SINGLE-REALIZATION RECOVERY OF A RANDOM SCHRÖDINGER EQUATION WITH UNKNOWN SOURCE AND POTENTIAL

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Abstract. In this paper, we study an inverse scattering problem associated with the stationary Schrödinger equation where both the potential and the source terms are unknown. The source term is assumed to be a generalised Gaussian random distribution of the microlocally isotropic type, whereas the potential function is assumed to be deterministic. The well-posedness of the forward scattering problem is first established in a proper sense. It is then proved that the rough strength of the random source can be uniquely recovered, independent of the unknown potential, by a single realisation of the passive scattering measurement. We develop novel techniques to completely remove a restrictive geometric condition in our earlier study [J. Li, H. Liu, and S. Ma, Comm. Math. Phys. 381 (2021), 527–556], at an unobjectionable cost of requiring the unknown potential to be deterministic. The ergodicity is used to establish the single realisation recovery, and the asymptotic arguments in our analysis are based on techniques from the theory of pseudo-differential operators and the stationary phase principle.

Keywords: random Schrödinger equation, inverse scattering, microlocally isotropic Gaussian distribution, single realisation, ergodicity, pseudo-differential operators

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1. Introduction

1.1. Statement of the main results. In this paper, we are mainly concerned with the quantum scattering problem governed by the following stationary Schrödinger equation (cf. [13, 14])

\[
\begin{align*}
(-\Delta - E + V(x))u(x, \sqrt{E}, \omega) &= f(x, \omega), \quad x \in \mathbb{R}^3, \\
\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - i\sqrt{E}u \right) &= 0, \quad r := |x|.
\end{align*}
\]

In (1.1a)–(1.1b), \(u\) is the scattered wave field generated by the interaction of the source \(f\) and the scattering potential \(V\), and \(E \in \mathbb{R}_+\) signifies the energy level. We write \(k := \sqrt{E}\), namely \(E = k^2\), which can be regarded as the wavenumber for the time-harmonic wave scattering. \(\omega\) in (1.1a) is the random sample belonging to \(\Omega\) with \((\Omega, \mathcal{F}, \mathbb{P})\) being a complete probability space. The limit (1.1b) is known as the Sommerfeld radiation condition (SRC) (cf. [9]), which holds uniformly in the angular variable \(\hat{x} := x/|x| \in \mathbb{S}^2\) that characterizes the outgoing nature of the scattered wave field \(u\).

In our study, \(V\) is assumed to be a deterministic smooth function, and \(f\) is assumed to be a compactly supported generalised Gaussian random distribution of the microlocally isotropic type (cf. [7, 20]), which is rigorously characterised as follows for the self-containedness of our study. First, it means that \(f(\cdot, \omega)\) is a random distribution and the mapping

\[
\omega \in \Omega \mapsto \langle f(\cdot, \omega), \varphi \rangle \in \mathbb{C}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),
\]

is a Gaussian random variable whose probabilistic measure depends on the test function \(\varphi\). Here and also in what follows, \(\mathcal{S}(\mathbb{R}^n)\) stands for the Schwartz space. Since both \(\langle f(\cdot, \omega), \varphi \rangle\)
and $\langle f(\cdot, \omega), \psi \rangle$ are random variables for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, from a statistical point of view, the covariance between these two random variables,

$$\mathbb{E}_\omega((f(\cdot, \omega) - \mathbb{E}(f(\cdot, \omega)), \varphi) \langle f(\cdot, \omega) - \mathbb{E}(f(\cdot, \omega)), \psi \rangle),$$

(1.2)
can be understood as the covariance of $f$, where $\mathbb{E}_\omega$ means to take expectation on the argument $\omega$. Formula (1.2) defines an operator $\mathcal{C}_f$,

$$\mathcal{C}_f : \varphi \in \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{C}_f(\varphi) \in \mathcal{S}'(\mathbb{R}^n),$$
in a way that $\mathcal{C}_f(\varphi) : \psi \in \mathcal{S}(\mathbb{R}^n) \mapsto (\mathcal{C}_f(\varphi))(\psi) \in \mathbb{C}$ where

$$(\mathcal{C}_f(\varphi))(\psi) := \mathbb{E}_\omega((f(\cdot, \omega) - \mathbb{E}(f(\cdot, \omega)), \varphi) \langle f(\cdot, \omega) - \mathbb{E}(f(\cdot, \omega)), \psi \rangle).$$

The operator $\mathcal{C}_f$ is called the covariance operator of $f$.

**Definition 1.1.** A generalized Gaussian random distribution $f$ on $\mathbb{R}^3$ is called microlocally isotropic with rough order $-m$ and rough strength $\mu(x)$ in a bounded domain $D$, if the following conditions hold:

1. the expectation $\mathbb{E}(f)$ is in $C^\infty(\mathbb{R}^3)$ with $\text{supp} \mathbb{E}(f) \subset D$;
2. $f$ is supported in $D$ a.s. (namely, almost surely);
3. the covariance operator $\mathcal{C}_f$ is a classical pseudodifferential operator of order $-m$;
4. $\mathcal{C}_f$ has a principal symbol of the form $\mu(x)|\xi|^{-m}$ with $\mu \in C^\infty_c(\mathbb{R}^3; \mathbb{R})$, $\text{supp} \mu \subset D$ and $\mu(x) \geq 0$ for all $x \in \mathbb{R}^3$.

In what follows, we abbreviate a microlocally isotropic Gaussian random distribution as an m.i.g.r. function. Let $f$ be an m.i.g.r. function. We consider the corresponding forward and inverse scattering problems associated with the Schrödinger equation (1.1a)–(1.1b). For the forward scattering problem, we shall show that there exists a well-defined scattering map in a proper sense as follows:

$$(f, V) \rightarrow u^\infty(\hat{x}, k, \omega), \quad \hat{x} \in \mathbb{S}^2, k \in \mathbb{R}^+ \subset \Omega,$$

where $u^\infty$ is a random distribution on the unit sphere and is called the far-field pattern. That is, for a given pair of $(f, V)$, by solving the forward scattering system (1.1a)–(1.1b), one can obtain the far-field pattern in a proper sense. It is noted that the far-field pattern is generated through the interaction of the source $f$ and the scattering potential $V$, and hence it carries the information of $f$ and $V$. The inverse scattering problem is concerned with recovering the unknown $f$ or and $V$ by knowledge of the far-field pattern, namely,

$$\{ u^\infty(\hat{x}, k, \omega) : \hat{x} \in \mathbb{S}^2, k \in \mathbb{R}^+, \omega \in \Omega \} \rightarrow (f, V).$$

(1.3)

It is noted that the measured far-field pattern in (1.3) is produced by the unknown source, and it is referred to as the passive measurement in the scattering theory. This is in difference to the active measurement, where one exerts certain known wave sources to generate the scattered waves in order to recover the unknown objects.

In what follows, we shall impose the following mild regularity assumption on the potential $V$:

$$V \in C^5(\mathbb{R}^3), \quad V \in L^2_{3/2+\epsilon}(\mathbb{R}^3) \text{ and } \partial^\alpha V \in L^{\infty}_{1/2+\epsilon}(\mathbb{R}^3) \text{ for } \forall \alpha : |\alpha| \leq 2.$$

(1.4)

It is emphasized that the above $C^5$-regularity requirement is mainly a technical condition, which shall be needed in our subsequent stationary phase argument (cf. (3.42)).

**Theorem 1.1.** Let $f$ be an m.i.g.r. distribution such that $\text{supp}(f) \subset D_f$ where $D_f$ is a bounded domain in $\mathbb{R}^3$ and $V$ satisfies (1.4). Let $\mu$ be the rough strength of $f$. Suppose that $f$ is of order $-m$ with $2 < m < 3$. Then the far-field data $u^\infty(\hat{x}, k, \omega)$ for all $(\hat{x}, k) \in \mathbb{S}^2 \times \mathbb{R}^+$ and a fixed $\omega \in \Omega$ can uniquely recover $\mu$ almost surely, independent of $V$. 
Remark 1.1. Theorem 1.1 indicates that a single realisation of the passive scattering measurement can uniquely recover the rough strength of the unknown source, independent of the scattering potential and the expectation of the source. In fact, our arguments in what follows in proving the theorem actually yield an explicit formula in recovering $\mu$ by the given far-field data (cf. formula (4.3)). It is emphasized we do not assume $V$ to be compactly supported. This is in sharp difference to our earlier study [24], where $V$ was also assumed to be compactly supported, and $\text{supp} V$ and $\text{supp} f$ are assumed to be well separated in the sense that their convex hulls stay a positive distance away from each other. We shall discuss more about this point in Section 1.2.

Remark 1.2. In Theorem 1.1, we only consider the recovery of the rough strength of the source, which is independent of the expectation of the source and the scattering potential, both of them being unknown. It is pointed out that in essence one can also recover the expectation of the source, but would need to make use of the full-realisation of the passive scattering measurement. Moreover, if active scattering measurement is further used, one may also be able to recover the potential by following similar arguments in [24]. However, in our view, the result presented in Theorem 1.1 is the most significant advancement in understanding the inverse scattering problem associated with the random Schrödinger equation (1.1a)–(1.1b).

1.2. Discussion of our results and literature review. Inverse scattering theory is a central topic in the mathematical study of inverse problems and on the other hand, it is the fundamental basis for many industrial and engineering applications, including radar/sonar, geophysical exploration and medical imaging. It is concerned with the recovery of unknown/inaccessible scattering objects by knowledge of the associated wave scattering measurements away from the objects. The scattering object could be a passive inhomogeneous medium or an active source. The scattering measurement might be generated by the underlying unknown source, referred to as the passive measurement, or by exerting a certain known wave field, referred to as the active measurement. Both the inverse medium scattering problem and the inverse source scattering problem in the deterministic settings have been intensively and extensively investigated in the literature; see e.g. [2,3,8,9,12,17,29,30,33] for some recent related studies and the references cited therein. The simultaneous recovery of an unknown source as well as the material parameter of an inhomogeneous medium by the associated passive measurement was considered [18,25], which arises in the photoacoustic and thermoacoustic tomography as an emerging medical imaging modality. Similar inverse problems were also considered in [10,11] associated with the magnetohydrodynamical system and in [12] associated with the Maxwell system that are related to the geomagnetic anomaly detection and the brain imaging, respectively. Inverse scattering problems in the random settings have also received considerable attentions in the literature; see e.g. [1,4–6,19–21,23,24,26,32] and the references therein. In [27], the second author of the present paper gives a review on recent progress of single-realization recoveries of random Schrödinger systems, and discuss some key ideas in [20] and [24].

Among the aforementioned studies of the random inverse problems, we are particularly interested in the case where a single random sample is used to recover the unknowns. Papapicolaou [4–6] studied the single realization recoveries that are more engineering-oriented. In [19,20], Lassas et. al. considered the inverse scattering problem for the two-dimensional random Schrödinger system, and recovered the rough strength of the potential by using the near-field data under a single random sample. In [23,24], we studied the random Schrödinger system in a different setting and recovered the rough strength under a single random sample. In [21], Li et. al. considered the inverse scattering problem of recovering a random source under a single random sample. It is emphasized that the recovery of the
potential is comparably more challenging than the recovery of the source. In this paper, we shall consider the case that both the source and the potential are unknown, making the corresponding study radically more challenging.

Recently, the m.i.g.r. model has been under an intensive study; see [7, 19–22] and the references cited therein. Two important parameters of the m.i.g.r. distribution are its rough order and rough strength. Roughly speaking, the rough order determines the degree of spatial roughness of the m.i.g.r., and the rough strength indicates its spatial correlation length and intensity. The rough strength also captures the micro-structure of the object of interest [20].

The current article is a continuation of our study in two recent works [23, 24] on the inverse scattering problem (1.3) associated with the Schrödinger system (1.1a)-(1.1b). The major connections and differences among those studies can be summarised as follows.

1. In [23], we considered the case that the random part of the source is a spatial Gaussian white noise, whereas the potential term is deterministic. It is proved that a single realisation of the passive scattering measurement can uniquely recover the variance of the random source, independent of the potential. However, in this paper, we derive a similar unique recovery result, but for the random source being a much more general m.i.g.r. distribution. As shall be seen in our subsequent analysis, the m.i.g.r source makes the corresponding analysis radically much more challenging.

2. In [24], both the source and potential terms were assumed to be random of the m.i.g.r. type. It was proved that a single realisation of the passive scattering measurement can uniquely recover the rough strength of the source, independent of the potential. However, in order to achieve such a unique recovery result, a restrictive geometric condition is critically required that the convex hulls of the supports of the source and potential are well separated. In this paper, we completely remove this geometric condition without imposing any assumption on the bounded supports of the source and the potential. As shall be seen in our subsequent study, the removal of this geometric condition makes the relevant analysis much more challenging and technical, and we develop novel mathematical techniques to handle this general geometric situation. On the other hand, it is remarked that the cost of removing this restrictive geometric condition is that we need to require the unknown potential to be deterministic. According to our intricate and subtle estimates in establishing the determination results in [24] and Theorem 1.1 in the present paper, we believe that such a cost is unobjectionable.

3. In both [23] and [24], it was shown that if full scattering measurement is used, namely both passive and active measurements are used, then both the source and the potential can be recovered. In this paper, we only consider the recovery of the source by using the associated passive measurement. Nevertheless, it is remarked that if full measurement is used, then one can also establish the recovery of both the source and the potential by following similar arguments to those in [23] and [24]; see Remark 1.2 as well.

The rest of the paper is organized as follows. In Section 2 we present the well-posedness of the direct scattering problem. Section 3 establishes several critical asymptotic estimates. In Section 4 we prove the unique recovery of the rough strength of the random source.

