Sample “optimal Quantum identity testing via Pauli Measurements

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In this paper, we show that \( \Theta(\text{poly}(n) \cdot 2^{\frac{n}{\epsilon^2}}) \) is the sample complexity of testing whether two \( n \)-qubit quantum states \( \rho \) and \( \sigma \) are identical or \( \epsilon \)-far in trace distance using two-outcome Pauli measurements.

INTRODUCTION

At this stage of development of quantum computation, testing the properties of new devices is of fundamental importance [1]. Quantum state tomography is to decide how many copies of an unknown mixed quantum state \( \rho \in \mathcal{D}(\mathbb{C}^d) \) is necessary and sufficient to output a good approximation of \( \rho \) in trace distance, with high probability. A sequence of work [2–6] is showed that \( \Theta(d^3 \epsilon^2) \) copies are sufficient and sufficient for quantum state tomography in a trace distance of no more than \( \epsilon \), by performing measurements on each copy of \( \rho \) independently. If one is allowed to measure many copies of \( \rho \) simultaneously, the sample complexity of state tomography is \( \Theta(d^2 \epsilon^2) \) in [6–8].

As one of the most fundamental problem in the general framework of quantum state property testing, the problem “quantum identity testing” has received much attention: Given many copies of unknown quantum states \( \rho \) and \( \sigma \), and the goal is to distinguish the case that they are identical and that they are \( \epsilon \)-far in trace distance. Here the goal is to derive a measurement scheme to learn some non-trivial property, using as few as possible copies.

This problem is a quantum analog of the identity testing of probabilistic distributions, a central problem of distribution testing. The idea of identity testing has been extensively explored in studying other property testing problems [9–16].

For practical purposes, the results from cases where \( \sigma \) is a known pure state have been extensively studied in the independent measurement setting [17–19]. For the general mixed states case, [20] solved the problem, in the joint measurement setting, where \( \sigma \) is a maximally mixed state case by showing that \( \Theta(d^2) \) copies are necessary and sufficient. Importantly, the sample complexity of the general problem was proven to be the same in [21]. In [22], we provide an independent measurement scheme using \( \Theta(d^2) \) copies for this problem. Interestingly, [23] showed that the sample complexity of this problem in the independent measurement setting, where \( \sigma \) is a maximally mixed state case, is \( \Theta(d^{3/2}) \) with nonadaptive measurements, and \( \Omega(d^{3/3}) \) with adaptive measurements.

To achieve the optimal complexity in the general mixed state scenario, [6–8, 20, 21] employ highly entangled measurements. The other side of these entangled measurements is the difficulty in the implementation: all \( \Theta(d^{3/2}) \) copies must be stored in a noiseless environment; for \( n \)-qubit system, the needed measurement is within dimension \( d^{1.2} = 2^\Omega(n^{1.2}) \), a double exponential function of the number of qubits. The complexity \( \Theta(d^{3/2}) \) for independent measurement of [23] requires \( n \)-qubit random unitary, which is highly entangled in \( n \)-qubit system.

This observation leads to the question: What if entangled measurements are not allowed? As the most important class of quantum measurements, Pauli measurements are of great interest.

Our results

The following quantum identity testing problem was extensively studied in [20, 21, 23]

**Problem 1.** Given two unknown \( n \)-qubit quantum mixed states \( \rho \) and \( \sigma \), and \( \epsilon > 0 \), the goal is to distinguish the two cases

\[
||\rho - \sigma||_1 > \epsilon \quad (1)
\]

and

\[
\rho = \sigma \quad (2)
\]

How many copies of \( \rho \) and \( \sigma \) are needed to achieve this goal, with high probability?
In this paper, we focus on the sample complexity of this problem using Pauli measurements. In particular, we show that

**Theorem 1.** The sample complexity of the quantum identity testing problem is $\Theta\left(\frac{\text{poly}(n) \cdot 4^n}{\epsilon^2}\right)$ using two-outcome Pauli measurements.

**PRELIMINARIES**

A positive-operator valued measure (POVM) on a finite dimensional Hilbert space $\mathcal{H}$ is a set of positive semi-definite matrices $\mathcal{M} = \{M_i\}$ such that

$$\sum M_i = I_{\mathcal{H}}.$$  

We use $\sigma_I$, $\sigma_X$, $\sigma_Y$ and $\sigma_Z$ to denote Pauli matrices,

$$\sigma_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

For $U \in \{\sigma_X, \sigma_Y, \sigma_Z\}$ with $U = |\psi_0\rangle\langle \psi_0| - |\psi_1\rangle\langle \psi_1|$, we use Pauli measurement to denote the following measurement,

$$M_0 = |\psi_0\rangle\langle \psi_0|, M_1 = |\psi_1\rangle\langle \psi_1|.$$  

The following is widely known,

**Observation 1.** For two unknown binary distributions $p$ and $q$, the sample complexity of distinguishing the two cases $p = q$ and $||p - q||_2 > \epsilon$ with probability at least $1 - \delta$ is $\Theta\left(\frac{\log(1 - \delta)}{\epsilon^2}\right)$.

