STOCHASTIC RESONANCE IN SPATIALLY EXTENDED SYSTEMS: THE ROLE OF FAR FROM EQUILIBRIUM POTENTIALS

H. S. Wio, S. Bouzat and B. von Haeften

1) Grupo de Física Estadística
Centro Atómico Bariloche (CNEA) and Instituto Balseiro (CNEA and UNC)
8400 San Carlos de Bariloche, Argentina

2) Universidad Nacional de Mar del Plata, Deán Funes 3350, 7600 Mar del Plata, Argentina.

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Abstract

Previous works have shown numerically that the response of a “stochastic resonator” is enhanced as a consequence of spatial coupling. Also, similar results have been obtained in a reaction-diffusion model by studying the phenomenon of stochastic resonance (SR) in spatially extended systems using nonequilibrium potential (NEP) techniques. The knowledge of the NEP for such systems allows us to determine the probability for the decay of the metastable extended states, and approximate expressions for the correlation function and the signal-to-noise ratio (SNR). Here, exploiting known forms of the NEP, we have investigated the role of NEP’s symmetry on SR, the enhancement of the SNR due to a selectivity of the coupling or diffusion parameter, and discussed competition between local and nonlocal (excitatory) coupling.

*Member of CONICET, Argentina; Electronic Address: wio@cab.cnea.gov.ar
†http://www.cab.cnea.gov.ar/CAB/invbasica/FisEstad/estadis.htm
I. INTRODUCTION

Many recent theoretical and experimental studies have provided a large body of evidence on the essentially constructive role played by fluctuations in a variety of intriguing noise induced phenomena. Some key-examples are: problems of self-organization and dissipative structures, noise induced transitions, noise induced phase transitions, thermal ratchets or Brownian motors, coupled Brownian motors, noise sustained patterns and stochastic resonance [1–3].

This last phenomenon, that is stochastic resonance (SR), has attracted considerable interest due, among other aspects, to its potential technological applications for optimizing the transmission of information in nonlinear dynamical systems. For a comprehensive recent review see Ref. [3], that shows the large number of applications in science and technology, ranging from paleoclimatology, to electronic circuits, lasers, and noise-induced information flow in sensory neurons in living systems, to name a few. Several recent papers have aimed at achieving an enhancement of the output SNR by means of the coupling of several SR units [4,5] in what conforms an “extended medium” [5].

The relevance of pattern-formation phenomena to several areas of science and technology is a very well established fact. Accounts of the state of the art can be found in many reviews and books [1], collecting the results obtained over the last couple of decades regarding the description of pattern formation and propagation phenomena in self-organizing systems.

It is a common belief that the nontrivial spatio-temporal behaviour occurring for instance in the Complex Ginzburg-Landau Equation (CGLE), reaction-diffusion (RD) schemes, and other nonequilibrium systems, originates from the non-potential (or non-variational) character of the dynamics, meaning that there is no Lyapunov functional (LF) for the dynamics. However, Graham and co-workers, have shown that a Lyapunov-like functional does exist for the CGLE [6].

In order to fix ideas, we start by discussing the existance of LF in two different dynamical situations. The simplest case, in which a LF exists, corresponds to a gradient flow system...
such as
\[ \dot{x}_j = -\frac{\partial}{\partial x_j} V(x_1, \ldots, x_n). \] (1)

The fixed points will correspond to the extrema of the “potential function” \( V(x_1, \ldots, x_n) \), and the system will evolve towards these minima of \( V(x_1, \ldots, x_n) \) following trajectories corresponding to the line of *steepest descent*. Clearly \( V(x_1, \ldots, x_n) \) is a LF as it also satisfies
\[ \frac{dV}{dt} = \sum \frac{\partial V}{\partial x_j} \frac{dx_j}{dt} = -\sum \left( \frac{\partial V}{\partial x_j} \right)^2 \leq 0. \] (2)

Now we consider an example corresponding to a non-gradient flow. We take the following case
\[ \dot{x}_j = -\sum (T)_{jl} \frac{\partial V}{\partial x_l}, \] (3)

where \( T \) is an arbitrary, positive definite matrix. We can separate it into a symmetric (\( S \)) and an antisymmetric (\( A \)) part
\[
T = S + A
\]
\[ S = \frac{1}{2}(T + T^T), \quad S = S^T \]
\[ A = \frac{1}{2}(T - T^T), \quad A = -A^T. \] (4)

The fixed points are given by the extrema of \( V \). On the other hand we have that \( V \) also fulfills
\[ \frac{dV}{dt} = \sum (S)_{jl} \frac{\partial V}{\partial x_j} \frac{\partial V}{\partial x_l} - \sum (A)_{jl} \frac{\partial V}{\partial x_j} \frac{\partial V}{\partial x_l} \leq 0, \] (5)
as, clearly, the first term on the rhs is \( \leq 0 \), while the second one is \( = 0 \), hence, \( V \) is a LF.

