ON SOME SUBCLASS OF HARMONIC CLOSE-TO-CONVEX MAPPINGS

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Abstract. Let $\mathcal{H}$ denote the class of harmonic functions $f$ in $D := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) − 1$. For $\alpha \geq 0$, we consider the following class $W_0^\alpha(\mathcal{H}) := \{f = h + g \in \mathcal{H}: \text{Re}(h'(z) + \alpha zh''(z)) > |g'(z) + \alpha zg''(z)|, \ z \in \mathbb{D}\}$. In this paper, we first prove the coefficient conjecture of Clunie and Sheil-Small for functions in the class $W_0^\alpha(\mathcal{H})$. We also prove growth theorem, convolution, convex combination properties for functions in the class $W_0^\alpha(\mathcal{H})$. Finally, we determine the value of $r$ so that the partial sums of functions in the class $W_0^\alpha(\mathcal{H})$ are close-to-convex in $|z| < r$.

1. Introduction

Let $\mathcal{H}$ be the class of complex-valued harmonic functions $f$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) − 1$. Any function $f$ in $\mathcal{H}$ has the canonical representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$  \hspace{1cm} (1.1)

Here both $h$ and $g$ are analytic functions in $\mathbb{D}$ and are called analytic and co-analytic part of $f$ respectively. In particular, for $g(z) = 0$, the class $\mathcal{H}$ reduces to the class $\mathcal{A}$, consisting of analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. If $f = h + \overline{g}$ then the Jacobian $J_f(z)$ of $f$ is defined by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and we say $f$ is sense preserving if $J_f(z) > 0$ in $\mathbb{D}$. Let $S_\mathcal{H}$ be the subclass of $\mathcal{H}$ consisting of univalent (that is, one-to-one) and sense preserving harmonic mappings. If $g(z) = 0$ in $\mathbb{D}$ then the class $S_\mathcal{H}$ reduces to the class $S$, containing univalent analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Furthermore, if $f = h + \overline{g} \in S_\mathcal{H}$ then $|g'(0)| = |b_1| < 1$ (as $J_f(0) = 1 - |g'(0)|^2 = 1 - |b_1|^2 > 0$). Thus function

$$F(z) = \frac{f - b_1 \overline{f}}{1 - |b_1|^2}$$

belongs to the class $S_\mathcal{H}$. Clearly $F$ is univalent because $F$ is an affine mapping of $f$. A simple observation shows that $F_{\overline{f}}(0) = 0$. Thus we may restrict our attention to the following subclass

$$S_\mathcal{H}^0 := \{f \in S_\mathcal{H} : b_1 = \overline{f(0)} = 0\}.$$
Hence for any function \( f = h + g \) in \( S^0_H \), its analytic and co-analytic parts can be represented by

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n.
\]

The family \( S^0_H \) is known to be compact and normal, whereas \( S_H \) is normal but not compact. In 1984, Clunie and Sheil-Small \([6]\) investigated the class \( S_H \) and its geometric subclasses. Since then, the class \( S_H \) and its subclasses have been extensively studied (see \([4], [5], [6], [14], [32]\)). A domain \( \Omega \) is called starlike with respect to a point \( z_0 \in \Omega \) if the line segment joining \( z_0 \) to any point in \( \Omega \) lies in \( \Omega \). In particular if \( z_0 = 0 \) then \( \Omega \) is called starlike domain. A complex-valued harmonic mapping \( f \in \mathcal{H} \) is said to be starlike if \( f(D) \) is starlike domain with respect to the origin. The class of harmonic starlike functions in \( D \) is denoted by \( S^* \). A domain \( \Omega \) is said to be convex domain if it is starlike with respect to every point in \( \Omega \). A function \( f \) in \( \mathcal{H} \) is said to be convex if \( f(D) \) is convex. The class of harmonic convex mappings in \( D \) is denoted by \( K \). A domain \( \Omega \) is said to be close-to-convex if the complement of \( \Omega \) can be written as union of non-intersecting half lines. A function \( f \in \mathcal{H} \) is called close-to-convex in \( D \) if \( f(D) \) is close-to-convex domain. The class of harmonic close-to-convex mappings in \( D \) is denoted by \( C \). If \( f(z) = 0 \) then the classes \( S^*_H \), \( K \) and \( C \) reduce to \( S^*_0 \), \( K^*_0 \) and \( C^*_0 \) respectively. In analytic case, \( S^*(K \) and \( C \) respectively) is a subclasses of \( S \) that contains functions \( f \) such that \( f(D) \) is starlike (convex and close-to-convex respectively) domain.

Clunie and Sheil-Small \([6]\) proved the following result which gives a sufficient condition for a harmonic function \( f \) to be close-to-convex.

**Lemma 1.3.** \([6]\) Suppose \( h \) and \( g \) are analytic in \( D \) with \( |g'(0)| < |h'(0)| \) and \( h + \epsilon g \) is close-to-convex analytic function for each \( \epsilon \) (\( |\epsilon| = 1 \)). Then \( f = h + g \) is harmonic close-to-convex in \( D \).

A classical problem for functions in the class \( S \) where functions are of the form

\[
f(z) = z + \sum_{n=0}^{\infty} a_n z^n
\]

is to find the sharp upper bound for the absolute value of the coefficients \( a_n \) for \( n \geq 2 \). In 1916, Bieberbach \([2]\) proved that if \( f \in \mathcal{S} \) then \( |a_2| \leq 2 \) and conjectured that \( |a_n| \leq n \) for \( n \geq 2 \). In 1985, de Branges \([3]\) proved this conjecture affirmatively. The following analogous type of conjecture was proposed by Clunie and Sheil-Small \([6]\) for functions in the class \( S^*_H \).

**Conjecture 1.** Let \( f = h + g \) belong to \( S^*_H \), where the representation of \( h \) and \( g \) are given by \((1.2)\). Then \( |a_n| \leq A_n \) and \( |b_n| \leq B_n \) for \( n \geq 2 \) where

\[
A_n = \frac{(2n + 1)(n + 1)}{6} \quad \text{and} \quad B_n = \frac{(2n - 1)(n - 1)}{6}.
\]

Clunie and Sheil-Small \([6]\) verified Conjecture \(1\) for typically real functions. In 1990, Sheil-Small \([30]\) proved Conjecture \(1\) for functions \( f \) in \( S^*_H \) such that \( f(D) \) is starlike with respect to the origin or \( f(D) \) is convex in one direction. In 2001,
Wang and Liang [32] proved Conjecture 1 for functions in the class \( C_0^H \). However, Conjecture 1 is still open for the class \( S_0^H \). Equality occurs in (1.4) for the harmonic Koebe function

\[
K(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} + \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3}.
\]

A harmonic function \( f \in H \) is said to be fully starlike (resp. fully convex) if each \(|z| < r\) is mapped onto a starlike (resp. convex) domain. In 2013, Nagpal and Ravichandran [16] proved the following theorem.

**Theorem A.** [16] A sense preserving harmonic function \( f = h + \phi \) is fully starlike in \( D \) if the analytic functions \( h + \epsilon g \) are starlike in \( D \) for each \( \epsilon \) (\(|\epsilon| = 1\)).

