.13) Each $w \in W$ can be written uniquely as $w = yv$ with $y \in W_M$ and $v \in W^M$ such that $l(w) = l(y) + l(v)$.

Corollary 3.3. ([8], equation (5.13.2)) Let $w = yv$ be the decomposition of $w \in W$ given above. Then $\Phi_w = y(\Phi_v) \bigcup \Phi_y$.

Consider the Schubert cell $X_w \cong U^w$. Suppose $w \in W$ has decomposition $w = yv$ with $y \in W_M$ and $v \in W^M$. Then by Corollary 3.3

$$U^w \cong u^w \cong \bigoplus_{\gamma \in y(\Phi_v)} g_\gamma \bigoplus \bigoplus_{\gamma \in \Phi_y} g_\gamma = \hat{y} \cdot u^v \oplus u^y.$$  

Now, $\mu$ yields a $\mathbb{C}^*$-action on $u_Q$ with strictly positive weights. Therefore, $(U^w)^\mu \cong (\hat{y} \cdot u^v)^\mu \oplus (u^y)^\mu = u^y \cong U^y$, since $u^y \subseteq m = g_0(\mu)$ and $\hat{y} \cdot u^v \subseteq u_Q$.

Remark 3.4. The isomorphism of each connected component of $B^Z$ with $B(M)$ given in Proposition 3.1 can be described explicitly on each Schubert cell $X_w$ by

$$X_w^Z = X_w^\mu \rightarrow X_y; \ u\hat{y} \cdot b \mapsto u\hat{y} \cdot b_M$$

for all $u \in U^y$.

Since $X_w \cong \hat{y} \cdot u^v \oplus u^y \cong \hat{y} \cdot u^v \times X_w^\mu$ we get a trivial vector bundle structure

$$\hat{y} \cdot u^v \times X_w^\mu \longrightarrow X_w^\mu \xrightarrow{\pi_w} X_w^\mu$$  

(3.1)

where the base space $X_w^\mu$ can be naturally identified with the Schubert cell in the flag variety of $M$ corresponding to $y \in W_M$.

Remark 3.5. The fiber of the vector bundle $\pi_\mu : X_w \rightarrow X_w^\mu$ is a subset of $u_Q$, so the $\mathbb{C}^*$-action induced by $\mu$ acts with strictly positive weights on the fiber.

Remark 3.6. If $Q$ is a Borel subgroup, then $Z$ is a maximal torus and the corresponding 1-parameter subgroup $\mu : \mathbb{C}^* \rightarrow T$ is regular with respect to $\Phi$, i.e. $\langle \alpha, \mu \rangle \neq 0$ for all $\alpha \in \Phi$. In this case, $X_w^\mu = \{ \hat{w} \cdot b \}$ and the fiber of $\pi_\mu$ is $u^w$.

We will show that for certain elements $X \in g$, the intersection $X_w \cap B(X, H)$ is affine for all $w \in W$. Our general method of proof will be to apply Lemma 2.9 to the vector bundle in equation (3.1). To apply the Lemma, however, we need to show that the intersection $X_w \cap B(X, H)$ is smooth. We can do this provided we have some understanding of the Adjoint $U$-orbit of $X$ in $g$, $U \cdot X$. 
Proposition 3.7. (see [4], Proposition 3.2) Let $X \in \mathfrak{g}$ have Jordan decomposition $X = S + N$, and assume $X$ is in standard position with respect to $(\mathfrak{b}, \mathfrak{t})$. Suppose $U \cdot X = X + \mathcal{V}$ where $\mathcal{V} = \bigoplus_{\gamma \in \Phi(\mathcal{V})} \mathfrak{g}_\gamma \subset \mathfrak{u}$ and $X \notin \mathcal{V}$. Fix a Hessenberg space $H$ of $\mathfrak{g}$ with respect to $\mathfrak{b}$. Then for all $w \in W$, $X_w \cap B(X, H) \neq \emptyset$ if and only if $w \in \hat{w} \cdot H$. If $X_w \cap B(X, H) \neq \emptyset$ then it is smooth and $\dim (X_w \cap B(X, H)) = |\Phi_w| - \dim \mathcal{V} / \mathcal{V} \cap \hat{w} \cdot H$.

Proof. First, we identify the nonempty intersections. Note that $S \in \hat{w} \cdot H$ for all $w \in W$ since $S \in \mathfrak{s} \subset \hat{w} \cdot H$. Thus if $N \in \hat{w} \cdot H$, then $X = N + S \in \hat{w} \cdot H$ and $X_w \cap B(X, H)$ is nonempty. Alternatively, say $u \hat{w} \cdot b \in X_w \cap B(X, H)$ for some $u \in U$ where $u^{-1} \cdot X = X + Y$ with $Y \in \mathcal{V}$. Write $N = \sum_{\gamma \in \Phi^{+}} c_{\gamma} E_{\gamma}$ and $Y = \sum_{\gamma \in \Phi^{+}} d_{\gamma} E_{\gamma}$ for some $c_{\gamma}, d_{\gamma} \in \mathbb{C}$. Since $X \notin \mathcal{V}$ and $Y \in \mathcal{V}$, $c_{\gamma} = 0$ for all $\gamma \in \Phi(\mathcal{V})$ and $d_{\gamma} = 0$ for all $\gamma \notin \Phi(\mathcal{V})$. Therefore

$$u^{-1} \cdot X \in \hat{w} \cdot H \Rightarrow X + Y \in \hat{w} \cdot H \Rightarrow S + N + Y \in \hat{w} \cdot H \Rightarrow N \in \hat{w} \cdot H$$

since $N$ and $Y$ do not have any components in common with respect to this root space decomposition.

