RADII AND BINDING ENERGIES OF NUCLEI IN THE
ALPHA-CLUSTER MODEL

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Abstract

The α-cluster model is based on two assumptions that the proton-neutron pair interactions
are responsible for the adherence between α-clusters and that NN-interactions in the α-cluster
structure are isospin independent. It allows one to estimate the Coulomb energy and the short
range inter-cluster bond energy in dependence on the number of clusters. Charge radii are
calculated on the number of α-clusters too. Unlike the Weizsäcker formula in this model the
binding energies of alpha-clusters and excess neutrons are estimated separately. The calculated
values are in a good agreement with the experimental data.

keywords: nuclear structure; alpha-cluster model; Coulomb energy; Surface tension energy,
binding energy; charge radius

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1 Introduction

The liquid drop model is explicitly formulated by two phenomenological formulas, the Weizsäcker
formula [1] to calculate the nuclear binding energy and the formula \( constA^{1/3} \) to calculate the
nuclear radii. It is known [2] that the idea of alpha clustering has developed together with the
liquid drop model since the beginning of the nuclear physics. Nowadays the alpha-cluster model
is becoming more popular. Recently it has been shown that a large body of data on the α-decay
is described by a simple cluster model with using a notion of an α-core potential [3]. There are
several attempts to incorporate the alpha-cluster model into the shell model [2,4]. To elucidate the
optical potential in nucleus-nucleus collisions an idea of bigger clusters like nucleon molecules of
\(^{12}\)C has been used [5]. Currently the idea of alpha-clustering is actively used in the microscopic
studies of the nuclear structure of the light nuclei in the framework of the few-body approach [6],
as well as in the microscopic approaches to the effective nucleon-nucleon interaction [7]. Also there
have been some studies recently to apply an alpha-cluster model to facilitate the description of
some characteristics of nuclei like the α-particle separation energy [8] and the nuclear radii [9] for
a wide range of nuclei.

There is another strong evidence in favor of an alpha-cluster model of nuclear structure. It
comes from the recent findings [10,11] that the radii of nuclei can be described independently of
the number of excess neutrons by the formula \( R = const(Z)^{1/3} \) fm. It can be considered as a
consequence of a well known fact that the charge density in nuclei is almost constant (see, for
example, Table 3.7 [12]).

In this connection it is interesting to obtain the formulas to calculate the charge radii and
binding energies of nuclei on the basis of the idea that the nuclear structure is defined by its
elements, α-clusters. In such an approach, unlike the Weizsäcker formula, the formula to calculate
the nuclear binding energy should consist of two separate parts. One is responsible for the binding
energy of α-clusters and the other should estimate the binding energy of excess neutrons.

The core starts growing with \( Z \) from the nucleus with \( Z \geq 10 \) [10] together with the surface
tension energy. The distance between the nearby clusters is determined by their charge radii, which
are bigger than their matter radii. It was shown [10,11] that the β-stable nuclei have a particular number of excess neutrons needed to fill in the difference in the volumes occupied by the charge and the matter of the nucleons of the α-clusters in the core. Then the radius \( R_m \) of a β-stable isotope can be estimated by the volume occupied by the matter of the core and the volume of the charge of a few peripheral clusters. It has been shown that the condition \( R_m = R \) determines the narrow β-stability path and its width. The binding energy of these excess neutrons can be calculated as a sum of the energies of \( nn \)-pairs.

The hypothesis that the \( nn \)-pairs are placed in the core explains its high mass density. In addition, it provides an explanation [10] why the experimental values of the most abundant isotopes \( R_{\exp \alpha} \) are well described by both functions, \( \sim A^{1/3} \) in the liquid drop model and \( \sim Z^{1/3} \) in the α-cluster model. Then it is expected that the NN-interaction in the regular α-cluster structure is different from that in excess neutron matter.

It is approved by a recent investigation of the optical potential for the elastic channels in reaction \( ^{48}\text{Ca}(p, n)^{48}\text{Sc} \) [13]. It has been found that the projectile proton is involved in the NN-interactions in \( p + ^{48}\text{Ca} \) channel somewhat different from NN-interactions of the outgoing excess neutron in \( n + ^{48}\text{Sc} \) channel. The proton belongs to the α-cluster structure and the neutron is from the excess neutrons. To elucidate the isospin independence of NN-interactions between the nucleons of the α-cluster structure the same investigation should be done for the different reaction \( ^{41}\text{Ca}(p, n)^{41}\text{Sc} \) where the both proton and neutron belong to one \( pn \)-pair.

The energy of the short-range inter-cluster bonds \( \epsilon_{\alpha\alpha} \) is taken to be equal \( \epsilon_{\alpha\alpha} = \epsilon_{\alpha\alpha}^{\text{nuc}} - \epsilon_{\alpha\alpha}^{C} \) where \( \epsilon_{\alpha\alpha}^{\text{nuc}} \) denotes the nuclear force energy and \( \epsilon_{\alpha\alpha}^{C} \) denotes the Coulomb repulsion energy between two nearby clusters. The old term 'nuclear force energy' means here the energy of the attraction between nearby clusters due to the strong inter-nucleon potential. The values of \( \epsilon_{\alpha\alpha} \) are \( 2.425 \) MeV, \( \epsilon_{\alpha\alpha}^{\text{nuc}} = 4.350 \) MeV and \( \epsilon_{\alpha\alpha}^{C} = 1.925 \) MeV have been found from analysis of the experimental data of the lightest nuclei with \( Z \leq 6 \) [14] under two assumptions that the proton-neutron (\( pn \)) pair interactions are responsible for the adherence between α-clusters and that the proton and the neutron belonging to one pair have equivalent single-particle nuclear bound state potentials, the EPN requirement [14,15]. The latter allows one to take the difference between the single-particle binding energies of the proton and the neutron of the last pair \( \Delta E_{pn} \) as the Coulomb energy of the proton's interaction with the other protons of the nucleus.

It was shown [14,15] that the experimental binding energies \( E_{\exp} \) of the α-cluster nuclei with \( N = Z \geq 6 \), known for the nuclei with \( Z \leq 28 \), are well described by the formula \( E^b = N_a \epsilon_a + 3(N_a - 2) \epsilon_{\alpha\alpha} \) where \( \epsilon_a = \epsilon_{\text{He}} = 28.296 \) MeV. The number of bonds \( 3(N_a - 2) \) means that every new cluster brings three bonds with the nearby clusters. This idea was discussed before [2]. In the nuclei heavier than \( ^{16}\text{O} \) the long-range part of the Coulomb interaction of the last α-cluster with the remote clusters starts growing. It must be compensated by its surface tension energy \( E^{\text{st}}_{\alpha} \). Then the empirical values of the surface tension energy of the last α-cluster \( E^{\text{st}}_{\alpha} \) can be obtained from Eq. \( E^{\text{st}}_{\alpha} = \Delta E_{\alpha} - (\epsilon_{\alpha}^{C} + 3 \epsilon_{\alpha\alpha}^{C}) \) where \( \epsilon_{\alpha}^{C} \) stands for the internal Coulomb energy of one α-cluster, which equals the Coulomb energy of the nucleus \( ^{4}\text{He} \) \( \epsilon_{\alpha}^{C} = 0.764 \) MeV, and \( \Delta E_{\alpha} = \sum_{2}^{N_a} \Delta E_{pn} \) of two \( pn \)-pairs belonging to one cluster [14,16,17].

