NON-SINGULAR SPHERICALLY SYMMETRIC SOLUTION IN EINSTEIN-SCALAR-TENSOR GRAVITY

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Abstract

A static spherically symmetric metric in Einstein-scalar-tensor gravity theory with a scalar field potential \( V[\phi] \) is non-singular for all real values of the coordinates. It does not have a black hole event horizon and there is no essential singularity at the origin of coordinates. The weak energy condition \( \rho_\phi > 0 \) fails to be satisfied for \( r \lesssim 1.3r_S \) (where \( r_S \) is the Schwarzschild radius) but the strong energy condition \( \rho_\phi + 3p_\phi > 0 \) is satisfied. The classical Einstein-scalar-tensor solution is regular everywhere in spacetime without a black hole event horizon. However, the violation of the weak energy condition may signal the need for quantum physics anti-gravity as \( r \to 0 \). The non-singular static spherically symmetric solution is stable against the addition of ordinary matter.

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1 Introduction

The problems associated with information loss of black holes have been a source of controversy for more than thirty years [1]. The maximal extension of the Schwarzschild spacetime by the Kruskal diagram [2], completing the space of geodesics except for the essential singularity at the origin, has led to a general acceptance by the physics community of the existence of black holes. There is evidence from the study of the measured motions of stars in the close vicinity of Sgr A* with a mass \( M \sim 3.7 \times 10^6 M_\odot \) that a black hole exists at the center of the Galaxy. However, due to the difficulty of actually detecting a black hole event horizon as predicted by general relativity (GR) the observational evidence remains circumstantial and controversial [3 4 5 6].
A non-singular solution for cosmology [7] has been obtained from a scalar-tensor-vector gravity (STVG) [8]. In the following, we shall give an example of a scalar-tensor gravity theory which can yield a static spherically symmetric solution which is free of an essential singularity and coordinate event horizon. A scalar field potential energy allows for a more general spherically symmetric solution than obtained in previous calculations in GR. The non-singular solution of the field equations leads to a violation of the weak energy condition \( \rho_\phi > 0 \) for \( r \lesssim 1.3r_S \), where \( r_S = 2G_NM \) is the Schwarzschild radius, avoiding the Hawking-Penrose singularity theorem [9, 10, 11]. We attribute the violation of the weak energy condition for \( r \lesssim 1.3r_S \) to the onset of quantum gravity or quantum repulsive exotic energy.

It is shown that when ordinary matter is added to the non-singular solution, then the absence of a singularity and an event horizon is maintained, for the scalar field barrier to singularity formation grows as the mass is added to the solution.

Because our non-singular spherically symmetric solution has neither an event horizon nor a singularity at the origin, standard black hole Hawking radiation is absent for a static, spherically symmetric astrophysical object and there is no Hawking information loss paradox. The simpler Einstein scalar-tensor gravity theory with a scalar field potential that can yield a non-singular static spherically symmetric solution can be generalized to a non-singular solution of the STVG field equations [8].

### 2 The Action and the Field Equations

The action takes the form:

\[
S = S_{\text{Grav}} + S_\phi + S_M, \tag{1}
\]

where

\[
S_{\text{Grav}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} (R + 2\Lambda), \tag{2}
\]

\[
S_\phi = -\int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V[\phi] \right]. \tag{3}
\]

Here, \( R \) is the Ricci scalar \( R = g^{\mu\nu}R_{\mu\nu} \), \( \Lambda \) is Einstein’s cosmological constant, \( \phi \) is a scalar field and \( V[\phi(r)] \) denotes a \( \phi \) field potential energy. We use the metric signature \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) where \( \eta_{\mu\nu} \) denotes the Minkowski metric tensor and we choose (unless otherwise stated) units with the speed of light \( c = 1 \).

