Ils ne savent pas ce qu’ils perdent
Tous ces sacrés cabotins.
Sans le latin, sans le latin,
La messe nous emmerde.

To Seraina and Theres

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A CYCLOTOMIC INVESTIGATION OF THE CATALAN – FERMAT CONJECTURE. DRAFT

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ABSTRACT. With give some new, simple results on the equation \( x^p + y^p = z^q \) using classical methods of cyclotomy.

1. Introduction

Consider the equation

\[ x^p + y^p = z^q, \quad \text{with} \quad x, y, z \in \mathbb{Z}, \quad (x, y, z) = 1 \quad \text{and} \quad p, q \text{ odd primes}. \]  

This is the natural generalization the equations \( x^p + y^p + z^p = 0 \) of Fermat and \( x^p - y^q = 1 \) of Catalan, and we shall denote it by Fermat - Catalan equation. This name is used by some authors for the more general

\[ x^p + y^q = z^r, \]

while others \([Za]\) use the term of super - Fermat equation, for (2). We shall also adopt this terminology in this paper. There has been an increasing literature on the subject in the last decade, yet general results are still scarce. Thus F. Buekers enumerates all the solutions of (2) for \( \chi = 1/p + 1/q + 1/r \geq 1 \), while for \( \chi < 1 \), Darmon and Granville \([DG]\) have proved that there are at most finitely many solutions for a fixed set of exponents. Specific curves are used for fixed triples of exponents by Darmon and coauthors \([Da1], [DM]\), Ellenberg \([E]\), Bruin \([Br]\), Poonen et. al. \([PSS]\). We refer the reader to \([Be]\) for a nice survey of the topic and a comprehensive overview of the cases known up to fall 2004.

A program generalizing the Wiles proof of Fermat’s Last Theorem was proposed by Darmon in \([Da]\); it suggests replacing elliptic curves by hyperelliptic curves or even surfaces, in the general case. A particular case we shall consider is the rational Catalan equation:

\[ X^p + Y^q = 1, \quad \text{with odd primes} \quad p, q \quad \text{and} \quad X, Y \in \mathbb{Q}; \]

it is easily shown (see below) that this is equivalent to the special case \( x^p + y^q = z^{pq} \) of (1). Tijdeman and Shorey emitted in \([ST]\), Chapter XII, the conjecture, that \([3]\) has at most finitely many rational solutions. We shall find conditions on \( (p, q) \) for which that equation has no non-trivial rational solutions at all.

Our purpose in this paper is to investigate (1) separately, using classical cyclotomic approaches. Unsurprisingly, the results obtained are partial, but the set of exponents \( p, q \) for which they are valid are unbounded. The results are ordered in increasing order of conditions on the (odd prime) exponents \( p, q \). Based on a simple relative class number divisibility condition we reduce first (1) to a Fermat - type equation (with only one exponent) over a totally real field:
Theorem 1. Let $p, q$ be odd primes with $p > 3$, $q \nmid h_p$ with $h_p$ the class number of the $p$-th cyclotomic extension - and for which the Fermat - Catalan equation (1) has a non trivial solution. Then the equation

$$aX^q + bY^q + cZ^q = 0 \quad X \in \mathbb{Z}; \quad Y, Z \in \mathbb{Z}[\zeta + \zeta^{-1}]$$

has a non-trivial solution. Here $b, c$ are units, while $a$ is either unit or the principal ideal $(a)$ is a power of the ramified prime ideal above $p$.

After deriving the distinction of the first and second case in (1), and the analogs of the Barlow - Abel formulae [R], we prove a theorem, which allows a useful additional case distinction:

Theorem 2. If the equation (1) has a solution $(x, y, z; p, q)$ and the exponents are such that $\max\{p, p^{(p-20)/16}\} > q$ and $q \nmid h_p$, with $h_p$ the relative class number of the $p$-th cyclotomic extension, then

$$x + f \cdot y \equiv 0 \mod q^2 \quad f \in \{-1, 0, 1\}.$$ 

This leads to a delicate case by case analysis (six cases in total). For five of the six cases we are able to find some simple algebraic conditions, while one of the cases ($z \not\equiv 0 \mod p$ and $x \equiv 0 \mod q^2$) remains unsolved. The main results of this paper are the following:

Theorem 3. Let $p, q$ be odd primes such that $q \nmid h(p, q)$; $p \not\equiv 1 \mod q$ and $\max\{p, h(p-20)/16\} > q$. Then the equation

$$x^p + y^p = z^q, \quad (x, y, z) = 1$$

has no non-trivial solutions for

$$\max\{|x|, |y|\} \geq \frac{1}{2} \left( \frac{1}{p(p-1)} \cdot \left( \frac{q-1}{2} \right)^{p-2} \right)^q .$$

Here $h(p, q)$ is a divisor of the relative class number $h_p^-$ which is explicitly computable by means of generalized Bernoulli numbers and will be defined below. Assuming the stronger condition $q \nmid h_{pq}^-$, the relative class number of the $pq$-th cyclotomic field, we have:

Theorem 4. Let $p, q > 3$ be primes such that (1) has a solution and suppose that $-1 < p \mod q >$, $\max\{p, h(p-20)/16\} > q$ and $q \nmid h_{pq}^-$. Then either

$$a^{q-1} \equiv 1 \mod q^2 \quad \text{for some} \quad a \in \{2, p, 2^{p-1}, p^q\},$$

or

A. $p \nmid z$ and $q^2|xy$ if $q \not\equiv 1 \mod p$ and $q^3|xy$, if $q \equiv 1 \mod p$.

B. If $q \not\equiv 1 \mod p$, then

$$\min\{|x|, |y|\} > c_1(q) \left( \frac{q^{p-1}}{p} \right)^{q-2}, \quad \text{if} \quad q \not\equiv 1 \mod p,$$

and

$$\min\{|x|, |y|\} > c_1(q) \left( \frac{q^{2(p-1)}}{p} \right)^{q-2},$$

otherwise. Here $c_1(q)$ is an effectively computable, strictly increasing function with $c_1(5) > 1/2$. 


An immediate consequence of this Theorem is the following generalization of
Catalan’s conjecture:

Corollary 1. Let $p, q$ be odd primes such that

1. $-1 \not\equiv p \mod q > 0$ and $q \not\equiv h_{pq}$,
2. $\max\{p, \frac{p(p-20)}{16}\} > q$,
3. $a^{q-1} \not\equiv 1 \mod a^2$ for $a \in \{2, p, 2p^{-1} \cdot p^q\}$.

Then the equation

$$X^p + C^q = Z^q$$

has no integer solution for fixed $C$ with $|C| < \frac{1}{2} \cdot \left(\frac{2^{(p-1)}}{p}\right)^{q-2}$. If $q \equiv 1 \mod p$, there are no solutions with $|C| < \frac{1}{2} \cdot \left(\frac{2^{(p-1)}}{p}\right)^{q-2}$.

Proof. The premises allow us to apply Theorem 4 and the claim follows from (8) and (9). □

Finally, by restricting the results above to the rational Catalan equation (3), we are able to give a criterion which only depends on the exponents, namely the following:

Theorem 5. Let $p, q > 3$ be distinct primes for which the following conditions are true:

1. $-1 \not\equiv p \mod q > 0$ and $-1 \not\equiv q \mod p > 0$,
2. $pq \equiv 1 \mod pq$,
3. $2^{q-1} \not\equiv 1 \mod 2^p$ and $2^{q-1} \not\equiv 1 \mod q^2$,
4. $(2^{p-1}p^q)^{q-1} \not\equiv 1 \mod q^2$ and $(2^{q-1}q^p)^{p-1} \not\equiv 1 \mod p^2$,
5. $p^{q-1} \not\equiv 1 \mod q^2$ and $q^{p-1} \not\equiv 1 \mod p^2$,
6. $\max\{p, \frac{p(p-20)}{16}\} > q$ and $\max\{q, \frac{q(q-20)}{16}\} > p$.

Then the equation $X^p + Y^q = 1$ has no rational solutions.

2. Generalities

The following lemma of Euler is used in both equations of Fermat and Catalan:

Lemma 1. Let $x, y$ be coprime integers and $n > 1$ be odd. Then

$$\left(\frac{x^n + y^n}{x + y}, x + y\right) | n.$$ (10)

Proof. Write $x = (x + y) - y$ and develop

$$\frac{x^n + y^n}{x + y} = \frac{(x + y)^n + \sum_{k=1}^{n-2} \binom{n}{k}(x + y)^{n-k} \cdot (-y)^k + n(x + y)y^{n-1} - y^n + y^n}{x + y}$$

$$= K \cdot (x + y) + n \cdot y^{n-1}, \quad \text{with} \quad K \in \mathbb{Z}. \quad (11)$$

The common divisor in (10) is consequently

$$D = \left(\frac{x^n + y^n}{x + y}, x + y\right) = (K \cdot (x + y) + n \cdot y^{n-1}, x + y) = (n \cdot y^{n-1}, x + y).$$

But since $x, y$ are coprime, and consequently also $(x + y, y) = 1$, it follows plainly that $D = (n, x + y)|n$. □
Like in Fermat’s equation, the above lemma leads to the following case distinction for (1):

Case I: The case in which \( p \nmid z \). Then
\[
x^p + y^p = (x + y) \cdot \frac{x^p + y^p}{x + y} = z^q,
\]
and since by Lemma 1 the two factors have no common divisor, they must simultaneously be \( q \)-th powers. Thus
\[
(12) \quad x + y = A^q, \quad \frac{x^p + y^p}{x + y} = B^q \quad \text{with} \quad A, B \in \mathbb{Z},
\]
for this case.

Case II: The case in which \( p \mid z \). Then
\[
x^p + y^p \equiv x + y \equiv z^q \equiv 0 \mod p.
\]
We show that in this case
\[
v_p \left( \frac{x^p + y^p}{x + y} \right) = 1.
\]
The development in (11) yields
\[
\frac{x^p + y^p}{x + y} = p \cdot y^{p-1} + \sum_{k=2}^{p-1} \binom{p}{k} (x + y)^{k-1} \cdot (-y)^{p-k} + (x + y)^{p-1}.
\]
The binomial coefficients in the above sum are all divisible by \( p \). Since \((y, x) = (y, x + y) = 1\), it follows also that \((y, p) = 1\) and thus \( v_p(py^{p-1}) = 1 \). But \( p|x + y \), so all the remaining terms in the expansion of \( \frac{x^p + y^p}{x + y} \) are divisible (at least) by \( p^2 \), which confirms our claim. In the second case thus, a \( q \)-th power of \( p \) is split between the factors \( x + y \) and \( \frac{x^p + y^p}{x + y} \) in such a way that the latter is divisible exactly by \( p \). In the second case we have herewith:
\[
(13) \quad x + y = p^{nq-1} \cdot A^q = (A')^q / p^{e}, \quad \frac{x^p + y^p}{x + y} = p \cdot B^q
\]
with \((A, B, p) = 1\) and \( n = v_p(z) \geq 1 \).

The relations (12) and (13) are the analogs of the Barlow - Abel relations for the Fermat - Catalan equations. Fermat’s equation is homogeneous; thus if \( x^p + y^p + z^p = 0 \) is a solution in which \((x, y, z)\) are not coprime, one may divide by the \( p \)-th power of the common divisor, thus obtaining a solution with coprime \((x, y, z)\). The equation (1) is not homogeneous and thus one may ask whether the requirement that \((x, y, z)\) be coprime is not restrictive.

The following lemma addresses this question and shows that one can construct arbitrary many solutions of \( x^p + y^p = z^q \), if common divisors are allowed. It appears that the condition \((x, y, z) = 1\) is thus plausible.

**Lemma 2.** Let \( x, y \) be coprime integers and \( p, q \) be distinct odd primes. Then there is an integer \( D \in \mathbb{Z} \) such that
\[
(D \cdot x)^p + (D \cdot y)^p = z^q, \quad \text{with} \quad z \in \mathbb{Z}.
\]
Furthermore, every integer solution of \( X^p + Y^p = Z^q \) with \((X, Y, Z) > 1\) arises in this way.

\(^{11}\)I owe David Masser the observation that the condition \((x, y, z) = 1\) is not obvious for the equation (1).
Proof. Let $x, y$ be coprime integers and
\[ x^p + y^p = C \cdot z^q, \]
where $C \in \mathbb{Z}$ and $z$ is the largest $q$–th power dividing the left hand side ($z = 1$ is possible); i.e. $C$ is $q$–th power - free. Let $\ell | C$ be a prime and $n = v_\ell(C)$. We show that there is an integer $D(\ell)$ such that $x' = D(\ell) \cdot x$ and $y' = D(\ell) \cdot y$ verify $x'^p + y'^p = C' \cdot z'^q$ and $z|z', C'|C$ while $(C', \ell) = 1$. Indeed, let $a \in \mathbb{N}$ be such that $n + a \cdot p \equiv 0 \mod q$; such an integer exists since $(p, q) = 1$. We define $D(\ell) = \ell^a$ and $b = (n + ap)/q$. Then
\[ x'^p + y'^p = \ell^ap \cdot C \cdot z'^q = \ell^{ap+n} \cdot C/\ell^a \cdot z'^q = \frac{C}{\ell^a} \cdot (z \cdot \ell^b)^q. \]
Setting $C' = C/\ell^a$ and $z' = z \cdot \ell^b$, the claim follows. By repeating the procedure recursively for all the prime divisors of $C$ one obtains $D = \prod_{\ell|C} D(\ell)$ for which the claim of the Lemma holds.

Conversely, let $x^p + y^p = z^q$ hold for a triple with $(x, y, z) = G$. For each prime $\ell|G$ let $a = v_\ell(x, y)$. Then $ap \leq q \cdot v_\ell(z)$. If $G' = (x, y)$ it follows that $w = z^q/G'^p$ is an integer. The integers $x' = x/G', y' = y/G'$ are coprime; if $C$ is the $q$–th power - free part of $w$ and $z'^q = w/C$, then
\[ (14) \quad x'^p + y'^p = C \cdot z'^q. \]
But then the initial equality can be derived from (14) by the procedure described above. Thus all non trivial solutions of $x^p + y^p = z^q$ have coprime $x, y, z$. \qed

We finally prove the relation between (3) and (1):

Lemma 3. Let $p, q$ be odd primes. The equation (3) has non trivial rational solutions if and only if
\[ (15) \quad x^p + y^q = z^{pq}, \quad (x, y, z) = 1 \quad \text{and} \quad x, y, z \in \mathbb{Z} \]
has non trivial solutions.

Proof. Suppose first that (15) has some non trivial solution $x, y, z$. Then one easily verifies that (3) has the solution $X = x/z^q, Y = y/z^p$.

Conversely, let $X = a/c, Y = b/d$ be a non trivial solution of (3) with $(a, c) = (b, d) = 1; a, b, c, d \in \mathbb{Z}$. Clearing denominators we find
\[ a^p d^q + b^q c^p = c^p d^q. \]
Since $(a, c) = (b, d) = 1$, by comparing the two sides of the identity, we find $c^p | d^q$ and $d^q | c^p$, thus $c^p = d^q$. But $p$ and $q$ are distinct primes, thus for each prime $\ell|c$, we have $pq|v_\ell(c)$ and we may write $c^p = d^q = u^{pq}$. The equation (3) becomes $a^p + b^q = u^{pq}$, as claimed. \qed

3. Cyclotomy and Fermat - Catalan

We start by fixing some notations which shall be used throughout the rest of this paper.
3.1. Notation. We shall let \( p, q \) be two odd primes and \( \zeta, \xi \in \mathbb{C} \) be primitive \( p \)-th and \( q \)-th roots of unity and \( \mathbb{K} = \mathbb{Q}(\zeta), \mathbb{K}' = \mathbb{Q}(\xi) \), \( L = \mathbb{Q}(\zeta, \xi) \) the respective cyclotomic fields. Furthermore, the Galois groups will be

\[
G_p = \text{Gal} \left( \mathbb{K}/\mathbb{Q} \right) = \langle \sigma \rangle, \quad G_q = \text{Gal} \left( \mathbb{K}'/\mathbb{Q} \right) = \langle \tau \rangle \quad \text{and} \quad G = \text{Gal} \left( \mathbb{L}/\mathbb{Q} \right) = \langle \sigma \tau \rangle = \langle \sigma \rangle \times \langle \tau \rangle.
\]

Unless stated otherwise in some particular context, \( \sigma, \tau \) are thus generators of the Galois groups \( G_p, G_q \), respectively. Furthermore, if \( 0 < a < p; 0 < b < q \) we shall use the notation \( \sigma_a \in G_p, \tau_b \in G_q \) for the elements of the Galois groups given by \( \zeta \mapsto \zeta^a \) and \( \xi \mapsto \xi^b \), respectively. The map \( G_p \to \mathbb{Z}/p \mathbb{Z}^* \) given by \( \sigma_a \mapsto a \) will be denoted by \( \bar{\sigma}_a = a \), and likewise for the analog for \( G_q \). Complex conjugation is denoted also by \( \ast \in G \), while \( j_p, j_q \) are the complex conjugation maps of \( G_p, G_q \) respectively, lifted to \( G \). Thus \( j_p \) acts on \( \zeta \) but fixes \( \xi \) and \( j_q \) does the reverse.

Finally, the ramified primes are \( \varrho = (1 - \zeta), q = (1 - \xi) \). We use the notations \( \lambda = (\xi - \overline{\xi}) \) and \( \lambda' = (\zeta - \overline{\zeta}) \) for generators of these ramified primes; this is due to the nice behavior under complex conjugation. In several contexts it will come natural to use the classical \( \lambda = (1 - \xi) \), etc.; this deviation from the general use will be mentioned in place.

We now start with some classical results, adapted to the present equation (1).

3.2. First Consequences of Class Field Theory. We assume \( q \not| h_p \) and deduce some consequences, starting from a presumed non-trivial solution \((x, y, z)\) of (1). With \( e \) defined like above, we shall let

\[
\alpha = \frac{x + \zeta \cdot y}{(1 - \xi)^e} \quad \text{and} \quad \mathfrak{A} = \left( \alpha, \frac{x^p + y^p}{p^e(x + y)} \right) \subset \mathcal{O}(\mathbb{K})^\times = \mathbb{Z}[\zeta]^\times.
\]

Lemma 4. Let \( x, y, z \) be coprime integers verifying (1). Then

\[
(\sigma(\alpha), \sigma'(\alpha)) = 1 \quad \forall \sigma, \sigma' \in G_p, \sigma \neq \sigma'
\]

\[
\mathfrak{A}^q = (\alpha).
\]

Proof. We cumulate the two Cases of the Fermat - Catalan equation in

\[
x + y = A^q/p^e \quad \text{and} \quad \frac{x^p + y^p}{x + y} = p^e \cdot B^q \quad \text{with} \quad e \in \{0, 1\}
\]

Here \( e = 0 \) corresponds to the First Case and \( e = 1 \) to the Second Case. If \( e = 1 \), it is understood that \( p|A \), so the right hand side in (18) is an integer. Then (19) is equivalent to

\[
N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\alpha) = B^q \quad \text{and} \quad \mathfrak{A} = (\alpha, B).
\]

We write \( P = \{1, 2, \ldots, p - 1\} \) and note that, for \( c, d \in P \) we have

\[
\Delta(c, d) = (\sigma_c(\alpha), \sigma_d(\alpha)) = (1)
\]

whenever \( c \neq d \). Indeed

\[
y(\zeta^c - \zeta^d) = (1 - \zeta^c)^e \sigma_c(\alpha) - (1 - \zeta^d)^e \sigma_d(\alpha) \in \Delta(c, d) \quad \text{and} \quad x(\zeta^{-c} - \zeta^{-d}) = \zeta^{c} (1 - \zeta^c)^e \sigma_c(\alpha) - \zeta^{d} (1 - \zeta^d)^e \sigma_d(\alpha) \in \Delta(c, d).
\]
Since \((x, y) = 1\) it follows that \(\Delta(c, d) \supset \wp\), the ramified prime above \(p\) in \(K\). But \(\alpha = (x + \zeta y)/(1 - \zeta)^e = \frac{x+y}{1-\zeta} - y \cdot (1 - \zeta)^{1-e}\). If \(e = 0\), the first term is coprime to \(\wp\) and the second is not. If \(e = 1\), the second term is coprime to \(\wp\) and the first is not. Thus in both cases, \((\alpha, \wp) = 1\) and \(\Delta(c, d) = (1)\), as claimed. The second relation in (17) follows now easily from the definition of \(A\):

\[
A^q/(\alpha) = (\alpha^{q-1}, B \cdot \alpha^{q-2}, \ldots, B^{q-1}, N(\alpha)/\alpha),
\]

and one verifies that the integer ideal on the right hand side is equal to the ideal \((\alpha, N(\alpha)/\alpha) = (1)\).

