Higher-order estimates of the chromomagnetic moment of a heavy quark

A.G. Grozin
Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia

M. Neubert
Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract

The leading $\beta_0^{n-1} \alpha_s^n$ terms in the Wilson coefficient and anomalous dimension of the chromomagnetic operator in the heavy-quark effective Lagrangian are summed to all orders of perturbation theory. The perturbation series for the anomalous dimension is well behaved, while that for the Wilson coefficient exhibits a divergent behaviour already in low orders, caused by a nearby infrared renormalon singularity. The resulting ambiguity is commensurate with terms of order $1/m^2$ in the effective Lagrangian, whose corresponding ultraviolet renormalons are identified. An excellent approximation for the scheme-invariant Wilson coefficient at next-to-next-to-leading order in renormalization-group improved perturbation theory is obtained.

(Submitted to Nuclear Physics B)
1 Introduction

The properties of hadronic bound states containing heavy quarks are characterized by a large separation of energy scales. Effects associated with the heavy-quark mass \( m \) are perturbative and can be controlled once they have been separated from other, long-distance effects. This separation is most conveniently done using an effective low-energy theory. In the case of hadrons containing a single heavy quark, the relevant effective theory is the heavy-quark effective theory (HQET) (see [1] for a review). Its Lagrangian is

\[
\mathcal{L} = \bar{h} v \cdot D h + \frac{1}{2m} \bar{h} v (iD_\perp)^2 h + \frac{C(m, \mu)}{4m} \bar{h} v g_s \sigma_{\mu\nu} G^{\mu\nu} h + O(1/m^2),
\]

where \( v \) is the velocity of the hadron containing the heavy quark, \( h \) the velocity-dependent heavy-quark field, and \( D_\perp \) the covariant derivative orthogonal to \( v \). The operators arising at order \( 1/m \) in the effective Lagrangian correspond to the kinetic energy of the heavy quark and its chromomagnetic interaction [2, 3]. Hadronic matrix elements of these operators appear in many applications of HQET. The coefficient of the kinetic-energy operator is fixed by reparametrization invariance, an invariance under infinitesimal changes of the velocity [4]. Hence, the only non-trivial short-distance coefficient in the effective Lagrangian at next-to-leading order in \( 1/m \) is the coefficient \( C(m, \mu) \) of the chromomagnetic operator.

The dependence of the Wilson coefficient on the large scale \( m \) can be factorized using the renormalization group. In general,

\[
C(m, \mu) = C(m, m) \exp \int \frac{\alpha_s(m)}{\alpha_s(\mu)} \frac{\gamma(\alpha_s)}{\beta(\alpha_s)} \equiv \hat{C}(m) K(\mu),
\]

where

\[
\beta(\alpha_s) = -\frac{d\alpha_s}{d\ln \mu} = 2\alpha_s \left[ \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \ldots \right], \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f
\]

is the \( \beta \)-function, and

\[
\gamma(\alpha_s) = \gamma_0 \frac{\alpha_s}{4\pi} + \gamma_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \ldots = C_A \alpha_s \left[ 1 + \frac{13\beta_0}{6} - \frac{25C_A}{4\pi} \frac{\alpha_s}{4\pi} + \ldots \right]
\]

the anomalous dimension of the chromomagnetic operator (given in the MS scheme). The two-loop coefficient \( \gamma_1 \) of the anomalous dimension was calculated in [5, 6], and the two-loop initial condition \( C(m, m) \) in [7]. The corresponding one-loop expressions were obtained much earlier in [8]. The factor \( K(\mu) \) in (2) compensates the scheme and scale dependence of the chromomagnetic operator in the effective Lagrangian (1). All short-distance physics relevant to physical observables is contained in the renormalization-scheme invariant function \( \hat{C}(m) \) which, up to an \( m \)-independent normalization, is given by

\[
\hat{C}(m) = [\alpha_s(m)]^{\gamma_0/2\beta_0} \left[ 1 + c(m) \right], \quad c(m) = \sum_{n=1}^{\infty} c_n \left( \frac{\alpha_s(m)}{4\pi} \right)^n.
\]
In the \( \overline{\text{MS}} \) scheme, the next-to-leading correction is given by \( [5] \)

\[
c_1 = \left( \frac{25}{6} C_A + 2 C_F \right) - \left( \frac{55}{6} C_A + 3 C_F \right) \frac{C_A}{\beta_0} + \left( 7 C_A + 11 C_F \right) \frac{C_A^2}{\beta_0^2}.
\] (6)

In the above equations, \( C_F = \frac{1}{2}(N-1/N) \), \( C_A = N \), and \( T_F = 1/2 \) are the colour factors for an SU(\( N \)) gauge group.

As an application, consider the mass splitting between the ground-state pseudoscalar and vector mesons containing a heavy quark. Its leading contribution comes from the chromomagnetic operator in the effective Lagrangian, which is the first term breaking the heavy-quark spin symmetry. Including terms of order \( 1/m^2 \), one obtains \( [7] \)

\[
M_V - M_P = 2 \frac{C(m, \mu)}{3 \hat{\mu}_G^2(\mu)} + \frac{1}{3m^2} \left[ C(m, \mu) \hat{\rho}^3_{\pi G}(\mu) + C^2(m, \mu) \rho^3_A(\mu) - C_{LS}(m, \mu) \rho^3_{LS}(\mu) \right] ,
\] (7)

