ON SPHERICALLY SYMMETRIC SOLUTIONS WITH HORIZON IN MODEL WITH MULTICOMPONENT ANISOTROPIC FLUID

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Abstract

A family of spherically symmetric solutions in the model with $m$-component multicomponent anisotropic fluid is considered. The metric of the solution depends on parameters $q_s > 0$, $s = 1, \ldots, m$, relating radial pressures and the densities and contains $(n-1)m$ parameters corresponding to Ricci-flat “internal space” metrics and obeying certain $m(m-1)/2$ (“orthogonality”) relations. For $q_s = 1$ (for all $s$) and certain equations of state ($p_s = \pm \rho^q$) the metric coincides with the metric of intersecting black brane solution in the model with antisymmetric forms. A family of solutions with (regular) horizon corresponding to natural numbers $q_s = 1, 2, \ldots$ is singled out. Certain examples of “generalized simulation” of intersecting $M$-branes in $D = 11$ supergravity are considered. The post-Newtonian parameters $\beta$ and $\gamma$ corresponding to the 4-dimensional section of the metric are calculated.

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1 Introduction

This paper is devoted to spherically-symmetric solutions with a horizon in the multidimensional model with multicomponent anisotropic fluid defined on product manifolds $\mathbb{R} \times M_0 \times \ldots \times M_n$. These solutions in certain cases may simulate black brane solutions [1, 2, 3] (for a review on $p$-brane solutions see [4] and references therein).

We remind that $p$-brane solutions (e.g. black brane ones) usually appear in the models with antisymmetric forms and scalar fields (see also [5]-[15]). Cosmological and spherically symmetric solutions with $p$-branes are usually obtained by the reduction of the field equations to the Lagrange equations corresponding to Toda-like systems [14]. An analogous reduction for the models with multicomponent "perfect" fluid was done earlier in [18, 19].

For cosmological models with antisymmetric forms without scalar fields any $p$-brane is equivalent to an multicomponent anisotropic perfect fluid with the equations of state:

\[ p_i = -\rho, \quad \text{or} \quad p_i = \rho, \quad (1.1) \]

when the manifold $M_i$ belongs or does not belong to the brane world-volume, respectively (here $p_i$ is the pressure in $M_i$ and $\rho$ is the density, see Section 2).

In this paper we find a new family of exact spherically-symmetric solutions in the model with $m$-component anisotropic fluid for the following equations of state (see Appendix for more familiar form of eqs. of state):

\[ p_s^r = -\rho^s(2q_s - 1)^{-1}, \quad p_s^0 = \rho^s(2q_s - 1)^{-1}, \quad (1.2) \]

and

\[ p_s^i = \left(1 - \frac{2U_s^i}{d_i}\right)\rho^s/(2q_s - 1), \quad (1.3) \]

$i > 1$, $s = 1, \ldots, m$, where for $s$-th component: $\rho^s$ is a density, $p_s^r$ is a radial pressure, $p_s^i$ is a pressure in $M_i$, $i = 2, \ldots, n$. Here parameters $U_s^i$ ($i > 1$) and the parameters $q_s = U_1^s > 0$ obey the following “orthogonality” relations (see also Section 2 below)

\[ B_{sl} = 0, \quad s \neq l \quad (1.4) \]

where

\[ B_{sl} \equiv \sum_{i=1}^{n} \frac{U_s^i U_l^i}{d_i} + \frac{1}{2 - D} \left(\sum_{i=1}^{n} U_s^i\right)\left(\sum_{j=1}^{n} U_l^j\right), \quad (1.5) \]
$q_s \neq 1/2$; and $s, l = 1, \ldots, m$. The manifold $M_0$ is $d_0$-dimensional sphere in our case and $p_0^s$ is the pressure in the tangent direction.

The one-component case was considered earlier in [1]. For special case with $q_s = 1$ see [2] and [3] (for one-component and multicomponent case, respectively).

The paper is organized as follows. In Section 2 the model with multicomponent (anisotropic or “perfect”) fluid is formulated. In Section 3 a subclass of spherically symmetric solutions (generalizing solutions from [3]) is presented and solutions with (regular) horizon corresponding to integer $q_s$ are singled out. Section 4 deals with certain examples of two-component solutions in dimension $D = 11$ containing for $q_s = 1$ intersecting $M_2 \cap M_2$, $M_2 \cap M_5$ and $M_5 \cap M_5$ black brane metrics. In Section 5 the post-Newtonian parameters for the 4-dimensional section of the metric are calculated. In the Appendix a class of general spherically symmetric solutions in the model under consideration is presented.

