The many body problem in relativistic quantum mechanics

Marcos Moshinsky* and Anatoly Nikitin

Instituto de Física
Universidad Nacional Autónoma de México
Apartado Postal 20-364, 01000 México D.F., México
moshi@fisica.unam.mx

a Permanent Address: Institute of Mathematics
Ukrainian Academy of Sciences,
3 Tereshchenkivska Str., 01601 Kyiv, Ukraine

Abstract

We discuss a relativistic Hamiltonian for an n-body problem in which all the masses are equal and all spins take value 1/2. In the frame of reference in which the total momentum \( \mathbf{P} = 0 \), the Foldy-Wouthuysen transformation is applied and the positive energy part of the Hamiltonian is separated. The Hamiltonian with unharmonic oscillator potential is applied to describe mass differences for charmonium and bottomonium states.

*Member of El Colegio Nacional and Sistema Nacional de Investigadores
1 Introduction

In the development of non-relativistic quantum mechanics the contribution of Schrödinger was not only in writing a single particle wave equation that, for a Coulomb potential, leads to the spectrum of the hydrogen atom, but also that his formalism could be extended immediately to a system of n-particles in configuration space. Thus wave mechanics from the very beginning was able to deal with systems of many particles in interaction as happens for electron in atoms or nucleons in nuclei.

The non-relativistic many body Hamiltonian, in the absence of spins, was composed of the sum of the kinetic energies $\sum_{s=1}^{n}(p_s^2/2m_s)$ of the $n$ particles together with an appropriate potential interaction between them. Many refined mathematical methods [1] were developed to obtain the eigenvalues and eigenstates of these Hamiltonians.

The corresponding development did not take place in the theory of many body problems in relativistic quantum mechanics. It is true that almost simultaneously to the development of non-relativistic quantum mechanics Dirac [2] introduced his famous relativistic wave equation for a single particle. The extension of this equation to many bodies found two main obstacles. One of them was the need to formulate the wave equation in a form invariant under the transformations of the Poincaré group, thus guaranteeing its relativistic character. In particular relativistic invariance presupposes a certain symmetry between space and time variables, which causes the appearance of multi-time variables in an n-body theory and thus is an aspect that we will correct without violating Poincaré invariance.

The other obstacle was that in relativity the relation between energy
\(E\) and momentum \(p\) of a particle of mass \(m\) is

\[
E^2 = p^2c^2 + m^2c^4
\]

(1)

instead of the \(E = p^2/2m\) of the non-relativistic case.

Equation (1) implies that both positive and negative energies are possible as taking the square root we have

\[
E = \pm \sqrt{p^2c^2 + m^2c^4}
\]

(2)

This sign ambiguity makes itself felt immediately in the Dirac equation and led him to propose that vacuum was actually a state in which all negative energy levels were filled by particles obeying Fermi statistics and thus unable to accept another particle of this type. From this point of view relativistic quantum mechanics turned to a field theoretical description and a procedure similar to that of non-relativistic quantum mechanics for relativistic many body problems was essentially abandoned.

There are many different formulations of the relativistic many-body problem [3]. Conventionally they can be subdivided to three classes. Namely, we can indicate manifestly covariant approaches based on the Bethe-Salpeter equation and its generalizations, theories with direct interaction and the mass shell constraints approach.

The Bethe-Salpeter equation [4], [5] presents powerful and elegant tools for construction of various two body theories which are transparently relativistic invariant. However, a generalization of this equation to \(n\)-body case seems to be too complicated if at all possible as far as practical applications are concerned., see, e.g., reference [6] for the case \(n = 3\).

A global receipt to overcome difficulties with multi time variables was proposed by Dirac [7] who proposed to use one time formulation for
n-body models but to ask for existence of realization of the Poincaré algebra on the set of solutions of the equations of motion. The price paid for the absence of extra time variables was the absence of manifestly relativistic invariance.

The mass shell constraints approach [8] shares with the Bethe-Salpeter equation manifestly relativistic invariance. This approach is much more easy to handle than ones based on the Bethe-Salpeter equation. However, the problem of separation of extra time variables is by no means trivial and was effectively solved only for two- and three-body problems [9].

In this paper we want to return to the possibility of discussing relativistic many body problems in a framework similar to that of the non-relativistic quantum mechanics. For this purpose we have to deal with two problems.

a) How can we formulate a wave equation explicitly invariant under the Poincaré group but which, in an appropriate frame of reference, involves only one time.

b) Once objective (a) is achieved how can we separate the positive and negative parts of its solution as only the first one will be of physical interest.

