Energy and Angular Momentum Densities of Stationary Gravity Fields

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Abstract

We give physical explanations of explicit invariant expressions for the energy and angular momentum densities of gravitational fields in stationary space-times. These expressions involve non-locally defined conformal factors. In certain coordinates these become locally defined in terms of the metric. These results are derived via expressions for total gravitational potential energy from the difference between the total energy and the mechanical energy. The latter involves kinetic energy seen in the frame of static observers.

When in the axially symmetric case we consider zero angular momentum observers (who move orthogonally to surfaces of constant time), we find that the angular momentum they attribute to the gravitational field is solely due to their motion.

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1. INTRODUCTION

The aim of this paper is to show that in the special case of stationary asymptotically flat spacetimes there are invariantly defined expressions for both the energy density of the gravitational field and for the angular momentum density. Furthermore, these expressions are the gravitational analogues of $(E^2 + B^2)/(8\pi)$ and $[\vec{r} \times (\vec{E} \times \vec{B})]/(4\pi)$ for electromagnetism. However the gravitational expressions do involve the gradient of a conformal factor which is defined non-locally using a geometrical structure depending on the Killing vector. We hope that our expressions can be generalized to slowly varying systems in general relativity. We do not aim to introduce any new generally defined concept of energy or quasi-local energy such as those of Penrose, Hawking or Hayward [1]. We are concerned to isolate the gravitational potential energy in the stationary case and reexpress it in terms of ‘fields’ whereas those quasi-local expressions contain material energy as well as gravitational energy and go over to the ADM mass at infinity.

Misner, Thorne and Wheeler (1973) [2], hereafter MTW, deny the existence of localised gravitational field energy-density (with somewhat strident rhetoric). Nevertheless they give an expression for the localised gravitational energy in the exceptional case of spherical symmetry (quoting Misner & Sharp (1964) [3] to include time dependence). Katz (2005) [4] has recently given an expression for it in the more general case of conformastationary metrics. As our expression (which agrees with his) differs from that of MTW even for the spherically symmetric case, we shall start by considering their special case in this introduction.

Landau & Lifshitz (1966) [5] show that the general spherically symmetric metric can be put in the form (setting $c^2 = 1$ and writing $\xi^2$ for their $e^\nu$)

$$
\begin{equation}
\frac{\partial m}{\partial r} = 4\pi G r^2 T^0_0, \text{ so } m(r, t) = m_\infty - \int_r^\infty 4\pi G r^2 T^0_0(r, t) dr \tag{1}
\end{equation}
$$
$T_0^0$ is the energy density of the matter in the rest frame $(r, \theta, \phi)$ constant. When $r = 2m(r, t)$ at some radius $r_*(t)$ a central black hole of that radius and mass $M_*(t) = r_*/2G = m_*/G$ is present. Such a case is considered in [6] but here we shall confine ourselves to global complete spacetimes.

To see exactly what equation (1) means consider for example a fluid of dust. Then $T^{\mu\nu} = \rho u^\mu u^\nu$ where $\rho$ is the density of the fluid in its rest frame and $u^\mu$ is its four velocity which is of the form $(u^0, u^r, 0, 0)$ by symmetry.

Let $w^\mu$ be the 4-velocity of static observers with $(r, \theta, \phi)$ constants, so these are unit vectors and in their coordinates have only one component $w^\mu = (\xi^{-1}, 0, 0, 0)$, $w_\mu = (\xi, 0, 0, 0)$. Since the fluid moves radially in our frame, they see it to have a greater density due to Lorentz contraction. This increased density of rest mass is $\rho u_\mu w^\mu \equiv \rho u \cdot w$ and the mechanical energy density in any frame now including kinetic energy is $\frac{\rho (u \cdot w)}{\sqrt{1 - v^2/c^2}}$. This is the time component of

$$\rho u \cdot w u^\nu = T^\nu_{\mu} w^\mu, \quad (2)$$

which is the flow vector of mechanical energy and momentum seen by the static observers.

The mechanical energy is its flux through a space-like hypersurface such as $t = \text{constant}$. The part of that flux through $t = \text{constant}$ that lies within $r = \text{constant}$ will be

$$E_M(r) = \int_0^r T^0_{\mu} w^\mu \sqrt{-g} \, d^3x = \int_0^r T^0_{\mu} 4\pi r^2 \left(1 - \frac{2m(r, t)}{r}\right)^{-1/2} \, dr. \quad (3)$$

$E_M(r)$ is a special case of the energy seen by observers moving orthogonally to a hypersurface like that geometrically defined by Wald [12]. Although for illustrative clarity we considered only dust above, this is still the mechanical energy within $r$ whatever constitutes the $T_{\mu\nu}$. Indeed for a perfect fluid it includes the internal energy as well as rest mass and kinetic energy,

$$T^0_0 = \frac{\rho + \rho v^2}{1 - v^2}, \quad (4)$$

where $v$ is the 3-velocity of the fluid in their frame.

A comparison between equations (1) and (3) illustrates that (1) is seductively like the classical relationship between density and mass but in fact conceals all the complications beneath a cloak of apparent clarity.

Firstly $T^0_0$ is not the density in these coordinates unless $v$ is zero; secondly $4\pi r^2 dr$ is not the volume element which is rather

$$4\pi r^2 [1 - 2m(r, t)/r]^{-1/2} dr = dV. \quad (5)$$
Outside the matter $m$ becomes constant and gives $G$ times the total mass-energy seen from infinity. MTW, quoting Misner & Sharp [2], argue that $M(r,t) = G^{-1}m(r,t)$ is the total mass-energy within $r$ at time $t$ and is only less than $E_M(r,t)$ because of $E_G(r,t)$ the gravitational energy which is negative. Indeed, since $E_M$ (equation (5)) contains the rest-mass energy, the kinetic energy and the internal energy within the matter, the difference

$$E_M(r,t) - G^{-1}m(r,t) = \int_0^r \left\{ \left[ 1 - \frac{2m(r,t)}{r} \right]^{-1/2} - 1 \right\} T_0^0 4\pi r^2 dr = -E_G(r,t)$$

must be due to gravitation.

