Improved Approximations for Multiprocessor Scheduling Under Uncertainty

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This paper presents improved approximation algorithms for the problem of multiprocessor scheduling under uncertainty (SUU), in which the execution of each job may fail probabilistically. This problem is motivated by the increasing use of distributed computing to handle large, computationally intensive tasks. In the SUU problem we are given \( n \) unit-length jobs and \( m \) machines, a directed acyclic graph \( G \) of precedence constraints among jobs, and unrelated failure probabilities \( q_{ij} \) for each job \( j \) when executed on machine \( i \) for a single timestep. Our goal is to find a schedule that minimizes the expected makespan, which is the expected time at which all jobs complete.

Lin and Rajaraman gave the first approximations for this NP-hard problem for the special cases of independent jobs, precedence constraints forming disjoint chains, and precedence constraints forming trees. In this paper, we present asymptotically better approximation algorithms. In particular, we give an \( O(\log \log(\min\{m, n\})) \)-approximation for independent jobs (improving on the previously best \( O(\log n) \)-approximation). We also give an \( O(\log(n + m) \log(\log(\min\{m, n\})) \)-approximation algorithm for precedence constraints that form disjoint chains (improving on the previously best \( O\left(\log(n) \log(m) \frac{\log(n + m)}{\log \log(n + m)}\right) \)-approximation by a \((\log n / \log \log n)^2\) factor when \( n = m^{\Theta(1)} \). Our algorithm for precedence constraints forming chains can also be used as a component for precedence constraints forming trees, yielding a similar improvement over the previously best algorithms for trees.

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1 INTRODUCTION

Our work concerns approximation algorithms for multiprocessor scheduling under uncertainty, first introduced in [12]. This model extends the classical construction of machine scheduling to handle cases where machines run jobs for discrete timesteps and succeed in processing them only probabilistically. Our motivation stems from the increasing use of distributed computing to handle large, computationally intensive tasks. Projects like Seti@Home [1] divide computations into smaller jobs of relatively uniform length, which are then executed on unreliable machines (e.g., of volunteers).

Scheduling multiple machines to process the same job at once can help overcome the problem of unreliable machines, but many machines processing a single job can also slow down overall throughput. The situation is exacerbated when precedence constraints among jobs are present, which is often the case for sophisticated computations; here a single job failing may delay the start of many others. Note that the special case of having no precedence constraints retains practical significance. Google’s MapReduce architecture [3], for example, generates jobs whose dependencies form a complete bipartite graph, which is equivalent to two phases of independent jobs.

Motivated by these examples, we study the multiprocessor scheduling under uncertainty (SUU) problem. An SUU instance is comprised of a set of $n$ unit-time jobs and a set of $m$ machines. For each machine $i$ and job $j$, we are given a failure probability $q_{ij}$, which is the chance that job $j$ does not complete when run on machine $i$ for a single timestep. Any precedence constraints are modeled as a directed acyclic graph (dag). Our objective is to construct a schedule assigning machines to eligible jobs at each timestep, minimizing the expected time until all jobs have successfully completed. In contrast to many other scheduling problems, SUU allows multiple machines to execute the same job in a single timestep.

Related work

Malewicz’s initial presentation of SUU [12] includes a polynomial-time dynamic-programming solution for instances where both the number of machines and the width of the precedence dag are constant. If either of these constraints is relaxed, he proves that the problem becomes NP-Hard. Furthermore, when both constraints are removed, there is no polynomial-time approximation algorithm for the problem achieving an approximation ratio lower than $5/4$, unless $P = NP$. This work does not include approximation algorithms for the general (NP-Hard) problem.

Lin and Rajaraman present the first (and, to date, only) approximation algorithms for SUU [11]. Using a greedy approximation algorithm to maximize the chance of success across all jobs, they give an $O(\log n)$-approximation when all jobs are independent. More sophisticated techniques, including LP-rounding and random delay [9,15], yield a variety of $O(\text{poly log}(n + m))$ approximations when precedence graphs are constrained to form only disjoint chains, collections of in- or out-trees, and directed forests. For these settings, our algorithms improve their approximation ratios by a $(\log n/\log \log n)^2$ factor, when $n = m^{\Theta(1)}$. See Table 1 for a complete comparison.

The wider field of machine scheduling is an established and well-studied area of research with a large number of variations on its core theme (see [6] for a survey). There are three main differences between SUU and problems studied in the literature. First, in SUU, jobs may run on multiple machines in the same timestep. Second, each job has a chance of failing to complete on any machine that processes it. Third, jobs must be scheduled at unit granularity.

Of deterministic scheduling problems, SUU most closely resembles $R|\text{prec, pmtn}|C_{\text{max}}$ [7,9,15], the problem of preemptively scheduling jobs with precedence constraints on unrelated parallel machines so as to minimize makespan. Instead of failure probabilities $q_{ij}$, there is a deterministic processing time $p_{ij}$, denoting how long it takes for machine $i$ to complete job $j$. In contrast to SUU, however, machines never fail, and jobs may only run on one machine at a time. As in [11], techniques
for this problem play an important role in our approximations. We also borrow techniques from “job-shop scheduling” [5] for our SUU algorithms for precedence constraints, but the particulars of that setting are not very similar to the ones we consider here.

There is also a large body of work in stochastic scheduling (see [14, Part 2] for a representative sample). The majority of the work in this area considers how to schedule jobs whose input lengths are not known, but instead given as random variables distributed according to some probability distribution. Particular attention has been paid to the case when these distributions are restricted to exponential families, which is similar in some respects to SUU (see Appendix C). However, we know of no approximation algorithms, even with this restriction, when machines are unrelated.

**Table 1: Improved approximation ratios**

| Precedence Constraints | Lin and Rajaraman [11] | This work |
|------------------------|------------------------|-----------|
| Independent            | $O(\log(n))$           | $O(\log(\min\{m, n\}))$ |
| Disjoint Chains        | $O\left(\frac{\log(m) \log(n) \log(n+m)}{\log\log(n+m)}\right)$ | $O(\log(n+m) \log \log(\min\{m, n\}))$ |
| Directed Forests       | $O\left(\frac{\log(m) \log^2(n) \log(n+m)}{\log\log(n+m)}\right)$ | $O(\log(n+m) \log(n) \log \log(\min\{m, n\}))$ |

**Our results**

We give improved approximation algorithms for SUU when jobs are independent (there are no precedence constraints), and when the precedence constraints form disjoint chains. For independent jobs, we give an $O(\log(\log(\min\{m, n\})))$-approximation algorithm. One component of this algorithm is based on an LP relaxation.

Our analysis for independent jobs relies on a competitive analysis [17]. Essentially, we show that our algorithm is $O(\log(p_{\text{max}}/p_{\text{min}}))$-competitive for a deterministic scheduling problem (similar to $R|\text{pmtn}|C_{\text{max}}$) in which each machine has a deterministic speed, but processing times for jobs are chosen arbitrarily by an adversary, with minimum and maximum values $p_{\text{min}}$ and $p_{\text{max}}$. This competitive result is interesting in its own right.

When the precedence constraints form a collection of disjoint chains, we have an $O(\log(n + m) \log \log(\min\{m, n\}))$-approximation algorithm. Our disjoint-chains algorithm uses an LP relaxation similar to the one used for independent jobs. We also apply techniques from network-flow theory and prior work on the SUU problem for chains [11]. The $\log \log(\min\{m, n\})$ factor arises from the independent-jobs algorithm — therefore improving that algorithm immediately yields a better algorithm for chains.

Our algorithm for disjoint chains can be extended to yield an $O(\log(n+m) \log(n) \log \log(\min\{m, n\}))$-approximation for directed forests using the chain-decomposition techniques of [7, 11].

We also show how to apply our algorithms to similar variants in the problem of stochastic scheduling, where jobs have stochastic processing times. To the best of our knowledge, these are the first approximation algorithms for stochastic scheduling with the expected-completion-time objective and unrelated machines.

