On initial inverse problem for nonlinear couple heat with Kirchhoff type

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Abstract
The main objective of the paper is to study the final model for the Kirchhoff-type parabolic system. Such type problems have many applications in physical and biological phenomena. Under some smoothness of the final Cauchy data, we prove that the problem has a unique mild solution. The main tool is Banach’s fixed point theorem. We also consider the non-well-posed problem in the Hadamard sense. Finally, we apply truncation method to regularize our problem. The paper is motivated by the work of Tuan, Nam, and Nhat [Comput. Math. Appl. 77(1):15–33, 2019].

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1 Introduction
In this paper, we consider the following Kirchhoff-type problem for parabolic equation systems:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) \Delta u, & (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial t} &= \mathcal{L}(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) \Delta v, & (x, t) \in \Omega \times (0, T), \\
u(x, t) &= v(x, t) = 0, & x \in \partial \Omega, t \in (0, T],
\end{align*}
\]

(1.1)

with the following terminal condition

\[
u(x, T) = f(x), \quad v(x, T) = g(x), \quad x \in \Omega,
\]

(1.2)

where \(f, g \in L^2(\Omega) \times L^2(\Omega)\) is the Cauchy terminal data, and \(\mathcal{L}\) is defined in Sect. 2. In recent years, partial differential equations concerning Kirchhoff terms have practical applications in continuum mechanics, phase transition phenomena, and population dynamics and attracted many authors; see, for example, [1–9].

The parabolic equations with nonlocal diffusion arise in a variety of physical and biological applications; see, for example, [10–15] and the references therein. To study interactions of two or more biological species, systems of parabolic equations have been proposed. For
example, Almeida [16] studied the following system of two population densities \( u, v \):

\[
\begin{aligned}
&u_t - D_1 (\ell_1(u)(t), \ell_2(v)(t)) \Delta u + \lambda_1 |u|^{p-2} u = F(x, t), \quad (x, t) \in Q_T, \\
v_t - D_2 (\ell_3(u)(t), \ell_4(v)(t)) \Delta v + \lambda_2 |v|^{p-2} v = G(x, t), \quad (x, t) \in Q_T, \\
&u(x, t) = 0, \quad v(x, t) = 0, \quad (x, t) \in S_T, \\
&u(x, 0) = u_0, \quad v(x, 0) = v_0, \quad x \in \Omega,
\end{aligned}
\]

where the death in species \( u \) is proportional to \( |u|^{p-2} u \) by the factor \( \lambda_1 \), the death in species \( v \) is proportional to \( |v|^{p-2} v \) by the factor \( \lambda_2 \), and \( f_1, f_2 \) are the supplies of external sources. The author obtained the results on the existence, uniqueness, and long-time behavior of a smooth global solution of the system. Ferreira [17] also proved the well-posedness of the system of nonlocal reaction–diffusion equations with both homogeneous Dirichlet or Neumann boundary conditions.

Since model (1.1) is a system having a gradient element, perhaps, the techniques for this problem are more complex. To the best of our knowledge, there is no result concerning problem (1.1)–(1.2). The current main applications of backward in time parabolic equations are hydrological inversion and image processing. The parabolic equation with terminal conditions plays an important role in physics and engineering, especially with thermal conductivity dependent on both time and space. We refer the reader to some interesting papers [18–21]. Our paper is motivated by the recent results of Baleanu et al. [21], Nam et al. [19], and Tuan et al. [20]. The techniques of this paper are based on the previous paper [22].

The main tool in the paper is the Fourier series technique in \( H^p \) spaces, combined with Banach’s fixed point theorem. The main and novel contributions of the paper are as follows:

- The first contribution result is the proof of the existence and uniqueness of a solution of our backward problem. To this end, we had to increase the smoothness properties of the input Cauchy data.
- The second result is showing that our backward problem is ill-posed in the Hadamard sense. Furthermore, we also regularize our inverse problem using the Fourier truncation method. We then obtain an error assessment between the regularized and exact solutions.

This paper is organized as follows. In Sect. 2, we introduce some preliminaries and the mild solution of problem (1.1)–(1.2). Using the Banach fixed point theorem, we show that our problem has a unique mild solution. In Sect. 3, we prove the ill-posedness of our problem. Applying the Fourier truncation method, we give a stability estimate of logarithmic type between the regularized and exact solutions.

2 Some preliminaries and the mild solution of problem (1.1)–(1.2)

In this section, we introduce some properties of the eigenvalues of the operator \(-\Delta\); see, for example, [6]. We have the equality

\[
-\Delta e_n(x) = -\lambda_n e_n(x), \quad x \in \Omega; \quad e_n(x) = 0, \quad x \in \partial \Omega, n \in \mathbb{N},
\] (2.1)
where \( \{\lambda_n\}_{n=1}^{\infty} \) is the set of eigenvalues of \(-\Delta\) satisfying

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\]  
(2.2)

and \( \lim_{n \to \infty} \lambda_n = \infty \). Let us recall the following Hilbert scale space for \( \nu > 0 \):

\[
H^\nu(\Omega) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\nu} \langle \psi, e_n \rangle^2 < +\infty \right\},
\]  
(2.3)

associated with the norm

\[
\|u\|_{H^\nu(\Omega)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2\nu} |\langle u, e_n \rangle|^2 \right)^{\frac{1}{2}}.
\]  
(2.4)

Let \( L \in C^1(\mathbb{R}^2) \) be a function such that:

- There exist two positive constants \( M_0, M_1 \) such that
  \[
  M_0 \leq L(z_1, z_2) \leq M_1 \quad \forall (z_1, z_2) \in \mathbb{R}^2.
  \]  
(2.5)

- There exists a positive constant \( K_l > 0 \) such that
  \[
  \|L(z_1, z_2) - L(\bar{z}_1, \bar{z}_2)\| \leq K_l (|z_1 - \bar{z}_1| + |z_2 - \bar{z}_2|).
  \]  
(2.6)

Due to conditions (2.5) and (2.6), we have the following lemma.

