A precise calculation of the fundamental string tension in $SU(N)$ gauge theories in $2+1$ dimensions

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Abstract

We use lattice techniques to calculate the continuum string tensions of $SU(N)$ gauge theories in $2+1$ dimensions. We attempt to control all systematic errors at a level that allows us to perform a precise test of the analytic prediction of Karabali, Kim and Nair. We find that their prediction is within 3% of our values for all $N$ and that the discrepancy decreases with increasing $N$. When we extrapolate our results to $N = \infty$ we find that there remains a discrepancy of $\simeq 1\%$, which is a convincing $\sim 6\sigma$ effect. Thus, while the Karabali-Nair analysis is remarkably accurate at $N = \infty$, it is not exact.

PACS numbers:
I. INTRODUCTION

Finding an analytic solution to confinement and the mass spectrum of four dimensional $SU(N)$ Yang-Mills gauge theories continues to be a challenge for theoretical physics. It is in the large-$N$ limit \cite{1} that the relation between gauge field theories and string theory is most natural \cite{2}, and this relation is strengthened by the AdS/CFT correspondence. Indeed a solution to the problem of confinement may shed light on string theories with moderately strong coupling.

In lower dimensions, gauge theories are of interest for the same theoretical reasons. In $D = 1 + 1$ all dynamical degrees of freedom can be removed by gauge fixing, and the theory can be analytically solved, with linear confinement arising trivially from the Coulomb potential. Moving up one dimension to $D = 2 + 1$, makes the theory much more complicated. Gluons become dynamical, a linearly confining potential appears to be dynamically generated, and the theory appears to be as analytically intractable as in the four dimensional case. This is unfortunate since the $D = 2 + 1$ theory is also phenomenologically interesting: through dimensional reduction, it approximates the high temperature limit of the four-dimensional theory. Indeed, an analytic solution in $D = 2 + 1$ dimensions would be a significant step forward, and would perhaps bring us closer to a solution of the $D = 3 + 1$ case.

While an exact solution to $D = 2 + 1$ gauge theories is not available, Karabali and Nair have claimed to obtain a very good approximation through studying the continuum Hamiltonian, and expressing it in terms of color-singlet fields \cite{4}. Truncating the Schrödinger equation for their ground state functional, they obtained the following prediction for the tension $\sigma$ of the string that binds distant static sources in the fundamental representation:

$$\frac{\sqrt{\sigma}}{g^2 N} = \sqrt{\frac{1 - 1/N^2}{8\pi}}. \quad (1.1)$$

Remarkably, this prediction turns out to be within $\sim 3\%$ of the lattice calculated values \cite{5, 6} for all values of $N$.

If one replaces these sources with ones in a general representation, $R$, then the analysis of \cite{4} predicts that the tension $\sigma_R$ of the string between these sources, obeys

$$\sigma_R = \sigma C_R, \quad (1.2)$$

where $C_R$ is the quadratic Casimir of the representation $R$. In Eq. (1.2) there are no restrictions on $R$, and so this clearly cannot be exact in general: in $D = 2 + 1$ gluons screen color charge and will, for example, break the string if $R$ has trivial $N$-ality. However in the limit $N \to \infty$ screening vanishes and, moreover, the lattice calculations \cite{5, 6} show a decreasing discrepancy with Eq. (1.1) as $N$ increases. Indeed the extrapolated $N = \infty$ lattice value is only about $\sim 1\%$ below the prediction of Eq. (1.1). At this level of accuracy there are a number of systematic errors that were inadequately controlled in these lattice calculations, and so this leaves open the tantalising possibility that the calculation of \cite{4} might become exact at $N = \infty$. The purpose of the present paper is to provide a lattice calculation in which all the systematic errors are controlled at a level that allows us to test this possibility.

One of these systematic errors has to do with corrections to the string energy as a function of length. This is an interesting theoretical subject of its own, having to do with the nature of the effective string theory that describes the confining flux tube. We expand in detail on
this issue in a companion publication [7] and here will only quote a few relevant results. A related issue is how the string tension varies as a function of the representation of the flux. We extend a similar control of systematic errors to these calculations in a second companion paper [8] where we test how well the Casimir scaling in Eq. (1.2) is satisfied – in particular by stable $k$-strings.

In the next section we describe the basic lattice setup. We then describe the methodology used in the calculation of the string tension, what are the important systematic errors, and how we control them. We then provide our results, extrapolate them to the continuum limit, and then extrapolate to $N = \infty$. We finish with our conclusions.

