RESOLVABILITY IN C.C.C. GENERIC EXTENSIONS

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Abstract. Every crowded space \( X \) is \( \omega \)-resolvable in the c.c.c generic extension \( V^{Fn(|X|,2)} \) of the ground model.

We investigate what we can say about \( \lambda \)-resolvability in c.c.c-generic extensions for \( \lambda > \omega \)?

A topological space is \textit{monotonically} \( \omega_1 \)-\textit{resolvable} if there is a function \( f : X \rightarrow \omega_1 \) such that

\[
\{ x \in X : f(x) \geq \alpha \} \subset \text{dense } X
\]

for each \( \alpha < \omega_1 \).

We show that given a \( T_1 \) space \( X \) the following statements are equivalent:

(1) \( X \) is \( \omega_1 \)-resolvable in some c.c.c-generic extension,
(2) \( X \) is monotonically \( \omega_1 \)-resolvable.
(3) \( X \) is \( \omega_1 \)-resolvable in the Cohen-generic extension \( V^{Fn(\omega_1,2)} \).

We investigate which spaces are monotonically \( \omega_1 \)-resolvable.

We show that if a topological space \( X \) is c.c.c, and \( \omega_1 \leq \Delta(X) \leq |X| < \omega_\omega \), where \( \Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\} \), then \( X \) is monotonically \( \omega_1 \)-resolvable.

On the other hand, it is also consistent, modulo the existence of a measurable cardinal, that there is a space \( Y \) with \( |Y| = \Delta(Y) = \aleph_\omega \) which is not monotonically \( \omega_1 \)-resolvable.

The characterization of \( \omega_1 \)-resolvability in c.c.c generic extension raises the following question: is it true that crowded spaces from the ground model are \( \omega \)-resolvable in \( V^{Fn(\omega,2)} \)?

We show that (i) if \( V = L \) then every crowded c.c.c. space \( X \) is \( \omega \)-resolvable in \( V^{Fn(\omega,2)} \), (ii) if there is no weakly inaccessible cardinals, then every crowded space \( X \) is \( \omega \)-resolvable in \( V^{Fn(\omega_1,2)} \).

On the other hand, it is also consistent, modulo a measurable cardinals, that there is a crowded space \( X \) with \( |X| = \Delta(X) = \omega_1 \) such that \( X \) remains irresolvable after adding a single Cohen real.

\[\text{Date: February 2, 2017.}\]

2010 Mathematics Subject Classification. 54A35, 03E35, 54A25.

Key words and phrases. resolvable, monotonically \( \omega_1 \)-resolvable, measurable cardinal.

The second author was supported by Fulbright Scholar Program.

The preparation of this paper was supported by OTKA grant no. K113047.
1. Introduction

Notion of resolvability was introduced and studied first by E. Hewitt, \cite{4}, in 1943. A topological space \( X \) is \( \kappa \)-resolvable if it can be partitioned into \( \kappa \) many dense subspaces. \( X \) is resolvable iff it is 2-resolvable, and irresolvable otherwise. Irresolvable spaces with many interesting extra properties were constructed, but there are no “absolute” examples for crowded irresolvable spaces, because if \( X \) is a crowded space, then clearly

\[
V^{\text{Fn}(\|X\|,2)} \models X \text{ is } \omega\text{-resolvable.}
\]

In this paper we investigate what we can say about \( \lambda \)-resolvability in c.c.c-generic extensions for \( \lambda > \omega \)?

To characterize spaces which are \( \omega_1 \)-resolvable in some c.c.c-generic extension we introduce the notion of monotonically \( \kappa \)-resolvable.

**Definition 1.1.** Let \( \kappa \) be an infinite cardinal. A topological space \( X \) is monotonically \( \kappa \)-resolvable\(^\dagger\) if there is a function \( f : X \to \kappa \) such that

\[
\{ x \in X : f(x) \geq \alpha \} \subset^\text{dense} X
\]

for each \( \alpha < \kappa \). We will say that \( f \) witnesses that \( X \) is monotonically \( \kappa \)-resolvable.

Clearly a space \( X \) is monotonically \( \kappa \)-resolvable iff \( X \) has a partition \( \{ X_\zeta : \zeta < \kappa \} \) of \( X \) such that

\[
\text{int} \left( \bigcup \{ X_\zeta : \zeta < \xi \} \right) = \emptyset
\]

for all \( \xi < \kappa \).