**Lemma 2.1** If \( u_1, u_2, v_1, v_2 \) belong to the space \( H^1(\Omega) \), then

\[
L(\|\nabla u_1\|_{L^2(\Omega)}, \|\nabla u_2\|_{L^2(\Omega)}) - L(\|\nabla v_1\|_{L^2(\Omega)}, \|\nabla v_2\|_{L^2(\Omega)}) \leq K_1 (\|\nabla (u_1 - v_1)\|_{L^2(\Omega)} + \|\nabla (u_2 - v_2)\|_{L^2(\Omega)}).
\]  
(2.7)

### 3 The existence and ill-posedness of our backward problem

Let us first investigate the formula of a mild solution of problem (1.1)–(1.2). Multiplying both sides of the main equation of problem (1.1)–(1.2) by \( e_n \) and integrating by parts, we get that

\[
\begin{align*}
\frac{d}{dt} \langle u, e_n \rangle &= \langle L(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) \Delta u, e_n \rangle, \\
\frac{d}{dt} \langle v, e_n \rangle &= \langle L(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) \Delta u, e_n \rangle.
\end{align*}
\]  
(3.1)

This equality immediately gives that

\[
\begin{align*}
\frac{d}{dt} \langle u, e_n \rangle &= \lambda_n \langle L(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) u, e_n \rangle, \\
\frac{d}{dt} \langle v, e_n \rangle &= \lambda_n \langle L(\|\nabla u\|_{L^2}, \|\nabla v\|_{L^2}) v, e_n \rangle.
\end{align*}
\]  
(3.2)

It is easy to see that system (3.2) allows us to get that the equalities

\[
u_n(t) = \exp\left( -\lambda_n \int_0^t L(\|\nabla u(\cdot, \tau)\|_{L^2}, \|\nabla v(\cdot, \tau)\|_{L^2}) d\tau \right) u_n(0)
\]  
(3.3)
and
\[ v_n(t) = \exp\left(-\lambda_n \int_0^t \mathcal{L}\left(\|\nabla u(\tau, t)\|_{L^2}, \|\nabla v(\tau, t)\|_{L^2}\right) d\tau\right) v_n(0). \]  
(3.4)

Due to the terminal condition (1.2), we get that
\[ u_n(T) = (f, e_n), \quad v_n(T) = (g, e_n). \]

By a simple calculation we get the Fourier coefficients of \( u \) and \( v \):
\[ u_n(t) = \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u(\tau, t)\|_{L^2}, \|\nabla v(\tau, t)\|_{L^2}\right) d\tau\) (f, e_n), \]
\[ v_n(t) = \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u(\tau, t)\|_{L^2}, \|\nabla v(\tau, t)\|_{L^2}\right) d\tau\) (g, e_n), \]
(3.5)

which allow us to get that
\[ u(x, t) = \sum_{n=1}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u(\tau, s)\|_{L^2}, \|\nabla v(\tau, s)\|_{L^2}\right) ds\) (f, e_n) e_n(x) \]
(3.6)

and
\[ v(x, t) = \sum_{n=1}^{\infty} \exp\left(\lambda_n \int_t^T \mathcal{L}\left(\|\nabla u(\tau, s)\|_{L^2}, \|\nabla v(\tau, s)\|_{L^2}\right) ds\) (g, e_n) e_n(x). \]
(3.7)

**Theorem 3.1** Let the Cauchy terminal data \((f, g) \in L^2(\Omega) \times L^2(\Omega)\) be such that
\[ \sum_{n=1}^{\infty} \lambda_n^2 \mathcal{L}^{2T^2} \lambda_n^2 = B_f, \quad \sum_{n=1}^{\infty} \lambda_n^2 \mathcal{L}^{2T^2} \lambda_n^2 = B_g \]
(3.8)

for two constants \(B_f, B_g > 0\). Then problem (1.1) has a mild solution on the space
\[ (L^\infty(0, T; H^1(\Omega)))^2. \]

**Proof** To show the existence of a mild solution, we define the operator \(Q(u, v)(t) = (Q_1(u, v)(t), Q_2(u, v)(t))\) and show that \(Q\) has a fixed point in the space \((L^\infty(0, T; H^1(\Omega)))^2\).