An immediate consequence is:

**Corollary 2.** If \(p, q\) are odd primes with \(q \not| h_p\), the class number of \(\mathbb{Q}(\zeta_p)\), \(x, y, z\) verify (17) and \(\alpha = (x + \zeta y)/(1 - \zeta)^e\) with \(e\) as above, then

\[
\alpha = \varepsilon \cdot \rho^q \quad \text{for some} \quad \varepsilon \in \mathbb{Z}[\zeta + \overline{\zeta}]^\times, \ \rho \in \mathbb{Z}[\zeta].
\]

and if only \(q \not| h_p^-\) holds, then

\[
\frac{x + \zeta y}{x + \zeta^q y} = \pm \left(\frac{\rho_1}{\rho_2}\right)^q, \quad \text{for some} \quad \rho_i \in \mathbb{Z}[\zeta].
\]

The two algebraic integers \(\rho, \rho_1\) may, but need not be equal.

**Proof.** The statement (22) is a direct consequence of (17), since the second relation implies that \(A\) is principal if \(q \not| h_p\). Since \(A^q = (\alpha)\) by (17), if \(q \not| h_p^-\), then there is a real ideal \(B \subset \mathbb{Z}[\zeta]\) together with an algebraic number \(\nu \in K\) such that \(A = (\nu) \cdot B\). By dividing through the complex conjugate of this identity, one finds

\[
\left(\frac{A}{\overline{A}}\right)^q = (\alpha/\overline{\alpha}) = (\nu/\overline{\nu})^q,
\]

and there is a unit \(\eta\) such that \(\alpha/\overline{\alpha} = \eta(\nu/\overline{\nu})^q\) and thus

\[
\frac{x + \zeta y}{x + \zeta^q y} = \left(\frac{1 - \zeta}{1 - \zeta}\right)^e \cdot \eta \cdot \left(\frac{\nu}{\overline{\nu}}\right)^q = \eta' \cdot \left(\frac{\nu}{\overline{\nu}}\right)^q.
\]

But then \(\eta' \cdot \overline{\nu} = 1\) and Dedekind’s unit Theorem implies that \(\nu'\) is a root of unity of \(K\). Since all roots of unity of this field have order dividing \(2p\), and \((2p, q) = 1\), the statement (23) follows.

We now prove a lemma concerning ideals related to the above \(A\), in a more general setting.

**Lemma 5.** Let \(k \subset L\) be some field such that \(q \not| h(k)\), the class number of the field \(k\). Let \(\phi_i \in O(k), i = 1, 2, \ldots, n\) be such that \((\phi_i, \phi_j) = (1)\) for \(1 \leq i \neq j < n\). Suppose that for some \(m \geq 0\) all \(\phi_i, i = m + 1, m + 2, \ldots, m\) are units, while \(\phi_j, j = 1, 2, \ldots, m\) are units. Furthermore, there is a \(C \in k, (C, pq) = 1\) such that

\[
(24) \quad \prod_{i=1}^n \phi_i = C^{\eta_i}.
\]

Then there are \(\eta_i \in O(k)^\times\) and \(\mu_i \in O(k)\) such that

\[
(25) \quad \phi_i = \eta_i \cdot \mu_i^q \quad \text{for} \quad i = m + 1, m + 2, \ldots, n.
\]
Proof. Let $\mathfrak{A}_i = (\phi_i, C)$ be ideals in $O(k)$. For $i > m$ these ideals are not trivial, while for $i \leq m$ they are equal to $O(k)$. We assume thus $i > m$ and claim that

\begin{equation}
\mathfrak{A}_i^q = (\phi_i).
\end{equation}

Indeed,

\begin{equation}
\mathfrak{A}_i^q / (\phi_i) = \left( \phi_i^q, C \cdot \phi_i^{q-2}, \ldots, C^{q-1}, C^q / \phi_i \right).
\end{equation}

It follows from (24) that the right hand side ideal is integer and $(\phi_i) | \mathfrak{A}_i^q$. On the other hand, since $(\phi_i, \phi_j) = (1)$ for $i \neq j$, we have $(C^q / \phi_i, \phi_i) = (1)$ and thus $\mathfrak{A}_i^q = (\phi_i)$, as claimed. Furthermore, $q \not| \text{h}(k)$ implies that the ideals $\mathfrak{A}_i, i > m$ must be principal. There are $\mu_i \in O(k)$ such that $\mathfrak{A}_i = (\mu_i)$. It follows then from (26) that

\begin{equation}
(\mu_i^q) = \mathfrak{A}_i^q = (\phi_i) \quad \text{and} \quad \phi_i = \eta_i \cdot \mu_i^q \quad \text{for some} \quad \eta_i \in (O(k))^\times.
\end{equation}

This completes the proof of the lemma. \hfill \Box

4. Fermat Equations and Proof of Theorem 1

Suppose that $x, y, z$ is a non trivial solution of (1) and $q \not| \text{h}(\mathcal{K})$. Then (22) holds by Corollary 2. This leads to a reduction of the initial Fermat-Catalan equation (1) to a Fermat-like equation (i.e. involving only one prime exponent) in extension fields. It is likely that this reduction may bring some progress in the general program announced by Darmon [Da] for the solution of (1). Indeed, in this programatic paper, Darmon suggests that in order to solve general cases of (1), "one is naturally led to replace elliptic curves by certain 'hypergeometric Abelian varieties', so named because their periods are related to values of hypergeometric functions". Our result shows however that there is a solid region of the $(p, q)$ plane, in which the simple elliptic curves, albeit defined over totally real (cyclotomic) extensions of $\mathbb{Q}$, can still do the job².

The result in this direction was enounced in Theorem 1 of which we give a proof below.

Proof. We shall treat Case I and Case II separately, using the definitions in (13), (19). Suppose first that $e = 0$ (Case I). Then

$$\alpha \cdot \overline{\alpha} = (x + \zeta y)(x + \overline{\zeta} y) = (x + y)^2 - \mu xy = A^{2q} - \mu xy = \delta \cdot \nu^q,$$

where $\delta = \varepsilon \cdot \tau$, $\nu = \rho \cdot \overline{\tau}$ and $\mu = (1 - \zeta)(1 - \overline{\zeta})$ is the ramified prime above $p$ in $\mathbb{K}^+$. Since $p > 3$ there is at least one non trivial automorphism $\sigma \in G_p$; we may apply this automorphism to the above equation and eliminate $xy$ from the resulting two identities:

$$\sigma(\mu) - \mu \cdot A^q = \sigma(\mu) \cdot \delta \cdot \nu^q - \mu \cdot \sigma(\delta \cdot \nu^q).$$

² I thank Jordan Ellenberg for pointing out to me that it is important to have Fermat equations defined over totally real fields, thus opening a door to the use of Hilbert modular forms. Consequently, the Fermat equations which we deduce here will have this property and be defined over the simplest totally real fields available in the context. It should be mentioned here, that a large variety of Fermat-like equations can be deduced from our results, there are reasons to believe that the ones we display may be the best point of departure for further (non-cyclotomic) investigations.
If \( \sigma(\zeta + \zeta) = \zeta^c + \zeta^c \), we note that
\[
\mu - \sigma(\mu) = (\zeta - \zeta^c) + \zeta - \zeta^c = (1 - \zeta^{c-1}) \cdot (\zeta - \zeta^c) = \delta_1 \mu,
\]
with \( \delta_1 \in \mathbb{Z}[\zeta + \zeta^\infty] \). After division by \( \mu \) in the previous identity, we find there are three units \( \delta_1, \delta_2 = -\delta \cdot \frac{\sigma(\mu)}{\mu}, \delta_3 = \sigma(\delta) \) such that
\[
\delta_1 A^{2q} + \delta_2 \nu^q + \delta_3 \sigma(\nu)^q = 0.
\]
In this Case, (1) holds with \( a, b, c \) being all units.

Suppose now that \( e = 1 \), so \( x + y = p^{q-1} \cdot A^q \) and \( \alpha \cdot (1 - \zeta) = x + \zeta y \). In this case,
\[
\mu \cdot \alpha \cdot \bar{\tau} = (x + \zeta y)(x + \zeta y) = (x + y)^2 - \mu xy = A^{2q} \cdot p^{2(q-1)} - \mu xy = \mu \cdot \delta \cdot \nu^q,
\]
where again \( \delta = \varepsilon \cdot \bar{\tau} \) and \( \nu = p \cdot \bar{\tau} \). We eliminate, like previously, the term in \( xy \), thus obtaining:
\[
(\sigma(\mu) - \mu) \cdot A^{2q} p^{2(q-1)} = (\mu \cdot \sigma(\mu)) \cdot (\delta \cdot \nu^q - \sigma(\delta \cdot \nu^q)),
\]
and with the same \( \delta_1 \) as above, upon division by \( \mu \cdot \sigma(\mu) \),
\[
-\delta_1 A^{2q} \cdot \frac{p^{2(q-1)}}{\sigma(\mu)} = \delta \cdot \nu^q - \sigma(\delta \cdot \nu^q).
\]
In this Case, (1) holds with \( b, c \) being units, while \( a = \delta_1 \cdot \frac{p^{2(q-1)}}{\sigma(\mu)} \), so \( (a) \) is a power of the ramified prime above \( p \). This completes the proof of the Theorem. \( \square \)

In the Second Case, one may wish a Fermat equation with all - units coefficients. This can be achieved at the cost of imposing \( p \geq 7 \) and the fact that all three unknowns will be non - rational. With this one has the following

**Proposition 1.** In the premises of Theorem 1 and assuming that \( e = 1 \) (the Second Case, thus), let \( p \geq 7 \) and \( K = \mathbb{Q}(\zeta_p), A = \mathcal{O}(K) \). Then for any \( \sigma \in Gal \, (K/\mathbb{Q}) \), there are three units \( \varepsilon_j \in A \) and \( a, \nu \in \mathfrak{A} \), such that the equation:
\[
(28) \quad \varepsilon_1 \cdot X^q + \varepsilon_2 Y^q + \varepsilon_3 Z^q = 0
\]
has the solution \( (X, Y, Z) = (\nu, \sigma(\nu), \sigma^2(\nu)) \in A^3 \).

**Proof.** We start form the identity
\[
\delta_1 A^{2q} \cdot \frac{p^{2(q-1)}}{\sigma(\mu)} = \delta \cdot \nu^q - \sigma(\delta \cdot \nu^q)
\]
derived above for this Case, in the proof of the Theorem 1. We shall need precise information about the units, and thus trace them back in the proof. Let \( \sigma \) be fixed and \( \chi = \mu^{q-1} \); then \( \delta_1 = \chi - 1 \) is also a unit. The unit \( \delta \) is fixed by its \( q \) -adic expansion, but we shall not require more detail here.

Now apply \( \sigma \) to the previous identity and use the definition of \( \chi \):
\[
\frac{(\chi - 1)^{-1} \cdot (\delta \cdot \nu^q - \sigma(\delta \cdot \nu^q))}{C/\sigma(\mu)} = C/\sigma(\mu),
\]
\[
\sigma \left( \frac{(\chi - 1)^{-1} \cdot (\delta \cdot \nu^q - \sigma(\delta \cdot \nu^q))}{C/\sigma(\mu)} \right) = C/\sigma(\mu) \times \sigma(\chi^{-1}),
\]
and notice the crucial identity among units:
\[
(29) \quad \frac{1}{\chi - 1} - \sigma \left( \frac{\chi}{\chi - 1} \right) = \frac{1}{\sigma(\chi^{-1}) - 1} \in \mathfrak{A}^\times =: \Delta \in \mathfrak{A}^\times.
\]
Thus, a linear combination of the previous equations yields:
\[ \varepsilon_1 \nu^q + \varepsilon_2 \sigma(\nu)_q + \varepsilon_3 \sigma^2(\nu)_q = 0, \]
with the units:
\[ \varepsilon_1 = \frac{\delta}{\chi - 1}, \quad \varepsilon_2 = \frac{\sigma(\chi \cdot \delta)}{\sigma(\chi) - 1}, \quad \varepsilon_3 = \sigma \left( \frac{\sigma(\delta)}{\chi - 1} \right). \]
This completes the proof. \( \square \)

**Remark 1.** One may use above two distinct Galois actions \( \sigma, \tau \) rather than just one and its square. The corresponding linear combination of units in (29) remains a unit and one thus obtains a more general Fermat equation with conjugate solutions.

4.1. **The Case** \( p = 3 \). According to Beukers [Be], this case has been solved for \( n = 4, 5, 17 \leq n \leq 10000 \) by N. Bruin [Br2] and A. Kraus [Kr], respectively (note that here, composite exponents are taken into consideration too). Since this leaves the general case open, it may be interesting to deduce the associated Fermat equations. They are given by the following:

**Proposition 2.** Let \( q > 3 \) be a prime for which the equation \( x^3 + y^3 = z^q \) has non-trivial coprime solutions in the integers. If \( K = \mathbb{Q}(\sqrt[-3]{3}) \) is the third cyclotomic field, \( E \subset K \) are the Eisenstein integers and \( \rho \in E \) is a third root of unity, then there is a \( \beta \in E \) such that one of the following alternatives hold:

(30) \[ \beta^q + \overline{\beta}^q = A^q, \]
(31) \[ 3(\rho - \rho^2) \cdot (\beta^q - \overline{\beta}^q) = A^q, \]
where \( A \in \mathbb{Z} \).

**Proof.** The alternative above corresponds to the two cases of (11). In the first case, \( x + y = A^q \) and \( \mathfrak{A} = (x + \rho y, z) \) is a principal ideal. Since all the units of \( E \) are (sixth) roots of unity, and thus \( q \)-th powers, it follows that there is a \( \beta \in E \) such that \( \beta^q = px + \overline{\rho}y \). Then \( \beta^q + \overline{\beta}^q = (\rho + \overline{\rho})(x + y) = -A^q \), which proves (30). In the second case \( x + y = A^q/3 \) and (31) follows by a similar computation, with details left to the reader. \( \square \)

It is also useful to know that the case \( q|z \) can be ruled out in both (30) and (31) by using a generalized form of Kummer descent. We shall give details for \( p > 3 \) in a later section, leaving this case as an open remark for the reader.

5. **Consequences of Class Field Theory**

In this section we shall deduce some consequences of class number conditions in the cyclotomic fields of our interest. Of most value for our investigation, these conditions give some control on local properties of units of the \( pq \)-th field and its subfields.

5.1. **Primary Numbers and Reflection.** We shall be interested in the sequel in the algebraic integers of the field \( L = \mathbb{Q}(\zeta, \xi) \) and its subfields. Let \( R \) be one of the rings of integers \( \mathbb{Z}[\zeta], \mathbb{Z}[\xi], \mathbb{Z}[\zeta, \xi] \) and \( K \) its quotient-field. An element \( \alpha \in R \) is \( q \)-primary if it is a \( q \)-adic \( q \)-th power. The \( q \)-primary numbers build a subring \( R_q \subset R \) and it is immediately verified that \( R^q \subset R_q \). If \( E(R) = E = R^* \subset R \) are
the units of the respective field, then we write $E_q = E \cap \mathbb{R}_q$. The ring $\mathbb{R}_q$ induces the following equivalence relation:

$$ \alpha \equiv_q \beta \iff \exists \mu, \nu \in \mathbb{R}_q : \mu \cdot \alpha = \nu \cdot \beta. $$

If $q = (q)$ or $q = (1 - \xi)$, depending on whether $\xi \notin \mathbb{R}$ or $\xi \in \mathbb{R}$, we let $S = \mathbb{R} \setminus (q)$ and $\mathbb{R}' = S^{-1}\mathbb{R}$ be the corresponding localization. One may extend the definition of $q$-primary numbers to $\mathbb{R}'$, thus obtaining the ring $\mathbb{R}_q' \subset \mathbb{R}'$. The equivalence relation in (32) may then also be written as

$$ \alpha \equiv_q \beta \iff \alpha = \gamma \cdot \beta \quad \text{for some } \gamma \in \mathbb{R}_q'. $$

A number $\alpha \in \mathbb{R}'$ is called $q$-singular if there is a non-principal ideal $\mathfrak{B} \subset \mathbb{K}$ such that $(\alpha) = \mathfrak{B}^q$ as ideals. Let $\mathbb{R}''_q$ be the ring of the $q$-primary numbers which are also singular. The degenerate case $\mathfrak{B} = \mathbb{R}$ suggests allowing $E_q \subset \mathbb{R}_q''$. By class field theory, the singular primary numbers $\alpha \in \mathbb{R}''_q$ have the property that the extension $\mathbb{Q}(\zeta, \xi)[\alpha^{1/q}]$ is unramified Abelian. There is thus, by Hilbert’s Theorem 94, an ideal of order $q$ of $\mathbb{Q}(\zeta, \xi)$ which capitulates in this extension (see e.g. [Wa], Exercise 9.3).

We now define the number $h(p, q), h_{pq}$ enounced in the introduction. The definition involves an explicit use of Leopoldt’s reflection theorem (see e.g. [Lo], [Mi2]). The number $h(p, q)$ will be defined so that the condition $(q, h(p, q)) = 1$ becomes the tightest easily computable condition which implies $q \not| h_p$. Let $X = \{\chi_0 : G_p \rightarrow \mathbb{F}_q\}$ be the set of Dirichlet characters of order $q$ and $\omega_0 : G_q \rightarrow F_q$ the Teichmüller character. Then $\psi_0 = \chi_0^{-1} \cdot \omega_0$ is a well defined Dirichlet character of $G_{pq}$ in $\mathbb{F}_q$ which corresponds by reflection to $\chi$. If $k = F_q[\Im(X)]$ in the obvious sense (with $\Im(\chi)$ being the image of the character $\chi$ in $\mathbb{Q}(\zeta_{q-1})$), then one may chose a subfield $\mathbb{K} \subset \mathbb{Q}(\zeta_{q-1})$ of the $q-1$-th cyclotomic extension and an integer ideal $\Omega$ in this field, in such a way that $k = \mathcal{O}(\mathbb{K})/\Omega$. This allows to lift $\psi_0, \chi_0$ and $\omega_0$ to characters $\chi, \psi, \omega$ with images in $k$ [Mi2]. In particular, the generalized Bernoulli number $B_{1, \psi}$ is given by [Wa]:

$$ B_{1, \psi} = \frac{1}{(p-1)(q-1)} \sum_{(a, pq) = 1; 0 < a < pq} a \psi^{-1} \gamma_a, $$

where $\gamma_a \in G$ with $\gamma_a(\zeta \xi) = (\zeta \xi)^a$. With this, we define

$$ h_{\omega} = \prod_{\chi_0 \in X} B_{1, \chi_0^{-1}, \omega} $$

and

$$ h(p, q) = h_{pq} \cdot B_{\omega}. $$

Note that the Bernoulli numbers can be computed explicitly and in general $B_{1, \psi} \notin \mathbb{Z}$, but $B_{\omega} \in \mathbb{Z}$, since it is the norm of an algebraic integer in $\mathbb{Q}(\zeta_{q-1})$. It is also true [Mi2], that $q \not| h_{pq}$ implies $q \not| B_{\omega}$.

We shall see that certain ideals of $\mathbb{L}$ occurring in the subsequent proofs have order dividing $q$; they are thus principal is $q \not| h_{pq}$ (by reflection then $q \not| h_p$); if the ideals belong to $\mathbb{K}$, then they are already principal if $q \not| h(p, q)$. The following Proposition reflects these and further useful consequences of the above class number conditions.

**Proposition 3.** Let $E, E_q \subset \mathbb{L}$ be the units, respectively the $q$-primary units of the $pq$-th cyclotomic extension and $E', E'_q \subset \mathbb{K}$ be the respective sets in the $p$-th cyclotomic extension. If $q \not| h_{pq}$ then $E_q = E^q$ and in particular, if $\varepsilon = q 1$ is a
unit, then it is a $q$–th power. Likewise, if $q \not| h(p,q)$, then $E'_q = E^{qt}$ and all $q$–
primary units in $E'$ are $q$–th powers. Furthermore, $q \not| h_{pq}$ in the first case and
$q \not| h_p$ in the second.

Proof. We start with the implication $q \not| h_{pq} \Rightarrow q \not| h_pq$: this follows directly by
reflection in $\mathbb{L}$. Let $q \not| h_{pq}$ and $\epsilon \in E_q \setminus E^q$ be a $q$– primary unit, which is not a
$q$–th power. Then $K = \mathbb{L}(\epsilon^{1/q})$ is an Abelian unramified extension (see e.g. [Wa],
lemma 9.1, 9.2) and $[K : \mathbb{L}] = q$; by Hilbert’s Theorem 94, there is an ideal of order
$q$ from $\mathbb{L}$ which capitulates in $K$, in contradiction with $q \not| h_{pq}$.

