LOCAL UNIQUENESS PROBLEM FOR A NONLINEAR
ELLIPITC EQUATION

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Abstract. In this paper, we consider the following nonlinear Schrödinger
equation

\[-\epsilon^2 \Delta u + u = K(x)u^{p-1}\text{ in } \mathbb{R}^N,\]

where \(N \geq 3\) and \(2 < p < 2N/(N - 2)\). Under mild assumptions on the
function \(K\) and using the local Pohozaev identity method developed by Deng,
Lin and Yan [10], we show that multi-peak solutions to the above equation are
unique for \(\epsilon > 0\) sufficiently small.

1. Introduction and main result.

1.1. Introduction. In this paper, we consider the nonlinear elliptic problem

\[
\begin{cases}
-\epsilon^2 \Delta u + u = K(x)u^{p-1} & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

(1.1)

where \(\epsilon > 0\) is a parameter, \(N \geq 3\) and \(2 < p < 2N/(N - 2)\), and \(K\) is a bounded
positive continuous function in \(\mathbb{R}^N\).

Problem (1.1) and its variants arise in many applications such as chemotaxis,
population genetics, chemical reactor theory. It is also a typical case of the more
general problem

\[-\epsilon^2 \Delta u + V(x)u = K(x)u^{p-1},\]

(1.2)

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which, for instance, stems from the study of standing waves with the type
\[ \psi(x, t) = e^{-iEt/\epsilon}u(x) \]
to the nonlinear Schrödinger equation
\[ i\epsilon \frac{\partial \psi}{\partial t} = -\frac{\epsilon^2}{2m} \Delta_x \psi + (V(x) + E)\psi - K(x)|\psi|^{p-2}\psi, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}. \]

In physics concerning e.g. nonlinear optics, plasma, condensed matter, the function \( V(x) \) represents the potential acting on the particle and \( K(x) \) is a particle-interaction term, which avoids spreading of the wave packets in the time-dependent version of the above equation. Furthermore, to describe the transition from quantum to classical mechanics, we let \( \epsilon \to 0 \) and thus the existence and uniqueness of solutions \( \psi \) for small \( \epsilon \) has an important physics interest. Following Oh [21] and many others in the literature, we call solutions of (1.2) semiclassical states hereafter.

Beginning from the pioneering paper by A. Floer and A. Weinstein [13], where (1.2) is considered with \( N = 1, p = 4 \) and \( K \equiv 1 \), a great deal of work has been devoted to the study of existence of problem (1.2). It would be impossible to list all the significant contributions since then, we mainly mention e.g. [1, 2, 3, 5, 9, 11, 12, 16, 20, 21, 22, 23] and refer the interested readers to the references therein. Since we are concerned with the case \( K \not\equiv 1 \), we mention e.g. [1, 2, 3, 5, 9, 20]. In the work [5], Bartch and Peng considered (1.2) in the case when both \( V \) and \( K \) are radial functions. They proved the existence of positive radially symmetric solutions concentrating simultaneously on multiple spheres. In the works [1, 2, 3, 9], the authors constructed multipeak semiclassical solutions to problem (1.2) under quite general assumptions on \( V \) and \( K \). In particular, \( V \) and \( K \) are allowed to be unbounded or vanishing at infinity. In the work [20], Noussair and Yan considered problem (1.1). Under the assumption that (i) \( K \) is a bounded continuous positive function in \( \mathbb{R}^N \) and \( K \) has a strict local minima \( x_0 \), (ii) \( K \) is Hölder continuous in a neighborhood of \( x_0 \), Noussair and Yan [20] constructed \( k \)-peak solutions concentrating at \( x_0 \) for every \( k \geq 1 \).

In this paper, we are concerned with local uniqueness of positive concentrating solutions to problem (1.1). Here, by local uniqueness, it means that if \( u^1_\epsilon, u^2_\epsilon \) are two semiclassical solutions of equation (1.1) concentrating at the same family of concentration points, then \( u^1_\epsilon \equiv u^2_\epsilon \) for \( \epsilon \) sufficiently small. As to the definition of concentrating solutions, it is standard and we refer to e.g. Cao and Heinz [6].

Let us now review some known results in the respect of local uniqueness. It seems that the first result in this respect is the uniqueness of solutions concentrating at one point for Dirichlet problems with critical nonlinearity on bounded domains given by Glangetas [14]. By calculating the number of single-bump(single-peak) solutions to (1.2) (with \( K \equiv 1 \)), Grossi [15] proved that there is one solution concentrating at any nondegenerate critical point of \( V(x) \). We remark that the uniqueness of single-bump(single-peak) solutions concentrating at some degenerate critical point of \( V(x) \) is true in [15] as well. Later, Cao and Heinz [6] proved the uniqueness of multi-bump solutions to (1.2) (with \( K \equiv 1 \)) which concentrate at the nondegenerate critical points of \( V \). The results in [6, 14] are obtained by using the topological degree. Recently, Deng, Lin and Yan [10] proved the local uniqueness and periodicity for the solutions with infinitely many bumps of the prescribed scalar curvature problem which involves the critical Sobolev exponent by the technique of Pohozaev identity. Using the idea of Deng et al. [10], Cao, Li and Luo [7] established local uniqueness of multi-peak solutions to (1.2) (with \( K \equiv 1 \)). In particular, in the work [7], they
do not need to assume $V$ is nondegenerate at concentration points. The same idea has also been used by Guo, Peng and Yan [17] to study uniqueness of solutions concerning polyharmonic operators.

We conclude from the above review that great progress on local uniqueness of semiclassical solutions to problem (1.2) in the case $K \equiv 1$ have been obtained. However, in the case $K \not\equiv 1$, even though there have been many works on the existence of semiclassical solutions as aforementioned, there seems to have no results for problem (1.2) in the respect of local uniqueness. This is the direct motivation of this work. However, to avoid too much involved in the arguments, we will restrict ourselves to the typical case $V \equiv 1$ and $K \not\equiv 1$ in problem (1.2). That is, we will only consider problem (1.1) and leave the more general case to interested readers.

1.2. Main result and strategy of proof. To state our main results, let us introduce some notations first. Denote
\[
\langle u, v \rangle_{\epsilon} = \int_{\mathbb{R}^N} (\epsilon^2 \nabla u \cdot \nabla v + uv), \quad \|u\|_{\epsilon}^2 = \langle u, u \rangle_{\epsilon},
\]
and
\[
H_{\epsilon} = \{ u \in H^1(\mathbb{R}^N) : \|u\|_{\epsilon} < \infty \}.
\]
Write
\[
u_{\epsilon,a}(x) = u((x - a)/\epsilon).
\]
Let \{a_1, \ldots, a_k\} \subset \mathbb{R}^N. Let $U^j \in H^1(\mathbb{R}^N)$ be the unique positive radial solution to equation
\[
-\Delta U^j + U^j = K(a_j)(U^j)^{p-1} \quad \text{in } \mathbb{R}^N.
\]
It is well known (see e.g. Cao and Peng [9]) $U^j$ can be represented by scaling the unique (Kwong [18]) positive radial solution $U$ of the problem
\[
-\Delta U + U = U^{p-1}, \quad U > 0 \quad \text{in } \mathbb{R}^N.
\]
However, to simplify the notations, we keep using the notation $U^j$ instead of a scaling of $U$.

For any $\epsilon > 0$, $b_i = (b_{i,1}, \ldots, b_{i,N}) \in \mathbb{R}^N$ ($1 \leq i \leq k$) and $B = (b_1, \ldots, b_k) \in R^{Nk}$, define
\[
E_{\epsilon,B} = \left\{ v \in H_{\epsilon} : \left\langle v, \frac{\partial U^j_{b_i}}{\partial b_{i,\alpha}} \right\rangle_{\epsilon} = 0 \quad \text{for all } 1 \leq i \leq k, 1 \leq \alpha \leq N \right\}.
\]
Then, following the scheme of Cao and Peng [9], one can construct solutions to equation (1.1) in the form
\[
\nu_{\epsilon}(x) = \sum_{j=1}^{k} \alpha(a_{j,\epsilon}) U^j \left( \beta(a_{j,\epsilon}) \left( \frac{x - a_{j,\epsilon}}{\epsilon} \right) \right) + \omega_{\epsilon}
\]
for some numbers $\alpha(a_{j,\epsilon}), \beta(a_{j,\epsilon}) \to 1$ as $\epsilon \to 0$, where $a_{j,\epsilon} \in \mathbb{R}^N$ satisfying
\[
a_{j,\epsilon} \to a_j \quad \text{as } \epsilon \to 0,
\]
and
\[
\omega_{\epsilon} \in E_{\epsilon,A_{\epsilon}} \quad \text{with } \|\omega_{\epsilon}\|_{\epsilon}^2 = o(\epsilon^N)
\]
for $A_{\epsilon} = (a_{1,\epsilon}, \ldots, a_{k,\epsilon})$. However, it is noted that by combining the improved Lyapunov-Schimdt reduction (see e.g. Li et al. [19]), one can in fact construct a
family of solutions concentrating at \( \{ a_1, \ldots, a_k \} \subset \mathbb{R}^N \) for equation (1.1) in the more brief form

\[
u_\epsilon(x) = \sum_{j=1}^{k} U(j) \left( \frac{x - a_{j, \epsilon}}{\epsilon} \right) + \omega_\epsilon(x),
\]

where \( a_{j, \epsilon} \) and \( \omega_\epsilon \) are defined as in the above.

