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CONSISTENT APPROXIMATIONS FOR THE OPTIMAL CONTROL
OF CONSTRAINED SWITCHED SYSTEMS—PART 1:
A CONCEPTUAL ALGORITHM*

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Abstract. Switched systems, or systems whose control parameters include a continuous-valued
input and a discrete-valued input which corresponds to the mode of the system that is active at
a particular instance in time, have shown to be highly effective in modeling a variety of physical
phenomena. Unfortunately, the construction of an optimal control algorithm for such systems has
proved difficult since it demands some form of optimal mode scheduling. In a pair of papers, we
construct a first order optimization algorithm to address this problem. Our approach, which we prove
in this paper converges to local minimizers of the constrained optimal control problem, first relaxes
the discrete-valued input, performs traditional optimal control, and then projects the constructed
relaxed discrete-valued input back to a pure discrete-valued input by employing an extension to the
classical chattering lemma that we formalize. In the second part of this pair of papers, we describe
how this conceptual algorithm can be recast in order to devise an implementable algorithm that
constructs a sequence of points by recursive application that converge to local minimizers of the
optimal control problem for switched systems.

Key words. optimal control, switched systems, chattering lemma, consistent approximations

AMS subject classifications. Primary, 49J21, 49J30, 49J52, 49N25, 93C30; Secondary,
49M25, 90C11, 90C30

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1. Introduction. Let \( Q \) and \( J \) be finite sets and \( U \subset \mathbb{R}^m \) a compact, convex,
nonempty set. Let \( f : [0,1] \times \mathbb{R}^n \times U \times Q \rightarrow \mathbb{R}^n \) be a vector field over \( \mathbb{R}^n \) that defines
a switched system, i.e., for each \( i \in Q \), the map \( (t,x,u) \mapsto f(t,x,u,i) \) is a classical
controlled vector field. Let \( \{ h_j \}_{j \in J} \) be a collection of functions, \( h_j : \mathbb{R}^n \rightarrow \mathbb{R} \) for each
\( j \in J \), and \( h_0 : \mathbb{R}^n \rightarrow \mathbb{R} \). Given an initial condition \( x_0 \in \mathbb{R}^n \), we are interested in
solving the following optimal control problem:

\[
\inf \left\{ h_0(x(1)) \mid \dot{x}(t) = f(t,x(t),u(t),d(t)), \ x(0) = x_0, \ h_j(x(t)) \leq 0, \right. \\
\left. x(t) \in \mathbb{R}^n, \ u(t) \in U, \ d(t) \in Q, \ \text{for a.e. } t \in [0,1] \forall j \in J \right\}.
\]

The optimization problem defined in (1.1) is numerically challenging because the function \( d : [0,1] \rightarrow Q \) maps to a finite set rendering the immediate application of derivative-based algorithms impossible. This paper, which is the first in a two-part
series, presents a conceptual algorithm, Algorithm 4.1, to solve the problem described in (1.1) and a theorem proving the convergence of the algorithm to points satisfying
a necessary condition for optimality, Theorem 5.16.

1.1. Related work. Switched systems, a particular class of hybrid systems,
have been used in a variety of modeling applications including automobiles and

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locomotives employing different gears [15, 26], manufacturing systems [8], and situations where a control module switches its attention among a number of subsystems [18, 33]. Given their utility, there has been considerable interest in devising algorithms to perform optimal control for such systems. Since the discrete mode of operation of switched systems is completely controlled, the determination of an optimal control for such systems is challenging due to the combinatorial nature of calculating an optimal discrete mode schedule. The theoretical underpinnings for the optimal control of switched systems have been considered in various forms including a sufficient condition for optimality via quasi-variational inequalities by Branicky, Borkar, and Mitter [7] and extensions of the classical maximum principle for hybrid systems by Piccoli [21], Riedinger, Iung, and Kratz [25], and Sussmann [31].

Implementable algorithms for the optimal control of switched systems have been pursued in numerous forms. Xu and Antsaklis [36] proposed one of the first such algorithms, wherein a bilevel optimization scheme that fixes the mode schedule iterates between a low-level algorithm that optimizes the continuous input within each visited mode and a high-level algorithm that optimizes the time spent within each visited mode. The Gauss pseudospectral optimal control software developed by Rao et al. [24] applies a variant of this approach. Shaikh and Caines [28] modified this bilevel scheme by designing a low-level algorithm that simultaneously optimizes the continuous input and the time spent within each mode for a fixed mode schedule by employing the maximum principle and a high-level algorithm that modifies the mode schedule by using the Hamming distance to compare nearby mode schedules. Xu and Antsaklis [37] and Shaikh and Caines [29] each extended their approach to systems with state-dependent switching. Alamir and Attia [1] proposed a scheme based on the maximum principle. Considering the algorithm constructed in this paper, the most relevant approach is the one described by Bengea and DeCarlo [4], which relaxes the discrete-valued input into a continuous-valued input over which it can apply the maximum principle to perform optimal control and then uses the chattering lemma [5] to approximate the optimal relaxation with another discrete-valued input. However, no constructive method to generate such an approximation is included. Importantly, the numerical implementation of optimal control algorithms that rely on the maximum principle for nonlinear systems including switched systems is fundamentally restricted due to their reliance on approximating needle variations with arbitrary precision as explained in [20].

Several authors [9, 16] have focused on the optimization of autonomous switched systems (i.e., switched systems without a continuous input) by fixing a mode sequence and optimizing the amount of time spent within each mode using implementable vector space variations. Axelsson et al. [3] extended this approach by employing a variant of the bilevel optimization strategy proposed in [28]. In the low level they optimized the amount of time spent within each mode using a first order algorithm, and in the high level they modified the mode sequence by employing a single-mode insertion technique. There have been two major extensions to this result. First, Wardi and Egerstedt [35] extended the approach by constructing an algorithm that allowed for the simultaneous insertion of several modes each for a nonzero amount of time. This reformulation transformed the bilevel optimization strategy into a single step optimization algorithm but required exact computation in order to prove convergence. Wardi, Egerstedt, and Twu [34] addressed this deficiency by constructing an adaptive-precision algorithm that balanced the requirement for precision with limitations on computation. Second, Gonzalez et al. [12, 13], extended the bilevel optimization approach to constrained nonautonomous switched systems. Though these insertion
techniques avoid the computational expense of considering all possible mode schedules during optimization, they restrict the possible modifications of the existing mode schedule, which may terminate the execution in undesired stationary points.

1.2. Our contribution and organization. In this paper, we devise and implement a conceptual first order algorithm for the optimal control of constrained nonlinear switched systems. In sections 2 and 3, we introduce the notation and assumptions used throughout the paper and formalize the optimization problem we solve. Our approach, which is formulated in section 4, computes a step by treating the discrete-valued input as a continuous-valued one and then uses a gradient descent technique on this continuous-valued input. Using an extension of the chattering lemma which we develop, a discrete-valued approximation to the output of the gradient descent algorithm is constructed. In section 5, we prove that the sequence of points generated by recursive application of our algorithm asymptotically satisfies a necessary condition for optimality. The second part of this two-paper series [32] describes an implementable version of our conceptual algorithm and includes several benchmarking examples to illustrate our approach.

2. Notation. We present a summary of the notation used throughout the paper.

\[ \begin{align*}
\| \cdot \|_p & \quad \text{standard vector space } p\text{-norm} \\
\| \cdot \|_{1,p} & \quad \text{matrix induced } p\text{-norm} \\
\| \cdot \|_{L^p} & \quad \text{functional } L^p\text{-norm} \\
Q & = \{1, \ldots, q\} \quad \text{set of discrete mode indices (section 1)} \\
\mathcal{J} & = \{1, \ldots, N_c\} \quad \text{set of constraint function indices (section 1)} \\
U & \subset \mathbb{R}^m \quad \text{compact, convex, nonempty set (section 1)} \\
\Sigma^q & \quad q\text{-simplex in } \mathbb{R}^q \text{ (equation (3.5))} \\
\Sigma^q_p & = \{e_i\}_{i \in Q} \quad \text{corners of the } q\text{-simplex in } \mathbb{R}^q \text{ (equation (3.6))} \\
\mathcal{D}_p & \quad \text{relaxed discrete input space (section 3.2)} \\
\mathcal{P}_p & \quad \text{pure discrete input space (section 3.2)} \\
\mathcal{U} & \quad \text{continuous input space (section 3.2)} \\
(X, \| \cdot \|_X) & \quad \text{general optimization space and its norm (section 3.2)} \\
X_r & \quad \text{relaxed optimization space (section 3.2)} \\
X_p & \quad \text{pure optimization space (section 3.2)} \\
N(\xi, \varepsilon) & \quad \varepsilon\text{-ball around } \xi \text{ in the } X\text{-norm (Definition 4.1)} \\
N_\text{w}(\xi, \varepsilon) & \quad \varepsilon\text{-ball around } \xi \text{ in the weak topology (Definition 4.3)} \\
d_i & \quad \text{ith coordinate of } d: [0, 1] \to \mathbb{R}^q \\
f: [0, 1] \times \mathbb{R}^n \times U \times Q & \to \mathbb{R}^n \quad \text{switched system (section 1)} \\
V: L^2([0, 1], \mathbb{R}^n) & \to \mathbb{R} \quad \text{total variation operator (equation (3.3))} \\
h_\text{t}: \mathbb{R}^n & \to \mathbb{R} \quad \text{terminal cost function (section 3.3)} \\
\{h_j: \mathbb{R}^n \to \mathbb{R}\}_{j \in \mathcal{J}} & \quad \text{collection of constraint functions (section 3.3)} \\
x(t): [0, 1] \to \mathbb{R}^n & \quad \text{trajectory of the system given } \xi \text{ (equation (3.8))} \\
\phi_t: X_r & \to \mathbb{R}^n \quad \text{flow of the system at } t \in [0, 1] \text{ (equation (3.9))} \\
J: X_r & \to \mathbb{R} \quad \text{cost function (equation (3.10))} \\
\Psi: X_r & \to \mathbb{R} \quad \text{constraint function (equation (3.11))} \\
\psi_{j,t}: X_r & \to \mathbb{R} \quad \text{component constraint functions (equation (3.12))} \\
P_p & \quad \text{optimal control problem (equation (3.13))} \\
P_r & \quad \text{relaxed optimal control problem (equation (4.1))} \\
D & \quad \text{derivative operator (equation (4.2))} \\
\theta: X_p & \to (-\infty, 0] \quad \text{optimality function (equation (4.3))} \\
g: X_p & \to X_r \quad \text{descent direction function (equation (4.3))}
\end{align*} \]
three spaces: the

\[(3.4)\]

optimality function objective (equation (4.4))

\[\mu: \mathcal{X}_p \to \mathbb{N}\]

step size function (equation (4.5))

\[\mathbb{I}_A: [0, 1] \to \mathbb{R}\]

indicator function of \(A \subset [0, 1]\), \(\mathbb{I} = \mathbb{I}_{[0, 1]}\) (section 4.4)

\[\mathcal{F}_N: \mathcal{D}_r \to \mathcal{D}_r\]