In Section 2 we deal with the first problem through a method we developed in previous publications and in section 3 with the second one employing a generalization of the Foldy-Wouthuysen (FW) transformation.
2 A Poincaré invariant n-body wave equation which, in a particular frame of reference, involves only one time

We mentioned in the introduction how we can formulate the wave equation for a n-body non-relativistic problem starting from the corresponding one body expression for the free particle. In the relativistic case we have also the wave equation of Dirac [2] for a single free particle given by

\[ \left( -i \frac{\partial}{\partial t} + \mathbf{\alpha} \cdot \mathbf{p} + \beta m \right) \psi = 0 \]  

(3)

where we use the usual relativistic units \( \hbar = c = 1 \), with \( \mathbf{p} \) being the momentum three vector, \( m \) the mass of the particle and the matrices \( \alpha, \beta \) are defined as in reference [2].

If we have \( n \) particles of the same mass \( m \), we add the index \( s = 1, 2 \ldots n \) to all the variables and an obvious Poincaré invariant n-body equation can be written in the form

\[ \sum_{s=0}^{n} (\gamma_\mu^s p_\mu + m) \psi = 0 \]  

(4)

where repeated index \( \mu \) are summed over the values \( \mu = 0, 1, 2, 3 \) with \( p_0 = -i \partial / \partial t_s, \, p_j = -i \partial / \partial x_{js}, j = 1, 2, 3 \) and the \( \gamma_\mu^s \) are matrices related to \( \mathbf{\alpha}_s \) and \( \beta_s \) by [2]

\[ \gamma_0^s = \beta_s, \, \gamma_i^s = \beta_s \alpha_{is}, \, s = 1, \ldots n, \, i = 1, 2, 3 \]  

(5)

The \( \gamma_\mu^s, p_\mu \) are respectively contravariant and covariant expressions [2] so that \( \gamma_\mu^s p_\mu \) is a Poincaré scalar and thus Eq. (4) is certainly a Poincaré invariant \( n \) particle wave equation, but it is not satisfactory because introduces \( n \) times through \( p_0 = -i \partial / \partial t_s, s = 1, 2, \ldots , n \).

How can we find a formulation of many body problem, still invariant
under the Poincaré group but, in an appropriate system of reference, involving only one time?

We start by denoting by $u_{\mu}$ unit time like four vector which implies that there is a reference frame in which it takes the form

$$(u_{\mu}) = (1, 0, 0, 0)$$

(6)

With the help of the four vector (6) we can define the Lorentz scalars

$$\Gamma = \prod_{r=1}^{n} (\gamma_{r}^{\mu} u_{\mu}), \quad \Gamma_{s} = (\gamma_{s}^{\mu} u_{\mu})^{-1} \Gamma$$

(7)

where $(\gamma_{s}^{\mu} u_{\mu})^{-1}$ eliminates the corresponding term in $\Gamma$ so that $\Gamma_{s}$ is still in product form.

Instead of Eq. (4) we propose now the following Lorentz invariant one [10]

$$\sum_{s=1}^{n} \Gamma_{s} (\gamma_{s}^{\mu} p_{\mu s} + m) \psi = 0.$$  

(8)

We now introduce the total energy-momentum four vector

$$P_{\mu} = \sum_{s=1}^{n} p_{\mu s}$$

(9)

and with its help define our four vector $u_{\mu}$ as

$$u_{\mu} = P_{\mu} (-P_{\tau} P^{\tau})^{\frac{1}{2}}.$$  

(10)

We immediately see that when the center of mass of our n-body system is at rest, i.e., $P_{i} = 0, i = 1, 2, 3$ our four vector $u_{\mu}$ takes the form Eq. (6) in which the wave equation (8) becomes

$$\left[ \Gamma^{0} \sum_{s=1}^{n} p_{0s} + \sum_{s=1}^{n} \Gamma^{0}_{s} (\gamma_{s} \cdot p_{s}) \right] \psi = 0$$

(11)

where bold face letters mean three dimensional vectors and

$$\Gamma^{0} = \prod_{r=1}^{n} \gamma_{r}^{0}, \quad \Gamma^{0}_{s} = (\gamma_{s}^{0})^{-1} \Gamma^{0}.$$  

(12)
Multiplying Eq. (11) by $\Gamma^0$ and making use of Eqs. (5), (9), (12) we obtain

$$[-P^0 + \sum_{s=1}^{n}(\alpha_s \cdot p_s + m\beta_s)]\psi = 0$$  \hspace{1cm} (13)

where we used a metric in which $P_0 = -P^0$ and the latter is the zero component of the four vector $P^\mu$, \textit{i.e.}, the total energy of the system.

So far we have obtained a Poincaré invariant wave equation for a system of non-interacting particles which in the frame of reference in which $\mathbf{P} = 0$ takes the form (13).