**Theorem 1.2.** Let \( X \) be a \( T_1 \) topological space. The following statements are equivalent:

1. \( X \) is \( \omega_1 \)-resolvable in some c.c.c-generic extension,
2. \( X \) is monotonically \( \omega_1 \)-resolvable,
3. \( X \) is \( \omega_1 \)-resolvable in the Cohen generic extension \( V^{\text{Fn}(\omega_1,2)} \).

Which spaces are monotonically \( \omega_1 \)-resolvable?

**Theorem 1.3.** If a topological space \( X \) is c.c.c, and \( \omega_1 \leq \Delta(X) \leq |X| < \omega_\omega \), then \( X \) is monotonically \( \omega_1 \)-resolvable.

**Theorem 1.4.** If \( \kappa \) is a measurable cardinal, then there is a space \( X \) with \( |X| = \Delta(X) = \kappa \) which is not monotonically \( \omega_1 \)-resolvable.

\(\dagger\)In \cite{13} a “monotonically \( \omega \)-resolvable” space is called “almost-\( \omega \)-resolvable”. However, in \cite{12} a space \( X \) is almost-\( \kappa \)-resolvable if it contains a family of \( \kappa \) dense sets with pairwise nowhere dense intersections.
What about spaces of cardinality $\omega_\omega$?

**Theorem 1.5.** It is consistent, modulo the existence of a measurable cardinals, that there is a space $X$ with $|X| = \Delta(X) = \omega_\omega$ which is not monotonically $\omega_1$-resolvable.

Do we really need to add $|X|$-many Cohen reals to make $X$ resolvable?

**Theorem 1.6.** (1) It is consistent, modulo a measurable cardinal, that there is a crowded space $X$ with $|X| = \Delta(X) = \omega_1$ (so $X$ is monotonically $\omega_1$-resolvable) such that $V^{\text{Fn}(\omega, 2)} \models "X \text{ is irresolvable."}$

(2) If $V = L$, then every crowded space with $|X| = \Delta(X) = \text{cf}(|X|)$ is monotonically $\omega$-resolvable, and so it is $\omega$-resolvable in $V^{\text{Fn}(\omega, 2)}$.

(3) If the cardinality of a crowded c.c.c space $X$ is less than the first weakly inaccessible cardinal, then $X$ is $\omega$-resolvable in $V^{\text{Fn}(\omega_1, 2)}$.

The almost resolvability of c.c.c spaces was investigated by Pavlov in [11]: on page 53 Pavlov writes that – mimicked Malykhin’s method by using Ulam matrices – he showed that every crowed ccc space of cardinality $\omega_1$ is almost resolvable. In [3, Theorem 2.22] a stronger result was proved: a crowded c.c.c. space is almost resolvable, if its cardinality is less than the first weakly inaccessible cardinal. Theorem 1.6(2) is a further improvement of this result because monotonically $\omega$-resolvability implies almost resolvability.

In [1] 3.12 Problem (2) the authors ask if every space with countable cellularity and cardinality less than the first inaccessible non-countable cardinal almost-$\omega$-resolvable? As we will see Theorem 1.6(3) gives a positive answer to a weakening of this question.

### 2. Characterization of $\omega_1$-resolvability in c.c.c extensions.

Instead of Theorem 1.2 we prove the following stronger result.

**Theorem 2.1.** Assume that $X$ is a crowded topological space and $\kappa$ is an infinite cardinal. If $\kappa = \text{cf} \left( [\kappa]^\omega, \subset \right)$ then following statements are equivalent.

1. $X$ is $\kappa$-resolvable in some c.c.c-generic extension,
2. there is a function $h : X \rightarrow [\kappa]^\omega$ such that $\bigcup h''U = \kappa$ for each non-empty open $U \subset X$.
3. $X$ is $\kappa$-resolvable in the Cohen-generic extension $V^{\text{Fn}(\kappa, 2)}$.

\[\text{Note: } \omega_1 \text{ is not a misprint here}\]
We say that a function $g : X \to \kappa$ witnesses that $X$ is $\kappa$-resolvable if
\[ \{ x \in X : g(x) = \alpha \} \subseteq^{\text{dense}} X \]
for each $\alpha < \kappa$.