The long-range part of the Coulomb energy of the last cluster’s interaction with \( N_a - 4 \) remote clusters is well approximated by expression \( 2(Z - 8) \epsilon_{\alpha}^2/(1.2R_p) \) where \( R_p \) is the radius of the last proton position in the nucleus. Then it is expected that \( E^{\text{st}}_{\alpha} = 2(Z - 8) \epsilon_{\alpha}^2/(1.2R_p) = \gamma_1 R_p^2 \), which determines \( R_p \). To find the Coulomb radius \( R_C \), the approximation \( E^{\text{st}}_{\alpha} = \gamma_1 R_p^2 = \gamma_2 N_a^{2/3} \) is used. From the analysis of the empirical values of \( \Delta E_{\alpha} \) known for the nuclei with \( Z \leq 22 \) with an accuracy of a few KeV the values of \( \gamma_1 \) and \( \gamma_2 \) have been obtained together with the formulas to calculate \( R_p \) and \( R_C \) on the number \( N_a \).

Thus, the formula to calculate the binding energy with using the nuclear force energy of α-clusters \( E^{\text{nuc}}_{\alpha} = N_a \epsilon_{\alpha}^{\text{nuc}} + 3(N_a - 2) \epsilon_{\alpha\alpha}^{\text{nuc}} \), the Coulomb energy \( E^C = 3/5(Ze)^2/R_C \) and the surface tension energy \( E^{\text{st}} = \sum_{5}^{N_a} E^{\text{st}}_{\alpha} \) plus the binding energy of the excess neutrons \( E_{\Delta N} \) estimated as a sum of separation energies of \( nn \)-pairs describes the binding energy of the nuclei with \( N_a \geq 5 \), with the errors close to those of the Weizsäcker formula [16,17].
It is evident that the model can be applied only for the nuclei with the number of excess neutrons \( \Delta N \geq 0 \), so that every proton is coupled with a neutron. This set of nuclei includes all stable nuclei and the unstable nuclei around the \( \beta \)-stability path. The model has been developed for the last few years and the main formulas have been published [10,11,14,15,16,17]. In the most recent papers [16,17] the surface tension energy was calculated with using the empirical values of \( E^s_\alpha \) for \( Z \leq 28 \), \( N_\alpha \leq 14 \). In this paper the value \( E^s \) is calculated by one formula for \( 9 \leq Z \leq 118 \). Besides, the phenomenological formula to calculate the excess neutron binding energy has been extended for the cases of odd number of excess neutrons.

2 Nuclear Binding Energy

2.1 Nuclear force energy, Coulomb energy and surface tension energy

The picture given in Fig. 1 shows the \( pn \)-pair bonds in the lightest nuclei. The distance between clusters is supposed to be constant for all nuclei except the nucleus \(^8\)Be. In this case one bond is not enough to keep two positively charged clusters close. Adding one more \( pn \)-pair causes two new bonds with the pairs of two alpha-clusters with the energy \( 2\varepsilon_{pnm} \) in the nucleus \(^{10}\)B, and the clusters get closer with the distance proper for the alpha-cluster liquid. The binding energy of the lightest nuclei \( E^b \) is calculated in accordance with Fig. 1 by the following equations:

\[
\begin{align*}
E^b(4\text{He}) &= \varepsilon_\alpha, \\
E^b(6\text{Li}) &= \varepsilon_\alpha + \varepsilon_{pnm} + \varepsilon_{pm}, \\
E^b(10\text{B}) &= 2\varepsilon_\alpha + \varepsilon_{aa} + 2\varepsilon_{pnm} + \varepsilon_{pn}, \\
E^b(12\text{C}) &= 3\varepsilon_\alpha + 3\varepsilon_{aa},
\end{align*}
\]

where \( \varepsilon_{aa} \) denotes the binding energy between two nearby alpha-clusters, \( \varepsilon_{pnm} \) denotes the binding energy between the single \( pn \)-pair and a \( pn \)-pair of the nearby \( \alpha \)-clusters and \( \varepsilon_{pn} \) stands for the binding energy of the proton and the neutron in the single pair. Solving these four equations with the values of \( E^b \) equal to the experimental values \( E_{\exp} \) [18] with an accuracy of 1 KeV (at experimental measurement error within 2-3 KeV) gives the values \( \varepsilon_\alpha = 28.296 \text{ MeV} \), \( \varepsilon_{aa} = 2.425 \text{ MeV} \), \( \varepsilon_{pnm} = 2.037 \text{ MeV} \), \( \varepsilon_{pn} = 1.659 \text{ MeV} \).

The experimental values of the other nuclei including \(^{12}\)C are well approximated by means of the following equations for the nuclei with even \( Z \) with the mass number \( A \):

\[
E^b = N_\alpha\varepsilon_\alpha + 3(N_\alpha - 2)\varepsilon_{aa},
\]

and for the nuclei with odd \( Z_1 = Z + 1 \) with the mass number \( A_1 = A + 2 \)
\[ E_1^{b} = E^{b} + 6\varepsilon_{pm} + \varepsilon_{pn}. \]  

The energy of nucleus \( A_1 \) is determined by the energy of the nucleus \( A \) plus the energy of six bonds between the single \( pm \)-pair and the six \( pm \)-pairs of the three nearby clusters plus the binding energy of the pair itself. The values of \( E^{b} \) are presented in Table 1 in comparison with the experimental values \( E_{exp} \).

