THE LIMIT OF THE HERMITIAN-YANG-MILLS FLOW ON REFLEXIVE SHEAVES

JIAYU LI, CHUANJING ZHANG, AND XI ZHANG

Abstract. In this paper, we study the asymptotic behavior of the Hermitian-Yang-Mills flow on a reflexive sheaf. We prove that the limiting reflexive sheaf is isomorphic to the double dual of the graded sheaf associated to the Harder-Narasimhan-Seshadri filtration, this answers a question by Bando and Siu.

1. Introduction

Let \((M, \omega)\) be a compact Kähler manifold and \(\mathcal{E}\) a coherent sheaf on \(M\). The \(\omega\)-degree and the \(\omega\)-slope of \(\mathcal{E}\) are defined by

\[
\text{deg}_\omega(\mathcal{E}) := \int_X c_1(\mathcal{E}) \wedge \frac{\omega^{n-1}}{(n-1)!},
\]

and

\[
\mu_\omega(\mathcal{E}) := \frac{\text{deg}_\omega(\mathcal{E})}{\text{rank}(\mathcal{E})},
\]

where \(c_1(\mathcal{E})\) is the first Chern class of \(\mathcal{E}\). We say that a torsion free coherent sheaf \(\mathcal{E}\) is \(\omega\)-stable (\(\omega\)-semi-stable) in the sense of Mumford-Takemoto if for every proper coherent sub-sheaf \(\mathcal{F} \hookrightarrow \mathcal{E}\) we have

\[
\mu_\omega(\mathcal{F}) < (\leq) \mu_\omega(\mathcal{E}).
\]

We denote by \(\Sigma_\mathcal{E}\) the set of singularities where \(\mathcal{E}\) is not locally free. It is well known that the coherent sheaf \(\mathcal{E}\) can be seen as a holomorphic vector bundle on \(M \setminus \Sigma_\mathcal{E}\). A Hermitian metric \(H\) on the sheaf \(\mathcal{E}\) is called admissible if it is a Hermitian metric which is defined on the holomorphic vector bundle \(\mathcal{E}|_{M \setminus \Sigma_\mathcal{E}}\) and satisfies: (1) \(|F_H|_{H, \omega}\) is square integrable; (2) \(|\Lambda_\omega F_H|_H\) is uniformly bounded, where \(F_H\) is the curvature tensor of Chern connection \(D_H\) with respect to \(H\), and \(\Lambda_\omega\) denotes the contraction with the Kähler metric \(\omega\). A Hermitian metric \(H\) on the holomorphic vector bundle \(\mathcal{E}|_{M \setminus \Sigma_\mathcal{E}}\) is said to be \(\omega\)-Hermitian-Einstein if it satisfies the following Einstein condition on \(M \setminus \Sigma_\mathcal{E}\), i.e.

\[
\sqrt{-1} \Lambda_\omega F_H = \lambda_{\mathcal{E}, \omega} \text{Id}_{\mathcal{E}},
\]

where \(\lambda_{\mathcal{E}, \omega} = \frac{2\pi}{\text{vol}(M, \omega)} \mu_\omega(\mathcal{E}).\)

The Donaldson-Uhlenbeck-Yau theorem ([37] [14] [15] [45]) states that, if \(\mathcal{E}\) is locally free on the whole \(M\), i.e. \(\Sigma_\mathcal{E} = \emptyset\), the \(\omega\)-stability of \(\mathcal{E}\) implies the existence of \(\omega\)-Hermitian-Einstein metric on \(\mathcal{E}\). This theorem has several interesting and important generalizations and extensions ([29] [20] [89] [7] [18] [13] [21] [2] [3] [25] [27] [28] [30] [34] [55], etc.). In [7], Bando and Siu introduced
the notion of admissible Hermitian metrics on torsion-free sheaves, and proved the Donaldson-Uhlenbeck-Yau theorem on stable reflexive sheaves. In fact, they obtained a long time solution $H(t)$ of the Hermitian-Yang-Mills flow on $(M \setminus \Sigma_\varepsilon) \times [0, +\infty)$, i.e. $H(t)$ satisfies:

$$
\begin{cases}
H^{-1}(t) \frac{\partial H(t)}{\partial t} = -2(\sqrt{-1} \Lambda_\omega F_{H(t)} - \lambda_\varepsilon \omega \text{Id}_\varepsilon), \\
H(0) = H,
\end{cases}
$$

(1.2)

where $\tilde{H}$ is an initial metric which will be described in section 2 in details. The above Hermitian-Yang-Mills flow was introduced and studied by Donaldson in [14, 15]. Bando and Siu have shown that $H(t)$ is admissible for every $t > 0$. Furthermore, they proved that: if the reflexive coherent sheaf $E$ is $\omega$-stable, then along the Hermitian-Yang-Mills flow, $H(t)$ converges to $H_\infty$ subsequently in weak $L^2_{\omega,\text{loc}}$-topology and $H_\infty$ is an admissible $\omega$-Hermitian-Einstein metric. There are also some results on the existence of approximate solution of Hermitian-Einstein equation (1.1) on a semi-stable holomorphic bundle and a semi-stable Higgs bundle, see references [26, 23, 10, 11, 31] for details. Recently, the authors ([33]) obtain the existence of admissible approximate $\omega$-Hermitian-Einstein structure on an $\omega$-semi-stable reflexive sheaf, i.e. they proved that, if the reflexive sheaf $E$ is $\omega$-semi-stable, along the Hermitian-Yang-Mills flow (1.2), we have

$$
\sup_{x \in M \setminus \Sigma_\varepsilon} |\sqrt{-1} \Lambda_\omega F_{H(t)} - \lambda_\varepsilon \omega \text{Id}_\varepsilon|_{H(t)}(x) \to 0,
$$

(1.3)
as $t \to +\infty$.

For an unstable torsion-free coherent sheaf $E$, one can associate a filtration ([26, 5]) by sub-sheaves

$$
0 = E_0 \subset E_1 \subset \cdots \subset E_t = E,
$$

(1.4)
such that every quotient sheaf $Q_i = E_i/E_{i-1}$ is torsion-free and $\omega$-stable, which is called the Harder-Narasimhan-Seshadri filtration of the reflexive sheaf $E$ (abbr. HNS-filtration). Moreover, $\mu(Q_i) \geq \mu(Q_{i+1})$ and the associated graded object

$$
\text{Gr}^{\text{HNS}}(E) = \oplus_{i=1}^l Q_i
$$

(1.5)
is uniquely determined by the isomorphism class of $E$ and the Kähler class $[\omega]$.

If the reflexive sheaf $E$ is not stable, Bando and Siu ([22]) proved that: there exists a subsequence $H(t_i)$ along the Hermitian-Yang-Mills flow (1.2) such that $\int_M |\nabla_\omega F_{H(t_i)}|_{H(t_i)}^2 \rightarrow 0$ as $t_i \to +\infty$. By Uhlenbeck’s theorem ([14, 15]), taking suitable complex gauge transformations one can choose a subsequence $t_i \to +\infty$ so that Chern connections $A(t_i) \to A_\infty$ weakly in $L^2_\varepsilon$-topology outside a closed subset $\tilde{\Sigma} \subset M$ of Hausdorff codimension at least 4. Since $\sqrt{-1} \Lambda_\omega F_{A_\infty}$ is parallel, we can decompose $E_\infty$ according to the eigenvalues of $\sqrt{-1} \Lambda_\omega F_{A_\infty}$ on $M \setminus \tilde{\Sigma}$. Then we obtain a holomorphic orthogonal decomposition

$$
E_\infty = \bigoplus_{i=1}^l E^i_\infty,
$$

(1.6)
every $E^i_\infty$ admits a Hermitian-Einstein metric and can be extended to a reflexive sheaf. In [22], Bando and Siu propose an interesting question: whether

$$
\bigoplus_{i=1}^l E^i_\infty \cong \text{Gr}^{\text{HNS}}(E)^{**}.
$$

(1.7)

Atiyah and Bott ([11]) first raised the same question for Riemann surfaces case, which has been proved by Daskalopoulos ([12]). When $E$ is locally free on the whole $M$, the conjecture was confirmed by Daskalopoulos and Wentworth ([13]) for Kähler surfaces case; by Jacob ([24])
Theorem 1.1. Let \( E \) be a reflexive sheaf on a compact Kähler manifold \((M, \omega)\), and \( H(t) \) be the solution of the Hermitian-Yang-Mills flow \((1.2)\) on \( E \) with the initial metric \( \hat{H} \). We have a family of integrable connections
\[
A(t) = g(t)(\hat{A}) \tag{1.8}
\]
on \( E|_{M \setminus \Sigma_E} \) for \( t \in [0, +\infty) \), where \( g(t) \in C^\infty \) satisfies \( g^*H_0(t) \circ g(t) = h(t) = H_0^{-1}H(t) \), \( \Sigma_E \) is the singularity set of \( E \) and \( \hat{A} \) is the Chern connection with respect to the initial metric \( \hat{H} \), such that:

1. For every sequence \( \{t_i\} \) there exists a subsequence \( \{t_j\} \) such that, \( A(t_j) \) converges, modulo gauge transformations, to a Yang-Mills connection \( A_\infty \) on a Hermitian vector bundle \((E_\infty, H_\infty)\) over \( M \setminus \Sigma \) in \( C^\infty \) topology as \( t_j \to +\infty \), where \( \Sigma \subset M \) is a closed set of Hausdorff codimension at least 4. Furthermore, the limiting \((E_\infty, \Omega A_\infty)\) can be extended to the whole \( M \) as a reflexive sheaf with a holomorphic orthogonal splitting
\[
(E_\infty, H_\infty, \Omega A_\infty) = \bigoplus_{i=1}^l (E^i_\infty, H^i_\infty), \tag{1.9}
\]
where \( H^i_\infty \) is an admissible Hermitian-Einstein metric on the reflexive sheaf \( E^i_\infty \).

2. Moreover, the extended reflexive sheaf is isomorphic to the double dual of the graded sheaf associated to the HNS-filtration of \( E \), i.e. we have
\[
(E_\infty, \Omega A_\infty) \cong Gr^{HNS}(E)^{**}. \tag{1.10}
\]

We now give an overview of our proof. The conclusion in the the part (1) of Theorem 1.1 is stronger than that in Theorem 4 in [7], because we prove that the convergence holds not only for every sequence \( \{t_i\} \) but also in much stronger topology, i.e. in \( C^\infty \) topology. To prove the part (1), we follow Hong-Tian’s argument in [22]. Even though the global approach is similar, some key estimates require new analytical ideas because the base manifold \( M \setminus \Sigma \) in our case is not compact. For examples: to prove that \( \int_M |D_{H(t)}(A_{\omega_{H(t)}})F_{H(t)}|^2_{H(t)} \Xi t \to 0 \) as \( t \to +\infty \) in Proposition 2.3 to analyze the limiting behavior of the Yang-Mills flow on \( E|_{M \setminus \Sigma} \) in Theorem 3.2.

To prove the second part of Theorem 1.1 we will use the basic idea in [13] for a locally free sheaf in the Kähler surface case, but there are two points where we need new arguments for reflexive sheaves case. The first one is to prove that the HN type of the limiting sheaf is in fact equal to that of \( E \); and the second one is to construct a non-zero holomorphic map from any stable quotient sheaf in HNS-filtration of \( E \) to the limiting sheaf.

The first one is closely related to the existence of an \( L^p \)-approximate critical Hermitian metric (as defined in [13]). When \( E \) is locally free, Sibley ([38]) constructs a resolution of the HNS-filtration of \( E \) by subbundles, i.e. there exists a finite sequence of blow-ups with smooth centers such that the pullback bundle \( \pi^*E \) has a filtration by subbundles, where \( \pi : M \to \tilde{M} \) is the composition of the blow-ups involved in the resolution. The metric \( \pi^*\omega \) is degenerate along the exceptional divisor \( \pi^{-1}(\Sigma_{HNS}) \), where \( \Sigma_{HNS} \) is the singularity set of the HNS-filtration of \( E \), and it can be approximated by a family of Kähler metrics \( \omega_\epsilon \) on \( \tilde{M} \). Since every quotient subbundle is \( \omega_\epsilon \)-stable for small \( \epsilon \), one can use Donaldson-Uhlenbeck-Yau theorem to take the direct sum of the Hermitian-Einstein metrics on quotient subbundles in the resolution. By choosing any fixed smooth Hermitian metric \( H_0 \) on \( \pi^*E \) over a neighborhood of \( \pi^{-1}(\Sigma_{HNS}) \)
such that $| \Lambda_\omega F_{H_0}|_{H_0}$ is uniformly bounded, Sibley uses Daskalopoulos and Wentworth’s cut-off argument \cite{14} to obtain a smooth $L^p$-approximate critical Hermitian metric on the locally free sheaf $E$. In our case, $E$ is only reflexive, we can not find such smooth metric $H_0$. So we can not use Sibley’s result directly, and need new arguments to obtain a smooth $L^p$-approximate critical Hermitian metric, see Proposition 4.2 and Proposition 4.5 for details. Furthermore, in Lemma 4.2 we prove the continuous dependence of the Hermitian-Yang-Mills flow \cite{14} on initial metrics, this is fully nontrivial for noncompact base manifolds case. Then we can follow Daskalopoulos and Wentworth’s trick (Lemma 4.3 in \cite{13}) to prove that the HN type of the limiting sheaf is in fact equal to that of $E$.

For the second one, we use Donaldson’s idea \cite{15} to construct a nonzero holomorphic map to the limiting bundle as the limit of the sequence of gauge transformations defined by the Yang-Mills flow. There are many difficulties to obtain uniform estimates, because we have no uniform $L^\infty$-bound on the mean curvature (i.e. $|\sqrt{-1}\Lambda_\omega F_A|$) of the induced connection for subsheaves. Using the resolution of singularities, we can pull back the HNS-filtration to $M$ by subbundles. Evolving the induced Hermitian metric on the subbundle by the Hermitian-Yang-Mills flow with respect to the Kähler metric $\omega_\epsilon$, by the result in \cite{7}, we can get a uniform $L^\infty$-bound on the mean curvature and a local uniform $C^0$-estimate of the evolved Hermitian metrics. Using these estimates and following the argument in Proposition 4.1 in \cite{32}, we can obtain a local uniform $C^0$-estimate of a sequence of holomorphic maps and then construct a nonzero holomorphic map to the limiting bundle. It should be pointed out that in Proposition 4.1 in \cite{32}, we need the assumption that the pulling back geometric objects including the complex gauge transformations and induced metrics on the subsheaves can be extended smoothly on the whole $\tilde{M}$. This assumption may not be satisfied in our case, so we modify the argument in \cite{32} suitably to the case that the geometric object we consider can be approximated by a sequence of smooth ones, see Proposition 4.1 for details.

This paper is organized as follows. In Section 2, we recall Bando and Siu’s regularization on the reflexive sheaf and some basic estimates for the Hermitian-Yang-Mills flow, and we prove that along the Hermitian-Yang-Mills flow, $\int_M |D_{H(t)} \Lambda_\omega F_{H(t)}|^2_{H(t)} \frac{\omega_t^n}{n!} \to 0$ as $t \to +\infty$. In section 3, we analyze the limiting behavior of the Yang-Mills flow on $(M \setminus \Sigma, E|_{M \setminus \Sigma}, \omega)$ and give a proof for the part (1) of Theorem 1.1. In section 4 and section 5, we obtain an $L^p$-approximate critical Hermitian metric and prove that the HN type of the limiting sheaf is in fact equal to that of the initial one. In the last section, we construct a non-zero holomorphic map between sheaves and complete the proof of Theorem 1.1.

2. Analytic preliminaries and basic estimates

In this section, we first recall Bando and Siu’s regularization on the reflexive sheaf, and then give some basic estimates for the Hermitian-Yang-Mills flow. Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$, and $E$ be a reflexive sheaf on $M$. The singularity set of $E$ will be denoted by $\Sigma_E$. Bando and Siu \cite{12} proved that: there is a regularization on the reflexive sheaf $E$, by successively blowing up $\pi_i : M_i \to M_{i-1}$ with smooth center $Y_{i-1} \subset M_{i-1}$ finite times such that the pull-back of $E$ to $M_k$ modulo torsion is locally free and the composition

$$\pi = \pi_1 \circ \cdots \circ \pi_k : \tilde{M} \to M$$

(2.1)

is biholomorphic outside $\Sigma_E$, where $i = 1, \cdots, k$, $M_0 = M$ and $\tilde{M} = M_k$. It is easy to see that the holomorphic vector bundle $\tilde{E} = \pi^* E/tor(\pi^* E)$ is isomorphic to $E$ on $\tilde{M} \setminus \pi^{-1}(\Sigma_E)$, where $tor(\pi^* E)$ is the torsion sheaf of $\pi^* E$. 


It is well known that every $M_i$ is Kähler ([17]). As in [7], we fix arbitrary Kähler metrics $\eta_i$ on $M_i$ and set
\[
\omega_{i,e} = \pi_i^* \omega + \epsilon_i \eta_i, \quad \omega_{i,e} = \pi_i^* \omega_{i-1,e} + \epsilon_i \eta_i
\] (2.2)
for all $1 \leq i \leq k$, where $0 < \epsilon_i \leq 1$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_k)$. Bando and Siu (Lemma 3 in [7]) derived a uniform Sobolev inequality for $(\tilde{M}, \omega_e)$, by using Cheng and Li’s estimate ([9]), they obtained the following uniform upper bounds of the heat kernels.

**Proposition 2.1. (Proposition 2 in [7])** Let $(M, \omega)$ be a compact Kähler manifold, and $\pi : \tilde{M} \to M$ be a single blow-up with smooth centre. Fix a Kähler metric $\eta$ on $\tilde{M}$ and set $\omega_e = \pi^* \omega + \epsilon \eta$, where $0 < \epsilon \leq 1$. Let $K_e$ be the heat kernel with respect to the metric $\omega_e$. Then, for every $x, y$ such that
\[
\tau > 0,
\]
for any $\tau > 0$, there exists a constant $C_K(\tau)$ independent of $\epsilon$, such that
\[
0 \leq K_e(x, y, t) \leq C_K(\tau)(t^{-n} \exp (-\frac{(d_{\omega_e}(x, y))^2}{(4 + \tau)t}) + 1)
\] (2.3)
for every $x, y \in \tilde{M}$ and $0 < t < +\infty$, where $d_{\omega_e}(x, y)$ is the distance between $x$ and $y$ with respect to the metric $\omega_e$. There also exists a constant $C_G$ such that
\[
G_e(x, y) \geq -C_G
\] (2.4)
for every $x, y \in \tilde{M}$ and $0 < \epsilon \leq 1$, where $G_e$ is the Green function with respect to the metric $\omega_e$.

Given a smooth Hermitian metric $\hat{H}$ on the bundle $E$, we denote the corresponding Chern connection by $D_{\hat{H}}$, and the corresponding curvature form by $F_{\hat{H}}$.
\[
|\Lambda_{\omega_{k,e}} F_{\hat{H}}|_{\hat{H}} \frac{\omega_{k,e}^n}{\eta_k^n} = n \left| \frac{F_{\hat{H}} \wedge \omega_{k,e}^{n-1}}{\eta_k^n} \right|_{\hat{H}} \leq \tilde{C}_0 \sup_{\tilde{M}} |F_{\hat{H}}|_{\hat{H}},
\] (2.5)
where $\tilde{C}_0$ is a uniform constant independent of $\epsilon$. So there exists a uniform constant $\tilde{C}_0$ such that
\[
\int_{\tilde{M}} |\Lambda_{\omega_{k,e}} F_{\hat{H}}|_{\hat{H}} \frac{\omega_{k,e}^n}{n!} \leq \tilde{C}_0,
\] (2.6)
for all $\epsilon$.

We consider the evolving metric $H_{k,e}(t)$ along the Hermitian-Yang-Mills flow ([12]) on the holomorphic bundle $E$ over $M$ with the fixed smooth initial metric $\hat{H}$ and with respect to the Kähler metric $\omega_{k,e}$, i.e. it satisfies
\[
\begin{align*}
H_{k,e}^{-1}(t) \frac{\partial H_{k,e}(t)}{\partial t} &= -2(\sqrt{-1} \Lambda_{\omega_{k,e}} F_{H_{k,e}(t)} - \lambda_{k,e} \text{Id}_E), \\
H_{k,e}(0) &= \hat{H},
\end{align*}
\] (2.7)
where $\lambda_{k,e} = \frac{2\pi}{\text{Vol}(\tilde{M}, \omega_{k,e})} \mu_{\omega_{k,e}}(E)$. For simplicity, set:
\[
\theta(H, \omega) = \sqrt{-1} \Lambda_{\omega} F_{\hat{H}} - \lambda_{\omega} \text{Id}_E.
\] (2.8)

Along the heat flow (2.7), we have the following estimates (the proof can be found in Siu’s lecture notes [11]):
\[
(\Delta_{k,e} - \frac{\partial}{\partial t}) \text{tr} (\theta(H_{k,e}(t), \omega_{k,e})) = 0,
\] (2.9)
\[
(\Delta_{k,e} - \frac{\partial}{\partial t}) |\theta(H_{k,e}(t), \omega_{k,e})|_{H_{k,e}(t)}^2 = 2 |D_{H_{k,e}(t)}(\theta(H_{k,e}(t), \omega_{k,e}))|_{H_{k,e}(t), \omega_{k,e}},
\] (2.10)
\[
(\Delta_{k,e} - \frac{\partial}{\partial t}) |\theta(H_{k,e}(t), \omega_{k,e})|_{H_{k,e}(t)} \geq 0,
\] (2.11)
\[
|\frac{\partial}{\partial t} \ln(\text{tr}(H_{k,\varepsilon}^{-1}(t)H_{k,\varepsilon}(t)) + \text{tr}(H_{k,\varepsilon}^{-1}(t)H_{k,\varepsilon}(t)))| \leq 2|\theta(H_{k,\varepsilon}(t), \omega_{k,\varepsilon})|_{H_{k,\varepsilon}(t)}.
\] (2.12)

Using the maximum principle and the above inequalities, we derive

\[
\int_M |\theta(H_{k,\varepsilon}(t), \omega_{k,\varepsilon})|_{H_{k,\varepsilon}(t)} \frac{\omega^n}{n!} \leq \int_M |\theta(\tilde{H}, \omega_{k,\varepsilon})|_{\tilde{H}} \frac{\omega^n}{n!} \leq \tilde{C}_1,
\] (2.13)

\[
|\theta(H_{k,\varepsilon}(t), \omega_{k,\varepsilon})|_{H_{k,\varepsilon}(t)}(x) \leq \int_M K_{k,\varepsilon}(x, y, t)|\theta(\tilde{H}, \omega_{k,\varepsilon})|_{\tilde{H}} \frac{\omega^n}{n!},
\] (2.14)

and

\[
|\theta(H_{k,\varepsilon}(t+1), \omega_{k,\varepsilon})|_{H_{k,\varepsilon}(t+1)}(x) \leq \int_M K_{k,\varepsilon}(x, y, t)|\theta(H_{k,\varepsilon}(t), \omega_{k,\varepsilon})|_{H_{k,\varepsilon}(t)} \frac{\omega^n}{n!},
\] (2.15)

for all \(x \in \tilde{M}\) and \(t > 0\).

