Shear hardening in frictionless amorphous solids near the jamming transition

Deng Pan, Fanlong Meng, and Yuliang Jin

Abstract

The jamming transition, generally manifested by a rapid increase of rigidity under compression (i.e. compression hardening), is ubiquitous in amorphous materials. Here we study shear hardening in deeply annealed frictionless packings generated by numerical simulations, reporting critical scalings absent in compression hardening. We demonstrate that hardening is a natural consequence of shear-induced memory destruction. Based on an elasticity theory, we reveal two independent microscopic origins of shear hardening: (i) the increase of the interaction bond number and (ii) the emergence of anisotropy and long-range correlations in the orientations of bonds—the latter highlights the essential difference between compression and shear hardening. Through the establishment of physical laws specific to anisotropy, our work completes the criticality and universality of jamming transition, and the elasticity theory of amorphous solids.

Keywords: shear hardening, jamming transition, critical scaling, elasticity theory, amorphous solid

Significance Statement

Under isotropic compression, a packing of spheres undergoes a jamming transition, at which rigidity emerges. Upon further compression, the rigidity increases due to the growth of inter-particle contact number—this phenomenon is called compression hardening, which can be statistically described by critical scaling laws. Here we investigate the anisotropic counterpart of compression hardening—shear hardening near the jamming transition in frictionless packings. We discover new jamming scaling laws specific to shear deformations from simulation data, and reveal microscopic origins of shear hardening by an elasticity theory. These results will be useful for understanding the mechanisms of related phenomena in granular materials and colloidal suspensions, including the dilatancy effect and shear jamming.

The jamming transition occurs in a wide range of soft materials, ranging from granular matter to colloidal suspensions to glasses. As a non-equilibrium, athermal phase transition, its criticality is specified by a set of scaling laws (1–4). In particular, the scaling relationship (under the harmonic approximation of the inter-particle interaction) \( G \sim \Delta Z \), between the shear modulus \( G \) of the jammed phase and the excess coordination number \( \Delta Z \), has been derived by the mean-field replica glass theory in infinite dimensions (5) and microscopic elasticity theories (6, 7), with a promising agreement to data from isotropic compression simulations (2). Here we demonstrate that this relationship cannot fully account for the shear hardening behavior—the influence of fabric anisotropy (quantified by the macroscopic friction coefficient \( \mu = \sigma / \tau \), which is the ratio of shear stress \( \sigma \) to pressure \( \tau \)) is significant. We report numerically a new scaling law unique to shear, \( \mu \sim \sigma^\beta \) with \( \beta \approx 0.25 \), between \( \mu \) and \( \sigma \), and derive theoretically an additional anisotropic contribution to the elasticity, \( G_{\text{AI}} \sim \mu^2 \), verified by our simulation data.

Shear hardening, meaning that \( G \) increases by shear, is closely related to several other interesting phenomena. The vestige of shear hardening in the unjammed phase is the phenomenon of shear jamming (the onset of rigidity under a fixed-volume shear deformation) observed in many experiments (8–12) and simulations (13–26). Shear hardening is generally accompanied by increasing pressure under fixed-volume conditions, which implies that under fixed-pressure conditions a dilatancy effect would occur (27–29). Recent studies have shown the possibility of complete decoupling between friction and shear jamming/dilatancy (30, 17, 20, 21). In this study, we show that friction is also not essential for shear hardening. Due to the lack of internal friction, the conventional sawtooth model (31) cannot be applied anymore, and frictionless mechanisms are demanded. To this end, here we provide a phenomenological explanation based on the generalized jamming phase diagram (21), and reveal microscopic origins of frictionless shear hardening from the elasticity theory of amorphous solids.
Our simulation and theoretical results demonstrate that fabric anisotropy makes important contributions to shear hardening, which resembles a similar connection between anisotropy and shear jamming (35). Note that the discussed shear hardening occurs in the solid phase (above jamming), which shall not be confused with shear thickening that presents in non-Newtonian fluids (below jamming) such as dense suspensions (36).

