Abstract

We give a combinatorial condition for the existence of efficient, LP-based FPT algorithms for a broad class of graph-theoretical optimisation problems. Our condition is based on the notion of biased graphs known from matroid theory. Specifically, we show that given a biased graph $\Psi = (G, B)$, where $B$ is a class of balanced cycles in $G$, the problem of finding a set $X$ of at most $k$ vertices in $G$ which intersects every unbalanced cycle in $G$ admits an FPT algorithm using an LP-branching approach, similar to those previously seen for VCSP problems (Wahlström, SODA 2014). Our algorithm has two parts. First we define a local problem, where we are additionally given a root vertex $v_0 \in V$ and asked only to delete vertices $X$ (excluding $v_0$) so that the connected component of $v_0$ in $G - X$ contains no unbalanced cycle. We show that this local problem admits a persistent, half-integral LP-relaxation with a polynomial-time solvable separation oracle, and can therefore be solved in FPT time via LP-branching, assuming only oracle membership queries for the class of balanced cycles in $G$. We then show that solutions to this local problem can be used to tile the graph, producing an optimal solution to the original, global problem as well. This framework captures many of the problems previously solved via the VCSP approach to LP-branching, as well as new generalisations, such as Group Feedback Vertex Set for infinite groups (e.g., for graphs whose edges are labelled by matrices). A major advantage compared to previous work is that it is immediate to check the applicability of the result for a given problem, whereas testing applicability of the VCSP approach for a specific VCSP, requires determining the existence of an embedding language with certain algebraically defined properties, which is not known to be decidable in general. Additionally, we study the approximation question, and show that every problem of this category admits an $O(\log \text{OPT})$-approximation.

1 Introduction

In recent years, we have seen a growing interest in the use of linear or integer linear programming methods (LP/ILP) in parameterized complexity \cite{DBLP:conf/soda/LokshtanovM15, DBLP:series/dagstuhl/2014, DBLP:conf/stoc/LokshtanovM14, DBLP:conf/focs/Wahlstrom14, DBLP:conf/stoc/Wahlstrom14}. The appeal is clear. On the one hand, linear programming and general continuous relaxations comes with a very powerful toolbox for theoretical investigation. This promises to be a powerful hammer, especially for optimisation problems; e.g., an FPT-size extended formulation, or more generally an FPT separation oracle for an integral polytope would provide a way towards FPT algorithms \cite{DBLP:conf/stoc/Wahlstrom14}. The same can be said for Lenstra’s algorithm and other methods for solving complex problems in...
few variables \cite{27,4}. On the other hand, it has been observed that ILP solvers, like SAT solvers, frequently perform much better in practice than can be currently be explained theoretically. It is appealing to study parameterizations of ILP problems, to find structural reasons for this apparent tractability \cite{22,14}.

Narrowing our focus, for many optimisation problems we have seen powerful FPT results based on running an ILP-solver with performance guarantees. These algorithms use LP-relaxations with strong structural properties (persistence and half-integrality), which allows you to run an FPT branching algorithm on top of the relaxation, typically getting a running time of $O^*(2^{O(k)})$ for a solution size of $k$. More powerfully, these algorithms also allow to bound the running time in terms of the relaxation gap, i.e., the additive difference between the value of the LP-optimum and the true integral optimum. For problems such as VERTEX COVER \cite{28} and MULTIWAY CUT \cite{8}, this gives us FPT algorithms that find an optimum in time $O^*(2^{O(k-\lambda)})$, where $k$ is the size of the optimum and $\lambda$ is the lower bound given by the LP-relaxation of the input. In this sense, these results can also be taken as a possible “parameterized explanation” of the success of ILP solvers for these problems, although this approach of course only works for certain problems as it is in general NP-hard to decide whether an ILP problem has a solution matching the value of the LP-relaxation.

The main restriction for this approach is to find LP-relaxations with the required structural properties (or even to find problems which admit such LP-relaxations). In the cases listed above, this falls back on classical results from approximation \cite{31,16}, but further examples proved elusive. A significant advancement was made relatively recently \cite{20} by recasting the search for LP-relaxations with the required properties in algebraic terms, using a connection between such LP-relaxations and so-called Valued CSPs (VCSPs; see below). Using this connection, FPT LP-branching algorithms were provided that significantly improved the running times for a range of problems, including SUBSET FEEDBACK VERTEX SET and GROUP FEEDBACK VERTEX SET in $O^*(4^k)$ time for solution size $k$, improving on previous records of $O^*(2^{O(k\log k)})$ time \cite{9,7}, and UNIQUE LABEL COVER in $O^*((|\Sigma|2^k)$ time for alphabet $\Sigma$ and solution size $k$, improving on a previous result of $O^*(|\Sigma|^{O(k^2 \log k)})$ \cite{2}. GROUP FEEDBACK VERTEX SET in particular is a meta-problem that includes many independently studied problems as special cases.

However, powerful though these results may be, the nature of the framework makes it difficult to apply to a given combinatorial problem. To do so would involve two steps. First, the problem must be phrased as a VCSP. A VCSP asks to minimise the value of an objective function $f(\phi)$ over assignments $\phi : V \to D$ from a finite domain $D$, where the objective $f(\phi)$ is in turn usually given as a sum of bounded-size cost functions $\sum_{i=1}^{m} f_i(\phi)$. To find such a formulation is not always easy, and sometimes it may be impossible to capture a problem precisely. As an example, it is possible to phrase FEEDBACK VERTEX SET as an instance of GROUP FEEDBACK VERTEX SET and hence arguably as a VCSP \cite{20}, but to discover such a formulation from first principles is not easy. Second, given a desired VCSP formulation, it must be determined whether the VCSP admits a discrete relaxation, so that the framework can be applied \cite{20}. It is not known in general how to decide the existence of such a relaxation.\footnote{The results of \cite{20} are obtained by working backwards from known relaxations with the required properties, a list which includes separable $k$-submodular relaxations and arbitrary bisubmodular relaxations, and with a small extension can be made to cover a class of problems also including so-called skew bisubmodular functions \cite{13}. However, again, even given a specific target class it is usually at least intuitively non-obvious whether a VCSP can be relaxed into the class or not.}

In short, although the results are powerful, they are also somewhat inscrutable.\footnote{Since the publication of the preliminary version of this paper \cite{35}, there have been a few further developments in this direction. We note two in particular. Iwata, Yamaguchi and Yoshida \cite{21} provide a combinatorial algorithm that replaces the LP-solving step in the above applications, implying linear-time FPT algorithms for all of the problems cited above. It is however currently not known whether such an algorithm exists for the general BIASED
In this paper, we instead give a combinatorial condition under which a graph-theoretical problem admits an LP-relaxation with the required properties, and hence an efficient FPT algorithm parameterized either by solution size, or in particular cases by a relaxation gap parameter. Our condition is based on the class of so-called biased graphs, which are combinatorial objects of importance especially to matroid theory [39, 40]. We review these next.

Biased graphs. A biased graph is a pair \( \Psi = (G = (V, E), \mathcal{B}) \) of a graph \( G \) and a set \( \mathcal{B} \subseteq 2^E \) of simple cycles in \( G \), referred to as the balanced cycles of \( G \). with the property that if two cycles \( C, C' \in \mathcal{B} \) form a theta graph (i.e., a collection of three internally vertex-disjoint paths with shared endpoints), then the third cycle of \( C \cup C' \) is also contained in \( \mathcal{B} \). A cycle class \( \mathcal{B} \) with this property is referred to as linear. Dually, and more important to the present paper, a simple cycle \( C \) is unbalanced if \( C \notin \mathcal{B} \). The definition is equivalent to saying that if \( C \) is an unbalanced cycle, and if \( P \) is a path with endpoints in \( C \) which is internally edge- and vertex-disjoint from \( C \), then at least one of the two new cycles formed by \( C \cup P \) is also unbalanced. We refer to a collection \( \mathcal{C} \) of cycles as a co-linear cycle class if the complement of \( C \) is a linear class. We say that an induced subgraph \( G[S] \) of \( G \) is balanced if \( G[S] \) contains no unbalanced cycles.

The basic problem considered in this paper is now defined as follows: Given a biased graph \( \Psi = (G = (V, E), \mathcal{B}) \) and an integer \( k \), find a set of vertices \( S \subseteq V \) with \( |S| \leq k \) such that \( G - S \) is balanced. We refer to this as the Biased Graph Cleaning problem. Our main result in this paper is that Biased Graph Cleaning is FPT by \( k \), with a running time of \( O^*(4^k) \), assuming only access to a membership oracle for the class \( \mathcal{B} \) (i.e., for a cycle \( C \), given as a set of edges, we can determine whether \( C \in \mathcal{B} \) with an oracle call).

An important example of biased graphs are group-labelled graphs. Let \( G = (V, E) \) be an oriented graph, and let the edges of \( G \) be labelled by elements from a group \( \Gamma = (D, \cdot) \), such that if an edge \( uv \in E \) has label \( \gamma \in D \), then the edge \( vu \) (i.e., \( uv \) traversed in the opposite direction) has label \( \gamma^{-1} \). Then the balanced cycles of \( G \) are the cycles \( C \) such that the product of the edge labels of the cycle, read in the direction of their traversal, is equal to the identity element \( 1_D \) of \( \Gamma \). Note that the orientation of the edges serves only to make the group-labelling well-defined, and has no bearing on which cycles we consider. It is easy to verify that this defines a linear class of cycles, and hence gives rise to a biased graph \( \Psi \).

The problem Group Feedback Vertex Set corresponds exactly to Biased Graph Cleaning when \( \Psi \) is defined by a group-labelled graph. However, not all biased graphs can be defined via group labels, and moreover, some group-labelled graphs can only be defined via an infinite group \( \Gamma \) [10]. More examples of biased graphs follow below.

