Measurable genuine tripartite entanglement of \((2 \otimes 2 \otimes n)\)-dimensional quantum states via only two simultaneous copies

Chang-shui Yu, Bao-qing Guo, and Si-ren Yang

School of Physics and Optoelectronic Technology,
Dalian University of Technology, Dalian 116024, China

(Dated: March 7, 2022)

Usually, the three-tangle of a tripartite pure state of qubits can be directly measured with the simultaneous preparation of a not-less-than-four-fold copy of the state. We show that the exact genuine tripartite entanglement for \((2 \otimes 2 \otimes n)\)-dimensional pure quantum states can be measured in a similar manner, provided that only two simultaneous copies of the state are available. Lower bounds are also proposed for more convenient experimental operations. As an example, a comprehensive demonstration of the scheme is provided for the three-tangle of a three-qubit state.

PACS numbers: 03.67.Mn, 42.50.-p

I. INTRODUCTION

Quantum entanglement is the combination of quantum superposition and the tensor product structure of quantum state space. It is one of the most fundamental features of quantum mechanics which distinguishes the quantum from classical world, while it serves as an important physical resource in quantum information processing tasks. In past decades, quantification of entanglement, one of the key subjects in entanglement theory, has attracted much interest and a lot of remarkable entanglement measures have been proposed [1]. However, quantum entanglement, in general, does not correspond to an observable due to the unphysical operations such as the complex conjugation for concurrence [2] and the partial transpose for negativity [3, 4]. This means that entanglement can not be directly measured in experiment. So the usual method to measuring entanglement is reconstructing the density matrix to be considered by the state tomography [5-7] which is fit for small systems. As an example, a comprehensive demonstration of the projective measurements are performed locally, which is the same as the previous schemes. As an example, we give a detailed demonstration and analysis on directly measuring the polarization entanglement of three photons in the frame of linear optics. This shows the feasibility of our scheme. In addition, the lower bounds of the genuine tripartite entanglement which is especially sufficient for detecting this type of entanglement are also provided for less adjustments of practical operations, but the cost, besides the lower bound, is that more projective measurement outcomes are needed. Finally, the influence of the imperfect experiment on our scheme is also discussed.

II. THE GENUINE TRIPARTITE ENTANGLEMENT

Unlike bipartite entanglement, multipartite entanglement can be divided into many inequivalent entanglement classes. For example, three qubits can be entangled in two ways [23] and four qubits can be entangled in nine ways [26]. Therefore, usually a single scalar can only effectively characterize the entanglement of a single class or for some particular purposes. Even though multipartite entanglement of several quantum states has been well
classified, three-tangle was first presented by Coffman, et al. [27] and is the most remarkable and widely accepted entanglement monotone for a general state (instead of the states of given class) to quantify the Greenberger-Horne-Zeilinger (GHZ) type entanglement of qubits. GHZ type entanglement describes genuine tripartite inseparability. It is distinguished from its opposite tripartite entanglement class (W type entanglement) that lies in the different robustness of the residual two-qubit entanglement against losing the third qubit. In fact, GHZ type entanglement can also be understood by the maximal extra average two-qubit entanglement induced by measurements on the third qubit with classical communication [28]. Furthermore, 3-tangle can be naturally generalized to a \((2 \otimes 2 \otimes n)\)-dimensional quantum state in terms of concurrence and the minimal average concurrence is given by the concurrence of qubits A and B which is characterized by the localizable concurrence [29, 30] which is given by

\[
C_a(\langle \psi \rangle_{ABC}) = \sum_{i=1}^{4} \lambda_i
\]

and the minimal average concurrence is given by the concurrence [2] of \(\rho_{AB}\), that is,

\[
C(\rho_{AB}) = \max\{0, 2\lambda_1 - C_a(\langle \psi \rangle_{ABC})\}
\]

where \(\lambda_i\) denotes the square root of the eigenvalues of the matrix,

\[
R = \rho_{AB}(\sigma_y \otimes \sigma_y) \rho^*_{AB}(\sigma_y \otimes \sigma_y)
\]

in decreasing order with \(\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\). Thus the genuine tripartite entanglement monotone can be defined by [28]

\[
\tau(\langle \psi \rangle_{ABC}) = \sqrt{C^2_a(\langle \psi \rangle_{ABC}) - C^2(\rho_{AB})}
\]