Thus, a natural question is whether solutions constructed in this way is unique. To answer this question, we shall assume

(K1) There exist \( m > 1 \), \( k_{i, \alpha} \neq 0 \) and \( \delta > 0 \) such that

\[
\begin{align*}
K(x) &= K(a_i) + k_{i, \alpha} |x - a_i|^m + O(|x - a_i|^{m+1}), \quad x \in B_\delta(a_i), \\
K_{x, \alpha}(x) &= mk_{i, \alpha} |x - a_i|^{m-2}(x_i - a_i) + O(|x - a_i|^{m}), \quad x \in B_\delta(a_i)
\end{align*}
\]

for all \( 1 \leq i \leq k \) and \( 1 \leq \alpha \leq N \). Then, our main result reads as

**Theorem 1.1.** Assume that \( K \) is a bounded positive continuous function satisfying the assumption (K1). Assume that \( u_\epsilon^{(i)}, i = 1, 2, \) are two positive solutions of problem (1.1) concentrating at \( k \) different points \( \{ a_1, \ldots, a_k \} \) in \( \mathbb{R}^N \). Then, \( u_\epsilon^{(1)} = u_\epsilon^{(2)} \) for \( \epsilon \) sufficiently small. Moreover,

\[
u_\epsilon^{(1)} = \sum_{j=1}^{k} \alpha(a_{j, \epsilon}) U(j) \left( \frac{x - a_{j, \epsilon}}{\epsilon} \right) + \omega_\epsilon, \quad (1.8)
\]

where \( \alpha(a_{j, \epsilon}), \beta(a_{j, \epsilon}) \to 1 \) as \( \epsilon \to 0 \), \( a_{j, \epsilon} \in \mathbb{R}^N \) and \( \omega_\epsilon \in H^1(\mathbb{R}^N) \) satisfying, as \( \epsilon \to 0 \),

\[|a_{j, \epsilon} - a_j| = o(\epsilon) \quad \text{and} \quad \|\omega_\epsilon\|_\epsilon = O(\epsilon^{N/2+m}). \]

Since in the work of Cao, Li and Luo [7], they have proved this type of results for equation (1.2) in the case \( K \equiv 1 \), we will follow the line of Cao et al. [7]. To control the size of this work, we now explain the strategy of proving Theorem 1.1 and devote the rest sections to the proof of the main steps (see Theorem 1.2).

As the first step of proving Theorem 1.1, one can prove by a standard argument (see e.g. Cao, Li and Luo [7] or Cao and Heinz [6]) that if \( u_\epsilon \) is a positive solution of problem (1.1) concentrating at \( k \) different points \( \{ a_1, \ldots, a_k \} \) in \( \mathbb{R}^N \), then \( u_\epsilon \) can be written in the form (1.8), where \( \alpha, \beta \) converges to 1, and \( \omega_\epsilon \in \cap_{j=1}^{k} E_{\epsilon, a_{j, \epsilon}}, \) where

\[
E_{\epsilon, a_{j, \epsilon}} = \left\{ v \in H^1 : \left\langle v, U(j) \left( \frac{x - a_{j, \epsilon}}{\epsilon} \right) \right\rangle_\epsilon = \left\langle v, \partial_{x, \alpha} U(j) \left( \frac{x - a_{j, \epsilon}}{\epsilon} \right) \right\rangle_\epsilon = 0, 1 \leq \alpha \leq N \right\}.
\]

Then, as the second step, one notices that it is sufficient to assume that \( \alpha(a_{j, \epsilon}) = \beta(a_{j, \epsilon}) = 1 \) by the arguments as that of Cao, Li and Luo [7, Proposition 2.5], and assume that \( \omega_\epsilon \) belongs to the simpler set \( \cap E_{\epsilon, a_{j, \epsilon}}, \) where \( E_{\epsilon, a_{j, \epsilon}}, \) is defined as in (1.4), and \( \omega_\epsilon \) satisfies

\[
\int_{\mathbb{R}^N} \left( \epsilon^2 |\nabla \omega_\epsilon|^2 + \omega_\epsilon^2 - (p-1) \sum_{j=1}^{k} K(a_j) (U(j)^p - (x - a_{j, \epsilon})/\epsilon) \omega_\epsilon^2 \right) \geq \rho \|\omega_\epsilon\|_\epsilon^2 \quad (1.9)
\]

for some constant \( \rho > 0 \). Indeed, solutions satisfying \( \alpha(a_{j, \epsilon}) = \beta(a_{j, \epsilon}) = 1 \) and \( \omega_\epsilon \in \cap E_{\epsilon, a_{j, \epsilon}}, \) can be constructed directly using the Lyapunov-Schmidt reduction (see e.g. Li et al. [19]). Furthermore, (1.9) can be proved using the same argument as that of Cao, Li and Luo [7] (see the section “Analysis of \( w_\epsilon \) and \( v_\epsilon \)” in their Appendix) or using the arguments of Cao et al. [8]. In the work of Li et al.
[19], estimates of type (1.9) has also been proved directly. We remark that in this step, the important well known nondegeneracy result (Kwong [18]) is used: If \( \varphi \in H^1(\mathbb{R}^N) \) satisfies

\[
-\Delta \varphi + \varphi = (p - 1)K(a_j)(U^j)^{p-2}\varphi,
\]

then \( \varphi = \sum_{a=1}^N d_a U^j_{x_a} \) for some constants \( d_a \in \mathbb{R} \). With the help of the above explanation, we are reduced to prove the following theorem.

**Theorem 1.2.** Assume that \( K \) is a bounded positive continuous function satisfying the assumption (K1). Assume that \( u^{(i)}_\epsilon \), \( i = 1, 2 \), are two positive solutions of problem (1.1) in the form (1.7) satisfying (1.5) and (1.6). Moreover, assume that \( \omega_\epsilon \in \cap E_{,a_j,\epsilon} \), where \( E_{,a_j,\epsilon} \) is defined as in (1.4), and \( \omega_\epsilon \) satisfies (1.9). Then, \( u^{(1)}_\epsilon = u^{(2)}_\epsilon \) for \( \epsilon \) sufficiently small. Moreover, let

\[
u^{(1)}_\epsilon = u^{(2)}_\epsilon = \sum_{i=1}^k U^{i}_{,\epsilon,a_i,\epsilon} + \omega_\epsilon.
\]

Then as \( \epsilon \to 0 \), there hold

\[|a_{j,\epsilon} - a_j| = o(\epsilon) \quad \text{and} \quad \|\omega_\epsilon\|_\epsilon = O(\epsilon^{N/2+m}).\]

Before closing the introduction, let us briefly explain our proof for the above simplified theorem. To prove Theorem 1.2, we use a contradiction argument as that of Cao, Li and Luo [7]. More precisely, if \( u^{(i)}_\epsilon \), \( i = 1, 2 \), are two distinct solutions of problem (1.1) as stated in Theorem 1.2, then it is clear that the function

\[
\xi_\epsilon = (u^{(1)}_\epsilon - u^{(2)}_\epsilon) / \|u^{(1)}_\epsilon - u^{(2)}_\epsilon\|_{L^\infty(\mathbb{R}^N)}
\]

is nonzero and \( \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^N)} = 1 \). However, we will use the equations satisfied by \( \xi_\epsilon \) to show that \( \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( \epsilon \to 0 \). This gives a contradiction, and thus the uniqueness result is obtained. To deduce the contradiction, delicate estimates on the asymptotic behaviors of solutions and the concentrating point \( a_{j,\epsilon} \) will be derived by a local Pohozaev type identity following the idea of Deng, Lin and Yan [10].