Shear hardening has been directly observed in previous studies (12, 22, 37, 38). Simulations report a hardening regime on the stress–strain curve of an individual sample obtained in a single realization of simulation, the coexistence of solid-like ($\sigma > 0$ and $G > 0$) and liquid-like ($\sigma = G = 0$) states can be identified (39). (ii) Shear hardening, manifested in the increasing function of $G(\gamma)$, appears in deeply annealed systems with $\phi_0 = \phi_1$, at small $P_0$ (see Fig. 1B). Note that in this case, there is a narrow shear softening regime (i.e. $G(\gamma)$ is a descending function) before shear hardening, which can only be identified in log scales (see Fig. 3 and Supplementary Material, Fig. S1). (iii) If the systems are over-compressed well above the jamming transition, then shear hardening disappears and only shear softening is observed independent of $\phi_0$ (see Fig. 1C).

From the above analysis, one finds that shear hardening emerges near the jamming transition ($P_0 \to 0$) in deeply annealed packings ($\phi_0 \gg \phi_1$), under constant-volume AQS. These two conditions are demonstrated more quantitatively in Fig. 2. The shear hardening behavior can be only observed in a stress window $\sigma_0 < \sigma < \sigma_1$, where $\sigma_0$ is the onset stress of hardening (see below for how $\sigma_0$ is determined). For a fixed $\phi_0$ with increasing $P_0$, $\sigma_1$ increases faster than $\sigma_0$, as shown in Fig. 2A. Above $P_0^*$, where $\sigma_0(P_0^*) \approx \sigma_1(P_0^*)$, the shear hardening regime disappears. For a fixed $P_0$, $\sigma_1$ is nearly independent of $\phi_0$ (see Fig. 2B), while $\sigma_0$ decreases with decreasing $\phi_0$ and vanishes when $\phi_0 \to \phi_1$ (45, 39) (a more detailed study on $\sigma_0(\phi_0)$ can be found in Ref. (21)). Thus the scaling regime of shear hardening also vanishes as $\phi_0 \to \phi_1$. The finite-size effects are discussed in Supplementary Material, Sec. 5 (see Fig. S5). In this study, we do not consider other factors, such as finite temperatures or finite shear rates (46).

**Critical scaling laws**

Below we focus on deeply annealed samples with a large jamming density $\phi_0 = 0.69$, which are quenched from dense equilibrium liquids at $\phi_{eq} = 0.643$. Based on the log–log plots in Fig. 3, we identify consecutively (before yielding) the elastic, shear softening and shear hardening regimes, separated by two crossover points at $\sigma_0$ and $\sigma_1$.

The data of shear modulus in the elastic and shear softening regimes can be described by a scaling function $G(P_0, \sigma)/G_0 = G(\sigma/\sigma_0)$, where $G_0 = G(\epsilon_0) = \sigma_0$.

![Fig. 1. Stress–strain curves. Plotted are stress–strain curves of A) a rapidly quenched system near the jamming transition, B) a deeply annealed system near the jamming transition, and C) a deeply annealed system over-compressed well above the jamming transition. Data are obtained from constant-volume AQS simulations of 3D frictionless soft spheres (SSs). The dashed and solid lines represent averaged and single-realization curves, respectively. The inset in panel B is an enlarged view of the single-realization curve.](image-url)
The scaling function $G(x)$ was initially proposed in Ref. (47) to describe both linear and softening behavior in rapidly quenched packings. We find that $G(x)$ agrees well with our data of deeply annealed systems in the linear and softening regimes, but deviate from the data starting from $\sigma = \sigma_0$, which is the onset point of shear hardening. The scaling function $G(x)$ allows us to determine numerically the crossover stress $\sigma_i = cP^{5/4}_0$, where the prefactor $c$ is chosen to be 0.02. The $\sigma_i$ estimated in this way (squares) reasonably separates linear and softening behavior, as can be seen in Fig. 3A. From $G(x)$, the scaling behavior of linear and softening regimes can be extracted. In the linear regime ($\sigma < \sigma_0$ or $x = \sigma/\sigma_0 \rightarrow 0$), $G(x \rightarrow 0) = 1$, which means that the shear modulus is simply a constant $G = G_0$. In the softening regime ($\sigma \gg \sigma_0$ or $x \rightarrow \infty$), $G(x \rightarrow \infty) = x^{-2}$, which means that $G \sim P_0^{-3/4}$ (where we have used the well know scaling $G_0 \sim P_0^3$ for unstrained systems (2)). These results are fully consistent with the findings in Ref. (47). Our data show that the presence of a strong shear hardening effect does not change the scalings in linear and softening regimes. It is of interest to provide a theoretical explanation of the softening exponent $-2$ in $G(x \rightarrow \infty) = x^{-2}$ in future studies.

