Axiomatizing first-order consequences in independence logic

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Abstract

Independence logic, introduced in [5], cannot be effectively axiomatized. However, first-order consequences of independence logic sentences can be axiomatized. In this article we give an explicit axiomatization and prove that it is complete in this sense. The proof is a generalization of the similar result for dependence logic introduced in [8].

1 Introduction

Independence logic [5] is a recent variant of dependence logic that extends first-order logic by formulas

\[ t_1 \perp t_3, t_2 \]

where \( t_i \) is a tuple of terms. The intuitive meaning of this formula is that the sets of values of \( t_1 \) and \( t_2 \) are independent of each other for a fixed value of \( t_3 \). Dependence logic [10] adds to first-order logic formulas

\[ \neq(t_1, \ldots, t_n) \]

where \( t_i \) is a term. Intuitively, this formula says that the values of \( t_1, \ldots, t_{n-1} \) determine the value of \( t_n \). As the notions of dependence and independence are not interesting for single assignments, the semantics of these two logics are defined for sets of assignments, called teams.

Historically these logics are preceded by partially ordered quantifiers (Henkin quantifiers) of Henkin [6] and Independence-Friendly (IF) logic of Hintikka and Sandu [7]. Dependence logic is a variant of these two and equivalent in expressive power whereas independence logic is a bit more general formalism. Dependence logic sentences can be translated to existential second-order logic (ESO) sentences and vice versa. From the point of view of descriptive complexity theory, this means that dependence logic captures all the classes of models in NP. Still, on the level of formulas, dependence logic is weaker in expressive power than ESO. Dependence logic formulas correspond to the ESO sentences that define a downwards closed class of teams [9].

This restriction does not apply to independence logic because it is not downwards closed. Galliani has showed that in expressive power independence logic is equivalent to ESO both on the level of formulas and sentences [3]. It follows that all the NP classes of teams are also definable in independence logic.

In this article we consider only first-order consequences of independence logic. The reason for this restriction is that independence logic cannot be effectively axiomatized. In independence logic it is possible to describe infinity. Using this and going a little further, there is an independence logic
formula $\Theta$ in the language of arithmetic saying that some elementary axioms of number theory fail or else some number has infinitely many predecessors. Now let $\phi$ be any first-order formula in the language of arithmetic. We show that the following claims are equivalent:

1. $\phi$ is true in $(\mathbb{N}, +, \times, <)$.
2. $\Theta \lor \phi$ is valid (true in every model) in independence logic.

Suppose first (1) holds. Let $M$ be an arbitrary model of the language of arithmetic. If $M \not\models \Theta$, then $M \models \phi$ when $M \models \Theta \lor \phi$. For the converse, suppose (2) holds. Since $(\mathbb{N}, +, \times, <) \not\models \Theta$, we have that $(\mathbb{N}, +, \times, <) \models \phi$.

The above shows that the truth in $(\mathbb{N}, +, \times, <)$ can be reduced to validity in independence logic. By Tarski’s Undecidability of Truth, validity in independence logic is non-arithmetical. Therefore, independence logic cannot have any effective complete axiomatization.

The above result of non-axiomatizability holds also for dependence logic. However, this is not an end point of research in this area. There are at least two directions left. One is to modify the semantics in order to get a complete axiomatization. A good example of this is Henkin semantics for second order logic, and for independence logic Galliani has taken this direction in [4]. Another is to restrict your attention to some fragment of the logic you are considering. In dependence logic this direction has been taken in [8]. In that article Kontinen and Väänänen gave an explicit axiomatization of dependence logic and showed that, although it could not be fully complete, it was complete in respect of first-order consequences of dependence logic sentences.

In this paper, using [8] as a background, we will generalize this result to independence logic. Although independence logic is strictly stronger than dependence logic, on the level of sentences these two logics coincide. Independence logic sentences can be translated to dependence logic sentences via ESO [5]. So we already know that at least somehow this generalization can be done.

Another background for this article is [3] where Galliani studied variants of independence logic and different ways of defining semantics for these logics. One of these definitions will be both reasonable and useful for our purposes and will therefore be used in this paper. The semantics we will use is called LAX semantics in Galliani’s work. Using it we can secure that only the variables occurring free in a formula will affect to the truth value of that formula. With LAX semantics we will be able to construct, for every independence formula, an equivalent formula in prenex normal form, and furthermore, an equivalent formula in a precise conjunctive normal form. This may be interesting in itself, although the constructions will be presented as parts of the completeness proof.

The structure of this paper is the following. In the next section we will go through some preliminaries that are necessary for this topic. In Section 3 the axioms and the rules of inference are introduced. In Section 4 we will show that our new deduction system is sound, and in Section 5 we will show that it is also complete in respect of first-order consequences of independence logic sentences. At the end of the paper an example and some further questions will be presented.

## 2 Preliminaries

In this section we introduce independence logic ($\mathcal{L}$) and go through some results that are needed in this paper.

At first few remarks on notations are needed. The most important one is that there will not be any notational distinction between tuples and singles. For example, $x$ can refer either to the single variable $x$ or to the tuple of variables $x = (x_1, \ldots, x_k)$. However, it is always mentioned in the text if we are considering tuples instead of singles at the time. Also if $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_l)$ are
tuples of variables, then by \( xy \) we denote the tuple \( (x_1, \ldots, x_k, y_1, \ldots, y_l) \). If \( A \) and \( B \) are sets of
tuples, then \( A \setminus B \) denotes the set \( \{ ab \mid a \in A \text{ and } b \in B \} \).

**Definition 1.** Formulas of \( \mathcal{I} \) are defined recursively as follows:

1. If \( \phi \in \text{FO} \), then \( \phi \in \mathcal{I} \).
2. If \( t_1, t_2 \) and \( t_3 \) are finite (or empty) tuples of terms, then \( t_1 \perp t_2 t_2 \in \mathcal{I} \).
3. If \( \phi, \psi \in \mathcal{I} \), then \( \phi \lor \psi \in \mathcal{I} \) and \( \phi \land \psi \in \mathcal{I} \).
4. If \( \phi \in \mathcal{I} \) and \( x \) is a variable, then \( \exists x \phi \in \mathcal{I} \) and \( \forall x \phi \in \mathcal{I} \).

So we allow negation only in front of first-order formulas. Also notice that in the independence atom, we allow any \( t_i \) to be empty. In the case of \( t_3 = \emptyset \), \( t_1 \perp \emptyset t_2 \) is denoted by \( t_1 \perp t_2 \).

In order to define the semantics of \( \mathcal{I} \), we first need to define the concept of a team. Let \( M \) be a
model. An assignment \( s \) of \( M \) is a finite mapping from a set of variables to the domain of \( M \). (In this text \( M \) can refer either to the model itself or its domain. It will be always clear from the context which one is under consideration.) Let \( \{ x_1, \ldots, x_k \} \) be a set of variables. A team \( X \) of \( M \) with
\( \text{Dom}(X) = \{ x_1, \ldots, x_k \} \) is a set of assignments \( s \) of \( M \) with \( \text{Dom}(s) = \{ x_1, \ldots, x_k \} \). The value of a
term \( t \) in an assignment \( s \) is denoted by \( t^M(s) \). If \( t = (t_1, \ldots, t_i) \) where \( t_i \) is a term, then by \( t^M(s) \) we
denote \( (t_1^M(s), \ldots, t_i^M(s)) \). By \( s(a/x) \), for a variable \( x \) and \( a \in M \), we denote the assignment which
(with domain \( \text{Dom}(s) \cup \{ x \} \)) agrees with \( s \) everywhere except that it maps \( x \) to \( a \). Then by \( X(M/x) \)
we denote the duplicated team \( \{ s(a/x) \mid s \in X \text{ and } a \in M \} \). If \( F : X \to \mathcal{P}(M) \), then \( X(F/x) \)
denotes to the supplemented team \( \{ s(a/x) \mid s \in X \text{ and } a \in F(s) \} \). Note that it can be the case that
\( X(M/x) = X(F/x) \).

The set \( \text{Fr}(\phi) \) of free variables of a formula \( \phi \in \mathcal{I} \) is defined as for first-order logic, except that we
now have the new case
\[
\text{Fr}(t_1 \perp t_2 t_3) = \text{Var}(t_1) \cup \text{Var}(t_2) \cup \text{Var}(t_3)
\]
where \( \text{Var}(t_i) \) is the set of variables occurring in the term tuple \( t_i \). If \( \text{Fr}(\phi) = \emptyset \), then we call \( \phi \) a sentence.

Now we are ready to define the semantics of \( \mathcal{I} \). In the definition, \( M \models_s \phi \) refers to the Tarskian
satisfaction relation of first-order logic.

**Definition 2.** Let \( M \) be a model, \( \phi \in \mathcal{I} \) and \( X \) a team of \( M \) such that \( \text{Fr}(\phi) \subseteq X \). The satisfac-
tion relation \( M \models_X \phi \) is defined as follows:

1. If \( \phi \in \text{FO} \), then \( M \models_X \phi \) iff \( M \models_s \phi \) for all \( s \in X \).
2. If \( \phi = t_1 \perp t_2 t_3 \), then \( M \models_X \phi \) iff for all \( s, s' \in X \) with \( t_1^M(s) = t_1^M(s') \), there is some \( s'' \in X \)
such that \( t_1^M(s'') t_2^M(s'') = t_2^M(s) t_2^M(s') \) and \( t_2^M(s'') = t_3^M(s') \).
3. If \( \phi = \psi \lor \theta \), then \( M \models_X \phi \) iff \( M \models_Y \psi \) and \( M \models_Z \theta \) for some \( Y, Z \subseteq X, Y \cup Z = X \).
4. If \( \phi = \psi \land \theta \), then \( M \models_X \phi \) iff \( M \models_X \psi \) and \( M \models_X \theta \).
5. If \( \phi = \exists x \psi \), then \( M \models_X \phi \) iff \( M \models_X(F/x) \psi \) for some \( F : X \to \mathcal{P}(M) \).
6. If \( \phi = \forall x \psi \), then \( M \models_X \phi \) iff \( M \models_X(M/x) \psi \).
In the case of $t = \emptyset$ occurring in an independence atom, we let $t^M(s) = t^M(s')$ for every $s, s' \in X$. Therefore,

$$M \models X \emptyset \perp_{t_1 \ldots t_n} t_2$$

and $M \models X t_1 \perp_{t_1 \ldots t_n} \emptyset$ for all $X$.

If it is the case that we have to verify a formula of the form $\forall x_1 \ldots \forall x_k \phi$, then instead of the notation $X(M/x_1) \ldots (M/x_k)$, we will often use the abbreviation $X(M^k/x_1 \ldots x_k)$. Also if we have to verify a formula of the form $\exists x_1 \ldots \exists x_k \phi$, then by the definition, we have to find witnessing functions $F_1 : X \rightarrow \mathcal{P}(M), F_2 : X(F_1/x_1) \rightarrow \mathcal{P}(M), \ldots, F_k : X(F_1/x_1) \ldots (F_{k-1}/x_{k-1}) \rightarrow \mathcal{P}(M)$ such that

$$M \models X(F_1/x_1) \ldots (F/x_k) \phi.$$ 

Clearly in this case it is equivalent to find a single function $F : X \rightarrow \mathcal{P}(M^k)$ such that

$$M \models X(F/x_1 \ldots x_k) \phi.$$ 

In addition to independence atoms, there are also many other type of atomic formulas that are relevant in team semantics setting. Dependence atom was already introduced but also inclusion and exclusion atoms will be useful for our purposes. The syntax of these atoms is the following:

- dependence: $= (t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ is a term.
- inclusion: $t_1 \subseteq t_2$, where $t_1$ and $t_2$ are tuples of terms of the same length.
- exclusion: $t_1 \mid t_2$, where $t_1$ and $t_2$ are tuples of terms of the same length.

The semantics of these atoms is defined as:

- dependence: $M \models (t_1, \ldots, t_n)$ iff for all $s, s' \in X$ with $t^M_1(s) = t^M_1(s'), \ldots, t^M_{n-1}(s) = t^M_{n-1}(s')$, it holds that $t^M_n(s) = t^M_n(s').$
- inclusion: $M \models (t_1 \subseteq t_2)$ iff for every $s \in X$, there is $s' \in X$ such that $t^M_1(s) = t^M_2(s').$
- exclusion: $M \models (t_1 \mid t_2)$ iff for every $s, s' \in X$, $t^M_1(s) \neq t^M_2(s').$

If we replace independence atom with one these atoms in Definition 1, then the resulting logic is called dependence logic, inclusion logic or exclusion logic.