2. Well-posedness of the direct problem

In this section, the unique existence of a mild solution shall be established to the random Schrödinger system (1.1).
We first fix some notations that shall be used throughout the rest of the paper. We write $\mathcal{L}(A, B)$ to denote the set of all the bounded linear mappings from a normed vector space $A$ to a normed vector space $B$. For any mapping $K \in \mathcal{L}(A, B)$, we denote its operator norm as $\|K\|_{\mathcal{L}(A, B)}$. We use $C$ and its variants, such as $C_D$, $C_{D, f}$, to denote some generic constants whose particular values may change line by line. For two quantities $P$ and $Q$, we write $P \lesssim Q$ to signify $P \leq CQ$ and $P \simeq Q$ to signify $CQ \leq P \leq CQ$, for some generic positive constants $C$ and $\bar{C}$. We may write “almost everywhere” as “a.e.” and “almost surely” as “a.s.” for short. We use $|S|$ to denote the Lebesgue measure of any Lebesgue-measurable set $S$. The Fourier transform and its inverse of a function $\varphi$ are defined respectively as

$$
\mathcal{F} \varphi(\xi) = \hat{\varphi}(\xi) := (2\pi)^{-n/2} \int e^{-i \xi \cdot x} \varphi(x) \, dx,
$$

$$
\mathcal{F}^{-1} \varphi(\xi) := (2\pi)^{-n/2} \int e^{i \xi \cdot x} \varphi(x) \, dx.
$$

Write $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$, $n \geq 1$. We introduce the following weighted $L^p$-norm and the corresponding function space over $\mathbb{R}^n$ for any $\delta \in \mathbb{R}$,

$$
\|\varphi\|_{L^p_\delta(\mathbb{R}^n)} := \|\langle \cdot \rangle^\delta \varphi(\cdot)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \langle x \rangle^{\delta p} |\varphi|^p \, dx \right)^{\frac{1}{p}},
$$

$$
L^p_\delta(\mathbb{R}^n) := \{ \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) : \|\varphi\|_{L^p_\delta(\mathbb{R}^n)} < +\infty \}. \tag{2.1}
$$

We also define $L^p_\delta(S)$ for any subset $S$ in $\mathbb{R}^n$ by replacing $\mathbb{R}^n$ in (2.1) with $S$. In what follows, we may write $L^p_\delta(\mathbb{R}^3)$ as $L^2_\delta$ for short without ambiguities.

Next, we present some basics about the random model and some other preliminaries for the subsequent use.

2.1. Random model and preliminaries. The following lemma shows the precise relationship between the regularity of $h$ and its rough order.

**Lemma 2.1.** Let $h$ be a m.i.g.r. of rough order $-m$ in $D_h$. Then $h \in H^{s,p}(\mathbb{R}^n)$ almost surely for any $1 < p < +\infty$ and $s < (m-n)/2$.

**Proof.** See [7, Proposition 2.4]. \hfill \Box

By the Schwartz kernel theorem [15, Theorem 5.2.1], there exists a kernel $K_h(x, y)$ with $\text{supp} K_h \subset D_h \times D_h$ such that

$$
(\mathcal{E}_h \varphi)(\psi) = \mathbb{E}_\omega((h(\cdot, \omega) - \mathbb{E}(h(\cdot, \omega)), \varphi(h(\cdot, \omega) - \mathbb{E}(h(\cdot, \omega)), \psi))
$$

$$
= \iint K_h(x, y) \varphi(x) \psi(y) \, dx \, dy, \tag{2.2}
$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. It is easy to show that $K_h(x, y) = \overline{K_h(y, x)}$. Denote the symbol of $\mathcal{E}_h$ as $c_h$. It can be verified that the following identities hold in the distributional sense (cf. [7]),

$$
\begin{align}
K_h(x, y) &= (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} c_h(x, \xi) \, d\xi, \tag{2.3a} \\
c_h(x, \xi) &= \int e^{-i(x-y) \cdot \xi} K_h(x, y) \, dy, \tag{2.3b}
\end{align}
$$

where the integrals shall be understood as oscillatory integrals. Despite the fact that $h$ usually is not a function, intuitively speaking, however, it is helpful to keep in mind the following correspondence,

$$
K_h(x, y) \sim \mathbb{E}_\omega(h(x, \omega)h(y, \omega)).
$$
We recall the domain $D_f$ in Theorem 1.1. Throughout the rest of the paper, for notational consistence, we let $D$ be a bounded open domain in $\mathbb{R}^3$ such that
\[ \overline{D_f} \subseteq D. \tag{2.4} \]
For a generalized Gaussian random field $f$, we define the so-called resolvent $\mathcal{R}_k f(x)$ as
\[ \mathcal{R}_k f(x) := \langle f, \Phi_k(x, \cdot) \rangle, \tag{2.5} \]
where $\Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$, and we may abbreviate $\Phi_k$ as $\Phi$ if no ambiguity occurs. We may also express $\mathcal{R}_k f(x)$ as an integral form $\int_{\mathbb{R}^3} \Phi_k(x, y) f(y) \, dy$. The following lemma shows some preliminary properties of $\mathcal{R}_k f$. Note that the $\mu$ is the rough strength of $f$.

**Lemma 2.2.** We have $\mathcal{R}_k f \in L^2_{-1/2-\epsilon}$ for any $\epsilon > 0$ almost surely, and $\mathbb{E}(\| \mathcal{R}_k f \|_{L^2(D)}) < C < +\infty$ for some constant $C$ independent of $k$.

**Proof.** We split $\mathcal{R}_k f$ into two parts, $\mathcal{R}_k(\mathbb{E} f)$ and $\mathcal{R}_k(f - \mathbb{E} f)$. [23, Lemma 2.1] gives $\mathcal{R}_k(\mathbb{E} f) \in L^2_{-1/2-\epsilon}$.

For $\mathcal{R}_k(f - \mathbb{E} f)$, by using (2.2), (2.3) and (2.5), one can compute
\[
\mathbb{E}(\| \mathcal{R}_k(f - \mathbb{E} f)(\cdot, \omega) \|_{L^2_{-1/2-\epsilon}}^2)
\]
\[= \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} \mathbb{E}\left( \left(\int_{\mathbb{R}^3} \Phi_k(x, y) (f - \mathbb{E} f, \Phi_k(x) ) \, dy \right)^2 \right) \, dx
\]
\[= \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} (2\pi)^{-3} \int e^{i(y-z)} \xi c_f(y, \xi) \cdot \Phi_k(x, z) \, dz \, d\xi \, dx
\]
\[\simeq \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} \int_{D_f} \left( \int_{\mathbb{R}^3} \frac{I(y, z) e^{-ik|x-z|}}{|x-z| \cdot |y-z|^2} \, dz \right) \frac{e^{ik|x-y|}}{|x-y|} \, dy \, dx, \tag{2.6} \]
where $c_f(y, \xi)$ is the symbol of the covariance operator $C_f$ and
\[I(y, z) := \int_{\mathbb{R}^3} |y-z|^2 e^{i(y-z) \cdot \xi} c_f(y, \xi) \, d\xi. \]
When $y = z$, we know $I(y, z) = 0$ because the integrand is zero. Thanks to the condition $m > 2$, when $y \neq z$ we have
\[ |I(y, z)| = \left| \sum_{j=1}^{3} \int_{\mathbb{R}^3} (y_j - z_j)^2 e^{i(y-z) \cdot \xi} c_f(y, \xi) \, d\xi \right| = \left| \sum_{j=1}^{3} \int_{\mathbb{R}^3} e^{i(y-z) \cdot \xi} (\partial_{\xi_j}^2 c_f)(y, \xi) \, d\xi \right|
\]
\[\leq \sum_{j=1}^{3} \int_{\mathbb{R}^3} C_j |\xi|^{-m+2} \, d\xi \leq C_0 < +\infty, \tag{2.7} \]
for some constant $C_0$ independent of $y$ and $z$. Note that $D_f \subset \mathbb{R}^3$ is bounded, so for $j = 1, 2$ we have
\[\int_{D_f} |x-y|^{-j} \, dy \leq C_f_j \langle x \rangle^{-j}, \quad \forall x \in \mathbb{R}^3, \tag{2.8} \]
for some constant $C_{f,j}$ depending only on $f, j$ and the dimension. The notation $\langle x \rangle$ in (2.8) stands for $(1 + |x|^2)^{1/2}$ and readers may note the difference between the $\langle \cdot \rangle$ and the $\langle \cdot, \cdot \rangle$ appeared in (2.5). With the help of (2.7) and (2.8) and Hölder’s inequality, we can continue (2.6) as
\[\mathbb{E}(\| \mathcal{R}_k(f - \mathbb{E} f)(\cdot, \omega) \|_{L^2_{-1/2-\epsilon}}^2). \]
Moreover, by Hölder’s inequality we can have
\[ \int (x)^{-1-2\epsilon}(\int_{D_f \times D_f} (|x-z|^{-1} \cdot |y-z|^{-1})(|y-z|^{-1} \cdot |x-y|)^{-1} \, dz \, dy) \, dx \]
\[ \leq \int (x)^{-1-2\epsilon} [C \int_{D_f} \int_{D_f} |y-z|^{-2} \, dy] |x-z|^{-2} \, dz 
\quad \cdot \int_{D_f} \int_{D_f} |y-z|^{-2} \, dy \cdot |x-y|^{-2} \, dy)^{1/2} \, dx \]
\[ = \int (x)^{-1-2\epsilon} (C_f \int_{D_f} |x-z|^{-2} \, dz \cdot \int_{D_f} |x-y|^{-2} \, dy)^{1/2} \, dx \quad \text{(by (2.8))} \]
\[ = \int (x)^{-1-2\epsilon} C_f (x)^{-2} \, dx \leq C_f < +\infty, \]
which gives
\[ \mathbb{E}(\|R_k(f - \mathbb{E}(f))(\cdot, \omega)\|_{L_{1/2-\epsilon}^2}) \leq C_f < +\infty. \quad (2.9) \]

By the Hölder inequality applied to the probability measure, we obtain from (2.9) that
\[ \mathbb{E}\|R_k(f - \mathbb{E}(f))\|_{L_{1/2-\epsilon}^2} \leq \mathbb{E}(\|R_k(f - \mathbb{E}(f))\|_{L_{1/2-\epsilon}^2})^{1/2} \leq C_f^{1/2} < +\infty, \quad (2.10) \]
for some constant $C_f$ independent of $k$. The formula (2.10) gives that $R_k(f - \mathbb{E}(f)) \in L^2_{1/2-\epsilon}$ almost surely, and hence
\[ R_k f \in L^2_{1/2-\epsilon} \quad \text{a.s.} \]
By replacing $\mathbb{R}^3$ with $\mathcal{D}$ and deleting the term $(x)^{-1-2\epsilon}$ in the derivation above, one easily arrives at $\mathbb{E}\|R_k f\|_{L^2(\mathcal{D})} < +\infty$. The proof is complete. \qed

The following resolvent estimate
\[ \|R_k \varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \lesssim k^{-1} \|\varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)} \quad (2.11) \]
is known to the literature (cf. [13, 16]). In what follows, we shall also need some variations of it for our arguments.

**Lemma 2.3.** There exists a constant $k_0 > 0$ depending on $\epsilon$ and $V$ such that for $\forall k > k_0$ and multi-index $\alpha$: $|\alpha| \leq 2$, we have
\[ \|R_k (\partial^\alpha V) \varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \leq C k^{-1} \|\varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)}, \quad (2.12) \]
\[ \|(\partial^\alpha V) R_k \varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)} \leq C k^{-1} \|\varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)}, \quad (2.13) \]
\[ \|(\partial^\alpha V) R_k \varphi\|_{L^1(\mathbb{R}^3)} \leq C k^{-1} \|\varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)}. \quad (2.14) \]

**Proof.** Recall the assumption on $V$. We only show the case where $\alpha = 0$. With the help of (2.11), we can have
\[ \|R_k V \varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \lesssim k^{-1} \|V\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \|\varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \lesssim \|\varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)}. \]
and
\[ \|V R_k \varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)} \leq \|V\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)} \|R_k \varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \lesssim k^{-1} \|\varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)}. \]
Moreover, by Hölder’s inequality we can have
\[ \|V R_k \varphi\|_{L^1(\mathbb{R}^3)} \leq \|V\|_{L^\infty(\mathbb{R}^3)} \|R_k \varphi\|_{L_{1/2-\epsilon}^2(\mathbb{R}^3)} \lesssim k^{-1} \|\varphi\|_{L_{1/2+\epsilon}^2(\mathbb{R}^3)}. \]
The proof is complete. \qed
2.2. The well-posedness of the direct problem. For a particular realization of the random sample \( \omega \in \Omega \), the regularity of an m.i.g.r. \( f \) could be very rough; see Lemma 2.1. Due to this reason, the classical second-order elliptic PDE theory may no longer be applicable to (1.1). To that end, the notion of the mild solution is introduced for random PDEs (cf. [1, 23]). In what follows, we introduce the mild solution for our problem setting (1.1), and we show that this mild solution and the corresponding far-field pattern are well-defined in a proper sense.

Reformulating (1.1) into the Lippmann-Schwinger equation formally (cf. [9]), we have

\[
(I - \mathcal{R}_k V) u = -\mathcal{R}_k f,
\]

(2.15)

where the term \( \mathcal{R}_k f \) is defined by (2.3). Suppose that \( k \) is large enough, then we know the operator \( I - \mathcal{R}_k V \) is an invertible mapping from \( L^2_{-1/2-\epsilon} \) to \( L^2_{2/1-2-\epsilon} \). Moreover, by Lemma 2.2 we know that the right-hand side of (2.15) belongs to \( L^2_{2/1-2-\epsilon} \) almost surely. We are now in a position to present one of the results concerning the direct scattering problem.

**Theorem 2.1.** When \( k \) is large enough such that \( \|\mathcal{R}_k V\|_{L(L^2_{-1/2-\epsilon}, L^2_{2/1-2-\epsilon})} < 1 \), there exists a unique stochastic process \( u(\cdot, \omega) : \mathbb{R}^3 \to \mathbb{C} \) such that \( u(x) \) satisfies (2.15) a.s.. Moreover, \( u(\cdot, \omega) \in L^2_{2/1-\epsilon} \) a.s. for any \( \epsilon \in \mathbb{R}_+ \). Then \( u(x) \) is called the mild solution to the random scattering problem (1.1).

