Electroweak symmetry breaking in other terms

Dennis D. Dietrich
Institut for Fysik og Kemi, Syddansk Universitet, Odense, Danmark
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We analyse descriptions of electroweak symmetry breaking in terms of ultralocal antisymmetric tensor fields and gauge-singlet geometric variables, respectively; in particular, the Weinberg–Salam model and, ultimately, dynamical electroweak symmetry breaking by technicolour theories with enhanced symmetry groups. Our motivation is to unveil the manifestly gauge invariant structure of the different realisations. We find, for example, parallels to different types of torsion.

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I. INTRODUCTION

Massive gauge bosons belong to the fundamental concepts we use for picturing nature. Apart from electroweak symmetry breaking, which here is our main interest, other prominent examples are superconductivity and confinement. The Drosophila for electroweak symmetry breaking is the Weinberg–Salam model which is at the basis of the standard model. It relies on breaking the electroweak symmetry by the coupling to an elementary scalar particle, the Higgs. Theoretical shortcomings of the basic model provide the incentive for looking for modifications beyond the standard model. One of those, which shall be studied here is known as technicolour [19]. In technicolour the electroweak symmetry is broken by chiral symmetry breaking among fermions (techniquarks) in an additional strongly interacting sector which replaces the Higgs sector in the standard model.

Even more than in the massless case, gauge invariance is a severe constraint for the construction of massive gauge field theories. Usually, additional fields beyond the original gauge field have to be included in order to obtain gauge invariant expressions. Technically, this is linked to the fact that the gauge field changes inhomogeneously under gauge transformations and encodes also spurious degrees of freedom arising from the construction principle of gauge invariance. All this complicates the extraction of physical quantities and might shroud essentials of the physical content of the theory. A variety of approaches has been developed in order to deal with this situation. Wilson loops [1], for example, represent gauge invariant but non-local variables. Alternatively, there exist decomposition techniques like the one due to Cho, Faddeev, and Niemi [2]. Here, we first pursue a reformulation [3] in terms of antisymmetric gauge algebra valued tensor fields $B_{\mu \nu}^a$ (Sect. II) and subsequently continue with a representation in terms of geometric variables (Sect. III).

Before we treat the Weinberg–Salam model and technicolour models in Sect. II B 2, we begin our discussion with the less involved set-ups of massless and massive field theories with simple gauge groups. In Sect. II A we review the massless case. The antisymmetric tensor field transforms homogeneously under gauge transformations. This fact already makes it simpler to account for gauge invariance. In Sect. II B 1 the generalisation to the massive case is presented. In the $B_{\mu \nu}^a$ field representation the (non-Abelian) St"uckelberg fields, which are commonly present in massive gauge field theories and needed there in order to keep track of gauge invariance, factor out completely. In other words, no scalar fields are necessary for a gauge invariant formulation of massive gauge field theories in terms of antisymmetric tensor fields. The case of a constant mass is linked to sigma models (gauged and ungauged) in different respects. Sect. II B 1 contains the generalisation to a position dependent mass, which corresponds to introducing the Higgs degree of freedom. In Sect. II B 2 non-diagonal mass terms are admitted. This is necessary to accommodate the electroweak symmetry breaking pattern. The Weinberg–Salam model and technicolour models are studied as particular cases. Technicolour models with enhanced symmetry groups have additional “pions” beyond the ones corresponding to the aforementioned St"uckelberg degrees of freedom.

Sect. III presents a description of the massive case in terms of geometric variables. In this step the remaining gauge degrees of freedom are eliminated. The resulting description is in terms of local colour singlet variables. Finally, Sect. III A is concerned with the geometric representation of the Weinberg–Salam model and Sect. III B with that of technicolour. In Sect. III C we make the link between the different variants of symmetry breaking and different contributions to torsion.

The Appendix treats the Abelian case. It allows to better interpret and understand several of the findings in the non-Abelian settings. Of course, in the Abelian case already the $B_{\mu \nu}$ field is gauge invariant. One also sees that the $m \rightarrow 0$ limit of the gauge propagator for the $B_{\mu \nu}$ fields is well-defined as opposed to the ill-defined limit for the $A_{\mu}$ field propagator.

Sect. IV summarises the paper.

II. ANTISYMMETRIC TENSOR FIELDS

A. Massless

Before we investigate massive gauge field theories let us recall some details about the massless case. The partition function of a massless non-Abelian gauge field theory
without fermions is given by

\[ P := \int [dA] \exp \{ i \int d^4 x \mathcal{L} \}, \tag{1} \]

with the Lagrangian density

\[ \mathcal{L} = \mathcal{L}_0 := -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \tag{2} \]

and the field tensor

\[ F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{3} \]

\( A^a_\mu \) stands for the gauge field, \( f^{abc} \) for the antisymmetric structure constant, and \( g \) for the coupling constant. \[ 32 \]

Variation of the classical action with respect to the gauge field gives the classical Yang–Mills equations

\[ D_{\mu}^a (A) F^{b\mu\nu} = 0, \tag{4} \]

where the covariant derivative is defined as \( D_{\mu}^a (A) := \delta_{\mu}^a \partial_\mu + f^{abc} A^b_\mu \). The partition function in the first-order formalism can be obtained after multiplying Eq. (1) with \( \delta \)

\[ \delta \]

field gives the classical Yang–Mills equations

\[ \text{Variation of the classical action with respect to the gauge field gives the classical Yang–Mills equations} \]

\[ D_{\mu}^a (A) F^{b\mu\nu} = 0, \tag{5} \]

(“\( \approx \)” indicates that in the last step the normalisation of the partition function has been changed.) Subsequently, the field \( B_{\mu\nu}^a \) is shifted by \( \frac{1}{g^2} \tilde{F}^a_{\mu\nu} \), where the dual field tensor is defined as \( \tilde{F}^a_{\mu\nu} := \frac{1}{g^2} \epsilon_{\mu\nu\kappa\lambda} F^{a\kappa\lambda} \),

\[ P \approx \int [dA][dB] \exp \{ i \int d^4 x [\mathcal{L}_0 - \frac{g^2}{4} B^a_{\mu\nu} B^{a\mu\nu}] \}. \tag{6} \]

In this form the partition function is formulated in terms of the Yang–Mills connection \( A^a_\mu \) and the antisymmetric tensor field \( B^a_{\mu\nu} \) as independent variables. Variation of the classical action with respect to these variables leads to the classical equations of motion

\[ g^2 B^a_{\mu\nu} = -\tilde{F}^a_{\mu\nu} \quad \text{and} \quad D_{\mu}^a (A) B^{b\mu\nu} = 0, \tag{7} \]

where \( \tilde{B}^a_{\mu\nu} := \frac{1}{g^2} \delta_{\mu\nu} \epsilon_{\kappa\lambda\mu\nu} B^{a\kappa\lambda} \). By eliminating \( B^a_{\mu\nu} \) the original Yang–Mills equation (1) is reproduced. Every term in the classical action in the partition function (6) contains at most one derivative as opposed to two in Eq. (1). This explains the name ”first-order” formalism. The classical action in Eq. (6) is invariant under simultaneous gauge transformations of the independent variables according to

\[ A^{a\mu} T^a := A^{a\mu} \quad \text{to} \quad U [A^{a\mu} - i U (\partial^\mu U)] U^\dagger \tag{8} \]

\[ B^{a\mu\nu} T^a := B^{a\mu\nu} \quad \text{to} \quad U B^{a\mu\nu} U^\dagger, \tag{9} \]

or infinitesimally,

\[ \delta A^a_\mu = \partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c \]

\[ \delta B^a_{\mu\nu} = f^{abc} B^b_{\mu\nu} \theta^c. \tag{10} \]

The \( T^a \) stand for the generators of the gauge group. From the Bianchi identity \( D_{\mu}^a (A) F^{b\mu\nu} = 0 \) follows a second symmetry of the \( BF \) term alone: Infinitesimally, for unchanged \( A^a_\mu \),

\[ \delta B^a_{\mu\nu} = \partial_\mu \theta^a_\nu - \partial_\nu \theta^a_\mu + f^{abc} (A^b_\mu \theta^c_\nu - A^b_\nu \theta^c_\mu). \tag{11} \]

A particular combination of the transformations (10) and (11), \( \theta^a = n^a \theta^a \) and \( \theta^c = n^c B^a_{\mu\nu} \), corresponds to the transformation of a tensor under an infinitesimal local coordinate transformation \( x^\mu \rightarrow x^\mu - n^\mu (x) \),

\[ \delta B^a_{\mu\nu} = B_{\lambda\nu} \partial_\mu n^\lambda + B_{\mu\lambda} \partial_\nu n^\lambda + n^\mu \partial_\lambda B_{\mu\nu}, \tag{12} \]

that is a diffeomorphism. Hence, the \( BF \) term is diffeomorphism invariant, which explains why this theory is also known as \( BF \) gravity. The \( BB \) term is not diffeomorphism invariant and, hence, imposes a constraint. The combination of the two terms amounts to an action of Plebsanski type which are studied in the context of quantum gravity [4,5].

