Persymmetric subspace adaptive detection and performance analysis

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Abstract: This study deals with the problem of detecting a subspace signal in coloured Gaussian noise, where the subspace signal belongs to a known subspace, but with unknown coordinates. The authors exploit the persymmetric structure of the covariance matrix by a unitary transform and then devise a persymmetric subspace detector based on two-step design procedure. By exploiting the persymmetric structure of the covariance matrix, the proposed detector can reduce training data requirements. Additionally, approximate expressions for the probabilities of false alarm and detection of the proposed detector are derived. Numerical results demonstrate that the proposed detector can offer significantly enhanced detection performance in comparison with the conventional counterparts when the amount of training data is limited.

1 Introduction

Adaptive detection of a signal in the presence of noise with an unknown covariance matrix has received increased attention in radar, sonar, wireless communications, and others. Lots of adaptive detection algorithms have been proposed, including the generalised likelihood ratio test (GLRT) [1], the two-step design procedure GLRT [2], and variants of many detectors based on the GLRT decision rule, e.g. [3–5]. All above solutions are devised for rank-one signal model that the actual signal steering vector is exactly matched to the assumed one.

However, mismatches of the target steering vector often occur in practice, which are caused by pointing errors, calibration, and wavefront distortions. In addition, rank-one signal model may be not suitable for the target signal that is naturally multi-rank in many applications, e.g. the signal observed from multiple channels in polarisation radars or by multipath propagation [6, 7]. As a result, a subspace signal model, which belongs to a known subspace but with unknown coordinates, receives considerable attention in adaptive target detection for its potential to improve the robustness of the rank-one signal. Therefore, many important work on subspace signal detection have been developed, such as subspace GLRT detection [8, 9], subspace Rao test [10], performance analysis of subspace detectors [11, 12], and references therein.

Frequently, adaptive target detection assumes that a set of training data are available to estimate the unknown noise covariance matrix. Generally, the training data require at least two times of the system dimension to guarantee an acceptable detection performance [13]. However, a large number of training data are not available because of the real-world complexity (e.g. varying terrain, dense target environments, non-linear array, and shadowing). A strategy to cope with such a problem is to exploit the persymmetric structure of the covariance matrix. A covariance matrix has the persymmetric structure means it is Hermitian about its diagonal and persymmetric about its cross-diagonal [14]. In [15], the persymmetric property is exploited to devise a persymmetric GLRT detector. By two-step procedure, the persymmetric two-step GLRT detector has been proposed in [16]. Following this work, many persymmetric adaptive detectors have been provided in [17–20]. However, all persymmetric detection methods mentioned above are only for rank-one signal.

In this paper, we focus on the detection of a subspace signal that belongs to a multi-rank subspace with unknown coordinates. A persymmetric subspace adaptive matched filter (PS-AMF) is proposed by a unitary transform to incorporate the persymmetric structure information into the detector design, which can reduce the training requirement.

In addition, approximate expressions for the probabilities of false alarm and detection of the PS-AMF are derived, which are verified by using Monte Carlo (MC) trials. Numerical results show that the proposed PS-AMF outperforms the conventional detector in training-limited scenarios.

Notations: the conjugate, transpose, and conjugate transpose operations are denoted by (·)' , (·)T , and (·)' , respectively. ℜ(·)' indicates the real part and j = √−1. det (·) represents the determinant of a square matrix and I denotes the modulus of a complex number. ‘ ∼ CN’ denotes ‘with complex Gaussian distribution’ and ‘ ∼ χ2(μ)’ denotes ‘with non-centrality Chi-square distribution with n degrees of freedom and non-centrality parameter μ ’.

2 Problem formulation

Consider a subspace signal model of the test data as follows

\[ x = Hα + n \]  

(1)

where the columns of the known full-rank matrix \( H \) span the subspace where the target signal lies, \( α \) denotes the unknown coordinate vector with size \( r \times 1 \), \( n \) denotes the complex Gaussian noise vector with zero mean and unknown covariance matrix \( M \). Here, the size of \( H \) is \( N \times r \), where \( r \) denotes the rank of the subspace. Additionally, a set of training data free of target components is available, namely, \( x_k \sim \text{CN}(0,M), k = 1, ..., K \) and \( K > N \).