The system evolves to the minima following trajectories different from the steepest descent ones, determined by \( S \), since the antisymmetric part of \( T \) induces a flow in the system that keeps the LF constant. A thorough (and didactic) discussion on the LF and different kinds of dynamical flows can be found in [9].

The use of the LF concept (as well as the related one of nonequilibrium potential for stochastic systems) in relation to far from equilibrium systems is scarce. However, there are
recent papers that have made use of these ideas in relation not only with CGLE \[7\], but also for reaction-diffusion systems. Some of these works concerned the derivation of effective equations for the evolution of fronts \[11\]; while others were related to the determination of the global stability of the resulting patterns and the possibility of exchanging the relative stability between attractors \[8,10\].

In the next Section we introduce the concept of nonequilibrium potential (NEP). When the NEP can be obtained, such an approach offers an alternative way of confronting a problem that has attracted considerable attention, both experimentally and theoretically. Namely, the relative stability of the different attractors, corresponding to spatially extended states, and the possibility of transitions among them due to the effect of fluctuations \[7\]. The latter aspect which is of great relevance in the study of SR in spatially extended systems, is the objective of this work.

The organization of the paper is as follows. As indicated, in the next section we briefly discuss some basic notions about nonequilibrium potentials, and show a few relevant examples for reaction-diffusion systems. In Section III we present the results for the SNR in some of the previous examples. Section IV contains a final discussion and some conclusions.

II. NONEQUILIBRIUM POTENTIAL

A. Brief Review

Loosely speaking, the notion of non-equilibrium potential (NEP) corresponds to an extension of the notion of equilibrium thermodynamical potential to non-equilibrium situations. In order to introduce such NEP, we consider a general form of non-linear stochastic equations, admitting the possibility of multiplicative noises. In particular, we consider equations of the form

\[
\dot{q}^\nu = K^\nu(q) + g_i^\nu(q) \xi_i(t), \quad \nu = 1, ..., n;
\]  

(6)
where repeated indices are summed over. Equation (6) is stated in the sense of Itô. The \( \{ \xi_i(t), (i = 1, ..., m \leq n) \} \) are mutually independent sources of Gaussian white noise with typical strength \( \eta \). The Fokker-Planck equation corresponding to Eq.(6) takes the form

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial q^\nu} K^\nu(q) P + \frac{\eta}{2} \frac{\partial^2}{\partial q^\nu \partial q^\mu} Q^{\nu\mu}(q) P
\]

where \( P(q,t; \eta) \) is the probability density of observing \( q = (q_1, ..., q_n) \) at time \( t \) for noise intensity \( \eta \), and \( Q^{\nu\mu}(q) = g^\nu_i(q) g^\mu_i(q) \) is the matrix of transport coefficients of the system, which is symmetric and non-negative. In the long time limit \( (t \to \infty) \), the solution of Eq.(7) tends to the stationary distribution \( P_{stat}(q) \). According to [6], \( \Phi(q) \), the NEP associated to Eq.(7), is defined by

\[
\Phi(q) = -\lim_{\eta \to 0} \eta \ln P_{stat}(q, \eta).
\]

In other words

\[
P_{stat}(q) d^nq = Z(q) \exp \left[ -\frac{\Phi(q)}{\eta} + O(\eta) \right] d\Omega_q,
\]

where \( \Phi(q) \) is the NEP of the system and \( Z(q) \) is defined as the limit

\[
\ln Z(q) = \lim_{\eta \to 0} \left[ \ln P_{stat}(q, \eta) + \frac{1}{\eta} \Phi(q) \right].
\]

Here \( d\Omega_q = \frac{d^nq}{\sqrt{G(q)}} \) is the invariant volume element in the \( q \)-space and \( G(q) \) is the determinant of the contravariant metric tensor (for the Euclidean metric it is \( G = 1 \)). It was shown [3] that \( \Phi(q) \) is the solution of a Hamilton-Jacobi like equation (HJE)

\[
K^\nu(q) \frac{\partial \Phi}{\partial q^\nu} + \frac{1}{2} Q^{\nu\mu}(q) \frac{\partial \Phi}{\partial q^\nu} \frac{\partial \Phi}{\partial q^\mu} = 0,
\]

and \( Z(q) \) is the solution of a linear first-order partial differential equation depending on \( \Phi(q) \) (not shown here).

Equation (8) and the normalizability condition ensure that \( \Phi \) is bounded from below. Furthermore, from the separation of the streaming velocity of the probability flow in the steady state into conservative and dissipative parts, it follows that
\[
\frac{d\Phi(q)}{dt} = K^\nu(q) \frac{\partial \Phi(q)}{\partial q^\nu} = - \frac{1}{2} Q^{\mu\nu}(q) \frac{\partial \Phi}{\partial q^\nu} \frac{\partial \Phi}{\partial q^\mu} \leq 0,
\]
i.e. \(\Phi\) is a LF for the dynamics of the system when fluctuations are neglected. Under the deterministic dynamics: \(\dot{q}^\nu = K^\nu(q)\), \(\Phi\) decreases monotonically and takes a minimum value on attractors. In particular, \(\Phi\) must be constant on all extended attractors (such as limit cycles or strange attractors) \cite{6}.

