Retarded potentials for wave propagation in elastic trusses

Dominik Pölz1,∗, Michael Gfrerer1,2, and Martin Schanz1
1 Institute of Applied Mechanics, Graz University of Technology, Technikerstraße 4/II, 8010 Graz, Austria
2 Felix Klein Zentrum für Mathematik, University of Kaiserslautern, Paul Ehrlich Straße 31, 67663 Kaiserslautern, Germany

A space-time boundary element method for the dynamic simulation of simplified elastic truss systems is proposed. The considered truss systems consist of elastic rods whose dynamic behaviour is governed solely by the 1D wave equation. By using time domain boundary integral equations the problem reduces to the nodes of the truss system and therefore no spatial discretization is necessary. The discretization employs an adaptive mesh refinement scheme enabling the accurate resolution of non-smooth solutions.

© 2018 The Authors. PAMM published by Wiley-VCH Verlag GmbH & Co. KGaA Weinheim.

1 Linearly elastic rod

We consider a linearly elastic rod with length \( L > 0 \), extensional stiffness \( EA > 0 \), wave speed \( c > 0 \), and a fixed simulation end time \( T > 0 \). The rod is loaded only at its end points and assumed to be subjected to sufficiently small longitudinal displacements \( u \). In this case the inner force function is \( p = EA \frac{\partial u}{\partial x} \). Starting at a quiescent state the longitudinal displacement is a solution of the homogeneous 1D elastodynamic wave equation

\[
c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{in } (0, L) \times (0, T), \quad \frac{\partial u}{\partial x}(x, 0) = 0 \quad \text{in } (0, L) .
\]

(1)

Solutions of this problem are given by the representation formula

\[
2u(x, t) = \frac{1}{EA} \int_0^{c^{-1}} p(0, s) \, ds + \frac{1}{EA} \int_0^{c^{-1}} L_0 x \int_0^t p(L, s) \, ds + u \left( 0, t - \frac{x}{c} \right) + u \left( L, t - \frac{L - x}{c} \right) .
\]

(2)

By performing the trace, i.e. \( x \to 0 \) and \( x \to L \), one can find an explicit representation of the Dirichlet-to-Neumann map [2]

\[
\text{DtN} : (u(0, \cdot), u(L, \cdot)) \mapsto (p(0, \cdot), p(L, \cdot)) .
\]

(3)

From a mechanical point of view this operator maps the displacement at the end points to the corresponding boundary forces.

2 Truss systems

A truss is a mechanical assembly of rod elements, forming a structure in \( d \)-dimensional space, where \( d \in \{ 2, 3 \} \). The individual rods are connected to each other via hinges at their end points. Thus, the relative rotation of the rods connected by a hinge is not constrained. At least \( d \) of these nodes are supported, restraining their movement. Figure 1 shows an illustration of such a system.

In this work, transversal forces and bending moments are neglected. Lateral and rotational inertia effects are also omitted. Although this poses a severe restriction of the model, it enables a thorough study of the performance of time domain boundary integral equations for this simplified model.

The set of all truss nodes is denoted \( \mathcal{N} \), the supported nodes \( \mathcal{N}_S \), and the unsupported nodes \( \mathcal{N}_F \). The nodal displacement field of the truss is denoted \( u : \mathcal{N} \times (0, T) \to \mathbb{R}^d \). Admissible displacement fields \( u \in \mathbb{V} \) vanish at nodes in \( \mathcal{N}_S \). External loads are applied only at the nodes of the structure and \( f_k \) denotes the force applied to node \( k \in \mathcal{N}_F \). Using the local DtN of each rod member in conjunction with kinematic coupling conditions a displacement-to-force map is established. For each node \( k \in \mathcal{N}_F \) it maps the nodal displacement field to the sum of interior forces at \( k \)

\[
P_k : u \mapsto \sum_{\ell \in \beta_k} p_{\ell}|_k
\]

(4)

where \( \beta_k \) is the set of rod members connected to node \( k \in \mathcal{N} \). The operator \( P_k \) is used to reformulate the nodal equilibrium at nodes in \( \mathcal{N}_F \) in terms of the displacement field \( u \). The proposed variational formulation is to find \( u \in \mathbb{V} \) such that

\[
\sum_{k \in \mathcal{N}_F} \left( P_k \, u, \frac{\partial v}{\partial t} \right)_{L_2^2(0, T)} = \sum_{k \in \mathcal{N}_F} \left( f_k, \frac{\partial v}{\partial t} \right)_{L_2^2(0, T)} \quad \forall v \in \mathbb{V} .
\]

(5)

* Corresponding author: e-mail poelz@tugraz.at
Based on the energy argument of Aimi et al. [1] one can show that there exists a $C(T) > 0$ such that

$$\sum_{k \in \mathcal{N}} (P_k \mathbf{u}, \partial_t \mathbf{u}|_k)_{L^2(0,T)} \geq C(T) \sum_{k \in \mathcal{N}} ||\partial_t \mathbf{u}|_k||^2_{L^2(0,T)} = C(T) ||\mathbf{u}||^2_V, \quad \mathbf{u} \in V$$

(6)

holds if $T$ is sufficiently small. In [2] this property is exploited to construct stable boundary element methods.

3 Discretization and numerical experiment

The implemented method uses hat functions to approximate the displacement field. To drive an adaptive mesh refinement scheme an error indicator is necessary. By using the representation formula, the numerical solution always satisfies the exact wave equation. Since the only error occurs in the nodal equilibrium the residual of this equation is used as an error indicator.

A numerical experiment is conducted for the tripod structure shown in Figure 1. The node $k_4$ is loaded by a step function. Figure 2 displays the approximate displacement solution and the markers indicate the elements. One can observe that the adaptive mesh refinement focuses on the kinks of the solution. In Figure 3 and Figure 4 the inner force of a rod member is plotted. Both approximations use roughly the same amount of elements. However, in Figure 3 a uniform mesh is used, while in Figure 4 the adaptive scheme is employed. The adaptive method resolves the discontinuities remarkably well.

4 Concluding remark

This work shows that boundary element methods are a powerful tool for the dynamic simulation of elastic trusses. Treating the physically complete model with time domain boundary integral equations may yield viable alternatives to existing methods.

References

[1] A. Aimi, M. Diligenti, C. Guardasoni, I. Mazzieri, and S. Panizzi, Int. J. Numer. Methods. Engrg. 80, 1196–1240(2009).

[2] D. Pölz, M. H. Gfrerer, M. Schanz, Wave propagation in elastic trusses: An approach via retarded potentials, Wave Motion, in press.