Then, the detection problem can be formulated as the following binary hypothesis test:

\[
\begin{cases}
H_0: x = n, \\ H_1: x = Hα + n, \end{cases} \quad \text{where } k = 1, ..., K.
\]

(2)

When a linear array is symmetrically spaced with respect to the phase centre or the pulse train is uniformly spaced, \( H \) has persymmetric structure, i.e.

\[ M = JM^\dagger J. \]

(3)
where $J$ is the permutation matrix whose cross-diagonal elements are ones and others are zeros.

Based on the persymmetric structure, we exploit the persymmetric property by following the unitary matrix [16] to transform the complex $M$ and $H$ into real ones:

$$
E = \begin{bmatrix}
I_{N/2} & J_{N/2} \\
\sqrt{2}J_{N/2} & -J_{N/2}
\end{bmatrix}
\begin{cases}
I_{N/2} & \text{for } N \text{ even} \\
0 & \text{for } N \text{ odd}
\end{cases}
$$

where $I$ denote the identity matrix.

Then, the transformed test data $z_{pk}$, the transformed training data $y_{pk}$, the transformed noise vectors $c_p$ and $c_{pk}$, and the transformed subspace matrix $H_p$ are given by

$$
x_p = Ex, \quad y_{pk} = Ex_{pk}, \quad H_p = EH, \\
\begin{bmatrix}
n_p \\
n_{pk}
\end{bmatrix} = En_p, \quad \begin{bmatrix}n_p \\
n_{pk}
\end{bmatrix} = En_{pk}.
$$

where $x_p$ and $x_{pk}$ are complex Gaussian vectors with zero mean and covariance matrix $R = EME^T$.

It is shown in [16] that the maximum likelihood estimation of the real covariance matrix $R$ is

$$
\hat{R}_p = \Re(E^*ME^T)
$$

where $M$ denotes the sample covariance matrix.

Thus, the original problem (2) can be equivalently reformulated as

$$
\begin{bmatrix}
H_0: x_p = n_p, \quad x_{pk} = n_{pk} \\
H_1: x_p = H_1 \alpha + n_p, \quad x_{pk} = n_{pk}
\end{bmatrix}
\begin{cases}
\begin{bmatrix}H_0 \\
\sqrt{2}
\end{bmatrix} & \text{for } N \text{ even} \\
0 & \text{for } N \text{ odd}
\end{cases}
$$

where $k = 1, \ldots, K$.

### 3 Proposed detector

#### 3.1 PS-AMF

We resort to the two-step design procedure to derive the PS-AMF detector. Assuming the noise covariance matrix is known, we first obtain the GLRT by maximising over other unknown parameters. The estimate of the noise covariance matrix based on the secondary data alone is then substituted into this test in place of the true noise covariance. The GLRT with known noise covariance matrix is expressed as

$$
\max_{\alpha} \frac{f(x_p; \alpha, H_1)}{f(x_p; H_0)} \frac{1}{\det(R)} \left| x_p - H_1 \alpha \right| \frac{1}{\det(R)} x_p^2 \left| (x_p - H_1 \alpha) \right|^T
$$

where

$$
f(x_p; H_0) = \frac{1}{\pi^N \det(R)} \exp(-x_p^2) .
$$

and

$$
f(x_p; \alpha, H_1) = \frac{\exp(-(x_p - H_1 \alpha)^T (x_p - H_1 \alpha))}{\pi^N \det(R)} .
$$

Substituting the probability density functions (PDFs) (10) and (11) yields

$$
\max_{\alpha} \frac{\exp\left[\frac{1}{\xi}(x_p - H_1 \alpha)^T x_p \left(1/H_1 \alpha\right)^T \right]}{\xi} .
$$

Maximising the left-hand-side of (12) is equivalent to

$$
\min_{\alpha} \frac{\exp\left[\frac{1}{\xi}(x_p - H_1 \alpha)^T x_p \left(1/H_1 \alpha\right)^T \right]}{\xi} .
$$