Haar wavelet operator (equation (4.7))

\[\mathcal{P}_N: \mathcal{D}_r \to \mathcal{D}_p\]

pulse-width modulation operator (equation (4.8))

\[\rho_N: \mathcal{X}_p \to \mathcal{X}_p\]

projection operator (equation (4.9))

\[\nu: \mathcal{X}_p \to \mathbb{N}\]

frequency modulation function (equation (4.10))

\[\Gamma: \mathcal{X}_p \to \mathcal{X}_p\]

algorithm recursion function (equation (4.11))

\[\Phi^{(q)}: [0, 1] \times [0, 1] \to \mathbb{R}^{n \times n}\]

state transition matrix given \(\xi\) (equation (5.2))

3. Preliminaries. We begin by formulating the problem we solve in this paper.

3.1. Norms and functional spaces. We focus on the optimization of functions with finite \(L^2\)-norm and bounded variation. Given \(n \in \mathbb{N}\), \(f: [0, 1] \to \mathbb{R}^n\) belongs to \(L^2([0, 1], \mathbb{R}^n)\) if

\[(3.1)\]

\[\|f\|_{L^2} = \left( \int_0^1 \|f(t)\|^2_2 \, dt \right)^{\frac{1}{2}} < \infty.\]

Similarly, \(f: [0, 1] \to \mathbb{R}^n\) belongs to \(L^\infty([0, 1], \mathbb{R}^n)\) if

\[(3.2)\]

\[\|f\|_{L^\infty} = \inf \{ \alpha \geq 0 \mid \|f(x)\|_2 \leq \alpha \text{ for almost every } x \in [0, 1] \} < \infty.\]

Both definitions are formulated with respect to the Lebesgue measure on \([0, 1]\). In order to define the space of functions of bounded variation, we first define the total variation of a function. Given \(P\), the set of all finite partitions of \([0, 1]\), we define the total variation of \(f: [0, 1] \to \mathbb{R}^n\) by

\[(3.3)\]

\[V(f) = \sup \left\{ \sum_{j=0}^{m-1} \|f(t_{j+1}) - f(t_j)\|_1 \mid \{t_k\}_{k=0}^m \in P, \ m \in \mathbb{N} \right\}.\]

Note that the total variation of \(f\) is a seminorm, i.e., it satisfies the triangle inequality but does not separate points. \(f\) is of bounded variation if \(V(f) < \infty\), and we define \(BV([0, 1], \mathbb{R}^n)\) to be the set of all functions of bounded variation from \([0, 1]\) to \(\mathbb{R}^n\). The following result, which is an extension of Propositions X.1.3 and X.1.4 in [17] for vector-valued functions, is critical to our analysis of functions of bounded variation.

**Proposition 3.1.** If \(f \in BV([0, 1], \mathbb{R}^n)\) and \(v: [0, 1] \to \mathbb{R}\) is bounded and differentiable almost everywhere, then

\[(3.4)\]

\[\left| \int_0^1 f(t) \dot{v}(t) \, dt \right| \leq \|v\|_{L^\infty} V(f).\]

3.2. Optimization spaces. To formalize the optimal control problem, we define three spaces: the pure discrete input space, \(D_p\), the relaxed discrete input space, \(D_r\), and the continuous input space, \(U\). Let the \(q\)-simplex, \(\Sigma_q^q\), be defined as

\[(3.5)\]

\[\Sigma_q^q = \left\{ (d_1, \ldots, d_q) \in [0, 1]^q \mid \sum_{i=1}^q d_i = 1 \right\},\]

and let the corners of the \(q\)-simplex, \(\Sigma_p^q\), be defined as

\[(3.6)\]

\[\Sigma_p^q = \left\{ (d_1, \ldots, d_q) \in \{0, 1\}^q \mid \sum_{i=1}^q d_i = 1 \right\}.\]
Note that $\Sigma^q_r \subset \Sigma^q_d$. Since there are exactly $q$ elements in $\Sigma^q_d$, we write $\Sigma^q_d = \{e_1, \ldots, e_q\}$. We employ the elements in $\Sigma^q_d$ to index the vector fields defined in section 1, i.e., for each $d \in \Sigma^q_d$, we write $f(\cdot, \cdot, e_i)$ for $f(\cdot, \cdot, i)$. Also note that for each $d \in \Sigma^q_d$, $d = \sum_{i=1}^q d_i e_i$.

Using this notation, we define the pure discrete input space, $D_p = L^2([0, 1], \Sigma^q_d) \cap BV([0, 1], \Sigma^q_d)$, the relaxed discrete input space, $D_r = L^2([0, 1], \Sigma^q_d) \cap BV([0, 1], \Sigma^q_d)$, and the continuous input space, $U = L^2([0, 1], U) \cap BV([0, 1], U)$. Let the general optimization space be $\mathcal{X} = L^2([0, 1], \mathbb{R}^m) \times L^2([0, 1], \mathbb{R}^n)$ endowed with the norm $\|\xi\|_\mathcal{X} = \|u\|_L^2 + \|d\|_L^2$ for each $\xi = (u, d) \in \mathcal{X}$. We combine $U$ and $D_p$ to define the pure optimization space, $\mathcal{X}_p = U \times D_p$, and we endow it with the same norm as $\mathcal{X}$. Similarly, we combine $U$ and $D_r$ to define the relaxed optimization space, $\mathcal{X}_r = U \times D_r$, and endow it with the $\mathcal{X}$-norm too. Note that $\mathcal{X}_p \subset \mathcal{X}_r \subset \mathcal{X}$. Note that we emphasize the fact that our input spaces are subsets of $L^2$ to motivate the appropriateness of the $\mathcal{X}$-norm.

### 3.3. Trajectories, cost, constraint, and the optimal control problem.

Given $\xi = (u, d) \in \mathcal{X}_r$, for convenience throughout the paper we let

$$
(3.7) \quad f(t, x(t), u(t), d(t)) = \sum_{i=1}^q d_i(t) f(t, x(t), u(t), e_i).
$$

Note that for points belonging to the pure optimization space, the bounded variation assumption restricts the number of switches between modes to be finite.

We make the following assumptions about the dynamics.

**Assumption 3.2.** For each $i \in \mathcal{Q}$ and $t \in [0, 1]$, the map $(x, u) \mapsto f(t, x, u, e_i)$ is differentiable. Also, for each $i \in \mathcal{Q}$ the vector field and its partial derivatives are Lipschitz continuous, i.e., there exists $L > 0$ such that given $t_1, t_2 \in [0, 1]$, $x_1, x_2 \in \mathbb{R}^n$, and $u_1, u_2 \in U$,

1. $\|f(t_1, x_1, u_1, e_i) - f(t_2, x_2, u_2, e_i)\|_2 \leq L(\|t_1 - t_2\| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2),$
2. $\|\frac{\partial f}{\partial t}(t_1, x_1, u_1, e_i) - \frac{\partial f}{\partial t}(t_2, x_2, u_2, e_i)\|_i, 2 \leq L(\|t_1 - t_2\| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2),$
3. $\|\frac{\partial f}{\partial x}(t_1, x_1, u_1, e_i) - \frac{\partial f}{\partial x}(t_2, x_2, u_2, e_i)\|_i, 2 \leq L(\|t_1 - t_2\| + \|x_1 - x_2\|_2 + \|u_1 - u_2\|_2).

Given $x_0 \in \mathbb{R}^n$, a trajectory of the system corresponding to $\xi \in \mathcal{X}_r$ is the solution to

$$
(3.8) \quad \dot{x}(t) = f(t, x(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, 1], \quad x(0) = x_0,
$$

and denote it by $x(\xi) : [0, 1] \to \mathbb{R}^n$ to emphasize its dependence on $\xi$. We also define the flow of the system, $\phi_t : \mathcal{X}_r \to \mathbb{R}^n$ for each $t \in [0, 1]$ as

$$
(3.9) \quad \phi_t(\xi) = x(\xi)(t).
$$

The next result is a straightforward extension of the classical existence and uniqueness theorem for nonlinear differential equations (see Lemma 5.6.3 and Proposition 5.6.5 in [22]).

**Theorem 3.3.** For each $\xi = (u, d) \in \mathcal{X}$, differential equation (3.8) has a unique solution. Moreover, there exists $C > 0$ such that for each $\xi \in \mathcal{X}_r$ and $t \in [0, 1]$, $\|x(\xi)(t)\| \leq C$. 

