On the Yamabe problem on contact Riemannian manifolds

Wei Wang · Feifan Wu

Abstract
A contact Riemannian manifold, whose complex structure is not necessarily integrable, is the generalization of the notion of a pseudohermitian manifold in CR geometry. The Tanaka–Webster–Tanno connection plays the role of the Tanaka–Webster connection for a pseudohermitian manifold. Conformal transformations and the Yamabe problem are also defined naturally in this setting. By using special frames and normal coordinates on a contact Riemannian manifold, we prove that if the complex structure is not integrable, the Yamabe invariant on a contact Riemannian manifold is always less than the Yamabe invariant of the Heisenberg group. So the Yamabe problem on a contact Riemannian manifold is always solvable.

Keywords The Yamabe problem · Contact Riemannian manifolds · The Yamabe functional · Asymptotic expansion · Almost complex structure · The Tanno tensor

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Supported by National Nature Science Foundation in China (Nos. 11171298, 11571306).

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1 Introduction

The Yamabe problem in Riemannian geometry was completely solved during 60–80 s (cf. [2,16,18,22,28] and references therein). For the analogous CR Yamabe problem, Jerison and Lee proved [12,13] that there is a numerical CR invariant $\lambda(M)$ called the Yamabe invariant, and for any compact, strictly pseudoconvex $(2n+1)$-dimensional CR manifold $M$, it is always less than or equal to the Yamabe invariant $\lambda(\mathcal{H}^n)$ of the Heisenberg group $\mathcal{H}^n$. Furthermore if $\lambda(M) < \lambda(\mathcal{H}^n)$, then $M$ admits a pseudohermitian structure with constant scalar curvature. In [14], Jerison and Lee proved that $\lambda(M) < \lambda(\mathcal{H}^n)$ holds if $n > 2$ and $M$ is not locally CR equivalent to $\mathbb{S}^{2n+1}$. The remaining case was solved by Gamara and Yacoub [7,8]. The purpose of this paper is to solve the Yamabe problem on general contact Riemannian manifolds.

A $(2n+1)$-dimensional manifold $(M, \theta)$ is called a contact manifold if it has a real 1-form $\theta$ such that $\theta \wedge d\theta^n \neq 0$ everywhere on $M$. $\theta$ is called a contact form. There exists a unique vector field $T$, the Reeb vector field, such that $\theta(T) = 1$ and $T \cdot d\theta = 0$. It is known that given a contact manifold $(M, \theta)$, there exist a Riemannian metric $h$ and a $(1, 1)$-tensor field $J$ on $M$ such that

$$h(X, T) = \theta(X), \quad J^2 = -Id + \theta \otimes T, \quad d\theta(X, Y) = h(X, JY),$$

(1.1)

for any vector field $X, Y$ (cf. [5,21]). Such a metric $h$ is said to be associated with $\theta$, and $J$ is called an almost complex structure. Given a contact form $\theta$, $J$ is uniquely determined once $h$ is fixed, and vice versa. $(M, \theta, h, J)$ is called a contact Riemannian manifold. $HM := \text{Ker}(\theta)$ is called the horizontal subbundle of the tangent bundle $TM$. On a contact Riemannian manifold, there exists a distinguished connection called the Tanaka–Webster–Tanno connection $\nabla$ (or TWT connection briefly). In the CR case, this connection is exactly the Tanaka–Webster connection (cf. [20] and [26]). Tanno constructed this connection for general contact Riemannian manifolds in [21]. Since there is no obstruction to the existence of an almost complex structure $J$, contact Riemannian structures exist naturally on any contact manifold and analysis on them has potential application to geometry of contact manifolds (cf., e.g., [17,19] and [27]).

Let $\mathbb{C}TM$ be the complexification of $TM$. $\mathbb{C}TM$ has a unique subbundle $T^{(1,0)}M$ such that $JX = iX$ for any $X \in \Gamma(T^{(1,0)}M)$. Here and in the following, $\Gamma(S)$ denotes the space of all smooth sections of a vector bundle $S$. Set $T^{(0,1)}M = \overline{T^{(1,0)}M}$. Then for any $X \in T^{(0,1)}M$, $JX = -iX$. $J$ is called integrable if

$$[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subset \Gamma(T^{(1,0)}M).$$

(1.2)
If \( J \) is integrable, \( J \) is called a CR structure and \((M, \theta, h, J)\) is called a pseudohermitian manifold. By [21], the integrable condition holds if and only if the Tanno tensor \( Q = \nabla J = 0 \). In general, a contact Riemannian manifold is not a CR manifold.

Under a conformal transformation of a contact Riemannian manifold, which is given by \( \tilde{\theta} = f\theta \), for some positive function \( f \), we have the transformation formula \((\theta, J, T, h) \rightarrow (\tilde{\theta}, \tilde{J}, \tilde{T}, \tilde{h}) \) given by

\[
\tilde{T} = \frac{1}{f}(T + \xi),
\]
\[
\tilde{h} = fh - f(\theta \otimes \omega + \omega \otimes \theta) + f(f - 1 + ||\xi||^2)\theta \otimes \theta,
\]
\[
\tilde{J} = J + \frac{1}{2f}\theta \otimes (\nabla f - T(f)T),
\]
(c.f. [5, (12)] or [21, Lemma 9.1]), where \( \xi = \frac{1}{2T}J\nabla f \) and \( \omega \) satisfies \( \omega(X) = h(X, \xi) \) for \( X \in TM \).

The contact Riemannian Yamabe problem is that given a compact contact Riemannian manifold \((M, \theta, h, J)\), find \( \tilde{\theta} \) conformal to \( \theta \) such that its scalar curvature is constant. It is known that (cf. [21] and [5, p. 337]) if we write the conformal transformation \( \tilde{\theta} = f\theta \) with \( f = u^{2/n} \), the scalar curvature \( \tilde{R} \) of the contact Yamabe flow for \( u \) is the volume form. A solution to the contact Yamabe flow for \( K \)-contact manifolds.

The Yamabe functional is defined as

\[
\mathcal{Y}_{\tilde{\theta}, h}(u) = \frac{\int_M (p|du|^2_H + Ru^2)dV_\theta}{(\int_M u^p dV_\theta)^{2/p}}, \quad p = b_n = 2 + \frac{2}{n},
\]

where \( |du|^2_H \) is the norm of the horizontal part of \( du \) and \( dV_\theta \) is the volume form. A solution to the Yamabe problem is a critical point of the Yamabe functional \( \mathcal{Y}_{\tilde{\theta}, h} \). The Yamabe invariant is defined as

\[
\lambda(M) = \inf_u \mathcal{Y}_{\tilde{\theta}, h}(u).
\]

Equivalently,

\[
\lambda(M) = \inf \{ A_{\tilde{\theta}, h}(u) : B_{\tilde{\theta}, h}(u) = 1 \},
\]

where \( A_{\tilde{\theta}, h}(u) = \int_M (b_n|du|^2_H + Ru^2)dV_\theta, B_{\tilde{\theta}, h}(u) = \int_M |u|^p dV_\theta \). Our main result in this paper is

**Theorem 1.1** Suppose \((M, \theta, h, J)\) is a compact contact Riemannian manifold of dimension \( 2n + 1 \) with \( n \ge 2 \). If the almost complex structure \( J \) is not integrable, then \( \lambda(M) < \lambda(\mathcal{H}^n) \).

Here we require \( n \ge 2 \) because the almost complex structure on a 3-dimensional contact Riemannian manifold is automatically integrable in the sense of (1.2). It is proved in [25] that the Yamabe invariant of a compact contact Riemannian manifold is always less than or equal to \( \lambda(\mathcal{H}^n) \), and if the invariant is strictly less than \( \lambda(\mathcal{H}^n) \), the Yamabe problem has a solution.
**Corollary 1.1** If the contact Riemannian manifold $M$ is not integrable, then the infimum (1.6) is attained by a positive $C^\infty$ solution $u$ to (1.4). Thus the contact form $\tilde{\theta} = u^{n-2}\theta$ has constant scalar curvature $R \equiv \lambda(M)$.

It is well known that the function
\[
\Phi(z, t) = \frac{1}{|w + i|^{n}}, \quad w = t + i|z|^2, \quad z \in \mathbb{C}^{2n}, \quad t \in \mathbb{R},
\]
(1.8)
is an extremal for the Yamabe functional on the Heisenberg group (cf. [13]). For each $\varepsilon > 0$, $\Phi^\varepsilon := \varepsilon^{-n}\delta^i_1 \Phi = \varepsilon^n|w + i\varepsilon^2|^{-n}$ is also an extremal. As in the CR case [14], we use test functions
\[
f^\varepsilon(z, t) = \psi(w)\Phi^\varepsilon(z, t),
\]
to calculate the asymptotic expansion for $\mathcal{Y}_{0,h}(f^\varepsilon)$ as $\varepsilon \to 0$, where $\psi \in C^\infty_0(M)$ is supported in the set $\{|w| < 2\kappa\}$ and $\psi(w) = 1$ for $|w| < \kappa$ for $\kappa > 0$.

To solve the CR Yamabe problem, Jerison and Lee [14] constructed pseudohermitian normal coordinates by parabolic geodesics and parabolic exponential map. On a contact Riemannian manifold, we also have parabolic geodesics analogous to the CR case. For a normal coordinates by parabolic geodesics and parabolic exponential map. On a contact

\[
\text{a general contact Riemannian manifold, the complex structure is not preserved under the parallel translation, and so $T^{(1,0)}M$ and $T^{(0,1)}M$ are preserved. But on a general contact Riemannian manifold, the complex structure is not preserved under the parallel translation. Namely, the special frame $\{W_a\}_{a=1}^n$ is not a $T^{(1,0)}M$ frame even if it is a basis of $T_q^{(1,0)}M$ at point $q$. This is our main difficulty.}

Since the Yamabe invariant (1.5) is conformally invariant, we can choose a conformal factor so that conformal change of contact Riemannian structure causes many tensor invariants to vanish, namely at the center of normal coordinates, some Ricci type and torsion tensors vanish. This will simplify our calculation greatly.

With the help of normal coordinates, following the method in [14], asymptotic expansion of $\mathcal{Y}_{0,h}(f^\varepsilon)$ can be calculated explicitly by using certain invariants at the origin. These invariants are constructed by the curvature, Webster torsion and Tanno tensors. In the CR case, besides the first term, the first nonzero term of the Yamabe functional $\mathcal{Y}_{0,h}(f^\varepsilon)$ is $O(\varepsilon^4)$, which depends only on the Weyl scalar conformal curvature. Because our frame $\{W_a\}$ is not holomorphic or anti-holomorphic, our expansion of $\mathcal{Y}_{0,h}(f^\varepsilon)$ is much more complicated than that in the CR case. Notably, we have to expand the almost complex structure $J$ asymptotically near $q$, while in the CR case $J$ is constant. But fortunately, if the Tanno tensor is non-vanishing at point $q$, the second-order term of the Yamabe functional $\mathcal{Y}_{0,h}(f^\varepsilon)$ is already nonzero. This makes calculation much easier than we expected. Moreover, because we expand the functional only up to the order 2, we do not need the scalar invariant theory.

The Tanno tensor plays an important role in the analysis of contact Riemannian manifolds (see also [27]). In Sect. 3, we construct the invariant
\[
\Omega = \sum_{\alpha, \beta, \gamma} Q^\varepsilon_{\alpha\beta}(q)^2,
\]
where $Q_{\alpha\beta}$ is components of the Tanno tensor with respect to a special frame. The Tanno tensor is nonzero at point $q$ if and only if $\Omega$ is strictly positive at this point. In Sect. 4, as in the CR case (cf. [14, Sect. 3]), for a fixed contact form $\theta$, we can make certain components of the Ricci curvature tensor and the Webster torsion tensor vanish at point $q$ after a suitable conformal transformation. This will make our calculation easier. Notably,

$$R(q) = -\frac{\Omega}{2}$$

(cf. (5.20)), while in the CR case, we can make $R(q) = 0$ after a suitable conformal transformation. This makes the second term in the numerator of the Yamabe functional (1.5) negative in the non-integrable case, and so the Yamabe invariant becomes smaller compared to the CR case.

In Sect. 5, we calculate asymptotic expansion for $\mathcal{Y}(f^\varepsilon)$ explicitly. By the preparation in Sects. 3 and 4, we finally find

$$\mathcal{Y}(f^\varepsilon) = \lambda(\mathcal{K}^n) \left(1 - \frac{3n - 1}{12(n - 1)n(n + 1)} \Omega \varepsilon^2 \right) + O(\varepsilon^3).$$

(1.9)

So if the complex structure is not integrable, we prove the main theorem.

In “Appendix A”, we discuss the transformation formulae for connection coefficients, the Webster torsion and curvature tensors under a conformal transformation. In “Appendix B”, we give details of calculation of second-order terms of the Yamabe functional $\mathcal{Y}(f^\varepsilon)$.

One of motivations of this work is the Yamabe problem on quaternionic contact manifolds (cf. [10,11,23]), on which there exist 3 not necessarily integrable complex structures. But it was solved recently by Ivanov and Petkov [9] for non-spherical quaternionic contact manifolds by asymptotic expansion of the Yamabe functional. Another interesting problem is the Yamabe problem for differential forms on Riemannian manifolds [24].

## 2 Normal coordinates

### 2.1 The TWT connection

**Proposition 2.1** (cf. (7)–(9) in [5]) On a contact Riemannian manifold $(M, \theta, h, J)$, there exists a unique linear connection $\nabla$ such that

$$\nabla \theta = 0, \quad \nabla T = 0, \quad \nabla h = 0,$$

$$\tau(X, Y) = 2d\theta(X, Y)T, \quad X, Y \in \Gamma(HM),$$

$$\tau(T, JZ) = -J\tau(T, Z), \quad Z \in \Gamma(TM),$$

for $X, Y \in \Gamma(TM)$, where $\tau$ is the torsion of $\nabla$, i.e., $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

This connection is called the **TWT connection**. The $(1, 2)$-tensor field $Q$ defined by

$$Q(X, Y) := (\nabla_Y J)X, \quad X, Y \in \Gamma(TM),$$

(2.2)

is called the **Tanno tensor** (cf. [5, (10)]). Tanno proved that a contact Riemannian manifold is a CR manifold if and only if $Q \equiv 0$ (cf. [21, Proposition 2.1]). The curvature tensor of TWT connection is $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. The Ricci tensor of the TWT connection is defined by $Ric(Y, Z) = tr\{X \mapsto R(X, Z)Y\}$, for any $X, Y, Z \in TM$. The scalar curvature is $R = tr(Ric)$. 

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We extend $h$, $J$ and $\nabla$ to the complexified tangent bundle trivially by $\mathbb{C}$-linear extension
\begin{equation}
\begin{aligned}
    h(X_1 + iY_1, X_2 + iY_2) &:= h(X_1, X_2) - h(Y_1, Y_2) + i(h(X_1, Y_2) + h(X_2, Y_1)), \\
    J(X_1 + iY_1) &:= JX_1 + iJY_1, \\
    \nabla_{(X_1+iY_1)}(X_2 + iY_2) &:= \nabla X_1 X_2 - \nabla Y_1 Y_2 + i(\nabla X_1 Y_2 + \nabla Y_1 X_2),
\end{aligned}
\end{equation}
for any $Z_j = X_j + iY_j \in \mathbb{C}TM$, $j = 1, 2$.

**Remark 2.1** The Riemannian metric $h$, the almost complex structure $J$, the TWT connection, the torsion and curvature tensors are preserved under the complex conjugation, i.e.,
\begin{align*}
    \overline{h(Z_1, Z_2)} &= h(\overline{Z_1}, \overline{Z_2}), \\
    J\overline{Z_1} &= JZ_1, \\
    \nabla_{\overline{Z_1}}Z_2 &= \nabla_{Z_1}\overline{Z_2}, \\
    \tau(Z_1, Z_2) &= \tau(\overline{Z_1}, \overline{Z_2}), \\
    R(Z_1, Z_2)Z_3 &= R(\overline{Z_1}, \overline{Z_2})\overline{Z_3},
\end{align*}
for any $Z_1, Z_2, Z_3 \in \mathbb{C}TM$.

### 2.2 The structure equation

**Notation 2.1** In this paper, we adopt the following index conventions:
\begin{align*}
    a, b, c, d, e, \ldots &\in \{1, 2, \ldots, 2n\}, \\
    j, k, l, r, s, \ldots &\in \{0, 1, \ldots, 2n\}, \\
    \alpha, \beta, \gamma, \rho, \lambda, \mu, \ldots &\in \{1, \ldots, n\}, \\
    \tilde{\alpha} &= \alpha + n.
\end{align*}

The order of index $j$ is defined to be $o(j) = 2$ if $j = 0$, and $o(j) = 1$ otherwise. For a multi-index $J = (j_1, \ldots, j_s)$, we denote $J^s = s$, $o(J) = o(j_1) + \cdots + o(j_s)$, $x^J = x^{j_1} \cdots x^{j_s}$, $Z_J = Z_{j_1} \cdots Z_{j_s}$, and $\partial_J = \partial^s / \partial x^{j_s} \cdots \partial x^{j_1}$.

In this subsection, we consider the structure equation with respect to a general frame $\{W_j\}$, where $W_a$’s are horizontal and $W_0 = T$ is the Reeb vector field. Let $U_q$ be a neighborhood of a point $q$ where this frame is defined. It’s easy to see that $h(T, T) = \theta(T) = 1$ and $h(W_a, T) = \theta(W_a) = 0$ by (1.1). On horizontal space, we set $h(W_a, W_b) = h_{ab}$ and use $h_{ab}$ and its inverse matrix to lower and raise indices. The Einstein summation convention will be used.

Let $\{\theta^i\}$ be the coframe dual to $\{W_j\}$. Write $\nabla W_j = \omega^k_j \otimes W_k$, with the TWT-connection 1-forms $\omega^k_j = \Gamma^k_{ij} \theta^i$. For the almost complex structure $J$, we write $J = J^k_j \partial^k \otimes W_l$ or equivalently $JW_k = J^l_k W_l$.

**Proposition 2.2**
\begin{align}
    \omega^0_j &= 0, & \omega^0_0 &= 0, & \Gamma^0_{ij} &= 0, & \Gamma^0_{ij} &= 0, \\
    J^0_j &= 0, & J^0_0 &= 0, & J_{ab} &= -J_{ba}. \tag{2.4}
\end{align}

**Proof** $\omega^0_0 = 0$ follows from $\nabla T = 0$. By $\theta(\nabla X) = 0$ for any $X \in HM$, we have $\omega^0_0 = 0$. $\omega^0_j = 0$ follows. $\Gamma^0_0 = 0$ and $\Gamma^j_0 = 0$ follow from $\omega^0_0 = 0$ and $\omega^0_j = 0$, respectively.

Note that (1.1) implies some useful relations (cf. [21, p. 351]):
\begin{align}
    JT &= 0, & \theta(JX) &= 0, \\
    h(X, Y) &= h(JX, JY) + \theta(X)\theta(Y), & d\theta(X, JY) &= -d\theta(JX, Y), \tag{2.5}
\end{align}
for any $X, Y \in TM$. $JT = 0$ implies $J^k_0 = 0$, and $\theta(JW_j) = 0$ in (2.5) implies $J^0_j = 0$.

Since
\begin{equation}
    h(W_a, JW_b) = h(W_a, J^c_b W_c) = h_{ac}J^c_b = J_{ab}, \tag{2.6}
\end{equation}
and \( h(X, Y) = d\theta(X, Y) = -d\theta(Y, X) = -h(Y, JX) \) holds for any \( X, Y \in TM \), we get \( J_{ab} = h(W_a, JW_b) = -h(W_b, JW_a) = -J_{ba}. \) \hfill \( \square \)

The Webster torsion is defined by \( \tau_a(X) = \tau(T, X) \), \( X \in TM \), (cf. \([5, p. 279]\)). We have the following lemma for the Webster torsion.

**Lemma 2.1** Let \((M, \theta, h, J)\) be a contact Riemannian manifold and \( T \) be the Reeb vector. Then:

1. (cf. \([5, Lemma 1]\)) (a) \( \tau_a(T) = 0 \), (b) \( \tau_a \circ J + J \circ \tau_a = 0 \), (c) \( \tau_a TM \subset HM \), (d) \( \tau_a T^{(1,0)} M \subset T^{(0,1)} M, \tau_a T^{(0,1)} M \subset T^{(1,0)} M \).

2. (cf. \([5, Lemma 3]\)) The Webster torsion \( \tau_a \) is self-adjoint, i.e., \( h(\tau_a X, Y) = h(X, \tau_a Y) \) for any \( X, Y \in TM \).

By (c) in Lemma 2.1 (1), we can write \( \tau_a(W_a) = A_{ab}^b W_b \) and define \( \tau^a := A_{ab}^b \theta^b \). We also write \( R(W_k, W_l) W_j = \nabla_{W_k} \nabla_{W_l} W_j - \nabla_{W_l} \nabla_{W_k} W_j - \nabla_{[W_k, W_l]} W_j = R^k_{jkl} W_k \), as components of the curvature tensor.

Recall that we have the following identities for wedge and exterior derivative

\[
\phi \wedge \psi(X, Y) = \frac{1}{2} \left( \phi(X)\psi(Y) - \psi(X)\phi(Y) \right),
\]

\[
X \cdot (\phi \wedge \psi) = 2(\phi \wedge \psi)(X, \cdot) = \phi(X)\psi - \psi(X)\phi,
\]

\[
2(d\phi)(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]) = (\nabla_X \phi) Y - (\nabla_Y \phi) X + \phi(\tau(X, Y)),
\]

(2.7)

where \( \phi \) and \( \psi \) is any 1-form. The Lie derivation of a differential form \( \phi \) is given by

\[
\mathcal{L}_X \phi = X \cdot d\phi + d(X \phi).
\]

(2.8)

Note that here we use the definition of exterior derivative with a factor \( \frac{1}{2} \). The reason we use this definition is that the \( n \)-form defined is this way has the property that \( dx^1 \wedge \cdots \wedge dx^n \) equal to the Lebesgue measure on \( \mathbb{R}^n \). We may refer to \([4, Sect. 4]\) for these identities.

