Strings in Noncompact Spacetimes:
Boundary Terms and Conserved Charges

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We study some of the novel properties of conformal field theories with noncompact target spaces as applied to string theory. Standard CFT results get corrected by boundary terms in the target space in a way consistent with the expected spacetime physics. For instance, one-point functions of general operators on the sphere and boundary operators on the disk need not vanish; we show that they are instead equal to boundary terms in spacetime. By applying this result to vertex operators for spacetime gauge transformations with support at infinity, we derive formulas for conserved gauge charges in string theory. This approach provides a direct CFT definition of ADM energy-momentum in string theory.

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1. Introduction

Historically, the solutions of string theory attracting the most attention have been of the form $M^4 \times K$, where $M^4$ is four dimensional Minkowski spacetime and $K$ is some compact space. Abstractly, $K$ is described by a compact, unitary, (super)conformal field theory of appropriate central charge. $M^4$, on the other hand, is described by a noncompact, nonunitary free CFT.

There are many instances in which one is interested in a more general class of solutions, namely those corresponding to noncompact interacting CFTs. The study of string solitons, cosmological spacetimes and anti-de Sitter spacetimes are just a few examples. Our focus here will be on the new features that arise due to the noncompactness of the underlying CFT.

Compactness and unitarity lead to some familiar theorems which must be reexamined in the more general context. An example is Zamolodchikov’s c-theorem [1], which establishes the existence of a function that monotonically decreases under RG flow and is stationary with respect to marginal perturbations. The proof of the theorem is based on properties of two-point energy-momentum tensor correlators $\langle T_{ab}T_{cd} \rangle$, but these correlators are generically ill-defined in a noncompact theory. And indeed, the theorem does not hold in the noncompact setting [2,3,4].

Another example — of direct interest to us here — are the conformal Ward identities obeyed by correlators in a compact CFT. These follow from the operator product expansion

\[
T(z')O(z, \bar{z}) \sim \ldots \frac{hO(z, \bar{z})}{(z' - z)^2} + \frac{\partial O(z, \bar{z})}{z' - z},
\]

and holomorphicity of the energy-momentum tensor

\[
\frac{\partial}{\partial z'} \langle T(z')O_1(z_1, \bar{z}_1) \ldots O_n(z_n, \bar{z}_n) \rangle = 0, \quad z' \neq z_i.
\]

An elementary consequence is the vanishing of one-point functions of operators on the sphere

\[
\langle O(z, \bar{z}) \rangle_{S_2} = 0, \quad (h, \tilde{h}) \neq (0,0),
\]

and of one-point functions of boundary operators on the disk

\[
\langle O_{\text{bndy}}(z, \bar{z}) \rangle_{D_2} = 0, \quad h \neq 0.
\]

In the noncompact case (1.2) need not be true, and with it the consequences (1.3) and (1.4). For example, nonzero one-point functions appear in Liouville theory; see, e.g., [3].
To understand this claim, note that in a CFT with noncompact target space there are two classes of states and operators, distinguished by their normalizability properties. There does not exist a normalizable $SL(2)$ invariant vacuum state; normalizable states instead correspond to inserting operators which fall off sufficiently rapidly at infinity in the target space. These operators correspond, roughly speaking, to normalizable wavefunctions on the target space. Correlation functions of such operators will obey the standard theorems (1.2)-(1.4).

However, the remaining “nonnormalizable operators” play a crucial role in noncompact theories. In the AdS/CFT correspondence correlation functions in the boundary theory are related to worldsheet CFT correlation functions of such nonnormalizable operators [6,7]. Therefore, these vertex operators are central in the study of string theory in AdS$_3$ [8,9,10]. Since what distinguishes the normalizable and nonnormalizable operators is their behavior at infinity, one might expect that violations of (1.2)-(1.4) can be expressed as boundary terms at infinity in the target space. We will show that this is the case, giving explicit formulas for the resulting boundary integrals in terms of CFT correlators. For instance, in the general Weyl invariant bosonic nonlinear sigma model (1.3) is replaced by

$$
\langle O(z, \bar{z})\rangle_{S^2} = -\frac{1}{2\hbar} (V_M)^{D/2-1} \int d^{D-1}S_\mu \int d^2z' (z' - z) e^{2\omega(\bar{z}', \bar{z})} \langle \partial X^\mu(z', \bar{z}') O(z, \bar{z}) \rangle'_{S^2}.
$$

(1.5)

Here $e^{2\omega(\bar{z}', \bar{z})}$ is the worldsheet conformal factor, $V_M$ is the volume of the worldsheet, and $\langle \ldots \rangle'$ denotes a path integral with the constant mode integration omitted.

These nonzero one-point functions play an important and desirable role from the spacetime point of view. Consider the spacetime action $S$ whose Euler-Lagrange equations reproduce the Weyl invariance conditions of the sigma model. The variation of $S$ with respect to the spacetime fields $\phi_i$ is equal to a bulk term, which vanishes on solutions to the equations of motion, plus a boundary term:

$$
\delta S = \int d^Dx \{ = 0 \text{ by e.o.m.} \} + \int d^{D-1}S^\mu \Pi^i_\mu \delta \phi_i.
$$

(1.6)

The statement that the spacetime action varies by a boundary term is directly related to the CFT results discussed above. In the case of open string theory at the level of disk amplitudes, the relation follows from the fact that the on-shell spacetime action is proportional to the partition function of the corresponding CFT on the disk [11,12,13,14]. The variation (1.6) is then the one-point function of an operator on the boundary of the
disk, so the statement that the one-point function is nonzero and equal to a boundary term in spacetime is consistent with the result (1.6). In closed string theory the connection is not quite as precise, as the spacetime action does not seem to be proportional to the sphere partition function; we will have more to say about this at the end of the paper.

The boundary terms in (1.6) are of interest from several points of view. In the AdS/CFT correspondence they define correlators of the boundary theory, as we have already mentioned. Another important application, which we develop here, is in defining conserved charges associated with gauge symmetries [15]. The idea is to consider (1.6) with \( \delta \phi_i \) corresponding to a gauge transformation with support at infinity. Being a gauge transformation, we must of course have \( \delta S = 0 \) for such a variation. On the other hand, after integration by parts the boundary term takes the form of a time derivative of a charge, so one learns that this charge is conserved. This same procedure can be carried out in the string path integral, leading to a direct CFT definition of conserved charges in string theory. These charges include electromagnetic and anti-symmetric tensor charges, as well as those associated with the gravitational field. In asymptotically flat spacetime the latter correspond to mass, momentum, and angular momentum. For the conserved energy-momentum we find the result

\[
P^\mu \propto \int d^{D-2}S \int d^2 z \int d^2 z' (z' - z) e^{2\omega(z', \vec{z}')}(\partial X^i(z', \vec{z}')(\partial X^0 \overline{\partial} X^\mu + \partial X^\mu \overline{\partial} X^0)(z, \vec{z})) \langle \partial X_i(z, \vec{z}) \partial X_0(z, \vec{z}) \rangle_{S_2}.
\]

(1.7)

(We suppressed some corrections from the dilaton; see section 6.) This will be shown to coincide with the standard ADM definition. As far as we know, a direct CFT definition of ADM energy and momentum has not been given before.

The alternative approach to deriving conserved charges in string theory is to first derive the low energy effective action in the \( \alpha' \) expansion, and then to proceed as in field theory. In asymptotically flat spacetime the two derivative approximation to the action is usually sufficient, since derivatives become small at infinity. Therefore, our results for conserved charges in these backgrounds will reproduce expected results. But more generally one could consider cases where higher \( \alpha' \) corrections contribute. An example is AdS with radius of curvature comparable to the string scale. It would be interesting to apply our approach to such an example.

This paper is organized as follows. In section 2 we show how boundary terms arise in the path integral. This leads to expressions for one-point functions on the sphere and disk in section 3. A simple example of the open string with constant gauge field strength

is given in section 4. In section 5 we review how conserved charges can be derived from the spacetime action, and this is extended to string theory in section 6. Section 7 contains some discussion of open questions.

2. Boundary terms from the string path integral

2.1. Path integral preliminaries

Although we could presumably be more general, for definiteness we will consider the general renormalizable bosonic sigma model, following the conventions of [16],

$$S = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \ g^{1/2} \left[ (g^{\alpha\beta} G_{\mu\nu}(X) + i\epsilon^{\alpha\beta} B_{\mu\nu}(X)) \partial_\alpha X^\mu \partial_\beta X^\nu + T_c(X) + \alpha' R \Phi(X) \right]$$

$$+ \int_{\partial M} ds \left[ iA_\mu(X) \frac{dX^\mu}{ds} + T_\alpha(X) + \frac{k}{2\pi} \Phi(X) \right].$$

(2.1)

We will usually be working in \(D = 25 + 1\) dimensions, but it is helpful to keep a general \(D\) in most of our formulas. It will be important for us to work on a compact worldsheet,

$$V_M = \int d^2\sigma \ g^{1/2} = \text{finite}. \quad (2.2)$$

We are interested in correlation functions of local operators,

$$\langle O_1 \ldots O_n \rangle = \int D X \ O_1 \ldots O_n \ e^{-S}. \quad (2.3)$$