Consider first dependence logic. Dependence atom $= (t_1, \ldots, t_n)$ express functional dependence between $t_n$ and the tuple $t_1 \ldots t_{n-1}$, and it can be expressed in independence logic as

$$t_n \perp_{t_1 \ldots t_{n-1}} t_n.$$ 

For the other direction, there is no translation of independence atom in dependence logic. Dependence logic is downwards closed (meaning that $M \models Y \phi$ whenever $M \models X \phi$ and $Y \subseteq X$) whereas independence logic is not. Consider for example independence atom $x \perp y$. This atom is true for the team

| $x$ | $y$ |
|-----|-----|
| $s_0$ | 0 | 0 |
| $s_1$ | 0 | 1 |
| $s_2$ | 1 | 0 |
| $s_3$ | 1 | 1 |
but not for the team

|   | x | y |
|---|---|---|
| s_0 | 0 | 0 |
| s_1 | 0 | 1 |
| s_2 | 1 | 0 |

Thus it cannot be expressed in dependence logic, and hence independence logic is expressively strictly
stronger on the level of formulas. On the level of sentences though, these logics coincide [5].

One should also mention that the semantics of all the other dependence logic formulas are not
normally defined entirely the same way as we did in Definition 2 for independence logic. There is
usually one exception concerning existential formulas. For $\exists x \phi$, it is usually required that each value
of the function $F : X \rightarrow \mathcal{P}(M)$ is singleton. Still, dependence logic is downwards closed with both
semantics, and thus with Axiom of Choice, these two definitions coincide.

A direct consequence of the example above is that we cannot adopt the rule $\forall x \phi \vdash \phi$ into our
inference system because it is not sound for independence logic. If $M$ is a model with domain \{0, 1\},
and $X$ is the team (4), then $X(M/x)$ is the team (3), and thus $M \models_X \forall x \perp y$ and $M \not\models_X x \perp y$.

Consider then inclusion and exclusion atoms. Galliani has showed that inclusion/exclusion logic
(first-order logic added with inclusion and exclusion atoms) is translatable to independence logic and
vice versa [3]. There the following independence logic translation of inclusion atom was presented.

**Proposition 5** ([3]). Let $t_1$ and $t_2$ be tuples of terms of the same length. Then the inclusion atom
$t_1 \subseteq t_2$ is equivalent to the independence formula

$$
\forall v_1 \forall v_2 \forall z ((\neg v_1 = t_1 \land \neg v_1 = t_2) \lor (v_1 = v_2 \land \neg v_1 = t_2) \lor ((v_1 = v_2 \lor v_2 = t_2) \land z \perp v_1 v_2))
$$

where $v_1$ and $v_2$ are variables and $z$ is a variable tuple of the same length than $t_i$, and none of the
variables in $v_1 v_2 z$ occur in $t_1 t_2$.

We will use dependence and inclusion atoms in our deduction system, and there every such an
occurrence should be understood as an independence logic translation of the form introduced here.

Before going to the proof, one important result need yet to be introduced.

**Definition 6.** Let $T$ be a set of formulas of independence logic with only finitely many free variables.
The formula $\phi$ is a logical consequence of $T$,

$$
T \models \phi,
$$

if for all models $M$ and teams $X$, with $\text{Fr}(\phi) \cup \bigcup_{\psi \in T} \text{Fr}(\psi) \subseteq \text{Dom}(X)$, and
$M \models_X T$, we have $M \models_X \phi$. The formulas $\phi$ and $\psi$ are logically equivalent,

$$
\phi \equiv \psi,
$$

if $\phi \models \psi$ and $\psi \models \phi$.

Let $X$ be a team with domain \{x_1, \ldots, x_k\} and $V \subseteq \{x_1, \ldots, x_k\}$. By $X \upharpoonright V$ we denote the team
$\{s \upharpoonright V \mid s \in X\}$. If $u$ is a tuple of variables such that $\text{Var}(u) = V$, then by $X \upharpoonright u$ we denote the team
$X \upharpoonright V$. The following result is important [3].

**Proposition 7** (Locality). Suppose $V \supseteq \text{Fr}(\phi)$. Then $M \models_X \phi$ if $M \models_{X \upharpoonright V} \phi$.

For a logic in team semantics setting, this is not an obvious fact. IF logic lacks this property, and
the same holds for independence logic if the semantics of $\exists x \phi$ is defined in the standard dependence
logic way (requiring that the witnessing $F$ maps the assignments of $X$ to singletons of $\mathcal{P}(M)$).
3 A system of natural deduction

In this section we introduce inference rules that allow us to derive all the first-order consequences of sentences of independence logic. Many of the rules below are just the same than the dependence logic rules introduced in [8]. Still some major differences occur in this system partly due the semantic differences between independence and dependence atomic formulas and partly due the fact that independence logic is not downwards closed.

The rules we are about to adopt are listed below in Figure 3 and in Definition 8. If \( A \) is a formula, \( t \) is a term and \( x \) is a variable, then \( A(t/x) \) denotes the formula \( A \) where all the free occurrences of \( x \) are replaced by \( t \). When using this notation we presume that no variable in \( t \) becomes bound in the substitution.

**Definition 8.** 1. Disjunction substitution:
\[
\frac{[B]}{A \lor B}
\]
\[
\vdots
\]
\[
A \lor B \quad \overset{\text{C}}{\rightarrow} \\
A \lor C
\]

2. Commutation and associativity of disjunction:
\[
\frac{B \lor A}{A \lor B}
\]
\[
\frac{(A \lor B) \lor C}{A \lor (B \lor C)}
\]

3. Extending scope:
\[
\frac{\forall x A \land B}{\forall x (A \land x \perp y) \lor B}
\]
where \( y \) is a tuple listing the variables in \( \text{Fr}(A \lor B) \) – \( \{x\} \) and the prerequisite for applying this rule is that \( x \) does not appear free in \( B \).

4. Extending scope:
\[
\frac{\exists x A \lor B}{\exists x (A \lor B)}
\]
where the prerequisite for applying this rule is that \( x \) does not appear free in \( B \).

5. Universal substitution:
\[
\frac{A(y/x)}{\vdots}
\]
\[
\forall x A \quad \overset{\text{y}}{\rightarrow} \\
\forall y B
\]
where the prerequisite for applying this rule is that \( y \) does not appear free in \( \forall x A \) and in any non-discharged assumption used in the derivation of \( B \), except in \( A(y/x) \).

6. Independence distribution: Let
\[
A = \exists x_0 (\bigwedge_{1 \leq i \leq m} u_i \perp v_i, v_i \land C)
\]
| Operation       | Introduction          | Elimination          |
|-----------------|-----------------------|----------------------|
| Conjunction     | \[ \frac{A \land B}{A \land B} \land I \] | \[ \frac{A \land B}{A} \land E \] \[ \frac{A \land B}{B} \land E \] |
| Disjunction     | \[ \frac{A}{A \lor B} \lor I \] \[ \frac{B}{A \lor B} \lor I \] | \[ \frac{[A]}{A \lor C} \lor\ 
|                 |                       | \[ \frac{[B]}{C \lor \ ldots} \lor \] \[ \frac{A \lor B}{C} \lor E \] |
|                 |                       | Condition 1.         |
| Negation        | \[ \frac{B \land \neg B}{\neg A} \neg I \] | \[ \frac{\neg A}{A} \neg E \] |
|                 |                       | Condition 2.         |
| Universal quantifier | \[ \frac{A}{\forall x_1 A} \forall I \] | \[ \frac{\forall x_1 A}{A(t/x_1)} \forall E \] |
|                 |                       | Condition 3.         |
| Existential quantifier | \[ \frac{A(t/x_1)}{\exists x_1 A} \exists I \] | \[ \frac{[A]}{\exists x_1 A} \exists E \] \[ \frac{\exists x_1 A}{B} \exists E \] |
|                 |                       | Condition 4.         |

Condition 1. \( C \) is first-order.
Condition 2. The formulas are first-order.
Condition 3. The variable \( x_1 \) cannot appear free in any non-discharged assumption used in the derivation of \( A \).
Condition 4. The variable \( x_1 \) cannot appear free in \( B \) and in any non-discharged assumption used in the derivation of \( B \), except in \( A \).

Figure 1: The first set of rules.
and

\[ B = \exists x_1 ( \bigwedge_{m+1 \leq i \leq m+n} u_i \downarrow w_i \land v_i \land D) \]  \hspace{1cm} (10) \]

be formulas where \( x_0 \) is a tuple of variables that do not appear in \( B \); \( x_1 \) is a tuple of variables that do not appear in \( A \); \( u_i; s, v_i; s \) and \( w_i; s \) are tuples of bound variables; \( C \) and \( D \) are first-order formulas.

Let

\[ E = \forall \alpha \vee \beta \exists x_0 \exists x_1 \exists z_0 \exists z_1 \exists E \left[ \bigwedge_{1 \leq i \leq m+n} u_i \downarrow w_i \land v_i \land \bigwedge_{i=0} = (z_i) \land (-z_0 = z_1 \lor \alpha = \beta) \land ((C \land r = z_0) \lor (D \land r = z_1)) \right] \]

where \( \alpha, \beta, z_0, z_1 \) and \( r \) are variables that do not appear in formula \( A \lor B \). Then

\[ \frac{A \lor B}{E} \]

Note that the logical form of this rule is

\[ \exists x_0 (\bigwedge_{1 \leq i \leq m} u_i \downarrow w_i \land v_i \land C) \lor \exists x_1 (\bigwedge_{m+1 \leq i \leq m+n} u_i \downarrow w_i \land v_i \land D) \]

\[ \forall \alpha \lor \beta \exists x_0 \exists x_1 \exists z_0 \exists z_1 \exists E \left[ \bigwedge_{1 \leq i \leq m+n} u_i \downarrow w_i \land v_i \land \bigwedge_{i=0} = (z_i) \land (-z_0 = z_1 \lor \alpha = \beta) \land ((C \land r = z_0) \lor (D \land r = z_1)) \right] \]

7. Independence introduction:

\[ \frac{\exists x \forall y A}{\forall y \exists x (A \land x \perp y)} \]

where \( z \) is a tuple listing the variables in \( Fr(A) - \{x, y\} \).

8. Independence transmission: Let

\[ A = \forall x_0, \exists y_0, \exists x_{0,1}, \ldots, \exists x_{0,p} \exists y_{0,p} \left( \bigwedge_{1 \leq i \leq m} u_{0,0} \downarrow w_{0,0} ^i \land v_{0,0} ^i \right) \land \bigwedge_{1 \leq i \leq p} x_{0,i} \land y_{0,i} \land \bigwedge_{0 \leq i \leq p} C_{0,i} \land D \]  

and

\[ B = \forall x_0, \exists y_0, \exists x_{0,1}, \ldots, \exists x_{0,p} \exists y_{0,p} \left( \bigwedge_{0 \leq i \leq p} C_{0,i} \land D \land \bigwedge_{1 \leq i \leq m} u_{i,1} \downarrow w_{i,1} \land v_{i,1} \right) \land \bigwedge_{0 \leq i \leq p} x_{1,1} \land y_{1,1} \land \bigwedge_{0 \leq i \leq p} C_{1,i} \land \bigwedge_{0 \leq i \leq p} x_{1,i} = x_{0,i} \land y_{0,i} \land \bigwedge_{0 \leq i \leq p} \left( E_{i,j,k} \land \bigvee_{p \leq l \leq p'} u_{i,j,k} ^l \land w_{i,j,k} ^l = u_{i,j,k} ^l \land v_{i,j,k} ^l \right) \land \bigwedge_{1 \leq i \leq m} \right) \]

be formulas where
• \( x_{j,k} \)'s are variable tuples of same length, \( y_{j,k} \)'s are variable tuples of same length, and the variables in these tuples are quantified only once and do not occur free in formula \( D \).

• \( u^i \)'s, \( v^i \)'s and \( w^i \)'s are tuples of variables from \( y_{0,0} \).

• \( C \) is a quantifier-free first-order formula with variables from \( x_{0,0}, y_{0,0} \).

• \( e_{j,k}^i = e^i(x_{j,k}y_{j,k}/x_{0,0}y_{0,0}) \), for \( e \in \{ u, v, w \} \).

• \( C_{j,k} = C(x_{j,k}y_{j,k}/x_{0,0}y_{0,0}) \).

• \( E_{n,j,k}^i = \begin{cases} \top & \text{if } w^i \text{ is empty. (Note: } \top \text{ is first-order.)} \\ ¬w^i_{n,j} = w^i_{n,k} & \text{otherwise.} \end{cases} \)

• \( p \geq 0 \) and \( p' = p + m(p + 2)^2 \).