**Proposition 2.3** With \( J_{ab}, A_{ab}, R^b_{a\,cd} \) defined as above, we have the following structure equations.

\[
d\theta = J_{ab} \theta^a \wedge \theta^b + 2J_{a\beta}^b \theta^a \wedge \theta^\beta + J_{ab}^\alpha \theta^a \wedge \theta^\beta,
\]

\[
d \omega^a = \theta^a \wedge \omega^a + \theta \wedge \tau = \theta^a \wedge \omega^a + A^a_b \theta \wedge \theta^b,
\]

\[
d \omega^b_a - \omega^c_a \wedge \omega^b_c = R^b_a \lambda \mu \theta^\lambda \wedge \theta^\mu + \frac{1}{2} R^b_a \lambda \mu \theta^\lambda \wedge \theta^\mu + \frac{1}{2} R^b_a \lambda \mu \theta^\lambda \wedge \theta^\mu,
\]

\[
R(X, Y) W_a = 2(d \omega^a_b - \omega^c_a \wedge \omega^b_c)(X, Y) W_b.
\]

(2.9)

**Proof** By (1.1) and (2.6), we have \( d\theta(W_a, W_\beta) = h(W_a, J W_\beta) = J_{a\beta}, d\theta(W_\alpha, W_\beta) = J_{a\beta}, d\theta(W_\alpha, W_\beta) = J_{a\beta}. \) We also have \( d\theta(T, \cdot) \equiv 0 \). So the first identity in (2.9) follows.

Substituting \( \phi = \theta^a, X = W_c \) and \( Y = W_d \) in (2.7) and using \( (\nabla_X \phi) Y = X(\phi(Y)) - \phi(\nabla_X Y) \) for any 1-forms \( \phi \), we get

\[
2d \omega^a(X, Y) = (\nabla_{W_c} \omega^a) W_d - (\nabla_{W_d} \omega^a) W_c + \omega^a(\tau(W_c, W_d))
\]

\[
= -\omega^a(\nabla_{W_c} W_d) + \omega^a(\nabla_{W_d} W_c)
\]

\[
= -\Gamma^a_{cd} + \Gamma^a_{dc} = 2(\theta^b \wedge \omega^a_b + A^a_b \theta \wedge \theta^b)(W_c, W_d).
\]

(2.10)
by (2.1) and (2.7). And similarly we get
\[ 2d\theta^a(T, W_d) = (\nabla_T \theta^a)W_d - (\nabla_{W_d} \theta^a)T + \theta^a(\tau(T, W_d)) = -\theta^a(\nabla_T W_d) + A^a_d \]
\[ = -\Gamma^a_{\theta d} + A^a_d = 2\left(\theta^b \wedge \omega^a_b + A^a_b \theta \wedge \theta^b\right)(T, W_d). \]
So the second identity in (2.9) holds.

For the fourth identity of (2.9), we have

\[ R(X, Y)W_a = \nabla_X \nabla_Y W_a - \nabla_Y \nabla_X W_a - \nabla_{[X,Y]}W_a \]
\[ = \nabla_X (\omega^b_a(Y) W_b) - \nabla_Y (\omega^b_a(X) W_b) - \omega^b_a([X, Y]) W_b \]
\[ = X(\omega^b_a(Y)) W_b - Y(\omega^b_a(X)) W_b - \omega^b_a([X, Y]) W_b \]
\[ + \omega^b_a(Y) \omega^b_b(X) W_c - \omega^b_a(X) \omega^b_b(Y) W_c \]
\[ = 2(d \omega^b_a \wedge \omega^b_c(X, Y) W_b), \]
by (2.7). The third identity of (2.9) follows from applying the fourth identity of (2.9) to
\[ X = W_j, Y = W_k. \]

**Remark 2.2** Note that the structure equations (13), (14) and (39) in [5] are the special case of (2.9) with respect to a \( T^{(1,0)} M \)-frame. Consequently, we have

\[ R_{\alpha \beta \gamma}^b = 2(d \omega^b_\alpha W_b - 2 \omega^c_\alpha \wedge \omega^b_\gamma W_c, W_d) \]
\[ = (\nabla_{W_b} \omega^b_\alpha)(W_d) - (\nabla_{W_d} \omega^b_\alpha)(W_b) + \omega^b_\tau(\tau(W_b, W_d)) - \Gamma^b_{\alpha b} \Gamma^b_{\alpha c} + \Gamma^b_{\alpha a} \Gamma^b_{\alpha c} \]
\[ = W_c \Gamma^b_{\alpha b} - W_d \Gamma^b_{\alpha c} - \omega^b_\alpha (\nabla_{W_c} W_d) + \omega^b_\alpha (\nabla_{W_d} W_c) + \omega^b_\alpha(2h(W_c, J W_d)T) \]
\[ - \Gamma^b_{\alpha b} \Gamma^b_{\alpha c} \Gamma^b_{\alpha d} + \Gamma^b_{\alpha a} \Gamma^b_{\alpha c} \Gamma^b_{\alpha d} \Gamma^b_{\alpha e} \Gamma^b_{\alpha f} - \Gamma^b_{\alpha b} \Gamma^b_{\alpha c} \Gamma^b_{\alpha e} \Gamma^b_{\alpha f} \Gamma^b_{\alpha d} \Gamma^b_{\alpha e} + 2 \Gamma^b_{\alpha a} J_{cd}, \quad (2.11) \]
by (2.1), (2.7) and the fourth identity in (2.9).

### 2.3 Special frames and normal coordinates

\( \mathcal{H}^n = \mathbb{C}^n \times \mathbb{R} \) with coordinates \( x = (z, t) \) has the structure of the Heisenberg group. The Heisenberg norm is \( |x| = (|z|^4 + t^2)^{1/4} \). We choose the standard contact form \( \Theta = dt - iz^a d\bar{z}^a + i z^a d\bar{z}^a \) on \( \mathcal{H}^n \) and set \( \Theta^a = dz^a \). Their dual are

\[ Z_0 = \frac{\partial}{\partial t}, \quad Z_a = \frac{\partial}{\partial z^a} - iz^a \frac{\partial}{\partial t}. \]

On \( \mathcal{H}^n \), the orbit of the parabolic dilation is a parabola through \( 0 \in \mathcal{H}^n \). Recall that in Riemannian geometry, the classical exponential map sends radial lines in the tangent space to geodesics. Similarly in the CR geometry, Jerison and Lee [14] defined the parabolic exponential map, which sends a parabola in the tangent space to a parabolic geodesic. On a contact Riemannian manifold, we need the parabolic exponential map analogous to the CR case, which has been developed by Dragomir and Perrone [6]. A smooth curve \( \gamma(s) \) on a contact Riemannian manifold \( M \) is a **parabolic geodesic** if it satisfies ODE (see also [6, (2.12)])

\[ \nabla_{\dot{\gamma}} \dot{\gamma} = 2c T, \quad (2.12) \]
for some \( c \in \mathbb{R} \), where \( \nabla \) is the TWT connection and \( T \) is the Reeb vector field. A vector field \( X \in M \) is called **parallel** along a curve \( \gamma(s) \) if it satisfies \( \nabla_{\dot{\gamma}} X = 0 \). The parabolic exponential
map exists by the following propositions. Kunkel actually proved these two propositions in a general setting (cf. [15, Theorem 3.1 and Lemma 3.2]).

**Proposition 2.4** (cf. [6, Theorem 2.2, 2.4, Definition 2.3]) Let \((M, \theta, h, J)\) be a contact Riemannian manifold and \(q \in M\). For any \(W \in H_q M\) and \(c \in \mathbb{R}\), let \(\gamma = \gamma_{W,c}\) denote the solution to the ODE (2.12) with initial conditions \(\gamma(0) = q\) and \(\dot{\gamma}(0) = W\). We call \(\gamma\) the parabolic geodesic determined by \(W\) and \(c\). Define the parabolic exponential map 
\[\Psi : T_q M \to M\]
by
\[\Psi(W + cT) = \gamma_{W,c}(1)\]
Then \(\Psi\) maps a neighborhood of \(0\) in \(T_q M\) diffeomorphically to a neighborhood of \(q\) in \(M\), and sends \(sW + s^2cT\) to \(\gamma_{W,c}(s)\).

**Proposition 2.5** (cf. [6, Lemma 2.5]) Let \(X_0 \in T_q M\). Suppose \(X\) is a vector field defined in a neighborhood of \(q\) on \(M\) which is parallel along each curve \(\gamma_{W,c}\) with \(X|q = X_0\). Then \(X\) is smooth near \(q\).

As introduced before, the complexified vector space \(\mathbb{C}T_q M\) at point \(q\) has a unique subspace \(T_q^{(1,0)} M\) such that \(JX = iX\) for any \(X \in T_q^{(1,0)} M\). Set \(T_q^{(0,1)} M = \overline{T_q^{(1,0)} M}\), and so for any \(X \in T_q^{(0,1)} M\) we have \(JX = -iX\). Furthermore, we choose an orthonormal basis of the horizontal space with respect to the metric \(h\) at fixed point \(q\) as in the following lemma.

**Lemma 2.2** We can choose \(W_{\alpha;u} \in T_q^{(1,0)} M\) and \(W_{\bar{\alpha};q} \in \overline{T_q^{(0,1)} M}\), such that
\[h(W_{\alpha;u}, W_{\tilde{\beta};q}) = \delta_{\alpha\tilde{\beta}}, \quad h(W_{\alpha;u}, W_{\overline{\alpha};q}) = 0.\]

**Proof** Choose a real vector \(\{X_1\}\) on \(H_q M\) such that \(h(X_1, X_1) = X_1, X_{n+1} := JX_1\). Then \(X_{n+1}\) is orthogonal to \(X_1\) by \(h(X_1, X_{n+1}) = h(X_1, JX_1) = d\theta(X_1, X_1) = 0\), and \(h(X_{n+1}, X_{n+1}) = h(JX_1, X_1) = h(X_1, X_1) = 2\). We can choose \(X_2\) orthogonal to \(\text{span}\{X_1, JX_1\}\), and define \(X_{n+2} := JX_2\). Repeating the procedure, we can choose an orthogonal basis \(X_1, \ldots, X_{2n}\) with \(h(X_a, X_b) = 2\delta_{ab}\) and \(JX_a = X_{a+n}\). Now define
\[W_{\alpha;u} := \frac{1}{2}(X_a - iX_{a+n}), \quad W_{\bar{\alpha};q} := \overline{W_{\alpha;u}}.\]
We see that \(W_{\alpha;u} \in T_q^{(1,0)} M\) and \(W_{\bar{\alpha};q} \in \overline{T_q^{(0,1)} M}\). Then by (2.13) and \(\mathbb{C}\)-linear extension (2.3), we have \(h(W_{\alpha;u}, W_{\tilde{\beta};q}) = \delta_{\alpha\tilde{\beta}}\) and \(h(W_{\alpha;u}, W_{\overline{\alpha};q}) = 0.\)

We extend \(\{X_a\}\) by parallel translation along each parabola \(\gamma_{W,c}\), which is also denoted by \(\{X_a\}\). If set \(W_{\alpha} = \frac{1}{2}(X_a - iX_{a+n})\), we have \(\nabla_{\gamma} W_{\alpha} = 0\). Let \(W_{\tilde{\alpha}} = \overline{W_{\alpha}}\), so \(W_{\tilde{\alpha}}\) is also parallel along \(\gamma_{W,c}\). \(T\) is automatically parallel along each curve \(\gamma_{W,c}\) by \(\nabla_{\gamma} T = 0\) in (2.1). Since every point in some punctured neighborhood \(U\) near \(q\) is in a unique curve \(\gamma_{W,c}\), the frame \(\{W_{\alpha}, W_{\tilde{\alpha}}, T\}\) is well defined and smooth near \(q\) by Proposition 2.5. We call such a frame a special frame centered at point \(q\).

Let \(\{\theta^\beta, \theta^\bar{\beta}, \theta^0\}\) denote the coframe dual to \(\{W_{\alpha}, W_{\tilde{\alpha}}, T\}\), i.e., \(\theta^\beta(W_{\alpha}) = \delta^\beta_\alpha, \theta^\bar{\beta}(W_{\tilde{\alpha}}) = \theta^\beta(T) = 0, \text{ and } \theta(W_{\alpha}) = \theta(W_{\tilde{\alpha}}) = \theta(T) = 1\). From now on, we denote
\[W_0 := T, \quad \theta^0 := \theta.\]
Since \(\nabla_{\gamma} T = 0\), we have \(\nabla_{\gamma} W_{k} = 0\). So \(0 = \nabla_{\gamma}(\theta^j(W_k)) = (\nabla_{\gamma}\theta^j)(W_k) + \theta^j(\nabla_{\gamma} W_k) = (\nabla_{\gamma}\theta^j)(W_k)\) holds for each geodesic \(\gamma\), i.e., \(\nabla_{\gamma}\theta^j = 0\) along each \(\gamma\). Thus \(\theta^j\) are also parallel along each \(\gamma\). We call such a coframe a special coframe. Define an isomorphism...
\[ \iota : T_q M \to \mathcal{K}^n \] by \( \iota(V) = (\theta^a(V), \theta^b(V), \theta(V)) = (z^a, z^\alpha, t) \), which determines a coordinate chart \( \iota \circ \Psi^{-1} \) in a neighborhood of \( q \).

We call this chart the normal coordinates centered at point \( q \) determined by \( \{W_j\} \).

**Remark 2.3** (1) In the CR case, Jerison and Lee chose a \( T^{(1,0)}M \)-frame at \( q \) with norm \( h(W_{\alpha q}, W_{\beta q}) = 2\delta_{\alpha \beta}, h(W_{\alpha \bar{q}}, W_{\beta \bar{q}}) = 0 \) to construct a special frame, since they used the structure equation \( d\theta = i h_{\alpha \bar{\beta}} \theta^a \wedge \bar{\theta}^\beta \) (cf. [14, p. 307]). But here in contact Riemannian case, we use the structure equation \( d\theta = -2i h_{\alpha \bar{\beta}} \theta^a \wedge \bar{\theta}^\beta \) (cf. [5, (13)]) for a \( T^{(1,0)}M \)-frame at \( q \). That’s why we choose a \( T^{(1,0)}M \)-frame at \( q \) with norm as Lemma 2.2 to construct a special frame.

(2) By Remark 2.1, since \( W_{\bar{a}} = \overline{W_a} \) holds for our special frame, the complex conjugation can be reflected in indices of components: \( \alpha_{\bar{a}}, h_{ab}, J_{\bar{a}}^b, A_{ab}, R_{abcd} \) and their covariant derivatives, e.g.,

\[ \overline{\omega_\alpha} = \omega_{\bar{\alpha}}, \quad \overline{J_{\alpha}^\beta} = J_{\bar{\alpha}}^\beta, \quad \overline{h_{\alpha \bar{\beta}}} = h_{\bar{\alpha} \beta}. \]

**Proposition 2.6** A special frame \( \{W_j\} \) centered at point \( q \) is parallel along each parabolic geodesic starting from point \( q \) and satisfies

\[
\begin{align*}
J_{\alpha}^\beta(q) &= i \delta_{\alpha}^\beta, \\
J_{\alpha}^\bar{\beta}(q) &= -i \delta_{\alpha}^\bar{\beta}, \\
J_{\alpha \bar{\beta}}(q) &= 0, \\
J_{\alpha \bar{\beta}}(q) &= -J_{\alpha \bar{\beta}}(q).
\end{align*}
\]  

(2.14)

**Proof** By \( \nabla h = 0 \), we see that

\[ \frac{d}{ds}(h_{ab}(\gamma(s))) = h(\nabla_{\gamma} W_a, W_b) + h(W_a, \nabla_{\gamma} W_b) = 0, \]

along each \( \gamma \). So \( h_{\alpha \bar{\beta}} = \delta_{\alpha \bar{\beta}} \) and \( h_{\alpha \beta} = 0 \) hold near \( q \).

\[ J_{\alpha}^\beta(q) = i \delta_{\alpha}^\beta \text{ and } J_{\alpha}^\bar{\beta}(q) = 0 \] follows from Lemma 2.2 by our choice of the special frame at \( q \). Then by (2.6), \( J_{\alpha \bar{\beta}}(q) = h_{\alpha \bar{\gamma}} J_{\bar{\beta}}^\gamma(q) = -i \delta_{\alpha \bar{\beta}} \) and \( J_{\alpha \beta}(q) = h_{\alpha \gamma} J_{\beta}^\gamma(q) = 0 \) hold. For the last identity in (2.14), we have

\[ J_{\alpha \bar{\beta}} = h_{\alpha \bar{\gamma}} J_{\bar{\beta}}^\gamma = J_{\alpha \beta} = -J_{\alpha \bar{\beta}} = -h_{\beta \gamma} J_{\alpha}^\gamma = -J_{\alpha}^\bar{\beta} \text{ by } h_{\alpha \bar{\gamma}} = \delta_{\alpha \bar{\gamma}} \text{ and the antisymmetry of } J_{ab} \text{ in Proposition 2.2}. \]

**Remark 2.4** Recall that when \( (M, \theta, h, J) \) is a CR manifold, \( Q = \nabla J = 0 \), and so \( J \) is also parallel along each parabolic geodesic, i.e., \( J_{\alpha}^b \) is a constant matrix near \( q \). But in general \( J_{\alpha}^b \) may not be constant near \( q \). We only know that \( J_{\alpha}^\beta(q) = i \delta_{\alpha}^\beta \) and \( J_{\alpha}^\bar{\beta}(q) = 0 \) at the point \( q \).

The following corollary follows from Lemma 2.1.

**Corollary 2.1** With respect to a special frame centered at point \( q \), we have

\[ A_{\alpha \bar{\beta}}(q) = 0, \quad A_{\alpha \beta}(q) = 0, \quad A_{ab} = A_{ba}. \]

The parabolic dilation in this coordinate \( (z, t) \) is \( \delta_z(z, t) = (sz, s^2t) \) for \( s > 0 \), and the generator of the parabolic dilation is the vector field

\[ P(z, t) = z^\alpha \frac{\partial}{\partial z^\alpha} + z^\bar{\alpha} \frac{\partial}{\partial z^\bar{\alpha}} + 2t \frac{\partial}{\partial t}. \]  

(2.15)
A tensor field $\varphi$ is called homogeneous of degree $m$ if $\mathcal{L}_P \varphi = m \varphi$. For any tensor $\varphi$, we denote $\varphi_{(m)}$ as the part of its Taylor’s series that is homogeneous of degree $m$ in terms of the parabolic dilations. So $\mathcal{L}_P \varphi_{(m)} = m \varphi_{(m)}$. If $\varphi$ is a differential form, we denote $\varphi_{(m)}$ as the part of its Taylor’s series that is homogeneous of degree $m$ in terms of the parabolic dilations. So $\mathcal{L}_P \varphi_{(m)} = m \varphi_{(m)}$.

If $\varphi$ is a differential form, $\varphi_{(m)} = 1/m (P \cdot d \varphi + d (P \cdot \varphi))_{(m)} = 1/m (P \cdot d \varphi + d (P \cdot \varphi))_{(m)}$.

Remark 2.5 In this paper, if indices $\alpha$ and $\bar{\alpha}$ both appear in low (or upper) indices, then the index $\alpha$ will be taken summation, e.g.,

$$\Theta = dt - i z^\alpha dz^\bar{\alpha} + i z^\bar{\alpha} dz^\alpha.$$

Theorem 2.1 On a contact Riemannian manifold $(M, \theta, h, J)$, suppose $F$ is a smooth function defined near $q$. Then with respect to the normal coordinates centered at point $q$, for any $m$, we have

$$F_{(m)}(x) = \sum_{\alpha(K)=m} \frac{1}{(\# K)!} z^K Z_K F(q).$$

The notations of the multi-index are defined as in Notation 2.1.

This theorem can be proved exactly in the same way as [14, Lemma 3.10] since the integrability of $J$ is not used there.

**2.4 Homogeneous parts of special (co)frames and connection coefficients**

As in the CR case [14], there exists a simple relation between the Euler vector field $P$ and the special coframe.

Lemma 2.3 Under normal coordinates $(z^\alpha, z^\bar{\alpha}, t)$, we have

$$\theta(P) = 2t, \quad \theta^\alpha(P) = z^\alpha, \quad \omega^\alpha_b(P) = 0,$$

where $P$ is the Euler vector field. Equivalently, $P = z^\alpha W_\alpha + z^\bar{\alpha} W_\bar{\alpha} + 2t T$.

Proof This lemma can be proved in the same way as [14, Lemma 2.4] since the integrability of $J$ is not used. We mention it briefly.
by Proposition 2.4. Note that by the definition (2.15), $P_{\gamma(s)} = P_{(sw^a z^c - 2s \sum \partial_a \frac{\partial^a}{\partial z^a} + 2s^2 c \frac{\partial}{\partial t})}$. By direct computation, we have $\hat{\gamma}(s) = \sum \partial_a w^a \frac{\partial}{\partial z^a} + 2sc \frac{\partial}{\partial t} = s^{-1} P_{\gamma(s)}$ for $s \neq 0$, and so

$$\frac{d}{ds} \left( \theta(\hat{\gamma}(s)) \right) = \theta(\nabla_{\hat{\gamma}} \hat{\gamma}(s)) = \theta(2cT) = 2c,$$

by $\nabla \theta = 0$ along $\gamma$. Hence, $\theta(\hat{\gamma}(s)) = 2cs$ by $\theta(\hat{\gamma}(0)) = 0$. Then $\theta(P) = \theta(s \hat{\gamma}(s)) = 2s^2 c = 2t$. Similarly by using $\nabla_{\hat{\gamma}} \theta(a) = 0$,

$$\frac{d}{ds} \theta^a(\hat{\gamma}(s)) = \theta^a(\nabla_{\hat{\gamma}} \hat{\gamma}(s)) = \theta^a(2cT) = 0.$$

Note that $\theta^a(\hat{\gamma}(0)) = \theta^a(W) = w^a$. We get $\theta^a(\hat{\gamma}(s)) = w^a$ all along $\gamma$. So $\theta^a(P) = \theta^a(s \hat{\gamma}(s)) = sw^a = z^a$.