In 2013, Li and Ponnusamy [11] investigated the properties of functions in the class

\[
P_0^H := \{ f = h + \phi \in H : \text{Re}(h'(z)) > |g'(z)|, \quad z \in D \}.
\]

The class \( P_0^H \) is closely related to the class \( R := \{ f \in S : \text{Re}(f'(z)) > 0, \quad z \in D \} \) which was introduced by MacGregor [15]. It has been proved that a harmonic function \( f = h + \phi \) belongs to the class \( P_0^H \) if and only if the analytic functions \( h + \epsilon g \) belong to \( R \) for each \( \epsilon \) (\(|\epsilon| = 1\)) (see [11], [12]). Using this property, Li and Ponnusamy [11] have obtained the coefficient bounds and radius of convexity for the functions in the class \( P_0^H \).

In 1977, Chichra [7] studied the subclass \( W(\alpha) \), consisting of \( f \in A \) such that \( \text{Re} \left( f'(z) + \alpha zf''(z) \right) > 0 \) for \( z \in D \). It has been proved that for \( \text{Re} \alpha \geq 0 \), the members of \( W(\alpha) \) are univalent in \( D \) and \( W(\alpha) \) is a subclass of the class of analytic close-to-convex functions. In 2010, the regions of variability for functions in the class \( W(\alpha) \) was studied by Ponnusamy and Vasudevarao [20]. In 1982, R. Singh and S. Singh [28] proved that \( W(1) \) is a subclass of analytic starlike functions. In 2014, Nagpal and Ravichandran [17] studied the following class

\[
W_0^H := \{ f = h + \phi \in H : \text{Re}(h'(z) + zh''(z)) > |g'(z) + zg''(z)|, \quad z \in D \}
\]

which is the harmonic analogue of the class \( W(1) \). It is known that \( W_0^H \) is a subclass of \( S_0^H \) and \( P_0^H \). In particular, the members of \( W_0^H \) are fully starlike in \( D \). The sharp coefficient bounds and the growth theorem for functions in the class \( W_0^H \) have been investigated in [17]. It has been proved that the class \( W_0^H \) is closed under convolution and convex combinations.

For two analytic functions

\[
F_1(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad F_2(z) = \sum_{n=0}^{\infty} b_n z^n,
\]

the convolution (or Hadamard product) is defined by

\[
F_1 \ast F_2 = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in D.
\]
Analogously, for harmonic functions \( f_1 = h_1 + \overline{g_1} \) and \( f_2 = h_2 + \overline{g_2} \) in \( \mathcal{H} \), the convolution is defined as
\[
f_1 \ast f_2 = h_1 * h_2 + \overline{g_1} * \overline{g_2}.
\]

In 1973, Ruscheweyh and Sheil-Small [21] proved that the class of convex analytic functions is closed under convolution. It is well-known that if \( f \in \mathcal{C} \) and \( g \in \mathcal{S}^* \) then \( f \ast g \in \mathcal{S}^* \) and the convolution of convex function with close-to-convex function is also close-to-convex. However, the class of analytic starlike functions is not closed under the convolution. In 1984, Clunie and Sheil-Small [6] proved that if \( f \) is harmonic close-to-convex function and \( \phi \) is analytic convex function, then the function \( f * (\phi + \alpha \overline{\phi}) \) is harmonic close-to-convex function for all \( \alpha (|\alpha| < 1) \). For the extensive study on convolution of harmonic mappings we refer to [8], [9] and [10].

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be in the class \( \mathcal{S} \). Then the \( n^{\text{th}} \) partial sum of \( f(z) \) is defined by
\[
s_n(f) = \sum_{k=0}^{n} a_k z^k \quad \text{for } n \in \mathbb{N}
\]
where \( a_0 = 0 \) and \( a_1 = 1 \). Analogously in harmonic case, the p, q-th section/partial sum of harmonic function \( f = h + \overline{g} \) given by (1.1) is defined as follows:
\[
s_{p,q} = s_p(h) + s_q(g)
\]
where \( s_p(h) = \sum_{k=1}^{p} a_k z^k \) and \( s_q(g) = \sum_{k=1}^{q} b_k z^k \), \( p, q \geq 1 \) with \( a_1 = 1 \).

In 1928, Szegö [31] proved a remarkable result which asserts that every section \( s_n(f) \) of a function \( f \in \mathcal{S} \) is univalent in the disk \( |z| < 1/4 \). The number \( 1/4 \) is the best possible as is evident from the second partial sum of Koebe function \( k(z) = z/(1-z)^2 \). For \( f \in \mathcal{S} \), determining the exact radius of univalence \( r_n \) of \( s_n(f) \) remains an open problem. However, many related problems concerning sections have been solved for various geometric subclasses of \( \mathcal{S} \). In 1941, Robertson [23] studied the partial sums of multivalently starlike functions (see also [22]). In 1988, Ruscheweyh [25] proved a strong result by showing that the partial sums, \( s_n(f) \) are starlike in the disk \( |z| < 1/4 \) not only for the functions \( f \) in the class \( \mathcal{S} \) but also for the closed convex hull of \( \mathcal{S} \). For many interesting results on sections of analytic functions we refer to [18], [21] and [26] (also see [1], [19], [27]). In 2013, Li and Ponnusamy [12] discussed the properties of sections of functions in the class
\[
\mathcal{P}_h^0(\alpha) := \{ f = h + \overline{g} : \text{Re} \left( h'(z) - \alpha \right) > |g'(z)| \quad \text{for} \quad z \in \mathbb{D} \}.
\]
In 2015, Li and Ponnusamy [13] investigated the properties of sections of stable harmonic convex functions.

In this paper we introduce the following class \( \mathcal{W}_h^0(\alpha) \) (for \( \alpha \geq 0 \))
\[
\mathcal{W}_h^0(\alpha) := \{ f = h + \overline{g} \in \mathcal{H} : \text{Re} \left( h'(z) + \alpha h''(z) \right) > |g'(z)| + \alpha g''(z) \quad \text{for} \quad z \in \mathbb{D} \}
\]
with \( g'(0) = 0 \). The organization of this paper as follows: In section 2, we prove that the class \( \mathcal{W}_h^0(\alpha) \) is a subclass of close-to-convex harmonic mappings. We also obtain the sharp coefficient bounds and growth theorem for functions in the class \( \mathcal{W}_h^0(\alpha) \). In Section 3, we obtain the convolution and convex combination properties
of functions in the class $W^0_H(\alpha)$. Finally, in section 4, we determine $r$ so that the partial sums $s_{p,q}(f)$ of $f \in W^0_H(\alpha)$ are close-to-convex in the disk $|z| < r$.

2. Main Result

**Theorem 2.1.** A harmonic mapping $f = h + \overline{g}$ is in $W^0_H(\alpha)$ if and only if the analytic function $F = h + \epsilon g$ belongs to $W(\alpha)$ for each $|\epsilon| = 1$.