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To define the cost function, we assume that we are given a terminal cost function, \( h_0: \mathbb{R}^n \rightarrow \mathbb{R} \). The cost function, \( J: \mathcal{X}_r \rightarrow \mathbb{R} \), for the optimal control problem is then defined as

\[
J(\xi) = h_0(x^{(\xi)}(1)).
\]

(3.10)

If the problem formulation includes a running cost, then one can extend the existing state vector by introducing a new state and modifying the cost function to evaluate this new state at the final time, as shown in section 4.1.2 in [22]. Employing this modification, observe that each mode of the switched system can have a different running cost associated with it. Given \( \xi \in \mathcal{X}_r \), the state \( x^{(\xi)} \) is said to satisfy the constraint if \( h_j(x^{(\xi)}(t)) \leq 0 \) for each \( t \in [0, 1] \) and each \( j \in \mathcal{J} \). We compactly describe all the constraints by defining the constraint function \( \Psi: \mathcal{X}_r \rightarrow \mathbb{R} \) by

\[
\Psi(\xi) = \max \{ h_j(x^{(\xi)}(t)) \mid t \in [0, 1], \ j \in \mathcal{J} \}
\]

(3.11)

since \( h_j(x^{(\xi)}(t)) \leq 0 \) for each \( t \) and \( j \) if and only if \( \Psi(\xi) \leq 0 \). To ensure clarity in the ensuing analysis, it is useful to sometimes emphasize the dependence of \( h_j(x^{(\xi)}(t)) \) on \( \xi \). Therefore, we define component constraint functions \( \psi_{j,t}: \mathcal{X}_r \rightarrow \mathbb{R} \) for each \( t \in [0, 1] \) and \( j \in \mathcal{J} \) as

\[
\psi_{j,t}(\xi) = h_j(\phi_t(\xi)).
\]

(3.12)

With these definitions, we can state the optimal control problem, which is a restatement of the problem defined in (1.1):

\[
\inf \{ J(\xi) \mid \Psi(\xi) \leq 0, \ \xi \in \mathcal{X}_r \}. \quad (P_p)
\]

We introduce the following assumption to ensure the well-posedness of \( P_p \).

**Assumption 3.4.** The functions \( h_0 \) and \( \{ h_j \}_{j \in \mathcal{J}} \), and their derivatives, are Lipschitz continuous, i.e., there exists \( L > 0 \) such that given \( x_1, x_2 \in \mathbb{R}^n \),

1. \( |h_0(x_1) - h_0(x_2)| \leq L \|x_1 - x_2\|_2 \),
2. \( \left\| \frac{\partial h_0}{\partial x}(x_1) - \frac{\partial h_0}{\partial x}(x_2) \right\|_2 \leq L \|x_1 - x_2\|_2 \),
3. \( |h_j(x_1) - h_j(x_2)| \leq L \|x_1 - x_2\|_2 \),
4. \( \left\| \frac{\partial h_j}{\partial x}(x_1) - \frac{\partial h_j}{\partial x}(x_2) \right\|_2 \leq L \|x_1 - x_2\|_2 \).

4. **Optimization algorithm.** In this section, we describe our optimization algorithm to solve \( P_p \), which proceeds as follows: first, we treat a given pure discrete input as a relaxed discrete input by allowing it to belong \( D_r \); second, we perform optimal control over the relaxed optimization space; and finally, we project the computed relaxed input into a pure input. We begin with a brief digression to explain why such a roundabout construction is required in order to devise a first order numerical optimal control scheme to solve \( P_p \) defined in (3.13).

4.1. **Directional derivatives.** To appreciate why the construction of a numerical scheme to find the local minima of \( P_p \) defined in (3.13) is difficult, suppose that the optimization in the problem took place over the relaxed optimization space rather than the pure optimization space. The relaxed optimal control problem is then defined as

\[
\inf \{ J(\xi) \mid \Psi(\xi) \leq 0, \ \xi \in \mathcal{X}_r \}. \quad (P_r)
\]
The local minimizers of this problem are then defined as follows.

**Definition 4.1.** Let an $\varepsilon$-ball in the $X$-norm around $\xi$ be defined as $N(\xi, \varepsilon) = \{ \xi \in X \mid \|\xi - \bar{\xi}\|_X < \varepsilon \}$. A point $\xi \in X$ is a local minimizer of $P_r$ if $\Psi(\xi) \leq 0$ and there exists $\varepsilon > 0$ such that $J(\xi) \geq J(\xi')$ for each $\xi' \in N(\xi; \varepsilon) \cap \{ \xi \in X \mid \Psi(\xi) \leq 0 \}$.

To concretize how to formulate an algorithm to find minimizers in $P_r$, we introduce some additional notation. Given $\xi \in X$ and any function $G : X \to \mathbb{R}^n$, the directional derivative of $G$ at $\xi$, denoted $DG(\xi; \cdot) : X \to \mathbb{R}^n$, is computed as

$$DG(\xi; \xi') = \lim_{\lambda \downarrow 0} \frac{G(\xi + \lambda \xi') - G(\xi)}{\lambda}$$

when the limit exists. Using the directional derivative $DJ(\xi; \xi')$, the first order approximation of the cost at $\xi$ in the $\xi' \in X$ direction is $J(\xi + \lambda \xi') \approx J(\xi) + \lambda DJ(\xi; \xi')$ for $\lambda$ sufficiently small. Similarly, using the directional derivative $D\psi_{j,t}(\xi; \xi')$ one can construct a first order approximation of $\psi_{j,t}$ at $\xi$ in the $\xi' \in X$ direction.

It follows from the first order approximation of the cost that if $DJ(\xi; \xi')$ is negative, then it is possible to decrease the cost by moving in the $\xi'$ direction. In an identical fashion, one can argue that if $\xi$ is infeasible and $D\psi_{j,t}(\xi; \xi')$ is negative, then it is possible to reduce the infeasibility of $\psi_{j,t}$ by moving in the $\xi'$ direction. Employing these observations, one can utilize the directional derivatives of the cost and component constraints to construct both a descent direction and a necessary condition for optimality for $P_r$. Unfortunately, trying to exploit the same strategy for $P_p$ is untenable, since it is unclear how to define directional derivatives for $D_p$ which is not a vector space. However, as we describe next, in an appropriate topology over $X_p$ there exists a way to exploit the directional derivatives of the cost and component constraints in $X_r$ while reasoning about the optimality of points in $X_p$ with respect to $P_p$.

### 4.2. The weak topology on the optimization space and local minimizers

To motivate the type of required relationship, we begin by describing the chattering lemma.

**Theorem 4.2** (Theorem 4.3 in [5]). For each $\xi_r \in X_r$ and $\varepsilon > 0$ there exists a $\xi_p \in X_p$ such that for each $t \in [0, 1]$, $\|\phi_t(\xi_r) - \phi_t(\xi_p)\|_2 \leq \varepsilon$.

**Theorem 4.2** says that any trajectory produced by an element in $X_r$ can be approximated arbitrarily well by a point in $X_p$. Unfortunately, the relaxed and pure points prescribed as in **Theorem 4.2** need not be near one another in the metric induced by the $X$-norm. However, these points can be made arbitrarily close in a different topology.

**Definition 4.3.** The weak topology on $X_r$ induced by differential equation (3.8) is the smallest topology on $X_r$ such that the map $\xi \mapsto x^{(\xi)}$ is continuous with respect to the $L^2$-norm. Moreover, an $\varepsilon$-ball in the weak topology around $\xi$ is defined as $N_w(\xi, \varepsilon) = \{ \xi \in X_r \mid \|x^{(\xi)} - x^{(\epsilon)}\|_{L^2} < \varepsilon \}$.

More details about the weak topology of a space can be found in section 3.8 in [27]. To understand the relationship between $N$ and $N_w$, observe that $\phi_t$ is Lipschitz continuous for all $t \in [0, 1]$, say, with constant $L > 0$ as is proved in Lemma 5.1. It follows that for any $\xi \in X_r$ and $\varepsilon > 0$, if $\xi \in N(\xi, \varepsilon)$, then $\xi \in N_w(\xi, L\varepsilon)$. However,

\footnote{The theorem as proved in [5] is not immediately applicable to switched systems, but an extension as proved in Theorem 1 in [4] makes that feasible. A particular version of this existence theorem can also be found in Lemma 1 in [30].}
the converse is not true in general. Since the weak topology, in contrast to the $\mathcal{X}$-norm induced topology, places points that generate nearby trajectories next to one another, and using the fact that $\mathcal{X}_p \subset \mathcal{X}$, we can define the concept of a local minimizer for $P_p$.

**Definition 4.4.** We say that $\xi \in \mathcal{X}_p$ is a local minimizer of $P_p$ if $\Psi(\xi) \leq 0$ and there exists $\varepsilon > 0$ such that $J(\xi) \geq J(\xi')$ for each $\xi \in \mathcal{N}_w(\xi, \varepsilon) \cap \{\xi \in \mathcal{X}_p \mid \Psi(\xi) \leq 0\}$.

With this definition of local minimizer, we can exploit Theorem 4.2, as an existence result, along with the notion of directional derivative over the relaxed optimization space to construct a necessary condition for optimality for $P_p$.

### 4.3. An optimality condition.

Motivated by the approach in [22], we define an optimality function, $\theta : \mathcal{X}_p \to (-\infty, 0]$, which determines whether a given point is a local minimizer of $P_p$, and a corresponding descent direction, $g : \mathcal{X}_p \to \mathcal{X}_r$,

$$\theta(\xi) = \min_{\xi' \in \mathcal{X}_r} \zeta(\xi, \xi') + V(\xi' - \xi), \quad g(\xi) = \arg\min_{\xi' \in \mathcal{X}_r} \zeta(\xi, \xi') + V(\xi' - \xi),$$

where

$$\zeta(\xi, \xi') = \begin{cases} \max \left\{ \max_{(j,t) \in J \times [0,1]} D\psi_{j,t}(\xi; \xi' - \xi) + \gamma\Psi(\xi), DJ(\xi; \xi' - \xi) \right\} \\ + \|\xi' - \xi\|_X \end{cases} \quad \text{if } \Psi(\xi) \leq 0,$$

$$\zeta(\xi, \xi') = \begin{cases} \max \left\{ \max_{(j,t) \in J \times [0,1]} D\psi_{j,t}(\xi; \xi' - \xi), DJ(\xi; \xi' - \xi) - \Psi(\xi) \right\} \\ + \|\xi' - \xi\|_X \end{cases} \quad \text{if } \Psi(\xi) > 0,$$

with $\gamma > 0$ a design parameter, and where without loss of generality for each $\xi = (u, d) \in \mathcal{X}_r$, $V(\xi) = V(u) + V(d)$. The well-posedness of $\theta(\xi)$ and $g(\xi)$ for each $\xi \in \mathcal{X}_p$ follows from Theorem 5.5. Note that $\theta(\xi) \leq 0$ for each $\xi \in \mathcal{X}_p$, as we show in Theorem 5.6. Also note that the directional derivatives in (4.4) consider directions $\xi' - \xi$ in order to ensure that they belong to $\mathcal{X}_r$. Our definition of $\zeta$ does not include the total variation term since it does not satisfy the same properties as the directional derivatives or the $\mathcal{X}$-norm, such as Lipschitz continuity (as shown in Lemma 5.4). In fact, the total variation term is added in order to make sure that $g(\xi) - \xi$ is of bounded variation.

