Nanhua Xi

Kazhdan–Lusztig basis and a geometric filtration of an affine Hecke algebra, II

Received February 15, 2008 and in revised form November 25, 2008

Abstract. An affine Hecke algebra can be realized as an equivariant K-group of the corresponding Steinberg variety. This gives rise naturally to some two-sided ideals of the affine Hecke algebra by means of the closures of nilpotent orbits of the corresponding Lie algebra. In this paper we will show that the two-sided ideals are in fact the two-sided ideals of the affine Hecke algebra defined through the two-sided cells of the corresponding affine Weyl group after the two kinds of ideals are tensored by $\mathbb{Q}$. This proves a weak form of a conjecture of Ginzburg proposed in 1987.

0. Introduction

Let $H$ be an affine Hecke algebra over the ring $\mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in an indeterminate $v$ with integer coefficients. The affine Hecke algebra has a Kazhdan–Lusztig basis. The basis has many remarkable properties and plays an important role in representation theory. Also, Kazhdan and Lusztig and Ginzburg gave a geometric realization of $H$, which is the key to the proof by Kazhdan and Lusztig of the Deligne–Langlands conjecture on classification of irreducible modules of affine Hecke algebras over $\mathbb{C}$ at non-roots of 1. This geometric construction of $H$ has some two-sided ideals defined naturally by means of the nilpotent variety of the corresponding Lie algebra. The two-sided ideals form a nice filtration of the affine Hecke algebra. In [G2] Ginzburg conjectured that the two-sided ideals are in fact the two-sided ideals of the affine Hecke algebra defined through two-sided cells of the corresponding affine Weyl group (see also [L6, T2]). The conjecture is known to be true for the trivial nilpotent orbit $\{0\}$ (see Corollary 8.13 in [L6] and Theorem 7.4 in [X1]) and for type $A$ [TX]. Other evidence is showed in [L6, Corollary 9.13]. We will prove the two kinds of two-sided ideals coincide after they are tensored by $\mathbb{Q}$ (see Theorem 1.5 in Section 1). This proves a weak form of Ginzburg’s conjecture.

1. Affine Hecke algebra

1.1. Let $G$ be a simply connected simple algebraic group over the complex number field $\mathbb{C}$. The Weyl group $W_0$ acts naturally on the character group $X$ of a maximal torus $\mathcal{T}$.
of $G$. The semidirect product $W = W_0 \ltimes X$ with respect to this action is called an (extended) affine Weyl group. Let $H$ be the associated Hecke algebra over the ring $A = \mathbb{Z}[v, v^{-1}]$ (we think of $v$ as an indeterminate) with parameter $v^2$. Thus $H$ has an $A$-basis $\{T_w \mid w \in W\}$ and its multiplication is defined by the relations $(T_s - v^2)(T_s + 1) = 0$ if $s$ is a simple reflection and $T_wT_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, where $l$ is the length function of $W$.

1.2. Let $g$ be the Lie algebra of $G$, $\mathcal{N}$ the nilpotent cone of $g$, and $B$ the variety of all Borel subalgebras of $g$. The Steinberg variety $Z$ is the subvariety of $\mathcal{N} \times B \times B$ consisting of all triples $(n, b, b')$, $n \in \mathcal{N} \cap b' \cap B$, $b', b \in B$. Let $\Lambda = \{(n, b) \mid n \in \mathcal{N} \cap b, b \in B\}$ be the cotangent bundle of $B$. Clearly $Z$ can be regarded as a subvariety of $\Lambda \times \Lambda$ via the imbedding $Z \rightarrow \Lambda \times \Lambda$, $(n, b, b') \mapsto (n, b, b')$. Define a $G \times \mathbb{C}^*$-action on $\Lambda$ by $(g, z) : (n, b) \mapsto (z^{-2}ad(g)n, ad(g)b)$. Let $G \times \mathbb{C}^*$ act on $\Lambda \times \Lambda$ diagonally; then $Z$ is a $G \times \mathbb{C}^*$-stable subvariety of $\Lambda \times \Lambda$. For $1 \leq i < j \leq 3$, let $p_{ij}$ be the projection from $\Lambda \times \Lambda \times \Lambda$ to its $(i, j)$-factor. Note that the restriction of $p_{13}$ gives rise to a proper morphism $p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z) \rightarrow Z$. Let $K^{G \times \mathbb{C}^*}(Z) = K^{G \times \mathbb{C}^*}(\Lambda \times \Lambda; Z)$ be the Grothendieck group of the category of $G \times \mathbb{C}^*$-equivariant coherent sheaves on $\Lambda \times \Lambda$ with support in $Z$. We define the convolution product

\[ * : K^{G \times \mathbb{C}^*}(Z) \times K^{G \times \mathbb{C}^*}(Z) \rightarrow K^{G \times \mathbb{C}^*}(Z), \quad \mathcal{F} \ast \mathcal{G} = (p_{13})_*(p_{12}^*\mathcal{F} \otimes \mathcal{O}_{\Lambda \times \Lambda}(Z) p_{23}^*\mathcal{G}), \]

where $\mathcal{O}_{\Lambda \times \Lambda \times \Lambda}$ is the structure sheaf of $\Lambda \times \Lambda \times \Lambda$. This endows $K^{G \times \mathbb{C}^*}(Z)$ with an associative algebra structure over the representation ring $R^G_{G \times \mathbb{C}^*}$ of $G \times \mathbb{C}^*$. We shall regard the indeterminate $v$ as the representation $G \times \mathbb{C}^* \rightarrow \mathbb{C}^*$, $(g, z) \mapsto z$. Then $R^G_{G \times \mathbb{C}^*}$ is identified with $A \otimes_{\mathbb{Z}} R^G_G$. In particular, $K^{G \times \mathbb{C}^*}(Z)$ is an $A$-algebra. Moreover, as an $A$-algebra, $K^{G \times \mathbb{C}^*}(Z)$ is isomorphic to the Hecke algebra $H$ (see [G1], [KL2] or [CG], [L2]). We shall identify $K^{G \times \mathbb{C}^*}(Z)$ with $H$.

