A version of scale calculus and the associated Fredholm theory

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Abstract

This article provides a version of scale calculus geared towards a notion of (nonlinear) Fredholm maps between certain types of Fréchet spaces, retaining as many as possible of the properties Fredholm maps between Banach spaces enjoy, and the existence of a constant rank theorem for such maps. It does so by extending the notion of linear Fredholm maps from [HWZ14] and [Weh12] to a setting where the Nash-Moser inverse function theorem can be applied and which also encompasses the necessary examples such as the reparametrisation action and (nonlinear) elliptic partial differential operators.

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0 Introduction

One of the more notorious problems in fields of geometric analysis such as the study of holomorphic curves in symplectic geometry is the fact that while the maps between spaces of functions (or rather their completions to Banach manifolds of maps of, say, some Sobolev class) defined by the partial differential operator under question, such as a nonlinear Cauchy-Riemann operator, are usually Fréchet differentiable, the same cannot be said about reparametrisation actions. As a standard example one can even take the action $S^1 \times W^{k,p}(S^1, \mathbb{R}) \to W^{k,p}(S^1, \mathbb{R})$, $(g, f) \mapsto (h \mapsto f(g \cdot h))$, which is continuous but not Fréchet differentiable. $W^{k,p}(S^1, \mathbb{R})$ here denotes the Banach space of maps $S^1 \to \mathbb{R}$ of Sobolev class $(k, p)$ for some $k \in \mathbb{N}$ and $1 < p < \infty$. In recent years, one very promising approach has been the sc-calculus that is part of the theory of polyfolds developed by H. Hofer, K. Wysocki and E. Zehnder, see the most recent [HWZ14] and the references therein. There the authors develop a new structure on a Banach space, called an sc-structure, and differentiability for maps between subsets of such spaces, called sc-differentiability.

The notion of sc-structure builds on the older notion of a scale of Banach spaces, i.e. a sequence $(E_i)_{i \in \mathbb{N}_0}$ of Banach spaces together with continuous inclusions $\iota_i : E_{i+1} \to E_i$, each of which has dense image. Asking in addition for each $\iota_i$ to be compact results in an sc-scale. Examples of such sc-scales would be $E_i := C^i(S^1, \mathbb{R})$ or $F_i := W^{1+i,p}(S^1, \mathbb{R})$ for some $p > 1$.

Under this new notion of differentiability, reparametrisation actions such as the one above become smooth. This sc-calculus also comes with a notion of linear and nonlinear Fredholm map, see [Weh12] for a closer examination of these.

Unfortunately, the notions of sc-Banach spaces and of sc-Fredholm operators and maps have a few not so desirable features. For example, the sc-scales $(E_i)_{i \in \mathbb{N}_0}$ and $(F_i)_{i \in \mathbb{N}_0}$ above satisfy $\bigcap_{i \in \mathbb{N}_0} E_i = C^\infty(S^1, \mathbb{R}) = \bigcap_{i \in \mathbb{N}_0} F_i$ and both these equalities turn $C^\infty(S^1, \mathbb{R})$ into the same Fréchet space. On the other hand, these sc-scales are not equivalent under the standard notion of an isomorphism between sc-Banach spaces.

Also, the notion of a nonlinear Fredholm map is fairly complicated, making questions such as whether or not the composition of two Fredholm maps is Fredholm quite intransparent.

In contrast in the setting of Banach spaces, one possible definition is that a continuous linear operator is Fredholm iff it is invertible modulo compact operators. This definition has the advantage that isomorphisms are obviously Fredholm and since compact operators form an ideal in the space of all bounded operators, it is very easy to see that the composition of two Fredholm operators is Fredholm again.

The most straightforward generalisation of this to Fréchet differentiable nonlinear maps between Banach spaces is that such a map is Fredholm iff its differential at every point is a Fredholm operator. It then is obvious that every diffeomorphism is Fredholm and by the chain rule the composition of two Fredholm maps is Fredholm again, resulting in a naturally coordinate invariant notion of a Fredholm map.

Because for maps between Banach spaces there exists a very strong inverse function theorem, there consequently also exists a constant rank theorem for such Fredholm maps.

Unfortunately, the situation drastically changes when moving from Banach to Fréchet spaces, where the invertibility of the differential of a map at a point does no longer imply the existence of a local inverse for the map itself. And even invertibility of the differential at all points does not suffice. So while the theory of Fredholm operators between Fréchet spaces largely mirrors the theory of Fredholm operators between Banach spaces, defining a Fredholm map to be a map whose differential at every point is Fredholm results in a class of maps that does not allow a constant rank
Fortunately, the situation is not completely hopeless, for at least for a certain class of Fréchet spaces and a class of maps between them that satisfy a certain boundedness condition, there does exist the famous Nash-Moser inverse function theorem (see the excellent article [Ham82] for the theorem, its proof, and plenty of (counter-)examples), guaranteeing the existence of a local inverse, provided the differential is invertible as a family of operators, within this given class of maps.

In this article I will join these two theories, of sc-calculus and the setting where the Nash-Moser inverse function theorem applies, and develop a theory of nonlinear Fredholm maps that follows the one in the setting of Banach spaces as closely as possible. This also involves a natural equivalence relation (basically a more formalised version of tame equivalence from [Ham82]) on sc-scales that makes the above two examples equivalent, removing a lot of ambiguity in the choice of a concrete sc-scale.

More concretely, after a brief excursion in Section 1 into differentiation in locally convex topological vector spaces, where the main definitions and results used in the remainder of this text are collected, first the linear theory of Fredholm operators is developed in Section 2. After defining the types of spaces, called sc-Fréchet spaces, and morphisms between them that are the basic elements of the theory, this theory of Fredholm operators starts by defining what a compact operator is in this setting. In short, the definition of these operators is as a strengthening of the definition of an sc+-operator from [HWZ14] that behaves well under equivalence, called a strongly smoothing operator, and which subsequently are shown to form an ideal. The theory then mirrors the Banach space setting by defining a Fredholm operator as an operator that is invertible modulo strongly smoothing operators, making the Fredholm property evidently stable under composition and under perturbation by strongly smoothing operators. It culminates in the main structure theorem on Fredholm operators, characterising them as operators with the property that the kernel and cokernel are finite dimensional, split the domain and target space, respectively, and s.t. the restriction of the operator to a complement of the kernel defines an isomorphism onto the image.

In Section 3, which is fairly technical and consists mainly of results on well definedness and independence of choices for the definitions, the notions of differentiability from Section 1 are applied to a scale setting.

In Section 4, in preparation for the nonlinear theory, augmented versions of an sc-Fréchet space are defined, which join the previous notion with the tameness conditions from [Ham82] and [LZ79]. Subsequently a version of the Nash-Moser inverse function theorem (extending the theorem from [Ham82] by results from [LZ79]) that is adapted to these notions is stated and proved.

In Section 5, the nonlinear Fredholm theory is then built by following the same scheme as in Section 2 but in a family version: First a notion of strongly smoothing family of operators is defined and shown to be an ideal under composition of families of operators. Then a Fredholm family of operators is defined to be one that is invertible modulo families of strongly smoothing operators and shown to have the standard properties: Compositions of Fredholm families are Fredholm, strongly smoothing perturbations of Fredholm families are Fredholm, and the Fredholm index is locally constant, behaves additively under composition and is invariant under strongly smoothing perturbations. And finally a Fredholm map is a map whose differential is a Fredholm family of operators.

Last but not least it is then shown that the Nash-Moser inverse function theorem implies the main theorems on Fredholm maps by virtually identical proofs as for maps between open subsets of Banach spaces: The constant rank theorem, finite
dimensional reduction and the Sard-Smale theorem. In the final Section, the results from the previous parts are summarised and collected into an application friendly framework. To get a more detailed overview of the main definitions, results and examples of this article it might actually be advisable to check out this part of the article first.
1 Notions of differentiability

In this section I will give a quick overview over some of the notions and results about differentiability in locally convex vector spaces. This is not meant as an exhaustive treatment and will cover only the elements of the theory that are used in later parts. For a more complete picture of the topic and its history the reader is referred e.g. to the textbooks [Kel74] or [Yam74].

One of the first things any student attending a calculus course hears when it comes to differentiability is that a function is differentiable if near every point (after a shift) it can be approximated by a linear function (equivalently, if it can be approximated by an affine function). Formalising this, let $X$ and $Y$ be Hausdorff locally convex topological vector spaces, let $U \subseteq X$ be an open subset and let $f : U \to Y$ be a continuous function. For a point $x \in U$ consider the shifted function $\tilde{f} : U \to Y$ $x' \mapsto f(x') - f(x)$.

The goal is to “approximate” this function by a linear function $L \in L_c(X,Y)$, which only becomes meaningful after one has given a precise definition of “approximate”. So let $V \subseteq X$ be a convex balanced neighbourhood of 0 s.t. $x + V \subseteq U$. For every $t \in (0,1]$, one can consider the rescaling map $\phi_t : V \times Y \to U \times Y$ $(u,y) \mapsto (x + tu, ty)$ and pull back both $\tilde{f}$ and $L$ by $\phi_t$:

$$\phi_t^* \tilde{f} : V \to Y$$

$$u \mapsto \frac{1}{t} (f(x + tu) - f(x)),$$

whereas $\phi_t^* L = L|_V$. “Approximate” then means that the difference $r^f_x(\cdot, t) := \phi_t^* \tilde{f} - \phi_t^* L : V \to Y$

$$u \mapsto \frac{1}{t} (f(x + tu) - f(x)) - Lu$$

“goes to zero as $t \to 0$”.

One can now follow several strategies to give precise meaning to this phrase. As a first possibility, one can consider this as a one parameter family of continuous functions $(r^f_x(\cdot, t))_{t \in (0,1]} \subseteq C(V,Y)$ and after equipping the space $C(V,Y)$ with a concrete topology one arrives at a precise definition of “approximate” is supposed to mean: Setting $r^f_x(\cdot, 0) \equiv 0$, $r^f_x(\cdot, t) \to r^f_x(\cdot, 0)$ as $t \to 0$ in the given topology on $C(V,Y)$. Alternatively, one can consider this one parameter family as a function $r^f_x : V \times [0,1] \to Y$ and ask for this to be either a continuous or a uniformly continuous function, eliminating the need for a choice of topology on $C(V,Y)$.

As a third possibility one can ask for $r^f_x$ to be either continuous or uniformly continuous when restricted to subsets of $V \times [0,1]$ of the form $A \times [0,1]$, where $A$ comes from a chosen class of subsets such as finite, compact or bounded subsets. Some of these choices will now be explored over the next few sections.
1.1 Definitions and various characterisations

1.1.1 Nonlinear maps between topological vector spaces

I will use the terminology from [Osb14]:

Let $X$ and $Y$ be Hausdorff locally convex topological vector spaces over a ground field $k$, which is $\mathbb{R}$ or $\mathbb{C}$. $L_c(X,Y)$ denotes the space of continuous linear maps and will be equipped with the topology of bounded convergence (or bounded-open topology). It is generated (i.e. these sets form a neighbourhood basis of 0) by the subsets

$$N(A,U) := \{ L \in L_c(X,Y) \mid L(A) \subseteq U \},$$

where $A \subseteq X$ is a bounded subset (i.e. for every neighbourhood $V \subseteq X$ of 0 there exists $c \in k$ s.t. $A \subseteq cV$) and $U \subseteq Y$ is a convex balanced ($\forall c \in k$ with $|c| \leq 1$, $cU \subseteq U$) neighbourhood of 0. If $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ are normed spaces (so in particular for Banach spaces), this coincides with the probably more familiar operator norm topology. I will denote the associated operator norm by

$$\|\cdot\|_{L_c(X,Y)} : L_c(X,Y) \to [0,\infty) \quad L \mapsto \sup \{ \|Lx\|_Y \mid x \in X, \|x\|_X = 1 \}.$$ 

I will also repeatedly use the following properties for subsets of a locally convex topological vector space and (not necessarily linear) maps between such subsets:

**Definition 1.1.** Let $X$ and $Y$ be locally convex topological vector spaces, let $U \subseteq X$ be a subset and let $f : U \to Y$ be a map.

1. A subset $A \subseteq X$ is called **bounded** iff for every neighbourhood $V \subseteq X$ of 0 there exists $c \in k$ s.t. $A \subseteq cV$. A subset $A \subseteq U$ is called bounded if it is bounded as a subset of $X$.
2. $X$ is called **locally bounded** iff there exists a bounded neighbourhood of 0 in $X$.
3. $f$ is called **bounded** if it maps bounded subsets of $U \subseteq X$ to bounded subsets of $Y$.
4. $f$ is called **locally bounded** if for every $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ in $U$ s.t. $f|_V : V \to Y$ is bounded.
5. $f$ is said to have **bounded image** if $f(U) \subseteq Y$ is a bounded subset.
6. $f$ is said to have **locally bounded image** if for every $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ in $U$ s.t. $f|_V : V \to Y$ has bounded image.

**Remark 1.1.**

1. In finite dimensions, a subset $A \subseteq X$ is compact iff it is bounded and closed. This follows by combining Propositions 2.8 and 3.33 in [Osb14].
2. $X$ is locally bounded iff it is normable.
3. The space of smooth sections of a vector bundle with its Fréchet topology (i.e. the inverse limit of the topologies defined by the $C^k$-norms, cf. Section 1.3) is not locally bounded. This follows immediately from the existence, for any $k \in \mathbb{N}_0$, of a sequence $(u_n^k)_{n \in \mathbb{N}_0}$ of smooth sections whose $C^k$-norms are bounded (independently of
\[ n \in \mathbb{N}_0, \] but whose \( C^{k+1} \)-norms go to infinity when \( n \) goes to infinity. The construction is completely analogous to the construction of the section \( u^n \) from Example 1.2.

4. In finite dimensions, if \( U \) is closed, then any continuous map \( f : U \to Y \) is bounded.

For a bounded subset of \( U \) then has compact closure in \( U \), hence its image is contained in a compact, hence bounded (cf. [Osb14], Theorem 4.28), subset of \( Y \), so is bounded itself.

5. In finite dimensions, any continuous map \( f : U \to Y \) is locally bounded, but obviously not every continuous map has bounded image.

6. Any continuous linear map \( L : X \to Y \) is bounded (cf. [Osb14], Proposition 2.18).

7. If \( f \) has (locally) bounded image, then \( f \) is (locally) bounded.

8. If \( X \) is locally bounded, then \( f \) has locally bounded image iff \( f \) is locally bounded.

**Definition 1.2** (and Lemma). Let \( X \) and \( Y \) be locally convex topological vector spaces and let \( U \subseteq X \) be an open subset.

1. \( \mathcal{C}(U,Y) \) is the vector space of all continuous functions from \( U \) to \( Y \).

2. \( \mathcal{C}^b(U,Y) \) is the vector space of all bounded continuous functions from \( U \) to \( Y \).

3. \( \mathcal{C}^\text{bi}(U,Y) \) is the vector space of all continuous functions from \( U \) to \( Y \) with bounded image.

4. The subsets of the form

\[ N(C,W) := \{ g \in \mathcal{C}(U,Y) \mid g(C) \subseteq W \}, \]

where \( W \subseteq Y \) is a convex balanced neighbourhood of 0 in \( Y \) and \( C \subseteq U \) is a finite subset, form a neighbourhood base at 0 of a topology on \( \mathcal{C}(U,Y) \) called the \textit{finite-open} topology.

It turns \( \mathcal{C}(U,Y) \) into a Hausdorff locally convex topological vector space denoted \( \mathcal{C}_{fo}(U,Y) \).

5. The subsets of the form

\[ N(K,W) := \{ g \in \mathcal{C}(U,Y) \mid g(K) \subseteq W \}, \]

where \( W \subseteq Y \) is a convex balanced neighbourhood of 0 in \( Y \) and \( K \subseteq U \) is a compact subset, form a neighbourhood base at 0 of a topology on \( \mathcal{C}(U,Y) \) called the \textit{compact-open} topology.

It turns \( \mathcal{C}(U,Y) \) into a Hausdorff locally convex topological vector space denoted \( \mathcal{C}_{co}(U,Y) \).

6. The subsets of the form

\[ N(B,W) := \{ g \in \mathcal{C}^b(U,Y) \mid g(B) \subseteq W \}, \]

where \( W \subseteq Y \) is a convex balanced neighbourhood of 0 in \( Y \) and \( B \subseteq U \) is a bounded subset, form a neighbourhood base at 0 of a topology on \( \mathcal{C}^b(U,Y) \) called the \textit{bounded-open} topology.

It turns \( \mathcal{C}^b(U,Y) \) into a Hausdorff locally convex topological vector space denoted \( \mathcal{C}^b_{bo}(U,Y) \).
7. The subsets of the form
\[ N(A,W) := \{ g \in \mathcal{C}^{bi}(U,Y) \mid g(A) \subseteq W \}, \]
where \( W \subseteq Y \) is a convex balanced neighbourhood of 0 in \( Y \) and \( A \subseteq U \) is a bounded subset, form a neighbourhood base at 0 of a topology on \( \mathcal{C}^{bi}(U,Y) \) called the arbitrary-open topology.

It turns \( \mathcal{C}^{bi}(U,Y) \) into a Hausdorff locally convex topological vector space denoted \( \mathcal{C}^{bi}_{uo}(U,Y) \).

**Proof.** The proofs that the above topologies are well defined with the stated properties all use \([Osb14], \) Theorem 3.2.

For let \( A \subseteq U \) be a (finite, compact, bounded) subset and let \( W \subseteq Y \) be a convex balanced neighbourhood of 0. Then \( \tilde{N}(A,W) \) is convex and balanced because \( W \) is. It is also absorbent: If \( g \in \mathcal{C}(U,Y) \) \((g \in \mathcal{C}^{bi}(U,Y), g \in \mathcal{C}^{bi}(U,Y))\), then \( g(A) \subseteq Y \) is bounded, because it is finite or compact (cf. \([Osb14], \) Theorem 4.28), or by assumption. So there exists \( c \in \mathbb{K} \) s. t. \( g(A) \subseteq cW \). Hence \( g \in \tilde{N}(A,cW) = c\tilde{N}(A,W) \) and \( \tilde{N}(A,W) \cap \tilde{N}(A',W') \subseteq \tilde{N}(A \cup A',W \cap W') \). Finally, \( \bigcap \mathcal{B}_0 = \{0\} \), for let \( 0 \not= g \in \mathcal{C}(U,Y) \) \((g \in \mathcal{C}^{bi}(U,Y), g \in \mathcal{C}^{bi}(U,Y))\). Choose \( x \in U \) s. t. \( g(x) \not= 0 \) and let \( A := \{x\} \).\( W := Y \setminus \{g(x)\} \) is an open neighbourhood of 0 so by \([Osb14], \) Proposition 3.1, there exists a convex balanced neighbourhood \( U \subseteq Y \) of 0 s. t. \( W \subseteq W' \). Then \( g \not\in \tilde{N}(A,W) \).

**Definition 1.3.** Let \( X \) and \( Y \) be locally convex topological vector spaces, let \( U \subseteq X \) be a subset and let \( f : U \to Y \) be a map.

1. \( f : U \to Y \) is called **uniformly continuous** if for every neighbourhood \( W \) of 0 in \( Y \) there exists a neighbourhood \( V \) of 0 in \( U \) s. t. \( f(x) - f(y) \in W \) for all \( x,y \in U \) with \( x - y \in V \).

2. \( f \) is called **locally uniformly continuous** if for every \( x \in U \) there exists a neighbourhood \( V \subseteq U \) of \( x \) in \( U \) s. t. \( f|_V : V \to Y \) is uniformly continuous.

3. \( \mathcal{C}^{uc}(U,Y) \), \( \mathcal{C}^{uc,b}(U,Y) \) and \( \mathcal{C}^{uc,bi}(U,Y) \) are the vector spaces of all function from \( U \) to \( Y \) that are uniformly continuous, bounded uniformly continuous and uniformly continuous with bounded image, respectively.

If equipped with the corresponding finite-open, compact-open, bounded-open and arbitrary-open topologies, they are Hausdorff locally convex topological vector spaces denoted \( \mathcal{C}^{uc}_{fo}(U,Y) \), \( \mathcal{C}^{uc}_{co}(U,Y) \), \( \mathcal{C}^{uc,b}_{ho}(U,Y) \) and \( \mathcal{C}^{uc,bi}_{uo}(U,Y) \), respectively.

**Remark 1.2.** Obviously, every (locally) uniformly continuous map is continuous.

An important result about uniform continuity is the following rather well known (at least for metric spaces) statement relating continuity and uniform continuity on compact subsets.

**Proposition 1.1.** Let \( X \) and \( Y \) be Hausdorff locally convex topological vector spaces, let \( K \subseteq X \) be a compact subset and let \( g : K \to Y \) be a continuous function. Then \( g \) is uniformly continuous.

**Proof.** Consider the function
\[ G : K \times K \to Y \]
\[ (x, y) \mapsto g(x) - g(y) \]
and the embedding

\[ \Phi : K \times K \to K \times X \]
\[ (x, y) \mapsto (x, x - y) \]

and define \( \tilde{K} := \Phi(K \times K) \) and \( \tilde{G} := G \circ \Phi^{-1}|_{\tilde{K}} : \tilde{K} \to Y \). Let \( W \subseteq Y \) be an arbitrary neighbourhood of 0. \( \tilde{G}(K \times \{0\}) = \{0\} \subseteq W \), so by continuity of \( g \) and hence \( \tilde{G} \), for every \( x \in K \) there exists a neighbourhood (which can be assumed to be of the form) \( V'_1 \times V'_2 \subseteq K \times X \) of \( (x, 0) \) s.t. \( \tilde{G}(K \cap V'_1 \times V'_2) \subseteq W \). Now as in the proof of Lemma 1.1, there exist finitely many \( x_1, \ldots, x_r \in K \) s.t. \( K \subseteq \bigcup_{i=1}^r V'_{x_i} \). Define \( V := \bigcap_{i=1}^r V_{x_i} \). Then \( \tilde{G}(K \cap V \times X) \subseteq \tilde{G}(K \cap \bigcup_{i=1}^r V'_{x_i} \times V) = \bigcup_{i=1}^r \tilde{G}(K \cap V_{x_i} \times V) \subseteq W \). On the other hand, \( \tilde{G}(K \cap K \times V) = G(\Phi^{-1}(K \cap K \times V)) = G(\{(x, y) \in K \times K \mid x \in K, x - y \in V\}) = \{g(x) - g(y) \mid (x, y) \in K \times K, x - y \in V\} \). Or in other words, \( g(x) - g(y) \in W \) for all \( x, y \in K \) with \( x - y \in V \), so \( g \) is uniformly continuous. \( \square \)

**Corollary 1.1.** Let \( X \) and \( Y \) be Hausdorff locally convex topological vector spaces, let \( U \subseteq X \) be an open subset and let \( f : U \to Y \) be a map. 
If \( X \) is finite dimensional, then \( f \) is locally uniformly continuous iff \( f \) is continuous.

**Proof.** One direction is trivial. And in the other direction, because \( X \) is finite dimensional, every point in \( U \) has a compact neighbourhood \( K \). So if \( f \) is continuous, by Proposition 1.1, \( f|_K \) is uniformly continuous. \( \square \)

### 1.1.2 Weak Fréchet differentiability

**Definition 1.4.** Let \( X \) and \( Y \) be Hausdorff locally convex topological vector spaces, let \( U \subseteq X \) be an open subset and let \( f : U \to Y \) be a continuous map. 
Given a point \( x \in U \), a continuous linear map \( L \in L_c(X, Y) \) and a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. \( x + V \subseteq U \), define a function

\[
r^f_x : V \times [0, 1] \to Y \\
(u, t) \mapsto \begin{cases} 
\frac{t}{2} (f(x + tu) - f(x)) - Lu & t > 0 \\
0 & t = 0
\end{cases}.
\]

\( f \) is called

1. **Gâteaux, or pointwise weakly Fréchet, differentiable at \( x \) with derivative \( Df(x) := L \)** iff there exist \( L \in L_c(X, Y) \) and \( V \subseteq X \) as above s.t. for every finite subset \( C \subseteq V \) the map

\[
r^f_x|_{C \times [0, 1]} : C \times [0, 1] \to Y
\]

is continuous.

2. **compactly weakly Fréchet differentiable at \( x \) with derivative \( Df(x) := L \)** iff there exist \( L \in L_c(X, Y) \) and \( V \subseteq X \) as above s.t. for every compact subset \( K \subseteq V \) the map

\[
r^f_x|_{K \times [0, 1]} : K \times [0, 1] \to Y
\]

is continuous.
3. weakly Fréchet differentiable at $x$ with derivative $Df(x) := L$ iff there exist $L \in L_c(X,Y)$ and $V \subseteq X$ as above s.t. the map

$$r_f^L : V \times [0, 1] \to Y$$

is continuous.

If one of the above holds for some $V \subseteq X$, then it also holds for all $V' \subseteq V$.

If an $L$ exists s.t. 1., 2. or 3. above holds, then it is unique and $f$ is simply called pointwise, compactly or just weakly Fréchet differentiable at $x$, respectively.

**Remark 1.3.** In the definition of weak Fréchet differentiability, asking just for continuity of $r_f^L$ along $V \times \{0\}$, this is also called Leslie differentiability, cf. [DGV15].

**Remark 1.4.** Another natural class of subsets, along with the finite, compact or arbitrary ones used in the definition above, is the class of bounded subsets. So the obvious question is why not repeat the above definition for this class of subsets? This will not be examined in this text any further because of the following reasons:

- First, it would not be used in this text anyway, outside of this section.
- Second, for locally bounded (i.e. normable) Hausdorff locally convex topological vector spaces it coincides with weak Fréchet differentiability.
- And third, for not locally bounded spaces it seems that further distinctions/assumptions need to be made for the analogous results to those in this and the subsequent sub-sections to hold, which tipped the scale in favour of excluding it.

**Remark 1.5.** Note that the continuity of $r_f^L|_{V \times \{0\}} : V \times [0, 1] \to Y$ is automatic by virtue of the continuity assumption on $f$. One could also drop this assumption and instead ask for continuity of $r_f^L$ at the points of $A \times \{0\}$, where $A \subseteq V$ is finite, compact, or all of $V$, respectively. But for the purposes of this article this just presents an unnecessary complication (the question of whether differentiability in various forms implies continuity will not be addressed here).

**Remark 1.6.** By [Osb14], Proposition 3.1, the existence of a convex balanced neighbourhood $V$ of 0 in the previous definition is not a restriction and one can furthermore w.l.o.g. assume that $V$ is open or that $V$ is closed.

**Example 1.1.** This is a standard example that can be found in many calculus textbooks.

Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto \begin{cases} \frac{x^3 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$f$ is continuous and Gâteaux differentiable at $(0, 0)$ but not weakly Fréchet and hence by Theorem 1.3 also not compactly weakly Fréchet or Fréchet differentiable.

We have $f(0, y) = 0$ and for $x \neq 0$, $|f(x, y)| = |x| \frac{|y/x^2|}{1 + |y/x^2|} \leq |x|$, so $f$ is continuous. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be the zero map. Then

$$r_f^L((x, y), t) = t \frac{x^3 y}{t^2 x^2 + y^2}$$

$r_f^L((x, 0), t) \equiv 0$ and for $y \neq 0$, $|r_f^L((x, y), t)| \leq t|x^3/y| \to 0$ as $t \to 0$, so $f$ is Gâteaux differentiable at $(0, 0)$ with derivative $Df(0, 0) = 0$. On the other hand, $r_f^L((1, 0), t) = t \frac{1}{t^2 + t^2} = \frac{1}{2}$ whereas $r_f^L((1, 0), t) \equiv 0$. So $r_f^L((0, 0))$ is not continuous and $f$ is not weakly Fréchet differentiable at $(0, 0)$. 9
Definition 1.5. Let \( X \) and \( Y \) be Hausdorff locally convex topological vector spaces, let \( U \subseteq X \) be an open subset and let \( f : U \to Y \) be a continuous map. For a subset \( A \subseteq U \), if \( f \) is (pointwise, compactly) weakly Fréchet differentiable at \( x \) for all \( x \in A \), then \( f \) is called (pointwise, compactly) weakly Fréchet differentiable on \( A \), respectively.

Let a topological space \( \tilde{U} \) together with a continuous map \( \iota : \tilde{U} \to U \) be given. If \( f \) is (pointwise, compactly) Fréchet differentiable on \( \iota(\tilde{U}) \) and if furthermore the map

1. \[ \tilde{U} \to L_c(X,Y) \]
   \[ x \mapsto Df(\iota(x)) \]

is continuous, then \( f \) is called strongly continuously (pointwise, compactly) weakly Fréchet differentiable along \( \iota \), respectively.

In case \( \iota = \text{id}_U \), \( f \) is just called strongly continuously (pointwise, compactly) weakly Fréchet differentiable.

2. \[ \tilde{U} \times X \to Y \]
   \[ (x,u) \mapsto Df(\iota(x))u \]

is continuous, then \( f \) is called weakly continuously (pointwise, compactly) weakly Fréchet differentiable along \( \iota \), respectively.

In case \( \iota = \text{id}_U \), \( f \) is just called weakly continuously (pointwise, compactly) weakly Fréchet differentiable.

3. If

   \[ \tilde{U} \to L_c(X,Y) \]
   \[ x \mapsto Df(\iota(x)) \]

is locally bounded, then \( f \) is said to have locally bounded derivative along \( \iota \).

In case \( \iota = \text{id}_U \), \( f \) is just said to have locally bounded derivative.

Theorem 1.1 (Chain rule). Let \( X, Y \) and \( Z \) be Hausdorff locally convex topological vector spaces and let \( U \subseteq X \) and \( V \subseteq Y \) be open subsets. Let furthermore \( f : U \to Y \) and \( g : V \to Z \) be continuous functions with \( f(U) \subseteq V \). Given \( x \in U \) and a topological space \( \tilde{U} \) together with a continuous function \( \iota : \tilde{U} \to U \), the following hold:

1. If \( f \) is (compactly) weakly Fréchet differentiable at \( x \) and \( g \) is (compactly) weakly Fréchet differentiable at \( f(x) \), then the composition \( g \circ f : U \to Z \) is (compactly) weakly Fréchet differentiable at \( x \) with

   \[ D(g \circ f)(x) = Dg(f(x)) \circ Df(x). \]

2. If \( f \) is weakly continuously (compactly) weakly Fréchet differentiable along \( \iota \) and \( g \) is weakly continuously (compactly) weakly Fréchet differentiable along \( f \circ \iota \), then \( g \circ f \) is weakly continuously (compactly) weakly Fréchet differentiable along \( \iota \).

   If \( Y \) is locally bounded, then the same holds for “strongly continuously” in place of “weakly continuously”.

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Proof. 1. Note that for $t > 0$, $r^f_x(u,t)$ (and analogously for $r^g$ and $r^{g,f}$) is uniquely determined by the equation

$$f(x + tu) = f(x) + t(Df(x)u + r^f_x(u,t)).$$

One hence calculates for $t > 0$.

$$g \circ f(x + tu) = g(f(x) + t(Df(x)u + r^f_x(u,t))) = g(f(x)) + t[Dg(f(x)) (Df(x)u + r^f_x(u,t)) + r^g_{f(x)}(Df(x)u + r^f_x(u,t), t)] = g(f(x)) + t[Dg(f(x)) \circ Df(x)u + Dg(f(x))r^f_x(u,t) + r^g_{f(x)}(Df(x)u + r^f_x(u,t), t)].$$

It follows that

$$r^{g,f}_x(u,t) = Dg(f(x))r^f_x(u,t) + r^g_{f(x)}(Df(x)u + r^f_x(u,t), t),$$

whenever both sides of the equation are defined.

1. Let $f$ and $g$ be weakly Fréchet differentiable at $x$ and $f(x)$, respectively. For the right hand side in the above formula to be well defined, if $r^g_{f(x)}$ is defined on $V' \times [0,1]$ and $r^f_x$ is defined on $V \times [0,1]$, one has to find a convex balanced neighbourhood $V'' \subseteq V \subseteq X$ of 0 s. t. $Df(x)u + r^f_x(u,t) \in V'$ for all $(u,t) \in V'' \times [0,1]$. $r^{g,f}_x$ is then defined on $V'' \times [0,1]$ and satisfies the above formula.

Now the function

$$\phi : V \times [0,1] \to Y$$

$$(u,t) \mapsto Df(x)u + r^f_x(u,t)$$

is continuous and satisfies $\{0\} \times [0,1] \subseteq \phi^{-1}(0) \subseteq \phi^{-1}(V')$. By Lemma 1.1 below there hence exists a neighbourhood $V'' \subseteq V \subseteq X$ of 0 s. t. $V'' \times [0,1] \subseteq \phi^{-1}(V')$ and by [Osb14], Proposition 3.1, one can assume that $V''$ is convex and balanced.

If $f$ and $g$ are weakly Fréchet differentiable at $x$ and $f(x)$, respectively, then the right hand side of the above formula for $r^{g,f}_x$ extends continuously by 0 to $t = 0$. So $r^{g,f}_x$ is continuous as a composition of continuous functions.

2. Let $f$ and $g$ be compactly weakly Fréchet differentiable at $x$ and $f(x)$, respectively, and let $V$ and $V'$ be convex balanced neighbourhoods of 0 in $X$ and $Y$, respectively, as in Definition 1.4 After possibly making $V$ smaller (and again using [Osb14], Proposition 3.1), one can also assume that $f(x + V) \subseteq f(x) + V'$ and $Df(x)(V) \subseteq V'$, using continuity of $f$ and $Df(x)$, respectively. Let $K \subseteq V$ be compact. The goal is to show that $r^{g,f}_x|_{K \times [0,1]}$ is continuous. Since $r^{g,f}_x|_{K \times [0,1]}$ is already continuous by definition, it suffices to show that $r^{g,f}_x|_{K \times [0,0,\delta]}$ is continuous for an arbitrarily small $\delta = \delta(K) > 0$. By assumption, the map

$$\phi : K \times [0,1] \to Y$$

$$(u,t) \mapsto Df(x)u + r^f_x(u,t)$$

is continuous and satisfies $\phi(K \times \{0\}) = Df(x)(K) \subseteq Df(x)(V) \subseteq V'$, hence $K \times \{0\} \subseteq \phi^{-1}(V')$. By Lemma 1.1 there exists $\delta = \delta(K) > 0$
Lemma 1.1. Let \( K \times [0, 2\delta] \subseteq \phi^{-1}(V') \). This implies that in the above formula for \( r_x^g \) the right hand side is well defined for \((u, t) \in K \times [0, 2\delta] \). The map 

\[
K \times [0, \delta] \to Y
\]

\[(u, t) \mapsto Df(x)u + r_x^f(u, t)\]

is continuous because \( Df(x) \) is and because \( r_x^f : K \times [0, 1] \to Y \) is by assumption. It hence has compact image and one can conclude as before that \( r_x^f|_{K \times [0, \delta]} \) is continuous as a composition of continuous functions.

2. For weak continuity, the map 

\[
\tilde{U} \times X \to Y
\]

\[(x, u) \mapsto (g(f(u)))u\]

is given by the composition

\[(x, u) \mapsto (f(u)), Df(u))u \mapsto Dg(f(u))Df(u)u,

hence continuous.

If \( Y \) is locally bounded, then the composition \( L_c(X, Y) \times L_c(Y, Z) \to L_c(X, Z) \) is continuous, which shows the last statement.

\[ \square \]

Lemma 1.1. Let \( A, B \) and \( K \) be topological spaces with \( K \) compact.

1. Let \( \phi : A \times K \to B \) be a continuous function, let \( a_0 \in A \) and let \( V \subseteq B \) be an open subset with \( \{a_0\} \times K \subseteq \phi^{-1}(V) \). Then there exists a neighbourhood \( U \subseteq A \) of \( a_0 \) s.t. \( U \times K \subseteq \phi^{-1}(V) \).

2. If \( \phi : A \times K \to \mathbb{R} \) is a continuous function, then so is the function 

\[
\max_K \phi : A \to \mathbb{R}
\]

\[ a \mapsto \max\{\phi(a, k) \mid k \in K\}. \]

Proof. 1. Let \( \phi, a_0 \) and \( V \) be as in the statement of the lemma. Because \( \phi \) is continuous, for every \( k \in K \) there exist neighbourhoods \( U_k \subseteq A \) of \( a_0 \) and \( W_k \subseteq K \) of \( k \) with \( U_k \times W_k \subseteq \phi^{-1}(V) \). Because \( K \) is compact, one can choose finitely many \( k_1, \ldots, k_d \in K \) s.t. \( W_{k_1}, \ldots, W_{k_d} \) cover \( K \). Let \( U := \bigcap_{j=1}^d U_{k_j} \).

Then \( U \times K \subseteq U \times \bigcup_{j=1}^d W_{k_j} \subseteq \bigcup_{j=1}^d U_{k_j} \times W_{k_j} \subseteq \phi^{-1}(V) \).

2. Let \( a_0 \in A \), \( b_0 := \max_K \phi(a_0) \) and let \( \varepsilon > 0 \). By definition \( \{a_0\} \times K \subseteq \phi^{-1}((-\infty, b_0 + \varepsilon)) \) and hence by \([4] \) there exists a neighbourhood \( U' \subseteq A \) of \( a_0 \) s.t. \( U' \times K \subseteq \phi^{-1}((-\infty, b_0 + \varepsilon)) \). Hence \( \max_K \phi(a) < b_0 + \varepsilon \) for all \( a \in U' \).

Now pick \( k_0 \in K \) s.t. \( b_0 = \phi(a_0, k_0) \). By continuity of \( \phi \) there exists a neighbourhood \( U'' \subseteq A \) of \( a_0 \) s.t. \( |\phi(a, k_0) - \phi(a_0, k_0)| < \varepsilon \) for all \( a \in U'' \). Hence \( \max_K \phi(a) \geq \phi(a, k_0) > \phi(a_0, k_0) - \varepsilon = b_0 - \varepsilon \) for all \( a \in U'' \). \( U := U' \cap U'' \) then satisfies \( \max_K \phi(a) - \max_K \phi(a_0) < \varepsilon \) for all \( a \in U \).

\[ \square \]
1.1.3 Fréchet differentiability

**Definition 1.6.** Let $X$ and $Y$ be Hausdorff locally convex topological vector spaces, let $U \subseteq X$ be an open subset and let $f : U \to Y$ be a continuous map. Given a point $x \in U$, a continuous linear map $L \in Lc(X,Y)$ and a convex balanced neighbourhood $V \subseteq X$ of 0 s.t. $x + V \subseteq U$, define a function

$$r^f_x : V \times [0,1] \to Y$$

$$(u,t) \mapsto \begin{cases} 
\frac{1}{t} (f(x + tu) - f(x)) - Lu & t > 0 \\
0 & t = 0 
\end{cases}$$

$f$ is called

1. **Gâteaux, or pointwise Fréchet, differentiable at** $x$ with derivative $Df(x) := L$ **iff** there exist $L \in Lc(X,Y)$ and $V \subseteq X$ as above s.t. the map

$$[0,1] \to C_{bo}(V,Y)$$

$$t \mapsto r^f_x(\cdot,t)$$

is continuous.

2. **compactly Fréchet differentiable at** $x$ with derivative $Df(x) := L$ **iff** there exist $L \in Lc(X,Y)$ and $V \subseteq X$ as above s.t. the map

$$[0,1] \to C_{co}(V,Y)$$

$$t \mapsto r^f_x(\cdot,t)$$

is continuous.

3. If $f$ has locally bounded image, then $f$ is called **Fréchet differentiable at** $x$ with derivative $Df(x) := L$ **iff** there exist $L \in Lc(X,Y)$ and $V \subseteq X$ as above s.t. the map

$$[0,1] \to C_{bo}(V,Y)$$

$$t \mapsto r^f_x(\cdot,t)$$

is well defined and continuous.

If one of the above holds for some $V \subseteq X$, then it also holds for all $V' \subseteq V$.

If an $L$ exists s.t. 1.2 or 3. above holds, then it is unique and $f$ is simply called pointwise, compactly or just Fréchet differentiable at $x$, respectively. The notions of weakly and strongly continuously (pointwise, compactly) Fréchet differentiable, as well as of locally bounded derivative, are defined in complete analogy to Definition 1.5.

**Remark 1.7.** Note that, similar to Remark 1.5, one could also ask for $t \mapsto r^f_x(\cdot,t)$ to be continuous just at $t = 0$, instead of continuity for all $t \in [0,1]$. Indeed, at least for Fréchet differentiability, this is equivalent only under the assumption that $f$ is locally uniformly continuous (cf. also Remark 1.8). But for the purposes of this article, the distinction is immaterial since Fréchet differentiability won’t be used anyway.

At first glance, the notions of weak Fréchet differentiability and Fréchet differentiability might seem quite different, but the following proposition shows that in suitable reformulations these notions are quite similar.
Proposition 1.2. In the notation of Definition 1.6,

1. The following are equivalent:
   (a) \( f \) is pointwise Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. for all finite subsets \( C \subseteq V \) the map
       \[ r^f_x|_{C \times [0,1]} : C \times [0,1] \to Y \]
       is uniformly continuous.

2. The following are equivalent:
   (a) \( f \) is compactly Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. for all compact subsets \( K \subseteq V \) the map
       \[ r^f_x|_{K \times [0,1]} : K \times [0,1] \to Y \]
       is uniformly continuous.

3. If \( f \) has locally bounded image and is locally uniformly continuous, then the following are equivalent:
   (a) \( f \) is Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. the map
       \[ r^f_x : V \times [0,1] \to Y \]
       is uniformly continuous.

Proof. The proofs in all three cases are actually almost identical, so will be treated all at once:
Let \( V \subseteq X \) be a convex balanced neighbourhood of 0 either as in the definition of Fréchet differentiability or as in the statement of the proposition and let \( A \subseteq V \) be a (finite, compact) subset. Furthermore, assume that \( f|_A \) is uniformly continuous and has bounded image, either by choosing \( V \) appropriately (possible since it was assumed that \( f \) has locally bounded image and is locally uniformly continuous) or by using Proposition 1.1 in case \( A \) is (finite or) compact.

Claim. \( r^f_x|_{A \times [\delta_0,1]} : A \times [\delta_0,1] \to Y \) is uniformly continuous for every \( \delta_0 \in (0,1] \).

Proof. One computes for \( (u,t),(u',t') \in (0,1] \times V \) that
\[
    r^f_x(u,t) - r^f_x(u',t') = \frac{1}{t}(f(x + tu) - f(x)) - Lu - \frac{1}{t'}(f(x + t'u') - f(x)) + Lu' = \frac{t - t'}{tt'} f(x) + L(u' - u) + \frac{1}{tt'}(t'f(x + tu) - tf(x + t'u')) = \frac{t - t'}{tt'} f(x) + L(u' - u) +
\]

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so

\[
\begin{align*}
\frac{1}{t} (t'f(x+tu) - t'f(x+t'u')) + \\
\frac{1}{t} (f(x+tu) - f(x)) + L(u' - u) + \\
\frac{1}{t} (f(x+tu) - f(x+t'u')).
\end{align*}
\]

Given any convex balanced neighbourhood \(W \subseteq Y\) of 0 one can find convex balanced neighbourhoods \(W_1, W_2, W_3 \subseteq Y\) of 0 s.t. \(x_1 + x_2 + x_3 \in W\) for all \((x_1, x_2, x_3) \in W_1 \times W_2 \times W_3\) (just by continuity of addition). Now if \(t, t' \in [\delta_0, 1]\), then \(\frac{1}{t'}\) and \(\frac{1}{t}\) are bounded below away from 0. Also, \(\{f(x+t'u') - f(x) \mid (u', t') \in A \times [0, 1]\} \subseteq Y\) is a bounded subset by the assumption that \(f|_A\) has bounded image. Consequently, \(M_1 := \{\frac{1}{t'} (f(x+t'u') - f(x)) \mid t, t' \in [\delta_0, 1], u' \in A\} \subseteq Y\) is a bounded subset and there exists \(\delta_1 \) s.t. \((t-t')M_1 \subseteq W_1\) for all \(t, t' \in [0, 1]\) with \(|t - t'| < \delta_1\).

\(L\) is a continuous and hence uniformly continuous map, so there exists \((0, 1] \to \mathbb{C}^{\partial_0}(V, Y) (\mathbb{C}^{\partial_0}(V, Y), \mathbb{C}^{\partial_0}(V, Y)), t \mapsto r^t(\cdot, t), \) is continuous.

\[\text{Claim.} \quad (0, 1] \to \mathbb{C}^{\partial_0}(V, Y) (\mathbb{C}_{\partial_0}(V, Y), \mathbb{C}_{\partial_0}(V, Y)), t \mapsto r^t(\cdot, t), \]

It remains to show that the first claim above also holds for \(\delta_0 = 0\). So let again \(W \subseteq Y\) be a convex balanced neighbourhood of 0 and choose a convex balanced neighbourhood \(W' \subseteq Y\) of 0 s.t. \(W' + W' \subseteq W\). By the assumption of (pointwise, compact) Fréchet differentiability there exists \(\delta' \in (0, 1] \) s.t. \(r^t(\cdot, t) \in N(A, W')\) for all \(t \in [0, \delta']\) or in other words that \(r^t(\cdot, t) \subseteq W'\) for all \((u, t) \in A \times [0, \delta']\). Observe that for all \(t, t' \in [0, 1]\) with \(|t - t'| < \delta'/2\) either \(t, t' \in [0, \delta']\) and/or \(t', t' \in [\delta'/2, 1]\). Also, by the claim, \(r^t|_{A \times [\delta'/2, 1]} : A \times [\delta'/2, 1] \to Y\) is uniformly continuous, so there exists \(\delta'' > 0\) and a convex balanced neighbourhood \(V \subseteq X\) of 0 s.t. \(r^t(u, t) - r^t(u', t') \in W\) for all \((u, t), (u', t') \in V \times (\delta''/2, 1] \) with \(|u - u'|, t - t'| \in V \times (\delta''/2, 1]\). Then \(\delta := \min\{\delta'/2, \delta''\}\). It follows that if \((u, t), (u', t') \in A \times [0, \delta']\), then \(r^t(u, t) - r^t(u', t') \in W' + W' \subseteq W\). And if \((u, t), (u', t') \in A \times [\delta'/2, 1]\) with \(|u - u'|, t - t'| \in V \times (\delta'', \delta)\), then \(r^t(u, t) - r^t(u', t') \in W\). This shows that \(r^t|_{A \times [0, 1]}\) is uniformly continuous.
“Uniform continuity ⇒ Fréchet differentiability”:
This part of the proof is identical to the proof of the second claim, where now one can use that \( t \mapsto r_x^f(\cdot, t) \) is uniformly continuous for all \( t \in [0, 1] \) (and not just for \( t \in [\delta_0, 1] \), for some \( \delta_0 \in (0, 1] \)).

**Remark 1.8.** The condition that \( f \) be locally uniformly continuous in Proposition 1.2 might seem somewhat strong and arbitrary, considering that the definition of Fréchet differentiability did not assume it. But note (e.g. by taking \( u = u' \) in Equation (2)) that the proof of the first and hence also of the second claim in the proof of Proposition 1.2 does make use of this assumption. So for maps that are not locally uniformly continuous, continuity of \([0, 1] \to C_{\text{bio}}(V,Y), t \mapsto r_x^f(\cdot, t)\), for all \( t \in [0,1] \) is a priori stronger than just continuity at \( t = 0 \).

**Theorem 1.2** (Chain rule). Let \( X, Y \) and \( Z \) be Hausdorff locally convex topological vector spaces and let \( U \subseteq X \) and \( V \subseteq Y \) be open subsets. Let furthermore \( f : U \to Y \) and \( g : V \to Z \) be continuous functions with \( f(U) \subseteq V \). In the following, where Fréchet differentiability is concerned, assume furthermore that \( f \) has locally bounded image and is locally uniformly continuous.
Given \( x \in U \) and a topological space \( \tilde{U} \) together with a continuous function \( \iota : \tilde{U} \to U \), the following hold:

1. If \( f \) is (compactly) Fréchet differentiable at \( x \) and \( g \) is (compactly) Fréchet differentiable at \( f(x) \), then the composition \( g \circ f : U \to Z \) is (compactly) Fréchet differentiable at \( x \) with
   \[
   D(g \circ f)(x) = Dg(f(x)) \circ Df(x).
   \]
2. If \( f \) is weakly continuously (compactly) Fréchet differentiable along \( \iota \) and \( g \) is weakly continuously (compactly) Fréchet differentiable along \( f \circ \iota \), then \( g \circ f \) is weakly continuously (compactly) Fréchet differentiable along \( \iota \).
   If \( Y \) is locally bounded, then the same holds for “strongly continuously” in place of “weakly continuously”.

**Proof.** Using Proposition 1.2, the proof is identical to the proof of Theorem 1.1 just replacing the word “continuous” by the words “uniformly continuous”.

\[ \square \]
1.2 Interrelations

**Proposition 1.3.** In the notation of Definitions 1.4 and 1.6, assume that the topology on \( Y \) is given by a chosen translation invariant metric \( d \).

1. The following are equivalent:
   (a) \( f \) is pointwise Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. for all \( u \in V \)
       \[
       \lim_{t \to 0} d \left( 0, r_x^f(u,t) \right) = 0.
       \]
   (c) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t.
       \[
       \forall u_0 \in V, \varepsilon > 0 \ \exists \delta = \delta(u_0, \varepsilon) > 0 : d \left( 0, r_x^f(u_0,t) \right) < \varepsilon \ \forall t < \delta.
       \]

2. The following are equivalent:
   (a) \( f \) is compactly Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t. for all compact subsets \( K \subseteq V \)
       \[
       \lim_{t \to 0} \sup \left\{ d \left( 0, r_x^f(u,t) \right) \mid u \in K \right\} = 0.
       \]
   (c) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t.
       \[
       \forall K \subseteq V \text{ compact}, \varepsilon > 0 \ \exists \delta = \delta(K, \varepsilon) > 0 : d \left( 0, r_x^f(u,t) \right) < \varepsilon \ \forall u \in K, t < \delta.
       \]

3. The following are equivalent:
   (a) \( f \) is weakly Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t.
       \[
       \forall u_0 \in V, \varepsilon > 0 \ \exists \delta = \delta(u_0, \varepsilon) > 0, V' \subseteq V \text{ an open neighbourhood of } u_0 : d \left( 0, r_x^f(u,t) \right) < \varepsilon \ \forall u \in V', t < \delta.
       \]

4. Assume that \( f \) has locally bounded image and is locally uniformly continuous. Then the following are equivalent:
   (a) \( f \) is Fréchet differentiable at \( x \).
   (b) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t.
       \[
       \lim_{t \to 0} \sup \left\{ d \left( 0, r_x^f(u,t) \right) \mid u \in V \right\} = 0.
       \]
   (c) There exists a convex balanced neighbourhood \( V \subseteq X \) of 0 s.t.
       \[
       \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 : d \left( 0, r_x^f(u,t) \right) < \varepsilon \ \forall u \in V, t < \delta.
       \]

**Proof.** All of the statements in the proposition are obtained by simply writing out the definitions in terms of the metric on \( Y \). \( \square \)

**Remark 1.9.** For the following, remember that an embedding of topological vector spaces is an injective continuous linear map. It does not need to be an embedding of topological spaces or have closed image.

For the definition of a compact linear operator, see Definition 2.12.
Proposition 1.4. Let $X$ and $Y$ be Hausdorff locally convex topological vector spaces, let $A \subseteq U \subseteq X$ be subsets with $U$ open and let $f : U \to Y$ be a continuous map. Let furthermore $X_1$ be another Hausdorff locally convex topological vector space that comes with a compact embedding of topological vector spaces $\kappa : X_1 \hookrightarrow X$ and define $U_1 := \kappa^{-1}(U)$, $A_1 := \kappa^{-1}(A)$.

If $f$ is compactly Fréchet differentiable on $A$, then $f_1 := f \circ \kappa|_{U_1} : U_1 \to Y$ is Fréchet differentiable on $A_1$ with

$$Df_1(x) = Df(\kappa(x)) \circ \kappa \quad \forall x \in A_1.$$ 

Proof. For simplicity I will drop $\kappa$ from the notation and consider $A_1, U_1, X_1$ as subsets of $X$, although not with the induced topologies. For $x \in U_1$, by compactness of $\kappa$ there exists a neighbourhood $U'_1 \subseteq U_1$ of $x$ s.t. $K := \overline{U'_1} \subseteq U \subseteq X$ is compact. By Proposition 1.1, $f|_K$ is uniformly continuous and it has compact, hence bounded, image. Since $\kappa$, as a continuous linear map, is uniformly continuous, this shows that $f_1 = f \circ \kappa|_{U_1}$ is locally uniformly continuous and has locally bounded image. It is clear from the definition that for all $x \in A_1$ and some convex balanced neighbourhood $V_1 \subseteq X_1$ of 0

$$r^f_2 = r^f_2|_{V_1 \times [0,1]}.$$ 

Denote by $K$ the closure of $V_1$ in $X$, which can be assumed to be compact by compactness of $\kappa$, and $K \subseteq V$ for $V_1$ small enough. By assumption, the function $r^f_2|_{K \times [0,1]} : K \times [0,1] \to Y$ is continuous and hence by Proposition 1.1 it is uniformly continuous. It follows that $r^f_2$ is uniformly continuous and so by Proposition 1.2, $f_1$ is Fréchet differentiable.

Theorem 1.3. Let $X$ and $Y$ be locally convex topological vector spaces, let $U \subseteq X$ be an open subset, $x \in U$, and let $f : U \to Y$ be a continuous map.

1. $f$ weakly Fréchet differentiable at $x$ $\Rightarrow$ $f$ compactly weakly Fréchet differentiable at $x$ $\Rightarrow$ $f$ pointwise weakly Fréchet differentiable at $x$.

2. If the topology on $X$ is compactly generated, in particular if $X$ is first countable (equivalently, if $X$ is metrisable), then

$$f \text{ compactly weakly Fréchet differentiable at } x \iff f \text{ weakly Fréchet differentiable at } x.$$ 

3. Each of the other implications “$f$ pointwise weakly Fréchet differentiable at $x$ $\Rightarrow$ $f$ (compactly) weakly Fréchet differentiable at $x$” in general is false.

4. $f$ pointwise weakly Fréchet differentiable at $x$ $\iff$ $f$ pointwise Fréchet differentiable at $x$.

5. $f$ compactly weakly Fréchet differentiable at $x$ $\iff$ $f$ compactly Fréchet differentiable at $x$.

6. Assume that $f$ has locally bounded image. Then $f$ Fréchet differentiable at $x$ $\Rightarrow$ $f$ weakly Fréchet differentiable at $x$, but the converse implication in general is false.
7. If $X$ is finite dimensional, then $f$ Fréchet differentiable at $x$ $\iff$ $f$ weakly Fréchet differentiable at $x$.

**Proof.** 1. This is completely obvious.

2. $X$ metrisable and $X$ first countable are equivalent by [Osh14], Theorem 3.35, and first countable spaces are compactly generated, cf. [Mun75], Lemma 46.3. By [Mun75], Lemma 46.4, it remains to show that if, in the notation of Definition 1.4, $\tilde{K} \subseteq V \times [0,1]$ is compact, then $r^f_{c} : \tilde{K} \rightarrow Y$ is continuous. Denoting $K := \text{pr}_1(\tilde{K}) \subseteq V$, which is compact, by definition, $r^f_{c} |_{K \times [0,1]} : K \times [0,1] \rightarrow Y$ is continuous. Since $\tilde{K} \subseteq K \times [0,1]$, the claim follows.

3. This was already shown by way of Example 1.1.

4. This is immediate from Proposition 1.2 and Proposition 1.1.

5. Ditto.

6. Let $V \subseteq X$ be as in the definition of $f$ Fréchet differentiable. Then continuity of $r^f_{c} : V \times [0,1] \rightarrow Y$ at points $(u,t) \in V \times (0,1]$ is automatic by virtue of its definition (and because $f$ is assumed to be continuous). And to show continuity at $(u,0)$ for some $u \in V$, let $W \subseteq Y$ be a convex balanced neighbourhood of 0. By definition of the topology on $\ell^{bi}(V,Y)$, there exists $\delta > 0$ s.t. $r^f_{c}((-1,0)) = r^f_{c}((0,0)) = \mathcal{N}(V,W)$ for all $t' \in [0,1]$ with $|t'-0| = \delta$. But this implies in particular that for all $(u',t') \in V \times [0,1]$ with $(u',t') - (u,0) \in X \times [0,\delta]$, $r^f_{c}(u',t') - r^f_{c}(u,0) = r^f_{c}(u',t')\in W$. So $r^f_{c}$ is continuous at $(u,t)$.

That the converse implication in general is false will be shown by virtue of an example in Section 1.3.

7. Apply Proposition 1.4 with $X_1 := X$, $\kappa := \text{id}_X$, which is compact precisely in finite dimensions to show “compactly weakly Fréchet” $\Rightarrow$ “Fréchet”.

**Proposition 1.5.** Let $X$, $Y_0$, $Y_1$ and $Z$ be Hausdorff locally convex topological vector spaces, let $\kappa : Y_1 \rightarrow Y_0$ be a compact embedding of topological vector spaces, let $U \subseteq X$ be an open subset and let $\Phi : U \times Y_0 \rightarrow Z$ be a continuous map that is linear in the second factor. I.e. for all $(x,v) \in U \times Y_0$, $\Phi(x,v) = \Phi_{x}(v)$, where $\Phi_{x} \in L_c(Y_0,Z)$.

Then the map

$$U \rightarrow L_c(Y_1,Z)$$

$$x \mapsto \Phi_{x} \circ \kappa$$

is continuous.

**Proof.** Let $A$ be a bounded subset of $Y_1$ and let $V$ be a convex balanced neighbourhood of 0 in $Z$. Also let $N(A,V) = \{L \in L_c(Y_1,Z) \mid L(A) \subseteq V \} \subseteq L_c(Y_1,Z)$ be the corresponding neighbourhood of 0 in $L_c(Y_1,Z)$. It has to be shown that there exists a neighbourhood $W \subseteq U$ of $x_0$ s.t. $\Phi_{x} \circ \kappa - \Phi_{x_0} \circ \kappa \in N(A,V)$, i.e. $(\Phi_{x} - \Phi_{x_0})(\kappa(A)) \subseteq V$, for all $x \in W$. 