**Proof.** First we show that (1) $\rightarrow$ (2). Assume that $\mathbb{P}$ is a c.c.c. poset such that there is a function $g \in V^\mathbb{P}$ witnessing the $\kappa$-resolvability of $X$.

For each $x \in X$ define
\[ h(x) = \{ \alpha < \kappa : \exists p^x_\alpha \in \mathbb{P}(p^x_\alpha \Vdash \dot{g}(\dot{x}) = \dot{\alpha}) \}. \]
Since the conditions $\{ p^x_\alpha : \alpha \in h(x) \}$ are pairwise incomparable and $\mathbb{P}$ is c.c.c., the set $h(x)$ is countable.

We now show that the function $h$ defined above satisfies (2). Fix $\alpha < \kappa$ and $U$ an open subset of $X$. We need to show that there exists $x \in U$ such that $\alpha \in h(x)$. Since
\[ V^\mathbb{P} \models g^{-1}(\{ \alpha \}) \subseteq^{\text{dense}} X \]
it follows that there is $x \in U$ such that
\[ V^\mathbb{P} \models g(x) = \alpha. \]
Thus, there exists $p \in \mathbb{P}$ such that
\[ p \models \text{“} \dot{g}(\dot{x}) = \dot{\alpha}. \text{”} \]
Then $\alpha \in h(x)$.

Next we now show that (2) $\rightarrow$ (3). Let $\mathcal{A}$ be a cofinal subset of $[\kappa]^\omega$ with $|\mathcal{A}| = \kappa$.

Let $\{ A_\alpha : \alpha < \kappa \}$ be an enumeration of $\mathcal{A}$, and for each $x \in X$ pick
\[ h^*(x) \in \mathcal{A} \text{ such that } h^*(x) \supseteq \bigcup_{\alpha \in h(x)} A_\alpha. \]

Then for all non-empty open $U$
\[ \{ h^*(x) : x \in U \} \text{ is cofinal in } [\kappa]^\omega. \quad (+) \]

Next we note that forcing with $\text{Fn}(\kappa, 2)$ is the same as forcing with $\text{Fn}(\kappa, \omega)$. Further, $\text{Fn}(\kappa, \omega)$ is isomorphic to
\[ \mathbb{P} = \{ p \in \text{Fn}(\mathcal{A}, \kappa) : \forall A \in \text{dom}(p) \ p(A) \in A \}. \]
Indeed, for each $A \in \mathcal{A}$ fix a bijection $\rho_A : \omega \to A$, and then for $q \in \text{Fn}(\kappa, \omega)$ define $\varphi(q) \in \mathbb{P}$ as follows:
\begin{enumerate}
\item $\text{dom}(\varphi(q)) = \{ A_\alpha : \alpha \in \text{dom}(q) \}$, and
\item $\varphi(q)(A_\alpha) = \rho_{A_\alpha}(q(\alpha))$ for $A_\alpha \in \text{dom}(\varphi(q))$.
\end{enumerate}
Then $\varphi$ is clearly an isomorphism between $\text{Fn}(\kappa, \omega)$ and $\mathbb{P}$.

We will proceed using $\mathbb{P}$.

Let $G$ be a $\mathbb{P}$-generic filter, and let $g = \bigcup G$. Then $g \in V^\mathbb{P}$ and $g : \mathcal{A} \to \kappa$ such that $g(A) \in A$.

We claim that $f = g \circ h^*$ witnesses that $X$ is $\kappa$-resolvable.

Fix $\alpha < \kappa$ and an open $U \subset X$.

Let $q \in \mathbb{P}$ be arbitrary. Then, by $(\hyperlink{[+]}{[+]}), there is $x \in U$ such that

$$\{\alpha\} \cup \bigcup \text{dom}(q) \subsetneq h^*(x).$$

Then $h^*(x) \notin \text{dom}(q)$, and $\alpha \in h^*(x)$, so

$$p = q \cup \{\langle h^*(x), \alpha \rangle\} \in P_1,$$

and

$$p \Vdash (g \circ h^*)(\check{x}) = \check{\alpha}.$$ 

Thus, by genericity, there is $p \in G$ and $x \in U$ such that

$$p \Vdash (g \circ h^*)(\check{x}) = \check{\alpha}.$$ 

Hence

$$V^\mathbb{P} \models X \text{ is } \kappa\text{-resolvable}.$$ 

Finally $(3) \to (1)$ is trivial. $\square$

**Problem 2.2.** Can we drop the assumption $\kappa = \text{cf}(\leq \kappa)$ from Theorem 2.1?

3. ON MONOTONICALLY $\omega_1$-RESOLVABILITY OF C.C.C SPACES

We start with an easy observation.