We now would like to eliminate the Yang–Mills connection by integrating it out. For fixed \( B^a_{\mu\nu} \) the integrand of the path integral is not gauge invariant with respect to gauge transformations of the gauge field \( A^a_\mu \) alone; the field tensor \( F^a_{\mu\nu} \) transforms homogeneously and the corresponding gauge transformations are not absorbed if \( B^a_{\mu\nu} \) is held fixed. Therefore, the integral over the gauge group is in general not cyclic which otherwise would render the path integral ill-defined. The term in the exponent linear in the gauge field \( A^a_\mu \), \( A^a_\mu \partial_\nu B^a_{\mu\nu} \), is obtained by carrying out a partial integration in which surface terms are ignored. Afterwards it is absorbed by shifting \( A^a_\mu \)

\[ \text{by} \quad (\mathbb{B}^{-1})^a_{\mu\nu} \partial^\mu B^{a\nu}, \]

where \( \mathbb{B}^{ab}_{\mu\nu} := f^{abc} B^c_{\mu\nu} \). In general its inverse \( (\mathbb{B}^{-1})^{ab}_{\mu\nu} \) defined by \( (\mathbb{B}^{-1})^{ab}_{\mu\nu} \mathbb{B}^{bc}_{\mu\nu} = \delta^{ab} \delta_{\mu\nu} \), exists in three or more space-time dimensions [33]. We are left with a Gaussian integral in \( A^a_\mu \) giving the inverse square-root of the determinant of \( \mathbb{B}^{ab}_{\mu\nu} \),

\[ \text{Det}^{-\frac{1}{2}} \mathbb{B} := \prod_x \text{det}^{-\frac{1}{2}} \mathbb{B} \cong \]

\[ \cong \int [da] \exp \{ -\frac{1}{2} \int d^4 x a^a \mathbb{B}^{ab}_{\mu\nu} a^{b\nu} \}. \tag{13} \]

In the last expression \( \mathbb{B}^{ab}_{\mu\nu} \) appears in the place of an inverse gluon propagator, that is sandwiched between two gauge fields. This analogy carries even further: Interpreting \( \partial_\mu B^{a\mu\nu} \) as a current, \( (\mathbb{B}^{-1})^{ab}_{\mu\nu} \partial_\mu B^{a\nu} \), the current together with the ”propagator” \( (\mathbb{B}^{-1})^{ab}_{\mu\nu} \partial_\mu B^{a\nu} \), is exactly the abovementioned term to be absorbed in the gauge field \( A^a_\mu \). Finally, we obtain,

\[ P \cong \int [dB] \text{Det}^{-\frac{1}{2}} \mathbb{B} \exp \{ i \int d^4 x [-\frac{g^2}{4} B^a_{\mu\nu} B^{a\mu\nu} - \frac{1}{2} (\partial_\mu B^{a\mu\nu} (\mathbb{B}^{-1})^{ab}_{\mu\nu} \partial_\nu B^{b\nu})] \}. \tag{14} \]
This result is known from [6, 8, 9]. The exponent in the previous expression corresponds to the value of the \([dA]\) integral at the saddle-point value \(A^a_\mu\) of the gauge field. It obeys the classical field equation (7). Using \(A^a_\mu(B) = (\mathcal{B}^{-1})^{ab}(\partial_\lambda \tilde{B}^{b\lambda\nu})\) the second term in the above exponent can be rewritten as \(-\frac{1}{2} \int d^4x \tilde{B}_{\mu\nu}^a F^{a\mu\nu}[\tilde{A}(B)]\), which involves an integration by parts and makes its gauge invariance manifest. The fluctuations \(a^a_\mu\) around the saddle point \(A^a_\mu\), contributing to the partition function (9), are Gaussian because the action in the first-order formalism is only of second order in the gauge field \(A^a_\mu\). They give rise to the determinant (13). What happens if a zero of the determinant is encountered can be understood by looking at the Abelian case discussed in Appendix A. There the \(BF\) term does not fix a gauge for the integration over the gauge field \(A_\mu\) because the Abelian field tensor \(F^{\mu\nu}\) is gauge invariant. If it is performed nevertheless one encounters a functional \(\delta\) distribution which enforces the vanishing of the current \(\partial_\mu B^{\mu\nu}\). In this sense the zeros of the determinant in the non-Abelian case arise if \(\tilde{B}_{\mu\nu}^a\) is such that the \(BF\) term does not totally fix a gauge for the \([dA]\) integration, but leaves behind a residual gauge invariance. It in turn corresponds to vanishing components of the current \(\partial_\mu \tilde{B}^{a\mu\nu}\). (Technically, there then is at least one flat direction in the otherwise Gaussian integrand. The flat directions are along those eigenvectors of \(\mathcal{B}\) possessing zero eigenvalues.)

When incorporated with the exponent, which requires a regularisation \(10\), the determinant contributes a term proportional to \(\frac{1}{4} \ln \det \mathcal{B}\) to the action. This term together with the \(BB\) term constitutes the effective potential, which is obtained from the exponent in the partition function after dropping all terms containing derivatives of fields. The effective potential becomes singular for field configurations for which \(\det \mathcal{B} = 0\). It is gauge invariant because all contributing addends are gauge invariant separately.

The classical equations of motion obtained by varying the action in Eq. (13) with respect to the dual antisymmetric tensor field \(\tilde{B}^{a\mu\nu}\) are given by

\[ g^2 \tilde{B}_{\mu\nu}^a = \left( g^a_\mu g^a_\nu - g^a_\mu g^a_\sigma \right) \partial_\rho (\mathcal{B}^{-1})^{ab}_{\sigma\nu} (\partial_\lambda \tilde{B}^{b\lambda\rho}) - \left( -\partial_\rho \tilde{B}^{d\rho\sigma} (\mathcal{B}^{-1})^{db}_{\mu\nu} f^{abc} (\mathcal{B}^{-1})^{c\nu}_{\rho\lambda} (\partial_\sigma \tilde{B}^{e\sigma\lambda}) \right), \]

which coincides with the first of Eqs. (7) with the field tensor evaluated at the saddle point of the action, \(F^{a\mu\nu}[\tilde{A}(B)]\). Taking into account additionally the effect due to fluctuations of \(A_\mu^a\) contributes a term proportional to \(\frac{\text{det} \mathcal{B}^a}{\text{det} \mathcal{B}_{\mu\nu}} \mathcal{B}^{-1}\) to the previous equation.

### B. Massive

In the massive case the prototypical Lagrangian is of the form \(\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m\), where \(\mathcal{L}_m := \frac{m^2}{2} A^a_\mu A^{a\mu}\). (Due to our conventions the physical mass is given by \(m_{\text{phys}} := \frac{m}{\sqrt{g}}\).) This contribution to the Lagrangian is of course not gauge invariant. Putting it, regardlessly, into the partition function, gives

\[ P = \int [dA][dU] \exp \left\{ i \int d^4x \left[ \mathcal{L}_0 + \frac{m^2}{2} A^a_\mu A^{a\mu} \right] \right\}, \]

which can be interpreted as the unitary gauge representation of an extended theory. In order to see this let us split the functional integral over \(A^a_\mu\) into an integral over the gauge group \([dU]\) and gauge inequivalent field configurations \([dA]'\). Usually this separation is carried out by fixing a gauge according to

\[ \int [dA]' := \int [dA] [\delta f^a(A) - C^a] \Delta_f(A). \]

\(f^a(A) = C^a\) is the gauge condition and \(\Delta_f(A)\) stands for the Faddeev–Popov determinant defined through

\[ 1 \frac{1}{\Delta_f(A)} \int [dA] [\delta f^a(A) - C^a] \Delta_f(A). \]

Introducing this reparametrisation into the partition function (13) yields,

\[ P = \int [dA]' [dU] \exp \left\{ i \int d^4x \left[ -\frac{1}{2} F^{a\mu\nu} F_{a\mu\nu} + \frac{m^2}{2} \left( A_\mu - iU^\dagger (\partial_\mu U) \right) \left( A^\mu - iU^\dagger (\partial_\mu U) \right) \right] \right\}. \]

\(\mathcal{L}_0\) is gauge invariant in any case and remains thus unaffected. In the mass term the gauge transformations appear explicitly (11). We now replace all of these gauge transformations with an auxiliary (gauge group valued) scalar field \(\Phi, U^\dagger \rightarrow \Phi\), obeying the constraint

\[ \Phi^\dagger \Phi \equiv 1. \]

The field \(\Phi\) can be expressed as \(\Phi := e^{-i\theta}\), where \(\theta := \theta^a T^a\) is the gauge algebra valued non-Abelian generalisation of the Stuckelberg field (12). For a massive gauge theory they are a manifestation of the longitudinal degrees of freedom of the gauge bosons. In the context of symmetry breaking they arise as Goldstone modes (“pions”). In the context of the Thirring model these observations have been made in (13). There it was noted as well that the \(\theta\) is also the field used in the canonical Hamiltonian Batalin–Fradkin–Vilkovisky formalism (14). We can extract the manifestly gauge invariant classical Lagrangian

\[ \mathcal{L}_{\text{cl}} := -\frac{1}{4g} F^{a\mu\nu} F_{a\mu\nu} + m^2 \text{tr}[(D_\mu \Phi)^\dagger (D^\mu \Phi)], \]

where \(\Phi\) fields have been rearranged making use of the product rule of differentiation and the cyclic property of the trace and where \(D_\mu \Phi := \partial_\mu \Phi - i A_\mu \Phi\). Eq. (20) resembles the Lagrangian density of a non-linear gauged sigma model. In the Abelian case the fields \(\theta\) decouple from the dynamics. For non-Abelian gauge groups they do not and one would have to deal with the non-polynomial coupling to them. In the following we show that these spurious degrees of freedom can be absorbed when making the transition to a.
formulation based on the antisymmetric tensor field $B_{\mu \nu}^a$. Introducing the antisymmetric tensor field into the corresponding partition function, like in the previous section, results in,

$$P \cong \int [dA][d\Phi][dB] \exp(i \int d^4x \times \left\{ \frac{g^2}{4} B_{\mu \nu}^a B^{\alpha \beta \mu \nu} - \frac{i}{2} F_{\mu \nu}^a B^{a \mu \nu} + m_\Phi^2 [A_\mu - i \Phi (\partial_\mu \Phi^\dagger)]^2 + i \Phi (\partial^\mu \Phi^\dagger) F_{\mu \nu}^a \right\}).$$

Removing the gauge scalars $\Phi$ from the mass term by a gauge transformation of the gauge field $A_\mu^a$ makes them explicit in the $BF$ term,

$$P = \int [dA][dB] \exp(i \int d^4x \left\{ \frac{g^2}{4} B_{\mu \nu}^a B^{a \mu \nu} - \text{tr}(\Phi F_{\mu \nu}^a \Phi^\dagger B^{a \mu \nu}) + m_\Phi^2 A_\mu^a A_\mu^a \right\}).$$

In the next step we would like to integrate over the Yang–Mills connection $A_\mu^a$. Already in the previous expression, however, we can perceive that the final result will only depend on the combination of fields $\Phi B_{\mu \nu}^a \Phi$. The field can also be made explicit in the $BB$ term in form of the constraint \((19)\). Therefore, the functional integral over $\Phi$ only covers multiple times the range which is already covered by the $[dB]$ integration. Hence the degrees of freedom of the field $\Phi$ have become obsolete in this formulation and the $[d\Phi]$ integral can be factored out. Thus, we could have performed the unitary gauge calculation right from the start. In either case, the final result reads,

$$P \cong \int [dB] \det^{-\frac{1}{2}} M \exp \left( i \int d^4x \left\{ -\frac{g^2}{4} B_{\mu \nu}^a B^{a \mu \nu} - \frac{i}{2} (\partial_\mu B^{a \mu \nu}) (M^{-1})^{ab}_{\mu \nu} (\partial_\nu B^{b \alpha \mu}) \right\} \right),$$

where $M_{\mu \nu}^{ab} := B_{\mu \nu}^{ab} - m^2 \delta^{ab} g_{\mu \nu}$, which coincides with \((12)\). $M_{\mu \nu}^{ab}$ and hence $(M^{-1})^{ab}_{\mu \nu}$ transform homogeneously under the adjoint representation. In Eq. \((14)\) the central matrix $(B^{-1})_{\mu \nu}^{ab}$ in the analogous term transformed in exactly the same way. There this behaviour ensured the gauge invariance of this term’s contribution to the classical action. Consequently, the classical action in the massive case has the same invariance properties. In particular, the aforementioned gauge invariant classical action describes a massive gauge theory without having to resort to additional scalar fields. For $\det B \neq 0$, the limit $m \to 0$ is smooth. For $\det B = 0$ the conserved current components alluded to above would have to be separated appropriately in order to recover the corresponding $\delta$ distributions present in these situations in the massless case.