After obtaining local uniform \(C^\infty\)-bounds on \(H_{k,\varepsilon}(x, t)\), Bando and Siu ([7]) get the following lemma.

**Lemma 2.2.** ([7]) By choosing a subsequence, \(H_{k,\varepsilon}(x, t)\) converges successively to a long time solution \(H(x, t)\) of the Hermitian-Yang-Mills flow (1.2) on \(M \setminus \Sigma_\varepsilon \times [0, +\infty)\) in \(C^\infty_{\text{loc}}\)-topology as \((\varepsilon_1, \cdots, \varepsilon_k) \to 0\). Furthermore, \(H(x, t)\) is admissible and satisfies:

\[
\int_M |\theta(H(t), \omega)|_{H(t)} \frac{\omega^n}{n!} \leq \int_M |\theta(\tilde{H}, \omega)|_{\tilde{H}} \frac{\omega^n}{n!} \leq \tilde{C}_1,
\] (2.16)

\[
|\theta(H(t+\tilde{t}), \omega)|_{H(t+\tilde{t})}(x) \leq \int_M K_{\omega}(x, y, t)|\theta(H(\tilde{t}), \omega)|_{H(\tilde{t})} \frac{\omega^n}{n!},
\] (2.17)

for all \(x \in M \setminus \Sigma_\varepsilon\), \(t > 0\) and \(\tilde{t} \geq 0\).

Denote by \(D_A\) the Chern connection on the holomorphic bundle \(E|_{M \setminus \Sigma_\varepsilon}\) with respect to the initial metric \(\tilde{H}\). Let \(h(t) = \tilde{H}^{-1}H(t)\), using the identities

\[
\partial H(t) - \partial \tilde{H} = h^{-1}(t)\partial \tilde{H} h(t),
\]

\[
F_H(t) - F_{\tilde{H}} = \bar{\partial} A(h^{-1}(t)h(t)),
\] (2.18)

then we can rewrite (1.2) as

\[
\frac{\partial h(t)}{\partial t} = -2\sqrt{-1} h(t) \Lambda_\omega(F_{\tilde{H}} + \bar{\partial} A(h^{-1}(t)h(t))) + 2\lambda_{\omega, \varepsilon} h(t).
\] (2.19)

Let’s consider the Hermitian vector bundle \((E|_{M \setminus \Sigma_\varepsilon}, \tilde{H})\). We denote by \(A_{\tilde{H}}\) the space of connections of \(E|_{M \setminus \Sigma_\varepsilon}\) compatible with \(\tilde{H}\), by \(A_{\tilde{H}}^{1,1}\) the space of unitary integrable connections of \(E|_{M \setminus \Sigma_\varepsilon}\), i.e.

\[
A_{\tilde{H}}^{1,1} = \{A \in A_{\tilde{H}}|F_{A}^{0,2} = F_{A}^{2,0} = 0\},
\] (2.20)

and by \(G^C\) (resp. \(G\), where \(G = \{\sigma \in G^C|\sigma^*\tilde{H}\sigma = \text{Id}\}\) the complex gauge group (resp. unitary gauge group) of the Hermitian vector bundle \((E|_{M \setminus \Sigma_\varepsilon}, \tilde{H})\). \(G^C\) acts on the space \(A_{\tilde{H}}\) as follows: let \(\sigma \in G^C\) and \(A \in A_{\tilde{H}}\),

\[
\bar{\partial}_{\sigma(A)} = \sigma \circ \bar{\partial}_A \circ \sigma^{-1}, \quad \partial_{\sigma(A)} = (\sigma^*\tilde{H})^{-1} \circ \partial_A \circ \sigma^*\tilde{H}.
\] (2.21)

In [15], Donaldson has shown that the Hermitian-Yang-Mills flow (1.2) is formally gauge-equivalent to the Yang-Mills flow, i.e. we have the following proposition:
Proposition 2.3. There is a family of complex gauge transformations $\sigma(t) \in G^C$ satisfying $\sigma^*H(t)\sigma(t) = h(t) = H^{-1}H(t)$, where $H(t)$ is the long time solution of the Hermitian-Yang-Mills flow (1.2) with the initial metric $H$, such that $A(t) = \sigma(t)\hat{A}$ is a long time solution of the Yang-Mills flow with the initial connection $\hat{A}$, i.e. it satisfies:

$$
\begin{cases}
\frac{\partial A(t)}{\partial t} = -D^*_{A(t)}F_{A(t)}, \\
A(0) = \hat{A}.
\end{cases}
$$

(2.22)

It is well known that

$$
\sigma^{-1}(t) \circ F_{\hat{A}(t)} \circ \sigma(t) = F_A + \overline{\partial}A(h^{-1}(t)\partial h(t)) = F_{H(t)},
$$

(2.23)

and then

$$
|F_{H(t)}|^2_{H(t)} = |F_{\hat{A}(t)}|^2_H,
$$

(2.25)

and

$$
|D_{H(t)}(\Lambda_\omega F_{H(t)})|^2_{H(t)} = |D_{\hat{A}(t)}(\Lambda_\omega F_{\hat{A}(t)})|^2_H.
$$

(2.26)

For simplicity, set

$$
\theta(A(t), \omega) = \sqrt{-1}\Lambda_\omega F_{A(t)} - \lambda_\omega \omega \text{Id},
$$

(2.27)

and

$$
I(t) = \int_M |D_{A(t)}\theta(A(t), \omega)|^2 \frac{\omega^n}{n!} = \int_M |D_{H(t)}\theta(H(t), \omega)|^2 \frac{\omega^n}{n!}.
$$

(2.28)

In the following we will prove that $I(t) \to 0$ as $t \to +\infty$. When $\mathcal{E}$ is locally free, i.e. $\Sigma\mathcal{E} = \emptyset$, this was proved by Donaldson and Kronheimer ([16]). In the case that $\mathcal{E}$ is only reflexive, we need new arguments because the base manifold $M \setminus \Sigma\mathcal{E}$ is non-compact.

Proposition 2.4. Let $H(t)$ be the long time solution of the Hermitian-Yang-Mills flow (1.2) with the initial metric $H$, then $I(t) \to 0$ as $t \to +\infty$.

Proof. As that in the beginning of this section, there is a finite sequence of blowing up $\pi_i : M_i \to M_{i-1}$ with smooth center, where $i = 1, \ldots, k$, such that $E = \pi^*\mathcal{E}/\text{tor}(\pi^*\mathcal{E})$ is locally free on $\hat{M}$, where $\pi : \hat{M} \to M$ is the composition of the sequence of blow-ups. The initial Hermitian metric $H$ is a smooth metric on $E$. By induction, we can assume that there is just one blow-up, i.e. $k = 1$. Set $\omega_\epsilon = \pi^*\omega + \epsilon\eta$, where $\eta$ is a fixed Kähler metric on $\hat{M}$. Let $H_\epsilon(t)$ be the long time solution of the Hermitian-Yang-Mills flow (1.2) on the holomorphic bundle $E$ over $\hat{M}$ with the fixed smooth initial metric $H$ and with respect to the Kähler metric $\omega_\epsilon$, i.e. it satisfies

$$
\begin{cases}
H^{-1}_\epsilon(t)\frac{\partial H_\epsilon(t)}{\partial t} = -2\sqrt{-1}\Lambda_\omega F_{H_\epsilon(t)} - \lambda_\omega \text{Id}, \\
H_\epsilon(0) = H.
\end{cases}
$$

(2.29)

Lemma 2.2 says that $H_\epsilon(x, t)$ converges to the long time solution $H(x, t)$ of the Hermitian-Yang-Mills flow (1.2) on $M \setminus \Sigma\mathcal{E} \times [0, +\infty)$ in $C^\infty_{loc}$-topology as $\epsilon \to 0$. We also denote by $\hat{M}$ the Chern connection on the holomorphic vector bundle $E$ with respect to the smooth metric $H$. Let $A_\epsilon(t)$ be the long time solution of the Yang-Mills flow on the Hermitian vector bundle $(E, \hat{H})$ over the Kähler manifold $(\hat{M}, \omega_\epsilon)$, i.e.

$$
\begin{cases}
\frac{\partial A_\epsilon(t)}{\partial t} = -D^*_{A_\epsilon(t)}F_{A_\epsilon(t)}, \\
A_\epsilon(0) = \hat{A}.
\end{cases}
$$

(2.30)
Set
\[ I_c(t) = \int_M |D_{A_c(t)}(A_c(t), \omega_c)|^2 \frac{\omega^n}{n!} = \int_M |D_{H_c(t)}(H_c(t), \omega_c)|^2 \frac{\omega^n}{n!}. \] (2.31)

By the uniform bound on the heat kernel (2.3) and (2.14), there exists a uniform constant 
\( C \) such that
\[ \sup_{M} \| \theta(A_c(t), \omega_c) \|_{H^2} \leq C. \] (2.32)

for any \( 0 < t_0 \leq t \) and \( 0 \leq \epsilon \leq 1 \). Direct computations show that
\[ \frac{dI_c(t)}{dt} = -2 \int_M |D_{A_c(t)}D_{A_c(t)}(A_c(t), \omega_c)|^2 \frac{\omega^n}{n!} \]
\[ + 2Re \int_M \left[ \left( \phi(A_c(t), \omega_c) - \phi(A_c(t), \omega_c), \theta(A_c(t), \omega_c) \right), D_{A_c(t)}(A_c(t), \omega_c) \right] \frac{\omega^n}{n!} \]
\[ \leq 2I_c(t), \]
(2.33)
where \( C \) is a uniform constant. So we know that there exists a uniform constant such that
\[ I_c(t) \leq e^{C(t-s)} I_c(s), \] (2.34)

for any \( 0 < t_0 \leq s \leq t \) and \( 0 \leq \epsilon \leq 1 \). Of course the formula (2.10) yields
\[ \int_M |\sqrt{-1}A \omega \cdot F_{A_c(t)}(t) - \lambda d| H^2 \frac{\omega^n}{n!} + 2 \int_{t_0}^t \int_M |D_{A_c(s)}(A_c(t), \omega_c)|^2 \frac{\omega^n}{n!} ds \]
\[ = \int_M |\sqrt{-1}A \omega \cdot F_{A_c(t_0)}(t) - \lambda d| H^2 \frac{\omega^n}{n!}. \] (2.35)

According to Fatou’s lemma, we get
\[ \int_{M \setminus \Sigma_{\epsilon}} |\sqrt{-1}A \omega \cdot F_{A(t)} - \lambda d|^2 
\frac{\omega^n}{n!} + 2 \int_{t_0}^t \int_{M \setminus \Sigma_{\epsilon}} \left| \frac{\partial A(s)}{\partial s} \right|^2 \frac{\omega^n}{n!} ds \]
\[ \leq \int_{M \setminus \Sigma_{\epsilon}} |\sqrt{-1}A \omega \cdot F_{A(t_0)} - \lambda d|^2 
\frac{\omega^n}{n!}. \] (2.36)

This implies that \( \int_M |\sqrt{-1}A \omega \cdot F_{A_c(t)}(t) - \lambda d| H^2 \frac{\omega^n}{n!} \) and \( \int_{M \setminus \Sigma_{\epsilon}} |\sqrt{-1}A \omega \cdot F_{A(t)} - \lambda d|^2 \frac{\omega^n}{n!} \) both are monotonically nonincreasing with respect to \( t \). Then we must have
\[ \left[ \int_M |\sqrt{-1}A \omega \cdot F_{A_c(t)}(t) - \lambda d|^2 \frac{\omega^n}{n!} \right] - \left[ \int_M |\sqrt{-1}A \omega \cdot F_{A_c(t_0) + t} - \lambda d|^2 \frac{\omega^n}{n!} \right] \rightarrow 0 \] (2.37)

\[ \left[ \int_{M \setminus \Sigma_{\epsilon}} \left| \sqrt{-1}A \omega \cdot F_{A(t)}(t) - \lambda d \right|^2 H^2 \frac{\omega^n}{n!} \right] - \left[ \int_{M \setminus \Sigma_{\epsilon}} \left| \sqrt{-1}A \omega \cdot F_{A(t_0) + t} - \lambda d \right|^2 H^2 \frac{\omega^n}{n!} \right] \rightarrow 0, \] (2.38)
as \( t \to +\infty \).

For any \( m \geq t_0 > 0 \), there exists \( t_m \in [m, m+1] \), such that \( I_c(t_m) = \int_{t_m}^{t_{m+1}} I_c(t) dt \). From the formula (2.35), it follows that
\[ I_c(t) \leq e^{2C} I_c(t_m) = e^{2C} \int_{t_m}^{t_{m+1}} I_c(t) dt \]
\[ = \frac{e^{2C}}{2} \left( \int_M |\sqrt{-1}A \omega \cdot F_{A(0)}(t) - \lambda d|^2 H^2 \frac{\omega^n}{n!} - \int_M |\sqrt{-1}A \omega \cdot F_{A(m+1)}(t) - \lambda d|^2 H^2 \frac{\omega^n}{n!} \right), \] (2.39)
for any $t \in [m + 1, m + 2]$. Applying Fatou’s lemma again, we derive
\[
I(t) \leq \lim_{\epsilon \to 0} I_{\epsilon}(t) \leq \lim_{\epsilon \to 0} \frac{e^{2C}}{2} \left( \int_M |\sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}}(m) - \lambda \| \right) \left( \int_M |\sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}}(m+1) - \lambda \| \right)
\]
\[
= \lim_{\epsilon \to 0} \frac{e^{2C}}{2} \left( \int_M |\sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}}| - \lambda \| \right) \left( \int_M |\sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}}| - \lambda \| \right),
\]
for any $t \in [m + 1, m + 2]$. This together with (2.40) means that $I(t) \to 0$, as $t \to +\infty$.

Now we recall other Hermitian-Yang-Mills type functionals which are introduced in [13]. For any $a \in \mathfrak{u}(R)$, let $\varphi_{a}(\alpha) = \sum_{j=1}^{R} |\lambda_j|^\alpha$, where $\mathfrak{u}(R)$ is the Lie algebra of the unitary group $U(R)$, $\sqrt{-1}\Lambda_j$ are the eigenvalues of $a$, and $\alpha \geq 1$ is a real number. For a given real number $N$, define the Hermitian-Yang-Mills type functionals as follows:
\[
HYM_{\alpha,N}(A, M, \omega) = \frac{1}{\text{Vol}(M, \omega)} \int_M \varphi_{a}(\frac{1}{2\pi} \Lambda_{\omega} F_{A} - \sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}}| - \lambda \|^\omega^n_n!).
\]
Let $A_{\epsilon}(t)$ be the long time solution of the Yang-Mills flow (2.30) on the Hermitian vector bundle $(E, \hat{M}, \omega_{\epsilon})$ over the Kähler manifold $(\hat{M}, \omega_{\epsilon})$. For any smooth convex ad-invariant function $\varphi$, we have
\[
(\Delta_{\omega_{\epsilon}} - \frac{\partial}{\partial t}) \varphi \left( \frac{1}{2\pi} \Lambda_{\omega_{\epsilon}} F_{A_{\epsilon}(t)} - \sqrt{-1}\Lambda_{\omega_{\epsilon}} F_{A_{\epsilon}(t)} \right) \geq 0,
\]
whose proof can be found in [13] (Proposition 2.25). From [1] (Proposition 12.16), we know that $\varphi_{a}$ is a convex function on $\mathfrak{u}(R)$ and it can be approximated by a family of smooth convex ad-invariant functions $\varphi_{a, \rho}$ as $\rho \to 0$. Integrating (2.42) gives that $t \to HYM_{\alpha,N}(A_{\epsilon}(t), \hat{M}, \omega_{\epsilon})$ is nonincreasing along the Yang-Mills flow, for any $0 < \epsilon \leq 1$. Since $H_{\epsilon}(x, t)$ converges to the long time solution $H(x, t)$ of the Hermitian-Yang-Mills flow (1.2) outside $\Sigma_M$ in $C^\infty_{\text{loc}}$-topology as $\epsilon \to 0$, and $|\Lambda_{\omega} F_{A_{\epsilon}(t)}|_{\hat{M}}$ is uniformly bounded for any $0 < \epsilon \leq 1$ and $0 < t_{\infty} \leq t$, it is easy to see that $HYM_{\alpha,N}(A_{\epsilon}(t), \hat{M}, \omega_{\epsilon}) \to HYM_{\alpha,N}(A(t), M, \omega)$ as $\epsilon \to 0$ and $t \to HYM_{\alpha,N}(A(t), M, \omega)$ is also nonincreasing. So we obtain the following lemma.

**Lemma 2.5.** Let $A(t)$ be the long time solution of the Yang-Mills flow (2.22) on the Hermitian vector bundle $(E_{\hat{M}} \delta \Sigma, \hat{M}, \omega)$, then $t \to HYM_{\alpha,N}(A(t), M, \omega)$ is nonincreasing.

Clearly Fatou’s lemma tells us
\[
4\pi^2 \int_M (2c_2(\mathcal{E}) - c_1(\mathcal{E}) \wedge c_1(\mathcal{E})) \wedge \frac{\omega_{\epsilon}^{n-2}}{(n-2)!} \leq \lim_{\epsilon \to 0} 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega_{\epsilon}^{n-2}}{(n-2)!} \leq \lim_{\epsilon \to 0} \int_M (|F_{A_{\epsilon}(t)}|^2_{\hat{M}, \omega_{\epsilon}} - |\Lambda_{\omega} F_{A_{\epsilon}(t)}|^2_{\hat{M}}) \frac{\omega_{\epsilon}^n}{n!} \leq \int_M (|F_{A(t)}|^2_{\hat{M}, \omega} \frac{\omega^n}{n!} - \int_M (|\sqrt{-1}\Lambda_{\omega} F_{A_{\epsilon}(t)}|^2_{\hat{M}} \frac{\omega^n}{n!}).
\]
and then it holds that
\[
\int_{M \setminus \Sigma_\epsilon} |F_{A(t)}|^2 \frac{\omega^n}{n!} \leq \int_{M \setminus \Sigma_\epsilon} \left| \sqrt{-1} \Lambda \omega F_{A(t)} \right|^2 \frac{\omega^n}{n!} + 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!} + 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!},
\]
for all \(0 < t_0 \leq t\). For simplicity, in the sequel we set
\[
HYM(A(t_0)) = \int_{M \setminus \Sigma_\epsilon} |\sqrt{-1} \Lambda \omega (F_{A(t_0)}) - \lambda \text{Id}|^2 \frac{\omega^n}{n!} + \lambda^2 \text{rank} E \int_{M \setminus \Sigma_\epsilon} \frac{\omega^n}{n!} + 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]
Let \(f\) be a smooth function with support in \(M \setminus \Sigma_\epsilon\), we have
\[
\frac{d}{dt} \left( \int_M f^2 |F_{A(t)}|^2 \frac{\omega^n}{n!} \right) = -2 \int_M f^2 \left( \left| \frac{\partial A(t)}{\partial t} \right|^2 \right) \omega^n \frac{n!}{n!} + 2 \text{Re} \int_M \left( \sqrt{-1} \Lambda \omega ((\overline{\partial} - \partial)(f^2) \wedge (F_{A(t)})), \frac{dA(t)}{dt} \right) \frac{\omega^n}{n!} - 2 \text{Re} \int_M \left( \sqrt{-1} \Lambda \omega F_{A(t)} (\overline{\partial} - \partial)(f^2), \frac{dA(t)}{dt} \right) \frac{\omega^n}{n!}.
\]
Integrating over \([s, \tau]\) with respect to \(t\) on both sides of (2.46) and using the inequality (2.44), we deduce the following local energy estimate.