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Let $K$ be the closure of $\kappa(A)$ in $Y_0$, which by assumption is compact and let $x_0 \in U$. By assumption, the map $\Phi : U \times Y_0 \to Z$ is continuous and hence so is the map

$$\phi : U \times K \to Z$$

$$(x, v) \mapsto \Phi(x)(v) - \Phi(x_0)(v).$$

It then obviously suffices to find a neighbourhood $W \subseteq U$ of $x_0$ s. t. $W \times K \subseteq \phi^{-1}(V)$. But $\phi(x_0) = \{0\} \subseteq V$, so by Lemma 1.1 the existence of such a neighbourhood $W \subseteq U$ of $x_0$ follows.

**Proposition 1.6.** Let $X$ and $Y$ be Hausdorff locally convex topological vector spaces.

Let furthermore $X_1$ be another Hausdorff locally convex topological vector space that comes with a compact embedding of topological vector spaces $\kappa : X_1 \hookrightarrow X$ and define $U_1 := \kappa^{-1}(U)$.

If $\tilde{U}$ is a topological space together with a continuous map $\iota_1 : \tilde{U} \to U_1$ s. t. $f$ is weakly continuously compactly weakly Fréchet differentiable along $\iota := \kappa \circ \iota_1$, then $f_1 \circ \kappa | U_1 : U_1 \to Y$ is strongly continuously Fréchet differentiable along $\iota_1$.

**Proof.** For simplicity I will drop $\kappa$ from the notation and consider $U_1, X_1$ as subsets of $X$, although not with the induced topologies.

By Proposition 1.4, $f_1$ is Fréchet differentiable on $\iota_1(\tilde{U})$ and $Df_1(x) = Df(\kappa(x)) \circ \kappa$.

By assumption, the map

$$\phi : \tilde{U} \times X \to Y$$

$$(x, u) \mapsto Df(\iota(x))u$$

is continuous. Proposition 1.5 then implies that the map

$$\tilde{U} \to L_c(X_1, Y)$$

$$x \mapsto Df(\iota(x)) \circ \kappa = Df_1(\iota_1(x))$$

is continuous, i. e. $f_1$ is strongly continuously Fréchet differentiable along $\iota_1$. 

\[\square\]
1.3 Main example: The reparametrisation action

Let \((\Sigma, g)\) be a closed n-dimensional Riemannian manifold and let \(\pi : F \to \Sigma\), \(\pi_1 : F_1 \to \Sigma\) and \(\pi_2 : F_2 \to \Sigma\) be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections. Let furthermore \(B \subseteq \mathbb{R}^r\) (for some \(r \in \mathbb{N}_0\)) be an open subset and let

\[
\phi : B \times \Sigma \to B \times \Sigma
\]

\[(b, z) \mapsto (b, \phi_b(z))\]

and

\[
\Phi : B \times F_1 \to B \times F_2
\]

\[(b, \xi) \mapsto (b, \Phi_b(\xi))\]

be smooth families of maps, where \(\phi\) is a family of diffeomorphisms and \(\Phi\) covers \(\phi^{-1}\). I.e. \(\phi\) and \(\Phi\) are smooth, \(\phi_b \in \text{Diff}(\Sigma)\) and \(\Phi_b \in \mathcal{C}^\infty(F_1, F_2)\) for all \(b \in B\), and

\[
\begin{array}{ccc}
B \times F_2 & \xrightarrow{\Phi} & B \times F_1 \\
\downarrow \text{id}_B \times \pi & \circ & \downarrow \text{id}_B \times \pi \\
B \times \Sigma & \xrightarrow{\phi} & B \times \Sigma
\end{array}
\]

commutes. Furthermore, assume that \(\Phi\) is linear in the fibres of \(F_1\) and \(F_2\), i.e. for every \(b \in B\), \(\Phi_b : F_1 \to F_2\) defines a vector bundle morphism covering \(\phi_b^{-1} : \Sigma \to \Sigma\).

For \(k \in \mathbb{N}_0\), let \(\Gamma^k(F)\) be the space of \(k\)-times continuously differentiable sections of \(F\) equipped with the \(\mathcal{C}^k\)-topology. It can be defined for example via the (complete) norm

\[
\| \cdot \|_k : \Gamma^k(F) \to [0, \infty)
\]

\[u \mapsto \sum_{j=0}^k \sup\{ |\nabla^j u(z)| \mid z \in \Sigma\} \]

Also, denote by \(\Gamma(F)\) the (Fréchet) space of smooth sections equipped with the inverse limit of the \(\mathcal{C}^k\)-topologies, for \(k \in \mathbb{N}_0\).

Finally, for \(k \in \mathbb{N}_0\) and \(1 < p < \infty\), let \(W^{k,p}(F)\) be the Sobolev space of sections of \(F\) of class \((k,p)\). For concreteness' sake, let it be defined as the completion of \(\Gamma(E)\) w.r.t. the norm

\[
\| \cdot \|_{k,p} : \Gamma^k(F) \to [0, \infty)
\]

\[u \mapsto \sum_{j=0}^k \left( \int_\Sigma |\nabla^j u(z)|^p \, \text{dvol}_g \right)^{1/p} \]

Also assume that \(kp > \dim \Sigma\), so that the Sobolev embedding and multiplication theorems hold, in particular one can identify \(W^{k,p}(F)\) with a subset of \(\Gamma^0(F)\) and consider the elements of \(W^{k,p}(F)\) as continuous sections of \(F\).

Define

\[
\Gamma_F := B \times \Gamma(F), \quad \Gamma^k_B := B \times \Gamma^k(F) \quad \text{and} \quad W^{k,p}_B(F) := B \times W^{k,p}(F)
\]

and set for \(b \in B\)

\[
\Gamma_b := \{ b \} \times \Gamma(F), \quad \Gamma^k_b := \{ b \} \times \Gamma^k(F) \quad \text{and} \quad W^{k,p}_b(F) := \{ b \} \times W^{k,p}(F).
\]
All of these are equipped with the product topologies and one has vector bundles (via projection onto the first factor)

\[ \rho : \Gamma_B(F) \to B, \quad \rho^k : \Gamma_B^{k}(F) \to B \quad \text{and} \quad \rho^{k,p} : W_B^{k,p}(F) \to B \]

Note that \( \Gamma_B(F) \), \( \Gamma_B^{k}(F) \) and \( W_B^{k,p}(F) \) are open subsets of the Hausdorff locally convex topological vector spaces \( \Gamma_{R^c}(F) \), \( \Gamma_{R^c}^{k}(F) \) and \( W_{R^c}^{k,p}(F) \), respectively.

There then are vector bundle morphisms

\[ \Psi : \Gamma_B(F_1) \to \Gamma_B(F_2), \quad \Psi^k : \Gamma_B^{k}(F_1) \to \Gamma_B^{k}(F_2) \quad \text{and} \quad \Psi^{k,p} : W_B^{k,p}(F_1) \to W_B^{k,p}(F_2), \]

defined in all three cases by

\[ (b, u) \mapsto (b, \Phi_k^* u), \]

where

\[ \Phi_k^* u := \Phi_b \circ u \circ \phi_b. \]

**Proposition 1.7.** In the above notation,

1. \( \Psi^k : \Gamma_B^{k}(F_1) \to \Gamma_B^{k}(F_2) \) is continuous and has locally bounded image, but in general is not locally uniformly continuous.

2. \( \Psi^{k,p} : W_B^{k,p}(F_1) \to W_B^{k,p}(F_2) \) is continuous and has locally bounded image, but in general is not locally uniformly continuous.

3. \( \Psi : \Gamma_B(F_1) \to \Gamma_B(F_2) \) is continuous and locally bounded, but in general has neither locally bounded image nor is it locally uniformly continuous.

**Proof.**

**Claim.** There exist continuous functions \( C^k, C^{k,p} : B \to (0, \infty) \) s.t.

\[ \| \Phi_k^* u \|_k \leq C^k(b) \| u \|_k \quad \forall u \in \Gamma^k(F_1) \]

and

\[ \| \Phi_k^* u \|_{k,p} \leq C^{k,p}(b) \| u \|_{k,p} \quad \forall u \in W^{k,p}(F_1). \]

If for every \( b \in B \), \( \Phi_b \) is a diffeomorphism (i.e. a vector bundle isomorphism covering \( \phi_b^{-1} \)), then there exist continuous functions \( c^k, c^{k,p}, C^k, C^{k,p} : B \to (0, \infty) \) s.t.

\[ c^k(b) \| u \|_k \leq \| \Phi_k^* u \|_k \leq C^k(b) \| u \|_k \quad \forall u \in \Gamma^k(F_1) \]

and

\[ c^{k,p}(b) \| u \|_{k,p} \leq \| \Phi_k^* u \|_{k,p} \leq C^{k,p}(b) \| u \|_{k,p} \quad \forall u \in W^{k,p}(F_1). \]

**Proof.** For a fixed \( b \in B \) and \( u \in \Gamma^k(F_1) \), a simple calculation gives \( \| \Phi_k^* u \|_k \leq C^k(b) \| u \|_k \), where \( C^k(b) \) is a universal expression in the terms \( \max \{ |\nabla^j \Phi_b(z)| : z \in \Sigma \} \), for \( j = 0, \ldots, k \). By Lemma 1.1, this depends continuously on \( b \in B \).

For the other inequality, apply the same reasoning to \( \Phi_b^{-1} \).

The case \( b \in B \) and \( u \in W^{k,p}(F_1) \) is identical. \[\square\]
1. The claim that $\Psi^k$ is in general not locally uniformly continuous will be shown by way of Example [12] below. If $K \subseteq B$ is a compact subset, let $C^k(K) := \max\{C^k(b) \mid b \in K\} \in (0, \infty)$. By the above claim, for any $\rho > 0$,

$$
\Psi^k(K \times B^{t_k}(F_1)(0, \rho)) \subseteq K \times B^{t_k(F_2)}(0, C^k(K) \rho),
$$

which is bounded. Since every point in $\Gamma^k_B(F_1)$ has a neighbourhood of the form $K \times B^{t_k(F_2)}(0, \rho)$ for some compact subset $K \subseteq B$ and some $\rho > 0$, this shows that $\Psi^k$ has locally bounded image.

For the proof of continuity of $\Psi^k$ 1 will follow [HWZ10], esp. Section 2.2. So let $(b_0, u_0) \in \Gamma^k_B(F_1)$ and let $\varepsilon > 0$. For any $(b, u) \in \Gamma^k_B(F_1)$,

$$
\|\Psi^k(b, u) - \Psi^k(b_0, u_0)\| = \|(b - b_0, \Phi^*_b u - \Phi^*_{b_0} u_0)\| = |b - b_0| + \|\Phi^*_b u - \Phi^*_{b_0} u_0\|_k
$$

and for any $u'_0 \in \Gamma(F)$,

$$
\|\Phi^*_b u - \Phi^*_{b_0} u_0\|_k \leq \|\Phi^*_b u - \Phi^*_b u'_0\|_k + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u_0\|_k
= |\Phi^*_b(u - u'_0)||_k + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u_0\|_k
\leq C^k(b) \|u - u'_0\|_k + C^k(b_0) \|u'_0 - u_0\|_k + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u_0\|_k
\leq (C^k(b) + C^k(b_0))(\|u - u'_0\|_k + \|u'_0 - u_0\|_k) + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u_0\|_k.
$$

Now let any $\varepsilon > 0$ be given. Let $K \subseteq B$ be any compact convex neighbourhood of $b_0$ and let $\delta' := \frac{\varepsilon}{12C^k(K)}$. Pick $u'_0 \in \Gamma(F)$ s.t. $\|u'_0 - u_0\| < \delta'$. Then for any $u \in \Gamma^k(F)$ with $\|u - u_0\| < \delta'$ and any $b \in K$,

$$
\|\Phi^*_b u - \Phi^*_{b_0} u_0\|_k \leq 2C^k(K)(2\delta') + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u'_0\|_k
\leq \frac{\varepsilon}{3} + \|\Phi^*_b u'_0 - \Phi^*_{b_0} u'_0\|_k.
$$

Now

$$
\|\Phi^*_b u'_0 - \Phi^*_{b_0} u'_0\|_k = \left\|\int_0^1 \frac{d}{dt}\Phi^*_{b_0 + t(b - b_0)} u'_0 \right\|_k
\leq \int_0^1 \|\frac{d}{dt}\Phi^*_{b_0 + t(b - b_0)} u'_0\|_k dt
\leq D^k \|u'_0\|_{k+1} (b - b_0),
$$

where $D^k > 0$ is a universal expression in the supremum over the $C^{k+1}$-norms of $\Phi_v$, for $b \in K$.

Choose $\delta'' > 0$ so small that $B^B(b_0, \delta'') \subseteq K$ and set

$$
\delta := \min\left\{\delta'', \frac{\varepsilon}{6D^k \|u'_0\|_{k+1}}, \frac{\varepsilon}{2} \delta', \frac{\varepsilon}{2} \delta''\right\}.
$$

The above then shows that for all $(b, u) \in \Gamma^k_B(F_1)$ with $\|(b, u) - (b_0, u_0)\| < \delta$, $\|\Psi^k(b, u) - \Psi^k(b_0, u_0)\| < \varepsilon$.

2. Completely analogous to 1.

3. That $\Psi$ is continuous follows immediately from 1 because for $i = 1, 2$, $\Gamma_B(F_i)$ is the inverse limit of the spaces $\Gamma^k_B(F_i)$ and $\Psi$ is the inverse limit of the maps $\Psi^k$. 

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That $\Psi$ in general is not locally uniformly continuous is shown in Example 1.2 below.

To show that $\Psi$ in general does not have locally bounded image, assume that $F_1 = F_2 = F$ and that $\Psi_b$ is a diffeomorphism for all $b \in B$, e. g. as in Example 1.2 below. Let $U \subseteq \Gamma_B(F)$ be any neighbourhood of $(b_0, 0)$ for some $b_0 \in B$. Then $U$ contains a subset of the form $K \times (B^{k\infty}(F)(0, \rho) \cap \Gamma(F))$ for a compact neighbourhood $K \subseteq B$ of $b_0$ and some $\rho > 0$. The claim above then shows that $\Psi(U)$ contains a subset of the form $K \times (B^{k\infty}(F)(0, c^k(K)\rho) \cap \Gamma(F))$, which is not bounded in $\Gamma_B(F)$, cf. Remark 1.1.

To show that $\Psi$ is locally bounded, let $K \subseteq B$ be any compact subset. Then $\Psi|_{\Gamma_K(F)} : \Gamma_K(F) \to \Gamma_K(F)$ is bounded:

For if $C \subseteq \Gamma(F)$ is a bounded subset, then $K \times C \subseteq \Gamma_K(F)$ is bounded and if $A \subseteq \Gamma_K(F)$ is a bounded subset, then $C := pr_2(A) \subseteq \Gamma(F)$ is bounded (for $pr_2 : \mathbb{R}^\times \times \Gamma(F) \to \Gamma(F)$ is a continuous linear map) and $A \subseteq K \times C$. It hence suffices to show that for any bounded subset $C \subseteq \Gamma(F)$, $\Psi(K \times C) \subseteq \mathbb{R}^\times \times \Gamma(F)$ is bounded. Now $\Psi(K \times C) = \bigcup_{b \in K} \bigcup_{c \in C} (b, \Phi^*_k u) \subseteq K \times \bigcup_{b \in K} \Phi^*_k C$. It hence suffices to show that $\bigcup_{b \in K} \Phi^*_k C \subseteq \Gamma(F)$ is bounded for any bounded subset $C \subseteq \Gamma(F)$, i.e. it has to be shown that for any neighbourhood $U \subseteq \Gamma(F)$ of 0 there exists $c \in \mathbb{K}$ s.t. $\bigcup_{b \in K} \Phi^*_k C \subseteq c'U$. So let such a neighbourhood $U \subseteq \Gamma(F)$ be given. As has been used several times before, there then exists a $k \in \mathbb{N}_0$ and a $\rho > 0$ s.t. $B^{k\infty}(F)(0, \rho) \cap \Gamma(F) \subseteq U$. Because $C$ is bounded, there exists $c' \in \mathbb{K}$ s.t. $C \subseteq c'(B^{k\infty}(F)(0, \rho) \cap \Gamma(F)) = B^{k\infty}(F)(0, |c'|\rho) \cap \Gamma(F)$. By the claim above, for $b \in B$, then $\Phi^*_k C \subseteq B^{k\infty}(F)(0, C^k(b)|c'|\rho) \cap \Gamma(F)$. Set $c := C^k(K)|c'|$, where $C^k(K)$ is as in 1. above. Then $\bigcup_{b \in K} \Phi^*_k C \subseteq B^{k\infty}(F)(0, C^k(K)|c'|\rho) \cap \Gamma(F) = cB^{k\infty}(F)(0, \rho) \cap \Gamma(F) \subseteq c'U$.

\[\square\]

**Example 1.2.** Let $B := \mathbb{R}$, $\Sigma := S^1$ with the standard metric, $F := \Sigma \times \mathbb{R}$ with the trivial connection and standard fibre metric and let

$\phi : B \times \Sigma \to B \times \Sigma$

$s, e^t \mapsto (s, e^{t(s)})$

$\Phi : B \times F \to B \times F$

$s, (e^t, \xi) \mapsto (s, (e^{t(s)}, \xi))$.

It will now be shown that the corresponding maps $\Psi$, $\Psi^k$ and $\Psi^{k,p}$ as above are not locally uniformly continuous. To do so, in the following I will construct, for any $k \in \mathbb{N}_0$ and $1 < p < \infty$, sequences of smooth sections $(u^k_n)_{n \in \mathbb{N}} \subseteq \Gamma^k(F) \cap \Gamma(F)$ and $(u^{k,p}_n)_{n \in \mathbb{N}} \subseteq W^{k,p}(F) \cap \Gamma(F)$ s.t. the following hold:

1. There exist constants $c_k > 0$ s.t. $\|u^k_n\|_k \leq c_k$ and $\|u^{k,p}_n\|_{k,p} \leq c_k$ for all $n \in \mathbb{N}$.
2. For any $t > 0$,

$$\liminf_{n \to \infty} \|\Phi^*_k u^k_n - u^k_n\|_k \geq 2$$

and

$$\liminf_{n \to \infty} \|\Phi^{k,p}_n u^{k,p}_n - u^{k,p}_n\|_{k,p} \geq 2.$$
From this it follows immediately that $\Psi$, $\Psi^k$ and $\Psi^{k,p}$ are not uniformly continuous in any neighbourhood of $(0, 0) \in \Gamma_B(F)$.

First, restrict to $\Psi^k$, for the proof for $\Psi^{k,p}$ is the same almost ad verbatim, just replacing $u_n^k$ by $u_n^{k,p}$, etc.

Second, given any neighbourhood $U \subseteq \Gamma_B^k(F)$ of $(0, 0)$, there exists $\delta > 0$ s.t $(\delta, \delta) \times B^{\mathbb{R}^k}(0, \delta) \subseteq U$, where $B^{\mathbb{R}^k}(0, \delta)$ is the ball of radius $\delta$ around 0 in $\Gamma^k(F)$. Then the sequence of sections $(\tilde{u}_n^k)_{n \in \mathbb{N}} \subseteq \Gamma^k(F)$, $\tilde{u}_n^k := \frac{1}{\left| \partial_n \right|^k} u_n^k$ is contained in $B^{\mathbb{R}^k}(0, \delta)$ any hence $(t, \tilde{u}_n^k) \in U$ for all $t \in (\delta, \delta)$. By the above, for any $t \in (\delta, \delta)$ with $t \neq 0$, there exists $n_0(t) \in \mathbb{N}$ s.t.

$$||\Psi^k(t, \tilde{u}_n^k) - \Psi^k(0, \tilde{u}_n^k)|| = |t| + ||\Phi^*_k u_n^k - \tilde{u}_n^k||_k$$

$$= |t| + \frac{\delta}{c_k}||\Phi^*_k u_n^k - \tilde{u}_n^k||_k$$

$$\geq \frac{\delta}{c_k}$$

for all $n \geq n_0(t)$. On the other hand, for any neighbourhood $V \subseteq \Gamma_B^k(F)$ of 0, $(t, \tilde{u}_n^k) - (0, \tilde{u}_n^k) = (t, 0) \in V$ for all $t \neq 0$ small enough, independent of $\tilde{u}_n^k$. Now let $W := B^{\mathbb{R}^k}(0, \frac{\delta}{c_k}) \subseteq \Gamma_B^k(F)$. Then by the above, for any neighbourhood $V \subseteq \Gamma_B^k(F)$ of 0 we can find points $x, y \in U$ of the form $x = (t, \tilde{u}_n^k)$ and $y = (0, \tilde{u}_n^k)$ for some $t \neq 0$ small enough and some $n \in \mathbb{N}$ large enough with $x - y \in V$, but $\Psi^k(x) - \Psi^k(y) \notin W$, which contradicts uniform continuity.

Third, to show that $\Psi$ is not uniformly continuous in any neighbourhood of $(0, 0)$, observe that if $U \subseteq \Gamma_B(F)$ is any neighbourhood of $(0, 0)$, because $\Gamma_B(F)$ as a topological space is the inverse limit of the spaces $\Gamma_{B_n}(F)$, there exists $k \in \mathbb{N}_0$ and $\delta > 0$ s.t $(\delta, \delta) \times (B^{\mathbb{R}^k}(0, \delta) \cap \Gamma(F)) \subseteq U$. Then the points $(t, \tilde{u}_n^k)$, where $\tilde{u}_n^k$ is defined exactly as before, are contained in $U$. Define $W := B^{\mathbb{R}^k}(0, \frac{\delta}{c_k}) \cap \Gamma(F)$. Then by the same calculation as before, for any neighbourhood $V \subseteq \Gamma_B(F)$, $(t, \tilde{u}_n^k) - (0, \tilde{u}_n^k) = (t, 0) \in V$, for $t \neq 0$ small enough, but $\Psi(t, \tilde{u}_n^k) - \Psi(0, \tilde{u}_n^k) \notin W$ for $n \in \mathbb{N}$ large enough.

It remains to show the existence of the sections $u_n^k, u_n^{k,p}$ as above.

It follows from a standard exercise done in Calculus I courses that for every $1 < p < \infty$ and $n \in \mathbb{N}$ there exist smooth functions $f_n^0, f_n^{k,p} : \mathbb{R} \to \mathbb{R}$ s.t.

$$\chi_{[1/2-1/4n,1/2+1/4n]}(s) \leq f_n^0(s) \leq 2\chi_{[1/2-1/2n,1/2+1/2n]}(s),$$

$$(2n)^{1/p}\chi_{[1/2-1/4n,1/2+1/4n]}(s) \leq f_n^{k,p}(s) \leq 2n^{1/p}\chi_{[1/2-1/2n,1/2+1/2n]}(s),$$

where $\chi_A : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of a subset $A \subseteq \mathbb{R}$. Define inductively for $k \in \mathbb{N}_0$, $f_n^{k+1}(s) := f_n^0 f_n^{k}(r) \, dr$ and $f_n^{k+1,p}(s) := f_n^0 f_n^{k,p}(r) \, dr$.

Claim. The functions $f_n^0, f_n^{k,p} : \mathbb{R} \to \mathbb{R}$, for all $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $1 \leq p < \infty$, have the following properties:

1. $1 \leq \int (f_n^0)^p \, dr, ||f_n^{k,p}||_{W^{0,p}} \leq 2.$
2. For all $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $1 \leq p < \infty$, $f_n^k$ and $f_n^{k,p}$ are nonnegative and monotone increasing with $f_n^k(s) = f_n^{k,p}(s) = 0$ for all $s \leq 0$.
3. $\frac{d^\ell}{dr^\ell} f_n^k = f_n^{k-\ell}$ and $\frac{d^\ell}{dr^\ell} f_n^{k,p} = f_n^{k-\ell,p}$ for all $0 \leq \ell \leq k$.
4. For all $k \in \mathbb{N}_0$ there exist polynomials $P^k \in \mathbb{R}[s]$ of degree $k - 1$ s.t.

$$f_n^k(1+s), f_n^k(1+s) \leq P^k(s)$$

for all $s \geq 0$, $n \in \mathbb{N}$ and $1 < p < \infty$. 

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Proof. 1 to 3 are immediate from the definition. For 4, note that for $k = 0$ the statement obviously holds with $P^0 := 0$. For $k = 1$ and $s \geq 0$,

$$f_n^{1,p}(1 + s) = \int_0^1 f_n^{0,p}(r) \, dr + \int_1^{1+s} f_n^{0,p}(r) \, dr = \int_0^1 f_n^{0,p}(r) \, dr \leq 2n^{1/p} \cdot 1/n = 2/n^{1-1/p} \leq 2 =: P^1$$

and similarly for $f_n^1(1 + s) \leq 2 = P^1$. Now for $k \geq 1$ and $s \geq 0$,

$$f_n^{k+1,p}(1 + s) = \int_0^1 f_n^{k,p}(r) \, dr + \int_1^{1+s} f_n^{k,p}(r) \, dr \leq \int_0^1 P^k(0) \, dr + \int_1^{1+s} P^k(r) \, dr =: P^{k+1}(s).$$

Pick a smooth cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ with $\text{supp} \rho \subseteq [-1, 2]$ and $\rho|_{[0,1]} \equiv 1$. Define $g_n^k, g_n^{k,p} : \mathbb{R} \to \mathbb{R}$, for $k \in \mathbb{N}_0$, $1 < p < \infty$ and $n \in \mathbb{N}$ by $g_n^k := \rho f_n^k$ and $g_n^{k,p} := \rho f_n^{k,p}$. It follows from the above claim that there are constants $c_k > 0$ s.t. $\|g_n^k\|_{C^k}, \|g_n^{k,p}\|_{W^{k,p}} \leq c_k$.

Furthermore, for any $t \in \mathbb{R}$, $g_n^0(t + \cdot)$ and $g_n^0$ have disjoint supports for all $n \geq 1/t$ and likewise for $g_n^{0,p}(t + \cdot)$ and $g_n^{0,p}$. It follows easily that for all $n \in \mathbb{N}$ with $n \geq 1/t$,

$$\|g_n^k(t + \cdot) - g_n^k\|_{C^0} \geq \|g_n^0(t + \cdot) - g_n^0\|_{C^0} \geq \|g_n^0(t + \cdot)\|_{C^0} + \|g_n^0\|_{C^0} \geq 2\|g_n^0\|_{C^0} \geq 2$$

and likewise

$$\|g_n^{k,p}(t + \cdot) - g_n^{k,p}\|_{W^{k,p}} \geq 2\|g_n^{0,p}\|_{W^{n,p}} \geq 2.$$

Now define

$$u_n^k : \Sigma \to F$$

$$e^{st} \mapsto \begin{cases} (\epsilon_{x,t}, g^k(t)) & t \in [-1, 2] \\ (\epsilon_{x,t}, 0) & \text{else} \end{cases}$$

and

$$u_n^{k,p} : \Sigma \to F$$

$$e^{st} \mapsto \begin{cases} (\epsilon_{x,t}, g^{k,p}(t)) & t \in [-1, 2] \\ (\epsilon_{x,t}, 0) & \text{else} \end{cases}.$$
In the following proposition, since \( \Psi \) is not uniformly continuous and does not have locally bounded image, asking for Fréchet differentiability of \( \Psi \) does not make much sense anyway, at least as defined in Definition \([1,6]\) but the proof easily shows that also the weaker definition of Fréchet differentiability from Remark \([1,7]\) is not satisfied either. The assumption of locally bounded image is used to show that \( \mathcal{C}^0(U,Y) \) is a Hausdorff locally convex topological vector space, cf. Definition \([1,2]\). Without it one still gets a topological space \( \mathcal{E}^0(U,Y) \).

In preparation for stating the next proposition about differentiability of \( \Psi \), \( \Psi^k \) and \( \Psi^{k,p} \), it is helpful for expressing the differential to introduce a further bit of notation: For \( (b,u) \in \Gamma_B(F_1) \) and \( (e,v) \in \Gamma_{B^*}(F_1) \) let

\[
\nabla_e \phi_b : \Sigma \to T \Sigma \quad \text{and} \quad \nabla_e \Phi_b : F_1 \to F_2
\]

\[
\nabla_e \phi_b (z) := \left. \frac{d}{dt} \right|_{t=0} \phi_{b+te}(z) \quad \nabla_e \Phi_b := \left. \frac{d}{dt} \right|_{t=0} (\Phi_{b+te} \circ P_{\gamma^t}).
\]

Here, \( \gamma^t \) is the family of paths

\[
\gamma^t : [0,1] \times \Sigma \to \Sigma
\]

\[
(s,z) \mapsto \gamma^t_z := \phi_{b+ste}(z)
\]

and \( P_{\gamma^t} \) denotes the family of parallel transports along \( \gamma^t \), i.e. for \( z \in \Sigma \), one has parallel transport \( P_{\gamma^1} : (F_1)_{\phi_b(z)} \to (F_1)_{\phi_{b+ste}(z)} \) along \( \gamma^t_z \).

Define

\[
\hat{B} := TB \quad \hat{F} := \text{Hom}(T \Sigma, F_1)
\]

\[
\hat{\phi} : \hat{B} \times \Sigma \to \hat{B} \times \Sigma
\]

\[
\hat{\phi}_{(b,c)} := \phi_b
\]

\[
\hat{\Phi} : \hat{B} \times F_1 \to \hat{B} \times F_2
\]

\[
((b,e), u) \mapsto ((b,e), (\nabla_e \phi_b)(u)).
\]

\( \hat{\Phi}_{(b,c)} \) and \( \hat{\Phi}_{(b,c)} \) cover \( \hat{\phi}_{(b,c)}^{-1} \).

**Proposition 1.8.** In the above notation,

1. \( \Psi : \Gamma_B(F_1) \to \Gamma_B(F_2) \) is weakly continuously weakly Fréchet differentiable with locally bounded derivative given by

\[
D \Psi(b,u)(e,v) = \Phi_b \circ \nabla_e \phi_b u + (\nabla_e \Phi_b) \circ u \circ \phi_b + \Phi^*_b v
\]

for \( (b,u) \in \Gamma_B(F_1) \) and \( (e,v) \in \Gamma_{B^*}(F_1) \).

2. \( \Psi : \Gamma_B(F_1) \to \Gamma_B(F_2) \) is in general neither strongly continuously weakly Fréchet differentiable nor Fréchet differentiable.

3. \( \Psi^k : \Gamma^k_B(F_1) \to \Gamma^k_B(F_2) \) is weakly continuously weakly Fréchet differentiable along the canonical inclusion

\[
i_k : \Gamma^{k+1}(F_1) \hookrightarrow \Gamma^k(F_1),
\]

with locally bounded derivative along \( i_k \), given by the same formula as for \( \Psi \).

4. \( \Psi^{k,p} : \Gamma^{k,p}_B(F_1) \to \Gamma^{k,p}_B(F_2) \) is in general neither weakly Fréchet differentiable everywhere, nor strongly continuously weakly Fréchet differentiable along \( i_k \), nor Fréchet differentiable along \( i_k \).
5. $\Psi^{k,p} : W^{k,p}_B(F_1) \to W^{k,p}_B(F_2)$ is weakly continuously weakly Fréchet differentiable along the canonical inclusion

$$t_{k,p} : W^{k+1,p}(F_1) \hookrightarrow W^{k,p}(F_1),$$

with locally bounded derivative along $t_{k,p}$, given by the same formula as for $\Psi$.

6. $\Psi^{k,p} : W^{k,p}_B(F_1) \to W^{k,p}_B(F_2)$ is in general neither weakly Fréchet differentiable everywhere, nor strongly continuously weakly Fréchet differentiable along $t_{k,p}$, nor Fréchet differentiable along $t_{k,p}$.

Proof. The following calculation holds equally for all three cases, so let $f = \Psi$, $f = \Psi^k$, $(b,u) \in \Gamma_B(F_1)$, $(b,u) \in \Gamma^{k+1}_B(F_1)$ or $(b,u) \in W^{k+1,p}(F_1)$ and $(e,v) \in \Gamma_V(F_1)$, $(e,v) \in \Gamma^k_V(F_1)$ or $(e,v) \in W^{k,p}_V(F_1)$, respectively, where $V \subseteq \mathbb{R}^r$ is a convex balanced neighbourhood of $0$ s.t. $b+e \in B$ for all $e \in V$.

One then calculates for $t \in (0,1]$, that

$$f(b+te,u+tv) = f(b,u) +$$

$$+ t \left[ \Phi_b \circ \nabla_{\nabla \phi_b} u + (\nabla_e \Phi_b) \circ u \circ \phi_b + \Phi^*_b v +
+ \Phi_b \left( \frac{1}{t} \left( P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} - u \circ \phi_b \right) - \nabla_{\nabla \phi_b} u \right) +
+ \left( \frac{1}{t} (\Phi_{b+te} \circ P_{\gamma t}^{-1} - \Phi_b) \right) \circ P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} +
+ (\nabla_e \Phi_b) \circ \left( P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} - u \circ \phi_b \right) +
+ \Phi^*_b v \right]$$

and hence

$$r^f_{(b,u)}((e,v),t) = \Phi_b \circ \left( \frac{1}{t} \left( P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} - u \circ \phi_b \right) - \nabla_{\nabla \phi_b} u \right) +$$

$$=: (I_1)t(e) +$$

$$+ \left( \frac{1}{t} (\Phi_{b+te} \circ P_{\gamma t}^{-1} - \Phi_b) \right) \circ P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} +$$

$$=: (I_2)(e) +$$

$$+ (\nabla_e \Phi_b) \circ \left( P_{\gamma t}^{-1} \circ u \circ \phi_{b+te} - u \circ \phi_b \right)$$

$$=: (III_1)(e) +$$

$$+ \Phi^*_b v$$

Note that (at least once $[1,3]$ and $[5]$ have been shown) for $(b,u) = (b,0)$,

$$D\Psi(b,0)(e,v) = (e,\Phi^*_b v)$$

and

$$r^\Psi_{(b,0)}((e,v),t) = (0,\Phi^*_b v)$$

and similarly for $\Psi^k$ and $\Psi^{k,p}$. Using this and the above formula for $r^f_2$, $[3]$, and $[5]$ follow immediately from Proposition $1.7$. 

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We now examine the terms (I₁)–(IV) first for $\Psi^k$ and $\Psi^{k,p}$. (I₁) is obviously independent of $e$, $v$ and $t$ so there is nothing to show about it.

For (I₂), define

$$
\tilde{B} := (-1, 1) \times V \\
\tilde{F} := \text{Hom}(T\Sigma, F_1)
$$

$$
\tilde{\Phi} : \tilde{B} \times \tilde{F} \to \tilde{B} \times \tilde{F} \\
((s, e), h) \mapsto ((s, e), P^{-1}_{\gamma} \circ h \circ P^\Sigma_{\gamma})
$$

$$
\tilde{\phi} : \tilde{B} \times \Sigma \to \tilde{B} \times \Sigma \\
((s, e), z) \mapsto ((s, e), \phi_{b+se}(z)).
$$

$P^\Sigma_{\gamma}$ here denotes parallel transport in $T\Sigma$. Furthermore, define a smooth family $(X_{(s,e)})_{(s,e) \in \tilde{B}}$ of vector fields $X_{(s,e)} \in \mathfrak{X}(\Sigma)$ by $X_{(s,e)}(z) := (P^\Sigma_{\gamma})^{-1} \frac{d}{ds}\phi_{b+se}(z)$. Then

$$
\frac{d}{ds} P^{-1}_{\gamma} \circ u \circ \phi_{b+se} = \left( \tilde{\Phi}_{(s,e)}^* \nabla u \right) (X_{(s,e)})
$$

and

$$(I₂)(t, e) = \frac{1}{t} \int_0^t \frac{d}{ds} P^{-1}_{\gamma} \circ u \circ \phi_{b+se} \ ds - \frac{d}{ds} \bigg|_{s=0} P^{-1}_{\gamma} \circ u \circ \phi_{b+se}
$$

$$
= \frac{1}{t} \int_0^t \left( \tilde{\Phi}_{(s,e)}^* \nabla u \right) (X_{(s,e)}) - \left( \tilde{\Phi}_{(0,e)}^* \nabla u \right) (X_{(0,e)}) \ ds
$$

$$
= \frac{1}{t} \int_0^t \left( \tilde{\Phi}_{(s,e)}^* \nabla u - \tilde{\Phi}_{(0,e)}^* \nabla u \right) (X_{(0,e)}) \ ds +
$$

$$
= (I₂₁)(s,e)
$$

This equality holds pointwise (over $\Sigma$) if $u \in \Gamma^{k+1}(F_1)$ or $u \in W^{k+1,p}(F_1)$. It is used here that the maps

$$
\Gamma^{k+1}(F_1) \to \Gamma^k(\tilde{F}) \hspace{1cm} W^{k+1,p}(F_1) \to W^{k,p}(\tilde{F})
$$

$$
u \mapsto \nabla u
$$

are continuous and that $kp > \dim \Sigma$, so the Sobolev embedding theorem applies. Also using that the maps $(\mathfrak{X}^k(\Sigma) := \Gamma^k(T\Sigma))$

$$
\Gamma^{k}(\tilde{F}) \times \mathfrak{X}^k(\Sigma) \to \Gamma^{k}(F_1) \hspace{1cm} W^{k,p}(\tilde{F}) \times \mathfrak{X}^k(\Sigma) \to W^{k,p}(F_1)
$$

$$
(h, X) \mapsto h(X)
$$

are continuous, one sees that if $K \subseteq V$ is a compact subset and $\| \cdot \|$ denotes either $\| \cdot \|_k$ or $\| \cdot \|_{k,p}$ as appropriate, then by Proposition 1.7 and Lemma 1.1

- There exist continuous functions (also depending on $(b,u)$, which are fixed) $c^k(\cdot, K), c^{k,p}(\cdot, K) : [0, 1] \to [0, \infty)$ with $c^k(0, K) = c^{k,p}(0, K) = 0$ s.t.

$$
\| (I₂₁)(s,e) \|_k \leq \left\| \tilde{\Phi}_{(s,e)}^* \nabla u - \tilde{\Phi}_{(0,e)}^* \nabla u \right\|_k \| X_{(0,e)} \|_k \leq c^k(s,K)
$$

$$
\| (I₂₁)(s,e) \|_{k,p} \leq \left\| \tilde{\Phi}_{(s,e)}^* \nabla u - \tilde{\Phi}_{(0,e)}^* \nabla u \right\|_{k,p} \| X_{(0,e)} \|_k \leq c^{k,p}(s,K)
$$

for all $e \in K$. 29
• There exist constants $C^k(K), C^{k,p}(K) \in \mathbb{R}$ and a continuous function $\hat{c}^k(\cdot, K) : [0, 1] \to [0, \infty)$ with $\hat{c}^k(0, K) = 0$ s.t.

$$\|((I_2))_{(s, e)}\|_k \leq \left\| \Phi^*_{(s, e)} \nabla u \right\|_k \|X_{(s, e)} - X_{(0, e)}\|_k \leq C^k(K)\hat{c}^k(s, K)$$

$$\|((I_2))_{(s, e)}\|_{k, p} \leq \left\| \Phi^*_{(s, e)} \nabla u \right\|_{k, p} \|X_{(s, e)} - X_{(0, e)}\|_k \leq C^{k,p}(K)\hat{c}^k(s, K)$$

In conclusion, for all $e \in K$,

$$\|((I_2))_{(t, e)}\|_k \leq \frac{1}{t} \int_0^1 \|((I_2))_{(s, e)}\|_k ds + \frac{1}{t} \int_0^1 \|((I_2))_{(s, e)}\|_{k, p} ds$$

$$\leq \frac{1}{t} \int_0^1 \hat{c}^k(s, K) ds + \frac{1}{t} \int_0^1 C^k(K)\hat{c}^k(s, K) ds$$

$$\leq \max\{\hat{c}^k(s, K) + C^k(K)\hat{c}^k(s, K) \mid s \in [0, t]\}$$

$$\to 0 \quad \text{for } t \to 0.$$  

and similarly

$$\|((I_2))_{(t, e)}\|_{k, p} \leq \max\{\hat{c}^{k,p}(s, K) + C^{k,p}(K)\hat{c}^k(s, K) \mid s \in [0, t]\}$$

$$\to 0 \quad \text{for } t \to 0.$$  

Using that $\mathbb{R}^r$ is locally compact, the above shows that $(I_2) : [0, 1] \times V \to \Gamma^k(F_1)$ (or $(I_2) : [0, 1] \times V \to W^{k,p}(F_1)$) is a continuous function.

Completely analogous (somewhat simpler, in fact) arguments using Proposition 1.7 for the terms $(II_2)(t, e)$ and $(III_2)(t, e)$ (applied to a new reparametrisation action $\Phi_{(t, e)}(u) := P_{\bullet}^{-1} \circ u \circ \phi_{b+te}$) show that the terms $(II_1)(t, e)$ and $(III_2)(t, e)$ are continuous functions in $(t, e)$ as well.

Finally, $(IV)(t, e, v)$ is continuous in $(t, e, v)$ directly by Proposition 1.7.

This shows that $\Psi^k$ and $\Psi^{k,p}$ are weakly Fréchet differentiable on $\Gamma_{B}^{k+1}(F_1)$ and $W^{k+1,p}_{B}(F_1)$, respectively.

That they are also weakly continuously weakly Fréchet differentiable along the inclusions of $\Gamma_{B}^{k+1}(F_1)$ and $W^{k+1,p}_{B}(F_1)$ with locally bounded derivative, respectively, follows immediately from the explicit formula for their differential and Proposition 1.7.

Finally, the corresponding statements for $\Psi^k$ by virtue of the definition of the topologies on $\Gamma(F_1)$ and $\Gamma(F_2)$. For example, for $(h, u) \in \Gamma_p(F_1)$, $r^y_{(h, u)} : \Gamma^p(F_1) \times [0, 1] \to \Gamma_{R^1}(F_2)$ is continuous because it is the inverse limit of the maps $r^y_{(h, u)} : \Gamma^y_{F_1}(F_1) \times [0, 1] \to \Gamma^y_{R^1}(F_2)$, etc. □
2 The linear theory of \(\text{sc}-\text{Fréchet spaces}\)

2.1 Chains of Banach spaces

There are various names for the following objects, here I will follow mainly \[Omo97\] and \[HWZ07\]. The main reason to not use “sc-Banach space” is to avoid putting emphasis on the fixed Banach space \(E_0\). In fact, most of the setup presented here is motivated by the goal of shifting focus from the Banach space \(E_0\) in an sc-Banach space in the sense of \[HWZ07\], or the more recent \[HWZ10\], to the Fréchet space \(E_\infty\), which contains the objects one is usually more interested in, and treat the remaining structures as variable additional choices more akin to charts in a manifold.

The nonspecific “Banach space” also means that one has decided to either work with real or complex Banach spaces and real or complex linear maps, once and for all. The chosen ground field will be denoted by \(k\).

**Definition 2.1** ([Omo97], Definition 1.1 and [HWZ07], Definition 2.1).

1. An \(\text{ILB-chain} E\) is a sequence \((\left\{E_k, \|\cdot\|_k\right\}, \iota_k)_{k \in \mathbb{N}_0}\), where
   (a) \((E_k, \|\cdot\|_k)\) is a Banach space and
   (b) \(\iota_k : E_{k+1} \hookrightarrow E_k\) is a continuous embedding with dense image.
   (c) \(\|\iota_k\|_{L_c(E_{k+1}, E_k)} \leq 1\).

One regards the \(E_k\) as a nested sequence of linear subspaces of \(E_0\) and denotes the (dense) linear subspace \(E_\infty := \bigcap_{k \in \mathbb{N}_0} E_k \subseteq E_0\) as a topological vector space with the weakest topology s.t. all inclusions \(E_\infty \hookrightarrow E_k\) are continuous. For \(k, \ell \in \mathbb{N}_0\) with \(\ell > k\) one denotes

\[
\iota^k_\ell := \iota_{\ell-1} \circ \iota_{\ell-2} \circ \cdots \circ \iota_{k+1} \circ \iota_k : E_\ell \hookrightarrow E_k
\]

and \(\iota^k_\infty := \text{id}_{E_k}\). \(\iota^k_\infty : E_\infty \hookrightarrow E_k\) is defined as the canonical inclusion.

2. An \(\text{sc-chain} E\) is an ILB-chain \(E\) s.t. all the inclusions \(\iota_k\) are compact.

**Remark 2.1.**

1. “Continuous embedding” here means that \(\iota_k : E_{k+1} \to E_k\) is an injective continuous linear map. It does not mean that \(\iota_k\) has closed image or that \(\iota_k\) is an embedding of topological spaces (i.e. that \(E_{k+1}\) carries the subspace topology inherited from \(\iota_k(E_{k+1}) \subseteq E_k\)).

2. Condition \((c)\) is not as restrictive as it might look. For first of all in the main examples such as Example 2.1 and Example 2.2 it is automatically satisfied. And second if condition \((c)\) is dropped, then one can define new norms

\[
\|\cdot\|_{k+1} := \sum_{i=0}^{k} \|\iota^k_i(e)\|_i,
\]

i.e. \(\|\cdot\|_{k+1} = \|\iota_k(\cdot)\|_k + \|\cdot\|_{k+1}\). From this it follows immediately that for \(e \in E_{k+1}\)

\[
\|\iota_k(e)\|_k \leq \|\iota_k(e)\|_k + \|e\|_{k+1} = \|e\|_{k+1},
\]

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Define an ILB-chain as a monotone increasing sequence and let 1

\[ \text{Example 2.1.} \] Let \((\Sigma, g)\) be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let

\[ \pi : (\Sigma, g) \to \Sigma \]

be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let

\[ (\Sigma, g) \]

be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let

\[ \pi = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} C_i \| \cdot \|_{k_i} \]

be s. t.

\[ \left( \frac{\| \cdot \|_{k_i}(x-y)}{1 + \| \cdot \|_{k_i}(x-y) \} \right) \]

where \( r > 0 \) and \((c_k)_{k \in \mathbb{N}_0} \subseteq (0, \infty)\) is any sequence s. t. \( \sum_{k=0}^{\infty} c_k < \infty \).

\[ \text{Example 2.2.} \] Let \((\Sigma, g)\) be a closed n-dimensional Riemannian manifold and let

\[ \pi : F \to \Sigma \]

be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let \( 1 = (l_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) be a strictly monotone increasing sequence and let \( 1 < p < \infty \) be s. t. \( l_{0p} > n \).

Define an ILB-chain

\[ \mathcal{W}^{1, p}(F) := \left( \left( W^{l_k, p}(F), \| \cdot \|_{l_k}, t_k \right) \right)_{k \in \mathbb{N}_0}, \]

where

\[ \left( W^{l_k, p}(F), \| \cdot \|_{l_k, p} \right) \]

is the Sobolev space of sections of \( F \) of class \( W^{l_k, p} \), realised as the completion of \( \Gamma(F) \) w. r. t. the norm \( \| \cdot \|_{l_k, p} \), as in Section [3.3]. By

the Sobolev embedding theorem, \( W^{l_k, p}(F) \)

will be regarded as a subset of \( \Gamma(F) \).

\[ t_k : W^{l_k, p}(F) \to W^{l_k, p}(F) \]

is the canonical inclusion.

By the Rellich-Kondrachov theorem this ILB-chain is an sc-chain called the Sobolev chain of class \((1, p)\).

Furthermore, by the Sobolev embedding theorem,

\[ W^{1, p}(F) = \Gamma(F) \]

equipped with the \( C^\infty \)-topology.

\[ \text{Example 2.3.} \] Let \((\Sigma, g)\) be a closed n-dimensional Riemannian manifold and let

\[ \pi : F \to \Sigma \]

be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let \( 1 = (l_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) be a strictly monotone increasing sequence.

Define an ILB-chain

\[ \mathcal{I}^1(F) := \left( \left( \Gamma^{l_k}(F), \| \cdot \|_{l_k}, t_k \right) \right)_{k \in \mathbb{N}_0}, \]

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where \((\Gamma^k(F), \|\cdot\|_k)\) is the space of \(k\)-times continuously differentiable sections of \(F\) equipped with the \(C^k\)-norm \(\|\cdot\|_k\), as in Section 1.3. \(\iota_k : \Gamma^{k+1}(F) \hookrightarrow \Gamma^k(F)\) is the canonical inclusion.

By the Theorem of Arzela-Ascoli this ILB-chain is an sc-chain called the \textit{chain of continuously differentiable sections of class I}.

By definition, \(\Gamma^1(F) = \Gamma(F)\) equipped with the \(C^\infty\)-topology.

**Definition 2.2** ([HWZ07], Definition 2.6). Let \(E, E'\) and \(E''\) be ILB- or sc-chains.

1. A \textit{continuous linear operator} \(T : E \to E'\) is a sequence
   \[
   (T_k : (E_k, \|\cdot\|_k) \to (E'_k, \|\cdot\|'_k)_{k \in \mathbb{N}_0})_{k \in \mathbb{N}_0}
   \]
of continuous linear operators \(s. t.
   \[
   T_k \circ \iota_k = \iota'_k \circ T_{k+1} : E_{k+1} \to E'_k \quad \forall k \in \mathbb{N}_0.
   \]
   It induces a well defined continuous map \(T_\infty : E_\infty \to E'_\infty\) via \(T^\infty_k(T_\infty(u)) : T_k(T_\infty(u))\) for any \(k \in \mathbb{N}_0\).

2. For continuous linear operators \(T : E \to E'\) and \(T' : E' \to E''\), their \textit{composition} \(T' \circ T : E \to E''\) is the continuous linear operator \(T''\) with \(T''_k := T'_k \circ T_k\).

3. For continuous linear operators \(T, T' : E \to E'\) and \(\lambda, \mu \in k\), one defines \(\lambda T + \mu T' : E \to E'\) as the continuous linear operator \(T''\) with \(T''_k := \lambda T'_k + \mu T'_k\).

4. A continuous linear operator \(T : E \to E'\) is called an \textit{embedding} if \(T_k : E_k \to E'_k\) is a continuous embedding of Banach spaces for all \(k \in \mathbb{N}_0\).

**Lemma 2.1.** Let \(E, E'\) and \(E''\) be ILB- or sc-chains.

1. For continuous linear operators \(T : E \to E'\) and \(T' : E' \to E''\), if \(T'' := T' \circ T\), then \(T''_\infty = T'_\infty \circ T_\infty : E_\infty \to E''_\infty\).

2. There is a \(1-1\) correspondence between continuous linear operators \(T : E \to E'\) and continuous linear operators \(T : E_\infty \to E'_\infty\) \(s. t.\) \(i^\infty_k \circ T : E_\infty \to E'_k\) is a bounded operator on \((E_\infty, \|\cdot\|_{E_\infty})\).

**Proof.**

1. Trivial.

2. In one direction, one maps \(T : E \to E'\) to \(T_\infty : E_\infty \to E'_\infty\).

   In the other direction, to \(T : E_\infty \to E'_\infty\), one assigns the continuous linear operator \(\tilde{T} : E \to E'\) with \(T_k : E_k \to E'_k\) the unique completion of \(i^\infty_k \circ T : E_\infty \to E'_k\). Because \(E_\infty\) is dense in \(E_k\) and because \(i^\infty_k \circ T\) by assumption is bounded w.r.t. \(\|\cdot\|_k\), this completion exists and is unique.

   To verify that with this definition \(T_k \circ \iota_k = i'_k \circ T_{k+1}\), by density of \(E_\infty\) in \(E_{k+1}\) it suffices to verify this on \(E_\infty\), where \(T_{k+1} \circ \iota_k = T_\infty \circ \iota_k = \iota'_k \circ T = \iota'_{k+1} \circ i^\infty_k \circ T = T_{k+1} \circ i'_k \circ T = T_{k+1} \circ i'_k \circ T_{k+1}\).

\(\square\)

**Example 2.3.** Let \((\Sigma, g)\) be a closed \(n\)-dimensional Riemannian manifold and let \(\pi : F \to \Sigma\) and \(\pi' : F' \to \Sigma\) be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections.

Let \(k_0 \in \mathbb{N}_0\) and \(1 < p < \infty\) be s.t. \(k_0 p > n\). Given \(m \in \mathbb{N}_0\), define \(k := (k_0 + k)_{k \in \mathbb{N}_0}\) and \(k + m := (k_0 + m + k)_{k \in \mathbb{N}_0}\).
Let $P : \Gamma(F) \to \Gamma(F')$ be a partial differential operator (with smooth coefficients) of class $m$. Then for all $k \in \mathbb{N}_0$, the completion $P_{k_0+k,p} : W^{k_0+k+m,p}(F) \to W^{k_0+k,p}(F')$ of $P$ defines a continuous linear operator and hence

$$
P_{k,p} := (P_{k_0+k,p})_{k \in \mathbb{N}_0} : W^{k+m,p}(F) \to W^{k,p}(F')
$$
defines a continuous linear operator with $P_\infty = P : \Gamma(F) \to \Gamma(F')$ between Sobolev chains from Example 2.1.

Similarly, one can also look at the completions $P_k : \Gamma^{k+m}(F) \to \Gamma^k(F')$ of $P$ to give a continuous linear operator $(\text{id} = (j)_{j \in \mathbb{N}_0})$

$$
P_{\text{id}} := (P_k)_{k \in \mathbb{N}_0} : \Gamma^{k+m}(F) \to \Gamma^{k}(F')
$$
between chains of continuously differentiable sections from Example 2.2.

**Definition 2.3.** $\text{scChains}^k$ is the (preadditive) category with objects the sc-chains of $k$-Banach spaces and morphisms the continuous linear operators.

It comes with faithful underlying functors ($\text{Banach}^k$ and $\text{Frechet}^k$ are the categories of Banach- and Fréchet spaces over $k$, respectively)

$$
U_k : \text{scChains}^k \to \text{Banach}^k
\quad E \mapsto E_k
\quad (T : E \to E') \mapsto T_k
$$

$$
U_\infty : \text{scChains}^k \to \text{Frechet}^k
\quad E \mapsto E_\infty
\quad (T : E \to E') \mapsto T_\infty.
$$

and the maps $\iota_{\ell}^k$ for $0 \leq \ell \leq k \leq \infty$ provide natural transformations between these ($\text{Banach}^k$ comes with a natural forgetful functor to $\text{Frechet}^k$).
2.2 Rescaling, weak morphisms and equivalence

Definition 2.4. Let $E$ and $E'$ be ILB- or sc-chains and let $k = (k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ be a strictly monotone increasing sequence.

1. The ILB- or sc-chain
   \[ E^k = \left( (E^j_k, \| \cdot \|_j^k), j \in \mathbb{N}_0 \right) \]
   is called a rescaling of $E$ (by $k$). It satisfies $E^k_{\infty} = E_{\infty}$ as topological vector spaces.

2. A strictly monotone increasing sequence of the form $k = (k_0 + j)_{j \in \mathbb{N}_0}$ for some $k_0 \in \mathbb{N}_0$ is called a shift.
The rescaling of $E^k$ of $E$ by $k$ is called a shifted ILB- or sc-chain.

3. Let $l = (l_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ be another strictly monotone increasing sequence with $k \geq l$, i.e. $k_j \geq l_j$ for all $j \in \mathbb{N}_0$.
   Then there is a canonical continuous linear operator $l^k_{l^j} : E^k \to E^l$ given by
   \[ (l^k_{l^j})_{j \in \mathbb{N}_0} := \left( l_j^k : E_{k_j}^j \to E_{l_j}^j \right) \]
   One abbreviates $l^k := l^k_{1^0}$, where $1^0 := (j)_{j \in \mathbb{N}_0}$.
The $l^k$ satisfy $(l^k)^{-j} = l^{k^{-j}}_{1^0}$.

4. If $T : E \to E'$ is a continuous linear operator, then there is an induced continuous linear operator $T^k : E^k \to E'^k$ defined by
   \[ T^k_j := T_{k_j} : E^k_{k_j} \to E'^k_{k_j} \]
   It satisfies $T^k_{\infty} = T_{\infty} : E^k_{\infty} \to E'^k_{\infty}$.
   Conversely, $T^k$ is uniquely determined by $k$ and $T_{\infty}$.

5. A weak morphism between $E$ and $E'$ is a continuous linear operator $T : E_{\infty} \to E'_{\infty}$ s.t. there exist strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ and a continuous linear operator $T : E^k \to E^l$ with $T = T_{\infty}$.
   In this case $T : E^k \to E^l$ is said to extend (or be an extension of) $T : E_{\infty} \to E'_{\infty}$.

6. A weak morphism $T : E_{\infty} \to E'_{\infty}$ is called a weak embedding if there exist strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ and an embedding $T : E^k \to E^l$ s.t. $T = T_{\infty}$.

Remark 2.2. Note that by Lemma 2.1 in the definition of “weak morphism” for given choices of $k, l \subseteq \mathbb{N}_0$ the extension $T : E^k \to E^l$, if it exists, is uniquely determined by $T$.

Lemma 2.2. Let $E$, $E'$ and $E''$ be ILB- or sc-chains.

1. Let $k, l, m \subseteq \mathbb{N}_0$ be strictly monotone increasing sequences.
   \[ (E^k)^l = E^{k \circ l} \]
   is a rescaling of $E$ by $k \circ l := (k_j l_j)_{j \in \mathbb{N}_0}$.
   $l^{k \circ l} = l^k \circ (l^l)^{1^0}$.
   If $k \geq l \geq m$, then $l^k_m = l^l_m \circ l^k_m$.
   If $T : E \to E'$ is a continuous linear operator, then
   \[ T^l \circ l^k = T^l \circ l^{k \circ l} . \]
2. Let \( k, l \subseteq N_0 \) be strictly monotone increasing sequences with \( k \geq l \).

   \[ \|t_k\|_1 : E_k \to E_1 \] is an embedding.

   If \( T : E \to E' \) is an embedding, then so is \( T^k : E_k \to E^k \).

