Inverse spectral problem for a third-order differential operator with non-local potential

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Abstract. Spectral problem for a self-adjoint third-order differential operator with non-local potential on a finite interval is studied. Elementary functions that are analogues of sines and cosines for such operators are described. Direct and inverse problems for third-order operators with non-local potential are solved.

Key words: third-order differential operator, non-local potential, inverse problem.

Introduction

Direct and inverse problems for second-order differential operators (Sturm – Liouville) are well studied and play an important role in mathematical physics and in theory of nonlinear equations [1, 2]. As a rule, these studies are based on transformation operators [1, 2]. Absence of transformation operators for differential operators higher than the second-order does not allow generalizing the formalism of the Sturm – Liouville operators to this class of operators [3].

Construction of the Lax L-A pair for nonlinear Camassa-Holm and Degasperis-Procesi equations leads to a third-order differential operator which is interpreted as an operator describing vibrations of a cubic string [4]. Therefore, the statement of inverse problems is natural for such operators.

This work deals with direct and inverse spectral problems for third-order self-adjoint operators of the form

\[(L_\alpha y)(x) = iy'''(x) + \alpha \int_0^l y(t)\overline{v(t)}dtv(x)\]

where \(\alpha \in \mathbb{R}\), \(v \in L^2(0, l)\) \((0 < l < \infty)\), domain of which is formed by functions \(y \in W^3_2(0, l)\) such that \(y(0) = 0\), \(y'(0) = y'(l)\), \(y(l) = 0\). Non-local potentials play an important role in mathematical physics [5]. Works [6, 7, 8] study second-order differential operators with non-local potentials.

The paper consists of 4 sections. First section deals with spectral analysis of the self-adjoint operator \(L_0\) of the third derivative. New class of functions (similar to sines and cosines) is introduced and its properties are described. In terms of these functions, eigenfunctions of the operator \(L_0\) are described and characteristic function is obtained. Spectrum of the operator \(L_0\) is described.
Section 2 studies operator $L_\alpha$ and constructs its eigenfunctions and calculates its characteristic function. Section 3 describes spectral characteristics of the operator $L_\alpha$. Section 4 presents solution to the inverse problem and specifies technique of recovery of one-dimensional perturbation by spectral data of the operator $L_\alpha$.

1 Characteristic function and eigenfunctions of the operator $L_0$

1.1. Denote by $L_0$ the self-adjoint differential operator

$$
(L_0 y)(x) \overset{\text{def}}{=} i D^3 y(x) \quad \left( D = \frac{d}{dx} \right)
$$

(1.1)

acting in the space $L^2(0, l)$ ($0 < l < \infty$). Domain $\mathcal{D}(L_0)$ of the operator $L_0$ is formed by the functions $y \in W^2_2(0, l)$ that satisfy boundary conditions

$$
y(0) = 0, \quad y'(0) = y'(l), \quad y(l) = 0.
$$

(1.2)

Let $\{\zeta_k\}_{1}^{3}$ be roots of the equation $\zeta^3 = 1$,

$$
\zeta_1 = 1, \quad \zeta_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \zeta_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2},
$$

(1.3)

then

$$
\zeta_1 + \zeta_2 + \zeta_3 = 0, \quad \zeta_1 \zeta_2 \zeta_3 = 1, \quad \zeta_1 - \zeta_2 = i\sqrt{3} \zeta_3, \quad \zeta_2 - \zeta_3 = i\sqrt{3} \zeta_1, \quad \zeta_3 - \zeta_1 = i\sqrt{3} \zeta_2.
$$

(1.4)

The equation

$$
i D^3 y(x) = \lambda^3 y(x) \quad (\lambda \in \mathbb{C})
$$

(1.5)

has three linearly independent solutions $\{e^{i \lambda \zeta_k x}\}_{1}^{3}$ and each solution to equation (1.5) is their linear combination. In particular, solution to the Cauchy problem

$$
i D^3 y(x) = \lambda^3 y(x) + f(x); \quad y(0) = y_0, \quad y'(0) = y_1, \quad y''(0) = y_2
$$

(1.6)

for $f(x) \equiv 0$ is

$$
y_0(\lambda, x) = \frac{1}{3} \sum_k \left( y_0 + \frac{y_1}{i \lambda \zeta_k} + \frac{y_2}{(i \lambda \zeta_k)^2} \right) e^{i \lambda \zeta_k x}.
$$

(1.7)

Hence, by the method of variation of constants, one finds solution to the Cauchy problem (1.6) when $f(x) \neq 0$,

$$
y(\lambda, x) = \frac{1}{3} \sum_k \left( y_0 + \frac{y_1}{i \lambda \zeta_k} + \frac{y_2}{(i \lambda \zeta_k)^2} \right) e^{i \lambda \zeta_k x} - i \int_0^x \frac{1}{3} \sum_k e^{i \lambda \zeta_k (x-t)} f(t) dt.
$$

(1.8)
Integral term in (1.8) is a solution to equation (1.6) and satisfies zero initial data $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ due to (1.4).

Instead of the exponents $\{e^{i\lambda_k x}\}_{k=1}^3$, one can take another system of fundamental solutions to equation (1.5), for example, $c(i\lambda x)$, $s(i\lambda x)$, $d(i\lambda x)$, where

$$c(z) = \frac{1}{3} \sum_k e^{z\zeta_k}; \quad s(z) = \frac{1}{3} \sum_k \frac{1}{\zeta_k} e^{z\zeta_k}; \quad d(z) = \frac{1}{3} \sum_k \frac{1}{\zeta_k^2} e^{z\zeta_k} \quad (1.9)$$

($z \in \mathbb{C}$). Functions (1.9) are similar to cosines and sines for a second-order equation.

**Lemma 1.1** Entire functions of exponential type (1.9) satisfy the relations:

(i) $s'(z) = c(z)$, $d'(z) = s(z)$, $c'(z) = d(z)$;

(ii) $c(z) = c(\overline{z})$, $s(z) = s(\overline{z})$, $d(z) = d(\overline{z})$;

(iii) $c(z\zeta_2) = c(z)$, $s(z\zeta_2) = \zeta_2 s(z)$, $d(z\zeta_2) = \zeta_2^2 d(z)$;

(iv) Euler’s formula

$$e^{z\zeta_p} = c(z) + \zeta_p s(z) + \zeta_p^2 d(z) \quad (1 \leq p \leq 3);$$

(v) the functions (1.9) are solutions to the equation $D^3y = y$ and satisfy the initial data at zero:

$$c(0) = 1, \quad c'(0) = 0, \quad c''(0) = 0; \quad (1.10)$$

$$s(0) = 0, \quad s'(0) = 1, \quad s''(0) = 0;$$

$$d(0) = 0, \quad d'(0) = 0, \quad d''(0) = 1;$$

(vi) the main identity

$$c^3(z) + s^3(z) + d^3(z) - 3c(z)s(z)d(z) = 1;$$

(vii) the summation formulas

$$c(z + w) = c(z)c(w) + s(z)d(w) + d(z)s(w);$$

$$s(z + w) = c(z)s(w) + s(z)c(w) + d(z)d(w);$$

$$d(z + w) = c(z)d(w) + s(z)s(w) + d(z)c(w),$$

(viii)

$$3c^2(z) = c(2z) + 2c(-z);$$

$$3s^2(z) = d(2z) + 2d(-z),$$

$$3d^2(z) = s(2z) + 2s(-z),$$

(ix)

$$s^2(z) - d(z)c(z) = d(-z);$$
\[ d^2(z) - s(z)c(z) = s(-z); \]
\[ c^2(z) - s(z)d(z) = c(-z); \]

(x) Taylor’s formulas

\[ c(z) = 1 + \frac{z^3}{3!} + \frac{z^6}{6!} + ..., \]
\[ s(z) = z + \frac{z^4}{4!} + \frac{z^7}{7!} + ..., \]
\[ d(z) = \frac{z^2}{2!} + \frac{z^5}{5!} + \frac{z^8}{8!} + .... \]

Proof of the lemma follows from the formulas (1.4).

The roots \( \{\zeta_p\}_1 \) divide the plane \( \mathbb{C} \) into three sectors:

\[ S_p = \left\{ z \in \mathbb{C} : \frac{2\pi}{3} (p - 1) < \arg z < \frac{2\pi p}{3} \right\} \quad (1 \leq p \leq 3). \]  

(1.11)

Relations (iii) (1.10) imply that it is sufficient to know functions (1.9) in one of the sectors \( S_p \).

Lemma 1.2 Zeros of the functions \( c(z), s(z), d(z) \) (1.9) lie on the rays formed by bisectors of the sectors \( S_p \) (1.11) and they correspondingly are given by

\[ \{-\zeta_2^l x_c(k)\}_{k=1}^{\infty}; \quad \{-\zeta_2^l x_s(k)\}_{k=1}^{\infty}; \quad \{-\zeta_2^l x_d(k)\}_{k=1}^{\infty} \]

where \( l = -1, 0, 1 \) and \( x_c(k), x_s(k), x_d(k) \) are nonnegative numbers in ascending order.

Besides, \( x_c(k) \) are simple positive roots of the equation

\[ \cos \frac{\sqrt{3}}{2} x = -\frac{1}{2} e^{-\frac{3}{2} x} \quad (x_c(1) > 0) \]  

(1.12)

and the numbers \( x_s(k) \) and \( x_d(k) \) are nonnegative simple roots of the equations

\[ \sin \left( \frac{\sqrt{3}}{2} x + \frac{\pi}{6} \right) = \frac{1}{2} e^{-\frac{3}{2} x} (x_s(1) = 0); \quad \sin \left( \frac{\sqrt{3}}{2} x - \frac{\pi}{6} \right) = -\frac{1}{2} e^{-\frac{3}{2} x} (x_d(1) = 0). \]  

(1.13)

The sequence \( x_d(k) \) is interlacing with the sequence \( x_s(k) \), which, in its turn, is interlacing with the sequence \( x_c(k) \).