**B. Examples of Nonequilibrium Potentials**

1. **Scalar Reaction-Diffusion Model**

As a first example we focus on a one-dimensional, one-component model of an electrothermal instability \cite{1}, which corresponds to an approximation to the continuous limit of the coupled system studied by Lindner et al \cite{4}. For this model, the effect of boundary conditions (b.c.) in pattern selection, the *global stability* of nonhomogeneous structures, and the critical-like behaviour due to the coalescence of two patterns \cite{8}, have been studied.

The RD model that we work with describes the time evolution of a field \(\phi(x,t)\) which represents the temperature profile in the so-called “hot spot model” in superconducting microbridges (or ballast resistor) \cite{8,10}. The evolution of \(\phi\) is given by

\[
\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} - \phi + \theta[\phi - \phi_c] + \xi(x,t),
\]

where \(\xi(x,t)\) is a white noise in space and time, in the bounded domain \(x \in [-L, L]\) with Dirichlet b.c. at both ends, *i.e.* \(\phi(\pm L, t) = 0\). \(\theta[x]\) is the step function. We restrict our discussion to the parameter range where the (associated deterministic) system has two stable attractors (patterns). The piecewise linear approximation of the reaction term, mimicking a cubic polynomial, was chosen in order to find analytical expressions for the spatially symmetric solutions of Eq. (10). It is clear that the trivial solution \(\phi_0(x) = 0\), which is linearly stable, exists for the whole range of parameters. Besides this solution we find only one stable nonhomogeneous structure, \(\phi_s(x)\), which presents an excited \((\phi_s(x) > \phi_c)\) central
zone, and another similar unstable structure, \( \phi_u(x) \), with a smaller excited central zone. The latter pattern corresponds to the saddle separating both attractors \( \phi_0(x) \) and \( \phi_s(x) \) \[8,10\].

The indicated patterns are minima of the NEP of our system. For the present case, such a NEP reads \[10\]

\[
\Phi[\phi, \phi_c] = \int_{-L}^{+L} \left\{ -\int_0^\phi (-\phi + \theta|\phi - \phi_c|) \, d\phi + \frac{D}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \right\} \, dx .
\]

(11)

It can be shown that \( \frac{\partial \Phi}{\partial \phi} = -\frac{\delta \Phi}{\delta \phi} \) and also \( \dot{\Phi} = -\int \left( \frac{\delta \Phi}{\delta \phi} \right)^2 \, dx \leq 0 \). This functional offers us the possibility to study both the local and global stability of the patterns as well as the changes associated to variations of model parameters. \[8,10\]

In Fig. 1 we depict \( \Phi[\phi, \phi_c] \) evaluated at the stationary patterns \( \phi_0 (\Phi[\phi_0] = 0) \), \( \phi_s(x) \) \( (\Phi^s = \Phi[\phi_s]) \) and \( \phi_u(x) \) \( (\Phi^u = \Phi[\phi_u]) \), for a system size \( L = 1 \), as a function of \( \phi_c \) for a given value of \( D \). The upper branch is the NEP for \( \phi_u(x) \), where \( \Phi \) attains an extremum (as a matter of fact it is a saddle). On the lower branch, for \( \phi_s(x) \), and also for \( \phi_0(x) \), the NEP has local minima. The curves exist up to a certain critical value of \( \phi_c \) at which both branches collapse \[8,10\]. It is interesting to note that, since the NEP for \( \phi_u(x) \) is always positive and, for \( \phi_s(x) \), \( \Phi^s > 0 \) for “large” values of \( \phi_c \) and also \( \Phi^s < 0 \) for “small” values of \( \phi_c \), \( \Phi^s \) vanishes for an intermediate value \( \phi_c = \phi^*_c \), where \( \phi_s(x) \) and \( \phi_0(x) \) exchange their relative stability.

2. Three Component Activator-Inhibitor Model

Here we present an exact form of the nonequilibrium potential for a three component system of the activator-inhibitor type (with one activator and two inhibitors) in a particular parameter region. Such a three component system provides the adequate framework for a minimal model describing pattern formation in high-pressure or low-pressure chemical reactors \[12\]. The system we consider is described by the following set of equations

\[
\frac{\partial u(x,t)}{\partial t} = D \nabla^2 u(x,t) + f(u(x,t)) - v(x,t) - w(x,t) + g_1^u \xi_1(x,t) + g_2^u \xi_2(x,t)
\]
\[
\begin{align*}
\epsilon_1 \frac{\partial v(x, t)}{\partial t} &= \nu_1 \nabla^2 v(x, t) + \beta u(x, t) - \gamma v(x, t) + g_1^v \xi_1(x, t) + g_2^v \xi_2(x, t) \\
\epsilon_2 \frac{\partial w(x, t)}{\partial t} &= \nu_2 \nabla^2 w(x, t) + \beta' u(x, t) - \gamma' w(x, t) + g_1^w \xi_1(x, t) + g_2^w \xi_2(x, t),
\end{align*}
\]