Here the operators \(Q_1\) and \(Q_2\) are defined as follows:
\[
\begin{cases}
Q_1(u, v)(t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u(\tau, s)\|_{L^2}, \|\nabla v(\tau, s)\|_{L^2}) ds) (f, e_n) e_n(x), \\
Q_2(u, v)(t) = \sum_{n=1}^{\infty} \exp(\lambda_n \int_t^T \mathcal{L}(\|\nabla u(\tau, s)\|_{L^2}, \|\nabla v(\tau, s)\|_{L^2}) ds) (g, e_n) e_n(x).
\end{cases}
\]
(3.9)

We will prove by induction that if \((u_1, v_1) \in (L^\infty(0, T; H^1(\Omega)))^2\) and \((u_2, v_2) \in (L^\infty(0, T; H^1(\Omega)))^2\), then
\[
\|Q^m(u_1, v_1)(\cdot, t) - Q^m(u_2, v_2)(\cdot, t)\|_{H^1(\Omega)}^2 \\
\leq \frac{(T - t)^m}{m!} (2C(B_f + B_g)K_f^2)^m \|u_1, v_1 - u_2, v_2\|_{(L^\infty(0, T; H^1(\Omega)))^2}^2.
\]
(3.10)
For $m = 1$, using the inequality $(c + d)^2 \leq 2c^2 + 2d^2$, we have

$$
\|Q^m(u_1, v_1)(\cdot, t) - Q^m(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
\leq 2\|Q^m(u_1, v_1)(\cdot, t) - Q^m(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
+ 2\|Q^m_2(u_1, v_1)(\cdot, t) - Q^m_2(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)},
$$

(3.11)

Applying Lemma 2.1 and the inequality $|e^r - e^q| \leq |r - q| \max(e^r, e^q)$ for $r, q \in \mathbb{R}$, we have

$$
\|Q_1(u_1, v_1)(\cdot, t) - Q_1(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
= \sum_{n=1}^{\infty} \lambda_n \left[ \exp \left( \lambda_n \int_t^T \mathcal{L} \left( \|\nabla u_1(s)\|_{L^2}, \|\nabla v_1(s)\|_{L^2} \right) ds \right) \\
- \exp \left( \lambda_n \int_t^T \mathcal{L} \left( \|\nabla u_2(s)\|_{L^2}, \|\nabla v_2(s)\|_{L^2} \right) ds \right) \right]^2 \langle f, e_n \rangle^2 \\
\leq \sum_{n=1}^{\infty} \lambda_n \left[ \int_t^T \mathcal{L} \left( \|\nabla u_1(s)\|_{L^2}, \|\nabla v_1(s)\|_{L^2} \right) ds \\
- \int_t^T \mathcal{L} \left( \|\nabla u_2(s)\|_{L^2}, \|\nabla v_2(s)\|_{L^2} \right) ds \right]^2 \exp(2(T-t)\lambda_n) \langle f, e_n \rangle^2 \\
\leq B_1K_1^2 \left( \int_t^T \|\nabla (u_1 - u_2)(s)\|_{L^2(\Omega)} ds + \int_t^T \|\nabla (v_1 - v_2)(s)\|_{L^2(\Omega)} ds \right)^2 \\
\leq CB_1K_1^2 \int_t^T \|u_1(s) - u_2(s)\|^2_{H^1(\Omega)} ds,
$$

(3.12)

where in the above line, we applied the inequality $\|\nabla \psi\|_{L^2(\Omega)} \leq C\|\psi\|_{H^1(\Omega)}$. In a similar way, we get that

$$
\|Q_2(u_1, v_1)(\cdot, t) - Q_2(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
\leq CB_2K_2^2 \int_t^T \|u_1(s) - u_2(s)\|^2_{H^1(\Omega)} ds.
$$

(3.13)

Combining (3.11), (3.12), and (3.13), we deduce that

$$
\|Q(u_1, v_1)(\cdot, t) - Q(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
\leq 2(CB_1 + B_2)K_1^2(T - t)\|(u_1, v_1) - (u_2, v_2)\|_{L^\infty(0,T;H^1(\Omega))^2}.
$$

(3.14)

Let (3.10) hold for $m = p$. We will show that (3.10) holds for $m = p + 1$. Indeed, we have

$$
\|Q^{p+1}_1(u_1, v_1)(\cdot, t) - Q^{p+1}_1(u_2, v_2)(\cdot, t)\|^2_{H^1(\Omega)} \\
= \|Q_1(Q^p(u_1, v_1)(\cdot, t)) - Q_1(Q^p(u_2, v_2)(\cdot, t))\|^2_{H^1(\Omega)} \\
\leq CB_1K_1^2 \int_t^T \|Q^p(u_1, v_1)(\cdot, s) - Q^p(u_2, v_2)(\cdot, s)\|^2_{H^1(\Omega)} ds
$$
and
\[
\left\| Q_n^{p+1}(u_1, v_1)(t) - Q_n^{p+1}(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 = \left\| Q_n^p(u_1, v_1)(t) - Q_n^p(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 \leq C B_k K_1^2 \int_t^T \left\| Q_n^p(u_1, v_1)(s) - Q_n^p(u_2, v_2)(s) \right\|_{H^1(\Omega)}^2 ds.
\]

From two above observations we find that
\[
\left\| Q_n^{p+1}(u_1, v_1)(t) - Q_n^{p+1}(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 \leq 2 \left\| Q_n^{p+1}(u_1, v_1)(t) - Q_n^{p+1}(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 + 2 \left\| Q_n^{p+1}(u_1, v_1)(t) - Q_n^{p+1}(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 \\
\leq 2C(B_f + B_g)K_1^2 \int_t^T \left\| Q_n^p(u_1, v_1)(s) - Q_n^p(u_2, v_2)(s) \right\|_{H^1(\Omega)}^2 ds.
\]

By the induction assumption on (3.10), from (3.15) it follows that
\[
\left\| Q_n^{m+1}(u_1, v_1)(t) - Q_n^{m+1}(u_2, v_2)(t) \right\|_{H^1(\Omega)}^2 \leq \frac{(T-t)^{m+1}}{(m+1)!} \left(2C(B_f + B_g)K_1^2\right)^{m+1} \left\| (u_1, v_1) - (u_2, v_2) \right\|_{L^\infty(0, T; H^1(\Omega))^2}^2.
\]

