On the GL(2n) eigenvariety: branching laws, Shalika families and $p$-adic $L$-functions

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Abstract

In this paper, we prove that a GL(2n)-eigenvariety is étale over the (pure) weight space at non-critical Shalika points, and construct multi-varibled $p$-adic $L$-functions varying over the resulting Shalika components. Our constructions hold in tame level 1 and Iwahori level at $p$, and give $p$-adic variation of $L$-values (of regular algebraic cuspidal automorphic representations, or RACARs, of GL(2n) admitting Shalika models) over the whole pure weight space. In the case of GL(4), these results have been used by Loeffler and Zerbes to prove cases of the Bloch–Kato conjecture for GSp(4).

Our main innovations are: a) the introduction and systematic study of ‘Shalika refinements’ of local representations of GL(2n), evaluating their attached local twisted zeta integrals; and b) the $p$-adic interpolation of representation-theoretic branching laws for GL(n) × GL(n) inside GL(2n). Using (b), we give a construction of many-varibled $p$-adic functionals on the overconvergent cohomology groups for GL(2n), interpolating the zeta integrals of (a). We exploit the resulting non-vanishing of these functionals to prove our main arithmetic applications.

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1. Introduction

1.1. Motivation

The Bloch–Kato conjectures are amongst the most important open problems in modern algebraic number theory, and predict a deep link between arithmetic and analysis. Through decades of research, a fruitful approach to Bloch–Kato has been to find and prove $p$-adic reinterpretations; for every special case of the Bloch–Kato conjecture, there should be an analogous $p$-adic Iwasawa Main Conjecture relating $p$-adic arithmetic data to a $p$-adic $L$-function. These $p$-adic reinterpretations are usually more tractable than the original conjectures – for example, the Iwasawa Main Conjecture for elliptic curves has been proved in many cases (for example in [Kat04, SU14], but also in many other works). Moreover, understanding the $p$-adic picture can lead to proofs of special cases of Bloch–Kato.

Crucial to proofs of Bloch–Kato/Iwasawa Main Conjectures is a good understanding of $p$-adic $L$-functions, eigenvarieties, and $p$-adic $L$-functions over eigenvarieties. In this paper, we prove new results about these objects for $GL_{2n}/\mathbb{Q}$. In particular, let $\pi$ be a regular algebraic symplectic cuspidal automorphic representation (RASCAR) of $GL_{2n}(\mathbb{A})$ that is everywhere spherical; here symplectic implies $\pi$ admits a Shalika model (i.e. is a functorial transfer from $\text{GSpin}_{2n+1}(\mathbb{A})$). This ensures there is an integer $w$ such that $\pi \cong \pi^w \otimes \cdot^w$, i.e. $\pi$ is essentially self-dual. Let $\hat{\pi}$ be a non-critical slope Iwahoric $p$-refinement. By [BSDW, Thm. A], there is a $p$-adic $L$-function $L_p(\hat{\pi})$ attached to $\hat{\pi}$, that is, a locally analytic distribution on $\mathbb{Z}_p^\times$ of controlled growth that interpolates its Deligne-critical $L$-values. In this paper:

(A) we prove that that (full, Iwahoric) $GL_{2n}$-eigenvariety is étale over the pure weight space at $\hat{\pi}$, and that the unique $(n+1$ dimensional) connected component $\mathcal{C}$ through $\hat{\pi}$ contains a Zariski-dense set $\mathcal{C}_{\text{sha}}$ of classical Shalika points; and

(B) we construct an $(n+2)$-variable $p$-adic $L$-function $L_p^\mathcal{C}$ interpolating $L_p(\hat{\pi})$ for $y \in \mathcal{C}_{\text{sha}}$.

We describe an application. In the special case of $GL_4$, part (B) fulfills the forward compatibility required by Loeffler and Zerbes in their recent tour-de-force work [LZb] proving new cases of the Bloch–Kato conjecture for $GSp_4$; in particular, the present paper is the ‘forthcoming work’ mentioned in §17.5 op. cit., where this special case was first announced.

1.2. Set-up and previous work

Let $\pi$ be as above, and let $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ be its weight, a dominant algebraic character of the diagonal torus $T \subset G := GL_{2n}$. Then $w = \lambda_n + \lambda_{n+1}$ is the purity weight of $\lambda$ (see §2.2). Let $L(\pi, s)$ be the standard $L$-function of $\pi$, normalized so that for $j \in \mathbb{Z}$, the value $L(\pi, j + \frac{1}{2})$ is Deligne-critical if and only if

$$j \in \text{Crit}(\lambda) := \{ j \in \mathbb{Z} : -\lambda_{n+1} \geq j \geq -\lambda_n \}. \quad (1.1)$$

Let $K = Iw_G \prod_{l \neq p} G(\mathbb{Z}_l) \subset G(\mathbb{Z})$, where $Iw_G$ is the Iwahori subgroup at $p$. Let $S_K$ be the locally symmetric space for $G$ of level $K$. As $\pi$ is regular algebraic, it contributes to the compactly supported cohomology of $S_K$ with coefficients in $\mathcal{V}_\chi^\vee$ in degrees $n^2, n^2 + 1, \ldots, n^2 + n - 1$. Here $V_\chi$ is the algebraic representation of $G$ of highest weight $\lambda$, and $\mathcal{V}_\chi$ is the local system on $S_K$ attached to its dual. Let $t = n^2 + n - 1$ (the top degree).

Our work builds on ideas of Grobner–Rahuram [GR14], of Dimitrov–Janzumszewski–Raghubram [DJR20], and particularly of Barrera–Dimitrov–Williams [BSDW], all of which worked in the $Q$-parahoric setting, for $Q$ the $(n, n)$-parabolic subgroup of $G$. As op. cit., our methods are built upon the existence of evaluation maps, functionals on Betti cohomology groups. In particular:

- In [GR14], the authors constructed $\mathbb{C}$-valued evaluation maps

$$E_{j,y}^{\lambda,GR} : H^j_c(S_K, \mathcal{V}_\chi^\vee(\mathbb{C})) \to \mathbb{C}$$

and used them to prove algebraicity for the Deligne-critical $L$-values $L(\pi, j + \frac{1}{2})$. 

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In [DJR20], for $\chi$ finite-order of conductor $p^3$, the authors used $p$-adic analogues

$$\text{Ev}_{\chi,j}^{\lambda,M}: H_p^j(S_K, \mathcal{F}_\chi(Q_p)) \to \overline{Q}_p$$

to construct $p$-adic $L$-functions for ordinary ‘Shalika’ $Q$-parahoric $p$-refinements $\tilde{\pi}_Q$ of $\pi$.

In [BSDW], the authors constructed (parahoric) overconvergent evaluations

$$H_p^j(S_K, \mathcal{G}_Q^j) \to \mathcal{D}(Z_p^\times, \mathcal{O}_Q),$$

where $\Omega$ is a (2-dimensional) parahoric $p$-adic family of weights, $\mathcal{G}_Q^j$ is a space of parahoric distributions, and $\mathcal{D}(Z_p^\times, R)$ is the space of $R$-valued locally analytic distributions on $Z_p^\times$; so $\mathcal{D}(Z_p^\times, \mathcal{O}_Q)$ is a space of 3-variable distributions. These interpolated the $\text{Ev}_{\chi,j}^{\lambda,M}$ for varying $\lambda, \chi$ and $j$. They were used to construct $p$-adic $L$-functions $L_p(\tilde{\pi}_Q)$ attached to finite slope $Q$-parahoric Shalika $p$-refinements $\tilde{\pi}_Q$, to construct 2-dimensional (parahoric) $p$-adic families through $\tilde{\pi}_Q$, and to vary $L_p(\tilde{\pi}_Q)$ over these families.

These papers work with $GL_{2n}$ over totally real fields and do not require $\pi$ everywhere spherical; for a detailed summary of these works, we refer the reader to [BSDW, Intro.]. However, the $Q$-parahoric setting considered op. cit. cannot see variation in more than 2 weight variables, meaning our present results are necessary for the application to Bloch–Kato in [LZb].

### 1.3. New input

In our generalisation to Iwahoric families, substantial new ideas are required in two particular places: one automorphic (the computation of local zeta integrals at $p$), and one $p$-adic (the $p$-adic interpolation of classical representation-theoretic branching laws).

#### 1.3.1. Shalika $p$-refinements and local zeta integrals

To study local zeta integrals attached to the (unramified) representation $\pi_p$, we introduce Shalika $p$-refinements of $\pi_p$. In this paper, we consider Iwahoric $p$-refinements, rather than the $Q$-parahoric $p$-refinements of [DJR20, BSDW].

Let $H_p^G$ be the Hecke algebra at $p$ (see §2.4.3). An (Iwahoric) $p$-refinement of $\pi_p$ is a system of Hecke eigenvalues

$$\alpha: H_p^G \longrightarrow \overline{Q}$$

appearing in $\pi_p^{\text{Iw.g}}$; we write $\tilde{\pi}_p = (\pi_p, \alpha)$. If $\pi_p$ has regular semisimple Satake parameter, there are $2n!$ such $p$-refinements, indexed by elements of the Weyl group $W_G = S_{2n}$.

Attached to any $p$-refinement $\tilde{\pi}_p$ is a certain family of twisted local zeta integrals at $p$. We call $\tilde{\pi}_p$ a Shalika $p$-refinement if one of these local zeta integrals is non-vanishing.

In [DJR20, BSDW], it is implicitly predicted that a $p$-refinement should be Shalika if and only if it lies in a certain class of ‘spin’ $p$-refinements, i.e. those that interact well with the Shalika model. In the parahoric case, an ad-hoc definition of a spin refinement – there called a ‘$Q$-regular $Q$-refinement’ – is given in [DJR20, §3.3], inspired by [AG94]; and the relevant zeta integrals are shown to be non-vanishing.

In §6, we give a much more conceptual definition of spin $p$-refinements, generalising and justifying [DJR20, §3.3]. Since $\pi_p$ admits a Shalika model, it is the functorial transfer of a representation $\Pi_p$ of $G(Q_p)$, where $G = GSpin_{2n+1}$. Via a careful study of the root systems of $G$ and $G$, we construct a map $j^\vee: H_p^G \rightarrow H_p^G$ of Hecke algebras at $p$. We then say $\alpha$ is a spin $p$-refinement if there exists an eigensystem $\alpha^G$ for $G$ such that $\alpha$ factors as

$$\alpha: H_p^G \xrightarrow{j^\vee} H_p^G \xrightarrow{\alpha^G} \overline{Q},$$

and show the system $\alpha^G$ then appears in $\Pi_p^{\text{Iw.g}}$. There are $2^n n!$ such spin $p$-refinements.

We prove that the family of local zeta integrals attached to a spin $p$-refinement is non-vanishing, and hence that spin $p$-refinements are Shalika $p$-refinements. We hope to prove the converse (that Shalika refinements are spin refinements) in a sequel.

We actually compute the local zeta integral in two different ways, with different benefits.


• In §5, we compute the local zeta integral at Iwahori level, with the restriction that our method works only for ramified characters.

• The unramified case at Iwahori level appears to be very difficult. We get around this in §9 by instead computing the unramified integral at parahoric level, via a totally different method.

The latter result strengthens the results of [DJR20, BSDW] by proving that the $p$-adic $L$-functions they construct satisfy the expected interpolation property at the trivial character, a fundamental case these works omitted.

(In later sections, we use interpolation at ramified characters to prove that the $p$-adic $L$-functions at Iwahori level and parahoric level agree; and thus we obtain interpolation at the trivial character for the $p$-adic $L$-functions of the present paper.)

All of these local results are proved in §5–§9.

1.3.2. $p$-adic interpolation of branching laws. Over $Q$, for any $n$ the evaluation maps of [BSDW] are valued in a space of distributions in only 3 variables. In the present paper we construct evaluation maps in the full expected $n+2$ variables.

Our key input is a $p$-adic interpolation, in $(n+2)$-variables, of classical representation-theoretic branching laws. More precisely: in [GR14, DJR20, BSDW] the subgroup $H = GL_n \times GL_n$ (embedded diagonally in $GL_{2n}$) plays a distinguished role. For $j_1, j_2 \in Z$, let $V^{H}_{j_1, j_2}(\pi)$ denote the $H$-representation $\det^{j_1} \det^{j_2}$, the algebraic representation of highest weight $(j_1, \ldots, j_1, j_2, \ldots, j_2)$. Recall $w$ is the purity weight of $\lambda$. Then we have the following reinterpretation of the Deligne-critical $L$-values (1.1):

\[(\dagger) \quad \text{Branching law: } j \in \text{Crit}(\lambda) \iff V^{H}_{(-j, w+j)}(\lambda) \subset V^H_{\lambda}|_H \text{ with multiplicity one.}\]

Example. Let $G = GL_2$, $H = GL_1 \times GL_1$, and let $\pi$ be a RACAR of weight $(k, 0)$, corresponding to a classical newform $f$ of weight $k+2$. Then (in our normalisations) $\pi$ has Deligne-critical $L$-values $L(\pi, j + \frac{1}{2})$ with $-k \leq j \leq 0$. Here $w = k$ and $V^H_{\lambda} = \text{Sym}^k(C^2)$, the space of homogeneous polynomials in two variables $X,Y$ of degree $k$. We have $V^H_{\lambda}|_H = \oplus_{j=-k}^{0}[C \cdot X^{-j}Y^{k+j}]$.

The summand at $j$ is the character $(*, *) \mapsto s^{-j}Y^{k+j}$ of $H$, and corresponds to the Deligne-critical $L$-value $L(\pi, j + \frac{1}{2})$.

Attached to $H$ are a family of automorphic cycles $\{X_\beta\}_{\beta \geq 1}$, which are modified locally symmetric spaces for $H$ which crucially have dimension $t$. For $j \in \text{Crit}(\lambda)$, the evaluation maps $E v_{\lambda, j}^{H, \text{DJR}}$ of [DJR20] were constructed as a composition

\[H^t(K, Y_\lambda) \to H^t(X_\beta, Y_\lambda^H|_H) \to H^t(X_\beta, Y_\lambda^{H}_{(-j, w+j)}) \to \bigoplus_{(Z/p^\infty)^x} \mathcal{O}_p \to \mathcal{O}_p,\]

where:

• the first map is a twisted pullback under the natural inclusion $X_\beta \to S_K$,

• the second is projection of the coefficients via the branching law $(\dagger)$,

• the third is integration of scalar-valued classes (of degree $t$) over the connected components of $X_\beta$ (which have dimension $t$),

• and the last map is ‘evaluation at $\chi’$, where $\chi$ has conductor $p^\beta$.

To interpolate these maps, in [BSDW] the authors gave a $p$-adic interpolation of $(\dagger)$. There were two aspects to this: the interpolation for a fixed $\lambda$ as $j$ varies, proved in §5.2 op. cit., used to prove existence of $L_p(\pi)$; and the interpolation as $\lambda$ varies in a (2-dimensional) $Q$-parabolic weight family $\mathcal{W}^Q$ in §6.2, used to construct parabolic families of RASCARs and 3-variable $p$-adic $L$-functions for these families.
The full pure weight space \( \mathcal{W}_0^G \) for \( GL_{2n} \) has dimension \( n+1 \), whilst the families of [BSDW] are only 2-dimensional. The trade-off made op. cit. was that variation in lower dimension allowed weaker assumptions on \( \tilde{\pi} \), giving an ‘optimal’ notion of non-criticality for \( \tilde{\pi} \). However, if one assumes stronger conditions on \( \tilde{\pi} \), then one expects to be able to vary the \( p \)-adic \( L \)-function over the full \( (n+1) \)-dimensional pure weight space; and it is this, higher-dimensional, variation that is required for the application to the Bloch–Kato conjecture of [LZb].

To extend the results of [BSDW] to get full variation, one needs to interpolate the branching law (1) as \( \lambda \) varies over \( \mathcal{W}_0^G \). The approach of [BSDW] has the parahoric, hence 2-dimensional, setting baked into it, so to interpolate in higher dimension requires new ideas.

In Proposition 11.12, we give a full interpolation of (1) over \( \mathcal{W}_0^G \) (in the language of \( p \)-adic distributions). This result occupies the entirety of §11. Our approach exploits properties of spherical varieties, and should apply in much more general settings.

In §12, via §4, we use our \( p \)-adic branching laws to construct \( (n+2) \)-variabled evaluation maps

\[
Ev_{\lambda,j}^\Omega : H^i_c(S_K, \mathcal{F}_\Omega) \rightarrow D(\mathbb{Z}_p^\times, \mathcal{O}_\Omega),
\]

for an \( (n+1) \)-dimensional affinoid \( \Omega \subset \mathcal{W}_0^G \). These maps interpolate (Iwahoric analogues of) \( Ev_{\lambda,j}^{L,DR} \) as \( \lambda \) varies in \( \Omega \), \( j \) varies over \( \text{Crit}(\lambda) \), and \( \chi \) varies over finite-order Hecke characters of conductor \( p^\beta \).

### 1.4. Main results

We give two main applications of these results. Using the Shalika refinements of §7, we give simple automorphic criteria for the non-vanishing of the evaluation maps (1.2). This non-vanishing puts tight restrictions on the structure of \( H^i_c(S_K, \mathcal{F}_\Omega) \), and thus – via [Han17] – on the structure of the \( GL_{2n} \)-eigenvariety \( \mathcal{E}_{GL_{2n}} \). Exploiting ideas developed in [BSDW], we prove:

**Theorem A.** Suppose \( \lambda \) is regular, and let \( \tilde{\pi} \) be a regular non-critical slope \( p \)-refinement of \( \pi \). Then the \( GL_{2n} \)-eigenvariety \( \mathcal{E}_{GL_{2n}} \) is étale over \( \mathcal{W}_0^G \) at \( \tilde{\pi} \). There exists a neighbourhood of \( \tilde{\pi} \) in \( \mathcal{E}_{GL_{2n}} \) containing a very Zariski-dense set of classical points corresponding to RASCARs.

In other words, there exists an \( (n+1) \)-dimensional neighbourhood \( \Omega \subset \mathcal{W}_0^G \) of \( \lambda \) such that:

(a) \( \tilde{\pi} \) varies in a unique \( p \)-adic family \( \mathcal{C} \subset \mathcal{E}_{GL_{2n}} \) over \( \Omega \),

(b) \( \mathcal{C} \) contains a very Zariski-dense set \( \mathcal{C}^{sha} \) of classical points corresponding to RASCARs,

(c) and the weight map \( \mathcal{C} \rightarrow \Omega \) is an isomorphism.

To prove this result, we observe that regularity of \( \lambda \) implies existence of a non-vanishing Deligne-critical \( L \)-value. Since \( Ev_{\lambda}^\Omega \) interpolates these \( L \)-values, it is therefore non-vanishing. We use this twice: once to produce existence of an \( (n+1) \)-dimensional family, and again to prove existence of a Zariski-dense set of classical points attached to RASCARs.

Our second main result, under the same hypotheses, is the construction of an \( (n+2) \)-variable \( p \)-adic \( L \)-function over \( \mathcal{C} \). We show that \( Ev_{\lambda+1}^\Omega = Ev_{\lambda}^\Omega \circ U_p^\alpha \), where \( U_p^\alpha \) is the full (normalised) Iwahori Hecke operator at \( p \). We thus use (1.2) to attach a well-defined distribution\( \mu_i^\Omega(\Phi) := (\alpha_p^\lambda)^{-i}Ev_{\lambda}^\Omega(\Phi) \) to any finite-slope eigenclass \( \Phi \in H^i_c(S_K, \mathcal{F}_\Omega) \) with \( U_p^\alpha \Phi = \alpha_p^\lambda \Phi \). Note this is independent of \( \beta \). We show existence of a distinguished eigenclass \( \Phi_\varphi \in H^i_c(S_K, \mathcal{F}_\Omega) \) attached to the family \( \mathcal{C} \), and then define \( L_\varphi^\Omega := \mu_i^\Omega(\Phi_\varphi) \in D(\mathbb{Z}_p^\times, \mathcal{O}_\Omega) \). Under the Amice transform, and via (b), we view \( L_\varphi^\Omega \) as a rigid analytic function \( L_\varphi^\Omega \) on \( \mathcal{C} \times X(\mathbb{Z}_p^\times) \), where \( X(\mathbb{Z}_p^\times) \) is the \( \mathbb{Q}_p \)-rigid space of characters on \( \mathbb{Z}_p^\times \). We show:

**Theorem B.** For all \( y \in \mathcal{C}^{sha} \), there exist \( c_{\varphi y}^\pm \in \mathbb{Q}_p^\times \) such that for all \( \chi \in \mathcal{X}(\mathbb{Z}_p^\times) \) with \( \chi(-1) = \pm 1 \), we have

\[
L_\varphi^\Omega(y, \chi) = c_{\varphi y}^\pm \cdot L_p(\tilde{\pi}_y, \chi),
\]

where \( L_p(\tilde{\pi}_y, -) \) is the \((1\text{-variable}) \ p \)-adic \( L \)-function from [BSDW].
In other words, $\mathcal{L}_p^y$ interpolates the $p$-adic $L$-functions from [BSDW] as $y$ varies in $\mathcal{C}$.

1.5. Application: Bloch–Kato for $\text{GSp}(4)$

In [LZb], Loeffler and Zerbes prove new cases of the Bloch–Kato conjecture for Galois representations attached to Siegel modular forms of genus 2 (i.e. for automorphic representations of $\text{GSp}_4$). More precisely, if $\mathcal{F}_{\text{new}}$ is a Siegel modular form of level 1 and sufficiently high weight, and $\mathcal{F}$ is an ordinary $p$-stabilisation, they prove the Bloch–Kato conjecture holds for the spin Galois representation attached to $\mathcal{F}$ in analytic rank 0. This has also led to new understanding of the Bloch–Kato conjecture for symmetric cube modular forms [LZc] and of Iwasawa theory for quadratic Hilbert modular forms [LZa].

In [LZb], the authors built on previous joint works with Skinner and Pilloni [LSZ, LPSZ] constructing Euler systems and $p$-adic $L$-functions for $\text{GSp}_4$. For applications to Bloch–Kato in analytic rank 0, one wants to show the Euler system of [LSZ] is non-trivial. The main new input in [LZb] was an explicit reciprocity law relating the Euler system of [LSZ] to a specific value of the $p$-adic $L$-function of [LPSZ]. If this $p$-adic $L$-value does not vanish, then the Euler system is non-trivial and can be used to bound a Selmer group.

This non-vanishing is delicate, since the $p$-adic $L$-value seen by the explicit reciprocity law is outside the region of interpolation (so it does not directly relate to a Deligne-critical classical $L$-value). In [LZb, §17], Loeffler–Zerbes deform this into the region of interpolation – and thus prove the Bloch–Kato conjecture – conditional on the existence of a family of $p$-adic $L$-functions on $\text{GL}_4$, stated as Theorem 17.6.2 op. cit. This theorem, whose proof was deferred to ‘forthcoming work’ of the present authors, is a special case of Theorem B.

1.6. Remarks on assumptions

We restrict to base field $\mathbb{Q}$ and $\pi$ of tame level 1, a setting where all of our key new ideas are already present. These assumptions drastically simplify the notation and reduce technicality, allowing for a shorter, more conceptual article, whilst still including the results required by [LZb]. We indicate which of our various assumptions could be relaxed.

Firstly, all of these results can be modified in a conceptually straightforward (but notationally awkward) way to work for $\text{GL}_{2n}$ over an arbitrary totally real field $F$. This was the setting treated in [BSDW]; the reader could consult that paper for the extra details occurring in this case.

Our most serious assumption is that $\pi$ has tame level 1. This is certainly not necessary to $p$-adically interpolate branching laws and evaluation maps, and our results in this direction make no assumption on the tame level. We can also entirely remove this assumption in proving Theorem A, following [BSDW, §7]. However, to prove Theorem B requires much more control: we need not only that systems of Hecke eigenvalues vary $p$-adically, but also that we can vary certain local test vectors in a family. This is unconditionally possible in tame level 1. This could be relaxed to the assumption that $\pi$ admits parahoric-fixed vectors at every finite place using [DJ]. For a discussion on possible generalisations beyond this, see [BSDW, §8].

1.7. Structure of the paper

This paper falls into three parts.

In Part I (§2–3), we fix notation and recall relevant automorphic results. In §4, we generalise the abstract construction of evaluation maps from [BSDW, §4], showing that these evaluation maps compute classical $L$-values of RASCARs.

In Part II (§5–§9), we develop the theory of spin and Shalika refinements, and compute local zeta integrals (in two different ways). In these sections we prove all the results described in §1.3.1. Our Iwahoric results are summarised in §8.
In Part III (§10–§13), we build our $p$-adic machine on this automorphic foundation, reinterpreting the above in the context of overconvergent cohomology. The heart of is §11-12, where we give our main technical results on $p$-adic interpolation of branching laws. In §13, we obtain our main arithmetic applications, following strategies developed in [BSDW].

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PART I. AUTOMORPHIC RESULTS

2. Set-up and notation

2.1. Notation

Let $n \geq 1$ and let $G := GL_{2n}$. We write $B = B_{2n}$ for the Borel subgroup of upper triangular matrices, $\overline{B} = \overline{B}_{2n}$ for the opposite Borel of lower triangular matrices and $T = T_{2n}$ for the maximal split torus of diagonal matrices. We have decompositions $B = TN$ and $\overline{B} = \overline{N}T$ where $N = N_{2n}$ and $\overline{N} = \overline{N}_{2n}$ are the unipotent radicals of $B$ and $\overline{B}$. We also let $G_n = GL_n$ with $B_n, T_n, N_n$ etc. the analogous subgroups. Let $H := GL_n \times GL_n$, with an embedding $\iota : H \hookrightarrow G$, $\iota(h_1, h_2) = \left( \begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix} \right)$.

Let $W_G = S_{2n}$ (resp. $W_n = S_n$) be the Weyl group of $G$ (resp. $G_n$), identified with the permutation subgroup of $G(\mathbb{Z})$ (resp. $G_n(\mathbb{Z})$). We write $w_{2n}$ and $w_n$ for the longest Weyl elements (i.e. the antidiagonal matrices with $1$s on the antidiagonal).

Let $K_\infty = C_\infty Z_\infty$, where $Z_\infty$ is the center and $C_\infty$ is the maximal compact subgroup of $G(\mathbb{R})$. For any reductive real Lie group $A$ we let $A^\circ$ denote the connected component of the identity.

Fix a rational prime $p$ and an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We fix a (non-canonical) extension of $i_p$ to an isomorphism $i_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Let $\mathbb{Q}^{\text{p}}\text{c}$ be the maximal abelian extension of $\mathbb{Q}$ unramified outside $p\text{c}$, and let $\text{Gal}_p : = \text{Gal}(\mathbb{Q}^{\text{p}}\text{c}/\mathbb{Q}) \cong \mathbb{Z}_p^\times$ be its Galois group.

Throughout, we work in ‘tame level’, that is, with the open compact level subgroup

$$K = \text{Iw}_G \cdot \prod_{\ell \neq p}^{} G(\mathbb{Z}_\ell) \subset G(\mathbb{A}_f),$$

so away from $p$ we take maximal hyperspecial level and at $p$ we take Iwahori level

$$\text{Iw}_G := \{ g \in G(\mathbb{Z}_p) : g \pmod{p} \in B(\mathbb{F}_p) \} \subset G(\mathbb{Z}_p).$$

Let $\delta_B : T(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be the standard modulus character

$$t = (t_1, \ldots, t_{2n}) \mapsto |t_1|^{2n-1}|t_2|^{2n-3} \cdots |t_{2n-1}|^{3-2n}|t_{2n}|^{1-2n}.$$

We repeatedly used that if $\pi_t$ is a smooth irreducible representation of $GL_{2n}(\mathbb{Q}_p)$ that is spherical, i.e. $\pi_t^{GL_{2n}(\mathbb{Q}_p)} \neq 0$, then there is an unramified character $\theta : T(\mathbb{Q}_p) \rightarrow \mathbb{C}^\times$ such that

$$\pi_t = \text{Ind}_B^G \theta,$$
the normalised parabolic induction of $\theta$ to $G$.

If $\pi$ is a regular algebraic cuspidal automorphic representation (RACAR) of $G(A)$, then we write $L(\pi, s)$ for its standard $L$-function.

All our group actions will be on the left. If $M$ is a $R$-module, with a left action of a group $\Gamma$, then we write $M^\prime = \text{Hom}_R(M, R)$, with associated left dual action $(\gamma \cdot \mu)(m) = \mu(\gamma^{-1} \cdot m)$.

In later sections we work extensively with affinoid rigid spaces. For such a space $X$, we write $\mathcal{O}_X$ for the ring of rigid functions on $X$, so $X = \text{Sp}I(\mathcal{O}_X)$.

### 2.2. Algebraic weights

Let $X^*(T)$ be the set of algebraic characters of $T$. Each element of $X^*(T)$ corresponds to an integral weight $\lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \mathbb{Z}^{2n}$. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$, we say $\lambda$ is dominant, and write $X^+_n(T) \subset X^*(T)$ for the subset of dominant weights. We say that $\lambda$ is pure if there exists $w \in \mathbb{Z}$, the purity weight of $\lambda$, such that $\lambda_i + \lambda_{2n-i+1} = w$ for all $1 \leq i \leq n$; we write $\text{pure}(\lambda) := w$. We write $X^+_n(T) \subset X^+_n(T)$ for the subset of pure $B$-dominant integral weights, which are exactly those supporting cuspidal cohomology [Cl09, Lem. 4.9]. We say $\lambda$ is regular if $\lambda_{n+i} > \lambda_i$ for all $i$ (emphasising that trivial cohomological weight $\lambda = (0, \ldots, 0)$ is not regular).

For $\lambda \in X^+_n(T)$, let $V^\lambda_n$ be the algebraic irreducible representation of $G$ of highest weight $\lambda$, and let $V^\lambda_n\vee$ denote its linear dual, with its (left) dual action. We have an isomorphism $V^\lambda_n \cong V^{\lambda_n}_\vee$ where $\lambda_n = (-\lambda_{2n}, \ldots, -\lambda_1)$. Given a pure dominant algebraic weight $\lambda \in X^+_n(T)$, let

$$\text{Crit}(\lambda) := \{ j \in \mathbb{Z} : -\lambda_n \leq j \leq -\lambda_{n+1} \}. \quad (2.5)$$

If $\pi$ is a RACAR for $G(A)$ of weight $\lambda$ (which we take to mean cohomological with respect to $V^\lambda_n\vee$), then $j \in \text{Crit}(\lambda)$ if and only if the $L$-value $L(\pi, j + \frac{1}{2})$ is Deligne-critical (see [GR14, §6.1]).

### 2.3. Shalika models

Our main results come in the setting of RACARs that admit Shalika models. We recall relevant definitions and properties (see e.g. [GR14, §1.3.1], [BSDW, §2.6]).

#### 2.3.1. Definition of Shalika models

Let $S = \{ s = (h \cdot 1_n) : h \in \text{GL}_n, X \in M_n \}$ be the Shalika subgroup of $\text{GL}_{2n}$. Let $\psi$ be the standard non-trivial additive character of $\mathbb{Q}\backslash A$ from [DJR20, §4.1], and let $\eta$ be a character of $\mathbb{Q}\backslash A \times \mathbb{Q}_\ell$. A cuspidal automorphic representation $\pi$ of $G(A)$ is said to have an $(\eta, \psi)$-Shalika model if there exist $\varphi \in \pi$ and $g \in G(A)$ such that

$$S^\eta_\pi(\varphi)(g) := \int_{S\text{GL}_{2n}(\mathbb{Q}_{\ell}) \backslash S(A)} \varphi(s(g)) (\eta \otimes \psi)^{-1}(s)ds \neq 0. \quad (2.6)$$

This forces $\eta^{\infty}$ to be the central character of $\pi$, and $\eta = \eta_0 \cdot \eta^{\infty}$, where $\eta_0$ has finite order and $w = \text{pure}(\lambda)$. Let $\eta \otimes \psi$ be the character of $S\text{GL}_{2n}(\mathbb{Q}_{\ell})$ given by $(\eta \otimes \psi)(s) = \eta(\text{det}(h))\psi(\text{tr}(X))$. If (2.6) holds, then $S^\eta_\pi$ defines an intertwining $\pi \leftrightarrow \text{Ind}_{S\text{GL}_{2n}(\mathbb{Q}_{\ell})}^{G(A)}(\eta \otimes \psi)$. If $\pi$ has an $(\eta, \psi)$-Shalika model, then for each prime $\ell$ the local component $\pi_\ell$ has a local $(\eta_\ell, \psi_\ell)$-Shalika model [GR14, §3.2], that is, we have (non-canonical) intertwinings

$$S^\eta_{\psi_\ell} : \pi_\ell \leftrightarrow \text{Ind}_{S\text{GL}_{2n}(\mathbb{Q}_{\ell})}^{G(A)}(\eta_\ell \otimes \psi_\ell). \quad (2.7)$$

We fix a choice of intertwining $S^\eta_{\psi_\ell}$ of $\pi_\ell$ (or equivalently, via (2.6), an intertwining $S^{\psi_\ell}_{\psi_\infty}$ of $\pi_\infty$).

By [AG94, Prop. 1.3], if $\pi_\ell$ is spherical then it admits a $(\eta_\ell, \psi_\ell)$-Shalika model if and only if $\pi_\ell = \pi_\ell \otimes \eta_\ell^{-1}$, in which case case $\eta_\ell$ is unramified.

We recall another characterisation of Shalika models:
2.4.2. The Hecke algebra at $p$.

Then define operators $\pi$. The Definition 2.3. Attached to $\pi$ is $t$.

3.13]). Attached to $\pi$ then $\zeta$.

2.4.1. A way from $K$.

Recall we took $\mathcal{C}_G$.

By [JS90], having a global Shalika model is equivalent to a (partial) exterior square $L$-function having a pole at $s = 1$. But in [AS14] this is shown to be equivalent to being such a functorial transfer. (For further details see [GR14, Prop. 3.1.4]).