Throughout this paper, we use \( B_R(x) \) (and \( B_R^{(x)} \)) to denote open (and close) balls in \( \mathbb{R}^N \) centered at \( x \) with radius \( R \). Unless otherwise stated, we write \( \int u = \int_{\mathbb{R}^N} u(x)dx \) to denote Lebesgue integral of an integrable function \( u \) over \( \mathbb{R}^N \), and denote by \( \|u\|_s \) the \( L^s \)-norm of a function \( u \) in \( L^s(\mathbb{R}^N) \) for any \( 1 \leq s \leq \infty \). We will use the same \( C \) to denote various generic positive constants, and use \( O(t) \) and \( o(t) \) to mean \( |O(t)| \leq C|t| \) and \( o(t)/t \to 0 \) as \( t \to 0 \) respectively. We also use \( o(1) \) or \( o(\epsilon) \) to denote quantities that tend to 0 as \( \epsilon \to 0 \).

2. **Preliminary estimates.** In this section we prove the following estimates.

**Proposition 1.** Assume that

\[
u_\epsilon(x) = \sum_{j=1}^k U^{j}\left(\frac{x - a_{j,\epsilon}}{\epsilon}\right) + \omega_\epsilon(x)
\]

is a solution of equation (1.1) with \( a_{j,\epsilon} \) and \( \omega_\epsilon \) satisfying the assumptions of Theorem (1.2). Then,

\[|a_{j,\epsilon} - a_j| = o(\epsilon) \quad \text{(2.1)} \]

and

\[\|\omega_\epsilon\|_\epsilon = O(\epsilon^{N/2+m}). \quad \text{(2.2)}\]
For simplicity of notations and calculations, we hereafter assume without loss of generality that $k = 2$. The general cases follow easily. We first prove (2.2).

**Proof of (2.2).** Recall that for the linear operator $\mathcal{L}$ defined by

$$
\mathcal{L}\omega = -\epsilon^2 \Delta \omega + \omega - (p - 1)K(x) \left( \sum_{i=1}^{2} U_{\epsilon,a_i} \right)^{p-2} \omega,
$$

there exist $\epsilon, \delta > 0$ sufficiently small, and $\rho > 0$ such that

$$
\langle \mathcal{L}\omega_{\epsilon}, \omega_{\epsilon} \rangle \geq \rho \|\omega_{\epsilon}\|_2^2
$$

for $\omega_{\epsilon} \in E_{\epsilon,A_{\epsilon}}$, where $A_{\epsilon} = (a_{1,\epsilon}, a_{2,\epsilon})$.

On the other hand, by the equation (1.1) of $u_{\epsilon}$ and the equation (1.3) of $U_{i}$ ($i = 1, 2$), $\omega_{\epsilon}$ must satisfy

$$
\mathcal{L}\omega_{\epsilon} = \sum_{i=1}^{3} f_i(x),
$$

with

$$
f_1 = K(x) \left( \left( \sum_{i=1}^{2} U_{\epsilon,a_i} \right)^{p-1} - \left( \sum_{i=1}^{2} U_{\epsilon,a_i} \right)^{p-1} - (p-1) \left( \sum_{i=1}^{2} U_{\epsilon,a_i} \right)^{p-2} \omega_{\epsilon} \right),
$$

$$
f_2 = K(x) \left( \left( \sum_{i=1}^{2} U_{\epsilon,a_i} \right)^{p-1} - \sum_{i=1}^{2} \left( U_{\epsilon,a_i} \right)^{p-1} \right),
$$

$$
f_3 = \sum_{i=1}^{2} \left( (K(x) - K(a_i)) \left( U_{\epsilon,a_i} \right)^{p-1} \right).
$$

Hence,

$$
\rho \|\omega_{\epsilon}\|_2^2 \leq \langle \mathcal{L}\omega_{\epsilon}, \omega_{\epsilon} \rangle = \sum_{i=1}^{3} \int f_i \omega_{\epsilon}.
$$

(2.4)

By the same argument as that of Cao and Peng [9, Lemma 3.1] and by the assumption (1.6), we have

$$
\int f_1 \omega_{\epsilon} = O(\epsilon^{\frac{N(a_2-2)}{2}} \|\omega_{\epsilon}\|_p^p) = o(1) \|\omega_{\epsilon}\|_2^2.
$$

(2.5)

Since $a_1 \neq a_2$ and $U^i$ decays exponentially, we have

$$
\int f_2 \omega_{\epsilon} = O(\epsilon^\gamma) \|\omega_{\epsilon}\|_2
$$

(2.6)

for any given $\gamma > 0$. To estimate the last term, we use the assumption (K1). Note that

$$
\int f_3 \omega_{\epsilon} = \sum_{i=1}^{2} \int (K(x) - K(a_i)) \left( U_{\epsilon,a_i} \right)^{p-1} \omega_{\epsilon}
$$

$$
= \sum_{i=1}^{2} \left( \int_{B_\delta(a_i,\epsilon)} + \int_{B_\delta(a_i,\epsilon)} \right) (K(x) - K(a_i)) \left( U_{\epsilon,a_i} \right)^{p-1} \omega_{\epsilon}.
$$
Proof of (2.1)

Let \( \phi \) holds for all \( i \).

By (K1),
\[
\int_{B_{\delta}(a_{i,\epsilon})} (K(x) - K(a_{i})) \left( U_{\epsilon,a_{i,\epsilon}} \right)^{p-1} \omega_{\epsilon} = \epsilon^{N} \int_{B_{\delta}(0)} (K(\epsilon z + a_{i,\epsilon}) - K(a_{i})) (U^{i}(z))^{p-1} \omega_{\epsilon} (\epsilon z + a_{i,\epsilon})
\]
\[
= \epsilon^{N} O \left( \int_{B_{\delta}(0)} (\epsilon^{m} |z|^{m} + |a_{i,\epsilon} - a_{i}|^{m}) (U^{i})^{p-1} \omega_{\epsilon} (\epsilon z + a_{i,\epsilon}) \right)
\]
\[
= \epsilon^{N/2} O (\epsilon^{m} + |a_{i,\epsilon} - a_{i}|^{m}) \|\omega_{\epsilon}\|_{\epsilon}.
\]
In the last step, we used H"older’s inequality and the assumption (1.6). By the boundedness of \( K \) and the exponential decay of \( U^{i} \), we have
\[
\int_{B_{\delta}(a_{i,\epsilon})} (K(x) - K(a_{i})) \left( U_{\epsilon,a_{i,\epsilon}} \right)^{p-1} \omega_{\epsilon} = O(\epsilon^{\gamma}) \|\omega_{\epsilon}\|_{\epsilon}
\]
for any given \( \gamma > 0 \). Hence, taking \( \gamma > N/2 + m \), we derive
\[
\int f_{\delta} \omega_{\epsilon} = \epsilon^{N/2} O (\epsilon^{m} + |a_{i,\epsilon} - a_{i}|^{m}) \|\omega_{\epsilon}\|_{\epsilon}.
\]
(2.7)

Now, combining (2.4)-(2.7) and using \( \epsilon \)-Young’s inequality \( ab \leq \epsilon a^{2} + C_{b} b^{2} \) for any \( a, b > 0 \), we deduce
\[
\|\omega_{\epsilon}\|_{\epsilon} = \epsilon^{N/2} O (\epsilon^{m} + |a_{i,\epsilon} - a_{i}|^{m}).
\]
(2.8)

By using a Pohozaev type identity, the assumption (K1), and (2.8), we will prove in below that (2.1) holds. This in turn implies that (2.2), and thus finishes the proof of (2.2).

Next we prove (2.1). We will use the following Pohozaev type identity.

**Proposition 2.** Let \( u \) be a positive solution of equation (1.1) and \( \Omega \) a bounded smooth domain in \( \mathbb{R}^{N} \). Then, for every \( \alpha = 1, \ldots, N \), there holds
\[
- \frac{2}{p} \int_{\Omega} \frac{\partial K}{\partial x_{\alpha}} u^{p} = \epsilon^{2} \int_{\partial \Omega} \left( |\nabla u|^{2} \nu_{\alpha} - 2 \frac{\partial u}{\partial x_{\alpha}} \frac{\partial u}{\partial x_{\alpha}} \right) + \int_{\partial \Omega} u^{2} \nu_{\alpha} - \frac{2}{p} \int_{\partial \Omega} K(x) u^{p} \nu_{\alpha}.
\]
(2.9)

Here \( \nu = (\nu_{1}, \ldots, \nu_{N}) \) is the unit outward normal of \( \partial \Omega \).