Some of our preliminary results, both from this paper and from [7], have been presented in [9].

II. LATTICE SETUP

The theory is defined on a discretized periodic Euclidean three dimensional space-time lattice, with spacing $a$ and, typically, with $L_s^2 L_t$ sites. The Euclidean path integral is given by

$$ Z = \int DU \exp (-\beta S_W), \quad (2.1) $$

where $\beta$ is the dimensionless lattice coupling, and is related to the dimensionful coupling $g^2$ by

$$ \lim_{a \to 0} \beta = \frac{2N}{ag^2}. \quad (2.2) $$

In the large-$N$ limit, the ’t Hooft coupling $\lambda = g^2 N$ is kept fixed, and so we must scale $\beta = 2N^2/\lambda \propto N^2$ in order to keep the lattice spacing fixed (up to $O(1/N^2)$ corrections). The action we choose to use is the standard Wilson action

$$ S_W = \sum_P \left[ 1 - \frac{1}{N} \text{ReTr} U_P \right], \quad (2.3) $$

where $P$ is a lattice plaquette index, and $U_P$ is the plaquette variable obtained by multiplying link variables along the circumference of a fundamental plaquette. We calculate observables by performing Monte-Carlo simulation of Eq. (2.1), in which we use a mixture of Kennedy-Pendelton heat bath and over-relaxation steps for all the $SU(2)$ subgroups of $SU(N)$.

III. METHODOLOGY

To obtain the string tension, $\sigma$, we calculate the energy of the lightest flux tube that winds around one of the spatial tori. Extracting the mass from the correlation function is the first area in which we need to control systematic errors, as described below.

In the confining phase such a winding flux tube cannot break. Since there are no sources here (in contrast to Wilson loop calculations of the static potential) there are no extraneous contributions to the energy (such as the Coulomb potential) and one can hope to obtain a string description for all lengths $l$ of the flux tube. There is of course a smallest possible length $l_c = 1/T_c$, where $T_c$ is the deconfining temperature, below which there are no winding flux tubes. However for $N \geq 4$ the transition is first order [10] and so for larger $N$ we can hope
to have a string description for any \( l \geq l_c \). Such a description should become particularly simple at \( N = \infty \) where mixing and decay vanish. We have performed a careful study of the way the string energy depends on its length, which will be published elsewhere \[7\]. Here we use those results to bound the theoretical uncertainties in extracting the asymptotic string tension from the string energy, so controlling the associated systematic errors.

There are also systematic errors in extrapolating to the continuum \( a = 0 \) limit and, subsequently, to the \( N = \infty \) limit. These will be discussed below as well.

### A. Extracting string masses from correlation functions

We calculate the energy of the winding flux tube from the correlator of suitably smeared \( \vec{P} = 0 \) Polyakov loops that wind around a spatial torus. This is a standard technique \[5, 6\] with the smearing/blocking designed to enhance the projection of our operators onto the ground states. (We use a scheme that is the obvious dimensional reduction of the one in \[11\].) We calculate with several blocking levels and construct the full correlation matrix. From this we obtain best estimates for the ground and excited string states using a variational method applied to the transfer matrix \( \hat{T} = e^{-aH} \) – again a standard technique \[5, 6, 11, 12\].

In practice, our best operator for the string ground state has an overlap \( \sim 99\% \) onto that state so that the normalised ‘ground state’ correlation function satisfies

\[
C(t) = (1 - |\epsilon|) \exp\{-M_0(l)t\} + |\epsilon_1| \exp\{-M_1(l)t\} + \ldots \quad \sum_i |\epsilon_i| = |\epsilon| \sim 0.01 \tag{3.1}
\]

where \( M_0, M_1 \) are the ground and first excited state string energies. (Since our time-torus is finite, we use \( \cosh \) fits rather than simple exponentials, although in practice we use \( L_t \) large enough for any contributions around the ‘back’ of the torus to be negligible.) To extract \( M_0 \) from this correlator one can fit with a single exponential for \( t \geq t_0 \), discarding the lower \( t \) values so that a statistically acceptable fit is obtained. This is a reasonable approach and one followed in \[5, 6\]. However it neglects the systematic error arising from the fact that there is certainly some excited state contribution as demonstrated, for example, by the fact that one cannot obtain a good fit with a single exponential from \( t = 0 \). To control this systematic error we also perform fits with two exponentials, with a fixed mass \( M^* \) for the excited state, resulting in a mass \( M_0(M^*) \) for the ground state. Typically \( M_0(M^*) \) is smallest when \( M^* \) is as small as possible, i.e. \( M^* = M_1 \), and is largest when \( M^* = \infty \), i.e. effectively a single-exponential fit. So the true value typically satisfies:

\[
M_0(M_1) \leq M_0^{\text{true}} \leq M_0(\infty). \tag{3.2}
\]