We now consider the case $q \not| h(p,q)$. The crucial remark here is that even Dirichlet
characters of $\mathbb{K}$ correspond by reflection to the characters indexing Bernoulli
numbers which divide $B_{2n}$ above. The claims for this case follow then in analogy to
the ones for $q \not| h_{pq}$. □

5.2. Units. We start the analysis of local properties of units in $\mathbb{O}(\mathbb{L})$ under certain
eventual restriction on the class number, by several simple basic Lemmata.

Lemma 6. Let $\delta = < -\zeta \cdot \xi >$ be a root of unity of $\mathbb{L}$. If $\delta \equiv 1 \mod q$, then $\delta = 1$.

Furthermore, if $\epsilon \in \mathbb{Z}[:\zeta, \xi^\times]$ is a unit such that $\epsilon = a + b(q) + O(q^\lambda)$ and $a \in \mathbb{Z}$
and $b(\zeta) \in \mathbb{Z}[\zeta]$; then $b(\zeta) = b(\zeta)$.

Proof. Let $\delta^2 = \zeta^a \cdot \xi^b$, with $0 \leq a < p$, $0 \leq b < q$ - squaring cancels the sign. If
$a \cdot b \neq 0$, then $\delta^2$ is a primitive $pq$–th root of unity and thus

$$P(G) = \prod_{\psi \in G} (1 - \psi(\delta)) = \Phi_{pq}(1) = 1.$$ 

But since $\delta^2 \equiv 1 \mod q$ we should have $P(G) \equiv 0 \mod q^{\psi(\psi\zeta)}$, so $a \cdot b \neq 0$ is
impossible. In the cases $a = 0, b \neq 0$ and $a \neq 0, b = 0$, the root $\delta^2$ is primitive of
order $q$, resp. $p$ and the value of $P(G)$ is $q^{p-1}$ and $p^{q-1}$, respectively. In both cases
$P(G) \neq 0 \mod q^{\psi(\psi\zeta)}$, so we must have plainly $\delta^2 = 1$ and since $\delta \equiv 1 \mod q$, also
$\delta = 1$.

If $\epsilon$ is like in the claim of the lemma, then $\delta = \epsilon/\sigma \equiv 1 \mod q$ is a root of unity,
and thus $\epsilon = \sigma$. The claim on $b(\zeta)$ follows. □

Lemma 7. Let the primes $p, q$ be odd primes with $p \not| h_{pq}$, $q \not| (q-1)$ and suppose that
$\epsilon \in \mathbb{O}(\mathbb{L})^\times$ is a unit such that $\epsilon = q a$, with $a \in \mathbb{Z}$. Then $\epsilon$ is a $q$–th power.

Proof. Let $\sigma \in G = \text{Gal}\left(\mathbb{Q}(\zeta, \xi)/\mathbb{Q}\right)$. Then $\delta = \epsilon^{1 - \sigma} = q 1$ and by Proposition it is a $q$–th power. Since this holds for all $\sigma$,

$$\epsilon^{(p-1)(q-1)} = \prod_{\sigma \in G} \delta | \mathbb{L}^\times.$$ 

But $(q, (p-1)(q-1)) = 1$ by hypothesis and consequently $\epsilon$ is in a $q$–th power, as
claimed. □

Finally, we have:

Lemma 8. Let $p, q$ be primes and $\epsilon \in \mathbb{Z}[\zeta_p]^\times$ be a unit such that $\epsilon = q c(1-\zeta)$, with
c $\in \mathbb{Q}$. If $p \not| 1 \mod q$ and $(p-1)m \equiv 1 \mod q$, then $p = q$ and

$$\epsilon = q \left( \frac{1 - \zeta}{p^m} \right) = q \left( \frac{(1 - \zeta)^{p-1}}{p} \right)^m = \gamma \in \mathbb{Z}[\zeta]^\times.$$
Proof. Let $\sigma \in G = \text{Gal} \left( \mathbb{Q}(\zeta_p)/\mathbb{Q} \right)$ be a generator and $\Omega = \sum_{i=0}^{p-3} (p - 2 - i)\sigma^i \in \mathbb{Z}[G]$, so that $(\sigma - 1)\Omega + (p - 1) = \mathbb{N}_{\mathbb{Q}(\zeta)/\mathbb{Q}}$. By hypothesis, $\varepsilon^{p-1} \equiv q \eta = (1 - \zeta)^{p-1}$, so

$$1 = \mathbb{N}(\varepsilon) = \varepsilon^{p-1 + \Omega(\sigma - 1)} = q \varepsilon^{p-1} \cdot \eta^{\Omega},$$

and thus $\varepsilon^{p-1} \equiv q \eta^{-\Omega}$ and $\varepsilon \equiv q \eta^{-m\Omega}$. Note also that $\eta^{\Omega} = (1 - \zeta)^{(\sigma - 1)\Omega} = (1 - \zeta)^{N_{\mathbb{Q}(\zeta)/\mathbb{Q}} - 1}$, a simple expression for this unit. □

The deeper results on local properties of units in $\mathbb{L}$ (and subfields), given class number restraints, are summarized in the following Proposition. The proof of the proposition is quite lengthy and involves, along with the previous class field results, some interesting properties of cyclotomic units in fields of composite order. The result is interesting in itself, but it shall be used only for improving an estimate in Theorem 4 for the special case when $q \equiv 1 \mod p$. Given the lack of generality of its application, the reader who is more interested in an overview of the main ideas and proofs, can thus skip to the next section.

**Proposition 4.** Let $p, q$ be odd primes with $q \not\equiv h_p^p$, $p \neq 1 \mod q$ and $L = \mathbb{Q} (\zeta, \xi)$ be the $pq$-th cyclotomic extension of $\mathbb{Q}$. If $\varepsilon \in L$ is a unit for which there is a $a \in \mathbb{Z}[\zeta]$ such that $\varepsilon \equiv a \mod q$, then there is a unit $\delta \in \mathbb{Z}[\zeta]$ such that $\varepsilon = q \delta$.

We first prove some Lemmata:

**Lemma 9.** Let $p, q$ be odd primes with $p \neq 1 \mod q$ and $q \not\equiv h_p^p$; let $\zeta \in \mathbb{C}$ be a primitive $p-1$-th root of unity. Then $\gamma_c = 1/(1 - \zeta^c), c = 1, 2, \ldots, p - 1$ form a basis for the Galois ring $\mathbb{Z}[\zeta]/(q \cdot \mathbb{Z}[\zeta])$.

Proof. We must show that $\gamma_c$ are linear independent. For this, we shall show that the discriminant $\Delta$ of the $\mathbb{Z}$-module $M = [\gamma_1, \gamma_2, \ldots, \gamma_{p-1}]$ is coprime with $q$ under the given conditions. Let $\sigma$ be a generator of $G_p = \text{Gal} \left( \mathbb{Q}(\zeta)/\mathbb{Q} \right)$ and the matrix $A = \left( \sigma^{i+j} \left( \frac{1}{1 - \zeta} \right) \right)_{i,j=1}^{p-1}$; then $\Delta = \text{det}^2(A)$. The matrix $A$ is a circulant matrix; if $\omega \in \mathbb{C}$ is a primitive $p-1$-th root of unity, then the vectors $\vec{f}_k = (\omega^{jk})_{j=0}^{p-2}$, for $k = 0, 1, \ldots, p - 2$ are eigenvectors of $A$. In the base spanned by these vectors, $A$ is diagonal. The base transform matrix having $\vec{f}_k$ as columns is a Vandermonde matrix with discriminant $D = \prod (\omega^i - \omega^j)$, which is a power of $p - 1$ and thus a unit modulo $q$, since $p \neq 1 \mod q$. The base transform is thus regular modulo $q$ and $\Delta$ is invertible modulo $q$ iff its conjugate matrix in the eigenvector base is so.

We now compute the determinant of the conjugate diagonal matrix $D$ of $A$. If $\chi : \mathbb{Z}/(p \cdot \mathbb{Z})^\times \rightarrow \omega$ is a character with $\chi(\sigma) = \omega$ - where $\sigma(\zeta) = \zeta^q$ - then one verifies that the representation of $A$ in the eigenvector base is

$$A \sim \text{Diag} \left( \left( \tau' (\chi^j) \right)_{j=0}^{p-2} \right),$$

where, for $\psi \in \chi >$, we defined the Lagrange resolvents

$$\tau'(\psi) = \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \frac{\psi(x)}{1 - \zeta^x}.$$ 

Let $\tau(\psi) = \sum_{x=0}^{p-1} \psi(x)\zeta^x$ be a regular Gauss sum; a known formula (see e.g. [La]) implies $\psi^{-1}(i)\tau(\psi) = \sum_{x=1}^{p-1} \psi(x)\zeta^{ix}$. By an easy calculation, $-p/(1 - \zeta^x) = \sum_{i=1}^{p-1} i$. 


\(\zeta^{ix}\). Finally, assembling these formulae we find:

\[
-p \cdot \tau'(\psi) = \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \psi(x) \cdot \sum_{i=1}^{p-1} i \cdot \zeta^{xi} = \sum_{i=1}^{p-1} i \cdot \left( \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \psi(x) \cdot \zeta^{xi} \right) = \tau(\psi) \cdot \sum_{i=1}^{p-1} i \cdot \psi^{-1}(i).
\]

But \(B_{1,\psi^{-1}} = \frac{1}{p} \sum_{i=1}^{p-1} i \cdot \psi^{-1}(i)\) is a generalized Bernoulli number and \(N(B_{1,\psi^{-1}}) | h_p^\gamma\).

But since \(q \not| h_p^\gamma\), this is a unit modulo \(q\). Furthermore, \(\tau(\psi) \cdot \tau(\psi) = p\), so \(\tau(\psi)\) is a unit too. Finally \(\tau'(\psi) = -\tau(\psi) \cdot B_{1,\psi^{-1}}\) is a unit modulo \(q\) for all \(\psi \in \chi\).

But

\[
det(A) = det \left( \text{Diag} \left( \tau'(\chi^j) \right)_{j=0}^{p-2} \right) = \prod_{j=0}^{p-2} \tau'(\chi^j).
\]

Since all the factors have been shown to be units modulo \(q\), it follows that \((\Delta, q) = (\det^2(A), q) = 1\), which completes the proof.

We study next the cyclotomic units of \(\mathbb{L}\). Let \(\delta = (1 - \zeta \xi)\) and \(C_1 = \langle - \xi \rangle\) be the \(\mathbb{Z}[G]\) module generated by the unit \(\delta\) together with the roots of unity. If \(q \neq 1\) mod \(p\) we let \(C_2 = \{1\}\); otherwise, let \(C_2 = \mathbb{Z}[G_p] \cdot (1 - \zeta)(1 + j)\) for \(\zeta\) modulo \(K^+\) generated by the cyclotomic unit \(\eta = (1 - \zeta)^{\sigma-1}\). Thus \(C_2\) is in this case essentially equal to the cyclotomic units of \(K^+\) \([W_a]\), Chapter VIII; it has in fact an index 2 in this group, which is of no importance in our context, since we are focusing on \(q\) - parts of unit groups.

Note that for \(q \equiv 1\) mod \(p\), the norm \(N_{\mathbb{L}^+/K^+}(\delta) = \frac{1 - \zeta}{1 - \zeta} = 1\) and we always have

\[C_1 \cap C_2 = \{1\} \text{.} \tag{35}\]

For \(q \neq 1\) mod \(p\), the statement is trivial. Suppose now that \(p|q - 1\) and let \(\varepsilon = \gamma_1 \cdot \gamma_2 \in C_1 \cap C_2\), with \(\gamma_1 \in C_1\). Since \(\varepsilon \in C_1\), \(N_{\mathbb{L}^+/K^+}(\varepsilon) = 1 = \gamma_2^{-1} \gamma_1^{-1}\) and since \(\gamma_2\) is real, it follows that \(\gamma_2 = \pm 1\). But \(\varepsilon \in C_2\) implies, by taking norms again, that \((\varepsilon/\gamma_2)^{-1} = \gamma_1^{-1} = 1\) and eventually \(\varepsilon = \gamma_1 \cdot \gamma_2 = 1\), as claimed.

Lemma 10. Let \(p, q\) be odd primes with \(q \not| h_p^\gamma\) and \(p \neq 1\) mod \(q\). If \(C' = C_1\) for \(q \neq 1\) mod \(p\) and \(C' = C_1 \cdot C_2\) otherwise, then \(C'\) has finite index in \(E\), the group of units of \(\mathbb{L}\) and \(q \not| \kappa = [E : C']\). In particular,

\[E = C' \cdot E^q\] \(\text{.} \tag{36}\]

Proof. In the case \(q \neq 1\) mod \(p\), there are no multiplicative dependencies in \(C_1\) and the claims are a direct consequence of \([W_a]\) Corollary 8.8 (note that both \(C'\) and \(E\) contain the same torsion, the roots of unity of \(\mathbb{L}\)). Indeed, since \(p \neq 1\) mod \(q\) and \(q \neq 1\) mod \(p\), the Euler factor in this corollary is not vanishing and also coprime to \(q\). Thus \(E/E^q\) and \(C'/C'^q\) have the same rank and annihilators and the subsequent claims follow from this observation, since \(C_2\) is trivial in this case.

We now consider the case \(q \equiv 1\) mod \(p\), for which we apply the Theorem 8.3 in \([W_a]\). Note that the Ramachandra units are, up to roots of unity and an index 4, exactly \(C' = C_1 \cdot C_2\) in this case. By the Theorem of Ramachandra, it follows that \(q \not| \kappa = [E : C']\), which also implies \((36)\). \(\square\)
Next we investigate the structure of the cyclotomic units as group ring modules. For this we note that the map \( \iota : \mathbb{Z}[G] \to \mathbb{Z} = \mathbb{Z}[X,Y]/(X^{p-1} - 1, Y^{q-1} - 1) \) given by \( \sigma \mapsto X \) and \( \tau \mapsto Y \) where \( \sigma, \tau \) are generators of \( G_p, G_q \), as usual - is an isomorphism of rings. We shall consider next various restrictions of this map to subrings and quotient rings of \( \mathbb{Z}[G] \), without changing the notation.

The image of \( \mathbb{Z}[G]^+ \) under this map is \( \mathbb{Z}^+/(X^{(p-1)/2} - Y^{(q-1)/2}) \), since the partial conjugations \( p_\iota = \iota q \) in the real subfield.

We are interested in the \( q \) - parts \( W' = \mathbb{C}'/\mathbb{C}^{rq} \) and the components \( W_i = \mathbb{C}_i/\mathbb{C}_i^q \), for \( i = 1, 2 \). These are obviously \( \mathbb{F}_q[G]^+ \) - modules and as a consequence of (35) we also have

\[
W' = W_1 \oplus W_2.
\]

Since the \( \mathbb{N}_{L^+}/Q \) annihilates the units, they are also \( \mathbb{R} = \mathbb{F}_q[G]^+ / (\mathbb{N}_{L^+}/Q) \) - modules. We have

\[
\mathbb{R} \cong \mathbb{Z}^+ / \left( q, \frac{X^{p-1} \cdot Y^{q-1} - 1}{X \cdot Y - 1} \right)
\]

\[
\cong \mathbb{F}_q[X,Y]/\left( X^{p-1} - 1, Y^{q-1} - 1, X^{(p-1)/2} - Y^{(q-1)/2}, \frac{X^{p-1} \cdot Y^{q-1} - 1}{X \cdot Y - 1} \right)
\]

under the isomorphism \( \iota \). The above isomorphism illustrates that \( \mathbb{R} \) is a semi-simple module, which is not cyclic.

Suppose now that \( q \not\equiv 1 \pmod{p} \) so \( C' = C_1 \), a cyclic \( \mathbb{R} \) - module. By comparing ranks in (36), it follows in fact that \( C_1 = \mathbb{R} \cdot \delta \) in this case. If \( q \equiv 1 \pmod{p} \) then \( \mathbb{N}_{L^+}/(\mathbb{F}_q[G]^+ \cdot R) \) \( \equiv \mathbb{Z}^+ / (X^{(p-1)/2} - 1) \), then one verifies that \( W_1 = \mathbb{R}_1 \cdot \delta \) in this case; in order to keep a uniform notation, we shall also write \( W_1 = W_1 \), if \( q \not\equiv 1 \pmod{p} \), so that \( W_1 = \mathbb{R}_1 \cdot \delta \) in both cases.

As to \( W_2 \), by [Wa], Theorem 8.11, one simply has \( W_2 = \mathbb{F}_q[G_p]^+ \cdot \eta \). Note that the ranks of \( \mathbb{R}_1 \) and \( \mathbb{F}_q[G_p]^+ \) add up to \( (p-1)(q-1)/2 = \text{rank}(W') \). We now apply the gained structure for analyzing some particular cyclotomic units.

**Lemma 11.** The notations being like above, let \( \delta_1 \in C_1 \) be a unit which verifies \( \delta_1^{q-\hat{\tau}} \equiv \hat{\tau} \pmod{C_1} \). Then \( \delta_1^{q-\tau} = \delta_{\Theta} \) for some \( \Theta \in \mathbb{R}_1 \) such that

\[
\Theta = \varepsilon_1 \cdot \theta, \quad \text{with} \quad \theta \in \mathbb{F}_q[G_p]^+,
\]

and \( \varepsilon_1 = \frac{1}{q-1} \sum_{b=1}^{q-1} b \cdot \tau_b^{-1} \in G_q \) is the first orthogonal idempotent of \( \mathbb{F}_q[G_q] \). Furthermore,

\[
\delta_1 \in C_1 \quad \text{and} \quad \delta_1^{q-\tau} \in C_1 \quad \Rightarrow \quad \delta_1 \cdot \mathbb{N}(\delta_1) \subseteq C_1.
\]

**Proof.** We let \( \tilde{\delta}_1 = \delta_1 \) mod \( C_1 \) be the image in \( W_1 \). Then the hypothesis on \( \delta_1 \) translates to \( \tilde{\delta}_1^{q-\hat{\tau}} = 1 \). If \( \delta_1 = \delta_{\Theta_0} \) for some \( \Theta_0 \in \mathbb{Z}(G)^+ \), then \( \Theta(\tau - \hat{\tau}) \) lays thus in the kernel of the map \( \mathbb{Z}(G)^+ \rightarrow \mathbb{R}_1 \).

We shall have, like usual, to distinguish whether \( q \equiv 1 \pmod{p} \) or not. In the latter, simple case, we know that \( W_1 = \mathbb{R} \cdot \delta \) and the previous remark on \( \Theta \) implies that

\[
\iota(\Theta_0) \cdot (\tau - \hat{\tau}) \equiv 0 \pmod{\left( q, \frac{X^{p-1} \cdot Y^{q-1} - 1}{X \cdot Y - 1} \right)}.
\]

In the second case, we have

\[
\iota(\Theta_0) \cdot (\tau - \hat{\tau}) \equiv 0 \pmod{\left( q, \frac{Y^{q-1} - 1}{Y - 1} \right)}.
\]
We let \( \Theta = \Theta_0 \cdot (\sigma \tau - 1) \). The second generators of the ideals in the kernels of the last two congruences are images of norms and they are annihilated in \( \mathbb{Z}^+ \) by \( \nu(\sigma \tau - 1) \). It follows that \( \Theta \cdot (\tau - \hat{\tau}) \equiv 0 \mod q \). Let \( \theta_i \in \mathbb{Z}[G_p], i = 0, 1, 2, \ldots, q - 2 \) be such that

\[
\Theta = \sum_{n=1}^{q-1} \tau^n \cdot \theta_{n-1} \quad \text{and} \quad \Theta \cdot (\tau - \hat{\tau}) = \sum_{n=1}^{q-1} (\theta_{n-2} - \hat{\tau} \theta_{n-1}) \tau^n \equiv 0 \mod q,
\]

where the indices in the last sum are taken modulo \( q \). Since \( \tau^n \) are independent modulo \( q - 1 \), the sum vanishes modulo \( q \) if all of the coefficients do. Thus, inductively,

\[
\theta_n \equiv (\tau)^{-n} \cdot \theta_0, \quad n = 1, 2, \ldots, q - 2.
\]

But then

\[
\Theta \equiv -\theta_0 \cdot \varepsilon_1 \mod q\mathbb{Z}[G^+],
\]

with \( \varepsilon_1 \equiv -\sum_{a=1}^{q-1} (\tau/\theta)^a = -\sum_{a=1}^{q-1} a\tau^{-1} \mod q \) being the first orthogonal idempotent of \( \mathbb{Z}/(q \cdot \mathbb{Z})[G_q] \).