We have

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{4}
\]

where \( T_{\mu\nu} \) is the total stress-energy momentum tensor

\[
T_{\mu\nu} = T_{M\mu\nu} + T_{\phi\mu\nu}, \tag{5}
\]
and \( T_{M\mu\nu} \) and \( T_{\phi\mu\nu} \) denote the matter and scalar field energy-momentum tensors, respectively. We have

\[
T_{\phi\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi - V[\phi] \right). \tag{6}
\]

The gravitational field equations are given by

\[
G_{\mu\nu} - g_{\mu\nu} \Lambda = 8\pi G_N T_{\mu\nu}, \tag{7}
\]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. From the Bianchi identities \( \nabla_\nu G^{\mu\nu} = 0 \) we obtain the conservation law:

\[
\nabla_\nu T^{\mu\nu} = 0, \tag{8}
\]

where \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \). The field \( \phi \) satisfies the equation of motion

\[
\nabla^\mu \nabla_\mu \phi + \frac{\partial V[\phi]}{\partial \phi} = 0. \tag{9}
\]

3 Non-Singular Static Spherically Symmetric Solution

The line element is of the standard form for a time-dependent spherically symmetric metric:

\[
ds^2 = B(r, t) dt^2 - A(r, t) dr^2 - r^2 d\Omega^2, \tag{10}\]

where

\[
d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \tag{11}\]

The field equations for \( T_{M\mu\nu} = 0 \) and \( \Lambda = 0 \) are given by

\[
R_{\mu\nu} = 8\pi G_N (T_{\phi\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\phi}), \tag{12}\]

where \( T_{\phi} = g^{\mu\nu} T_{\phi\mu\nu} \). We have

\[
T_{\phi\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\phi} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} V[\phi]. \tag{13}\]

The Ricci tensor is given by

\[
R_{00} = \frac{B'' - \ddot{A}}{2A} + \frac{\dot{A} \dot{B} - B''}{4AB} + \frac{\dot{\Lambda}^2 - \dot{A}' \dot{B}'}{4A^2} + \frac{B'}{Ar}, \tag{14}\]

\[
R_{rr} = \frac{\ddot{A} - B''}{2B} + \frac{B^2 - \ddot{A} \dot{B}}{4B^2} + \frac{A' \dot{B}' - \dot{A} \dot{B}'}{4AB} + \frac{A'}{Ar}, \tag{15}\]

\[
R_{r0} = R_{0r} = \frac{\ddot{A}}{Ar}, \tag{16}\]

\[
R_{\theta\theta} = \frac{1}{\sin^2 \theta} R_{\phi\phi} = 1 - \frac{1}{A} + \frac{A' r}{2A^2} - \frac{B' r}{2AB}. \tag{17}\]
We have for the right-hand side of (12):

\[ T_{\phi\mu\nu} = \frac{1}{2} g_{\mu\nu} T_{\phi} \]

\[ = \begin{pmatrix}
\dot{\phi}^2 - g_{00} V[\phi] & \dot{\phi} \phi' & 0 & 0 \\
\phi' \phi' & \phi^2 - g_{rr} V[\phi] & 0 & 0 \\
0 & 0 & -g_{\theta\theta} V[\phi] & 0 \\
0 & 0 & 0 & -g_{\phi\phi} V[\phi]
\end{pmatrix}. \tag{19} \]

In the static case \( \dot{A} = \dot{B} = 0 \) and \( \dot{\phi} = 0 \), we obtain

\[ R_{00} = -8\pi G_N g_{00} V[\phi(r)], \tag{20} \]
\[ R_{\theta\theta} = -8\pi G_N g_{\theta\theta} V[\phi(r)], \tag{21} \]

or

\[ \frac{R_{00}}{R_{\theta\theta}} = \frac{g_{00}}{g_{\theta\theta}}. \tag{22} \]

This equation can be shown to give

\[ \frac{2ABB''r - AB^2r - A'B'Br + 4ABB'}{4A^2B - 4AB + 2A'B'r - 2AB'r} = -\frac{B}{r}. \tag{23} \]