where \( x \) represents an \( n \)-dimensional spatial coordinate. The \( \xi_i(x, t) \)'s are gaussian white-noise sources of zero mean satisfying

\[
\langle \xi_i(x, t) \xi_j(x', t') \rangle = \eta \delta_{ij} \delta(t-t') \delta(x-x'),
\]

where \( \eta \) is again a small parameter measuring the noise intensity. All the parameters and fields shall be considered as dimensionless (scaled) quantities. We shall only consider situations where the noise terms in the third of Eqs. (12) are negligible, and we set \( g_1^w = g_2^w = 0 \).

Furthermore, we will analyze the system in the limit \( \nu_1 = \epsilon_2 = 0 \) with \( \epsilon_1 = 1 \) and \( \nu_2 = \nu \), where for the now *temporally slaved inhibitor* \( w \) we have

\[
w(x, t) = \int dx' G(x, x') u(x', t),
\]

where \( G(x, x') \) is the Green function of the third of Eqs. (12) in the indicated limit [8]. Hence the system can be reduced to an effective two component system with a *nonlocal interaction* [8,12].

When the relation

\[
\gamma = \frac{Q_u \beta + Q_v}{2Q_{uv}},
\]

holds, the two equations for such an effective two component system adopt the form

\[
\begin{pmatrix}
\frac{\partial u(x, t)}{\partial t} \\
\frac{\partial v(x, t)}{\partial t}
\end{pmatrix} = -\mathcal{T} \begin{pmatrix}
\frac{\delta \Phi}{\delta u(x, t)} \\
\frac{\delta \Phi}{\delta v(x, t)}
\end{pmatrix} + \begin{pmatrix}
g_1^v \xi_1(x, t) + g_2^v \xi_2(x, t) \\
g_1^w \xi_1(x, t) + g_2^w \xi_2(x, t)
\end{pmatrix}.
\]

Here the matrix \( \mathcal{T} \), which has the form of the matrix \( \mathcal{T} \) in Eq. (4) is given by

\[
\mathcal{T} = \frac{1}{2} \begin{pmatrix}
Q_u & 2Q_{uv} \\
0 & Q_v
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
Q_u & Q_{uv} \\
Q_{uv} & Q_v
\end{pmatrix} + \begin{pmatrix}
0 & \frac{Q_{uv}}{2} \\
-\frac{Q_{uv}}{2} & 0
\end{pmatrix} = \mathcal{S} + \mathcal{A},
\]

where
and the functional $\Phi[u(x, t), v(x, t)]$ by

$$\Phi[u(x, t), v(x, t)] = \int dx \left[ \frac{D}{Q_u} (\nabla u(x, t))^2 + V(u(x, t), v(x, t)) + \frac{1}{Q_u} \int dx' G(x, x') u(x, t) u(x', t) \right],$$

(18)

where

$$V(u, v) = -2 \frac{Q_{uv} \beta}{Q_u Q_v} u^2 + \frac{\gamma}{Q_v} v^2 - 2 \frac{\beta}{Q_v} uv.$$  

(19)

When the symmetric matrix $S$ is positive definite (when $Q_u Q_v > Q_{uv}^2$ holds), the functional $\Phi$, that in the associated deterministic system decreases monotonously with time, is the NEP of the system in Eqs. (16) satisfying the HJE (Eq. (9)). Equation (15), which resembles a detailed balance condition, arises in this context as a mathematical constraint necessary for $\Phi$, as defined by Eq. (18), to be the solution of the HJE above mentioned.

We limited the analysis to the parameter region where Eq. (15) is valid, the matrix $S$ is positive definite, and hence the nonequilibrium potential is given by Eq. (18). Although these conditions impose restrictions on the range of application of our treatment, it is worth noting that, after choosing the values of the $g_i^{\nu}$'s satisfying the condition of positive definiteness, there is still a wide spectrum of interesting situations to analyze [12,13]. Furthermore the nonlinear function $f(u)$ is still arbitrary, opening a wealth of possibilities. When plotting $\Phi$ vs. $\phi_c$, with $\Phi$ evaluated on the stationary patterns, we see a result similar to the one shown in Fig. 1.

III. STOCHASTIC RESONANCE IN EXTENDED MEDIA

A. Preliminaries

The calculation of the SNR proceeds, for the spatially extended problem, through the evaluation of the space-time correlation function $\langle \phi(y, t) \phi(y', t') \rangle$. To do that we have used a simplified point of view, based on the two-state approach [14], which allows us to apply some known results almost directly. We consider a random system described by a discrete
dynamical variable $x$ adopting two possible values: $c_1$ and $c_2$, with probabilities $n_{1,2}(t)$ respectively. Such probabilities satisfy the condition $n_1(t) + n_2(t) = 1$. The equation governing the evolution of $n_1(t)$ (with a similar one for $n_2(t) = 1 - n_1(t)$) is

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt} = W_2(t)n_2(t) - W_1(t)n_1(t) = W_2(t) - [W_2(t) + W_1(t)]n_1,$$  \hspace{1cm} (20)

where the $W_{1,2}(t)$ are the transition rates out of the $x = c_{1,2}$ states. For the bistable extended system in which we are interested, such states correspond to the spatially extended attractors (see Refs. [5,16] for details).