We now prove (38). For this we note the following decomposition in \( \mathbb{Z}[G_q] \):

\[
\mathbb{N} = \mathbb{N}_{K'/\mathbb{Q}} = \mathbb{N}_{L/K} = (\tau - 1) \cdot \Omega + (q - 1), \text{ for some } \Omega \in \mathbb{Z}[G_q]; \text{ the verification is a simple computation and is left to the reader. But then, given } \delta_1 \text{ in (38), we have:}
\]

\[
\mathbb{N}(\delta_1) = \delta_1^{(\tau - 1)\Omega + \delta_1} \quad \text{and} \quad \delta_1 \cdot \mathbb{N}(\delta_1) \in \mathbb{C}_1^q.
\]

This completes the proof. \( \square \)

The main result towards the proof of the Proposition is the following:

**Lemma 12.** Let \( p, q \) be odd primes with \( q \not| h_{pq} \), \( p \neq 1 \mod q \), \( L = \mathbb{Q}(\zeta, \xi) \) be the \( pq \)-th cyclotomic extension and \( G = \text{Gal}(L/\mathbb{Q}) = G_p \times G_q \) with \( G_p = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}), G_q = \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}). \) If \( \varepsilon \equiv 1 \mod q \lambda \) is a unit of \( L \), then \( \varepsilon \) is a \( q \)-th power. In particular, if \( \varepsilon = 1 + a\tau + O(q\lambda^2) \), with \( a \in \mathbb{Z}[\zeta] \), then there is a \( \beta \in \mathbb{Z}[\zeta] \) such that

\[
\varepsilon \equiv \sigma_q(\beta) - \beta \mod q.
\]

**Proof.** Let \( E \subset L \) be the real units, \( C' \subset E \) the cyclotomic units defined above and \( \varepsilon \equiv 1 + a\tau + O(q\lambda^2) \mod q \lambda^2 \), for \( a \in \mathbb{Z}[\zeta] \). If \( \kappa = [E : C'] \in \mathbb{N} \), then \( \varepsilon^\kappa \in C' \) is a unit with the same type of \( \lambda \)-expansion as \( \varepsilon \), since \( (\kappa, q) = 1 \). We may thus assume, for simplicity, that \( \varepsilon \in C' \) to start with and thus

\[
\varepsilon \equiv \delta_1 \cdot \delta_2, \quad \text{with} \quad \delta_i \in C_i.
\]

Note that \( \varepsilon^{\tau - 1} = \delta' = \delta_1^{\tau - 1} \equiv 1 \mod q \lambda \). Since \( \tau(\lambda) \equiv \hat{\tau} \cdot \lambda \mod \lambda^2 \), we have

\[
\psi = \delta^{\tau - \hat{\tau}} = \varepsilon^{(\tau - 1)(\tau - \hat{\tau})} \equiv \frac{1 + a' \cdot q \cdot \tau(\lambda)}{1 + a' \cdot q \cdot \hat{\tau} \cdot \lambda} \cdot \lambda \equiv 1 \mod q\lambda^2,
\]

where \( a' = \hat{\tau} - 1 \in \mathbb{Z}^\times \); thus \( \psi = q \) and so \( \psi \in C'^q \cap C_1 = C_1^q \) by Proposition 30 and (39).

We are thus in the context of Lemma 11 which implies that

\[
\delta' = \delta^{-\varepsilon_1 \cdot \theta} \equiv 1 \mod q \lambda.
\]
We now estimate the unit $\delta^{\varepsilon_1}$ up to $\lambda^2$ and compare the result with the above.

$$
\delta = 1 - \zeta \xi = (1 - \zeta) + \zeta(1 - \xi) = (1 - \zeta) \cdot \left(1 - \frac{1 - \xi}{1 - \zeta}\right), \quad \text{so}
$$

$$
\delta^{-\varepsilon_1} = (1 - \zeta)^{-\varepsilon_1} \cdot \left(1 - \sum_{a=1}^{q-1} a \cdot \tau_a^{-1} \left(\frac{1 - \xi}{1 - \zeta}\right) + O(\lambda^2)\right)
$$

$$
= (1 - \zeta)^{-\varepsilon(q-1)/2} \cdot \left(1 + \frac{1 - \xi}{1 - \zeta} + O(\lambda^2)\right) \mod q\mathbb{Z}[\xi, \zeta].
$$

If $\theta = \sum_{c=1}^{p-1} n_c \sigma_c$ and $A = (1 - \zeta)^{-\varepsilon(q-1)/2} \in \mathbb{Z}[\zeta]$, then

$$
\delta^{-\varepsilon_1} = A \left(1 + \frac{1 - \xi}{1 - \zeta} + O(\lambda^2)\right) \quad \text{and}
$$

$$
\delta^\theta = \delta^{\varepsilon_1} \theta = A^\theta \left(1 + \frac{1 - \xi}{1 - \zeta} + O(\lambda^2)\right)^\theta
$$

$$
= A^\theta \left(1 + (1 - \xi) \cdot \sum_{c=1}^{p-1} \frac{n_c}{1 - \zeta - \varepsilon} + O(\lambda)^2\right).
$$

Lemma 9 implies that the sum in the last equation only vanishes modulo $q$ if all the coefficients $n_c$ vanish, so $\theta \equiv 0 \mod q$ and also $\Theta \equiv 0 \mod q$, to start with. But then $\delta_1^{-1} = \delta^\theta = \delta^{\varepsilon_1}$ is a $q$-th power. If $q \equiv 1 \mod p$, then (88) implies that $\delta_1$ is a $q$-th power, and since we have already shown that $\delta_2$ is a $q$-th power, we have $\varepsilon = \delta_1 \delta_2 \in E^q$. Oddly, the case $q \equiv 1 \mod p$ requires now more attention - this is not an intrinsic problem, but rather a consequence of the build up of the auxiliary Lemmata, where the load was taken away from the second case. For the case $q \not\equiv 1 \mod p$ we have thus, again using (88), that $\varepsilon = \psi \cdot \gamma^q$, with $\psi = N(\delta_1)^{-1} \in \mathbb{Z}[\zeta]$ and $\gamma \in C_1 = C'$, if $\gamma = a + b\lambda$, with $a \in \mathbb{Z}[\zeta]$ and $b \in \mathbb{Z}[\zeta, \xi]$, then the definition of $\varepsilon$ implies that

$$
\psi \equiv a^{-q} \mod q\lambda,
$$

and since $x \equiv 0 \mod \lambda$ implies $x \equiv 0 \mod q$ for $x \in \mathbb{Z}[\zeta]$, it follows that $\psi = q$ 1 and by Proposition 3 it follows that $\psi$ is a $q$-th power.

We still have to prove (89). Let $\gamma^q = \varepsilon = 1 + aq\lambda + O(q\lambda^2)$. Then $\gamma = 1 + b\lambda + O(\lambda^2)$, with $b \in \mathbb{Z}[\zeta]$ and raising to the $q$-th power we find that

$$
a \equiv (b^q\lambda^{q-1}/q + b) \mod \lambda.
$$

But $\frac{\lambda^{q-1}}{q} = \prod_{i=1}^{q-1} \frac{1 - \xi}{1 - \zeta}^q = \prod_{i=1}^{q-1} (1/i) \equiv -1 \mod \lambda$, where the last congruence is derived from Wilson’s theorem. Thus $a \equiv b^q - b \mod \lambda$ and since $a, b \in \mathbb{Z}[\zeta]$, it follows that $a \equiv b^q - b \mod q$. If $\Omega$ is a prime of $\mathbb{Z}[\zeta]$ over $q$, then it is fixed by $\sigma_q$ and $b^q \equiv \sigma_q(b) \mod \Omega$, so the previous equivalence implies $a \equiv \sigma_q(b) - b \mod q$. This holds for all primes $\Omega|(q)$ uniformly, and it follows that $a \equiv \sigma_q(b) - b \mod q$. This completes the proof.

We can now prove Proposition 4.

**Proof.** By the hypothesis of the Proposition, one can write $\varepsilon = a + b \cdot q + q\nu$, with $a, b \in \mathbb{Z}[\zeta]$ and $\nu \in \mathbb{Z}$; then

$$
\varepsilon = C \cdot (1 + q\lambda\nu'), \quad \text{with} \quad C = a + b \cdot q \in \mathbb{Z}[\zeta], \quad \nu' = \nu/C.
$$
We let \( \delta^{-1} = N_{L/K}(\varepsilon) = C^{q-1} \cdot N_{L/K}(1 + q\lambda\nu') \equiv q^{-1} C^{-1} \) and thus \( \delta \equiv q^{-1} C \equiv \varepsilon \mod q\). Obviously, \( \delta \in \mathbb{Z}[\zeta]^\times \) and \( \varepsilon/\delta = 1 + q\lambda\nu' \) is a unit verifying the hypothesis of Lemma 12. The first claim follows by applying the Lemma to \( \varepsilon/\delta \) (note that the claim is trivial if \( \varepsilon \in \mathbb{Z}[\zeta] \)).

\[\Box\]

6. An Improved Case Distinctions

In this section we derive some easy consequences from the conditions deduced in the previous one. Finally, the methods developed in this section will be sharpened in the next one, thus leading to a proof of Theorem 1. In this Theorem, the two cases discussed above, and which depend on congruences modulo \( p \), are analogous to the Abel-Barlow Cases in the classical Fermat equation, split into three additional cases each, and these additional cases rely upon congruences modulo \( q \).

**Lemma 13.** Let \( p,q \) be odd primes and \( x, y \) coprime integers with \( x \cdot y \not\equiv 0 \mod q \) and such that there is a \( \beta \in \mathbb{Q}(\zeta) \) with

\[
\frac{x + \zeta^q \cdot y}{x + \zeta \cdot y} = \pm \left( \frac{\beta}{\beta} \right)^q.
\]

Then

\[
-(\zeta^q - \zeta^{-q}) \varphi(t) \equiv \sum_{k=1}^{q-1} \frac{t^k - t^{2-k}}{k} \cdot (\zeta^k - \zeta^{-k}) \mod q.
\]

**Proof.** A development of (41) up to the second power of \( q \) yields:

\[
\frac{x + \zeta^q \cdot y}{x + \zeta \cdot y} \equiv \left( \frac{x + \zeta \cdot y}{x + \zeta^q \cdot y} \right)^q \mod q\mathbb{Z}[\zeta].
\]

Combining with (40) we find

\[
\pm \frac{\beta}{\beta} = \frac{x + \zeta \cdot y}{x + \zeta^q \cdot y} + q \cdot \mu,
\]

with \( \mu \in \mathbb{Q}(\zeta) \) being a \( q \) -adic integer. Raising to the power \( q \), it follows that in fact

\[
\frac{x + \zeta^q \cdot y}{x + \zeta \cdot y} \equiv \pm \left( \frac{x + \zeta \cdot y}{x + \zeta^q \cdot y} \right)^q \mod q^2\mathbb{Z}[\zeta].
\]

We write \( \varphi(a) = a^{q^2-q} \mod q \), for \( (a,q) = 1 \) and let \( t \equiv -y/x \mod q^2 \), so \( -(y/x)^q \equiv t + q\varphi(t) \mod q^2 \). Now

\[
(x + \zeta \cdot y)^q \equiv x^q \cdot (1 - t \cdot \zeta)^q \equiv (x + q\varphi(x)) \cdot (1 - t \cdot \zeta)^q \equiv (x + q\varphi(x)) \cdot (1 - t\zeta^q + qf(\zeta)) \mod q^2
\]

where

\[
f(\zeta) = -\zeta^q \cdot \varphi(t) + \sum_{k=1}^{q-1} \frac{q}{k} (-t\zeta)^k \equiv -\left( \zeta^q \cdot \varphi(t) + \sum_{k=1}^{q-1} \frac{t^k \zeta^k}{k} \right) \mod q.
\]

Writing \( x + \zeta^q y = x(1 - t\zeta^q) = x \cdot \alpha \) and eliminating denominators in (42) we find that

\[
\alpha \cdot (x + \varphi(x)) \left( \alpha + q \cdot f(\zeta) \right) \equiv \alpha \cdot (x + \varphi(x)) \cdot (\alpha + q \cdot f(\zeta)) \mod q^2 \quad \text{and}
\]

\[
\alpha \cdot f(\zeta) \equiv \alpha \cdot f(\zeta) \mod q.
\]
We let $S = \sum_{k=1}^{q-1} \frac{t^k}{k}$ and regroup the terms, finding:

$$(1 - t \zeta^k) \cdot (\varphi(t) \cdot \zeta^q + S) \equiv (1 - t \zeta^q) \cdot (\varphi(t) \cdot \zeta^q + S) \mod q,$$

and

$$-(\zeta^q - \zeta^q) \varphi(t) \equiv (1 - t \zeta^q)S - (1 - t \zeta^q)S \mod q,$$

and

$$-(\zeta^q - \zeta^q) \varphi(t) \equiv \sum_{k=1}^{q-1} \frac{t^k}{k}(\zeta^k - \zeta^k) - \sum_{k=1}^{q-1} \frac{t^{k+1}}{k}(\zeta^{k-q} - \zeta^{k-q}) \mod q.$$

We regroup the powers of $\zeta$ using $q - k \equiv -k \mod q$, so $\zeta^{k-q} \equiv -\zeta^{-k} / (q - k)$, which can be applied in the above for $k = 1, 2, \ldots, q - 1$:

$$-(\zeta^q - \zeta^q) \varphi(t) \equiv \sum_{k=1}^{q-1} \frac{t^k - t^{2-k}}{k} \cdot (\zeta^k - \zeta^k) \mod q,$$

the statement of (41).

Lemma 13 yields essentially a system of equations modulo $q$ in the unknown $t$. It turns out that under some additional conditions on $p$ and $q$, there are only three possible values for $t$ (one of which is $t = 0$). The light version of this condition was presented in [M1]; it reflects the main ideas which will subsequently lead, by a more in depth study of the system (41), to a sharper inequality between $p$ and $q$. The light result is the following:

**Proposition 5.** Assume that $p > q$ are odd primes and there is a $\beta \in \mathbb{Q}(\zeta)$ such that (41) holds. Then

$$x + f \cdot y \equiv 0 \mod q^2 \quad \text{for some} \quad f \in \{-1, 0, 1\}.$$

**Proof.** Assume first that $x \equiv 0 \mod q$ and $x = qu$ with $(u, q) = 1$. Since $(x, y) = 1$ and $p \neq q$, it follows that $(x + \zeta^q y, q) = 1$, so the right hand side of (41) is a $q$-adic integer. The equation is Galois-invariant, so we can replace $\zeta$ by $\zeta^q$. Thus (41) becomes

$$\frac{y + q\zeta^qu}{y + q\zeta^qu} = \gamma^q,$$

with $\gamma = \pm \zeta^2 \cdot \beta / \overline{\beta}$. Obviously the above implies $\gamma \equiv 1 \mod q$, so $\gamma^q \equiv 1 \mod q^2$ and $y + qu\zeta^2 \equiv y + qu\zeta^2 \mod q^2$, so $u \cdot (\zeta^2 - \overline{\zeta}^2) \equiv 0 \mod q$. This is only possible if $u \equiv 0 \mod q$ and thus $x \equiv 0 \mod q^2$. Since we can interchange $x$ and $y$, this proves that if $x$ or $y$ is divisible by $q$, then it is divisible by $q^2$, which takes care of $f = 0$ in this case.

We may now assume that $x \cdot y \not\equiv 0 \mod q$ and use the previous lemma, which implies that (41) holds under the given premises. Since the set $\{\zeta, \zeta^2, \ldots, \zeta^{q-1}\}$ builds a base of the algebra $\mathbb{Z}[\zeta] / (q \cdot \mathbb{Z}[\zeta])$, the coefficients of the single powers in the above identity must all vanish and $p > q + 1$ implies that the coefficient of $\zeta$ is $a_1 = t(1 - t^{-3})$ and thus

$$t^4 \equiv 1 \mod q$$

must hold. Furthermore, if $q + 2 < p$, then the coefficient of $\zeta^2$ is

$$2 \cdot a_2 = (t^2 - t^{-4}) \equiv 0 \quad \text{hence} \quad t^6 - 1 \equiv 0 \mod q.$$

The last two congruences in $t$ have the only common solution $t^2 = 1 \mod q$. One easily verifies that if this holds, then the right hand side in (41) vanishes and thus $\varphi(t) \equiv 0 \mod q$. This leads to the possible solution $x \pm y \equiv 0 \mod q^2$; inserting
the value back shows that this is indeed a solution of (40). If \( p = q + 2 \), then we still have \( a_1 = t^{-3}(t^4 - 1) \) so \( t^4 \equiv 1 \mod q \). If \( t^2 - 1 \equiv 0 \mod q \), we find the previous solution. So let us assume that \( t^2 \equiv -1 \mod q \) and consider the second coefficient: but \( \varphi(t)\zeta^2 = \varphi(t)\zeta^2 \) has in this case a contribution to \( a_2 \). We estimate this coefficient by using \( t^2 \equiv -1 \mod q \):

\[
2 \cdot a_2 = t^2 - t^{-4} - 2\varphi(t) \equiv -t^{-4} \left(t^6 - t^2 + t^2 - 1 + 2t^4\varphi(t)\right)
\]

\[
\equiv t^2 - 1 + 2\varphi(t) \equiv (2\varphi(t) - 1) \mod q,
\]

a congruence which is satisfied by \( \varphi(t) \equiv 1 \mod q \). We have to consider also

\[
3 \cdot a_3 = (t^3 - t^{-5}) - (t^3 - t^{-q-1}) \equiv 0 \mod q \iff 0 = t^{-5}(t^8 - 1) - (1 - t^{-2}) \mod q.
\]

But if \( t^2 \equiv -1 \mod q \), then the first term vanishes while the second is \( -2 \neq 0 \mod q \), so \( t^2 \equiv -1 \mod q \) is not possible. This takes care also of the case \( p = q + 2 \), thus completing the proof of the proposition.

It follows from Corollary 2 that

**Corollary 3.** If \( p > q > 3 \) are odd primes for which \( (7) \) has non trivial solutions and such that \( q \nmid h_p \), then \( (43) \) holds.

**Proof.** The premises of Corollary 2 are given and thus \( (23) \) holds. By setting \( \beta = \rho_1 \) in this equation, we find that the hypotheses of Proposition 5 also hold, and by its proof it follows that \( (43) \) must be true. \( \square \)

6.1. **Sharpening.** Let \( k \) be a field and \( T \) be the space of sequences on \( k(t) \). We define the following operators on \( T \):

\[
b_n = \theta_+(a_n) = a_n - t \cdot a_{n-1} \\
c_n = \theta_-(a_n) = t \cdot a_n - a_{n-1} \\
d_n = \Theta(a_n) = \theta_+(\theta_-(a_n)).
\]

Furthermore we let \( \Delta \) be, classically, the forward difference operator \( \Delta \cdot a_n = a_n - a_{n-1} \) and \( n^k = n \cdot (n-1) \cdots (n-k+1) \) be the \( k \)-th falling power of \( n \), so \( \Delta n^k = k \cdot (n-1)^{k-1} \). With the main properties of the operators in (44) are given by

**Lemma 14.** The operators \( \theta_+, \theta_- \) are linear and they commute, thus \( \Theta = \theta_+ \circ \theta_- = \theta_- \circ \theta_+ \). Furthermore,

\[
\theta_+(t^n) = 0 \quad \text{and} \quad \theta_+(t^{-n}) = (1 - t^2)t^{-n},
\]

\[
\theta_-(t^n) = 0 \quad \text{and} \quad \theta_-(t^{-n}) = -(1 - t^2)t^{-n},
\]

\[
\theta_+(n^k \cdot t^n) = \frac{k}{k!} \cdot (n-l)^{k-l} \cdot t^n,
\]

\[
\theta_-(n^k \cdot t^n) = \frac{k}{k!} \cdot (n-l)^{k-l} \cdot t^{-(n-l)},
\]

where we set \( a^{\frac{k-l}{l}} = 0 \) if \( k < l \). In particular, we have:

\[
\theta_+(n^k \cdot t^n) = k! \cdot t^n,
\]

\[
\theta_+(n^k \cdot t^{-n}) = k! \cdot t^{-(n-k)},
\]

\[
\Theta(n^k \cdot t^n) = k! \cdot (t^2 - 1)^k \cdot t^{n-k},
\]

\[
\Theta(n^k \cdot t^{-n}) = k! \cdot (-1)^k \cdot (t^2 - 1)^k \cdot t^{-(n-k)}.
\]
Proof. Commutativity follows by a straightforward computation from
\[
\theta_+ \circ \theta_-(a_n) = \theta_- \circ \theta_+(a_n) = t \cdot (a_n + a_{n-2}) - (t^2 + 1)a_{n-1}.
\]
The rules \((44)\) are also easily verified and they yield \((45)\) by induction on \(k\). Finally, the first two actions in \((46)\) are obtained by setting \(l = k\) in \((44)\), while the action of \(\Theta\) is obtained due to commutativity, by setting \(\Theta^k = \theta_+^k \circ \theta_-^k\) or \(\Theta^k = \theta_-^k \circ \theta_+^k\), depending whether the operand is \(t^n\) or \(t^{-n}\). Note that \(k + 1\) consecutive values of \(a_n\) are necessary for applying \(\theta^k\), while \(\Theta^k\) requires \(2k + 1\) consecutive values. \(\square\)

The task we pursue is to improve our estimates on pairs \(p, q\) for which the system \((41)\) has no other solutions except \((43)\); in particular, we are concerned with \(p < q\) - since Proposition 5 deals already with \(p > q\). We choose to analyze one of the intervals \(0 < k < \nu\). Let \(\delta \leq k \leq \nu\) be the value for which the equations are not homogeneous. Second the number of terms in the sums of the right hand side changes between 0 < \(k < \nu\) and this allows one consider \((41)\) as a linear system modulo \(q\). Let \(\zeta = \zeta^p\) be the value for which \(\zeta \equiv 1\) mod \(p\) or \(\nu \equiv -q\) mod \(p\); then, with \(\delta_{ij}\) the Kronecker \(\delta\), the above remark yields the equations:
\[
\delta_{ij} \cdot \varphi(t) = \sum_{j \geq 0; jp + k < q} \frac{t^{k+j} - t^{-2-(k+j)}}{pj + k} p \equiv \sum_{j \geq 0; jp + (p-k) < q} \frac{t^{p+j} - t^{-2-(p+j)}}{p - k + jp} \mod q.
\]
The index value \(\nu\) is singular for the equations above; first, it is the only index for which the equations are not homogeneous. Second the number of terms in the sums of the right hand side changes between 0 < \(k < \nu\) and \(p/2 \geq k > \nu\). In these two intervals \((47)\) yields homogeneous equations which manifest in the vanishing of polynomials of fixed degree in \(k\). This suggests the use of the difference operators defined above. Let 5 ≤ \(p < q\) be primes. We shall take the approach of choosing the one of the intervals 0 < \(k < \nu\) or \(\nu < k < p/2\), which has more elements: in these intervals \((47)\) translates into polynomial equations of the type \(f_q(k; t) = 0\). Having a contiguous interval on which this equation holds, one can use the iteration of \(\Theta\) in order to reduce the degree in \(k\) of the polynomial \(f_q\). We have thus to distinguish the cases \(\nu < p/4\) and \(\nu > p/4\)\(^3\).