We wish now to consider interactions and for simplicity we will consider them to depend only on the relative coordinates

$$x_{s \mu}^{st} \equiv x_{\mu s} - x_{\mu t}, \quad \mu = 0, 1, 2, 3$$  \hspace{1cm} (14)

We want that these relative coordinates become purely spatial ones in the frame of reference where the total momentum $\mathbf{P} = 0$ and this is easily achieved through the use of the $u_\mu$ four vector of (6) by defining

$$x_{s \mu}^{\perp} \equiv x_{s \mu}^{st} - (x_{s \tau}^{st} u^\tau) u_\mu$$  \hspace{1cm} (15)

because when $u_\mu$ takes the form (6) the $x_{s \mu}^{st}$ becomes

$$x_0^{st} = 0, \quad x_i^{st} = x_{is} - x_{it}, \quad i = 1, 2, 3$$  \hspace{1cm} (16)

As we want our potential to be Poincaré invariant it is sufficient to make it a function of $x_{s \mu}^{st} x_{s' \mu}^{st'}$ where repeated index $\mu$ are summed over their values $\mu = 0, 1, 2, 3$. Thus we restrict ourselves to potentials of the form

$$V = V(x_{s \mu}^{st} x_{s' \mu}^{st'})$$  \hspace{1cm} (17)

which in the frame of reference where $\mathbf{P} = 0$ becomes $V(\mathbf{x}^{st} \cdot \mathbf{x}^{st'})$ with the bold face letters indicating spatial relative vectors, \textit{i.e.}

$$\mathbf{x}^{st} = \mathbf{x}_s - \mathbf{x}_t$$  \hspace{1cm} (18)
We are now in a position to write of Poincaré invariant wave equation for as of \( n \) particle of the same mass and of spin \( \frac{1}{2} \) as

\[
\left[ \sum_{s=1}^{n} \Gamma_{s} \cdot (\gamma_{\mu} p_{\mu s} + m) + \Gamma V(x_{s}^{t} x_{s}^{t'}) \right] \psi = 0 \tag{19}
\]

which in the frame of reference in which the center of mass of the system is at rest, \( i.e., \mathbf{P} = 0 \) leads to the Hamiltonian equation

\[
H \psi \equiv \left[ \sum_{s=1}^{n} (\alpha_{s} \cdot \mathbf{p}_{s} + m \beta_{s}) + V(x^{st} \cdot x^{s't'}) \right] \psi = E \psi \tag{20}
\]

where we replaced \( P^{0} \) by the total energy \( E \). So we can denote the square bracket in (20) as our Hamiltonian.

Equation (20) is not the end of our story because even for the one particle case it involves both positive and negative energies and it is the former ones that will be of interest to us. Fortunately Foldy and Wouthuysen [11] gave us a procedure to separate the positive and negative energy parts for the one particle case and in the next section we proceed to generalize it for the \( n \) particle system.

3 The FW transformation for the many body Hamiltonian

To be able to derive the FW transformed Hamiltonian for the \( n \) body system we need to review briefly the corresponding analysis for the one body case.

The Hamiltonian is then [12]

\[
H \equiv \mathcal{O} + \mathcal{E} + V \tag{21}
\]

where the odd part \( \mathcal{O} \) is \( \alpha \cdot \mathbf{p} \), the even one \( \mathcal{E} = \beta m \) and \( V \) is the potential being a scalar function of the one variable \( \mathbf{x} \).
We now follow FW [11] and, in particular, the book of Bjorken and Drell [12] which states that it is possible to find a unitary operator

\[ U = \exp(iS) \] (22)

that allows us to transform \( H \) in a series of inverse powers of the mass \( m \) corresponding to the positive energy part of \( H \).

The \( S \) in (22) is given as

\[ S = -\frac{i\beta}{2m}(\mathcal{O} + \mathcal{O}' + \mathcal{O}'') \] (23)

with

\[ \mathcal{O} = \alpha \cdot p, \quad \mathcal{O}' = \beta \frac{[\alpha \cdot p, V]}{2m}, \quad \mathcal{O}'' = -(\alpha \cdot p) p^2 \] (24)

and the new positive energy Hamiltonian \( H' \) is now given as

\[ H' = \hat{H} + V, \quad \hat{H} = \beta \left( m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \right) \]
\[ + \frac{1}{4m^2} \mathbf{s} \cdot \left[ (p \times \mathbf{E}) - (\mathbf{E} \times p) \right] + \frac{1}{8m^2} \nabla^2 V \] (25)

where one makes the expansion only through order \((\text{kinetic energy}^2/m^3)\) and \([\text{(energy)}(\text{field energy})]/m^2\). The field strength \( \mathbf{E} \) is given by

\[ \mathbf{E} = -\nabla V \] (26)

and \( \mathbf{s} \) is the spin of the particle \( \mathbf{s} = -i/4\alpha \times \alpha \).