Table 1. Binding energies calculated for the nuclei with \( N = Z \). The values are given in MeV. Empirical values are given to three places of decimals and calculated ones are given to one place of decimals.

| \( Z \) | \( E_{exp}[18] \) | \( E^{b}(1,2) \) | \( \Delta E_{pm} \) | \( \Delta E_{\alpha} \) | \( E^{C}(3,4) \) | \( E_{\alpha}^{nuc}(5) \) | \( E^{st}(6) \) | \( \Delta E^{b}(7) \) | \( \Delta E^{Wz}(10) \) |
|---|---|---|---|---|---|---|---|---|---|
| 2  | 28.296 | 28.296 | .764 | .764 | 29.060 | 0.000 | 0.0 | -.1 |
| 3  | 31.992 | 31.992 | 1.007 | 1.771 | 33.753 | 0.000 | 0.0 | 8.7 |
| 4  | 56.498 | 56.498 | 1.644 | 2.651 | 3.415 | 59.912 | 0.000 | 0.0 | 0.2 |
| 5  | 64.750 | 64.750 | 1.850 | 5.303 | 70.205 | 0.000 | 0.0 | 4.2 |
| 6  | 92.163 | 92.163 | 2.764 | 4.614 | 8.067 | 100.230 | 0.000 | 0.0 | 2.4 |
| 7  | 104.659 | 106.0 | 3.003 | 11.070 | 117.1 | 0.000 | -1.3 | 6.7 |
| 8  | 127.620 | 127.7 | 3.536 | 6.539 | 14.606 | 142.3 | 0.000 | -0.1 | 2.4 |
| 9  | 137.371 | 141.6 | 3.544 | 18.150 | 159.2 | 0.541 | 0.2 | 2.0 |
| 10 | 160.646 | 163.3 | 4.021 | 7.565 | 22.171 | 184.5 | 1.026 | 2.2 | -0.6 |
| 11 | 174.146 | 177.2 | 4.328 | 26.499 | 201.3 | 2.351 | 2.0 | 1.4 |
| 12 | 198.258 | 198.9 | 4.838 | 9.166 | 31.337 | 226.6 | 3.653 | 2.7 | -2.7 |
| 13 | 211.896 | 212.8 | 5.042 | 36.379 | 243.4 | 5.692 | 1.8 | 2.1 |
| 14 | 236.536 | 234.4 | 5.593 | 10.635 | 41.972 | 268.7 | 7.749 | 2.7 | -2.7 |
| 15 | 250.603 | 248.3 | 5.731 | 47.703 | 285.6 | 10.477 | 3.1 | 4.0 |
| 16 | 271.780 | 270.0 | 6.227 | 12.958 | 53.930 | 310.8 | 13.169 | 1.1 | 1.9 |
| 17 | 285.573 | 283.9 | 6.351 | 60.281 | 327.7 | 16.517 | 1.3 | 2.6 |
| 18 | 306.719 | 305.6 | 6.746 | 13.097 | 67.027 | 352.9 | 19.727 | -0.4 | 1.1 |
| 19 | 320.634 | 319.5 | 6.923 | 73.950 | 369.8 | 23.647 | 0.0 | 1.8 |
| 20 | 342.056 | 341.2 | 7.286 | 14.209 | 81.236 | 395.0 | 27.397 | -1.0 | 1.1 |
| 21 | 354.710 | 355.0 | 7.277 | 88.513 | 411.9 | 31.671 | -1.8 | 0.4 |
| 22 | 375.587 | 376.7 | 7.615 | 14.892 | 96.128 | 437.1 | 35.750 | -2.9 | -0.2 |
| 23 | 390.350 | 390.6 | 8.165 | 15.9 | 112.0 | 479.2 | 45.1 | -1.8 | 1.5 |
| 24 | 411.720 | 412.3 | 8.496 | 496.1 | 506.0 | 521.3 | -0.2 | 3.0 |
| 25 | 426.659 | 427.9 | 8.4  | 14.209 | 81.236 | 395.0 | 27.397 | -1.0 | 1.1 |
| 26 | 447.669 | 447.9 | 8.8  | 154.8 | 580.3 | 66.1 | 1.6 | 6.4 |
| 27 | 462.726 | 462.3 | 9.259 | 154.8 | 580.3 | 66.1 | 1.6 | 6.4 |

The other values in Table 1 have been defined in Introduction and are provided by the number of the corresponding formula. 

The EPN requirement comes from isospin independence of NN-interactions in \( \alpha \)-clusters. It provides a possibility to estimate the total Coulomb energy of a nucleus, \( E^{C} = \sum \Delta E_{pn} \), where \( \Delta E_{pm} \) is the differences between the binding energies of the proton and neutron belonging to one \( pm \)-pair. If a sum of the values \( \Delta E_{pm} \) of two \( pm \)-pairs belonging to one \( \alpha \)-cluster is presented as the Coulomb energy of the \( \alpha \)-cluster \( \Delta E_{\alpha} = \sum \Delta E_{pm} \), then
\[ E_C = \sum_1^{N_\alpha} \Delta E_\alpha + \delta, \]  
(3)

where, the value of \( \delta \) comes from taking into account that two clusters in \(^8\)Be are at bigger distance than in \(^{10}\)B, see Fig. 1. For the nucleus \( A_1 \) the corresponding equation is

\[ E_C^1 = \sum_1^{N_\alpha} \Delta E_\alpha + \delta + \Delta E_{pn}. \]  
(4)

The values of \( \Delta E_{pn} \) for the nuclei with \( N = Z \) are obtained by using the experimental binding energies taken from [18,19]. The binding energies of the nuclei with \( (Z - 1, N) \) and \( (Z, N - 1) \) needed for calculation of \( \Delta E_{pn} \) for the nuclei with \( 3 \leq Z \leq 8 \) and \( Z = 24, 25 \) have been calculated from the mass deficiency [19]. The energy of \(^4\)He 28.296 MeV and of \(^5\)He 27.338 MeV are taken from [18]. For the nuclei with \( Z = 23, 26, 27, 28 \) the values of \( \Delta E_{pn} \) cannot be obtained by this procedure due to the lack of experimental data for the nuclei with \( (Z, N - 1) \). The values have been estimated in the framework of this model by equations (17)/(18) with using the values of the radius of the last proton position \( R_{p/p1} \) (21), see Section 3.

The value of the Coulomb repulsion energy between two nearby clusters \( \varepsilon_{\alpha\alpha}^C = 1.925 \) MeV has been found from the equation for the Coulomb energy of one cluster in \(^{12}\)C \( \Delta E_\alpha(^{12}\)C) = \( \varepsilon_\alpha^C + 2\varepsilon_{\alpha\alpha}^C \), where \( \varepsilon_\alpha^C \) denotes the internal Coulomb energy of one cluster and \( \varepsilon_{\alpha\alpha}^C = E_{\alpha\alpha}^C - 0.764 \) MeV. This value perfectly fits the corresponding equation for \(^{16}\)O, which is \( \Delta E_\alpha(^{16}\)O) = \( \varepsilon_\alpha^C + 3\varepsilon_{\alpha\alpha}^C \). The value of the Coulomb repulsion energy between the single \( p\)-\( n \) pair and one nearby alpha-cluster in case of odd \( Z_1 \) is found from the data for the nuclei \(^6\)Li and \(^{14}\)N, \( \varepsilon_{\alpha\alpha}^C = 1.001(6) \) MeV.

