A New Look at
Convexity
and Pseudoconvexity

Steven G. Krantz

0 Introduction

Convexity is a classical idea. Archimedes used a version of convexity in his considerations of arc length. Yet the idea was not formalized until 1934 in the monograph of Bonneson and Fenchel [BOF].

The classical definition of convexity is this: An open domain \( \Omega \subseteq \mathbb{R}^N \) is convex if, whenever \( P, Q \in \Omega \), then the segment \( PQ \) connecting \( P \) to \( Q \) lies in \( \Omega \). We call this the synthetic definition of convexity. It has the advantage of being elementary and accessible (see [VAL]). The disadvantages are that it is non-quantitative and non-analytic. It is of little use in situations of mathematical analysis where it is most likely to arise.

The analytic definition of convexity is a bit more recondite. Let \( \Omega \subseteq \mathbb{R}^N \) have \( C^2 \) boundary. For us this means that there exists a \( C^2 \) function \( \rho \) defined in a neighborhood \( U \) of \( \partial \Omega \) such that

\[
\Omega \cap U = \{ x \in U : \rho(x) < 0 \}
\]

and further that \( \nabla \rho \neq 0 \) on \( \partial \Omega \). We call \( \rho \) a defining function for \( \Omega \). Let \( P \in \partial \Omega \). We say that a vector \( w \in \mathbb{R}^N \) is a tangent vector to \( \partial \Omega \) at \( P \), and we write \( w \in T_P(\partial \Omega) \), if

\[
\sum_{j=1}^N \frac{\partial \rho}{\partial x_j}(P)w_j = 0.
\]

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The domain $\Omega$ is said to be *analytically convex* at $P$ if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P)w_jw_k \geq 0$$

for all $w \in T_P(\partial \Omega)$.

A moment’s thought reveals that the condition (*) simply mandates that the second partial derivative of $\rho$ in the direction $w$ be nonnegative. This is the classical “convex up” condition from calculus. This analytic definition of convexity has the advantage that it can be localized to individual boundary points, and it is *quantitative*. It is a straightforward exercise (see [KRA1]) to see that the analytic definition of convexity is equivalent to the synthetic definition of convexity.

The notion of pseudoconvexity has a slightly different ontology. Discovered by E. E. Levi in the study of domains of holomorphy, this idea was first formulated in its analytic form. Let $\Omega \subset \mathbb{C}^n$ have $C^2$ boundary. Let $\rho$ be a $C^2$ defining function for $\Omega$ as in our earlier discussion of convexity. Let $P \in \partial \Omega$. We say that $\xi \in \mathbb{C}^n$ is a *complex tangent vector* at $P$, and we write $\xi \in T_P(\partial \Omega)$, if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P)\xi_j = 0.$$  

The point $P$ is said to be a point of *Levi pseudoconvexity* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P)\xi_j\overline{\xi_k} \geq 0$$

for all $\xi \in T_P(\partial \Omega)$.

It is not a simple matter to give an elementary geometrical interpretation to the expression (*). Part of the purpose of the present paper is to come to some basic geometric understanding of this notion of pseudoconvexity.

It is appropriate to record in passing a classical, alternative notion of pseudoconvexity. Let $\Omega \subset \mathbb{C}^n$ be any domain (smoothly bounded or not). We say that $\Omega$ is *Hartogs pseudoconvex* if, with $\delta_\Omega$ denoting the function of Euclidean distance to the boundary, we have that $-\log \delta_\Omega$ is plurisubharmonic on $\Omega$. It is known—see [KRA1]—that a domain $\Omega$ with $C^2$ boundary is Levi pseudoconvex if and only if it is Hartogs pseudoconvex.

While the notion of Hartogs provides a sort of synthetic idea of pseudoconvexity, it is not strictly analogous to the idea that is used in classical
convexity theory. Convexity has played an ever more prominent role in the function theory of several complex variables in recent years (see [LEM] and [MCN1], [MCN2]). Thus it is worthwhile to be able to develop in further detail the analogy between classical convexity theory and modern pseudo-convexity theory. That is our first purpose in the present paper.