1.3. Let $C$ and $C'$ be two $G$-orbits in $\mathcal{N}$. We say that $C \leq C'$ if $C$ is in the closure of $C'$. This defines a partial order on the set of $G$-orbits in $\mathcal{N}$. Given a locally closed $G$-stable subvariety of $\mathcal{N}$, we set $Z_Y = \{(n, b, b') \in Z \mid n \in Y\}$.

If $Y$ is closed, then the inclusion $i_Y : Z_Y \rightarrow Z$ induces a map $(i_Y)_* : K^{G \times \mathbb{C}^*}(Z_Y) \rightarrow K^{G \times \mathbb{C}^*}(Z)$ (see [G1], [KL2]). The image $H_Y$ of $(i_Y)_*$ is in fact a two-sided ideal of $K^{G \times \mathbb{C}^*}(Z)$ (see [L4], Corollary 9.13), which is generated by $G \times \mathbb{C}^*$-equivariant sheaves supported on $Z_Y$. It is conjectured that this ideal is spanned by elements in a Kazhdan–Lusztig basis (see [G2], [L2], [L1]).

1.4. Let $C_w = v^{-l(w)} \sum_{y \leq w} P_{y,w}(v^2) T_y$, where $P_{y,w}$ are the Kazhdan–Lusztig polynomials. Then the elements $C_w (w \in W)$ form an $A$-basis of $H$, called a Kazhdan–Lusztig basis of $H$. Define $w \leq_L u$ if $a_w \neq 0$ in the expression $hC_w h' = \sum_{z \in W} a_z C_z (a_z \in A)$ for some $h, h' \in H$. This defines a preorder on $W$. The corresponding equivalence classes are called two-sided cells and the preorder gives rise to a partial order $\leq_L$ on the set of two-sided cells of $W$. (See [KL1].) For an element $w \in W$ and a two-sided cell $c$ of $W$ we shall write $w \leq_L c$ if $w \leq_L u$ for some (equivalent any) $u$ in $c$.

Lusztig established a bijection between the set of $G$-orbits in $\mathcal{N}$ and the set of two-sided cells of $W$ (see [L4], Theorem 4.8). Lusztig’s bijection preserves the partial orders we have defined; this was conjectured by Lusztig and verified by Bezrukavnikov (see [B]).
Theorem 4(b)). Perhaps this bijection is at the heart of the theory of cells in affine Weyl groups; many deep results are related to it. Now we can state the main result of this paper.

**Theorem 1.5.** Let $C$ be a $G$-orbit in $\mathcal{N}$ and $c$ the two-sided cell of $W$ corresponding to $C$ under Lusztig’s bijection. Then the elements $C_w$ ($w \leq_{LR} c$) form a $\mathbb{Q}[v, v^{-1}]$-basis of $H^*_C \otimes_{\mathbb{Z}} \mathbb{Q}$, where $C$ denotes the closure of $C$ and $H^*_C$ is the image of the map $(i_C)_*: K^{G \times C'}(Z_C) \to K^{G \times C'}(Z) = H$. 

**Remark.** In [B] Bezrukavnikov established a closely related result, which involves affine flag manifolds, derived categories and the Springer resolution (see Theorem 4(a) there). Bezrukavnikov’s result deals with canonical left cells and suggests a very nice possible approach to Theorem 1.5. We will discuss this approach in Section 3. I am very grateful to the referee for pointing out this approach.

### 2. Proof of the theorem

#### 2.1. Before proving the theorem we need to recall some results about representations of an affine Hecke algebra. Let $H = \mathbb{C}[v, v^{-1}] \otimes_{\mathcal{A}} H$ and for any nonzero complex number $q$ set $H_q = H \otimes_{\mathbb{C}[v, v^{-1}]} \mathbb{C}$, where $\mathbb{C}$ is regarded as a $\mathbb{C}[v, v^{-1}]$-algebra by specializing $v$ to a square root of $q$.

For any $G$-stable locally closed subvariety $Y$ of $\mathcal{N}$ we set $K_{G \times C'}(Z_Y) = K^{G \times C'}(Z_Y) \otimes \mathbb{C}$. If $Y$ is closed, then the inclusion $i_Y: Z_Y \to Z$ induces an injective map $(i_Y)_*: K^{G \times C'}(Z_Y) \hookrightarrow K^{G \times C'}(Z) = H$. If $Y$ is a closed subset of $\mathcal{N}$, we shall identify $K^{G \times C'}(Z_Y)$ with the image of $(i_Y)_*$, which is a two-sided ideal of $H$. See [KL2] 5.3 or [L6 Corollary 9.13].

Let $s$ be a semisimple element of $G$, and $n$ a nilpotent element in $\mathcal{N}$ such that $ad(s)n = qn$, where $q$ is in $\mathbb{C}^*$. Let $B_n^r$ be the subvariety of $B$ consisting of the Borel subalgebras containing $n$ and fixed by $s$. Then the component group $A(s, n) = C_G(s, n)/C_G(s, n)^+$ of the simultaneous centralizer in $G$ of $s$ and $n$ acts on the total complex homology group $H_*(B_n^r)$. Let $\rho$ be a representation of $A(s, n)$ appearing in the space $H_*(B_n^r)$. It is known that if $w \in W_n, q^{l(w)} \neq 0$ then the isomorphism classes of irreducible representations of $H_q$ are in one-to-one correspondence to the $G$-conjugacy classes of all the triples $(s, n, \rho)$, where $s \in G$ is semisimple, $n \in \mathcal{N}$ satisfies $ad(s)n = qn$, and $\rho$ is an irreducible representation of $A(s, n)$ appearing in $H_*(B_n^r)$. See [KL2] 3.3.

