A generalization of Fueter’s theorem

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September 21, 2018

Abstract

We show that Fueter’s theorem holds for a more general class of quaternionic functions than those constructed by the Fueter’s method.

Description of result

Let $f(z)$ be a holomorphic function defined on the upper complex plane. If we write $z = x + iy$ and $f(z) = u(z) + iv(z)$ we can construct a quaternionic function $f(p) = u(t, r) + iv(t, r)$ where $p = t + xi + yj + zk$, $r = \sqrt{x^2 + y^2 + z^2}$ and

$$
\tau = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}} \tag{1}
$$

If we denote $D_l$ and $D_r$ the left and right Fueter (or Dirac) operator then Fueter’s theorem asserts that

$$
D_l \Delta f = D_r \Delta f = 0 \tag{2}
$$

such functions are never regular unless they are real constants, instead they satisfy:

$$
D_l(f) = D_r(f) = \frac{-2v(t, r)}{r} \tag{3}
$$

One can easily prove Fueter’s theorem observing that:

$$
-\frac{1}{2} \Delta(f) = D_l\left(\frac{v}{r}\right) = -\frac{\tau f}{r} \frac{\partial}{\partial t} + \frac{\tau}{r^2} v(t, r) \tag{4}
$$

which is an axial symmetric function that can be written in the form $\tilde{f} = \tilde{u} + i\tilde{v}$ and that such functions are regular if and only if they satisfy

$$
\left(\frac{\partial}{\partial t} + \frac{\tau}{r^2}\right)\tilde{f} = \frac{2\tilde{v}}{r} \tag{5}
$$

Our result points out that if a quaternionic function of Class $C^2$ can be written as $f(p) = u(p) + iv(p)$ and satisfy

$$
D_l(f) = \frac{-2v}{r} \tag{6}
$$

then its laplacian will be left- and right-regular. Note that we are dropping the condition that the function is obtained by the Fueter’s method and so the resulting functions need not be axial symmetric.
II. Preliminaries

Let \( f(p) \) be a quaternionic, \( C^2 \) function that satisfy \( f(p)p = pf(p) \). Then there exists real functions \( u, v \) such that

\[
f(p) = u(p) + iv(p) \tag{7}
\]

Notice that complex functions obtained by the Fueter’s method are of this form. We write the quaternion \( p \) as

\[
p = t + ri \tag{8}
\]

And we parametrize \( i \) by spherical coordinates.

\[
i = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta) \tag{9}
\]

Let \( D_l \) denote the left-Fueter operator, so

\[
D_l := \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \tag{10}
\]

And we are interested in functions of the form (7) that satisfy

\[
D_l f = -\frac{2v}{r} \tag{11}
\]

The Fueter operator in \((t, r, \alpha, \beta)\) coordinates is written as

\[
D_l = \frac{\partial}{\partial t} + i \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \alpha} \tag{12}
\]

where the symbol divided by \( r \) is defined as:

\[
\frac{\partial}{\partial \alpha} := (r^{-1}) \frac{\partial}{\partial \alpha} + (r^{-1}) \frac{\partial}{\partial \beta} \tag{13}
\]

and \( \iota_\alpha \) and \( \iota_\beta \) denote the partial derivative of \( i \) respect of \( \alpha, \beta \).

As a function that satisfy (11) is holomorphic on the variables \( t, r \) the we conclude that it must hold

\[
\frac{\partial f}{\partial \iota_l} = 2v \tag{14}
\]

which occurs if and only if \( u \) and \( v \) satisfy the following Cauchy-Riemann type system:

\[
\frac{1}{\sin \beta} \frac{\partial u}{\partial \alpha} = \frac{\partial v}{\partial \beta} \tag{15}
\]

\[
\frac{1}{\sin \beta} \frac{\partial v}{\partial \alpha} = -\frac{\partial u}{\partial \beta} \tag{16}
\]
II. Proof of result

We calculate explicitly that if $f = u + \iota v$ satisfy (11) then:

$$\nabla_v v = 0$$  \hspace{1cm} (17)

Observe that

$$D_l\left(\frac{v}{r}\right) = -\frac{\iota}{r} \frac{\partial f}{\partial t} + \frac{\iota}{r^2}v + \frac{1}{r^2} \frac{\partial v}{\partial l}$$  \hspace{1cm} (18)

Before proceeding, we will remark some properties that will be helpful.

The first remark is that if $f$ satisfy (11) then $\frac{\partial f}{\partial t}$ will also satify (11). The second remark is to note that, for all functions $f$

$$\frac{\partial}{\partial l} (\iota f) = 2f - \frac{\partial f}{\partial l}$$  \hspace{1cm} (19)

The third remark is that if $f = u + \iota v$

$$\frac{\partial f}{\partial l} = 2v$$  \hspace{1cm} (20)

is equivalent to

$$\frac{\partial u}{\partial l} = \frac{\partial v}{\partial l}$$  \hspace{1cm} (21)

The fourth remark is that if

$$\frac{\partial f}{\partial l} = 2v$$  \hspace{1cm} (22)

then

$$\frac{\partial}{\partial l} (\iota f) = 2u$$  \hspace{1cm} (23)

