Dynamical stability of running solitary waves in fluid-filled elastic membrane tubes

A T Il’ichev\textsuperscript{1,2}

\textsuperscript{1}Steklov Mathematical Institute, Gubkina str. 8, 119991 Moscow, Russia
\textsuperscript{2} National Research Nuclear University “MePHI”, Kashirskoye sh. 31, 115409 Moscow, Russia

E-mail: ilichev@mi.ras.ru

Abstract. We examine the problem of stability of solitary waves, propagating in a fluid-filled membrane tube. We consider waves with speeds starting from those given by the linear dispersion relation (it is known that there may exist four families of solitary waves having such speeds), i.e. the waves of a finite amplitude bifurcating from the quiescent state of the system. It is shown that if a solitary wave speed is bounded away from zero the solitary wave itself is orbitally stable, when the fluid initially stationary (the mean flow is absent).

1. Introduction

The governing equations for quasi-one dimensional motion of the perfect fluid in the axisymmetric membrane tube were obtained in [1] by means of straightforward derivation. Study of spectral stability of a branch of steady solitary-wave solutions (so-called aneurysm solutions) in the absence of the fluid inside the tube (pressure-controlled case) is given in [2]. A bifurcation parameter was the inflation pressure, and the authors found that all family of solitary waves is always spectrally unstable (i.e. a perturbation of a wave form exponentially grows with time). In [3] stability of the whole branch aneurysm solutions is studied when the fluid inside the tube is present, but a mean flow (a constant speed of the fluid at infinity) is zero. It was found there that the aneurysm is still unstable, but the presence of fluid has a strong stabilizing effect. The authors of [4] undertook a stability analysis of aneurysm solution in the presence of the mean flow and found that if a speed of the fluid at infinity is bounded away from zero, then the aneurysm is spectrally stable.

In the present paper we examine the problem of stability of solitary waves, propagating with a non-zero speed in a fluid-filled axisymmetric membrane tube. In this case a bifurcation parameter is not the inflation pressure any more (it may take arbitrary values), but a solitary wave speed. We consider only waves with speeds close to those given by the linear dispersion relation, i.e. the waves of a small (but finite) amplitude bifurcating from the quiescent state of the system. In other words we adopt weakly nonlinear description of solitary waves.

2. Formulation of the problem

We model the tube as incompressible, isotropic, hyperelastic, cylindrical membrane. The tube has a constant undeformed radius $R$ and a constant undeformed thickness $H$. The tube is assumed to be infinitely long, and end conditions are imposed at infinity. We use cylindrical coordinates, and undeformed configuration is given by coordinates $R, \Theta, Z$. 
We assume that the axisymmetry remains throughout the entire deformation; the deformed configuration is expressed using cylindrical polar coordinates \( r, \theta, z \), where \( r = r(Z,t), \theta = \theta(Z,t), z = z(Z,t) \), and \( t \) denotes time.

The principal directions of the deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface, and the principal stretches are given by

\[
\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r^2 + z^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H},
\]

where the indices 1, 2, 3 are used for the circumferential, axial and radial directions respectively, a prime represents differentiation with respect to \( Z \), and \( h \) denotes the deformed thickness.

The principal Cauchy stresses \( \sigma_1, \sigma_2, \sigma_3 \) in the deformed configuration for an incompressible material are given by

\[
\sigma_i = \lambda_i W_i - p, \quad i = 1, 2, 3 \quad \text{(no summation)},
\]

where \( W = W(\lambda_1, \lambda_2, \lambda_3) \) is the strain-energy function, \( W_i = \partial W / \partial \lambda_i \), and \( p \) is a Lagrange multiplier, associated with the constraint of incompressibility. Utilizing the incompressibility constraint \( \lambda_1 \lambda_2 \lambda_3 = 1 \) and the membrane assumption of no stress through the thickness direction \( \sigma_3 = 0 \), we find

\[
\sigma_i = \lambda_i \dot{W}_i, \quad i = 1, 2
\]

where \( \dot{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1} \lambda_2^{-1}) \) and \( \dot{W}_1 = \partial \dot{W} / \partial \lambda_1 \) etc. The pipe material has a density \( \rho \).

As an example we give here frequently used three strain-energy functions, the Varga, Ogden and Gent materials, given respectively by,

\[
W = 2\mu (\lambda_1 + \lambda_2 + \lambda_3 - 3), \quad (2)
\]

\[
W = \mu \sum_{r=1}^{3} \mu_r (\lambda_1^r + \lambda_2^r + \lambda_3^r - 3)/\alpha_r, \quad (3)
\]

\[
W = -\frac{1}{2} \mu J_m \ln (1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_m}), \quad (4)
\]

where \( \mu \) is the shear modulus for infinitesimal deformations, \( J_m > 0 \) is a material constant representing the maximum stretch of the material and \( \alpha_1 = 1.3, \alpha_2 = 5.0, \alpha_3 = -2.0, \mu_1 = 1.491, \mu_2 = 0.003, \mu_3 = -0.023 \). The Ogden and Gent materials were proposed in [6] and [7] respectively, and are popularly used to model rubber.