Again the effective action is dominated by the term proportional to $\frac{1}{2} \det M$. The contribution from the mass to $M$ shifts the eigenvalues from the values obtained for $B$. Hence the singular contributions are typically obtained for eigenvalues of $B$ of the order of $m^2$. The effective potential is again gauge invariant, for the same reason as in the massless case.

The classical equations of motion obtained by variation of the action in Eq. \((21)\) are given by,

$$g^2 B_{\mu \nu}^a = -F_{\mu \nu}^a,$$

$$D_{\mu}^{ab}(A) \tilde{B}_{\mu \nu}^{ab} = -m^2 [A^\nu - i \Phi (\partial^\nu \Phi^\dagger)]^a,$$

$$0 = \frac{\delta}{\delta \Phi} \int d^4x \left( [A_\mu - i \Phi (\partial_\mu \Phi^\dagger)]^a \right)^2.$$

In these equations a unique solution can be chosen, that is a gauge fixe, by selecting the scalar field $\Phi$. $\Phi \equiv 1$ gives the unitary gauge, in which the last of the above equations drops out. The general non-Abelian case is difficult to handle already on the classical level, which is one of the main motivations to look for an alternative formulation. In the non-Abelian case, the equation of motion obtained from Eq. \((23)\) resembles strongly the massless case,

$$g^2 \tilde{B}_{\mu \nu}^{ab} = ( \hat{g}^a \hat{g}^a - \hat{g}^a \hat{g}^a ) \partial_\rho (M^{-1})^{ab}_{\rho \nu} (\partial_\nu \tilde{B}^{b \lambda \lambda}) - (\partial_\rho \tilde{B}^{b \lambda \lambda}) (M^{-1})^{ab}_{\rho \nu} (\partial_\nu \tilde{B}^{b \lambda \lambda}),$$

insofar as all occurrences of $(B^{-1})_{\mu \nu}^{ab}$ have been replaced by $(M^{-1})_{\mu \nu}^{ab}$. Incorporation of the effect of the Gaussian fluctuations of the gauge field $A_\mu^a$ would give rise to a contribution proportional to $\frac{\det}{\det M} \det^{-1} \det M$ in the previous equation.

Before we go over to more general cases of massive non-Abelian gauge field theories, let us have a look at the weak coupling limit: There the $BB$ term in Eq. \((21)\) is neglected. Subsequently, integrating out the $B_{\mu \nu}^a$ field enforces $F_{\mu \nu}^a = 0$. [This condition also arises from the classical equation of motion \((21)\) for $g=0$.] Hence, for vanishing coupling exclusively pure gauge configurations of the gauge field $A_\mu^a$ contribute. They can be combined with the $\Phi$ fields and one is left with a non-linear realisation of a partition function,

$$P \cong \int [d\Phi] \exp \left( i m^2 \int d^4x \tr (\partial_\mu \Phi^\dagger (\partial^\mu \Phi) \right),$$

of a free massless scalar \((15)\). Setting $g = 0$ interchanges with integrating out the $B_{\mu \nu}^a$ field from the partition function \((21)\). Thus, the partition function \((25)\) with $g = 0$ is equivalent to \((26)\). That a scalar degree of freedom can be described by means of an antisymmetric tensor field has been noticed in \((16)\).

1. Position-dependent mass and the Higgs

One possible generalisation of the above set-up is obtained by softening the constraint \((19)\). This can be seen as allowing for a position dependent mass. The new degree of freedom ultimately corresponds to the Higgs. When introducing the mass $m$ as new degree of freedom (as "mass scalar") we can restrict its variation by introducing a potential term $V(m)$, which remains to be specified, and a kinetic term $K(m)$, which we choose in
its canonical form $K(m) = \frac{1}{4} (\partial_\mu m)(\partial^\mu m)$. It gives a penalty for fast variations of $m$ between neighbouring space-time points. The fixed mass model is obtained in the limit of an infinitely sharp potential with its minimum located at a non-zero value for the mass. Putting together the partition function in unitary gauge leads to,

$$P = \int [dA][dm] \text{exp}\{i \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu + K(m) + V(m^2) \right] \},$$

where we have introduced the normalisation constant $N := \text{dim } \mathbb{R}$, with $\mathbb{R}$ standing for the representation of the scalars. This factor allows us to keep the canonical normalisation of the mass scalar $m$. We can now repeat the same steps as in the previous section in order to identify the classical Lagrangian,

$$\mathcal{L}_{cl} := -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + N^{-1} \text{tr}[(D_\mu \phi)^\dagger (D^\mu \phi)] + V(|\phi|^2),$$

where now $\phi := m\Phi$. In order to reformulate the partition function in terms of the antisymmetric tensor field we can once more repeat the steps in the previous section. Again the spurious degrees of freedom represented by the field $\Phi$ can be factored out. Finally, this gives [17],

$$P \cong \int [dB][dm] \text{Det}^{-\frac{1}{8}} M \text{exp}\{i \int d^4x \left[ -2 g F_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \left( \partial_\mu \tilde{B}^{\alpha\nu} \right) \left( \mathcal{M}^{-1} \right)_{\alpha\mu}^{\beta\nu} \left( \partial_\lambda \tilde{B}^{\beta\lambda} \right) + K(m) + V(m^2) \right] \},$$

where $\mathcal{M}_{\mu\nu} = \mathbb{B}_{\mu\nu} - m^2 N^{-1} \delta^4 g_{\mu\nu}$ depends on the space-time dependent mass $m$. The determinant can as usual be included with the exponent in form of a term proportional to $\frac{1}{8} \text{det } M$, the pole of which will dominate the effective potential. As just mentioned, however, $M$ is also a function of $m$. Hence, in order to find the minimum, the effective potential must also be varied with respect to the mass $m$.

Carrying the representation in terms of antisymmetric tensor fields another step further, the partition function containing the kinetic term $K(m)$ of the mass scalar can be expressed as Abelian version of Eq. (27),

$$\int [db][da] \text{exp}\{i \int d^4x \left[ -\frac{1}{2} b_{\mu\nu} F^{\mu\nu} + \frac{1}{2} a_\mu a^\mu \right] \} = \int [dm] \text{exp}\{i \int d^4x \left[ \frac{1}{2} (\partial_\mu m)(\partial^\mu m) \right] \},$$

where here the mass scalar $m$ is identified with the Abelian gauge parameter. Combining the last equation with the partition function [27] all occurrences of the mass scalar $m$ can be replaced by the phase integral $m \rightarrow \int dx^4 a_\mu$. The $b_f$ term enforces the curvature $f$ to vanish which constrains $a_\mu$ to pure gauges $\partial_\mu m$ and the aforementioned integral becomes path-independent.

2. Non-diagonal mass term and the Weinberg–Salam model

The mass terms investigated so far had in common that all the bosonic degrees of freedom they described possessed the same mass. A more general mass term would be given by $L_m := \frac{m_1^2}{2} A_\mu^a A^{a\mu}$. Another similar approach is based on the Lagrangian $\mathcal{L}_m := \frac{1}{2} \mathbf{m}^2 \text{tr} \{ A_\mu^a A^{a\mu} \}$ where $\Psi$ is group valued and constant. We shall begin our discussion with this second variant and limit ourselves to a $\Psi$ with real entries and $\text{tr} \Psi = 1$, which, in fact, does not impose additional constraints. Using this expression in the partition function (26) and making explicit the gauge scalars yields,

$$P = \int [dA][dm] \text{exp}\{i \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu + \frac{1}{2} \text{tr} \{ (D_\mu \phi)^\dagger (D^\mu \phi) \} + V(|\phi|^2) \right] \},$$

Expressed in terms of the antisymmetric tensor field $B_{\mu\nu}^{ab}$, the corresponding partition function coincides with Eq. (27) but with $\mathbf{M}_{ab}^{\mu\nu}$ replaced by $\mathbf{M}_{\mu\nu}^{ab} := \mathbb{B}_{\mu\nu}^{ab} - m^2 \text{tr} \{ T^a T^b \} g_{\mu\nu}$.

Let us now consider directly the $SU(2) \times U(1)$ Weinberg–Salam model. Its partition function can be expressed as,

$$P = \int [dA][d\psi] \text{exp}\{i \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \psi^\dagger \left( \tilde{D}_\mu + iA_\mu \right) \psi + V(|\psi|^2) \right] \},$$

where $\psi$ is a complex scalar doublet, $A_\mu := \frac{1}{2} T^a A_\mu^a$, with $a \in \{0; \ldots; 3\}$, $T^a$ here stands for the generators of $SU(2)$ in fundamental representation, and, accordingly, $T^a$ for $\frac{g_0}{2}$ times the $2 \times 2$ unit matrix, with the $U(1)$ coupling constant $g_0$. The partition function can be reparametrised with $\psi = m \Phi \psi_1$, where $m = \sqrt{|\psi|^2}$, $\Phi$ is a group valued scalar field as above, and $\psi$ is a constant doublet with $|\psi|^2 = 1$. The partition function then becomes,

$$P = \int [dA][d\Phi][dm] \text{exp}\{i \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{m_1^2}{2} \text{tr} \{ (\tilde{D}_\mu + iA_\mu)^a (\tilde{D}^\mu - iA^\mu) \Psi \} \right] \},$$

where

$$\Psi = \psi \otimes \psi_1^\dagger.$$
As in the previous case, a gauge transformation of the gauge field \( A_a^\mu \) can remove the gauge scalar of \( \Phi \) from the mass term (despite the matrix \( \Psi \)). Thereafter \( \Phi \) only appears in the combination \( \Phi^I B_{\mu I} \Phi \) and the integral \([d\Phi]\) merely leads to repetitions of the \([dB]\) integral. [The \( U(1) \) part drops out completely right away.] Therefore the \([d\Phi]\) integration can be factored out,

\[
P \cong \int [dA][dB][dm] \exp\{i \int dt x[-\frac{1}{4} F_{\mu \nu}^a B_{\mu \nu}^a - \frac{1}{2} \tilde{F}_{\mu \nu}^a B_{\mu \nu}^a + m^2 \delta_{\mu \nu} \delta_{ab} + K(m) + V(m^2)]\}.
\]

The subsequent integration over the gauge fields \( A_{\mu}^{ab} \) leads to

\[
P \cong \int [dB][dm] \det \tilde{\mathbf{M}} \exp\{i \int dt x[-\frac{1}{4} F_{\mu \nu}^a B_{\mu \nu}^a + m^2 \delta_{\mu \nu} \delta_{ab} + K(m) + V(m^2)\},
\]

(33)

where \( \mathbf{M}^{ab} := \mathbf{B}^{ab} - m^2 T(T^a T^b) \delta_{\mu \nu} \) and \( \mathbf{B}^{ab} = \mathbf{B}^{ab} \), \( \forall a, b \neq 0 \) and \( \mathbf{B}^{ab} = 0 \) otherwise.