It is obvious that \(\tau^2(\langle \psi \rangle_{ABC})\) will become three-tangle for \(n = 2\). Our first result shows that \(\tau(\langle \psi \rangle_{ABC})\) can be directly measured in experiment provided that only two-fold copy of \(\langle \psi \rangle_{ABC}\) is available. Our second result shows that the lower bound of \(\tau(\langle \psi \rangle_{ABC})\) can also be directly measured under the same condition with much simpler practical operations.

### III. THE MEASURABLE TRIPARTITE ENTANGLEMENT WITH TWO-FOLD COPY

It can be found that the matrix \(R\) given in Eq. (4) is the key to obtaining the genuine tripartite entanglement monotone \(\tau(\langle \psi \rangle_{ABC})\). Next we will construct another measurable matrix that can extract all of the useful information related to \(\tau(\langle \psi \rangle_{ABC})\) from the matrix \(R\).

Considering a set of orthonormal basis \(\{|\phi_k\rangle\}\) in \(H_{AC}\), \(\langle \psi \rangle_{ABC}\) can always be rewritten by \(\langle \psi \rangle_{ABC} = \sum_{k=0}^{n-1} |\phi_k\rangle_{AB} |\phi_k\rangle_{C}\), with \(|\phi_k\rangle_{AB}\) denoting the bipartite pure state without normalization. Based on these \(|\phi_k\rangle\), one can easily construct the following symmetric matrix \(M\):

\[
M_{ij} = \langle \phi_i | AB (\sigma_y \otimes \sigma_y) | \phi_j \rangle_{AB}
\]

Thus one can find that the following lemma holds.

**Lemma 1.** The set of the nonzero singular values of the matrix \(M\) is completely equal to the set of the square root of the eigenvalues of the matrix \(R\).

**Proof.** At first, we note that the reduced density matrix \(\rho_{AB}\) can be written as \(\rho_{AB} = \sum_{k=0}^{n-1} |\phi_k\rangle_{AB} \langle \phi_k |\). So we can construct an \(n\)-dimensional matrix \(\Psi\) such that

\[
\Psi = \{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_{n-1}\rangle\}
\]

where we have omitted the subscripts \((AB)\). Thus it can be easily found that \(\rho_{AB} = \Psi \Psi^\dagger\), and \(M\) can be rewritten as \(M = \Psi^T (\sigma_y \otimes \sigma_y) \Psi\). In order to find the singular values, we will have to calculate the eigenvalues of \(\Psi \Psi^T (\sigma_y \otimes \sigma_y) \Psi\). It is obvious that \(\Psi \Psi^T (\sigma_y \otimes \sigma_y) \Psi\) has the same eigenvalue set as the matrix \(\sigma_y \otimes \sigma_y \Psi \Psi^T (\sigma_y \otimes \sigma_y) \Psi\). This finishes the proof.

On the basis of Lemma 1, one can draw the conclusion that the genuine tripartite entanglement \(\tau(\langle \psi \rangle_{ABC})\) will be completely determined once the matrix \(M\) is known. Considering the two-fold copy of \(\langle \psi \rangle_{ABC}\), i.e., \(\langle \psi \rangle_{AB_1C_1} \otimes \langle \psi \rangle_{AB_2C_2}\) in the Hilbert space \(H_1 \otimes H_2 = (H_{AB_1} \otimes H_{B_2} \otimes H_{C_1} \otimes H_{AB_2} \otimes H_{B_2} \otimes H_{C_2})\), one can define the projectors in the anti-symmetric subspace \(H_{im} \wedge H_{nj}\).
Lemma 2.- The entries of $M$ can be given, subject to the two-fold copy of $|\psi\rangle_{ABC}$, by

$$M_{ij} = 2 \left( \langle \Psi^-_{A_2} | (i_{C_2}) \right) \left( |\psi\rangle_{A_1B_1C_1} |\psi\rangle_{A_2B_2C_2} \right),$$

where $|i_{C_1}\rangle$ and $|i_{C_2}\rangle$ are the basis in $H_{C_1}$ and $H_{C_2}$, respectively.