This would lead to a gravitational energy density of

$$\frac{dE_G(r)}{dV} = - \left[ 1 - \left( 1 - \frac{2m(r,t)}{r} \right)^{1/2} \right] T_0^0 = - \left[ 1 - \left( 1 - \frac{2m(r,t)}{r} \right)^{1/2} \right] \left( \frac{1}{4\pi Gr^2} \right) \frac{\partial m}{\partial r},$$

which is only non-zero inside matter.

To see that this deduction might not be water-tight, we turn to the electrostatic analogue in flat space. The electrical energy of a spherical charge distribution, $Q(r)$ being within $r$, can be calculated by starting at the centre and imagining the distribution to be built up by adding shells of charge consecutively. The shell with charge $dQ$ is added when the potential at $r$ is $Q(r)/r$ so the electrical energy up to $r$ is

$$E_{em}(r) = \int_0^r \frac{Q(r)}{r} \frac{dQ}{dr} dr.$$

As further shells are added we merely extend the upper bound of the integral so the part within $r$ does not change. However it would be wrong to deduce that the energy density in the electric field at $r$ in flat space is

$$\frac{1}{4\pi r^2} \frac{dE_{em}(r)}{dr} = \frac{Q}{4\pi r^3} \frac{dQ}{dr}.$$

An alternative is found from the formula

$$E^*_{em}(r) = \int_0^r \frac{1}{2} \phi(dQ/dr) dr$$

where $\phi$ is the electrical potential $\phi = \int_r^\infty (Q/r^2) dr$. This formula takes account of the fact that the electrical potential changes from $Q/r$ as outer shells are added. However it is also wrong to imagine that the electrical energy density is $(4\pi r^2)^{-1}(dE^*_{em}/dr) = (8\pi r^2)^{-1}\phi(dQ/dr)$. Notice that both of the above formulae only give contributions from within the charge distribution $Q(r)$.
whereas the true answer due to Maxwell is \((8\pi)^{-1} \int E^2 dV = (8\pi)^{-1} \int (Q/r^2)^2 dV\). Of course all three expressions integrate over space to the same total electrical energy, what is in dispute is the distribution of that energy. This electrical analogy suggests that it might be appropriate to evaluate \((6)\) to infinity and integrate the result by parts. Then it might be possible to express the result as the volume integral of a perfect square which might be interpreted as in the electrical case.

Doing this we write \(x = 2m(r,t)/r\) and use \((1)\)

\[-GE_G = \int_0^\infty [(1-x)^{-1/2} - 1] \partial m/\partial r \, dr = \frac{1}{2} \int_0^\infty [(1-x)^{-1/2} - 1] \left(x + r \frac{\partial x}{\partial r}\right) \, dr \]

\[= \frac{1}{2} \int_0^\infty \left\{[(1-x)^{-1/2} - 1]x + r \frac{\partial}{\partial r} \left[2 - 2(1-x)^{1/2} - x\right]\right\} \, dr .\]

Integrating by parts and remembering that \(x \to 0\) as \(r \to \infty\) the two terms recombine to give us just such a perfect square integrated over the volume \(dV\) given by \((5)\) viz

\[-E_G = \int \frac{1}{8\pi G r^2} \left[1 - \left(1 - \frac{2m(r,t)}{r}\right)^{1/2}\right]^2 \, dV = \frac{1}{8\pi G} \int F^2 dV . \tag{7} \]

where we have introduced a gravitational field strength \(F\) given by

\[F = \frac{1}{r} \left[1 - \left(1 - \frac{2m}{r}\right)^{1/2}\right] = -(-g_{rr})^{1/2} \frac{\partial \Phi}{\partial r} ,\]

where \(\Phi\) is a gravitational potential. Equation \((7)\) may be seen as the gravitational analogue of Maxwell’s \(\int E^2 dV/(8\pi)\) but in the special case of spherical symmetry. For the geometrical interpretation of \(\Phi\) see below. An expression equivalent to \((7)\), but expressed in isotropic coordinates was given in [4]. Denoting by \(\to\) values at large \(r\),

\[F_r = -\partial \Phi/\partial r = \frac{1}{r} \left[1 - \left(1 - \frac{2m}{r}\right)^{1/2}\right] \to \frac{m}{r^2} .\]

Evidently

\[\Phi = \int_r^\infty \frac{1}{r} \left\{\left[1 - \frac{2m(r,t)}{r}\right]^{1/2} - 1\right\} \, dr \to \frac{m}{r} . \tag{8} \]

What physical meaning should be ascribed to this gravitational field and what is the source of \(\Phi\)?

\[\nabla^2 \Phi = \frac{1}{r^2} \left(1 - \frac{2m}{r}\right)^{1/2} \frac{\partial}{\partial r} \left[\left(1 - \frac{2m}{r}\right)^{1/2}r^2 \frac{\partial \Phi}{\partial r}\right] = -\frac{1}{r^2} \frac{\partial m}{\partial r} + \frac{1}{r^2} \left[\sqrt{1 - \frac{2m}{r}} - \left(1 - \frac{m}{r}\right)\right] ,\]
so
\[- \nabla^2 \Phi = \frac{1}{2} \left( \kappa T_0^0 - F^2 \right) = \frac{1}{2} \left[ \kappa T_0^0 - (\nabla \Phi)^2 \right], \tag{9} \]
where \( \kappa = 8\pi G \). This shows that the negative gravitational field energy acts alongside the positive \( T_0^0 \) as a source for \( \Phi \). In this last form it is natural to generalise (9) out of spherical symmetry and this is indeed what has already been done in [4] for conformastatic and conformastationary systems.

We now turn to the interpretation of \( \Phi \). All spherical spaces have conformally flat spatial metrics. If we rewrite our spatial metric in isotropic form,
\[ d\sigma^2 = e^{2\Lambda} (d\bar{r}^2 + \bar{r}^2 d\hat{r}^2), \]
then, at constant \( t \),
\[ \left( 1 - \frac{2m(r,t)}{r} \right)^{-1/2} dr = e^\Lambda d\bar{r} \]
and
\[ r = e^\Lambda \bar{r}, \]
so
\[ \left[ \left( 1 - \frac{2m}{r} \right)^{-1/2} - 1 \right] \frac{dr}{r} = d\ln \bar{r} - d\ln r = d\ln \left( \frac{r}{\bar{r}} \right). \]
Hence, taking \( \frac{\bar{r}}{r} \to 1 \) at infinity, we find, cf equation (8),
\[ e^{-\Lambda} = \frac{\bar{r}}{r} = \exp \left\{ - \int_r^\infty \left[ \left( 1 - \frac{2m}{r} \right)^{-1/2} - 1 \right] \frac{dr}{r} \right\} = e^{-\Phi}. \tag{10} \]
So \( \Phi \) is \( \Lambda \) and \( e^{2\Phi} \) is \(-g_{11}\) in isotropic coordinates. For time dependent spherical metrics \( r \) is geometrically defined as a real radius and the surfaces of constant \( t \) are geometrically defined as those symmetric cuts of spacetime orthogonal to \( r \).