A second purpose of this paper is to examine the concept of finite type. Originating with the seminal paper [KOH] of Kohn, and later developed by Catlin [CAT1], [CAT2] and D’Angelo [DAN], this idea has become a central and influential artifact in complex function theory and partial differential equations. It is always worthwhile to find new ways to understand finite type, and we explore some of these in the present paper.

1 Analytic Discs and Pseudoconvexity

The results that we present here have a history. Certainly they are related to the classical Kontinuitätssatz, for which see [KRA1]. But the proofs, of necessity, are different.

Let $D \subseteq \mathbb{C}$ be the unit disc. An analytic disc in $\mathbb{C}^n$ is a holomorphic mapping $\varphi : D \to \mathbb{C}^n$. A closed analytic disc in $\mathbb{C}^n$ is a continuous mapping $\psi : \overline{D} \to \mathbb{C}^n$ such that $\psi|_D$ is holomorphic. In practice we may refer to either of these simply as an “analytic disc”. The boundary of a closed analytic disc is just $\psi(\partial D)$ whenever this expression makes sense. It will frequently be convenient to confuse the mapping $\varphi$ or $\psi$ with the image disc $\varphi(D)$ or $\psi(D)$ (or the boundary map $\psi(\partial D)$). We do so without further comment. The center of an analytic disc is $\varphi(0)$ or $\psi(0)$.

In this paper we shall think of the boundary of a closed analytic disc as the complex-analytic analogue of two points $P$ and $Q$ in the classical theory of convex sets. We shall think of the (image) analytic disc $\psi(\overline{D})$ as the complex-analytic analogue of the segment $PQ$ that connects $P$ and $Q$.

Thus we should like to have a characterization of pseudoconvexity, in terms of analytic discs, that is parallel to the synthetic characterization of convexity in terms of segments. It is the following.

**Proposition 1** Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with $C^2$ boundary. Then $\Omega$ is Levi pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : \overline{D} \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than $\delta_0$, and if $\psi(\partial D) \not\subseteq \partial \Omega$, then $\psi(\overline{D}) \not\subseteq \overline{\Omega}$. 
Proof: Let dist denote Euclidean distance. Choose $\epsilon_0 > 0$ so that

$$U_{\epsilon_0} \equiv \{z \in \mathbb{C}^n : \text{dist}(z, \partial \Omega) < \epsilon_0 \}$$

is a tubular neighborhood of $\partial \Omega$ (see [HIR]). Let $\delta_0 = \epsilon_0/100$. Let $\psi$ be a closed analytic disc as in the statement of the proposition. It follows immediately from the triangle inequality that the (image of the) closed analytic disc lies entirely inside the tubular neighborhood $U_{\epsilon_0}$. Now there are two cases:

Some point of $\psi(D)$ lies outside $\overline{\Omega}$. In this case let $p_0 \equiv \psi(\zeta_0)$ be the point of $\psi(D)$ that lies outside $\overline{\Omega}$ and furthest from $\partial \Omega$. Let $\nu$ be the unique normal vector from $\partial \Omega$ out to $p_0$. Say that $\nu$ emanates from the base point $q_0 \in \partial \Omega$. Then the domain

$$\hat{\Omega} \equiv \Omega - \nu = \{z - \nu : z \in \Omega \}$$

has the property that the disc $\psi(D)$ is tangent to $\partial \hat{\Omega}$ at $q_0 - \nu$ and the punctured disc $\psi(D) \setminus \{q_0 - \nu\}$ lies entirely in $\hat{\Omega}$. But of course $\hat{\Omega}$ is Levi pseudoconvex with $C^2$ boundary. So this last is impossible (see [KRA1]) by the classical Kontinuitätssatz. We have eliminated this case.

All points of $\psi(D)$ lie in $\Omega$. In this case $\psi(D) \subset \overline{\Omega}$ and we are done.

An argument similar to the one just presented, but even simpler, gives the following result. It is closer to the spirit of the classical synthetic definition of convexity.