**Proof.** If $f = h + \overline{g} \in W^0_H(\alpha)$ then for each $|\epsilon| = 1$,

$$
\text{Re}(F'(z) + \alpha z F''(z)) = \text{Re}((h'(z) + \alpha z h''(z)) + \epsilon (g'(z) + \alpha z g''(z)))
$$

$$
> \text{Re}((h'(z) + \alpha z h''(z)) - |(g'(z) + \alpha z g''(z))|)
$$

$$
> 0 \quad \text{for} \quad z \in \mathbb{D}.
$$

Hence $F = h + \epsilon g \in W(\alpha)$ for each $|\epsilon| = 1$. Conversely, if $F \in W(\alpha)$ then

$$
\text{Re}((h'(z) + \alpha z h''(z)) + \epsilon (g'(z) + \alpha z g''(z))) > 0 \quad \text{for} \quad \mathbb{D}
$$

or, equivalently,

$$
\text{Re}((h'(z) + \alpha z h''(z))) > \text{Re}(-\epsilon (g'(z) + \alpha z g''(z))) \quad \text{for} \quad \mathbb{D}.
$$

Since $\epsilon (|\epsilon| = 1)$ is arbitrary, for an appropriate choice of $\epsilon$ we obtain

$$
\text{Re}((h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)| \quad \text{for} \quad z \in \mathbb{D}.
$$

Consequently $f \in W^0_H(\alpha)$. \hfill \square

Note that for $\alpha \geq 0$, each function in the class $W(\alpha)$ is close-to-convex. Hence by Lemma 1.3 $W^0_H(\alpha)$ is a subclass of $C^0_H$ for $\alpha \geq 0$. In 1977, Chichra \cite{7} proved that if $0 \leq \beta < \alpha$ then $W(\alpha) \subset W(\beta)$. Thus $W^0_H(\alpha) \subset W(\beta)$ if $0 \leq \beta < \alpha$. In view of this, $W(\alpha)$ is starlike for $\alpha \geq 1$ because the class $W(1)$ is starlike. For $\alpha \geq 1$, if $f \in W^0_H(\alpha)$ then $h + \epsilon g \in W(\alpha)$ is starlike function in $\mathbb{D}$ for each $\epsilon (|\epsilon| = 1)$. Hence by Theorem A $W^0_H(\alpha) \subset S^0_H$ for all $\alpha \geq 1$. In particular, for $\alpha \geq 1$ each member of $W^0_H(\alpha)$ is fully starlike.

The following theorems give sharp coefficient bounds for functions in the class $W^0_H(\alpha)$.

**Theorem 2.2.** Let $f = h + \overline{g} \in W^0_H(\alpha)$ for $\alpha \geq 0$ be of the form (1.2). Then for $n \geq 2$,

$$
|b_n| \leq \frac{1}{\alpha n^2 + n(1 - \alpha)}.
$$

The result is sharp for the function $f(z)$ which is given by $f(z) = z + \frac{1}{\alpha n^2 + n(1 - \alpha)} z^n$.

**Proof.** Let $f \in W^0_H(\alpha)$. Then

$$
\text{Re}(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)| \quad \text{for} \quad z \in \mathbb{D}.
$$
Using the series expansion of $g(z)$ and \((2.4)\) we have
\[
 r^{n-1}(\alpha n^2 + (1-\alpha)n)|b_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |g'(re^{i\theta}) + \alpha re^{i\theta} g''(re^{i\theta})| d\theta \\
\leq \frac{1}{2\pi} \int_0^{2\pi} \Re(h'(re^{i\theta}) + \alpha(re^{i\theta}h''(re^{i\theta}))) d\theta \\
= 1.
\]

Letting $r \to 1^-$ we obtain the desired bound. To show the bound in \((2.3)\) is sharp, we consider $f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\alpha n^2 + n(1-\alpha)} z^n$. It is easy to see that $f \in \mathcal{W}_H^0(\alpha)$ and $|b_n(f)| = \frac{1}{\alpha n^2 + n(1-\alpha)}$.

\[\square\]

**Theorem 2.5.** Let $f = h + \overline{f} \in \mathcal{W}_H^0(\alpha)$ for $\alpha \geq 0$ be of the form \((1.2)\). Then for any $n \geq 2$,

(i) $|a_n| + |b_n| \leq \frac{2}{\alpha n^2 + n(1-\alpha)}$;

(ii) $||a_n| - |b_n|| \leq \frac{2}{\alpha n^2 + n(1-\alpha)}$;

(iii) $|a_n| \leq \frac{2}{\alpha n^2 + n(1-\alpha)}$.

All these results are sharp for the function $f(z)$ which is given by $f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\alpha n^2 + n(1-\alpha)} z^n$.

**Proof.** It is sufficient to prove only the first inequality as the rest follow from it. Since $f = h + \overline{f} \in \mathcal{W}_H^0(\alpha)$ by Theorem \(2.1\), $h + \epsilon \overline{g}$ is in the class $\mathcal{W}(\alpha)$ for $\epsilon (|\epsilon| = 1)$. Thus for each $|\epsilon| = 1$ we have
\[
\Re((h + \epsilon \overline{g})' + \alpha z(h + \epsilon \overline{g})'') > 0 \quad \text{for} \quad z \in \mathbb{D}.
\]

This implies there exists an analytic function $p(z)$ which is of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ with $\Re p(z) > 0$ in $\mathbb{D}$ such that
\[
(2.6) \quad h'(z) + \alpha z h''(z) + \epsilon (g'(z) + \alpha z g''(z)) = p(z).
\]

Comparing the coefficients on the both sides of \((2.6)\) we obtain
\[
(2.7) \quad (\alpha n^2 + (1-\alpha)n)(a_n + \epsilon b_n) = p_{n-1} \quad \text{for} \quad n \geq 2.
\]

Since $|p_n| \leq 2$ for $n \geq 1$ and $\epsilon (|\epsilon| = 1)$ is arbitrary, it follows from \((2.7)\) that
\[
(2.8) \quad (\alpha n^2 + (1-\alpha)n)(|a_n| + |b_n|) \leq 2
\]
which yields the first inequality.

To prove the sharpness of \((2.8)\), we consider the function $f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\alpha n^2 + n(1-\alpha)} z^n$. It is easy to see that $f \in \mathcal{W}_H^0(\alpha)$ and all the three inequalities are sharp. \[\square\]
Remark 2.1. (i) For $\alpha = 0$, the class $\mathcal{W}_H^0(\alpha)$ reduces to $\mathcal{R}_H^0$ which was studied by Li and Ponnusamy [12]. If we put $\alpha = 0$ in Theorem 2.2 and Theorem 2.5, we obtain the results [12, Theorems 1 and 2] as a particular case.

(ii) For $\alpha = 1$, the class $\mathcal{W}_H^0(\alpha)$ reduces to $\mathcal{W}_H^0(1) = \{f = h + g : \text{Re}(h'(z) + zh''(z)) > |g'(z) + zg''(z)|\}$ which was studied by Nagpal and Ravichandran [17]. If we put $\alpha = 1$ in Theorem 2.2 and Theorem 2.5, we obtain the result [17, Theorem 3.5] as a particular case.

Theorem 2.9. Let $f = h + g \in \mathcal{W}_H^0(\alpha)$ be as in (1.2) with $0 < \alpha \leq 1$. Then

\[
|z| + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}|z|^n}{\alpha n^2 + n(1 - \alpha)} \leq |f(z)| \leq |z| + 2 \sum_{n=2}^{\infty} \frac{|z|^n}{\alpha n^2 + n(1 - \alpha)}.
\]

This result is sharp for the function $f(z) = z + \sum_{n=2}^{\infty} \frac{2}{\alpha n^2 + n(1 - \alpha)} z^n$ and its rotations.