Asymptotic behavior of \( x_c(k), x_s(k), x_d(k) \) as \( k \to \infty \) can be obtained from the equations (1.12), (1.13).
Formulas (1.7), (1.8) in terms of the functions (1.9) are given by
\[ y_0(\lambda, x) = y_0c(i\lambda x) + y_1 \frac{s(i\lambda x)}{i\lambda} + y_2 \frac{d(i\lambda x)}{(i\lambda)^2}; \]  
(1.14)
\[ y(\lambda, x) = y_0c(i\lambda x) + y_1 \frac{s(i\lambda x)}{i\lambda} + y_2 \frac{d(i\lambda x)}{(i\lambda)^2} + i \int_0^x \frac{d(i\lambda(x - t))}{\lambda^2} f(t) dt. \]  
(1.15)

1.2. Find eigenfunctions of the operator \( L_0 \) (1.1), (1.2). Relation (1.14) implies that the function
\[ y_0(\lambda, x) = y_1 \frac{s(i\lambda x)}{i\lambda} + y_2 \frac{d(i\lambda x)}{(i\lambda)^2} \]  
(1.16)is the solution to equation (1.5) and satisfies the first boundary condition \( y(0) = 0 \). Since
\[ y'_0(\lambda, x) = y_1c(i\lambda x) + y_2 \frac{s(i\lambda x)}{(i\lambda)}, \]  
due to (i) (1.10), then the boundary condition \( y'(0) = y'(l) \) gives
\[ y_1 = y_1c(i\lambda l) + y_2 \frac{s(i\lambda l)}{i\lambda}. \]  
(1.17)

**Remark 1.1** If \( \lambda = 0 \), then (1.17) implies that \( y_2 = 0 \), thus, \( y_0(0, x) = y_1x \), using the third boundary condition (1.2), one obtains that \( y_1 = 0 \), i.e., \( y_0(0, x) = 0 \) and thus the eigenfunction \( y_0(0, x) \) is trivial.

Assuming that \( \lambda \neq 0 \), from (1.17) one finds that
\[ y_1 = \frac{y_2}{i\lambda} \cdot \frac{S(i\lambda l)}{1 - c(i\lambda l)} \]
and thus function (1.16) is
\[ y_0(\lambda, x) = -\frac{y_2}{\lambda^2(1 - c(i\lambda l))} \{ s(i\lambda x)s(i\lambda l) + d(i\lambda x)(1 - c(i\lambda l)) \}. \]  
(1.18)
The third boundary condition \( y(l) = 0 \) implies
\[ \frac{s^2(i\lambda l) + d(i\lambda l)(1 - c(i\lambda l))}{\lambda^2(1 - c(i\lambda l))} = 0 \quad (\lambda \neq 0). \]  
(1.19)

**Remark 1.2** Numerator and denominator of fraction (1.19), as \( \lambda \neq 0 \), do not vanish simultaneously. If this is not the case, then \( c(i\lambda l) = 1 \) and \( s(i\lambda l) = 0 \) and thus \( d(i\lambda l) = 0 \), due to (vi) (1.10). This and (iv) (1.10) imply that \( e^{i\lambda l}\zeta_k = 0 \) \((k = 1, 2, 3)\), which is impossible.
The function
\[ 
\Delta(0, \lambda) \equiv \{ s^2(i\lambda l) + d(i\lambda l)(1 - c(i\lambda l)) \} \tag{1.20} 
\]
is said to be the characteristic function of the operator \( L_0 \) \( (1.1), (1.2) \).

**Remark 1.3** Equality \( (1.17) \) and boundary condition \( y(l) = 0 \) for \( y_0(\lambda, x) \) \( (1.16) \) lead to the system of linear equations
\[
\begin{align*}
\begin{cases}
y_1(c(i\lambda l) - 1) + y_2 \frac{s(i\lambda l)}{i\lambda} & = 0; \\
y_1 \frac{s(i\lambda l)}{i\lambda} - y_2 \frac{d(i\lambda l)}{\lambda^2} & = 0
\end{cases}
\end{align*}
\tag{1.21}
\]
relative to \( y_1, y_2 \). Solution to system \( (1.21) \) is non-trivial if its determinant \( D(\lambda) = \Delta(0, \lambda) \) \( (1.20) \) vanishes, \( D(\lambda) = 0 \).

Using (ix) \( (1.10) \), one writes characteristic function \( \Delta(0, \lambda) \) \( (1.20) \) as
\[ 
\Delta(0, \lambda) = \frac{1}{\lambda^2} (d(i\lambda l) + d(-i\lambda l)) = \frac{2}{3\lambda^2} \sum_k \zeta_k \cos \lambda \zeta_k l. \tag{1.22} 
\]

**Lemma 1.3** The function \( \Delta(0, \lambda) \) \( (1.23) \) is an even entire function of exponential type and
\[ 
\overline{\Delta(0, \lambda)} = \Delta(0, \lambda); \quad \Delta(0, \zeta_2 \lambda) = \Delta(0, \lambda). \tag{1.23} 
\]
Zeros of \( \Delta(0, \lambda) \) are given by \( \{ \pm \zeta_k^0 \lambda_k(0) \}_{k=1}^\infty \) \( l = 0, 1, 2 \) where \( \{ \lambda_k(0) \}_{k=1}^\infty \) are positive simple zeros of \( \Delta(0, \lambda) \) on \( \mathbb{R}_+ \), \( 0 < \lambda_1(0) < \lambda_2(0) < ... \), each zero \( \lambda_k(0) \) lies in the interval \( \left( \frac{\pi}{l}(2k - 1), \frac{\pi}{l}(2k + 1) \right) \) \( k = 1, 2, ... \) and the asymptotic
\[ 
\lambda_k(0) = \frac{2\pi k}{l} - \frac{\pi}{3l} + \delta_k \quad (\delta = o \left( \frac{1}{k} \right), k \to \infty) \tag{1.24} 
\]
is true.

**Proof.** Due to evenness and symmetry \( \Delta(0, \zeta_2 \lambda) = \Delta(0, \lambda) \) \( (1.24) \), it is sufficient to find zeros of \( \Delta(0, \lambda) \) that are situated on \( \mathbb{R}_+ \). Since \( \Delta(0, 0) = -2l^2 \), then \( \lambda = 0 \) is not a zero of the function \( \Delta(0, \lambda) \) \( (1.22) \). Therefore the equation \( \Delta(0, \lambda) = 0 \) is equivalent to the equality
\[
\sum_k \zeta_k \cos \lambda \zeta_k l = 0
\]
or, taking into account \( (1.3) \),
\[
\cos \lambda l - \cos \frac{\lambda}{2} \frac{\lambda \sqrt{3l}}{2} \frac{3l}{2} \frac{\lambda \sqrt{3l}}{2} - \sqrt{3} \sin \frac{\lambda}{2} \frac{3l}{2} \frac{\lambda \sqrt{3l}}{2} \frac{3l}{2} = 0,
\]
\[
\cos \frac{\lambda l}{2} \left( \cos \frac{\lambda l}{2} - \text{ch} \frac{\lambda \sqrt{3}l}{2} \right) = \sin \frac{\lambda l}{2} \left( \sin \frac{\lambda l}{2} + \sqrt{3} \text{sh} \frac{\lambda \sqrt{3}l}{2} \right).
\] (1.25)

It is necessary to find positive zeros \( \lambda (>0) \) of this equation. Since \( \sin \frac{\lambda l}{2} + \sqrt{3} \text{sh} \frac{\lambda \sqrt{3}l}{2} > 0 \), for \( \lambda > 0 \), then equation (1.25) is equivalent to the equality

\[
\tan \frac{\lambda l}{2} = f(\lambda) \quad (\lambda > 0),
\]

where

\[
f(\lambda) = \frac{\cos \frac{\lambda l}{2} \text{ch} \frac{\lambda \sqrt{3}l}{2}}{\sin \frac{\lambda l}{2} + \sqrt{3} \text{sh} \frac{\lambda \sqrt{3}l}{2}}.
\]

\( f(\lambda) < 0 \), as \( \lambda > 0 \), and \( f(0) = 0 \). Using

\[
f'(\lambda) = \frac{1 - \sqrt{3} \sin \frac{\lambda l}{2} \text{sh} \frac{\lambda \sqrt{3}l}{2} - \cos \frac{\lambda l}{2} \text{ch} \frac{\lambda \sqrt{3}l}{2}}{\left( \sin \frac{\lambda l}{2} + \sqrt{3} \text{sh} \frac{\lambda \sqrt{3}l}{2} \right)^2} < 0
\]

and obvious inequalities \( \sin x < \text{sh} \sqrt{3}x, \cos x < \text{ch} \sqrt{3}x \) that are true for all \( x > 0 \), one obtains that

\[
f'(\lambda) < \frac{1 - \sqrt{3} \sin^2 \frac{\lambda l}{2} - \cos^2 \frac{\lambda l}{2}}{\left( \sin \frac{\lambda l}{2} + \sqrt{3} \text{sh} \frac{\lambda \sqrt{3}l}{2} \right)^2} < 0.
\]

Consequently, the function \( f(\lambda) \) decreases on \( \mathbb{R}_+ \) and \( f(\lambda) \rightarrow -\frac{1}{\sqrt{3}} \), as \( \lambda \rightarrow \infty \), therefore \( f(\lambda) \) monotonously decreases starting from \( f(0) = 0 \) and up to the horizontal asymptote \( y \rightarrow -\frac{1}{\sqrt{3}} \). Hence it follows that in each interval \( \left( \frac{\pi}{l}(2k - 1), \frac{\pi}{l}(2k + 1) \right) \) \((k \in \mathbb{N})\) there lies exactly one root \( \lambda_k(0) \) of the equation \( \tan \frac{\lambda l}{2} = f(\lambda) \) \((\lambda \in \mathbb{R}_+)\) and

\[
\lambda_k(0) = \frac{2\pi k}{l} - \varepsilon_k \quad (0 < \varepsilon_k < \frac{\pi}{l}).
\] (1.26)
Numbers \( \lambda_k(0) \) are enumerated in accordance with the intervals \( \left( \frac{\pi}{l}(2k - 1), \frac{\pi}{l}(2k + 1) \right) \) \((k \in \mathbb{N})\) and are in ascending order.