To understand how the optimality function behaves, consider several cases. First, if $\theta(\xi) < 0$ and $\Psi(\xi) = 0$, then there exists a $\xi' \in \mathcal{X}_r$ such that both $DJ(\xi; \xi' - \xi)$ and $D\psi_{j,t}(\xi; \xi' - \xi)$ are negative for all $j \in J$ and $t \in [0,1]$; thus, using the first order approximation argument in section 4.1 and Theorem 4.2, $\xi$ is not a local minimizer of $P_p$. Second, if $\theta(\xi) < 0$ and $\Psi(\xi) < 0$, an identical argument shows that $\xi$ is not a local minimizer of $P_p$. Finally, if $\theta(\xi) < 0$ and $\Psi(\xi) > 0$, then there exists a $\xi' \in \mathcal{X}_r$ such that $D\psi_{j,t}(\xi; \xi' - \xi)$ is negative for all $j \in J$ and $t \in [0,1]$. It is clear that $\xi$ is not a local minimizer of $P_p$ since $\Psi(\xi) > 0$, but it also follows by the first order approximation argument in section 4.1 and Theorem 4.2 that it is possible to locally reduce the infeasibility of $\xi$. In this case, the addition of the DJ term in $\zeta$ is a heuristic to ensure that the reduction in infeasibility does not come at the price of an undue increase in the cost.

These observations are formalized in Theorem 5.6, where we prove that if $\xi$ is a local minimizer of $P_p$ as in Definition 4.4, then $\theta(\xi) = 0$, i.e., $\theta(\xi) = 0$ is a necessary condition for the optimality of $\xi$. Recall that in first order algorithms for finite-dimensional optimization the lack of negative directional derivatives at a point is a
PART 1: A CONCEPTUAL ALGORITHM

4.4. Choosing a step size and projecting the relaxed discrete input.

Theorem 4.2 is unable to describe how to exploit the descent direction, \( g(\xi) \), since its proof provides no means to construct a pure input that approximates the behavior of a relaxed input while controlling the quality of the approximation. We extend Theorem 4.2 by devising a scheme that allows for the development of a numerical algorithm for optimal control over the relaxed optimization space.

The argument that minimizes \( \xi \) is a “direction” along which to move the inputs in order to reduce the cost in the relaxed optimization space, but first we require an algorithm to choose a step size. We employ a line search algorithm similar to the traditional Armijo algorithm used in finite-dimensional optimization in order to choose a step size [2]. Let \( \alpha, \beta \in (0, 1) \); then the step size for \( \xi \) is chosen as \( \beta^{\mu(\xi)} \), where the step size function, \( \mu : \mathcal{X}_p \rightarrow \mathbb{N} \), is defined as

\[
\mu(\xi) = \begin{cases} 
\min \left\{ k \in \mathbb{N} \mid J(\xi + \beta^k (g(\xi) - \xi)) - J(\xi) \leq \alpha \beta^k \theta(\xi) \right\}, & \text{if } \Psi(\xi) \leq 0, \\
\min \left\{ k \in \mathbb{N} \mid \Psi(\xi + \beta^k (g(\xi) - \xi)) - \Psi(\xi) \leq \alpha \beta^k \theta(\xi) \right\}, & \text{if } \Psi(\xi) > 0.
\end{cases}
\]

Lemma 5.13 proves that if \( \theta(\xi) < 0 \) for some \( \xi \in \mathcal{X}_p \), then \( \mu(\xi) < \infty \), which means that we can construct a point \( (\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)) \in \mathcal{X}_r \) that produces a reduction in the cost (if \( \xi \) is feasible) or a reduction in the infeasibility (if \( \xi \) is infeasible).

Now, we define a projection that takes this constructed point in \( \mathcal{X}_r \) to a point belonging to \( \mathcal{X}_p \) while controlling the quality of approximation in two steps. First, let \( \lambda : \mathbb{R} \rightarrow \mathbb{R} \) be the Haar wavelet (section 7.2.2 in [19]):

\[
\lambda(t) = \begin{cases} 
1 & \text{if } t \in \left[ 0, \frac{1}{2} \right), \\
-1 & \text{if } t \in \left[ \frac{1}{2}, 1 \right), \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \mathbb{I}_A : [0, 1] \rightarrow \mathbb{R} \) be the indicator function of \( A \subset [0, 1] \), and let \( \mathbb{I} = \mathbb{I}_{[0,1]} \). Given \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, 2^k - 1\} \), let \( b_{k,j} : [0, 1] \rightarrow \mathbb{R} \) be defined by \( b_{k,j}(t) = \lambda(2^k t - j) \).

The Haar wavelet operator, \( \mathcal{F}_N : \mathcal{D}_r \rightarrow \mathcal{D}_r \), computing the \( N \)th partial sum Haar wavelet approximation is defined as

\[
[\mathcal{F}_N(d)]_j(t) = \langle d, \mathbb{I} \rangle + \sum_{k=0}^{N} \sum_{j=0}^{2^k-1} \langle d, b_{k,j} \rangle \frac{b_{k,j}(t)}{\|b_{k,j}\|_{L^2}}.
\]

The inner product here is the traditional inner product in \( L^2 \). Without loss of generality we apply \( \mathcal{F}_N \) to continuous inputs in \( \mathcal{U} \), noting that the only change is the domain of the operator. Lemma 5.7 proves that for each \( N \in \mathbb{N} \) and \( d \in \mathcal{D}_r \), \( \mathcal{F}_N(d) \in \mathcal{D}_r \).

We can project the output of \( \mathcal{F}_N(d) \), which is piecewise constant, to a pure discrete input by employing the pulse-width modulation operator, \( \mathcal{P}_N : \mathcal{D}_r \rightarrow \mathcal{D}_p \), which computes the pulse-width modulation of its argument with frequency \( 2^{-N} \):

\[
\mathcal{P}_N(d) = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{if } d \geq \frac{1}{2}, \\
\frac{1}{2} & \text{if } d < \frac{1}{2}, \\
0 & \text{otherwise}.
\end{array} \right.
\]
\[
[p_N(d)](t) = \begin{cases} 
1 & \text{if } t \in [S_{N,k,i-1}, S_{N,k,i}), \; k \in \{0,1,\ldots,2^N-1\}, \\
0 & \text{otherwise},
\end{cases}
\]

where \(S_{N,k,i} = 2^{-N}(k + \sum_{j=1}^{i} d_j \left(\frac{1}{2^j}\right))\). Lemma 5.7 also proves that for each \(N \in \mathbb{N}\) and \(d \in \mathcal{D}_r\), \(p_N(F_N(d)) \in \mathcal{D}_p\), as desired. Given \(N \in \mathbb{N}\), we compose the two operators and define the projection operator, \(\rho_N: \mathcal{X}_r \to \mathcal{X}_p\), as
\[(4.9)\]
\[\rho_N(u,d) = (F_N(u), p_N(F_N(d))).\]

As shown in Theorem 5.10, this projection allows us to extend Theorem 4.2 by providing a rate of convergence for the approximation error between \(\xi_r \in \mathcal{X}_r\) and \(\rho_N(\xi_r) \in \mathcal{X}_p\). In our algorithm, we control this approximation error by choosing the frequency at which we perform the pulse-width modulation with a rule inspired in the Armijo algorithm. Let \(\bar{\alpha} \in (0,\infty), \bar{\beta} \in \left(\frac{1}{\sqrt{2}}, 1\right), \text{and } \omega \in (0,1)\); then we modulate a point \(\xi\) with frequency \(2^{-\nu(\xi)}\), where the frequency modulation function, \(\nu: \mathcal{X}_p \to \mathbb{N}\), is defined as
\[(4.10)\]
\[\nu(\xi) = \begin{cases} 
\min \left\{ k \in \mathbb{N} \mid \xi' = \xi + \beta^{\mu(\xi)}(g(\xi) - \xi), \right. \\
J(\rho_k(\xi')) - J(\xi) \leq (\alpha^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^k)\theta(\xi), \\
\left. \Psi(\rho_k(\xi')) \leq 0, \quad \bar{\alpha}\bar{\beta}^k \leq (1 - \omega)\alpha^{\mu(\xi)} \right\} \\
\min \left\{ k \in \mathbb{N} \mid \xi' = \xi + \beta^{\mu(\xi)}(g(\xi) - \xi), \right. \\
\left. \Psi(\rho_k(\xi')) - \Psi(\xi) \leq (\alpha^{\mu(\xi)} - \bar{\alpha}\bar{\beta}^k)\theta(\xi), \quad \bar{\alpha}\bar{\beta}^k \leq (1 - \omega)\alpha^{\mu(\xi)} \right\} \quad \text{if } \Psi(\xi) \leq 0,
\end{cases}\]

Lemma 5.14 proves that if \(\theta(\xi) < 0\) for some \(\xi \in \mathcal{X}_p\), then \(\nu(\xi) < \infty\), which implies that we can construct a new point \(\rho_{\nu(\xi)}(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)) \in \mathcal{X}_p\) that produces a reduction in the cost (if \(\xi\) is feasible) or a reduction in the infeasibility (if \(\xi\) is infeasible).