**Proof.** By Lemma 2.2, we obtain

\[
F := -\mathcal{R}_k f \in L^2_{-1-\epsilon}.
\]

According to (2.12) we have \( \|\mathcal{R}_k V\|_{L(L^2_{-1/2-\epsilon}, L^2_{2/1-2-\epsilon})} < 1 \). Hence, \( \sum_{j=0}^{\infty} (\mathcal{R}_k V)^j \) is well-defined. Therefore, \( \sum_{j=0}^{\infty} (\mathcal{R}_k V)^j F \in L^2_{-1/2-\epsilon} \). Because \( \sum_{j=0}^{\infty} (\mathcal{R}_k V)^j = (I - \mathcal{R}_k V)^{-1} \), we see \( (I - \mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon} \). Let \( u := (I - \mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon} \), then \( u \) fulfills the requirements. Hence, the existence of a mild solution is proven. The uniqueness and stability of the mild solution follows easily from the inequality

\[
\|u\|_{L^2_{2/1-\epsilon}} \leq \sum_{j=0}^{\infty} \|\mathcal{R}_k V\|_{L(L^2_{-1/2-\epsilon}, L^2_{2/1-2-\epsilon})}^j \|\mathcal{R}_k f\|_{L^2_{2/1-2-\epsilon}} \leq C \| \mathcal{R}_k f \|_{L^2_{2/1-2-\epsilon}}.
\]

The proof is complete. \( \square \)

Next we show that the far-field pattern is well-defined in the \( L^2 \) sense. Assume that \( k \) is large enough. From (2.15) we deduce that

\[
u = -(I - \mathcal{R}_k V)^{-1}(\mathcal{R}_k f) = -\mathcal{R}_k(I - V\mathcal{R}_k)^{-1}(f).
\]

Therefore, we define the far-field pattern of the scattered wave \( u(x, k, \omega) \) formally in the following manner,

\[
u(\hat{x}, k, \omega) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y}(I - V\mathcal{R}_k)^{-1}(f)(y) dy, \quad \hat{x} \in \mathbb{S}^2.
\]

(2.16)

**Theorem 2.2.** Define the far-field pattern of the mild solution as in (2.10). When \( k \) is large enough, there is a subset \( \Omega_0 \subset \Omega \), with zero measure \( \mathbb{P}(\Omega_0) = 0 \), such that it holds

\[
u(\hat{x}, k, \omega) \in L^2(\mathbb{S}^2), \quad \forall \omega \in \Omega \setminus \Omega_0.
\]

**Proof of Theorem 2.2.** By [23, Lemma 2.4], we have

\[
\|V\mathcal{R}_k\|_{L(L^2(D), L^2(D))} \leq Ck^{-1} < 1,
\]

when \( k \) is sufficiently large. Therefore, it holds that

\[
\int_{\mathbb{S}^2} |\nu(\hat{x}, k, \omega)|^2 dS(\hat{x})
\]
\[
\begin{align*}
\lesssim & \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} (I - V) (f) \, dy \right|^2 \, dS(\hat{x}) \\
\lesssim & \int \left| \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} \sum_{j \geq 1} (V\mathcal{R}_k)^j (f) \, dy \right|^2 \, dS(\hat{x}) + \int \left| \langle f, e^{-ik\hat{x} \cdot -} \rangle \right|^2 \, dS(\hat{x}) \\
= & f_1(\hat{x}, k, \omega) + f_2(\hat{x}, k, \omega). \tag{2.17}
\end{align*}
\]

Next, we derive estimates on these terms \( f_j \) \((j = 1, 2)\) in (2.17). We have

\[
\left| \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} \sum_{j \geq 1} (V\mathcal{R}_k)^j (f) \, dy \right| \leq \sum_{j \geq 0} \|V\|_{L^{1/2, \infty} (\mathbb{R}^3)} \| (\mathcal{R}_k V)^j \mathcal{R}_k f \|_{L^{2, 1/2}_{-2} (\mathbb{R}^3)}
\]

where we have used the assumption (1.4), eq. (2.12) and Lemma 2.2. Therefore,

\[
f_1(\hat{x}, k, \omega) \lesssim \int \|\mathcal{R}_k f\|^2_{L^{2, 1/2}_{-2} (\mathbb{R}^3)} \, dS(\hat{x}) < C < +\infty,
\]

for some constant \( C \) independent of \( k \).

By (2.2) and Fubini’s theorem, the expectation of \( f_2(\hat{x}, k, \omega) \) can be computed as

\[
\begin{align*}
\mathbb{E} f_2(\hat{x}, k, \omega) &= \mathbb{E} \left| \langle f, e^{-ik\hat{x} \cdot -} \rangle \right|^2 \, dS(\hat{x}) = \int \mathbb{E} \left| \langle f, e^{-ik\hat{x} \cdot -} \rangle \right|^2 \, dS(\hat{x}) \\
&= \int \|\mathcal{C}_f (\chi_D f e^{-ik\hat{x} \cdot -})\|_{L^2 (\mathbb{R}^3)} \cdot \|\chi_D f e^{-ik\hat{x} \cdot -}\|_{L^2 (\mathbb{R}^3)} \, dS(\hat{x}) \\
&\leq \int \|\mathcal{C}_f (\chi_D f e^{-ik\hat{x} \cdot -})\|_{L^2 (\mathbb{R}^3)} \cdot \|\chi_D f e^{-ik\hat{x} \cdot -}\|_{L^2 (\mathbb{R}^3)} \, dS(\hat{x}) + C_f,
\end{align*}
\]

where the constant \( C_f \) is independent of \( \hat{x} \) and \( k \). The symbol of the pseudo-differential operator is of order \( -m < 0 \), thus \( \mathcal{C}_f \) is a bounded operator from \( L^2 (\mathbb{R}^3) \) to \( L^2 (\mathbb{R}^3) \); see [31, Theorem 11.7]. Hence

\[
\begin{align*}
\mathbb{E} f_2(\hat{x}, k, \omega) &\leq C \int \|\chi_D f e^{-ik\hat{x} \cdot -}\|_{L^2 (\mathbb{R}^3)} \cdot \|\chi_D f e^{-ik\hat{x} \cdot -}\|_{L^2 (\mathbb{R}^3)} \, dS(\hat{x}) + C_f \\
&\leq C \int \|\chi_D f\|_{L^2 (\mathbb{R}^3)} \cdot \|\chi_D f\|_{L^2 (\mathbb{R}^3)} \, dS(\hat{x}) + C_f \\
&\leq C_f < +\infty,
\end{align*}
\]

for some constant \( C_f \) independent of \( \hat{x} \) and \( k \). Thus, \( f_2(\hat{x}, k, \omega) < +\infty \) almost surely.

Combining the estimates on \( f_j(\hat{x}, \omega) \) \((j = 1, 2)\), we conclude that

\[
\int_{\mathbb{S}^2} |u^\infty(\hat{x}, k, \omega)|^2 \, dS(\hat{x}) < \infty
\]

almost surely. The proof is complete. \( \square \)

### 3. Several critical asymptotic estimates

In this section we shall establish a method to recover rough strength of \( f \) through the following quantity

\[
\frac{1}{K} \int_K^{2K} u^\infty(\hat{x}, k, \omega) \cdot u^\infty(\hat{x}, k + \tau, \omega) \, dk. \tag{3.1}
\]
As an auxiliary critical step in justifying (3.1), we need to first consider the following recovery formula

$$\frac{1}{K} \int_{-K}^{2K} \left[ u^\infty(\hat{x}, k) - \mathbb{E}(u^\infty(\hat{x}, k)) \right] \cdot \left[ u^\infty(\hat{x}, k) - \mathbb{E}(u^\infty(\hat{x}, k)) \right] \, dk. \quad (3.2)$$

It is noted that $\mathbb{E}(u^\infty(\hat{x}, k))$ requires the full realization of the random samples. We would like to emphasise that $\mathbb{E}(u^\infty(\hat{x}, k))$ shall play an auxiliary role in our analysis and we shall develop techniques to remove it from the recovery procedure.

To analyze the behaviour of (3.2), we shall derive several critical asymptotic estimates in this section. Henceforth, we use $k^*$ to signify the maximum value between the quantity $k_0$ from Lemma 2.3 and the quantity

$$\sup_{k \in \mathbb{R}^+} \{k : \|R_k V\|_{L^2_{1/2-\epsilon}, L^2_{1/2-\epsilon}} \geq 1\} + 1.$$

Assume that $k > k^*$, then we can expand $\sum_{j=0}^{+\infty} (R_k V)^j$ into Neumann series and obtain

$$u^\infty(\hat{x}, k, \omega) - \mathbb{E}(u^\infty(\hat{x}, k)) = -\frac{1}{4\pi} \sum_{j=0}^{+\infty} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} (R_k V)^j (f - \mathbb{E}(f))(y) \, dy, \quad \hat{x} \in \mathbb{S}^2$$

$$:= -\frac{1}{4\pi} \left[ F_0(k, \hat{x}) + F_1(k, \hat{x}) \right], \quad (3.3)$$

where

$$\begin{align*}
F_0(k, \hat{x}, \omega) &:= (f - \mathbb{E}(f), e^{-ik\hat{x} \cdot \cdot}) \\
F_1(k, \hat{x}, \omega) &:= \sum_{j \geq 1} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} (V R_k)^j (f - \mathbb{E}(f))(y) \, dy.
\end{align*} \quad (3.4)$$

The expectation $\mathbb{E}(u^\infty(\hat{x}, k))$ in (3.3) can be expressed as

$$\mathbb{E}(u^\infty(\hat{x}, k)) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} ((I - VR_k)^{-1} \mathbb{E}(f))(y) \, dy, \quad \hat{x} \in \mathbb{S}^2. \quad (3.5)$$

**Lemma 3.1.** For all $k > k^*$, there exists a constant $C$ independent of $\hat{x}$ and $k$ such that

$$|\mathbb{E}(u^\infty(\hat{x}, k))| \leq Ck^{-2}. \quad (3.6)$$

**Proof.** Note that $\mathbb{E}f \in C^\infty_c(\mathbb{R}^3)$ (cf. Definition 1.1). The function $R_k(\mathbb{E}f)$ is a convolution and thus is a $C^\infty$-smooth function. For $k > k^*$ we denote $F(y) := \sum_{j=0}^{+\infty} (VR_k)^j (\mathbb{E}f)(y)$ for simplicity. By using (3.3) and Lemma 2.3, we can compute

$$\mathbb{E}(u^\infty(\hat{x}, k)) \leq \int |(ik^{-1} \hat{x} \cdot \nabla)^2 e^{-ik\hat{x} \cdot y}| F(y) \, dy + \sum_{j \geq 2} \int e^{-ik\hat{x} \cdot y} (VR_k)^j (\mathbb{E}f)(y) \, dy$$

$$\leq Ck^{-2} \int \sum_{|\alpha| = 2} C_{\alpha} \partial_{y}^\alpha F(y) \, dy + \|V\|_{L^{2}_{1/2+\epsilon}} \sum_{j \geq 1} \|(R_k V)^j R_k(\mathbb{E}f)\|_{L^{2}_{1/2-\epsilon}}$$

$$\leq Ck^{-2} \int \sum_{|\alpha| = 2} C_{\alpha} \partial_{y}^\alpha F(y) \, dy + Ck^{-2} \cdot \|\mathbb{E}f\|_{L^{2}_{1/2+\epsilon}(\mathbb{R}^3)};$$

where the constant $C$ is independent of $\hat{x}$ and $k$. Note that $\mathbb{E}f \in C^\infty_c$ so $\partial_j(\mathbb{E}f) \in C^\infty_c$ and $\partial_j R_k(\mathbb{E}f) = R_k \partial_j (\mathbb{E}f)$, thus by Lemma 2.3 and the assumption 1.4 we have

$$\int \sum_{|\alpha| = 2} C_{\alpha} \partial_{y}^\alpha F(y) \, dy \leq \sum_{|\alpha| = 2} \|\partial^\alpha \mathbb{E}f\|_{L^1(\mathbb{R}^3)} + \sum_{|\alpha| + |\beta| = 2} \|(\partial^\alpha V) R_k (\partial^\beta \mathbb{E}f)\|_{L^1(\mathbb{R}^3)} < +\infty.$$

The proof is complete. \qed
By substituting (3.3), (3.4) into (3.2), we obtain several crossover terms between $F_0$ and $F_1$. The asymptotic estimates of these crossover terms are the main purpose of Sections 3.1 and 3.2. Section 3.1 focuses on the estimate of the leading-order term while the estimates of the higher-order terms are presented in Section 3.2.

3.1. Asymptotics of the leading-order term. The following lemma is needed for the study of the asymptotics of the aforementioned leading-order term.

**Lemma 3.2.** When $\mu \in C_c^\infty(D)$, $\tau \in \mathbb{R}$ and $\hat{x} \in S^2$, we have

\[
\frac{(2\pi)^3}{K^2} \int_K^{2K} \int_K^{2K} |\hat{\mu}((k_1 - k_2)\hat{x})|^2 \, dk_1 \, dk_2 \leq CK^{-1},
\]

(3.7)

\[
\frac{(2\pi)^3}{K^2} \int_K^{2K} \int_K^{2K} |\hat{\mu}((k_1 + k_2 + \tau)\hat{x})|^2 \, dk_1 \, dk_2 \leq CK^{-1},
\]

(3.8)

for some constant $C$ independent of $\tau$ and $\hat{x}$. Here $\hat{\mu}$ signifies the Fourier transform of $\mu$.