Asymptotic (1.24) follows from \( f(\lambda) \to -\frac{1}{\sqrt{3}} (\lambda \to \infty) \).

Function \( \Delta(0, \lambda) \) has no complex zeros lying inside sector \( S_1 \) (as in \( S_2, S_3 \)). In other case, if \( \Delta(0, \mu) = 0 \) and \( \mu \in S_1 \), then \( y(\mu, x) \) (1.18) is an eigenfunction of the operator \( L_0 \) corresponding to the eigenvalue \( \mu^3 \) which is impossible in view of self-adjointness of \( L_0 \) since \( \mu^3 \) is a complex number.

**Theorem 1.1** Spectrum \( \sigma(L_0) \) of the operator \( L_0 \) (1.1), (1.2) is simple and

\[
\sigma(L_0) = \{ \pm \lambda_k^3(0) : \lambda_k(0) > 0, k \in \mathbb{N} \} 
\]

(1.27)

where \( \lambda_k(0) \) are positive zeros of characteristic function \( \Delta(0, \lambda) \) (Lemma 1.3). Eigenfunctions corresponding to \( \lambda_k(0) \) are

\[
u(0, \lambda_k(0), x) = \frac{1}{u_k} \{ s(i\lambda_k(0)x) s(i\lambda_k(0)l) + d(i\lambda_k(0)x)[1 - c(i\lambda_k(0)l)] \}
\]

(1.28)

where \( u_k = \|u(0, \lambda_k(0), x)\|_{L^2} \).

1.2. Calculate the resolvent \( R_{L_0}(\lambda^3) = (L_0 - \lambda^3I)^{-1} \) of the operator \( L_0 \) (1.1), (1.2) and let \( R_{L_0}(\lambda^3)f = y \), then \( L_0y = \lambda^3y + f \) and according to (1.15) the function

\[
y(\lambda, x) = y_1 \frac{s(i\lambda x)}{i\lambda} - y_2 \frac{d(i\lambda x)}{\lambda^2} + i \int_0^x \frac{d(i\lambda(x - t))}{\lambda^2} f(t)dt \]

(1.29)

is a solution to equation (1.6) and satisfies the first boundary condition \( y(\lambda, 0) = 0 \). Since

\[
y'(\lambda, x) = y_1 c(i\lambda x) - iy_2 \frac{s(i\lambda x)}{\lambda} - \int_0^x \frac{s(i\lambda(x - t))}{\lambda} f(t)dt,
\]

then the boundary conditions \( y'(0) = y'(l) \) and \( y(l) = 0 \) yield the system of linear equations for \( y_1, y_2, \)

\[
\begin{align*}
y_1(c(i\lambda l - 1)) - iy_2 s(i\lambda l) & = \int_0^l \frac{s(i\lambda(l - t))}{\lambda} f(t)dt, \\
y_1 \frac{s(i\lambda l)}{i\lambda} - y_2 \frac{d(i\lambda l)}{\lambda^2} & = -i \int_0^l \frac{d(i\lambda(l - t))}{\lambda^2} f(t)dt.
\end{align*}
\]

(1.30)
coinciding with system (1.21) as \( f = 0 \). Using the fact that determinant of this system \( D(\lambda) \) equals \( \Delta(0, \lambda) \) (Remark 1.3), one obtains that

\[
y_1 = \frac{1}{\lambda^3 D(\lambda)} \int_0^l \{s(i\lambda)d(i\lambda(l - t)) - d(i\lambda)s(i\lambda(l - t))\}f(t)dt;
\]

\[
y_2 = \frac{i}{\lambda^2 D(\lambda)} \int_0^l \{[1 - c(i\lambda)]d(i\lambda(l - t)) + s(i\lambda)s(i\lambda(l - t))\}f(t)dt.
\]

Hence one finds \( y(\lambda, x) \) (1.29),

\[
y(\lambda, x) = \frac{i}{\lambda^4 \Delta(0, \lambda)} \left\{ s(i\lambda x) \int_0^l [d(i\lambda)s(i\lambda(l - t)) - s(i\lambda)
\times d(i\lambda(l - t))]f(t)dt - d(i\lambda x) \int_0^l [(1 - c(i\lambda))d(i\lambda(l - t)) + s(i\lambda)
\times s(i\lambda(l - t))]f(t)dt + \int_0^x d(i\lambda(x - t))[s^2(i\lambda l) + d(i\lambda)(1 - c(i\lambda))]f(t)dt \right\}.
\]

To simplify this expression, one use the statement.

**Lemma 1.4** For all \( z \), the identities

(a) \( d(zl)s(z(l - t)) - s(zl)d(z(l - t)) = s(-zl)d(-zt) - d(-zl)s(-zt) \);

(b) \( (c(zl) - 1)d(z(l - t)) - s(zl)s(z(l - t)) = -(d(zl) + d(-zl))c(-zt) - s(zl)s(-zt) - (c(zl) - c(-zl))d(-zt) \);

(c) \( s(zx)[d(zl)s(z(l - t)) - s(zl)d(z(l - t))] + d(zx)[(c(zl) - 1)d(z(l - t) - s(zl)s(z(l - t))] = d(z(x - l))d(-zt) - d(zx)d(z(l - t)) - d(z(x - t))d(-zl) \)

are true.

**Proof.** Using the summation formulas (vii) (1.10), one obtains

\[
d(zl)s(z(l - t)) - s(zl)d(z(l - t)) = d(zl)[c(zl)s(-zt) + s(zl)c(-zt) + d(zl)d(-zt) - s(zl)[c(zl)d(-zt) + s(zl)s(-zt) + d(zl)c(-zt)]
\]

\[
= s(-zt)[c(zl)d(zl) - s^2(zl)] + d(-zt)[d^2(zl) - s(zl)c(zl)]
\]

\[
= -s(-zt)d(-zl) + d(-zt)s(-zl),
\]


due to (ix) \[(1.10)\], which proves (a) \[(1.33)\]. The second equality (b) \[(1.33)\] is similarly proved. Substituting (a), (b) into expression for (c), one obtains

\[
s(zx)d(-zt)s(-zl) - s(zx)s(-zt)d(-zl) - d(zx)c(-zt)[dzl]
\]

\[
-d(zl)] - d(zx)s(-zt)s(zl) - d(zx)d(-zt)[c(zl) - c(-zl)]
\]

\[
= -s(zx)s(-zt) + d(zx)c(-zt))d(-zl) - d(zx)c(-zt)d(zl)
\]

\[
+ [s(zx)s(-zl) + d(zx)c(-zl)]d(-tz)
\]

\[
= -d(z(x-t))d(-zl) - d(zx)(z(l-t)) + d(z(x-l))d(-zt),
\]

due to (vii) \[(1.10)\].

Substituting (c) \[(1.33)\] into \[(1.32)\], one obtains

\[
y(x, \lambda) = \frac{i}{\lambda^4 \Delta(0, \lambda)} \left\{ \int_0^l [d(i\lambda(x - l))d(-i\lambda t) - d(i\lambda x)d(i\lambda(l - t))
\]

\[
-d(i\lambda(x - t))d(i\lambda l)]f(t)dt + \int_0^x d(i\lambda(x - t))][d(i\lambda l) + d(-i\lambda l)]f(t)dt \right\}
\]

(1.34)

hence the statement follows.

**Theorem 1.2** Resolvent \(R_{L_0}(\lambda^3) = (L_0 - \lambda^3 I)^{-1}\) of the operator \(L_0\) \[(1.2), (1.3)\] is

\[
(R_{L_0}(\lambda^3)f(x) = \frac{i}{\lambda^4 \Delta(0, \lambda)} \left\{ \int_0^x [d(i\lambda(x - l))d(-i\lambda t) + d(i\lambda(x - t))d(i\lambda l)
\]

\[
-d(i\lambda x)d(i\lambda(l - t))] + \int_x^l [d(i\lambda(x - l))d(-i\lambda t) - d(i\lambda(x - t))d(-i\lambda l)]f(t)dt \}
\]

(1.35)

where \(\Delta(0, \lambda)\) is the characteristic function \[(1.23)\] of the operator \(L_0\).

2 Operator \(L_\alpha\)

2.1 Consider a self-adjoint operator \(L_\alpha (= L_\alpha(v))\) in \(L^2(0,l)\) which is a one-dimensional perturbation of \(L_0\) \[(1.1), (1.2)\],

\[
(L_\alpha y)(x) \overset{\text{def}}{=} iD^3y(x) + \alpha \int_0^l y(t)\overline{v}(t)dtv(x)
\]

(2.1)
where \( \alpha \in \mathbb{R} \) and \( v \in L^2(0, l) \). Without loss in generality, assume that \( \|v\|_{L^2} = 1 \). Domains of the operators \( L_\alpha \) and \( L_0 \) coincide, \( \mathcal{D}(L_\alpha) = \mathcal{D}(L_0) \).