Suppose $X_w \cap B(X, H) \neq \emptyset$. The stabilizer of $\hat{w} \cdot b$ in $U$ is $U_w = \hat{w} U \hat{w}^{-1} \cap U$ and $U^w \cong U / U_w$. Now

$$X_w \cap B(X, H) = \{ u \hat{w} \cdot b : u^{-1} \cdot X \in \hat{w} \cdot H \} \subset U / U_w.$$ 

Since $X_w \cap B(X, H)$ is the image of

$$U(X, \hat{w} \cdot H) = \{ u \in U : u^{-1} \cdot X \in \hat{w} \cdot H \}$$

under the quotient map $U \to U / U_w$, it is enough to show that $U(X, \hat{w} \cdot H)$ is smooth and has dimension $\dim U - \dim \mathcal{V} / (\mathcal{V} \cap \hat{w} \cdot H)$.

Consider the morphism $\phi : U \to X + \mathcal{V}$ given by $\phi(u) = u^{-1} \cdot X$. Since $U \cdot X = X + \mathcal{V}$, $\phi$ can be identified with the quotient morphism $U \to U / Z_U(X)$, where $Z_U(X)$ denotes the centralizer of $X$ in $U$, and is therefore a smooth morphism of relative dimension $\dim Z_U(X)$. Let $i : \mathcal{V} \cap \hat{w} \cdot H \to X + \mathcal{V}$ be the map given by $Y \mapsto X + Y$ for all $Y \in \mathcal{V} \cap \hat{w} \cdot H$. By [3], Proposition III.10.1(b), the morphism $\hat{i}$ induced by the base change given in the Cartesian diagram

$$\begin{array}{ccc}
U \times \mathcal{V} (\mathcal{V} \cap \hat{w} \cdot H) & \longrightarrow & U \\
\phi \downarrow & & \phi \\
\mathcal{V} \cap \hat{w} \cdot H & \longleftarrow & X + \mathcal{V}
\end{array}$$

is smooth of relative dimension $\dim Z_U(X)$. Since $\mathcal{V} \cap \hat{w} \cdot H \subset \mathfrak{g}$ is a linear subspace it is a smooth variety, and the projection of $\mathcal{V} \cap \hat{w} \cdot H$ onto a point is smooth of relative dimension $\dim (\mathcal{V} \cap \hat{w} \cdot H)$. Since the composition of smooth morphisms is smooth ([3], Proposition III.10.1(c)), $U \times \mathcal{V} (\mathcal{V} \cap \hat{w} \cdot H)$ is smooth and has dimension $\dim Z_U(X) + \dim (\mathcal{V} \cap \hat{w} \cdot H)$. But

$$U \times \mathcal{V} (\mathcal{V} \cap \hat{w} \cdot H) = \{(u, Y) \in U \times (\mathcal{V} \cap \hat{w} \cdot H) : \phi(u) = i(Y)\}$$

$$\cong \{ u \in U : u^{-1} \cdot X = X + Y \in X + (\mathcal{V} \cap \hat{w} \cdot H)\}$$

$$= \{ u \in U : u^{-1} \cdot X \in \hat{w} \cdot H \}$$

$$= U(X, \hat{w} \cdot H).$$

Thus $U(X, \hat{w} \cdot H)$ is indeed smooth and has dimension

$$\dim Z_U(X) + \dim (\mathcal{V} \cap \hat{w} \cdot H) = \dim U - \dim \mathcal{V} + \dim (\mathcal{V} \cap \hat{w} \cdot H)$$

$$= \dim U - \dim \mathcal{V} / (\mathcal{V} \cap \hat{w} \cdot H)$$

as required. \qed
4. The nilpotent case, $N \in \mathfrak{g}$

In the section above, we proved that when the $U$-orbit through $X \in \mathfrak{g}$ is an affine subspace of $\mathfrak{g}$ then $X_w \cap \mathcal{B}(X, H)$ is smooth. This will allow the application of Lemma 2.9 to the vector bundle in equation (3.1). In this section we do this for a nilpotent element $N \in \mathfrak{g}$ which is regular in some Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$. To understand the orbit $U \cdot N$ we utilize of the theory of the parabolic subgroup associated to $N$.

Suppose $N$ is nilpotent and regular in a Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ corresponding to Levi subgroup $M$ of $G$. Since $N$ is regular in $\mathfrak{m}$ it is conjugate to a sum of simple root vectors in $\mathfrak{m}$. Fix a standard Borel subalgebra $\mathfrak{b}$ with respect to which $\mathfrak{m}$ is standard and

$$N = \sum_{\alpha \in \Delta_M} E_\alpha.$$  

Let $\dot{\lambda} : \mathbb{C}^* \to T$ be the 1-parameter subgroup associated to $N$ as in section 2.4. Note that $\dot{\lambda}$ may not be dominant with respect to $\Phi^+$, but there exists $w_1 \in W$ such that $\dot{w}_1 \cdot \dot{\lambda}$ is dominant. Let $P$ be the standard parabolic subgroup whose Lie algebra is $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$ where $\mathfrak{l} = \mathfrak{g}_0(\dot{w}_1 \cdot \dot{\lambda})$ and $\mathfrak{u}_P = \oplus_{i > 0} \mathfrak{g}_i(\dot{w}_1 \cdot \dot{\lambda})$. Let $L$ be the Levi subgroup of $G$ with Lie algebra $\mathfrak{l}$.

**Lemma 4.1.** If $\dot{w}_1 \cdot \dot{\lambda}$ is dominant then $\dot{v}_1 \cdot \dot{\lambda}$ is dominant, where $w_1 = y_1v_1$ with $y_1 \in W_L$ and $v_1 \in W^L$.