**Proof.** To conclude (3.7), we make a change of variable,

\[
\begin{cases}
  s = k_1 - k_2, \\
  t = k_2.
\end{cases}
\]

Write $Q = \{(s,t) \in \mathbb{R}^2 \mid K \leq s + t \leq 2K, K \leq t \leq 2K\}$, which is illustrated in Fig. 1. Recall that supp $\mu \subseteq D_f$. Then we have

\[
\frac{1}{K^2} \int_K^{2K} \int_K^{2K} |\hat{\mu}((k_1 - k_2)\hat{x})|^2 \, dk_1 \, dk_2 = \frac{1}{K^2} \int_Q |\hat{\mu}(s\hat{x})|^2 \, ds \, dt
\]

\[
= \frac{1}{K^2} \int_{-K}^{0} (K + s)|\hat{\mu}(s\hat{x})|^2 \, ds + \frac{1}{K^2} \int_{0}^{K} (K - s)|\hat{\mu}(s\hat{x})|^2 \, ds
\]

\[
\leq \frac{2}{K} \int_{\mathbb{R}} |\hat{\mu}(s\hat{x})|^2 \, ds.
\]

(3.9)

Recall that $\mu \in C_c^\infty(\mathbb{R}^3) \subset \mathcal{S}(\mathbb{R}^3)$, thus $\hat{\mu}(x)$ decays faster than the reciprocal of any polynomials, especially, $|\hat{\mu}(s\hat{x})| \leq C(s)^{-1}$ for all $s \in \mathbb{R}$, thus

\[
\frac{1}{K^2} \int_K^{2K} \int_K^{2K} |\hat{\mu}((k_1 - k_2)\hat{x})|^2 \, dk_1 \, dk_2 \leq \frac{2}{K} \int_{\mathbb{R}} C(s)^{-2} \, ds \leq CK^{-1},
\]

which is (3.7). To prove (3.8), again we make a change of variable:

\[
\begin{cases}
  s = k_1 + k_2 + \tau, \\
  t = k_2.
\end{cases}
\]
Write $Q' = \{(s, t) \in \mathbb{R}^2 | K \leq s - t - \tau \leq 2K, K \leq t \leq 2K\}$. One can compute
\[
\frac{1}{K^2} \int_{K}^{2K} \int_{K}^{2K} |\hat{\mu}((k_1 + k_2 + \tau) \hat{x})|^2 \, dk_1 \, dk_2 = \frac{1}{K^2} \int_{Q'} |\hat{\mu}(s \hat{x})|^2 \, ds \, dt
\]
\[
= \frac{1}{K^2} \int_{2K+\tau}^{3K+\tau} (s - 2K - \tau)|\hat{\mu}(s \hat{x})|^2 \, ds + \frac{1}{K^2} \int_{3K+\tau}^{4K+\tau} (4K + \tau - s)|\hat{\mu}(s \hat{x})|^2 \, ds
\]
\[
\leq \frac{2}{K} \int_{2K-\tau}^{2K+\tau} |\hat{\mu}(s \hat{x})|^2 \, ds = \frac{2}{K} \int_{\mathbb{R}} |\hat{\mu}(s \hat{x})|^2 \, ds \leq \frac{C}{K} \int_{\mathbb{R}} \langle s \rangle^{-2} \, ds \leq \frac{C}{K},
\]
which gives (3.8). The proof is complete. \(\square\)

For notational convenience, we shall use \(\{K_j\} \in P(t)\) to signify a sequence \(\{K_j\}_{j \in \mathbb{N}}\) satisfying \(K_j \geq C j^t\) (\(j \in \mathbb{N}\)) for some fixed constant \(C > 0\). Throughout the rest of the paper, \(\gamma\) stands for a fixed positive real number. The next lemma gives the asymptotic estimate of the leading-order term.

**Lemma 3.3.** Let \(F_j(k, \hat{x})\) \((j = 0, 1)\) be defined as in (3.4). Write
\[
X_{0,0}(K, \tau, \hat{x}) = \frac{1}{K} \int_{K}^{2K} k^m \mathbb{E}[\hat{F}(k, \hat{x}) F_0(k + \tau, \hat{x})] \, dk.
\]
Assume that \(\{K_j\} \in P(1 + \gamma)\). Then for any \(\tau > 0\), we have
\[
\lim_{j \to +\infty} X_{0,0}(K_j, \tau, \hat{x}) = (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}) \quad \text{a.s. .} \quad (3.10)
\]

The proof of Lemma 3.3 utilizes ergodicity. In what follows, we may denote \(X_{0,0}(K, \tau, \hat{x})\) as \(X_{0,0}\) for short if it is clear in the context.

**Proof of Lemma 3.3** By (2.2), (2.3) and (3.4), we can compute \(\mathbb{E}[\hat{F}(k, \hat{x}) F_0(k + \tau, \hat{x})]\) as follows,
\[
\mathbb{E}(\hat{F}(k, \hat{x}) F_0(k + \tau, \hat{x}))
\]
\[
= \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} K_f(y, z) e^{-ik \hat{x}(y-z)} \, dz \right) e^{-i\tau \hat{x} \cdot y} \, dy
\]
\[
= \int_{\mathbb{R}^3} c_T(y, k \hat{x}) e^{-i\tau \hat{x} \cdot y} \, dy = (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}) k^{-m} + \int_{\mathbb{R}^3} a(y, k \hat{x}) e^{i\tau \hat{x} \cdot y} \, dy. \quad (3.11)
\]

Note that \(a(y, k \hat{x})\) is compactly supported in \(y\) and \(|a(y, k \hat{x})| \lesssim k^{-m-1}\). Therefore,
\[
\mathbb{E}(X_{0,0}) = \frac{1}{K} \int_{K}^{2K} k^m \mathbb{E}[(\hat{F}(k, \hat{x}) F_0(k + \tau, \hat{x}))] \, dk
\]
\[
= \frac{1}{K} \int_{K}^{2K} [(2\pi)^{3/2} \hat{\mu}(\tau \hat{x}) + O(K^{-1})] \, dk
\]
\[
= (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}) + O(K^{-1}), \quad K \to +\infty. \quad (3.12)
\]

By Isserlis’ Theorem and (3.12), and noting that \(\overline{F_j(k, \hat{x})} = F_j(-k, \hat{x}), F_0(-k, -\hat{x}) = F_0(k, \hat{x})\), one can compute
\[
\mathbb{E}[(X_{0,0} - (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}))^2]
\]
\[
= \frac{1}{K^2} \int_{K}^{2K} \int_{K}^{2K} \mathbb{E} \left( [k_1^m F_0(k_1 + \tau, \hat{x}) F_0(k_1, \hat{x}) - (2\pi)^{3/2} \hat{\mu}(\tau \hat{x})] \right.\]
\[
\left. \times [k_2^m F_0(k_2 + \tau, \hat{x}) F_0(k_2, \hat{x}) - (2\pi)^{3/2} \hat{\mu}(\tau \hat{x})] \right) \, dk_1 \, dk_2
\]
Fixing an integer $K_0 > 0$ and by Chebyshev’s inequality and (3.14) we have
\[
P\left( \bigcup_{j \geq K_0} \{|X_{0,0}(K_j) - (2\pi)^{3/2} \hat{\mu}(\tau \hat{x})| \geq \epsilon\} \right) \leq \frac{1}{\epsilon^2} \sum_{j \geq K_0} \mathbb{E}(|X_{0,0}(K_j) - (2\pi)^{3/2} \hat{\mu}(\tau \hat{x})|^2) \leq \frac{1}{\epsilon^2} \sum_{j \geq K_0} K_j^{-1} \leq \frac{1}{\epsilon^2} \sum_{j \geq K_0} j^{-1-\gamma} \leq \frac{1}{\epsilon^2} \int_{K_0}^{+\infty} (t-1)^{-1-\gamma} dt = \frac{1}{\epsilon^2\gamma}(K_0-1)^{-\gamma}. \tag{3.15}\]

Here $X_{0,0}(K_j)$ stands for $X_{0,0}(K_j, \tau, \hat{x})$. By [23, Lemma 3.2], formula (3.15) implies that for any fixed $\tau \geq 0$ and $\hat{x} \in S^2$, one has
\[
X_{0,0}(K_j, \tau, \hat{x}) \to (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}) \quad \text{a.s.}.
\]
The proof is complete. \hfill \Box

### 3.2. Asymptotics of the higher-order terms.

**Lemma 3.4.** Define $F_j(k, \hat{x})$ ($j = 0, 1$) as in (3.4). For every $\hat{x} \in S^2$ and every $k_1, k_2 \geq k$, when $k \to +\infty$, we have the following estimates:
\[
\left| \mathbb{E}(F_1(k_2, \hat{x})F_0(k_1, \hat{x})) \right| = \mathcal{O}(k^{-m-1}), \quad \left| \mathbb{E}(F_0(k_1, \hat{x}) \cdot F_1(k_2, \hat{x})) \right| = \mathcal{O}(k^{-m-1}), \tag{3.16}
\]
uniformly for all $\hat{x}$.

**Proof.** We only prove the first asymptotic estimate in (3.16) and the second one can be proved by following similar arguments. For simplicity, we may use $D_y$ to stand for $D$ to indicate that the argument $y$ is integrated over this domain.

In what follows we let $\hat{x}_1, \hat{x}_2 \in S^2$. In this proof we may drop the arguments $k, \hat{x}$ if it is clear in the context. Write
\[
G_0(k, \hat{x}) := \langle f - \mathbb{E}f, e^{-ik\hat{x}(\cdot)} \rangle, \quad G_j(k, \hat{x}) := \int e^{-ik\hat{x}(\cdot)}((VR_k)^j(f - \mathbb{E}f)) (y) dy, \quad r_j(k, \hat{x}) := \sum_{s \geq j} G_s(k, \hat{x}), \quad j = 1, 2, \cdots \tag{3.17}
\]
thus $F_0 = G_0$ and $F_1 = r_1 = G_1 + r_2$, so we have
\[
\mathbb{E}(F_0(k_1, \hat{x}_1) \cdot F_1(k_2, \hat{x}_2)) = \mathbb{E}(G_0(k_1, \hat{x}_1) \cdot G_1(k_2, \hat{x}_2)) + \mathbb{E}(G_0(k_1, \hat{x}_1) \cdot r_2(k_2, \hat{x}_2)). \tag{3.18}
\]
To prove (3.16), we need to estimate $\mathbb{E}(G_0G_1)$ and $\mathbb{E}(G_0r_2)$. One can compute
\[
\mathbb{E}(G_0(k_1, \hat{x}_1) \cdot G_1(k_2, \hat{x}_2)) = \int_{D_y} e^{-ik_1\hat{x}_1}(f - \mathbb{E}f)(y) dy \times \int_{D_z} e^{-ik_2\hat{x}_2}V(z) \int_{D_t} \Phi(z,t)(f - \mathbb{E}f)(t) dt dz \tag{3.19}
\]
By taking the conjugate of (3.23), we arrive at (3.21). Then to prove (3.22) one can compute
\[ e^{-ik_1\hat{x}_1 y} \int e^{-ik_1\hat{x}_1 y} \Phi(z, t) dt \, dz = |e^{ik_2\hat{x}_1 z} \mathbf{v}(z) \cdot e^{-ik_1\hat{x}_1 y} \Phi(z, t) dt \, dz | \]
\[ = |e^{ik_2\hat{x}_1 z} \mathbf{v}(z) \cdot (e^{ik_1\hat{x}_1 y} K_f(t, y) \Phi(z, t) dt) \, dz | \]
\[ = |e^{ik_2\hat{x}_1 z} \mathbf{v}(z) \cdot (e^{ik_1\hat{x}_1 y} K_f(t, y) \Phi(z, t) dt) \, dz | \]
\[ = ||V R_{k_2}(\mu(t)k_1^{-m} + a(-k_1\hat{x}_1)) e^{ik_1\hat{x}_1} \Phi_k(z, t) dt) \, dz | \]
\[ \leq k_2^{-1} ||(\mu(t)k_1^{-m} + a(-k_1\hat{x}_1)) e^{ik_1\hat{x}_1} \Phi_k(z, t) dt) \, dz | \]
\[ \leq k_2^{-1} k_1^{-m}, \quad k \to +\infty. \quad (3.20) \]

To estimate \( E(G_0(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2)) \) we first prove for \( j > 1 \),
\[ E(G_0(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2)) = \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz. \quad (3.21) \]
\[ E(G_1(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2)) = \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz. \quad (3.22) \]

We have
\[ E(G_0(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2)) = E \left( (f - E f, e^{ik_1\hat{x}_1}) \cdot \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz \right) \]
\[ = \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz. \quad (3.23) \]

By taking the conjugate of (3.23), we arrive at (3.21). Then to prove (3.22) one can compute
\[ E(G_1(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2)) = E \left( (f - E f, e^{ik_1\hat{x}_1}) \cdot \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz \right) \]
\[ = \int e^{-ik_2\hat{x}_2 z} (V R_{k_2}) \chi_D((f - E f)(y), \Phi_k(y, \cdot)) \, dz. \quad (3.24) \]

We arrive at (3.22) by taking the conjugate of (3.24). By applying (3.21), we have
\[ |E(G_0(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2))| \leq \sum_{j \geq 2} |E(G_0(k_1, \hat{x}_1) \cdot G_j(k_2, \hat{x}_2))| \]
\[ \leq \sum_{j \geq 2} ||V R_{k_2} \chi_D((f - E f)(y), \Phi_k(y, \cdot)) ||_{L^1(\mathbb{R}^3)} \]
\[ \leq C k_1^{-m} \sum_{j \geq 2} k_2^{-j} ||k_1^{-m} (f - E f)(y), \Phi_k(y, \cdot) ||_{L^{1/2+}(\mathbb{R}^3)} \]
\[ = O(k_1^{-m} k_2^{-2}), \quad k \to +\infty. \quad (3.25) \]

By (3.19), (3.20), and (3.25), the formula (3.16) is proved.
Before we analyze the behavior of $\mathbb{E}(F_1(k_2, \hat{x})F_1(k_1, \hat{x}))$ in terms of $k_1$ and $k_2$, we first present an auxiliary lemma that shall be useful in the proof of Lemma 3.6. In the sequel, we denote $\text{diam} \Omega := \sup \{|x - x'|\}$.