Multiplying both sides of equation (1.1) by \( \partial_{x_{\alpha}} u \) for each \( 1 \leq \alpha \leq N \) and then integrating by parts, Proposition 2 can be proved. We omit the details, see Cao, Li and Luo [7, Proposition 2.3].

The following type of Sobolev inequality will be used repeatedly: For any \( 2 \leq q \leq 6 \) there exists a constant \( C > 0 \) depending only on \( n, V, a, q \) and \( \epsilon \), but independent of \( \epsilon \), such that
\[
\|\varphi\|_{L^{q}(\mathbb{R}^{n})} \leq C_{\epsilon}^{\frac{q}{2} - \frac{n}{2}} \|\varphi\|_{\epsilon}
\]
holds for all \( \varphi \in H_{\epsilon} \). For a proof, see e.g. (3.6) of Li et al. [19].

**Proof of (2.1).** We prove (2.1) for \( i = 1 \). The rest can be proved similarly. Apply the identity (2.9) to \( \omega_{\epsilon} \) with \( \Omega = B_{\delta}(a_{1,\epsilon}) \), where \( 0 < \delta < \delta \) is chosen in such a way (see e.g. Cao, Li and Luo [7, Lemma 4.5] for details) that
\[
\int_{\partial B_{\delta}(a_{1,\epsilon})} (\epsilon |\nabla \omega_{\epsilon}|^{2} + |\omega_{\epsilon}|^{2} + |\omega_{\epsilon}|^{p}) = O \left( \|\omega_{\epsilon}\|_{\epsilon}^{2} + \int |\omega_{\epsilon}|^{p} \right).
\]
(2.11)
We have
\[
\frac{-2}{p} \int_{B_d(a_1, \epsilon)} \frac{\partial K}{\partial x_\alpha} u_\epsilon^p = \epsilon^2 \int_{\partial B_d(a_1, \epsilon)} \left( |\nabla u_\epsilon|^2 \nu_\alpha - 2 \frac{\partial u_\epsilon}{\partial \nu} \frac{\partial u_\epsilon}{\partial x_\alpha} \right) + \int_{\partial B_d(a_1, \epsilon)} u_\epsilon^2 \nu_\alpha - \frac{2}{p} \int_{\partial B_d(a_1, \epsilon)} K(x) u_\epsilon^p \nu_\alpha.
\] (2.12)

We estimate (2.12) term by term.

By an elementary inequality, we have
\[
|\nabla u_\epsilon|^2 \leq C \sum_{i=1}^{2} |\nabla U_{\epsilon,a_i,\epsilon}|^2 + C |\omega_\epsilon|^2.
\]

Since \(U_i, i = 1, 2,\) decay exponentially at infinity and \(|a_1 - a_2| > \delta > d,\) we have
\[
\int_{\partial B_d(a_1, \epsilon)} \sum_{i=1}^{2} |\nabla U_{\epsilon,a_i,\epsilon}|^2 = O(\epsilon^\gamma),
\]
for any given \(\gamma > 0.\) Thus, by the choice (2.11) of \(d,\) we have
\[
\epsilon^2 \int_{\partial B_d(a_1, \epsilon)} \left( |\nabla u_\epsilon|^2 \nu_\alpha - 2 \frac{\partial u_\epsilon}{\partial \nu} \frac{\partial u_\epsilon}{\partial x_\alpha} \right) = O(\epsilon^\gamma + ||\omega_\epsilon||^2_\epsilon).
\]

Similarly, we derive
\[
\int_{\partial B_d(a_1, \epsilon)} u_\epsilon^2 \nu_\alpha = O(\epsilon^\gamma + ||\omega_\epsilon||^2_\epsilon)
\]
and
\[
\int_{\partial B_d(a_1, \epsilon)} K(x) u_\epsilon^p \nu_\alpha = O(\epsilon^\gamma + C^N - m/2 ||\omega_\epsilon||^p_\epsilon) = O(\epsilon^\gamma + ||\omega_\epsilon||^2_\epsilon),
\]
where we have used (2.11) and (2.10).

Combining the above estimate and (2.12), we obtain
\[
\int_{B_d(a_1, \epsilon)} \frac{\partial K}{\partial x_\alpha} u_\epsilon^p = O(\epsilon^\gamma + ||\omega_\epsilon||^2_\epsilon) \quad (2.13)
\]
for any given \(\gamma > 0.\)

To estimate the left hand side of the above equation, we apply the assumption (K1) to get
\[
\int_{B_d(a_1, \epsilon)} \frac{\partial K}{\partial x_\alpha} u_\epsilon^p = k_{1,\epsilon} \int_{B_d(a_1, \epsilon)} |x_\alpha - a_{1,\alpha}|^{m-2} (x_\alpha - a_{1,\alpha}) u_\epsilon^p \quad (2.14)
\]

Using a scaling argument, we obtain
\[
\int_{B_d(a_1, \epsilon)} |x - a_1|^m u_\epsilon^p \leq C \int_{B_d(a_1, \epsilon)} |x - a_1|^m \left( \sum_{i=1}^{2} U_i^{i,a_i,\epsilon} \right)^p + C \int_{\mathbb{R}^N} |\omega_\epsilon|^p.
\]
By the exponential decay of $U^i$ and the fact that $|a_{1,\epsilon} - a_{2,\epsilon}| \geq d$, we obtain

$$\int_{B_a(0)} |\epsilon z + a_{1,\epsilon} - a_1|^m \left( (U^i)^p + \left( U^2 \left( z + \frac{a_{1,\epsilon} - a_{2,\epsilon}}{\epsilon} \right) \right)^p \right) = O(m^m + |a_{1,\epsilon} - a_1|^m).$$

From (2.10), we have

$$\int_{\mathbb{R}^N} |\omega_\epsilon|^p = O \left( \epsilon^{N - \frac{2N}{p}} \|\omega_\epsilon\|^p \right).$$

Hence,

$$\int_{B_d(a_{1,\epsilon})} |x - a_1|^m u_\epsilon^p = \epsilon^N O(m^m + |a_{1,\epsilon} - a_1|^m + \epsilon^{-\frac{2N}{p}} \|\omega_\epsilon\|^p).$$

Combining this estimate together with (2.13) and (2.14), we deduce

$$\int_{B_d(a_{1,\epsilon})} |x_{\alpha} - a_{1,\alpha}|^{m-2} (x_{\alpha} - a_{1,\alpha}) u_\epsilon^p$$

$$= O \left( \epsilon^N (m^m + |a_{1,\epsilon} - a_1|^m + \epsilon^{-\frac{2N}{p}} \|\omega_\epsilon\|^p) \right).$$

$$= O \left( \epsilon^N (m^m + |a_{1,\epsilon} - a_1|^m + \epsilon^{-N} \|\omega_\epsilon\|^p) \right),$$

since $p > 2$ and $\epsilon^{N/2} \|\omega_\epsilon\| \to 0$.

Note that

$$\int_{B_d(a_{1,\epsilon})} |x_{\alpha} - a_{1,\alpha}|^{m-2} (x_{\alpha} - a_{1,\alpha}) u_\epsilon^p$$

$$= \epsilon^N \int_{B_d(0)} |\epsilon z_{\alpha} + a_{1,\epsilon,\alpha} - a_{1,\alpha}|^{m-2} (\epsilon z_{\alpha} + a_{1,\epsilon,\alpha} - a_{1,\alpha}) u_\epsilon^p (\epsilon z + a_{1,\epsilon}).$$