Biased graphs were originally defined in the context of matroid theory. Although this connection is not important to the present paper, we nevertheless give a brief review. Each biased graph \( \Psi \) gives rise to two matroids, the frame matroid and the lift matroid of \( \Psi \). These are important examples in structural matroid theory. The Dowling geometries \( Q_n(\Gamma) \) for a group \( \Gamma \), originally defined by Dowling [11], are equivalent to frame matroids of complete \( \Gamma \)-labelled multigraphs. For more on matroids for biased graphs, see Zaslavsky [39, 40], as well as the series of blog posts on the Matroid Union weblog [32, 33, 34].

Our approach. Inspired by the previous algorithm for Group Feedback Vertex Set [20], our approach for Biased Graph Cleaning consists of two parts, the local problem and the global problem. In the local problem, the input is a biased graph \( \Psi = (G = (V, E), \mathcal{B}) \) together with a root vertex \( v_0 \in V \) and an integer \( k \), and the task is restricted to finding a set \( X \subseteq V \) of
vertices, $|X| \leq k$ and $v_0 \notin X$, such that the connected component of $v_0$ in $G - X$ is balanced. Equivalently, the local problem can be defined as finding a set $S \subseteq V$ with $v_0 \in S$ such that $G[S]$ is balanced and connected, and $|NG(S)| \leq k$. We refer to this local problem as Rooted Biased Graph Cleaning.

We show that Rooted Biased Graph Cleaning can be solved via an LP which is half-integral and has a stability property similar to persistence. This LP uses a formulation where the constraints correspond to rooted cycles we refer to as balloons. The formulation of this slightly unusual LP is critical to the tractability of the problem. The possibly more natural approach of letting the obstacles of the LP simply be unbalanced cycles would not work as well; for instance, although Feedback Vertex Set is an instance of Biased Graph Cleaning (with balanced cycles $B = \emptyset$), it is known that the natural cycle-hitting LP has an integrality gap of a factor of $\Theta(\log n)$ [3].

The properties of the LP further imply that Rooted Biased Graph Cleaning has a 2-approximation, even for weighted instances, and can be solved (in the unweighted case) in time $O^*(4^k - \lambda)$ where $\lambda$ is the value of the LP-optimum, assuming access to a membership oracle for the class $B$ as above. In particular, Rooted Biased Graph Cleaning can be solved in time $O^*(2^k)$. We note that several independently studied problems arise as special cases of Rooted Biased Graph Cleaning; see below.

In order to solve the global problem, Biased Graph Cleaning, we show that the local LP obeys a strong persistence-like property, analogous to the important separator property frequently used in graph separation problems [29], which allows us to identify “furthest-reaching” local connected components when solving the local problem, such that the connected components produced by the algorithm for the local problem can be used to “tile” the original graph in a solution to the global problem. A $O^*(4^k)$-time algorithm for Biased Graph Cleaning follows (although the “above lower bound” perspective does not carry over to solutions for the global problem).

**Results and applications.** We summarise the above statements in the following theorems. Let $\Psi = (G = (V, E), B)$ be a biased graph, where $B$ is defined via a membership oracle that takes as input a simple cycle $C$, provided as an edge set, and tests whether $C \in B$. Then the following apply.

**Theorem 1.** Assuming a polynomial-time membership oracle for the class of balanced cycles, Rooted Biased Graph Cleaning admits the following algorithmic results:

- A polynomial-time 2-approximation;
- An FPT algorithm with a running time of $O^*(4^k - \lambda)$, where $\lambda \geq k/2$ is the value of the LP-relaxation of the problem;
- An FPT algorithm with a running time of $O^*(2^k)$.

The 2-approximation holds even for weighted graphs.

**Theorem 2.** Assuming a polynomial-time membership oracle for the class of balanced cycles, Biased Graph Cleaning admits an FPT algorithm with a running time of $O^*(4^k)$.

To illustrate the flexibility of the notion, let us consider some classes of biased graphs.

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3 Although formulations of LPs for Feedback Vertex Set exist with an integrality gap of 2 [4], we are not aware of any similarities between these formulations and our local LP. Moreover, other special cases of Biased Graph Cleaning, e.g., Odd Cycle Transversal, admit no constant-factor approximation unless the Unique Games Conjecture fails.
• If $B = \emptyset$, then **Biased Graph Cleaning** corresponds simply to **Feedback Vertex Set**.

• If $\Psi$ arises as a $\Gamma$-labelled graph, then **Biased Graph Cleaning** corresponds to **Group Feedback Vertex Set**. If $\Gamma$ is finite, then the result is equivalent to the previous LP-based algorithm [20].

• If $\Psi$ is $\Gamma$-labelled for an infinite group $\Gamma$, e.g., a matrix group, then previous results do not apply, since they assume the existence of an underlying VCSP presentation (in the correctness proofs, if not in the algorithms). However, the problem is still FPT, assuming, essentially, that the word problem for $\Gamma$ can be solved.

• To show a case that does not obviously correspond to group-labelled graphs, let $G$ be (improperly) edge-coloured, and let a cycle be balanced if and only if it is monochromatic. It is not difficult to see that this defines a biased graph.

• Finally, Zaslavsky [39] notes another case that in general does not admit a group-labelled representation. Let $G$ be a copy of $C_n$, with two parallel copies of every edge. Let $B$ be a class of “isolated” Hamiltonian cycles of $G$, in the sense that for any $C \in B$, switching the edge used between $u$ and $v$ for any pair of consecutive vertices of $G$ results in an unbalanced cycle. It is not hard to verify that this defines a biased graph. To keep $G$ as a simple graph, we simply subdivide the edges; this does not affect the collection of cycles (although it means that the term “Hamiltonian” fails to apply).

However, the corresponding problem can be very difficult, e.g., $B$ may consist of only exactly one of the $2^n$ candidate cycles, giving an oracle lower bound of $\Omega(2^k)$ against any algorithm for **Biased Graph Cleaning**.

Finally, we show as promised that the algorithm for **Rooted Biased Graph Cleaning** has independent applications.

• Let $G = (V, E)$ be any graph, and add an apex vertex $v_0$ to $G$. Let $B = \emptyset$. Then **Rooted Biased Graph Cleaning** corresponds to **Vertex Cover**, hence Theorem 1 is an $O^*(4^k)$-time FPT algorithm for the problem **Vertex Cover Above Matching**, which encompasses the more commonly known problem **Almost 2-SAT** [30, 35].

• Let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals. Duplicate each terminal $t \in T$ into $d(t)$ copies, forming a set $T'$, and add a vertex $v_0$ to $G$ with $N(v_0) = T'$. Let a cycle be **unbalanced** if and only if it passes through $v_0$ and two vertices $t, t' \in T'$ which are copies of distinct terminals in $T$. Then **Rooted Biased Graph Cleaning** corresponds to **Multiway Cut**, and Theorem 1 reproduces the known 2-approximation and $O^*(2^k)$-time algorithm [16] [8].

We note that the linear class condition prevents us from representing any other cut problem this way; if terminal-terminal paths from $t$ to $t'$ and from $t$ to $t''$ are allowed in $G$, for some $t, t', t'' \in T$, then also the path from $t$ to $t''$ must be allowed. Hence the set of allowed paths induces an equivalence relation on $T$.

In general, we find that the notion of biased graphs is surprisingly subtle, and corresponds surprisingly well to the class of (natural) problems for which LP-branching FPT algorithms are

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4 Yoichi Iwata (personal communication) noted that this does correspond to a group-labelled graph, using a quotient group of $Z_2^E$.

5 To be precise, it gives a lower bound of $2^k$ against the edge-deletion version of the problem, but it is not difficult to transfer such a lower bound to a lower bound against **Biased Graph Cleaning**.
known. We also note, without further study at the moment, that there is significant similarity between the (local or global) problems that can be expressed via biased graphs this way and the class of graph separation problems for which the existence of polynomial kernels is either known, or most notoriously open [24].

Finally, although our results do include some cases which were not previously known to be FPT, we feel that the most significant advantage of the present work is the transparency and naturalness of the definition. The existence of a purely combinatorial condition also arguably brings us closer to the existence of a purely combinatorial algorithm for these problems. Since the results in this paper rely on using an LP-solver with a separation oracle, a purely combinatorial algorithm could significantly decrease the hidden polynomial factor in the running times.

Additional results: Approximation. As an additional result in the current paper (not present in the preliminary version [38]), we consider the approximation properties of Biased Graph Cleaning. Using properties of the LP developed in the rest of the paper, and using standard region growing arguments used for Multicut and Sparsest Cut [26, 15], we show the following result.

**Theorem 3.** Assuming a polynomial-time membership oracle for the class of balanced cycles, the weighted version of Biased Graph Cleaning admits a polynomial-time $O(\log k)$-approximation, where $k$ is the cost of an optimal solution.

This problem does not appear to have been previously studied in the approximation literature, even in the natural special case of Group Feedback Vertex Set.

We note that although ratios of $O(\log n)$ and $O(\log k)$ appear similar from an approximation perspective, the latter are strongly preferable for applications in parameterized complexity. In particular, $f(k)$-type approximation ratios have been used repeatedly in polynomial kernelization to bootstrap the kernelization procedure, including for Odd Cycle Transversal [23], and for Almost 2-SAT and Group Feedback Vertex Set [24]. In all of these cases, the size of the kernelization depends on the approximation ratio when expressed as a function of $k$, and improved ratios $f(k)$ would immediately improve the corresponding kernel sizes. Using Theorem 3 would improve the kernel size computed in [24] for Group Feedback Vertex Set, from $O(k^{2s+2})$ vertices to $O((k \log k)^{s+1})$ vertices, where $s$ is the size of the group. We do not pursue this connection further in the present paper, because in preparation of a journal version of [24] (in preparation, 2019), Kratsch and Wahlström already noted an $O(s \log (sk) \log \log (sk))$-approximation by reducing to Symmetric Multicut and using results by Even et al. [12]. Thus, for $s = O(1)$, the difference in approximation ratio is minor. However, for $s = \omega(1)$, Theorem 3 provides a significant improvement in the available approximation guarantee.

