Z₃ Symmetry and Neutrino Mixing in Type II Seesaw

Bo Hu¹,², Feng Wu² and Yue-Liang Wu²

¹ Department of Physics, Nanchang University, 330031, China and
² Kavli Institute for Theoretical Physics China (KITPC),
Institute of Theoretical Physics, Chinese Academy of Sciences,
P.O. Box 2735, Beijing 100080, China

Abstract

Neutrino mixing matrix satisfying the current experimental data can be well described by the
HPS tri-bimaximal mixing matrix. We propose that its origin can be understood within the seesaw
framework by a hidden condition on the mass matrix of heavy right-handed neutrinos under the
transformation of the Abelian finite group Z₃ on the flavor basis. Ignoring CP phases, we show
that it can lead to the generic form of the effective light neutrino mass matrix from which the HPS
mixing matrix appears naturally, as well as an experimentally allowed non-zero sinθ₁₃. We show
that the model based on our proposal is in good agreement with the current experimental data.

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*Electronic address: bohu@ncu.edu.cn
†Electronic address: fengwu@itp.ac.cn
‡Electronic address: ylwu@itp.ac.cn
In the standard model (SM), the neutrinos are massless. The results of neutrino oscillation experiments indicate that neutrinos are massive. Due to the smallness of neutrino masses, the mechanism in the SM which gives quark and charged lepton masses is unnatural for neutrinos. Therefore, the observed neutrino oscillations are considered to be the first convincing evidences of new physics beyond the SM and have been discussed extensively in literatures (for recent reviews, see [1, 2]).

Besides neutrino masses, the global fit of current experimental data shows that, unlike the mixing angles in the quark sector, two of the three mixing angles are large and one of them might be maximal. As a matter of fact, $30^\circ < \theta_{\text{sol}} < 38^\circ$, $36^\circ < \theta_{\text{atm}} < 54^\circ$, and $\theta_{\text{CHOOZ}} < 10^\circ$ at the 99% confidence level [3]. To understand this peculiar property is also an interesting theoretical issue. In fact, these mixing angles can be well described by the HPS mixing matrix [4] where $\sin^2 \theta_{\text{sol}} = \frac{1}{3}$, $\sin^2 \theta_{\text{atm}} = \frac{1}{2}$, and $\theta_{\text{CHOOZ}} = 0$. The HPS mixing matrix can be considered as the lowest order approximation. Efforts on revealing its origin may help understanding not only the neutrino physics, but also the physics beyond the SM, such as the new symmetries at high energy scales.

With the assumption of the existence of right-handed neutrinos, seesaw mechanism provides a simple way to understand the smallness of neutrino masses and has long been considered as the leading candidate of neutrino mass generating mechanism. However, by its own seesaw mechanism cannot explain the observed neutrino mixing pattern.

In many models with right-handed neutrinos, e.g. SO(10) or SU(2)$_L \times$ SU(2)$_R \times U(1)_{R-L}$ based models, effective light neutrino mass matrix is given by Type II seesaw relation [5]

$$M_\nu = M_L - M_\nu^D M_R^{-1} (M_\nu^D)^T.$$  \hspace{1cm} (1)

where $M_L$, $M_R$ are the majorana mass matrices for left-handed and right-handed neutrinos and $M_\nu^D$ the Dirac mass matrix. Given that the neutrino mass is generated by type II seesaw, as shown in (1), the observed neutrino mixing can provide important information about the structure of $M_\nu$ and thus the physics behind $M_L$, $M_R$ and $M_\nu^D$.

In this Letter we make the natural assumptions that there are three Majorana neutrinos and consider the case where the flavour symmetry is only broken in $M_R$ sector. Currently none of the CP violating phases has been observed. In the following discussion, we assume vanishing CP phases and focus on the mixing pattern. The case with non-vanishing CP phases will be discussed elsewhere.
It is natural to expect that symmetries can lead to specific neutrino mass matrix. This idea has been pursued in many works. In particular, discrete symmetries including $S_3$ (e.g. [6]), $S_4$ (e.g. [7]), $A_4$ (e.g. [8]) etc., have been discussed extensively in literatures (for recent review, please see [1] and references therein). Also appropriate flavour symmetries can also lead to desired neutrino mixing. Without the CP violating phases, there are six free parameters in $M_R$ in the second term of (1), the effective neutrino mass matrix in Type I see-saw. In this letter, we propose a hidden condition on $M_R$ under the transformation of the Abelian finite group $Z_3$ on the flavor basis, which will reduce the independent parameters down to three. We then use the resultant mass matrix to explain the observed mixing pattern.