Now, for the sake of readability we will apply $D_l$ only to the first summand in (18) and we have:

$$D_l\left(-\frac{\iota}{r^2} \frac{\partial f}{\partial t}\right) = \frac{1}{r^2} \left( -\frac{\partial f}{\partial t} + \frac{\partial}{\partial l} (\iota \frac{\partial f}{\partial t}) \right) = \frac{1}{r^2} \left( -\frac{\partial f}{\partial t} + \frac{2}{r^2} \frac{\partial u}{\partial l} \right)$$  \hspace{1cm} (24)

$$= \frac{1}{r^2} \left( \frac{\partial u}{\partial l} - \iota \frac{\partial v}{\partial l} \right)$$  \hspace{1cm} (25)

We apply $D_l$ to the second summand in (18) and we have:

$$D_l\left(\frac{\iota}{r^2} v\right) = \frac{\iota}{r^2} \frac{\partial v}{\partial l} + \iota \left( -\frac{2}{r^3} v + \frac{\iota}{r^2} \frac{\partial v}{\partial r} \right) + \frac{1}{r^3} \frac{\partial}{\partial l} (\iota v)$$  \hspace{1cm} (26)

$$= \frac{1}{r^2} \left( \frac{\partial v}{\partial l} - \frac{\partial v}{\partial r} \right) + \frac{1}{r^3} \frac{\partial v}{\partial l}$$  \hspace{1cm} (27)

We apply $D_l$ to the third summand in (18) and we have:

$$D_l\left(\frac{1}{r^2} \frac{\partial v}{\partial l}\right) = \frac{1}{r^2} \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial l} \right) + \iota \left( -\frac{2}{r^3} \frac{\partial v}{\partial l} + \frac{1}{r^2} \frac{\partial v}{\partial r} \right) + \frac{1}{r^3} \frac{\partial^2 v}{\partial l^2}$$  \hspace{1cm} (28)

$$= \frac{1}{r^2} \frac{\partial}{\partial l} \left( \frac{\partial v}{\partial l} \right) + \frac{\partial}{\partial l} \left( \frac{\partial v}{\partial l} \right) + \frac{1}{r^3} \left( -\frac{2}{\partial l} \frac{\partial v}{\partial l} - \frac{\partial^2 v}{\partial l^2} \right)$$  \hspace{1cm} (29)
Now observe that in the first summand in (29)
\[
\frac{\partial}{\partial t} \frac{\partial v}{\partial r} + \frac{\partial}{\partial r} \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial r} + \lambda \frac{\partial v}{\partial l} \right) = 0
\]
(30)
because of the third remark. We also observe that the sum of the first two summands in (25) and (27) is zero because \( f \) is holomorphic in \( t, r \). So, for now we have established that
\[
\nabla^2 v = \frac{1}{r^3} \left( \frac{\partial v}{\partial l} + \lambda \frac{\partial v}{\partial l} \right) = \frac{1}{r^3} \left( \frac{\partial v}{\partial l} + \lambda \frac{\partial v}{\partial l} \right)
\]
(31)
so we now must show that
\[
\frac{\partial^2 v}{\partial l^2} = -\theta \frac{\partial v}{\partial l} - \frac{1}{\sin^2 \beta} \frac{\partial^2 v}{\partial \beta^2} - \cot \beta \frac{\partial v}{\partial \beta}
\]
(43)
So in order for our result to hold, we must have

\[
\frac{1}{\sin^2 \beta} \frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial^2 v}{\partial \beta^2} = -\cot \beta \frac{\partial v}{\partial \beta}
\]  

(44)

but this can be easily deduced from (15) and (16) together. We have determined that if a function \( f \) is such that \( u, v \) are \( C^2 \) and satisfy (6) then its laplacian will be left regular, we now prove that its laplacian will be right-regular also. Fortunately, the hard word is already done and so this is almost immediate. First we observe that the operators \( D_l, D_r, \bar{D}_l \) and \( \bar{D}_r \) all commute with each other. It is also true that for any scalar function, say, \( g \) we have:

\[
D_l g = D_r g
\]  

(45)

Now we use the fact that the right hand side of (6) is an scalar function. We already know that

\[
0 = D_l \bar{D}_l D_l f = D_l \bar{D}_l (-2\frac{v}{r}) = \bar{D}_l D_l (-2\frac{v}{r}) = \bar{D}_l D_r (-2\frac{v}{r})
\]  

(46)

\[
= D_r \bar{D}_l (-2\frac{v}{r}) = D_r \bar{D}_l D_l f
\]  

(47)

And so have we proved our

**Theorem 1** Let \( f \) be a quaternionic, \( C^2 \) function that can be written as \( f = u + \iota v \), where \( u, v \) are real functions and that satisfy

\[
D_l f = -\frac{2v}{r}
\]  

(48)

Then it holds that

\[
D_l \Delta f = D_r \Delta f = 0
\]  

(49)

**Remark** The author suspects that the condition \( C^2 \) can be relaxed to be \( C^1 \), and that the \( C^2 \) can be deduced from the holomorphism in \( t, r \). But he doesn’t have a proof yet.