κ-DEFORMED COVARIANT PHASE SPACE AND QUANTUM-GRAVITY UNCERTAINTY RELATIONS

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Abstract

We describe the deformed covariant phase space corresponding to the so-called κ-deformation of $D = 4$ relativistic symmetries, with quantum “time” coordinate and Heisenberg algebra obtained according to the Heisenberg double construction. The associated modified uncertainty relations are analyzed, and in particular it is shown that these relations are consistent with independent estimates of quantum-gravity limitations on the measurability of space-time distances.

1. Introduction

Recently there has been a lot of interest (see, e.g., [1-10]) in the possibility that classical ideas about space-time structure might fail to describe physics below some minimal length $l_{\text{min}}$. In general relativity the metric is a dynamical quantity, determined by the presence of matter and energy. Inserting in the classical Einstein equations the quantum-mechanical indeterminacy of energy the response of space-time implies that below the Planck length $l_{\text{min}} \approx 10^{-33} \text{cm}$ the classical concept of space-time is not applicable. In string theories, the appearance of minimal distances follows from the analysis of string collisions at Planckian energies, which are found to be characterized by the following modified uncertainty relation (see [2])

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha G \Delta p,$$

(1.1)

where $G = \frac{l_p^2}{\hbar}$ is the gravitational coupling (Newton) constant and $\alpha$ is a constant related with the string tension (Regge slope). The relation implies a lower bound on the measurability of distances

$$\min \Delta x \sim \sqrt{\hbar \alpha G}.$$

(1.2)
It is quite plausible that at very short distances some sort of quantization of space-time coordinates could provide an algebraic abstraction of this or similar measurability bounds. Similar ideas have been discussed since the late 1940s (see, *e.g.*, [11, 12]) but new possibilities appeared quite recently with the development of the theory of quantum groups and quantum deformations of Lie algebras (see, *e.g.*, [13-17]). The notion of quantum group as a Hopf algebra permits to consider deformed symmetries; in fact, the Hopf algebra axioms provide simultaneously an algebraic generalization of the definition of Lie group as well as of Lie algebra. The phase space containing the coordinate and momentum sectors can be described in the quantum-deformed case by the Heisenberg double construction (see, *e.g.*, [18, 19]). An important part of the definition of the Heisenberg double is the duality relation between quantum Lie groups and quantum Lie algebras, which generalizes the Fourier transform relation between the ordinary coordinate and momentum spaces.

The standard form of covariant fourdimensional Heisenberg commutation relations, describing quantum-mechanical covariant phase space looks as follows:

\[
[x_\mu, p_\nu] = i\hbar g_{\mu\nu}, \quad g_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \tag{1.3}
\]

Our aim here is to investigate the deformation of the phase space commutation relations (1.3) following from the so-called quantum $\kappa$-deformations [20-23] of relativistic symmetries. By using the algebraic scheme of Heisenberg doubles, we introduce the $\kappa$-deformation of covariant phase space relations \(^1\), and comment on some desirable features of such a deformation. In particular, we investigate the implications of this deformation for the measurability of space-time distances, and, generalizing the results reported recently in [10], we find overall consistency with earlier heuristic analyses of measurability bounds in quantum gravity. These findings provide support for the possibility that the covariant $\kappa$-deformed phase space give an effective description (at scales below the Planck scale) of certain quantum-gravity or string-theory effects [10].

### 2 Quantum Deformations of $D = 4$ Relativistic Phase Space

The space-time coordinates $x_\mu$ ($\mu = 0, 1, 2, 3$) can be identified with the translation sector of the Poincaré group, and the fourmomenta $p_\mu$ ($\mu = 0, 1, 2, 3$) are given by the translation generators of the Poincaré algebra. In considering quantum deformations of relativistic symmetries as describing the modification of space-time structure one is lead to the study of the possible quantum Poincaré groups. The classification of quantum deformations of $D = 4$ Poincaré groups in the framework of Hopf algebras was given by Podleś and Woronowicz ([26]; see also [27]) and provides the most general class of noncommutative space-time coordinates $\hat{x}_\mu$ allowed by the quantum-group formalism. One obtains the following algebraic relations (we put $\hbar = c = 1$)

\[
(R - 1)_{\mu\rho}^{\nu\tau}(\hat{x}_\rho \hat{x}_\tau + \frac{1}{\kappa} T_{\mu\rho} \hat{x}_\rho + \frac{1}{\kappa^2} C_{\mu\nu}) = 0, \tag{2.1}
\]

where

- the matrix $R$ satisfies the relation $R^2 = 1$ and describes the allowed class of quantum Lorentz groups;

- $T_{\mu\nu}^\rho$, $C_{\mu\nu}$ are numerical dimensionless tensors, satisfying suitable conditions [26].