**Lemma 2.6. (Lemma 5 in [22])** Let \(A(t)\) be the long time solution of the Yang-Mills flow (2.22) on the Hermitian vector bundle \((\mathcal{E}|_{M \setminus \Sigma_\epsilon}, H)\). For any \(x_0\) with \(B_{2R}(x_0) \subset M \setminus \Sigma_\epsilon\) and for any two finite numbers \(s, \tau \geq t_0 > 0\), we have
\[
\int_{B_{2R}(x_0)} |F_{A(t)}|^2 \omega^n \left( s, \tau \right) \leq \int_{B_{2R}(x_0)} |F_{A(t)}|^2 \omega^n \left( s, \tau \right) + 2 \int_{\text{min} \{s, \tau\}}^{\text{max} \{s, \tau\}} \int_{B_{2R}(x_0)} \left( \left| \frac{\partial A(t)}{\partial t} \right|^2 + \left| \frac{\partial \phi}{\partial t} \right|^2 \right) \omega^n dt,
\]
where \(C\) is a uniform constant.

### 3. The Limit Behaviour of the Yang-Mills Flow

In this section, we consider the limit behaviour of the Yang-Mills flow (2.22) on the Hermitian bundle \((\mathcal{E}|_{M \setminus \Sigma_\epsilon}, \hat{H})\). We first recall the monotonicity inequality and the \(\varepsilon\)-regularity theorem obtained by Hong and Tian in [22]. For a fixed point \(u_0 = (x_0, t_0) \in M \times \mathbb{R}_+\), denote
\[
T_r(x_0, t_0) = \{ u = (x, t) : t_0 - 4r^2 < t < t_0 - r^2, x \in X \},
\]
\[
P_r(u_0) = B_r(x_0) \times [t_0 - r^2, t_0 + r^2].
\]
The fundamental solution of (backward) heat equation with singularity at \((z_0, t_0) \in C^n \times \mathbb{R}_+\) is

\[
\tilde{G}_{(z_0,t_0)}(z,t) = \frac{1}{(4\pi(t_0-t))^n} \exp\left(-\frac{|z-z_0|^2}{4(t_0-t)}\right), \quad (t < t_0).
\]  

(3.2)

Denote the exponential map centered at \(x_0 \in (M, \omega)\) by \(\exp_{x_0}\), and set

\[
G_{u_0}(x,t) = \tilde{G}_{(0,t_0)}(\exp^{-1}_{x_0}(x), t).
\]

(3.3)

In the following, we denote \(d_{x_0} = \min\{\text{dist}(x_0, \Sigma_\mathcal{E}), i(M)\}\), where \(\text{dist}(x_0, \Sigma_\mathcal{E})\) is the distance from \(x_0\) to the closed set \(\Sigma_\mathcal{E}\), \(i(M)\) is the injective radius of \((M, \omega)\). Let \(\varphi_{x_0} \in C_0^\infty(B_{d_{x_0}}(x_0))\) be a cut-off function such that \(\varphi_{x_0} \equiv 1\) on \(B_{d_{x_0}/2}(x_0)\), \(\varphi_{x_0} \equiv 0\) outside \(B_{d_{x_0}}(x_0)\) and \(|\nabla \varphi_{x_0}| \leq 4/d_{x_0}\). Let \(A(t)\) be the long time solution of the Yang-Mills flow \([2,12]\) on the Hermitian vector bundle \((\mathcal{E}|_M, \Sigma_\mathcal{E}, \mathring{H})\) with initial value \(\mathring{A}\). Set

\[
\Phi(r; A) = r^2 \int_{T_r(x_0,t_0)} \varphi_{x_0}^2 G_{u_0} |F_{A(t)}|^2_{H, \omega} \frac{\omega_n}{n!} dt.
\]

(3.4)

The same argument in \([22]\), only replacing the energy inequality by the above inequality (3.4), concludes the following monotonicity inequalities.

**Theorem 3.1. (Theorem 2 and 2’ in [22])** Let \(A(t)\) be the long time solution of the Yang-Mills flow \([2,12]\) with initial connection \(\mathring{A}\) on \((M \setminus \Sigma_\mathcal{E}, \mathcal{E}|_{M \setminus \Sigma_\mathcal{E}}, \omega)\). Then for any fixed \(t^* > 0\), \(u_0 = (x_0, t_0) \in M \setminus \Sigma_\mathcal{E} \times [t^*, T]\), and for \(r_1\) and \(r_2\) with \(0 < r_1 \leq r_2 < \min\{d_{x_0}, \sqrt{t_0 - t^*}/2\}\), we have

\[
\Phi(r_1; A) \leq C \exp(C(r_2 - r_1)) \Phi(r_2; A) + C(r_2^2 - r_1^2) \text{HYM}(A(t^*)),
\]

where \(C\) is a positive constant which depends only on \(\text{dist}^{-1}(x_0, \Sigma_\mathcal{E})\) and the geometry of \((M, \omega)\). Furthermore, if \(R \leq d_{x_0}\) and \(f_{x_0,R} \in C_0^\infty(B_R(x_0))\) is a cut-off function satisfying \(0 \leq f_{x_0,R} \leq 1\), \(f_{x_0,R} \equiv 1\) on \(B_R/2(x_0)\), \(|\nabla f_{x_0,R}| \leq 8/R\) on \(B_R(x_0) \setminus B_{R/2}(x_0)\), then we have

\[
r^2_1 \int_{T_{r_1}(x_0,t_0)} |F_{A(t)}|^2_{H, \omega} f_{x_0,R} G_{u_0} \omega_n/n! dt \leq C \exp(C(r_2 - r_1)) r^2_2 \int_{T_{r_2}(x_0,t_0)} |F_{A(t)}|^2_{H, \omega} f_{x_0,R} G_{u_0} \omega_n/n! dt + C (r_2^2 - r_1^2) \text{HYM}(A(t^*))
\]

(3.6)

for any \(0 < r_1 \leq r_2 \leq \min\{R/2, \sqrt{t_0 - t^*}/2\}\), where \(C\) is a positive constant depending only on the geometry of \((M, \omega)\).

Using the above monotonicity inequality (3.6), Hong and Tian obtain the following \(\varepsilon\)-regularity theorem.

**Theorem 3.2. (Theorem 4’ in [22])** Let \(A(t)\) be the long time solution of the Yang-Mills flow \([2,12]\) with initial connection \(\mathring{A}\) on \((M \setminus \Sigma_\mathcal{E}, \mathcal{E}|_{M \setminus \Sigma_\mathcal{E}}, \omega)\), and \(t^*\) be a positive number. There exist positive constants \(\varepsilon_0, \delta_0 < 1/4\) such that for any \(x_0 \in M \setminus \Sigma_\mathcal{E}\), if it holds that

\[
R^{2-2n} \int_{P_{r_0}(x_0,t_0)} |F_{A(t)}|^2_{H, \omega} \frac{\omega_n}{n!} dt \leq \varepsilon_0,
\]

(3.7)

where \(0 < R < \min\{d_{x_0}, \sqrt{t_0 - t^*}/2\}\), then for any \(\delta \in (0, \delta_0)\), we have

\[
\sup_{P_{r_0}(x_0,t_0)} |F_{A(t)}|^2_{H, \omega} \leq C(\delta R)^{-4},
\]

(3.8)

where \(C\) depends only on the geometry of \((M, \omega)\), \(\delta_0^{-1}\) and \(\text{HYM}(A(t^*))\).
Using the above $\varepsilon$-regularity theorem, we can analyze the limiting behavior of the Yang-Mills flow \( \frac{\partial A}{\partial t} \) on \( (M \setminus \Sigma_E, \mathcal{E}|_{M \setminus \Sigma_E}, \omega) \). We will modify Tian’s argument (Proposition 3.1.2 in \cite{Tian2}) and Hong-Tian’s argument (Proposition 6 in \cite{HongTian}) to be suitable for the non-compact case.

**Theorem 3.3.** Let \( A(t) \) be the long time solution of the Yang-Mills flow \( \frac{\partial A}{\partial t} \) on the Hermitian vector bundle \( (\mathcal{E}|_{M \setminus \Sigma_E}, H) \) over \( (M \setminus \Sigma_E, \omega) \). Then for every sequence \( t_k \to +\infty \), there exists a subsequence \( \{t_j\} \) such that as \( t_j \to +\infty \), \( A(t_j) \) converges, modulo gauge transformations, to a solution \( A_\infty \) of the Yang-Mills equation on a Hermitian vector bundle \( (E_\infty, H_\infty) \) in \( C^\infty_{0\text{loc}} \)-topology outside \( \Sigma \subset M \), where \( \Sigma \) is a closed set of Hausdorff complex codimension at least 2 and \( \Sigma \subset \Sigma_E \subset \Sigma_E \).

**Proof.** By Proposition \ref{2.4} we know that \( \int_M \partial_A^n \frac{\partial A}{\partial t}^2 \frac{\omega^n}{n!} dt \to 0 \) as \( t \to +\infty \), and then

\[
\int_{t_k-a}^{t_k+a} \int_M \partial_A^n \frac{\partial A}{\partial t}^2 \frac{\omega^n}{n!} dt \to 0, \tag{3.9}
\]

as \( t_k \to +\infty \), for any \( a > 0 \). Choosing \( r_0 \) small enough and assuming that

\[
r_0^{4-2n} \int_{B_{r_0}(x_0)} |F_{A(t_k)}|^2\frac{\omega^n}{n!} < \varepsilon_1, \tag{3.10}
\]

where \( \varepsilon_1 \) is determined later. Using the local energy estimate \( \eqref{2.17} \) gives us that \( \forall t_k - r_0^2 \leq s \leq t_k + r_0^2 \), it holds that

\[
\int_{B_{r_0/2}(x_0)} |F_{A(s)}|^2\frac{\omega^n}{n!} \leq \int_{B_{r_0}(x_0)} |F_{A(t_k)}|^2\frac{\omega^n}{n!} + 2 \int_{t_k-r_0^2}^{t_k+r_0^2} \int_{B_{r_0}(x_0)} |\partial_A|^2\frac{\omega^n}{n!} dt \tag{3.11}
\]

\[
+ C(\text{HYM}(A(t_0))) \int_{t_k-r_0^2}^{t_k+r_0^2} \int_M |\partial_A|^2\frac{\omega^n}{n!} dt \bigg( \frac{\omega^n}{n!} \bigg)^{\frac{1}{2}}.
\]

and

\[
\int_{t_k-r_0^2}^{t_k+r_0^2} \int_{B_{r_0/2}(x_0)} |F_{A(s)}|^2\frac{\omega^n}{n!} ds \leq \frac{1}{2} r_0^2 \left( \int_{B_{r_0}(x_0)} |F_{A(t_k)}|^2\frac{\omega^n}{n!} + 2 \int_{t_k-r_0^2}^{t_k+r_0^2} \int_{B_{r_0}(x_0)} |\partial_A|^2\frac{\omega^n}{n!} dt \tag{3.12}
\]

\[
+ C(\text{HYM}(A(t_0))) \int_{t_k-r_0^2}^{t_k+r_0^2} \int_M |\partial_A|^2\frac{\omega^n}{n!} dt \bigg( \frac{\omega^n}{n!} \bigg)^{\frac{1}{2}}. \]
Then (3.10) implies that, for sufficiently large $t_k$,
\[
\left(\frac{r_0}{2}\right)^{2\nu-2n} \int_{P_{r_0/2}(x_0,t_k)} |F_{A(t)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} \, dt \\
\leq 2^{2n-3}r_0^{4-2n} \int_{B_{r_0}(x_0)} |F_{A(t_k)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} \, dt \\
+ 2^{2n-2}r_0^{4-2n} \int_{t_k-r_0^2}^{t_k+r_0^2} \int_{B_{r_0}(x_0)} \frac{\partial A}{\partial t} \left| \frac{\omega^n}{n!} \right| \, dt \\
+ 2^{n-3}r_0^{4-2n} \left(\text{HYM}(A(t_0))\right) \int_{t_k-r_0^2}^{t_k+r_0^2} \int_{M \setminus \Sigma} \left| \frac{\partial A}{\partial t} \right| \left(\frac{\omega^n}{n!}\right)^{\frac{1}{2}} \\
\leq 2^{2n-1} \varepsilon_1,
\]
where we choose $2^{2n-1} \varepsilon_1 \leq \varepsilon_0$ and $\varepsilon_0$ is the constant determined in Theorem 3.2. Therefore, we obtain
\[
\sup_{P_{3r_0/2}(x_0,t_k)} |F_{A(t)}|^2 \delta_{\omega \omega} \leq C(\delta r_0)^{-4}
\]
for any $\delta \in (0, \delta_0)$ and sufficiently large $k$, where $C$ is a uniform constant.

Applying (3.11), (3.14) and Moser’s parabolic estimate to the following inequality
\[
(\Delta - \frac{\partial}{\partial t})|F_{A(t)}|^2 \delta_{\omega \omega} \geq -C(|F_{A(t)}| \delta_{\omega \omega} + |Rm(\omega)|\omega)|F_{A(t)}|^2 \delta_{\omega \omega},
\]
we derive for sufficiently large $k$,
\[
\sup_{(x,t) \in P_{\frac{3r_0}{2}}(x_0,t_k)} |F_{A(t)}|^2 \delta_{\omega \omega} (x) \leq C(\delta r_0)^{-4-2n} \int_{P_{\frac{3r_0}{2}}(x_0,t_k)} |F_{A(t)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} \, dt \\
\leq C(\delta r_0)^{-2-2n} \sup_{t_k-(\frac{4n}{3}) \leq t \leq t_k+\frac{(4n-2)}{3}} \int_{B_{2r_0}(x_0)} |F_{A(t)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} \\
\leq C(\delta r_0)^{-2-2n} \int_{B_{2r_0}(x_0)} |F_{A(t)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} + C(\delta r_0)^{-2-2n} \varepsilon_1 \\
\leq C(\delta r_0)^{-2-2n} (\delta r_0)^{-6} + (\delta r_0)^{-2-2n} \varepsilon_1,
\]
and then
\[
(\delta r_0)^{-4-2n} \int_{B_{2r_0}(x_0)} |F_{A(t_k)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} \\
\leq (\delta r_0)^{-4-2n} \text{Vol}(B_{\delta r_0}(x_0))C(\delta r_0)^{-4-2n} (\delta r_0)^{-6} + (\delta r_0)^{-2-2n} \varepsilon_1 \\
\leq \varepsilon_1,
\]
where $\delta \leq r_0^{n+1}$ and $r_0$ is small enough. Setting
\[
r_i = (r_{i-1})^{n+2} = r_0^{(n+2)i},
\]
and repeating the above argument, we know that (3.10) implies
\[
(r_i)^{4-2n} \int_{B_{r_i}(x_0)} |F_{A(t_k)}|^2 \delta_{\omega \omega} \frac{\omega^n}{n!} < \varepsilon_1,
\]
for $i \geq 1$ and sufficiently large $t_k$.

We set
\[
d_x = \text{dist}(x, \Sigma \varepsilon), \quad U_d = \{x \in M : d_x < d\},
\]
\[ \tilde{\Sigma}_{k,j,i} = \{ x \in M \setminus U_{r_j} : r_j^{4-2n} \int_{B_{r_j}(x)} |F_A(t_k)|^2_{H,\omega} \frac{\omega^n}{n!} \geq \varepsilon_1 \}, \tag{3.21} \]

for any \( k \geq 1 \) and \( i \geq j \geq 1 \). By the standard diagonal process, we can choose a subsequence which also is denoted by \( \{ t_k \} \) such that for each \( j \leq i \), \( \tilde{\Sigma}_{k,j,i} \) converges to a closed subset \( \Sigma_{j,i} \) as \( k \to +\infty \). From (3.19), it is easy to see that \( \Sigma_{j,i_1} \subset \Sigma_{j,i_2} \) for \( i_1 \geq i_2 \). Define

\[ \Sigma_j = \bigcap_i \Sigma_{j,i}, \Sigma_{an} = \bigcup_j \Sigma_j, \quad \Sigma = \Sigma \varepsilon \bigcup \Sigma_{an}. \tag{3.22} \]

**Claim 3.1.** \( \Sigma \) is closed.

**Proof.** Suppose \( x_0 \in M \setminus \Sigma \) and set \( d_0 = dist(x_0, \Sigma \varepsilon) > 0 \). For any \( r_j < d_0 \), we have \( x_0 \notin \Sigma_j \), \( x_0 \notin \Sigma_{j,i} \) for \( i \) sufficiently large, and \( x_0 \notin \tilde{\Sigma}_{k,j,i} \) for \( t_k \) sufficiently large. Then it follows that

\[ \liminf_{k \to \infty} r_j^{4-2n} \int_{B_{r_j}(x_0)} |F_A(t_k)|^2_{H,\omega} \frac{\omega^n}{n!} < \varepsilon_1, \tag{3.23} \]

for \( i \) and \( k \) sufficiently large. Together with (3.19), fixing small \( r_{i_0} \), for any \( x \in B_{\frac{\delta_0}{2} r_0}(x_0) \subset M \setminus \Sigma \varepsilon \), we get

\[ r_j^{4-2n} \int_{B_{r_j}(x)} |F_A(t_k)|^2_{H,\omega} \frac{\omega^n}{n!} < \varepsilon_1, \tag{3.24} \]

when \( r \) is small enough and \( k \) is large enough. Clearly (3.24) implies that, for \( k \) and \( i \) sufficiently large, \( B_{\frac{\delta_0}{2} r_0}(x_0) \cap \{ \Sigma \varepsilon \cup \tilde{\Sigma}_{k,j,i} \} = \emptyset \), and

\[ B_{\frac{\delta_0}{2} r_0}(x_0) \cap \{ \Sigma \varepsilon \cup \Sigma_{j,i} \} = \emptyset \tag{3.25} \]

for all \( j \). Then \( B_{\frac{\delta_0}{2} r_0}(x_0) \cap \{ \Sigma \varepsilon \cup \Sigma_j \} = \emptyset \) for all \( j \), this means that \( B_{\frac{\delta_0}{2} r_0}(x_0) \subset M \setminus \Sigma \) and concludes the proof of Claim 3.1.

**Claim 3.2.** The Hausdorff codimension of \( \Sigma \) is at least 4.

**Proof.** Since the sheaf \( \varepsilon \) is torsion-free, it is well known that the Hausdorff codimension of \( \Sigma \varepsilon \) is at least 4 and the \((2n-4)\) dimensional Hausdorff measure is finite (i.e. \( H^{2n-4}(\Sigma \varepsilon) < +\infty \)). The definition says that

\[ H^{2n-4}_\delta(\Sigma \varepsilon) = \inf \{ \sum_{\alpha} (r_\alpha)^{2n-4} | \cup\alpha B_{x_\alpha}(r_\alpha) \supset \Sigma \varepsilon, r_\alpha < \delta \}, \tag{3.26} \]

\[ H^{2n-4}(\Sigma \varepsilon) = \lim_{\delta \to 0} H^{2n-4}_\delta(\Sigma \varepsilon). \tag{3.27} \]

Because \( H^{2n-4}_\delta \) is monotonically nonincreasing with respect to \( \delta \), \( H^{2n-4}_\delta(\Sigma \varepsilon) \leq H^{2n-4}(\Sigma \varepsilon) \) for all \( \delta \). Since \( \Sigma \varepsilon \) is compact, for an arbitrary \( \delta_0 > 0 \), there exists a finite cover of \( \Sigma \varepsilon \), \( \{ B_{R_\alpha}(x_\alpha) \} \), such that \( \sum_{\alpha} (R_\alpha)^{2n-4} < H^{2n-4}(\Sigma \varepsilon) + 1 \) and \( R_\alpha < \delta_0 \). Then we can find a positive number \( \delta' < \delta_0 \) such that \( M \setminus \cup\alpha B_{R_\alpha}(x_\alpha) \subset M \setminus U_{\delta'} \). So it follows that \( \Sigma \cap (M \setminus \cup\alpha B_{R_\alpha}(x_\alpha)) \subset \Sigma \cap (M \setminus U_{\delta'}) \) and \( \Sigma \cap (M \setminus U_{\delta'}) \) is closed. Set

\[ \tilde{\Sigma}_{\delta'} = \Sigma \cap (M \setminus U_{\delta'}) = \Sigma_{an} \cap (M \setminus U_{\delta'}). \tag{3.28} \]

Suppose that \( r_i \leq \frac{1}{2} \delta' < r_{i-1} \). Let \( r = r_i > 0 \), we can find a finite collection of geodesic balls \( \{ B_{x_\beta}(r_\beta) \} \) such that \( \{ B_{x_\beta}(r_\beta) \} \) is a cover of \( \tilde{\Sigma}_{\delta'} \), \( x_\beta \in \tilde{\Sigma}_{\delta'} \) for all \( i \), and \( B_{2r_\beta}(x_\beta) \cap B_{2r_\beta}(x_\beta) = \emptyset \) for \( \beta_1 \neq \beta_2 \). For every point \( x_\beta \), suppose that \( x_\beta \in \Sigma \) and take \( i \) large enough such that \( r_i < r \), then for \( k \) sufficiently large, there are \( y_\beta \in \Sigma_{k,j,i} \) such that \( dist(x_\beta, y_\beta) < r \). It is easy to see that \( \{ B_{r_\beta}(y_\beta) \} \) is a finite covering of \( \tilde{\Sigma}_{\delta'} \) and \( B_{r}(y_{\beta_1}) \cap B_{r}(y_{\beta_2}) = \emptyset \) for \( \beta_1 \neq \beta_2 \).
Choosing sufficiently large \( k, y_\beta \in \hat{\Sigma}_{k,j,t} \), we know

\[
(r_i \hat{\omega}_i)^{4-2n} \int_{B_{r_i}(y_\beta)} |F_{A(t_k)}|^2_{\hat{H},\omega} \frac{\omega^n}{n!} \geq \varepsilon_1,
\]

(3.29)

for every \( \beta \). Repeating the argument in the proof of (3.19) yields

\[
(r_i \hat{\omega}_i)^{4-2n} \int_{B_{r_i}(y_\beta)} |F_{A(t_k)}|^2_{\hat{H},\omega} \frac{\omega^n}{n!} \geq \varepsilon_1,
\]

(3.30)

for every \( \beta \). Summing over \( \beta \) and using the inequality (2.44), we get

\[
\sum_\beta \varepsilon_1 < \sum_\beta (r_i \hat{\omega}_i)^{4-2n} \int_{B_{r_i}(y_\beta)} |F_{A(t_k)}|^2_{\hat{H},\omega} \frac{\omega^n}{n!} \leq (r_i \hat{\omega}_i)^{4-2n} \int_M |F_{A(t_k)}|^2_{\hat{H},\omega} \frac{\omega^n}{n!}
\]

(3.31)

and then

\[
\sum_\alpha R_\alpha^{2n-4} + \sum_\beta (5\beta)^{2n-4} < H^{2n-4}(\Sigma_\beta) + 1 + \frac{1}{\varepsilon_1} 5^{2n-4} \text{HYM}(A(t_1)).
\]

(3.32)

It implies that

\[
H^{2n-4}(\Sigma) < H^{2n-4}(\Sigma_\beta) + 1 + \frac{1}{\varepsilon_1} 5^{2n-4} \text{HYM}(A(t_1)).
\]

(3.33)

Letting \( \delta_0 \to 0 \), we obtain \( H^{2n-4}(\Sigma) \) is finite. This concludes the proof of Claim 3.2.