The path integral measure has to be treated with some care. We desire a measure preserving worldsheet diffeomorphism invariance and spacetime gauge symmetries, and consistent with the “ultralocality” principle [17,18] of being expressible as a pointwise product over the worldsheet. We can formally define the measure in terms of a norm on the tangent space,

$$||\delta X||^2 = \int_M d^2\sigma \ g^{1/2} G_{\mu\nu}(X) \delta X^\mu \delta X^\nu, \quad (2.4)$$

or as the pointwise product

$$D X = \prod_\sigma [-G(X(\sigma))]^{1/2} d^D X(\sigma) \quad = e^{\frac{i}{2} \int_M d^2\sigma \ g^{1/2} \delta(0) \ln[-G(X(\sigma))] \prod_\sigma d^D X(\sigma). \quad (2.5)$$
To give meaning to (2.5) we need to define $\delta^{(2)}(0)$ as part of our regularization procedure. For instance, in heat kernel regularization,

$$\delta^{(2)}(0) \Rightarrow K_\epsilon(\sigma, \sigma) = \frac{1}{4\pi \epsilon} + \frac{1}{24\pi} R + O(\epsilon). \quad (2.6)$$

Therefore — and more generally given ultralocality — the measure factors take the form of tachyon and dilaton couplings and so can be absorbed in (2.1) [17,18,19]. Assuming this has been done, in a given regularization scheme the tachyon and dilaton might therefore transform in an unconventional way under spacetime coordinate transformations in order to compensate for the noninvariance of the measure. This can be rectified by a field redefinition; that is by adding additional counterterms to (2.1). We will assume that the counterterms implicit in (2.1) are such that spacetime symmetry transformations act in the usual way. For instance, this can be made manifest by working with the covariant background field expansion.

When the target spacetime is noncompact the path integral (2.3) can be ill-defined due to the large volume integration. To isolate this feature we separate out the integral over the constant mode of $X^\mu$ from the nonconstant modes. We therefore expand $X^\mu$ in a complete set of modes

$$X^\mu(\sigma) = x^\mu + \sum_{n \neq 0} x^\mu_n X_n(\sigma), \quad (2.7)$$

with

$$\int d^2 \sigma \ g^{1/2} X_n X_m = \delta_{nm}, \quad \int d^2 \sigma \ g^{1/2} X_n = 0, \quad n, m \neq 0. \quad (2.8)$$

We will sometimes use the notation $X^\mu = x^\mu + \tilde{X}^\mu$. In terms of the mode coefficients the measure is

$$\mathcal{D}X = (V_M)^{D/2} d^D x \prod_{n \neq 0} d^D x_n \equiv (V_M)^{D/2} d^D x \mathcal{D}X'. \quad (2.9)$$

The powers of $V_M$ are due to the different normalization of $x^\mu$ compared to $x^\mu_n$; these factors are familiar from computations in flat spacetime. The normalization of the path integral is in fact fixed by Weyl invariance and ultralocality of the measure [17,18].

Functional derivatives are defined by

$$\frac{\delta}{\delta X^\nu(\sigma')} X^\nu(\sigma') = g^{-1/2} \delta^{(2)}(\sigma - \sigma') \delta^\nu, \quad \text{or in terms of modes,}$$

$$\frac{\delta}{\delta X^\mu(\sigma)} = V_M^{-1} \frac{\partial}{\partial x^\mu} + \sum_{n \neq 0} X_n(\sigma) \frac{\partial}{\partial x^\mu_n}. \quad (2.10)$$
2.2. Appearance of boundary terms

In the compact case the classical equations of motion hold inside correlation functions since the path integral of a total derivative vanishes,

$$\langle \frac{\delta S}{\delta X^\mu(\sigma)} O_1(\sigma_1) \cdots O_n(\sigma_n) \rangle = - \int D X \frac{\delta}{\delta X^\mu(\sigma)} \{ O_1(\sigma_1) \cdots O_n(\sigma_n) e^{-S} \} = 0, \quad (2.11)$$

(modulo contact terms if $\sigma = \sigma_i$). In the noncompact case (2.11) can be modified by boundary terms in spacetime. The point is that the factor $e^{-S}$ typically decays exponentially for large $|x^\mu|$, but this need not be so for large $|x^\mu|$, for instance as occurs for the case of the trivial Minkowski vacuum. Therefore, we should only assume that the path integral of a total derivative with respect to a nonconstant mode vanishes, and so we use (2.10) to write

$$\int D X \frac{\delta}{\delta X^\mu(\sigma)} \{ O_1 \cdots O_n e^{-S} \} = (V_M)^{-1} \int D X \frac{\partial}{\partial x^\mu} \{ O_1 \cdots O_n e^{-S} \} = (V_M)^{D/2-1} \int d^D x \frac{\partial}{\partial x^\mu} \langle O_1 \cdots O_n \rangle', \quad (2.12)$$

where $\langle \cdots \rangle'$ denotes the path integral with respect to the nonconstant modes.

It is useful to relate the boundary terms to nonholomorphicity of the energy-momentum tensor. The action (2.1) is invariant under the infinitesimal worldsheet diffeomorphism

$$\delta \xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a, \quad \delta \xi X^\mu = \xi^a \partial_a X^\mu. \quad (2.13)$$

Using the definition of the energy-momentum tensor,

$$T^{ab} = 4\pi \frac{\delta S}{\delta g_{ab}}, \quad (2.14)$$

and integrating the variation of the action by parts, we find

$$\nabla^a T_{ab} = 2\pi \frac{\delta S}{\delta X^\mu} \partial_b X^\mu. \quad (2.15)$$

(2.15) is of course just the statement that the energy-momentum tensor is conserved when the equations of motion are satisfied. (2.15) holds as an operator equation once we define products of fields appropriately. In terms of the path integral, the right hand side becomes

$$-2\pi \int D X \frac{\delta}{\delta X^\mu(\sigma)} \{ \partial_b X^\mu(\sigma) O_1 \cdots O_n e^{-S} \}. \quad (2.16)$$
We are assuming that none of the operators $O_i$ is at $\sigma$. We have also absorbed a delta function contribution into the definition of the operator $\delta S_{\delta X^\mu} \partial_\mu X^\mu$. Indeed, in free field theory this precisely corresponds to the standard normal ordering prescription \[16\].

Now we use (2.12) to write

$$\langle \nabla^a T_{ab}(\sigma)O_1 \ldots O_n \rangle = -2\pi (V_M)^{D/2-1} \int d^D x \frac{\partial}{\partial x^\mu} \langle \partial_\mu X^\mu(\sigma)O_1 \ldots O_n \rangle'.$$

(2.17)

Despite appearances, (2.17) is invariant under spacetime diffeomorphisms since, according to our measure conventions, $\langle \partial_\mu X^\mu(\sigma)O_1 \ldots O_n \rangle'$ transforms like $(-G)^{1/2}$ times a spacetime vector.

One is used to saying that the left hand side of (2.17) should vanish by worldsheet diffeomorphism invariance. But as we have shown, this conclusion only follows if the classical equations of motion hold inside correlators, and this can be violated by boundary terms. (2.17) is consistent with worldsheet diffeomorphism invariance.

3. One point functions on the sphere and disk

3.1. The sphere

We now restrict to Weyl invariant theories of the form (2.1), including also the Fadeev-Popov determinant to cancel the matter central charge. We work in conformal gauge

$$ds^2 = e^{2\omega(z,\bar{z})} dz d\bar{z}.$$

(3.1)

Now consider a local operator of scaling dimension $(h, \tilde{h})$ obeying the standard OPE

$$T(z', \bar{z}') O(z, \bar{z}) \sim \ldots + \frac{h O(z, \bar{z})}{(z' - z)^2} + \frac{\partial O(z, \bar{z})}{z' - z},$$

(3.2)

and similarly for $\tilde{T}(z', \bar{z}')$. It is important to emphasize that (3.2) should hold on a curved worldsheet. On a curved worldsheet operators of different engineering dimension can mix via appearance of factors of $R_{z\bar{z}}$, so an operator of definite scaling dimension on a flat worldsheet need not have definite scaling dimension on a curved worldsheet.

Let $C$ be a small contour circling $z$. Using the OPE we have for the one point function with $h \neq 0$:

$$\langle O(z, \bar{z}) \rangle_{S^2} = \frac{1}{h} \int_C \frac{dz'}{2\pi i} (z' - z) \langle T(z', \bar{z}') O(z, \bar{z}) \rangle_{S^2}.$$  

(3.3)
We would now like to deform the contour, eventually contracting it to zero by “sliding it off the opposite pole of the sphere”. In a compact CFT the stress tensor is holomorphic, and therefore the correlator is independent of $C$ provided that no other operators are encountered. This is the standard logic by which one concludes that all one point functions of operators with $h \neq 0$ vanish on the sphere. But in a noncompact CFT the stress tensor can be non-holomorphic as in (2.17), and so we will pick up a contribution from deforming the contour. In particular, we use the divergence theorem
\[
\int_R d^2z \left( \partial v^z + \partial v^\bar{z} \right) = i \oint_{\partial R} (v^z d\bar{z} - v^\bar{z} dz),
\]
(3.4)
to write
\[
\langle O(z, \bar{z}) \rangle_{S^2} = \frac{1}{2\pi h} \int_{S^2} d^2z' (z' - z) (\overline{\partial T(z', \bar{z})} O(z, \bar{z}))_{S^2}.
\]
(3.5)
Then using (2.17) we arrive at
\[
\langle O(z, \bar{z}) \rangle_{S^2} = -\frac{1}{2h} (V_M)^{D/2-1} \int d^Dx \partial_\mu \left\{ \int d^2z' (z' - z) e^{2\omega(z', \bar{z})} \langle \partial X^\mu(z', \bar{z}) O(z, \bar{z}) \rangle_{S^2} \right\}.
\]
(3.6)
(3.6) gives our desired result: it expresses a one point function in the CFT as a boundary term in spacetime.

