MEASURABLE LOWER BOUNDS ON CONCURRENCE

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Received (received date)
Revised (revised date)

We derive measurable lower bounds on concurrence of arbitrary mixed states, for both
bipartite and multipartite cases. First, we construct measurable lower bounds on the
purely algebraic bounds of concurrence [F. Mintert et al. (2004), Phys. Rev. lett., 92,
167902]. Then, using the fact that the sum of the square of the algebraic bounds is a
lower bound of the squared concurrence, we sum over our measurable bounds to achieve
a measurable lower bound on concurrence. With two typical examples, we show that our
method can detect more entangled states and also can give sharper lower bounds than
the similar ones.

Keywords: Measuring entanglement, Concurrence
Communicated by: to be filled by the Editorial

1. Introduction
Recently, many studies have been focused on the experimental quantification of entangle-
ment [1]. Bell inequalities and entanglement witnesses [1, 2] can be used to detect entangled
states experimentally, but they don’t give any information about the amount of entangle-
ment. In addition, quantum state tomography [3], determination of the full density operator
$\rho$ by measuring a complete set of observables, is only practical for low dimensional systems
since the number of measurements needed for it grows rapidly as the dimension of the sys-
tem increases. Fortunately, several methods have been introduced which let one to estimate
experimentally the amount of the entanglement of an unknown $\rho$ with no need to the full
tomography [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 29, 19, 30, 20, 26, 27, 28, 21,
22, 23, 24, 25]. A simple and straightforward method is the one introduced in [8, 14, 18] for
finding measurable lower bounds on an entanglement measure, namely the concurrence [31].
These lower bounds are in terms of the expectation values of some Hermitian operators with
respect to two-fold or one-fold copy of $\rho$. It is worth noting that these bounds work well for
weakly mixed states [32, 8, 14, 18, 5].

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In this paper we will use a similar procedure as [8, 14] to construct measurable lower bounds on the purely algebraic bounds of concurrence [33, 31]. In addition, using a theorem in Sec. II, we show that the sum of our measurable bounds leads to a measurable lower bound on the concurrence itself. Then, we show that this method gives better results than those introduced in [8, 14] for two typical examples.

The paper is organized as follows. In Sec. II, the concurrence and its $MKB$ (Mintert-Kus-Buchleitner) lower bounds [33] are introduced. In Secs. III and IV, we propose measurable lower bounds on the purely algebraic bounds of concurrence [33], which are a special class of $MKB$ bounds. The generalization to the multipartite case is given in Sec. V and we end this paper in Sec. VI with a summary and discussion.

### 2. Concurrence and its $MKB$ Lower Bounds

For a pure bipartite state $|\Psi\rangle$, $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, concurrence is defined as [31]:

$$C(\Psi) = \sqrt{2[\langle \Psi | \Psi \rangle^2 - \text{tr} \rho_r^2]},$$  \hspace{1cm} (1)

where $\rho_r$ is the reduced density operator obtained by tracing over either subsystems A or B. It is obvious that iff $|\Psi\rangle$ is a product state, i.e. $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, then $C(\Psi) = 0$. Interestingly, $C(\Psi)$ can be written in terms of the expectation value of an observable with respect to two identical copies of $|\Psi\rangle$ [31, 11, 12]:

$$C(\Psi) = \sqrt{\langle AB | AB \langle A | A | AB | \Psi \rangle_{AB} | AB \rangle_{AB}},$$  \hspace{1cm} (2)

where $P_A^A (P_B^B)$ is the projector onto the antisymmetric subspace of $\mathcal{H}_A \otimes \mathcal{H}_A$ ($\mathcal{H}_B \otimes \mathcal{H}_B$).

A possible decomposition of $A$ is

$$A = \sum_{\alpha} |\chi_{\alpha}\rangle \langle \chi_{\alpha}|,$$

$$|\chi_{\alpha}\rangle = (|x\rangle - |y\rangle)_A (|p\rangle - |q\rangle)_B,$$  \hspace{1cm} (3)

where $|x\rangle$ and $|y\rangle$ ($|p\rangle$ and $|q\rangle$) are two different members of an orthonormal basis of the A (B) subsystem. For mixed states the concurrence is defined as follows [31]:

$$C(\rho) = \min \sum_i p_i C(\Psi_i),$$

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1,$$  \hspace{1cm} (4)

where the minimum is taken over all decompositions of $\rho$ into pure states $|\Psi_i\rangle$. It is appropriate to write $C(\rho)$ in terms of the subnormalized states $|\psi_i\rangle$ rather than the normalized ones $|\Psi_i\rangle$:

$$C(\rho) = \min \sum_i \sqrt{\langle \psi_i | (A_i | \psi_i \rangle | \psi_i \rangle},$$

$$|\psi_i\rangle = \sqrt{p_i} |\Psi_i\rangle, \quad \rho = \sum_i |\psi_i\rangle \langle \psi_i|;$$  \hspace{1cm} (5)

since all decompositions of $\rho$ into subnormalized states are related to each other by unitary matrices [3]: consider an arbitrary decomposition of $\rho = \sum_j |\varphi_j\rangle \langle \varphi_j|$ (As a special case,
one can choose $|\varphi_j\rangle = \sqrt{\lambda_j}|\Phi_j\rangle$, where $|\Phi_j\rangle$ and $\lambda_j$ are eigenvectors and eigenvalues of $\rho$ respectively: $\rho = \sum_j \lambda_j |\Phi_j\rangle \langle \Phi_j|$. for any other decomposition of $\rho = \sum_i |\psi_i\rangle \langle \psi_i|$ we have [3]:

$$|\psi_i\rangle = \sum_j U_{ij} |\varphi_j\rangle, \quad \sum_i U_{ki}^\dagger U_{ij} = \delta_{jk}.$$  