2 The model

Here, we consider a family of spherically symmetric solutions to Einstein equations with an multicomponent anisotropic fluid matter source

$$R^M_N - \frac{1}{2} \delta^M_N R = k T^M_N$$  \hspace{1cm} (2.1)$$

defined on the manifold

$$M = R_{\text{radial variable}} \times (M_0 = S^{d_0}) \times (M_1 = R) \times M_2 \times \ldots \times M_n,$$  \hspace{1cm} (2.2)$$

with the block-diagonal metrics

$$ds^2 = e^{2\gamma(u)} du^2 + \sum_{i=0}^{n} e^{2X^i(u)} h^{(i)}_{m_i n_i} dy^{m_i} dy^{n_i}.$$  \hspace{1cm} (2.3)$$

Here $R_i = (a, b)$ is interval. The manifold $M_i$ with the metric $h^{(i)}$, $i = 1, 2, \ldots, n$, is the Ricci-flat space of dimension $d_i$:

$$R_{m_i n_i} [h^{(i)}] = 0,$$  \hspace{1cm} (2.4)$$

and $h^{(0)}$ is standard metric on the unit sphere $S^{d_0}$.
\[ R_{\mu_0\alpha_0}[h^{(0)}] = (d_0 - 1)h_{\mu_0\alpha_0}^{(0)}, \quad (2.5) \]

\( u \) is radial variable, \( \kappa \) is the multidimensional gravitational constant, \( d_1 = 1 \) and \( h^{(1)} = -dt \otimes dt \).

The energy-momentum tensor is adopted in the following form

\[ T^M_N = \sum_{s=1}^{m} T^{(s)M}_N, \quad (2.6) \]

where

\[ T^{(s)M}_N = \text{diag}(-2q_s-1)^{-1}\rho^s, (2q_s-1)^{-1}\rho^s \delta_{k_0}^m, -\rho^s, p^s_2 \delta_{k_2}^m, \ldots, p^s_n \delta_{k_n}^m), \quad (2.7) \]

\( q_s > 0 \) and \( q_s \neq 1/2 \). The pressures \( p^s_i \) and the density \( \rho^s \) obeys the relations \( (1.3) \) with constants \( U_i^s, i > 1 \).

The “conservation law” equations

\[ \nabla_M T^{(s)M}_N = 0 \quad (2.8) \]

are assumed to be valid for all \( s \).

In what follows we put \( \kappa = 1 \) for simplicity.

3 Exact solutions

Let us define

1. \[ U_0^s = 0, \quad (3.1) \]
2. \[ U_1^s = q_s, \quad (3.2) \]
3. \[ (U^s, U^l) = U_i^s G^{ij} U_j^l \quad (3.3) \]

where \( U^s = (U_i^s) \) is \((n + 1)\)-dimensional vector and

\[ G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D} \quad (3.4) \]

are components of the matrix inverse to the matrix of the minisuperspace metric \[ (G_{ij}) = (d_i \delta_{ij} - d_id_j), \quad (3.5) \]
\(i, j = 0, \ldots, n\), and \(D = 1 + \sum_{i=0}^{n} d_i\) is the total dimension.

In our case the scalar products (3.3) are given by relations:

\[
(U^s, U^l) = B_{kl}
\]  

(3.6)

with \(B_{kl}\) from (1.5) and hence due to (1.4) vectors \(U^s\) are mutually orthogonal, i.e.

\[
(U^s, U^l) = 0, \quad s \neq l.
\]  

(3.7)

It is proved in Appendix that the relation 1° implies

\[
(U^s, U^s) > 0,
\]  

(3.8)

for all \(s\).