Proposition 2 Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$ boundary. Then $\Omega$ is pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : D \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than $\delta_0$, and if $\psi(\partial D) \subset \overline{\Omega}$, then $\psi(D) \subset \overline{\Omega}$.

Yet another variant is this:

\[\text{The purpose of forcing the analytic disc to lie inside a tubular neighborhood (and to have small diameter) is to guarantee that the disc does not form the basis of a homology class in the boundary.}\]
Proposition 3 Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$ boundary. Then $\Omega$ is pseudoconvex in the classical sense if there is a number $\delta_0 > 0$ so that, whenever $\psi : \mathcal{D} \to \mathbb{C}^n$ is a closed analytic disc in $\mathbb{C}^n$ with diameter less than $\delta_0$, and if $\psi(\partial \mathcal{D}) \subset \overline{\Omega}$, then $\psi(\mathcal{D}) \subset \overline{\Omega}$.

2 The Concept of Finite Type

The idea of finite type was first conceived in the paper [KOH] of Kohn. Kohn’s idea was to measure the complex-analytic flatness of a boundary point of a domain in $\mathbb{C}^2$; this was conceived as an obstruction to subelliptic estimates for the $\overline{\partial}$-Neumann problem.

Later, Bloom and Graham [BLG] generalized Kohn’s work to higher dimensions. Perhaps more significantly, they isolated two very interesting definitions of finite type and proved them to be equivalent. We now briefly review these two definitions.

Definition 1 Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^\infty$ boundary. Let $\rho$ be a smooth defining function for $\Omega$. Let $P \in \partial \Omega$. Let $m$ be a positive integer. We say that $P$ has geometric type at least $m$ if there is a nonsingular analytic disc $\varphi : D \to \mathbb{C}^n$ such that $\varphi(0) = P$ and

$$|\rho(\varphi(\zeta))| \leq C|\zeta|^m. \quad (\dagger)$$

The greatest $m$ for which this is true is called the type of the point $P$. If there is no greatest $m$ then the point $P$ is said to be of infinite type.

Of course a point $P \in \partial \Omega$ has complex tangent space $T_P(\partial \Omega)$ (see [KRA1]). If $V$ is a small neighborhood of $P$ in $\partial \Omega$, then we may write down a collection $L_1, \ldots, L_{n-1}$ of tangent holomorphic vector fields on $V$ that are linearly independent at each point of $V$. A commutator (or Poisson bracket) $[L_j, L_k]$ or $[\overline{L}_j, L_k]$ or $[\overline{L}_j, \overline{L}_k]$ is called a second-order commutator. If $M$ is a $p^{th}$-order commutator, then an expression of the form $[M, L_j]$ or $[M, \overline{L}_j]$ is called a $(p + 1)^{st}$-order commutator. For convenience, we refer to the individual vector fields $L_1, \ldots, L_{n-1}$ as first-order commutators.

Definition 2 Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^\infty$ boundary. Let $\rho$ be a smooth defining function for $\Omega$. Let $P \in \partial \Omega$. Let $m$ be an integer exceeding 1. We say that $P$ has geometric type at least $m$ if there is a nonsingular analytic disc $\varphi : D \to \mathbb{C}^n$ such that $\varphi(0) = P$ and

$$|\rho(\varphi(\zeta))| \leq C|\zeta|^m. \quad (\dagger)$$

The greatest $m$ for which this is true is called the type of the point $P$. If there is no greatest $m$ then the point $P$ is said to be of infinite type.

3 And we say that a disc satisfying condition $(\dagger)$ is tangent to $\partial \Omega$ at $P$ to order $m$. 

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say that \( P \) has \textit{commutator type} \( m \) if any commutator \( N \) of order \( m - 1 \) or less satisfies
\[
\langle N, \partial \rho \rangle = 0
\]
but there is some commutator \( N' \) of order \( m \) that satisfies
\[
\langle N', \partial \rho \rangle \neq 0.
\]

The theorem of Bloom and Graham [BLG] says that, in \( \mathbb{C}^2 \), a point \( P \in \partial \Omega \) is of geometric type \( m \) if and only if it is of commutator type \( m \) (see [KRA1, pp. 463–464] for a quick and elegant proof). In higher dimensions this equivalence is still not fully understood, although there has been heartening recent progress by Fornaess and Lee [FOL].