Using the elementary inequality

$$|b|^m \leq m|a + b|^{m-2}(a + b) + C \left( |a|^m + |b|^{m-m^*}|a|^{m^*} \right)$$

for any $a, b \in \mathbb{R}$, where $m^* = \min\{m, 2\}$ and $C$ is positive constant depending only on $m$, and using the same argument as that of Cao, Li and Luo [7, Lemma 2.1], we conclude that

$$\int_{\mathbb{R}^N} |\epsilon z_{\alpha} + a_{1,\epsilon,\alpha} - a_{1,\alpha}|^{m-2} (\epsilon z_{\alpha} + a_{1,\epsilon,\alpha} - a_{1,\alpha}) (U^1(z))^p$$

$$= O \left( \epsilon^m + |a_{1,\epsilon} - a_1|^m + \epsilon^{-N} \|\omega_\epsilon\|^2 \right)$$

and

$$|a_{1,\epsilon,\alpha} - a_{1,\alpha}|^m = O \left( \left( \epsilon^m + |a_{1,\epsilon} - a_1|^m + \epsilon^{-N} \|\omega_\epsilon\|^2 \right) |a_{1,\epsilon,\alpha} - a_{1,\alpha}| \right)$$

$$+ O \left( \epsilon^m + \epsilon^{m^*} |a_{1,\epsilon} - a_1|^m \right).$$

Since $|a_{1,\epsilon,\alpha} - a_{1,\alpha}| = o_\epsilon(1)$ by assumption, using Young’s inequality, we deduce

$$|a_{1,\epsilon,\alpha} - a_{1,\alpha}|^m = O \left( \epsilon^m + \left( \epsilon^{-N}/(\|\omega_\epsilon\|_\epsilon) \right)^{m/m^*} \right).$$
To conclude, recall (2.8). We obtain
\[
\sum_i |a_{i,\epsilon} - a_i| = O \left( \epsilon + \left( \epsilon^m + \sum_i |a_{i,\epsilon} - a_i|^m \right)^{\frac{2}{m+1}} \right)
\]
\[
= O \left( \epsilon + \left( \sum_i |a_{i,\epsilon} - a_i|^m \right)^{\frac{2}{m+1}} \right).
\]
This implies
\[
\sum_i |a_{i,\epsilon} - a_i| = O(\epsilon).
\]
To further prove (2.1), we assume, on the contrast, that there exist \( \epsilon_k \to 0 \) such that \( |a_{1,\epsilon} - a_1|/\epsilon \to A \neq 0 \). Then, from (2.15) we deduce
\[
\int |z_\alpha + A|^{m-2}(z_\alpha + A)(U^1(z))^p = 0.
\]
However, since \( U^1 \) is radially symmetric-decreasing, this is impossible unless \( A = 0 \). We reach a contradiction. The proof of (2.1) is complete. Thus complete the proof of Proposition 1.

3. **Proof of Theorem 1.2.** In this section, we prove Theorem 1.2. We use a contradiction argument. As in Cao, Li and Luo [7], we assume that
\[
u^{(i)}_\epsilon = \sum_{j=1}^2 U^{j,\epsilon}_{a,\epsilon}(i) + \omega^{(i)}_\epsilon, \quad i = 1, 2
\]
are two distinct solutions of equation (1.1). Set
\[
\xi_\epsilon = \frac{\nu^{(1)}_\epsilon - \nu^{(2)}_\epsilon}{\|\nu^{(1)}_\epsilon - \nu^{(2)}_\epsilon\|_{L^\infty(\mathbb{R}^N)}}.
\]
It is clear that \( \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^N)} = 1 \). To obtain a contradiction, in the rest of this section, we prove
\[
\|\xi_\epsilon\|_{L^\infty(\mathbb{R}^N)} = o(1) \quad \text{as} \quad \epsilon \to 0. \quad (3.1)
\]
Then, Theorem 1.2 follows from the above proposition. To prove (3.1), we will first prove that \( \xi_\epsilon = o(1) \) holds locally, and then prove that it holds at infinity. The proof is lengthy. We split it into several lemmas and propositions.

Note that \( \xi_\epsilon \) satisfies equation
\[
-\epsilon^2 \Delta \xi_\epsilon + \xi_\epsilon = C(x)\xi_\epsilon \quad \text{in} \quad \mathbb{R}^N, \quad (3.2)
\]
where
\[
C_\epsilon(x) = (p-1)K(x)\int_0^1 \left( t\nu^{(1)}_{\epsilon}(x) + (1-t)\nu^{(2)}_{\epsilon}(x) \right)^{p-2} dt.
\]
Our first estimate reads as

**Lemma 3.1.** There holds
\[
\|\xi_\epsilon\| = O(\epsilon^{N/2}).
\]
Proof. By (3.2), we have

$$\|\xi_\epsilon\|^2 = \int C(x)\xi_\epsilon^2.$$

Since $K$ is bounded, we have

$$C_\epsilon \leq C \sum_{i,j=1}^2 \left( U_{\epsilon,a_j}^{3,\epsilon} \right)^{p-2} + \sum_{i=1}^2 \left( \omega_\epsilon^{(i)} \right)^{p-2}.$$

Direct computation gives

$$\int \left( U_{\epsilon,a_j}^{3,\epsilon} \right)^{p-2} \xi_\epsilon^2 = O(\epsilon^N)$$

since $\|\xi_\epsilon\|_{L^\infty(\mathbb{R}^3)} = 1$, and

$$\int \left( \omega_\epsilon^{(i)} \right)^{p-2} \xi_\epsilon^2 \leq \|\omega_\epsilon^{(i)}\|_p^{-p} \|\xi_\epsilon\|^2 \leq C \left( \epsilon^{-\frac{N}{p}} N \|\omega_\epsilon^{(i)}\|_p \right)^2 \left( \epsilon^{-\frac{N}{p}} N \|\xi_\epsilon\|_p \right)^2,$$

where we have used (2.10) and (1.6). Hence,

$$\|\xi_\epsilon\|^2 = \int C(x)\xi_\epsilon^2 = O(\epsilon^N) + o(1)\|\xi_\epsilon\|^2.$$

The desired result follows from the above directly.

Next set

$$\bar{\xi}_\epsilon(x) = \xi_\epsilon \left( \epsilon x + a_{1,\epsilon}^{(i)} \right).$$

We study the asymptotic behavior of $\bar{\xi}_\epsilon$.

**Proposition 3.** There exist $d_\alpha \in \mathbb{R}$, $\alpha = 1, \ldots, N$, such that (up to a subsequence)

$$\bar{\xi}_\epsilon \to \sum_{\alpha=1}^N d_\alpha \frac{\partial U^1}{\partial x_\alpha} \text{ in } C^1_{\text{loc}}(\mathbb{R}^N).$$

**Proof.** The idea is to prove that the limiting function of $\bar{\xi}_\epsilon$ belongs to the kernel of a linear operator associated to $U^1$.

Note that $\bar{\xi}_\epsilon$ satisfies

$$-\Delta \bar{\xi}_\epsilon + \bar{\xi}_\epsilon = C_\epsilon \left( \epsilon x + a_{1,\epsilon}^{(i)} \right) \bar{\xi}_\epsilon \text{ in } \mathbb{R}^N.$$

Since $|\bar{\xi}_\epsilon| \leq 1$, the elliptic regularity theory shows that for any $R > 0$, $\bar{\xi}_\epsilon$ is uniformly bounded in $C^{1,\theta}(\mathbb{B}_R(0))$ for some $\theta \in (0, 1)$ with respect to $\epsilon$. Hence, there exists $\bar{\xi} \in C^1(\mathbb{R}^N)$ such that

$$\bar{\xi}_\epsilon \to \bar{\xi} \text{ in } C^1_{\text{loc}}(\mathbb{R}^N).$$

We claim that $\bar{\xi}$ satisfies the equation

$$-\Delta \bar{\xi} + \bar{\xi} = (p-1)K(a_1)(U^1)^{p-2}\xi \text{ in } \mathbb{R}^N.$$

To this end, we only need to prove, for any function $\varphi \in C^1(\mathbb{R}^N)$ with compact support, there holds

$$\int C_\epsilon \left( \epsilon x + a_{1,\epsilon}^{(i)} \right) \bar{\xi}_\epsilon \varphi \to \int (p-1)K(a_1)(U^1)^{p-2}\xi \varphi,$$  \hspace{1cm} (3.3)
Note that
\[ C_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) = (p - 1)K(x + a^{(1)}_{1,\varepsilon}) \int_0^1 \left( tu^{(1)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) + (1 - t)u^{(2)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) \right)^{p - 2} \, dt, \]
where
\[ u^{(1)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) = U^1(x) + U^2 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}}{\varepsilon} \right) + \omega^{(1)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right), \]
\[ u^{(2)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) = U^1 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}}{\varepsilon} \right) + U^2 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}}{\varepsilon} \right) + \omega^{(2)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) \]
and
\[ K(x + a^{(1)}_{1,\varepsilon}) \rightarrow K(a_1) \text{ locally uniformly. (3.4)} \]

By Proposition 1, we have
\[ \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}}{\varepsilon} = o(1). \]

Since \( a^{(1)}_{1,\varepsilon} \in B_\delta(a_1) \) and \( a^{(2)}_{1,\varepsilon} \in B_\delta(a_2) \), it follows that
\[ \frac{|a^{(1)}_{1,\varepsilon} - a^{(1)}_{2,\varepsilon}|}{\varepsilon} \geq \frac{\delta}{4\varepsilon} \rightarrow \infty \quad \text{and} \quad \frac{|a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}|}{\varepsilon} \geq \frac{\delta}{4\varepsilon} \rightarrow \infty. \]

Thus
\[ t \left( U^1(x) + U^2 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{2,\varepsilon}}{\varepsilon} \right) \right) + (1 - t) \left( U^1 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{1,\varepsilon}}{\varepsilon} \right) + U^2 \left( x + \frac{a^{(1)}_{1,\varepsilon} - a^{(2)}_{2,\varepsilon}}{\varepsilon} \right) \right) \rightarrow U^1 \]
locally uniformly.