Hence (3.10) holds for any positive integer \(m\). As a consequence, we derive that
\[
\left\| Q_n^m(u_1, v_1) - Q_n^m(u_2, v_2) \right\|_{L^\infty(0, T; H^1(\Omega))^2}^2 \leq \frac{T^m(2C(B_f + B_g)K_1^2)^m}{m!} \left\| (u_1, v_1) - (u_2, v_2) \right\|_{L^\infty(0, T; H^1(\Omega))^2}^2.
\]

Since
\[
\frac{T^m(2C(B_f + B_g)K_1^2)^m}{m!} \to 0, \quad m \to \infty,
\]

there exists a positive constant \(m_0\) such that the term \(\frac{T^m(2C(B_f + B_g)K_1^2)^m}{(m_0)^m} < 1\). Using the Banach fixed point theorem, we conclude that \(Q^m\) has a fixed point \((u^*, v^*)\) on the space \((L^\infty(0, T; H^1(\Omega)))^2\). It is easy to get that \((u^*, v^*)\) is also a solution of the equation \(Q(u, v) = (u, v)\).

**Theorem 3.2** Problem (1.1) is ill-posed in the sense of Hadamard.

**Proof** Let us illustrate by an example that the solution of problem (1.1) is not stable according to the input data. We take the input Cauchy data \(f_m, g_m\) with
\[
f_m(x) = g_m(x) = \frac{\phi_m(x)}{\lambda_m}.
\]
for natural \( m \geq 1 \). It is easy to check that
\[
\sum_{n=1}^{\infty} \lambda_m^3 e^{2T\lambda_m} (f_n, e_n)^2 = \sum_{n=1}^{\infty} \lambda_m^3 e^{2T\lambda_m} (g_n, e_n)^2 = \lambda_m^2 e^{2T\lambda_m}.
\] (3.18)

Under the Cauchy terminal data \((f_m, g_m)\) as above, by Theorem (3.1) we get that problem (1.1) has a mild solution \((u_m, v_m) \in (L^\infty(0, t; H^1(\Omega))^2 \), which is given by
\[
\begin{align*}
\{ u_m(x, t) &= \sum_{n=1}^{\infty} \exp(\lambda_n) \int_0^T \mathcal{L}(\| \nabla u_m(\cdot, s) \|_{L^2}, \| \nabla v_m(\cdot, s) \|_{L^2}) ds \} (f_n, e_n) e_n(x), \\
v_m(x, t) &= \sum_{n=1}^{\infty} \exp(\lambda_n) \int_0^T \mathcal{L}(\| \nabla u_m(\cdot, s) \|_{L^2}, \| \nabla v_m(\cdot, s) \|_{L^2}) ds \} (g_n, e_n) e_n(x).
\end{align*}
\] (3.19)

Recalling the lower bound of \( \mathcal{L} \), we derive that for any \( 0 \leq t \leq T \),
\[
\| u_m(\cdot, t) \|_{L^2(\Omega)}^2 \geq \frac{\exp(2\lambda_m) M_0(T-t)}{\lambda_m^2}.
\] (3.20)

By a similar argument we also obtain that
\[
\| v_m(\cdot, t) \|_{L^2(\Omega)}^2 \geq \frac{\exp(2\lambda_m) M_0(T-t)}{\lambda_m^2}.
\] (3.21)

From two recent observations we arrive at
\[
\| (u_m, v_m) \|_{(L^\infty(0, T; H^1(\Omega))^2) \geq \frac{\exp(\lambda_m) M_0(T-t)}{\lambda_m} = \frac{\exp(\lambda_m) M_0 T}{\lambda_m}.
\]

Taking the limits as \( m \to +\infty \), we get
\[
\lim_{m \to +\infty} \| (f_m, g_m) \|_{L^2(\Omega)} = \lim_{m \to +\infty} \frac{1}{\lambda_m} = 0,
\]
\[
\lim_{m \to +\infty} \| (u_m, v_m) \|_{(L^\infty(0, T; H^1(\Omega))^2) = \lim_{m \to +\infty} \frac{\exp(\lambda_m) M_0 T}{\lambda_m} = +\infty.
\]

Therefore, problem (1.1) is ill-posed in the sense of Hadamard.

Remark 3.1 We expect to extend our model to noninteger derivatives according to papers [23–30].

4 Fourier truncation method and error estimate

Let us define the regularized solution by the Fourier truncation method as
\[
\begin{align*}
\{ u^{N, \lambda}(x, t) &= \sum_{n=1}^{N} \exp(\lambda_n) \int_0^T \mathcal{L}(\| \nabla u^{N, \lambda}(\cdot, s) \|_{L^2}, \| \nabla v^{N, \lambda}(\cdot, s) \|_{L^2}) ds \} (f_n, e_n) e_n(x), \\
v^{N, \lambda}(x, t) &= \sum_{n=1}^{N} \exp(\lambda_n) \int_0^T \mathcal{L}(\| \nabla u^{N, \lambda}(\cdot, s) \|_{L^2}, \| \nabla v^{N, \lambda}(\cdot, s) \|_{L^2}) ds \} (g_n, e_n) e_n(x),
\end{align*}
\] (4.1)
where \( N := N(\delta) \) is a regularization parameter. Here the function \((f^3, g^3) \in L^2(\Omega) \times L^2(\Omega)\) satisfies
\[
\|f^3 - f\|_{L^2(\Omega)} + \|g^3 - f\|_{L^2(\Omega)} \leq \delta. \tag{4.2}
\]