**Paper organization**

In Section 2 we give formal definitions of the SUU problem, the scheduling algorithms that we apply to it, and an equivalent formulation of SUU that plays an important role in our approximation algorithms. A full treatment of this reformulation is given in Appendix A. Section 3 presents our algorithms for independent jobs, and Section 4 shows how to extend them to handle precedence constraints forming chains. We defer tree-like precedence constraints and stochastic scheduling to Appendices B and C, respectively.
2 PRELIMINARIES

In this section, we give a formal statement of our problem and then define what we mean by a schedule. Most of our notation is consistent with that of Malewicz [12] or Lin and Rajaraman [11]. We then present a reformulation of the SUU problem, which we use in subsequent sections to simplify both our algorithms and the analysis involved.

The SUU problem

An instance \( I = (J, M, \{q_{ij}\}, G) \) of the SUU problem includes a set \( J \) of unit-step jobs and a set \( M \) of machines. Throughout this paper, we let \( n = |J| \) be the number of jobs and \( m = |M| \) be the number of machines. For each machine \( i \) and job \( j \), we are given a failure probability \( q_{ij} \), which is the probability that job \( j \) does not complete when run on machine \( i \) for one unit step; these probabilities are independent. Without loss of generality, we assume that for each job \( j \), there exists a machine \( i \) such that \( q_{ij} < 1 \).

An SUU instance also includes a set of precedence constraints comprising a directed acyclic graph (dag) \( G \) with jobs as vertices. We say that a job \( j \) is eligible for execution at time \( t \) if all jobs preceding \( j \) (i.e., jobs having a directed path to \( j \)) in the dag have successfully completed before time \( t \). If a job \( j \) is eligible at time \( t \), a schedule may assign multiple machines \( M_{j,t} \subseteq M \) to execute \( j \) in parallel. As all machine/job failures are independent, the probability that \( j \) does not complete in that timestep is \( \prod_{i \in M_{j,t}} q_{ij} \).

Failure probabilities are difficult to work with because they multiply. Instead, we define the log failure of job \( j \) on machine \( i \), denoted by \( \ell_{ij} \), as \( \ell_{ij} = -\log q_{ij} \). Here and throughout the paper, we use log to mean a base-2 log. Note that by definition, \( q_{ij} = 1/2^{\ell_{ij}} \), and hence \( \prod_{i \in M_{j,t}} q_{ij} = 1/2^{\sum_{i \in M_{j,t}} \ell_{ij}} \). We define the log mass of an assignment to job \( j \) in step \( t \) as the sum of the log failures, given by \( \sum_{i \in M_{j,t}} \ell_{ij} \). We also use log mass to refer to the sum of log masses across multiple timesteps; that is, the log mass accrued from step \( t_1 \) through step \( t_2 \) is given by \( \sum_{t=t_1}^{t_2} \sum_{i \in M_{j,t}} \ell_{ij} \).

Our work focuses on finding scheduling algorithms that minimize expected makespans for restricted classes of precedence constraints. If there are no precedence constraints, we say that the jobs are independent, and refer to the problem as SUU-I. When the precedence constraints form a collection of chains, we call the problem SUU-C. When the constraints form a collection of disjoint trees, we call the problem SUU-T. We use sans serif fonts to refer to problem variants, whereas serif fonts refer to algorithms/schedules for the problem.

Schedules

A schedule \( \Sigma \) is a policy for assigning machines to (uncompleted) jobs. Jobs must be scheduled at a unit granularity, but the schedule may assign multiple machines to the same job. A schedule may base its decisions on any of its history, but we concern ourselves with only schedules that can be computed in polynomial time. More formally, a schedule is a function \( \Sigma : (H \times \mathbb{N}) \rightarrow (M \rightarrow J \cup \{\bot\}) \) that, given a history\(^1\) \( h \in H \) and time \( t \in \mathbb{N} \), returns a function assigning machines to jobs. We use the symbol \( \bot \) to indicate that the machine remains idle. To allow for more concise schedules, the assignment function returned by \( \Sigma(h, t) \) may map a machine to a job that has already completed.

We define an execution of \( \Sigma \) as follows. Suppose that \( h \in H \) is the history of the execution up to time \( t \). Then \( \Sigma \) assigns machine \( i \) to job \( j = \Sigma(h, t)(i) \) at step \( t \). If \( j \) has been completed when

\(^1\)A full history for a deterministic schedule can be captured by the sets of remaining jobs at each timestep prior to the current timestep \( t \). More formally, let \( H_t = \{(S_1, S_2, \ldots, S_t) | J = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_t \} \) denote the set of all feasible ordered sets of remaining jobs at timesteps 1, 2, \ldots, \( t \). Then valid histories are given by the set \( H = \bigcup_{t=1}^{\infty} H_t \). Note that compact representations of the history exist, so a polynomially computable schedule may consider the entire history.
it is scheduled to run, \( i \) is assigned to \( \bot \). Since \( J, M, \{q_{ij}\}, \) and \( G \) are invariant over a problem instance, we allow \( \Sigma \) to reference those implicitly.

Whenever the schedule \( \Sigma \) is such that it assigns machines to jobs depending only on the current time and the initial set of jobs, not the jobs that have completed (i.e., for all \( t, \Sigma(h, t) = \Sigma(h', t) \) for all \( h, h' \in H \)), we say that the schedule is oblivious. An oblivious schedule has finite length if it is only defined for \( t \leq t_o \), for some \( t_o \).

We say that a schedule is semioblivious if it can be decomposed into “rounds” such that the assignments within each round are characterized by finite oblivious schedules. Thus, while executing a step contained in a particular round, the assignment of machines to jobs depends only on the initial set of jobs when the round began and the number of steps the round has been running.

We let \( T_\Sigma \) be a random variable denoting the length of the execution of schedule \( \Sigma \), which is the number of steps before all jobs have completed. Our objective is to minimize \( E[T_\Sigma] \) (denoted by \( E[C_{\text{max}}] \) in much of the scheduling literature). We refer to a schedule that has minimum expected makespan as \( \Sigma_{\text{OPT}} \), and its expected makespan, which is finite [12], as \( E[T_{\text{OPT}}] \). For any SUU instance, \( \Sigma_{\text{OPT}} \) exists, and can be computed (inefficiently) by selecting the assignment of jobs to machines on a particular timestep that minimizes the expected makespan of the remaining jobs.

In this paper, we consider schedules that are polynomial-time computable (in \( n, m, \) and \( \log E[T_{\text{OPT}}] \)) and whose expected makespans approximate \( E[T_{\text{OPT}}] \). We say that \( \Sigma \) is an \( \alpha \)-approximation if \( E[T_\Sigma] \leq \alpha E[T_{\text{OPT}}] \) for all choices of probabilities \( \{q_{ij}\} \).

Throughout the remainder of this paper, we use algorithm and schedule interchangeably. Moreover, we generally do not give the schedule explicitly as a function assigning machines to jobs. Instead, we describe it algorithmically.

**Problem reformulation**

We now describe a new, and equivalent formulation of the SUU problem, which we refer to as SUU". Because of their equivalence, we refer to both problems as SUU later in the paper. A full treatment is presented in Appendix A.

An SUU" instance \( I = (J, M, \{q_{ij}\}, G) \) has the same structure as an SUU instance. The difference is that rather than considering the success or failure of a job as it runs on machines in each timestep, we use the Principle of Deferred Decisions [13] to view the problem as one of deterministically scheduling jobs with randomly distributed lengths.

Instead of failure probability, in SUU" we view \( \ell_{ij} = -\log q_{ij} \) as an amount of “work” that a machine does towards a job completion in each unit timestep. As in SUU, machines must be scheduled at a unit granularity. At the start of a schedule’s execution, we draw for each job \( j \) a single random variable \( r_j \) chosen uniformly at random from the \((0,1)\) interval. A job \( j \) completes when the total work done (or, “log mass” accrued) on \( j \) exceeds \(-\log r_j \). That is, \( j \) completes at the first step \( t \) in which \( \sum_{k=1}^{t} \sum_{i \in M_j} \ell_{ij} \geq -\log r_j \). As schedules are oblivious to these \( r_j \), they behave the same way on SUU and SUU" instances.

