Singular Bifurcations: a Regularization Theory

Alexander Farutin\textsuperscript{1} and Chaouqi Misbah\textsuperscript{1,2}

\textsuperscript{1}Université Grenoble Alpes, CNRS, LIPhy, F-38000 Grenoble, France

(Dated: 10 janvier 2022)

Several nonlinear and nonequilibrium driven as well as active systems (e.g. microswimmers) show bifurcations from one state to another (for example a transition from a non motile to motile state for microswimmers) when some control parameter reaches a critical value. Bifurcation analysis relies either on a regular perturbative expansion close to the critical point, or on a direct numerical simulation. While many systems exhibit a regular bifurcation such as a pitchfork one, other systems undergo a singular bifurcation not falling in the classical nomenclature, in that the bifurcation normal form is not analytic. We present a swimmer model which offers an exact solution showing a singular normal form, and serves as a guide for the general theory. We provide an adequate general regularization theory that allows us to handle properly the limit of singular bifurcations, and provide several explicit examples of normal forms of singular bifurcations. This study fills a longstanding gap in bifurcations theory.

Introduction.— Nonequilibrium driven systems constitute a large branch of science which has been the subject of active research in the last decades [1–4]. Typical examples are Bénard convection [5], Turing patterns [3,6,7], crystal growth [8–10] and so on. By varying a control parameter (e.g. Rayleigh number in convection) the system exhibits a bifurcation from one state (e.g. quiescent fluid) into a new state (the fluid shows convection rolls) when a control parameter reaches a critical value. If \( \mu \) designates the distance of the control parameter from the bifurcation point, the amplitude of the field of interest, say convection amplitude \( A \), behaves as \( A \sim \pm \mu^{1/2} \), known as a pitchfork (classical) bifurcation. In other situations, like driven fluids in a pipe, a transition from a laminar to a non-laminar (turbulent) flow takes place beyond a critical Reynolds number in the form of a saddle-node bifurcation [11]. This behavior is also generic for many pattern-forming systems [1–4]. The dynamics of the amplitude \( A \) (known as normal form) of these two bifurcations (pitchfork and saddle-node) read respectively

\[
\dot{A} = \mu A - A^3, \quad \dot{A} = \mu - A^2 \tag{1}
\]

where dot designates time derivative. Other types of bifurcations are also common, such as transcritical, subcritical [1–4] and so on. A hallmark of classical bifurcations theory is the regular (analytic) expansion in powers of \( A \) in Eq. (1). The same holds also in catastrophe theory `a la René Thom [12].

More recently, active matter, a subject of great topicality, has revealed several bifurcations from a non-motile state to a motile one when activity reaches a critical value [13–21]. In its simplest version this consists of a particle emitting/absorbing a solute which diffuses and is advected in the suspending fluid. The advection-diffusion equations read as

\[
\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = D \Delta c, \tag{3}
\]

where \( c \) is the solute concentration, \( D \) is the diffusion constant, \( \mathbf{u} \) and \( p \) are the velocity and pressure fields, obeying Stokes equations. The associated boundary conditions of surface activity and the swimming speed (which will be taken to be along the \( z \)-direction) are

\[
D \frac{\partial c}{\partial r}(1, \theta, t) = -\mathcal{A}, \quad V_0 = -\frac{\mathcal{M}}{a} \int_0^1 \mu c(r = a, \mu, t) d\mu \tag{4}
\]

with \( \mu = \cos(\theta) \), where \( \mathcal{A} \) is the azimuthal angle in spherical coordinates, \( a \) is the particle radius \( \mathcal{A} \) is the emission rate (\( \mathcal{A} > 0 \): emission, \( \mathcal{A} < 0 \): adsorption), \( \mathcal{M} \) is a mobility factor (which can be either positive or negative); see [13] for more

This means that the regular amplitude expansion ceases to be valid, as manifested by the non analytical term \( |A| \). Numerical simulations [17] of this system are, in contrast, consistent with a classical pitchfork bifurcation, \( A \sim \pm \mu^{1/2} \). We will see that this is due to finite size in numerical simulations.

Examples of singular nature have been also encountered in crystal growth. It has been shown that the usual perturbative scheme in terms of the crystal surface deformation amplitude is not legitimate [24]. Besides these examples, the emergence of singular bifurcations is likely to be abundant, and has been probably overlooked in many numerical simulations (see also conclusion). The purpose of this Letter is to fill this gap. We will show how to handle singular bifurcations from the usual commonly used regular perturbative scheme. We will first illustrate the theory on an explicit example of microswimmer for which an exact analytical solution is obtained. We then present a systematic method on how to properly treat singular bifurcations.