A useful relationship between \( A(r) \) and \( B(r) \) can be derived:

\[ 4(A - 1) + \left( 2 \frac{A'}{A} + 2 \frac{B'}{B} \right) r + \left( 2 \frac{B''}{B} - \frac{B^2}{B^2} - \frac{A'}{A} \frac{B'}{B} \right) r^2 = 0. \tag{24} \]

Let us take

\[ B(r) = 1/A(r). \tag{25} \]

We obtain from Eq. (24) the solution

\[ B(r) = 1 + \frac{C_1}{r} + C_2 r^2. \tag{26} \]

Choosing \( C_1 = -r_S = -2G_N M \) and \( C_2 = 0 \), we arrive at the Schwarzschild metric:

\[ ds^2 = \left( 1 - \frac{r_S}{r} \right) dt^2 - \frac{1}{1 - \frac{r_S}{r}} dr^2 - r^2 d\Omega^2. \tag{27} \]

We note that the pressure terms in the energy-momentum tensor \( T_{\phi\mu\nu} \) for the Einstein-scalar theory of gravity are not isotropic.

Let us now consider the case when

\[ B(r) \neq 1/A(r). \tag{28} \]
We see that even though $\dot{A} = 0$ from Eqs. (16) and (19), we still have $\dot{B} \neq 0$. Therefore, when (28) holds the solution for the metric does not satisfy the Birkhoff theorem [12] as in the case when (25) is satisfied for the Schwarzschild solution.

Using $\alpha = \ln A$ and $\beta = \ln B$ such that $A' / A = \alpha'$, $B' / B = \beta'$, and $B'' / B = \beta'' + \beta'^2$, we get

$$4(\exp(\alpha) - 1) + (2r - \beta' r^2) \alpha' + 2\beta' r + (2\beta'' + \beta'^2) r^2 = 0. \tag{29}$$

This equation is solvable for $\alpha$. We have

$$f(r)\alpha'(r) + \exp(\alpha(r)) + g(r) = 0, \tag{30}$$

from which

$$\alpha(r) = -\int dr \frac{g(r)}{f(r)} - \ln \left\{ C + \int dr \frac{\exp \left[ -\int dr \frac{g(r)}{f(r)} \right]}{f(r)} \right\}, \tag{31}$$

where $C$ is a constant. Given

$$f(r) = \frac{1}{2} r - \frac{1}{4} \beta'r^2, \tag{32}$$

$$g(r) = \frac{1}{2} \beta' r + \frac{1}{2} \beta'' r^2 + \frac{1}{4} \beta'^2 r^2 - 1, \tag{33}$$

we get

$$\alpha(r) = \int dr \frac{4 - 2\beta'r - 2\beta''r^2 - \beta'^2 r^2}{r(2 - \beta'r)} \ln \left\{ C + 4 \int dr \frac{\exp \left[ \int dr \frac{4 - 2\beta'r - 2\beta''r^2 - \beta'^2 r^2}{r(2 - \beta'r)} \right]}{r(2 - \beta'r)} \right\}. \tag{34}$$

Further,

$$\alpha' = \beta' + \frac{2}{r} - \frac{2\beta''r + \beta'}{2 - \beta' r} - \frac{e^{\beta}(2r - \beta' r^2)}{4 \beta'^2} + \int dr e^{\beta}(2r - \beta' r^2). \tag{35}$$

Therefore,

$$A(r) = \frac{(2B(r)r - B'(r)r^2)^2}{B(r)(C + 8 \int dr (B(r)r - B'(r)r^2))}. \tag{36}$$

Let us consider the ansatz for $g_{00}(r) = B(r)$:

$$B(r) = (1/a)[a - 1 + \exp (-arS/r)]. \tag{37}$$

We see that $B(r)$ does not have an event horizon for any real value of $r$ in the range $0 \leq r \leq \infty$ for $a > 1$. Moreover, $B(r)$ is non-singular at $r = 0$:

$$B(0) = \frac{a - 1}{a}. \tag{38}$$
Moreover, we have

\[ B(r) \sim 1 - \frac{r S}{r} \]  

so that \( B(r) \) satisfies the Schwarzschild solution for large \( r \), and the metric line element satisfies the Minkowski spacetime boundary condition as \( r \to \infty \).