If we make the conformal transformation \( d\bar{\sigma}^2 = e^{-2\Phi} d\sigma^2 \), then
\[ d\bar{\sigma}^2 = (d\bar{r}^2 + \bar{r}^2 d\hat{r}^2). \tag{11} \]

Thus the interpretation of \( \Phi \) for the spherically symmetric case is that \( e^{-2\Phi} \) is the conformal factor of the transformation which makes the spatial metric flat. In general, spatial metrics of general spaces cannot be made flat, most of those that can have already been explored in [4]. It is worth noticing that the energy (7) or (3) depends solely on the spatial metric, \( \xi \) is not involved.
Although we can not generally make a conformal transformation to give a flat 3-space, we can in practice make a conformal transformation to a 3-space which has no scalar 3-curvature. This turns out to be a crucial step in determining the invariant energy density of general stationary gravitational fields.

In Katz, Lynden-Bell & Bičák (2006) [6] this technique is employed to give energy densities according to both static and hypersurface orthogonal observers, such as ZAMOs (Bardeen (1970) [7]).

Here we give a more physical exposition of that work and extend it to include angular momentum density. This leads us to a new and different expression for gravitational field energy $\mathcal{V}$.

For stationary space times we write our metric in the alternative forms

$$ds^2 = \xi^2(dt - A_k dx^k)^2 - \gamma_{kl} dx^k dx^l = g_{\mu\nu} dx^\mu dx^\nu = \zeta^2 dt^2 - \tilde{\gamma}_{kl}(dx^k - W^k dt)(dx^l - W^l dt)$$

where Greek indices run from 0 to 3 and latin ones from 1 to 3. Evidently

$$A_k = -\frac{g_{0k}}{g_{00}} \quad \text{and} \quad \xi^2 A_k = -g_{0k} = -\tilde{\gamma}_{kl} W^l = -\xi_k$$

where $\xi^\mu$ is the ‘stationary’ Killing vector. We define

$$\gamma = \det(\gamma_{kl})$$

and the equivalent for $\tilde{\gamma}$.

Landau and Lifshitz [5] show that for a given spacetime there are many stationary metrics of the form (12) since an arbitrary function of the $x^k$ can be added to $t$ without destroying the stationary form of the metric. Thus there are many different slicings of the given spacetime into time and space. However, as discussed in [6], we can get a unique geometrically defined slicing by demanding that it be maximal in the sense that the trace of the the external curvature, $K$, of the constant time slices be zero. This choice of time slicing is clearly a good one in that it picks out Boyer-Lindquist time in Kerr spacetime. We shall thereafter make this choice of the time coordinate so that the space on each constant time slice is a well defined geometrical concept.
II. GRAVITATIONAL FIELD ENERGY DENSITIES

In [4] it was realised that the definition of mechanical energy used in spherical symmetry by MTW [1] could be extended to any stationary space-time. If $w^\mu = \xi^\mu / \xi$, then the density of rest mass in a dust fluid seen by a static observer is $\rho = \sqrt{1 - v^2}$ and the energy density on any spacelike surface element $d\Sigma_\mu$ is $\rho (u^\mu w_\nu - \gamma_{\mu\nu}) = T_{\mu\nu}^0 w_\nu \sqrt{-g} d\Sigma_\mu$ [12]. If we have a more complicated $T_{\mu\nu}$ such as a gas or a plasma this last expression still measures the mechanical energy density as seen by the static observers but it now includes internal energy (including rest mass energy) and the kinetic energy of motion relative to the observers. If we sum these local contributions we get the total mechanical energy with no contribution from the gravitational binding energy

$$E_M = \int T_{\mu\nu}^0 w_\nu \sqrt{-g} d\Sigma_\mu = \int T_0^0 dV, \quad \text{where} \quad dV = \sqrt{\gamma} d^3 x. \quad (14)$$

We notice that this expression agrees with that given for spherical symmetry (3) and (6). However, this is not generally the same as Wald’s expression because $\xi^\mu$ and $w^\mu$ are not generally hypersurface orthogonal, so are not normal to the hypersurface. When black holes are absent we may now define the total gravitational energy of any stationary space-time as

$$E_G = M - E_M, \quad \text{where} \quad M$$

is the total mass. We get an interesting expression for $E_G$ by using Einstein’s equations to transform $T_{\mu\nu}^0$ into ‘field’ quantities via integrations by parts analogously to Maxwell’s treatment in electrodynamics.

We write $\nabla$ for the vector operator in the 3-space with the $\gamma_{kl}$ metric and then follow Lynden-Bell & Nouri-Zonoz (1998) [8] (see also Natário (2000) [9]) in writing $B^k = \eta^{ijk} \partial_j A_k$, where $\eta^{ijk} = (\sqrt{\gamma})^{-1} \epsilon^{ijk}$. The 4-vector $B^\lambda$ is invariently defined by $B^\lambda = -\eta^{\lambda\mu\nu} \partial_\mu (\xi_\nu / \xi^2) \xi_\sigma \xi = (0, B^k)$. Then $B^2 = \frac{1}{2} \partial_j A_k \nabla^j A^k$ where square brackets around indices denote the anti-symmetric part. Landau & Lifshitz [5] denote $\partial_i A_j - \partial_j A_i$ by $f_{ij}$ so, translating their expression of Einstein’s equations to our notation and writing $\mathcal{E} = -\nabla \ln \xi$, we get

$$\xi^2 \left( \xi^{-1} \nabla^2 \xi + \frac{1}{2} \xi^2 B^2 \right) = R_{00}, \quad (15)$$