where \( \mu_G^2(\mu) \) is the matrix element of the chromomagnetic operator between ground-state mesons. Similarly, the hadronic parameters \( \rho^3_A(\mu) \) are defined in terms of the matrix elements of operators appearing at order \( 1/m^2 \) in the effective Lagrangian \( [7] \): \( \rho^3_{LS}(\mu) \) parametrizes the spin-orbit interaction, while \( \rho^3_{\pi G}(\mu) \) and \( \rho^3_A(\mu) \) are bilocal matrix elements of kinetic-chromomagnetic and chromomagnetic-chromomagnetic insertions. (We omit the contributions from 4-quark operators \( [8] \), which are irrelevant to our discussion.) These parameters are independent of the heavy-quark mass, but they depend on the renormalization scale in such a way that the scale dependence of the Wilson coefficients is compensated. The coefficient of the spin-orbit term is related by reparametrization invariance to that of the chromomagnetic operator \( [10] - [12] \) (see also the appendix of \( [9] \)): \( C_{LS}(m, \mu) = 2C(m, \mu) - 1 \). Introducing then the renormalization-scheme invariant parameters

\[
\hat{\rho}^2_G = K(\mu) \mu^2_G(\mu) , \\
\hat{\rho}^3_{\pi G} = K(\mu) \rho^3_{\pi G}(\mu) + 2[1 - K(\mu)] \rho^3_{LS}(\mu) , \\
\hat{\rho}^3_A = K^2(\mu) \rho^3_A(\mu) , \\
\hat{\rho}^3_{LS} = \rho^3_{LS}(\mu) ,
\] (8)

we find

\[
M_V - M_P = 2 \frac{\hat{C}(m)}{3m} \hat{\rho}^2_G + \frac{1}{3m^2} \left[ \hat{C}(m) (\hat{\rho}^3_{\pi G} - 2\hat{\rho}^3_{LS}) + \hat{C}^2(m) \hat{\rho}^3_A + \hat{\rho}^3_{LS} \right] .
\] (9)

All short-distance effects in this relation are contained in the single coefficient \( \hat{C}(m) \), which is the object of our study.

---

1 A similar mass formula has been derived in \( [8] \); however, the term proportional to \( \rho^3_A \) is missing there.
2 All-order results in the large-\(\beta_0\) limit

The perturbation series for \(C(m, \mu)\) can be arranged as

\[
C(m, \mu) = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_n^{(L)}(m/\mu) \beta_0^n \alpha_s^L(\mu), \tag{10}
\]

where \(L\) is the number of loops, and \(\beta_0\) the leading coefficient of the \(\beta\)-function in (3). In this letter, we sum the terms of order \(\beta_0^{L-1}\alpha_s^L\) to all orders of perturbation theory. In other words, we consider the limit of large \(\beta_0\) for fixed \(\beta_0\alpha_s\) and calculate the coefficient \(C(m, \mu)\) to order \(1/\beta_0\), neglecting terms of order \(1/\beta_0^2\) and higher. Strictly speaking, there is no sensible limit of QCD in which \(\beta_0\) may be considered a large parameter (except, maybe, \(n_f \to -\infty\)); however, retaining only the leading \(\beta_0\) terms often gives a good approximation to exact multi-loop results (see, e.g., [13]), in particular in cases when there is a nearby infrared renormalon [14]. At the least, it will provide us with some information about the summability of the perturbation series.

The coefficients \(a_n^{(L)}\) of the terms with the highest degree of \(\beta_0\) in (10) are determined by diagrams with \(L-1\) light-quark loops, which are rather straightforward to calculate. We work in dimensional regularization with \(d = 4 - 2\varepsilon\) space-time dimensions and adopt the \(\overline{\text{MS}}\) subtraction scheme. At first order in \(1/\beta_0\), multiplicative renormalization simply amounts to a subtraction of the \(1/\varepsilon\) poles, and coupling-constant renormalization is given by

\[
\beta_0 g_0^2/(4\pi)^2 = \bar{\mu}^2 \varepsilon \frac{b}{1 + b/\varepsilon}, \quad b = \frac{\beta_0 \alpha_s(\mu)}{4\pi} = \frac{1}{2 \ln(\mu/\Lambda_{\overline{\text{MS}}})}. \tag{11}
\]

The perturbation series for \(C(m, \mu)\) can then be written as

\[
C(m, \mu) = 1 + \sum_{L=1}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left( \frac{b}{\varepsilon + b} \right)^L - (\text{minimal subtractions}) + O(1/\beta_0^2). \tag{12}
\]

The function \(F(\varepsilon, u)\) is regular at \(\varepsilon = u = 0\). Following the methods of [13, 14] (used also in [13]), we now expand \(F(\varepsilon, u)\) in powers of \(\varepsilon\) and \(u\), and \([b/(\varepsilon + b)]^L\) in powers of \(b/\varepsilon\), to obtain a quadruple sum in (12). Combinatorial identities relate the \(1/\varepsilon\) terms, and hence the anomalous dimension of the chromomagnetic operator, to the Taylor coefficients of \(F(\varepsilon, 0)\) [15]:

\[
\gamma = \frac{2b}{\beta_0} F(-b, 0) + O(1/\beta_0^2). \tag{13}
\]

The finite terms, which determine the Wilson coefficient itself, receive contributions from the Taylor coefficients of both \(F(\varepsilon, 0)\) and \(F(0, u)\) [16]:

\[
C(m, \mu) = 1 + \frac{1}{\beta_0} \int_{-b}^{0} d\varepsilon \frac{F(\varepsilon, 0) - F(0, 0)}{\varepsilon} \\
+ \frac{1}{\beta_0} \int_{0}^{\infty} du e^{-u/b} \frac{F(0, u) - F(0, 0)}{u} + O(1/\beta_0^2). \tag{14}
\]
In this expression, all dependence on the heavy-quark mass resides in the function 
\[ F(0, u) \sim (\mu/m)^{2u}, \]
while a dependence on the renormalization scale also enters through the coupling 
\[ b \sim \alpha_s(\mu). \]
By separating in (14) terms depending on the two scales \( m \) and \( \mu \), we find that the next-to-leading logarithmic correction \( c(m) \) in (14) is given by

\[
c(m) = \frac{1}{\beta_0} \int_0^\infty du \ e^{-u[1/b(m)+\kappa]} \ S(u) + O(1/\beta_0^2),
\]

\[
S(u) = e^{\kappa u} \frac{F(0, u) - F(0, 0)}{u} \bigg|_{\mu=m}.
\]