Proof. Let $f = h + g$ be in the class $\mathcal{W}_H^0(\alpha)$. Then $F = h + \epsilon g$ is in the class $\mathcal{W}(\alpha)$ and for each $|\epsilon| = 1$ we have

\[
\text{Re} (F'(z) + \alpha z F''(z)) > 0 \quad \text{for} \quad z \in \mathbb{D}.
\]

Thus there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$ such that

\[
F'(z) + \alpha z F''(z) = \frac{1 + \omega(z)}{1 - \omega(z)}.
\]

A simplification of (2.12) gives

\[
z^{1/\alpha} F'(z) = \frac{1}{\alpha} \int_0^{|z|} \xi^{1/\alpha - 1} \frac{1 + \omega(\xi)}{1 - \omega(\xi)} d\xi
\]

\[
= \frac{1}{\alpha} \int_0^{|z|} (te^{i\theta})^{1/\alpha - 1} \frac{1 + \omega(te^{i\theta})}{1 - \omega(te^{i\theta})} e^{i\theta} dt.
\]

Therefore we have

\[
|z^{1/\alpha} F'(z)| = \left| \frac{1}{\alpha} \int_0^{|z|} (te^{i\theta})^{1/\alpha - 1} \frac{1 + \omega(te^{i\theta})}{1 - \omega(te^{i\theta})} e^{i\theta} dt \right|
\]

\[
\leq \frac{1}{\alpha} \int_0^{|z|} t^{1/\alpha - 1} \frac{1 + t}{1 - t} dt
\]

and
Further computation gives

\begin{equation}
|\frac{z^{1/\alpha} F'(z)}{}| = \left| \frac{1}{\alpha} \int_0^{|z|} (t e^{i\theta})^{\frac{1}{\alpha} - 1} \frac{1 + \omega(t e^{i\theta})}{1 - \omega(t e^{i\theta})} e^{i\theta} dt \right| \\
\geq \frac{1}{\alpha} \int_0^{|z|} t^{\frac{1}{\alpha} - 1} \Re \frac{1 + \omega(t e^{i\theta})}{1 - \omega(t e^{i\theta})} dt \\
\geq \frac{1}{\alpha} \int_0^{|z|} t^{\frac{1}{\alpha} - 1} \Re \frac{1 - t}{1 + t} dt.
\end{equation}

The equality in (2.10) holds for the function \( f(z) \) which is given by

\[ f(z) = z + \sum_{n=2}^{\infty} \frac{2}{\alpha n^2 + n(1 - \alpha)} z^n \]

and its rotations.
The following result is a sufficient condition for functions to be in the class $W_{H}^{0}(\alpha)$.

**Theorem 2.14.** Let $f = h + \overline{g} \in S_{H}^{0}$ be of the form (1.2) and satisfies the condition

$$
\sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)(|a_n| + |b_n|) < 1.
$$

Then $f \in W_{H}^{0}(\alpha)$.

**Proof.** Let $f = h + \overline{g} \in S_{H}^{0}$. Using the series representation of $h(z)$ given by (1.2), we obtain

$$
\text{Re} \left( h'(z) + \alpha z h''(z) \right) = 1 + \text{Re} \left( \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)a_n z^{n-1} \right)
$$

$$
\geq 1 - \left| \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)a_n z^{n-1} \right|
$$

$$
\geq 1 - \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)|a_n|.
$$

In view of (2.15), the inequality (2.16) reduces to

$$
\text{Re} \left( h'(z) + \alpha z h''(z) \right) > \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)|b_n|
$$

$$
\geq \left| \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n)b_n \right|
$$

$$
= |g'(z) + \alpha z g''(z)|
$$

and hence $f \in W_{H}^{0}(\alpha)$. \qed

### 3. Convex combinations and Convolution

In this section, we show that the class $W_{H}^{0}(\alpha)$ is closed under convex combinations. Also we show that $W_{H}^{0}(\alpha)$ is closed under convolution.

**Theorem 3.1.** The class $W_{H}^{0}(\alpha)$ is closed under convex combinations.

**Proof.** Let $f_i := h_i + \overline{g_i} \in W_{H}^{0}(\alpha)$ for $i = 1, 2, \ldots n$ and $\sum_{i=1}^{n} t_i = 1$ ($0 \leq t_i \leq 1$). The convex combination of $f_i$’s can be written as

$$
f(z) = \sum_{i=1}^{n} t_i f_i(z) = h(z) + \overline{g(z)}
$$
where \( h(z) = \sum_{i=1}^{n} t_i h_i(z) \) and \( g(z) = \sum_{i=1}^{n} t_i g_i(z) \). Then \( h \) and \( g \) both are analytic in \( \mathbb{D} \) with \( h(0) = g(0) = h'(0) - 1 = g'(0) = 0 \). A simple computation yields

\[
\text{Re} \left( h'(z) + \alpha zh''(z) \right) = \text{Re} \left( \sum_{i=1}^{n} t_i (h'_i(z) + \alpha z h''_i(z)) \right) > \sum_{i=1}^{n} t_i |g'_i(z) + \alpha z g''_i(z)| \geq |g'(z) + \alpha z g''(z)|.
\]

This shows that \( f \in W^0_\mathbb{H}(\alpha) \). \( \square \)

A sequence \( \{c_n\}_{n=0}^\infty \) of non-negative numbers is said to be a convex null sequence if \( c_n \to 0 \) as \( n \to \infty \) and

\[
c_0 - c_1 \geq c_1 - c_2 \geq c_2 - c_3 \geq \ldots \geq c_{n-1} - c_n \geq \ldots \geq 0.
\]

To prove the convolution results on the class \( W^0_\mathbb{H}(\alpha) \) we need the following lemmas:

**Lemma 3.2.** [29] Let \( \{c_n\}_{n=0}^\infty \) be a convex null sequence. Then the function

\[
q(z) = c_0 + \sum_{n=1}^{\infty} c_n z^n
\]

is analytic and \( \text{Re} q(z) > 0 \) in \( \mathbb{D} \).

**Lemma 3.3.** [29] Let \( p(z) \) be an analytic function in the unit disk \( \mathbb{D} \) with \( p(0) = 1 \) and \( \text{Re} (p(z)) > 1/2 \) in \( \mathbb{D} \). Then for any analytic function \( f \) in \( \mathbb{D} \), the function \( p * f \) takes values in the convex hull of the image of \( \mathbb{D} \) under \( f \).

Using Lemma 3.2 and Lemma 3.3 we prove the following interesting lemma.