**Proof.** By assumption $\langle \gamma, \dot{w}_1 \cdot \dot{\lambda} \rangle \geq 0$ for all $\gamma \in \Phi^+$. Recall that if $\gamma \in \Phi(\mathfrak{u}_P)$ then $y_1(\gamma) \in \Phi(\mathfrak{u}_P)$ and if $\gamma \in \Phi_+^L$ then $y_1(\gamma) \in \Phi_L$. Thus for all $\gamma \in \Phi^+$

$$\langle \gamma, \dot{v}_1 \cdot \dot{\lambda} \rangle = \langle y_1(\gamma), \dot{v}_1 \cdot \dot{\lambda} \rangle = \langle y_1(\gamma), \dot{w}_1 \cdot \dot{\lambda} \rangle \geq 0,$$

so $\dot{v}_1 \cdot \dot{\lambda}$ is dominant. \qed

Set $\lambda := \dot{v}_1 \cdot \dot{\lambda}$. Note that $\lambda$ defines the same parabolic subgroup $P$ as $y_1 \cdot \lambda = w_1 \cdot \dot{\lambda}$, since conjugation by an element of $W_L$ preserves $P$. Therefore we can replace $P$ by its $y_1^{-1}$-conjugate, i.e. we let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$ where $\mathfrak{l} = \mathfrak{g}_0(\lambda)$ and $\mathfrak{u}_P = \oplus_{i > 0} \mathfrak{g}_i(\lambda)$. Replace $N$ by its $v_1$-conjugate. Abusing notation we denote it by $N$, so

$$N = \sum_{\alpha \in \Phi_N} E_\alpha$$

where $\Phi_N = v_1(\Delta_M)$. Then $\lambda$ is the 1-parameter subgroup associated to $N$ and $P$ is the standard parabolic subgroup associated to $N$.

**Remark 4.2.** Since $N \in \mathfrak{g}_2(\lambda)$, $\langle \alpha, \lambda \rangle = 2$ for all $\alpha \in \Phi_N$. Therefore $\langle \gamma, \lambda \rangle \geq 2$ for all $\gamma \in v_1(\Phi_M)$; in particular, $\Phi_L \cap v_1(\Phi_M) = \emptyset$.

Define $\Phi^+(V) = \{ \gamma \in \Phi^+ : \gamma = \alpha + \beta \text{ for some } \alpha \in \Phi_N \text{ and } \beta \in \Phi_+^\perp \}$ and let $V = \bigoplus_{\gamma \in \Phi^+(V)} \mathfrak{g}_\gamma$. Then $V \subset \mathfrak{g}_2(\lambda)$ is well-defined subspace with respect to this basis. Similarly, let $\Phi^-(V) = \{ \gamma \in \Phi^+ : \gamma = \alpha + \beta \text{ for some } \alpha \in \Phi_N \text{ and } \beta \in \Phi_-^\perp \}$ and $V^- = \bigoplus_{\gamma \in \Phi^-(V)} \mathfrak{g}_\gamma$. Our reason for defining these subspaces of $\mathfrak{g}_2(\lambda)$ is to analyze the adjoint action of $N$ on $\mathfrak{I} = \mathfrak{u}_L \oplus \mathfrak{t} \oplus \mathfrak{u}_L$. Indeed, given $E_\beta \in \mathfrak{g}_\beta \subset \mathfrak{u}_L$ we have

$$ad_N E_\beta = [N, E_\beta] = \sum_{\alpha \in \Phi_N} [E_\alpha, E_\beta] \in V$$

since $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha + \beta}$ whenever $\alpha + \beta \in \Phi$. Similarly for all $E_\beta \in \mathfrak{g}_\beta \subset \mathfrak{u}_L$, $ad_N E_\beta \in V^-$. 

Remark 4.3. We have just shown $[Y, N] \in V$ for all $Y \in u_L$. By construction, $ad_Y : V \to V$ as well.

Lemma 4.4. There is a direct sum decomposition of $g_2(\lambda)$,

$$g_2(\lambda) = V^- \oplus \bigoplus_{\alpha \in \Phi_N} g_\alpha \oplus V.$$ 

Proof. By Lemma 2.8, the map $ad_N : I \to g_2(\lambda)$ is onto. Since $ad_N(u_L) \subset V$, $ad_N(u_L^\perp) \subset V^-$ and $ad_N(t) \subset \bigoplus_{\alpha \in \Phi_N} g_\alpha$, it is certainly the case that $g_2(\lambda)$ is a sum of these subspaces. We must show that their pairwise intersection is $\{0\}$. To do so, we show that the corresponding subsets of roots are pairwise disjoint.

First, suppose there exists $\gamma \in \Phi^+(V) \cap \Phi^-(V)$. Then there are roots $\alpha_1, \alpha_2 \in \Phi_N$ and $\beta_1, \beta_2 \in \Phi^+_I$ so that

$$\alpha_1 + \beta_1 = \gamma = \alpha_2 - \beta_2.$$ 

Recall that $\Phi_N = v_1(\Delta_M)$, so we rewrite this equality as

$$v_1^{-1}(\alpha_1) + v_1^{-1}(\beta_1) = v_1^{-1}(\alpha_2) - v_1^{-1}(\beta_2).$$ 

Since $v_1 \in W^-$ and $\beta_1, \beta_2 \in \Phi^+_I$, we get that $v_1^{-1}(\beta_1), v_1^{-1}(\beta_2) \in \Phi^+$. By assumption $v_1^{-1}(\alpha_1)$ and $v_1^{-1}(\alpha_2)$ are simple roots. Therefore $v_1^{-1}(\alpha_1) + v_1^{-1}(\beta_1) \in \Phi^+$ and $v_1^{-1}(\alpha_2) - v_1^{-1}(\beta_2) \in \Phi^-$. The two cannot be equal, giving a contradiction.