Furthermore, note that
\[ \int \left( \omega^{(1)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) \right)^{p - 2} \xi \varphi \leq \left\| \omega^{(1)}_\varepsilon \left( x + a^{(1)}_{1,\varepsilon} \right) \right\|^{p - 2}_p \left\| \xi \right\|_{L^p(\text{supp } \varphi)} \left\| \varphi \right\|_p \]
\[ = o(1) \left\| \xi \right\|_{L^p(\text{supp } \varphi)} \left\| \varphi \right\|_p = o(1) \]
since \( \|\omega^{(i)}_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2}) \) and \( \xi \rightarrow \hat{\xi} \) in \( C^1(\text{supp } \varphi) \).

By (3.4)-(3.6), we conclude that (3.3) holds. The claim is proved.

Finally, by the fact that \( U^1 \) is nondegenerate, we infer that
\[ \hat{\xi} = \sum_{\alpha=1}^N d_\alpha \frac{\partial U^1}{\partial x_\alpha} \]
for some constants \( d_\alpha \in \mathbb{R}, \ 1 \leq \alpha \leq N \). The proof is complete. \( \square \)

Next we prove

**Lemma 3.2.** Let \( d_\alpha, \ 1 \leq \alpha \leq N \) be the coefficients given in Lemma 3. Then,
\[ d_\alpha = 0, \quad 1 \leq \alpha \leq N. \]
Proof. We will combine Proposition 1 and the identity (2.9) to prove that
\[ d_\alpha \int |x_\alpha|^{m-2} x_\alpha U^1(x) \frac{\partial U^1}{\partial x_\alpha} = 0. \] (3.7)
Once we prove (3.7), then \( d_\alpha = 0 \) since \( U^1 = U^1(|x|) \) is a radially symmetric decreasing function, which implies that \( x_\alpha U_{x_\alpha} \) is positive except at the origin.

We prove (3.7) as follows. By (2.2) and Lemma 3.1, we can choose \( d > 0 \) such that
\[ \int_{\partial B_d(a_1^{(i)})} \left( \epsilon^2 |\nabla \omega_1^{(i)}|^2 + |\omega_1^{(i)}|^2 \right) = O(\epsilon^{N+2m}). \] (3.8)
From Lemma 3.1, we have
\[ \int_{\partial B_d(a_1^{(i)})} \left( \epsilon^2 |\nabla \xi|^2 + |\xi|^2 \right) = O(\epsilon^N). \] (3.9)

Applying the Pohozaev type identity (2.9) to \( u_\epsilon^{(i)} \), \( i = 1, 2 \), in \( \Omega = B_d \left( a_1^{(i)} \right) \), we deduce
\[
-2 \int_{B_d(a_1^{(i)})} \frac{\partial K}{\partial x_\alpha} A_\epsilon \xi_e = \int_{\partial B_d(a_1^{(i)})} \left( \epsilon^2 \nabla \left( u_\epsilon^{(1)} + u_\epsilon^{(2)} \right) \cdot \nabla \xi_e + \left( u_\epsilon^{(1)} + u_\epsilon^{(2)} \right) \cdot \xi_e \right) \nu_\alpha \\
-2 \int_{\partial B_d(a_1^{(i)})} \epsilon^2 \left( \frac{\partial \xi_e}{\partial \nu} \frac{\partial u_\epsilon^{(1)}}{\partial x_\alpha} + \frac{\partial \xi_e}{\partial x_\alpha} \frac{\partial u_\epsilon^{(2)}}{\partial \nu} \right) - 2 \int_{\partial B_d(a_1^{(i)})} K(x) A_\epsilon \xi_e \nu_\alpha,
\] (3.10)
where \( 1 \leq \alpha \leq N \) and
\[ A_\epsilon(x) = \int_0^1 \left( t u_\epsilon^{(1)}(x) + (1 - t) u_\epsilon^{(2)}(x) \right)^{p-1}. \]

Using the same computation as in the proof of (2.1), (3.9) and (3.8), we have
\[
\int_{\partial B_d(a_1^{(i)})} \epsilon^2 |\nabla \xi_e| |\nabla u_\epsilon^{(1)}(x)| \leq \left( \int_{\partial B_d(a_1^{(i)})} \epsilon^2 |\nabla \xi_e|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_d(a_1^{(i)})} \epsilon^2 |\nabla u_\epsilon^{(1)}(x)| \right)^{\frac{1}{2}} \\
= O \left( \epsilon^{N/2} \left( \epsilon^{\gamma} + \|\omega_\epsilon^{(i)}\|_2^\gamma \right) \right) \\
= O(\epsilon^{N+m})
\]
by taking \( \gamma \geq N/2 + m \). Thus,
\[
\int_{\partial B_d(a_1^{(i)})} \left( \epsilon^2 \nabla \left( u_\epsilon^{(1)} + u_\epsilon^{(2)} \right) \cdot \nabla \xi_e + \left( u_\epsilon^{(1)} + u_\epsilon^{(2)} \right) \cdot \xi_e \right) \nu_\alpha \\
-2 \int_{\partial B_d(a_1^{(i)})} \epsilon^2 \left( \frac{\partial \xi_e}{\partial \nu} \frac{\partial u_\epsilon^{(1)}}{\partial x_\alpha} + \frac{\partial \xi_e}{\partial x_\alpha} \frac{\partial u_\epsilon^{(2)}}{\partial \nu} \right) = O(\epsilon^{N+m}).
\]
As to \( A_\epsilon(x) \), we have
\[ A_\epsilon(x) = O \left( \epsilon^1 + \sum_{i=1}^2 |\omega_\epsilon^{(i)}|^{p-1} \right) \text{ on } \partial B_d \left( a_1^{(i)} \right) \]
for any given $\gamma > 0$ since $U^i$, $i = 1, 2$, decay exponentially at infinity. Hence we can deduce as in the above that

$$\int_{\partial B_d(a^{(1)}_{1,\epsilon})} K(x) A_{\epsilon} \xi_{\epsilon} \nu_{\epsilon} \leq C \epsilon^\gamma \left( \int_{\partial B_d(a^{(1)}_{1,\epsilon})} |\xi_{\epsilon}|^2 \right)^{\frac{1}{2}}$$

$$+ C \sum_{i=1}^2 \left( \int_{\partial B_d(a^{(1)}_{1,\epsilon})} |\omega_{\epsilon}^{(i)}| p \right)^{\frac{p-1}{p}} \left( \int_{\partial B_d(a^{(1)}_{1,\epsilon})} |\xi_{\epsilon}|^p \right)^{\frac{1}{p}}$$

$$\leq C \epsilon^{\gamma + N/2} + O \left( \sum_{i=1}^2 \left( \epsilon^{\frac{p}{p-1}} \|\omega_{\epsilon}^{(i)}\|_p \right)^{\frac{p-1}{p}} \epsilon^{\frac{p}{p-2}} \|\xi_{\epsilon}\|_p \right)$$

$$= O \left( \epsilon^{N+(p-1)m} \right)$$

by taking $\gamma$ sufficiently large, where we have used (2.10) again. Hence, by the above estimates, there holds the right hand side of (3.10) $= O(\epsilon^{N+m})$.