**Lemma 3.4.** Let \( F \) be in the class \( W(\alpha) \). Then \( \text{Re} \left( \frac{F(z)}{z} \right) > 1/2 \).

**Proof.** Let \( F \in W(\alpha) \) be given by \( F(z) = z + \sum_{n=2}^{\infty} A_n z^n \). Since \( F \in W(\alpha) \), \( \text{Re} \left( F'(z) + \alpha z F''(z) \right) > 0 \) in \( \mathbb{D} \), which is equivalent to

\[
\text{Re} \left( 1 + \sum_{n=2}^{\infty} A_n (n^2 \alpha + n(1-\alpha)) z^{n-1} \right) > 0 \quad \text{for} \quad z \in \mathbb{D}.
\]

Therefore \( \text{Re} p(z) > 1/2 \) in \( \mathbb{D} \), where

\[
p(z) = 1 + \frac{1}{2} \sum_{n=2}^{\infty} A_n (n^2 \alpha + n(1-\alpha)) z^{n-1}.
\]

Consider a sequence \( \{c_n\}_{n=0}^\infty \) defined by \( c_0 = 1 \) and \( c_{n-1} = \frac{2}{n^2 \alpha + n(1-\alpha)} \) for \( n \geq 2 \). Then it is easy to see that \( c_n \to 0 \) as \( n \to \infty \) and

\[
c_0 - c_1 \geq c_1 - c_2 \geq c_2 - c_3 \geq \ldots \geq c_{n-1} - c_n \geq \ldots \geq 0.
\]
Thus the sequence \( \{c_n\}_{n=0}^{\infty} \) is a convex null sequence. In view of Lemma 3.2, the function
\[
q(z) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2}{n^2 \alpha + n(1 - \alpha)} z^{n-1}
\]
is analytic and \( \text{Re}(q(z)) > 0 \) in \( \mathbb{D} \). A simple computation shows that
\[
\frac{F(z)}{z} = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} = p(z) \ast \left( 1 + \sum_{n=2}^{\infty} \frac{2}{n^2 \alpha + n(1 - \alpha)} z^{n-1} \right).
\]
An application of Lemma 3.3 yields \( \text{Re} \left( \frac{F(z)}{z} \right) > 1/2 \) for \( z \in \mathbb{D} \).

**Lemma 3.5.** Let \( F_1 \) and \( F_2 \) be in \( \mathcal{W}(\alpha) \). Then the Hadamard product \( F_1 \ast F_2 \) is in \( \mathcal{W}(\alpha) \).

**Proof.** Let \( F_1(z) = z + \sum_{n=2}^{\infty} A_n z^n \) and \( F_2(z) = z + \sum_{n=2}^{\infty} B_n z^n \). Then the convolution of \( F_1 \) and \( F_2 \) is given by
\[
F(z) = (F_1 \ast F_2)(z) = z + \sum_{n=2}^{\infty} A_n B_n z^n.
\]
Since \( zf''(z) = zF'_1(z) \ast F'_2(z) \), a computation shows that
\[
(3.6) \quad F''(z) + \alpha z F''(z) = (F'_1(z) + \alpha zF'_2(z)) \ast \left( \frac{F_2(z)}{z} \right).
\]
Since \( F_1, F_2 \in \mathcal{W}(\alpha) \), \( \text{Re} \left( F'_1(z) + \alpha z F'_2(z) \right) > 0 \) for \( z \in \mathbb{D} \). In view of Lemma 3.3, \( \text{Re} \left( \frac{F_2(z)}{z} \right) > 1/2 \) in \( \mathbb{D} \). An application of Lemma 3.3 to (3.6) gives \( \text{Re} \left( F''(z) + \alpha z F''(z) \right) > 0 \) in \( \mathbb{D} \). Hence \( F = F_1 \ast F_2 \) belongs to the class \( \mathcal{W}(\alpha) \).

Using Lemma 3.3, we shall prove the following interesting result.

**Theorem 3.7.** If \( f_1 \) and \( f_2 \) are in \( \mathcal{W}^0_H(\alpha) \) then \( f_1 \ast f_2 \) is in \( \mathcal{W}^0_H(\alpha) \).

**Proof.** Let \( f_1 = h_1 + \overline{g}_1 \) and \( f_2 = h_2 + \overline{g}_2 \) be two functions in the class \( \mathcal{W}^0_H(\alpha) \). The convolution of \( f_1 \) and \( f_2 \) is given by \( f_1 \ast f_2 = h_1 \ast h_2 + \overline{g}_1 \ast \overline{g}_2 \). To show \( f_1 \ast f_2 \) is in the class \( \mathcal{W}^0_H(\alpha) \), it is sufficient to prove that \( F = h_1 \ast h_2 + \epsilon(g_1 \ast g_2) \) is in \( \mathcal{W}(\alpha) \) for each \( \epsilon (|\epsilon| = 1) \). Since by Lemma 3.5, \( \mathcal{W}(\alpha) \) is closed under convolution, for each \( \epsilon (|\epsilon| = 1) \), \( h_i + \epsilon g_i \in \mathcal{W}(\alpha) \) for \( i = 1, 2 \). Hence the following functions
\[
F_1 = (h_1 - g_1) \ast (h_2 - \epsilon g_2)
\]
and
\[
F_2 = (h_1 + g_1) \ast (h_2 + \epsilon g_2)
\]
are in the class $\mathcal{W}(\alpha)$. Again, since $\mathcal{W}(\alpha)$ is closed under convex combination, the function
\[ \frac{1}{2}(F_1 + F_2) = (h_1 * h_2) + \epsilon(g_1 * g_2) = F \]
is in the class $\mathcal{W}(\alpha)$. Hence $W_H^0(\alpha)$ is closed under convolution.

In 2002, Goodloe [10] considered the Hadamard product of harmonic function with an analytic function defined as follows:
\[ f \ast \phi = h \ast \phi + g \ast \phi \]
where $f = h + \overline{g}$ is harmonic and $\phi$ is analytic function in $\mathbb{D}$.

**Theorem 3.8.** Let $f \in W_H^0(\alpha)$ and $\phi \in \mathcal{A}$ be such that $\text{Re} \left( \frac{\phi(z)}{2} \right) > 1/2$ for $z \in \mathbb{D}$. Then $f \ast \phi \in W_H^0(\alpha)$.

**Proof.** Let $f = h + \overline{g}$ be in the class $W_H^0(\alpha)$. Then we have
\[ f \ast \phi = h \ast \phi + g \ast \phi. \]
In view of Theorem 2.1, to prove $f \ast \phi$ is in the class $W_H^0(\alpha)$, it suffices to prove that $F(z) = h \ast \phi + \epsilon(g \ast \phi)$ belongs to the class $\mathcal{W}(\alpha)$ for each $\epsilon (|\epsilon| = 1)$. Since $f \in W_H^0(\alpha)$, for each $\epsilon (|\epsilon| = 1)$ the function $F_1(z) = h + \epsilon g$ belongs to $\mathcal{W}(\alpha)$. Therefore $F = F_1 \ast \phi$ and
\[ F'(z) + \alpha z F''(z) = (F'_1(z) + \alpha z F''_1(z)) + \frac{\phi(z)}{z}. \]
Since $\text{Re} \left( \frac{\phi(z)}{2} \right) > 1/2$ and $\text{Re}(F'_1(z) + \alpha z F''_1(z)) > 0$ in $\mathbb{D}$, in view of Lemma 3.3, it can be seen that $F \in \mathcal{W}(\alpha)$. \(\square\)

As a consequence of Theorem 3.8, we obtained the following result.

**Corollary 3.9.** Suppose $f$ belongs to $W_H^0(\alpha)$ and $\phi \in \mathcal{K}$. Then $f \ast \phi \in W_H^0(\alpha)$.