**4.5. Switched system optimal control algorithm.** Algorithm 4.1 describes our numerical method to solve \(P_p\). For analysis purposes, we define the algorithm recursion function, \(\Gamma: \mathcal{X}_p \to \mathcal{X}_p\), by
\[(4.11)\]
\[\Gamma(\xi) = \rho_{\nu(\xi)}(\xi + \beta^{\mu(\xi)}(g(\xi) - \xi)).\]

We say \(\{\xi_j\}_{j \in \mathbb{N}}\) is a sequence generated by Algorithm 4.1 if \(\xi_{j+1} = \Gamma(\xi_j)\) for each \(j \in \mathbb{N}\) such that \(\theta(\xi_j) = 0\), and \(\xi_{j+1} = \xi_j\) otherwise. We prove several important properties about the sequence generated by Algorithm 4.1. First, Lemma 5.15 proves that if there exists \(i_0 \in \mathbb{N}\) such that \(\Psi(\xi_{i_0}) \leq 0\), then \(\Psi(\xi_i) \leq 0\) for each \(i \geq i_0\). Second, Theorem 5.16 proves that \(\lim_{j \to \infty} \theta(\xi_j) = 0\), i.e., Algorithm 4.1 converges to a point that satisfies the optimality condition.

**5. Algorithm analysis.** In this section, we derive the various components of Algorithm 4.1 and prove that Algorithm 4.1 converges to a point that satisfies our optimality condition. Our argument proceeds as follows: first, we construct the components of the optimality function and prove that these components satisfy various properties that ensure the well-posedness of the optimality function; second, we prove that we can control the quality of approximation between the trajectories generated by a relaxed discrete input and its projection by \(\rho_N\) as a function of \(N\); and finally, we prove the convergence of our algorithm.
ALGORITHM 4.1. Conceptual optimization algorithm for \( P_p \).

**Require:** \( \xi_0 \in \mathcal{X}_p \), \( \alpha, \beta, \omega \in (0, 1) \), \( \alpha, \gamma \in (0, \infty) \), \( \beta \in \left(\frac{1}{\sqrt{2}}, 1\right) \).

1: Set \( j = 0 \).
2: loop
3: Compute \( \theta(\xi_j) \), \( g(\xi_j) \), \( u(\xi_j) \), and \( \nu(\xi_j) \).
4: if \( \theta(\xi_j) = 0 \) then
5: return \( \xi_j \).
6: end if
7: Set \( \xi_{j+1} = \rho(\xi_j) (\xi_j + \beta\mu(\xi_j)(g(\xi_j) - \xi_j)) \).
8: Replace \( j \) by \( j + 1 \).
9: end loop

5.1. Derivation of algorithm terms. We begin by noting the Lipschitz continuity of the solution to differential equation (3.8) with respect to \( \xi \). The argument is a simple extension of Lemma 5.6.7 in [22], and therefore we omit the details.

**Lemma 5.1.** There exists a constant \( L > 0 \) such that for each \( \xi_1, \xi_2 \in \mathcal{X}_r \) and \( t \in [0, 1] \), \( \| \phi(\xi_1) - \phi(\xi_2) \|_2 \leq L \| \xi_1 - \xi_2 \|_2 \).

Next, we formally derive the components of the optimality function. This result could be proved using an argument similar to Theorem 4.1 in [6], but we include the proof for the sake of thoroughness and in order to draw comparisons with its corresponding argument, Lemma 4.4, in the second of this pair of papers.

**Lemma 5.2.** Let \( \xi = (u, d) \in \mathcal{X}_r \) and \( \xi' = (u', d') \in \mathcal{X} \). Then the directional derivative of \( \phi_1 \), as defined in (4.2), is given by

\[
\mathbf{D} \phi_1(\xi; \xi') = \int_0^t \Phi^{(t)}(\tau, \tau) \left( \frac{\partial f}{\partial u}(\tau, \phi_1(\xi), u(\tau), d(\tau)) u'(\tau) + f(\tau, \phi_1(\xi), u(\tau), d(\tau)) \right) d\tau,
\]

where \( \Phi^{(t)}(\tau, \tau) \) is the unique solution of the following matrix differential equation:

\[
\frac{\partial \Phi}{\partial t}(t, \tau) = \frac{\partial f}{\partial x}(t, \phi_1(\xi), u(t), d(t)) \Phi(t, \tau) \quad \forall t \in [0, 1], \quad \Phi(t, \tau) = I.
\]

**Proof.** For notational convenience, let \( x^{(\lambda)} = x^{(\xi + \lambda \xi')} \), \( u^{(\lambda)} = u + \lambda u' \), and \( d^{(\lambda)} = d + \lambda d' \). Then, if we define \( \Delta x^{(\lambda)} = x^{(\lambda)} - x^{(\xi)} \), and using the mean value theorem,

\[
\Delta x^{(\lambda)}(t) = \int_0^t \lambda \sum_{i=1}^q d_i(\tau) f(\tau, x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), e_i) d\tau
\]

\[
+ \int_0^t \frac{\partial f}{\partial x}(\tau, x^{(\xi)}(\tau) + \nu_x(\tau) \Delta x^{(\lambda)}(\tau), u^{(\lambda)}(\tau), d(\tau)) \Delta x^{(\lambda)}(\tau) d\tau
\]

\[
+ \int_0^t \lambda \frac{\partial f}{\partial u}(\tau, x^{(\xi)}(\tau), u(\tau) + \nu_u(\tau) \lambda u'(\tau), d(\tau)) u'(\tau) d\tau,
\]

where \( \nu_u, \nu_x : [0, t] \to [0, 1] \). Let \( z(t) \) be the unique solution of the following differential equation:

\[
\dot{z}(\tau) = \frac{\partial f}{\partial x}(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)) z(\tau) + \frac{\partial f}{\partial u}(\tau, x^{(\xi)}(\tau), u(\tau), d(\tau)) u'(\tau)
\]

\[
+ \sum_{i=1}^q d_i(\tau) f(\tau, x^{(\xi)}(\tau), u(\tau), e_i), \quad \tau \in [0, t], \quad z(0) = 0.
\]
We want to show that \( \lim_{\lambda \to 0} \| \frac{1}{\lambda} \Delta x(\lambda)(t) - z(t) \|_2 = 0 \). To prove this, consider the inequalities that follow from condition (2) in Assumption 3.2,

\[
(5.5) \quad \left\| \int_0^t \frac{\partial f}{\partial x}(\tau, x(\xi)(\tau), u(\tau), d(\tau)) z(\tau) \right\|_2 \\
+ \left\| \int_0^t \left( \frac{\partial f}{\partial u}(\tau, x(\xi)(\tau), u(\tau), \nu(\tau) \Delta x(\lambda)(\tau), u^{\lambda}(\tau), d(\tau)) \right) u'(\tau) d\tau \right\|_2 \\
\leq L \int_0^t \left\| z(\tau) - \frac{\Delta x(\lambda)(\tau)}{\lambda} \right\|_2 d\tau + L \int_0^t \left( \| \Delta x(\lambda)(\tau) \|_2 + \| u'(\tau) \|_2 \right) \| z(t) \|_2 d\tau,
\]

from condition (3) in Assumption 3.2,

\[
(5.6) \quad \left\| \int_0^t \left( \frac{\partial f}{\partial x}(\tau, x(\xi)(\tau), u(\tau), d(\tau)) \\
- \frac{\partial f}{\partial u}(\tau, x(\xi)(\tau), u(\tau) + \nu(\tau) \lambda u'(\tau), d(\tau)) \right) u'(\tau) d\tau \right\|_2 \\
\leq L \int_0^t \lambda \| \nu(\tau) u'(\tau) \|_2 \| u'(\tau) \|_2 d\tau \leq L \int_0^t \lambda \| u'(\tau) \|_2^2 d\tau,
\]

and from condition (1) in Assumption 3.2,

\[
(5.7) \quad \left\| \int_0^t \sum_{i=1}^q d'_i(\tau) (f(\tau, x(\xi)(\tau), u(\tau), e_i) r - f(\tau, x(\lambda)(\tau), u^{\lambda}(\tau), e_i)) d\tau \right\|_2 \\
\leq L \int_0^t \sum_{i=1}^q d'_i(\tau) (\| \Delta x(\lambda)(\tau) \|_2 + \| u'(\tau) \|_2) d\tau.
\]

Then, using the Bellman–Gronwall inequality,

\[
(5.8) \quad \left\| \frac{\Delta x(\lambda)(t)}{\lambda} - z(t) \right\|_2 \leq e^{Lt} L \left( \int_0^t \left( \| \Delta x(\lambda)(\tau) \|_2 + \| u'(\tau) \|_2 \right) \| z(t) \|_2 + \| u'(\tau) \|_2^2 \\
+ \sum_{i=1}^q d'_i(\tau) (\| \Delta x(\lambda)(\tau) \|_2 + \| u'(\tau) \|_2) d\tau \right),
\]

but note that every term in the integral above is bounded, and \( \Delta x(\lambda)(\tau) \to 0 \) for each \( \tau \in [0, t] \) since \( x(\lambda) \to x(\xi) \) uniformly as follows from Lemma 5.1. Thus, by the dominated convergence theorem and by noting that \( D\phi_t(\xi; \xi') \) is exactly the solution of differential equation (5.4) we get

\[
(5.9) \quad \lim_{\lambda \to 0} \left\| \frac{\Delta x(\lambda)(t)}{\lambda} - z(t) \right\|_2 = \lim_{\lambda \to 0} \frac{1}{\lambda} \| x^{(\xi+\lambda \xi')}(t) - x^{(\xi)}(t) - D\phi_t(\xi; \lambda \xi') \|_2 = 0.
\]

The result of the lemma then follows. \( \square \)
We now construct the directional derivative of the cost and component constraint functions, which follows from the chain rule and Lemma 5.2.