Using the functions \( c(z) \), \( s(z) \), \( d(z) \) (1.9), define Fourier transforms

\[
\tilde{v}_c(\lambda) \overset{\text{def}}{=} \langle v(x), c(i\lambda x) \rangle = \frac{1}{3} \sum_k \tilde{v}_k(\lambda);
\]

\[
\tilde{v}_s(\lambda) \overset{\text{def}}{=} \langle v(x), s(i\lambda x) \rangle = \frac{1}{3} \sum_k \zeta_k^{-1} \tilde{v}_k(\lambda);
\] (2.2)

\[
\tilde{v}_d(\lambda) \overset{\text{def}}{=} \langle v(x), d(i\lambda x) \rangle = \frac{1}{3} \sum_k \zeta_k \tilde{v}_k(\lambda)
\]

where

\[
\tilde{v}_k(\lambda) = \int_0^l e^{-i\lambda \zeta_k x} v(x) dx \quad (1 \leq k \leq 3).
\] (2.3)

Introduce the operation of involution

\[
f^*(\lambda) = f(\overline{\lambda}),
\] (2.4)

then

\[
\tilde{v}^*_c(\lambda) = \langle c(i\lambda x), v(x) \rangle = \frac{1}{3} \sum_k \tilde{v}^*_k(\lambda);
\]

\[
\tilde{v}^*_s(\lambda) = \langle s(i\lambda x), v(x) \rangle = \frac{1}{3} \sum_k \zeta_k \tilde{v}^*_k(\lambda);
\] (2.5)

\[
\tilde{v}^*_d(\lambda) = \langle d(i\lambda x), v(x) \rangle = \frac{1}{3} \sum_k \zeta_k^{-1} \tilde{v}^*_k(\lambda),
\]

besides,

\[
\tilde{v}^*_1(\lambda) = \int_0^l e^{i\lambda \zeta_1 x} \overline{v}(x) dx; \quad \tilde{v}^*_2(\lambda) = \int_0^l e^{i\lambda \zeta_2 x} \overline{v}(x) dx; \quad \tilde{v}^*_3(\lambda) = \int_0^l e^{i\lambda \zeta_3 x} \overline{v}(x) dx.
\] (2.6)

According to the Cauchy problem (1.6), equation \( L_\alpha y = \lambda^3 y \) has the solution (see (1.15))

\[
y(\lambda, x) = y_1 \frac{s(i\lambda x)}{i\lambda} - y_2 \frac{d(i\lambda x)}{\lambda^2} - i\alpha \langle y, v \rangle \int_0^x \frac{d(i\lambda (x-t))}{\lambda^2} v(t) dt
\] (2.7)

satisfying the boundary condition \( y(0) = 0 \). Multiplying equality (2.7) by \( \overline{v}(x) \) and integrating it from 0 to \( l \), one obtains

\[
iy_1 \frac{\tilde{v}^*_1(\lambda)}{\lambda} + y_2 \frac{\tilde{v}^*_2(\lambda)}{\lambda^2} + \langle y, v \rangle \left( 1 + \frac{i\alpha}{\lambda^2} m(x) \right) = 0
\] (2.8)
where \( \tilde{v}_s^*(\lambda) \) and \( \tilde{v}_d^*(\lambda) \) are given by (1.5) and

\[
m(\lambda) \overset{\text{def}}{=} \left( \int_0^x d(i\lambda(x-t))v(t)\,dt, v(x) \right) = \frac{1}{3} \sum \zeta_k \psi_k(\lambda); \tag{2.9}
\]

\[
\psi_k(\lambda) \overset{\text{def}}{=} \int_0^l dx e^{i\lambda_k x} \bar{v}(x) \int_0^x \,dt e^{-i\lambda_k t}v(t) \quad (1 \leq k \leq 3).
\]

**Remark 2.1** The functions \( \tilde{v}_k(\lambda) \) (2.3) and \( \tilde{v}_s^*(\lambda) \) (2.6) (and thus the functions (2.2) and (2.5) also) are entire functions of exponential type.

**Lemma 2.1** The function \( m(\lambda) \) (2.9) is an entire function of exponential type and

\[
m(\lambda) + m^*(\lambda) = \tilde{v}_d(\lambda)\tilde{v}_c^*(\lambda) + \tilde{v}_s(\lambda)\tilde{v}_s^*(\lambda) + \tilde{v}_c(\lambda)\tilde{v}_d^*(\lambda). \tag{2.10}
\]

**Proof.** Since \( \psi_k(\lambda) \) (2.9) is the Fourier transform of the convolution

\[
\psi_k(\lambda) = \int_0^l dx \int_0^x \,dt e^{i\lambda_k(x-t)}v(t)\bar{v}(x) = \int_0^l \,ds e^{i\lambda_k s} \int_0^l v(x-s)\bar{v}(x)\,dx,
\]

then \( \psi_k(\lambda) \) is an entire function of exponential type, thus \( m(\lambda) \) also has this property.

Relation (2.9) implies that

\[
m(\lambda) + m^*(\lambda) = \frac{1}{3} \{ \zeta_1(\psi_1(\lambda) + \psi_1^*(\lambda)) + \zeta_2(\psi_2(\lambda) + \psi_2^*(\lambda)) + \zeta_3(\psi_3(\lambda) + \psi_3^*(\lambda)) \}.
\]

Integrating by parts, one obtains

\[
\psi_k(\lambda) = \int_0^l \,dt e^{-i\lambda_k t}v(t) \int_0^l dx e^{i\lambda_k x} \bar{v}(x) - \int_0^l \,dt e^{-i\lambda_k t}v(t) \int_0^t dx e^{i\lambda_k x} \bar{v}(x),
\]

therefore

\[
\psi_1(\lambda) = \tilde{v}_1(\lambda)\bar{v}_1^*(\lambda) - \psi_1^*(\lambda), \quad \psi_2(\lambda) = \tilde{v}_2(\lambda)\bar{v}_2^*(\lambda) - \psi_2^*(\lambda), \quad \psi_3(\lambda) = \tilde{v}_3(\lambda)\bar{v}_3^*(\lambda) - \psi_3^*(\lambda).
\]

So,

\[
m(\lambda) + m^*(\lambda) = \frac{1}{3} \{ \zeta_1 \tilde{v}_1(\lambda)\bar{v}_1^*(\lambda) + \zeta_2 \tilde{v}_2(\lambda)\bar{v}_2^*(\lambda) + \zeta_3 \tilde{v}_3(\lambda)\bar{v}_3^*(\lambda) \}
\]

\[
= \int_0^l dx \int_0^l \,dt d(i\lambda(x-t))\bar{v}(x)v(t) = \int_0^l dx \int_0^l \,dt d\bar{v}(x)v(t) \{ c(i\lambda x) d(-i\lambda t)
\]
yield the system of linear equations for $y$

function of the operator $D$

Solutions to system (2.11) are non-trivial then and only then when its determinant

due to (vii) (1.10) and according to (2.2), (2.5),

$\alpha, \lambda$

$\Delta(\alpha, \lambda)$

Equality (2.10) is an analogue of a well-known statement on the Fourier

transform of a convolution.

Formula (2.8) and boundary conditions $y'(0) = y'(l)$, $y(l) = 0$ for $y(\lambda, x)

(2.7) yield the system of linear equations for $y_1$, $y_2$, $\langle y, v \rangle$,

\[
\begin{cases}
  y_1 \frac{\widetilde{v}_s(\lambda)}{\lambda} + y_2 \frac{\widetilde{v}_d(\lambda)}{\lambda^2} + \langle y, v \rangle \left(1 + \frac{i\alpha}{\lambda^2} m(\lambda)\right) = 0; \\
  y_1(1 - c(i\lambda l)) + y_2 \frac{s(i\lambda l)}{\lambda} - \alpha \langle y, v \rangle \int_0^l \frac{s(i\lambda(l-t))}{\lambda} v(t) dt = 0, \\
  y_1 \frac{is(i\lambda l)}{\lambda} + y_2 \frac{d(i\lambda l)}{\lambda^2} + \alpha i \langle y, v \rangle \int_0^l \frac{d(i\lambda(l-t))}{\lambda^2} v(t) dt = 0.
\end{cases}
\] (2.11)

Solutions to system (2.11) are non-trivial then and only then when its determinant

$D_\alpha(\lambda)$ vanishes. The function $\Delta(\alpha, \lambda) = D_\alpha(\lambda)$ is said to be characteristic

function of the operator $L_\alpha$ (2.1) and equals

\[
\Delta(\alpha, \lambda) \overset{\text{def}}{=} \left(1 + \frac{\alpha i}{\lambda^2} m(\lambda)\right) \Delta(0, \lambda) + \frac{i\alpha}{\lambda^4} \left\{ \widetilde{v}_d^*(\lambda) \int_0^l [(c(i\lambda l) - 1)d(i\lambda(l-t))

- s(i\lambda l)s(i\lambda(l-t))] v(t) dt + \widetilde{v}_s^*(\lambda) \int_0^l [d(i\lambda l)s(i\lambda(l-t))

- s(i\lambda l)d(i\lambda(l-t))] v(t) dt \right\}
\] (2.12)

Using (1.22) and (a), (b) (1.33), one has

\[
\Delta(\alpha, \lambda) - \Delta(0, \lambda) = \frac{i\alpha}{\lambda^4} \{m(\lambda)(d(i\lambda l) + d(-i\lambda l)) - \widetilde{v}_d^*(\lambda)\widetilde{v}_c(\lambda)

\times (d(i\lambda l) + d(-i\lambda l)) - \widetilde{v}_d(\lambda)\widetilde{v}_s(-\lambda)s(i\lambda l) - \widetilde{v}_d^*(\lambda)\widetilde{v}_d(\lambda)(c(i\lambda l) - c(-i\lambda l))

+ \widetilde{v}_s^*(\lambda)\widetilde{v}_d(\lambda)s(-i\lambda l) - \widetilde{v}_s^*(\lambda)v_s(\lambda)d(-i\lambda l)\}.
\]

Expressing $\widetilde{v}_s^*(\lambda)\widetilde{v}_s(\lambda)$ from (2.10) and substituting it into this expression, one obtains

\[
\Delta(\alpha, \lambda) - \Delta(0, \lambda) = \frac{i\alpha}{\lambda^4} \{m(\lambda)d(i\lambda l) + \widetilde{v}_d(\lambda)[\widetilde{v}_c(\lambda)d(-i\lambda l)