**Proof.** Since $\phi$ is convex, $\text{Re} \left( \frac{\phi(z)}{2} \right) > 1/2$ for $z$ in $\mathbb{D}$ and hence the desired result follows from Theorem 3.8. \(\square\)

### 4. Partial Sums of Functions in $W_H^0(\alpha)$

**Lemma 4.1.** Let $f = h + \overline{g}$ be in the class $W_H^0(\alpha)$ with $\alpha \geq 0$. Then for $|\epsilon| = 1$ and $|z| < 1/2$, we have
\[ \text{Re} \left( (s_2(h) + \epsilon s_2(g))' + \alpha z(s_2(h) + \epsilon s_2(g))'' \right) > 1/4. \]

**Proof.** Let $f = h + \overline{g} \in W_H^0(\alpha)$. Then by Theorem 2.1, $h + \epsilon g$ is in the class $\mathcal{W}(\alpha)$ for $\epsilon (|\epsilon| = 1)$. Then $\text{Re} F_\epsilon(z) > 0$ where
\[ F_\epsilon(z) = (h + \epsilon g)' + \alpha z(h + \epsilon g)'' = 1 + \sum_{k=1}^\infty c_k z^k. \]
It is easy to see that
\[ |2c_2 - c_1^2| \leq 4 - |c_1|^2. \]
Now let \(2c_2 - c_1^2 = c\). Then \(c_2 = c/2 + c_1^2/2\) and \(|c| \leq 4 - |c_1|^2\). Let \(c_1z = \alpha + i\beta\) and \(\sqrt{c}z = \gamma + i\delta\), where \(\alpha, \beta, \gamma\) and \(\delta\) are real numbers. Then for \(|z| < 1/2\) it is easy to see that
\[ \alpha^2 + \beta^2 = |c_1|^2 |z|^2 < \frac{|c_1|^2}{4} \]
and
\[ \gamma^2 + \delta^2 = |c||z|^2 < \frac{|c|}{4} \leq 1 - \frac{|c_1|^2}{4} \leq 1 - (\alpha^2 + \beta^2). \]
Therefore from (1.2) and \(c_2 = c/2 + c_1^2/2\), we obtain
\[ \text{Re} ((s_3(h) + \epsilon s_3(g))' + \alpha z(s_3(h) + \epsilon s_3(g))'') = \text{Re} (1 + c_1 z + c_2 z^2) \]
\[ = \text{Re} (1 + c_1 z + \frac{c}{2} z^2 + \frac{c_1^2}{2} z^2) \]
\[ = 1 + \alpha + \left( \frac{\gamma^2}{2} - \frac{\delta^2}{2} \right) + \left( \frac{\alpha^2}{2} - \frac{\beta^2}{2} \right) \]
\[ > 1 + \alpha + \left( \frac{\gamma^2}{2} - 1 - (\alpha^2 + \beta^2 + \gamma^2) \right) + \left( \frac{\alpha^2}{2} - \frac{\beta^2}{2} \right) \]
\[ = \frac{1}{4} + \left( \alpha + \frac{1}{2} \right)^2 + \gamma^2 \geq \frac{1}{4} \]
which yields the desired result. \( \square \)

**Theorem 4.3.** Let \( f \in \mathcal{W}_H^0(\alpha) \). Then for each \( q \geq 2 \), \( s_{1,q}(f) \in \mathcal{W}_H^0(\alpha) \) for \(|z| < 1/2\).

**Proof.** Let \( f = h + \overline{g} \) be in the class \( \mathcal{W}_H^0(\alpha) \), where \( h \) and \( g \) are of the form (1.2). Clearly,
\[ s_{1,q}(f)(z) = s_2(h)(z) + s_q(g)(z) = z + \sum_{n=2}^{q} b_n z^n. \]
and \( \text{Re} (s_1(h)'(z) + \alpha z(s_1(h)''(z))) = 1 \). An application of Theorem 2.2 shows that
\[ |s_q(g)'(z) + \alpha z s_q(g)''(z)| = \left| \sum_{n=2}^{q} (n^2 \alpha + n(1 - \alpha)) b_n z^{n-1} \right| \]
\[ \leq \sum_{n=2}^{q} (n^2 \alpha + n(1 - \alpha)) |b_n| |z|^{n-1} \leq \sum_{n=2}^{q} |z|^{n-1} \]
\[ = |z|(1 - |z|^{q-1}) < \frac{|z|}{1 - |z|} < 1 \]
\[ = \text{Re} (s_1(h)'(z) + \alpha z(s_1(h)''(z))) \]
when \(|z| < 1/2\). Therefore \( s_{1,q}(f) \in \mathcal{W}_H^0(\alpha) \) in \(|z| < 1/2\).
Theorem 4.4. Let \( f \in W_0^0(\alpha) \) and \( p \) and \( q \) satisfy one of the following conditions:

(i) \( 3 \leq p < q \),
(ii) \( p = q \geq 2 \),
(iii) \( p > q \geq 3 \),
(iv) \( p = 3 \) and \( q = 2 \).

Then \( s_{p,q}(f) \in W_0^0(\alpha) \) in \( |z| < 1/2 \).

**Proof.** Let \( f = h + g \) be in the class \( W_0^0(\alpha) \), where \( h \) and \( g \) are of the form (1.2). So \( \sigma_p(h)(z) = \sum_{k=p+1}^\infty a_k z^k \) and \( \sigma_q(g)(z) = \sum_{k=q+1}^\infty b_k z^k \). Then \( h = \sigma_p(h) + \sigma_p(h) \) and \( g = \sigma_q(g) \). To prove \( s_{p,q}(f) \in W_0^0(\alpha) \) it suffices to prove that \( s_{p,q}(h) + \epsilon s_{q,q}(g) \) is in the class \( W(\alpha) \) for each \( \epsilon \) (\( |\epsilon| = 1 \)). If \( f \in W_0^0(\alpha) \) then

\[
\text{Re} \left( (s_p(h) + \epsilon s_q(g))' + \alpha z(s_p(h) + \epsilon s_q(g))'' \right) \\
\geq \text{Re} \left( (h + \epsilon g)' + \alpha z(h + \epsilon g)'' - \left| ((\sigma_p(h) + \epsilon \sigma_q(g))' + \alpha(z\sigma_p(h) + \epsilon \sigma_q(g))'' \right| \right)
\]

\[
\geq \frac{1 + |z|}{1 - |z|} - \left| ((\sigma_p(h) + \epsilon \sigma_q(g))' + \alpha(z\sigma_p(h) + \epsilon \sigma_q(g))'' \right| \right|
\]

**Case (i):** \( 3 \leq p < q \)

An application of Theorem 2.2 and Theorem 2.5 gives

\[
|((\sigma_p(h) + \epsilon \sigma_q(g))' + \alpha(z\sigma_p(h) + \epsilon \sigma_q(g))'')|
\]

\[
= \left| ((p + 1) + \alpha(p + 1))a_{p+1}z^p + ... + (q + \alpha q(q - 1))a_qz^{q-1}\right|
\]

\[
+ \sum_{k=q+1}^\infty (k + \alpha k(k - 1))(a_k + \epsilon b_k)z^{k-1}
\]

\[
\leq 2(|z|^p + ... + |z|^{q-1} + 2 \sum_{k=q+1}^\infty |z|^{k-1}) = \frac{2|z|^p}{1 - |z|}
\]

Using (4.6) in (4.5) we obtain

\[
\text{Re} \left( (s_p(h) + \epsilon s_q(g))' + \alpha z(s_p(h) + \epsilon s_q(g))'' \right)
\]

\[
\geq \frac{1 + |z|}{1 - |z|} - \frac{2|z|^p}{1 - |z|}.
\]