From hereon we continue our discussion based on the mass matrix

\[
m_{ab} := \frac{1}{2} \text{tr}(\{T^a, T^b\})
\]

(35)

which had already been mentioned at the beginning of Sect. [122]. \( m_{ab} \) is real and has been chosen to be symmetric. (Antisymmetric parts are projected out by the contraction with the symmetric \( A_{\mu}^{ab} A_{\mu}^{ab} \).) Thus it possesses a complete orthonormal set of eigenvectors \( \mu_{ab} \) with the associated real eigenvalues \( m_j \), \( m_{ab} \mu_j \frac{1}{2} = \Sigma_j m_{ab} \mu_j \frac{1}{2} \). With the help of these normalised eigenvectors one can construct projectors \( \pi_{2ab} := \Sigma_j \mu_j \frac{1}{2} \mu_j \frac{1}{2} \) and decompose the mass matrix, \( m_{ab} = \mu_j \frac{1}{2} \mu_j \frac{1}{2} \). The projectors are complete, \( \mathbf{b}^{ab} = \sum_j \pi_{2ab} \), idempotent \( \sum_j \pi_{2ab} \) and \( \pi_{2ab} = \pi_{2ab} \). For the matrix \( \mathbf{B}^{ab} \), the antisymmetric tensor field \( B_{\mu \nu}^a \), and the gauge field \( A_a^\mu \) can also be decomposed with the help of the eigenvectors: \( \mathbf{B}^{ab} = \mu_j \frac{1}{2} \mu_j \frac{1}{2} \mu_j \frac{1}{2} \), where \( \mu_j \frac{1}{2} := \mu_j \frac{1}{2} \mu_j \frac{1}{2} \mu_j \frac{1}{2} \), \( \mathbf{b}^{ab} = \mathbf{b}^{ab} \), where \( \mathbf{b}^{ab} := \mathbf{b}^{ab} \), \( \mathbf{B}^{ab} = \mathbf{B}^{ab} \), and \( A_a^\mu = a_a^\mu \). Using this decomposition in the partition function \( \Sigma \) leads to

\[
P \cong \int [db][dm] \det \tilde{\mathbf{M}} \exp\{i \int dt x[-\frac{1}{4} b_{\mu \nu} b^{\mu \nu} - \frac{1}{2} (\partial_\lambda b^{\mu \nu}) (\partial_\lambda b^{\mu \nu}) + K(m) + V(m^2)\},
\]

(36)

where \( m_{ab} := m_{ab} - 2 \sum_i m_{ab} \delta_i \delta_i g_{\mu \nu} \). Making use of the concrete form of \( m_{ab} \) given in Eq. (35), inserting \( \Psi \) from Eq. (34), and subsequent diagonalisation leads to the eigenvalues \( 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \). These correspond to the photon, the two W bosons and the heavier Z boson, respectively. The thus obtained tree-level Z to W mass ratio squared consistently reproduces the cosine of the Weinberg angle in terms of the coupling constants, \( \frac{\cos^2 \theta_W}{g^2 + g_0^2} \). Due to the masslessness of the photon one addend in the sum over \( i \) in the expression \( m_{ab} \) above does not contribute. Still, the total \( m_{ab} \) does not vanish like in the case of a single massless Abelian gauge boson (see Appendix A). Physically this corresponds to the coupling of the photon to the W and Z bosons.

3. Technicolour

In technicolour theories the standard model minus the Higgs is supplemented by an additional strongly interacting sector containing fermions (techniquarks) transforming under a given representation of the technicolour gauge group and also charged under the electroweak gauge group. The electroweak symmetry is broken by chiral symmetry breaking in the technicolour sector. From the point of view of modern collider experiments the most visible manifestations are those at relatively low energies, that is below the electroweak scale. A standard method to describe these signals is the construction of the corresponding low-energy theory. Its basic degrees of freedom are technicolour singlet fields like (pseudo)scalars and (axial) vectors. In connection with the present investigation we are most interested in the pseudoscalar sector and leave, for example the spin-one sector for a later study. The simplest breaking pattern of the flavour symmetry is \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \), which is realised for two techniflavours transforming under a non-(pseudo)-real representation of the technicolour group, gives rise to the three pions which become the longitudinal degrees of freedom of the W and Z bosons. In this respect the model’s low-energy Lagrangian looks just like the standard model which has already been discussed above. The breaking pattern becomes richer—and the number of pions larger—either by increasing the number of flavours \( [SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V] \) and/or in the presence of techniquarks transforming under real \([SU(2)_f \rightarrow SO(2)_f]\) or pseudoreal \([SU(2)_f \rightarrow Sp(2)_f]\) representations; always assuming a breaking down to the maximal diagonal subgroup. (For a survey of phenomenologically viable technicolour models of these types see, e.g. Ref. [19].) It turns out that the technicolour models which are most favoured by electroweak precision data are walking [20] (that is quasi-conformal) technicolour models which feature techniquarks in higher dimensional representations of the gauge group. So also the minimal walking technicolour model with two flavours in the adjoint representation of \( SU(2) \) [21]. The adjoint representation is real which leads to the enhanced flavour symmetry \( SU(4) \) which breaks to \( SO(4) \) yielding nine pions. Let us discuss this setup along the lines of Ref. [22]. In the effective
low-energy theory, the kinematic term of these pions together with their scalar chiral partner provides the mass term for the gauge bosons,
\[ \mathcal{L}_{TC} = \frac{1}{2} \text{tr}[(D_\mu M)(D^\mu M^\dagger)]. \]  
(37)

(We do not scale out the pion decay constant \( f_\pi \) as in \[22\].) Here \( M \) transforms like the techniquark bilinear \( M_{ij} \sim Q_i^a Q_j^b \epsilon_{\alpha\beta} \) and in terms of low-energy fields can be parametrised according to
\[ M = \left[ \frac{1}{2}(\sigma + i\theta) + \sqrt{2}((\Pi^\alpha + \Pi^\alpha_n)X^n) \right]E. \]  
(38)
The matrix
\[ E := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  
(39)
characterises the condensate \( Q_i^a Q_j^b \epsilon_{\alpha\beta} E^{ij} \) with the expectation value \( 2(M) = vE \) in the basis
\[ Q := \begin{pmatrix} U_L & D_L \\ -i\sigma^2 U_R & -i\sigma^2 D_R^\dagger \end{pmatrix}, \]  
(40)
where \( U_{L/R} \) and \( D_{L/R} \) denote the left-/right-handed up and down techniquarks, respectively. The \( X^a \) are the nine generators of \( SU(4) \) which do not commute with the condensate \( \sim E \). \( D_\mu \) stands for the electroweak covariant derivative. The electroweak group must be embedded in \( SU(2)_L \times U(1)_Y \) to \( U(1)_{em} \). This is achieved by the choice \( \sqrt{2}L^a := S^a + X^a \) for the generators of \( SU(2)_L \), \( \sqrt{2}Y := (S^3 - X^3)^T + 2Y_V S^4 \) for the hypercharge generator and \( \sqrt{2}Q = S^3 = 2Y_V S^4 \). \( Y_V \) parametrises the hypercharge assignments for the techniquarks. The \( S^a \) are the four generators which leave the vacuum invariant, that is which commute with \( E \). (For explicit expressions for the generators see Ref. \[23\].) Then the covariant derivative reads,
\[ D_\mu M = \partial_\mu M - i(G_\mu M + MG_\mu^T), \]  
(41)
where
\[ G_\mu = A_\mu L^a + \frac{2a}{g} A_\mu^0 Y. \]  
(42)

Due to the enhanced symmetry, the pion fields cannot be completely absorbed by a gauge transformation of the gauge potential and the antisymmetric tensor field. (This is also evident from another viewpoint; they carry non-zero technibaryon number which clearly incompatible with the electroweak gauge fields.) We therefore have,
\[ \frac{m_\pi^2}{2} m^{ab} A_\mu^a A_\mu^b := \text{tr}[(G_\mu M + MG_\mu^T)(G_\mu M + MG_\mu^T)^\dagger]. \]

Further, let us define the current,
\[ mJ_\mu^a A_\mu^b := \frac{1}{2} \text{tr}[(G_\mu M + MG_\mu^T)(-\partial_\mu M)^\dagger + (+\partial_\mu M)(G_\mu M + MG_\mu^T)^\dagger]. \]  
(43)
Structurally, the matrix \( m^{ab} \) has the same eigenvector decomposition as explained after Eq. (35). The radial degree of freedom, denoted as above by \( m^2 \), is given by the sum over all the squares of the fields present in Eq. (38). In particular, it receives contributions from the extra pion fields, which do not directly participate in the breaking of the electroweak symmetry. Further, one of the eigenvalues of \( m^{ab} \) is still zero, accounting for the massless photon. Also the current can be decomposed in the eigenbasis of the mass matrix, \( \delta J_\mu^a = J_\mu^a \delta M_\mu^a \), where \( \delta J_\mu^a = J_\mu^a \delta M_\mu^a \). The Lagrangian density corresponding to the one in the exponent of Eq. (30) up to kinetic and potential terms for the (pseudo)scalars then reads,
\[ \mathcal{L}_{TC} \cong -\frac{\delta^2}{4} b_{\mu\nu} b^{\mu\nu} - \frac{1}{2} \text{tr}[(\partial_\mu b^{\mu\nu} + m J^{\mu\nu})(\partial_\nu b^{\mu\nu} + m J^{\mu\nu})]. \]

We have here analysed the phenomenologically most preferred setting for technicolour with an enhanced symmetry breaking pattern, two techniflavours in the adjoint representation of \( SU(2) \). The runner up, what viability is concerned, is a model with two techniflavours in the two-index symmetric representation of \( SU(3) \). It possesses the simple breaking pattern which is covered by the discussion of the Weinberg–Salam model. In \[19\] other possibilities with larger symmetries are listed. They will differ from the concrete example studied here by a different number of extra pions. The main difference with respect to the simple (Weinberg–Salam) patterns will, however, be the appearance of the extra momentum dependent current.