Theory— It is instructive to begin with an explicit model revealing a singular bifurcation. We first introduce the full model, before considering a simplified version which can be handled fully analytically. The model consists [14] of a rigid particle (taken to be a sphere with radius \( a \)), which emits/absorbs a solute that diffuses and is advected by the flow. The advection-diffusion equations read as

\[
D \frac{\partial c}{\partial r}(1, \theta, t) = -\mathcal{A}, \quad V_0 = -\frac{\mathcal{M}}{a} \int_0^1 \mu c(r = a, \mu, t) d\mu \tag{4}
\]

with \( \mu = \cos(\theta) \), where \( \mathcal{A} \) is the azimuthal angle in spherical coordinates, \( a \) is the particle radius \( \mathcal{A} \) is the emission rate (\( \mathcal{A} > 0 \): emission, \( \mathcal{A} < 0 \): adsorption), \( \mathcal{M} \) is a mobility factor (which can be either positive or negative); see [13] for more
details. This model has been studied numerically \cite{14,17,18}, coming to the conclusion that for \( Pe \) (with \( Pe = |\alpha|/a|D^2| \)) sufficiently small the only solution is the non-moving state of the particle, with a concentration field which is symmetric around the particle. When \( Pe \) exceeds a critical value it is shown that the concentration field loses its spherical symmetry and a concentration comet develops, resulting in a motion of the particle with a constant velocity \( V_0 \). It is found numerically \cite{17} that \( V_0 \) is well represented by \( V_0 \sim \sqrt{Pe- Pe_1} \), where \( Pe_1 \) is the critical value of \( Pe \) at which the transition from a non motile to a motile state occurs. The determination of the critical condition has also been analyzed by linear stability analysis \cite{14,17,18}. Analytical asymptotic perturbative studies \cite{15,22,23} (for an infinite system size) revealed that the velocity of the swimmer follows in fact the following singular behavior \( |V_0| \sim (Pe- Pe_1) \).

Exactly solvable model—The main simplification adopted here is to disregard the fluid, in that the variable \( u \) is ignored in what follows. A justification of this is the fact that the singular behavior is associated with the concentration field at long distance \cite{15,22,23}, while the velocity field vanishes at infinite distance from the swimmer. We consider a particle moving at constant speed \( V_0 \). A further simplification is that we assume that the particle size is small in comparison to length scales of interest. The only length scale is given by \( D/V_0 \), so our assumption corresponds to assuming \( a \ll D/V_0 \). Under this assumption the particle can be taken as a quasi-material point. With these assumptions the corresponding simplified model reads (in the laboratory frame)

\[
\frac{\partial c}{\partial t} - D \Delta c = S \delta(r - V_0 t)
\]

where \( S \) is the emission rate (related to \( \alpha \), by \( \alpha = S/(4\pi a^2) \)).

Using the diffusion propagator the solution is given by

\[
c(r,t) = \int_0^\infty d\tau \frac{S}{4\pi D \tau^{3/2}} \exp\left\{ - \frac{(r + V_0 \tau - V_0 t)^2}{4D\tau} \right\},
\]

Expression (6) can be integrated to yield

\[
c(\bar{r}) = \frac{S}{4\pi D \bar{r}} \exp\left\{ - \frac{\bar{r} V_0 + V_0 \bar{\rho}}{2D} \right\},
\]

with \( \bar{r} = r - V_0 t \) (the coordinate in the frame attached to the particle). Along \( \bar{r} \), it is clear that the concentration decays exponentially with distance ahead of the particle, while it decays only algebraically at the rear (\( c \) has front-back symmetry). Indeed, the emitted solute is advected (by swimming speed) backwards, enriching the rear zone, whereas ahead of the particle only diffusion can be effective.

Using (4), only the first spherical harmonics enters the expression of velocity, and we obtain \( V_0 = -Mc_1/(a\sqrt{3\pi}) \), \( c_1 \) being the first harmonic amplitude, obtained by projection of (6) on that harmonic, so that the velocity satisfies

\[
V_0 = 4Pe e^{-|V_0|/2} \frac{\bar{V}_0 \cosh(V_0/2) - 2 \sinh(V_0/2)}{V_0^2}, \quad \bar{V}_0 = \frac{aV_0}{D}
\]

Expanding for small \( \bar{V}_0 \) we obtain

\[
\bar{V}_0 = \frac{Pe}{Pe_1} (1 - 2|\bar{V}_0|)
\]

where \( \alpha = S/(4\pi a^2) \), and \( Pe_1 = 3 \) is the critical Péclet number. In the full model \( Pe_1 = 4 \) \cite{14}. Including hydrodynamics close to particle surface we can capture analytically this result \cite{25}. The result (9) has been also obtained thanks to a singular perturbative scheme \cite{15,22,23}. We see from (9) that \( V_0 = 0 \) always exists. When \( Pe > Pe_1 \), there exists another solution given by

\[
|V_0| \sim \frac{1}{6}(Pe - Pe_1)
\]

Expression (10) corresponds to a pitchfork bifurcation (and not transcritical \cite{22}) where the \( V_0 = 0 \) solution becomes unstable in favor of two symmetric solutions, \( V_0 \sim \pm (Pe - Pe_1) \). This is, however, an atypical behavior of a pitchfork solution, and is traced back to the infinite system size (as seen below). We refer to this bifurcation as singular pitchfork bifurcation. The term ‘singular’ refers to the non analytic nature \( |V_0| \).