Now for \( 4 \leq p < q \) and \( |z| = 1/2 \), the inequality (4.7) yields

\[
\text{Re} \left( (s_p(h) + \epsilon s_q(g))' + \alpha z(s_p(h) + \epsilon s_q(g))'' \right) \geq \frac{1}{3} - \frac{1}{4} > 0.
\]

Since \( \text{Re} \left( (s_p(h) + \epsilon s_q(g))' + \alpha z(s_p(h) + \epsilon s_q(g))'' \right) \) is harmonic, it assumes the minimum value on the circle \( |z| = 1/2 \). Therefore for \( 4 \leq p < q \), \( s_{p,q}(f) \in W_0^0(\alpha) \) in \( |z| < 1/2 \).
If \( p = 3 < q \) then an application of Theorem 2.2 and Lemma 4.1 shows that

\[
\Re \left( (s_3(h) + \varepsilon s_q(g))' + \alpha z(s_3(h) + \varepsilon s_q(g))'' \right)
= \Re \left( (s_3(h) + \varepsilon s_3(g))' + \alpha z(s_3(h) + \varepsilon s_3(g))'' + \varepsilon \sum_{k=4}^{q} (k + \alpha k(k - 1)) b_k z^{k-1} \right).
\]

\[
\geq \frac{1}{4} - \frac{|z|^3}{1 - |z|}.
\]

It is easy to see that

\[
\Re \left( (s_3(h) + \varepsilon s_q(g))' + \alpha z(s_3(h) + \varepsilon s_q(g))'' \right) > 0
\]

for \( |z| < 1/2 \) and hence \( s_{3,q}(f) \in W^0_H(\alpha) \) for \( |z| < 1/2 \).

**Case (ii):** \( p = q \geq 2 \)

If \( p = q \geq 4 \), then the inequality (4.4) gives \( s_{p,q} \in W^0_H(\alpha) \) in \( |z| < 1/2 \) and Lemma 4.1 implies that \( s_{3,3} \in W^0_H(\alpha) \) in \( |z| < 1/2 \). For \( p = q = 2 \), \( s_{2,2}(f)(z) = z + a_2 z^2 + b_2 z^2 \).

An application of Theorem 2.2 and Theorem 2.5 shows that

\[
\Re \left( (s_2(h) + \varepsilon s_2(g))' + \alpha z(s_2(h) + \varepsilon s_2(g))'' \right)
= 1 + 2(1 + \alpha) \Re (a_2 + \varepsilon b_2) z
\geq 1 - 2(1 + \alpha) |(a_2 + \varepsilon b_2) z|
\geq 1 - 2|z| > 0
\]

in the disk \( |z| < 1/2 \). Hence, \( s_{2,2}(f) \in W^0_H(\alpha) \) for \( |z| < 1/2 \).

**Case (iii):** \( p > q \geq 3 \)

By Theorem 2.2 and Theorem 2.5 we obtain

\[
|\sigma_p(h) + \varepsilon \sigma_q(g)' + \alpha z(\sigma_p(h) + \varepsilon \sigma_q(g))''|
= \left| (q + 1) + \alpha q(q + 1)\right| b_{q+1} z^q + \ldots + \left( p + \alpha p(p - 1)\right) b_p z^{p-1}
\]

\[
+ \sum_{k=p+1}^{\infty} (k + \alpha k(k - 1)) (a_k \varepsilon b_k) z^{k-1}
\]

\[
= \frac{|z|^q - |z|^p}{1 - |z|} + 2 \frac{|z|^p}{1 - |z|} = \frac{|z|^q + |z|^p}{1 - |z|}.
\]

Using (4.8) in (4.5) we obtain

\[
\Re \left( s_p(h) + \varepsilon s_q(g)' + \alpha z(s_p(h) + \varepsilon s_q(g))'' \right)
\geq \frac{1 - |z|}{1 + |z|} - \frac{|z|^q + |z|^p}{1 - |z|}.
\]
If $q \geq 4$ then as in Case (i) we obtain
\[
\Re ((s_p(h) + \varepsilon s_q(g))' + \alpha z(s_p(h) + \varepsilon s_q(g))'') \geq \frac{1 + |z|}{1 - |z|} - \frac{2|z|^q}{1 - |z|} > 0.
\]
Hence for $q \geq 4$ and $p > q$, $s_{p,q} \in \mathcal{W}_H^0(\alpha)$ for $|z| < 1/2$.

If $q = 3$ and $|z| = 1/2$, from (4.13) it follows that
\[
\Re ((s_p(h) + \varepsilon s_3(g))' + \alpha z(s_p(h) + \varepsilon s_3(g))'') \geq \frac{1}{12} - \frac{1}{2^{p-1}}.
\]
If $p \geq 5$ then the last estimate shows that
\[
\Re ((s_p(h) + \varepsilon s_3(g))' + \alpha z(s_p(h) + \varepsilon s_3(g))'') > 0
\]
and hence $s_{p,q} \in \mathcal{W}_H^0(\alpha)$ for $|z| < 1/2$.

If $q = 3$ and $p = 4$, it follows that
\[
\Re ((s_4(h) + \varepsilon s_3(g))' + \alpha z(s_4(h) + \varepsilon s_3(g))'')
= \Re ((s_4(h) + \varepsilon s_3(g))' + \alpha z(s_4(h) + \varepsilon s_3(g))'') + \Re ((4 + 12\alpha)a_4 z^3)
\geq \Re ((s_4(h) + \varepsilon s_3(g))' + \alpha z(s_4(h) + \varepsilon s_3(g))'') - |(4 + 12\alpha)a_4 z^3|.
\]
Using Theorem 2.5 and Lemma 4.1 in (4.10) we obtain
\[
\Re ((s_4(h) + \varepsilon s_3(g))' + \alpha z(s_4(h) + \varepsilon s_3(g))'')
> \frac{1}{4} - 2|z|^3 > \frac{1}{4} - \frac{1}{2^2} = 0 \quad \text{for } |z| = \frac{1}{2}.
\]
This implies $s_{4,3} \in \mathcal{W}_H^0(\alpha)$ for $|z| < 1/2$ and hence for $p > q \geq 3$, $s_{p,q} \in \mathcal{W}_H^0(\alpha)$ for $|z| < 1/2$.

Case (iv): $p = 3$ and $q = 2$

An application of Theorem 2.2 and Lemma 4.1 shows that
\[
\Re ((s_3(h) + \varepsilon s_2(g))' + \alpha z(s_3(h) + \varepsilon s_2(g))'')
= \Re ((s_3(h) + \varepsilon s_2(g))' + \alpha z(s_3(h) + \varepsilon s_2(g))'') - \Re (((3 + 6\alpha)\epsilon)b_3 z^2)
\geq \frac{1}{4} - |z|^2 > \frac{1}{4} - \frac{1}{2^2} = 0 \quad \text{for } |z| = 1/2.
\]
Hence $s_{3,2} \in \mathcal{W}_H^0(\alpha)$ for $|z| < 1/2$. This completes the proof.