The inclusion of the corresponding technivector (technirho-s) and axial vector fields would make the structure even richer. It is possible to incorporate them in such a way that they can be second quantised (see for example \[22\]) which would allow us to treat them analogous to the electroweak gauge bosons. This point, however, shall not be discussed here.

### III. GEOMETRIC REPRESENTATION

The fact that the antisymmetric tensor field \( B_\mu^a \) transforms homogeneously represents already an advantage over the formulation in terms of the inhomogeneously transforming gauge fields \( A_\mu^a \). Still, \( B_\mu^a \) contains degrees of freedom linked to the gauge transformations \( \Phi \). These can be eliminated by making the transition to a
formulation in terms of geometric variables. In this section we provide a description of different massive gauge field theories in terms of geometric variables in Euclidean space for two colours by adapting Ref. [24] to include mass. The first-order action is quadratic in the gauge-field $A^a_{\mu}$. Thus the evaluation of the classical action at the saddle point yields the expression equivalent to the different exponents obtained after integrating out the gauge field $A^a_{\mu}$ in the various partition functions in the previous section. In Euclidean space the classical massive Yang–Mills action in the first order formalism reads

$$S := \int d^4x (\mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{AA}),$$

where

$$\mathcal{L}_{BB} = -\frac{e^2}{4} B^a_{\mu\nu} B^a_{\mu\nu},$$

$$\mathcal{L}_{BF} = +\frac{e}{2} \epsilon^{\mu\nu\lambda\rho} F^a_{\mu\nu} F^a_{\lambda\rho},$$

$$\mathcal{L}_{AA} = -\frac{m^2}{2} A^a_{\mu} A^a_{\mu}.$$ (47)

At first we will investigate the situation for the unitary gauge mass term $\mathcal{L}_{AA}$ and study the role played by the scalars $\Phi$ afterwards.

As starting point it is important to note that a metric can be constructed that makes the tensor $B^a_{\mu\nu}$ self-dual [7]. In order to exploit this fact, it is convenient to define the antisymmetric tensor ($j \in \{1; 2; 3\}$)

$$T^j_{\mu\nu} := \eta_{AB} \epsilon^A_{\mu} \epsilon^B_{\nu},$$

with the self-dual 't Hooft symbol $\eta_{AB}$ [22] and the tetrad $\epsilon^A_{\mu}$. From there we construct a metric $g_{\mu\nu}$ in terms of the tensor $T^j_{\mu\nu}$,

$$g_{\mu\nu} = \epsilon^A_{\mu} \epsilon^A_{\nu} = \frac{1}{6} \epsilon^{jkl} T^j_{\mu\nu} T^{kl\lambda} T^l_{\lambda\nu},$$

where

$$T^{j\mu\nu} := \frac{1}{2\sqrt{g}} \epsilon^{\mu\nu\lambda\rho} T^j_{\lambda\rho}.$$ (50)

and

$$(\sqrt{g})^3 := \frac{1}{12} \epsilon^j_{kl} T^j_{\mu_1\nu_1} T^k_{\mu_2\nu_2} T^l_{\mu_3\nu_3} \times
\epsilon^j_{j'k'} T^{j'}_{\mu_1'\nu_1'} T^{k'}_{\mu_2'\nu_2'} T^{l'}_{\mu_3'\nu_3'} \times
\epsilon^j_{\mu_1\nu_1\lambda_1} \epsilon^k_{\mu_2\nu_2\lambda_2} \epsilon^l_{\mu_3\nu_3\lambda_3}.$$ (51)

Subsequently, we introduce a triad $d^a_j$ such that

$$B^a_{\mu\nu} =: d^a_j T^j_{\mu\nu}.$$ (52)

This permits us to reexpress the $BB$ term of the classical Lagrangian,

$$\mathcal{L}_{BB} = -\frac{e^2}{4} T^j_{\mu\nu} h_{jk} T^k_{\mu\nu},$$

where $h_{jk} := d^a_j d^a_k$. Putting Eqs. (52) and (50) into the saddle point condition

$$\frac{1}{2} \epsilon^{\lambda\mu\nu} D^b_{\mu}(\hat{A}) B^b_{\kappa\lambda} = +im^2 \hat{A}^a_{\nu}.$$ (54)

gives

$$D^b_{\mu}(\hat{A})(\sqrt{g} d^a_j T^{j\mu\nu}) = +im^2 \hat{A}^a_{\nu}.$$ (55)

In the following we define the connection coefficients $\gamma_{\mu|j}^k$ as expansion parameters of the covariant derivative of the triads at the saddle point in terms of the triads,

$$D^a_{\mu}(\hat{A}) d^a_j := \gamma_{\mu|j}^k d^a_k.$$ (56)

This would not be directly possible for more than two colours, as then the set of triads is not complete. The connection coefficients allow us to define covariant derivatives according to

$$\nabla_{\mu|j}^k := \partial_{\mu} \delta^k_j + \gamma_{\mu|j}^k.$$ (57)

These, in turn, permit us to rewrite the saddle point condition [24] as

$$d^a_j \nabla_{\mu|j}^k (\sqrt{g} T^{j\mu\nu}) = im^2 \hat{A}^a_{\nu},$$ (58)

and the mass term in the classical Lagrangian becomes

$$\mathcal{L}_{AA} = \frac{1}{8\pi^2} \left| \nabla_{\mu|j}^k (\sqrt{g} T^{j\mu\nu}) \right| h_{kl} |\nabla_{\nu|l}^k (\sqrt{g} T^{j\mu\nu})|.$$ (59)

In the limit $m \to 0$ this term enforces the covariant conservation condition $\nabla_{\mu|j}^k (\sqrt{g} T^{j\mu\nu}) \equiv 0$, known for the massless case. It results also directly from the saddle point condition [25]. Here $d^a_j \nabla_{\mu|j}^k (\sqrt{g} T^{j\mu\nu})$ are the direct analogues of the Abelian currents $e^{\mu\nu\kappa} \partial_{\nu} B_{\kappa\lambda}$, which are conserved in the massless case [see Eq. (A3)] and distributed following a Gaussian distribution in the massive case [see Eq. (A8)].

The commutator of the above covariant derivatives yields a Riemann-like tensor $R^k_{\mu|j\nu}$,

$$R^k_{\mu|j\nu} := [\nabla_{\mu|j}, \nabla_{\nu|k}].$$ (60)

By evaluating, in adjoint representation (marked by $\hat{\cdot}$), the following difference of double commutators $[D^k_{\mu}(\hat{A}), [D^\nu_{\nu}(\hat{A}), d_j]] - (\mu \leftrightarrow \nu)$ in two different ways, one can show that

$$i[d_j, \hat{F}_{\nu}(\hat{A})] = \hat{d}_k R^k_{\mu|j\nu},$$ (61)

or in components,

$$F^a_{\mu|\nu}(\hat{A}) = \frac{i}{e} \epsilon^{abc} d^b_j d^c_k R^k_{\mu|j\nu},$$ (62)

where $d^b_j d^c_k := \delta^c_j$ defines the inverse triad, $d^a_j = h_{jk} d^a_k$. Hence, we are now in the position to rewrite the remaining $BF$ term of the Lagrangian density. Introducing Eqs. (52) and (62) into Eq. (10) results in

$$\mathcal{L}_{BF} = i \sqrt{g} T^{j\mu\nu} R^k_{\mu|j\nu} \epsilon^{jmk} h_{lm}.$$ (63)

Let us now repeat the previous steps with a mass term in which the gauge scalars $\Phi$ are explicit,
In that case the saddle point condition \(\text{[54]}\) is given by,
\[
\frac{1}{2} \epsilon^{\kappa \lambda \mu \nu} D^a_\mu (\hat{A}) B^b_\kappa = i m^2 [\hat{A}_\nu - i \Phi (\partial_\nu \Phi^\dagger)]^a ,
\]
or in the form of Eq. \(\text{[58]}\), that is with the left-hand side replaced,
\[
d_k^a \nabla_\mu [\sqrt{g} T^{j \mu \nu}] = im^2 [\hat{A}_\nu - i \Phi (\partial_\nu \Phi^\dagger)]^a .
\]
Reexpressing \(L_{\hat{A}^a}^b\) with the help of the previous equation reproduces exactly the unitary gauge result \(\text{[59]}\) for the mass term.

Finally, the tensor \(B\) appearing in the determinant \(\text{[13]}\), which accounts for the Gaussian fluctuations of the gauge field \(A_\mu^a\), formulated in the new variables reads \(B_{bc}^\mu = \sqrt{g} f^{abc} d^b_\mu T^{j \mu \nu}\).

The last ingredients required to put together the partition function is the quantum measure in gauge invariant variables. Let us choose \(W_{\mu \nu \lambda} := T_{\mu \nu} h_{jk} T_{\lambda}^{jk}\). They are antisymmetric in the first and the second pair of indices and symmetric under exchange of the first pair with the second pair. In three space-time dimensions they would suffice exactly to parametrize the six gauge invariant degrees of freedom. The Jacobian required for the change of variables would be,
\[
\int [dB]_3 \equiv \int [dW]_3 \text{Det} \left( \frac{1}{2} J_3 \right),
\]
where
\[
J_3 := \begin{pmatrix}
W_{1212} & W_{1223} & W_{1231} \\
W_{2112} & W_{2123} & W_{2131} \\
W_{3112} & W_{3123} & W_{3131}
\end{pmatrix}.
\]
In four space-time dimensions for \(SU(2)\) not all \(W_{\mu \nu \lambda}\) are independent and we have to select a subset. One possible choice leads to
\[
\int [dB]_4 \equiv \int [dW]_4 \text{Det} \left( \frac{1}{2} J_4 \right),
\]
where
\[
J_4 := J_3 (31) J_3 (41) J_3 (42) J_3 (43),
\]
the index pair “31” in Eq. \(\text{[58]}\) is each time replaced by the index pair in brackets, and the functional integral runs over all 15 components of \(W_{\mu \nu \lambda}\) contained in the Jacobian.