Similarly, suppose $\gamma \in \Phi^+(V) \cap \Phi^+_N$. Then there exists $\alpha_1 \in \Phi_N$ and $\beta_1 \in \Phi^+_I$ so that $\gamma = \alpha_1 + \beta_1$, implying $v_1^{-1}(\gamma) = v_1^{-1}(\alpha_1) + v_1^{-1}(\beta_1)$. Since $v_1 \in W^-$, $v_1^{-1}(\beta_1) \in \Phi^+$ and by assumption $v_1^{-1}(\gamma)$ and $v_1^{-1}(\alpha_1)$ are simple. But this means that simple root $v_1^{-1}(\gamma)$ can be written as the sum of positive roots $v_1^{-1}(\alpha_1)$ and $v_1^{-1}(\beta_1)$, which is a contradiction.

Finally, $\Phi^-(V) \cap \Phi_N = \emptyset$ by a similar argument. $\square$

Recall that our goal is to understand the Adjoint $U$-orbit of $N, U \cdot N$. To do so, we need a few facts about unipotent groups. Let $\tilde{U}$ be a unipotent subgroup with Lie algebra $\tilde{u}$. First, since $\tilde{U}$ is unipotent the exponential map $exp : \tilde{u} \to \tilde{U}$ is a diffeomorphism. Recall that for all $Y \in \tilde{u}$,

$$exp(Y) \cdot X = X + [Y, X] + \frac{1}{2} [Y, [Y, X]] + \cdots = X + \sum_{i=1}^{\infty} \frac{ad_Y^i(X)}{i!}.$$ 

Thus if $ad_Y : V \to V \subset g$ and $[Y, X] \in V$ for all $Y \in \tilde{u}$, we get $\tilde{U} \cdot X \subset X + V$.

Next, suppose $\tilde{U}$ acts on an irreducible affine variety $Y$. Given $y \in Y$ the $\tilde{U}$-orbit of $y$ is closed (17. Exercise 17.8). Therefore if the dimension of the orbit is equal to the dimension of $Y$, $Y = \tilde{U} \cdot y$. We can apply this to our situation as follows.

Remark 4.5. Let $\tilde{U}$ be a unipotent subgroup such that $\tilde{U} \cdot X \subset X + V \subset g$ for $X \in g$ and $\dim \tilde{U} - \dim Z_G(X) = \dim \tilde{U} \cdot X = \dim V$. Then $\tilde{U} \cdot X = X + V$.

Lemma 4.6. Recall that $U_L$ is the unipotent subgroup of $G$ with Lie algebra $u_L$. Then $U_L \cdot N = N + V$.

Proof. First, Remark 4.3 implies $U_L \cdot N \subset N + V$. By Remark 4.5, we have only to show that $\dim U_L - \dim Z_{U_L}(N) = \dim V$.

Since $ad_{N} : I \to g_2$ is surjective, for all $X \in V \subset g_2$ there exists $Y \in I$ such that $[N, Y] = X$. Using the decomposition $I = u_L^\perp \oplus t \oplus u_L$, there exists $Y_- \in u_L^\perp$, $S \in t$ and $Y_+ \in u_L$ such that $Y = Y_- + S + Y_+$. Therefore

$$[N, Y_-] + [N, S] + [N, Y_+] = [N, Y] = X \in V.$$
But \([N,Y_+] \in V^-\) and \([N,S] \in \bigoplus_{\alpha \in \Phi_N} \mathfrak{g}_\alpha\). So by Lemma 4.4 \([N,Y_+] = [N,S] = 0\). Thus for all \(X \in V\) there exists an element \(Y_+ \in \mathfrak{u}_L\) such that

\[
\text{ad}_N(Y_+) = [N,Y_+] = [N,Y] = X.
\]

Since \(\text{ad}_N : \mathfrak{u}_L \to V\) is surjective, \(\dim V = \dim \mathfrak{u}_L - \dim \mathfrak{u}_L^N = \dim \mathfrak{u}_L - \dim Z_{\mathfrak{u}_L}(N).\) \(\square\)

**Corollary 4.7.** \(U \cdot N = N + V\) where \(V = V \oplus \bigoplus_{i \geq 3} \mathfrak{g}_i(\lambda).\)

**Proof.** First, since \(u = \mathfrak{u}_L \oplus \mathfrak{u}_P\), \(\text{ad}_N : \mathfrak{u}_L \to V\), and \(\text{ad}_N : \mathfrak{u}_P \to \bigoplus_{i \geq 3} \mathfrak{g}_i(\lambda)\) we conclude that \([Y,N] \in V\) for all \(Y \in \mathfrak{u}\). Additionally, \(\text{ad}_\mathfrak{v} : \mathfrak{v} \to \mathfrak{v}\) for all \(Y \in \mathfrak{v}\), therefore \(U \cdot N \subset N + V\). Note that \(\mathfrak{u}_L^N = \mathfrak{u}_L^N \oplus \mathfrak{u}_P^N\) (since \(V \cap \bigoplus_{i \geq 3} \mathfrak{g}_i(\lambda) = \{0\}\)) so

\[
\dim U - \dim Z_U(N) = \dim \mathfrak{u} - \dim \mathfrak{u}_L^N
= \dim \mathfrak{u}_L - \dim \mathfrak{u}_L^N + \dim \mathfrak{u}_P - \dim \mathfrak{u}_P^N
= \dim V + \dim \bigoplus_{i \geq 3} \mathfrak{g}_i(\lambda)
= \dim V
\]

where the third equality follows by Lemma 2.8. Hence \(U \cdot N = N + V\) by Remark 4.5. \(\square\)

**Remark 4.8.** If \(N \in \mathfrak{g}\) is regular then \(N = \sum_{\alpha \in \Delta} E_\alpha\) with respect to the choice of Borel subalgebra above. Then \(\lambda\) is dominant and regular, i.e. \(\lambda = \lambda\). In particular, \(1 = \mathfrak{g}_0(\lambda) = t\) so \(\mathfrak{v} = \{0\} = V^-\) and \(\mathfrak{v} = \bigoplus_{i \geq 3} \mathfrak{g}_i(\lambda) = \bigoplus_{\gamma \in \Phi^+ \setminus \Delta} \mathfrak{g}_\gamma\).

