Abstract

The two-step graphs are revisited by studying their chromatic numbers in this paper. We observe that the problem of coloring of two-step graphs is equivalent to the problem of vertex partitioning of graphs into open packing sets. With this remark in mind, it can be considered as the open version of the well-known 2-distance coloring problem as well as the dual version of total domatic problem.

The minimum $k$ for which the two-step graph $N(G)$ of a graph $G$ admits a proper coloring assigning $k$ colors to the vertices is called the open packing partition number $p_o(G)$ of $G$, that is, $p_o(G) = \chi(N(G))$. We give some sharp lower and upper bounds on this parameter as well as its exact value when dealing with some families of graphs like trees. Relations between $p_o$ and some well-know graph parameters have been investigated in this paper. We study this vertex partitioning in the Cartesian, direct and lexicographic products of graphs. In particular, we give an exact formula in the case of lexicographic product of any two graphs. The NP-hardness of the problem of computing this parameter is derived from the mentioned formula. Graphs $G$ for which $p_o(G)$ equals the clique number of $N(G)$ are also investigated.

Keywords: Open packing partition, two-step graphs, 2-distance coloring, lexicographic product, direct product, Cartesian product, Nordhaus-Gaddum inequality, NP-complete.

1 Introduction and preliminaries

Throughout this paper, we consider $G$ as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [19] as a reference for terminology and notation which are not explicitly defined here. The open neighborhood of a vertex $v$ is denoted by $N_G(v)$, and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The minimum and maximum degrees of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Given the subsets $A, B \subseteq V(G)$, by $[A, B]$ we mean the set of all edges with one end point in $A$ and the other in $B$. Finally, for a given set $S \subseteq V(G)$, by $G[S]$ we represent the subgraph of $G$ induced by $S$. 


For all four standard products of graphs $G$ and $H$, the vertex set of the product is $V(G) \times V(H)$. Their edge sets are defined as follows.

- In the Cartesian product $G \Box H$ two vertices are adjacent if they are adjacent in one coordinate and equal in the other.
- In the direct product $G \times H$ two vertices are adjacent if they are adjacent in both coordinates.
- The edge set of the strong product $G \boxtimes H$ is the union of $E(G \Box H)$ and $E(G \times H)$.
- Two vertices $(g, h)$ and $(g', h')$ are adjacent in the lexicographic product $G \circ H$ if either $gg' \in E(G)$ or $"g = g'"$ and $hh' \in E(H)$.

Note that all four products are associative and only the first three ones are commutative, while the lexicographic product is not (see [8]).

A subset $B \subseteq V(G)$ is a packing (or packing set) in $G$ if for every pair of distinct vertices $u, v \in B$, $N_G[u] \cap N_G[v] = \emptyset$ (equivalently, $B$ is a packing in $G$ if $|N_G[v] \cap B| \leq 1$ for all $v \in B$). The packing number $\rho(G)$ is the maximum cardinality of a packing in $G$.

A subset $S \subseteq V(G)$ is a dominating set (total dominating set) if each vertex in $V(G) \setminus S$ has at least one neighbor in $S$. The domination number (total domination number) $\gamma(G)$ ($\gamma_t(G)$) is the minimum cardinality of a dominating set (total dominating set) in $G$. For more information on domination theory, the reader can consult [9] and [10].

The study of distance coloring was initiated by Kramer and Kramer ([12] and [13]) in 1969. A 2-distance coloring (or, 2DC for short) of a graph $G$ is a mapping of $V(G)$ to a set of colors (nonnegative integers for convenience) in such a way that any two vertices of distance at most 2 have different colors. The minimum number of colors (nonnegative integers) $k$ for which there is a 2DC of $G$ is called the 2-distance chromatic number $\chi_2(G)$ of $G$.

For a given graph $G$, the closed neighborhood graph of $G$, denoted by $\mathcal{N}_c(G)$, has the vertex set $V(G)$; two distinct vertices $u$ and $v$ are adjacent in $\mathcal{N}_c(G)$ if and only if $N_G[u] \cap N_G[v] \neq \emptyset$ (see [3]). In fact, a 2-distance coloring of a graph $G$ is the same as coloring of its closed neighborhood graph. Furthermore, it is easily seen that $\mathcal{N}_c(G)$ is isomorphic to the square $G^2$ of $G$. Therefore, we observe that

\[
\chi_2(G) = \chi(G^2) = \chi(\mathcal{N}_c(G)),
\]

in which $\chi$ is the classic chromatic number. Note that the problem of 2DC is equivalent to the problem of vertex partitioning of a graph into packing sets. In fact, $\chi_2(G)$ equals the minimum cardinality of such vertex partition of $G$.

A subset $B \subseteq V(G)$ is said to be an open packing (or open packing set) in a graph $G$ if for any distinct vertices $u, v \in B$, $N_G[u] \cap N_G(v) = \emptyset$. The open packing number, denoted by $\rho_o(G)$, is the maximum cardinality of an open packing sets in $G$ (see [11]). It is well-known that $\rho(G) \leq \gamma(G)$ for any graph $G$, and $\rho_o(G) \leq \gamma_t(G)$ for any graph $G$ with no isolated vertices (see [10] and [17], respectively).

When dealing with “domination” (“total domination”) instead of “packing”, the problem of finding the maximum cardinality of a vertex partition of a graph $G$ into dominating sets (total dominating sets) has been widely investigated in literature. The study of this parameter, the domatic number $d(G)$ (total domatic number $d_t(G)$) of the graph $G$, was initiated by Cockayne and Hedetniemi in [6] (Cockayne et al. in [5]).
A partition $P = \{P_1, \cdots, P_p\}$ of the vertex set of a graph $G$ is called an open packing partition (OPP for short) if $P_i$ is an open packing for each $1 \leq i \leq |P|$. The open packing partition number $p_o(G)$ is the minimum cardinality of an OPP of $G$. In this paper, we study and investigate this kind of vertex partitioning of graphs. An equivalent definition for this parameter can be stated by using functions. A function $f : V(G) \to \{1, \ldots, k\}$ is an open packing partitioning function, or OPP-function for short, if no vertex $v$ is adjacent to two vertices $u$ and $w$ with $f(u) = f(w)$. The minimum $k$ for which a graph $G$ admits an OPP-function $f : V(G) \to \{1, \ldots, k\}$ represents the open packing partition number of $G$, and is denoted $p_o(G)$. Note that $\{U_1^f, \ldots, U_k^f\}$, in which $U_i^f = \{v \in V(D) \mid f(v) = i\}$ for each $1 \leq i \leq k$, is an OPP of $G$.