The same result can also be derived by evaluating (2) in the large-\( \beta_0 \) limit. Here 
\[ b(m) = \frac{\beta_0 \alpha_s(m)}{4\pi}, \]
and the constant \( \kappa \) is introduced to compensate the scheme dependence of 
\( 1/b(m) \). The most natural choice is to have \( \kappa = -5/3 \) in the \( \overline{\text{MS}} \) scheme. Then the combination

\[
\frac{1}{b(m)} + \kappa = 2 \ln(m/\Lambda_{\overline{\text{MS}}}) - \frac{5}{3} \equiv 2 \ln(m/\Lambda_{V})
\]

defines the inverse coupling in the so-called V scheme \([17]\), with \( \Lambda_{V} = e^{5/6} \Lambda_{\overline{\text{MS}}} \). The function \( S(u) \) is the Borel transform of the perturbation series for \( c(m) \) in that scheme.

The coefficient \( C(m, \mu) \) is obtained by matching scattering amplitudes of an on-shell heavy quark in an external field in QCD and HQET, including terms of order \( 1/m \) \([2, 6]\). As mentioned above, our focus is on \( L \)-loop diagrams with \( L - 1 \) light-quark loops. All HQET diagrams vanish because they contain no mass scale. The relevant QCD diagrams are shown in Fig. 1. They must be supplemented by the wave-function renormalization of the external quark fields \([13]\). We find that in the \( \overline{\text{MS}} \) scheme the result for the function 
\[ F(\varepsilon, u) \] in (12) has the form

\[
F(\varepsilon, u) = \left( \frac{\mu}{m} \right)^{2u} e^{\gamma_E} \frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u-\varepsilon)} D(\varepsilon)^{u/\varepsilon-1} \left[ C_F N_F(\varepsilon, u) + C_A N_A(\varepsilon, u) \right],
\]

where

\[
D(\varepsilon) = 6 e^{\gamma_E} \frac{\Gamma(1+\varepsilon)\Gamma^2(2-\varepsilon)}{\Gamma(4-2\varepsilon)} = 1 + \frac{5\varepsilon}{3} + \ldots
\]

is related to the contribution of a light-quark loop to the gluon self-energy, and

\[
N_F(\varepsilon, u) = 4u(1 + u - 2\varepsilon u),
\]

\[
N_A(\varepsilon, u) = \frac{2 - u - \varepsilon}{2(1 - \varepsilon)} \left[ (2 - 5\varepsilon + 2\varepsilon^2) + (3 - 6\varepsilon + 4\varepsilon^2)u \right].
\]

These formulae reproduce the known \( L = 1 \) \([2]\) and \( L = 2 \) \([3]\) results, where \( L = u/\varepsilon \).

Having obtained an explicit result for the function \( F(\varepsilon, u) \), we are ready to derive all-order results for the anomalous dimension and Wilson coefficient of the chromomagnetic
operator in the large-\(\beta_0\) limit. From (13), we obtain
\[
\gamma = \frac{2 C_A}{\beta_0} \frac{b(1 + 2b)\Gamma(5 + 2b)}{24(1 + b)\Gamma^2(2 + b)\Gamma(1 - b)} + O(1/\beta_0^2)
\]
\[
= \frac{C_A \alpha_s}{2\pi} \left[ 1 + \frac{13}{6} \frac{\beta_0 \alpha_s}{4\pi} - \frac{1}{2} \left( \frac{\beta_0 \alpha_s}{4\pi} \right)^2 - 2\zeta(3) \left( \frac{\beta_0 \alpha_s}{4\pi} \right)^3 + \ldots \right],
\]
(20)
which reproduces correctly the leading (in \(\beta_0\)) term of the exact two-loop result (4). The radius of convergence of the perturbation series in (20) is \(\beta_0 |\alpha_s| < 4\pi\).

Next, setting \(\mu = m\) in (14), we find that the perturbative expansion of the Wilson coefficient at the scale \(\mu = m\) reads
\[
C(m, m) = 1 + \frac{\alpha_s(m)}{4\pi} \left\{ 2(C_F + C_A) + \left[ \frac{25}{3} C_F + \left( \frac{299}{36} + \frac{\pi^2}{3} \right) C_A \right] \frac{\beta_0 \alpha_s(m)}{4\pi} \right. \\
+ \left[ \left( \frac{317}{9} + \frac{4}{3} \pi^2 \right) C_F + \left( \frac{3535}{162} + \frac{25}{9} \pi^2 + \frac{14}{3} \zeta(3) \right) C_A \right] \left( \frac{\beta_0 \alpha_s(m)}{4\pi} \right)^2 + \ldots \right\},
\]
(21)
which again reproduces the leading term of the exact two-loop result (4). Unlike the case of the anomalous dimension, here retaining only the leading term in \(1/\beta_0\) gives a reasonable approximation; for \(N = 3\) colours, the exact two-loop result is
\[
C(m, m) = 1 + \frac{13}{6} \frac{\alpha_s(m)}{\pi} + \left( 2.869\beta_0 - 9.761 \right) \left( \frac{\alpha_s(m)}{\pi} \right)^2,
\]
(22)
showing that the sub-leading term is about one third of the leading one.