The deviation of the simulation data from the master curve $G(x)$ in Fig. 4 is followed by the shear hardening behavior. The scaling of $\sigma_i$ is determined based on the rescaled plot of macroscopic friction $\mu$ in Fig. 4B, with a scaling function $\mu(P_0, \sigma) / \mu_0 = U(\sigma/\sigma_0)$, where $U(x \rightarrow 0) \sim -x$, $U(x \rightarrow \infty) \sim x^{3/4}$, $\sigma_i \sim P_0^{3/4}$ and $\mu_0 \sim P_0^{3/4}$. Fig. 4B shows that elastic and softening data follow a universal scaling $\mu \sim \sigma$. Previously, Kawasaki and Miyazaki (22) report two scalings $\mu \sim \sqrt{x}$ and $\mu \sim x^{1/2}$, respectively, in the elastic and softening regimes for a two-dimensional model, which, together with the shear softening scaling $\sigma \sim \sqrt{x}$, give a consistent result $\mu \sim \sigma$.

In the shear hardening regime, $\sigma_i < \sigma < \sigma_{\infty}$, the system’s behavior is characterized by two scaling laws. Numerical fitting gives $\mu \sim \sigma^{\beta}$ with $\beta = 0.248 \pm 0.006$ and $\Delta Z \sim \sigma^\nu$ with $\nu = 0.411 \pm 0.005$ in the shear hardening regime (Fig. 3). The scaling of $\Delta Z$ is consistent with the ansatz $\Delta Z \sim \sigma^{3/5}$ proposed in Ref. (4). Note that this scaling is examined in (4) by looking at the stress variance $\sigma^2$ of isotropically compressed configurations with $\sigma_0 = \phi_0$, where the mean stress is zero. Here we provide direct verification of the ansatz, thanks to the emergence of shear hardening in deeply annealed packings ($\phi_i \gg \phi_0$). The scaling $\mu \sim \sigma^{1/4}$ (or equivalently $P \sim \sigma^{3/4}$), is absent in the framework of Ref. (4), and to our knowledge, has never been reported previously. Based on the elasticity theory (see below for

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**Fig. 3.** Shear hardening scalings. Simulation results in 3D of A) the shear modulus $G$, B) the excess coordination number $\Delta Z$ and C) the macroscopic friction coefficient $\mu$ as functions of stress $\sigma$, for a few different $P_0$. The solid lines represent scaling laws in the shear hardening regime, $G \sim \sigma^{2/5}$, $\Delta Z \sim \sigma^{3/5}$ and $\mu \sim \sigma^{1/4}$. The crossovers at $\sigma_i$ and $\sigma_{\infty}$, which are determined in Fig. 4, are marked by black solid squares and pentagons respectively. D) Universal relationship between $\sigma$ and $\sigma_0$ in 3D samples prepared by cyclic shear ($\phi_i = 0.662$), cyclic compression ($\phi_i = 0.663$), and swap thermal annealing ($\phi_i = 0.672, 0.677, 0.682, 0.690$), as well as in two and four dimensions ($P_0 = 10^{-6}$ in all cases).

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**Fig. 4.** Data collapsing in shear softening and hardening regimes. Rescaled plots of A) shear modulus $G$ and B) macroscopic friction $\mu$ as functions of stress $\sigma$, where $P_0$ and $G_0$ are unstrained pressure and shear modulus. The crossover stresses are numerically defined as $\sigma_i = \phi_i P_0^{3/4} = 2 \times 10^{-2}$ (solid black square) in A) and $\sigma_i = \phi_i P_0^{3/4} = 2$ (solid black pentagon) in B). The bold lines represent the master curves $y = \mu / \mu_0$ and $y = x$ in A) and B), respectively.
Under constant-volume shear, it is easy to see from Fig. 5 that the system has to gain rigidity, because both the pressure and the number of contacts increase. The increase of anisotropy $\mu$ makes an additional contribution to the shear modulus, as demonstrated below. Shear hardening is thus a natural consequence of the memory erasing process caused by shear. Different unstrained states must evolve to the same stationary state that is independent of the initial condition. Initial states that are dense packings ($\phi \gg 0.1$) created by deep annealing should also be brought back by shear to the generic packings at $\phi$ (corresponding to the rapid quench limit (2))—this process is accompanied by the increase of pressure and shear modulus under the constant-volume condition.