Finite size regularizes the bifurcation and turns the singular bifurcation into a classical pitchfork bifurcation (see \cite{25}). Another way to regularize the model is via a consumption of solute in the bulk. In that case we modify Eq.(5) by adding \( \beta c \) on the left hand side, where \( \beta \) is the consumption rate. We have in mind the possibility that the emitted solute reacts in the bulk and is consumed by another reaction, giving rise to some secondary product. The solution for \( c \) becomes

\[
c(\bar{r}) = \frac{Sa}{4\pi D} \exp\left\{ - \frac{(\bar{r}/a) \sqrt{\bar{V}_0^2 + \bar{\rho}^2/a}}{2} \right\},
\]

with \( \bar{\rho} = \frac{\alpha D \beta}{D} \). The equation for \( \bar{V}_0 \) becomes

\[
\bar{V}_0 = 4Pe e^{-\sqrt{\bar{V}_0^2 + \bar{\rho}^2}/2} \left[ \frac{\bar{V}_0 \cosh(\bar{V}_0/2) - 2 \sinh(\bar{V}_0/2)}{\bar{V}_0^2} \right]
\]

For \( \varepsilon = 0 \) we recover the singular bifurcation solution, and for \( \varepsilon \neq 0 \) we obtain a regular pitchfork bifurcation. Expansion for small \( \bar{V}_0 \) provides

\[
\bar{V}_0 = Pe\bar{V}_0 e^{-\varepsilon/2} \left\{ \frac{1}{3} + \frac{\varepsilon - 10}{120\varepsilon} \bar{V}_0^2 + O(\bar{V}_0^4) \right\}
\]

Besides the trivial solution \( \bar{V}_0 = 0 \), we have \( \bar{V}_0 \sim \pm (Pe - Pe_1)^{1/2} \) which is a classical pitchfork bifurcation, with \( Pe_1 = 3 \). Consumption has turned the singular bifurcation into a regular bifurcation. Regularization theory—The expression of type (13) is the one that one would usually obtain by an analytical expansion in \( \bar{V}_0 \) in the absence of an exact solution. By trying to compare it to the exact solution (12) in the vicinity of bifurcation where
$V_0$ is small (Fig.1) one realizes that the smaller $\epsilon$ is the worse
the approximation \[ (13) \] is, and a fortiori this expression can
in no way account for the singular limit $\epsilon = 0$, a limit where
the coefficients of the series \[ (13) \] diverge. One could then be
tempted to say that \[ (13) \] is of little practical interest for small
$\epsilon$. However, and this is the main point, we will be able, in a
way that may seem a little surprising, to extract from analysis
of a regular expansion \[ (13) \] the singular behavior $\mid V_0 \mid$ (for $\epsilon \to
0$) dictated by the exact calculation \[ (12) \], without any a priori
knowledge on an exact solution. Moreover, we will regularize
the expression \[ (13) \] in such a way that it represents correctly
the exact behavior when $\epsilon$ is nonzero but small.

The crux of our theory is the observation that the singular
behavior in the above model is due to the existence of a singular
point in the complex plane, namely $V_0 = i\epsilon$, arising from
$\sqrt{V_0^2 + \epsilon^2}$ in \[ (12) \]. This model will serve as a precious guide,
but the theory can be made general. We assume that the tri-
vial solution always exist ($V_0$ in the above model), so that the
search for nontrivial solutions amounts to setting in \[ (12) \] the
r.h.s. divided by $V_0$ (to be denoted below as $f(V_0,\epsilon)$) equal
to unity. We focus on the behavior of $f(V_0,\epsilon)$. We use below the
notation $f(x,\epsilon)$ to present the general theory. Suppose, with-
out restriction, that singularity is located on the imaginary
axis at $x = i\epsilon$. We propose the following transformation
\[ \epsilon = x_0(1 - s), \quad x^2 = x_0^2(2s - s^2) \] \[ (14) \]
with $x_0$ a real positive number. Thanks to this transformation
$x^2 + \epsilon^2 = x_0^2$ remains constant. $s$ is a parametrization, and the
singular limit corresponds to $s = 1$. The above transformation
means that instead of taking the singular limit $\epsilon \to 0$ at a
given $x$, we move in the plane $(x,\epsilon)$ along the circle of radius
$x_0$. This transformation renders the expansion in terms of $s$ re-
gular since $x^2 + \epsilon^2$ is constant along the circle. Another way
to appreciate our choice is that the singularity in the original
coordinate, $x^2 = -\epsilon^2$, reads $x_0^2(1 - (s - 1)^2) = -x_0^2(1 - s)^2$
which has no solution meaning that in terms of $s$-variable the
original singularity has been moved to infinity. This guaran-
tees absolute convergence of series in term of $s$. The procedure
consists now in substituting in the regular expansion
\[ f(x,\epsilon) = \sum_{k=0}^{\infty} a_k(\epsilon)x^{2k} \] \[ (15) \]
x and $\epsilon$ as functions of $s$ and $x_0$ (Eq.\[ (14) \]) and expand in a
Taylor series in terms of $s$ as
\[ f(x(s),\epsilon(s)) = \sum_{k=0}^{\infty} a_k[x_0(1 - s)](2s - s^2)^k x_0^{2k} = \sum_{k} b_k(x_0)s^k \] \[ (16) \]
The relation between $b_k$ and $a_k$ is easily deduced (see [25]).