First consider a finite group $G$. Each element $U_i$ of $G$ satisfies $U_i^{n_i} = 1$ for some non-zero integer $n_i$. Under an unitary transformation of $G$ on the flavor basis $\nu_f = (\nu_e, \nu_\mu, \nu_\tau)$, we propose that for each $U_i$ belongs to $G$, the mass matrix $M_R$ in the new basis satisfies

$$U_i M_R U_i^T = U_i' M_R.$$

(2)

We show below that $U_i'$ is strongly constrained and any choice of $U_i'$ satisfying the constraint will further restrict the possible form of $M_R$. In particular, we show that if the finite group $G$ is chosen to be $Z_3$, $U_i' = U_i^2$ will lead to a phenomenologically interesting $M_\nu$ and thus provides a possible origin of the observed neutrino mixing angles.

To see that $U_i'$ cannot be arbitrary, consider that

$$M_R = (U_i)^{n_i} M_R (U_i^T)^{n_i} = (U_i)^{n_i-1} U_i' M_R (U_i^T)^{n_i-1}$$

$$= (U_i)^{n_i-1} U_i' (U_i^T U_i)^{n_i-2} = \ldots$$

$$= (U_i)^{n_i-1} U_i' (U_i^T U_i)^{n_i-1} M_R = (U_i^T U_i)^{n_i} M_R.$$

(3)

From (3) we find that (2) requires $(U_i^T U_i)^{n_i} = 1$. Consequently, we obtain $U_i' = e^{i2\pi m/n_i} U_i^k$ with $m$ some integer and $k = 0, 1, \ldots, n_i - 1$. Note that $m = 0$ when $U_i$ and $M_R$ are real. Moreover, $k$ can be different for different group element $U_i$. Based on simplicity, we assume $k$ is universal for all group elements. It is obvious that $(U_i^T U_i)^{n_i} = 1$ is only a necessary condition for (2) to be held. Given $U_i'$, (2) will restrict the form of $M_R$. In general, different choice of $k$ will lead to different $M_R$. We show this below in the case where the finite group $G$ is the cyclic group $Z_3$. 

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The group $Z_3$ contains only 3 elements, thus $n_i \leq 3$. Therefore, the only possible choices for $U_i'$ are $U_i' = I$, or $U_i' = U_i$, or $U_i' = U_i^2$. The first choice, demanding $M_R$ to be invariant under $Z_3$ on the flavour basis, leads to an unrealistic mass matrix with $\nu_e - \nu_\mu - \nu_\tau$ symmetry. Another choice that one might think interesting is the case where $U_i' = U_i$. One of the necessary conditions in this case requires $M_R$ to be non-invertible so that at least one of the mass eigenvalues is zero. For the cyclic group $Z_3$, the symmetric mass matrix $M_R$ turns out to be democratic in this case and there is only one non-zero eigenvalue. We won’t pursue these in this Letter.

In the following we focus on the case $U_i' = U_i^2$. $M_R$ built in this way will give interesting phenomenology. In fact, the resultant $M_R$ can be expressed as linear combinations of elements in one of the two cosets of $Z_3$ in the non-Abelian symmetric group $S_3$. Our bottom-up approach ends up with the proposal that under some finite group $G$, $M_R U_i^T = U_i M_R$, $\forall U_i \in G$.

To be explicit, consider the following three dimensional unitary representation of $S_3 = \{I_i|i = 1 \sim 6\}$:

$$
I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
I_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
I_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
I_6 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
$$

The four non-trivial subgroups $\{I_1, I_2\}$, $\{I_1, I_3\}$, $\{I_1, I_4\}$, and $Z_3 = \{I_1, I_5, I_6\}$ are all Abelian. Different from the other three subgroups, the cyclic group $Z_3$ is the only non-trivial invariant subgroup of $S_3$. $\{I_1, I_5, I_6\}$ form a regular representation of $Z_3$. It is straightforward to solve that the mass matrix $M_R$ which satisfies (2) with $U_i' = U_i^2$ has the following form

$$
M_R = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = aI_2 + bI_3 + cI_4.
$$

Note that $\{I_2, I_3, I_4\}$ is a coset of $Z_3$ in $S_3$.