**Theorem 4.1** Let \( f \in L^2(\Omega) \) be such that
\[
\sum_{n=1}^{\infty} (f, e_n)^2 \lambda_n^{1+\delta} e^{2M_1 T\lambda_n} \leq E
\]
for some constants \( E > 0 \) and \( \delta > 0 \). Let us choose \( N := N(\delta) \) such that
\[
\lim_{\delta \to 0} N(\delta) = +\infty, \quad \lim_{\delta \to 0} \delta^2 \lambda_N e^{2M_1 \lambda_N} = 0.
\]

Then we have the estimate
\[
\|u^{N\delta}(\cdot, t) - u(\cdot, t)\|_{H^1(\Omega)}^2 + \|v^{N\delta}(\cdot, t) - v(\cdot, t)\|_{H^1(\Omega)}^2 \leq (6\lambda_N \exp(2M_1 T\lambda_N)\delta^2 + 6E(N(\delta))\exp(6\lambda_N \delta) EC(T - t)). \tag{4.3}
\]

**Remark 4.1** It is obvious that \( \lambda_N \sim N^{\frac{\delta}{2}} \). So we can choose a natural number \( N \) such that
\[
\frac{1 - \nu}{2TM_1} \ln \left( \frac{1}{\delta} \right) \leq \lambda_N \leq \frac{1 - \nu}{MB_1} \ln \left( \frac{1}{\delta} \right), \quad 0 < \nu < 1.
\]

Then the error \( \|u^{N\delta}(\cdot, t) - u(\cdot, t)\|_{H^1(\Omega)}^2 + \|v^{N\delta}(\cdot, t) - v(\cdot, t)\|_{H^1(\Omega)}^2 \) is of logarithmic order
\[
\max \left( \left[ \ln \left( \frac{1}{\delta} \right) \right]^{-2} \delta^{2\nu} \ln \left( \frac{1}{\delta} \right) \right).
\]

**Proof** To show the existence of a mild solution, we define the operator \( R_m(u, v)(t) = (R_{1,\delta}(u, v)(t), R_{2,\delta}(u, v)(t)) \) and show that \( R_m \) has a fixed point in the space \((L^\infty(0, T; H^1(\Omega)))^2\). Here the operators \( R_{1,\delta} \) and \( R_{2,\delta} \) are defined as follows:
\[
\begin{aligned}
R_{1,\delta}(u, v)(t) &= \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_0^T \mathcal{L}(\|\nabla u^{N\delta}(\cdot, s)\|_{L^2}, \|\nabla v^{N\delta}(\cdot, s)\|_{L^2}) ds) (f^3, e_n) e_n(x), \\
R_{2,\delta}(u, v)(t) &= \sum_{n=1}^{N(\delta)} \exp(\lambda_n \int_0^T \mathcal{L}(\|\nabla u^{N\delta}(\cdot, s)\|_{L^2}, \|\nabla v^{N\delta}(\cdot, s)\|_{L^2}) ds) (g^3, e_n) e_n(x).
\end{aligned} \tag{4.4}
\]

We will prove by induction that if \((u_1, v_1) \in (L^\infty(0, T; H^1(\Omega)))^2\) and \((u_2, v_2) \in (L^\infty(0, T; H^1(\Omega)))^2\), then
\[
\|R_m(u_1, v_1)(\cdot, t) - R_m(u_2, v_2)(\cdot, t)\|_{H^1(\Omega)}^2 \leq \frac{(2D(\delta, f, g))^m(T - t)^m}{m!}. \tag{4.5}
\]
For \( m = 1 \), using the inequality \((c + d)^2 \leq 2c^2 + 2d^2\), we get that
\[
\|R_m(u_1, v_1)(\cdot, t) - R_m(u_2, v_2)(\cdot, t)\|_{H^1(\Omega)}^2 \leq 2 \|R_m(u_1, v_1)(\cdot, t) - R_{1,\delta}(u_2, v_2)(\cdot, t)\|_{H^1(\Omega)}^2 \leq 2 \|R_{1,\delta}(u_1, v_1)(\cdot, t) - R_{1,\delta}(u_2, v_2)(\cdot, t)\|_{H^1(\Omega)}^2.
\]
By applying Lemma 2.1 and the inequality $|e' - e^f| \leq |r - q| \max(e', e^f)$ for $r, q \in \mathbb{R}$, we have

$$\| \mathcal{R}_{1,0}^m (u_1, v_1)(\cdot, t) - \mathcal{R}_{1,0}^m (u_2, v_2)(\cdot, t) \|^2_{H^1(\Omega)}$$