Close to the bifurcation point $x_0$ is small, so we will retain
only $b_0$, $b_1$, and $b_2$. Let us illustrate the study on the phoretic
system. Taylor expansion of \[ (12) \] to order $x^4$ (in the form \[ (15) \])
yields
\[ a_0(\epsilon) = \frac{e^{-\epsilon/2}}{3}, \quad a_1(\epsilon) = e^{-\epsilon/2} \frac{1 - 10/\epsilon}{120}, \]
\[ a_2(\epsilon) = \frac{\epsilon^3 - 28\epsilon^2 + 140\epsilon + 280}{13440\epsilon^3} e^{-\epsilon/2}. \] \[ (17) \]
from which $b'_0 s$ are determined and $f(V_0,\epsilon)$ reads
\[ f(V_0,0) = Pe\{1/3 - |V_0|/6 + V_0^2/20 + O(V_0)\} \] \[ (18) \]

A remarkable feature is that due to our regularization theory
we are able to extract, by using the traditional analytical ex-
pansion \[ (15) \], the singular behavior exhibiting the absolute va-
ue $\mid V_0 \mid$. Referring to the exact result obtained in the limit
$\epsilon = 0$ (Eq.\[ 8 \]), we can check that to leading order in $V_0$ we
obtain exactly the result \[ (13) \] (recall we omit the trivial solution
$V_0 = 0$). This shows the consistency of the theory. Another
virtue of the theory is that it allows to transform the expansion
\[ (13) \], which has a small radius of convergence of order $\epsilon$,
into a form having a wider radius of convergence by ap-
plying the method above (used for \[ (13) \] after expansion in $V_0$ to order 2). Figure 1 summarizes the results.

Use of expansion \[ (13) \] –dotted lines in Fig.1– fails to capture
properly the bifurcation from obtained from the exact result
(Eq.\[ 12 \], represented by solid lines in Fig.1), and this be-
comes worst as $\epsilon$ goes to zero. In contrast \[ (19) \] –dashed lines in Fig.1– impres-sively captures the exact result (Eq.\[ 12 \], so-
lidi lines in Fig.1). The regularization theory does not only
account properly for the singular limit ($\epsilon = 0$; Eq.\[ 13 \]) but
also it offers a precious way to approach this limit (Eq.\[ 19 \]).

Generally, in nonlinear systems an exact solution is the ex-
ception. The traditional way is then to expand the model equa-
tions in power series in an amplitude (denoted here as $x$) to ob-
tain the final result in the form \[ (15) \]. The present study shows that
wee can extract from the traditional expansion the results \[ (15) \] (18) and \[ (19) \], the correct singular behavior and the
appropriate regularized form when $\epsilon$ is small but finite. This
highlights the generality of the method and its application to
various nonlinear systems with a hidden singularity.

Let us finally briefly classify singular bifurcations on the
basis of the behavior of the general traditional expansion \[ (15) \].
Suppose that the singularity is due to the presence of terms of
the form \((x^2 + \varepsilon^2)^\alpha\) where \(\alpha\) is real non integer positive number such that \(\alpha < 1\). Following the general procedure presented above, we straightforwardly obtain to leading order

\[
f(x_0, 0) = \beta - |x_0|^{2\alpha}
\]

where \(\beta\) is a real number, and where we have rescaled \(x_0\) so that the coefficient in front of the singular term can be set to unity. If \(\alpha > 1\) the first dominant term is \(x_0^2\) and to leading order the expansion is regular. Note that we have assumed the first nonlinear term to saturate the linear growth, this is why we set its coefficient to be negative. In the opposite case higher order terms (such as \(x_0^3\)) must be taken into account. This question is beyond our scope here. In terms of a dynamical system, and by remembering that we assume \(x_0 = 0\) to exist always as a solution, the corresponding normal form is

\[
\dot{A} = \mu A - A|A|^{2\alpha}
\]

with \(\mu = 1 - \beta\). Equation (21) constitutes the generic normal form for singular bifurcation. We used here the notation \(A\), as often adopted in bifurcation theory. The nontrivial fixed point behaves as \(A \sim \pm \mu^{1/(2\alpha)}\). The bifurcation structure is qualitatively different depending on whether \(\alpha > 1/2\) or \(\alpha < 1/2\). In the first case the bifurcation diagram is similar to a pitchfork bifurcation with infinite slope at \(\mu = 0\), whereas in the second case the slope vanishes for \(\mu = 0\). \(\alpha = 1/2\) is a special case with finite slope. Finally for \(\alpha < 0\) the normal form is

\[
\dot{A} = \mu A + A|A|^{2\alpha}
\]