$$- \xi^{-1} \nabla^k \nabla^l \xi + \frac{1}{2} \xi^2 (\gamma^{kl} B^2 - B^k B^l) + P^{kl} = R^{kl}, \quad (16)$$

$$- \frac{1}{2} \xi^2 [\nabla \times B - 3 \mathcal{E} \times B]^k = R_0^k, \quad (17)$$

where $P^{kl}$ is the curvature tensor of the $\gamma_{kl}$ 3-space formed from $\gamma_{kl}$ just as $R^{\mu\nu}$ is formed from $g_{\mu\nu}$ in 4-space. Now $R = g_{\mu\nu} R^{\mu\nu} = (w_\mu w_\nu - \gamma_{\mu\nu}) R^{\mu\nu} = \xi^{-2} R_{00} - \gamma_{kl} R^{kl}$, where
\( \gamma_{\mu\nu} = w_{\mu}w_{\nu} - g_{\mu\nu} \) is a 4-dimensional covariant version of \( \gamma_{kl} \). From (15) and (16)

\[
2\xi^{-1}\nabla^2\xi - \frac{1}{2}\xi^2B^2 - P = R,
\]

from (15) and (18)

\[
\kappa T_{00} = R_{00} - \frac{1}{2}\xi^2R = \xi^2\left(\frac{3}{4}\xi^2B^2 + \frac{1}{2}P\right).
\]

Now in the introduction we found that an important step in making the field energy a perfect square was the introduction of a conformal transformation which yielded a flat 3-space. We can not do that generally but we can transform so that our new 3-space has a vanishing scalar curvature. We write \( \gamma_{kl} = e^{-2\Phi}\gamma_{kl} \) and use the relationships between the curvature of two conformally related spaces given, e.g., in Stephani et al. (2003) [10], equation 3.85 (contracted)

\[
e^{-2\Phi}\overline{P} = P + 4\nabla^2\Phi - 2(\nabla\Phi)^2.
\]

So the transformation that makes \( \overline{P} \) zero obeys

\[
-2\nabla^2\Phi + (\nabla\Phi)^2 = \frac{1}{2}P,
\]

which we may rewrite like a Schrodinger equation by setting \( \overline{\psi} = e^{-\frac{1}{2}\Phi} \) so that

\[
\nabla^2\overline{\psi} - \frac{1}{8}P\overline{\psi} = 0.
\]

Equation (20) is the generalisation of equation (9) of the introduction and using (19) takes the form

\[
-\nabla^2\Phi = \frac{1}{2}\left[\xi^{-2}\kappa T_{00} - \frac{3}{4}\xi^2B^2 - |\nabla\Phi|^2\right],
\]

which shows how both gravomagnetic and gravitational field strengths subtract from the ‘source’ of \( \Phi \). However previously that source was written in terms of \( \kappa T_0^0 \) not \( \kappa T_{00} \) and since our expression (14) for mechanical energy also involves \( \kappa T_0^0 \) we now rewrite our basic equations (15)–(18) in terms of \( \kappa T_0^0, \kappa T_k^k \) and \( \kappa T^{kl} \). To do this we use equations (19) and (17) and aim to get only second derivative terms on the left \( \kappa T_0^0 = \xi^{-2}\kappa T_{00} + A_k R_0^k \), so

\[
\frac{3}{4}\xi^2B^2 + \frac{1}{2}P - \frac{1}{2}\xi^2A \cdot (\nabla \times B - 3E \times B) = \kappa T_0^0.
\]

Using (20) for \( P \) and \(-\frac{1}{2}\xi^2A \cdot \nabla \times B = \frac{1}{2}\nabla \cdot (\xi^2A \times B) - \frac{1}{2}B \cdot \nabla \times (\xi^2A) \), we find on simplifying the right-hand side of (24)

\[
\nabla \cdot \left(\frac{1}{2}\xi^2A \times B - 2\nabla\Phi\right) \equiv \nabla \cdot \mathbf{D} = \kappa(T_0^0 + \rho_G),
\]
where

\[ \kappa \rho G = -\frac{1}{4} \xi^2 B^2 - |\nabla \Phi|^2 - \frac{1}{2} \xi^2 A \cdot (E \times B) ; \quad (25) \]

and from (17)

\[ (\nabla \times B)^k = -2\kappa \xi^{-2} T^k_0 + 3(E \times B)^k . \quad (26) \]

Notice that all quantities on the right of (25) are defined in terms of a Killing vector and (the square of the gradient of) a conformal factor which, being the solution of the ‘invariant’ elliptic equation (20), depends only on the slicing, its geometry and on the boundary conditions.

Equations (24) and (26) may be thought of as a non-linear generalisation of Maxwell’s equations and indeed we shall see presently that the terms of (25) constitute the energy density of the gravitational field.

The final equation of this trio comes from (16) and (18) and reads

\[ \xi^{-1}(\gamma^{kl} \nabla^2 \xi - \nabla^k \nabla^l \xi) = \kappa T^{kl} - \left( P^{kl} - \frac{1}{2} \gamma^{kl} P \right) - \frac{1}{2} \xi^2 \left( \frac{1}{2} \gamma^{kl} B^2 - B^k B^l \right) . \quad (27) \]

Now in making our conformal transformation we recognise that our space metric will tend to the Schwarzschild form at infinity with \( m = GM \) being the Schwarzschild asymptotic mass parameter. Since Schwarzschild space is conformally 3-flat we may impose the boundary condition \( \Phi \to O(1/r) \) on the solution \( \Phi \) of (20). Then \( \Phi \) tends to the corresponding Schwarzschild value \( m/r \) found in equation (8) of the introduction as \( r \to \infty \), that is we identify the coefficient of \( 1/r \) as the total mass parameter. When we integrate equation (24) over all space so as to generate the mechanical energy \( E_M \) from the first term on the right, we find on the left

\[ \int \nabla \cdot \left( \frac{1}{2} \xi^2 A \times B - 2 \nabla \Phi \right) dV = \int \left( \frac{1}{2} \xi^2 A \times B - 2 \nabla \Phi \right) \cdot dS, \]

where the \( dS \) integral is to be evaluated over the sphere at infinity still assuming no black hole is present. Now \( B \) is \( O(1/r^3) \) there, the triad component of \( A \) will be \( O(1/r^2) \) like the classical vector potential and \( \xi^2 \to 1 \) so the integral of the first term on the right vanishes; however \( \Phi \to m/r \), so the second term gives \( 8\pi m = \kappa M \). Using this result, the integration of (24) over all space yields

\[ E_G = M - E_M = \int \rho G dV , \quad (28) \]
with $\rho_G$ given in (23).