Proposition 6. Let 5 ≤ \(p < q\) be primes such that \((47)\) holds and \(\nu\) be defined above. Suppose that \(\nu > p/4\); if additionally, \(q < \frac{p^2}{16}\), then \((43)\) holds.

Proof. Let \(n = [q/p]\). The equation \((47)\) yields on the interval 0 < \(k < \nu\):
\[
\sum_{0 \leq j \leq n} \frac{t^{k+j} - t^{-2-(k+j)}}{pj + k} \equiv \sum_{0 \leq j < n} \frac{t^{p+j} - t^{-2-(p+j)}}{p - k + jp} \mod q.
\]

\(^3\) One may also take the approach of considering the whole interval 0 < \(k < p/2\); in this case the polynomials \(f_q(k; t)\) change the degree and shape when \(k\) passes the "singular" value \(k = \nu\). The computations become more intricate, for a gain of a factor at most 2. We choose to analyze here the simpler approach.
After eliminating denominators, this yields a polynomial equation:

\[(−1)^n k^{2n} \cdot \sum_{0 \leq j \leq n} \left( t^{k+j} - t^{2-(k+j)} \right) + O(k^{2n-1}) \equiv 0 \]

\[(−1)^{n−1} k^{2n} \cdot \sum_{0 \leq j < n} \left( t^{p-k+j} - t^{2-(p-k+j)} \right) + O(k^{2n-1}). \]

In order to eliminate the lower order terms in \(k\), we may take \(\Theta^{2n}\) on both sides of the congruence. This requires at least \(2(2n) + 1\) contiguous points, so \(1 \leq k - 2n < k + 2n < p/4\), which means \(2(2n) + 1 < p/4\). If this is provided, the equation reduces, after simplifying by \((-1)^n \cdot (2n)! \cdot (1 - t^2)^{2n}\), to:

\[
\begin{align*}
\sum_{0 \leq j \leq n} \left( t^{k+j-2n} - t^{2-(k+j-2n)} \right) + \sum_{0 \leq j < n} \left( t^{p-k+j} - t^{2-(p-k+j)} \right) &\equiv 0 \mod q. \\
\end{align*}
\]

If \(t \notin \{-1, 0, 1\}\) then we can apply \(\theta_+\) and \(\theta_-\) independently to the above congruence. This yields:

\[
0 \equiv \sum_{0 \leq j \leq n} t^{2-(k+j-2n)} - \sum_{0 \leq j < n} t^{p-k+2n+j} \quad \text{and} \\
0 \equiv \sum_{0 \leq j \leq n} t^{k+j-2n} - \sum_{0 \leq j < n} t^{2-(p+2n-k+j)},
\]

and, upon multiplication by the lowest power of \(t\),

\[
\begin{align*}
0 &\equiv \sum_{0 \leq j \leq n} tp^j - \sum_{0 \leq j < n} t^{p(n+1) - 2 + pj} \mod q \quad \text{and} \\
0 &\equiv \sum_{0 \leq j \leq n} t^{pn+j-2} - \sum_{0 \leq j < n} t^{pj} \mod q.
\end{align*}
\]

Adding up the two congruences, we obtain \(t^{pn} \equiv -t^{pn-2} \mod q\) with the solutions \(t \equiv 0 \mod q\) and \(t^2 \equiv -1 \mod q\). We show that the latter solution is impossible by reinserting it in \((\star)\); this yields, after simple computations, \(t^k + t^{-k} \equiv 0 \mod q\). Since we assumed \(t \neq 0\), it follows that \((-1)^k + 1 \equiv 0 \mod q\). It suffices to take \(k\) even in order to reach a contradiction. Let us finally examine all the conditions on \(\nu\) (and thus on \(p\) and \(q\)), which allowed us to reach this contradiction. Adding the points necessary for the final application of \(\theta_\pm\) together with the condition that \(k\) be even, we find:

\[2n + 1 \leq k \leq p/4 - (2n + 1),\]

condition which is satisfied by the even value \(k = 2(n + 1)\), provided that \(4n + 3 < p/4\). On the other hand, we find from the definition of \(\nu\) and the fact that \(\nu > p/4\), that \(p(4n + 3) > 4q\), and thus

\[p^2/4 > p(4n + 3) > 4q,\]

as claimed. \(\square\)

**Proposition 7.** Let \(5 \leq p < q\) be primes such that \((\star)\) holds and \(\nu\) be defined above. Suppose that \(\nu < p/4\); if additionally, \(q < \frac{p(p-20)}{16}\), then \((\star)\) holds.
Proof. The proof of this proposition follows the same line as the previous, but raises few particular obstructions. We shall let
\[ n = \begin{cases} \lfloor q/p \rfloor & \text{if } (q \mod p) < p/4, \\ \lfloor q/p \rfloor + 1 & \text{if } (q \mod p) > 3p/4. \end{cases} \]
The equation (47) yields now on the interval \( \nu < k < p/4 \):
\[
\sum_{0 \leq j \leq n} \frac{t^{k+j} - t^{-(k+j)}}{p^2 + k} \equiv \sum_{0 \leq j \leq n} \frac{t^{p-k+j} - t^{-(p-k+j)}}{p - k + j} \mod q.
\]
Note that there are equally many terms in the sums of both sides of the above congruences, unlike the case of the previous proposition. This perpetuates down to the analog of (49), in which the two congruences become identical; they both yield the condition
\[
\text{either } t^{p(n+1)} \equiv 1 \mod q \text{ or } t^{p(n+1)} \equiv t^2 \mod q,
\]
whose deduction is left to the reader. Note that this condition is equivalent to applying any of \( \Theta_+ \Theta^{2n+1} \) or \( \Theta_- \Theta^{2n+1} \) to the original system (47).

In order to draw a contradiction we shall have to consider lower order terms in \( k \). Let
\[
\sigma_j = t^{k+j} - t^{-(k+j)} \quad \text{and} \quad \tau_j = t^{p-k+j} - t^{2-(p-k+j)},
\]
with some additional work, the first congruence yields, after elimination of denominators:
\[
\sum_{0 \leq j \leq n} \sigma_j \cdot (k^{2n+1} - [(n+j+1)p - (2n+1)n] \cdot k^{2n}) + \sum_{0 \leq j \leq n} \tau_j \cdot (k^{2n+1} - [(n-j)p - (2n+1)n] \cdot k^{2n}) + O(k^{2n-1}) \equiv 0 \mod q.
\]
We apply \( \Theta^{2n} \) to the above and let
\[
\sigma_j' = t^{k-2n+p+j} - t^{2-(k-2n+p+j)} \quad \text{and} \quad \tau_j' = t^{p-k+2n+p+j} - t^{2-2n-(p-k+p+j)}.
\]
With this we obtain
\[
\sum_{j=0}^{n} \sigma_j' ((2n+1)k - [(n+j+1)p - (2n+1)n]) + \sum_{j=0}^{n} \tau_j' ((2n+1)k - [(n-j)p - (2n+1)n]) \equiv 0 \mod q.
\]
We now apply \( \Theta_+^2 \) to the above relation; this cancels the terms in \( t^k \) and modifies the terms in \( t^{-k} \). Note that by commutativity, \( \Theta_- \Theta_+^2 \Theta^{2n} = \Theta_+ \Theta^{2n+1} \), which yields (50). But applying \( \Theta_- \) after \( \Theta_+^2 \) to (51) yields to a cancellation of all but the terms in \( \Theta_+^2 (kt^{-k}) \); conversely, it is precisely these terms which are canceled if the condition (50) holds. Since \( \Theta_+^2 \) \( t^{-k} \neq 0 \mod q \) if \( t \mod q \notin \{-1,0,1\} \), this eventually leads to the congruence:
\[
p \cdot t^{2n} \cdot \left( \sum_{j=0}^{n} ((j+1)t^{2-p+j} + jt^{p+p+j}) \right) \Theta_+^2 (t^{-k}) \equiv 0 \mod q, \quad \text{so}
\]
\[
\sum_{j=0}^{n} ((j+1)t^{2-p+j} + jt^{p+p+j}) \equiv 0 \mod q, \quad \text{and}
\]
\[
t^{2-p+2n(n+1)} \cdot \sum_{j=0}^{n} t^{pj} + t^{p} \cdot (1 - t^{2-p+2n+1}) \cdot \sum_{j=0}^{n} jt^{pj} \equiv 0 \mod q.
\]
We can now reintroduce the alternative (50) in the last congruence above. If \( t^{p(n+1)} \equiv 1 \mod q \), then the first sum \( \sum_{j=0}^{n} tp^j \) vanishes; furthermore, since \( (n, q) = 1 \), one easily verifies that \( \sum_{j=0}^{n} tp^j \equiv \sum_{j=0}^{n} jtp^j \equiv 0 \mod q \) cannot simultaneously hold, and thus it follows that \( t^{2-p(n+1)} \equiv 1. \) Since we also assumed \( t^{(n+1)p} \equiv 1 \mod q \), it follows that \( t^2 \equiv 1 \mod q \), as required. Suppose now that in (50) it is the condition \( t^{p(n+1)} \equiv t^2 \mod q \) which holds; by inserting this in the last congruence above, we find (since \( t(n+1) \neq 0 \mod q \)) that \( \sum_{j=0}^{n} tp^j \equiv 0 \mod q \) and we are in the previous case. Both ways, it follows that \( t \mod q \in \{ -1, 0, 1 \} \).

We finally have to derive the inequality between \( p \) and \( q \), for which the proof above holds. The condition is that the interval \( (p/4, p/2) \) contains sufficient contiguous points for applying both \( \theta \equiv \Theta^{2n+1} \) and \( \theta^2 \Theta^{2n} \); i.e. \( 4n + 5 < p/4. \) Note that by definition of \( n \), we always have \( np > q \) and thus the previous inequality amounts to \( 4q^2 < 4np < p(p/4 - 5) \) and thus

\[
p(p - 20)/16 > q.
\]

This completes the proof of the proposition.

### 6.2. Proof of Theorem 2

The statement of Theorem 2 follows directly from Corollary 3 together with the sharpening Propositions 6 and 7.

### 6.3. The Resulting Case Analysis

We suppose that the Fermat - Catalan equation (11) has a solution for odd primes \( p, q \) with \( p \neq 1 \mod q \), \( q \not| h_n \) and \( \max\{p, (p-20)/16\} > q \). Then Theorem 2 holds and we are reduced to investigate the case \( e = 0 \) (Case I) or \( e = 1 \) (Case II) each with three subcases: \( f = -1 \) (case a), \( f = 0 \) (case b) and \( f = 1 \) (case c); together, this yields the Table 1, with six cases. Furthermore, if either \( q \not| h(p, q) \) or \( q \not| h_{pq} \), Corollary 2 holds and in particular the identity (23).

We aim next to eliminate the unit \( \varepsilon \) in this identity, using the fact that, by Proposition 3, the \( q \) - primary units of \( \mathbb{Q}(\zeta) \) are global \( q \)-th powers. This shall be done by a case by case study. In view of Lemma 8 we let \( 0 < m < q \) with \( m(p-1) \equiv 1 \mod q \) and \( \gamma = (1-\zeta^{p-1}/p)^m \) be the unit in \( \mathbb{Q}(\zeta) \).

Suppose first that \( e = 0 \) and thus \( \alpha = x + y\zeta \). In case a, \( x \equiv y \mod q^2 \) and

\[
\varepsilon \cdot \rho^q = \alpha \equiv x(1 + \zeta) \mod q^2.
\]

But then \( \delta = \frac{x}{1+\zeta} = q^{-1}x \) and by Lemma 7 it follows that \( \delta \in \mathbb{Z}[\zeta] \); consequently \( \alpha = (1 + \zeta)\rho^q \) in this case. If \( f = 0 \) (case b)), then \( x \equiv 0 \mod q^2 \) and \( \varepsilon = q^{-1}y \) and Lemma 7 shows that \( \varepsilon \) is a \( q \)-th power, so \( \alpha = \rho^q \) in this case. Finally, if \( f = 1 \), then \( \varepsilon = -y(1 - \zeta) \). Since \( \delta = \varepsilon : \gamma = q^{-1}y \), the Lemma 8 implies that \( \delta \) is a \( q \)-th power. It follows that \( \alpha = \gamma \cdot \rho^q \) in this case.

We assume next that \( e = 1 \) and consider the three possible values of \( f \). In case a, \( \alpha = \frac{x+y\zeta}{1-\zeta} \equiv x(1+\zeta)/(1-\zeta) \) and combining the Lemmata 7 and 8 we find that \( \alpha = (1 + \zeta)/\gamma \rho^q \). Likewise, \( \alpha = \rho^q/\gamma \) in case b) and \( \alpha = \rho^q \) in case c. We combine
Table 2. Values of the unit $\varepsilon$ in the six Cases

|   | a   | b   | c   |
|---|-----|-----|-----|
| I | $1+\zeta$ | 1   | $\gamma$ |
| II| $\frac{1+\zeta}{1/\gamma}$ | 1   | 1   |

all these results in Table 2 and the following

**Proposition 8.** Let $p, q$ be odd primes with $p \neq 1 \mod q$, $q \not| h(p, q)$ and suppose that $\max\{p, \frac{p(p-20)}{15}\} > q$ and the Fermat - Catalan equation (1) has a non trivial solution. Furthermore, let $0 < m < q$ be an integer with $m(p-1) \equiv 1 \mod p$ and $\gamma = \left(\frac{(1-\zeta)^{p-1}}{p}\right)^m \in \mathbb{Z}[\zeta]^\times$. Then for $e \in \{0,1\}$ and $f \in \{-1,0,1\}$ like in Table 1, the following identity holds (with $\delta_{a,b}$ being the Kronecker $\delta$ symbol):

$$\alpha = \frac{x + \zeta y}{1 - \zeta^e} = (1 + \zeta)^{\delta_{f,-1}} \cdot \zeta^{\delta_{f,1-e}} \cdot \rho^e, \quad \text{with} \quad \rho \in \mathbb{Q}(\zeta).$$

(52)

We proceed with a case by case analysis of possible solutions in the above six cases. The results come in different levels of complexity and require different class number conditions - essentially the two possibilities $q \not| h(p, q)$ or $q | h - pq$, mentioned above.

The simplest fact is that in three out of six cases, one deduces some Wieferich - type local conditions, involving only the exponents $p$ and $q$. This is the topic of the next section. In the following section, keeping the same class number condition, we show that one can give lower bounds on $\max\{|x|, |y|\}$: this is Theorem 3. We sharpen subsequently the class number condition to $q \not| h - pq$ and prove, by a generalization of Kummer descent - as used by Kummer in his Theorem on the Second Case of Fermat’s Last Theorem, [Wa] - that two additional cases ($f = -1$) are impossible. This leaves on last case - which we called the Astérisque - Case - untreated by conditions involving only the exponents $p, q$. By using the sharper class number condition, we are able to improve the lower bound in this case to one on the minimum $\min\{|x|, |y|\}$: this is Theorem 4 which is a first generalization of Catalan’s conjecture. Finally, by applying this Theorem together with an additional consequence of the Kummer descent, we prove the Theorem 5 on the rational case of Catalan’s equation, which is the most exhaustive result of this paper, since it shows the lack of solutions of Catalan’s equation in the rationals, provided some conditions hold, which are related only to the exponents.

7. The Wieferich Cases

We assume in this section that (1) has non-trivial solutions for odd prime exponents $p, q$ for which the premises of Theorem 2 hold. Based on this theorem, we can thus assume that the solutions are in one of the cases given in the above tables. The three simplest cases lead to some Wieferich - type (see [Kr]) condition.

**Proposition 9.** Notations being as above, if $e = 0$ and $f = -1$, then

$$2^{q-1} \equiv 1 \mod q^2.$$

Furthermore, $X = q \ Y = q \ 1$. 
Proof. Since \( e = 0 \), we are in Case I and \( X + Y = A^q \); also, \( f = -1 \) means \( X - Y \equiv 0 \mod q^2 \), so \( X = q Y \) and \( X + Y = q A^q = q 1 \). But from (I), \( X^p + Y^p = Z^q = q 2X = q 1 \). Dividing the last two relations, we find \( X^{p-1} = q 1 \) and since \( q \nmid p - 1 \) by hypothesis, it follows also that \( X = q 1 \). Combined with \( 2X = q 1 \) this yields the statement of the proposition. Since \( X = q Y \) by definition of this case and \( 2X = 2 = q 1 \), the second statement follows too. \( \square \)

**Proposition 10.** Notations being as above, if \( e = 1 \) and \( f = 0 \), then

\[ p^{q-1} \equiv 1 \mod q^2 \]

and \( Y = q 1 \).

**Proof.** Since \( e = 1 \), we are in Case II and \( \frac{X^p + Y^p}{X + Y} = p \cdot B^q = q p \); also, \( f = 0 \) means \( X \equiv 0 \mod q^2 \), so

\[ \frac{X^p + Y^p}{X + Y} = q Y^{p-1} = q p. \]

But \( X^p + Y^p = Z^q = q Y^p = q 1 \) and since \( (p, q) = 1 \), we must have \( Y = q 1 \), which is the second statement of the Proposition. Combined with the previous equivalence, this yields \( Y = q p = q 1 \), which leads to the first claim. \( \square \)

The third Wieferich case has a more complex statement. This is:

**Proposition 11.** Notations being as above, if \( e = 1 \) and \( f = -1 \), then

\[ (2^{p-1} \cdot p^p)^{q-1} \equiv 1 \mod q^2 \]

and \( X = q Y = q p^n \), where \( m(p - 1) \equiv 1 \mod q \).

**Proof.** Since \( e = 1 \), we are still in Case II, so \( \frac{X^p + Y^p}{X + Y} = p \cdot B^q = q p \); also, \( f = -1 \) implies \( X = q Y \) and \( \frac{X^p + Y^p}{X + Y} = q X^{p-1} = q p \), the second claim of the Proposition. Furthermore, \( X^p + Y^p = q 2X^p = q 2^{p+1} = q 1 \). Raising this to the power \( p - 1 \) and the preceding equivalence to the power \( p \), we find after division that:

\[ 2^{p-1} X^{p(p-1)} = q 2^{p-1} - p = q 1. \]

This is the first statement of the Proposition and completes the proof. \( \square \)

8. Lower Bounds and Proof of Theorem \( \Box \)

We assume in this section that (I) has non trivial solutions for odd primes \( p, q \) with \( p \neq 1 \mod q, q < \max\{\frac{2p-20}{k}\} \) and such that \( q \nmid h(p, q) \). The purpose of this section is to prove Theorem \( \Box \)

The following \( q \)-adic expansion will serve for gaining estimates in all six cases under investigation.