From (13), the estimation of $\alpha$ is given by

$$
\hat{\alpha} = (H_1^T H_1)^{-1} H_1^T x_p .
$$

Inserting (14) into (12) yields

$$
x_p^2 (H_1^T H_1)^{-1} H_1^T x_p < \xi
$$

Finally, substituting the sample covariance matrix $\hat{R}_p$ in (7) into (15) yields the final PS-AMF detector

$$
x_p \hat{R}_p^{-1} H_1 (H_1^T H_1)^{-1} H_1^T \hat{R}_p x_p < \xi .
$$

#### 3.2 Performance analysis

In this section, the approximate expressions for the probabilities of false alarm and detection of the PS-AMF are derived. First, the test vectors are whiten as $\tilde{x}_p = R^{-1/2} x_p$ and $\tilde{x}_{pk} = R^{-1/2} x_{pk}$. Then, we have

$$
x_p \tilde{R}_p^{-1} x_p = 2K \tilde{x}_p Q^{-1} \tilde{x}_p .
$$

where

$$
Q = (R^{-1/2} (2K \hat{R}_p) R^{-1/2} T
$$

follows real Wishart distributed with $2KS$ degrees of freedom and matrix $I_\nu$. Therefore, the equivalent statistical form of $Q$ can be expressed as $Q = \sum_{k=1}^{K} \gamma_{pk} y_{pk}$, where $y_{pk}$ is a real Gaussian vector with zero mean and covariance matrix $I_\nu$.

Let $U$ be as a unitary matrix such that

$$
U \begin{bmatrix}
\hat{R}_p^{-1/2} H_1 (H_1^T H_1)^{-1/2} \hat{R}_p \end{bmatrix} = [I_\nu, 0]^T .
$$

where $[I_\nu, 0]^T$ is denoted as $T$.

After the same unitary transform, $\tilde{x}_p, x_{pk}$ and $Q$ become

$$
\tilde{x}_p = U \tilde{x}_p, \tilde{y}_{pk} = U y_{pk}, \hat{Q} = U Q .
$$

Using $T$, $\tilde{x}_p$, and $\hat{Q}$, the statistic for PS-AMF in (16) can be rewritten as

$$
A_0 = 2K \tilde{x}_p^T \hat{Q}^{-1} T (\hat{Q}^{-1} T)^{-1} \hat{Q}^{-1} \tilde{x}_p .
$$

Then, $\tilde{x}_p$ and $\tilde{y}_{pk}$ are rewritten as

$$
\tilde{x}_p = [\tilde{x}_A]^T \tilde{x}_B^T
$$

and

$$
\tilde{y}_{pk} = [\tilde{y}_{A_k}]^T \tilde{y}_{B_k}^T
$$

where $\tilde{x}_A$ and $\tilde{y}_{A_k}$ are $r \times 1$ vectors.

Similarly, $\hat{Q}$ can be rewritten as
Therefore, the test statistic (21) of the PS-AMF can be equivalently expressed as

\[ \Lambda = z^T \left[ \frac{1}{k} \sum_{i=1}^{2} \tilde{z}_i \tilde{z}_i^T \right]^{-1} \frac{H}{\xi^K} \frac{1}{\sum_{i=1}^{k}} \]

(27)

where

\[ z = \tilde{x}_A - \tilde{Q}_A \tilde{Q}_B \tilde{x}_B \]

(28)

\[ \tilde{z}_i = \tilde{y}_A - \tilde{Q}_A \tilde{Q}_B \tilde{z}_B \]

(29)

Note that \( z \) is a complex vector but \( \tilde{z}_i \) is a real vector. Thus, it is intractable to find the exact statistical distribution of (27) so far. In the following, we provide an approximate solution to this problem.

Let \( \beta = \frac{1}{L \left( 1 + \sum_{i=1}^{k} \tilde{z}_i \tilde{z}_i^T \right) K} \), which is a random variable with PDF (the derivation is similar to (26) [17])

\[ f_\beta(\beta) = \frac{\beta^{(N-1)/2}}{\Gamma(N/2) \Gamma(N/2)} \left( 1 - \mu \right)^{N/2 - 1} \int_0^1 \right. \left( 1 - \delta x_3 - \sum_{i=1}^{k} \right) \frac{\delta x_3}{\mu} d\gamma \]

(30)

Denote \( \tilde{\beta} = \beta^{1/2} \). Then, the test statistic (16) becomes

\[ \Lambda = \tilde{\beta}^T \left[ \frac{1}{k} \sum_{i=1}^{2} \frac{1}{\tilde{z}_i \tilde{z}_i^T} \right]^{-1} \frac{H}{\xi^K} \frac{1}{\sum_{i=1}^{k}} \]

(31)

By some calculation, the statistical distribution of (25) is given by

\[ 2 \Lambda \sim \left[ \frac{\tilde{X}_0}{\tilde{X}_0 - K - N + 1} \right] \left[ \frac{\tilde{\chi}_2}{\tilde{\chi}_2 - N + 1} \right] \left[ \frac{\tilde{\chi}_2}{\tilde{\chi}_2 - N + 1} \right] \]