For more on kernelization, see the book of Fomin, Lokshtanov, Saurabh and Zehavi [13]. We will not consider this topic any further in the present paper.

**Preliminaries.** We assume familiarity with the basic notions of graph theory, parameterized complexity, and the basics of combinatorial optimisation. For a reference on parameterized complexity, see Cygan et al. [6]: for all necessary material on linear programming and combinatorial optimisation, see Schrijver [37]. Other notions will be introduced as they are used.

## 2 Biased graphs and the local LP

In this section, we define the local LP, used to solve the local problem, and give some results about the structure of min-weight obstacles in it. We will also show that we can optimise over
the LP in polynomial time by providing a separation oracle. In subsequent sections, we will derive the properties of half-integrality and persistence, and show Theorems 1 and 2.

We first introduce some additional terminology. Recall the definition of biased graphs from Section 1. Let \( \Psi = (G = (V,E),\mathcal{B}) \) be a biased graph. For a simple cycle \( C \), a chord path for \( C \) is a simple path with end vertices in \( C \) and internal vertices and edges disjoint from \( C \). If \( C \) is an unbalanced cycle in \( \Psi \), a reconfiguration of \( C \) by \( P \) refers to an unbalanced cycle \( C' \) formed from \( C \) and \( P \), with \( C' \) containing \( P \), as is guaranteed by the definition of biased graphs. Note that a chord path can consist of a single edge and no internal vertices; however, it is also possible that a chord path is non-induced, e.g., for structural purposes we may reconfigure a cycle \( C \) using a chord path \( P \) that contains internal vertices, even if there is a direct edge in \( G \) connecting the end points of \( P \). A membership oracle for \( \mathcal{B} \) is a (black-box) algorithm which, for every set of edges forming a simple cycle \( C \) in \( G \), will respond whether \( C \in \mathcal{B} \).

For simplicity, we assume that \( G \) is a simple graph, by subdividing edges if \( G \) contains parallel edges. Let us observe that this can be done without loss of generality.

**Proposition 1.** Let \( \Psi = (G,\mathcal{B}) \) be a biased graph and let \( C' \) be the result of subdividing some edges in \( G \). Then there is a bijection between simple cycles of \( G \) and simple cycles of \( C' \). Furthermore, we can define a biased graph \( \Psi' = (G',\mathcal{B}') \) where \( \mathcal{B}' \) contains edge sets of simple cycles in \( G' \) which correspond to balanced cycles in \( G \); and given a membership oracle for \( \mathcal{B} \) we can define a membership oracle for \( \mathcal{B}' \).

As a warm-up, and to illustrate the kind of arguments we will be using, we give a simple lemma that shows why co-linearity is a useful structural property from the perspective of the local problem.

**Lemma 1.** Let \( \Psi = (G = (V,E),\mathcal{B}) \) be a biased graph, and let \( V_R \subseteq V \) be a set of vertices such that \( G[V_R] \) is balanced and connected. Let \( C \) be an unbalanced cycle. Then either \( C \) intersects \( N(V_R) \) in at most one vertex, or there exists an unbalanced cycle \( C' \) such that \( C' \) intersects \( V_R \) in a non-empty simple path, \( C' \) intersects \( N(V_R) \) in at most two vertices, and \( (V(C') \setminus V_R) \subseteq (V(C) \setminus V_R) \).

**Proof.** Assume that \( C \) intersects \( N(V_R) \) at least twice, as otherwise there is nothing to show. Assume first that \( C \) is disjoint from \( V_R \), and let \( u \) and \( v \) be two distinct members of \( V(C) \cap N(V_R) \). Let \( P \) be a uv-path with internal vertices in \( V_R \). Reconfiguring \( C \) using \( P \) as a chord path results in an unbalanced cycle \( C' \) which intersects \( V_R \), and whose intersection with \( V \setminus V_R \) is a subset of that of \( C \), hence we may assume that the cycle \( C \) intersects \( V_R \).

Now consider \( V(C) \cap V_R \). Since \( V_R \) is balanced, this is a collection of paths (rather than the entire cycle). Let \( P_C \) be one such path, and let \( u \) and \( w \) be the vertices in \( N(V_R) \) that it terminates at. Now, if possible, let \( P \) be a shortest path in \( V_R \) connecting \( P_C \) to a vertex of either \( (N(V_R) \cap V(C)) \setminus \{u,w\} \) or \( (V(C) \setminus V(P_C)) \cap V_R \). In both cases, using \( P \) as a chord path results in a new unbalanced cycle \( C' \) whose intersection with \( N(V_R) \) is strictly decreased.

The only remaining case is that \( V(C) \cap V_R \) forms a single path, and \( V(C) \cap N(V_R) \) consists of exactly two vertices (necessarily the attachment points of the path), and we are done. \( \square \)

Properties and arguments similar to this will be used extensively in the arguments concerning the behaviour of the LP.

### 2.1 The LP relaxation

We now define the LP-relaxation used for the **Rooted Biased Graph Cleaning**. The LP uses constraints we refer to as **balloons**; we will see that balloons can equivalently be thought of as pairs of paths rooted in \( v_0 \), or as a cycle connected to \( v_0 \) by a path.
Let $\Psi = (G = (V, E), B)$ be a biased graph and $v_0 \in V$ a distinguished vertex. Let $C$ be the corresponding class of unbalanced simple cycles. The Rooted Biased Graph Cleaning problem asks for a set $S \subseteq V$ with $v_0 \in S$ such that $G[S]$ is balanced and connected, and $|N(S)|$ is minimum. We consider the following LP-relaxation for it. The variables are $\{x_v : v \in V\}$, with $0 \leq x_v \leq 1$ and $x_{v_0} = 0$. For $C \subseteq S$, a $v_0$-$C$-path is a simple path $P = v_0 \ldots v_0$ where $v_0 \in S$ and $v_0 \notin C$ for $1 \leq i \leq \ell$. If $v_0 \in C$, then $P$ consists of the single vertex $v_0$ and no edges. We define the weights of $C$ and $P$ as $w(C) = \sum_{v \in C} x_v$ and $w(P) = \sum_{uv \in P, v \neq v_0} x_v + \frac{1}{2}x_{v_0}$, i.e., $w(P)$ assigns coefficient $\frac{1}{2}$ to the endpoint of $P$ and 1 to the internal vertices of $P$. A $(v_0)$-balloon is a pair $B = (P, C)$ where $C \subseteq S$ and $P$ is a $v_0$-$C$-path; the weight of $B$ is $w(B) = 2w(P) + w(C)$. We call the endpoint $v_\ell$ of $P$ the knot vertex of $B$. We let $V(B)$ (respectively $V(C)$, $V(P)$) denote the set of vertices occurring in $B$ (respectively in $C$, in $P$); hence $V(C) \cap V(P) = \{v_\ell\}$ is the knot vertex. The edges used in $B$, $E(B)$, is the set of edges required for $B$ to be a balloon, i.e., the edges $v_i v_{i+1}$ of $P$ and the edges of $C$. Note that this does not necessarily include all edges of $G[V(B)]$. The edges used in $P$ (in $C$) is defined correspondingly.

Having fixed $v_0$ and $C$ as above, we define a polytope $P$ by constraints

$$w(B) \geq 1 \text{ for every } v_0\text{-balloon } B = (P, C).$$

with $x_v \geq 0$ for every vertex $v$ and $x_{v_0} = 0$. We refer to this as the local LP.

Given an optimisation goal $\min c^T x$ for the above LP, the dual of the system asks to pack balloons, at weight 1 for every balloon, subject to every vertex $v$ having a capacity $c_v$, and a balloon $B = (P, C)$ using capacity from $v$ in proportion to the coefficient of $v$ in $w(B)$ (which is 2 if $v \in V(P)$, and 1 otherwise).

### 2.2 Balloons and path pairs

We now give some observations that will simplify the future arguments regarding the local LP, and in particular will allow us to change perspectives between viewing the constraints as pairs of paths, or as rooted unbalanced cycles.

Let $x : V \rightarrow [0, 1]$ be a fractional assignment. We first observe that our weights $w(P)$ and $w(C)$ for balloons $B = (P, C)$ can be recast as edge weights: For any edge $uv \in E$, we let the length of $uv$ under $x$ be $\ell_x(uv) = (x_u + x_v)/2$. For a path $P$, we let $\ell_x(P) = \sum_{uv \in E(P)} \ell_x(uv)$ be the length of $P$ under this metric, and similarly for simple cycles. We also define $z_x(v) = \min_P \ell_x(P)$ ranging over all $v_0$-$v$-paths $P$ as the distance to $v$ (under $x$, from $v_0$). This metric agrees well with the notion of the weight of a balloon that we use in the LP, as we will see. Note that the end points of a path $P$ contribute only half their weight to the length $\ell_x(P)$ of a path, as in $w(P)$.

Now observe that for any balloon $B = (P, C)$, and any vertex $v \in C$ other than the knot vertex, it is possible to form two paths $P_1, P_2$ from $v_0$ to $v$, such that $P_1 \cup P_2$ covers $B$, and such that the weight of the balloon equals $\ell_x(P_1) + \ell_x(P_2)$. Indeed, this sum gives coefficient 2 to every vertex of $P$, and coefficient 1 to every vertex of $V(C) \setminus V(P)$, including the vertex $v$. We refer to this as a path decomposition of $B$. We also observe two alternative decompositions.