2.3.2. Friedberg–Jacquet integrals.

Theorem 2.1

Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A})$, and $\chi$ a finite order Hecke character for $F$. For $W \in S^0_\eta(\pi)$ consider the Friedberg–Jacquet zeta integral

$$
\zeta(s, W, \chi) = \int_{\mathcal{G}_n(\mathbb{A})} W \left( \begin{pmatrix} h & \ 0 \\ 0 & I_n \end{pmatrix} \right) \chi(\det(h)) |\det(h)|^{s-\frac{1}{2}} \, dh,
$$

converging absolutely in a right-half plane and extending to a meromorphic function in $s \in \mathbb{C}$. When $W = \bigotimes_{\ell \leq s} W_\ell$ for $W_\ell \in S^0_{\psi_\ell}(\pi_\ell))$, this decomposes into a product of local zeta integrals $\zeta_\ell(s, W_\ell, \chi_\ell)$. Suppose $\pi$ is a RACAR admitting a $(\eta, \psi)$-Shalika model. If $\pi_\ell$ is spherical, then $\pi_\ell^{G(Z_\ell)}$ is a line; let $W_\ell \in S^0_{\psi_\ell}(\pi_\ell^{G(Z_\ell)})$ be the spherical test vector normalised so that $W_\ell(1) = 1$. Then by [FJ93, Props. 3.1.3.2] $W_\ell$ is a Friedberg–Jacquet test vector, i.e. for all unramified quasi-characters $\chi_\ell : \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ we have

$$
\zeta_\ell \left( s + \frac{1}{2}, W_\ell^\circ, \chi_\ell \right) = L \left( \pi_\ell \otimes \chi_\ell, s + \frac{1}{2} \right). \quad (2.8)
$$

We apply this to choose local test vectors at all finite $\ell \neq p$.

2.4. The Hecke algebra and $p$-refinements

Recall we took $K = Iw_G \cdot \prod_{\ell \neq p} G(\mathbb{Z}_\ell)$.

2.4.1. Away from $p$.

If $\nu \in X_+^e(T)$ is a dominant cocharacter, and $\ell \neq p$ a prime, define $T_{\nu, \ell} : = [G(\mathbb{Z}_\ell) \cdot \nu(\ell) \cdot G(\mathbb{Z}_\ell))]$, a double coset operator.

Definition 2.2. The spherical Hecke algebra is the commutative algebra $\mathcal{H}^\ell(\mathcal{G}_n)$. Attached to $\pi$ there is a homomorphism $\psi_\ell : \mathcal{H}^\ell \otimes E \to E$ sending $T_{\nu, \ell}$ to its eigenvalue on $\pi_\ell^{G(Z_\ell)}$. We define $m_\pi := \ker(\psi_\ell)$, a maximal ideal in $\mathcal{H}^\ell \otimes E$. If $M$ is a module on which $\mathcal{H}^\ell \otimes E$ acts, we write $M_\pi$ for its localisation at $m_\pi$.

2.4.2. The Hecke algebra at $p$.

For $i = 1, \ldots, 2n - 1$ define matrices $t_{p,r} \in T(\mathbb{Q}_p)$ by

$$
t_{p,1} = \text{diag}(p, 1, \ldots, 1), \quad t_{p,2} = \text{diag}(p, p, 1, \ldots, 1), \ldots, \quad t_{p,2n-1} = \text{diag}(p, \ldots, p, 1), \quad (2.9)
$$

and let

$$
t_p = t_{p,1} \cdots t_{p,2n-1} = \text{diag}(p^{2n-1}, p^{2n-2}, \ldots, p, 1) \in T(\mathbb{Q}_p). \quad (2.10)
$$

Then define operators $U_{p,r} = [Iw_G \cdot t_{p,r} \cdot Iw_G]$ and $U_p = [Iw_G \cdot t_p \cdot Iw_G] = U_{p,1} \cdots U_{p,2n-1}$.

Definition 2.3. The Hecke algebra at $p$ is $\mathcal{H}_p := \mathbb{Z}[U_{p,r} : r = 1, \ldots, 2n - 1]$, and the full Hecke algebra is $\mathcal{H} = \mathcal{H}^\ell \otimes \mathcal{H}_p$. 


2.4.3. $p$-refinements. Recall $\pi_p$ is spherical. In particular $\pi_p^{\text{IwG}} \neq 0$.

**Definition 2.4.** A $p$-refinement of $\pi_p$ is a pair $\tilde{\pi}_p = (\pi_p, \alpha)$, where $\alpha : \mathcal{H}_p \rightarrow \overline{\mathbb{C}}$ is a system of $\mathcal{H}_p$-eigenvalues appearing in $\pi_p^{\text{IwG}}$; i.e. if we set $\alpha_{p,r} = \alpha(U_{p,r})$, then there is an eigenvector $\varphi_p \in \pi_p^{\text{IwG}}$ with $U_{p,r} \varphi_p = \alpha_{p,r} \varphi_p$ for each $r$. Such a $\tilde{\pi}_p$ is regular if the attached generalised eigenspace satisfies

$$\dim \pi_p^{\text{IwG}} [U_{p,r} - \alpha_{p,r} : r = 1, \ldots, 2n - 1] = 1.$$  \hspace{1cm} (2.11)

We will write $\varphi \in \tilde{\pi}_p$ as shorthand for $\varphi \in \pi_p^{\text{IwG}} [U_{p,r} - \alpha_{p,r} : r = 1, \ldots, 2n - 1]$.

Let $\tilde{\pi}$ be a $p$-refinement. After possibly extending $E$, we extend $\psi_z$ to a homomorphism

$$\psi_z : \mathcal{H} \otimes E \rightarrow E, \quad U_{p,r} \mapsto \alpha_{p,r}.$$  \hspace{1cm} (2.12)

We let $\mathfrak{m}_z := \ker(\psi_z)$ be the corresponding maximal ideal of $\mathcal{H} \otimes E$. If $M$ is a module upon which $\mathcal{H} \otimes E$ acts, we write $M_z$ for the localisation of $M$ at $\mathfrak{m}_z$. If $L$ is a field containing $E$ and $M$ is a finite-dimensional $L$-vector space, $M_z$ is the generalised eigenspace upon which $\mathcal{H} \otimes L$ acts by $\psi_z$.

We recall the standard classification of $p$-refinements for spherical $\pi_p$. Let $\theta$ be an unramified character such that $\tilde{\pi}_p = \text{Ind}_B^G \theta$ (as in (2.4)). Recall $\mathcal{W}_G = S_{2n}$ and $\delta_B$ from §2.1.

**Proposition 2.5.** [Che04, Lem. 4.8.4].

(i) The semisimplification of $\pi_p^{\text{IwG}}$ as a $\mathcal{H}_p$-module is isomorphic to $\bigoplus_{\sigma \in \mathcal{W}_G} (\delta_B^{1/2} \theta^\sigma) \circ \text{ev}_p$, where $\text{ev}_p$ is the map sending $[\text{Iw}_G : \mu \cdot \text{Iw}_G] \in \mathcal{H}_p$ to $\mu(p)$, where $\mu \in \hat{X}_e(T)$. Thus if $\tilde{\pi}_p = (\pi_p, \alpha)$ is a $p$-refinement, then there exists $\sigma \in \mathcal{W}_G$ such that for each $r$,

$$\alpha_{p,r} = \alpha(U_{p,r}) = \left(\delta_B^{1/2} \theta^\sigma\right)^{w_2n} (t_{p,r}) = \prod_{j=1}^{r} \left( p^{-2n-2j+1} \theta_{\sigma(2n+1-j)}(p) \right).$$

(ii) There are equivalences

$$\pi_p \text{ admits a regular } p\text{-refinement} \iff \text{its Satake parameter is regular semisimple} \iff \text{every } p\text{-refinement of } \pi_p \text{ is regular.}$$

In this case, the choice of $\sigma$ in (i) is unique, and via this correspondence there are exactly $(2n)!$ $p$-refinements of $\pi_p$, all of which are regular.

**Remark 2.6.** (i) This normalisation, with $w_2n$, matches [DJR20] but might appear strange. Cheenevier uses antidominant cocharacters, and switching to dominant characters is equivalent to conjugating by $w_2n$. This normalisation will be convenient in §7.

(ii) When the Satake parameter of $\pi_p$ is regular semisimple, there is a bijection

$$\Delta_\theta : \{ p\text{-refinements of } \pi_p \} \rightarrow \mathcal{W}_G$$

induced by the above. This is not canonical, depending on the choice of character $\theta$ from which we induce. For $\tau \in \mathcal{W}_G$, replacing $\theta$ by $\theta^\tau$ conjugates the image of $\Delta_\theta$ by $\tau$.

(iii) When $\pi_p$ is the local component of a RACAR, the eigenvalues $\alpha_{p,r}$ are algebraic but not $p$-integral. To account for this, we make the following definition.

**Definition 2.7.** Let $\tilde{\pi}_p = (\pi_p, \alpha)$ be a $p$-refinement. Define integral normalisations

$$U_{p,r}^\circ := \lambda(t_{p,r}) U_{p,r}, \quad \alpha_{p,r}^\circ := \lambda(t_{p,r}) \alpha_{p,r} = p^{\lambda_1 + \cdots + \lambda_r} \alpha_{p,r}.$$  \hspace{1cm}

The $\alpha_{p,r}^\circ$ are $p$-integral (see Remark 10.4 or [BSW21, Rem. 3.23]).
3. Automorphic cohomology classes

For $K \subset G(\mathbb{A}_f)$ open compact, the locally symmetric space of level $K$ is

$$S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_\infty^\circ.$$  

It is a $(2n-1)(n+1)$-dimensional real orbifold. We will now recall how to realise RACARs in the compactly supported Betti cohomology of $S_K$.

### 3.1. Local systems

We recall standard facts about local systems on $S_K$ (e.g. [Urb11, §1], [BSDW, §2.3]). If $M$ is a left $G(\mathbb{Q})$-module such that the centre of $G(\mathbb{Q}) \cap K K_\infty^\circ$ acts trivially, let $\mathcal{M}$ be the local system on $S_K$ given by locally constant sections of $G(\mathbb{Q}) \backslash \{G(\mathbb{A}) \times M\} / K K_\infty^\circ$, with action $\gamma(g, m) k z = (\gamma g k z, \gamma \cdot m)$. We denote such local systems with calligraphic letters.

If $M$ is a left $K$-module, let $\mathcal{M}$ (with a script letter) be the local system on $S_K$ given by locally constant sections of $G(\mathbb{Q}) \backslash \{G(\mathbb{A}) \times M\} / K K_\infty^\circ$ with action $\gamma(g, m) k z = (\gamma g k z, k^{-1} \cdot m)$.

If $M$ is a left $G(\mathcal{A})$-module, then it has actions of the subgroups $G(\mathbb{Q})$ and $K$, and there is an isomorphism $\mathcal{M} \rightarrow \mathcal{M}$ of associated local systems given by $(g, m) \mapsto (g, g^{-1} \cdot m)$. The key example of such $M$ for this paper is $M = V_\lambda^\vee$, whence $V_\lambda^\vee \rightarrow \mathcal{T}_\lambda^\vee$.

### 3.2. Hecke operators

Let $\gamma \in G(\mathbb{A}_f)$ and $M$ be a left $G(\mathbb{Q})$-module (resp. $K$-module). We suppose $\gamma$ acts on $M$.

We have a natural projection map $p_{K, \gamma}: S_{\gamma, K^{-1} \cap K} \rightarrow S_K$, and a double coset operator $[K \gamma K]$ on $H^\bullet_c(S_K, \mathcal{M})$ (resp. $H^\bullet_c(S_K, \mathcal{A})$) defined as the composition

$$[K \gamma K] := \text{tr}(p_{K, \gamma}) \circ [\gamma] \circ p_{K, \gamma}^\ast,$$

where tr is the trace and $[\gamma]: H^\bullet_c(S_{\gamma, K^{-1} \cap K}, \mathcal{M}) \rightarrow H^\bullet_c(S_{\gamma, K^{-1} \cap K}, \mathcal{M})$ is given on local systems by $(g, m) \mapsto (g^{-1}, \gamma \cdot m)$ (and similarly for $\mathcal{A}$).

#### 3.2.1. Localisation at RACARs

Recall $K = Iw_{G} \cdot \prod_{\mathfrak{d} \neq \mathfrak{d}} G(\mathbb{Z}_\mathfrak{d})$ and $\mathcal{H}$ from §2.4. For appropriate $\mathcal{M}$ (e.g. $\mathcal{M} = V_\lambda^\vee$), this acts on $H^\bullet_c(S_K, \mathcal{M})$ and $H^\bullet_c(S_K, \mathcal{A})$ via the process above. If $\pi$ is a RACAR with $\pi^K_\lambda \neq 0$, it therefore makes sense to localise $H^\bullet_c(S_K, \mathcal{M})$ at $\pi^K_\lambda$ as in §2.4.1. We denote the localisation by $H^\bullet_c(S_K, -)_\pi$.

#### 3.2.2. The action at infinity

We have $K_\infty / K_\infty^\circ = \{\pm 1\}$. This group has two characters $\varepsilon^\pm: K_\infty / K_\infty^\circ \rightarrow \{\pm 1\}$, where $\varepsilon^\pm$ sends $-1$ to $\pm 1$. If $M$ is a module on which $K_\infty / K_\infty^\circ$ acts and 2 acts invertibly – for example, the cohomology of $S_K$ over a field – then we have $M = M^+ + M^-$, where $M^\pm$ are the eigenspaces where $K_\infty / K_\infty^\circ$ acts via $\varepsilon^\pm$. We obtain a (Hecke-equivariant) decomposition of the cohomology groups $H^\bullet_c(S_K, -)$ into $\pm$-submodules (as the action of $K_\infty / K_\infty^\circ$ commutes with the $G(\mathbb{A}_f)$-action).

#### 3.2.3. Integral normalisations

The module $V_\lambda^\vee$ comes equipped with the natural (algebraic) action of $GL_{2n}$, which we have been denoting with $\epsilon$. As we have already remarked, the resulting Hecke operators $U_{p,r} = U_{p,r}^{\ast} = [K_{p}^{\ast} p_{r} K_{p}]$ on the cohomology of $V_\lambda^\vee$ are not integrally normalised.

In §10.3, we will equip $V_\lambda^\vee(\mathbb{Q}_p)$ with another natural action of $GL_{2n}(\mathbb{Z}_p)$ and $t_{p,r}$, denoted $\ast$. Concretely, we will have $t_{p,r} \ast \mu = \lambda(t_{p,r})(t_{p,r} \cdot \mu)$. In light of Definition 2.7, if we let $U^\ast_{p,r}$ be the Hecke operator defined via (3.1) with the $\ast$-action instead of the $\cdot$-action, then $U^\ast_{p,r} = U^\circ_{p,r}$ is integrally normalised. This is all explained in detail in [BSDW, Rem. 3.13].

### 3.3. Cohomology classes attached to RACARs

Let $t = n^2 + n - 1$, which is the top degree of cohomology to which RACARs for $G(\mathbb{A})$ contribute. In particular, let $\pi$ be a RACAR of $G(\mathbb{A})$; then we recall that there exists a a
Hecke-equvariant isomorphism
\[ \pi^K_f \rightsquigarrow \mathbb{H}_c^i(S_K, \mathcal{Y}_\chi'(Q_p))^\pm, \tag{3.2} \]
for a unique \( \lambda \in X^*_0(T) \). The isomorphism (3.2) is non-canonical, depending on our fixed choice of \( t_p : C \rightarrow Q_p \) and a choice of basis \( \Xi \in \mathbf{C} \) of the 1-dimensional \( \mathbf{C} \)-vector space \( \mathbb{H}^i(g_{\infty}, K_{\infty}^2; \pi_{\infty} \otimes V^\vee_N(C))^\pm \), where \( g_{\infty} := \text{Lie}(G_{\infty}) \). This is all standard, explained e.g. in [BSDW, Prop. 2.3].

Suppose \( \pi \) admits an \((\eta, \psi)\)-Shalika model, and recall we chose an intertwining \( S_{\psi_f}^{ij} : \pi_f \rightarrow S^{ij}_{\psi_f}(\pi_f) \). Combining with (3.2), we get a (non-canonical) Hecke-equvariant isomorphism
\[ \Theta^\pm : S_{\psi_f}^{ij}(\pi^K_f) \rightarrow \mathbb{H}_c^i(S_K, \mathcal{Y}_\chi'(Q_p))^\pm. \tag{3.3} \]

Possibly enlarging the number field \( E \), there is a natural \( E \)-rational subspace \( S^{ij}_{\psi_f}(\pi^K_f), E \subset S^{ij}_{\psi_f}(\pi^K_f) \). As in [GR14, Prop. 4.2.1] (cf. [BSDW, §2.10]), there exist \( \Omega_{\pm}^{ij} \in \mathbf{C}^\times \) (canonical up to \( E^\times\)-multiple) and finite \( L/Q_p \) such that \( \Theta^\pm /_{p}(\Omega_{\pm}^{ij}) \) maps \( S_{\psi_f}^{ij}(\pi^K_f), E \) into \( \mathbb{H}_c^i(S_K, \mathcal{Y}_\chi'(L))^\pm \). Moreover, for \( \ell \neq p \) the spherical test vector \( W_\ell^\pi \) is \( E \)-rational.

\[4. \] Evaluation maps

Evaluation maps were crucial to the methods of [GR14, DJR20, BSDW]. We give constructions of abstract evaluation maps, generalising [BSDW] and [DJR20].

\[4.1. \] Automorphic cycles and abstract evaluation maps

In this section we generalise the abstract theory in [BSDW, §4], where the evaluation maps were defined with respect to the parabolic \( Q \) with Levi \( H \). These ‘parahoric’ evaluation maps can be interpolated over 2-dimensional parabolic subsets of weight space, but are not suitable for our goal of interpolation in \((n + 1)\)-weight variables. We now construct evaluation maps defined with respect to any standard parabolic \( P \subset Q \). For the Iwahoric case, we are most interested in \( P = B \). The proofs of [BSDW, §4] go through almost identically with these modifications, so we are terse with details here.

**Remark 4.1.** Since the notation is heavy, we sketch the differences between our new definitions and those of [BSDW]. Firstly, to better suit the more general theory, we replace the twisting operator \( \xi = \left( \begin{smallmatrix} 1 & * \\ 0 & w_a \end{smallmatrix} \right) \) of [BSDW, Def. 4.2] with \( u^{-1} \), where \( u = \left( \begin{smallmatrix} 1 & w_n \\ 0 & 1 \end{smallmatrix} \right) \in G(\mathbf{Z}_p) \). Unlike \( \xi \), the element \( u^{-1} \) lies in \( Iw_G \). We will show that the definitions/results of [BSDW] are essentially unchanged with this switch.

In [DJR20, BSDW], the evaluation maps for \( Q \) used the matrix \( t_{p,n} = \text{diag}(p, ..., p, 1, ..., 1) =: t_Q \) and operator \( U_{p,n} = [K_p t_{p,n} K_p] =: U_{Q_p,n}^\circ \), a \( Q \)-controlling operator (in the sense of [BSW21, §2.5]). For a general parabolic \( P \), we instead use a different matrix \( t_P \in \text{GL}_{2n}(Q_p) \) (see Definition 4.2), giving a Hecke operator \( U_{p,n}^\circ \) attached to \( P \).

\[4.1.1. \] Automorphic cycles.

Automorphic cycles are coverings of locally symmetric spaces for \( H \) that have real dimension equal to \( t \), the top degree of cohomology to which RACARs for \( G \) contribute. This ‘magical numerology’ was exploited in [GR14, DJR20] to define classical evaluation maps and to give a cohomological interpretation of the Deligne-critical \( L \)-values of RACARs.

**Definition 4.2.** Let \( P \subset Q \subset \text{GL}_{2n} \) be a standard parabolic with Levi \( \text{GL}_{m_1} \times \cdots \times \text{GL}_{m_r} \).

Define a diagonal matrix
\[ t_P = \text{diag}(p^{r-1}I_{m_1}, p^{r-2}I_{m_2}, \cdots, pI_{m_{r-1}}, I_{m_r}). \]

Note \( t_P \in T_p^{++} \) as in [BSW21, §2.5]; and since \( P \subset Q \), the first \( n \) diagonal entries are all a positive power of \( p \).
For example, we have \( t_Q = t_{p,n} \) (from (2.9)), and
\[
t_B = \text{diag}(p^{2n-1}, p^{2n-2}, \ldots, p, 1).
\]

**Definition 4.3.** Fix \( m \in \mathbb{Z}_{>0} \) prime to \( p \), and let \( K = K_p K^p \subset G(A_f) \) be open compact. For \( \beta \in \mathbb{Z}_{>0} \), define an open compact subgroup \( L^p_\beta = L^p_\beta L^p \subset H(A_f) \) by setting:

(i) \( L^p_\beta := H(Z_p) \cap K_p \cap (u^{-1} t_p^\beta) K_p (u^{-1} t_p^\beta)^{-1} \), and

(ii) \( L^p := \{ h \in H(Z^{(p)}) : h \equiv 1 \pmod{m} \} \), the principal congruence subgroup of level \( m \).

The automorphic cycle of level \( L^p_\beta \) is
\[
X^p_\beta := H(Q) \backslash H(A) / L^p L^p_\beta,
\]
where \( L_\infty = H_\infty \cap K_\infty \) for \( H_\infty = H(R) \). This is a real orbifold of dimension \( t \) [DJR20, (23)].

We will always take \( m \) to be the smallest positive integer such that \( L^p \subset K^p \cap H(A_f) \) and \( H(Q) \cap hL^p_\beta h^{-1} = Z_G(Q) \cap L^p_\beta L^\infty_\beta \) for all \( h \in H(A) \) and for both \( P = B, Q \) (compare [BSDW, (4.1),(4.2)]). This means \( X^p_\beta \) is a real manifold [DJR20, (21)]. The impact of changing \( m \) is discussed in [BSDW, §4.1].

**Lemma 4.4.** We have
\[
\text{vol}(L^p_\beta) = \delta_B(t_p^\beta) \cdot A_P,
\]
where \( A_P = \delta_B(t_p^{-1}) \text{vol}(L^p_\beta) \) is a constant independent of \( \beta \).

**Proof.** Let \( N \subset G \) be the upper unipotent subgroup, and let \( N^\beta := t_p^\beta N(Z_p) t_p^{-\beta} \subset N(Z_p) \). By [Loe22, Lem. 4.4.1], for \( \beta \geq 1 \) we have
\[
[L^p_\beta : L^p_{\beta+1}] = [N^\beta : N^{\beta+1}] = \delta_B(t_p^{-1}),
\]
by definition of the modulus character \( \delta_B \). It follows that \( \text{vol}(L^p_\beta) = \delta_B(t_p^{-1}) \text{vol}(L^p_\beta) \), from which the result follows.

**Lemma 4.5.** If \( (\ell_1, \ell_2) \in L^p_\beta \), then \( \ell_2 \equiv w_n \ell_1 w_n \pmod{p^\beta} \). Hence there is an isomorphism
\[
\det(L^p_\beta) \cong (1 + p^\beta Z_p) \times Z_p^\times, \quad (x, y) \mapsto (xy^{-1}, y).
\]

**Proof.** Similar to [DJR20, Lem. 2.1]. First, compute that for \( (\ell_1, \ell_2) \in L^p_\beta \), we need
\[
t_p^{-\beta} u(\ell_1, \ell_2) u^{-1} t_p^{-\beta} = t_p^{-\beta} \left( \ell_1 w_n(\ell_2 - w_n \ell_1 w_n) \right) t_p^{-\beta} \in K_p.
\]
Since \( P \subset Q \), each of the first \( n \) diagonal entries of \( t_p^\beta \) is congruent to 0 (mod \( p^\beta \)). In particular, after expanding we see \( p^{-\beta}(\ell_2 - w_n \ell_1 w_n) \in GL_n(Z_p) \), so \( \ell_2 \equiv w_n \ell_1 w_n \pmod{p^\beta} \), giving the first statement. We then have \( \det(\ell_1) \equiv \det(\ell_2) \pmod{p^\beta} \), so to see the isomorphism, it suffices to prove surjectivity. But given \((a, b) \in (1 + p^\beta Z_p) \times Z_p^\times \), we see \( \ell_1 = \left( \begin{smallmatrix} 1 & -ab \\ ab & 1 \end{smallmatrix} \right), \ell_2 = \left( \begin{smallmatrix} b & 1 \\ a & b \end{smallmatrix} \right) \) works (for any \( P \)).

**Corollary 4.6.** We have \( L^p_\beta \subset L^Q_\beta \).

**Proof.** By the proof of Lemma 4.5 (or [DJR20, Lem. 2.1]), we deduce \( L^Q_\beta = \{ (\ell_1, \ell_2) \in K_p : \ell_2 \equiv w_n \ell_1 w_n \pmod{p^\beta} \} \}. But by Lemma 4.5, any element of \( L^p_\beta \) satisfies this.
By Lemma 4.5 and strong approximation for $H$, via the map

$$\pi_0(X^P_\beta) := \mathcal{C}^+(\ell^\beta m) \times \mathcal{C}^+(m)$$

(cf. [DJR20, (22)]). Here for an ideal $I \subset \mathbb{Z}$, let $\mathcal{U}(I) := \{x \in \hat{\mathbb{Z}}^\times : x \equiv 1 (\text{mod } I)\} \subset \hat{\mathbb{Z}}^\times$ and let

$$\mathcal{C}^+(\mathcal{U}(I)) = \mathbb{Q}^\times \setminus \mathbb{A}^\times \mathcal{U}(I)\mathcal{R}_{>0} \cong (\mathbb{Z}/I)^\times$$

(4.2) be the narrow ray class group of conductor $I$. For $\delta \in H(A_f)$, we write $[\delta]$ for its associated class in $\pi_0(X^P_\beta)$ and denote the corresponding connected component

$$X^P_\beta[\delta] := H(Q) \setminus H(Q)_\delta L_\beta^P H^\circ_{\infty}/L^P_\beta L_{\infty}.$$  

As $L^P \subset K^P \cap H(A_f)$, by definition of $L^P_\beta$, there is a proper map (see [Ash80, Lemma 2.7])

$$\iota^P_\beta : X^P_\beta \longrightarrow \mathcal{S}_K, \quad [h] \longmapsto [\iota(h)u^{-1}\iota^P_\beta].$$

(4.3)

4.1.2. Abstract evaluation maps. Define $J_P \subset GL_{2n}(\mathbb{Z}_p)$ be the parahoric subgroup for $P$. Define $\Delta_P \subset GL_{2n}(\mathbb{Q}_p)$ to be the semigroup generated by $J_P$ and the matrices

$$t_{p,m_1}, t_{p,m_1+m_2}, \ldots, t_{p,m_1+\ldots+m_{r-1}},$$

where $P$ has Levi $GL_{m_1} \times \cdots \times GL_{m_r}$ (and recalling $t_{p,r}$ from (2.9)). For example:

- $J_B = Iw_G$, and $\Delta_B$ is generated by $Iw_G$ and $t_{p,1}, t_{p,2}, \ldots, t_{p,2n-1}$.
- $J_Q = \{g \in GL_{2n}(\mathbb{Z}_p) : g \text{ (mod } p) \in Q(F_p)\}$ and $\Delta_Q$ is generated by $J_Q$ and $t_{p,n}$.

Let $K \subset G(A_f)$ be open compact such that $N_Q(\mathbb{Z}_p) \subset K_p \subset J_P$. Let $M$ be a left $\Delta_P$-module, with action denoted $\cdot$. Then $K$ acts on $M$ via its projection to $K_p \subset \Delta_P$, giving a local system $\mathcal{M}$ on $S_K$ via §3.1. The notation is suggestive: as in §3.2.3, using this $\cdot$-action in (3.1) we get 'integrated normalised' Hecke operators $U^P_{\gamma^\beta \cdot \mathcal{M}}$ on the cohomology $H^1_\mathcal{M}(S_K, \mathcal{M})$.

The constructions here are almost identical to those of [BSDW, §4.2] where they are motivated and explained in great detail; thus we give only the briefest description here.

For $\beta \in \mathbb{Z}_{\beta \delta}$ and $\delta \in H(A_f)$, define a congruence subgroup

$$\Gamma^P_{\beta, \delta} := H(Q) \cap \delta L^P_{\beta} H^\circ_{\infty} \delta^{-1}.$$  

(4.4)

This acts on $M$ via

$$\gamma \cdot_{\Gamma^P_{\beta, \delta}} m := (\delta^{-1}\gamma\delta)_{f \cdot m}.$$  

(4.5)

Let $M_{\Gamma^P_{\beta, \delta}} := \{\gamma \cdot_{\Gamma^P_{\beta, \delta}} m : m \in M, \gamma \in \Gamma^P_{\beta, \delta}\}$ be the coinvariants of $M$ by $\Gamma^P_{\beta, \delta}$.

Definition 4.7. The evaluation map for $M$ and $P$ of level $p^3$ at $\delta$ is the composition

$$\text{Ev}^M_{\Gamma^P_{\beta, \delta}} : H^1_\mathcal{M}(S_K, \mathcal{M}) \overset{\tau^{P,\mathcal{M}}_{\beta, \delta}}{\longrightarrow} H^1_\mathcal{M}(X^P_\beta, \mathcal{M}) \overset{\iota^P_{\beta, \delta}}{\longrightarrow} H^1_\mathcal{M}(\mathcal{M}) \overset{\text{coinv}^{P,\mathcal{M}}_{\beta, \delta}}{\longrightarrow} H^1_\mathcal{M}(\mathcal{M}) \overset{\cong}{\longrightarrow} M_{\Gamma^P_{\beta, \delta}}.$$  

(4.6)

where:

- $\iota^P_{\beta, \delta}$ is the map from (4.3), and $\tau^{P,\mathcal{M}}_{\beta, \delta}$ is the map $(\iota^P_{\beta, \delta})^* \mathcal{M} \rightarrow \mathcal{M}$ of local systems on $X^P$ induced by $(h,m) \mapsto (h, u^{-1}\iota^P_{\beta, \delta} \cdot m)$;
• \( \Gamma_{\beta,\delta}^P \) acts on \( \mathcal{X}_H = H^\infty_\infty / L^\infty_\infty \) by left translation, and there is an isomorphism

\[
c_\delta : \Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H \to X_\beta^P[\delta] \subset X_\beta, \quad [h_\infty]_{\beta,\delta} \mapsto [\delta h_\infty],
\]

where if \([h_\infty] \in \mathcal{X}_H\), we write \([h_\infty]_{\beta,\delta}\) for its image in \(\Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H\);
• \(\text{coinv}_{\beta,\delta}^P\) is the quotient map \(M \to M_{\Gamma_{\beta,\delta}^P}\), which induces a map on cohomology with image in the cohomology of the trivial local system on \(\Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H\) attached to \(M_{\Gamma_{\beta,\delta}^P}\);
• and \((- \cap \theta_\delta^P\) is induced from cap product \((- \cap \theta_\delta^P\) : \(H^i(\Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H, \mathbb{Z}) \to \mathbb{Z}\), for \(\theta_\delta^P\) a fundamental class in the Borel–Moore homology \(H^0_{BM}(\Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H, \mathbb{Z}) \cong \mathbb{Z}\).

We choose the classes \(\theta_\delta^P\) compatibly in \(\delta\) and \(P\). Let \(\theta_\delta^Q\) be exactly as in [BSDW, §4.2.3]. We have a natural map \(pr_B^\beta : \Gamma_{\beta,\delta}^P \backslash \mathcal{X}_H \to \Gamma_{\beta,\delta}^Q \backslash \mathcal{X}_H\); we let \(\theta_{\beta,\delta}^B = (pr_B^\beta)^*\theta_\delta^Q\).

Exactly as in [BSDW, §4.3], we can track dependence of these maps as we allow \(M, \beta\) and \(\delta\) to vary. Each of the following results is proved exactly as their given counterpart op. cit.:

**Lemma 4.8.** (Variation in \(M\); [BSDW, Lem. 4.6]) \(\alpha : M \to N\) be a \(\Delta P\)-module map. There is a commutative diagram

\[
\begin{array}{ccc}
H^\alpha_\infty(S_K, \mathcal{M}) & \xrightarrow{\text{Ev}_{P,\beta,\delta}^\alpha} & M_{\Gamma_{\beta,\delta}^P} \\
\downarrow{\kappa} & & \downarrow{\kappa} \\
H^\alpha_\infty(S_K, \mathcal{N}) & \xrightarrow{\text{Ev}_{P,\beta,\delta}^\alpha} & N_{\Gamma_{\beta,\delta}^P}.
\end{array}
\]

**Proposition 4.9.** (Variation in \(\delta\); [BSDW, Prop. 4.9]) \(\beta\) be a left \(H(A)\)-module, with action denoted \(*\), such that \(H(Q)\) and \(H^\infty_\infty\) act trivially. Let \(\alpha : M \to N\) be a map of \(L^\infty_\beta\)-modules (with \(N\) an \(L^\infty_\beta\)-module by restriction). Then

\[
\text{Ev}_{P,\beta,\delta}^{M,K} := \delta \star [\alpha \circ \text{Ev}_{P,\beta,\delta}^M] : H^\alpha_\infty(S_K, \mathcal{M}) \to N
\]

is well-defined and independent of the representative \(\delta\) of \([\delta]\).

To vary \(\beta\), we have a natural projection \(\alpha_{\beta+1} : X^P_{\beta+1} \to X^P_{\beta}\), inducing a projection \(\alpha_{\beta} : \pi_0(X^P_{\beta+1}) \to \pi_0(X^P_{\beta})\). The action of \(t_P\) on \(M\) yields an action of \(U^P_\beta\) on \(H^\alpha_\infty(S_K, \mathcal{M})\), where \(U^P_\beta = U^P_{\beta,n}\) and \(U^Q_\beta = U^Q_{\beta,n}\).

For compatibility in \(\beta\), we need to assume additionally that \(K_p = J_P\) is the parahoric for \(P\).