Then we let

\[
\frac{A}{B}
\]

9. Identity axiom: If \( x \) is a variable, then \( x = x \) is an axiom.

10. Identity rule: If \( x \) and \( y \) are variables, then we let

\[
\frac{x = y}{y = x}
\]

11. Identity rule: If \( t \) is a term and \( x \) and \( y \) are variables, then we let

\[
\frac{x = y}{t(x/y) = t}
\]

12. Identity rule: If \( A \) is a formula and \( x \) and \( y \) are variables, then we let

\[
\frac{A \land x = y}{A(x/y)}
\]

Disjunction elimination rule is not sound for independence logic, so we introduce rules 1-4 for disjunction. These rules are all derivable in first-order logic. Also similar rules for conjunction are easily derivable in this system with an exception that we can derive the correspondent of rule 3 without this new independence atom \( x \perp y \) occurring in the derived formula. As mentioned before, universal elimination rule does not hold for independence logic, so we introduce rule 5 here which is also derivable in first-order logic. Rules 3, 4, 6 and 7 preserve logical equivalence.

4 The Soundness Theorem

In this section we will show that the previous system of natural deduction is sound. First we prove that rules 3, 4, 6 and 7 (plus the conjunctive versions of rules 3 and 4 which are denoted by 3' and 4') preserve logical equivalence.

**Lemma 11** (Rules 3, 4, 7 and the conjunctive versions 3' and 4'). *The following equivalences hold for formulas of independence logic:*

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Proof. (3) Follows from locality of the semantics.

(3′) Follows from locality of the semantics.

∃

∀

(7) As above it is enough to prove the equivalence for models $M$ and teams $X$ with $\text{Dom}(X) = \text{Fr}(\varphi \lor \psi) - \{x\}$. So assume that $M \models x \forall x((\varphi \land \neg x) \lor \psi)$. Then we can find $Y, Z \subseteq X(M/x)$, $Y \cup Z = X(M/x)$, such that $M \models Y \varphi \land y \perp x$ and $M \models Z \psi$. There are two options:

(i) For all $s \in X$, there is some $a \in M$ such that $s(a/x) \in Z$. Then by locality, $M \models x \varphi$ when $M \models x \forall x \varphi \lor \psi$.

(ii) For some $s \in X$, $s(a/x) \in Y$ for all $a \in M$. Then because $M \models y \perp x$, we conclude that $Y(M/x) = Y$, and hence $M \models y \perp x$. If $Y' = Y \setminus \text{Dom}(X)$ and $Z' = Z \setminus \text{Dom}(X)$, then by locality, $M \models Y' \forall x \varphi$ and $M \models Z' \psi$. Now $X = Y' \cup Z'$, so we conclude that $M \models x \forall x \varphi \lor \psi$.

For the converse, assume that $M \models x \forall x \varphi \lor \psi$. Let $Y, Z \subseteq X$, $Y \cup Z = X$, be such that $M \models Y(M/x) \varphi$ and $M \models Z \psi$. Clearly $M \models Y(M/x) \varphi \land y \perp x$, and by locality, $M \models Z(M/x) \psi$. So $M \models X(M/x) \varphi \land y \perp \psi$ and hence $M \models x((\varphi \land y \perp x) \lor \psi)$.

(3′) Follows from locality of the semantics.

(4) If $M \models x \exists x(\varphi \lor \psi)$, and $F : X \rightarrow \mathcal{P}(M)$ is such that $M \models x(F(x)/x) \varphi \lor \psi$, then we can find $Y, Z \subseteq X(F(x))$, $Y \cup Z = X(F(x))$, so that $M \models Y \varphi$ and $M \models Z \psi$. Define

$$Y' = \{ s \in X \mid s(a/x) \in Y \text{ for some } a \in F(s) \}$$

and

$$Z' = \{ s \in X \mid s(a/x) \in Z \text{ for some } a \in F(s) \}.$$  

Then $M \models Z' \psi$, and if $F' : Y' \rightarrow \mathcal{P}(M)$ is the function $s \mapsto \{ a \in F(s) \mid s(a/x) \in Y \}$, then $M \models Y'(F'(x)/x) \varphi$ and thus $M \models Y' \exists x \varphi$. Hence $M \models x \exists x \varphi \lor \psi$.

If $M \models x \exists x \varphi \lor \psi$, then for some $Y, Z \subseteq X$, $Y \cup Z = X$, $M \models Y \exists x \varphi$ and $M \models Z \psi$. If $F : Y \rightarrow \mathcal{P}(M)$ is such that $M \models Y(F(x)/x) \varphi$, choose $F' : X \rightarrow \mathcal{P}(M)$ so that $F' \mid Y = F$ and

$$F' \mid (X - Y)$$

is some constant function. Then $Y(F'(x)/x) \cup Z(F'(x)/x) = X(F'(x)/x), M \models Y(F'(x)/x) \varphi$ and by locality, $M \models Z(F'(x)/x) \psi$. So $M \models X(F'(x)/x) \varphi \lor \psi$ and hence $M \models x \exists x(\varphi \lor \psi)$.

(4′) Follows from locality of the semantics.

(7) As above it is enough to prove the equivalence for models $M$ and teams $X$ with $\text{Dom}(X) = \text{Fr}(\varphi) - \{x, y\}$. Assume first $M \models x \forall x \exists y((x \perp_z y) \land \phi)$. Then there is $F : X(M/x) \rightarrow \mathcal{P}(M)$ such that $X' = X(M/x)(F/y)$, then $M \models x \perp_z y \land \phi$. If now $b \in M$ is such that there are $a \in M$ and $s \in X$ with $s(a/x)(b/y) \in X'$, then the independence atom guarantees that $s(a/x)(b/y) \in X'$ for all $a \in M$. Therefore, if we define $F' : X \rightarrow \mathcal{P}(M)$ so that

$$F'(s) = \{ b \in M \mid s(a/x)(b/y) \in X' \text{ for some } a \in M \},$$

then $X(F'(y)/y)(M/x) = X(M/x)(F/y)$. Hence $M \models x \exists y \forall x \phi$. 

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For the converse, assume that \( M \models X \forall x \phi \). Then there is \( F : X \rightarrow \mathcal{P}(M) \) such that if \( X' = X(F/y)(M/x) \), then \( M \models X' \phi \). Clearly \( M \models X' x \land y \) holds also. If we define \( F' : \{ M/x \} \rightarrow \mathcal{P}(M) \) so that \( F'(s(a/x)) = F(s) \) for all \( s \in X \) and \( a \in M \), then \( X(M/x)(F'/y) = X(F/y)(M/x) \). Hence \( M \models X \forall x \exists y(x \land y \land \phi) \).

\[ \square \]

**Example 12.** Generally it is not true that \( M \models X \forall x (\varphi \land \psi) \Leftrightarrow M \models X \forall x \varphi \land \psi \) if \( x \) does not occur free in \( \psi \). Let \( \varphi := x \leq y \land (x = 1 \vee y = 1) \) and \( \psi := y = 0 \). If \( M \) is a model with domain \( \{0, 1\} \) and \( X = \{(y, 0), (y, 1)\} \), then \( M \models X \forall x (\varphi \land \psi) \) but \( M \not\models X \forall x \varphi \land \psi \). On the other hand, we can now see that \( M \not\models X (\forall x (\varphi \land x \leq y) \land \psi) \).

**Lemma 13 (Rule 6).** Let

\[ \phi_0 = \exists x_0 \left( \bigwedge_{1 \leq i \leq m} u_i \land v_i \land \theta_0 \right) \]  

(14)

and

\[ \phi_1 = \exists x_1 \left( \bigwedge_{m+1 \leq i \leq m+n} u_i \land v_i \land \theta_1 \right) \]  

(15)

be formulas where \( x_0 \) is a tuple of variables that do not occur in \( \phi_0 \); \( u_i : s \) and \( v_i : s \) and \( w_i : s \) are tuples of bound variables; \( \theta_0 \) and \( \theta_1 \) are first-order formulas. Let \( \alpha, \beta, z_0, z_1 \) and \( r \) be variables that do not appear in formula \( \phi_0 \lor \phi_1 \). Then if we define

\[ \varphi = \forall \alpha \forall \beta \exists x_0 \exists x_1 \exists z_0 \exists z_1 \exists r \left[ \bigwedge_{1 \leq i \leq m+n} u_i \land v_i \land \theta_0 \land \bigwedge_{i \leq 0, 1} (z_i) \land (\neg z_0 = z_1 \lor (\alpha = \beta) \land ((\theta_0 \land r = z_0) \lor (\theta_1 \land r = z_1))) \right], \]

we have that \( \phi_0 \lor \phi_1 \equiv \varphi \).

**Proof.** We divide the proof into two parts. First we prove that the equivalence holds for models \( M \) with \( |M| = 1 \) and then for models with larger domain. By locality of the semantics, we can without loss of generality assume that \( X \) is always a team with \( \text{Dom}(X) = \text{Fr}(\phi_0 \lor \phi_1) \). For notational simplicity we can without loss of generality assume that \( x_0 \) and \( x_1 \) are both of same length \( l \).

1. Suppose \( M \) is a model \( |M| = 1 \) and \( X \) is a team. If \( M \models X \phi_0 \lor \phi_1 \), then \( M \models X \phi_0 \) or \( M \models X \phi_1 \). Now if we evaluate all the quantified variables in \( \varphi \) by the only possible way, we have that \( (\theta_0 \land r = z_0) \) or \( (\theta_1 \land r = z_1) \) holds in \( X \). Also \( \alpha = \beta \) must be true, so \( (\neg z_0 = z_1 \lor \alpha = \beta) \) holds in \( X \). All the independence atoms are trivially true, so \( M \models X \varphi \).

Suppose then \( M \models X \varphi \). Then \( X \) extended with values for \( x_0, x_1 \) must have \( \theta_0 \) or \( \theta_1 \) true. In either case independence atoms hold trivially, so \( M \models X \phi_0 \) or \( M \models X \phi_1 \). Hence \( M \models X \phi_0 \lor \phi_1 \).

2. Suppose now \( M \) is a model with \( |M| > 1 \) and \( X \) is a team. Let \( 0 \) and \( 1 \) be some distinct members of \( M \).

Assume first that \( M \models X \phi_0 \lor \phi_1 \). Then there are \( Y, Z \subseteq X \), \( Y \cup Z = X \), such that \( M \models Y \phi_0 \) and \( M \models Z \phi_1 \). Let \( F_Y : Y \rightarrow \mathcal{P}(M^l) \) and \( F_Z : Z \rightarrow \mathcal{P}(M^l) \) be functions witnessing this. Now we want to form a function \( F : X(M^l(\alpha \beta)) \rightarrow \mathcal{P}(M^{2l+3}) \) so that if \( X' = X(M^l(\alpha \beta))(F/x_0 z_0 x_1 z_1 r) \), then \( M \) and \( X' \) satisfy the quantifier-free part of \( \varphi \). First we
define sets of tuples as follows:

Let \( s \in X(M^2/\alpha\beta) \). Define

\[
A_{s,z_0} = \{0\} \text{ and } A_{s,z_1} = \{1\}
\]

and let

\[
B_{s,x_0} = F_Y(s \upharpoonright \text{Dom}(X)), \quad B_{s,x_1} = \{0\} \text{ and } B_{s,r} = \{0\} \text{ if } s \upharpoonright \text{Dom}(X) \in Y,
\]

\[
B_{s,x_0} = B_{s,x_1} = B_{s,r} = \emptyset \text{ otherwise.}
\]

\[
C_{s,x_0} = \{0\}, \quad C_{s,x_1} = F_Z(s \upharpoonright \text{Dom}(X)) \text{ and } C_{s,r} = \{1\} \text{ if } s \upharpoonright \text{Dom}(X) \in Z,
\]

\[
C_{s,z_0} = C_{s,y_1} = C_{s,r} = \emptyset \text{ otherwise.}
\]

Then define

\[
B_s = B_{s,x_0} \prec B_{s,x_1} \prec A_{s,z_0} \prec A_{s,z_1} \prec B_{s,r} \text{ and }
\]

\[
C_s = C_{s,x_0} \prec C_{s,x_1} \prec A_{s,z_0} \prec A_{s,z_1} \prec C_{s,r}
\]

and let \( F(s) = B_s \cup C_s \).