\times (d(i\lambda l) + d(-i\lambda l)) - \widetilde{v}_d(\lambda)\widetilde{v}_s(-\lambda)s(i\lambda l) - \widetilde{v}_d^*(\lambda)\widetilde{v}_d(\lambda)(c(i\lambda l) - c(-i\lambda l))

+ \widetilde{v}_s^*(\lambda)\widetilde{v}_d(\lambda)s(-i\lambda l) - \widetilde{v}_s^*(\lambda)v_s(\lambda)d(-i\lambda l)\}.
\]
\[ + \tilde{c}(\lambda)s(-i\lambda l) + \tilde{v}_s(\lambda)c(-i\lambda l) - m^*(\lambda)d(-i\lambda l) - \tilde{a}(\lambda) \]
\[ \times [\tilde{c}(\lambda)d(i\lambda l) + \tilde{v}_s(\lambda)s(i\lambda l) + \tilde{a}(\lambda)c(i\lambda l)]. \]

Taking (vii) \((1.10)\) into account, one finds
\[
\tilde{c}(\lambda)d(i\lambda l) + \tilde{v}_s(\lambda)s(i\lambda l) + \tilde{a}(\lambda)c(i\lambda l) = \int_0^l d(i\lambda(l-x))v(x)dx = \int_0^l d(i\lambda t)v(l-t)dt = \tilde{w}_d(-\lambda)
\]
where \(\tilde{w}_d(\lambda)\) is the Fourier transform \(\tilde{w}_d(\lambda) = \langle w(x), d(i\lambda x) \rangle \) \((2.2)\) of the function
\[
w(x) \overset{\text{def}}{=} v(l-x).
\]

Lemma 2.2 Characteristic function \(\Delta(\alpha, \lambda) \) \((2.12)\) of the operator \(L_{\alpha} \) \((2.1)\) is
\[
\Delta(\alpha, \lambda) - \Delta(0, \lambda) = \frac{i\alpha}{\lambda^4}\{F(\lambda) - F^*(\lambda)\}
\]
where
\[
F(\lambda) \overset{\text{def}}{=} m(\lambda)d(i\lambda l) + \tilde{a}(\lambda)\tilde{w}_d(\lambda),
\]
besides, \(\tilde{w}_d(\lambda)\) and \(m(\lambda)\) are given by \((2.5)\), \((2.9)\), \(\tilde{w}_d(\lambda)\) is the Fourier transform \((2.2)\) of the function \(w(x) \) \((2.13)\). Moreover, \(\Delta(\alpha, \lambda) \) \((2.14)\) is a real entire function of exponential type and
\[
\Delta(\alpha, \lambda_2) = \Delta(\alpha, \lambda), \quad \Delta^*(\alpha, \lambda) = \Delta(\alpha, \lambda).
\]

2.2. Find eigenfunctions of the operator \(L_{\alpha} \) \((2.1)\). Boundary conditions \(y'(0) = y'(l)\) and \(y(l) = 0\) for the function \(y(\lambda, x) \) \((2.7)\) yield system of equations formed by the two last equalities in \((2.11)\)
\[
\begin{cases}
y_2i\frac{s(i\lambda l)}{\lambda} - \frac{\alpha\langle y, v \rangle}{\lambda} \int_0^l s(i\lambda(l-t))v(t)dt = -y_1(1 - c(i\lambda l)), \\
y_2\frac{d(i\lambda l)}{\lambda} + \frac{\alpha i\langle y, v \rangle}{\lambda} \int_0^l d(i\lambda(l-t))v(t)dt = -y_1is(i\lambda l).
\end{cases}
\]

Hence we find that
\[
\frac{y_2}{\lambda} = \frac{iy_1}{a(\lambda)} \int_0^l [(c(i\lambda l) - 1)d(i\lambda(l-t)) - s(i\lambda l)s(i\lambda(l-t))]v(t)dt;
\]
\[
\frac{\alpha\langle y, v \rangle}{\lambda} = \frac{y_1}{a(\lambda)}(s(i\lambda l)^2 + d(i\lambda l))(1 - c(i\lambda l)),
\]

\[\frac{y_2}{\lambda} = \frac{iy_1}{a(\lambda)} \int_0^l [(c(i\lambda l) - 1)d(i\lambda(l-t)) - s(i\lambda l)s(i\lambda(l-t))]v(t)dt;\]
besides,

\[ a(\lambda) = \int_0^l [d(i\lambda l)s(i\lambda(l-t)) - s(i\lambda t)d(i\lambda(l-t))]v(t)dt. \]

Substituting expressions (2.17) into (2.7), one obtains

\[ y(\lambda, x) = -\frac{iy_1}{\lambda a(\lambda)} \left\{ \int_0^l \{ s(i\lambda x)[d(i\lambda l)s(i\lambda(l-t)) - s(i\lambda l)d(i\lambda(l-t))] \right. \\
+ d(i\lambda x)[(ci\lambda l - 1)d(i\lambda(l-t)) - s(i\lambda l)s(i\lambda(l-t))] \} v(t)dt \\
\left. \right\} \]

and according to (c) (1.33),

\[ y(\lambda, x) = -\frac{iy_1}{\lambda a(\lambda)} \left\{ \int_0^l [d(i\lambda(x-l))d(-i\lambda t) - d(i\lambda x)d(i\lambda(l-t))] \\
- d(i\lambda(x-t))d(-i\lambda l)]v(t)dt + (d(i\lambda l) + d(-i\lambda l)) \int_0^x d(i\lambda(x-t))v(t)dt \right\} \}

**Lemma 2.3** Eigenfunctions \( u(\alpha, \lambda, x) \) of the operator \( L_\alpha \) (2.1) are given by

\[ u(\alpha, \lambda, x) = \frac{1}{u(\alpha, \lambda)} \left\{ d(i\lambda(x-l))\tilde{v}_d(\lambda) - d(i\lambda x)\tilde{w}_d(-\lambda) \\
+ d(i\lambda l) \int_0^x d(i\lambda(x-t))v(t)dt - d(-i\lambda l) \int_x^l d(i\lambda(x-t))v(t)dt \right\} \]

where \( \lambda = \lambda_k(\alpha) \) is a zero of the characteristic function \( \Delta(\alpha, \lambda) \) (2.14), \( \tilde{w}_d(\lambda) \) is the Fourier transform (2.2) of the function \( w(x) \) (2.13); \( u(\alpha, \lambda) = \|u(\alpha, \lambda, x)\|_{L^2} \)

and

\[ u(\alpha, \lambda\zeta_2, x) = u(\alpha, \lambda, x). \] (2.19)

### 3 Abstract problem

**3.1.** Consider an abstract problem of one-dimensional perturbation of a self-adjoint operator with simple spectrum. Let \( L_0 \) be a self-adjoint operator acting in a Hilbert space \( H \) with simple discrete spectrum

\[ \sigma(L_0) = \{ z_n : z_n \in \mathbb{R}, n \in \mathbb{Z} \}. \] (3.1)
Its eigenfunctions $u_n (L_0 u_n = z_n u_n)$ are orthonormal, \( \langle u_n, u_m \rangle = \delta_{n,m} \), and form an orthonormal basis in the space $H$. Resolvent $R_{L_0}(z) = (L_0 - zI)^{-1}$ of the operator $L_0$ is

$$R_{L_0}(z) = \sum_n \frac{f_n}{z_n - z} u_n$$

(3.2)

where $f_n = \langle f, u_n \rangle$ are Fourier coefficients of the vector $f$ in the basis $\{u_n\}$.

By $L_\alpha$ denote a self-adjoint operator which is a one-dimensional perturbation of $L_0$ (cf. (2.1)),

$$L_\alpha \overset{\text{def}}{=} L_0 + \alpha \langle ., v \rangle v$$

(3.3)

where $\alpha \in \mathbb{R}$ and $v$ is a fixed vector from $H$ such that $\|v\| = 1$. Domains of $L_\alpha$ and $L_0$ coincide, $\mathcal{D}(L_\alpha) = \mathcal{D}(L_0)$.

**Lemma 3.1** Resolvent $R_{L_\alpha}(z) = (L_\alpha - zI)^{-1}$ of the operator $L_\alpha$ (3.3) is

$$R_{L_\alpha}(z)f = R_{L_0} - \alpha \frac{\langle R_{L_0}(z)f, v \rangle}{1 + \alpha \langle R_{L_0}(z)v, v \rangle} \cdot R_{L_0}(z)v$$

(3.4)

where $R_{L_0}(z) = (L_0 - zI)^{-1}$ is the resolvent of operator $L_0$ and $f \in H$.

**Proof.** Let $y = R_{L_0}(z)f$, then

$$f = (L_0 - zI)y + \alpha \langle y, v \rangle v,$$

and thus

$$R_{L_0}(z)f = y + \alpha \langle y, v \rangle R_{L_0}(z)v.$$ (3.5)

Scalar multiplying this equality by $v$, one obtains

$$\langle R_{L_0}(z)f, v \rangle = \langle y, v \rangle (1 + \alpha \langle R_{L_0}(z)v, v \rangle).$$

From here expressing $\langle y, v \rangle$ and substituting it into (3.5), one arrives at formula (3.4). ■

Formulas (3.2), (3.4) imply

$$R_{L_\alpha}(z)f = \sum_n \frac{f_n}{z_n - z} u_n - \alpha \frac{\sum_k \frac{f_k \overline{v}_k}{z_k - z}}{1 + \alpha \sum_k \frac{|v_k|^2}{z_k - z}} \cdot \sum_n \frac{v_n}{z_n - z} u_n$$

(3.6)

where $f_n = \langle f, u_n \rangle$ and $v_n = \langle v, u_n \rangle$ are Fourier coefficients of the vectors $f$ and $v$ correspondingly.