For the last identity in (2.20), note that we have $\omega^b_a(P) \theta^b = 0$ by $\nabla P \theta^a = \nabla s \gamma(\hat{\gamma}) \theta^a = 0$ since $\theta^a$'s are parallel along each parabolic geodesic. □

Then by using Lemma 2.3, we get the following proposition:

**Proposition 2.7** With respect to a special frame and under the normal coordinates defined as above, we have

$$\omega^b_a(m) = \frac{1}{m} \left( R^b_{ac} \omega^c_d \theta^d + 2t R^b_{a0d} \theta^d + R^b_{a e} c^e \theta \right)_{(m)} ,$$

$$\theta^b_a(m) = \frac{1}{m} \left( z^a \omega^a_d + 2t A^a_b \theta^a + d z^b \right)_{(m)} ,$$

$$\theta^a(m) = \frac{1}{m} \left( 2J_{ab} z^a \theta^b + 2dt \right)_{(m)} .$$

**Proof** We have

$$\omega^b_a(m) = \frac{1}{m} \left( P_{da} \omega^b_a(m) \right) = \frac{1}{m} \left( P_{d} \left( R^b_{a \lambda \mu} \theta^\lambda \wedge \theta^\mu + \frac{1}{2} R^b_{a \lambda \mu} \theta^\lambda \wedge \theta^\mu + \frac{1}{2} R^b_{a \lambda \mu} \theta^\lambda \wedge \theta^\mu \right) \right)_{(m)}$$

$$= \frac{1}{m} \left( R^b_{a \lambda \mu} \left( z^\lambda \theta^\mu - \bar{z}^\mu \theta^\lambda \right) + \frac{1}{2} R^b_{a \lambda \mu} \left( z^\lambda \theta^\mu - \bar{z}^\mu \theta^\lambda \right) + \frac{1}{2} R^b_{a \lambda \mu} \left( z^\lambda \theta^\mu - \bar{z}^\mu \theta^\lambda \right) \right)_{(m)}$$

$$= \frac{1}{m} \left( R^b_{a cd} z^c \theta^d + 2t R^b_{a0d} \theta^d + R^b_{a e} c^e \theta \right)_{(m)},$$

by (2.7), (2.9), (2.16) and $\omega^b_a(P) = 0$ in (2.20). Here we also use the relation $R^b_{a fkj} = -R^b_{a jk}$. Similarly we get

$$\theta^b_a(m) = \frac{1}{m} \left( P_{da} \theta^b_a \theta^d + \left( \theta^b_a \theta^d \right) \right)_{(m)} = \frac{1}{m} \left( P_{d} \left( \theta^a \wedge \omega^a_d + A^b_a \theta^a \wedge \theta^a \right) + d z^b \right)_{(m)}$$

$$= \frac{1}{m} \left( z^a \omega^a_d + 2t A^a_b \theta^a + A^b_a z^a \theta + d z^b \right)_{(m)},$$

by (2.7), (2.9), (2.16) and (2.20).
Noting that $J_{ab}$ is antisymmetric, we get
\[
\theta_{(m)} = \frac{1}{m} (P \omega d\theta + d(\theta(P)))_{(m)}
\]
\[
= \frac{1}{m} \left( P(J_{ab} \theta^a \land \theta^b - 2J_{a\beta} \theta^a \land \theta^\beta + J_{a\beta} \theta^a \land \theta^\beta) + 2dt \right)_{(m)}
\]
\[
= \frac{1}{m} \left( J_{ab} (\bar{z}^a \theta^b - z^a \theta^b) + 2J_{a\beta} (\bar{z}^a \theta^\beta - z^a \theta^\beta) + 2dt \right)_{(m)}
\]
\[
= \frac{1}{m} \left( 2J_{a\alpha} z^\alpha \theta^b + 2J_{a\beta} \theta^a \theta^\beta + 2J_{a\beta} \theta^a \theta^\beta + 2dt \right)_{(m)}
\]
\[
= \frac{1}{m} (2J_{ab} z^a \theta^b + 2dt)_{(m)}.
\]
by (2.7), (2.9), (2.16) and (2.20). Proposition 2.7 is proved. \hfill \Box

Then we have the following corollary for lower-order homogeneous parts.

**Corollary 2.2** With respect to a special frame centered at point $q$, we have
\[
\omega_a^{b(1)} = 0, \quad \omega_a^{b(2)} = \frac{1}{2} R_a^{b c d} (q) z^c dz^d,
\]
\[
\theta_a^{b(1)} = dz^b, \quad \theta_a^{b(2)} = 0,
\]
\[
\theta_a^{b(3)} = \frac{1}{6} R_a^{b c d} (q) z^c z^e dz^d, \quad \text{mod } \mathcal{A},
\]
\[
\theta_a^{(2)} = \Theta, \quad \theta_a^{(3)} = \frac{2}{3} J_{ab(1)} z^a dz^b,
\]
\[
\theta_a^{(4)} = \frac{1}{12} J_{ab}(q) R_e^{b c d} (q) z^c z^e dz^d + \frac{1}{2} J_{ab(2)} z^a dz^b, \quad \text{mod } \mathcal{A},
\]
where $\mathcal{A}$ means terms linearly depending on $A^b_a(q)$.

**Proof** By the first identity in (2.21), it’s obvious that $\omega_a^{b(1)} = 0$. Then it follows from the second identity in (2.21) and $\omega_a^{b(1)} = 0$ that $\theta_a^{b(1)} = dz^b$ and $\theta_a^{b(2)} = 0$.

By the third identity in (2.21) for $m = 2$ and (2.14), we get
\[
\theta_a^{(2)} = dt + J_{a\beta}(q) z^a dz^\beta + J_{a\alpha}(q) z^\alpha dz^a = dt - i \delta_{a\beta} z^a dz^\beta + i \delta_{a\beta} z^\beta dz^a = \Theta.
\]
By the third identity in (2.21) for $m = 3$ and $\theta_a^{b(2)} = 0$, we get $\theta_a^{(3)} = \frac{2}{3} J_{ab(1)} z^a dz^b$. By (2.21) for $m = 2$ and $\theta_a^{b(1)} = dz^d$, we find that $\omega_a^{b(2)} = \frac{1}{2} R_a^{b c d} (q) z^c dz^d$. Hence
\[
\theta_a^{b(3)} = \frac{1}{3} z^a \omega_a^{b(2)} = \frac{1}{6} R_a^{b c d} (q) z^c z^e dz^d, \quad \text{mod } \mathcal{A},
\]
holds by (2.21). And so we also have
\[
\theta_a^{(4)} = \frac{1}{4} (2J_{ab}(q) z^a \theta_a^{b(3)} + 2J_{ab(2)} z^a dz^b)
\]
\[
= \frac{1}{12} J_{ab}(q) R_e^{b c d} (q) z^c z^e dz^d + \frac{1}{2} J_{ab(2)} z^a dz^b, \quad \text{mod } \mathcal{A},
\]
by (2.14) and (2.21). \hfill \Box
Remark 2.6  (1) In the CR case, Jerison and Lee [14] used the identity $d\theta(X, Y) = h(JX, Y)$. But in contact Riemannian case, people usually use $d\theta(X, Y) = h(X, JY) = -h(JX, Y)$ (cf. [5] or [21]). So $J_{ab}$ is different from that in [14] by a factor $-1$. That’s why we choose $\Theta$ as (2.18), which coincides with standard contact form in [14] up to signs.

(2) We choose $dV_b = (-1)^n\theta \wedge (d\theta)^n$ as the volume form on $(M, \theta, h, J)$. And we will see later in Sect. 5 that the volume form $dV = (-1)^n\Theta \wedge (d\Theta)^n$ on the Heisenberg group is positive (cf. (5.5)).

(3) Recall that in the CR case, $\theta(3)$ vanishes by the integrability of $J$ (cf. [14, Proposition 2.5]), while in the general case, $\theta(3)$ may not vanish.

2.5 Asymptotic expansion of a special frame

Let us examine the Taylor series of $W_j$ in terms of $Z_j$’s in normal coordinates $(z, t)$. Write

$$W_j = s_j^k Z_k = s_j^\alpha Z_\alpha + s_j^\beta Z_\beta + s_j^0 Z_0,$$

for some functions $s_j^k$. Since as mentioned above, $Z_j$ is homogenous of degree $-\alpha(j)$, we can examine the Taylor series of $W_j$ by the Taylor series of their coefficient functions $s_j^k$. We denote $\psi \in \mathcal{O}_m$ if all the terms in the Taylor series of $\psi$ in normal coordinates are homogeneous of degree $\geq m$. It’s easy to see that if $\psi \in \mathcal{O}_m$ and $\psi \in \mathcal{O}_n$, then $\psi \psi \in \mathcal{O}_{m+n}$.

Proposition 2.8  In normal coordinates centered at point $q$, we have:

$$s_j^\alpha(0) = s_j^\beta(0) = \delta_j^\alpha, \quad s_j^\beta(0) = s_j^\alpha(0) = 0, \quad s_j^0(1) = 0,$$

$$s_j^a(0) = -\frac{1}{6} R_{db}^a(q) z^c z^d, \quad \text{mod } \mathcal{A},$$

$$s_j^0(0) = 0, \quad s_j^0(0) = 1, \quad s_j^0(0) = 0.$$

Proof  By $\theta_j^\alpha = dz^a$ and $\theta_j^\beta = \Theta$ in Corollary 2.2, $\theta_j^a(W_b) = \delta_j^a$ and $\theta_j(W_b) = 0$ lead to $W_{\beta(-1)} = Z_\beta$, $W_{\beta(-1)} = Z_\beta$. By $\theta_j^a(W_b) = \delta_j^a$ and $\theta_j^a(2) = 0$, we get

$$\delta_j^a = (dz^a + \Theta_3) \left(Z_b + s_{b(1)}^0 Z_c + s_{b(2)}^0 Z_0 + \Theta_1\right)$$

$$= \delta_j^a + dz^a \left(s_{b(1)}^0 Z_c\right) + \Theta_2 = \delta_j^a + s_{b(1)}^0 + \Theta_2.$$

Namely we can write

$$W_{b(0)} = s_{b(1)}^0 Z_a + s_{b(2)}^0 Z_0 = 0, \quad \text{mod } Z_0.$$

Also we have

$$0 = \delta_j^a(2) = \left(\left(dz^a + \Theta_3 + \Theta_4\right) \left(Z_b + W_{b(0)} + W_{b(1)} + \Theta_2\right)\right)_{(2)}$$

$$= \theta_j^a(3)(Z_b) + dz^a(W_{b(1)}) = \theta_j^a(3)(Z_b) + s_{b(2)}^0,$$

and so

$$s_{b(2)}^a = -\theta_j^a(Z_b) = -\frac{1}{6} R_{db}^a(q) z^c z^d, \quad \text{mod } \mathcal{A}.\]
By \( \theta(W_b) = 0 \) and (2.25), we get \( 0 = \Theta(W_b(0)) + \theta_3(W_b(-1)) = s_{b(2)}^0 + \theta_3(Z_b) \), i.e.,
\[
s_{b(2)}^0 = -\theta_3(Z_b) = -\frac{2}{3} J_{ab(1)} \zeta^a = \frac{2}{3} J_{ba(1)} \zeta^a.
\]
Also
\[
0 = \theta_4(W_b(-1)) + \theta_3(W_b(0)) + \Theta(W_b(1))
\]
\[
= \theta_4(Z_b) + \theta_3(s_{b(2)}^0 Z_0) + \Theta(s_{b(2)}^0 Z_a + s_0^0 Z_0)
\]
by (2.25), which gives us
\[
s_{b(3)}^0 = -\theta_4(Z_b) = -\frac{1}{12} J_{af}(q) R_{eb}(q) \zeta^a \zeta^c \zeta^e
- \frac{1}{2} J_{ab(2)} \zeta^a, \quad \text{mod } \mathcal{A},
\]
by (2.14) and (2.22). The identity for \( s_{b(3)}^0 \) follows.

By \( \theta(T) = 1 \), we get \( 1 = \theta_2(W_0(-2)) = \Theta(s_{0(0)}^0 \frac{\partial}{\partial t}) = s_{0(0)}^0 \), and \( W_0(-2) = \frac{\partial}{\partial t} \). By the fact that \( \theta_3 \) has no \( dt \) term (cf. (2.22)), we get
\[
0 = \theta_3(W_0(-2)) + \theta_2(W_0(-1)) = \Theta(s_{0(0)}^0 Z_a + s_{0(1)}^0 \frac{\partial}{\partial t}) = s_{0(1)}^0.
\]
By \( \theta^a(T) = 0 \) and \( \theta^a_{(2)} = 0 \) in (2.22), we get
\[
0 = \theta(a_2) W_0(-2) + \theta_{(1)}(a)(W_0(-1)) = d\zeta^a \left( s_{0(0)}^0 Z_b \right) = s_{a(0)}^a.
\]
We finish the proof of (2.24).

By Proposition 2.8, we have
\[
W_a = Z_a + \mathcal{O}_0 = Z_a + \frac{2}{3} J_{ab(1)} \zeta^b \frac{\partial}{\partial t} + \mathcal{O}_1
\]
\[
= Z_a + \frac{2}{3} J_{ab(1)} \zeta^b \frac{\partial}{\partial t} + s_{a(2)}^b Z_b + s_{a(3)}^0 Z_0 + \mathcal{O}_2,
\]
(2.26)
\[
W_0 = \frac{\partial}{\partial t} + \mathcal{O}_0,
\]
where \( s_{a(2)}^b \) and \( s_{a(3)}^0 \) are given by (2.24). In our case, \( W_a \) has an extra term \( \frac{2}{3} J_{ab(1)} \zeta^b \frac{\partial}{\partial t} \), which vanishes in the CR case (cf. [14, p. 314]).

**Corollary 2.3** With respect to a special frame centered at \( q \), connection coefficients vanish at \( q \), i.e.,
\[
\Gamma^i_{jk}(q) = 0, \quad \Gamma^i_{jk} \in \mathcal{O}_1.
\]
(2.27)

**Proof** By \( \omega^b_{a(1)} = 0 \) in (2.22), we get \( \Gamma^b_{ca}(q) = \omega^b_{a(1)} W_c(-1) = 0 \). Again by (2.22), \( \omega^b_{a(1)} = 0 \) and \( \omega^b_{a(2)} \) having no \( dt \) term, we see that \( \Gamma^b_{ba}(q) = \omega^b_{a(2)} W_0(-2) + \omega^b_{a(1)} W_0(-1) = 0 \). We also have \( \Gamma^i_{j0} = 0 \) by \( \nabla T = 0 \) and \( \Gamma^0_{ab} = 0 \) by \( \theta(\nabla_X Y) = 0 \) for any \( X, Y \in HM \). So (2.27) follows.

**3 Asymptotic expansion of the almost complex structure, curvature and Tanno tensors**

In this section, we give asymptotic expansion of curvature tensors, Tanno tensors and the almost complex structure with respect to a special frame centered at \( q \).
3.1 The Tanno tensor at a point $q$

For the Tanno tensor, we write $Q (W_j, W_k) = (\nabla_{W_j} J) W_j = Q^l_{jk} W_l$. Components of the Tanno tensor can be written as

$$Q^l_{jk} = W_k J^l_j - \Gamma^l_{kj} J^l_s + \Gamma^l_{ks} J^s_j.$$  \hfill (3.1)

So at the point $q$, we have

$$Q^l_{jk} (q) = W_k J^l_j (q),$$  \hfill (3.2)

by (2.27). And noting that $\overline{Q^\gamma_{ab}} = Q^\gamma_{a\beta}$ by definition, we set

$$\Omega := Q^\gamma_{a\beta} (q) Q^\gamma_{a\bar{\beta}} (q) = \sum_{\alpha, \beta, \gamma} |Q^\gamma_{a\beta} (q)|^2.$$  \hfill (3.3)

Proposition 3.1 With respect to a special frame centered at $q$, $J^\gamma_{a(1)} = 0$, $J_{a\bar{\beta}(1)} = 0$.

Proof Since $J^2 W_\alpha = -W_\alpha$ means $J^b J^\alpha = -\delta^\gamma_\alpha$, we get

$$0 = J^\beta_{a(0)} J^\gamma_{\bar{\beta}(1)} + J^\beta_{a(1)} J^\gamma_{\bar{\beta}(0)} + J^\bar{\beta}_{a(0)} J^\gamma_{\beta(1)} + J^\bar{\beta}_{a(1)} J^\gamma_{\beta(0)} = 2i J^\gamma_{a(1)},$$

by (2.14). Hence $J^\gamma_{a(1)} = 0$. And so $J_{a\bar{\beta}(1)} = h_{a\bar{\gamma}} J^\gamma_{\bar{\beta}(1)} = 0$. \hfill $\square$

Proposition 3.2 With respect to a special frame centered at $q$, components of the Tanno tensor $Q$ at $q$ satisfy

$$Q^0_{ab} = Q^0_{i0j} = Q^0_{i0} = 0, \quad Q^\gamma_{a\beta} (q) = Q^\gamma_{a\bar{\beta}} (q) = Q^\gamma_{\bar{\beta}a} (q) = 0, \quad Q^\gamma_{a\beta} (q) Q^\gamma_{a\bar{\beta}} (q) = \frac{1}{2} \Omega.$$  \hfill (3.4)

In particular, $Q (q) \neq 0$ if and only if $\Omega > 0$.

Proof $Q^0_{ab} = 0$ follows from

$$\theta (Q (X, Y)) = \theta ((\nabla_Y J) X) = h (T, (\nabla_Y J) X) = h (T, \nabla_Y (JX)) - h (T, J \nabla_Y X) = \theta (\nabla_Y (JX)) - \theta (J \nabla_Y X) = 0,$$

for any $X, Y \in HM$, by (1.1) and horizontality of $JX, \nabla_Y (JX)$ and $J \nabla_Y X$.

$Q^0_{i0j} = 0$ follows from $Q (T, Y) = (\nabla_Y J) T = \nabla_Y (JT) - J \nabla_Y T = 0$ by $JT = 0$ and $\nabla T = 0$. By (3.2) and Proposition 3.1, $Q^\gamma_{a\beta} (q) = W_{\beta} J^\gamma_{a(1)} = W_{\beta} (J^\gamma_{a(1)} + \nabla_{\bar{\beta}a(0)} J^\gamma_{a(0)}) = 0$. Similarly, $Q^\gamma_{a\beta} (q) = 0$.

To prove $Q^0_{i0} = 0$ and $Q^\gamma_{a\beta} (q) = 0$, recall that

$$2h (Q (X, Y), Z) = h \left( N^1 (X, Z) - \theta (X) N^1 (T, Z) - \theta (Z) N^1 (X, T), JY \right),$$  \hfill (3.5)

for any $X, Y, Z \in TM$ (cf. [5, (15)]), where

$$N^1 = [J, J] + 2 (d \theta) \otimes T,$$

$$[J, J] (X, Y) = J^2 [X, Y] + [JX, JY] - J [JX, Y] - J [X, JY].$$

$Q^0_{i0} = 0$ is equivalent to $Q (X, T) = 0$, for any $X \in TM$. Apply (3.5) to $Y = T$ to get $h (Q (X, T), Z) = 0$ for any $X, Z \in TM$ by $JT = 0$. 

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Substituting $X = W_\beta, Y = W_\alpha, Z = W_\rho$ into (3.5), and noting that $h(JX, JY) = h(X, Y)$ for any $X, Y \in HM$, we get

$$2h_{Y\rho} Q^{Y}_{\beta a} = h(N^{1}(W_\beta, W_\beta), JW_\alpha) = h([J, J](W_\beta, W_\beta), JW_\alpha)$$

$$= h(J^2[W_\beta, W_\beta] + [JW_\beta, JW_\beta] - [JW_\beta, JW_\beta], JW_\alpha)$$

$$= -h([W_\beta, W_\beta], JW_\alpha) + h(JW_\beta, JW_\beta), JW_\alpha)$$

$$- h([JW_\beta, W_\rho], W_\alpha) - h([W_\beta, JW_\beta], W_\alpha),$$

(3.6)

by using (1.1). Note that $[X, Y] = \nabla_X Y - \nabla_Y X - \tau(X, Y)$ and $\tau(X, Y) = 0 \mod T$, for any $X, Y \in HM$. We find that

$$[W_\beta, W_\beta] = \nabla W_\beta W_\beta - \nabla W_\beta W_\beta = \left(\Gamma^c_{\beta \rho} - \Gamma^c_{\rho \beta}\right) W_\rho, \quad \mod T,$$

$$[JW_\beta, JW_\beta] = \nabla Jc W_\beta (J^d_{\rho} W_d) \nabla Jc W_\rho (J^d_{\rho} W_d) - Jc_{\beta} Jc_{\rho} \Gamma^d_{\beta \rho} W_d, \quad \mod T,$$

$$[JW_\beta, W_\beta] = -W_\beta J^d_{\rho} \cdot W_d + Jc_{\beta} \Gamma^d_{\beta \rho} W_d - Jc_{\rho} \Gamma^d_{\beta \rho} W_d, \quad \mod T,$$

$$[W_\beta, JW_\beta] = W_\beta J^d_{\rho} \cdot W_d + Jc_{\beta} \Gamma^d_{\beta \rho} W_d - Jc_{\rho} \Gamma^d_{\beta \rho} W_d, \quad \mod T.$$

Then (3.6) becomes

$$2Q^\rho_{\beta a} = \left(-\Gamma^d_{\beta \rho} + \Gamma^d_{\rho \beta}\right) J^f_{\alpha} h_{df}$$

$$+ \left(J^c_{\beta} \cdot W_c J^d_{\rho} + Jc_{\beta} Jc_{\rho} \Gamma^d_{\alpha \rho} W_d - Jc_{\rho} Jc_{\beta} \Gamma^d_{\alpha \rho} W_d - Jc_{\beta} \Gamma^d_{\alpha \rho} W_d\right) h_{\alpha d}$$

$$+ \left(W_\beta J^d_{\rho} - Jc_{\beta} \Gamma^d_{\beta \rho} W_d - Jc_{\rho} \Gamma^d_{\beta \rho} W_d\right) h_{\alpha d} + \left(-W_\beta J^d_{\rho} - Jc_{\beta} \Gamma^d_{\beta \rho} W_d - Jc_{\rho} \Gamma^d_{\beta \rho} W_d\right) h_{\alpha d},$$

(3.7)

by writing $JW_\alpha = J^f_{\alpha} W_f$ and $h_{Y\rho} = \delta_{Y\rho}$. (3.7) will be used later.