### 3 INDEPENDENT JOBS

This section describes an \( O(\log \log n) \)-approximation algorithm for SUU-I, the SUU problem with independent jobs. We first give an oblivious \( O(\log n) \)-approximation algorithm for SUU-I, based on scheduling an (approximation of an) integer linear program. We then modify this algorithm into a semioblivious solution consisting of \( O(\log \log n) \) nearly optimal phases.

**An oblivious \( O(\log n) \)-approximation**

We now describe an \( O(\log n) \)-approximation for SUU-I, called SUU-I-OBL. Our approach constructs a schedule of length \( O(E[T_{\text{OPT}}]) \), based on an integer linear program, such that each job has no more
than a constant probability of failure upon completion. This finite oblivious schedule is repeated until all jobs have completed. Using Chernoff bounds, we conclude that the expected number of repetitions is $O(\log n)$, yielding an $O(\log n)$-approximation.

We use the following integer linear program for SUU-I-OBL. Let $x_{ij}$ denote the number of steps during which machine $i$ is assigned to job $j$. Recall $\ell_{ij} = -\log q_{ij}$ is the log failure of job $j$ on machine $i$. Let $L$ be a fixed positive real, representing a target log mass for each job, and let $J' \subseteq J$ be a subset of jobs that need to achieve that log mass. For now, think of $L$ as being fixed at $L = 1/2$, and $J' = J$. We assign different values to $L$ later for the semioblivious $O(\log \log n)$-approximation.

\begin{equation}
(LP1) \quad \min t \\
\text{s.t.} \quad \sum_{i \in M} \ell_{ij} x_{ij} \geq L \quad \forall j \in J' \\
\sum_{j \in J} x_{ij} \leq t \quad \forall i \in M \\
x_{ij} \in \mathbb{N} \cup \{0\} \quad \forall i \in M, j \in J.
\end{equation}

Here Equation (1) enforces that every job in $J$ has a failure probability no greater than $1/\sqrt{2}$, and Equation (3) guarantees that all jobs are scheduled for an integral number of steps on each machine. We use (LP1) to refer to this integer linear program generically, and $LP1(J', L)$ to refer to it with particular values of $J'$ and $L$. We denote the optimal value for $LP1(J', L)$ by $t_{LP1(J', L)}$.

A solution for $LP1(J', L)$ naturally generalizes to a finite oblivious schedule, denoted by $\Sigma_{LP1(J', L)}$, with length $t_{LP1(J', L)}$ as follows. Consider a machine $i$, and consider each job $j$ in arbitrary order. Assign machine $i$ to job $j$ for $x_{ij}$ timesteps. To finish our description of this schedule, we first claim that $t_{LP1(J,1/2)}$ approximates $E[T_{OPT}]$ (the proof appears in Appendix D). Then we show how to approximate (LP1) in polynomial time.

**Lemma 1** $t_{LP1(J,1/2)} = O(E[T_{OPT}])$

The following lemma states that, in polynomial time, we can find an integral assignment that approximates (LP1) to within a constant factor. Some aspects of the proof are similar to [11, Theorem 4.1], but we add several steps that improve the approximation ratio.

**Lemma 2** There exists a polynomial-time algorithm that computes a feasible solution to $LP1(J', L)$ having value $O(t_{LP1(J', L)})$.

**Proof.** We relax our integer linear program to a linear program, and then show that the relaxed LP can be rounded to yield an integral $\{\hat{x}_{ij}\}$ solution with value $O(T_{LP1(J', L)})$.

First, let $\ell'_{ij} = \min\{\ell_{ij}, L\}$. Then we replace each $\ell_{ij}$ in Equation (1) with $\ell'_{ij}$, yielding the constraint $\sum_{i \in M} \ell'_{ij} x_{ij} \geq L, \forall j \in J'$. Note that since assignments are restricted to be integral, this change has no effect on either the feasibility or the value of an assignment. Next we remove Equation (3) and solve the relaxed linear program. Letting $\{x^{*}_{ij}, t^{*}\}$ be an optimal solution, we note that $t^{*} \leq t_{LP1(J', L)}$, because integral solutions are feasible.

Our goal now is to round the LP solution to an integral solution, while not increasing its value by very much. We proceed in three steps. First, we group machines having similar $\ell'_{ij}$ for a job $j$, yielding a single assignment for the whole group. Then, we round those assignments to integers. Finally, we show that the rounded assignments satisfy (LP1), using an integral flow network.

For each job $j$, we group machines having $\ell'_{ij}$ values within a factor of 2, and determine the total assignment to that group. More formally, for each $j$ and integer $k$, we let $D^{*}_{jk} = \sum_{i : \lfloor \log \ell'_{ij} \rfloor = k} x^{*}_{ij}$ be the total assignment of machines with $\ell'_{ij} \in [2^{k}, 2^{k+1})$ to job $j$. It should be clear that $\sum_{i \in M} \ell'_{ij} x^{*}_{ij} \geq \sum_{k} D^{*}_{jk} 2^{k} \geq L/2$, for all $j \in J'$.
We next round the value of $D^*_{jk}$ up to $\lfloor 6D^*_{jk} \rfloor$. We claim that $\sum_k |6D^*_{jk}| 2^k \geq L, \forall j \in J'$. Note that since $\ell'_{ij} \neq L$, the maximum value of $k$ having nonzero $D^*_{jk}$ is $\lfloor \log L \rfloor$. Thus, the claim follows because $\sum_k |6D^*_{jk}| 2^k \geq 3(2 \sum_k D^*_{jk} 2^k) - (\sum_{k \leq \log L} 2^k) \geq 3(L) - (\sum_{k = 0}^{\infty} L/2^k) \geq 3L - 2L = L$. In other words, rounding the group assignments down to integers can only cause us to lose a log mass of at most $2L$. We thus need an assignment giving $3L \log$ mass to the job.

To complete the integral assignment, we construct a network-flow instance as follows. For each job $j$ and integer $k$, we have a node $u_{jk}$. For each machine $i$, we have a node $v_i$. We also add a source-node $s$ and a sink-node $w$. For each $u_{jk}$, we add a directed edge $(s, u_{jk})$ with capacity $\lfloor 6D^*_{jk} \rfloor$. For each $v_i$, we add a directed edge $(v_i, w)$ with capacity $\lfloor 6t^* \rfloor$. Finally, we add a directed edge $(u_{jk}, v_i)$ with infinite capacity, for any $j, k, i$ such that $|\log \ell'_{ij}| = k$. Note that for a given $j$ and $i$, there is exactly one $k$ such that $(u_{jk}, v_i)$ exists. We refer to this edge as edge $(j, i)$.

Note that if we make the capacity of edges $(s, u_{jk})$ be $6D^*_{jk}$ instead, then a flow of demand $\sum_{jk} 6D^*_{jk}$ exists in this network. Thus, a flow of capacity $\sum_{jk} |6D^*_{jk}|$ exists when we lower the capacity of edge $(s, u_{jk})$ to $\lfloor 6D^*_{jk} \rfloor$.

Ford-Fulkerson’s theorem [2, 4] states that an integral max flow exists whenever the capacities are integral, as they are here. We therefore take the flow across the edges $(j, i)$ as our integral assignments $\hat{x}_{ij}$. Moreover, by construction, $\hat{x}_{ij}$ satisfy $\sum_{j \in J} \hat{x}_{ij} \leq \lfloor 6t^* \rfloor$ for all $i \in M$, and $\sum_{i \in M} \ell'_{ij} \hat{x}_{ij} \geq \sum_k |6D^*_{jk}| 2^k \geq L$ for all $j \in J$. We thus have an integral feasible solution $\{\hat{x}_{ij}, 6t^*\}$. Noting that $6t^* \leq 6T_{LP1(J, L)}$ completes the proof.