**Lemma 15.** Let \( \rho \in \mathcal{O}(L)^\times \) be an algebraic integer with \( q \)-adic expansion

\[ \rho = a \cdot \sum_{m=0}^{\infty} \left( \frac{1/q}{m} \right) (\mu b)^m, \quad \mu \in \{\zeta, 1/(1 \pm \zeta)\}, \quad a \in \mathbb{Z}_q, \quad b \in \mathbb{Q}^\times, \quad \nu_q(b) = k \geq 2. \]

Furthermore, suppose there is a real number such that \( |\sigma(\rho)| \geq M > 0 \) for all \( \sigma \in G_p \). Then

\[ M \geq \frac{1}{p - 1} \cdot \frac{q^{(p-2)(k - \frac{1}{2})}}{2^{p-2}}. \]
Proof. Let 
\[ \nu = (\zeta^2 - \zeta) \begin{cases} \frac{1}{\mu - (p - 3)} & \text{if } \mu = \zeta \\ \mu & \text{otherwise.} \end{cases} \]

It is an easy verification, that \( \text{Tr}_{K/q}(\nu \cdot \mu^j) = 0 \) for \( j = 0, 1, \ldots, p - 3 \). The case \( \mu = \zeta \) is trivial, since \( \text{Tr}\left(\zeta^{i+2} - \zeta^{i+1}\right) = (-1) - (-1) = 0 \) for \( 0 \leq i \leq p - 2 \). If \( \mu = 1/(1 \pm \zeta) \), then
\[ \text{Tr}(\nu \cdot \mu^j) = \sum_{i=0}^{p-3-j} (1)^i \binom{p-3-j}{i} \cdot \text{Tr}\left((\zeta^2 - \zeta)\zeta^j\right) = 0, \]
as claimed. Let now \( \delta = \nu \cdot \rho \in \mathcal{O}(K)^x \). The \( q \)-adic expansion of \( \rho \) together with the above remark on the trace of \( \nu \cdot \mu^j \) shows that the first \( p - 2 \) terms in the \( q \)-adic expansion of \( \Delta = \text{Tr}_{K/q}(\delta) \in \mathbb{Z} \) vanish. Thus
\[ \Delta = \left(\frac{1}{p - 2}\right)b^{p-2} \cdot \left(\text{Tr}(\mu^{p-2}) + O(q)\right). \]

Note that
\[ v_q\left(\frac{1}{n!}\right) = v_q\left(\frac{1}{q^n} \cdot \frac{(1-q) \cdots (1-(n-1)q)}{n!}\right) = -n - v_q(n!), \]

and since \( v_q(b) \geq k \), it follows that
\[ v_q\left(\frac{1}{n!}\right) \geq k(p-2) - (p-2) - v_q((p-2)!). \]
\[ > (p-2)(k-1 - 1/(q-1)) = (p-2)\left(k - \frac{q}{q-1}\right). \]

We now show that \( \Delta \neq 0 \). Assume first that \( \mu = \zeta \). Then from (54) we find that \( \Delta = \left(\frac{1}{p - 2}\right)b^{p-2}(p + o(q)) \). Since \( b \neq 0 \) the first \( q \)-adic term is non vanishing, and so \( \Delta \neq 0 \). The proof is similar for \( \mu = 1 \pm \zeta \). Finally, \( \Delta \) is a rational integer and by assembling all the information, we find:
\[ q^{(p-2)(k-1 - 1/(q-1))} \leq |\Delta| \leq \sum_{\sigma \in \mathcal{G}_p} |\sigma(\nu \cdot \rho)| \leq 2^{p-2} \cdot (p - 1) \cdot M. \]
The claim follows from these inequalities.

The proof of Theorem follows now from the Lemma and the fact that \( x + fy \equiv 0 \mod q^2 \).

Proof. We shall show in a case by case analysis, that a \( q \)-adic expansion such as required by Lemma exists. If \( f = 1 \), then (22) and (23) yield:
\[ \frac{x + \zeta y}{(1 - \zeta)^e} = q \frac{1 + \zeta}{\gamma^e} = p^{(m-q)e} \frac{1 + \zeta}{(1 - \zeta)^e} \cdot \rho^e, \]

and thus \( \rho \in \mathbb{Z}[\zeta] \) with
\[ \rho^e = p^{(q-m)e} \cdot y \left(1 + \frac{x - y}{y(1 + \zeta)}\right). \]
If \( e = 0 \), the expansion follows by Proposition [10] since the leading term is \( y = q \); if \( e = 1 \) the leading term is \( p(q-m)x = q \), by Proposition [11]. We still have to deduce the bound \( M \) from the expression of \( \rho \). But

\[
|\rho| = \left| p^{(q-m)e} \cdot \frac{x + \zeta y}{1 + \zeta} \right|^{1/q} \leq p^{(1-m)q/e} \cdot \left( \frac{|x| + |y|}{1/p} \right)^{1/q} \\
\leq \left( 2p^{1+(q-m)e} \cdot \max\{|x|,|y|\} \right)^{1/q}.
\]

the last estimate is obviously Galois invariant, so we can replace \( M \) in the Lemma [15] by this value. It follows that

\[
M = \left( 2p^{1+(q-m)e} \cdot \max\{|x|,|y|\} \right)^{1/q} \geq \frac{1}{p-1} \cdot \frac{q(p-2)(\frac{q-2}{2})^{p-2}}{2^{p-2}},
\]

and

\[
\max\{|x|,|y|\} \geq \frac{1}{2} \cdot \left( \frac{1}{p(p-1)} \cdot \left( \frac{q+2}{2} \right)^{-2} \right)
\]

for both cases, as claimed in the first inequality of (16).

Let now \( f = 0 \) and \( y \equiv 0 \mod q \), to fix the ideas. Then \( \frac{x+\zeta y}{1-\zeta} = \gamma^{-e} \rho \) and \( \rho^q = p^{(q-m)e}(x + \zeta y) \). If \( e = 1 \), the leading term is \( p^{(q-m)e}x = q \) by Proposition [11] otherwise, \( A^q = x + y = q \; x = q \), so the expansion of \( \rho \) follows in both cases. Furthermore,

\[
|\rho| = \left| p^{(q-m)e} \cdot (x + \zeta y) \right|^{1/q} \leq \left( p^{(q-m)e} \cdot 2 \cdot \max\{|x|,|y|\} \right)^{1/q} = M.
\]

Like before, by applying the Lemma [15] we find that \( \rho \) holds.

Finally, if \( f = 1 \), some usual computations yield

\[
\rho^q = \frac{x + \zeta y}{1-\zeta} \cdot p^{me} = -y \cdot p^{me} \cdot \left( 1 - \frac{x + y}{1-\zeta} \right).
\]

It follows immediately from the fact that the parenthesis on the right hand side of the last identity is a \( q \)-adic \( q \)-th power (since \( x + y \equiv 0 \mod q^2 \)) that so must then be the cofactor \( yp^{me} \); this shows the existence of the \( q \)-adic expansion of \( \rho \) required by Lemma [15]. The details for the estimation of \( M \) are analogous to the case \( f = -1 \) and are left to the reader. \( \square \)

9. KÜMMER DESCENT

We shall prove in this section the following main Theorem, which generalizes Kummer’s descent method to the present context.

**Theorem 6.** Let \( p, q > 3 \) be primes such that \(-1 \in \langle p \mod q \rangle \), \( \zeta, \xi \in \mathbb{C} \) be respectively \( p \)-th and \( q \)-th primitive roots of unity, \( L = \mathbb{Q}(\zeta, \xi) \) and \( \mathbb{L}^{++} \) the fixed field of the partial complex conjugations \( \rho_p, \rho_q \). Suppose that the equation

\[
X^q + Y^q = \varepsilon \cdot \lambda^N \cdot \lambda' M \cdot Z^q
\]

admits solutions with \( X, Y, Z \in \mathcal{O}(\mathbb{L}^{++}) \), \( X \cdot Y \cdot Z \neq 0 \) and \( (X \cdot Y \cdot Z, p \cdot q) = (1) \) and \( X, Y \) are not units. Here \( \lambda = (\xi - \bar{\zeta}), \lambda' = (\zeta - \bar{\xi}) \) and \( \varepsilon \in \mathcal{O}(\mathbb{L}^{++}) \). \( M, N \) are integers with \( N > 2q, N \) even and \( M = 0 \) or \( M \geq 2 \). Then \( Z \) is not a unit and \( \sqrt{p/q} \).
The next Lemma will explain the condition $-1 \in \mathfrak{p} \mod q >$.

**Lemma 16.** Let $p, q$ be odd primes and $\mathbb{K}' = \mathbb{Q}(\xi)$ be the $q$–th cyclotomic extension. Then $p$ splits in $\mathbb{K}'$ in real prime ideals iff $-1 \in \mathfrak{p} \mod q >.$

**Proof.** This is a direct consequence of Kummer’s Theorem on the splitting of primes in extensions with a power base for the ring of algebraic integers [14]. Let $\Phi_q(X) = \prod_{i=1}^{n-1} (X - \xi^i)$ be the $q$–th cyclotomic polynomial, $n = \text{ord}_q(p) = | < p \mod q > |$ and $F(X) = \prod_{j=0}^{n-1} (X - \xi^j) \in \mathcal{Z}[\xi][X]$. If $\mathbf{k} = \mathbb{F}_p^n$ is the finite field with $p^n$ elements, then $\mathbf{k}$ is the smallest field of characteristic $p$ which contains a non trivial $q$–th root of unity. Let $\rho \in \mathbf{k}$ be such a root of unity. Then there is a natural map $\iota : \mathcal{O}(\mathbb{K}') \to \mathbf{k}$ given by $\xi \mapsto \rho$. Let then $\tilde{f}(X) = \iota(F(X)) \in \mathbb{F}_p(X)$ and $f \in \mathcal{Z}[X]$ be some polynomial with $\tilde{f} = f \mod p$. Then $\tilde{f} \in \mathbb{F}_p[X]$ is an irreducible factor of $\Phi(X) \mod p$; if $\mathfrak{p} = (f(\xi), p)$, then $\mathfrak{p}$ is a prime above $(p)$ and each prime above $(p)$ arises in this way, by a choice of $\rho \in \mathbf{k}$. In particular, $\xi \mod \mathfrak{p} = \rho$ and $\iota$ is in fact the reduction $\mod \mathfrak{p}$ map. Furthermore, $\tilde{f} = \iota(F(X)) = f(X) \mod p$, where in general only the second polynomial has rational integer coefficients.

After this exposition of Kummer’s Theorem, we can proceed with the proof of our Lemma. First note that since $\mathbb{K}'/\mathbb{Q}$ is a CM Galois extension, all the primes above $(p)$ are simultaneously real or not real. Let us first suppose that $\mathfrak{p} = (f(\xi), p)$ is a real ideal. Since $f(X) = F(X)$ mod $\mathfrak{p}$ and $\overline{\mathfrak{p}} = \mathfrak{p}$, it follows that $F(X) = \overline{F(X)}$ and under the action of $\iota$,

$$f(X) = \prod_{i=0}^{n-1} (X - \rho^i) = \prod_{i=0}^{n-1} (X - \rho^{-i})$$

But $\rho \in \mathbf{k}$, which is a field in which $\tilde{f}(X)$ has unique decomposition. Thus $\rho^{-1} \in \{\rho^i : i = 0, 1, 2, \ldots, n - 1\}$ and by the definition of $n$ it follows that $-1 \in \mathfrak{p} \mod q >$, as claimed. Conversely, if $-1 \in \mathfrak{p} \mod q >$, then $F(X) = \overline{F(X)}$ and it follows that $\mathfrak{p}$ is invariant under complex conjugation. \qed

We proceed with the proof of the Theorem, assume that $q \not\mid h_{pq}$ holds under the given hypotheses and will derive a contradiction. The two statements of the theorem are apparently contradictory: if we show that $q \not\mid h_{pq}$ is impossible, then it is irrelevant whether $Z$ is a unit or not. For technical reasons, however, it will be useful to show that under the given premises, if $X, Y$ are not units, then neither is $Z$. Note that by Proposition 8 it follows that $E_q = E^q$ and $\mathcal{C} = \mathcal{C}^q$, with $\mathcal{C}$ the ideal class group of $\mathbb{L}$. The quite lengthy proof is a straightforward adaption of the descent method used by Kummer in the proof of his fundamental Theorem on the Second Case of Fermat’s Last Theorem (see [14], Chapter 9). The additional problems are linked to the fact that we work in a larger field.

We start with a simple fact:

**Lemma 17.** Let $X, Y \in \mathbb{L}^{++}$ verify the premises of the theorem; in particular, suppose that $X^q + Y^q \equiv 0 \mod \lambda^N, \lambda^M$ and $v_\lambda(X^q + Y^q) = N, v_\lambda(X^q + Y^q) = M$. Then $v_\lambda(X + Y) = N - (q - 1)$ and $v_\lambda(X + Y) = M$.

**Proof.** Any integer $\gamma \in \mathcal{O}(\mathbb{L})$ has the $\lambda$ - development

$$\gamma = \sum_{i=0}^{G} g_i \cdot \lambda^i, \text{ for some } G \in \mathbb{N},$$
and
\[ g_i = \sum_{j=1}^{p-1} g_{i,j} \xi^j \in \mathbb{Z}[\xi], \quad 0 \leq g_{i,j} < q. \]

Let \( X = x_0 + x_1 \cdot \lambda + O(\lambda^2) \). Since \( J_0(\lambda) = -\lambda \) and \( X \in \mathbb{L}^{++} \), so \( J_0(X) = X \), we must have \( x_1 = 0 \), so \( X = x_0 + O(\lambda^2) \). Likewise, \( Y = y_0 + O(\lambda^2) \). Thus (59) implies that \( x_0^q + y_0^q \equiv 0 \mod \lambda^2 \) and since \((X \cdot Y, q) = 1\), we have \((x_0/y_0)^q \equiv -1 \mod \lambda^2\). Since \( \mathbb{Z}[\xi]/(q\mathbb{Z}[\xi]) \) contains no \( q \)-th roots of unity except 1, it follows that \( x_0/y_0 \equiv -1 \mod \lambda^2 \) and \( x_0 + y_0 \equiv X + Y \equiv 0 \mod \lambda^2 \). The algebraic integers
\[
\phi_i = \xi^i X + \bar{\xi} Y \in \mathbb{L}, \quad \text{for } i = 1, 2, \ldots, q-1
\]
have the common divisor
\[
(\phi_i, \phi_j) = \left( (\xi^i - \xi^j) \cdot Y, (\bar{\xi}^i - \bar{\xi}^j) \cdot X \right) = (\lambda).
\]

But \( (X + Y) \cdot \prod_{i=1}^{q-1} \phi_i = X^q + Y^q \equiv 0 \mod \lambda^N \). Thus \( v_\lambda(X + Y) = N - (q - 1) \), as claimed.

Due to \((\phi_i, \phi_j) = (\lambda)\), it follows also that if \( \mathfrak{P}[\lambda'] \) is a prime ideal of \( \mathbb{L}^{++} \) with \( \mathfrak{P}[(X + Y)] \), then \( \mathfrak{P}[\lambda'][(X + Y)] \). The primes above \((p) \in \mathbb{L}\) are \((p, (1 - \xi))\) for some prime \( p \in \mathbb{K} \) and the hypothesis together with Lemma 17 imply that they are real primes. Suppose that \( M > 0 \) (there is nothing to prove for \( M = 0 \)); then for \( \mathfrak{P}[(p)] \) we have \( X^q + Y^q \equiv 0 \mod \mathfrak{P} \) and \(-X/Y \equiv 1 \mod \mathfrak{P} \). Since \( \mathfrak{P} \) is a prime ideal, \( O(\mathbb{L}^{++})/\mathfrak{P} \) is a field and there is an integer \( 0 \leq a < q \) such that \(-X/Y \equiv \xi^a \mod \mathfrak{P} \). Taking complex conjugates - under consideration of the fact that \( X/Y \) is invariant under conjugation - we also have \(-X/Y \equiv \xi^{-a} \mod \mathfrak{P} \). But since \( \mathfrak{P} = \mathfrak{P} \), it follows that \( \xi^{2a} \equiv 1 \mod \mathfrak{P} \) and \( a = 0 \). This holds for all primes above \( (p) \) and together with the previous remark implies that \( v_\lambda(X + Y) = M \), as claimed.

We wish to normalize the algebraic integers defined in (57), eliminating all primes above \( q \) and \( p \). Using the result of the Lemma 17 this can be done as follows:

\[
\phi_i = \frac{\xi^i X + \bar{\xi} Y}{\xi^i - \bar{\xi}}, \quad \text{for } i = 1, 2, \ldots, q-1
\]
\[
\phi_0 = \frac{q(X + Y)}{\varepsilon \cdot \lambda^N \cdot \lambda^M}.
\]
It follows from the Lemma 17 that \((\phi_i, p \cdot q) = (1)\) for \( i = 0, 1, \ldots, q-1 \) and
\[
\prod_{i=0}^{q-1} \phi_i = \frac{1}{q} \cdot \phi_0 \cdot \prod_{i=1}^{q-1} \phi_i = \frac{q}{q \cdot \varepsilon \cdot \lambda^N \cdot \lambda^M} \cdot (X + Y) \cdot \prod_{i=1}^{q-1} \phi_i = \frac{X^q + Y^q}{\varepsilon \cdot \lambda^N \cdot \lambda^M}.
\]
Finally, this yields
\[
\prod_{i=0}^{q-1} \phi_i = Z^q.
\]

If \( \mathbb{L}_p \subset \mathbb{L} \) is the subfield fixed by \( J_p \), the definition of \( \phi_i \) implies \( \phi_i \in \mathbb{L}_p \) for \( i > 0 \) and (60) shows that this holds also for \( \phi_0 \):

\[
\phi_i \in \mathbb{L}_p \quad i = 0, 1, \ldots, q-1.
\]
From \((\phi_i', \phi_j') = (\lambda)\) and the definition of \(\phi_i\) we deduce that

\[(62) \quad (\phi_i, \phi_j) = 1 \quad \text{for} \quad i, j \in \{0, 1, \ldots, q-1\}, \quad \text{and} \quad i \neq j.\]

We want to show that \(\phi_i\) are not units. This implies that \(Z\) is not a unit, as a consequence of \((60)\). For \(\phi_0\), this fact will follow indirectly, with more work. We prove it first only for \(i > 0\) and investigate the \(q\)-expansion of \(\phi_i\):

\[(63) \quad \phi_i = \frac{\xi^i X + \xi^{-i} Y}{\xi^i - \xi^{-i}} = \frac{\xi^i (X + Y) - (\xi^i - \xi^{-i}) \cdot Y}{\xi^i - \xi^{-i}} = -Y + \frac{\xi^i (X + Y)}{\xi^i - \xi^{-i}} = q \cdot -Y.\]

Note that \(\phi_{q^{-i}} = -\overline{\phi_i}\). If \(\phi_i\) is a unit, then \(\delta = \phi_i / \phi_{q^{-i}} = \phi_i / \overline{\phi_i}\) is a root of unity and by \((63)\), since \(Y \in \mathbb{R}\), it follows that \(\delta = q\). By Lemma \(18\) it follows that \(\delta = \pm \zeta^a\) for some \(a \in \mathbb{Z}/(p \cdot \mathbb{Z})\). Then \(\zeta^{-a/2} \phi_i = \pm \zeta^{a/2} \phi_{q^{-i}}\) and a short computation shows that

\[X = -Y \cdot \frac{\zeta^{a/2} \cdot \xi \mp \zeta^{a/2} \cdot \xi}{\zeta^{a/2} \cdot \xi \mp \zeta^{a/2} \cdot \xi} = \gamma \cdot Y.\]

But \(\gamma\) is a unit and \(X = \gamma Y\) is a contradiction to \((X, Y) = (1)\), since \(X\) and \(Y\) are not units. The contradiction confirms our claim that \(\phi_i\) are not units, for \(i > 0\); thus \(Z\) is not a unit.