At the point $q$, (3.7) for $a = \alpha$ becomes

$$2Q^\rho_{\beta a} (q) = \left(Jc_{\beta}(q) \left(W_c J^d_{\beta}(q) - Jc_{\rho}(q) \left(W_c J^d_{\rho}(q)\right)\right) J^f_{\alpha}(q) h_{df}\right.$$

$$+ \left(W_\beta J^d_{\rho}(q) h_{\alpha d} - W_\beta J^d_{\beta}(q) h_{\alpha d}\right)$$

$$= W_\beta J^\mu_{\beta}(q) h_{\alpha \mu} - W_\beta J^\mu_{\rho}(q) h_{\alpha \mu} + W_\beta J^\mu_{\beta}(q) h_{\alpha \mu} - W_\beta J^\mu_{\rho}(q) h_{\alpha \mu} = 0,$$

by Proposition 2.6. So $Q^Y_{\beta a} (q) = 0$.

It remains to prove the last identity in (3.4). Similarly at the point $q$, (3.7) for $a = \bar{\alpha}$ becomes

$$2Q^\rho_{\beta \bar{\alpha}} (q) = 2h_{Y\bar{\rho}} Q^Y_{\beta \bar{\alpha}} (q) = Jc_{\beta}(q) \left(W_c J^d_{\beta}(q)\right) J^f_{\alpha}(q) h_{df}$$

$$+ \left(W_\beta J^d_{\rho}(q) h_{\alpha d} - W_\beta J^d_{\beta}(q) h_{\alpha d}\right)$$

$$= -W_\beta J^\mu_{\rho}(q) h_{\alpha \mu} + W_\beta J^\mu_{\rho}(q) h_{\alpha \mu} + W_\beta J^\mu_{\beta}(q) h_{\alpha \mu} - W_\beta J^\mu_{\rho}(q) h_{\alpha \mu}$$

$$= -2Q^\mu_{\beta \rho}(q) h_{\alpha \mu} + 2Q^\mu_{\rho \beta}(q) h_{\alpha \mu} = 2Q^\mu_{\beta \rho}(q) - 2Q^\mu_{\rho \beta}(q).$$
by Proposition 2.6 and (3.2). Taking conjugate on the both sides of last equation, we get
\[ Q^\alpha_{\beta\rho}(q) = Q^\alpha_{\rho\beta}(q) - Q^\alpha_{\rho\beta}(q). \]
Then we get
\[
\sum_{\alpha,\beta,\rho} |Q^\alpha_{\beta\rho}(q)|^2 = \sum_{\alpha,\beta,\rho} Q^\alpha_{\beta\rho}(q)Q_{\alpha\beta}(q) = \sum_{\alpha,\beta,\rho} \left( Q^\alpha_{\beta\rho}(q) - Q^\alpha_{\rho\beta}(q) \right) \left( Q^\alpha_{\beta\rho}(q) - Q^\alpha_{\rho\beta}(q) \right)
\]
\[= 2 \sum_{\alpha,\beta,\rho} |Q^\alpha_{\beta\rho}(q)|^2 - 2 \sum_{\alpha,\beta,\rho} Q^\alpha_{\beta\rho}(q)Q_{\alpha\beta}(q). \]
By relabeling indices, we get
\[2Q^\gamma_{\alpha\beta}(q)Q^\gamma_{\beta\alpha}(q) = \sum_{\alpha,\beta,\gamma} |Q^\gamma_{\alpha\beta}(q)|^2 = \Omega. \]

\[\square\]

**Remark 3.1** For the Tanno tensor \(Q\), if we choose a local \(T^{(1,0)}\) frame, only \(Q^\gamma_{\alpha\beta}\) and its conjugate are non-vanishing (cf. [5, (16)–(18)]). Here we have the similar property at point \(q\).

### 3.2 Asymptotic expansion of the almost complex structure \(J\) at point \(q\)

**Proposition 3.3** With respect to a special frame centered at \(q\), we have
\[J_{\alpha\beta}(1) = J^\beta_{\alpha}(1) = Q^\alpha_{\beta\gamma}(q)z^\gamma,\]
\[J_{\alpha\beta}(2) = \frac{i}{2} Q^\gamma_{\alpha\lambda}(q)Q^\gamma_{\beta\mu}(q)z^\lambda z^\mu.\]

**Proof** By \(Q^\alpha_{\beta\gamma}(q) = 0\) in (3.4) and the expansion (2.19), we get
\[J_{\alpha\beta}(1) = h_{\alpha\beta}J^\rho_{\beta\rho}(1) = J^\rho_{\beta\rho}(1) = z^c Z_c J^\rho_{\beta\rho}(q) = z^c Q^\rho_{\beta\rho}(q)z^\gamma.\]

It follows from \(J^\beta_{\alpha}(1) = -\delta^\beta_{\alpha}\) that
\[0 = J^\beta_{\alpha}(0)J^\gamma_{\beta}(2) + J^\beta_{\alpha}(1)J^\gamma_{\beta}(1) + J^\beta_{\alpha}(2)J^\gamma_{\beta}(0) + J^\beta_{\alpha}(0)J^\gamma_{\beta}(2) + J^\beta_{\alpha}(1)J^\gamma_{\beta}(1) + J^\beta_{\alpha}(2)J^\gamma_{\beta}(0).\]

By (2.14) and Proposition 3.1, we get
\[2i J^\gamma_{\alpha}(2) + J^\beta_{\alpha}(1)J^\gamma_{\beta}(2) = 0, \text{ i.e., } J^\gamma_{\alpha}(2) = \frac{i}{2} J^\beta_{\alpha}(1)J^\gamma_{\beta}(1).\]

So
\[J_{\alpha\beta}(2) = h_{\alpha\beta}J^\gamma_{\beta}(2) = -\frac{i}{2} J^\rho_{\beta\rho}(1)J^\gamma_{\beta}(1) = \frac{i}{2} J^\rho_{\beta\rho}(1)J^\gamma_{\beta}(1) = \frac{i}{2} Q^\rho_{\alpha\lambda}(q)Q^\rho_{\beta\mu}(q)z^\lambda z^\mu,\]

by (3.8) and \(J^\rho_{\beta\rho}(1) = -J^\rho_{\alpha\lambda}(q)\) in (2.14). We complete the proof of this proposition. \(\square\)

Proposition 3.1 and Proposition 3.3 lead to the following corollary.

**Corollary 3.1** \(s^0_{\beta}(2)\) in (2.24) in Proposition 2.8 can be rewritten as
\[s^0_{\beta}(2) = \frac{2}{3} Q^\beta_{\alpha\gamma}(q)z^\alpha z^\gamma, \quad s^0_{\beta}(2) = \frac{2}{3} Q^\beta_{\alpha\gamma}(q)z^\alpha z^\gamma.\]

The following relation shows that \(J^\gamma_{\beta}(1)\) is independent of \(t\).

**Proposition 3.4** With respect to a special frame centered at \(q\), we have
\[J^\gamma_{\beta}(2) = \frac{1}{2} z^c Z_c J^\gamma_{\beta}(q), \quad J_{\alpha\beta}(2) = \frac{1}{2} z^c Z_c J_{\alpha\beta}(q).\]
Proof By expansion (2.19), \( J_{\tilde{\rho}(2)}^\gamma = \frac{1}{2} z_{c,\tilde{\rho}} z_{\tilde{d}} Z_{c,\tilde{d}} J_{\tilde{\rho}}^\gamma (q) + \frac{\partial J_{\tilde{\rho}}^\gamma}{\partial t}(q) \). To get this proposition, we need to prove that \( \frac{\partial}{\partial t} J_{\tilde{\rho}}^\gamma (q) = 0 \).

By (3.2) and (3.4), \( 0 = Q_{\tilde{\rho}0}^\gamma(q) = W_0 J_{\tilde{\rho}}^\gamma(q) \). Noting that \( W_0(-2) = \frac{\partial}{\partial t}, \ W_0(-1) = 0 \) (cf. (2.26)) and \( J_{\tilde{\rho}(0)}^\gamma = 0 \) (cf. (2.14)), we get

\[
0 = W_0 J_{\tilde{\rho}}^\gamma(q) = W_0(-2) \left( J_{\tilde{\rho}(2)}^\gamma \right) + W_0(-1) \left( J_{\tilde{\rho}(1)}^\gamma \right) + W_0(0) \left( J_{\tilde{\rho}(0)}^\gamma \right) = \frac{\partial}{\partial t} J_{\tilde{\rho}}^\gamma(q).
\]

By \( h_{ab} \) being constant, we get \( \frac{\partial}{\partial t} J_{\alpha\beta}(q) = h_{\alpha\gamma} \frac{\partial}{\partial t} J_{\gamma\beta}(q) = 0 \). Proposition 3.4 is proved. \( \square \)

### 3.3 Asymptotic expansion of curvature tensors

**Proposition 3.5** With respect to a special frame centered at \( q \), we have

\[
R_{\alpha\beta\lambda\mu}(q) = \frac{1}{4} \left( Q_{\mu\alpha}^\gamma(q) - Q_{\lambda\mu}^\gamma(q) \right) \left( Q_{\alpha\beta}^\gamma(q) - Q_{\beta\alpha}^\gamma(q) \right).
\]

In particular,

\[
R_{\alpha\beta\lambda\mu}(q) = -\frac{1}{4} \Omega.
\]

Proof At the point \( q \), it follows from (2.11) and (2.27) that

\[
R_{\alpha\beta\lambda\mu}(q) = W_\lambda \Gamma_{\mu\beta}(q) - W_\mu \Gamma_{\lambda\beta}(q) = Z_\lambda \left( \Gamma_{\mu\beta(1)}^\alpha - \Gamma_{\lambda\beta(1)}^\alpha \right).
\]

Let us calculate \( Z_c(\Gamma_{\alpha\beta\lambda}^{(1)}) \). By (3.1), we have \( Q_{\beta\alpha}^\gamma(1) = W_\alpha J_{\beta\alpha}(q) - \Gamma_{\alpha\beta\lambda}^{(1)} J_{\lambda\alpha}(q) + \Gamma_{\alpha\beta\lambda}^{(1)} J_{\lambda\alpha}(q) = \left( W_\alpha J_{\beta\alpha}(1) - 2i \Gamma_{\alpha\beta}(1) \right) \).

\[
Q_{\beta\alpha}(1) = \left( W_\alpha J_{\beta\alpha}(1) - 2i \Gamma_{\alpha\beta}(1) \right).
\]

i.e., \( \Gamma_{\beta\alpha}(1) = \frac{i}{2} Q_{\beta\alpha}(1) \).

Firstly we deal with the term \( Q_{\beta\alpha}(1) \) in (3.14). Take index \( a \) to be \( \alpha \) in (3.7) and consider homogeneous parts of degree 1. Noting \( J_{\alpha\beta}(q) = i \delta_{\alpha\beta}, J_{\alpha\beta}(q) = 0 \) by Proposition 2.6, \( J_{\alpha\beta}(1) = 0 \) by Proposition 3.1 and \( Q_{\beta\alpha}(q) = 0 \) by (3.4) in Proposition 3.2, we get

\[
\left( -\Gamma_{\beta\alpha}^{(1)} + \Gamma_{\beta\alpha}^{(1)} \right) J_{\alpha\beta}(1) = (-\Gamma_{\beta\alpha}^{(1)} + \Gamma_{\beta\alpha}^{(1)}) J_{\alpha\beta}(1) h_{\alpha\beta} = -i \Gamma_{\beta\alpha}^{(1)} + i \Gamma_{\beta\alpha}^{(1)},
\]

and

\[
\left( \left( W_\beta J_{\beta\alpha}^{(1)} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} + J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \right) h_{\alpha\beta} \right) = \left( W_\beta J_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \delta_{\alpha\beta} + J_{\beta\alpha}^{(1)} \left( \Gamma_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} \}
\]

and

\[
\left( \left( W_\beta J_{\beta\alpha}^{(1)} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} + J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \right) h_{\alpha\beta} \right) = \left( W_\beta J_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \delta_{\alpha\beta} + J_{\beta\alpha}^{(1)} \left( \Gamma_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} \}
\]

and

\[
\left( \left( W_\beta J_{\beta\alpha}^{(1)} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} + J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \right) h_{\alpha\beta} \right) = - \left( W_\beta J_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} - J_{\beta\alpha}^{(1)} \Gamma_{\beta\alpha}^{(1)} \delta_{\alpha\beta} + J_{\beta\alpha}^{(1)} \left( \Gamma_{\beta\alpha}^{(1)} \right) \delta_{\alpha\beta} \}
\]
and
\[
\left\{ J^\gamma_\beta(W^c_\lambda J^d_\mu) + J^\gamma_\beta J^c_\rho \Gamma^d_{ce} - J^\gamma_\rho J^c_\beta \Gamma^d_{ec} \right\} J^d_\delta \{ f_{\delta} \} \tag{3.18}
\]

\[
= i \delta_{\alpha \delta} \left( J^\gamma_\beta(W_\lambda J^c_\mu(q) + J^\gamma_\beta(q) \left( W_\lambda J^c_\mu(q) \right) \right. \\
\left. + J^\gamma_\beta(q) J^\gamma_\beta(q) \Gamma^\mu_{\lambda \delta}(1) \right)
\]

\[
- J^\gamma_\rho(q) W_\sigma J^\gamma_\beta(q) - J^\gamma_\rho(q) \left( W_\sigma J^\gamma_\beta(q) \right) \right. \\
\left. + \left( J^\gamma_\beta(q) W_\lambda J^\mu_\rho(q) - J^\gamma_\rho(q) W_\sigma J^\mu_\beta(q) \right) J^\gamma_\alpha(1) h_{\mu \gamma}
\]

\[
= i J^\gamma_\beta(Q^\alpha_{\rho \lambda}(q)) + \left( W_\beta J^\alpha_\rho(1) \right) - i \Gamma^\alpha_{\rho \beta}(1) - i J^\gamma_\rho(1) Q^\alpha_{\rho \alpha}(q) - \left( W_\rho J^\alpha_\beta(1) \right) + i \Gamma^\alpha_{\rho \beta}
\]

\[
= \left( W_\beta J^\alpha_\rho(1) \right) - i \Gamma^\alpha_{\rho \beta}(1) + i Q^\gamma_{\rho \beta}(q) J^\gamma_\alpha(1) + i Q^\gamma_{\rho \beta}(q) J^\gamma_\alpha(1).
\]

Substitute the summation of (3.15)–(3.18) into (3.7) to get
\[
Q_\rho^\alpha \equiv \frac{i}{2} \left( Q^\gamma_{\rho \beta}(q) - Q^\gamma_{\rho \beta}(q) \right) J^\gamma_\alpha(1). \tag{3.19}
\]

Now we deal with \((W_\alpha J^\gamma_\beta(1)\) in (3.14). By expansion (2.19), \(J^\gamma_\beta(1) = z^{c} Z_\lambda J^\gamma_\beta(q)\) is independent with \(t\). And note that \(W(0) = z^{0} \frac{\partial q}{\partial t} \) by (2.26). So \(W(0)(J^\gamma_\beta(1) = 0\). By \(J^\gamma_\beta(1) = 0\) (cf. (2.14)) and \(J^\gamma_\beta(2)\) being independent with \(t\) in Proposition 3.4, we get
\[
\left( W_\alpha J^\gamma_\beta(1) \right) = \left( Z_\alpha + W_\alpha(0) + W_\alpha(1) + \mathcal{O}_{2} \right) \left( J^\gamma_\beta(1) + J^\gamma_\beta(2) + \mathcal{O}_{3} \right)
\]

\[
= W_\alpha(0) (J^\gamma_\beta(1)) + Z_\alpha (J^\gamma_\beta(2)) = Z_\alpha (J^\gamma_\beta(2))
\]

\[
= \frac{i}{2} \left( z^{c} Z_\lambda Z_\alpha J^\gamma_\beta(q) + z^{c} Z_\lambda Z_\theta J^\gamma_\beta(q) \right). \tag{3.20}
\]

Now substitute (3.19) and (3.20) into (3.14) to get
\[
Z_\lambda \left( \Gamma^\alpha_{\mu \beta}(1) \right) = \frac{i}{2} Z_\lambda \left( Q^\alpha_{\beta \mu}(1) - (W_\mu J^\alpha_\beta(1) \right)
\]

\[
= -\frac{1}{4} Z_\lambda \left( \left( Q^\alpha_{\beta \mu}(q) - Q^\alpha_{\beta \mu}(q) \right) \right. \\
\left. - J^\gamma_\mu(1) \right)
\]

\[
- \frac{i}{4} Z_\lambda \left( z^{c} Z_\mu Z_\lambda J^\alpha_\beta(q) + z^{c} Z_\lambda Z_\mu J^\alpha_\beta(q) \right)
\]

\[
= \frac{1}{4} \left( \left( Q^\alpha_{\beta \mu}(q) - Q^\alpha_{\beta \mu}(q) \right) Q^\alpha_{\mu \lambda}(q) \right) - \frac{i}{4} Z_\mu Z_\lambda J^\alpha_\beta(q) - \frac{i}{4} Z_\lambda Z_\mu J^\alpha_\beta(q).
\]

In the last identity, we use the relation \(Q^\alpha_{\mu \lambda}(q) = W_\lambda J^\alpha_\mu(q) = Z_\lambda (J^\alpha_\mu(1)) = Z_\lambda (J^\alpha_\mu(1))\) by (3.2) and \(J^\gamma_\mu(q) = 0\). Similarly, by exchanging \(\lambda\) and \(\mu\), we get
\[
Z_\mu \left( \Gamma^\alpha_{\lambda \beta}(1) \right) = \frac{1}{4} \left( \left( Q^\alpha_{\beta \mu}(q) - Q^\alpha_{\beta \mu}(q) \right) Q^\alpha_{\beta \mu}(q) \right) - \frac{i}{4} Z_\mu Z_\lambda J^\alpha_\beta(q) - \frac{i}{4} Z_\lambda Z_\mu J^\alpha_\beta(q).
\]
Substituting (3.21)–(3.22) into (3.13), we get (3.11). In particular,
\[ R^α_{β \bar{α} β}(q) = \frac{1}{4} \left( Q_{β α}(q) Q^α_{β \bar{α}}(q) - Q_{β α}(q) Q^α_{β \bar{α}}(q) - Q_{α β}(q) Q^α_{α \bar{β}}(q) + Q_{α β}(q) Q^α_{α \bar{β}}(q) \right) \]
\[ = \frac{1}{2} \left( Q_{β α}(q) Q^α_{β \bar{α}}(q) - |Q^α_{α β}(q)|^2 \right) = -\frac{1}{4} |Q^α_{α β}(q)|^2 = -\frac{1}{4} \Omega, \]
by (3.4).

\[ \square \]

**Proposition 3.6** (A Bianchi identity) With respect to a special frame associated with \( \theta \), components of the curvature tensor satisfy the following relation:
\[ R^α_{β μ λ} + R^α_{δ λ β} + R^α_{μ β λ} = 0, \mod A \cup Γ. \] (3.23)

\( Γ \) means terms depending on \( Γ^l_{jk} \) linearly or quadratically.

**Proof** Differentiate the second identity of the structure equation (2.9) to get
\[ 0 = dθ^β \wedge ω^α_{β} - θ^β \wedge dω^α_{β} + dθ^{\bar{β}} \wedge ω^α_{\bar{β}} - θ^{\bar{β}} \wedge dω^α_{\bar{β}} + dθ \wedge τ^α, \mod θ. \] (3.24)

By the definition \( ω^α_{β} = Γ^b_{j a β} \theta^j \) and \( τ^α \), we have
\[ dθ^b \wedge ω^α_{b} = 0, \mod Γ, \text{ and } dθ \wedge τ^α = 0, \mod A, \]
and by the third identity in (2.9)
\[ dω^α_{b} = R^b_{a λ μ} \theta^λ \wedge dθ^μ + \frac{1}{2} R^b_{a λ μ} \theta^λ \wedge θ^μ + \frac{1}{2} R^b_{a λ μ} \theta^λ \wedge θ^{\bar{μ}}, \mod θ \cup Γ. \]
Consequently, by substituting the above identities into (3.24), we find that
\[ 0 = -R^α_{β λ μ} dθ^β \wedge θ^λ \wedge θ^{\bar{μ}} - \frac{1}{2} R^α_{β λ μ} dθ^{\bar{β}} \wedge θ^λ \wedge θ^{\bar{μ}}, \mod A \cup Γ. \]
Consequently,
\[ 0 = -R^α_{β λ μ} + R^α_{β λ μ} - \frac{1}{2} R^α_{λ β μ} + \frac{1}{2} R^α_{\bar{μ} λ β} = -R^α_{β λ μ} + R^α_{λ β μ} + R^α_{\bar{μ} λ β}, \mod A \cup Γ. \]
\[ \square \]

**4 Normalized special frames**

As in the CR case (cf. [14, Sect. 3]), to simplify the calculation of asymptotic expansion of the Yamabe functional, we choose a conformal contact form
\[ \hat{θ} = e^{2u} θ, \] (4.1)
so that certain components of the Webster torsion and curvature tensors vanish at the point \( q \).

The main theorem of this section is the following.