**Lemma 2.** Let $B = (P, C)$ be a $v_0$-balloon. Each of the following is an equivalent decomposition of the weight of $B$.

1. $w(B) \geq \ell_x(C) + \min_{v \in V(C)} 2z_x(v)$, with equality achieved if $B$ is a min-weight $v_0$-balloon.
2. $w(B) = \ell_x(P_1) + \ell_x(P_2)$, for any decomposition of $B$ into two paths $P_1, P_2$ ending at a non-knot vertex $v \in V(C)$.
the prefix of two cases. See Figure 1 for an illustration.

Let Lemma 3.

Note that the first decomposition here implies that the knot vertex \( v \) of a min-weight balloon \( B \) will be chosen for minimum \( z_w \)-value.

In general, we view constraints as pairs of paths when deriving simple properties of the LP, but will need to revert to the view of biased graphs when arguing persistence in the next section.

2.3 Structure of min-weight balloons

We now work closer towards a separation oracle, by showing properties of balloons \( B \) with respect to an assignment \( x : V \to [0, 1] \), as this will help us finding the most violated constraint of an instance of the local LP.

Our first lemma is a structural result that will be independently useful in the next section.

Lemma 3. Let \( B = (P, C) \) be a min-weight balloon with respect to an assignment \( x : V \to [0, 1] \). Then for every vertex \( v \) in \( B \), \( B \) contains a shortest \( v_0-v \)-path under the metric \( \ell_x \).

Proof. Clearly, \( P \) must be a shortest path, by decomposition 1 of Lemma 2 hence the claim holds for every vertex of \( P \). Let \( v \in C \setminus P \), and let \( P_v \) be a shortest \( v_0-v \)-path such that both paths to \( v \) along \( B \) (as in decomposition 2) are longer than \( P_v \); let \( v \) be chosen with minimum \( z_w(v) \)-value, subject to these conditions. Choose \( P_v \) to maximize the length of the prefix of \( P_v \) that follows \( B \); in particular, \( P_v \cap P \) is a common prefix of \( P \) and \( P_v \). We may also assume that \( P_v \) “rejoins” \( B \) only exactly once, i.e., after \( P_v \) has followed its first edge not present in \( B \), then no further vertex of \( P_v \) other than \( v \) is contained in \( B \). Indeed, let \( w \) be the first vertex of \( B \) that \( P_v \) encounters after having departed from \( B \). Then by assumption the prefix of \( P_v \) up to \( w \) is shorter than both paths to \( w \) along \( B \), and instead of \( (v, P_v) \) above we may choose \( (w, P_w) \) where \( P_w \) is the aforementioned prefix of \( P_v \). Note that \( P_w \) now has all the properties assumed of \( P_v \). Let \( u \) denote the departure point of \( P_v \) from \( B \) (i.e., \( u \) is the last vertex of \( P_v \) such that the prefix of \( P_v \) from \( v_0 \) to \( u \) follows edges of \( B \)). Let \( v_k \) be the knot vertex of \( B \). We split into two cases. See Figure 1 for an illustration.

If \( u \) lies in \( P \), then we reason as follows. The two paths from \( u \) to \( v_k \) (in \( P \)) respectively \( v \) (in \( P_v \)) form a chord path for \( C \). Let \( C' \) be a reconfiguration of \( C \) by this path, and use \( P' = P \cap P_v \) as path to attach \( C' \) to \( v_0 \), defining a balloon \( B' = (P', C') \). Decompose \( B \) as \( P_1 + P_2 \), where \( P_1 \) and \( P_2 \) both end in \( v \), and similarly decompose \( B' \) as \( P'_1 + P'_2 \), ending in \( v \). Observe that
since \( u \) is the new knot vertex, for both possible choices of \( C' \) it holds that one of the paths \( P'_i \) will be identical to \( P_i \), while the other will be either \( P_1 \) or \( P_2 \). Since \( z(P_i) < z(P_1), z(P_2) \), the new balloon \( B' \) represents a constraint with a smaller value than \( B \).

Otherwise, the departure point \( u \) lies in \( C \setminus P \). The suffix of \( P \), from \( u \) to \( v \) forms a chord path; reconfiguring by this chord path leaves two options for the new cycle \( C' \), with \( C' \) either including \( v_k \) or excluding \( v_k \). If \( C' \) includes \( v_k \), then we may choose \( v_k \) as our new knot vertex. As a result, we may use the same argument as in the previous paragraph, noting that one of the two paths in the decomposition of \( B' \) was also present in the decomposition of \( B \), whereas the other path is shorter than both previous paths. Otherwise, finally, the knot vertex will be \( u \), and again one of the two paths in the decomposition of \( B' \) is present in the decomposition in \( B \), while the other is the new shortest path to \( v \).

We observe a particular consequence useful for finding min-weight balloons.

**Corollary 1.** Any minimum-weight balloon constraint \( B = (P, C) \) can be decomposed as \( w(B) = z_x(u) + z_x(v) + (x(u) + x(v))/2 \), for some \( u, v \in V \) with \( u v \in E \).

**Proof.** Let \( B = (P, C) \) be a minimum-weight balloon. Recall that \( |V(C)| \geq 3 \) since the input graph is simple (by Prop. 1). If there is a vertex \( u \in V(C) \setminus V(P) \) such that both paths to \( u \) in \( B \) are shortest paths under the metric \( \ell_x \), then the result follows. Indeed, decompose \( B \) into two paths \( P_1, P_2 \) ending at \( u \), and truncate \( P_2 \) by one edge to end at some vertex \( v \neq u \), where \( v \) is chosen not to be the knot vertex. Then this creates shortest paths to \( u \) and to \( v \), and by Lemma 2, decomposition \( \exists \) these paths and the edge \( u v \) form a decomposition of \( B \).

Otherwise, every vertex in \( V(C) \setminus V(P) \) has exactly one shortest path in \( B \) under \( \ell_x \), by Lemma 3. There is then (precisely) one edge \( u v \in E(C) \) that is not used by any shortest path in \( B \), and neither of \( u \) or \( v \) is the knot vertex. In this scenario, again, the shortest paths in \( B \) to \( u \) and to \( v \) and the edge \( u v \) form a decomposition of \( B \) as in Lemma 2 decomposition \( \exists \) and we are done.

### 2.4 The separation oracle

We now finish the results of this section by constructing a polynomial-time separation oracle over the local LP, assuming a membership oracle for the class \( B \). The result essentially follows from Corollary 1 by slightly perturbing edge lengths so that shortest paths are unique.

**Theorem 4.** Let \( \Psi = (G = (V, E), B) \) be a biased graph, and let \( v_0 \in V \) be the root vertex. Assume that we have access to a polynomial-time membership oracle for \( B \), that for every simple cycle \( C \) of \( G \) can inform us whether \( C \in B \) or not. Then there is a polynomial-time separation oracle for the local LP rooted in \( v_0 \).

**Proof.** Let \( x : V \rightarrow [0, 1] \) be a fractional assignment, where we want to decide whether \( x \) is feasible for the LP, i.e., we wish to decide whether there is a \( v_0 \)-balloon \( B \) such that \( w(B) < 1 \) under \( x \). Order the edges of \( G \) as \( E = \{ e_1, \ldots, e_m \} \). We will associate with each edge \( e_i \in E \) a tuple \( (\ell_x(e_i), 2^i) \), and modify our distance measure to work componentwise over these tuples, and order distances over such tuples in lexicographical order. This is to deterministically simulate the perturbation of all weights by a random infinitesimal amount. Note that all paths and cycles (in fact, all sets of edges) have unique lengths in this way. In particular, for every vertex \( u \) there is a unique shortest path \( P_u \) from \( v_0 \) to \( u \). Also observe that computations over these weights (and the computations of shortest paths) can be done in polynomial time.

Assume that there is a balloon \( B \) with \( w(B) < 1 \). Then the modified weight of \( B \) will be \((w(B), \alpha)\) for some \( \alpha \in \mathbb{N} \), where clearly \( w(B) < 1 \) still holds. Also note that Corollary 1 still
applies to the modified weights, since the proof consists only of additions and comparisons. Hence, if \( B \) is the (unique) min-weight balloon under the modified weights, then \( B \) can be decomposed into two shortest paths \( P_u, P_v \) and an edge \( uv \), so that \( w(B) = \ell_x(P_u) + \ell_x(P_v) + \ell_x(uv) = z_x(u) + z_x(v) + \ell_x(uv) \). Since shortest paths are unique, we can find the components \( P_u, P_v \) and \( uv \) defining \( B \), if starting from the edge \( uv \). From these paths, it is easy to reconstruct the cycle \( C \) and verify via an oracle query that \( C \notin B \). By iterating the procedure over all edges \( uv \in E \), we will find the min-weight balloon. \( \square \)

### 3 Half-integrality and persistence of the local LP

We now proceed to use the insights gained in the previous section to show that the local LP is in fact half-integral and obeys a strong persistence property. The approach of the proof is closely related to that of Guillemot for variants of Multiway Cut \[17\].

#### 3.1 Half-integrality

Let \( c : V \to \mathbb{Q} \) be arbitrary vertex weights, let \( x^* \) be an optimal solution to the above LP, and let \( y^* \) be an optimum solution to the dual. By complementary slackness, if \( y^*_B > 0 \) then \( w(B) = 1 \) under \( x^* \), and if \( x^*_B > 0 \) then the packing \( y^* \) saturates \( v \) to capacity \( c_v \). Let \( V_R \) denote the set of vertices reachable from \( v_0 \) at distance 0 (also implying that \( x^*_v = 0 \) for every \( v \in V_R \)). We let \( V_1 = \{ v \in V : x_v = 1 \} \) and \( V_{1/2} = N(V_R) \setminus V_1 \). We claim that \( \frac{1}{2}V_{1/2} + V_1 \) is a new LP-optimum. This will follow relatively easily from the “path pair perspective” on balloons.