There is another useful definition for an OPP of graphs. For a given graph $G$, the two-step graph $\mathcal{N}(G)$ of $G$ is the graph having the same vertex set as $G$ with an edge joining two vertices in $\mathcal{N}(G)$ if and only if they have a common neighbor in $G$. These graphs were introduced in [1] and investigated later in [2], [3] and [4]. Taking into account the fact that a vertex subset in $\mathcal{N}(G)$ is independent if and only if it is an open packing in $G$, we observe that

$$p_o(G) = \chi(\mathcal{N}(G)),$$

which is an open analogue of [1].

By a $p_o(G)$-partition, $\chi_2(G)$-coloring and $p_o(G)$-set we mean an/a OPP, 2DC and open packing set of $G$ of cardinality $p_o(G)$, $\chi_2(G)$ and $p_o(G)$, respectively. We also mean an OPP-function assigning $p_o(G)$ labels to $V(G)$ by a $p_o(G)$-function.

## 2 Lexicographic product

The aim of this section is to obtain an exact formula for $p_o(G \circ H)$. This in particular results in the NP-completeness of the decision problem associated with $p_o$.

**Theorem 1.** Let $G$ be a connected graph of order at least two and let $H$ be any graph. Then,

$$p_o(G \circ H) = \chi_2(G)|V(H)| - i_H(\chi_2(G) - p_o(G)).$$

Here $i_H$ is the number of isolated vertices of $H$.

**Proof.** Let $A = \{A_1, \cdots, A_{p_o(G)}\}$, $B = \{B_1, \cdots, B_{\chi_2(G)}\}$ and $I_H$ be a $p_o(G)$-partition, a $\chi_2(G)$-coloring and the set of isolated vertices of $H$, respectively. We set

$$P = \{A_i \times \{h\} \mid 1 \leq i \leq p_o(G), h \in I_H\} \cup \{B_i \times \{h\} \mid 1 \leq i \leq \chi_2(G), h \in V(H) \setminus I_H\}.$$

Clearly, $P$ is a vertex partition of $G \circ H$.

Suppose that there exists a vertex $(g, h')$ adjacent to two vertices $(g', h), (g'', h) \in A_i \times \{h\}$, for some $1 \leq i \leq p_o(G)$ and $h \in I_H$. Note that $g$ is not simultaneously adjacent to both $g'$ and $g''$ due to the fact that $A_i$ is an open packing in $G$. So, we may assume that $g = g''$. This shows that $hh' \in E(H)$, a contradiction to the fact that $h$ is an isolated vertex of $H$.

If $(g, h')$ is adjacent to two vertices $(g', h), (g'', h) \in B_i \times \{h\}$ for some $1 \leq i \leq \chi_2(G)$ and $h \in V(H) \setminus I_H$, then $g', g'' \in N_G[g] \cap B_i$. This is a contradiction since $B_i$ is a packing in $G$. So, we have concluded that $P$ is an OPP of $G \circ H$. Therefore,

$$p_o(G \circ H) \leq |P| = p_o(G)i_H + \chi_2(G)(|V(H)| - i_H) = \chi_2(G)|V(H)| - i_H(\chi_2(G) - p_o(G)).$$

(3)
Let $\mathbb{P}^t$ be a $p_o(G \circ H)$-partition. Suppose that $B \cap \{(g) \times V(H)\}$ contains two distinct vertices $(g,h)$ and $(g,h')$ for some $B \in \mathbb{P}^t$. Since $G$ is a connected graph on at least two vertices, there is a vertex $g'$ adjacent to $g$ in $G$. Therefore, $(g',h')$ is adjacent to both $(g,h),(g,h') \in B$, which is impossible. This implies that $|B \cap \{(g) \times V(H)\}| \leq 1$, for each $B \in \mathbb{P}^t$ and $g \in G$.

Suppose now that $f$ is a $p_o(G \circ H)$-function. Consider two packing sets $B_i, B_j \in \mathbb{B}$. Because of the minimality of $\mathbb{B}$, there exists a vertex $g \in N_G[g_i] \cap N_G[g_j]$ for some $g_i \in B_i$ and $g_j \in B_j$. Let $f\left(\{(g_i) \times (V(H) \setminus I_H)\}\right)$ and $f\left(\{(g_j) \times (V(H) \setminus I_H)\}\right)$ be the sets of labels assigned to the vertices in $\{(g_i) \times (V(H) \setminus I_H)\}$ and $\{(g_j) \times (V(H) \setminus I_H)\}$ by $f$, respectively. Suppose that $f\left((g_i, h_i)\right) = f\left((g_j, h_j)\right)$ for some $h_i, h_j \in V(H) \setminus I_H$. Because $h_i$ is not an isolated vertex in $H$, it follows that $h' h'' \in E(H)$ for some $h'' \in V(H) \setminus I_H$. Since $g \in N_G[g_i] \cap N_G[g_j]$, we may assume that $g g'' \in E(H)$. It is now easy to see that $(g, h''')$ is adjacent to both $(g_i, h_i)$ and $(g_j, h_j)$. This is impossible as $f$ is an OPP-function of $G \circ H$. Consequently, $f\left(\{(g_i) \times (V(H) \setminus I_H)\}\right) \cap f\left(\{(g_j) \times (V(H) \setminus I_H)\}\right) = \emptyset$ for all $g_i \in B_i$ and $g_j \in B_j$. This fact together with the discussion given in the previous paragraph show that $f$ assigns at least $|V(H) \setminus I_H| = |V(H)| - i_H$ labels to the vertices in $B_i \times (V(H) \setminus I_H)$ so that none of them appears on any vertex in $B_j \times (V(H) \setminus I_H)$, for $1 \leq i \neq j \leq \chi_2(G \circ H)$. Therefore,

$$|f(V(G) \times (V(H) \setminus I_H))| \geq \chi_2(G)(|V(H)| - i_H).$$

(4)

A similar argument shows that

$$|f(V(G) \times I_H))| \geq p_o(G)i_H.$$  

(5)

Together the inequalities (4) and (5) imply that

$$p_o(G \circ H) = |f(V(G) \times (V(H) \setminus I_H))| + |f(V(G) \times I_H))| \geq \chi_2(G)|V(H)| - i_H \left(\chi_2(G) - p_o(G)\right).$$

(6)

Now the desired equality follows from (3) and (6). \hfill \Box

McCormick [15] showed that the decision problem associated with the 2-distance chromatic number is NP-complete. Taking this fact into account, we prove that the OPP-problem is also NP-complete as an application of Theorem 1. More formally, we analyze the following decision problem.