Now it is enough to show that the quantifier-free part of \( \varphi \) holds for \( M \) and \( X' \). So let us go through it part by part:

- \( \bigwedge_{1 \leq i \leq m+n} u_i \perp w_i, v_i' \): Let \( i \leq m+n \) and \( t, t' \in X' \) be such that \( t(w_i) = t'(w'_i) \). If they both evaluate \( r \) as, say \( 0 \), then by the definition of \( F' \), \( t \upharpoonright (\text{Dom}(X) \cup \text{Var}(x_0)), t' \upharpoonright (\text{Dom}(X) \cup \text{Var}(x_0)) \in Y(F_Y/x_0) \). If \( i \leq m \), then this team satisfies \( u_i \perp w_i, v_i \), and there is an assignment in \( Y(F_Y/x_0) \) agreeing with \( t \) for \( u_i, w_i \) and with \( t' \) for \( v_i \). Now we can extend it to an assignment \( t'' \) of \( X' \) such that \( t''(r) = 0 \). Then this \( t'' \) is as wanted. Suppose \( i > m \). Then all the variables in tuples \( u_i, v_i \) and \( w_i \) are from tuple \( x_1 \) and \( t(x_1) = t'(x_1) = 0 \). Thus we can choose \( t'' = t \).

The case where \( t(r) = t'(r) = 1 \) is analogous.

- \( \bigwedge_{i=0,1} = (z_i) \): Follows from the definition of \( F \).

- \( \neg z_0 = z_1 \lor \alpha = \beta \): Clearly \( M \models X \neg z_0 = z_1 \).

- \( (\theta_0 \land r = z_0) \lor (\theta_1 \land r = z_1) \): Simply divide \( X' \) to \( Y' \) and \( Z' \) so that in \( Y', r = 0 \) and in \( Z', r = 1 \). Then \( Y' \upharpoonright (\text{Dom}(X) \cup \text{Var}(x_0)) = Y(F_Y/x_0) \), so \( \theta_0 \) holds in \( Y' \). Also \( r = z_0 \) holds trivially and hence \( M \models Y, \theta_0 \land r = z_0 \). Similarly \( M \models Z, \theta_1 \land r = z_1 \).

Assume then that \( M \models X \varphi \) and let \( F : X(M^2/\alpha\beta) \to \mathcal{P}(M^{2+\beta}) \) be a function witnessing this. Then if \( X' = X(M^2/\alpha\beta)(F/x_0,x_1,z_0,z_1) \) we have that the quantifier-free part of \( \varphi \) is true for \( M \) and \( X' \). Now define

\[
Y = \{ s \in X \mid \exists t \in X'[t \upharpoonright \text{Dom}(X) = s \text{ and } t(r) = t(z_0)] \} \text{ and }
\]

\[
Z = \{ s \in X \mid \exists t \in X'[t \upharpoonright \text{Dom}(X) = s \text{ and } t(r) = t(z_1)] \}.
\]

Note that \( M \models t \uparrow r = z_0 \lor r = z_1 \) for all \( t \in X' \), so \( Y \cup Z = X \). Define also functions \( F_Y : Y \to \mathcal{P}(M^t) \) and \( F_Z : Z \to \mathcal{P}(M^t) \) by

\[
F_Y(s) = \{ t(x_0) \mid t \in X', t \upharpoonright \text{Dom}(X) = s \text{ and } t(r) = t(z_0) \}
\]

and

\[
F_Z(s) = \{ t(x_1) \mid t \in X', t \upharpoonright \text{Dom}(X) = s \text{ and } t(r) = t(z_1) \}.
\]
It is enough to show that

\[ M \models Y(F_Y/x_0) \bigwedge_{1 \leq i \leq m} u_i \perp_{w_i} v_i \land \theta_0 \]  

(16)

and

\[ M \models Z(F_X/x_1) \bigwedge_{m+1 \leq i \leq m+n} u_i \perp_{w_i} v_i \land \theta_1. \]  

(17)

For (16) assume first that \( 1 \leq i \leq m \) and \( s, s' \in Y(F_Y/x_0) \) are such that \( s(w_i) = s'(w_i) \). By the definition of \( F_Y \), these assignments are extended by some \( t, t' \in X' \) such that \( t(r) = t(z_0) \) and \( t'(r) = t'(z_0) \). Atom \( = (z_0) \) holds in \( X' \), so \( t(r) = t'(r) \). Also \( u_i \perp_{w_i} v_i \) holds in \( X' \), so there is \( t'' \in X' \) such that \( t''(u_i w_i r) = t(u_i w_i r) \) and \( t''(v_i) = t'(v_i) \). Now also \( t''(r) = t''(z_0) \), so \( t'' \) extends some \( s'' \in Y(F_Y/x_0) \). Then \( s''(u_i w_i) = t''(u_i w_i) = t(u_i w_i) = s(u_i w_i) \) and \( s''(v_i) = t''(v_i) = t'(v_i) = s'(v_i) \), and hence \( s'' \) is as wanted.

Then let us show that \( M \models Y(F_Y/x) \theta_0 \). Consider this extension \( t \) of \( s \) such that \( t(r) = t(z_0) \). First notice that \( \alpha = \beta \) cannot hold in whole \( X' \) because \( \alpha \) and \( \beta \) were universally quantified and \( |M| > 1 \). Therefore, for some assignment in \( X' \), \( \neg z_0 = z_1 \) holds. But in \( X' \) \( z_0 \) and \( z_1 \) are constants, so \( \neg z_0 = z_1 \) holds in whole \( X' \). Hence \( t(r) \neq t(z_1) \), and so \( t \) belongs to the part of \( X' \) where \( \theta_0 \land r = z_0 \) holds. Therefore \( M \models s \theta_0 \), and because \( \theta_0 \) is first-order, we have by definition that \( M \models Y(F_Y/x_0) \theta_0 \).

The proof of (17) is analogous. Hence \( M \models X \phi_0 \lor \phi_1 \).

Notice that in the previous lemma parameters \( \alpha \) and \( \beta \) were needed only for the case \( |M| = 1 \). If we forget these trivial models, rule 6 can be simplified.

Before going to the soundness proof, we need the following lemma. Recall that the notation \( \phi(x_{i_1}/x_1) \ldots (x_{i_n}/x_n) \) presumes that none of the variables \( x_{i_1}, \ldots, x_{i_n} \) become bound in the substitution.

**Lemma 18** (Change of free variables). *Let the free variables of \( \phi \) be \( x_1, \ldots, x_n \). Let \( i_1, \ldots, i_n \) be distinct. If \( X \) is a team with \( \text{Dom}(X) = \{x_1, \ldots, x_n\} \), let \( X' \) consist of the assignments \( x_{i_j} \mapsto s(x_j) \) where \( s \in X \). Then*

\[ M \models X \phi \iff M \models X'. \phi(x_{i_1}/x_1) \ldots (x_{i_n}/x_n) \]

**Proof.** Easy induction on the complexity of the formula. □

**Proposition 19.** *Let \( T \cup \{\psi\} \) be a set of formulas of independence logic. If \( T \vdash X \psi \), then \( T \models \psi \).*

**Proof.** We will prove this claim by induction on the length of derivation. First notice that the previous lemmas provide the soundness of rules 3, 4, 6 and 7. Rules 1, 2, 9, 10, 11, 12, \( \land I \), \( \land E \), \( \lor I \) and \( \lor E \) are obviously sound. Also rules \( \lor E \), \( \neg I \), \( \forall I \), \( \exists I \) and \( \exists E \) are identical to the corresponding rules in the dependence logic case and the proof for these goes the same way as in [8]. Rule \( \forall E \) is restricted version of the corresponding dependence logic rule and also here the proof introduced in [8] suffices. We are then left to prove induction steps for rules 5 and 8 only.

**Rule 5** Assume that we have a natural deduction proof of \( \forall y B \) from the assumptions

\[ \{A_1, \ldots, A_k\} \]

with last rule 5. Let \( M \) and \( X \) be such that \( M \models X A_i \) for \( i = 1, \ldots, k \). By the assumption, we have a shorter proof of \( \forall x A \) from the assumptions \( \{A_1, \ldots, A_k\} \). Then by the induction
assumption, $M \models_{x} \forall x A$. Let $V = \text{Dom}(X) - \{x, y\}$ and $X' = X \mid V$. Variables $x$ and $y$
do not occur free in $\forall x A$, so also $M \models_{x} \forall x A$ and hence $M \models_{X(M/y)} A$. By Lemma 18, $M \models_{X(M/y)} A(y/x)$. Because $X'(M/y) = X(M/y) \mid (V \cup \{y\})$ and $x$ does not occur free in $A(y/x)$, we have that $M \models_{X(M/y)} A(y/x)$. Also by the assumption, we have a shorter proof of $B$ from the assumptions

$$\{A(y/x), A_{1}, \ldots, A_{i}\}$$

where $\{A_{1}, \ldots, A_{i}\} \subseteq \{A_{1}, \ldots, A_{k}\}$ and $y$ does not occur free in $A_{j}$ for $j = 1, \ldots, i$. Hence $M \models_{X(M/y)} A_{i}$ for $j = 1, \ldots, i$, so by the induction assumption, $M \models_{X(M/y)} B$. Hence $M \models_{X} \forall y B$.

Rule 8 Let

$$A = \forall x_{0,0} \exists y_{0,0} \exists x_{0,1} \exists y_{0,1} \ldots \exists x_{0,p} \exists y_{0,p}(\bigwedge_{1 \leq i \leq m} u_{0,0}^{i} \parallel w_{0,i}^{i} v_{0,0}^{i})$$

$$\land \bigwedge_{1 \leq i \leq p} x_{0,i} y_{0,i} \subseteq x_{0,0} y_{0,0} \land \bigwedge_{0 \leq i \leq p} C_{0,i} \land D\)$$

and

$$B = \forall x_{0,0} \exists y_{0,0} \exists x_{0,1} \exists y_{0,1} \ldots \exists x_{0,p} \exists y_{0,p}(\bigwedge_{0 \leq i \leq p} C_{0,i} \land D \land$$

$$\bigwedge_{0 \leq i \leq p'} x_{1,i} y_{1,i} \subseteq x_{1,-1} y_{1,-1} \land \bigwedge_{-1 \leq i \leq p'} C_{1,i} \land \bigwedge_{0 \leq i \leq p} x_{1,i} y_{1,i} = x_{0,i} y_{0,i} \land$$

$$\bigwedge_{-1 \leq j,k \leq p} (E_{1,j,k}^{i} \lor \bigvee_{p' \leq p} u_{1,j}^{i} v_{1,i,k}^{i} w_{1,i,j}^{i} = u_{1,j}^{i} v_{1,i,k}^{i} w_{1,i,j}^{i}))$$

be formulas where

- $x_{j,k}$'s are variable tuples of same length and $y_{j,k}$'s are variable tuples of same length.
- Variables in these tuples are quantified only once and do not occur free in formula $D$.
- $u^{i}$, $v^{i}$ and $w^{i}$ are tuples of variables from $y_{0,0}$.
- $C$ is a quantifier-free first-order formula with variables from $x_{0,0} y_{0,0}$.
- $e_{j,k}^{i} = e^{i}(x_{j,k} y_{j,k} / x_{0,0} y_{0,0})$, for $e \in \{u, v, w\}$.
- $C_{j,k}^{i} = C(x_{j,k} y_{j,k} / x_{0,0} y_{0,0})$.
- $\mathcal{E}_{n,j,k}^{i} = \{\parallel w_{n,j}^{i} = w_{n,k}^{i} \text{ if } w^{i} \text{ is empty}\}$
- Otherwise
- $p \geq 0$ and $p' = p + m(p + 2)^{2}$.