For a position-dependent mass the above discussion does not change materially. The potential and kinematic term for the mass scalar \(m\) have to be added to the action.

Contrary to the massless case the \(A_\mu^a\) dependent part of the Euclidean action is genuinely complex. Without mass only the T-odd and hence purely imaginary \(BF\) term was \(A_\mu^a\) dependent. With mass there contributes the additional T-even and thus real mass term. Therefore the saddle point value \(\hat{A}_\mu^a\) for the gauge field becomes complex. This is a known phenomenon and in this context it is essential to deform the integration contour of the path integral in the partition function to run through the saddle point \(\text{[26]}\). For the Gaussian integrals which are under consideration here, in doing so, we do not receive additional contributions. The imaginary part \(L_{\hat{A}^a}\) of the saddle point value of the gauge field transforms homogeneously under gauge transformations. The complex valued saddle point of the gauge field which is integrated out does not affect the real-valuedness of the remaining fields, here \(B_{bc}^\mu\). In this sense the field \(B_{bc}^\mu\) represents a parameter for the integration over \(A_\mu^a\). The tensor \(T^{j \mu \nu}\) is real-valued by definition and therefore the same holds also for the triad \(d_j^a\) [see Eq. \(\text{[52]}\)]. \(h_{kl}\) is composed of the triads and, consequently, real-valued as well. The imaginary part of the saddle point value of the gauge field, \(L_{\hat{A}^a}\), enters the connection coefficients \(\text{[59]}\). Through them it affects the covariant derivative \(\text{[67]}\) and the Riemann-like tensor \(\text{[60]}\). More concretely the connection coefficients \(\gamma_\mu^i j^k\) can be decomposed according to
\[
D_{\mu}^{ab} (R \hat{A}) d_j^b = (R \gamma_{\mu j}^k) d_j^b,
\]
and the obvious consequences for the covariant derivative,
\[
\nabla_{\mu j}^k = R \nabla_{\mu j}^k + 2 \nabla_{\mu j}^k,
\]
\[
R \nabla_{\mu j}^k = \partial_\mu d_{j}^k + R \gamma_{\mu j}^k,
\]
\[
\nabla_{\mu j}^k = \tilde{\nabla}_{\mu j}^k.
\]
This composition reflects in the mass term,
\[
\mathcal{L}_{AA} = \frac{1}{2 m^2} \left[ (R \nabla_{\mu}^k (\sqrt{g} T^{j \mu \nu})) h_{kl} [R \nabla_{\nu}^l (\sqrt{g} T^{j \mu \nu})] - (R \gamma_{\mu l}^k (\sqrt{g} T^{j \mu \nu})) h_{kl} [R \gamma_{\nu l}^j (\sqrt{g} T^{j \mu \nu})] \right]
\]
\[
\mathcal{L}_{AA} = \frac{1}{2 m^2} \left[ R \nabla_{\mu}^k (\sqrt{g} T^{j \mu \nu}) h_{kl} [\nabla_{\nu j}^l (\sqrt{g} T^{j \mu \nu})] \right]
\]
on one hand, and in the Riemann-like tensor,
\[
\mathcal{R} R_{j \mu \nu} = [R \nabla_{\mu}, R \nabla_{\nu}]^k - [R \nabla_{\mu} \tilde{\nabla}_{\nu}]^k,
\]
\[
\mathcal{I} R_{j \mu \nu} = [R \nabla_{\mu}, \nabla_{\nu}]^k + [R \nabla_{\mu} \nabla_{\nu}]^k.
\]
on the other. The connection to the imaginary part of \(A_\mu^a\) is more direct in Eq. \(\text{[62]}\) which yields,
\[
\mathcal{R} F_{a \mu} (\hat{A}) = \frac{1}{2} e^{abcdef} d_j^k \mathcal{R} R_{j \mu \nu}^k,
\]
\[
\mathcal{I} F_{a \mu} (\hat{A}) = \frac{1}{2} e^{abcdef} d_j^k \mathcal{I} R_{j \mu \nu}^k.
\]
Finally, the \(BF\) term becomes,
\[
\mathcal{R} L_{BF} = - \frac{1}{2} \sqrt{g} T^{j \mu \nu} \epsilon_{mnk} h_{lmn} \mathcal{R} R_{l \mu \nu}^k,
\]
\[
\mathcal{I} L_{BF} = + \frac{1}{2} \sqrt{g} T^{j \mu \nu} \epsilon_{mnk} h_{lmn} \mathcal{I} R_{l \mu \nu}^k.
\]
Summing up, at the complex saddle point of the \([dA]\) integration the emerging Euclidean \(\mathcal{L}_{AA}\) and \(\mathcal{L}_{BF}\) are both complex, whereas before they were real and purely imaginary, respectively. Both terms together determine the saddle point value \(\hat{A}_\mu^a\). Therefore, they become coupled...
and cannot be considered separately any longer. This was already expected from the analysis in Minkowski space in Sect. [I] where the matrix $M_{\mu\nu}^{ab}$ combines T-odd and T-even contributions, which originate from $\mathcal{L}_{AA}$ and $\mathcal{L}_{BF}$, respectively. There the different contributions become entangled when the inverse $(\mathbf{M}^{-1})_{\mu\nu}^{ab}$ is calculated.

### A. Weinberg–Salam model

Now, let us reformulate the Weinberg–Salam model in geometric variables. We omit here the kinematic term $K(m)$ and the potential term $V(m^2)$ for the sake of brevity because they do not interfere with the calculations and can be reinstated at every time. The remaining terms of the classical action are

$$S := \int d^4x(\mathcal{L}_{BB}^{Abel} + \mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{AA}),$$

$$\mathcal{L}_{AA} := -\frac{m^2}{2}m_{ab}A^a_\mu A^b_\mu,$$  \hspace{1cm} (71)

$$\mathcal{L}_{BB}^{Abel} := -\frac{g^2}{4}B^{\mu\nu}_\mu B^{0\nu}_\mu,$$ \hspace{1cm} (72)

$$\mathcal{L}_{BF}^{Abel} := +\frac{g}{2}i\kappa^{\mu\nu\lambda\rho}B^0_{\mu\nu}F^0_{\lambda\rho},$$  \hspace{1cm} (73)

and $\mathcal{L}_{BB}$ as well as $\mathcal{L}_{BF}$ have been defined in Eqs. (45) and (46), respectively.

The saddle point conditions for the $[dA]$ integration with this action are given by

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}D_{\mu}(\bar{A})B^b_{\lambda\nu} = +im^2m_{ab}A^a_\mu,$$  \hspace{1cm} (74)

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}\partial_{\mu}B^0_{\kappa\lambda} = +im^2m_{ab}A^a_\mu.$$  \hspace{1cm} (75)

For the following it is convenient to use linear combinations of these equations, which are obtained by contraction with the eigenvectors $\mu^a_\mu$ of the matrix $m^{ab}$—defined between Eqs. (55) and (61)—

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}[\mu^a_{\mu}D_{\mu}(\bar{A})B^b_{\lambda\nu} + \mu^a_{\mu}\partial_{\mu}B^0_{\kappa\lambda}] = im^2\mu^a_\mu m^{ab}A^b_\mu.$$  \hspace{1cm} (76)

The non-Abelian term on the left-hand side can be rewritten using the results from the first part of Sect. [III]. The right-hand side may be expressed in terms of eigenvalues of the matrix $m^{ab}$. We find (no summation over $l$),

$$\mu^a_\mu X^{ab} = im^2m_{ab}a^a_\mu,$$  \hspace{1cm} (77)

where

$$X^{ab} := d_{a}^{\mu} \nabla_{\mu}^{\prime}(\sqrt{\kappa}T_{\mu\nu}) + \frac{1}{2}\epsilon^{\lambda\mu\nu}\mu^a_\mu\partial_{\mu}B^0_{\kappa\lambda}.$$  \hspace{1cm} (78)

The mass term can be decomposed in the eigenbasis of $m^{ab}$ as well and, subsequently, be formulated in terms of the geometric variables,

$$\mathcal{L}_{AA} = -\frac{m^2}{2}\sum_l m_{ab}a^l_\mu a^l_\mu =$$

$$\frac{1}{2m^2}(\bar{m}^{-1})^{ab}X^{ab}X^{ab},$$  \hspace{1cm} (79)

where

$$\bar{m}^{-1} := \sum_l \gamma^0 m^{-1}_{ab}a^l_\mu a^l_\mu,$$  \hspace{1cm} (80)

Taking the mass eigenvalues to zero this addend leads to the covariant conservation of the composite current in $X^{ab} = 0$, just as previously observed in the Abelian case in terms of gauge invariant antisymmetric tensor fields and in the non-Abelian case with simple mass term in geometric variables. Like in those situations for finite mass eigenvalues the magnitude of the aforementioned current components follow a Gaussian distribution. The mixture of Abelian and non-Abelian currents is caused by the symmetry breaking pattern $SU(2)_L \times U(1)_Y \rightarrow U(1)_em$ which leaves unbroken $U(1)_Y$ and not the $U(1)_Y$, which is a symmetry in the unbroken phase.

It should be emphasised that in the present geometric representation on the classical level the introduction of a Higgs doublet does not inevitably suggest itself, as its non-radial degrees of freedom are not needed to ensure gauge invariance. In the quantised form the radial degree of freedom of the Higgs takes course care of perturbative renormalisability.

With the help of the above relations and the results from the beginning of Sect. [III] we are now in the position to express the classical action in geometric variables: The mass term is given in the previous expression. The Abelian antisymmetric fields $B^0_{\mu\nu}$ in $\mathcal{L}_{BB}^{Abel}$ are gauge invariant and we leave $\mathcal{L}_{BB}^{Abel}$ as defined in Eq. (72). In geometric variables $\mathcal{L}_{BB}$ is given by Eq. (63) and $\mathcal{L}_{BF}$ by Eq. (69). At the end the kinetic term $K(m)$ and the potential term $V(m^2)$ should be reinstated.