John D’Angelo and David Catlin have demonstrated the importance of the concept of finite type, both for function theory and for the study of the \( \overline{\partial} \)-Neumann problem (see [DAN] and references therein). See also the work of Baouendi, Ebenfelt, and Rothschild [BER] for applications to the study of mappings. It is worthwhile to be able to understand points of finite type from a variety of different geometric points of view.

Our goal here is to understand the concept of finite type from the point of view of analytic discs, analogous to our understanding of pseudoconvexity in the last section. We continue to let \( \text{dist} \) denote Euclidean distance. We also let \( \text{H-dist} \) denote the Hausdorff distance on sets. The result we are about to present is certainly related to the work of Dwilewicz and Hill [DWH1], [DWH2]. These authors announce their results in all dimensions; but in the end they only prove them in dimension two. The results of the present paper are valid in all dimensions.

**Proposition 4** Fix a domain \( \Omega \subseteq \mathbb{C}^n \) with smooth (that is, \( C^\infty \)) boundary. Let \( P \in \partial \Omega \). Fix an integer \( m > 1 \). If \( P \) has geometric type \( m \) then there is a sequence \( \varphi_j : D \to \Omega \) of analytic discs satisfying

(a) \( \varphi_j(0) \to P \) as \( j \to \infty \).

(b) \( \text{diam}(\varphi_j(D)) \equiv \delta_j \to 0 \) as \( j \to \infty \).

(c) \( \text{H-dist}(\varphi_j(D), \partial \Omega) \leq \delta_j^m \).
Proof: Let $\varphi : D \to \mathbb{C}^n$ be an analytic disc that is tangent to $\partial \Omega$ to order $m$ at $p$. Let $\nu$ be the unit outward normal vector to $\partial \Omega$ at $P$. Then the discs

$$\varphi_j = \varphi - \frac{1}{j} \nu$$

(with an obvious slight adjustment to the domain of each disc) will satisfy the three conclusions of the proposition. \qed

For the next result, which is the central one for this exposition, we need to lay some groundwork. First some background concepts and notation (see [KRA3]). If $g$ is a function on domain $U$ in $\mathbb{R}^N$, $x \in U$, and $h \in \mathbb{R}^N$ is small, then we let

$$\Delta_h g(x) = \Delta_1^h g(x) \equiv g(x + h) - g(x)$$

whenever this expression makes sense. Iteratively, we set (for $j \in \mathbb{N}$)

$$\Delta_j^h g(x) \equiv \Delta_h (\Delta_{j-1}^h g)(x).$$

So, for example,

$$\Delta_2^h g(x) = g(x + 2h) - 2g(x + h) + g(x)$$

and

$$\Delta_3^h g(x) = g(x + 3h) - 3g(x + 2h) + 3g(x + h) - g(x).$$

We say “whenever this expression makes sense” because we must ensure that $x, x + h$, etc., lie in $U$.

Definition 3 Let $g$ be a continuous function on an open domain $U \subseteq \mathbb{R}^N$. Let $\alpha > 0$ and let $j$ be an integer that exceeds $\alpha$. We say that $g$ is Lipschitz $\alpha$ on $U$, and write $g \in \Lambda_\alpha(U)$, provided that $g$ is bounded on $U$ and there is a constant $C > 0$ such that

$$\left| \Delta_j^h g(x) \right| \leq C \cdot |h|^{\alpha}.$$

The paper [KRA3] discusses the equivalence of this definition with other standard definitions of Lipschitz/Hölder functions.

Proposition 5 Fix a domain $\Omega \subset \mathbb{C}^2$ with smooth (that is, $C^\infty$) boundary. Let $P \in \partial \Omega$. Fix an integer $m > 0$. Assume that there is a sequence $\varphi_j : D \to \mathbb{C}^n$ of analytic discs satisfying
(a) \( \varphi_j(0) \to P \) as \( j \to \infty \).