**Lemma 3.1.** Let $X$ be a topological space and $B \subset \mathcal{P}(X)$. If every $B \in B$ is monotonically $\kappa$-resolvable, then so is $\bigcup B$. So every space contains a greatest monotonically $\kappa$-resolvable subspace (that subspace can be empty, of course).

**Corollary 3.2.** Let $X$ be a topological space. Let $Z$ be a dense subset of $X$. If $Z$ is monotonically $\kappa$-resolvable, then $X$ is also monotonically $\kappa$-resolvable.

Before proving Theorem 3.3 we prove the following “stepping-down” theorem.

**Theorem 3.3.** If $X$ is a $\kappa$-c.c., monotonically $\kappa^+$-resolvable space, then $X$ is monotonically $\kappa$-resolvable as well.

The proof uses ideas from $[8].$
Proof. Since an open subspace of a $\kappa$-c.c., monotonically $\kappa^+$-resolvable space is also $\kappa$-c.c. and monotonically $\kappa^+$-resolvable, by Lemma 3.1 it is enough to show that

$(\ast)$ every $\kappa$-c.c., monotonically $\kappa^+$-resolvable space $X$ has a monotonically $\kappa$-resolvable non-empty open subset.

Ulam [14] proved that there is a “matrix”

$$\langle M_{\alpha, \zeta} : \alpha < \kappa^+, \zeta < \kappa \rangle \subset \mathcal{P}(\kappa^+)$$

such that

(i) $M_{\alpha, \xi} \cap M_{\beta, \xi} = \emptyset$ for $\{\alpha, \beta\} \in [\kappa^+]^2$ and $\xi \in \kappa$,

(ii) $M_{\alpha, \xi} \cap M_{\alpha, \zeta} = \emptyset$ for $\alpha \in \kappa^+$ and $\{\xi, \zeta\} \in [\kappa]^2$,

(iii) and $|M_{\alpha^\leftarrow}| \leq \kappa$, where $M_{\alpha^\leftarrow} = \kappa^+ \setminus \bigcup_{\zeta < \kappa} M_{\alpha, \zeta}$ for $\alpha < \kappa^+$.

Fix a partition $\{Y_\eta : \eta < \kappa^+\}$ witnessing that $X$ is monotonically $\kappa^+$-resolvable.

Let

$$Z_{\alpha, \zeta} = \bigcup \{Y_\eta : \eta \in M_{\alpha, \zeta}\}$$

for $\alpha < \kappa^+$ and $\zeta < \kappa$, and let

$$Z_\alpha = \bigcup_{\zeta < \kappa} Z_{\alpha, \zeta}.$$ 

Since $Z_\alpha = \bigcup\{Y_\eta : \eta \in \kappa^+ \setminus M_{\alpha^\leftarrow}\}$, assumption (iii) implies that every $Z_\alpha$ is dense in $X$.

Case 1. There is $\alpha < \kappa^+$ such that for all $\zeta < \kappa$

$$\bigcup_{\zeta \leq \xi} Z_{\alpha, \xi} \subset \text{dense} \ Z_\alpha.$$ 

Then $(Z_{\alpha, \zeta})_{\zeta < \kappa}$ witnesses $Z_\alpha$ is monotonically $\kappa$-resolvable and so by corollary 3.2, $X$ is also monotonically $\kappa$-resolvable.

Case 2. For all $\alpha < \kappa^+$ there is $\zeta_\alpha < \kappa$ and there is an non-empty open set $U_\alpha \in \tau_X$ such that

$$\bigcup_{\zeta \leq \xi} Z_{\alpha, \xi} \cap U_\alpha = \emptyset. \quad (\dagger)$$

Then there is a set $I \in [\kappa^+]^{\kappa^+}$ and there is an ordinal $\zeta < \kappa$ such that $\zeta_\alpha = \zeta$ for all $\alpha \in I$.