Given a compact subset \( \Omega \subset M \setminus \Sigma \), we suppose \( \Omega \subset M \setminus U_{d_0} \) for some \( d_0 > 0 \). For any point \( x_0 \in \Omega \), as that in the proof of Claim 3.2 we know that, there is \( r_{i_0} \) such that

\[
\sup_{B_{r_{i_0}}(x_0)} |F_{A(t_k)}|^2_{\hat{H},\omega} \leq C(\delta_0 r_{i_0})^{-4}
\]

(3.34)

for \( t_k \) sufficiently large. Since \( \Omega \) is compact, we can cover it by a finite union of balls such that every ball satisfies the above estimate (3.34). So it follows that \( \sup_\Omega |F_{A(t_k)}|^2_{\hat{H},\omega} \) is uniformly bounded. Uhlenbeck’s Theorem (Theorem 3.6 in [13]) implies that there exists a subsequence of \( \{ A(t_k) \} \), modulo gauge transformations, converging to a connection \( A_\infty \) weakly in \( L^2_{1,\text{loc}} \) topology outside \( \Sigma \), where \( A_\infty \) is a solution of the Yang-Mills equation on a Hermitian vector bundle \( (E_\infty, H_\infty) \) which is isometric to \( (\mathcal{E}|_M, \mathcal{H}) \) outside \( \Sigma \). Furthermore, by standard parabolic regularity techniques and using Hong-Tian’s argument (Proposition 6 in [22]), we know that \( A(t_k) \) converges to \( A_\infty \) in \( C^\infty_{\text{loc}} \)-topology outside \( \Sigma \). This concludes the proof of Theorem 3.3.

\[ \square \]

From the estimates (2.16) and (2.17), we see that \( |\theta(A(t), \omega)|_{\hat{H}} \) is uniformly bounded for \( t \geq t_0 > 0 \). Through the same argument as that in Corollary 2.12 in [13] (or Corollary 3.12 in [30]), we have the following corollary.

**Corollary 3.4.** Let \( A(t_k) \) be a sequence of connections along the Yang-Mills flow (2.22) with the limit \( A_\infty \), then:

1. \( \theta(A(t_k), \omega) \to \theta(A_\infty, \omega) \) strongly in \( L^p \) as \( k \to +\infty \) for all \( 1 \leq p < \infty \), and consequently

\[
\lim_{t \to +\infty} \int_M |\theta(A(t), \omega)|^2_{\hat{H},\omega} \frac{\omega^n}{n!} = \int_M |\theta(A_\infty, \omega)|^2_{\hat{H},\omega} \frac{\omega^n}{n!},
\]

(3.35)
(2) \( \| \theta(A_\infty, \omega) \|_{L^\infty} \leq \| \theta(A(t_k), \omega) \|_{L^\infty} \leq \| \theta(A(t_0), \omega) \|_{L^\infty} \) for \( 0 < t_0 \leq t_k \).

In the sequel, we call \( A_\infty \) an Uhlenbeck limit of \( A(t) \). Since \( A_\infty \) is a solution of the Yang-Mills equation, i.e. it satisfies
\[
D_{A_\infty}^* F_{A_\infty} = 0,
\]
by the Kähler identity, we have
\[
D_{A_\infty} \theta(A_\infty, \omega) = 0,
\]
i.e. \( \theta(A_\infty, \omega) \) is parallel. On the other hand, \( (\sqrt{-1} \theta(A_\infty, \omega))^* H_\infty = \sqrt{-1} \theta(A_\infty, \omega) \), we can decompose \( E_\infty \) according to the eigenvalues of \( \sqrt{-1} \theta(A_\infty, \omega) \) and obtain a holomorphic orthogonal decomposition: \( E_\infty = \bigoplus_{i=1}^t E_i^\infty \) on \( M \setminus \Sigma \). Let \( \lambda_i \) be the eigenvalues of \( \sqrt{-1} \theta(A_\infty, \omega) \), \( H_i^\infty \) be the restrictions of \( H_\infty \) to \( E_i^\infty \) and \( A_i^\infty = A_\infty |_{E_i} \), it is easy to see that \( A_i^\infty \) is a Hermitian-Einstein connection on \( (E_i^\infty, H_i^\infty) \), i.e. it satisfies
\[
\sqrt{-1} \Lambda \omega F_{A_i^\infty} = \lambda_i \text{Id}_{E_i^\infty}.
\]
Of course (2.41) means that
\[
\int_{M \setminus \Sigma} |F_{A_\infty}|^2_{H_\infty} \frac{\omega^n}{n!} \leq C < \infty.
\]
Since the singularity set \( \Sigma \) is of Hausdorff codimension at least 4, by Theorem 2 in Bando and Siu’s paper [7], we know that every \( (E^i_\infty, \overline{\partial}_{A_i^\infty}) \) can be extended to the whole \( M \) as a reflexive sheaf (which is also denoted by \( (E^i_\infty, \overline{\partial}_{A_i^\infty}) \) for simplicity), and \( H_i^\infty \) can be smoothly extended over the place where the sheaf \( (E_\infty, \overline{\partial}_{A_\infty}) \) is locally free. Therefore, we deduce the following proposition.

**Proposition 3.5.** The limiting \( (E_\infty, \overline{\partial}_{A_\infty}) \) can be extended to the whole \( M \) as a reflexive sheaf with a holomorphic orthogonal splitting
\[
(E_\infty, \overline{\partial}_{A_\infty}, H_\infty) = \bigoplus_{i=1}^t (E_i^\infty, \overline{\partial}_{A_i^\infty}, H_i^\infty),
\]
and \( H_i^\infty \) is an admissible Hermitian-Einstein metric on the reflexive sheaf \( (E_i^\infty, \overline{\partial}_{A_i^\infty}) \) for any \( 1 \leq i \leq t \).

4. \( L^p \)-approximate critical Hermitian metric

In this section, we first recall the Harder-Narasimhan-Seshadri filtration of reflexive sheaves ([26], v.7.15, 7.17, 7.18; or [5], section 7). Then we prove the existence of \( L^p \)-approximate critical Hermitian metric. We will modify Daskalopoulos and Wentworth’s cut-off argument ([13]) and Sibley’s trick ([38]) to be suitable for the reflexive sheaf case.

Let \( \mathcal{E} \) be a reflexive sheaf over a compact Kähler manifold \( (M, \omega) \). If \( \mathcal{E} \) is not \( \omega \)-stable, there is a filtration of \( \mathcal{E} \) by coherent sub-sheaves
\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E},
\]
such that the quotients \( Q_j = \mathcal{E}_j / \mathcal{E}_{j-1} \) are torsion-free, \( \omega \)-semi-stable and \( \mu_\omega(Q_j) > \mu_\omega(Q_{j+1}) \). We call it the Harder-Narasimhan filtration (abbr. HN-filtration) of \( \mathcal{E} \). The associated graded sheaf \( Gr^{HN}(\mathcal{E}) = \bigoplus_{j=1}^k Q_j \) is uniquely determined by the isomorphism class of \( \mathcal{E} \) and the Kähler class \( [\omega] \).
**Definition 4.1.** For a reflexive sheaf \( E \) of rank \( R \) over a compact Kähler manifold \((M,\omega)\), construct a nonincreasing \( R \)-tuple of numbers

\[
\bar{\mu}(E) = (\mu_1, \omega, \cdots, \mu_R, \omega)
\]  
(4.2)
from the HN-filtration by setting: \( \mu_i, \omega = \mu_{\omega}(Q_j) \), for \( \text{rank}(E_{i-1}) + 1 \leq i \leq \text{rank}(E_j) \). We call \( \bar{\mu}(E) \) the Harder-Narasimhan type of \( E \).

**Remark:** For a pair \( \bar{\mu}, \bar{\lambda} \) of \( R \)-tuple’s satisfying \( \mu_1 \geq \cdots \geq \mu_R, \lambda_1 \geq \cdots \geq \lambda_R \), and \( \sum_{i=1}^R \mu_i = \sum_{i=1}^R \lambda_i \), we define:

\[
\bar{\mu} \leq \bar{\lambda} \iff \sum_{i \leq k} \mu_i \leq \sum_{i \leq k} \lambda_i, \quad \text{for all } k = 1, \cdots, R.
\]  
(4.3)

Moreover, for every \( \omega \)-semistable quotient sheaf \( Q_j \), there is a further filtration, which is called by the Seshadri filtration, by subsheaves

\[
0 = E_{j,0} \subset E_{j,1} \subset \cdots \subset E_{j,k_j} = Q_j,
\]  
(4.4)
such that the quotients \( Q_{j,a} = E_{j,a}/E_{j,a-1} \) are torsion-free and \( \omega \)-stable, \( \mu_{\omega}(Q_{j,a}) = \mu_{\omega}(Q_j) \) for each \( \alpha \). We call this double filtration \( \{E_{j,a}\} \) the Harder-Narasimhan-Seshadri filtration (abbr. HNS-filtration) of the sheaf \( E \). The associated graded sheaf: \( Gr^{HNS}(E) = \bigoplus_{j=1}^k \bigoplus_{\alpha=1}^{k_j} Q_{j,a} \) is uniquely determined by the isomorphism class of \( E \) and the Kähler class \([\omega]\).

In the following, we denote the Harder-Narasimhan-Seshadri filtration (or HNS-filtration) of \( E \) simply by:

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E,
\]  
(4.5)
where each \( E_i \) is a saturated subsheaf of \( E \). Set

\[
\Sigma_{HNS} = \bigcup_{i=1}^l (\Sigma_{Q_i} \cup \Sigma_E),
\]  
(4.6)
and refer to it as the singularity set of the HNS-filtration, where \( Q_i = E_i/E_{i-1} \) for each \( 1 \leq i \leq l \).

Since every \( Q_i \) is torsion-free, it is well known that \( \Sigma_{HNS} \) is a complex analytic subset of complex codimension at least two.

By Hironaka’s flattening theorem ([19] or [7]), there is a finite sequence of blowing ups along compact sub-manifolds such that, if we denote by \( \pi : \tilde{M} \to M \) the composition of all the blowing ups, then \( \pi^*E/\text{tor}(\pi^*E) \) is locally free.

**Proposition 4.2.** Let \( E = \pi^*E/\text{tor}(\pi^*E) \), then we can get a filtration \( \tilde{E} = \{\tilde{E}_i\}_{i=1}^l \) of \( E \) from the HNS-filtration of \( E \):

\[
0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = E_{\tilde{M}}
\]  
(4.7)
such that, for every \( 1 \leq i \leq l \), \( \tilde{E}_i \) is a reflexive sheaf, \( \tilde{Q}_i = \tilde{E}_i/\tilde{E}_{i-1} \) is torsion free and isomorphic to the sheaf \( Q_i \) outside \( \pi^{-1}(\Sigma_{HNS}) \). Furthermore, every quotient sheaf \( Q_i \) in the filtration \( \{\tilde{E}_i\} \) is \( \omega_i \)-stable for any \( 0 < \epsilon < \epsilon^* \approx 1 \), and \( \text{lim}_{\epsilon \to 0} \deg_{\omega_\epsilon}(\tilde{Q}_i) = \deg_{\omega}(Q_i) \).

**Proof.** Pulling back the following exact sequences:

\[
0 \to E_i \to E \to Q_i = E/E_i \to 0, \quad \text{over } M,
\]
\[
0 \to E_{i-1} \to E_i \to Q_i = E_i/E_{i-1} \to 0, \quad \text{over } M,
\]
we get
\[
\begin{align*}
\pi^*\mathcal{E}_i &\xrightarrow{f_i} \pi^*\mathcal{E} \rightarrow \pi^*\mathcal{G}_i \rightarrow 0, & \text{over } \tilde{M}, \\
\pi^*\mathcal{E}_{i-1} &\xrightarrow{f_{i-1}} \pi^*\mathcal{E} \rightarrow \pi^*\mathcal{G}_{i-1} \rightarrow 0, & \text{over } \tilde{M}, \\
\pi^*\mathcal{E}_{i-1} &\xrightarrow{g_i} \pi^*\mathcal{E}_i \rightarrow \pi^*\mathcal{Q}_i \rightarrow 0, & \text{over } \tilde{M}.
\end{align*}
\]

Set \( T_i = \text{tor}(\text{Im}f_i) = \text{tor}(\text{Im}\{\pi^*\mathcal{E}_i \rightarrow \pi^*\mathcal{E}\}) \), \( T = \text{tor}(\pi^*\mathcal{E}) \). Then we can obtain the following exact sequence:
\[
0 \rightarrow \text{Im}f_i/T_i \rightarrow \pi^*\mathcal{E}/T \rightarrow \tilde{\mathcal{G}}_i \triangleq \frac{\pi^*\mathcal{E}/T}{\text{Im}f_i/T_i} \rightarrow 0, \quad \text{over } \tilde{M}.
\]

Set \( E = \pi^*\mathcal{E}/T, \tilde{\mathcal{E}}_i = \text{Sat}_E(\text{Im}f_i/T_i) \) and then \( \tilde{\mathcal{E}}_i \) is reflexive. Clearly the definition gives the following exact sequences:
\[
\begin{align*}
0 \rightarrow \tilde{\mathcal{E}}_i \rightarrow E &\xrightarrow{\tilde{g}_i} \tilde{\mathcal{G}}_i \triangleq \frac{\tilde{\mathcal{G}}_i}{\text{tor}(\tilde{\mathcal{G}}_i)} \rightarrow 0, \quad \text{over } \tilde{M}, \\
0 \rightarrow \tilde{\mathcal{E}}_{i-1} \rightarrow E &\xrightarrow{\tilde{g}_{i-1}} \tilde{\mathcal{G}}_{i-1} \triangleq \frac{\tilde{\mathcal{G}}_{i-1}}{\text{tor}(\tilde{\mathcal{G}}_{i-1})} \rightarrow 0, \quad \text{over } \tilde{M}.
\end{align*}
\]

Consider
\[
\begin{array}{ccccccccc}
\pi^*\mathcal{E}_{i-1} &\xrightarrow{g_i} &\pi^*\mathcal{E}_i &\xrightarrow{f_i} &\pi^*\mathcal{E} &\xrightarrow{\pi^*}\pi^*(\mathcal{E}/\mathcal{E}_i) &\xrightarrow{\pi^*}\pi^*(\mathcal{E}/\mathcal{E}_{i-1}) &\xrightarrow{0}.
\end{array}
\]

Of course \( \text{Im}f_{i-1} = \text{Im}(f_i \circ g_i) \) means that \( \text{Im}f_{i-1} \) is a subsheaf of \( \text{Im}f_i \). Hence the following commutative diagram holds (all the horizontal sequences are exact):
\[
\begin{array}{ccccccccc}
0 &\rightarrow & T_{i-1} &\rightarrow & \text{Im}f_{i-1} &\rightarrow & \text{Im}f_{i-1}/T_{i-1} &\rightarrow & 0, \\
0 &\rightarrow & T_i &\rightarrow & \text{Im}f_i &\rightarrow & \text{Im}f_i/T_i &\rightarrow & 0,
\end{array}
\]

where we define the map \( h_i \) by the commutation (it is easy to check that \( h_i \) is well-defined).

Moreover, a simple diagram shows that \( h_i \) is injective. Noting that \( \tilde{\mathcal{E}}_{i-1} = \text{Ker}\{E \rightarrow \tilde{\mathcal{G}}_{i-1}/\text{tor}(\tilde{\mathcal{G}}_{i-1})\} \) and \( \tilde{\mathcal{E}}_i = \text{Ker}\{E \rightarrow \tilde{\mathcal{G}}_i/\text{tor}(\tilde{\mathcal{G}}_i)\} \), considering the following sequences:
\[
\begin{align*}
E &\rightarrow \tilde{\mathcal{G}}_{i-1} \rightarrow \tilde{\mathcal{G}}_{i-1}/\text{tor}(\tilde{\mathcal{G}}_{i-1}) \rightarrow 0, & \text{over } \tilde{M}, \\
E &\rightarrow \tilde{\mathcal{G}}_i \rightarrow \tilde{\mathcal{G}}_i/\text{tor}(\tilde{\mathcal{G}}_i) \rightarrow 0, & \text{over } \tilde{M},
\end{align*}
\]
we can see \( \tilde{\mathcal{E}}_{i-1} \) is a subsheaf of \( \tilde{\mathcal{E}}_i \).
Consider the following commutative diagram (all the vertical and horizontal sequences are exact):

\[
\begin{array}{cccccc}
0 & \rightarrow & \tilde{E}_{i-1} & \rightarrow & E & \rightarrow & \tilde{G}_{i-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \tilde{E}_{i} & \rightarrow & E & \rightarrow & \tilde{G}_{i} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \tilde{E}_{i}/\tilde{E}_{i-1} & \rightarrow & E & \rightarrow & \tilde{G}_{i}/\tilde{G}_{i-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

where we define the map \( p_i \) by the commutation and it is easy to check \( p_i \) is well-defined (moreover, \( p_i \) is surjective). The snake lemma tells us \( \tilde{E}_{i}/\tilde{E}_{i-1} \cong \text{Ker}(p_i) \) and then \( \tilde{Q}_i \) is torsion free (because \( \tilde{G}_{i-1} \) is torsion free). By a similar argument as that in Theorem 4.9, Proposition 4.10, Proposition 4.12 in [38], it is easy to see that \( \lim_{\epsilon \to 0} \deg_\omega(\tilde{Q}_i) = \deg_\omega(Q_i) \), and \( \tilde{Q}_i \) in the filtration (4.7) is \( \omega_\epsilon \)-stable for any \( 0 < |\epsilon| \leq \epsilon^* \ll 1 \) and \( 1 \leq i \leq l \).

\[\Box\]

Let \( H \) be a smooth Hermitian metric on the holomorphic bundle \( E \), and \( \mathcal{F} = \{ F_i \}_{i=1}^l \) be a filtration of \( E \) by saturated subsheaves:

\[ 0 = F_0 \subset F_1 \subset \cdots \subset F_{l-1} \subset F_l = E. \]

For each \( F_i \) and the metric \( H \), we have the associated unitary projection \( \pi_i^H : E \rightarrow E \) onto \( F_i \), where \( \pi_i^H \) is an \( L^2_1 \)-bounded Hermitian endomorphism. For convenience, set \( \pi_0^H = 0 \). Given real numbers \( \mu_1, \ldots, \mu_l \) and a filtration \( \mathcal{F} \), we define an \( L^2_1 \)-bounded Hermitian endomorphism of \( E \) by

\[ \Psi(\mathcal{F}, (\mu_1, \ldots, \mu_l), H) = \sum_{i=1}^l \mu_i (\pi_i^H - \pi_{i-1}^H). \]

(4.8)

The Harder-Narasimhan projection \( \Psi^H(E_i, H) \) is the \( L^2_1 \)-bounded Hermitian endomorphism defined above in the particular case where \( \mathcal{F} \) is the HN-filtration \( \mathcal{F}^H = \{ F_i^H(E) \}_{i=1}^l \) and \( \mu_i = \mu_\omega(F_i^H(E)/F_{i-1}^H(E)) \).

**Definition 4.3.** Fix \( \delta > 0 \) and \( 1 \leq p \leq \infty \). An \( L^p-\delta \)-approximate critical Hermitian metric on a holomorphic bundle \( E \) over a compact Kähler manifold \((M, \omega)\) is a smooth metric \( H \) such that

\[ \left\| \frac{\sqrt{-1}}{2\pi} \Lambda_\omega(F_{A_H}) - \Psi^H_E(E, H) \right\|_{L^p(\omega)} \leq \delta, \]

where \( A_H \) is the Chern connection determined by \((\bar{\partial}_E, H)\).
Now recall the following lemma which was proved by Sibley in [38] (Lemma 5.3).