The pressure in the ideal fluid filling tube is denoted by \( P \), the fluid itself has a velocity \( v_f \) equal to \( v_{f,\infty} \) at infinity, the density of the fluid is denoted by \( \rho_f \).

The equations of motion for the tube can be derived using the linear momentum balance applied to an infinitesimal material form of tube's wall [1]. We shall non-dimensionalize the governing equations using the following scales: \( R \) for \( Z, z \) and \( r \), \( \mu \) for the Cauchy stresses, \( \mu / (R \bar{H}) \) for \( P \), \( \sqrt{\rho / \mu} \) for \( v_f \), and \( R \sqrt{\rho / \mu} \) for the time. Using the same notation for the scaled variables, we have

\[
\left[ \frac{\sigma_2 z' \lambda_2'}{\lambda_2} \right]' - P r r' = \ddot{z}, \quad \left[ \frac{\sigma_2 r'}{\lambda_2} \right]' - \frac{\sigma_1}{\lambda_1} + P r z' = \ddot{r}, \quad (5)
\]
where
\[ z = z_\infty Z + u(Z, t), \quad r = r_\infty + w(Z, t), \quad u, w \to 0, \text{ as } Z \to \infty, \]
and dot denotes the differentiation with respect to dimensionless time.

Expressed in terms of the Lagrangian coordinate \( Z \) and the time \( t \) the dimensionless equations of motion for the fluid inside the tube read [1], (for derivation, see [3, 4])
\[ \dot{r}z' - r' \dot{z} + v_f r' + \frac{1}{2} rv'_f = 0, \quad b_f [\dot{v}_f z' - v'_f \dot{z} + v_f v'_f] + P' = 0, \]
where \( b_f \) is defined by
\[ b_f = \frac{\rho_f R}{\rho H}. \]

The governing equations (5) and (6) admit the uniform solution
\[ r = r_\infty, \quad z' = z_\infty = \lambda_2, v_f = v_f, P = P_\infty = \frac{W_1(c, r_\infty, \lambda_2)}{r_\infty \lambda_2}. \]

We look for a general localized traveling wave solution for which the dependence on \( Z \) and \( t \) is through \( Z - c t \), where \( c \) denotes the wave speed of the wave. Localization means that as \( Z - c t \to \pm \infty \), the fluid-filled tube is in a uniform state given by (7).

We shall also need to refer to the dispersion relation for small-amplitude traveling waves superimposed on the uniform state 7. Assuming that the small-amplitude perturbations are proportional to then the scaled wavenumber \( k \) and wave speed \( c = \hat{c} \lambda_2 \) satisfy the dispersion relation [4]
\[ (k^2 m + 2) c^4 - 4v_f c^3 - (m \alpha_0 k^2 + m \gamma_1 k^2 - 2v_f c^2 - m \beta_0 + m \beta_1 + 2 \gamma_1) c^2 + 4v_f \gamma_1 c - 2v_f \gamma_1 + m \gamma_1 (k^2 \alpha_0 + \beta_1 - \beta_0) - m (\alpha_1 - \beta_0)^2 = 0, \]
where \( m = 1/(b_f r_\infty^2 \lambda_2) \), and the expressions for the quantities \( \alpha_0, \alpha_1, \gamma_1, \beta_0, \beta_1 \) are given in [5] (with remarks in [4]). When \( v_f = 0 \), the above equation becomes a bi-quadratic on \( c \) at the limit \( k \to 0 \), and the roots can be written as
\[ 4c^2 = 2 \gamma_1 + m (\beta_1 - \beta_0) \pm \sqrt{[2 \gamma_1 - m (\beta_1 - \beta_0)^2 + 8m (\alpha_1 - \beta_0)^2}, \]
so that the four roots are all real. These characteristic speeds correspond to four branches of long waves: two of them propagate to the right, the other two propagate symmetrically to the left.