First, we give a structural lemma.

**Lemma 4.** Let \( B = (P, C) \) be a balloon with \( w(B) = 1 \) with respect to the vertex weights \( x^* \). Then \( B \) is of one of the following types:

1. \( B \) is contained within \( G[V_R + v] \) for some vertex \( v \), where \( v \in V_1 \cap (V(C) \setminus V(P)) \);

2. \( B \cap N(V_R) \) is a single vertex \( v \in V_{1/2} \), and \( v \in V(P) \);

3. \( B \cap N(V_R) \) consists of two vertices \( u, v \) which cut \( C \) “in half” as follows: \( C \) consists of two \( uv \)-paths of which one is contained in \( V_R \) and contains at least one internal vertex \( v' \) which is the knot vertex of \( B \), and the other path is disjoint from \( V_R \).

**Proof.** Since \( x^* \) is a feasible solution for the local LP, any balloon with \( w(B) = 1 \) is a min-weight balloon. Thus by Corollary[11] we can decompose \( B \) as \( P_u + P_v + uv \) where \( P_u \) and \( P_v \) are shortest paths. In particular, both \( P_u \) and \( P_v \) contain a prefix in \( V_R \), make at most one visit to \( N(V_R) \), and proceed to subsequently not revisit \( N[V_R] \) at all. Also note that \( B \) necessarily intersects \( N(V_R) \), since \( B \) is a connected subgraph rooted in \( v_0 \) but not contained in \( V_R \).

First assume that \( B \) intersects a vertex of \( v_1 \in V_1 \). Then since \( w(B) = 1 \), only one of the paths \( P_u, P_v \) contains \( v_1 \), whereas the other path is entirely contained in \( V_R \). But then we must have \( v_1 \in \{ u, v \} \), as otherwise the edge \( uv \) cannot exist. We conclude that in this case, \( B \) is a balloon of type 1.

Next, assume that \( B \) intersects \( N(V_R) \) in a single vertex \( v' \in V_{1/2} \). We claim that \( B \) must intersect \( v' \) with a coefficient of 2: Indeed, if not, then one of the paths \( P_u \) and \( P_v \), say \( P_u \), must be entirely contained in \( V_R \), in which case \( v \in N(u) \) must be contained in \( N(V_R) \) and we are back in the previous case (contradicting \( v' \in V_{1/2} \)). Thus \( v' \in V(P) \), and \( B \) is a balloon of type 2.

Finally, assume that \( B \) intersects \( N(V_R) \) in at least two vertices. Then in fact \( B \) intersects \( N(V_R) \) in exactly two vertices \( u', v' \), by the properties of \( P_u \) and \( P_v \), and neither of these vertices...
are in $V_1$, since $w(B) = 1$. Furthermore, since each of $P_u$ and $P_v$ intersects $N(V_R)$ only once, these vertices $u'$, $v'$ lie after the common part of $P_u$ and $P_v$, i.e., in $V(C) \setminus V(P)$. Then indeed the knot vertex lies in $V_R$, whereas the path from $u'$ to $v'$ via $uv$ lies outside of $V_R$, as required.

We now show that complementary slackness implies that the new fractional assignment is actually an LP-optimum.

**Lemma 5.** The assignment $V_1 + \frac{1}{2}V_{1/2}$ is an LP-optimum for the local LP.

**Proof.** We first show that $V_1 + \frac{1}{2}V_{1/2}$ is a valid LP-solution. Assume towards a contradiction that $w(B) < 1$ for some balloon $B = (P, C)$ under the proposed weights. Note that $V_1 \cup V_{1/2}$ intersects every balloon since $G[V_R]$ is balanced. If $w(B) < 1$ we must thus have $B \cap (V_1 \cup V_{1/2}) = \{v\}$ for some $v \in V_{1/2}$, where $v \in (V(C) \setminus V(P))$. But then $V(B) \subseteq V_R \cup \{v\}$ (since otherwise $C$ passes the “border” $N(V_R)$ in at least two locations), which implies $x^*_v = 1$, contrary to assumptions.

We now show optimality. By complementary slackness, $y^*$ is a packing of balloons saturating every $v \in N(V_R)$ to its capacity $c_v$, with $w(B) = 1$ for every balloon $B$ in the support of $y^*$; hence $B$ will be of one of the three types of Lemma 4. For $i = 1, 2, 3$, let $B_i$ be the set of balloons of type $i$ from the support of $y^*$. By optimality, $c^T x^* = \sum_i y^*_B$. Note that a balloon of type 1 intersects one vertex in $V_1$ with coefficient 1 and no other vertex in $V_1 \cup V_{1/2}$, while a balloon of type 2 or 3 intersects one vertex in $V_{1/2}$ with coefficient 2, respectively two vertices in $V_{1/2}$ with coefficient 1 each, and no other vertex from $V_1 \cup V_{1/2}$. Since every vertex of $V_1 \cup V_{1/2}$ is in the support of $x^*$, these vertices are saturated by the packing $y_B$. For a balloon $B$ and vertex $v$, let $c(B, v) \in \{0, 1, 2\}$ be the coefficient of $v$ in the constraint $w(B) \geq 1$. We thus get

$$\sum_{v \in V_1} c_v = \sum_{v \in V_1} \sum_{B \in B_1} c(B, v) \cdot y^*_B = \sum_{B \in B_1} y^*_B \sum_{v \in V_1} c(B, v) = \sum_{B \in B_1} y^*_B$$

and

$$\sum_{v \in V_{1/2}} c_v = \sum_{v \in V_{1/2}} \sum_{B \in B_2 \cup B_3} c(B, v) \cdot y^*_B = \sum_{B \in B_2 \cup B_3} y^*_B \sum_{v \in V_{1/2}} c(B, v) = \sum_{B \in B_2 \cup B_3} 2y^*_B,$$

hence

$$c^T x^* = \sum_B y^*_B = c^T (V_1 + \frac{1}{2}V_{1/2})$$

which shows that $V_1 + \frac{1}{2}V_{1/2}$ is an LP-optimum. \qed

### 3.2 Persistence and the tiling property

Finally, we reach the statement concerning the persistence of the local LP. Since the statement is somewhat intricate, let us walk through it. First of all, the basic persistence property is similar to that used in Multiway Cut \[16, 17, 8\]. Let $V_R$ be defined as before, and for a solution $X \subseteq V$ to Rooted Biased Graph Cleaning (hence $v_0 \notin X$) define $S_X$ as the set of vertices of the connected component of $G - X$ containing $v_0$. Then persistence dictates that there is an optimal solution $X$ such that if $z_X(v) = 0$, i.e., if $v \in V_R$ then $v \in S_X$, and if $v \in V_1$ then $v \in X$. We note that both these properties hold for the computed set $S'$ in the below lemma.

However, the lemma also gives a useful tiling property, for the purposes of solving the global problem: Let $X$ be a solution to the full, global Biased Graph Cleaning problem, with $v_0 \notin X$, and let $S$ be the vertices reachable from $v_0$ in $G - X$. Then it “does not hurt” the global solution to assume that the induced solution $X \cap N(S)$ to the local problem also observes the persistence properties as above. This is implied by the closed-neighbourhood condition
Lemma 6. Let \( X \) be the global problem out of pieces computed for instances of the local problem. \( X \) in second picture) and \( S \) such that all vertices of \( S \) there is a set of vertices \( S \) and \( N \) closed region

Proof. Observe that \( S \) be an unbalanced cycle contained in \( G[S^+] \) respects persistence properties while not being more expensive than the original solution \( X \). See Figure 2 for an illustration of the modification of \( S \). This allows us to assemble a solution to the global problem out of pieces computed for instances of the local problem.

Lemma 6. Let \( x = V_1 + \frac{1}{2}V_{1/2} \) be the half-integral optimum from above, and let \( V_R \) be the corresponding reachable region. Let \( S \) be a balanced set with \( v_0 \in S \). Then we can grow the closed region \( N[S] \to N[S \cup V_R] \) without paying a larger cost for deleting vertices. More formally, there is a set of vertices \( S^+ \) and a set \( S' \subseteq S^+ \) such that \( G[S'] \) is balanced and the following hold.

1. \( S^+ = N[S \cup V_R] \);
2. \( N[S'] \subseteq S^+ \);
3. \( V_R \subseteq S' \);
4. \( V_1 \subseteq (S^+ \setminus S') \);
5. \( c(S^+ \setminus S') \leq c(N(S)) \).

Proof. Let \( U \) be the connected component of \( v_0 \) in \( G[S \cap V_R] \). We define the sets \( S^+ = N[S \cup V_R] \), and \( S' = V_R \cup (N(U) \cap V_{1/2} \cap S) \cup (S \setminus N[V_R]) \).