Finally, we quote the result for the next-to-leading logarithmic correction \(c(m)\) in (3). Expanding the integral in (13) in powers of the coupling constant, we find the expansion coefficients
\[
c_1 = 2C_F + \frac{25}{6} C_A + O(1/\beta_0),
\]
where the numerical values refer to $N$ that the higher-order coefficients obtained in the large-$\beta$ limit to be well behaved; in fact, this series is rapidly convergent. Although we do not expect the contrary, the coefficients in the expansion of the anomalous dimension in (20) appear which renders the perturbation series for $c$ divergent and not Borel summable. On the contrary, the coefficients in the expansion of the anomalous dimension in (20) appear to be well behaved; in fact, this series is rapidly convergent. Although we do not expect that the higher-order coefficients obtained in the large-$\beta$ limit accurately represent the exact values, we do trust in the qualitative features of the results. This allows us to derive an approximation for the next-to-next-to-leading order (NNLO) correction $c_2$ in (3).

Expanding (2) to second order in $\alpha_s(m)$, and inserting the known expansion coefficients for the three-loop $\beta$-function [13], the two-loop anomalous dimension [5, 6], and the two-loop matching coefficient $C(m, m)$ [3], we obtain an exact expression for $c_2$ depending only on the unknown three-loop coefficient of the anomalous dimension, for which we write $\gamma_2 = C_A(-\beta_0^2 + \eta_1 \beta_0 + \eta_0)$, where $\eta_1$ and $\eta_0$ are unknown. This parametrization takes into account that the anomalous dimension vanishes in the abelian limit $C_A = 0$ [13]. We then expand $c_2 = d_{-1} \beta_0 + \sum_{n=0}^{4} d_n (C_A/\beta_0)^n$ and find

$$d_{-1} = \frac{25}{3} C_F + \left( \frac{145}{18} + \frac{\pi^2}{3} \right) C_A = \frac{635}{18} + \pi^2,$$

$$d_0 = \left( -31 + \frac{20}{3} \pi^2 \right) C_F^2 + \left( -\frac{23}{24} + \frac{4}{3} \pi^2 \right) C_F C_A + \left( \frac{49}{48} - \frac{28}{9} \pi^2 \right) C_A^2 + \left( \frac{476}{9} - \frac{16}{3} \pi^2 \right) C_F T_F + \left( -\frac{298}{27} + \pi^2 \right) C_A^T_F + \frac{\eta_1}{4} C_A + \frac{4}{3} \pi^2 \ln 2 - 2\zeta(3) \right) (C_A^2 + C_F C_A - 6 C_F^2) = \frac{21353}{432} - \frac{695}{54} \pi^2 + \frac{28}{9} \pi^2 \ln 2 - \frac{14}{3} \zeta(3) + \frac{3}{4} \eta_1,$$

$$d_1 = -\frac{3}{4} C_F^2 + \frac{275}{18} C_F C_A + \frac{\eta_0}{4} = -\frac{370}{3} + \frac{1}{4} \eta_0,$$

$$d_2 = -\frac{37}{4} C_F^2 + \frac{5}{4} C_F C_A + \frac{6491}{144} C_A^2 = \frac{56771}{144},$$

$$d_3 = \frac{55}{2} C_F^2 - \frac{269}{6} C_F C_A - \frac{119}{3} C_A^2 = -\frac{4387}{9},$$

$$d_4 = \frac{121}{2} C_F^2 + 77 C_F C_A + \frac{49}{2} C_A^2 = \frac{11449}{18},$$

where the numerical values refer to $N = 3$ colours. Because of the good convergence
of the perturbation series for the anomalous dimension indicated by our analysis of the large-$\beta_0$ limit, it is conservative to assume that $|\eta_1 \beta_0 + \eta_0| < \beta_0^2$, which is equivalent to the statement that the true value of the three-loop anomalous dimension differs from its value in the large-$\beta_0$ limit by less than 100%. Under this assumption, we obtain

$$c_2(n_f = 4) = 210.08 \pm 6.25, \quad c_2(n_f = 3) = 238.04 \pm 6.75. \quad (25)$$

The uncertainty due to the unknown terms in the three-loop anomalous dimension is negligible compared with the overall size of the coefficient. Using this result together with the exact one-loop coefficient $c_1$ in (17), we find for $n_f = 4$

$$\bar{C}(m) = [\alpha_s(m)]^{9/25} \left[ 1 + 0.672\alpha_s(m) + (1.33 \pm 0.04)\alpha_s^2(m) + \ldots \right]. \quad (26)$$

In the context of the heavy-quark expansion, this is the most precisely known Wilson coefficient to date.

### 3 Borel summation and renormalon ambiguities

As a consequence of its divergent behaviour, the perturbation series for $c(m)$ must be truncated close to its minimal term, and the perturbative result for the Wilson coefficient is intrinsically ambiguous. Although we have explored this feature only in the large-$\beta_0$ limit, it is believed to be of a rather general nature (see, e.g., [19]). In regularization schemes without an explicit infrared cutoff (such as dimensional regularization with minimal subtraction), the perturbative calculation of the coefficients $c_n$ involves an integration over all gluon momenta, including long-distance contributions from soft gluons. High orders in the expansion probe the region of increasingly smaller gluon momenta, a regime where perturbation theory is bound to fail. Remarkably, the perturbation series knows about its deficiency and signals it through the divergent behaviour of the expansion coefficients. We will now investigate this phenomenon in more detail.

