Shannon capacity of nonlinear regenerative channels

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We compute Shannon capacity of nonlinear channels with regenerative elements. Conditions are found under which capacity of such nonlinear channels is higher than the Shannon capacity of the classical linear additive white Gaussian noise channel. We develop a general scheme for designing the proposed channels and apply it to the particular nonlinear sine-mapping. The upper bound for regeneration efficiency is found and the asymptotic behavior of the capacity in the saturation regime is derived.

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Information theory provides underlying concepts for major areas of science and technology from communications and computer science to biology and economics. The seminal Shannon’s result for capacity of the linear additive white Gaussian noise (AWGN) channel: $C = B \log_2(1 + S/N)$ is arguably one of the few most known equations in the history of science. In practical communication channels that are close to the linear AWGN the Shannon capacity can be approached very closely by using low density parity check codes and turbo codes. The linear AGWN channel is nowadays a textbook material and is virtually wholly understood. There is, however, a growing interest in studies of a capacity of more complex communication channels including nonlinear channels for which the limits have yet to be defined. A well known and extremely important for practical applications example is the optical fibre channel that is inherently nonlinear due to intensity-dependent refractive index in silica at high enough signal powers and very small fibre core where signal is guided over long distances.

The definition of the Shannon capacity for arbitrary channel (in what follows, capacity $C$ is per unit bandwidth) involves maximizing the mutual information functional [1]:

$$C = \max_{P(x)} \int \int P(x)P(y|x) \log_2 \frac{P(y|x)}{\int P(x)P(y|x)} \, dx \, dy,$$

over all valid input probability distributions $P(x)$ subject to power constraint $\int P(x) |x|^2 \leq S$. Here statistical properties of the channel are given by the conditional input-output probability density function (PDF) $P(y|x)$. Thus, from the view point of the information theory there is not much difference between linear or nonlinear channels as long as PDF $P(y|x)$ is known. Yet, the modern information theory is mostly developed for linear channels, simply because it is often technically challenging to derive $P(y|x)$ for practical nonlinear channels. This reflects both difficulty of analysis of nonlinear systems with noise and the fact that there are varieties of nonlinear communication channels that hardly can be described by any single generic theory. An important new feature introduced by nonlinearity is the possibility of nonlinear filtering and/or signal regeneration. Whenever the nonlinear transformation has multiple fixed points, the consequent interleaving of the accumulating noise with nonlinear filter will produce effective suppression of the noise. In other words the nonlinear filter will attract a signal to the closest fixed points and suppress the effective signal diffusion caused by the noise. Similar idea has been discussed in multiple contexts from physical systems to the interpretation of biological memory effects in terms of potentials with multiple minima [2]. In quantum theory a qualitatively similar phenomena is known under the name of the Zeno effect where continuous measurement of the quantum system and associated von Neumann collapse of wave function prevents the natural dispersion of the wave function and causes the quantum system to remain in the same state [3]–[8]. Capacity of nonlinear transmission channels with "hard decision" regenerators have been examined in [7].

In this Letter we introduce new method – regenerative mapping for designing nonlinear information channels with capacity exceeding Shannon capacity for linear AWGN channel. We start from substantial expansion of analysis presented in [7] and comprehensively quantify improvement in the Shannon capacity that various nonlinear channels with regenerators can provide over the linear AWGN channel. Next we introduce new type of channels with smooth nonlinear transfer functions that are principally distinct from a "hard decision" regeneration considered in [7]. We stress that the proposed channel model is fundamentally different from Decode-and-Forward channel model and does not assume any decoder/encoder pair to be used with in-line elements. The proposed regenerative element acts like nonlinear filter on the stochastic signal distortions through an effective periodic potential creating attraction regions in the signal mapping.

Consider the nonlinear regenerative channel model with $R$ identical nonlinear filters placed in the transmission line. The signal transmission is distorted by stochastic process which is modeled as AWGN uniformly distributed along the line. The noise term incorporates the stochastic effects from different sources. Depending on the model application, the stochastic process can be considered as an analogue of the random force in the time-continuous case or noise that adds up to signal transmission through the media. This term also includes an
additive noise that is originated from by the nonlinear device itself. Here we develop the general mathematical method of constructing and optimizing the set of transfer functions $y = T(x)$ (see e.g. Fig. 1) that can increase nonlinear transmission system capacity beyond the capacity of the linear AWGN channel.