(6)

So Eq. (5) can be written as:

$$C(\rho) = \min_U \sum_i \sqrt{\sum_{jklm} U_{ij} U_{ik} A_{jk}^{lm} U_{lm}^\dagger},$$

$$A_{jk}^{lm} = \langle \varphi_i (|\varphi_m\rangle \langle \varphi_j|) \varphi_k \rangle.$$

(7)

From the definition of $C(\rho)$ in Eq. (4) it is obvious that $C(\rho) = 0$ iff $\rho$ can be decomposed into product states. In other words, $C(\rho) = 0$ iff $\rho$ is separable. In addition, it can be shown that the concurrence is an entanglement monotone [34]. An entanglement monotone is a function of $\rho$ which does not increase, on average, under LOCC and vanishes for separable states [35]. But, except for the two-qubit case [36], $C(\rho)$ can not be computed in general; i.e., in general, one can not find the $U$ which minimizes Eq. (7). Any numerical method for finding the $U$ which minimizes Eq. (7) leads to an upper bound for $C(\rho)$. So, finding lower bounds on $C(\rho)$ is desirable. So far, several lower bounds for $C(\rho)$ have been introduced [33, 31, 37, 38, 39, 40, 41, 42, 43, 5, 8, 13, 14, 18, 19, 21, 22, 23, 24]. One of them is that introduced by F. Mintert et al. in [33, 31]. Now, we redrive their lower bounds in a slightly different form to make them more suitable for finding measurable lower bounds in the following sections.

Assume that the decomposition of $\rho$ which minimizes Eq. (5) is $\rho = \sum_j |\xi_j\rangle \langle \xi_j|$, then from Eqs. (3) and (5), we have:

$$C(\rho) = \sum_j \sqrt{\sum_{\alpha} |\langle \chi_\alpha | \xi_j \rangle|^2} \geq \sum_j |\langle \chi_\beta | \xi_j \rangle| \geq \min_{\{|\psi_i\rangle\}} \sum_{\alpha} |\langle \chi_\beta | \psi_i \rangle| |\psi_i \rangle|,$$

(8)

where $|\chi_\beta\rangle \in \{|\alpha\rangle\}$, and the minimum is taken over all decompositions of $\rho$ as $\rho = \sum_i |\psi_i\rangle \langle \psi_i|$. Now, using Eq. (6), we have:

$$\min_{\{|\psi_i\rangle\}} \sum_i |\langle \chi_\beta | \psi_i \rangle| = \min_U \sum_i |\sum_j U_{ij} T_{jk}^\beta U_{ki}^\dagger| = \min_U \sum_i |[UT^\beta U^\dagger]_{ii}|,$$

$$T_{jk}^\beta = \langle \chi_\beta (|\varphi_j\rangle \langle \varphi_k|) \rangle.$$

(9)

Since $T_{jk}^\beta$ is a symmetric matrix, the minimum in Eq. (9) can be computed and we have [31]:

$$\min_U \sum_i |[UT^\beta U^\dagger]_{ii}| = \max\{0, S_1^\beta - \sum_{i>1} S_i^\beta\},$$

(10)

where $S_i^\beta$ are the singular values of $T_{jk}^\beta$, in decreasing order. The above expression is what was named purely algebraic lower bound of concurrence in [31, 33] and we will refer to it as $ALB(\rho)$.

Let us define

$$|\tau\rangle = \sum_{\alpha} z_{\alpha}^* |\chi_\alpha\rangle, \quad \sum_{\alpha} |z_\alpha|^2 = 1.$$

(11)
Obviously, $|\tau\rangle$ is an element of another (normalized to 2) basis of $P^A \otimes P^B$, $\{|\chi'^\alpha\rangle\}$. Then:

$$A = \sum_\alpha |\chi_\alpha\rangle\langle\chi_\alpha| = |\tau\rangle\langle\tau| + \sum_{\alpha > 1} |\chi'^\alpha\rangle\langle\chi'^\alpha|.$$  

(12)

Again, as the inequality (8), we have:

$$C(\rho) = \sum_j \sqrt{\sum_\alpha |\langle\chi'^\alpha_j|\xi_j\rangle|^2} \geq \sum_j |\langle\tau|\xi_j\rangle|$$

$$\geq \min_{\{|\psi_i\rangle\}} \sum_i |\langle\tau|\psi_i\rangle|$$

$$= \min_T \sum_j |[UTU^T]_{ji}| = \max\{0, S_T - \sum_{l > 1} S_T^l\},$$

$$T_{jk} = \langle\tau|\varphi_j\rangle\langle\varphi_k| = \sum_\alpha \tau_{jk}^\alpha,$$

(13)

where $S_T^l$ are the singular values of $T$, in decreasing order. The above expression is the general form of the lower bounds introduced in [33, 31] and we call it $LB(\rho)$.