**Remark.** In the proof of this section we shall often use arguments from [KL2] although the setting there is different from ours. In [KL2] equivariant topological K-homology $K_{top}(\_)$ is considered, while we consider equivariant algebraic K-theory $K(\_)$.

We explain why the arguments of [KL2] worked in the present paper. Besides the fact that algebraic K-theory and topological K-theory share many properties (one may compare [KL2] with [Th1] [Th2] [CG]), the key reason is that $K(B_n^r) \simeq K_{top}(B_n^r)$ and $K(B_n^r) \otimes \mathbb{C} \simeq K_{top}(B_n^r) \otimes \mathbb{C}$, as explained in [L5 p. 80]. (The isomorphisms rely on the results in [DL1]). One may see that the properties of $K_{top}(B_n^r) \otimes \mathbb{C}$ play a key role in the arguments of [KL2].

#### 2.2. From now on we assume that $q$ is not a root of 1. Let $L_q(s, n, \rho)$ be an irreducible representation of $H_q$ corresponding to the triple $(s, n, \rho)$. Kazhdan and Lusztig con-
structured a standard module $M(s, n, q, ρ)$ over $H_q$ such that $L_q(s, n, ρ)$ is the unique simple quotient of $M(s, n, q, ρ)$ (see [KL2, 5.12(b) and Theorem 7.12]). We shall write $M_q(s, n, ρ)$ for $M(s, n, q, ρ)$. The following simple fact will be needed.

(a) Let $C$ be a $G$-orbit in $\mathcal{N}$. Then the image $H_C$ of $(i_C)_*$ acts on $M_q(s, n, ρ)$ and $L_q(s, n, ρ)$ by zero if $n$ is not in $\tilde{C}$.

Proof. Clearly $Y = \tilde{C} \cup (G.n - G.n)$ is closed. If $n$ is not in $\tilde{C}$, then the complement in $X = \tilde{C} \cup G.n$ of $Y$ is $G.n$. Recall that $K^{G \times C^+}(Z_Y)$ is regarded as a two-sided ideal of $H$ for any closed subset $Y'$ of $\mathcal{N}$ (see 2.1). According to [KL2, 5.3(c), (d) and (e)], the inclusions $i : Y \hookrightarrow X$ and $j : G.n \hookrightarrow X$ induce an exact sequence of $H$-bimodules

$$0 \to K^{G \times C^+}(Z_Y) \to K^{G \times C^+}(Z_X) \to K^{G \times C^+}(Z_{G.n}) \to 0.$$ 

Using [KL2 5.3(e)] we know the inclusion $k : \tilde{C} \hookrightarrow Y$ induces an injective $H$-bimodule homomorphism $k_* : K^{G \times C^+}(Z_{\tilde{C}}) \to K^{G \times C^+}(Z_Y)$. Since $M_q(s, n, ρ)$ is a quotient module of $K^{G \times C^+}(Z_{G.n})$ (cf. proof of 5.13 in [KL2]), the statement (a) then follows from the exact sequence above.

2.3. Let $J_c$ be the based ring of a two-sided cell $c$ of $W$, which has a $\mathbb{Z}$-basis $\{t_w \mid w \in c\}$. Let $D_c$ be the set of distinguished involutions in $c$. For $x, y \in W$, we write $C_x C_y = \sum_{w \in W} h_{x,y,z} C_z$, $h_{x,y,z} \in \mathcal{A}$. The map

$$\varphi_c(C_w) = \sum_{d \in D_c, u \in W} h_{w,d,u} a_u, \quad w \in W,$$

defines an $\mathcal{A}$-algebra homomorphism $H \to J_c \otimes_{\mathbb{Z}} \mathcal{A}$, where $a : W \to \mathbb{N}$ is the $a$-function defined in [L1 2.1]. The homomorphism $\varphi_c$ induces a $\mathcal{C}$-algebra homomorphism $\varphi_c : H_q \to J_c = I_c \otimes_{\mathbb{Z}} \mathcal{C}$. If $E$ is a $J_c$-module, then through $\varphi_c$, $E$ gets an $H_q$-module structure, which will be denoted by $E_q$. See [L2 L3].

Let $C$ be the nilpotent orbit corresponding to $c$. According to [L4] Theorems 4.2 and 4.8, the map $E \to E_q$ defines a bijection between the isomorphism classes of simple $J_c$-modules and the isomorphism classes of standard modules $M_q(s, n, ρ)$ with $n$ in $C$. The following fact will be needed.

(a) Let $c$ be a two-sided cell of $W$ and $C$ the corresponding nilpotent class. Let $M_q(s, n, ρ)$ be a standard module with $n$ in a nilpotent class $\mathcal{C}'$. If $C_w M_q(s, n, ρ) \neq 0$ for some $w \in c$, then $\mathcal{C}' \subseteq \tilde{C}$.

Proof. Let $c'$ be the two-sided cell corresponding to $\mathcal{C}'$. Then $M_q(s, n, ρ)$ is isomorphic to $E_q$ for some simple $J_{c'}$-module $E$. Thus $C_w M_q(s, n, ρ) \neq 0$ implies that $\varphi_{c'}(C_w) E \neq 0$. So $h_{w,d,u} \neq 0$ for some distinguished involution $d \in c'$ and some $u \in c'$. We then have $c' \subseteq_{LR} c$. By [L3 Theorem 4(b)] we know that $\mathcal{C}' \subseteq \tilde{C}$. The statement is proved.

Now we start to prove Theorem 1.5.
2.4. We first show that $H_C$ is contained in the two-sided ideal $H \leq^C$ of $H$ spanned by all $C_w (w \leq_{LR} C)$.