Before proceeding to the discussion of seesaw mechanism, we would like to point out
another interesting feature of (2). In fact, before considering the constraints from symmetry, if one uses a novel mechanism to generate the most general mass matrix which does not necessarily to be symmetric, the non-trivial fact is that, with our proposal ($U'_i = U_i^2$), the mass matrix will still be in the form of (4) under $Z_3$ group. This, however, is not true for the case $U'_i = I$ or $U'_i = U_i$. Starting with the most general $M$ with nine parameters, for the case $U'_i = I$, one gets

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = aI_1 + bI_5 + cI_6,$$

while for the case that $U'_i = U_i$ one gets

$$M = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}.$$

But in the symmetric case, these two matrices will become a $\nu_e - \nu_\mu - \nu_\tau$ symmetric one and a democratic one, respectively, as discussed above. That is, the requirement for $M$ to be symmetric will further reduce the number of free parameters in $M$. On the other hand, unlike the above two cases, without any assumption on $M_R$, starting from our simple proposal $M_R U'_i^T = U_i M_R \forall U_i \in Z_3$ and the most general $M_R$ with nine parameters, one still arrives at the unique form of $M_R$ as given by (4).

Assuming that $M_L = m_0 I_1$ and $M^D = m_d I_1$ which are invariant trivially under $Z_3$, from (1) it can be shown that the effective neutrino mass can be written as

$$M_\nu = mI_1 + m_d^2 \begin{pmatrix} B + C & -B & -C \\ -B & A + B & -A \\ -C & -A & C + A \end{pmatrix},$$  \hspace{1cm} (5)$$

where $m = m_0 - m_d^2(A + B + C)$, and

$$A = \frac{a^2 - bc}{R}, \quad B = \frac{b^2 - ac}{R}, \quad C = \frac{c^2 - ab}{R}$$  \hspace{1cm} (6)$$

with $R = a^3 + b^3 + c^3 - 3abc$.

This particular form of mass matrix can be diagonalized by the tri-bimaximal mixing followed by a pure 1-3 rotation. It is worth to mention that any real symmetric mass matrix
which is diagonalized by the tri-bimaximal mixing followed by a pure 1-3 rotation can always be written in the form of (5). Therefore what we derive here is a novel way to understand the phenomenological Majorana neutrino mass matrix with vanishing CP phases that one can construct from the current neutrino data.

Note that the form of $M_{\nu}$ in (5) is coincident with the one in Friedberg-Lee (FL) model \[9\] in which a new symmetry, i.e. the invariance of the neutrino mass terms under the transformation

$$\nu_e \rightarrow \nu_e + z, \quad \nu_\mu \rightarrow \nu_\mu + z, \quad \nu_\tau \rightarrow \nu_\tau + z,$$

is proposed to explain the observed neutrino mixings. Although more works are necessary in order to understand the origin of this symmetry and its breaking mechanism leading to the first term in the right-hand side of (3), Friedberg and Lee’s work provides an illuminating example showing neutrino physics is a great arena for exploring new physics, which is also what we pursue here. Although sharing the same motivation to explain neutrino data, ideas presented in this letter and the physics discussed here are very different. For example, what Friedberg and Lee discussed are Dirac neutrinos, but here we consider Majorana neutrinos. Moreover, based on $Z_3$ symmetry and the seesaw mechanism, we provide a simple but new way which can lead to not only the desired neutrino mass matrix, but also the small neutrino masses.

Before proceed, let’s discuss another way to implement $Z_3$ symmetry. Consider the $Z_3$ transformation which is realized in the following way

$$\nu_{1R} \rightarrow \nu_{1R}, \quad \nu_{2R} \rightarrow e^{i4\pi/3}\nu_{2R}, \quad \nu_{3R} \rightarrow e^{i2\pi/3}\nu_{3R}$$

and

$$\phi_1 \rightarrow e^{i4\pi/3}\phi_1, \quad \phi_2 \rightarrow \phi_2, \quad \phi_3 \rightarrow e^{i2\pi/3}\phi_3$$

where $\phi_i$ are gauge singlet scalar fields. The invariant majorana mass terms are

$$\begin{pmatrix} \phi_2 & \phi_3 & \phi_1 \\ \phi_3 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}.$$

The VEVs of $\phi_i$ will lead to a mass matrix as the one given in (4). This is equivalent to constructing the following mass term

$$\begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix} \begin{pmatrix} \phi_{3j} \nu_{jR} \end{pmatrix}.$$
where $\phi_{ij} = \phi_{(i+j) \mod 3}$.

We show above that the desired mass matrix can be obtained via $Z_3$ symmetry. Although more works are necessary to build a complete model and in particular, appropriate assignment of the charges of gauge symmetries are needed, here we concentrate on possible consequences of $Z_3$ in the neutrino sector and assume that other symmetries will not spoil our discussion. For example, we require any $U(1)$ symmetry or other symmetries, if exist, will not forbid the required mass terms under discussion.