**Lemma 3.2** If $v_p \neq 0$, then point $z_p (\in \sigma(L_0))$ does not belong to the spectrum $\sigma(L_\alpha)$ operator $L_\alpha$ (3.3).
Proof. Really, residue of the resolvent $R_{L\alpha}(z)$ (3.6) vanishes at the point $z_p,$
\[ \lim_{z \to z_p} (z_p - z) R_{L\alpha}(z) = f_p u_p - \frac{\alpha f_p v_p}{\alpha v_p u_p} \cdot v_p u_p = 0. \]
\[ \blacksquare \]

Hence it follows that it is natural to divide the set $\sigma(L_0)$ (3.1) into two disjoint subsets $\sigma(L_0) = \sigma_0 \cup \sigma_1$ where
\[ \sigma_0 \overset{\text{def}}{=} \{ z_n^0 = z_n \in \sigma(L_0) : v_n = 0 \}, \quad \sigma_1 \overset{\text{def}}{=} \{ z_n^1 = z_n \in \sigma(L_0) : v_n \neq 0 \}. \tag{3.7} \]

Assume that elements from $\sigma(L_0), \sigma_0, \sigma_1$ are numbered in ascending order. Partition $\sigma(L_0) = \sigma_0 \cup \sigma_1$ corresponds to the decomposition of the space $H$ into orthogonal sum $H = H_0 \oplus H_1$ where
\[ H_0 \overset{\text{def}}{=} \text{span}\{ u_n : z_n \in \sigma_0 \}, \quad H_1 \overset{\text{def}}{=} \text{span}\{ u_n : z_n \in \sigma_1 \}. \tag{3.8} \]

The subspaces $H_0, H_1$ reduce the operator $L_\alpha$ (3.3) and
\[ L_\alpha|_{H_0} = L_{\alpha,0}; \quad L_\alpha|_{H_1} = L_{\alpha,1}, \tag{3.9} \]
therefore
\[ R_{L\alpha}(z) = R_{L\alpha,0}(z) \oplus R_{L\alpha,1}(z) \]

where
\[ R_{L\alpha,0}(z)f = \sum_{z_n^0 \in \sigma_0} \frac{f_n}{z_n^0 - z} u_n; \]
\[ R_{L\alpha,1}(z)f = \frac{1}{Q(z)} \sum_{z_n^1 \in \sigma_1} \frac{1}{z_n^1 - z} \{ f_n Q(z) - \alpha v_n P(z,f) \} u_n, \tag{3.10} \]
besides,
\[ Q(z) \overset{\text{def}}{=} 1 + \alpha \sum_{z_n^1 \in \sigma_1} \frac{|v_n|^2}{z_n^1 - z}; \quad P(z, f) \overset{\text{def}}{=} \sum_{z_n^1 \in \sigma_1} \frac{f_n v_n}{z_n^1 - z}. \tag{3.11} \]

Lemma 3.2 implies that the resolvent $R_{L_{\alpha,1}}(z)$ (3.10) does not have singularities at the points $z_n^1 \in \sigma_1$ but can have singularities at zeros of the function $Q(z)$ (3.11).

The function
\[ G(z) = \sum_{z_n^1 \in \sigma_1} \frac{|v_n|^2}{z_n^1 - z} \tag{3.12} \]
monotonously increases for all $z \in \mathbb{R} \setminus \sigma_1$ (since $F'(z) > 0$ for all $z \in \mathbb{R} \setminus \sigma_1$). Therefore equation $1 + \alpha F(z) = 0$, which is equivalent to $Q(z) = 0$, has only simple roots.
Lemma 3.3 Zeros of \( Q(z) \) \((3.11) \) are real, simple and alternate with numbers \( z_n^1 \in \sigma_1 \) \((3.7) \).

Define the set
\[
Q(L_{\alpha,1}) \overset{\text{def}}{=} \{ \mu_n : Q(\mu_n) = 0 \} \tag{3.13}
\]
where \( Q(z) \) is given by \((3.11) \) and numbers \( \mu_n = \mu(\alpha) \) depend on the parameter \( \alpha \). Resolvent \( R_{L_{\alpha,1}}(z) \) \((3.10) \) cannot have removable singularities at the points \( \mu_n \in \sigma(L_{\alpha,1}) \). Thus singularities of \( R_{L_{\alpha,1}}(z) \) are the simple poles at the points \( \mu_n \in \sigma(L_{\alpha,1}) \).

3.2 Find eigenfunctions of the operator \( L_{\alpha,1} \). Formula \((3.10) \) implies that residue of the resolvent \( R_{L_{\alpha,1}}(z) \) at the point \( z = \mu_p \) is
\[
c_p(f) \overset{\text{def}}{=} \lim_{z \to \mu_p} (\mu_p - z) R_{L_{\alpha,1}}(z) = \frac{\alpha}{Q'(\mu_p)} \sum_{\substack{z_n^1 \in \sigma_1 \atop z_n^1 - \mu_p \neq 0}} \frac{v_n}{z_n^1 - \mu_p} \left( \sum_{\substack{z_k^1 \in \sigma_1 \atop z_k^1 - \mu_p \neq 0}} \frac{f_k \bar{v}_k}{z_k^1 - \mu_p} \right) u_n. \tag{3.14}
\]

Note that
\[
Q'(\mu_p) = \alpha \sum_{\substack{z_n^1 \in \sigma_1 \atop z_n^1 - \mu_p \neq 0}} \frac{|v_n|^2}{z_n^1 - \mu_p} = \alpha G'(\mu_p)(\neq 0)
\]
where \( G(z) \) is given by \((3.12) \) and \( G'(\mu_p) > 0 \). Define the vectors
\[
\tilde{u}_p \overset{\text{def}}{=} \frac{1}{\sqrt{G'(\mu_p)}} \sum_{\substack{z_n^1 \in \sigma_1 \atop z_n^1 - \mu_p \neq 0}} \frac{v_n}{z_n^1 - \mu_p} u_n. \tag{3.15}
\]
This series converges in \( H \) and \( ||\tilde{u}_p|| = 1 \). Moreover, \( \tilde{u}_p \) are eigenfunctions of the operator \( L_{\alpha,1}, L_{\alpha,1} \tilde{u}_p = \mu_p \tilde{u}_p \). Really,
\[
L_{\alpha,1} \tilde{u}_p = \frac{1}{\sqrt{G'(\mu_p)}} \sum_{\substack{z_n^1 \in \sigma_1 \atop z_n^1 - \mu_p \neq 0}} \frac{v_n}{z_n^1 - \mu_p} z_n^1 u_n + \frac{\alpha}{\sqrt{G'(\mu_p)}} \sum_{\substack{z_n^1 \in \sigma_1 \atop z_n^1 - \mu_p \neq 0}} \frac{|v_n|^2}{z_n^1 - \mu_p} v
\]
\[
= \mu_p \tilde{u}_p + \frac{1}{\sqrt{G'(\mu_p)}} Q(\mu_p)v = \mu_p \tilde{u}_p
\]
due to \( Q(\mu_p) = 0 \). Formula \((3.15) \) implies that \( c_p(f) \) \((3.14) \) equal \( c_p(f) = \langle f, \tilde{u}_p \rangle \tilde{u}_p \). Therefore resolvent \( R_{L_{\alpha,1}}(z) \) \((3.11) \) is given by
\[
R_{L_{\alpha,1}}(z)f = \sum_{\mu_p \in \sigma(L_{\alpha,1})} \frac{f_p}{\mu_p - z} \tilde{u}_p \tag{3.16}
\]
where \( f_p = \langle f, \tilde{u}_p \rangle \) are Fourier coefficients of the vector \( f \) in the basis \( \{ \tilde{u}_p \} \) \((3.15) \) of the subspace \( H_1 \).
Lemma 3.4 Spectrum of the operator \( L_{\alpha,1} \) (3.9) is simple and coincides with the set \( \sigma(L_{\alpha,1}) \) (3.13). Eigenvectors of \( L_{\alpha,1} \) corresponding to eigenvalues \( \mu_p \) are given by (3.15) and resolvent \( R_{L_{\alpha,1}}(z) \) is given by formula (3.16).

It can turn out that a number \( \mu_p \in \sigma(L_{\alpha,1}) \) coincides with \( z_0^n \in \sigma_0 \) (3.7), then the proper subspace of the operator \( L_{\alpha} \) (3.3) is two-dimensional and is generated by the vectors \( \{u_n, \tilde{u}_p\} \).

Theorem 3.1 Spectrum of the operator \( L_{\alpha} \) (3.3) is
\[
\sigma(L_{\alpha}) = \sigma_0 \cup \sigma(L_{\alpha,1})
\] (3.17)
where \( \sigma_0 \) and \( \sigma(L_{\alpha,1}) \) are given by (3.7) and (3.13) correspondingly. Spectrum of the operator \( L_{\alpha} \) is simple, excluding points belonging to \( \sigma_0 \cap \sigma(L_{\alpha,1}) \) where it is of multiplicity 2.

Remark 3.1 Let the set \( \sigma(L_0) \) (3.1) be such that
\[
\lim_{n \to \infty} z_n = \infty; \quad \min_{n,s} |z_n - z_s| = d > 0,
\] (3.18)
then the intersection \( \sigma_0 \cap \sigma(L_{\alpha,1}) \) is finite. Thus, when conditions (3.18) hold, the operator \( L_{\alpha} \) (3.3) has finite number of points of the spectrum of multiplicity 2.

Really, if \( \sigma_0 \cap \sigma(L_{\alpha,1}) \) is infinite and consists of points \( \{z_0^p\}_1^\infty \), then in view of \( Q(z_0) = 0 \)
\[
1 + \alpha \sum_{z_0^p \in \sigma_1} \frac{|v_n|^2}{z_0^p - z_0^p} = 0 \quad (\forall p).
\]
Formula (3.18) and \( \sum |v_n|^2 = 1 \) imply that this series converges uniformly. Passing to the limit as \( p \to \infty \) in this equality and taking into account (3.18), one obtains \( 1 = 0 \) which is absurd.