For future use, we restate Corollary 4.7 as follows.

**Corollary 4.9.** Suppose \(N \in \mathfrak{g}\) is regular in some Levi subalgebra of \(\mathfrak{g}\). Then there exists a Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\) so that \(N \in \mathfrak{b}\) and \(U \cdot N = N + V\) where \(V \subset \mathfrak{u}\) is a direct sum of root spaces such that \(N \notin \mathfrak{v}\).

**Proof.** Pick a Borel subalgebra \(\mathfrak{b} \subset \mathfrak{g}\) so that the parabolic subgroup \(P = LU_P\) associated to \(N\) is standard and \(N = \sum_{\alpha \in \Phi_N} E_\alpha\) where \(\Phi_N \subset \Phi^+\) has the property that there exists \(v_1 \in W^L\) such that \(v_1^{-1}(\Phi_N) \subset \Delta\). Such a Borel subalgebra can be found by following the process given above. The statement now follows from Corollary 4.7. \(\square\)

**Theorem 4.10.** Suppose \(N \in \mathfrak{g}\) is regular in some Levi subalgebra \(\mathfrak{m}\) of \(\mathfrak{g}\). Then \(\mathcal{B}(N,H)\) is paved by affines.

**Proof.** Let \(H\) be a Hessenberg space with respect to \(\mathfrak{b}\). By Corollary 4.7, the \(U\)-orbit of \(N\) is \(N + V\) where \(V\) is a direct sum of root spaces such that \(N \notin \mathfrak{v}\). Therefore by Proposition 3.7, \(X_w \cap \mathcal{B}(N,H) \neq \emptyset\) if and only if \(N \in w \cdot H\) and when \(X_w \cap \mathcal{B}(N,H) \neq \emptyset\), the intersection is smooth. Recall that equation (3.1) exhibits a vector bundle \(\pi_\lambda : X_w \to X^\lambda_w\) with a fiber preserving the strictly positive \(\mathbb{C}^*\)-action induced by \(\lambda\). In addition, \(X_w \cap \mathcal{B}(N,H)\) is stable under this action. Indeed, if \(wu^{-1} \cdot \mathfrak{b} \in X_w \cap \mathcal{B}(N,H)\) then for all \(z \in \mathbb{C}^*\)

\[
(\lambda(z)u^{-1} \cdot N = u^{-1} \cdot (\lambda(z^{-1}) \cdot N) = z^{-2}u^{-1} \cdot N \in w \cdot H.
\]

Applying Lemma 2.9 we get a vector sub-bundle \(\pi_\lambda : X_w \cap \mathcal{B}(N,H) \to X^\lambda_w \cap \mathcal{B}(N,H)\). Write \(w = yv\) where \(y \in W_L\) and \(v \in W^L\). Then \(X^\lambda_w \cong X_y\) where \(X_y\) is the Schubert cell in \(\mathcal{B}(L)\) corresponding to \(y \in W_L\) (see Remark 4.4).

Consider the torus \(Z = Z_G(v_1 M v_1^{-1})\). Let \(\mu : \mathbb{C}^* \to T\) be a dominant 1-parameter subgroup such that \(\mathcal{B}_Z = \mathcal{B}_0\). Then \(\mu\) is regular with respect to \(\Phi_L^\mathfrak{m}\) by Remark 4.2 and \(\mu(z) \cdot N = N\) for all
Let $\mu \in \mathbb{C}^*$ since $N \in \mathfrak{v}_1 \cdot \mathfrak{m}$. Apply Remark 3.6 to get a vector bundle $\pi_{\mu} : X_y \to X^\mu_y = \{y \cdot b_L\}$ with fiber preserving a strictly positive $\mathbb{C}^*$-action induced by $\mu$. For all $w \cdot b_L \in X_y \cap \mathcal{B}(N, H)$, $$(\mu(z)u)^{-1} \cdot N = u^{-1} \cdot (\mu(z^{-1}) \cdot N) = u^{-1} \cdot N \in \hat{w} \cdot H$$ so $X_y \cap \mathcal{B}(N, H)$ is stable under this $\mathbb{C}^*$-action. Now $X_y \cap \mathcal{B}(N, H)$ is smooth since $X_y \cap \mathcal{B}(N, H) \cong X^\lambda_y \cap \mathcal{B}(N, H)$, the $\mathbb{C}^*$-fixed points of smooth variety $X_y \cap \mathcal{B}(N, H)$. Thus we can apply Lemma 2.9 to $\pi_{\mu}$ to get a trivial vector sub-bundle $\pi_{\mu} : X_y \cap \mathcal{B}(N, H) \to \{y \cdot b_L\}$. Using the identification $X_y \cong X^\lambda_w$ there is a tower of vector bundles $$X_w \cap \mathcal{B}(N, H) \xrightarrow{\pi_{\lambda}} X^\lambda_w \cap \mathcal{B}(N, H) \xrightarrow{\pi_{\mu}} \{\hat{w} \cdot b\}$$ over the fixed point $\hat{w} \cdot b$. The composition must be trivial, so $X_w \cap \mathcal{B}(N, H) \cong \mathbb{C}^d$ for some $d \in \mathbb{Z}$. The result now follows from Remark 2.6.