Fix an arbitrary $K \in [I]^{\kappa}$. By (iii) we can find $\rho < \kappa^+$ such that

$$\bigcup_{\alpha \in K} M_{\alpha^\leftarrow} \subset \rho.$$ 

Let $Z = \bigcup_{\rho < \eta} Y_\eta$. Then $Z \subset \text{dense} \ X$ and $Z \subset Z_\alpha$ for all $\alpha \in K$. 
Claim. If $L \in [K]^\kappa$ then
\[ \bigcap_{\alpha \in L} U_\alpha \cap Z = \emptyset. \]

Proof of the Claim. Assume on the contrary that $z \in \bigcap_{\alpha \in L} U_\alpha \cap Z$. Then $z \in Y_\eta$ for some $\rho < \eta$.

Let $\alpha \in L$. Then $\eta \in \kappa^+ \setminus \rho \subset \bigcup_{\xi < \kappa} M_{\alpha, \xi}$. Pick $\xi_\alpha < \kappa$ with $\eta \in M_{\alpha, \xi_\alpha}$. Then $Y_\eta \subset Z_{\alpha, \xi_\alpha}$, so $Z_{\alpha, \xi_\alpha} \cap U_\alpha \neq \emptyset$, so $\xi_\alpha < \zeta_\alpha = \zeta$ by (†).

Since $\zeta < \kappa = |L|$, there are $\alpha \neq \beta \in [L]^2$ such that $\xi_\alpha = \xi_\beta$. Thus $\eta \in M_{\alpha, \xi_\alpha} \cap M_{\beta, \xi_\beta}$ which contradicts (i) because $\xi_\alpha = \xi_\beta$. \[ \square \]

Fix an enumeration $K = \{ \chi_\xi : \xi < \kappa \}$, and let $V_\zeta = \bigcup_{\xi < \zeta} U_{\chi_\xi}$. Then the sequence $\{ V_\zeta : \zeta < \kappa \}$ is decreasing and
\[ \bigcap_{\zeta < \kappa} V_\zeta \cap Z = \emptyset \]
by the Claim.

Since $X$ is $\kappa$-c.c. there is $\xi < \kappa$ such that $\overline{V_\xi} = V_\xi$ for all $\xi < \zeta < \kappa$.

We can assume that $\xi = 0$. Let
\[ T_\zeta = \begin{cases} V_0 \setminus Z & \text{if } \zeta = 0, \\ ((\bigcap_{\xi < \zeta} V_\xi) \setminus V_\zeta) \cap Z & \text{if } \zeta > 0. \end{cases} \]

Then
\[ \bigcup_{\xi < \zeta} T_\zeta \supset V_\xi \cap Z \subset^\text{dense} V. \]

Thus the partition $\{ T_\zeta : \zeta < \kappa \}$ witnesses that $V$ is monotonically $\kappa$-resolvable. \[ \square \]

Proof of Theorem 1.3. Let $\mathcal{Y} = \{ Y \in \tau_X : |Y| = \Delta(Y) \}$.

Then $\bigcup \mathcal{Y}$ is dense in $X$, and every open subset of every $Y \in \mathcal{Y}$ is also in $\mathcal{Y}$. Thus by lemma 3.1 it is enough to prove that a c.c.c. space $Y$ with $\omega_1 \leq |Y| = \Delta(Y) < \omega$ is monotonically $\omega_1$-resolvable.

Let $Y \in \mathcal{Y}$ such that $\omega_1 = |Y|$. Clearly, $Y$ is monotonically $\omega_n$-resolvable as $|Y| = \Delta(Y) = \omega_n$. Since $Y$ is c.c.c. then $Y$ is $\omega_{n-1}$-c.c.. By theorem 3.3 $Y$ is monotonically $\omega_{n-1}$-resolvable. By continually applying theorem 3.3 we conclude that $Y$ is monotonically $\omega_1$-resolvable. \[ \square \]

Problem 3.4. Is it true that every crowded c.c.c space with $\Delta(X) \geq \omega_1$ is monotonically $\omega_1$-resolvable?
4. Spaces which are not monotonically \(\omega_1\)-resolvable.