As KLB [6] pointed out the mechanical and gravitational energy densities defined above fail for systems with ergospheres because they involve $w^\mu = \xi^\mu/\xi$ in their definition and $\xi$ is zero on the ergosphere. However such difficulties can be circumvented by using the mechanical energy as estimated by hypersurface orthogonal observers such as Bardeen’s ZAMOs. We take such observers moving orthogonally to surfaces of constant time with 4-velocities $\tilde{w}^\mu = \zeta^{-1}(1, W^k)$. The appropriate form of metric is in terms of lapse and shift written as the final expression of (12). The metric components and their inverses were given in equations 2.37 and 2.38 of KLB [6] and we notice that $W^k = \xi^k$:

$$g_{00} = \zeta^2 - W^2, \quad g_{0l} = W_l = \tilde{\gamma}_{kl} W^k, \quad g_{kl} = -\tilde{\gamma}_{kl},$$

$$g^{00} = \zeta^{-2}, \quad g^{0l} = \zeta^{-2} W^l, \quad g^{kl} = -\tilde{\gamma}^{kl} + \zeta^{-2} W^k W^l,$$

$$\sqrt{-g} = \zeta \sqrt{\tilde{\gamma}}, \quad \tilde{\gamma} = \text{det}\tilde{\gamma}_{kl}. \quad (29)$$

The Einstein equations are

$$\kappa T^{00} = G^{00} = \frac{1}{2} \zeta^{-2} (\tilde{P} + K^2 - K^{kl} K_{kl}), \quad (30)$$

$$\kappa T^k_l = G^k_l = -\zeta^{-1} \tilde{\nabla}_l (K^k_l - \delta_l^k K), \quad (31)$$

$$- \kappa T = R = 2 (\zeta^2 G^{00} - \tilde{\gamma}^{kl} R_{kl}), \quad (32)$$

$$\kappa T_{kl} = R_{kl} = \tilde{P}_{kl} + \tilde{\nabla}_{(k} \tilde{E}_{l)} - \tilde{E}_k \tilde{E}_l + 2 \zeta K^m_{(l} \nabla_{[m} W_{l)} - \zeta^{-1} \tilde{E}_l W^m K_{kl} + \nabla_m (W^m K_{kl}), \quad (33)$$

where $\tilde{P}_{kl}$ is the Ricci tensor of the spatial metric $\tilde{\gamma}_{kl}$, $\tilde{P} = \tilde{\gamma}^{kl} \tilde{P}_{kl}$ is the corresponding 3-scalar curvature and $K_{kl}$ is the second fundamental form of the hypersurface $t =$constant. Thus

$$K_{kl} = \zeta^{-1} \tilde{\nabla}_k (W_l), \quad K = K^k_k = \tilde{\gamma}^{kl} K_{kl}, \quad \text{and} \quad \tilde{E}_k = -\partial_k \ln \zeta. \quad (34)$$

The mechanical energy density on any hypersurface $\Sigma$ of normal $n^\mu$ is for dust $\rho(u.n)^2$ so for our observers moving orthogonally to the cut $n^\mu = \tilde{w}^\mu$ and for general $T^{\mu\nu}$ the mechanical energy on an element of hypersurface $\sqrt{-g} d^3x$ is $T^{\mu\nu} \tilde{w}_\nu \sqrt{-g} d^3x = \zeta^2 T^{00} \sqrt{\tilde{\gamma}} d^3x$. This is given directly by equation (30). Once again $\tilde{P}$ is determined by using a conformal transformation $\tilde{\Phi}$ such that the transformed $\tilde{\gamma}_{kl}$ has vanishing 3-curvature. Analogously to (20) this leads to

$$-2 \tilde{\nabla}^2 \tilde{\Phi} + |\tilde{\nabla} \tilde{\Phi}|^2 = \frac{1}{2} \tilde{\Phi}. \quad (35)$$
Transporting second derivative terms to the left this yields in place of \((30)\)

\[
\kappa \zeta^2 T^{00} + 2 \nabla^2 \Phi = |\nabla \Phi|^2 + \frac{1}{2} (K^2 - K_{kl} K^{kl}) .
\]  

(36)

Integrating and using the boundary condition that \(\nabla \Phi \to (GM/r^2) \hat{r}\) at infinity we find for maximal slices \((K = 0)\)

\[
- \tilde{E}_G = \tilde{E}_M - M = \frac{1}{\kappa} \int \left[ |\nabla \Phi|^2 - \frac{1}{2} K_{kl} K^{kl} \right] dV .
\]  

(37)

The mechanical energy density \(\tilde{E}_M\) has real advantages over \(E_M\). Not only can it be measured within the ergosphere but also the two ‘\(\gamma\)’ factors \((u.n)\) are the same and correspond to what a hypersurface orthogonal observer would see. It also coincides with the geometrical expression of Wald alluded to earlier. However there are strong arguments against it also. The observers move relative to static observers and worse still they move relative to each other. As seen from infinity they circulate and for the axially symmetric case with azimuthal Killing vector \(\eta^\mu\) they rotate about the axis at angular velocity \(\omega = \xi.\eta/(\eta.\eta)\) which depends on position. In this case the space-time cuts correspond to surfaces of constant Boyer-Lindquist time. Now in classical physics observers who rotate at angular velocity \(\Omega\) see as their energy not the true energy \(E\) relative to observers at rest at infinity but rather the Jacobi constant \(E - \Omega \cdot J\), where \(J\) is the total angular momentum. For observers such as ZAMOs in differential rotation we expect \(E - j \int \omega dV\), where \(j\) is the angular momentum density about the symmetry axis. It is interesting that our expression for \(\tilde{E}_M\) takes just such a form (writing \(d\tilde{V} = \sqrt{\gamma} d^3x\)):

\[
\tilde{E}_M = \int T^{00} \tilde{w}_\nu\sqrt{-g} d\Sigma_\mu = \int T^{00}_\nu \zeta^\nu d\tilde{V} = \int T^{00}_\nu (\tilde{\zeta}^\nu + \omega \eta^\nu) d\tilde{V} = \mathcal{T} - \int \omega j d\tilde{V},
\]  