Recall that Lemma 1 shows that $t_{LP1(J, 1/2)} = O(E[T_{OPT}])$. Then Lemmas 1 and 2 in concert state that in polynomial time, we can find a schedule $\Sigma$ of length $O(E[T_{OPT}])$, such that every job has at most a constant probability of failure. Repeating $\Sigma$ until all jobs complete gives our oblivious schedule SUU-I-OBL. The proof appears in Appendix D.

**Theorem 3** Let $T_{SUU-I-OBL}$ denote the random variable corresponding to the amount of time it takes for an execution of SUU-I-OBL to complete all jobs. Then $E[T_{SUU-I-OBL}] = O(E[T_{OPT}] \log n)$.

A semioblivious $O(\log \log(\min\{m, n\}))$-approximation

We construct our semioblivious schedule SUU-I-SEM as follows. The schedule is divided into “rounds.” The first round corresponds to an execution of the schedule suggested by the (rounded) solution to $LP1(J, 1/2)$. In each following round, ($LP1$) is applied to all remaining jobs with doubling targets. If $J_k \subseteq J_{k-1} \subseteq J$ are the set of jobs left at the start of round $k$, then for that round we find an approximate solution to $LP1(J_k, 2^{k-2})$, and schedule obliviously according to it.

SUU-I-SEM runs at most $K = \lfloor \log \log \min\{m, n\} \rfloor + 3$ of these rounds. If uncompleted jobs remain after the $K$th round, one of two things is done. If $n \leq m$, SUU-I-SEM runs each job one at a time on all machines, until all jobs are completed. If $m < n$, SUU-I-SEM simply repeats the schedule $\Sigma_{LP1(J_k, 2^{k-2})}$ given by the $K$th round until all jobs complete.

The following theorem states that SUU-I-SEM achieves an $O(\log \log(\min\{m, n\}))$-approximation. The key aspect of our analysis is viewing SUU-I-SEM as an “online algorithm” to solve the SUU problem over the hidden input $\{r_j\}$. We compare the length of SUU-I-SEM’s schedule against that of an optimal offline algorithm, called OFF, that knows the values $\{r_j\}$. In particular, we show that if OFF takes total time $t$ on input $\{r_j\}$, then each round of SUU-I-SEM takes time $O(t)$ on the same input. This part of our proof is essentially a competitive analysis [17].

**Theorem 4** Let $K = \lfloor \log \log \min\{m, n\} \rfloor + 3$ and let $T_{SUU-I-SEM}$ denote the random variable corresponding to the amount of time it takes for an execution of SUU-I-SEM to complete all jobs. Then $E[T_{SUU-I-SEM}] = O(E[T_{OPT}] \cdot K)$. 

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PROOF. First, we show that for any fixed set of random values \( \{r_j\} \), each round of SUU-I-SEM takes time proportional to the optimal strategy \( \text{OFF} \). Then, we show that the expected time to complete the first \( K \) rounds is \( O(E[T_{\text{OPT}}] \cdot K) \). Finally, we analyze the cases when SUU-I-SEM does not complete in \( K \) rounds.

Consider an optimal offline strategy \( \text{OFF} \) that knows the random values of \( \{r_j\} \), and let \( T_{\text{OFF}}(\{r_j\}) \) denote \( \text{OFF} \)'s makespan given the values \( \{r_j\} \) (i.e., \( T_{\text{OFF}}(\{r_j\}) \) is the minimum over all strategies for a fixed \( \{r_j\} \)). For each \( k \in \{2, 3, \ldots, K\} \), let \( J_k \subseteq J \) be the subset of jobs such that \( -\log r_j > 2^{k-3} \). Thus, \( \text{OFF} \) must assign machines to job \( j \in J_k \) such that \( \sum_{ij \in \mathcal{M}} x_{ij} \ell_{ij} \geq 2^{k-3} \). Hence, jobs executed in the \( k \)th round are a subset of those defined by \( J_k \) above. By Lemma 2, the \( k \)th round takes time \( O(T_{LP1}(J_k, 2^{k-3})) \). Observing that \( T_{LP1}(J_k, 2^{k-3}) \leq 2T_{LP1}(J_k, 2^{k-3}) \), we conclude that SUU-I-SEM’s \( k \)th round takes time \( O(T_{OFF}(\{r_j\})) \).

We thus have that for any particular \( \{r_j\} \), if SUU-I-SEM completes in \( d + 1 \leq K \) rounds, then it takes total time \( O(E[T_{OPT}] + dT_{OFF}(\{r_j\})) \). The \( E[T_{OPT}] \) term comes from the time it takes to execute the first round (as in Lemma 1). Since \( T_{OPT} \geq E[T_{OFF}] \), where \( T_{OFF} \) is the time taken by \( \text{OFF} \) on a randomly selected \( \{r_j\} \), we conclude that the expected time for SUU-I-SEM’s first \( d + 1 \leq K \) rounds is \( O(E[T_{OPT}] \cdot d) = O(E[T_{OPT}] \cdot K) \).

We now consider the case when SUU-I-SEM has not completed after \( K \) rounds. Let \( F_K \) be Bernoulli r.v. that is 1 if all jobs have completed after the \( K \)th round and 0 otherwise. Clearly, \( E[T_{SUU-I-SEM} | F_K = 1] = O(E[T_{OPT}] \cdot K) \). To complete the proof, we must show that \( \Pr\{F_K = 0\} E[T_{SUU-I-SEM} | F_K = 0] \leq O(E[T_{OPT}] \cdot K) \). We consider the remainder of the proof in two cases, depending on whether \( n \leq m \) or \( m < n \). The case of \( m < n \) appears in Appendix D.

Suppose \( n \leq m \) and \( F_K = 0 \). Recall that after the \( K \)th round, SUU-I-SEM runs jobs one after the other. Recall also that remaining jobs must have \( -\log r_j \geq 2^{K-2} \geq 2 \log \log n + 1 \geq 2 \log n \). This event occurs when \( r_j < 1/n^2 \), which, by the union bound, happens with probability no more than \( 1/n \). Running jobs one at a time is trivially an \( O(n) \)-approximation, so we conclude that \( \Pr\{F_K = 0\} E[T_{SUU-I-SEM} | F_K = 0 \land n \leq m] \leq O(E[T_{OPT}] K) \).

4 JOBS WITH CHAIN-LIKE PRECEDENCE CONSTRAINTS

This section gives our \( O(\log(n + m) \log \log(\min\{m, n\})) \)-approximation for SUU-C, the case when precedence constraints form a collection of disjoint chains. Our algorithm may be used as a subroutine for SUU-T, the more general case where precedence constraints form disjoint trees (see Appendix B).

In SUU-C, the dependency graph \( G \) is a collection disjoint chains \( G = \{C_1, C_2, \ldots, C_{z_2}\} \), where each \( C_k \) gives a total order on a subset of jobs. If job \( j_1 \) precedes \( j_2 \) in a chain, we write \( j_1 < j_2 \).

Our algorithm for disjoint chains is similar to Lin and Rajaraman’s algorithm [11], but we achieve a better approximation ratio through various improvements. We first give an overview of the algorithm. We provide more details later in the section.

To construct our schedule, we first find assignment \( \{x_{ij}\} \) of machines to jobs (where \( x_{ij} \) is an integral number of steps for which machine \( i \) is assigned to job \( j \) ), giving each job one unit of log mass, such that the “length” and “load” of the assignment are bounded by \( O(E[T_{OPT}]) \). The load of a machine is the number of timesteps for which any job is assigned to it (i.e., \( \sum_j x_{ij} \)), and the load of the assignment is the maximum across all machines. The length of a chain is the sum of the length of the jobs in the chain. The length of a job \( j \), denoted by \( d_j \), is the maximum number
of steps for which \( j \) is assigned to a single machine (i.e., \( d_j = \max_i x_{ij} \)). Clearly, a schedule taking time \( T \) must have a length and load no more than \( T \).