\(^1\)The relations characterizing this deformation were previously available only in the conference reports [23, 24].
\( \kappa \) describes the fundamental mass parameter

If we assume that the quantum deformation does not affect the nonrelativistic kinematics, i.e. we preserve the nonrelativistic \( O(3) \) rotations classical, the only consistent class of noncommuting space-time coordinates is described by the relations of the \( \kappa \)-deformed Minkowski space (see [28, 23]), obtained from (2.1) by putting \( R = \tau \) (classical Lorentz symmetry) where \( \tau \) is a flip operator \( \tau (a \otimes b) = b \otimes a \), \( T^\rho_{\mu\nu} = \delta^0_\mu \delta^\rho_\nu - \delta^0_\nu \delta^\rho_\mu \) and \( C_{\mu\nu} = 0 \), i.e. \( (i = 1, 2, 3) \)

\[
\begin{align*}
[\hat{x}_0, \hat{x}_i] &= \frac{i}{\kappa} \hat{x}_i , \quad \text{(2.2a)} \\
[\hat{x}_i, \hat{x}_j] &= 0 . \quad \text{(2.2b)}
\end{align*}
\]

The relations (2.2a)–(2.2b) describe the translation sector of \( \kappa \)-deformed \( D = 4 \) Poincaré group, and in particular (2.2b) reflects the classical nature of the space coordinates.

We choose the basis for \( \kappa \)-Poincaré algebra, with the following \( \kappa \)-deformed Hopf algebra of fourmomentum \( P_\kappa (k = 1, 2, 3) \)

\[
\begin{align*}
[\hat{p}_0, \hat{p}_k] &= 0 \quad \text{(2.3a)} \\
\Delta(\hat{p}_0) &= \hat{p}_0 \otimes 1 + 1 \otimes \hat{p}_0 \\
\Delta(\hat{p}_k) &= \hat{p}_k \otimes 1 + e^{\frac{i\hbar}{\kappa}} \otimes \hat{p}_k \quad \text{(2.3b)}
\end{align*}
\]

and the antipode and counit is given by

\[
\begin{align*}
S(\hat{p}_\mu) &= -\hat{p}_\mu \quad \epsilon(\hat{p}_\mu) = 0, \quad \text{(2.4)}
\end{align*}
\]

where the fundamental constant \( c \) (light velocity) is inserted in (2.3b).

Using the duality relations with second fundamental constant \( \hbar \) (Planck’s constant)

\[
\langle \hat{x}_\mu, \hat{p}_\nu \rangle = -i\hbar g_{\mu\nu} \quad g_{\mu\nu} = (-1, 1, 1, 1) \quad \text{(2.5)}
\]

we obtain the noncommutative \( \kappa \)-deformed configuration space \( X_\kappa \) as a Hopf algebra with the following algebra and coalgebra structure

\[
\begin{align*}
[\hat{x}_0, \hat{x}_k] &= \frac{i\hbar}{\kappa c} \hat{x}_k , \quad \text{(2.6a)} \\
[\hat{x}_k, \hat{x}_l] &= 0 \\
\Delta(\hat{x}_\mu) &= \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu, \quad \text{(2.6b)} \\
S(\hat{x}_\mu) &= -\hat{x}_\mu \quad \epsilon(\hat{x}_\mu) = 0 \quad \text{(2.6c)}
\end{align*}
\]

The \( \kappa \)-deformed phase space can be considered as the Heisenberg double i.e a vector space \( X_\kappa \otimes P_\kappa \) with product

\[
(x \otimes p)(\bar{x} \otimes \bar{p}) = x(p_{(1)} \triangleright \bar{x}) \otimes p_{(2)} \bar{p} \quad \text{(2.7)}
\]

where left action is given by

\[
p \triangleright x = \langle p, x_{(2)} \rangle x_{(1)} \quad \text{(2.8)}
\]

The product (2.7) can be rewritten as the commutators between coordinates and momenta using the obvious isomorphism \( x \sim x \otimes 1, p \sim 1 \otimes p \), which applied to the case of \( \kappa \)-Poincaré algebra provides the following relations

\[
\begin{align*}
[\hat{x}_k, \hat{p}_l] &= i\hbar \delta_{kl} , \quad [\hat{x}_k, \hat{p}_0] = 0 , \quad [\hat{x}_0, \hat{p}_k] = -i\hbar \delta_{kl} , \quad \text{and} \quad [\hat{x}_0, \hat{p}_0] = -i\hbar . \quad \text{(2.9)}
\end{align*}
\]
The set of relations (2.2a)-(2.4) and (2.9) describes the $\kappa$-deformed relativistic quantum phase space.