We adopted the positive sign in front of the nonlinear term to guarantee a stable branch for \(A \neq 0\). Note that this does not affect the bifurcation diagram topology. The nontrivial fixed point is given \(A \sim \pm (\mu)^{1/(2\alpha)}\). Figure 2 summarizes the results. We note four different singular bifurcations (in blue in Fig.2) corresponding to (i) \(\alpha > 1/2\), (ii) \(\alpha < 1/2\), (iii) \(\alpha = 1\), (iv) \(\alpha < 0\). We refer to these four singular bifurcations as (i) fold, (ii) cusp, (iii) angular and (iv) unbounded. When these bifurcations are regularized, they all fall into a pitchfork bifurcation (Figure 2). We may refer to the above bifurcations as \textit{singular pitchfork bifurcations} as well, albeit the singular limits have different behaviors. It must be noted that the above classification does not exhaust by far all kinds of singularities. For example, the 2D phoretic model provides an example of transcendental singularity where the velocity behaves as \(\dot{V}_0 \sim e^{-1/\mu}\).
We thank CNES (Centre National d’Etudes Spatiales) for financial support and for having access to data of microgravity, and the French-German university programme “Living Fluids” (Grant CFDA-Q1-14) for financial support.

* chaouqi.misbah@univ-grenoble-alpes.fr

[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).

[2] R. Hoyle, Pattern Formation An Introduction to Methods (Cambridge University Press, 2010).

[3] C. Misbah, Complex Dynamics and Morphogenesis (Springer, Berlin, 2017).

[4] M. Cross and H. Greenside, Pattern Formation and Dynamics in Nonequlibrium Systems (Cambridge University Press, 2012).

[5] D. Goluskin, Internally Heated Convection and Rayleigh-Benard Convection (Springer Berlin, 2015).

[6] A. M. Turing, Phil. Trans. R. Soc. Lond. B 273, 37 (1951).

[7] P. Bourgine and A. Lesne, Morphogenesis Origin of Shape and Patterns (Springer, 2011).

[8] K. Kassner, Pattern Formation in Diffusion-Limited Crystal Growth: Beyond the Single Dendrite (World Scientific, 1996).

[9] Y. Saito, Statistical Physics Of Crystal Growth (World Scientific, 1996).

[10] C. Misbah, O. Pierre-Louis, and Y. Saito, Rev. Mod. Phys. 82, 981 (2010).

[11] B. Hof, C. W. H. van Doorne, J. Westerweel, F. T. M. Nieuwstadt, H. Faisst, B. Eckhardt, H. Wedin, R. R. Kerswell, and F. Waleffe, Science 305, 1594 (2004).

[12] R. Thom, Structural Stability and Morphogenesis (CRC Press, 2018).

[13] Z. Izri, M. N. Van Der Linden, S. Michelin, and O. Dauchot, Phys. Rev. Lett. 113, 248302 (2014).

[14] S. Michelin, E. Lauga, and D. Bartolo, Phys. Fluids 25, 061701 (2013).

[15] A. Y. Rednikov, Y. S. Ryazantsev, and M. G. Velarde, Physics of Fluids 6, 451 (1994).

[16] C. Jin, C. Krüger, and C. C. Maass, Proceedings of the National Academy of Sciences 114, 5089 (2017).

[17] W. F. Hu, T. S. Lin, S. Rafai, and C. Misbah, Phys. Rev. Lett. 123, 238004 (2019).

[18] M. Morozov and S. Michelin, J. Chem. Phys. 150, 044110 (2019).

[19] A. Izzet, P. G. Moerman, P. Gross, J. Groenewold, A. D. Hollingsworth, J. Bibette, and J. Brujic, Phys. Rev. X 10, 021035 (2020).

[20] B. V. Hokmabad, R. Dey, M. Jalaal, D. Mohanty, M. Almukambetova, K. A. Baldwin, D. Lohse, and C. C. Maass, Phys. Rev. X 11, 011043 (2021).

[21] Y. Chen, K. L. Chong, L. Liu, R. Verzicco, and D. Lohse, Journal of Fluid Mechanics 919, A10 (2021).

[22] M. Morozov and S. Michelin, Journal of Fluid Mechanics 860, 711 (2019).

[23] S. Saha, E. Yariv, and O. Schnitzer, Journal of Fluid Mechanics 916, A47 (2021).

[24] O. Pierre-Louis, C. Misbah, Y. Saito, J. Krug, and P. Politi, Phys. Rev. Lett. 80, 4221 (1998).

[25] See supplemental material at [URL will be inserted by publisher].

[26] M. Sauzade, G. J. Ifring, and E. Lauga, Physica D 240, 1567 (2011).

[27] A. Farutin, O. Aouane, and C. Misbah, Phys. Rev. E 85, 061922 (2012).

[28] A. Farutin and C. Misbah, Phys. Rev. Lett. 110, 108104 (2013).
Supplemental Materials: Singular Bifurcations: a Regularization Theory

We provide here the regularization solution for the phoretic model for finite size in 3D. We also present the singular behavior in 2D, which is quite distinct from that in 3D. More details about the results discussed in the main text are also presented.