If the system is subject (through one of its parameters) to a time dependent signal of the form $A\cos(\omega_s t)$, up to first order in the amplitude (assumed to be small) the transition rates may be expanded as

$$W_1(t) = \mu_1 - \alpha_1 A \cos(\omega_s t)$$

$$W_2(t) = \mu_2 + \alpha_2 A \cos(\omega_s t),$$  \hspace{1cm} (21)

where the constants $\mu_{1,2}$ and $\alpha_{1,2}$ depend on the detailed structure of the system under study. Here we remark that the $\mu_i$'s, which are the (time independent) values of the $W_i$'s without signal, are in general different from each other as a consequence of the different stability of the two states, and the same happens to the $\alpha_i$’s. With the indicated modulation the system becomes nonstationary but we make an adiabatic assumption considering small signal frequencies that makes the NEP valid at each time for the corresponding value of the signal.

Using Eq. (21) we can integrate Eq. (20) with the initial condition $x(t_0) = x_0$ and obtain the conditional probability $n_1(t \mid x_0, t_0)$. This result allows us to calculate the autocorrelation function, the power spectrum and finally the SNR. The details of the calculation were shown in Ref. [5,16]. When the symmetrical case is considered all the results reduce to those in [14]. For the SNR, up to the relevant (second) order in the signal amplitude $A$, we find for the SNR the result [16]

$$\mathcal{R} = \frac{A^2\pi(\alpha_2 \mu_1 + \alpha_1 \mu_2)^2}{4\mu_1 \mu_2 (\mu_1 + \mu_2)},$$  \hspace{1cm} (22)
In order to evaluate the transition rates between both states we discretize the space and the fields as
\[ x \rightarrow x_i, \quad (u(x), v(x)) \rightarrow (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_N, \tilde{v}_1, ..., \tilde{v}_N) \] (23)
and use the Kramers-like formula [15]
\[ W_{U_i \rightarrow U_j} \equiv W_i = \frac{\lambda_+}{2\pi} \sqrt{|\Phi''_i|} \exp \left[ -\frac{(\Phi_m - \Phi_i)}{\eta} \right], \] (24)
where \( \lambda_+ \) is the unstable eigenvalue of the deterministic flux at the unstable state \( U_m \), \( \Phi''_i \) and \( \Phi''_m \) indicate the determinants of the matrix of second order derivatives of the NEP with respect to the discretized fields in the states \( U_i \) and \( U_m \) respectively, and \( \Phi_i \) and \( \Phi_m \) are the values of the NEP evaluated at the stationary states \( U_i \) and \( U_m \), \( i = 1, 2 \). Finally, in order to compute the SNR as indicated above, we calculate the parameters \( \mu_i \) and \( \alpha_i \) numerically as
\[ \mu_i = W_i \big|_{S(t)=0}; \quad \alpha_i = -\frac{dW_i}{dS(t)} \big|_{S(t)=0}. \] (25)

B. Role of the NEP symmetry

To fix ideas we consider the system given by Eq. (16) in one dimension, with the spatial coordinate \( x \) varying between \(-L \) and \( L \), and assuming Dirichlet boundary conditions for the three fields. We adopt the following piecewise linearized form for the nonlinear function \( f(u) \)
\[ f(u) = -u + \theta(u - a) \] (26)
where \( \theta(u) \) is the step function. We fix \( g_1^u = 0, g_2^u = .02, g_1^v = .01 \) and \( g_2^v = 1 \). This leads us to a situation in which we have essentially only one noise source \( \xi_2 \) acting on \( v \). With this choice we have \( Q_u \ll Q_v \) and \( \gamma \) results approximately independent of \( \beta \).

The piecewise linearized form of \( f(u) \) allows us to calculate analytically the stationary inhomogeneous patterns of the associated deterministic system as linear combinations of
We search for solutions symmetric with respect to \( x = 0 \), with a central activated region \((u > a)\) surrounded by a non activated region \((u < a)\). Depending on the values of \( a, D, \beta \) and \( \beta' \) we find four (two stable and two unstable), two (one stable and one unstable) or zero stationary inhomogeneous solutions. Furthermore the homogeneous null solution \((u = v = w = 0)\) always exists and is stable. We call these solutions \( s_1 \) and \( ns_1 \) (existing in the regions were there are two or four inhomogeneous solutions), \( s_2 \) and \( ns_2 \) (existing only in the region of four solutions) and \( s_0 \) (the null solution). The \( s_i \)'s are the stable solutions and the \( ns_i \)'s are the unstable ones.