3.2. The disk

Now consider a worldsheet of disk topology. A conformal field theory on the disk has an energy-momentum tensor obeying the boundary condition
\[
n^a t^b T_{ab}|_{\partial D^2} = 0,
\]
(3.7)
where $n^a$ and $t^a$ are normal and tangent to the boundary. The conformal symmetry is generated by a single copy of the Virasoro algebra. To derive the form of the Virasoro generators it is convenient to start with a representation of the disk as the upper half $w$ plane, with a metric chosen such that the worldsheet volume is finite. We again work in conformal gauge (3.3). The boundary conditions are then
\[
T(w) = \bar{T}(\bar{w}), \quad w = \bar{w}.
\]
(3.8)
Let $C$ be any contour in the $w$ plane with endpoints on the real axis. Then, assuming for the moment holomorphicity of $T$, the following charges are independent of the contour $C$ provided that no other operators are encountered,
\[
L_n = \int_C \left[ \frac{dw'}{2\pi i} (w')^{n+1} T(w') - \frac{d\bar{w}'}{2\pi i} (\bar{w}')^{n+1} \bar{T}(\bar{w}') \right].
\]
(3.9)
For calculational purposes it can be convenient to use a flat worldsheet metric. We also want a finite coordinate range, so we rewrite the above charges after transforming to the disk $|z'| \leq 1$. Let $z$ be a point on the boundary, $|z| = 1$, and let the map from the $w'$-plane to the $z'$-plane be

$$z' = \left(\frac{i-w'}{i+w'}\right) z. \quad (3.10)$$

The Virasoro generators then take the form

$$L_n = \int_C \left[ \frac{dz'}{4\pi z} (z + z')^2 \left( i \frac{z - z'}{z + z'} \right)^n \frac{T(z')}{4\pi z} (\bar{z} + \bar{z}')^2 \left( -i \frac{\bar{z} - \bar{z}'}{\bar{z} + \bar{z}} \right)^{n+1} \tilde{T}(\bar{z}') \right]. \quad (3.11)$$

$C$ is now any contour with endpoints on the boundary of the disk.

In CFTs with noncompact target spaces, the charges $L_n$ need not be independent of $C$ since the energy-momentum tensor need not be holomorphic. As we did for the sphere, we use this to give a formula for one point functions of boundary operators in terms of surface integrals in the target space. So consider a local boundary operator $O(z, \bar{z})$ with scaling dimension $h \neq 0$. If we let $C$ be a tiny semi-circular contour around $z$ then we can use the OPE to write

$$L_0 O(z, \bar{z}) = h O(z, \bar{z}). \quad (3.12)$$

Using the divergence theorem gives

$$\langle O(z, \bar{z}) \rangle_{D_2} = \frac{1}{2\pi h} \int_{D_2} \frac{d^2z'}{2z} \left\{ \left( z' + \bar{z} \right) \left( z' - z \right) \langle \partial T(z', \bar{z}') O(z, \bar{z}) \rangle_{D_2} \right. \right.
\left. + \left( \bar{z}' + \bar{z} \right) \left( \bar{z}' - \bar{z} \right) \langle \partial \tilde{T}(z', \bar{z}') O(z, \bar{z}) \rangle_{D_2} \right\}. \quad (3.13)$$

Then (2.17) gives the final result

$$\langle O(z, \bar{z}) \rangle_{D_2} = -\frac{1}{2h} (V_M)^{D/2-1} \int d^Dx \partial_\mu \left\{ \int_{D_2} d^2z' e^{2\omega(z', \bar{z})} \left( z' + \bar{z} \right) \left( z' - z \right) \langle \partial X^\mu(z', \bar{z}') O(z, \bar{z}) \rangle_{D_2} \right. \right.
\left. + \left( \bar{z}' + \bar{z} \right) \left( \bar{z}' - \bar{z} \right) \langle \partial \bar{X}^\mu(z', \bar{z}') O(z, \bar{z}) \rangle_{D_2} \right\}. \quad (3.14)$$

4. Example: open string with constant field strength

The simplest illustration of our result is for the open string with constant field strength. We work on the unit disk $|z| \leq 1$ with flat metric $ds^2 = dzd\bar{z}$. The sigma model action is

$$S = \frac{1}{2\pi\alpha'} \int_{D_2} d^2z \eta_{\mu\nu} \partial X^\mu \overline{\partial X}^\nu + \frac{i}{2} \int_{\partial D_2} d\theta F_{\mu\nu} X^\mu \frac{\partial X^\nu}{\partial \theta}. \quad (4.1)$$
The propagator
\[ G_{\mu\nu}(z', \bar{z}'; z, \bar{z}) = \frac{\langle X^\mu(z', \bar{z}') X^\nu(z, \bar{z}) \rangle_D}{\langle 1 \rangle_D} - x^\mu x^\nu \] (4.2)
obeys
\[ \partial \bar{\partial} G_{\mu\nu} = -\pi \alpha' \left[ \delta^{(2)}(z' - z) - \frac{1}{2\pi} \right] \eta_{\mu\nu}, \] (4.3)
with boundary condition
\[ \left[ \frac{\partial}{\partial r'} G_{\mu\nu} + 2\pi i \alpha' F^\mu_\alpha \frac{\partial}{\partial \theta'} G^{\alpha\nu} \right]_{r'=1} = 0. \] (4.4)

We wrote \( z = re^{i\theta} \). The solution is [20],
\[ G_{\mu}(z', \bar{z}'; z, \bar{z}) = \frac{\alpha'}{2} \left[ -\ln |z' - z|^2 - \frac{1 + F}{1 - F} \ln(1 - z'\bar{z}) - \frac{1 - F}{1 + F} \ln(1 - \bar{z}'\bar{z}) + z'\bar{z}' + z\bar{z} + c \right]^\mu_\nu \] (4.5)
where \( F = 2\pi \alpha' F \). The constant \( c \) is fixed by requiring \( \int d^2 z G_{\mu\nu} = 0 \), but we will not need its value. From (4.3) it is straightforward to see that \( T = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \) is not holomorphic inside a general correlation function.

Now we use our formula (3.14) to compute the following one-point function
\[ F_{\mu\nu} \langle X^\mu \partial X^\nu / \partial \theta \rangle_D. \] (4.6)

This is clearly a “nonnormalizable operator”, since it corresponds to a gauge field which becomes arbitrarily large in spacetime. Of course, it is easy to compute (4.6) directly from (4.3), but we use (3.14) for illustration. Acting with \( \partial_\mu \) we find
\[ F_{\mu\nu}\langle X^\mu \partial X^\nu / \partial \theta \rangle_D = -\frac{1}{2} (V_M)^{D/2-1} \langle 1 \rangle_D' F_{\mu\nu} \int d^D x \int_{D_2} d^2 z' \left\{ \left( \frac{z' + z}{2z} \right) (z' - z) \frac{\partial}{\partial z'} G_{\mu\nu} + \left( \frac{\bar{z}' + \bar{z}}{2\bar{z}} \right) (\bar{z}' - \bar{z}) \frac{\partial}{\partial \bar{z'}} G_{\mu\nu} \right\}. \] (4.7)

Using (4.5) and performing the \( z' \) integral gives
\[ F_{\mu\nu}\langle X^\mu \partial X^\nu / \partial \theta \rangle_D = \frac{i}{4\pi} \eta_{\mu\nu} \left[ \frac{(2\pi \alpha' F)^2}{1 - (2\pi \alpha' F)^2} \right]^{\mu\nu} \langle 1 \rangle_D. \] (4.8)

We have used
\[ \langle 1 \rangle_D = (V_M)^{D/2} \int d^D x \langle 1 \rangle_D', \] (4.9)
From (4.1) it follows that
\[
F_{\mu\nu} \langle X^\mu \frac{\partial X^\nu}{\partial \theta} \rangle_{D^2} = \frac{i}{2\pi} F_{\mu\nu} \delta F_{\mu\nu} \langle 1 \rangle_{D^2}. \tag{4.10}
\]
Integrating then gives
\[
\langle 1 \rangle_{D^2} = N \int d^D x \sqrt{-\det[\eta_{\mu\nu} + 2\pi \alpha' F_{\mu\nu}]} \tag{4.11}
\]
The Born-Infeld action (4.11) is indeed the expected result for the partition function in the presence of a constant field strength [21], so our result (4.8) for the one-point function is correct.

This example is “trivial” in the sense that (4.1) is a free theory and so all correlators are easily computed without use of (3.14). More generally, we will have an interacting CFT which simplifies at infinity in spacetime, in which case (3.14) is needed. We exploit this below in our derivation of conserved charges.

5. Spacetime action and conserved charges

Conserved charges associated with gauge symmetries appear as surface integrals at spatial infinity. This is most easily seen by considering the variation of the spacetime action with respect to gauge transformations supported at infinity [15,22].

