A note on non-Robba p-adic differential equations

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Abstract: Let $\mathcal{M}$ be a differential module, whose coefficients are analytic elements on an open annulus $I$ ($\subset \mathbb{R}_{>0}$) in a valued field, complete and algebraically closed of inequality characteristic, and let $R(\mathcal{M}, r)$ be the radius of convergence of its solutions in the neighborhood of the generic point $t_r$ of absolute value $r$, with $r \in I$. Assume that $R(\mathcal{M}, r) < r$ on $I$ and, in the logarithmic coordinates, the function $r \rightarrow R(\mathcal{M}, r)$ has only one slope on $I$. In this paper, we prove that for any $r \in I$, all the solutions of $\mathcal{M}$ in the neighborhood of $t_r$ are analytic and bounded in the disk $D(t_r, R(\mathcal{M}, r)^{-1})$.

Key words: $p$-adic differential equations; Frobenius antecedent theorem.

1. Notations and Preliminaries. Let $p$ be a prime number, $\mathbb{Q}_p$ the completion of the field of rational numbers for the $p$-adic absolute value $|.|$, $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$, and $\Omega_p$ a $p$-adic complete and algebraically closed field containing $\mathbb{C}_p$ such that its value group is $\mathbb{R}_{\geq 0}$ and the residue class field is strictly transcendental over $\mathbb{F}_p$. For any positive real $r, t_r$ will denote a generic point of $\Omega_p$ such that $|t_r| = r$. Let $I$ be a bounded interval in $\mathbb{R}_{\geq 0}$. We denote by $\mathcal{A}(I)$ the ring of analytic functions, on the annuli $C(I) := \{a \in \mathbb{Q}_p \mid |a| \in I\}$, $\mathcal{A}(I) = \{\sum_{n \in \mathbb{Z}} a_n x^n \in C_p[[x, 1/x]] \mid \lim_{n \to +\infty} |a_n| r^n = 0, \forall r \in I\}$, and by $\mathcal{H}(I)$ the completion of the ring of rational fractions $f$ of $C_p(x)$ with no pole in $C(I)$ with respect to the norm $\|f\|_I := \sup_{r \in I} |f(t_r)|$. It is well known that $\mathcal{H}(I) \subseteq \mathcal{A}(I)$, with equality if $I$ is closed. We define, for any $r \in I$, the absolute value $|.|_r$ over $\mathcal{A}(I)$ by $\left|\sum_{n \in \mathbb{Z}} a_n x^n\right|_r = \sup_{n \in \mathbb{Z}} |a_n| r^n$.

Let $R(I)$ denotes $\mathcal{A}(I)$ or $\mathcal{H}(I)$. A free $R(I)$-module $\mathcal{M}$ of finite rank $\mu$ is said to be $R(I)$-differential module if it is equipped with a $R(I)$-linear map $D: \mathcal{M} \to \mathcal{M}$ such that $D(am) = \partial(a)m + aD(m)$ for any $a \in R(I)$ and any $m \in \mathcal{M}$ where $\partial = d/dx$. To each $R(I)$-basis $\{e_i\}_{1 \leq i \leq \mu}$ of $\mathcal{M}$ over $R(I)$ corresponds a matrix $G = (G_{ij}) \in M_{\mu}(R(I))$ satisfying $D(e_i) = \sum_{j=1}^\mu G_{ij} e_j$, called the matrix of $\partial$ with respect to the $R(I)$-basis $\{e_i\}_{1 \leq i \leq \mu}$ or simply an associated matrix to $\mathcal{M}$, together with a differential system $\partial X = GX$ where $X$ denotes a column vector $\mu \times 1$ or $\mu \times \mu$ matrix (see [2,3]). If $G' \in M_{\mu}(R(I))$ is the matrix of $\partial$ with respect to another $R(I)$-basis $\{e'_i\}_{1 \leq i \leq \mu}$ of $\mathcal{M}$ and if $H = (H_{ij}) \in GL_{\mu}(R(I))$ is the change of basis matrix defined by $e'_i = \sum_{i=1}^\mu H_{ij} e_j$ for all $1 \leq i \leq \mu$, it is known that:

- the matrices $G$ and $G'$ are related by the formula $G' = HGH^{-1} + \partial(H)H^{-1}$. The matrix $HGH^{-1} + \partial(H)H^{-1}$ is denoted $H[G]$.
- if $Y$ is a solution matrix for the system $\frac{d}{dx} X = GX$ with coefficients in a differential field extension of $R(I)$, then the matrix $HY$ is a solution matrix for $\frac{d}{dx} X = H[G]X$.