Assume that we have a natural deduction proof of $B$ from the assumptions

$$\{A_{1}, \ldots, A_{k}\}$$
with last rule 8 and last formula \( A \) preceding \( B \) in the proof. Let \( M \) and \( X \) be such that \( M \models X A_i \) for \( i = 1, \ldots, k \). By the assumption, we have a shorter proof of \( A \) from the assumptions \( \{ A_1, \ldots, A_k \} \). So by the induction assumption, \( M \models X A \). Also variables in tuples \( x_{j,k} \) and \( y_{j,k} \), for \( j = 0, 1 \) and \( k = -1, \ldots, p' \), do not occur free in \( A \) and \( B \), so we can without loss of generality assume that these variables are not in \( \text{Dom}(X) \). Let \( r \) and \( r' \) be the lengths of tuples \( x_{0,0} \) and \( y_{0,0} \), respectively. Then there is a function \( F : X(M'/x_{0,0}) \to \mathcal{P}(M^{(r+r'+r')}) \) such that if \( X' = X(M'/x_{0,0})(F/y_{0,0}x_{0,1} \ldots y_{0,p}) \), then

\[
M \models X' \bigwedge_{1 \leq i \leq m} u_{i,0}^i \downarrow w_{i,0}^i \wedge \bigwedge_{1 \leq i \leq p} x_{0,i}y_{0,i} \subseteq x_{0,0}y_{0,0} \wedge \bigwedge_{0 \leq i \leq p} C_{0,i} \wedge D. \tag{20}
\]

It suffices to show that

\[
M \models X' \forall x_{1,-1} \exists y_{1,-1} \exists x_{1,0} \exists y_{1,0} \ldots \exists x_{1,p'} \exists y_{1,p'} ( \bigwedge_{1 \leq i \leq m} u_{i,1,-1}^i \downarrow w_{i,1,-1}^i \wedge \bigwedge_{0 \leq i \leq p'} x_{1,i}y_{1,i} \subseteq x_{1,-1}y_{1,-1} \wedge \bigwedge_{-1 \leq i \leq p'} C_{1,i} \wedge \bigwedge_{0 \leq i \leq p} x_{1,i}y_{1,i} = x_{0,i}y_{0,i} \wedge \bigwedge_{1 \leq i \leq m} E_{1,i,k} \bigwedge_{p < l \leq p'} \bigwedge_{-1 \leq j,k \leq p} u_{1,j}^i v_{1,k}^i u_{1,l}^j = u_{1,j}^i v_{1,k}^i u_{1,l}^j ) .
\]

We will first define a function \( F' : X'(M'/x_{1,-1}) \to \mathcal{P}(M^{(p+1)(r+r')}) \) so that if \( Y = X'(M'/x_{1,-1})(F/y_{1,1}x_{1,0} \ldots y_{1,p}) \), then \( M \) and \( Y \) satisfy the part of the conjunction that have no variables from tuples \( x_{1,j}, y_{1,j} \), for \( j = p + 1, \ldots, p' \), occurring free in it. So we want that

\[
M \models Y \bigwedge_{1 \leq i \leq m} u_{i,1,-1}^i \downarrow w_{i,1,-1}^i \wedge \bigwedge_{0 \leq i \leq p} x_{1,i}y_{1,i} \subseteq x_{1,-1}y_{1,-1} \wedge \bigwedge_{-1 \leq i \leq p} C_{1,i} \wedge \bigwedge_{0 \leq i \leq p} x_{1,i}y_{1,i} = x_{0,i}y_{0,i}.
\]

Let \( s \in X' \) and \( a \in M' \). If \( s' = s \upharpoonright \text{Dom}(X) \), we let

\[
A_{1,-1}^y = \{ b \in M' \mid \exists c \in M^{p(r+r')} [bc \in F(s'(a/x_{0,0}))] \},
\]

\[
A_{1,i}^x = \{ s(x_{0,i}) \} \text{ for } i = 0, \ldots, p \text{, and}
\]

\[
A_{1,i}^y = \{ s(y_{0,i}) \} \text{ for } i = 0, \ldots, p.
\]

Then we let

\[
F'(s(a/x_{1,-1})) = A_{1,-1}^y \wedge A_{1,0}^x \wedge \cdots \wedge A_{1,p}^x \wedge A_{1,p}^y.
\]

By the construction, \( Y \upharpoonright x_{1,-1}y_{1,-1} \) consist of functions \( x_{1,-1} \mapsto s(x_{0,0}), y_{1,-1} \mapsto s(y_{0,0}) \) where \( s \in X' \upharpoonright x_{0,0}y_{0,0} \). By Lemma 18,

\[
M \models Y \bigwedge_{1 \leq i \leq m} u_{i,1,-1}^i \downarrow w_{i,1,-1}^i \wedge C_{1,-1}. \tag{21}
\]

By the construction,

\[
M \models Y \bigwedge_{0 \leq i \leq p} x_{1,i}y_{1,i} = x_{0,i}y_{0,i}, \tag{22}
\]

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and thus by the third conjunct of (20) and Lemma 18,

\[ M \models Y \bigwedge_{0 \leq i \leq p} C_{1,i}. \]

Also from (22) and the second conjunct of (20), it follows by the construction of \( Y \) that

\[ M \models Y \bigwedge_{0 \leq i \leq p} x_{1,i}y_{1,i} \subseteq x_{1,-1}y_{1,-1}. \]  

(23)

Now it suffices to show that there is \( G : Y \rightarrow \mathcal{P}(M(p'-p)(r+r')) \) such that if \( Y' = Y(G/x_1, p+1 \ldots x_1, p' y_1, p') \), then

\[ M \models Y' \bigwedge_{p+1 \leq i \leq p'} x_{1,i}y_{1,i} \subseteq x_{1,-1}y_{1,-1} \land \bigwedge_{p+1 \leq i \leq p'} C_{1,i} \land \bigwedge_{1 \leq i \leq m} \bigvee_{-1 \leq j, k \leq p} (E_{i,j,k}^i \lor \bigcup_{p<\ell \leq p'} u_{1,j,k}^i v_{1,j,k}^i w_{1,j,k}^i = u_{1,i}^i v_{1,i}^i w_{1,i}^i). \]

(24)

Let \( s \in Y \) and consider the last conjunct of the formula. Notice that the set \{ \((i, j, k) \mid 1 \leq i \leq m, -1 \leq j, k \leq p\) \} is of size \( p' - p \). Because of this and (23) and the first conjunct of (21), it is possible to extend \( s \) up to \( x_1, p' y_1, p' \) in such a way that

\[ \bigwedge_{1 \leq i \leq m} \bigvee_{-1 \leq j, k \leq p} (E_{i,j,k}^i \lor \bigcup_{p<\ell \leq p'} u_{1,j,k}^i v_{1,j,k}^i w_{1,j,k}^i = u_{1,i}^i v_{1,i}^i w_{1,i}^i). \]

holds. We just pick the values of \( x_1, y_{1,1} \), for \( p < l \leq p' \), from \( Y \upharpoonright x_{1,-1}y_{1,-1} \). This construction method guarantees that this extended assignment satisfies

\[ \bigwedge_{p+1 \leq i \leq p'} x_{1,i}y_{1,i} \subseteq x_{1,-1}y_{1,-1} \land \bigwedge_{p+1 \leq i \leq p'} C_{1,i} \]

also. Clearly the required \( G \) is now definable. We do not even need Axiom of Choice here because we can define \( G(s) \) as the set of tuples

\[ a_{p+1}b_{p+1} \ldots a_{p'}b_{p'} \in M(p'-p)(r+r') \]

where the assignment \( s(a_i b_i / x_1, y_{1,1}) \), for \( i = p+1, \ldots, p' \), satisfies (24) and the assignments \( t(x_1, -1) = a_i, t(y_1, -1) = b_i \), for \( i = p+1, \ldots, p' \), are in \( Y \upharpoonright x_{1,-1}y_{1,-1} \).

\( \square \)

5 The Completeness Theorem

In this section we will show that using our system of natural deduction we can derive all the first-order consequences of sentences of independence logic. Our proof is analogous to the proof of the corresponding dependence logic theorem in Kontinen and Väänänen [8] which in turn builds on the earlier work of Barwise [1] by using first-order approximations in the completeness proof.
5.1 The roadmap for the proof

1. First we will show that from any independence logic sentence $\phi$ it is possible to derive an equivalent sentence of the form

$$\phi' = \forall x \exists y (\bigwedge_{1 \leq i \leq m} u_i \perp w_i \land \theta)$$  \hspace{1cm} (25)$$

where $x$ and $y$ are tuples of variables where each variable is quantified only once; $u_i$, $v_i$ and $w_i$ are tuples of existentially quantified variables and $\theta$ is a quantifier-free first-order formula.

2. The sentence $\phi'$ can be shown to be equivalent, in countable models, to the game expression

$$\Phi := \bigwedge_{-n \leq i \leq p_n} \theta_{n,i} \land \bigwedge_{-n+1 \leq i \leq p_{n-1}} x_{n,i} y_{n,i} = x_{n-1,i} y_{n-1,i} \land$$

$$\bigwedge_{1 \leq i \leq m} \bigvee_{-n \leq j,k \leq p_{n-1}} (\pi_{n,j,k} \lor \bigvee_{p_{n-1} < l \leq p_n} u_{n,j}^i v_{n,k}^i w_{n,j}^i = u_{n,l}^i v_{n,l}^i w_{n,l}^i)$$

In the game expression, $\Psi^0 := \theta_{0,0}$, and for $n \geq 1$,

$$\Psi^n := \bigwedge_{-n \leq i \leq p_n} \theta_{n,i} \land \bigwedge_{-n+1 \leq i \leq p_{n-1}} x_{n,i} y_{n,i} = x_{n-1,i} y_{n-1,i} \land$$

$$\bigwedge_{1 \leq i \leq m} \bigvee_{-n \leq j,k \leq p_{n-1}} (\pi_{n,j,k}^i \lor \bigvee_{p_{n-1} < l \leq p_n} u_{n,j}^i v_{n,k}^i w_{n,j}^i = u_{n,l}^i v_{n,l}^i w_{n,l}^i)$$

where

- $x_{j,k}$ and $x$ are tuples of same length and $y_{j,k}$ and $y$ are tuples of same length such that each variable in these tuples is quantified only once,
- $\theta_{j,k} = \theta(x_{j,k} y_{j,k} / xy)$,
- $e_{n,j}^i = e_i(x_{n,j} y_{n,j} / x y)$ for $e \in \{u, v, w\}$,
- $\pi_{n,j,k}^i = \begin{cases} \perp & \text{if } w^i \text{ is empty} \\ -w_{n,j}^i = w_{n,k}^i & \text{otherwise} \end{cases}$,
- $p_0 = 0$ and $p_n = p_{n-1} + m(p_{n-1} + n + 1)^2$, for $n \geq 1$.

The idea behind the game expression is that at level $n$, $x_{n, -n} y_{n, -n}$ introduces a new tuple of $M$, tuples $x_{n,i} y_{n,i}$, for $i = -n + 1, \ldots, p_{n-1}$, copy all the tuples introduced at the previous level and tuples $x_{n,i} y_{n,i}$, for $i = p_{n-1} + 1, \ldots, p_n$, confirm that the independence atoms hold between all the tuples $x_{n,i} y_{n,i}$ and $x_{n,j} y_{n,j}$ where $-n \leq i, j \leq p_{n-1}$.
3. The game expression $\Phi$ can be approximated by the first-order formulas

$$\Phi^n := \forall x_0, 0 \exists y_0, 0 (\Psi^n \land$$

$$\forall x_1, -1 \exists y_1, -1 \exists x_1, 0 \exists y_1, 0 \ldots \exists x_{1,p_1} \exists y_1, p_1 (\Psi^1 \land$$

$$\forall x_2, -2 \exists y_2, -2 \exists x_2, -1 \exists y_2, -1 \ldots \exists x_{2,p_2} \exists y_2, p_2 (\Psi^2 \land$$

$$\ldots$$

$$\ldots$$

$$\forall x_{n,-n} \exists y_{n,-n} \exists x_{n,-n+1} \exists y_{n,-n+1} \ldots \exists x_{n,p_n} \exists y_{n,p_n} (\Psi^n) \ldots))$$

4. Then we will show that these approximations can all be deduced from $\phi'$.

5. Then we note that for recursively saturated (or finite) models $M$, it holds that

$$M \models \Phi \leftrightarrow \bigwedge_n \Phi^n.$$  

6. At last we show that for any $T \subseteq I$ and $\phi \in \text{FO}$:

$$T \models \phi \iff T \vdash \phi.$$ 

Suppose $T \not\models \phi$. If $T^*$ consist of the first-order approximations of sentences of $T$, then $T^* \not\models \phi$ and $T^* \cup \{\neg \phi\}$ is deductively consistent in first-order logic. Taking some countable recursively saturated model of $T^* \cup \{\neg \phi\}$, we have a model of $T \cup \{\neg \phi\}$ and hence $T \not\models \phi$.

**5.2 From $\phi$ to $\phi'$**

In this section we are going to prove that from $\phi$ one can derive an equivalent formula $\phi'$ of the form

$$\forall x \exists y (\bigwedge_{1 \leq i \leq m} u_i \perp v_i, v_i \land \theta)$$

(26)

where $x$ and $y$ are tuples of variables where each variable is quantified only once; $u_i$, $v_i$ and $w_i$ are tuples of existentially quantified variables and $\theta$ is a quantifier-free first-order formula.