Let \( A_p \) and \( \phi_q \) denote some collection of gauge and matter fields, with all spacetime indices suppressed. Consider a gauge invariant spacetime action for these fields,
\[
S = \int_V d^D x \mathcal{L}_{\text{bulk}}(A_p, \phi_q) + \int_{\partial V} d^{D-1} x \mathcal{L}_{\text{bndy}}(A_p, \phi_q). \tag{5.1}
\]
Both \( \mathcal{L}_{\text{bulk}} \) and \( \mathcal{L}_{\text{bndy}} \) are allowed to depend on first and higher derivatives acting on the fields. In general, the boundary term is required for two reasons. First, the Euler-Lagrange equations should imply stationarity of the action with respect to variations that vanish at the boundary; if \( \mathcal{L}_{\text{bulk}} \) contains second or higher derivatives then an appropriate \( \mathcal{L}_{\text{bndy}} \) will be needed to cancel terms arising from integration by parts. A familiar example is the Gibbons-Hawking boundary term [23] which is added to the Einstein-Hilbert action. Second, the action might diverge as the boundary is taken to infinity within the class of field configurations that one wishes to include in the theory. If so, one can try to
add an additional boundary term to cancel this divergence, though this boundary term should depend only on the boundary values of the fields and not their normal derivatives, otherwise the Euler-Lagrange equations will no longer follow. Boundary terms of this sort arise naturally in anti-de Sitter spacetime \[24,25,26,27\].

Given an acceptable spacetime action, consider a general field variation about some configuration satisfying the Euler-Lagrange equations. The bulk term vanishes by assumption, leaving the boundary term

\[
\delta S = \int_{\partial V} d^{D-1}S \{ \pi_{Ap} \delta A_p + \pi_{\phi q} \delta \phi_q \}.
\]

(5.2)

If \(\partial V\) is a constant time hypersurface then \(\pi_{Ap}\) and \(\pi_{\phi q}\) define the usual canonical momenta; more generally they are functionals of the fields and their normal and tangential derivatives.

To define conserved charges let \(\partial V\) be a timelike surface at spatial infinity, and consider a gauge transformation parameterized by \(\xi\). In general, both the gauge and matter fields will contribute a nonzero variation. However, one usually imposes asymptotic conditions on the matter fields such that

\[
\int_{\partial V} d^{D-1}S \pi_{\phi q} \delta \xi \phi_q = 0.
\]

(5.3)

Assuming that this is the case, then the gauge invariance of the action, \(\delta \xi S = 0\), implies

\[
\int_{\partial V} d^{D-1}S \pi_{Ap} \delta \xi A_p = 0.
\]

(5.4)

Now take \(\xi\) to depend only on time with respect to some asymptotic timelike Killing vector, and to be tangent to \(\partial V\) (when \(\xi\) has spacetime indices). Integrating by parts so that \(\xi\) appears without derivatives we will arrive at an expression of the form

\[
\int dt \, \xi(t) \frac{dQ}{dt} = 0.
\]

(5.5)

Since \(\xi(t)\) is arbitrary we obtain the conserved charge \(Q\).

As a rather elementary example consider an abelian gauge field minimally coupled to a complex scalar field,

\[
S = \int d^Dx \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (D^\mu \phi)^* D_\mu \phi - V(\phi^* \phi) \right\}.
\]

(5.6)

When the equations of motion are satisfied the variation of the action is

\[
\delta S = \int dt d^{D-2}S_i \left\{ -F^{\mu\nu} \delta A_\mu + \frac{1}{2} (D^i \phi)^* \delta \phi + \frac{1}{2} D^i \phi \delta \phi^* \right\}.
\]

(5.7)
Consider a gauge transformation, \( \delta \xi A_\mu = \partial_\mu \xi, \delta \xi \phi = i \xi \phi \), with \( \xi = \xi(t) \). Then, assuming that the matter current \( J^i = i[\phi^* D^i \phi - \phi(D^i \phi)^*]/2 \) falls off faster than \( 1/r^{D-2} \), we obtain the usual expression for electric charge

\[
\delta S = - \int dt d^{D-2} S_i F_{i0}^a \frac{d\xi}{dt} \quad \Rightarrow \quad Q \propto \int d^{D-2} S_i^a F_{i0}^a. \tag{5.8}
\]

As our main example consider the low energy action for the bosonic string

\[
S = S_V + S_{\partial V} + S_{ct}
\]

where

\[
S_V = \frac{1}{2\kappa_0^2} \int d^D x (-G)^{1/2} e^{-2\Phi} \left\{ R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial_\mu \Phi \partial^\mu \Phi \right\}. \tag{5.10}
\]

is the standard bulk action, and we have to add the Gibbons-Hawking surface term

\[
S_{\partial V} = -\frac{1}{\kappa_0^2} \int_{\partial V} d^{D-1} x (-\gamma)^{1/2} e^{-2\Phi} \Theta. \tag{5.11}
\]

\( S_{ct} \) is required for finiteness of the action, but its precise form will not be needed; see [24,25,26,27] for more details. Here

\[
\gamma_{\mu\nu} = G_{\mu\nu} - n_\mu n_\nu
\]

is the induced metric on the (assumed to be timelike) boundary \( \partial V \), and the extrinsic curvature tensor and scalar

\[
\Theta_{\alpha\beta} = -\gamma_{\alpha\mu} \nabla^\mu n_\beta, \quad \Theta = -\gamma^{\mu\nu} \nabla_\mu n_\nu \tag{5.13}
\]

are defined in the standard way, see [15], [28]. Indices are raised and lowered with the original metric \( G_{\mu\nu} \), and \( n^\mu \) is the outward unit normal vector to \( \partial V \) (\( n^\mu \) is spacelike). The extrinsic curvature satisfies \( \Theta_{\alpha\beta} n^\beta = \Theta_{\beta\alpha} n^\beta = 0 \), from which follows \( \Theta_{\alpha\beta} = \Theta_{\beta\alpha} \) (we also assume \( n^\alpha \nabla_\alpha n_\beta = 0 \); for further details see [28]). Then the definition (5.13) agrees with the manifestly symmetric expression

\[
\Theta^{\alpha\beta} = -\frac{1}{2} (\nabla^\alpha n^\beta + \nabla^\beta n^\alpha) \tag{5.14}
\]

used in [25] and elsewhere. Finally, the volume elements on \( V \) and \( \partial V \)

\[
(d^D x)(-G)^{1/2} = \epsilon_{\mu_1 \ldots \mu_D} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D},
\]

\[
(d^{D-1} x)(-\gamma)^{1/2} = \tilde{\epsilon}_{\mu_2 \ldots \mu_D} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_D}, \tag{5.15}
\]

\[
\frac{1}{D} \xi_{\mu_1 \mu_2 \ldots \mu_D} = n_{[\mu_1} \tilde{\xi}_{\mu_2 \ldots \mu_D]}.
\]
are related through Gauss’s law
\[ \int_V d^D x (-G)^{1/2} \nabla_{\mu} v^\mu = \int_{\partial V} d^{D-1} x (-\gamma)^{1/2} n_{\mu} v^\mu \]
\[ = \int_{\partial V} d^{D-1} S_{\mu} v^\mu. \] (5.16)

Setting \( S_{\text{ct}} = 0 \) for the moment, the variation of the action (5.9) is a bulk term which vanishes when the equations of motion

\[
0 = R - \frac{1}{12} H^2 + 4 \nabla^2 \Phi - 4 (\nabla \Phi)^2 \\
0 = \frac{1}{2} \nabla^\alpha H_{\mu \nu \alpha} - \nabla^\alpha \Phi H_{\mu \nu \alpha} \\
0 = R_{\mu \nu} - \frac{1}{4} H_{\alpha \beta \mu} H^{\alpha \beta \nu} + 2 \nabla_{\mu} \nabla_{\nu} \Phi
\] (5.17)
are satisfied, plus the boundary term

\[
\delta S = \frac{1}{2 \kappa_5^2} \int_{\partial V} d^{D-1} x (-\gamma)^{1/2} e^{-2\Phi} \left\{ 4 \delta \Phi (\Theta + 2 n^\mu \nabla_{\mu} \Phi) - \frac{1}{2} \delta B_{\alpha \beta} n_\gamma H^{\alpha \beta \gamma} \\
- \delta G_{\alpha \beta} \left[ \gamma^{\alpha \beta} (\Theta + 2 n^\mu \nabla_{\mu} \Phi) - \Theta^{\alpha \beta} \right] \right\}. \] (5.18)

There are conserved gauge charges associated with diffeomorphisms as well as antisymmetric tensor gauge transformations. Consider first \( \delta \Phi = \delta G_{\mu \nu} = 0 \), and the gauge transformation

\[
\delta \Lambda_{\mu \nu} = \nabla_{\mu} \Lambda_{\nu} - \nabla_{\nu} \Lambda_{\mu} = \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu}. \] (5.19)

Then (5.18) gives

\[
\delta S = \frac{1}{2 \kappa_5^2} \int_{\partial V} d^{D-1} S_{\gamma} e^{-2\Phi} \left\{ - (\nabla_\alpha \Lambda_{\beta}) H^{\alpha \beta \gamma} \right\} 
\] (5.20)

We take the metric to approach the Minkowski metric asymptotically, so that it makes sense to talk about time at spatial infinity. Assuming this is the case, we can take our gauge parameter to be a function of only \( t \) at spatial infinity, so demanding that \( \delta S = 0 \) gives rise to the conserved charges

\[
Q_j \propto \int d^{D-2} S_i e^{-2\Phi} H_{0ij}. \] (5.21)

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Here and in the following, \( D - 2 \) dimensional integrals are evaluated on a constant \( t \) slice of \( \partial V \).