For the equations of state (1.2) and (1.3) with parameters obeying (1.4) we have obtained the following spherically symmetric solutions to the Einstein equations (2.1) (see Appendix)

\[
ds^2 = J_0 \left( \frac{dr^2}{1 - \frac{2\phi}{r^d}} + r^2 d\Omega^2_{d_0} \right) - J_1 \left( 1 - \frac{2\mu}{r^d} \right) dt^2 + \sum_{i=2}^{n} J_i h_{m_i,n_i} dy^{m_i} dy^{n_i},
\]

\[
\rho^s = \frac{(2q_s - 1)(dq_s)^2 P_s (P_s + 2\mu)(1 - 2\mu r^{-d}) q_s^{-1}}{2(U^s, U^s)(\prod_{s=1}^{m} H_s)^2 J_0 r^{2d_0}},
\]

(3.9)

(3.10)

by methods similar to obtaining \(p\)-brane solutions [14]. Here \(d = d_0 - 1\), \(d\Omega^2_{d_0} = h_{m_0,n_0} dy^{m_0} dy^{n_0}\) is spherical element, the metric factors

\[
J_i = \prod_{s=1}^{m} H_s^{-2U^{s_i}/(U^s, U^s)};
\]

(3.11)

\[
H_s = 1 + \frac{P_s}{2\mu} \left[ 1 - \left( 1 - \frac{2\mu}{r^d} \right)^{q_s^*} \right];
\]

(3.12)

\(P > 0, \mu > 0\) are constants and

\[
U^{s_i} = G^{s_i} U_j^s = \frac{U^s_j}{d_i} + \frac{1}{2 - D} \sum_{j=0}^{n} U_j^s;
\]

(3.13)
Using (3.13) and $U_s^0 = 0$ we get

$$U_s^0 = \frac{1}{2 - D} \sum_{j=0}^{n} U_j^s$$

(3.14)

and hence one can rewrite (3.9) as follows

$$ds^2 = J_0 \left[ \frac{dr^2}{1 - \frac{2\mu}{r^d}} + r^2 d\Omega_{d_b}^2 - \left( \prod_{s=1}^{m} H_s^{-2q_s/(U_s^*,U_s^*)} \right) \left( 1 - \frac{2\mu}{r^d} \right) dt^2 + \sum_{i=2}^{n} \left( \prod_{s=1}^{m} H_s^{-2q_s/(d_i(U_s^*,U_s^*))} \right) h^{(i)}_{m_i n_i} dy_{m_i} dy_{n_i} \right].$$

(3.15)

These solutions are special case of general solutions spherically symmetric solutions obtained in Appendix by method suggested in [19].

**Black holes for natural $q_s$.**

For natural

$$q_s = 1, 2, \ldots,$$

(3.16)

the metric has a horizon at $r^d = 2\mu = r_h^2$. Indeed, for these values of $q_s$ the functions $H_s(r) > 0$ are smooth in the interval $(r_s, +\infty)$ for some $r_s < r_h$.

For odd $q_s = 2m_s + 1$ (for all $s$) one get $r_s = 0$.

A global structure of the black hole solution corresponding to these values of $q_s$ will be a subject of a separate publication.

It was shown in [1] that in one-component case for $2U_s^0 \neq -1$ and $0 < q_s < 1$ one get singularity at $r^d \rightarrow 2\mu$.

**Remark.** For non-integer $q_s > 1$ the function $H_s(r)$ have a non-analytical behavior in the vicinity of $r^d = 2\mu$. In this case one may conject that the limit $r^d \rightarrow 2\mu$ corresponds to the singularity (in general case) but here a separate investigation is needed.

4 **Examples: generalized simulation of intersecting black branes**

The solutions with a horizon from the previous section allow us to simulate the intersecting black brane solutions [4] in the model with antisymmetric forms without scalar fields [2] when all $q_s = 1$. 

6
These solutions may be also generalized to the case of general natural
\( q_s \in \mathbb{N} \). In this case the parameters \( U_i^s \) and the pressures have the following form

\[
U_i^s = q_s d_i, \quad p_i^s = \begin{cases} 
-\rho^s, & i \in I_s; \\
(2q_s - 1)^{-1} \rho^s, & i \notin I_s.
\end{cases}
\]

(4.1)

Here \( I_s = \{i_1, \ldots, i_k\} \in \{1, \ldots, n\} \) is the index set corresponding to “brane” submanifold \( M_{i_1} \times \ldots \times M_{i_k} \).