**OPEN PACKING PARTITION PROBLEM (OPP-problem for short)**

**INSTANCE:** A connected graph $G$ of order $n \geq 2$ and an integer $1 \leq r \leq n$.

**QUESTION:** Is $p_o(G) \leq r$?

**Corollary 2.** The OPP-problem is NP-complete.

**Proof.** The problem is clearly in NP since checking that a given vertex partition is an OPP of cardinality at most $r$ can be done in polynomial time.

Assume that $G$ is a connected graph on at least two vertices. Let $G' = G \circ K_2$. We also set $r' = 2r$. Using the equation given in Theorem 1 for $H = K_2$, we get $p_o(G') = 2\chi_2(G)$. So, $p_o(G') \leq r'$ if and only if $\chi_2(G) \leq r$. Since the 2DC problem is NP-complete, it follows that the OPP-problem is. \hfill \Box

As a consequence of the result above, we conclude that the problem of computing the open packing partition number is NP-hard. So, it is worth bounding $p_o$ or giving its exact value for some special classes of graphs in terms of several invariants of the graphs.
3 Cartesian, direct and corona products

Since the Cartesian product $G \square H$ contains copies of $G$ and $H$ as subgraphs, it follows that $p_o(G \square H) \geq \max\{p_o(G), p_o(H)\}$. By the way, the difference between $p_o(G \square H)$ and this lower bound can be arbitrarily large.

**Theorem 3.** For any graphs $G$ and $H$, $p_o(G \square H) \leq \min\{p_o(G)\chi_2(H), \chi_2(G)p_o(H)\}$. This bound is sharp.

Proof. Let $\mathbb{A} = \{A_1, \ldots, A_{p_o(G)}\}$ and $\mathbb{B} = \{B_1, \ldots, B_{\chi_2(H)}\}$ be a $p_o(G)$-partition and a $\chi_2(H)$-coloring, respectively. Clearly, $\mathbb{P} = \{A_i \times B_j \mid 1 \leq i \leq p_o(G), 1 \leq j \leq \chi_2(H)\}$ is a partition of $V(G \square H)$. Suppose that $(g, h) \in V(G \square H)$ is adjacent to two distinct vertices $(g_1, h_1), (g_2, h_2) \in A_i \times B_j$, for some $1 \leq i \leq p_o(G)$ and $1 \leq j \leq \chi_2(H)$. Because $h \in N_H[h_1] \cap N_H[h_2]$ and since $B_j$ is a packing in $H$, it follows that $h_1 = h_2$. In particular, $g_1 \neq g_2$. Note that $h = h_1 = h_2$, otherwise $g = g_1 = g_2$, which is impossible. This implies that $g \in N_G(g_1) \cap N_G(g_2)$. This contradicts the fact that $A_i$ is an open packing in $G$. So, $A_i \times B_j$ is an open packing in $G \square H$ for all $1 \leq i \leq p_o(G)$ and $1 \leq j \leq \chi_2(H)$. Therefore, $p_o(G \square H) \leq p_o(G)\chi_2(H)$. Interchanging the roles of $G$ and $H$ yields to $p_o(G \square H) \leq p_o(H)\chi_2(G)$. This results in the desired upper bound.

In what follows, we prove the sharpness of the bound. Consider $C_{4m} \square K_n$ with the set of vertices $V_{4m,n} = \{v_{ij} \mid 1 \leq i \leq 4m, 1 \leq j \leq n\}$ in a matrix form, for any integers $m \geq 1$ and $n \geq 3$. Since $p_o(C_{4m}) = 2$ and $\chi_2(K_n) = n$ for all integers $m \geq 1$ and $n \geq 3$, we have $p_o(C_{4m} \square K_n) \leq 2n$ by the proved upper bound. It is readily observed that $B = \{v_{4k-1}v_{4k-2} \mid 1 \leq k \leq m\}$ is a $p_o(C_{4m} \square K_n)$-set. On the other hand, $p_o(G) \geq |V(G)|/p_o(G)$ for any graph $G$ (see Part (i) of Proposition 6). Therefore, $p_o(C_{4m} \square K_n) \geq 4mn/|B| = 2n$. This completes the proof.

When dealing with the direct product, it is clear that $E(G \times H) = \emptyset$ (and hence $p_o(G \times H) = 1$) if and only if $E(G) = \emptyset$ or $E(H) = \emptyset$. So, we may assume that both $E(G)$ and $E(H)$ are nonempty.

**Theorem 4.** Let $G$ and $H$ be two nonempty graphs. Then,

\[
\max\{p_o(G), p_o(H)\} \leq p_o(G \times H) \leq p_o(G)p_o(H).
\]

These bounds are sharp.

Proof. Let $\mathbb{A} = \{A_1, \ldots, A_{p_o(G)}\}$ and $\mathbb{B} = \{B_1, \ldots, B_{p_o(H)}\}$ be a $p_o(G)$-partition and a $p_o(H)$-partition, respectively. Clearly, $\mathbb{P} = \{A_i \times B_j \mid 1 \leq i \leq p_o(G), 1 \leq j \leq p_o(H)\}$ is a partition of $V(G \times H)$. If a vertex $(g, h)$ have two neighbors $(g', h'), (g'', h'') \in A_i \times B_j$ for some $1 \leq i \leq p_o(G)$ and $1 \leq j \leq p_o(H)$, then $|N_G(g) \cap A_i| \geq 2$ or $|N_G(h) \cap B_j| \geq 2$. This is a contradiction due to the fact that $A_i$ and $B_j$ are open packing sets in $G$ and $H$, respectively. Therefore, $\mathbb{P}$ is an OPP of $G \times H$. So, $p_o(G \times H) \leq |\mathbb{P}| = p_o(G)p_o(H).