**Lemma 3.5.** Assume $\Omega$ is a bounded domain in $\mathbb{R}^n$. For all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < n$ and $\beta < n$, and for all $p \in \mathbb{R}^n \setminus \{0\}$, there exists a constant $C_{\alpha,\beta}$ independent of $p$ and $\Omega$ such that

$$\int_{\Omega} |t|^{-\alpha} |t - p|^{-\beta} dt \leq C_{\alpha,\beta} \times \begin{cases} |p|^{n-\alpha-\beta} + (\text{diam} \Omega)^{n-\alpha-\beta}, & \alpha + \beta \neq n, \\ \ln |p| + \ln(\text{diam} \Omega) + C_{\alpha,\beta}, & \alpha + \beta = n. \end{cases} \quad (3.26)$$

**Remark 3.1.** Formula (3.26) also holds when $p = 0$ and $\alpha + \beta \neq n$. When $p \neq 0$ and $\alpha + \beta \geq n$, the upper bound of the integral in (3.26) goes to infinity as $p$ approaches the origin. When $p = 0$ and $\alpha + \beta \geq n$, the integral is ill-defined, i.e. the Cauchy principal value of the integral is infinity. Hence formula (3.26) gives a description about how fast (in terms of $|p|$) the integral goes to infinity as $p$ approaches the origin.

**Proof of Lemma 3.5.** We use $B(0, \text{diam} \Omega)$ to signify the ball centering at the point 0 and of radius $\text{diam} \Omega$. We divide $\Omega$ into three parts: $\Omega_1 := B(p, |p|/2)$, $\Omega_2 := B(0, 2|p|) \setminus \Omega_1$, and $\Omega_3 := \Omega \setminus (\Omega_1 \cup \Omega_2)$. Noting that $\beta < n$, we can compute

$$\int_{\Omega_1} |t|^{-\alpha} |t - p|^{-\beta} dt \leq \int_{\Omega_1} |p/2|^{-\alpha} |t - p|^{-\beta} dt = 2^\alpha |p|^{-\alpha} \int_{B(0, |p|/2)} |t|^{-\beta} dt$$

$$= C_{\alpha,\beta} |p|^{n-\alpha-\beta}. \quad (3.27)$$

Then we compute the integral over $\Omega_2$ as follows (noting that $\alpha < n$),

$$\int_{\Omega_2} |t|^{-\alpha} |t - p|^{-\beta} dt \leq \int_{\Omega_2} |t|^{-\alpha} |p/2|^{-\beta} dt = 2\alpha |p|^{-\alpha} \int_{\Omega_2} |t|^{-\beta} dt$$

$$\leq 2\beta |p|^{-\beta} \int_{B(0, 2|p|)} |t|^{-\alpha} dt = C_{\alpha,\beta} |p|^{n-\alpha-\beta}. \quad (3.28)$$

We claim that $|t|/2 \leq |t - p| \leq 3|t|/2$ for all $t \in \Omega_3$. This can be seen in the following way: fix a quantity $T > 2|p|$, then $p$ is an inner point of the ball $B(0, T)$. The distance between $t$ and $p$ is $|t - p|$. For every $t$ such that $|t| = T$, the longest distance between $t$ and $p$ is $T + |p|$ while the shortest distance is $T - |p|$, thus $T - |p| \leq |t - p| \leq T + |p|$ holds. Because $T > 2|p|$ and $|t| = T$, we obtain $|t|/2 \leq |t - p| \leq 3|t|/2$ for all $t \in \Omega_3$. The quantity $\text{diam} \Omega$ is finite because $\Omega$ is bounded. Therefore, the integral over $\Omega_3$ can be computed as follows,

$$\int_{\Omega_3} |t|^{-\alpha} |t - p|^{-\beta} dt \leq \int_{\Omega_3} |t|^{-\alpha} (|t|/2)^{-\beta} dt \leq 2\beta |p|^{-\alpha} \int_{\{2|p| \leq |t| \leq \text{diam} \Omega\}} |t|^{-\alpha} dt$$

$$\leq \begin{cases} (\text{diam} \Omega)^{n-\alpha-\beta} - |p|^{n-\alpha-\beta}, & \alpha + \beta \neq n, \\ 2\beta |\ln |p| + \ln(\text{diam} \Omega)|, & \alpha + \beta = n, \end{cases} \leq C_{\alpha,\beta} \times \begin{cases} |p|^{n-\alpha-\beta} + (\text{diam} \Omega)^{n-\alpha-\beta}, & \alpha + \beta \neq n, \\ \ln|p| + \ln(\text{diam} \Omega) - 2, & \alpha + \beta = n. \end{cases} \quad (3.29)$$

Summing up (3.27), (3.28) and (3.29), we obtain (3.26). The proof is complete. \hfill \Box

**Lemma 3.6.** Define $F_j(k, \hat{x})$ ($j = 0, 1$) as in (3.4). For every $\hat{x} \in \mathbb{S}^2$ and every $k_1, k_2 \geq k$, when $k \to +\infty$, we have the following estimates:

$$\mathbb{E}(F_1(k_2, \hat{x})F_1(k_1, \hat{x})) = O(k^{-3}), \quad \mathbb{E}(F_1(k_1, \hat{x}) \cdot F_1(k_2, \hat{x})) = O(k^{-3}), \quad (3.30)$$

uniformly for all $\hat{x}$. 
Thus, (3.32) is proved.

In (3.32), with some abuse of notations, we use “1” (resp. “2”) to represent the variable that the operator \( V R_{k_1} \) (resp. \( V R_{k_2} \)) acts on.

To prove (3.31), one can compute

\[
[(VR_{k_j})f](x) = [(VR_{k_j})^{j-1}(VR_{k_j}f)](x) = [(VR_{k_j})^{j-1}(V\cdot)(f(s), \Phi_k(s,\cdot))](x)
\]

\[
= (f(s), [(VR_{k_j})^{j-1}(V\cdot)(\Phi_k(s,\cdot))](x)).
\]

By (3.17) and (3.31), we arrive at (3.32). To prove (3.32), one can compute

\[
\begin{align*}
\mathbb{E}(G_j(k_1, \tilde{x}_1) \cdot G_\ell(k_2, \tilde{x}_2)) &= \int e^{-ik\tilde{x}_1 \cdot y} [(VR_{k_1})^{j-1}(V\cdot)(\Phi_k(s,\cdot))](y) dy \\
&\cdot \int e^{-ik\tilde{x}_2 \cdot z} [(VR_{k_2})^{\ell-1}(V\cdot)(\Phi_k(t,\cdot))](z) dz \\
&= \int e^{-ik\tilde{x}_1 \cdot y} [(VR_{k_1})^{j-1}(K(s, t)V\cdot)(\Phi_k(s,\cdot))](y) dy \\
&\cdot \int e^{ik\tilde{x}_2 \cdot z} [(VR_{k_2})^{\ell-1}(V\cdot)(\Phi_k(t,\cdot))](z) dz ds dt \\
&= \int e^{ik\tilde{x}_2 \cdot z} \left( (VR_{k_2})^{\ell-1} \left( \int e^{-ik\tilde{x}_1 \cdot y} [(VR_{k_1})^{j-1}(V(1)V(2))](y) dy \right) (y) dy \right) (z) dz.
\end{align*}
\]

Thus, (3.32) is proved.

Note that

\[
\mathbb{E}(F_1(k_1, \tilde{x}_1) \cdot F_1(k_2, \tilde{x}_2)) = \mathbb{E}(G_1(k_1, \tilde{x}_1) \cdot G_1(k_2, \tilde{x}_2)) + \sum_{j+\ell \geq 1} \mathbb{E}(G_j(k_1, \tilde{x}_1) \cdot G_\ell(k_2, \tilde{x}_2)).
\]

Next we estimate \( \mathbb{E}(G_1G_1) \) and \( \mathbb{E}(G_jG_\ell) \) (\( j + \ell \geq 3 \), \( j, \ell \geq 1 \)) in different manners.

Recall the definition of \( D \) given in (2.2). We denote \( \tilde{D} := \{ x + x', x - x' \mid x, x' \in D \} \). To estimate \( \mathbb{E}(G_1G_1) \), we fix real-valued cut-off functions \( \eta_i \in C_c^\infty(\mathbb{R}^3) \) (\( i = 1, 2 \)) satisfying

\[
\begin{align*}
\text{supp} \eta_i &\subset \tilde{D}, \ i = 1, 2, \\
\eta_1 &\equiv 1 \text{ in } D_f, \\
\eta_2 &\equiv 1 \text{ in } \{ s + t \in \mathbb{R}^3 \mid s, t \in D_f \}.
\end{align*}
\]

With the help of (3.32) and (3.33) and by using [16, Lemma 18.2.1] repeatedly, one have

\[
\mathbb{E}(G_1(k_1, \tilde{x}_1) \cdot G_1(k_2, \tilde{x}_2))
\]
\[
\int_{\mathbb{R}^2} e^{ik_2 \hat{x}_2 z} \int e^{-ik_1 \hat{x}_1 y} V(y) V(z) I(z, y) \, dy \, dz \\
\approx \int_{\mathbb{R}^2} \eta_2(s + t) \eta_1(s) ( \int e^{i(s-t) \xi} c_f(s, \xi) \, d\xi \cdot ( \int e^{-ik_1 \hat{x}_1 (y - \hat{x}_1)} V(y) |y - s| \, dy ) \\
\cdot ( \int e^{ik_2 \hat{x}_2 - |z - t|} V(z) |z - t| \, dz ) \, ds \, dt \\
= \int_{\mathbb{R}^2} \eta_2(s + t) ( \int e^{i(s-t) \xi} \hat{c}(s, t, \xi) \, d\xi ) e^{-ik_1 \hat{x}_1 s} e^{ik_2 \hat{x}_2 t} G(s, k_1, \hat{x}_1) \hat{G}(t, k_2, \hat{x}_2) \, ds \, dt \\
= \frac{1}{2} \int_{\mathbb{R}^2} \eta_2(T) e^{i\theta_2 T} e^{-i\theta_1 S} ( \int e^{iS \xi} G\left(\frac{T + S}{2}, k_1, \hat{x}_1\right) \hat{G}\left(\frac{T - S}{2}, k_2, \hat{x}_2\right) c_2(T, \xi) \, d\xi ) \, dS \, dT \\
= \frac{1}{2} \int_{\mathbb{R}^2} \eta_2(T) e^{i\theta_2 T} ( \int e^{-i\theta_1 S} ( \int e^{iS \xi} c_3(S, T, \xi) \, dS ) \, dT \\
\approx \int_{\mathbb{R}^2} \eta_2(T) e^{i\theta_2 T} c_3(T, \theta_1) \, dT, \quad (3.37)
\]

where
\[
G(s, k, \hat{x}) := \int_{\mathbb{R}^2} e^{-ik(\hat{x}y - |y|)} V(y + s) |y| \, dy,
\]
and
\[
\begin{aligned}
\theta_1 &:= (k_1 \hat{x}_1 + k_2 \hat{x}_2) / 2 \\
\theta_2 &:= (k_1 \hat{x}_1 - k_2 \hat{x}_2) / 2 \\
S &:= s - t \\
T &:= s + t
\end{aligned}
\]
and
\[
\begin{aligned}
\hat{c}(s, t, \xi) &:= \eta_1(s) \eta_1(t) c_f(s, \xi), \\
c_2(T, \xi) &:= \hat{c}(T/2, T/2, \xi) + S^{-m-1} = (\eta_1(T/2))^2 c(T/2, \xi) + S^{-m-1}, \\
c_3(S, T, \xi) &:= \hat{c}_3(S, T, \xi; k_1, \hat{x}_1, k_2, \hat{x}_2) := G\left(\frac{T + S}{2}, k_1, \hat{x}_1\right) \hat{G}\left(\frac{T - S}{2}, k_2, \hat{x}_2\right) c_2(T, \xi), \\
c_3(T, \xi) &:= \hat{c}_3(0, T, \xi) + S^{-m-1}.
\end{aligned}
\]

Here the notation \( S^{-m-1} \) stands for the set of symbols of pseudo-differential operators of order \(-m - 1\); see e.g. [31] for more details about pseudo-differential operators. Therefore,
\[
c_3(T, \xi) = G(T/2, k_1, \hat{x}_1) \hat{G}(T/2, k_2, \hat{x}_2) \, c_2(T, \xi) + S^{-m-1} \\
= (\eta_1(T/2))^2 G(T/2, k_1, \hat{x}_1) \hat{G}(T/2, k_2, \hat{x}_2) c(T/2, \xi) \\
+ G(T/2, k_1, \hat{x}_1) \hat{G}(T/2, k_2, \hat{x}_2) c_2(T, \xi) \cdot S^{-m-1} \\
= (\eta_1(T/2))^2 G(T/2, k_1, \hat{x}_1) \hat{G}(T/2, k_2, \hat{x}_2) c(T/2, \xi) + S^{-m-1}. \quad (3.38)
\]

Set \( \hat{x} = \hat{x}_1 = \hat{x}_2 \) and recall that \(|S|\) signifies the Lebesgue measure of any Lebesgue-measurable set \( S \), from (3.37) and (3.38) we obtain
\[
|E\left(G_1(k_1, \hat{x}) \cdot \hat{G}_1(k_2, \hat{x})\right)| \leq C|\text{supp} \eta_2| \cdot \sup_{T \in \text{supp} \eta_3} |c_3(T, \theta_1)| \\
\leq C|\text{supp} \eta_2| |\theta_1|^{-m} ( \sup_{T \in \text{supp} \eta_2} \left|G(T/2, k_1, \hat{x}_1)\right| \cdot \left|\hat{G}(T/2, k_2, \hat{x}_2)\right| + C|\text{supp} \eta_2| \cdot |\theta_1|^{-1}) \\
\leq C_f \sup_{T \in \text{supp} \eta_2} \left|G(T/2, k_1, \hat{x})\right| \cdot \left|\hat{G}(T/2, k_2, \hat{x})\right| \cdot k^{-m} + C_f k^{-m-1}, \quad (3.39)
\]
where the constant \( C_f \) is independent of \( k_1, k_2 \) and \( \hat{x} \).