The “orthogonality” relations lead us to the following dimension of intersection of brane submanifolds

\[
d(I_s \cap I_l) = d(I_s) d(I_l) / D - 2, \quad s \neq l,
\]

(4.2)

where \( d(I_s) = \sum_{i \in I_s} d_i \) is dimension of \( p \)-brane worldvolume.

**Remark.** The set of Diophantus equations was solved explicitly in [20] for so-called “flower” Ansatz from [21]. The solution in this case takes place for infinite number of dimensions \( D = 6, 10, 11, 14, 18, 20, 26, 27, \ldots \) etc.

As an example, here we consider a “generalized simulation” of intersecting \( M2 \cap M5, M2 \cap M2 \) and \( M5 \cap M5 \) black branes in \( D = 11 \) supergravity. In what follows functions \( H_s, s = 1, 2, \) are defined in (3.12).

\( a \). For an analog of intersecting \( M2 \cap M5 \) branes the metric reads:

\[
ds^2 = H_1^{1/(3q_1)} H_2^{2/(3q_2)} \left[ \frac{dr^2}{1 - 2\mu/r^d} + r^2 d\Omega_0^2 \right.
\]

\[
- H_1^{-1/q_1} H_2^{-1/q_2} \left\{ \left( 1 - \frac{2\mu}{r^d} \right) dt^2 + dy^{m_2} dy^{m_2} \right\}
\]

\[
+ H_2^{-1/q_2} h_{m_3n_3}^{(3)} dy^{m_3} dy^{n_3} + H_1^{-1/q_1} dy^{m_4} dy^{m_4} + h_{m_5n_5}^{(5)} dy^{m_5} dy^{n_5} \right] ,
\]

where \( M2 \)-brane includes three one-dimensional spaces: \( M_2, M_4 \) and the time manifold \( M_1 \); and \( M5 \)-brane includes \( M_1, M_2 \) and \( M_3 \) (\( d_3 = 4 \)).

\( b \). An analog of two electrical \( M2 \) branes intersecting on the time manifold has the following metric

\[
ds^2 = H_1^{1/(3q_1)} H_2^{1/(3q_2)} \left[ \frac{dr^2}{1 - 2\mu/r^d} + r^2 d\Omega_0^2 \right.
\]

\[
- H_1^{-1/q_1} H_2^{-1/q_2} \left( 1 - \frac{2\mu}{r^d} \right) dt^2
\]

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\[ + H_1^{-1/q_1} h_{m_2 n_2}^{(2)} dy^{m_2} dy^{n_2} + H_2^{-1/q_2} h_{m_3 n_3}^{(3)} dy^{m_3} dy^{n_3} + h_{m_4 n_4}^{(4)} dy^{m_4} dy^{n_4} \],

where \( d_2 = d_3 = 2 \).

c). For an analog of two intersecting M5 branes the dimension of intersection is 4 and the metric reads

\[ ds^2 = H_1^{2/(3q_1)} H_2^{2/(3q_2)} \left[ \frac{dr^2}{1 - 2\mu/r} + r^2 d\Omega_2^2 \right] \]

(4.5)

\[ - H_1^{-1/q_1} H_2^{-1/q_2} \left\{ \left( 1 - \frac{2\mu}{r} \right) dt^2 + h_{m_2 n_2}^{(2)} dy^{m_2} dy^{n_2} \right\} \]

\[ + H_1^{-1/q_1} h_{m_3 n_3}^{(3)} dy^{m_3} dy^{n_3} + H_2^{-1/q_2} h_{m_4 n_4}^{(4)} dy^{m_4} dy^{n_4} \].

Here \( d_0 = d_3 = d_4 = 2 \) and \( d_2 = 3 \).

For the density of \( s \)-th component we get in any of these three cases

\[ \rho_s = \frac{(2q_s - 1)d^2 P_s (P_s + 2\mu)(1 - 2\mu r^{-d})^{q_s - 1}}{4(H_1 H_2)^2 J_0 r^{2d_0}} \]

(4.6)

where

\[ J_0 = \prod_{s=1}^{2} H_s^{d(I_s)/(9q_s)} \]

(4.7)

and \( d(I_s) = 3, 6 \) for M2, M5 branes, respectively.