**Microscopic origins of shear hardening**

Previous studies have developed an elasticity theory for athermal disordered solids near the jamming transition, attributing compression hardening to the increase of interaction bonds via the relation (6, 7, 35),

$$G = G_t + G_{AI},$$

where $c_t = 1/30$ (we have set both the bond stiffness $k$ and the unstressed bond length $r_0 = 1$ to one). While this theoretical result agrees well with the simulation data of sphere packings under isotropic compression (7), it cannot fully account for our shear hardening data (see Fig. 6A).

To understand the microscopic origin of this deviation, we consider a generalized formula (6) (see Supplementary Material for details),

$$G = c_t \Delta G,$$

where $c_t = 1/30$ (we have set both the bond stiffness $k$ and the unstressed bond length $r_0 = 1$ to one). While this theoretical result agrees well with the simulation data of sphere packings under isotropic compression (7), it cannot fully account for our shear hardening data (see Fig. 6A).

To understand the microscopic origin of this deviation, we consider a generalized formula (6) (see Supplementary Material for details),

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for $P = 0$, where $G_{AI} = \frac{1}{30} \sum_{j} f_j \sum_{i} f_i h_i h_j h_k h_l h_m h_n$ depends on the normalized forces $f_j$ and bond orientations (represented by...
unit vectors $\mathbf{n}_b$), $N_b$ is the number of contacts and $N_{nnb}$ is the number of non-rattler particles (rattlers have fewer than $d + 1$ contacts in $d$ dimensions). In isotropic packings, the lack of long-range spatial correlations between bond orientations, $(\mathbf{n}_{b1}^x \mathbf{n}_{b1}^y \mathbf{n}_{b1}^z \mathbf{n}_{b2}^x \mathbf{n}_{b2}^y \mathbf{n}_{b2}^z) = (\mathbf{n}_{b2}^x \mathbf{n}_{b2}^y \mathbf{n}_{b2}^z \mathbf{n}_{b1}^x \mathbf{n}_{b1}^y \mathbf{n}_{b1}^z) = 0$, suggests that $G_{AI} \approx 0$ (note that $G_{AI} = \left< \mathbf{f}_{b1} \cdot \mathbf{f}_{b2} \right>$ under the effective media approximation (7), and all contact forces $\mathbf{f}_b$ are positive). In sheared packings, the bonds tend to align in shear-preferred directions and are orientationally long-range correlated (16); the correlation function, $Q_b(r) = \langle n_{b1}^x n_{b2}^y n_{b2}^z n_{b1}^x n_{b1}^z n_{b2}^y \rangle$, where $r_b$ is the position of bond $b$, decays to a finite value at large $r$ (see Fig. 7). Consequently, $G_{AI}$ is non-zero in such a case.

Our theoretical analysis predicts a simple relation between $G_{AI}$ and the macroscopic friction $\mu$,

$$G_{AI} = c_{AI} (\Delta Z) \mu^2,$$

where

$$c_{AI} (\Delta Z) = c_0 - a \Delta Z,$$

with two constants $c_0$ and $a$. In the jamming limit ($\Delta Z \approx 0$), the coefficient $c_0$ can be evaluated from the formula, $c_0 = \langle f^2 \rangle / \langle f \rangle^2$, where $\langle f \rangle$ and $\langle f^2 \rangle$ are the first two moments of the force distribution. In our system, we obtain from simulations $\langle f^2 \rangle / \langle f \rangle^2 \approx 0.5$, which gives $c_0 \approx 0.17$ (see Supplementary Material, Fig. S6). Above jamming ($\Delta Z > 0$), $G_{AI}$ is lowered by a higher-order correction term $\sim \Delta Z \mu^2$.