Proof. Expand $\sigma_x \otimes \sigma_y$ in the computational basis, and we have $\sigma^A_x \otimes \sigma^B_y = -|00\rangle_{AB}(11) + |01\rangle_{AB}(10) + |10\rangle_{AB}(01) - |11\rangle_{AB}(00)$, where we use the indices $A$ and $B$ to mark different action objects. Therefore, $M_{ij}$ can be rewritten as

$$M_{ij} = \langle \phi^x_i \rangle_{AB} \left( \sigma^x_y \otimes \sigma^y_y \right) |\phi^y_j\rangle_{AB}$$

and

$$M_{ij} = 2 \left( \langle \Psi^-_{A_2} | (i_{C_2}) \right) \left( |\phi^x_{i_{A_2}} \rangle_{A_1B_1} |\phi^y_{j_{A_2}} \rangle_{A_2B_2} \right),$$

which is exactly Eq. (9). Substitute $|\phi^x_i \rangle_{A_2B_2C_2}$ into Eq. (12) and one will easily obtain Eq. (10).

Theorem 1.- The absolute value $|M_{ij}|$ can be directly measured by local projective measurements, i.e.,

$$|M_{ij}| = 2 \sqrt{\langle \phi^x_{i_{A_1}} | \langle \phi^x_{i_{A_1}} | \langle \phi^x_{i_{A_1}} | \langle \phi^x_{i_{A_1}} |}$$

with $A$ a factorizable observable given by

$$A_{ijk} = P^{(A_1A_2)}_{-} \otimes P^{(B_1B_2)}_{-} \otimes P^{(C_1)}_{-} \otimes P^{(C_2)}_{-},$$

where the subscripts $\varepsilon_k$ denote the index $A_kB_kC_k$ and $P^{(C_k)}_{-} = |m\rangle_{C_k} \langle m|$ are the projectors subject to the basis $|m\rangle$ in $H_{C_k}$. Let $M = U_1U_2^T$, with $U$ unitary and $\Lambda$ diagonal and positive. Then $\tau(\langle \psi\rangle_{ABC})$ can be directly measured via $|M_{ij}|$ with the optimal choice $P^{(C_k)}_{+} = U^* |i\rangle_{C_k}$.

Proof. Eq. (13) is a direct and obvious result of Lemma 2. This proof is omitted here. Next we will show that the genuine tripartite entanglement $\tau(\langle \psi\rangle_{ABC})$ can be obtained by the measurable $|M_{ij}|$. From Ref. [31], one can find that any operation $Q$ operated on the qubit $C$ can be equivalently described as $M = QTMQ$. Based on Takagi decomposition [22] of a complex symmetric matrix, one can always write $M = U\Lambda^T$ where $U$ is a unitary matrix and $\Lambda$ is a diagonal matrix with the diagonal entries corresponding to the singular values of $M$. In this sense, one can always select a proper local unitary operation $Q$ on qubit $C$ such that $QTU = I$ which corresponds to $P_{-}^{(C_1)} = U^* |i\rangle_{C_1}$ and $P_{-}^{(C_2)} = U^* |j\rangle_{C_2}$.

With such a choice, one will obtain that $\hat{M} = |\hat{M}| = \Lambda$.

In other words, so long as we choose the optimal projective measurements on $C_1$ and $C_2$, $\hat{M}_{ii}$ just corresponds to $\lambda_i$, i.e., the singular values of $M$. This means that $\tau(\langle \psi\rangle_{ABC})$ can be measured directly and locally based on Eq. (14).

The above proof implies very important contents. One can find that $M_{ij}$, $i \neq j$, vanish once the optimal projective measurements on $C_2$ are achieved. It means that, if the optimal projector on $C_1$ and $C_2$ are different, there will not be any output corresponding to the projective measurements $P^{(A_1A_2)}_{-}$ and $P^{(B_1B_2)}_{-}$. So this becomes an important index by which one can signal when the optimal measurement basis has been achieved in the practical adjusting procedure.