**Proposition 4.10.** (Variation in \(\beta\); [BSDW, Prop. 4.10]) \(\beta\) be as in Proposition 4.9. If \(\beta > 0\), then as maps \(H^\alpha_\infty(S_K, \mathcal{M}) \to N\) we have

\[
\sum_{[\nu] \in \text{pr}_{\beta,\delta}^P([\delta])} \text{Ev}_{P,\beta+1,\nu}^{M,K} = \text{Ev}_{P,\beta,\delta}^{M,K} \circ U^P_\beta.
\]

**Proof.** The proof follows almost exactly as in [BSDW]. There is a unique point at which more detail is required. The left-hand square of diagram (4.14) op. cit. generalises to

\[
\begin{array}{ccc}
S_K & \xrightarrow{i_{\beta,P}} & S_{K_\beta}(P) \\
\downarrow{\iota_{\beta}^P} & & \downarrow{\iota_{\beta,P,n}} \\
X^P_{\beta} & \xrightarrow{i_{\beta+1}^P} & X^P_{\beta+1}
\end{array}
\]

where \(K_\beta(P) = K \cap t_P K t_P^{-1}\), the map \(i_{\beta}^P\) is induced by the map \([h] \mapsto [\iota(h) w^{-1} t_{P,n}^\beta]\), and where the horizontal maps are the natural projections. We need to show that this square is Cartesian.

---

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in this generality. For this, since the vertical maps are embeddings, it is enough to show that the horizontal maps have the same degree; i.e., that \([K : K^0_p(p)] = [L^p,β : L^p,β+1]\) for any \(β \geq 1\). Let \(N_p\) denote the unipotent radical of \(P\) and set \(N_β = t_p^β N_p(Z_p) t_p^{-β}\). By the Iwahori decomposition for \(K_p\), we easily find that \([K : K^0_p(p)] = [N_P(Z_p) : N_1] = [N_β : N_β+1]\) for any \(β \geq 1\). On the other hand, the element \(x^{-1}\) is a representative of the (unique) Zariski dense \(H\)-orbit in the flag variety \(G/P\), where \(P\) denotes the opposite of \(P\). Therefore the proof of [Loe22, Lem. 4.4.1] implies that \([L^p,β : L^p,β+1] = [N_β : N_β+1]\). Hence we have

\([K : K^0_p(p)] = [L^p,β : L^p,β+1]\)

as required.

4.2. Classical evaluation maps, Shalika models and \(L\)-values

The classical evaluation maps \(E_{P,δ}^{j,w}\) of [DJR20] were rephrased in the abstract language of Definition 4.7 in [BSDW, §5]. We recap the construction, whilst again generalising it to parahoric level for a general parabolic \(P\). When \(P = Q\) this recovers [DJR20, BSDW]; in this paper we are primarily interested in \(P = B\) (which is new). We relate the values of these evaluation maps to critical \(L\)-values. Throughout, we assume \(K_p = J_p\).

The definition of \(E_{P,δ}^{j,w}\) fundamentally used the following branching law from [GR14, Prop. 6.3.1] and [BSDW, Lem. 5.2]. Let \(λ ∈ X^0_0(Γ)\) be a pure algebraic weight, with purity weight \(w\). For integers \(j_1, j_2\), let \(V^{H}_{(j_1, j_2)}\) denote the 1-dimensional \(H(Z_p)\)-representation given by the character

\[H(Z_p) \rightarrow \mathbb{Z}_p^×, \quad (h_1, h_2) \mapsto \det(h_1)^{j_1} \det(h_2)^{j_2}.\]

Lemma 4.11. Let \(j ∈ Z\). Then \(j ∈ \text{Crit}(λ)\) if and only if \(\dim \text{Hom}_{H(Z_p)}(V^{V}_{(j_1, w, j_2)}, V^{H}_{(j_1, w, j_2)}) = 1\).

For each \(j ∈ \text{Crit}(λ)\), fix some choice of non-trivial \(H(Z_p)\)-map \(κ_{λ,j} : V^{V}_{λ}(L) \rightarrow V^{H}_{(j_1, w, j_2)} \cong L\). We will make more precise choices in §11, but for now they can be arbitrary.

The \(p\)-adic cyclotomic character is

\[χ_{cyc} : Q^×/A^× \rightarrow \mathbb{Z}_p^×, \quad y \mapsto \text{sgn}(y_∞) \cdot |y_f| \cdot y_p.\]  

(4.7)

It is the \(p\)-adic character associated to the adelic norm [BSW19, §2.2.2]. It is trivial on \(F_p^0\).

It is simple to see that the \(H(Z_p)\)-representation \(V^{H}_{(j_1, j_2)}\) extends to \(H(A)\) via the character

\[H(A) \rightarrow \mathbb{Z}_p^×, \quad (h_1, h_2) \mapsto χ_{cyc} \left[\det(h_1)^{j_1} \det(h_2)^{j_2}\right].\]

Note the action of \(L_β \subset H(A_f)\) factors through projection to \(H(Z_p)\), so the map \(κ_{λ,j} : V^{V}_{λ}(L) \rightarrow L\), chosen above is a map of \(L_β\)-modules. Moreover, \(H(Q)\) and \(H_∞\) act trivially on \(V^{H}_{(j_1, j_2)}\), so we can use the formalism of Proposition 4.9.

Definition 4.12. Let \(L/Q_0\) be an extension. The classical evaluation map for \(P\) of level \(p^β\) at \(δ\) is

\[E_{P,δ}^{j,w} \in \text{Ev}_{P,δ}^{V^0,κ_{λ,j}} : H^1_{cl}(S_K, V^0_{λ}(L)) \rightarrow L.\]

Here \(\text{Ev}_{P,δ}^{V^0,κ_{λ,j}}\) was defined in Proposition 4.9, which shows \(E_{P,δ}^{j,w}\) is independent of the choice of \(δ\) representing the class \([δ]\). We introduce the notation \(E_{P,δ}^{j,w}\) for consistency with [DJR20, BSDW]; via [BSDW, Lem. 5.3] this definition is consistent with that in [DJR20].

Recall \(π_0(X^P_δ) = (Z/p^m)× \times (Z/m)×\) from (4.1). Write \(pr_1, pr_2\) for the projections of \(π_0(X^P_δ)\) onto the first and second factors respectively, and let \(pr_β\) denote the natural composition

\[pr_β : (Z/p^β m)× \times (Z/m)× \xrightarrow{pr_1} (Z/p^β m)× \rightarrow (Z/p^β)×.\]  

(4.8)
For \( \eta_0 \) be any character of \((\mathbb{Z}/m)^\times \) and \( d \in (\mathbb{Z}/p^3)^\times \), define

\[
\mathcal{E}_{P,\beta,d}^{j,\eta_0} = \sum_{[\delta] \in \text{pr}_s^{-1}(d)} \eta_0([\text{pr}_2([\delta])]) \mathcal{E}_{P,\beta,[\delta]}^{j,w} : H^j_c(S_K, \mathcal{Y}_c^\vee(L)) \to L.
\]

(In our main application, we will take \( \eta_0 \) trivial; but the obstructions to taking more general \( \eta_0 \) are automorphic, not \( p \)-adic, so we develop the theory in full generality here).

Let \( \chi \) be a finite-order Hecke character of conductor \( p^\beta \), let \( \beta = \max(1, \beta') \) and let \( L(\chi) \) be the smallest extension of \( L \) containing \( \text{Im}(\chi) \). For \( j \in \text{Crit}(\lambda) \), define

\[
\mathcal{E}_{P,\chi}^{j,\eta_0} = \sum_{[d] \in (\mathbb{Z}/p^3m)^\times} \chi(d) \mathcal{E}_{P,\beta,d}^{j,\eta_0} : H^j_c(S_K, \mathcal{Y}_c^\vee(L)) \to L(\chi),
\]

\[
\phi \mapsto \sum_{[\delta] \in (\mathbb{Z}/p^3m)^\times \times (\mathbb{Z}/m)^\times} \chi([\text{pr}_2([\delta])]) \cdot \eta_0([\text{pr}_2([\delta])]) \cdot \left( \delta - [\kappa_{\lambda,j} \circ \text{Ev}_{P,\beta,d}^V(\phi)] \right).
\]

Remark 4.14. We see \( \mathcal{E}_{P,\chi}^{j,\eta_0} \) is the composition

\[
H^j_c(S_K, \mathcal{Y}_c^\vee) \xrightarrow{\oplus \Xi_{P,\beta,d}^{\eta_0}} (V_c^\vee)_{L^P_{\beta,d}} \xrightarrow{\oplus \pi_j} \mathcal{S}_{\delta} \xrightarrow{\oplus \pi_j} \mathcal{S}_{\delta} \xrightarrow{\oplus \Xi_{P,\beta,d}^{\eta_0}} L(\chi),
\]

where the sums are over \([\delta] \in (\mathbb{Z}/p^3m)^\times \times (\mathbb{Z}/m)^\times \) or \( d \in (\mathbb{Z}/p^3)^\times \), related by \( d = \text{pr}_2([\delta]) \), and \( \Xi_{\beta,d}^{\eta_0} \) is the \( \eta_0 \)-averaging map

\[
\Xi_{\beta,d}^{\eta_0} : (m[\delta])[d] \mapsto \sum_{[\delta] \in \text{pr}_2^{-1}(d)} \eta_0([\text{pr}_2([\delta])]) \cdot m[\delta].
\]

We give two applications of these maps. Let \( \pi \) be any RACAR of weight \( \lambda \) with attached maximal ideal \( m_\pi \subset H^j \) as in \S 2.4.

Firstly, classical evaluation maps can detect existence of Shalika models:

**Proposition 4.15.** Suppose there exists \( \phi \in H^j_c(S_K, \mathcal{Y}_c^\vee(\mathbb{Q}_p))_{m_\pi} \) and some \( \chi, j \) and \( \eta_0 \) such that

\[
\mathcal{E}_{P,\chi}^{j,\eta_0}(\phi) \neq 0.
\]

Then \( \pi \) admits a global \( (\eta_0 \cdot [\omega, \psi]) \)-Shalika model, where \( \omega \) is the parity weight of \( \lambda \).

**Proof.** This is proved exactly as in [BSDW, Prop. 5.14]. Whilst \( \xi \) is replaced by \( \omega^{-1} \), and some non-zero volume factors change depending on \( P \), the argument of proof is identical.

Secondly, we generalise [DJR20, \S 4.], and show that up to a local zeta factor at \( p \), these maps compute \( L \)-values. Let \( K = \mathcal{J}_p \prod_{\ell \neq p} \text{GL}_{2n}(\mathbb{Z}_\ell) \) and let \( W = W_p \otimes \bigotimes_{\ell \neq p} W_{\ell} = S_{\omega,j}(\pi_p^j, E) \), where we choose the normalised spherical vector at each \( \ell \neq p \), and where \( W_p \) is an arbitrary \( E \)-rational vector in \( \pi_p^j \). Recall the map \( \Theta^\pm \) from (3.3), and the Friedberg–Jacquet integral \( \zeta(s, W, \chi) \) from \S 2.3.2.

**Theorem 4.16.** Let \( \chi \) be a finite-order Hecke character of conductor \( p^\beta \), let \( \beta = \max(1, \beta') \), and let \( j \in \text{Crit}(\lambda) \). If \( (-1)^j \chi \eta(-1) \neq \pm 1 \), then \( \mathcal{E}_{P,\chi}^{j,\eta_0}(\Theta^\pm(W)/\Omega^\pm) = 0 \). If \( (-1)^j \chi \eta(-1) = \pm 1 \), then

\[
\mathcal{E}_{P,\chi}^{j,\eta_0}(\Theta^\pm(W)/\Omega^\pm) = \delta_B(t_p^{-\beta}) T_p \cdot \lambda(t_p^\beta) \cdot \zeta(W^\pm) \times \frac{L(\pi \otimes \chi, j + 1/2)}{\Omega^\pm} \cdot \zeta_p \left( j + \frac{1}{2}, (u^{-1}t_p^\beta) \cdot W_p, \chi_p \right).
\]
Here \( \Upsilon_P \) is a non-zero rational volume constant independent of \( \chi \) and \( j \); we have

\[
\Upsilon_P = \gamma \cdot A_p^{-1} \cdot p^{n^2} \cdot [\# \text{GL}_n(\mathbb{Z}/p\mathbb{Z})]^{-1},
\]

where \( \gamma \) is the constant from [DJR20, (77)] and \( A_P \) is the constant from Lemma 4.4. (Note that when \( P = Q \) has Levi \( \text{GL}_n \times \text{GL}_n \), we have \( \Upsilon_Q = \gamma \)). Also, \( \zeta_j(W_\infty^\pm) \) is an archimedean zeta integral (in the notation of [DJR20, Thm. 4.7], our \( \zeta \)). As in [DJR20, Prop. 4.6], one first writes \( E \).

Proof. A rephrasing of [DJR20, Prop. 4.6, Thm. 4.7] in this language is described in [BSDW, 2.3] the integral equals a global Friedberg–Jacquet integral, which breaks into a product of local integrals. A way from that op. cit.; and at \( p \), by definition the zeta integral is the one in the statement of the theorem.

We will evaluate this local zeta integral for \( P = B \) and for specific choices of \( W_p \) over the next few sections.

**Part II. Local Theory: Shalika \( p \)-refinements**

For the remainder of the paper, unless otherwise specified we specialise to \( P = B \) and consider Iwahori level.

Let us summarise what we have done so far. We took \( \pi \) to be a RASCAR that is everywhere spherical. A \( p \)-refinement of \( \pi \) was a choice of Hecke eigenspace \( \tilde{\pi}_p \) in \( \pi_p^\text{IwG} \). To any choice of eigenvector \( W_p \in \tilde{\pi}_p \), before Theorem 4.16 we associated a (global) cohomology class in \( H^1_c(S,K,\mathcal{Y}_\chi') \). In that theorem, we computed its image under a scalar-valued functional, and showed that it took the form

\[
\left[\text{non-zero scalar}\right] \times \left[\text{critical } L\text{-value for } \pi\right] \times \zeta_p \left( j + \frac{1}{2}, (u^{-1}t_\beta^2) \cdot W_p, \chi_p \right).
\]

Over the next few sections, we compute the third term in this product – the local integral at \( p \) – for ‘nice’ choices of \( W_p \). This is a significant computation, spanning several sections, so we briefly sketch the steps.

- In §5, we compute \( \zeta_p(s, (u^{-1}t_\beta^2) \cdot W_p, \chi_p) \) as an explicit non-zero multiple of a specific value of \( W_p \) (depending on \( \beta \)).

- We are interested in finding \( p \)-refinements containing Hecke eigenvectors \( W_p \) for which this value is non-zero. We call such \( p \)-refinements *Shalika \( p \)-refinements*.

- In §6, we begin a systematic combinatorial study of \( p \)-refinements, and introduce ‘spin \( p \)-refinements’, a class of \( p \)-refinements for \( \text{GL}_{2n} \) that ‘come from \( \text{GSpin}_{2n+1} \).’

- In §7, we write down explicit eigenvectors attached to spin \( p \)-refinements. We precisely evaluate relevant values of these eigenvectors, and thus deduce that spin \( p \)-refinements are Shalika \( p \)-refinements. (In fact, we conjecture the converse is true; we hope to return to this in a sequel to this paper).

- In §8, we summarise all of the above, and fold the local theory back into the global results of Part I.

**Notation.** Since it will be entirely focused on local theory, in Part II we henceforth drop subscripts \( p \). In particular, we let \( \pi \) be a spherical representation of \( \text{GL}_{2n}(\mathbb{Q}_p) \) admitting an
Substituting this into (5.1), and using the Shalika transformation property, we reduce to

\[ \zeta(s, (u^{-1} t_p^β) \cdot W, χ) = \int_{GL_n(Q_p)} W \left[ \begin{pmatrix} x & \beta \\ 1 & 1 \end{pmatrix} \right] u^{-1} t_p^β \chi(\det(x)) |\det(x)|^{s-\frac{1}{2}} \, dx. \]  

(5.1)

The main aim of this section is Proposition 5.2, which computes this in terms of a specific choice of Hecke eigensystem \( α \) occurring in \( π^{Iw_G} \).

5. The local zeta integral at Iwahori level

We now give our first reduction of the local zeta integral

\[ \zeta(s, (u^{-1} t_p^β) \cdot W, χ) = \int_{GL_n(Q_p)} W \left[ \begin{pmatrix} x & \beta \\ 1 & 1 \end{pmatrix} \right] u^{-1} t_p^β \chi(\det(x)) |\det(x)|^{s-\frac{1}{2}} \, dx. \]

The main aim of this section is Proposition 5.2, which computes this in terms of a specific value of \( W \). First, we reduce the support of the zeta integral:

**Lemma 5.1.** Suppose \( W \in S_ρ^0(\pi) \) is fixed under the action of \( Iw_G \). Then the function

\[ GL_n(Q_p) \to C, \quad x \mapsto W(\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}) \]

is supported on \( M_n(Z_p) \cap GL_n(Q_p) \).

**Proof.** For \( y \in M_n(Z_p) \), right translation by \( \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix} \) in \( Iw_G \) gives

\[ W(\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}) = W \left( \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix} \right) = \psi(\text{tr}(xy)) W(\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}). \]

If \( x \notin M_n(Z_p) \cap GL_n(Q_p) \), then we can choose \( y \) such that \( \text{tr}(xy) \notin Z_p \), so \( \psi(\text{tr}(xy)) \neq 1 \). \( \square \)

Let \( d^\xi c \) be the Haar measure on \( Z_p^\xi \) with total measure 1. Let \( χ : Q_p^\times \to C^\times \) be a finite order character of conductor \( p^β \). In practice \( χ \) will be the local component at \( p \) of a Hecke character of \( p \)-power conductor, which forces \( χ(p) = 1 \); we thus impose this condition throughout. For such a character, denote its Gauss sum by

\[ \tau(χ) = p^β(1 - p^{-1}) \int_{Z_p^\xi} χ(c) \psi(p^{-β} c) d^\xi c. \]

(5.2)

Recall \( t_p = \text{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1) \). We write this in the form

\[ t_p = \begin{pmatrix} p^n z & \beta \\ z & 1 \end{pmatrix}, \quad z = \text{diag}(p^{n-1}, p^{n-2}, ..., p, 1) \in T_n(Q_p). \]

**Proposition 5.2.** Let \( χ \) have conductor \( p^β > 1 \), and let \( W \in S_ρ^0(π^{Iw_G}) \). Then

\[ \zeta(s, (u^{-1} t_p^β) \cdot W, χ) = \Upsilon(\chi(\det z^β)) \cdot \frac{β^{n^2+n}}{2} \cdot p^{βn(s-1/2)} \cdot τ(χ)^n \cdot χ(\det w_n) \cdot W \left( \begin{pmatrix} w_n z^{2β} & \beta \\ \beta & 1 \end{pmatrix} \right), \]

where \( \Upsilon = \text{vol}(Iw_n) \cdot (1 - p^{-1})^{-n} \cdot p^{(n^2-n)/2} \) is a scalar independent of \( W, χ \) and \( β \).

**Proof.** First, observe that

\[ \begin{pmatrix} x & \beta \\ 1 & 1 \end{pmatrix} u^{-1} t_p^β = \left( \begin{pmatrix} x & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z^β & \beta \\ \beta & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & -z^β x w_n z^β \\ z^β & 1 \end{pmatrix} \right) \left( \begin{pmatrix} p^{-nβ} z^{-β} x z^β & \beta \\ \beta & 1 \end{pmatrix} \right). \]

Substituting this into (5.1), and using the Shalika transformation property, we reduce to

\[ \zeta(s, (u^{-1} t_p^β) \cdot W, χ) = \eta(\det z^β) \]

\[ \times \int_{GL_n(Q_p)} \psi \left[ -\text{tr}(z^{-β} x w_n z^β) \right] W \left( \begin{pmatrix} p^{nβ} z^{-β} x z^β & \beta \\ \beta & 1 \end{pmatrix} \right) \chi(\det x) |\det(x)|^{s-1/2} \, dx. \]
We make the change of variables $y = p^{n β} z^{-β} x^β$. As $dx$ is a left and right Haar measure, we have $dy = dx$. Recalling that $χ(1) = 1$ and $|p| = 1/p$, we get
\[
\begin{aligned}
\zeta(s,(u^{-1}t^β) \cdot W,χ) &= \eta(\det z^β) p^{n^2 β(s-1/2)} \\
& \quad \times \int_{\text{GL}_n(\mathbb{Q}_p)} \psi\left(-\text{tr}(p^{-n β} z^β y z^{-β} w_n)\right) I(y) dy,
\end{aligned}
\]
where in the trace term, we have conjugated by $z^β$, and where we define
\[
I(y) := W \begin{pmatrix} y & 1 \\ \det y & \chi(\det y) \det y^{|s-1/2}. \end{pmatrix}
\]

We now cut down the support of this integral. Firstly, by Lemma 5.1 we can immediately reduce the support to $\text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$. To go further, we exploit Iwahori invariance of $W$.

**Notation 5.3.** (1) Let $A$ denote the set of all diagonal $n \times n$-matrices of the form
\[
\gamma = \text{diag}(c_{11},\ldots,c_{nn}), \quad c_{ii} \in \mathbb{Z}_p^×.
\]
(2) Let $B_β$ denote the additive group of all $n \times n$-matrices $δ$ with
\[
\delta_{i,j} = \begin{cases}
\begin{array}{ll}
c_{i,j} & \text{if } i > j \\
0 & \text{if } i = j \\
p^β c_{i,j} & \text{if } i < j
\end{array}
\end{cases}, \quad c_{ij} \in \mathbb{Z}_p,
\]
We will consider matrices of the form $α = γ + δ \in M_n(\mathbb{Z}_p)$ for $γ \in A$ and $δ \in B_β$. Note that $α$ is in the depth $p^β$ Iwahori subgroup $\text{Iw}_n(p^β) \subset \text{GL}_n(\mathbb{Z}_p)$ (the matrices that are upper-triangular modulo $p^β$). We set
\[
ε := \begin{pmatrix} α^{-1} \\ 1 \end{pmatrix} \in \text{Iw}_G \subset \text{GL}_{2n}(\mathbb{Z}_p).
\]
Now we translate the argument of the zeta integral by $ε$. By Iwahori invariance, we get
\[
\begin{aligned}
\zeta(s,(u^{-1}t^β) \cdot W,χ) &= \zeta(s,((u^{-1}t^βε) \cdot W,χ) \\
&= \eta(\det z^β) p^{n^2 β(s-1/2)} \\
& \quad \times \int_{\text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)} \psi\left[-\text{tr}(p^{-n β} z^β y z^{-β} w_n)\right] W \left(\begin{pmatrix} yα^{-1} \\ 1 \end{pmatrix} \chi(\det y) \det y^{|s-1/2/2} dy.
\end{aligned}
\]
Make the change of variables $x = yα^{-1}$; then this becomes
\[
\begin{aligned}
&= \eta(\det z^β) p^{n^2 β(s-1/2)} \chi(\det γ) \int_{\text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)} \psi\left[-\text{tr}(p^{-n β} z^β xα z^{-β} w_n)\right] I(x) dx.
\end{aligned}
\]
Here we used that $\det(α) = \det(γ) (\text{mod } p^β)$, that $χ$ has conductor $p^β$, and that $|\det(γ)| = 1$. Now we have
\[
\begin{align}
\psi\left[\text{tr}(-p^{-n β} z^β xα z^{-β} w)\right] &= \psi\left[\text{tr}(-p^{-n β} z^β xγ z^{-β} w)\right] \cdot \psi\left[\text{tr}(-p^{-n β} z^β xδ z^{-β} w)\right]
\end{align}
\]
We cut the support down first by averaging over $δ \in B_β$, then over $γ \in A$.

**Step 1: Average over $B_β$.** For $x \in \text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$, define
\[
F_x : B_β \rightarrow \mathbb{C}^X
\]
\[
δ \mapsto \psi\left[\text{tr}(-p^{-n β} z^β xδ z^{-β} w)\right].
\]
This is a group homomorphism by additivity of trace and $ψ$. 

**Lemma 5.4.** (i) There exists a finite index subgroup $B'_\beta \subset B_\beta$ such that $F_x$ is trivial on $B'_\beta$ for all $x \in \text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$.

(ii) For any fixed $x$, $F_x$ is the trivial function if and only if

$$x_{n+1-i,j} \in \left\{ \begin{array}{ll} p^{2\beta(n-i)+\beta}\mathbb{Z}_p & \text{if } i > j \\ p^{2\beta(n-i)}\mathbb{Z}_p & \text{if } i < j \end{array} \right. \quad (5.8)$$

**Proof.** (i) For $\delta$ sufficiently divisible by $p$, we have $F_\delta(\delta) = 1$ for all $x \in M_n(\mathbb{Z}_p)$.

(ii) Writing out the trace explicitly, one sees

$$\text{tr}[-p^{-n\beta}z^\beta x_\delta z^{-\beta}w] = -\sum_{i=1}^n \left( p^{\beta(1-2i)}x_{i,k} \left( \sum_{k<n+1-i} c_{k,n+1-i} + \sum_{k>n+1-i} p^{\beta}c_{k,n+1-i} \right) \right).$$

and uses the change of variables $i \mapsto n+1-i$. If (5.8) fails for some $(i,j)$, then $F_x$ will be non-trivial on the matrix $\delta$ which is zero apart from a 1 at $(i,j)$, so $F_x$ is not the trivial function. Conversely, if (5.8) does hold, then the trace above is always integral and $F_x$ is trivial. \qed

Let $M'_\beta \subset \text{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$ be the subset of matrices $x$ satisfying the conditions in (5.8).

**Corollary 5.5.** For any $\gamma \in A$, we have

$$\zeta(s, (u^{-1}l^\beta_p) \cdot W, \chi) = \eta(\det z)^2p^{2\beta(s-1/2)}\chi(\det \gamma) \int_{M'_\beta} \psi \left[ \text{tr}(p^{-n\beta}z^\beta x_\gamma z^{-\beta}w) \right] I(x)dx.$$ 

**Proof.** Using (5.5) (in the first equality) and (5.6) and (5.7) (in the second), we have

$$\zeta(s, (u^{-1}l^\beta_p) \cdot W, \chi) = \frac{1}{[B_\beta : B'_\beta]} \sum_{\delta \in B_\beta / B'_\beta} \zeta(s, (u^{-1}l^\beta_p(\gamma + \delta)) \cdot W, \chi)$$

$$= \eta(\det z)^2p^{2\beta(s-1/2)}\chi(\det \gamma) \int_{\text{Gl}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)} \psi \left[ -\text{tr}(p^{-n\beta}z^\beta x_\gamma z^{-\beta}w_n) \right]$$

$$\times \left[ \frac{1}{[B_\beta : B'_\beta]} \sum_{\delta \in B_\beta / B'_\beta} F_x(\delta) \right] I(x)dx.$$ 

When $F_x$ is non-trivial, the square-bracketed term (hence the integrand) vanishes by character orthogonality; and when $F_x \equiv 1$, it is identically 1. We conclude since by Lemma 5.4(ii), $F_x$ is trivial if and only if $x \in M'_\beta$. \qed

**Step 2: Average over $A$.** Equip $A \cong (\mathbb{Z}_p^\times)^n$ with the measure $d^\times A = \prod_{i=1}^n d^\times c_{i,i}$.

**Lemma 5.6.** We have

$$\chi(\det \gamma) \psi \left[ \text{tr}(p^{-n\beta}z^\beta x_\gamma z^{-\beta}w) \right] = \prod_{i=1}^n \chi(c_{i,i}) \psi \left[ -p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i} \right].$$

Therefore

$$\int_{\gamma \in A} \chi(\det \gamma) \psi \left[ \text{tr}(p^{-n\beta}z^\beta x_\gamma z^{-\beta}w) \right] d^\times A$$

$$= \prod_{i=1}^n \int_{\mathbb{Z}_p^\times} \chi(c_{i,i}) \psi \left[ -p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i} \right] d^\times c_{i,i}$$

$$= \left\{ \begin{array}{ll} \frac{\tau(\chi)^n}{\prod_{i=1}^n c_{i,i}} \prod_{i=1}^n \chi(x'_{n+1-i,i})^{-1} & \text{if } x_{n+1-i,i} \in p^{2\beta(n-i)}\mathbb{Z}_p^\times \forall i \\ 0 & \text{otherwise} \end{array} \right.$$

where $x'_{n+1-i,i} = x_{n+1-i,i}/p^{2\beta(n-i)}$ and $\tau(\chi)$ is the Gauss sum from (5.2).
Proof. We have
\[ \chi(\det \gamma) = \prod_{i=1}^{n} \chi(c_{i,i}) \]
and
\[ \psi\left[ \text{tr}(-p^{-n\beta}z^\beta \chi^z) \right] = \prod_{i=1}^{n} \psi\left(-p^{\beta(1-2i)}x_{i,n+1-i}c_{n+1-i,n+1-i} \right) \]
\[ = \prod_{i=1}^{n} \psi\left(-p^{-(2\beta(n-i)+\beta)}x_{n+1-i,i}c_{i,i} \right), \]
giving the first part. The rest follows from a simple change of variables.

Let \( M_{\beta} \subset M'_{\beta} \) be the subset where \( x_{n+1-i,i} \in p^{2\beta(n-i)}Z_p^\times \) for all \( i \). Note that
\[ M_{\beta} = \prod_{i=1}^{n} \chi(x'_{n+1-i,i}) \cdot Iw_{n}(p^\beta). \]

**Corollary 5.7.** We have
\[ \zeta(s, (u^{-1})_{p} \cdot W, \chi) = \eta(\det z^\beta)p^n z^{\beta(s-1/2)} \frac{\tau(\chi)}{\beta(1-p^{-1})} \int_{M_{\beta}} \prod_{i=1}^{n} \chi(x'_{n+1-i,i})^{-1} I(x) dx. \]

**Proof.** This is similar to Corollary 5.5. Integrate the expression of that corollary over \( A \), and reduce the support using Lemma 5.6.

Now using that fact that \( M_{\beta} = \prod_{i=1}^{n} x'_{n+1-i,i} \) modulo \( p^\beta \). Making the change of variables, we see that
\[ \int_{M_{\beta}} \prod_{i=1}^{n} \chi(x'_{n+1-i,i})^{-1} I(x) dx \]
\[ = \int_{Iw_{n}(p^\beta)} W\left(w_{n}z^{2\beta}x''\right) \prod_{i=1}^{n} \chi(x'_{n+1-i,i})^{-1} \chi(\det w_{n}z^{2\beta}x'') \det w_{n}z^{2\beta}|s-1/2| dx'' \]
\[ = \chi(\det w_{n}) \cdot p^{-\beta n(n-1)(s-1/2)} \int_{Iw_{n}(p^\beta)} W\left(w_{n}z^{2\beta}\right) dx'' \]
\[ = \chi(\det w_{n}) \cdot p^{-\beta n(n-1)(s-1/2)} \cdot \text{Vol}(Iw_{n}(p^\beta)) \cdot W\left(w_{n}z^{2\beta}\right). \]

In the penultimate equality, we have used that
\[ \det x'' \equiv \prod_{i=1}^{n} x'_{n+1-i,i} \text{ (mod } p^\beta) \],
\[ \chi(p) = 1, \ |\det w_{n}| = 1, \text{ and } |\det z| = p^{-n(n-1)/2}. \]
Finally note that
\[ \text{Vol}(Iw_{n}(p^\beta)) = p^{-(\beta-1)\frac{2^2-n}{2}} \text{Vol}(Iw_{n}). \]

Putting (5.10) and (5.9) into Corollary 5.7 completes the proof of Proposition 5.2.

**6. Spin \( p \)-refinements**

As highlighted in the introduction to Part II, we want to answer:
For which \( p \)-refinements \( \tilde{\pi} \) does there exist \( W \in S^0(\tilde{\pi}) \) with \( \zeta(s, (u^{-1}j_p^2)_\tau, t) \neq 0 \)?

Given Proposition 5.2, this is equivalent to asking when there exists \( W \in \tilde{\pi} \) and \( \beta \geq 1 \) such that \( W \left( w_{\alpha, 2\beta} \right) \neq 0 \). We expect this to be true only for a special class of \( p \)-refinements, those that ‘interact well with the Shalika model’.

In this section, we begin to make this assertion rigorous. We define ‘spin’ \( p \)-refinements as those ‘that come from \( GSpin_{2n+1} \)’, made precise in Proposition 6.13. In later sections we will show that spin \( p \)-refinements contain eigenvectors \( W \) such that \( W \left( w_{\alpha, 2\beta} \right) \neq 0 \). Our key application of spin \( p \)-refinements will ultimately be summarised in Corollary 8.2.

**Notation 6.1.** Let \( G = GSpin_{2n+1} \), where we take the split form as in [AS06]. Since it admits a Shalika model, we know from [AS06] that \( \pi \) is the functorial transfer of a representation \( \Pi \) on \( G(F) \).

We first give a concrete definition of spin \( p \)-refinement. We justify it in §6.4, and give several equivalent formulations in Propositions 6.8 and 6.13. Let \( \tilde{\pi} = (\pi, \alpha) \) be a \( p \)-refinement of \( \pi \) (as in §2.4.3), and for \( 1 \leq r \leq 2n - 1 \) write \( \alpha_{p,r} := \alpha(U_{p,r}) \).

**Definition 6.2.** We say \( \tilde{\pi} \) is a spin \( p \)-refinement if \( \alpha_{p,n+s} = \eta(p)^s \alpha_{p,n-s} \) for all \( 0 \leq s \leq n - 1 \).

We will show \( \tilde{\pi} \) is spin if and only if \( \alpha \) factors through an eigensystem occurring in \( \Pi^{Iw} \), for \( I_G \) the Iwahori subgroup of \( G \) (see §6.4).

### 6.1. Conventions for Shalika models

In the next two sections, there will be **two** equivalent but competing conventions for Shalika models – those of Ash–Ginzburg from [AG94], and of Asgari–Shahidi from [AS06]. For clarity, we highlight both here. In both cases the set \( \{1, \ldots, 2n\} \) is partitioned into pairs, and:

- Ash–Ginzburg identify \( i \) and \( n + i \), i.e. \( \{1, n + 1\}, \{2, n + 2\}, \ldots, \{n, 2n\} \). This is used in [AG94, §1] to give explicit intertwining operators from \( \pi \) into its Shalika model (see §7.2).
- Asgari–Shahidi identify \( i \) and \( 2n + 1 - i \), i.e. \( \{1, 2n\}, \{2, 2n - 1\}, \ldots, \{n, n + 1\} \). This interacts better with \( GSpin_{2n+1} \), and hence is more natural when discussing spin \( p \)-refinements.