Lemma 1.3 implies that the conditions (3.18) for the operator \( L_0 \) (1.1), (1.2) hold, therefore for the operator \( L_{\alpha} \) (2.1) Theorem 3.1 holds and there is finite number of points in \( \sigma(L_{\alpha}) \) of multiplicity 2.

3.3. Calculate \( Q(z) = 1 + \alpha \langle R_{L_0}(z)v, v \rangle \) for the operator \( L_0 \) (1.1), (1.2). Relation (1.35) implies that
\[
\langle R_{L_0}(\lambda^3)v, v \rangle = \frac{i}{\lambda^4 \Delta(0, \lambda)} \left\{ \int_0^l [d(i \lambda(x - l))d(-i \lambda t) - d(i \lambda x)d(i \lambda(l - t))] \right\}
\]
\[ \times v(t) dt, v(x) \rangle + \left\langle d(i\lambda l) \int_0^x d(i\lambda(x - t))v(t) dt, v(x) \right\rangle \\
- \left\langle d(-i\lambda l) \int_x^l d(i\lambda(x - t))v(t) dt, v(x) \right\rangle \right\}.

Taking into account (2.2), (2.5) and

\[ m(\lambda) = \left\langle \int_0^x d(i\lambda(x - t))v(t) dt, v(x) \right\rangle, \]

one obtains

\[ \langle R_{L_0}(\lambda^3)v, v \rangle = \frac{i}{\lambda^4\Delta(0, \lambda)} \{ m(\lambda)d(i\lambda l) - \tilde{v}_d^*(\lambda)\tilde{w}_d(-\lambda) \}
- m^*(\lambda)d(-i\lambda l) + \tilde{v}_d(\lambda)\tilde{w}_d^*(-\lambda) \} = \frac{i}{\lambda^4\Delta(0, \lambda)} \{ F(\lambda) - F^*(\lambda) \}
\]

where \( F(\lambda) \) is given by (2.15), using (2.14), one arrives at the equality

\[ \alpha \langle R_{L_0}(\lambda^3)v, v \rangle = \frac{\Delta(\alpha, \lambda) - \Delta(0, \lambda)}{\Delta(0, \lambda)}. \]

**Lemma 3.5** For the operator \( L_0 \) (1.1), (1.2), the identity

\[ Q(\lambda^3) = \frac{\Delta(\alpha, \lambda)}{\Delta(0, \lambda)} \] (3.19)

where \( Q(z) = 1 + \alpha \langle R_{L_0}(z)v, v \rangle \) and \( \Delta(0, \lambda) \) (1.23) and \( \Delta(\alpha, \lambda) \) (2.14) are characteristic functions of the operators \( L_0 \) (1.1), (1.2) and \( L_\alpha \) (2.1).

**4 Inverse problem**

4.1. To obtain multiplicative expansions of characteristic functions \( \Delta(0, \lambda) \) (1.23) and \( \Delta(\alpha, \lambda) \) (2.14), use the well-known Hadamard theorem on factorization [9, 10].

**Theorem 4.1 (Hadamard)** Let \( f(z) \) be an entire function of exponential type and \( \{z_n\}_1^\infty \) be a subsequence of its zeros where each zero is repeated according to its multiplicity, besides, \( z_1 \neq 0 \) and \( 0 < |z_1| \leq |z_2| \leq \ldots \). Then

\[ f(z) = cz^k e^{bz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \] (4.1)

where \( k \in \mathbb{Z}_+ ; c, b \in \mathbb{C} \).
Lemma 1.3 implies that zeros of the function \( \Delta(0, \lambda) \) come in triplets \( \{\zeta_l \lambda_n(0)\}_{l=1}^3 \) (and \( \{-\zeta_l \lambda_n(0)\}_{l=1}^3 \) \((\lambda_n(0) > 0)\), therefore

\[
\prod_{n=1}^3 \left(1 - \frac{\lambda}{\zeta_k \lambda_n(0)} \right) e^{\lambda/\zeta_k \lambda_n(0)} = 1 - \frac{\lambda^3}{\lambda_n^3(0)} \tag{4.2}
\]
due to (1.4) (similarly for \( \{-\zeta_l \lambda_n(0)\} \) also), and according to (4.1)

\[
\Delta(0, \lambda) = c \lambda^k e^{b \lambda} \prod_n \left(1 - \frac{\lambda^6}{\lambda_n^6(0)} \right). \tag{4.3}
\]

Infinite product (4.3) converges uniformly on every compact due to location of \( \lambda_n(0) \) (Lemma 1.3). Taylor formula (ix) (1.10) for \( d(z) \) and (1.23) imply the expansion

\[
\Delta(0, \lambda) = -l^2 + \frac{2}{8} \lambda^5 l^8 + \ldots \tag{4.4}
\]
and thus \( \Delta(0, 0) = -l^2 \,(\neq 0) \) and \( k = 0, c = -l^2 \) in the formula (4.3). So,

\[
\Delta(0, \lambda) = -l^2 e^{b \lambda} \prod_n \left(1 - \frac{\lambda^6}{\lambda_n^6(0)} \right). \tag{4.5}
\]

Differentiating this equality and assuming that \( \lambda = 0 \), one obtains that \( b = 0 \) since \( \Delta'(0, 0) = 0 \) due to (4.4). Thus, multiplicative expansion of \( \Delta(0, \lambda) \) is

\[
\Delta(0, \lambda) = -l^2 \prod_n \left(1 - \frac{\lambda^6}{\lambda_n^6(0)} \right). \tag{4.5}
\]

Property (2.16) of the characteristic function \( \Delta(\alpha, \lambda) \) (2.14) yields that its roots also form triplets \( \{\zeta_l \lambda_n(\alpha)\}_{l=1}^3 \) \((0 = \lambda_n(\alpha) \in \mathbb{R})\) and for them equality (4.2) is true, therefore, due to Theorem 4.1

\[
\Delta(\alpha, \lambda) = c \lambda^k e^{b \lambda} \prod_n \left(1 - \frac{\lambda^3}{\lambda_n^3(\alpha)} \right). \tag{4.6}
\]

This product also converges uniformly on every compact. Again using the Taylor formula (ix) (1.10) for \( d(z) \), one finds series expansion of the function \( F(\lambda) \) (2.15),

\[
F(\lambda) = \frac{\lambda^4}{4} a_1(v) - \frac{i \lambda^7}{2 \cdot 5!} a_2(v) + \ldots \tag{4.7}
\]
where

\[
a_1(v) = \left\langle i^2 \int_0^x (x - t)^2 v(t) dt + (l - x)^2 \int_0^l t^2 v(t) dt, v(x) \right\rangle. \tag{4.8}
\]
Substituting (4.4) and (4.7) into (2.14), one finds the expansion

\[ \Delta(\alpha, \lambda) = -l^2 + \alpha i 4(a_1(v) - \overline{a_1(v)}) + \frac{\lambda^3\alpha}{2 \cdot 5!}(a_2(v) - \overline{a_2(v)}) + \ldots \] (4.9)

Hence (4.6) implies that \( k = 0 \) and 
\[ c = -l^2 + \frac{\alpha i}{4}(a_1(v) - \overline{a_1(v)}) \neq 0. \] (4.10)

Take into account that \( \Delta'(\alpha, 0) = 0 \) (4.9), then (4.6) implies that \( b = 0 \). So,
\[ \Delta(\alpha, \lambda) = c \prod_{n} \left( 1 - \frac{\lambda^3}{\lambda_n^3(\alpha)} \right) \] (4.11)

where \( c \) is given by (4.10). Further, the number \( c \) will be found in terms of spectral data.

4.2. Use equality (3.19),
\[ Q(z) = \frac{\Delta(\alpha, z^{1/3})}{\Delta(0, z^{1/3})}, \] (4.12)

and note that in the quotient all terms in infinite products (4.5), (4.14) which correspond to \( z_n(0) = \pm \lambda_n^3(0) \) from \( \sigma_0 \) (3.7) reduce. Only terms corresponding to \( z_n(0) = \varepsilon \lambda_n^3(0) \in \sigma_1 \) (3.7) \( (\varepsilon = 1 \text{ or } -1) \) and corresponding to it in numerator \( z_n(\alpha) \) remain. So,
\[ 1 + \alpha \sum_{z_n \in \sigma_1} \frac{|v_p|^2}{z_n - z} = -\frac{c}{l^2} \prod_{z_n \in \sigma_1} \frac{z_n}{z_n(\alpha)} \left( 1 + \frac{z_n(\alpha) - z_n}{z_n - z} \right). \] (4.13)

Substituting \( z = iy \ (y \in \mathbb{R}) \) and passing to the limit as \( y \to \infty \), one obtains
\[ 1 = -\frac{c}{l^2} \prod_{z_n \in \sigma_1} \frac{z_n}{z_n(\alpha)}, \]

hence once finds the number \( c \) via the spectral data
\[ c = -l^2 \prod_{z_n \in \sigma_1} \left( 1 + \frac{z_n(\alpha) - z_n}{z_n} \right). \] (4.14)

Calculating residue at the point \( z = z_n \) in both parts of the equality (4.13), one has
\[ \alpha|v_p|^2 = -\frac{c}{l^2} \frac{z_p(\alpha) - z_n}{z_p(\alpha)} \prod_{z_n \neq z_p} \frac{z_n}{z_n(\alpha)} \left( 1 + \frac{z_n(\alpha) - z_n}{z_n - z_p} \right). \] (4.15)
Theorem 4.2 Using the formulas (4.15), the numbers $\alpha|v_p|^2$, where $c$ is given by (4.14), are found unambiguously from spectra $\sigma(L_0) = \{z_n\}$ and $\sigma(L_\alpha) = \{z_n(\alpha)\}$ of the operators $L_0$ (1.1), (1.2) and $L_\alpha$ (2.1).