(38)

where \(\mathcal{T} = \int T^{00}_0 \sqrt{\gamma} d^3x\) is the part of the mechanical energy independent of the angular velocity of the observer and \(j = -T^{00}_\nu \eta^\nu\) is the angular momentum density. If, following this line of thought, we identify \(\mathcal{T}\) with the mechanical energy rather than \(\tilde{E}_M\) then we need to evaluate \(T^{00}_0\) rather than \(T^{00}\). Einstein’s equations in mixed form follow from \((30)\) and \((31)\) by writing \(G^0_0 = (g^{00})^{-1}(G^{00} - g^{0k} G^0_k)\); hence using \((35)\) and putting all second derivatives on the left

\[
\kappa T^{00}_0 + 2 \nabla^2 \Phi - \nabla_l \left[ \zeta^{-1} W_k (K^{kl} - \zeta^{kl} K) \right] = |\nabla \Phi|^2 + \frac{3}{2} (K^2 - K^{kl} K_{kl}) - \zeta^{-1} W_k K^l_k - \delta^l_k K \tilde{E}_l .
\]  

(39)
On integration the third term vanishes over the sphere at infinity and in the absence of black holes we have, setting $K = 0$ for a maximal $t =$ constant slice,

$$ -V = T - M = \frac{1}{\kappa} \int \left[ |\nabla \tilde{\Phi}|^2 - \frac{3}{2} K^{kl} K_{kl} - \zeta^{-1} W_k K^{kl} \tilde{E}_l \right] d\tilde{V}, $$

which expresses the new gravitational potential energy $V$ in terms of ‘field’ quantities.

Advantages of $T$ as a mechanical energy are:

(i) like $\tilde{E}_M$ it can be evaluated for systems with ergospheres;

(ii) like $E_M$ it involves $m \mathbf{u} \cdot \mathbf{\xi}$ which is the energy for a dust particle, even for one that darts into and out from the ergosphere.

(iii) It has removed the part of $\tilde{E}_M$ which is related to the circulation of the ZAMOs around the axis.

However, even for dust it is not clear that it can be related to kinetic energy as seen by any chosen observers.

### III. MECHANICAL ANGULAR MOMENTUM DENSITIES

For axially symmetrical systems there is no difficulty in defining total angular momentum. It is $-\int T^\mu_\nu \eta' \sqrt{-g} \, d\Sigma_\mu$ and this translates into $-\int T^\mu_\nu \eta' \zeta \, d\tilde{V}$, or, for stationary systems without ergospheres $-\int T^\mu_\nu \eta' \xi \, dV$. However, whereas the split between mechanical energy and gravitational energy was clear, the split between mechanical angular momentum and field angular momentum caused us difficulty. Luckily considerable enlightenment comes from first studying the electrodynamic analogue in flat space, so to this we now turn. We consider a charged rotating fluid held together by some cohesive force such as surface tension; a charged oil drop might be a good example. We wish to split the total angular momentum into a part due to the mechanics of the fluid itself and an electromagnetic part. We start by considering a single particle of mass $m$ and charge $q$. Its 3-velocity will be $\mathbf{v}$ or in cylindrical polar coordinates $(\dot{R}, R\dot{\phi}, \dot{z})$, but that is written in local triad language; a relativist would use the 3-metric $d\sigma^2 = dR^2 + \gamma_{\phi\phi} d\phi^2 + dz^2$ and write the 3-velocity as $v^k = (\dot{R}, \gamma_{\phi\phi} \dot{\phi}, \dot{z})$ or $v_k = (\dot{R}, \gamma_{\phi\phi} \dot{\phi}, \dot{z})$, where $\gamma_{\phi\phi}$ is just $R^2$. As recorded by static observers the mechanical angular momentum of a particle is just $m \gamma_{\phi\phi} (d\phi/d\tau)$, where $d\tau = dt \sqrt{1 - v^2}$; however this is not the total angular momentum associated with the particle because it is charged and moves in an electromagnetic field. Its Lagrangian will be $-m \sqrt{1 - v^2} + q \mathbf{v} \cdot \mathbf{A} - q A_0$ where $\mathbf{A}$ is
the electromagnetic vector potential and \( A_0 \) is the electrostatic potential. Thus \( \partial L / \partial \dot{\phi} = p_\phi \) will have two pieces: \( m\gamma_{\phi\phi} \dot{\phi} / \sqrt{1 - \dot{v}^2} + qA_\phi \). The first is the mechanical angular momentum \( m\gamma_{\phi\phi}(d\phi/d\tau) \), considered above; the second is a piece of electromagnetic angular momentum that is to some extent associated with the motion of the particle and its charge, but it also depends on all the other charges that act together to give rise to the vector potential \( A \).

Notice this piece of momentum is not generally gauge invariant because we have not yet said anything about fixing the electromagnetic gauge. It is actually incorrect to believe that the sum over all particles of all their \( p_\phi \) gives the total angular momentum. It does not! The trouble arises just because the \( A_\phi \) is generated by other particles in the same assemblage — we encounter a similar problem when adding individual particle energies in which the electrical potential is included. Even classically the straight sum counts the electrostatic energy twice. What then is the correct procedure in the electrodynamic case? The answer is to separate the mechanical angular momentum which is additive and gauge invariant. We sum this over all the particles. Then quite separately we work out the total electromagnetic angular momentum from the gauge invariant expression \( J_{em} = \frac{1}{4\pi} \int [r \times (E \times H)] \cdot \hat{z} d^3x = \int M_\phi^0 \eta^\phi d^3x \), where \( M_\nu^\mu \) is the Maxwell stress-energy-momentum of the electromagnetic field and \( \eta^\mu \) is the angular Killing vector, so \( \eta^\phi = 1 \). When we sum \( J_M + J_{em} \) we find that we do indeed get the total angular momentum. This is obvious since the total \( T^{\mu\nu} \) is the sum of the mechanical and the electromagnetic energy momentum tensors. This final more obvious option is not open to us when we deal with the gravitational field’s angular momentum, as there is no known stress energy tensor for it; however, as we now show, the earlier argument for splitting the mechanical angular momentum from the field part can be followed in the gravitational case to which we now return.