(b) \( \text{diam}(\varphi_j(\mathcal{D})) \equiv \delta_j \to 0 \) as \( j \to \infty \).

(c) \( \text{H-dist}(\varphi_j(\mathcal{D}), \partial \Omega) \leq \delta_j^m \).

Then \( P \) has geometric type at least \( m \).

Proof: It is a standard fact—see [KRA1, p. 463]—that there is a defining function \( \rho \) for \( \Omega \) with the form

\[
\rho(z) = 2\text{Re}z_2 + \mu(z_1) + \mathcal{O}(\|z_1z_2| + |z_2|^2).
\]

In fact the first term of this expansion comes from normalizing the unit outward normal at \((0,0) \in \partial \Omega\) to coincide with the real \( z_2 \) direction. The second term comes from an application of E. Borel’s theorem. The remainder term comes from inspection of the Taylor expansion.

Now fix \( \delta_j > 0 \) as in the statement of the proposition and let \( \varphi_j : D \to \Omega \) be an analytic disc satisfying conditions (a), (b), (c). Certainly we may suppose that this disc lies in a tubular neighborhood of the boundary. We may as well suppose that this is a closed disc. Therefore we may take a point \( P = \varphi(\zeta_0) \) that is furthest from the boundary. There is no loss of generality to assume that \( \zeta_0 = 0 \) and we do so henceforth.

Let \( \eta = \text{dist}(P, \partial \Omega) \) and let \( \tilde{P} \) be the Euclidean projection of \( P \) to the boundary of \( \Omega \). Let \( \nu \) be the unit outward normal to \( \partial \Omega \) at \( \tilde{P} \). Now consider the analytic disc

\[
\tilde{\varphi}(\zeta) = \varphi(\zeta) + \eta \nu.
\]

Assuming for the moment that \( P \) is a nonsingular point of the analytic disc\(^4\) we now see that \( \tilde{\varphi} \) is tangent to \( \partial \Omega \) at \( \tilde{P} \). What is more, we may consider

\[
|\Delta_h \rho \circ \tilde{\varphi}(0)| \tag{\star}
\]

for \( h \) small (of size say \( \delta_j/10 \)). Condition (c) in the statement of the proposition tells us that the expression in (\star) is of size \( \delta_j^m \). The same can be said for

\[
|\Delta_h^\ell \rho \circ \tilde{\varphi}(0)| \tag{\star\star}
\]

\(^4\)In the case that the point is singular, we may perturb the analytic disc an arbitrarily small amount in space to arrange that this extremal point not be singular. After all, the set of singular points is discrete in the (image) analytic disc.
provided that $0 \leq \ell \leq m$. Thus we see immediately (because $\rho$ and $\mu$ are known in advance to be smooth functions) that the function $\mu$ in the asymptotic expansion for $\rho$ vanishes to order $m$. But this in turn says that the analytic variety $\{z : z_2 = 0\}$ has order of contact $m$ with the boundary of $\Omega$ at 0. So the point $0 \in \partial \Omega$ has type at least $m$. That is what we wished to prove.

\[ \square \]

**Remark 6** Certainly our Proposition 5 implies the result of Dwilewicz and Hill in [DWH1]. Our approach has the advantage that it does not use delicate calculations involving the Bishop equation, and it generalizes to higher dimensions.

The result of Proposition 5 is in complex dimension 2. We wish also to prove a result in complex dimension $n$ for any $n$. We cannot prove a sharp result at this time, but we can offer a useful characterization of a finite type condition in terms of analytic discs.

**Proposition 7** Fix a domain $\Omega \subset \mathbb{C}^n$ with smooth (that is, $C^\infty$) boundary. Let $P \in \partial \Omega$. Fix an integer $m > 0$. Assume that there is a sequence of positive numbers $\delta_j \to 0$ such that for any analytic disc $\varphi : D \to \mathbb{C}^n$ it holds that:

If $\text{dist}(\varphi(0), P) \approx \delta_j$ then we have $H\text{-dist}(\varphi(D), \partial \Omega) \leq \delta^m$.