To examine Eq. 3, we separate isotropic and anisotropic moduli in our simulation data as follows. Considering that $\Delta Z(\gamma)$ is increased during shear, at each $\gamma$ we first decompress the configurations keeping $\gamma$ unchanged, until $\Delta Z$ reaches $\Delta Z_0(\gamma = 0)$ that is fixed by the initial condition $\Delta Z_0 \sim f_{D,0}^{1/2}$. The decompressed configurations at different $\gamma$ thus now have the same $\Delta Z = \Delta Z_0$, but different $\mu$ (we note these configurations as a constant-$Z$ ensemble). In this way, we remove the contributions from the change of $\Delta Z$, left with $G_{AI}$ that changes with $\mu$. Fig. 6B shows that $G_{AI}$ is proportional to $\mu$ for any constant $\Delta Z$, as predicted by Eq. 3. The behavior of the slope $G_{AI}^{\text{sim}}(\Delta Z)$ obtained from fitting the simulation data is also consistent with our theoretical prediction Eq. 4. The remaining shear modulus $G_I = G - G_{AI}$ linearly depends on $\Delta Z$ (see Fig. 6A), with a $P_{0}$-independent slope in agreement with the analytical result $c_1 = 1/30$ as in the case of pure compression (7). Note that here $\Delta Z$ is increased by shear instead of compression. We find in our simulations that the pressure $P$ does not depend on $\mu$, but solely on $\Delta Z$. Thus, the constant-$Z$ ensemble considered here is in fact equivalent to a constant-$P$ ensemble.

Our results show that, within the first-order approximation, the excess coordination number and anisotropy make additive contributions to the shear modulus. At a large strain, the contact network becomes strongly anisotropic, in which case the anisotropic correction to $G$ is important. On the other hand, in the limit $\sigma \to 0$ (or $\gamma \to 0$), the anisotropic contribution is subleading: combining the scalings $\Delta Z \sim \sigma^{2/5}$ and $\mu \sim \sigma^{1/4}$ (Fig. 3B and C) with Eqs. 1, 2, 3, one obtains $G \sim \Delta Z^{1/2}$ (Fig. 3A).

For comparison, we also derive a theoretical expression of the bulk modulus $B$ near the jamming transition (see Supplementary Material, Sec. 6),

$$B = c_0 + \left( \frac{1}{18} \frac{c_0}{\bar{\sigma}} \right) \Delta Z,$$

where the constant $c_0 = \langle f^2 \rangle / \langle f \rangle^2$ is identical to the one appeared in Eq. 4 for the shear modulus. In the jamming limit $\Delta Z \to 0$, the bulk modulus is a constant $B(\Delta Z = 0) = c_0$. In other words, $B$ changes discontinuously at either compression or shear jamming, following a scaling $B \sim \Delta Z^0$ as previously reported (2, 16). Equation (5) further
In particular, the scaling of jamming transition and rheology of amorphous solids. Our finding raises several challenges to the theoretical understanding of the possibility of “state rejuvenation” since simulations suggest that the critical exponents are non-universal with different friction coefficients. In addition, the correction to the isotropic shear modulus should be even more significant, because friction amplifies spatial heterogeneity and orientational anisotropy of the contact network.

Materials and methods

Simulation models

We consider a 3D granular model of frictionless poly-disperse spheres with a continuous diameter distribution \( P(D) \sim D^{-3} \), for \( D_{\text{min}} \leq D \leq D_{\text{max}}/0.45 \) (20, 43, 44). The particles interact via a harmonic soft pair potential, \( v_{ij}(r) = \frac{1}{2}(1 - r_{ij}/D_{ij})^2 \) (zero if \( r_{ij} > D_{ij} \)), where \( r_{ij} \) is the inter-particle distance and \( D_{ij} = (D_i + D_j)/2 \) the mean diameter of particles \( i \) and \( j \). We set the volume average \( \bar{D} \) as the unit of length. The lowest jamming density, or the J-point density (2), of this model is \( \phi_J \approx 0.655 \) (20, 55). The reported results are obtained from systems of \( N = 2000 \) particles, averaged over 192 independent samples. In addition, we study 2D and 4D models with the same kind of interaction in Supplementary Material.