We shall use

\[ \beta(r) = \ln B(r) = \ln \left\{(1/a)\left[a - 1 + \exp \left(-ar S/r\right)\right]\right\}, \]

(40)

to obtain from Eq. (36) the solution

\[ A(r) = \frac{\left[2(a - 1)r + (2r - ar S)\exp \left(-\frac{ar S}{r}\right)\right]^2}{aB(r) \left\{C + 4ar^2B(r) + 8ar S\left[ar S E_1 \left(\frac{ar}{r}\right) - r \exp \left(-\frac{ar S}{r}\right)\right]\right\}}, \]

(41)

with \( E_1 \) denoting the exponential integral of the first kind.

If \( a = 2 \) and the integration constant \( C = 0 \), we get

\[ \lim_{r \to +0} A(r) = 1. \]

(42)

To first order, \( A(r) \) evaluates to

\[ A(r) \simeq \frac{1}{1 - \frac{r S}{r}}, \]

(43)

and our metric satisfies the Schwarzschild solution for large values of \( r \). Plots of the metric components \( A(r) \) and \( B(r) \) are shown in Fig.1. We observe that \( A(r) \) and \( B(r) \) are non-singular for all real values in the range \( 0 \leq r \leq \infty \). Because the lowest order behavior of \( B(r) \) and \( A(r) \) for large values of \( r \) is the same as the Schwarzschild solution and the higher order contributions are small, then our solution agrees with all the classical gravitational experiments in the solar system and the binary pulsar observations.

By using the components of the Einstein tensor:

\[ G_{0}^{0} = \frac{1}{r A} \left(\frac{A'}{A} + \frac{1}{r}\right) + \frac{1}{r^2}, \quad G_{r}^{r} = -\frac{1}{r A} \left(\frac{B'}{B} + \frac{1}{r}\right) + \frac{1}{r^2}, \]

(44)

we can evaluate \( \rho_\phi \) and \( \rho_\phi + 3p_\phi \) from the field equations and the energy-momentum tensor [19]. We find that the non-singular solution violates the weak energy condition \( \rho_\phi \geq 0 \) for \( r \lesssim 1.3 r_s \) but not the strong energy condition \( \rho_\phi + 3p_\phi \geq 0 \). Plots of \( \rho_\phi \) and \( \rho_\phi + 3p_\phi \) are shown in Fig. 2.
The Ricci scalar $R$ and the Kretschmann curvature invariant given by
\[ K = R^\mu\nu\rho\sigma R_{\mu\nu\rho\sigma} \tag{45} \]
also remain well-behaved at $r = 0$ as shown in Figs. 3 and 4.

The total mass of the static spherically symmetric solution is calculated from the following integral:
\[ M = \frac{1}{2G_N} \int_0^\infty dr r^2 G_0^0(r), \tag{46} \]
where the Einstein tensor components $G_0^0$ are given by (44). This integral is quite complicated and cannot be evaluated in closed form. However, when evaluated numerically using $C = 0$ and $a \geq 3/2$ for arbitrary values of $r_S$, we get
\[ M = \frac{1}{2G_N} r_S. \tag{47} \]

The geodesic equations for a test particle are given by
\[ 0 = \frac{B'}{B} \frac{dr}{ds} + \frac{d^2t}{ds^2}, \tag{48} \]
Figure 4: This displays the Kretschmann curvature invariant \( K \).

\[
0 = \frac{B'}{2A} \left( \frac{dt}{ds} \right)^2 + \frac{A'}{2A} \left( \frac{dr}{ds} \right)^2 + \frac{d^2 r}{d^2 s} - \frac{r}{A} \left[ \left( \frac{d\theta}{ds} \right)^2 + \left( \frac{d\phi}{ds} \right)^2 \sin^2 \theta \right],
\]