**Proposition 27.** Let $\phi$ be a sentence of independence logic. Then $\phi \vdash \phi'$ where $\phi$ and $\phi'$ are logically equivalent and $\phi'$ is of the form (26).

**Proof.** We will prove the claim in several steps. Without loss of generality we may assume that in $\phi$ each variable is quantified only once.

**Step 1** We derive from $\phi$ an equivalent sentence in prenex normal form

$$Q^1 x_{i_1} \ldots Q^n x_{i_n} \theta$$

(28)

where $Q^i \in \{\exists, \forall\}$ and $\theta$ is a quantifier-free formula.

We will prove this for every formula $\phi$ satisfying the assumption made in the beginning of the proof and the assumption that no variable appears both free (if $\phi$ has free variables) and bound in the formula. Now if $\phi$ is atomic or first-order formula, then the claim clearly holds. (In the latter case we know that our deduction system covers the natural first-order deduction system.
and in that system we can derive an equivalent formula in prenex normal form.) Also the cases of
universal and existential quantifications are trivial. So we need only to consider the cases of
disjunction and conjunction. We prove these cases by simultaneous induction.

Assume \( \phi = \psi \lor \theta \) or \( \phi = \psi \land \theta \). By the induction assumption, we have derivations \( \psi \vdash I \psi^* \) and \( \theta \vdash I \theta^* \) where

\[
\psi^* = Q^1 x_{i_1} \ldots Q^n x_{i_n} \psi_0, \quad \theta^* = Q^{n+1} x_{i_{n+1}} \ldots Q^{n+m} x_{i_{n+m}} \theta_0,
\]

and \( \psi \equiv \psi^* \) and \( \theta \equiv \theta^* \). If \( \phi = \psi \lor \theta \), we can derive \( \psi^* \lor \theta^* \) from \( \phi \) using applications of rules 1 and 2. If \( \phi = \psi \land \theta \), we can derive \( \psi^* \land \theta^* \) from \( \phi \) using applications of rules \( \land I \) and \( \land E \).

Next we prove by induction on \( n \) that from \( \psi^* \land \theta^* \) we can derive an equivalent formula

\[
Q^1 x_{i_1} \ldots Q^n x_{i_n} Q^{n+1} x_{i_{n+1}} \ldots Q^{n+m} x_{i_{n+m}} (\psi_0 \land \theta_0)
\]

(29)

and from \( \psi^* \lor \theta^* \) we can derive an equivalent formula

\[
Q^1 x_{i_1} \ldots Q^n x_{i_n} Q^{n+1} x_{i_{n+1}} \ldots Q^{n+m} x_{i_{n+m}} (\psi_1 \lor \theta_1)
\]

(30)

where \( \psi_1 \) and \( \theta_1 \) are quantifier-free formulas. Let \( n = 0 \). We prove this case also by induction, this time on \( m \). For \( m = 0 \) the claim holds. Suppose \( m = k + 1 \) and the claim holds for \( k \). We consider only the case where the connective is \( \lor \) and \( Q^1 = \forall \). The other cases are analogous, except that they are a bit easier. The following deduction shows the claim:

1. \( \psi_0 \lor Q^1 x_{i_1} \ldots Q^m x_{i_m} \theta_0 \)
2. \( Q^1 x_{i_1} \ldots Q^m x_{i_m} \theta_0 \lor \psi_0 \) (rule 2)
3. \( Q^1 x_{i_1} ((Q^2 x_{i_2} \ldots Q^m x_{i_m} \theta_0 \land x_{i_1} \bot y) \lor \psi_0) \) (rule 3)
4. \( Q^1 x_{i_1} Q^2 x_{i_2} \ldots Q^m x_{i_m} (\psi_1 \lor \theta_1) \) (rule 5 and D1)

where D1 is the derivation

1. \( (Q^2 x_{i_2} \ldots Q^m x_{i_m} \theta_0 \land x_{i_1} \bot y) \lor \psi_0 \)
2. \( \psi_0 \lor (Q^2 x_{i_2} \ldots Q^m x_{i_m} \theta_0 \land x_{i_1} \bot y) \) (rule 2)
3. \( \psi_0 \lor Q^2 x_{i_2} \ldots Q^m x_{i_m} (x_{i_1} \bot y \lor \theta_0) \) (rule 1 and D2)
4. .
5. .
6. .
7. \( Q^2 x_{i_2} \ldots Q^m x_{i_m} (\psi_1 \lor \theta_1) \) (induction assumption)

where D2 is the derivation

1. \( Q^2 x_{i_2} \ldots Q^m x_{i_m} \theta_0 \land x_{i_1} \bot y \)
2. \( x_{i_1} \bot y \lor Q^2 x_{i_2} \ldots Q^m x_{i_m} \theta_0 \) (\( \land E \) and \( \land I \))
3. .
4. .
5. .
6. $Q^2x_1 \ldots Q^mx_m(x_i \perp y \land \theta_0)$ (induction assumption)

We can use the induction assumption in the deduction because $x_i$'s are all different from each other and none of them are in tuple $y$. This concludes the proof for the case $n = 0$.

Assume then that $n = l + 1$ and that the claim holds for $l$. We show the claim in the case where the connective is $\lor$ and $Q^1 = \lor$. The other cases are again analogous.

1. $Q^1x_1 \ldots Q^nx_n \psi_0 \lor Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0$
2. $Q^1x_1((Q^2x_2 \ldots Q^nx_n \psi_0 \land x_1 \perp y) \lor Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0)$ (rule 3)
3. $Q^1x_1 \ldots Q^{n+m}x_{n+m}(\psi_1 \lor \theta_1)$ (rule 5 and D3)

where D3 is the derivation

1. $(Q^2x_2 \ldots Q^nx_n \psi_0 \land x_1 \perp y) \lor Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0$
2. $Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0 \lor (Q^2x_2 \ldots Q^nx_n \psi_0 \land x_1 \perp y)$ (rule 2)
3. $Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0 \lor Q^2x_2 \ldots Q^nx_n(\psi_0 \land x_1 \perp y)$ (rule 1 and D4)
4. $Q^2x_2 \ldots Q^nx_n(\psi_0 \land x_1 \perp y) \lor Q^{n+1}x_{n+1} \ldots Q^{n+m}x_{n+m} \theta_0$ (rule 2)
5. 
6. 
7. 
8. $Q^2x_2 \ldots Q^{n+m}x_{n+m}(\psi_1 \lor \theta_1)$ (induction assumption)

where D4 is the derivation

1. $Q^2x_2 \ldots Q^nx_n \psi_0 \land x_1 \perp y$
2. 
3. 
4. 
5. $Q^2x_2 \ldots Q^nx_n(\psi_0 \land x_1 \perp y)$ (induction assumption)

This concludes the proof.

Step 2 Next we show that from a quantifier-free formula $\theta$ one can derive an equivalent formula of the form

$$\forall y_1 \ldots \forall y_l \exists y_{l+1} \ldots \exists y_{l+t'}(\bigwedge_{1 \leq i \leq m} u_i \bot w_i \land v_i \land \theta^*)$$

(31)

where $\theta^*$ is a quantifier-free first-order formula and $u_i, v_i$ and $w_i$ are tuples of existentially quantified variables. We do this by induction on the complexity of the formula. If $\theta$ is first-order formula, then the claim holds. Assume that $\theta = t \bot t'$ where $t, t$ and $t'$ are term tuples $(s_1, \ldots, s_k), (s_k+1, \ldots, s_{k+k'})$ and $(s_{k+k'+1}, \ldots, s_{k+k+\nu})$, respectively. Let $l = k + k' + \nu$.

Assume that $0 \leq n < l$ and we have already derived

$$\exists y_1 \ldots \exists y_n(t_n \bot t'_n \land y_1 = s_1 \land \ldots \land y_n = s_n)$$

(32)

where $t_i$ refers to the tuple $t(y_1/s_1) \ldots (y_i/s_i)$ and tuples $t'_i$ and $t''_i$ are defined analogously.

Let D5 be the derivation
1. \( t_n \land t'_{n+1} \land y_1 = s_1 \land \ldots \land y_n = s_n \)

2. \( t_n \land t'_{n+1} \land y_1 = s_1 \land \ldots \land y_n = s_n \land s_{n+1} = s_{n+1} \) (rules 9, 11 and \( \land \))

3. \( \exists y_{n+1}(t_{n+1} \land t'_{n+1} \land y_1 = s_1 \land \ldots \land y_n = s_n \land y_{n+1} = s_{n+1}) \) (\( \exists I \))

The last step can be done if we interpret the second formula as \( \phi(s_{n+1}/y_{n+1}) \) for

\[ \phi = t_{n+1} \land t'_{n+1} \land y_1 = s_1 \land \ldots \land y_n = s_n \land y_{n+1} = s_{n+1}. \]

Using \( n \) times rule \( \exists E \), once \( D_5 \) and \( n \) times rule \( \exists I \), we can derive

\[ \exists y_1 \ldots \exists y_{n+1} (t_{n+1} \land t'_{n+1} \land y_1 = t_1 \land \ldots \land y_{n+1} = s_{n+1}) \]

from (32).

So from \( \theta \) one can derive

\[ \exists y_1 \ldots \exists y_l (t_l \land t'_{l+1} \land y_1 = s_1 \land \ldots \land y_l = s_l) \] (33)

which is clearly equivalent to \( \theta \) and of the required form.

Assume then that \( \theta = \phi \lor \psi \). By the induction assumption, we have derivations \( \phi \vdash \phi^* \) and \( \psi \vdash \psi^* \) where

\[ \phi^* = \forall y_1 \exists y_2 (\bigwedge_{1 \leq i \leq m_1} u_i \land v_i \land \phi_0), \] (34)

\[ \psi^* = \forall y_1' \exists y_2' (\bigwedge_{1 \leq i \leq m_2} u_i' \land v_i' \land \psi_0) \] (35)

such that \( \phi \equiv \phi^* \), \( \psi \equiv \psi^* \), \( \phi_0 \) and \( \psi_0 \) are quantifier-free first-order formulas, \( y_i \) and \( y_i' \), for \( i = 1, 2 \), are tuples of bound variables such that none of these variables occur in both formulas or are quantified more than once, \( e_i \) is a tuple of existentially quantified variables for \( e \in \{u, v, w, u', v', w'\} \).

Now \( \theta \vdash \phi^* \lor \psi^* \). First we show by induction on the length of \( y_1 \) that from \( \phi^* \lor \psi^* \) one can derive an equivalent formula of the form

\[ \forall y_1 \forall y'_1 (\exists y_3 (\bigwedge_{1 \leq i \leq m_3} u_i \land v_i \land \phi_1) \lor \exists y_3' (\bigwedge_{1 \leq i \leq m_4} u_i' \land v_i' \land \psi_1)) \] (36)

where \( \phi_1 \) and \( \psi_1 \) are quantifier-free first-order formulas, \( y_3 \) and \( y_3' \) are tuples of bound variables such that none of these variables are quantified more than once or occur free in the formula, \( e_i \) is a tuple of existentially quantified variables for \( e \in \{u, v, w, u', v', w'\} \).

Assume first that \( \text{len}(y_1) = 0 \). We show this case by induction on the length of \( y'_1 \). The case \( \text{len}(y'_1) = 0 \) is clear. Suppose \( \text{len}(y'_1) = k + 1 \). Let \( y'_1 = xy_4 \) where \( \text{len}(y_4) = k \) and let \( y \) be a tuple listing the free variables in \( \phi^* \lor \psi^* \). The following deduction shows the claim.