Let $C = G.n$ and recall that $H_C$ stands for the image of $(i_C)_* : K^{G \times C^*}(Z_C) \rightarrow K^{G \times C^*}(Z) = H$. If $H_C$ were not contained in the $A$-submodule $H \leq^C$ of $H$, we could find $x \in W$ such that $x \notin_{LR} C$ and $C_x$ appears in $H_C$. (We say that $C_x$ appears in $H_C$ if there exists an element $\sum_{w \in W} a_w C_w (a_w \in A)$ in $H_C$ such that $a_x \neq 0$.) Choose $x \in W$ such that $C_x$ appears in $H_C$, $x \notin_{LR} C$ and $x$ is highest with respect to the preorder $\leq_{LR}$ and to $H_C$ in the following sense: whenever $C_w$ appears in $H_C$, then either $w$ and $x$ are in the same two-sided cell or $x \notin_{LR} w$. Let $c'$ be the two-sided cell containing $x$. We then have $c' \notin_{LR} C$.

Choose an element $h = \sum_{w \in W} a_w C_w (a_w \in A)$ in $H_C$ such that $h_{c'} = \sum_{w \in c'} a_w C_w$ is nonzero. We have $\phi_c(h) = \phi_c(h_{c'})$.

We claim that $\phi_c(h_{c'})$ is nonzero. Let $u \in c'$ be such that $a_u$ has the highest degree (as a Laurent polynomial in $v$) among all $a_w, w \in c'$. Let $d$ be the distinguished involution such that $d$ and $u$ are in the same left cell. It is known that for any distinguished involution $d'$, the degree $h_{w,d',u}$ is less than the degree of $h_{u,d,u}$ if either $w \neq u$ or $d' \neq d$ (see [L2, Theorems 1.8 and 1.10]). Thus the degree of $a_u h_{w,d',u}$ is less than the degree of $a_u h_{u,d,u}$ if either $w \neq u$ or $d' \neq d$. Hence $\phi_c(h_{c'})$ is nonzero.

Clearly, there are only finitely many $q$ such that $\phi_{c',q}(h_{c'})$ is zero after specializing $v$ to a square root of $q$. According to [BO] Theorem 4], the ring $J_{c'}$ is semisimple, that is, its Jacobson radical is zero. So we can find a nonzero $q$ in $C$ of infinite order and a simple $J_{c'}$-module $E'$ such that $\phi_{c',q}(h) = \phi_{c',q}(h_{c'})$ is nonzero and its action on $E'$ is nonzero.

According to [L4] Theorems 4.2 and 4.8, $E'_q$ is isomorphic to a standard module $M_q(s', n', \rho)$ with $n'$ in the nilpotent orbit $C'$ corresponding to $c'$. Since $c' \leq_{LR} C, C'$ is not in the closure of $C$ (see [B] Theorem 4(b)), so by 2.2(a), the image $H_C$ of $(i_C)_*$ acts on $E'_q$ by zero. This contradicts that the action of $\phi_{c',q}(h)$ on $E'$ is nonzero. Therefore $H_C$ is contained in the two-sided ideal $H \leq^C$.

2.5. In this subsection all tensor products are over $\mathbb{Z}$ except when other specifications are given.

Now we show that $H \leq^C \otimes \mathbb{Q}$ is equal to $H_C \otimes \mathbb{Q}$. If $C$ is regular, then $C$ is the whole nilpotent cone and the corresponding two-sided cell $c$ contains the neutral element $e$; in this case, both $H_C$ and $H \leq^C$ are the whole Hecke algebra.

We use induction on the partial order $\leq_{LR}$ in the set of all two-sided cells of $W$. Assume that for all $c'$ with $c \leq_{LR} c'$ and $c' \neq c$, we have $H_{C'} \otimes \mathbb{Q} = H \leq^{c'} \otimes \mathbb{Q}$, where $C'$ is the nilpotent orbit corresponding to $c'$.

We need to show $H_c \otimes \mathbb{Q} = H \leq^C \otimes \mathbb{Q}$. Let $c'$ be a two-sided cell different from $c$ such that $c \leq_{LR} c'$ but there is no two-sided cell $c''$ between $c$ and $c'$, i.e. no $c''$ such that $c \leq_{LR} c'' \leq_{LR} c'$ and $c \neq c'' \neq c'$.

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{C}(v)$. We first show that $\mathbb{F} \otimes_A H_C = \mathbb{F} \otimes_A H \leq^C$. Assume this were not true. Note that $\mathbb{F}$ is isomorphic to $\mathbb{C}$ (noncanonically), so we can apply the results in [KL2]. By 2.4 and induction hypothesis, there would exist $w \in c$ such that $C_w$ is contained in $\mathbb{F} \otimes_A H_C$, but not in $\mathbb{F} \otimes_A H \leq^C$. 
We claim that $C_w$ is not contained in $\mathbb{F} \otimes_A H^F_{C_i-C_j}$. Let $C_{i_j} (i = 1, \ldots, k)$ be nilpotent classes such that $C' - C_j$ is the union of $C_{i_1}, \ldots, C_{i_k}$ and $C_{i_j} \not\subseteq C_{i_j}$ whenever $1 \leq i \neq j \leq k$. By the choice of $C_j$, we have $C = C_i$ for some $i$. It is known that $\mathbb{F} \otimes_A H^F_{C_i} = \mathbb{F} \otimes_A H^F_{C_i-C_j}$ is the sum of all $\mathbb{F} \otimes_A H^F_{C_i}, 1 \leq i \leq k$ (see [KL2, 5.3(e)]).