Alternatively, consider a diffeomorphism. The fields transform as

\[
\begin{align*}
\delta \xi G_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \\
\delta \xi \Phi &= \xi^\alpha \nabla_\alpha \Phi \\
\delta \xi B_{\mu\nu} &= \xi^\alpha \nabla_\alpha B_{\mu\nu} + \nabla_\mu \xi^\alpha B_{\alpha\nu} + \nabla_\nu \xi^\alpha B_{\mu\alpha} \\
&= \xi^\alpha H_{\alpha\mu\nu} + \left[ \nabla_\mu (\xi^\alpha B_{\alpha\nu}) - \nabla_\nu (\xi^\alpha B_{\alpha\mu}) \right].
\end{align*}
\]

The term in square brackets in the last line of (5.22) is of the form (5.19). It is a gauge variation of the \( B_{\mu\nu} \) field and so can be dropped. Now (5.18) becomes

\[
\delta S = -\frac{1}{2\kappa_0^2} \int_{\partial V} d^{D-1}x (-\gamma)^{1/2} e^{-2\Phi} \left[ 2\nabla_\alpha \xi_\beta \left[ \gamma^{\alpha\beta} (\Theta + 2n^\mu \nabla_\mu \Phi) - \Theta^{\alpha\beta} \right] \\
- \xi_\beta \left[ 4\nabla_\beta \Phi (\Theta + 2n^\mu \nabla_\mu \Phi) - \frac{1}{2} H^{\beta\mu\nu} n^\gamma H_{\mu\nu\gamma} \right] \right].
\]

(5.23)

If \( \nabla_\mu \Phi \) and \( H_{\mu\nu\lambda} \) vanish sufficiently rapidly at infinity they will not contribute to the surface integrals (this is the condition that there is no matter flux out through spatial infinity), and the second line in (5.23) vanishes. A conserved energy-momentum vector is obtained by taking \( \xi_\mu = \xi_\mu(t) \) at the boundary,

\[
P^\beta \propto -\int d^{D-2}x (-\gamma)^{1/2} e^{-2\Phi} \left[ \gamma^{0\beta} (\Theta + 2n^\mu \nabla_\mu \Phi) - \Theta^{0\beta} \right].
\]

(5.24)

A conserved angular momentum can be obtained in a similar way by allowing for spatial dependence in \( \xi_\mu \) at infinity.

Actually, (5.24) diverges even for empty Minkowski space as the boundary is taken to infinity, which is why one needs to include the counterterm action \( S_{ct} \) in (5.3). The form of \( S_{ct} \) depends on the asymptotic boundary conditions that have been chosen; examples for asymptotically flat and asymptotically AdS spacetimes can be found in \( \text{[24,25,26,27]} \).

In the asymptotically flat case the effect is to simply subtract from (5.24) the terms which are nonvanishing in empty Minkowski space.

It is instructive to compare expressions written in the string frame, such as (5.23), with their counterparts in the Einstein frame. Metrics in the string and Einstein frames are related as

\[
G_{\mu\nu}^E = e^{2\omega} G_{\mu\nu}, \quad \omega \equiv - \frac{2\tilde{\Phi}}{D - 2}, \quad \tilde{\Phi} \equiv \Phi - \Phi_0, \quad \kappa \equiv \kappa_0 e^{\Phi_0}.
\]

(5.25)
This filters through to the definitions of the unit normal vector to \( \partial V \) and the connection
\[
n^E_{\mu} = e^{\omega} n_{\mu}
\]
\[
\Gamma^E_{\alpha\mu\nu} = \Gamma^E_{\mu\nu} + \left( \delta^\alpha_\mu \partial_\nu \omega + \delta^\alpha_\nu \partial_\mu \omega - G_{\mu\nu} G^{\alpha\beta} \partial_\beta \omega \right).
\]
From this, we see that the extrinsic curvatures in the two frames are related as
\[
\Theta^E_{\mu\nu} = e^{\omega} \left( \Theta_{\mu\nu} - \gamma_{\mu\nu} n^\beta \nabla_\beta \omega \right),
\]
\[
\Theta^E = e^{-\omega} \left( \Theta - (D - 1) n^\beta \nabla_\beta \omega \right)
\]
and indices are raised and lowered with the metric appropriate for a given frame. The expression for the gravitational stress-energy in Einstein frame converts to string frame as
\[
\Theta^E_{\gamma\mu\nu} - \Theta^E_{\mu\nu} = e^{\frac{\phi}{D-2}} \left( \Theta_{\gamma\mu\nu} - \Theta_{\mu\nu} + 2\gamma_{\mu\nu} n^\beta \nabla_\beta \Phi \right).
\]
Finally, diffeomorphism parameters in the two frames are related as
\[
\xi^E_{\mu} = e^{2\omega} \xi_{\mu}
\]
\[
\delta \xi G^E_{\mu\nu} = e^{2\omega} \left( \delta \xi G_{\mu\nu} + 2G_{\mu\nu} \xi^\beta \partial_\beta \omega \right).
\]
Hence the first line of (5.23) reads
\[
\delta S = -\frac{1}{2\kappa^2} \int_{\partial V} d^{25} x (-\gamma)^{1/2} e^{-2\Phi} \left\{ 2\nabla_\alpha \xi_\beta \left[ \gamma^{\alpha\beta} (\Theta + 2n^\mu \nabla_\mu \Phi) - \Theta^{\alpha\beta} \right] \right\}
\]
\[
= -\frac{1}{2\kappa^2} \int_{\partial V} d^{25} x (-\gamma_E)^{1/2} \left\{ 2\nabla_\alpha^E \xi_\beta^E \left[ \gamma_E^{\alpha\beta} \Theta_E - \Theta^{\alpha\beta}_E \right] + 4 \xi^E_\mu \nabla_\mu \Phi \Theta_E \right\}.
\]
The last term is again a flux, and does not contribute to the conserved charges provided \( \nabla_\mu \Phi \) falls off sufficiently rapidly at spatial infinity. From the term proportional to \( \nabla_\alpha^E \xi_\beta^E \) we can read off the energy-momentum vector in the Einstein frame, and this is known to agree with the standard ADM definition. Indeed, the definition of energy-momentum is unique if one demands that \( P_{\mu} \) is conserved, transforms like a Lorentz vector under asymptotic Lorentz transformations, and is additive for distant subsystems \([29]\).

### 6. Conserved charges in string theory

We would like to repeat the analysis of the previous section in the context of the string path integral. The basic idea is to examine the behavior of the partition function under spacetime gauge transformations.
6.1. Open string

We first consider the case of the open string with a nontrivial gauge field

\[ S = \frac{1}{2\pi\alpha'} \int_{D_2} d^2z \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + i \oint_{\partial D_2} d\theta A_\mu(X) \frac{\partial X^\mu}{\partial \theta}. \]  

(6.1)

We are working on the unit disk with flat metric and boundary coordinate \( \theta \). Spacetime gauge invariance corresponds to the fact that \( \delta \xi A_\mu = \partial_\mu \xi \) simply adds a total derivative to the worldsheet Lagrangian, which then integrates to zero provided one chooses a suitable regularization scheme.

Now, consider a gauge field variation such that \( \mathcal{O} = \delta A_\mu(X) \frac{\partial X^\mu}{\partial \theta} \) is a dimension \( h = 1 \) boundary operator; i.e. such that \( \delta A_\mu \) satisfies the linearized spacetime equations of motion expanded around \( A_\mu \). (3.14) gives us a formula for the one point function of \( \mathcal{O} \). Since the one point function is expressed as a boundary term at infinity, what matters is the behavior of \( A_\mu \) and \( \delta A_\mu \) for large values of the constant modes \( x^i \). Therefore we expand in powers of nonconstant modes, writing \( X^\mu = x^\mu + \tilde{X}^\mu \), where \( \tilde{X}^\mu \) are the nonconstant modes as in (2.7). At leading order we can write

\[ \delta A_\mu(X) \frac{\partial X^\mu}{\partial \theta} = \delta A_\mu(x) \frac{\partial \tilde{X}^\mu}{\partial \theta} + \ldots. \]  

(6.2)

Higher order terms in the expansion will be seen to give a vanishing contribution provided we assume that a derivative expansion is valid at spatial infinity (that is, that \( \partial_\mu \sim 1/r \)). Applying (3.14) we have

\[ \langle \delta A_\mu(X) \frac{\partial X^\mu}{\partial \theta}(z, \bar{z}) \rangle_{D_2} = -\frac{1}{2} \langle V_M \rangle_0^{D/2-1} \int d^{D-1} \delta A_\mu(x) \int d^2z' \]

\[ \left\{ \left( z' + \frac{z}{2z} \right) (z' - z) \langle \partial X^i(z', \bar{z}') \frac{\partial \tilde{X}^\mu}{\partial \theta}(z, \bar{z}) \rangle_{D_2}' + \left( z' + \frac{\bar{z}'}{2\bar{z}} \right) (\bar{z}' - \bar{z}) \langle \bar{\partial} X^i(z', \bar{z}') \frac{\partial \tilde{X}^\mu}{\partial \theta}(z, \bar{z}) \rangle_{D_2}' \right\}. \]  

(6.3)

Now we take \( \delta A_\mu = \partial_\mu \xi(X^0) \). The left hand side of (6.3) vanishes since it is the \( \theta \) derivative of a \( \theta \) independent quantity. Upon integrating the right hand side by parts, we get an equation stating that the following charge is conserved

\[ Q = \int d^{D-2} z' \int d^2z' \left\{ \left( z' + \frac{z}{2z} \right) (z' - z) \langle \partial X^i(z', \bar{z}') \frac{\partial \tilde{X}^0}{\partial \theta}(z, \bar{z}) \rangle_{D_2}' \right. \]

\[ + \left( z' + \frac{\bar{z}'}{2\bar{z}} \right) (\bar{z}' - \bar{z}) \langle \bar{\partial} X^i(z', \bar{z}') \frac{\partial \tilde{X}^0}{\partial \theta}(z, \bar{z}) \rangle_{D_2}' \right\}. \]  