(4.6)

$$= \sum_{n=1}^{N(b)} \left[ \exp \left( \lambda_n \int_t^T \mathcal{L} \left( \| \nabla u_1(s) \|_{L^2}, \| \nabla v_1(s) \|_{L^2} \right) ds \right) - \exp \left( \lambda_n \int_t^T \mathcal{L} \left( \| \nabla u_2(s) \|_{L^2}, \| \nabla v_2(s) \|_{L^2} \right) ds \right) \right]^2 \left[ f^3, e_n \right]^2$$

$$\leq \sum_{n=1}^{N(b)} \lambda_n^3 e^{2(T-t)\mathcal{M}_1+\lambda_n} \left[ \int_t^T \mathcal{L} \left( \| \nabla u_1(s) \|_{L^2}, \| \nabla v_1(s) \|_{L^2} \right) ds - \int_t^T \mathcal{L} \left( \| \nabla u_2(s) \|_{L^2}, \| \nabla v_2(s) \|_{L^2} \right) ds \right]^2 \left[ f^3, e_n \right]^2$$

$$\leq \left| N(b) \right|^3 e^{2TM_1N(b)K_t^2} \left\| f^3 \right\|_{L^2(\Omega)}^2 \times \left( \int_t^T \| \nabla (u_1 - u_2)(s) \|_{L^2(\Omega)} ds + \int_t^T \| \nabla (v_1 - v_2)(s) \|_{L^2(\Omega)} ds \right)^2$$

$$\leq C \left| N(b) \right|^3 e^{2TM_1N(b)K_t^2} \left\| g^4 \right\|_{L^2(\Omega)}^2 \int_t^T \left( (u_1, v_1)(s) - (u_2, v_2)(s) \right)^2_{H^1(\Omega)} ds,$$  (4.7)

where in the last line, we used the inequality $\| \nabla \psi \|_{L^2(\Omega)} \leq C \| \psi \|_{H^1(\Omega)}$. By a similar argument we obtain that

$$\| \mathcal{R}_{2,0}^m (u_1, v_1)(\cdot, t) - \mathcal{R}_{2,0}^m (u_2, v_2)(\cdot, t) \|^2_{H^1(\Omega)}$$

$$\leq C \left| N(b) \right|^3 e^{2TM_1N(b)K_t^2} \left\| g^4 \right\|_{L^2(\Omega)}^2 \int_t^T \left( (u_1, v_1)(s) - (u_2, v_2)(s) \right)^2_{H^1(\Omega)} ds,$$  (4.8)

Combining (4.6), (4.7), and (4.8), we find that

$$\| \mathcal{R}_3(u_1, v_1)(\cdot, t) - \mathcal{R}_3(u_2, v_2)(\cdot, t) \|^2_{H^1(\Omega)}$$

$$\leq \mathcal{D}(\delta, f, g) \int_t^T \left( (u_1, v_1)(s) - (u_2, v_2)(s) \right)^2_{H^1(\Omega)} ds$$

$$\leq \mathcal{D}(\delta, f, g) (T-t) \left( (u_1, v_1)(-1) - (u_2, v_2)(-1) \right)^2_{L^\infty(0,T;H^1(\Omega))},$$  (4.9)

where

$$\mathcal{D}(\delta, f, g) = 2C \left| N(b) \right|^3 e^{2TM_1N(b)K_t^2} \left( \left\| f^3 \right\|_{L^2(\Omega)}^2 + \left\| g^4 \right\|_{L^2(\Omega)}^2 \right).$$

This implies (4.5). Assume that (4.5) holds for $m = j$. We will check that (4.5) holds for $m = j + 1$. Indeed, by similar arguments as before, we also get the following two bounds:

$$\| \mathcal{R}_{1,j+1}^m (u_1, v_1)(\cdot, t) - \mathcal{R}_{1,j+1}^m (u_2, v_2)(\cdot, t) \|^2_{H^1(\Omega)}$$

$$= \| \mathcal{R}_{1,j} (\mathcal{R}_{1,j+1}^j (u_1, v_1)(\cdot, t)) - \mathcal{R}_{1,j} (\mathcal{R}_{1,j+1}^j (u_2, v_2)(\cdot, t)) \|^2_{H^1(\Omega)}$$
From two above observation we find that

\[
\| \mathcal{R}_{2, \delta}^{i+1}(u_1, v_1)(t) - \mathcal{R}_{2, \delta}^{i+1}(u_2, v_2)(t) \|_{H^1(\Omega)}^2 
\leq 2 \| \mathcal{R}_{1, \delta}^{i+1}(u_1, v_1)(t) - \mathcal{R}_{1, \delta}^{i+1}(u_2, v_2)(t) \|_{H^1(\Omega)}^2 
+ 2 \| \mathcal{R}_{2, \delta}^{i+1}(u_1, v_1)(t) - \mathcal{R}_{2, \delta}^{i+1}(u_2, v_2)(t) \|_{H^1(\Omega)}^2 
\leq 2 D(\delta, f, g) \int_t^T \| \mathcal{R}_{i}^{i+1}(u_1, v_1)(s) - \mathcal{R}_{i}^{i+1}(u_2, v_2)(s) \|_{H^1(\Omega)}^2 ds. 
\]  

(4.10)

Using the induction assumption of (3.10), from (3.15) it follows that

\[
\| \mathcal{R}_{i}^{i+1}(u_1, v_1)(t) - \mathcal{R}_{i}^{i+1}(u_2, v_2)(t) \|_{H^1(\Omega)}^2 
\leq (2 D(\delta, f, g))^{i+1} \| (u_1, v_1) - (u_2, v_2) \|_{L^\infty(0, T; H^1(\Omega))}^2 \int_t^T \frac{(T - s)^{i+1}}{s^i} ds 
\leq \frac{(T - t)^{i+1}}{(j + 1)!} (2 D(\delta, f, g))^{i+1} \| (u_1, v_1) - (u_2, v_2) \|_{L^\infty(0, T; H^1(\Omega))}^2. 
\]  