Preparation of initial states

Two approaches are used to prepare ultra-stable jammed configurations. (i) Swap thermal annealing. The thermal annealing is realized by applying an efficient swap Monte Carlo algorithm (43, 44), which generates equilibrium hard sphere (HS) configurations at \( \phi_{\text{eq}} \) above the mode-coupling theory (MCT) crossover density \( \phi_{\text{MCT}} \approx 0.594 \) (56). After that, we set the temperature to zero in the remaining simulations, and switch to the HS potential \( v_{ij} \) for the fire energy minimization algorithm (57) as described in (21, 55, 58), we obtain jammed configurations at \( \phi_{\text{eq}} \)-dependent jamming densities \( \phi_J \) (20, 55, 59). In the main text, we focus on the case of \( \phi_J = 0.69 \) unless otherwise specified, corresponding to \( \phi_{\text{eq}} = 0.643 \). (ii) Mechanical training. The method is realized by cyclic AQs (21, 22, 23, 60) or cyclic AQ (13). Random initial configurations are rapidly compressed over \( \phi_J \), and become unjammed after a sufficient number of cyclic AQs or AQ under the constant-volume condition (21), meaning that the jamming density is increased to \( \phi_J > \phi_J \). In both approaches, \( \phi_J \) correlates to the stability of the jammed states, and reflects the degree of annealing.

The above procedures generate packings at \( \phi = \phi_J \), and \( \bar{P} = 6 \times 10^{-3} \), which are extremely close to the jamming/unjamming transition. They are then compressed quasi-statically to over-jammed states at a series of pressures \( P_0 \) up to \( \bar{P}(\phi_J) \). The pressure \( P_0 \) characterizes the distance to the jamming point, via the well-known scaling between \( P_0 \) and the density \( \phi_J \) for harmonic SSSs (2), \( P_0 \sim (\phi_J - \phi_J) \). In order to remove residual stresses, the shear stabilization method (61) is applied, and these configurations with zero residual stresses are referred to as the unstrained states (\( \sigma = 0 \)). According to this definition, in finite-size unstrained states, the
stress $\sigma$ is always zero, while the strain $\gamma$ can fluctuate around zero from sample to sample. In our scaling analysis (see Fig. 3), it is better to use $\sigma$ than $\gamma$ as the independent variable, since the latter can introduce additional uncertainties. Of course, in the limit of large sample size and/or of larger number of samples, this sample-to-sample fluctuation disappears. In fact, a similar issue exists in the case of compression jamming: the jamming transition is defined at the zero pressure, while the jamming density can fluctuate around the mean value in finite-size simulations. Thus practically it is better to use the pressure than the density as the independent variable in scaling analyses.

Compression and shear protocols

Compression and decompression procedures are realized by re-scaling particle diameters with energy minimization. Starting from the unstrained states, we apply constant-volume, simple AQs in the x-z plane (21, 20). At each step, the shear strain $\gamma$ is increased by $\delta \gamma$, followed by the FIRE energy minimization, where $\delta \gamma$ is logarithmically increased from $10^{-8}$ to $2 \times 10^{-4}$.

The stress and pressure are calculated using the virial formula

$$\sigma = \frac{1}{bN_{iso}} \sum_{i=1}^{bN_{iso}} r_i^2 f_i$$

and

$$P = \frac{1}{bN_{iso}} \sum_{i=1}^{bN_{iso}} r_i \cdot f_i,$$

where $r_i$ and $f_i$ are the contact vector and force on bond $b$, $N_{iso}$ the number of non-rattler particles (rattlers have fewer than $d+1$ contacts in $d$ dimensions), and $N_{iso}$ the total number of bonds. The shear modulus is defined as $G = (\partial \sigma / \partial \gamma) + (P)$, averaged over 192 independent samples, where $\delta \gamma$ is between $10^{-10}$ and $10^{-8}$. The term $\partial \sigma / \partial \gamma$ corresponds to the slope of a single-realization stress–strain curve, which is piecewise linear (63) (see the inset of Fig. 1B). The correction term $(P)$, which is typically three orders smaller than the first term, is added so that $G$ is equivalent to the xxzz component of the stiffness matrix. We have checked that the value of $G$ obtained in this way is identical to that directly calculated using the explicit expressions derived from the elastic theory (34). The bulk modulus is computed by the formula $B = -\langle \partial P / \partial \epsilon_V \rangle - \frac{1}{V}(P)$, where $\epsilon_V$ is the volumetric strain and $P \leq 2 \times 10^{-9}$.