(49)

\[
0 = \frac{d^2 \theta}{d s^2} + 2 \frac{d r}{d s} \frac{d \theta}{d s} - \left( \frac{d \phi}{d s} \right)^2 \sin \theta \cos \theta,
\]

(50)

\[
0 = \frac{d^2 \phi}{d s^2} + 2 \frac{d r}{d s} \frac{d \phi}{d s} + 2 \frac{d \theta}{d s} \frac{d \phi}{d s} \cot \theta.
\]

(51)

We have for \( a = 2 \): \( A(0) = 1, B(0) = 1/2, A'(0) = 0 \) and \( B'(0) = 0 \), and the above equations reduce to the geodesic equations:

\[
0 = \frac{d^2 t}{d s^2},
\]

(52)

\[
0 = \frac{d^2 r}{d s^2} - r \left[ \left( \frac{d\theta}{d s} \right)^2 + \left( \frac{d\phi}{d s} \right)^2 \sin^2 \theta \right],
\]

(53)

\[
0 = \frac{d^2 \theta}{d s^2} + 2 \frac{d r}{d s} \frac{d \theta}{d s} - \left( \frac{d \phi}{d s} \right)^2 \sin \theta \cos \theta,
\]

(54)

\[
0 = \frac{d^2 \phi}{d s^2} + 2 \frac{d r}{d s} \frac{d \phi}{d s} + 2 \frac{d \theta}{d s} \frac{d \phi}{d s} \cot \theta.
\]

(55)

for the Minkowski metric in spherical polar coordinates:

\[
ds^2 = dt^2 - dr^2 - r^2 d\Omega^2.
\]

(56)

In other words, at \( r = 0 \) the geodesic equations are the vacuum geodesic equations. The test particle equations for the non-singular metric components \( B(r) \) and \( A(r) \) are geodesically complete.

Let us now consider the \( V[\phi(r)] \) that can be obtained from the field equations:

\[
R_{\theta\theta} = -8\pi G_N g_{\theta\theta} V[\phi],
\]

(57)

or

\[
V[\phi(r)] = -\frac{R_{\theta\theta}}{8\pi G_N r^2}.
\]

(58)

We have

\[
R_{rr} = 8\pi G_N \left( \phi'^2 - g_{rr} V[\phi] \right) = 8\pi G_N \left( \phi'^2 + A \frac{R_{\theta\theta}}{8\pi G_N r^2} \right),
\]

(59)
from which we obtain
\[
\phi' = \sqrt{\frac{1}{8\pi G_N} \left( R_{rr} - \frac{A}{r^2} R_{\theta\theta} \right)}
\]  
(60)
\[
= \sqrt{\frac{1}{8\pi G_N} \left( -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{A'}{Ar} - \frac{A}{r^2} + 1 - \frac{A}{2Ar} + \frac{B'}{2Br} \right)}
\]  
(61)
\[
= \sqrt{\frac{-1}{32\pi G_N r^2} \left[ 4(A - 1) - \left( \frac{2A'}{A} + \frac{2B'}{B} \right) r + \left( \frac{2B''}{B} - \frac{B'^2}{B^2} - \frac{A'B'}{AB} \right) r^2 \right]}
\]  
(62)
Comparing this result with Eq. (24) allows us to eliminate many terms, leaving us with
\[
\phi' = \sqrt{\frac{1}{8\pi G_N r^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) r} = \sqrt{\frac{(\ln AB)'}{8\pi G_N r}}.
\]  
(63)
From this, it can be seen that unless the product \(AB\) is monotonically increasing, \(\phi'\) becomes imaginary. We can also use Eq. (36) to obtain
\[
\phi' = \sqrt{\frac{\ln \left( \frac{(2B'r-B'^2)^2}{C+0} \right)'}{8\pi G_N r}.
\]  
(64)
In the case of \(B(r) = (1/a)[a - 1 + \exp(-ar_S/r)]\), we get the explicit result,
\[
\phi' = 2ar_S \left\{ \left[ \left( r^2 + ar_S - \frac{1}{2} a^2 r_S^2 \right) E_1 \left( \frac{ar_S}{r} \right) - \frac{(a-1)r^2}{4} \right] \exp \left( \frac{-ar_S}{r} \right) \right. \\
+ \left[ \left( \frac{ar_S}{2} - \frac{11r}{8} \right) \exp \left( \frac{-2ar_S}{r} \right) + (a-1)r E_1 \left( \frac{ar_S}{r} \right) \right] r^{1/2} \\
\left. \times \left( r(r - 2ar_S) \exp \left( \frac{-ar_S}{r} \right) + 2a^2 r_S^2 E_1 \left( \frac{ar_S}{r} \right) + (a-1)r^2 \right) \right] \\
\times \left[ \left( r - \frac{ar_S}{2} \right) \exp \left( \frac{-ar_S}{r} \right) + (a-1)r \right]^{-1/2} \right\}.
\]  
(65)
In Fig. 4, we plot \(\phi'^2\) versus \(r\) for \(a = 2\) and \(r_S = 1\).