Remark 4.1 The number $\alpha$ is defined by the numbers $\alpha|v_p|^2$ (4.15) from the condition $\sum |v_p|^2 = 1$. Thus, $\{\alpha, |v_p|^2\}$ is unambiguously recovered by the spectra $\sigma(L_0)$ and $\sigma(L_\alpha)$.

Relation (1.28) implies that Fourier coefficient equals

$$v_p = \langle v(x), u(0, \lambda_p(0), x) \rangle = \frac{1}{u_p} \{s(-i\lambda_p(0)l)\tilde{v}_s(\lambda_p(0))$$

$$+(1 - c(-i\lambda_p(0)l))\tilde{v}_d(\lambda_p(0))\}$$

where $\tilde{v}_s(\lambda)$ and $\tilde{v}_d(\lambda)$ are given by (2.2). For the function $g(x) = l - x$, Fourier coefficients are calculated explicitly,

$$g_p = \langle g(x), u(0, \lambda_p(0), x) \rangle = \frac{il}{\lambda_p(0)}(c(-i\lambda_p(0)) - 1).$$

Theorem 4.3 The number $\alpha$ and the function $v(x)$ ($\|v\| = 1$) are unambiguously recovered from the spectra $\sigma(L_0), \sigma(L_\alpha(v)), \sigma(L_\alpha(v + g)), \sigma(L_\alpha(v + ig))$ where $L_\alpha(v)$ is given by (2.1) and $g(x) = l - x$.

Proof. Theorem 4.2 implies that numbers $\alpha|v_p|^2$ are unambiguously defined by $\sigma(L_0)$ and $\sigma(L_\alpha(v))$. Exactly in the same way from $\sigma(L_0)$ and $\sigma(L_\alpha(v + g))$ one finds

$$\alpha|v_p + g_p|^2 = \alpha|v_p|^2 + 2\text{Re}(\alpha v_p g_p) + |g_p|^2$$

where $g_p$ are given by (4.17). Hence, the numbers $\text{Re}(\alpha v_p g_p)$ are unambiguously calculated by the three spectra $\sigma(L_0), \sigma(L_\alpha(v)), \sigma(L_\alpha(v + g))$. Similarly, from $\sigma(L_0), \sigma(L_\alpha(v)), \sigma(L_\alpha(v + ig))$ $\text{Im}(\alpha v_p g_p)$ are defined. Thus, $\alpha v_p g_p$, and so $\alpha v_p$ also, are unambiguously calculated from the four spectra $\sigma(L_0), \sigma(L_\alpha(v)), \sigma(L_\alpha(v + g)), \sigma(L_\alpha(v + ig))$. Finally, numbers $\alpha$ and $v_p$ are found from the normalization condition $\sum |v_p|^2 = 1$. Thereafter, the function $v(x)$ is defined by its Fourier series,

$$v(x) = \sum_p v_p u(0, \lambda_p(0), x).$$

4.3. Proceed to description of the data of inverse problem. Denote by $\sigma_1$ a subset in $\sigma(L_0) = \{z_n(0, \varepsilon) = \varepsilon \lambda_n^3(0) : \Delta(0, \lambda_n(0)) = 0, \varepsilon = \pm 1\}$,

$$\sigma_1 \overset{\text{def}}{=} \{z_n^1(0, \varepsilon) : z_n^1(0, \varepsilon) \in \sigma(L_0)\} \quad (4.18)$$
where the numbers \( z_n^1(0, \varepsilon) \) are enumerated in ascending order (the case of \( \sigma_1 = \sigma(L_0) \) is not excluded). And let
\[
\sigma_0 \overset{\text{def}}{=} \sigma(L_0) \setminus \sigma_1 = \{ z_n^0(0, \varepsilon) : z_n^0(0, \varepsilon) = \sigma(L_0) \}. \tag{4.19}
\]
Consider the set
\[
\sigma(\mu) \overset{\text{def}}{=} \{ \mu_n \in \mathbb{R} : n \in \mathbb{N} \} \tag{4.20}
\]
assuming that sets \( \sigma(\mu) \) and \( \sigma_1 \) interlace, moreover, \( \mu_k \) are enumerated in ascending order, \( \mu_k \neq \mu_s \) \((k \neq s)\), \( \mu_1 \neq 0 \) \((\sigma(\mu) \cap \sigma(1) = \emptyset)\).

**Condition 1** The intersection \( \sigma(\mu) \cap \sigma_0 \) is finite.

Since the number \( c \) \((4.14)\) is finite, then one arrives at the second condition.

**Condition 2** The sets \( \sigma_1 \) and \( \sigma(\mu) \) are such that the series
\[
\sum_n \ln \left( 1 + \frac{\mu_n - z_n^1(0, \varepsilon)}{z_n^1(0, \varepsilon)} \right) < \infty \tag{4.21}
\]
which is equivalent to the convergence of the product
\[
\prod_n \left( 1 + \frac{\mu_n - z_n^1(0)}{z_n^1} \right). \tag{4.22}
\]

Numbers \( v_p \in l^2(\mathbb{N}) \), therefore the following requirement is natural, due to \((4.15)\).

**Condition 3** The numerical series
\[
\sum_p \frac{z_p^1(0, \varepsilon)}{\mu_p} (\mu_p - z_p^1(0, \varepsilon)) \left[ \prod_{n \neq p} \frac{z_n^1(0, \varepsilon)}{\mu_n} \left( 1 + \frac{\mu_n - z_n^1(0, \varepsilon)}{z_n^1(0) - z_p^1(0, \varepsilon)} \right) \right] < \infty \tag{4.23}
\]
converges.

Form the infinite product
\[
b_{\sigma_1}(z) \overset{\text{def}}{=} \prod_{z_n^1 \in \sigma_1} \left( 1 - \frac{z}{z_n^1(0, \varepsilon)} \right) \tag{4.24}
\]
converging in any circle \( C_R = \{ z \in \mathbb{C} : |z| < R \} \) due to \( z_n \in \sigma(L_0) \). Similarly, set
\[
b_{\sigma(\mu)}(z) \overset{\text{def}}{=} \prod_{z_n^1 \in \sigma_1} \left( 1 - \frac{z}{\mu_n} \right). \tag{4.25}
\]
Consider a meromorphic function
\[ Q(z) = k \frac{b_{\sigma(\mu)}(z)}{b_{\sigma_1}(z)} \quad (k \in \mathbb{R}). \]
Residue of \( Q(z) \) at the point \( z^1_p(0, \varepsilon) \) equals

\[ c_p = \lim_{z \to z^1_p(0)} (z^1_p(0, \varepsilon) - z) Q(z) = k z^1_p(0) \frac{b_{\sigma(\mu)}(z^1_n(0, \varepsilon))}{\prod_{n \neq p} \left(1 - \frac{z^1_p(0, \varepsilon)}{z^1_n(0, \varepsilon)}\right)}. \tag{4.26} \]

**Remark 4.2** Interlacing of sequences \( z^1_n(0) \) and \( \mu_n \) implies that all the numbers \( c_p \) are of the same sign, \( \text{sign} c_p = \text{sign} c_1 \ (\forall p) \).

If Condition 3 is met, then defining the number
\[ \frac{1}{\alpha} \stackrel{\text{def}}{=} \sum_p z^1_p(0, \varepsilon) \frac{\mu_p - z^1_p(0, \varepsilon)}{\mu_p} \prod_{n \neq p} z^1_n(0, \varepsilon) \left(1 + \frac{\mu_n - z^1_n(0, \varepsilon)}{z^1_n(0) - z^1_p(0, \varepsilon)}\right) \]
one obtains that the numbers
\[ |v_p|^2 \stackrel{\text{def}}{=} \frac{c_p}{\alpha} \quad (p \in \mathbb{N}) \]
are positive and \( \sum |v_p|^2 = 1 \).

**Theorem 4.4** Let \( \sigma \) be a countable set on \( \mathbb{R} \) and \( \sigma = \sigma_0 \cup \sigma(\mu) \) where \( \sigma_0 \subset \sigma(L_0) \) \( (4.19) \) \( (\sigma(L_0) = \{z_n(0, \varepsilon) = \varepsilon \lambda^3_n(0) : \Delta(0, \lambda_n(0)) = 0, \varepsilon = \pm 1\} \) and \( \sigma(\mu) \) \( (4.20) \) consists of pairwise different points \( \mu_n \) \( (\mu_1 \neq 0) \), \( n \in \mathbb{N} \), enumerated in ascending order and interlacing with \( z^1_n(0, \varepsilon) \in \sigma_1 = \sigma(L_0)\setminus\sigma_0 \) and \( \sigma_1 \cap \sigma(\mu) = \emptyset \).

In order that \( \sigma \) coincide with the spectrum \( \sigma(L_\alpha) \) of the operator \( L_\alpha \) \( (2.1) \) it is necessary and sufficient that
(a) \( \sigma(\mu) \cap \sigma_0 \) is finite;
(b) the series
\[
\sum \ln \left(1 + \frac{\mu_n - z^1_n(0, \varepsilon)}{z^1_n(0, \varepsilon)}\right) < \infty; \\
\sum \frac{z^1_p(0, \varepsilon)}{\mu_p} (\mu_p - z^1_p(0, \varepsilon)) \prod_{n \neq p} \left(1 - \frac{\mu_n - z^1_n(0, \varepsilon)}{z^1_n(0) - z^1_p(0, \varepsilon)}\right) < \infty \tag{4.27} \]
converge.

Description of the spectrum \( \sigma(L_\alpha) \) of the operator \( L_\alpha \) \( (2.1) \) one can give in terms of characterization of the class of functions \( \Delta(\alpha, \lambda) \) that are characteristic for \( L_\alpha \) \( [7] \).
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