Shannon capacity (Eq. 1) of the considered systems is a function of signal to noise ratio (SNR), the number of nonlinear filters $R$ and the parameters of nonlinear mapping. SNR is defined here as the ratio of the input signal power $S$ to the noise added linearly to the signal during transmission at each node $k = 1, ..., R$:

$$SNR = \frac{S}{N}, \quad N = \sum_{k=0}^{R} N_k$$

Note that in the nonlinear communication system due to the mixing of signal with noise during propagation the definition of in-line SNR is a nontrivial issue. The introduced SNR has the meaning of the signal to noise ratio in the respective linear system in the absence of nonlinear in-line elements. This allows us to compare performance of the considered system with the corresponding linear AWGN channel having the same noise level. Evidently, that the effect of noise squeezing is enhanced with the number of regenerators/nonlinear filters. Therefore, to evaluate the system performance improvement one need to take into account the accumulative effect of nonlinear signal and noise transformation along the line. To quantify the overall effect, we need to study capacity of the source-destination transmission as a function of signal power ratio to the sum of power of all added noise at the source-destination link.

We start the capacity analysis with the system of the ideal regenerators that is used further as a reference system. In this case, we can derive analytical expressions for the maximum achievable regenerative capacity gain. Next, we introduce the general method of designing and optimization for the class of regenerative channels that guarantee capacity improvement without requirement of decoder/encoder pair. Finally, we illustrate the proposed method by presenting new nonlinear channels with capacity above the Shannon capacity of the linear AWGN channel.

A physical model of regenerative mapping for increasing information capacity can find applications in various fields. The considered model can be applied to the evolution of a stochastic system in time with $k = 0, 1, ..., R$ being a temporal index (see [8], [9]) or to analysis of a spatial dynamics, when signal propagates through the cascaded scheme of regenerative filters. The latter model, for instance, can be applied to signal propagation and regeneration in fiber-optic communication (see e.g. [10]-[16] and references therein). The ring packing regeneration was recently demonstrated by phase sensitive amplification [17]. We would like to emphasise in this Letter generic features of the model, important for both spatial and temporal applications.

We start the analysis with the ideal regenerators that assign each transmitted symbol to the closest element of the given alphabet (in Fig. 1 the corresponding stepwise transfer function is plotted by the dashed blue line). The step function defines a maximum regeneration capability and, consequently, determines the maximum achievable capacity gain due to nonlinear filtering. The conditional pdf of such a system is defined through matrix elements $T$:

$$P(y = x_k|x_l) = \int_{S_k} dx' P_C(x'|x_l) = W_{kl}, \quad (2)$$

where transition matrix is defined as follows

$$W_{kl} = \frac{1}{2} \left( \text{erf}(\Delta_{kl}^+) - \text{erf}(\Delta_{kl}^-) \right), \quad \Delta_{kl}^\pm = (x_k + x_{k \pm 1} - 2x_l) \sqrt{\frac{R}{8N}},$$

Due to Markovian nature of the stochastic system, the overall transition matrix after $R$ regenerative segments reads as $W^R$.

First, consider the rectangular constellations (also known as quadrature amplitude modulation). At low SNR range the channel is binary in each of $n$ dimension.
Therefore, capacity is well approximated by the following expression:

\[ C_R \approx C_{R \text{opt}} = n(1 + m_+ \log_2 m_+) + m_+ \log_2 m_+ \] (3)

with the transition matrix elements (denote SNR as \( \rho \)):

\[ m_+ = m_- = \frac{1}{2R} \left( 1 + \text{erf} \left( \frac{R \rho}{2} \right) \right)^R, \quad m_+ = 1 - m_- \]