To prove the lower bound, we let $k$ be a $p_o(G \times H)$-function and let $hh^* \in E(H)$. We define $f : V(G) \to \{1, \ldots, p_o(G \times H)\}$ by $f(g) = k((g, h^*))$. Suppose that $g \in V(G)$ is adjacent to two vertices $g_1$ and $g_2$ of $G$. This implies that $(g, h)$ is adjacent to both $(g_1, h^*)$ and $(g_2, h^*)$ in $G \times H$. We then have $k((g_1, h^*)) = k((g_2, h^*))$ since $k$ is an OPP-function of $G \times H$, and hence $f(g_1) = f(g_2)$. Therefore, $f$ is an OPP-function of $G$. Thus, $p_o(G) \leq p_o(G \times H)$. Moreover, we have $p_o(H) \leq p_o(G \times H)$ by a similar fashion. This results in the lower bound.

It is known that $G \times K_2 \cong 2G$ for each bipartite graph $G$. So, $p_o(G \times K_2) = p_o(G) = \max\{p_o(G), p_o(K_2)\} = p_o(G)p_o(K_2)$. So, both the lower and upper bounds are sharp.
We believe that the upper bound given in Theorem 4 usually gives the exact value for \( p_o \) in the case of direct product of two graphs.

Let \( u \in V(G) \) be a vertex of maximum degree and let \( \mathcal{B} = \{B_1, \ldots, B_{p_o(G)}\} \) be a \( p_o(G) \)-partition. Suppose that \( u \in B_i \). Since \( B_i \) is an open packing in \( G \), \( u \) has at most one neighbor in \( B_i \) and hence it has at least \( \deg_G(u) - 1 = \Delta(G) - 1 \) neighbors such that no two such neighbors belong to one open packing \( B_j \) for \( i \neq j \). So,
\[
    p_o(G) \geq \Delta(G). \tag{7}
\]
This simple but important inequality will turn out to be useful in some places in this paper.

Let \( G \) and \( H \) be graphs and \( V(G) = \{v_1, \ldots, v_n\} \). We recall that the corona product \( G \circ H \) of graphs \( G \) and \( H \) is obtained from the disjoint union of \( G \) and \( n \) disjoint copies of \( H \), say \( H_1, \ldots, H_n \), such that for all \( i \in \{1, \ldots, n\} \), the vertex \( v_i \in V(G) \) is adjacent to every vertex of \( H_i \). We next present a closed formula for \( p_o(G \circ H) \).

**Theorem 5.** For any graphs \( G \) and \( H \), \( p_o(G \circ H) = \max\{p_o(G), |V(H)| + \Delta(G)\} \).

**Proof.** Clearly, any \( p_o(G \circ H) \)-function assigns at least \( p_o(G) \) labels to the vertices of \( G \) since \( G \) is a subgraph of \( G \circ H \). So, \( p_o(G \circ H) \geq p_o(G) \). On the other hand, \( p_o(G \circ H) \geq \Delta(G \circ H) = |V(H)| + \Delta(G) \) by the inequality (7). So, it suffices to prove that \( p_o(G \circ H) \leq p_o(G) \) or \( p_o(G \circ H) \leq |V(H)| + \Delta(G) \).

For each \( 1 \leq i \leq |V(G)| \), we let \( V(H_i) = \{u_{i1}, \ldots, u_{i|V(H)|}\} \) be the vertex set of the copy of \( H \) for which \( v_i \) is adjacent to all vertices. Let \( \mathcal{A} = \{A_1, \ldots, A_{p_o(G)}\} \) be a \( p_o(G) \)-partition. Let \( f \) assign the labels \( 1, \ldots, p_o(G) \) to the vertices in \( A_1, \ldots, A_{p_o(G)} \), respectively. We consider two cases depending on \( p_o(G) \).

**Case 1.** \( |V(H)| \leq p_o(G) - \Delta(G) \). We choose a vertex \( v_i \in A_k \). If \( v_i \) has no neighbor in \( A_k \), then we let \( f \) assign
- \( f(v_i) = k \) to \( u_{i1} \), and
- \( |V(H)| - 1 \) labels \( r \in \{1, \ldots, p_o(G)\} \) such that \( N_G[v_i] \cap A_r = \emptyset \) to the vertices \( u_{i2}, \ldots, u_{i|V(H)|} \) (since \( |V(H)| \leq p_o(G) - \deg_G(v_i) \), there are such labels).

If \( v_i \) is adjacent to (exactly) one vertex in \( A_k \), then \( f \) assigns \( |V(H)| \) labels \( r \in \{1, \ldots, p_o(G)\} \) such that \( N_G[v_i] \cap A_r = \emptyset \) to the vertices \( u_{i1}, \ldots, u_{i|V(H)|} \) (this also is possible because \( |V(H)| \leq p_o(G) - \deg_G(v_i) \)).

By iterating the process above for all vertices in \( V(G) \), we can easily see that \( f \) is an OPP-function of \( G \circ H \) assigning \( p_o(G) \) to the vertices of \( G \circ H \). Therefore, \( p_o(G \circ H) \leq p_o(G) \).

**Case 2.** \( |V(H)| \geq p_o(G) - \Delta(G) + 1 \). This shows that the vertices of \( G \circ H \) can not be labeled with \( p_o(G) \) labels. Therefore, \( p_o(G \circ H) > p_o(G) \).

Let \( v_j \) be a vertex of maximum degree in \( G \). Let \( v_i \in A_k \). If \( |V(H)| \leq p_o(G) - \deg_G(v_i) \), then the vertices \( u_{i1}, \ldots, u_{i|V(H)|} \) can be labeled with the labels in \( \{1, \ldots, p_o(G)\} \) by \( f \) similar to the process described in Case 1. So, we may assume that \( |V(H)| \geq p_o(G) - \deg_G(v_i) + 1 \).