Once we have the above well known results we pass to obtaining the corresponding ones for the n-body problem.

We start with the two body case where

\[ H = H_1 + H_2 + V(x_1, x_2) \] (27)

with

\[ H_s = \alpha_s \cdot p_s + \beta_s m, \quad s = 1, 2 \] (28)
Rewriting $H$ as

$$H = [H_1 + V(x_1, x_2)] + H_2$$  \hspace{1cm} (29)

we can first apply to it the unitary transformation

$$U_1 = \exp(iS_1)$$  \hspace{1cm} (30)

where $S_1$ is given by $S$ of (23) where $\beta, \alpha, p$ have the index 1 i.e. $\beta_1, \alpha_1, p_1$ to indicate its dependence on particle 1. As $S_1$ only depends on the variables of particle 1 it is clear that it does not affect $H_2$ in (29) and for $(H_1 + V)$ it gives the result of (25) with an index 1 for the variable i.e.

$$\exp(iS_1)H\exp(-iS_1) = \hat{H}_1 + (\hat{H}_2 + V)$$  \hspace{1cm} (31)

where $\hat{H}_1$ is given by (25) with index one for all the variables.

We now apply the unitary transformation of the second particle i.e.

$$U_2 = \exp(iS_2)$$  \hspace{1cm} (32)

where $S_2$ is given by $S$ of (23) where $\beta, x, p$ have the index 2 i.e. $\beta_2, \alpha_2, p_2$. It is clear that the unitary transformation $U_2$ has no effect on $\hat{H}_1$ as the only terms it could affect are $(\nabla_1^2 V/8m^2)$ and $\frac{1}{4m^2} S \cdot \left[ (p \times E) - (E \times p) \right]$ and this would be of higher order that the ones we accept. We have then that $U_2$ only acts on $(\hat{H}_2 + V)$ giving us

$$U_2(H_2 + V)U_2^\dagger = \hat{H}_2 + V$$  \hspace{1cm} (33)

where operator $\hat{H}_2$ is the one given in (25) where all the variables have index 2. Thus if consider $U_2U_1$ as our unitary transformation we have

$$U_2U_1(H_1 + H_2 + V)U_1^\dagger U_2^\dagger = \hat{H}_1 + \hat{H}_2 + V$$  \hspace{1cm} (34)

The procedure for the two particle problem immediately suggests that for the n-body case where $V(x_1, x_2, \ldots x_n)$ we can carry out the transformation
\[ H' = U_n U_{n-1} \cdots U_2 U_1 H U_1^\dagger U_2^\dagger \cdots U_n^\dagger = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_n + V \] (35)

where

\[
H_t = \beta_t \left( m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \right) + \frac{1}{4m^2} s_t \cdot (p_t \times E_t - E_t \times p_t) + \frac{1}{8m^2} \nabla^2 V, \quad t = 1, 2, \cdots n \] (36)

We must still keep in mind that in our equation (33) we have to take into account that the total momentum \( P = 0 \), which we can achieve by passing from our coordinate system to the Hamilton-Jacobi one as will be indicated in the examples discussed in the following section.

Thus we extend the approximate FW transformation to the case of multi particle relativistic wave equations. For exact FW transformations for one- and two-particle systems see refs. [13].

### 4 The two-body problem

In the case of two particles when the total momentum \( P = 0 \) we have that

\[ p_1 + p_2 = 0, \quad \text{or} \quad p_1 = -p_2 \equiv p \] (37)

and we shall denote the corresponding relative coordinate vector as \( r = r_1 - r_2 \). Besides, for simplicity, we shall take the potential \( V \) as of harmonic oscillator one, \( i.e., V = \frac{m \omega^2 r^2}{4} \).

From (36) and using the notation \( p, r \) introduced in the above we obtain that

\[
H' = (\beta_1 + \beta_2) \left( m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \right) + V \\
+ \frac{1}{4m^2} (s_1 + s_2) \cdot \left[ (p \times E) - (E \times p) \right] + \frac{1}{4m^2} \nabla^2 V \] (38)

as

\[ E_1 = E_2 = E = -\nabla V = -\frac{m \omega^2 r}{2} \quad \text{and} \quad \nabla^2 V = \nabla^2 V = \frac{3m \omega^2}{2} \] (39)
when $V = \frac{m\omega^2 r^2}{4}$. Introducing then the total orbital angular momentum $L = r \times p$ and using (39) we reduce $H'$ to the form

$$H' = (\beta_1 + \beta_2) \left( m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \right) + \frac{m\omega^2 r^2}{4} + \frac{\omega^2}{4m} S \cdot L + 3\frac{\omega^2}{8m}$$

where

$$S = s_1 + s_2$$

is the total spin vector and so the spin values can only be 1 or 0.