These normalisations are interchanged by the element \( \tau = (1_n, w_n) \in W_G \), recalling \( w_n \in W_n \) is the longest Weyl element (that interchanges \( [n + 1 \leftrightarrow 2n], [n + 2 \leftrightarrow 2n - 1], \) etc.).

As \( \pi \) is spherical, it can be written as a (normalised) induction from the upper-triangular Borel, i.e. there exists an unramified character \( \theta : T(F) \to \mathbb{C}^\times \) such that \( \pi = \text{Ind}_B^G \theta \).

Since each normalisation has its own advantages and disadvantages, we will use each in their favoured setting, and carefully track the differences between them (as e.g. in Remark 6.11). In particular:

- For the root systems and spin refinements in the present section, we adopt Asgari–Shahidi’s normalisation.
- For convenience in §7.4, for the rest of the paper we always adopt the Ash–Ginzburg normalisation on \( \theta \), i.e.:

**Proposition 6.3.** [AG94, Prop. 1.3]. \( \pi \) admits an \( (\eta, \psi) \)-Shalika model if and only if we may normalise \( \theta \) so that \( \theta, \theta_{n+1} = \eta \).
6.2. Root systems for $GL_{2n}$ and $GSpin_{2n+1}$

Recall the space of algebraic characters and cocharacters of the torus $T \subset G = GL_{2n}$ are given by
\[
X = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \mathbb{Z}e_{2n}, \quad X^\vee = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \oplus \cdots \mathbb{Z}e_{2n}^*.
\]
Write $(-,-)_G$ for the natural pairing on $X \times X^\vee$. The corresponding root system is $A_{2n-1}$, with roots $R = \{ \pm (e_i - e_j) : 1 \leq i < j \leq 2n \}$. The Weyl group $W_G = S_{2n}$ acts by permuting the $e_i$, with longest Weyl element $w^0_{G}$ the permutation that sends $e_i \mapsto e_{2n-i+1}$ for all $i$.

Let $X_0 \subset X$ be the space of pure characters $X_0 = \{ \lambda \in X : \exists \omega \in \mathbb{Z} \text{ such that } \lambda_i + \lambda_{2n-i+1} = \omega \}$, and let
\[
W^0_G := \{ \sigma \in W_G : \forall \lambda \in X_0, \ \lambda^\sigma \in X_0 \} \subset W_G. \tag{6.1}
\]

Now fix a standard upper Borel subgroup $B$ and maximal split torus $T$ in $G = GSpin_{2n+1}$. This has rank $n+1$ [Asg02, Thm. 2.7]. We use calligraphic letters to denote objects for $GSpin$, and otherwise maintain the same notational conventions as before.

**Proposition 6.4.** The root system for $G$ is $(X,R,X^\vee,R^\vee)$, where
\[
X = \mathbb{Z}f_0 \oplus \mathbb{Z}f_1 \oplus \cdots \mathbb{Z}f_n, \quad X^\vee = \mathbb{Z}f_0^* \oplus \mathbb{Z}f_1^* \oplus \cdots \mathbb{Z}f_n^*,
\]
with roots $R = \{ \pm f_i \pm f_j : 1 \leq i < j \leq n \} \cup \{ f_i : 1 \leq i \leq n \}$ and positive roots $\{ f_i : 1 \leq i \leq n \} \cup \{ f_i \pm f_j : 1 \leq i < j \leq n \}$. The Weyl group $W_G$ has size $2^n \cdot n!$, generated by permutations $\sigma \in S_n$ and sign changes $\text{sgn}_{i}$, which act on roots and coroots respectively as (for $j \neq i$)
\[
\begin{align*}
\sigma f_0 &= f_0, & \sigma f_i &= f_{\sigma(i)}, & \text{sgn}_{f_0} &= f_0 + f_i, & \text{sgn}_{i} f_i &= - f_i, & \text{sgn}_{j} f_i &= f_i, \\
\sigma f^*_0 &= f_0^*, & \sigma f_i^* &= f_{\sigma(i)}^*, & \text{sgn}_{f^*_0} &= f_0^* + f_i^*, & \text{sgn}_{i} f_i^* &= f_0^* - f_i^*, & \text{sgn}_{j} f_i^* &= f_i^*.
\end{align*}
\tag{6.2}
\]

**Proof.** The first part is [Asg02, Prop. 2.4], and the second [HS16, Lem. 13.2.2].

Write $(-,-)_G$ for the natural pairing on $X \times X^\vee$.

6.3. The maps $j$ and $j^\vee$

There is a natural injective map $j : X \hookrightarrow X$ given by
\[
f_i \mapsto e_i - e_{2n-i+1} \text{ for } 1 \leq i \leq n, \quad f_0 \mapsto e_{n+1} + \cdots + e_{2n}.
\]
We may identify $X$ with cocharacters of $GSpin_{2n}$, and $X$ with cocharacters of $GL_{2n}$. The map $j$ is then the natural map on cocharacters induced by the inclusion $T_{GSpin_{2n}} \subset T$ of tori.

**Proposition 6.5.** We have $X_0 = j(X)$.

**Proof.** Any linear combination of the $j(f_i)$ is a pure weight with purity weight 0, and any such weight arises in this form; and scaling the purity weight to $w$ corresponds to adding $j(wf_0)$.

**Proposition 6.6.** There is a map $W_G \to W_G$ of Weyl groups, which we also call $j$, such that:

(i) $j$ induces an isomorphism $W_G \cong W^0_G \subset W_G$;

(ii) for all $\sigma \in W_G$ and $\mu \in X$, we have $j(\mu^\sigma) = j(\mu)^{j(\sigma)}$.

**Proof.** We know $W_G$ is generated by permutations and sign changes in $\{ f_1, \ldots, f_n \}$, and $W_G$ is permutations in $\{ e_1, \ldots, e_{2n} \}$. If $\sigma \in S_n \subset W_G$ is a permutation, then define
\[
\begin{align*}
\sigma(f_i(1), \ldots, f_i(n), f_{2n-\sigma(n)+1}, \ldots, e_{2n-\sigma(1)+1}) = j(\sigma) := (e_{\sigma(1)}, \ldots, e_{\sigma(n)}, e_{2n-\sigma(n)+1}, \ldots, e_{2n-\sigma(1)+1})
\end{align*}
\]
and if $\sigma = e_i$ is the sign change at $i \in \{ 1, \ldots, n \}$, define $j(e_i)$ to be the permutation switching $e_i$ and $e_{2n-i+1}$. A simple check shows this induces a well-defined homomorphism.
Suppose \( \lambda \in X \) is pure with all the \( \lambda_i \) distinct. Let \( \sigma \in W_G = S_n \); if \( \lambda^\sigma \) is pure, then \( \sigma \) must preserve the relative positions of each pair \( \{ \lambda_i, \lambda_{2n-i+1} \} \). The only way to do this is to permute \( i \in \{1, \ldots, n\} \) or to switch \( \lambda_i, \lambda_{2n-i+1} \). These are exactly the permutations in \( j(W_G) \), giving (i).

Part (ii) is a simple explicit check. \[ \square \]

Dually, define also a map \( j^\sigma : X^\sigma \to X^\sigma \) by sending \( \nu \in X^\sigma \) to

\[
j^\sigma(\nu) = \sum_{i=0}^n (j(f_i), \nu)_G \cdot f_i^\sigma.
\]

Then for all \( \mu \in X \), we have

\[
(\mu, j^\sigma(\nu))_G = (j(\mu), \nu)_G
\]

by construction. (Again, this map arises from our explicit realisation of \( T_{\text{GSpin}_G} \subset T \).) Also let \( j^\sigma : W_G \to W_G \) denote the inverse to \( j : W_G \cong W_G^0 \).

**Proposition 6.7.** For all \( \nu \in X^\sigma \) and \( \sigma \in W_G^0 \), we have \( j^\sigma(\nu^\sigma) = j^\sigma(\nu)^{(\sigma)} \).

**Proof.** If \( \sigma \in W_G^0 \), then \( \sigma = j(\rho) \) for some \( \rho \in W_G \), and \( j^\sigma(\sigma) = \rho \). Then compute that

\[
j^\sigma(\nu^\sigma) = \sum_{i=0}^n (j(f_i), \nu^\sigma)_G f_i^\sigma = \sum_{i=0}^n (j(f_i)(e_i^{-1}), \nu)_G f_i^\sigma = \sum_{i=0}^n (j(f_i)^{-1}, \nu)_G f_i^\sigma = \sum_{i=0}^n (j(f_i), \nu)_G (f_i^\sigma)^\rho = j^\sigma(\nu)^{(\sigma)},
\]

where the penultimate equality is a simple check. \( \square \)

### 6.4. Spin refinements via GSpin

Via the Satake isomorphism, we may describe Hecke operators in terms of cocharacters. Note \( U_{p,r} \) is attached to the cocharacter

\[
[v_r : x \mapsto (e_1^r + \cdots + e_r^r)(x)] \in X^\sigma,
\]

since \( t^G_{p,r} := t_{p,r} = \nu_r(p) \). Define \( t^G_{p,r} := j^\sigma(\nu_r(p)) \in T(F); \) by definition, for \( 0 \leq s \leq n - 1 \)

\[
j^\sigma(\nu_{n-s}) = f_1^s + \cdots + f_{n-s}^s, \quad j^\sigma(\nu_{n+s}) = j^\sigma(\nu_{n-s}) + s f_0^s.
\]

Let \( Iw_G \subset G(Z_p) \) be the Iwahori subgroup for \( G \), and for \( 1 \leq r \leq n \), let \( U_{p,r} := [Iw_G t^G_{p,r}, Iw_G] \).

Also let \( \mathcal{V}_p := [Iw_G f_0^s(p) Iw_G] \). From (6.5), we see \( U_{p,n+s} = \mathcal{V}_p \cdot U_{p,n-s} \).

Let \( \mathcal{H}_{p,r} := Z_p[U_{p,r}, U_{p,r} : 1 \leq r \leq n] \) be the Hecke algebra for \( G \). Then \( j^\sigma \) induces a map

\[
\mathcal{H}_p \longrightarrow \mathcal{H}_{p,r}^0, \quad U_{p,n-s} \longmapsto U_{p,n-s}, \quad U_{p,n+s} \longmapsto \mathcal{V}_p \cdot U_{p,n-s},
\]

for each \( 0 \leq s \leq n - 1 \). From the definitions, we obtain:

**Proposition 6.8.** A \( p \)-refinement \((\pi, \alpha)\) is a spin \( p \)-refinement if and only if \( \alpha \) factors through

\[
\mathcal{H}_p \longrightarrow \mathcal{H}_{p,r}^0 \stackrel{\alpha^G}{\longrightarrow} \overline{Q},
\]

for some character \( \alpha^G \) with \( \alpha^G(\mathcal{V}_p) = \eta(p) \).

**Remark 6.9.** We could add the operator \( U_{p,2n} = [Iw_G \text{diag}(p, \ldots, p)Iw_G] \) to \( \mathcal{H}_p \); it acts by \( \text{diag}(p, \ldots, p) \), so acts on \( \pi \) by \( \eta(p)^n \) (since \( \pi \) has central character \( \eta^n \)). We would then have \( j^\sigma(U_{p,2n}) = \mathcal{V}_p^n \). In particular, the requirement that \( \alpha^G(\mathcal{V}_p) = \eta(p) \) is natural.
Recall $\pi$ on $G(\mathbb{Q}_p)$ is the transfer of $\Pi$ on $\mathcal{G}(\mathbb{Q}_p)$, and:

From now on, we assume that the Satake parameter of $\pi$ is regular semisimple. (6.6)

We now show in this case that if $(\pi, \alpha)$ is a spin refinement, then $\alpha^G$ occurs in $\Pi$.

Recall in §6.1 we fixed an unramified character $\theta$ of $T(\mathbb{Q}_p)$ such that $\pi = \text{Ind}_T^G \theta$ (cf. [DJR20, (43)]), where $\theta = | \cdot |^{-1/2}$. Recall $\tau = (1_{w_0}) \in \mathcal{W}_G$. From [AS06, p.177(i)] and [AS14, Prop. 5.1], we see:

**Proposition 6.10.** There is an unramified character $\theta_G$ of $T(\mathbb{Q}_p)$ such that:

(i) $\Pi = \text{Ind}_T^G \theta_G$ is a (normalised) parabolic induction,

(ii) we have $j(\theta_G) = \theta^\tau$.

**Remark 6.11.** The $\tau$ is necessary as we are using the normalisation of Ash–Ginzburg (see §6.1). From this, we see that $\theta^\tau$ is a more natural normalisation for $G$Spin computations. If we normalised $\theta$ as in Asgari–Shahidi, we could remove $\tau$ from (ii) and henceforth in this section.

By Remark 2.6(ii) and (6.6), our fixed choice of $\theta$ (hence $\theta^\tau$) fixes a bijection

$$\Delta_{\theta^\tau} : \{ \text{p-refinements of } \pi \} \rightarrow \mathcal{W}_G.$$

**Lemma 6.12.** A p-refinement $\tilde{\pi} = (\pi, \alpha)$ is spin if and only if $\Delta_{\theta^\tau}(\tilde{\pi}) \in \mathcal{W}_G^0$.

**Proof.** Conjugating Proposition 6.3 by $\tau$, we see for each $i$, we have $\theta_i^\tau \cdot \theta_{n+1-i}^\tau = \eta$ as characters of $F^\times$. Let $\sigma = \Delta_{\theta^\tau}(\tilde{\pi})$. By definition of $\mathcal{W}_G^0$, we see if $\sigma \in \mathcal{W}_G^0$ then

$$\theta_{\sigma(i)}^\tau \cdot \theta_{\sigma(n+1-i)}^\tau = \eta,$$

whilst as the Satake parameter is regular semisimple, if $\sigma \notin \mathcal{W}_G^0$, then (6.7) fails for some $i$. Thus (6.7) holds for all $i$ if and only if $\sigma \in \mathcal{W}_G^0$.

From the explicit description of $\alpha_{p,r} = \alpha(U_{p,r})$ from Proposition 2.5, we see that $\alpha_{p,n+s} = \eta(p)^s \alpha_{p,n-s}$ if and only if (6.7) holds for all $i$. The result follows. \qed

A p-refinement of $\Pi$ is a tuple $\tilde{\Pi} = (\Pi, \alpha^G)$, where $\alpha^G : \mathcal{H}_p^G \rightarrow \mathcal{Q}$ is a system of Hecke eigenvalues appearing in $\Pi^{w_{\mathfrak{p}}}$. We say $\tilde{\Pi}$ is regular if this system of eigenvalues appears in $\Pi^G$ without multiplicity, i.e. the generalised eigenspace is a line. As in Proposition 2.5, after fixing the unramified character $\theta_G$, such p-refinements correspond to elements $\sigma \in \mathcal{W}_G$.

The following is our main motivation for the definition of spin p-refinement.

**Proposition 6.13.** Suppose the Satake parameter of $\pi$ is regular semisimple, and let $\tilde{\pi} = (\pi, \alpha)$ be a p-refinement. Then $\tilde{\pi}$ is a spin p-refinement if and only if there exists a p-refinement $(\Pi, \alpha^G)$ of $\Pi$ such that $\alpha = \alpha^G \circ j^\mathfrak{p}$ as characters $\mathcal{H}_p \rightarrow \mathcal{Q}$.

**Proof.** By Proposition 6.8, $\tilde{\pi}$ is spin if and only if $\alpha$ factors through some $\alpha^G$; so suffices to show that in this case, the system $\alpha^G$ occurs in $\Pi^{w_{\mathfrak{p}}}$. Let $\sigma = \Delta_{\theta^\tau}(\tilde{\pi})$. By Lemma 6.12, $\sigma \in \mathcal{W}_G^0$.

Denote half the sum of the positive roots for $G$ and $\mathcal{G}$ by

$$\rho_G = \left( \frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{2n-3}{2}, \frac{(2n-1)}{2} \right), \quad \rho_\mathcal{G} = \frac{2n-1}{2} f_1 + \frac{2n-3}{2} f_2 + \cdots + \frac{1}{2} f_n. \quad (6.8)$$

Note $j(\rho_G) = \rho_G$. By rewriting the formulation of Proposition 2.5, the $U_{p,r}$-eigenvalue of $\tilde{\pi}$ can be written as

$$\alpha_{p,r} = q^{(\rho_G, \nu^G_{\mathfrak{p}, r})} p^{(\theta^\tau, \nu^G_{\mathfrak{p}, r})},$$

where we identify $\theta^\tau(\nu^G_{\mathfrak{p}, r}(p)) = p^{(\theta^\tau, \nu^G_{\mathfrak{p}, r})}$ under the natural extension of $\langle -,- \rangle_G$.
Since \( \sigma \in \mathcal{W}_G^0 \), by Proposition 6.6 it is of the form \( j(\omega) \) for some \( \omega \in \mathcal{W}_G \). Let \( \Pi = (\Pi, \tilde{\alpha}) \) be the \( p \)-refinement corresponding to \( \omega \); then by considering the characteristic polynomial of \( \mathcal{U}_{p,r} \) on \( \Pi^p_{\mathcal{W}_G} \) (see [OST, Prop. 4.3]), we see that the \( \mathcal{U}_{p,r} \)-eigenvalue attached to \( \Pi \) is

\[
\tilde{\alpha}(\mathcal{U}_{p,r}) = q^{\langle \rho_G \circ \mathcal{U}_{p,r}, \mathcal{U}_{p,r} \rangle} p^{\langle \mathcal{U}_{p,r}, \mathcal{U}_{p,r} \rangle} \alpha = q^{\langle \rho_G, \mathcal{U}_{p,r} \rangle} p^{\langle \mathcal{U}_{p,r}, \mathcal{U}_{p,r} \rangle} \alpha_p = \alpha_{p,r},
\]

where in the second equality we have used \( j(\rho_G) = \rho_G \) with (6.3), and in the third we have used Proposition 6.7 with (6.3). In particular, \( \tilde{\alpha}(\mathcal{U}_{p,r}) = \tilde{\alpha}(\mathcal{U}_{p,r}) \) for all \( r \).

It remains to show \( \tilde{\alpha}(\mathcal{V}_p) = \tilde{\alpha}(\mathcal{V}_p) \) by Proposition 6.8. Also, \( f_1^p(p) \) is central in \( \mathcal{G}(\mathbb{Q}_p) \) by [AS06, Prop. 2.3], and the central character of \( \Pi \) is \( \eta \) by p.178 op. cit. Hence \( \mathcal{V}_p \) acts on \( \Pi \) by \( \eta(p) \). It follows that \( \tilde{\alpha}(\mathcal{V}_p) = \eta(p) \), and we conclude that \( \tilde{\alpha} = \alpha^G \), as required.

Remark 6.14. We finally indicate how spin \( p \)-refinements relate to the notion of \( Q \)-regular \( Q \)-refinement in [DJR20, Def. 3.5]. This was defined to be an element \( T \in \mathcal{W}_G \), equivalent to a choice of \( n \)-element subset \( S_T \subset \{ 1, \ldots, 2n \} \), satisfying two conditions. Their condition (i) is our definition of regularity, and their condition (ii) guarantees that \( T \) lies in the image of the composition \( \mathcal{W}_G^0 \to \mathcal{W}_G \to \mathcal{W}_G \). One sees that spin \( p \)-refinements are in bijection with \( T \in \mathcal{W}_G \) satisfying (ii) together with an ordering of \( S_T \).

7. Shalika \( p \)-refinements

We now define another class of \( p \)-refinements. Let \( \tilde{\pi} = (\pi, \alpha) \) be a \( p \)-refinement. Recall we write \( f \in \tilde{\pi} \) as shorthand for \( f \in \pi^w \mathcal{U}_{p,r} \) \( : 1 \leq r \leq 2n - 1 \) (and similarly for \( W \in \mathcal{S}_0^\alpha(\tilde{\pi}) \)). Recall that \( z = \text{diag}(p^{n-1}, p^{n-2}, \ldots, p, 1) \).

Definition 7.1. We say \( \tilde{\pi} \) is a Shalika \( p \)-refinement if there exist \( W \in \mathcal{S}_0^\alpha(\tilde{\pi}) \) and \( \beta \geq 1 \) such that

\[
W \begin{pmatrix} w_n z^{2\beta} & 1 \\ & \end{pmatrix} \neq 0. \tag{7.1}
\]

Note that via Proposition 5.2, this implies that the local zeta integral \( \zeta(s, (u^{-1}) \cdot W, \chi) \) is non-vanishing for any smooth character \( \chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times \) of conductor \( p^\beta \). Accordingly this generalises the condition [BSDW, Cond. 2.9(C2)] required to construct a \( p \)-adic \( L \)-function for \( \tilde{\pi} \) via the methods of [BSDW].

Our main results of this section (Proposition 7.12 and Corollary 7.13) show that all regular spin \( p \)-refinements are Shalika \( p \)-refinements, and precisely compute the value (7.1).

7.1. Explicit \( p \)-refinements

We now write down an explicit eigenvector in any given \( p \)-refinement.

Recall that we fixed an identification \( \pi \cong \text{Ind}_{B}^G \theta \), for \( \theta \) in the Ash–Ginzburg normalisation (i.e. \( \theta_i \theta_{i+1} = \eta \) for all \( i \) as in §6.1). Recall also that our choice of \( \theta \) fixes a bijection \( \Delta_{\theta} : \{ \text{rfmns of } \pi \} \to \mathcal{W}_G \) (see Remark 2.6(ii)).

For \( \sigma \in \mathcal{W}_G \), let \( \tilde{\pi}_\sigma = (\pi, \alpha_\sigma) := \Delta_{\theta}^{-1}(\sigma) \); then every \( p \)-refinement is of the form \( \tilde{\pi}_\sigma \) for some \( \sigma \). Let

\[
\sigma_0 \in \text{Ind}_{B}^G \theta^\sigma \cong \pi
\]

be the unique function that is:

- Iwahori-invariant,
- supported on the big Bruhat cell \( B(\mathbb{Q}_p) \cdot w_{2n} \cdot \text{Iw}_G \), and
• normalised so that \( f_0^\sigma(w_{2n}) = p^{n(n-1)}. \)

**Proposition 7.2.** We have \( f_0^\sigma \in \hat{\pi}_\sigma \). Further, \( f_0^\sigma \) is a Hecke eigenvector.

**Proof.** We must show that for each \( r = 1, \ldots, 2n-1 \), we have \( U_{p,r} f_0^\sigma = \alpha_\sigma(U_{p,r}) f_0^\sigma \). For such \( r \), let \( P_r \) be the maximal standard parabolic subgroup with Levi \( \text{GL}_r \times \text{GL}_{2n-r} \), and let \( J_r \) be the associated parahoric subgroup. Parahoric decomposition gives \( J_r = N_{P_r}(Z_p) \cdot (P_r^-)(Z_p) \cap J_r \), where \( N_{P_r} \subset P_r \) is the unipotent radical. Intersecting with \( Iw_G \) shows \( Iw_G = N_{P_r}(Z_p) \cdot (P_r^-)(Z_p) \cap Iw_G \). As \( t_{p,r}^{-1}(P_r^-)(Z_p) \cap Iw_G \times (P_r^-)(Z_p) \cap Iw_G \) is a normal subgroup of \( Iw_G \), we deduce

\[
Iw_G t_{p,r} Iw_G = N_{P_r}(Z_p)(P_r^-)(Z_p) \cap Iw_G \cdot t_{p,r} \cdot Iw_G = N_{P_r}(Z_p) \cdot t_{p,r} \cdot Iw_G,
\]

and in particular we can decompose the double coset into single cosets via

\[
Iw_G t_{p,r} Iw_G = \bigsqcup_{m \in M_{r,2n-1}(Z_p)/M_{r,2n-1}(p)} \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) \cdot t_{p,r} \cdot Iw_G. \tag{7.2}
\]

We have Bruhat decomposition

\[
\text{GL}_n(Q_p) = \bigsqcup_{\rho \in W_n} B_n(Q_p) \cdot \rho \cdot Iw_n, \tag{7.3}
\]

so it suffices to compute \( U_{p,r} f_0^\sigma(\rho) \) for \( \rho \in W_G \). By (7.2), we have

\[
U_{p,r} f_0^\sigma(\rho) = \sum_{m \in M_{r,2n-1}(Z_p)/M_{r,2n-1}(p)} \rho \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) t_{p,r}.
\]

**Claim 7.3.** \( \rho \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) t_{p,r} \in B(Q_p) \cdot w_{2n} \cdot Iw_G \) if and only if \( \rho = w_{2n} \) and \( m \in M_{r,2n-1}(p) \).

**Proof of claim:** Let \( t_{p,r}' = w_{2n} t_{p,r} w_{2n} \in T(Q_p) \). Then

\[
B(Q_p) w_{2n} Iw_G = B(Q_p) t_{p,r}' w_{2n} Iw_G = B(Q_p) w_{2n} t_{p,r} Iw_G = B(Q_p) w_{2n} Iw_{G}^{-r_{p,r}} t_{p,r},
\]

where \( Iw_{G}^{-r_{p,r}} = t_{p,r} Iw_{G} t_{p,r}^{-1} \). Thus

\[
\rho \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) t_{p,r} \in B(Q_p) w_{2n} Iw_G \iff \rho \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) \in B(Q_p) w_{2n} Iw_{G}^{-r_{p,r}}.
\]

Conjugating (7.3) by \( t_{p,r} \), we obtain \( \text{GL}_{2n}(Q_p) = \bigsqcup_{\rho' \in W_G} B(Q_p) \rho' Iw_{G}^{-r_{p,r}} \). It follows immediately that \( \rho \left( \begin{smallmatrix} 1_r & m \\ 0 & 1_{2n-r} \end{smallmatrix} \right) \) is in the cell \( w_{2n} Iw_{G}^{-r_{p,r}} \), and the claim follows.

We return to Proposition 7.2. The claim implies \( U_{p,r} f_0^\sigma(\rho) = 0 \) unless \( \rho = w_{2n} \), and

\[
U_{p,r} f_0^\sigma(w_{2n}) = f_0^\sigma(w_{2n} t_{p,r}) = f_0^\sigma(t_{p,r} w_{2n}) = \theta^\sigma(t_{p,r}) f_0^\sigma(w_{2n}) = \theta^\sigma(t_{p,r}) f_0^\sigma(w_{2n}) = \alpha_\sigma(U_{p,r}) f_0^\sigma(w_{2n}),
\]

where the last equality is the equation for \( \alpha_\sigma(U_{p,r}) \) in Proposition 2.5.

### 7.2. Local Shalika models `a la Ash–Ginzburg

We’ve written down explicit eigenvectors in \( \pi \cong \text{Ind}^G_B \theta^\sigma \). However, we really want to study the corresponding eigenvectors in the Shalika model of \( \pi \). To describe these precisely, we recall results of Ash–Ginzburg, who in [AG94] defined and studied an explicit intertwining of \( \text{Ind}^G_B \theta \) into its Shalika model. This intertwining is the major reason we normalise \( \theta \) as in Ash–Ginzburg (see §6.1).
For $f \in \text{Ind}^G_{B} \theta$ and $g \in \text{GL}_{2n}(\mathbb{Q}_p)$, as in [AG94, (1.3)] we let

$$S_\phi^0(f)(g) := \int_{\text{GL}_{n}(\mathbb{Z}_p)} \int_{M_n(\mathbb{Q}_p)} f \left[ (1 \ 1) \ (1 \ X) \ (k \ k) \ g \right] \psi^{-1}(\text{tr}(X)) \eta^{-1}(\text{det}(k)) dX dk.$$  \hspace{1cm} (7.4)

The map $S_\phi^0$ converges absolutely only for certain characters $\theta$ (described before [AG94, Lem. 1.4]). For $\theta$ such that it converges, $S_\phi^0$ is $\text{GL}_{2n}(\mathbb{Q}_p)$-equivariant and (by [BFG92]) we have $S_\phi^0(f) \in S_\phi^0(\pi)$. For $\sigma \in W_G$, let

$$P(\sigma) := \prod_{r=1}^{n} \prod_{s=r+1}^{2n} (\theta_{\sigma|}(p) - \theta_{\sigma|(s)}(p)) \in \mathbb{C}. \hspace{1cm} (7.5)$$

Across Lemmas 1.4–1.6 of [AG94], Ash–Ginzburg prove:

**Proposition 7.4.** The function $P(1) \cdot S_\phi^0(f)$ converges absolutely for all characters $\theta$. Thus when $P(1) \neq 0$, the map $S_\phi^0$ defines a non-trivial intertwining $S_\phi^0 : \text{Ind}^G_{B} \theta \rightarrow S_\phi^0(\pi)$.

Recall $\tilde{\pi}_\sigma = \Delta_{\phi}^{-1}(\sigma)$. By Lemma 6.12, $\tilde{\pi}_\sigma$ is a spin $p$-refinement. For later use, we record:

**Lemma 7.5.** If the Satake parameter of $\pi_p$ is regular semisimple, then $S_\phi^0$ defines a non-trivial intertwining $S_\phi^0 : \text{Ind}^G_{B} \theta \rightarrow S_\phi^0(\pi)$.

**Proof.** Since the Satake parameter is regular semisimple, all the $\theta_i(p)$ are pairwise distinct, so $\theta_{r}(p) - \theta_{s}(p) \neq 0$ for all $r \neq s$. We see $P(1) \neq 0$, and conclude by Proposition 7.4. \qed

### 7.3 Spin $p$-refinements under intertwining maps

For $\theta$ unramified, we have shown:

- for any $\sigma \in W_G$, an Iwahori-invariant function in $\pi_\sigma = \text{Ind}^G_{B} \theta^\sigma$ supported only on the big Bruhat cell is always a Hecke eigenvector in the generalised eigenspace for $\tilde{\pi}_\sigma$.
- for $\sigma \in W^0_G$, when $\tilde{\pi}_\sigma$ is regular we have an explicit intertwining $\text{Ind}^G_{B} \theta^\sigma \rightarrow S_\phi^0(\pi)$.
- by Lemma 6.12, $\tilde{\pi}_\sigma$ is a spin $p$-refinement if and only if $\sigma \in W^0_G$.

For $n > 1$, the last two cases are disjoint: there is a gap between normalisations where we can write down explicit eigenvectors in the principal series, and normalisations where we have explicit intertwinings into the Shalika model. We now bridge this gap: for $\sigma \in W^0_G$, we transfer the explicit eigenvector from $\text{Ind}^G_{B} \theta^\sigma$ into $\text{Ind}^G_{B} \theta$, where we can then compute its local zeta integral via the Ash–Ginzburg intertwining.

Given $\sigma = \left( \begin{smallmatrix} 1 & \rho \end{smallmatrix} \right) \in \text{W}_{2n}$ with $\rho \in \text{W}_n$, there is an isomorphism

$$M_\rho : \text{Ind}^G_{B} \theta^\sigma \rightarrow \text{Ind}^G_{B} \theta,$$

which is unique up-to-scaler by Schur’s Lemma. We are most interested in

$$M_{w_\nu} : \text{Ind}^G_{B} \theta^\sigma \rightarrow \text{Ind}^G_{B} \theta.$$

**Definition 7.6.** For each $w, \nu \in W_G$:

- Let $f^\nu_w \in \text{Ind}^G_{B} \theta^\nu$ be the unique $\text{Iw}_G$-invariant function supported on $B(\mathbb{Q}_p)\text{Iw}_G$ such that $f^\nu_w(w) = p^\nu(n-1)$.
- For $\rho \in \text{W}_n$ (and $\sigma$ as above), let

$$F^\nu_{\rho, w} := f^\nu_{\left( \rho w_\nu \right)^{-1}} = f^\nu_{\sigma w_\nu}.$$
Lemma 7.7. After possibly renormalising $M_\rho$, we have
\[ M_\rho(f^\nu) = F_\rho + \sum_{\ell(\delta) < \ell(\rho)} c_\delta F_\delta, \quad \text{with } c_\delta \in \mathbb{C}. \] (7.6)

Proof. For any simple reflection $s \in W_n$ and $\nu \in \mathcal{W}_n$, we have an intertwining isomorphism
\[ M_\nu^s : \text{Ind}^G_B \theta^\nu \rightarrow \text{Ind}^G_B \theta^s\nu. \]

By [Cas80, Thm. 3.4] (see [DJ] for more details) this can be normalised so that for any $\delta \in \mathcal{W}_n$, there is a constant $c_{\delta,s} \in \mathbb{C}$ (depending also on $\theta$) such that
\[ M_\nu^s(F^{\delta,s}) = \begin{cases} F^{s\nu} + c_{\delta,s} F^{s\nu}_{\delta}, & \text{if } \ell(s) = \ell(\delta) + 1, \\ p^{-1} F^{s\nu}_{\delta} + c_{\delta,s} F^{s\nu}_{\delta}, & \text{if } \ell(s) = \ell(\delta) - 1. \end{cases} \]

Writing $\rho = s_1 \cdots s_r$, $M_\rho$ is the composition $M_{s_{r-1}} \cdots s_1 \circ \cdots \circ M_{s_1}$. The lemma is then obtained by induction on $\ell(\rho)$, using the basic properties of the Bruhat length.

Now finally we map into the Shalika model. For $\delta \in \mathcal{W}_n$, let
\[ W_\delta := S^\nu_\delta(F_\delta) \in S^\nu_\delta(\pi), \]
for $S^\nu_\delta$ the intertwining of (7.4). This is well-defined, since $F_\delta \in \text{Ind}^G_B \theta$ and we normalised $\theta$ as in Ash–Ginzburg.

Corollary 7.8. Let $\tilde{\pi}$ be a regular spin $p$-refinement. Up to possibly renormalising $\theta$ by $\tau^{-1} \mathcal{W}^0_{\mathcal{G},\tau}$ (so that it still satisfies the Ash–Ginzburg condition), $S^\nu_\delta(\tilde{\pi})$ is spanned by an eigenvector of the form
\[ W_\delta = W_{w_n} + \sum_{w_n \neq \delta \in \mathcal{W}_n} c_\delta W_\delta \in S^\nu_\delta(\text{Ind}^G_B \theta). \] (7.7)

Proof. Let $\sigma = \Delta_\rho(\tilde{\pi})$; then by Lemma 6.12 we have $\sigma = \sigma' \tau \in \mathcal{W}^0_{\mathcal{G},\tau}$. After renormalising $\theta$ by $\tau^{-1}(\sigma')^{-1} \tau$, which preserves the Ash–Ginzburg normalisation of §6.1, we may without loss of generality assume $\sigma' = 1$, so $\sigma = \tau$.