There is no difference for 2-norm (the Euclidean norm) and 1-norm (total variation) in binary distribution setting.

**QUANTUM IDENTITY TESTING**

In this section, we provide a measurement scheme which uses $O\left(\frac{n^{4}4^n}{\epsilon^2}\right)$ copies to solve Problem 1. The main tool is an algorithm for testing the property of probability collections.

**Upper bound**

The property testing of collections of discrete distributions was studied in [16, 24]. In this section, we study the identity testing of discrete distributions in the query model. Suppose we are given two collection of binary distributions in a query model, with $m$ binary distributions $p_1, \ldots, p_m$ for the first collection and $q_1, \ldots, q_m$ for the second collection such that for any given index $i$ that we can choose to access $p_i$ and $q_i$ and obtain samples. The goal is to distinguish the case of $p_i = q_i$ for all $1 \leq i \leq m$ from the case

$$\frac{1}{m} \sum_{i=1}^{m} ||p_i - q_i||_2^2 > \epsilon^2$$

**Algorithm 1: An Identity Test for Collections with a Query Model**

**Input:** Access to binary distributions $p_1, \ldots, p_m$ and $q_1, \ldots, q_m$ with $\epsilon > 0$.

**Output:** "Yes" with a probability of at least $\frac{2}{3}$ if $p_i = q_i$ for all $1 \leq i \leq m$; and "No" with a probability of at least $\frac{2}{3}$ if $\frac{1}{m} \sum_{i=1}^{m} ||p_i - q_i||_2^2 > \epsilon^2$.

Let $L$ be a sufficiently large constant;

for $k^2 \leftarrow 0$ to $[\log_2 m]$ do

| Select $2^k \cdot (k^2 + 1) \cdot L$ uniformly random elements $1 \leq i \leq m$; |
| For each selected $i$, distinguish between $p_i = q_i$ and $||p_i - q_i||_2^2 > 2^{k-1}\epsilon^2$ with a failure probability of at most $L^{-2}6^{-k}$; |
| If any of these testers returned "No", return "No"; |
| Return "Yes"; |
If $p_i = q_i$ for all $1 \leq i \leq m$, then the probability of success is at least

$$1 - \sum_{k=0}^{\lfloor \log_2 m \rfloor} (1 - p_k),$$

where

$$p_k \geq (1 - 2^k \cdot (k^2 + 1) \cdot L \cdot L^{-2} - k^2) = 1 - \frac{k^2 + 1}{3^k L}.$$ 

Then

$$1 - \sum_{k=0}^{\lfloor \log_2 m \rfloor} (1 - p_k) \geq 1 - \sum_{k=0}^{\lfloor \log_2 m \rfloor} \frac{k^2 + 1}{3^k L} \geq 1 - O(\frac{1}{L}).$$

On the other hand, if $\frac{1}{m} \sum_{i=1}^{m} ||p_i - q_i||_2^2 > \epsilon^2$, we observe the following

$$\frac{1}{m} \sum_{i=1}^{m} ||p_i - q_i||_2^2 < \{i : ||p_i - q_i||_2^2 < \frac{\epsilon^2}{2}\} \frac{\epsilon^2}{2} + \sum_{k=0}^{\lfloor \log_2 m \rfloor} \{i : 2^k \cdot x^2 \leq ||p_i - q_i||_2^2 < 2^k \cdot \epsilon^2\} 2^k \epsilon^2$$

Thus,

$$\frac{\epsilon^2}{2} + \sum_{k=0}^{\lfloor \log_2 m \rfloor} \{i : 2^k \cdot x^2 \leq ||p_i - q_i||_2^2 < 2^k \cdot \epsilon^2\} 2^k \epsilon^2 > \epsilon^2 + \sum_{k=0}^{\lfloor \log_2 m \rfloor} \frac{1}{100(k^2 + 1)} (2^k \cdot \epsilon^2) 2^k \epsilon^2$$

Therefore, there exists some $k \geq 0$ such that

$$\{i : 2^k \cdot \epsilon^2 \leq ||p_i - q_i||_2^2\} > \frac{1}{2^k \cdot 100(k^2 + 1)^m}.$$ 

Actually, there is always such a $k \leq k_0 = \log_2 m$: If we find some $k > k_0$ with the above property, then the above property is also true for $k_0$,

$$\{i : 2^{k-1} \cdot \epsilon^2 \leq ||p_i - q_i||_2^2\} \geq \frac{1}{2^k \cdot 100(k^2 + 1)^m}.$$ 

Here, the probability of selecting some $i$ with this property is at least

$$1 - (1 - \frac{1}{100 \cdot 2^k \cdot (k^2 + 1)})^{2^k \times (k^2 + 1) \times L} \geq 1 - O(e^{-L/100}).$$

After this, the corresponding tester will return ”No” with high probability.