**Lemma 4.4.** Let \((M, \omega)\) be a compact Kähler manifold of complex dimension \(n\), and \(\pi : \overline{M} \to M\) be a blow-up along a smooth complex sub-manifold \(\Sigma\) of complex co-dimension \(k\) where \(k \geq 2\). Let \(\eta\) be a Kähler metric on \(\overline{M}\), and consider the family of Kähler metrics \(\omega_\eta = \pi^*\omega + \epsilon \eta\), where \(0 < \epsilon \leq 1\). Then for any \(0 \leq \gamma < \frac{1}{n-1}\), we have \(\frac{\eta^n}{\omega^n} \in L^\gamma(\overline{M}, \eta)\), and the \(L^\gamma(\overline{M}, \eta)\)-norm of \(\frac{\eta^n}{\omega^n}\) is uniformly bounded in \(\epsilon\), i.e. there is a positive constant \(C^*\) such that

\[
\int_{\overline{M}} \left(\frac{\eta^n}{\omega^n}\right)^\gamma \frac{n!}{n!} \leq C^*
\]

for all \(\epsilon\).

Fixing a number \(\tilde{\gamma} < \frac{1}{n-1}\), for \(0 < \gamma \ll \tilde{\gamma}\), using the Hölder inequality, we have

\[
\int_{\overline{M}} \left(\frac{\eta^n}{\omega^n_{k,\epsilon}}\right)^\gamma \frac{n!}{n!} \leq \int_{\overline{M}} \left(\frac{\eta^n_{k,\epsilon}}{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}\right)^{1+\gamma} \left(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k\right)^n \frac{n!}{n!} \leq \left(\int_{\overline{M}} \left(\frac{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}{\omega^n_{k,\epsilon}}\right) \left(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k\right)^n \frac{n!}{n!}\right)^{(1+\gamma)\tilde{\gamma}}.
\]

Taking limit \(\epsilon_k \to 0\) in (4.10) and using (4.9) yield

\[
\int_{\overline{M}} \left(\frac{\eta^n_{k,\epsilon}}{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}\right)^\gamma \frac{n!}{n!} = \lim_{\epsilon_k \to 0} \int_{\overline{M}} \left(\frac{\eta^n_{k,\epsilon}}{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}\right)^\gamma \frac{n!}{n!} \leq \lim_{\epsilon_k \to 0} \left(C^*\right)^{1+\gamma} \left(\int_{\overline{M}} \left(\frac{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}{\omega^n_{k,\epsilon}}\right) \left(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k\right)^n \frac{n!}{n!}\right)^{(1+\gamma)\tilde{\gamma}}.
\]

Repeating the argument in (4.10) and taking limit \(\epsilon_i \to 0\) successively, we know that there exists a positive constant \(a^* \ll \frac{1}{n-1}\) such that

\[
\lim_{\epsilon_i \to 0} \cdots \lim_{\epsilon_k \to 0} \int_{\overline{M}} \left(\frac{\eta^n_{k,\epsilon}}{(\pi_k^n \eta_{k-1} + \epsilon_k \eta_k)^n}\right)^\gamma \frac{n!}{n!} \leq C^*
\]

for all \(0 < \gamma \leq a^*\), where \(\widetilde{C}\) is a uniform constant.

**Proposition 4.5.** Let \(E\) be a reflexive sheaf on a smooth compact Kähler manifold \((M, \omega)\), and \(\mathcal{F}^{HNS} = \{E_i\}_{i=1}^{\mathcal{F}}\) be the HNS-filtration of \(E\) by saturated subsheaves:

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E,
\]

where every quotient \(Q_i = E_i/E_{i-1}\) is torsion-free and \(\omega\)-stable. Let \(\pi : \tilde{M} \to M\) be the composition of a finite sequence of blowing ups along compact sub-manifolds such that \(E = \cdots \subset E_l \subset E_{l-1} \subset \cdots \subset E_1 \subset E_0 = E\).
π∗E/tor(π∗E) is locally free. Then there exists a positive constant ̄a∗ such that, for any δ > 0 and any 1 ≤ p < 1 + ̄a∗, there is a smooth metric H on E such that
\[
\| \frac{1}{2\pi} \Lambda_ω(F(\xi, H)) - \Psi(\xi, (μ_1, \omega), H) \|_{L^p(M, ω)} \leq δ,
\]
where μ_iω is the ω-slope of E_i in (E, H) is the Chern connection on E with respect to the metric H.

Proof. Consider the filtration \( \tilde{F} = \{ \tilde{E}_n \}_{n=1} \) of E over \( \tilde{M} \) which is constructed in Proposition 14.2. Every quotient sheaf \( \tilde{Q} \) in the filtration is \( ω_\ast \)-stable for all 0 < |ε| ≤ ε∗. Following Sibley’s argument (38), we can construct a resolution of the filtration such that the pullback bundle has a filtration by subbundles, which away from the exceptional divisor is precisely the filtration. Using Daskalopoulos and Wentworth’s cut-off argument (13) and Sibley’s trick (38), we can obtain an \( L^p \)-approximate critical Hermitian metric on the holomorphic vector bundle E over the Kähler manifold \( \tilde{M}, ω_\ast \) (Theorem 5.12 in [38]). Given δ > 0 and any 1 ≤ p' < +∞, for every small ε', there exists a smooth Hermitian metric \( H_{ε'} \) on E such that
\[
\| \frac{1}{2\pi} \Lambda_ω(F(\xi, \pi_{E}, H_{ε'})) - \Psi(\xi, (μ_1, \omega), H_{ε'}) \|_{L^{p'}(\tilde{M}\setminus\pi^{-1}(\Sigma_{HNS}), ω_{ε'})} \leq δ'.
\]
We choose a smooth metric \( H_{ε'} \) satisfying (4.15) for some ε' and p' which will be chosen later. For simplicity, denote \( Θ_1 = \frac{1}{2\pi}(F(\xi, \pi_{E}, H_{ε'})) \). A straightforward computation shows that
\[
\| \frac{1}{2\pi} \Lambda_ω(F(\xi, \pi_{E}, H_{ε'})) - \Psi(\xi, (μ_1, \omega), H_{ε'}) \|_{L^p(\tilde{M}, ω_\ast)} 
\leq \| Λ_ω(Θ_1 - \frac{ω_{ε'}}{n} Ψ(\xi, (μ_1, \omega), H_{ε'})) \|_{L^p(\tilde{M}, ω_\ast)} 
+ \| \frac{1}{n} Λ_ω(ω_{ε'} - ω_\ast) Ψ(\xi, (μ_1, \omega), H_{ε'}) \|_{L^p(\tilde{M}, ω_\ast)} 
+ \| Ψ(\xi, (μ_1, \omega), H_{ε'}) - Ψ(\xi, (μ_1, \omega), (μ_1, \omega)) \|_{L^p(\tilde{M}, ω_\ast)}.
\]
Clearly the Chern-Weil theory implies
\[
\int_{\tilde{M}} (2c_2(E) - c_1(E) \wedge c_1(E)) + \frac{ω_{n-2}^{n-2}}{(n-2)!}(n-2)
= \int_{\tilde{M}} |Θ_1|_{H_{ε'}}^2 - |Λ_ω(Θ_1)|_{H_{ε'}}^2 \frac{ω_{n}^{n}}{n!},
\]
Setting \( Θ_2 = Θ_1 - \frac{ω_{n-2}}{ω_\ast} Ψ(\xi, (μ_1, \omega), (μ_1, \omega)) \), we know that \( \| Θ_2 \|_{L^2(\tilde{M}, ω_\ast)} \) is bounded uniformly. In the sequel, we always assume that 1 ≤ p < 1 + \( \frac{n}{2+4\ast} \). From the definition of Λ_ω, it follows that
\[
\| Λ_ω(Θ_2) \|_{L^p(\tilde{M}, ω_\ast)} = \| \frac{ω_\ast^{n}}{ω_{n}^{n}}(Λ_ω(Θ_2 + n Θ_2 ∧ (ω_{n-2}^{n-2} - ω_{n-2}^{n-2})) \|_{L^p(\tilde{M}, ω_\ast)} 
\leq \frac{ω_\ast^{n}}{ω_{n}^{n}}(Λ_ω(Θ_2)) \|_{L^p(\tilde{M}, ω_\ast)} 
+ n \sum_{β=1}^{k} |ε_β - ε'_β| \| Θ_2 ∧ η_β ∧ (\sum_{i=0}^{n-2} ω_{n-2}^{n-2} ∧ ω_{n-2}^{n-2}) \|_{L^p(\tilde{M}, ω_\ast)}.
\]
Direct calculations show that

$$\left\| \frac{\omega_\nu^n}{\omega_\nu^n} (\Lambda_{\omega_\nu} \Theta_2) \right\|_{L^p(M, \omega_\nu)}^p = \int_M |\Lambda_{\omega_\nu} \Theta_2|^p \left( \frac{\omega_\nu^n}{\omega_\nu^n} \right)^{p-1} \frac{\omega_\nu^n}{n!} \leq \left( \int_M |\Lambda_{\omega_\nu} \Theta_2|^p \frac{\omega_\nu^n}{n!} \right)^{\frac{1}{p}} \left( \int_M \left( \frac{\omega_\nu^n}{\omega_\nu^n} \right)^{(p-1)b} \frac{\omega_\nu^n}{n!} \right)^{\frac{1}{p}},$$

where $p \cdot a = p', \frac{1}{1 - \frac{p}{p'}(p-1)} < \tilde{p}$ and $\frac{1}{a} + \frac{1}{b} = 1$, we have $(p-1)b < a^*$. Let $U$ be a neighborhood of the singularity set $\Sigma_\varepsilon$. Since $\pi^* \omega$ is degenerate only along $\pi^{-1}(\Sigma_\varepsilon)$, there must exist a positive constant $C_U$ depending only on $U$ such that $\pi^* \omega \geq C_U \eta_\beta$ on $M \setminus U$ for all $1 \leq \beta \leq k$. On the other hand, we can suppose that $\pi^* \omega \leq C_M \eta_k$ on $M$ for some positive constant $C_M$. Now suppose that $\epsilon < \epsilon'$, then we get

$$\int_{M \setminus U} \left| \Theta_2 \wedge \eta_\beta \wedge \left( \sum_{i=0}^{n-2} \omega_\nu^\cdot \omega_\nu^\cdot \omega_\nu^\cdot \right) \right|^p \frac{\omega_\nu^n}{n!} = \int_{M \setminus U} \left| \Theta_2 \wedge \eta_\beta \wedge \left( \sum_{i=0}^{n-2} \omega_\nu^{n-i-2} \wedge \omega_\nu^i \right) \right|^p \frac{\omega_\nu^n}{n!} \leq C(n) C_U^{-p} \int_{M \setminus U} \left| \Theta_2 \right|^p \frac{\omega_\nu^n}{n!} \leq C(n) C_U^{-(n+1)p+n} (C_M + |\epsilon'|)^{n(p-1)} \int_{M \setminus U} \left| \Theta_2 \right|^p \frac{\omega_\nu^n}{n!},$$

where $C(n)$ is a uniform constant. On the other hand, we know

$$\int_U |\epsilon_\beta - \epsilon'_\beta|^p \left| \Theta_2 \wedge \eta_\beta \wedge \left( \sum_{i=0}^{n-2} \omega_\nu^\cdot \omega_\nu^\cdot \omega_\nu^\cdot \right) \right|^p \frac{\omega_\nu^n}{n!} = \int_U |\epsilon_\beta - \epsilon'_\beta|^p \left| \Theta_2 \wedge \eta_\beta \wedge \left( \sum_{i=0}^{n-2} \omega_\nu^{n-i-2} \wedge \omega_\nu^i \right) \right|^p \frac{\omega_\nu^n}{n!} \leq C(n) \left( \int_U \left| \Theta_2 \right|^2 \left( \frac{\omega_\nu^n}{n!} \right)^{\frac{2-p}{p}} \left( \int_U \left( \frac{\omega_\nu^n}{\omega_\nu^n} \right)^{2p-2} \frac{\omega_\nu^n}{n!} \right)^{\frac{2-p}{p}} \leq C(n) (\text{Vol}(U, \omega_\nu)) \frac{2p-2}{2p} (\int_U \left| \Theta_2 \right|^2 \frac{\omega_\nu^n}{n!}) \frac{2-p}{2p},$$

where $\tilde{b} = \frac{1}{2} \left( \frac{2p-2}{2p} + a^* \right) \cdot \left( \frac{2p-2}{2p} \right)^{-1}$, and note that the condition on $p$ gives us $\frac{2p-2}{2p} \cdot \tilde{b} < a^*$. Combining (14.18), (14.19), (14.20) and (14.21), we derive

$$\left\| \Lambda_{\omega_\nu} \Theta_2 \right\|_{L^p(M, \omega_\nu)} \leq \sum_{\beta=1}^k \tilde{C}_1 |\epsilon_\beta - \epsilon'_\beta| C_U^{-(n+1)+\tilde{b}} \left\| \Theta_2 \right\|_{L^p(M, \omega_\nu)},$$

where $\tilde{C}_2 = C_2 (\text{Vol}(U, \omega_\nu))^C(p, a^*) \left\| \Theta_2 \right\|_{L^2(M, \omega_\nu)} \left( \int_U \left( \frac{\omega_\nu^n}{\omega_\nu^n} \right)^{(p-1)b} \frac{\omega_\nu^n}{n!} \right)^{\frac{1}{p}}$.
where \( C(p,a^*) = \frac{1-p}{2} - \frac{(p-1)(2-p)}{2p+2-2a(2-p)} \). This together with (4.12) implies

\[
\lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_k \to 0} \|\Lambda_{\omega_1} \Theta_2\|_{L^p(M,\omega_1)} \leq \sum_{j=1}^{k} \hat{C}_1 |\epsilon_j|^{(n+1)+1 \over \bar{p}} \|\Theta_2\|_{L^p(M,\omega_1)} + C_2 \left( \|\Lambda_{\omega_1} \Theta_2\|_{L^p(M,\omega_1)} + (\text{Vol}(U,\omega_1))^{(p,a^*)} \|\Theta_2\|_{L^2(M,\omega_1)} \right)
\]

(4.23)

where \( \hat{C}_1 \) is a uniform constant. We may choose \( U \) such that \( \text{Vol}(U,\omega_1) \) small enough first, and then \( \delta \) and \( \epsilon \) both sufficiently small so that

\[
\lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_k \to 0} \|\Lambda_{\omega_1} \Theta_2\|_{L^p(M,\omega_1)} \leq \frac{\delta}{3}.
\]

(4.24)

By (4.12) and the fact that \( \mu_{i,\omega_0} \to \mu_{i,\omega} \) as \( \epsilon \to 0 \), we may choose \( \epsilon \) small enough so that the second and third terms in (4.19) are both smaller than \( \frac{\delta}{9} \), hence it follows that

\[
\|\sqrt{\frac{\omega}{2\pi}} \Lambda_{\omega}(F(\mathcal{E}_t)) - \Psi(f_t, (\mu_1,\ldots, \mu_\ell), H)\|_{L^p(M,\omega)} \leq \delta.
\]

(4.25)

\( \square \)

5. The HN Type of the Uhlenbeck Limit

Let \( \mathcal{E} \) be a reflexive sheaf on a smooth Kähler manifold \( (M,\omega) \), \( H(t) \) be a solution of the Hermitian-Yang-Mills flow \( (1.2) \) on \( \mathcal{E}|_{M \setminus \Sigma} \) with the initial metric \( \bar{H} \), and \( A(t) \) be the related Yang-Mills flow \( (2.22) \) on the Hermitian vector bundle \( (\mathcal{E}|_{M \setminus \Sigma}, \bar{H}) \). Let \( A_\infty \) be an Uhlenbeck limit. From Theorem 3.3 we know that \( A_\infty \) is a smooth Yang-Mills connection on the Hermitian bundle \( (E_\infty, H_\infty) \) over \( M \setminus (\Sigma_\infty \cup \Sigma_{an}) \), and \( \theta(A_\infty,\omega) \) is parallel, then the constant eigenvalues vector \( \vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R) \) of \( \sqrt{\frac{\omega}{2\pi}} \Lambda_{\omega}A_\infty \) is just the HN type of the extended Uhlenbeck limit sheaf \( \mathcal{E}_\infty = (E_\infty, \mathcal{O}_{A_\infty}) \). Denote by \( \vec{\mu}_0 = (\mu_1, \ldots, \mu_R) \) the HN type of the reflexive sheaf \( \mathcal{E} \). In this section, we will show that the HN type of the limiting sheaf for the Hermitian-Yang-Mills flow \( (1.2) \) is in fact equal to the HN type of the reflexive sheaf \( \mathcal{E} \), i.e. \( \vec{\lambda}_\infty = \vec{\mu}_0 \).

**Lemma 5.1.** Let \( A(t) \) be the long time solution of the Yang-Mills flow \( (2.22) \) on a complex vector bundle \( \mathcal{E}|_{M \setminus \Sigma} \) of rank \( R \) with a Hermitian metric \( \bar{H} \). Let \( S \) be a coherent subsheaf of \( \mathcal{E} \). Suppose there is a sequence \( \{A_j\} \), modulo gauge transformations, such that \( \sqrt{-1} \Lambda_{\omega}(F_{A_j}) \to B \) in \( L^1 \) as \( j \to +\infty \), where \( B \in L^1(\sqrt{-1}u(E)) \), and the eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_R \) of \( \sqrt{-1} \Lambda_{\omega}F_{A_\infty} \) are constant almost everywhere. Then: \( \text{deg}_{\omega}(S) \leq \sum_{i \leq \text{rank}(S)} \lambda_i \).

**Proof:** Because \( \text{deg}_{\omega}(S) \leq \text{deg}_{\omega}(\text{Sat}_E(S)) \), we may assume that \( S \) is saturated. As before, let \( \pi : M \to M \) be the composition of a finite sequence of blowups resolving the sheaf \( \mathcal{E} \), i.e. such that \( E = \pi^* \mathcal{E}/\text{tor}(\pi^* \mathcal{E}) \) is locally free. Considering the exact sequence

\[
0 \to S \to \mathcal{E} \to Q \to 0, \quad \text{over} \ M,
\]

(5.1)

we get the following exact sequences

\[
\pi^* S \overset{f_1}{\to} \pi^* \mathcal{E} \to \pi^* Q \to 0, \quad \text{over} \ M,
\]

(5.2)

and

\[
0 \to \text{Im}(f_1)/\text{tor}(\text{Im}(f_1)) \to E = \pi^* \mathcal{E}/\text{tor}(\pi^* \mathcal{E}) \to \bar{Q} \to 0, \quad \text{over} \ M,
\]

(5.3)

where \( \bar{Q} = \frac{E}{\text{Im}(f_1)/\text{tor}(\text{Im}(f_1))} \). Setting \( \bar{S} = \text{Sat}_E(\text{Im}(f_1)/\text{tor}(\text{Im}(f_1))) \), since \( \pi \) is biholomorphic outside \( \Sigma_{\infty} \) and \( \text{codim}(\Sigma_{\infty}) \geq 3 \), we have \( (\pi)_* \bar{S} = S \) on \( M \setminus \Sigma_{\infty} \).
Let $H_{k, \epsilon}(t)$ be the long time solution of the Hermitian-Yang-Mills flow (2.7) on the holomorphic bundle $E$ over $\mathcal{M}$ with the fixed smooth initial metric $\tilde{\mathcal{H}}$ and with respect to the Kähler metric $\omega_{k, \epsilon}$. Clearly Lemma 2.22 and Proposition 2.23 say that $H_{k, \epsilon}(x, t)$ converges successively to the long time solution $H(x, t)$ of the Hermitian-Yang-Mills flow (1.2) as $\epsilon \to 0$, $A(t) = \sigma(t)(\mathcal{D}_R)$ and $\sigma(t)^* \circ \sigma(t) = \tilde{\mathcal{H}}^{-1}H$. Let $\pi^{H(t)}$ (resp. $\pi^{H_{k, \epsilon}(t)}$) denote the orthogonal projection onto $\mathcal{S}$ (resp. $\tilde{\mathcal{S}}$) with respect to the Hermitian metric $H(t)$ (resp. $H_{k, \epsilon}(t)$). Using the Gauss-Codazzi equation and Fatou’s lemma, we derive

$$\deg_\omega(S) = \int_M c_1(\det S) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_M c_1(\pi_*(\det \tilde{\mathcal{S}})) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$= \int_M c_1(\det S) \wedge \frac{\pi^* \omega^{n-1}}{(n-1)!} = \lim_{\epsilon \to 0} \int_M c_1(\tilde{\mathcal{S}}) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\pi M} (\text{tr}(\sqrt{-1} \Lambda_\omega(F_{H(t)})) \pi^{H(t)}) - |\bar{\partial}_E \pi^{H(t)}|^2 \frac{\omega^n}{n!}$$

$$\leq \frac{1}{2\pi} \int_{\pi M} (\text{tr}(\sqrt{-1} \Lambda_\omega(F_{H(t)})) \pi^{H(t)}) - |\bar{\partial}_E \pi^{H(t)}|^2 \frac{\omega^n}{n!}$$

$$\leq \frac{1}{2\pi} \int_{\pi M} \text{tr}(\sqrt{-1} \Lambda_\omega(F_{H(t)})) \pi^{H(t)}) \frac{\omega^n}{n!}$$

$$= \frac{1}{2\pi} \int_{\pi M} \text{tr}(\sqrt{-1} \Lambda_\omega(F_{H(t)})) \pi^{H(t)}) \frac{\omega^n}{n!}$$

$$= \frac{1}{2\pi} \int_{\pi M} \text{tr}((\sqrt{-1} \Lambda_\omega F_{A(t)} - B)(\sigma(t) \circ \pi^{H(t)} \circ \sigma^{-1}(t))) \frac{\omega^n}{n!}$$

$$+ \frac{1}{2\pi} \int_{\pi M} \text{tr}(B(\sigma(t) \circ \pi^{H(t)} \circ \sigma^{-1}(t))) \frac{\omega^n}{n!}$$

for $t > 0$. By a result from linear algebra (Lemma 2.20 in [13]), we obtain $\frac{1}{2\pi} \text{tr}(B(\sigma(t) \circ \pi^{H(t)} \circ \sigma^{-1}(t))) \leq \sum_{\lambda_i \leq \text{rank}(S)} \lambda_i$. So it holds that $\deg_\omega(S) \leq \sum_{\lambda_i \leq \text{rank}(S)} \lambda_i + \frac{1}{2\pi} \|\sqrt{-1} \Lambda_\omega(F_{A(t)}) - B\|_{L^1}$. Letting $j \to \infty$ concludes the proof of the lemma.