We use an LP relaxation (similar to (LP1) in Section 3) to generate our assignment. Details appear later in the section. As in Section 3, our LP relaxation achieves an \( O(1) \)-approximation. Note that this assignment does not immediately yield a schedule.

As we transform our assignment into an adaptive schedule, we treat long and short jobs differently. We say that a job is \textit{short} if the length of its assignment is at most some value \( \gamma \), to be defined later, and the job is \textit{long} otherwise. To simplify presentation, suppose for now that all jobs are short. We later describe how to deal with long jobs.

We then transform the assignment into an adaptive schedule \( \Sigma_k \) for each chain \( C_k \). The schedule \( \Sigma_k \) considers the next eligible (uncompleted) job \( j \) in \( C_k \), and (obliviously) schedules the next \( d_j \) timesteps according to the assignment \( \{x_{ij}\} \). Specifically, if \( \Sigma_k \) begins executing job \( j \) at time \( t \), then it schedules \( j \) from time \( t \) to \( t + x_{ij} \) on machine \( i \). (Machine \( i \) remains idle from time \( t + x_{ij} \) to \( t + d_j \).) After the \( d_j \) timesteps, \( \Sigma_k \) again considers the next eligible job in the chain (which may be the same job if it failed). We note that each time job \( j \) is obliviously scheduled, it has a constant probability of success.

We then combine all the \( \Sigma_k \) in a straightforward manner, yielding a “pseudoschedule” for the SUU-C instance, denoted by \( \{\Sigma_k\} \). In particular, a \textit{pseudoschedule} runs all \( \Sigma_k \) “in parallel,” possibly assigning multiple jobs to the same machine in each timestep. To avoid confusion, we call each of the timesteps of a pseudoschedule a \textit{superstep}, and we call the number of jobs assigned to a single machine during a superstep \( t \) the \textit{congestion} at that superstep, denoted by \( c(t) \). We “flatten” each superstep to \( c(t) \) timesteps by arbitrarily ordering the jobs assigned to each machine, thus yielding a schedule called SUU-C. If \( c_{\text{max}} \) is the maximum congestion over all supersteps, and \( Z \) is the maximum length of any chain, then SUU-C comprises \( O(c_{\text{max}}Z) \) timesteps.

To reduce congestion, we apply a random-delay technique [9, 16], also used by Lin and Rajaraman [11]. We also utilize the fact that when chains consist of sufficiently many (short) jobs, the number of supersteps spanned by \( \Sigma_k \) is near the expected length of \( \Sigma_k \), with high probability.

To deal with long jobs, we run SUU-I-SEM \( O(\log(n + m)) \) times, which dominates the runtime, yielding the \( O(\log(n + m) \log \log(\min(m, n))) \)-approximation.

**Finding an assignment with low load and length.**

As in Section 3, we use an integer linear program to optimize for the constraints. This integer linear program for chains matches that used in [11, LP1].

\[
\text{(LP2)} \quad \begin{align*}
\min \ t \\
\text{s.t.} & \quad \sum_{i \in M} \ell_{ij} x_{ij} \geq 1 \quad \forall j \in J \quad (4) \\
& \quad \sum_{j \in J} x_{ij} \leq t \quad \forall i \in M \quad (5) \\
& \quad \sum_{j \in C_k} d_j \leq t \quad \forall C_k \in G \quad (6) \\
& \quad 0 \leq x_{ij} \leq d_j \quad \forall i \in M, j \in J \\
& \quad d_j \geq 1 \quad \forall j \in J \\
& \quad x_{ij} \in \mathbb{N} \cup \{0\} \quad \forall i \in M, j \in J . 
\end{align*}
\]

Equations (4), (5), and (9) correspond to Equations (1), (2), and (3), respectively, in (LP1). Equation (5) bounds the load of each machine. Equation (6) bounds the length of each chain, and Equations (7) and (8) determines the length of each job.

The following lemma, proven in [11, Lemma 4.2], states that the optimal value for (LP2) is a lower bound on \( \mathbb{E}[T_{\text{OPT}}] \).
Lemma 5 Let \( t_{(LP2)} \) be the optimal value for \((LP2)\). Then \( t_{(LP2)} = O(E[T_{OPT}]) \).

The next lemma exhibits an \( O(1) \)-approximation to \((LP2)\). Lemmas 5 and 6 together imply a polynomial-time algorithm giving an integral assignment \( \{x_{ij}\} \) of machines to jobs, such that the load and length are both \( O(E[T_{OPT}]) \).

Lemma 6 Let \( t_{(LP2)} \) be the optimal value for \((LP2)\). There exists a polynomial-time algorithm that computes a feasible solution to \((LP2)\) having value \( O(t_{(LP2)}) \).

Proof. The rounding proceeds as in Lemma 2, starting by removing Equation (9) and replacing Equation (4) by \( \sum_{i \in M} t_{ij}^* x_{ij} \geq 1 \), for \( t_{ij}^* = \min\{t_{ij}, 1\} \). The only major difference is in the capacity of some edges in the flow network. Instead of giving edge \((j, i)\) an infinite capacity, we restrict the capacity of edge \((j, i)\) to \([6d_j^*]\), where \( d_j^* \) is the assignment given by the optimal solution to the relaxed linear program. We note that the length of a chain \( C_k \) may increase up to at most \( 6 \sum_{j \in C_k} d_j^* + |C_k| \leq 7 \sum_{j \in C_k} d_j^* \). \( \square \)

Reducing congestion of SUU-C

As described thus far, SUU-C may have \( \Theta(n) \) congestion. We take advantage of a random-delay technique [9, 16] to reduce congestion to \( O(\frac{\log(n+m)}{\log \log(n+m)}) \), with high probability. Essentially, we modify SUU-C to simply delay the start time of each chain by a value chosen uniformly at random from \( \{0, 1, \ldots, H\} \), where \( H \) is the load of SUU-C.

The delay technique is summed up by the following theorem, proof omitted (as similar theorems appear elsewhere). It originates in [9], and Lin and Rajaraman [11, Section 4.1] outline the necessary proof as applied to SUU-C.

Theorem 7 Consider a pseudoschedule \( \{\Sigma_k\} \) with total load \( H \), where \( H \) is polynomially bounded in \( n \) and \( m \). Consider the pseudoschedule \( \{\Sigma_{k'}\} \) generated by randomly shifting, or “delaying,” the start time of each chain schedule \( \Sigma_k \) by a value chosen uniformly at random from \( \{0, 1, \ldots, H\} \). Then \( \{\Sigma_{k'}\} \) has congestion at most \( O(\frac{\log(n+m)}{\log \log(n+m)}) \), with high probability with respect to \( n \) and \( m \). \( \square \)

Notice that whenever the load of and length of \( \{\Sigma_k\} \) are bounded by \( O(E[T_{OPT}]) \), it follows that the length of \( \{\Sigma_{k'}\} \) is at most \( O(E[T_{OPT}]) \) supersteps, with high probability.

Since \( \Sigma_k \) repeats the assignment for some jobs, the load and length of the pseudoschedule \( \{\Sigma_{k'}\} \) are random variables. We note, however, that the random successes and failures of jobs (and hence load and length) are independent of the initial random delay selected. Thus, as long as our random execution yields a load and length of \( O(E[T_{OPT}]) \), then Theorem 7 implies that SUU-C consists of \( O(E[T_{OPT}]) \) supersteps, for a total time of \( O(E[T_{OPT}]c_{max}) = O(E[T_{OPT}]\frac{\log(n+m)}{\log \log(n+m)}) \) steps.