Remark 4.11. If $G = SL_n(\mathbb{C})$ then Theorem 4.10 proves that $\mathcal{B}(N, H)$ is paved by affines for each Hessenberg space $H$ and nilpotent element $N \in \mathfrak{sl}_n(\mathbb{C})$. Indeed, given $N$ let $d_1 \geq \cdots \geq d_k$ be the size of its Jordan blocks. Then when $N$ is in Jordan form, it is regular in the standard Levi subalgebra $\mathfrak{sl}_d(\mathbb{C}) \times \cdots \times \mathfrak{sl}_d(\mathbb{C})$ of $\mathfrak{g}$.

Remark 4.12. There are nilpotent elements in simple Lie algebras $\mathfrak{g}$, not of type $A$, which are not regular in a Levi subalgebra, such as any distinguished element of $\mathfrak{g}$ which is not regular. When $\mathfrak{g}$ is the complex symplectic algebra of dimension $2n$ then a nilpotent element is distinguished if the sizes of its Jordan blocks consist of distinct even parts and regular if its Jordan form consists of a single block of dimension $2n$. In general, if a nilpotent element is distinguished but not regular in $\mathfrak{g}$, or in a Levi subalgebra of $\mathfrak{g}$, it will not satisfy the assumptions of Theorem 4.10. Therefore while this theorem generalizes the results of Tymoczko in [10] to a larger collection of nilpotent elements, there are still interesting cases to be considered.

To illustrate our method, we compute the dimension of the affine cells paving the Hessenberg variety associated to a regular nilpotent element.

Corollary 4.13. Let $N$ be a regular nilpotent element of $\mathfrak{g}$. Fix a Hessenberg space $H$ with respect to $\mathfrak{b}$. Then for all $w \in W$, $X_w \cap \mathcal{B}(N, H)$ is nonempty if and only if $\Delta \subset w(\Phi_H)$. When $X_w \cap \mathcal{B}(N, H)$ is nonempty, $$\dim (X_w \cap \mathcal{B}(N, H)) = |\Phi_w \cap w(\Phi_H)|,$$ where $\Phi_H = \Phi^- \cap \Phi_H$.

Proof. First $N \in \hat{w} \cdot H$ if and only if $X_w \cap \mathcal{B}(N, H)$ is nonempty. But $N$ is the sum of all simple root vectors with respect to the fixed Borel subalgebra $\mathfrak{b}$, so $N \in \hat{w} \cdot H$ if and only if $\Delta \subset w(\Phi_H)$. To calculate the dimension of the nonempty set $X_w \cap \mathcal{B}(N, H)$, recall that in this
The arbitrary case, $X \in \mathfrak{g}$

We extend the affine paving result of the previous section to many Hessenberg varieties $\mathcal{B}(X, H)$ where $X \in \mathfrak{g}$ is not necessarily nilpotent. Let $X = S + N$ be the Jordan decomposition of $X$ and $M = Z_G(S)$. Then $M$ is a Levi subgroup of $G$ whose Lie algebra $\mathfrak{m}$ contains $X$.

Suppose $N$ is regular in some Levi subalgebra of $\mathfrak{m}$. By Corollary 4.9 there exists a standard Borel subalgebra $\mathfrak{b}_M$ of $\mathfrak{m}$ so that

$$U_M \cdot N = N + \mathcal{V}_N$$

where $N \in \mathfrak{u}_M$ and $\mathcal{V}_N \subset \mathfrak{u}_M$ is a direct sum of root spaces so that $N \notin \mathcal{V}_N$. Since $S$ is in the center of $\mathfrak{m}$, $S \in \mathfrak{t}$ where $\mathfrak{t}$ is a fixed standard Cartan subalgebra of $\mathfrak{m}$.

Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ so that $\mathfrak{m}$ is standard and $\mathfrak{b} \cap \mathfrak{m} = \mathfrak{b}_M$ is the standard Borel subalgebra of $\mathfrak{m}$ above. Then $X = S + N$ is in standard position with respect to $(\mathfrak{b}, \mathfrak{t})$. Let $Q = MU_Q$ denote the standard parabolic associated to $M$ in this basis.

**Lemma 5.1.** $U \cdot X = X + \mathcal{V}$ where $\mathcal{V} = \mathcal{V}_N \oplus u_Q$.

**Proof.** The $U$-orbit of $S$, the semisimple part of $X$, is $u_Q$. Indeed, $\mathfrak{m} = \mathfrak{g}^S$ and $u_Q = \bigoplus_{\gamma \in \Phi^+, \gamma(S) \neq 0} \mathfrak{g}_\gamma$. For all $Y \in \mathfrak{u}$, $\text{ad}_Y : u_Q \to u_Q$ and $[Y, S] \in u_Q$, so $U \cdot S \subset S + u_Q$. In addition,

$$\dim U - \dim Z_U(S) = \dim u - \dim u^S = \dim u - \dim (\mathfrak{m} \cap u) = \dim u - \dim u_M = \dim u_Q$$

implying $U \cdot S = S + u_Q$ by Remark 4.5.