(6.4)

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(6.4) gives our result for the conserved electric charge in string theory. To check it we now compute \( Q \) in a background such that all gauge invariant combinations of fields die off at spatial infinity; \textit{i.e.} a localized charge/current distribution. At infinity we can therefore expand \( A_\mu \) as

\[
\oint_{\partial D_2} d\theta A_\mu(X) \frac{\partial X_\mu}{\partial \theta} = \frac{1}{2} F_{\mu\nu}(x) \oint_{\partial D_2} d\theta \tilde{X}_\mu \frac{\partial \tilde{X}_\nu}{\partial \theta} + \ldots, \tag{6.5}
\]

Discarded terms will not contribute to \( Q \). Since \( F_{\mu\nu}(x) \) goes to zero at infinity, at the boundary it makes sense to expand in powers of \( F_{\mu\nu}(x) \), hence we need only compute correlation functions in the free CFT on the disk. The contribution to \( Q \) at zeroth order in \( F_{\mu\nu} \) is easily seen to vanish, so the leading nonzero contribution is

\[
Q = \frac{1}{2} \int d^{D-2}S^i \int d^2 \theta' I \int d^2 \theta'' \left\{ \left( \frac{z' + z}{2z} \right) (z' - z) \langle \partial X^i(z', \bar{z}') \frac{\partial \tilde{X}^0}{\partial \theta}(z, \bar{z}) \tilde{X}^\alpha(z'', \bar{z}'') \frac{\partial X^\beta}{\partial \theta}(z'', \bar{z}'') \rangle_{0, D_2} \\
+ \left( \frac{\bar{z}' + \bar{z}}{2\bar{z}} \right) (z' - z) \langle \partial X^i(z', \bar{z}') \frac{\partial \tilde{X}^0}{\partial \theta}(z, \bar{z}) \tilde{X}^\alpha(z'', \bar{z}'') \frac{\partial X^\beta}{\partial \theta}(z'', \bar{z}'') \rangle_{0, D_2} \right\} \\
= 4\pi^2 (\alpha')^2 \langle 1 \rangle_{D_2} D^2 \int d^{D-2}S^i \ F^{0i}(x), \tag{6.6}
\]

where \( \langle \ldots \rangle_0 \) indicates that expectation values are with respect to the first term of (6.1). We therefore find a conserved charge of the same form as in (5.8):

\[
Q \propto \int d^{D-2}S^i \ F^{0i}. \tag{6.7}
\]

For this to be finite \( Fr^0 \) must fall off as \( 1/r^{D-2} \), and so all higher powers in the expansion will give vanishing contributions. Not surprisingly, the conserved charge (6.7) derived from string theory has exactly the same form as in field theory. We should emphasize that to reach this conclusion we assumed that all gauge invariant fields and derivatives die off asymptotically; if these conditions are relaxed one would expect to find corrections to low energy field theory.

\textbf{6.2. Closed string}

In the closed string the relevant symmetries are anti-symmetric tensor gauge transformations and spacetime coordinate transformations. We consider asymptotically flat
backgrounds in which \( H_{\mu\nu\lambda}, \, T_c, \) and \( \nabla_\mu \Phi \) go to zero asymptotically. Any asymptotically flat spacetime has – by definition – a metric that can be brought to the form

\[
ds^2 = -\left(1 - \frac{\mu}{r^{D-3}} + O\left(\frac{1}{r^{D-2}}\right)\right)dt^2 - \left(\frac{A^{ij}x^i}{r^{D-1}} + O\left(\frac{1}{r^{D-2}}\right)\right)dx^j dt
+ \left[1 + \frac{\mu}{r^{D-3}} + O\left(\frac{1}{r^{D-2}}\right)\right]\delta_{ij} + \frac{\epsilon^{ij}}{r^{D-3}} + O\left(\frac{1}{r^{D-2}}\right)\right]dx^i dx^j.
\]

(6.8)

\( \mu \) is proportional to the ADM mass of the spacetime. The anti-symmetric tensor \( A^{ij} \) is proportional to the angular momentum, and the symmetric traceless tensor \( \epsilon^{ij} \) represents gravitational radiation. For an isolated system, \( \mu \) and \( A^{ij} \) are constants. In the case of string theory, the above statements hold for the Einstein frame metric.

We will assume that we have a nonradiating system, and that a derivative expansion is valid in the asymptotic region. The latter assumption means that \( \partial_\mu \) acting on any field brings down at least one power of \( 1/r \). The case with radiation present would be interesting to study further, but is much more involved.

In the asymptotic region we will write \( G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and expand the action (2.1) as

\[
S = 2\Phi(\infty) + \frac{1}{2\pi\alpha'} \int d^2 z \, \eta_{\mu\nu} \partial \tilde{X}^\mu \partial \tilde{X}^\nu \\
+ \frac{1}{2\pi\alpha'} \int d^2 z \, \left\{ \partial_\lambda h_{\mu\nu}(x) + \frac{1}{3} H_{\mu\nu\lambda}(x) \right\} \partial \tilde{X}^\mu \partial \tilde{X}^\nu \tilde{X}^\lambda + \frac{\epsilon^{2\omega(z, \tau)}}{4} \left[ \partial_\lambda T_c(x) + \alpha' R \partial_\lambda \Phi(x) \right] \tilde{X}^\lambda \right\} \\
+ \ldots.
\]

(6.9)

6.3. Anti-symmetric tensor

For the anti-symmetric tensor consider the operator

\[
\mathcal{O} = [\partial_\mu \Lambda_\nu(X) - \partial_\nu \Lambda_\mu(X)] \partial \tilde{X}^\mu \partial \tilde{X}^\nu = \partial \left[ \Lambda_\mu(X) \partial \tilde{X}^\mu \right] - \overline{\partial} \left[ \Lambda_\mu(X) \partial \tilde{X}^\mu \right].
\]

(6.10)

Since \( \mathcal{O} \) is a total derivative, it can be added to the worldsheet action without effect; this is a gauge transformation of \( B_{\mu\nu} \). To derive the corresponding conserved charge we will use the fact that \( \langle \mathcal{O} \rangle_{S_2} \) vanishes when integrated over the worldsheet.

Now, in the free theory defined by the first line of (6.9) \( \mathcal{O} \) is a dimension \( (1, 1) \) operator and we can define the product of \( X \)'s by the standard normal ordering procedure. However, both of these statements are modified in the interacting theory with nontrivial spacetime
fields. Fortunately, we are only interested in the structure of $\mathcal{O}$ in the asymptotic region, and this can be deduced by including the terms in the second line of (5.9).

In particular, when we compute correlation functions with insertions of $\mathcal{O}$ we will find divergences from collisions with $\partial \bar{X}^{\mu} \bar{\partial} \bar{X}^{\nu} \bar{X}^{\lambda}$. Using the OPE to compute the required counterterm we find the following renormalized operator in the asymptotic region (restricting attention to the asymptotic region means we only keep terms with at most two spacetime derivatives)

$$
\mathcal{O} = \partial \left[ \Lambda_{\mu}(X) \partial \bar{X}^{\mu} - c \ln \Lambda \left( \partial_{\lambda} h_{\mu\nu}(x) + \frac{1}{3} H_{\mu\nu\lambda}(x) \right) \partial^{\mu} \Lambda^{\lambda}(x) \partial \bar{X}^{\nu} \right] - \bar{\partial} \left[ \Lambda_{\mu}(X) \partial \bar{X}^{\mu} - c \ln \Lambda \left( \partial_{\lambda} h_{\mu\nu}(x) + \frac{1}{3} H_{\mu\nu\lambda}(x) \right) \partial^{\nu} \Lambda^{\lambda}(x) \partial \bar{X}^{\mu} \right].
$$

(6.11)

Now we use (3.6). We need only compute to first order in spacetime fields to get the nonvanishing surface integrals. So consider

$$
\int d^2 z' \left( z' - z \right) e^{2\omega(z', \bar{z}')} n_{i}(\partial_{\mu} \Lambda_{\nu}(x) - \partial_{\nu} \Lambda_{\mu}(x)) \langle \partial X^{i}(z', \bar{z}') \partial X^{\mu}(z, \bar{z}) \partial \bar{X}^{\nu}(z, \bar{z}) \delta L(z'', \bar{z}'') \rangle_{0, S_2},
$$