It can be easily shown [1] that the fluid equations (6) in this case can be integrated to yield
\[ P = P_\infty + v_f \left(1 - \frac{r_\infty^4}{r^4}\right), \quad v_f = \frac{v_f r_\infty^2}{r^2}, \]
where the constant \( v_f \) is defined by
\[ v_f = \frac{1}{2} b_f (v_f - c)^2, \quad c = \hat{c} \lambda_2. \]

It is also known [1, 4] that the equations in (5) together with (10) have two integrals. They are given by
\[ W - \lambda_2 W_2 + \frac{1}{2} c^2 \lambda_2^2 = C_1, \quad \frac{W_2 z'}{\lambda_2} - \frac{1}{2} P' r^2 - c^2 z' = C_2, \]
where a prime again denotes differentiation with respect to \( Z - \hat{c} t \),
\[ P' = P_\infty + v_f \left(1 + \frac{1}{r_\infty^4}\right), \]
and the constants \( C_1 \) and \( C_2 \) can be determined by evaluating the corresponding left hands at the uniform state (7).

3
3. Running solitary waves

The two equations (11) can be written as a system of first-order ordinary differential equations [8, 9]:

\[
\begin{align*}
\lambda_1' &= \lambda_2 \sin \phi, \\
\lambda_2' &= \frac{W_1 - \lambda_2 W_{12}}{W_2 - \hat{c}^2} \sin \phi, \\
\phi' &= \frac{W_1}{W_2 - \hat{c}^2 \lambda_2} \cos \phi - \frac{P \lambda_1 \lambda_2}{W_2 - \hat{c}^2 \lambda_2},
\end{align*}
\]

where the prime denotes differentiation with respect to \( \xi = Z - \hat{c} t \), \( \phi \) is the angle between the meridian and the \( z \)-axis (so that \( \sin \phi = r'/\lambda_2 \), \( \cos \phi = z'/\lambda_2 \)). Without loss of generality, we may assume that the center of the symmetric localized travelling wave is located at \( \xi = 0 \) so that \( \phi(0) = 0 \). Then if \( \lambda_1(0) \) and \( \lambda_2(0) \) are also known, the solitary wave solution can be determined by integrating the above system as an initial value problem.

According to (9) at fixed values of \( r_\infty, z_\infty \) we determine the bifurcation value of the speed \( c \) and compute numerically a solitary wave solution for \( c \) varying. Hence, we obtain the solitary wave family, parameterized by \( c \). There are two such families for solitary waves running in each direction, and these families begin with corresponding bifurcation values of \( c \) and end with some finite values of \( c \).

4. Spectral stability

We denote such fully nonlinear bulging solutions of (3) by \( r = \bar{r}(\xi), z = \bar{z}(\xi), P = \bar{P}(Z), v_f = \bar{v}_f(\xi) \). To study their stability, we consider axisymmetric perturbations, and write

\[
\begin{align*}
\bar{r}(\xi, t) &= \bar{r}(\xi) + \Phi(\xi)e^{\eta t}, \\
\bar{z}(\xi, t) &= \bar{z}(\xi) + \Phi(\xi)e^{\eta t}, \\
\bar{P}(\xi, t) &= \bar{P}(\xi) + \Pi(\xi)e^{\eta t}, \\
\bar{v}_f(\xi, t) &= \bar{v}_f(\xi) + V(\xi)e^{\eta t},
\end{align*}
\]

where the mode functions \( \Phi(\xi), \Phi(\xi), \Pi(\xi), V(\xi) \) and the growth rate \( \eta \) are to be determined. The equations for perturbations can be written in the form

\[
y' = \mathcal{M}y,
\]

where dot denotes differentiation with respect to \( \xi \), \( y = (\Phi, \Phi', \Psi, \Psi', \Pi, V)^T \) and \( \mathcal{M} \) is a 6 \( \times \) 6 matrix whose components are (numerically) known functions of \( \xi \) and \( \eta \). This system of equations are to be solved subject to the decay conditions \( y \to 0 \) as \( Z \to \pm \infty \). Denoting by \( \mathcal{M}_{\infty} \) the limit of \( \mathcal{M} \) as \( \xi \to \pm \infty \), and substituting a trial solution of the form \( y = e^{\hat{k}\xi} \mathbf{r} \) into \( y' = \mathcal{M}_{\infty} y \), we obtain the eigenvalue problem

\[
(\mathcal{M}_{\infty} - \hat{k} I) \mathbf{r} = \mathbf{0},
\]

where \( I \) is the 6 \( \times \) 6 identity matrix. Denote by \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \) the three eigenvalues of (2) with negative real parts, and by \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) the associated right eigenvectors. There then exist solutions \( y_i(\xi) \) and of (1) such that

\[
\lim_{\xi \to \infty} e^{-\hat{k}_i \xi} y_i(\xi) = \mathbf{r}_i, \quad i = 1, 2, 3.
\]

Alternatively, we may consider the exterior system [10]

\[
y^{\wedge'} = \mathcal{M}^{\wedge} y^{\wedge},
\]
and its adjoint system
\[ \mathbf{x}^{\wedge\prime} = -\mathbf{x}^{\wedge}(\mathcal{M}^{\wedge}), \] (16)