Theorem 4.11. Let $f \in \mathcal{W}_H^0(\alpha)$. If $p = 2 < q$, then $s_{2,q}(f) \in \mathcal{W}_H^0(\alpha)$ in $|z| < \frac{3 - \sqrt{5}}{2}$.
If $p \geq 4$ and $q = 2$, then $s_{p,2}(f) \in \mathcal{W}_H^0(\alpha)$ in $|z| < r_0$ where $r_0 \approx 0.433797$ is the unique real root of the equation $1 - 2r - r^3 - r^4 - r^5 = 0$.

Proof. Let $p = 2 < q$. Then $s_{2,q}(f) = s_2(h) + \varepsilon s_q(g) = z + \alpha_2 z^2 + \sum_{k=2}^{q} b_k z^k$. So
\[
(s_2(h) + \varepsilon s_q(g))' + \alpha z(s_2(h) + \varepsilon s_q(g))''
= 1 + 2\alpha_2 z(1 + \alpha) + \epsilon \sum_{k=2}^{q} (k + \alpha k(k - 1)) b_k z^{k-1}.
\]
An application of Theorem 2.5 shows that
\[
|((s_2(h) + \epsilon s_2(g))' + \alpha z(s_2(h) + \epsilon s_2(g)))'' - 1| \\
= \left| 2a_2 z(1 + \alpha) + \epsilon \sum_{k=2}^{q} (k + \alpha k(1 - k)) b_k z^{k-1} \right| \\
= \left| 2(1 + \alpha (a_2 + \epsilon b_2) z) + \epsilon \sum_{k=3}^{q} (k + \alpha k(1 - k)) b_k z^{k-1} \right| \\
< 2|z| + \frac{|z|^2}{1 - |z|} < 1
\]
for \(|z| < \frac{3 - \sqrt{5}}{2}\). Hence \(s_{2,q}(f) \in W_{H}^0(\alpha)\) for \(|z| < \frac{3 - \sqrt{5}}{2}\).

Consider the case \(p \geq 4\) and \(q = 2\). Then using (4.9) we obtain
\[
\text{Re}\left((s_p(h) + \epsilon s_2(g))' + \alpha z(s_p(h) + \epsilon s_2(g)))''\right) \\
\geq \frac{1 - |z|}{1 + |z|} - \frac{|z|^p + |z|^q}{1 - |z|} \\
\geq \frac{1 - |z|}{1 + |z|} - \frac{|z|^2 + |z|^4}{1 - |z|} \\
= \frac{1 - 2|z| - |z|^3 - |z|^4 - |z|^5}{1 - |z|^2} > 0
\]
for \(|z| < r_0 \approx 433797\). Hence \(s_{p,2}(f) \in W_{H}^0(\alpha)\) for \(|z| < 0.433797\).

\[\square\]

REFERENCES

[1] S. V. Bharanedhar and S. Ponnusamy, Uniform close-to-convexity radius of sections of functions in the close-to-convex family, J. Ramanujan Math. Soc. 29 (2014), 243-251.
[2] L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche ein e schlichte Abbildung des Einheitskreises vermitteln, Sitzungsber. Preuss. Akad. Wiss. (1916), 940–955.
[3] L. de Branges, A proof of the Bieberbach conjecture. Acta Math. 154 (1985), no. 1-2, 137–152.
[4] D. Bshouty and A. Lyzzaik, Close-to-convexity criteria for planar harmonic mappings, Complex Anal. Oper. Theory 5 (2011), 767–774.
[5] D. Bshouty, S. S. Joshi and S. B. Joshi, On close-to-convex harmonic mappings, Complex Var. Elliptic Equ. 58 (2013), 1195–1199.
[6] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A.I. 9 (1984), 3–25.
[7] P. N. Chichra, New subclass of the class of close-to-convex function, Proc. Amer. Math. Soc. 62 (1977), 37–43.
[8] M. Dorff, Convolutions of planar harmonic convex mappings, Complex Variables Theory Appl. 45 (2001), 263–271.
[9] M. Dorff, Convolutions of harmonic convex mappings, Complex Var. Elliptic Equ. 57 (2012), 489–503.
[10] R. M. Goodloe, Hadamard products of convex harmonic mappings, Complex Var. Theory Appl. 47 (2002), 81–92.
[11] L. Li and S. Ponnusamy, Disk of convexity of sections of univalent harmonic functions, *J. Math. Anal. Appl.* **408** (2013), 589–596.

[12] L. Li and S. Ponnusamy, Injective section of univalent harmonic mappings, *Nonlinear Analysis* **89** (2013), 276–283.

[13] L. Li and S. Ponnusamy, Sections of stable harmonic convex functions, *Nonlinear Analysis* **123/124** (2015), 178–190.

[14] D. Kalaj, S. Ponnusamy and M. Vuorinen, Radius of close-to-convexity and full starlikeness of harmonic mappings, *Complex Var. Elliptic Equ.* **59** (2014), 539–552.

[15] T. H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.* **104** (1962), 532–537.

[16] S. Nagpal and V. Ravichandran, Fully starlike and fully convex harmonic mappings of order α, *Ann. Polon. Math.* **108** (2013), 85–107.

[17] S. Nagpal and V. Ravichandran, Construction of subclasses of univalent harmonic mappings, *J. Korean Math. Soc.* **53** (2014), 567–592.

[18] M. Obradovic and S. Ponnusamy, Injectivity and starlikeness of sections of a class of univalent functions, *Complex analysis and dynamical systems V*, 195–203, Contemp. Math., **591**, Amer. Math. Soc., Providence, RI, 2013, 195–203.

[19] M. Obradovic and S. Ponnusamy, Starlikeness of sections of univalent functions, *Rocky Mountain J. Math.* **44** (2014), 1003–1014.

[20] S. Ponnusamy and A. Vasudevarao, Region of variability for functions with positive real part, *Ann. Polon. Math.* **99** (2010), 225–245.

[21] S. Ponnusamy, S. K. Sahoo and H. Yanagihara, Radius of convexity of partial sums of functions in the close-to-convex family, *J. Korean Math. Soc.* **53** (2014), 567–592.

[22] M. S. Robertson, On the theory of univalent functions, *Ann. of Math.* **37** (1936), 374–408.

[23] M. S. Robertson, The partial sums of multivalently star-like functions, *Ann. of Math.* **42** (1941), 829–838.

[24] S. Ruscheweyh and T. Sheil-Small, Corrigendum: Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.* **48** (1973), 119–135.

[25] S. Ruscheweyh, Extension of Szegő’s theorem on the sections of univalent functions, *SIAM J. Math. Anal.* **19** (1988), 1442–1449.

[26] H. Silverman, Radii problems for sections of convex functions, *Proc. Amer. Math. Soc.* **104** (1988), 1191–1196.

[27] R. Singh, Radius of convexity of partial sums of a certain power series, *J. Austral. Math. Soc.* **bf 11** (1970), 407–410.

[28] S. Singh and R. Singh, Starlikeness of close-to-convex function, *Indian. J. pure. appl. Math.* **13** (1982), 190–194.

[29] R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.* **106** (1989), 145–152.

[30] T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc.* **42** (1990), 237–248.

[31] G. Szegő, Zur Theorie der schlichten Abbildungen, *Math. Ann.* **100** (1928), 188–211.

[32] X-T. Wang and X-Q. Liang, Precise coefficient estimates for close-to-convex harmonic univalent mappings, *J. Math. Anal. Appl.* **263** (2001), 501–509.

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