This gives,

$$\int_{B_\epsilon(a^{(1)}_{1,\epsilon})} \frac{\partial K}{\partial x_\alpha} A_{\epsilon} \xi_{\epsilon} = O(\epsilon^{N+m}).$$

(3.11)

Now we estimate the left hand side of (3.11). By the assumption (K1), we have

$$\int_{B_\epsilon(a^{(1)}_{1,\epsilon})} \frac{\partial K}{\partial x_\alpha} A_{\epsilon} \xi_{\epsilon} = k_{1,\alpha} \int_{B_\epsilon(a^{(1)}_{1,\epsilon})} |x_\alpha - a_{1,\alpha}|^{m-2} (x_\alpha - a_{1,\alpha}) A_{\epsilon} \xi_{\epsilon}$$

$$+ O \left( \int_{B_\epsilon(a^{(1)}_{1,\epsilon})} |x - a_1|^m A_{\epsilon} \xi_{\epsilon} \right)$$

$$=: l_1 + l_2.$$

Note that

$$A_{\epsilon} = O \left( \epsilon^{\epsilon(1)}_{\epsilon} \right)^{p-1} + \epsilon^{\epsilon(2)}_{\epsilon} \right)^{p-1}.$$

Thus, we have

$$l_2 = \sum_{i=1}^2 O \left( \epsilon^N \int_{B_{\epsilon}^{(0)}} |\epsilon x + a^{(1)}_{1,\epsilon} - a_1|^{m} \epsilon^{\epsilon(1)}_{\epsilon} \epsilon^{\epsilon(1)}_{\epsilon} \right)^{p-1} \xi_{\epsilon}$$

$$= \sum_{i=1}^2 O \left( \epsilon^{N+m} \int_{B_{\epsilon}^{(0)}} (|x|^m + 1) \epsilon^{\epsilon(1)}_{\epsilon} \epsilon^{\epsilon(1)}_{\epsilon} \right)^{p-1} \xi_{\epsilon}$$

$$= O(\epsilon^{N+m}),$$

where we have used (2.1) in the second equality and the exponential decay of $U^i$ ($i = 1, 2$, (2.2) and (3.1) in the last equality. Hence, combining the estimate of $l_2$ and (3.11), together with the assumption $k_{1,\alpha} \neq 0$, we have

$$\int_{B_\epsilon(a^{(1)}_{1,\epsilon})} |x_\alpha - a_{1,\alpha}|^{m-2} (x_\alpha - a_{1,\alpha}) A_{\epsilon} \xi_{\epsilon} = O(\epsilon^{N+m}).$$
Equivalently, we obtain
\[
\int_{B^2_{\alpha}(0)} \left| x_\alpha + \frac{a^{(1)}_{1,\epsilon,\alpha} - a_{1,\alpha}}{\epsilon} \right|^{m-2} \left( x_\alpha + \frac{a^{(1)}_{1,\epsilon,\alpha} - a_{1,\alpha}}{\epsilon} \right) \partial U^1(x) \, \partial x_\beta = 0.
\]
By the similar arguments as in (3.3) and note that \( |a^{(1)}_{1,\epsilon,\alpha} - a_{1,\alpha}| = o(\epsilon) \) by (2.1), we deduce from the above estimate that
\[
\sum_{\beta=1}^N d_\beta \int |x_\alpha|^{m-2} x_\alpha U^1(x) \, \partial U^1(x) \, \partial x_\beta = 0.
\]
Since \( U^1 \) is radially symmetric, we infer from the above that (3.7) holds. Thus, \( d_\alpha = 0 \) for all \( 1 \leq \alpha \leq N \). The proof of Lemma (3.2) holds.

Now the following result is natural.

**Proposition 4.** For any fixed \( R > 0 \), there holds
\[
\| \xi_\epsilon \|_{L^\infty(B_{R^2}(a^{(1)}_{1,\epsilon})))} = o(1) \quad \text{as} \; \epsilon \to 0.
\]
**Proof.** By Lemma 3 and Lemma 3.2, we deduce that \( \xi_\epsilon \to 0 \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \). Hence, for any fixed \( R > 0 \), there holds
\[
\| \xi_\epsilon \|_{L^\infty(B_R(0))} = o(1) \quad \text{as} \; \epsilon \to 0.
\]
That is,
\[
\| \xi_\epsilon \|_{L^\infty(B_{R_{a^{(1)}_{1,\epsilon}}})))} = o(1) \quad \text{as} \; \epsilon \to 0.
\]
Similarly, we can prove that \( \| \xi_\epsilon \|_{L^\infty(B_{R_{a^{(1)}_{1,\epsilon}}})))} = o(1) \) holds also. The proof is complete.

Next, we estimate \( \xi_\epsilon \) in \( \mathbb{R}^N \setminus \bigcup_{i=1}^2 B_{R_{a^{(1)}_{1,\epsilon}}} \). We need the following estimate.

**Lemma 3.3.** Assume that \( K \in L^\infty(\mathbb{R}^N) \) and \( u_\epsilon = \sum_{i=1}^2 U_{i,\epsilon,\alpha} + \omega_\epsilon \) is a positive solution to equation (1.1) with \( \| \omega_\epsilon \|_{\infty} = o(\epsilon^{N/2}) \). Then
\[
\| \omega_\epsilon \|_{\infty} = o(1) \quad \text{as} \; \epsilon \to 0.
\]
**Proof.** We divide the proof into several steps.

**Claim 1:** \( \| \omega_\epsilon \|_{\infty} = O(1) \).
To prove Claim 1, it suffices to prove that \( \| u_\epsilon \|_{\infty} = O(1) \) since \( \omega_\epsilon = u_\epsilon - \sum_{i=1}^2 U_{i,\epsilon,\alpha} \) and \( U_i \), \( i = 1, 2 \), are bounded. We use \( L^p \)-theory for elliptic equations. For simplicity, assume that \( N \geq 3 \). Let \( \tilde{u}_\epsilon(x) = u_\epsilon(\epsilon x + a_1, \epsilon) \). Then \( \tilde{u}_\epsilon \) satisfies
\[
-\Delta \tilde{u}_\epsilon + \tilde{u}_\epsilon = \tilde{K}(x)(\tilde{u}_\epsilon(x))^{p-1} \quad \text{in} \; \mathbb{R}^N
\]
with \( \tilde{K}(x) = K(\epsilon x + a_1, \epsilon) \). Since \( \tilde{K} \in L^\infty \) and \( \| \tilde{u}_\epsilon \|_{H^1(\mathbb{R}^N)} \) is uniformly bounded with respect to \( \epsilon \) and \( 2 < p < 2^* \), we know that \( \tilde{K}(\tilde{u}_\epsilon)^{p-1} \in L^{p_1}(\mathbb{R}^3) \) with \( p_1 = 2^*/(p-1) \), and \( \| \tilde{u}_\epsilon \|_{p_1} \) is uniformly bounded. Hence, the standard \( L^p \)-theory for elliptic equations implies that \( \tilde{u}_\epsilon \in W^{2,p_1}(\mathbb{R}^N) \) with estimates
\[
\| \tilde{u}_\epsilon \|_{W^{2,p_1}(\mathbb{R}^N)} \leq C_{n,p} (\| u_\epsilon \|_{p_1} + \| K(\tilde{u}_\epsilon)^{p-1} \|_{p_1}) \leq C_{n,p}
\]
uniformly with respect to \( \epsilon \). If \( 2p_1 < N \), the Sobolev embedding theorem implies that \( u_\epsilon^{1} \in L^q(\mathbb{R}^N) \) with \( q^{-1} = p_1^{-1} - 2/N \) and \( \| u_\epsilon \|_q \) is uniformly bounded with respect to \( \epsilon \). Thus \( \tilde{K}(\tilde{u}_\epsilon)^{p-1} \in L^{p_2}(\mathbb{R}^N) \) with \( p_2 = q/(p-1) > p_1 \). We can iterate
until \( \bar{u}_\varepsilon \in W^{2,r}(\mathbb{R}^N) \) for some \( r > N/2 \), which implies that \( \bar{u}_\varepsilon \in C^\alpha(\mathbb{R}^N) \) for some \( \alpha \in (0, 1) \) with uniform boundedness. In particular, this implies that \( u_\varepsilon \in L^\infty(\mathbb{R}^N) \) and \( \|u_\varepsilon\|_\infty = O(1) \).