Generic radius of convergence. Let $\mathcal{M}$ be an $R(I)$-differential module of rank $\mu$, $G = (G_{ij}) \in M_{\mu}(R(I))$ an associated matrix to $\mathcal{M}$, $(G_n)_{n}$ a sequence of matrices defined by $G_0 = I_{\mu}$ and $G_{n+1} = \partial(G_n) + G_n G$, and $\|G\|_\infty = \max |G_{ij}|$ be the norm of $G$ associated to the absolute value $|.|_\infty$. For any $r \in I$, the quantity $R(\mathcal{M}, r) = \min(r, \liminf \|G_n\|^{-1/n})$ represents the radius of convergence in the generic disc $D(t_r, r^{-1})$ of the solution matrix.
of the system $\frac{dy}{dx} = X$ with $X(t_r) = I_r$. We know that the function $r \mapsto R(M, r)$ is independent of the choice of basis and the ring $R(I)$ [3, Proposition 1.3], and the graph of the map $\rho \mapsto \log R(M, \exp(\rho))$, on any closed subinterval of $I$, is a concave polygon with rational slopes [5, Theorem 2]. This graph is called the generic polygon of the convergence of $M$. The system $\partial X = G X$ is said to have an analytic and bounded solution in the disk $D(t_r, R(M, r)^{-1})$ if

$$\sup_{n \geq 0} \left\| \frac{G_n}{n!} \right\|_r R(M, r)^n < \infty.$$ 

The $R(I)$-differential module $M$ is said to be non-Robba if $R(M, r) < r$ for all $r \in I$.

**Robba.** Let $\varphi : C(I) \to C(P)$ be the Frobenius ramification $x \mapsto x^p$, where $P$ is the image of $I$ by the map $x \mapsto x^p$. A $R(P)$-differential module $N$ is said to be a Frobenius antecedent of an $R(I)$-differential module $M$ if $M$ is isomorphic to the $R(I)$-differential module $N$ such that $px^p - F(x^p)$ is a matrix of $d/dz$ in some $R(I)$-basis of $M$. The existence of such a Frobenius antecedent depends of the values of the function $r \mapsto R(M, r)$. Recall the Frobenius structure theorem of Christol-Mebkhout [4, Theorem 4.1-4] where $\pi = p^{-1/p - 1}$.

**Theorem 1.1.** Let $h$ be a positive integer and let $M$ be an $R(\hat{I})$-differential module such that $R(M, r) > r^{1/p - 1}$ for all $r \in I$. Then, there exists an $R(I)$-differential module $N_h$ such that $(\varphi^h)^* N_h \cong M$ and $R(M, r)^p = R(N_h, r^p)$ for any $r \in I$, and $N_h$ is the Frobenius antecedent of order $h$ of $M$.

In particular, if a $R(I)$-differential module $M$ satisfies $R(M, r) > r\pi$ for all $r \in I$, it has a Frobenius antecedent.

**2. Main theorem.** In this section, $I$ denotes an open interval in $R_{\geq 0}$ and $M$ a non-Robba $A(I)$-differential module associated to some matrix $G \in M_p(A(I))$.

**Theorem 2.1.** Assume that the generic polygon of convergence of $M$ has only one slope. Then

$$\sup_{n \geq 0} \left\| \frac{G_n}{n!} \right\|_r R(M, r)^n < \infty \quad \text{for all} \quad r \in I.$$

The proof of this theorem is based on the following lemmas:

**Lemma 2.2.** Assume $R(M, r) > r\pi$ for all $r \in I$ and let $N$ be a Frobenius antecedent of $M$. Let $F$ be an associated matrix to $N$ and assume there exists a real $r_0 \in I$ such that $\sup_{n \geq 0} \left\| \frac{F_n}{n!} \right\|_r R(N, r_0)^n < \infty$. Then