A convenient tool to study the ambiguities encountered in the attempt to resum an asymptotic perturbation series is provided by the Borel image of that series, given by the function $S(u)$ in the integral representation (15). Using (17), we find that

$$S(u) = \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \left[ 4u(1+u)C_F + \frac{(2-u)(2+3u)}{2}C_A \right] - e^{-5u/3} \frac{C_A}{u}. \quad (27)$$

If it existed, the Laplace integral in (15) would define the Borel sum of the perturbation series for $c(m)$ in the large-$\beta_0$ limit. However, the presence of singularities along the integration contour (i.e. for positive values of $u$) makes the integral ill-defined. In the large-$\beta_0$ limit, the function $S(u)$ has pole singularities at half-integer values of $u$ called infrared renormalons [20, 21]. Any attempt to resum perturbation theory involves an arbitrary choice of how to deal with these singularities. A measure of the ambiguity in the value of the Borel sum, which is of the same size as the minimal term in the series, is
provided by the residue of the nearest singularity, in the present case located at \( u = 1/2 \). Using (16), we obtain

\[
\Delta[c(m)] = \frac{1}{\beta_0} \left( 2C_F + \frac{7}{4} C_A \right) \frac{\Lambda_V}{m},
\]

with \( \Lambda_V = e^{5/6} \Lambda_{\text{MS}} \). The situation is similar to the well-known case of the infrared renormalon ambiguity in the definition of the pole mass of a heavy quark [22, 23]. The perturbation series relating the pole mass to a short-distance mass is also affected by an infrared renormalon at \( u = 1/2 \), and the corresponding ambiguity is \( \Delta m = 2C_F \Lambda_V/\beta_0 \). Taking into account that the chromomagnetic operator appears in the effective Lagrangian (1) multiplied by a power of the inverse pole mass, we find that

\[
\Delta \left[ \frac{\hat{C}(m)}{m} \right] = \frac{7C_A}{8C_F} \frac{\Delta m}{m^2} + O(1/\beta_0^3).
\]

Hence, the infrared renormalon ambiguity in the product \( \hat{C}(m)/m \) is purely non-abelian, and commensurate with the contributions of higher-dimensional operators in the \( 1/m \) expansion. Indeed, in all predictions for physical quantities, the infrared renormalon ambiguities must cancel against corresponding ambiguities in the long-distance matrix elements of some higher-dimensional operators.

When the heavy-quark expansion is applied to calculate a physical quantity, e.g. the mass splitting in (17), the resulting expressions contain short-distance Wilson coefficients and long-distance hadronic matrix elements. As mentioned above, in regularization schemes without a hard momentum cutoff the Wilson coefficients also contain contributions from large distances, where perturbation theory is ill-defined, and these contributions produce infrared renormalon ambiguities. Likewise, the hadronic matrix elements (which are not calculable perturbatively) contain contributions from small distances, which lead to ultraviolet renormalon ambiguities. In other words, in such schemes the separation of short- and long-distance contributions into Wilson coefficients and matrix elements is intrinsically ambiguous. Only when all contributions are combined to form a physical quantity, an unambiguous result is obtained. In the context of the HQET, the cancelations between infrared and ultraviolet renormalon ambiguities have been traced in detail in [24]. Here we consider the particular case of the mass splitting.

The ultraviolet contributions to the hadronic parameters \( \rho_i^3 \) are independent of the nature of the external states and thus may be calculated using quark and gluon rather than hadron states. At one-loop order, the relevant diagrams are shown in Fig. 2. On dimensional grounds, they are linearly divergent in the ultraviolet region. In dimensional regularization such power divergences are not seen (by definition), but they reflect themselves in a factorial divergence of the perturbation series when higher-order diagrams are taken into account. We shall again analyse these contributions in the large-\( \beta_0 \) limit, by inserting an arbitrary number of light-quark loops in all possible ways into the diagrams of Fig. 2. The relevant matrix elements are then expanded to linear order in the gluon momentum \( q \) and projected onto the structure of the chromomagnetic operator. The external quarks are kept off-shell in order to provide for an infrared regulator. The Borel
Figure 2: Diagrams contributing to the ultraviolet renormalons contained in the parameters $\rho_{aG}^3$ (a), $\rho_{A}^3$ (b), and $\rho_{LS}^3$ (c). Insertions of the chromomagnetic, kinetic and spin-orbit operators are indicated by squares, triangles and circles, respectively. It is implied that light-quark loops are inserted in all possible ways, for example as shown in the case of diagram (b). Also implied are mirror copies of the diagrams if appropriate.
transforms of these matrix elements contain ultraviolet renormalon poles at \( u = 1/2 \). For dimensional reasons, the residues are proportional to \( \Lambda_V \) times the matrix element \( \mu_G^2 \) of the chromomagnetic operator, which is the only lower-dimensional operator contributing to the mass splitting. Therefore, the leading ultraviolet renormalon ambiguities of the \( \rho_i^3 \) parameters are proportional to \( \mu_G^2 \Delta m \). Note that diagrams in which the external gluon is attached to a heavy-quark line, or to a two-gluon kinetic or chromomagnetic vertex, do not contribute. The diagram (c1) does not contribute to the chromomagnetic structure since the spin-orbit vertex only couples to the chromoelectric field. Moreover, with a suitable choice of the external momenta one can achieve that the diagrams (c2) and (c3) vanish. (Their sum vanishes with any choice of momenta.) A subtlety that needs to be taken into account is that there exists an ultraviolet renormalon at \( u = 1/2 \) in the \( 1/m \) suppressed contribution to the quark wave-function renormalization with a kinetic-energy insertion, shown in diagrams (d1) and (d2); the corresponding ambiguity \( \Delta Z_Q = \frac{3}{2} \Delta m/m \) in the normalization of \( \mu_G^2 \) contributes to \( \Delta \rho_{3G}^2 \) and cancels the \( C_F \) terms in the diagrams (a1) and (a2). From a direct evaluation of the non-vanishing diagrams, using the method described above, we obtain for the ultraviolet renormalon ambiguities in the large-\( \beta_0 \) limit

\[
\Delta \rho_{3G}^2 + \Delta \rho_A^3 - \Delta \rho_{LS}^3 = -\frac{7C_A}{4C_F} \mu_G^2 \Delta m + O(1/\beta_0^2). \tag{30}
\]

It precisely cancels the infrared renormalon ambiguity in the leading term.