It follows there is a complex tangential direction $\tau$ along which $P$ has geometric type at least $m$. More precisely, there is a complex tangent vector $\tau$ and an analytic disc $\varphi : D \to \mathbb{C}^n$ such that

(i) $\varphi(0) = P$;

(ii) $\varphi'(0) = c\tau$ for some complex constant $c$;

(iii) The disc $\varphi$ has order of contact at least $m$ with $\partial \Omega$ at $P$: $|\rho \circ \varphi(P + \zeta \tau)| \leq C|\zeta|^m$.

**Remark 8** It is natural to wonder under what circumstance the hypothesis of Proposition 6 holds, near a given boundary point of $\Omega$, for a sequence of $\delta$s tending to zero. It follows from elementary contact geometry that the hypothesis will hold at any smooth, pseudoconvex boundary point with $m = 2$. Thus any pseudoconvex boundary point has type at least two, and that assertion is well known.

\[ \square \]
Remark 9 It is worth noting that, in dimension $n = 2$, the hypothesis of Proposition 6 is sufficient to imply the conclusion of Proposition 5. In that case there is only one complex tangential direction, so the conclusion of the proposition is considerably more elegant. The proof is the same.

Proof of Proposition 6: Let $\delta_j \to 0$ be chosen as in the statement of the proposition and $\varphi_j$ be corresponding analytic discs.

We begin, just as in the proof of Proposition 5, by writing a defining function for $\Omega$ as

$$\rho(z) = 2\text{Re} z_n + \mu(z_1, \ldots, z_{n-1}) + \mathcal{O}(|z_1 z_n| + |z_2 z_n| + \cdots |z_{n-1} z_n| + |z_n|^2). \quad (\star)$$

This, again, can be done by a normalization of coordinates (for the first term) and an application of E. Borel’s theorem (for the second term). Arguing as before (and using the notation of the proof of Proposition 5), we see that

$$|\Delta^\ell_h \rho \circ \tilde{\varphi}_j(0)| \leq C|h|^m$$

for $|h| \approx \delta$ and $0 \leq \ell \leq m$. We may select a subsequence $\varphi_{j_k}$ such that the tangent vectors $\varphi'_{j_k}(0)/\|\varphi'_{j_k}(0)\|$ converge to a limit vector $\tau$. It follows, just because $\rho$ and $\mu$ are smooth functions, that

$$\left(\frac{\partial}{\partial \tau}\right)^j \mu(P) = 0 \quad \text{for } 0 \leq j \leq m.$$

By our normalization $(\star)$ of the defining function, this gives the conclusion of the proposition with the variety having contact $m$ being a complex line in the complex tangent plane at $P$.

3 Other Geometric Conditions Involving Analytic Discs

Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}^n$. Let $P \in \partial \Omega$. We now consider the following definition.

Definition 4 Suppose that $\psi : \overline{D} \to \overline{\Omega}$ is a closed analytic disc. Assume that whenever $\psi(\partial D) \cap \partial \Omega$ then $\psi(D) \cap \partial \Omega = \emptyset$. Then we say that every point of $\partial \Omega$ is complex analytically extreme.
This definition is analogous to the classical notion of “extreme point” from the theory of convex sets (see, for example, [VAL]). It is not the case that a domain satisfying the condition of this last definition must have the property that every boundary point is finite type. The example
\[
\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + 2e^{-1/|z_2|^2} < 1\}
\]
illustrates this observation.

An analytically extreme boundary point is *analytically isolated* in the traditional sense of that terminology. This idea is important for the study of boundary orbit accumulation points of automorphism group actions (for which see [GK1], [GK2], and the survey [ISK]).