By Proposition 7.2, we know $f^\nu_0 \in \text{Ind}^G_B \theta^\tau$ is an eigenvector in $\tilde{\pi}_\tau$. By Lemma 7.7, its image in $\text{Ind}^G_B$ under the intertwining $\tau$ has the form
\[ F_{w_n} + \sum_{w_n \neq \delta \in \mathcal{W}_n} c_\delta F_\delta \in \text{Ind}^G_B \theta. \]

By definition the image of this under $S^\nu_\delta$ is (7.7), which hence gives a non-zero eigenvector in $S^\nu_\delta(\tilde{\pi})$. By regularity this space is a line, so this eigenvector spans.

7.4. The local zeta integral at Iwahori level, II

From Corollary 7.8 we’ve written down an explicit eigenvector $W_0 \in S^\nu_\delta(\pi)$ in any spin $p$-refinement. We now show $W_0 \left( w_n z^n_{-1} \right) \neq 0$, recalling $z = \text{diag}(p^{n-1}, p^{n-2}, \ldots, 1)$, and deduce that all spin $p$-refinements are Shalika $p$-refinements. Moreover, we compute the value exactly, and hence – via Proposition 5.2 – complete the computation of the local zeta integral for $W_0$.

We first show that $W_0 \left( w_n z^n_{-1} \right) = 0$ unless $\delta = w_n$ (which means, by Proposition 5.2, that the local zeta integral vanishes for $W_\delta$ unless $\delta = w_n$). To do so, we examine when the integrand of (7.4) lies in the support of $F_\delta \in \text{Ind}^G_B(\theta)$. 

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Proposition 7.9. Let $\delta \in W_n$, $X \in M_n(Q_p)$ and $k \in \text{GL}_n(Z_p)$, and let $\beta \geq 1$. Then
\[
\begin{pmatrix}
1 & 1 \\
1 & X \\
k & k \\
w_n z^{2\beta} & 1
\end{pmatrix} \in B_n(Q_p) \begin{pmatrix}
\delta w_n \\
w_n
\end{pmatrix} \mathbb{I}_G
\] (7.8)
if and only if:
\begin{itemize}
  \item $\delta = w_n$ is the longest Weyl element,
  \item $k \in B_n(Z_p) w_n I_n$ and
  \item $k^{-1} X \in w_n z^{2\beta} M_n(Z_p)$.
\end{itemize}

Proof. Suppose (7.8) holds, and write
\[
\begin{pmatrix}
1 & 1 \\
1 & X \\
k & k \\
w_n z^{2\beta} & 1
\end{pmatrix} = \begin{pmatrix} A & B \\ D & D \end{pmatrix} \begin{pmatrix} \delta w_n \\
w_n \\
1 \\
\end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\] (7.9)

where 
\[A, D \in B_n(Q_p), \ B \in M_n(Q_p), \ a, d \in I_n, \ c \in p M_n(Z_p), \ b \in M_n(Z_p).\]

Expanding this out, we get the equality of matrices:
\[
\begin{pmatrix} k \end{pmatrix} = \begin{pmatrix} B \delta w_n a + A w_n c \\ D \delta w_n b + A w_n d \end{pmatrix}.
\]

This implies the following:

1. $B = -A w_n c a^{-1} w_n^{-1} \delta^{-1}$ (from the top left entry).
2. $-A w_n (c a^{-1} b - d) = k (1)$ and top right), whence $A \in B_n(Z_p)$ and 
\[k \in B_n(Z_p) \cdot w_n \cdot I_n.\]
3. $D \delta w_n a = k w_n z^{2\beta}$ (bottom left) which implies 
\[k^{-1} D = w_n z^{2\beta} a^{-1} w_n \delta^{-1} \in w_n z^{2\beta} \text{GL}_n(Z_p).
\]
4. $D \delta w_n b k^{-1} = X$ (bottom right), which by (3) implies 
\[k^{-1} X = k^{-1} D \cdot \delta w_n b k^{-1} \in w_n z^{2\beta} M_n(Z_p).
\]

We treat the cases $\delta \neq w_n$ and $\delta = w_n$ separately.

Case 1: $\delta \neq w_n$. Suppose there exist $X$ and $k$ such that (7.8) holds. We will derive a contradiction.

For $r \in \{1, \ldots, 2n - 1\}$, let $P_r$ be the parabolic with Levi $\text{GL}_r \times \text{GL}_{n-r}$, with associated (opposite) parahoric subgroup $\mathcal{J}_r \subset \text{GL}_{2n}(Z_p)$.

Claim 7.10. If $\delta \neq w_n$, there exists $r \in \{1, \ldots, 2n - 1\}$ such that 
\[B_n(Q_p) \cdot \delta w_n \cdot \mathcal{J}_r \cap B_n(Q_p) \cdot \mathcal{J}_r = \varnothing.\]

Proof of claim: Let $r := \min(i : \delta w_n(i) \neq i)$, which exists since $\delta w_n \neq 1$. Let $W_{r,n-r}$ be the Weyl group of $\text{GL}_r \times \text{GL}_{n-r}$: it is the subgroup of $W_n$ that preserves both $\{1, \ldots, r\}$ and $\{r+1, \ldots, n\}$. In particular, $\delta w_n \not\in W_{r,n-r}$.
We have the opposite Bruhat decomposition

\[ G(\mathbb{Q}_p) = \bigsqcup_{\sigma \in W_t / W_{t,n-r}} B(\mathbb{Q}_p) \sigma \mathcal{J}_r. \]

Since \( \delta w_n \notin W_{t,n-r} \), the cells \( B(\mathbb{Q}_p) \delta w_n \mathcal{J}_r \) and \( B(\mathbb{Q}_p) \mathcal{J}_r \) are disjoint, giving the claim.

For \( r \) as in the claim, let \( \mu \) be such that \( z^{2\beta} = t_{p,r} \cdot \mu \), recalling \( t_{p,r} = \text{diag}(p, \ldots, p, 1, \ldots, 1) \) with \( r \) lots of \( p \). We see that

\[ t_{p,r} Iw_n t_{p,r}^{-1} \subset \mathcal{J}_r. \]

Note that the valuation of \( \mu \) under any positive root is non-negative, so \( \mu^{-1} \mathcal{N}_n(\mathbb{Z}_p) \mu \subset \mathcal{N}_n(\mathbb{Z}_p) \), where \( \mathcal{N}_n \) is the lower triangular unipotent.

We now analyse (3) from the list above. Multiply both sides by \( t_r^{-1} \) to get

\[ D \delta w_n a t_r^{-1} = k w_n \mu. \]

We know from (2) that we can write \( k = \alpha \cdot w_n \cdot \beta \), with \( \alpha \in B_n(\mathbb{Z}_p) \) and \( \beta \in Iw_n \). We have

\[ k w_n \mu = \alpha w_n \beta w_n \mu \in B_n(\mathbb{Z}_p) \cdot \mathcal{W}_n \cdot \mu. \]

But by the Iwahori decomposition, we see that

\[ B_n(\mathbb{Z}_p) \cdot \mathcal{W}_n \cdot \mu \subset B_n(\mathbb{Q}_p) \cdot \mathcal{W}_n \subset B_n(\mathbb{Q}_p) \cdot \mathcal{J}_r. \]

We also have

\[ k w_n \mu = D \delta w_n a t_r^{-1} \in B_n(\mathbb{Q}_p) \cdot \delta w_n t_r^{-1} \cdot \mathcal{J}_r \subset B_n(\mathbb{Q}_p) \cdot \mathcal{J}_r, \]

as the element \( \delta w_n \) normalises the torus.

We must therefore have

\[ k w_n \mu \in (B_n(\mathbb{Q}_p) \cdot \mathcal{J}_r) \cap (B_n(\mathbb{Q}_p) \cdot \mathcal{J}_r) \neq \emptyset, \]

a contradiction. In particular, there do not exist \( X \) and \( k \) such that (7.8) holds if \( \delta \neq w_n \).

**Case 2:** \( \delta = w_n \). We have shown that if (7.8) holds, then

\[ k \in B(\mathbb{Z}_p) w_n Iw_n \quad \text{and} \quad k^{-1} X \in w_n z^{2\beta} M_n(\mathbb{Z}_p). \]  

\[ (7.10) \]

Conversely, suppose (7.10), and write \( k = A w_n d \), with \( A \in B_n(\mathbb{Z}_p) \) and \( d \in Iw_n \). Via the Iwahori decomposition, we may assume \( d \in N_n(\mathbb{Z}_p) \) is upper unipotent.

For \( A \) and \( d \) as above, set \( B = 0 \) and \( D = A z^{2\beta} \in B_n(\mathbb{Q}_p) \). Also set \( c = 0 \) and \( a = z^{-2\beta} w_n \delta w_n z^{2\beta} \in Iw_n \) (since \( \beta \geq 1 \)). If we set \( b = D^{-1} X k \in M_n(\mathbb{Q}_p) \), then (7.9) holds. Clearly \( \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \) \( \in B_n(\mathbb{Q}_p) \), so we are done if we can show \( \left( \begin{smallmatrix} a & b \\ z & 0 \end{smallmatrix} \right) \in Iw_G \). This will hold if \( b \in M_n(\mathbb{Z}_p) \). Observe

\[ b = D^{-1} X k = z^{-2\beta} A^{-1} X k = z^{-2\beta} w_n d k^{-1} X k \]

\[ \in z^{-2\beta} w_n d \cdot w_n z^{2\beta} M_n(\mathbb{Z}_p) k = a M_n(\mathbb{Z}_p) k \subset M_n(\mathbb{Z}_p), \]

where in the second step we substitute \( D \), the third we substitute \( A^{-1} = w_n dk \), in the third we use (7.10), and in the fourth we substitute \( a = z^{-2\beta} w_n d w_n z^{2\beta} \). Thus \( \left( \begin{smallmatrix} a & b \\ z & 0 \end{smallmatrix} \right) \in Iw_G \), completing the proof. \( \square \)

From the proof, we also see the following:
Corollary 7.11. Let \( \Theta \) be any unramified character of \( T(\mathbb{Q}_p) \), extended trivially to \( B(\mathbb{Q}_p) \). If we have

\[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
X & k \\
k & k
\end{pmatrix}
\begin{pmatrix}
w_n z^{2\beta} & 1 \\
1 & 1
\end{pmatrix}
= \mathcal{B}
\begin{pmatrix}
w_n & 1 \\
1 & 1
\end{pmatrix}
\mathcal{I} \in B_n(\mathbb{Q}_p)
\begin{pmatrix}
w_n & 1 \\
1 & 1
\end{pmatrix}
\text{Iw}_G,
\]  

(7.11)

then

\[
\Theta(\mathcal{B}) = \Theta \begin{pmatrix} 1 \\ z^{2\beta} \end{pmatrix}.
\]

Proof. The proposition gave necessary and sufficient conditions for (7.11). When they hold, we wrote down explicit values of \( \mathcal{B} \) and \( \mathcal{I} \). By definition \( A \in B_n(\mathbb{Z}_p) \) and \( D = A z^{2\beta} \). The result follows easily. \( \Box \)

We now put this all together. Recall that (without loss of generality) we have normalised \( \pi \cong \text{Ind}_{D}^{\text{GL}_n} \theta \), fixing an identification \( \Delta_\theta : \{p\text{-refinements}\} \rightarrow \mathcal{W}_G \) so that for our fixed spin \( p \)-refinement \( \tilde{\pi} \), we have \( \Delta_\theta(\tilde{\pi}) = \tau = \left( \begin{smallmatrix} 1 & w_n \\ \end{smallmatrix} \right) \). By Proposition 2.5, the Hecke eigenvalue of \( U_{p,r} \) on \( \tilde{\pi} \) is

\[
\alpha_{p,r} = \left[ \mathcal{B}^{1/2} \theta \right]^{w_{n}}_{p}(l_{p,r}).
\]

Note also that since \( \tilde{\pi} \) is a spin \( p \)-refinement, by Definition 6.2 we have \( \alpha_{r,n+s} = \eta(p)^{s} \alpha_{n,n-s} \) for \( 0 \leq s \leq n-1 \). We also have the \( U_p \)-eigenvalue \( \alpha_p = \alpha_{p,1} \cdots \alpha_{p,2n-1} \).

Proposition 7.12.

(i) If \( \delta \neq w_n \), then \( W_\delta \left( \begin{pmatrix} w_n z^{2\beta} & 1 \\ 1 & 1 \end{pmatrix} \right) = 0 \).

(ii) We have

\[
W_\delta \left( \begin{pmatrix} w_n z^{2\beta} & 1 \\ 1 & 1 \end{pmatrix} \right) = W_{w_n} \left( \begin{pmatrix} w_n z^{2\beta} & 1 \\ 1 & 1 \end{pmatrix} \right)
\]

\[
= \mathcal{T}'' \cdot p^{n-2} \frac{\delta_{p}(l_{p})^{\beta}}{\eta(\text{det} z^{2\beta})} \left( \frac{\alpha_p}{\alpha_{p,n}} \right)^{\beta} \neq 0,
\]

where \( \mathcal{T}'' = \text{vol} \left[ B_n(\mathbb{Z}_p) \cdot w_n \cdot \text{Iw}_n \right] \cdot p^{n(n-1)} \) is a constant independent of \( \beta \).

Proof. In (7.4), we gave an integral representation of

\[
W_\delta \left( \begin{pmatrix} w_n z^{2\beta} & 1 \\ 1 & 1 \end{pmatrix} \right) = S_\psi^\delta(F_\delta) \left( \begin{pmatrix} z^{2\beta} & 1 \\ 1 & 1 \end{pmatrix} \right)
\]

\[
= \int_{\text{GL}_n(\mathbb{Z}_p)} \int_{M_n(\mathbb{Q}_p)} F_{w_n} \left[ \left( \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} X \end{pmatrix} \begin{pmatrix} k \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \right) \psi^{-1}(\text{tr}(X)) \eta^{-1}(\text{det}(k)) dX dk.
\]

(i) If \( \delta \neq w_n \), then Proposition 7.9 shows the domain of \( F_\delta \) in the integral is disjoint from the support of \( F_\delta \), hence the integrand (thus the integral) vanishes.

(ii) The first equality in (ii) follows from (7.7) and (i). Now, if \( \delta = w_n \), then Proposition 7.9 means we can restrict the domain of the integral to \( k \in B_n(\mathbb{Z}_p) w_n \text{Iw}_n \) and \( X \in k w_n z^{2\beta} M_n(\mathbb{Z}_p) \).

Moreover:

- If \( k \in B_n(\mathbb{Q}_p) w_n \text{Iw}_n \) and \( X \in k w_n z^{2\beta} M_n(\mathbb{Z}_p) \), then

\[
F_{w_n} \left[ \left( \begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} X \end{pmatrix} \begin{pmatrix} k \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \right) \right] = \left( \delta_{B}^{1/2} \theta \right) \left[ \begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} z^{2\beta} \end{pmatrix} \right] F_{w_n} \left( \begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} w_n \end{pmatrix} \begin{pmatrix} k \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \right)
\]

using Corollary 7.11, Iwahori-invariance of \( F_{w_n} \), and Definition 7.6.

- Since \( X \in M_n(\mathbb{Z}_p) \), we have \( \psi(\text{tr}(X)) = 1 \).
• \( \eta(\det(k)) = 1 \) as \( \eta \) is unramified.

In particular, the integral collapses to

\[
W_{w_n} \left( w_n z^\beta \right) = (\delta_B^{-1}) \left( \left( \begin{array}{c} 1 \\ z^\beta \end{array} \right) \right) p^{\beta(n-3)} \cdot \frac{1}{B_{n}(\mathbb{Z}_p)w_n \cdot \text{Vol} \left( B_{n}(\mathbb{Z}_p) \cdot w_n \cdot \text{Vol} \left( z^\beta M_n(\mathbb{Z}_p) \right) \right)} dX dk,
\]

where in the last step we made the change of variables \( X \mapsto w_n k^{-1} X k w_n \) before integrating. We easily see that

\[
\text{Vol} \left( z^\beta M_n(\mathbb{Z}_p) \right) = p^{\beta(n^2-3^3)} = p^{\beta n^2} \cdot \delta_B \left( \left( \begin{array}{c} p^\beta \\ 1 \end{array} \right) \right).
\]

Now, note that \( w_{2n} \tau \left( \left( \begin{array}{c} 1 \\ z \end{array} \right) \right) = \left( \begin{array}{c} 1 \\ z \end{array} \right) (t_{p,1} \cdots t_{p,n-1}) \) (from (2.9)). In particular,

\[
(\delta_B^{-1}) \left( \left( \begin{array}{c} 1 \\ z^\beta \end{array} \right) \right) = \delta_B^{-1} \left( \left( \begin{array}{c} 1 \\ z^\beta \end{array} \right) \right) \cdot \delta_B \left( \left( \begin{array}{c} 2^\beta \\ t_{p,1} \cdots t_{p,n-1} \end{array} \right) \right) \cdot \frac{1}{B_{n}(\mathbb{Z}_p)w_n \cdot \text{Vol} \left( z^\beta M_n(\mathbb{Z}_p) \right)} dX dk,
\]

Here in the second equality we use that \( B_{2n}^{-1} = \delta_B^{-1} \), and in the third that \( \eta(\det z^\beta) - \eta(\det(z) \alpha_{p,1} \cdots \alpha_{p,n-1}) = \eta(\det z) \alpha_{p,1} \cdots \alpha_{p,n-1} \). Finally, we obtain (ii) after combining (7.12), (7.13) and (7.14), as

\[
\delta_B(t_p)^\beta = \delta_B \left( \left( \begin{array}{c} p^\beta \\ 1 \end{array} \right) \right) \cdot \delta_B \left( \left( \begin{array}{c} z^\beta \\ z^\beta \end{array} \right) \right).
\]

\( \square \)

**Corollary 7.13.** Any regular spin \( p \)-refinement is a Shalika-refinement.

**Proof.** In Proposition 7.12, we started with an arbitrary regular spin \( p \)-refinement \( \tilde{\pi} \), and exhibited in (ii) an eigenvector \( W_0 \in S^0_\psi(\tilde{\pi}) \) with \( W_0 \left( \left( \begin{array}{c} w z^\beta \\ 1 \end{array} \right) \right) = 0 \). By definition, \( \tilde{\pi} \) is thus a Shalika refinement.

\( \square \)

### 8. Interlude: an automorphic summary

Everything so far has been classical/automorphic in nature. The rest of the paper is concerned with a \( p \)-adic interpolation of the previous sections. For the benefit of the reader, we now collect our running automorphic assumptions in one place, and summarise our main classical results.

Throughout the rest of the paper, we work with base field \( \mathbb{Q} \), fixes \( K = \text{Iw}_G \prod_{\ell \neq p} \text{GL}_{2n}(\mathbb{Z}_\ell) \), and let \( \pi \) be a \( \text{RACAR} \) of \( \text{GL}_{2n}(\mathbb{A}) \) of weight \( \lambda \) such that:

**Conditions 8.1.**

(C1) \( \pi \) admits a global \( (\eta, \psi) \)-Shalika model, for a Hecke character \( \eta \);
(C2) \( \pi_p \) is spherical and admits a regular Shalika \( p \)-refinement \( \tilde{\pi}_p = (\pi_p, \alpha) \);
(C3) for each \( \ell \neq p \), \( \pi_\ell \) is spherical;
(C4) for each \( r = 1, \ldots, 2n - 1 \), letting \( \alpha_{p,r} = \alpha(U_{p,r}) \) and \( \alpha_{p,r}^2 = \lambda(t_{p,r}) \alpha_{p,r} \), we have

\[
v_p(\alpha_{p,r}^2) = \left[ v_p(\alpha_{p,r}) - \sum_{j=1}^{r} \lambda_{2n+1-j} \right] < \lambda_r - \lambda_{r+1} + 1.
\]
In this case we write $\tilde{\pi} = (\pi, \alpha)$ and call it a $p$-refined RACAR satisfying (C1-4).

(C2) and (C3) imply that $\eta$ is everywhere unramified, so $\eta = | \cdot |^w$, where $w$ is the purity weight of $\lambda$. (C4) ensures $\tilde{\pi}$ is a non-critical slope $p$-refinement, which we will explain in §10.4.

If $\tilde{\pi}$ satisfies (C1-4), choose

$$W_f = \otimes \pi_f \in S^\Omega_\phi(\pi_f, E)$$

as follows: for each $\ell \neq p$, let $W_{f,\ell} = W_{f,\ell}^\alpha$ be the spherical test vector, and at $p$, let $W_p$ be the vector $W_0 \in S^\Omega_{\phi_p}(\pi_p, E)$ from Corollary 7.8. Define

$$\phi_\pm = \Theta^\pm(W_f) / \Omega_\phi^\pm \in H^\pm_\phi(S_K, \mathcal{Y}_\chi^\gamma(L))^\pm,$$

where $\Theta^\pm/\Omega^\pm_\phi$ is defined in (3.3), and $(-)_\pm$ is the localisation at $\mathfrak{m}_\pm$ (see §2.4.3). Now recall:

- In Theorem 4.16, we showed that for $\chi$ of conductor $p^\beta$ and $\beta = \max(\beta', 1)$,

$$E_{B,\chi}^{\gamma, \nu}(\phi_\pm) = \delta_B(t_p^\beta) \cdot \frac{L(\pi \otimes \chi, j + 1/2)}{\Omega_\phi} \cdot \zeta_p(j + \frac{1}{2}, (u^{-1}i_B^\beta) \cdot W_p, \chi_p).$$

- In Proposition 5.2, we showed that if $\chi_p$ is ramified,

$$\zeta_p\left(j + \frac{1}{2}, (u^{-1}i_B^\beta) \cdot W_p, \chi_p\right) = \mathcal{Y}' \cdot \eta(\det z^\beta) \cdot p^{-\frac{n^2 + n}{2}} \cdot p^{\beta n_j} \cdot \tau(\chi)^n \cdot (\det w_\alpha) \cdot W_p\left(w_n z^\beta \cdot 1\right).$$

- In Proposition 7.12, we showed

$$W_p\left(w_n z^\beta \cdot 1\right) = \mathcal{Y}' \cdot p^{\beta n_j} \cdot \frac{\delta_B(t_p^\beta)}{\eta(\det z^\beta)} \cdot \left(\frac{\alpha_p}{\alpha_{p,n}}\right)^\beta.$$ 

Combining all of these, and letting $\gamma(pm) := \mathcal{Y} \cdot \mathcal{Y}' \cdot \mathcal{Y}$, we deduce:

**Corollary 8.2.** Let $\chi$ be a Dirichlet character of conductor $p^\beta$, with $\beta \in \mathbb{Z}_{\geq 1}$, and let $j \in \text{Crit}(\lambda)$. If $(-)^j \chi \eta(-1) \neq \pm 1$, then $E_{B,\chi}^{\gamma, \nu}(\phi_\pm) = 0$. If $(-)^j \chi \eta(-1) = \pm 1$, we have

$$E_{B,\chi}^{\gamma, \nu}(\phi_\pm) = \gamma(pm) \cdot \chi(\det w_\alpha) \cdot \left(\frac{\alpha_p}{\alpha_{p,n}}\right)^\beta \cdot Q'(\tau, \chi, j) \cdot \zeta_p(W_{\infty}^{\pm}) \cdot \frac{L(\pi \otimes \chi, j + 1/2)}{\Omega_\phi^\pm},$$

where

$$Q'(\tau, \chi, j) = p^{\nu_j(n_j + n/2)} \tau(\chi)^n.$$

**Proof.** We use $\alpha_p = \lambda(t_p) \alpha_p$. The rest is book-keeping.

Finally, we record one more relevant result:

**Proposition 8.3.** The $L$-vector space $H^\pm_\phi(S_K, \mathcal{Y}_\chi^\gamma(L))^\pm$ is 1-dimensional, generated by $\phi_\pm$.

**Proof.** The generalised eigenspace in $\pi_{\mathcal{H}}^K$ where $\mathcal{H}$ acts by $\psi_\pi$ is 1-dimensional; locally, at $\ell \neq p$ we have $\pi_{\mathcal{H}}^{\gamma}(\mathbb{A})$ is a line as $\pi_{\mathcal{H}}$ is spherical, and at $p$, this is (C2). This line is generated by $W_f$ by construction. The result now follows from Hecke-equivariance of $\Theta^\pm$. 


9. The local zeta integral at parahoric level

The local zeta integral we computed in Part II required the twisting character \( \chi \) to be ramified. This is similar to previous papers [DJR20, BSDW] on this topic, where \( p \)-adic \( L \)-functions were only shown to have the required interpolation property at ramified characters. However, this excludes the trivial character, which is typically the most interesting one. We finish Part II by computing the local zeta integral again in a different way which allows us to also handle unramified characters. Doing this at Iwahoric level appears to be very difficult. Instead, for this section only, we work at \( Q \)-parahoric level, for \( Q \subset GL_{2n} \), the parabolic with Levi \( GL_n \times GL_n \) (the setting treated in [DJR20, BSDW]). This allows us to directly strengthen the results of [DJR20, BSDW] (see Proposition 14.3 below). In §12.4 we’ll exploit the interpolation properties of \( p \)-adic \( L \)-functions to deduce the result at Iwahoric level, completing the present paper.

For compatibility with [DJR20, BSDW], we work over a general local field rather than just over \( \mathbb{Q}_p \). In particular, for this section only we adopt the following notation.

**Notation 9.1.** Let \( F/\mathbb{Q}_p \) be a finite extension, with ring of integers \( \mathcal{O} \) and maximal ideal \( p = \# \mathcal{O}/p \). Let \( \varpi^\mathcal{O} \) be the different of \( F/\mathbb{Q}_p \).

Let \( \pi \) be a spherical representation of \( GL_{2n}(F) \) admitting an \( (\eta, \psi) \)-Shalika model, for \( \eta : F^\times \to \mathbb{C}^\times \) a smooth character and \( \psi : F \to \mathbb{C}^\times \) the usual additive character (e.g. [DJR20, §4.1]). We will assume that (7.4) gives a non-trivial intertwining \( S^\eta_\psi \) for an unramified character \( \theta \) in the Ash–Ginzburg normalisation (cf. Lemma 7.5).

Let \( \tilde{\pi} = (\pi, \alpha) \) be a regular spin \( p \)-refinement of \( \pi \) (to Iwahori level), normalised without loss of generality as in the proof of Corollary 7.8. At parahoric level:

- Rather than \( Iw_G \)-invariant vectors we use \( J \)-invariant vectors.
- Attached to \( \tilde{\pi} \) is a parahoric refinement \( \tilde{\pi}Q = (\pi, \alpha(U_{p,n})) \), which is a \( Q \)-regular \( Q \)-refinement in the sense of [BSDW, Def. 3.5]. (In the notation of [DJR20], it corresponds to the set \( \tau = \{ n+1, \ldots, 2n \} \).
- For a vector \( W_p \in \pi_p^J \), the relevant twisted local zeta integral arising in Theorem 4.16 is \( \zeta_p(j + \frac{1}{2}, (u^{-1}t_p^\delta) \cdot W_p, \chi_p) \).

**Definition 9.2.** Let \( \mathcal{F}_0 \in Ind_{H}^{G} \theta \) be the unique \( J \)-invariant function supported on \( B(F) \cdot w_{2n} \cdot J \) such that \( \mathcal{F}_0(w_{2n}) = q^{\alpha(n-1)}. \) By [DJR20, Lem. 3.6], we have \( \mathcal{F}_0 \in \hat{\pi}^Q \) is a generator.

Let \( W_0 = S^\eta_\psi(\mathcal{F}_0) \in S^\eta_\psi(\pi) \) be the Shalika vector associated with \( \mathcal{F}_0 \) (7.4).

Let \( d^\psi c \) be the Haar measure on \( \mathcal{O}^\times \) of total measure 1. Let \( \chi : F^\times \to \mathbb{C}^\times \) be a finite order character of conductor \( p^{\beta} \), and denote its Gauss sum by

\[
\tau(\chi) = \tau(\chi, \psi) = q^{\beta}(1 - q^{-1}) \int_{\mathcal{O}^\times} \chi(c \varpi^{-\beta - \delta}) \psi(c \varpi^{-\beta - \delta}) d^\psi c.
\]

Our normalisation means that if \( \chi \) is a Dirichlet character, this recovers the usual Gauss sum

\[
\tau(\chi) = \sum_{c \in (\mathbb{Z}/p^{\beta})^\times} \chi(c) e^{2\pi ic/p^{\beta}}.
\]

In the rest of this section, we prove:

**Proposition 9.3.** Let \( \beta = \max(1, \beta') \). Then

\[
\zeta\left(s, (u^{-1}t_p^\beta) \cdot W_0, \chi\right) = q^{\beta\alpha(s+\frac{1}{2})+\delta n(s-1)} \prod_{i=n+1}^{2n} \theta_i(\varpi)^{-\delta} Q(\pi, \chi, s)
\]
where

\[
Q(\tau, \chi, s) = \begin{cases} 
q^{-\beta n} \frac{q^n}{(q^n)^{\frac{1}{2n}}} \cdot \tau(\chi)^n & : \chi \text{ ramified}, \\
\frac{\chi(\varpi)^{-n(\delta + 1)}}{(1-q)^n} \prod_{i=n+1}^{\infty} \frac{1-\theta_i(\varpi)q^{1-s} - \theta_i(\varpi)q^{s}}{1-\theta_i(\varpi)q^{s}} & : \chi \text{ unramified}.
\end{cases}
\]

**Proof.** We first rewrite the local zeta integrals. For simplicity, we write \(G_n = \text{GL}_n(F)\), \(B_n = \text{B}_n(F)\), and \(M_n = \text{M}_n(F)\). Since \(\eta\) is unramified, we have

\[
\zeta(s, (u^{-1}t_{p,n}) \cdot W, \chi) = \int_{G_n} \int_{K_n} \int_{M_n} F_0 \left[ \begin{array}{cc} 1 & 1 \\ h & X \\ k & 1 \\ 1 & 1 \end{array} \right] \chi(\det h) |\det h|^{-s+1/2} dh \\
\times \chi(\det h) |\det h|^{-s+1/2} \eta^{-1} (\det k) \psi( - \text{tr} X ) dX dk dh
\]

Since \(G_n = \text{B}_n K_n\) (with \(B_n \cap K_n\) of volume 1) we may rewrite the above integral using \(h = b\ell\) with \(b \in B_n\) and \(\ell \in K_n\) (so \(|\det h| = |\det b|\)):

\[
= q^{-\beta n(s-1/2)} \int_{B_n} \int_{K_n} \int_{K_n} \int_{M_n} F_0 \left[ \begin{array}{cc} 1 & 1 \\ b\ell & k \\ 1 & 1 \end{array} \right] \chi(\det b) |\det b|^{-s+1/2} \eta^{-1} (\det b) \psi( - \text{tr} b\ell Xk^{-1} + \omega^{-\beta} b\ell w_n k^{-1} ) dX d\ell dk db
\]

Using \(J\)-invariance of \(F_0\), we see that

\[
F_0 \left[ \begin{array}{cc} 1 & 1 \\ \ell & k \end{array} \right] = F_0 \left[ \begin{array}{cc} 1 & 1 \\ 1 & \ell Xk^{-1} \end{array} \right] \]

The proof of [DJR20, Lemma 3.6] implies that this vanishes unless \(X \in M_n(O)\), in which case it equals \(F_0 \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] = q^{(n-1)}\). The above integral becomes

\[
= q^{-\beta n(s-1/2)+n(n-1)} \int_{B_n} \int_{K_n} \int_{K_n} \delta_{\ell}^{1/2} \theta \left[ \begin{array}{cc} 1 \\ b \end{array} \right] \chi(\det k^{-1} b\ell \omega^{-\beta}) \\
\times |\det b|^{-s+1/2} \psi( - \text{tr} \omega^{-\beta} b\ell w_n k^{-1} ) \left( \int_{M_n(O)} \psi( - \text{tr} b\ell Xk^{-1} ) dX \right) d\ell dk db
\]

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Note that \( \int_{M_n(O)} \psi(tr AX) dX = 0 \) unless \( A \in \varpi^{-\delta} M_n(O) \), in which case the integral is \( \text{vol}(M_n(O)) = 1 \). We conclude that the above inside integral vanishes unless \( k^{-1}b \ell \in \varpi^{-\delta} M_n(O) \), i.e., if \( b \in B_n \cap \varpi^{-\delta} M_n(O) \). The integral becomes

\[
= q^\beta n(s-1/2)+n(n-1) \int_{B_n \cap \varpi^{-\delta} M_n(O)} \int_{K_n} \int_{K_n} \left( \begin{array}{c} 1 \\ b \end{array} \right) \delta_B^{1/2} \theta^{-\delta}(1, b) \times \chi \left( \det b \ell w^{-\delta} \right) |det b n+n+s-1/2 | \psi \left( - \text{tr} \left[ \varpi^{-\delta} b \ell w_n^{-1} \right] \right) dkd \ell db.
\]

Changing variables so that \( \ell w_n k^{-1} \) becomes \( \ell \), and integrating out \( k \), the integral becomes

\[
= q^\beta n(s-1/2)+n(n-1) \sum_{\rho \in W_n} \int_{B_n \cap \varpi^{-\delta} M_n(O)} \int_{N_n(O)} \int_{N_n^{-\delta}(\varpi)} \int_{B(O)} \delta_B^{1/2} \theta^{-\delta}(1, b) \times \chi \left( \det b b_1 \rho \ell w_n \right) |det b n+n+s-1/2 | \psi \left( - \text{tr} \left[ \varpi^{-\delta} b n \rho \ell w_n^{-1} \right] \right) db_1 d \ell db.
\]

Changing variables so that \( b_1 b n \) becomes \( b \) doesn’t change the Haar measure, as \( b_1 \in B(O) \). Integrating out \( n \) and \( b_1 \), the integral becomes

\[
= q^\beta n(s-1/2)+n(n-1) \sum_{\rho \in W_n} \chi(\det \rho w_n \varpi^{-\delta}) \int_{B_n \cap \varpi^{-\delta} M_n(O)} \int_{N_n^{-\delta}(\varpi)} \int_{B(O)} \delta_B^{1/2} \theta^{-\delta}(1, b) \times \chi \left( \det b \right) |det b n+n+s-1/2 | \psi \left( - \text{tr} \left[ \varpi^{-\delta} b \rho \ell w_n^{-1} \right] \right) d \ell db.
\]

Write

\[
\varpi = \left( \begin{array}{cccc} 1 & & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ x_{ij} & \ddots & \ddots & 1 \\ & \cdots & \cdots & \ddots \end{array} \right), \quad b = \varpi^{-\delta} \left( \begin{array}{cccc} t_1 & & & u_{ij} \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & \cdots & \cdots & t_n \end{array} \right),
\]

where \( t_i \in O - 0, u_{ij} \in O \) and \( x_{ij} \in \varpi O \). In this case, \( db = \prod |t_i|^{i-n-1} \prod dt_i \prod du_{ij} \) and \( d \ell = \prod dx_{ij} \).