Since $C \not\subseteq C_i$ for $i \neq 1$, by [B] Theorem 4(b) we know that $C_w$ is not in $H^{\Sigma_1}$, where $c_i$ is the two-sided cell corresponding to $C$. By 2.4 we see that $\mathbb{F} \otimes_A H^F_{C_i} (i \geq 1)$ does not contain $C_w$. Assume that $C_w$ were contained in $\mathbb{F} \otimes_A H^F_{C_i-C_j}$. Then there would exist a subset $J \subseteq \{1, \ldots, k\}$ and $h_i \in \mathbb{F} \otimes_A H^F_{C_i} (i \in J)$ such that $C = \sum_{i \in J} h_i$ for different $i, j$ in $J$. We may choose such a $J$ so that $\sum_{i \in J} i$ is minimal possible. Let $j$ be the largest number in $J$. Then $j > 1$ since $C_w$ is not contained in $\mathbb{F} \otimes_A H^F_{C_i}$ (recall that $C_1 = C$).

Let $C_j$ be a nilpotent class in $C_j$ such that $h_j$ is in $\mathbb{F} \otimes_A H^F_{C_j}$ but not in $\mathbb{F} \otimes_A H^F_{C_i-C_j}$. Thus the image in $M_{C_j} \equiv \mathbb{F} \otimes_A K^{G \times C_i} (Z_{C_j}) = \mathbb{F} \otimes_A H^F_{C_j}/\mathbb{F} \otimes_A H^F_{C_i-C_j}$ is nonzero. According to [KL2, Corollary 5.9], the action of each nonzero element in $\mathbb{F} \otimes_A H^F_{C_j} \setminus \mathbb{F} \otimes_A H^F_{C_i-C_j}$ on $M_C$ is nonzero. The argument for [KL2, Proposition 5.13] implies that each nonzero element in $M_{C_j}$ would have nonzero image in some standard quotient module of $M_{C_j}$. Thus the action of $h_j$ on some standard quotient module $M_{C_j}(s, n_j', \rho')$ of $M_{C_j}$ is nonzero, where $n_j' \in C_j$. Note that $C_j \not\subseteq C_i$ for any $i \in J$ with $i \neq j$ since $h_j$ is not in $\mathbb{F} \otimes_A H^F_{C_i}$ if $i \neq j$. By 2.2(a), $h_i$ acts on $M_{C_j}(s, n_j', \rho')$ by zero if $i \neq j$. So $C_w M_{C_j}(s, n_j', \rho') = h_1 M_{C_j}(s, n_j', \rho') = 0$.

By 2.3(a), we get $C_j \not\subseteq C_i$. This contradicts that $\sum_{i \in J} i$ is minimal and $j > 1$. Therefore $C_w$ is not contained in $\mathbb{F} \otimes_A H^F_{C_i-C_j}$.

Thus the image in $M_C \equiv \mathbb{F} \otimes_A K^{G \times C_i} (Z_C) = \mathbb{F} \otimes_A H^F_{C_j}/\mathbb{F} \otimes_A H^F_{C_i-C_j}$ of $C_w$ is nonzero. According to [KL2, Corollary 5.9], the action of each nonzero element in $\mathbb{F} \otimes_A H^F_{C_i} \setminus \mathbb{F} \otimes_A H^F_{C_i-C_j}$ on $M_C$ is nonzero. The argument for [KL2, Proposition 5.13] implies that each nonzero element in $M_{C_j}$ would have nonzero image in some standard quotient module of $M_{C_i}$. Thus the action of $C_w$ on some standard quotient module $M_{C_j}(s, n_j', \rho)$ of $M_{C_j}$ is nonzero, where $n_j' \in C_j$. According to 2.3(a), we have $C_j \not\subseteq C_i$. By Theorem 4(b) in [B], we get $C_j \leq C_j$. This contradicts our assumption $C_j \neq C \leq C_j$. So we have $\mathbb{F} \otimes_A H^F_{C_j} = \mathbb{F} \otimes_A H^F_{C_j}$.

Thus for each $w \in C$, we can find a nonzero $a \in \mathbb{F}$ such that $a C_w$ is in $H^F_{C_j}$. Clearly, we must have $a \in A$. Now we show that $K^{G \times C_i}(Z_Y)$ is a free $\mathbb{C}[v, v^{-1}]$-module for any $G$-stable locally closed subvariety $Y$ of $N$. According to [KL2, 5.3] we may assume that $Y$ is a nilpotent orbit $C$. It is enough to show that the completion of $K^{G \times C_{i_j}}(Z_Y)$ at any semisimple class in $G \times C_{i_j}$ is free over $\mathbb{C}[v, v^{-1}]$. Using [KL2, 5.6] it is enough to show that the right hand side of 5.6(a) in [KL2] is free. This follows from [KL2 (13)]; the assumption there is satisfied by [KL2, 4.1]. Using [KL2, 5.3] we know that as a free $\mathbb{C}[v, v^{-1}]$-module, $H^F_{C_j}$ is a direct sum of $H^F_{C_j} \otimes \mathbb{C}$ and $K^{G \times C_j}(Z_{C_j-C_i})$. By assumption, $H^F_{C_j} \otimes \mathbb{Q} = H^F_{C_j} \otimes \mathbb{Q}$, thus $H^F_{C_j} \otimes \mathbb{Q}$ is a free $\mathbb{Q}[v, v^{-1}]$-module and contains $C_w$. These imply that if $a C_w$ is in $H^F_{C_j}$ for some nonzero $a \in A$ then $C_w$ is in $H^F_{C_j} \otimes \mathbb{C}$. Therefore we
can find a nonzero complex number $a$ such that $aC_w$ is in $H_G$. Obviously $a \in \mathbb{Z}$. Thus $H^{\leq c} \otimes \mathbb{Q}$ is contained in $H_C \otimes \mathbb{Q}$. By 2.4 we then have $H^{\leq c} \otimes \mathbb{Q} = H_C \otimes \mathbb{Q}$. Theorem 1.5 is proved.