We note also that in Eq. (40) besides terms familiar in non-relativistic quantum mechanics there are the matrices $\beta_1$ and $\beta_2$. They can be chosen in the form of the direct products [10]

$$\beta_1 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

where $I_2$ is the $2 \times 2$ unit matrix, and so

$$\beta_1 + \beta_2 = 2 \begin{pmatrix} I_4 & 0_8 \\ 0_8 & -I_4 \end{pmatrix}$$

when $I_4$ and $0_8$ are the $4 \times 4$ unit matrix and $8 \times 8$ zero matrix correspondingly.

Our only interest in the positive energy part of the wave function which implies that $\beta_1 + \beta_2$ should be replaces by 2 and thus finally we have to deal with the expression

$$H' = \left( 2m + 3\frac{\omega^2}{8m} \right) + \left( \frac{p^2}{m} + \frac{m\omega^2 r^2}{4} + \frac{\omega^2}{4m} S \cdot L \right) - \frac{p^4}{4m^3}$$

As the second parenthesis correspond to an harmonic oscillator with spin-orbit coupling whose eigenfunctions and eigenvalues are very well known we can use the former as a complete basis to convert $H'$ into a numerical matrix.
Before proceeding to give a procedure to calculate the eigenvalues for $H'$ of (44) it is convenient to make the canonical transformation

$$
\pi = \sqrt{\frac{2}{m\omega}} p, \quad \rho = \sqrt{\frac{m\omega}{2}} r
$$

(45)
to write $H'$ as

$$
H' = 2m + 3\frac{\omega^2}{8m} + \frac{\omega}{2}(\pi^2 + \rho^2) + \frac{\omega^2}{4m} S \cdot L - \frac{\omega^2}{16m} \pi^4
$$

(46)
where in the second parenthesis we have the oscillator hamiltonian of unit frequency and the spin-orbit term remains uncharged as $r \times p = \rho \times \pi$ while $p^4$ is replaced by $\frac{1}{4}m^2\omega^2\pi^4$.

To convert $H'$ into a numerical matrix we can use the states of the harmonic oscillator and states of squared total angular momentum $J^2$, total orbital momentum $L^2$ and total spin $S^2$, i.e.,

$$
|nl, (\frac{1}{2} \frac{1}{2}) S; j, m > = \sum_{\mu, \sigma} |l\mu, S\sigma| jm > |nl\mu > |(\frac{1}{2} \frac{1}{2}) S\sigma >
$$

(47)
where $J^2$, $L^2$ and $S^2$ commute with the Hamiltonian (46) and so are integrals of motion. The kets $|nl\mu >$ are those of the harmonic oscillator of unit frequency [14] and $|(\frac{1}{2} \frac{1}{2}) S\sigma >$ are those of the total spin.

The numerical matrix we want to determine has then the elements

$$
< n'l, (\frac{1}{2} \frac{1}{2}) S; j, m | H' | nl, (\frac{1}{2} \frac{1}{2}) S; j, m > = \\
\left( 2m + \frac{3\omega^2}{8m} + \omega \left( 2n + l + \frac{3}{2} \right) + \frac{\omega^2}{8m} [j(j+1) - l(l+1) - s(s+1)] \right) \delta_{nn'} - \frac{\omega^2}{16m} < n''l | \pi^4 | nl >
$$

(48)
The last term in (48) can then be defined starting with the relation $< n'l|\pi^4|nl > = < n'l|\pi^2|n''l > < n''l|\pi^2|n'l >$ and using the expression for $< n'l|\pi^2|nl >$ given in p. 7, Eq. (3.11) of reference [14]. Thus we
get that

\[ < n'|\pi^4|nl> = \sqrt{n(n-1)(n+l+1/2)(n+l-1/2)}\delta_{n'}_{n-2} \]
\[ + (4n + 2l + 1)\sqrt{n(n + l + 1/2)}\delta_{n'}_{n-1} \]
\[ + [(2n + l + 3/2)(2n + l + 5/2) + 2n(n + l + 1)]\delta_{n'}_{n} \]
\[ + (4n + 2l + 5)\sqrt{(n + 1)(n + l + 3/2)}\delta_{n'}_{n+1} \]
\[ + \sqrt{(n + 1)(n + 2)(n + l + 3/2)(n + l + 5/2)}\delta_{n'}_{n+2} \]  \hspace{1cm} (49)

We note that as we are using units in which \( \hbar = c = 1 \), the dimensionless term \( \omega/m \) becomes in the c.g.s. units

\[ \frac{\omega}{m} \rightarrow \frac{h\omega}{mc^2} \]  \hspace{1cm} (50)

and can be treated as a small parameter. In following section we compare the results of our analysis with the experimental spectrum of the bottomonium and charmonium masses.