Observe that \( S' \subseteq S \cup V_R \) and that \( V_1 \subseteq N(S') \). We first show that \( G[S'] \) is balanced. Assume not, and let \( C \) be an unbalanced cycle contained in \( G[S'] \). We may assume that \( C \) is reduced as by Lemma 1 with respect to \( V_R \). Observe that \( C \) must intersect \( V_R \), as otherwise \( V(C) \subseteq S \) contradicting that \( G[S] \) is balanced. By Lemma 1 this intersection takes the form of a simple path \( P_{ab} \) connecting two vertices \( a, b \in N(V_R) \). Furthermore, we have \( a, b \in N(U) \cap V_{1/2} \cap S \), and the path \( P_{ab} \) intersects \( V_R \setminus S \), i.e., \( P_{ab} \) contains internal vertices not contained in \( U \). Let \( P_a \) respectively \( P_b \) be the prefix respectively suffix of \( P_{ab} \) contained in \( U \), if any, together with the vertices \( a \) respectively \( b \). Let \( P'_{ab} \) be a new chord path connecting \( P_a \) and \( P_b \) in \( U \), and let \( C' \) be a new cycle resulting from the reconfiguration of \( C \) by \( P'_{ab} \). Since \( C' \) cannot be contained
in $S$, this cycle must be formed from $P_{ab} + P_{ab}'$, and since $C'$ cannot be contained in $V_R$ it must still contain the vertices $a$ and $b$ (i.e., we had $P_a = a$ and $P_b = b$; recall that $a, b \in V_{1/2}$). But then $V(C') \cap V_R$ consists of two distinct paths; use a chord path $P''$ between these paths to reconfigure $C'$ into a new cycle $C''$. Then we may see that $|C'' \cap N(V_R)| = 1$, namely only of the vertices $\{a, b\}$, contradicting that $a, b \in V_{1/2}$. We conclude that $S'$ is balanced.

Items 1–4 in the lemma hold by definition or are easy, so it remains to show that $c(S^+ \setminus S') \leq c(N(S))$. Let us break down this expression. Writing $S^+ = V_R \cup S \cap N(V_R) \cup N(S)$, first note that $S^+ \setminus S' \subseteq N(V_R) \cup N(S)$ by definition of $S'$; more carefully,

$$S^+ \setminus S' = (N(S) \setminus S') \cup (N(V_R) \setminus S').$$

Vertices of $N(S) \setminus S'$ contribute equally to both sides of the inequality and can be ignored, hence we are left with vertices of $N(V_R) \setminus (S' \cup N(S))$ contributing to the left hand side and vertices of $N(S) \cap S' = N(S) \cap V_R$ contributing to the right hand side. Splitting $N(V_R) = V_1 \cup V_{1/2}$, the former set simplifies to $(V_1 \setminus N(S)) \cup (V_{1/2} \setminus (N(U) \cup N(S)))$. Relaxing slightly we define

$$Z := (V_1 \setminus N(S)) \cup (V_{1/2} \setminus N(U))$$

and

$$Y := N(U) \cap V_R;$$

it will suffice to show $c(Z) \leq c(Y)$. This will occupy the rest of the proof.

Let $y^*$ be the dual optimum, i.e., a fractional packing of balloons which saturates $v$ for every $v \in Z$, with each balloon $B$ in the support being of types 1–3 of Lemma 3.2 (by complementary slackness). Note that every vertex of $Z$ is in the support of $x$. Let $B_1$ contain the balloons from the support of $y^*$ which intersect $Z$ with a total coefficient of 1 (i.e., $B_1$ contains balloons of type 1, and balloons of type 3 which intersect $Z$ in only one vertex), and let $B_2$ contain those which intersect $Z$ with a total coefficient of 2 (i.e., balloons of type 2, and balloons of type 3 which intersect $Z$ in two vertices). Note that no balloon from the support of $y^*$ intersects $Z \subseteq N(V_R)$ with a total coefficient of more than 2. Then

$$c(Z) = \sum_{B \in B_1} y_B^* + \sum_{B \in B_2} 2y_B^*.$$

We need to show that every $B \in B_1$ intersects $Y$ with a total coefficient of at least 1, and every $B \in B_2$ intersects $Y$ with a total coefficient of at least 2. The inequality will follow.

First consider $v \in V_1 \cap Z$, and let $B \in B_1$ intersect $v$. Then $B$ is of type 1, hence contained in $G[V_1 + v]$. If $B - v \subseteq U$, then $B \cap N(S) = \{v\}$, but $v \in V_1 \cap Z$ implies $v \notin N(S)$; hence not all of $B - v$ is contained in $U$, and $B$ intersects $Y$.

Next, consider a vertex $v \in V_{1/2} \cap Z$ and a balloon $B \in B_1$ intersecting $v$ with coefficient 1; hence $B$ is of type 3. Since $v \in Z$ we have $v \notin N(U)$; since $B$ connects $v_0 \in U$ with $v \notin N(U)$ using internal vertices in $V_R$, there must exist some vertex $u \in B$ contained in $N(U) \cap V_R = Y$. Hence $B$ intersects $Y$.

Thirdly, consider some $v \in V_{1/2} \cap Z$ intersecting some $B \in B_2$ with a coefficient of 2. Again $v \notin N(U)$, and $B$ contains a path from $v_0$ to $v$ with internal vertices in $V_R$; furthermore every vertex on this path has coefficient 2 in $B$. Thus $B$ intersects $Y$ with a coefficient of at least 2.

Finally, consider a balloon $B \in B_2$ of type 3, intersecting two vertices $v, v' \in V_{1/2} \cap Z$. Then $B$ traces two paths from $v_0$ to $v, v'$, and either both these paths intersect $Y$ in a single vertex (which then has coefficient 2), or two distinct vertices of $Y$ intersect $B$ (for a total coefficient of 2).

This shows that every balloon $B$ in the support of $y^*$ intersects $Y$ with at least as large a total coefficient as it intersects $Z$. Since $y^*$ is a packing that saturates $Z$ and does not oversaturate any vertex, we have $c(Y) \geq \sum_B y_B^* + \sum_B 2y_B^* = c(Z)$ as promised. This finishes the proof.

\[\square\]
4 The FPT algorithms

We finally wrap up by giving our main results.

4.1 Rooted Biased Graph Cleaning

We use the results of the previous section to finalise Theorem 1. We proceed by lemmas. Throughout, we assume access to a membership oracle for the biased graph so that we can optimise the LP.

**Lemma 7.** Rooted Biased Graph Cleaning admits a 2-approximation, even for weighted graphs.

*Proof.* Let \( x^* \in [0,1]^V \) be an LP-optimum computed via Theorem 4. Compute the sets \( V_R, V_1 \) and \( V_{1/2} \) from \( x^* \) as in Section 3.1, forming a half-integral optimum \( x \). It is now clear that the set \( X = V_1 \cup V_{1/2} \) is an integral solution and a 2-approximation to the problem. \( \square \)

**Lemma 8.** Rooted Biased Graph Cleaning for unweighted graphs can be solved in \( O^*(4^k-\lambda) \) time, where \( \lambda \) is the optimum of the local LP.

*Proof.* Assume \( k < n \), as otherwise we may simply accept the instance. We will execute a branching process, repeatedly selecting half-integral vertices and recursively “forcing” their variables to take values \( x_v = 0 \) or \( x_v = 1 \). More precisely, we will use the terms \( \text{fix} v = 0 \) and \( \text{fix} v = 1 \) for the following procedures. To fix \( v = 0 \), we simply set \( c_v = 2n \); since the LP is half-integral in the presence of vertex weights, this implies that either \( x_v = 0 \) in a half-integral optimum or the LP-optimum costs at least \( n > k \). To fix \( v = 1 \), we similarly set \( c_v = 0 \). Let us make a quick observation about this procedure.

**Claim 1.** Consider a situation where we have fixed \( v = 0 \) for a set of vertices \( A_0 \), and fixed \( v = 1 \) for a set of vertices \( A_1 \), where \( A_0, A_1 \subseteq V \). Let \( \lambda \) be the resulting optimal value of the LP-relaxation. Then the following hold.

1. If there is a set \( S \subseteq V \setminus (A_0 \cup A_1) \) such that \( A_1 \cup S \) is a solution to Rooted Biased Graph Cleaning, then \( |S| \geq \lambda \).
2. If there is an integral solution to the LP of cost \( \lambda < n \), setting \( x_v = 1 \) for some set of vertices \( S \subseteq V \), then \( S \cap A_0 = \emptyset \), and \( S \cup A_1 \) is a solution to Rooted Biased Graph Cleaning of cardinality \( |A_1| + \lambda \).

*Proof of claim.* The only difference incurred by fixing vertices is a change of vertex weights \( c_v \). Therefore, in the first case, if there is a set \( S \) such that \( S \cup A_1 \) is a solution, then setting \( x_v = 1 \) for \( v \in S \cup A_1 \) is a feasible solution to the LP of cost \( |S| \), hence \( |S| \geq \lambda \). In the second case, as observed above, any half-integral solution of cost \( \lambda < n \) must set \( x_v = 0 \) for \( v \in A_0 \). Therefore, the statement follows since the set of solutions is upwards closed. \( \square \)

We now describe the branching process. Consider a generic situation, where some sets \( A_0 \) resp. \( A_1 \) of vertices have been fixed to \( v = 0 \) resp. \( v = 1 \), and where we have remaining budget \( k \) to spend on non-fixed vertices. Compute a half-integral optimum \( V_1 + \frac{1}{2} V_{1/2} \) for the local LP as above, and reject the instance if \( \lambda > k \). Adjust this optimum such that the corresponding set \( V_R \) is as large as possible, by repeatedly fixing \( v = 0 \) for non-fixed vertices \( v \in V_{1/2} \), and keeping \( v \) fixed if the cost of the LP does not increase. Assume first that this leaves \( V_{1/2} \) with only fixed vertices Then we must have \( V_{1/2} \subseteq A_1 \), since otherwise we would have \( \lambda > k \), and
the solution where \( x_v = 1 \) for every \( v \in V_1 \cup V_{1/2} \cup A_1 \) is an integral LP-optimum since fixed vertices have cost \( c_v = 0 \). In particular, we can compute the set \( V_R \) of reachable vertices around \( v_0 \) as previously, and \( X = N(V_R) \) will be a possibly smaller solution of at most \( k + |A_1| \) vertices.