The following lemma implies that most executions of SUU-C result in low load and length. In this lemma, \( y_j \) is the random variable indicating the number of repetitions of job \( j \)’s assignment used to complete \( j \), and \( d_j \) denotes the length of job \( j \)’s assignment. In the SUU-C context, \( \eta = n + m \). The value \( W \) here represents the load or the length of the assignment. The lemma states that whenever a job has length (or causes load) that is logarithmically smaller than the total, then the length of the chain or (load on a machine) is close to the expectation, with high probability. Union bounding over all \( O(n) \) chains (or \( m \) machines) implies that the total length (and load) of SUU-C schedule is close to the expectation, with high probability. The proof appears in Appendix D.

Lemma 8 Consider \( y_j \in \mathbb{N} \) drawn from the geometric distribution \( \Pr\{y_j = k \in \mathbb{N}\} = (1/2)^k \), and let \( 1 \leq d_j \leq W/\log \eta \) be a weight associated with each \( y_j \) for any values such that \( W \geq \sum_j 2d_j \) and \( \log \eta \leq W \). Then \( \sum_j y_j d_j \leq O(cT) \) with probability at least \( 1 - 1/\eta^c \).
We conclude that if jobs are short, where short jobs have length at most \( \gamma = t_{LP2} / \log(n + m) \), and if \( t_{LP2} \) is polynomial in \( n \) and \( m \), then SUU-C takes time \( O(E[T_{OPT}] \log(n + m) \log \log(n + m)) \) with high probability. To get this bound in expectation, we simply modify SUU-C to run the \( O(n) \)-approximation (as for SUU-I-SEM) whenever congestion, load, or length exceed the desired bounds, which occurs with probability at most \( 1/n \).

## Handling long jobs

We now extend SUU-C to handle jobs having length more than \( t_{LP2} / \log(n + m) \). In the chain schedule \( \Sigma_k \), we replace each longer job by a “pause” of length \( t_{LP2} / \log(n + m) \). Specifically, no job from the chain is scheduled until \( t_{LP2} / \log(n + m) \) supersteps later. We then divide our schedule SUU-C into \( O(\log(n + m)) \) segments of length \( t_{LP2} / \log(n + m) \) supersteps. Note that by construction, there is at most one pause per chain per segment. After executing each segment, SUU-C executes SUU-I-SEM on the jobs corresponding to the pauses starting in that segment (suspending the rest of the chains until completion). Once those long jobs complete, SUU-I-SEM continues to the next segment.

All of our previous analyses (that assume jobs are short) still hold. In particular, we satisfy the requirement that all long relevant long jobs complete before the short jobs are scheduled again. Since there are \( O(\log(n + m)) \) executions of SUU-I-SEM, it follows that the total expected time increases to \( O(E[T_{OPT}] \cdot \log(n + m) \log \log(\min\{m, n\})) \), and hence we have an \( O(\log(n + m) \log \log(\min\{m, n\})) \)-approximation.

## Extending to nonpolynomial \( t_{LP2} \)

We now address the requirement in Theorem 7 that load and length be polynomially bounded in \( n \) and \( m \). We make use of a trick from [16, Section 3.1], also used in [11]. Consider the chain schedule \( \Sigma_k \) (having length \( O(t_{LP2}) \), with high probability) before the random delay is applied. We round each assignment \( x_{ij} \) down to the nearest multiple of \( t_{LP2}/nm \). We thus treat the assignments as integers in the range \( \{0, 1, \ldots, O(nm)\} \). We can then apply the random-delay technique (from Theorem 7) to these rounded assignments.

The issue now is that the rounding may have decreased many assignments, so we reinsert steps into the schedule. In particular, whenever executing job \( j \), we reinsert steps (not supersteps) into the execution, executing only job \( j \) during those steps. Specifically, the execution of job \( j \) may result in reinserting at most an expected \( 2t_{LP2}/nm \) steps for each machine, and hence \( 2t_{LP2}/n \) steps in total. Summing across all \( n \) jobs gives an expected \( 2t_{LP2} \) steps, thereby increasing the total length of SUU-C by \( O(E[T_{OPT}]) \).

**Theorem 9** Let \( T_{SUU-C} \) denote the random variable indicating the time at which an execution of SUU-C completes all jobs. Then \( E[T_{SUU-C}] = O(E[T_{OPT}] \cdot \log(n + m) \log \log(\min\{m, n\})) \). \( \square \)

## 5 CONCLUSION

In this paper, we have presented improved approximation algorithms for multiprocessor scheduling under uncertainty. We believe that our bounds are not tight. In particular, we believe that a fully adaptive schedule should be able to trim an \( O(\log \log(\min\{m, n\})) \) factor from our bounds. It would also be interesting if a greedy heuristic could achieve the same bounds. Finally, we would be interested in developing nontrivial approximations for more general precedence constraints. At first glance, however, it seems like any technique for SUU and arbitrary precedence constraints may generalize to \( R|pmtn, prec|C_{max} \), which remains unsolved.
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A PROBLEM REFORMULATION

This section presents a full description of our reformulation of the SUU problem. We use this new formulation to simplify both our algorithms and the analyses involved.

To disambiguate between the original statement of the SUU problem given in Section 2 and the new one described herein, in this section we refer to the new formulation as SUU*. Since we show that they are equivalent, we refer to both problems as SUU later in the paper.

As with SUU, an SUU* instance includes a set J of jobs, a set M of machines, and a set of precedence constraints forming a dag. Also as before, the instance specifies a real q_{ij} \in [0,1] for each job j and machine i. In SUU, q_{ij} specifies a failure probability. In SUU*, however, we do not view q_{ij} as a probability; instead we view \ell_{ij} = -\log q_{ij} as an amount of “work” that a machine does towards a job completion in each unit timestep. As in SUU, machines must be scheduled at a unit granularity.

We model the stochastic nature of the problem in SUU* by associating with each job j a single random variable r_j chosen uniformly at random from the (0,1) interval. We say that a job j completes once the total work done (or log mass accrued) on j exceeds -\log r_j. More formally, let M_{j,t} be the set of the machines assigned to job j by a schedule on time t. Then j completes during the first step t in which \sum_{k=1}^t \sum_{i \in M_{j,k}} \ell_{ij} \geq -\log r_j, or equivalently \prod_{k=1}^t \prod_{i \in M_{j,k}} q_{ij} \leq r_j.

Note that a schedule \Sigma is oblivious to the random value r_j. Instead, it is only aware of whether a job completes in each timestep. Thus, a schedule must make its decisions for assignments in step t based only on the surviving sets of jobs S_1, S_2, \ldots, S_t for each of the preceding timesteps. Hence, the same schedule may be applied to both SUU and SUU*.

The following theorem states that SUU and SUU* have the same distribution over states of uncompleted jobs in each timestep.

**Theorem 10** Consider executions X and X* of schedule \Sigma on SUU-instance I = (J,M,\{q_{ij}\},G) and corresponding SUU* -instance I* = (J,M,\{q_{ij}\},G), respectively, that run for t - 1 timesteps. We define the history of the execution X (and X*) after t - 1 steps, denoted by h_t (and h_t*), as a sequence of job subsets h_t = \langle S_1, S_2, \ldots, S_t \rangle, where S_k \subseteq J is the set of uncomplete jobs remaining at the start of step k in the execution. For any state \langle S_1, S_2, \ldots, S_t \rangle, we have Pr \{ h_t = \langle S_1, S_2, \ldots, S_t \rangle \} = Pr \{ h_t* = \langle S_1, S_2, \ldots, S_t \rangle \}.

**Proof.** By induction on time t. Initially, Pr \{ h_1 = \langle J \rangle \} = Pr \{ h_1* = \langle J \rangle \} = 1.

Suppose h_t = h_t* = \langle S_1, S_2, \ldots, S_t \rangle. Then \Sigma makes the same decisions for assigning machines to jobs in step t in both executions (i.e., \Sigma(h_t,t) = \Sigma(h_t*,t)). Let M_{j,t} be the set of machines assigned to j by \Sigma(h_t,t). Let S_{t+1} and S_{t+1}* denote the random variables indicating the subsets of jobs remaining after executing step t in X and X*, respectively. We will show that S_{t+1} and S_{t+1}* have the same distribution.