We proceed to show that \( G(T/2, k, \hat{x}) = O(k^{-1}) \). For any \( \hat{x} \in \mathbb{S}^2 \), we can always find two unit vectors \( \hat{x}^{\perp, 1}, \hat{x}^{\perp, 2} \in \mathbb{S}^2 \) such that the set \( \{ \hat{x}, \hat{x}^{\perp, 1}, \hat{x}^{\perp, 2} \} \) forms an orthonormal basis.
Write the $3 \times 3$ matrix $\Phi = (\hat{x}, \hat{x}^\perp, \hat{x}^\parallel)$, then $\Phi^T \hat{x} = (1, 0, 0)^T =: e_1$. Denoting
\[ \hat{V}(y, s) := \langle y \rangle^{1+\sigma} V(y + s), \]
where the value of $\sigma$ shall be determined later, we know $\hat{V}(y, s) \in C^3$ in $y$ variable. We have
\[
\mathcal{G}(s, k, \hat{x}) = \int_{\mathbb{R}^3} e^{-ik(\hat{x} - y - |y|)} |y|^{-1} \langle y \rangle^{-1-\sigma} \hat{V}(y, s) \, dy
= \mathcal{O}(k^{-1}) + \int_{k^{-1/2}}^{+\infty} r(r)^{-1-\sigma} e^{irk \hat{x} \cdot w} \hat{V}(rw, s) \, ds(r) \cdot \int_{\mathbb{S}^2} e^{ikr \hat{w} \cdot \hat{V}(r \Phi w, s) \, dS(w), \quad k \to +\infty.}
\]

We cover the unit sphere $\mathbb{S}^2$ by six (relative) open parts:
\[ \Gamma_{p,q} := \{(w_1, w_2, w_3) \in \mathbb{R}^3; \sum_{j=1}^{3} w_j^2 = 1, (-1)^j w_p > \sqrt{3}/6\}, \quad p = 1, 2, 3, \quad q = 0, 1. \]

It is straightforward to verify that $\{\Gamma_{p,q}\}$ is an open covering of $\mathbb{S}^2$, i.e. $\mathbb{S}^2 \subset \cup_{p,q} \Gamma_{p,q}$. There exists a partition of unity $\{\rho_{p,q}\}$ subject to the open covering $\{\Gamma_{p,q}\}$, and we write
\[ g_{p,q}(r, k, \hat{x}, s) := \int_{\Gamma_{p,q}} e^{ikr \hat{w} \cdot \hat{V}(r \Phi w, s) \, dS(w). \]

Hence,
\[
\mathcal{G}(s, k, \hat{x}) = \mathcal{O}(k^{-1}) + \sum_{p,q} \int_{k^{-1/2}}^{+\infty} r(r)^{-1-\sigma} e^{irk \rho_{p,q}(r, k, \hat{x}, s) \, dr. (3.40)}
\]

We proceed to analyze $g_{1,0}$ and $g_{3,0}$. The analysis of $g_{1,1}$ is similar to that of $g_{1,0}$, and $g_{p,q}$ ($p = 2, 3, q = 0, 1$) is similar to $g_{3,0}$, so we skip the analyses of these terms.

In what follows, we write $w = (w_1, w_2, w_3)^T \in \mathbb{S}^2$ as a vertical vector. Noticing that in $\Gamma_{1,0}$ the $w_1$ is uniquely determined by the $w_2$ and $w_3$, so there exists a unique function $\phi \in C^\infty$ such that $w_1 = \phi(w_2, w_3)$, and with a slight abuse of notation, we may write $w = w(w_2, w_3) = (\phi(w_2, w_3), w_2, w_3)^T$. Denote the projection of $\Gamma_{1,0}$ onto the $(w_2, w_3)$-coordinate as $\Pi_{1,0}$. We know $\Pi_{1,0} \subset (-1, 1)^2$. We have
\[ \phi(w_2, w_3) \in (\sqrt{3}/6, 1), \quad \forall (w_2, w_3) \in \Pi_{1,0}. \]

We can fix some $\rho_{1,0} \in C^\infty_{c}((-1, 1)^2)$ such that $\rho_{1,0} \equiv 1$ in $\Pi_{1,0}$. Then
\[
g_{1,0} = \int_{\mathbb{R}^2} e^{ikr \phi(w_2, w_3)} \rho_{1,0}(w_2, w_3) \hat{V}(r \Phi w, s) \cdot \sqrt{\det[\langle \partial_{w_2}w, \partial_{w_3}w \rangle^T (\partial_{w_2}w, \partial_{w_3}w)]} \, dw_2 \, dw_3.
\]

According to $\phi^2 + w_2^2 + w_3^2 = 1$ we have
\[
\begin{align*}
\phi w_2 &= -w_2/\phi \\
\phi w_3 &= -w_3/\phi
\end{align*}
\]
and
\[
\begin{align*}
\phi w_2 w_3 &= -(1 + \phi^2) / \phi \\
\phi w_2^2 &= -\phi w_2^2 w_3^2 / \phi \\
\phi w_3^2 &= -(1 + \phi^2) / \phi
\end{align*}
\]
Note that $\phi > \sqrt{3}/6$. Hence, we have that $|\nabla \phi| = 0$ only when $w_2 = w_3 = 0$ and that $\det[\langle \partial_{w_2}w, \partial_{w_3}w \rangle^T = (1 + \phi^2 w_2^2 + \phi^2 w_3^2) / \phi^2 \neq 0$. This means that $(0, 0)$ is the only critical point of
the phase function \( kr \phi(w_2, w_3) \) in (3.41), when \( w \in \Gamma_{1,0} \). According to the stationary phase lemma [34, Chapter 3], we have

\[
\begin{align*}
g_{1,0}(r, k, \hat{x}, s) &= \left( \frac{2\pi}{kr} \right) C_1 \left( C_2 + O((kr)^{-1}) \right) = O((kr)^{-1}), \quad k \to +\infty. \quad (3.42)
\end{align*}
\]

Note that in order to use the stationary phase lemma to obtain the high-order term with \(-1\) order decay, the integrand should have \( C^0 \)-smoothness, which is guaranteed by (3.41).

Next we analyze \( g_{3,0} \). We may write \( w = w(w_1, w_2) = (w_1, w_2, \phi(w_1, w_2))^T \). It holds that

\[
\begin{align*}
g_{3,0}(r, k, \hat{x}, s) &= \int_{\mathbb{R}^2} e^{ikrw_1} \rho_{3,0}(w_1, w_2) \tilde{V}(r \Phi w, s) \\
&= \left. \frac{1}{ikr} \int_{\mathbb{R}^2} \partial_r e^{ikrw_1} \rho_{3,0}(w) \tilde{V}(r \Phi w, s) C_1(w_1, w_2) \, dw_1 \, dw_2 \right|_{r \to +\infty}
\end{align*}
\]

where \( C_1 \) and \( C_2 := \rho_{3,0}(w) \tilde{V}(r \Phi w, s) C_1(w_1, w_2) \) are two functions such that \( C_1 \in C^\infty \) and \( C_2 \in C_0^2((-1,1)^2) \) because \( \tilde{V}(\cdot, s) \in C^3 \), and \( \rho_{3,0} \) is chosen in the same manner as \( \rho_{1,0} \). Therefore the partial derivative of the function \( C_2 \) is bounded above, and hence

\[
|g_{3,0}(r, k, \hat{x}, s)| \leq C(kr)^{-1}. \quad (3.43)
\]

Combining (3.40) with (3.42) and (3.43), one can compute

\[
\begin{align*}
|G(s, k, \hat{x})| &\leq O(k^{-1}) + \sum_{m=1}^{\infty} \int_{k^{-1/2}}^{+\infty} r^{-\sigma} \left[ C_1(kr)^{-1} + C_2(kr)^{-2} + O((kr)^{-3}) \right] \, dr \\
&= O(k^{-1+\sigma/2}), \quad k \to +\infty,
\end{align*}
\]

where the asymptotics is uniform in terms of \( s \) and \( \hat{x} \). Note that we used \( r(r)^{-1-\sigma} \leq r^{-\sigma} \). By (3.39) and (3.44), we arrive at

\[
|\mathbb{E}(G_1(k_1, \hat{x}_1) \cdot \widehat{G_1(k_2, \hat{x}_2)})| \leq Ck^{-m-2+\sigma} + C \cdot k^{-m-1} = O(k^{-m-1}), \quad k \to +\infty,
\]

where the last inequality is by taking \( \sigma = 1/3 \). Because \( m > 2 \), we obtain

\[
|\mathbb{E}(G_1(k_1, \hat{x}_1) \cdot \widehat{G_1(k_2, \hat{x}_2)})| \leq O(k^{-3}), \quad k \to +\infty. \quad (3.45)
\]

To estimate \( \mathbb{E}(G_{1, \ell}) \) for \( j + \ell \geq 3, j, \ell \geq 1 \), we first estimate \( I(z, y) \) which is defined in (3.33). Choose \( \eta_1, \eta_2 \in C_0^\infty(\mathbb{R}^3) \) as in (3.36). It follows that

\[
\begin{align*}
I(z, y) &= \int_{D_1 \times D_1} \int_{D_1 \times D_1} K(s, t) \eta_1(s) \eta_1(t) \Phi(s - y) \Phi(t - z) \, ds \, dt \\
&\approx \int_{D_1 \times D_1} \mathcal{F}^{-1} \{ c(s, \cdot) \} (s - t) \cdot \eta_1(s) \eta_1(t) \Phi(s - y) \Phi(t - z) \, ds \, dt \\
&\approx \int_{D_1 \times D_1} e^{ik_1|s-y|+ik_2|t-z|} (|s-y|^{-1}|t-z|^{-1}) \int e^{i(s-t) \cdot \xi} C_1(s, t, \xi) \, d\xi \, ds \, dt,
\end{align*}
\]

where \( c_1(s, t, \xi) := c(s, \xi) \eta_1(s) \eta_1(t) \). Define two differential operators

\[
L_1 := \frac{(s-y) \cdot \nabla_s}{ik_1 |s-y|} \quad \text{and} \quad L_2 := \frac{(t-z) \cdot \nabla_t}{-ik_2 |t-z|}.
\]
It can be verified that
\[ L_1 L_2(e^{ik_1|s-y| - ik_2|t-z|}) = e^{ik_1|s-y| - ik_2|t-z|}. \]
Hence, noting that the integrand is compactly supported in \( D_f \times D_f \) and by using integration by part, we can continue (3.46) as
\[
|I(z, y)| \simeq \left| \int_{D_f} \int_{D_f} L_1 L_2(e^{ik_1|s-y| - ik_2|t-z|}) |(s-y)|^{-1}|(t-z)|^{-1} \int e^{i(s-t)\cdot \xi} c_1(s, t, \xi) d\xi | ds dt \right|
\]
\[
\simeq k_1^{-1} k_2^{-1} \int_{D_f} \int_{D_f} e^{ik_1|s-y| - ik_2|t-z|} | \nabla_s \int e^{i(s-t)\cdot \xi} c_1(0) \, d\xi |
\] 
\[
+ |s-y|^{-1} |(t-z)|^{-1} \int_{D_f} \int_{D_f} e^{i(s-t)\cdot \xi} c_1(0) \, d\xi |
\]
\[
\lesssim k_1^{-1} k_2^{-1} \int_{D_f} \int_{D_f} |(s-y)|^{-1}|(t-z)|^{-2} J_0 + |s-y|^{-1}|(t-z)|^{-1}(\max a \, J_{1;a})
\]
\[
+ |s-y|^{-1}|(t-z)|^{-2}(\max a \, J_{2;a,b}) + |s-y|^{-1}|(t-z)|^{-1}(\max a, b \, J_{2;a,b}) \, d\xi \, dt, \quad (3.47)
\]
where \( a, b \) are indices running from 1 to 3, and
\[
J_0 := | \int e^{i(s-t)\cdot \xi} c_1(s, t, \xi) \, d\xi |, \quad J_{1;a} := | \int e^{i(s-t)\cdot \xi} c_1(s, t, \xi) \, d\xi |,
\]
\[
J_{2;a,b} := | \int e^{i(s-t)\cdot \xi} c_1(s, t, \xi) \, d\xi |.
\]
Because of the condition \( m > 2 \), we can find a number \( \tau \in (0, 1) \) satisfying the inequalities
\[ 3 - m < \tau < 1. \]
Therefore, we have
\[
\begin{align*}
-3m - \tau &< -3, \quad \text{(3.48a)} \\
-2 - \tau &> -3. \quad \text{(3.48b)}
\end{align*}
\]
By using [24, Lemmas 3.1 and 3.2], these quantities \( J_0, J_{1;a} \) and \( J_{2;a,b} \) can be estimated as follows:
\[
J_0 = |s-t|^{-\tau} \cdot \int (-\Delta)^{\tau/2}(e^{i(s-t)\cdot \xi} c_1(s, t, \xi)) d\xi |
\]
\[
= |s-t|^{-\tau} \cdot \int e^{i(s-t)\cdot \xi} (-\Delta)^{\tau/2}(c_1(s, t, \xi)) d\xi |
\]
\[
\lesssim |s-t|^{-\tau} \cdot \int (\xi)^{-m-\tau} d\xi \lesssim |s-t|^{-\tau}. \quad (3.49)
\]
The last inequality in (3.49) makes use of the fact (3.48a). We estimate \( J_{1;a} \) as follows,
\[
J_{1;a} = | \int \left( \frac{s-t}{|s-t|^2} \cdot \nabla_s (e^{i(s-t)\cdot \xi} c_1(s, t, \xi)) d\xi | 
\]
\[
= \frac{|s-t|}{|s-t|^{2+\tau}} \int e^{i(s-t)\cdot \xi} (-\Delta)^{\tau/2}(\nabla_s c_1(s, t, \xi)) d\xi |
\]
\[ \leq C|s-t|^{-1-\tau} \int (\xi)^{-m+1-1-\tau} \, d\xi \leq C|s-t|^{-1-\tau}, \quad (3.50) \]

where the constant \( C \) is independent of the index \( a \). Similarly, we have

\[ J_{2;a,b} = |s-t|^{-2-\tau} \int \Delta_\xi (-\Delta_\xi)^{7/2} (e^{i(s-t)\cdot\xi}) \xi_a \xi_b c_1(s,t,\xi) \, d\xi \]

\[ \leq C|s-t|^{-2-\tau} \int (\xi)^{-m+3/2-2-\tau} \, d\xi \leq C|s-t|^{-2-\tau}, \quad (3.51) \]

where the constant \( C \) is independent of the indices \( a \) and \( b \). Combining (3.47), (3.49), (3.50) and (3.51), we can rewrite (3.47) as

\[ k_1 k_2 |I(z,y)| \lesssim \int_{D_f \times D_f} \left[ |s-y|^{-2} |t-z|^{-2} |s-t|^{-\tau} + |s-y|^{-2} |t-z|^{-1} |s-t|^{-1-\tau} \right. \]
\[ + |s-y|^{-1} |t-z|^{-2} |s-t|^{-1-\tau} + |s-y|^{-1} |t-z|^{-1} |s-t|^{-2-\tau} \] \[ \left. \right] \, ds \, dt \]
\[ =: I_1 + I_2 + I_3 + I_4. \quad (3.52) \]