3. An approach based on Theorem 4(a) in [B]

In this section we discuss a nice possible approach to the main result of the present paper based on Theorem 4(a) in [B]: this was suggested by the referee. Let $\Gamma$ be the union of all canonical left cells of $W$, and $I$ the left ideal of $H$ generated by all $C_w$, $w \notin \Gamma$. Then $M = H/I$ is the anti-spherical module. Moreover, the images in $M$ of all $C_w$, $w \in \Gamma$, form a basis of $M$. For each two-sided cell $c$ of $W$, let $M_{\leq c}$ be the submodule of $M$ spanned by the images of all $C_w$, $w \in \Gamma$ and $w \leq_{LR} c$.

According to Arkhipov and Bezrukavnikov (see Subsection 1.1.2 in [AB]), as an $H$-module, $M$ is isomorphic to $K^{G \times C^*}(\Lambda)$ (see Subsection 10.1 in [L6] for the definition of the $H$-module structure on $K^{G \times C^*}(\Lambda)$). Let $C$ be the nilpotent class corresponding to the two-sided cell $c$ under Lusztig’s bijection. Let $\Lambda_C = \{(N, b) \in \Lambda \mid N \in \tilde{C}\}$. Then the inclusion $j_C^0 : \Lambda_C \rightarrow \Lambda$ induces an $H$-module homomorphism $(j_C^0)_* : K^{G \times C^*}(\Lambda_C) \rightarrow K^{G \times C^*}(\Lambda)$. A variation of Theorem 4(a) in [B] implies that the image $\text{Im}(j_C^0)_*$ of $(j_C^0)_*$ is $M_{\leq c}$ if we identify $M$ with $K^{G \times C^*}(\Lambda)$.

Since each left cell in a two-sided cell has a nonempty intersection with any right cell in the same two-sided cell, we see that for a two-sided cell $c$, the two-sided ideal $H^{\leq c}$ of $H$ spanned by all $C_w$ ($w \leq_{LR} c$) is the annihilator of $M/M_{\leq c}$.

Let $\tilde{C}$ be the nilpotent class corresponding to the two-sided cell $c$. Then naturally one hopes to prove that the image $\text{Im}(i_{\tilde{C}})_*$ of the map $(i_{\tilde{C}})_* : K^{G \times C^*}(Z_{\tilde{C}}) \rightarrow K^{G \times C^*}(Z) = H$ coincides with the two-sided ideal $H^{\leq c}$ by using the above characterizations for $M_{\leq c}$ and $H^{\leq c}$. A natural way to reach this coincidence is to prove the following two statements:

(a) $K^{G \times C^*}(\Lambda \setminus \Lambda_C)$ is isomorphic to $K^{G \times C^*}(\Lambda)/\text{Im}(j_C^0)_*$.

(b) If $x \in K^{G \times C^*}(Z)$ annihilates $K^{G \times C^*}(\Lambda)/\text{Im}(j_C^0)_*$, then $x \in \text{Im}(i_{\tilde{C}})_*$.

(a) implies that the image $\text{Im}(i_{\tilde{C}})_*$ is in $H^{\leq c}$, and (b) implies that this image contains $H^{\leq c}$.

Unfortunately, the author has not been able to prove these two statements. See comments in Subsection 4.2 for some ideas.

4. Some comments

4.1. If one can show that $K^{G \times C^*}(Z_{\tilde{C}})$ is a free $\mathbb{Z}$-module for any nilpotent orbit $\tilde{C}$, then the argument in 2.5 shows that the image of $(i_{\tilde{C}})_*$ in $H = K^{G \times C^*}(Z)$ contains $H^{\leq c}$, where $c$ is the two-sided cell corresponding to $\tilde{C}$. Then Ginzburg’s conjecture would be proved. In fact, it seems that one can expect more. More precisely, it is likely the following result is true.
(a) $K^{G \times C^*}(Z_C)$ is a free $A$-module and $K^{G \times C^*}(Z_C) = 0$ for all nilpotent orbits $C$. (We refer to [CG] Section 5.2 and [Q] for the definition of the functor $K^G_i$.)

If (a) is true, then we also have

(b) The map $(i_C)_*: K^{G \times C^*}(Z_C) \to K^{G \times C^*}(Z)$ is injective.

We explain some evidence for (a) and prove it for $G = GL_m(C)$, $Sp_{2m}(C)$ and type $G_2$. Let $N$ be a nilpotent element in $C$, and $B_N$ be the variety of Borel subalgebras of $g$ containing $N$. By the Jacobson–Morozov theorem, there exists a homomorphism $\varphi: SL_2(C) \to G$ such that $d\varphiegin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N$. For $z \in C^*$, let $d_z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$. Following Kazhdan and Lusztig [KL2, 2.4], we define $Q_N = \{(g, z) \in G \times C^* \mid \text{ad}(g)N = z^2N\}$. Then $Q_N$ is a closed subgroup of $G \times C^*$. Let $x = (g, z) \in Q_N$ act on $(G \times C^*) \times B_N \times B_N$ by $x(y, b, b') = (yx^{-1}, \text{ad}(g)b, \text{ad}(g)b')$. Then $Z_C$ is isomorphic to the quotient space $Q_N/(G \times C^*) \times B_N \times B_N$. Thus we have $K^{G \times C^*}(Z_C) = K^{Q_N}_i(B_N \times B_N)$ (see [KL2] 5.5 and [B1] Prop. 6.2). It is known that $Q_\varphi = \{(g, z) \in G \times C^* \mid \varphi(x)g^{-1} = \varphi(d_zxd_z^{-1})\}$ for all $x \in SL_2(C)$ is a maximal reductive subgroup of $Q_N$ (see [KL2] 2.4(d)). So we have $K^{Q_N}_i(B_N \times B_N) = K^{Q_\varphi}_i(B_N \times B_N)$ (see [CG] 5.2.18).