**Theorem 4.1** For a contact Riemannian manifold \( (M, \theta, h, J) \), there exists \( (\hat{M}, \hat{θ}, \hat{h}, \hat{J}) \) with \( \hat{θ} = e^{2u} θ \) such that
\[ \hat{R}^γ_{α γ β}(q) = 0, \quad \hat{A}^α β(q) = 0, \]
where $\tilde{R}^\gamma_{\alpha\gamma\beta}(q)$ and $\tilde{A}_{\alpha\beta}(q)$ are the components of the curvature tensor and the Webster torsion tensor with respect to the special frame $\{\tilde{W}_a, \tilde{T}\}$ of $(M, \tilde{\theta}, \tilde{\eta}, \tilde{J})$ centered at $q$.

Only in this section, as in the CR case, we will work over the frame $\{W_a, \tilde{T}\}$ under the conformal transformation. We abuse notations to denote components of the Webster torsion tensor and the curvature tensor with respect to $\{W_a, \tilde{T}\}$ also by $\tilde{A}_{\alpha\beta}(q)$ and $\tilde{R}^\gamma_{\alpha\gamma\beta}(q)$. We will explain why we can do so later in Lemma 4.3.

### 4.1 Transformation formulae under a conformal transformation

We do not change the special frame $\{W_a\}$ in the horizontal space. As mentioned in (1.3), under the conformal transformation, we have $(\theta, J, T, h) \rightarrow (\tilde{\theta}, \tilde{J}, \tilde{T}, \tilde{h})$ with

$$\tilde{T} = e^{-2u}(T + J_b^a u^a W_b), \quad \tilde{h}_{ab} = e^{2u} h_{ab}, \quad \tilde{J}_b^a = J_b^a,$$

(4.2)

where $u_a = W_a u$ and $u^a = h_{ab} u_b$. Consequently, $\tilde{J}_{ab} = e^{2u} J_{ab}$.

Let $\{\theta^b, \theta\}$ denote the special coframe. Noting that we do not change $\{W_a\}$, we require $\tilde{\theta}^b(W_a) = \delta^a_a$ and $\tilde{\theta}^b(\tilde{T}) = 0$. So $\tilde{\theta}^b$ changes as

$$\tilde{\theta}^a = \theta^a - J_b^a u^b \theta. \quad (4.3)$$

**Lemma 4.1** Under the conformal transformation (4.1) with $u \in \mathcal{O}_m$, we have

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} - i Z_{\alpha} Z_{\beta} u + \phi_{m-1}, \quad (4.4)$$

$$\tilde{R}^\gamma_{\alpha\gamma\beta} = R^\gamma_{\alpha\gamma\beta} - \frac{n}{2} (Z_{\alpha} Z_{\beta} u + Z_{\beta} Z_{\alpha} u) + \frac{1}{2} \tilde{h}_{\alpha\beta} L_0 u + \phi_{m-1}, \quad (4.5)$$

where we set $L_0 := -(Z_{\alpha} Z_{\alpha} + Z_{\alpha} Z_{\alpha})$.

This lemma will be proved in “Appendix A”. The above transformation formulae under a conformal transformation are similar to the CR case (cf. [14, Lemma 3.6]), but with error terms $\phi_{m-1}$ instead of $\phi_m$. This is sufficient for our purpose.

**Lemma 4.2** Under the conformal transformation (4.1) with $u \in \mathcal{O}_m$, $m \geq 2$, the connection 1-forms of the TWT connections change as

$$\tilde{\omega}^b_a = \omega^b_a + \phi_m. \quad (4.6)$$

**Proof** First note that by (A.6) we have

$$\tilde{\Gamma}_{ca}^b = \Gamma_{ca}^b + \phi_{m-1}, \quad \tilde{\Gamma}_{0a}^b = \Gamma_{0a}^b + \phi_{m-2}. \quad (4.6)$$

By (4.1), (4.3) and $e^{2u} = 1 + \phi_m$, we get

$$\tilde{\theta} = (1 + \phi_m) \theta = \theta + \phi_{m+2}, \quad \tilde{\theta}^a = \theta^a + \phi_{m+1}.$$

So $\tilde{\omega}^b_a = \tilde{\Gamma}_{ca}^b \tilde{\theta}^c + \tilde{\Gamma}_{0a}^b \tilde{\theta}^0 = \omega^b_a + \phi_m. \quad \square$

### 4.2 The conformal contact form with vanishing $R^\gamma_{\alpha\gamma\beta}(q)$ and $A_{\alpha\beta}(q)$

As in the CR case (cf. [14, p. 320]), we define the tensor $S_{ab} \theta^a \otimes \theta^b$ whose components are given by

$$S_{ab} = S_{\alpha\beta} = - (n + 2) i A_{\alpha\beta}(q), \quad S_{\alpha\beta} = S_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}(q).$$

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Proposition 4.1 \( S_{ab} \) is a symmetric tensor.

**Proof** \( S_{\alpha \beta} \) and \( S_{\bar{\alpha} \bar{\beta}} \) are symmetric following directly from the self-adjointness of \( A_{ab} \) (see Lemma 2.1). So we need to prove \( S_{a \bar{\beta}} = S_{\bar{\alpha} \beta} \). Recall that

\[
h(R(X, Y)Z, W) = h(R(W, Z)Y, X) + h((LW \wedge LYZ), X) - h((LX \wedge LYZ), W), \quad (4.7)
\]

(cf. [5, (38)]), for any vector field \( X, Y, Z, W \), where \( (X \wedge Y)Z = h(X, Y)Z - h(Y, Z)X \) and \( L = J - \tau_s \). Now apply (4.7) to \( X = W_\gamma, Y = W_\beta, Z = W_\alpha, W = W_\bar{\mu} \) to get

\[
h(R(W_\gamma, W_\beta)W_\alpha, W_\bar{\mu}) = h(R(W_\bar{\mu}, W_\alpha)W_\beta, W_\gamma) + h((LW_\bar{\mu} \wedge LW_\alpha)W_\beta, W_\gamma)
\]

\[
- h((LW_\gamma \wedge LW_\beta)W_\alpha, W_\bar{\mu}). \quad (4.8)
\]

On the other hand, \( h(LW_\alpha, W_\beta) = h((J - \tau_s)W_\alpha, W_\beta) = h(J^c_\alpha W_\gamma - A^c_\alpha W_\gamma, W_\beta) = J_{ba} - A_{ab} \) by the definition of \( L \). Then we get

\[
h((LW_\bar{\mu} \wedge LW_\alpha)W_\beta, W_\gamma)|_q = h\left( h \left( LW_\bar{\mu}, W_\beta \right) LW_\alpha, W_\gamma \right)|_q
\]

\[
- h \left( h \left( LW_\alpha, W_\beta \right) LW_\bar{\mu}, W_\gamma \right)|_q
\]

\[
= (J_{\bar{\beta} \bar{\mu}} - A_{\bar{\beta} \bar{\mu}})(J_{\gamma \alpha} - A_{\gamma \alpha})|_q - (J_{\bar{\beta} \bar{\alpha}} - A_{\bar{\beta} \bar{\alpha}})(J_{\gamma \bar{\mu}} - A_{\gamma \bar{\mu}})|_q
\]

\[
= A_{\bar{\alpha} \bar{\beta}}(q)J_{\alpha \gamma}(q) - J_{\bar{\alpha} \bar{\beta}}(q)J_{\gamma \bar{\mu}}(q), \quad (4.9)
\]

and

\[
h \left( \left( LW_\gamma \wedge LW_\beta \right) W_\alpha, W_\bar{\mu} \right)|_q = h \left( h \left( LW_\gamma, W_\alpha \right) LW_\beta, W_\bar{\mu} \right)|_q
\]

\[
- h \left( h \left( LW_\beta, W_\alpha \right) LW_\gamma, W_\bar{\mu} \right)|_q
\]

\[
= (J_{\gamma \alpha} - A_{\gamma \alpha})|_q (J_{\bar{\beta} \bar{\alpha}} - A_{\bar{\beta} \bar{\alpha}})|_q
\]

\[
- (J_{\gamma \beta} - A_{\gamma \beta})|_q (J_{\bar{\alpha} \bar{\beta}} - A_{\bar{\alpha} \bar{\beta}})|_q
\]

\[
= A_{\bar{\alpha} \bar{\beta}}(q)J_{\alpha \gamma}(q) - J_{\bar{\alpha} \bar{\beta}}(q)J_{\gamma \bar{\mu}}(q), \quad (4.10)
\]

at the point \( q \) by using Proposition 2.6 and Corollary 2.1. Substitute (4.9)–(4.10) into (4.8) to get

\[
R_{\bar{\alpha} \bar{\beta} \gamma \alpha}(q) = R_{\bar{\beta} \gamma \bar{\alpha} \bar{\alpha}}(q) \quad (\text{at } q).
\]

Hence \( S_{\alpha \beta} = R_{\alpha \gamma \bar{\beta} \alpha}(q) = h_{\gamma \bar{\mu}}R_{\alpha \bar{\mu} \gamma \beta}(q) = R_{\bar{\gamma} \beta \gamma \alpha}(q) = R_{\bar{\gamma} \beta \gamma \alpha}(q) = S_{\bar{\alpha} \bar{\beta}} \). So the tensor \( S_{ab} \) is symmetric. □

**Proof of Theorem 4.1** If \( u = u(z) \) is a polynomial homogeneous of degree \( m \) but independent of \( t \), we denote \( u \in \mathcal{P}_m \). We assume \( u \in \mathcal{P}_2 \) in the conformal transformation (4.1).

For the symmetric tensor \( S_{ab} \) as above, we define the polynomial \( S = S_{ab}z^a z^b \). By Lemma 4.1, for \( u \in \mathcal{P}_2 \), we have

\[
\hat{S}_{a \bar{\beta}} = S_{a \bar{\beta}} - (n + 2)Z_\alpha Z_\beta u, \quad \hat{S}_{\bar{a} \beta} = S_{\bar{a} \beta} - \frac{n + 2}{2} (Z_\bar{\beta} Z_\alpha u + Z_\alpha Z_\bar{\beta} u) + \frac{1}{2} h_{\alpha \bar{\beta}} L_0 u.
\]

Now let \( \hat{S} = \hat{S}_{ab}z^a z^b \), we get

\[
\hat{S} = (S_{a \beta} - (n + 2)Z_\alpha Z_\beta u) z^a z^b + \left( S_{\alpha \beta} - \frac{n + 2}{2} (Z_\bar{\beta} Z_\alpha u + Z_\alpha Z_\bar{\beta} u) + \frac{1}{2} h_{\alpha \bar{\beta}} L_0 u \right) z^\alpha z^\bar{\beta}
\]

\[
+ \left( S_{\bar{a} \beta} - \frac{n + 2}{2} (Z_\bar{\beta} Z_\bar{a} u + Z_\bar{a} Z_\bar{\beta} u) + \frac{1}{2} h_{\beta \bar{\alpha}} L_0 u \right) z^\bar{\alpha} z^\beta
\]

\[
= S - (n + 2)z^a Z_a Z_b u + |z|^2 L_0 u. \quad (4.11)
\]

\( \hat{S} \) is a symmetric polynomial. Springer
Note that
\[ m^2 u = p^2 u = (z^a Z_a + 2z^0 Z_0)^2 u = z^a z^b Z_a Z_b u + 4z^0 z^a Z_0 Z_a u + 4z^0 z^0 Z_0 Z_0 u + 2z^0 Z_0 u + Pu, \]
(cf. [14, p. 320]). Thus for \( u \in \mathcal{R}_2 \), we have \( z^a z^b Z_a Z_b u = 2u \). Therefore by (4.11), \( \hat{S} = S - 2(n + 2) u + |z|^2 L_0 u \). The operator \(-2(n + 2) + |z|^2 L_0 \) is invertible on \( \mathcal{R}_2 \) by \( |z|^2 L_0 \) having no positive eigenvalues on the Heisenberg group (cf. [14, Lemma 3.9]). So we can find \( u = u_0 \in \mathcal{R}_2 \) such that \( \hat{S}(u_0) = 0 \). Namely \( u_0 \) satisfies \(-2(n + 2) u_0 + |z|^2 L_0 u_0 = -S(u_0) = -z^a z^b S_{ab}(q) \). Under the conformal transformation \( \hat{\theta} = e^{2u_0} \theta \) for such \( u_0 \), we have
\[ \hat{R}_{\alpha' \beta' \gamma' \delta'}(q) = 0, \quad \hat{A}_{\alpha' \beta'}(q) = 0, \tag{4.12} \]
with respect to a special frame \( \{ W_a, \hat{T} \} \) of \( (M, \theta, h, J) \) centered at \( q \).

**Corollary 4.1** \( \hat{Q}^\flat_{\beta \alpha}(q) = \tilde{Q}^\flat_{\beta \alpha}(q) \).

This is because \( \hat{Q}^\flat_{\beta \alpha}(q) = W_a \hat{\mathcal{T}}^\flat_{\beta \alpha}(q) = W_a J^\flat_{\beta \alpha}(q) \) by (3.2) and (4.2).

**Lemma 4.3** Under the conformal transformation (4.1) with \( u \in \mathcal{R}_2 \), changing the special frame \( \{ W_a \} \) of \( (M, \theta, h, J) \) centered at \( q \) to a special frame \( \{ \hat{W}_a \} \) of \( (M, \hat{\theta}, \hat{h}, \hat{J}) \) centered at \( q \) makes the value of the curvature tensor and the Webster torsion tensor at \( q \) invariant. So we can abuse notations to write \( \hat{R}_{\alpha' \beta' \gamma' \delta'}(q) \) and \( \hat{A}_{\alpha' \beta'}(q) \) no matter they are with respect to \( \{ W_a, \hat{T} \} \) or \( \{ \hat{W}_a, \hat{T} \} \).

**Proof** Since \( \{ W_a \} \) and \( \{ \hat{W}_a \} \) are both horizontal, we write \( \hat{W}_a = v^b_a W_b \) for some invertible matrix \( \{ v^b_a \} \). The value of \( \{ W_a \} \) and \( \{ \hat{W}_a \} \) at the point \( q \) is decided by relation
\[ \hat{h}(\hat{W}_a, \hat{W}_\beta) = \delta_{a \beta} = h(W_a, W_\beta), \quad \hat{h}(\hat{W}_a, \hat{W}_\beta) = 0 = h(W_a, W_\beta), \]
at \( q \). By (4.2), if \( u \in \mathcal{R}_2 \), we have \( h_{ab} = h(W_a, W_b) = \hat{h}(\hat{W}_a, \hat{W}_b) = (1 + \omega_2) v^c_a v^d_b h(W_c, W_d) = v^c_a v^d_b (1 + \omega_2) h_{cd} \). So the special frame satisfies \( \hat{W}_a = v^b_a W_b \) with \( v^b_a = \delta^b_a + \omega_1 \).

It is well known that \( R_{abcd} \) and \( A_{ab} \) are covariant tensors. So with changing \( W_a \rightarrow \hat{W}_a = v^b_a W_b \), these components change as \( R_{abcd} \rightarrow v^a_i v^b_j v^c_k v^d_l R_{i j k l}, \quad A_{ab} \rightarrow A_{ai} b^a v^b_1 \). So their value at \( q \) does not change since \( v \) is the identity transformation at \( q \). □

So (4.12) also holds with respect to a special frame of \( (M, \hat{\theta}, \hat{h}, \hat{J}) \) centered at \( q \).

5 Proof of the main theorem

5.1 Asymptotic expansion of the Yamabe functional

**Lemma 5.1** (cf. [21, Theorem 11.3]) For a contact Riemannian manifold \((M, \theta, h, J)\), the Yamabe functional \( \mathcal{Q}_{\theta, h}(u) \) in (1.5) is invariant under a conformal transformation.

Consider a contact Riemannian manifold \((M, \theta, h, J)\) with \( J \) not integrable. There exists a point \( q \) such that the Tanno tensor \( Q(q) \neq 0 \). By Proposition 3.2, we must have \( \mathcal{Q} > 0 \) at point \( q \). By Lemma 5.1, we can choose \((M, \hat{\theta}, \hat{h}, \hat{J})\) conformal to \((M, \theta, h, J)\) such that the components of curvature and Webster torsion tensors satisfy Theorem 4.1. \( \mathcal{Q} > 0 \) also holds with respect to \((M, \hat{\theta}, \hat{h}, \hat{J})\) by Corollary 4.1. We denote this \((M, \hat{\theta}, \hat{h}, \hat{J})\) as \((M, \theta, h, J)\) in this section.

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Proposition 5.1 We can choose \((M, \theta, h, J)\) in its conformal class such that with respect to a special frame \(\{W_\alpha, T\}\) of \((M, \theta, h, J)\) centered at \(q\), we have
\[
A_{ab}(q) = 0, \quad R^y_{\alpha y \beta}(q) = 0, \quad R^\alpha_{\beta \alpha \beta}(q) = -\frac{1}{4} \Omega, \quad R^\alpha_{\beta \beta \gamma}(q) = -\frac{1}{4} \Omega.
\]

Proof We only need to show the last identity since the others are given by Corollary 2.1, Theorem 4.1 and (3.12) in Proposition 3.5. Since we already have \(A_{ab}(q) = 0, R^\alpha_{\beta \alpha \beta}(q) = 0\) and \(\Gamma^i_{jk}(q) = 0\) (by (2.27)), the Bianchi-type identity (3.23) at \(q\) gives us
\[
0 = -R^\alpha_{\beta \alpha \mu}(q) + R^\alpha_{\beta \mu \alpha}(q) + R^\alpha_{\mu \alpha \beta}(q) = R^\alpha_{\beta \beta \mu}(q) + R^\alpha_{\mu \alpha \beta}(q),
\]
which implies \(R^\alpha_{\beta \beta \mu}(q) = -R^\alpha_{\beta \alpha \beta}(q) = \frac{1}{4} \Omega. \quad \Box \)

Then we have the main theorem of this section.

Theorem 5.1 For a contact Riemannian manifold \((M, \theta, h, J)\) such that \(Q(q) \neq 0\) for some point \(q\). If we choose the normalized contact form and the special frame as Proposition 5.1, then (1.9) holds. In particular, there exists \(\epsilon > 0\) such that \(\mathcal{F}_\theta(f^\epsilon) < \lambda(\mathcal{H}^n)\).

We write the volume form of the contact manifold \(dV_\theta = (-1)^n \theta \wedge d\theta^n\) as
\[
dV_\theta = (v_0 + v_1 + v_2 + \theta_3) dV,
\]
where \(v_j\) is a homogeneous polynomial of degree \(j = 0, 1, 2\) and \(dV = (-1)^n \theta \wedge d\theta\). By \(\theta_{(2)} = \emptyset\) in (2.22) we find \(v_0 = 1\).

Proposition 5.2 On the contact Riemannian manifold \((M, \theta, h, J)\), we have
\[
\int_M |f^\epsilon|^p dV_\theta = a_0(n) + a_1(n) \epsilon + a_2(n) \epsilon^2 + O(\epsilon^3),
\]
\[
\int_M |d f^\epsilon|^2_{\theta} dV_\theta = b_0(n) + b_1(n) \epsilon + b_2(n) \epsilon^2 + O(\epsilon^3),
\]
\[
\int_M |f^\epsilon|^2 dV_\theta = c_2(n) \epsilon^2 + O(\epsilon^3),
\]
where
\[
a_m(n) = \int_{\mathcal{H}^n} |\Phi|^p v_m dV, \quad b_m(n) = 2 \int_{\mathcal{H}^n} v_m^{jk} Z_j \Phi Z_k \Phi dV,
\]
\[
c_2(n) = \int_{\mathcal{H}^n} R(q) |\Phi|^2 dV.
\]
\(m = 0, 1, 2\). Here \(v_m\) is given by (5.1) and
\[
v_m^{jk} = \sum_{m_0 + m_1 + m_2 = m \atop m_i \geq 0} v_{m_0} s_{j}^{\gamma} \beta^{(m_1 + o(j) - 1)} \gamma_{\beta(m_2 + o(k) - 1)}.
\]

Proof The estimates (5.2) is similar to the CR case (cf. [14, Proposition 4.2]), but the third identity of (5.2) is \(O(\epsilon^3)\) in the CR case with \(R(q) = 0\). We sketch the proof here. First note that if a function \(|\phi| \leq CF(\rho)\), then \(\int_{a < \rho < b} \phi dV = O\left(\int_a^b F(\rho) \rho^{2n+1} d\rho\right)\). If we replace
(z, t) by $\delta_\varepsilon(z, t) = (\varepsilon z, \varepsilon^2 t)$, we have $\delta^*_\varepsilon \Phi^\varepsilon = \varepsilon^{-n} \Phi$, $\delta^*_\varepsilon (dV) = \varepsilon^{2n+2} dV$. We also note that $\Phi \leq \mathcal{C}(1 + \rho)^{-2n}$ (cf. [14, p. 330]). So

$$
\int_M |f^\varepsilon|^p dV_\theta = \int_{\mathcal{H}^n} |\psi|^p |\Phi^\varepsilon|^p (1 + v_1 + v_2 + O(\rho^3)) dV
$$

= \int_{\rho < \varepsilon / \kappa} |\Phi|^p (1 + \varepsilon v_1 + \varepsilon^2 v_2 + O(\varepsilon^3 \rho^3)) dV + O\left( \int_{\varepsilon / \kappa < \rho < 2\varepsilon / \kappa} |\Phi|^p dV \right)

= \int_{\mathcal{H}^n} |\Phi|^p (1 + \varepsilon v_1 + \varepsilon^2 v_2) dV + O\left( \int_{\varepsilon / \kappa} \sum_{i=0}^{2} \varepsilon^i \rho^i (1 + \rho)^{-4n-4} \rho^{2n+1} d\rho \right)

+ O\left( \int_{\varepsilon / \kappa} (1 + \rho)^{-4n-4} \rho^{2n+1} d\rho \right) + O\left( \int_{\varepsilon / \kappa} (1 + \rho)^{-4n-4} \rho^{2n+1} d\rho \right)