We will focus our analysis on a region of parameters where the system has two stationary stable patterns (stationary linearly stable solutions of Eqs. (12) for \( u, v, \) and \( w \)) and one stationary unstable pattern (stationary linearly unstable solution of Eqs. (12)). In Fig. 3 of Ref. [12] the \( u \)-fields for the three patterns for a particular choice of the parameters were shown. We call \( U_1 \) the large stable pattern which has a central activated region \((u > a)\), \( U_2 \) the small stable pattern which reduces to the homogeneous null solution when \( S \) is set equal to zero, and \( U_m \) the unstable pattern. A complete study of the pattern formation of this system can be found in Ref. [12].

In the region of only two stable patterns we are considering, the deterministic dynamics given by Eqs. (16) drives the system toward one of the patterns (selected depending on the initial condition) which is reached asymptotically. If small fluctuations are present in the system the fields fluctuate around one of the stable patterns and transitions between the two patterns become possible. Note that the \( g_i'\) in Eqs. (16) are constants that couple the noises to the system while the intensity of the fluctuations is determined by the parameter \( \eta \).

It is worth mentioning that the NEP given in Eq. (18) is valid for the system in Eq. (18) for arbitrary number of spatial dimensions, for an arbitrary nonlinear function \( f(u) \), and with the parameter region of validity being independent of the choice of \( f(u) \) [12]. The consideration of only one spatial dimension and the particular election of \( f(u) \) are in order to simplify the calculations, particularly regarding pattern formation. The signal is introduced
as a (slow) modulation in a parameter $S$ by setting $S = S(t) = A \cos(\omega_s t)$ added to the autocatalytic function $f(u)$.

We now analyze the SR phenomenon in our spatially extended system using the theory discussed before. To proceed with such an analysis we identify the two stable patterns ($U_1$ and $U_2$) with the states of the two-state-theory. Hence, the discrete variable $x$ will adopt values $c_1$ and $c_2$ according to whether the system is in the states $U_1$ or $U_2$, yielding the result for the SNR in Eq. (22). The changes induced in the patterns and their stability by the variation of some model parameter will be reflected in changes in the values of $\mu_i$ and $\alpha_i$ and, accordingly, will affect the results for the SNR.

We fix $L = 1, \beta = \beta' = 1, \gamma = 10.026, \gamma' = \nu = 10, g_1^u = 1, g_2^u = 0, g_1^v = .05$ and $g_2^v = .01$, and leave $D, a$ and $\eta$ (the noise intensity) as free parameters. Note that with the chosen values for the $g_i^\mu$’s, the only relevant noise term in the system (Eq. (16)) is $g_1^u \xi_1(t)$ in the equation for $u$ that appears added to the signal (hence it can be considered as coming together with the signal). The parameters $g_1^v$ and $g_2^v$ are set different from zero to keep the system inside the parameter region where $\Phi[u, v]$ as defined in Eq. (18) is valid as a nonequilibrium potential [16].

In Fig. 2 we show the results for the SNR ($R$) as a function of the noise intensity for different values of $D$ and $a$. We see that while keeping $a$ constant ($a = .25$) (Fig. 2 a), the largest values of $R$ are those for $D = D_s$, which is the symmetric situation. Also, if we fix $D = D_s$ (Fig. 2 b), any departure of $a$ from the value .25 (that is any departure from the symmetric situation) lowers the values of $R$. Hence, the symmetric situation is found to be the most favorable one concerning the improvement of SNR. Note that the maximum of the $R$ vs. $\eta$ curve (for fixed values of $a$ and $D$), that we will call $R_{max}$, increases with symmetry and reaches its largest value for the symmetric situation. In Fig. 3 we show $R_{max}$ plotted as a function of $D$ for $a = .25$, where it is apparent that the optimum value of diffusion is $D = D_s$ corresponding, as indicated, to the symmetric case.

A fact that arises from these results is that, while keeping all the other parameters of the system fixed, there exists an optimal value of diffusion (coupling of the distributed system)
that maximizes SNR. The interesting aspect is that this optimal value is the one that makes
the potential symmetric.

It is worth mentioning that these results do not contradict but complete those in [3] where
enhancement due to coupling was found, since in that work only symmetric situations were
analyzed. Roughly speaking, the main result in [3] can be summarized saying that, given
two different symmetric situations (each one necessarily having different values of $D$ and
$a$), the one with the higher value of $D$ produces higher values of SNR. However, we must
keep in mind that for a too large value of $D$, some of the approximations involved in the
calculations may break down [3]. Also, it is worth here pointing out that the same thing
may happen for too large asymmetries. For example, consider an extremely asymmetric
situation where the barrier for, say the transition from state $U_1$ to $U_2$, is much larger than
the barrier for the opposite transition. In such a case, the values of the noise intensity
leading to reasonable jumping rates from $U_1$ to $U_2$ will be far beyond the validity of the
Kramers-like approximation for the inverse transition.