If \(X\) is a topological space, and \(\mathcal{D} \subset \mathcal{P}(D)\), write

\[
\overline{\mathcal{D}} = \{ \overline{D} : D \in \mathcal{D} \}
\]

**Lemma 4.1.** Let \(X\) be a topological space. Assume that \(\overline{\mathcal{D}}\) is point-countable for each point-countable family \(\mathcal{D} \subset \mathcal{P}(X)\). Then \(X\) is not contain any monotonically \(\omega_1\)-resolvable subspace \(Y\).

**Proof.** Assume that \(\{Y_\zeta : \zeta < \omega_1\}\) is a partition of \(Y\). Let \(D_\xi = \bigcup\{Y_\zeta : \xi < \zeta\}\) for \(\xi < \omega_1\). Then the family \(\mathcal{D} = \{D_\xi : \xi < \omega_1\}\) is point-countable. So \(\overline{D}\) is also point-countable. So \(D_\xi\) is not dense in \(Y\) for all but countably many \(\xi\). So the partition \(\{Y_\zeta : \zeta < \omega_1\}\) does not witness that \(Y\) is monotonically \(\omega_1\)-resolvable. \(\square\)

To prove Theorems 1.4 and 1.5 we should recall some definitions and results from [6] and [5].

**Definition 4.2 ([6, Definition 3.1])**. Let \(\kappa\) be an infinite cardinal, and let \(\mathcal{F}\) be a filter on \(\kappa\). Let \(T\) be the tree \(\kappa < \omega\). A topology \(\tau_\mathcal{F}\) is defined on \(T\) by

\[
\tau_\mathcal{F} = \{ V \subset T : \forall t \in V \{ \alpha \in \kappa : t^{-} \alpha \in V \} \in \mathcal{F} \},
\]

and the space \(\langle T, \tau_\mathcal{F} \rangle\) is denoted by \(X(\mathcal{F})\).

**Proof of Theorem 1.4.** Let \(\mathcal{U}\) be a \(\kappa\)-complete non-principal ultrafilter on \(\kappa\).

The space \(X = X(\mathcal{U})\) is monotonically normal by [6, Theorem 3.1].

An ultrafilter \(\mathcal{U}\) is \(\lambda\)-descendingly complete if \(\bigcap\{U_\zeta : \zeta < \lambda\} \neq \emptyset\) for each decreasing sequence \(\{U_\zeta : \zeta < \lambda\} \subset \mathcal{U}\).

A \(\sigma\)-complete ultrafilter is clearly \(\omega\)-descendingly-complete. In the proof of [6, Theorem 3.5] the authors prove Lemma 3.6 which claims that \(\overline{\mathcal{D}}\) is point-countable for each point-countable family \(\mathcal{D} \subset \mathcal{P}(X(\mathcal{F}))\) provided that \(\mathcal{F}\) is a \(\omega\)-descendingly complete ultrafilter. So \(\overline{\mathcal{D}}\) is point-countable for each point-countable family \(\mathcal{D} \subset \mathcal{P}(X)\), and so \(X\) is not monotonically \(\omega_1\)-resolvable by Lemma 4.1. \(\square\)

Instead of Theorem 1.5 we prove the following theorem which is a slight improvement of [5, Theorem 5].

**Theorem 4.3.** If it is consistent that there is a measurable cardinal, then it is also consistent that there is an \(\omega\)-resolvable monotonically normal space \(X\) with \(|X| = \Delta(X) = \omega_\omega\) such that if a family \(\mathcal{D} \subset \mathcal{P}(X)\) is point-countable, then the family \(\overline{\mathcal{D}} = \{ \overline{D} : D \in \mathcal{D} \}\) is also point.
countable. Hence $X$ does not contain any monotonically $\omega_1$-resolvable subspace.

Proof. In [5, page 665] the authors write that "starting from one measurable, Woodin ([15]) constructed a model in which $\mathcal{K}_\omega$ carries an $\omega_1$-descendingly complete uniform ultrafilter. Woodin’s model $V_1$ can be embedded into a bigger ZFC model $V_2$ so that the pair of models $(V_1, V_2)$ with $\kappa = \mathcal{K}_\omega$ satisfies the two models situation”, i.e.

1. $\omega_1^{V_1} = \omega_1^{V_2}$,
2. there is a countable subset $A$ of $\omega_\omega$ in $V_2$ such that no $B \in V_1$ of cardinality $< \omega_\omega$ covers $A$;
3. for the filter $\mathcal{G}$ on $\omega_\omega$ defined in $V_2$ by $B \in \mathcal{G}$ iff $A - B$ is finite, we have $\mathcal{G} \cap V_1 \in V_1$.