One can estimate the Coulomb repulsion energy between two alpha clusters in the \(^8\)Be as follows, \( \varepsilon_{\alpha\alpha}^C(^8\)Be) = \( E_{\alpha\alpha}^C - 2\varepsilon_\alpha = 1.887 \) MeV, which is less than \( \varepsilon_{\alpha\alpha}^C \) by the value of \( \delta = 0.038 \) MeV.

Eqs. (3) and (4) give the empirical values of the total Coulomb energy \( E_C \) obtained on the basis of the values known from the experimental data with an accuracy of a few KeV. Having obtained the total Coulomb energy of nuclei as a sum of \( \Delta E_{pn} \) of the pairs, one can easily test the validity of the hypothesis of the alpha-cluster structure on the nuclei with few clusters \( N_\alpha \leq 4 \), that is the nuclei where each cluster is in touch with everyone of the other clusters. In this case the Coulomb energy can be calculated by the number of clusters and their bonds.

For \(^8\)Be according to Fig. 1. \( E_C(^8\)Be) = 2\varepsilon_\alpha + \varepsilon_{\alpha\alpha}^C(^8\)Be). For \(^{12}\)C and \(^{16}\)O \( E_C(^{12}\)C) = 3\varepsilon_\alpha + 3\varepsilon_{\alpha\alpha} \) and \( E_C(^{16}\)O) = 4\varepsilon_\alpha + 6\varepsilon_{\alpha\alpha} \). In case of \(^6\)Li \( E_C(^6\)Li) = \varepsilon_\alpha + \varepsilon_{\alpha\alpha} \) and for \(^{1\text{B}}\) \( E_C(^{10}\)B) = 2\varepsilon_\alpha + \varepsilon_{\alpha\alpha} + \Delta E_{pn} \). For \(^{14}\)N \( E_C(^{14}\)N) = \( E_C(^{12}\)C) + 3\varepsilon_{\alpha\alpha} \). For these nuclei the equations give values of \( E_C \) in agreement with the values obtained by (3) and (4), see Table 1, with an accuracy of 1 KeV, for \(^6\)Li with an accuracy of 6 KeV. It confirms the validity of the cluster structure of the nuclei with the quantities of \( \varepsilon_{\alpha\alpha} \) and \( \varepsilon_{\alpha\alpha} \), as well as the equations (3) and (4) are confirmed as the reliable formulas for obtaining the empirical values of the Coulomb energy for the nuclei with \( Z \leq 22 \) with the sure accuracy of few tens KeV. It also approves the statement about isospin independence of NN-interaction.

After adding the values of \( \Delta E_{pn} \) for the nuclei with \( Z = 26, 28 \) and \( Z = 29 \) estimated in the framework of the model with an accuracy of several tenth MeV (the accuracy is determined by the difference in the values obtained by the other possible way, by using (19)/(20), see section 3), the accuracy of the value of \( E_C \) for \(^{29}\)Cu is expected to be about 1 - 2 MeV.

One can estimate the nuclear force energy of \( \alpha \)-clusters from Eq. \( \varepsilon_{\alpha\alpha}^{\text{nuc}} = \varepsilon_\alpha + \varepsilon_{\alpha\alpha}^C = 29.060 \) MeV and \( \varepsilon_{\alpha\alpha}^{\text{nuc}} = \varepsilon_\alpha + \varepsilon_{\alpha\alpha}^C = 2.425\)MeV + 1.925MeV = 4.350MeV. This consideration allows one to estimate \( \varepsilon_{\alpha\alpha}^{\text{nuc}}(^8\)Be) = \( (E_{\text{exp}}(^8\)Be) - 2\varepsilon_\alpha + \varepsilon_{\alpha\alpha}^C(^8\)Be) = 1.792 \) MeV.

The empirical values of nuclear force energy of \( \alpha \)-clusters \( E_{\alpha\alpha}^{\text{nuc}} \) for the lightest nuclei \( Z \leq 6 \), \( N_\alpha \leq 3 \), are obtained from \( E_{\text{exp}} = E^{\text{nuc}} + E^C \) and they are presented in Table 1. For the nuclei with \( Z \geq 7 \), \( N_\alpha \geq 3 \), one gets

\[ E_{\alpha\alpha}^{\text{nuc}} = N_\alpha \varepsilon_{\alpha\alpha}^{\text{nuc}} + 3(N_\alpha - 2)\varepsilon_{\alpha\alpha}^{\text{nuc}}; E_{\alpha1}^{\text{nuc}} = E^{\text{nuc}} + 6\varepsilon_{\alpha\alpha}^{\text{nuc}} + \varepsilon_{\alpha\alpha}. \]  
(5)
where $6\varepsilon_{pnpn} + \varepsilon_{pn} = 6\varepsilon_{pmpm} + 3\varepsilon_C + \varepsilon_{pn} = 16.884 \text{ MeV.}$

The Coulomb energy for the nuclei with $N_\alpha > 4$ consists not only of the energies of bonds between nearby clusters. It must contain the long-range Coulomb interaction part. That fact that Eq. (1)/(2) has a good agreement with the experimental values allows one to suggest that the long-range part of the Coulomb energy $E_\alpha^{Clr}$ between the last $\alpha$-cluster and the $N_\alpha - 4$ remote clusters is compensated by the surface tension energy, which means that the $E^{st} \neq 0$ for the nuclei with $Z \geq 9$. It gives an opportunity to obtain the empirical values $E^{st}$ from the values of $\Delta E_\alpha$ and $\Delta E_{pn}$, known with a good accuracy.

The Coulomb energy of the last $\alpha$-cluster for the even nuclei with $N_\alpha \geq 5$ is defined as $\Delta E_\alpha = \varepsilon_C - 3\varepsilon_C$ and it is suggested that the surface tension energy of the last cluster $E^{st}_\alpha = E^{Clr}_\alpha$. The Coulomb energy of the single $pn$-pair in case of odd $Z_1 \Delta E_{pn} = 3\varepsilon_C + E^{Clr}_{pn}$ and the surface tension energy of the $pn$-pair $E^{st}_{pn} = E^{Clr}_{pn}$. Therefore, the empirical values of the surface tension energy are obtained from

$$E^{st} = \sum_{\alpha} E^{st}_{\alpha}; E^1 = E^{st} + E^{st}_{pn}, \quad (6)$$

where $E^{st}_\alpha = \Delta E_\alpha - \varepsilon_C - 3\varepsilon_C$ and $E^{st}_{pn} = \Delta E_{pn} - 3\varepsilon_C$. The values are presented in Table 1. This way to obtain empirical values of $E^C$ and $E^{st}$ makes the total binding energy equal to $E^b$ (1)/(2), see Table 1.