Claim 2: For any \( \eta > 0 \) sufficiently small, there exists \( R_0 > 1 \) such that
\[
|\omega_\varepsilon(x)| < \eta, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{2} B_{R_0}(a_{i,\varepsilon}).
\] (3.13)

We first prove that (3.13) holds with \( \omega_\varepsilon \) replaced by \( u_\varepsilon \). That is,
\[
|u_\varepsilon(x)| < \eta, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{2} B_{R_0}(a_{i,\varepsilon}).
\] (3.14)

Still consider \( \bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + a_{1,\varepsilon}) \). For any \( \eta > 0 \) sufficiently small, since \( \|\omega_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2}) \), we find that
\[
\int_{\mathbb{R}^N \setminus \bigcup_{i=1}^{2} B_{R_0}(a_{1,\varepsilon})} u_\varepsilon^2 \leq \sum_{i=1}^{2} \int_{\mathbb{R}^N \setminus B_{R_0}(a_{1,\varepsilon})} \left( U^1_{\varepsilon, a_{1,\varepsilon}} \right)^2 + \|\omega_\varepsilon\|_2^2
\]
\[
\leq \varepsilon^N \left( \sum_{i=1}^{2} \int_{\mathbb{R}^N \setminus B_{R_0}(0)} (U^i_\varepsilon)^2 + o(1) \right)
\]
\[
< \eta \varepsilon^N
\]
by taking \( R_0 > 1 \) sufficiently large. Hence, for such fixed \( R_0 \), we have
\[
\int_{\mathbb{R}^N \setminus (B_{R_0}(0) \cup B_{R_0}((a_{2,\varepsilon} - a_{1,\varepsilon})/\varepsilon))} \bar{u}_\varepsilon^2 = o(1).
\] (3.15)

Keep in mind that \( |a_{1,\varepsilon} - a_{2,\varepsilon}|/\varepsilon \to \infty \) as \( \varepsilon \to 0 \). Applying the local boundedness estimates of elliptic equations to \( \bar{u}_\varepsilon \) in equation (3.12), using (3.15) and Claim 1, we obtain, for any \( x \in \mathbb{R}^N \setminus (B_{2R_0}(0) \cup B_{2R_0}((a_{2,\varepsilon} - a_{1,\varepsilon})/\varepsilon)) \),
\[
\bar{u}_\varepsilon^2(x) \leq \max_{B_1(x)} \bar{u}_\varepsilon^2 \leq C \left( \int_{B_2(x)} \bar{u}_\varepsilon^2 + \|K \bar{u}_\varepsilon^{q-1}\|_{L^q(B_2(x))}^2 \right)
\]
\[
\leq C \left( \int_{B_2(x)} \bar{u}_\varepsilon^2 + \left( \int_{B_2(x)} \bar{u}_\varepsilon^2 \right)^{2/q} \right)
\]
\[
= o_\varepsilon(1)
\]

since \( B_2(x) \subset \mathbb{R}^N \setminus (B_{R_0}(0) \cup B_{R_0}((a_{2,\varepsilon} - a_{1,\varepsilon})/\varepsilon)) \). This shows that
\[
\bar{u}_\varepsilon(x) \leq \eta, \quad x \in \mathbb{R}^N \setminus (B_{R_0}(0) \cup B_{R_0}((a_{2,\varepsilon} - a_{1,\varepsilon})/\varepsilon)),
\]
which is equivalent to (3.14). Next (3.13) follows from \( \omega_\varepsilon = u_\varepsilon - \sum_{i=1}^{2} U^i_{\varepsilon, a_{i,\varepsilon}} \) and \( U^i, i = 1, 2 \), decay exponentially at infinity. Claim 2 is proved.

Claim 3: Let \( R_0 \) be given by Claim 2. Then
\[
\|\omega_\varepsilon\|_{L^\infty(\bigcup_{i=1}^{2} B_{R_0}(a_{i,\varepsilon}))} = o(1)
\]
as \( \varepsilon \to 0 \). We use Moser’s iteration argument. Note that \( \bar{\omega}_\varepsilon(x) = \omega_\varepsilon(\varepsilon x + a_{1,\varepsilon}) \) satisfies equation
\[
-\Delta \bar{\omega}_\varepsilon(x) + \bar{\omega}_\varepsilon(x) = f \quad \text{in} \ B_{2R_0}(0),
\]
Proposition 5. Let 
\[ f = \tilde{K} \left( \left( U^1 + U^2 \left( x + \frac{a_1 \epsilon - a_2 \epsilon}{\epsilon} \right) + \omega^1_\epsilon \right) \right)^{p-1} \]
and \( \tilde{K}(x) = K(\epsilon x + a_1 \epsilon) \). For any fixed \( \delta > 0 \) sufficiently small, using the elementary inequality
\[ |(a+b)^{p-1} - a^{p-1}| \leq \delta a^{p-1} + C \delta b^{p-1}, \quad a, b > 0, p > 2 \]
we obtain
\[ |f(x)| \leq \delta (U^1(x))^{p-1} + C \delta \left( \left( U^2 \left( x + \frac{a_1 \epsilon - a_2 \epsilon}{\epsilon} \right) \right) \right)^{p-1} + (\tilde{\omega}_\epsilon(x))^{p-1} \]
Note that \( |a_1 \epsilon - a_2 \epsilon|/\epsilon \to \infty \) as \( \epsilon \to 0 \) and \( \|\omega_\epsilon\|_\infty = O(1) \) by Claim 1. Consequently, for any fixed \( q > N/2 \), there holds
\[ \left( \int_{B_{2R}(0)} |f|^q \right)^{1/q} \leq C \delta + C \delta o_\epsilon(1) \]
since \( U^2 \) decays exponentially and \( \int_{\mathbb{R}^N} |\tilde{\omega}_\epsilon|^2 = o_\epsilon(1) \). Hence, by the local boundedness estimates of elliptic equations, we find
\[ \|\tilde{\omega}_\epsilon\|_{L^\infty(B_{R_0}(0))} \leq C \left( \|\tilde{\omega}_\epsilon\|_{L^2(B_{R_0}(0))} + \|f\|_{L^q(B_{2R_0}(0))} \right) \leq C \delta + C \delta o_\epsilon(1). \]
For any fixed \( \eta > 0 \), we can first choose \( \delta > 0 \) sufficiently small such that \( C \delta < \eta/2 \), and then choose \( \epsilon_0 > 0 \) small enough such \( C \delta o_\epsilon(1) < \eta/2 \). This proves that
\[ \|\tilde{\omega}_\epsilon\|_{L^\infty(B_{R_0},(a_1 \epsilon))} = o(1) \]
as \( \epsilon \to 0 \). Similarly, we can prove
\[ \|\tilde{\omega}_\epsilon\|_{L^\infty(B_{R_0},(a_2 \epsilon))} = o(1). \]
This proves Claim 3.

Combining Claim 2 and Claim 3 finishes the proof. \( \square \)

Now we can prove the following estimate.

Proposition 5. Let \( R > 0 \) be sufficiently large. There holds
\[ \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^N \setminus \bigcup_{i=1}^2 B_{R_\epsilon}(a_{i,\epsilon}))} = o(1) \quad \text{as} \ \epsilon \to 0. \]

Proof. Recall that \( \xi_\epsilon \) satisfies equation (3.2). By Lemma 3.3 and the exponential decay of \( U^1 \) at infinity, we infer that
\[ C_\epsilon(x) = o_\epsilon(1) + a_\epsilon(1), \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^2 B_{R_\epsilon}(a_{i,\epsilon}), \]
where \( C_\epsilon(x) \) is the coefficient in equation (3.2), and \( a_\epsilon(1) \to 0 \) as \( R \to \infty \). Then, Proposition 4 implies that \( \xi_\epsilon \) satisfies
\[
\begin{cases}
-\epsilon^2 \Delta \xi_\epsilon + (1 - C_\epsilon(x)) \xi_\epsilon = 0, & x \in \mathbb{R}^N \setminus \bigcup_{i=1}^2 B_{R_\epsilon}(a_{i,\epsilon}), \\
\xi_\epsilon(x) = a_\epsilon(1), & x \in \partial \left( \bigcup_{i=1}^2 B_{R_\epsilon}(a_{i,\epsilon}) \right) \\
\xi_\epsilon(x) \to 0 & \text{as} \ |x| \to 0.
\end{cases}
\]
We may take $R$ sufficiently large and $\epsilon$ sufficiently small such that
$$1 - C(1)_{\epsilon}(x) \geq 1/2, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{2^N} B_{R\epsilon}(a^{(1)}_{\epsilon}, \epsilon)$$
for $\epsilon$ sufficiently small. Hence, applying the maximum principle to $\xi_\epsilon$ in the above problem, we obtain the desired result. The result is complete.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 4 and Proposition 5, we conclude that 3.1 holds. However, this contradicts $\|\xi_\epsilon\|_\infty = 1$. The proof of Theorem 1.2 is complete.

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