It is interesting that the requirement of a cancelation of renormalon ambiguities in physical quantities allows us to derive further information about the asymptotic behaviour of the expansion coefficients \( c_n \) in (5) without any additional dynamical input [25, 26]. The point is that, beyond the large-\( \beta_0 \) limit, the different terms in (9) contain different powers of \( [\alpha_s(m)]^{\gamma_0/2\beta_0} \), since they contain different powers of the Wilson coefficient \( \hat{C}(m) \). These leading logarithms are exactly known to all orders of perturbation theory. They multiply the renormalon ambiguities of the various terms. In order to maintain the cancelation between infrared and ultraviolet renormalon ambiguities in the presence of the leading logarithms, the Borel transform for the Wilson coefficient in (27) has to be modified. In the vicinity of \( u = 1/2 \), the simple pole in

\[
S(u) = \left(2C_F + \frac{7}{4}C_A\right) \frac{1}{\frac{3}{2} - u} + \ldots \tag{31}
\]

must be replaced by a sum of branch points,

\[
S(u) = \frac{1}{(\frac{3}{2} - u)^{1+\beta_1/2\beta_0}} \left[2C_FK_1 - \frac{1}{3}C_AK_2 + \frac{19}{12} (\frac{1}{2} - u)^{-\gamma_0/2\beta_0} + \frac{1}{2} (\frac{1}{2} - u)^{\gamma_0/2\beta_0} \right], \tag{32}
\]

where \( \gamma_0 = 2C_A \) is the one-loop coefficient of the anomalous dimension of the chromomagnetic operator given in (4), and \( \beta_1 \) denotes the two-loop coefficient of the \( \beta \)-function. The normalization constants \( K_i = 1 + O(1/\beta_0) \) are undetermined beyond the large-\( \beta_0 \)
limit \([20]\). Up to corrections of order \(1/n\), the corresponding asymptotic behaviour of the expansion coefficients \(c_n\) is

\[
c_{n+1} = (2\beta_0)^n n! n^{\beta_1/2\beta_0} \left[ 4C_F K_1 - \frac{2}{3} C_A K_2 + \frac{19}{6} C_A K_3 n^{-\gamma_0/2\beta_0} + C_A K_4 n^{\gamma_0/2\beta_0} \right]. \quad (33)
\]

For very large \(n\), the last term gives the dominant behaviour (since \(\gamma_0/\beta_0 > 0\)); however, for moderate values of \(n\) all contributions are of similar importance.

In order to evaluate the Borel integral \([15]\), it is useful to rewrite it in the form \([14]\)

\[
c(m) = \frac{1}{\beta_0} \int_{-\infty}^{\infty} d\ln \tau \frac{w(\tau)}{\ln \tau + \ln (m/\Lambda_V)^2} + O(1/\beta_0^2)
\]

\[
= \int_0^\infty \frac{d\tau}{\tau} w(\tau) \frac{\alpha_s(\sqrt{\tau} e^{-5/6} m)}{4\pi} + O(1/\beta_0^2). \quad (34)
\]

The function

\[
w(\tau) = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du S(u) \tau^u = C_F w_F(\tau) + C_A w_A(\tau); \quad 0 < u_0 < \frac{1}{2},
\]

which is the inverse Mellin transform of the Borel image \(S(u)\), describes the distribution of gluon virtualities in the one-loop diagrams in Fig. 1, which contribute to the calculation of \(c(m)\). Using the methods developed in \([14]\), we obtain

\[
w_F(\tau) = \frac{4}{\sqrt{1 + 4/\tau}} - 4\tau + 2\tau^2 \left( \sqrt{1 + 4/\tau} - 1 \right),
\]

\[
w_A(\tau) = \frac{5\tau}{4} \left( \sqrt{1 + 4/\tau} - 1 \right) - \frac{3}{2\sqrt{1 + 4/\tau}} - \Theta(\tau - e^{5/3}). \quad (36)
\]

These functions are shown in Fig. 3. Note that the integral in \([34]\) runs over the Landau pole in the running coupling constant, located at \(\tau_L = (\Lambda_V/m)^2\). This is how infrared renormalons make their appearance. As in the case of the original Borel integral, we must specify an arbitrary prescription of how to deal with the Landau singularity. The renormalon ambiguity is given by the residue of the pole, i.e. \(\Delta[c(m)] = w(\tau_L)/\beta_0\). From an expansion of the distribution function in the region \(\tau \ll 1\), we readily recover the result \([28]\).

The representation \([34]\) makes explicit that perturbative calculations contain long-distance contributions from the region of low momenta in Feynman diagrams. At the same time, it provides for a convenient way to separate these long-distance contributions from the short-distance ones by introducing a hard separation scale \(\lambda\). We may then define the short-distance coefficient \(c_{sd}(m, \lambda)\) as

\[
c_{sd}(m, \lambda) = \int_{\lambda^2}^{\infty} \frac{d\mu^2}{\mu^2} w(\mu^2/m^2) \frac{\alpha_s(e^{-5/6} \mu)}{4\pi} + O(1/\beta_0^2). \quad (37)
\]
As long as $\lambda$ is chosen large enough, this coefficient can be reliably calculated in perturbation theory and is free of renormalon ambiguities. The long-distance contribution eliminated by this procedure must be combined with the non-perturbative power corrections in the heavy-quark expansion.