The formula to calculate the nuclear binding energy is written as follows

$$E^b = E^{nuc} - E^C + E^{st} + E_{\Delta N}, \quad (7)$$

For the lightest nuclei with $Z \leq 8$, $N_\alpha \leq 4$, $E^{st} = 0$. The values of $E^{nuc}$ and $E^C$ in this case are calculated in accordance with Fig. 1 by adding the corresponding energy portions of the $\alpha$-clusters and the $pn$-pair. In case of the nuclei not having a core the binding energy of excess neutrons $E_{\Delta N}$ stays out of consideration.

For the other nuclei with $Z \geq 9$ $E^{nuc}_\alpha$ is calculated by (5), $E^C$ is calculated by a well known formula for a spherical charged body with radius $R_C$, for odd $Z$ nuclei $R_{C1}$ ($R_C$ and $R_{C1}$ are determined by (22) on the number of $\alpha$-clusters in the nucleus, see Section 3)

$$E^C = \frac{3 Z^2 e^2}{5 R_C}, \quad (8)$$

$E^{st}$ is calculated by (6) with $E^{st}_\alpha = \gamma_1 R^2_{C1}$, so Eq.(6) is rewritten as the following

$$E^{st} = E^5 + \sum_6 \gamma_1 R^2_{C1}; E^1 = E^{st} + \gamma_1 R^2_{p1}/2, \quad (9)$$

where $E^5 = 1.026 \text{ MeV}$ (see Table 1 for $Z = 10$, $N_\alpha = 5$), $\gamma_1 = 0.471 \text{ MeV/fm}^2$, $R_{p1}$ stands for the radius of the $pn$-pair (21), see Section 3. The binding energy of excess neutrons $E_{\Delta N}$ is calculated by (13), see Subsection 2.2.

The values of $E^b$ (7) for the nuclei with $N = Z$, $E_{\Delta N} = 0$, are presented in Table 1 in the column of $\Delta E^b(7)$.

The well known Weizsäcker formula [20] is

$$E^{Wz} = \alpha A - \beta A^{2/3} - \frac{\gamma Z^2}{A^{1/3}} \pm \frac{\delta}{A^{3/4}} - \frac{\epsilon (A - Z)^2}{A}. \quad (10)$$

The values $\Delta E^b$ (7) depends on $N_\alpha$ and $\Delta N$, $\Delta E^{Wz}$ (10) depends on $A$ and $Z$. The least differences between the values of the Coulomb energy estimated in these approaches are for the nuclei with $N = Z$, see Table 1 [17]. With $Z$ growing the difference increases. Calculation of $E^b$ (7) for $^{164}Gd$, $E_{exp} = 1338 \text{ MeV}$, gives $3\frac{Z^2 e^2}{R_C} = 627 \text{ MeV}$ and the nuclear force energy of all
inter-nucleon interactions $E_{\alpha}^{\text{nuc}} + E^{\text{st}} + E_{\Delta N} = 1964$ MeV, so $\Delta E^{b} = 1$ MeV. Calculation of $E^{Wz}$ (10) gives $\frac{Z^2}{A^{1/3}} = 565$ MeV and $\alpha A - \beta A^{2/3} + \frac{\delta}{A^{1/3}} - \frac{\epsilon(A/2-Z)^2}{A} = 1902$ MeV, so $\Delta E^{Wz} = 1$ MeV. In despite of the fact that $\Delta E^{Wz} = \Delta E^{b}$, these two approaches have different values of the Coulomb energy and the nuclear force energy of all inter-nucleon interactions.

The values of $E^{b}$ (7) and $E^{Wz}$ (10) calculated for the nuclei with different number of excess neutrons are presented in Fig 2. Every nucleus is presented in the graphs ‘$E^{b}$’ and ‘$E^{Wz}$’ by two dots, one for $E_{\exp}$ and the other for the calculated value.

![Figure 2: The values of $E^{b}$ (7) and the Weizsäcker formula $E^{Wz}$ (10) calculated for the nuclei $A(Z)$ indicated in the corresponding graph.](image)

The widths of the graphs contain both the width of $E_{\exp}$ distribution on $A$ and the deviations between experimental values and calculated ones. For the nuclei with $Z > 100$ the values calculated by the Weizsäcker formula have been used instead of unknown experimental values. The width of graph ‘$E^{b}$’ in the part corresponding to the nuclei with $Z > 100$ consists of the both the width of ‘$E^{Wz}$’ and the deviations $E^{Wz} - E^{b}$, which is within a few MeV. The width of the part of ‘$E^{b}$’ is a little bit bigger than that of the corresponding part of ‘$E^{Wz}$’ graph where $\Delta E^{Wz} = 0$. The equal widths of the two graphs for the nuclei with $Z \leq 100$ clearly show that the deviations $\Delta E^{b}$ and $\Delta E^{Wz}$ are close.

To test the validity of formulas (5), (8) and (9) independently of the energy of excess neutrons, the alpha-particle separation energy $E_{\alpha}^{\text{sep}} = E^{b}(A, Z) - E^{b}(A - 4, Z - 2)$ as well as the deuteron separation energy $E_{d}^{\text{sep}} = E^{b}(A, Z) - E^{b}(A - 2, Z - 1)$ for the nuclei with even and odd number of excess neutrons $\Delta N$ has been calculated. In Fig. 3 the calculated values are given in comparison with the experimental data.

The formula to calculate $E_{\alpha}^{\text{sep}}$ does not contain $E_{\Delta N}$ neither for even nor for odd excess neutrons. Neither does the formula to calculate $E_{d}^{\text{sep}}$ in case of even number of excess neutrons. The deuteron separation energy in the nuclei with even number of excess neutrons splits into two lines in dependence on whether the nucleus is of even $Z$ or odd $Z$. It is clearly seen in the left graph of Fig. 3.

In case of odd number of excess neutrons the experimental energy of the single excess neutron is bigger for odd $Z$ nuclei than that of even $Z$ nuclei. It is taken into account in the formula to calculate the energy of the last excess neutron (14), see Subsection 2.2. Then the formula to calculate the deuteron separation energy must have $-0.2E_{nn}$ for even $Z$ and $+0.2E_{nn}$ for odd $Z$. The nucleon where $E_{nn}$ is the binding energy of the $nn$-pair the single neutron belongs to. This is in
agreement with the experimental data, see the right graph in Fig. 3.

One can see a periodical structure in the experimental data, which refers to the shell effects connected with the spin-orbit correlations of the last nucleons, so called valent nucleons. The calculated values are not sensitive to the details. They describe the α-particle and deuteron separation energies for all nuclei with an average deviation from the experimental values in 2.5 MeV, which is a small percentage of the total binding energy.