5 Physical parameters

5.1 Gravitational mass and PPN parameters

Here we put \( d_0 = 2 \) (\( d = 1 \)). Let us consider the 4-dimensional space-time section of the metric \((3.15)\). Introducing a new radial variable by the relation:

\[ r = R \left( 1 + \frac{\mu}{2R} \right)^2 \]

(5.1)

we rewrite the 4-section in the following form:

\[ ds^2_{(4)} = \left( \prod_{s=1}^{m} H_s^{-2U_s^0/(U_s^0 U_s)} \right) \left[ - \left( \prod_{s=1}^{m} H_s^{-2q_s/(U_s^0 U_s)} \right) \left( 1 - \frac{\mu}{2R} \right)^2 dt^2 \right] \]

\[ + \left( 1 + \frac{\mu}{2R} \right)^4 \delta_{ij} dx^i dx^j \]

(5.2)
$i, j = 1, 2, 3$. Here $R^2 = \delta_{ij}x^i x^j$.

The parameterized post-Newtonian (Eddington) parameters are defined by the well-known relations

$$g^{(4)}_{00} = -(1 - 2V + 2\beta V^2) + O(V^3), \quad (5.3)$$

$$g^{(4)}_{ij} = \delta_{ij} (1 + 2\gamma V) + O(V^2), \quad (5.4)$$

$i, j = 1, 2, 3$. Here

$$V = \frac{GM}{R} \quad (5.5)$$

is the Newtonian potential, $M$ is the gravitational mass and $G$ is the gravitational constant.

From (5.2)-(5.4) we obtain:

$$GM = \mu + \sum_{s=1}^{m} \frac{P_s q_s (q_s + U_s^0)}{(U_s^*, U_s)} \quad (5.6)$$

and

$$\beta - 1 = \sum_{s=1}^{m} \frac{|A_s|}{(GM)^2} (q_s + U_s^0), \quad (5.7)$$

$$\gamma - 1 = -\sum_{s=1}^{m} \frac{P_s q_s}{(U_s^*, U_s) GM} (q_s + 2U_s^0), \quad (5.8)$$

where

$$|A_s| = \frac{1}{2} q_s^2 P_s (P_s + 2\mu)/(U_s^*, U_s) \quad (5.9)$$

(see Appendix) or, equivalently,

$$P_s = -\mu + \sqrt{\mu^2 + 2|A_s|(U_s^*, U_s) q_s^2} > 0. \quad (5.10)$$

For fixed $U_s^*$ the parameter $\beta - 1$ is proportional to the ratio of two quantities: the weighted sum of multicomponent anisotropic fluid density parameters $|A_s|$ and the gravitational radius squared $(GM)^2$.  

9
5.2 Hawking temperature

The Hawking temperature of the black hole may be calculated using the well-known relation [22]

\[ T_H = \frac{1}{4\pi\sqrt{-g_{tt}g_{rr}}} \frac{d(-g_{tt})}{dr} \bigg|_{\text{horizon}}. \]  

Equation (5.11)

We get

\[ T_H = \frac{d}{4\pi(2\mu)^{1/d}} \prod_{s=1}^{m} \left( 1 + \frac{P_s}{2\mu} \right)^{-q_s/(U^s, U^s)}. \]  

Equation (5.12)

Here all \( q_s \) are natural numbers.

For any of \( D = 11 \) metrics from Section 4 the Hawking temperature reads

\[ T_H = \frac{d}{4\pi(2\mu)^{1/d}} \prod_{s=1}^{2} \left( 1 + \frac{P_s}{2\mu} \right)^{-1/(2q_s)}. \]

6 Conclusions

In this paper, using the methods developed earlier for obtaining perfect fluid and p-brane solutions, we have considered a family of spherically symmetric solutions in the model with \( m \)-component anisotropic fluid when the equations of state (1.2)–(1.4) are imposed. The metric of any solution contains \((n-1)\) Ricci-flat “internal” space metrics and depends upon a set of parameters \( U^s, i > 1. \)

For \( q_s = 1 \) (for all \( s \)) and certain equations of state (with \( p_s = \pm \rho^s \)) the metric of the solution coincides with that of intersecting black brane solution in the model with antisymmetric forms without dilatons [3]. For natural numbers \( q_s = 1, 2, \ldots \) we have obtained a family of solutions with regular horizon.

Here we have considered three examples of solutions with horizon, that simulate (by fluids) binary intersecting \( M2 \) and \( M5 \) black branes in \( D = 11 \) supergravity.