3. A continuous linear operator \( T : E_\infty \to E'_\infty \) defines a weak morphism if and only if there exists a strictly monotone increasing sequence \( k \subseteq N_0 \) and a continuous linear operator \( T : E^k \to E' \) s.t. \( T = T_\infty \).

4. A weak morphism \( T : E_\infty \to E'_\infty \) is an embedding \( E \to E' \) if and only if there exists a strictly monotone increasing sequence \( k \subseteq N_0 \) and an embedding \( T : E^k \to E' \) s.t. \( T = T_\infty \).

5. Let \( T : E_\infty \to E'_\infty \) be a continuous linear operator and \( k, l \subseteq N_0 \). Then there exists a strictly monotone increasing sequence \( m \subseteq N_0 \) s.t. \( T \circ \Pi_m^k = S \circ \Pi_l^m : \Pi_m \to E' \).

6. \( \text{id}_{E_\infty} : E_\infty \to E_\infty \) is a weak embedding.

7. If \( T : E_\infty \to E'_\infty \) and \( T' : E'_\infty \to E''_\infty \) be weak morphisms. Then their composition \( T' \circ T : E_\infty \to E''_\infty \) is a weak morphism as well.

   If \( T \) and \( T' \) are weak embeddings then so is \( T' \circ T \).

8. Let \( T, T' : E_\infty \to E'_\infty \) be weak morphisms and let \( \lambda, \mu \in k \). Then \( \lambda T + \mu T' : E_\infty \to E'_\infty \) is a weak morphism as well.

**Proof.**

1. Obvious.

2. Obvious.

3. One direction is trivial, taking \( l := (j)_{j \in N_0} \). For the other direction, assume that \( T : E_\infty \to E'_\infty \) is a weak morphism and let \( T : E^k \to E^d \) be an extension of \( T_\infty \), where \( k, l \subseteq N_0 \) are strictly monotone increasing sequences. Then \( T' := I^d \circ T : E^k \to E' \) is a continuous linear operator with \( T'_\infty = I^d_\infty \circ T_\infty = \text{id}_{E^k} \circ T = T \).

4. This follows from the proof of \( \square \) together with \( \square \).

5. For existence of such a sequence \( m \) set \( m := (\max\{k_j, l_j\})_{j \in N_0} \). Now if \( T' := T \circ \Pi_m^k \) and \( S' := S \circ \Pi_l^m \), then \( T'_\infty = S'_\infty \) and the claim follows from Lemma \( \square \).

6. Obvious.

7. Using \( \square \) let \( T : E^k \to E' \) and \( T' : E^l \to E'' \) be extensions of \( T_\infty \) and \( T'_\infty \), respectively, for strictly monotone increasing sequences \( k, l \subseteq N_0 \). Then \( T'' := T' \circ T^1 : E^k \to E'' \) is a continuous linear operator with \( T''_\infty = T'_\infty \circ T \) i.e. an extension of \( T' \circ T \).

   This formula also shows the last statement about embeddings.

8. Let \( T : E^k \to E' \) and \( T' : E^l \to E' \) be extensions of \( T \) and \( T' \), respectively, for strictly monotone increasing sequences \( k, l \subseteq N_0 \). Define \( m := (\max\{k_j, l_j\})_{j \in N_0} \). Then \( \lambda T + \mu T' : E^m \to E' \) is an extension of \( \lambda T + \mu T' : E_\infty \to E'_\infty \).

\[ \square \]
Example 2.4. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection.

Let $k_0 \in \mathbb{N}_0$ and $1 < p < \infty$ be s.t. $k_0 p > n$. Let $k := (k_0 + k)_{k \in \mathbb{N}_0}$ and let $I \subseteq \mathbb{N}_0$ be a strictly monotone increasing sequence with $I_0 \geq k_0$. Define $I^p := (I_j - k_0)_{j \in \mathbb{N}_0}$.

Then the Sobolev chains from Example 2.1 satisfy $\mathcal{W}^k_p(F) = (\mathcal{W}^{k+p}(F))^Y$.

Given $p \leq q < \infty$, for every $j \in \mathbb{N}_0$, the Sobolev embedding theorem there is a canonical continuous embedding $J_j : \mathcal{W}^{k,j}_q(F) \hookrightarrow \mathcal{W}^{k,j}_p(F)$ that is the identity on $\Gamma(F)$. These fit together to an embedding $J : \mathcal{W}^{k,q}_p(F) \hookrightarrow \mathcal{W}^{k,p}_p(F)$ with $J_\infty = \text{id}_{\Gamma(F)}$.

Example 2.5. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ and $\pi' : F' \to \Sigma$ be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections.

Let $k_0 \in \mathbb{N}_0$ and $1 < p < \infty$ be s.t. $k_0 p > n$ and define $k := (k_0 + k)_{k \in \mathbb{N}_0}$.

Let $P : \Gamma(F) \to \Gamma(F')$ be a partial differential operator (with smooth coefficients) of class $m$. Then $P : \mathcal{W}^{k,p}(F) = \Gamma(F) \to \Gamma(F') = \mathcal{W}^{k,p}(F')_\infty$ defines a weak morphism between $\mathcal{W}^{k,p}(F)$ and $\mathcal{W}^{k,p}(F')$. For putting $1 := (m + j)_{j \in \mathbb{N}_0}$, the continuous linear operator $\mathbb{P}^{k,p} : \mathcal{W}^{k+m,p}(F) = (\mathcal{W}^{k,p}(F))_1 \to \mathcal{W}^{k,p}(F')$ from Example 2.3 extends $P$. In just the same way, the operator $\mathbb{P}^{k} : \mathcal{W}^{id+m}(F) = (\mathcal{W}^{id}(F))_1 \to \mathcal{W}^{id}(F')$ extends $P : \mathcal{W}^{id}(F) = \Gamma(F) \to \Gamma(F') = \mathcal{W}^{id}(F')_\infty$, showing that $P$ defines a weak morphism between $\mathcal{W}^{id}(F)$ and $\mathcal{W}^{id}(F')$ as well.

Definition 2.5. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be ILB- or sc-chains. A weak embedding $J : \mathcal{E}_\infty \to \tilde{\mathcal{E}}_\infty$ is called an equivalence if there exists another weak embedding $K : \tilde{\mathcal{E}}_\infty \to \mathcal{E}_\infty$ s.t. $J \circ K = \text{id}_{\mathcal{E}_\infty}$ and $K \circ J = \text{id}_{\mathcal{E}_\infty}$.

$\mathcal{E}$ and $\tilde{\mathcal{E}}$ are called equivalent if there exists an equivalence between them.

Remark 2.3. Note that the above definition of equivalence of ILB- or sc-chains does not mean that one can find continuous linear operators $J : \mathcal{E}^k \to \tilde{\mathcal{E}}^1$ and $K : \tilde{\mathcal{E}}^1 \to \mathcal{E}^k$ between rescalings s.t. $J \circ K : \tilde{\mathcal{E}}^1 \to \tilde{\mathcal{E}}^1$ and $K \circ T : \mathcal{E}^k \to \mathcal{E}^k$ are the identity.

Instead, at least for sc-chains, by Lemma 2.3 below and (the proof of) Lemma 2.2 one can find $J : \mathcal{E}^k \to \tilde{\mathcal{E}}$ and $K : \mathcal{E}^1 \to \mathcal{E}$ s.t.

$$K \circ J^1 = \mathbb{P}^{k,1} : \mathcal{E}^{k,1} \to \mathcal{E}$$

and

$$J \circ K^k = \mathbb{P}^{1,k} : \mathcal{E}^{1,k} \to \tilde{\mathcal{E}}.$$

See e.g. Example 2.6 below.

Because of the previous remark and the following lemma, in the future I will restrict almost exclusively to sc-chains and there is no such thing as an $\text{ILB}$-Fréchet space appearing in this text.

Lemma 2.3. Let $\mathcal{E}$ be an sc-chain, let $k, l \subseteq \mathbb{N}_0$ be strictly monotone increasing sequences and let $J : \mathcal{E}^k \to \mathcal{E}^l$ be a morphism with $J_\infty = \text{id}_{\mathcal{E}_\infty}$.

1. If $\dim E_\infty < \infty$, then $E_k = E_\infty$, $\iota_k = \text{id}_{E_\infty}$ and $J_k = \text{id}_{E_\infty}$ for all $k \in \mathbb{N}_0$.

2. If $\dim E_\infty = \infty$, then $k \geq l$ and $J = \mathbb{P}^{k,l}$.
Proof. 1. If \( \dim E_\infty < \infty \), because \( E_\infty \subseteq E_k \) is dense for all \( k \in \mathbb{N}_0 \), \( \dim E_k < \infty \) for all \( k \in \mathbb{N}_0 \) and \( \iota^\infty_k : E_\infty \to E_k \) is the identity. Hence also \( E^k = E_\infty = E^j \) for all \( j \in \mathbb{N}_0 \) and it follows that \( J_k = \text{id}_{E_\infty} \) for all \( k \in \mathbb{N}_0 \) directly from the axioms for a continuous linear operator between sc-chains.

2. \( J_j : E^k \to E^j \) is a continuous linear operator between Banach spaces with \( J_j|_{E_\infty} = \iota^\infty_j \). Since \( E_\infty \subseteq E_k \) is dense, this uniquely determines \( J_j \). If \( j \leq l \), then \( (\iota^k_j)_j = \iota^\infty_j \) is a continuous linear operator with \( \iota^\infty_j|_{E_\infty} = \iota^\infty_j \); hence \( J_j = (\iota^k_j)_j \).

Let \( j < l \) and assume that there exists a continuous linear operator \( J_j : E_k \to E_l \) with \( J_j|_{E_\infty} = \iota^\infty_j \). Then \( K := \iota^l_j \circ J_j : E_k \to E_k \) is a compact linear operator with \( K|_{E_\infty} = \iota^\infty_j \). By uniqueness it follows that \( K = \text{id}_{E_k} \) is compact, hence \( \dim E_k < \infty \) and also \( \dim E_\infty < \infty \), a contradiction to the assumption \( \dim E_\infty = \infty \).

\( \square \)

**Lemma 2.4.** Let \( E \) and \( E' \) be ILB- or sc-chains.

1. Equivalence of ILB- or sc-chains is an equivalence relation.

2. If \( k \subseteq \mathbb{N}_0 \) is a strictly monotone increasing sequence, then \( E^k \) and \( E \) are equivalent.

3. Let \( \hat{E} \) and \( \hat{E}' \) be ILB- or sc-chains that are equivalent to \( E \) and \( E' \), respectively. Then there is a \( 1 \leftarrow 1 \) correspondence between weak morphisms \( T : E_\infty \to E'_\infty \) and weak morphisms \( \hat{T} : E_\infty \to \hat{E}'_\infty \), mapping the identity to the identity (for \( E = E' \) and \( \hat{E} = \hat{E}' \)), weak embeddings to weak embeddings and that is compatible with composition of weak morphisms.

Proof. 1. \( E \) is equivalent to \( \hat{E} \) via the identity (as a weak morphism) and equivalence is a symmetric relation by definition. Transitivity follows because compositions of weak embeddings are weak embeddings by Lemma 2.2.

2. The weak embeddings in both directions are just tautologically given by the identity \( E^k \to E^k \).

3. Let \( J : E_\infty \to \hat{E}_\infty \), \( K : \hat{E}_\infty \to E_\infty \) and \( J' : E'_\infty \to \hat{E}'_\infty \), \( K' : \hat{E}'_\infty \to E'_\infty \) be as in the definition of equivalence for \( E, \hat{E} \) and \( E', \hat{E}' \), respectively. Then for a weak morphism \( T : E_\infty \to E'_\infty \) define \( \hat{T} : \hat{E}_\infty \to \hat{E}'_\infty \) by \( \hat{T} := J' \circ T \circ K \).

Conversely, for a weak morphism \( \hat{T} : \hat{E}_\infty \to \hat{E}'_\infty \) define \( T : E_\infty \to E'_\infty \) by \( T := K' \circ T \circ J \).

This has the desired properties by Lemma 2.2.

\( \square \)

**Example 2.6.** Let \( (\Sigma, g) \) be a closed \( n \)-dimensional Riemannian manifold and let \( \pi : F \to \Sigma \) be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection.

Let \( k_0 \in \mathbb{N}_0 \) and \( 1 < p < \infty \) be s. t. \( k_0 p > n \). Let \( k := (k_0 + k)_{k \in \mathbb{N}_0} \) and let \( k \subseteq \mathbb{N}_0 \) be a strictly monotone increasing sequence with \( l_0 \geq k_0 \).

By Example 2.4, \( \mathbb{W}^{k,p}(F) \) and \( \mathbb{W}^{l,p}(F) \) are equivalent.

We want to show that for \( 1 < p \leq q < \infty \), \( \mathbb{W}^{k,p}(F) \) and \( \mathbb{W}^{k,q}(F) \) are equivalent as
well. From Example 2.4, we already have the embedding \( J : \mathcal{W}^{1,q}(F) \to \mathcal{W}^{1,p}(F) \) with \( J_{\infty} = \text{id}_{\Gamma(F)} \). So (using Example 2.4) one is left to construct another embedding \( K : \mathcal{W}^{1,p}(F) \to \mathcal{W}^{k,q}(F) \), for some strictly monotone increasing sequence \( 1 \subseteq N_0 \), satisfying \( K_{\infty} = \text{id}_{\Gamma(F)} \).

To do so, pick any \( m \in N_0 \) with \( m > n^2/p \) and define \( l := (k_0 + m/k)k \in N_0 \). Then by the Sobolev embedding theorem, for any \( j \in N_0 \) there is a canonical continuous embedding \( K_j : W^{l,j,p}(F) \to W^{k,q}(F) \) that is the identity on \( \Gamma(F) \). These fit together to an embedding \( K : \mathcal{W}^{1,p}(F) \to \mathcal{W}^{k,q}(F) \) with \( K_{\infty} = \text{id}_{\Gamma(F)} \).

This, together with transitivity of equivalence of \( \text{sc}\)-chains, shows that the \( \mathcal{W}^{k,p}(F) \) for different choices of \( k \) and \( p \) are all equivalent.

**Example 2.7.** Let \( (\Sigma, g) \) be a closed \( n \)-dimensional Riemannian manifold and let \( \pi : F \to \Sigma \) be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection.

Let \( K_0 \in N_0 \) and 1 < \( p < \infty \) be s.t. \( k_0p > n \) and let \( k := (k_0 + k)k \in N_0 \). Then \( \mathcal{W}^{1,p}(F) \) and \( \mathcal{W}^{k,p}(F) \) are equivalent:

Because \( \Sigma \) was assumed closed, there is a canonical inclusion \( \Gamma(F) = \Gamma^{k_0+j}(F) \to W^{k,p}(F) \). In the other direction, the Sobolev embedding theorem gives a canonical inclusion \( W^{k,p}(F) \to \Gamma(F) = \Gamma^{k_0+j}(F) \). Both of these inclusions are the identity on \( \Gamma(F) \).

Since for any strictly monotone increasing sequence \( 1 \subseteq N_0 \), \( \mathcal{W}^{1}(F) = (\mathcal{W}^{1})^{1} \), this, together with Example 2.6, shows that the chains \( \mathcal{W}^{k,p}(F) \) and \( \mathcal{W}^{1}(F) \) are all equivalent, for any choice of strictly monotone increasing sequence \( 1 \subseteq N_0 \) and any \( 1 < p < \infty \) (s.t. \( k_0p > n \)).

### 2.3 \text{sc}\-structures

**Definition 2.6.** Let \( E \) be a topological vector space.

1. An \text{sc}\-structure on \( E \) is a pair \( (\mathbb{E}, \phi) \), where \( \mathbb{E} \) is an \( \text{sc}\)-chain and \( \phi : E_{\infty} \to E \) is an isomorphism of topological vector spaces.

2. Two \text{sc}\-structures \( (\mathbb{E}, \phi) \) and \( (\tilde{\mathbb{E}}, \tilde{\phi}) \) on \( E \) are called equivalent if there exists an equivalence (of \( \text{sc}\)-chains) \( J : E_{\infty} \to \tilde{E}_{\infty} \) with \( \phi = \tilde{\phi} \circ J \).

Equivalently, if there exists an equivalence (of \( \text{sc}\)-chains) \( K : \tilde{E}_{\infty} \to E_{\infty} \) with \( \tilde{\phi} = \phi \circ K \).

$$
\begin{array}{ccc}
E_{\infty} & \xrightarrow{J} & \tilde{E}_{\infty} \\
\phi & \circ & \tilde{\phi} \\
E & \xrightarrow{\phi} & \tilde{E}
\end{array}
\quad
\begin{array}{ccc}
\tilde{E}_{\infty} & \xrightarrow{K} & E_{\infty} \\
\tilde{\phi} & \circ & \phi \\
\tilde{E} & \xrightarrow{\tilde{\phi}} & E
\end{array}
$$

Note that in this case \( J = \tilde{\phi}^{-1} \circ \phi \) and \( K = J^{-1} = \phi^{-1} \circ \tilde{\phi} \) as continuous linear operators between topological vector spaces.

3. An \text{sc}\-Fréchet space is a topological vector space \( E \) together with an equivalence class of \text{sc}\-structures on it. The \( \text{sc}\)-chains in this equivalence class are then called compatible.

4. A morphism between \text{sc}\-Fréchet spaces \( E \) and \( E' \) is a continuous linear operator \( T : E \to E' \) between \( E \) and \( E' \) as topological vector spaces s.t. there exist compatible \text{sc}\-structures \( (\mathbb{E}, \phi) \) and \( (\mathbb{E}', \phi') \) on \( E \) and \( E' \), respectively.
Let \( T : E \to E' \) be a continuous linear operator \( T : E \to E' \) defines a morphism of \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces if there exists a pair of compatible \( \mathfrak{p}\mathcal{r} \)-structures \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) on \( E \) and \( E' \), respectively, and a continuous linear operator \( T \) \( T : E \to E' \) s.t. \( \tilde{T}_\infty = \phi'^{-1} \circ T \circ \phi \) defines a weak morphism.

A morphism \( T : E \to E' \) between \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces is called an embedding if there exist compatible \( \mathfrak{p}\mathcal{r} \)-structures \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) on \( E \) and \( E' \), respectively, s.t. \( \phi'^{-1} \circ T \circ \phi : E_\infty \to E'_\infty \) defines a weak embedding.

**Remark 2.4.** Note that the underlying topological vector space of an \( \mathfrak{p}\mathcal{r} \)-Fréchet space is automatically a Fréchet space.

**Lemma 2.5.** Let \( E, E' \) and \( E'' \) be \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces.

1. A continuous linear operator \( T : E \to E' \) defines a morphism of \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces if there exists a pair of compatible \( \mathfrak{p}\mathcal{r} \)-structures \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) on \( E \) and \( E' \), respectively, and a continuous linear operator \( T \) \( T : E \to E' \) s.t. \( \tilde{T}_\infty = \phi'^{-1} \circ T \circ \phi \) defines a weak morphism.

2. A continuous linear operator \( T : E \to E' \) defines an embedding of \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces if and only if there exist compatible \( \mathfrak{p}\mathcal{r} \)-structures \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) on \( E \) and \( E' \), respectively, \( \tilde{T}_\infty := \phi'^{-1} \circ T \circ \phi \) defines an embedding.

3. A continuous linear operator \( T : E \to E' \) defines a morphism of \( \mathfrak{p}\mathcal{r} \)-Fréchet spaces if for any pair of compatible \( \mathfrak{p}\mathcal{r} \)-structures \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) on \( E \) and \( E' \), respectively, \( \tilde{T}_\infty := \phi'^{-1} \circ T \circ \phi \) defines a weak morphism.

4. \( \text{id}_E : E \to E \) is a morphism and embedding.

5. Let \( T, T' : E \to E' \) be morphisms and let \( \lambda, \mu \in \mathbb{k} \). Then \( \lambda T + \mu T' : E \to E' \) is a morphism as well.

6. Let \( T : E \to E' \) and \( T' : E' \to E'' \) be morphisms. Then \( T' \circ T : E \to E'' \) is a morphism as well.

7. Let \( T : E \to E' \) and \( T' : E' \to E'' \) be embeddings. Then \( T' \circ T : E \to E'' \) is an embedding as well.

**Proof.**

1. Let \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) be as in the definition of a morphism. That \( \tilde{T}_\infty := \phi'^{-1} \circ T \circ \phi \) defines a weak morphism means by definition that there exist strictly monotone increasing sequences \( k, l \subseteq \mathbb{N}_0 \) an continuous linear operator \( T : \mathbb{E}^k \to \mathbb{E}^l \) extending \( T_\infty \). But if \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) are compatible \( \mathfrak{p}\mathcal{r} \)-structures on \( E \) and \( E' \), respectively, then so are \((\mathfrak{E}, \phi)\) \((\mathfrak{E}', \phi')\) \( \mathfrak{p}\mathcal{r} \)-structures.

2. Let \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) be as in the definition of a morphism and let \( K : \mathfrak{E}_\infty \to \mathfrak{E}_\infty \) and \( J : \mathfrak{E}_\infty \to \mathfrak{E}_\infty \) be as in the definition of equivalence for \((\mathfrak{E}, \phi)\) with \((\mathfrak{E}, \phi)\) and \((\mathfrak{E}', \phi')\) with \((\mathfrak{E}', \phi')\), respectively. Then the claim follows from the following commutative diagram, where the composition of the upper row is given by \( \tilde{T}_\infty \).

\[
\begin{array}{ccc}
\mathfrak{E}_\infty & \xrightarrow{\phi} & \mathfrak{E}_\infty \\
\phi \downarrow & & \phi' \downarrow \\
E & \xrightarrow{T} & E'
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{E}_\infty' & \xrightarrow{\phi} & \mathfrak{E}_\infty' \\
\phi' \downarrow & & \phi' \downarrow \\
E' & \xrightarrow{T'} & E'
\end{array}
\]
3. Follows as in 2. because in the above diagram $K$ and $J'$ are weak embeddings.

4. Trivial.

5. Immediate from 2. and Lemma 2.2

6. Because $T$ and $T'$ are morphisms, by using 2. for any compatible $\mathcal{F}$-structures $((E, \phi), (E', \phi'))$ and $((E'', \phi''), (E''', \phi'''))$ on $E$, $E'$, and $E''$, respectively, $T_\infty := \phi'^{-1} \circ T \circ \phi : E_\infty \to E'_\infty$ and $T'_\infty := \phi'''^{-1} \circ T' \circ \phi' : E'_\infty \to E''''_\infty$ define weak morphisms. Then so does $T''_\infty := T'_\infty \circ T'' = \phi'''^{-1} \circ (T' \circ T) \circ \phi : E_\infty \to E''''_\infty$ by Lemma 2.2. Hence $T' \circ T$ is a morphism.

7. Follows from the proof of 6. and Lemma 2.2.

Example 2.8. Let $E$ be a finite dimensional (real or complex) vector space. By Lemma 2.3, $E$ defines an $\mathcal{F}$-Fréchet space in a unique way, where a compatible $\mathcal{F}$-structure on $E$ is given by $(E, \text{id}_E)$, where $E_j := E$, $\iota_j := \text{id}_E$ and $\| \cdot \|_j$ is an arbitrary norm on $E$.

Example 2.9. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ be a real (or complex) vector bundles equipped with a euclidean (or hermitian) metric and metric connection. Let $\Gamma(F)$ be the space of smooth sections of $F$ equipped with the $C^\infty$-topology. By Examples 2.1 to 2.6 for any $1 < p < \infty$ and any strictly monotone increasing sequence $k \subseteq \mathbb{N}$, $(W^{k,p}(F), \text{id}_{\Gamma(F)})$ defines an $\mathcal{F}$-structure on $\Gamma(F)$ and these $\mathcal{F}$-structures are all equivalent. $\Gamma(F)$ together with the $\mathcal{F}$-Fréchet space structure given by the equivalence class these $\mathcal{F}$-structures define will be denoted by $W(F)$ and called the Sobolev $\mathcal{F}$-space of sections of $F$.

Example 2.10. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ be a real (or complex) vector bundles equipped with a euclidean (or hermitian) metric and metric connection. Let $\Gamma(F)$ be the space of smooth sections of $F$ equipped with the $C^\infty$-topology. By Example 2.2 for any strictly monotone increasing sequence $k \subseteq \mathbb{N}$, $(\Gamma^k(F), \text{id}_{\Gamma(F)})$ defines an $\mathcal{F}$-structure on $\Gamma(F)$ and these $\mathcal{F}$-structures are all equivalent. By abuse of notation, $\Gamma(F)$ together with the $\mathcal{F}$-Fréchet space structure given by the equivalence class these $\mathcal{F}$-structures define will be denoted by $\Gamma(F)$ and called the $\mathcal{F}$-space of sections of $F$.

Example 2.11. By Example 2.7

$$W(F) = \Gamma(F)$$

as $\mathcal{F}$-spaces and this is an actual identity, not just an isomorphism.

Example 2.12. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ and $\pi' : F' \to \Sigma$ be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections. Let $P : \Gamma(F) \to \Gamma(F')$ be a partial differential operator (with smooth coefficients). Then by Examples 2.3 and 2.5 $P$ defines a morphism $P : W(F) \to W(F')$ between the Sobolev $\mathcal{F}$-spaces of sections of $F$ and $F'$. Which is the same as saying that $P : \Gamma(F) \to \Gamma(F')$ defines a morphism between the $\mathcal{F}$-spaces of sections of $F$ and $F'$. 

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\textbf{Definition 2.7.} $\mathcal{SFrechet}^k$ is the (preadditive) category with objects the $\mathcal{SF}$-Fréchet spaces and morphisms the morphisms of $\mathcal{SF}$-Fréchet spaces. It comes with a faithful underlying functor $U : \mathcal{SFrechet}^k \to \text{Frechet}^k$.

\section{Subspaces and direct sums}

\textbf{Definition 2.8} (\cite{HWZ07}, Definition 2.5). Let $E$, $E'$ and $E^1, \ldots, E^k$, for some $k \in \mathbb{N}_0$, be ILB- or sc-chains.

1. $E'$ is called a subchain of $E$ if for all $j \in \mathbb{N}_0$, $E'_j \subseteq E_j$ is a closed linear subspace, $\| \cdot \|_j = \| \cdot \|_{E'_j}$ and $\iota'_j = \iota_{E'_j}$.

2. The direct sum (also: biproduct, direct product) of the $E^i$, $i = 1, \ldots, k$, is the ILB- or sc-chain $E^1 \oplus \cdots \oplus E^k := E''$ with

$$
E''_j := E^1_\pm \cdots \oplus E^k_\pm,
\| \cdot \|''_j := \| \cdot \|_1^\pm + \cdots + \| \cdot \|_k^\pm
\text{ and }
\iota''_j := \iota^1_\pm \cdots \oplus \iota^k_\pm.
$$

It satisfies $E''_\pm = E^1_{\pm} \times \cdots \times E^k_{\pm}$ (with the product topology) and comes with the continuous linear operators

$$
P^j_i : E^1 \oplus \cdots \oplus E^k \to E^i
$$

\begin{align*}
P^j_i(e^1, \ldots, e^k) &:= e^i & \forall (e^1, \ldots, e^k) &\in E^1_\pm \oplus \cdots \oplus E^k_\pm, \\
J^j_i : E^i \to E^1 \oplus \cdots \oplus E^k &:= (0, \ldots, 0, e, 0, \ldots, 0) & \forall e &\in E^i_\pm
\end{align*}

for $i = 1, \ldots, k$, called the projection and injection operators, respectively.

3. A subchain $E'$ of $E$ is said to split if there exists another subchain $E''$ of $E$ s.t. the canonical continuous linear operator

$$
J : E' \oplus E'' \to E
$$

\begin{align*}
J_j(e, e') &:= e + e' & \text{for } (e, e') &\in E'_j \oplus E''_j
\end{align*}

is an isomorphism.

\textbf{Remark 2.5.} Note that a subchain $E'$ of an ILB- or sc-chain $E$ is uniquely determined by the closed linear subspace $E'_0 \subseteq E_0$, for $E'_j = (\iota^0_j)^{-1}(E'_0)$ for all $j \in \mathbb{N}_0$ and $\| \cdot \|'_j$ and $\iota'_j$ are simply defined via restriction.

\textbf{Example 2.13.} Let $E$ be an ILB- or sc-chain and let $C \subseteq E_\infty$ be a finite dimensional subspace (as vector spaces). Then $E' := ((C, \| \cdot \|_C), \text{id}_C)_{j \in \mathbb{N}_0}$ is a split subchain of $E$. For a proof, see \cite{HWZ07}, Proposition 2.7.

\textbf{Example 2.14.} Let $E$ be an ILB- or sc-chain and let $E'$ be a subchain of $E$. If $k \subseteq \mathbb{N}_0$ is a strictly monotone increasing sequence, then $E^k$ is a subchain of $E^k$.

\textbf{Example 2.15.} Let $E$ and $E'$ be ILB- or sc-chains and let $T : E \to E'$ be a continuous linear operator. Let furthermore $\tilde{E}'$ be a subchain of $E'$. Then for all $j \in \mathbb{N}_0$, $\tilde{E}_j := T_j^{-1}(E'_j)$ is a closed linear subspace of $E_j$ and from $\iota'_j \circ T_{j+1} = T_j \circ \iota_j : E_{j+1} \to E'_j$ it follows that $\iota^{-1}_j \left(T_j^{-1}(\tilde{E}_j)\right) = T_{j+1}^{-1} \left(\iota'_j \circ T_{j+1}(\tilde{E}_j)\right) = T_{j+1}^{-1} \left(\tilde{E}_{j+1}\right)$.

Unfortunately, one cannot a priori guarantee that $\tilde{E}_\infty$ lies dense in $\tilde{E}_j$ for all $j \in \mathbb{N}_0$, so the $\left(\tilde{E}_j\right)_{j \in \mathbb{N}_0}$ need not form a subchain of $E$. 

\textbf{Example 2.15.}
Example 2.16. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection. Also let $l = (l_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ be a strictly monotone increasing sequence and let $m \in \mathbb{N}_0$ and $1 < p < \infty$ be s.t. $l_0p > n$ and $mp > n$.

For a closed submanifold $C \subseteq \Sigma$, by the Sobolev trace theorem, there exists a continuous linear operator

$$\mathcal{W}^{l_1+m,p}(F) \to \mathcal{W}^{l,p}(F|_C)$$

between sc-chains from Example 2.1.

There is then a well-defined subchain $\mathcal{W}^{l_1+m,p}(F; C)$, defined via the closed linear subspaces

$$W^{l_1+m,p}(F; C) := \{ u \in W^{l_1+m,p}(F) | u|_C = 0 \}.$$

Analogously, for an arbitrary strictly monotone increasing sequence $l = (l_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0$, there exists a continuous linear operator

$$\Gamma^{l_1}(F) \to \Gamma^{l_1}(F|_C)$$

between sc-chains from Example 2.2.

There is then a well-defined subchain $\Gamma^{l_1}(F; C)$, defined via the closed linear subspaces

$$\Gamma^{l_1}(F; C) := \{ u \in \Gamma^{l_1}(F) | u|_C = 0 \}.$$

Lemma 2.6. $\text{scChains}^k$ is an additive category.

Proof. This is a straightforward verification of the axioms, showing that the biproduct from Definition 2.8 is indeed a biproduct.

Definition 2.9. Let $E$ be an $\text{sc}$-Fréchet space. A closed linear subspace $E' \subseteq E$ is called an $\text{sc}$-subspace if there exists a compatible $\text{sc}$-structure $(E, \phi)$ on $E$ and a subchain $E'$ of $E$ s.t. $\phi(E'_\infty) = E'$.

Lemma 2.7. Let $E$ be an $\text{sc}$-Fréchet space and let $E' \subseteq E$ be a closed linear subspace. The following are equivalent:

1. $E'$ is an $\text{sc}$-subspace.

2. For every compatible $\text{sc}$-structure $(E, \phi)$ on $E$ there exists a shift $k \subseteq \mathbb{N}_0$ s.t. for the compatible $\text{sc}$-structure $(E^k, \phi)$ on $E$, the shifted sc-chain $E^k$ has a subchain $E'$ with $\phi(E^k_\infty) = E'$.

Proof. The direction 2 $\Rightarrow$ 1 is trivial.

In the other direction, let $(\tilde{E}, \tilde{\phi})$ be a compatible $\text{sc}$-structure on $E$. By Remark 2.5 it suffices to find $k_0 \in \mathbb{N}_0$ and a closed linear subspace $E'_{k_0} \subseteq E_{k_0}$ s.t. $\phi \left( (i_{k_0}^{-1}(E'_{k_0})) \right) = E'$.

Because $E'$ is an $\text{sc}$-subspace, there exists a compatible $\text{sc}$-structure $(\tilde{E}, \tilde{\phi})$ on $E$ and a subchain $\tilde{E}'$ of $\tilde{E}$ s.t. $\tilde{\phi}(E'_\infty) = E'$. By compatibility of $(E, \phi)$ and $(\tilde{E}, \tilde{\phi})$
there exists a strictly monotone increasing sequence \( k \subseteq \mathbb{N}_0 \) and a continuous linear operator \( J : \mathbb{R}^k \to \mathbb{R} \) s. t. \( J_\infty^{-1} = \phi^{-1} \circ \phi \). Define \( E'_{k_0} := J_0^{-1}(E'_0) \subseteq E'_{k_0} = E_{k_0} \). Then
\[
\phi \left( (i_{k_0}^\infty)^{-1}(E'_{k_0}) \right) = \phi \left( (i_{k_0}^\infty)^{-1}(J_0^{-1}(E'_0)) \right)
= \phi \left( (J_0 \circ i_{k_0}^\infty)^{-1}(E'_0) \right)
= \phi \left( J_\infty^{-1}(E'_0) \right)
= \phi \circ \phi^{-1} \circ \phi \left( E'_0 \right)
= E'.
\]

**Definition 2.10.** Let \( E^1, \ldots, E^k \), for some \( k \in \mathbb{N}_0 \), be \( \mathbb{R} \)-Fréchet spaces. The direct sum (also: biproduct, direct product) of the \( E^i \), \( i = 1, \ldots, k \), is the \( \mathbb{R} \)-Fréchet space \( E^1 \oplus \cdots \oplus E^k \) where the underlying Hausdorff locally convex topological vector space is \( E^1 \times \cdots \times E^k \) with the product topology and the equivalence class of compatible \( \mathbb{R} \)-structures on \( E^1 \times \cdots \times E^k \) is generated by the following \( \mathbb{R} \)-structures on \( E^1 \times \cdots \times E^k \):

For each \( i = 1, \ldots, k \) pick a compatible \( \mathbb{R} \)-structure \((E^i, \phi^i)\) on \( E^i \). Then \( \mathbb{E} := \mathbb{E}^1 \oplus \cdots \oplus \mathbb{E}^k \) is an \( \mathbb{R} \)-chain and \( \phi := \phi^1 \times \cdots \times \phi^k : \mathbb{E}_\infty = E^1_\infty \times \cdots \times E^k_\infty \to E^1_\infty \times \cdots \times E^k_\infty \) is a homeomorphism. \((\mathbb{E}, \phi)\) defines an \( \mathbb{R} \)-structure on \( E^1 \times \cdots \times E^k \) and for any two such choices the resulting \( \mathbb{R} \)-structures are compatible by a straightforward verification. The resulting equivalence class of \( \mathbb{R} \)-structures on \( E^1 \times \cdots \times E^k \) then defines the \( \mathbb{R} \)-Fréchet space \( E^1 \oplus \cdots \oplus E^k \).

It comes with the morphisms
\[
P^i : E^1 \oplus \cdots \oplus E^k \to E^i
\]
\[
P^i(e^1, \ldots, e^k) := e^i \quad \forall (e^1, \ldots, e^k) \in E^1 \times \cdots \times E^k
\]
\[
J^i : E^1 \to E^1 \oplus \cdots \oplus E^k
\]
\[
J^i(e) := (0, \ldots, 0, e, 0, \ldots, 0) \in E^1 \oplus \cdots \oplus E^k \quad \forall e \in E^i
\]
for \( i = 1, \ldots, k \), called the projection and injection morphisms, respectively.

**Example 2.17.** Let \((\Sigma, g)\) be a closed \( n \)-dimensional Riemannian manifold, let \( C \subseteq \Sigma \) be a closed submanifold and let \( \pi : F \to \Sigma \) be a real (or complex) vector bundle equipped with a euclidean (or hermitian) metric and metric connection.

By Example 2.16 there is a well-defined morphism
\[
W(F) \to W(F|_C)
\]
\[
\Gamma(F) \ni u \mapsto u|_C \in \Gamma(F|_C)
\]

between \( \mathbb{R} \)-Fréchet spaces from Example 2.9 and also a well-defined \( \mathbb{R} \)-subspace \( W(F; C) \subseteq W(F) \) with underlying topological vector space \( \Gamma(F; C) := \{ u \in \Gamma(F) \mid u|_C = 0 \} \).

Now let \( T_{\mathbb{C}} \) be a metric tubular neighbourhood of \( C \), i.e. \( T_{\mathbb{C}} \) is the diffeomorphic image of an \( \varepsilon \)-neighbourhood, for some \( \varepsilon > 0 \), of the zero section in \( T_{\mathbb{C}}^1 \Sigma := \{ \xi \in T_{\mathbb{C}} \Sigma \mid z \in C, (\xi, v) = 0 \quad \forall v \in T_z C \} \)

under the map
\[
T_{\mathbb{C}}^1 \Sigma \to \Sigma
\]
\[
\xi \mapsto \exp(\xi).
\]
It comes with a canonical projection $\pi^T_C : T_C \to C$ which corresponds to the vector bundle projection $T^\perp_C \Sigma \to C$ under the above diffeomorphism and a map $\rho : T_C \to [0, \varepsilon)$, $\rho(\exp_z(\xi)) := |\xi|$. Furthermore, there is a canonical identification

\[
(\pi^T_C)\ast F|_C \to F|_{T_C}
\]

\[
(e_z, \exp_z(\xi)) \mapsto P_z e_z,
\]

where $P_z : F \to F_{\exp_z(\xi)}$ denotes parallel transport along the geodesic $[0, 1] \to \Sigma$, $t \mapsto \exp_z(t\xi)$.

Given a smooth map $\phi : \mathbb{R} \to \mathbb{R}$ s.t. $\phi(t) \equiv 1$ for $t \leq \varepsilon/3$ and $\phi(t) \equiv 0$ for $t \geq 2\varepsilon/3$, using the above one can define

\[
\Phi : \Gamma(F) \to \Gamma(F|_C) \times \Gamma(F; C)
\]

\[
u \mapsto (\nu|_C, \nu - (\phi \circ \rho) \cdot (\pi^T_C)\ast \nu|_C)
\]

and

\[
\Psi : \Gamma(F|_C) \times \Gamma(F; C) \to \Gamma(F)
\]

\[
(u_0, u_1) \mapsto u_1 + (\phi \circ \rho) \cdot (\pi^T_C)\ast u_0.
\]

These maps are inverse to each other and have extensions

\[
W^{1+m,p}(F) \to W^{1,p}(F|_C) \oplus W^{1,p}(F; C)
\]

and

\[
W^{1,p}(F|_C) \oplus W^{1,p}(F; C) \to W^{1,p}(F),
\]

where $l \subseteq \mathbb{N}_0$ is a strictly monotone increasing sequence, $m \in \mathbb{N}_0$ and $1 < p < \infty$ are s.t. $lOp > n$ and $mp > n$.

This shows that there is an $\mathfrak{sc}$-splitting

\[
W(F) \cong W(F|_C) \oplus W(F; C).
\]

In just the same way as above, $\Phi$ and $\Psi$ also have extensions

\[
\Gamma^1(F) \to \Gamma^1(F|_C) \oplus \Gamma^1(F; C)
\]

and

\[
\Gamma^1(F|_C) \oplus \Gamma^1(F; C) \to \Gamma^1(F),
\]

where $l \subseteq \mathbb{N}_0$ is any strictly monotone increasing sequence.

This shows that there is an $\mathfrak{sc}$-splitting

\[
\Gamma(F) \cong \Gamma(F|_C) \oplus \Gamma(F; C)
\]

using the $\mathfrak{sc}$-structures on $\Gamma(F) = W(F)$, etc., defined via chains of continuously differentiable sections.
Lemma 2.8. \( \text{scFrechet}^k \) is an additive category.

Proof. This is a straightforward verification of the axioms, showing that the biproduct from Definition 2.10 is indeed a biproduct.

Definition 2.11. Let \( E \) be an \( \text{sc} \)-Fréchet space. An \( \text{sc} \)-subspace \( E' \) of \( E \) is said to split if there exists another \( \text{sc} \)-subspace \( E'' \) of \( E \) s.t. the canonical morphism

\[
J : E' \oplus E'' \to E \quad (e, e') \mapsto e + e'
\]

is an isomorphism.

Example 2.18. Let \( E \) be an \( \text{sc} \)-Fréchet space and let \( C \subseteq E \) be a finite dimensional subspace (as vector spaces). Then \( C \) is a split \( \text{sc} \)-subspace. This follows immediately from Example 2.13.

2.5 (Strongly) smoothing operators

Definition 2.12. Let \( E \) and \( E' \) be Hausdorff locally convex topological vector spaces and let \( K : E \to E' \) be a continuous linear operator. \( K \) is called compact if there exists a neighbourhood \( U \subseteq E \) of 0 s.t. the closure \( \overline{K(U)} \) of \( K(U) \) in \( E' \) is compact.

Remark 2.6. If \( K : E \to E' \) is a compact linear operator and \( A \subseteq E \) is a bounded subset, then \( \overline{K(A)} \subseteq E' \) is compact.

For by definition of \( K \) compact there exists a neighbourhood \( U \subseteq E \) of 0 s.t. \( \overline{K(U)} \subseteq E' \) is compact and by definition of \( A \) bounded there exists \( c \in k \) s.t. \( A \subseteq cU \). Hence \( K(A) \subseteq cK(U) = cK(U) \) and \( cK(U) = cK(U) \) is compact. So \( \overline{K(A)} \subseteq cK(U) \) is compact as a closed subset of a compact set.

Lemma 2.9. Let \( E, E' \) and \( E'' \) be Fréchet spaces and let \( K, K' : E \to E', S : E'' \to E \) and \( T : E \to E'' \) be continuous linear operators with \( K \) and \( K' \) compact. Let furthermore \( \lambda, \mu \in k \). Then the following operators are compact:

1. \( K \circ S : E'' \to E' \).
2. \( T \circ K : E \to E'' \).
3. \( \lambda K + \mu K' : E \to E' \).

Proof. See [Osb14], Chapter 5, Exercises 26 and 27.

Definition 2.13. Let \( E \) and \( E' \) be sc-chains.

A continuous linear operator \( K : E \to E' \) is called

1. smoothing if there exists a strictly monotone increasing sequence \( k \subseteq \mathbb{N}_0 \) and a continuous linear operator \( K_k : E \to E^{k} \) s.t. \( k > \text{id} \) and \( k = \text{id} \circ K_k \).
2. strongly smoothing if for all strictly monotone increasing sequences \( k \subseteq \mathbb{N}_0 \) there exists a continuous linear operator \( K_k : E \to E^{k} \) s.t. \( k = \text{id} \circ K_k \).
Example 2.19. Let $\mathbb{E}$ be an sc-chain and let $k \subseteq \mathbb{N}_0$ be strictly monotone increasing sequences. If $k > \text{id}$, then $\|k\| : \mathbb{E}^k \to \mathbb{E}$ is smoothing but not strongly smoothing.

Remark 2.7. Note that a continuous linear operator $K : \mathbb{E} \to \mathbb{E}'$ is smoothing if there exists a continuous linear operator $\tilde{K} : \mathbb{E} \to \mathbb{E}'$, where $1 := (k + 1)_{k \in \mathbb{N}_0}$. Or in other words, $K_j = t_j \circ \tilde{K}_j$, for a continuous linear operator $\tilde{K}_j : E_j \to E'_{j+1}$, for all $j \in \mathbb{N}_0$.

For one direction is trivial and in the other direction, $k > \text{id}$ means precisely that $k \geq 1$. So given $K : \mathbb{E} \to \mathbb{E}'$, define $\hat{K} := \|k\| : \mathbb{E} \to \mathbb{E}'$.

Lemma 2.10. Let $\mathbb{E}$ and $\mathbb{E}'$ be sc-chains and let $K : \mathbb{E} \to \mathbb{E}'$ be a continuous linear operator. The following are equivalent:

1. $K$ is strongly smoothing.
2. For every shift $k \subseteq \mathbb{N}_0$ there exists a continuous linear operator $\hat{K}_k : \mathbb{E} \to \mathbb{E}'$ s.t. $K = \|k\| \circ \hat{K}_k$.
3. There exists a continuous linear operator $\mathcal{K}_0 : E_0 \to E'_{0\infty}$ s.t.
   \[
   K_j = t_j^\infty \circ \mathcal{K}_0 \circ t_j^0 \quad \forall j \in \mathbb{N}_0 \cup \{\infty\}.
   \]
4. $K^\infty : (E_{\infty}, \|\cdot\|_{E_{\infty}}) \to E'_{\infty}$ is continuous, i.e. $K^\infty : (E_{\infty}, \|\cdot\|_{E_{\infty}}) \to (E'_{\infty}, \|\cdot\|_{E'_{\infty}})$ is continuous for all $j \in \mathbb{N}_0$.

In particular there is a $1-1$ correspondence between strongly smoothing operators $K : \mathbb{E} \to \mathbb{E}'$ and continuous linear operators $\mathcal{K}_0 : E_0 \to E'_{0\infty}$.

Proof. I will show $[1] \Rightarrow [2] \Rightarrow [4] \Rightarrow [3] \Rightarrow [1]$.

1. $[1] \Rightarrow [2]$ Clear.

2. $[2] \Rightarrow [4]$ Given $j \in \mathbb{N}_0$, let $j := (k + j)_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0$. By assumption there exists a continuous linear operator $\hat{K}_j : \mathbb{E} \to \mathbb{E}'$ s.t. $K = \|j\| \circ \hat{K}_j$. Then $K^\infty = (\hat{K}_j)^\infty$ and $(\hat{K}_j)_0 = (\hat{K}_j)_0 |_{E_{\infty}}$. Now $\mathcal{K}_j : E_0 \to E'_{0\infty} = E'_{j\infty}$ is continuous and hence $K^\infty : (E_{\infty}, \|\cdot\|_{E_{\infty}}) \to (E'_{\infty}, \|\cdot\|_{E'_{\infty}})$ is continuous.

4. $[4] \Rightarrow [3]$ For each $j \in \mathbb{N}_0$ let $D_j \in (0, \infty)$ be the operator norm of $K^\infty : (E_{\infty}, \|\cdot\|_{E_{\infty}}) \to (E'_{\infty}, \|\cdot\|_{E'_{\infty}})$ and set $C_j := \max\{1, D_j\} \in [1, \infty)$. Then $d : E'_{\infty} \times E'_{\infty} \to [0, \infty)$
   \[
   (x, y) \mapsto \sum_{j=0}^{\infty} \frac{1}{C_j^{2^j}} \min\{1, \|x - y\|_j^j\}
   \]
   is a complete metric on $E'_{\infty}$ inducing the given topology on $E'_{\infty}$. And for $x, y \in E_{\infty}$,
   \[
   d(K^\infty x, K^\infty y) = \sum_{j=0}^{\infty} \frac{1}{C_j^{2^j}} \min\{1, \|K^\infty(x - y)\|_j^j\}
   \leq \sum_{j=0}^{\infty} \frac{1}{C_j^{2^j}} \min\{1, D_j \|x - y\|_0\}
   \leq \sum_{j=0}^{\infty} \frac{D_j}{C_j^{2^j}} \|x - y\|_0
   \leq 2\|x - y\|_0.
   \]

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So $K_\infty : (E_\infty, \| \cdot \|_{E_\infty}) \to (E_\infty', d)$ is Lipschitz and hence has a unique continuous completion $\tilde{K}_0 : E_0 \to E_\infty'$. It is straightforward to see (by the usual density argument) that this $\tilde{K}_0$ satisfies 2.

Given a strictly monotone increasing sequence $k \subseteq N_0$, define $\tilde{K}_j := \cup_{j=0}^{\infty} \circ \tilde{K}_0 \circ \iota_j^k : E_j \to E_j'$. The operator $\tilde{K}_k : E \to E'_{\infty}$ defined by the $\tilde{K}_j$ then has the required properties.

Proof.

Corollary 2.1. Let $E$ and $E'$ be sc-chains and let $k \subseteq N_0$ be strictly monotone increasing sequences with $k_0 = 0$. If $K : E^k \to E'$ is a strongly smoothing continuous linear operator, then there exists a strongly smoothing continuous linear operator $\tilde{K} : E \to E'$ s.t. $K = \Pi^k \circ \tilde{K}_k$.

Corollary 2.2. Let $E$ and $E'$ be sc-chains and let $K : E \to E'$ be a continuous linear operator. If $K$ is strongly smoothing, then $K_\infty : E_\infty \to E'_\infty$ is a compact operator.

Example 2.20. Let $E$ be an sc-chain. $\id_E : E \to E$ is strongly smoothing iff $E$ is the constant sc-chain on a finite dimensional vector space $E_\infty$.

Remark 2.8. The analogous statement of the previous corollary for smoothing instead of strongly smoothing operators is evidently false, for the operator $\Pi^1 : E^1 \to E$ is smoothing, but $I^1_\infty : E^1_\infty = E_\infty \to E_\infty$ is the identity, which is compact only if $\dim E_\infty < \infty$.

This is one of the reasons the notion of a smoothing operator does not transfer well to weak morphisms and $\sr$-Fréchet spaces (the operators $\Pi^k$ all induce the identity on the $\sr$-Fréchet space $E_\infty$).

Another reason is that if $K : E \to E'$ is smoothing and $k \subseteq N_0$ is a strictly monotone increasing sequence, then $K^k$ need not be smoothing. The analogous statement for strongly smoothing on the other hand does hold, as is shown in the following lemma.

Lemma 2.11. Let $E$, $E'$ and $E''$ be sc-chains and let $K, K' : E \to E'$, $S : E'' \to E$ and $T : E' \to E''$ be continuous linear operators. Let furthermore $\lambda, \mu \in k$ and let $l \subseteq N_0$ be a strictly monotone increasing sequence.

1. If $K$ and $K'$ are smoothing, then so are
   (a) $\lambda K + \mu K' : E \to E'$,
   (b) $K \circ S : E'' \to E'$ and
   (c) $T \circ K : E \to E''$.

2. If $K$ and $K'$ are strongly smoothing, then so are
   (a) $\lambda K + \mu K' : E \to E'$,
   (b) $K \circ S : E'' \to E'$,
   (c) $T \circ K : E \to E''$ and
   (d) $K^l : E^l \to E^l$.

Proof. 1. Because $K$ is smoothing, there exists a strictly monotone increasing sequence $k \subseteq N_0$ with $k > \id$ and a continuous linear operator $\tilde{K} : E \to E'_{\infty}$ with $K = \Pi^k \circ \tilde{K}$.
Proof. I will show 1.

(a) Straightforward.

(b) \( K \circ S = {}^1 k \circ K \circ S = {}^1 k \circ K \), where \( K := K \circ S : E' \to E'k \). So \( K \circ S \) is smoothing.

(c) \( T \circ K = T \circ {}^1 k \circ K = {}^1 k \circ Tk \circ K = {}^1 k \circ \overline{K} \), where \( \overline{K} := Tk \circ \overline{K} : E \to E'k \). So \( T \circ K \) is smoothing.

2. Let \( k \subseteq N_0 \) be any strictly monotone increasing sequence. Because \( K \) is strongly smoothing, there exists a continuous linear operator \( K_k : E \to E'k \) with \( K = {}^1 k \circ K_k \).

(a) Straightforward.

(b) \( K \circ S = {}^1 k \circ K_k \circ S = {}^1 k \circ \overline{K}_k \), where \( \overline{K}_k := K \circ S : E'' \to E'k \). So \( K \circ S \) is strongly smoothing.

(c) \( T \circ K = T \circ {}^1 k \circ K_k = {}^1 k \circ Tk \circ K_k = {}^1 k \circ \overline{K}_k \), where \( \overline{K}_k := Tk \circ \overline{K}_k : E \to E'k \). So \( T \circ K \) is strongly smoothing.

(d) Because \( K \) is strongly smoothing, there exists a continuous linear operator \( \tilde{K}_{l,k} : E \to E'\tilde{k} = \left( (E')^l \right)^k \) with \( k = {}^1\tilde{k} \circ \tilde{K}_{l,k} \). Define \( \tilde{K}_{k}^l := \tilde{K}_{l,k} \circ {}^1 l : E^l \to \left( (E')^l \right)^k \). Then \( \tilde{K}_{k}^l = {}^1\tilde{k} \circ \tilde{K}_{k}^l : E^l \to E^l \). For if \( \tilde{K} := {}^1\tilde{k} \circ \tilde{K}_{k}^l \), then \( \tilde{K}_\infty = K_{\tilde{k}}^1 \). \( \square \)

Definition 2.14. Let \( E \) and \( E' \) be sc-chains. A weak morphism \( K : E_\infty \to E'_\infty \) is called strongly smoothing if it has a strongly smoothing extension \( K : E^k \to E'^l \) for some strictly monotone increasing sequences \( k, l \subseteq N_0 \).

Remark 2.9. Note that by Corollary 2.2, a strongly smoothing weak morphism \( K : E_\infty \to E'_\infty \) between sc-chains \( E \) and \( E' \) is a compact operator between the Fréchet spaces \( E_\infty \) and \( E'_\infty \).

Lemma 2.12. Let \( E \) and \( E' \) be sc-chains and let \( K : E_\infty \to E'_\infty \) be a weak morphism. Then the following are equivalent:

1. \( K \) is strongly smoothing.

2. \( K \) has a strongly smoothing extension \( K : E^k \to E'^l \) for some strictly monotone increasing sequence \( k \subseteq N_0 \).

3. For every extension \( K : E^k \to E'^l \) of \( K \) there exists a shift \( m \subseteq N_0 \) s. t. the shifted extension \( K^m : E^{k+m} \to E'^{l+m} \) of \( K \) is strongly smoothing.

Proof. I will show \( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \) and \( 3 \Rightarrow 1 \)

\( 1 \Rightarrow 2 \). If \( K : E^k \to E'^l \) is a strongly smoothing extension of \( K \), then by Lemma 2.11 so is \( {}^l \circ K : E^k \to E' \).

\( 2 \Rightarrow 1 \). Trivial.

\( 3 \Rightarrow 1 \). Because \( K \) is strongly smoothing, one can find a strongly smoothing extension \( \overline{K} : E^n \to E' \) of \( K \). By Lemma 2.10 this is equivalent to

\[
K : \left( E^n_\infty, \| | \cdot |_n_{E^n_\infty} \right) = \left( E_\infty, \| | \cdot |_{n_\infty} \right) \to \left( E'_\infty, \| | \cdot |_{l'_\infty} \right)
\]
being continuous for all \(j \in \mathbb{N}_0\). Let \(n_0 := \min\{j \mid k_j \geq n_0\}\) and \(m := (m_0 + j)_{j \in \mathbb{N}_0}\). Then
\[
\begin{align*}
K : (E^\text{kom}_\infty, \| \cdot \|_{0}^{\text{kom}} | E^\text{kom}_\infty) & \to (E^\infty, \| \cdot \|_{n_0}^{\text{kom}} | E^\infty) \\
E^\prime_\infty & \to (E^\text{kom}_\infty, \| \cdot \|_{0}^{\text{kom}} | E^\text{kom}_\infty)
\end{align*}
\]
is continuous for all \(j \in \mathbb{N}_0\), because since \(k_{m_0} \geq n_0\) by definition,
\[
\text{id}_{E^\infty} : (E^\infty, \| \cdot \|_{n_0}^{\text{kom}} | E^\infty) \to (E^\infty, \| \cdot \|_{n_0}^{\text{kom}} | E^\infty)
\]
is continuous. And by Lemma 2.10, \(K^m : E^\text{kom}_\infty \to E^\text{kom}_\infty\) is strongly smoothing.

\[ \implies \text{If } K : E^k \to E^l \text{ is an extension of } K, \text{ then so is } K^m : E^\text{kom}_m \to E^\text{kom}_m. \]

\[ \square \]

**Lemma 2.13.** Let \(E, E'\) and \(E''\) be sc-chains, let \(K, K' : E^\infty \to E'_\infty\), \(S : E''_\infty \to E^\infty\) and \(T : E'_\infty \to E''_\infty\) be weak morphisms and let \(\lambda, \mu \in \mathbb{k}\). If \(K\) and \(K'\) are strongly smoothing, then so are

1. \(\lambda K + \mu K' : E^\infty \to E'_\infty\),
2. \(K \circ S : E''_\infty \to E^\infty\) and
3. \(T \circ K : E^\infty \to E''_\infty\).

**Proof.** This is a corollary to Lemma 2.11. For if \(K : E^k \to E', S : E'^m \to E\) and \(T : E''^m \to E''\) are extensions of \(K, S\) and \(T\), respectively, then \(K \circ S^k\) and \(T \circ K^m\) are extensions of \(K \circ S\) and \(T \circ K\), respectively.

And similarly for \(\lambda K + \mu K'\): If \(K : E^k \to E'\) and \(K' : E^l \to E'\) are extensions of \(K\) and \(K'\), respectively, then \(\lambda K \circ S^m + \mu K' \circ \phi^{m} \) is an extension of \(\lambda K + \mu K'\), where \(m := \max\{k_j, l_j\}_{j \in \mathbb{N}_0}\).

\[ \square \]

**Definition 2.15.** Let \(E\) and \(E'\) be \(\mathcal{P}\)-Fréchet spaces. A morphism \(K : E \to E'\) is called strongly smoothing if there exist compatible \(\mathcal{P}\)-structures \((\mathcal{E}, \phi)\) and \((\mathcal{E}', \phi')\) on \(E\) and \(E'\), respectively, s.t. \(K^\infty := \phi'^{-1} \circ K \circ \phi : E^\infty \to E'_\infty\) defines a strongly smoothing weak morphism.