$$\sup_{n \geq 0} \left\| \frac{G_n}{n!} \right\|_r R(M, r_0)^n < \infty.$$

**Proof.** The matrix $V(z) = (V_{ij}(z))_{ij} = V_{F,G}(z) = \sum_{n \geq 0} \frac{F_n(z)}{n!} (z - t_r)^n$ is the solution matrix of the differential system $\frac{d}{dz} V(z) = F(z) V(z)$ in the neighborhood of $t_r^p$ for all $z = x^p$ and $V(t_x) = I_p$. The change of variables leads to $\frac{d}{dz} V(x^p) = px^{p-1} F(x^p) V(x^p)$. In addition, since $R(M, r_0) > r\pi$, the map $x \mapsto x^p$ sends the closed disk $D(t_r^p, R(M, r_0))$ into $D(t_r^p, R(M, r_0)^p) = D(t_r^p, R(N, r_0)^p)^p$ [1, Lemma 2.1], and $\sup_{n \geq 0} \left\| \frac{F_n(z)}{n!} \right\|_r (x^p - t_r^p)^n = \sup_{n \geq 0} \left( \left\| \frac{F_n(z)}{n!} \right\|_r x^{p-1} + x^{p-1} t_n^p + \ldots + x^{-1} t_n^p \right) < \infty$ for all $x \in D(t_r^p, R(M, r_0))$. In the neighborhood of $t_r^p$, the matrix $V_{F,G}(x^p)$ can be written as $V(x^p) = \sum_{n \geq 0} B_n(x - t_r^p)$ where $B_n = (B_n(i, j))_{ij}$ are $\nu \times \nu$ matrices with entries in $\Omega$. In that case, we have $\lim_{\rho \to 0} |B_n(i, j)| \rho^\nu = 0$ for any $\rho < R(M, r_0)$, and therefore

$$\sup_{n \geq 0} |B_n(i, j)| \rho^\nu = \sup_{x \in D(t_r^p, \rho)} |V_{ij}(x^p)| \leq \sup_{z \in D(t_r^p, \rho^p)} |V_{ij}(z)| = \sup_{n \geq 0} \left\| \frac{F_n(z)}{n!} \right\|_r \rho^p.$$ 

Since the matrix $V(z)$ is analytic and bounded in $D(t_r^p, R(M, r_0)^p)$, there exists a positive constant $C > 0$, by [2, Proposition 2.3.3], such that

$$\sup_{n \geq 0} \left\| \frac{F_n(z)}{n!} \right\|_r \rho^p < C$$

for any $\rho < R(M, r_0)$ and close to $R(M, r_0)$. Combining (2.1) and (2.2), and using again [2, Proposition 2.3.3], we find $\sup_{n \geq 0} |B_n(i, j)| R(M, r_0)^n < \infty$ for all $1 \leq i, j \leq \nu$, and therefore, the matrix $V(x^p)$ is analytic and bounded in the disk $D(t_r^p, R(M, r_0)^p)$. In addition, since the matrix $px^{p-1} F(x^p)$ is associated
to $\mathcal{M}$, then there exists an invertible matrix $H \in \text{GL}_{n}(\mathcal{A}(J))$ (hence $H$ is analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$) such that 

$$G = H[p_{x} \sigma^{\rho} - F(x^{\rho})].$$

Thus, by [2, Proposition 2.3.2], the matrix $HV(x^{\rho})$ is a solution to the system $\partial X = GX$ in the neighborhood of $t_{\rho}$, and moreover it is analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$. This means that $U_{G,J}(x) = HV(x^{\rho})H(t_{\rho})^{-1}$ is also analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$. \qed

**Lemma 2.3.** The set of reals $r$ in $I$ for which 

$$\sup_{x \geq 0} \left\| \frac{G_{n}}{n!} R(M, r)^{n} \right\| < \infty$$

is dense in $I$.

**Proof.** Let $J$ be a closed subinterval of $I$ not reduced to a point and let $\rho$ be a real number in the interior of $J$. Then, by hypothesis, $R(M, \rho)/\rho < 1$ and therefore there exists an integer $k$ such that $\pi^{1/\rho^{k}} < R(M, \rho)/\rho < \pi^{1/\rho^k}$. Since the function $r \mapsto R(M, r)$ is continuous on $J$, there exists an open subinterval $J' \subset J$ containing $\rho$ such that $\pi^{1/\rho^{k}} < R(M, r) < \pi^{1/\rho^k}$ for all $r \in J'$.