We have also calculated (for possible estimations of observable effects of extra dimensions) the post-Newtonian parameters \( \beta \) and \( \gamma \) corresponding to the 4-dimensional section of the metric and the Hawking temperature as well.
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Appendix

A Lagrange representation

It is more convenient for finding of exact solutions, to write the stress-energy tensor in cosmological-type form

\[
(T^{(s)M}_N) = \text{diag}(-\hat{\rho}^s, \hat{p}_0^s \delta_{k_0}^m, \hat{p}_1^s \delta_{k_1}^m, \ldots, \hat{p}_n^s \delta_{k_n}^m),
\]

(A.1)

where \(\hat{\rho}^s\) and \(\hat{p}_i^s\) are "effective" density and pressures of \(s\)-th component, respectively, depending upon the radial variable \(u\) and the physical density \(\rho^s\) and pressures \(p_i^s\) are related to the effective ("hat") ones by formulas

\[
\rho^s = -\hat{\rho}_1^s, \quad p_r^s = -\hat{\rho}^s, \quad p_i^s = \hat{p}_i^s, \quad (i \neq 1),
\]

(A.2)

\(s = 1, \ldots, m\).

The equations of state may be written in the following form

\[
\dot{\hat{\rho}}_i = \left(1 - \frac{2U_i^s}{d_i}\right)\hat{\rho}^s,
\]

(A.3)

where \(U_i^s\) are constants, \(i = 0, 1, \ldots, n\). It follows from (A.2), (A.3) and \(U_1^s = q_s\) that

\[
\rho^s = (2q_s - 1)\hat{\rho}^s.
\]

(A.4)

The "conservation law" equations \(\nabla_M T^{(s)M}_N = 0\) may be written, due to relations (2.3) and (A.1) in the following form:

\[
\dot{\hat{\rho}}^s + \sum_{i=0}^{n} d_i X^i (\hat{\rho}^s + \hat{p}_i^s) = 0.
\]

(A.5)

Using the equation of state (A.3) we get

\[
\hat{\rho}^s = -A_s e^{2U_i^s X^i - 2\gamma_0},
\]

(A.6)
where $\gamma_0(X) = \sum_{i=0}^{n} d_i X^i$ and $A_s$ are constants.

The Einstein equations (2.1) with the relations (A.3) and (A.6) imposed are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} e^{-\gamma + \gamma_0(X)} G_{ij} \dot{X}^i \dot{X}^j - e^{\gamma - \gamma_0(X)} V,$$

(A.7)

where

$$V = \frac{1}{2} d_0 (d_0 - 1) \exp(2 U_0^0 X^i) + A_s \exp(2 U_s^s X^i)$$

(A.8)

is the potential and the components of the minisupermetric $G_{ij}$ are defined in (3.5),

$$U_0^0 X^i = -X^0 + \gamma_0(X), \quad U_s^0 = -\delta^0_i + d_i,$$

(A.9)

$i = 0, \ldots, n$ (for cosmological case see [13, 19]).

For $\gamma = \gamma_0(X)$, i.e. when the harmonic time gauge is considered, we get the set of Lagrange equations for the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{X}^i \dot{X}^j - V,$$

(A.10)

with the zero-energy constraint imposed

$$E = \frac{1}{2} G_{ij} \dot{X}^i \dot{X}^j + V = 0.$$

(A.11)

It follows from the restriction $U_0^0 = 0$ that

$$(U_0^0, U_s^s) \equiv U_0^0 G^{ij} U_j^s = 0.$$  

(A.12)

Indeed, the contravariant components $U_{0i}^0 = G^{ij} U_j^0$ are the following ones

$$U_{0i}^0 = -\frac{\delta^0_i}{d_0}.$$  

(A.13)

Then we get $(U_0^0, U_s^s) = U_{0i}^0 U_i^s = -U_0^0 / d_0 = 0$. In what follows we also use the formula

$$(U_0^0, U_0^0) = \frac{1}{d_0} - 1 < 0$$

(A.14)

for $d_0 > 1$.