If \( v_i \in A_k \) has no neighbor in \( A_k \), we let \( f \) assign
- \( f(v_i) = k \) to \( u_{i1} \),
- \( t(v_i) = p_o(G) - \deg_G(v_i) - 1 \) labels \( r \in \{1, \ldots, p_o(G)\} \) for which \( N_G[v_i] \cap A_r = \emptyset \) to the vertices \( u_{i2}, \ldots, u_{i(t(v_i) + 1)} \), and
- \( s(v_i) = |V(H)| - t(v_i) - 1 \) new labels \( 1', \ldots, s(v_i)' \) to the other vertices of \( H_i \).
If \( v_i \in A_k \) is adjacent to (exactly) one vertex in \( A_k \), we let \( f \) assign

- \( a(v_i) = p_o(G) - \deg_G(v_i) \) labels \( r \in \{1, \cdots, p_o(G)\} \) for which \( N_G[v_i] \cap A_r = \emptyset \) to the vertices \( u_{i1}, \cdots, u_{it} \), and
- \( b(v_i) = |V(H)| - a(v_i) \) new labels \( 1', \cdots, b(v_i)' \) to the other vertices of \( H_i \).

It is easy to observe that \( f \) is an OPP-function of \( G \odot H \). Moreover, \( t(v_j) = p_o(G) - \Delta(G) - 1 \leq t(v_i) \), and hence

\[
|V(H)| - p_o(G) + \Delta(G) = s(v_j) \geq s(v_i).
\]  

We also have \( a(v_j) = p_o(G) - \Delta(G) \leq a(v_i) \), and hence

\[
|V(H)| - p_o(G) + \Delta(G) = b(v_j) \geq b(v_i).
\]

Together the inequalities (8) and (9) imply that \( f \) assign \(|V(H)| + \Delta(G)\) labels to the vertices of \( G \odot H \). Thus, \( p_o(G \odot H) \leq |V(H)| + \Delta(G) \). This completes the proof.

\[\square\]

4 Bounds and exact values for some classes of graphs

It is shown in [18] that “for any graph \( G \) on at least three vertices, \( \rho_o(G) = 1 \) if and only if \( \text{diam}(G) \leq 2 \) and every edge of \( G \) lies on a triangle”. It is readily observed that this is also a necessary and sufficient condition for \( p_o(G) = |V(G)| \).

**Proposition 6.** Let \( G \) be a graph of order \( n \). Then,

1. \( n/\rho_o(G) \leq p_o(G) \leq n - \rho_o(G) + 1 \).
2. \( \chi_2(G)/2 \leq p_o(G) \leq \chi_2(G) \).

These bounds are sharp.

**Proof.** (i) Let \( B = \{B_1, \cdots, B_{\#B}\} \) be a \( p_o(G) \)-partition. Since every \( B_i \) is an open packing set in \( G \), we have \( n = \sum_{1 \leq i \leq \#B} |B_i| \leq |B| \rho_o(G) = p_o(G) \rho_o(G) \). Hence, \( p_o(G) \geq n/\rho_o(G) \). On the other hand, if \( B \) is a \( \rho_o(G) \)-set, then \( \{B\} \cup \{\{g\} \mid g \in V(G) \setminus B\} \) is an OPP of \( G \) of cardinality \( n - \rho_o(G) + 1 \). So, \( p_o(G) \leq n - \rho_o(G) + 1 \).

That the lower bound is sharp was already shown in the proof of Theorem 8. In fact, \( p_o(C_{4m} \Box K_n) = 2n = 4mn/\rho_o(C_{4m} \Box K_n) \) for all integers \( m \geq 1 \) and \( n \geq 3 \). The upper bound is sharp for \( K_n \) as well as \( K_{1,n-1} \) on \( n \geq 3 \) vertices.

(ii) Any \( \chi_2(G) \)-coloring is an OPP of \( G \) by the definition. This results in the upper bound. Assume now that \( \mathbb{P} \) is a \( p_o(G) \)-partition. Note that the subgraph of \( G \) induced by each open packing \( B \in \mathbb{P} \) contains a disjoint union of copies of \( P_1 \) and \( P_2 \). Therefore, the open packing \( B \) can be written as the disjoint union of at most two independent sets \( B' \) and \( B'' \) such that one of them (if any) contains precisely one vertex from each \( P_2 \)-copy. It is easy to observe that both \( B' \) and \( B'' \) are packing sets in \( G \). Iterating this process for all open packing sets in \( \mathbb{P} \), we get a 2-distance coloring of \( G \) of cardinality at most \( 2|\mathbb{P}| = 2p_o(G) \). This leads to the lower bound.

The upper bound is sharp for any graph \( G \) of order at least three for which \( \text{diam}(G) \leq 2 \) and every edge lies on a triangle. In such a situation, we have \( p_o(G) = |V(G)| = \chi_2(G) \). The lower bound is sharp for the complete bipartite graph \( K_{n,n} \) as \( \chi_2(K_{n,n}) = 2n = 2p_o(K_{n,n}) \).
In order to give the characterization of all graphs attaining the lower bound given in the next theorem, we introduce the family $\Psi$ as follows. Let $H$ be an $r$-partite graph of order $n \equiv 0 \pmod{2r}$ satisfying the following properties:

1. $|X_i| = n/r$ for each partite set $X_i$, and
2. For each $1 \leq i \neq j \leq r$, the subgraph induced by $X_i \cup X_j$ is isomorphic to $(n/r)P_2$.

Let $G$ be obtained from $H$ by making a perfect matching using the vertices in $X_i$ for each $1 \leq i \leq r$. Finally, let $\Psi$ be the family of all such graphs $G$.

**Theorem 7.** For any graph connected $G$ of order $n \geq 2$ and size $m$,

$$p_o(G) \geq \left(1 + \sqrt{1 + 4(2m - n)/\rho_o(G)}\right)/2.$$  

The equality holds if and only if $G \in \Psi$.

**Proof.** Let $P = \{P_1, \cdots, P_{|P_o(G)|}\}$ be a $p_o(G)$-partition of $G$. Relabeling the subscripts if necessary, we may assume that $|P_1| \leq \cdots \leq |P_{|P_o(G)|}|$. By definition, every vertex in $P_i$ has at most one neighbor in $P_j$ for all $1 \leq i < j \leq |P_o(G)|$. Moreover, the subgraph of $G$ induced by $P_i$, for each $1 \leq i \leq |P_o(G)|$, has at most $|P_i|/2$ edges. We therefore conclude that,

$$m = \sum_{1 \leq i < j \leq |P_o(G)|} |P_i, P_j| + \sum_{1 \leq i \leq |P_o(G)|} |P_i, P_i| \leq \sum_{i=1}^{p_o(G)-1} |P_i| (p_o(G) - i) + \frac{n}{2} \leq |P_{|P_o(G)|}| \sum_{i=1}^{p_o(G)-1} (p_o(G) - i) + \frac{n}{2} \leq \left(\frac{p_o(G)(p_o(G)-1)}{2}\right) p_o(G) + \frac{n}{2}.$$  

Solving the inequality chain (10) for $p_o(G)$, we get $p_o(G) \geq \left(1 + \sqrt{1 + 4(2m - n)/\rho_o(G)}\right)/2$.