## 5 Energy spectrum of the two body problem

We start with the eigenvalue problem

\[ H'\psi = E_{nl}\psi \]  \hspace{1cm} (51)

for the two-body hamiltonian (48).

To find the related energy spectrum we are supposed to diagonalize matrix (49) which can be done using numerical methods. Moreover, for sufficiently small coupling constant (50) it is possible to apply the standard perturbation theory and express the eigenvalues of \( H' \) (48) in power series of \( \nu = \omega/m \):

\[ E_{nl} - 2m - \frac{3\omega^2}{16m} = E_{nl}^0 + E_{nl}^1 + E_{nl}^2 + \cdots \]  \hspace{1cm} (52)
where the non-perturbed levels $E_{nl}^0$ are linear in $\nu$,

$$E_{nl}^0 = m\nu \left(2n + l + \frac{3}{2}\right)$$  \hspace{1cm} (53)

while $E_{nl}^1$ and $E_{nl}^2$ are quadratic and cubic in $\nu$ respectively. Moreover, the first and second perturbations of energy spectrum can be expressed via the elements of the perturbing matrix $<n'l'j'|K|jnl> = \frac{\omega^2}{8m} [j(j+1) - l(l+1) - s(s+1)] \delta_{nn'} \delta_{ll'} - \frac{\omega^2}{16m} <n'l'|\pi^4|nl> \delta_{jj'}$ as follows [1]

$$E_{nlj}^1 = <nlj|K|nlj>, \quad E_{nlj}^2 = \sum_{n'\neq n, l'\neq l} \frac{<n'l'j'|K|n'l'j>^2}{E_{nl}^0 - E_{n'l'}^0}$$ \hspace{1cm} (54)

Then, using (49), (52) and (53) we obtain

$$E_{nl} = 2m + \frac{3\omega^2}{16m} + \omega(2n + l + \frac{3}{2}) + \frac{\omega^2}{8m} [j(j+1) - l(l+1) - s(s+1)]
- \frac{\omega^2}{16m^2} \left(2n + l + \frac{3}{2}\right) \left(2n + l + \frac{3}{2}\right) + 2n(n + l + \frac{1}{2})
- \frac{\omega^2}{16m^2} \left(2n + l + \frac{3}{2}\right) \left(9n(n + l + \frac{1}{2}) + 2n(n + l + \frac{5}{2})\right) \left(2n + l + \frac{11}{4}\right)$$ \hspace{1cm} (55)

Formula (55) describes the spectrum of relativistic two body systems with the harmonic oscillator potential. We compare it with the energy spectrum of two quark systems (mesons) and find that it presents a rather realistic qualitative distribution of bottonium and charmonium masses. For example, setting $m = 4.7 \text{ GEV}$ (i.e., supposing $m$ be equal to the bottonium quark mass) and choosing the dimensionless coupling constant $\nu = 0.19$ we obtain from (55) the following values for the mass differences of the bottonium states: $\xi_{b0} - \Upsilon(1S) = 0.411 \text{ GEV}, \xi_{b1}(1P) - \Upsilon(1S) = 0.423 \text{ GEV}, \xi_{b2}(1P) - \Upsilon(1S) = 0.443 \text{ GEV}$ while the experimental data are [15] 0.400, 0.432 and 0.453 GEV respectively.

To obtain a better agreement with experimental data we consider anharmonic oscillator potential

$$V = \frac{m\omega^2 \gamma^2}{4} + V', \quad V' = -\frac{\alpha m\omega^4 r^4}{64}$$ \hspace{1cm} (56)

where $\alpha$ is a dimensionless interaction constant. In addition, we take into account relativistic corrections up to order $\nu^3$ for the approximate Hamiltonian $H'$ which needs continuation of the FW reduction.
We omit the routine calculations which are analogous to ones given in Section 3 and present the resulting transformed Hamiltonian using variables (45):

\[
H' = 2m + \frac{3\omega^2}{8m} + \frac{\omega}{2}\left(\pi^2 + \rho^2 \right) + \frac{\omega^2}{4m} \mathbf{S} \cdot \mathbf{L} - \frac{\omega^2}{16m}(\pi^4 + \alpha \rho^4) + \frac{\omega^3}{32m^2}\left(\frac{\pi^6}{2} + (2 - 5\alpha)\rho^2 - (3\pi^2 + 2\rho^2)\mathbf{S} \cdot \mathbf{L} - (\mathbf{S} \cdot \mathbf{\pi})^2 + (\mathbf{S}^2 - 8)\pi^2 \right)
\]