We also calculate \(V[\phi(r)]\):
\[
V[\phi(r)] = a^2 r_S^2 \left\{ \left( \frac{r}{2} + \frac{ar_S}{8} \right)^2 r \exp \left( \frac{-2ar_S}{r} \right) \\
+ \left[ (a-1)(ar_S - r) - ar_S E_1 \left( \frac{ar_S}{r} \right) \right] r \exp \left( \frac{-ar_S}{r} \right) \\
+ (a-1) \left[ ar_S (ar_S - r) E_1 \left( \frac{ar_S}{r} \right) + \frac{a - 1}{2} r^2 \right] \right\} \\
\times \left\{ \pi G_N \left[ (ar_S - 2r) \exp \left( \frac{-ar_S}{r} \right) - 2(a-1)r \right] r^3 \exp \left( \frac{ar_S}{r} \right) \right\}^{-1}
\]  
(66)
A plot of $V[\phi(r)]$ with $a = 2$ and $r_S = 1$ is shown in Fig.5:

We observe that $\phi'(r)$ becomes imaginary for $r < r_S$, while $V[\phi(r)]$ is negative corresponding to a repulsive potential and approaches zero as $r \to \infty$. The imaginary behavior of $\phi'(r)$ for $r < r_S$ means that a quantum behavior of the scalar field energy is required to explain the exotic form of the scalar field density $\rho_\phi$ needed to remove a singularity at $r = 0$. This could be quantum gravity or some form of quantum repulsive scalar field energy. On the other hand, all the physical properties of our regular solution, such as $\rho_\phi$ and $V[\phi(r)]$ are bounded, so this may mean that we do not require quantum gravity to obtain non-singular solutions in gravity theory.

In the range $0 \leq r \leq \infty$, the strong energy condition $\rho_\phi + 3p_\phi \geq 0$ or $R_{\mu\nu}U^\mu U^\nu \geq 0$ for a null or timelike vector $U^\mu$ is not violated. However, the weak energy condition: $\rho_\phi \geq 0$ is violated as we approach the Schwarzschild radius $r_S$. The Hawking-Penrose theorems [9, 10, 11] state that if both the weak and strong energy conditions for matter are satisfied and there exists an apparent event horizon, then the static spherically symmetric solution in GR must be singular. In particular, the focussing of null and time-like geodesics produces an event horizon at $r_S$ and an essential singularity at $r = 0$. Since our non-singular static spherically symmetric metric violates the weak energy condition for $0 \leq r \lesssim 1.3r_S$, we do not contradict the Hawking-Penrose theorems. Because of the absence of an event horizon our exterior solution does not possess a trapped surface and the geodesics of the spacetime metric are complete [9, 10, 11].