= \int_{\mathcal{H}^n} |\Phi|^p (1 + \varepsilon v_1 + \varepsilon^2 v_2) dV + O(\varepsilon^3),

for $n \geq 2$. So we get the first identity in (5.2). Noting that $|df^\varepsilon|_H^2 = (W_\alpha f^\varepsilon \theta^\alpha + W_\beta f^\varepsilon \theta^\beta + W_{\bar{\beta}} f^\varepsilon \bar{\beta}^\beta) = h\bar{\beta}^\beta W_\alpha f^\varepsilon W_\beta f^\varepsilon + h\bar{\alpha}^\alpha W_\alpha f^\varepsilon W_\beta f^\varepsilon = 2W_\alpha f^\varepsilon W_\beta f^\varepsilon$, we can write

$$
\int_M |df^\varepsilon|_H^2 dV_\theta = 2 \int_M W_\beta f^\varepsilon W_\bar{\beta} f^\varepsilon dV_\theta
$$

= 2 \int_{\mathcal{H}^n} s^j_{\bar{\beta}} Z_j (\psi \Phi^\varepsilon) s^k_{\bar{\beta}} Z_k (\psi \Phi^\varepsilon) (1 + v_1 + v_2 + \cdots) dV

= 2 \int_{\rho < \kappa} (v_0^{jk} + v_1^{jk} + v_2^{jk} + O(\rho^{1+o(jk)})) Z_j \Phi Z_k \Phi dV

+ O\left( \int_{\kappa < \rho < 2\kappa} (|Z_j \Phi| |Z_k \Phi| + |Z_j \Phi|^2 |\Phi| + |\Phi|^2) dV \right),

by setting $v_m^{jk}$ as (5.4), which is a homogeneous polynomial of degree $m + o(jk) - 2$. Noting that $\delta^*_\varepsilon (Z_j \Phi^\varepsilon) = \varepsilon^{-n-o(j)} Z_j \Phi$ and $|Z_j \Phi| \leq \mathcal{C}(1 + \rho)^{-2n-o(j)}$ (cf. Lemma 5.5), we get

$$
\int_M |df^\varepsilon|_H^2 dV_\theta = 2 \int_{\rho < \varepsilon / \kappa} \sum_{m=0}^{2} \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi dV
$$

+ $O\left( \int_{v_0}^{\varepsilon / \kappa} \sum_{i=2}^{4} \varepsilon^i \rho^{1+i} (1 + \rho)^{-4n-i} \rho^{2n+1} d\rho \right)

+ O\left( \int_{\varepsilon / \kappa} \sum_{i=0}^{4} \varepsilon^{2-i} (1 + \rho)^{-4n-i} \rho^{2n+1} d\rho \right)

= 2 \int_{\mathcal{H}^n} \sum_{m=0}^{2} \varepsilon^m v_m^{jk} Z_j \Phi Z_k \Phi dV

+ O\left( \int_{\varepsilon / \kappa} \sum_{m=0}^{2} \varepsilon^m \sum_{i=2}^{4} \rho^{m+i-2} (1 + \rho)^{-4n-i} \rho^{2n+1} d\rho \right) + O(\varepsilon^3),

from which we get the second identity in (5.2). The third identity in (5.2) follows from

$$
\int_M R |f^\varepsilon|^2 dV_\theta = \left( \int_{\mathcal{H}^n} R(q) |\Phi|^2 dV \right) \varepsilon^2 + O(\varepsilon^3),
$$
Note that the volume form of the Heisenberg group $dV = (-1)^n \Theta \wedge d\Theta^n$ can be written as
\[
dV = (-1)^n \Theta \wedge d\Theta^n = (-1)^n \Theta \wedge d\Theta^n = (-1)^n dt \wedge (-2idz^\alpha \wedge dz^\bar{\alpha})^n
\]
\[
= 2^n n! dt \wedge (idz^\alpha \wedge dz^\bar{\alpha}) \wedge \cdots \wedge (idz^n \wedge dz^{\bar{n}}) = 4^n n! dt \wedge dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n
\]
\[
= 4^n n! d\mu(z) = 4^n n! 2^{n-1} d\nu(\xi) dr dt,
\]
(5.5)
where $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}^n$, and $d\nu$ is the surface measure on $S^{2n-1} = \{ z \in \mathbb{C}^n : |z| = 1 \}$, normalized so that if we write $z = r \xi$, $\xi \in S^{2n-1}$, then $d\mu(z) = r^{2n-1} dr d\nu(\xi)$.

To calculate $a_m(n)$, $b_m(n)$, $m = 0, 1, 2$, explicitly, we need the following lemmas.

**Lemma 5.2** For a real two-form $\omega = m_{\alpha\beta} dz^\alpha \wedge dz^\beta + 2im_{\alpha\bar{\beta}} dz^\alpha \wedge dz^\bar{\beta} + m_{\bar{\alpha}\beta} dz^\bar{\alpha} \wedge dz^\beta$, we have
\[
n\Theta \wedge \omega \wedge d\Theta^{n-1} = -\delta^\alpha_{\bar{\beta}} m_{\alpha\beta} \Theta \wedge d\Theta^n,
\]
\[
n(n-1)\Theta \wedge \omega^2 \wedge d\Theta^{n-2} = ((\delta^\alpha_{\bar{\beta}} \delta^\rho_{\bar{\sigma}} - \delta^\alpha_{\bar{\sigma}} \delta^\rho_{\bar{\beta}}) m_{\alpha\bar{\beta}} m_{\rho\bar{\sigma}}
+ \frac{1}{2} (\delta^\alpha_{\bar{\beta}} \delta^\rho_{\bar{\sigma}} - \delta^\alpha_{\bar{\sigma}} \delta^\rho_{\bar{\beta}}) m_{\alpha\bar{\beta}} m_{\rho\bar{\sigma}}) \Theta \wedge d\Theta^n.
\]

**Proof** This is essentially [14, Lemma 5.1] up to signs. To avoid confusion, we will not use the summation convention in the proof of this lemma. Note that
\[
n\Theta \wedge \omega \wedge d\Theta^{n-1} = n dt \wedge \left( 2im_{\alpha\bar{\beta}} dz^\alpha \wedge dz^\bar{\beta} \right) \wedge \left( -2i \sum_{\gamma} dz^\gamma \wedge dz^\bar{\gamma} \right)^{n-1}
\]
\[
= (-2i)^n n! dt \wedge \left( -\sum_{\alpha, \beta} m_{\alpha\bar{\beta}} dz^\alpha \wedge dz^\bar{\beta} \right) \wedge \left( \sum_{\gamma} dz^1 \wedge dz^\bar{1} \wedge \cdots \wedge dz^n \wedge dz^\bar{n} \wedge \cdots \wedge dz^\bar{n} \wedge dz^n \right)
\]
\[
= (-2i)^n n! \sum_{\alpha} ( - m_{\alpha\alpha} dt \wedge dz^1 \wedge dz^\bar{1} \wedge \cdots \wedge dz^n \wedge dz^\bar{n} \wedge \cdots \wedge dz^\bar{n} \wedge dz^n = -\delta^\alpha_{\bar{\beta}} m_{\alpha\bar{\beta}} \Theta \wedge d\Theta^n.
\]

Here $\widehat{dz^\gamma}$ means it does not appear in the product. And for the second identity, we can prove in the same way as the second identity in [14, Lemma 5.1].

**Corollary 5.1**
\[
v_1 = 0; \quad v_2 = -\frac{1}{6} R_{\bar{\rho}}^\alpha \alpha \mu (q) z^\bar{\rho} z^\mu - \frac{1}{6} R_{\alpha}^{\bar{\rho}} \bar{\alpha} \bar{\mu} (q) z^\beta z^\bar{\mu}, \quad mod \quad z^\beta z^\bar{\mu}, \quad z^\bar{\beta} z^\mu.
\]
(5.6)

**Proof** By the definition of $v_1$, $v_1 \Theta \wedge d\Theta^n = (-1)^n (dV_\theta)_{(2n+3)} = (\theta \wedge d\theta^n)_{(2n+3)} = (\theta_{(3)} \wedge d\theta^n + n \Theta \wedge (d\theta)_{(3)} \wedge d\Theta^{n-1})$.

By (2.22), $\theta_{(3)}$ has no $dt$ term and so $\theta_{(3)} \wedge d\Theta^n$ vanishes. By (2.9) and Proposition 3.1,
\[
(\theta_{(3)}) = J_{\alpha\beta(i)} dz^\alpha \wedge dz^\beta + J_{\alpha\bar{\beta}(i)} dz^\alpha \wedge dz^\bar{\beta},
\]
(5.7)
which has no $dz^\alpha \wedge dz^\beta$ term. So by Lemma 5.2, $\Theta \wedge (d\theta)_{(3)} \wedge d\Theta^{n-1} = 0$. Hence $v_1 = 0$. 

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By the definition of $v_2$ in (5.1), we see that

$$
v_2dV = (-1)^n v_2 \Theta \land d\Theta^n = (dV_\theta)_{(2n+4)} = (-1)^n (\theta \land d\theta^n)_{(2n+4)}$$

$$= (-1)^n (\theta_4) \land d\Theta^n + n\Theta \land (d\theta)_4 \land d\Theta^{n-1} + n\theta_{(3)} \land (d\theta)_3 \land d\Theta^{n-1}$$

$$+ \frac{n(n-1)}{2} \Theta \land ((d\theta)_3)^2 \land d\Theta^{n-2}.$$ 

By (2.22) and $A_{ab}(q) = 0$, $\theta_4$ has no $dt$ term, and so the first term of the right-hand side vanishes. By (2.22) and (5.7), $\theta_{(3)}$ and $(d\theta)_3$ have no $dt$ term, and so the third term of the right-hand side vanishes. Now apply Lemma 5.2 to $\omega = (d\theta)_3$ in (5.7) to get

$$\frac{n(n-1)}{2} \Theta \land ((d\theta)_3)^2 \land d\Theta^{n-2} = \frac{1}{4} (J_{ab}(1)J_{ab}(1) - J_{ab}(1)J_{ab}(1))\Theta \land d\Theta^n$$

$$= \frac{1}{2} J_{ab}(1)J_{ab}(1)\Theta \land d\Theta^n = \frac{1}{2} Q_{\beta\gamma}(q)Q_{\beta\mu}(q)z^\gamma z^\mu \Theta \land d\Theta^n,$$

by $J_{ab} = -J_{ba}$ in (2.4) and (3.8). Noting that by the structure equation (2.9), $J_{ab}(q) = 0$ by (2.14), $J_{ab}(1) = 0$ and $\theta_{a(2)} = 0$ by (2.22), we have

$$(d\theta)_4 = (J_{ab}\theta^a \land \theta^b + 2J_{ab}\theta^a \land \theta^b + J_{ab}\theta^a \land \theta^b)_{(4)}$$

$$= 2J_{ab}(2)\bar{z}^\alpha \land d\bar{z}^\beta + 2J_{ab}(1)\theta^a_{(3)} \land d\bar{z}^\beta + 2J_{ab}(1)\theta^a \land d\Theta^n \theta_{(3)}$$

$$= 2J_{ab}(2)\bar{z}^\alpha \land d\bar{z}^\beta - \frac{i}{3} R^\alpha_{\beta \gamma}(q)z^\gamma d\bar{z}^\alpha \land d\bar{z}^\beta - \frac{i}{3} R^\alpha_{\beta \mu}(q)z^\mu d\bar{z}^\alpha \land d\bar{z}^\beta$$

$$+ 2i \left(-i J_{ab}(2)\bar{z}^\alpha \land d\bar{z}^\beta + \frac{1}{6} R^\alpha_{\beta \gamma}(q)z^\gamma \bar{z}^\mu d\bar{z}^\alpha \land d\bar{z}^\beta + \frac{1}{6} R^\alpha_{\beta \mu}(q)z^\mu \bar{z}^\gamma d\bar{z}^\alpha \land d\bar{z}^\beta \right)$$

$$\mod d\bar{z}^\alpha \land d\bar{z}^\beta, \quad d\bar{z}^\alpha \land d\bar{z}^\beta, \quad \bar{z}^\alpha z^\beta, \quad \bar{z}^\alpha z^\beta, \quad \bar{z}^\alpha\bar{z}^\beta,$$

by using Corollary 2.2. Now apply Lemma 5.2 to $\omega = (d\theta)_4$ to get

$$n\Theta \land (d\theta)_4 \land d\Theta^{n-1} = \left( i J_{a\tilde{a}(2)} - \frac{1}{6} R^a_{\beta a\mu}(q)z^\beta z^\mu - \frac{1}{6} R^a_{\beta \tilde{a}(2)}(q)z^\beta z^\mu \right) \Theta \land d\Theta^n,$$

$$= \left(-\frac{1}{2} Q_{ab}(q)Q_{a\mu}(q)z^\beta z^\mu - \frac{1}{6} R^a_{\beta a\mu}(q)z^\beta z^\mu - \frac{1}{6} R^a_{\beta \tilde{a}(2)}(q)z^\beta z^\mu \right) \Theta \land d\Theta^n, \mod z^\alpha z^\beta, \quad \bar{z}^\alpha\bar{z}^\beta.$$  

Here we have used (3.9) for $J_{a\tilde{a}(2)}$ and Proposition 5.1 for $R^a_{\beta a\mu}(q) = R^a_{\beta \tilde{a}(2)}(q) = 0$. So we conclude that

$$v_2 \Theta \land d\Theta^n = n\Theta \land (d\theta)_4 \land d\Theta^{n-1} + \frac{n(n-1)}{2} \Theta \land ((d\theta)_3)^2 \land d\Theta^{n-2}$$

$$= \left(-\frac{1}{6} R^a_{\beta a\mu}(q)z^\beta z^\mu - \frac{1}{6} R^a_{\beta \tilde{a}(2)}(q)z^\beta z^\mu \right) \Theta \land d\Theta^n, \mod z^\beta z^\mu, \quad z^\beta z^\mu.$$

We finish the proof of this corollary.  

\(\square\)

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5.2 Calculation of some integrals

Lemma 5.3 (cf. [14, Proposition 5.3]) Let $A = (\alpha_1, \ldots, \alpha_m)$, $B = (\beta_1, \ldots, \beta_m)$ be the multi-indices with $1 \leq \alpha_i, \beta_i \leq n$, let $\delta(A, B) = 1$ if $A = B$ and 0 otherwise. Then

$$\int_{S^{2n-1}} z^{\alpha_1} \ldots z^{\alpha_m} \tilde{z}^{\beta_1} \ldots \tilde{z}^{\beta_m} dV = \frac{2\pi^n}{(n + m - 1)!} \sum_{\sigma \in S_m} \delta(A, \sigma B).$$

Lemma 5.3 leads to following corollary.

Corollary 5.2 If $F$ is a function of $r$ and $t$, denote $\mathcal{F}_m := \int_{-\infty}^{\infty} \int_0^\infty F(r, t)r^m dr dt$. We have

$$\int_{\mathcal{H}^n} Q_{\mathcal{a}}^{\mathcal{p}}(q) Q_{\mathcal{b}}^{\mathcal{p}}(q) z^\mathcal{a} \tilde{z}^\mathcal{b} F dV = \frac{(4\pi)^n}{2} \mathcal{F}_{2m+1},$$

$$\int_{\mathcal{H}^n} R_{\mathcal{p}}^{\mathcal{a}}(q) z^\mathcal{a} F dV = \int_{\mathcal{H}^n} R_{\mathcal{p}}^{\mathcal{a}}(q) \tilde{z}^\mathcal{a} F dV = \frac{(4\pi)^n}{2} \mathcal{F}_{2m+1},$$

$$\int_{\mathcal{H}^n} R_{\mathcal{p}}^{\mathcal{a}}(q) z^\mathcal{a} F dV = \int_{\mathcal{H}^n} R_{\mathcal{p}}^{\mathcal{a}}(q) \tilde{z}^\mathcal{a} F dV = 0,$$

$$\int_{\mathcal{H}^n} Q_{\mathcal{a}}^{\mathcal{p}}(q) Q_{\mathcal{b}}^{\mathcal{p}}(q) z^\mathcal{a} \tilde{z}^\mathcal{b} F dV = \frac{3(4\pi)^n}{n + 1} \mathcal{F}_{2m+3},$$

$$\int_{\mathcal{H}^n} J_{\mathcal{p}}(\mathcal{a}(2)) z^\mathcal{a} \tilde{z}^\mathcal{b} F dV = 0,$$
where by (3.3) and the last identity in (3.4). By (3.9), we get

\[ J_{\alpha \beta}(2)z^{\alpha}z^{\beta} F dV = \frac{i}{2} \int_{\mathcal{M}} Q_{\alpha \lambda}(q) Q_{\beta \mu}(q)z^{\alpha}z^{\lambda}z^{\beta}z^{\mu} F dV = \frac{3(4\pi)^n}{2(n+1)} \Omega_{2n+3}. \]

And by Proposition 3.4, we get

\[ \int_{\mathcal{M}} J_{\alpha \beta}(2)z^{\alpha}z^{\beta} F dV = \frac{1}{2} \int_{\mathcal{M}} z^{\alpha}z^{\beta}z^{d}Z_{c}Z_{d}J_{\alpha \beta}(q) F dV = 0 \]

by the antisymmetry of \( J_{\alpha \beta} \). Taking conjugation, we get \( \int_{\mathcal{M}} J_{\alpha \beta}(2)z^{\alpha}z^{\beta} F dV = 0. \quad \square \)

**Lemma 5.4** (cf. [14, Lemma 5.5]) Suppose that \( \alpha, \gamma + 1, \beta + 1 \) and \( \alpha - \gamma - 1 \) are positive real numbers. If \( 2\alpha - 2\gamma - \beta > 3 \), then

\[ \int_{-\infty}^{\infty} \int_{0}^{\infty} |t + i(1 + r^2)|^{-\alpha}r^\beta|t|^{\gamma} dr dt = N_{1}(\alpha, \beta, \gamma), \]

where

\[ N_{1}(\alpha, \beta, \gamma) = \frac{\Gamma \left( \frac{1}{2}(\beta + 1) \right) \Gamma \left( \alpha - \gamma - \frac{1}{2}\beta - \frac{3}{2} \right) \Gamma \left( \frac{1}{2}(\gamma + 1) \right) \Gamma \left( \frac{1}{2}(\alpha - \gamma - 1) \right)}{2\Gamma(\alpha - \gamma - 1)\Gamma(\frac{\beta}{2})}. \]

By the expression of \( N_{1}(\alpha, \beta, \gamma) \) above, we get

\[
\begin{align*}
N_{1}(2n, 2n - 1, 0) &= \frac{\Gamma(n)\Gamma(n - 1)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n - 1}{2} \right)}{2\Gamma(2n - 1)\Gamma(n)} = \frac{\Gamma(n - 1)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n - 1}{2} \right)}{2\Gamma(2n - 1)}, \\
N_{1}(2n + 2, 2n - 1, 0) &= \frac{\Gamma(n)\Gamma(n + 1)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)\Gamma(n + 1)} = \frac{\Gamma(n)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)}, \\
N_{1}(2n + 2, 2n + 1, 0) &= \frac{\Gamma(n + 1)\Gamma(n)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)\Gamma(n + 1)} = \frac{\Gamma(n)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)}, \\
N_{1}(2n + 2, 2n + 3, 0) &= \frac{\Gamma(n + 2)\Gamma(n - 1)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)\Gamma(n + 1)} = \frac{\Gamma(n - 1)\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)}, \\
N_{1}(2n + 4, 2n + 3, 2) &= \frac{\Gamma(n + 2)\Gamma(n - 1)\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)\Gamma(n + 2)} = \frac{\Gamma(n - 1)\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{2n + 1}{2} \right)}{2\Gamma(2n + 1)}.
\end{align*}
\]
which satisfy $2\alpha - 2\gamma - \beta > 3$ for $n > 2$. So we can find that

$$
\frac{N_1(2n, 2n - 1, 0)}{N_1(2n + 2, 2n + 1, 0)} = \frac{4n}{n - 1},
$$

$$
\frac{N_1(2n + 2, 2n + 1, 0)}{N_1(2n + 2, 2n - 1, 0)} = 1,
$$

$$
\frac{N_1(2n + 2, 2n + 3, 0)}{N_1(2n + 4, 2n + 3, 2)} = \frac{n + 1}{n - 1},
$$

$$
\frac{N_1(2n + 2, 2n + 1, 0)}{2(n - 1)}.
$$

(5.9)

### 5.3 Calculation of constants $a_m(n)$ and $b_m(n)$

**Lemma 5.5** For $Z_j$ given by (2.17), we have

$$
Z_\alpha \Phi = i nz^\alpha |t + i(z^2 + 1)|^{-1}, \quad Z_\beta \Phi = -inz^\beta |t - i(z^2 + 1)|^{-1}, \quad Z_0 \Phi = -n |t| w + i|t|^{n+2},
$$

**Proof** By $|w + i|^{-n} = (t^2 + (|z|^2 + 1)^2)^{-\frac{n}{2}}$, we have

$$
\frac{\partial}{\partial t}(|w + i|^{-n}) = -nt|w + i|^{-n-2}, \quad \frac{\partial}{\partial z^\alpha}(|w + i|^{-n}) = -nz^\alpha |w + i|^{-n-2}(|z|^2 + 1).
$$

The result follows. \(\square\)

Note that we have

$$
\int_{S^{2n-1}} d\nu = \frac{2\pi^n}{(n - 1)!},
$$

by the case $m = 0$ in Lemma 5.3, and by (5.9) we have

$$
N_1(2n + 2, 2n + 1, 0) = N_1(2n + 2, 2n - 1, 0) = \frac{4^{n-1}\pi}{2n},
$$

(5.10)

(cf. [14, p. 341] for the second identity). Hence by (5.3), (5.5) and $v_0 = 1$ we get

$$
a_0(n) = \int_{F_{2n}} A^{\alpha} G dV = 4^n n! \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{S^{2n-1}} d\nu \frac{r^{2n-1}}{|t + i(1 + r^2)|^{2n+2}} dt dr
d\nu
$$

$$
= (4\pi)^n (2n) N_1(2n + 2, 2n - 1, 0) = \pi^{n+1}.
$$

(5.11)

Note that

$$
v_0^{ik} Z_j \Phi Z_k \Phi = \sum_{m_0 = m_1 = m_2 = 0, \beta} s^{j}_{\beta(m_1 + o(j) - 1)} s^{k}_{\beta(m_2 + o(k) - 1)} v_{m_0} Z_j \Phi Z_k \Phi
$$

$$
= s^{\alpha}_{\beta(0)} s^{\gamma}_{\beta(0)} Z_\alpha \Phi Z_\gamma \Phi = Z_\beta \Phi Z_\beta \Phi.
$$

(5.12)
by (2.24) for $s^j_{\beta(0)} = \delta^j_{\beta}$ and $s^j_{\beta(1)} = 0$. So we get

$$b_0(n) = 2 \int_{\mathcal{H}^n} v_1^{ij} Z_j \Phi Z_k \Phi dV = 2 \int_{\mathcal{H}^n} Z_{\beta} \Phi Z_{\beta} \Phi dV = 2 \int_{\mathcal{H}^n} n^2 |z|^2 |\omega + i|^{-2n-2} dV$$

$$= 2n^2 q^2 n! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{S^{2n-1}} \frac{r^{2n+1}}{|t + i(1 + r^2)|^{2n+2}} dr dt$$

$$= 4n^3 (4\pi)^n N_1 (2n + 2, 2n + 1, 0) = 2n^2 \pi^{n+1},$$

(5.13)

by (5.5). By (5.3) and Corollary 5.1, we have

$$a_1(n) = \int_{\mathcal{H}^n} |\Phi|^p v_1 dV = 0.$$  

(5.14)

By Proposition 2.8 for $s^j_{\beta}$ and $v_1 = 0$ in (5.6), we can get

$$v_1^{ik} Z_j \Phi Z_k \Phi = \sum_{m_0 + m_1 + m_2 = 1, \beta} s^j_{\beta(m_1+o(j)-1)} s^k_{\beta(m_2+o(k)-1)} v_{m_0} Z_j \Phi Z_k \Phi$$

$$= s^a_{\beta(0)} s^b_{\beta(2)} Z_a \Phi Z_0 \Phi + s^0_{\beta(2)} s^\gamma_{\beta(0)} Z_0 \Phi Z_\gamma \Phi = z^a z^b z^c F_{abc}(r, t),$$

for some functions $F_{abc}$ only depending on $r$ and $t$, by Proposition 2.8, 3.1, 3.3. So

$$b_1(n) = 2 \int_{-\infty}^{\infty} \int_{0}^{\infty} \left( \int_{S^{2n-1}} z^a z^b z^c dV \right) F_{abc}(r, t) r^3 dr dt = 0,$$

(5.15)

by Lemma 5.3. $a_2(n)$ is given by following lemma.