C. Enhancement due to selective coupling

The model we discuss next is an extension of the one-component RD model discussed
in Section 2, but now the diffusive parameter depends on the field $\phi(x,t)$. As a matter of
fact, since in the ballast resistor the thermal conductivity is a function of the energy density,
the resulting equation for the temperature field includes a temperature-dependent diffusion
coefficient in a natural way.

By adequate rescaling of the field, space-time variables and parameters, we get a dimen-
sionless time-evolution equation for the field $\phi(x,t)$

$$\partial_t \phi(x,t) = \partial_x (D(\phi) \partial_x \phi) + f(\phi) + \xi(x,t),$$

(27)

where $\xi(x,t)$ is a white noise in space and time, and $f(\phi) = -\phi + \theta(\phi - \phi_c)$, $\theta(x)$ is the step
function. All the effects of the parameters that keep the system away of equilibrium (such
as the electric current in the electrothermal devices or some external reactant concentration in chemical models) are included in $\phi_c$.

As was done for the reaction term $[5,8,10]$, a simple choice that retains however the qualitative features of the system is to consider the following dependence of the diffusion term on the field variable

$$D(\phi) = D_0(1 + h \theta[\phi - \phi_c]),$$

(28)

For simplicity, here we choose the same threshold $\phi_c$ for the reaction term and the diffusion coefficient.

We assume the system to be limited to a bounded domain $x \in [-L, L]$ with Dirichlet boundary conditions at both ends, i.e. $\phi(\pm L, t) = 0$. As before, the piecewise-linear approximation of the reaction term in Eq.(27) was chosen in order to find analytical expressions for its stationary spatially-symmetric solutions. In addition to the trivial solution $\phi_0(x) = 0$ (which is linearly stable and exists for the whole range of parameters) we find another linearly stable nonhomogeneous structure $\phi_s(x)$—presenting an excited central zone (where $\phi_s(x) > \phi_c$) for $-x_c \leq x \leq x_c$—and a similar unstable structure $\phi_u(x)$, which exhibits a smaller excited central zone. The form of these patterns is analogous to what has been obtained in previous related works $[8,10]$. The difference is that in the present case $d\phi/dx|_{x_c}$ is discontinuous and the area of the “activated” central zone depends on $h$.

The indicated patterns are extrema of the NEP. In fact, the unstable pattern $\phi_u(x)$ is a saddle-point of this functional, separating the attractors $\phi_0(x)$ and $\phi_s(x)$. For the case of a field-dependent diffusion coefficient $D(\phi(x, t))$ as described by Eq. (27), the NEP reads $[16]$

$$\Phi[\phi] = \int_{-L}^{+L} \left\{ - \int_{0}^{\phi} D(\phi') f(\phi') \, d\phi' + \frac{1}{2} \left( D(\phi) \frac{\partial \phi}{\partial x} \right)^2 \right\} \, dx.$$  

(29)

Given that $\partial_t \phi = -(1/D(\phi))\delta \Phi/\delta \phi$ one finds $\dot{\Phi} = -\int (\delta \Phi/\delta \phi)^2 \, dx \leq 0$, thus warranting the LF property.

For a given threshold value $\phi_c^*$, both wells corresponding in a representation of the NEP to the linearly stable states have the same depth (i.e. both states are equally stable). Figure
4 shows the dependence of $\Phi[\phi]$ on the parameter $\phi_c$. As in previous cases, we analyze only the neighborhood of $\phi_c = \phi^*_c$ [5]. Here we also consider the neighborhood of $h = 0$, where the main trends of the effect can be captured.

In Fig.5 we depict the dependence of $R$ on the noise intensity $\gamma$, for several (positive) values of $h$. These curves show the typical maximum that has become the fingerprint of the stochastic resonance phenomenon. Figure 6 is a plot of the value $R_{max}$ of these maxima as a function of $h$. The dramatic increase of $R_{max}$ (several dB for a small positive variation of $h$), is apparent and shows the strong effect that the selective coupling (or field-dependent diffusivity) has on the response of the system.

It must be noted that the only two approximations made in order to derive Eq.(22)—namely the Kramers-like expression in Eq.(24) and the two-level approximation used for the evaluation of the correlation function [3,16]—break down for large positive values of $h$ because for increasing selectivity the curves of $\Phi[\phi]$ vs. $\phi_c$ in Fig. 4 shift towards the left, which in turn means that the barrier separating the attractors at $\phi^*_c$ tends to zero. This effect is basically the same as the one discussed in Ref. [5] in connection with global diffusivity $D_0$. It is also worth noting that except for the two aforementioned approximations, all the previous results (e.g. the profiles of the stationary patterns and the corresponding values of the non-equilibrium potential) are analytically exact.

IV. CONCLUSIONS

In this work we have presented the basic ingredients for studying SR in coupled or extended media ($stochastic resonant medium$ [3]) by means of nonequilibrium potential techniques. We have shown some examples of NEP for RD systems and the way we can exploit them to get information about the system’s response (the SNR) to weak potential modulations.