Otherwise, fix \( v = 0 \) for every vertex \( v \in V_R \), fix \( v = 1 \) for every vertex \( v \in V_1 \), and reduce \( k \) by \( |V_1 \setminus A_1| \). As observed previously, since \( V_1 + \frac{1}{2}V_{1/2} \) is an LP-optimum of a persistent LP, these vertices can all be fixed without losing any integral optimum. Select an unfixed vertex \( v \in V_{1/2} \), which exists by assumption, and branch recursively on fixing \( v = 0 \) and fixing \( v = 1 \), in the latter branch reducing our budget \( k \) by one, and solve the problem recursively. Then in the branch \( v = 0 \), \( \lambda \) increases by at least \( 1/2 \), since the LP-relaxation has half-integral costs and fixing \( v = 0 \) increases the cost; and in the branch \( v = 1 \), \( \lambda \) decreases by \( 1/2 \) due to the change of vertex cost, whereas \( k \) decreases by at least \( 1 \). Hence in both branches the value of \( k - \lambda \) decreases by at least \( 1/2 \), which means that after a branching depth of at most \( 2(k - \lambda) \), each branch has either terminated or produced a solution. Thus the total running time is \( O^*(2^{2(k-\lambda)}) \) as promised.

\[ \square \]

**Lemma 9.** Rooted Biased Graph Cleaning for unweighted graphs can be solved in \( O^*(2^k) \) time.

**Proof.** If \( \lambda \leq k/2 \), then we may produce a solution of size at most \( k \) by rounding up the LP-optimum. Otherwise \( \lambda > k/2 \) and the result follows from Lemma 8 since \( 4^{k-\lambda} = 2^{2(k-\lambda)} < 2^{2(k-k/2)} = 2^k \).

This concludes the proof of Theorem 4.

### 4.2 Biased Graph Cleaning

We now finally show the full solution to Biased Graph Cleaning.

**Proof of Theorem 4.** Select an arbitrary vertex \( v_0 \in V \) and branch over two options: either delete \( v_0 \), or decide that \( v_0 \notin X \) and proceed to solve the local problem rooted in \( v_0 \). In the latter case, compute a half-integral optimum \( V_1 + \frac{1}{2}V_{1/2} \) of the local LP for which the set \( V_R \) of reachable vertices is maximal, as in Lemma 8. We claim that under the assumption that there is an optimum \( X \subseteq V \) to the global problem with \( v_0 \notin X \), there is such an optimum with \( V_1 \subseteq X \) and \( V_R \subseteq V(H) \) for some connected component \( H \) of \( G - X \).

For this, let \( Y \) be an optimum with \( v_0 \notin Y \), and let \( H \) be the connected component of \( G - Y \) for which \( v_0 \in V(H) \). Applying Lemma 8 to \( V(H) \) gives us the sets \( S^+ = N[V(H)] \cup V_R \) and \( S^+ \geq V_R \). Let \( Y' = (Y \setminus S^+) \cup (S^+ \setminus S') \). Since \( N[V(H)] \subseteq S^+ \) we have \( N(V(H)) \subseteq Y \cap S^+ \), and by Lemma 8 we have \( |N(V(H))| \geq |S^+ \setminus S'| \), hence \( |Y'| \leq |Y| \). We also have \( V_1 \subseteq Y' \), and \( G - Y' \) contains a connected component \( H' \) with \( V_R \subseteq V(H') \). We claim that \( Y' \) is a solution. Assume to the contrary that there is some unbalanced cycle \( C \) with \( V(C) \cap Y' = \emptyset \). Then \( C \) intersects \( Y \) in \( S^+ \setminus Y' = S' \). But since \( N(S') \subseteq Y' \) this contradicts that \( G[S'] \) is balanced. Hence \( Y' \) is also an optimal solution, and the claim is shown.

Hence, we may fix \( v = 0 \) for every \( v \in V_R \), and \( v = 1 \) for every \( v \in V_1 \), and proceed as in Lemma 8 until the vertices fixed to 0 contain a connected component containing \( v_0 \), surrounded entirely by vertices fixed to 1. In such a case, we simply proceed as above with a new starting vertex \( v_0 \) in a non-balanced connected component of \( G \), until we either exceed our budget \( k \) or discover an integral solution \( X \), and we are done. As in Lemma 8 while branching on a local LP the gap between lower bound and remaining budget decreases in both branches, whereas branching on a new vertex \( v_0 \) will certainly increase the solution cost, since the previous solution at this point does not account for any vertices in the connected component of \( v_0 \). Hence we...
have a tree with a branching factor of 2 and a height of at most 2k, implying a total size and running time of $O^*(4^k)$, and we are done.

4.3 Lower bounds

We show two lower bounds for Biased Graph Cleaning, one unconditional in the black-box oracle model, and one conditional on SETH (the strong exponential-time hypothesis). More concretely, let us say that an algorithm solves Biased Graph Cleaning in the black-box oracle model if it takes as input $(\Psi = (G = (V, E), B), k)$ where $\Psi$ is a biased graph and $B$ is provided purely in the form of a membership oracle. The result is simply the more carefully worked-out version of the lower bound informally announced in the introduction.

**Theorem 5.** Every algorithm that solves Biased Graph Cleaning in the black-box oracle model needs to make $\Omega(2^k)$ membership queries in the worst case, both in the edge- and vertex-deletion versions.

**Proof.** We show the result for the edge-deletion version. It is standard to reduce this to the vertex-deletion version by a suitable transformation.

Describe a graph $G_k$ as follows. Start with the simple cycle $C_k$, with vertex set $V = \{v_1, \ldots, v_k\}$ and edges $v_iv_{i+1}$, where $v_{k+1} = v_1$ (i.e., vertex numbers are read cyclically along $C_k$). Then replace every edge $uv$ in $C_k$ by a $C_4$, on vertices $u-x^1_{uv}-v-x^2_{uv}-u$, where $x^1_{uv}, x^2_{uv}$ are new vertices. Pick an arbitrary vector $b \in \{0, 1\}^k$. We next define a biased graph $\Psi_{b,k} = (G_k, B_b)$ where $B_b$ consists of the single cycle $C_b$ on vertex set $V(C_b) = \{v_1, \ldots, v_k\} \cup \{x^b_{v_i,v_{i+1}} : i \in [k]\}$, where again $v_{k+1} = v_1$. That is, precisely one out of all $2^k$ ways of traversing the original cycle $C_k$ is considered balanced. It is easy to see that $\Psi_{b,k}$ is a biased graph, since every unbalanced cycle $C$ and chord path $P$ for $C$ contains two distinct possible reconfigurations of $C$, one of which is distinct from $C_b$. We claim that $(\Psi_{b,k}, k)$ is positive for every vector $b$, but has an essentially unique solution. Indeed, since $G_k$ contains $k$ pairwise edge-disjoint unbalanced $C_4$’s, every deletion set of size $k$ must contain precisely one edge for each $C_4$, and this leaves precisely one cycle $C_{b'}$ spanning $C_k$, where $C_{b'}$ is defined as $C_b$ according to some vector $b' \in \{0, 1\}^k$. Since the choice of $b$ was arbitrary, the only way to detect in a black-box fashion which choice of deletions is feasible is to probe each such cycle $C_{b'}$ until the choice $b' = b$ is found.

Via known lower bounds for Unique $k$-SAT, we can also show a similar lower bound for explicitly represented instances assuming SETH (the Strong Exponential-Time Hypothesis) [11][19][6]. Recall that SETH is the hypothesis that for every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $k$-SAT cannot be solved in time $O^*(2^{(1-\varepsilon)n})$.

**Corollary 2.** Unless SETH is false, there is no algorithm that solves Biased Graph Cleaning in time $O^*((2-\varepsilon)^k)$ for any $\varepsilon > 0$, even if the class of balanced cycles of the input graph is provided through an explicit circuit.

**Proof.** Unique $k$-SAT is the promise problem where the input is a $k$-CNF formula $F$ with the promise that $F$ has at most one satisfying assignment, and the question is whether $F$ is satisfiable. Calabro et al. [11] show that under SETH, $k$-SAT and Unique $k$-SAT have the same asymptotic complexity. In other words, for every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that if Unique $k$-SAT can be solved in time $O^*(2^{(1-\varepsilon)n})$, then SETH is false.

Assume that Biased Graph Cleaning can be solved in time $O^*((2-\varepsilon)^k)$ for some $\varepsilon > 0$, and let $q \in \mathbb{N}$ be such that Unique $q$-SAT cannot be solved in $O^*(2^{(1-\varepsilon)n})$ time under SETH. Let $F$ be an instance of Unique $q$-SAT, and let $k$ be the number of variables in $F$. Fix an arbitrary ordering on the variables of $F$. Construct the graph $G_k$ as in Theorem 5 and let the
class of balanced cycles $\mathcal{B}$ be provided as a membership oracle, where a cycle $C_b$, $b \in \{0,1\}^k$ is balanced if and only the assignment $b$ satisfies $F$ (interpreted according to the ordering of variables of $F$). Then clearly, $\mathcal{B}$ can be provided as a polynomial-time membership oracle, and due to the promise property of UNIQue-qsAT the resulting class $\mathcal{B}$ is a balanced class of cycles. Furthermore, the instance $(\Psi = (G_k, \mathcal{B}), k)$ has a solution if and only if $F$ has a satisfying assignment.

Finally, regarding the special case of Group Feedback Vertex Set (GFVS), we note that Theorem 5 can easily be extended to give the same lower bound for GFVS when the group is represented in a black-box manner using the group $\mathbb{Z}_2^n$ (see Cygan et al. [7] for a discussion on black-box representations of GFVS instances). However, we are not aware of any lower bound for the running time of GROUP FEEDBACK VERTEX SET in any explicit group representation.