Let $P$ be the parabolic subgroup of $G$ associated to $N$ (see [DLP] 1.12). Then we know that the intersection $B_{N, \Theta}$ of $B_N$ with any $P$-orbit $\Theta$ on $B$ is smooth. The torus $D = \{\varphi(d_z) \mid z \in C^*\}$ is a subgroup of $P$ and acts on $B_{N, \Theta}$, and $B_{N, \Theta}$ is a vector bundle over the $D$-fixed point set $B_{N, \Theta}^D$ (see [DLP] 3.4(d)). Since the action of $Q_\varphi$ on $B_{N, \Theta}$ commutes with the action of $D$, according to [BB], this vector bundle is isomorphic to a $Q_\varphi$-stable subbundle of $T(B_{N, \Theta})|_{B_{N, \Theta}^D}$, where $T(B_{N, \Theta})$ is the tangent bundle of $B_{N, \Theta}$.

Thus the vector bundle is $Q_\varphi$-equivariant, so that the computation of $K^{Q_\varphi}_i(B_N \times B_N)$ is reduced to the computation of $K^{Q_\varphi}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D)$ for various $P$-orbits $\Theta, \Theta'$ on $B$ (see Theorems 2.7 and 4.1 in [B1], or Theorems 5.4.17 and 5.2.14 in [CG]). Note that $C_\varphi = \{\varphi(d_z^{-1}) \mid (g, z) \in Q_\varphi\}$ is a maximal reductive subgroup of the centralizer $C_G(N)$ of $N$ (see [BY] 2.4) and the map $(g, z) \mapsto (\varphi(d_z^{-1}), z)$ defines an isomorphism from $Q_\varphi$ to $C_\varphi \times C^*$. Thus we have $K^{Q_\varphi}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D) = K^{C_\varphi \times C^*}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D)$.

Now the factor $C_\varphi$ and the group $D$ act on $B_{N, \Theta}^D \times B_{N, \Theta}^D$ trivially, we therefore have $K^{Q_\varphi}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D) = K^{C_\varphi}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D) \otimes R_{C_\varphi}$ (see [CG] 5.2.4), the argument there works for higher $K$-groups). Note that we have identified $R_{C_\varphi}$ with $A = Z[v, v^{-1}]$.

Thus the statement (a) is equivalent to the following one.

(c) $K^{C_\varphi}_i(B_{N, \Theta}^D \times B_{N, \Theta}^D)$ is a free $Z$-module for $i = 0$ and is 0 for $i = 1$.

The statement (c) seems much easier to access. The variety $B_{N, \Theta}^D$ and its fixed point set $B_{N, \Theta}^{s, D}$ for any semisimple element $s$ in $C_\varphi$ are smooth and have good homology properties. See [DLP].

4.2. Replacing $Z$ by $\Lambda$, we can state the counterparts of 4.1(a), 4.1(b) and 4.1(c) as follows.
(a) \( K^{G \times C^*}(A_C) \) is a free \( A \)-module and \( K^{G \times C^*}_1(A_C) = 0 \) for all nilpotent orbits \( C \).

If (a) is true, then we have

(b) The map \((i_C)_* : K^{G \times C^*}(A_C) \to K^{G \times C^*}(A) \) is injective.

As in 4.1, the statement (a) is equivalent to the following one:

(c) \( K^{C^*}_1(B^D_{N,\varnothing}) \) is a free \( \mathbb{Z} \)-module for \( i = 0 \) and is 0 for \( i = 1 \).

It is easy to check that the statement (a) implies 3(a). Also (a) is helpful to understand the statement 3(b).

**Proposition 4.3.** The statements 4.1(a) and 4.2(a) are true for \( GL_n(\mathbb{C}) \), \( Sp_4(\mathbb{C}) \) and type \( G_2 \). In particular, Ginzburg’s conjecture is true in these cases.

**Proof.** We only need to prove statements 4.1(c) and 4.2(c). For \( G = GL_n(\mathbb{C}) \), we know that \( B^D_{N,\varnothing} \) has an \( r \)-partition into subsets which are affine space bundles over the flag variety \( B' \) of \( C_\varnothing \) (see Theorems 2.2 and 2.4(a) in [X2]). In this case, 4.1(a) and 4.2(a) are true since we are reduced to computing \( K^{C^*}_i(B' \times B') \) and \( K^{C^*}_i(B') \) (cf. [CG, Lemma 5.5.1] and the argument for [L7, Lemma 1.6]). For \( G = Sp_4(\mathbb{C}) \) or type \( G_2 \), we know that \( B^D_{N,\varnothing} \) is either empty or the flag variety of \( C_\varnothing \) if \( N \) is not subregular (see Prop. 4.2(i) and Section 4.4 in [X2]). In this case, we are also reduced to computing \( K^{C^*}_i(B' \times B') \) and \( K^{C^*}_i(B') \) (loc.cit.), so 4.1(a) and 4.2(a) are true. If \( N \) is subregular, then \( B_N \) is a Dynkin curve and it is easy to see that \( B^D_{N,\varnothing} \) is either a projective line or a finite set (see Prop. 4.2(ii) and Section 4.4 in [X2] for a computable description of \( B_N \)). The computation for \( K^{C^*}_i(B^D_{N,\varnothing} \times B^D_{N,\varnothing}) \) and \( K^{C^*}_i(B^D_{N,\varnothing}) \) is easy, they are free \( \mathbb{Z} \)-modules for \( i = 0 \) (see 4.3(b) and 4.4 in [X2]), and are 0 for \( i = 1 \) (since this is true for a projective line and a finite set). The proposition is proved.