(the “two model situation” is defined in [5, Theorem 4.5]).

Let $\mathcal{F} = \mathcal{G} \cap V_1$ and consider the space $X = X(\mathcal{F})$. As it was observed in [6], spaces obtained as $X(\mathcal{H})$ from some filter $\mathcal{H}$ are monotonically normal and $\omega$-resolvable.

In [5, Theorem 4.1] Juhász and Magidor showed that the space $X(\mathcal{F})$ is actually hereditarily $\omega_1$-irresolvable. They proved the following lemma:

**Lemma 4.2 from [5].** For any $D \subset X(\mathcal{F})$ and $t \in \overline{D}$ there is a finite sequence $s$ of members of $A$ such that $t \smallsetminus s \in D$.

Using this lemma we show that $\overline{D}$ is point-countable for each point-countable family $D \subset \mathcal{P}(X)$, and so $X$ is not monotonically $\omega_1$-resolvable by Lemma 4.1.

Indeed, let $\mathcal{D} \subset \mathcal{P}(X)$ be an uncountable family such that $t \in \bigcap_{D \in \mathcal{D}} \overline{D}$. Then, by [5, Lemma 4.3], for each $D \in \mathcal{D}$ we can pick a finite sequence $s_D$ of members of $A$ such that $t \smallsetminus s_D \in D$. Since there are only countable many finite sequences of elements of $A$ there is $s$ such that $s_D = s$ for uncountably many $D \in \mathcal{D}$. Then $t \smallsetminus s$ is in uncountably many elements of $\mathcal{D}$, so $\mathcal{D}$ is not point-countable.

So we proved that no subspace of $X$ is monotonically $\omega_1$-resolvable. \hfill \Box

5. $\omega$-resolvability after adding a single Cohen reals

Before proving Theorem 1.6 we need some preparation.

The notion of almost resolvability was introduced by Bolstein ([2]) in 1973: a topological space is *almost-resolvable* if it is a countable union of sets with empty interiors. The notion of monotonically $\omega$-resolvability was first considered in [13] under the name almost-$\omega$-resolvability.
Clearly almost $\omega$-resolvable (i.e. monotonically $\omega$-resolvable) spaces are almost resolvable.

**Lemma 5.1.** Let $X$ be a crowded topological space.

(1) If $X$ is monotonically $\omega$-resolvable, then $X$ is $\omega$-resolvable in $V^{Fn(\omega, 2)}$.

(2) If $X$ is resolvable in $V^{Fn(\omega, 2)}$, then $X$ is almost-resolvable.

**Proof of Lemma 5.1.** (1) Assume that the function $f: X \to \omega$ witnesses the monotonically $\omega$-resolvability of $X$.

If $\mathcal{G}$ is the $V$-generic filter in $Fn(\omega, \omega)$, and $g = \bigcup \mathcal{G}$, then the function $h = g \circ f$ witnesses that $X$ is $\omega$-resolvable.

We need to show that \{ $y \in X: (g \circ f)(y) = n$ \} is dense in $X$.

Indeed, let $p \in Fn(\omega, \omega)$, $\emptyset \neq U \in \tau_X$. Since $f: X \to \omega$ witnesses the monotonically $\omega$-resolvability of $X$ there is $y \in U$ such that $f(y) > \max \text{dom}(p)$.

Let $q = p \cup \{(f(y), n)\}$.

Then $q \leq p$ and $g \Vdash (g \circ f)(y) = n$.

So we proved that $g \circ f$ witnesses that $X$ is $\omega$-resolvable in the generic extension.

(2) Assume $V^{Fn(\omega, 2)} \models \text{"}X \text{ has a partition } \{D_0, D_1\} \text{ into dense subsets."}$

For all $p \in Fn(\omega, 2)$ and $i < 2$ let

\[ D_i^p = \{ x \in X: p \Vdash x \in \dot{D}_i \}. \]

Then $X = \bigcup \{ D_i^p : p \in Fn(\omega, 2), i < 2 \}$, and we claim that $\text{int } D_i^p = \emptyset$ for each $p \in Fn(\omega, 2)$, and $i < 2$.