**Remark 2.10.** By Remark 2.9 and Lemma 2.9, a strongly smoothing morphism \(K : E \to E'\) between \(\mathcal{P}\)-Fréchet spaces is a compact operator between the underlying Fréchet spaces.

**Corollary 2.3.** Let \(E, E'\) and \(E''\) be \(\mathcal{P}\)-Fréchet spaces.

1. A continuous linear operator \(K : E \to E'\) defines a strongly smoothing morphism of \(\mathcal{P}\)-Fréchet spaces iff for any pair of compatible \(\mathcal{P}\)-structures \((\mathcal{E}, \phi)\) and \((\mathcal{E}', \phi')\) on \(E\) and \(E'\), respectively, \(K^\infty := \phi'^{-1} \circ K \circ \phi\) defines a strongly smoothing weak morphism.

2. Let \(K, K' : E \to E'\) be strongly smoothing morphisms and let \(\lambda, \mu \in \mathbb{k}\). Then \(\lambda K + \mu K' : E \to E'\) is a strongly smoothing morphism as well.
3. Let $K : E \rightarrow E'$, $S : E'' \rightarrow E$ and $T : E' \rightarrow E''$ be morphisms with $K$ strongly smoothing. Then $K \circ S : E'' \rightarrow E$ and $T \circ K : E \rightarrow E''$ are strongly smoothing morphisms as well.

Proof. This is a corollary to Lemma 2.13.

Corollary 2.4. Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces and let $(\mathfrak{E}, \phi)$ be a compatible $\mathfrak{sc}$-structure on $E$. Then there is a $1:1$ correspondence between the set of strongly smoothing morphisms $K : E \rightarrow E'$ and

$$
\lim_{i} L_{c}(E_{i}, E'),
$$

where the limit is taken over the direct system $(L_{c}(E_{i}, E'))_{i \in \mathbb{N}_{0}}$ with structure maps given by

$$
L_{c}(E_{i}, E') \rightarrow L_{c}(E_{j}, E') \quad K \mapsto K \circ \iota_{j}^{i},
$$

for $i, j \in \mathbb{N}_{0}, i \leq j.$

The correspondence is given by the direct system of maps

$$
L_{c}(E_{i}, E') \rightarrow L_{c}(E_{i}, E) \quad K \mapsto K \circ \iota_{\infty}^{i} \circ \phi^{-1}.
$$

Proof. This is a corollary to Lemmas 2.10 and 2.12.

Remark 2.11. Above and below, when writing $L_{c}(E_{i}, E')$, $E'$ is considered as just its underlying Fréchet space.

Example 2.21. 1. Let $E$ be an $\mathfrak{sc}$-Fréchet space. $\text{id}_{E} : E \rightarrow E$ is strongly smoothing iff $E$ is finite dimensional.

2. Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces and let $F : E \rightarrow E'$ be a morphism. If there exists a finite dimensional $\mathfrak{sc}$-Fréchet space $C$ and morphisms $F_{1} : E \rightarrow C$, $F_{2} : C \rightarrow E'$ s.t. $F = F_{2} \circ F_{1}$, then $F$ is strongly smoothing.

Definition 2.16. Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces. Then $\text{Sm}(E, E')$ denotes the space of strongly smoothing morphisms from $E$ to $E'$ together with the topology defined by the direct limit of topological spaces

$$
\text{Sm}(E, E') \cong \lim_{i} L_{c}(E_{i}, E'),
$$

for some compatible $\mathfrak{sc}$-structure $(\mathfrak{E}, \phi)$ on $E$, as in Corollary 2.4.

Remark 2.12. By Corollary 2.4 and the definition of the direct limit topology, $\text{Sm}(E, E')$ is a Hausdorff locally convex topological vector space that comes with a canonical continuous inclusion $\text{Sm}(E, E') \hookrightarrow L_{c}(E, E')$. But a priori the topology on $\text{Sm}(E, E')$ is stronger than the subspace topology induced by the bounded-open topology on $L_{c}(E, E')$.

Example 2.22. Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces with $E$ finite dimensional. Then $\text{Sm}(E, E') \cong L_{c}(E, E')$ as topological spaces.
Proposition 2.1. Let $E$ and $E'$ be $\mathfrak{F}$-Fréchet spaces and let $F : E \to E'$ be an isomorphism. Given any compatible $\mathfrak{F}$-structure $(\mathfrak{E}, \phi)$ on $E$, there exists $i_0 \in \mathbb{N}_0$ and for every $i \geq i_0$ a neighbourhood $U_i \subseteq L_c(E_i, E_i') \subseteq \text{Sm}(E, E')$ of 0 s. t. $F + K : E \to E'$ is an isomorphism for all $K \in U_i$. Furthermore, $(F + K)^{-1} = F^{-1} + K'$ for a unique $K' \in \text{Sm}(E', E)$.

Proof. By the same arguments used before, one can assume that $(\mathfrak{E}, \phi)$ and $(\mathfrak{E}', \phi')$ are compatible $\mathfrak{F}$-structures on $E$ and $E'$, respectively. Let $\mathcal{K}$ be a sequence in the Banach space $U_i$ for some strictly monotone increasing sequence $l \subseteq \mathbb{N}_0$, with $F = \phi' \circ F' \circ \phi^{-1}$, $F^{-1} = \phi \circ F'_\infty \circ \phi'^{-1}$, $F \circ \mathfrak{F}' = \mathfrak{F}_I$ and $\mathfrak{F}' \circ \mathfrak{F}_I = 1$. To simplify notation, identify $E$ with $E_\infty$ and $E'$ with $E'_\infty$. For $i \in \mathbb{N}_0$, let

$$U_i := \{ K \in L_c(E_i, E'_\infty) \mid \| \iota_i \circ F' \circ K \|_{L_c(E_i, E_i)} < 1 \}.$$ 

Then $U_i \subseteq L_c(E_i, E'_\infty)$ is an open subset. Given $K \in U_i$, define (via geometric series in the Banach space $L_c(E_i, E_i)$)

$$K' := -F'_\infty \circ K \circ \sum_{k=0}^{\infty} (-\iota_i \circ F'_\infty \circ K)^k \circ F'_i : E'_i \to E_\infty.$$

$\mathcal{K} := K \circ \iota_i \circ \phi^{-1} : E \to E'$ and $\overline{\mathcal{K}} := K' \circ (\iota_i')^\infty \circ \phi'^{-1} : E' \to E$ define strongly smoothing operators. Two completely straightforward calculations, done pointwise over $E$ and $E'$, show that $(F' + \overline{\mathcal{K}}) \circ (F + \mathcal{K}) = \text{id}_E$ and $(F + \mathcal{K}) \circ (F' + \overline{\mathcal{K}}) = \text{id}_{E'}$, respectively.

That $\mathcal{K}$ is unique is immediate from $\overline{\mathcal{K}} = (F + \mathcal{K})^{-1} - F^{-1}$. □

2.6 $\mathfrak{F}$-Fredholm operators

Proposition 2.2. Let $E$ and $E'$ be Fréchet spaces and let $F : E \to E'$ be a continuous linear operator. Then the following are equivalent:

1. $F$ is invertible modulo compact operators, i.e. there exist continuous linear operators $F' : E' \to E$, $K : E \to E$ and $K' : E' \to E'$ with $K$ and $K'$ compact s. t.

$$F' \circ F = \text{id}_E + K$$

$$F \circ F' = \text{id}_{E'} + K'.$$

2. $\dim \ker F < \infty$, $\text{im} F \subseteq E'$ is closed and $\dim \text{coker} F < \infty$.

3. $\dim \ker F < \infty$ and $\dim \text{coker} F < \infty$.

Proof. I will show $[1] \Rightarrow [2] \Rightarrow [3] \Rightarrow [1]$.

$[1] \Rightarrow [3]$. $F' \circ F = \text{id}_E + K$ implies $F' \circ F|_{\ker F} = \text{id}_E|_{\ker F} + K|_{\ker F} = \text{id}_{\ker F} + K|_{\ker F}$, i.e. $\text{id}_{\ker F} = -K|_{\ker F}$ is a compact operator. In other words ker $F$ is a locally compact topological vector space, hence finite dimensional (cf. [Osb14], Corollary 2.11). $F \circ F' = \text{id}_{E'} + K'$, i.e. $-K' = \text{id}_{E'} - F \circ F'$ implies that $-K'$ maps im $F$ to im $F$ and hence the well-defined map $\tilde{K} : \text{ker} F \to \text{ker} F$ which $-K'$ induces is compact and equal to the identity. Hence as before $F$ is finite dimensional.
ker \( F \subseteq E \) is closed, and hence by replacing \( E \) with \( E/\ker F \) (which is a Fréchet space by [Osb14] Corollary 1.36 and Corollary 3.36) and \( F \) by the induced map \( E/\ker F \to E' \) one can assume that \( F \) is injective. Let \( C \subseteq E' \) be a finite dimensional, hence in particular closed, complement of \( \text{im} F \). The continuous linear map

\[
T : E \oplus C \to E'
\]

\[
(e, c) \mapsto F(e) + c
\]

is an isomorphism of vector spaces hence by the open mapping theorem ([Osb14] Theorem 4.35) it is a homeomorphism. Thus \( \text{im} F = T(E \oplus \{0\}) \) is closed.

ker \( F \) is finite dimensional hence it has a closed complement \( X \subseteq E \). Since \( \text{coker} F \) is finite dimensional, \( \text{im} F \) has a finite dimensional, hence closed (cf. [Osb14], Corollary 2.10), complement \( C \subseteq E' \). \( X \), \( \ker F \), \( \text{im} F \) and \( C \) are Fréchet spaces (cf. [Osb14], Corollary 1.36 and Corollary 3.36) and \( E = X \oplus \ker F \), \( E' = \text{im} F \oplus C \). \( F|_X : X \to \text{im} F \) is a continuous linear map that is an isomorphism of vector spaces, hence by the open mapping theorem ([Osb14] Theorem 4.35) it is a homeomorphism. Let \( R := (F|_X)^{-1} \) and define \( F' : E' \to E \) by \( F' := \iota_X^E \circ R \circ \text{pr}_{\text{im} F}^{E'} \), where \( \iota_X^E : X \hookrightarrow E \) is the inclusion and \( \text{pr}_{\text{im} F}^{E'} : E' \to \text{im} F \) is the projection along \( C \). Then \( F' \circ F = 1 - \text{pr}_{\ker F}^E \), where \( \text{pr}_{\ker F}^E : E \to \ker F \) is the projection along \( X \), and \( F \circ F' = 1 - \text{pr}_{C}^{E'} \), where \( \text{pr}_{C}^{E'} : E' \to C \) is the projection along \( \text{im} F \). \( K := -\text{pr}_{\ker F}^E \) and \( K' := -\text{pr}_{C}^{E'} \) are finite rank, hence in particular compact, operators.

Definition 2.17. Let \( X, Y \) be Fréchet spaces

1. If a continuous linear operator \( F : X \to Y \) satisfies any (hence all) of the conditions of Proposition 2.2 then \( F \) is called a Fredholm operator. A continuous linear operator \( F' : Y \to X \) as in Proposition 2.2 is called a Fredholm inverse to \( F \).

2. The number

\[
\text{ind} F := \dim \ker F - \dim \text{coker} F
\]

is called the (Fredholm) index of \( F \).

Proposition 2.3. Let \( E, E' \) and \( E'' \) be Fréchet spaces, let \( F : E \to E' \) and \( F' : E' \to E'' \) be Fredholm operators and let \( K : E \to E' \) be a compact operator.

1. \( F' \circ F : E \to E'' \) is Fredholm with \( \text{ind}(F' \circ F) = \text{ind} F' + \text{ind} F \).

2. \( F + K : E \to E' \) is Fredholm with \( \text{ind}(F + K) = \text{ind} F \).

Proof. 1. That \( F' \circ F \) is Fredholm is immediate from the characterisation as invertible modulo compact operators and Lemma 2.9.

For the index formula, one uses the following result from linear algebra:

Claim. Let

\[
0 = V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} V_{n-1} \xrightarrow{f_{n-1}} V_n = 0
\]

with \( \text{im} f_i \subseteq \ker f_{i+1} \). Then

\[
\text{ind}(F' \circ F) = \text{ind} F' + \text{ind} F.
\]
be an exact sequence of finite dimensional vector spaces. Then
\[ \sum_{j=0}^{n} (-1)^j \dim V_j = 0. \]

**Proof.** Exercise. \( \square \)

The index formula then follows easily by applying this claim to the exact sequence
\[ 0 \rightarrow \ker F \rightarrow \ker(F' \circ F) \rightarrow \ker F' \rightarrow \coker F \rightarrow \coker(F' \circ F) \rightarrow 0, \]
where \( \iota : \ker F \hookrightarrow \ker(F' \circ F) \) is the inclusion and \( q : E' \supseteq \ker F' \rightarrow E'/\text{im} F = \text{coker} F \) and \( q' : \text{coker}(F' \circ F) = E''/\text{im}(F' \circ F) \rightarrow E''/\text{im} F' = (E''/\text{im}(F' \circ F))/(\text{im} F'/\text{im}(F' \circ F)) \) are the quotient maps.

2. See [Osb14], Chapter 5, Exercises 28–31. \( \square \)

**Remark 2.13.** Because obviously for any Fréchet space \( \text{id}_E : E \rightarrow E \) is Fredholm, by the above Lemma any operator of the form \( \text{id}_E + K \), where \( K : E \rightarrow E \) is compact, is Fredholm of index 0. Also, if \( F : E \rightarrow E' \) and \( F' : E' \rightarrow E \) are Fredholm inverses to each other, then \( \text{ind} F = -\text{ind} F' \).

**Definition 2.18.** Let \( E \) and \( E' \) be sc-chains.

1. A continuous linear operator \( F : E \rightarrow E' \) is called **Fredholm** iff \( F \) is invertible modulo strongly smoothing operators, i.e. iff there exist continuous linear operators \( F' : E' \rightarrow E, K : E \rightarrow E \) and \( K' : E' \rightarrow E' \) with \( K \) and \( K' \) strongly smoothing s.t.
\[ F' \circ F = \text{id}_E + K, \]
\[ F \circ F' = \text{id}_{E'} + K'. \]

Such a continuous linear operator \( F' : E' \rightarrow E \) is then called a **Fredholm inverse** of \( F \).

2. The **Fredholm index** \( \text{ind} F \) of a Fredholm morphism \( F : E \rightarrow E' \) is defined as the Fredholm index of \( F_{\infty} \) as a Fredholm operator between the Fréchet spaces \( E_{\infty} \) and \( E'_{\infty} \).

**Remark 2.14.** Note that if \( F : E \rightarrow E' \) is Fredholm and \( F' : E' \rightarrow E \) is a Fredholm inverse to \( F \), then given any strongly smoothing continuous linear operator \( K : E' \rightarrow E, F'' := F' + K \) is a Fredholm inverse to \( F \) as well, by Lemma 2.11.

**Definition 2.19.** Let \( E \) and \( E' \) be sc-chains and let \( F : E \rightarrow E' \) be a continuous linear operator. \( F \) is called

1. **regularising** iff for all \( e \in E_0 \) and \( m \in \mathbb{N}_0 \), if \( F_0(e) \in E'_m \) then \( e \in E_m \).
2. weakly regularising iff there exists an $m_0 \in \mathbb{N}_0$ s. t. for all $e \in E_{m_0}$, if $F_{m_0}(e) \in \ker F_{m_0}$, then $e \in E_{m_0}$.

Equivalently, iff there exists an $m_0 \in \mathbb{N}_0$ s. t. for all $m \geq m_0$ and $e \in E_m$, if $F_m(e) \in E_m'$ then $e \in E_m$.

**Remark 2.15.** If $F : \mathbb{E} \to \mathbb{E}'$ is weakly regularising, then there exists $m_0 \in \mathbb{N}_0$ s. t. $\ker F_j = \ker F_{k} \subseteq E_{m_0}$ for all $j, k \in \mathbb{N}_0$ with $j, k \geq m_0$.

**Proposition 2.4.** Let $\mathbb{E}$ and $\mathbb{E}'$ be sc-chains and let $F : \mathbb{E} \to \mathbb{E}'$ be a continuous linear operator. Then the following are equivalent:

1. $\mathbb{F}$ is Fredholm.
2. $\mathbb{F}$ is invertible modulo smoothing operators, i. e. there exist continuous linear operators $\mathbb{F}' : \mathbb{E}' \to \mathbb{E}$, $\mathbb{K} : \mathbb{E} \to \mathbb{E}$ and $\mathbb{K}' : \mathbb{E}' \to \mathbb{E}'$ with $\mathbb{K}$ and $\mathbb{K}'$ smoothing s. t.

$$
\mathbb{F}' \circ \mathbb{F} = \text{id}_{\mathbb{E}} + \mathbb{K}
$$

$$
\mathbb{F} \circ \mathbb{F}' = \text{id}_{\mathbb{E}'} + \mathbb{K}'.
$$

3. $\mathbb{F}$ is sc-Fredholm in the sense of [Weh12], Definition 3.1, i. e.

   (a) $\mathbb{F}$ is regularising.
   (b) $F_0 : E_0 \to E_0'$ is a Fredholm operator (between Banach spaces).

4. $\mathbb{F}$ is HWZ-Fredholm in the sense of [Weh12], Definition 3.4, i. e.

   (a) $\dim \ker F_0 < \infty$, $\ker F_0 \subseteq E_\infty$ and there exists a subchain $X \subseteq \mathbb{E}$ s. t. $\mathbb{E} = \ker F_0 \oplus X$.

   (b) $\text{im} \mathbb{F} := (\text{im} F_j, \iota)_{|\text{im} F_{j+1}} \in \mathbb{N}_0$ is a well-defined sc-chain and there exists a finite dimensional subspace $C \subseteq E_\infty'$ with $\mathbb{E}' = \text{im} \mathbb{F} \oplus C$.

   (c) For any subchain $X \subseteq \mathbb{E}$ as in (a)

$$
\pi_{\text{im} \mathbb{F}} \circ \mathbb{F} \circ \iota_X : X \to \text{im} \mathbb{F}
$$

is an isomorphism of sc-chains.

**Proof.** I will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

1 $\Rightarrow$ 2. Trivial.

2 $\Rightarrow$ 3. (a) Let $\mathbb{F}'$, $\mathbb{K}$ and $\mathbb{K}'$ be as in 2. One can assume (cf. Remark 2.7) that $\mathbb{K} = I_1 \circ \mathbb{K}'$ for some $\mathbb{K}' : \mathbb{E} \to \mathbb{E}'$. Let $e \in E_0$ with $F_0(e) \in E_m'$, for some $m \in \mathbb{N}_0$. Then $E_m \ni F_0' \circ F_0(e) = e + K_0e = e + \iota_1 \tilde{K}_0(e)$,

$$
e \in E_1
$$

hence $e = F_0' \circ F_0(e) - \iota_1 \tilde{K}_0(e) \in E_1$. Hence $\iota_1 \tilde{K}_0(e) \in E_2$, so $e \in E_2$.

Proceeding inductively one shows that $e \in E_m$, hence $\mathbb{F}'$ is regularising.

(b) Written out, one has $F_0' F_0 = \text{id}_{E_0} + K_0$ and $F_0' F_0 = \text{id}_{E_0} + K_0'$, since $\iota_1$ and $\iota_1'$ are compact, so are $K_0$ and $K_0'$, so $F_0$ is invertible modulo compact operators, hence Fredholm (cf. Proposition 2.2 and Definition 2.17).

3 $\Rightarrow$ 4. See [Weh12], Lemmas 3.5 and 3.6.

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This follows formally in precisely the same way as the direction $2 \Rightarrow 1$ in the proof of Proposition 2.2 noting that the projections $E = \ker F_0 \oplus \mathcal{X} \to \ker F_0 \subseteq E_{\infty}$ and $E' = \im \mathcal{F} \oplus C \to C \subseteq E'_{\infty}$ are strongly smoothing.

\[ \square \]

Remark 2.16. The condition \( \text{(c)} \) in Proposition 2.4.4 is slightly more general than that in \[Weh12\], Definition 3.4. But a look at the proof of Lemma 3.6 in \[Weh12\] immediately shows that one can assume this to hold as well.

Remark 2.17. In particular, by Proposition 2.4, if $F : \mathcal{E} \to \mathcal{E}'$ is Fredholm, then for every $k \in \mathbb{N}_0$, $\iota_k^\infty: \ker F_{\infty} \to \ker F_k$ is an isomorphism and if $C \subseteq E'_{\infty}$ is a complement to $\im F_{\infty}$, i.e. if the quotient projection $C \to \coker F_{\infty} = E'_{\infty}/\im F_{\infty}$ is an isomorphism, then $\iota_k^\infty|_C : C \to E_k'$ induces an isomorphism $C \cong \coker F_k = E_k'/\im F_k$ for every $k \in \mathbb{N}_0$. In particular, $F_k : E_k \to E_k'$ is Fredholm and $\text{ind } F_k = \text{ind } F_l$ for all $k, l \in \mathbb{N}_0 \cup \{\infty\}$.

Example 2.23. Let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi : F \to \Sigma$ and $\pi' : F' \to \Sigma$ be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections.

Let $k_0 \in \mathbb{N}_0$ and $1 < p < \infty$ be s.t. $k_0 p > n$ and define $k := (k_0 + j)_{j \in \mathbb{N}_0}$, $k + m := (k_0 + m + j)_{j \in \mathbb{N}_0}$. Define $E := \mathcal{W}^{k+m,p}(F)$ and $E' := \mathcal{W}^{k,p}(F')$.

Let $P : \Gamma(F) \to \Gamma(F')$ be an elliptic partial differential operator (with smooth coefficients) of class $m$. Then the continuous linear operator $\mathcal{F} : \mathcal{W}^{k+m,p}(F) = E \to E' = \mathcal{W}^{k,p}(F')$ from Example 2.5 is Fredholm:

By standard elliptic theory there exist pseudodifferential operators
\[
Q : \Gamma(F') \to \Gamma(F),
\]
\[
R : \Gamma(F) \to \Gamma(F)
\]
and
\[
R' : \Gamma(F') \to \Gamma(F')
\]
of orders $-m$, $-1$ and $-1$, respectively, s.t.
\[
Q \circ P = \text{id} + R
\]
\[
P \circ Q = \text{id} + R'.
\]

Furthermore, $Q$, $R$ and $R'$ induce continuous linear operators
\[
Q_j : W^{k_0+p}(F') \to W^{k_0+m+p}(F'),
\]
\[
R_j : W^{k_0+m+p}(F) \to W^{k_0+m+1+p}(F)
\]
and
\[
R'_j : W^{k_0+p}(F') \to W^{k_0+1+p}(F')
\]
for all $j \in \mathbb{N}_0$. These define continuous linear operators
\[
Q : \mathcal{W}^{k,p}(F') \to \mathcal{W}^{k+m,p}(F),
\]
\[
R : \mathcal{W}^{k+m,p}(F) \to \mathcal{W}^{k+m+1,p}(F)
\]
and
\[
R' : \mathcal{W}^{k,p}(F') \to \mathcal{W}^{k+1,p}(F')
\]

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s.t.

\[ Q \circ P = \text{id} + \mathbb{R} \]
\[ P \circ Q = \text{id} + \mathbb{R}' \]

and \( \mathbb{R} \) and \( \mathbb{R}' \) are smoothing.

**Definition 2.20.** Let \( E \) and \( E' \) be \( \mathfrak{S} \)-Fréchet spaces.

1. A morphism \( F : E \to E' \) is called *Fredholm* iff \( F \) is invertible modulo strongly smoothing operators i.e. iff there exists a morphism \( F' : E' \to E \) and strongly smoothing morphisms \( K : E \to E \) and \( K' : E' \to E' \) s.t.

\[
F' \circ F = \text{id}_E + K \\
F \circ F' = \text{id}_{E'} + K'.
\]

Such a morphism \( F' : E' \to E \) is then called a *Fredholm inverse* of \( F \).

2. The *Fredholm index* \( \text{ind} F \) of a Fredholm morphism \( F : E \to E' \) is defined as the Fredholm index of \( F \) as a Fredholm operator between the underlying Fréchet spaces.

**Remark 2.18.** That a Fredholm morphism between \( \mathfrak{S} \)-Fréchet spaces defines a Fredholm operator between the underlying Fréchet spaces is immediate from Remark 2.10 and Definition 2.17.

**Remark 2.19.** Note that if \( F : E \to E' \) is Fredholm, \( F' : E' \to E \) is a Fredholm inverse to \( F \) and \( K : E' \to E \) is strongly smoothing, then \( F'' := F' + K \) is a Fredholm inverse to \( F \) as well, by Corollary 2.3.

**Proposition 2.5.** Let \( E, E' \) and \( E'' \) be \( \mathfrak{S} \)-Fréchet spaces, let \( F : E \to E' \) and \( F' : E' \to E'' \) be Fredholm morphisms and let \( K : E \to E' \) be a strongly smoothing morphism.

1. \( F' \circ F : E \to E'' \) is Fredholm with \( \text{ind}(F' \circ F) = \text{ind} F' + \text{ind} F \).
2. \( F + K : E \to E' \) is Fredholm with \( \text{ind}(F + K) = \text{ind} F \).

**Proof.** Immediate from the definition using Corollary 2.3 and Remark 2.10.

For the following theorem remember that a Fredholm morphism between \( \mathfrak{S} \)-Fréchet spaces defines a Fredholm operator between the underlying Fréchet spaces (cf. Remark 2.18) and hence its kernel is finite dimensional (cf. Proposition 2.2). Consequently (cf. Example 2.18) its kernel is a split subspace. Maps of the form \( pr_Z^W \) denote a projection w.r.t. some splitting \( Z = W \oplus W' \) and maps of the form \( \iota^W_Z \) denote inclusions of a subspace \( W \subseteq Z \).

The theorem is stated in a fairly complicated manner because it then provides several technical results which will be used later on. For the gist of the statement it may be advisable to read its Corollary 2.5 first.
Theorem 2.1. Let $E$ and $E'$ be $\overline{\pi}$-Fréchet spaces and let $F : E \to E'$ be a Fredholm morphism. Then $F$ is a Fredholm operator between the Fréchet spaces underlying $E$ and $E'$, i.e. $\dim \ker F < \infty$ and $\dim \coker F < \infty$, and the following holds: Given any Fredholm inverse $F' : E' \to E$ to $F$ and any splitting $E = \ker F \oplus X$, for some $\overline{\pi}$-subspace $X \subseteq E$, there exist the following:

1. finite dimensional subspaces $A, B \subseteq X$ with images
   
   $A' := F(A) \subseteq \ker F'$, $B' := F_\infty(B) \subseteq C'$ and $C := F'(C') \subseteq \ker F$,

2. finite dimensional subspaces $\overline{A} \subseteq \ker F'$, $\overline{B} \subseteq C'$ and $\overline{C} \subseteq \ker F$,

3. $\overline{\pi}$-subspaces $Y \subseteq E$ and $Y' \subseteq E'$,

4. and splittings

   
   $$E = C \oplus \overline{C} \oplus A \oplus B \oplus Y$$

   $$E' = A' \oplus \overline{A}' \oplus B' \oplus \overline{B}' \oplus Y'.$$

   

   $\text{pr}_{A'}^E \circ F \circ \iota_E^A : A \to A'$

   and

   
   $\text{pr}_{B'}^E \circ F \circ \iota_E^B : B \to B'$

   are isomorphisms of finite dimensional vector spaces. Consequently one can define

   
   $$F'' := F' - F' \circ \iota_E^C \circ \text{pr}_C^E +$$

   $$+ \iota_E^A \circ (\text{pr}_{A'}^E \circ F \circ \iota_E^A)^{-1} \circ \text{pr}_{A'}^E +$$

   $$+ \iota_E^B \circ (\text{pr}_{B'}^E \circ F \circ \iota_E^B)^{-1} \circ \text{pr}_{B'}^E$$

   and with

   
   $H := F'' \circ F$ and $H' := F \circ F''$

   the following hold:

5. $F'' : E' \to E$ defines a Fredholm inverse to $F$.

6. $\text{im } F \subseteq E'$ and $\text{im } F'' \subseteq E$ are split $\overline{\pi}$-subspaces with

   $\ker F = \ker H$ \hspace{1cm} $\ker F'' = \ker H'$

   $\text{im } F = \text{im } H'$ \hspace{1cm} $\text{im } F'' = \text{im } H$

   

   $E = \ker F \oplus \text{im } F''$ \hspace{1cm} $E' = \ker F'' \oplus \text{im } F$

   $= \ker H \oplus \text{im } H$ \hspace{1cm} $= \ker H' \oplus \text{im } H'$

   $= \ker F \oplus X$. 

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7. $F|_X : X \to \text{im } F$, $F''|_X : \text{im } F \to X$, $H|_X : X \to X$ and $H'|_{\text{im } F} : \text{im } F \to \text{im } F$ are isomorphisms with

$$(F|_X)^{-1} = (H|_X)^{-1} \circ F''|_{\text{im } F} = F'' \circ (H'|_{\text{im } F})^{-1}$$

$$(F''|_X)^{-1} = (H'|_{\text{im } F})^{-1} \circ F|_X = F \circ (H|_X)^{-1}.$$  

Proof. That $F : E \to E'$ defines a Fredholm operator between Fréchet spaces has already been remarked in Remark 2.18.

Let $K : E \to E$ and $K' : E' \to E'$ be strongly smoothing with $F' \circ F = \text{id}_E + K$, $F \circ F' = \text{id}_{E'} + K'$.

After choosing $\varphi$-structures $(E, \phi)$ and $(E', \phi')$ there exist (cf. Lemma 2.5) strictly monotone increasing sequences $K, I \subseteq \mathbb{N}_0$ and continuous linear operators $F : \mathbb{E}^k \to \mathbb{E}'$ and $F' : \mathbb{E}' \to \mathbb{E}$ with $F_{\infty} = \phi'^{-1} \circ F \circ \phi$ and $F_{\infty}' = \phi^{-1} \circ F' \circ \phi'$.

Furthermore, one can assume that there is a splitting $E = \ker F_{\infty} \oplus X$ s.t. $\phi(X_{\infty}) = X$ and analogously that there is a splitting $E' = \ker F_{\infty}' \oplus X'$ by Example 2.13.

Replacing $\mathbb{E}$ by $\mathbb{E}^k$, $\mathbb{E}'$ by $\mathbb{E}^k$ and $X$ by $\mathbb{X}^k$, one can assume that $K = (j)_{j \in \mathbb{N}_0}$, so $F : E \to E'$ and $F' : E' \to E$.

Using Lemma 2.3, $K := \mathbb{F}' \circ \mathbb{F} = \mathbb{I}'_{\mathbb{E}'} - \mathbb{I}' : \mathbb{E}' \to \mathbb{E}$ is an extension of $K_{\infty} = \phi'^{-1} \circ K \circ \phi = F_{\infty}' \circ F_{\infty} - \text{id}_{\mathbb{E}_\infty}$ and using Lemmas 2.11 and 2.12 to modify $\mathbb{F}$ and $\mathbb{F}'$, one can assume that $K$ is strongly smoothing. And similarly for an extension $K' = \mathbb{F} \circ \mathbb{F}' - \mathbb{I}' : \mathbb{E} \to \mathbb{E}'$ of $K_{\infty}' = \phi^{-1} \circ K' \circ \phi' = F_{\infty}' \circ F_{\infty} - \text{id}_{\mathbb{E}_\infty}$.

Claim. $F$ is weakly regularising.

Proof. From $\mathbb{F}' \circ \mathbb{F} = \mathbb{I}'_{\mathbb{E}'_\infty} + \mathbb{K}$ one has that $F_{\infty}' \circ F_{\infty} = \mathbb{I}'_{\mathbb{E}'} + \mathbb{K}_{\mathbb{E}'} = \mathbb{E}_{\mathbb{E}'} \to \mathbb{E}_{\mathbb{E}'}$.

So if $e \in \mathbb{E}_{\mathbb{E}'}$ with $F_{\infty}(e) \in \mathbb{E}_\infty$, then $F_{\infty}'(F_{\infty}(e)) \in \mathbb{E}_\infty$ and $K_{\infty}(e) \in \mathbb{E}_\infty$ because $K$ is strongly smoothing. Hence $F_{\infty}'(F_{\infty}(e)) - K_{\infty}(e) \in \mathbb{E}_\infty$, so $e \in \mathbb{E}_\infty$. 

Now replace $\mathbb{E}$ by $\mathbb{E}^m$, $F$ by $\mathbb{I}'_{\mathbb{E}'} \circ \mathbb{F}^m$ and $F'$ by $\mathbb{F}^m$.

Then (cf. Remark 2.13) for all $j \in \mathbb{N}_0$, $\ker F_j = \ker F_{\infty} \subseteq \mathbb{E}_\infty$ which is finite dimensional since $F : E \to E'$ and hence $F_{\infty} : E_\infty \to E'_\infty$ is Fredholm.

Applying the same reasoning to $F'$, one can also assume that $\ker F_j' = \ker F_{\infty}' \subseteq \mathbb{E}'_\infty$ for all $j \in \mathbb{N}_0$.

Because $K$ and $K'$ are strongly smoothing, there are continuous linear operators $K : E^j \to E^j$ and $K' : E^j \to E^j$ s.t. $\mathbb{K} = \mathbb{I}'_{\mathbb{E}'} \circ K$ and $K' = \mathbb{I}'_{\mathbb{E}'} \circ K'$. Furthermore, by Corollary 2.1, one can assume that $\mathbb{K} = \mathbb{K}'$ for a strongly smoothing continuous linear operator $\mathbb{K} : E \to E$ and similarly $\mathbb{K}' = \mathbb{K}'$ for a strongly smoothing continuous linear operator $\mathbb{K} : E' \to E'$.

In conclusion,

$$\mathbb{F}' \circ \mathbb{I} = \mathbb{I}' \circ (\text{id}_E + \mathbb{K})$$

$$\mathbb{F} \circ \mathbb{F}' = \mathbb{I}' \circ (\text{id}_{\mathbb{E}'} + \mathbb{K}' )$$

Because $G_{\infty} = F_{\infty}' \circ F_{\infty}$, one has ker $F_{\infty} \subseteq$ ker $G_{\infty}$ and denoting

$$\tilde{A} := (F_{\infty}|_{X_{\infty}})^{-1} (\ker F_{\infty}') ,$$

ker $G_{\infty} = \ker F_{\infty} \oplus \tilde{A}$, which is finite dimensional as well, because $G_{\infty}$ is Fredholm as an operator between Fréchet spaces. By Example 2.13 again, there hence exists a splitting $X = \tilde{A} \oplus \tilde{A}$ and hence $E = \ker F_{\infty} \oplus \tilde{A} \oplus \tilde{A}$, where ker $G_{\infty} = \ker F_{\infty} \oplus \tilde{A}$ and
\[ F_\infty(A_\infty) \cap \ker F_\infty' = \{0\}. \] Applying the same reasoning to \( G' \), one obtains a similar splitting \( \mathbb{E}' = \ker F_\infty' \oplus \tilde{A} \oplus A' \), where \( \ker G_\infty' = \ker F_\infty' \oplus \tilde{A} \) and \( F_\infty'(A_\infty') \cap \ker F_\infty = \{0\} \). Note that \( F_\infty \) maps \( \tilde{A} \) injectively into \( \ker F_\infty' \) and \( F_\infty' \) maps \( \tilde{A} \) injectively into \( \ker F_\infty \). Hence, one can further decompose

\[
\begin{align*}
\mathbb{E} & = \left( \ker F_\infty \oplus \tilde{A} \oplus A \right) + \left( \ker G_\infty \oplus A' \right) \oplus A \\
\mathbb{E}' & = \left( \ker F_\infty' \oplus \tilde{A} \oplus A' \right) + \left( \ker G_\infty' \oplus A \right) \oplus A' \mathbb{E}.
\end{align*}
\]

W. r. t. this decomposition, one can then write in matrix form

\[
F_\infty = \begin{pmatrix}
F_\infty'(\tilde{A}) & \tilde{A} & A_\infty \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & a_1 & 0 \\
0 & 0 & 0 & a_2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
F_\infty' = \begin{pmatrix}
F_\infty'(\tilde{A}) & \tilde{A} & A_\infty \\
0 & 0 & \beta & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_1 \\
0 & 0 & 0 & b_2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
G_\infty = \begin{pmatrix}
F_\infty'(\tilde{A}) & \tilde{A} & A_\infty \\
0 & 0 & \beta a_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_1 a_2 \\
0 & 0 & 0 & b_2 a_2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
G_\infty' = \begin{pmatrix}
F_\infty'(\tilde{A}) & \tilde{A} & A_\infty \\
0 & 0 & \alpha b_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 b_2 \\
0 & 0 & 0 & a_2 b_2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

where by the choices of the above splittings, \( \alpha = \text{pr}_{F_\infty'(\tilde{A})}^{E'} \circ F_\infty \circ \text{id}_{F_\infty} \) is invertible as a linear map between finite dimensional vector spaces. Now define \( F'' : \mathbb{E}' \rightarrow \mathbb{E} \) by

\[
F'' := F' - F' \circ \text{id}_{\tilde{E}'} \circ \text{pr}_{\tilde{E}'}^{F_\infty'(\tilde{A})} \circ F_\infty \circ \text{id}_{F_\infty} \circ \text{id}_{\tilde{E}'} \circ \left( \text{pr}_{F_\infty'(\tilde{A})}^{F_\infty} \circ F_\infty \circ \text{id}_{F_\infty} \right)^{-1} \circ \text{pr}_{F_\infty'(\tilde{A})},
\]

where \( \text{id}_{\tilde{E}'} \) is the inclusion and the maps of the form \( \text{pr}_{\tilde{E}'} \) are the projections w. r. t. the splittings above. Note that the two summands on the right are strongly smoothing operators, by Example 2.21, so \( F'' \) defines a Fredholm inverse \( F'' : \mathbb{E}' \rightarrow \mathbb{E} \) to \( F' \).
Written in matrix form,

\[
F'' = \begin{pmatrix}
F_\infty(\bar{A}) & \bar{A}' & \bar{A}_\infty \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\alpha^{-1} & 0 & 0 & b_1 \\
0 & 0 & 0 & b_2
\end{pmatrix}
\begin{pmatrix}
\bar{A}' \\
\bar{A} \\
\bar{A}_\infty
\end{pmatrix}
\]

Now again \(E'' \circ F' = H \circ F\) and \(F \circ E'' = F'' \circ H'\), where

\[
H_\infty = \begin{pmatrix}
F_\infty(\bar{A}') & \bar{A}' & \bar{A}_\infty \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Note that \(H_\infty|_{A_\infty} = G_\infty|_{A_\infty}\) and hence \(ker H_\infty|_{A_\infty} = ker G_\infty|_{A_\infty} = \{0\}\), by choice of \(A\). Just the same, \(F''|_{A_\infty} = F_\infty|_{A_\infty}\) and hence \(ker F''|_{A_\infty} = ker F_\infty|_{A_\infty} = \{0\}\), by choice of \(A\). It follows that \(ker F''|_{A_\infty} = \bar{A} \oplus \bar{A}' = ker H_\infty\).

Also, from the above matrix representations one sees that \(ker H_\infty = F_\infty(\bar{A}') \oplus \bar{A} \oplus ker H_\infty|_{A_\infty}\) and \(H_\infty|_{A_\infty} = F''|_{A_\infty} \circ pr_{A_\infty}^{E''} \circ F_\infty|_{A_\infty} \circ a_2\), hence

\[
ker H_\infty|_{A_\infty} = ker a_2, \text{ because } F''|_{A_\infty} \text{ is injective by choice of } A.
\]

Define

\[
\bar{A}'' := ker pr_{A_\infty}^{E''} \circ F_\infty|_{A_\infty} \subseteq A_\infty.
\]

Then \(F_\infty(\bar{A}'') = a_1(\bar{A}'') \subseteq \bar{A}' \text{ and } pr_{F_\infty(\bar{A}'')}^{E''} \circ F_\infty \circ t_{E_\infty}^{\bar{A}''} : \bar{A}'' \to F_\infty(\bar{A}'')\) is a bijective linear map between finite dimensional vector spaces. Hence one can define \(F'' : E'' \to E\) by

\[
F'' := F'' + t_{E''}^{\bar{A}''} \circ \left( pr_{F_\infty(\bar{A}'')}^{E''} \circ F_\infty \circ t_{E_\infty}^{\bar{A}''} \right)^{-1} \circ pr_{\bar{A}''}
\]

If one now applies the same reasoning as above, replacing \(E''\) by \(F''\), then in the decompositions \(\bar{A}' = {0}\) one gets after this replacement, \(\bar{A}' = \{0\}\). A simple calculation with the matrix representations of \(F_\infty\) and \(F''\) w.r.t. this new decomposition then shows that

\[
ker F_\infty = ker (F'' \circ F_\infty)
\]
and
\[ \ker F'''' = \ker (F_{\infty} \circ F'''). \]

Using the above, replacing \( F' \) by \( F'' \), one can hence assume that \( \ker G_\infty = \ker F_\infty \) and \( \ker G'_\infty = \ker F'_\infty \). Also note that \( G : E \to E \) and \( G' : E' \to E' \) are Fredholm operators of index 0. Also, from \( G_\infty = F'_\infty \circ F_\infty \) and \( G'_\infty = F_\infty \circ F''_\infty \), it follows that \( \im G_\infty \subseteq \im F''_\infty \) and \( \im G'_\infty \subseteq \im F'_\infty \), so \( \dim \coker G_\infty \geq \dim \coker F''_\infty \) and \( \dim \coker G'_\infty \geq \dim \coker F'_\infty \).

Consequently,
\[
\dim \ker F_\infty = \dim \ker G_\infty = \dim \ker F'_\infty \\
\geq \dim \coker F''_\infty
\]

and
\[
\dim \ker F'_\infty = \dim \ker G'_\infty = \dim \coker G_\infty \\
\geq \dim \coker F'_\infty.
\]

Using that furthermore, because \( F_\infty \) and \( F'_\infty \) are Fredholm inverses to each other, that by Proposition 2.3 \( \ind F_\infty = - \ind F'_\infty \), it follows that
\[
0 \leq \dim \ker F_\infty - \dim \ker F''_\infty = \dim \ker F'_\infty - \dim \ker F''_\infty \leq 0,
\]

hence
\[
\dim \ker F_\infty = \dim \ker F'_\infty
\]

and
\[
\dim \coker F_\infty = \dim \ker F''_\infty.
\]

By the above then also \( \dim \ker G_\infty = \dim \ker G'_\infty \) and \( \dim \ker F_\infty = \dim \ker F'_\infty = \dim \ker F''_\infty \). Because \( G : E \to E \) and \( G' : E' \to E' \) are Fredholm operators, by Proposition 2.4 \( \im G \subseteq E \) and \( \im G' \subseteq E' \) are well defined subchains. Furthermore, if \( e' \in \im F_\infty \cap \ker F''_\infty, e' = F_\infty(e) \) for some \( e \in E_\infty \), then \( 0 = F'_\infty \circ F_\infty(e) = G_\infty(e) \), hence \( e \in \ker G_\infty = \ker F_\infty \) and so \( e' = 0 \). It follows that \( \im G_\infty \cap \ker F''_\infty = \im F_\infty \cap \ker F'_\infty = \{0\} \) and similarly \( \im G'_\infty \cap \ker F_\infty = \im F'_\infty \cap \ker F''_\infty = \{0\} \). So, summarising,
\[
\begin{align*}
\ker G_\infty & = \ker F_\infty & \ker G'_\infty & = \ker F'_\infty \\
\im G_\infty & = \im F''_\infty & \im G'_\infty & = \im F'_\infty \\
E_\infty & = \ker F_\infty \oplus \im F''_\infty & E'_\infty & = \ker F'_\infty \oplus \im F'_\infty \\
& = \ker G_\infty \oplus \im G_\infty & & = \ker G'_\infty \oplus \im G'_\infty.
\end{align*}
\]

Using the above, define \( \mathcal{X}'' := \im \mathcal{H}' \). The above shows that \( X = \phi(X_\infty) \) and \( \im F = \phi(X''_\infty) \) are split \( \overline{\mathcal{R}} \)-subspaces of \( E \) and \( E' \) with \( \overline{\mathcal{R}} \)-complements \( \ker F \) and \( \ker F'' \), respectively.

Define
\[
A := \phi(\tilde{A}) \quad B := \phi(\tilde{A}'')
\]

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and

\[ C' := \phi'(\tilde{A}'). \]

From the above,

\[ \tilde{G} := G|_{X} : X \to X \quad \text{and} \]
\[ \tilde{G}' := G'|_{X''} : X'' \to X'' \]

are well-defined isomorphisms, and

\[ \tilde{F} := F|_{X} : X \to X \quad \text{and} \]
\[ \tilde{F}' := F'|_{X''} : X'' \to X \]

are well-defined and satisfy

\[ \tilde{F}' \circ \tilde{F} = \mathbb{1} \circ \tilde{G}' \]
\[ \tilde{F} \circ \tilde{F}' = \mathbb{1} \circ \tilde{G} \]

\[ H := \tilde{F}' \circ \left(\tilde{G}'^{-1}\right)^{-1} : X'' \to X \]

is a well-defined continuous linear operator which extends

\[ H_\infty = \tilde{F}'_\infty \circ \left(\tilde{G}'_\infty^{-1}\right)^{-1} = \tilde{F}'_\infty \circ \left(\tilde{F}'_\infty \circ \tilde{F}'_\infty^{-1}\right)^{-1} = \tilde{F}'_\infty^{-1} \]

and defines a weak morphism that is inverse to \( \tilde{F}'_\infty : X_\infty \to X_\infty'' \). Hence \( F|_{X} : X \to X'' = \operatorname{im} F \) is an isomorphism of \( \mathfrak{s}-\text{Fréchet} \) spaces.

**Corollary 2.5.** Let \( E \) and \( E' \) be \( \mathfrak{s}-\text{Fréchet} \) spaces and let \( F : E \to E' \) be a morphism. Then the following are equivalent:

1. \( F \) is Fredholm.

2. (a) \( F \) is a Fredholm operator between the Fréchet spaces underlying \( E \) and \( E' \), i.e. \( \dim \ker F < \infty \) and \( \dim \operatorname{coker} F < \infty \).

   (b) There exist splittings of \( \mathfrak{s}-\text{Fréchet} \) spaces

\[ E = \ker F \oplus X \quad \text{and} \quad E' = \operatorname{im} F \oplus C. \]

In particular \( \operatorname{im} F \) defines a split \( \mathfrak{s} \)-subspace of \( E' \) and the quotient projection \( E' \to \operatorname{coker} F = E'/\operatorname{im} F \) is a well defined morphism between \( \mathfrak{s}-\text{Fréchet} \) spaces.

(c) For any pair of splittings as in (b)

\[ \operatorname{pr}_{\operatorname{im} F}^{E'} \circ F \circ \iota_{X} : X \to \operatorname{im} F \]

is an isomorphism of \( \mathfrak{s}-\text{Fréchet} \) spaces.

**Proof.**

1. \( \Rightarrow \) 2. Immediate from Theorem 2.1.

2. \( \Rightarrow \) 1. This again follows formally in the same way as the direction \( 2. \Rightarrow 1. \) in the proof of Proposition 2.2, noting that the projections \( E = \ker F \oplus X \to \ker F \) and \( E' = \operatorname{im} F \oplus C \to C \) are strongly smoothing.
Example 2.24. If $E$ and $E'$ are $\mathfrak{sc}$-Fréchet spaces and $(\mathcal{E}, \phi)$ and $(\mathcal{E}', \phi')$ are compatible $\mathfrak{sc}$-structures on $E$ and $E'$, respectively, then any Fredholm operator $\mathcal{F}: \mathcal{E} \to \mathcal{E}'$ induces a Fredholm operator $F: E \to E'$.

In particular, let $(\Sigma, g)$ be a closed $n$-dimensional Riemannian manifold and let $\pi: F \to \Sigma$ and $\pi': F' \to \Sigma$ be real (or complex) vector bundles equipped with euclidean (or hermitian) metrics and metric connections. Then by Example 2.23 any elliptic differential operator $P: \Gamma(F) \to \Gamma(F')$ defines a Fredholm operator $P: W(F) \to W(F')$.
3 Nonlinear maps between $\overline{sc}$-Fréchet spaces

3.1 Envelopes and $\overline{sc}$-continuity

**Definition 3.1.** Let $E$ and $E'$ be ILB- or sc-chains, let $A \subseteq E_\infty$ be a subset, and let $f : A \to E'_\infty$ be a map.

1. An *envelope of $A$ (in $E$)* is given by a sequence $\mathcal{U} = (U_k)_{k \in \mathbb{N}_0}$, where
   
   (a) $U_k \subseteq E_k$ is an open subset,
   (b) $\iota_k(U_{k+1}) \subseteq U_k$ for all $k \in \mathbb{N}_0$, and
   (c) $A = \bigcap_{k \in \mathbb{N}_0} (\iota_k)^{-1}(U_k)$.

   It is called *strict* if $U_{k+1} = (\iota_k)^{-1}(U_k)$ for all $k \in \mathbb{N}_0$.

2. A (continuous) *envelope of $f$ (in $E$ and $E'$)* is given by a sequence $(f_k : U_k \to E'_k)_{k \in \mathbb{N}_0}$, where

   (a) $\mathcal{U} = (U_k)_{k \in \mathbb{N}_0}$ is an envelope of $A$ (in $E$) and
   (b) the $f_k : U_k \to E'_k$ are continuous maps s.t.
   
   i. $f_k \circ \iota_k^\infty|A = \iota_k^\infty \circ f$ for all $k \in \mathbb{N}_0$,
   ii. $f_k \circ \iota_k|U_{k+1} = \iota_k^f \circ f_{k+1}$ for all $k \in \mathbb{N}_0$.

   It will be denoted by $\mathcal{F} : \mathcal{U} \to \mathcal{E}'$, or just $\mathcal{F}$ if the choice of $\mathcal{U}$ and $\mathcal{E}'$ is clear, and it will be called *strict* if $\mathcal{U}$ is a strict envelope of $A$.

3. Given two envelopes $\mathcal{U} = (U_k)_{k \in \mathbb{N}_0}$ and $\tilde{\mathcal{U}} = (\tilde{U}_k)_{k \in \mathbb{N}_0}$ of $A$, $\tilde{\mathcal{U}}$ is said to be a *refinement of $\mathcal{U}$* if $\tilde{U}_k \subseteq U_k$ for all $k \in \mathbb{N}_0$.

4. Given an envelope $\mathcal{F} = (f_k : U_k \to E'_k)_{k \in \mathbb{N}_0}$ and a refinement $\tilde{\mathcal{U}}$ of $\mathcal{U}$, the restriction $\mathcal{F}|_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \to \mathcal{E}'$ is the envelope $(f_k|_{\tilde{U}_k} : \tilde{U}_k \to E'_k)_{k \in \mathbb{N}_0}$ of $f$.

5. Given two envelopes $\mathcal{F}$ and $\tilde{\mathcal{F}}$ of $f$, $\tilde{\mathcal{F}}$ is said to be a *refinement of $\mathcal{F}$*, if $\tilde{\mathcal{U}}$ is a refinement of $\mathcal{U}$ and $\mathcal{F} = \mathcal{F}|_{\tilde{\mathcal{U}}}$.

6. Two envelopes of $f$ are called *equivalent* if they have a common refinement.

**Lemma 3.1.** Let $E$ and $E'$ be ILB- or sc-chains, let $A \subseteq E_\infty$ be a subset, and let $f : A \to E'_\infty$ be a map.

$f$ is continuous and uniquely determined by the maps $f_k$ for $k \in \mathbb{N}_0$ in the following sense:

Let $\mathcal{U} = (U_k)_{k \in \mathbb{N}_0}$ be an envelope of a subset $A \subseteq E_\infty$ and let $\mathcal{F} = (f_k : U_k \to E'_k)_{k \in \mathbb{N}_0}$ be a sequence of continuous maps s.t. $f_k \circ \iota_k = \iota_k^f \circ f_{k+1}$ for all $k \in \mathbb{N}_0$. Then there exists a unique continuous map $f : A \to E'_\infty$ s.t. $\mathcal{F}$ is an envelope of $f$.

**Proof.** That $f$ is continuous follows immediately from Definition 3.1.2(b) and 3.1.4 because $f$ is the inverse limit of the (continuous) maps $f_k$. Given $a \in A$, $f_k(\iota_k^\infty(a)) = f_k(\iota_k \circ \iota_k^\infty(a)) = \iota_k^f f_k(\iota_k^\infty(a)) = \cdots = \iota_k^f f_k(\iota_k^\infty(a))$ for all $k, \ell \in \mathbb{N}_0$ with $\ell > k$. So $f_k(\iota_k^\infty(a)) \in \text{im} \iota_k^f$ for all $k, \ell \in \mathbb{N}_0$ with $\ell > k$ and hence $f_k(\iota_k^\infty(a)) \in \text{im} \iota_k^\infty$. Defining $f(a) := (\iota_k^\infty)^{-1}(f_k(\iota_k^\infty(a)))$, the above shows that this is well defined, independent of the choice of $k \in \mathbb{N}_0$ and thus defines map $f : A \to E'_\infty$. Furthermore, by definition, $f_k \circ \iota_k^\infty|A = \iota_k^f \circ f$ for all $k \in \mathbb{N}_0$. It remains to show that $f$
is continuous. By definition of the topology on $E'_\infty$, $f$ is continuous if and only if the maps $\iota^\infty_k \circ f$ are continuous for all $k \in N_0$. But the formula $\iota^\infty_k \circ f \big|_A$ shows that they are compositions of continuous functions, hence continuous. \hfill \square

**Remark 3.1.** Note that if $\mathcal{U}$ is a strict envelope of $A$, then $A = (\iota^\infty_k)^{-1}(U_k) \subseteq E_\infty$ is open and $U_\ell = (\iota^\infty_k)^{-1}(U_k)$ for all $\ell \geq k$.

**Lemma 3.2.** Let $E$ and $E'$ be ILB- or sc-chains, let $A \subseteq E_\infty$ be a subset, and let $f : A \to E'_\infty$ be a map. Then:

1. Any two envelopes of $A$ have a common refinement.
2. A strict envelope of $A$ is a refinement of any other envelope of $A$. In particular, $A$ has at most one strict envelope.
3. A strict envelope of $f$ is a refinement of any other envelope of $f$. In particular, $f$ has at most one strict envelope.

**Proof.**

1. Let $\mathcal{U} = (U_k)_{k \in N_0}$ and $\mathcal{U}' = (U'_k)_{k \in N_0}$ be two envelopes of $A$. Then $\mathcal{U} \cap \mathcal{U}' := (U_k \cap U'_k)_{k \in N_0}$ is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$.

2. If $(U_k)_{k \in N_0}$ is a strict envelope of $A$, then $A = (\iota^\infty_k)^{-1}(U_k)$ for any $k \in N_0$. Now if $e \in U_k$, because $\iota^\infty_k(E_\infty)$ is dense in $E_k$, there exists a sequence $(e_j)_{j \in N_0} \subseteq E_\infty$ s.t. $\iota^\infty_k(e_j) \to e$ in $E_k$. Since $U_k$ is open, for $j$ large enough, $e_j \in U_k$ and hence $e_j \in (\iota^\infty_k)^{-1}(U_k) = A$. So one can assume that $(e_j)_{j \in N_0} \subseteq A$ and hence it follows that $\iota^\infty_k(A)$ is dense in $U_k$. Now if $(U'_k)_{k \in N_0}$ is another envelope of $A$, then for each $k \in N_0$, both $U_k$ and $U'_k$ contain the same subset $\iota^\infty_k(A)$, and since $\iota^\infty_k(A)$ is dense in $U_k$ hence $U_k \subseteq U'_k$.

3. If $\mathcal{F} = (f_k : U_k \to E'_k)_{k \in N_0}$ and $\mathcal{F}' = (f'_k : U'_k \to E'_k)_{k \in N_0}$ are two envelopes of $f$ with $\mathcal{F}$ strict, then by *2* for each $k \in N_0$, $U_k \subseteq U'_k$ and $f_k|_{\iota^\infty_k(A)} = \iota^\infty_k \circ f \circ (\iota^\infty_k)^{-1}|_{\iota^\infty_k(A)} = f'_k|_{\iota^\infty_k(A)}$, so by continuity $f_k = f'_k|_{U_k}$. \hfill \square

**Definition 3.2.** Let $E$, $\tilde{E}$, $E'$ and $\tilde{E}'$ be ILB- or sc-chains, let $A \subseteq E_\infty$ be a subset and let $f : A \to E'_\infty$ be a continuous map. Let furthermore $k \subseteq N_0$ be a strictly monotone increasing sequence, and let $T : \tilde{E} \to E$ and $S : E' \to \tilde{E}'$ be continuous linear operators.

1. Given an envelope $\mathcal{U} = (U_k)_{k \in N_0}$ of $A$,
   
   (a) the **rescaling of $\mathcal{U}$ (by $k$)** is the envelope
   
   $\mathcal{U}^k := (U_{kj})_{j \in N_0}$
   
   of $A$ in $E^k$
   
   (b) The **pullback of $\mathcal{U}$ by $T$** is the envelope
   
   $\mathcal{T}^*\mathcal{U} := (T^{-1}(U_k))_{k \in N_0}$
   
   of $T^{-1}_k(A) := T^{-1}_k(A) \subseteq \tilde{E}_\infty$.