Denote \( D := \{ x + x', x - x' ; x, x' \in \tilde{D} \} \). Then we apply Lemma 3.3 to estimate \( I_j \) \((j = 1, 2, 3, 4)\) as follows,

\[ I_1 = \int_{D_f \times D_f} |s-y|^{-2} |t-z|^{-2} |s-t|^{-\tau} \, ds \, dt \]
\[ \leq \int_D |s|^{-2} \left( \int_D |t|^{-2} |t - (s + y - z)|^{-\tau} \, dt \right) \, ds \]
\[ \lesssim \int_D |s|^{-2} \left[ |s - (z - y)|^{3-2-\tau} + (\text{diam } D)^{3-2-\tau} \right] \, ds \]
\[ = C_f + \int_D |s|^{-2} |s - (z - y)|^{-(\tau-1)} \, ds \]
\[ \lesssim |z - y|^{2-\tau} + C_f. \quad (3.53) \]

Note that in (3.53) we used Lemma 3.5 twice. Similarly,

\[ I_2, I_3, I_4 \lesssim |z - y|^{2-\tau} + C_f. \quad (3.54) \]

Recall that \( \tau \in (0, 1) \). By (3.52), (3.53) and (3.54), we arrive at

\[ |I(z,y)| \leq C k_1^{-1} k_2^{-1} |z - y|^{2-\tau} + C, \quad (3.55) \]

where the constant \( C \) is independent of \( y \), \( z \) and \( k \).

Recall \( V \in L^2_{3/2+\epsilon}(\mathbb{R}^3) \) stipulated in [1.4], so it follows \( \|V\|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} < +\infty \). This will be used in the next computation. Combining (3.32) and (3.55) (and without loss of generality) assuming \( \ell \geq 2 \), one can compute

\[ |E(G_j(k_1, \tilde{x}_1) \cdot G\ell(k_2, \tilde{x}_2))| \]
\[ = \left| \int e^{ik_2 \tilde{x}_2 \cdot z} \left( (\mathcal{R}_j k_1)^{-1} \left( \int e^{-ik_1 \tilde{x}_1 \cdot y} ([\mathcal{R}_j k_1]^{-j-1} V(1) \bar{V}(2) I(2,1)) \right) (y) \, dy \right) (z) \, dz \right| \]
\[ \leq C V \| \mathcal{R}_j k_2^{-1} \| \| \mathcal{R}_j k_1^{-j-1} V(1) \bar{V}(2) I(2,1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \]
\[ \leq C V k_2^{-1+j+1} \| \mathcal{R}_j k_1^{-j-1} V(1) \bar{V}(2) I(2,1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \]
\[ \lesssim k_2^{-\ell+1} k_1^{-j+1} \| V(1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \| \bar{V}(2) I(2,1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \]
\[ \lesssim k_2^{-\ell+1} k_1^{-j+1} \| V(1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \| \bar{V}(2) I(2,1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \]
\[ \lesssim k_2^{-\ell+1} k_1^{-j+1} \| V(1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \| \bar{V}(2) I(2,1) \|_{L^2_{1/2+\epsilon}(\mathbb{R}^3)} \]
By substituting (3.55) into the computation above, we can continue
\[ |\mathbb{E}(G_j(k_1, \hat{x}_1) \cdot \hat{G}_\ell(k_2, \hat{x}_2))| \]
\[ \lesssim k_2^{-\ell+1}k_1^{-1+j} \left( \int \langle y \rangle^{1+2\tau} \langle z \rangle^{1+2\tau} |V(y)V(z)|^2 \, dy \, dz \right)^{1/2} \]
\[ \leq k_2^{-\ell}k_1^{-j} \left( \int \langle y \rangle^{1+2\tau} \langle z \rangle^{1+2\tau} |V(y)V(z)|^2 \, dy \, dz \right)^{1/2} \] (by (3.55))
\[ \leq k_2^{-\ell}k_1^{-j} \left( \int \langle y \rangle^{\tau+2\tau} \langle z \rangle^{\tau+2\tau} |V(y)V(z)|^2 \, dy \, dz + C\|V\|_{L^2_{\tau+2\tau}}^2 \right)^{1/2} \]
\[ = Ck_2^{-\ell+1}k_1^{-j} \|V\|_{L^2_{\tau+2\tau}}^2 < Ck_2^{+\ell-2}k_1^{-j}, \]
where in the last inequality we used \( \|V\|_{L^2_{\tau+2\tau}}^2 < +\infty \) guaranteed by (1.4). We also used \( |z - y| \leq |z - y| \leq \langle z \rangle \langle y \rangle \). Therefore,
\[ \left| \sum_{j + \ell \geq 1, j \geq 1} \mathbb{E}(G_j(k_1, \hat{x}_1) \cdot \hat{G}_\ell(k_2, \hat{x}_2)) \right| \]
\[ \lesssim \sum_{j \geq 1} k_2^{-\ell+1}k_1^{-j} + \sum_{j \geq 2} k_2^{-\ell+1}k_1^{-j} \lesssim k_2^{-2}k_1^{-1} + \sum_{j \geq 2} k_2^{-2}k_1^{-j} \lesssim k^{-3}, \quad k \to +\infty. \quad (3.56) \]

Finally, by combining (3.35), (3.45) and (3.56), we conclude (3.30), which completes the proof. \( \square \)

The following lemma is the ergodic version of Lemmas 3.4 and 3.6.

**Lemma 3.7.** Define \( F_j(k, \hat{x}) \) (\( j = 0, 1 \)) as in (3.4). Write
\[ X_{p,q}(K, \tau, \hat{x}) = \frac{1}{K} \int_k^{2K} k^m F_q(k, \hat{x}) \cdot F_p(k + \tau, \hat{x}) \, dk, \text{ for } (p, q) \in \{(0, 1), (1, 0), (1, 1)\}. \]
Then for any \( \hat{x} \in \mathbb{S}^2 \) and any \( \tau \geq 0 \), when \( K \to +\infty \), we have the following estimates:
\[ |\mathbb{E}(X_{p,q}(K, \tau, \hat{x}))| = O(K^{-1}), \quad |\mathbb{E}(|X_{p,q}(K, \tau, \hat{x})|^2)| = O(K^{-3/2}), \quad (p, q) \in \{(0, 1), (1, 0)\}, \quad (3.57) \]
\[ |\mathbb{E}(X_{1,1}(K, \tau, \hat{x}))| = O(K^{-m-3}), \quad |\mathbb{E}(|X_{1,1}(K, \tau, \hat{x})|^2)| = O(K^{2(m-3)}). \quad (3.58) \]
Let \( \{K_j\} \in P(\max\{2/3, (3 - m)^{-1}/2\} + \gamma) \), then for any \( \tau \geq 0 \), we have
\[ \lim_{j \to +\infty} X_{p,q}(K_j, \tau, \hat{x}) = 0 \quad a.s., \quad (3.59) \]
for every \( (p, q) \in \{(0, 1), (1, 0), (1, 1)\} \).

We may denote \( X_{p,q}(K, \tau, \hat{x}) \) as \( X_{p,q} \) for short if it is clear in the context.

**Proof of Lemma 3.7.** According to Lemmas 3.4 and 3.6, we have
\[ \mathbb{E}(X_{0,1}) = \frac{1}{K} \int_k^{2K} k^m \mathbb{E}(F_1(k, \hat{x}) \cdot F_0(k + \tau, \hat{x})) \, dk = \frac{1}{K} \int_k^{2K} O(k^{-1}) \, dk \]
\[ = O(K^{-1}), \quad K \to +\infty. \quad (3.60) \]
By formula (3.12), Isserlis’ Theorem and Lemma 3.2, we compute the secondary moment of \( X_{0,1} \),
\[ \mathbb{E}(|X_{0,1}|^2) \]
\[ = \mathbb{E} \left( \frac{1}{K} \int_k^{2K} k_1^m F_0(k_1 + \tau, \hat{x}) \cdot F_1(k_1, \hat{x}) \, dk_1 \cdot \frac{1}{K} \int_k^{2K} k_2^m F_0(k_2 + \tau, \hat{x}) \cdot F_1(k_2, \hat{x}) \, dk_2 \right) \]
\[ \frac{1}{K^2} \int_K^{2K} \left[ \mathcal{O}(K^{-2}) + 2(\pi)^{3/2} \tilde{\mu}((k_1 - k_2)\hat{x}) \cdot \mathcal{O}(K^{-1}) + \mathcal{O}(K^{-2}) \right] \, dk_1 \, dk_2 \]
\[ = \frac{1}{K^2} \int_K^{2K} \int_K^{2K} (2\pi)^{3/2} \tilde{\mu}((k_1 - k_2)\hat{x}) \, dk_1 \, dk_2 \cdot \mathcal{O}(K^{-1}) + \mathcal{O}(K^{-2}) \]
\[ = \mathcal{O}(K^{-1/2}) \cdot \mathcal{O}(K^{-1}) + \mathcal{O}(K^{-2}) \text{ (Hölder ineq. and (3.7))} \]
\[ = \mathcal{O}(K^{-3/2}), \quad K \to +\infty. \quad (3.61) \]

From (3.60) and (3.61) we obtain (3.57) for \((p, q) = (0, 1)\). Similarly, formula (3.57) for \((p, q) = (1, 0)\) can be proved.

By Chebyshev’s inequality and (3.61), for any \(\epsilon > 0\), we have
\[
P\left( \bigcup_{j \geq K_0} \{|X_{0,1}(K_j, \tau, \hat{x}) - 0| \geq \epsilon\} \right) \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} K_j^{-3/2} \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} j^{-1-3\gamma/2} \]
\[ \leq \frac{C}{\epsilon^2} \int_{K_0}^{+\infty} (t-1)^{-1-3\gamma/2} \, dt \to 0, \quad K_0 \to +\infty. \quad (3.62) \]

According to [23, Lemma 3.3], (3.62) implies (3.59) for \((p, q) = (0, 1)\). Similarly, formula (3.59) for \((p, q) = (1, 0)\) can be proved.

Now we prove (3.58). We have:
\[
\mathbb{E}(X_{1,1}) = \frac{1}{K} \int_K^{2K} k^m \mathbb{E}(F_1(k, \hat{x}) \cdot F_1(k + \tau, \hat{x})) \, dk = \frac{1}{K} \int_K^{2K} \mathcal{O}(K^{-m}) \, dk = \mathcal{O}(K^{-m}). \quad (3.63) \]

Compute the secondary moment:
\[
\mathbb{E}(|X_{1,1}|^2) = \mathbb{E}\left( \frac{1}{K} \int_K^{2K} k^m F_1(k + \tau, \hat{x}) \cdot F_1(k + \tau, \hat{x}) \, dk \right) = \frac{1}{K} \int_K^{2K} \mathcal{O}(K^{-m}) \cdot \mathcal{O}(K^{-m}) \, dk 
\]
\[ = \mathcal{O}(K^{2(m-3)}), \quad K \to +\infty. \quad (3.64) \]

Formulae (3.63) and (3.64) gives (3.58).

By Chebyshev’s inequality and (3.64), for any \(\epsilon > 0\), we have
\[
P\left( \bigcup_{j \geq K_0} \{|X_{1,1} - 0| \geq \epsilon\} \right) \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} K_j^{2(m-3)} \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} j^{-1-\gamma'} \]
\[ \leq \frac{C}{\epsilon^2} \int_{K_0}^{+\infty} (t-1)^{-1-\gamma'} \, dt \to 0, \quad K_0 \to +\infty., \quad (3.65) \]

where \(\gamma'\) is some positive constant depending on \(m\). According to [23, Lemma 3.3], (3.65) implies (3.59) for \((p, q) = (1, 1)\). The proof is complete. \(\square\)

4. The recovery of the rough strength

In this section we focus on the recovery of the rough strength \(\mu(x)\) of the random source. We employ only a single-realisation of the passive scattering measurement, namely the random sample \(\omega\) is fixed. The data set \(\{u^\infty(\hat{x}, k, \omega) \mid \hat{x} \in S^2, k \in \mathbb{R}_+\}\) is utilized to achieve the unique recovery result. In what follows, we present the main results of recovering \(\mu(x)\) in Section 4.1 and put the corresponding proofs in Section 4.2. The auxiliary lemmas derived in Section 3.2 shall play a key role to the proofs in Section 4.2.
4.1. Main unique recovery results. The first main recovery result is given as follows.

**Theorem 4.1.** We have the following asymptotic identity,
\[
4\sqrt{2\pi} \lim_{k \to +\infty} \mathbb{E}(k^m [u^\infty(\hat{x}, k) - \mathbb{E}u^\infty(\hat{x}, k)] \cdot [u^\infty(\hat{x}, k + \tau) - \mathbb{E}u^\infty(\hat{x}, k + \tau)]) = \hat{\mu}(\tau \hat{x}),
\]
\[m \geq 0, \; \hat{x} \in \mathbb{S}^2, \; \text{and } \hat{\mu} \text{ is the Fourier transform of } \mu.
\]

Theorem 4.1 clearly yields a recovery formula for the rough strength \(\mu\). However, it requires many realizations and is lack of practical usefulness. The result in Theorem 4.1 can be improved by using the ergodicity as follows.