EFFECT OF HYDRODYNAMICS ON CRITICAL CONDITION

The goal of this section is to introduce the corrections into the exactly solvable model in order to account for the finite size of the particle. These corrections are evaluated for small propulsion velocity and provide quantitatively correct value of the critical Peclet number. There are two finite-size effects that are neglected in the main model: First, the near-field flow disturbance due to a translating spherical particle is neglected, and second, the particle emission is represented by a point source, while the finite-size particle should be represented by a homogeneous distribution of sources along the particle surface. Both of these two effects are essential for quantitative evaluation of the concentration field close to the critical point.

This problem is solved in the reference frame comoving with the particle. The concentration evolution equation is then written as

$$\dot{c}(r) + \nabla \cdot (u(r)c(r)) = D\nabla^2 c(r) + A(r), \quad (S1)$$

where $u(r)$ is the fluid velocity relative to the particle, $A(r)$ represents a distribution of sources and source dipoles on the particle surface which accounts for the concentration emission or consumption, and $r$ is the position vector relative to the particle center. It is known that the velocity field in the comoving frame can be written as

$$u(r) = -V_0 + \frac{a^3}{2r^3} \left[ 3 \frac{r(r \cdot V_0)}{r^2} - V_0 \right] \quad (S2)$$

for a rigid force-free spherical particle or radius $a$, moving with velocity $V_0$ relative to the laboratory frame. The flow field in eq. (S2) can be written in potential representation $u(r) = \nabla \phi(r)$, where

$$\phi(r) = -(V_0 \cdot r) \left( 1 + \frac{a^3}{2r^3} \right). \quad (S3)$$

We also have $\nabla^2 \phi(r) = 0$ for $r > 0$ due to the flow incompressibility.

We focus on the steady-state solution of Eq. (S1). Multiplying eq. (S1) by $\exp[-\phi(r)/(2D)]$, yields

$$D\nabla^2 \tilde{c}(r) - \frac{u(r)^2}{4D} \tilde{c}(r) + \tilde{A}(r) = 0, \quad (S4)$$

where $\tilde{c}(r) = c(r) \exp[-\phi(r)/(2D)]$ and $\tilde{A}(r) = A(r) \exp[-\phi(r)/(2D)]$.

The original model corresponds to setting $u(r)^2$ to $V_0^2$, $\phi(r)$ to $-V_0 \cdot r$, and $\tilde{A}(r)$ to a point source in eq. (S4). Here we still simplify $u(r)^2$ to $V_0^2$ because this term is quadratic in velocity and thus should be small close to the critical point. We keep, however, the full expression for $\phi$ and replace the $\tilde{A}(r)$ term with a combination of a point source and a point source dipole. The amplitude of the dipole source is chosen in a way that corresponds to an isotropic emission rate at distance $a$ from the particle center.

We thus consider the following equation

$$D\nabla^2 \tilde{c}(r) - \frac{V_0^2}{4D} \tilde{c}(r) + 4\pi a^2 \tilde{A}[\delta(r) + b(V_0 \cdot \nabla)\delta(r)] = 0. \quad (S5)$$

This equation can be solved analytically, yielding

$$\tilde{c}(r) = \frac{a^2 A}{Dr} \exp \left( -\frac{V_0 r}{2D} \right) + b(V_0 \cdot \nabla) \left\{ \frac{a^2 A}{Dr} \exp \left( -\frac{V_0 r}{2D} \right) \right\}. \quad (S6)$$

The constant $b$ is found by taking the concentration field $c(r) \equiv \tilde{c}(r) \exp[\phi(r)/(2D)]$ and setting the first harmonic of $r \cdot \nabla c(r)$ to zero:

$$b = \frac{9a^2}{2D} \frac{\xi + 3}{\xi^2 + 6\xi + 18 (\xi^2 - 4\xi + 8)e^\xi - (\xi^2 + 4\xi + 8)}, \quad (S7)$$

where $\xi$ is the Péclet number.
where \( \xi = 3V_0a/(2D) \). Substituting eq. (S7) into eq. (S6) yields the corrected concentration field. We extract the first harmonic of the concentration for \( r = a \) from this solution, which gives us the following expression of the swimming velocity

\[
V_0 = -\frac{18AM\left[(\xi-2)e^\xi + \xi + 2\right]}{D\xi^2(\xi^2 + 6\xi + 18)}\left(-\frac{5\xi}{6}\right) = \frac{-AMaV_0}{4D^2}\left[1 - aV_0/D + O\left((aV_0/D)^2\right)\right]
\]  

(S8)

Dividing both sides of eq. (S8) by \( V_0 \) and setting \( V_0 \) to 0 yields \(-AMa/D^2 \equiv Pe = 4\) for the critical Peclet number, which agrees with the previous works.