### 4 Some Examples, and Comparison with Harmonic Discs

**EXAMPLE 5** Let

\[
A = \{\zeta \in \mathbb{C} : 1/2 < |\zeta| < 2\}
\]

and set

\[
\Omega = A \times D.
\]

Then \(\Omega\) is a pseudoconvex domain in \(\mathbb{C}^2\). It is a standard result—see [KRA1]—that \(\Omega\) may be exhausted by an increasing union of smoothly bounded, strongly pseudoconvex domains \(\Omega_j\). Thus we may choose \(j\) so large that

\[
\text{dist}(\partial \Omega_j, \partial \Omega) < 10^{-10}.
\]

Now it is the case that the analytic disc

\[
\psi : D \to \Omega_j
\]

\[
\zeta \mapsto (\zeta, 0)
\]

has the property that \(\partial \psi\) lies in \(\Omega_j\) while the entire disc does not. \(\square\)

This example does not contradict our characterization of pseudoconvexity with analytic discs (Proposition 1) because in that proposition we take the discs sufficiently small that they do not generate any nontrivial homology classes (see particularly the footnote on the next page).
EXAMPLE 6 Let

\[ A = \{ \zeta \in \mathbb{C} : 1/2 < |\zeta| < 2 \} \]

and set

\[ \tilde{\Omega} = (A \times D) \cup \left( D(0, 2) \times \{ \zeta \in D : \text{Re} \zeta < -3/4 \} \right). \]

The disc

\[ \psi : D \rightarrow \tilde{\Omega} \]
\[ \zeta \mapsto (\zeta, 0) \]

still has the property that \( \partial \psi(D) \subset \tilde{\Omega} \). Yet the full disc does not lie in \( \tilde{\Omega} \).

However note that this \( \Omega \) is \emph{not} pseudoconvex. Indeed the smallest pseudoconvex domain that contains \( \Omega \) is \( D(0, 2) \times D \). Thus, even though the disc \( \psi \) has boundary curve that is homotopic to a point, the example is insignificant because the domain is not pseudoconvex.

It is natural to wonder whether analytic discs are the right device to use to measure pseudoconvexity. Perhaps harmonic discs—which we use in effect to recognize plurisubharmonic functions—are more appropriate. They would certainly be more flexible. Here by a \emph{harmonic disc} we mean the following. Let \( \eta : \{ e^{i\theta} : 0 \leq \theta \leq 2\pi \} \rightarrow \mathbb{C}^n \) be a continuous function with \( \eta(e^{i\theta}) = \eta(e^{i2\pi}) \). Now solve the Dirichlet problem with boundary data \( \eta \) to obtain a harmonic function

\[ u : D \rightarrow \mathbb{C}^n \]

that is continuous up to the boundary and has boundary function \( \eta \). Is there a characterization of pseudoconvex domain using harmonic discs that is analogous to Proposition 1? The answer is “no”, as the following example illustrates.

EXAMPLE 7 Let

\[ A = \{ \zeta \in \mathbb{C} : |\text{Re} \zeta| < 3, |\text{Im} \zeta| < 3 \} \setminus \{ \zeta \in \mathbb{C} : |\text{Re} \zeta| \leq 1, |\text{Im} \zeta| \leq 1 \} \]

and define

\[ \Omega = D(0, 3) \times A. \]
Now consider the curve $\gamma$, for $\epsilon > 0$ small, that is traced out by

$$
\gamma(t) = \begin{cases} 
(2 - \epsilon, 1 + \epsilon + i(1 + \epsilon)t) & \text{if } 0 \leq t \leq 1 \\
(2 - \epsilon, 2 + 2\epsilon + i(1 + \epsilon) - t(1 + \epsilon)) & \text{if } 1 < t \leq 2 \\
(2 - \epsilon, i(1 + \epsilon) - 2(1 + \epsilon) + t(1 + \epsilon)) & \text{if } 2 < t \leq 3 \\
(2 - \epsilon, 1 + \epsilon + 4i(1 + \epsilon) - ti(1 + \epsilon)) & \text{if } 3 < t \leq 4.
\end{cases}
$$

This is certainly a closed curve in $\Omega$. Yet the harmonic disc with this curve as boundary (i.e., the solution of the Dirichlet problem with boundary data given by $\gamma$) certainly has in it the point which is the average over the curve. A simple calculation shows that that point is $(2-\epsilon, 3/4(1+\epsilon)+3i/4(1+\epsilon)) \not\in \Omega$.

Thus, even though $\Omega$ is obviously pseudoconvex, the harmonic disc condition fails.

\[\square\]
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