For each job j \in S_t, the probability that X does not complete j in step t of the SUU execution X is given by Pr \{ j \in S_{t+1}|h_t \} = \prod_{i \in M_{j,t}} q_{ij}.

We now consider the probability that X does not complete j \in S_t in step t of the SUU* execution X*. By definition, j completes if \prod_{k=1}^t \prod_{i \in M_{j,k}} q_{ij} \leq r_j. By assumption, since j \in S_t, we have \prod_{k=1}^{t-1} \prod_{i \in M_{j,k}} q_{ij} > r_j. Thus,

\[
Pr \{ j \in S_{t+1}|h_t \} = Pr \left\{ r_j \leq \prod_{k=1}^t \prod_{i \in M_{j,k}} q_{ij} \right\} = Pr \left\{ r_j \leq \left( \prod_{i \in M_{j,t}} q_{ij} \right) \left( \prod_{k=1}^{t-1} \prod_{i \in M_{j,k}} q_{ij} \right) \right\} = Pr \left\{ r_j \leq \left( \prod_{k=1}^{t-1} \prod_{i \in M_{j,k}} q_{ij} \right) \right\}
\]
which applies here. The time they take to run by \( \sum \) is just as in one machine at the same time. An instance \( \lambda \) of \( i \) machine only is given in an instance, and \( j \) is not revealed until the job completes. Finally for each job \( j \), \( p_j \) is given in an instance, and \( p_j \) is not revealed until the job completes. Theorem 12 If precedence constraints form a directed forest, there exists a polynomially computable schedule with expected makespan \( O(\text{E}[T_{\text{OPT}}] \cdot \log(n) \log(n + m) \log(\log(\min\{m, n\})) \).

C STOCHASTIC SCHEDULING

This section shows how our algorithms from Sections 3 and 4 apply to the problem of preemptively scheduling jobs whose lengths are given by random variables on unrelated parallel machines. Specifically, we give polynomial time algorithms for problems of the form \( R|\text{pmtn}, \text{prec}, p_j \sim \text{stoch}|\text{E}[C_{\text{max}}] \) with approximation ratios for each type of precedence constraint identical identical to those of previous sections, so long as job lengths are drawn from exponential distributions with known means.

First, we review the stochastic model and how it differs from SUU. Then, we overview an \( O(\log \log n) \)-approximation for \( R|\text{pmtn}, p_j \sim \text{stoch}|\text{E}[C_{\text{max}}] \), which we designate STOCH-I. Finally, we discuss briefly how to generalize to cases with precedence constraints.

Preliminaries

An instance \( I_{\text{stoch}} = (J, M, \{\lambda_j\}, \{v_{ij}\}) \) of STOCH-I contains a set of jobs \( J \) and a set of machines \( M \) just as in SUU. For every job \( j \), \( \lambda_j \) specifies the rate parameter of the exponential distribution from which \( j \)'s length, denoted by the random variable \( p_j \), is drawn. That is, \( \Pr\{p_j \leq c\} = 1 - e^{-\lambda_j} \). Only \( \lambda_j \) is given in an instance, and \( p_j \) is not revealed until the job completes. Finally for each machine \( i \) and job \( j \), \( v_{ij} \) specifies the speed with which machine \( i \) processes job \( j \). Specifically, let \( x_{ij} \) be the amount of time during which machine \( i \) processes job \( j \). Then \( j \) completes once \( \sum_i x_{ij}v_{ij} \geq p_j \). This inequality should look very similar to SUU*, which is why our earlier algorithms apply here.

In STOCH-I, \( x_{ij} \) need not be integral, but we do require that no job be processed by more than one machine at the same time.

We continue to refer to optimal algorithms by OPT and (the random variable denoting) the time they take to run by \( T_{\text{OPT}} \).
An $O(\log \log n)$-approximation for STOCH-I

We now show how to provide an $O(\log \log n)$-approximation for STOCH-I, using arguments very similar to those in Section 3 and a constant-factor approximation algorithm for $R|\text{pmtn}|C_{\text{max}}$.

Our algorithm STC-I operates similarly to SUU-I-SEM. In particular, STC-I operates in $K = \lceil \log \log n + 3 \rceil$ rounds, each corresponding to an oblivious schedule $\Sigma_k$. We construct the oblivious $\Sigma_k$ such that any job having (stochastically chosen) $p_j \leq 2^{k-2}/\lambda_j$ completes. Specifically, $\Sigma_k$ corresponds to (approximately) solving the deterministic analog $R|\text{pmtn}|C_{\text{max}}$, setting the length of job $j$ to $2^{k-2}/\lambda_j$. Any jobs remaining after the end of these $K$ rounds is run one at a time on the fastest possible machine.

We use the algorithm from Lawler and Labetoulle [8] to compute an $O(1)$-approximation for $R|\text{pmtn}|C_{\text{max}}$ in polynomial time, giving us each of our $\Sigma_k$.

The following theorem states that STC-I approximates STOCH-I. Proof (omitted) is similar to Theorem 3 and Lemma 1.

**Theorem 13** Let $T_{\text{STC-I}}$ be the random variable denoting the time it takes for an execution of STC-I to complete all jobs. Then $E[T_{\text{STC-I}}] = O(E[T_{\text{OPT}}])$.

**Proof Sketch.** The full proof includes a component similar to Lemma 1, showing that the first round (solving for deterministic lengths $1/(2\lambda_j)$) approximates $E[T_{\text{OPT}}]$, and a component similar to Theorem 3, using an offline-algorithm argument to prove that the $K-1$ subsequent rounds take expected time $O(E[T_{\text{OPT}}] \cdot K)$.

To complete the proof, we note that when $p_j$ are bounded above by $2 \log n/\lambda_j$, then all jobs complete during (or before) $\Sigma_K$. Since the $p_j$ are exponentially distributed, $\Pr \{\exists j \text{ s.t. } p_j > 2 \log n/\lambda_j\} \leq 1/n$. Running jobs sequentially is an $n$-approximation, but this sequential run occurs only with probability at most $1/n$.

A virtually identical algorithm gives an $O(\log \log(n))$-approximation to the slightly weaker setting, $R|\text{restart}, p_j \sim \text{stoch}|E[C_{\text{max}}]$. In this setting, a job must run fully on a single machine, but it may be restarted on a different machine. Here, job lengths are stochastically chosen only once. The only necessary change to the algorithm is substitute the $k$th round with the corresponding solution to $R||C_{\text{max}}$, in lieu of $R|\text{pmtn}|C_{\text{max}}$.

**Other results**

Similar analysis yields an $O(\log \log m)$-approximation for STOCH-I. Substituting assignments generated from algorithms for $R|\text{pmtn}, \text{chains}|C_{\text{max}}$ [10] for those given by (LP2) gives an $O(\log(n + m) \log \log(\min \{m, n\}))$-approximation when precedence constraints form chains, and an $O(\log(n) \log(n + m) \log \log(\min \{m, n\}))$-approximation when they form directed forests, using the same algorithms and techniques.

**D PROOFS**

**Lemma 1** $t_{LP1(J,1/2)} = O(E[T_{\text{OPT}}])$

**Proof.** Let $t$ be the optimum solution to $LP1(J,1/2)$. Consider any subset $U \subseteq J$, and its complement $U$. Then $LP1(U,1/2) + LP1(U,1/2) \geq t$, since we can construct a solution to $LP1(J,1/2)$ by adding a solution to $LP1(U,1/2)$ and a solution to $LP1(U,1/2)$. Now recall our view of the problem in terms of $\text{SUU}^*$: there is an $r_j$ chosen uniformly at random from $(0,1)$ for each job $j$ such that job $j$ completes only if $\sum_{i \in M} t_{ij}x_{ij} \geq -\log r_j$. For any sample from the event space, let $U$ be the set of jobs $j$ for which $r_j > 1/2$, and let $U$ be the complement set of jobs $j$ for which $r_j < 1/2$ (note $r_j = 1/2$ with probability 0 so can be ignored). By definition,
each job is in $U$ independently with probability $1/2$. Next observe that however OPT generates its schedule, it must allocate at least $1/2$ unit of “work” to each job in $U$; in other words, Equation (1) of (LP1) must hold for every $j \in U$. Thus, the optimum schedule contains a feasible solution to $LP1(U, 1/2)$.