Now we prove that $(U_s^s, U_s^s) > 0$ for all $s > 0$. Indeed, minisupermetric has the signature $(-, +, \ldots, +)$ [16, 17], vector $U_0^0$ is time-like and orthogonal to any vector $U_s^s \neq 0$. Hence any vector $U_s^s$ is space-like.
B General spherically symmetric solutions

When the orthogonality relations (A.12) and (3.7) are satisfied the Euler-Lagrange equations for the Lagrangian (A.10) with the potential (A.8) have the following solutions (see relations from [19] adopted for our case):

\[ X^i(u) = -\sum_{\alpha=0}^{m} \frac{U^\alpha_i}{(U^\alpha, U^\alpha)} \ln |f_\alpha(u - u_\alpha)| + c^i u + \bar{c}^i, \quad (B.1) \]

where \( u_\alpha \) are integration constants; and vectors \( c = (c^i) \) and \( \bar{c} = (\bar{c}^i) \) are dually-orthogonal to co-vectors \( U^\alpha = (U^\alpha_i) \), i.e. they satisfy the linear constraint relations

\[ U^0(c) = U^0_i c^i = -c^0 + \sum_{j=0}^{n} d_j c^j = 0, \quad (B.2) \]
\[ U^0(\bar{c}) = U^0_i \bar{c}^i = -\bar{c}^0 + \sum_{j=0}^{n} d_j \bar{c}^j = 0, \quad (B.3) \]
\[ U^s(c) = U^s_i c^i = 0, \quad (B.4) \]
\[ U^s(\bar{c}) = U^s_i \bar{c}^i = 0. \quad (B.5) \]

Here

\[ f_\alpha(\tau) = \begin{cases} R_\alpha \frac{\sinh(\sqrt{C_\alpha} \tau)}{\sqrt{C_\alpha}}, & C_\alpha > 0, \quad \eta_\alpha = +1, \\ R_\alpha \frac{\cosh(\sqrt{C_\alpha} \tau)}{\sqrt{C_\alpha}}, & C_\alpha > 0, \quad \eta_\alpha = -1, \\ R_\alpha \frac{\sin(\sqrt{|C_\alpha|} \tau)}{\sqrt{|C_\alpha|}}, & C_\alpha < 0, \quad \eta_\alpha = +1, \\ R_\alpha \tau, & C_\alpha = 0, \quad \eta_\alpha = +1, \end{cases} \quad (B.6) \]

\( \alpha = 0, \ldots, m; \) where \( R_0 = d_0 - 1, \eta_0 = 1, R_s = \sqrt{2|A_s|(U^s, U^s)}, \eta_s = -\text{sign}A_s \) (\( s = 1, \ldots, m \)).

The zero-energy constraint, corresponding to the solution (B.1) reads

\[ E = \frac{1}{2} \sum_{\alpha=0}^{m} \frac{C_\alpha}{(U^\alpha, U^\alpha)} + \frac{1}{2} G_{ij} c^i c^j = 0. \quad (B.7) \]
Special solutions. The (weak) horizon condition (i.e. infinite time of propagation of light for \( u \to +\infty \)) lead us to the following integration constants

\[
\bar{c}^i = 0, \quad (B.8)
\]

\[
c^i = \mu \sum_{\alpha=0}^{m} \frac{U_1^\alpha U^\alpha_i}{(U^\alpha, U^\alpha)} - \bar{\mu} \delta^i_1, \quad (B.9)
\]

\[
C_\alpha = (U_1^\alpha)^2 \bar{\mu}^2, \quad (B.10)
\]

where \( \bar{\mu} > 0, \alpha = 0, \ldots, m \). For analogous choice of parameters in \( p \)-brane case see [13, 14, 4].

We also introduce a new radial variable \( r = r(u) \) by relations

\[
\exp(-2\bar{\mu}u) = 1 - \frac{2\mu}{r^d}, \quad \mu = \bar{\mu}/d > 0, \quad d = d_0 - 1, \quad (B.11)
\]

and put \( u_s < 0 \) and \( A_s < 0 \) for all \( s \) and also \( u_0 = 0 \).

The relations of the Appendix imply the formulae (B.9) and (B.10) for the solution from Section 3 with

\[
H_s = \exp(-\bar{\mu}q_s u) f_s(u - u_s), \quad A_s = -\frac{(dq_s)^2}{2(U^s, U^s)} P_s (P_s + 2\mu), \quad (B.12)
\]

where \( P_s > 0 \).
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