**Lemma 5.3.** Let $\xi \in X_r$, $\xi' \in X_j$, $j \in J$, and $t \in [0, 1]$. The directional derivatives of the cost $J$ and the component constraint $\psi_{j,t}$ in the $\xi'$ direction are

$$DJ(\xi'; \xi') = \frac{\partial h_0}{\partial x}(\phi_1(\xi)) D\phi_1(\xi; \xi') \quad \text{and} \quad D\psi_{j,t}(\xi'; \xi') = \frac{\partial h_j}{\partial x}(\phi_t(\xi)) D\phi_t(\xi; \xi').$$

**Lemma 5.4.** There exists an $L > 0$ such that for each $\xi_1, \xi_2 \in X_r$, $\xi' \in X_j$, and $t \in [0, 1],$

1. $\|D\phi_t(\xi_1; \xi') - D\phi_t(\xi_2; \xi')\|_2 \leq L \|\xi_1 - \xi_2\|_X \|\xi'\|_X$,
2. $\|D\psi_{j,t}(\xi_1; \xi') - D\psi_{j,t}(\xi_2; \xi')\| \leq L \|\xi_1 - \xi_2\|_X \|\xi'\|_X$, and
3. $|\xi(\xi_1; \xi') - \xi(\xi_2; \xi')| \leq L \|\xi_1 - \xi_2\|_X$.

**Proof.** Condition (1) follows after proving the boundedness of the vector fields, the Lipschitz continuity of the vector fields and their derivatives, and the boundedness and Lipschitz continuity of the state transition matrix. Conditions (2) and (3) follow immediately from condition (1).

To prove condition (4), first notice that for $\{x_i\}_{i=1}^k, \{y_i\}_{i=1}^k \subseteq \mathbb{R},$

$$\max_{i \in \{1, \ldots, k\}} x_i - \max_{i \in \{1, \ldots, k\}} y_i \leq \max_{i \in \{1, \ldots, k\}} |x_i - y_i|.$$

Let $\Psi^+(\xi) = \max\{0, \Psi(\xi)\}$ and $\Psi^-(\xi) = \max\{0, -\Psi(\xi)\}$. Then,

$$|\xi_1; \xi') - \xi(\xi_2; \xi')| \leq \max \left\{ \left| DJ(\xi_1; \xi' - \xi_1) - DJ(\xi_2; \xi' - \xi_2) \right| + |\Psi^+(\xi_2) - \Psi^+(\xi_1)|, \right.$$

$$\left. \max_{j \in J, t \in [0, 1]} \left| D\psi_{j,t}(\xi_1; \xi' - \xi_1) - D\psi_{j,t}(\xi_2; \xi' - \xi_2) \right| \right\}$$

$$+ \left| \xi_1; \xi') - \xi(\xi_2; \xi') \right| + \|\xi' - \xi_1\|_X - \|\xi' - \xi_2\|_X \|.$$
Theorem 5.5. For each $\xi \in X_p$, the map $\xi' \mapsto \zeta(\xi, \xi') + V(\xi' - \xi)$, with domain $X_p$, is strictly convex and has a unique minimizer.

Proof. The map is strictly convex since it is the sum of the maximum of linear functions of $\xi' - \xi$ (which is convex), the total variation of $\xi' - \xi$ (which is convex), and the $\mathcal{X}$-norm of $\xi' - \xi$ (which is strictly convex).

Next, let $\theta(\xi) = \{\zeta(\xi, \xi') + V(\xi' - \xi) \mid \xi' \in L^2([0, 1], U) \times L^2([0, 1], \Sigma_r)\}$. Note that $|\zeta(\xi, \xi')| \to \infty$ as $\|\xi'\|_{L^2} \to \infty$. Then, by Proposition II.1.2 in [10], there exists a unique minimizer $\hat{\xi}^*$ of $\theta(\xi)$. But $\hat{\xi}^* \in X_p$, since $\theta(\xi)$ and $\zeta(\xi, \xi^*)$ are bounded. The result follows after noting that $X_p \subset L^2([0, 1], U) \times L^2([0, 1], \Sigma_r)$; thus $\hat{\xi}^*$ is also the unique minimizer of $\theta(\xi)$.

Finally, we prove that $\theta$ encodes a necessary condition for optimality.

Theorem 5.6. The function $\theta$ is nonpositive valued. Moreover, if $\xi \in X_p$ is a local minimizer of $P_p$ as in Definition 4.4, then $\theta(\xi) = 0$.

Proof. Notice that $\zeta(\xi, \xi) = 0$ and $\theta(\xi) \leq \zeta(\xi, \xi)$, which proves the first part.

To prove the second part, we begin by making several observations. Given $\xi' \in X_r$ and $\lambda \in [0, 1]$, using the mean value theorem and condition (2) in Lemma 5.4 we have that there exists $L > 0$ such that

\begin{equation}
J(\xi + \lambda(\xi' - \xi)) - J(\xi) \leq \lambda D J(\xi; \xi' - \xi) + L\lambda^2 \|\xi' - \xi\|^2_X.
\end{equation}

Similarly, if $A(\xi) = \arg \max \{h_j(\phi_t(\xi)) \mid (j, t) \in J \times [0, 1]\}$, then there exists a pair $(j, t) \in A(\xi + \lambda(\xi' - \xi))$ such that using condition (3) in Lemma 5.4,

\begin{equation}
\Psi(\xi + \lambda(\xi' - \xi)) - \Psi(\xi) \leq \lambda D \psi_{j,t}(\xi; \xi' - \xi) + L\lambda^2 \|\xi' - \xi\|^2_X.
\end{equation}

Also, by condition (1) in Assumption 3.4,

\begin{equation}
\Psi(\xi + \lambda(\xi' - \xi)) - \Psi(\xi) \leq L \max_{t \in [0, 1]} \|\phi_t(\xi + \lambda(\xi' - \xi)) - \phi_t(\xi)\|_2.
\end{equation}

We proceed by contradiction, i.e., we assume that $\theta(\xi) < 0$ and show that for each $\epsilon > 0$ there exists $\xi \in X_p \cap \{\xi \in X_p \mid \Psi(\xi) \leq 0\}$ such that $J(\xi) < J(\xi')$. Note that since $\theta(\xi) < 0$, $g(\xi) \neq \xi$, and also, as a result of Theorem 4.2, for each $(\xi + \lambda(g(\xi) - \xi)) \in X_r$ and $\epsilon' > 0$ there exists $\lambda \in X_p$ such that $\|x^{(\xi_0)} - x^{(\xi + \lambda(g(\xi) - \xi))}\|_{L^2} < \epsilon'$. Now, letting $\epsilon' = -\frac{\lambda(\xi)}{2L} > 0$, using Lemma 5.1, and adding and subtracting $x^{(\xi + \lambda(g(\xi) - \xi))}$,

\begin{equation}
\|x^{(\xi_0)} - x^{(\xi)}\|_{L^2} \leq \left(-\frac{\theta(\xi)}{2L} + L\|g(\xi) - \xi\|_X\right)\lambda.
\end{equation}

Also, by (5.14), after adding and subtracting $J(\xi + \lambda(g(\xi) - \xi))$,

\begin{equation}
J(\xi_0) - J(\xi) \leq \frac{\theta(\xi)\lambda}{2} + 4A^2L\lambda^2.
\end{equation}
where $A = \max\{\|u\|_2 + 1 \mid u \in U\}$ and we used the fact that $\|\xi - \xi'\|^2_X \leq 4A^2$ and $D(J;\xi',\xi) \leq \theta(\xi)$. Hence for each $\lambda \in (0, \frac{\theta(\xi)}{8A^2L})$, $J(\xi_{\lambda}) - J(\xi) < 0$. Using a similar argument, and after adding and subtracting $\Psi(\xi + \lambda(g(\xi) - \xi))$, we have

$$
\Psi(\xi_{\lambda}) \leq L \max_{t \in [0,1]} \|\phi_t(\xi_{\lambda}) - \phi_t(\xi + \lambda(g(\xi) - \xi))\|_2 + \Psi(\xi) + (\theta(\xi) - \gamma \Psi(\xi)) \lambda + 4A^2 L \lambda^2
$$

where we used the fact that $D(\xi_{\lambda})$ takes elements from the relaxed optimization space to the pure optimization space $\|\cdot\|_\infty$. This holds since $\frac{\theta(\xi)}{8A^2L}, \Psi(\xi_{\lambda}) \leq (1 - \gamma \lambda)\Psi(\xi) \leq 0$.

Summarizing, suppose $\xi \in \mathcal{X}_p$ is a local minimizer of $P_p$ and $\theta(\xi) < 0$. For each $\varepsilon > 0$, by choosing any

$$
\lambda \in \left(0, \min\left\{\frac{-\theta(\xi)}{8A^2L}, \frac{1}{\gamma}, \frac{2L}{\gamma 2L^2 \|g(\xi) - \xi\|_\infty} \right\}\right],
$$

we can construct a $\xi_{\lambda} \in \mathcal{X}_p$ such that $\xi_{\lambda} \in N_w(\xi, \varepsilon)$, $J(\xi_{\lambda}) < J(\xi)$, and $\Psi(\xi_{\lambda}) \leq \Psi(\xi)$. Therefore, $\xi$ is not a local minimizer of $P_p$, which is a contradiction.

5.2. Approximating relaxed inputs. In this subsection, we prove that $\rho_N$ takes elements from the relaxed optimization space to the pure optimization space with a known bound for the quality of approximation.

**Lemma 5.7.** For each $N \in \mathbb{N}$ and $d \in D_r$, $\mathcal{F}_N(d) \in D_r$, and $\mathcal{P}_N(\mathcal{F}_N(d)) \in D_p$.

**Proof.** Note first that $[\mathcal{F}_N(d)]_i(t) \in [0,1]$ for each $t \in [0,1]$ due to the result in section 3.3 in [14]. Next, observe that $\sum_{i=1}^q [\mathcal{F}_N(d)]_i(t) = 1$ for each $t \in [0,1]$ since the wavelet approximation is linear and

$$
\sum_{i=1}^q [\mathcal{F}_N(d)]_i = \sum_{i=1}^q \left( \langle d_i, \mathbb{1} \rangle + \sum_{k=0}^{N-1} \sum_{j=0}^{2^k-1} \langle d_{i, k,j}, b_{k,j} \rangle \frac{b_{k,j}}{\|b_{k,j}\|_{L^2}^2} \right)
$$

$$
= \langle \mathbb{1}, \mathbb{1} \rangle + \sum_{k=0}^{N-1} \sum_{j=0}^{2^k-1} \langle \mathbb{1}, b_{k,j} \rangle \frac{b_{k,j}}{\|b_{k,j}\|_{L^2}^2} = 1,
$$

where the last equality holds since $\langle \mathbb{1}, b_{k,j} \rangle = 0$ for each $k, j$.