(4.11)

Hence, (3.10) holds for any positive integer \( m \). As a consequence, we conclude that (4.5) holds for any \( m \in \mathbb{N} \). Since

\[
\lim_{j \to +\infty} \frac{T^{i+1}}{(j + 1)!} (2 D(\delta, f, g))^{i+1} = 0, 
\]

there exists a positive constant \( j_0 \) such that

\[
\frac{T^{i+1}}{(j + 1)!} (2 D(\delta, f, g))^{i+1} < 1. 
\]

Using the Banach fixed point theorem (see [31–33]), we conclude that \( \mathcal{R}^i_{\delta} \) has a fixed point \( (u^*, v^*) \) on the space \( L^\infty(0, T; H^1(\Omega))^2 \). It is easy to get that \( (u^*, v^*) \) is also a solution of the nonlinear equation

\[
\mathcal{R}^i_{\delta}(u^*, v^*) = (u^*, v^*). 
\]
From (3.6) and (4.1) we find that

\[ u^{N,\delta}(x, t) - u(x, t) \]

\[ = \sum_{n=1}^{N(\delta)} \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u^{N,\delta}(\cdot, s) \|_{L^2}, \| \nabla v^{N,\delta}(\cdot, s) \|_{L^2} \right) ds \right) \left[ f^\delta, e_n \right] e_n(x) \]

\[ - \sum_{n=1}^{\infty} \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u(\cdot, s) \|_{L^2}, \| \nabla v(\cdot, s) \|_{L^2} \right) ds \right) \left( f, e_n \right) e_n(x). \]  

(4.12)

By a simple calculation we find that

\[ u^{N,\delta}(x, t) - u(x, t) \]

\[ = \sum_{n=1}^{N(\delta)} \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u^{N,\delta}(\cdot, s) \|_{L^2}, \| \nabla v^{N,\delta}(\cdot, s) \|_{L^2} \right) ds \right) \left[ f^\delta - f, e_n \right] e_n(x) \]

\[ + \sum_{n=1}^{N(\delta)} \left( f, e_n \right) e_n(x) \left[ \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u(\cdot, s) \|_{L^2}, \| \nabla v(\cdot, s) \|_{L^2} \right) ds \right) \right] \]

\[ - \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u^{N,\delta}(\cdot, s) \|_{L^2}, \| \nabla v^{N,\delta}(\cdot, s) \|_{L^2} \right) ds \right) \]

\[ + \sum_{n=N(\delta)}^{\infty} \exp\left( \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u(\cdot, s) \|_{L^2}, \| \nabla v(\cdot, s) \|_{L^2} \right) ds \right) \left( f, e_n \right) e_n(x) \]

\[ = \text{Error}_1 + \text{Error}_2 + \text{Error}_3. \]  

(4.13)

First of all, let us look at the first term. Using Parseval’s equality and noting that \( \mathcal{L}(z_1, z_2) \leq \mathcal{M}_1 \) for all \((z_1, z_2) \in \mathbb{R}^2\), \( \text{Error}_1 \) is bounded by

\[ \| \text{Error}_1 \|^2_{H^1(\Omega)} \leq \sum_{n=1}^{N(\delta)} \lambda_n \exp\left( 2 \lambda_n \int_t^T \mathcal{L}\left( \| \nabla u^{N,\delta}(\cdot, s) \|_{L^2}, \| \nabla v^{N,\delta}(\cdot, s) \|_{L^2} \right) ds \right) \left[ f^\delta - f, e_n \right]^2 \]

\[ \leq \sum_{n=1}^{N(\delta)} \lambda_{N(\delta)} \exp(2M_1T)n_{N(\delta)} \sum_{n=1}^{N(\delta)} \left[ f^\delta - f, e_n \right]^2 \]

\[ \leq \lambda_{N(\delta)} \exp(2M_1Tn_{N(\delta)}) \left\| f^\delta - f \right\|^2 \leq \lambda_{N(\delta)} \exp(2M_1Tn_{N(\delta)}) \delta^2, \]  

(4.14)

which allows us to derive that

\[ \| \text{Error}_1 \|_{H^1(\Omega)} \leq \sqrt{\lambda_{N(\delta)}} \exp(\mathcal{M}_1Tn_{N(\delta)})\delta. \]  

(4.15)

Next, we treat \( \text{Error}_2 \). Using Parseval’s equality, by the inequality \(|e^c - e^d| \leq |c - d| \max(e^c, e^d)\), we get

\[ \| \text{Error}_2 \|^2_{H^1(\Omega)} \leq \sum_{n=1}^{N(\delta)} \left[ f, e_n \right]^2 \lambda_n e^{2M_1Tn} \]

\[ \times \left( \int_t^T \mathcal{L}\left( \| \nabla u^{N,\delta}(\cdot, s) \|_{L^2}, \| \nabla v^{N,\delta}(\cdot, s) \|_{L^2} \right) ds \right) \]

\[ \leq \lambda_{N(\delta)} \exp(2M_1Tn_{N(\delta)}) \left\| f^\delta - f \right\|^2 \leq \lambda_{N(\delta)} \exp(2M_1Tn_{N(\delta)}) \delta^2, \]  