We end this section by proving a useful theorem: if $\{|\psi'^\alpha_i\rangle\} = \{\psi'^\alpha_0\}$ be an orthogonal (normalized to 2) basis of $P^A \otimes P^B$, i.e. $A = \sum_\alpha |\chi'^\alpha\rangle\langle\chi'^\alpha|$, then:

$$C^2(\rho) = \sum_{ij} \sqrt{\sum_\alpha |\langle\chi'^\alpha_i|\xi_i\rangle|^2} \sqrt{\sum_\alpha |\langle\chi'^\alpha_j|\xi_j\rangle|^2}$$

$$\geq \sum_{ij} \sum_\alpha |\langle\chi'^\alpha_i|\xi_i\rangle||\langle\chi'^\alpha_j|\xi_j\rangle|$$

$$= \sum_\alpha \left(\sum_i |\langle\chi'^\alpha_i|\xi_i\rangle|\right)^2 \geq \sum_\alpha \sum_i [LB_\alpha(\rho)]^2,$$

$$LB_\alpha(\rho) = \min_{\{|\psi_i\rangle\}} \sum_i |\langle\chi'^\alpha_i|\psi_i\rangle|.$$  

(14)

In proving the above theorem we have used the Cauchy-Schwarz inequality. Obviously, any entangled $\rho$ which can not be detected by $LB_\alpha$, can not be detected by $\sum_\alpha [LB_\alpha(\rho)]^2$ either; i.e., $\sum_\alpha [LB_\alpha(\rho)]^2$ is not a more powerful criteria than $LB_\alpha$, but, quantitatively, it may lead to a better lower bound for $C(\rho)$.

It should be mentioned that the above theorem is, in fact, the generalization of what has been proved in [42]. There, it was shown that:

$$\tau(\rho) = \sum_\alpha C^2_{mn}(\rho) \leq C^2(\rho),$$

$$C_{mn}(\rho) = \min_{\{|\psi_i\rangle\}} \sum_i |\langle\psi_i|L_\alpha A \otimes L_\beta B |\psi_i^\alpha\rangle|,$$

(15)

where $L_\alpha A$ and $L_\beta B$ are generators of $SO(d_A)$ and $SO(d_B)$ respectively ($d_A/B = \dim(\mathcal{H}_A/B)$), and $|\psi_i^\alpha\rangle$ is the complex conjugate of $|\psi_i\rangle$ in the computational basis. In this basis $L_\alpha A$ and $L_\beta B$ are [44]:

$$L_\alpha A = |x\rangle_A \langle y| - |y\rangle_A \langle x|, \quad L_\beta B = |p\rangle_B \langle q| - |q\rangle_B \langle p|.$$
For an arbitrary $|\psi\rangle$, according to the definition of $|\chi_\alpha\rangle$ in Eq. (3), it can be seen that:

$$|\langle \psi | L_{m_A} \otimes L_{n_B} |\psi\rangle|^2 = |\langle \chi_\alpha | \psi \rangle|^2.$$  

(16)

So:

$$C_{mn}(\rho) = ALB_\alpha(\rho),$$

$$ALB_\alpha(\rho) = \min_{\{|\psi_i\rangle\}} \sum_i |\langle \chi_\alpha | \psi_i \rangle|^2.$$  

(17)

So what was proved in [42] is, in fact, the special case of $|\chi_\alpha'\rangle = |\chi_\alpha\rangle$ in expression (14). In addition, since $ALB_\alpha$ can detect bound entangled states [33, 31], this claim of [42] that any state for which $\tau(\rho) > 0$ is distillable, is not correct.

3. Measurable Lower Bounds in terms of Two Identical Copies of $\rho$

As we have seen in Eq. (2) the concurrence of a pure state $|\Psi\rangle$ can be written in terms of the expectation value of the observable $A$ with respect to two identical copies of $|\Psi\rangle$. For an arbitrary mixed state $\rho_{AB}$, it has been shown that [8]:

$$C^2(\rho_{AB}) \geq tr(\rho_{AB} \otimes \rho_{AB} V_{(i)}),$$

$$V_{(1)} = 4 (P_A^A - P_A^A) \otimes P_B^B, \quad V_{(2)} = 4 P_A^A \otimes (P_B^B - P_B^B),$$  

(18)

where $P_A^A$ ($P_B^B$) is the projector onto the symmetric subspace of $\mathcal{H}_A \otimes \mathcal{H}_A$ ($\mathcal{H}_B \otimes \mathcal{H}_B$). The above expression means that measuring $V_{(i)}$ on two identical copies of $\rho$, i.e. $\rho \otimes \rho$, gives us a measurable lower bound on $C^2(\rho)$. It is worth noting that if the entanglement of $\rho$ can be detected by $V_{(i)}$, then $\rho$ is distillable [24].

As one can see from expression (13), the LB of a pure state $|\Psi\rangle$ can also be written in terms of the expectation value of the observable $|\tau\rangle\langle \tau|$ with respect to two identical copies of $|\Psi\rangle$. Now, for an arbitrary mixed state $\rho$, can we find an observable $V$ such that the following inequality holds?

$$LB^2(\rho) \geq tr(\rho \otimes \rho V),$$  

(19)

Fortunately for the special case of $|\tau\rangle = |\chi_\alpha\rangle$, where $|\chi_\alpha\rangle$ are defined in Eq. (3), we can do so.

Assume that the decomposition of $\rho$ which gives the minimum in Eq. (9) is $\rho = \sum_i |\theta_i^\alpha\rangle\langle \theta_i^\alpha|$, i.e.:

$$ALB_\alpha(\rho) = \sum_i |\langle \chi_\alpha | \theta_i^\alpha \rangle|^2.$$  

(20)

In addition, assume that for a Hermitian operator $V_\alpha$, which acts on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, and arbitrary $|\psi\rangle$ and $|\varphi\rangle$, $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\varphi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have:

$$|\langle \chi_\alpha | \psi \rangle | |\langle \chi_\alpha | \varphi \rangle | \geq |\langle \psi | V_\alpha | \psi \rangle | |\langle \varphi | V_\alpha | \psi \rangle |.$$  

(21)