Next, observe that $[\mathcal{P}_N(\mathcal{F}_N(d))]_i(t) \in [0,1]$ for each $t \in [0,1]$ due to the definition of $\mathcal{P}_N$. Also note that $\sum_{i=1}^q [\mathcal{P}_N(\mathcal{F}_N(d))]_i(t) = 1$ for each $t \in [0,1]$. This follows because of the piecewise continuity of $\mathcal{F}_N(d)$ over intervals of size $2^{-N}$ and because $\sum_{i=1}^q [\mathcal{F}_N(d)]_i(t) = 1$, which implies that $\{[\mathcal{S}_{N,k,i-1}, \mathcal{S}_{N,k,i}]\}_{i \in \{1, \ldots, q\} \times \{0, \ldots, 2^N - 1\}}$ forms a partition of $[0,1]$. Our desired result follows.

The next two lemmas lay the foundations of our extension to the chattering lemma and rely upon a bounded variation assumption in order to construct a rate of approximation, which is critical while choosing a value for the parameter $\tilde{\beta}$ in order to ensure that if $\theta(\xi) < 0$, then $\nu(\xi) < \infty$. This is proved in Lemma 5.14 and guarantees that step 7 of Algorithm 4.1 is well-defined.

**Lemma 5.8.** Let $f \in L^2([0,1], \mathbb{R}) \cap BV([0,1], \mathbb{R})$; then

$$
\|f - \mathcal{F}_N(f)\|_{L^2} \leq \frac{1}{2^{\sqrt{2}}} V(f).
$$
Proof. Since $L^2$ is a Hilbert space and the collection $\{b_{k,j}\}_{k,j}$ is a basis, then

\begin{equation}
(5.23) \quad f = \langle f, 1 \rangle + \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} \frac{\langle f, b_{k,j} \rangle}{\|b_{k,j}\|^2_{L^2}} b_{k,j}.
\end{equation}

Note that $\|b_{k,j}\|^2_{L^2} = 2^{-k}$ and that $|f^t_{0} b_{k,j}(s)ds| \leq 2^{-k-1}$ for each $t \in [0,1]$. Thus, by Proposition 3.1, $|\langle f, b_{k,j} \rangle| \leq 2^{-k} V(f \mathbb{1}_{(j+1)2^{-k}})$. Finally, Parseval’s identity for Hilbert spaces (Theorem 5.27 in [11]) implies that

\begin{equation}
(5.24) \quad \|f - \mathcal{F}_N(f)\|_{L^2}^2 \leq \sum_{k=N+1}^{\infty} 2^{-k-2} \sum_{j=0}^{2^k-1} (V(f \mathbb{1}_{(j+1)2^{-k}}))^2 \leq 2^{-N-2} V(f)^2,
\end{equation}

as desired, where the last inequality follows by the definition in (3.3).

Lemma 5.9. There exists $K > 0$ such that for each $d \in D_r$ and $f \in L^2([0,1], \mathbb{R}^q) \cap BV([0,1], \mathbb{R}^q)$,

\begin{equation}
(5.25) \quad \|\langle d - \mathcal{P}_N(\mathcal{F}_N(d)), f \rangle\| \leq K \left( \left( \frac{1}{\sqrt{2}} \right)^N \|f\|_{L^2} V(d) + \left( \frac{1}{2} \right)^N V(f) \right).
\end{equation}

Proof. To simplify our notation, let $p_{i,k} = \mathcal{F}_N(d)_i \left( \frac{t}{2^k} \right)$, $S_i = \sum_{j=1}^{i} p_{j,k}$, and $A_{i,k} = [kS_{i-1,k}, kS_{i,k}]$. Note that $|\int_{0}^{1} t \neq \mathcal{P}_N(\mathcal{F}_N(d))_i(t) dt| \leq K \neq_{\frac{1}{2}}$ for each $t \in [0,1]$. Thus, by Proposition 3.1, the definition in (3.3), and Hölder’s inequality,

\begin{equation}
(5.26) \quad \|\langle \mathcal{F}_N(d) - \mathcal{P}_N(\mathcal{F}_N(d)), f \rangle\| = \sum_{k=0}^{N-1} \sum_{i=1}^{q} \int_{\frac{k}{2^k}}^{\frac{k+1}{2^k}} (p_{i,k} - \mathbb{1}_{A_{i,k}}(t)) f_i(t) dt \leq \frac{1}{2^N} V(f).
\end{equation}

The final result follows from the inequality above, Lemma 5.8, and the Cauchy–Schwarz inequality.

The following theorem is an extension of the chattering lemma for functions of bounded variation. As explained earlier, this theorem is key to our result since it provides a rate of convergence for the approximation of trajectories.

Theorem 5.10. There exists $K > 0$ such that for each $\xi \in \mathcal{X}$ and $t \in [0,1]$,

\begin{equation}
(5.27) \quad \|\phi_t(\rho_N(\xi)) - \phi_t(\xi)\|_2 \leq K \left( \frac{1}{\sqrt{2}} \right)^N (V(\xi) + 1).
\end{equation}

Proof. Let $\xi = (u,d)$. To simplify our notation, let us denote $u_N = \mathcal{F}_N(u)$ and $d_N = \mathcal{P}_N(\mathcal{F}_N(d))$; thus $\rho_N(\xi) = (u_N, d_N)$. First note that since $f$ is Lipschitz continuous,

\begin{equation}
(5.28) \quad \|x(u_N,d_N)(t) - x(u,d)(t)\|_2 \leq L \int_{0}^{1} \|x(u_N,d_N)(s) - x(u,d)(s)\|_2 + \|u_N(s) - u(s)\|_2 ds \leq \frac{L e^{k\sqrt{2}}}{2} \left( \frac{1}{\sqrt{2}} \right)^N V(u),
\end{equation}

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where the last inequality follows by Bellman–Gronwall's inequality together with Lemma 5.8. On the other hand,

\begin{equation}
\|x^{(u,d_N)}(t) - x^{(u,d)}(t)\|_2 \leq e^L \left\| \int_0^1 \sum_{i=1}^q (|d_N|_i(s) - d_i(s)) f(t, x^{(u,d)}(s), u(s), e_i) ds \right\|_2,
\end{equation}

where again we used the Lipschitz continuity of \( f \) and Bellman–Gronwall's inequality. Let \( v_{k,i}(t) = [f(t, x^{(u,d)}(t), u(t), e_i)]_k \) and \( v_k = (v_{k,1}, \ldots, v_{k,q}) \); then \( v_k \) is of bounded variation. Indeed, \( x^{(\ell)} \) is of bounded variation (by Corollary 3.33 and Lemma 3.34 in [11]); thus there exists \( C > 0 \) such that \( V(x^{(\ell)}) \leq C \) and \( \|v_{k,i}\|_{L^2} \leq C \), and using the Lipschitz continuity of \( f \), we get that for each \( i \in Q \), \( V(v_{k,i}) \leq L(1 + C + V(u)) \). Hence, Lemma 5.9 implies that there exists \( K > 0 \) such that for each \( k \in \{1, \ldots, n\} \),

\begin{equation}
|\langle d - d_N, v_k \rangle| \leq K \left( \left( \frac{1}{\sqrt{2}} \right)^N CV(d) + \left( \frac{1}{2} \right)^N (1 + C + V(u)) \right).
\end{equation}

The result follows easily from these three inequalities.

5.3. Convergence of the algorithm. To prove the convergence of our algorithm, we employ a technique similar to the one prescribed in section 1.2 in [22]. Summarizing the technique, one can think of an algorithm as a discrete-time dynamical system, whose desired stable equilibria are characterized by the stationary points of its optimality function, i.e., points \( \xi \in X_p \), where \( \theta(\xi) = 0 \), since we know from Theorem 5.6 that all local minimizers are stationary. Before applying this line of reasoning to our algorithm, we present a simplified version of this argument for a general unconstrained optimization problem in the interest of clarity.

**Definition 5.11** (Definition 2.1 in [3]). Let \( S \) be a metric space, and consider the problem of minimizing the cost function \( J: S \to \mathbb{R} \). A function \( \Gamma: S \to S \) has the uniform sufficient descent property with respect to an optimality function \( \theta: S \to (-\infty, 0] \) if for each \( C > 0 \) there exists a \( \delta_C > 0 \) such that for every \( x \in S \) with \( \theta(x) < -C \), \( J(\Gamma(x)) - J(x) \leq -\delta_C \).

A sequence of points generated by an algorithm satisfying the uniform sufficient descent property can be shown to approach the zeros of the optimality function.

**Theorem 5.12** (Proposition 2.1 in [3]). Let \( S \) be a metric space and \( J: S \to \mathbb{R} \) be a lower bounded function. Suppose that \( \Gamma: S \to S \) satisfies the uniform sufficient descent property with respect to \( \theta: S \to (-\infty, 0] \). If \( \{x_j\}_{j \in \mathbb{N}} \) is constructed such that

\begin{equation}
x_{j+1} = \begin{cases} 
\Gamma(x_j) & \text{if } \theta(x_j) < 0, \\
x_j & \text{if } \theta(x_j) = 0,
\end{cases}
\end{equation}

then \( \lim_{j \to \infty} \theta(x_j) = 0 \).