2. Given an envelope $\mathcal{F} = (f_k : U_k \to E'_k)_{k \in N_0}$ of $f$, 

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Lemma 3.3. Let $E$, $E'$, $E''$ and $E'''$ be ILB- or sc-chains, let $k,l,l',m \leq N_0$ be strictly monotone increasing sequences with $k \geq l \geq l' \geq m$, and let $T : E \to E'$ and $S : E' \to E''$ be continuous linear operators. Let furthermore $A \subseteq E_\infty$ be a subset and let $f : A \to E_\infty$ be a map together with envelopes $\mathcal{U} = (U_k)_{k \in N_0}$ and $\mathcal{F} = (f_k : U_k \to E_k')_{k \in N_0}$ of $A$ and $f$, respectively. Then the following holds:

1. The envelope $(\mathcal{F}^*)^\mathcal{U}$ of $A$ in $E^k$ is a refinement of $(\mathcal{F}^*)^\mathcal{U}'$ and the envelope $\mathcal{V}_m \circ (\mathcal{F}^*)^\mathcal{V}$ of $f$ in $E^k$ and $E^m$ is a refinement of $\mathcal{V}_m \circ (\mathcal{F}^*)^\mathcal{V}$. In particular, up to equivalence, the envelopes $(\mathcal{F}^*)^\mathcal{U}$ and $\mathcal{V}_m \circ (\mathcal{F}^*)^\mathcal{V}$ are independent of $1$.

2. Given another envelope $\mathcal{F} : \mathcal{U} \to E'$, $\mathcal{V}_m \circ \mathcal{F}$ and $\mathcal{V}_m \circ \mathcal{F}^k$ are equivalent iff $\mathcal{F}^k$ and $\mathcal{F}^k$ are equivalent.

3. If $\mathcal{U}$ is strict, then so are $\mathcal{U}^k$ and $T^* \mathcal{U}$.

4. If $\mathcal{F}$ is strict, then so are $\mathcal{F}^k$, $T^* \mathcal{F}$ and $S \circ \mathcal{F}$.

Proof. 1. Denote $\mathcal{U} = (\mathcal{F}^*)^\mathcal{U}$ and $\mathcal{U}' = (\mathcal{F}^*)^\mathcal{U}'$. Similarly, denote $\mathcal{F} := \mathcal{V}_m \circ (\mathcal{F}^*)^\mathcal{V}$ and $\mathcal{F} := \mathcal{V}_m \circ (\mathcal{F}^*)^\mathcal{V}$. Then for any $j \in N_0$,

$$
\begin{align*}
\tilde{U}_j &= (t_{ij})^{-1} U_j = (t_{ij} \circ t_{ij})^{-1} U_j \\
&= (t_{ij})^{-1} (t_{ij})^{-1} U_j \\
&= (t_{ij})^{-1} U_j \\
&\geq \tilde{U}_j \\
&= \tilde{U}_j
\end{align*}
$$

and

$$
\begin{align*}
\tilde{F}_j &= \tilde{f}_{ij} \circ f_{ij} \circ t_{ij}^k | \tilde{U}_j \\
&= \tilde{f}_{ij} \circ t_{ij} \circ t_{ij}^k | \tilde{U}_j \\
&= \tilde{f}_{ij} \circ t_{ij}^k | \tilde{U}_j
\end{align*}
$$

and
\[
\tilde{f}_j|_{\tilde{U}_j} = \iota_{\tilde{m}_j}^* \circ f_j \circ \iota_j|_{\tilde{U}_j} = \tilde{f}_j.
\]

2. Straightforward because all the \(i_{m_j}^k\) for \(j \in \mathbb{N}_0\) are injective.

3. That \(\mathbb{U}^k\) is strict is almost tautological and if \(\tilde{U} = T^* U\), then for any \(k \in \mathbb{N}_0\),
\[
\tilde{U}_{k+1} = T_{k+1}^{-1}(U_{k+1}) = T_{k+1}^{-1}(\iota_k^{-1}(U_k)) = (T_k \circ \iota_{k+1})^{-1}(U_k) = \iota_k^{-1}(T_k^{-1}(U_k)) = \iota_k^{-1}(\tilde{U}_k).
\]

4. Immediate from \(3\) and the definition of “strict”.

**Definition 3.3.** Let \(E\) and \(E'\) be ILB- or sc-chains, let \(A \subseteq E_\infty\) be a subset, and let \(f : A \to E_\infty\) be a map. Two envelopes \(\mathcal{F}\) and \(\tilde{\mathcal{F}}\) of \(f\) are said to be **weakly equivalent** if there exists a strictly monotone increasing sequence \(k \subseteq \mathbb{N}_0\) s. t. \(\mathcal{F}^k\) and \(\tilde{\mathcal{F}}^k\) are equivalent.

**Lemma 3.4.** Weak equivalence of envelopes is an equivalence relation.

**Proof.** Symmetry and reflexivity are clear. For transitivity, let \(E\) and \(E'\) be ILB- or sc-chains, let \(A \subseteq E_\infty\) be a subset and let \(f : A \to E_\infty\) be a continuous map. Let envelopes \(\mathcal{F}\), \(\mathcal{F}'\) and \(\tilde{\mathcal{F}}\) of \(f\) be given s. t. \(\mathcal{F}\) is weakly equivalent to \(\tilde{\mathcal{F}}\) and \(\mathcal{F}'\) is weakly equivalent to \(\tilde{\mathcal{F}}\). Then there exist strictly monotone increasing sequences \(k, k' \subseteq \mathbb{N}_0\) s. t. \(\mathcal{F}^k\) is equivalent to \(\tilde{\mathcal{F}}^k\) and \(\mathcal{F}'^k\) is equivalent to \(\tilde{\mathcal{F}}^k\). Then \(\mathbb{U}^k \circ \mathcal{F}^k\) and \(\mathbb{U}^k \circ \tilde{\mathcal{F}}^k\) are equivalent and by Lemma 3.3 hence so are \((\mathbb{U}^k)^* \mathcal{F}\) and \((\mathbb{U}^k)^* \tilde{\mathcal{F}}\). Similarly \((\mathbb{U}^{k'})^* \mathcal{F}'\) and \((\mathbb{U}^{k'})^* \tilde{\mathcal{F}}\) are equivalent. Define \(m = (\max(k, k'))_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0\). Then \((\mathbb{U}^m)^* \mathcal{F} = (\mathbb{U}^m)^* (\mathbb{U}^k)^* \mathcal{F} = \mathcal{F}_m = \mathcal{F}_{m'}\) which in turn is equivalent to \((\mathbb{U}^m)^* \tilde{\mathcal{F}} = (\mathbb{U}^m)^* (\mathbb{U}^{k'})^* \tilde{\mathcal{F}} = \mathcal{F}_{m'}\). So \((\mathbb{U}^m)^* \mathcal{F}\) is equivalent to \((\mathbb{U}^m)^* \tilde{\mathcal{F}}\) and by Lemma 3.3 \(\mathbb{U}^m \circ \mathcal{F}_m\) is equivalent to \(\mathbb{U}^m \circ \mathcal{F}_{m'}\). Again by Lemma 3.3 \(\mathcal{F}_m\) is equivalent to \(\mathcal{F}_{m'}\).

**Definition 3.4.** Let \(E\) and \(E'\) be \(\mathfrak{F}\)-Fréchet spaces, let \(A \subseteq E\) be a subset, and let \(f : A \to E'\) be a map.

1. An envelope of \(A\) is given by a pair \((\mathbb{E}, \phi), \mathbb{U}\), where \((\mathbb{E}, \phi)\) is a compatible \(\mathfrak{F}\)-structure on \(E\) and \(\mathbb{U}\) is an envelope of \(\phi^{-1}(A) \subseteq E_\infty\) in \(E\). It is called strict if \(\mathbb{U}\) is strict.

2. An envelope of \(f\) is given by a triple \(((\mathbb{E}, \phi), (\mathbb{E}', \phi')), \mathcal{F} : \mathbb{U} \to \mathcal{F}'\), where \((\mathbb{E}, \phi)\) and \((\mathbb{E}', \phi')\) are compatible \(\mathfrak{F}\)-structures on \(E\) and \(E'\), respectively, and \(\mathcal{F} : \mathbb{U} \to \mathcal{F}'\) is an envelope of \(\phi'^{-1} \circ f \circ \phi : \phi^{-1}(A) \to E_\infty\) in \(E\) and \(E'\). It is called strict if \(\mathcal{F}\) is strict.

3. Let \(\mathfrak{F} = ((\mathbb{E}, \phi), (\mathbb{E}', \phi'), \mathcal{F})\) and \(\mathfrak{F}' = ((\mathbb{E}', \phi'), (\mathbb{E}'', \phi''), \mathcal{F}')\) be two envelopes of \(f\). By compatibility of \((\mathbb{E}, \phi)\) with \((\mathbb{E}', \phi')\) and of \((\mathbb{E}', \phi')\) with \((\mathbb{E}'', \phi'')\), \(J = \phi'^{-1} \circ \phi' : E_\infty \to E''\) and \(K' = \phi'^{-1} \circ \phi' : E' \to E''\) are weak equivalences. \(\mathfrak{F}\) and \(\mathfrak{F}'\) are said to be equivalent iff for any extensions \(J : \mathbb{F}^k \to \mathbb{E}\) and \(K' : \mathbb{F}'^k \to \mathbb{E}'\) of \(J\) and \(K\), respectively, \(\mathbb{F}_k \circ \mathcal{F}_k\) and \(\mathbb{F}'_k \circ (J \circ \mathcal{F}')\) are weakly equivalent.
Lemma 3.5. Equivalence of envelopes is a well defined (i.e. independent of choices) equivalence relation.

Proof. Straightforward.

Proposition 3.1. Let $E$ and $E'$ be $\varpi$-Fréchet spaces, let $A \subseteq E$ be a subset, and let $f : A \to E'$ be a map.

If $A \subseteq E$ is open, then for any $a \in A$ there exists a neighbourhood $B \subseteq A$ of $a$ in $E$ s.t. the following holds:

1. If $((E, \phi), \mathcal{U})$ is any envelope of $B$, then there exists a shift $k \subseteq N_0$ s.t. $\mathcal{U}^k$ has a refinement by a strict envelope.

2. If $((E, \phi), (E', \phi'), \mathcal{F})$ is any envelope of $f|_B$, then there exists a shift $k \subseteq N_0$ s.t. $\mathcal{F}^k$ has a refinement by a strict envelope. Furthermore, $k$ only depends on $B$ and $(E, \phi)$.

3. Every envelope $((E, \phi), (E', \phi'), \mathcal{F})$ of $f|_B$ is equivalent to a strict envelope.

4. Any two envelopes of $f|_B$ are equivalent.

Proof. Let $(E, \phi)$ be any compatible $\varpi$-structure on $E$ and denote $\tilde{A} := \phi^{-1}(A) \subseteq E_\infty$ and $\tilde{a} := \phi^{-1}(a) \in \tilde{A}$. Since $\tilde{A}$ is open, by definition of the Fréchet topology on $E_\infty$, there exists a $k_0 \in N_0$ and an open neighbourhood $\tilde{B}_0 \subseteq E_{k_0}$ of $i_\infty^k(\tilde{a})$ in $E_{k_0}$ s.t. $\tilde{B} := (i_{k_0}^{-1})^{-1}(\tilde{B}_0) \subseteq \tilde{A}$ is an open neighbourhood of $\tilde{a}$ in $E_\infty$. Set $V_j := ((i_{k_0+j})^{-1}(\tilde{B}_0)) \subseteq E_{k_0+j}$ and $V := (V_j)_{j \in N_0}$. $V$ is a strict envelope of $\tilde{B}$ in $E^k$, where $k := (k_0 + j)_{j \in N_0}$.

Set $B := \phi(\tilde{B})$.

1. If $((E, \phi), \mathcal{U})$ is any envelope of $B$ (with $(E, \phi)$ still fixed), i.e. $\mathcal{U}$ is an envelope of $\tilde{B}$, then $V$ is a refinement of $\mathcal{U}^k$ by Lemma 3.2.

Now if $(\tilde{E}, \tilde{\phi})$ is any other compatible $\varpi$-structure on $E$, then there exists an equivalence $K : \tilde{E}_\infty \rightarrow E_\infty$ with $\tilde{\phi} = \phi \circ K$. Using Lemma 2.2 there exists a continuous linear operator $K^* : E^1 \rightarrow E^k$ for some strictly monotone increasing sequence $l \subseteq N_0$ with $K_{\infty} = K$. Then $K^*V$ is a strict envelope of $\tilde{\phi}^{-1}(B)$ in $\tilde{E}^1$ by Lemma 3.3 and by Lemma 3.2 $K^*V$ is a refinement of $\tilde{U}_l$ of $B$.

2. Immediate from 1. by restriction of $\mathcal{F}^k$.

3. Immediate from 2. and the definition of equivalence.

4. Let $\mathcal{F} = ((E, \phi), (E', \phi'), \mathcal{F})$ and $\tilde{\mathcal{F}} = ((\tilde{E}, \tilde{\phi}), (\tilde{E}', \tilde{\phi}'), \tilde{\mathcal{F}})$ be two envelopes of $f|_B$. By compatibility of $(E, \phi)$ with $(\tilde{E}, \tilde{\phi})$ and of $(E', \phi')$ with $(\tilde{E}', \tilde{\phi}')$, $J = \phi' \circ \phi : E_\infty \rightarrow \tilde{E}_\infty$ and $K' = \phi^{-1} \circ \phi' : \tilde{E}_\infty \rightarrow E_\infty$ are weak equivalences, hence there exist extensions $J : E^k \rightarrow \tilde{E}$ and $K' : \tilde{E}^n \rightarrow E'$ of $J$ and $K$, respectively. $\mathcal{F}$ is equivalent to $\mathcal{F}' := ((E^k, \phi), (E', \phi'), \mathcal{F}')$, where $\mathcal{F}' := J^k \circ \mathcal{F}_k$ and $\tilde{\mathcal{F}}$ is equivalent to $\tilde{\mathcal{F}}' := ((\tilde{E}^k, \tilde{\phi}), (\tilde{E}', \tilde{\phi}'), \tilde{\mathcal{F}}')$, where $\tilde{\mathcal{F}}' := \tilde{K}' \circ (J^* \tilde{\mathcal{F}}^k)^1$.

By 2. and Lemma 3.2 after a further rescaling, $\mathcal{F}'$ and $\tilde{\mathcal{F}}'$ have a common refinement.

\[\square\]
Definition 3.5. Let $E$ and $E'$ be $\mathcal{F}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a map.

1. $f$ is called $\mathcal{F}$-continuous or $\mathcal{F}^0$ iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. $f|_V : V \to E'$ has an envelope.

2. $f$ is called $\mathcal{F}$-continuous or $\mathcal{F}^0$ iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ and compatible $\mathcal{F}$-structures $\mathcal{F}$ on $E$ and $\mathcal{F}'$ on $E'$ s.t. $\tilde{f} := \phi^{-1} \circ f|_V \circ \phi : E_\infty \to E'_\infty$ has the following properties:

(a) $\phi^{-1}(V) \subseteq (E_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_j)$ is open for all $j \in \mathbb{N}_0$ and

(b) $\tilde{f} : (E_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_j) \supseteq \phi^{-1}(V) \to (E'_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_{j'}^\infty)$ is continuous for all $j \in \mathbb{N}_0$.

Lemma 3.6. Let $E$ and $E'$ be $\mathcal{F}$-Fréchet spaces and let $F : E \to E'$ a linear map. Then $F$ is a morphism of $\mathcal{F}$-Fréchet spaces iff $F$ is $\mathcal{F}^0$.

Proof. The implication that a morphism of $\mathcal{F}$-Fréchet spaces is $\mathcal{F}^0$ is trivial. So let $F$ be linear and $\mathcal{F}^0$. Then by definition there exists a neighbourhood $V \subseteq E$ of $0$ and compatible $\mathcal{F}$-structures $\mathcal{F}$ on $E$ and $\mathcal{F}'$ on $E'$ s.t. $\tilde{F} := \phi^{-1} \circ F \circ \phi : E_\infty \to E'_\infty$ has the property that the linear map $\iota_j^\infty \circ \tilde{F} : (E_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_j) \to (E'_\infty, \parallel \cdot \parallel_{j'})^*$ is continuous on the open neighbourhood $\phi^{-1}(V)$ of $0$ in the normed space $(E_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_j)$. But $\tilde{F}$ being linear, this implies that $\iota_j^\infty \circ \tilde{F} : (E_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_j) \to (E'_\infty, \parallel \cdot \parallel_{j'})$ is continuous everywhere and hence has a unique extension to a continuous linear map $\tilde{F}_j : E_j \to E'_j$. It is easy to see that the maps $(\tilde{F}_j)_{j \in \mathbb{N}_0}$ define a continuous linear operator $\tilde{F} : E \to E'$, which shows that $F : E \to E'$ defines a morphism of $\mathcal{F}$-Fréchet spaces.

Lemma 3.7. Let $E$ and $E'$ be $\mathcal{F}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a map.

1. If $f$ is $\mathcal{F}$-continuous, then given $x \in U$ and compatible $\mathcal{F}$-structures $\mathcal{F}$ and $\mathcal{F}'$ on $E$ and $E'$, respectively, there exists a neighbourhood $V \subseteq U$ of $x$ and a strictly monotone increasing sequence $k \subseteq \mathbb{N}_0$ s.t. $f|_V$ has an envelope of the form $((E_k^\mathcal{F}, \mathcal{F}), (E'_k, \mathcal{F}')) : V \to E'$.

2. If $f$ is $\mathcal{F}$-continuous, then given $x \in U$ and compatible $\mathcal{F}$-structures $\mathcal{F}$ and $\mathcal{F}'$ on $E$ and $E'$, respectively, there exists a neighbourhood $V \subseteq U$ of $x$ and a strictly monotone increasing sequence $k \subseteq \mathbb{N}_0$ s.t. $\tilde{f} := \phi^{-1} \circ f|_V \circ \phi : E_\infty^\mathcal{F} \supseteq \phi^{-1}(V) \to E'_\infty$ has the following properties:

(a) $\phi^{-1}(V) \subseteq (E_k^\mathcal{F}, (\iota_j^k)^* \parallel \cdot \parallel_k^\mathcal{F})$ is open for all $j \in \mathbb{N}_0$ and

(b) $\tilde{f} : (E_k^\mathcal{F}, (\iota_j^k)^* \parallel \cdot \parallel_k^\mathcal{F}) \supseteq \phi^{-1}(V) \to (E'_\infty, (\iota_j^\infty)^* \parallel \cdot \parallel_{j'}^\infty)$ is continuous for all $j \in \mathbb{N}_0$.

Proof. Straightforward, using Proposition 3.1.
Proposition 3.2. Let $E$, $E'$ and $E''$ be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets and let $f : U \to E'$ and $f' : U' \to E''$ be maps with $f(U) \subseteq U'$.

1. If $f$ is $\mathfrak{sc}$-continuous, then $f$ is continuous (as a map between Fréchet spaces).
2. If $f$ is $\mathfrak{sc}$-continuous, then $f$ is $\mathfrak{sc}$-continuous.
3. If $f$ and $f'$ are $\mathfrak{sc}$-continuous, then so is $f \circ f'$.
4. If $f$ and $f'$ are $\mathfrak{sc}$-continuous, then so is $f' \circ f$.

Proof. 1. Clear from the definitions.

2. Dito.

3. Let $x \in U$ and $x' := f(x) \in U'$. By assumption there exist neighbourhoods $V \subseteq U$ and $V' \subseteq U'$ of $x$ and $x'$, respectively together with envelopes $\mathfrak{F} = ((\mathfrak{E}, \phi), ((\mathfrak{E}', \phi') : \mathcal{U} \to \mathfrak{E}')' and \mathfrak{F}' = ((\mathfrak{E}', \phi'), ((\mathfrak{E}'', \phi'')) : \mathcal{U}' \to \mathfrak{E}'')$ of $f|V$ and $f'|V'$, respectively. Since $(\mathfrak{E}, \phi')$ and $(\mathfrak{E}', \phi')$ are equivalent $\mathfrak{sc}$-structures, there exists a weak equivalence $K : \mathfrak{E}' \to \mathfrak{E}'$ with extension $\mathfrak{K} : \mathfrak{E}_k \to \mathfrak{E}'$ for some strictly monotone increasing sequence $k \subseteq \mathbb{N}_0$. Replacing $\mathfrak{F}$ by $((\mathfrak{E}_k, \phi), ((\mathfrak{E}_k, \phi'), ((\mathfrak{E}''', \phi''), \mathfrak{F}' \circ \mathfrak{K})$, one can assume that $(\mathfrak{E}', \phi') = (\mathfrak{E}', \phi')$. Using Proposition 3.1, after shrinking $V$ and $V'$ and rescaling, one can furthermore assume that $\mathfrak{F}$ and $\mathfrak{F}'$ are strict. Now $f_0 : U_0 \to E_0'$ is continuous, $U_0' \subseteq E_0'$ is open and $f_0(\iota_0^E(\phi(x))) \subseteq U_0'$. Shrink $U_0$ to a smaller neighbourhood $U_0$ of $\iota_0^E(x)$ s.t. $f_0(\iota_0^E(\phi(x))) \subseteq U_0'$. Let $\mathcal{U}_j := (\iota_0^E(x))^{-1}(U_0)$ for $j \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{V}_j := \phi_j^{-1}(\mathcal{U}_0 \cap \mathcal{V}_j)$ Then $\mathcal{U}_j := (\mathcal{U}_j)_j \in \mathbb{N}_0$ is a strict envelope for the open neighbourhood $\mathcal{U}_0 \cap \mathcal{V}_j$ of $\phi_j^{-1}(x)$ and $\mathfrak{F}|\mathcal{U}_j$ is a strict envelope for $f_0|\mathcal{U}_j$, where $f_0 := \phi_j^{-1} \circ f \circ \phi$. Furthermore, because $\mathcal{U}_j$ and $\mathcal{V}_j$ are strict, $f_j(\mathcal{U}_j) \subseteq U_j'$ for all $j \in \mathbb{N}_0 \cup \{\infty\}$ and $f(\mathcal{V}) \subseteq \mathcal{V}'$. Hence there is a well defined envelope $\mathfrak{F}'' : \mathcal{U} \to \mathfrak{E}''$ of $\phi''^{-1} \circ (f' \circ f|\mathcal{V}) \circ \phi^{-1}$ defined by $f_j' := f_j' \circ f_j|\mathcal{U}_j$ for all $j \in \mathbb{N}_0$.

4. If $(\mathfrak{E}, \phi)$ and $(\mathfrak{E}', \phi')$ are compatible $\mathfrak{sc}$-structures on $E$ and $E'$, respectively, s.t. the condition on $f$ in Definition 3.3 is satisfied, then the same holds for $(\mathfrak{E}_k, \phi)$ and $(\mathfrak{E}_k, \phi')$, where $k, l \subseteq \mathbb{N}_0$ are strictly monotone increasing sequences with $k \geq l$.

Also, if $f$ is $\mathfrak{sc}$-continuous, then given arbitrary compatible $\mathfrak{sc}$-structures $(\mathfrak{E}, \phi)$ and $(\mathfrak{E}', \phi')$ on $E$ and $E'$, respectively, there exists a strictly monotone increasing sequence $k \subseteq \mathbb{N}_0$ s.t. Definition 3.3 is satisfied, when $(\mathfrak{E}, \phi)$ is replaced by $(\mathfrak{E}_k, \phi)$. These two facts together easily imply the claim.

Example 3.1. I continue Section 1.3.

Then in the notation of that Subsection, by Proposition 1.7, the reparametrisation action $\Psi : \Gamma_{B}(F_1) \to \Gamma_{B}(F_2)$ is $\mathfrak{sc}$-continuous, where an envelope of $\Psi$ is provided by either the maps $\Psi^k : \Gamma_{B}^k(F_1) \to \Gamma_{B}^k(F_2)$, $k \in \mathbb{N}_0$, or the maps $\Psi^{k,p} : W_{B}^{k,p}(F_1) \to W_{B}^{k,p}(F_2)$, for some fixed $1 < p < \infty$ and $k \in \mathbb{N}_0$ s.t. $kp > n$. 71
3.2 \textit{sc}-differentiability

\textbf{Definition 3.6.} Let \( E \) and \( E' \) be \textit{sc}-Fréchet spaces, let \( U \subseteq E \) be an open subset, and let \( f : U \to E' \) be a continuous map.

1. If \( f \) is weakly Fréchet differentiable as a continuous map between open subsets of the Fréchet spaces \( E \) and \( E' \) and both \( f \) and the differential of \( f \),
   \[
   Df : E \oplus E \supseteq U \times E \to E' \\
   (x, u) \mapsto Df(x)u,
   \]

are

(a) \textit{sc}-continuous, then \( f \) is called \textit{continuously sc-differentiable} or \( \text{sc}^1 \).

(b) \textit{sc}-continuous, then \( f \) is called \textit{continuously \textit{sc}-differentiable} or \( \text{sc}^1 \).

2. If \( f \) is pointwise weakly Fréchet differentiable as a continuous map between open subsets of the Fréchet spaces \( E \) and \( E' \) and both \( f \) and the differential of \( f \),
   \[
   Df : E \oplus E \supseteq U \times E \to E' \\
   (x, u) \mapsto Df(x)u,
   \]

are

(a) \textit{sc}-continuous, then \( f \) is called \textit{continuously pointwise \textit{sc}-differentiable} or \( \text{p-sc}^1 \).

(b) \textit{sc}-continuous, then \( f \) is called \textit{continuously pointwise \textit{sc}-differentiable} or \( \text{p-sc}^1 \).

\textbf{Definition 3.7.} Let \( E \) and \( E' \) be \textit{sc}-Fréchet spaces, let \( U \subseteq E \) be an open subset, and let \( f : U \to E' \) be a map.

1. An envelope \( ((E, \phi), (E', \phi'), \mathcal{F} : U \to \mathbb{E}') \) of \( f \) is called \textit{continuously \textit{sc}-differentiable} or \( \text{sc}^1 \) iff \( f_k : U_k \to E'_k \) is weakly continuously weakly Fréchet differentiable along \( \iota_k \mid_{U_{k+1}} : U_{k+1} \to U_k \) for every \( k \in \mathbb{N}_0 \).

2. \( f \) is called \textit{continuously \textit{sc}-differentiable} or \( \text{sc}^1 \) iff \( f \) is \( \textit{sc}\)-continuous and for any \( x \in U \) there exists a neighbourhood \( V \subseteq U \) of \( x \) s.t. \( f|_V : V \to E' \) has a continuously \textit{sc}-differentiable envelope.

\textbf{Remark 3.2.} There is no “pointwise” version of \textit{sc}-differentiability due to the lack of a chain rule: Theorem [1.1] does not hold for pointwise weak Fréchet differentiability. There is however a chain rule for maps that are, in the present terminology, weakly continuously pointwise weakly Fréchet differentiable (along the identity), cf. [Ham82], Section I.3.3 (esp. Theorem 3.3.4). This suffices for a chain rule for continuously pointwise \textit{sc}-differentiable and \textit{sc}\-differentiable maps.

\textbf{Remark 3.3.} Note that in the definition of a continuously \textit{sc}-differentiable envelope it is not required that \( f \) itself is weakly continuously weakly Fréchet differentiable as a map between Fréchet spaces.
Proposition 3.3. Let $E$ and $E'$ be $p$-Fréchet spaces, let $U \subseteq E$ be an open subset, and let $f : U \to E'$ be a map. Then the following implications hold:

\[
\begin{align*}
& f \overset{sc}{\subseteq}^1 \quad \Downarrow \quad f \quad \overset{sc}{\subseteq}^1 \quad \Downarrow \quad f \quad p\text{-}\overset{sc}{\subseteq}^1 \\
& f \quad p\text{-}\overset{sc}{\subseteq}^1 \quad \Downarrow \quad f \quad \overset{sc}{\subseteq}^1 \quad \Downarrow \quad f \quad p\text{-}\overset{sc}{\subseteq}^1
\end{align*}
\]

Proof. I will only show the implication $f \overset{sc}{\subseteq}^1 \Rightarrow f \overset{sc}{\subseteq}^1$. The remaining implications are either shown analogously or are immediate from the definitions, Proposition 3.2 and Theorem 1.3.

So let $f$ be $\overset{sc}{\subseteq}^1$. Then for any $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ and a continuously differentiable envelope $\hat{\gamma} = ((\mathbb{E}, \phi), (\mathbb{E}', \phi'))$ of $V$ using Proposition 3.1, after possibly making $V$ smaller, rescaling and restriction, one can assume that $\hat{\gamma}$ is strict. Define $1 := (k + 1)\in \mathbb{R}_0$. Then $U^1 \oplus \mathbb{E} := (U_{k+1} \times E_k)_{k\in \mathbb{R}_0} \subseteq \mathbb{E}^1 \oplus \mathbb{E}$ is an envelope of $U_\infty \times E_\infty = (\phi \circ \phi)^{-1}(V \times E)$, hence $((\mathbb{E}^1 \oplus \mathbb{E}, \phi \circ \phi), U^1 \oplus \mathbb{E})$ a strict envelope of $V \times E \subseteq E \oplus E$. Now define $D\gamma : U^1 \times \mathbb{E} \to \mathbb{E}'$ by the sequence of maps

\[
Df_k : U_{k+1} \times E_k \to E_k'
(y, u) \mapsto Df_k(t_k(y))u.
\]

Each $Df_k$ is a well defined continuous map because $\hat{\gamma}$ is differentiable by assumption, i.e. each of the maps $f_k : U_k \to E_k'$ in the envelope $\gamma$ of $f_\infty := \phi^{-1} \circ f|_V \circ \phi : U_\infty \to E_\infty'$ is weakly continuously weakly Fréchet differentiable along $t_k$. It will now be proved that $f_\infty$ is weakly continuously weakly Fréchet differentiable, and hence so is $f|_V$, and that $D\gamma$ is an envelope of $Df_\infty : U_\infty \times E_\infty \to E_\infty'$. This finishes the proof.

First, note that from $f_k \circ t_k|_{U_{k+1}} = t_k \circ f_{k+1}$, by the chain rule, Theorem 1.1 it follows that $Df_k \circ (t_{k+1}|_{U_{k+1}} \circ t_k) = t_k \circ Df_{k+1} : U_{k+2} \times E_{k+1} \to E_k'$. Lemma 3.1 shows that there exists a unique continuous map $\Phi : U_\infty \times E_\infty \to E_\infty'$ s.t. $D\gamma$ is an envelope of $\Phi$, i.e. $Df_k \circ (t_{k+1}|_{U_{k+1}} \circ t_k) = t_k \circ \Phi$. It remains to show that $f_\infty$ is weakly Fréchet differentiable with differential $Df_\infty = \Phi$. For this, one computes for $(y, u) \in U_\infty \times E_\infty$, $t \in (0, 1]$ and

\[
r_{f_\infty}^t(u, t) = \frac{1}{t} (f_\infty(y + tu) - f_\infty(u)) - \Phi(y, u)
\]

that

\[
i_k \circ r_{f_\infty}^t(u, t) = \frac{1}{t} (i_k \circ f_\infty(y + tu) - i_k \circ f_\infty(y)) - i_k \circ \Phi(y, u)
= \frac{1}{t} (f_k(i_k(y) + t_k(y)) - f_k(t_k(y))) - Df_k((i_k \times t_k)(y, u))
= r_{i_k}^t(u, t)
\]

and hence (note that $i_k(y) = i_k(i_k(y)) \in \text{im } t_k$)

\[
i_k \circ r_{f_\infty} = r_{i_k}^t \circ (i_k \times \text{id}_{[0,1]}).\]
By the definition of the topology on $E'_\infty$, $r^\infty_y$ is continuous iff $r^\infty_k \circ r^\infty_y$ is continuous for all $k \in \mathbb{N}_0$. And by the above equality, these maps are continuous as compositions of continuous functions. So by definition, $f_\infty$ is weakly Fréchet differentiable with differential given by $\Phi$. \hfill \Box

**Definition 3.8.** Let $E$ and $E'$ be sc-chains, let $U_\infty \subseteq E_\infty$ be an open subset, and let $\mathcal{F} : U \to E'$ be a continuously sc-differentiable envelope of a continuous map $f_\infty : U_\infty \to E'_\infty$. The envelope $D\mathcal{F} : U^1 \oplus E \to E'$ of $Df_\infty : U_\infty \times E_\infty \to E'_\infty$ given by the sequence of maps

$$Df_k : U_{k+1} \times E_k \to E'_k$$

$$(y, u) \mapsto Df_k(t_k(y))u$$

is called the differential of $\mathcal{F}$.

**Definition 3.9.** Let $E$ and $E'$ be $\infty$-Fréchet spaces and let $U \subseteq E$ be an open subset. Let furthermore $k \in \mathbb{N}_0$, $k \geq 2$.

1. An
   
   i. $\infty$-continuous
   
   ii. $\infty$-continuous
   
   iii. $\infty$-continuous
   
   iv. $\infty$-continuous

   map $f : U \to E'$ is called
   
   i. $k$-times continuously $\infty$-differentiable or $\infty^k$
   
   ii. $k$-times continuously sc-differentiable or sc$^k$
   
   iii. $k$-times continuously pointwise $\infty$-differentiable or $p-\infty^k$
   
   iv. $k$-times continuously pointwise sc-differentiable or $p\text{-sc}^k$

   iff
   
   i. $f$ is $\infty^k$ and $Df$ is $\infty^{k-1}$. It is called $\infty^\infty$ iff it is $\infty^k$ for all $k \in \mathbb{N}_0$.
   
   ii. $f$ is sc$^k$ and $Df$ is sc$^{k-1}$. It is called sc$^\infty$ iff it is sc$^k$ for all $k \in \mathbb{N}_0$.
   
   iii. $f$ is $p\text{-sc}^k$ and $Df$ is $p\text{-sc}^{k-1}$. It is called $p\text{-sc}^\infty$ iff it is $p\text{-sc}^k$ for all $k \in \mathbb{N}_0$.
   
   iv. $f$ is $p\text{-sc}^k$ and $Df$ is $p\text{-sc}^{k-1}$. It is called $p\text{-sc}^\infty$ iff it is $p\text{-sc}^k$ for all $k \in \mathbb{N}_0$.

2. An envelope $(E, \phi), (E', \phi'), (\mathcal{F} : U \to E')$ of $f$ is called $k$-times continuously $\infty$-differentiable or $\infty^k$ iff $\mathcal{F}$ is $\infty^1$ and $D\mathcal{F}$ is $\infty^{k-1}$. It is called $\infty^\infty$ iff it is $\infty^k$ for all $k \in \mathbb{N}_0$.

3. An $\infty$-continuous map $f : U \to E'$ is called $k$-times continuously $\infty$-differentiable or $\infty^k$, for $k \in \mathbb{N}_0 \cup \{\infty\}$, iff for any $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. $f|_V : V \to E'$ has a $k$-times continuously differentiable envelope.

**Remark 3.4.** As usual, if $f : U \to E'$ is $k$-times continuously differentiable in one of the versions above, then the repeated differentials provide maps

$$D^kf : E \oplus E^\otimes k \supseteq U \times E^k \to E'$$

$$(x, u_1, \ldots, u_k) \mapsto D^kf(x)(u_1, \ldots, u_k)$$

and one can express $D(Df)$ in terms of $D^1f$ and $D^2f$, etc.
Theorem 3.1. (Chain rule). Let $E$, $E'$ and $E''$ be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets and let $f : U \to E'$ and $f' : U' \to E''$ be maps with $f(U) \subseteq U'$. For any $k \in \mathbb{N}_0 \cup \{\infty\}$, if $f$ and $f'$ are $\mathfrak{sc}^k$, $\mathfrak{sc}^k$, $\mathfrak{pc}^k$, $\mathfrak{pp}^k$ or $\mathfrak{sc}^k$, then so is $f' \circ f : U \to E''$. In all cases, for $k \geq 1$,

$$D(f' \circ f)(x) = Df'(f(x)) \circ Df(x) \quad \forall x \in U.$$ 

Proof. This is a corollary to Theorem 1.1 [Ham82], Part I, Theorem 3.3.4, and Proposition 3.2.

Proposition 3.4. Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset, let $f : U \to E'$ be a map, and let $((E, \phi), (E', \phi'), \mathcal{F} : U \to \mathbb{R}^k)$ be an envelope of $f$. If $\mathcal{F} : U \to \mathbb{R}^k$ is strict and $\mathfrak{sc}^k$ for $k \in \mathbb{N}_0 \cup \{\infty\}$, then $f_0 : U_0 \to E_0'$ defines an $\mathfrak{sc}^k$-map in the sense of [HWZ10], Definition 1.8.

Proof. This is a corollary to Propositions 1.4 and 1.6.

Example 3.2. I continue Section 1.3 and Example 3.1. Then in the notation of that Subsection, by Proposition 1.8 the reparametrisation action $\Psi : \Gamma_B(F_1) \to \Gamma_B(F_2)$ is $\mathfrak{sc}^\infty$, where an envelope of $\Psi$ is provided by either the maps $\Psi^k : \Gamma^k_B(F_1) \to \Gamma^k_B(F_2)$, $k \in \mathbb{N}_0$, or the maps $\Psi^{k,p} : W^{k,p}_B(F_1) \to W^{k,p}_B(F_2)$, for some fixed $1 < p < \infty$ and $k \in \mathbb{N}_0$ s.t. $kp > n$. Also, by Proposition 1.8 the differential of $\Psi$ is composed of maps of the same type as $\Psi$:

- $\Psi(b, u) = (b, \Psi_2(b, u))$
- $\Psi_2(b, u) = \Phi_0^* u$
- $D\Psi(b, u)(e, v) = (e, D\Psi_2(b, u)(e, v))$
- $D\Psi_2(b, u)(e, v) = \Phi_{(b,e)}^* \nabla u + \Phi_{(b,e)}^* u + \Phi_0^* v$.

Showing that $\Psi$ is $\mathfrak{sc}^k$ for all $k \in \mathbb{N}_0 \cup \{\infty\}$ is now a simple matter of induction, repeatedly using Proposition 1.8.
4 The Nash-Moser inverse function theorem

In this section, I first give a quick overview of the necessary terminology, adapted to the present notation, from [Ham82] (mainly Chapter II), before stating the famous Nash-Moser inverse function theorem. This is intended to set up notation and does not aim to give a general pedagogic introduction to the Nash-Moser inverse function theorem. It assumes knowledge of at least the main points of the article [Ham82], or Section 51 of the textbook [KM97]. Subsequently, a version that is better adapted to the present notation and needed to prove the constant rank theorem, finite dimensional reduction, and the Sard-Smale theorem for nonlinear Fredholm maps in the next section is stated and proved.

4.1 Tame $\mathfrak{m}$-structures and morphisms

**Lemma 4.1.** Let $k = (k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ be a strictly monotone increasing sequence. Then for all $i, j \in \mathbb{N}_0$ with $i \leq j$,

$$k_j - k_i \geq j - i$$

and the following are equivalent:

1. There exists $c \in \mathbb{N}_0$ s.t. $k_j - k_i \leq j - i + c$ for all $i, j \in \mathbb{N}_0$ with $i \leq j$.
2. There exists $c \in \mathbb{N}_0$ s.t. $k_j \leq j + c$ for all $j \in \mathbb{N}_0$, i.e. there exists a shift $1 \subseteq \mathbb{N}_0$ with $k \leq 1$.
3. There exists $j_0 \in \mathbb{N}_0$ s.t. $k_j - k_i = j - i$ for all $j \geq i \geq j_0$.
4. There exists $j_0 \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$ s.t. $k_j = j + r$ for all $j \geq j_0$.
5. There exists a shift $1$ s.t. $k \circ 1$ is a shift again.

**Proof.** $k$ strictly monotone increasing means that $k_1 - k_{l-1} \geq 1$ for all $l \in \mathbb{N}_0$. Hence

$$k_j - k_i = \sum_{l=i+1}^{j} (k_l - k_{l-1}) \geq \sum_{l=i+1}^{j} 1 = j - i.$$

I will show $1 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 1 \Rightarrow 3$ and $1 \Rightarrow 5 \Rightarrow 2$.

$1 \Rightarrow 2$: Set $i = 0$.

$2 \Rightarrow 3$: It is $k_j - j = k_0 + \sum_{l=1}^{j} (k_l - k_{l-1} - 1)$ and $k_j - k_{j-1} - 1 \geq 0$ because $k$ is strictly monotone increasing. So $k_j - j \leq c$ for all $j \in \mathbb{N}_0$ implies that $k_j - k_{j-1} - 1 \neq 0$ only for finitely many $j \in \mathbb{N}_0$ and hence $k_j - j$ is eventually constant. Set $r := \lim_{j \to \infty} (k_j - j)$.

$3 \Rightarrow 1$: By assumption, if $j_0 \leq i \leq j$, then $k_j - k_i = j + r - (i + r) = j - i$. If $j \geq j_0$ and $i \leq j_0$, then $k_j - k_i = j + r - k_i \leq j + r \leq j - i + j_0 + r$. If $0 \leq i \leq j \leq j_0$, then $k_j - k_i \leq j \leq i + k_{j_0}$. So for arbitrary $i, j \in \mathbb{N}_0$ with $i \leq j$, $k_j - k_i \leq j - i + \max\{k_{j_0}, j_0 + r\}$.

$3 \Rightarrow 5$: Set $i = j_0$. Then $k_j = j + (k_i - i)$.
4. \( \Rightarrow \) 3.: For \( j \geq i \geq j_0, k_j - k_i = j + r - (i + r) = j - i. \)

4. \( \Rightarrow \) 5.: Define \( l := (j_0 + j)_{j \in \mathbb{N}_0}. \) Then \( m := k \circ l \) satisfies \( m_j = k_{j_0+j} = j_0 + j + r = j + (j_0 + r). \)

5. \( \Rightarrow \) 2.: It is \( k \circ l \geq k. \)

**Definition 4.1.** Let \( k = (k_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) be a strictly monotone increasing sequence and let \( E \) and \( E' \) be ILB- or sc-chains.

1. \( k \) is called *tame* iff any/all of the conditions in Lemma 4.2 are satisfied.
2. The rescaling \( E^k \) of \( E \) is called *tame* iff \( k \) is tame.
3. A weak morphism \( T : E_\infty \rightarrow E'_\infty \) is called *tame* iff it has an extension \( T : E^k \rightarrow E' \) s. t. \( E^k \) is a tame rescaling.
4. An equivalence \( J : E_\infty \rightarrow E'_\infty \) is called *tame* iff \( J \) is tame and it has a tame inverse, i.e. if there exists a tame weak embedding \( K : E'_\infty \rightarrow E_\infty \) s. t. \( J \circ K = \text{id}_{E'_\infty} \) and \( K \circ J = \text{id}_{E_\infty}. \)
5. \( E \) and \( E' \) are called *tamely equivalent* iff there exists a tame equivalence between them.

**Example 4.1.** Every shift is tame. The only rescalings explicitly chosen in any lemma/proposition/theorem so far, such as Lemma 2.7, Lemma 2.12 or Proposition 3.1, have been shifts, in particular tame.

**Example 4.2.** Let \( E \) be a tame ILB- or sc-chain and let \( k, l \subseteq \mathbb{N}_0 \) be tame strictly monotone increasing sequences with \( k \geq l. \) Then \( l^k : E^k \rightarrow E^l \) is tame.

**Lemma 4.2.** Let \( k, l \subseteq \mathbb{N}_0 \) be strictly monotone increasing sequences. If \( k \) and \( l \) are tame, then so is \( k \circ l. \)

**Proof.** Obvious. \( \square \)

**Definition 4.2.** Let \( E \) be a topological vector space.

1. Two \( \mathfrak{F} \)-structures \( (E, \phi) \) and \( (\tilde{E}, \tilde{\phi}) \) on \( E \) are called *tamely equivalent* if there exists a tame equivalence \( J : E_\infty \rightarrow \tilde{E}_\infty \) with \( \phi = \tilde{\phi} \circ J. \)
2. A *pre-tame \( \mathfrak{F} \)-Fréchet space* is a topological vector space \( E \) together with a tame equivalence class of \( \mathfrak{F} \)-structures on it. The sc-chains in this equivalence class are then called *tamely compatible*.
3. A *tame morphism* between pre-tame \( \mathfrak{F} \)-Fréchet spaces \( E \) and \( E' \) is a continuous linear operator \( T : E \rightarrow E' \) between \( E \) and \( E' \) as topological vector spaces s. t. there exist tamely compatible \( \mathfrak{F} \)-structures \( (E, \phi) \) and \( (E', \phi') \) on \( E \) and \( E' \), respectively, s. t. \( T_\infty := \phi'^{-1} \circ T \circ \phi : E_\infty \rightarrow E'_\infty \) defines a tame weak morphism.

\[
\begin{array}{ccc}
E_\infty & \xrightarrow{T_\infty} & E'_\infty \\
\phi \downarrow & \circ & \downarrow \phi' \\
E & \xrightarrow{T} & E'
\end{array}
\]
Remark 4.1. So the relation between sc-Fréchet spaces and pre-tame sc-Fréchet spaces is as follows:
Every sc-Fréchet space has its associated class of compatible sc-structures. This class of sc-structures then is further decomposed into the equivalence classes of tame equivalence. Picking one such equivalence class produces a pre-tame sc-Fréchet space.

Remark 4.2. One also has the corresponding notions of (split) subspaces, direct sums, (strongly) smoothing morphisms and Fredholm operators for pre-tame sc-Fréchet spaces just as in Sections 2.4 to 2.6. This generalisation is completely straightforward and would repeat the definitions and results from Sections 2.4 to 2.6 almost word by word, just inserting the word “(pre-)tame” over and over again, so it will be skipped.

Definition 4.3. Let \((X, \| \cdot \|_X)\) be a Banach space. The chain of exponentially decreasing sequences in \(X\) is the following ILB-chain \(\Sigma(X) = \{(\Sigma_k(X), \| \cdot \|_k, \iota_k)_{k \in \mathbb{N}_0}\}:
\[
\Sigma_k(X) := \left\{ x = (x_j)_{j \in \mathbb{N}_0} \in X^{\mathbb{N}_0} \left| \sum_{j=0}^{\infty} e^{kj} \| x_j \|_X < \infty \right. \right\}
\]
\[\|x\|_k := \sum_{j=0}^{\infty} e^{kj} \| x_j \|_X\]
\[\iota_k \text{ is the canonical inclusion.}\]

Remark 4.3. \(\Sigma(X)\) is not an sc-chain unless \(X\) is finite dimensional. To see this take a sequence \(x^i \subseteq \Sigma_\infty(X)\), where \(x^i_j = 0\) for all \(j > 0\) and \((x^i_0)_{i \in \mathbb{N}_0}\) is any bounded sequence in \(X\).

Definition 4.4. Let \(E\) be an ILB- or sc-chain. \(E\) is called tame if there exist tame weak morphisms \(J : E_\infty \to \Sigma_\infty(X)\) and \(K : \Sigma_\infty(X) \to E_\infty\) between \(E\) and \(\Sigma(X)\) for some Banach space \(X\) s.t. \(K \circ J = \text{id}_{E_\infty}\).

Lemma 4.3. Let \(E\) and \(E'\) be ILB- or sc-chains and let \(k, l \subseteq \mathbb{N}_0\) be strictly monotone increasing sequences.

1. Assume that there exist tame weak morphisms \(J : E'_\infty \to E_\infty\) and \(K : E_\infty \to E'_\infty\) s.t. \(K \circ J = \text{id}_{E'_\infty}\). If \(E\) is tame, then so is \(E'\).
2. If \(E\) and \(E'\) are tamely equivalent, then \(E\) is tame iff \(E'\) is tame. In particular, if \(E\) is tame, then so is every tame rescaling of \(E\).

Proof. All of these are immediate from the definitions. \(\square\)

Example 4.3. All the ILB-chains or sc-chains and morphisms and equivalences between them seen so far in the previous sections are tame, cf. [Ham82], esp. Chapter II, Corollary 1.3.9.

Definition 4.5. Let \(E\) be an ILB- or sc-chain.

1. \(E\) is said to admit smoothing operators if there exists a continuous family of strongly smoothing operators \((S_t : E \to E)_{t \in [0, \infty)}\), i.e. there exists a continuous map
\[
\mathcal{S} : [0, \infty) \times E_0 \to E_\infty
\]
\[(t, e) \mapsto S_t e,\]
with each $S_t^k : E_0 \to E_\infty$, linear, s. t. the following hold:
For $k, l \in \mathbb{N}_0$, denote $S^k_\infty := \iota^k_\infty \circ \overline{S}_t \circ \iota^k_0 : E_k \to E_l$. Then $S_t = (S^k_{t,k})_{k \in \mathbb{N}_0} : E \to E$.
Furthermore there exist constants $p \in \mathbb{N}_0$, $C^k_l \in [0, \infty)$ for $k, l \in \mathbb{N}_0$ s. t.

\[ \| S^k_{t,l} \|_{L_c(E_k, E_l)} \leq C^k_l \left(1 + e^{(p+(l-k))t}\right) \quad \forall k, l \in \mathbb{N}_0, t \geq 0 \]
\[ \| S^k_{t,l} - S^k_{0,l} \|_{L_c(E_k, E_l)} \leq C^k_l e^{(p-(k-l))t} \quad \forall k, l \in \mathbb{N}_0, k - l \geq p, t \geq 0. \]

2. $E$ is called weakly tame if there exists a tame rescaling $E^k$ of $E$ that admits smoothing operators.

**Remark 4.4.** Note that for $k, l \in \mathbb{N}_0$ with $k - l \geq p$ the first inequality follows from the second (modulo replacing $C^k_l$ by $C^k_l + 1$), because $\| S^k_{t,l} \|_{L_c(E_k, E_l)} = \| t^k - S^k_{t,l} \|_{L_c(E_k, E_l)} \leq \| t^k \|_{L_c(E_k, E_l)} + \| t - S^k_{t,l} \|_{L_c(E_k, E_l)} \leq 1 + C^k_l e^{(p-(k-l))t} \leq 1 + C^k_l$, since $p - (k - l) \leq 0$. And $C^k_l \left(1 + e^{p+(l-k)}t\right) \geq C^k_l$.

And for $k - l < p$, i.e. $p + (l - k) > 0$, one has $e^{(p+(l-k))t} \leq 1 + e^{(p+(l-k))t} \leq 2e^{(p+(l-k))t}$. Hence one could also replace the first inequality by

\[ \| S^k_{t,l} \|_{L_c(E_k, E_l)} \leq C^k_l e^{(p+(l-k))t} \quad \forall k, l \in \mathbb{N}_0, k - l < p, t \geq 0 \]

**Remark 4.5.** Defining $p := (p + j)_{j \in \mathbb{N}_0}$, $\tilde{S}_t := \overline{S}_t \circ \iota^0_0 : E^p_0 = E_p \to E_\infty$, one can consider the $\tilde{S} : [0, \infty] \times E^p_0 \to E_\infty$ as defining a continuous family of strongly smoothing operators $\tilde{S}_t = E^p \to E_\infty$. Then the continuous linear operators $\tilde{S}_t : = \iota^\infty \circ \tilde{S}_t \circ (p_0^j) : E^p \to E_l$ satisfy $\tilde{S}_t^k = S_{k,t}^{p+k}$ and satisfy the inequalities

\[ \| \tilde{S}_t^k \|_{L_c(E^p_k, E_l)} \leq \tilde{C}^k_l \left(1 + e^{(l-k)}t\right) \quad \forall k, l \in \mathbb{N}_0, t \geq 0 \]
\[ \| t^k - \tilde{S}_t^k \|_{L_c(E^p_k, E_l)} \leq \tilde{C}^k_l e^{-(k-l)}t \quad \forall k, l \in \mathbb{N}_0, k - l \geq 0, t \geq 0, \]

where $\tilde{C}^k_l : = C^k_l + k$.

**Remark 4.6.** Note that by Proposition 1.5 the map

\[ [0, \infty) \to L_c(E_1, E_\infty) \quad t \mapsto \overline{S}_t \circ \iota^1_0 \]

is continuous. So if $E$ is a weakly tame sc-chain, then there exists a tame rescaling $E^k$ of $E$ that admits smoothing operators that a fortiori have the property that the map

\[ S : [0, \infty) \to L_c(E_0, E_\infty) \quad t \mapsto \overline{S}_t \]

is continuous.

**Lemma 4.4.** Let $E$ and $E'$ be ILB- or sc-chains and let $k \subseteq \mathbb{N}_0$ be a strictly monotone increasing sequence.

1. If $k$ is tame and $E$ admits smoothing operators, then $E^k$ admits smoothing operators.

2. Assume that there exist tame weak morphisms $J : E'_\infty \to E_\infty$ and $K : E_\infty \to E'_\infty$ s. t. $K \circ J = \text{id}_{E'_\infty}$. If $E$ is weakly tame, then so is $E'$.  

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3. If $\mathcal{E}$ and $\mathcal{E}'$ are tamely equivalent, then $\mathcal{E}$ is weakly tame if $\mathcal{E}'$ is weakly tame. In particular, if $\mathcal{E}$ is weakly tame, then so is every tame rescaling of $\mathcal{E}$.

Proof. 1. In the notation of Definition 4.5 define $\mathcal{S}' : [0, \infty) \times E_0^k \to E_\infty^k$ by $\mathcal{S}'_{\mathcal{E}} := \mathcal{S}'_{\mathcal{E}} \circ \iota_0^{k_0} : E_0^k = E_{k_0} \to E_\infty^k = E_{\infty}^k$. Then

\[ S_{n,t}^{m,k} = (\iota_{n})_{n} \circ \mathcal{S}' \circ (\iota_{n})_{0}^{m} \]

Using that by Lemma 4.2 there exists $c \in \mathbb{N}_0$ s.t. $j - i \leq k_j - k_i \leq j - i + c$ or $i - j - c \leq k_i - k_j \leq j - i$, for $j \geq i$, the required estimates for $S_{n,t}^{m,k}$ follow from those for the $S_{n,t}^{m,k}$ with a constant $p' := p + c$.

2. As in Remark 4.3 one can find tame strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ and extensions $K : E^j \to E'$ and $J : E'^k \to E^l$ with $K \circ J = 1^k : E'^k \to E'$ and $J \circ K = 1^l : E^l \to E^j$. Furthermore, using [1] and Lemma 4.2, one can assume that $k$ and $l$ are shifts, i.e. $k = (k_0 + 1)_j \in \mathbb{N}_0$ and $l = (l_0 + 1)_j \in \mathbb{N}_0$ for some $k_0, l_0 \in \mathbb{N}_0$. Then $E'^k$ admits smoothing operators:

Let $\mathcal{S} : [0, \infty) \times E_0 \to E_\infty$ and $p \in \mathbb{N}_0$, $C^m_\mathcal{E}_\mathcal{E} \in \mathbb{N}_0$ be as in Definition 4.5 Define

\[ p' := p + k_0, \]

\[ C^m_{\mathcal{S}} := C^m_{\mathcal{S}_n} \circ K_{k_0+n} \circ L_n(E_0^k, E_{k_0+n}^l) \circ J_n \circ L_n(E_{k_0+n}^l, E_{k_0+n}^l), \]

and define $\mathcal{S}' : [0, \infty) \times E_0^k \to E_{\infty}^k$ by

\[ \mathcal{S}'_{\mathcal{E}} := K_{\infty} \circ \mathcal{S}_{\mathcal{E}} \circ \iota_0^{l_0} \circ J_0 : E_0^k \to E_{\infty}^k \]

Now one calculates for $n, m \in \mathbb{N}_0$

\[ S_{n,t}^{m,k} = (\iota_{n})_{n} \circ \mathcal{S}' \circ (\iota_{n})_{0}^{m} \]

And if $l_0 + m \geq k_0 + l_0 + n + p$, i.e. $m \geq n + k_0 + p = n + p'$, then one can furthermore calculate

\[ K_{k_0+n} \circ \iota_{k_0+n}^{l_0+m} \circ J_m = K_{k_0+n} \circ (\iota_{k_0+n})_{k_0+n}^{m} \circ J_m \]

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hence

\[(i^k)_n - S^m_{n,t} = K_{k_0+n} \circ (i^{l_0+n}_{k_0+l_0+n} - S^l_{k_0+l_0+n,t}) \circ J_m.\]

Applying the estimates for the \(S^k_{l,t}\) and \(i^k_l - S^k_{l,t}\) to the above formulas for \(S^m_{n,t}\) and \((i^k)_n - S^m_{n,t}\) immediately yields the result.

3. This is immediate from 2.

Proposition 4.1. Let \(E\) be an ILB- or sc-chain. If \(E\) is tame, then \(E\) is weakly tame.

Proof. By [Ham82], Part II, Section 1.4, \(\Xi(X)\), for \((X, \| \cdot \|_X)\) a Banach space, is weakly tame. The result now follows immediately from the definition of tame and Lemma 4.4, 2.

Definition 4.6. Let \(E\) be a pre-tame \(\mathcal{S}\)-Fréchet space.

1. A tamely compatible \(\mathcal{S}\)-structure \((E, \phi)\) on \(E\) is called tame if \(E\) is tame.

2. \(E\) is called tame if one/any tamely compatible \(\mathcal{S}\)-structure on \(E\) is tame.

3. A tame \(\mathcal{S}\)-Fréchet space is a tame pre-tame \(\mathcal{S}\)-Fréchet space.

4. A tamely compatible \(\mathcal{S}\)-structure \((E, \phi)\) on \(E\) is called weakly tame if \(E\) is weakly tame.

5. \(E\) is called weakly tame if one/any tamely compatible \(\mathcal{S}\)-structure on \(E\) is weakly tame.

6. A weakly tame \(\mathcal{S}\)-Fréchet space is a weakly tame pre-tame \(\mathcal{S}\)-Fréchet space.

Example 4.4. All the \(\mathcal{S}\)-Fréchet spaces that appeared so far define tame \(\mathcal{S}\)-Fréchet spaces and all the morphisms between them that appeared so far are tame.

Proposition 4.2. Let \(E, E'\) and \(E^1, \ldots, E^k\), for some \(k \in \mathbb{N}_0\), be pre-tame \(\mathcal{S}\)-Fréchet spaces.

1. If \(E\) is (weakly) tame, \(E' \subseteq E\) is a pre-tame \(\mathcal{S}\)-subspace, and if there exists a tame morphism \(P : E \to E'\) s.t. \(P \circ J = \text{id}_{E'}\), where \(J : E' \hookrightarrow E\) is the canonical inclusion, then \(E'\) is (weakly) tame.