Now, from the expressions (20) and (21), we have:

$$ALB_\alpha^2(\rho) = \sum_{ij} |\langle \chi_\alpha | \theta_i^\alpha | \theta_j^\alpha \rangle |^2 |\langle \chi_\alpha | \theta_i^\alpha | \theta_j^\alpha \rangle | \geq \sum_{ij} |\langle \theta_i^\alpha | V_\alpha | \theta_j^\alpha \rangle | |\langle \theta_i^\alpha | V_\alpha | \theta_j^\alpha \rangle | = tr(\rho \otimes \rho V_\alpha).$$  

(22)
So, for any $V_\alpha$ satisfying inequality (21), measuring $V_\alpha$ on two identical copies of $\rho$ gives a lower bound on $ALB_\alpha^2(\rho)$. We can prove that the inequality (21) holds for (see the Appendix):

$$V_\alpha = V_{(1)\alpha} = M V_{(1)\alpha} M,$$

$$V_\alpha = V_{(2)\alpha} = M V_{(2)\alpha} M,$$

$$M_A = |x\rangle\langle x| + |y\rangle\langle y|,$$

$$M_B = |p\rangle\langle p| + |q\rangle\langle q|,$$  \hspace{1cm} (23)

where $|x\rangle$, $|y\rangle$, $|p\rangle$, $|q\rangle$ are introduced in Eq. (3) (note that $|\chi_\alpha\rangle\langle \chi_\alpha| = MAM$). In addition, for any $V_\alpha$ such as

$$V_\alpha = c_1 V_{(1)\alpha} + c_2 V_{(2)\alpha}, \quad c_1 \geq 0, \quad c_2 \geq 0, \quad c_1 + c_2 = 1,$$  \hspace{1cm} (24)

inequalities (21) and, consequently, (22) also hold.

According to the definition of $V_\alpha$ in Eqs. (23) and (24), we have:

$$tr(\rho \otimes \rho V_\alpha) = tr(\varrho \otimes \varrho V_\alpha),$$

$$\varrho = M_A \otimes M_B \varrho_A \otimes \varrho_M,$$  \hspace{1cm} (25)

which means that if $V_\alpha$ detects the entanglement of $\rho$, it has, in fact, detected the entanglement of a two-qubit submatrix of $\rho$. Any $\rho$ which has an entangled two-qubit submatrix is distillable [45]. So any $\rho$ which is detected by $V_\alpha$ is distillable.

The right hand side of the inequality (18) is invariant under local unitary transformations [8]:

$$tr(\rho \otimes \rho V_{(i)}) = tr(\rho' \otimes \rho' V_{(i)}),$$

$$\rho' = U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger,$$  \hspace{1cm} (26)

where $U_A$ and $U_B$ are arbitrary unitary operators. This is so because $U_A^\dagger \otimes U_B^\dagger P_A^U \otimes U_A = P_A^U$ and $U_B^\dagger \otimes U_B^\dagger P_B^U \otimes U_B = P_B^U$. So, the choices of local bases in the definition of $V_{(i)}$ in (18) are not important since all the choices lead to the same result. But, according to the definition of $V_\alpha$ in Eqs. (23) and (24), the right hand side of the inequality (22) is not invariant under local unitary transformations. It is however expected since the $ALB_\alpha(\rho)$ is not invariant under such transformations either.

Using Eqs. (23) and (24), it can be shown simply that the right hand side of the inequality (22) is invariant under the following transformations:

$$tr(\rho \otimes \rho V_\alpha) = tr(\rho' \otimes \rho' V_\alpha),$$

$$\rho' = u_A \otimes u_B \rho u_A^\dagger \otimes u_B^\dagger,$$

$$M_A u_A M_A = u_A, \quad u_A u_A^\dagger = u_A^\dagger u_A = M_A,$$

$$M_B u_B M_B = u_B, \quad u_B u_B^\dagger = u_B^\dagger u_B = M_B,$$

$$\Rightarrow tr(\rho') \leq 1.$$  \hspace{1cm} (27)

$|\chi_\alpha\rangle$ is also invariant, up to a phase, under the above transformations, i.e. $u_A \otimes u_A \otimes u_B \otimes u_B |\chi_\alpha\rangle = e^{i\beta} |\chi_\alpha\rangle$ and $0 \leq \beta \leq 2\pi$, but it is not so for the $ALB_\alpha(\rho)$. Consider the decomposition of $\rho$ into pure states as $\rho = \sum_i |\theta_i^\alpha\rangle\langle \theta_i^\alpha|$. From Eq. (27) we know that there is a decomposition of $\rho'$ into pure states as $\rho' = \sum_i |\theta'_i\rangle\langle \theta'_i|$, where $|\theta'_i\rangle = u_A \otimes u_B |\theta_i\rangle$. So, using Eq. (20):

$$\sum_i |\langle \chi_\alpha|\theta_i^\alpha\rangle| |\theta'_i\rangle = \sum_i |\langle \chi_\alpha|\theta_i^\alpha\rangle| |\theta_i\rangle = ALB_\alpha(\rho).$$  \hspace{1cm} (28)
But

\[ \sum_i |\langle \chi_\alpha | \hat{q}_i^\alpha \rangle |^2 \geq \min_{\{\hat{q}_j^\beta\}} \sum_j |\langle \chi_\alpha | \hat{p}_j^\beta \rangle |^2 = ALB_\alpha(\rho'), \tag{29} \]

where the minimum is taken over all decompositions of \(\rho'\) into pure states: \(\rho' = \sum_j |\psi_j^\beta\rangle\langle \psi_j^\beta|\).

So:

\[ ALB_\alpha(\rho') \leq ALB_\alpha(\rho). \tag{30} \]

Note that expressions (22), (27) and (30) show that \(\text{tr}(\rho \otimes \rho V_\alpha)\) bounds the amount of \(ALB_\alpha^2(\rho')\), for all possible \(\rho'\) in Eq. (27), from below.