Combining (2.9), Lemma 2.22 and Corollary 3.4, we know

$$\int_{\pi M} \text{tr}(\theta(H_{k, \epsilon}(t), \omega_{k, \epsilon})) \frac{\omega^n}{n!} = \int_{\pi M} \text{tr}(\theta(\tilde{\mathcal{H}}, \omega_{k, \epsilon})) \frac{\omega^n}{n!}$$

and then

$$\deg_\omega(\mathcal{E}) = \int_M \text{tr}(\frac{\sqrt{-1}}{2\pi} \Lambda_\omega F_{\tilde{\mathcal{H}}}) \frac{\omega^n}{n!} = \int_M \text{tr}(\frac{\sqrt{-1}}{2\pi} \Lambda_\omega F_{A}) \frac{\omega^n}{n!} = \deg_\omega(\mathcal{E}_\infty, \mathcal{D}_{A_\infty})$$

i.e.

$$\sum_{\alpha=1}^R \mu_\alpha = \sum_{\alpha=1}^R \lambda_\alpha.$$

(5.7)
Let $\{\mathcal{E}_i\}_{i=1}^n$ be the HNS-filtration of the reflexive sheaf $\mathcal{E}$. Applying Lemma 5.1 yields:

$$\sum_{\alpha \leq \text{rank} \mathcal{E}_i} \mu_\alpha = \text{deg}_\omega(\mathcal{E}_i) \leq \sum_{\alpha \leq \text{rank} \mathcal{E}_i} \lambda_\alpha$$  \hspace{1cm} (5.8)

for all $i$. Of course Lemma 2.3 in [13] means

$$\bar{\mu}_0 \leq \bar{\lambda}_\infty.$$  \hspace{1cm} (5.9)

For further consideration, we show the continuous dependence of the Hermitian-Yang-Mills flow (1.2) on initial metrics.

**Lemma 5.2.** Let $\hat{H}_1$ and $\hat{H}_2$ be two smooth metrics on the holomorphic bundle $E$ over $\tilde{M}$, and $\hat{\delta} = \sup_{\gamma \in \tilde{M}} (\text{tr } \hat{H}_1^{-1} \hat{H}_2 + \text{tr } \hat{H}_2^{-1} \hat{H}_1 - 2 \text{rank}(E))$. If $H_i(t)$ is the long time solution of the Hermitian-Yang-Mills flow (1.2) on $\mathcal{E}|_{\tilde{M}\setminus \Sigma_E}$ with the initial metric $\hat{H}_i$ respectively for $i = 1, 2$, then for any $t > 0$,

$$\int_{\tilde{M}} \sqrt{-1} \Lambda_\omega F_{H_2(t)} - \sqrt{-1} \Lambda_\omega F_{H_1(t)} \frac{\omega^n}{n!} dt \leq f_t(\hat{\delta}),$$  \hspace{1cm} (5.10)

where $f_t: R^+ \to R^+$ is a continuous function satisfying $f_t(x) \to 0$ as $x \to 0$.

**Proof.** Let $H_{i(\cdot)}(t)$ be the long time solution of the Hermitian-Yang-Mills flow (1.2) on the holomorphic bundle $E$ over $\tilde{M}$ with the smooth initial metric $\hat{H}_i$ and with respect to the Kähler metric $\omega_\epsilon$, where $i = 1, 2$. Set

$$\hat{h}_\epsilon(t) = H_{1(\cdot)}^{-1}(t) H_{2(\cdot)}(t).$$  \hspace{1cm} (5.11)

It is easy to check that

$$\text{tr} \left( \sqrt{-1} \hat{h}_\epsilon(t) (\Lambda_\omega F_{H_{2(\cdot)}}(t) - \Lambda_\omega F_{H_{1(\cdot)}}(t)) \right) = -\frac{1}{2} \Lambda_\omega \text{tr} \hat{h}_\epsilon(t) + \text{tr} \left( -\sqrt{-1} \Lambda_\omega \bar{\partial} \hat{h}_\epsilon(t) \hat{h}_\epsilon^{-1}(t) \partial H_1(t) \hat{h}_\epsilon(t) \right),$$  \hspace{1cm} (5.12)

$$\text{tr} \left( \sqrt{-1} \hat{h}_\epsilon^{-1}(t) (\Lambda_\omega F_{H_{1(\cdot)}}(t) - \Lambda_\omega F_{H_{2(\cdot)}}(t)) \right) = -\frac{1}{2} \Lambda_\omega \text{tr} \hat{h}_\epsilon^{-1}(t) + \text{tr} \left( -\sqrt{-1} \Lambda_\omega \bar{\partial} \hat{h}_\epsilon^{-1}(t) \hat{h}_\epsilon^{-1}(t) \partial H_1(t) \hat{h}_\epsilon(t) \right),$$  \hspace{1cm} (5.13)

$$\text{tr} \left( \sqrt{-1} (\hat{h}_\epsilon(t) - \hat{h}_\epsilon^{-1}(t)) (\Lambda_\omega F_{H_{2(\cdot)}}(t) - \Lambda_\omega F_{H_{1(\cdot)}}(t)) \right)$$

$$= -\frac{1}{2} \Lambda_\omega \left( \text{tr} \hat{h}_\epsilon(t) + \text{tr} \hat{h}_\epsilon^{-1}(t) + \text{tr} \left( -\sqrt{-1} \Lambda_\omega \bar{\partial} \hat{h}_\epsilon(t) \hat{h}_\epsilon^{-1}(t) \partial H_1(t) \hat{h}_\epsilon(t) \right) \right)$$

$$+ \text{tr} \left( -\sqrt{-1} \Lambda_\omega \bar{\partial} \hat{h}_\epsilon^{-1}(t) \hat{h}_\epsilon(t) \partial H_1(t) \hat{h}_\epsilon^{-1}(t) \right),$$  \hspace{1cm} (5.14)

and

$$\left( \Lambda_\omega - \frac{\partial}{\partial t} \right) \left( \text{tr} \hat{h}_\epsilon(t) + \text{tr} \hat{h}_\epsilon^{-1}(t) - 2 \text{rank}(E) \right) \geq 0.$$  \hspace{1cm} (5.15)

The inequality (5.15) together with the maximum principle gives us

$$\sup_{x \in \tilde{M}} \left( \text{tr} \hat{h}_\epsilon(t) + \text{tr} \hat{h}_\epsilon^{-1}(t) - 2 \text{rank}(E) \right) \leq \sup_{x \in \tilde{M}} \left( \text{tr} \hat{h}_\epsilon(0) + \text{tr} \hat{h}_\epsilon^{-1}(0) - 2 \text{rank}(E) \right) = \hat{\delta}.\hspace{1cm} (5.16)$$

In the following, we assume that $\hat{\delta}$ is small enough. Suppose $\lambda_i$ is the eigenvalue of $\hat{h}_\epsilon(t)$ for $1 \leq i \leq n$, then

$$1 - \sqrt{\bar{\delta} \bar{\lambda} + 4n} \leq \lambda_i \leq 1 + \sqrt{\bar{\delta} \bar{\lambda} + 4n},$$  \hspace{1cm} (5.17)

$$1 - \sqrt{\bar{\delta} \bar{\lambda} + 4n} \leq \frac{1}{\lambda_i} \leq 1 + \sqrt{\bar{\delta} \bar{\lambda} + 4n},$$  \hspace{1cm} (5.18)
and

\[ -2\sqrt{\delta(\delta + 4n)}\text{Id} \leq \hat{h}_\epsilon(t) - \hat{h}_\epsilon^{-1}(t) \leq 2\sqrt{\delta(\delta + 4n)}\text{Id}. \quad (5.19) \]

A direct computation shows that

\[
|\text{tr} \left( \sqrt{-1}(\hat{h}_\epsilon(t) - \hat{h}_\epsilon^{-1}(t))(\Lambda_{\omega, F_{H(2,\epsilon)}}(t) - \Lambda_{\omega, F_{H(1,\epsilon)}}(t)) \right) |
\leq 2\sqrt{n} \sqrt{\delta(\delta + 4n)}|\Lambda_{\omega, F_{H(2,\epsilon)}}(t) - \Lambda_{\omega, F_{H(1,\epsilon)}}(t)|_{H(1,\epsilon)}(t) \leq \sqrt{\delta(\delta + 4n)} \left( \sqrt{n} + \sqrt{\delta(\delta + 4n)} \right) |\Lambda_{\omega, F_{H(2,\epsilon)}}(t)|_{H(1,\epsilon)}(t) + |\Lambda_{\omega, F_{H(1,\epsilon)}}(t)|_{H(1,\epsilon)}(t). \quad (5.20)
\]

Set \( \hat{T}_\epsilon(t) = \hat{h}_\epsilon^{-1}(t)\partial_{H(1,\epsilon)}(t)\hat{h}_\epsilon(t) \). Clearly (5.18) implies that

\[
|\hat{T}_\epsilon(t)\|^2_{H(1,\epsilon),\omega} \leq \sqrt{n} \left( 1 + \sqrt{\delta(\delta + 4n)} \right) \text{tr} \left( -\sqrt{-1}\Lambda_{\omega, \partial\hat{h}_\epsilon(t)}\hat{h}_\epsilon^{-1}(t)\partial_{H(1,\epsilon)}(t)\hat{h}_\epsilon(t) \right). \quad (5.21)
\]

From (2.13), it follows that there exists a uniform constant \( C \) such that

\[
\int_{\tilde{M}} |\Lambda_{\omega, F_{H(2,\epsilon)}}(t)|_{H(2,\epsilon)}(t) \omega^\mathsf{n} \frac{\omega^\mathsf{n}}{n!} + \int_{\tilde{M}} |\Lambda_{\omega, F_{H(1,\epsilon)}}(t)|_{H(1,\epsilon)}(t) \omega^\mathsf{n} \frac{\omega^\mathsf{n}}{n!} \leq C, \quad (5.22)
\]

for any \( t \geq 0 \) and \( 0 < \epsilon \leq 1 \). Combining (5.14) and (5.19), we get

\[
\int_{\tilde{M}} |\hat{T}_\epsilon(t)|^2_{H(1,\epsilon),\omega} \omega^\mathsf{n} \frac{\omega^\mathsf{n}}{n!} \leq C n \left( 1 + \sqrt{\delta(\delta + 4n)} \right)^2 \sqrt{\delta(\delta + 4n)} \leq C \sqrt{\delta}. \quad (5.23)
\]

By straightforward calculations, we deduce

\[
\frac{\partial}{\partial t} \hat{T}_\epsilon(t) = -\hat{h}_\epsilon^{-1}(t) \frac{\partial \hat{h}_\epsilon(t)}{\partial t} \hat{h}_\epsilon^{-1}(t)\partial_{H(1,\epsilon)}(t)\hat{h}_\epsilon(t) + \hat{h}_\epsilon^{-1}(t)\partial_{H(1,\epsilon)}(t) \left( \hat{h}_\epsilon(t) \frac{\partial \hat{h}_\epsilon(t)}{\partial t} \right) \\
+ \hat{h}_\epsilon^{-1}(t) \frac{\partial}{\partial t} \left( \partial_{H(1,\epsilon)}(t) \hat{h}_\epsilon(t) \right) \\
= \partial_{H(2,\epsilon)}(t) \left( \hat{h}_\epsilon^{-1}(t) \frac{\partial \hat{h}_\epsilon(t)}{\partial t} \right) + \hat{h}_\epsilon^{-1}(t) \frac{\partial}{\partial t} \left( H_{-1}(\epsilon)(t) \partial H_{1}(\epsilon)(t) \right) \hat{h}_\epsilon(t) \\
- \hat{h}_\epsilon^{-1}(t) \hat{h}_\epsilon(t) \frac{\partial}{\partial t} \left( H_{-1}(\epsilon)(t) \partial H_{1}(\epsilon)(t) \right) \\
= \partial_{H(2,\epsilon)}(t) \left( \hat{h}_\epsilon^{-1}(t) \frac{\partial \hat{h}_\epsilon(t)}{\partial t} \right) + \partial_{H(2,\epsilon)}(t) \left( \hat{h}_\epsilon^{-1}(t) \left( H_{-1}(\epsilon)(t) \frac{\partial H_{1}(\epsilon)(t)}{\partial t} \right) \hat{h}_\epsilon(t) \right) \\
- \partial_{H(1,\epsilon)}(t) \left( H_{-1}(\epsilon)(t) \frac{\partial H_{1}(\epsilon)(t)}{\partial t} \right) \\
= 2\partial_{H(1,\epsilon)}(t) \left( -\sqrt{-1}\Lambda_{\omega, F_{H(1,\epsilon)}}(t) - \sqrt{-1}\Lambda_{\omega, F_{H(2,\epsilon)}}(t) \right) \\
- 2\hat{T}_\epsilon(t) \sqrt{-1}\Lambda_{\omega, F_{H(2,\epsilon)}}(t) + 2\sqrt{-1}\Lambda_{\omega, F_{H(2,\epsilon)}}(t) \hat{T}_\epsilon(t) \\
= -2\partial_{H(1,\epsilon)}(t) \left( \sqrt{-1}\Lambda_{\omega, \partial E}\hat{T}_\epsilon(t) \right) - 2\hat{T}_\epsilon(t) \sqrt{-1}\Lambda_{\omega, F_{H(2,\epsilon)}}(t) + 2\sqrt{-1}\Lambda_{\omega, F_{H(2,\epsilon)}}(t) \hat{T}_\epsilon(t), \quad (5.24)
\]
and
\[ \frac{\partial}{\partial t} |\hat{T}_e(t)|_{H^{(1,\omega)}(t)}^2 - \frac{\partial}{\partial t} \sqrt{-1} \Lambda_\omega \tr (\hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \wedge \hat{T}_e(t) H^{-1}_{(1,\omega)}(t)) \]
\[ = 2 \Re \sqrt{-1} \Lambda_\omega \tr \left( \frac{\partial}{\partial t} \hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \wedge \hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \right) \]
\[ + \sqrt{-1} \Lambda_\omega \tr (\hat{T}_e(t) (-H^{-1}_{(1,\omega)}(t) \frac{\partial H}{\partial t}_{(1,\omega)}(t)) \wedge \hat{T}_e(t) H_{(1,\omega)}(t)) \]
\[ + \sqrt{-1} \Lambda_\omega \tr \left( \hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \wedge \hat{T}_e(t) \frac{\partial H_{(1,\omega)}(t)}{\partial t} \right) \]
\[ \leq -4 \Re \sqrt{-1} \Lambda_\omega \partial t \tr \left( \sqrt{-1} \Lambda_\omega \bar{\partial} \hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \wedge \hat{T}_e(t) H_{(1,\omega)}(t) \right) \]
\[ + 4 \Re \sqrt{-1} \Lambda_\omega \tr \left( \sqrt{-1} \Lambda_\omega \bar{\partial} E \hat{T}_e(t) H^{-1}_{(1,\omega)}(t) \wedge \hat{T}_e(t) H_{(1,\omega)}(t) \right) \]
\[ + 8 |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t)|_{H_{(1,\omega)}(t)} |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 + 4 |\sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)} |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \]  
(5.25)

Recall that
\[ F_{H(2,\omega)}(t) - F_{H(1,\omega)}(t) = \bar{\partial}_E (\bar{h}_e^{-1}(t) \partial H_{(1,\omega)}(t) \bar{h}_e(t)), \]
(5.26)
we have
\[ \frac{\partial}{\partial t} \int_M |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \]
\[ \leq -4 \int_M |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t) - \sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \]
\[ + \int_M \left( 8 |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t)|_{H_{(1,\omega)}(t)} + 4 |\sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)} \right) |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!}. \]
(5.27)

Then it holds that
\[ \frac{\partial}{\partial t} \left( t^n \int_M |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \right) \]
\[ = n t^{n-1} \int_M |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} + t^n \frac{\partial}{\partial t} \int_M |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \]
\[ \leq n t^{n-1} \int_M |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} - 4 t^n \int_M \left| \sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t) - \sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t) \right|^2 \frac{\omega^n}{n!} \]
\[ + t^n \int_M \left( 8 |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t)|_{H_{(1,\omega)}(t)} + 4 |\sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)} \right) |\hat{T}_e(t)|_{H_{(1,\omega)}(t)}^2 \frac{\omega^n}{n!}. \]
(5.28)

The inequalities (5.24), (5.26) and (5.28) tell us that there must exist uniform constants \( \hat{C}_1 \) and \( \hat{C}_1 \) such that
\[ \sup_M t^n |\sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)} \leq \hat{C}_1, \]
(5.29)
\[ \sup_M t^n |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t)|_{H_{(1,\omega)}(t)} \leq \hat{C}_2, \]
(5.30)
for \( t \in [0, 1] \), and
\[ \sup_M |\sqrt{-1} \Lambda_\omega \tr F_{H(2,\omega)}(t)|_{H_{(1,\omega)}(t)} \leq \hat{C}_1, \]
(5.31)
\[ \sup_M |\sqrt{-1} \Lambda_\omega \tr F_{H(1,\omega)}(t)|_{H_{(1,\omega)}(t)} \leq \hat{C}_2, \]
(5.32)
for $t \geq 1$. The above inequalities together with (5.28) yield

$$
\int_0^1 t^n \int_M \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) - \sqrt{-1} A \omega_t F_{H(1,\omega)}(t)^2 |H_{(1,\omega)}(t)| \frac{\omega^n}{n!} dt
$$

$$
\leq - \frac{1}{4} \int_M [\tilde{T}_e(1)]^2 |H_{(1,\omega)}(t)| \frac{\omega^n}{n!} dt + \frac{n}{4} \int_0^1 t^{n-1} \int_M [\tilde{T}_e(t)]^2 |H_{(1,\omega)}(t)| \frac{\omega^n}{n!} dt
$$

$$
+ \frac{1}{4} \int_0^1 t^n \int_M (8 \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) \partial_{H(1,\omega)} )^2 |H_{(1,\omega)}(t)| |\tilde{T}_e(t)|^2 |H_{(1,\omega)}(t)| \frac{\omega^n}{n!} dt
$$

$$
\leq \hat{C} \hat{\delta},
$$

(5.33)

where $\hat{C}$ is a uniform constant. For simplicity, set

$$
\tilde{\Phi}_e(t) = \sqrt{-1} A \omega_t (F_{H(2,\omega)}(t) - F_{H(1,\omega)}(t)) = \sqrt{-1} A \omega_t \partial E (\tilde{h}_e^{-1}(t) \partial H_{(1,\omega)}(t) \tilde{h}_e(t))
$$

(5.34)

Using (5.24), we obtain

$$
\frac{\partial}{\partial t} \tilde{\Phi}_e(t) = \sqrt{-1} A \omega_t \partial E \left( \frac{\partial}{\partial t} \tilde{T}_e(t) \right)
$$

$$
= \sqrt{-1} A \omega_t \partial E \left( -2 \partial H_{(1,\omega)}(t) (\sqrt{-1} A \omega_t \partial E \tilde{T}_e(t)) - 2 \tilde{T}_e(t) \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) 
$$

$$
+ 2 \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) \tilde{T}_e(t) 
$$

$$
= - 2 \sqrt{-1} A \omega_t \partial E (\partial H_{(1,\omega)}(t) \tilde{\Phi}_e(t)) + \tilde{T}_e(t) \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) - \sqrt{-1} A \omega_t F_{H(2,\omega)}(t) \tilde{T}_e(t).
$$

(5.35)

On the other hand, we have

$$
\Delta \omega_t |\tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)} = 4 R e (\sqrt{-1} A \omega_t \partial E \partial H_{(1,\omega)}(t) \tilde{\Phi}_e(t), \tilde{\Phi}_e(t))_{H_{(1,\omega)}(t)}
$$

$$
+ 2 |\partial H_{(1,\omega)}(t) \tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)} + 2 |\partial E \tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)}
$$

$$
- 2 \sqrt{-1} A \omega_t (\tilde{\Phi}_e(t) H_{(1,\omega)}^{-1}(t) [F_{H(1,\omega)}(t), \tilde{\Phi}_e(t)] H_{(1,\omega)}(t)).
$$

(5.36)

and then

$$
(\Delta \omega_t - \frac{\partial}{\partial t}) |\tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)}
$$

$$
= 2 |\partial H_{(1,\omega)}(t) \tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)} + 2 |\partial E \tilde{\Phi}_e(t)|^2_{H_{(1,\omega)}(t)}
$$

$$
- 4 R e r (\sqrt{-1} A \omega_t \partial E (\sqrt{-1} A \omega_t F_{H(2,\omega)}(t), \tilde{T}_e(t)))_{H_{(1,\omega)}(t)} [\tilde{\Phi}_e(t) H_{(1,\omega)}^{-1}(t), \tilde{\Phi}_e(t)] H_{(1,\omega)}(t)
$$

$$
+ 4 (\sqrt{-1} A \omega_t [F_{H(1,\omega)}(t), \tilde{\Phi}_e(t)] [\tilde{\Phi}_e(t)] H_{(1,\omega)}(t).
$$