After simplifying Eqs. (5) and (8) one gets

\[ E_{\alpha}^{\text{nuc}} = 42.110 N_{\alpha} - 26.100 \text{MeV}, \]

\[ E_{1}^{\text{nuc}} = E_{\alpha}^{\text{nuc}} + 16.884 \text{MeV}, \]

\[ E_{C} = 1.848 N_{\alpha}^{5/3}, \]

for odd \( Z \) \( E_{C} = 1.848(N_{\alpha} + 0.5)^{5/3} \).

For the nuclei with \( Z \leq 29 \) the empirical values of \( E_{C} \) and \( E_{st} \) have been obtained with a good accuracy, Table 1. It is good to have a simple function to calculate \( E_{st} \) instead of the sum (9). It can be successfully approximated in accordance with the representation of the nucleus as a core plus four peripheral clusters like it is used in calculations of radii (24), see Section 3. Then the function consists of the surface tension energy of the core clusters \( E_{st}^{\text{core}} \) and the energy of four peripheral clusters placed at a distance \( R_{p} \) (21) from the center of mass

\[ E_{st} = E_{st}^{\text{core}} + 4 \gamma_{1} R_{p}^{2} E_{1}^{st} = E_{st} + \gamma_{1} R_{p1}^{2}/2, \]  

(11)

where \( E_{st}^{\text{core}} = (N_{\alpha} - 5.28)(N_{\alpha} - 5)^{2/3} - 11.5 \text{MeV} \). The approximation is good for the nuclei with \( Z \geq 22 \).

A simple function to approximate \( E_{st} \) for \( Z \geq 30, N_{\alpha} \geq 15 \), is the following

\[ E_{st} = (N_{\alpha} + 1.7)(N_{\alpha} - 4)^{2/3}; E_{1}^{st} = (N_{\alpha} + 2.8)(N_{\alpha} - 4)^{2/3}. \]  

(12)

The values of \(|\Delta E^{b}|\) of \( E^{b} \) (7) with using \( E_{st}^{b} \) (9), (11) and (12) as well as the values of \(|\Delta E^{Wz}|\) for the most of the cases are less than 0.5% of the total binding energy.

### 2.2 Excess neutron binding energy

As it was shown [10] the core starts growing with \( N_{\alpha} \) from the nucleus with \( N_{\alpha} = 5 \). It is suggested that the value of the binding energy \( E_{\Delta N} \) of the number of excess neutrons \( \Delta N \) placed in the core
and bound in $M$ pairs, $\Delta N = 2M + 1$ is calculated as follows:

$$E_{\Delta N} = \sum_{i=1}^{M} E_{nn_i} + \epsilon_n,$$

(13)

where $E_{nn_i}$ stands for the energy of $i$th $nn$-pair, $\epsilon_n$ is the energy of the single excess neutron and

$$\epsilon_n = E_{nn_{M+1}}/2 \pm 0.1E_{nn_{M+1}},$$

(14)

where '+' stands for odd $Z$ nuclei and '-' for even $Z$ ones. The phenomenological formula takes into account that fact that a sum of experimental energies of single excess neutrons of the nuclei with odd $Z$ and even $Z$ approximately equals the energy $E_{nn_{M+1}}$ of the $M + 1$th $nn$-pair which they belong to. In case of $\Delta N = 1$, $M = 0$ $E_{nn_{M+1}} = E_{nn_1}$.

The experimental values of $E_{nn_i}$ in MeV are fitted by an equation

$$E_{nn_i} = 22.5 - 1.358i^{2/3},$$

(15)

It is known that the experimental values of separation energy of $nn$-pairs have deviations within up to 5 MeV determined by what nucleus loses the pair, which certainly refers to the shell effects. Despite the relatively big deviation of the values, the empirical values of the energies of all excess $nn$-pairs $E_{\Delta N(\text{exp})}$ known only for the nuclei with $N = Z$ from the following equation:

$$E_{\Delta N(\text{exp})} = E_{\exp}(Z, N + \Delta N) - E_{\exp}(Z, N),$$

(16)

are restricted within relatively narrow corridor. For example, for the nuclei with $21 \leq Z \leq 29$ the value of the separation energy of two excess neutrons varies within $E_{2(\text{exp})} = 21 \div 23$ MeV, for four excess neutrons $E_{4(\text{exp})} = 42 \div 45$ MeV, $E_{6(\text{exp})} = 61 \div 63$ and $E_{8(\text{exp})} = 77 \div 80$ MeV. Therefore, the parameters in (15) have been obtained by fitting both the experimental values of $nn$-pair separation energies known [18] for 27 $nn$-pairs and the values of $E_{\Delta N(\text{exp})}$. The values of $E_{\Delta N}$ (13) have been used here to calculate the binding energies of the excess neutrons filling the core, which exists only in the nuclei with $Z \geq 10$, $N_\alpha \geq 5$. It is evident that Eq. (13) is not proper for the exotic nuclei like, for example $^6\text{He}$ or $^{11}\text{Li}$.

3 Radius of the last proton position, Coulomb radius and charge radii of nuclei

To estimate the value of the radius of the last proton position $R_p$ in a nucleus $A$ the value of $\Delta E_{pn}$ is taken as the Coulomb energy of the last proton. The center of mass in the approach is supposed to be in the center of the Coulomb field. The energy of the last proton in the nucleus with an even $Z$ consists of the energy of its interaction with the other proton of the $\alpha$-cluster $\varepsilon_{\alpha}^C$ and the Coulomb energy of its interaction with the other $Z - 2$ protons of the nucleus

$$\Delta E_{pn} = \varepsilon_{\alpha}^C + (Z - 2)e^2/R_p,$$

(17)

and for odd $Z$

$$\Delta E_{pn} = (\varepsilon_{p\alpha} - \varepsilon_{\alpha}^C) + (Z - 1)e^2/R_{p1},$$

(18)

where $R_{p1}$ stands for the radius of the position of the single $pn$-pair.