Now, using inequalities (14) and (22):

\[ C^2(\rho) \geq \sum_\alpha ALB^2_\alpha(\rho) \geq \sum_\alpha \text{tr}(\rho \otimes \rho V_\alpha), \tag{31} \]

where the summation is only over those \(\alpha\) for which \(\text{tr}(\rho \otimes \rho V_\alpha) \geq 0\).

**Example 1.** In a \(d \times d\) dimensional Hilbert space, isotropic states are defined as [2]:

\[ \rho_F = \frac{1-F}{d^2-1} (I - |\phi^+\rangle\langle \phi^+|) + F |\phi^+\rangle\langle \phi^+|, \]

\[ |\phi^+\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}} |i_A i_B\rangle, \]

\[ 0 \leq F \leq 1, \quad C(\rho_F) = \max \left\{ 0, \sqrt{\frac{2d}{d-1} \left( F - \frac{1}{d} \right)} \right\}. \tag{33} \]

The concurrence of \(\rho_F\) is known and we have [34]:

\[ C(\rho_F) = \max \left\{ 0, \sqrt{\frac{2d}{d-1} \left( F - \frac{1}{d} \right)} \right\}. \tag{33} \]

If we rewrite \(\rho_F\) as

\[ \rho_F = \frac{1-F}{d^2-1} I + \frac{F d^2-1}{d^2-1} |\phi^+\rangle\langle \phi^+| \equiv g I + h |\phi^+\rangle\langle \phi^+|, \]

then:

\[ \text{tr}(\rho_F \otimes \rho_F V_\alpha) = 2d(d-1) \left[ \frac{h^2}{d^2} - 2g^2 - \frac{2}{d} gh \right]. \tag{34} \]

In Eq. (23), if we choose \(\{x = p, y = q\}\), then:

\[ \text{tr}(\rho_F \otimes \rho_F V_\alpha) = 4 \left[ \frac{h^2}{d^2} - 2g^2 - \frac{2}{d} gh \right], \]

and the expectation values of other \(V_\alpha\) are not positive. Since the case \(\{x = p, y = q\}\) occurs \(n = \frac{d(d-1)}{2}\) times in a \(d \times d\) dimensional system, we have:

\[ \text{tr}\left(\rho_F \otimes \rho_F \sum_\alpha V_\alpha\right) = 2d(d-1) \left[ \frac{h^2}{d^2} - 2g^2 - \frac{2}{d} gh \right], \tag{35} \]
where the summation is only over those $V_\alpha$ for which $\{x = p, y = q\}$. For $d > 2$, Eq. (35) gives a better result than Eq. (34) (Fig. 1). For $d = 2$ both give the same result, as we expect from Eq. (23).

4. Measurable Lower Bounds in terms of One Copy of $\rho$

From the experimental point of view, any lower bound which is defined in terms of the expectation value of an observable with respect to two identical copies of $\rho$, encounters, at least, two problems. First, for measuring $V(i)$ or $V_\alpha$ we need to measure in an entangled basis in both parts A and B. Measuring in an entangled basis is more difficult than measuring in a separable one [12]. Second, it is not clear that the state which enters the measuring devices is really as $\rho \otimes \rho$ even if we can produce such state at the source place [46, 10]. So, having lower bounds in terms of the expectation value of an observable with respect to one copy of $\rho$ is more desirable.

Using:

$$C(\rho)C(\sigma) \geq \text{tr} (\rho \otimes \sigma V(i)) , \quad i = 1, 2;$$

$$\Rightarrow C(\rho) \geq \frac{1}{C(\sigma)} \text{tr} (\rho \otimes \sigma V(i)) ,$$

for arbitrary $\rho$ and $\sigma$, F. Mintert has introduced the following measurable lower bound on $C(\rho)$ [14]:

$$C(\rho) \geq -\text{tr} (\rho W_\sigma) , \quad W_\sigma = \frac{-1}{C(\sigma)} \text{tr}_2 (I \otimes \sigma V(i)) ,$$

where $\sigma$ is a pre-determined entangled state and the partial trace is taken over the second copy of $\mathcal{H}_A \otimes \mathcal{H}_B$. If $C(\sigma)$ is not computable, which is the case for almost all mixed $\sigma$, an upper bound of $C(\sigma)$ can be used in the definition of $W_\sigma$. From inequality (37), it is obvious...
that for any separable state: \( \text{tr} (\rho \sigma W) \geq 0 \). If, at least, for one entangled state \( \text{tr} (\rho \sigma W) < 0 \), then \( W \) is an entanglement witness [2].

We can, also, construct measurable lower bounds in terms of one copy of \( \rho \) by using inequality (21). Suppose that the decomposition of \( \sigma \) which gives the minimum in Eq. (9) is \( \sigma = \sum_j |\gamma_j^\alpha \rangle \langle \gamma_j^\alpha | \), i.e.:

\[
ALB_\alpha (\sigma) = \sum_j |\langle \chi_\alpha | \gamma_j^\alpha \rangle |^2 \tag{38}
\]

Using expressions (20), (21) and (38):

\[
[ALB_\alpha (\rho)] [ALB_\alpha (\sigma)] = \sum_{ij} (\langle \chi_\alpha | \theta_i^\alpha \rangle |\theta_i^\alpha \rangle \langle \chi_\alpha | \gamma_j^\alpha \rangle |\gamma_j^\alpha \rangle)
\geq \sum_{ij} (|\theta_i^\alpha \rangle |\gamma_j^\alpha \rangle V_\alpha |\theta_i^\alpha \rangle |\gamma_j^\alpha \rangle) = -\text{tr} (\rho W'_{\sigma \alpha} ) ,
\]

\[
W'_{\sigma \alpha} = -\text{tr}_2 (I \otimes \sigma V_\alpha) .
\]