Suppose that the lower bound holds with equality. Therefore, all three inequalities in (10) necessarily hold with equality. In particular, together the second and third resulting equalities show that $|P_1| = \cdots = |P_{|P_o(G)|}| = p_o(G)$. The first one implies that any vertex in $P_1$ has precisely one neighbor in any other $P_j$, and $|P_i, P_j| = |P_i|/2$ for all $1 \leq i \leq p_o(G)$. This shows that the edges of each subgraph $G[P_i]$ form a perfect matching. It is now a simple matter to observe that the resulting subgraph of $G$ by removing $\bigcup_{i=1}^{p_o(G)} [P_i, P_i]$ is isomorphic to the graph $H$ (described in the process of introducing $\Psi$) for $r = p_o(G)$ and $X_i = P_i$, for each $1 \leq i \leq r$. Thus, $G \in \Psi$.

Conversely, let $G \in \Psi$. By the structure of $G$, the vertex partition $X = \{X_1, \cdots, X_r\}$ is an OPP of $G$. So, $p_o(G) \leq r$. Since $|X_1|$ is both an open packing and a total dominating set in $G$, it follows that $|X_1| \leq p_o(G) \leq \gamma_t(G) \leq |X_1|$, and so $p_o(G) = |X_1| = n/r$ (it is known that $\rho_o(G) \leq \gamma_t(G)$ for every graph $G$ with no isolated vertices. See [17] for example). Moreover, we have $r = 2m/n$ as $2m = \sum_{v \in V(G) \{deg_G(v) = r \}} n$. A simple calculation then shows that $(1 + \sqrt{1 + 4(2m - n)/\rho_o(G)})/2 = 2m/n = r \geq p_o(G)$. This implies the equality in the lower bound. \qed

Nordhaus and Gaddum [16] in 1956, gave lower and upper bounds on the sum and product of the chromatic numbers of a graph and its complement in terms of the order. Since then, inequalities involving the sum or product of a parameter for a graph and its complement are known as Nordhaus-Gaddum type relations. For more information about this kind of inequalities, the reader can consult [2].
Theorem 8. For any graph $G \notin \{C_4, 2P_2\}$ of order $n$, $p_o(G) + p_o(\overline{G}) \geq n$. Moreover, this bound is sharp.

Proof. By the inequality (7), we have

$$p_o(G) + p_o(\overline{G}) \geq \Delta(G) + \Delta(\overline{G}) = \Delta(G) + n - 1 - \delta(G) \geq n - 1.$$  \hfill (11)

Suppose that $p_o(G) + p_o(\overline{G}) = n - 1$ for a graph $G \notin C_4, 2P_2$, and that $\mathbb{B} = \{B_1, \ldots, B_{p_o(G)}\}$ be a $p_o(G)$-partition. We necessarily have $p_o(G) = \Delta(G)$, $p_o(\overline{G}) = \Delta(\overline{G})$ and $\Delta(G) = \delta(G)$ by the equality in (11). In particular, $G$ is a regular graph. Suppose that there exists a vertex $v \in B_i$ having no neighbor in $B_i$. Since $\text{deg}_G(v) = \Delta(G)$ and $\mathbb{B}$ is an OPP of $G$, $v$ has precisely one neighbor in any $\Delta(G)$ open packing sets in $\mathbb{B} \setminus \{B_i\}$. Therefore, $p_o(G) \geq \Delta(G) + 1$. This is a contradiction. Thus, every vertex $v \in B_i$ has precisely one neighbor in $B_i$ for each $1 \leq i \leq p_o(G)$.

In particular, this implies that every open packing class $B_i \in \mathbb{B}$ has an even number of vertices. Moreover, for any vertex $v \in V(G)$ and $B_i$ containing $v$, the vertex $v$ has precisely one neighbor in each $B_j$ for $i \neq j$.

We now consider two subsets $B_i$ and $B_j$ for any $1 \leq i \neq j \leq p_o(G)$. By the above argument, there exists a one-to-one correspondence between the vertices in $B_i$ and the vertices in $B_j$. In particular, $|B_i| = \cdots = |B_{p_o(G)}| = n/p_o(G)$. If $|B_i| = 1$, then $G$ is isomorphic to the complete graph $K_n$ for $n \notin 2$. So, $p_o(K_n) + p_o(\overline{K_n}) \geq n$ (equals $n + 1$ when $n \geq 3$). This is a contradiction. Therefore, $|B_i| \geq 2$.

Suppose first that $|B_i| = 2$. This implies that $p_o(G) = n/2$. Therefore, $p_o(\overline{G}) = n/2 - 1$ by the equality $p_o(G) + p_o(\overline{G}) = n - 1$. Suppose that $\mathbb{B}' = \{B'_1, \ldots, B'_{p_o(\overline{G})}\}$ be a $p_o(\overline{G})$-partition. We deduce, by a similar argument, that every open packing set in $\mathbb{B}'$ has an even number of vertices and that $|B'_1| = \cdots = |B'_{p_o(\overline{G})}| \geq 2$. Now if $|B'_1| \geq 4$, then $n = \sum_{i=1}^{p_o(\overline{G})} |B'_i| \geq 4p_o(\overline{G})$. Therefore, $n - 1 = p_o(G) + p_o(\overline{G}) \leq 3n/4$. This implies that $n \in \{1, 2, 3, 4\}$. Taking now into account the fact that $G \notin \{C_4, 2P_2\}$, it implies that $p_o(G) + p_o(\overline{G}) \geq n - 1$. This is a contradiction.

Suppose now that $|B'_1| \geq 4$. This implies that, $p_o(G) \leq n/4$. We have again $|B'_1| = \cdots = |B'_{p_o(\overline{G})}| \geq 2$. Therefore, $p_o(G) + p_o(\overline{G}) \leq n/2$ and hence $n - 1 = p_o(G) + p_o(\overline{G}) \geq 3n/4$, the same contradiction. Consequently, $p_o(G) + p_o(\overline{G}) \geq n$.