We can now apply Lemma \(18\) with \(n = q, C = Z, \mathbb{L}' = \mathbb{L}_p\), thus obtaining:

**Lemma 18.** Let the premises of Theorem \(7\) hold, the normalized elements \(\phi_i, i > 0\) be defined by \((58)\) and the ideals \(\mathfrak{A}_i = (\phi_i, Z)\). If \(\mathbb{L}_p \subset \mathbb{Q}(\zeta, \xi)\) is the subfield fixed by \(j_p\), then \(\mathfrak{A}_i\) are principal and there are \(\mu_i' \in \mathcal{O}(\mathbb{L}_p)\) and \(\eta_i \in (\mathcal{O}(\mathbb{L}_p))^\times\), such that \((27)\) holds.

Note that if \(\phi_0\) is not a unit, the result of Lemma \(18\) also holds for \(\phi_0\). We proceed our proof, allowing for both possibilities. It will turn out that the same computations which allow descent also imply in the long run that \(\phi_0\) is not a unit.

From \((60)\), \(\phi_0 = \frac{Z^\gamma}{\prod_{i \neq 0} \phi_i} = q \cdot (-Y)^{1-q} = q \cdot -Y\). If \(\phi_0\) is not a unit, we saw that we can write \(\phi_0 = \eta_0 \cdot \mu_0^q\), with \(\eta_i, \mu_i\) like in \((27)\); otherwise we may set \(\phi_0 = \eta_0\). In both cases, the unit \(\eta_0\) is defined and \(\eta_0 = q \cdot \phi_0 = q \cdot -Y\).

Thus, for all \(i = 0, 1, \ldots, q-1\), we have \(\eta_i = q \cdot -Y\); the Lemma \(7\) implies that \(\eta_i\) must be \(q\)-th powers, so

\[(64) \quad \phi_i = \mu_i^q, \quad \text{for} \quad i = 0, \ldots, q - 1.\]

If \(\phi_0\) is a unit, then the previous remarks imply that \(\phi_0 = \mu_0^q\), with \(\mu_0\) a unit of the same field. Otherwise, by the same reasoning as in the case \(i > 0\), \(\mu_0 \in \mathcal{O}(\mathbb{L}_p)\).
We are prepared for the main computations which will allow to perform the descent. We evaluate $\phi_i \times \phi_{-i}$ for $i > 0$, using the identity in (63):

$$
\psi_i = \phi_i \times \phi_{-i} = \left(-Y + \frac{\xi^i(X + Y)}{\xi^i - \xi^{-i}}\right) \cdot \left(-Y + \frac{\xi^{-i}(X + Y)}{\xi^{-i} - \xi^i}\right) = \left(\frac{X + Y}{1 - \xi^{2i}} - Y\right) \cdot \left(\frac{X + Y}{1 - \xi^{-2i}} - Y\right) = Y^2 + \frac{(X + Y)^2}{(1 - \xi^{2i})} - Y \cdot (X + Y) \cdot \left(\frac{1}{1 - \xi^{2i}} + \frac{1}{1 - \xi^{-2i}}\right)
$$

The last equation above shows that $\psi_i \in \mathcal{O}(L^+)$.

By subtracting the values of $\psi$ for two indices $i \neq \pm j \mod q$ we find $\psi_i - \psi_j = \delta_{i,j} \cdot (X + Y)^2$, with $\delta_{i,j} = 1/[(1 - \xi^{2i}) - 1/(1 - \xi^{2j})]$. For the choice of such indices we need here that $q \geq 5$. We claim that $\lambda^2 \cdot \delta_{i,j} = \eta_{i,j} \in \mathcal{O}(L^+)^\times$. Indeed,

$$
\lambda^2 \cdot \delta_{i,j} = \frac{\lambda^2}{|(1 - \xi^{2i})(1 - \xi^{2j})|^2} \cdot \left((2 - \xi^{2i} - \xi^{-2i}) - (2 - \xi^{2j} - \xi^{-2j})\right)
$$

In our definition $\lambda = \xi - \overline{\xi}$ is an imaginary number, so $\lambda^2$ is real and so is $\lambda^2 \cdot \delta_{i,j}$. The last equality above shows that $v_\eta \left(\lambda^2 \cdot \delta_{i,j}\right) = 0$ and since it is real and invariant under $\eta$ it follows that $\eta_{i,j} \in \mathcal{O}(L^+)^\times$, as claimed.

We now substitute the definition (64) of $\phi_i$ and (64) in the recent results, finding:

$$
\psi_i - \psi_j = \eta^2 \cdot \left((\mu_i \cdot \mu_{-q-i})^q - (\mu_j \cdot \mu_{-q-j})^q\right) = \eta_{i,j} \cdot \lambda^{-2} \cdot (X + Y)^2 = \eta_{i,j} \cdot \lambda^{-2} \cdot \left(\frac{\varepsilon \cdot \lambda^N \cdot \lambda^{2M} \cdot \phi_0}{q}\right)^2
$$

After division by $\eta^2$ this yields:

$$
(\mu_i \cdot \mu_{-q-i})^q - (\mu_j \cdot \mu_{-q-j})^q = \eta' \cdot \lambda'^M \cdot \lambda'^N \cdot \mu_0^{2q},
$$

(67)
where \( \eta' = \left( \delta_{i,j} \cdot \left( \frac{\lambda_{i-1}}{q} \right)^2 \right) \in \mathcal{O}(L^{++}) \) and \( N' = 2(N - q) = N + (N - 2q) > N \) is even, \( M' = 2M \). Also, by (65), the numbers occurring at the \( q \)-th power in (67) are elements of \( L^{++} \).

We have shown that for \( i > 0 \), \( \phi_i \) are not units, and thus the \( q \)-th powers on the left hand side of (67) are neither units. It was shown that Theorem 6 are verified. The equation (68) is then a reformulation of (67).

Proof. The Proposition 12 can be applied recursively to (68), thus generating an infinite sequence

\[
(Z) = (Z^{(0)}) \subset (Z^{(1)}) \subset (Z^{(2)}) \subset \ldots \subset (Z^{(k)}) \subset \ldots,
\]

such that \( \omega(Z^{(k)}) > \omega(Z^{(k+1)}) \) for all \( k \geq 0 \). But \( Z = Z^{(0)} \) has only a finite number of prime factors and the function \( \omega \) is positive integer valued, so it cannot decrease indefinitely. This is a contradiction which shows that the hypothesis \( q \not| h_{pq} \) of Proposition 12 is untenable, thus proving Theorem 6.
10. Case Analysis

We consider two primes \( p, q > 3 \) such that \( q \nmid h_{pq}^- \) and \(-1 \in \langle p \mod q \rangle \) and suppose that (11) holds for these values of \( p, q \). The Barlow - Abel relations imply then that

\[
\frac{X^p + Y^p}{X + Y} = p^e \cdot A^q,
\]

for some \( e \in \{0, 1\} \) and \( A \in \mathbb{Z} \). By theorem 2

\[
x + f \cdot y \equiv 0 \mod q^2 \quad \text{with} \quad f \in \{-1, 0, 1\}.
\]

Together this yields six cases, three of which have been dealt with above, by means of Wieferich relations. We shall investigate below the remaining cases.

10.1. The Descent Cases.

Theorem 7. Notations being as above and assuming the premises of Theorem 4, the equation (1) has no solution with \( e = 1, f = -1 \).

Proof. Assume that (11) has a solution with \( e = 1, f = -1 \). Then \((X + Y)/p\) is a \( q\)-th power, divisible by \( p \cdot q \). Let \( v_q(X + Y) = nq \) and \( v_p(X + Y) = mq - 1 \), so

\[
X + Y = p^{mq-1} \cdot q^{nq} \cdot C^q \quad C \in \mathbb{Z}, \quad (C, pq) = 1.
\]

By 52 we have in the present case:

\[
(69) \quad \frac{x + \zeta y}{1 - \zeta} = -y + \frac{x + y}{1 - \zeta} = \rho^a.
\]

Note that

\[
\alpha - \alpha = \frac{x + y}{1 - \zeta} - \frac{x + y}{1 - \zeta} = \frac{(x + y)(1 + \zeta)}{1 - \zeta} = -\frac{(x + y)(\zeta^a + \zeta^{-a})}{\zeta^a - \zeta^{-a}} \quad \text{so}
\]

\[
\rho^a - \rho^a = \prod_{j=0}^{q-1} (\zeta^j \cdot \rho - \zeta^{-j} \cdot \rho) = -C_q \cdot (p^m \cdot q^n)^q \cdot \frac{\zeta^a + \zeta^{-a}}{p(\zeta^a - \zeta^{-a})},
\]

with \( a = (p + 1)/2 \).

We define the following system of normed divisors of \( C^q \):

\[
(70) \quad \phi_i = \frac{\xi^i \rho - \xi \rho^a}{\xi^i - \xi}
\]

\[
(71) \quad \phi_0 = -\frac{\rho - \rho^a}{p^{mq-1} \cdot q^{nq-1}} \cdot \frac{\zeta^a - \zeta^{-a}}{\zeta^a + \zeta^{-a}}.
\]

Let

\[
\varepsilon_1 = \prod_{c=1}^{p-1} \frac{\zeta^c - \zeta^{-c}}{\lambda^c} \in \mathbb{Z}[\zeta + \zeta^{-1}]^{\times} \subset \mathcal{O}(L^+)^{\times},
\]

\[
\varepsilon_2 = \prod_{c=1}^{q-1} \frac{\zeta^c - \zeta^{-c}}{\lambda} \in \mathbb{Z}[\zeta + \zeta^{-1}]^{\times} \subset \mathcal{O}(L^+)^{\times}.
\]

Then \( p = \varepsilon_1 \cdot \lambda^{p-1} \) and \( q = \varepsilon_2 \cdot \lambda^{q-1} \) and (71) can be rewritten as

\[
\rho - \rho^a = \varepsilon \cdot \lambda^{(q-1)(nq-1)} \cdot \lambda^{(p-1)(mq-1) - 1} \cdot \phi_0,
\]
with
\[ \varepsilon = \varepsilon_1^{mq-1} \cdot \varepsilon_2^{nq-1} \cdot \frac{(\zeta^a + \overline{\zeta})^t}{\zeta^a - \zeta} \in \mathcal{O}(\mathbb{L}^+)^\times. \]

Finally, with \( N = (q-1)(nq-1) \) and \( M = (p-1)(mq-1) - 1 \), we have
\[ (72) \quad \rho - \overline{\rho} = \varepsilon \cdot \lambda^N \cdot \lambda^M \cdot \phi_0 \quad \text{with} \quad \varepsilon \in \mathcal{O}(\mathbb{L}^+)^\times. \]

The rest of the proof goes through a series of steps which were proved in detail in the previous section, so we list the arguments, leaving it to the reader to check the details.

We have by construction \( \prod_{i=0}^{q-1} \phi_i = C^q \) and since \((C, pq) = 1\), a fortiori \((\phi_i, pq) = (1)\). Since \((\rho, \overline{\rho}) = (1)\), one verifies that \((\phi_i, \phi_j) = (1)\) for \( 0 \leq i \neq j < q \). We can apply Lemma 5 to \( \phi_i \), with \( m = q, L' = L_p \), and find
\[ \phi_i = \eta_i \cdot \mu_i^q \quad \eta_i \in \mathcal{O}(L_p)^\times, \quad \mu_i \in \mathcal{O}(L_p). \]

We have from (69) that \( \rho^q = q - y \); then there is an integer \( t \) with \( t^q \equiv -y \mod q^{nq} \) and the \( q \)-adic development based on (69) yields \( \rho \equiv \overline{\rho} \equiv t \mod q^{nq-1} \). But for \( i > 0 \) we have \( \eta_i = q \phi_i = q \) and since \( \phi_0 = C^q / \prod_{i>0} \phi_i \), we also have \( \eta_0 = q \phi_0 = q \).

Consequently, we may assume that
\[ \phi_i = \eta_0 \mu_i^q, \]
and \( \eta_0 \in \mathbb{Z}[\zeta + \overline{\zeta}]^\times \subset \mathcal{O}(\mathbb{L}^+)^\times \).

Finally we define \( \psi_i = \phi_i \cdot \phi_{q-i} \in \mathcal{O}(\mathbb{L}^+) \) and verify that for \( i \neq \pm j \mod q \) we have
\[
\psi_i + \psi_j = \eta_0^2 \cdot ((\mu_i \cdot \mu_{q-i})^q + (\mu_j \cdot \mu_{q-j})^q) = \eta_{i,j} \cdot \lambda^{-2} \cdot (\rho - \overline{\rho})^2
\]
where \( \eta_{i,j} = \lambda^2 \cdot \delta_{i,j} \) is the unit in (66). After dividing by \( \eta_0^2 \), we set \( X = \mu_i \cdot \mu_{q-i}, Y = \mu_j \cdot \mu_{q-j}, Z = \mu_0^2, N' = 2(N - 1), M' = 2M, \varepsilon' = \varepsilon^2 \cdot \eta_{i,j} \) and find:
\[ X^q + Y^q = \varepsilon' \cdot \lambda^{N'} \cdot \lambda^{M'} \cdot Z^n. \]

The hypotheses \((X, Y, Z) = (XYZ, pq) = (1), X, Y, Z, \varepsilon' \in \mathbb{L}^+, N > 2q \) is even and \( M \geq 0 \) being all fulfilled, as has been showed above, we can apply the Kummer descent Theorem 3. This raises a contradiction to \( q \nmid h_{pq} \), which proves the statement of this Proposition. \( \square \)

Next we treat the case \( p \not| z, q|x + y: \)

**Theorem 8.** Notations being as above and assuming the premises of Theorem 3, the equation (1) has no solution with \( e = 0, f = 1. \)

**Proof.** This is a case with \((X + Y, p) = (1) \) and \( M = 0 \) in the descent theorem. By Corollary 2, we have \( \alpha = \varepsilon \cdot \rho^q \). The \( q \)-adic development of \( \rho \) is more delicate in this case and we shall work it out in detail - the rest of the proof being exempt of surprises. We are in the First Case and
\[
(73) \quad x + y \equiv 0 \mod q^9, \]
so
\[
\frac{x^p + y^p}{x + y} = q \quad \Rightarrow \quad B^q = q \cdot 1. \]
If \( m(p - 1) = 1 + nq \), \([52]\) yields in this case, for a \( \rho \) twisted by a root of unity:
\[
\rho^q = -\zeta^{-m/2} \cdot (1 - \zeta)^{-nq} \cdot ypp^n \cdot \left(1 - \frac{x + y}{(1 - \zeta)ypp^n}\right).
\]
Since the cofactor of \( ypp^n \) is a \(-q\)-adic \(-q\)-th power, it follows from the above equation that \( ypp^n \equiv 1 \) if there is a \( t \in \mathbb{Z} \) with \( t^q \equiv -ypp^n \bmod q^2 \). The ring \( \mathbb{Z}/(q \cdot \mathbb{Z})[\zeta] \) contains no non trivial \(-q\)-th roots of unity (since \( q \) in not ramified in \( \mathbb{Z}_q[\zeta] \)), so the resulting \(-q\)-adic extension of \( \rho \) starts as follows:
\[
\rho = \frac{t}{\xi^{-m/2q}(1 - \zeta)^m} \left(1 - \frac{x + y}{(1 - \zeta)ypp^n} + O(q^{2(q-1)})\right).
\]
From the definition it follows that \( n \) is odd and one verifies that \( \rho \) verifies the necessary condition \( (\rho/\mathcal{R})^q \equiv -\xi \bmod q^2 \).

We now investigate an adequate factoring of \( x + y = B^q \). We have
\[
(\zeta^{-1/2} + \zeta^{1/2})(x + y) = \zeta^{-1/2}(\alpha + \zeta^{1/2}t) = \varepsilon \cdot \left(\zeta^{-1/2} \rho^q + \zeta^{1/2} \mathcal{R}\right).
\]
Defining \( \rho_1 = \zeta^{1/2q} \cdot \rho \), we have \( \rho_1/\mathcal{R} \equiv -1 \bmod q^{q-1} \) and
\[
\varepsilon \cdot (\rho_1 + \mathcal{R}) = (\zeta^{1/2} + \zeta^{1/2}) \cdot B^q.
\]
this looks like a good starting point. Let \( q^{aq}\| (x + y) \), \( C = B/q^n \) with \( (C, pq) = 1 \) and define
\[
\phi_i = \frac{\xi^i \rho_1 + \overline{\xi} \overline{\rho}_1}{\xi^i - \overline{\xi}^i} \quad \text{for} \quad i = 1, 2, \ldots, q - 1 \quad \text{and} \quad \phi_0 = \frac{\varepsilon (\rho_1 + \mathcal{R})}{q^{aq-1} \cdot (\zeta^{1/2} + \overline{\zeta}^{1/2})}.
\]
Note that \( \phi_0 \) is an algebraic integer, since \( \rho_1/\mathcal{R} \equiv -1 \bmod q^{aq-1} \). Then
\[
\prod_{i=0}^{q-1} \phi_i = \left(\frac{\rho_1 + \mathcal{R}}{q}\right) \times \left(\frac{\varepsilon}{q^{aq-1} \cdot (\zeta^{1/2} + \overline{\zeta}^{1/2})}\right) = (B/q^n)^q = C^q
\]
According to the usual frame, one verifies that \( (\phi_i, \phi_j) = 1 \) for \( 0 \leq i \neq j < q \) and by Lemma \([6]\) it follows that
\[
\phi_i = \eta_i \cdot \mu_i^q, \quad \text{with} \quad \eta_i \in \mathcal{O}(L_p)^\times, \quad \mu_i \in \mathcal{O}(L_p).
\]
Since \( \phi_i \equiv -\overline{\rho}_1 \bmod \left(\frac{\rho_1 + \mathcal{R}}{L}\right) \) and \( \phi_0 = C^q/\prod_{i=0}^{q-1} \phi_i \), it follows that \( \phi_i = \eta_i \cdot \mu_i^q = -\rho_1 \) and, using Lemma \([6]\) one deduces, after eventual modification of \( \mu_i \), that
\[
\phi_i = \eta_0 \cdot \mu_i^q \quad i = 0, 1, \ldots, q - 1.
\]
Next we choose \( i \neq \pm j \bmod q \), let
\[
\psi_i = \phi_i \cdot \phi_{q-i} \quad \text{and} \quad \psi_j = \phi_j \cdot \phi_{q-j},
\]
and verify
\[
\psi_i - \psi_j = \eta_{i,j} \cdot \lambda^{-2} \cdot (\rho_1 + \overline{\rho}_1)^2
\]
\[
= \eta_{i,j} \cdot \lambda^{-2} \cdot (\varepsilon q^{aq-1} \cdot (\zeta^{1/2} + \overline{\zeta}^{1/2}))^2.
\]
We let \( X = \mu_i \cdot \mu_{q-i}, Y = -\mu_j \cdot \mu_{q-j} \) and \( Z = \mu_{q}. \) While \( \mu_i \) are imaginary numbers, \( X, Y \) are real, so \( X, Y \in \mathbb{L}_p \cap \mathbb{R} = \mathbb{L}^+. \) \( \mathbb{L}^+ \) trivially, \( Z \in \mathbb{Z}[\zeta + \bar{\zeta}] \subset \mathbb{L}^+. \) Now write \( q = \varepsilon_1 \cdot \lambda^{r-1} \) for the obvious real unit \( \varepsilon_1 \in \mathcal{O}(\mathbb{L}^+)\times, \) set \( N = 2((q-1)(nq-1) - 1) \) and
\[
\delta = \eta_{i,j} \cdot \left( \frac{\varepsilon_1^{1/2} + \varepsilon_1^{-1/2}}{\varepsilon_1} \right)^2 \cdot \varepsilon_1^N \in \mathcal{O}(\mathbb{L}^+)\times.
\]
Inserting these new notations in (74) leads, after division by \( \eta_0^2 \), to:
\[
X^q + Y^q = \delta \cdot \lambda^N \cdot Z^q.
\]
Once again we can apply Theorem 1 obtaining a contradiction with \( q \not| h_{pq}^- \). This completes the proof of this case. \( \Box \)

10.2. The Astérisque Case. We denote the case \( e = f = 0 \) by astérisque case.
This is the only case in which our results still depend on \( x \) and \( y \) - the obstruction to a more general result.