So:

\[
ALB_\alpha (\rho) \geq -\text{tr} (\rho W_{\sigma \alpha} ) , \quad W_{\sigma \alpha} = \frac{1}{ALB_\alpha (\sigma)} W'_{\sigma \alpha} , \tag{39}
\]

where \( \sigma \) is a pre-determined entangled state for which \( ALB_\alpha (\sigma) > 0 \). Note that, in contrast to \( C(\sigma) \), \( ALB_\alpha (\sigma) \) is always computable, so we never need to use an upper bound of it in the definition of \( W_{\sigma \alpha} \). In addition, it can be shown simply that

\[
\text{tr} (\rho W_{\sigma \alpha} ) = \text{tr} (\rho W_{\sigma \alpha} ) , \tag{40}
\]

where \( \rho \) is defined in Eq. (25). So any \( \rho \) which is detected by \( W_{\sigma \alpha} \) is distillable. Also, using inequalities (14) and (39):

\[
C^2 (\rho) \geq \sum_\alpha [ALB_\alpha (\rho)]^2 \geq \sum_\alpha [\text{tr} (\rho W_{\sigma \alpha} )]^2 , \tag{41}
\]

where the summation is over those \( \alpha \) for which \( \text{tr} (\rho W_{\sigma \alpha} ) \leq 0 \).

For isotropic states, using expressions (37) or (41) (by choosing \( \sigma = |\phi^+ \rangle \langle \phi^+ | \)) gives the exact value of \( C(\rho_\phi) \) for arbitrary \( d \). In the following, we give an example for which the expression (41) gives better results than the expression (37).

**Example 2.** Consider a two-qutrit system which is initially in the pure state

\[
|\Phi \rangle = \sqrt{\lambda_0} |01 \rangle + \sqrt{\lambda_1} |12 \rangle + \sqrt{\lambda_2} |20 \rangle , \tag{42}
\]

and its time evolution is given by the following Master equation [14]:

\[
\dot{\rho} = \mathcal{L} \rho ,
\]

\[
\mathcal{L} = \mathcal{L}_A \otimes 1 + 1 \otimes \mathcal{L}_B , \tag{43}
\]

where \( \mathcal{L}_{A/B} \), for a one-qutrit \( \rho_{A/B} \), is

\[
\mathcal{L}_{A/B} = \frac{\Gamma}{2} (2 \gamma |\rho_{A/B} \rangle \langle \rho_{A/B} | - |\rho_{A/B} \rangle \langle \gamma | - |\gamma \rangle \langle \rho_{A/B} |) ,
\]
and $\gamma$ is the coupling matrix for the spontaneous decay:

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

To construct $W_{\sigma\alpha}$ in expression (39) and $W_{\sigma}$ in expression (37), we choose

$$\sigma = |\Phi_{ME}\rangle \langle \Phi_{ME}|,$$

$$$|\Phi_{ME}\rangle = \frac{1}{\sqrt{3}} (|01\rangle + |12\rangle + |20\rangle).$$

(44)

It can be shown simply that for three $|\chi_\alpha\rangle$, for which

$$\{p = x \oplus 1, q = y \oplus 1\} \oplus (1 \equiv \text{sum modulo 3}), \text{ALB}_\alpha(\sigma) = 2/3, \text{and \ ALB}_\alpha(\sigma) = 0 \text{for other} \ |\chi_\alpha\rangle.$$

Thus, using expression (39), we can construct three $W_{\sigma\alpha}$ as

$$W_{\sigma\alpha} = |x,y\rangle \langle x,y| + |y,x\rangle \langle y,x| - |x,x\rangle \langle x,x|.$$  

(45)

Also, using expression (37), we can show that:

$$W_{\sigma} = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} W_{\sigma\alpha}.$$

(46)

As we can see from Eqs. (45) and (46), the number of local observables needed for measuring $W_{\sigma}$ or three $W_{\sigma\alpha}$ is the same and is equal to 12, which is less than what is needed for a full tomography. Also, note that $\{|l,m\rangle\}$ is an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$. So, at least from the theoretical point of view, all the observables $|l,m\rangle \langle l,m|$ can be measured using only one set up. In such cases, for measuring $W_{\sigma}$ or three $W_{\sigma\alpha}$, we only need 7 different set up of local measurements. The comparison of the results of inequalities (37) and (41), for two typical $\{l_i\}$, is given in Fig. 2.

5. Extending to Multipartite Systems

In a bipartite system, any Hermitian operator which, for arbitrary $|\psi\rangle$ and $|\varphi\rangle$, satisfies the inequality

$$C(\psi)C(\varphi) \geq \langle \psi|\langle \varphi|V|\psi\rangle|\varphi\rangle,$$

(47)

gives a measurable lower bound on $C^2(\rho)$, i.e. $C^2(\rho) \geq tr (\rho \otimes \rho V)$ [10]. This can be proved simply by writing $\rho$ in terms of its extremal decomposition $\rho = \sum_j |\xi_j\rangle \langle \xi_j|$. In [18] it was shown how to use such $V$ to construct measurable lower bounds for multipartite concurrence. Following a similar procedure, we construct measurable lower bounds on multipartite concurrence using $V_\alpha$. As the previous sections, we will use the inequality (21) instead of the inequality (47). In other words, we will work with the algebraic lower bounds of $C(\rho)$ rather than the concurrence itself.