In order to show that the lower bound of the theorem is sharp, we construct an infinite family of graphs attaining the bound. For any integer $k \geq 3$, let $G$ be a graph on the set of vertices $\{v_1, v_2, \cdots, v_{2k}\}$ such that $N_G(v_{2i-1}) = \{v_{2j}\}_{j=1}^k$ and $N_G(v_{2i}) = \{v_{2j-1}\}_{j=1}^k$ for any $1 \leq i \leq k$.

It is easy to check that $\mathbb{P} = \{\{v_{2i-1}, v_{2i}\}\}_{i=1}^k$ is an OPP of $G$. Moreover, $\overline{G}$ is isomorphic to the disjoint union of two copies of the complete graph $K_k$ (see Figure 1 when $k = 4$). Therefore, $p_o(G) + p_o(\overline{G}) \leq |\mathbb{P}| + k = 2k = n$. Consequently, $p_o(G) + p_o(\overline{G}) = n$. This completes the proof. \hfill $\square$

We recall that the eccentricity of a vertex $v$ in a graph $G$, written $\varepsilon_G(v)$, is the maximum of distances from $v$ to the other vertices of $G$. The following theorem shows that the simple lower bound given in (7) gives the exact values of $p_o$ when dealing with nontrivial trees. Notice that Theorem 9 can be easily derived from Theorem 10. However, it is worth giving a proof for it that can be implemented as a polynomial-time algorithm in order to obtain an optimal OPP of each tree.
Theorem 9. For any tree $T$ on at least two vertices, $p_o(T) = \Delta(T)$.

Proof. We have $p_o(T) \geq \Delta(T)$ by the inequality (7). Therefore, it suffices to construct an OPP of $T$ of cardinality $\Delta(T)$.

Let $r$ be a vertex of maximum degree in $T$. We root $T$ at $r$. Then, any vertex at distance $\varepsilon_T(r)$ from $r$ is a leaf. We assign $1$ to $r$ and the labels $1, \ldots, \Delta(T)$ to the children of $r$ so that any of them takes a unique label. If $T$ is a star, then we are done. So, let $v$ be a child of $r$ labeled with $i \in \{1, \ldots, \Delta(T)\}$ which is not a leaf. Since $\deg(v) \leq \deg(r) = \Delta(T)$, it follows that $v$ has at most $\Delta(T) - 1$ children. We now assign $\deg(v) - 1$ labels among $\{1, \ldots, \Delta(T)\} \setminus \{1\}$ to its children so that any of these $\deg(v) - 1$ labels appears on only one such a child. This process is continued until all descendants of $v$ are assigned labels among $\{1, \ldots, \Delta(T)\}$. Iterating this process for any other child of $r$, any vertex of $T$ takes a label from $\{1, \ldots, \Delta(T)\}$ so that no vertex is adjacent to two vertices having the same labels. Therefore, the subsets $V_i = \{v \in V(T) \mid v \text{ is labeled with } i\}$ for $1 \leq i \leq \Delta(T)$ give an OPP of $T$ of cardinality $\Delta(T)$. This completes the proof. \hfill \Box

5 On graphs $G$ with $\chi(N(G)) = \omega(N(G))$

By the structure of the two-step graph $N(G)$ of a graph $G$, we observe that $\omega(N(G))$ equals the maximum number of vertices of $G$ such that any two of them have a common neighbor.

Theorem 10. If a graph $G$ contains no even cycle, then $\chi(N(G)) = \omega(N(G))$.

Proof. We prove that $N(G)$ is a chordal graph. Suppose to the contrary that $C_k : v_1v_2\cdots v_kv_1$ is a chordless cycle in $N(G)$ for some $k \geq 4$. Therefore, there exist $u_1, u_2, \ldots, u_k \in V(G)$ such that $\{v_1, v_2\} \subseteq N(u_1), \ldots, \{v_{k-1}, v_k\} \subseteq N(u_{k-1}), \{v_k, v_1\} \subseteq N(u_k)$. If the vertices $u_i$ are pairwise distinct, then $v_1u_1v_2\cdots v_{k-1}u_{k-1}v_kv_kv_1$ is a cycle in $G$ on $2k$ vertices, a contradiction. Therefore, $u_s = u_t$ for some $1 \leq s < t \leq k$. If $t = s + 1$, then $v_tv_{s+2}$ is a chord of $C_k$ in $N(G)$. If $t > s + 1$, then $v_sv_{t+1}$ is a chord of $C_k$ in $N(G)$. Each leads to a contradiction. Therefore, $N(G)$ is a chordal. In particular, $N(G)$ is a perfect graph. Thus, $\chi(N(G)) = \omega(N(G))$. \hfill \Box

Let $T$ be a tree and $Q$ be a maximum clique in $N(T)$. So, any two vertices of $Q$ have a common neighbor in $T$. Since $T$ is a tree, $Q$ is independent in $T$. So, a unique vertex is adjacent to all vertices in $Q$. This implies that $p_o(T) = \omega(N(T)) = \Delta(T)$. Consequently, Theorem 9 is an immediate result of Theorem 10 as already mentioned.
Note that the condition “being $C_{2k}$-free for each $k \geq 2$” cannot be removed in Theorem 10.

To see this, consider the cycle $C_{4t+2}$ for $t \geq 2$. It is easy to see that $N(C_{4t+2}) \cong 2C_{2t+1}$, and so $\chi(N(C_{4t+2})) = 3 \neq 2 = \omega(N(C_{4t+2}))$. In fact, this example shows that the equality in Theorem 10 does not necessarily hold even if $G$ is a perfect graph. However, it remains true for several infinite families of graphs containing even cycles as induced subgraphs.