From (6), we have $A + B + C = 1/(a + b + c)$, and

$$a = \frac{A^2 - BC}{R'}, \quad b = \frac{B^2 - AC}{R'}, \quad c = \frac{C^2 - AB}{R'},$$

where $R' = A^3 + B^3 + C^3 - 3ABC$. Now from any set of $A$, $B$ and $C$ which satisfies experimental data, the corresponding $a$, $b$ and $c$ can be found by (7). For heavy right-handed neutrinos, $m \simeq m_0$.

Under tri-bimaximal rotation, we have

$$(U_0)^T M_\nu U_0$$

$$= m I_1 + m_d^2 \begin{pmatrix}
\frac{3(B+C)}{2} & 0 & \frac{\sqrt{3}}{2} (-B + C) \\
0 & 0 & 0 \\
\frac{\sqrt{3}}{2} (-B + C) & 0 & \frac{1}{2} (4A + B + C)
\end{pmatrix}$$

where

$$U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix}
2 & \sqrt{2} & 0 \\
-1 & \sqrt{2} & \sqrt{3} \\
-1 & \sqrt{2} & -\sqrt{3}
\end{pmatrix}$$

is the tri-bimaximal mixing matrix.

The current experimental bound on the matrix element $U_{13}$ given by $\sin \theta_{13}$ in the standard parameterization is $\sin^2 \theta_{13} \leq 0.040$ at 3$\sigma$ C.L. (please see the latest arXiv version of [11]). This can be satisfied if

$$\left( \frac{B - C}{A - C} \right)^2 = \left( \frac{b - c}{a - c} \right)^2 \ll 1.$$ 

Without loss of generality, assume $A > C$. The neutrino masses are found to be

$$m_1 \simeq m + \frac{3}{2} m_d^2 (B + C),$$

$$m_2 = m,$$

$$m_3 \simeq m + 2m_d^2 A + \frac{1}{2} m_d^2 (B + C)$$
One can examine that appropriately chosen $a$, $b$ and $c$ can satisfy the current experimental data. As an example,

\[ m = 0.01 \text{ eV}, \quad m_d = 100 \text{ GeV}, \]
\[ a = 4.7 \times 10^{14} \text{ GeV}, \quad b = 5.7 \times 10^{13} \text{ GeV}, \]
\[ c = 3.0 \times 10^{13} \text{ GeV} \]

will lead to

\[ |U_{13}| = 0.022, \]

and

\[ \Delta m_{21}^2 = m_2^2 - m_1^2 = 7.9 \times 10^{-5} \text{ eV}^2, \]
\[ \Delta m_{31}^2 = m_3^2 - m_1^2 = 2.6 \times 10^{-3} \text{ eV}^2, \]

which are in good agreement with the current neutrino experimental data, i.e.

\[ 7.1 \times 10^{-5} \text{ eV}^2 < \Delta m_{21}^2 < 8.9 \times 10^{-5} \text{ eV}^2 \]
\[ 2.0 \times 10^{-3} \text{ eV}^2 < \Delta m_{31}^2 < 3.2 \times 10^{-3} \text{ eV}^2 \]

at 3\sigma C.L. \[11\].

In addition, one can show that this model can account for the case of nearly degenerate neutrinos. As an example,

\[ m = 0.25 \text{ eV}, \quad m_d = 15 \text{ GeV}, \]
\[ a = 8.93 \times 10^{13} \text{ GeV}, \quad b = 2.92 \times 10^{12} \text{ GeV}, \]
\[ c = 9.01 \times 10^{11} \text{ GeV} \]

will lead to the same squared mass differences as given above and $|U_{13}| = 0.008$.

In conclusion, we show that $Z_3$ symmetry can lead to observed neutrino mixing. We find that if one requires $M_R U_i^T = U_i M_R$, $\forall U_i \in Z_3$, $M_R$ must be in a cyclic permuted form, as shown in \[4\]. This will lead to tri-bimaximal mixing followed by an additional 1-3 rotation. Another way is based on the invariance of the mass terms under $Z_3$ transformations, similar to the usual $Z_2$ R-parity transformations. In the seesaw framework, this will lead to a possible explanation to both the smallness of neutrino masses and the origin of the neutrino mixing. It can be easily shown that $\theta_{13} = 0$ requires $b = c$ in \[4\], i.e., the $\nu_\mu - \nu_\tau$ symmetry. Therefore, from naturalness principle, the smallness of $\theta_{13}$ is presumably protected by the
symmetry. However, what we ignored here is the $\nu_\mu - \nu_\tau$ symmetry breaking mechanism leading to the smallness of $\sin \theta_{13}$, and other possible phenomena including lepton flavor violations (LFV), which is worth further studies in the future.

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