**FINITE SIZE EFFECT**

We consider the same phoretic model except that the size is finite. We focus here only on steady state solutions in the co-moving frame with velocity \( \mathbf{V}_0 \). The concentration field obeys in this frame

\[
D\Delta c + \mathbf{V}_0 \cdot \nabla c = -S\delta(r)
\]

(S9)

The particle is taken to move along the \( z \)---direction. Making the substitution \( c = \bar{c}e^{-\sqrt{\xi}r} \) we find

\[
\Delta \bar{c} - k^2 \bar{c} = -\frac{S}{D} \delta(r), \quad k^2 = \frac{V_0^2}{(4a^2)}
\]

(S10)

This is the so-called screened Helmholtz equation with a delta source term. The associated Green’s function is defined as

\[
\Delta G(\mathbf{r}, \mathbf{r'}) - k^2 G(\mathbf{r}, \mathbf{r'}) = \delta(\mathbf{r} - \mathbf{r'}),
\]

(S11)

We consider the domain to be finite and bounded by a sphere with radius \( r = R \) (counted from the point source). The boundary condition is taken as \( \bar{c}(r = R) = 0 \). We use the eigenfunctions of the Laplacian in order to express the Green’s function. The Laplacian eigenfunctions are spherical harmonics \( Y_m^\ell(\theta, \phi) \) times spherical Bessel functions \( j_\ell(r) \). Let \( \beta_m^\ell \) define the zero's of \( j_\ell \), we have \( j_\ell(\beta_m^\ell) = 0 \). The Laplacian eigenfunction which vanishes at \( r = R \) can be written as

\[
\psi_{n\ell m}(r, \theta, \phi) = A_{n\ell} Y_m^\ell(\theta, \phi) j_\ell(\beta_m^\ell r/R)
\]

(S12)

Then making use of the classical method to express the Green’s function in terms of eigenfunctions, we obtain

\[
G(\mathbf{r}, \mathbf{r'}) = -\sum_{n\ell m} \frac{2}{R^3} \frac{1}{j_{\ell+1}^2(\beta_m^\ell)} \frac{Y_m^\ell(\theta, \phi) j_\ell(\beta_m^\ell r/R) Y_m^\ell(\theta', \phi') j_\ell(\beta_m^\ell r'/R)}{k^2 + (\beta_m^\ell R)^2}
\]

(S13)

Note that the eigenvalues of the Laplacian are \( (\beta_m^\ell/R)^2 \), meaning that the eigenvalues of the full operator in (S10) are \( k^2 + (\beta_m^\ell/R)^2 \). The above Green’s function can be rewritten as

\[
G(\mathbf{r}, \mathbf{r'}) = -\frac{2}{R^3} \sum_{n\ell m} \frac{2\ell + 1}{4\pi} P_\ell(\cos(\gamma)) \frac{1}{j_{\ell+1}^2(\beta_m^\ell)} \frac{j_\ell(\beta_m^\ell r'/R) j_\ell(\beta_m^\ell r/R)}{k^2 + (\beta_m^\ell R)^2}
\]

(S14)

after having used the addition theorem for spherical harmonics, where \( P_\ell \) is the Legendre polynomial of order \( \ell \) and \( \cos(\gamma) = \cos(\theta) \cos(\theta') + \sin(\theta) \sin(\theta') \cos(\phi - \phi') \). Since the source term is assumed to be at the center, we set \( r' = 0 \), so that \( j_0(\beta_m^\ell r'/R) = j_0(0) \). Due to the properties of \( j_0 \) only \( \ell = 0 \) survives in the sum. Using the definition of \( j_0 \) and \( j_1 \) functions, we obtain (upon using that \( \beta_{n0} = n\pi \)) that the concentration field can be written as

\[
c(r, \theta) = \frac{A}{4\pi DR} e^{-\sqrt{\xi}r} \frac{\cos(\theta)}{\sqrt{\xi}} \text{csch}(|k|(R-r))
\]

(S15)

where we have used the result \( \sum_{n=0}^{\infty} n \sin(na)/(n^2 + b^2) = \pi \text{csch}(\pi |b|) \sinh((\pi - a) |b|) \), \( \text{csch} \) being the hyperbolic cosecant function. Projecting \( c(r, \theta) \) on the first spherical harmonic, and using the condition that \( V_0 = -Mc_1/(a\sqrt{3\pi}) \) (recall that \( c_1 \) is the concentration contribution of the first harmonic at \( r = a \)) we find

\[
\bar{V}_0 = 4Pe \left[ \frac{V_0 \cosh(V_0/2) - 2 \sinh(V_0/2)}{V_0^2} \right] \text{csch}(|V_0|R/2) \sinh(|V_0|(R-1)/2),
\]

(S16)
where $\bar{R} \equiv R/a$. Expanding this result for small $\bar{V}_0$ we obtain to cubic order

$$\bar{V}_0 = \frac{Pe}{3} \bar{V}_0 \left[ 1 - \bar{R}^{-1} - \frac{\bar{V}_0^2}{24} (2\bar{R} - 3 + \bar{R}^{-1}) \right]$$  \hspace{1cm} (S17)

We see that the expansion is regular; the finite size has regularized the singular pitchfork behavior. The solution $\bar{V}_0 = 0$ always exists. Beyond a certain critical value $Pe = Pe_1$ there exists another solution behaving as $\bar{V}_0 \sim \pm (Pe - Pe_1)^{1/2}$, with $Pe_1 = 3/(1 - \bar{R}^{-1})$. If we take first the limit $\bar{R} \to \infty$ in Eq. (S16), we obtain

$$\bar{V}_0 = -\frac{Pe}{2} \left[ \sinh(2\bar{V}_0) - \cosh(2\bar{V}_0) \right] e^{-2|\bar{V}_0|},$$  \hspace{1cm} (S18)

yielding the same expression as in the main text for infinite size. The function $\text{csch}(|\bar{V}_0|\bar{R}/2)$ has an infinite and countable set of singularities on the imaginary axis, $\bar{V}_0 = i\pi/(2\bar{R})$, $n$ being an integer.