As SNR increases, the closest neighbors distance reaches the optimal cell size which is defined by the noise variance and the number of in-line regenerators, \( d_{\text{opt}}^2 = 16N/R \Omega[e^2 R^2/(16\pi)] \), where \( \Omega \) is the so-called Lambert function. Thus, at high SNR the system is characterized by the optimal decision boundaries that are sufficiently large compared with the noise variance to suppress noise effectively. Therefore, with the growing signal power the amplitude distribution \( x_i \) remains constant (i.e. equidistant with the closest neighbors distance \( d_{\text{opt}} \)), whereas the maximum entropy principle defines Maxwell-Boltzmann distribution as the optimal pdf for a fixed average energy constraint. Thereafter, the output pdf can be well approximated as \( q_i = \nu e^{-x_i^2} \), here constants are chosen to satisfy conditions \( \sum_{i=1}^{M} q_i = 1 \) and \( \sum_{i=1}^{M} q_i x_i^2 = S + N/R (\lambda = 1/[2(S + N/R)] \) for the large number of constellation symbols). In the limit of high SNR and/or large number of regenerators, so that \( \Delta = \sqrt{2\Omega[e^2 R^2/(16\pi)]} \gg 1 \), the noise is sufficiently squeezed and the faulty decision occurs only between the nearest neighbors. The symmetry of the modulation format allows us to consider one-dimensional \( M \) points problem, the resulted capacity is multiplied by the number of dimensions \( n \). Finally, the channel capacity in the limit of high SNR for \( M \)-rectangular discrete modulation format is found as

\[ C_{R \text{opt}} = n \sum_{i=1}^{M} q_i \log_2 q_i + nR \frac{e^{-\Delta^2}}{\Delta \sqrt{\pi}} \log_2 \left[ R \frac{e^{-\Delta^2}}{4\Delta \sqrt{\pi}} \right] \] (4)

Hence, with growing SNR one can observe a constant gap (that quantifies improvement) between the regenerative channel and linear AWGN channel capacities. The capacity improvement factor is defined by the noise variance and the number of regenerators:

\[ \Delta C_R = \frac{n}{2} \log_2 \left( \frac{\nu e R}{4\Delta} \right) + nR \frac{e^{-\Delta^2}}{\Delta \sqrt{\pi}} \log_2 \left[ R \frac{e^{-\Delta^2}}{4\Delta \sqrt{\pi}} \right] \] (5)

Thus, the maximum capacity gain due to regeneration (i.e. the maximum regeneration efficiency) is observed for the binary channel. This result emphasizes efficiency of a simple binary channel that might be important for practical design consideration. The minimum SNR value, when \( d_{\text{opt}} \) is achieved, defines the maximum capacity ratio to its linear analogue, i.e.

\[ \text{SNR}_{\text{opt}} = \frac{d_{\text{opt}}^2}{4N} = \frac{4}{R} \Omega \left( \frac{e^2 R^2}{16\pi} \right) \]

Also, at this SNR value two analytic formula Eq. 3 and 4 can be interpolated to describe capacity at the full range of SNR values.

The analytical approximations shown in Fig. 2 by black line demonstrate an excellent agreement with numerics are shown by black (dash-dotted line) and dotted line) lines. The inset shows mutual information gain for \( M^2 \) - rectangular constellation.

FIG. 2: (Color online) Gain, the capacity ratio to the Shannon formula \( C_L = \log_2 (1 + \rho) \), for the different number of regenerators. The analytical results demonstrating an excellent agreement with numerics are shown by black (dash-dotted line) and dotted line) lines. The inset shows mutual information gain for \( M^2 \) - rectangular constellation.

FIG. 3: (Color online) Capacity compared to MI for discrete \( M^2 \) - rectangular (left panel) and \( M \) - points in the ring (right panel) alphabets. The arrows show the constant capacity increase.
(left panel) and ring (right panel) packing (in communications [17], those constellations correspond to the so-called quadrature amplitude modulation and phase-shift-keying modulation formats, respectively). Here the role of optimization is demonstrated: the optimal constellation choice depends on the SNR value and the number of the in-line regenerators. It is important to note, that with the non-optimal format one can get the resulting nonlinear capacity lower than the linear AWGN channel Shannon capacity (due to non-optimal value of the cell size), whereas the correct format choice can give dramatic gain in capacity.