**Lemma 5.6**

$$a_2(n) = \frac{\pi^{n+1}}{12n} \Omega.$$  

(5.16)

**Proof** By (5.3), (5.6), the second line of (5.8) and Lemma 5.4, we have

$$a_2(n) = \int_{\mathcal{H}^n} |\Phi|^p v_2 dV = \int_{\mathcal{H}^n} \left( - \frac{1}{6} R^a_{\beta \alpha \mu} (q) z^\beta z^\mu \right) \left( - \frac{1}{6} R^a_{\beta \alpha \mu} (q) z^\beta z^\mu \right) |w + i|^{-2n-2} dV$$

$$= \frac{1}{6} (4\pi)^n \Omega \int_{0}^{\infty} \int_{-\infty}^{\infty} |t + i(1 + r^2)|^{-2n-2} r^{2n+1} dr dt$$

$$= \frac{1}{6} (4\pi)^n \Omega N_1 (2n + 2, 2n + 1, 0) = \frac{\pi^{n+1}}{12n} \Omega.$$  

We finish the proof of Lemma 5.6. \qed

To calculate $b_2(n)$, we need the following results.

**Lemma 5.7**

$$\int_{\mathcal{H}^n} v_2^{ab} Z_a \Phi Z_b \Phi dV = \frac{n^4 + 2n^3 + 2n^2}{6(n - 1)(n + 1)} (4\pi)^n N_1 (2n + 2, 2n + 1, 0) \Omega,$$

$$\int_{\mathcal{H}^n} \left( v_2^{0a} Z_a \Phi Z_0 \Phi + v_2^{0a} Z_0 \Phi Z_a \Phi \right) dV = \frac{-5n^2}{6(n - 1)(n + 1)} (4\pi)^n N_1 (2n + 2, 2n + 1, 0) \Omega,$$

$$\int_{\mathcal{H}^n} v_2^{00} Z_0 \Phi Z_0 \Phi dV = \frac{2n^2}{3(n - 1)(n + 1)} (4\pi)^n N_1 (2n + 2, 2n + 1, 0) \Omega.$$  

(5.17)
This lemma will be proved in “Appendix B”, from which we get

\[ b_2(n) = 2 \int_{\mathcal{H}^n} v_{ij}^{jk} Z_j \Phi Z_k \Phi dV = \frac{n^2(n+1)}{3(n-1)} (4\pi)^n N_1(2n + 2, 2n + 1, 0) \Omega \]
\[ = \frac{n(n+1)\pi^{n+1}}{6(n-1)} \Omega. \]  

(5.18)

We also have

**Lemma 5.8**

\[ c_2(n) = -\frac{2n\pi^{n+1}}{n-1} \Omega. \]

(5.19)

**Proof** Applying \( X = W_0, Y = W_{\bar{\mu}} \) and taking index \( a = \beta \) in the last identity of (2.9), we get \( R_{\beta 0 \bar{\mu}}^0 = \theta (R(W_0, W_{\bar{\mu}})W_{\beta}) = 0 \). Hence by definition we have \( R_{\beta \bar{\mu}} = R_{\beta \bar{\mu}}^0 + R_{\beta \bar{\mu}}^\alpha \), and so we get

\[ R(q) = h^{jk} R_{jk}(q) = h^{0\bar{\mu}} R_{0\bar{\mu}}(q) + h^{\beta \bar{\mu}} R_{\beta \bar{\mu}}(q) \]
\[ = \delta^{\beta \bar{\mu}} (R_{\beta \bar{\mu}}^0(q) + R_{\beta \bar{\mu}}^\alpha(q)) + \delta^{\beta \bar{\mu}} (R_{\beta \bar{\mu}}^0(q) + R_{\beta \bar{\mu}}^\alpha(q)) \]
\[ = R_{\beta \bar{\mu}}^\alpha(q) + R_{\beta \bar{\mu}}^\alpha(q) = -\frac{1}{2} \Omega, \]  

(5.20)

by Proposition 5.1. Noting that

\[ \int_{\mathcal{H}^n} |\Phi|^2 dV = \int_{\mathcal{H}^n} w + i|^{-2n} dV = \int_0^\infty d\nu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4n! \nu^{2n-1}}{|t+i(1+r^2)|^{2n}} dr \]
\[ = 2n(4\pi)^n N_1(2n, 2n - 1, 0) = \frac{8n^2}{n-1} (4\pi)^n N_1(2n + 2, 2n + 1, 0), \]

we get

\[ c_2(n) = -\frac{\Omega}{2} \int_{\mathcal{H}^n} |\Phi|^2 dV = -\frac{4n^2}{n-1} (4\pi)^n N_1(2n + 2, 2n + 1, 0) \Omega = -\frac{2n\pi^{n+1}}{n-1} \Omega \]

by (5.3). \( \square \)

**Proposition 5.3** The extremal of the Yamabe functional on the Heisenberg group is

\[ \lambda(\mathcal{H}^n) = 2pn^2 \pi. \]

**Proof** Recall that on \( \mathcal{H}^n, \Theta = dt - iz^\alpha d\bar{z}^\alpha + iz^\bar{\alpha} dz^\alpha \). Then \( d\Theta = -2idz^\alpha \wedge d\bar{z}^\alpha \). So \(-i\delta_{\alpha \bar{\beta}} = d\Theta(Z_\alpha, Z_{\bar{\beta}}) = h(Z_\alpha, JZ_{\bar{\beta}}) = -ih(Z_\alpha, Z_{\bar{\beta}})\), namely \( h(Z_\alpha, Z_{\bar{\beta}}) = \delta_{\alpha \bar{\beta}} \).

Hence it induces the dual norm \( |\cdot, \cdot|_H \) by \( (\Theta^\alpha, \Theta^\bar{\beta})_H = \delta_{\alpha \bar{\beta}} \). Then \( |d\Phi|^2_H = (Z_\alpha \Phi \Theta^\alpha + Z_{\bar{\beta}} \Phi \Theta^\bar{\beta})_H = 2Z_{\bar{\beta}} \Phi Z_\alpha \Phi \). Since the curvature tensor \( R \equiv 0 \) on the Heisenberg group, we have

\[ \lambda(\mathcal{H}^n) = \frac{\int_{\mathcal{H}^n} |d\Phi|^2_H dV}{(\int_{\mathcal{H}^n} |\Phi|^p dV)^{\frac{2}{p}} dV} = \frac{2p \int_{\mathcal{H}^n} Z_\beta \Phi Z_{\bar{\beta}} \Phi dV}{(\int_{\mathcal{H}^n} |\Phi|^p dV)^{\frac{2}{p}}} = 2pn^2 \pi, \]

(5.21)

by (5.11) and (5.13). \( \square \)
Remark 5.1 Recall that in [14], Jerison and Lee used the structure equation \( d\theta = i\hbar_{\alpha\beta} \Theta^\alpha \wedge \Theta^\beta \) (cf. [14, p. 307]). So in Heisenberg case, \( d\Theta = i\hbar_{\alpha\beta} \Theta^\alpha \wedge \Theta^\beta = i\hbar_{\alpha\beta} dz^\alpha \wedge dz^\beta \). On the other hand, \( \Theta = dt + iz^\alpha dz^\alpha - i\bar{z}^\alpha dz^\alpha \) leads to \( d\Theta = 2idz^\alpha \wedge dz^\alpha \), i.e., \( h_{\alpha\beta} = 2\delta_{\alpha\beta} \) on the Heisenberg group in [14]. Thus \( |d\Phi|^2_H \) differs from that in [14] by a factor 2, and so does the Yamabe invariant.

Proof of Theorem 5.1 Substituting (5.11), (5.14) and (5.16) to the first identity in (5.2), we get
\[
\int_M |f^\varepsilon|^p dV_\Theta = \pi^{n+1} \left( 1 + \frac{1}{12n} \Omega \varepsilon^2 \right) + O(\varepsilon^3),
\]
and so
\[
\left( \int_M |f^\varepsilon|^p dV_\Theta \right)^{-\frac{2}{p}} = \pi^{-n} \left( 1 - \frac{1}{12(n+1)} \Omega \varepsilon^2 \right) + O(\varepsilon^3). \tag{5.22}
\]
Substituting (5.13), (5.15) and (5.18) to the second identity in (5.2) leads to
\[
\int_M p|d f^\varepsilon |^2_H dV_\Theta = 2pn^2 \pi^{n+1} \left( 1 + \frac{n+1}{12(n-1)n} \Omega \varepsilon^2 \right) + O(\varepsilon^3). \tag{5.23}
\]
By the third identity in (5.2) and Lemma 5.8, we get
\[
\int_M R|f^\varepsilon|^2 dV_\Theta = 2pn^2 \pi^{n+1} \frac{-1}{2(n-1)(n+1)} \Omega \varepsilon^2 + O(\varepsilon^3). \tag{5.24}
\]
Finally, substituting (5.22), (5.23) and (5.24) to the definition of \( \mathcal{Y}_\theta(f^\varepsilon) \) in (1.5), we get
\[
\mathcal{Y}_\theta(f^\varepsilon) = \left( \int_M |f^\varepsilon|^p dV_\Theta \right)^{-\frac{2}{p}} \left( \int_M p|d f^\varepsilon |^2_H dV_\Theta + \int_M R|f^\varepsilon|^2 dV_\Theta \right)
\]
\[
= 2pn^2 \pi \left( 1 + \frac{(n+1)^2 - 6n - n(n-1)}{12(n-1)n(n+1)} \Omega \varepsilon^2 \right) + O(\varepsilon^3)
\]
\[
= 2pn^2 \pi \left( 1 - \frac{3n-1}{12(n-1)n(n+1)} \Omega \varepsilon^2 \right) + O(\varepsilon^3).
\]
As \( \Omega > 0 \) and \( n > 2 \), it is \( \lambda(\mathcal{H}^n) \) for sufficiently small \( \varepsilon \). Theorem 5.1 is proved. \( \Box \)

The Corollary 1.1 now follows from the existence of the extremal for the subcritical case of the Yamabe problem in [25]. See also [1] for the existence of the extremal for the subcritical case of the similar Yamabe problem on Dirichlet spaces.

Appendix A: Transformation formulae under a conformal transformation

In this appendix, we will discuss conformal transformations and prove Lemma 4.1. We write components of the tensors after conformal transformation with respect to \( \{W_a, \widehat{T}\} \) and \( \{\widehat{\Theta}^a, \widehat{\Theta}\} \) satisfying (4.1)–(4.3), e.g., \( \tau(\widehat{T}, W_a) = \widehat{A}^b_a W_b, \Gamma^c_{0b} = \omega^c_0(\widehat{T}) \). See [5, Lemma 10] for a version for a local \( T^{(1,0)}M \)-frame.
A.1 The transformation formula for connection coefficients

Lemma A.1 We have
\begin{align*}
2h(\nabla_X Y, Z) &= X(h(Y, Z)) + Y(h(X, Z)) - Z(h(X, Y)) \\
&\quad - 2h(X, JZ)\theta(Y) - 2h(Y, JZ)\theta(X) + 2h(X, JY)\theta(Z) \\
&\quad - h([X, Z], Y) - h([Y, Z], X) + h([X, Y], Z),
\end{align*}
(A.1)
for any \( X, Y, Z \in TM \). And also we have
\begin{align*}
2h(\nabla_T Y, Z) &= T(h(Y, Z)) - h([T, Z], Y) + h([T, Y], Z),
\end{align*}
(A.2)
for any \( Y, Z \in HM \).

Proof We refer to [5, p. 334] for (A.1). For (A.2), we have
\[
T(h(Y, Z)) = h(\nabla_T Y, Z) + h(Y, \nabla_T Z) = h(\nabla_T Y, Z) + h(Y, [T, Z]) + h(\tau_s Z)
\]
\[
= h(\nabla_T Y, Z) + h(Y, [T, Z]) + h(\tau_s Y, Z)
\]
\[
= 2h(\nabla_T Y, Z) + h([T, Z], Y) - h([T, Y], Z),
\]
by \( \nabla T = 0 \), the definition of the Webster torsion \( \tau_s \) and its self-adjointness (cf. Lemma 2.1).

\begin{corollary}
Corollary A.1 With respect to a frame \( \{W_a, T\} \) with \( \{W_a\} \) horizontal, we have
\[
\Gamma^c_{ab} = \frac{1}{2} h^{cd} \left( W_a(h_{bd}) + W_b(h_{ad}) - W_d(h_{ab}) \right)
\]
\[
- h([W_a, W_d], W_b) - h([W_b, W_d], W_a) + h([W_a, W_b], W_d) \right),
\]
(A.3)
and
\[
\Gamma^c_{0b} = \frac{1}{2} h^{cd} \left( Th_{bd} - h([T, W_d], W_b) + h([T, W_b], W_d) \right).
\]
(A.4)
\end{corollary}

Proof (A.3) follows from substituting \( X = W_a, Y = W_b, Z = W_d \) into (A.1). (A.4) follows from substituting \( Y = W_b, Z = W_d \) into (A.2).

Lemma A.2 Under the conformational transformation (4.1) with \( u \in \mathcal{C}_m \), we have
\[
[\hat{T}, W_\beta] = [T, W_\beta] - i Z_\beta Z_{a\mu} u W_\alpha + i Z_\beta Z_{a\mu} u W_{\bar{\alpha}} + \mathcal{C}_{m-1}(W), \tag{A.5}
\]
where \( \mathcal{C}_{m-1}(W) \) denotes the linear combination of \( W_j \)'s with coefficients in \( \mathcal{C}_{m-1} \).

Proof We have
\[
[\hat{T}, W_\beta] = [e^{-2u}(T + J^c_a u^a W_c), W_\beta]
\]
\[
= e^{-2u}[T, W_\beta] + e^{-2u}[J^c_a u^a W_c, W_\beta] + 2e^{-2u}u_\beta(T + J^c_a u^a W_c)
\]
\[
= [T, W_\beta] - (W_\beta u^\alpha)J^c_a W_\alpha + \mathcal{C}_{m-1}(W)
\]
\[
= [T, W_\beta] - (W_\beta u^\alpha)J^c_a W_\alpha + (W_\beta u^\alpha)h^{\mu\bar{\alpha}} J^\rho_{\alpha}(q) W_\rho + \mathcal{C}_{m-1}(W)
\]
\[
= [T, W_\beta] - i Z_\beta Z_{a\mu} u W_\alpha + i Z_\beta Z_{a\mu} u W_{\bar{\alpha}} + \mathcal{C}_{m-1}(W),
\]
by \( h_{\alpha\beta} = \delta_{\alpha\beta}, \ J^\rho_{\alpha}(q) = i\delta^\rho_{\alpha} \) in (2.14) and \( u_\alpha \in \mathcal{C}_{m-1} \) for \( u \in \mathcal{C}_m \). (A.5) follows.  \( \square \)
Proposition A.1 Under the conformal transformation (4.1), connection coefficients of $T W T$ connections change as

$$
\hat{\Gamma}^c_{\alpha \beta} = \Gamma^c_{\alpha \beta} + u_a \delta^c_b + u_b \delta^c_a - u^c h_{ab}.
$$

(A.6)

Proof By Corollary A.1, we get

$$
\hat{\Gamma}^c_{\alpha \beta} = \frac{1}{2} e^{-2u} h^{cd} \left[ W_a \left( e^{2u} h_{bd} \right) + W_b \left( e^{2u} h_{ad} \right) - W_d \left( e^{2u} h_{ab} \right) \right] - e^{2u} h \left[ (W_a, W_d), (W_b, W_d) \right] + e^{2u} h \left[ (W_a, W_b), (W_d) \right] = \Gamma^c_{\alpha \beta} + u_a \delta^c_b + u_b \delta^c_a - u^c h_{ab}.
$$

Note that

$$
\hat{\Gamma}^\rho_{\alpha \beta} = \frac{1}{2} \hat{h}^{\rho \mu} \left( \hat{T} h_{\rho \mu} - \hat{h} \left( \left[ \hat{T}, W_\mu \right], W_\beta \right) + \hat{h} \left( \left[ \hat{T}, W_\beta \right], W_\mu \right) \right).
$$

(A.7)

For the first term in the right-hand side of (A.7), by (4.2), we have

$$
\frac{1}{2} \hat{h}^{\rho \mu} \hat{T} h_{\rho \mu} = \frac{1}{2} e^{-4u} h^{\rho \mu} \left( T + J^e u^a W_e \right) \left( e^{2u} h_{\rho \mu} \right) = \frac{1}{2} h^{\rho \mu} T h_{\rho \mu} + u_0 \delta^\rho_\beta + \mathcal{O}_{m-1}.
$$

Take conjugation on both sides of (A.5) to get $\left[ \hat{T}, W_\mu \right] = \left[ T, W_\mu \right] + i Z_\mu Z_\alpha u W_\alpha - i Z_\mu Z_\alpha u W_\alpha + \mathcal{O}_{m-1}(W)$ and so

$$
- \frac{1}{2} \hat{h}^{\rho \mu} \hat{h} \left( \left[ \hat{T}, W_\mu \right], W_\rho \right) = - \frac{1}{2} h^{\rho \mu} h \left( \left[ T, W_\mu \right], W_\rho \right) - i Z_\mu Z_\beta u + \mathcal{O}_{m-1},
$$

and

$$
\frac{1}{2} \hat{h}^{\rho \mu} \hat{h} \left( \left[ \hat{T}, W_\beta \right], W_\mu \right) = \frac{1}{2} h^{\rho \mu} h \left( \left[ T, W_\beta \right], W_\mu \right) - i Z_\mu Z_\beta u + \mathcal{O}_{m-1}.
$$

So (A.7) becomes $\hat{\Gamma}^\rho_{\alpha \beta} = \Gamma^\rho_{\alpha \beta} + u_0 \delta^\rho_\beta - \frac{i}{2} (Z_\beta Z_{\beta} u + Z_{\beta} Z_{\beta} u) + \mathcal{O}_{m-1}$. \qed

A.2. Transformation formulae for curvature and Webster torsion tensors

Proof of Lemma 4.1 By $\nabla T = 0$ and $\tau_s W_a = \tau (T, W_a) = \nabla T W_a - [T, W_a]$, we get

$$
A_{ab} = h (A^c_a W_c, W_b) = h (\tau_s W_a, W_b) = h (\nabla T W_a - [T, W_a], W_b)
$$

$$
= T (h_{ab}) - h (W_a, \nabla T W_b) - h ([T, W_a], W_b)
$$

$$
= T (h_{ab}) - h (W_a, \tau_s W_b + [T, W_b]) - h ([T, W_a], W_b)
$$

$$
= T (h_{ab}) - A_{ba} - h ([W_a, [T, W_b]]) - h ([T, W_a], W_b).
$$

Since the tensor $A$ is self-adjoint by Lemma 2.1, we get

$$
A_{ab} = \frac{1}{2} \left( T (h_{ab}) - h (W_a, [T, W_b]) - h ([T, W_a], W_b) \right).
$$