The sample complexity of this algorithm is

$$\sum_{k=0}^{k_0} 2^k \cdot (k^2 + 1) \times O(\frac{1}{2^{k-1} \cdot \epsilon^2} \times \log(L^2 \cdot 6^k)) = O(\frac{\log^4 m}{\epsilon^2}).$$

Although it looks that in the left handside $2^{k_0} \cdot k_0^2 >> m$, but this term $2^{k_0} \cdot k_0^2$ appears only if $2^{k-1} \cdot \epsilon^2 < 1$.

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1 Here, we have not attempted to optimize the exponent of $\log m$. It is direct to obtain $O(\frac{\log^{4+\mu} m}{\epsilon^2})$ for any $\mu > 0$ by selecting $2^k \cdot k^{1+\mu} \cdot C$ uniformly random elements $1 \leq i \leq m$.  

This bound is tight up to the poly(log m) factor: Even for the case m = 1, \( \frac{\log(\frac{1}{\epsilon})}{\epsilon^2} \) samples are needed to achieve successful probability \( 1 - \delta \).

Back to the quantum identity testing problem: Suppose we are given two n-qubit quantum states \( \rho \) and \( \sigma \), and we guarantee that either \( \rho = \sigma \) or \( ||\rho - \sigma||_1 > \epsilon \), and the goal is to distinguish them.

Without loss of generality, we can let

\[
\rho = \frac{\sum P \alpha_P P}{2^n},
\]

\[
\sigma = \frac{\sum P \beta_P P}{2^n},
\]

where in the summation \( P \) ranges over \( \{\sigma_I, \sigma_X, \sigma_Y, \sigma_Z\} \). Then,

\[
||\rho - \sigma||_2^2 = \sum P (\alpha_P - \beta_P)^2 \geq \frac{||\rho - \sigma||_1^2}{2^n}.
\]

Therefore, if \( \rho = \sigma \), \( \alpha_i = \beta_i \) for all \( i \).

If \( ||\rho - \sigma||_1 > \epsilon \), we have

\[
\sum_i (\alpha_P - \beta_P)^2 \geq \epsilon^2.
\]

If we measure \( \rho \) and \( \sigma \) for Pauli \( P \), the output of \( \rho \) would be a sample of probability distribution \( p_P = (\frac{1+\alpha_P}{2}, \frac{1-\alpha_P}{2}) \), and the output of \( \sigma \) would be a sample of probability distribution \( q_P = (\frac{1+\beta_P}{2}, \frac{1-\beta_P}{2}) \). That is

\[
||p_P - q_P||_2^2 = \frac{(\alpha_P - \beta_P)^2}{2}.
\]

Therefore, to solve the quantum identity testing problem, we only need to solve the “Identity testing for distribution collections” in a query model with \( m = 4^n \) and \( \frac{\epsilon^2}{2^{4n}} \). It can be solved using

\[
\mathcal{O}\left(\frac{n^3 \cdot 4^n}{\epsilon^2}\right)
\]

number of samples.

**Lower bound**

We use the quantum mixedness problem testing problem, a special case of Problem 1 by letting \( \rho = \frac{I_{2^n}}{2^n} \), to provide lower bound.

If we regard Pauli measurement corresponding to \( Q = Q_+ - Q_- \in \{\sigma_I, \sigma_X, \sigma_Y, \sigma_Z\} \) as a two-outcome measurement \( \{Q_+, Q_-\} \), we can observe that \( \Omega\left(\frac{n^3}{\epsilon^2}\right) \) copies of \( \sigma \) are necessary to solve the quantum mixedness testing. To see this, for any \( P \in \{\sigma_I, \sigma_X, \sigma_Y, \sigma_Z\} \), we let

\[
\sigma_P = \frac{I_{2^n} + \epsilon P}{2^n}
\]

Then,

\[
||\sigma_P - \frac{I_{2^n}}{2^n}||_1 = \epsilon
\]

Of course, to distinguish \( \sigma_P \) from \( \frac{I_{2^n}}{2^n} \), at least \( \frac{1}{\epsilon^2} \) copies are needed to be measured in Pauli measurement corresponding to \( P \).

Therefore, to distinguish \( \frac{I_{2^n}}{2^n} \) from the uniform distribution over \( \sigma_P \)s, we need at least

\[
(4^n - 1) \frac{1}{\epsilon^2} = \Omega\left(\frac{4^n}{\epsilon^2}\right)
\]

copies.
Adaptively chosen Pauli measurement would not provide any advantage here.

As recently noticed in [25], Pauli measurements is usually a $2^n$ outcome measurements. One can obtain a lower bound $\Omega(\frac{3^n}{2^n})$ by using

$$\sigma_P = \frac{I^{2^n} + \epsilon P}{2^n}$$

for any $P \in \{\sigma_X, \sigma_Y, \sigma_Z\}^{\otimes n}$.

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