Fix \( \rho \in W_n \). Writing \( \rho \varpi = (m_{ij}) \) we see that

\[
\text{tr} b \rho \ell w_n = \varpi^{-\delta} \left( \sum_{i=1}^{n} m_i t_i + \sum_{i<j} u_{ij} m_{ij} \right).
\]

If \( \rho \neq 1 \), there exist indices \( i < j \) such that \( m_{ji} = 1 \). Indeed, this is equivalent to there existing \( j = \tau(i) > i \) where \( \tau \) is the permutation of \( \rho \). In this case, from the inner integral we may factor

\[
\int_{\varpi} \psi(-\varpi^{-\delta} m_{ji} u_{ij}) du_{ij} = 0,
\]

as \( \beta \geq 1 \).
We conclude that all the terms of the sum vanish, except for the term $\rho = 1$. We see that the zeta integral becomes

$$
q^{\beta n(s-1/2)+n(n-1)}\chi(\det \omega_n) \prod_{i=1}^{n} \prod_{j} \psi \left[ -\omega^{-\delta} x_j \right] \prod dx_j dt_i du_{ij},
$$

where $x_{ij} \in \mathcal{O}$, $u_{ij} \in \mathcal{O}$, $t_i \in \mathcal{O} = 0$.

The integral $\int_{\mathcal{O}} \psi \left[ -\omega^{-\delta} x_j u_{ij} \right] du_{ij}$ vanishes unless $x_{ij} \in \omega^{\beta} \mathcal{O}$, in which case the integral is 1, so the zeta integral is

$$
q^{\beta n(s-1/2)+n(n-1)-\beta(z)}\chi(\det \omega_n) \prod_{i=1}^{n} \prod_{j} \psi \left[ -\omega^{-\delta} x_j \right] \prod dx_j dt_i \cdot \prod_i dt_i.
$$

When $\chi$ is ramified with conductor $\beta$ then

$$
\int_{\mathcal{O} = 0} \theta |^{-1} \chi(t\omega^{-\delta}) \psi(t\omega^{-\delta-1}) dt = \sum_{k=0}^{\infty} q^{-k(\beta-\delta)(s-1)} \theta(\omega)^{k-\delta} \chi(\omega)^{k-\delta-1} \int_{\mathcal{O} \times \mathcal{O}} \psi(t\omega^{k-\delta-1}) dt\times t
$$

Since the integral vanishes unless $k = 0$, whence it is the Gauss sum. If $\chi$ is unramified, then $\beta = 1$ and, dropping indices for simplicity,

$$
\int_{\mathcal{O} = 0} \theta |^{-1} \chi(t\omega^{-\delta}) \psi(t\omega^{-\delta-1}) dt = \sum_{k=0}^{\infty} \frac{q^{-k(\beta-\delta)(s-1)} \theta(\omega)^{k-\delta} \chi(\omega)^{k-\delta-1}}{1-q} \int_{\mathcal{O} \times \mathcal{O}} \psi(t\omega^{k-\delta-1}) dt\times t
$$

using that $\int_{\mathcal{O} \times \mathcal{O}} \psi(t\omega^{k-\delta-1}) dt\times t$ is 0 if $k < 0$, is 1 if $k > 0$, and is $1/(1-q)$ if $k = 0$.

Putting everything together we obtain the desired result.

**Part III. $p$-adic Interpolation**

10. Overconvergent cohomology

We recap the theory of overconvergent cohomology in $p$-adic families, as developed for example [Urb11, Han17]. All of this material is explained in depth *op. cit.*, so we are terse with details.
10.1. Weight spaces

Recall $X^*(T)$, $X^+_0(T)$, $X^*(H)$ and $X^+_0(H)$ from §2.2. If $K \subset G(A_f)$ is an open compact subgroup, let $\overline{Z}_K$ be the $p$-adic closure of $Z(G(Q)) \cap K$.

**Definition 10.1.** (i) The weight space for $G$ of level $K$ is $\mathcal{W}_K^G := \text{Spf}(\mathcal{W}_K^G)$.

This is a rigid analytic space whose $L$-points, for $L \subset C_p$ any extension of $Q_p$, are given by

$$\mathcal{W}_K^G(L) = \text{Hom}_{\text{cont}}(\text{Hom}(\mathcal{W}_K^G)/\mathcal{Z}_K, L^\times).$$

(ii) Any element of $\mathcal{W}_K^G(L) \cap X^+_0(T)$ is called an algebraic weight.

(iii) The pure weight space $\mathcal{W}_K^G(0)$ is the Zariski-closure of $X^+_0(T) \cap \mathcal{W}_K^G$ in $\mathcal{W}_K^G$.

We view all of these weights as characters of $T(Z_p)$ trivial on $\overline{Z}_K$. If $\lambda$ is pure of weight $w$ and $z \in \overline{Z}_K$, then $\lambda(z) \subset \{ \pm 1 \}$, so if $\lambda$ is trivial on $\overline{Z}_K$, then this is also true of all weights in a neighbourhood of $\lambda$ in the pure weight space. Since the level subgroup will always be fixed, we will henceforth always drop it from notation.

A weight $\lambda \in \mathcal{W}_K^G$ decomposes as $\lambda = (\lambda_1, \ldots, \lambda_{2n})$, where each $\lambda_i$ is a character of $Z_p^\times$. We see $\lambda \in \mathcal{W}_K^G$ if and only if there exists $w_\lambda \in \text{Hom}_{\text{cts}}(Z_p^\times, L^\times)$ such that $\lambda_1 \cdot \lambda_i = w_\lambda$ for all $1 \leq i \leq n$. The space $\mathcal{W}_K^G$ has dimension $n + 1$ (corresponding to changing $\lambda_1, \ldots, \lambda_n, w_\lambda$).

If $\Omega \subset \mathcal{W}_K^G$ is an affinoid in the pure weight space, then $\Omega$ is equipped with a character $\chi: T(Z_p) \to \mathcal{O}_\Omega^\times$ such that for any $\lambda \in \Omega(L)$, the composition $T(Z_p) \to \mathcal{O}_\Omega^\times \to L^\times$ is the character attached to $\lambda$, where sp$_\lambda$ is evaluation at $\lambda$. Moreover, write $\chi_\Omega = (\chi_{\Omega, 1}, \ldots, \chi_{\Omega, 2n})$, where each $\chi_{\Omega, i}$ is a character of $Z_p^\times$; then since $\Omega$ is pure, there exists a character $w_\Omega: Z_p^\times \to \mathcal{O}_\Omega^\times$ such that $w_\Omega(z) = \chi_{\Omega, i}(z) \cdot \chi_{\Omega, 2n+1-i}(z)$ for all $z \in Z_p^\times$.

10.2. Algebraic and analytic induction

For $\lambda \in X^+_0(T)$, recall $V_\lambda$ is the algebraic representation of $G$ of highest weight $\lambda$. The $L$-points of $V_\lambda$ can be described explicitly as the algebraic induction, whose points are algebraic functions $f: G(Q_p) \to L$ such that

$$f(n^{-1}g) = \lambda(t)f(g) \quad \forall n^{-1} \in \mathcal{N}(Q_p), t \in T(Q_p), g \in G(Q_p).$$

The action of $\gamma \in G(Q_p)$ on $f \in V_\lambda(L)$ is by $(\gamma \cdot f)(g) := f(\gamma g)$.

As $G(Z_p)$ is Zariski-dense in $G(Q_p)$, we can identify $V_\lambda(L)$ with the set of algebraic $f: G(Z_p) \to L$ satisfying (10.2). We have an integral subspace $V_\lambda(\mathcal{O}_L)$ of $f$ such that $f(G(Z_p)) \subset \mathcal{O}_L$, and we let $V_\lambda'(\mathcal{O}_L) = \text{Hom}_{\mathcal{O}_L}(V_\lambda(\mathcal{O}_L), \mathcal{O}_L)$.

If $I$ is a $p$-adic Lie group and $R$ a $\mathcal{Q}_p$-algebra, let $\mathcal{A}(I, R)$ denote the space of locally analytic functions $I \to R$. Let $\mathcal{I}_G \subset G(Z_p)$ be the Iwahori subgroup, and $\Omega \subset \mathcal{W}_K^G(L)$ an affinoid, with attached character $\chi_\Omega$; we allow $\Omega = \{ \lambda \}$, whence $\chi_\Omega = \lambda$. Recall the analytic induction spaces:

**Definition 10.2.** Let $A_\Omega$ be the space of $f \in \mathcal{A}(\mathcal{I}_G, \mathcal{O}_\Omega)$ such that

$$f(n^{-1}g) = \chi_\Omega(t)f(g) \quad \forall n^{-1} \in \mathcal{N}(\mathcal{Z}_p), t \in T(Z_p), g \in \mathcal{I}_G.$$

Via Iwahori decomposition and (10.3), restriction to $N(Z_p)$ identifies $A_\Omega(L)$ with $\mathcal{A}(N(Z_p), \mathcal{O}_\Omega)$.

As any $f \in V_\lambda(L)$ is determined by its restriction to the Zariski-dense subgroup $\mathcal{I}_G$, we see $V_\lambda(L)$ is the (finite Banach) subspace of $f \in A_\lambda$ that are algebraic on $\mathcal{I}_G$ (e.g. [Urb11, §3.2.8]).

**Definition 10.3.** Define $\mathcal{D}_\Omega := \text{Hom}_{\text{cont}}(A_\lambda, \mathcal{O}_\lambda)$, a compact Fréchet $\mathcal{O}_\Omega$-module.

If $\Sigma \subset \Omega$ is a closed affinoid, then $\mathcal{D}_\Omega \otimes_{\mathcal{O}_\Omega} \mathcal{O}_\Sigma \cong \mathcal{D}_\Sigma$. 
10.3. Hecke actions and slope decompositions

Let $K \subset G(\mathbb{A}_f)$ be open compact such that $K_p \subset Iw_G$. Let $\Omega \subset \mathcal{W}_0^G$ be an affinoid. Recall $t_p = \text{diag}(p^{2n-1}, p^{2n-2}, ..., p, 1)$, and let $\Delta_p \subset G(\mathbb{Q}_p)$ be the semigroup generated by $Iw_G$ and $t_p$. There is a left-action of $\Delta_p$ on $\mathcal{D}_\Omega$ as follows:

- $k \in Iw_G$ acts on $f \in \mathcal{A}_\Omega$ by $(k * f)(g) = f(gk)$, inducing a dual left action on $\mu \in \mathcal{D}_\Omega$ by $(k * \mu)(f(g)) = \mu(f(gk^{-1}))$.
- $t_p$ acts on the left on $B(\mathbb{Z}_p)$ by $t_p * h = t_p b t_p^{-1}$. Since any $f \in \mathcal{A}_\Omega$ is uniquely determined by its restriction to $B(\mathbb{Z}_p)$, this induces a left action of $t_p$ on $\mu \in \mathcal{D}_\Omega$ by $(t_p * \mu)(f(h)) = \mu(f(t_p b t_p^{-1}))$.

As $\bigcap_{r \geq 0} t_p^r N(\mathbb{Z}_p)t_p^{-r} = 1$, we have $t_p \in T^{++}$ in the notation of [Han17, §2]. Thus we get an $\mathcal{O}_\Omega$-linear controlling operator $U^\circ_p := [K_p t_p K_p]$ on the cohomology groups $\check{H}_c^\bullet(S_K, \mathcal{D}_\Omega)$. Up to shrinking $\Omega$, the $\mathcal{O}_\Omega$-module $\check{H}_c^\bullet(S_K, \mathcal{D}_\Omega)$ admits a slope decomposition with respect to $U_p^\circ$ (see [Han17, Def. 2.3.1]). For $h \in \mathcal{Q}_{\geq 0}$ we let $\check{H}_c^\bullet(S_K, \mathcal{D}_\Omega)^{\leq h}$ denote the subspace of elements of slope at most $h$, and note that it is an $\mathcal{O}_\Omega$-module of finite type.

**Remark 10.4.** The operator $U^\circ_p$ preserves the integral structure $\check{H}_c^\bullet(S_K, \mathcal{D}_\Omega) \subset \check{H}_c^\bullet(S_K, \mathcal{D}_\Omega(L))$. We also have a $*$-action of $\Delta_p$ on $V^\vee_c(L)$, defined identically, giving an operator $U^\circ_p$ on $\check{H}_c^\bullet(S_K, V^\vee_c(L))$ that preserves its natural integral subspace. If $U_p$ denotes the automorphic Hecke operator from §2.4.3, one may check $U_p^\circ = \lambda(t_p) \cdot U_p$. This is all explained in [BSDW, Rem. 3.13].

10.4. Non-critical slope refinements

Let $\lambda \in X^\bullet_0(T)$ be a pure dominant integral weight, $K$ as above, and $L/\mathbb{Q}_p$ a finite extension. The natural inclusion of $\mathcal{V}_\lambda(L) \subset \mathcal{A}_\lambda(L)$ induces dually a surjection $r_\lambda : D_\lambda(L) \twoheadrightarrow V^\vee_\lambda(L)$, which is equivariant for the $*$-actions of $\Delta_p$. This induces a map

$$r_\lambda : \check{H}_c^\bullet(S_K, \mathcal{D}_\Omega(L)) \twoheadrightarrow \check{H}_c^\bullet(S_K, V^\vee_\lambda(L)),$$

(10.4) equivariant for the $*$-actions of $\Delta_p$ (hence $U_p^\circ$) on both sides.

**Definition 10.5.** Let $\pi = (\pi, \alpha)$ be a $p$-refined RACAR of $G(\mathbb{A})$ of weight $\lambda$. We say $\pi$ is non-critical if $r_\lambda$ restricts to an isomorphism

$$r_\lambda : \check{H}_c^\bullet(S_K, \mathcal{D}_\Omega(L))_\pi \cong \check{H}_c^\bullet(S_K, V^\vee_\lambda(L))_\pi$$

of generalised eigenspaces. We say $\pi$ is strongly non-critical if this is true with $\check{H}_c^\bullet$ replaced with $\check{H}^\bullet$ (i.e., if $\pi$ is non-critical for $\check{H}^\bullet$ and for $\check{H}_c^\bullet$ as in [BSW21, Rem. 4.6]).

**Definition 10.6.** Let $\pi$ be a $p$-refinement of $\pi$. For $1 \leq r \leq 2n - 1$, let $\alpha_{p,r} = \lambda(t_p) \alpha_{p,r}$, the corresponding $U_p^\circ$-eigenvalue. We say $\pi$ has non-critical slope if for each $1 \leq r \leq 2n - 1$, we have

$$v_p(\alpha_{p,r}) < \lambda_r - \lambda_{r+1} + 1.$$

**Theorem 10.7** (Classicality). If $\pi$ has non-critical slope, then it is strongly non-critical.

*Proof.* This is [BSW21, Thm. 4.4, Rem. 4.6], explained in Examples 4.5 op. cit. 

11. $p$-adic interpolation of branching laws

A branching law describes how an irreducible representation of $G$ decomposes upon restriction to $H$. Of particular interest to us is the branching law given by Lemma 11.1, which provides a representation-theoretic interpretation of the Deligne-critical range. In this section, we provide a $p$-adic interpolation of the classical branching law Lemma 11.1.
11.1. Classical branching laws, revisited

Let $\lambda \in X^*_0(T)$ be a pure algebraic weight, with purity weight $w$. By dualising, we get the following equivalent formulation of Lemma 4.11:

**Lemma 11.1.** For $j \in \mathbb{Z}$, we have $j \in \text{Crit}(\lambda)$ if and only if $\dim \text{Hom}_H(Z_p)(V^H_{(−j,w+j)}, V_\lambda) = 1$, that is if and only if $V_\lambda | H(Z_p)$ contains $V^H_{(−j,w+j)}$ with multiplicity 1.

We now give a conceptual description of the generators of $V^H_{(−j,w+j)} \subset V_\lambda | H(Z_p)$. Define weights

$$\alpha_i = (1, 0, \ldots, 0, -1), \quad \alpha_2 = (1, 1, 0, \ldots, 0, -1, -1), \quad \ldots \quad \alpha_{n-1} = (1, \ldots, 1, 0, -1, \ldots, -1), \quad \alpha_0 = (1, \ldots, 1, 0, \ldots, 0).$$

(When $n = 1$, we just have $\alpha_0 = (1, 1)$ and $\alpha_1 = (1, 0)$). If $\lambda = (\lambda_1, \ldots, \lambda_{2n}) \in X^*_0(T)$ is a dominant pure algebraic weight, then we easily see that

$$\lambda = [\lambda_1 - \lambda_2] \alpha_1 + [\lambda_2 - \lambda_3] \alpha_2 + \cdots + [\lambda_n - \lambda_{n+1}] \alpha_n + \lambda_{n+1} \alpha_0,$$

where each coefficient is a non-negative integer except perhaps $\lambda_{n+1}$, which can be negative.

**Notation 11.2.** (i) If $1 \leq i \leq n - 1$, then $\text{Crit}(\alpha_i) = \{0\}$. Since the purity weight of $\alpha_i$ is 0, by Lemma 11.1 there is a non-zero vector $v_{(i)} \in V_{\alpha_i}(Q_p)$ upon which the action of $H(Z_p)$ is trivial, and $v_{(i)}$ is unique up to $Q_p^*$-multiple.

(ii) We have $\text{Crit}(\alpha_n) = \{-1, 0\}$, and the purity weight of $\alpha_n$ is 1. By Lemma 11.1, there exist non-zero vectors $v_{(n,1)}, v_{(n,2)} \in V_{\alpha_n}(Q_p)$ such that the action of $(h_1, h_2) \in H(Z_p)$ on $v_{(n)}$ is by $\det(h_i)$. Again, these vectors are unique up to $Q_p^*$-multiple.

(iii) The space $V_{\alpha_n}$ is a line, with basis $v_{(0)}(g) = \det(g)$.

We view all of the elements $v_{(i)}$ as explicit algebraic functions $G(Z_p) \to Q_p$.

**Proposition 11.3.** Let all $\lambda \in X^*_0(T)$ be pure algebraic weights, and let $j \in \text{Crit}(\lambda)$. Then

$$v_{\lambda,j} = [v_{(0)}(1)] \cdot [v_{(2)}^{\lambda_2 - \lambda_1}] \cdots [v_{(n-1)}^{\lambda_{n-1} - \lambda_n}] \cdot [v_{(n,1)}^{\lambda_n - \lambda_{n+1} - j}] \cdot [v_{(n,2)}^{\lambda_{n+1}}] \cdot [v_{(0)}^{\lambda_{n+1}}]$$

is a generator of the line $V^H_{(−j,w+j)}(Q_p)$ inside the $H(Z_p)$-representation $V_\lambda(Q_p) | H(Z_p)$.

Proof. To see this function is algebraic, note the $v_{(i)}$ and $v_{(n),i}$ are algebraic by construction, and:

- for $1 \leq i \leq n - 1$, we have $\lambda_i - \lambda_{i+1} \geq 0$, so $v_{(i)}^{\lambda_i - \lambda_{i+1}}$ is algebraic;
- since $\lambda_n \geq -j \geq \lambda_{n+1}$, we have $v_{(n,1)}^{\lambda_n - \lambda_{n+1} - j}$ and $v_{(n,2)}^{\lambda_{n+1}}$ are algebraic;
- and $v_{(0)}^{\lambda_{n+1}}$ is algebraic.

Thus their product is algebraic. If $t \in T(Z_p)$, then by (11.2) we see

$$\lambda(t) = \alpha_1^{\lambda_1 - \lambda_2(t)} \cdots \alpha_{n-1}^{\lambda_{n-1} - \lambda_n(t)} \cdot \alpha_n(t)^{\lambda_n - \lambda_{n+1} - j} \cdot \alpha_{n+1}^{\lambda_{n+1}} \cdot \alpha_0^{\lambda_{n+1}}(t),$$

so for $n^{-} \in \overline{N}(Z_p)$ and $g \in G(Z_p)$, we see $v_{\lambda,j}(n^{-} t g) = \lambda(t) v_{\lambda,j}(g)$ by multiplying together the analogous relations for the $v_{(i)}$. Finally, by Notation 11.2 we see $(h_1, h_2) \in H(Z_p)$ acts on $v_{\lambda,j}$ by

$$\det(h_1)^{\lambda_{n+1} - j} \cdot \det(h_2)^{\lambda_{n+1}} \cdot \det(h_1 h_2)^{\lambda_{n+1}} = \det(h_1)^{-j} \det(h_2)^{w+j},$$

as required (using $w = \lambda_n + \lambda_{n+1}$ in the final step). ☐
11.2. Support conditions for branching laws

Let $w_n$ denote the antidiagonal $n \times n$ matrix with $(w_n)_{ij} = \delta_{i,n+1-j}$, and recall
\[ u = \begin{pmatrix} 1_n & w_n \\ 0 \end{pmatrix} \in G(Z_p). \quad (11.3) \]

For $\beta \in \mathbb{Z}_{\geq 1}$, let
\[ N^\beta(Z_p) := N(p^\beta Z_p) \cdot u = \{ n \in N(Z_p) : n \equiv \begin{pmatrix} 1_n & w_n \\ 0 \end{pmatrix} \pmod{p^\beta} \}. \]

Note this is not a subgroup of $N(Z_p)$. We also emphasise that $N^\beta(Z_p)$ is not the set of $Z_p$-points of an algebraic group, and hope the notation does not cause confusion.

Let $\lambda \in X^*_0(T)$ be pure dominant algebraic, let $j \in \text{Crit}(\lambda)$, and let $v$ be a generator of the line $V_H^{(-j, w+j)}$ inside the $H(Z_p)$-representation $V_\lambda(Q_p)$, viewed as an explicit algebraic function $G(Z_p) \to Q_p$. The key examples we consider are $\lambda = \alpha_i$ from (11.1), with $v = v(i)$ from Notation 11.2. The aim of this subsection is to prove:

**Proposition 11.4.** Possibly rescaling $v \in V_H^{(-j, w+j)}$, we have $v(N^\beta(Z_p)) \subset 1 + p^\beta Z_p$ for all $\beta \geq 1$.

Recall $G_n = \text{GL}_n$ and its subgroups $B_n, \overline{U}_n, N_n$, and $\overline{N}_n$ from §2.1. Let $Iw_n$ be the Iwahori subgroup of $G_n(Z_p)$, and let $J^\beta := \{ g \in G_n(Z_p) : g \pmod{p^\beta} \in T_n(Z/p^\beta) \} \subset Iw_n$. By the Iwahori factorisation, any element $1_n + p^\beta Y \in J^\beta$ has an Iwahori factorisation $1_n + p^\beta Y = RS$, where $R \in N_n(Z_p)$ and $S \in \overline{B}_n(Z_p)$.

**Lemma 11.5.** If $X \in M_n(Z_p)$, there exist $\overline{u} \in \overline{B}_n(Z_p), h \in H(Z_p)$ with $\overline{u}, h \equiv 1_{2n} \pmod{p^\beta}$ and
\[ \begin{pmatrix} 1_n & w_n + p^\beta X \\ 0 \\ 1_n \end{pmatrix} \equiv \overline{u} \cdot h. \]

**Proof.** Fix $R \in B_n(Z_p), S \in \overline{B}_n(Z_p)$ such that $1 + p^\beta w_n X = RS$. A simple check shows $R, S \equiv 1 \pmod{p^\beta}$. We see
\[ (w_n R w_n S^{-1}) \in \overline{B}_n(Z_p), \quad \begin{pmatrix} (w_n R w_n S^{-1}) \\ 1_n \end{pmatrix} \in H(Z_p), \]
and
\[ \begin{pmatrix} (w_n R w_n S^{-1}) \\ 1_n \end{pmatrix} \equiv \begin{pmatrix} 1_n & w_n + p^\beta X \\ 0 \\ 1_n \end{pmatrix}, \]
which has the claimed form. \( \square \)

Now let $v$ be as in Proposition 11.4. Since $v$ transforms by $\lambda(\overline{u}) \in Z_p^\times$ under left translation by $\overline{u} \in \overline{B}_n(Z_p)$, and transforms like $\det(h_1)^{-j} \det(h_2)^{w+j} \in Z_p^\times$ under right translation by $h = (h_1, h_2) \in H(Z_p)$, we see
\[ v[\overline{B}_n(Z_p) \cdot u \cdot H(Z_p)] \subset Z_p^\times \cdot v(u). \]

Suppose $v(u) = 0$; then $v$ vanishes on the cell $\overline{B}_n(Z_p) \cdot u \cdot H(Z_p)$. This cell is open and dense in $G(Z_p)$ (e.g. [Loe22, §5.1.3]), forcing $v = 0$, which contradicts our assumptions. Thus $v(u) \neq 0$, and we are free to rescale it by an element of $Q_p^\times$ so that $v(u) = 1$. Further if $\overline{u}, h \equiv 1_{2n} \pmod{p^\beta}$, then $\lambda(\overline{u}) \det(h_1)^{-j} \det(h_2)^{w+j} \equiv 1 \pmod{p^\beta}$. Combining all of this with Lemma 11.5, we deduce that for any $X \in M_n(Z_p)$, we have
\[ v \left( \begin{pmatrix} 1_n & w_n + p^\beta X \\ 0 \\ 1_n \end{pmatrix} \right) \in 1 + p^\beta Z_p. \quad (11.4) \]

**Proof.** (Proposition 11.4). A general element of $N^\beta(Z_p)$ looks like $n = \begin{pmatrix} A & w_n + p^\beta Y \\ 0 & B \end{pmatrix}$, where $A, B \in N_n(Z_p)$ with $A \equiv B \equiv 1_n \pmod{p^\beta}$ and $Y \in M_n(Z_p)$. Letting
\[ X := Y - w_n \begin{pmatrix} B^{-1} & 0 \\ p^\beta \end{pmatrix} B^{-1} \in M_n(Z_p), \]
we have
\[ \begin{pmatrix} A & w_n + p^\beta Y \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1_n & w_n + p^\beta X \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix}. \]
Then $v(n) = \det(A)^{-j} \det(B)^{w+j} \cdot v \left( \begin{pmatrix} 1_n & w_n + p^\beta Y \\ 0 \\ 1_n \end{pmatrix} \right) \in 1 + p^\beta Z_p$ by (11.4). \( \square \)
11.3. $p$-adic interpolation of branching laws

Now we $p$-adically interpolate, with $\beta = 1$. Assume the choices $v_{(0)}, v_{(1)}, \ldots, v_{(n-1)}, v_{(n)}$ in §11.1 were all normalised so $v_{(i)}(u) = 1$, whence $v_{(i)}(1) \subset 1 + p \mathbb{Z}_p \subset \mathbb{Z}_p^\times$, as in Proposition 11.4 (and its proof).

If $R$ is a $\mathbb{Q}_p$-algebra and $\chi = (\chi_1, \ldots, \chi_{2n}) : T(\mathbb{Z}_p) \to R^\times$ is a character, then define a function

$$w_\chi : N^1(\mathbb{Z}_p) \to R^\times, \quad \chi \mapsto (11.5)$$

$$g \mapsto v_{(0)}(g)^{\chi_0} \cdot \prod_{i=1}^{n-1} v_{(i)}(g)^{\chi_i - \chi_{i+1}} \cdot v_{(n),1}(g)^{\chi_{n+1}} \cdot v_{(n),2}(g)^{\chi_n},$$

where if $x \in \mathbb{Z}_p^\times$ we write $x^\chi$ as shorthand for $\chi(x)$. If $\beta \geq 1$, let

$$Iw^\beta := \bar{N}(p \mathbb{Z}_p) \cdot T(\mathbb{Z}_p) \cdot \mathbb{N}^\beta(\mathbb{Z}_p) \subset Iw^\beta_G,$$

which again is not a subgroup. We may extend $w_\chi$ to a function $w_\chi : Iw^1_{\mathbb{Q}_p} \to R^\times$ via (10.3):

$$w_\chi(\tau \cdot t \cdot n) = \chi(t) \cdot w_\chi(n), \quad \tau \in \bar{N}(p \mathbb{Z}_p), t \in T(\mathbb{Z}_p), n \in N^1(\mathbb{Z}_p).$$

For $\beta \geq 1$, let

$$Iw^\beta_H = H(\mathbb{Z}_p) \cap u^{-1}Iw^\beta_G.$$ 

If $g \in Iw^1_H$ and $h \in Iw^1_H$, then a simple check shows $gh \in Iw^1_G$, so we can consider $w_\chi(gh)$. The following will be important in the sequel: recalling $L^\beta_B$ from Definition 4.3, note that for appropriate choices of $K_p$ (e.g. if $K_p = Iw^1_G$) we have $L^\beta_B \subset Iw^\beta_H$.

**Lemma 11.6.** If $h = (h_1, h_2) \in Iw^1_H$, then $w_\chi(gh) = \det(h_2)^{\chi_0 + \chi_1} \cdot w_\chi(g)$.

**Proof.** By definition, $w_\chi(gh)$ is a product of terms involving $v_{(i)}(gh)$, $v_{(n),i}(gh)$. By construction $v_{(i)}(gh) = v_{(i)}(g)$ for $1 \leq i \leq n - 1$, and $v_{(0)}(gh) = \det(h_1) \det(h_2)v_{(0)}(g)$, and $v_{(n),i}(gh) = \det(h_1)v_{(n),i}(g)$ for $i = 1, 2$. We get factors of $\det(h_1)^{\chi_0 + \chi_1}$ $\det(h_2)^{\chi_1}$ from $v_{(0)}$, $\det(h_1)^{-\chi_{n+1}}$ from $v_{(n),1}$, and $\det(h_2)^{\chi_n}$ from $v_{(n),2}$, so

$$w_\chi(gh) = \det(h_1)^{\chi_0 + \chi_1} \cdot \det(h_2)^{\chi_1} \cdot \det(h_1)^{-\chi_{n+1}} \cdot \det(h_2)^{\chi_n} \cdot w_\chi(g) = \det(h_2)^{\chi_0 + \chi_1} \cdot w_\chi(g). \quad \Box$$

**Remark 11.7.** The definition of $w_\chi$ is heavily motivated by Proposition 11.3. Indeed we see that if $\lambda \in X^\chi_G(T)$ is a pure dominant algebraic weight and $j \in \text{Crit}(\lambda)$, then for any $g \in N^1(\mathbb{Z}_p)$ we have

$$v_{\lambda,j}(g) = w_\chi(g) \cdot \left[ \frac{v_{(n),2}(g)}{v_{(n),1}(g)} \right]^j. \quad (11.9)$$

To $p$-adically vary branching laws, we take $R = O_\Omega$, for $\Omega \subset \mathfrak{O}_G^\times$ an affinoid in the pure weight space, equipped with a character $\chi : T(\mathbb{Z}_p) \to O_\Omega^\times$ as in §10.1. We allow $\Omega = \{ \lambda \}$ to be a point.

**Definition 11.8.** Let $f \in \mathcal{A}(\mathbb{Z}_p^\times, O_\Omega)$ be a locally analytic function. Define a function $v_\Omega(f) : N(\mathbb{Z}_p) \to O_\Omega$ by

$$v_\Omega(f)(g) = \begin{cases} w_\chi(g) \cdot f \left( \frac{v_{(n),2}(g)}{v_{(n),1}(g)} \right) & \text{if } g \in N^1(\mathbb{Z}_p), \\ 0 & \text{otherwise} \end{cases}.$$

This is well-defined by the normalisations fixed at the start of §11.3, by Proposition 11.4, and the definition (11.5). Under the transformation law (10.3) and Iwahori decomposition, this extends to a unique element $[v_\Omega(f) : Iw^1_G \to O_\Omega] \in \mathcal{A}_\Omega$, with support on $Iw^1_H$ from (11.6).
Lemma 11.9. If \( \lambda \in X_0^r(T) \) and \( g \in Iw_1^G \), then for all \( j \in \text{Crit}(\lambda) \) we have

\[
[v_\lambda(z \mapsto z^j)](g) = v_{\lambda,j}(g).
\]

Proof. For \( g \in N^1(Z_p) \), this follows by combining the definition of \( v_\lambda \) with (11.9). Both sides satisfy the same transformation law (11.7) to extend to \( Iw_1^1 \), so we have equality on the larger group too.

Recall \( \omega_1 : Z_p^* \to \Omega_1^* \) from (10.1). Let \( h = (h_1, h_2) \in H(Z_p) \) act on \( f \in \mathcal{A}(Z_p^*, \Omega_1) \) by

\[
(h * f)(z) = \det(h_2)^{\omega_1} \cdot f \left( \frac{\det(h_2)}{\det(h_1)} \cdot z \right). \tag{11.10}
\]

Recall \( Iw_H^1 \) from (11.8); this acts on \( \mathcal{A}(Z_p^*, \Omega_1) \) and \( \mathcal{A}_\Omega \) by its embeddings into \( H(Z_p) \) and \( Iw_G^1 \).

Lemma 11.10. The map \( \nu_1 : \mathcal{A}(Z_p^*, \Omega_1) \to \mathcal{A}_\Omega \) is \( Iw_H^1 \)-equivariant.