There are two cases to consider:

**Case 1:** $h \leq 0$.

Let $H(J')$ be the quotient field of $H(J')$. By cyclic vector lemma, we can associate $H(J') \otimes \mathcal{M}$ to a differential equation $\Delta(H(J) \otimes \mathcal{M}) = \partial^{\rho} + q_{I}(x)\partial^{\rho-1} + \cdots + q_{0}(x)$, where $q_{i}(x)$ is in $H(J')$ for $i = 1, \ldots, \mu$. Now pick a nonempty subinterval $J''$ of $J'$ such that $q_{i}(x) \in H(J'')$ for $i = 1, \ldots, \mu$, and let $\rho$ be a real number in the interval $J''$ and $\lambda(\rho)$ be the minimum of the $p$-adic absolute values of the roots of the polynomial $\Delta(H(J) \otimes \mathcal{M}) = \lambda^{\rho} + q_{0}(x)\lambda^{\rho-1} + \cdots + q_{\mu}(x)$. Since $R(M, r_{0}) = R(H(J) \otimes \mathcal{M}, r_{0})$. By virtue of [6, Theorem 3.1], we have $\log(R(M, r_{0}) = \frac{1}{\lambda^{\rho}} = \log(\lambda(\rho))$ and all the solutions $u_{1}, \ldots, u_{\mu}$ of $\Delta(H(J) \otimes \mathcal{M})$ in the neighborhood of $t_{\rho}$ are analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$. Now let $W$ be the wronskian matrix of $(u_{1}, \ldots, u_{\mu})$. Then, $W$ is a solution of the system $\partial X = A_{\Delta(H(J) \otimes \mathcal{M})}X$ where

$$A_{\Delta(H(J) \otimes \mathcal{M})} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-q_{\mu} & -q_{\mu-1} & -q_{\mu-2} & \cdots & -q_{1}
\end{bmatrix}$$

Moreover, by [2, Proposition 2.3.2], the matrix $W$ is analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$. Since $G$ and $A_{\Delta(H(J) \otimes \mathcal{M})}$ are associated to $H(J') \otimes \mathcal{M}$, there exists a matrix $H \in \text{GL}_{n}(H(J'))$ such that $G = H[A_{\Delta(H(J) \otimes \mathcal{M})}]$. Since $R(M, r_{0}) < r_{0}$, the matrix $H$ is analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0})^{-})$. Hence, by [2, Proposition 2.3.2], the matrix $U_{G,J}(x) = HV(x^{\rho})H(t_{\rho})^{-1}$ is also analytic and bounded in the disk $D(t_{\rho}, R(M, r_{0}))$. This ends the proof of the lemma in this case.

**Case 2:** $h > 0$.

Applying Theorem 1.1 to $H(J') \otimes \mathcal{M}$, there exists a $H(J')$-differential module $\mathcal{M}$, which is a Frobenius antecedent of order $h$ of $H(J') \otimes \mathcal{M}$. Moreover, $R(N_{h}, \rho) < \pi \rho$ for all $\rho \in J^\rho$. Let $b^{h}F$ be an associated matrix of $\mathcal{N}_{h}$. Then, by case 1, there exists $r_{0} \in J'$ such that $b^{h}F$ is analytic and bounded in the disk $D(t_{\rho}, R(N_{h}, r_{0}))$. The proof of the lemma in this case can be concluded by iteration of Lemma 2.2. \qed

**Proof of Theorem 2.1.** By hypothesis, the generic polygon of convergence of $\mathcal{M}$ has only one slope. This slope is a rational number by [5, Theorem 2]. Thus, we may assume there exist $\alpha \in C_{0}$ and $\beta \in Q$ such that $R(M, r) = |\alpha| r^{\beta}$ for all $r \in I$.

Let now $r$ be a real in the interior of $J$. Then, by Lemma 2.3, there exist two reals $r_{1}, r_{2} \in I$ such that $r_{1} < r < r_{2}$ with $\sup_{n \geq 0} \left\| \frac{G_{n}}{n!} R(M, r_{1})^{n} \right\| < \infty$ and $\sup_{n \geq 0} \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\| < \infty$, which are equivalent to $\sup_{n \geq 0} \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{1}} < \infty$ and $\sup_{n \geq 0} \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{2}} < \infty$. Since all the matrices $\alpha^{n} x^{n} G_{n}$ have all their entries in $H[r_{1}, r_{2}]$, and for any element $f \in H([r_{1}, r_{2}])$, we have $|f| \leq \max(|f_{r_{1}}, |f_{r_{2}}|)$, then for any integer $n \geq 0$, we have

$$\left\| \frac{G_{n}}{n!} R(M, r)^{n} \right\|_{r} \leq \max \left( \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{1}}, \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{2}} \right).$$

Hence, for

$$\sup_{n \geq 0} \left\| \frac{G_{n}}{n!} R(M, r)^{n} \right\|_{r} \leq \max \left( \sup_{n \geq 0} \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{1}}, \sup_{n \geq 0} \left\| \frac{G_{n}}{n!} \alpha^{n} x^{n} \right\|_{r_{2}} \right) < \infty.$$

\qed
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