From these relations follow the modified covariant Heisenberg uncertainty relations. Introducing the dispersion of the observable $a$ in quantum mechanical sense by

$$\Delta(a) = \sqrt{\langle a^2 \rangle - \langle a \rangle^2}$$

we have

$$\Delta(a)\Delta(b) \geq \frac{1}{2} |\langle c \rangle| \quad \text{where} \quad c = [a, b] \quad (2.11)$$

We obtain $\kappa$-deformed uncertainty relations

$$\Delta(\hat{t})\Delta(\hat{x}_k) \geq \frac{\hbar}{2\kappa c^2} |\langle \hat{x}_k \rangle| \quad (2.12a)$$
$$\Delta(\hat{p}_k)\Delta(\hat{x}_l) \geq \frac{1}{2}\hbar\delta_{kl} \quad (2.12b)$$
$$\Delta(\hat{E})\Delta(\hat{t}) \geq \frac{\hbar}{2}\frac{1}{c} \quad (2.12c)$$
$$\Delta(\hat{p}_k)\Delta(\hat{t}) \geq \frac{\hbar}{2\kappa c^2} |\langle \hat{p}_k \rangle| = \frac{1}{2}\frac{1}{\kappa c^2} (1 - \frac{P_0^2}{2\kappa c} + O(\frac{1}{\kappa^2})) \quad (2.12d)$$

where $l_\kappa = \frac{\hbar}{\kappa c}$ describes the fundamental length at which the time variable should already be considered noncommutative. In the recent estimates $\kappa > 10^{12} GeV$ (see e.g. [30]) i.e. $l_\kappa < 10^{-26}\text{cm}$; in particular one can put $\kappa$ equal to the Planck mass what implies that $l_\kappa = l_p \simeq 10^{-33}\text{cm}$.

The relations (2.3a)-(2.3b) describe the fourmomentum sector of the $\kappa$-deformed Poincaré algebra, written in the bicrossproduct basis [22, 23] with classical Lorentz algebra.

$$[M_{\mu\nu}, M_{\rho\tau}] = i(g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau}) \quad (2.13a)$$

and the following deformed covariance relations ($M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$, $N_i = M_{i0}$)

$$[M_i, P_j] = i\epsilon_{ijk}P_k \quad [M_i, P_0] = 0 \quad (2.13b)$$
$$[N_i, P_j] = -i\delta_{ij} \left[\frac{\kappa c}{2}(1 - e^{2\kappa cP_0}) + \frac{1}{2\kappa c}(\bar{P}^2)\right] + i\frac{\kappa c}{2}P_i P_j \quad \text{and} \quad [N_i, P_0] = iP_i, \quad (2.13b)$$

where the translation generators $P_\mu$ should be identified with the $\kappa$-relativistic phase space coordinates $\hat{p}_\mu$.

The $\kappa$-deformed mass Casimir takes the form

$$C_2^{\kappa} = \frac{1}{c^2} \bar{P}^2 e^{-\frac{P_0}{\kappa c}} - (2\kappa \sinh \frac{P_0}{2\kappa c})^2 = -M^2, \quad (2.14)$$

where $M$ denotes the $\kappa$-invariant mass parameter. In particular for $M = 0$ ($\kappa$-deformed photons) from (2.14) one obtains that

$$P_0 = \kappa c \ln(1 + \frac{\bar{P}}{\kappa c}) = |\bar{P}| - \frac{|\bar{P}|^2}{2\kappa c} + O(\frac{1}{\kappa^2}) \quad (2.15)$$

and in particular the velocity formula for massless $\kappa$-deformed quanta looks as follows ($E = cP_0$)