Additional contributions from fluctuations give rise to an addend (on the level of the Lagrangian) proportional to $\frac{1}{2}\ln \det m$, where $m$ can be expressed in the new variables, $m_{\mu\nu}^{bc} = f_{abc}d_\mu^b d_\nu^c T^{0\lambda\mu} - \frac{m^2}{2}\sum_l m_{\mu\nu}^{0l} d_l^a d_l^b g^{\kappa\lambda}$. In order to reconstruct the partition function only the measure of the functional integral has to be translated into gauge invariant variables. The Abelian antisymmetric field $B^0_{\mu\nu}$ is already gauge invariant and can be kept as variable. The integral over the non-Abelian fields can be reexpressed like in Eq. (69).

Repeating the entire calculation not in unitary gauge, but with explicit gauge scalars $\Phi$, yields exactly the same result because the mass term and the saddle point condition change in unison, such that Eq. (77) is obtained again. This has already been demonstrated explicitly for a massive Yang–Mills theory just before Sect. [III].

### B. Technicolour

We begin by replacing Eq. (71) by Eq. (87). Then the saddle point conditions for the variation of the corresponding classical action with respect to the gauge potentials $A^a_\mu$ reads,

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}D_{\mu}(\bar{A})B^b_{\lambda\nu} = +im^2m_{ab}A^a_\mu + mJ^0_\mu,$$  \hspace{1cm} (81)

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}\partial_{\mu}B^0_{\kappa\lambda} = +im^2m_{ab}A^a_\mu + mJ^0_\mu.$$  \hspace{1cm} (82)
The linear combinations analogous to Eq. (70) are given by,

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}[\mu_\lambda B^{\mu}_{\nu}(A)B_{\lambda}^{\nu} + \mu_\nu B^{\mu}_{\lambda} - \mu_\mu B^{\nu}_{\lambda}] = \mu^{\parallel}(im^2 m^{-1})^a A^a_\mu + m J^a_\mu,$$

Reexpressing the covariant derivative on the left-hand side in terms of gauge singlet variables as described in Sect. [11] we obtain (no summation over l),

$$\mu^{\parallel}_\mu(X^{\mu\nu} - m J^\mu_{\nu}) = im^2 m^{-1}a^a_\mu,$$

(83)

where $X^{\mu\nu}$ has been defined in Eq. (78). Evaluating the Lagrangian at the saddle point for the gauge potential we find,

$$\mathcal{L}_{AA} + \mathcal{L}_{1A} = -\frac{a^2}{2}\sum_i m_i a^i_\mu + im J^i_\mu a^i_\mu \rightarrow$$

$$\rightarrow -\frac{a^2}{2m^2}(m^{-1})^{ab} (X^{ab} X^{\mu\nu} - m^2 J^{ab} J^{\mu\nu}),$$

with $(m^{-1})^{ab}$ from Eq. (79). Fluctuations of the gauge field induce the determinant term which on the Lagrangian level is proportional to $\frac{1}{2}\ln \det m$, where again

$$m^{-1}_{\mu\nu} := f^{abc}d^a_\mu \mu^b \sqrt{F_{\mu\nu}} - m^2 \sum_i m_i \delta^{ij} \delta_{\mu\nu}.$$  

As gauge invariant quantum measure for the gauge field sector we can use the expression from Eq. (69).

Hence, outwardly it at first looks as if, apart from the combination in the radial degree of freedom $m^2$, the degree of freedom in the current $X^{\mu\nu}$ decouple from those in the current $J^{\mu\nu}$ at tree level, as long as we do not take into account potential terms for the $F$ field. This is, however, not the case because the current $J^{\mu\nu}$ enters the connection coefficient $\gamma^{\mu\nu}_{ij}$ through the saddle point condition [22] and from there feeds into the current $X^{\mu\nu}$ and the tensor $F^\mu_{\nu\lambda}$. This corresponds to the coupling of said current to the saddle point expression of the vector potential alluded to before in the representation based on antisymmetric tensor fields. The self interaction between the currents is the same as in that case.

**C. Link to torsion**

Here we interpret the above result by pointing out that in three space-time dimensions a non-zero mass leads to the presence of torsion in the geometrical description of SU(2) gauge groups. In three dimensions we introduce a one-form $E$ instead of the two-form $B$ because the dual of the two-form will be the required two-form. Thus, the Lagrangian density reads,

$$\mathcal{L} := \mathcal{L}_{EE} + \mathcal{L}_{EF} + \mathcal{L}_{AA},$$

(84)

where

$$\mathcal{L}_{EE} := -\frac{a^2}{2} F^a_\mu F^{a\mu},$$

(85)

$$\mathcal{L}_{EF} := +\frac{a^2}{2} \epsilon^{\mu\nu\lambda} F^a_\mu F^{a\mu \lambda},$$

(86)

$$\mathcal{L}_{AA} := -\frac{a^2}{2} A^a_\mu A^{\mu a},$$

(87)

We stick to a simple diagonal mass term which is sufficient to understand the reasoning. As next step, we reexpress the mass term through the introduction of a Lagrange multiplier field $C^a_\mu$,

$$\mathcal{L}_{AA} \rightarrow -\frac{1}{4} C^a_\mu C^{a\mu} + im C^a_\mu A^a_\mu.$$  

(88)

For the action constructed from the resulting total Lagrangian the saddle point condition reads,

$$\epsilon^{\mu\nu\lambda} D^{\mu}_\nu (A) E^{a\nu \lambda} = m C^a_\lambda.$$  

(89)

In three dimensions we can interpret the $E^a_\mu$ directly as, in general, complete set of dreibein and define the connection coefficients $\Gamma^a_{\mu \nu}$,

$$D^{\mu}_{ab}(A) E^{a\nu}_\nu := \Gamma^a_{\mu \nu} E^a_\nu.$$  

(90)

The general solution of Eq. (89) for vanishing mass $m$ is given by symmetric connection coefficients,

$$\epsilon^{\mu\nu\lambda} \Gamma^{|a|}_{\mu \nu} = 0.$$  

(91)

For finite mass $m$, $C^a_\mu$ is equal to the antisymmetric part of the connection coefficient,

$$\epsilon^{\mu\nu\lambda} \Gamma^{|a|}_{\mu \nu} E^a_\nu = m C^a_\lambda,$$  

(92)

that is the torsion. The relation obtained by variation with respect to $C^a_\mu$ reads,

$$C^a_\mu = im A^a_\mu.$$  

(93)

Combined with Eq. (92) this yields,

$$\epsilon^{\mu\nu\lambda} \Gamma^{|a|}_{\mu \nu} E^a_\nu = m^2 A^a_\lambda,$$  

(94)

which is the equivalent of Eq. (58).

Adding another term to the total Lagrangian density, $\mathcal{L} \rightarrow \mathcal{L} + \mathcal{L}_{1A}$, like the one arising from the extra pions in technicolour with enhanced symmetry,

$$\mathcal{L}_{1A} := im J^a_\mu A^a_\mu,$$  

(95)

yields another contribution to the torsion,

$$\epsilon^{\mu\nu\lambda} \Gamma^{|a|}_{\mu \nu} E^a_\nu = m C^a_\lambda + m J^a_\lambda.$$  

(96)

The two contributions differ insofar as $C^a_\mu$ does not contain derivatives of the underlying fields and $J^a_\mu$ is exclusively first order in derivatives. Therefore, at very low energies $J^a_\mu$ is suppressed relative to $C^a_\mu$. Hence, there the $C^a_\mu$ contribution to torsion dominates. At higher scales the momentum dependent contribution from $J^a_\mu$ becomes increasingly important.

For electroweak symmetry breaking the mass matrix $m^2$ or its restricted inverse $(m^{-1})^{ab}$ have to be reinstated together with the contributions from the hypercharge field. Neither of these two steps, however, does fundamentally alter what was just said. Going to four space-time dimensions necessitates a generalisation of the concept of torsion. This is most directly perceptible by comparing the connection coefficients $\Gamma^a_{\mu \nu}$ and $\gamma^{\mu\nu}_{ij}$. The latter feature a mismatch between the dimensions of the two lower indices, which inhibits the standard definition of torsion. What for four space-time dimensions replaces the antisymmetric part of $\Gamma^{|a|}_{\mu \nu}$ is the current $X^{\mu\nu}$ given in Eq. (75).
IV. SUMMARY

We have here derived manifestly gauge-invariant formulations for theories breaking the electroweak symmetry. Namely, we have studied the standard model case, that is the Weinberg–Salam model, and dynamical electroweak symmetry breaking through technicolour models. For each approach the derivation proceeded through two stages. The outset was always the standard formulation in terms of Yang–Mills gauge potentials. This field transforms inhomogeneously under gauge transformations. From there we introduced antisymmetric tensor fields and subsequently eliminated the Yang–Mills potential. The antisymmetric tensor fields transform ultralocally under gauge transformation. In this way Stückelberg degrees of freedom, which were required for a gauge invariant formulation of a massive gauge field theory in terms of Yang–Mills potentials, become obsolete. Still, the antisymmetric tensor fields are no gauge singlets. Therefore, in the second stage we have introduced gauge singlet variables, which lead to a formulation in terms of geometric quantities. In this framework we have linked the presence of massive gauge bosons to the presence of torsion in the geometric representation. More precisely, in three space-time dimensions and for a three-dimensional gauge group like the relevant $SU(2)_L$, a massless theory corresponds to a torsionless geometric description. When mass is included the torsion becomes non vanishing, but follows a Gaussian distribution centred around zero and the width of which is given by the mass. For other combinations of the number of space-time dimensions and the dimension of the gauge group a generalisation of the concept of torsion is necessary.

The mass-generation for the gauge bosons of the electroweak interactions possesses a number of non-standard features: It exhibits a non-diagonal breaking pattern, $SU(2)_L \times U(1)_Y \rightarrow U(1)_em$, and the related non-diagonal mass term with, on top, a zero eigenvalue for the massless photon, on one hand, and the position-dependent mass, that is the Higgs degree of freedom needed for perturbative renormalisability. In order to disentangle which characteristic of the alternative formulations arises from which trait of the massive and non-Abelian gauge theory, we first present the translation for a massless non-Abelian theory. We then continue with massive non-Abelian theories with diagonal mass terms and constant mass. The next generalisation is mandated by the requirement of perturbative renormalisability and leads to a position-dependent mass also known as Higgs degree of freedom. In order to be able to accommodate the phenomenologically relevant breaking pattern the generalisation to a non-diagonal mass term has to be performed. What is referred to as the Higgs doublet is a combination of this radial degree of freedom and the aforementioned Stückelberg degree of freedom. In the treatment with Yang–Mills potential one commonly picks an "expectation value" for the Higgs doublet field, however, misleading as in reality it is not even necessarily different from zero $\langle 0 \rangle$. This discord can also be avoided in the standard formulation. In the alternative formulations spelled out here, though, this is automatic.