The concurrence of an N-partite pure state $|\Psi\rangle$, $|\Psi\rangle \in \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}$, is defined as [31]:

$$C(\Psi) = 2^{-1/2} \sqrt{\sum_{l} C^2_l(\Psi)},$$

(48)
Fig. 2. Comparing the lower bounds given by (37), solid line, and (41), dashed line, for two typical \( \{\lambda_i\} \): a) \( \lambda_i = 1/3 \); b) \( \{\lambda_0 = 1/12, \lambda_1 = 5/6, \lambda_2 = 1/12\} \). When the lower bound given by \( W_\sigma \) is less than zero, we set it to zero.
where $\sum_{l}$ is the summation over all possible subdivisions of $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}$ into two subsystems, and $C_l$ is the related bipartite concurrence. For example, for a 3-partite system we have three $C_l$, namely $C_{1,23}, C_{12,3}$ and $C_{13,2}$. As before we have:

$$C_l^2(\Psi) = \langle \Psi | (A_l | \Psi \rangle | \Psi \rangle, \quad A_l = \sum_{\alpha_l} | \chi_{\alpha_l} \rangle \langle \chi_{\alpha_l} |,$$

(49)

where $| \chi_{\alpha_l} \rangle$ are the same as $| \chi_\alpha \rangle$ which have been defined in Eq. (3). Obviously, they are constructed according to the related subdivision denoted by $l$. So:

$$C(\Psi) = 2^{1-\frac{N}{2}} \sqrt{\sum_{l,\alpha_l} | \langle \chi_{\alpha_l} | \Psi \rangle |^2} = 2^{1-\frac{N}{2}} \sqrt{\sum_{\gamma} | \langle \chi_{\gamma} | \Psi \rangle |^2},$$

(50)

where instead of $l$ and $\alpha_l$ we have used a collective index $\gamma$. From now on, everything is as the bipartite case, except that we deal with the summation over $\gamma$ instead of $\alpha$. The definition of concurrence for mixed states is as follows:

$$C(\rho) = \min_{\{ | \psi_i \rangle \}} \sum_{i} C(\psi_i) = \min_{\{ | \psi_i \rangle \}} \sum_{i} 2^{1-\frac{N}{2}} \sqrt{\langle \psi_i | (A' | \psi_i \rangle \langle \psi_i \rangle), \quad A' = \sum_{\gamma} | \chi_{\gamma} \rangle \langle \chi_{\gamma} |,$$

(51)

where the minimization is over all decompositions of $\rho$ into subnormalized states $| \psi_i \rangle$: $\rho = \sum_{i} | \psi_i \rangle \langle \psi_i |$. It is worth noting that $C(\rho)$, as defined in Eq. (51), is an entanglement monotone for the multiparticle case too [47].

If we define $| \chi'_{\gamma} \rangle = \sum_{\gamma} U_{\gamma'} | \chi_{\gamma} \rangle$, where $U'$ is a unitary matrix, then $A' = \sum_{\gamma} | \chi_{\gamma} \rangle \langle \chi_{\gamma} | = \sum_{\gamma} | \chi'_{\gamma} \rangle \langle \chi'_{\gamma} |$. So, by similar reasoning leading to inequality (13), we have:

$$C(\rho) \geq LB_{\gamma}(\rho) = \min_{\{ | \psi_i \rangle \}} \sum_{i} 2^{1-\frac{N}{2}} | \langle \tau | \psi_i \rangle | \langle \psi_i | \rangle, \quad | \tau \rangle \equiv | \chi'_{\gamma} \rangle = \sum_{\gamma} z_{\gamma}^* | \chi_{\gamma} \rangle, \quad \sum_{\gamma} | z_{\gamma} |^2 = 1.$$

(52)

As before, in contrast to $C(\rho)$, $LB_{\gamma}(\rho)$ is always computable. We also have:

$$C^2(\rho) \geq \sum_{\gamma} | LB_{\gamma}(\rho) |^2, \quad LB_{\gamma}(\rho) = \min_{\{ | \psi_i \rangle \}} 2^{1-\frac{N}{2}} \sum_{\gamma} | \langle \chi'_{\gamma} | \psi_i \rangle | \langle \psi_i | \rangle.$$

(53)

The above expression is the counterpart of the inequality (14) for the multiparticle case. What was proved in [43], neglecting an unimportant constant in the definition of $C(\rho)$, is, in fact, the inequality (53) for the special case of $| \chi'_{\gamma} \rangle = | \chi_{\gamma} \rangle$ (see Eqs. (16) and (17)).

According to the inequality (21), for any $| \chi_{\gamma} \rangle$:

$$| \langle \chi_{\gamma} | \psi \rangle | \langle \psi | \varphi \rangle \rangle | \geq | \langle \psi | V_{\gamma} | \psi \rangle \rangle | \langle \varphi | \varphi \rangle \rangle |,$$

(54)

where $V_{\gamma}$ are the same as $V_{\alpha}$ introduced in Eqs. (23) and (24), defined according to the related $| \chi_{\gamma} \rangle$. So:

$$C^2(\rho) \geq 2^{2-N} \sum_{\gamma} tr (\rho \otimes \rho V_{\gamma}) \,,$$

(55)
where the summation is over those $\gamma$ for which $tr(\rho \otimes \rho V_\gamma) \geq 0$. Also, we have:

$$C^2(\rho) \geq \sum_{\gamma} [tr(\rho W_{\sigma\gamma})]^2, \quad W_{\sigma\gamma} = \frac{-2^{2-N}}{ALB_\alpha(\sigma)} tr_2 (I \otimes \sigma V_\gamma), \quad (56)$$

where $\sigma$ is a pre-determined density operator with $ALB_\gamma(\sigma) > 0$, and the summation is over those $\gamma$ for which $tr(\rho W_{\sigma\gamma}) \leq 0$.