Indeed, fix $p$ and $i$ and let $U$ be an arbitrary non-empty open subset. Then $p \not\Vdash U \cap D_{1-i} \neq \emptyset$, so there is $q \leq p$ and $y \in U$ such that $q \Vdash y \in D_{1-i}$. Then $q \Vdash y \notin \dot{D}_i$, so $p \not\Vdash y \in \dot{D}_i$, and so $y \notin D_i^p$. Thus $U \nsubseteq D_i^p$. Since $U$ was arbitrary, we proved that $\text{int } D_i^p = \emptyset$. \hfill \Box

After this preparation we can prove Theorem 1.6.

**Proof of Theorem 1.6.** (1) Kunen [7] proved that it is consistent, modulo a measurable cardinal, that there is a maximal independent family $\mathcal{A} \subset \mathcal{P}(\omega_1)$ which is also $\sigma$-independent.

In [9, Theorems 3.1 and 3.2] the authors proved that if there is a maximal independent family $\mathcal{A} \subset \mathcal{P}(\omega_1)$ which is also $\sigma$-independent, then there is a Baire space $X$ with $|X| = \Delta(X) = \omega_1$ such that every open subspace of $X$ is irresolvable, i.e. the space $X$ is $OHI$.
It is well-known that a crowded OHI Baire space \( X \) is not almost resolvable: if \( X = \bigcup_{n \in \omega} X_n \), then \( \text{int} X_n \neq \emptyset \) for some \( n \in \omega \).

Indeed, if \( \text{int} X_n = \emptyset \), then \( X \setminus X_n \) is dense, so \( U_n = \text{int}(X \setminus X_n) \) is dense in \( X \) because every open subset of \( X \) is irresolvable. Thus \( \bigcap_{n \in \omega} U_n \neq \emptyset \) because \( X \) is Baire. However

\[
\bigcap_{n \in \omega} U_n \subset \bigcap_{n \in \omega} (X \setminus X_n) = X \setminus \bigcup_{n \in \omega} X_n = \emptyset,
\]

which is a contradiction.

Thus \( X \) is not almost resolvable, so it is not \( \omega \)-resolvable in the model \( V^\text{Fn}(\omega, 2) \) by Lemma 5.1(2).

(2) In [10] the authors proved that if \( V = L \), then there are no crowded Baire irresolvable spaces. Hence, by [13], if \( V = L \), then every crowded space \( X \) is almost-\( \omega \)-resolvable (i.e. monotonically \( \omega \)-resolvable).

So these spaces are \( \omega \)-resolvable in the model \( V^\text{Fn}(\omega, 2) \) by Lemma 5.1(1).

Proof of Theorem 1.6(3). Let \( X \) be a crowded c.c.c space.

We can assume that \( |X| = |\Delta(X)| \).

By induction we define a strictly decreasing sequence of cardinals:

\[
\kappa_0, \kappa_1, \ldots, \kappa_n \ldots
\]
as follows.

(i) \( \kappa_0 = \Delta(X) \),

(ii) if \( \kappa_i \) is singular, then \( \kappa_{i+1} = \text{cf}(\kappa_i) \),

(iii) if \( \kappa_i > \omega \) is regular, then \( \kappa_i = \lambda^+ \) (because \( |X| \) is below the first weakly inaccessible cardinal,) and let \( \kappa_{i+1} = \lambda \),

(iv) if \( \kappa_i = \omega \) or \( \kappa_i = \omega_1 \), then we stop.

Assume that the construction stopped in the \( n \)th step.

Then we can prove, by finite induction, then \( X \) is monotonically \( \kappa_i \)-resolvable for all \( i \leq n \) by theorem 3.3 Thus \( X \) is monotonically \( \omega \)-resolvable or monotonically \( \omega_1 \)-resolvable, and so either \( X \) is \( \omega \)-resolvable in \( V^\text{Fn}(\omega, 2) \) by by Lemma 5.1(1), or \( X \) is \( \omega_1 \)-resolvable in \( V^\text{Fn}(\omega_1, 2) \) by Thereon 2.1.

Problem 5.2 ([13 Questions 5.2.]). Are almost resolvability and almost-\( \omega \)-resolvability equivalent in the class of irresolvable spaces?

Problem 5.3. Is there, in ZFC, a crowded topological space \( X \) which is irresolvable in the Cohen generic extension \( V^\text{Fn}(\omega, 2) \).
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