Technicolour models which pass the constraints from currently available electroweak precision data have, in general, larger flavour symmetries than the minimally necessary $SU(2)_L \times SU(2)_R$ which by breaking to $SU(2)_V$ provides the three necessary longitudinal degrees of freedom for the W and Z bosons. For this reason, they have a richer low-energy particle content, among which are the additional pseudoscalars of importance to this investigation. The extras do not correspond to Stückelberg degrees of freedom of the electroweak gauge symmetry and can accordingly not be absorbed in the antisymmetric tensor fields and also appear explicitly in the geometric formulation. In the $B$-field formulation currents constructed from these additional fields act as source for the saddle point expression of the vector potential expressed in terms of the $B$-fields. There, and later on in the geometric representation a term quadratic in the currents appears. It is of fourth order in the fields and of second order in derivatives. In general, because said current contains one derivative, it decouples at small momenta. In the geometric description it enters in the definition of the connection coefficients and from there the Riemann-like tensor.

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APPENDIX A: ABELIAN

1. Massless

The partition function of an Abelian gauge field theory without fermions is given by

$$ P := \int [dA] \exp(i \int d^4x \mathcal{L}) $$

(A1)

with the Lagrangian density

$$ \mathcal{L} = \mathcal{L}_0 := -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} $$

(A2)

and the field tensor

$$ F_{\mu\nu} := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. $$

(A3)
$g$ stands for the coupling constant. The transition to the first-order formalism can be performed just like in the non-Abelian case, which is treated in the main body of the paper. We find the partition function,

$$P = \int [dA][dB] \times \exp\{i \int d^4x \left[ -\frac{1}{2} \tilde{F}_{\mu\nu} \tilde{B}^{\mu\nu} - \frac{g^2}{4} B_{\mu\nu} B^{\mu\nu} \right] \}.$$  

Here the antisymmetric tensor $B_{\mu\nu}$, like the field tensor $F_{\mu\nu}$, is gauge invariant. The classical equations of motion are given by

$$\partial_\mu \tilde{B}^{\mu\nu} = 0 \quad \text{and} \quad g^2 B_{\mu\nu} = -\tilde{F}_{\mu\nu}, \quad (A4)$$

which after elimination of $B_{\mu\nu}$ reproduce the Maxwell equations one would obtain from Eq. (A2). Now we can formally integrate out the gauge field $A_\mu$. As no gauge is fixed by the $BF$ term because the Abelian field tensor $F_{\mu\nu}$ is gauge invariant this gives rise to a functional $\delta$ distribution. This constrains the allowed field configurations to those for which the conserved current $\partial_\mu \tilde{B}^{\mu\nu}$ vanishes,

$$P \cong \int [dB] \delta(\partial_\mu \tilde{B}^{\mu\nu}) \exp\{i \int d^4x \left[ -\frac{g^2}{4} B_{\mu\nu} B^{\mu\nu} \right] \}. \quad (A5)$$

2. Massive

In the massive case the Lagrangian density becomes $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$, where $\mathcal{L}_m := \frac{m^2}{2} A_\mu A^\mu$. First, we here repeat some steps carried out above in the non-Abelian case: We can directly write down the partition function in unitary gauge. Regauging like in Eq. (15) leads to

$$P = \int [dA][dU] \exp\{i \int d^4x \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} [A_\mu - i U^\dagger (\partial_\mu U)][A^\mu - i U^\dagger (\partial^\mu U)] \right]\}.$$  

The corresponding gauge-invariant Lagrangian then reads,

$$\mathcal{L}_c := -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} (D_\mu \Phi)^\dagger (D^\mu \Phi), \quad (A7)$$

with the constraint $\Phi^\dagger \Phi \equiv 1$. Constructing a partition function in the first-order formalism from the previous Lagrangian yields,

$$P \cong \int [dA][d\Phi][dB] \times \exp\{i \int d^4x \left[ -\frac{1}{2} B_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{g^2}{4} B_{\mu\nu} B^{\mu\nu} + \frac{m^2}{2} [A_\mu - i \Phi (\partial_\mu \Phi)] [A^\mu - i \Phi (\partial^\mu \Phi^\dagger)] \right]\}.$$  

The $\Phi$ fields can be absorbed entirely in a gauge-transformation of the gauge field $A_\mu$. The integration over $\Phi$ decouples. This can also be seen by putting the parametrisation $\Phi = e^{-i\theta}$ into the previous equation and carrying out the $[dA]$ integration,

$$P \cong \int [dB][d\theta] \exp\{i \int d^4x \left[ -\frac{g^2}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2m} (\partial_\mu \tilde{B}^{\mu\nu}) g_{\mu\nu} (\partial_\lambda \tilde{B}^{\lambda\nu}) - (\partial_\mu \theta)(\partial^\mu B^{\kappa\nu}) \right] \}. \quad (A8)$$

The only $\theta$ dependent term in the exponent is a total derivative and drops out, leading to a factorisation of the $\theta$ integral. Contrary to the non-Abelian case there arises no fluctuation determinant depending on dynamical fields. Hence, in the Abelian case starting with the classical action in Eq. (A8) or the non-linear sigma model (A6) is equivalent (29).

A third way which yields the same final result, starts by integrating out the $\theta$ field first. This gives a transverse mass term $\sim 4 \mu (g_{\mu\nu} - \frac{\partial_\mu \theta}{\partial^\nu \theta}) A^\nu$. Integration over $\mu$ then leads to the same result as before.

Instead of a vanishing current $\partial_\mu B^{\mu\nu}$ like in the massless case, in the massive case the current has a Gaussian distribution. The distribution’s width is proportional to the mass of the gauge boson.

$$m \to 0 \ \text{limit}$$

In the gauge-field representation the massless limit for the classical actions discussed above are smooth. In terms of the $B_{\mu\nu}$ field the mass $m$ ends up in the denominator of the corresponding term in the action. Together with the $m$ dependent normalisation factors arising form the integrations over the gauge-field in the course of the derivation of the $B_{\mu\nu}$ representation, however, the limit $m \to 0$ still yields the $m = 0$ result for the partition function (A8).

Still, it is known that the perturbative propagator for a massive photon is ill-defined if the mass goes to zero: Variation of the exponent of the Abelian massive partition function in unitary gauge with respect to $A_\mu$ and $A_\lambda$ gives the inverse propagator for the gauge fields,

$$(G^{-1})^{\kappa\lambda} = [(p^2 - m_{\text{phys}}^2)g^{\kappa\lambda} - p^\kappa p^\lambda], \quad (A9)$$

which here is already transformed to momentum space. The corresponding equation of motion,

$$(G^{-1})^{\kappa\lambda} G_{\lambda\mu} \equiv g_{\mu}^{\nu}, \quad (A10)$$

is solved by

$$G_{\lambda\mu} = \frac{g_{\lambda\mu}}{p^2 - m_{\text{phys}}^2} - \frac{1}{m_{\text{phys}}^2} \frac{p_\lambda p_\mu}{p^2 - m_{\text{phys}}^2}, \quad (A11)$$

with boundary conditions (an $\epsilon$ prescription) to be specified and $m_{\text{phys}} := mg$. This propagator diverges in the limit $m \to 0$. 


In the representation based on the antisymmetric tensor fields, variation of the exponent of the partition function \(\Delta_S\) with respect to the fields \(\tilde{B}_{\mu\nu}\) and \(\tilde{B}_{\kappa\lambda}\) yields the inverse propagator
\[
(G^{-1})^{\mu\nu|\kappa\lambda} = g^\mu\rho g^\kappa\lambda - g^\mu\kappa g^\rho\lambda + \frac{m_{\text{phys}}^2}{p^2} (\partial^\mu \partial^\rho g^\kappa\lambda - \partial^\mu \partial^\kappa g^\rho\lambda - \partial^\rho \partial^\kappa g^\mu\lambda + \partial^\rho \partial^\mu g^\kappa\lambda),
\]
(A12)
already expressed in momentum space. Variation with respect to \(\tilde{B}_{\mu\nu}\) instead of \(B_{\mu\nu}\) corresponds only to a reshuffling of the Lorentz indices and gives an equivalent description. The antisymmetric structure of the inverse propagator is due to the antisymmetry of \(\tilde{B}_{\mu\nu}\). The equation of motion is then given by
\[
(G^{-1})^{\mu\nu|\kappa\lambda} G_{\kappa\lambda|\rho\sigma} = \frac{1}{g^\mu\rho} g^\nu\sigma - g^\mu\sigma g^\nu\rho
\]
(A13)
and solved by
\[
2G_{\kappa\lambda|\rho\sigma} = (g_{\kappa\rho} g_{\lambda\sigma} - g_{\kappa\sigma} g_{\lambda\rho}) - \frac{1}{p^2 - m_{\text{phys}}^2} \times
\]
\[
\times (p_\kappa p_\rho g_{\lambda\sigma} - p_\kappa p_\sigma g_{\lambda\rho} - p_\lambda p_\rho g_{\kappa\sigma} + p_\lambda p_\sigma g_{\kappa\rho}).
\]
(A14)
Here we observe that the limit \(m \to 0\) is well-defined,
\[
2G_{\kappa\lambda|\rho\sigma} \xrightarrow{m \to 0} g_{\kappa\rho} g_{\lambda\sigma} - g_{\kappa\sigma} g_{\lambda\rho} - \frac{1}{p^2} (p_\kappa p_\rho g_{\lambda\sigma} - p_\kappa p_\sigma g_{\lambda\rho} - p_\lambda p_\rho g_{\kappa\sigma} + p_\lambda p_\sigma g_{\kappa\rho}).
\]
(A15)
This is due to the consistent treatment of the gauge degrees of freedom in the second approach.

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[31] There are other descriptions of massive theories involving antisymmetric tensor fields not arising from a reformulation [2].
[32] The functional integral over a gauge field is ill-defined as long as no gauge is fixed. We have to keep this fact in mind at all times and will discuss it in detail when a field is really integrated out.
[33] Singular configurations can be linked to the Wu–Yang ambiguity [7].
[34] Due to the possible occurrence of what is known as Gribov copies [27] this method might not achieve a unique splitting of the two parts of the integration. For our illustrative purposes, however, this is not important.
[35] For two and/or three colours and four space-time dimensions there exist also other treatments of the massless setting [28], different from the one which here is extended to the massive case.