(5.37)
Integrating this over $\tilde{M}$ gives
\[
-\frac{\partial}{\partial t} \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} = 2 \int_M (|\partial_{H^{(1,\omega)}}(t)\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} + |\partial_E \tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \\
- 4\text{Re} \int_M (\sqrt{-1}A_{\omega_e} F_{H^{(2,\omega)}(t)}(t), \tilde{T}_e(t)) + \partial_{H^{(1,\omega)}(t)} \tilde{\Phi}_e(t))_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \\
+ 4 \int_M (\sqrt{-1}A_{\omega_e} [F_{H^{(1,\omega)}(t)}(t), \tilde{F}_e(t)]_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!}) \\
\geq - 8 \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(2,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{T}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \\
- 8 \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(1,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{F}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!},
\] (5.38)

and then
\[
-\frac{\partial}{\partial t} \left( t^{2n} \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \right) \geq - 2n^{2n-1} \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \\
- 8t^{2n} \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(2,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{T}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} \\
- 8t^{2n} \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(1,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{F}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!}.
\] (5.39)

By (5.33), (5.29), (5.30) and (5.23), we immediately get that
\[
t^{2n} \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \leq 2n \int_0^1 t^{2n-1} \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} dt \\
+ 8 \int_0^1 t^{2n} \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(2,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{T}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} dt \\
+ 8 \int_0^1 t^{2n} \int_M |\sqrt{-1}A_{\omega_e} F_{H^{(1,\omega)}(t)}(t)|_{H^{(1,\omega)}(t)}^2 |\tilde{F}_e(t)|_{H^{(1,\omega)}(t)}^2 \frac{\omega^n}{n!} dt \\
\leq 2n \int_0^1 t^n \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} dt + 8\hat{C}_2 \int_0^1 \int_M |\tilde{T}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} dt \\
+ 8\hat{C}_1 \int_0^1 t^n \int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} dt \\
\leq \hat{C}_4 \sqrt{\delta}
\] (5.40)

for any $t \in [0, 1]$, where $\hat{C}_4$ is a uniform constant. For any $t > 1$, (5.38), (5.31), (5.32), (5.23) and (5.40) imply that
\[
\int_M |\tilde{\Phi}_e(t)|^2_{H^{(1,\omega)}(t)} \frac{\omega^n}{n!} \leq \sqrt{\delta} \exp(\hat{C}_5 t),
\] (5.41)

where $\hat{C}_5$ is a uniform constant. Since $H_{i,e}(x, t)$ converges to the long time solution $H_i(x, t)$ outside $\Sigma_\epsilon$ in $C_{loc}^{\infty}$-topology as $\epsilon \to 0$, (5.40) and (5.41) mean the inequality (5.10).
From Lemma 2.3, we see that $t \mapsto HYM_{\alpha,N}(A(t), M, \omega)$ is nonincreasing along the Yang-Mills flow (2.22). Note that Corollary 3.4 says we can choose a sequence $t_j \to +\infty$, such that

$$HYM_{\alpha,N}(A(t_j), M, \omega) = HYM_{\alpha,N}(A_{\infty}, M, \omega).$$

Then we have

$$\lim_{t \to \infty} HYM_{\alpha,N}(A(t), M, \omega) = HYM_{\alpha,N}(A_{\infty}, M, \omega)$$

for any $\alpha \geq 1$ and any $N$. In the following we assume that $\text{Vol}(M, \omega) = 1$, and set $HYM_{\alpha,N}(\overline{\mu}) = HYM(\overline{\mu} + N) = \varphi_{\alpha}(\sqrt{-1}T(\overline{\mu} + N))$, where $\overline{\mu} + N = \text{diag}(\mu_1 + N, \ldots, \mu_R + N)$. Using Proposition 4.5, Lemma 5.2 and following the argument in Theorem 4.1 in [13], we can obtain $\hat{\lambda}_{\infty} = \hat{\mu}_0$.

We give a proof briefly for readers’ convenience.

**Theorem 5.3.** Let $\mathcal{E}$ be a reflexive sheaf on a smooth Kähler manifold $(M, \omega)$, $H(t)$ be a solution of the Hermitian-Yang-Mills flow (1.2) on $\mathcal{E}|_{M \setminus \Sigma_{\delta}}$ with the initial metric $\hat{H}$, and $A(t)$ be the related Yang-Mills flow (2.22) on the Hermitian vector bundle $E|_{M \setminus \Sigma_{\delta}}$. Assume that $A_{\infty}$ is an Uhlenbeck limit of $A(t)$, and $(E_{\infty}, H_{\infty})$ is the corresponding Hermitian vector bundle defined on $M \setminus (\Sigma_{\delta} \cup \Sigma_{\delta_0})$. Then there is a constant $p_0 > 1$ such that

$$HYM_{\alpha,N}(A_{\infty}, M, \omega) = \lim_{t \to \infty} HYM_{\alpha,N}(A(t), M, \omega) = HYM_{\alpha,N}(\overline{\mu}_0),$$

for all $1 \leq \alpha < p_0$ and all $N \in \mathbb{R}$; and the HN type of the reflexive sheaf $(E_{\infty}, \mathcal{I}_{A_{\infty}})$ is the same as that of $\mathcal{E}$, i.e. $\hat{\lambda}_{\infty} = \hat{\mu}_0$.

**Proof** As before, let $\pi : \tilde{M} \to M$ be the composition of a finite sequence of blowups resolving the sheaf $\mathcal{E}$, i.e. such that $E = \pi^*\mathcal{E}/\text{tor}(\pi^*\mathcal{E})$ is locally free. Firstly, since the norm $(\int_M \varphi_{\alpha}(u) dv)_{L^2}$ is equivalent to the $L^2$-norm on $u(E)$, we have

$$|(HYM_{\alpha,N}(\mathcal{E}, M, \omega))_{L^2} - (HYM_{\alpha,N}(\overline{\mu}_0))_{L^2}|$$

$$\leq \left( \int_M |(\varphi_{\alpha}(\frac{\sqrt{-1}}{2\pi} \Lambda_{\omega}(F(\mathcal{E}, H)) + N\text{Id}_E))_{L^2} - (\varphi_{\alpha}(\overline{\mu} + N))_{L^2}|^n d\omega \right)^{\frac{1}{n}}$$

$$\leq \left( \int_M \varphi_{\alpha}(\frac{\sqrt{-1}}{2\pi} \Lambda_{\omega}(F(\mathcal{E}, H)) - \Psi(F, (\mu_1, \ldots, \mu_l, H))_{L^2} \right)^{\frac{1}{n}}$$

$$\leq C(\alpha)^{\frac{1}{2}} \Lambda_{\omega}(F(\mathcal{E}, H)) - \Psi(F, (\mu_1, \ldots, \mu_l, H))|_{L^2}.$$ 

This together with Proposition 3.3 gives us that for any $\delta > 0$ and any $1 \leq \alpha < p_0$ there is $H$ on the bundle $E$ such that

$$HYM_{\alpha,N}(\mathcal{E}, M, \omega) \leq HYM_{\alpha,N}(\overline{\mu}_0) + \delta.$$

For fixed $\alpha$ and $N$, since the image of the degree map on line bundles is discrete, we can define $\delta_0 > 0$ such that

$$2\delta_0 + HYM_{\alpha,N}(\overline{\mu}_0) = \min\{HYM_{\alpha,N}(\overline{\mu}) : HYM_{\alpha,N}(\overline{\mu}) > HYM_{\alpha,N}(\overline{\mu}_0), \overline{\mu} \}.$$ 

(5.46)

where $\overline{\mu}$ runs over all possible HN types of torsion-free sheaves on $M$ with the rank of $\mathcal{E}$.

Let $\hat{H}$ be a smooth Hermitian metric on the holomorphic vector bundle $E$, $H(t)$ be the solution of the Hermitian-Yang-Mills flow (1.2) on $\mathcal{E}|_{M \setminus \Sigma_{\delta}}$ with the initial metric $\hat{H}$ and $A\hat{H}(t)$ be the solution of the related Yang-Mills flow (2.22) on the Hermitian vector bundle $(\mathcal{E}|_{M \setminus \Sigma_{\delta}}, \hat{H})$.
with the initial connection \( \hat{A} = (E, H) \). Let \( A^\infty \) be an Uhlenbeck limit along the Yang-Mills flow \((2.22)\). Assume that \( H_0 \) satisfies:

\[
HYM_{\alpha,N}(E, H_0, M, \omega) \leq HYM_{\alpha,N}(\mu_0) + \delta_0.
\]

(5.47)

Combining \((5.48)\), Lemma \((2.24)\) and \((5.49)\), we obtain:

\[
HYM_{\alpha,N}(\mu_0) \leq HYM_{\alpha,N}(A^\infty_0, M, \omega) \leq HYM_{\alpha,N}(\mu_0) + \delta_0.
\]

Hence we must have \( HYM_{\alpha,N}(A^\infty_0, M, \omega) = HYM_{\alpha,N}(\mu_0) \). This shows that the result holds if the metric \( H_0 \) satisfies \((5.47)\).

For any fixed \( \delta \leq \frac{\delta_0}{2} \), we denote by \( H_\delta \) the set of smooth Hermitian metrics on \( E \) satisfying that, there is \( T \geq 0 \) such that

\[
HYM_{\alpha,N}(A^\infty(t), M, \omega) < HYM_{\alpha,N}(\mu_0) + \delta,
\]

(5.48)

for all \( t \geq T \). From \((5.45)\) and the discussion above, we see \( H_\delta \) is nonempty. In Lemma \((5.2)\) we have proved the continuous dependence of the Hermitian-Yang-Mills flow \((1.2)\) on initial metrics, this implies the openness of \( H_\delta \). By Lemma \((2.2)\) and \((2.4)\), \( \| A_\omega F_{A^\infty(t)} \|_{L^\infty} \) and \( \| F_{A^\infty(t)} \|_{L^2} \) are uniformly bounded along the Yang-Mills flow \((2.22)\) for \( t \geq t_0 > 0 \). On the other hand, the Uhlenbeck compactness theorem (Theorem 5.2 in \([45]\)) is also valid for the non-compact case, i.e. on the non-compact Kähler manifold \((M \setminus \Sigma_E, \omega)\). So we can follow the argument in Lemma 4.3 in \([13]\) to show that \( H_\delta \) is closed. The proof is exactly the same, we omit it. Since the space of smooth metrics on \( E \) is connected, we conclude that every metric is in \( H_\delta \). Then it follows that \( HYM_{\alpha,N}(A^\infty, M, \omega) = \lim_{t \to +\infty} HYM_{\alpha,N}(A^\infty(t), M, \omega) = HYM_{\alpha,N}(\tilde{\mu}_0) \) for any metric \( \tilde{H} \) on \( E \). With Proposition 2.24 in \([13]\), we know \( \tilde{\mu}_0 = \tilde{\lambda}_\infty \). This concludes the proof of Theorem \(5.3\). \( \Box \)

Let \( H(t) \) be the long time solution of the Hermitian-Yang-Mills flow \((1.2)\) with the initial metric \( H \), and \( A(t) \) be the solution of the related Yang-Mills flow \((2.22)\) with the initial connection \( \hat{A} \). As that in Proposition \((2.3)\) we have \( A(t) = \sigma(t)(\hat{A}) \), where \( \sigma(t) \) is a family of gauge transformations satisfying \( \sigma^*H(t)H(t) = \hat{H} \). Consider the following HN-filtration of \( E \) by saturated sheaves

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = E.
\]

(5.49)

Let \( \pi^H_\alpha(t) \) be the orthogonal projection onto \( \mathcal{E}_\alpha \) with respect to the Hermitian metric \( H(t) \), and \( \pi^H_\alpha(t) = \sigma(t) \circ \pi^H_\alpha \circ \sigma^{-1}(t) \). It is easy to check that: \( \text{Id} - \pi^H_\alpha(t) = 0; \) \( \pi^H_\alpha(t)^2 = \pi^H_\alpha(t) \), \( \| \mathring{\nabla}_{\hat{A}(t)} \pi^H_\alpha(t) \|_{H(t)} = \| \mathring{\nabla}_{\hat{A}(t)} \pi^H_\alpha(t) \|_{H(t)} \). From \((5.4)\), it can be seen that \( \pi^H_\alpha(t) \in L^2(\text{End}(E)) \).

Using Theorem \((5.3)\) and following the same argument in \([13]\) (Proposition 4.5), we deduce the following lemma.

**Lemma 5.4.** Let \( E \) be a reflexive sheaf on a smooth Kähler manifold \((M, \omega)\), and satisfy the same assumptions as that in Theorem \(5.3\). Assume that \( A_\infty \) is an Uhlenbeck limit of \( A(t) \), and \((E_\infty, H_\infty) \) is the corresponding Hermitian vector bundle defined on \( M \setminus (\Sigma_E \cup \Sigma_{an}) \).

1. Let \( \{ \pi^H_\alpha \} \) be the HN-filtration of the reflexive sheaf \((E_\infty, \mathcal{F}_{A_\infty})\), then there is a sequence of \( \{ \pi^H_\alpha(t) \} \) which converges to \( \{ \pi^\infty_\alpha \} \) strongly in \( L^p \cap L^2_{1,\text{loc}} \) outside \( \Sigma_E \cup \Sigma_{an} \) as \( j \) tends to \(+\infty\).

2. Assume the sheaf \( E \) is semi-stable and \( \{ \mathcal{E}_\alpha \} \) is the Seshadri filtration of \( E \), then \( \{ \pi^H_\alpha(t) \} \) converges to a filtration \( \{ \pi^\infty_\alpha \} \) strongly in \( L^p \cap L^2_{1,\text{loc}} \) outside \( \Sigma_E \cup \Sigma_{an} \) as \( j \) tends to \(+\infty\), the rank and degree of \( \pi^\infty_\alpha \) is equal to the rank and degree of \( \pi^H_\alpha(t) \) for all \( \alpha \) and \( j \).
6. Proof of Theorem 1.1.

In this section, we will prove the part (2) of Theorem 1.1 inductively on the length of the HNS-filtration. The inductive hypotheses are following:

**Inductive hypotheses:** Let $Q$ be a torsion-free sheaf on a compact Kähler manifold $(M, \omega)$, $S$ be a saturated sub-sheaf of $Q$.

1. There is a sequence of connections $A_j^Q \in A_{Q, H_0}^{1,1}$ on the Hermitian bundle $(Q|_{M \setminus \Sigma}; H_0)$ such that $A_j^Q \to A_{\infty}^Q$ in $C^\infty_{\text{top}}$-topology off $\Sigma$ as $j \to +\infty$, where $\Sigma$ is a complex analytic subset of $M$ of complex codimension at least 2 and satisfies $\Sigma_Q \cup \Sigma_S \subset \Sigma$.

2. $A_j^Q = g_j(A_{H_0}^Q)$ for some complex gauge transformations $g_j \in G^\infty(Q)$ and $\|\sqrt{-1}\Lambda_\omega(F_{A_{H_0}^Q})\|_{L^1(\omega)}$ is uniformly bounded in $j$, where $A_{H_0}^Q$ is the Chern connection on $Q$ with respect to the metric $H_0 = H_0g_j^{-1}g_j$.

3. There exists a sequence of blow-ups with smooth center: $\pi_i : M_i \to M_{i-1}$, $i = 1, \ldots, r$, and an exact sequence of holomorphic vector bundles

$$0 \to \tilde{S} \to \tilde{Q} \to \tilde{W} \to 0$$

(6.1)

over $\tilde{M}$, such that the composition $\pi = \pi_r \circ \cdots \circ \pi_1 : \tilde{M} \to M$ is biholomorphic outside $\tilde{\Sigma}$, $\tilde{S}$ and $\tilde{Q}$ are isomorphic to $S$ and $Q$ outside $\tilde{\Sigma}$ respectively, where $M_0 = M$, $\tilde{M} = M_r$.

4. Set $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ and define Kähler metrics $\omega_\epsilon$ on $M_r$ as that in (2.2). For every $j$, there exists a sequence of metrics $H_{j,\epsilon}$ on $\tilde{Q}$ such that $H_{j,\epsilon} \to H_j$ in $C^\infty_{\text{top}}$-topology outside $\tilde{\Sigma}$ as $\epsilon \to 0$, $\|\sqrt{-1}\Lambda_\omega(F_{A_{H_{j,\epsilon}}^Q})\|_{L^1(\omega_\epsilon)}$ is uniformly bounded, and $\sup_{M_r}(\text{tr}(H_{j,\epsilon}^{-1}H_{j,\epsilon}) + \text{tr}(H_{j,\epsilon}^{-1}H_{1,\epsilon})) < C_j$, where $C_j$ is a constant independent of $\epsilon$. Furthermore, $\|\sqrt{-1}\Lambda_\omega(F_{A_{H_{1,\epsilon}}^Q})\|_{L^1(\omega_\epsilon)}$ is uniformly bounded, where $\tilde{H}_{1,\epsilon}$ is the induced Chern connection on $\tilde{S}$.

5. Two torsion-free sheaves $Q$ and $(Q_\infty, \mathcal{F}_{A_{Q_\infty}})$ have the same HN type.

Now we construct non-zero holomorphic maps from subsheaves in the HNS-filtration of $E$ to the limiting reflexive sheaf $(E_\infty, \mathcal{F}_\infty)$. We get a non-zero holomorphic map which we need by limiting a sequence of holomorphic maps. The key problem is to obtain local uniform estimate of this sequence of holomorphic maps. We will follow the argument in Proposition 4.1 in [32] to handle this problem. There is a difference in the assumption for our case, so we write a proof briefly of the following proposition for readers’ convenience.

**Proposition 6.1.** Let $Q$ be a torsion-free sheaf on a compact Kähler manifold $(M, \omega)$, $S$ be a saturated sub-sheaf of $Q$. Assume that the conditions (1), (2), (3), (4) in the above inductive hypotheses are satisfied. Let $i_0 : S \to Q$ be the holomorphic inclusion, then there is a subsequence of $g_j \circ i_0$, up to rescale, converges to a non-zero holomorphic map $f_\infty : S \to (Q_\infty, \mathcal{F}_{A_{Q_\infty}}) \subset C^\infty_{\text{top}}$-topology off $\tilde{\Sigma}$ as $j \to +\infty$.

**Proof.** By induction, we can assume that $\pi : \tilde{M} \to M$ is a single blow-up with smooth centre. Fix a Kähler metric $\eta$ on $\tilde{M}$ and set $\omega_\epsilon = \pi^* \omega + c_\eta$ for $0 < \epsilon \leq 1$. On the blow-up $\tilde{M}$, let $H_{j,\epsilon}(t)$ and $H_{1,\epsilon}(t)$ be the solutions of the following Hermitian-Yang-Mills flow on holomorphic bundles $Q$ and $S$ with the fixed initial metrics $H_{j,\epsilon}$ and $H_{1,\epsilon}$ and with respect to the metric $\omega_\epsilon$, i.e. they satisfy the following heat equation

$$H^{-1}(t) \frac{\partial H(t)}{\partial t} = -2\sqrt{-1}\Lambda_\omega(F_H(t)), \quad (6.2)$$

where
where $H_{j,\epsilon}$ is defined in condition (4) among the inductive hypotheses. A direct computation yields
\[
(\Delta_\epsilon - \frac{\partial}{\partial t})|\Lambda_\omega(F_{H_{j,\epsilon}})_{H_{j,\epsilon}(t)}| \geq 0,
\]
(6.3)
and
\[
(\Delta_\epsilon - \frac{\partial}{\partial t})|\Lambda_\omega(F_{H_{j,\epsilon}^\delta})_{H_{j,\epsilon}(t)}| \geq 0,
\]
(6.4)
and
\[
(\Delta_\epsilon - \frac{\partial}{\partial t})|\iota_0|^2_{H_{j,\epsilon}^\delta, H_{j,\epsilon}(t)} \geq 0.
\]
(6.5)
The maximum principle implies that
\[
|\Lambda_\omega(F_{H_{j,\epsilon}})_{H_{j,\epsilon}(t)}(x)| \leq \int_M K_\epsilon(t - t_0, x, y)|\Lambda_\omega(F_{H_{j,\epsilon}(t_0)})|_{H_{j,\epsilon}(t_0)} \frac{\omega^n}{n!},
\]
(6.6)
and
\[
|\Lambda_\omega(F_{H_{j,\epsilon}^\delta})_{H_{j,\epsilon}^\delta}(t_0, t)\rangle \leq \int_M K_\epsilon(t - t_0, x, y)|\Lambda_\omega(F_{H_{j,\epsilon}^\delta}(t_0))|_{H_{j,\epsilon}^\delta(t_0)} \frac{\omega^n}{n!},
\]
(6.7)
and
\[
|\iota_0|^2_{H_{j,\epsilon}^\delta, H_{j,\epsilon}(t_0, t_0+\epsilon)} \leq \int_M K_\epsilon(t - t_0, x, y)|\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t_0)} \frac{\omega^n}{n!},
\]
(6.8)
for any $t > t_0 \geq 0$. By [7] (Lemma 4), the heat kernels $K_\epsilon(t, x, y)$ have a uniform bound for $0 < \epsilon \leq 1$. Following Bando and Siu’s argument ([7]), we could choose a subsequence of $H_{j,\epsilon}(t)$ (and the same for $H_{j,\epsilon}^\delta(t)$) which converges to $H_j(t)$ (resp. $H_j^\delta(t)$) of the Hermitian-Yang-Mills flow (6.2) on $Q$ (resp. $S$) over $M \setminus \bar{\Sigma}$ as $\epsilon$ tends to 0. Combining (6.6), (6.7), (6.8) and the condition (4), we derive
\[
2(|\Lambda_\omega(F_{H_j(t)})|_{H_j(t)} + |\Lambda_\omega(F_{H_j^\delta(t)})|_{H_j^\delta(t)}) \leq C_F
\]
(6.9)
and
\[
|\iota_0|^2_{H_{j,\epsilon}^\delta, H_{j,\epsilon}(t_0, t_0+\epsilon)} \leq \int_M K(t - t_0, x, y)|\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t_0)} \frac{\omega^n}{n!},
\]
(6.10)
for all $x \in \bar{\Sigma}$ and $t > t_0 \geq 0$, where $K(t, x, y)$ is the heat kernel of $(M, \omega)$ and $C_F$ is a uniform constant which is independent of $j$.