1. \( \exists y_2 (\bigwedge_{1 \leq i \leq m_2} u_i \land v_i \land \phi_1) \lor \forall y'_1 \exists y_3 (\bigwedge_{1 \leq i \leq m_2} u_i' \land v_i' \land \psi_0) \)

2. \( \forall y'_1 \exists y_2 (\bigwedge_{1 \leq i \leq m_2} u_i' \land v_i' \land \psi_0) \lor \exists y_2 (\bigwedge_{1 \leq i \leq m_1} u_i \land v_i \land \phi_0) \) (rule 2)

3. \( \forall x (\forall y'_1 \exists y_2 (\bigwedge_{1 \leq i \leq m_2} u_i' \land v_i' \land \psi_0) \land x \land y) \lor \exists y_2 (\bigwedge_{1 \leq i \leq m_1} u_i \land v_i \land \phi_0) \) (rule 3)

4. \( \forall y'_1 (\exists y_3 (\bigwedge_{1 \leq i \leq m_3} u_i \land v_i \land \phi_1) \lor \exists y_3' (\bigwedge_{1 \leq i \leq m_4} u_i' \land v_i' \land \psi_1)) \) (rule 5 and D6)
where D6 is the derivation

1. \((\forall y'_1 \exists y'_2(\bigwedge_{1 \leq i \leq m_2} u'_i \perp w'_i \psi_i) \land \phi_0) \land x \perp y) \lor \exists y_2(\bigwedge_{1 \leq i \leq m_1} u_i \perp w_i \psi_i)\) (rule 2)
2. \(\exists y_2(\bigwedge_{1 \leq i \leq m_1} u_i \perp w_i \psi_i) \lor (\forall y'_1 \exists y'_2(\bigwedge_{1 \leq i \leq m_2} u'_i \perp w'_i \psi'_i) \land x \perp y)\) (rule 2)
3. \(\exists y_2(\bigwedge_{1 \leq i \leq m_1} u_i \perp w_i \psi_i) \lor \forall y'_4 \exists a \exists b(\bigwedge_{1 \leq i \leq m_2} u'_i \perp w'_i \psi'_i \land a \perp b \land \psi_0 \land ab = xy))\) (rule 1 and D7)

This concludes the proof of this case.

Suppose then \(\text{len}(y_1) = n + 1\). Let \(y_1 = xy_4\) where \(\text{len}(y_4) = n\) and let \(y\) be a tuple listing the free variables in \(\phi^* \lor \psi^*\). The following deduction shows the claim

1. \(\forall y_1 \exists y_2(\bigwedge_{1 \leq i \leq m_1} u_i \perp w_i \psi_i) \lor \forall y'_1 \exists y'_2(\bigwedge_{1 \leq i \leq m_2} u'_i \perp w'_i \psi'_i)\) (rule 2)
2. \(\forall x((\forall y_4 \exists y_2(\bigwedge_{1 \leq i \leq m_1} u_i \perp w_i \psi_i) \land x \perp y) \lor \forall y'_4 \exists y'_2(\bigwedge_{1 \leq i \leq m_2} u'_i \perp w'_i \psi'_i)))\) (rule 3)
3. \(\forall y_1 \forall y'_4(\bigwedge_{1 \leq i \leq m_3} u_i \perp w_i \psi_i) \lor \exists y'_3(\bigwedge_{1 \leq i \leq m_4} u'_i \perp w'_i \psi'_i))\) (rule 5 and D8)

This concludes the proof of this case.
5. 
6. 
7. 
8. \(\forall y \forall y' (\exists y_3 (\bigwedge_{1 \leq i \leq m_3} u_i \bot w_i v_i \land \phi_1) \lor \exists y_3' (\bigwedge_{1 \leq i \leq m_4} u'_i \bot w'_i v'_i \land \psi_1))\) (induction assumption)

where D9 is a derivation similar to D7. This concludes the claim.

Consider then the existential part of (36) which is the formula

\[\exists y_3 (\bigwedge_{1 \leq i \leq m_3} u_i \bot w_i v_i \land \phi_1) \lor \exists y_3' (\bigwedge_{1 \leq i \leq m_4} u'_i \bot w'_i v'_i \land \psi_1)\]  

(37)

With one application of rule 6 we can derive from (37) an equivalent formula \(\theta'\) of the form

\[\forall \alpha \forall \beta \exists y_3 \exists y_3' \exists z_0 \exists z_1 \exists r (\bigwedge_{1 \leq i \leq m_3} u_i \bot w_i v_i \land \bigwedge_{1 \leq i \leq m_4} u'_i \bot w'_i v'_i \land (z_i \land (\neg z_0 = z_1) \lor (\theta_0 \land r = z_0) \lor (\theta_1 \land r = z_1))).\]

So together we can derive from (36) an equivalent formula of the required form

\[\forall y_1 \forall y'_1 \theta'.\]

This concludes the proof of the case \(\theta = \phi \lor \psi\).

Suppose then \(\theta = \phi \land \psi\). By the induction assumption, \(\phi \vdash \phi^*\) and \(\psi \vdash \psi^*\) where \(\phi^*\) and \(\psi^*\) are as in (34) and (35). Now \(\theta \vdash \phi^* \land \psi^*\), and using rule 5 and the first-order rules for \(\exists\) and \(\land\), it is possible to derive from \(\phi^* \land \psi^*\) an equivalent formula of the required form

\[\forall y_1 \forall y'_1 \exists y_2 \exists y_2' (\bigwedge_{1 \leq i \leq m_1} u_i \bot w_i v_i \land \bigwedge_{1 \leq i \leq m_2} u'_i \bot w'_i v'_i \land (\phi_0 \land \psi_0)).\]

Remembering items (3') and (4') in Lemma 11, it is obvious that the formulas are equivalent. This concludes the proof of Step 2.

Step 3 The deductions in Step 1 and 2 (from \(\phi\) to (28) and from \(\theta\) to (31)) can be combined to show that

\[\phi \vdash Q^1 x_{i_1} \ldots Q^n x_{i_n} \forall y_1 \ldots \forall y_{l+1} \exists y_{l+1} \ldots \exists y_{l+v} (\bigwedge_{1 \leq i \leq m} u_i \bot w_i v_i \land \theta^*).\]  

(38)

Step 4 At last we can derive an equivalent formula of the form (26) from the formula (38) above. Using rule 7 we can swap the places of existential and universal quantifiers which sit next to each other. Every swap gives us some new independence atom which we can push to conjunction

\[\bigwedge_{1 \leq i \leq m} u_i \bot w_i v_i.\]

Pushing every universal quantifier in front of the formula and the new independence atoms to the quantifier-free part, we have a formula which is almost of the required form; every new
independence atom has still variables that are not existentially quantified. We omit the proof of
this part here because it is essentially the same than the proof of Step 4 in [8]. Only exceptions
are that rule 7 is the independence logic version of the similar dependence logic rule and in place
of $\forall E$ and $\forall I$ we use rule 5. After finishing this part we replace all the universally quantified
variables in these new independence atoms as existentially quantified variables. This can be done
easily just as we did it in Step 2 in the case of independence atoms.

Steps 1-4 show that from a sentence $\phi$ a logically equivalent sentence of the form (26) can be deduced.

5.3 Derivation of the approximations $\phi^n$

In the previous section we proved that from every sentence $\phi$ we can derive a logically equivalent
sentence of the form

$$\forall x \exists y (\bigwedge_{1 \leq i \leq m} u_i \bot w_i vi \land \theta)$$

(39)

where $x$ and $y$ are tuples of variables; $u_i$, $v_i$ and $w_i$ are tuples of existentially quantified variables and
$\theta$ is a quantifier-free first-order formula. Next we will show that the approximations $\Phi^n$ of the game
expression $\Phi$ corresponding to the sentence (39) can be deduced from it.

The formulas $\Phi$ and $\Phi^n$ are defined as follows.

Definition 40. Let $\phi$ be the formula (39). For $j,k \in \mathbb{Z}$ and $1 \leq k \leq m$, we let:

- $x$ and $x_{j,k}$ be variable tuples of same length and $y$ and $y_{j,k}$ be variable tuples of same length such
  that each variable occurs at most once in these tuples.

- $\theta_{j,k} = \theta(x_{j,k} y_{j,k} / xy)$ and $e^i_{j,k} = e_i(x_{j,k} y_{j,k} / xy)$ for $e \in \{u, v, w\}$.

- $\pi_{n,j,k} = \begin{cases} 
  \bot & \text{if } w_i \text{ is empty} \\
  -w^i_{n,j} = w^i_{n,k} & \text{otherwise}
\end{cases}$

- $p_0 = 0$ and $p_n = p_{n-1} + m(p_{n-1} + n + 1)^2$ for $n \geq 1$.

Also for $n \geq 1$, we let $\Psi^n$ be the following formula:

$$\bigwedge_{-n \leq i \leq -p_n} \theta_{n,i} \land \bigwedge_{-n+1 \leq i \leq -p_{n-1}} x_{n,i} y_{n,i} = x_{n-1,i} y_{n-1,i} \land$$

$$\bigwedge_{1 \leq i \leq m} \bigvee_{p_{n-1} < t \leq p_n} (\pi_{n,j,k} \lor x_{n,j} y_{n,k} w_{n,j}^i = u^i_{n,j} v_{n,k} w_{n,j}^i) \lor$$

In the case $n = 0$, we let $\Psi^0 = \theta_{0,0}$.

- The infinitary formula $\Phi$ is now defined as:

$$\forall x_{0,0} \exists y_{0,0} (\Psi^0 \land$$

$$\forall x_{1,-1} \exists y_{1,-1} \exists x_{1,0} \exists y_{1,0} \ldots \exists x_{1,p_1} \exists y_{1,p_1} (\Psi^1 \land$$

$$\forall x_{2,-2} \exists y_{2,-2} \exists x_{2,-1} \exists y_{2,-1} \ldots \exists x_{2,p_2} \exists y_{2,p_2} (\Psi^2 \land$$

$$
\ldots

\ldots

\ldots

\ldots)\ldots)\ldots)\ldots).$$

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The \( n \):th approximation \( \Phi^n \) of \( \phi \) is defined as:
\[
\forall x_0, 0 \exists y_0, 0 (\Psi^0) \land \\
\forall x_1, 1 \exists y_1, 1 \exists x_1, 0 \exists y_1, 0 \ldots \exists x_1, p_1 \exists y_1, p_1 (\Psi^1) \land \\
\forall x_2, 2 \exists y_2, 2 \exists x_2, 1 \exists y_2, 1 \ldots \exists x_2, p_2 \exists y_2, p_2 (\Psi^2) \land \\
\ldots \\
\ldots \\
\forall x_n, n \exists y_n, n \exists x_n, n+1 \exists y_n, n+1 \ldots \exists x_n, p_n \exists y_n, p_n (\Psi^n (\ldots)).
\]

Next we will show that \( \phi \vdash \phi \Omega^n \) for natural numbers \( n \).

**Theorem 41.** Let \( \phi \) and \( \Phi^n \) be as in definition (40). Then \( \phi \vdash \phi \Omega^n \) for all \( n \geq 0 \).

**Proof.** First let
\[
\Omega^n = \bigwedge_{1 \leq i \leq m} u^n_{i, n} \perp w^n_{i, n} \wedge \bigwedge_{-n+1 \leq i \leq p_n} x_n, i y_n, i \subseteq x_n, -n y_n, -n.
\]
Notice that
\[
\Omega^0 = \bigwedge_{1 \leq i \leq m} u^0_{i, 0} \perp w^0_{i, 0}.
\]
We will prove a bit stronger claim saying that \( \phi \vdash \Omega^n \) where \( \Omega^n \) is defined otherwise as \( \Phi^n \) except that in the last line we also have the formula \( \Phi^n \). So we let \( \Omega^n \) be of the form
\[
\forall x_0, 0 \exists y_0, 0 (\Psi^0) \land \\
\forall x_1, 1 \exists y_1, 1 \exists x_1, 0 \exists y_1, 0 \ldots \exists x_1, p_1 \exists y_1, p_1 (\Psi^1) \land \\
\forall x_2, 2 \exists y_2, 2 \exists x_2, 1 \exists y_2, 1 \ldots \exists x_2, p_2 \exists y_2, p_2 (\Psi^2) \land \\
\ldots \\
\ldots \\
\forall x_n, n \exists y_n, n \exists x_n, n+1 \exists y_n, n+1 \ldots \exists x_n, p_n \exists y_n, p_n (\Phi^n (\ldots)).
\]
It is not hard to see that we can deduce \( \Phi^n \) from \( \Omega^n \) so proving this claim suffices. We prove the claim by induction on \( n \). For \( n = 0 \) the claim holds, since \( \phi = \Phi^0 \).