(4.16)
where we used that

\[
\sum_{n=1}^{N^*(\delta)} (f, e_n)^2 \lambda_ne^{2M_1T\lambda_n} \leq \sum_{n=1}^{\infty} (f, e_n)^2 \lambda_ne^{2M_1T\lambda_n} \leq \lambda_1^{-\gamma} E,
\]

and

\[
\exp\left(2\lambda_n \int_t^T L\left(\|\nabla u^{N\lambda}(s), \nabla v^{N\lambda}(s)\|_{L^2}\right) ds\right) \leq \exp\left(2\lambda_n \int_t^T M_1 ds\right) \leq e^{2M_1T\lambda_n}.
\]

Using Lemma (2.1), we get that

\[
\left| \int_t^T L\left(\|\nabla u^{N\lambda}(s), \nabla v^{N\lambda}(s)\|_{L^2}\right) ds - \int_t^T L\left(\|\nabla u(s), \nabla v(s)\|_{L^2}\right) ds \right|
\leq K_1 \int_t^T \left(\|\nabla u^{N\lambda}(s) - \nabla u(s)\|_{L^2} + \|\nabla v^{N\lambda}(s) - \nabla v(s)\|_{L^2}\right),
\]

and from the inequality \((c + d)^2 \leq 2c^2 + 2d^2, c, d \geq 0\), it follows that

\[
\begin{align*}
\left( \int_t^T L\left(\|\nabla u^{N\lambda}(s), \nabla v^{N\lambda}(s)\|_{L^2}\right) ds - \int_t^T L\left(\|\nabla u(s), \nabla v(s)\|_{L^2}\right) ds \right)^2 \\
\leq 2K_1 \int_t^T \|\nabla u^{N\lambda}(s) - \nabla u(s)\|_{L^2} ds + \int_t^T \|\nabla v^{N\lambda}(s) - \nabla v(s)\|_{L^2} ds.
\end{align*}
\]

Combining (4.16) and (4.20), we arrive at

\[
\|\text{Error}_2\|_{H^1(\Omega)}^2 \leq 2K_1\lambda_1^{-\gamma} E \left( \int_t^T \|\nabla u^{N\lambda}(s) - \nabla u(s)\|_{L^2} ds + \int_t^T \|\nabla v^{N\lambda}(s) - \nabla v(s)\|_{L^2} ds \right).
\]

The term Error_3 is bounded by

\[
\|\text{Error}_3\|_{H^1(\Omega)}^2
= \sum_{n>N^*(\delta)} \lambda_n \exp\left(2\lambda_n \int_t^T L\left(\|\nabla u(s), \nabla v(s)\|_{L^2}\right) ds\right) (f, e_n)^2.
\]
By the previous to equations, recalling that

\[
\text{By applying Grönwall's inequality we deduce that}
\]

\[
E(N(\delta))^{-\gamma}.
\]

Combining (4.14), (4.21), and (4.22), we find that

\[
\|U^N(t) - U(t)\|_{H^1(\Omega)}^2 \\
\leq 3\|\text{Error}_1\|^2_{H^1(\Omega)} + 3\|\text{Error}_2\|^2_{H^1(\Omega)} + 3\|\text{Error}_3\|^2_{H^1(\Omega)} \\
\leq 3\lambda_N(\delta)\exp(2\mathcal{M}_1 T\lambda_N(\delta)\delta^2 + 3E(N(\delta))^{-\gamma}) \\
+ 6K_1\lambda_1^{-\gamma}E\left(\int_t^T \|\nabla U^N(\cdot,s) - \nabla U(\cdot,s)\|_{L^2}^2 ds\right) \\
+ \int_t^T \|\nabla V^N(\cdot,s) - \nabla V(\cdot,s)\|_{L^2}^2 ds.
\]

By a similar argument we also get that

\[
\|V^N(t) - V(t)\|_{H^1(\Omega)}^2 \\
\leq 3\lambda_N(\delta)\exp(2\mathcal{M}_1 T\lambda_N(\delta)\delta^2 + 3E(N(\delta))^{-\gamma}) \\
+ 6K_1\lambda_1^{-\gamma}E\left(\int_t^T \|\nabla U^N(\cdot,s) - \nabla U(\cdot,s)\|_{L^2}^2 ds\right) \\
+ \int_t^T \|\nabla V^N(\cdot,s) - \nabla V(\cdot,s)\|_{L^2}^2 ds.
\]

By the previous to equations, recalling that \(\|\nabla \psi\|_{L^2(\Omega)} \leq C\|\psi\|_{H^1(\Omega)}\), we obtain the estimate

\[
\|U^N(t) - U(t)\|_{H^1(\Omega)}^2 + \|V^N(t) - V(t)\|_{H^1(\Omega)}^2 \\
\leq 6\lambda_N(\delta)\exp(2\mathcal{M}_1 T\lambda_N(\delta)\delta^2 + 6E(N(\delta))^{-\gamma}) \\
+ 6K_1\lambda_1^{-\gamma}EC\int_t^T \left(\|U^N(\cdot,s) - U(\cdot,s)\|_{H^1(\Omega)}^2 + \|V^N(\cdot,s) - V(\cdot,s)\|_{H^1(\Omega)}^2\right) ds.
\]

By applying Grönwall’s inequality we deduce that

\[
\|U^N(t) - U(t)\|_{H^1(\Omega)}^2 + \|V^N(t) - V(t)\|_{H^1(\Omega)}^2 \\
\leq (6\lambda_N(\delta)\exp(2\mathcal{M}_1 T\lambda_N(\delta)\delta^2 + 6E(N(\delta))^{-\gamma}) \exp(6K_1\lambda_1^{-\gamma}EC(T-t))).
\]

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Authors’ contributions
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