**Remark.** For \( GL_n(\mathbb{C}) \), this proposition also provides another proof for the main result of [TX], where results of [TH] are used.

**Proposition 4.4.** Assume that \( C_\varnothing \) is connected. Then

(a) \( K^{C^*}(B_N \times B_N) \) is a free \( \mathbb{Z} \)-module.

(b) \( Q^*(B_N \times B_N) \) is a free \( A \)-module. That is, \( K^{G \times C^*}(Z_{G,N}) \) is a free \( A \)-module.

**Proof.** Let \( T \) be a maximal torus of \( C_\varnothing \). According to [TH2] (1.11)), we have a split monomorphism \( K^{C^*}(B_N \times B_N) \to K^T(B_N \times B_N) \). Similar to the argument for [L7, Lemma 1.13(d)], we see that \( K^T(B_N \times B_N) \) is a free \( R_T \)-module. (a) follows.

The reasoning for (b) is similar since \( Q_\varnothing \) is isomorphic to \( C_\varnothing \times \mathbb{C}^* \) and the monomorphism \( Q^*(B_N \times B_N) \to K^T \times C^*(B_N \times B_N) \) is split. The proposition is proved.

**Remark.** If \( G = GL_n(\mathbb{C}) \), then all \( C_\varnothing \) are connected and have simply connected derived group. In this case \( K^{C^*}(B_N \times B_N) \) is a free \( R_{Q_\varnothing} \)-module since \( R_{Q_\varnothing} = R_{C_\varnothing} \otimes A \) and \( R^T \otimes C^* \) is a free \( R_{C_\varnothing} \otimes A \)-module. Combining this, Subsection 2.4 and the argument in Subsection 2.5 we obtain a different proof of the main result in [TX].
4.5. The $K$-groups $K^F(B_N)$ and $K^F(B_N \times B_N)$ are important in representation theory of affine Hecke algebras for $F$ being $Q$, $C\phi$ or a torus of $Q\phi$ (see [KL1],[L7]). For the nilpotent element $N$, in [L4] 10.5) Lusztig conjectured that there exists a finite $C\phi$-set $Y$ which plays a key role in understanding the based ring of the two-sided cell corresponding to $G.N$. It seems that as $RC\phi$-modules, $KC\phi(Y)$ and $KC\phi(Y \times Y)$ are isomorphic to $KC\phi(B_N)$ and $KC\phi(B_N \times B_N)$ respectively. Let $X = B_N$ or $B_N \times B_N$. In view of [L4] 10.5) one may hope to find a canonical $Z$-basis of $KC\phi(X)$ and a canonical $A$-basis of $KQ\phi(X)$ in the spirit of [L6],[L7]. Moreover, there should exist a natural bijection between the elements of the canonical basis of $K^F(B_N \times B_N)$ ($F = C\phi$ or $Q\phi$) and the elements of the two-sided cell corresponding to $G.N$.

Acknowledgments. I thank Professor G. Lusztig for very helpful correspondence and for providing the argument for the freeness of $K\times C\phi(Z_C)$ over $C[v, v^{-1}]$. I am grateful to Professor T. Tanisaki for helpful correspondence and to Professor Jianzhong Pan for a helpful conversation. I am indebted to the referee for very helpful comments and an insightful suggestion.

This research was partially supported by Natural Sciences Foundation of China (No. 10671193).

References

[AB] Arkhipov, S., Bezrukavnikov, R.: Perverse sheaves on affine flags and Langlands dual groups. Israel J. Math. 170, 135–183 (2009) Zbl pre05601711 MR 2506322
[BV] Barbasch, D., Vogan, D.: Unipotent representation of complex semisimple groups. Ann. of Math. 121, 41–110 (1985) Zbl 0582.22007 MR 0782558
[B] Bezrukavnikov, R.: Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group. Israel J. Math. 170, 185–206 (2009) Zbl pre05601712 MR 2506323
[BO] Bezrukavnikov, R., Ostrik, V.: On tensor categories attached to cells in affine Weyl groups, II. In: Representation Theory of Algebraic Groups and Quantum Groups, Adv. Stud. Pure Math. 40, Math. Soc. Japan, Tokyo, 101–119 (2004) Zbl 1078.20045 MR 2074591
[BB] Białynicki-Birula, A.: Some theorems on actions of algebraic groups. Ann. of Math. 98, 480–497 (1973) Zbl 0275.14007 MR 0366940
[CG] Chriss, N., Ginzburg, V.: Representation Theory and Complex Geometry. Birkhäuser Boston, Boston, MA (1997) Zbl 0871.22001 MR 1433132
[CL] De Concini, C., Lusztig, G., Procesi, C.: Homology of the zero-set of a nilpotent vector field on a flag manifold. J. Amer. Math. Soc. 1, 15–34 (1988) Zbl 0924700
[G1] Ginzburg, V.: Lagrangian construction of representations of Hecke algebras. Adv. Math. 63, 100–112 (1987) MR 0871082
[G2] Ginzburg, V.: Geometrical aspects of representation theory. In: Proc. International Congress of Mathematicians, Vol. 1 (Berkeley, CA, 1986), Amer. Math. Soc., Providence, RI, 840–848 (1987) Zbl 0667.14034 MR 0934285
[KL1] Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. 53, 165–184 (1979) Zbl 0499.20035 MR 05060412
[KL2] Kazhdan, D., Lusztig, G.: Proof of the Deligne–Langlands conjecture for Hecke algebras. Invent. Math. 87, 153–215 (1987) Zbl 0613.22004 MR 0862716
[L1] Lusztig, G.: Cells in affine Weyl groups. In: Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math. 6, Kinokuniya and North-Holland, 255–287 (1985) Zbl 0569.20032 MR 0803338