Now observe that by construction, $U$ is a uniformly random subset of $J$, meaning all subsets are equally likely. Thus,

$$E[T_{OPT}] = 2^{-n} \sum_{U} E[T_{OPT} | U] = 2^{-n} \cdot \frac{1}{2} (\sum_{U} E[T_{OPT} | U] + \sum_{U} E[T_{OPT} | \overline{U}])$$

$$\geq 2^{-n} \frac{1}{2} \sum_{U} (LP1(U, 1/2) + LP1(\overline{U}, 1/2))$$

$$\geq 2^{-n} \frac{1}{2} \sum_{U} LP1(J, 1/2)$$

$$= \frac{1}{2} LP1(J, 1/2)$$

Where the second line of this derivation follows from the first paragraph of this proof.

**Theorem 3** Let $T_{SUU-I-OBL}$ denote the random variable corresponding to the amount of time it takes for an execution of SUU-I-OBL to complete all jobs. Then $E[T_{SUU-I-OBL}] = O(E[T_{OPT}] \log n)$.

**Proof.** From Lemmas 1 and 2, we have a schedule $\Sigma$ of length $O(E[T_{OPT}])$ that gives each job a constant probability of success. Applying a Chernoff bound gives us that a particular job completes in $O(\log n)$ repetitions of $\Sigma$, with probability at least $1 - \frac{1}{n^{\Theta(1)}}$, where the constant exponent appears as a constant factor in the number of repetitions. Taking a union bound over all jobs gives that with probability at least $1 - \frac{1}{n^{\Theta(1)}}$, all jobs complete in $O(\log n)$ repetitions. Since this probability drops off dramatically as the number of repetitions increases, we have $E[T_{SUU-I-OBL}] = O(E[T_{OPT}] \log n)$$\square$

**Theorem 4** Let $K = \lceil \log \log \min\{m, n\} \rceil + 3$ and let $T_{SUU-I-SEM}$ denote the random variable corresponding to the amount of time it takes for an execution of SUU-I-SEM to complete all jobs. Then $E[T_{SUU-I-SEM}] = O(E[T_{OPT}] \cdot K)$.

**Proof.** This proof resumes where the same proof from Section 3 leaves off. In particular, we show here that SUU-I-SEM produces an $O(\log \log m)$-approximation by showing that repeating the $(\lceil \log \log m \rceil + 3)$rd round until completion takes time $O(E[T_{OPT}])$.

Suppose that $m < n$ and $F_K = 0$. Let $\Sigma_K = \Sigma_{LP1(J_k, 2^{k-2})}$ be the schedule computed at the end of the $K$th phase. Here, SUU-I-SEM repeats $\Sigma_K$ until all jobs complete. Define the load $H$ of a finite schedule to be the maximum number of timesteps during which any machine is assigned to an uncompleted job (i.e., $H = \max \sum_j x_{ij}$); a schedule can be compressed to run in exactly $H$ time-steps. We will analyze how long it takes for the load of our instance to drop from its initial expected value $O(E[T_{OPT}])$ (at the end of the $K$th phase) to 0 (when all jobs have completed).

Define $X$ as a random variable denoting the number of timesteps until the compressed load of $\Sigma_K$ drops to 0, and let $T_K$ be the (random variable) denoting the length of $\Sigma_K$. Then $X = \sum_{i=0}^{\log T_K} X_i T_i / 2^i = T_K \sum_{i=0}^{\log T_K} X_i / 2^i$, where $X_i$ is the random variable representing the number
of repetitions of $\Sigma_K$ necessary to drop its load from $T_K/2^i$ to $T_K/2^{i+1}$. By construction, a single execution of $\Sigma_K$ ensures that each job remains with probability at most $1/m^2$, and thus the expected load of each machine shrinks by at least a (multiplicative) factor of $m^2$. Markov’s inequality gives us that each machines load decreases by a a factor of $m/2$. Taking a union bound over all machines gives us that the load of all machines decreases by a factor of $m/2$. with probability at least $1/2$.

For $m > 4$, we now have that each repetition of $\Sigma_K$ decreases the remaining load by a factor of 2 with probability at least $1/2$, and hence $E[X_i] \leq 2$ for all $i$—requiring only an expected constant number of repetitions to reduce the load by a constant factor. We now have that $E\left[\sum_{i=0}^{\log T_K} X_i / 2^i\right] \leq \sum_{i=0}^{\infty} E[X_i] / 2^i \leq O(1)$. Since each $X_i$ and $T_K$ are independent, it follows that $E[\sum_{i=0}^{\infty} E[X_i] / 2^i = O(E[T_K])]$. Having $E[T_K] \leq O(E[\OPT])$ completes the proof.

Lemma 8 Consider $y_j \in \mathbb{N}$ drawn from the geometric distribution $\Pr\{y_j = k \in \mathbb{N}\} = (1/2)^k$, and let $1 \leq d_j \leq W/\log \eta$ be a weight associated with each $y_j$ for any values such that $W \geq \sum_j 2d_j$ and $\log \eta \leq W$. Then $\sum_j y_j d_j \leq O(cT)$ with probability at least $1 - 1/\eta^c$.

Proof. Round all the $d_j$ up to the next power of 2. Let $Z_k$ be the set of $j$ such that $T/(2^k \log \eta) < d_j \leq T/(2^{k-1} \log \eta)$, for $k \in \{1, 2, \ldots, \log(T/\log \eta)\}$. We apply a Chernoff bound over all jobs in $Z_k$ to show that the weighted sum of their $y_j$ is near the expectation, with high probability. In particular, let $B_k$ be the sum $b_k$ (to be decided later) Bernoulli random variables with probability $1/2$. Then clearly $\Pr\{B_k < |Z_k|\} = \Pr\{\sum_{j \in Z_k} y_j > b\}$. We therefore apply Chernoff bounds to the $B_k$. We will then union bound over all $B_k$ to complete the proof.

Since there are potentially many nonempty $Z_k$ sets, we choose the $b_k$ so as to give geometrically decreasing failure probabilities. In particular, we set $b_k = \alpha(c)(|Z_k| + \log \eta + k)$ for some constant $\alpha(c)$, with $(1 - 1/\alpha(c))^2 / 2 = c$. Then a Chernoff bound states that $\Pr\{B_k < |Z_k|\} < e^{-(\log \eta + k)(1-1/\alpha(c))^2/2} \leq \eta^{-c} e^{-ck}$. Taking a union bound over all $k$ gives $\Pr\{\text{any } B_k < |Z_k|\} \leq \sum_{k=1}^{\log(T/\log \eta)} \eta^{-c} e^{-ck} \leq \eta^{-c} \sum_{k=1}^{\infty} e^{-ck} \leq \eta^{-c}$ when $c \geq \log_2 e$. It follows that $\Pr\{\text{any } \sum_{j \in Z_k} y_j > b_k\} < \eta^{-c}$. And hence with probability at least $1 - 1/\eta^c$, we have $\sum_j d_j y_j \leq \sum_k (T/(2^k \log \eta)) \sum_{j \in Z_k} y_j) \leq \sum_k (Tb_k/(2^k \log \eta)) \leq O(c \sum_j d_j) + (T/\log \eta) \sum_k (\log \eta + k)/2^k = O(c \sum_j d_j) + O(T) = O(cT).$