An intuitive illustration of this scheme is sketched in Fig. 1. Suppose we have a pair of entangled tripartite pure states $|\psi\rangle_{A_1B_1C_1}$ and $|\psi\rangle_{A_2B_2C_2}$. The projective measurements $P^{(C_1)}_{i}$ and $P^{(C_2)}_{j}$ are performed on qubits $C_1$ and $C_2$, respectively. At the same time, joint projective measurement $P^{(A_1A_2)}_{-}$ and $P^{(B_1B_2)}_{-}$ is performed on $A_1$, $A_2$ and $B_1$, $B_2$. Adjust the measurement basis of $P^{(C_1)}_{i}$ and $P^{(C_2)}_{j}$ such that no signal is output from the measurement terminals $P^{(A_1A_2)}_{-}$ and $P^{(B_1B_2)}_{-}$ when $i \neq j$. At this moment, $\hat{M}_{ii}$ can be expressed by

$$\hat{M}_{ii} = \frac{P^{(C_1)}_{i}}{P^{(C_2)}_{j}} \frac{P^{(A_1A_2)}_{-}}{P^{(B_1B_2)}_{-}}$$

with $p_{\varepsilon_k}$ denoting the probability corresponding to the projective measurements on $\varepsilon_k$. So $\tau(\langle \psi\rangle_{ABC})$ can be easily obtained.

IV. MEASURING 3-TANGLE OF QUBITS IN LINEAR OPTICAL EXPERIMENT

We take the linear optical experiment of our scheme as an example. The experimental setup is briefly sketched in Fig. 2. Two entanglement resources are used to generate three entangled polarized photons with the state $|\psi\rangle_{A_1B_1C_1}$ and $|\psi\rangle_{A_2B_2C_2}$, respectively. As a demonstration, one can use the same setups as Ref. [33] to produce the polarized GHZ state of three photons. Each group of entangled photons are distributed into three paths represented by the labels of the corresponding qubits, respectively. Let photons $A_1$ and $A_2$ go through a beam splitter...
This is a good lower bound in that it is a sufficient and necessary condition for GHZ type inseparability of $|\psi\rangle_{ABC}$. It can be seen from that the lower bound vanishes if and only if the matrix $M$ is rank-one which is equivalent to $\tau$. Based on the upper bound of the singular value of a matrix \[52\], one can find that $\lambda_1$ can be well bounded by
\[
\lambda_1 \leq \sigma_U(q) = \left( \frac{\max_k \sum_j |M_{jk}|^{2q}}{\max_j \sum_k |M_{jk}|^{2(1-q)}} \right)^{1/2}
\]
for $q \in [0, 1]$. Thus we have
\[
\tau(|\psi\rangle_{ABC}) \geq 2 \sqrt{TrMM^\dagger - \min_{q \in [0, 1]} \sigma_U^2(q)}.
\]

Some simple bounds can be found when $q = 0, 1, 1$. It is obvious that $TrMM^\dagger = Tr\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}(\sigma_y \otimes \sigma_y) = Tr[\rho_{A_1B_1} \otimes \rho_{A_2B_2} P_{-}^{(A_1B_2)} \otimes P_{-}^{(B_1B_2)}]$ which shows that $TrMM^\dagger$ can be directly measured by local projective measurements with a two-fold copy of the state. In addition, $\sigma_{U}(q)$ given in Eq. (17) is described by $|M_{jk}|$, which can be obtained by the measurement statistics produced by $n(n+1)/2$ measurements including $P_{i}^{(C_1)}$ and $P_{j}^{(C_2)}$. From the lower bound point of view, it is not necessary to adjust the basis for $P_{i}^{(C_1)}$ and $P_{j}^{(C_2)}$. But the optimal choice of these two projectors can greatly improve the lower bound. So the lower bound is locally measurable provided that two copies of the states are available.