**Proof.** Suppose that \( \lim \inf_{j \to \infty} \theta(x_j) = -2\varepsilon < 0 \). Then there exists a subsequence \( \{x_{j_k}\}_{k \in \mathbb{N}} \) such that \( \theta(x_{j_k}) < -\varepsilon \) for each \( k \in \mathbb{N} \). Definition 5.11 implies that there exists \( \delta_\varepsilon \) such that for each \( k \in \mathbb{N} \), \( J(x_{j_k+1}) - J(x_{j_k}) \leq -\delta_\varepsilon \). But this is a contradiction, since \( J(x_{j+1}) \leq J(x_j) \) for each \( j \in \mathbb{N} \), and thus \( J(x_j) \to -\infty \) as \( j \to \infty \), even though \( J \) is lower bounded. \( \square \)

\(^2\)Versions of this result can also be found in [23, 35].
Note that Theorem 5.12 does not assume the existence of accumulation points of the sequence \( \{x_j\}_{j \in \mathbb{N}} \). This is critical in infinite-dimensional optimization problems where the level sets of a cost function may not be compact. Our proof of convergence of the sequence of points generated by Algorithm 4.1 does not make explicit use of Theorem 5.12 since \( \rho_N \) and the constraints require special treatment, but the main argument of the proof follows from a similar principle.

We begin by showing that \( \mu \) and \( \nu \) are bounded when \( \theta \) is negative.

**Lemma 5.13.** Let \( \alpha, \beta \in (0, 1) \). For each \( \delta > 0 \) there exists \( M_\delta^* < \infty \) such that if \( \theta(\xi) \leq -\delta \) for \( \xi \in \mathcal{X}_p \), then \( \mu(\xi) \leq M_\delta^* \).

**Proof.** Assume that \( \Psi(\xi) \leq 0 \). Recall that \( DJ(\xi; g(\xi) - \xi) \leq \theta(\xi) \); then using (5.14),

\[
J(\xi + \beta^k(g(\xi) - \xi)) - J(\xi) - \alpha \beta^k \theta(\xi) \leq -(1 - \alpha) \beta^k \theta(\xi) + 4A^2L\beta^{2k},
\]

where \( A = \text{max} \left\{ ||u||_2 + 1 \mid u \in U \right\} \). Hence, for each \( k \in \mathbb{N} \) such that \( \beta^k \leq \frac{(1-\alpha)\delta}{4A^2L} \) we have that \( J(\xi + \beta^k(g(\xi) - \xi)) - J(\xi) \leq \alpha \beta^k \theta(\xi) \). Similarly, since \( D\psi_{j,t}(\xi; g(\xi) - \xi) \leq \theta(\xi) \) for each \( (j, t) \in J \times [0, 1] \), using (5.15),

\[
\Psi(\xi + \beta^k(g(\xi) - \xi)) - \Psi(\xi) + \beta^k \left( \gamma \Psi(\xi) - \alpha \theta(\xi) \right) \leq -(1 - \alpha) \beta^k \theta(\xi) + 4A^2L\beta^{2k};
\]

hence, for each \( k \in \mathbb{N} \) such that \( \beta^k \leq \min \left\{ \frac{(1-\alpha)\delta}{4A^2L}, \frac{1}{\gamma} \right\} \), \( \Psi(\xi + \beta^k(g(\xi) - \xi)) - \alpha \beta^k \theta(\xi) \leq (1 - \beta^k \gamma) \Psi(\xi) \leq 0 \).

Now, if \( \Psi(\xi) > 0 \),

\[
\Psi(\xi + \beta^k(g(\xi) - \xi)) - \Psi(\xi) - \alpha \beta^k \theta(\xi) \leq -(1 - \alpha) \delta \beta^k + 4A^2L\beta^{2k}.
\]

Hence, for each \( k \in \mathbb{N} \) such that \( \beta^k \leq \frac{(1-\alpha)\delta}{4A^2L} \), \( \Psi(\xi + \beta^k(g(\xi) - \xi)) - \Psi(\xi) \leq \alpha \beta^k \theta(\xi) \).

Finally, let \( M_\delta^* = 1 + \text{max} \left\{ \log_{\beta} \left( \frac{(1-\alpha)\delta}{4A^2L} \right), \log_{\beta} \left( \frac{1}{\gamma} \right) \right\} \); then we get that \( \mu(\xi) \leq M_\delta^* \) as desired. \( \square \)

The following result relies on the rate of convergence result proved in Theorem 5.10 that follows from Lemmas 5.8 and 5.9 and from the assumption that the relaxed optimization space is of bounded variation.

**Lemma 5.14.** Let \( \alpha, \beta, \omega \in (0, 1) \), \( \bar{\alpha} \in (0, \infty) \), \( \bar{\beta} \in \left( \frac{1}{\sqrt{2}}, 1 \right) \), and \( \xi \in \mathcal{X}_p \). If \( \theta(\xi) < 0 \), then \( \nu(\xi) < \infty \).

**Proof.** To simplify our notation let us denote \( M = \mu(\xi) \) and \( \xi' = \xi + \beta^M (g(\xi) - \xi) \).

Theorem 5.10 implies that there exists \( K > 0 \) such that

\[
J(\rho_N(\xi')) - J(\xi') \leq K L 2^{-\frac{\bar{\alpha}}{2}M} (V(\xi') + 1),
\]

where \( L \) is the Lipschitz constant of \( h_0 \).

Let \( \mathcal{A}(\xi) = \text{arg max} \left\{ h_{j,t}(\phi_t(\xi)) \mid (j, t) \in J \times [0, 1] \right\} \); then given \( (j, t) \in \mathcal{A}(\rho_N(\xi')) \),

\[
\Psi(\rho_N(\xi')) - \Psi(\xi') \leq \psi_{j,t}(\rho_N(\xi')) - \psi_{j,t}(\xi') \leq K L 2^{-\frac{\bar{\alpha}}{2}M} (V(\xi') + 1).
\]

Hence, there exists \( N_0 \in \mathbb{N} \) such that \( K L 2^{-\frac{\bar{\alpha}}{2}M} (V(\xi') + 1) \leq -\bar{\alpha} \beta^N \theta(\xi) \) for each \( N \geq N_0 \). Also, there exists \( N_1 \geq N_0 \) such that for each \( N \geq N_1 \), \( \bar{\alpha} \beta^N \leq (1 - \omega)\alpha \beta^M \).
Now suppose that \( \Psi(\xi) \leq 0 \); then for each \( N \geq N_1 \) the following inequalities hold:

\[
\begin{align*}
(\text{5.37}) & \quad J(\rho_N(\xi')) - J(\xi) = J(\rho_N(\xi')) - J(\xi') + J(\xi') - J(\xi) \leq (\alpha \beta^M - \bar{\alpha} \beta^N) \theta(\xi), \\
(\text{5.38}) & \quad \Psi(\rho_N(\xi')) = \Psi(\rho_N(\xi')) - \Psi(\xi') + \Psi(\xi') \leq (\alpha \beta^M - \bar{\alpha} \beta^N) \theta(\xi) \leq 0.
\end{align*}
\]

Similarly, if \( \Psi(\xi) > 0 \), then, using the same argument as above, we have that

\[
(\text{5.39}) \quad \Psi(\rho_N(\xi')) - \Psi(\xi) \leq (\alpha \beta^M - \bar{\alpha} \beta^N) \theta(\xi).
\]

Therefore, from (5.37), (5.38), and (5.39), it follows that \( \nu(\xi) \leq N_1 \) as desired. \( \square \)

The following lemma proves that once Algorithm 4.1 finds a feasible point, every point generated afterward is also feasible. We omit the proof since it follows directly from the definition of \( \nu \).

**Lemma 5.15.** Let \( \{\xi_i\}_{i \in \mathbb{N}} \) be a sequence generated by Algorithm 4.1. If there exists \( i_0 \in \mathbb{N} \) such that \( \Psi(\xi_{i_0}) \leq 0 \), then \( \Psi(\xi_i) \leq 0 \) for each \( i \geq i_0 \).

Employing these preceding results, we can prove that every sequence produced by Algorithm 4.1 asymptotically satisfies our optimality condition.

**Theorem 5.16.** Let \( \{\xi_i\}_{i \in \mathbb{N}} \) be a sequence generated by Algorithm 4.1. Then \( \lim_{i \to \infty} \theta(\xi_i) = 0 \).

**Proof.** If the sequence produced by Algorithm 4.1 is finite, then the theorem is trivially satisfied, so we assume that the sequence is infinite. Suppose the theorem is not true; then \( \lim \inf_{i \to \infty} \theta(\xi_i) = -2\delta < 0 \) and therefore there exists \( k_0 \in \mathbb{N} \) and a subsequence \( \{\xi_{i_k}\}_{k \in \mathbb{N}} \) such that \( \theta(\xi_{i_k}) \leq -\delta \) for each \( k \geq k_0 \). Also, recall that \( \nu(\xi) \) was chosen such that given \( \mu(\xi) \), \( \alpha \beta^{\mu(\xi)} - \bar{\alpha} \beta^{\mu(\xi)} \geq \omega \alpha \beta^{\mu(\xi)} \).

From Lemma 5.13 we know that there exists \( M^*_\delta \), which depends on \( \delta \), such that \( \beta^{\mu(\xi)} \geq \beta^{M^*_\delta} \). Suppose that the subsequence \( \{\xi_{i_k}\}_{k \in \mathbb{N}} \) is eventually feasible; then by Lemma 5.15, the sequence is always feasible. Thus,

\[
(\text{5.40}) \quad J(\Gamma(\xi_{i_k})) - J(\xi_{i_k}) \leq (\alpha \beta^{\mu(\xi)} - \bar{\alpha} \beta^{\mu(\xi)}) \theta(\xi_{i_k}) \leq -\omega \alpha \beta^{\mu(\xi)} \delta \leq -\omega \alpha \beta^{M^*_\delta} \delta.
\]

This inequality, together with the fact that \( J(\xi_{i+1}) \leq J(\xi_i) \) for each \( i \in \mathbb{N} \), implies that \( \lim \inf_{k \to \infty} J(\xi_{i_k}) = -\infty \), but this is a contradiction since \( J \) is lower bounded, which follows from Theorem 3.3.

The case when the sequence is never feasible is analogous after noting that since the subsequence is infeasible, then \( \Psi(\xi_{i_k}) > 0 \) for each \( k \in \mathbb{N} \), establishing a similar contradiction. \( \square \)

As described in the discussion that follows Theorem 5.12, this result does not prove the existence of accumulation points to the sequence generated by Algorithm 4.1, but only shows that the generated sequence asymptotically satisfies a necessary condition for optimality.

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