2. If the \(E^i, i = 1, \ldots, k\), are (weakly) tame, then so is their direct sum \(E^1 \oplus \cdots \oplus E^k\) (as a pre-tame \(\mathcal{S}\)-Fréchet space) and the canonical projections \(P^i : E^1 \oplus \cdots \oplus E^k \to E^i\) and canonical inclusions \(J^i : E^i \to E^1 \oplus \cdots \oplus E^k\) are tame morphisms.

Proof. 1. This follows immediately from Lemmas 4.3 and 4.4.

2. The case that the \(E^i, i = 1, \ldots, k\), are tame follows from [Ham82], Part II, Lemma 1.3.4.

For the case that the \(E^i, i = 1, \ldots, k\), are ILB-chains which admit smoothing operators, then their direct sum \(E := E^1 \oplus \cdots \oplus E^k\) admits smoothing operators as well. So
assume this to be the case and let \((S^i_t : E^i \to E^i)_{t \in [0, \infty)}\) be the corresponding continuous families of strongly smoothing operators, given by continuous maps \(\overline{S}^i : [0, \infty) \times E^i_0 \to E^i_\infty\)
\[
(t, e) \mapsto \overline{S}^i_t e,
\]
as in Definition 4.5 Define \((S_t : E \to E)_{t \in [0, \infty)}\) via
\[
\overline{S} : [0, \infty) \times E_0 \to E_\infty,
\]
\[
(t, (e^1, \ldots, e^k)) \mapsto (\overline{S}^1_t e^1, \ldots, \overline{S}^k_t e^k).
\]

Then for \(m, n \in \mathbb{N}_0\) and \((e^1, \ldots, e^k) \in E_m\),
\[
S^m_{n,t}(e^1, \ldots, e^k) = (\overline{S}^1_{t,n} e^1, \ldots, \overline{S}^k_{t,n} e^k).
\]

Hence, if \(p^i, C_{i,m}^n \in [0, \infty)\) are the constants from Definition 4.5 for the \(E^i\), then
\[
\|S^m_{n,t}(e^1, \ldots, e^k)\|_n = \|\overline{S}^1_{t,n} e^1\|_1 + \cdots + \|\overline{S}^k_{t,n} e^k\|_k
\leq C_{i,m}^n (1 + e^{(p^i + (n-m))t}) \|e^1\|_1 + \cdots + C_{i,m}^n (1 + e^{(p^i + (n-m))t}) \|e^k\|_k
\leq C_{i,m}^n (1 + e^{(p^i + (n-m))t}) \|e^1\|_m + \cdots + C_{i,m}^n (1 + e^{(p^i + (n-m))t}) \|e^k\|_m,
\]
where \(C_{i,m}^n := \max\{C_{i,m}^n \mid i = 1, \ldots, k\}\) and \(p := \max\{p^i \mid i = 1, \ldots, k\}\). Hence
\[
\|S^m_{n,t}\|_{L_c(E_m, E_n)} \leq C_{i,m}^n (1 + e^{(p^i + (n-m))t}).
\]
And analogously for the estimate for \(e^m_n - S^m_{n,t}\).

**4.2 Tame nonlinear maps**

First, note that all the notions and results on envelopes from Section 4.1 carry over ad verbatim to pre-tame \sc-Fréchet spaces, just adding the word “tame” in the appropriate places. E.g. if \(E\) and \(E^t\) are pre-tame \sc-Fréchet spaces, \(A \subseteq E\) is a subset and \(f : A \to E^t\) is a map, then for an envelope \((\langle E, \phi, \tau \rangle, \mathfrak{U})\) of \(A\) it is assumed that \((\langle E, \phi \rangle, \mathfrak{U})\) is a tame compatible \sc-structure. And likewise for an envelope \((\langle E, \phi, (E', \phi'), \mathfrak{F} \rangle : \mathfrak{U} \to E')\) of \(f\) it is assumed that \((\langle E, \phi \rangle, (E', \phi')\) are tame compatible \sc-structures on \(E\) and \(E^t\), respectively, and all the rescalings appearing in the definition of equivalence of envelopes are assumed to be tame. To not unnecessarily bloat this text this will not be carried out explicitly but I will rely instead on this being sufficiently obvious.

**Definition 4.7.** Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be normed vector spaces, let \(A \subseteq X\) be a subset, and let \(f : A \to Y\) be a map.

\(f\) is said to satisfy a tame estimate if there exists a constant \(C \in [0, \infty)\) s.t.
\[
\|f(x)\|_Y \leq C(1 + \|x\|_X) \quad \forall x \in A.
\]

**Definition 4.8.** Let \(E\) and \(E^t\) be ILB- or sc-chains, let \(A \subseteq E_\infty\) be a subset, and let \(f : A \to E^t_\infty\) be a map.

1. \(f\) is called tame \((w. r. t. E\) and \(E^t)\) if for every \(k \in \mathbb{N}_0\),
\[
f : (E_\infty, (i_k^E)^\ast\|\cdot\|_k) \supseteq U \to (E^t_\infty, (i_k^E)^\ast\|\cdot\|_k)
\]
satisfies a tame estimate.
2. An envelope $\mathcal{F} : \mathcal{U} \rightarrow \mathcal{E}'$ of $f$ is called tame iff for every $k \in \mathbb{N}_0$,

$$f_k : (E_k, \| \cdot \|_k) \supseteq \mathcal{U}_k \rightarrow (E'_k, \| \cdot \|_k)$$

satisfies a tame estimate.

**Lemma 4.5.** Let $E$, $\tilde{E}$, $E'$ and $\tilde{E}'$ be ILB- or sc-chains, let $A \subseteq E'_\infty$ be a subset and let $f : A \rightarrow E'_\infty$ be a map, and let $T : \tilde{E} \rightarrow \tilde{E}$ and $\mathcal{S} : E' \rightarrow E$ be continuous linear operators.

1. If $f$ is tame as a map from $A \subseteq E_\infty$ to $E'_\infty$, then $f$ is tame as a map from $A \subseteq E'_\infty = \tilde{E}'_\infty$ to $E'_\infty$ for all strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ with $k \geq l$.

2. If $f$ is tame, then $f \circ T_\infty : \tilde{E}_\infty \supseteq T_\infty^{-1}(A) \rightarrow E'_\infty$ and $S_\infty \circ f : A \rightarrow \tilde{E}'_\infty$ are tame as well.

3. If $\mathcal{F} : \mathcal{U} \rightarrow \mathcal{E}'$ is a tame envelope of $f$, then so is any refinement of $\mathcal{F}$ as well as $\tilde{\mathcal{F}} \circ \mathcal{F} : U_k \rightarrow \mathcal{E}_k$ for all strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ with $k \geq l$.

4. If $\mathcal{F} : \mathcal{U} \rightarrow \mathcal{E}'$ is a tame envelope of $f$, then $\mathcal{S}^* \mathcal{F}$ and $\mathcal{S} \circ \mathcal{F}$ are tame envelopes of $f \circ T_\infty$ and $S_\infty \circ f$, respectively.

**Proof.** Straightforward. $\square$

**Definition 4.9.** Let $E$ and $E'$ be pre-tame $\mathfrak{F}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \rightarrow E'$ be a map.

1. If $f$ is $\mathfrak{F}^0$, then $f$ is called tame iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ and tamely compatible $\mathfrak{F}$-structures $(E, \phi)$ and $(E', \phi')$ on $E$ and $E'$, respectively, s.t. $\phi^{-1} \circ f \circ \phi^{-1}(V) \rightarrow E'_\infty$ is tame.

2. An envelope $((E, \phi), (E', \phi'), \mathcal{F})$ of $f$ is called tame iff $(E, \phi)$ and $(E', \phi')$ are tamely compatible $\mathfrak{F}$-structures on $E$ and $E'$, respectively, and $\mathcal{F}$ is tame.

3. If $f$ is $\mathfrak{F}^0$, then $f$ is called tame iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. $f|_V : V \rightarrow E'$ has a tame envelope.

**Proposition 4.3.** Let $E$ and $E'$ be pre-tame $\mathfrak{F}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \rightarrow E'$ be a map.

1. If $f$ is $\mathfrak{F}^0$, then $f$ is tame iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. for every pair of tamely compatible $\mathfrak{F}$-structures $(E, \phi)$ and $(E', \phi')$ on $E$ and $E'$, respectively, there exist tame strictly monotone increasing sequences $k, l \subseteq \mathbb{N}_0$ s.t. $\phi'^{-1} \circ f \circ \phi^{-1}(V) \rightarrow E'_\infty$ is tame.

2. If $f$ is $\mathfrak{F}^0$, then $f$ is tame iff for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. for every envelope $((E, \phi), (E', \phi'), \mathcal{F})$ there exists a tame strictly monotone increasing sequence $k \subseteq \mathbb{N}_0$ s.t. $((E^k, \phi), (E'^k, \phi', \mathcal{F}^k) \mathcal{F}^k)$ is tame.

3. If $f$ is $\mathfrak{F}^0$, then $f$ is tame as an $\mathfrak{F}^0$-map iff $f$ is tame as an $\mathfrak{F}^0$-map.
Proof. 1. This is a straightforward verification.

2. By Proposition 3.1 and Lemma 4.5 for every $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ and a tame strict envelope of $f|_V$. And by Proposition 3.1, any other envelope of $f|_V$ is equivalent to this tame strict envelope. Now one just needs to unravel the definition of equivalence of envelopes and apply Lemma 4.5

3. One direction is trivial and if $f$ is tame as an $\text{sc}^0$-map, then by the same arguments as above, for every point $x \in U$ there exists a neighbourhood $V \subseteq U$ of $x$ s.t. $f|_V$ has a strict envelope $((E, \phi), (E', \phi'), \mathcal{F} : U \to E')$ s.t. $f_\infty := \phi'^{-1} \circ f \circ \phi : E_\infty \supseteq \phi^{-1}(V) \to E'_\infty$ is tame. But because $\mathcal{F} : U \to E'$ is strict, $U_\infty = \phi^{-1}(V)$ is dense in $U_k$ for all $k \in \mathbb{N}_0$ and $f_k : U_k \to E'_k$ is the unique continuous extension of $f_\infty$. So the result follows from a straightforward continuity argument.

\[ \Box \]

**Proposition 4.4.** Let $E$, $E'$ and $E''$ be pre-tame $\text{sc}$-Fréchet spaces.

1. Let $T : E \to E'$ be a continuous linear map. Then $T$ defines a tame morphism iff $T$ defines a tame $\text{sc}$-continuous map.

2. Let $U \subseteq E$ and $U' \subseteq E'$ be open subsets, and let $f : U \to E'$ and $f' : U' \to E''$ be $\text{sc}^0$ or $\text{sc}^0$ with $f(U) \subseteq U'$.

If $f$ and $f'$ are tame, then so is $f' \circ f$.

**Proof.**

1. [Ham82], Chapter II, Theorem 2.1.5.

2. [Ham82], Chapter II, Theorem 2.1.6.

\[ \Box \]

**Definition 4.10.** Let $E$ and $E'$ be pre-tame $\text{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset, let $f : U \to E'$ be a map and let $k \in \mathbb{N}_0$.

1. If $f$ is $\text{sc}^k$ or $p-\text{sc}^k$, then $f$ is called tame up to order $k$ iff for every $0 \leq j \leq k$, $D^j f : U \times E^j \to E'$ is tame as an $\text{sc}$-continuous map.

2. If $f$ is $\text{sc}^k$ or $p-\text{sc}^k$, then $f$ is called tame up to order $k$ iff for every $0 \leq j \leq k$, $D^j f : U \times E^j \to E'$ is tame as an $\text{sc}$-continuous map.

3. A tame envelope $((E, \phi), (E', \phi'), \mathcal{F} : U \to E')$ of $f$ that is $\text{sc}^k$ is called tame up to order $k$ iff $D\mathcal{F} : U^1 \oplus E \to E'$ is tame up to order $k - 1$.

4. If $f$ is $\text{sc}^k$, then $f$ is called tame up to order $k$ iff $f$ has an envelope that is $\text{sc}^k$ and tame up to order $k$.

**Remark 4.7.** Note that the condition of tameness for the differentials $D^k f : U \times E^k$, if $f$ is $\text{sc}^k$, coincides with the condition of tameness for functions of more than one variable in [Ham82], p. 142f.

**Example 4.5.** I continue Section 1.3 and Examples 3.1 and 3.2.

In the notation from this Subsection, $\Psi : \Gamma_B(F_1) \to \Gamma_B(F_2)$ is tame up to arbitrary order: By Example 3.2, $\Psi$ is $\text{sc}^\infty$. That $\Psi$ is tame follows at once from the claim in the proof of Proposition 1.7. That $\Psi$ is tame to arbitrary order follows by a simple induction, repeatedly applying Proposition 1.8 and the claim in the proof of Proposition 1.7 to the terms on the right hand side of the equations in Example 3.2 for $\Psi$ and $D\Psi$.
Proposition 4.5. Let $E$ and $E'$ be pre-tame $\mathcal{P}$-Fréchet spaces, let $U \subseteq E$ be an open subset, let $f : U \to E'$ be a map and let $k \in \mathbb{N}_0$ with $k \geq 1$.

1. If $f$ is $\mathcal{P}^{k}$ and tame up to order $k$, then $f$ is $\mathcal{P}^{k-1}$ and tame up to order $k-1$.

2. If $f$ is $\mathcal{P}^{k}$ and tame up to order $k$, then $f$ is $\mathcal{P}^{k-1}$ and tame up to order $k-1$.

Proof. By induction and Proposition 3.3 it suffices to show the implication $f$ $\mathcal{P}^{k}$ and tame up to order $1 \Rightarrow f$ $\mathcal{P}^{0}$ and tame.

So let $f$ be $\mathcal{P}^{k}$ and tame up to order $1$ and let $x_0 \in U$. In the following, as usual, I will drop the inclusions in the sc-chains from the notation.

Now an easy argument with rescaling and the definition of tameness shows that one can choose a neighbourhood $V \subseteq U$ of $x_0$ and compatible tame $\mathcal{P}$-structures $(\mathbb{E}, \phi)$ and $(\mathbb{E}', \phi')$ on $E$ and $E'$, respectively, together with a strict envelope $\mathcal{V} \subseteq \mathbb{E}$ of $V_{\infty} := \phi^{-1}(V)$ s. t. $g := \phi' \circ f \circ \phi^{-1}$ $V_{\infty} : V_{\infty} \to E'_{\infty}$ is tame and so is $Dg : V_{\infty} \times E_{\infty} \to E'_{\infty}$. Because $\mathcal{V}$ is strict, one can assume w.l. o. g., after shrinking $V$, that $V_k = \{ x \in V_k \mid \| x \|_k \leq c \}$ for some $c > 0$. In particular, $V_{\infty}$ is dense in $V_{\infty}$ for all $k \in \mathbb{N}_0$.

By definition there then exist constants $C_k$, $D_k > 0$, for $k \in \mathbb{N}_0$, s. t.

$$
\| g(x) \|_k' \leq C_k(1 + \| x \|_k)
$$

$$
\| Dg(x)u \|_k' \leq D_k(1 + \| x \|_k + \| u \|_k)
$$

for all $x \in V_{\infty}$, $u \in E_{\infty}$ and $k \in \mathbb{N}_0$.

Now by [Ham82], Part II, Lemma 2.1.7, there exist constants $c_k > 0$, for $k \in \mathbb{N}_0$, s. t.

$$
\| Dg(x)u \|_k' \leq c_k(\| x \|_k \| u \|_0 + \| u \|_k)
$$

$$
\leq c_k(1 + \| x \|_k)\| u \|_k
$$

for all $x \in V_{\infty}$, $u \in E_{\infty}$ and $k \in \mathbb{N}_0$.

For $x, y \in V_{\infty}$, because

$$
g(y) - g(x) = \int_0^1 Dg(x + t(y - x))(y - x) \, dt,
$$

it hence follows that

$$
\| g(y) - g(x) \|_k \leq \int_0^1 \| Dg(x + t(y - x))(y - x) \|_k' \, dt
$$

$$
\leq \int_0^1 c_k(1 + \| x + t(y - x) \|_k)\| y - x \|_k \, dt
$$

$$
\leq c_k(1 + \| x \|_k + \| y \|_k)\| y - x \|_k.
$$

So $g : (V_{\infty}, \| \cdot \|_k) \to (E'_{\infty}, \| \cdot \|_k')$ is locally uniformly continuous and hence there exists a unique continuous continuation $g_k : V_k \to E'_{\infty}$ of $g$.

The $g_k$ form the desired envelope $\mathcal{G} : \mathcal{V} \to \mathbb{E}'$ of $g$ and hence $((\mathbb{E}, \phi), (\mathbb{E}', \phi'), \mathcal{G} : \mathcal{V} \to \mathbb{E}')$ forms an envelope of $f|_V$. Now because $g$ is tame, so is the envelope just constructed (cf. Proposition 4.3). \qed
4.3 The Nash-Moser inverse function theorem

Definition 4.11. Let $E$, $E'$ and $E^i$, $i = 1, \ldots, 3$, be $\mathfrak{c}$-Fréchet spaces, and let $U \subseteq E$ and $U' \subseteq E'$ be open subsets.

1. An $\mathfrak{c}^0$ family (over $U$) of morphisms (from $E^1$ to $E^2$) is an $\mathfrak{c}^0$ map

$$\phi : U \times E^1 \to E^2$$

$$(x, u) \mapsto \phi(x)u$$

that is linear in the second factor.

By abuse of notation, given a morphism $F : E^1 \to E^2$, the constant family

$$U \times E^1 \to E^2$$

$$(x, u) \mapsto F(u)$$

will be denoted by $F$ again (and will always be $\mathfrak{c}^0$).

2. Given $\mathfrak{c}^0$ families of morphisms $\phi : U \times E^1 \to E^2$ and $\psi : U \times E^2 \to E^3$, their composition is the $\mathfrak{c}^0$ family of morphisms

$$\psi \circ \phi : U \times E^1 \to E^3$$

$$(x, u) \mapsto \psi(x)\phi(x)u.$$ 

3. An $\mathfrak{c}^0$ family of morphisms $\phi : U \times E^1 \to E^2$ is called invertible if there exists an $\mathfrak{c}^0$ family of morphisms $\psi : U \times E^2 \to E^1$ with $\psi \circ \phi = \text{id}_{E^1}$ and $\phi \circ \psi = \text{id}_{E^2}$. $\psi$ will then be called the inverse to $\phi$ and denoted by $\phi^{-1} := \psi$.

4. Given an $\mathfrak{c}^0$ family of morphisms $\phi : U \times E^1 \to E^2$ and an $\mathfrak{c}^0$ map $f : U' \to U$, the pullback of $\phi$ by $f$ is the $\mathfrak{c}^0$ family of morphisms $f^* \phi := \phi \circ (f \times \text{id}_{E^1})$, i.e.

$$f^* \phi : U' \times E^1 \to E^2$$

$$(x, u) \mapsto \phi(f(x))u.$$ 

And analogously for $\phi$ $\mathfrak{c}^0$ and/or tame.

Remark 4.8. If $\phi : U \times E^1 \to E^2$ is an $\mathfrak{c}^0$ family of morphisms, then for every $x \in U$, $\phi(x) : E^1 \to E^2$ is a morphism of $\mathfrak{c}$-Fréchet spaces. For by assumption it is a linear map and as the composition of $\phi$ with the inclusion $E^1 \to U \times E^1$, $c \mapsto (x, c)$ it is $\mathfrak{c}^0$, hence $\mathfrak{c}^0$. So the claim follows from Lemma 3.6.

Theorem 4.1 (The Nash-Moser inverse function theorem). Let $E$ and $E'$ be pre-tame $\mathfrak{c}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a map that is $(p-)\mathfrak{c}^k$ and tame up to order $k$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq 2$.

If $E$ is weakly tame and $Df : U \times E \to E'$ is invertible with an inverse $\psi : U \times E' \to E$ that is $(p-)\mathfrak{c}^0$ and tame (up to order 0), then $f$ is locally invertible and each local inverse is $(p-)\mathfrak{c}^k$ and tame up to order $k$.

I.e. for each $x \in U$ there exists an open neighbourhood $V \subseteq U$ of $x$ s.t. $f(V) \subseteq E'$ is an open neighbourhood of $f(x)$ and there exists a map $g : f(V) \to V$ that is $(p-)\mathfrak{c}^k$ and tame up to order $k$, and that satisfies $g \circ f|_V = \text{id}_V$ and $f|_V \circ g = \text{id}_{f(V)}$. Also, $Dg = g^* \psi$. 

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4.3.1 The proof of the Nash-Moser inverse function theorem

There are two main differences between Theorem 4.1 and [Ham82], Part III, Theorem 1.1.1: In Theorem 4.1 it is only assumed that we are dealing with weakly tame \( \mathcal{SC} \)-Fréchet spaces instead of tame \( \mathcal{SC} \)-Fréchet spaces and furthermore, we are also dealing with maps of class \( \mathcal{SC}^k \) instead of only maps of class \( p-\mathcal{SC}^k \).

Now the proof of the Nash-Moser inverse function theorem in [Ham82] (or the nearly identical one in [KM97]), while also using the existence of smoothing operators, makes explicit use of the tameness condition, especially in the proof of local surjectivity. While there are alternative sources for a proof using only existence of smoothing operators, such as [LZ79], these use a definition of smoothing operators which requires the constant \( p \) from Definition 4.5 to be zero. But allowing \( p > 0 \) is crucial for the notion of an ILB-chain to admit smoothing operators to be well-behaved under tame rescalings and equivalences, cf. Lemma 4.4. The proof in [LZ79] also does not show tameness of the inverse map, cf. the definition of the quantity \( \sigma(n,b) \) in [LZ79], Lemma 4, which defines a non-tame rescaling.

As a first step, observe that it only needs to be shown that in case \( k = 2 \), around each point \( x \in U \) there exists a neighbourhood \( V \subseteq U \) of \( x \) and a local inverse \( g : f(V) \to V \) with \( Dg = g^* \psi \) that is \( \mathcal{SC}^1 \) (\( p-\mathcal{SC}^1 \)) and tame. For the cases of \( k \geq 2 \) then follow from Theorem 3.1, Lemma 4.6, and Proposition 4.4.

Lemma 4.6. Let \( E \) and \( E_i \), \( i = 1, 2 \), be \( \mathcal{SC} \)-Fréchet spaces, and let \( U \subseteq E \) be an open subset. Let furthermore \( \phi : U \times E^1 \to E^2 \) be an invertible \( (\mathcal{SC}^0 \) or \( \mathcal{SC}^0 \)) family of morphisms and let \( k \in \mathbb{N}_0 \cup \{ \infty \} \).

1. If \( \phi \) is \( \mathcal{SC}^k \) and \( \phi^{-1} \) is \( \mathcal{SC}^0 \), then \( \phi^{-1} \) is \( \mathcal{SC}^k \).
2. If \( \phi \) is \( \mathcal{SC}^k \) and \( \phi^{-1} \) is \( \mathcal{SC}^0 \), then \( \phi^{-1} \) is \( \mathcal{SC}^k \).
3. If \( \phi \) is \( p-\mathcal{SC}^k \) and \( \phi^{-1} \) is \( \mathcal{SC}^0 \), then \( \phi^{-1} \) is \( p-\mathcal{SC}^k \).
4. If \( \phi \) is \( p-\mathcal{SC}^k \) and \( \phi^{-1} \) is \( \mathcal{SC}^0 \), then \( \phi^{-1} \) is \( p-\mathcal{SC}^k \).

In all of these cases, if \( E \) and \( E^i \), \( i = 1, 2 \), are pre-tame \( \mathcal{SC} \)-Fréchet spaces, \( \phi \) is tame up to order \( k \) and \( \phi^{-1} \) is tame (up to order 0), then \( \phi^{-1} \) is tame up to order \( k \).

Proof. It will be shown that in all cases,

\[
D\phi^{-1} : (U \times E^2) \times (E \times E^2) \to E^1
D\phi^{-1}(x, u)(e, v) = \phi^{-1}(x)(v - D\phi(x, \phi^{-1}(x)u)(e, 0)).
\]

Steps 3 and 4 would actually follow from [Ham82], Part I, Theorem 5.3.1, and Part II, Theorem 3.1.1, in conjunction with Proposition 3.2 but will be explicitly shown below.

First, observe that once the above formula for \( D\phi^{-1} \) has been shown, it suffices to consider the case \( k = 1 \), for the other cases then follow by induction, using Theorem [3.1] and Proposition 4.4.
So let \((x,u),(e,v)\) \(\in (U \times E^2) \times (E \times E^2)\) and \(t \in (0, \infty)\). Then
\[
\phi^{-1}(x + te)(u + tv) - \phi^{-1}(x)u = \phi^{-1}(x + te)u - \phi^{-1}(x)u + t\phi^{-1}(x + te)v
\]
\[
= \phi^{-1}(x + te)(u - \phi(x + te)\phi^{-1}(x)u + tv)
\]
\[
= \phi^{-1}(x + te)(tv - (\phi(x + te) - \phi(x))\phi^{-1}(x)u)
\]
\[
= \phi^{-1}(x + te)(tv - tD\phi(x, \phi^{-1}(x)u)(e,0))
\]
\[
= t\phi^{-1}(x)(v - D\phi(x, \phi^{-1}(x)u)(e,0)) + \frac{t^2}{2}\phi^{-1}(x)u - (e,0), t),
\]
where
\[
r_{(x,u)}((e,v), t) = (\phi^{-1}(x + te) - \phi^{-1}(x))(v - D\phi(x, \phi^{-1}(x)u)(e,0)) - \frac{t^2}{2}\phi^{-1}(x)u - (e,0), t).
\]

From this it follows by definition that \(\phi^{-1}\) is (pointwise) weakly Fréchet differentiable provided that \(\phi\) is and provided that \(\phi^{-1}\) is weakly continuous (which just means that it is a continuous map \(U \times E^2 \to E^1\)).

The above formula for \(D\phi^{-1}\) then also shows (by Theorem 3.1) that \(D\phi^{-1}\) is \(\mathfrak{sc}^1\) or \(\mathfrak{sc}^0\), provided that \(\phi^{-1}\) and \(D\phi\) are, and (by Proposition 4.4) that \(D\phi^{-1}\) is tame, provided that \(\phi^{-1}\) and \(D\phi\) are.

Now assume that the assumptions of Theorem 4.1 hold. I.e. let \(E\) and \(E'\) be pre-tame \(\mathfrak{sc}\)-Fréchet spaces with \(E\) weakly tame, let \(U \subseteq E\) be an open subset and let \(f : U \to E'\) be a map that is \((p-)\mathfrak{sc}^2\) and tame up to order 2. Furthermore, assume that \(Df : U \times E \to E'\) is invertible with inverse \(\psi : U \times E' \to E\) that is \((p-)\mathfrak{sc}^0\) and tame. Given \(x \in U\) one has to show that there exists an open neighbourhood \(V \subseteq U\) s.t. \(f(V) \subseteq E'\) is an open neighbourhood of \(f(x)\) and there exists a map \(g : f(V) \to V\) which is \(\mathfrak{sc}^1\) and tame, and satisfies \(g \circ f|V = \text{id}_V\) and \(f|V \circ g = \text{id}_f|V\).

As usual in this type of proof, one can assume w.l.o.g. that \(x = 0\) and \(f(x) = 0\). For if \(x \neq 0\) or \(f(x) \neq 0\), then replace \(f\) by
\[
\tilde{f} : E \supseteq U - x \to E'
\]
\[
y \mapsto f(x+y) - f(x).
\]

Moving to a concrete choice of \(\mathfrak{sc}\)-chain on \(E\), directly from the definitions (Definitions 3.5, 3.6, 3.7, 4.7 and 4.8), and using (the analogue in the tame context of) Lemma 4.2, Lemma 3.7 and Proposition 4.3, there exists a neighbourhood \(U' \subseteq U\) and a compatible \(\mathfrak{sc}\)-structure \((E, \phi)\) on \(E\) that admits smoothing operators as well as a compatible \(\mathfrak{sc}\)-structure \((E', \phi')\) on \(E'\), s.t. the following holds: By abuse of notation, for \(j \in \mathbb{N}_0\), denote the norm \((t^\infty)^{-1} \| \cdot \|_j : E_\infty \to [0, \infty)\) by \(\| \cdot \|_j\) again and similarly for \(\| \cdot \|_{j'} : E'_\infty \to [0, \infty)\). And similarly if \(k \in \mathbb{N}_0\) is a strictly monotone increasing sequence, then \(E^k_\infty = E_\infty\) and we have the norms \(\| \cdot \|_k = \| \cdot \|_{k+j} : E_\infty \to [0, \infty)\) and \(\| \cdot \|_{j'} = \| \cdot \|_{k+j} : E'_\infty \to [0, \infty)\).

Define
\[
V' := \phi^{-1}(U') \subseteq E_\infty
\]
\[
f := \phi^{-1} \circ f|U' \circ \phi : V' \to E'_\infty
\]
\[
\tilde{\psi} := \phi^{-1} \circ \psi|U \times E \circ (\phi \times \phi') : V' \times E'_\infty \to E_\infty.
\]

Then there exist shifts \(k, l, m, n \in \mathbb{N}_0\) with the following properties:
1. $V' \subseteq (E_\infty, \| \cdot \|_j)$ is open for all $j \in \mathbb{N}_0$. Equivalently, $V' \subseteq (E_\infty, \| \cdot \|_0)$ is open.

2. $\tilde{f} : (E_\infty, \| \cdot \|_j^\infty) \supseteq V' \to (E_\infty', \| \cdot \|_j')$ is continuous and satisfies a tame estimate for all $j \in \mathbb{N}_0$.

3. $D\tilde{f} : (E_\infty \times E_\infty, \| \cdot \|_j^0 + \| \cdot \|_j^0) \supseteq V' \times E_\infty \to (E_\infty', \| \cdot \|_j')$ is continuous and satisfies a tame estimate for all $j \in \mathbb{N}_0$.

4. $D^2\tilde{f} : (E_\infty \times E_\infty \times E_\infty, \| \cdot \|_j^m + \| \cdot \|_j^m + \| \cdot \|_m) \supseteq V' \times E_\infty \times E_\infty \to (E_\infty', \| \cdot \|_j')$ is continuous and satisfies a tame estimate for all $j \in \mathbb{N}_0$.

5. $\tilde{\psi} : (E_\infty \times E_\infty', \| \cdot \|_j^k + \| \cdot \|_j^k) \supseteq V' \times E_\infty' \to (E_\infty, \| \cdot \|_j)$ is continuous and satisfies a tame estimate for all $j \in \mathbb{N}_0$.

Also, w.l.o.g. one can assume that $V'$ is convex. Now $n \circ l \circ m \geq n, l, m$, so one can replace $n, l$ and $m$ by $n \circ l \circ m$, i.e. one can assume that $n = l = m$. Assuming this to hold, by replacing $E$ by $\mathbb{P}^n$ and replacing $k$ by $k \circ n$ one can furthermore assume that $n = l = m = id = (j)_{j \in \mathbb{N}_0}$.

In the first part of the proof below $l$ will follow the proof from [Ham82] very closely, only modifying the proofs where necessary to accommodate for the changed assumptions in Theorem [7].

First, due the (bi-)linearity of $D\tilde{f}, \tilde{\psi}$ and $D^2\tilde{f}$ in the second (and third) factor, one has the following variant of the tame estimates:

**Lemma 4.7.** In the notation as above, there exist constants $\delta > 0, a_j, b_j, c_j, d_j \in [0, \infty)$, for $j \in \mathbb{N}_0, s. t.$

\[
\|f(x)\|_j^1 \leq a_j \|x\|_j \\
\|Df(x)\|_j^l \leq b_j \left(\|x\|_j \|e\|_0 + \|e\|_j\right) \\
\|D^2f(x)(e, \tilde{e})\|_j^l \leq c_j \left(\|x\|_j \|e\|_0 \|\tilde{e}\|_0 + \|e\|_j \|\tilde{e}\|_0 + \|e\|_0 \|\tilde{e}\|_j\right) \\
\|\tilde{\psi}(x)e'\|_j^l \leq d_j \left(\|x\|_j \|e'\|_0^k + \|e'\|_j^k\right)
\]

for all $x \in E_\infty$ with $\|x\|_j^k < \delta, e, \tilde{e} \in E_\infty, e' \in E_\infty'$, and $j \in \mathbb{N}_0$.

**Proof.** The first inequality is just the tameness condition for $\tilde{f}$. For the remaining ones apply [Ham82], Part II, Lemmas 2.1.7 and 2.1.8, to $L = D\tilde{f}$ and $b = r = s = 0$, $B = D^2\tilde{f}$ and $b = r = s = t = 0$, and $L = \tilde{\psi}$ and $b = 0, r = s = k_0$.

The following proposition settles local injectivity of $\tilde{f}$ in a neighbourhood of 0 and will also provide the main ingredient in the formula for the derivative of the local inverse, once surjectivity of $\tilde{f}$ onto a neighbourhood of 0 has been shown.

**Proposition 4.6.** In the setting as above, there exist constants $\delta > 0, c_j', c_j'' \in [0, \infty)$, for $j \in \mathbb{N}_0, s. t.$

\[
\|y - x\|_j \leq c_j' \left(\|x\|_j^k + \|y\|_j^k\right) \|\tilde{f}(y) - \tilde{f}(x)\|_0^k + \|\tilde{f}(y) - \tilde{f}(x)\|_j^k
\]

and

\[
\|y - x - \tilde{\psi}(x)(\tilde{f}(y) - \tilde{f}(x))\|_j \leq c_j'' \left(1 + \|x\|_0^{k0} + \|y\|_0^{k0}\right) \cdot \left(\|x\|_j^{k0} + \|y\|_j^{k0}\right) \|\tilde{f}(y) - \tilde{f}(x)\|_0^{k0} + \|\tilde{f}(y) - \tilde{f}(x)\|_j^{k0}
\]

for all $j \in \mathbb{N}_0$ and $x, y \in E_\infty$ with $\|x\|_0^k, \|y\|_0^k < \delta$.  

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Proof. This is an amalgamation of [HamS2, Part III, Theorem 1.3.1 and Corollaries 1.3.2 and 1.3.3].

Following the layout of the proof there, by Taylor’s theorem, for \(x, y \in V'\),

\[
\tilde{f}(y) = \tilde{f}(x) + D\tilde{f}(x)(y-x) + \int_0^1 (1-t) D^2\tilde{f}((1-t)x + ty)(y-x, y-x) \, dt
\]

and hence, using \(\tilde{\psi}(x)D\tilde{f}(x) = \text{id}_{E_n}\),

\[
y - x = \tilde{\psi}(x)(\tilde{f}(y) - \tilde{f}(x)) - \tilde{\psi}(x) \int_0^1 (1-t) D^2\tilde{f}((1-t)x + ty)(y-x, y-x) \, dt.
\]

Applying Lemma 4.7 for \(j \in \mathbb{N}_0\),

\[
\|\alpha(x,y)\|_j = \|\tilde{\psi}(x)(\tilde{f}(y) - \tilde{f}(x))\|_j
\leq d_j (\|x\|_j \|\tilde{f}(y) - \tilde{f}(x)\|_0^k + \|\tilde{f}(y) - \tilde{f}(x)\|_j^k).
\]  

(5)

Similarly, applying Lemma 4.7 twice, for \(j \in \mathbb{N}_0\),

\[
\|\beta(x,y)\|_j = \left\|\tilde{\psi}(x) \int_0^1 (1-t) D^2\tilde{f}((1-t)x + ty)(y-x, y-x) \, dt\right\|_j
\leq d_j \left(\|x\|_j^k \left\|\int_0^1 (1-t) D^2\tilde{f}((1-t)x + ty)(y-x, y-x) \, dt\right\|_0^k + \right.
+ \left. \left\|\int_0^1 (1-t) D^2\tilde{f}((1-t)x + ty)(y-x, y-x) \, dt\right\|_j^k \right)
\leq d_j \left(\|x\|_j^k \int_0^1 (1-t) \|D^2\tilde{f}((1-t)x + ty)(y-x, y-x)\|_{k_0} \, dt + \right.
+ \left. \int_0^1 (1-t) \|D^2\tilde{f}((1-t)x + ty)(y-x, y-x)\|_{k_0+j} \, dt \right)
\leq d_j \left(\|x\|_j^k \int_0^1 (1-t)c_{k_0}(\|(1-t)x + ty\|_{k_0}(\|y-x\|_0)^2 + \right.
+ \left. 2\|y-x\|_{k_0}\|y-x\|_0 + \right)
+ \left. \int_0^1 (1-t)c_{k_0+j}(\|(1-t)x + ty\|_{k_0+j}(\|y-x\|_0)^2 + \right.
+ \left. 2\|y-x\|_{k_0+j}\|y-x\|_0 \right)
\leq d_j \left((c_{k_0} + c_{k_0+j} + 2)(1 + \|x\|_0^k + \|y\|_0^k)(\|x\|_j^k + \|y\|_j^k) \cdot \right.
+ \left. \|y-x\|_0^k\|y-x\|_0 + 2\|y-x\|_0^k\|y-x\|_0 + \right)
+ \left. \int_0^1 (1-t)c_{k_0+j}(\|(1-t)x + ty\|_{k_0+j}(\|y-x\|_0)^2 + \right.
+ \left. 2\|y-x\|_{k_0+j}\|y-x\|_0 \right)
\leq d_j (c_{k_0} + c_{k_0+j} + 2)(1 + 2\delta) \cdot \right.
+ \left. ((\|x\|_j^k + \|y\|_j^k))\|y-x\|_0^k + \|y-x\|_j^k)\|y-x\|_0.
\]

So

\[
\|\beta(x,y)\|_j \leq d_j' \left((\|x\|_j^k + \|y\|_j^k)\|y-x\|_0^k + \|y-x\|_j^k)\|y-x\|_0.\right.
\]  

(6)
Applying this for \( j = 0 \), then given any \( \delta > 0 \), for all \( x, y \in V' \) with \( \|x\|_0^k, \|y\|_0^k < \delta \) one has the estimates
\[
\|\alpha(x, y)\|_0 \leq d_0(1 + \delta)\|\bar{f}(y) - \bar{f}(x)\|_0^k \\
\|\beta(x, y)\|_0 \leq d_0'(1 + 2\delta)\|y - x\|_0^k\|y - x\|_0 \\
\leq d_0'(1 + 2\delta)\|y - x\|_0.
\]
And hence from \( \|y - x\|_0 - \|\beta(x, y)\|_0 \leq \|\alpha(x, y)\|_0 \) it follows that
\[
(1 - 2d_0'(1 + 2\delta))\|y - x\|_0 \leq d_0(1 + \delta)\|\bar{f}(y) - \bar{f}(x)\|_0^k.
\]
Choosing \( \delta > 0 \) s.t. \( \delta < \frac{1}{12c_0} \), this implies that
\[
\|y - x\|_0 \leq 4d_0\|\bar{f}(y) - \bar{f}(x)\|_0^k. \tag{7}
\]
This shows the first estimate in the statement of the proposition, in case \( j = 0 \) with \( c_0 := 4d_0 \). From now on, \( \delta \) is fixed, satisfying the above assumptions. For general \( j \in \mathbb{N}_0 \) one can then combine this with the above estimate for \( \|\beta(x, y)\|_j \) to obtain
\[
\|\beta(x, y)\|_j \leq 4d_0d_j'(1 + 2\delta)(\|x\|_j^k + \|y\|_j^k)\|\bar{f}(y) - \bar{f}(x)\|_0^k.
\]
Setting \( c_j' := (1 + d_j')(1 + 4d_0d_j'(1 + 2\delta)) \), combining this inequality with (5) shows that
\[
\|y - x\|_j \leq \|\alpha(x, y)\|_j + \|\beta(x, y)\|_j \\
\leq c_j'((\|x\|_j^k + \|y\|_j^k)\|\bar{f}(y) - \bar{f}(x)\|_0^k + \|\bar{f}(y) - \bar{f}(x)\|_0^k) \tag{8}
\]
for all \( x, y \in V' \) with \( \|x\|_0^k, \|y\|_0^k < \delta \). This shows the first part of the proposition. For the second part, note that
\[
\|y - x - \tilde{\psi}(x)(\bar{f}(y) - \bar{f}(x))\|_j = \|\beta(x, y)\|_j
\]
and combining (6), (7) and (8) produces
\[
\|\beta(x, y)\|_j \leq 4d_0d_j'(\|x\|_j^k + \|y\|_j^k)\|y - x\|_0^k + \|\bar{f}(y) - \bar{f}(x)\|_0^k \\
\leq 4d_0d_j'((\|x\|_j^k + \|y\|_j^k)c_j\|\bar{f}(y) - \bar{f}(x)\|_0^k + \|\bar{f}(y) - \bar{f}(x)\|_0^k) \\
+ c_j'\|\bar{f}(y) - \bar{f}(x)\|_j^k + \|\bar{f}(y) - \bar{f}(x)\|_0^k \\
\leq 4d_0d_j'(c_j + c_{j+1})\|x\|_0^k + \|y\|_0^k \\
\leq 8d_0d_j'(c_j + c_{j+1})\|\bar{f}(y) - \bar{f}(x)\|_0^k.
\]
Setting \( c_j' := 8d_0d_j'(c_j + c_{j+1}) \) finishes the proof. \( \square \)
Corollary 4.1. In the setting as above, there exists a shift \(1 \subseteq \mathbb{N}_0\) and an open neighbourhood \(W \subseteq (E_\infty, \| \cdot \|_0)\) of 0 s. t.

\[ \bar{f}|_W : W \to E'_\infty \]

is injective.

Proof. Let \(W\) be a ball of radius \(\delta\) around zero in \((E_\infty, \| \cdot \|_0)\), where \(\delta\) is as in the proposition. Then for \(x, y \in W\),

\[ \|y - x\|_0 \leq c_0'(1 + 2\delta)\|\bar{f}(y) - \bar{f}(x)\|_0'. \]

To show local surjectivity of \(\tilde{f}\), I will follow the modified Newton’s method from [L79]. It is at this point that the condition of \(E\) being weakly tame comes into play. Concretely this means that one can assume that \(E\) admits smoothing operators, i.e. by Definition 4.5 and Remark 4.6 one can assume that there exists a continuous map

\[ S : [0, \infty) \to L_c(E_0, E_\infty) \]

\[ t \mapsto S_t \]

and constants \(p, C, n, m \in [0, \infty), n, m \in \mathbb{N}_0\), s. t.

\[ \|S^m_{n,t}\|_{L_c(E_n, E_m)} \leq C^m_n(1 + e^{(p+(n-m))t}) \quad \forall m, n \in \mathbb{N}_0, t \geq 0 \]

\[ \|n^m - S^m_{n,t}\|_{L_c(E_n, E_m)} \leq C^m_n e^{(m-n)t} \quad \forall m, n \in \mathbb{N}_0, m - n \geq p, t \geq 0, \]

where \(S^m_{n,t} := \psi^n \circ S_t \circ \iota^m_0 : E_m \to E_n\).

By abuse of notation, I will also denote by \(S_t : E_\infty \to E_\infty\) the map \(S_t \circ \iota^\infty_0\). Then the above inequalities read

\[ \|S_t e\|_n \leq C^m_n(1 + e^{(p+(n-m))t})\|e\|_m \quad \forall e \in E_\infty, m, n \in \mathbb{N}_0, t \geq 0 \]

\[ (\|\text{id}_{E_\infty} - S_t\| e\|_n \leq C^m_n e^{(m-n)t}\|e\|_m \quad \forall e \in E_\infty, m, n \in \mathbb{N}_0, m - n \geq p, t \geq 0. \]

Or, using Remark 4.5 these can also be written as

\[ \|S_t e\|_n \leq C^m_n(1 + e^{(n-m)t})\|e\|_m^p \quad \forall e \in E_\infty, m, n \in \mathbb{N}_0, t \geq 0 \]

\[ (\|\text{id}_{E_\infty} - S_t\| e\|_n \leq C^m_n e^{-(m-n)t}\|e\|_m^p \quad \forall e \in E_\infty, m, n \in \mathbb{N}_0, m - n \geq 0, t \geq 0, \]

where \(p = (p + j) j \in \mathbb{N}_0\).

The modified Newton’s procedure then goes as follows: Given \(y \in E_\infty\), inductively define a sequence \((x_r)_{r \in \mathbb{N}_0}\) by

\[
\begin{align*}
x_0 & := 0 \\
x_{r+1} & := x_r + \Delta x_r \\
\Delta x_r & := S_{t_r} \psi(x_r) z_r \\
z_r & := y - \bar{f}(x_r) \\
t_r & := \left(\frac{9}{2}\right)^r.
\end{align*}
\]
The goal is to show that there exists a shift $1 \in \mathbb{N}_0$ and constants $\tau_j \in [0, \infty)$ s.t.

$$\|x_r\|^k_j \leq \tau_j \|y\|^q_j$$

$$\|\Delta x_r\|^k_j \xrightarrow{r \to \infty} 0$$

$$\|z_r\|^k_j \xrightarrow{r \to \infty} 0.$$ 

This shows that the sequence $(x_r)_{r \in \mathbb{N}_0}$ is well defined, for $y \in E_\infty$ with $\|y\|^k_j < \delta$, where $\delta > 0$ is some constant. For taking the special case $j = 0$, $\|x_r\|^k_0 \leq \tau_0 \delta$, so $x_r \in V'$ for $\delta$ small enough. $\|\Delta x_r\|^k_j \xrightarrow{r \to \infty} 0$ for all $j \in \mathbb{N}_0$ then shows that $(x_r)_{r \in \mathbb{N}_0}$ converges in $(E_\infty, \| \cdot \|^k_j)$ for all $j \in \mathbb{N}_0$, i.e. there exists $x \in E_\infty$ with $x_r \to x$ and by definition of $z_r$ and since $\tilde{f}$ is continuous, $\|z_r\|^k_j \xrightarrow{r \to \infty} 0$ implies $\tilde{f}(x) = y$. So setting $g(y) := x$ defines the required local inverse of $\tilde{f}$, satisfying

$$\|g(y)\|^k_j \leq \tau_j \|y\|^q_j,$$

i.e. $g$ is tame.

This will be proved through a series of lemmas, where the first one, Lemma 4.8, roughly corresponds to Lemma 1 in [LZ79] and the second one, Lemma 4.9, to part of the proof of Lemma 2 in [LZ79].

The following points might be worth taking note of, since the proof relies quite heavily on them and it is actually quite important to know what each quantity/constant depends on:

- The quantity $\rho$ in Lemmas 4.8 and 4.9 only depends on $p$, the constant coming from the smoothing operators, and $k_0$, the constant defining the shift $k$ appearing in the tameness condition for $\psi$.
- In the estimates for $\|x_r\|^k_j$, $\|\Delta x_r\|^k_j$, $\|z_r\|^k_j$ and $\|z_{r+1}\|^k_j$, only products of a $\| \cdot \|^k_j$- and a $\| \cdot \|^k_j$-norm (such as $\|y\|^k_j \|z_r\|^k_j$ or $(\|z_r\|^k_j)^2$) appear but no products of two $\| \cdot \|^k_j$-norms (such as $(\|z_r\|^k_j)^2$).

**Lemma 4.8.** In the setting as above, given any $\delta_0 > 0$ small enough, there exist for every $j \in \mathbb{N}_0$ constants $\gamma_j, \gamma_j \in [1, \infty)$, depending on $\delta_0$, s.t. if $\|y\|^k_j < \delta_0$ and for some $r_0 \in \mathbb{N}_0$ also $\|x_r\|^k_0 < \delta_0$ for $r = 0, \ldots, r_0$, then

$$\|x_r\|^k_j \leq \gamma_j e^{\rho t_j} \|y\|^k_j$$

and

$$\|\Delta x_r\|^k_j \leq \gamma_j e^{2t_j} (\|y\|^k_j \|z_r\|^k_0 + \|z_r\|^k_j)$$

as well as

$$\|z_r\|^k_j \leq \gamma_j e^{\rho t_j} \|y\|^k_j$$

for all $r = 0, \ldots, r_0 + 1$, where $\rho := 2(p + k_0) + 1$.

**Proof.** For any $j \in \mathbb{N}_0$, if $\delta_0$ is small enough, then the assumption $\|x_r\|^k_0 < \delta_0$ implies $x_r \in V'$ and by Lemma 4.7 $\|z_r\|^k_j = \|y + f(x_r)\|^k_j \leq \|y\|^k_j + \alpha_j \|x_r\|^k_j$. Hence
for any \(i, j \in \mathbb{N}_0\) with \(j \geq i\), using the estimates on the smoothing operators (from Remark 4.4 and Lemma 4.7)
\[
\| \Delta x_r \|_j = \| S_{t_r} \tilde{\psi}(x_r) z_r \|_j \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \| \tilde{\psi}(x_r) z_r \|_{j-i} \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \left( \| x_r \|_{j-i} \| z_r \|_0^k + \| z_r \|_{j-i}^k \right) \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \left( \| x_r \|_{j-i} \left( \| y \|_0^k + a_{k_0} \| x_r \|_{k_0} \right) + \| y \|_{j-i}^k + a_{k+j_0} \| z_r \|_{j-i}^k \right) \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \left( 1 + a_{k_0} \right) \left( 1 + a_{k+j_0} \right) \left( 1 + \| y \|_0^k + \| x_r \|_{k_0} \right).
\]

Now take \(i = k_0\) and define \(\gamma_j' := C_{j-k_0}^{\gamma_j} d_{j-k_0} \left( 1 + a_{k_0} \right) \left( 1 + a_j \right) \left( 1 + 2d_0 \right)\). Then
\[
\| \Delta x_r \|_j \leq \gamma_j' e^{(p+k_0)t_r} \left( \| x_r \|_j + \| y \|_j \right).
\]

It is
\[
\| x_{r+1} \|_j + \| y \|_j' = \| x_r + \Delta x_r \|_j + \| y \|_j' \\
\leq \| x_r \|_j + \| \Delta x_r \|_j + \| y \|_j' \\
\leq \| x_r \|_j + \gamma_j' e^{(p+k_0)t_r} \left( \| x_r \|_j + \| y \|_j' \right) + \| y \|_j' \\
\leq \left( \gamma_j' + 1 \right) e^{(p+k_0)t_r} \left( \| x_r \|_j + \| y \|_j' \right),
\]
so by induction \(\| x_r \|_j \leq \| x_r \|_0 + \| y \|_j' \leq \left( \gamma_j' + 1 \right) e^{(p+k_0)t_r} \sum_{r=0}^{j-1} t_r \| y \|_r\).
By definition of \(t_r\), \(\sum_{r=0}^{j-1} \left( \frac{3}{2} \right)^r = \frac{\left( \frac{3}{2} \right)^j - 1}{\frac{3}{2} - 1} = 2 \left( \frac{3}{2} \right)^r - 1 \leq 2 \left( \frac{3}{2} \right)^r = 2t_r\). Hence
\[
\| x_r \|_j \leq \left( \gamma_j' + 1 \right) e^{2(p+k_0)t_r} \| y \|_j' = e^{r \ln \left( \gamma_j' + 1 \right) - t_r} e^{2(p+k_0)t_r} \| y \|_j'.
\]

Now
\[
r \ln \left( \gamma_j' + 1 \right) - t_r = r \ln \left( \gamma_j' + 1 \right) - \left( \frac{3}{2} \right)^r \to \infty,
\]
hence one can define \(\gamma_{j-k_0} := \max\left\{ 1 + e^{r \ln \left( \gamma_j' + 1 \right) - t_r} \mid r \geq 0 \right\} \in [1, \infty)\). This settles the first part of the statement. For the second part, by (10) and Lemma 4.7 for any \(i, j \in \mathbb{N}_0\) with \(j \geq i\), by [2]
\[
\| \Delta x_r \|_j = \| S_{t_r} \tilde{\psi}(x_r) z_r \|_j \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \| \tilde{\psi}(x_r) z_r \|_{j-i} \\
\leq C_{j-i}^{\gamma_j} e^{(p+i)t_r} \left( \| x_r \|_{j-i} \| z_r \|_0^k + \| z_r \|_{j-i}^k \right).
\]

In particular, if \(j \geq k_0\), then one can insert \(i = k_0\) and obtain
\[
\| \Delta x_r \|_j \leq C_{j-k_0}^{\gamma_j} d_{j-k_0} e^{(p+k_0)t_r} \left( \| x_r \|_{j-k_0} \| z_r \|_{k_0} + \| z_r \|_{j-k_0}^k \right) \\
= C_{j-k_0}^{\gamma_j} d_{j-k_0} e^{(p+k_0)t_r} \left( \| x_r \|_j \| z_r \|_{k_0} + \| z_r \|_{j}^k \right).
\]
Replacing \(j\) with \(j + k_0\) produces, via the first part of the statement,
\[
\| \Delta x_r \|_{j+k_0} \leq C_{j+k_0}^{\gamma_j} d_{j+k_0} e^{(p+k_0)t_r} \left( \| x_r \|_{j+k_0} \| z_r \|_{k_0} + \| z_r \|_{j+k_0}^k \right) \\
\leq C_{j+k_0}^{\gamma_j} d_{j+k_0} e^{(p+k_0)t_r} \left( \gamma_j e^{2rt_r} \| y \|_{j+k_0} \| z_r \|_{k_0} + \| z_r \|_{j+k_0}^k \right) \\
\leq C_{j+k_0}^{\gamma_j} d_{j+k_0} e^{2rt_r} \left( \| y \|_{j+k_0} \| z_r \|_{k_0}^k + \| z_r \|_{j+k_0}^k \right)
\]
from which the second part of the statement follows with \(\tilde{\gamma}_j := C_{j+k_0}^{\gamma_j} d_{j+k_0}\). For the third part of the statement, as before, one estimates, using the first part of the statement
\[
\| z_{r+1} \|_{j+k_0} \leq \| y \|_{j+k_0} + a_{k_0+j} \| x_r \|_{j+k_0} \\
\leq \| y \|_{j+k_0} + a_{k_0+j} \| y \|_{j+k_0}^k \\
\leq \left( 1 + a_{k_0+j} \right) \| y \|_{j+k_0}^k,
\]
from which the third part of the statement follows with \(\tilde{\gamma}_j := (1 + a_{k_0+j})\). \(\square\)
Lemma 4.9. In the setting as above, given any \( \delta_0 > 0 \) small enough and any \( \mu \in \mathbb{N}_0 \), for every \( j \in \mathbb{N}_0 \) there exists a constant \( \lambda_{\mu,j} \in [1, \infty) \), depending on \( \delta_0 \), s.t. if \( \|y\|_j^k < \delta_0 \) and for some \( r_0 \in \mathbb{N}_0 \) also \( \|x_r\|_0^k < \delta_0 \) for \( r = 0, \ldots, r_0 \), then

\[
\|z_{r+1}\|_j^k \leq \lambda_{\mu,j} \left( e^{-\mu t_{r+1}}\|y\|_j^{m_\mu} + e^{\gamma_{r_0}} \left( \|y\|_j^k (\|z_r\|_j^k)^2 + \|z_r\|_j^k \|z_r\|_j^k) \right) \right)
\]

for all \( r = 0, \ldots, r_0 + 1 \), where \( \rho := 2(p + k_0) + 1 \) as before and \( m_\mu := (p + 2k_0 + 3\rho + 2\mu + j)_{j \in \mathbb{N}_0} \).