To construct the Evans function we need to solve (15) and (16) subjected to the conditions at both infinities
\[ \lim_{\xi \to \infty} e^{-k^{\wedge}\xi} \mathbf{y}^{\wedge}(\xi) = \mathbf{r}^{\wedge}, \]
\[ \lim_{\xi \to \text{infty}} e^{k^{\wedge}\xi} \mathbf{x}^{\wedge}(\xi) = \mathbf{l}^{\wedge}, \] (17)

where \( k^{\wedge} = \hat{k}_1 + \hat{k}_2 + \hat{k}_3, \mathbf{r}^{\wedge} \) and \( \mathbf{l}^{\wedge} \) are right and left eigenvectors of \( \mathcal{M}^{\wedge}_{\infty} \) associated with the eigenvalue \( k^{\wedge} \), correspondingly. The Evans function is defined by
\[ D(\eta) = \mathbf{x}^{\wedge}(\eta, \xi) \cdot \mathbf{y}^{\wedge}(\eta, \xi), \] (18)

and it can be shown that the condition of existence of an unstable eigenvalue \( \eta_0 \) is equivalent to \( D(\eta_0) = 0 \).

The problem of finding the unstable discrete spectrum \( \eta \) (eigenvalues located in the right half complex \( \eta \)-plane \( \Omega^+ \)) is analogous to determining the zeroes of the Evans function \( D(\eta) \) lying in \( \Omega^+ \). The number of zeroes of \( D(\eta) \) can be computed with the help of the argument principle. The number of zeroes of \( D(\eta) \) in \( \Omega^+ \) is determined by the number of rotations of the image of the imaginary axis \{ \eta : \eta = ir, \tau \in \mathbb{R} \} under the mapping \( D(\cdot) \). Therefore, we need to construct the function \( D(\eta) \) on the imaginary axis, i.e., numerically solve the ordinary linear equations (3), (4) under the conditions (5) with \( \eta \) running in a considerably large segment of the imaginary axis [11].

Our numerical computations showed that for bifurcation values of \( c \) increasing from zero two existing unstable real eigenvalues in \( \Omega^+ \) converge, at some value \( c = c_0 \) coincide, and for \( c > c_0 \) cease to exist, so all solitary wave family, bifurcating from \( c \) is spectrally stable. This corresponds to the nonlinear orbital stability result of solitary waves of small amplitude [12], i.e. “beginners” of the respective family of solitary waves.

Acknowledgments
This work was supported by the Russian Science Foundation, project no. 16-19-00188.

References
[1] Epstein M and Johnston C On the exact speed and amplitude of solitary waves in fluid-filled elastic tubes 2001 Proc. Roy. Soc. Lond. A 457 1195-213
[2] Pearse S P and Fu Y B Characterization and stability of localized bulging/necking in inflated membrane tubes 2010 IMA J. Appl. Math. 75 581-602
[3] Il’ichev A T and Fu Y B Stability of aneurysm solutions in a fluid-filled elastic membrane tube 2012 Acta Mech. Sin. 28 1209-18
[4] Fu Y B and Il’ichev A T Localized standing waves in a hyperelastic membrane tube and their stabilization by a mean flow 2015 Math. Mech. Solids 20 1198-214
[5] Fu Y B and Il’ichev A T Solitary waves in fluid-filled elastic tubes: existence, persistence, and the role of axial displacement 2010 IMA J. Appl. Math. 75 257–68
[6] Ogden R W Large deformation isotropic elasticity-on the correlation of theory and experiment for incompressible rubber-like solids 1972 Proc. Roy. Soc. Lond. A 326 565–84
[7] Gent A N A new constitutive relation for rubber 1996 Rubber Chem. Nechnol. 69 59–61
[8] Fu Y B, Pearce S P and Liu K K Post-bifurcation analysis of a thin-walled hyperelastic tube under inflation 2008 Int. J. Non-linear Mech. 43 697–706
[9] Fu Y B and Xie Y X Effects of imperfections on localized bulging in inflated membrane tubes 2012 Phil. Trans. R. Soc. A 370 1896–911
[10] Alexander J C and Sachs R Linear instability of solitary waves of a Boussinesq-type equation: A computer assisted computation 1995 Nonlinear World 2 471–507
[11] Pego R L, Smereka P and Weinstein M I Oscillatory instability of traveling waves for a KdV-Burgers equation 1993 Physica D 67 45–65

[12] Il’ichev A T Stability of solitary waves in membrane tubes: A weakly nonlinear analysis 2017 Theoret. and Math. Phys. 193 1593–601