It is valid only for the nuclei with $Z \geq 10$, because the last proton Coulomb energy for the lightest nuclei must be calculated with a different function. For example, one can use the spherical function with the Coulomb radius $R_C$ like the Coulomb potential at the small radii less than $R_C$ in the Shrödinger equation for proton bound state wave function in a Distorted Wave Born Approximation.
Another way to estimate $R_p$ comes from an equation for the Coulomb energy for the last $\alpha$-cluster. The Coulomb energy of the cluster consists of the internal Coulomb energy of the cluster $\varepsilon_C^\alpha$, Coulomb energy of its interaction with the three nearby clusters $3\varepsilon_C^{\alpha_\alpha}$ and the long-range part of the Coulomb energy of its interaction with the other $N_\alpha - 4$ clusters of the nucleus

$$\Delta E_\alpha = \varepsilon_C^\alpha + 3\varepsilon_C^{\alpha_\alpha} + 2(Z - 8)e^2/R_{N_\alpha-4},$$  \hspace{1cm} (19)$$

where $R_{N_\alpha-4}$ stands for the distance between the mass center of the remote $N_\alpha - 4$ clusters and the cluster under consideration. The value $R_{N_\alpha-4}$ is approximated as $R_{N_\alpha-4} = 1.2R_p$ by fitting the empirical values of $\Delta E_\alpha$. For the nuclei with odd $Z$ in accordance with the same logic of taking into account the long-range Coulomb interaction, one obtains

$$\Delta E_{pn} = 3\varepsilon_C^{\alpha_\alpha} + (Z - 7)e^2/(1.2R_p + R_{p1} - R_p).$$  \hspace{1cm} (20)$$

The third way to estimate $R_p$ comes from the idea that the long-range Coulomb energy must be compensated by the surface tension energy $E_{\alpha}^{st}$. The latter is expected to be proportional to $R_p^2$. Therefore the last member in the sum (19) is taken equal to $E_{\alpha}^{st} = \gamma_1R_p^2 = 2(Z - 8)e^2/(1.2R_p)$, and the value of $\gamma_1 = 0.471$ MeV/fm$^2$ is obtained from fitting the empirical values of $\Delta E_\alpha$. As a result, one obtains an equation to calculate $R_p$ in dependence on $N_\alpha$ only.

$$R_p = 2.168(N_\alpha - 4)^{1/3}.\hspace{1cm} (21)$$

The value of $R_{p1}$ is calculated by the same formula with using $N_\alpha + 0.5$ instead of $N_\alpha$.

The values of $R_{p/p1}$ (21) have been used in (17)/(18) to estimate the values of $\Delta E_{pn}$ for the nuclei with $Z \geq 23, 26, 27, 28$. The values are given in Table 1. The accuracy of several tenth MeV is determined by the deviation between two possible ways to obtain $\Delta E_{pn}$ by (17)/(18) or (19)/(20).

Finally, the fourth way to calculate $R_p$ comes from the requirement that the surface tension energy is expected to be proportional to the number of clusters on the surface of the liquid drop $E_{\alpha}^{st} = \gamma_1R_p^2 = \gamma_2N_\alpha^{2/3}$, where $\gamma_2 = 1.645$ MeV is obtained by fitting the values of $\Delta E_\alpha$. The value is supposed to refer to the Coulomb radius $R_C$. Therefore the equation is

$$R_C = 1.869(N_\alpha)^{1/3}.\hspace{1cm} (22)$$

For odd $Z$ the value of $R_{C1}$ is calculated by (22) with $N_{\alpha 1} = N_\alpha + 0.5$.

The values $R_{p/p1}$ obtained by different ways differ within 0.3 fm for the nuclei with $16 \leq Z \leq 29$. For the nuclei with $9 \leq Z \leq 15$ the difference is bigger. For the nuclei with $Z > 22$, $N_\alpha > 11$, $R_C$(22) < $R_p$ (21).

It was shown [10] that the formula to calculate the radii of nuclei

$$R = R_\alpha(N_\alpha)^{1/3},$$  \hspace{1cm} (23)$$

successfully fits the experimental radii of the most abundant isotopes $R_{\alpha}^{exp}$ with three close values of $R_\alpha$. Using the value of $R_\alpha$ equal to $R_{\alpha He} = 1.710$ fm [22] fits the radii of the nuclei with few clusters $N_\alpha = 3, 4$. There is no core in these nuclei. In the nuclei with $5 \leq N_\alpha \leq 12$, where the number of clusters in the core is comparable with the number of peripheral clusters, the value $R_\alpha = 1.628$ fm.

In the nuclei with $N_\alpha \geq 12$ the radii $R_{\alpha}^{exp}$ are well described by the radius $R_\alpha = 1.600$ fm, which is the radius of the clusters of the core [10].

Varying the number of the peripheral clusters from 1 to 4 (or 5) in accordance with the comprehensive shell model is the point of the alpha-cluster shell model to calculate the root mean square radius $R_{shl}$ [14,17]. Then the formula is similar to that usually used in the framework of the single-particle potential approaches with using occupation numbers assumed in the shell model [23] in calculations of root mean square radii.

It was also shown [17] that the number of the peripheral clusters can be fixed at 4 for the nuclei with $N_\alpha \geq 12$, then the root mean square radius $R \approx R_{shl}$ where $R$ is calculated as follows.
\[ N_\alpha R^2 = (N_\alpha - 4)1.600^2(N_\alpha - 4)^{2/3} + 4R^2_p. \] (24)

For odd \( Z \) the charge radius \( R_1 \) is calculated by the equation
\[(N_\alpha + 0.5)R_1^2 = N_\alpha R^2 + 0.5R^2_p, \] (25)
where the radius of the position of the single \( pn \)-pair \( R_p \) (21) is weighted by factor 0.5. For the nuclei with \( 5 \leq N_\alpha \leq 10 \) \( R \) is calculated as \( R_{shl} \), see (15) in [17].

The radii calculated in the representation of the alpha-cluster shell model \( R_{shl} \) have deviation \(< \Delta^2 >^{1/2} = 0.034 \text{ fm} \) with (24) (for odd \( Z \) (25)) for the nuclei with \( 9 \leq Z \leq 118 \) [17].

A good agreement between the values of \( R \) (23) and \( R \) (24) (for odd \( Z \) (25)) and the experimental radii of the most abundant isotopes \( R_{exp}^{abn} \) [9,22,24] is shown in Fig. 4. From the figure one can see that for all nuclei with \( Z \geq 10, N_\alpha \geq 5 \), the Eqs. (23) and (24) give close to the experimental data values. The values as well as the calculated radii of the \( \alpha \)-decay nuclei stable to \( \beta \)-decay [21] with \( 83 \leq Z \leq 117 \) are presented in the tables in [10].

![Figure 4](image.png)

Figure 4: Left part: Experimental radii of stable isotopes \( R_{exp} \) in dependence on \( A \) and \( Z \). On the right part: Charge radii \( R \) (23) for the \( 5 \leq Z \leq 23 \) and (24)/(25) for the \( Z \geq 24 \) (solid line) and \( R = 1.600N_\alpha^{1/3} \) \text{ fm} (dashed line) in comparison with the experimental radii of the most abundant isotopes \( R_{exp}^{abn} \) (crosses). The charge radius of one nucleon of \( \alpha \)-clusters \( r_{ch} = R/(2Z)^{1/3} \) is indicated by the lines corresponding to \( R \).