Further, to demonstrate the constructive role of non-linearity without necessarily "hard decision", we propose the nonlinear filtering element that transforms the input x into the output y in the following way: $y = T(x) = x + \alpha \sin(\beta x)$. The model is commonly referred in theoretical physics as a sine-mapping (see [8], [9]). One of the possible ways to achieve sine-mapping is to apply sine-Fourier transformation $F_y[s(x)] = \int_{-\infty}^{\infty} dx \sin(\beta x) s(x)$ to each quadrature $z = (x_R, x_I)$ of the input signal described by the waveform $s = \sum_i \delta(z - z_i) f(t - IT_i)$, here summation is performed over the number of symbols and $T_s$ is a symbol period. Then we add up the original and the transformed signal amplified by $\alpha$. The resulted transformation for each quadrature $z = (x_R, x_I)$ is then given by

$$y = x + \alpha \sin(\beta x)$$

Note that the sine-mapping nonlinear transfer function is the continuous analogue (first approximation) of the Fourier series expansion of TF for the ideal stepwise regenerator.

The signal evolution in such system can be presented by the stochastic map – a discrete version of the Langevin equation for stochastic processes:

$$y_k = T(y_{k-1}) + \eta_k, \quad k = 1, ..., R, \quad y_0 = x + \eta_0, \quad y = y_R$$

Here $k$ is the discrete spatial/temporal index and $T$ is the transfer function of the regenerative filter (see the channel scheme in Fig. 1). The term $\eta_k$ models the Gaussian noise with zero mean and the variance given by $N_k$ added at $k$-th node. The model scheme is shown in Fig. 1. The nonlinear map has a set of special points that are optimal for nonlinear filtering. The sine-defined transformations result in the effective periodic potential which creates attraction regions in the signal mapping without making the hard decision. Whereas, the points are "attracted" to the alphabet, the alphabet should remain stable. This results in the following set of conditions imposed on the transfer function:

$$T[x^*] = x^*, \quad T''[x^*] = 0, \quad |T'[x^*]| < 1 \quad (6)$$

The first expression implies that the alphabet is defined by the stationary points $x^*$ of the mapping. Next, the transfer function should change curvature at the alphabet, i.e. the alphabet point is the inflection point of the transfer function. The third expression reflects stability, so that the signal points distortion is effectively suppressed. When the first derivative is equal to zero, the alphabet is superstable.

In the considered example of sine mapping, the alphabet is placed at the points $\pi(k + 1)/\beta$ where $k \in \mathbb{Z}$ that are stable if and only if $\alpha \beta \leq 1$. Note that, in particular, the system is superstable when $\alpha \beta = 1$.

The conditional pdf for the output at k-th node for each quadrature $y_k$ given the input $y_{k-1}$ is found as

$$P(y_k|y_{k-1}) = \frac{1}{\sqrt{\pi N_k}} \exp\left[-\frac{|y_k - T(y_{k-1})|^2}{N_k}\right]$$

Because of the Markovian property of the process, the conditional pdf of the received signal after propagation through $R$ links, $y_R$ given the input $x$ is expressed by a product of single-step conditional probabilities

$$P(y_R|x) = \int dy_{R-1} ... dy_0 P(y_R|y_{R-1}) ... P(y_0|x) \quad (7)$$

Consequently, when $N_k = N_0 = \text{const}$ the conditional pdf can be expressed through Onsager-Machlup
functional or action of the path given by $S = \Sigma_k (y_k - T(y_{k-1}))^2$ as follows

$$P(y_R|x) = \int \prod_{k=0}^{R-1} dy_k e^{-S/N_0}$$

The numerically calculated capacity is shown in Fig. 4. The proposed sine-mapping nonlinear channel demonstrates visible capacity gain over the Shannon capacity of the linear AWGN channel. Here we demonstrate that though TF with plateau is the most efficient, nevertheless, it is not necessary requirement for regeneration. One can see that suboptimal parameter values also provide a capacity increase. The set of conditions given by Eq. 5 defines optimization and design rules for implementation of such nonlinear regenerative channels.