4 Numerical results and conclusions

Our goal was to study the large-order behaviour of the Wilson coefficient of the chromomagnetic operator in the HQET Lagrangian. The renormalization-group invariant coefficient $\hat{C}(m)$ defined in (5) is known exactly at next-to-leading order [5]. Here, we have derived an excellent next-to-next-to-leading order approximation by combining the exact two-loop matching condition obtained in [6] with an approximation for the three-loop anomalous dimension. This estimate has been obtained by summing fermion-loop contributions to all orders in perturbation theory, thus deriving the leading term in an expansion in powers of $1/\beta_0$. We have also derived an all-order result for the Wilson coefficient in the large-$\beta_0$ limit, finding that the perturbation series for $\hat{C}(m)$ exhibits a divergent behaviour already in low orders, caused by a nearby infrared renormalon singularity. The resulting ambiguity is commensurate with terms of order $1/m^2$ in the effective Lagrangian, whose corresponding ultraviolet renormalons we have identified.

Let us now investigate the implications of our results by comparing the various approximations for the Wilson coefficient. To this end, we evaluate the partial sums

$$\hat{C}_{(N)}(m) = [\alpha_s(m)]^{\gamma_0/2\beta_0} \left[ 1 + \sum_{n=1}^{N} c_n \left( \frac{\alpha_s(m)}{4\pi} \right)^n \right]$$

\[ (38) \]
Table 1: Perturbative approximations for the Wilson coefficients $\hat{C}_n(m_b)$ and $\hat{C}_n(m_c)$

| $N$ | $\hat{C}_n(m_b)$/$\hat{C}_n(m_c)$ |
|-----|----------------------------------|
|     | \begin{tabular}{l|l|l|l|l} \hline $N$ & $\hat{C}_n(m_b)$ & $\hat{C}_n(m_c)$ & \hat{C}_n(m_b) / \hat{C}_n(m_c) \\ \hline 0 & 0.580 & 0.692 & 0.838 & 0.838 \\ 1 & 0.734 & 0.847 & 0.739 & 0.778 \\ 2 & 0.801 & 0.967 & 0.664 & 0.721 \\ 3 & 0.850 & 0.817 & 0.778 & 0.744 \\ 4 & 0.898 & 0.687 & 0.581 & 0.774 \\ 5 & 0.958 & 0.687 & 0.353 & 0.774 \\ Borel sum & 0.839 & 1.027 & 0.817 & 0.898 \\ min. term & 0.898 & 1.027 & 0.774 & 0.906 \\ $\mu > 1$ GeV & 0.774 & 0.906 & 0.854 & 0.906 \\ \hline \end{tabular} \end{tabular} |

for $n_f = 4$ light quark flavours, using $\alpha_s(m_b) = 0.22$ and $\alpha_s(m_c) = 0.36$ in the $\overline{\text{MS}}$ scheme. The ratio of the two coefficient functions at these scales (for $n_f = 4$) gives the perturbative correction to the ratio of mass splitting between the ground-state pseudoscalar and vector mesons in the bottom and charm systems [5]:

$$
\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \hat{C}(m_b) / \hat{C}(m_c) \left[ 1 + \Lambda_{\text{eff}} \left( \frac{1}{m_c} - \frac{1}{m_b} \right) + \ldots \right] \approx 0.89 ,
$$

where $\Lambda_{\text{eff}}$ is a combination of the hadronic parameters $\rho_i^3$ introduced in (7), and for simplicity we neglect short-distance corrections in the $1/m$ terms.

Our predictions for the Wilson coefficients $\hat{C}_n(m_b)$ and $\hat{C}_n(m_c)$ are given in Table 1. In the upper part of the Table, we show the partial sums obtained at different orders in perturbation theory, both in the large-$\beta_0$ limit and exactly, so far as the exact results are known. In the large-$\beta_0$ limit, the divergent behaviour of the perturbation series sets in already in low orders. At the scale $m_b$, the minimal term in the series is reached around $N = 4$, whereas it is reached around $N = 2$ at the scale $m_c$. As a consequence, the values of the coefficient functions at the two scales become more and more different for larger $N$, and their ratio drifts away from the experimental value of the ratio of mass splittings.

The reason for this behaviour lies in the nearby location of the first infrared renormalon. This can be understood intuitively by considering the distribution functions shown in Fig. 3, which are broad and extend far into the infrared region. Note that
the point $\mu = m$ corresponds to $\ln \tau = 5/3$ in the $\overline{\text{MS}}$ scheme. Hence, there are essentially no contributions from scales above $m$ (in the case of the $C_A$ terms those are subtracted by $\overline{\text{MS}}$ renormalization), but on the other hand the distribution functions decrease very slowly ($\sim \sqrt{\tau}$) in the infrared region $\tau \to 0$. The strength of this decrease is determined by the position of the nearest infrared renormalon singularity. In general, if the nearest infrared renormalon is located at $u = k$, then $w(\tau) \sim \tau^k$ for $\tau \to 0$ [14]. As a consequence, the dominant contributions come from scales significantly below the heavy-quark mass. In the lower portion of Table 1, we show the results for the Wilson coefficients obtained by taking the principal value of the Borel integral. The quoted errors reflect the renormalon ambiguities, which are sizeable, in particular, at the scale $m_c$. The Borel resummation gives values close to those obtained when the series is truncated at the minimal term (in the latter case the error is taken to be the size of the minimal term), as it is expected on general grounds. In the last row we show the short-distance contributions to the coefficient functions arising from virtualities above 1 GeV (in the $V$ scheme), as defined in (37). The portion of infrared contributions is much larger at the scale $m_c$ than it is at the scale $m_b$. This is clearly seen in Fig. 3, where the arrows indicate the location of the separation scale $\lambda = 1$ GeV (corresponding to 435 MeV in the $\overline{\text{MS}}$ scheme). If the infrared contributions are cut away, the resulting values of the coefficients are again close to their values obtained from low-order calculations.