The violation of the weak energy condition signals that the classical non-singular solution fails for $r \lesssim 1.3r_S$. As $r \to 0$ quantum physics must intervene to preserve the non-singular solution and prevent the existence of an event horizon and an
essential singularity at \( r = 0 \).

If we add ordinary matter density \( \rho_M \) to the right-hand side of our field equations:

\[
\rho = \rho_M + \rho_\phi,
\]

then we observe that the non-singular metric solutions \( B(r) \) and \( A(r) \) and the potential \( V[\phi(r)] \) only depend on \( M \) through the Schwarzschild radius \( r_S = 2G_NM \). Therefore, as mass \( \delta M \) is added:

\[
M = M + \delta M
\]

the size of the repulsive potential barrier and the negative contribution of \( \rho_\phi \) increase and continue to maintain a stable non-singular metric. Thus, the scalar field acts as an effective, quantum exotic matter-energy that prevents the existence of an essential singularity in the metric at \( r = 0 \) and the formation of a black hole event horizon.

4 Conclusions

We have proposed an exterior non-singular static spherically symmetric solution for Einstein-scalar field gravity. The scalar field allows for negative density \( \rho_\phi \) and the potential \( V[\phi(r)] \) is repulsive as a function of \( r \) and \( M \).

When the nuclear fuel burns out in cores of stars, then in standard GR the star can collapse to zero radius and form an event horizon with an essential singularity at \( r = 0 \). The Fermi pressure at the core of a collapsed star caused by the degenerate electron and neutron gas and Pauli’s exclusion principle stabilizes white dwarfs and neutron stars, respectively. For the stable white dwarfs and neutron stars the Chandrasekhar mass limits are: \( M_c \sim 1.4 M_\odot \) and \( M_c \sim 2 - 3 M_\odot \), respectively [13]. When the mass of the star satisfies \( M > 6 M_\odot \), then according to the singular solutions of GR the star collapses to a black hole. However, for our non-singular solution, even when the star has a mass \( M > 6 M_\odot \), the effective quantum field density \( \rho_\phi < 0 \) as \( r \to 0 \) and the repulsive potential \( V[\phi(r)] \) prevents the star from forming a singularity at \( r = 0 \) and a black hole event horizon. At the core of the collapsed star the density of negative field energy \( \rho_\phi \) can be sufficiently large to prevent the star from collapsing to a black hole as the mass \( M \) is increased. Thus, if an effective matter-energy \( \rho_\phi \) dominates as \( r \to 0 \), then we can expect that at the core of a compact star the energy density \( \rho_\phi \) can produce enough “antigravity” to form a stable “dark grey” or “black” star, even as we continue to increase the ordinary matter \( M \) for the star. An important issue justifying further investigation is whether a non-singular collapsed dark grey star is stable. The non-singular solution must also be extended to the case of a rotating grey or black star.

We can speculate that the scalar \( \phi \) field energy permeates all of spacetime as a form of “quintessence”. This would mean that the vacuum field equations in GR:

\[
R_{\mu\nu} = 0
\]
are never fulfilled and that if our potential \( V[\phi(r)] \) is valid for a collapsed star, then black holes as described by the Schwarzschild solution do not exist in nature.

Our classical solution to Einstein-scalar-tensor gravity is non-singular throughout spacetime with bounded curvature invariant \( K, \rho, p, \phi, \) and \( V[\phi(r)] \). On the other hand, the fact that \( \rho \) becomes negative as \( r \to 0 \) could signal the need for quantum gravity at short distances to yield a solution that is regular everywhere in spacetime.

Because our static spherically symmetric solution does not possess a black hole event horizon, the compact star will only radiate “normal” radiation. This radiation may be so small that the star appears to an outside observer to be “black”. Since Hawking radiation is intimately associated with a black hole event horizon in the Schwarzschild solution of GR and such an event horizon is absent in our exterior regular solution, then our dark grey star does not have an information loss problem.

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