$$V_i = c \frac{\partial P_0}{\partial P_i} = c \frac{P_i}{1 + |\bar{P}|^2/|\bar{P}|} \quad (2.16a)$$
This three-momentum-dependent (i.e. energy-dependent) “speed of light” is a completely novel phenomenon that arises in the formalism here considered. Interestingly, it has the same functional form (upon appropriate identification between $\kappa$ and the string scale) as the energy-dependent speed of light recently discussed [9] in the non-critical (“Liouville”) string literature. Both in the $\kappa$-Poincaré and in the string theory contexts the deviation from ordinary physics, while very significant at the conceptual level, is rather marginal from the phenomenological viewpoint. For example, for photons of energies of order 1 GeV the Eq. (2.16b) entails a minuscule $10^{-19}c$ correction with respect to the ordinary scenario with constant speed of light. As discussed in greater detail in [9], at least when $\kappa$ is identified with the Planck scale, the Eq.(2.16b) is completely consistent with available experimental data. As manifest in the relations (2.12a)-(2.12d), the $\kappa$-modifications of the covariant Heisenberg commutations relations are of quantum mechanical nature, i.e. proportional to the Planck constant $\hbar$. This suggests that the $\kappa$-deformation (together with its exotic energy-dependent speed of light) can be related with the quantum corrections to the classical dynamics of space-time.

3 Measurement process and covariant $\kappa$-deformed phase space

Recently one of the present authors has presented (see [8]) heuristic quantum-gravity arguments (based on combining quantum mechanics with general relativity) indicating that the measurability of distances may be bound by a quantity that grows with the time required by the measurement procedure, as needed for the decoherence mechanism discussed in [4]. Specifically, by observing that gravitational effects prevent one from relying on the availability of classical agents for the measurement procedure (since the limit of infinite mass leads to inconsistencies associated with the formation of horizons), the following bound is found for the measurability of a distance $L$ [8]:

$$\min [\Delta L] \sim l_p \sqrt{sT/s} \sim l_p \sqrt{L/s}, \quad (3.1)$$

where $l_p$ is the Planck length, $s$ is a length scale characterizing the spatial extension of the devices (e.g., clocks) used in the measurement, $T$ is the time needed to complete the procedure of measuring $L$, and on the right-hand-side we used the fact that typically $T \sim L$. Further analyses of this type of measurability bound have relied again on the gravitational effects associated to macroscopic devices or on the study of the light probes exchanged during the measurement; in particular, the dynamics of the light probes was shown to lead to a measurability bound of type (3.1) in the framework of Liouville noncritical string theories with the target time identified with the Liouville mode [9].

In this section we analyze the measurement of the distance $L$ between two bodies as it results from a plausible physical interpretation of the uncertainty relations (2.12a)-(2.12d). Like the related studies [4, 8, 9] we consider the procedure of measurement of distances set out by Wigner [29, 30], which relies on the exchange of a light probe/signal between the bodies. The distance is therefore measured as $L = cT/2$, where $T$ is the time spent by the probe to go from one body to the other and return. In general the quantum mechanical nature of the probe introduces uncertainties in the measurement.
of $L$, and in particular one finds that \[ \Delta L \geq \Delta x + c \Delta t + T \Delta v \] (3.2)

where $\Delta x$ and $\Delta t$ are the uncertainties on the space-time position of the probe at time $T$, while $\Delta v$ is the uncertainty on the velocity of the probe.

The analysis of these uncertainties within ordinary quantum mechanics would be trivial since $\Delta x$, $\Delta t$ and $\Delta v$ are not correlated in that framework. The $\kappa$ deformation induces such correlations. In particular, concerning the correlation between $\Delta x$ and $\Delta t$ we observe that (2.12a) implies (interpreting the $x$ on the right-hand-side of (1.3) as the distance travelled by the probe)

\[ \Delta t \geq \frac{\hbar L}{2 \kappa c^2 \Delta x}. \] (3.3)

Moreover, if the probe is massless with modified velocity (2.16b) one finds that

\[ |\Delta v| \sim \frac{\Delta P}{\kappa} \sim \frac{\hbar}{2 \kappa \Delta x}, \] (3.4)

where on the right-hand-side we used (2.12b).

Using (3.3) and (3.4) one can rewrite (3.2) as

\[ \Delta L \geq \Delta x + \frac{\hbar L}{2 \kappa c \Delta x} + \frac{\hbar T}{2 \kappa \Delta x}. \] (3.5)

Taking into account that $L = c T/2$ one finds that the minimal value of $\Delta L$ is obtained if $(\Delta x)^2 = \frac{\hbar L}{4 \kappa}$ and this implies that the minimal uncertainty in the measurement of the distance $L$ is

\[ \min[\Delta L] \sim \sqrt{\frac{\hbar L}{\kappa c}}. \] (3.6)

Remarkably this result reproduces (up to appropriate mapping between the scale $\kappa$ and the scales $l_p$ and $s$) the relation (3.1) that was derived within a completely independent analysis of quantum-gravity effects. The preliminary investigation reported in [10], which considered only the $\kappa$-deformed Minkowski coordinates sector, had raised the possibility that this might be the case; however, it is quite non-trivial that the present analysis taking into account the structure of the full covariantly $\kappa$-deformed phase space ultimately leads to (3.6). The form of the $\kappa$-deformation advocated here plays a rather central role in obtaining this result; in fact, the relation (2.16) ensures that the third term on the right-hand side of Eq. (3.2) (which was not considered in [10]) is of the same order (once the uncertainty relations are taken into account) as the first term.