The values of \( R \) calculated for two nuclei \( ^6\text{Li}, ^9\text{Be} \) are not in agreement with the experimental data. These nuclei do not have enough number of bonds between the clusters and the single \( pn \)-pair to provide a sufficient nuclear density. For the nuclei with \( 6 \leq Z \leq 83 \) the deviation of \( R \) (23) and (24)/(25) from the experimental values \( R_{exp} \) is \(< \Delta^2 >^{1/2} = 0.050 \text{ fm} \). The deviation between the two lines in Fig. 4 for the nuclei with \( 24 \leq Z \leq 118 \) is \(< \Delta^2 >^{1/2} = 0.028 \text{ fm} \). The values of experimental radii of 125 stable isotopes \( R_{exp} \) [9,22,24,25,26] are also given in dependence of \( A \) and \( Z \). The radii \( R_{exp} \) are given without indication of their errors, which in most of the cases are within \( 0.004 \div 0.060 \text{ fm} \). It visually approves that fact [10] that both distributions of radii on \( A \) and \( Z \) are well described by the functions \( \sim A^{1/3} \) and \( \sim Z^{1/3} \).

4 Conclusions

The approach based on the \( \alpha \)-cluster model proposes some formulas to calculate the binding energies and the charge radii of the nuclei of the \( \beta \)-stability path and around it. The formulas have been derived on the basis of the idea of isospin independence of inter-nucleon interactions in the \( \alpha \)-cluster structure.
The approach implies that the nucleus is a dense package of alpha-clusters. The inter-cluster distances are determined by the charge radii of the clusters. Some amount of excess neutrons fill in the gap between the matter bodies of the $\alpha$-clusters of the core [10]. The energy of these excess neutrons is described by a smooth function on the number of the $nn$-pairs (13). The formula to calculate the binding energy proper for the nucleus with five $\alpha$-clusters (7) turned out to be good for the other nuclei up to the most heavy ones. The same thing one can say about the formula to calculate the charge radii $R = r_{ch}(2Z)^{1/3}$ with the average charge radius of one nucleon of the $\alpha$-cluster structure $r_{ch} \sim 1.01$ fm for all nuclei with $N_{\alpha} \geq 5$, see Fig 4.

The formula to calculate the nuclear binding energy is evidently different from the well known Weizsäcker formula. These two approaches give different estimations of the total Coulomb energy and the energy due to all inter-nucleon interactions, but the values of the total binding energies of these approaches are close. To calculate the charge radii both the approaches propose successful but different formulas, one is $\sim A^{1/3}$ and the other $\sim Z^{1/3}$.

A few useful phenomenological formulas have been found in the approach. These are the formulas to calculate the root mean square charge radius, the Coulomb radius and the radius of the last proton’s position in dependence on the number of $\alpha$-clusters. Besides, the empirical values of the Coulomb energy and the surface tension energy with a good accuracy have been obtained for the nuclei with $N = Z$.

The values of the Coulomb repulsion energy and the nuclear force energy of the inter-cluster interaction, which are 1.925 MeV and 4.350 MeV, correspondingly, could be a good test in a microscopic description of the light nuclei $^{12}\text{C}$ and $^{16}\text{O}$ in the framework of the few-body approach.

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**References**

[1] W. C. Weizsäcker, *Z. Phys.* **96**, 431 (1935).
[2] P. D. Norman, *Eur. J. Phys.* **14**, 36 (1993).
[3] B. Buck, A. C. Merchant, M. J. Horner and S. M. Perez, *Nucl. Phys.* **A675**, 157 (2000).
[4] P. K. Kakanis and G. S. Anangostatos, *Phys. Rev.* **C54 N6**, 2996 (1996).
[5] B. Imanishi and W. von Oertzen, Physics Reports **155**, 29 (1987)
[6] D.V. Fedorov and A.S. Jensen, *Phys. Lett.* **B389**, 631 (1996).
[7] M. Dufour and P. Descouvemont, *Nucl. Phys.* **A750**, 2-4, 218 (2005).
[8] K. A. Gridnev, S. Yu. Torilov, V.G. Kartavenko, W.Greiner, D.K. Gridnev, J. Hamilton [arXiv:nucl-th/0408031v1 11Aug2004].
[9] S. Anangostatos, *International Journal of Modern Physics* **E5** 557 (1996)
[10] G. K. Nie, [arXiv:nucl-th/0512023v1] 07Dec2005, accepted for publication in MPLA.
[11] G. K. Nie *Uzbek Journal of Physics* **7**, N3 175 (2005).
[12] D. Blanc, *Nuclei, Particles, Nuclear Reactors* (MIR, Moscow, 1989), 336 p. Russian translation of the book D. Blanc *Noyaux, particules. Reacteurs nucléaires* (Masson, Paris, 1987).
[13] Dao T. Khoa, W. von Oertzen, H.G. Bohlen and H.S. Than, [arXiv:nucl-th/0510048v1 17Oct2005]
[14] G. K. Nie, *Uzbek Journal of Physics* **6**, N1 1 (2004).

[15] G. K. Nie, in *Proceedings of ICNRP '03, 4th International Conference "Nuclear and Radiation Physics", September, 15-17, 2003, Almaty, Kazakhstan* (Institute of Nuclear Physics, Almaty, Kazakhstan) p.147 and p.168.

[16] G. K. Nie, in *Izvestiya RAN, ser. Phys.* **70**, N5 (2006) to be published.

[17] G. K. Nie, [arXiv:nucl-th/0508026v1 13Aug2005](https://arxiv.org/abs/nucl-th/0508026)

[18] V. A. Kravcov, *Massy Atomov I Energii Seyasy Yader* (Moscow, Atomizdat, 1965) in Russian.

[19] C.M. Lederer and V. S. Shirley, *Table of Isotopes (seventh edition)*, Lawrence Berkeley Laboratory, University of California, Berkeley, John Wiley and Sons, Inc., New York, 1978.

[20] K. N. Mukhin, *Experimental’naya Yadernaya Physika. 1.Fizika Atommogo Yadra* (Moscow, Atomizdat, 1974) P. 128.

[21] I.P. Selinov, *Tables of Atoms, Atomic Nuclei and Subatomic particles* (Center of International Data and Atomic and Nuclides of Russian Academy of Sciences and the Ministry on Atomic Energy of the Russian Federation, 1994).

[22] C.W.De Jager, H. De Vries, and C. De Vries *Atomic Data and Nuclear Data Tables*

[23] P. E. Hodgson, *Hyperfine Interactions* **74** 75 (1992), **14**, 479 (1974).

[24] H. De Vries, C.W.De Jager, and C. De Vries *Atomic Data and Nuclear Data Tables* **36**, 495 (1987).

[25] G. Fricke et al, *Atomic Data and Nuclear Data Tables* **60**, 177-285 (1995).

[26] R. C. Barret and D. F. Jackson, *Nuclear Structure and Sizes*, Clarendon Press, Oxford 1977, Table 6.2.