We shall again consider a single particle but now it will be uncharged and in a stationary metric as in (12), i.e.,

\[
ds^2 = \xi^2(dt - A_k dx^k)^2 - \gamma_{kl}dx^kdx^l,
\]

but we shall take this metric to be axially symmetric and, to start with, we consider the still simpler case in which there is a \( \phi \to -\phi, \ t \to -t \) symmetry. In the latter case we write \( A_k dx^k = A_{,\phi} d\phi + A_{,L} dx^L \) and

\[
\gamma_{kl}dx^kdx^l = \gamma_{\phi\phi} d\phi^2 + \gamma_{KL} dx^K dx^L,
\]

where \( K \) and \( L \) run from 2 to 3 whereas \( k, l \) run from 1 to 3 and Greek suffices run from 0
IV. MECHANICAL ANGULAR MOMENTUM FROM STATIC OBSERVERS

To keep contact with directly observable quantities we introduce our set of static observers (see Section II). The speed of the fluid, \( v \), past the observers is given as \( (1 - v^2)^{-1/2} = w^\mu u_\mu = (\mathbf{w} \cdot \mathbf{u}) \) and the components of \( \frac{\mathbf{v}}{\sqrt{1-v^2}} \) are given by \( \mathbf{u} - (\mathbf{u} \cdot \mathbf{w}) \mathbf{w} \). Hence the mechanical momentum of the particle, unconnected with its field momentum, will be \( m[\mathbf{u} - (\mathbf{u} \cdot \mathbf{w}) \mathbf{w}] \) and the corresponding mechanical angular momentum is \( m(\mathbf{u} - (\mathbf{u} \cdot \mathbf{w}) \mathbf{w}) \cdot \mathbf{\eta} \), \( \mathbf{\eta} \) is the axial Killing vector. Since only the component of \( \mathbf{\eta} \) transverse to \( \xi_\mu \) is involved in this product, this angular momentum can also be re-expressed as \( m u^\mu \gamma_{\mu\nu} \eta^\nu \), where \( \gamma_{\mu\nu} = w_\mu w_\nu - g_{\mu\nu} \).

If we consider a stress-energy tensor made up of dust, then \( T_{\mu\nu} = \rho u^\mu u^\nu \) and \( \rho u^\mu \) is the rest mass flux vector; multiply by \( u^\nu \gamma_{\nu\sigma} \eta^\sigma \), the mechanical angular momentum per unit rest mass, and \( \rho u^\mu u^\nu \gamma_{\nu\sigma} \eta^\sigma = T^{\mu\nu} \gamma_{\nu\sigma} \eta^\sigma \) is the flux of mechanical angular momentum. We shall now find this by another method directly analogous to our electrodynamic calculation.

The Lagrangian for a free particle moving in the metric (41) is
\[
L = -m[\xi^2(1 - A_\phi \dot{\phi} - A_K \dot{x}^K)^2 - \gamma_{\phi\phi} \dot{\phi}^2 - \gamma_{KL} \dot{x}^K \dot{x}^L]^{1/2} = -m \frac{ds}{dt}.
\]
Its total angular momentum will be \( p_\phi = \partial L / \partial \dot{\phi}, \) so
\[
p_\phi = m[\xi^2(1 - A_\phi \dot{\phi} - A_K \dot{x}^K)A_\phi + \gamma_{\phi\phi} \dot{\phi}] / (ds/dt) ,
= m\xi^2(u^0 - A_\phi \dot{\phi} - A_K \dot{u}^K)A_\phi + m\gamma_{\phi\phi} \dot{\phi} .
\]
Now \( u^\mu = (dt/ds, dx^k/ds) \), so \( u_0 = \xi^2(u^0 - A_\phi \dot{\phi} - A_K \dot{u}^K) \) and \( (1 - v^2)^{-1/2} = w^\mu u_\mu = \xi^{-1} u_0, \) so reversing the order of the terms we get \( p_\phi = m\gamma_{\phi\phi} \dot{\phi} + m\xi(1 - v^2)^{-1/2} A_\phi \). This expression is in precise correspondence with the electromagnetic case with the mechanical angular momentum in the first term and the gauge dependent field term coming afterwards. When the system does not have \( \phi \rightarrow -\phi, t \rightarrow -t \) as a symmetry, there are \( d\phi dx^K \) terms in \( \gamma_{kl} dx^k dx^l \), and the form of \( p_\phi / m \) becomes
\[
p_\phi / m = \gamma_{\phi k} u^k + \xi(1 - v^2)^{-1/2} A_\phi .
\]
The mechanical angular momentum per unit mass is the first term and to get the angular momentum flux vector we multiply this by the rest mass flux \( \rho u^\mu \), so the mechanical angular momentum flux remains \( T^{\mu k} \gamma_{k\phi} = T^{\mu\nu} \gamma_{\nu\phi} = T^{\mu\nu} \gamma_{\nu\sigma} \eta^\sigma \).
Although for easy explanation we have adopted the simplest dust case to explain our points, nevertheless our final formulae still hold whatever the constitution of $T^{\mu\nu}$.

The total gravitational field angular momentum reckoned by static observers is the difference between the total angular momentum and the total mechanical angular momentum, so for a spacelike surface $\Sigma$

$$J_{GS} = J - J_{MS} = \int -T^{\mu\nu}(g_{\nu\sigma} + \gamma_{\nu\sigma})\eta^{\sigma}\sqrt{-g}d\Sigma_{\mu} = \int -T^{\mu\nu}w_{\nu}w_{\sigma}\eta^{\sigma}\sqrt{-g}d\Sigma_{\mu} = -\int T_{0}^{0}\xi A_{\phi}dV.$$ 

Now we have established this as the angular momentum they attribute to the gravitational field our next aim is to re-express it in terms of the squares of field quantities by using Einstein’s equations for $T_{\mu\nu}$ and performing integrations by parts. Multiplying (24) by $\xi A_{\phi}$, integrating over all space we find on smuggling the $\xi A_{\phi}$ into the divergence and then paying the duty

$$\int \xi A_{\phi}D \cdot dS - \int \nabla(\xi A_{\phi}) \cdot DdV = \kappa \left( \int T_{0}^{0}\xi A_{\phi}dV + \int \rho G \xi A_{\phi}dV \right).$$

Now $A_{\phi}$ is $O(1/r)$ and $D$ is $O(1/r^2)$, so the first term vanishes when integrated over the sphere at infinity while the first term on the right gives $-J_{GS}$. Hence,

$$J_{GS} = \int \left[ \rho G \xi A_{\phi} + \frac{1}{\kappa} \nabla(\xi A_{\phi}) \cdot D \right]dV,$$

where $D$ is given in (24) and $\rho G$ in (25). When $A$ has only a $\phi$ component, $\nabla_{k}A_{\phi} = -\eta_{\phi km}B^{m}$, so then $D \cdot \nabla \phi = -(D \times B)_{\phi}$ and the final term becomes \( \frac{\kappa}{\kappa}[-(D \times B)_{\phi} - D \cdot E A_{\phi}] \).