From (6.9), it follows that
\[
\left| \frac{\partial}{\partial t} \ln |\iota_0|^2_{H_{j,\epsilon}^\delta, H_{j,\epsilon}(t)} \right| \leq 2(|\Lambda_\omega(F_{H_j(t)})|_{H_j(t)} + |\Lambda_\omega(F_{H_j^\delta(t)})|_{H_j^\delta(t)}) \leq C_F,
\]
(6.11)
for all $x \in M \setminus \bar{\Sigma}$ and $t > t_0 > 0$. Then
\[
e^{-C_F \delta} \leq \frac{|\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t)} \frac{\omega^n}{n!} \langle x \rangle \leq e^{C_F \delta},
\]
(6.12)
and
\[
|\iota_0|^2_{H_{j,\epsilon}^\delta(t)} \leq e^{C_F \delta} |\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t)} \frac{\omega^n}{n!} \langle x \rangle \leq e^{C_F \delta} \int_M K_\epsilon(t, x, y)|\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t)} \frac{\omega^n}{n!} \langle x \rangle
\]
(6.13)
\[
\leq C_K e^{C_F \delta} (1 + \delta^{-n}) \int_M |\iota_0|^2_{H_{j,\epsilon}^\delta(t_0, t)} \frac{\omega^n}{n!} \langle x \rangle,
\]
for all $x \in M \setminus \bar{\Sigma}$ and $\delta > 0$.

Denote $h_{j,\epsilon}(t) = H_j^{-1}H_{j,\epsilon}(t)$, and then the heat equation (6.2) yields
\[
(\Delta_\epsilon - \frac{\partial}{\partial t}) \ln (tr (h_{j,\epsilon}(t))) + tr (h_{j,\epsilon}^{-1}(t))) \geq -2|\Lambda_\omega(F_{H_{j,\epsilon}})|_{H_{j,\epsilon}}.
\]
(6.14)
Integrating the above inequality and using the condition (4), we have
\[ \int_M \ln(tr(h_j, e(t)) + tr(h_j^{-1}(t))) \frac{\omega^n}{n!} - \ln 2 \text{rank}(\Omega) \text{Vol}(M, \omega) \leq t C_h \]  
(6.15)
and then
\[ \int_M \ln(tr(h_j(t)) + tr(h_j^{-1}(t))) \frac{\omega^n}{n!} - \ln 2 \text{rank}(\Omega) \text{Vol}(M, \omega) \leq t C_h. \]  
(6.16)
On the other hand, it holds that
\[ \Delta \ln(tr(h_j(t)) + tr(h_j^{-1}(t))) \geq -2|A_\omega(F_{H_j(1)}(t))| - 2|A_\omega(F_{H_j})|H_j \]  
(6.17)
on \( M \setminus \tilde{\Sigma} \), for all \( t > 0 \). Here, we should note that \( |A_\omega(F_{H_j})|H_j = |A_\omega(F_{A_j^Q})|H_0 \).

For any compact subset \( \Omega \subset M \setminus \tilde{\Sigma} \), the condition (1) implies that \( |A_\omega(F_{H_j})|H_j = |A_\omega(F_{A_j^Q})|H_0 \) is uniformly bounded on \( \Omega \). By (6.16), (6.17), and the Moser’s iteration, there must exist a uniform constant \( C_{1, Q} \) such that, for all \( j \),
\[ \sup_{x \in \Omega} \ln(tr(h_j(1)) + tr(h_j^{-1}(1))) \leq C_{1, Q}. \]  
(6.18)
Define the holomorphic map \( \tilde{\eta}_j : (\mathcal{S}|_{M \setminus \tilde{\Sigma}}, \partial_{A_j^Q}) \rightarrow (\mathcal{Q}|_{M \setminus \tilde{\Sigma}}, \partial_{A_j^Q}) \) by \( \tilde{\eta}_j = g_j \circ i_0 \), where \( A_0^Q \) is the induced connection on \( \mathcal{S} \) by the connection \( A_0^Q \). It is easy to check that
\[ |\tilde{\eta}_j|H^Q_{1, H_0} = |i_0|_{H^Q_{1, H_j}}, \]  
(6.19)
where \( H^Q_1 \) is the induced metric on \( \mathcal{S} \) by the metric \( H_1 \). Set
\[ f_j = \left( \int_M |i_0|_{H^Q_{1, H_j}}^2 \omega^n \right)^{\frac{1}{n}} \tilde{\eta}_j. \]  
(6.20)
Clearly (6.18) means that there is a constant \( C_a \) such that
\[ \sup_{x \in M \setminus \tilde{\Sigma}} \left( \int_M |i_0|_{H^Q_{1, H_j}}^2 \omega^n \right)^{\frac{1}{n}} |i_0|_{H^Q_{1, H_j}}(x) \leq C_a, \]  
(6.21)
for all \( j \). Using (6.21) and (6.18), we obtain a local uniform \( C^0 \)-estimate on \( f_j \), i.e. for any compact subset \( \Omega \subset M \setminus \tilde{\Sigma} \), there is a constant \( C_{\omega, f} \) such that
\[ \sup_{x \in \Omega} |f_j|H^Q_{1, H_0}(x) \leq C_{\omega, f} \]  
(6.22)
for all \( j \). By the above local uniform \( C^0 \)-bound of \( f_j \) and the assumption that \( A_j \rightarrow A_\infty \) in \( \mathcal{C}^\infty_{\text{loc}} \)-topology outside \( \tilde{\Sigma} \) as \( j \rightarrow +\infty \), the elliptic theory implies that there exists a subsequence of \( f_j \) (for simplicity, also denoted by \( f_j \)) such that \( f_j \) converges to a holomorphic map \( f_\infty : \mathcal{S} \rightarrow (Q_\infty, \partial_{A_\infty^Q}) \) in \( \mathcal{C}^\infty_{\text{loc}} \)-topology outside \( \tilde{\Sigma} \) as \( j \rightarrow +\infty \). Now we only need to prove that \( f_\infty \) is non-zero. Since \( \tilde{\Sigma} \) is of Hausdorff complex codimension at least 2, for any small \( \delta > 0 \), we can choose a compact subset \( \Omega_\delta \subset M \setminus \tilde{\Sigma} \) such that
\[ \int_{M \setminus \Omega_\delta} \frac{\omega^n}{n!} \leq \delta. \]  
(6.23)
Of course the local uniform estimate (6.18) gives us that there is a positive constant \( C_\delta \) such that
\[ C_\delta |i_0|_{H^Q_{1, H_j}}^2 \leq |i_0|_{H^Q_{1, H_j}}(x) \leq C_\delta^{-1} |i_0|_{H^Q_{1, H_j}}^2 (x), \]  
(6.24)
for all $x \in \Omega_\delta$ and $j$. Then

$$
\int_{M \setminus \Omega_\delta} |f_\infty|_{H^2_{ij}, H_0} \frac{\omega^n}{n!} = \lim_{j \to +\infty} \int_{M \setminus \Omega_\delta} |f_j|_{H^2_{ij}, H_0} \frac{\omega^n}{n!}
= \lim_{j \to +\infty} \left( \int_M |i_0|^2_{H^2_{ij}(1), H_j(1)} \frac{\omega^n}{n!} \right) ^{-1} \int_{M \setminus \Omega_\delta} \frac{\omega^n}{n!} \geq C_S \left( \int_M |i_0|^2_{H^2_{ij}(1), H_j(1)} \frac{\omega^n}{n!} \right) ^{-1} \int_{M \setminus \Omega_\delta} \frac{\omega^n}{n!}
\geq C_S (1 - \delta C_\alpha) > 0.
$$

(6.25)

Therefore $f_\infty$ is a non-zero holomorphic map. This concludes the proof of Proposition 6.1.

\[ \square \]

A proof of Theorem 1.1 Let $\{E_\alpha\}_{\alpha=1}^l$ be the Harder-Narasimhan-Seshadri filtration of $E$. $G_{H^{HNS}}(E) = \bigoplus_{\alpha=1}^l Q_\alpha$ be the associated graded object, where $Q_\alpha = E_\alpha/E_{\alpha-1}$ is torsion-free for each $1 \leq \alpha \leq l$. We refer to $\Sigma_{HNS}$ as the singularity set of the HNS-filtration, it is a complex analytic subset of $M$ with complex codimension at least 2.

According to Hironaka’s flattening theorem ([19]), there is a finite sequence of blowing ups $\{\pi_1\}^{l}_{i=1}$ along compact sub-manifolds such that if we denote by $\tilde{E}$ the pull back metric on the bundle $\tilde{E}$ and $E_j$ be the solution of the related Yang-Mills flow on the holomorphic vector bundle $(\tilde{E}, \tilde{\pi})$, then we have $\tilde{E} = E_j$.

(6.27)

by subbundles. If we write $\Lambda \tilde{E}_\alpha$, the image of $\tilde{\pi}^* \tilde{E}_\alpha \to \tilde{\sigma}^* E$, then $E_i = \text{Sat}_{\tilde{\pi}} (\Lambda \tilde{E}_\alpha)$.

If $Q_\alpha = E_\alpha/E_{\alpha-1}$, then we have $\tilde{\sigma} E_\alpha = \tilde{E}_\alpha$ and $Q_\alpha^* = (\tilde{\sigma} E_\alpha)^*$. Now set $\tilde{\pi} = \tilde{\sigma} \circ \pi : \tilde{M} \to M$. We know $\tilde{\pi}^* E_\alpha$ and $Q_\alpha^*$ is torsion-free. It is easy to see that $\tilde{\pi}$ is biholomorphic outside $\pi^{-1}(\Sigma_{HNS})$, $E_\alpha$ and $Q_\alpha$ are isomorphic to $\tilde{\pi}^* E_\alpha$ and $Q_\alpha$ outside $\tilde{\Sigma}$ respectively.

Let $H(t)$ be the long time solution of the Hermitian-Yang-Mills flow ([12] on the holomorphic vector bundle $E_{|\Sigma_E}$ with the initial metric $\tilde{H}$, and $A(t)$ be the solution of the related Yang-Mills flow ([22] on the Hermitian vector bundle $(E_{|\Sigma_E}, \tilde{H})$ with the initial connection $\tilde{A}$. We have $A(t) = \sigma(t) (\tilde{A})$, where $\sigma$ satisfies $\sigma^* \tilde{H}(t) \sigma(t) = \tilde{H}(t) \to H^{-1} H(t)$. Note that Lemma 2.2 says there is a sequence of heat flows $H_j(t)$ on the holomorphic vector bundle $E$ which converges successively to $H(t)$ in $C^{\infty}_{loc}$-topology outside $\Sigma_E$ as $(\epsilon_1, \cdots, \epsilon_k) \to 0$. In the sequel, we denote by $\overline{H}_t(t) = \tilde{\sigma}^* H_j(t)$ the pull back metric on the bundle $\tilde{E}$.

Theorem 3.3 and Proposition 5.6 imply the part (1) of Theorem 1.1. So we only need to prove the part (2) of Theorem 1.1. We assume there is a sequence of connections $A(t_j)$ which converges to $A_\infty$ in $C^{\infty}_{loc}$-topology outside $\Sigma$ as $j \to +\infty$. Let $\tilde{\mathcal{S}} = \tilde{E}_1$ be the first $\omega$-stable sub-sheaf corresponding to the above HNS-filtration, $\tilde{Q} = \tilde{E}_1$, and $g_j = \sigma(t_j)$. Using the formulas ([2], [12], [5.4], Lemma 2.2, Theorem 5.3) and considering the metrics $\overline{H}_t(t)$, one can check easily that the conditions (1), (2), (3), (4) in the above inductive hypotheses are satisfied. Based on Theorem 3.3 we suppose that there exists a sequence of isomorphisms

$$
a_j : (E_{|M \setminus \Sigma}, \tilde{H}) \to (E_{\infty}, H_\infty)
$$

(6.28)
such that \((a_j^{-1})^* (A(t_j)) \to A_{\infty}\) in \(C^\infty_{loc}-\text{topology}\) outside \(\Sigma\) as \(j \to +\infty\). Let \(i_0 : \mathcal{E}_1 \to \mathcal{E}\) be the holomorphic inclusion, by Proposition \ref{prop:4.1}, then there is a subsequence of \(f_j = a_j \circ g_j \circ i_0\), up to rescale, converging to a non-zero holomorphic map \(f_\infty : \mathcal{E}_1 \to (E_\infty, \overline{\partial}_{A_\infty})\) outside \(\Sigma_{HNS} \cup \Sigma_{an}\) as \(j \to +\infty\). Applying Hartog’s theorem, we can extend \(f_\infty\) to the whole \(M\) as a sheaf homomorphism.

Let \(\tilde{\pi}_1^{H(t)} : \mathcal{E} \to \mathcal{E}\) be the orthogonal projection onto \(\mathcal{E}_1\) with respect to the Hermitian metric \(H(t)\), and \(\pi_1^{H(t)} = \sigma(t) \circ \pi_1^{H(t)} \circ \sigma^{-1}(t)\). Set \(\tilde{\pi}_1^j = a_j \circ \pi_1^{H(t)} \circ a_j^{-1}\). From Lemma \ref{lem:5.2}, we know that \(\tilde{\pi}_1^j \to \tilde{\pi}_1^\infty\) strongly in \(L^p \cap L^2_{loc}\) outside \(\Sigma_{HNS} \cup \Sigma_{an}\) as \(j \to +\infty\), and \(\pi_1^\infty\) determines a subsheaf \(E_1^\infty\) of \((E_\infty, \overline{\partial}_{A_\infty})\), with rank\((E_1^\infty) = \text{rank}(\mathcal{E}_1)\) and \(\mu_\omega(E_1^\infty) = \mu_\omega(\mathcal{E}_1)\). Because \(\tilde{\pi}_1^j \circ f_j = f_j\) for all \(j\), we see that in the limit \(\pi_1^\infty \circ f_\infty = f_\infty\), and then

\[
 f_\infty : \mathcal{E}_1 \to E_1^\infty. \tag{6.29}
\]

Moreover, Theorem \ref{thm:5.3} tells us that \((E_\infty, \overline{\partial}_{A_\infty})\) and \(\mathcal{E}\) have the same HN type, and then the subsheaf \(E_1^\infty\) is \(\omega\)-semistable. Recalling that \(\mathcal{E}_1\) is \(\omega\)-stable, with the result in \[26\] (V.7.11; 7.12), we observe that the non-zero holomorphic map \(f_\infty\) must be injective, then

\[
 \mathcal{E}_1 \simeq E_1^\infty = f_\infty(\mathcal{E}_1), \tag{6.30}
\]

and \(E_1^\infty\) is an \(\omega\)-stable subsheaf of \((E_\infty, \overline{\partial}_{A_\infty})\).

Let \(\{e_\alpha\}\) be a local frame of \(\mathcal{E}_1\), and \(H_{j,\alpha}\beta = (f_j(e_\alpha), f_j(e_\beta))\). We derive

\[
 \tilde{\pi}_1^j(X) = (X, f_j(e_\beta)) \circ H_{j,\alpha}\beta f_j e_\alpha \tag{6.31}
\]

for any \(X \in \Gamma(E)\), where \((H_{j,\alpha}\beta)\) is the inverse of the matrix \((H_{j,\alpha}\beta)\). Because \(f_j \to f_\infty\) in \(C^\infty_{loc}-\text{topology}\) as \(j \to +\infty\), and \(f_\infty\) is injective, we can prove that \(\tilde{\pi}_1^j \to \pi_1^\infty\) in \(C^\infty_{loc}-\text{topology}\) off \(\Sigma_{an} \cup \Sigma_{HNS}\) as \(j \to +\infty\).

Consider the orthogonal holomorphic decomposition \((E_\infty, \overline{\partial}_{A_\infty}) = E_1^\infty \oplus Q_\infty\), where \(Q_\infty = (E_1^\infty)^\perp\). Let \(\hat{\pi}_1 : E_\infty \to E_\infty\) be the projection onto \(E_1^\infty\) with respect to the metric \(H_\infty\). Using Lemma \ref{lem:5.12} in \[12\], we can choose a sequence of unitary gauge transformations \(\hat{u}_j\) such that \(\hat{\pi}_1^{(j)} = \hat{u}_j \hat{\pi}_1 \hat{u}_j^{-1}\) and \(\hat{u}_j \to I_E\) in \(C^\infty_{loc}-\text{topology}\) on \(M \setminus (\Sigma_{HNS} \cup \Sigma_{an})\) as \(j \to +\infty\). It is easy to check that \(\hat{u}_j(Q_\infty) = \hat{u}_j((E_1^\infty)^\perp) = (\hat{\pi}_1^{(j)}(E_\infty))^\perp\), and the unitary gauge transformation \(\hat{u}_0 : E_\infty = E_\infty\) satisfies \(a_0^{-1} \circ \hat{u}_0(Q_\infty) = a_0^{-1} \circ \hat{u}_0((E_1^\infty)^\perp) = E_1^{1\perp}\).

Set \(Q = \mathcal{E}/\mathcal{E}_1\), then we have \(Gr_{HNS}(\mathcal{E}) = S \oplus Gr_{HNS}(Q)\). Denote by \(p^{1\perp} = I - \pi_1^{1\perp} : Q \to \mathcal{E}_1^{1\perp}\) the induced bundle isomorphisms on \(M \setminus \Sigma_{HNS}\), and consider the induced connections on \(Q\)

\[
 D_{A_j^Q} = (p_{1\perp})^{-1} \circ a_0^{-1} \circ \hat{u}_0 \circ \hat{\pi}_1^{1\perp} \circ \hat{u}_j^{-1} \circ a_j \circ D_{A_j} \circ a_j^{-1} \circ \hat{u}_j \circ \hat{\pi}_1^{1\perp} \circ \hat{u}_0^{-1} \circ a_0 \circ p_{1\perp}, \tag{6.32}
\]

and the complex gauge transformation

\[
 h_j = (p_{1\perp})^{-1} \circ a_0^{-1} \circ \hat{u}_0 \circ \hat{\pi}_1^{1\perp} \circ \hat{u}_j^{-1} \circ a_j \circ g_j \circ p_{1\perp} \in G^C(Q). \tag{6.33}
\]

Then it holds that

\[
 \overline{\partial}_{A_j^Q} = h_j \circ \overline{\partial}_{A_j} \circ h_j^{-1}, \tag{6.34}
\]

and

\[
 \partial_{A_j^Q} = (h_j^*)^{-1} \circ \partial_{A_j^Q} \circ h_j^*, \tag{6.35}
\]

where we have used the facts \((\pi_1^{(t_j)})^\perp \circ g_j = (\pi_1^{(t_j)})^\perp \circ g_j \circ (\pi_1^{R})^\perp \) and \(h_j^{-1} = (p_{1\perp})^{-1} \circ (\pi_1^{t_0})^\perp \circ g_j^{-1} \circ a_j^{-1} \circ \hat{u}_j \circ \hat{u}_0^{-1} \circ a_0 \circ p_{1\perp}\). By the definition, it is easy to check that \(((p_{1\perp})^{-1} \circ a_0^{-1} \circ \hat{u}_0)(A_j^Q) \to A_j^{Q\perp}\) in \(C^\infty_{loc}-\text{topology}\) as \(j \to +\infty\), and \(h_j^* h_j = (H^Q)^{-1} H^Q(t_j)\), where \(H^Q(t)\) denotes the
induced metric on the quotient \( Q \) by \( H(t) \). Combining (5.4) and Lemma 2.2, we get that 
\[ \|\sqrt{-1}\Lambda \omega (F^Q_{A_H(t)})\|_{L^1(\omega)} \] is uniformly bounded for \( t \geq t_0 > 0 \). So inductive hypotheses (1) and (2) are satisfied.

Let \( S = Q_2 = E_2/E_1 \), then (6.27) implies the inductive hypothesis (3). Considering the induced metric \( H^Q(t) \) on the quotient \( \hat{E}/E_1 \) by \( \sigma^*(H_\epsilon(t)) \), from the formulas (2.12), (5.4) and Lemma 2.2 we see that the inductive hypothesis (4) is valid. Using Theorem 5.3 and Lemma 5.4, one can check easily that the inductive hypothesis (5) is also valid. Repeating the above argument, we obtain an isomorphism

\[ f : (E_\infty, \overline{\mathcal{J}}_{A_\infty}) \to Gr^{HNS}(E) = \bigoplus_{\alpha=1}^l Q_\alpha \] (6.36)
on \( M \setminus (\Sigma_{HNS} \cup \Sigma_{an}) \). By the uniqueness of reflexive extension in [40], we know that \( f \) can be extended to a sheaf isomorphism on the whole \( M \). This completes the proof of Theorem 1.1.

\[ \blacksquare \]

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Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, and AMSS, CAS, Beijing, 100080, P.R. China,
E-mail address: jiayuli@ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P.R. China,
E-mail address: chjzhang@mail.ustc.edu.cn

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P.R. China,
E-mail address: mathzx@ustc.edu.cn