From here on, we refer to the single-cosh fitting procedure by ‘\( S \)’, and to the double-cosh fitting procedure, by ‘\( D \)’, and add these as superscripts to any relevant results (such as masses, string tensions, etc.). Consequently we shall have two continuum string tensions \( \sigma^S \), and \( \sigma^D \), that bracket the true string tension

\[
\sigma^D \leq \sigma^{\text{true}} \leq \sigma^S. \tag{3.3}
\]
B. Extracting string tensions from string energies

From the ground state string energy, \( M_0(l) \), we need to extract the tension \( \sigma \). Taking into account the Lüscher term [13], \( \sigma \) is given by

\[
\sigma = \frac{M_0(l)}{l} + \frac{\pi}{6l^2} + O\left(\frac{1}{l^4}\right). \tag{3.4}
\]

In practice using \( \sqrt{\sigma l} \geq 3 \) one can expect the neglected \( O(1/l^4) \) corrections to be small. However, they represent another systematic error that needs to be controlled. To do this we shall use the results of our study in [7], where we have calculated \( M_0(l) \) as a function of \( l \) for \( N = 3, 4, 6, 8 \). We have done so in the range \( 1.3 - 1.6 \lesssim l \sqrt{\sigma} \lesssim 3 - 6.2 \) and for different lattice spacings. We find that our results can be well encompassed by

\[
\left(\frac{M_0(l)}{\sigma l}\right)^2 = 1 - \frac{\pi}{3(\sqrt{\sigma l})^2} - \frac{0.2(1)}{\{\sqrt{\sigma l}\}^5}. \tag{3.5}
\]

Here the first two terms on the right hand side constitute the exact Nambu-Goto string prediction. It is believed that when one expands \( M_0(l)/\sigma l \) in inverse powers of \( \sqrt{\sigma l} \) the first two corrections are universal [13, 14] and equal to those for the Nambu-Goto string. (This is consistent with our numerical calculations in [7].) The next possible correction term corresponds to the last term on the right of Eq. (3.5). We observe that the fitted coefficient is much smaller than the \( O(1) \) coefficient characteristic of the other terms. This shows that Nambu-Goto provides a remarkably accurate description of the ground state winding string energy for all possible lengths.

The calculations of \( M_0(l) \) in this paper are performed for \( l \sqrt{\sigma} \geq 3 \). We extract the corresponding values of \( \sigma \) using Eq. (3.5). For such \( l \) the contribution of the correction terms that are additional to the Lüscher correction is in fact almost negligible, as was assumed in earlier calculations [5, 6]. However, now we are able to control the accuracy of that assumption.

C. Extrapolation to the continuum limit

To extrapolate to the continuum limit we need to choose a theoretically motivated fitting ansatz for the way \( a \sqrt{\sigma} \) depends on the bare lattice coupling \( \beta \). From Eq. (2.2) it is clear that

\[
\lim_{a \to 0} \frac{\beta}{2N^2} a \sqrt{\sigma} = \frac{\sqrt{\sigma}}{g^2 N}. \tag{3.6}
\]

The leading perturbative correction to this relation will be \( O(1/\beta) \). (Note that we do not have here a ratio of physical quantities for which the leading correction would be \( O(a^2) \).) Lattice perturbation theory is notoriously ill-behaved and to reduce the higher order corrections we replace \( \beta \) by the mean-field improved coupling [15]

\[
\beta_{MF} = \beta \times \langle \text{Tr} U_P \rangle \tag{3.7}
\]

as in [5, 6].

In [5, 6] the continuum extrapolation was performed with a leading \( O(1/\beta_{MF}) \) correction. The values of \( a \sqrt{\sigma} \) from the coarser values of \( a \) typically did not lie on the fit, and were
excluded. Although this is a sensible procedure, it ignores the (small) systematic error due to the neglect of higher order corrections in $1/\beta_{MF}$. Here we will control this error by fitting with an additional correction

$$
\frac{a\sqrt{\sigma}}{2N^2\beta_{MF}} = \left(\frac{\sqrt{\sigma}}{g^2N}\right)_{\text{continuum}} + \frac{a_1}{\beta_{MF}} + \frac{a_2}{\beta_{MF}^2}.
$$ (3.8)

By comparing these fits with linear fits where we constrain $a_2 = 0$, we shall have an estimate of the effect of higher order corrections. Clearly such a strategy is only possible where the calculations are of sufficient range and accuracy, as in the present paper.