The deformation of phase space here considered has also implications for the analysis of macroscopic devices in the measurement, but this does not lead to additional bounds to the measurability of $L$. To illustrate this, let us consider for example the clock used in the measurement. The uncertainties in the time indicated by the clock,2

\footnotetext[2]{Of course there are other contributions to $\Delta L$ (e.g., coming from the quantum mechanical nature of the other devices used in the experiment [8]); however, since they obviously contribute additively to the total uncertainty in the measurement of $L$, these uncertainties could only make stronger the bound derived in the following.}

\footnotetext[3]{As implicit in the terminology here adopted, the Wigner measurement procedure is essentially one-dimensional, and the only relevant spatial coordinate is the one along the axis passing through the bodies whose distance is being measured.
the position of the clock, and the velocity of the clock affect the measurement of $L$
according to
\[ [\Delta L]_{\text{clock}} \sim \Delta x_{\text{clock}} + v_{\text{clock}} \Delta t_{\text{clock}} + T \Delta v_{\text{clock}}. \]
(3.7)

Again the $\kappa$ deformation induces correlations between $\Delta x_{\text{clock}}, \Delta t_{\text{clock}}, \Delta v_{\text{clock}}$; in fact, using the $\kappa$-deformed uncertainty relations (2.12a)-(2.12d) (and considering an ideal non-relativistic clock with $p = Mv$) one finds that
\[ \Delta v_{\text{clock}} \geq \frac{\hbar |v_{\text{clock}}|}{2\kappa c^2 \Delta t_{\text{clock}}} \]
(3.8)
and
\[ \Delta x_{\text{clock}} \geq \frac{\hbar v_{\text{clock}} T}{2\kappa c^2 \Delta t_{\text{clock}}} \]
(3.9)

Therefore the contribution from the quantum-mechanical uncertainty in the kinematics of the clock satisfies
\[ [\Delta L]_{\text{clock}} \geq \frac{\hbar v_{\text{clock}} T}{2\kappa c^2 \Delta t_{\text{clock}}} + v_{\text{clock}} \Delta t_{\text{clock}} + \frac{\hbar v_{\text{clock}} T}{2\kappa c^2 \Delta t_{\text{clock}}} \]
(3.10)

For a given $v_{\text{clock}}$ one finds
\[ \min[\Delta L]_{\text{clock}} \sim v_{\text{clock}} \sqrt{\frac{\hbar T}{\kappa c^2}} = v_{\text{clock}} \sqrt{\frac{l_\kappa}{c T}}, \]
(3.11)

whereas, of course, in ordinary quantum mechanics is $\min[\Delta L]_{\text{clock}} = 0$. However, Eq. (3.11) does not lead to a genuine measurability bound, since the observer would naturally prepare the system with $v_{\text{clock}} = 0$.

It appears therefore that the covariant $\kappa$-deformation of phase space leads to a measurability bound in agreement with some of the heuristic quantum-gravity measurement analyses, and that this bound emerges from the nature of the kinematics of the light probe, whereas the kinematics of macroscopic devices does not contribute to the bound.

4 Closing Remarks

The covariant $\kappa$-deformed relativistic symmetries here considered, and the associated covariant $\kappa$-deformation of the Heisenberg algebra (2.9), has several appealing properties as a candidate for the high-energy modification of classical relativistic symmetries. As a dimensionful deformation it is relevant only to the description of processes characterized by energies of order $\kappa$ or higher. In addition, in an appropriate sense, it provides a rather moderate (at least in comparison with some of its alternatives) deformation of classical relativistic symmetries, which in particular reflects the reasonable expectation that, if any of the space-time coordinates is to be special, the special coordinate should be time. (Interestingly this intuition appears to be also realized in certain approaches to string theory, see e.g. [32].)

In conclusion, the fact that we have provided some evidence that the bounds on the measurability of distances associated with the uncertainty relations characterizing the $\kappa$-deformed covariant Heisenberg algebra (2.2a)-(2.2b) and (2.9) are consistent with independent analyses of such measurability bounds in quantum gravity suggests that the $\kappa$-deformation might provide an effective description at ultra-short distances (perhaps just above the Planck length) of certain quantum-gravity effects.
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