In the limit of large SNR and/or large number of nonlinear filters all regenerative schemes tend to the asymptotic behaviour, when the gain gap between regenerative and linear AWGN channel capacity is constant. The saturation effect occurs when noise is squeezed to such level that the stochastic distortion is small and the shift takes place within the plateau area. Thus, under such conditions the system with the nonlinear filter is equivalent to the considered above ideal regenerative system. In the limit of small noise the core in Eq. 7 approximates delta-function behavior and, consequently, reproduce the result of Eq. 2. Thus, the capacity gain of any optimized regenerative system considered here tends to the asymptotic value defined by Eq. 2. Using the method of steepest descent we derive the capacity gain for the sine transfer function with the sub-optimal parameters relation (see Eq. 6) $q = \alpha \beta \leq 1$:

$$\lim_{SNR \rightarrow \infty} \Delta C_S = \Delta C_R(R) - \frac{n}{2} \log_2 \left( \frac{1 - (1 - q)^{2(R+1)}}{1 - (1 - q)^2} \right)$$

We develop analytical model that proves the information capacity increase in stochastic systems with regenerative mapping. The gain is achieved by the noise squeezing due to introduced nonlinear filter design that creates attraction regions around the stable alphabet. We presented the design rules for implementation of nonlinear regenerative systems with increased channel capacity. The introduced classes of nonlinear devices can be used for construction of nonlinear communication channels with capacity exceeding the Shannon capacity of the linear AWGN channel. We anticipate that our results will lead to new insights into the Shannon capacity of nonlinear communications channels.

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[1] C. E. Shannon, Bell Syst. Tech. J. 27, 379 (1948); 27, 623 (1948).
[2] D. Angeli, J. F. Ferrell and E.D. Sontag, Proc. Nat. Acad. Sci. USA 101, 1822 (2004).
[3] E. C. G. Sudarshan and B. Misra, J. Math. Phys. 18, 756 (1977).
[4] K. Koshino and A. Shimizu, Phys. Rep. 412, 191 (2005).
[5] A. Degasperis, L. Fonda and G.C. Ghirardi, Nuovo Cimento 21 A, 471 (1974).
[6] K. Yamane, M. Ito, and M. Kitano, Opt. Commun. 192, 299 (2001).
[7] K. S. Turitsyn and S. K. Turitsyn, Opt. Lett. 37, 3600 (2012).
[8] J. R. Crutchfield, M. Nauenberg, and J. Rudnick, Phys. Rev. Lett. 46, 933 (1981).
[9] P. Reimann and P. Talkner, Hel. Phys. Acta 64, 946 (1991).
[10] F. Parmigiani, L. Provost, P. Petropoulos, D.J. Richardson, W. Freude, J. Leuthold, A.D. Ellis, I. Tomkos, IEEE J. Select. Topics in Quantum Electron. 18, 689 (2012).
[11] P. V. Mamysheva, in Proceedings of the European Optical Communications Conference (ECOC), (Optical Society of America, Madrid, Sep. 1998).
[12] M. Vasilyev and T. I. Lakoba, Opt. Lett. 30, 1458 (2005).
[13] T. I. Lakoba and M. Vasilyev, Opt. Exp. 15, 10061 (2007).
[14] O. Leclerc, B. Lavigne, E. Balmefrezol, P. Brindel, L. Pierre, D. Rouvillain, and F. Seguineau, J. Lightwave Technol. 21, 2779 (2003).
[15] J. Kakande, R. Slavik, F. Parmigiani, A. Bogris, D. Syvridis, L. Gruner-Nielsen, D. Phelan, P. Petropoulos, and D. J. Richardson, Nat. Phot. 5, 748 (2011).
[16] R. Slavik, F. Parmigiani, J. Kakande, C. Lundstrom, M. Sjodin, P. A. Andrekson, R. Weerasuriya, S. Sygletos, Andrew D. Ellis, L. Gruner-Nielsen, D. Jakobsen, S. Herstrom, R. Phelan, J. O’Gorman, A. Bogris, D. Syvridis, S. Dasgupta, P. Petropoulos, and D. J. Richardson, Nat. Phot. 4, 690 (2010).
[17] J. G. Proakis, Digital Communications (McGraw- Hill, New York, 2001).