It is important to note that this procedure is not without its ambiguities. The expansion in Eq. (3.8) is a weak coupling one which is functionally incorrect in the strong coupling region. If our fit includes one or more points in the strong coupling region, it is these points that may well determine our estimate of the coefficient $a_2$ in Eq. (3.8), in which case the estimate will be unrelated to its actual value. This problem is exacerbated by the fact that unlike the case in $D = 3 + 1$ (for $N \geq 5$), the separation between strong and weak coupling does not involve a clear-cut first order transition, but rather a crossover that turns into a smooth phase transition at $N = \infty$ [16], very much like the Gross-Witten transition in $D = 1 + 1$ [17]. This crossover peak increases with $N$ and lies in the range $\beta/2N^2 \sim 0.40 - 0.45$ [16], and so in our fits we shall avoid using any values obtained on the strong coupling side of this peak.

D. Extrapolation to $N = \infty$

The continuum value of $\sqrt{\sigma}/g^2N$ is expected to have a finite limit, with leading corrections of $O(1/N^2)$. Hence linear fits in $1/N^2$ can be used to extrapolate to $N = \infty$, as in [5, 6]. To control the neglected higher order corrections to the linear fit we shall also perform fits using the more general form

$$
\frac{\sqrt{\sigma}}{g^2N} = \left(\frac{\sqrt{\sigma}}{g^2N}\right)_{N=\infty} + \frac{b_1}{N^2} + \frac{b_2}{N^4}
$$ (3.9)

and compare the results to those of linear fits ($b_2 = 0$). There is no reason to believe that this expansion becomes functionally incorrect at small $N$, and we shall use it all the way down to $N = 2$.

IV. RESULTS

In Fig. 1 we plot the values of $a\sqrt{\sigma}\beta_{MF}/(2N^2)$, as obtained from both $S$ and $D$ fits, versus $N^2/\beta_{MF}$ for our SU(4) and SU(6) calculations. We display fits that are quadratic, as in Eq. (3.8). The $O(1/\beta^2)$ correction is always positive, except possibly for SU(2), so the quadratic fit leads to a significantly higher value than the linear one in the continuum limit. Despite this fact, we see from the figure that the extrapolated string tensions are still lower than the values predicted by Karabali, Kim, and Nair in Eq. (1.1).

We give the results of our continuum fits in Tables I-II where we also list for comparison the Karabali-Kim-Nair predictions.
FIG. 1: The dimensionless quantity $\beta_{\text{MF}} \frac{a\sqrt{\sigma}}{2N^2}$ as a function of the improved inverse coupling $1/\beta_{\text{MF}}$ for $N = 4, 6$. The error bars at $1/\beta_{\text{MF}} = 0$ denote the result of the continuum extrapolation, while the horizontal bars denote the values predicted by Karabali, Kim, and Nair [4].

| $N$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ | $N = 8$ |
|-----|---------|---------|---------|---------|---------|---------|
| $a_1$ | -0.0133(7) | -0.0158(4) | -0.0186(9) | -0.0175(9) | -0.0204(8) | -0.0186(8) |
| $a_1^D$ | -0.0122(12) | -0.0152(9) | -0.0168(19) | -0.0160(14) | -0.0200(11) | -0.0186(10) |
| $a_2$ | -0.0004(3) | +0.0005(2) | +0.0023(6) | +0.0018(6) | +0.0040(5) | +0.0025(5) |
| $a_2^D$ | -0.0009(5) | +0.0004(3) | +0.0012(12) | +0.0010(8) | +0.0039(7) | +0.0025(6) |

TABLE I: The parameters $a_{1,2}$ in the fit Eq. (3.8), which are obtained for the string tensions $\sigma^{S,D}$. The superscripts $S, D$ denote the way we fit the correlation function, and bracket the actual string tension (see Section IIIA).