In previous works we have shown, in agreement with numerical simulations [4] how the coupling enhances the system’s response. Here we have shown two other aspects: (a) the
role played by the symmetry of the NEP on such enhancement and; (b) the effect of selective coupling.

In the first case the results we have obtained clearly indicate the central role played by symmetry in improving the SNR. We studied the behavior of $R_{\text{max}}$, that is the maximum of the SNR vs. $\eta$ curve, as the different model parameters are (not simultaneously) varied, finding that $R_{\text{max}}$ always increases with the symmetry of the potential. This fact led us to our main result: the optimal values of the different model parameters (for instance diffusivity or threshold), as regards the maximization of $R_{\text{max}}$, correspond to those making the potential more symmetric in each situation. Besides the analysis of the influence of symmetry on stochastic resonance, it is important to remark that the mere consideration of asymmetric situations has its own relevance. This is because such bistable asymmetric models provide, for example, the appropriate framework for describing SR in voltage–dependent ion channels, as proposed in some biological experiments. In those systems, the conducting state is associated a higher-energy well than the non–conducting one [17].

In the second case, the results of our analysis of a scalar RD system (a generalization of the continuous limit of the model analyzed in [4,5]) indicate that a “selective” coupling (that is a field diffusion dependent constant) could offer a new enhancing mechanism in coupling systems. This prediction prompts to devise experiments (for instance, through electronic setups) as well as numerical simulations taking into account the indicated selective coupling.

One direction in which the present studies can be extended corresponds to the analysis of the competition between local and nonlocal couplings. Consider the same NEP described by Eq. (18), but with only one inhibitor, the temporally slaved $w$. For this system the NEP $\Phi$ has the form

$$
\Phi[u(x, t), v(x, t)] = \int dx \left[ \frac{D}{2} (\nabla u(x, t))^2 + C \int dx' G(x, x') u(x, t) u(x', t) \right],
$$

(30)

where the coupling constant $C$, that in one of the systems studied in [3] was positive (that is due to an inhibition interaction), is now assumed negative (that is an excitatory interaction).

The results of preliminary studies for the case of excitatory coupling, where we used the
same kind of approach indicated above, are shown in Fig. 6. There we have compared the results for the SNR vs. noise intensity for the FitzHugh-Nagumo model with inhibitory interaction, with those obtained in the above indicated case. It is apparent, on one hand, the SNR enhancement due to the excitatory interaction, while on the other, that small changes in the diffusion constant of the activator ("local coupling" constant) strongly affect the SNR. The analysis of the different aspects of this problem will be the subject of a forthcoming paper.

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FIGURES

FIG. 1. NEP Φ, evaluated at the stationary patterns, for Dirichlet b.c., as a function of φ_c, for L = 1 and D = 1. The bottom curve (1) corresponds to φ_s(x) and the top one (3) to φ_u(x). The bistability point φ^*_c, is indicated.

FIG. 2. a) SNR as a function of the noise intensity for a = .25 and different values of D. The solid line corresponds to the symmetric situation D = D_s, the long-dashed line to D = .35 and the short-dashed line to D = .25. b) SNR as a function of the noise intensity for D = D_s and different values of a. The solid line corresponds to the symmetric situation a = .25, the dotted line corresponds to a = .27 and the dot-dashed to a = .23.

FIG. 3. Maximum of SNR (R_{max}) as a function of the activator diffusion D for a = .25. The maximum of R_{max} occurs for the symmetric situation D = D_s.

FIG. 4. SNR (R) as a function of the noise intensity (here indicated by γ), for three values of h: h = 0.0 (full line), h = −0.25 (dashed line) and h = 0.25 (dotted line). We have fixed L = 1, D_0 = 1, δφ_c = 0.01 and Ω = 0.01.

FIG. 5. Maximum R_{max} of the SNR curve (Fig.4) as a function of h, for three values of D_0: D_0 = 0.9 (dashed line), D_0 = 1.0 (full line) and D_0 = 1.1 (dotted line). The arrows a and b indicate the response gain due to an homogeneous increase of the coupling and to a selective one respectively. The larger gain in the second case is apparent. The inset shows the dependence of R_{max} on D_0 for h = −0.25 (lower line), h = 0 and h = 0.25 (upper line).

FIG. 6. SNR for the case of competition between local and nonlocal interaction as a function of the noise intensity (here indicated by γ). All curves have D_v = 4 (nonlocal interaction). The following curves correspond to a nonlocal excitatory interaction with different values of D_u: the upper broken line D_u = 1.2, the continuous line D_u = 0.8, the lower broken line D_u = 1. The dotted line corresponds to the original (inhibitory) FitzHugh-Nagumo case D_u = 1.2).
|                |                  |
|----------------|------------------|
| $R_{\text{max}}$ (dB) |                  |
| $D_0$ (arb. units) |                  |

![Graph showing $R_{\text{max}}$ vs. $D_0$ and $h$ with labels a and b.]