**Theorem 4.2.** Let \(m^* = \max\{2/3, (3 - m)^{-1}/2\}\). Assume that \(\{K_j\} \in P(m^* + \gamma)\). Then \(\exists \Omega \subset \Omega: \mathbb{P}(\Omega) = 0, \Omega_0 \text{ depending only on } \{K_j\}_{j \in \mathbb{N}}, \text{ such that for any } \omega \in \Omega \setminus \Omega_0, \text{ there exists } S_\omega \subset \mathbb{R}^3: |S_\omega| = 0, \text{ it holds that for } \forall \tau \in \mathbb{R}_+ \text{ and } \forall \hat{x} \in \mathbb{S}^2 \text{ satisfying } \tau \hat{x} \in \mathbb{R}^3 \setminus S_\omega,
\[
4\sqrt{2\pi} \lim_{j \to +\infty} \frac{1}{K_j} \int_{\mathbb{R}^3} k^m [u^\infty(\hat{x}, k, \omega) - \mathbb{E}u^\infty(\hat{x}, k)] \cdot [u^\infty(\hat{x}, k + \tau, \omega) - \mathbb{E}u^\infty(\hat{x}, k + \tau)] \, dk = \hat{\mu}(\tau \hat{x}).
\]
\[\text{Proof of Theorem 4.3.} \text{ Theorem 4.3 provides a recovery formula for the local streng th}
\]
\[\text{and hence } \text{by one only realization of the passive scattering measurement, the } \mathbb{E}(u^\infty(\hat{x}, k)) \text{ should be further relaxed in (4.2), and this is done by Theorem 4.3 in the following.}
\]

**Theorem 4.3.** Under the same condition as in Theorem 4.2, we have
\[
4\sqrt{2\pi} \lim_{j \to +\infty} \frac{1}{K_j} \int_{\mathbb{R}^3} k^m [u^\infty(\hat{x}, k, \omega) - \mathbb{E}u^\infty(\hat{x}, k)] \cdot [u^\infty(\hat{x}, k + \tau, \omega) - \mathbb{E}u^\infty(\hat{x}, k + \tau)] \, dk = \hat{\mu}(\tau \hat{x}),
\]
\[\text{holds for } \forall \tau \in \mathbb{R}_+ \text{ and } \forall \hat{x} \in \mathbb{S}^2 \text{ satisfying } \tau \hat{x} \in \mathbb{R}^3 \setminus S_\omega.
\]

Now Theorem 4.1 becomes a direct consequence of Theorem 4.3.

**Proof of Theorem 4.1.** Theorem 4.3 provides a recovery formula for the local strength \(\mu\) by the far-field data \(\{u^\infty(\hat{x}, k, \omega)\}; \forall \hat{x} \in \mathbb{S}^2, \forall k \in \mathbb{R}_+\) with a single fixed \(\omega \in \Omega. \text{ \square}
\]

4.2. Proofs of the main theorems. In this subsection, we present the proofs of Theorems 4.1, 4.2, and 4.3.

**Proof of Theorem 4.1.** Let \(k \) be large enough s.t. \((I - \mathcal{R}_kV)^{-1} = \sum_{j=0}^{\infty}(\mathcal{R}_kV)^j\), and let \(\tau \in \mathbb{R}_+.\) According to the analysis at the beginning of Section 3 one can compute
\[
16\pi^2 \mathbb{E}
\]
\[\left(\mathbb{E}(u^\infty(\hat{x}, k)) - \mathbb{E}u^\infty(\hat{x}, k)\right)\left[u^\infty(\hat{x}, k + \tau) - u^\infty(\hat{x}, k)\right]
\]
\[= \sum_{j, \ell=0}^{\infty} \mathbb{E}\left(F_\ell(\hat{x}, k)F_j(k + \tau, \hat{x})\right) =: I_{0,0} + I_{0,1} + I_{1,0} + I_{1,1}.
\]
\[\text{From Lemmas 3.1 and 3.6 we have that } I_{0,1}, I_{1,0}, I_{1,1} \text{ are all of the order no less than } k^{-3},
\]
\[\text{hence}
\]
\[16\pi^2 \mathbb{E}\left(\mathbb{E}(u^\infty(\hat{x}, k)) - \mathbb{E}u^\infty(\hat{x}, k)\right)\left[u^\infty(\hat{x}, k + \tau) - u^\infty(\hat{x}, k)\right]\]
\[= k^{-m} I_{0,0} + O(k^{-m-3}),
\]
\[\text{as } k \text{ goes to infinity. Then, (3.10) gives}
\]
\[I_{0,0} = \mathbb{E}\left(F_0(\hat{x}, k)F_0(k + \tau, \hat{x})\right) = (2\pi)^{3/2} \hat{\mu}(\tau \hat{x})k^{-m} + \int_D a(y, k\hat{x})e^{i\tau\hat{x} \cdot y} \, dy.
\]
\[\text{The symbol } a \text{ is of order } -m - 1, \text{ and thus}
\]
\[\left| \int_D a(y, k\hat{x})e^{i\tau\hat{x} \cdot y} \, dy \right| \leq |D| \cdot |a(y, k\hat{x})| \leq |D| |C\langle k \hat{x} \rangle|^{-m-1} = |D| |C\langle k \rangle|^{-m-1}.
\]
\[\text{(4.6)}
\]
Lemma 3.7 implies the existence of the sets $\Omega_{p,q}$ for all probability measures such that $E_{\pi}(\Omega_{p,q}) = 0$, $\Omega_{p,q}$ depending on $\tau$ and $\hat{x}$, such that

$$\lim_{j \to +\infty} X_{0,0}(K_j, \tau, \hat{x}) = (2\pi)^{3/2} \tilde{\mu}(\tau \hat{x}), \quad \forall \omega \in \Omega \setminus \Omega_{\tau,\hat{x}}^{0,0}. \quad (4.10)$$

Lemma 3.7 implies the existence of the sets $\Omega_{p,q}^{\tau,\hat{x}}$ with zero probability measures such that $\forall \tau \geq 0$ and $\forall \hat{x} \in S^2$,

$$\lim_{j \to +\infty} X_{p,q}(K_j, \tau, \hat{x}) = 0, \quad \forall \omega \in \Omega \setminus \Omega_{\tau,\hat{x}}^{p,q}. \quad (4.11)$$

for all $(p, q) \in \{(0, 1), (1, 0), (1, 1)\}$. Write $\Omega_{\tau,\hat{x}} = \bigcup_{p,q=0,1} \Omega_{p,q}^{\tau,\hat{x}}$, then $\mathbb{P}(\Omega_{\tau,\hat{x}}) = 0$. From Lemmas 3.3 and 3.7 we note that $\Omega_{\tau,\hat{x}}^{p,q}$ also depends on $K_j$, so does $\Omega_{\tau,\hat{x}}$, but we omit this dependence in the notation. Write

$$Z(\tau \hat{x}, \omega) := \lim_{j \to +\infty} \frac{16\pi^2}{K_j} \int_{K_j} k^{2m} u^{\infty}(\hat{x}, k) u^{\infty}(\hat{x}, k + \tau) \, dk - (2\pi)^{3/2} \tilde{\mu}(\tau \hat{x})$$

for short. By (4.9), (4.10) and (4.11), we conclude that

$$\forall y \in \mathbb{R}^3, \exists \gamma_y \subset \Omega: \mathbb{P}(\gamma_y) = 0, \quad s.t. \forall \omega \in \Omega \setminus \gamma_y, Z(y, \omega) = 0. \quad (4.12)$$

To conclude (4.2) from (4.12), we need to exchange the order between $y$ and $\omega$. To achieve this, we utilize the Fubini’s Theorem. Denote the usual Lebesgue measure on $\mathbb{R}^3$ as $\mathbb{L}$ and the product measure $\mathbb{L} \times \mathbb{P}$ as $\mu$, and construct the product measure space $\mathbb{M} := (\mathbb{R}^3 \times \Omega, \mathcal{G}, \mu)$ in the canonical way, where $\mathcal{G}$ is the corresponding complete $\sigma$-algebra. Write

$$\mathcal{A} := \{(y, \omega) \in \mathbb{R}^3 \times \Omega: Z(y, \omega) \neq 0\},$$

then $\mathcal{A}$ is a subset of $\mathbb{M}$. Set $\chi_{\mathcal{A}}$ as the characteristic function of $\mathcal{A}$ in $\mathbb{M}$. By (4.12) we obtain

$$\int_{\mathbb{R}^3} \left( \int_{\Omega} \chi_{\mathcal{A}}(y, \omega) \, d\mathbb{P}(\omega) \right) \, d\mathbb{L}(y) = 0. \quad (4.13)$$

By (4.13) and [28, Corollary 7 in Section 20.1], we obtain

$$\int_{\mathbb{M}} \chi_{\mathcal{A}}(y, \omega) \, d\mu = \int_{\Omega} \left( \int_{\mathbb{R}^3} \chi_{\mathcal{A}}(y, \omega) \, d\mathbb{L}(y) \right) \, d\mathbb{P}(\omega) = 0. \quad (4.14)$$

Because $\chi_{\mathcal{A}}(y, \omega)$ is nonnegative, (4.14) implies

$$\exists \Omega_0: \mathbb{P}(\Omega_0) = 0, \quad s.t. \forall \omega \in \Omega \setminus \Omega_0, \int_{\mathbb{R}^3} \chi_{\mathcal{A}}(y, \omega) \, d\mathbb{L}(y) = 0. \quad (4.15)$$
The analysis to From Theorem 4.2 we obtain

\[ u \]

For convenience, we denote

Let

This is (4.2). The proof is complete.

Formula (4.15) further implies for every \( \omega \in \Omega \setminus \Omega_0 \),

\[ \exists S_\omega \subset \mathbb{R}^3 : \mathbb{L}(S_\omega) = 0, \text{ s.t. } \forall y \in \mathbb{R}^3 \setminus S_\omega, Z(y, \omega) = 0. \]  

(4.16)

This is (4.2). The proof is complete.

**Proof of Theorem 4.3.** Let \( \mathcal{E}_k \) be the averaging operator as defined in the proof of Theorem 4.2. For convenience, we denote \( u_\infty^\infty(\hat{x}, k) = u_\infty(\hat{x}, k) - \mathbb{E}u_\infty(\hat{x}, k) \), we write \( u_1^\infty(\hat{x}, k) = \mathbb{E}u_\infty(\hat{x}, k) \), thus \( u_\infty = u_0^\infty + u_1^\infty \). And we have

\[ 16\pi^2 \mathcal{E}_k(k^m u_\infty^\infty(\hat{x}, k) u_\infty(\hat{x}, k + \tau)) = 16\pi^2 \sum_{p,q=0,1} \mathcal{E}_k(k^m u_p^\infty(\hat{x}, k) u_q^\infty(\hat{x}, k + \tau)) \]

\[ =: J_{0,0} + J_{0,1} + J_{1,0} + J_{1,1}. \]

From Theorem 4.2 we obtain

\[ \lim_{j \to +\infty} J_{0,0} = 16\pi^2 \lim_{j \to +\infty} \int_{K_j}^{2K_j} k^m u_0^\infty(\hat{x}, k) \cdot u_0^\infty(\hat{x}, k + \tau) \, dk = (2\pi)^{3/2} \hat{\mu}(\tau \hat{x}), \]

\( \tau \hat{x} \) a.e. \( \in \mathbb{R}^3, \ \omega \) a.s. \( \in \Omega \).

Then we study \( J_{0,1} \),

\[ |J_{0,1}|^2 \simeq |\mathcal{E}_k(k^m u_0^\infty(\hat{x}, k) u_1^\infty(\hat{x}, k + \tau))|^2 = \left| \frac{1}{K_j} \int_{K_j}^{2K_j} k^m u_0^\infty(\hat{x}, k) u_1^\infty(\hat{x}, k + \tau) \, dk \right|^2 \]

\[ \leq \frac{1}{K_j} \int_{K_j}^{2K_j} k^m |u_0^\infty(\hat{x}, k)|^2 \, dk \cdot \frac{1}{K_j} \int_{K_j}^{2K_j} k^m |u_1^\infty(\hat{x}, k + \tau)|^2 \, dk. \]

(4.18)

Combining (4.18) with Theorem 4.2 and Lemma 3.1 we obtain

\[ |J_{1,2}|^2 \lesssim (\sigma^2(0) + o(1)) \cdot O(k^{m-4}) = o(1) \to 0, \quad j \to +\infty. \]

(4.19)

The analysis to \( J_{1,0} \) is similar to that of \( J_{0,1} \), so we skip the details.

Finally we analyze \( J_{1,1} \). By Lemma 3.1 we have

\[ |J_{1,1}|^2 \simeq |\mathcal{E}_k(k^m u_1^\infty(\hat{x}, k) u_1^\infty(\hat{x}, k + \tau))|^2 = \left| \frac{1}{K_j} \int_{K_j}^{2K_j} k^m u_1^\infty(\hat{x}, k) u_1^\infty(\hat{x}, k + \tau) \, dk \right|^2 \]

\[ \leq \frac{1}{K_j} \int_{K_j}^{2K_j} k^m |u_1^\infty(\hat{x}, k)|^2 \, dk \cdot \frac{1}{K_j} \int_{K_j}^{2K_j} k^m |u_1^\infty(\hat{x}, k + \tau)|^2 \, dk \]

\[ \leq \frac{1}{K_j} \int_{K_j}^{2K_j} k^m \sup_{\kappa \geq K_j} |u_1^\infty(\hat{x}, \kappa)|^2 \, dk \cdot \frac{1}{K_j} \int_{K_j}^{2K_j} k^m \sup_{\kappa \geq K_j + \tau} |u_1^\infty(\hat{x}, \kappa)|^2 \, dk \]

\[ = (2K_j)^m \sup_{\kappa \geq K_j} |u_1^\infty(\hat{x}, \kappa)|^2 \cdot \sup_{\kappa \geq K_j + \tau} |u_1^\infty(\hat{x}, \kappa)|^2 \to 0, \quad j \to +\infty. \]

(4.20)

Combining (4.17), (4.19) and (4.20), we can conclude (4.3). The proof is complete.

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