Since we are interested in the accuracy of the Karabali-Kim-Nair (KKN) prediction with increasing $N$, we define the ratio $r$ of that prediction to our lattice values

$$r \equiv \frac{(\sqrt{\sigma}/g^2N)_{\text{KKN}}}{(\sqrt{\sigma}/g^2N)_{\text{Lattice}}}. \quad (4.1)$$

We now fit $r^2$ with the form

$$r^2 = (r_{\infty})^2 + \frac{c_1}{N^2} + \frac{c_2}{N^4}, \quad (4.2)$$

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TABLE II: Our continuum values of $\sqrt{\sigma}/g^2N$ and the predictions of Karabali, Kim, and Nair (KKN) [4]. We present results from linear and quadratic extrapolations to the continuum ($a_2 = 0$ or $a_2$ as a free fit parameter).

| Type of fit | Source of data | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ | $N = 8$ |
|-------------|----------------|---------|---------|---------|---------|---------|---------|
| Linear      | $M^S$          | 0.1678(1) | 0.1837(2) | 0.1921(2) | 0.1932(2) | 0.1948(2) |
| Linear      | $M^D$          | 0.1675(3) | 0.1813(4) | 0.1886(3) | 0.1910(4) | 0.1927(3) | 0.1943(3) |
| Quadratic   | $M^S$          | 0.1675(3) | 0.1839(2) | 0.1902(2) | 0.1924(3) | 0.1934(3) | 0.1955(3) |
| Quadratic   | $M^D$          | 0.1666(6) | 0.1832(4) | 0.1893(7) | 0.1915(5) | 0.1939(4) | 0.1951(4) |
| KKN prediction |               | 0.17275 | 0.18806 | 0.19314 | 0.19544 | 0.19668 | 0.19791 |

TABLE III: The extrapolation of $r$ to the large-$N$ limit.

| Type of fit | Source of data | $r_{\infty}$ | $c_1$ | $c_2$ | $\chi^2/d.o.f.$ |
|-------------|----------------|--------------|-------|------|-----------------|
| Linear      | $M^S$          | 0.9902(12)   | -0.215(32) | -    | 0.84            |
| Linear      | $M^D$          | 0.9878(17)   | -0.245(56) | -    | 0.64            |
| Quadratic   | $M^S$          | 0.9908(14)   | -0.268(58) | 0.41(22) | 0.98            |
| Quadratic   | $M^D$          | 0.9886(21)   | -0.317(103) | 0.52(40) | 0.57            |

V. SUMMARY

We have calculated the tensions of strings in the fundamental representation of $SU(N)$ gauge theories in 2+1 dimensions. Our immediate goal was to test the prediction of Karabali-Nair in Eq. (1.1), particularly at $N = \infty$, where the screening effects that are clearly not incorporated in that scheme, vanish. Since earlier lattice calculations had already shown that any discrepancy was no more than a few percent, our calculation needed to control a number of previously neglected systematic errors that are small but could be significant at the $\sim 1\%$ level.

In this paper we described how we controlled the following errors. Firstly the contribution of excited states to our variationally selected ground state correlators, from which we extract the energy of the ground state winding flux loop. Secondly higher order string corrections in the relationship between this ground state energy and the asymptotic string tension. (Using the results of our companion publication [7] on the effective string theory describing winding flux loops.) Thirdly, higher order corrections in the continuum extrapolation, and fourthly, higher order corrections in the extrapolation in $N$ to $N = \infty$.

Our final results are similar to the ones of the older work [6] which assumed that the systematic errors that we control here, are negligible. We find that this assumption is, as it
happens, essentially correct. Our string tensions are $3\% - 1\%$ smaller than the prediction of Eq. (1.1), and a discrepancy persists at $N = \infty$, where our result is

$$\rho_{\text{Lattice}} = 0.1975 \pm 0.0002 - 0.0005.$$  

Here the first error is statistical, and the second comes from the difference $\delta \sigma = \sigma_S - \sigma_D > 0$. This error can only lower the string tension, away from Eq. (1.1). Consequently, our result is lower by $0.98\% - 1.2\%$ than Eq. (1.1)

$$\lim_{N \to \infty} \left( \frac{\sqrt{\sigma}}{g^2 N} \right)_{KKN} = \frac{1}{\sqrt{8\pi}} = 0.199471 \ldots.$$  

This difference while small is statistically significant at a compelling $8 - 5.4$ sigma level (depending on the details of the fit).

While it is clear that the leading term in the scheme of [4] is not exact at $N = \infty$, our results show that it is astonishingly accurate.

**Acknowledgements**

MT acknowledges very useful discussions with David Gross and V.P. Nair during the KITP ‘QCD and String Theory’ Programme in 2004. BB acknowledges the support of
PPARC. The computations were performed on machines funded primarily by Oxford and EPSRC.

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