Certainly $U \cdot X \subset X + \mathcal{V}$. Consider $u = u_M \oplus u_Q$. Since $X \in \mathfrak{m}$, $u^X = u_M^X \oplus u_Q^X$. By properties of the Jordan form, $\mathfrak{g}^X = \mathfrak{g}^S \cap \mathfrak{g}^N$. Thus $u_Q^X = u_Q^S \cap u_Q^N = \{0\}$ since $u_Q^S = \{0\}$. Similarly, $u_M^X = u_M^S \cap u_M^N = u_M \cap u_M^N = u_M^N$. Now,

$$\dim U - \dim Z_U(X) = \dim u - \dim u^X = \dim u_M - \dim u_M^X + \dim u_Q - \dim u_Q^X = \dim u_M - \dim u_M^X + \dim u_Q = \dim \mathcal{V}_N + \dim u_Q$$

so $U \cdot X = X + \mathcal{V}$. \hfill \square

**Proposition 5.2.** Let $H$ be a fixed Hessenberg space in $\mathfrak{g}$ with respect to $\mathfrak{b}$. Then for each $v \in W^M$, $H_v := v \cdot H \cap \mathfrak{m}$ is a Hessenberg space in $\mathfrak{m}$ with respect to $\mathfrak{b}_M$.\hfill \square
Proof. We have only to show that $\Phi^+ \subseteq \Phi_{H_+}$ and that $\Phi_{H_+}$ is closed under addition with roots from $\Phi^+$. Note that $\Phi_{H_+} = v(\Phi_H) \cap \Phi_M$. Let $\alpha \in \Phi^+_M$ and write $v^{-1}(\alpha) = \gamma$, so $\alpha = v(\gamma)$ for some $\gamma \in \Phi^+$ since $\Phi_v \cap \Phi^+_M = \emptyset$.

First, $\gamma \in \Phi^+ \subseteq \Phi_H$ and therefore $\alpha = v(\gamma) \in v(\Phi_H)$ and $\alpha \in \Phi_M$ implying that $\alpha \in \Phi_{H_+}$. Thus $\Phi^+_M \subseteq \Phi_{H_+}$. Next, let $v(\beta) \in \Phi_{H_+}$ such that $\alpha + v(\beta)$ is a root of $\Phi_M$. Then

$$\alpha + v(\beta) - v(\gamma) - v(\beta) - v(\gamma) \in v(\Phi_H)$$

since $\gamma \in \Phi^+ \in \Phi_H$ and $\Phi_H$ is closed under addition of roots from $\Phi^+$. So $\alpha + v(\beta) \in v(\Phi_H) \cap \Phi_M = \Phi_{H_+}$, i.e. $\Phi_{H_+}$ is closed with respect to addition with roots from $\Phi^+_M$.

Let $Z = Z_G(M)^0$ as in Section 3.4 and let $\mu : \mathbb{C}^* \to T$ be a dominant 1-parameter subgroup such that $B^Z = B^\mu$. The $Z$-action on $B$ restricts nicely to the Hessenberg variety in the following sense.

Proposition 5.3. Fix $X \in \mathfrak{g}$ with Jordan decomposition $X = S + N$. Let $H$ be a Hessenberg space of $\mathfrak{g}$ with respect to $\mathfrak{b}$ and let $w \in W$ have decomposition $w = yu$ where $y \in W_M$ and $u \in W_M$. Then the isomorphism $X^Z_w = X_w^\mathfrak{g} \cong X_y$ given in Remark 3.4 restricts to an isomorphism

$$X^Z_w \cap B(X, H) \rightarrow X_y \cap B(N, H_v)$$

$$uw \cdot b \mapsto uy \cdot b_M$$

where $B(N, H_v)$ is the Hessenberg variety in $B(M)$ associated to nilpotent element $N \in \mathfrak{m}$ and Hessenberg space $H_v$.

Proof. We must show that $uw \cdot b \in B(X, H)$ if and only if $uy \cdot b_M \in B(N, H_v)$ for all $u \in U^y$. We have

$$u^{-1} \cdot X \in \dot{w} \cdot H \iff \dot{y}^{-1}u^{-1} \cdot S + \dot{y}^{-1}u^{-1} \cdot N \in \dot{w} \cdot H$$

$$\iff S + \dot{y}^{-1}u^{-1} \cdot N \in \dot{w} \cdot H$$

$$\iff u^{-1} \cdot N \in \dot{y} \cdot H_v$$

since $S \in \dot{w} \cdot H$ for all $v \in W_M$ and $\dot{y}, u \in M, N \in \mathfrak{m}$ implies $\dot{y}^{-1}u^{-1} \cdot N \in \mathfrak{m}$.

Theorem 5.4. Suppose $X \in \mathfrak{g}$ has Jordan decomposition $X = S + N$ and $N$ is regular in some Levi subalgebra of $\mathfrak{m}$, where $\mathfrak{m}$ is the Lie algebra of Levi subgroup $M = Z_G(S)$. Then $B(X, H)$ is paved by affines.

Proof. Fix a Hessenberg space $H$ with respect to $\mathfrak{b}$. By Lemma 5.1 there exists a direct sum of root spaces $\mathcal{V} \subset u$ such that $X \notin \mathcal{V}$ and $U \cdot X = X + \mathcal{V}$. Therefore by Proposition 3.7 $X_w \cap B(X, H) \neq \emptyset$ if and only if $N \in \dot{w} \cdot H$ and when $X_w \cap B(X, H) \neq \emptyset$, the intersection is smooth. Recall that equation (5.3) exhibits a vector bundle $\pi_u : X_w \rightarrow X_u$ with fiber preserving a strictly positive $\mathbb{C}^*$-action induced by $\mu$. Since $\mathfrak{m} = g_0(\mu)$ and $X \in \mathfrak{m}$ this $\mathbb{C}^*$-action fixes $X$ and therefore the intersection $X_w \cap B(X, H)$ is $\mathbb{C}^*$-stable. Apply Lemma 2.9 to get a vector sub-bundle $\pi_u : X_w \cap B(X, H) \rightarrow X_u \cap B(X, H)$.