(32)

where \( \tilde{\mu} = \beta \left( \frac{f_{H}(R'H) \Gamma}{t} \right)^{1/2} \). Denote

\[ 2t_1 \sim N(0,1) \]

(33)

\[ t_2 \sim \left[ \left( \frac{\tilde{X}_0}{\tilde{X}_2 - N + 1} \right), \frac{H_0}{H_1} \right] \]

(34)

\[ \left( \frac{H_0}{H_1} \right) \]

where

\[ f(t; H_0) = \frac{1}{(r-1)!} \left( \frac{r}{r-1} \right)^{r-1} \exp(-t_1) \]

(35)

\[ f(t; H_1) = \frac{1}{(r-1)!} \left( \frac{r}{r-1} \right)^{r-1} \exp(-t_1) \sum_{m=0}^{\infty} \left( \frac{(r+1)!}{(r+1+m)!} \right) \]

(36)

with \( L = (2K - N - 1)/2 \).

By (30) and (32), the false alarm probability is given by

\[ P_{fa} = \int_0^{\infty} \int_0^{\infty} f(t; H_0) f(t; H_1) \frac{1}{t} \left( 1 + \frac{\xi \beta}{K} \right)^{r-1} \]

(38)

Next, we derive the detection probability. By (31) and (32), the detection probability conditioned on \( \beta \) is given by

\[ P_{d|\beta} = \int_0^{\infty} \int_0^{\infty} f(t; H_0) f(t; H_1) \frac{1}{t} \left( 1 + \frac{\xi \beta}{K} \right)^{r-1} \]

(39)

Therefore, the probability of detection can be obtained by averaging over \( \beta \), namely,

\[ P_d = \int_0^{\infty} \int_0^{\infty} f_{H}(\beta) f(t; \beta) d\beta \]

(40)

### 4 Numerical results

In this section, numerical examples are provided to assess the performances of the proposed PS-AMF. In all simulations, we set \( N=9 \) or \( N=10 \), \( r=3 \). The column vector of \( H \) is \( h_i = \left[ \sqrt{2} \left( 1, e^{j\pi/2}, \ldots, e^{j\pi(N-1)/2} \right) \right] \), where \( f_1 = 0.09 \), \( f_2 = 0.1 \), \( f_3 = 0.11 \). \( M \) is an exponentially correlated covariance matrix with one-lag correlation coefficient, \( \rho = 0.95 \), i.e. the \( (i,j) \)th element of \( M \) is \( \rho^{\|i-j\|} \). Then, the signal-to-noise ratio (SNR) is defined as

\[ \text{SNR} = \frac{a^H A R^H H a}{a^H A R^H H a} \]

(41)

To decrease the computation load, the probability of false alarm is set to \( P_{fa} = 10^{-6} \). The number of trials For the MC simulation is \( 100/P_{fa} \).

To evaluate the validity of the theoretical analysis about the PS-AMF, we compare the results from the proposed expression with that from MC trials. Fig. 1 shows the curves of the Pfa with respect to the threshold, and Fig. 2 shows the detection performance versus the SNR. Theoretical, the values of \( P_{fa} \) come from (38), and the probability of detection come from (40). As shown in Figs. 1 and 2, the analytical results are in good agreement with the MC results. In addition, the detection performance of PS-AMF increases as the size of training data becomes larger.

Then, the proposed detector is compared with the conventional subspace GLRT (GLRT), AMF (AMF), and Rao (Rao) detectors. Fig. 3 shows the detection performance of these detectors for different size of training data. From Fig. 3, it is seen that the proposed PS-AMF outperforms its counterparts for low sample support. For \( K=9 \), the conventional detectors suffer from performance degradation, while the PS-AMF maintains a higher probability of detection. As \( K \) increases, the performance gap between the PS-AMF and the conventional detectors becomes small. When \( K \) continues to increase to two times of the system dimension, the performance of the PS-AMF are almost identical as that of the conventional detectors.

### 5 Conclusion

In this paper, we addressed the detection of a subspace signal that belongs to a multi-rank subspace with unknown coordinates. A
A persymmetric subspace AMF detector was proposed by incorporating the persymmetric structure information into the detector design. Moreover, the approximate expressions for the probabilities of false alarm and detection of the proposed PS-AMF were derived, which were verified by using MC simulations. Numerical results illustrated that the proposed PS-AMF outperformed the conventional detector in training-limited scenarios.

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