Proof. As in the proof of Proposition 4.6

\[
\tilde{f}(x_{r+1}) = \tilde{f}(x_r) + D\tilde{f}(x_r)\Delta x_r + \int_0^1 (1-t)D^2\tilde{f}(x_r + t\Delta x_r)(\Delta x_r, \Delta x_r)\ dt,
\]

hence

\[
z_{r+1} = y - \tilde{f}(x_{r+1}) = y - \tilde{f}(x_r) - D\tilde{f}(x_r)\Delta x_r - \int_0^1 (1-t)D^2\tilde{f}(x_r + t\Delta x_r)(\Delta x_r, \Delta x_r)\ dt,
\]

so

\[
z_{r+1} = D\tilde{f}(x_r)(id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r - \int_0^1 (1-t)D^2\tilde{f}(x_r + t\Delta x_r)(\Delta x_r, \Delta x_r)\ dt. \tag{11}
\]

Now repeatedly using Lemma 4.7 applying the estimates for the smoothing operators \( S_{t_r} \) in the form provided by Equation (10) and using the estimates \( \|z_r\|_j \leq \|y\|_j + a_j\|x_r\|_j \) as in the proof of Lemma 4.8 one can estimate that

\[
\alpha_j(x_r, z_r) := \|D\tilde{f}(x_r)(id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r\|_j' \leq b_j (\|x_r\|_j (\|id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r\|_j + \|z_r\|_j + \|y\|_j)
\]

and for any constant \( \alpha \in \mathbb{N}_0 \),

\[
\|id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r\|_j \leq C_j^{\alpha+j} e^{-\alpha t} \|\hat{\psi}(x_r)z_r\|_{p+j}^{\alpha+j}
\]

\[
\leq C_j^{\alpha+j} e^{-\alpha t} d_{p+\alpha+j} (\|x_r\|_{\alpha+j} \|z_r\|_j \|y\|_{\alpha+j} \|x_r\|_{\alpha+j} + \|z_r\|_{\alpha+j} + \|y\|_{\alpha+j})
\]

\[
\leq C_j^{\alpha+j} e^{-\alpha t} d_{p+\alpha+j} (\|x_r\|_{\alpha+j} \|y\|_{\alpha+j} + \|z_r\|_{\alpha+j} + \|y\|_{\alpha+j})
\]

where \( \rho_{\alpha,j} := C_j^{\alpha+j} d_{1}(1 + a_{k_0} + a_{p+k_0+\alpha+j}) \). Combining this with Lemma 4.8 gives

\[
\|id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r\|_j \leq e^{-\alpha t} \rho_{\alpha,j} (1 + \gamma_0 e^{\gamma t} \|y\|_{\alpha+j}^k + \|y\|_{\alpha+j}^k).
\]

and hence furthermore

\[
\|x_r\|_j \|id_{E_{\infty}} - S_{t_r})\hat{\psi}(x_r)z_r\|_0 \leq e^{-\alpha t} \rho_{\alpha,0} (1 + \|y\|_{\alpha+j}^k).
\]

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provided that \( j \geq k_0 \). Replacing \( j \) by \( j + k_0 \) and combining the above gives

\[
\alpha_{k_0+j}(x_r, z_r) \leq e^{-\alpha t_r} \left( 1 + \|y\|_0^k \right)^2 \tau_{\alpha,j} e^{3\mu t_r} \|y\|_0^\mu \tau_{\alpha,j} e^{3\mu t_r} \|y\|_0^\mu
\]

\[
eq e^{-3(3\mu+2\mu)t_r} \left( 1 + \|y\|_0^k \right)^2 \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu
\]

\[
\leq e^{-\mu t_{r+1}} \left( 1 + \|y\|_0^k \right)^2 \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu
\]

\[
\leq e^{-\mu t_{r+1}} \left( 1 + \|y\|_0^k \right)^2 \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu
\]

\[
\leq e^{-\mu t_{r+1}} \left( 1 + \|y\|_0^k \right)^2 \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu \tau_{3\mu+2\mu,j} e^{3\mu t_r} \|y\|_0^\mu
\]

for any \( \mu \in \mathbb{N}_0 \), where

\[
\tau_{\alpha,j} := 4b_{k_0+j}(\rho_{\alpha,0} + \rho_{\alpha,k_0+j})(\gamma_0^\alpha \gamma_{p+k_0+\alpha+j} + \gamma_0^\alpha \gamma_{p+\alpha} \gamma_{k_0+j})
\]

\[
\tau_{\mu,j} := \tau_{3\mu+2\mu,j}
\]

\[
m_{\mu} := (p + 2k_0 + 3\rho + 2\mu + j)_{j \in \mathbb{N}_0}.
\]

Now the second term in Equation (11) can be estimated as follows, using Lemma 4.8 as before:

\[
\beta_j(x_r, z_r) := \left\| \int_0^1 (1 - t) D^2 \tilde{f}(x_r + t \Delta x_r) (\Delta x_r, \Delta x_r) \, dt \right\|_j
\]

\[
\leq \int_0^1 \left\| D^2 \tilde{f}(x_r + t \Delta x_r) (\Delta x_r, \Delta x_r) \right\|_j \, dt
\]

\[
\leq c_j \int_0^1 \left( (\|x_r + t \Delta x_r\|_j^2 + 2\|\Delta x_r\|_j) \|\Delta x_r\|_0 \right) \, dt
\]

\[
\leq c_j \int_0^1 \left( (\|x_r\|_j + t \|\Delta x_r\|_j) (\|\Delta x_r\|_j^2 + 2\|\Delta x_r\|_j) \|\Delta x_r\|_0 \right) \, dt
\]

\[
\leq 2c_j (\|x_r\|_j (\|\Delta x_r\|_0^2 + \|\Delta x_r\|_j \|\Delta x_r\|_0 (1 + \|\Delta x_r\|_0))
\]

Using Lemma 4.8 and the estimate \( \|z_r\|_j \leq \|y\|_j + a_j \|x_r\|_j \), as before, one can further estimate

\[
\|\Delta x_r\|_0^k \leq \tilde{\gamma}_0 e^{2\rho t_r} (1 + \|y\|_0^k) (\|x_r\|_0^k + \|\Delta x_r\|_0^k)
\]

\[
\leq \tilde{\gamma}_0 e^{2\rho t_r} \left( 1 + \|y\|_0^k \right)^2 (\|x_r\|_0^k + \|\Delta x_r\|_0^k)
\]

\[
\leq \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]

and consequently also

\[
1 + \|\Delta x_r\|_0^k \leq 2 \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]

Inserting this in the inequality above and using Lemma 4.8 gives

\[
\beta_{k_0+j}(x_r, z_r) \leq 2c_{j+k_0} (\|x_r\|_{j+k_0} (\|\Delta x_r\|_0^2 + \|\Delta x_r\|_{j+k_0} \|\Delta x_r\|_0 (1 + \|\Delta x_r\|_0))
\]

\[
\leq 2c_{j+k_0} (\|x_r\|_{j+k_0} (\|\Delta x_r\|_0^2 + \|\Delta x_r\|_{j+k_0} \|\Delta x_r\|_0 (1 + \|\Delta x_r\|_0))
\]

\[
\leq 2c_{j+k_0} (\tilde{\gamma}_0 e^{2\rho t_r} (\|y\|_0^k \|\Delta x_r\|_0^k + \|\Delta x_r\|_0^k) + \tilde{\gamma}_0 e^{2\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]

\[
\leq 2 \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]

\[
\leq 2 \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
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\leq \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
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\leq \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
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\[
\leq \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]

\[
\leq \tilde{\gamma}_0 (1 + a_{k_0}) e^{3\rho t_r} \left( 1 + \|y\|_0^k \right)^2
\]
where \( \tilde{\tau}_j := 2c_{j+k_0} \delta_0^2 (\gamma_j + 2\gamma_j (1 + \alpha_0)) \). Finally one can now combine the above estimates for \( \alpha_{k_0+j}(x_r, z_r) \) and \( \beta_{k_0+j}(x_r, z_r) \) to obtain

\[
\|z_{r+1}\|_k^k = \|z_{r+1}\|_{k_0+j}^k \\
\leq \alpha_{k_0+j}(x_r, z_r) + \beta_{k_0+j}(x_r, z_r) \\
\leq (\tau_{\mu,j} + \tilde{\tau}_j) \left( 1 + \|y\|_0^m \right)^3 \\
\cdot \left( e^{-\mu_{r+1} t_r} \|y\|_j^m e^{7\mu_{r}} \left( \|z_r\|_0^k \right)^2 + \|z_r\|_0^k \|z_r\|_j^k \right).
\]

This finishes the proof by defining \( \lambda_{\mu,j} := (\tau_{\mu,j} + \tilde{\tau}_j)(1 + \delta_0)^3 \).

**Proposition 4.7.** In the setting as above for \( \mu \in N_0 \) large enough (e.g. \( \mu \geq 16\rho \) where \( \rho := 2(p + k_0) + 1 \)) there exists a shift \( m \subseteq N_0 \) (e.g. \( m := (p + 2k_0 + 3p + 2\mu + j)_{j \in N_0} \)) s.t. for \( \delta_0 > 0 \) small enough (depending on \( \mu \)) there exist constants \( \tau_j, \tau'_j, \tau''_j \in [1, \infty) \) (depending on \( \delta_0 \)), for \( j \in N_0 \), s.t. if \( \|y\|_0^m < \delta_0 \) then \( x_r \) is well defined for all \( r \in N_0 \) and

\[
\|x_r\|_k^k \leq \tau_j \|y\|_j^m \\
\|z_r\|_k^k \leq \tau'_j e^{-\frac{\mu_{r+1}}{2} t_r} \|y\|_j^m \\
\|\Delta x_r\|_k^k \leq \tau''_j e^{-\frac{\mu_{r+1}}{2} t_r} \|y\|_j^m.
\]

**Proof.**

**Claim.** Given any \( \mu \geq 14\rho \), there exist constants \( \tau'_j, \tau''_j > 0 \) (arbitrarily large, depending on \( \mu \)) and \( \delta_0 > 0 \) arbitrarily small s.t. if \( \|y\|_0^m < \delta_0 \) and for some \( r_0 \in N_0 \) also \( \|x_r\|_k^k < \delta_0 \) for \( r = 0, \ldots, r_0 \), then

\[
\|z_r\|_0^k \leq \tau'_0 e^{-\mu_{r+1}} \|y\|_j^m, \\
\|\Delta x_r\|_k^k \leq \tau''_0 e^{-\frac{\mu_{r+1}}{2} t_r} \|y\|_j^m
\]

for all \( r = 0, \ldots, r_0 + 1 \).

**Proof.** Let \( \delta_0 > 0 \) be s.t. Lemma 4.9 applies. One obtains

\[
\|z_{r+1}\|_{r+1}^k \leq \lambda_{\mu,0} \left( e^{-\mu_{r+1} t_r} \|y\|_0^m + e^{7\mu_{r+1}} \left( 1 + \|y\|_0^k \right) \left( \|z_r\|_0^k \right)^2 \right) \\
\leq \lambda_{\mu,0} \left( 1 + \delta_0 \right) e^{-\mu_{r+1} t_r} \left( \|y\|_0^m + e^{7\mu_{r+1}} \left( \|z_r\|_0^k \right)^2 \right) \\
= \lambda_{\mu,0} \left( 1 + \delta_0 \right) \left( 1 + \gamma \right) e^{-\mu_{r+1} t_r} \|y\|_0^m \\
\leq \tau'^0 \|y\|_0^m,
\]

provided that \( \lambda_{\mu,0} \left( 1 + \delta_0 \right) \|y\|_0^m \leq \tau'_0 \). Now if \( \delta_0 \leq 1 \) then \( \lambda_{\mu,0} \left( 1 + \delta_0 \right) \|y\|_0^m \leq 2 \lambda_{\mu,0} \|y\|_0^m \leq \tau'_0 \). So \( \lambda_{\mu,0} \left( 1 + \delta_0 \right) \|y\|_0^m \leq \tau'_0 \) provided that \( \tau'_0 \leq 1 \) and assuming that \( \tau'_0 \geq 4 \lambda_{\mu,0} \) this holds if \( \delta_0 \leq \frac{\tau'_0}{4} \). Hence by induction, given \( \mu \geq 14\rho \), for \( \tau'_0 \geq 4 \lambda_{\mu,0} \) and \( \delta_0 > 0 \) small enough s.t. Lemma 4.9 applies, \( \delta_0 \leq 1 \)
and $\delta_0 \leq \frac{1}{2^{7/2}}$, the first part of the claim, the estimate on $\|z_r\|^k_0$, follows. For the second part combine this with Lemma 4.8 to see that

$$\|\Delta x_r\|^k_0 \leq \tau_0 e^{2\rho t_r} \left(1 + \|y\|^k_0\right) \|z_r\|^k_0$$

$$\leq \tau_0 (1 + \delta_0) \tau_0 e^{-\left(\mu - 2\rho\right)t_r} \|y\|^m_0$$

from which the claim follows with $\tau''_0 := 2\tau'\tau_0^2$, since $\mu - 2\rho \geq \mu - 7 \geq \mu/2$ provided that $\mu \geq 14\rho$.

\textbf{Claim.} Given any $\mu \geq 14\rho$ for $\delta_0 > 0$ small enough (depending on $\mu$) there exists a constant $\tau_0 > 0$ (arbitrarily large, depending on $\mu$) s.t. if $\|y\|^m_0 < \delta_0$, then $x_r$ is well defined for all $r \in \mathbb{N}_0$ and

$$\|x_r\|^k_0 \leq \tau_0 \|y\|^m_0.$$ 

\textbf{Proof.} Let $\delta_0 > 0$ and $\tau''_0 > 0$ be s.t. the previous claim applies. I will show by induction that the claim holds for $\tau_0 := 2\tau''_0$, after replacing $\delta_0$ by $\delta'_0 := \frac{\tau''_0}{2^{7/2}} > 0$. So let $\|y\|^m_0 < \delta'_0 \leq \delta_0$. By definition $x_0 = 0$, so the inequality in the claim holds for $r = 0$. Now assume that $x_{r-1}$ is well defined, lies in $V'$ and that the inequality holds for $x_s$, $s = 0, \ldots, r - 1$. By definition $x_r = \sum_{s=0}^{r-1} \Delta x_s$, and by the previous claim hence

$$\|x_r\|^k_0 \leq \sum_{s=0}^{r-1} \|\Delta x_s\|^k_0$$

$$\leq \sum_{s=0}^{r-1} \tau''_0 e^{-\frac{\mu}{2} t_r} \|y\|^m_0$$

$$\leq \tau''_0 \sum_{s=0}^{\infty} 2^{-s} \|y\|^m_0$$

$$= \tau''_0 \|y\|^m_0$$

since $t_r = (3/2)^r \geq r$ and $\mu/2 \geq 7\rho = 7(2(p + k_0) + 1) \geq 1$. In particular, $\|x_r\|^k_0 \leq 2\tau''_0 \delta'_0 = \frac{2\tau''_0}{2^{7/2}} \delta_0 < \delta_0$. Since for $\delta_0$ small enough $\|x_r\|^k_0 < \delta_0$ implies $x_r \in V'$, $x_r$ is well defined, lies in $V'$ and satisfies the inequality in the claim.

\textbf{Claim.} Let $\mu \geq 16\rho$ and let $\delta_0 > 0$ be small enough s.t. the previous two claims and Lemmas 4.8 and 4.9 hold. Then Proposition 4.7 holds as well.

\textbf{Proof.} $\tau_0, \tau'_0, \tau''_0 > 0$ s.t. the inequalities in Proposition 4.7 hold were already defined in the previous two claims. Using the first claim and Lemma 4.8 for general $j \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$,

$$\|z_{r+1}\|^k_j \leq \lambda_{\mu, j} \left(e^{-\mu t_{r+1}} \|y\|^m_j + \tau_0 e^{-\mu t_r} \|y\|^m_0\right)$$

$$\cdot \left(\|y\|^k_j e^{-\mu t_r} \|y\|^m_0 + \tilde{\tau}_j e^{\mu t_r} \|y\|^m_0\right)$$

$$\leq \lambda_{\mu, j} (1 + \tau_0 (\tau'_0 + \tilde{\tau}_j)) e^{\mu t_{r+1}} \left(1 + \tau_0 e^{-\mu t_r} \|y\|^m_0\right) \|y\|^m_j$$

$$\leq \lambda_{\mu, j} (1 + \tau_0 (\tau'_0 + \tilde{\tau}_j)) (1 + \|y\|^m_0)^2 e^{\mu t_{r+1}} \left(1 + \|y\|^m_0\right) \|y\|^m_j$$

$$\leq 2\lambda_{\mu, j} (1 + \tau_0 (\tau'_0 + \tilde{\tau}_j)) (1 + \|y\|^m_0)^2 e^{-\frac{\mu}{2}(\mu - 8\rho)t_{r+1}} \|y\|^m_j.$$ 

For $\mu \geq 16\rho$ one has $\frac{\mu}{2}(\mu - 8\rho) \geq \frac{1}{2}(\mu - 2\mu) = \frac{\mu}{2}$. So

$$\|z_r\|^k_j \leq \tau'_j e^{-\frac{\mu}{2} t_r} \|y\|^m_j.$$ 

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with \( \tau'_j := 2\lambda_\nu (1 + \tau'_0 (\tau'_0 + \tau_j))(1 + \delta_0)^2 \). Combining this with Lemma 4.8 yields
\[
\|\Delta x_r\|_j^k \leq \tau_j e^{2\rho \tau_0} (\|y\|_j^{m_0} \tau_0 e^{-\frac{\tau_j}{\rho} \|y\|_j^{m_0}} + \tau'_j e^{-\frac{\tau'_j}{\rho} \|y\|_j^{m_0}})
\leq \tau_j (\tau'_0 \|y\|_j^{m_0} + \tau'_j) e^{-\left(\tau_j + \delta_0\right) \|y\|_j^{m_0}}.
\]
For \( \mu \geq 16\rho \) one has \( \frac{\tau_j}{\mu} - 2\rho \geq \frac{\tau_j}{\mu} - \frac{\tau_j}{\mu} = \frac{\tau_j}{\mu} \). So
\[
\|\Delta x_r\|_j^k \leq \tau''_j e^{-\frac{\tau_j}{\mu} \|y\|_j^{m_0}}
\]
with \( \tau''_j := \tau_j (\tau'_0 \delta_0 + \tau'_j) \). Finally, from this and \( x_r = \sum_{s=0}^{r-1} \Delta x_s \) it follows that
\[
\|x_r\|_j^k \leq \sum_{s=0}^{r-1} \|\Delta x_s\|_j^k
\leq \tau''_j \sum_{s=0}^{\infty} e^{-\frac{\tau_j}{\mu} \|y\|_j^{m_0}}
\leq 2\tau''_j \|y\|_j^{m_0},
\]
since \( \mu \geq 16\rho \geq 16, t_s \geq 8 \) and hence \( e^{-\frac{\tau_j}{\mu} t_s} \leq 2^{-s} \). So
\[
\|x_r\|_j^k \leq \tau_j \|y\|_j^{m_0}
\]
for \( \tau_j := 2\tau''_j \). □

Combining Propositions 4.6 and 4.7 now easily finishes the proof of Theorem 4.1. By Corollary 4.1 there exists a neighbourhood \( W \subseteq E_\infty \) of 0 s. t. \( \tilde{f} \mid W : W \to E_\infty \) is injective. From the modified Newton’s procedure, in the way described before and because of Proposition 4.7 there exists a neighbourhood \( W' \subseteq E_\infty \) and a map \( g : W' \to W \) as well as a shift \( m \subseteq N_0 \) s. t. \( g \) satisfies the tameness conditions
\[
\|g(y)\|_j^k \leq \tau_j \|y\|_j^{m_0}
\]
for some constants \( \tau_j \in [0, \infty) \) and all \( j \in N_0 \). Also, w. l. o. g. one can assume that \( m \geq k \). Replacing \( W \) by \( \tilde{f}^{-1}(W') \), because \( \tilde{f} \mid W \) is injective one can assume that \( g \circ f \mid W = \text{id}_W \) and \( f \mid W \circ g = \text{id}_W \).

Now let \( y_1, y_2 \in W' \). By Proposition 4.6 and the tameness conditions for \( g \) above, applied to \( y = g(y_2) \) and \( x = g(y_1) \) one has for \( j \in N_0 \)
\[
\|g(y_2) - g(y_1)\|_j^k \leq c_j \left( (\|g(y_2)\|_j^m + \|g(y_1)\|_j^m) \|y_2 - y_1\|_j^k + \|y_2 - y_1\|_j^k \right)
\leq c_j (\tau_j (\|y_2\|_j^m + \|y_1\|_j^m) \|y_2 - y_1\|_j^k + \|y_2 - y_1\|_j^k)
\leq c_j (\tau_j + 1) (\|y_2\|_j^m + \|y_1\|_j^m) \|y_2 - y_1\|_j^m + \|y_2 - y_1\|_j^m).
\]
From this it is immediate that \( g : (E_\infty, \|\cdot\|_j^m) \supseteq W' \to W \subseteq (E_\infty, \|\cdot\|_j) \) is locally Lipschitz continuous for each \( j \in N_0 \).

Now let \( y' \in W' \), and let \( W \subseteq E_\infty \) be a convex balanced neighbourhood of 0 s. t. \( y' + u \in W' \) for all \( u \in W \). Let
\[
\tau_{y'}^g : W \times [0, 1] \to Y
\]
\[
(u, t) \mapsto \begin{cases} 
\frac{1}{t} (g(y' + tu) - g(y')) - \psi'(g(y'))u & t > 0 \\
0 & t = 0
\end{cases}
\]

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Given $t \in (0, 1]$, again applying Proposition 4.6, this time to $x = g(y')$ and $y = g(y' + t u)$, gives for $j \in \mathbb{N}_0$

\[
\|r^g_x(u, t)\|_j = \left\| \frac{1}{t} (g(y') - g(y' + tu)) - \tilde{\psi}(g(y')) u \right\|_j
\leq \frac{1}{t} c_j(t) \left( 1 + \|g(y')\|_0^{k_j} + \|g(y' + tu)\|_0^{k_j} \right) \cdot
\left( (\|g(y')\|_j^{k_j} + \|g(y' + tu)\|_j^{k_j}) \|tu\|_0^{k_j} + \|tu\|_j^{k_j} \right)
\leq t c_j(t) \tau_{k_0 + 1} \left( 1 + \|y'\|_0^{m_k} + \|y' + tu\|_0^{m_k} \right) \cdot
\left( (\|y'\|_j^{m_k} + \|y' + tu\|_j^{m_k}) \|u\|_0^{k_j} + \|u\|_j^{k_j} \right) \|u\|_0^{k_j}
\leq t c_j(t) \tau_{k_0 + 1} \left( 1 + 2 \|y'\|_0^{m_k} + \|u\|_0^{m_k} + \|u\|_0^{m_k} \right) \cdot
\left( (2\|y'\|_j^{m_k} + \|u\|_j^{m_k}) \|u\|_0^{m_k} + \|u\|_j^{m_k} \right) \|u\|_0^{m_k}.
\]

From this one can read off that $g : (E_\infty', \| \cdot \|_j^{m_k}) \supseteq W' \to W \subseteq (E_\infty, \| \cdot \|_j)$ is weakly Fréchet differentiable with derivative $Dg = g^* \tilde{\psi}$, for each $j \in \mathbb{N}_0$.
This finishes the proof of Theorem 4.1.
5 Nonlinear Fredholm maps

5.1 Strongly smoothing and Fredholm families of morphisms

In this section, family of morphisms will always mean \( \mathfrak{sc}\) family of morphisms. The words “and analogously in the tame context” will always mean that all topological vector spaces appearing are pre-tame \( \mathfrak{sc}\)-Fréchet spaces instead of \( \mathfrak{sc}\)-Fréchet spaces and all morphism, maps and rescalings are tame.

Definition 5.1. Let \( E, E^i, i = 1, 2, \) be \( \mathfrak{sc}\)-Fréchet spaces, let \( U \subseteq E \) be an open subset and let \( \kappa : U \times E^1 \to E^2 \) be a family of morphisms.

1. Let \( ((E \oplus E^1, \psi \oplus \psi^1), (E^2, \psi^2), \mathcal{K} : U \oplus E^1 \to E^2) \) be an envelope of \( \kappa \), where \( \mathcal{K} = (\kappa_k : U_k \times E^1_k \to E^2_k)_{k \in \mathbb{N}_0} \).

2. \( \mathcal{K} \) is called strongly smoothing if for every strictly monotone increasing sequence \( k \subseteq \mathbb{N}_0 \) there exists an envelope \( \mathcal{K}_k : U^k \oplus E^1 \to (E^2)^k \) of \( \kappa \) s.t.

\[
\mathcal{K}_1 \circ (\mathcal{I}^k \times \text{id}_{E^1}) = (E^2)^k \circ \mathcal{K}_k : U^k \times E^1 \to E^2.
\]

3. Given \( \kappa \) is called a strongly smoothing iff for every point \( x \in U \) there exists a neighbourhood \( V \subseteq U \) of \( x \) s.t. \( \kappa|_{V \times E^1} : V \times E^1 \to E^2 \) has a strongly smoothing envelope.

And analogously in the tame context.

Lemma 5.1. Let \( E, E^i, i = 1, 2 \) be \( \mathfrak{sc}\)-Fréchet spaces, let \( U \subseteq E \) be an open subset and let \( \kappa : U \times E^1 \to E^2 \) be a family of morphisms. Let furthermore \( ((E \oplus E^1, \psi \oplus \psi^1), (E^2, \psi^2), \mathcal{K} : U \oplus E^1 \to E^2) \) be an envelope of \( \kappa \), where \( \mathcal{K} = (\kappa_k : U_k \times E^1_k \to E^2_k)_{k \in \mathbb{N}_0} \).

1. If \( \mathcal{K} \) is strongly smoothing, then so is any refinement of \( \mathcal{K} \).

2. Given a strictly monotone increasing sequence \( k \subseteq \mathbb{N}_0 \), if \( \mathcal{K} \) is strongly smoothing, then so is \( \mathcal{K}^k : (U \oplus E^1)^k \to (E^2)^k \).

3. Given \( \mathcal{K} \) is strongly smoothing, then so is \( \mathcal{K} \circ (S \oplus S^1) \) for every point \( x \in U \) there exists a neighbourhood \( V \subseteq U \) of \( x \) with the following property: For every envelope \( \mathcal{K} \) of \( \kappa|_{V \times E^1} \) of product form as above there exists a refinement \( \mathcal{K}' \) of \( \mathcal{K} \) and shifts \( k \) with \( k \geq 1 \) s.t. \( (E^2)^k \circ \mathcal{K}' \) is strongly smoothing.

Proof. 1. Obvious.

2. Define \( (\mathcal{K}^k)_1 := \mathcal{K}_{k0} \circ (\text{id}_{E^1} \oplus (1)^k) \).

3. Define \( (\mathcal{T} \circ \mathcal{K} \circ (S \oplus S^1))_k := \mathcal{T}^k \circ \mathcal{K}_k \circ (S^k \oplus S^1) \).

4. Let \( V \subseteq U \) be a neighbourhood of \( x \) as in Proposition 3.1 and after possibly making \( V \) smaller, assume that \( \kappa|_{V \times E^1} \) has a strongly smoothing envelope \( \hat{\mathcal{K}} : V \oplus \hat{E}^1 \to E^2 \). Let \( \mathcal{K} : V \oplus E^1 \to E^2 \) be any envelope of \( \kappa|_{V \times E^1} \). By
Let \( F \) be a strongly smoothing family of morphisms, then \( F \circ (J^\ast K)^1 \) are weakly equivalent. After rescaling \( K \) (after which it is still strongly smoothing by \( 1 \) and \( 2 \)), \( (J^\ast k)^1 \) and \( K' \circ (J^\ast K)^1 \) can be assumed to be strict and equivalent, hence they coincide by Lemma \( 3.2 \) Since the latter is strongly smoothing by \( 2 \) and \( 3 \) so is the former.

\[ \square \]

**Example 5.1.** Let \( E, E^i, i = 1, 2 \), be \( \mathcal{F} \)-Fréchet spaces and let \( U \subseteq E \) be an open subset.

1. Given a strongly smoothing morphism \( K \in \text{Sm}(E^1, E^2) \), the constant family

\[
U \times E^1 \to E^2 \\
(x, u) \mapsto K(u)
\]

is strongly smoothing.

2. Given a family of morphisms \( \kappa : U \times E^1 \to E^2 \), if there exists a finite dimensional \( \mathcal{F} \)-Fréchet space \( C \) and families of morphisms \( \kappa_1 : U \times E^1 \to C \), \( \kappa_2 : U \times C \to E^3 \) s. t. \( \kappa = \kappa_2 \circ \kappa_1 \), then \( \kappa \) is strongly smoothing.

**Remark 5.1.** Note that if \( \kappa : U \times E^1 \to E^2 \) is a strongly smoothing family of morphisms, then \( \kappa(x) : E^1 \to E^2 \) is strongly smoothing in the sense of Definition \( 2.15 \) for all \( x \in U \).

**Proposition 5.1.** Let \( E, E' \) and \( E^i, i = 0, \ldots, 3 \), be \( \mathcal{F} \)-Fréchet spaces, let \( U \subseteq E \) and \( U' \subseteq E' \) be open subsets together with an \( \mathcal{F}^0 \) map \( f : U' \to U \) and let

\[
\kappa : U \times E^1 \to E^2 \\
\phi : U \times E^0 \to E^1 \\
\psi : U \times E^2 \to E^3
\]

be families of morphisms.

1. If \( \kappa \) is strongly smoothing then so are \( f^* \kappa, \kappa \circ \phi \) and \( \psi \circ \kappa \).

2. The map

\[
U \to \text{Sm}(E^1, E^2) \\
x \mapsto \kappa(x)
\]

is well defined and continuous.

And analogously in the tame context.

**Proof.** 1. First, observe that \( f^* \kappa = \kappa \circ (f \times \text{id}_{E^1}) \), \( \kappa \circ \phi = \kappa \circ (\text{id}_U, \phi) \) and \( \psi \circ \kappa = \psi \circ (\text{id}_U, \kappa) \), where \( (\text{id}_U, \phi) : U \times E^0 \to U \times E^1 \), \( (x, u) \mapsto (x, \phi(x, u)) \), etc. Given \( x \in U \), \( x' \in U' \), going through the proof of Proposition \( 3.2 \) and using Lemma \( 3.4 \) one can assume that there are open neighbourhoods \( V' \subseteq U' \) and \( V \subseteq U \) of \( x' \) and \( x \), respectively, s. t. \( f(V') \subseteq V \), together with strict envelopes \( \mathcal{T} : E' \supseteq V' \to E \) of \( f|_{V'} \), \( \mathcal{K} : E \oplus E^1 \supseteq V \oplus E^1 \to E^2 \) of
\( \kappa_{\mid V \times E^1} \), \( \mathcal{F} : \mathbb{E} \oplus \mathbb{E}^0 \supseteq \mathbb{V} \oplus \mathbb{E}^0 \rightarrow \mathbb{E}^1 \) of \( \phi_{\mid V \times E^0} \) and \( \mathcal{H} : \mathbb{E} \oplus \mathbb{E}^2 \supseteq \mathbb{V} \oplus \mathbb{E}^2 \rightarrow \mathbb{E}^2 \) of \( \psi_{\mid V \times E^2} \). Furthermore, \( \mathcal{H} \) can be assumed to be strongly smoothing.

Let \( k \subseteq \mathbb{N}_0 \) be a strictly monotone increasing sequence.

Then an envelope of \( f^* \kappa_{\mid V \times E^1} \) is given by \( \mathcal{F} \circ \mathcal{K} := \mathcal{K} \circ (\mathcal{F} \circ \text{id}_{E^k}) \) and one can define \( (f^\star \mathcal{K})_k := \mathcal{K}_k \circ (f^k \circ \text{id}_{E^k}) \).

An envelope of \( \kappa \circ \phi_{\mid V \times E^1} \) is given by \( \mathcal{K} \circ (\text{id}_V, \mathcal{F}) \) and one can define \((\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_k := \mathcal{K}_k \circ (\text{id}_V, \mathcal{F} \circ (f^k \circ \text{id}_{E^k})) \).

Similarly, an envelope of \( \psi \circ \kappa_{\mid V \times E^1} \) is given by \( \mathcal{H} \circ (\text{id}_V, \mathcal{X}) \) and one can define \((\mathcal{H} \circ (\text{id}_V, \mathcal{X}))_k := \mathcal{H}_k \circ (\text{id}_V, \mathcal{X}_k) \).

2. This is a local question, so one can assume that \( \kappa \) has a strongly smoothing envelope \( \mathcal{K} : \mathbb{U} \oplus \mathbb{E}^1 \rightarrow \mathbb{E}^2 \) as in Definition 5.1. For any strictly monotone increasing sequence \( k \subseteq \mathbb{N}_0 \) let \( \mathcal{K}_k : \mathbb{U}^k \oplus \mathbb{E}^1 \rightarrow (\mathbb{E}^2)^k \) be as in Definition 5.1.

For \( x \in \mathbb{U}_{\infty} \) and \( u \in E^1_0 \) and any \( k \in \mathbb{N}_0 \), let \( k := (k + j) \in \mathbb{N}_0 \). Then

\[
\kappa_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0(x, u) = \kappa_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0(x, u)
\]

\[
= (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0(x, u)
\]

\[
= (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0(x, u)
\]

Hence for every \( k \in \mathbb{N}_0 \), \( \kappa_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0 : U_{\infty} \times E^1_0 \rightarrow E^2_0 \) factors via a continuous map through \((i^2)^0 : E^2_0 \rightarrow E^2_0)\). So there exists a continuous map \( \pi_0 : U_{\infty} \times E^1_0 \rightarrow E^2_0 \) s.t. \( \kappa_0 \circ (\mathcal{K} \circ (\text{id}_V, \mathcal{F}))_0 = (i^2)^0 \circ \pi_0 \). By Proposition 1.5, the map \( U_{\infty} \rightarrow L_0 (E^1_1, E^2_0) \), \( x \mapsto \pi_0 \circ (i^2)^0 \), is continuous, hence defines the desired continuous map \( U \rightarrow \text{Sm}(E^1, E^2) \).

Proposition 5.2. Let \( E \) and \( E^i, i = 1, 2, \) be \( \pi \)-Fréchet spaces, let \( U \subseteq E \) be an open subset, let \( x_0 \in U \) and let \( \phi : U \times E^1 \rightarrow E^2 \) and \( \kappa : U \times E^1 \rightarrow E^2 \) be families of morphisms.

If \( \phi \) is invertible and \( \kappa \) is strongly smoothing with \( \kappa(x_0) = 0 \) then there exists a neighbourhood \( V \subseteq U \) of \( x_0 \) and a strongly smoothing family of morphisms \( \kappa' : V \times E^2 \rightarrow E^1 \) with \( \kappa'(x_0) = 0 \) s.t. \( (\phi + \kappa')|_V \) is invertible and \( (\phi + \kappa')|_V^{-1} = \phi^{-1} + \kappa' \).

And analogously in the tame context.

Proof.

Claim. One can assume w.l.o.g. that \( E^1 = E^2 =: E \) and \( \phi = \text{id}_E \).

Proof. \((\phi + \kappa') \circ \phi^{-1} = \phi \circ \phi^{-1} + \kappa \circ \phi^{-1} = \text{id}_E + \kappa \circ \phi^{-1} \) and \( \kappa \circ \phi^{-1} \) is strongly smoothing by Proposition 5.1 and satisfies \((\kappa \circ \phi^{-1})(x_0) = \kappa(x_0) \phi^{-1}(x_0) = 0 \). So assuming that the result holds for \( \phi = \text{id}_E \) there exists a neighbourhood \( V \subseteq U \) of \( x_0 \) and \( \kappa' : V \times E^2 \rightarrow E^1 \) s.t. \( \text{id}_E + \kappa \circ \phi^{-1}|_V \) is invertible with inverse \((\text{id}_E + \kappa \circ \phi^{-1})|_V^{-1} = \text{id}_E + \kappa' \). Then \((\phi + \kappa')|_V^{-1} = \phi^{-1} + \kappa \circ \phi^{-1}|_V \circ (\text{id}_E + \kappa \circ \phi^{-1})|_V^{-1} \). Define \( \kappa' := \phi^{-1} \circ (\text{id}_E + \kappa') \), which is strongly smoothing again by Proposition 5.1.

After applying this claim, an analogous argument to the one used in the proof of Proposition 5.1 choosing envelopes \( \mathcal{K} : \mathbb{U} \oplus E' \rightarrow E' \) and \( \mathcal{K}_k : \mathbb{U}^k \oplus E' \rightarrow (E')^k \) for \( k := (k + j) \in \mathbb{N}_0 \) (which is tame) and \( k \in \mathbb{N}_0 \) arbitrary, shows that \( \kappa \) defines and is defined by a sequence of continuous maps

\[
\kappa^k_j : U_k \times E'_j \rightarrow E'_k
\]
for $k \in \mathbb{N}_0$ and $j = 0, \ldots, k$ which satisfy the obvious compatibility conditions under the inclusions in $\mathbb{E}$ and $\mathbb{E}'$. W.l.o.g. one can also assume that $U$ is strict. By Proposition 1.5, for any $0 < j \leq k$, it follows from $\kappa^k_j = \kappa^k_{j-1} \circ (\text{id} \times \iota^j_{j-1})$ that the map

$$\tilde{\kappa}^k_j : U_k \to L_\infty(E'_j, E''_k)$$

$$x \mapsto \kappa^k_j(x)$$

is continuous. Also, by assumption, $\tilde{\kappa}^k_j(x_0) = 0$ for all $0 < j \leq k$, so in particular $\lambda := \tilde{\kappa}^k_1 : U_1 \to L_\infty(E'_1, E''_1)$ is continuous with $\lambda(x_0) = 0$.

Hence there is a neighbourhood $V_1 \subseteq U_1$ s.t. $\|\lambda(x)\| < \frac{1}{2}$ for all $x \in V_1$. $\|\lambda(x)\|$ here denotes the operator norm of $\lambda(x)$. It follows that the map

$$\mu : V_1 \to L_\infty(E'_1, E''_1)$$

$$x \mapsto \sum_{\ell=0}^{\infty} (-\lambda(x))^{\ell}$$

is well defined and continuous, and furthermore that $\|\mu(x)\| < 2$ for all $x \in V_1$.

Now define $V_k := (\kappa^k_1)^{-1}(V_1) \subseteq U_k$ and

$$\kappa^k_j : V_k \times E'_j \to E''_k$$

$$(x, u) \mapsto -\kappa^k_1(x)\mu(\iota^k_1(x))\iota^j_1(u)$$

for all $1 \leq j \leq k$. The $\kappa^k_j$ are obviously continuous as compositions of continuous functions and

$$(\text{id}_{E'_1} + \kappa^k_1)|^{-1} = \text{id}_{E''_1} + \kappa^k_1$$

as a straightforward calculation shows. Replacing $\mathbb{E}$ and $\mathbb{E}'$ by $\mathbb{E}^1$ and $\mathbb{E}'^1$, respectively (to account for the fact that one always needed $j \geq 1$ in the above), the $\kappa^k_j$ define strongly smoothing envelopes $\mathcal{K}^j : \mathbb{V} \oplus \mathbb{E}' \to \mathbb{E}'$ and $\mathcal{K}^k_j : \mathcal{V}^k \oplus \mathbb{E}' \to \mathbb{E}^k$, for all strictly monotone increasing sequences $k \subseteq \mathbb{N}_0$, as in the definition of a strongly smoothing family of morphisms $\kappa^j : \mathbb{V} \times \mathbb{E}' \to \mathbb{E}'$ with $(\text{id}_{E'_1} + \kappa)|^{-1} = \text{id}_{E''_1} + \kappa^j$.

Finally, one needs to show that in the tame context $\kappa^j$ is tame, which will follow by showing that $\mathcal{K}^j$ and the $\mathcal{X}^k_j$ are tame provided that $\mathcal{X}$ and the $\mathcal{X}_k$ are. It is fairly straightforward to see, directly from Definition 4.8, that this amounts to the existence of constants $C_k > 0$ s.t.

$$\|\kappa^j_k(x)u\|_k \leq C_k(\|x\|_k + \|u\|_1)$$

for all $0 \leq j \leq k$ and $(x, u) \in U_k \times E'_j$. Now for $1 \leq j \leq k$ and $(x, u) \in V_k \times E'_j$,

$$\|\kappa^j_k(x)u\|_k = \left\| -\kappa^j_1(x)\mu(\iota^j_1(x))\iota^j_1(u) \right\|_k$$

$$\leq C_k \left( \|x\|_k + \left\| \mu(\iota^j_1(x))\iota^j_1(u) \right\|_1 \right)$$

$$\leq C_k \left( \|x\|_k + 2\|\iota^j_1(u)\|_1 \right)$$

$$\leq 2C_k(\|x\|_k + \|u\|_1).$$

\[\square\]
Definition 5.2. Let $E, E^i, i = 1, 2,$ be Fréchet spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \to E^2$ be a family of morphisms.

1. $\phi$ is called a Fredholm iff it is locally invertible modulo strongly smoothing families of morphisms, i.e. iff for every $x_0 \in U$ there exist a neighbourhood $V \subseteq U$ of $x_0$ and families of morphisms

$$\psi : V \times E^2 \to E^1 \quad \kappa : V \times E^1 \to E^1 \quad \kappa' : V \times E^2 \to E^2$$

with $\kappa$ and $\kappa'$ strongly smoothing s.t.

$$\psi \circ \phi|_{V \times E^1} = \text{id}_{E^1} + \kappa$$

and

$$\phi|_{V \times E^1} \circ \psi = \text{id}_{E^2} + \kappa'.$$

$\psi$ is then called a local Fredholm inverse to $\phi$.

2. The map

$$\text{ind} : U \to \mathbb{Z}$$

$$x \mapsto \text{ind}(\phi(x)),$$

where $\text{ind}(\phi(x))$ denotes the Fredholm index of the Fredholm morphism $\phi(x) : E^1 \to E^2$ in the sense of Definition 2.20, is called the (Fredholm) index of $\phi$.

3. The map

$$\text{corank} \phi : U \to \mathbb{N}_0$$

$$x \mapsto \dim \text{coker}(\phi(x)),$$

where $\text{coker}(\phi(x))$ denotes the (finite dimensional) cokernel of the Fredholm morphism $\phi(x) : E^1 \to E^2$, is called the corank of $\phi$.

$\phi$ is said to have constant rank if $\text{corank} \phi : U \to \mathbb{N}_0$ is constant.

And analogously in the tame context.

Example 5.2. Every invertible family of morphisms $\phi$ is Fredholm of constant rank with $\text{corank} \phi \equiv 0$ and $\text{ind} \phi \equiv 0$.

Proposition 5.3. Let $E, E^i, i = 1, \ldots, 3,$ be Fréchet spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets together with an $\mathcal{E}^0$ map $f : U' \to U$ and let

$$\phi : U \times E^1 \to E^2,$$

$$\psi : U \times E^2 \to E^3$$

and

$$\kappa : U \times E^1 \to E^2$$

be families of morphisms with $\phi$ and $\psi$ Fredholm and $\kappa$ strongly smoothing.

1. $f^* \phi : U' \times E^1 \to E^2$ is Fredholm with $\text{ind}(f^* \phi) = f^* \text{ind}(\phi)$.

2. $\psi \circ \phi : U \times E^1 \to E^3$ is Fredholm with $\text{ind}(\psi \circ \phi) = \text{ind} \psi + \text{ind} \phi$.

3. $\phi + \kappa : U \times E^1 \to E^2$ is Fredholm with $\text{ind}(\phi + \kappa) = \text{ind} \phi$.

And analogously in the tame context.

Proof. Using Proposition [5.1], the proofs are completely straightforward. \qed
Lemma 5.2. Let $E$, $E'$, $i = 1, 2$, be $\mathfrak{F}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \to E^2$ be a family of morphisms. 

$\phi$ is Fredholm iff for every $x_0 \in U$ there exists a neighbourhood $V \subseteq U$ of $x_0$ and families of morphisms

$$
\psi : V \times E^2 \to E^1 \quad \kappa : V \times E^1 \to E^1 \quad \kappa' : V \times E^2 \to E^2
$$

with $\kappa$ and $\kappa'$ strongly smoothing s. t.

$$
\psi \circ \phi |_{V \times E^1} = \text{id}_{E^1} + \kappa \quad \text{and} \quad \phi|_{V \times E^1} \circ \psi = \text{id}_{E^2} + \kappa'
$$

and furthermore for any splitting

$$
E = \ker(\phi(x_0)) \oplus X
$$

one can assume that

$$
E' = \ker(\psi(x_0)) \oplus \text{im}(\phi(x_0))
$$

and

$$
\ker(\phi(x_0)) = \ker(\text{id}_{E^1} + \kappa(x_0))
$$

$$
\text{im}(\phi(x_0)) = \text{im}(\text{id}_{E^1} + \kappa'(x_0))
$$

$$
\ker(\psi(x_0)) = \ker(\text{id}_{E^2} + \kappa'(x_0))
$$

$$
\text{im}(\psi(x_0)) = \text{im}(\text{id}_{E^2} + \kappa(x_0))
$$

(12)

$$
= X.
$$

The same holds in the tame context.

Proof. For $x_0 \in U$ let $\psi : V \times E^2 \to E^2$ be a local Fredholm inverse to $\phi$, in a neighbourhood $V \subseteq U$ of $x_0$. Then $\phi(x_0) : E^1 \to E^2$ is a Fredholm morphism with Fredholm inverse $\psi(x_0) : E^2 \to E^1$. Choose any splitting $E' = \ker F' \oplus X'$. By Theorem 2.1 applied to $F := \phi(x_0)$ and $E' := \psi(x_0)$, there exist splittings

$$
E = C \oplus C' \oplus A \oplus B \oplus Y = \ker F
$$

$$
E' = A' \oplus A' \oplus B' \oplus B' \oplus Y', \quad = \ker F' = C'
$$

where all of these subspaces are finite dimensional with the exception of $Y$ and $Y'$. Furthermore, $F'' : E' \to E$ defined by

$$
F'' := \psi(x_0) - \psi(x_0) \circ \iota_{E'}^C \circ \text{pr}_{E'}^C + \\
+ \iota_{E'}^A \circ (\text{pr}_{A'}^E \circ \phi(x_0) \circ \iota_{E'}^A)^{-1} \circ \text{pr}_{A'}^E + \\
+ \iota_{E'}^B \circ (\text{pr}_{B'}^E \circ \phi(x_0) \circ \iota_{E'}^B)^{-1} \circ \text{pr}_{B'}^E
$$

is a well defined Fredholm inverse to $\phi(x_0)$ that satisfies the conditions [12] with $\psi(x_0)$, $\kappa(x_0)$ and $\kappa'(x_0)$ replaced by $F''$, $F'' \circ \phi(x_0) = \text{id}_{E^1}$ and $\phi(x_0) \circ F'' = \text{id}_{E^2}$, respectively. So if one can show that there exists a neighbourhood $V \subseteq U$ of $x_0$ s. t.

$$
\psi' : V \times E^2 \to E^2
$$

$$
\psi'(x) := \psi(x) - \psi(x) \circ \iota_{E'}^C \circ \text{pr}_{E'}^C + \\
+ \iota_{E'}^A \circ (\text{pr}_{A'}^E \circ \phi(x) \circ \iota_{E'}^A)^{-1} \circ \text{pr}_{A'}^E + \\
+ \iota_{E'}^B \circ (\text{pr}_{B'}^E \circ \phi(x) \circ \iota_{E'}^B)^{-1} \circ \text{pr}_{B'}^E
$$

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is a well-defined family of morphisms that is a Fredholm inverse to \( \phi \), then the proof is finished. But the last part is immediate, because by Example 5.1 2. \( \psi' - \psi |_{V \times E^2} \) is strongly smoothing. And from

\[
pr_{E^i_A} \circ \phi(x) \circ \iota^A_E = pr_{E^i_A} \circ \phi(x_0) \circ \iota^A_E + pr_{E^i_A} \circ (\phi(x) - \phi(x_0)) \circ \iota^A_E =: \lambda(x)
\]

\[
pr_{E^i_B} \circ \phi(x) \circ \iota^B_E = pr_{E^i_B} \circ \phi(x_0) \circ \iota^B_E + pr_{E^i_B} \circ (\phi(x) - \phi(x_0)) \circ \iota^B_E =: \mu(x)
\]

well definedness follows by Proposition 5.2 (or much more elementary because \( A, A' \) and \( B, B' \) are finite dimensional) because the constant families defined by \( pr_{E^i_A} \circ \phi(x_0) \circ \iota^A_B : A \to A' \) and \( pr_{E^i_B} \circ \phi(x_0) \circ \iota^B_E : B \to B' \) are invertible and the families \( \lambda : U \times A \to A' \) and \( \mu : U \times B \to B' \) are strongly smoothing by Example 5.1 2. and vanish at \( x_0 \).

**Theorem 5.1.** Let \( E, E^i, i = 1, 2 \), be \( \pi \)-Fréchet spaces, let \( U \subseteq E \) be an open subset and let \( \phi : U \times E^1 \to E^2 \) be a family of morphisms. Then the following are equivalent:

1. \( \phi \) is Fredholm.
2. (a) For every \( x_0 \in U \), \( \phi(x_0) : E^1 \to E^2 \) is Fredholm.
   (b) For one/any pair of splittings
      \[
      E^1 = X \oplus \ker \phi(x_0) \quad \text{and} \quad E^2 = \im \phi(x_0) \oplus C
      \]
      there exists a neighbourhood \( V \subseteq U \) of \( x_0 \) s. t.
      \[
      pr_{\im \phi(x_0)} \circ \phi \circ \iota^X_{E^1} |_{V \times X} : V \times X \to \im \phi(x_0)
      \]
      is invertible.

And analogously in the tame context.

**Proof.** The direction 1 \( \Rightarrow \) 2 \( \Rightarrow \) 2 will show that 2 \( \therefore \) holds for any pair of splittings of \( E^1 = X \oplus \ker \phi(x_0) \) and \( E^2 = \im \phi(x_0) \oplus C \) (which exist by Example 2.18 and Corollary 2.5) and the direction 2 \( \Rightarrow \) 1 \( \Rightarrow \) will show that if there exists one such pair of splittings s. t. 2 \( \therefore \) holds, then \( \phi \) is Fredholm.

1 \( \Rightarrow \) 2. Given \( x_0 \in U \), let \( V \subseteq U \), \( \psi : V \times E^2 \to E^1 \), \( \kappa : V \times E^1 \to E^1 \) and \( \kappa' : V \times E^2 \to E^2 \) be as in Lemma 5.2. Defining

\[
\bar{\phi} : V \times X \to \im \phi(x_0)
\]

\[
\hat{\phi}(x) := pr^{E^2}_{\im \phi(x_0)} \circ \phi(x) \circ \iota^X_{E^1},
\]

\[
\bar{\psi} : V \times \im \phi(x_0) \to \im \psi(x_0) = X
\]

\[
\hat{\psi}(x) := pr^{E^2}_{X} \circ \psi(x) \circ \iota^{\im \phi(x_0)}_{E^2},
\]

\[
\bar{\kappa} : V \times X \to X
\]

\[
\hat{\kappa}(x) := pr^{E^1}_{X} \circ \kappa(x) \circ \iota^{X}_{E^1},
\]

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and

\[ \bar{r}': V \times \text{im } \phi(x_0) \to \text{im } \phi(x_0), \]

\[ \bar{r}'(x) := \text{pr}_{\text{im } \phi(x_0)}^E \circ \bar{r}'(x) \circ t_{E^2}^{\text{im } \phi(x_0)}, \]

it has to be shown that, after possibly shrinking \( V, \), \( \bar{r} \) is invertible. Then by Corollary 5.1. Let \( \text{id}_{\text{im } \phi(x_0)} + \bar{r}(x_0) \) and \( \text{id}_{\text{im } \phi(x_0)} + \bar{r}'(x_0) \) are isomorphisms, so by Proposition 5.2 after shrinking \( V, \) one can assume that \( \text{id}_{\text{im } \phi(x_0)} + \bar{r} = \text{id}_{\text{im } \phi(x_0)} + \bar{r}(x_0) + (\bar{r} - \bar{r}(x_0)) \) and \( \text{id}_{\text{im } \phi(x_0)} + \bar{r}' = \text{id}_{\text{im } \phi(x_0)} + \bar{r}'(x_0) + (\bar{r}' - \bar{r}'(x_0)) \) are invertible. Then

\[
\begin{align*}
\bar{\phi} \circ \psi &= \text{pr}_{\text{im } \phi(x_0)}^E \circ \phi \circ t_{E^1}^{X} \circ \psi \circ t_{E^2}^{\text{im } \phi(x_0)} \\
&= \text{pr}_{\text{im } \phi(x_0)}^E \circ \phi \circ \psi \circ t_{E^2}^{\text{im } \phi(x_0)} + \\
&+ \text{pr}_{\text{im } \phi(x_0)}^E \circ \phi \circ \left( t_{E^1}^{X} \circ \text{pr}_{E^1}^E - \text{id}_{E^1} \right) \circ \psi \circ t_{E^2}^{\text{im } \phi(x_0)} \\
&= \text{id}_{\text{im } \phi(x_0)} + \bar{r}' + \text{pr}_{\text{im } \phi(x_0)}^E \circ \phi \circ \left( t_{E^1}^{X} \circ \text{pr}_{E^1}^E - \text{id}_{E^1} \right) \circ \psi \circ t_{E^2}^{\text{im } \phi(x_0)} \\
&\quad - \text{id}_{\text{im } \phi(x_0)} + \bar{r}' - \text{pr}_{\text{im } \phi(x_0)}^E \circ \phi \circ \text{id}_{E^1} \circ \text{pr}_{E^2}^E \circ \text{id}_{\text{im } \phi(x_0)} \circ \psi \circ t_{E^2}^{\text{im } \phi(x_0)} \\
&\quad = \bar{r}'.
\end{align*}
\]

\[ \bar{r}' : V \times \text{im } \phi(x_0) \to \text{im } \phi(x_0) \]

is strongly smoothing by Example 5.1, 2., and \( \bar{r}'(x_0) = 0 \) by choice of \( \psi. \) Again by Proposition 5.2 after possibly shrinking \( V, \) \( \bar{\phi} \circ \psi \) is invertible, so \( \bar{\phi} \) has a left inverse. By a completely analogous argument, involving \( \bar{r} \) instead of \( \bar{r}', \) one shows that, after possibly shrinking \( V, \) again, \( \psi \circ \bar{r} \) is invertible, hence \( \bar{r} \) has a left inverse. So \( \bar{r} \) is invertible.

24 \Rightarrow 11. Formally the same proof as that of the corresponding statements in Proposition 2.2, Proposition 2.4, or Corollary 2.5.

\[ \square \]

Corollary 5.1. Let \( E, E^i, i = 1, 2, \) be \( \tau \)-Fréchet spaces, let \( U \subseteq E \) be an open subset and let \( \phi : U \times E^1 \to E^2 \) be a family of Fredholm morphisms. Then the following hold:

1. The maps

\[ \dim \ker \phi : U \to \mathbb{N}_0 \]

\[ x \mapsto \dim \ker(\phi(x)) \]

and

\[ \text{corank } \phi : U \to \mathbb{N}_0 \]

\[ x \mapsto \dim \text{coker}(\phi(x)) \]

are upper semicontinuous.

2. The Fredholm index

\[ \text{ind } \phi = \dim \ker \phi - \text{corank } \phi : U \to \mathbb{Z}, \]

of \( \phi \) is continuous.

Proof. By Theorem 5.1, one can follow the standard proof found e.g. in the Lecture notes [Mro04] by T. Mrowka, Chapter 16, esp. Lemmas 16.18–16.20.
So let $x_0, x \in V \subseteq U$, $X \subseteq E^1$ and $C \subseteq E^2$ be as in Theorem 5.1 W. r. t. the decomposition

$$E^1 = X \oplus \ker \phi(x_0) \quad \text{and} \quad E^2 = \operatorname{im} \phi(x_0) \oplus C$$

one can write $\phi(x)$ in matrix form as

$$\phi(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix},$$

where

$$A(x) = \Pr_{\operatorname{im} \phi(x_0)}^E \circ \phi(x) \circ \iota_x^X, \quad C(x) = \Pr_{\operatorname{ker} \phi(x_0)}^E \circ \phi(x) \circ \iota_x^X,$$

$$B(x) = \Pr_{\operatorname{im} \phi(x_0)}^E \circ \phi(x) \circ \iota_x^{\ker \phi(x_0)}, \quad D(x) = \Pr_{\operatorname{im} \phi(x_0)}^E \circ \phi(x) \circ \iota_x^{\ker \phi(x_0)}.$$

$B(x_0), C(x_0)$ and $D(x_0)$ all vanish and $A(x)$ is invertible for all $x \in V$, so one can define

$$G(x) := \begin{pmatrix} \text{id}_X & -A(x)^{-1} \circ B(x) \\ 0 & \text{id}_{\ker \phi(x_0)} \end{pmatrix} \quad \text{and} \quad H(x) := \begin{pmatrix} \text{id}_{\operatorname{im} \phi(x_0)} & 0 \\ -C(x) \circ A(x)^{-1} & \text{id}_C \end{pmatrix}$$

with

$$G(x)^{-1} = \begin{pmatrix} \text{id}_X & A(x)^{-1} \circ B(x) \\ 0 & \text{id}_{\ker \phi(x_0)} \end{pmatrix} \quad \text{and} \quad H(x)^{-1} = \begin{pmatrix} \text{id}_{\operatorname{im} \phi(x_0)} & 0 \\ C(x) \circ A(x)^{-1} & \text{id}_C \end{pmatrix}.$$

Then

$$H(x) \circ \phi(x) \circ G(x) = \begin{pmatrix} A(x) & 0 \\ 0 & D(x) \end{pmatrix},$$

where

$$D'(x) := D(x) - C(x) \circ A(x)^{-1} \circ B(x) : \ker \phi(x_0) \to C.$$

It follows that $(\operatorname{ind} \phi)(x) = \operatorname{ind}(\phi(x)) = \operatorname{ind}(D'(x)) = \dim \ker \phi(x_0) - \dim C = \dim \ker \phi(x_0) - \dim \operatorname{coker} \phi(x_0) = \operatorname{ind}(\phi(x_0)) = (\operatorname{ind} \phi)(x_0)$ by the dimension formula for linear maps between finite dimensional vector spaces. This shows 2.

For 1. first observe that $\dim \ker \phi(x) = \dim \ker(D'(x))$. But $D' : V \times \ker \phi(x_0) \to C$ is a continuous family of linear maps between finite dimensional vector spaces, for which the result is known. The result about $\operatorname{corank} \phi$ then follows from the results for $\operatorname{ind} \phi$ and $\dim \ker \phi$.

\[\square\]

**Example 5.3.** Let $E$ and $E^i, i = 1, 2$, be $\infty$-Fréchet spaces and let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \to E^2$ be a family of morphisms.

Assume that for every $x_0 \in U$, there exists a neighbourhood $V \subseteq U$ of $x_0$ s. t. $\phi|_V$ has an envelope $((E \oplus E^1, \rho \oplus \rho^1), (E^2, \rho^2), F : V \oplus E^1 \to E^2)$ with the following properties (for simplicity the inclusions of the sc-chains $E$ and $E^i, i = 1, 2$, will be dropped from the notation):

1. $F = (\phi_k : V_k \times E^1_k \to E^2_k)$ is strict.
2. For all $k \in \mathbb{N}_0$, $\phi_k : V_k \times E^1_k \to E^2_k$ is strongly continuous, i. e.

$$\tilde{\phi}_k : V_k \to L_c(E^1_k, E^2_k),$$

$$x \mapsto \phi_k(x)$$

is continuous.
3. For all $x \in V_0$, $\phi_0(x) : E^1_0 \to E^2_0$ is Fredholm.

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4. $\mathcal{F}$ is regularising in the following sense: For all $k \in \mathbb{N}_0 \cup \{\infty\}$, if $x \in V_k$ and $u \in E^1_j$ with $\phi_0(x)u \in E^2_j$ for some $0 \leq j \leq k$, then $u \in E^1_j$.

Then $\phi$ is Fredholm.

Assume furthermore that $E, E^i, i = 1, 2$, are pre-tame $\mathbb{R}$-Fréchet spaces, that $\phi$ is tame, and that the following elliptic inequalities hold: For all $k \in \mathbb{N}_0$ there exist constants $c_k, d_k \in [0, \infty)$ s.t. if $x \in V_k$ and $u \in E^1_k$, then

$$||u||^2_k \leq c_k ||x||_0 ||\phi_k(x)u||^2_k + d_k ||x||_k ||\phi_k(x)u||_0^2.$$  

Then $\phi$ is a tame Fredholm family of morphisms.