The changes of two independent parameters $Z$ and $\mu$ reflect local rearrangements of particles. The increase in the number of contacts is quantified by the excess coordination number, $\Delta Z = Z - Z_{idio}$, where $Z_{idio} = 2d - 2d/N_{iso}$ is the coordination number required to satisfy the isostatic condition at the jamming transition in $d$ dimensions with a finite-size correction (64, 65), and rattlers are excluded in the computation of $Z$. The macroscopic friction coefficient $\mu = \sigma / P$ measures the degree of anisotropy caused by shear (66).

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Supplementary material

Supplementary material is available at PNAS Nexus online.

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Authors’ contributions

D.P. and Y.J. performed simulations and theoretical computations. All authors contributed to the writing of the manuscript and the discussion of the ideas behind it.

Data availability

All data are available in the main text or the supplementary information.

References

1. Makse HA, Johnson DL, Schwartz LM. 2000. Packing of compressible granular materials. Phys Rev Lett. 84:4160.
2. O’Hern CS, Silbert LE, Liu AJ, Nagel SR. 2003. Jamming at zero temperature and zero applied stress: the epitome of disorder. Phys Rev E. 68:011306.
3. Charbonneau P, Kurchan J, Parisi G, Urbani P, Zamponi F. 2014. Fractal free energy landscapes in structural glasses. Nat Commun. 5:3725.
4. Goodrich CP, Liu AJ, Sethna JP. 2016. Scaling ansatz for the jamming transition. Proc Natl Acad Sci USA. 113:9745–9750.
5. Yoshino H, Zamponi F. 2014. Shear modulus of glasses: results from the full replica-symmetry-breaking solution. Phys Rev E. 90:022302.
6. Wyart M. 2005. On the rigidity of amorphous solids. Ann Phys Fr. 30:1–96.
7. Zaccone A, Scossa-Romano E. 2011. Approximate analytical description of the nonaffine response of amorphous solids. Phys Rev B. 83:184205.
8. Bi D, Zhang J, Chakraborty B, Behringer RP. 2011. Jamming by shear. Nature. 480:355–8.
9. Ren J, Dijkstra J, Chakraborty B, Behringer RP. 2013. Reynolds pressure and relaxation in a sheared granular system. Phys Rev Lett. 110:018302.
10. Sarkar S, Bi D, Zhang J, Behringer R, Chakraborty B. 2013. Origin of rigidity in dry granular solids. Phys Rev Lett. 111:068301.
11. Wang D, Ren J, Dijkstra J, Zheng H, Behringer RP. 2018. Microscopic origins of shear jamming for 2D frictional grains. Phys Rev Lett. 120:208004.
12. Zhao Y, et al. 2022. Ultrastable shear-jammed granular material. Phys Rev X. 12:031021.
13. Kumar N, Leding S. 2016. Memory of jamming–multiscale models for soft and granular matter. Granul Mater. 18:58.
14. Bertrand T, Behringer RP, Chakraborty B, O’Hern CS, Shattuck MD. 2016. Protocol dependence of the jamming transition. Phys Rev E. 93:012901.
15. Vinutha HA, Sastry S. 2016. Disentangling the role of structure and friction in shear jamming. Nat Phys. 12:578–583.
16. Baity-Jesi M, Goodrich CP, Liu AJ, Nagel SR, Sethna JP. 2017. Emergent SO(3) symmetry of the frictionless shear jamming transition. J Stat Phys. 167:735–748.
17. Jin YL, Urbani P, Zamponi F, Yoshino H. 2018. A stability-reversibility map unifies elasticity, plasticity, yielding, and jamming in hard sphere glasses. Sci Adv. 4:12.
18. Seto R, Singh A, Chakraborty B, Denn MM, Morris JF. 2019. Shear jamming and fragility in dense suspensions. Granul Mater. 21:1–8.