(6.12)

where $\delta L(z'', \bar{z}'')$ stands for the operators appearing in the second line of (6.9). If we choose a homogeneous metric on $S_2$ then $\langle \mathcal{O}(z, \bar{z}) \rangle_{S_2}$ will be independent of $z$; it is convenient to take $z = 0$. We will take $\Lambda_{\mu}$ to depend only on $X^0$, since this is the dependence that is needed in order to derive conserved charges. Now, $\partial X^{i}(z', \bar{z}')$ can contract against either $\partial X^{\mu}(0)$, $\bar{\partial} X^{\nu}(0)$ or a field in $\delta L(z'', \bar{z}'')$. But the former case gives zero after performing the angular part of the $d^2 z'$ integral. Furthermore, $\partial X^{i}(z', \bar{z}')$ cannot contract against the tachyon and dilaton terms in $\delta L(z'', \bar{z}'')$, since this would leave a contraction between $\partial X^{\mu}(0)$ and $\bar{\partial} X^{\nu}(0)$ which vanishes when anti-symmetrized. This just leaves the contraction of $\partial X^{i}(z', \bar{z}')$ with the $h_{\mu\nu}$ and $H_{\mu\nu\lambda}$ terms. Performing the various contractions and integrals, and cancelling divergences against the counterterm in (6.11), we find

$$
\langle \mathcal{O} \rangle_{S_2} = \int dtd^{D-2} S^i e^{-2\Phi} \partial_{0} \Lambda_{j} \left\{ a_1 H_{0ij} + a_2 \partial_{0} h_{ij} + a_3 \partial_{j} h_{0i} \right\}.
$$

(6.13)

The coefficients $a_{1,2,3}$ can in principle be computed, but this is not needed. Assuming our standard falloff behavior the second and third terms do not contribute to the surface integral. The $a_2$ term vanishes because $\mu$ in (6.8) will be shown in the next subsection to be conserved, and since $\partial_0$ brings down at least one power of $1/r$ all other terms give vanishing surface integrals. The expansion (6.8) also immediately shows that the $a_3$ term does not contribute.
Now, the left hand side of \((6.13)\) vanishes upon integration over the worldsheet since \(O\) is a total derivative, so upon integrating by parts on the right hand side we find the conserved charges
\[
Q_j \propto \int d^{D-2}S^i e^{-2\Phi} H_{0ij}.
\] (6.14)
This agrees with the expected result from low energy field theory, \((7.21)\). A typical situation is to have translation invariance along some spatial direction \(x^j\), giving the conserved charge per unit length \(\int d^{D-3}S^i H_{0ij}\).

6.4. Gravitational field: ADM mass

Our goal here is to derive a conserved energy-momentum vector from string theory. More generally, we could also try to derive the conserved angular momentum tensor, include the effects of radiation, and so on. Our scope will be more limited, hopefully laying the groundwork for a more complete treatment in the future. Also, in string theory there are complications due to the dilaton which we do not entirely understand, as will be discussed.

The relevant symmetry to be considered is spacetime diffeomorphism invariance. In particular, \(\mu\) is the conserved charge associated with asymptotic time translation invariance. Let us then examine spacetime diffeomorphism invariance at the level of the sigma model action \((2.1)\), (considering only the spherical worldsheet). Consider the variations
\[
\begin{align*}
\delta_\xi G_{\mu\nu}(X) &= \nabla_\mu \xi_\nu(X) + \nabla_\nu \xi_\mu(X), \\
\delta_\xi B_{\mu\nu}(X) &= \xi^\lambda(X) \nabla_\lambda B_{\mu\nu}(X) + \nabla_\mu \xi^\lambda B_{\lambda\nu}(X) + \nabla_\nu \xi^\lambda B_{\mu\lambda}(X), \\
\delta_\xi T_c(X) &= \xi^\lambda(X) \partial_\lambda T_c(X), \\
\delta_\xi \Phi(X) &= \xi^\lambda(X) \partial_\lambda \Phi(X).
\end{align*}
\] (6.15)
The action \((2.1)\) is invariant under the combination of \((6.15)\) and
\[
\delta_\xi X^\mu = -\xi^\mu(X).
\] (6.16)
Therefore, the partition function
\[
Z_{S_2} [G, B, T, \Phi] = \langle 1 \rangle_{S_2} = \int \mathcal{D}X \ e^{-S}
\] (6.17)
obeys
\[
\delta_\xi Z_{S_2} [G, B, T, \Phi] = V_M \int \mathcal{D}X \ \xi^\mu(X) \frac{\delta S}{\delta X} e^{-S} = -(V_M)^{D/2} \int d^Dx \ \frac{\partial}{\partial x^\mu}(\xi^\mu(X))_{S_2}'
\] (6.18)
In arriving at the last line of (6.18) we have used the same chain of manipulations as in (2.12), and singularities in operator products have been removed as in (2.16). (6.18) is the expected behavior under a spacetime diffeomorphism. A generic diffeomorphism invariant functional

\[ S = \int d^Dx (-G)^{1/2} \mathcal{L} \]  

(6.19)

transforms under (6.15) as

\[ \delta_\xi S = \int d^Dx (-G)^{1/2} \nabla_\mu \{ \xi^\mu \mathcal{L} \} = \int d^Dx \frac{\partial}{\partial x^\mu} \left\{ \xi^\mu (-G)^{1/2} \mathcal{L} \right\}. \]  

(6.20)

(6.18) and (6.20) are consistent when we remember that \( \langle \xi^\mu \rangle' \) transforms as \( (-G)^{1/2} \) times a vector.

As usual, we will consider a boundary at spatial infinity. To obtain true symmetries of the partition function we can take \( \xi^\mu \) tangent to the boundary, \( n_\mu \xi^\mu = 0 \) so that \( \delta_\xi Z = 0 \). The conserved charges associated with these symmetries are mass, linear momentum, and angular momentum.

To show this, consider a Weyl invariant sigma model corresponding to some asymptotically flat field configuration. We take the matter fields to fall off sufficiently rapidly at infinity so that no energy-momentum flows through the boundary (we will make this more precise momentarily). As with the gauge field and anti-symmetric tensor, the idea is to consider some on-shell variation of the gravitational field so that the variation of the partition function is a boundary term. We will set the tachyon to zero in the following. With the variations in (6.15) define

\[ \mathcal{O} = \delta_\xi G_{\mu\nu}(X) \partial X^\mu \overline{\partial X}^\nu + \delta_\xi B_{\mu\nu}(X) \partial X^\mu \overline{\partial X}^\nu + \frac{\alpha'}{4} \epsilon^{2\omega} R \delta_\xi \Phi(X). \]  

(6.21)

Counterterms need to be added to define (6.21) as in (6.11), but we will not write them explicitly. Now, adding \( \mathcal{O} \) to the worldsheet Lagrangian preserves Weyl invariance, since its effect can be undone by the field redefinition (6.16). Therefore, \( \mathcal{O} \) must be a dimension (1, 1) operator, plus a total derivative on the worldsheet, plus a possible number times the Euler number density. The latter is absent, as seen by taking \( \xi^\mu = 0 \). We know from (6.18) (taking \( n_\mu \xi^\mu = 0 \) at the boundary) that \( \int d^2z \langle \mathcal{O}(z, \overline{z}) \rangle_{S^2} = 0 \). Combining this with (3.6) we have

\[ \int d^{D-1}S_1 \int d^2z \int d^2z' (z' - z) \langle \partial X^i(z', \overline{z}') \mathcal{O}(z, \overline{z}) \rangle'_{S^2} = 0. \]  

(6.22)
Which terms in (6.21) can contribute to the surface integral? Some contributions correspond to the time rate of change of the gravitational “charge” and others correspond to currents flowing through the boundary. The former are linear in the perturbations about flat space near the boundary, while the latter are at least quadratic and so give vanishing surface integrals provided one adopts standard falloff conditions. It is not hard to see that only the metric and dilaton can contribute linear terms. Therefore, we have

\[
\int d^{D-1}S_i \int \frac{d^2 z}{2} \int \frac{d^2 z'}{2} \left( z' - z \right) \left\langle \partial X^i(z', \bar{z}') \left\{ (\nabla_{\mu} \xi_\nu(X) + \nabla_{\nu} \xi_\mu(X)) \partial X^\mu \bar{\partial} X^\nu + \frac{\alpha'}{4} e^{2\omega(z, \bar{z})} R \xi_\mu \nabla_\mu \Phi(X) \right\} \right\rangle'_{S_2} = 0.
\] (6.23)

At the boundary we can replace \( \xi^\mu(X) \) by its constant mode part \( \xi^\mu(x) \). We take \( n^\nu \nabla_\nu \xi^\mu(x) = 0 \) at the boundary so that we can integrate by parts to get

\[
\int d^{D-1}S_i \xi_\mu(x) \nabla_\nu \left\{ \int d^2 z \int d^2 z' \left( z' - z \right) e^{2\omega(z', \bar{z})} \right\}
\left\langle \partial X^i(z', \bar{z}') \left\{ \partial X^\mu \bar{\partial} X^\nu + \partial X^\mu \bar{\partial} X^\nu - \frac{\alpha'}{4} e^{2\omega(z, \bar{z})} R \bar{\Phi}(X) \eta^\mu_\nu \right\} \right\rangle'_{S_2} = 0.
\] (6.24)

We defined \( \bar{\Phi} = \Phi - \Phi(\infty) \) since this is what contributes in (6.23), and replaced \( G_{\mu\nu} \) by the asymptotic Minkowski metric since deviations give vanishing surface integrals when multiplied by \( \bar{\Phi} \). This implies the following expression for the conserved energy-momentum

\[
P^\mu \propto \int d^{D-2}S_i \int \frac{d^2 z}{2} \int \frac{d^2 z'}{2} \left( z' - z \right) e^{2\omega(z', \bar{z})} 
\left\langle \partial X^i(z', \bar{z}') \left\{ \partial X^0 \bar{\partial} X^\mu + \partial X^\mu \bar{\partial} X^0 - \frac{\alpha'}{4} e^{2\omega(z, \bar{z})} R \bar{\Phi}(X) \eta^0_\mu \right\} \right\rangle'_{S_2}.
\] (6.25)

We now check our result for the mass \( P^0 \) for the general asymptotically flat background with vanishing tachyon. We work in coordinates such that the metric takes the form (6.8). The anti-symmetric tensor and dilaton can be taken to fall off as \( 1/r^{D-3} \). Since the dilaton adds some complications we will first consider the case in which the dilaton falls off faster than \( 1/r^{D-3} \), in which case we can disregard the dilaton term in (6.23).