V. MECHANICAL ANGULAR MOMENTUM WITH RESPECT TO ZAMO’S

The ZAMOs have a different concept of what constitutes space than the static observers and that concept is superior in that it extends within ergospheres. It is natural to replace the $w_{\nu}$ of the static observers in equation (14) by the $\tilde{w}_{\nu} = n_{\nu}$ of the ZAMOs. Indeed if we calculate $p_{\phi}$ in the appropriate ZAMOs coordinates, we find

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = m\tilde{\gamma}_{\phi k}(u^{k}W^{k}u^{0}),$$

and it is just the $u^{k}$ term that the ZAMOs consider to be the mechanical angular momentum with respect to them. Their estimate of mechanical angular momentum is therefore (with $\phi$ unsummed)

$$\tilde{J}_{M} = \int T_{\mu\nu}\tilde{\gamma}_{\nu\sigma}\eta^{\sigma}\sqrt{-g}d\Sigma_{\mu}.$$
But $J = -\int T^{\mu\nu}g_{\nu\sigma}\eta^\sigma \sqrt{-g}d\Sigma_\mu$ and $\eta^\sigma$ has only a $\phi$ component and since $\tilde{\gamma}_{\mu\nu} = -g_{\mu\nu} + \tilde{w}_\mu \tilde{w}_\nu$, it follows that $\tilde{J}_M - J = \int T^{\mu\nu}\tilde{w}_\nu \tilde{w}_\sigma \eta^\sigma \sqrt{-g}d\Sigma_\mu = 0$ because $\tilde{w}_\sigma \eta^\sigma = 0$.

Thus the total angular momentum is the same as the material angular momentum as seen in the local rest frames of the ZAMOs.

The mechanical angular momentum defined via static observers cannot be assessed for systems with ergospheres. The mechanical angular momentum defined via ZAMOs takes no account of their circulation around the axis. Is there a splitting into mechanical and field angular momentum that takes account of the ZAMOs motion but does not break down within ergospheres? Something like the split we found for energy by writing $\zeta^\mu = \xi^\mu + \omega \eta^\mu$ is needed.

For a dust fluid the angular momentum is $J = -\int \rho u^\mu u^\nu \eta^\nu \sqrt{-g}d\Sigma_\mu = -\int \rho u^0 u^\nu g_{\nu\phi}d\tilde{V}$. Now $u^\nu g_{\nu\phi} = u^0 g_{0\phi} + u^k g_{k\phi} = u^0 W_\phi - u^k \tilde{\gamma}_{k\phi}$. The first term on the right is due to the circulation of the ZAMOs, the last is most readily seen as an angular momentum per unit mass of the fluid when $u^k$ has only a $\phi$-component. Thus, following Bardeen (1970) [7], $J$ can be split as follows

$$J = -\int T^{00}W_\phi \zeta d\tilde{V} + \int T^{0k} \tilde{\gamma}_{k\phi} \zeta d\tilde{V}.$$  

For those who want this split expressed covariantly the first ‘field’ term may be rewritten

$$J_f = -\int T^{\mu\nu} \tilde{w}_\nu \zeta^{-1}(\xi, \eta) \sqrt{-g}d\Sigma_\mu.$$  

To evaluate the field angular momentum we rewrite (36) in a form suitable for constructing the required integral (setting $K^2 = 0$ for a maximal slice):

$$\kappa T^{00} W_\phi + 2\tilde{\nabla} \cdot (\zeta^{-1} W_\phi \tilde{\nabla} \Phi) = \zeta^{-1} W_\phi |\tilde{\nabla} \Phi|^2 + 2\tilde{\nabla} (\zeta^{-1} W_\phi) \cdot \tilde{\nabla} \Phi - \frac{1}{2} \zeta^{-1} W_\phi K_{kl}K^{kl} = -\kappa j_f.$$  

Integrating, we have (after division by $-\kappa$)

$$J_f = \int j_f d\tilde{V},$$

where $j_f$ is defined in equation (43). $\zeta$ and $W_\phi$ can be expressed in terms of the Killing vectors $\xi$ and the angular one $\eta$ via $W_\phi = \xi \cdot \eta$ and $\zeta^2 = \xi^2 - (W_\phi^2/\eta^2)$.  

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VI. CONCLUSIONS

We have elucidated the physics of three different expressions for the energy and angular momentum densities of gravitational fields. While the authors differ as to the relative importance of these expressions DL-B favours the third.

None of these densities make contributions to $T_{\mu\nu}$ although they do act as (negative) sources for the conformal field $\Phi$ which gives the scalar curvature of 3-space.

The fact that they do not contribute to the source $T_{\mu\nu}$ was one of MTW’s strongest arguments for the non-existence of a stress-energy tensor for gravity. That argument still stands. We have conducted our investigation within General Relativity. A different theory might have the field energy contributing to the source $T_{\mu\nu}$ but that is not General Relativity. Energy densities are only really useful when the system changes, whereas this paper is confined to stationary systems. Nevertheless, we believe these methods may be generalised to systems that change so slowly that no gravitational waves are emitted.

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[12] Here $d \Sigma_{\mu} = \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma$, $\epsilon_{\mu \nu \rho \sigma}$ is the permutation symbol in 4 dimensions with $\epsilon_{0123} = 1$ and in 3 dimensions it is $\epsilon_{klm}$ with $\epsilon_{123} = 1$. 