In fact, for three qubits, an alternative lower bound that could be relatively tight can be given by
\[
\tau_{\text{qubit}}(|\psi\rangle_{ABC}) = 2 \sqrt{|\det M|} \geq 2 \sqrt{|M_{00}| |M_{11}| - |M_{01}| |M_{10}|}.
\]

This bound can be easily proved by Eq. (20) for $i, j \leq 1$. So Eq. (5) can be directly related to the determinant of matrix $M$. It is obvious that all of the elements in the lower bound can be directly measured based on the above scheme, so the lower bound can be experimentally determined.

VI. DISCUSSIONS AND CONCLUSION

No experiment is perfect, so we have to know to what degree the measurement results are acceptable. In this scheme, in order to reduce the copies of the measured state, a key point is to adjust the projective measurements on $C_k$ such that no effective outputs corresponding to $P_{-}^{(A_1B_2)} \otimes P_{-}^{(B_1B_2)}$ are generated. However, there could be a small probability $\epsilon$ to detect photons in a practical
scenario. Thus, the practical $|M_{ii}|$ can always be formally given by $|M_{ii}| = \lambda_i + \epsilon \Delta$ in first order. This will lead to a small $(\sim \epsilon)$ deviation for the exact tripartite entanglement, but a little smaller lower bound for Eq. (19). In addition, the previous similar jobs tried to compensate for the experimental imperfection, which could imply that more prior information should be known. Here instead of doing this, we will mainly find out the potential errors. Without loss of generality, we only suppose that the prepared state is a quasi-pure state, that is, $\rho_{k_{1}k_{2}}(\psi)_{A_{1}B_{1}C_{1}} + \epsilon_{k} \rho_{k}$, where $\epsilon_{k} \ll 1$ and $\rho_{k}$ is a general tripartite density matrix with the subscript $k = 1, 2$ distinguishing different copies. Since the copy of the state is generated by another setup, it is reasonable to assume that 

$$|\psi\rangle_{A_{2}B_{2}C_{2}} = \sqrt{1 - \epsilon_{0}^{2}}|\psi\rangle_{A_{1}B_{1}C_{1}} + \epsilon_{0} |\phi\rangle$$  

where $\epsilon_{0} \ll 1$ and $\langle \psi | \phi\rangle = 0$. Substitute Eqs. (20) and (21) into Eq. (13), and one will find that $|M_{ij}'|$, corresponding to the imperfect preparation and copy, can be written in first order of $\epsilon_{i}$ as 

$$|M_{ij}'| \sim \sqrt{(1 - \epsilon_{1} - \epsilon_{2}) |M_{ij}|^2 + \epsilon N},$$

where $\tilde{\epsilon} = \max\{\epsilon_{0}, \epsilon_{1}, \epsilon_{2}\}$ and $N$ is not explicitly given here. All of the above analysis shows that the imperfect experiment will lead to a small deviation (about $\max\{\epsilon, \tilde{\epsilon}\}$) of the real value. However, if the entanglement of $|\psi\rangle$ is so small that $\tau \sim \epsilon$, that is, noise drowns out the signal, then this scheme cannot detect any entanglement which is similar to all the relevant jobs.

In summary, we have found that the high-dimensional tripartite entanglement can be locally measured with only a two-fold copy of the state. For simplicity, we also provide a good lower bound for entanglement which will simplify the practical operations but require more measurement outcomes. As a demonstration, we consider how to measure the three-tangle for three entangled qubits based on linear optical setups. The current scheme is only fit for a pure state which may not be so practical. However, needless to say, for the measurable entanglement with less copies of the state, even a simple lower bound for tripartite entanglement is not available in entanglement theory. We think this scheme could be an important step towards the more general cases.

VII. ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China, under Grants No.11375036 and No. 11175033, the Xinghai Scholar Cultivation Plan, and the Fundamental Research Funds for the Central Universities under Grants No. DUT15LK35 and No. DUT15TD47.
a few clicks (ineffective clicks) should be allowed for simplifying the practical operations.

[34] Ideally, no click should be recorded. However, every experiment has its tolerable error $\varepsilon$. Within such an error,

[35] J.-S. Xu, X.-Y. Xu, C.-F. Li, et al., Nat. Commun. 1, 7 (2010).