To evaluate the remaining correlation functions in (6.23) we need only use the first order perturbation of the sigma model around Minkowski space, since higher order terms will yield vanishing surface integrals. To compute \( P^0 \) we need to contract a \( n_i \partial X^i \), a \( \partial X^0 \), and a \( \bar{\partial} X^0 \) against the first order perturbations in the sigma model action. The
contribution of the dilaton will be proportional to \( n^i \partial_i \partial_0^2 \Phi(x) \) so we get a surface integral of a function falling off at least as rapidly as \( 1/r^D \), and this vanishes. Similarly, for the anti-symmetric tensor, by anti-symmetry at least one of \( \partial X^0 \) or \( \overline{\partial} X^0 \) must contract against an \( X \) in \( B_{\mu \nu}(X) \). Then together with the \( \partial X^i \) contraction we get the surface integral of a function falling as least as rapidly as \( 1/r^{D-1} \), which vanishes. This leaves only the metric perturbation. Examining the asymptotic form of the metric given in (6.8), we see that the only nonvanishing surface integral comes from an insertion of the operator \( G_{00}(X) \partial X^0 \overline{\partial} X^0 \). Asymptotically,

\[
G_{00}(X) \partial X^0 \overline{\partial} X^0 = -\left[ 1 - \frac{\mu}{r^{D-3}} + (D-3) \frac{\mu}{r^{D-1}} x^i \tilde{X}^i + O\left( \frac{1}{r^{2D-1}} \right) \right] \partial \tilde{X}^0 \overline{\partial} \tilde{X}^0 .
\] (6.26)

We therefore have

\[
n_i \langle \partial X^i(z', \overline{z}') \partial X^0 \overline{\partial} X^0(z, \overline{z}) \rangle_{S_2} \propto \frac{\mu}{r^{D-2}} .
\] (6.27)

Finally, from (6.25) we find

\[
P^0 \propto \mu ,
\] (6.28)

which is the desired result. There is a multiplicative factor relating \( \mu \) to the ADM mass and which is not fixed by our considerations since we are just looking for a conserved quantity. The numerical factor could be fixed by computing the explicit correlators in (6.27). We chose to work in the center of mass coordinate system (6.8) in which \( P^i = 0 \), but we could repeat the analysis in a boosted coordinate system. The result for \( P^\mu \) is of course fixed by asymptotic Lorentz invariance, but it might be useful to check this directly.

Now we generalize by allowing the dilaton to fall off as \( 1/r^{D-3} \). From the low energy field theory analysis, we know that the conserved energy-momentum is directly related to the asymptotic behavior of the Einstein metric; see (5.30). In particular, if the 00 component of the string metric and dilaton behave as

\[
G_{00}(x) \sim -1 + \frac{\mu_s}{r^{D-3}} ,
\]

\[
\Phi(x) \sim \Phi(\infty) + \frac{Q \Phi}{r^{D-3}} ,
\] (6.29)

then, since the Einstein metric is \( G^{E}_{\mu \nu} = e^{-4\Phi/(D-2)} G_{\mu \nu} \), the conserved energy is proportional to

\[
\mu = \mu_s + \frac{4}{D-2} Q \Phi .
\] (6.30)

On the other hand, it is easy to verify that with the assumed fall off conditions the formulas for momentum are unchanged by the presence of the dilaton.
On the worldsheet, we see the corresponding effect of the dilaton from (6.25), which clearly shifts the conserved energy but not the momentum. Furthermore, the shift is proportional to $Q_\Phi$, since $\langle \partial X^i \Phi \rangle$ acts as a radial derivative of $\Phi$, and then the surface integral picks out the leading piece proportional to $Q_\Phi$. The $\frac{4}{D-2}$ prefactor in (6.30) is more difficult to establish, as it involves computing relatively complicated correlators on a curved worldsheet. Presumably, the correct value of the prefactor follows from a general worldsheet principle, but we have not been able to identify this so far. This certainly deserves further study.

7. Discussion

The purpose of this work was to study the role of boundary terms in solutions to string theory with noncompact target spaces. The particular application developed here was defining conserved gauge charges as surface integrals at infinity. In particular, this led to an intrinsic CFT definition of the energy-momentum of an asymptotically flat spacetime. Finding such a definition was an outstanding challenge in early studies of string solitons (see, for example, [32]). We have succeeded in doing this here, though a number of details such as the dilaton dependence should certainly be developed more fully.

There are several interesting open questions and directions for further research.

7.1. Anti-de Sitter Spacetimes

In AdS, the conserved charges associated with diffeomorphisms are the generators of conformal transformation of the boundary. As shown by Brown and Henneaux [33], this is especially interesting in the case of AdS$_3$, where one gets two copies of the Virasoro algebra with central charge $c = 3\ell/2G$. This result holds for any asymptotically AdS$_3$ spacetime, with the conserved charges expressed as surface integrals. It is interesting to try to reproduce this result from string theory, and for pure AdS$_3$ (with NS-NS B-field) this was done in [8,34]. This should generalize to the asymptotically AdS$_3$ case using the

\footnote{Part of the complication here is the lack of a convenient regulator preserving the spacetime gauge symmetries, especially since we need to work on a curved worldsheet for finiteness of $V_M$. Dimensional regularization suffers from ambiguities [31] due to the appearance of $\lim_{d \to 2}(R_{ab} - \frac{1}{2}R g_{ab})/(d-2)$. Heat kernel regularization preserves the symmetries, but is awkward for extracting a finite part.}
approach developed here. If we write the metric in the form
\[ ds^2 = d\phi^2 + g_{\mu\nu}(\phi, x)dx^\mu dx^\nu, \] (7.1)

the conserved energy-momentum tensor can then be presumably be written as roughly (modulo dilaton terms)
\[ T^{\mu\nu} \sim \int d^2 z \int d^2 z' (z' - z)e^{2\omega(z', \bar{z}')} \langle \partial \phi(z', \bar{z}') \partial X^\mu \partial \bar{X}^\nu + \partial X^\nu \partial \bar{X}^\mu \rangle(z, \bar{z}) |_{S_2}. \] (7.2)

7.2. Spacetime action and the string partition function

Since in field theory conserved charges arise as symmetries of the action, it would seem that the most efficient way to pass to string theory would be to first give a CFT definition of the spacetime action. The idea that the spacetime action should be closely related to the string partition function goes back to work of Fradkin and Tseytlin [35] and has been investigated by many authors since, but is still not completely understood. Some of the confusions have to do with obtaining a suitable off-shell action, which is perhaps unnatural in a theory with gravity. Fortunately, to derive conserved charges one only needs the on-shell action, and it seems reasonable to suppose that this is equal to the string theory vacuum-to-vacuum amplitude. For the open string at the level of disk amplitudes this works nicely (and can be extended off-shell) [11,12,13,14]:
\[ S_{\text{open}} = \frac{1}{\text{Vol}_{SL(2,R)}} \left( \det'_{D_2} P_1^1 P_1 \right)^{1/2} e^{-\lambda} Z_{D_2}. \] (7.3)

The functional integrals can be computed unambiguously [36,37], \( e^{-\lambda} \) is related to the gravitational coupling \( \kappa \) by unitarity [16], and \( \text{Vol}_{SL(2,R)} \) can be assigned a finite “renormalized” value [38]. For instance, the correct value of the D-brane tension can be obtained this way. So for the open string we could have formulated things in this framework.

For the closed string the formula analogous to (7.3) is not as successful. The main problem is that \( \text{Vol}_{SL(2,C)} \) is divergent even after “renormalization” due to the appearance of a logarithmic divergence. Tseytlin [19] has proposed a spacetime action by essentially cancelling this divergence against a similar divergence in the sphere partition function. Unfortunately, this proposal does not seem to successfully reproduce the boundary terms in the spacetime action [39]. The latter are crucial, especially in our context; for instance they give the entire result in spacetimes with constant dilaton.
A better understanding of this issue is important for the AdS/CFT correspondence, since this is supposed to equate the boundary CFT partition function in the presence of sources to the spacetime action with prescribed boundary conditions. The former is generically nonzero, but the latter naively vanishes in string theory due to the division by $\text{Vol}_{SL(2,\mathbb{C})}$. It has been suggested that for AdS$_3$ there is a compensating divergence from the spacetime volume integration, given that Euclidean AdS$_3$ is the coset $SL(2,\mathbb{C})/SU(2)$. Besides the fact that this cancellation is specific to AdS$_3$, it does not give the correct result even in this case. A direct comparison can be made since the sphere partition function for AdS$_3$ is readily computed, and the partition function of the boundary theory (in the absence of sources but for a general boundary metric) is determined by the conformal anomaly.

We believe that the resolution is along the lines of the present paper. We have seen that in noncompact spacetimes worldsheet $SL(2,\mathbb{C})$ symmetry can be violated by boundary terms, so simply dividing all amplitudes by the divergent $\text{Vol}_{SL(2,\mathbb{C})}$ factor is not justified. Hopefully, the correct procedure leaves boundary terms in a way similar to what we have seen here for one-point functions.

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