**Theorem 11.** If $\overline{G}$ is a bipartite graph of order $n$, then $\chi(N(G)) = \omega(N(G))$.

**Proof.** Let $X$ and $Y$ be the partite sets of $\overline{G}$ with $|X| \leq |Y|$. Since $\overline{G}$ is bipartite, both $X$ and $Y$ are cliques in $G$. If $\overline{G}$ is a complete bipartite graph, then $G$ is isomorphic to the disjoint union $K_{|X|} + K_{|Y|}$. In such a situation, we have

$$N(G) \cong \begin{cases} K_n & \text{if } |Y| \leq 2, \\ K_{|X|} + K_{|Y|} & \text{if } |X| \leq 2 \text{ and } |Y| \geq 3, \\ K_{|X|} + K_{|Y|} & \text{if } |X| \geq 3. \end{cases}$$

In each case, $\chi(N(G)) = \omega(N(G))$.

So, in what follows we may assume that $\overline{G}$ is not complete bipartite. Therefore, $[X, Y] \subseteq E(G)$ is nonempty. We distinguish two cases depending on $|X|$.

**Case 1.** $|X| \geq 3$. Suppose first that $[X, Y] = \{x_1y_1, \ldots, x_ky_k\}$ is a matching in $G$. By the structure of $N(G)$, $x_i$ is adjacent to all vertices in $Y \setminus \{y_i\}$ in $N(G)$ for each $1 \leq i \leq k$. Moreover, $x_1y_1, \ldots, x_ky_k \not\in E(N(G))$. In such a situation, $\omega(N(G)) = |Y|$ because $|Y| \geq |X|$. Let $f$ assign the colors

- $1, \ldots, k$ to the vertices in $\{x_1, y_1\}, \ldots, \{x_k, y_k\}$, respectively,
- $k + 1, \ldots, |Y|$ to the other vertices of $Y$ (if any), and
- $k + 1, \ldots, |X|$ to the other vertices of $X$ (if any).

It is readily seen that $f$ is a coloring of $N(G)$ that assigns $|Y|$ colors to the vertices of $V(N(G)) = V(G)$. Therefore, $\chi(N(G)) \leq |Y| = \omega(N(G))$.

Suppose now that $[X, Y]$ is not a matching. This shows that $|N_G(x) \cap Y| \geq 2$ or $|N_G(y) \cap X| \geq 2$ for some $x \in X$ or $y \in Y$, respectively. Let $X' = \{x \in X \mid |N_G(x) \cap Y| \geq 2\}$ and $Y' = \{y \in Y \mid |N_G(y) \cap X| \geq 2\}$. By the adjacency rule of $N(G)$, we have $N_{N(G)}(x) = Y$ and $N_{N(G)}(y) = X$ for any $x \in X' \cup (N_G(Y') \cap X)$ and $y \in Y' \cup (N_G(X') \cap Y)$. It is not hard to see that both $Q_1 = X \cup Y' \cup (N_G(X') \cap Y)$ and $Q_2 = Y \cup X' \cup (N_G(Y') \cap X)$ are cliques in $N(G)$. We now let $X'' = X \setminus (X' \cup (N_G(Y') \cap X))$ and $Y'' = Y \setminus (Y' \cup (N_G(X') \cap Y))$. Note that each vertex $x \in X'' (y \in Y'')$ has at most one neighbor in $Y (X)$, and this neighbor belongs to $Y'' (X'')$, necessarily. This implies that $[X'', Y'']$ is a matching in $G$. Suppose that $[X'', Y''] = \{x''_1y''_1, \ldots, x''_ry''_r\}$. Again by using the adjacency rule of $N(G)$, we observe that none of the edges $x''_1y''_1, \ldots, x''_ry''_r$ appears in $N(G)$.

Suppose first that $|Q_1| \geq |Q_2|$. This implies that $|X''| \geq |Y''|$. Now let $g$ assign the colors

- $1, \ldots, r$ to the vertices in $\{x''_1, y''_1\}, \ldots, \{x''_r, y''_r\}$, respectively,
- $r + 1, \ldots, |X''|$ to the other vertices in $X''$ (if any),
- $|X''| + 1, \ldots, |X|$ to the vertices in $X \setminus X'$,
- $|X| + 1, \ldots, |Q_1|$ to the vertices in $Q_1 \setminus X$, and
- $r + 1, \ldots, |Y''|$ to the other vertices in $Y''$ (if any).

It is readily seen that $g$ is a coloring of $N(G)$ assigning $|Q_1|$ colors to the vertices of $N(G)$. Therefore, $\chi(N(G)) \leq |Q_1| \leq \omega(N(G))$. 


In a similar fashion, we deduce that $\chi(\mathcal{N}(G)) \leq |Q_2| \leq \omega(\mathcal{N}(G))$ when $|Q_2| \geq |Q_1|$. Thus, we get the desired equality when $|X| \geq 3$.

**Case 2.** $|X| \leq 2$. We now consider two possibilities depending on $|Y|$.

**Subcase 2.1.** $|Y| \geq 3$. Since $\overline{G}$ is not complete bipartite, there exists an edge $xy \in E(G)$ in which $x \in X$ and $y \in Y$. If $|X| = 1$, then it is easy to see that $\mathcal{N}(G) \in \{K_n, K_n - xy\}$. In both cases, the desired equality holds. Suppose that $x, x' \in X$ are two distinct vertices. If $N_G(x) \cap N_G(x') \neq \emptyset$, then $\mathcal{N}(G) = K_n$. So, we assume that $x$ and $x'$ have no common neighbor in $G$. In such a situation, we conclude that

$$\mathcal{N}(G) \cong \begin{cases} 
K_n - xx' & \text{if both } x \text{ and } x' \text{ have at least two neighbors in } G, \\
K_n - \{xx', x'y\} & \text{if } x \text{ has at least two neighbors and } x' \text{ has a unique neighbor } y' \text{ in } G, \\
K_{n-1} + x' & \text{if } x \text{ has at least two at least neighbors and } x' \text{ is an isolated vertex in } G, \\
(K_{n-1} - xy) + x' & \text{if } x \text{ has a unique neighbor } y \text{ and } x' \text{ is an isolated vertex in } G, \\
K_{n-2} + \{x, y\} & \text{otherwise}.
\end{cases}$$

In all above cases, we have $\chi(\mathcal{N}(G)) = \omega(\mathcal{N}(G))$.

**Subcase 2.2.** $|Y| \leq 2$. It is a simple matter to see that

$$\mathcal{N}(G) \in \{2K_1, K_3, K_2 + K_1, K_4, 2K_2, K_4 - xy\},$$

in which $x$ and $y$ are two vertices of $K_4$. In each case, we get the desired equality. This completes the proof.

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