In particular, $A_{ab} = -\frac{i}{2} (h (W_a, [T, W_b])) + h ([T, W_a], W_b)$. Applying Lemma A.2 with respect to the frame $\left\{ W_a, \hat{T} \right\}$, we get
\( \hat{A}_{\alpha\beta} = \frac{-1}{2} \left( \hat{h}(W_\alpha, [\hat{T}, W_\beta]) + \hat{h}([\hat{T}, W_\alpha], W_\beta) \right) = A_{\alpha\beta} - \frac{i}{2} Z_\alpha Z_\beta u - \frac{i}{2} Z_\beta Z_\alpha u + \mathcal{O}_{m-1} \)
\[ = A_{\alpha\beta} - i Z_\alpha Z_\beta u + \mathcal{O}_{m-1} , \]

by \( \hat{h} = (1 + \mathcal{O}_m) h \), (A.5) and \([Z_\alpha, Z_\beta] = 0\). By (2.11) with respect to frame \([W_\alpha, \hat{T}]\), we get
\[ \hat{R}^\gamma_{\alpha\gamma\beta} = W_\gamma \hat{\Gamma}^\gamma_{\beta\alpha} - W_\beta \hat{\Gamma}^\gamma_{\gamma\alpha} - \hat{\Gamma}^e_{\beta\gamma} \hat{\Gamma}^\gamma_{e\alpha} + \hat{\Gamma}^e_{\gamma\beta} \hat{\Gamma}^\gamma_{e\alpha} - \hat{\Gamma}^e_{\gamma\alpha} \hat{\Gamma}^\gamma_{e\beta} + \hat{\Gamma}^e_{\beta\alpha} \hat{\Gamma}^\gamma_{e\gamma} + 2 \hat{\Gamma}^\gamma_{0\alpha} \hat{\Gamma}^\gamma_{0\beta} . \]
(A.8)

By the first identity in (A.6) and (2.14), for \( u \in \mathcal{O}_m \), we have
\[ W_\gamma \hat{\Gamma}^\gamma_{\beta\alpha} = W_\gamma \left( \Gamma^\gamma_{\beta\alpha} + u_\beta \delta^\gamma_\alpha - u^\gamma h_{\alpha\beta} \right) = W_\gamma \Gamma^\gamma_{\beta\alpha} + W_\gamma (u_\beta) \delta^\gamma_\alpha - h_{\gamma\alpha} W_\gamma (u_\beta) h_{\alpha\beta} = W_\gamma \Gamma^\gamma_{\beta\alpha} + Z_\alpha Z_\beta u - \delta_{\alpha\beta} Z_\gamma Z_\alpha u + \mathcal{O}_{m-1} , \]
(A.9)

and
\[ W_\beta \hat{\Gamma}^\gamma_{\gamma\alpha} = W_\beta \left( \Gamma^\gamma_{\gamma\alpha} + u_\gamma \delta^\gamma_\alpha + u_\alpha \delta^\gamma_\beta \right) = W_\beta \Gamma^\gamma_{\gamma\alpha} + (n + 1) Z_\beta Z_\alpha u + \mathcal{O}_{m-1} . \]

Again by the first identity of (A.6), we have \( \hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \mathcal{O}_{m-1} \), and by (2.27), we have \( \Gamma^c_{ab} = \partial_1 \). So we get
\[ \hat{\Gamma}^c_{ab} \hat{\Gamma}^f_{de} = \Gamma^c_{ab} \Gamma^f_{de} + \mathcal{O}_m , \]
for any indices \( a, b, c, d, e, f \). By the second identity of (A.6), \( J^\gamma_{\beta\alpha} = -i \delta^\gamma_{\alpha\beta} \) in (2.14) and \( u_0 = Tu = \frac{\partial u}{\partial t} + \mathcal{O}_m \) by (2.26), we get
\[ 2 \hat{\Gamma}^\gamma_{0\alpha} \hat{\Gamma}^\gamma_{0\beta} = 2 \left( \Gamma^\gamma_{0\alpha} + u_\alpha \delta^\gamma_\alpha - \frac{i}{2} Z_\gamma Z_\alpha u - \frac{i}{2} Z_\alpha Z_\gamma u \right) J^\gamma_{\beta\alpha} + \mathcal{O}_{m-1} \]
\[ = 2 \Gamma^\gamma_{0\alpha} J^\gamma_{0\beta} - 2 i \frac{\partial u}{\partial t} \delta_{\alpha\beta} - Z_\beta Z_\alpha u - Z_\alpha Z_\beta u + \mathcal{O}_{m-1} , \]
(A.10)

Noting that \([Z_\alpha, Z_\beta] = 2 i \delta_{\alpha\beta} \frac{\partial}{\partial t} \), (A.8) leads to
\[ \hat{R}^\gamma_{\alpha\gamma\beta} = R^\gamma_{\alpha\gamma\beta} - 2 i \delta_{\alpha\beta} \frac{\partial u}{\partial t} - (n + 2) Z_\beta Z_\alpha u - \delta_{\alpha\beta} Z_\gamma Z_\alpha u + \mathcal{O}_{m-1} \]
\[ = R^\gamma_{\alpha\gamma\beta} - \frac{n + 2}{2} \left( Z_\beta Z_\alpha u + Z_\alpha Z_\beta u \right) + \frac{1}{2} \delta_{\alpha\beta} Z_0 u + \mathcal{O}_{m-1} , \]
with \( Z_0 = -(Z_\alpha Z_\alpha + Z_\alpha Z_\alpha) \).

\[ \square \]

**Appendix B: Calculation of \( a_2(n) \) and \( b_2(n) \)**

Recall that we choose special frames satisfying Proposition 5.1 over a contact Riemannian manifold \((M, \theta, h, J)\).
B.1. Calculation of $v_{jk}^2$

Lemma B.1 For $v_{2}^{jk}$ defined in (5.4), we have

$$v_{2}^{αγ} = -\frac{1}{6} R_{d(α)}^\gamma(q) z^d z^c,$$

$$v_{2}^{αγ} = -\frac{1}{6} \left( R_{d(α)}^\gamma(q) + R_{d(α)}^\gamma(q) \right) z^c z^c + δ^α γ v_2,$$

$$v_{2}^{α0} = -\frac{1}{2} J_{βα(2)} z^b + \frac{i}{12} R_{d(α)}^\delta(q) z^d z^c z^o - \frac{i}{12} R_{d(α)}^\delta(q) z^d z^c z^o,$$

$$v_{2}^{α0} = 0 = v_{2}^{0α},$$

Proof In the sequel, we will use Proposition 2.8 repeatedly, especially $s_{β(0)}^α = s_{β(0)}^α = δ^α β, s_{β(0)}^α = s_{β(0)}^α = 0, s_{β(0)}^α = 0, s_{β(0)}^α = 0,$ and we also have $v_0 = 1, v_1 = 0,$ by Corollary 5.1. We find that

$$v_{2}^{αγ} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^γ v_{m_0}$$

$$= s_{β(2)}^α s_{β(2)}^γ v_0 + s_{β(0)}^α s_{β(1)}^γ v_0 + s_{β(1)}^α s_{β(0)}^γ v_1 + s_{β(0)}^α s_{β(0)}^γ v_2$$

$$= δ^α β s_{β(2)}^γ v_0 = s_{β(2)}^γ v_0 = -\frac{1}{6} R_{d(α)}^\gamma(q) z^d z^c,$$

by (2.24) for $s_{β(2)}^α.$ Similarly we get

$$v_{2}^{αγ} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^γ v_{m_0} = s_{β(2)}^α s_{β(2)}^γ v_0 + δ^α β s_{β(2)}^γ v_2 = s_{β(2)}^α s_{β(0)}^γ v_2$$

$$v_{2}^{αγ} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^γ v_{m_0} = s_{β(2)}^α s_{β(2)}^γ v_0 = s_{β(2)}^α s_{β(0)}^γ v_2 = -\frac{1}{6} R_{d(α)}^\gamma(q) z^d z^c,$$

$$v_{2}^{α0} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^0 v_{m_0} = s_{β(2)}^α s_{β(2)}^0 v_0 = s_{β(2)}^α s_{β(0)}^0 v_2$$

$$v_{2}^{α0} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^0 v_{m_0} = s_{β(2)}^α s_{β(2)}^0 v_0 = s_{β(2)}^α s_{β(0)}^0 v_2 = -\frac{1}{6} R_{d(α)}^\gamma(q) z^d z^c,$$

$$v_{2}^{α0} = \sum_{m_0+m_1+m_2=2,β} s_{β(m_1)}^α s_{β(m_2)}^0 v_{m_0} = s_{β(2)}^α s_{β(2)}^0 v_0 = s_{β(2)}^α s_{β(0)}^0 v_2 = -\frac{1}{6} R_{d(α)}^\gamma(q) z^d z^c.$$
By Corollary 3.1 for \( s_{(2)} \) and Proposition 3.3, we get\( v_2^{00} = \sum_{m_0+m_1+m_2=2, \beta} s_{(m_1+1)}^{0} s_{(m_2+2, \beta)}^{0} v_{m_0} \)
\( = s_{(2)}^{\beta} s_{(2)}^{\beta} = \frac{4}{9} Q_{\gamma \lambda}^{\beta} (q) Q_{\sigma \mu}^{\beta} (q) z^{\gamma} z^{\lambda} z^{\sigma} z^{\mu} \). So we finish the proof of Lemma B.1.

\( \Box \)

**B.2 Proof of Lemma 5.7**

By the first identity in (B.1) for\( v_2^{\alpha \gamma} \) and (5.5), we get
\[
\int_{\mathcal{A}^n} v_2^{\alpha \gamma} Z_{\alpha} \Phi Z_{\gamma} \Phi dV = \int_{\mathcal{A}^n} \frac{n^2}{6} R_{\rho \alpha \gamma} (q) z^{\rho} z^{\alpha} z^{\gamma} \frac{t^2 + 2 i (|z|^2 + 1) t - (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= \int_{\mathcal{A}^n} \frac{n^2}{6} R_{\rho \alpha \gamma} (q) z^{\rho} z^{\alpha} z^{\gamma} \frac{t^2 + 2 i (|z|^2 + 1) t - (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= \frac{n^2 (4 \pi^n)}{12(n+1)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{t^2 + 2 i (|z|^2 + 1) t - (|z|^2 + 1)^2}{|1 + i (1 + r^2)|^{2n+4}} r^{2n+3} dt dr,
\]
(B.2)

where the last identity follows from the third identity in (5.8). Similarly we have
\[
\int_{\mathcal{A}^n} v_2^{\alpha \gamma} Z_{\alpha} \Phi Z_{\gamma} \Phi dV = \frac{n^2 (4 \pi^n)}{12(n+1)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{t^2 + 2 i (|z|^2 + 1) t - (|z|^2 + 1)^2}{|1 + i (1 + r^2)|^{2n+4}} r^{2n+3} dr dt,
\]
(B.3)

Then by (B.1) and (5.5), we get
\[
\int_{\mathcal{A}^n} v_2^{\alpha \gamma} Z_{\alpha} \Phi Z_{\gamma} \Phi dV
\]
\[
= \int_{\mathcal{A}^n} \left[ -\frac{n^2}{6} \left( R_{\rho \alpha \gamma} (q) + R_{\rho \alpha \gamma} (q) \right) z^{\rho} z^{\alpha} z^{\gamma} \frac{t^2 + 2 i (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= \int_{\mathcal{A}^n} \left[ -\frac{n^2}{6} \left( R_{\rho \alpha \gamma} (q) + R_{\rho \alpha \gamma} (q) \right) z^{\rho} z^{\alpha} z^{\gamma} \frac{t^2 + 2 i (|z|^2 + 1)^2}{|w + i|^{2n+4}} + v_2 n^2 |z|^2 \frac{t^2 + 2 i (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= \frac{n^2}{6(n+1)} (4 \pi^n)^2 \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{(t^2 + (r^2 + 1)^2) r^{2n+3}}{|t^2 + i (1 + r^2)|^{2n+4}} dr dt
\]
\[
+ \frac{n^2}{6} (4 \pi^n) \Omega_{1} (2n + 2, 2n + 3, 0).
\]

The last identity follows from the third and fourth identities in (5.8) and
\[
\int_{\mathcal{A}^n} v_2 n^2 |z|^2 \frac{t^2 + (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= -\frac{1}{6} \int_{\mathcal{A}^n} \left( R_{\beta \alpha \mu} (q) z^{\beta} z^{\mu} + R_{\beta \alpha \mu} (q) z^{\beta} z^{\mu} \right) n^2 |z|^2 \frac{t^2 + (|z|^2 + 1)^2}{|w + i|^{2n+4}} dV
\]
\[
= \frac{n^2}{6} (4 \pi^n) \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{r^{2n+3}}{|t + i (1 + r^2)|^{2n+2}} dr dt = \frac{n^2}{6} (4 \pi^n) \Omega_{1} (2n + 2, 2n + 3, 0).
\]
Taking summation of (B.2)–(B.5), we get

\[ \int_{\mathcal{H}^n} v_2^\nu Z_\alpha \Phi Z_\nu \Phi dV = 0. \quad (B.5) \]

By (B.1), we get

\[ \int_{\mathcal{H}^n} v_2^{ab} Z_a \Phi Z_b \Phi dV = \frac{n^2}{3(n+1)} (4\pi)^n \Omega \int_{-\infty}^{\infty} \int_0^{\infty} \frac{t^2 r^{2n+3}}{(t^2 + i(1 + r^2))^{2n+4}} dr dt \]

\[ + \frac{n^2}{6} (4\pi)^n \Omega N_1(2n + 2, 2n + 3, 0) \]

\[ = \frac{n^2}{3(n+1)} (4\pi)^n (2n + 4, 2n + 3, 2) \Omega + \frac{n^2}{6} (4\pi)^n N_1(2n + 2, 2n + 3, 0) \Omega \]

\[ = \frac{n^4 + 2n^3 + 2n^2}{6(n - 1)(n + 1)} (4\pi)^n N_1(2n + 2, 2n + 1, 0) \Omega, \]

by using the last two identities in (5.9). So the first identity in (5.17) follows.

By (B.1), we get

\[ \int_{\mathcal{H}^n} v_2^{0\alpha} Z_0 \Phi Z_\alpha \Phi dV = \int_{\mathcal{H}^n} v_2^{\tilde{\alpha}} Z_\alpha \Phi Z_0 \Phi dV = 0. \quad (B.6) \]

By (B.1), (5.5) and substituting identities in (5.8) for certain terms, we get

\[ \int_{\mathcal{H}^n} v_2^{0\alpha} Z_\alpha \Phi Z_0 \Phi dV = \int_{\mathcal{H}^n} \left( -\frac{1}{2} J_{b\alpha(2)} z^b + \frac{i}{12} R_{d\alpha c\beta(2)} (q) z^c z^d z^\beta - \frac{i}{12} R_{d\alpha c\beta(2)} (q) z^c z^d z^\beta \right) \]

\[ \cdot (-in^2 z^\alpha) \frac{t^2 + it(|z|^2 + 1)}{|w + i|^2 n+4} dV \]

\[ = n^2 \int_{\mathcal{H}^n} \left( \frac{i}{2} J_{\beta\alpha(2)} z^\beta z^\alpha + \frac{i}{12} J_{\beta\alpha(2)} z^\beta z^\alpha + \frac{1}{12} (R_{\beta\mu\alpha\beta}(q) z^\beta z^\mu z^\beta z^\alpha - R_{\beta\mu\alpha\beta}(q) z^\beta z^\mu z^\beta z^\alpha) \right) \]

\[ - \frac{1}{12} R_{\beta\mu\alpha\beta}(q) z^\beta z^\mu z^\beta z^\alpha \left( \frac{t^2 + it(|z|^2 + 1)}{|w + i|^2 n+4} dV \right) \]

\[ = \frac{n^2}{n + 1} (4\pi)^n \left( -\frac{3}{4} + 0 + 0 - \frac{1}{24} - \frac{1}{24} \right) \Omega \int_{-\infty}^{\infty} \int_0^{\infty} \frac{t^2 + it(r^2 + 1)}{|t^2 + i(1 + r^2)|^{2n+4} r^{2n+3}} dr dt \]

\[ = - \frac{5n^2}{6(n + 1)} (4\pi)^n \Omega \int_{-\infty}^{\infty} \int_0^{\infty} \frac{t^2 + it(r^2 + 1)}{|t^2 + i(1 + r^2)|^{2n+4} r^{2n+3}} dr dt. \quad (B.7) \]

By taking conjugation of (B.7), we get

\[ \int_{\mathcal{H}^n} v_2^{\tilde{\alpha}} Z_\alpha \Phi Z_0 \Phi dV = -\frac{5n^2}{6(n + 1)} (4\pi)^n \Omega \int_{-\infty}^{\infty} \int_0^{\infty} \frac{t^2 - it(r^2 + 1)}{|t^2 + i(1 + r^2)|^{2n+4} r^{2n+3}} dr dt. \quad (B.8) \]

Now taking summation of (B.6)–(B.8), we get

\[ \int_{\mathcal{H}^n} (v_2^{0\alpha} Z_\alpha \Phi Z_0 \Phi + v_2^{\tilde{\alpha}} Z_\alpha \Phi Z_0 \Phi) dV \]

\[ = -\frac{5n^2}{3(n + 1)} (4\pi)^n \Omega \int_{-\infty}^{\infty} \int_0^{\infty} \frac{r^{2n+3} t^2}{|t^2 + i(1 + r^2)|^{2n+4}} dr dt \]
\[ - \frac{5n^2}{3(n+1)} (4\pi)^n \Omega N_1(2n+4, 2n+3, 2) = - \frac{5n^2}{6(n+1)(n-1)} \Omega N_1(2n+2, 2n+1, 0), \]

by (5.9) and Lemma 5.4. So the second identity in (5.17) follows.

By (B.1), (5.9), (5.5), the fifth identity in (5.8) and Lemma 5.4, we get

\[
\int_{\mathbb{H}^n} v_0^0 Z_0 \Phi Z_0 \Phi dV = \int_{\mathbb{H}^n} \frac{4}{3(n+1)} O_{\gamma \lambda} (q) O_{\bar{\sigma} \bar{\mu}} (q) z^\gamma \bar{z}^\lambda \bar{z}^\sigma \bar{z}^\mu \frac{n^2 r^2}{w+i |2n+4|} dV = \frac{4n^2}{3(n+1)} (4\pi)^n N_1(2n+4, 2n+3, 2) \Omega = \frac{2n^2}{3(n+1)(n-1)} (4\pi)^n N_1(2n+2, 2n+1, 0) \Omega.
\]

So the third identity in (5.17) follows.

References

1. Akutagawa, K., Carron, G., Mazzeo, R.: The Yamabe problem on Dirichlet spaces. arXiv:1306.4373
2. Aubin, T.: Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. 55, 269–296 (1976)
3. Barletta, E., Dragomir, S.: Differential equations on contact Riemannian manifolds. Ann. Sc. Norm. Super. Pisa Cl. Sci. Ser. IV XXX(1), 63–96 (2001)
4. Blair, D.E., Dragomir, S.: Pseudohermitian geometry on contact Riemannian manifolds. Rendiconti di Matematica Serie VII 22, 275–341 (2002)
5. Bishop, R.L., Goldberg, S.: Tensor Analysis on Manifolds. Dover Publications Inc., New York (1980)
6. Dragomir, S., Perrone, D.: Levi harmonic maps of contact Riemannian manifolds. J. Geom. Anal. 24, 1233–1275 (2014)
7. Gamara, N.: The CR Yamabe conjecture the case \( n \) = 1. J. Eur. Math. Soc. 3, 105–137 (2001)
8. Gamara, N., Yacoub, R.: CR Yamabe conjecture-the conformally flat case. Pac. J. Math. 201, 121–175 (2001)
9. Ivanov, S.P., Petkov, A.: The qc Yamabe problem on non-spherical quaternionic contact manifolds. J. Math. Pures Appl. 118, 44–81 (2018)
10. Ivanov, S.P., Vassilev, D.N.: Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem. Imperial College Press Lecture Notes. World Scientific Publishing Co., NJ (2011)
11. Ivanov, S.P., Vassilev, D.N.: Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem. Mem. Amer. Math. Soc. 231(1086), vi+82 (2014)
12. Jerison, D., Lee, J.M.: The Yamabe problem on CR manifolds. J. Differ. Geom. 25, 167–197 (1987)
13. Jerison, D., Lee, J.M.: Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. Amer. Math. Soc. 1, 1–13 (1988)
14. Jerison, D., Lee, J.M.: Intrinsic CR normal coordinates and the CR Yamabe problem. J. Differ. Geom. 29, 303–343 (1989)
15. Kunkel, C.: Quaternionic contact normal coordinates, preprint, arXiv:0807.0465
16. Lee, J.M., Parker, T.: The Yamabe problem. Bull. Amer. Math. Soc. 17, 37–91 (1987)
17. Petit, R.: Spin^c-structures and Dirac operators on contact manifolds. Differential Geom. Appl. 22(2), 229–252 (2005)
18. Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geometry 20, 479–495 (1984)
19. Seshadri, N.: Approximately Einstein ACH metrics, volume renormalization, and an invariant for contact manifolds. Bull. Soc. Math. France 137(1), 63–91 (2009)
20. Tanaka, N.: A Differential Geometry Study on Strongly Pseudo-Convex Manifolds. Kinokuniya, Tokyo (1975)
21. Tanno, S.: Variational problems on contact Riemannian manifolds. Trans. Amer. Math. Soc. 314, 349–379 (1989)
22. Trudinger, N.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22, 265–274 (1968)
23. Wang, W.: The Yamabe problem on quaternionic contact manifolds. Ann. Mat. Pura Appl. 186(2), 359–380 (2007)
24. Wang, W.: On a conformally invariant variational problem on differential forms. Nonlinear Anal. 68(4), 828–844 (2008)
25. Wang, W., Wu, F.: On the existence of the Yamabe problem on contact Riemannian manifolds. Balkan J. Geom. Appl. 22(2), 101–128 (2017)
26. Webster, S.M.: Pseudohermitian structures on a real hypersurface. J. Differential Geometry 13, 25–41 (1978)
27. Wu, F., Wang, W.: The Bochner-type formula and the first eigenvalue of the sub-Laplacian on a contact Riemannian manifold. Differential Geom. Appl. 37, 66–88 (2014)
28. Yamabe, H.: On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12, 21–37 (1960)
29. Zhang, Y.: The contact Yamabe flow on $K$-contact manifold. Sci. China Ser. A 52(8), 1723–1732 (2009)

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