**RELATION BETWEEN $a_k$ AND $b_k$**

It is easy to obtain the general relation between $a_k$ and $b_k$. However here we only list the relations for the first three terms (generalization to arbitrary order is straightforward). The starting point is to write the Taylor expansion in terms of $a_k(x) e^{2k}$ and makes the substitution $\varepsilon = x_0(1 - s)$ and $x^2 = (2s - s^2)x_0^2$, so that we have

$$a_0[x_0(1 - s)] + a_1[x_0(1 - s)](2s - s^2)x_0^2 + a_2[x_0(1 - s)](2s - s^2)^2x_0^4 + \ldots$$  \hspace{1cm} (S19)

Then expanding $a_k[x_0(1 - s)]$ in Taylor series with respect to $s$, we obtain to leading order

$$b_0(x_0) + b_1(x_0)s + b_2(x_0)s^2 + \ldots$$  \hspace{1cm} (S20)

with the relations

$$b_0(x_0) = a_0, \quad b_1(x_0) = a_1 + 2a_1x_0^2,$$

$$b_2(x_0) = a_2^2 - a_1x_0^2 + 2a_1a_2x_0^4$$  \hspace{1cm} (S21)

where $a_k$ as well as $a_k'$ and $a_k''$, which designate first and second derivative with respect to $s$, are evaluated at $s = 0$.

**2D MODEL WITH CONSUMPTION**

In 2D we only need to substitute in the denominator of the propagator $\tau^{3/2}$ by $\tau$, so that the concentration field takes the form

$$c(r,t) = \int_0^\infty d\tau \frac{S}{4\pi D} \exp - \left\{ \frac{\left(r + V_0 \tau - V_0 t\right)^2}{4D\tau} \right\},$$  \hspace{1cm} (S22)

yielding

$$c(r,t) = \frac{S}{2\pi a D} K_0(\bar{R}\sqrt{V_0^2 + \varepsilon^2/4}) e^{-\bar{R}0 \cos(\theta)/2},$$  \hspace{1cm} (S23)

where $\bar{R} = \bar{r}/a$, and $K_0$ is the Bessel function of the second kind. Projecting Eq. (S17) on the first Fourier mode and using the equation fixing velocity as a function of concentration (see main text) we find $V_0 = -2Mc_1/(3a)$ (where $c_1$ is the amplitude of the first Fourier mode), obtaining finally

$$\bar{V}_0 = \frac{Pe}{3} I_1(\bar{V}_0/2) K_0\left(\sqrt{\bar{V}_0^2 + \varepsilon^2/4}\right)$$  \hspace{1cm} (S24)

where $I_1$ is the Bessel function of the first kind. Besides the trivial solution, this equation exhibits a pitchfork bifurcation. For $\varepsilon = 0$ the bifurcation becomes singular with $\bar{V}_0 \sim e^{-3/Pe}$; for a small argument $I_1 \sim \bar{V}_0$ and $K_0 \sim -\ln(\bar{V}_0)$. 

The radius of convergence of the expansion of $F$ in powers of $s$ as a function of the proportionality coefficients $\Delta_i$, setting the singular points of $F$ as a function of $\epsilon$. The black curves mark the boundary of the region of $\Delta$ for which the radius of convergence is greater than 1. Only this region is colored.

**SINGULARITIES OUTSIDE OF THE IMAGINARY AXIS**

Here we discuss the applicability of the method for the problems in which the singularities are not necessary on the imaginary axis. Suppose there is a function $F(x^2, \epsilon)$, where $x$ is the expansion parameter and $\epsilon$ is the regularization parameter, as in the Main Letter. The function $F$ is an analytic function of $x$ with exception of singular points $x_i(\epsilon) = \Delta_i \epsilon$. Here we allow $\Delta_i$ to be arbitrary complex numbers. Applying the transformation $\epsilon = x_0(1 - s)$ and $x^2 = (2s - s^2)x_0^2$, we obtain a function of $s$ and $x_0$. This function is an analytical function of $s$ with exception of singular points $s_i$ given by

$$s_i = 1 \pm \frac{1}{\sqrt{1 + \Delta_i^2}} \quad (S25)$$

The radius of convergence of the expansion of $F$ in powers of $s$ is governed by the singular point $s_i$ with the lowest absolute value. The success of the proposed method requires this radius of convergence to be greater than 1. The method thus works if all $\Delta_i$ are such that $|s_i| > 1$, where $s_i$ is given by (S25). Figure S1 shows the region of the complex plane which must contain $\Delta_i$ for all singular points of $F$ in order for the expansion in $s$ to converge for $s = 1$. 

**Figure S1**