Suppose then \( \phi \vdash \Omega^h \) and \( n = h + 1 \). By the induction assumption, it is enough to show that \( \Omega^n \) can be deduced from \( \Omega^h \). So let us first consider the last line of \( \Omega^h \) which is the following formula:
\[
\forall x_{h, -h} \exists y_{h, -h} \exists x_{h, -h+1} \exists y_{h, -h+1} \ldots \exists x_{h, p_n} \exists y_{h, p_n} (\Phi^h \land \Phi^h).
\]
(42)
If we interpret \( \theta \) as \( C \) and
\[
\bigwedge_{-h+1 \leq i \leq p_{h-1}} x_{h, i} y_{h, i} = x_{h, -1} y_{h, -1} \land \bigwedge_{1 \leq i \leq m} (\forall_{h, j, k} \land \\
\bigvee_{p_{h-1} < i \leq p_h} a_{h, j}^i v_{h, j}^i w_{h, j}^i = u_{h, j}^i v_{h, j}^i w_{h, j}^i)
\]

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as \( D \) (in the case \( h = 0 \) we interpret \( D \) as empty) we have that
\[
\Upsilon^h \land \Psi^h = \bigwedge_{1 \leq i \leq m} u^i_{h,-h} \downarrow w^i_{h,-h} v^i_{h,-h} \land \bigwedge_{-h+1 \leq i \leq p_h} x_{h,i} y_{h,i} \subseteq x_{h,-h} y_{h,-h} \land \bigwedge_{-h \leq i \leq p_h} C_{h,i} \land D.
\]

Now with one application of rule 8 we can derive from (42) the formula
\[
\forall x_{h,-h} \exists y_{h,-h} \exists x_{h,-h+1} \exists y_{h,-h+1} \ldots \exists x_{h,p_h} \exists y_{h,p_h} (\bigwedge_{-h \leq i \leq p_h} C_{h,i} \land D \land \\
\forall x_{n,-n} \exists y_{n,-n} \exists x_{n,-n+1} \exists y_{n,-n+1} \ldots \exists x_{n,p_n} \exists y_{n,p_n} (\bigwedge_{-n \leq i \leq p_n} C_{n,i} \land \\
\bigwedge_{-n+1 \leq i \leq p_n-1} x_{n,i} y_{n,i} = x_{n-1,i} y_{n-1,i} \land \\
\bigwedge_{-n \leq j,k \leq p_n-1} (E^j_{n,j,k} \land \\
\bigwedge_{p_n-1 < i \leq p_n} u^i_{n,j} v^i_{n,j} w^i_{n,j} = u^i_{n,j} v^i_{n,j} w^i_{n,j})).)
\]

But this formula is just the same than
\[
\forall x_{h,-h} \exists y_{h,-h} \exists x_{h,-h+1} \exists y_{h,-h+1} \ldots \exists x_{h,p_h} \exists y_{h,p_h} (\Psi^h \land \\
\forall x_{n,-n} \exists y_{n,-n} \exists x_{n,-n+1} \exists y_{n,-n+1} \ldots \exists x_{n,p_n} \exists y_{n,p_n} (\Upsilon^n \land \Psi^n)).
\]

Now we can easily derive \( \Omega^n \) from \( \Omega^h \). First we use repeatedly rules \( \exists E, \land E \) and the "elimination" part of rule 5 in order to reach the last line of \( \Omega^h \), namely (42), then from it we deduce (43) with one application of rule 8, and then for the reverse direction we use rules \( \exists I, \land I \) and the "introduction" part of rule 5 to get the formula \( \Omega^n \).

\[\square\]

6 Back from approximations

**Proposition 44.** Let \( \phi \) be as in (39) and \( \Phi \) as in (40). Then \( \phi \models \Phi \) and in countable models \( \Phi \models \phi \).

**Proof.** Assume that \( M \models \phi \). We show \( M \models \Phi \). The truth of \( \Phi \) in \( M \) means that there is a winning strategy for player II in the following game

\[
\begin{array}{cccccc}
I & a_{0,0} & a_{1,-1} & a_{2,-2} & a_{3,-3} & \ldots \\
H & b_{0,0} & b_{1,-1}a_{1,0}b_{1,0} & b_{1,-1}a_{1,0}b_{1,0} & b_{1,-1}a_{1,0}b_{1,0} & \ldots \\
\end{array}
\]

where \( a_{n,i}, b_{n,i} \) are tuples chosen from \( M \) and player II wins if the assignment \( s(x_{n,i}) = a_{n,i} \), \( s(y_{n,i}) = b_{n,i} \) satisfies \( \Psi^n \) in (40) for all \( n \).

Let \( x \) and \( y \) be tuples of sizes \( r \) and \( r' \), respectively. Since \( M \models \phi \), there is a function \( F : \{0\}(M'/x) \rightarrow \mathcal{P}(M') \) such that if \( X = \{0\}(M'/x)(F/y) \), then
\[
M \models X \bigwedge_{1 \leq i \leq m} u_i \downarrow w_i v_i \land \theta.
\]

(45)
We will now construct a winning strategy for player II recursively so that for each round \( n \) the assignment \( s(x) = a_{n,i}, s(y) = b_{n,i} \) is in \( X \).

- If \( n = 0 \) and player I has played \( a_{0,0} \), then player II chooses \( b_{0,0} \) to be any member of \( F(s) \) where \( s(x) = a_{0,0} \). The assignment \( s(x) = a_{0,0}, s(y) = b_{0,0} \) is in \( X \) and \( M \models X \theta \). Thus the assignment \( s(x_0) = a_{0,0}, s(y_0) = b_{0,0} \) satisfies \( \theta_{0,0} = \Psi^0 \).

- Suppose then \( n = h + 1 \) and tuples \( a_{h,i} \) and \( b_{h,i} \) have been played in the previous round successfully by player II and so that every assignment \( s(x) = a_{h,i}, s(y) = b_{h,i} \) is in \( X \). First player I chooses some assignment \( a_{n, -n} \). Then player II chooses \( b_{n, -n} \) to be some member of \( F(s) \), for \( s(x) = a_{n, -n} \), as above. Then II chooses \( a_{n,i} = a_{h,i} \) and \( b_{h,i} = b_{h,i} \) for \(-h \leq i \leq p_h \). By the construction and the assumption, the assignment \( s(x_{n,i}) = a_{n,i}, s(y_{n,i}) = b_{n,i} \) satisfies

\[
\bigwedge_{n + 1 \leq i \leq p_n} \theta_{n,i} \land \bigwedge_{-n + 1 \leq i \leq p_{n-1}} x_{n,i} = x_{n-1,i} \cup y_{n-1,i}.
\]

Now for each \( a_{n,i} b_{n,i} \) which have already been played i.e. the pairs with \(-n \leq i \leq p_h \), there is some assignment in \( X \) corresponding to it. So for each \( 1 \leq i \leq m \) and \(-n \leq j, k \leq p_h \), if \( s(w_{n,j}^i) = s(w_{n,k}^i) \) (or \( w_i \) is empty), then by (45), there is \( t \in X \) such that \( t(u_{i,w_i}) = s(w_{n,j}^i) \) and \( t(v_i) = s(v_{n,k}^i) \). The set

\[
\{(i, j, k) \mid 1 \leq i \leq m, -n \leq j, k \leq p_h\}
\]

is of size \( p_n - p_h \), so player II can play each remaining \( a_{n,i} \) and \( b_{n,i} \) as some \( t(x) \) and \( t(y) \) for some appropriate \( t \in X \) so that the formula

\[
\bigwedge_{1 \leq i \leq m} \bigvee_{-n \leq j, k \leq p_h} (\pi_{n,j,k} \land u_{n,k}^i v_{n,j}^i w_{n,j}^i = u_{n,i}^i v_{n,i}^i w_{n,i}^i)
\]

holds for the assignment \( s(x_{n,i}) = a_{n,i}, s(y_{n,i}) = b_{n,i} \). Then by (45) and the construction,

\[
\bigwedge_{p_h \leq t \leq p_n} \theta_{n,i}
\]

holds for \( s \) and thus \( M \models \Psi^p \).

Hence there is a winning strategy for player II.

Suppose then \( M \) is a countable model of \( \Phi \). We let \( a_{n, -n}, i < \omega \), be an enumeration of \( M^r \). We play the game \( G(M, \Phi) \) letting player I play the sequence \( a_{n, -n} \) as his \( n \)th move. Let \( s \) be the assignment determined by the play where player II follows her winning strategy. Let \( X \) be the team consisting of the assignments \( t(x) = s(x_{n,i}), t(y) = s(y_{n,i}) \), for \( n < \omega \), \(-n \leq i \leq p_n \). Every formula \( \theta_{n,i} \) holds for \( s \), so

\[
M \models X \theta.
\]

Suppose \( t, t' \in X \) and \( t(w_i) = t'(w_i) \) (or \( w_i \) is empty) for some \( 1 \leq i \leq m \). Then \( t \) and \( t' \) correspond to some \( a_{n,j} b_{n,j} \) and \( a_{n', k} b_{n', k} \). If \( h = \max \{n, n'\} + 1 \), then \( a_{n,j} b_{n,j} = a_{h,j} b_{h,j}, a_{n', k} b_{n', k} = a_{h,k} b_{h,k} \) and \(-h \leq j, k \leq p_{h-1} \). Because \( s \) satisfies the last conjunct of \( \Psi^h \) i.e. the formula

\[
\bigwedge_{1 \leq i \leq m} (\pi_{h,j,k} \land \bigvee_{-h \leq j, k \leq p_{h-1}} u_{h,k}^i v_{h,j}^i w_{h,j}^i = u_{h,i}^i v_{h,i}^i w_{h,i}^i),
\]

for
there is $t'' \in X$ corresponding to some $a_{b,h,l}b_{n,i}$ such that $t''(u_iw_i) = t(u_iw_i)$ and $t''(v_i) = t'(v_i)$. Hence

$$M \models X \bigwedge_{1 \leq i \leq m} u_i \perp_w v_i.$$ 

The team $X$ can now be presented as $\{\emptyset\}(M'/x)(F/y)$ for $F(t) = \{b_{n,i} \mid t(x) = a_{n,i}, n < \omega, -n \leq i \leq p_n\}$ where $F(t)$ is always non-empty for $t \in \{\emptyset\}(M'/x)$. Hence $M \models \phi$. □

Next we will define a concept of a recursively saturated model that will be important for our proof.

**Definition 47.** A model $M$ is recursively saturated if it satisfies

$$\forall \overline{x}(\bigwedge_n \exists y \bigwedge_{m \leq n} \phi_m(\overline{x}, y) \rightarrow \exists y \bigwedge_n \phi_n(\overline{x}, y))$$

whenever $\{\phi_n(\overline{x}, y) \mid n \in \mathbb{N}\}$ is recursive.

The following proposition is needed.

**Proposition 48 ([2]).** For every infinite model $M$, there is a recursively saturated countable model $M'$ such that $M \equiv M'$.

Over a recursively saturated model, we can replace the game expression $\Phi$ by a conjunction of its approximations $\Phi_n$.

**Proposition 49.** If $M$ is a recursively saturated (or finite) model, then

$$M \models \Phi \leftrightarrow \bigwedge_n \Phi_n.$$ 

*Proof.* The proof is analogous to the proof of Proposition 15 in [8]. □

**Corollary 50.** If $M$ is a countable recursively saturated (or finite) model, then

$$M \models \phi \leftrightarrow \bigwedge_n \Phi_n.$$ 

*Proof.* By propositions 44 and 49. □

Now we can prove the main result of this article.

**Theorem 51.** Let $T$ be a set of sentences of independence logic and $\phi \in FO$. Then

$$T \vdash_{I} \phi \iff T \models \phi.$$ 

*Proof.* Assume first that $T \not\vdash_{I} \phi$. Let $T^*$ consist of all the approximations of the independence sentences in $T$. Since the approximations are provable from $T$, we must have $T^* \not\vdash_{I} \phi$. Our deduction system covers all the first-order inference rules, so $T^* \not\vdash_{FO} \phi$ and thus $T^* \cup \{\neg \phi\}$ is deductively consistent in first-order logic. Let $M$ be a recursively saturated countable (or finite) model of this theory. By Corollary 50, $M \models T \cup \{\neg \phi\}$ and thus $T \not\models \phi$.

The other direction follows from Proposition 19. □
7 An example and open questions

In this section we go through an example and a couple of open questions regarding this topic.

Example 52. This is an example of using independence introduction rule in a context of uniformly continuous functions.

1. For every $\epsilon > 0$ there is $\delta > 0$ such that for every $x, y$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

2. Therefore, for every $\epsilon > 0$ and $x$ there is $\delta > 0$ such that $x$ and $\delta$ are independent of each other for fixed $\epsilon$, and for every $y$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

At the end we have some open questions.

- Is it possible to generalize our axiomatization so that it would cover all the first-order consequences of independence logic formulas? If we want to use first-order approximations in our proof, we would perhaps want to construct these approximations in a way that they would not have any new relation symbols in them.

- Suppose we have only independence atoms of the form $t_1 \perp t_2$ in our syntax. Is there a similar deductive system for this syntactical restriction? Very recently Galliani has showed in his notes that this non-relativized independence logic is expressively as strong as the relativized version.

- Our deduction system is still relatively weak. Can we somehow improve it in order to get for example all the atomic consequences of independence atomic formulas? In principle this should be possible because these independence atoms are essentially first-order expressions.

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