(57)

We need only diagonal matrix elements for terms of order \(\omega^3/m^2\), placed at the second line of equation (57). They can be easily found starting with matrix representation for \(\rho^2\) and \(\pi^2\) given in page 7, Eq. (3.11) of reference [14] and the representation for \(\mathbf{S} \cdot \mathbf{\pi}\) in the spherical spinor basis given in pages 422-423 of reference [16] :

\[
< nl|\rho^2|nl> =< nl|\pi^2|nl> = 2n + l + \frac{3}{2},
\]

\[
< nl|\rho^4|nl> = (2n + l + \frac{3}{2})(2n + l + \frac{5}{2}) + 2n(n + l + \frac{1}{2})],
\]

\[
< nl|\pi^6|nl> = 4(n + l + \frac{1}{2})(2n + l + 1)
\]

\[
+(2n + l + \frac{5}{2}) \left(2(n + 1)(n + l + \frac{3}{2}) + (2n + l + \frac{3}{2})^2 \right),
\]

\[
< nlj|(\mathbf{S} \cdot \mathbf{\pi})^2|nlj> = (1 - \frac{1}{2}(j - l)^2 + \frac{j-l}{2(2j+1)}) < nl|\pi^2|nl>
\]

and the related energy values \(E_{nlj}\) are described by the following formula

\[
E_{nlj} = 2m + \frac{(3-S^2)\omega^2}{8m} + \omega(2n + l + \frac{3}{2}) + \frac{\omega^2}{8m}[j(j + 1) - l(l + 1)]
\]

\[
-\frac{(1+\alpha)\omega^2}{16m}\left[\left(2n + l + \frac{3}{2}\right)\left(2n + l + \frac{5}{2}\right) + 2n\left(n + l + \frac{1}{2}\right)\right]
\]

\[
-\frac{\omega^2}{256m^2}\left(2n + l + \frac{3}{2}\right)\left(1-\alpha\right)^2\left(2\left(2n + l + \frac{5}{2}\right)^2 + 4n\left(2n + l + \frac{1}{2}\right)\right)
\]

\[
+(1+\alpha)^2\left((n + 1)\left(n + l + \frac{3}{2}\right) - \frac{1}{2}\left(2n + l + \frac{1}{2}\right)\right)
\]

\[
-\frac{\alpha\omega^3}{32m^2}\left(2n + l + \frac{3}{2}\right)(j(j + 1) - l(l + 1) + 5 - S^2)
\]

\[
+\frac{\omega^3}{64m^2}\left(4n(n + l + \frac{1}{2})(2n + l + 1) + (2n + l + \frac{3}{2})\left(2(n + 1)(n + l + \frac{3}{2})
\right)
\]

\[
+(2n + l + \frac{3}{2})^2)\right)
\]

\[
+\frac{\omega^3}{64m^2}(2n + l + \frac{3}{2}) \left[3l(l + 1) - 3j(j + 1) + (j - l)^2 + \frac{j-l}{2j+1} + 5S^2 - 14 \right].
\]

(58)

We compare the spectrum (58) with the experimental bottonium and charmonium mass spectra [15]. We set \(m = 4.7GEV\) for the bot-
tonium and $m = 1.4$ for charmonium cases and use MAPLE software to find the coupling constants $\omega$ and $\alpha$ which correspond to the minimal deviation of the spectrum (58) from experimental data. The results of our investigations are given in the following tables.

Table 1. Experimental spectra of the bottonium and our model results (in GEV), $m = 4.7, \omega = 0.378, \alpha = 4.7$

| $l = 0, j = 1$ | Experimental | Theoretical |
|----------------|--------------|-------------|
| $\Upsilon(2S) - \Upsilon(1S)$ | 0.563 | 0.574 |
| $\Upsilon(3S) - \Upsilon(1S)$ | 0.895 | 0.963 |
| $\Upsilon(4S) - \Upsilon(1S)$ | 1.119 | 1.142 |

| $l = 1, j = 0$ | Experimental | Theoretical |
|----------------|--------------|-------------|
| $\xi_{b0}(1P) - \Upsilon(1S)$ | 0.400 | 0.310 |
| $\xi_{b0}(2P) - \Upsilon(1S)$ | 0.772 | 0.797 |

| $l = 1, j = 1$ | Experimental | Theoretical |
|----------------|--------------|-------------|
| $\xi_{b1}(1P) - \Upsilon(1S)$ | 0.432 | 0.316 |
| $\xi_{b1}(2P) - \Upsilon(1S)$ | 0.794 | 0.801 |