6. Summary and Discussion

Inequality (21) is the main relation of this paper. Using this expression, we have constructed measurable lower bounds on concurrence in term of both one copy or two identical copies of $\rho$. We have proved that the inequality (21) holds for $V_\alpha$ introduced in Eq. (23). Now verifying whether it is possible to find $V_\alpha'$ for which (21) holds for arbitrary $|\chi_\alpha'\rangle$ is valuable.

Our measurable bounds are related to the $ALB_\alpha(\rho)$ rather than the concurrence itself, as we have seen in expressions (22) and (39). So we can use (14) to get the relations (31) and (41). Inequality (31)(Inequality (41)) has this advantage that we can omit the summation over those $\alpha$ for which $tr(\rho \otimes \rho V_\alpha) \leq 0$ ($tr(\rho W_{\sigma\alpha}) \geq 0$). This useful property can help us to achieve better results in detecting the entanglement. As an example, $W_\sigma$ in Eq. (46) is, up to a constant, the summation of three $W_{\sigma\alpha}$. Now, using expression (41), we can omit each $W_{\sigma\alpha}$ for which $tr(\rho W_{\sigma\alpha}) \geq 0$; But, using $W_\sigma$, we can not omit any $W_{\sigma\alpha}$ in Eq. (46). So, as it is shown in Fig. 2, the ability of $W_\sigma$ in detecting the entanglement reduces more rapidly than the three distinct $W_{\sigma\alpha}$.

Bounds obtained from $V_\alpha$ or $W_{\sigma\alpha}$ are always less than or equal to the $ALB_\alpha(\rho)$. In addition, we have shown that these bounds can not detect bound entangled states. So $ALB_\alpha(\rho) > 0$ and $N(\rho) > 0$ ($N(\rho)$ is the negativity of the system [48]) are two necessary conditions for detection of the entanglement by $V_\alpha$ or $W_{\sigma\alpha}$. However, the ability of these bounds and also comparing them with other observable bounds, especially those introduced in [8, 14], need further studies. For example, in the definition of $W_{\sigma\alpha}$, mixed states $\sigma$ can be used simply instead of pure states $\sigma$ since $ALB_\alpha(\sigma)$ is always computable. Studying the above case seems interesting.

At last, in section V, we have generalized our measurable bounds to the multipartite case. The applicability of these bounds also needs further studies.

Acknowledgements

We would like to thank the anonymous referees for their helpful suggestions and improvements.

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**Appendix A**

In this appendix, we prove inequality (21) for $V_\alpha$ introduced in Eq. (23). We prove it for $V_{(2)\alpha}$; the case of $V_{(1)\alpha}$ can be done analogously.

Any arbitrary $|\psi\rangle$ and $|\varphi\rangle$ can be decomposed in a separable basis of $\mathcal{H}_A \otimes \mathcal{H}_B$, like $|i_Aj_B\rangle$, as:

$$
|\psi\rangle = \sum_{ij} \psi_{ij} |i_Aj_B\rangle,
$$

$$
|\varphi\rangle = \sum_{ij} \varphi_{ij} |i_Aj_B\rangle.
$$
Now, from Eq. (23), we have:

\[
\langle \psi | \langle \varphi | V_{(2)\alpha} | \psi \rangle | \varphi \rangle = 2 \left[ -|\psi_{xq}\varphi_{yq} - \psi_{yq}\varphi_{xq}|^2 - |\psi_{xp}\varphi_{yp} - \psi_{yp}\varphi_{xp}|^2 + AA \right],
\]

\[
AA = -2Re (\psi_{xp}\varphi_{yq}\psi_{yq}^*\varphi_{xp}^*) - 2Re (\psi_{yp}\varphi_{xq}\psi_{xq}^*\varphi_{yp}^*) + 2Re (\psi_{xq}\varphi_{yp}\psi_{yp}^*\varphi_{xq}^*).
\] (A.1)

Also for \(|\chi_\alpha\rangle = (|xy\rangle - |yx\rangle)_A (|pq\rangle - |qp\rangle)_B\) we have:

\[
\|\langle \chi_\alpha | \psi \rangle||\langle \chi_\alpha | \varphi \rangle\| = 4| (\psi_{xp}\psi_{yq} - \psi_{xq}\psi_{yp})(\varphi_{xp}\varphi_{yq} - \varphi_{xq}\varphi_{yp}) |
\]

\[\equiv 4|BB|.
\] (A.2)

To get the inequality (21), we must show:

\[
AA \leq 2|BB| + |\psi_{xq}\varphi_{yq} - \psi_{yq}\varphi_{xq}|^2
\]

\[+ |\psi_{xp}\varphi_{yp} - \psi_{yp}\varphi_{xp}|^2.\]

(A.3)

If we have:

\[
AA \leq 2|BB| + 2|CC|,
\]

\[
CC = (\psi_{xp}\varphi_{yq} - \psi_{yq}\varphi_{xq})(\psi_{xp}\varphi_{yp} - \psi_{yp}\varphi_{xp}),
\]

(A.4)

then inequality (A.3) holds. To get the inequality (A.4), it is sufficient to have:

\[
\frac{AA}{2} \leq |BB + CC|
\]

\[= | (\psi_{xp}\varphi_{yq} - \psi_{yq}\varphi_{xq})(\psi_{yq}^*\varphi_{xp}^* - \psi_{xp}^*\varphi_{yq}^*) |.
\] (A.5)

But, the above expression holds since for any complex number \(z\), we have \(Re(z) \leq |z|\), which completes the proof.