**Proof.** I will check Theorem [5.1][2].

First, the proof of Lemma 3.5 in [Web12] shows that for all $k \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq j \leq k$, if $x \in V_k$ then $\ker \phi_j(x) = \ker \phi_0(x) \subseteq E^1_k$, $\im \phi_j(x) = \im \phi_0(x) \cap E^2_j$, and $\coker \phi_j(x) \cong \coker \phi_0(x)$ via the inclusion $(\iota^2)^j : E^2_j \rightarrow E^2_0$.

So in particular $\phi_j(x) : E^1_j \rightarrow E^2_j$ is Fredholm with $\text{ind} \phi_j(x) = \text{ind} \phi_0(x)$. By [3] and [4] together with Proposition 2.4, $\mathcal{F}(x) : E^1 \rightarrow E^2$ is a Fredholm operator between $\text{se-chains}$ for all $x \in V_\infty$. This shows Theorem [6.1][2] (0).

Let $x_0 \in V_\infty$. By the above there exist splittings $E^1 = \ker \phi_\infty(x_0) \oplus \mathbb{R}^2 = \im \mathcal{F}(x_0) \oplus \mathbb{R}^2$, where $C \subseteq \mathbb{R}^2$ is finite dimensional. In particular $E^1_0 = \ker \phi_0(x_0) \oplus X_0$ and $E^2_0 = \im \phi_0(x_0) \oplus C$, and $\text{pr}_{\im \phi_0(x_0)} \circ \phi_0(x_0) \circ \iota_{E^1_0} : X_0 \rightarrow \im \phi_0(x_0)$ is an isomorphism. Define, for $k \in \mathbb{N}_0 \cup \{\infty\}$,

$$\gamma_k : V_k \times X_k \rightarrow \im \phi_k(x_0)$$

$$x \mapsto \text{pr}_{\im \phi_k(x_0)} \circ \phi_k(x) \circ \iota_{E^1_k}$$

and

$$\tilde{\gamma}_k : V_k \rightarrow L_c(X_k, \im \phi_k(x_0))$$

$$x \mapsto \text{pr}_{\im \phi_k(x_0)} \circ \phi_k(x) \circ \iota_{E^1_k}.$$  

By [2] the maps $\tilde{\gamma}_k$ are continuous, hence there exists a neighbourhood $V'_0 \subseteq V_0$ of $x_0$ s.t. $\tilde{\gamma}_0(x)$ is invertible for all $x \in V'_0$ and the map

$$V'_0 \rightarrow L_c(\im \phi_0(x_0), X_0)$$

$$x \mapsto \tilde{\gamma}_0(x)^{-1}$$

is continuous. Define $\psi_0 : V'_0 \times \im \phi_0(x_0) \rightarrow X_0$ by $\psi_0(x, u) := \tilde{\gamma}_0(x)^{-1}u$ and, for $k \in \mathbb{N}_0 \cup \{\infty\}$, set $V'_k := (\psi_0)^{-1}(V'_0)$.

Then for every $k \in \mathbb{N}_0$ and $x \in V'_k$, $\tilde{\gamma}_k(x)$ is invertible: For given $u \in X_k \subseteq X_0$, $\tilde{\gamma}_k(x)u = 0$ means $\phi_k(x)u \in C$, but $\phi_k(x)u = \phi_0(x)u$, so $\tilde{\gamma}_0(x)u = 0$, hence $u = 0$ because $\tilde{\gamma}_0(x)$ is injective, and $\tilde{\gamma}_k(x)$ is injective. And given $v \in \im \phi_0(x_0) = \im \phi_0(x_0) \cap E^2_k$, because $\tilde{\gamma}_0(x)$ is surjective there exist $\tilde{v} \in C$ and $u \in X_0$ with $\phi_0(x_0)u = v - \tilde{v} \in E^2_k$. By [4] hence $u \in X_0 \cap E^2_k = X_k$ and hence $\phi_k(x)u = \phi_0(x_0)u = v - \tilde{v}$, so $\tilde{\gamma}_k(x)$ is surjective. It follows from the open mapping theorem that $\gamma_k(x) : X_k \rightarrow \im \phi_k(x_0)$ is an isomorphism. Denoting, for Banach spaces $A, B$, by $\text{Iso}(A, B) \subseteq L_c(A, B)$ the open subset of invertible operators, it follows that $\tilde{\gamma}_k : V'_k \rightarrow \text{Iso}(X_k, \im \phi_k(x_0))$ is well defined. Since the inversion $\text{Iso}(A, B) \rightarrow \text{Iso}(B, A)$, $\lambda \mapsto \lambda^{-1}$, is continuous, it follows that

$$\psi_k : V'_k \times \im \phi_k(x_0) \rightarrow X_k$$

$$(x, u) \mapsto \tilde{\gamma}_k(x)^{-1}u$$  

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is continuous. The maps \( \psi_k \) define an envelope for a (thus well defined) family of morphisms \( \psi : V' \times \text{im} \phi(x_0) \to X \), \( V' := \rho(V'_\infty) \), \( X := \rho^1(X_\infty) \), that is an inverse to \( \text{pr}^{E^2}_\text{im} \phi(x_0) \circ \phi \circ \iota_{E^1}^* |_{V' \times X} : V' \times X \to \text{im} \phi(x_0) \). This shows Theorem 5.1 2(b) so \( \phi \) is Fredholm.

Now assume that the elliptic inequalities hold. The goal is to show that then the family of morphisms \( \psi \) just constructed is tame. So let \( x \in V'_k \) and let \( v \in \text{im} \phi_k(x_0) \). Since one can write

\[
\phi_k(x) = \iota_{E^2_k} \circ \text{pr}^{E^2}_\text{im} \phi_k(x_0) \circ \phi_k(x) + \iota_{E^2_k} \circ \text{pr}^{E^2}_C \circ \phi_k(x) \circ \iota_{E^2_k},
\]

the elliptic inequalities give

\[
\|\psi_k(x)v\|_{X_k} \leq \|\iota_{E^2_k} \circ \text{pr}^{E^2}_\text{im} \phi_k(x_0) \circ \phi_k(x) + \iota_{E^2_k} \circ \text{pr}^{E^2}_C \circ \phi_k(x) \circ \iota_{E^2_k}v\|_{E^2_k}
\]

\[
\leq c_k \|x\|_0 \|\phi_k(x)\|_{E^2_k} + d_k \|x\|_k \|\phi_k(x)\|_{E^2_k} + \|\text{pr}^{E^2}_\text{im} \phi_k(x_0) \circ \phi_k(x) + \text{pr}^{E^2}_C \circ \phi_k(x) \circ \iota_{E^2_k}\|_{E^2_k}
\]

\[
\leq c_k \|x\|_0 \left( \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k} \right) + d_k \|x\|_k \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k}
\]

\[
\leq c_k \|x\|_0 \left( \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k} \right) + d_k \|x\|_k \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k}
\]

where the compatibility conditions of the \( \phi_k \) and \( \psi_k \) under the inclusions in \( E^1 \) and \( E^2 \) have been used (\( C \) defines an sc-scale where all the inclusions are the identity) as well as the fact that on the finite dimensional vector space \( C \) all the induced norms from the \( \|\cdot\|_{E^2_k} \) are equivalent. Hence

\[
\|\psi_k(x)v\|_{X_k} \leq \left( c_k \|x\|_0 + c_k \|\phi_k(x)\|_{E^2_k} \|\phi_k(x)\|_{E^2_k} \right) + \|\phi_k(x)\|_{E^2_k} \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k} + \|\phi_k(x)\|_{E^2_k}
\]

Now by choosing \( V'_0 \) a small enough bounded subset of \( E^0 \) one can assume (by strong continuity of \( \phi_0 \) and \( \psi_0 \) that the operator norms of \( \phi_0(x) \) and \( \psi_0(x) \) are bounded by a constant independent of \( x \in V'_0 \). Then for every bounded subset \( B_0 \subseteq \text{im} \phi_0(x_0) \) there exists a constant \( C > 0 \) s. t.

\[
\|\psi_k(x,v)\|_{X_k} \leq C(1 + \|x\|_k + \|v\|_{\text{im} \phi_k(x_0)})
\]

for all \((x,v) \in V'_k \times B_0 \) where \( B_k := B_0 \cap E^2_k \). So by definition the map \( \psi \) is tame. \( \square \)
5.2 Fredholm maps

**Definition 5.3.** Let $E$ and $E'$ be $\mathcal{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\mathcal{sc}^1$.

1. $f$ is called *Fredholm* iff $Df : U \times E \to E'$ is a Fredholm family of morphisms. If $E$ and $E'$ are pre-tame $\mathcal{sc}$-Fréchet spaces, then $f$ is called *tame Fredholm* iff $f$ is tame and $Df : U \times E \to E'$ is a tame Fredholm family of morphisms.

2. If $f$ is Fredholm, then the map

$$\text{ind } f : U \to \mathbb{Z},
\quad x \mapsto \text{ind } Df(x)$$

is called the *index* of $f$.

3. If $f$ is Fredholm, then the map

$$\text{corank } f : U \to \mathbb{N}_0,
\quad x \mapsto \dim \text{coker}(Df(x))$$

is called the *corank* of $f$. $f$ is said to have *constant rank* if $\text{corank } f : U \to \mathbb{N}_0$ is constant.

**Example 5.4.** Every $\mathcal{sc}$-diffeomorphism $f$ ($\mathcal{sc}^\infty$ map with an $\mathcal{sc}^\infty$ inverse) is Fredholm of constant rank with $\text{corank } f \equiv 0$ and $\text{ind } f \equiv 0$.

**Example 5.5.** Every (linear) Fredholm morphism defines a (nonlinear) Fredholm map, as does every affine map with linear part a Fredholm morphism.

**Example 5.6.** This example covers, in abstract form, applications such as nonlinear Cauchy-Riemann operators, for which the elliptic inequalities are very well studied (cf. [MS04], Appendix B). See also [Ham82], Sections II.2.2 and II.3.3.

Let $E$ and $E'$, be $\mathcal{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a map. Assume that for every $x_0 \in U$, there exists a neighbourhood $V \subseteq U$ of $x_0$ s.t. $f|_V$ has an envelope $((E, \rho), (E', \rho'), \mathcal{F} : V \to \mathcal{E}')$ with the following properties:

1. $\mathcal{F} = (f_k : V_k \to E'_k)$ is strict.
2. For all $k \in \mathbb{N}_0$, $f_k : V_k \to E'_k$ is strongly continuously weakly Fréchet differentiable.
3. The envelope $((E \oplus E, \rho \oplus \rho), (E', \rho'), \mathcal{D}\mathcal{F} : V \times \mathbb{E} \to \mathcal{E}')$ of $Df|_V : V \times E \to E'$ satisfies the conditions from Example 5.3.

Then $f$ is Fredholm.

**Proposition 5.4.** Let $E$, $E'$ and $E''$ be $\mathcal{sc}$-Fréchet spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets and let $f : U \to E'$ and $f' : U' \to E''$ be $\mathcal{sc}^1$ maps with $f(U) \subseteq U'$.

1. If $f$ is Fredholm, then $\text{ind } f : U \to \mathbb{Z}$ is continuous and $\text{corank } f : U \to \mathbb{N}_0$ is upper semicontinuous.
2. If $f$ and $f'$ are Fredholm, then so is $f' \circ f : U \to E''$ and

$$\text{ind } (f' \circ f) = \text{ind } f + f^* \text{ ind } f'.$$
The same holds in the tame context.

Proof. [1] follows from Corollary 5.1 and [2] follows from $D(f' \circ f) = (f^* Df') \circ Df : U \times E \to E''$ (cf. Theorem 3.1) and Proposition 5.3.

5.3 Applications of the Nash-Moser inverse function theorem

5.3.1 The constant rank theorem

Theorem 5.2 (Constant rank theorem). Let $E$ and $E'$ be weakly tame $\pi$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\pi^k$ and tame up to order $k$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq 2$.

Given $x_0 \in U$, if there exists a neighbourhood $V \subseteq U$ of $x_0$ s.t. $f|_V$ is tame Fredholm and has constant rank, then there exist open neighbourhoods $W \subseteq U$ of $x_0$ and $W' \subseteq E'$ of $f(x_0)$ with $f(W) \subseteq W'$, together with $\mathcal{C}^k$ diffeomorphisms, tame up to order $k$,

$$
\Phi : W \to \tilde{W} \subseteq \text{im } Df(x_0) \oplus \ker Df(x_0)
$$

$$
\Psi : W' \to \tilde{W}' \subseteq \text{im } Df(x_0) \oplus \text{coker } Df(x_0)
$$

s. t.

$$
\Psi \circ f|_W \circ \Phi^{-1} = (\text{id}_{\text{im } Df(x_0)} \oplus 0)|_{\tilde{W}}.
$$

Proof. First, one can w.l.o.g. assume that $V = U$, $x_0 = 0$ and $f(x_0) = 0$. Second, because $f$ is Fredholm, one can pick splittings

$$
E = X \oplus \ker Df(x_0) \quad \text{and} \quad E' = \text{im } Df(x_0) \oplus C',
$$

where $C' \subseteq E'$ is identified with $\text{coker } Df(x_0)$ via the quotient projection $E' \to \text{coker } Df(x_0)$. By Theorem 5.1, after shrinking $U$, one can assume that

$$
A : U \times X \to \text{im } Df(x_0)
$$

$$
A(x) := \text{pr}_{\text{im } Df(x_0)}^E \circ Df(x) \circ \iota_E
$$

has a tame inverse $A^{-1} : U \times \text{im } Df(x_0) \to X$. Now define

$$
\Phi' : U \to \text{im } Df(x_0) \oplus \ker Df(x_0)
$$

$$
x \mapsto (\text{pr}_{\text{im } Df(x_0)}^E (f(x)), \text{pr}_{\ker Df(x_0)}^E (f(x))).
$$

One can then write in matrix form

$$
Df(x) = \begin{pmatrix} X & \ker Df(x_0) \\
A(x) & B(x) \\
C(x) & D(x) \end{pmatrix} \begin{pmatrix} \text{im } Df(x_0) \\
C' \end{pmatrix}
$$

$$
D\Phi'(x) = \begin{pmatrix} X & \ker Df(x_0) \\
A(x) & B(x) \\
0 & \text{id}_{\ker Df(x_0)} \end{pmatrix} \begin{pmatrix} \text{im } Df(x_0) \\
\ker Df(x_0) \end{pmatrix}
$$

$$
D\Phi'(x)^{-1} = \begin{pmatrix} \text{im } Df(x_0) & \ker Df(x_0) \\
A^{-1}(x) & -A^{-1}(x) \circ B(x) \\
0 & \text{id}_{\ker Df(x_0)} \end{pmatrix} \begin{pmatrix} X \\
\ker Df(x_0) \end{pmatrix}
$$

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so $D\Phi' : U \times E \to \text{im} Df(x_0) \oplus \ker Df(x_0)$ has a tame inverse. By Theorem 4.1 there exist neighbourhoods $W \subseteq U$ and $\tilde{W} \subseteq \text{im} Df(x_0) \oplus \ker Df(x_0)$ s.t.

$$\Phi := \Phi|_W : W \to \tilde{W}$$

is an $\mathfrak{sc}^k$ diffeomorphism. Furthermore, for all $x \in W$,

$$D(f \circ \Phi^{-1})(\Phi(x)) = Df(x) \circ D\Phi'(x)^{-1}$$

$$= \begin{pmatrix}
\text{im} Df(x_0) & \ker Df(x_0) \\
\text{id}_{\text{im} Df(x_0)} & 0 \\
(C \circ A^{-1})(\Phi^{-1}(x)) & (D - C \circ A^{-1} \circ B)(x)
\end{pmatrix} \text{im} Df(x_0)$$

Since $\Phi$ is an $\mathfrak{sc}^k$ diffeomorphism, it follows that $\text{corank} f|_W = (\text{corank} f \circ \Phi^{-1}) \circ \Phi$, so $(\text{corank} f)(x) = \text{corank}(D - C \circ A^{-1} \circ B)(x)$ for all $x \in W$. Now by construction $(D - C \circ A^{-1} \circ B)(x_0) = 0$, so $\text{corank} f|_W$ is constant iff $(D - C \circ A^{-1} \circ B)(x) = 0$ for all $x \in W$. Thus, for all $\tilde{x} \in \tilde{W}$,

$$D(f \circ \Phi^{-1})(\tilde{x}) = \begin{pmatrix}
\text{im} Df(x_0) & \ker Df(x_0) \\
\text{id}_{\text{im} Df(x_0)} & 0 \\
(C \circ A^{-1})(\Phi^{-1}(\tilde{x})) & 0
\end{pmatrix} \text{im} Df(x_0)$$

So after possibly shrinking $\tilde{W}$, one can assume that $\tilde{W} = \tilde{W}_1 \times \tilde{W}_2$ for some neighbourhoods $\tilde{W}_1$, $\tilde{W}_2$ of 0 in $\text{im} Df(x_0)$ and $\ker Df(x_0)$, respectively, and

$$f \circ \Phi^{-1} : \text{im} Df(x_0) \oplus \ker Df(x_0) \supseteq \tilde{W}_1 \times \tilde{W}_2 \to \text{im} Df(x_0) \oplus C'$$

$$(u, z) \mapsto (u, \varphi(u))$$

where

$$\varphi : \tilde{W}_1 \to C'$$

$$\varphi := \text{pr}_{C'} \circ f \circ \Phi^{-1} \circ \text{id}_{\text{im} Df(x_0) \oplus \ker Df(x_0)}.$$ 

Now define $W' := W' := \tilde{W}_1 \times C'$ and

$$\Psi : \tilde{W}_1 \times C' \to \tilde{W}_1 \times C'$$

$$(u, y) \mapsto (u, y - \varphi(u)).$$

\[ \square \]

**Corollary 5.2.** Let $E$ and $E'$ be weakly tame $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\mathfrak{sc}^\infty$ and tame up to arbitrary order. If $f$ is tame Fredholm and both $\text{ind} f$ and $\text{corank} f$ are constant, then for any $y \in E'$, $f^{-1}(y) \subseteq U$ canonically carries a smooth structure.

In particular, if $f^{-1}(y)$ is 2nd-countable, then it is a smooth manifold of dimension $\dim f^{-1}(y) = \text{ind} f + \text{corank} f$. 

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5.3.2 Finite dimensional reduction and the Sard-Smale theorem

In this section it will be shown that the results for Fredholm maps between Banach spaces from [MS04], Sections A.4 and A.5, carry over almost ad verbatim. For brevities’ sake I will make certain simplifications like restricting to smooth maps. Note that for maps between tame $\mathfrak{m}$-Fréchet spaces that are tame up to arbitrary order, $\mathfrak{m}\mathfrak{c}$ and $\mathfrak{m}\mathfrak{c}$ coincide by Proposition [4.5]

**Theorem 5.3** (Finite dimensional reduction, cf. [MS04], Theorem A.4.3). Let $E$ and $E'$ be weakly tame $\mathfrak{m}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a tame Fredholm map that is $\mathfrak{m}\mathfrak{c}$ and tame up to arbitrary order. Given $x_0 \in U$, there exist the following:

1. A splitting $E' = \text{im} \, Df(x_0) \oplus C$,
2. open neighbourhoods $V, W \subseteq U$ of $x_0$,
3. an $\mathfrak{m}\mathfrak{c}$ diffeomorphism $g : V \to W$, tame up to arbitrary order, with $g(x_0) = x_0$ and $Dg(x_0) = \text{id}_E$,
4. an $\mathfrak{m}\mathfrak{c}$ map $\tilde{k} : V \to C$, tame up to arbitrary order, with $\tilde{k}(x_0) = 0$ and $D\tilde{k}(x_0) = 0$.

Furthermore, setting $k := i_{E'}^C \circ \tilde{k} : V \to E'$,

$$f \circ g(x) = f(x_0) + Df(x_0)(x - x_0) + k(x) \quad \forall \, x \in V.$$

**Proof.** First, by shifting, one can assume w.l.o.g. that $x_0 = 0$ and $f(x_0) = 0$. Second, let $D := Df(x_0)$ and pick splittings $E = \ker D \oplus X$ and $E' = \text{im} \, D \oplus C$ as in Corollary [2.5]. Then $\tilde{D} := \text{pr}_{\ker D}^E \circ D \circ i_X : X \to \text{im} \, D$ is an isomorphism and one can define $Q := i_E^X \circ \tilde{D}^{-1} \circ \text{pr}_{\ker D}^E : E' \to E$ and

$$\psi : U \to E$$

$$x \mapsto \text{pr}_{\ker D}^E \, x + Q \circ f(x).$$

$\psi$ is a tame Fredholm map, $\mathfrak{m}\mathfrak{c}$ and tame up to arbitrary order, satisfying $\psi(0) = 0$ and $D\psi(0)u = \text{pr}_{\ker D}^E \, u + QDu = \text{pr}_{\ker D}^E \, u + \text{pr}_{E'}^E \, u = u$ for all $u \in E$, i.e. $D\psi(0) = \text{id}_E$. So by Theorem [4.4] there exists a neighbourhood $V \subseteq U$ of 0 s. t. $D\psi|_{V \times E} : V \times E \to E$ is invertible with tame inverse. After possibly shrinking $V$ and setting $W := \psi(V)$, by the Nash-Moser inverse function theorem, Theorem [4.4] together with Proposition [4.5] there is a well-defined $\mathfrak{m}\mathfrak{c}$ diffeomorphism, tame up to arbitrary order,

$$g : V \to W$$

$$x \mapsto (\psi|_V)^{-1}(x).$$

Define $\tilde{k} := \text{pr}_{E'}^E \circ f \circ g$. One computes $D \circ \psi(x) = D \circ \text{pr}_{\ker D}^E \, x + D \circ Q \circ f(x) = D \circ Q \circ f(x)$, so $D = D \circ Q \circ f \circ g$ and hence

$$f \circ g = f \circ g \circ D + D$$

$$= f \circ g \circ D \circ Q \circ f \circ g + D$$

$$= (\text{id}_{E'} - D \circ Q) \circ f \circ g + D$$

$$= i_{E'}^C \circ \tilde{k} + D.$$

\qed
**Definition 5.4.** Let $E$ and $E'$ be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $k : U \rightarrow E'$ be $\mathfrak{sc}^1$.  

$k$ is called **strongly smoothing** iff $Dk : U \times E \rightarrow E'$ is a strongly smoothing family of morphisms.

And analogously in the tame context.

**Proposition 5.5.** Let $E$, $E'$ and $E''$ be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$, $V \subseteq E'$ and $W \subseteq E''$ be open subsets, and let $f : U \rightarrow V$, $g : V \rightarrow W$ and $k : U \rightarrow E'$ be $\mathfrak{sc}^1$.

1. If at least one of $f$ and $g$ is strongly smoothing, then so is $g \circ f$.

2. If $f$ is Fredholm and $k$ is strongly smoothing, then $f + k : U \rightarrow E'$ is Fredholm as well with $\ind(f + k) = \ind(f)$.

And analogously in the tame context.

**Proof.** Clear from the definitions and Propositions 5.1 and 5.3.

**Corollary 5.3.** Let $E$ and $E'$ be weakly tame $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \rightarrow E'$ be $\mathfrak{sc}^1$.

Then the following are equivalent:

1. $f$ is tame Fredholm, $\mathfrak{sc}^\infty$, and tame up to arbitrary order

2. for every $x_0 \in U$, $Df(x_0) : E \rightarrow E'$ is Fredholm and there exist

   (a) open neighbourhoods $V, W \subseteq U$ of $x_0$,
   
   (b) an $\mathfrak{sc}^\infty$ diffeomorphism $g : V \rightarrow W$, tame up to arbitrary order, with
   
   $g(x_0) = x_0$ and $Dg(x_0) = \id_E$,

   (c) a strongly smoothing $\mathfrak{sc}^\infty$ map $k : V \rightarrow E'$, tame up to arbitrary order,

   with

   $k(x_0) = 0$ and $Dk(x_0) = 0$,

   s. t.

   $f \circ g(x) = f(x_0) + Df(x_0)(x - x_0) + k(x)$ \hspace{1cm} $\forall x \in V$.

**Proof.** One direction is provided by Theorem 5.3 and Example 5.1, the other direction follows from Example 5.3, Proposition 5.5, Example 5.4 and Proposition 5.4, using that being Fredholm is a local property.

**Definition 5.5.** Let $E, E^i, i = 1, 2$, be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \rightarrow E^2$ be a Fredholm family of morphisms.

A point $x \in U$ is called a **regular point** of $\phi$ iff $\corank(\phi(x)) = 0$.

**Lemma 5.3.** Let $E, E^i, i = 1, 2$, be $\mathfrak{sc}$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \rightarrow E^2$ be a Fredholm family of morphisms.

Then the set of regular points of $\phi$ is an open subset of $U$.

**Proof.** This follows from Corollary 5.1.
Definition 5.6. Let $E$ and $E'$ be $\mathcal{C}^\infty$-Fréchet spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a Fredholm map.

1. A point $x \in U$ is called a regular point of $f$ iff $x$ is a regular point of $Df : U \times E \to E'$.

2. A point $y \in E'$ is called a regular value of $f$ iff $x \in U$ is a regular point of $f$ for every $x \in f^{-1}(y)$.

Theorem 5.4 (Sard-Smale, cf. [MS04], Theorem A.5.1). Let $E$ and $E'$ be weakly tame $\mathcal{C}^\infty$-Fréchet spaces with $E$ 2nd-countable, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be a tame Fredholm map that is $\mathcal{C}^\infty$ and tame up to arbitrary order. Then the set of regular values of $f$ is a generic subset of $E'$ (i.e. a countable intersection of open and dense subsets).

Proof. For an arbitrary subset $A \subseteq U$, denote

$$
\mathcal{V}_{\text{reg}}(f; A) := \{ y \in E' | \text{corank } f(x) = 0 \ \forall x \in f^{-1}(y) \cap A \}.
$$

I will follow the proof of Theorem A.5.1 in [MS04] and show the following:

Claim. For every point $x_0 \in U$ there exists a neighbourhood $V \subseteq U$ of $x_0$ s.t. for every subset $A \subseteq V$ s.t. $A$ is closed in $E$, $\mathcal{V}_{\text{reg}}(f; A) \subseteq E'$ is an open and dense subset of $E'$.

Proof. First, let $D := Df(x_0)$ and pick splittings $E = \ker D \oplus X$ and $E' = \text{im } D \oplus C$ as in Corollary 2.5. Shifting and using Theorem 5.3 one can assume w.l.o.g. that $x_0 = 0$ and $f(x_0) = 0$, and that there exists a neighbourhood $V \subseteq U$ of 0 s.t.

$$
f(x) = Dx + k(x) \ \forall x \in V,
$$

where $k : V \to E'$ can be written as $k = c_{E'}^f \circ \tilde{k}$ for some map $\tilde{k} : V \to C$ with $\tilde{k}(0) = 0$ and $\tilde{Dk}(0) = 0$. For simplicities sake I will drop the inclusion $c_{E'}^f$ from the notation and write $k = \tilde{k}$. Furthermore after possibly shrinking $V$, because $\ker D$ is finite dimensional, hence locally bounded, one can assume that $\text{pr}_{\ker D}(V) \subseteq \ker D$ is bounded.

It needs to be shown that $\mathcal{V}_{\text{reg}}(f; A) \subseteq E'$ is open and dense for every subset $A \subseteq V$ that is closed in $E$.

So let $A \subseteq V$ be such a subset, let $y \in E'$ and let $(y_n)_{n \in \mathbb{N}} \subseteq E' \setminus \mathcal{V}_{\text{reg}}(f; A)$ be a sequence with $y_n \to y$. Write $y_n = y_n^0 + y_n^1$ for $y_n^0 \in \text{im } D$ and $y_n^1 \in C$. By assumption there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ with $f(x_n) = y_n$ and $\text{corank } f(x_n) \geq 1$, and one can write $x_n = x_n^0 + x_n^1$ for $x_n^0 \in \ker D$ and $x_n^1 \in X$. Then

$$
y_n = y_n^0 + y_n^1 = f(x_n) = D(x_n) + k(x_n) = D(x_n^1) + k(x_n) \in \text{im } D \oplus C,
$$

so $D(x_n^1) = y_n^0$. By Corollary 2.5 $\tilde{D} := \text{pr}_{\ker D} \circ D \circ \iota_E^X : X \to \text{im } D$ is an isomorphism, hence $x_n^1 = \tilde{D}^{-1}(y_n^0) = \tilde{D}^{-1} \circ \text{pr}_{\ker D}(y_n) \in X$ and $(x_n^1)_{n \in \mathbb{N}}$ converges. Since $(x_n^0)_{n \in \mathbb{N}} \subseteq \text{pr}_{\ker D}(V)$ is bounded by assumption, after passing to a subsequence one can assume that $(x_n^0)_{n \in \mathbb{N}}$ is convergent as well. Since $A$ is closed, $x := \lim_{n \to \infty} x_n \in A$ and $f(x) = y$. Since $\text{corank } f(x_n) \geq 1$, by Proposition 5.4 also
corank $f(x) \geq 1$ and hence $y \in E' \setminus V_{\text{reg}}(f;A)$. So $V_{\text{reg}}(f;A)$ is open.

To show that $V_{\text{reg}}(f;A)$ is also dense in $E'$, it will a fortiori be shown that $V_{\text{reg}}(f;V)$ is dense in $E'$. To do so, let $y \in E'$ be arbitrary. The goal is to define a sequence $(y_n)_{n \in \mathbb{N}} \subseteq V_{\text{reg}}(f;V)$ with $y_n \to y$. As before, write $y = y^0 + y^1$ with $y^0 \in \text{im} \, D$ and $y^1 \in C$. Fixing $y^0$, the sequence we will construct will be of the form $y_n = y^0 + y^1_n$ for some sequence $(y^1_n)_{n \in \mathbb{N}} \subseteq C$ with $y^1_n \to y^1$. For $x \in V$ write $x = x^0 + x^1$ for $x^0 \in \ker D$ and $x^1 \in X$. Then as above, $f(x) = y_n$ implies $y^0 = D(x^1)$ and $y^1_n = k(x^0 + x^1)$, or $y^1_n = k(x^0 + D^{-1}(y^0))$. So $y_n = y^0 + y^1_n$ is a regular value of $f|_V$ if $y^1_n$ is a regular value of the map

$$
\ker D \supseteq (V - \tilde{D}^{-1}(y^0)) \cap \ker D \to C
\quad x \mapsto k(x + \tilde{D}^{-1}(y^0)).
$$

Since $\ker D$ and $C$ are finite dimensional, by Sard’s theorem the set of regular values of this map is dense in $C$. Hence one can find the required sequence $(y^1_n)_{n \in \mathbb{N}} \subseteq C$ with the above properties.

Using the above claim, the remainder of the proof follows the proof of Theorem A.5.1 in [MS04] ad verbatim: Since $E$ is 2nd-countable and metrisable, cover $U$ by countably many open subsets $V_i$, $i \in \mathbb{N}$, as in the claim, and each $V_i$ in turn by countably many subsets $A^j_i \subseteq V_i$, $j \in \mathbb{N}$, each of which is closed in $X$ (taken for example to be small enough closed balls w. r. t. some chosen compatible metric on $E$). Then the set of regular values of $f$ is

$$
V_{\text{reg}}(f) = \bigcap_{i,j \in \mathbb{N}} V_{\text{reg}}(f;A^j_i).
$$

\qed
6 Summary and a convenient category of Fréchet spaces for nonlinear Fredholm theory

The purpose of this last part is to collect the main definitions, theorems and examples of this article, packaged neatly into an application friendly framework. This section may actually serve as either a starting point to reading this article or a way to avoid reading most of this article.

Definition 6.1 (Definitions 2.6, 4.2 and 4.6).
1. A tame scale space, or tsc-space is a weakly tame sc-Fréchet space.
2. A morphism between tsc-spaces is a continuous linear map that defines a tame morphism between pre-tame sc-Fréchet spaces.
3. A morphism \( F : E \to E' \) between tsc-spaces \( E \) and \( E' \) is called an isomorphism if there exists a morphism \( G : E' \to E \) s.t. \( F \circ G = \text{id}_{E'} \) and \( G \circ F = \text{id}_E \).

Remark 6.1. In particular, every tsc-space is a pre-tame sc-Fréchet space and has an underlying sc-Fréchet space, which in turn has an underlying Fréchet space and every (iso-)morphism of tsc-spaces defines an (iso-)morphism between the underlying (sc-)Fréchet spaces.

Example 6.1 (Example 2.8). Every finite dimensional vector space (over \( k = \mathbb{R}, \mathbb{C} \)) uniquely defines a tsc-space.

Example 6.2 (Examples 2.1 to 2.7 and 2.9 to 2.12). Let \( \Sigma \) be a closed manifold and let \( \pi : F \to \Sigma \) be a (real or complex) vector bundle. Then \( \Gamma(F) \), the vector space of smooth sections of \( F \) with the \( \mathcal{C}^\infty \)-topology, canonically carries the structure of a tsc-space. If \( \pi' : F' \to \Sigma \) is another vector bundle and \( P : \Gamma(F) \to \Gamma(F') \) is a (pseudo)differential operator, then \( P \) canonically defines a morphism of tsc-spaces.

Proposition 6.1 (Lemma 2.5 and Remark 4.2). Let \( E, E' \) and \( E'' \) be tsc-spaces, let \( S, T : E \to E' \) and \( S' : E' \to E'' \) be morphisms between tsc-spaces, and let \( \lambda, \mu \in k \). Then
\[
\lambda S + \mu T : E \to E' \quad \text{and} \quad S' \circ S : E \to E''
\]
are morphisms between tsc-spaces as well.

Definition 6.2 (Definitions 2.9 and 2.10 and Remark 4.2). Let \( E, E^i, i = 1, \ldots, k \) for some \( k \in \mathbb{N}_0 \), be tsc-spaces
1. The direct sum of the \( E^i, i = 1, \ldots, k \), is the tsc-space \( E^1 \oplus \cdots \oplus E^k \) defined by the direct sum of the \( E^i, i = 1, \ldots, k \), as pre-tame sc-Fréchet spaces.
2. A subspace \( E' \) of a tsc-space \( E \) is a sc-subspace \( E' \subseteq E \) of pre-tame sc-Fréchet spaces s.t. there exists a tame morphism \( P : E \to E' \) with \( P \circ J = \text{id}_{E'} \), where \( J : E' \hookrightarrow E \) is the canonical inclusion.
3. A subspace \( E' \) of \( E \) is called split if there exists another subspace \( E'' \) of \( E \) s.t. the canonical morphism
\[
J : E' \oplus E'' \to E \\
(e, e') \mapsto e + e'
\]
is an isomorphism. 

$E''$ is then called a complement to $E'$ and vice versa.

**Remark 6.2.** That subspaces and direct sums of tsc-spaces are well defined tsc-spaces follows from Proposition 4.2.

**Example 6.3** (Examples 2.13 and 2.18). If $E$ is a tsc-space and $C \subseteq E$ is a finite dimensional subspace (as vector spaces), then $C$ canonically defines a split subspace of $E$.

**Example 6.4** (Examples 2.16 and 2.17). Let $\Sigma$ be a closed manifold, let $C \subseteq \Sigma$ be a closed submanifold and let $\pi : F \to \Sigma$ be a real (or complex) vector bundle. There is a well-defined morphism

$$R : \Gamma(F) \to \Gamma(F|_C)$$

$$u \mapsto u|_C$$

between tsc-spaces and also a well-defined subspace

$$\Gamma(F; C) := \ker R = \{u \in \Gamma(F) \mid u|_C = 0\}.$$  

$\Gamma(F; C)$ is a split subspace of $\Gamma(F)$ with a complement isomorphic to $\Gamma(F|_C)$, i.e.

$$\Gamma(F) \cong \Gamma(F; C) \oplus \Gamma(F|_C).$$

**Definition 6.3** (Definition 2.15 and Remark 4.2). Let $K : E \to E'$ be a morphism between tsc-spaces. $K$ is called strongly smoothing if it is strongly smoothing as a tame morphism between pre-tame Fréchet spaces.

**Remark 6.3.** Note that a morphism $K : E \to E'$ between tsc-spaces is strongly smoothing iff the underlying morphism between Fréchet spaces is strongly smooth in the sense of Definition 2.15. This follows from Lemma 2.10.

**Example 6.5** (Example 2.21). If $K : E \to E'$ is a morphism of tsc-spaces s.t. there exists a finite dimensional vector space $C$ and morphisms $L_1 : E \to C$ and $L_2 : C \to E'$ with $K = L_2 \circ L_1$, then $K$ is strongly smoothing.

**Proposition 6.2** (Corollary 2.3 and Remark 2.10). Let $E, E'$ and $E''$ be tsc-spaces, let $K, L : E \to E'$, $S : E'' \to E$ and $T : E' \to E''$ be morphisms and let $\lambda, \mu \in \mathbb{k}$.

1. If $K$ is strongly smoothing, then its underlying morphism between Fréchet spaces is compact.

2. If $K$ and $L$ are strongly smoothing, then so are

$$\lambda K + \mu L : E \to E', \quad K \circ S : E'' \to E \quad \text{and} \quad T \circ K : E \to E''.$$  

**Definition 6.4** (Definition 2.20). Let $E$ and $E'$ be tsc-spaces.

1. A morphism $F : E \to E'$ is called Fredholm iff $F$ is invertible modulo strongly smoothing morphisms i.e. iff there exists a morphism $F' : E' \to E$ and strongly smoothing morphisms $K : E \to E$ and $K' : E' \to E'$ s.t.

$$F' \circ F = \text{id}_E + K,$$

$$F \circ F' = \text{id}_{E'} + K'.$$

Such a morphism $F' : E' \to E$ is then called a Fredholm inverse of $F$.  

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2. The Fredholm index $\text{ind } F$ of a Fredholm morphism $F : E \to E'$ is defined as the Fredholm index of $F$ as a Fredholm operator between the underlying Fréchet spaces.

**Example 6.6** (Examples 2.23 and 2.24).
Let $\Sigma$ be a closed manifold, and let $\pi : F \to \Sigma$ and $\pi' : F' \to \Sigma$ be vector bundles. If $P : \Gamma(F) \to \Gamma(F')$ is an elliptic (pseudo)differential operator, then $P$ defines a Fredholm morphism of tsc-spaces.

**Proposition 6.3** (Proposition 2.5).
Let $E, E'$ and $E''$ be tsc-spaces, let $F : E \to E'$ and $F' : E' \to E''$ be Fredholm morphisms and let $K : E \to E'$ be a strongly smoothing morphism. Then:

1. $F' \circ F : E \to E''$ is Fredholm with $\text{ind}(F' \circ F) = \text{ind } F' + \text{ind } F$.
2. $F + K : E \to E'$ is Fredholm with $\text{ind}(F + K) = \text{ind } F$.

**Theorem 6.1** (Corollary 2.5).
Let $E$ and $E'$ be tsc-spaces and let $F : E \to E'$ be a morphism. Then the following are equivalent:

1. $F$ is Fredholm.
2. (a) $F$ is a Fredholm operator between the Fréchet spaces underlying $E$ and $E'$.
   (b) There exist splittings
   $$E = \ker F \oplus X \quad \text{and} \quad E' = \text{im } F \oplus C.$$  
   In particular, $\text{im } F$ defines a split subspace of $E'$ and the quotient projection $E' \to \text{coker } F = E'/\text{im } F$ is a well defined morphism between tsc-spaces.
   (c) For any splitting as above,
   $$\text{pr}_{\text{im } F}^E \circ F \circ \text{pr}_X^E : X \to \text{im } F$$  
   is an isomorphism of tsc-spaces.

**Definition 6.5** (Definitions 3.5, 3.6, 3.9, 4.8 and 4.9).
Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ and $V \subseteq E'$ be open subsets, and let $f : U \to V \subseteq E'$ be a map.

1. $f$ is called tsc$^\infty$ if $f$ is $\mathcal{S}\mathcal{C}^\infty$ and tame up to arbitrary order as a map between open subsets of pre-tame $\mathcal{S}\mathcal{C}$-Fréchet spaces.
2. $f$ is called a diffeomorphism if $f$ is tsc$^\infty$ and if there exists a tsc$^\infty$-map $g : V \to U$ s.t. $g \circ f = \text{id}_U$ and $f \circ g = \text{id}_V$.
3. $f$ is called a local diffeomorphism if $f$ is tsc$^\infty$ and if for every $x \in U$ there exist open neighbourhoods $U'' \subseteq U$ and $V'' \subseteq V$ of $x$ and $f(x)$, respectively, s.t. $f(U'') = V''$ and $f|_{U''} : U'' \to V''$ is a diffeomorphism.

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**Proposition 6.4** (Propositions 3.2 and 4.4 and Theorem 3.1). Let $E$, $E'$ and $E''$ be tsc-spaces, let $U \subseteq E$, $V \subseteq E'$ and $W \subseteq E''$ be open subsets, and let $f : U \rightarrow V$ and $f' : V \rightarrow W$ be maps. If $f$ and $f'$ are tsc$^\infty$, then so is $f' \circ f$.

**Remark 6.4.** A map is tsc$^\infty$ iff it is sc$^\infty$ and tame up to arbitrary order as a map between open subsets of pre-tame sc-Fr´echet spaces. This follows from Propositions 4.3 and 4.5.

**Remark 6.5.** In particular, every tsc$^\infty$-map defines an arbitrarily often weakly continuously weakly Fr´echet differentiable map between open subsets of the underlying Fr´echet spaces and hence has a well defined differential.

**Definition 6.6** (Definition 4.11). Let $E$, $E'$ and $E_i$, $i = 1, 2, 3$, be tsc-spaces, and let $U \subseteq E$ and $U' \subseteq E'$ be open subsets.

1. A tsc$^\infty$-family (over $U$) of morphisms (from $E^1$ to $E^2$) is a tsc$^\infty$-map

$$\phi : U \times E^1 \rightarrow E^2$$

$$(x,u) \mapsto \phi(x)u$$

that is linear in the second factor.

2. Given tsc$^\infty$-families of morphisms $\phi : U \times E^1 \rightarrow E^2$ and $\psi : U \times E^2 \rightarrow E^3$, their composition is the tsc$^\infty$-family of morphisms

$$\psi \circ \phi : U \times E^1 \rightarrow E^3$$

$$(x,u) \mapsto \psi(x)\phi(x)u.$$ 

3. A tsc$^\infty$-family of morphisms $\phi : U \times E^1 \rightarrow E^2$ is called invertible if there exists a tsc$^\infty$-family of morphisms $\psi : U \times E^2 \rightarrow E^1$ with $\psi \circ \phi = \text{id}_{E^1}$ and $\phi \circ \psi = \text{id}_{E^2}$. $\psi$ will then be called the inverse to $\phi$ and denoted by $\phi^{-1} := \psi$.

4. Given a tsc$^\infty$-family of morphisms $\phi : U \times E^1 \rightarrow E^2$ and a tsc$^\infty$-map $f : U' \rightarrow U$, the pullback of $\phi$ by $f$ is the tsc$^\infty$-family of morphisms $f^*\phi := \phi \circ (f \times \text{id}_{E^1})$, i.e.

$$f^*\phi : U' \times E^1 \rightarrow E^2$$

$$(x,u) \mapsto \phi(f(x))u.$$ 

**Remark 6.6.** If $\phi : U \times E^1 \rightarrow E^2$ is a tsc$^\infty$-family of morphisms, then $\phi(x) : E^1 \rightarrow E^2$ is a morphism of tsc-spaces for all $x \in U$. This follows directly from the definitions and Proposition 4.4.

**Proposition 6.5.** Let $E$, $E'$ and $E''$ be tsc-spaces, let $U \subseteq E$, $V \subseteq E'$ and $W \subseteq E''$ be open subsets, and let $f : U \rightarrow V$ and $g : V \rightarrow W$ be tsc$^\infty$-maps. Then:

1. $Df : U \times E \rightarrow E'$ is a tsc$^\infty$-family of morphisms.

2. $g \circ f : U \rightarrow W$ is a tsc$^\infty$-map with

$$D(g \circ f) = (f^*Dg) \circ Df.$$ 

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Example 6.7 (Section 1.3 and Examples 3.1, 3.2 and 4.5).

Let $\Sigma$ be a closed $n$-dimensional manifold and let $\pi : F \to \Sigma$, $\pi_1 : F_1 \to \Sigma$ and $\pi_2 : F_2 \to \Sigma$ be vector bundles. Let furthermore $B \subseteq \mathbb{R}^r$ (for some $r \in \mathbb{N}_0$) be an open subset and let

$$\phi : B \times \Sigma \to B \times \Sigma$$

$$(b, z) \mapsto (b, \phi_b(z))$$

and

$$\Phi : B \times F_1 \to B \times F_2$$

$$(b, \xi) \mapsto (b, \Phi_b(\xi))$$

be smooth families of maps, where $\phi$ is a family of diffeomorphisms and $\Phi$ covers $\phi^{-1}$. I.e. $\phi$ and $\Phi$ are smooth, $\phi_b \in \text{Diff}(\Sigma)$ and $\Phi_b \in \mathcal{C}^\infty(F_1, F_2)$ for all $b \in B$, and

$$\begin{array}{ccc}
B \times F_2 & \xleftarrow{\Phi} & B \times F_1 \\
\downarrow{\text{id}_B \times \pi} & \circ & \downarrow{\text{id}_B \times \pi} \\
B \times \Sigma & \xrightarrow{\phi} & B \times \Sigma
\end{array}$$

commutes. Furthermore, assume that $\Phi$ is linear in the fibres of $F_1$ and $F_2$, i.e. for every $b \in B$, $\Phi_b : F_1 \to F_2$ defines a vector bundle morphism covering $\phi_b^{-1} : \Sigma \to \Sigma$.

Define

$$\Gamma_B(F) := B \times \Gamma(F),$$

$$\rho := \text{pr}_1 : \Gamma_B(F) \to B$$

and set for $b \in B$

$$\Gamma_b(F) := \{b\} \times \Gamma(F) = \rho^{-1}(b).$$

Also, there is a $\text{tsc}^\infty$-family morphisms

$$\psi : B \times \Gamma(F_1) \to \Gamma(F_2)$$

$$(b, u) \mapsto \Phi^*_b u,$$

where

$$\Phi^*_b u := \Phi_b \circ u \circ \phi_b.$$ 

$\Gamma_B(F)$ is an open subset of the $\text{tsc}$-space $\mathbb{R}^r \oplus \Gamma(F)$ and there is a $\text{tsc}^\infty$-map

$$\Psi : \Gamma_B(F_1) \to \Gamma_B(F_2)$$

$$(b, u) \mapsto (b, \psi(b, u)).$$

Definition 6.7 (Definition 5.1).

Let $E_i, i = 1, 2$ be $\text{tsc}$-spaces, let $U \subseteq E$ be an open subset and let $\kappa : U \times E^1 \to E^2$ be a $\text{tsc}^\infty$-family of morphisms. $\kappa$ is called strongly smoothing iff it is strongly smoothing as a family of morphisms between pre-tame $\overline{\text{pc}}$-Fréchet spaces.
Proposition 6.6 (Proposition 5.1). Let $E$, $E'$ and $E_i$, $i = 0, \ldots, 3$, be tsc-spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets together with a tsc$^\infty$-map $f : U' \to U$ and let

$$
\kappa : U \times E^1 \to E^2 \quad \phi : U \times E^0 \to E^1 \quad \psi : U \times E^2 \to E^3
$$

be tsc$^\infty$-families of morphisms. If $\kappa$ is strongly smoothing then so are $f^* \kappa$, $\kappa \circ \phi$ and $\psi \circ \kappa$.

Remark 6.7. If $\kappa : U \times E^1 \to E^2$ is a strongly smoothing tsc$^\infty$-family of morphisms, then for every $x \in U$, $\kappa(x) : E^1 \to E^2$ is a strongly smoothing morphism of tsc-spaces.

Example 6.8 (Example 5.1). Let $E, E^i, i = 1, 2$, be tsc-spaces and let $U \subseteq E$ be an open subset.

1. Given a strongly smoothing morphism $K : E^1 \to E^2$, the constant family

$$
U \times E^1 \to E^2
$$

$(x, u) \mapsto K(u)$

is strongly smoothing.

2. Given a family of morphisms $\kappa : U \times E^1 \to E^2$, if there exists a finite dimensional vector space $C$ and families of morphisms $\kappa_1 : U \times E^1 \to C$, $\kappa_2 : U \times C \to E^2$ s.t. $\kappa = \kappa_2 \circ \kappa_1$, then $\kappa$ is strongly smoothing.

Definition 6.8 (Definition 5.2). Let $E, E^i$, $i = 1, 2$, be tsc-spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \to E^2$ be a tsc$^\infty$-family of morphisms.

1. $\phi$ is called a Fredholm if it is locally invertible modulo strongly smoothing tsc$^\infty$-families of morphisms, i.e. iff for every $x_0 \in U$ there exist a neighbourhood $V \subseteq U$ of $x_0$ and tsc$^\infty$-families of morphisms

$$
\psi : V \times E^2 \to E^1 \quad \kappa : V \times E^1 \to E^1 \quad \kappa' : V \times E^2 \to E^2
$$

with $\kappa$ and $\kappa'$ strongly smoothing s.t.

$$
\psi \circ \phi|_{V \times E^1} = \text{id}_{E^1} + \kappa
$$

and

$$
\phi|_{V \times E^1} \circ \psi = \text{id}_{E^2} + \kappa'.
$$

$\psi$ is then called a local Fredholm inverse to $\phi$.

2. The map

$$
\text{ind} : U \to \mathbb{Z}
$$

$$
x \mapsto \text{ind}(\phi(x)),
$$

where $\text{ind}(\phi(x))$ denotes the Fredholm index of the Fredholm morphism $\phi(x) : E^1 \to E^2$, is called the (Fredholm) index of $\phi$.

3. The map

$$
\text{corank} : U \to \mathbb{N}_0
$$

$$
x \mapsto \dim \text{coker}(\phi(x)),
$$

where $\text{coker}(\phi(x))$ denotes the (finite dimensional) cokernel of the Fredholm morphism $\phi(x) : E^1 \to E^2$, is called the corank of $\phi$.

$\phi$ is said to have constant rank if $\text{corank} : U \to \mathbb{N}_0$ is constant.
Theorem 6.2 (Theorem 5.1).

Let $E$, $E^i$, $i = 1, 2$, be tsc-spaces, let $U \subseteq E$ be an open subset and let $\phi : U \times E^1 \to E^2$ be a tsc$^\infty$-family of morphisms. Then the following are equivalent:

1. $\phi$ is Fredholm.

2. (a) For every $x_0 \in U$,
   \[ \phi(x_0) : E^1 \to E^2 \]
   is a Fredholm morphism of tsc-spaces.

(b) For one/any pair of splittings
   
   \[ E^1 = X \oplus \ker \phi(x_0) \quad \text{and} \quad E^2 = \im \phi(x_0) \oplus C \]

   there exists a neighbourhood $V \subseteq U$ of $x_0$ s. t.
   \[ \pr_{\im \phi(x_0)} \circ \phi \circ \iota_{E^1} : V \times X \to \im \phi(x_0) \]

   is invertible.

Proposition 6.7 (Proposition 5.3 and Corollary 5.1).

Let $E$, $E'$ and $E^i$, $i = 1, \ldots, 3$, be tsc-spaces, let $U \subseteq E$ and $U' \subseteq E'$ be open subsets together with a tsc$^\infty$-map $f : U' \to U$ and let

\[ \phi : U \times E^1 \to E^2, \]
\[ \psi : U \times E^2 \to E^3 \]

and

\[ \kappa : U \times E^1 \to E^2 \]

be tsc$^\infty$-families of morphisms with $\phi$ and $\psi$ Fredholm and $\kappa$ strongly smoothing. Then the following hold:

1. $f^* \phi : U' \times E^1 \to E^2$ is Fredholm with $\ind(f^* \phi) = f^* \ind(\phi)$.

2. $\psi \circ \phi : U \times E^1 \to E^3$ is Fredholm with $\ind(\psi \circ \phi) = \ind \psi + \ind \phi$.

3. $\phi + \kappa : U \times E^1 \to E^2$ is Fredholm with $\ind(\phi + \kappa) = \ind \phi$.

4. The maps

   \[ \dim \ker \phi : U \to \mathbb{N}_0 \]

   \[ x \mapsto \dim \ker(\phi(x)) \]

   and

   \[ \corank \phi : U \to \mathbb{N}_0 \]

   \[ x \mapsto \dim \coker(\phi(x)) \]

   are upper semicontinuous.

5. The Fredholm index $\ind \phi = \dim \ker \phi - \corank \phi : U \to \mathbb{Z}$ of $\phi$ is continuous.
Definition 6.9 (Definitions 5.3 and 5.4). Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be tsc$^\infty$.

1. $f$ is called Fredholm iff $Df : U \times E \to E'$ is Fredholm.

2. If $f$ is Fredholm, then the map

$$\text{ind } f : U \to \mathbb{Z}$$

$$x \mapsto \text{ind } Df(x)$$

is called the (Fredholm) index of $f$.

3. If $f$ is Fredholm, then the map

$$\text{corank } f : U \to \mathbb{N}_0$$

$$x \mapsto \dim \text{coker}(Df(x))$$

is called the corank of $f$. $f$ is said to have constant rank if corank $f : U \to \mathbb{N}_0$ is constant.

4. $f$ is called strongly smoothing iff $Df : U \times E \to E'$ is strongly smoothing.

Proposition 6.8 (Propositions 5.4 and 5.5). Let $E$, $E'$ and $E''$ be tsc-spaces, let $U \subseteq E$, $V \subseteq E'$ and $W \subseteq E''$ be open subsets, and let $f : U \to V$, $g : V \to W$ and $k : U \to E'$ be tsc$^\infty$.

1. If $f$ is a diffeomorphism, then $f$ is Fredholm.

2. If both $f$ and $g$ are Fredholm, then so is $g \circ f$. Furthermore, 

$$\text{ind}(g \circ f) = \text{ind } f + f^* \text{ind } g.$$

3. If $f$ is Fredholm, then $\text{ind } f$ is continuous and $\text{corank } f$ is upper semicontinuous.

4. If at least one of $f$ and $g$ is strongly smoothing, then so is $g \circ f$.

5. If $f$ is Fredholm and $k$ is strongly smoothing, then $f + k : U \to E'$ is Fredholm as well with $\text{ind}(f + k) = \text{ind}(f)$.

Theorem 6.3 (Nash-Moser inverse function theorem, Theorem 4.1). Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be tsc$^\infty$.

If $Df : U \times E \to E'$ is invertible, then $f$ is a local diffeomorphism.

More explicitly, if there exists a tsc$^\infty$-family of morphisms $\phi : U \times E' \to E$ s.t. $Df \circ \phi = \text{id}_{E'}$ and $\phi \circ Df = \text{id}_E$, then for each $x \in U$ there exists an open neighbourhood $V \subseteq U$ of $x$ s.t. $f(V) \subseteq E'$ is an open neighbourhood of $f(x)$ and there exists a tsc$^\infty$-map $g : f(V) \to V$ that satisfies $g \circ f|_V = \text{id}_V$ and $f|_V \circ g = \text{id}_{f(V)}$. Furthermore, $Dg = g^* \phi$. 

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Theorem 6.4 (Constant rank theorem, Theorem 5.2). Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\text{tsc}^\infty$. Given $x_0 \in U$, if there exists a neighbourhood $V \subseteq U$ of $x_0$ s.t. $f|_V$ is Fredholm and has constant rank, then there exist open neighbourhoods $W \subseteq U$ of $x_0$ and $W' \subseteq E'$ of $f(x_0)$ with $f(W) \subseteq W'$ together with $\text{tsc}^\infty$-diffeomorphisms

$$\Phi : W \to \tilde{W} \subseteq \text{im} \, Df(x_0) \oplus \text{ker} \, Df(x_0)$$

$$\Psi : W' \to \tilde{W}' \subseteq \text{im} \, Df(x_0) \oplus \text{coker} \, Df(x_0)$$

s.t.

$$\Psi \circ f|_W \circ \Phi^{-1} = (\text{id}_{\text{im} \, Df(x_0) \oplus 0})|_{\tilde{W}}.$$  

Corollary 6.1. Let $E$ and $E'$ be $2^{nd}$-countable tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\text{tsc}^\infty$. If $f$ is Fredholm, has constant rank, and $\text{ind} \, f$ is globally constant, then for any $y \in E'$, $f^{-1}(y) \subseteq E$ is a smooth manifold of dimension $\dim f^{-1}(y) = \text{ind} \, f \, + \, \text{corank} \, f$.

Theorem 6.5 (Finite dimensional reduction, Corollary 5.3). Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\text{tsc}^\infty$. Then the following are equivalent:

1. $f$ is Fredholm.
2. For every $x_0 \in U$, $Df(x_0) : E \to E'$ is Fredholm and there exist
   - (a) open neighbourhoods $V, W \subseteq U$ of $x_0$,
   - (b) a $\text{tsc}^\infty$-diffeomorphism $g : V \to W$ with
     $$g(x_0) = x_0 \quad \text{and} \quad Dg(x_0) = \text{id}_E,$$
   - (c) a strongly smoothing $\text{tsc}^\infty$-map $k : V \to E'$ with
     $$k(x_0) = 0 \quad \text{and} \quad Dk(x_0) = 0,$$

s.t.

$$f \circ g(x) = f(x_0) + Df(x_0)(x - x_0) + k(x) \quad \forall x \in V.$$  

Definition 6.10 (Definition 5.6). Let $E$ and $E'$ be tsc-spaces, let $U \subseteq E$ and $V \subseteq E'$ be open subsets and let $f : U \to V$ be $\text{tsc}^\infty$ and Fredholm.

1. A point $x \in U$ is called a regular point of $f$ iff $\text{corank} \, f(x) = 0$.
2. A point $y \in V$ is called a regular value of $f$ iff $x \in U$ is a regular point of $f$ for every $x \in f^{-1}(y)$.

Theorem 6.6 (Sard-Smale, Theorem 5.4). Let $E$ and $E'$ be $2^{nd}$-countable tsc-spaces, let $U \subseteq E$ be an open subset and let $f : U \to E'$ be $\text{tsc}^\infty$ and Fredholm. Then the set of regular values of $f$ is a generic subset of $E'$.
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