| $l = 1, j = 2$ | Experimental | Theoretical |
|----------------|--------------|-------------|
| $\xi_{b2}(1P) - \Upsilon(1S)$ | 0.453 | 0.326 |
| $\xi_{b1}(2P) - \Upsilon(1S)$ | 0.808 | 0.808 |
Table 2. Experimental spectra of the charmonium and our model results (in GEV), $\omega = 0.45, \alpha = 1.2, m = 1.4$

|                | Experimental | Theoretical |
|----------------|--------------|-------------|
| $l = 0, j = 1, s = 0$ | $\eta_c(2S) - \eta_c(1S)$ | 0.603 | 0.604 |
| $l = 0, j = 1, s = 1$ | $\Psi(2S) - J/\Psi(1S)$ | 0.589 | 0.589 |
|                | $\Psi(3S) - J/\Psi(1S)$ | 0.943 | 0.925 |
| $l = 1, j = 0, s = 1$ | $\xi_{c0}(1P) - J/\Psi(1S)$ | 0.308 | 0.372 |
| $l = 1, j = 1, s = 1$ | $\xi_{c1}(2P) - J/\Psi(1S)$ | 0.413 | 0.390 |
| $l = 1, j = 2, s = 1$ | $\xi_{c2}(3P) - J/\Psi(1S)$ | 0.459 | 0.422 |

We see that in average the mass spectrum predicted by our very simple model is in rather good accordance with the experimental data. It is possible to obtain a better agreement with experimental data changing the quark masses by effective ones which are additional free parameters.

6 Conclusion

We propose a relativistic Hamiltonian for an n-body problem in which all the masses are equal and all spins take value $1/2$. Discussing the problem in the frame of reference in which the total momentum $P = 0$, we were able to extend the FW transformation to n-body case and separate the positive energy part of the Hamiltonian. Examples of two body systems are discussed in more detail.

The proposed approach admits a straightforward generalization to the case of particles with different masses and spins and is valid for
more general form of the interaction potential. In the present paper we discuss only the main ideas and demonstrate it effectiveness using the simplest interaction model. Nevertheless, even this very straightforward model predicts a rather realistic bottomonium and charmonium spectra presented in the Appendix. It looks rather curiously that starting with an interaction potential which is not well grounded physically and using only two free parameters $\omega/m$ and $\alpha$ it is possible to obtain a good qualitative and also relatively good quantitative description of bottomonium and charmonium masses.

References

[1] K. T. Hecht. Quantum Mechanics, Shpringer (2000).

[2] P.A. M. Dirac. The principles of quantum mechanics, Oxford at the Clarendon Press, Third Edition (1947), pp. 252-274 and, in particular, pp. 268-271.

[3] Relativistic Action-at-Distance. Classical and quantum aspects. LNP, 162, Llosa Ed., Springer Verlag (1982)

[4] E. E. Salpeter and H. E. Bethe, Phys. Rev. 84 (1954) 1232.

[5] M. Gell-mann and F. Low, Phys. Rev. 84 (1954) 350.

[6] J. Bijtebier, J. Phys. G 26 (2000) 871-886.

[7] P. A. M. Dirac, Rev. Mod. Phys. 21 (1949) 392.

[8] Ph. Droz-Vincent, ROMP 8 (1975) 79, H. Leutwiller and J. Stern, Ann. Phys. 112 (1978) 94; L. Bel, Phys. Rev. D 28 (1983) 1308; H. Sadjian, Phys. Rev. D 33 (1986) 3401, Phys. Lett. B208 (1986) 470.
[9] Ph. Droz-Vincent, *Reduction of relativistic three-body cinematics*, hep-th 9905119 v.4, 2003.

[10] M. Moshinsky and G. Loyola, Foundations of Physics, **23**, (1993) 197.

[11] L. L. Foldy and S. S. Wouthuysen, Phys. Rev. **78** (1950) 29.

[12] J. D. Bjorken and S. D. Drell. *Relativistic Quantum Mechanics*. McGraw Gill Book Company, 1964, pp. 46-51.

[13] A. G. Nikitin, J. Phys. A **31** (1998) 3297;
    A. G. Nikitin and V. V. Tretynyk, Int. J. Mod. Phys. **12** (1997) 4369.

[14] M. Moshinsky and Yu. F. Smirnov. The harmonic oscillator in modern physics. Harwood Academic Publishers, 1996.

[15] Particle Data Group, Eur. Phys. Journ. C **15** (2000) 650.

[16] W. I. Fushchich and A. G. Nikitin. *Symmetries of Equations of Quantum Mechanics*. Allerton Press, N.Y., 1994.