Proof. Let \( h = (h_1, h_2) \in Iw_H^1 \) and \( g \in N^1(Z_p) \). First note that by (10.1) we know \( \omega_1 = \chi_{\Omega_1, \Omega} \). Now let \( f \in \mathcal{A}(Z_p^*, \Omega_1) \), and compute

\[
\nu_1(h * f)(g) = w_{\chi_\Omega}(g) \cdot (h * f) \left( \frac{v_{(n),2}(g)}{v_{(n),1}(g)} \right)
= \det(h_2)^{\omega_1} w_{\chi_\Omega}(g) \cdot f \left( \frac{\det(h_2) \cdot v_{(n),2}(g)}{\det(h_1) \cdot v_{(n),1}(g)} \right) = w_{\chi_\Omega}(gh) \cdot f \left( \frac{v_{(n),2}(gh)}{v_{(n),1}(gh)} \right).
\]

Since \( gh \in Iw_H^1 \), we may write \( gh = b'g' \), with \( b' \in T(pZ_p) \cdot B(Z_p) \) and \( g' \in N^1(Z_p) \). Then

\[
[h * \nu_1(f)](g) = \nu_1(f)(gh) = \nu_1(f)(b'g') = \chi_\Omega(b') \cdot \nu_1(f)(g') \tag{11.12}
= \chi_\Omega(b')w_{\chi_\Omega}(g')f \left( \frac{v_{(n),2}(g')}{v_{(n),1}(g')} \right) = w_{\chi_\Omega}(gh) f \left( \frac{v_{(n),2}(gh)}{v_{(n),1}(gh)} \right),
\]

where in the last equality we use that \( v_{(n),i}(b')^{-1}gh = \alpha_n(b')^{-1}v_{(n),i}(gh) \) for both \( i = 1, 2 \). Combining (11.11) and (11.12) yields \( \nu_1(h * f)(g) = [h * \nu_1(f)](g) \), as required.

11.4. Branching laws for distributions

The overconvergent cohomology groups we consider have coefficients in \( \mathcal{D}_\Omega \), not \( \mathcal{A}_\Omega \), so we now dualise the above. Finally, we collate everything we have proved in the main result of this section (Proposition 11.12).

By Lemma 4.11, for \( \lambda \in X_0^r(T) \) we have \( j \in \text{Crit}(\lambda) \) if and only if \( \text{Hom}_{H(Z_p)}(V_{\lambda}^\vee, V_H^1{(\lambda, -w-j)}) \) is a line; and moreover, the choices made in §11.1 fix a generator

\[
\kappa_{\lambda,j} : V_{\lambda}^\vee(Q_p) \to V_H^1{(\lambda, -w-j)}(Q_p) \Rightarrow Q_p, \quad \mu \mapsto \mu(v_{\lambda,j}). \tag{11.13}
\]

This is \( H(Z_p) \)-equivariant, as if \( h = (h_1, h_2) \in H(Z_p) \), then

\[
(h * \mu)(v_{\lambda,j}) = \mu(h^{-1} * v_{\lambda,j}) = \det(h_1)^j \det(h_2)^{-w-j} \mu(v_{\lambda,j}). \tag{11.14}
\]

We can base-extend this to any extension \( L/Q_p \) and consider \( \kappa_{\lambda,j} \) as a map \( V_{\lambda}^\vee(L) \to L \).

Similarly, we can dualise the map \( \nu_1 : \mathcal{A}(Z_p^*, \Omega_1) \to \mathcal{A}_\Omega \) from Definition 11.8 to get

\[
\kappa_{\lambda} : \mathcal{D}_\Omega \to \mathcal{D}(Z_p^*, \Omega), \quad \mu \mapsto \mu(v_{\lambda}(f)) \text{ for } f \in \mathcal{A}(Z_p^*, \Omega_1). \tag{11.15}
\]
Definition 11.11. Let $\beta \geq 1$. We say that $f \in \mathcal{A}_\Omega$ (resp. $\mu \in \mathcal{D}_\Omega$) has support on $Iw_G^\beta$ if $f(g) = 0$ for $g \notin Iw_G^\beta$ (resp. $\mu(f |_{Iw_G^\beta})$ for all $f \in \mathcal{A}_\Omega$). Let $\mathcal{A}_\Omega^\beta \subset \mathcal{A}_\Omega$ (resp. $\mathcal{D}_\Omega^\beta \subset \mathcal{D}_\Omega$) be the subspace of functions (resp. distributions) supported on $Iw_G^\beta$. We similarly write $\mathcal{A}_\lambda^\beta$, $\mathcal{D}_\lambda^\beta$, etc.

There is a natural map $s_\lambda : V_\lambda(L) \to \mathcal{A}_\lambda^\beta(L)$ given by

$$s_\lambda(f)(g) = \begin{cases} f(g) & : g \in Iw_G^\beta \\ 0 & : \text{else} \end{cases}$$

Abusing notation, for any $\beta$ we continue to write $r_\lambda : \mathcal{D}_\lambda^\beta \to V_\lambda^\vee$ for its dual.

Proposition 11.12. For each classical $\lambda \in \Omega$ and each $j \in \text{Crit}(\lambda)$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{D}_\Omega & \xrightarrow{s_{p_\lambda}} & \mathcal{D}_\lambda^1(L) \\
\downarrow & & \downarrow \\
\mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega) & \xrightarrow{s_{p_\lambda}} & \mathcal{D}(\mathbb{Z}_p^\times, L) \\
\end{array}
$$

Proof. The first square commutes directly from the definitions and the fact that $s_{p_\lambda} \circ \kappa_\Omega = \lambda$.

To see the second square commutes, let $\mu \in \mathcal{D}_\lambda^1(L)$; then

$$
\kappa_\lambda(\mu)(z \mapsto z^j) = \int_{\mathbb{Z}_p^\times} z^j \cdot d\kappa_\lambda(\mu) = \int_{Iw_G^1} v_\lambda(z^j) \cdot d\mu = \int_{Iw_G^1} v_\lambda(z^j) \cdot d\mu = \int_{Iw_G^1} v_{\lambda,j} \cdot d\mu = \kappa_{\lambda,j} \circ r_\lambda(\mu),
$$

where for clarity we write $\int_{\mathbb{Z}_p^\times} f \cdot d\mu$ for $\mu(f | \mathbb{Z}_p^\times)$, interpreted suitably for each term. Then the second equality is by definition of $\kappa_\lambda$, the third and fifth equalities follow as $\mu$ is supported on $Iw_G^1$, the fourth equality is Lemma 11.9, the sixth equality is by definition of $r_\lambda$ as $v_{\lambda,j} \in V_\lambda$, and the seventh equality is the definition of $\kappa_{\lambda,j}$.

Recall the $H(\mathbb{Z}_p)$-action on $\mathcal{A}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$ from (11.10). We equip $\mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$ with the left dual action. In the diagram of Proposition 11.12, recall the action of $H(\mathbb{Z}_p)$ on the bottom-right term is by $\det(h_1)^j \det(h_2)^{-w-j}$. Finally we note:

Lemma 11.13. Every map in the diagram of Proposition 11.12 is $Iw_G^1$-equivariant.

Proof. For $s_{p_\lambda}$ and $r_\lambda$ this follows straight from the definition; for $\kappa_\Omega$ and $\kappa_\lambda$ this is Lemma 11.10; for $\kappa_{\lambda,j}$ this is (11.14); and for evaluation at $z^j$, this follows since for $\mu \in \mathcal{D}(\mathbb{Z}_p^\times, L)$ and $h \in H(\mathbb{Z}_p)$, we have $(h \ast \mu)(z^j) = \mu(h^{-1} \ast z^j) = \mu(\det(h_1)^j \det(h_2)^{-w-j} \cdot z^j)$.

12. Distribution-valued evaluation maps

We now define distribution-valued evaluation maps on the overconvergent cohomology by combining the $p$-adic branching laws of §11 with the abstract evaluation maps of §4.

12.1. Combining evaluations and branching laws

Recall $H(\mathbb{Z}_p)$ acts on $\mathcal{A}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$ by (11.10). We have an isomorphism

$$
\mathbb{Z}_p^\times \cong \mathbb{G}_m^\times(Q):= Q^\times \setminus A^\times / \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \mathbb{R}_{>0},
$$

(12.1)
and we extend the $H(Z_p)$-action on $f \in A(Z_p^x, O_{\Omega})$ to an action of $h = (h_1, h_2) \in H(A)$ by

$$(h * f)(z) = \chi_{\text{cyc}}(\det(h_2))^{\mu_\Omega} \cdot f \left( \frac{\det(h_2)}{\det(h_1)} \cdot z \right),$$

where translation of $z \in Z_p^x$ under $\det(h_2)\det(h_1)^{-1} \in A^\times$ is defined by lifting $z$ to $A^\times$ under (12.1), translating, and projecting back to $Z_p^x$. Note $H_{x,\infty}^\bullet$ and $H(Q)$ again act trivially, and that any subgroup of $H(Z)$ acts through projection to $H(Z_p)$.

**Lemma 12.1.** If $\beta \geq 1$, then the map $\kappa_{\Omega}$ from (11.15) is $L_\beta$-equivariant.

**Proof.** Recall that $\kappa_{\Omega}$ is the dual of $v_{\Omega} : A(Z_p^x, O_{\Omega}) \to A_{\Omega}$, so it suffices to prove $v_{\Omega}$ is $L_\beta$-equivariant. Recall $\text{Iw}_H^1$ from (11.8), and that $v_{\Omega}$ is $\text{Iw}_H^1$-equivariant by Lemma 11.10. A simple check shows that $L_p^\beta \subset \text{Iw}_H^1$, so $v_{\Omega}$ is $L_p^\beta$-equivariant. But the $L_\beta$-action on both terms factors through projection to $L_p^\beta$, since $L_\beta \subset H(A)$ (for $A(Z_p^x, O_{\Omega})$) and by definition (for $A_{\Omega}$).

To define our distribution-valued evaluations, we take strong motivation from Definition 4.12, adapting it with the Borel $B$ in place of the parabolic $Q$. Let $\Omega \subset W^G$ be an affinoid in the pure weight space; we allow $\Omega = \{ \lambda \}$ a single weight. Lemma 12.1 allows us to make the following definition:

**Definition 12.2.** Let $\beta \in Z_{\geq 0}$ and $\delta \in H(A_f)$, representing $[\delta] \in \pi_0(X_{\beta})$. The overconvergent evaluation map of level $p^3$ at $[\delta]$ is the map

$$\text{Ev}_{B,\beta,[\delta]}^\Omega : \text{Ev}_{B,\beta,[\delta]}^{\Omega,\kappa_{\Omega}} = \delta \ast \left[ \kappa_{\Omega} \circ \text{Ev}_{B,\beta,[\delta]}^{\Omega} \right] : \text{H}_{c}^\bullet(S_K, O_{\Omega}) \to \mathcal{D}(Z_p^x, \Omega_{\Omega}).$$

This is well-defined and independent of the choice of $\delta$ representing $[\delta]$ by Proposition 4.9. Here we also use that $H(Q)$ and $H_{x,\infty}^\bullet$ act trivially on $\mathcal{D}(Z_p^x, O_{\Omega})$.

**Proposition 12.3.** Suppose $\beta \geq 1$. Then for every $\lambda \in \Omega$, and $j \in \text{Crit}(\lambda)$, we have a commutative diagram

$$
\begin{array}{ccc}
\text{H}_{c}^\bullet(S_K, \Omega_{\Omega}) & \xrightarrow{\text{sp}_{\lambda}} & \text{H}_{c}^\bullet(S_K, \Omega_{\lambda}(L)) \\
\xrightarrow{\text{Ev}_{B,\beta,[\delta]}^{\Omega}} & & \xrightarrow{r_{\lambda}} \text{H}_{c}^\bullet(S_K, \Omega_{\lambda}(L)) \\
\text{D}(Z_p^x, \Omega_{\Omega}) & \xrightarrow{\text{sp}_{\lambda}} & \text{D}(Z_p^x, L) \\
\xrightarrow{\text{Ev}_{B,\beta,[\delta]}^{\Omega}} & & \xrightarrow{E_{\lambda}^{V_{\lambda}}(L)} L
\end{array}
$$

**Proof.** First note that applying Lemma 4.8 first to $\text{D}_{\Omega} \xrightarrow{\text{sp}_{\lambda}} \text{D}_{\lambda}(L)$ and then $\text{D}_{\lambda}(L) \xrightarrow{\iota_{\lambda}} V_{\lambda}^\vee(L)$, we have a commutative diagram

$$
\begin{array}{ccc}
\text{H}_{c}^\bullet(S_K, \Omega_{\Omega}) & \xrightarrow{\text{sp}_{\lambda}} & \text{H}_{c}^\bullet(S_K, \Omega_{\lambda}(L)) \\
\xrightarrow{\text{Ev}_{B,\beta,[\delta]}^{\Omega}} & & \xrightarrow{r_{\lambda}} \text{H}_{c}^\bullet(S_K, \Omega_{\lambda}(L)) \\
(\text{D}_{\Omega})_{\Gamma_{\beta,\delta}} & \xrightarrow{\text{sp}_{\lambda}} & \text{D}_{\lambda}(L)_{\Gamma_{\beta,\delta}} \\
\xrightarrow{\text{Ev}_{B,\beta,[\delta]}^{\Omega}} & & \xrightarrow{E_{\lambda}^{V_{\lambda}}(L)_{\Gamma_{\beta,\delta}}(L)} V_{\lambda}^\vee(L)_{\Gamma_{\beta,\delta}}.
\end{array}
$$

Recall $\text{D}_{\Omega}^1 \subset \text{D}_{\Omega}$ from Definition 11.11. Since $\beta \geq 1$, we have $\delta^{-1}\Gamma_{\beta,\delta} \subset \text{Iw}_H^1$ from (11.8), so the action of $\Gamma_{\beta,\delta}$ on $\text{D}_{\Omega}$ preserves $\text{D}_{\Omega}^1$ and we can consider $(\text{D}_{\Omega})_{\Gamma_{\beta,\delta}}$. Moreover, $H(Q)$ (hence $\Gamma_{\beta,\delta}$) acts trivially on $\text{D}(Z_p^x, O_{\Omega})$ and $L$, and $\kappa_{\Omega}$ and $\kappa_{\lambda, j}$ are $\text{Iw}_H^1$-equivariant; combining, each of these maps factors through the coinvariants, giving maps $(\text{D}_{\Omega})_{\Gamma_{\beta,\delta}} \to \text{D}(Z_p^x, O_{\Omega})$ and $V_{\lambda}^\vee(L)_{\Gamma_{\beta,\delta}} \to L$. Then by Proposition 11.12 (for the top squares) and Lemma 11.13 (for the
bottom squares) we have a commutative diagram

\[
\begin{array}{cccc}
(D_{\Omega}t_{\beta,d})_{\kappa} & \rightarrow & D_{\Omega}r_{\beta,d} & \rightarrow & V^{\vee}(L)_{\Gamma_{\beta,d}} \\
\uparrow & & \uparrow & & \uparrow \\
D(Z_p^{\lambda}, \mathcal{O}_\Omega) & \rightarrow & D(Z_p^{\lambda}, L) & \rightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
D(Z_p^{\lambda}, \mathcal{O}_\Omega) & \rightarrow & D(Z_p^{\lambda}, L) & \rightarrow & L.
\end{array}
\]

Since \(\text{Ev}_{B,\beta,[\delta]}^\Omega = \delta \ast [\kappa_\Omega \circ \text{Ev}_{B,\beta,[\delta]}^D]\) and \(\mathcal{E}_{B,\beta,[\delta]}^{1+}\) we can complete the proof by combining (12.3) with (12.4). This is possible by Lemma 12.4 below, noting if \(\beta \geq 1\), then \(D_{\Omega}^\beta \subset D_{\Omega}^{1+}\).

**Lemma 12.4.** Let \(\beta \geq 1\) and \(\Phi \in H^1_c(S_K, \mathcal{O}_\Omega)\). We have \(\text{Ev}_{B,\beta,[\delta]}^D(\Phi) \in (D_{\Omega}^\beta)_{\Gamma_{\beta,d}}\).

**Proof.** Note that \(\text{Ev}_{B,\beta,[\delta]}^D(\Phi) \in [(u^{-1}t_p^\beta) \ast D_{\Omega}]_{\Gamma_{\beta,d}}\) by construction of evaluation maps. By definition, if \(\mu \in \mathcal{D}_\Omega, f \in \mathcal{A}_\Omega\), then the action of \(u^{-1}t_p^\beta\) is by

\[
[u^{-1}t_p^\beta \ast \mu](f(n)) = \mu[f((t_p^\beta n)^{-1}u)].
\]

It is easily seen that \(t_p^\beta n^{-1}t_p^{-\beta} \equiv 1_{2n}\) (mod \(p^\beta\)) (e.g. from [BSW21, §2.5, Rem. 4.19]). By definition of \(N^\beta(Z_p)\) we deduce \(t_p^\beta n^{-1}t_p^{-\beta} u = N^\beta(Z_p)\). Extending via (11.6) and (11.7), it follows \([(u^{-1}t_p^\beta) \ast \mu](f)\) depends only on \(f|_{\mathcal{W}_G}\), so \((u^{-1}t_p^\beta \ast \mathcal{D}_\Omega) \subset D_{\Omega}^\beta\), proving the lemma (and thus Proposition 12.3).

#### 12.2. Further support conditions

Our ultimate goal is to interpolate the classical evaluation maps \(\mathcal{E}_{B,N}^\beta\) from §4.2. In Remark 4.14, we described this map as a composition of four maps; and in Proposition 12.3, we have used the branching laws of §11 to interpolate the first two maps of this composition. We will combine over \(\beta\) and \(\delta\) to interpolate the final two maps in Remark 14.1. First, we give a more precise description of \(\text{Im}(\text{Ev}_{B,\beta,[\delta]}^D)\).

Note that for \(\beta \geq 1\), we have decompositions

\[
\mathcal{A}(Z_p^{\lambda}, \mathcal{O}_\Omega) = \bigoplus_{d \in \mathcal{I}(Z_p^{\lambda})^0} \mathcal{A}(d + p^\beta Z_p, \mathcal{O}_\Omega), \quad \mathcal{D}(Z_p^{\lambda}, \mathcal{O}_\Omega) = \bigoplus_{d \in \mathcal{I}(Z_p^{\lambda})^0} \mathcal{D}(d + p^\beta Z_p, \mathcal{O}_\Omega). \tag{12.5}
\]

As in Definition 11.11, a distribution \(\mu \in \mathcal{D}(Z_p^{\lambda}, \mathcal{O}_\Omega)\) lies in the summand \(\mathcal{D}(d + p^\beta Z_p, \mathcal{O}_\Omega)\) if and only if \(\mu(f) = \mu(f|_{d+p^\beta Z_p})\) for all \(f \in \mathcal{A}(Z_p^{\lambda}, \mathcal{O}_\Omega)\).

**Lemma 12.5.** If \(\mu \in \mathcal{D}_{\Omega}^\beta\), then \(\kappa_\Omega(\mu) \in \mathcal{D}(1 + p^\beta Z_p, \mathcal{O}_\Omega)\).

**Proof.** Let \(f \in \mathcal{A}(Z_p^{\lambda}, \mathcal{O}_\Omega)\). If \(g \in N^\beta(Z_p)\), then by Proposition 11.4, we know \(v_{\delta}(f)(g) \in 1 + p^\beta Z_p\). Thus by the definition of \(v_{\delta}(f)\), we see \(v_{\delta}(f)(g) = v_{\delta}(f|_{1+p^\beta Z_p})(g)\), that is,

\[
v_{\delta}(f)\big|_{N^\beta(Z_p)} = v_{\delta}(f|_{1+p^\beta Z_p})\big|_{N^\beta(Z_p)}.
\]

By the transformation law (10.3), the function \(v_{\delta}(f)|_{\mathcal{W}_G} \in \mathcal{A}_\Omega\) depends only on \(v_{\delta}(f)|_{N^\beta(Z_p)}\), so we deduce \(v_{\delta}(f)|_{\mathcal{W}_G} = v_{\delta}(f|_{1+p^\beta Z_p})|_{\mathcal{W}_G}\). Thus if \(\mu\) has support on \(1 + p^\beta Z_p\), for any \(f \in \mathcal{A}(Z_p^{\lambda}, \mathcal{O}_\Omega)\) we have

\[
\kappa_\Omega(\mu)(f) = \mu[v_{\delta}(f)] = \mu[v_{\delta}(f)|_{1+p^\beta Z_p}]
\]

\[
= \mu[v_{\delta}(f)|_{1+p^\beta Z_p}]|_{\mathcal{W}_G} = \mu[v_{\delta}(f|_{1+p^\beta Z_p})] = \kappa_\Omega(\mu)(f|_{1+p^\beta Z_p}),
\]

so \(\kappa_\Omega(\mu)\) is supported on \(1 + p^\beta Z_p\), as required.
Recall $\text{pr}_\beta : \pi_0(X_\beta) \to (\mathbb{Z}/p^3)^\times$ from (4.8).

**Corollary 12.6.** If $\Phi \in H^\ell_\ell(S_K, \mathcal{D}_\Omega)$, we have $\text{Ev}^\Omega_{B, \beta, [\delta]}(\Phi) \in \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega)$, where $d = \text{pr}_\beta([\delta])$.

**Proof.** Recall $\text{Ev}^\Omega_{B, \beta, [\delta]} : \mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$ by Lemma 12.4, we have $\text{Im}(\text{Ev}^\Omega_{B, \beta, [\delta]} \subset (\mathcal{D}_\Omega)_{r, \beta, 0}$. Since $\kappa_\Omega$ factors through the coinvariants, Lemma 12.5 implies that $\kappa_\Omega[(\mathcal{D}_\Omega)_{r, \beta, 0} \subset \mathcal{D}(1 + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega)$.

Finally the action of $\delta$ on $\mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$ is by $\delta^{-1}$ on $\mathcal{A}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)$, which includes translation on $z \in \mathbb{Z}_p^\times$ by $\det(\delta_1)$. As this is a representative of $d = \text{pr}_\beta([\delta]) \in (\mathbb{Z}/p^3)^\times$, translation by $\det(\delta_1)$ sends $1 + p^\beta \mathbb{Z}_p$ to $d + p^\beta \mathbb{Z}_p$. This induces a map $\delta^* : \mathcal{D}(1 + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega) \to \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega)$. Combining all of the above gives the corollary. \hfill \square

### 12.3. Interpolation of classical evaluations

Let $\eta_0$ be any character of $(\mathbb{Z}/m)^\times$. For $\beta \geq 1$ and $d \in (\mathbb{Z}/p^\beta \mathbb{Z})^\times$, and motivated by Definition 4.13, define a map

$$
\text{Ev}^\Omega_{B, \beta, d} : H^\ell_\ell(S_K, \mathcal{D}_\Omega) \longrightarrow \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega)
$$

$$
\Phi \longmapsto \sum_{[\delta] \in \text{pr}_\beta^{-1}(d)} \eta_0(\text{pr}_2([\delta])) \text{ Ev}^\Omega_{B, \beta, [\delta]}(\Phi).
$$

Combining under (12.5), we finally obtain an evaluation map

$$
\text{Ev}^\Omega_{B, \beta} \coloneqq \bigoplus_{d \in (\mathbb{Z}/p^\beta)^\times} \text{Ev}^\Omega_{B, \beta, d} : H^\ell_\ell(S_K, \mathcal{D}_\Omega) \longrightarrow \mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega)
$$

$$
\Phi \longmapsto \sum_{[\delta] \in \pi_0(X_\beta)} \eta_0(\text{pr}_2([\delta])) \times \left(\delta^* \left[\kappa_\Omega \circ \text{Ev}^\Omega_{B, \beta, [\delta]}(\Phi)\right]\right).
$$

**Remark 12.7.** We have an analogue of Remark 4.14; $\text{Ev}^\Omega_{B, \beta}$ is the composition

$$
\begin{aligned}
H^\ell_\ell(S_K, \mathcal{D}_\Omega) \xrightarrow{\text{Ev}^\Omega_{B, \beta, [\delta]}} & \mathcal{D}(\mathcal{D}_{\Omega})_{r, \beta, 0} \xrightarrow{\delta^* \kappa_\Omega} \mathcal{D}(\text{pr}_\beta([\delta]) + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega) \\
\xrightarrow{\bigoplus_{d \in (\mathbb{Z}/p^\beta)^\times} \text{Ev}^\Omega_{B, \beta, d}} & \mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega),
\end{aligned}
$$

where again $\Xi_{\beta, [\delta]}$ sends a tuple $(m[\delta], m[\delta])$ to $\sum_{[\delta] \in \text{pr}_\beta^{-1}(d)} \eta_0(\text{pr}_2([\delta])) \times m[\delta]$.

Combining all of the results of this section, we finally deduce:

**Proposition 12.8.** Suppose $\beta \geq 1$ and $\chi$ is a finite-order Hecke character of conductor $p^\beta$. Then for every $\lambda \in \Omega$, and $j \in \text{Crit}(\lambda)$, we have a commutative diagram

$$
\begin{array}{ccc}
H^\ell_\ell(S_K, \mathcal{D}_\Omega) & \xrightarrow{\text{sp}_\lambda} & H^\ell_\ell(S_K, \mathcal{D}_\lambda(L)) \\
\downarrow \text{Ev}^\Omega_{B, \beta} & & \downarrow \text{Ev}^\lambda_{B, \beta} \\
\mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}_\Omega) & \xrightarrow{\text{sp}_\lambda} & \mathcal{D}(\mathbb{Z}_p^\times, L) \\
\mu \rightarrow \mu[\chi(z)^{1/2}] & & \text{Ev}^\mu_{B, \lambda}
\end{array}
$$

**Proof.** By combining Proposition 12.3 with Corollary 12.6, and taking a direct sum over $[\delta] \in (\mathbb{Z}/p^3)^\times$.
π₀(X_β), there is a commutative diagram

\[
\begin{align*}
H'_c(S_K, \mathcal{O}_\Omega) & \xrightarrow{sp_A} H'_c(S_K, \mathcal{O}_\Omega(L)) \xrightarrow{r_A} H'_c(S_K, \mathcal{O}_\Omega(L)) \quad (12.8) \\
\bigoplus_{[\delta]} \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega) & \xrightarrow{sp_A} \bigoplus_{[\delta]} \mathcal{D}(d + p^\beta \mathbb{Z}_p, L) \xrightarrow{\mu \rightarrow \mu(z^j)} \bigoplus_{[\delta]} L,
\end{align*}
\]

where \(d = pr_{\beta}([\delta]) \in (\mathbb{Z}/p^\beta)\times\). Also, there is a commutative diagram

\[
\begin{align*}
\bigoplus_{[\delta]} \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega) & \xrightarrow{sp_A} \bigoplus_{[\delta]} \mathcal{D}(d + p^\beta \mathbb{Z}_p, L) \xrightarrow{\mu \rightarrow \mu(z^j)} \bigoplus_{[\delta]} L \\
\bigoplus d \mathcal{D}(d + p^\beta \mathbb{Z}_p, \mathcal{O}_\Omega) & \xrightarrow{sp_A} \bigoplus d \mathcal{D}(d + p^\beta \mathbb{Z}_p, L) \xrightarrow{\mu \rightarrow \mu(z^j)} \bigoplus d L \\
\bigoplus \mathcal{D}(Z^\alpha_\beta, \mathcal{O}_\Omega) & \xrightarrow{sp_A} \bigoplus \mathcal{D}(Z^\alpha_\beta, L) \xrightarrow{\mu \rightarrow \mu[\chi(z)z^j]} \bigoplus \mathcal{D}(\mathcal{D}(\mathcal{D}(\mathcal{O}_\Omega) \xrightarrow{(t_d) \rightarrow \sum c(d)t_d} \bigoplus c(d)z^j, \mu.
\end{align*}
\]

(12.9)

where the direct sums are over \([\delta] \in \pi_0(X_\beta)\) and \(d \in (\mathbb{Z}/p^\beta)\times\). Indeed the top squares and bottom-left square all commute directly from the definitions; and the bottom right square commutes since for any \(\mu \in \mathcal{D}(Z^\alpha_\beta, L)\), we have

\[
\int_{Z^\alpha_\beta} \chi(z)z^j \cdot d\mu = \sum_{d \in (\mathbb{Z}/p^\beta)\times} \chi(d) \int_{d + p^\beta \mathbb{Z}_p} z^j \cdot d\mu.
\]

Now, in line with Remarks 4.14 and 12.7, the proposition follows by combining (12.8) and (12.9).

12.4. \(p\)-adic \(L\)-functions attached to RASCARs

Proposition 12.9. If \(\beta \geq 1\), then \(E_{\Omega, \eta_0}^{\Omega, \eta_0} = Ev_{\beta, \eta_0} : U^\alpha_\beta : H'_c(S_K, \mathcal{O}_\Omega) \rightarrow \mathcal{D}(Z^\alpha_\beta, \mathcal{O}_\Omega)\).

Proof. This follows from Proposition 4.10 (cf. [BSDW, Prop. 6.16]).

Definition 12.10. Let \(\Phi \in H'_c(S_K, \mathcal{O}_\Omega)\) be a \(U^\alpha_\beta\)-eigenclass with eigenvalue \(\alpha^\alpha_\beta\), fix \(\beta \geq 1\), and define

\[
\mu_{\Omega, \eta_0}(\Phi) := (\alpha^\alpha_\beta)^{-\beta} \cdot Ev_{\beta, \eta_0}(\Phi).
\]

By Proposition 12.9, this is independent of the choice of \(\beta\).

Now let \(\tilde{\pi} = (\pi, \alpha)\) be a \(p\)-refined RACAR of weight \(\lambda\) satisfying Conditions 8.1; so it admits an \((\cdot, \psi)-(\cdot)\)-Shalika model, with \(w\) the purity weight of \(\lambda\). In particular, we have \(m_0 = 1\) trivial. By (C4), \(\tilde{\pi}\) is non-critical (Definition 10.5). For \(K\) as in (2.1), let \(\phi^\alpha_\beta \in H'_c(S_K, \mathcal{O}_\Omega)\) as (8.2). By non-criticality, \(\phi^\alpha_\beta\) lifts uniquely to an eigenclass \(\Phi^\alpha_\beta \in H'_c(S_K, \mathcal{O}_\Omega)\) with \(U^\alpha_\beta\)-eigenvalue \(\alpha^\alpha_\beta\), recalling \(\alpha^\alpha_\beta = \lambda(t)\alpha_\beta\).

For \(h \in \mathbb{Q}_{\geq 0}\), recall the notion of \(\mu \in \mathcal{D}(Z^\alpha_\beta, \mathcal{O}_\Omega)\) having growth of order \(h\) [BSDJ, Def. 3.10].

Definition 12.11. Let \(L^\lambda_{\alpha_\beta}(\tilde{\pi}) := \mu^{\lambda, 1}(\Phi^\alpha_\beta) \in \mathcal{D}(Z^\alpha_\beta, L)\). Let \(\Phi^\pm = \Phi^\alpha_\beta \mp \Phi^\beta_\beta\), and define the \(p\)-adic \(L\)-function attached to \(\tilde{\pi}\) to be \(L^\lambda_{\alpha}(\tilde{\pi}) = \mu^{\lambda, 1}(\Phi^\pm) = L^\lambda_{\alpha}(\tilde{\pi}) + L^\lambda_{\alpha}(\tilde{\pi})\). Let \(\mathcal{X} := (Spf Z^\alpha_\beta \times \mathbb{Q}_p)\). The Amice transform allows us to consider \(L^\lambda_{\alpha}(\tilde{\pi}) : \mathcal{X} \rightarrow \overline{\mathbb{Q}_p}\) as an element of \(\mathcal{O}(\mathcal{X})\).
Theorem 12.12. The distribution $L_p(\tilde{\pi})$ has growth order $h_p = v_p(\alpha_p^\pm)$. For every finite order character $\chi$ of $Q^\times \setminus A^\times$ of conductor $p^\beta$ with $\beta \in \mathbb{Z}_{\geq 0}$, and all $j \in \text{Crit}(\lambda)$, we have

$$L_p(\tilde{\pi}, \chi(z)z^j) := \int_{\mathbb{Z}_p^2} \chi(z)z^j \cdot dL_p(\tilde{\pi}) = \gamma(pm) \cdot e_p(\tilde{\pi}, \chi, j) \cdot \zeta_j(W_2^\pm) \cdot \frac{L(\pi \otimes \chi, j + \frac{1}{2})}{\Omega_2^\pm},$$

where $\chi(-1)(-1)^j = \pm 1$. Here if $\chi \neq 1$ we define

$$e_p(\tilde{\pi}, \chi, j) := \left( \frac{p^{\nu_j + \frac{n^2 - n}{2}}}{\alpha_{p,n}} \right)^{\beta} \tau(\chi)^n,$$

whilst if $\chi = 1$ we let $\alpha_{i,j} = \theta_i(p)/p^{j+1/2}$, and define

$$e_p(\tilde{\pi}, 1, j) = \prod_{i=n+1}^{2n} \frac{1 - p^{-1} \alpha_{i,j}^{-1}}{1 - \alpha_{i,j}}.$$

All further notation is as in Theorem 4.16 and Corollary 8.2.

Proof. For ramified characters $\chi$, this is analogous to [BSDW, Thm. 6.23]. A difference is as follows: via the methods op. cit., we get a factor of $(\alpha_p^\pm)^{-\beta}$. Recall that in Corollary 8.2 we had a factor $(\alpha_p^\pm/\alpha_{p,n})^2$; so these combine to leave only $(\alpha_{p,n})^{-\beta}$.

This leaves $\chi = 1$. We will complete the proof in this case in Proposition 14.3.

Remark 12.13. The factor $e_p(\tilde{\pi}, \chi, j)$ is consistent with the Coates–Perrin-Riou conjecture from [Coa89]; this is mostly explained in [AG94, §3]. We expect, but do not prove, that $\zeta_j(W_2^\pm)$ recovers the factor at infinity in [Con99]; similar questions are studied in [Jau]. We note that for GL$_4$, this could be proved in full by via comparison to the $p$-adic $L$-functions for GSp$_4$ from [LPSZ] (where the factor at infinity is known in full), via the same strategy as [LW, §9.5]. Finally, the global scalar $\gamma(pm)$ can be removed by absorbing into $\Omega_2^\pm$.