We suppose that \( x \equiv 0 \mod q^2 \) and since \( e = 0 \), then \( A^q = y + x = q \), so that \( y \) is a \( q \)-adic \( q \)-th power. By (52) there is in this case a \( \rho \in \mathbb{Z}[\zeta] \) such that \( \rho^q = y + x \cdot \zeta^{2q} \) and
\[
\frac{\zeta q \cdot \rho^q + \zeta q \cdot \rho^q}{\zeta q + \zeta q} = x + y = A^q.
\]
By the usual argument of Lemma 3 we have then
\[
\phi_i = \frac{\zeta q \cdot \rho + \zeta q \cdot \rho}{\zeta q + \zeta q} = \delta_i \cdot \mu_i^q,
\]
for some units \( \delta_i \in \mathbb{Z}[\zeta, \xi] \) and \( i = 1, 2, \ldots, q \). Next, we investigate these units \( q \)-adically. Let \( t^q \equiv y \mod q^N \), for some integer \( t \equiv A \mod q^{n\sigma(x)-1} \), or, likewise, \( t \) be a \( q \)-adic approximation of the \( q \)-th root of \( y \). Then
\[
\delta_i = q \cdot t \cdot \left( 1 + \frac{\zeta^{q-2q} \cdot \xi^i + \zeta^{q-2q} \cdot \xi^i}{\zeta^{q} + \zeta^{q}} \cdot \frac{x}{qy} \right).
\]
In particular
\[
\delta_0 = q \cdot t \cdot \left( 1 + \frac{\zeta^{q-2q} \cdot \xi^i + \zeta^{q-2q} \cdot \xi^i}{\zeta^{q} + \zeta^{q}} \cdot \frac{x}{qy} \right) \in \mathbb{Z}[\zeta].
\]
Let \( \lambda = (1 - \xi) \) (note the deviation from the usual definition of \( \lambda! \)), so \( \xi^i \equiv 1 - i \lambda \mod \lambda^2 \). A further investigation of the units \( \psi_i = \delta_i / \delta_0 \) shows that \( \psi_i = 1 + \frac{x}{qy} \cdot c(\zeta) \cdot \lambda + O(q\lambda^2) \), with
\[
\lambda c(\zeta) \equiv \frac{\zeta^{2q-1} \cdot \xi^i + \zeta^{2q-1} \cdot \xi^i - \zeta^{2q-1} \cdot \zeta^{2q-1}}{\zeta^{2q-1} + \zeta^{2q-1}} \equiv 2i \cdot \lambda \cdot \frac{\zeta^{2q} - \zeta^{2q}}{(\zeta + \bar{\zeta})^2} \mod \lambda^2,
\]
and thus \( c(\zeta) = 2i \cdot \frac{\zeta^{2q} + \zeta^{2q}}{(\zeta + \bar{\zeta})^2} \neq 0 \). Thus Lemma 12 implies that \( \psi_i \) must be a \( q \)-th power and \( c(\zeta) = \sigma_q(\beta) - \beta \mod q \), for some \( \beta \in \mathbb{Z}[\zeta] \). In particular, if \( q \equiv 1 \mod p \), then \( \sigma_q(\beta) \equiv \beta \mod \Omega \) for all degree one primes \( \Omega \) of \( \mathbb{Z}[\zeta] \). But then \( c(\zeta) \equiv 0 \mod q \), which is in contradiction with \( N_{\mathbb{Z}/\mathbb{Q}}(c(\zeta)) = (2p)^{p-1} \cdot p \neq 0 \mod q \). In this case we should have \( x/q \equiv 0 \mod q^2 \). A fortiori, \( \delta_0 \equiv t \mod q^2 \) and so by Proposition 4 it must be a \( q \)-th power. We have \( \delta_0 \equiv t \equiv A \mod q^2 \).
and, since it is a $q$-th power, also $\delta_0 = \gamma^q$ for some unit $\gamma$. But then $y \equiv \gamma^q \mod q^3$, and so $y$ is a $q$-adic $q^3$-th power with $y^{p-1} \equiv N(\gamma^q) \equiv 1 \mod q^3$.

One notes that if $-1 \not< q \mod p >$ and thus $q$ splits in real primes in $K$, then $c(\zeta)$ always verifies the condition $c = \sigma_q(\beta) - \beta$ - since in fact $c(\zeta) + \tau(\zeta) = 0$ and the congruence holds modulo the real primes above $q$. What can be said more generally, if $-1 \not< q \mod p >$? One answer is that one can prove the statement of Lemma 19 in this case, provided that additionally $p \equiv 1 \mod 4$. We have thus our first partial result:

**Lemma 19.** Suppose that (7) has a solution with $e = f = 0$ and $q \equiv 1 \mod p$ of $-1 \not< q \mod p >$ and $p \equiv 1 \mod 4$. Then $q^3|x$ and $y^{p-1} \equiv 1 \mod q^3$.

**Proof.** The case $q \equiv 1 \mod p$ was already explained above. If $-1 \not< q \mod p >$, then let $q$ generate $(\mathbb{Z}/p\mathbb{Z})^*$ and $H = \{g^i : 0 \leq i < (p-1)/2\}$ be a set of representatives of $(\mathbb{Z}/p\mathbb{Z})^*/\{-1, 1\}$, and $H' = (\mathbb{Z}/p\mathbb{Z})^* \backslash H$. Thus $x \in H \iff (p-x) \in H'$ and by hypothesis, $<q \mod p > \subset H$. Let $q$ be some prime above $p$, so $q \not\equiv \overline{p}$. The condition $c(\zeta) = \sigma_q(b) - b$ implies then $\sum_{x \in H} \sigma_x(c(\zeta)) = \sum_{x <q \mod p >} \sigma_x(c(\zeta)) \equiv 0 \mod q$. Let

\[ a = \frac{\zeta^{2q}}{(\zeta + \overline{\zeta})^2} - \frac{\zeta^{2(q+1)}}{(1 + \overline{\zeta})^2} = \sigma_2 \left( \frac{\zeta^{q+1}}{(1 + \overline{\zeta})^2} \right), \]

so that $c(\zeta) = a - \overline{p}$ and the previous condition amounts to

\[ (75) \sum_{x \in H} \sigma_x(a) = \sum_{x \in H'} \sigma_x(a) \mod q. \]

We shall show that this condition cannot be fulfilled if $p \equiv 1 \mod 4$. \hfill \Box

**Remark 2.** The above result also implies that for all $q$ we have $\phi_i = \delta_0 \cdot \mu_i^q$ and $\mu_0^q = \mu_i^q + \mu_{-i}^q$; this fact is noteworthy but leads unfortunately to no further descent.

It may also be observed that if $-1 \not< q \mod p >$ and thus $q$ splits in real primes in $K$, then $c(\zeta)$ always verifies the condition $c = \sigma_q(\beta) - \beta$. We have already shown that this is not the case if $q \equiv 1 \mod p$. What can be said more generally, if $-1 \not< q \mod p >$? One answer is that one can prove the statement of Lemma 19 in this case, provided that additionally $p \equiv 1 \mod 4$.

We proceed with some global estimates. We shall assume that $|x| > |y|$, which is allowed since $x, y$ are interchangeable for the global estimates; furthermore, we assume that $x > 0$. Note that this choice does not allow any more to choose which of $x$ and $y$ is divisible by $q^3$; this is of no relevance for the global estimates we are about to prove. We have

**Lemma 20.** Suppose that (1) has a solution with $e = f = 0$ and $x > |y| > 0$. Then

\[ (76) |y| > c(q) \cdot x^{1-2/q}, \]

for some absolutely computable, strictly increasing function $c(q)$ with $c(5) > 1$.

**Proof.** Suppose first that $y > 0$ and let $\psi = \rho \cdot \overline{p} = (x + \zeta y)(x + \overline{\zeta} y) \in \mathbb{R} \cap \mathbb{Z}[\zeta]$. Then

\[ \psi^q = ((x + y)^2 - \mu xy) = A^q \cdot \left( 1 - \frac{\mu \cdot xy}{(x + y)^2} \right) \quad \text{with} \quad \mu = (1 - \zeta)(1 - \overline{\zeta}). \]
Note that \( \left| \frac{\mu \cdot x^y}{(x+y)^2} \right| \leq \frac{|\mu|}{4} < 1 \) in this case, so there is a converging global binomial expansion of \( f(x,y) = \left( 1 - \frac{\mu \cdot x^y}{(x+y)^2} \right)^{1/q} \). The expressions \( \psi \) and \( A^2 f(x,y) \) have the same \( q \)-th power, so they differ by a \( q \)-th root of unity. But since they are both real, they must coincide: \( \psi = A^2 \cdot f(x,y) \). Furthermore, the series summation commutes with the action of \( \text{Gal} \left( \mathbb{Q}(\zeta)/\mathbb{Q} \right) \), for the same reason. Thus, for all \( \sigma \in G_p \),

\[
\sigma(\psi) = A^2 \cdot \left( 1 + \sum_{n=1}^{\infty} \left( \frac{1/4}{n} \cdot \left( \frac{-\sigma(\mu) \cdot xy}{(x+y)^2} \right)^n \right) \right).
\]

An easy computation (see e.g. \([\text{Mi}]\)) shows that the binomial coefficients are uniformly bounded by \( \left| \frac{1/q}{n} \right| < \frac{1}{4q} \), while \( |\sigma(\mu)| < 4 \) for all \( \sigma \in \text{Gal} \left( \mathbb{Q}(\zeta)/\mathbb{Q} \right) \). If \( R_2 \) is the second order remainder of the above series, one finds from these estimates that

\[
|\sigma R_2| < \left( \frac{4A \cdot xy}{(x+y)^2} \right)^2 \cdot 2q \ln \left( \frac{x+y}{x-y} \right),
\]

uniformly for \( \sigma \in \text{Gal} \left( \mathbb{Q}(\zeta)/\mathbb{Q} \right) \).

For a fixed \( \sigma_0 \in \text{Gal} \left( \mathbb{Q}(\zeta)/\mathbb{Q} \right) \), we now give a uniform estimate of the difference \( \delta = |\psi - \sigma_0(\psi)| \in \mathbb{Z}[\zeta] \). Since \( \delta(\psi^q - \sigma_0(\psi)^q) = (\mu - \sigma_0(\mu)) \cdot xy \neq 0 \), it follows that \( \delta \) is a non vanishing algebraic integer. Its absolute value is:

\[
|\delta| = \left| A^2 \cdot \frac{(\mu - \sigma_0(\mu)) \cdot xy}{q(x+y)^2} + (R_2 - \sigma_0(R_2)) \right|
\]

\[
< A^2 \cdot \frac{4xy}{q(x+y)^2} + 2 \left( \frac{4A \cdot xy}{(x+y)^2} \right)^2 \cdot 2q \ln \left( \frac{x+y}{x-y} \right)
\]

\[
= \frac{4A^2xy}{q(x+y)^2} \cdot \left( 1 + \frac{16xy}{(x+y)^2} \cdot \ln \left( \frac{x+y}{x-y} \right) \right).
\]

Note that the above estimate holds for all \( \sigma \delta \) uniformly; one then verifies that for \( |y| < c(q) \cdot |x|^{1-2/q} \) and, say, \( c^{-1}(q) = 4/q \cdot (1 + 16q^{-2} \cdot \ln (1 + 2/q^2)) \) (use the lower bound on \( |x| \), above!), then \( 0 < |\sigma \delta| < 1 \) and thus \( \mathbb{N}|\delta| < 1 \), in contradiction with the fact that \( \delta \) is a non vanishing algebraic integer. This completes the proof for \( y > 0 \).

If \( y < 0 \), one lets \( y' = -y \) in the previous proof and sets \( \mu = (1 + \zeta)(1 - \zeta) \). Concretely, we have \( \psi^q = (x+y')^2 \cdot (1 - \mu \cdot x^y/(x+y)^2) \). As a result, the factor \( (x+y')^2 \) is not the \( q \)-th power of an integer any more, one replaces \( A^2 \) by the real positive value of \( (x-y)^{2/q} \). This has no impact on the estimates of the algebraic integer \( \delta \), which are perfectly analog, and lead to the same result. \( \square \)

### 10.3. The Proof of Theorem \[\text{4}\]

**Proof.** Suppose that \([\text{11}]\) has a non trivial solution for odd primes \( p, q \) verifying the conditions of the Theorem. Then by Theorem \([\text{2}]\) it follows that \( p | z \) and \( x + fy \equiv 0 \mod q^2 \), for some \( f \in \{-1, 0, 1\} \). The cases \( f = 1 \) are impossible, as proved in Theorems \([\text{7}]\) and \([\text{8}]\). Three of the remaining cases are dealt by some Wieferich condition, as proved in Propositions \([\text{9}]\), \([\text{10}]\) and \([\text{11}]\) while for the Astérisque case we have shown in Lemma \([\text{10}]\) that \( q^3 \mid x \) if \( q \equiv 1 \mod p \). We still have to prove the lower bound \([\text{8}]\). \footnote{For more detail on this kind of argument, see for instance \([\text{Mi}]\)}
Since the lower bound on $\max\{|x|, |y|\}$ is slightly better in the particular case Astérisque then the general bounds in Theorem 3 we deduce this bound separately.

We assume now that $q^2|x$ and $k = v_q(x) \geq 2$ - thus dropping the assumption $|x| > |y|$. By letting $t^q = y$ as elements of $\mathbb{Z}_q$, the $q$ - adic expansion of $\rho = (y + \zeta x)^{1/q}$ is then

$$\rho = t \cdot \left(1 + \sum_{n=1}^{\infty} \left(\frac{1}{q^n}\right) \cdot (\zeta x/y)^n \right).$$

If $A^q = x + y$, we now consider the algebraic integer $\delta = A + \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta \rho) \in \mathbb{Z}$. Since $A^q = y(1 + x/y)$, one observes that the $q$-adic expansion of $A$ results from the one for $\rho$ by replacing $\zeta$ with 1. An easy computation yields (note the factor $\zeta$ of $\rho$ in the definition of $\delta$) the $q$ - adic expansion:

$$\delta = pt \cdot \left(\frac{1}{q} \right) \cdot (x/y)^{p-1} + O \left(\left(\frac{1}{q} \right) \cdot (x/y)^p \right).$$

Obviously, $\delta \neq 0$. In order to see this, note that $V(j) = -v_q \left(\left(\frac{1}{q} \right) j \right) = j + v_q(jl)$ by (55). Furthermore, let $W(j) = v_q \left(\left(\frac{1}{q} \right) j \right) \cdot (x/y)^j = k j - V(j)$ and thus $W(p) - W(p-1) = k + (V(p-1) - V(p)) = k - 1 + v_q((p-1)!) - v_q(p!) = k - 1 > 0$.

It follows that $\delta \equiv \left(\frac{1}{p-1} \right) \cdot (x/y)^{p-1} \mod q^W(p)$ and thus $\delta \neq 0$. It follows in particular that $\delta \equiv 0 \mod q^{(k-1)(p-1)}$. Let now $B = \max|\rho|$, then $A^q \leq B$ and $|\sigma(\zeta \rho)|^q < B$, so $|\delta| < qB^{1/q}$. The last inequalities combine to:

$$\max|\rho| > (|x| + |y|)/2 > \frac{1}{2} \cdot \left(\frac{q^{(k-1)(p-1)}}{q^2} \right)^{1/q}. \quad (77)$$

If $|x| > |y|$, then one can use the bound in (77) which is stronger than (8). Otherwise, (77) implies $|y| > \frac{1}{2} \cdot \left(\frac{q^{(k-1)(p-1)}}{q^2} \right)^{1/q}$ and by interchanging $x$ and $y$ in (76) (the maximum is now $|y|$), we obtain the claim (8), where $c_1(q) = c(q)/2$, with $c(q)$ from (76). If $q \neq 1 \mod p$, all we know is $k \geq 2$, which yields (8); otherwise, by Lemma 19 we have $k \geq 3$ and (9).

This result improves upon (9). It is however due to (76) that one obtains a lower bound on $\min\{|x|, |y|\}$, which allows to assert that $x^p + C^q = q^g$ has no solutions for $|C|$ below this lower bound, as we have explicitly shown in the Corollary 11.

11. THE EQUATION OF CATALAN IN THE RATIONALS

We have proved in Lemma 8 that the rational Catalan equation 8 is equivalent to (15):

$$X^p + Y^q = Z^{pq}.$$ 

Note that this equation is symmetric in $p, q$, in the sense that it splits in the two equations:

$$X^p + (−Z)^q = (−Y)^q, \quad (78)$$

$$Y^q + (−Z)^p = (−X)^p, \quad (79)$$

which are both of type (11). Thus Theorem 14 applies to both equations. The task we still have to achieve for proving Theorem 5 consists in eliminating the Astérisque Case, by using the symmetry in the above equations. This is a consequence of the following:
Proposition 13. Suppose that \( p, q \) are two odd primes verifying the premises of Theorem \( 4 \) and for which \( (78) \) holds. Then \( q|X \).

Proof. Under the given premises, Theorem \( 4 \) implies that \( q|(X \cdot Z^q) \). For clarity, we use the substitution \( x = X, y = (-Z)^q \) and \( z = -Y \) in order to bring \( (78) \) in the form of the reference Fermat - Catalan equation \( (1) \). We will show that the assumption \( q|Z \) – and thus \( q|y \) – leads to a contradiction. For this we use again the Descent Theorem and the fact \( y = (-Z)^q \) is a \( q \)-th power.

We assume thus that \( q|y = (-Z)^q \) and \( p \not| \) \( xyz \). From \( (52) \) we have in this case \( \rho^q = x + \zeta y \) and thus

\[
(\zeta - \zeta) y = -(\zeta - \zeta) Z^q = \rho^q - pQ.
\]

Let \( \phi_i' = \xi \rho - \xi' \bar{\eta} \) and \( \phi_0' = \rho - \bar{\eta} \). Then \( \prod_{i=0}^{q-1} \phi_i' = (\zeta - \zeta) y \) and since \( (y,p) = 1 \) and \( \phi_i' = \tau_i(\phi_i') \), while \( p \neq 1 \mod q \) and thus \( \varphi = (1 - \zeta) \) does not split completely in \( L/K \). It follows that \( \varphi|\phi_0' \) and \( (\phi_i', \varphi) = (1) \). Let \( y = q^{nq} \cdot C^q \), with \( (C,pq) = 1 \). By introducing the normalization

\[
\phi_0 = \frac{\rho - \bar{\eta}}{q^{nq-1} \cdot (\zeta - \zeta)} \quad \text{and} \quad \phi_i = \frac{\xi \rho - \xi' \bar{\eta}}{\zeta - \xi'},
\]

the arguments use in the proof of the Descent Theorem yield here:

\[
\prod_{i=0}^{q-1} \phi_i = C^q, \quad \text{and} \quad (\phi_i, p \cdot q) = (\phi_i, \phi_j) = (1), \quad i \neq j > 0.
\]

We can apply now Lemma 5 and find that \( \phi_i = \delta_i \cdot \mu_i^q \), for \( i = 0, 1, \ldots, q - 1 \). If \( t \in \mathbb{Z}_q \) is such that \( t^q = x \) (existence is provided by Proposition 5), then the \( q \)-adic expansion of \( \delta_i \), given that \( q^2|y \), yields: \( \delta_i = q t \). This must then be a \( q \)-th power, by Lemma 7 and it follows plainly that \( \phi_i = \mu_i^q \). The proof proceeds like in the one for the first descent case and shall be sketched here. We define \( \psi_i = \phi_i \cdot \phi_{q-i} = (\rho + \bar{\eta})/(\xi^i - \bar{\xi})^2 - \rho \cdot \bar{\eta} \) and find that

\[
\psi_i - \psi_j = (\mu_i \cdot \mu_{q-i})^q - (\mu_j \cdot \mu_{q-j})^q = \delta_{i,j} \cdot (\xi - \bar{\xi})^2 \cdot q^{2(nq-1)} \cdot \mu_0^{2q},
\]

with \( \delta_{i,j} \) defined in the proof of Theorem 7 so that \( (\xi - \bar{\xi})^2 \delta_{i,j} \in \mathbb{Z}^\times [\xi, \bar{\xi}] \). The descent argument is in place and the claim of our Theorem follows from the assumption by means of Theorem 50.

11.1. Proof of Theorem 5 Suppose that \( p, q \) have the same conditions as in Proposition 5, and for which \( (78) \) and \( (79) \) hold simultaneously. The additional conditions ensure that the premises of Theorem 4 hold for both equations, considered as equations of the type \( (1) \) (e.g. by substitutions like in the proof of the previous Proposition). We analyze the consequences of the six conditions in Theorem 5 for this we refer the reader to the case analysis made for the proof of Theorem 4.

The conditions 1., 2. and 6. are sufficient for eliminating the descent cases \( f = 1 \) in both equations \( (78) \) and \( (79) \). The conditions 3. and 4. then show that the cases with \( f = -1 \) cannot occur for either \( (78) \) or \( (79) \). The only cases left are thus the ones with \( f = 0 \). Finally, condition 5. implies that the case \( e = 1, f = 0 \) does also not occur and the only case left is the Astérisque case \( e = f = 0 \), for both \( (79) \) and \( (78) \). However, by Proposition 13 this implies that \( q|X \). But this is exactly the
case $e = 1$ in [79], which is granted not to have solutions by the same condition 5. The contradiction completes the proof of the Theorem. □

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