By Proposition 5.3 $X_w^\mathfrak{g} \cap B(X, H) \cong X_y \cap B(N, H_v)$ where $B(N, H_v)$ is the Hessenberg variety associated to the nilpotent element $N \in \mathfrak{m}$ and Hessenberg space $H_v$ with respect to $\mathfrak{b}_M$. By assumption, $N$ is regular in some Levi subalgebra of $\mathfrak{m}$. Therefore by the proof of Theorem 4.10 $X_y \cap B(N, H_v)$ is the total space of a trivial vector bundle over $\{\dot{y} \cdot \mathfrak{b}_M\}$. Using the identification
$X_w^\mu \cong X_y$ given in Remark 5.4, there is a tower of vector bundles

$$X_w \cap \mathcal{B}(X, H) \xrightarrow{\pi_w} X_w^\mu \cap \mathcal{B}(X, H) \xrightarrow{} \{w \cdot b\}$$

over the fixed point $w \cdot b$. The composition must be trivial, so $X_w \cap \mathcal{B}(X, H) \cong \mathbb{C}^d$ for some $d \in \mathbb{Z}$. Now the result follows from Remark 2.6.

**Corollary 5.5.** Suppose $X \in \mathfrak{g}$ has Jordan decomposition $X = S + N$ and satisfies the conditions of Theorem 5.4. Fix a Hessenberg space $H$ with respect to $b$. Then if $X_w \cap \mathcal{B}(N, H)$ is nonempty, it has dimension

$$\dim (X_y \cap \mathcal{B}(N, H_v)) + |y(\Phi_v) \cap w(\Phi_H^-)|$$

where $w = yv$ for $y \in W_M$ and $v \in W^M$.

**Proof.** To compute the dimension of the nonempty set $X_w \cap \mathcal{B}(N, H)$ recall that $\mathcal{V} = \mathcal{V}_N \oplus u_Q$. Now $\mathcal{V} \cap w \cdot H = (\mathcal{V}_N \cap \hat{y} \cdot H_v) \oplus (u_Q \cap w \cdot H)$ so by Proposition 3.7 and Corollary 3.8

$$\dim X_w \cap \mathcal{B}(X, H) = |\Phi_w| - \dim \mathcal{V}/(\mathcal{V} \cap w \cdot H)$$

$$= |\Phi_y| - \dim \mathcal{V}_N/(\mathcal{V}_N \cap \hat{y} \cdot H_v) + |y(\Phi_v)| - \dim u_Q/(u_Q \cap w \cdot H).$$

A second application of Proposition 3.7 yields the equality

$$|\Phi_y| - \dim \mathcal{V}_N/(\mathcal{V}_N \cap \hat{y} \cdot H_v) = \dim (X_y \cap \mathcal{B}(N, H_v)).$$

Finally, $|y(\Phi_v)| - \dim u_Q/(u_Q \cap w \cdot H) = |y(\Phi_v) \cap w(\Phi_H^-)|$ by a calculation similar to that in the proof of Corollary 4.13.

**Remark 5.6.** Theorem 5.4 applies to Hessenberg varieties $\mathcal{B}(X, H)$ when $X$ is a semisimple element and when $X$ is a regular element. Indeed, if $X$ is semisimple, then $N = 0$ is a regular element of $t \subset m$. If $X$ is regular, then $N$ is a regular element of $m$. In both cases $X \in \mathfrak{g}$ satisfies the assumptions of the Theorem.

**Remark 5.7.** Theorem 5.4 gives an affine paving of the Springer variety $\mathcal{B}^X = \mathcal{B}(X, b)$ when $X \in \mathfrak{g}$ satisfies the assumptions of the Theorem.

**Corollary 5.8.** Suppose $X \in \mathfrak{g}$ has Jordan decomposition $X = S + N$ and satisfies the assumptions of Theorem 5.4. Fix a Hessenberg space $H$ with respect to $b$. Then for all $w = yv$ where $y \in W_M$ and $v \in W^M$ we have the following.

1. If $N = 0$, i.e. if $X$ is a semisimple element, then $X_w \cap \mathcal{B}(X, H)$ is nonempty for all $w \in W$ and

   $$\dim (X_w \cap \mathcal{B}(X, H)) = |\Phi_y| + |y(\Phi_v) \cap w(\Phi_H^-)|.$$

2. If $N$ is regular in $m$, i.e. if $X$ is a regular element, then $X_w \cap \mathcal{B}(X, H)$ is nonempty if and only if $\Delta_M \subset y(\Phi_H^-)$. When $X_w \cap \mathcal{B}(N, H) \neq \emptyset$,

   $$\dim (X_w \cap \mathcal{B}(X, H)) = |\Phi_y \cap y(\Phi_H^-)| + |y(\Phi_v) \cap w(\Phi_H^-)|.$$
Proof. First, by Proposition 3.7, if $N \in w \cdot H$ then $X_w \cap B(X,H)$ is nonempty. When $X$ is semisimple, $N = 0 \in w \cdot H$ for all $w \in W$ and $B(N,H_w) = B(M)$, so (1) is a direct consequence of Corollary 5.5. For part (2), $N \in w \cdot H$ if and only if $N \in y \cdot H_v$ using the identification given in Proposition 5.3. Therefore $X_w \cap B(X,H)$ is nonempty if and only if $N \in y \cdot H_v$. The statement now follows from Corollary 4.13 and Corollary 5.5. □

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