Although the all-order results are instructive, we must not forget that they are obtained in a very questionable approximation scheme (the large-$\beta_0$ limit). Indeed, the few exact results available indicate that the perturbation series may be much better behaved than indicated by the large-$\beta_0$ limit. Still, we trust in the qualitative observations that the series start to diverge at some (low) order, and that the onset of the divergence is reached early the lower the scale $m$ is. We will now explore the implications of these results for the phenomenology of the heavy-quark expansion. For a consistent inclusion of power-suppressed effects the perturbative coefficients of the leading terms must be known with sufficient accuracy. Different truncation or resummation schemes imposed on the short-distance coefficients imply different definitions of the hadronic parameters appearing at higher order in the heavy-quark expansion, such as the parameter $\Lambda_{\text{eff}}$ in (39). Only when the perturbative coefficients are truncated close to their asymptotic value, the hadronic parameters are $m$ independent as they should. When applied to the particular case of the chromomagnetic operator, the conclusion is that there is not much to be gained by calculating $\hat{C}(m)$ beyond the first few orders of perturbation theory. Taking into account that the series for $\hat{C}(m_c)$ diverges earlier than that for $\hat{C}(m_b)$, we may argue that the optimal perturbative prediction for the ratio based on exact information is obtained by combining the NNLO result at the bottom scale with the NLO

\footnote{For the combined distribution function $w(\tau)$ the average value of $\ln \tau$ is $-1.310$, corresponding to the scale $\mu_{\text{BLM}} \approx 0.226 m$ in the $\overline{\text{MS}}$ scheme. This is precisely the scale obtained in the BLM scale-setting prescription [17]. In view of the low value of the average virtuality, the bad convergence of the perturbation series is not surprising.}
result at the charm scale. This gives
\[ \frac{\hat{C}_{(2)}(m_b)}{\hat{C}_{(1)}(m_c)} \approx 0.80. \] (40)

In the large-$\beta_0$ limit, the corresponding ratio equals 0.81 and is indeed very close to the principal value of the Borel integral. Using the result (40), we conclude that the power corrections in (39) give a contribution of about 11%, corresponding to a scale $\Lambda_{\text{eff}} \approx 220$ MeV. Thus, if the asymptotic behaviour of perturbation theory is carefully taken into account, it appears that power corrections in the heavy-quark expansion are well under control.

Acknowledgments: We are grateful to M. Beneke for useful discussions. A.G.G. thanks the CERN Theory Division for its hospitality during the main part of this work.

References

[1] M. Neubert, Phys. Rep. 245 (1994) 259; Int. J. Mod. Phys. A11 (1996) 4173.

[2] E. Eichten and B. Hill, Phys. Lett. B243 (1990) 427.

[3] A.F. Falk, B. Grinstein and M.E. Luke, Nucl. Phys. B357 (1991) 185.

[4] M. Luke and A.V. Manohar, Phys. Lett. B286 (1992) 348.

[5] G. Amorós, M. Beneke and M. Neubert, Phys. Lett. B401 (1997) 81.

[6] A. Czarnecki and A.G. Grozin, Preprint TTP 97-04 (1997), hep-ph/9704143, Phys. Lett. B, to appear.

[7] I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, Phys. Rev. D52 (1995) 196.

[8] T. Mannel, Phys. Rev. D50 (1994) 428.

[9] B. Blok, J.G. Körner, D. Pirjol and J.C. Rojas, Nucl. Phys. B496 (1997) 358.

[10] Y.Q. Chen, Y.P. Kuang and R. Oakes, Phys. Rev. D52 (1995) 264.

[11] C. Balzereit and T. Ohl, Phys. Lett. B386 (1996) 335.

[12] M. Finkemeier and M. McIrvin, Phys. Rev. D55 (1997) 377.

[13] D.J. Broadhurst and A.G. Grozin, Phys. Rev. D52 (1995) 4082.

[14] M. Neubert, Phys. Rev. D51 (1995) 5924.

[15] A. Palanques-Mestre and P. Pascual, Comm. Math. Phys. 95 (1984) 277.
[16] D.J. Broadhurst, Z. Phys. C58 (1993) 339.

[17] S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys. Rev. D28 (1983) 228.

[18] O.V. Tarasov, A.A. Vladimirov and A. Yu. Zharkov, Phys. Lett. B93 (1980) 429.

[19] A.H. Mueller, in: QCD – 20 Years Later, edited by P.M. Zerwas and H.A. Kastrup (World Scientific, Singapore, 1993), p. 162; Phys. Lett. B308 (1993) 355.

[20] B. Lautrup, Phys. Lett. B69 (1977) 109.

[21] G. ’t Hooft, in: The Whys of Subnuclear Physics, Proceedings of the 15th International School on Subnuclear Physics, Erice, Sicily, 1977, edited by A. Zichichi (Plenum Press, New York, 1979), p. 943.

[22] M. Beneke and V.M. Braun, Nucl. Phys. B426 (1994) 301.

[23] I.I. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. D50 (1994) 2234.

[24] M. Neubert and C.T. Sachrajda, Nucl. Phys. B438 (1995) 235.

[25] G. Parisi, Phys. Lett. B76 (1978) 65; Nucl. Phys. B150 (1979) 163.

[26] M. Beneke, Phys. Lett. B344 (1995) 341; M. Beneke, V.M. Braun and N. Kivel, Preprint CERN-TH/97-50 (1997), hep-ph/9703389; Phys. Lett. B, to appear.