5 Polynomial time approximation

In this section, we give a polynomial time $O(\log k)$-approximation algorithm for the weighted version of BIASED GRAPH CLEANING, where $k$ denotes the cardinality of the optimal solution. The algorithm uses the standard region growing argument used for MULTICUT and SPARSEST CUT [26, 15], with a simple observation that leads to bound the approximation ratio by $O(\log k)$ instead of $O(\log n)$.

Given a weighted biased graph $\Psi = (G = (V, E), \mathcal{B})$ with weights $c : V \rightarrow \mathbb{Q}^+$ and the corresponding class of unbalanced simple cycles $\mathcal{C}$, we consider the following global LP relaxation that has $\{x_v\}_{v \in V}$ as variables.

\[
\begin{align*}
\text{minimize} & \quad \sum_{v \in V} c_v x_v \\
\text{subject to} & \quad \sum_{v \in C} x_v \geq 1 \quad \forall C \in \mathcal{C} \\
& \quad \sum_{v \in V} x_v \leq k \\
& \quad x_v \geq 0 \quad \forall v \in V
\end{align*}
\]

Lemma 10. Given access to a polynomial-time membership oracle for $\mathcal{B}$, the global LP relaxation admits a polynomial-time separation oracle.

Proof. By Theorem 3 there is a separation oracle for the balloon constraints of the local LP, for any root vertex $v_0 \in V(G)$. We reuse these as constraints in the global LP as follows. Let $uv \in E(G)$ be an arbitrary edge, and let $v_0$ be a new vertex subdividing $uv$ in $G$. Set $x_{v_0} = 0$. We now run the separation oracle for the local LP rooted in $v_0$. Repeating this for every edge of $G$ will yield our separation oracle for the global LP. The oracle is clearly polynomial time; we show correctness.

By Lemma 2 for every $v_0$-balloon $B$ involved in a constraint $w(B) \geq 1$, the balloon $B$ contains an unbalanced cycle $C$ and $w(B) \geq w(C) := \sum_{v \in C} x_v$. Therefore every constraint $w(B) \geq 1$ of the local LP rooted in $v_0$ is a valid constraint for the global LP. In the other direction, let $C$ be an unbalanced cycle such that $w(C) < 1$ and pick a root $v_0$ on an edge $uv$ of $C$. Then the balloon $B = (v_0, C)$ is a valid $v_0$-balloon, where $v_0$ represents the path at $v_0$ with no edges. Thus $w(B) = w(C) < 1$, and the local LP rooted at $v_0$ contains a violated constraint $w(B) \geq 1$. One such constraint will be found by the separation oracle. \qed
We define quantities similar to previous sections in a slightly different manner. For a path $P$, we let \( \ell_x(P) = \sum_{v \in P} x_v \). For \( u, v \in V \), we let \( z_x(u, v) = \min_P \ell_x(P) \) ranging over all \( u \)-\( v \)-path \( P \). Note that the current definition of \( \ell_x(P) \) differs from the previous definition by \( \frac{1}{8} (x_u + x_v) \), where \( u, v \) are the endpoints of \( P \). We omit subscripts \( x \) if they are clear from the context. The region around \( v \) of radius \( r \) is denoted by \( R_{v, r} \) and consists of the following two components.

- Interior \( I_{v, r} = \{ u \in V : z(u, v) < r \} \).
- Boundary \( \partial R_{v, r} = \{ u \in V : z(u, v) - x_u < r \leq z(u, v) \} \).

The LP constraints ensure that an interior of small radius does not have an unbalanced cycle.

**Lemma 11.** For any \( v \in V \) and \( r \leq 1/4 \), \( I_{v, r} \) does not have any unbalanced cycle.

**Proof.** Consider the alternative assignment \( x' \) where \( x'_u = x_u \) for every \( u \in I_{v, r} \) and \( x'_u = 1 \) otherwise. Assume that \( I_{v, r} \) contains an unbalanced cycle, and let \( C \) be a cycle of minimum weight. We claim that for any pair of vertices \( u, w \in C \), there is a shortest \( u \)-\( w \)-path (under the distance \( z_x(u, w) \)) that follows \( C \). Assume to the contrary and let \( u, w \in V \) be vertices such that both \( u \)-\( w \)-paths along \( C \) are longer than \( z_x(u, w) \), with \( u, w \) chosen to minimise \( z_x(u, w) \).

Let \( P \) be a shortest \( u \)-\( w \)-path. By the choice of \( u, w \), only the endpoints of \( P \) lie in \( C \), i.e., \( P \) is a chord path for \( C \). But then both possible results of reconfiguring \( C \) using \( P \) yield a cycle \( C' \) shorter than \( C \). This contradicts the choice of \( C \).

Now pick a vertex \( u \in C \), and let \( w_1, w_2 \in C \) be neighbours in \( C \) such that \( C \) decomposes into a shortest \( u \)-\( w_1 \)-path and a shortest \( u \)-\( w_2 \)-path. Observe that \( w_1, w_2 \) must exist. Then the weight of \( C \) is \( w(C) := \sum_{u \in C} x_u = z_x(u, w_1) + z_x(u, w_2) - x_u \). But since \( I_{v, r} \) has radius less than \( 1/4 \), all distances within \( I_{v, r} \) are less than \( 1/2 \). Thus \( w(C) \leq z_x(u, w_1) + z_x(u, w_2) < 1 \).

Since the weight of \( C \) is equal under \( x \) and \( x' \), the cycle \( C \) represents a constraint violated in the LP. \( \square \)

For \( U \subseteq V \), let \( x(U) = \sum_{u \in U} x_u \), \( c(U) = \sum_{u \in U} c_u \), \( LP(U) = \sum_{u \in U} c_u x_u \). Intuitively, a region \( R_{v, r} \) entirely contains a vertex \( u \in I_{v, r} \) in its interior, and for \( w \in \partial R_{v, r} \), only a \( \frac{r - (z(w, v) - x_w)}{x_w} \) fraction of it is contained in \( R_{v, r} \). To be consistent with this intuition, let

\[
x(R_{v, r}) = x(I_{v, r}) + \sum_{w \in \partial R_{v, r}} (r - (z(w, v) - x_w))
\]

\[
LP(R_{v, r}) = LP(I_{v, r}) + \sum_{w \in \partial R_{v, r}} c_w (r - (z(w, v) - x_w)).
\]

After obtaining the LP solution \( x \), the rounding algorithm proceeds as follows. The graph \( G = (V, E) \) is modified throughout the algorithm. Let \( \text{opt} = LP(V) \) be the initial LP value. Other definitions including \( R_{v, r} \) are with respect to the current graph. Note that for any \( U \subseteq V \), the LP solution \( x \) restricted to \( U \) is a feasible solution for \( G[U] \).

1. Choose an arbitrary vertex \( v \in V \).
2. Find the smallest \( r \geq 1/8 \) such that \( c(\partial R_{v, r}) \leq (16 \ln k) \cdot (LP(R_{v, r}) + \text{opt}/k) \).
3. Delete \( \partial R_{v, r} \), which will separate \( I_{v, r} \) from the rest of the graph. Let \( V \leftarrow V \setminus (I_{v, r} \cup \partial R_{v, r}) \).
4. Repeat from Step 1 as long as \( V \neq \emptyset \).

The standard analysis of the region growing process ensures the following fact.

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Lemma 12. In Step 2, above, the smallest $r$ is always at most $1/4$.

Proof. Let $y_r := LP(R,v,r) + \text{opt}/k$, and note that $c(\partial R,v,r) = \frac{dy_r}{dr}$. If the smallest $r$ is greater than $1/4$, $\frac{dy_r}{dr} = c(\partial R,v,r) > (16 \ln k)y_r$ for $r \in [1/8, 1/4]$, which implies

$$y_{1/4} > e^{(16 \ln k) - (1/8 - 1/4)} \cdot y_{1/8} = k^2 \cdot y_{1/8}.$$  

Since $y_{1/4} \leq \text{opt}(1 + 1/k)$ and $y_{1/8} \geq \text{opt}/k$, it leads to contradiction when $k \geq 2$. \hfill \Box

Therefore, each interior $I_{v,r}$ separated from the graph has no unbalanced cycle. Furthermore, notice that $x(R,v,r) \geq r \geq 1/8$ and all regions are disjoint, so the constraint $\sum_v x_v \leq k$ implies that the above algorithm runs in at most $8k$ iterations. In each iteration, the weight of deleted vertices is at most $(16 \ln k) \cdot (LP(R,v,r) + \text{opt}/k)$. Since the sum of the first terms is at most $\text{opt}$ and there are at most $8k$ iterations, the total weight of deleted vertices is at most

$$(16 \ln k) \cdot \left(\text{opt} + 8k \cdot \text{opt}/k\right) \leq O(\ln k) \cdot \text{opt},$$

giving an $O(\log k)$-approximation algorithm for Biased Graph Cleaning.

6 Conclusions

We have shown that the combinatorial notion of biased graphs, especially the notion of co-linear cycle classes, allows us to formulate an LP-branching FPT algorithm for a surprisingly broad class of problems, including the full generality of the Biased Graph Cleaning parameterized by $k$, and Rooted Biased Graph Cleaning parameterized by relaxation gap. Compared to previous results [20], these algorithms are somewhat more general, and significantly more grounded in combinatorial notions. We also showed that Biased Graph Cleaning admits an $O(\log k)$-approximation, where $k$ is the solution size.

Open problems include completely combinatorial FPT algorithms, and settling the associated kernelization questions.

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