13. Shalika families and $p$-adic $L$-functions

We fix a sufficiently large coefficient field $L/Q_p$ and drop it from most of the notation. Let $\tilde{\pi}$ be a $p$-refined RACAR of weight $\lambda_\pi$ satisfying (C1-4) of Conditions 8.1. Recall $K = \text{Iw}_G \prod_{\ell \neq p} \text{GL}_{2n}(\mathbb{Z}_\ell)$.

13.1. Existence and étaleness of Shalika families

This entire subsection is dedicated to the proof of Theorem 13.6 below, in which we will use our evaluation maps to construct Shalika families. The proof closely follows the proofs of [BSDW, Thms. 7.6.8.14].

Definition 13.1. Define $T_{\Omega, \leq h}$ to be the image of $H \otimes \mathcal{O}_\Omega$ in $\text{End}_{\Omega}(H_\ell^\pm(S_K, \mathcal{D}_\Omega)^{\leq h})$. Define a rigid space $\mathcal{E}_{\Omega, \leq h} = \text{Sp}(T_{\Omega, \leq h})$, a rigid analytic space.

Let $w : \mathcal{E}_{\Omega, \leq h} \rightarrow \Omega$ be the weight map induced by the structure map $\mathcal{O}_\Omega \rightarrow T_{\Omega, \leq h}$. Also write $T_{\Omega, h}^\pm$ and $\mathcal{E}_{\Omega, h}^\pm$ for the analogues using $\pm$-parts of the cohomology. By [JN19, Thm. 3.2.1], $\mathcal{E}_{\Omega, h}^\pm$ embeds as a closed subvariety of $\mathcal{E}_{\Omega, h}$, and $\mathcal{E}_{\Omega, h} = \mathcal{E}_{\Omega, h}^+ \cup \mathcal{E}_{\Omega, h}^-$.

By definition, $\mathcal{E}_{\Omega, \leq h}$ is a rigid space whose $L$-points $y$ biject with non-trivial homomorphisms $T_{\Omega, \leq h} \rightarrow L$, i.e. with systems of eigenvalues of $\psi_y : H \rightarrow L$ appearing in $H_\ell^\pm(S_K, \mathcal{D}_\Omega)^{\leq h}$.
Definition 13.2. A point \( y \in \mathcal{E}_{\Omega,h} \) is classical if there exists a cohomological automorphic representation \( \pi_y \) of \( G(\mathbb{A}) \) of weight \( \lambda_y = w(y) \) such that \( \psi_y \) appears in \( \pi_y \), whence \( \pi_y = (\pi_y, \alpha_y) \) is a \( p \)-refined RACAR (where \( \alpha_y = \psi_y/\mu_y \)). A classical point \( y \) is cuspidal if \( \pi_y \) is. A \((1, \psi)\)-Shalika point is a classical cuspidal point \( y \) such that \( \pi_y \) admits an \((I, \psi, \nu, \psi)\)-Shalika model, for \( \nu_y \) the purity weight of \( \lambda_y \).

A classical family in \( \mathcal{E}_{\Omega,h} \) is an irreducible component \( \mathcal{F} \) in \( \mathcal{E}_{\Omega,h} \) containing a Zariski-dense set of classical points. A \((1, \psi)\)-Shalika family is a classical family containing a Zariski-dense set of \((1, \psi)\)-Shalika points.

Since \( \tilde{\tau} \) satisfies Conditions 8.1, it is strongly non-critical by (C4) and Theorem 10.7. Let \( \Lambda = \mathcal{O}_{\Omega,\lambda_\tau} \) be the algebraic localisation of \( \mathcal{O}_\Omega \) at \( \lambda_\tau \).

Lemma 13.3. We have (Hecke-equivariant) isomorphisms

\[
H^\pm_c(S_K, \mathcal{D}_\Omega) \otimes \Lambda/m_\lambda \cong H^\pm_c(S_K, \mathcal{D}_\lambda) \oplus H^\pm_c(S_K, \mathcal{D}_\lambda)^0 \cong H^\pm_c(S_K, \mathcal{Y}_\lambda^\gamma)^0.
\]

Proof. The first isomorphism is proved identically to [BSDW, Prop. 7.8] (then applying \( \pm \)-projectors). The second follows from non-criticality of \( \tilde{\tau} \).

Corollary 13.4. (i) There exist ideals \( I_\pm^\pm \subset \Lambda \) such that \( H^\pm_c(S_K, \mathcal{D}_\Omega) \cong \Lambda/I_\pm^\pm \).

(ii) Possibly shrinking \( \Omega \), there exist ideals \( I_\pm^\pm \subset \mathcal{O}_\Omega \) such that \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \cong \mathcal{O}_\Omega/I_\pm^\pm \).

Proof. (i) By Proposition 8.3, the right-hand side in Lemma 13.3 is a line. Since \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \cong H^\pm_c(S_K, \mathcal{D}_\lambda)^0 \) is finite over \( \Lambda \), we may use Nakayama’s lemma, whence \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \) is generated by one element over \( \Lambda \); but every cyclic \( \Lambda \)-module has the form \( \Lambda/I_\pm^\pm \) for some \( I_\pm^\pm \).

(ii) This follows from rigid delocalisation of (i) (cf. [BSDW, Prop. 8.16]).

Proposition 13.5. Suppose \( \lambda_{\tau,n} > \lambda_{\tau,n+1} \). Then, up to shrinking \( \Omega \):

(i) \( T^\pm \) is free of rank one over \( \mathcal{O}_\Omega \).

(ii) \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \) is free of rank one over \( T^\pm \).

Proof. By the weight condition, as in [BSDW, Lem. 7.4], there exist \( \beta \geq 1 \), \( j \in \text{Crit}(\lambda_\tau) \) and Hecke characters \( \lambda^\pm \) of conductor \( p^d \) with \( \chi^\pm (-1)^j = \pm 1 \) such that \( L(\pi \otimes \chi^\pm, j + \frac{1}{2}) \neq 0 \).

As the \( \mathcal{C}^\pm \) are connected components, there exist idempotents \( e^\pm \) such that \( T^\pm = e^\pm T_{\Omega,h} \).

and \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \cong H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \). Restricting \( \text{Ev}^\pm_{B,\beta} \) from (12.6), we get

\[
\text{Ev}^\pm_{B,\beta} : H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \rightarrow D(\mathbb{Z}_p^\times, \mathcal{O}_\Omega).
\]

Using Corollary 13.4, let \( \Phi^\pm_\mathcal{C} \) be a generator of \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \) over \( \mathcal{O}_\Omega \). Note that \( \text{Ann}_{\mathcal{O}_\Omega}(\Phi^\pm_\mathcal{C}) = I_\mathcal{C}^\pm \). Then via the interolation of Theorem 12.12, we have

\[
\int_{\mathbb{Z}_p^\times} \lambda^\pm(z) z^j \cdot d\text{Ev}^\pm_{B,\beta} = (* ) \cdot L(\pi \otimes \chi^\pm, j + \frac{1}{2}) \neq 0,
\]

where \( * \) is non-zero. Thus \( \text{Ev}^\pm_{B,\beta}(\Phi^\pm_\mathcal{C}) \neq 0 \). Since \( D(\mathbb{Z}_p^\times, \mathcal{O}_\Omega) \) is a torsion-free \( \mathcal{O}_\Omega \)-module, it follows that \( \text{Ann}_{\mathcal{O}_\Omega}(\Phi^\pm_\mathcal{C}) = 0 \) (cf. [BSDW, Prop. 7.11]). Thus \( I_\mathcal{C}^\pm = 0 \), and \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \cong \mathcal{O}_\Omega \).

We deduce \( T^\pm \) is the image of \( \mathcal{H} \) in \( \text{End}_{\mathcal{O}_\Omega} \). Since this image is non-zero we deduce (i). Finally since the actions of \( \mathcal{O}_\Omega \) and \( T^\pm \) are compatible on \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \), and both \( T^\pm \) and \( H^\pm_c(S_K, \mathcal{D}_\Omega)^0 \otimes T_{\Omega,n}^\pm \) are free rank one \( \mathcal{O}_\Omega \)-modules, we deduce (ii).
We finally arrive at the main result of this section.

**Theorem 13.6.** Let $\tilde{\pi}$ be a $p$-refined RACAR of weight $\lambda_\pi$ satisfying Conditions 8.1. Suppose $\lambda_{\pi,n} > \lambda_{\pi,n+1}$. Then:

(i) There exists a point $x_\pi \in \mathcal{E}_{\Omega,h}$ attached to $\tilde{\pi}$, and $w : \mathcal{E}_{\Omega,h} \to \Omega$ is étale at $x_\pi$.

(ii) The connected component $\mathcal{C} = \text{Sp}(T)$ in $\mathcal{E}_{\Omega,h}$ through $x_\pi$ contains a Zariski-dense set $\mathcal{C}_{nc}$ of classical points corresponding to $p$-refined RACARs $\tilde{\pi}_y$.

(iii) There exist Hecke eigenvalues $\Phi^\pm_\ell \in H^1_c(S_K, \mathcal{D}_\Omega)^\pm$ such that for every $y \in \mathcal{C}_{nc}$, the specialisation $\text{sp}_{\lambda_\pi}(\Phi^\pm_\ell)$ generates $H^1_c(S_K, \mathcal{D}_{\lambda_\pi})^\pm$, where $\lambda_y := w(y)$.

(iv) Up to shrinking $\Omega$, for each $y \in \mathcal{C}_{nc}$ the $p$-refined RACAR $\tilde{\pi}_y$ satisfies Conditions 8.1.

**Proof.** (i) First consider the ±-analogues. Lemma 13.3 and Proposition 8.3 show $m_\pi$ appears in $H^1_c(S_K, \mathcal{D}_\Omega)^\pm$, giving points $x_\pi^\pm \in \mathcal{E}_{\Omega,h}^\pm$. Moreover Proposition 13.5 shows $\mathcal{E}_{\Omega,h}^\pm \to \Omega$ is étale at $x_\pi^\pm$. Using strong non-criticality of $\tilde{\pi}$, we deduce that $\mathcal{E}_{\Omega,h}^\pm$ contain Zariski-dense sets $\mathcal{C}_{nc}^\pm$ of cuspidal non-critical slope classical points by [BSW21, Prop. 5.15]. Now, as in [BSDW, Prop. 8.20], we can exhibit a bijection between $\mathcal{C}_{nc}^\pm$ and $\mathcal{C}_{nc}$ and (via [JN19, Thm. 3.2.1]) a canonical isomorphism $\mathcal{C}^+ \simeq \mathcal{C}^-$, whence $\mathcal{C} \cong \mathcal{C}^+ \cong \mathcal{C}^-$ is independent of sign. Part (i) follows immediately.

(ii) We let $\mathcal{C}_{nc} = \mathcal{C}_{nc}^+ = \mathcal{C}_{nc}^-$ be the set used in (i).

(iii) Let $\Phi_\ell^\pm$ be $\mathcal{O}_{T_\ell}$-module generators of $H^1_c(S_K, \mathcal{D}_\Omega)^\pm \otimes \mathbb{T}_{\Omega,h}^\pm$. $T^\pm \subset H^1_c(S_K, \mathcal{D}_\Omega)^\pm$, which are well-defined Hecke eigenvalues by Proposition 13.5. For each $y \in \mathcal{C}_{nc}$, let $m_y \subset T^\pm$ be the attached maximal ideal. Reduction modulo $m_y$ induces a map

$$\text{sp}_{\lambda_y} : H^1_c(S_K, \mathcal{D}_\Omega)^\pm \otimes \mathbb{T}_{\Omega,h}^\pm \to H^1_c(S_K, \mathcal{D}_{\lambda_y})^\pm,$$

which is surjective by combining étaleness of $w$ at $y$ with Lemma 13.3. By Proposition 13.5, we deduce $H^1_c(S_K, \mathcal{D}_{\lambda_y})^\pm$ is a line, generated by $\text{sp}_{\lambda_y}(\Phi_\ell^\pm)$.

(iv) Every $y \in \mathcal{C}_{nc}$ has non-critical slope, hence satisfies (C4). Recall $\pi_\ell^K \cong H^1_c(S_K, \mathcal{F}_\chi^\pm)_y$. By (3.2). As in (iii), the right-hand side is a line (using non-criticality), so $\pi_\ell^K \neq 0$. For each $\ell \neq p$, this ensures $\pi_\ell^{G(Z)} \neq 0$, so $\pi_\ell$ is spherical, giving (C3).

Now, let $\beta, j, \chi^\pm$ be as in the proof of Proposition 13.5, and define a map

$$\text{Ev}_{\beta,\chi,\lambda}^\Omega : H^1_c(S_K, \mathcal{D}_\Omega)^\pm \to \mathcal{O}_\Omega$$

$$\Phi \mapsto \int_{Z_{\chi}} z^j \cdot d\text{Ev}_{\beta,\chi,\lambda}^\Omega(\Phi).$$

As in the proof of Proposition 13.5, we have $\text{Ev}_{\beta,\chi,\lambda}^\Omega(\Phi_\ell^\pm)(\lambda_\pi) = (\ast) \times L(\pi \otimes \chi^\pm, j + \frac{1}{2}) \neq 0$. Possibly shrinking $\Omega$, we may thus suppose $\text{Ev}_{\beta,\chi,\lambda}^\Omega(\Phi_\ell^\pm)$ is everywhere non-vanishing on $\Omega$. Let

$$\phi_x^\pm := r_{\lambda_y} \circ \text{sp}_{\lambda_y}(\Phi_\ell^\pm) \in H^1_c(S_K, \mathcal{D}_\chi^\pm)_y.$$

Then by Proposition 12.8, we have

$$\mathcal{E}^{\chi,1}(\phi_x^\pm) = \int_{Z_{\chi}} z^j \left[ \text{Ev}_{\beta,\chi,\lambda}^\Omega \circ \text{sp}_{\lambda_y}(\Phi_\ell^\pm) \right] = \text{Ev}_{\beta,\chi,\lambda}^\Omega(\Phi_\ell^\pm)(\lambda_\pi) \neq 0.$$

We deduce $\pi_\ell$ satisfies (C1) by Proposition 4.15, i.e. $\pi_\ell$ admits a $(| \cdot |^{\langle \psi^\vee, \psi \rangle})$-Shalika model.

It remains to show (C2). Firstly, since failure to be spherical at $p$ is a closed condition in the eigenspace, we can remove the points of $\mathcal{C}_{nc}$ corresponding to $\pi_\ell$ with $\pi_{y,p}$ is not spherical, and it remains Zariski-dense; so assume $\pi_{y,p}$ is spherical. Let $\alpha_{y,p} = \alpha_{y,p}(U_{y,p})$ be the $U_{y,p}$-eigenvalue of $\phi_x^\pm$. As $H^1_c(S_K, \mathcal{D}_\chi^\pm)_y$ is a line, by (3.2) again we deduce that $\mathcal{E}^{\chi,1}(\phi_x^\pm)(\pi_{y,p} - \alpha_{y,p}) : r = 1, \ldots, 2n - 1$ is a line, so $\pi_{y,p}$ is a regular p-refinement. Let $W_{y,p}$ be a generator; then we can relate $W_{y,p}(w_{\psi,\lambda}^{-1})$ to a non-zero multiple of $\text{Ev}_{\beta,\chi,\lambda}^\Omega(\phi_x^\pm)$ exactly as in the proof of [BSDW, Prop. 8.22]. In particular, $W_{y,p}(w_{\psi,\lambda}^{-1}) \neq 0$, so $\pi_\ell$ satisfies (C2).

\[\square\]
13.2. $p$-adic $L$-functions in Shalika families

We finally give our main construction, of the variation of $p$-adic $L$-functions of RASCARs in pure weight families. Let $\tilde{\pi}$ of weight $\lambda_\pi$ satisfy (C1-4) of Conditions 8.1, and suppose $\lambda_{\pi,n} > \lambda_{\pi,n+1}$. By Theorem 13.6(i), the eigenvariety for $G$ is étale over weight space at $\tilde{\pi}$, and by Theorem 13.6(iv) its connected component $\mathcal{C}$ through $\tilde{\pi}$ contains a Zariski-dense set $\mathcal{C}_{nc}$ of classical points satisfying Conditions 8.1. Let $\Phi_{\tilde{\pi}}^\pm \in \mathcal{H}^i_c(S_K, \mathcal{D}_\Omega)^\pm$ be the classes of Theorem 13.6(iii). Possibly rescaling by $\mathcal{O}_{\Omega_\pi}$, we may always assume $\text{sp}_{\lambda_\pi}(\Phi_{\tilde{\pi}}^\pm) = \Phi_{\tilde{\pi}}^\pm$.

**Definition 13.7.** Let $\mathcal{L}_p^{\Phi,\pm} := \mu^{\Omega,1}(\Phi_{\tilde{\pi}}^\pm)$. Also let $\Phi_y = \Phi_{\tilde{\pi}}^+ + \Phi_{\tilde{\pi}}^- \in \mathcal{H}^i_c(S_K, \mathcal{D}_{\Omega_{\pi}})$, a Hecke eigenclass, and define the $p$-adic $L$-function over $\mathcal{C}$ to be

$$\mathcal{L}_p^{\Phi}(y, \pm) \in \mathcal{D}(\text{Gal}_p, \mathcal{O}_{\Omega_{\pi}}).$$

Via the Amice transform (cf. Definition 12.11), after identifying $\mathcal{C}$ with $\Omega$ via $w$ we consider $\mathcal{L}_p^{\Phi}$ as a rigid function $\mathcal{C} \times \mathcal{X} \rightarrow \overline{\mathbb{Q}}_p$.

**Theorem 13.8.** Let $y \in \mathcal{C}_{nc}$ be a classical cuspidal point attached $p$-refined RACAR $\tilde{\pi}_y$ satisfying Conditions 8.1. There exist $p$-adic periods $c_{\tilde{\pi}_y}^\pm \in L^x$ such that

$$\mathcal{L}_p^{\Phi}(y, \mp) = c_{\tilde{\pi}_y}^\pm \cdot \mathcal{L}_p^{\Phi}(\tilde{\pi}_y, -)$$

as functions $\mathcal{X} \rightarrow \overline{\mathbb{Q}}_p$. In particular, $\mathcal{L}_p^{\Phi}$ satisfies the following interpolation: for any $j \in \text{Crit}(w(y))$, and for any finite-order Hecke character $\chi$ of conductor $p^d > 1$, we have

$$\mathcal{L}_p^{\Phi}(y, \chi(z)j) = c_{\tilde{\pi}_y}^\pm \cdot A^\pm(\tilde{\pi}_y, \chi, j) \cdot L(\pi_y \times \chi, j + \frac{1}{2})/\Omega_{\pi_y}^\pm,$$

(13.1) where $\chi(-1)(-1)^j = \pm 1$ and $A^\pm(-)$ is as defined in Theorem 12.12. Finally we have $c_{\tilde{\pi}_y}^\pm = 1$.

The ‘$p$-adic periods’ $c_{\tilde{\pi}_y}^\pm$ $p$-adically align the natural algebraic structures in $\{\tilde{\pi}_y : y \in \mathcal{C}_{nc}\}$.

**Proof.** Let $y$ be as in the theorem. Using that $y$ satisfies Conditions 8.1, let $W_{y,f} \in S_{K,\Omega_{\pi}}^0((\pi_y)^f)$ be as defined as in (8.1), and pick complex periods $\Omega_{\pi_y}^\pm$ as in §8. Since $y \in \mathcal{C}$ is defined over $L$, as in §8, there exists a class

$$\phi_y^\pm := \Theta(\Phi_y^\pm)/i_p(\Omega_{\pi_y}^\pm) \in \mathcal{H}^i_c(S_K, \mathcal{D}_\Omega(L))^\pm.$$ 

As $y$ is non-critical, we can lift $\phi_y^\pm$ to a non-zero class $\Phi_y^\pm \in \mathcal{H}^i_c(S_K, \mathcal{D}_\Omega(L))^\pm$. By Theorem 13.6(iii), this space is $L \cdot \text{sp}_{\lambda_\pi}(\Phi_y^\pm)$, so there exists $c_{\tilde{\pi}_y}^\pm \in L^x$ such that

$$\text{sp}_{\lambda_\pi}(\Phi_y^\pm) = c_{\tilde{\pi}_y}^\pm \cdot \Phi_y^\pm.$$ 

By definition, $\mathcal{L}_p^{\Phi}(\tilde{\pi}_y) = \mathcal{E}_{\lambda_\pi,1}(\Phi_y^\pm)$. As evaluation maps commute with weight specialisation (Proposition 12.8), we deduce $\text{sp}_{\lambda_\pi}(\mathcal{L}_p^{\Phi,\pm}) = c_{\tilde{\pi}_y}^\pm \cdot \mathcal{L}_p^{\Phi}(\tilde{\pi}_y)$, which when combined with Theorem 12.12 gives (13.1). Finally, our normalisation of $\Phi_y^\pm$ ensures $c_{\tilde{\pi}_y}^\pm = 1$. 

14. Comparison to existing constructions

We finally show that the $p$-adic $L$-functions we’ve constructed at Iwahori level agree with previous constructions, and deduce their interpolation property at unramified characters.

If $\tilde{\pi}$ is a non-critical regular $p$-refinement (to Iwahori level), let $\tilde{\pi}^Q := (\pi,\alpha_{p,n})$. This is a non-$Q$-critical $Q$-regular $Q$-refinement as in [BSDW, §2.7,§3.5], and Theorem 6.23 op. cit. attaches a $p$-adic $L$-function $L_p(\tilde{\pi}^Q) \in \mathcal{D}(\mathbb{Z}_p^*, L)$ to $\tilde{\pi}^Q$. 

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**Proposition 14.1.** There exists a constant $\Upsilon \in \mathbb{Q}$, independent of $\pi$, such that

$$L_p(\tilde{\pi}) = \Upsilon \cdot L_p(\tilde{\pi}^Q).$$

**Proof.** First suppose

$$v_p(\alpha^2_{p,n}) < \#\text{Crit}(\lambda_p),$$

where $\lambda_p$ is the weight of $\pi$. In this case, both distributions have sufficiently small growth that they are uniquely determined by their interpolation properties. Their respective interpolation properties agree exactly except for:

- Volume terms at $p$ (the term denoted $\gamma$ in [DJR20, BSDW] and $\gamma(pm)$ in the present paper). These explicit factors differ by a rational number $\Upsilon$ independent of $\pi$.
- The terms at infinity.

One can check, via a long and tedious comparison of conventions, that these factors agree. Alternatively, we can deduce this directly by appealing to the same strategy as [LW, §10.4]: if $v_p(\alpha^2_{p,n}) = 0$, then both $L_p(\tilde{\pi})$ and $L_p(\tilde{\pi}^Q)$ are measures, and rescaling the various arbitrary choices (e.g. the classes $\Xi_\infty^\pm$ from §3.3, hence the periods $\Omega^\pm$) we can dictate the terms at $\infty$ agree on finite order characters (i.e. when $j = 0$). In particular, the ratio of the two measures is constant, and we then deduce the terms at infinity agree for all higher $j$. But the terms at infinity are independent of the slope $v_p(\alpha^2_{p,n})$.

In general, when we drop assumption (14.1), we instead argue in families (using that points satisfying (14.1) are Zariski-dense).

Since $\tilde{\pi}$ is non-critical, Theorem 13.8 gives a $p$-adic $L$-function $L^C_p \in \mathcal{D}(\text{Gal}_p, \mathcal{O}_\Omega)$ in an Iwahoric Shalika family through $\tilde{\pi}$. Let $\Omega^Q$ denote the intersection of $\Omega$ with the (2-dimensional) $Q$-parahoric weight space through $\lambda_e$ (see [BSDW, Def. 3.4]), and let $\mathcal{C}^Q := \mathcal{C} \cap w^{-1}(\Omega^Q)$ be the resulting $Q$-parabolic Shalika family. This contains a Zariski-dense set of points $y$ where (14.1) is satisfied (by the same argument as Theorem 13.6(iv)). For these points, we have shown that $L_p(\tilde{\pi}_y) = \Upsilon \cdot L_p(\tilde{\pi}^Q)$.

It follows that the restriction of $L^C_p$ to $\mathcal{C}^Q$ satisfies the conditions of [BSDW, Prop. 8.28]; and by that Proposition, we see that we have

$$L^C_p(x_{\tilde{\pi}}, -) = \Upsilon \cdot L_p(\tilde{\pi}^Q).$$

(Note that the function $\mathcal{B}^e$ in the proof op. cit. is constant and equal to $\Upsilon$ here, so this result is on the nose, not just up to scalar). By Theorem 13.8 we also have

$$L^C_p(x_{\tilde{\pi}}, -) = \mathcal{L}_0(\tilde{\pi}).$$

The result follows.

**Remark 14.2.** That the $p$-adic $L$-function depends only on the $Q$-refinement, not the full Iwahori refinement, should be expected; it is predicted by the Panchiskin condition from [Pan94].

Now let $\tilde{\pi}^Q$ be any non-$Q$-critical $Q$-refinement of $\pi$, and let $L_p(\tilde{\pi}^Q)$ be the $p$-adic $L$-function attached by [BSDW, Thm. 6.23]. The following strengthens the results of [DJR20, BSDW].

**Proposition 14.3.** The $p$-adic $L$-function $L_p(\tilde{\pi}^Q)$ satisfies the interpolation

$$L_p(\tilde{\pi}, z^j) = \Upsilon^{-1} \cdot \gamma(pm) \cdot e_p(\tilde{\pi}, 1, j) \cdot \zeta_j(W^\pm_{\infty}) \cdot \frac{L(\pi, j + \frac{1}{2})}{\Omega^\pm_{\pi}},$$

for all $j \in \text{Crit}(\lambda_e)$. Here $(-1)^j = \pm 1$ and all other notation is as in Theorem 12.12.

In particular, $L_p(\tilde{\pi})$ satisfies the interpolation of Theorem 12.12 at $\chi = 1$.
Proof. Let $\phi_{Q \mathfrak{O}}^\pm$ be the class defined in [BSDW, \S 6.6]. Then, taking $\beta = 1$, we have

$$L_p(\bar{\pi}, z^2) = (\alpha_{p,n}^2)^{-1} \cdot \mathcal{E}_{Q,1}^{\mathfrak{O}}(\phi_{Q \mathfrak{O}}^\pm)$$

$$= \Upsilon_Q \cdot \frac{p^{a_z^2}}{\alpha_{p,n}^2} \cdot \zeta_p(W_{\infty}^\pm) \cdot \frac{L(\pi, j + \frac{1}{2})}{\Omega_{p,\pm}^j} \cdot \zeta_p\left(j + \frac{1}{2}, (u^{-1}\lambda_{p,n}^3) \cdot W_p, 1\right).$$

Here the first equality is shown in [BSDW, Thm. 6.23]. The second is Theorem 4.16, noting that $\delta_B(t_{Q \mathfrak{O}}^{-1}) = p^{a_z^2}$ and $\lambda(t_Q)/\alpha_{p,n}^2 = 1/\alpha_{p,n}^2$. This local zeta integral was computed in Proposition 9.3. We find that this exactly agrees with the claimed formula (as by definition, $\Upsilon$ tracks the level).

The final statement follows immediately from the first part and Proposition 14.1. \hfill \square

Remark 14.4. Exactly the same proof shows more generally that the $p$-adic $L$-functions of [DJR20, BSDW], for $GL_{2n}$ over a general totally real field and at arbitrary tame level, satisfy the interpolation formula at unramified characters predicted by Coates–Perrin-Riou and Panchishkin.

References

[AG94] Avner Ash and David Ginzburg. $p$-adic $L$-functions for GL(2n). Invent. Math., 116(1-3):27–73, 1994.

[AS06] Mahdi Asgari and Freydoon Shahidi. Generic transfer for general spin groups. Duke Math. J., 132(1):137–190, 2006.

[AS14] Mahdi Asgari and Freydoon Shahidi. Image of functoriality for general spin groups. Manuscripta Math., 144(3-4):609–638, 2014.

[Asg02] Mahdi Asgari. Local $L$-functions for split spinor groups. Canad. J. Math., 54(4):673–693, 2002.

[Ash80] Avner Ash. Non-square-integrable cohomology of arithmetic groups. Duke Math. J., 47(2):435–449, 1980.

[BFG92] Daniel Bump, Solomon Friedberg, and David Ginzburg. Whittaker-orthogonal models, functoriality, and the Rankin-Selberg method. Invent. Math., 109(1):55–96, 1992.

[BSDJ] Daniel Barrera Salazar, Mladen Dimitrov, and Andrei Jorza. $p$-adic $L$-functions of Hilbert cusp forms and the trivial zero conjecture. J. Euro. Math. Soc. To appear. DOI: 10.4171/JEMS/1165.

[BSDW] Daniel Barrera Salazar, Mladen Dimitrov, and Chris Williams. On $p$-adic $L$-functions for GL(2n) in finite slope Shalika families. Preprint: https://arxiv.org/abs/1903.10907.

[BSW19] Daniel Barrera Salazar and Chris Williams. $p$-adic $L$-functions for GL_2. Canad. J. Math., 71(5):1019–1059, 2019.

[BSW21] Daniel Barrera Salazar and Chris Williams. Parabolic eigenvarieties via overconvergent cohomology. Math. Z., 299(1-2):961–995, 2021.

[Cas80] W. Casselman. The unramified principal series of $p$-adic groups. I. The spherical function. Compositio Math., 40(3):387–406, 1980.

[Che04] Gaetan Chenevier. Familles $p$-adiques de formes automorphes pour GL_n. J. Reine Angew. Math., 570:143–217, 2004.

[Clo90] Laurent Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In Automorphic forms, Shimura varieties, and $L$-functions, Vol. I (Ann Arbor, MI, 1988), volume 10 of Perspect. Math., pages 77–159. Academic Press, Boston, MA, 1990.

[Coa89] John Coates. On $p$-adic $L$-functions attached to motives over $Q$. II. Bol. Soc. Brasil. Mat. (N.S.), 20(1):101–112, 1989.

[DJ] Mladen Dimitrov and Andrei Jorza. Parahoric level $p$-adic $L$-functions for automorphic representations of $GL(2n)$ with Shalika models. In preparation.

[DJR20] Mladen Dimitrov, Fabian Januszewski, and A. Raghuram. $L$-functions of GL(2n): $p$-adic properties and nonvanishing of twists. Compositio Math., 156(12):2437–2468, 2020.

[FJ93] Solomon Friedberg and Hervé Jacquet. Linear periods. J. Reine Angew. Math., 443:91–139, 1993.

[GR14] Harald Grobner and A. Raghuram. On the arithmetic of Shalika models and the critical values of $L$-functions for $GL_n$. Amer. J. Math., 136(3):675–728, 2014. With an appendix by Wee Teck Gan.

[Han17] David Hansen. Universal eigenvarieties, trianguline Galois representations and $p$-adic Langlands functoriality. J. Reine Angew. Math., 730:1–64, 2017.

[HS16] Joseph Hundley and Eitan Sayag. Descent construction for GSpin groups. Mem. Amer. Math. Soc., 243(1148):v+124, 2016.

[Jan] Fabian Januszewski. On period relations for automorphic $L$-functions II. Preprint: https://arxiv.org/abs/1604.04253.
On the GL_{2n} eigenvariety

Barrera Salazar, Dimitrov, Graham, Jorza and Williams

[JN19] Christian Johansson and James Newton. Irreducible components of extended eigenvarieties and interpolating Langlands functoriality. Math. Res. Lett., 26(1):159–201, 2019.

[JS90] Hervé Jacquet and Joseph Shalika. Exterior square \( L \)-functions. In Automorphic forms, Shimura varieties, and \( L \)-functions, Vol. II (Ann Arbor, MI, 1988), volume 11 of Perspect. Math., pages 143–226. Academic Press, Boston, MA, 1990.

[Kat04] Kazuya Kato. \( p \)-adic Hodge theory and values of zeta functions of modular forms. Astérisque, 295:117–290, 2004.

[Loe22] David Loeffler. Spherical varieties and norm relations in Iwasawa theory. J. Théor. Nombres Bordeaux, 34(3.2):669–733, 2022. (Iwasawa 2019 special issue).

[LPSZ] David Loeffler, Vincent Pilloni, Chris Skinner, and Sarah Livia Zerbes. Higher Hida theory and \( p \)-adic \( L \)-functions for \( \text{GSp}(4) \). Duke Math. J. To appear: https://arxiv.org/abs/1905.08779.

[LS] David Loeffler, Chris Skinner, and Sarah Livia Zerbes. Euler systems for \( \text{GSp}(4) \). J. Eur. Math. Soc. To appear: https://arxiv.org/abs/1706.00201.

[LW] David Loeffler and Chris Williams. \( p \)-adic \( L \)-functions for \( \text{GL}(3) \). Preprint.

[LZa] David Loeffler and Sarah Livia Zerbes. Iwasawa theory for quadratic Hilbert modular forms. Preprint: https://arxiv.org/abs/2006.14491.

[LZb] David Loeffler and Sarah Livia Zerbes. On the Bloch–Kato conjecture for \( \text{GSp}(4) \). Preprint: https://arxiv.org/abs/2003.05960.

[LZc] David Loeffler and Sarah Livia Zerbes. On the Bloch–Kato conjecture for the symmetric cube. J. Euro. Math. Soc. To appear: https://arxiv.org/abs/2003.05960.

[OST] Masao Oi, Ryotaro Sakamoto, and Hiroyoshi Tamori. New expression of unramified local \( L \)-functions by certain Hecke operators. Preprint: https://arxiv.org/abs/1903.07613.

[Pan94] Alexei A. Panchishkin. Motives over totally real fields and \( p \)-adic \( L \)-functions. Ann. Inst. Fourier (Grenoble), 44(4):989–1023, 1994.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa Main Conjectures for \( \text{GL}(2) \). Invent. Math., 195(1):1–277, 2014.

[Sun19] Binyong Sun. Cohomologically induced distinguished representations and cohomological test vectors. Duke Math. J., 168(1):85–126, 2019.

[Urb11] Eric Urban. Eigenvarieties for reductive groups. Ann. of Math. (2), 174(3):1685–1784, 2011.