A VARIATIONAL PRINCIPLE OF TOPOLOGICAL PRESSURE ON SUBSETS FOR AMENABLE GROUP ACTIONS

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Abstract. In this paper, we establish a variational principle for topological pressure on compact subsets in the context of amenable group actions. To be precise, for a countable amenable group action on a compact metric space, say $G \curvearrowright X$, for any potential $f \in C(X)$, we define and study topological pressure on an arbitrary subset and measure theoretic pressure for any Borel probability measure on $X$ (not necessarily invariant); moreover, we prove a variational principle for this topological pressure on a given nonempty compact subset $K \subseteq X$.

1. Introduction. As it is well known, topological entropy is a fundamental invariant for dynamical systems, which was introduced in [1] in a way similar to the Kolmogorov-Sinai’s picture for measure theoretic entropy. They are related by the variational principle which asserts that for a continuous map on a compact metric space the topological entropy equals to the supremum of the measure theoretic entropy taken over all invariant probability measures.

Topological pressure is a natural generalization of topological entropy. For a given continuous potential function, the associated topological pressure roughly measures the orbit complexity of iterated maps on the potential function. This notion was brought to the theory of dynamical systems by Ruelle in [20], where he introduced topological pressure of a continuous function for $\mathbb{Z}^n$-actions, assuming that the action is expansive and satisfies the specification condition. Later on, without these assumptions, Walters generalized the variational principle of pressures for a $\mathbb{Z}^+$-action in [24]. Misiurewicz gave an elegant proof of the variational principle for a $\mathbb{Z}^n_+$-action in [16].

The theory related to topological pressure plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see [4, 21, 24, 6, 12] etc.). In particular, due to the work of Bowen [5] and Ruelle [21], topological pressure turned into a basic tool in dimension theory related to dynamical systems. Following that line, Pesin and Pitskel in [19] introduced another way to define topological pressure on noncompact subsets in the context of $\mathbb{Z}$-actions; moreover, they proved a variational principle under some supplementary conditions.

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In the context of $\mathbb{Z}$-actions, Feng and Huang [6] studied the Bowen’s topological entropy on an arbitrary subset and measure theoretic entropy for Borel probability measures (not necessarily invariant); and established a variational principle for the Bowen topological entropy on a nonempty compact subset. Later on, Wang and Chen generalized this result to BS-dimension in [25]. Furthermore, still in the setting of $\mathbb{Z}$-actions, Tang, Cheng and Zhao [23] develop the theory for topological pressure on subsets, and proved a corresponding variational principle.

On the other hand, people develop dynamical theory for more general group actions. Ornstein and Weiss [17] develop entropy theory for countable amenable group actions, and beyond that Bowen [2] investigate entropy theory for sofic group actions. Kerr and Li obtained lots of results for sofic group actions, and in particular established a variational principle in this context, see [11], [12].

In this paper, we study the theory of pressure in the context of countable amenable group actions $G \curvearrowright X$, together with the consideration of subsets of the phase space $X$. More precisely, we define topological pressure on an arbitrary subset, and (local) measure theoretic pressure for Borel probability measures on $X$. Furthermore, we prove a variational principle for topological pressure on nonempty compact subsets.

Throughout this paper, we focus on dynamical systems $G \curvearrowright X$, where $(X, \rho)$ is a compact metric space and $G$ is a countable amenable group. Denote by $\mathcal{M}(X)$ the collection of Borel probability measures on $X$.

The main result of this paper reads as follows:

**Theorem 1.1.** Let there be given a countable amenable group $G$. Suppose $G$ acts continuously on a compact metric space $X$. Then for any $f \in C(X)$, and any nonempty compact subset $K \subseteq X$, one has

$$P(K, f) = \sup \{ P_\mu(G, f) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}.$$  

(The left hand side above is the Pesin-Pitskel topological pressure on subsets defined in Section 2 and the measure theoretic pressure $P_\mu(G, f)$ is defined in Section 3.)

The paper is organized as follows. In Section 2, we define the Pesin-Pitskel topological pressure on subsets and develop related properties. In Section 3, we define local measure theoretic pressures for any Borel probability measure on $X$. In Section 4, the proof of Theorem 1.1 is given, the crucial ingredients are Theorem 2.10 and Theorem 3.2.

2. Topological pressure on subsets.

2.1. Amenable group. In this subsection, we give some basic properties of amenable group.

Let $G$ be a discrete group. Denote by $\mathcal{F}(G)$ the set of all nonempty finite subsets of $G$. For $K \in \mathcal{F}(G)$ and $\delta > 0$, denote by $\mathcal{B}(K, \delta)$ the set of all $F \in \mathcal{F}(G)$ satisfying $|KF\backslash F| < \delta |F|$. The group $G$ is called amenable if $\mathcal{B}(K, \delta)$ is nonempty for every $(K, \delta)$.

This is equivalent to the existence of a sequence of nonempty finite subsets $\{F_n\}$ of $G$ which are asymptotically invariant, i.e.,

$$\lim_{n \to \infty} \frac{|F_n \Delta gF_n|}{|F_n|} = 0 \quad \text{for all} \quad g \in G.$$
Such sequences are called Følner sequences. For details on actions of amenable groups, one may refer to Ornstein and Weiss’s pioneering paper [17] or Kerr and Li’s book [12].

The collection of pairs \( \Lambda = \{ (K, \delta) : K \in \mathcal{F}(G), \delta > 0 \} \) forms a net where \( (K', \delta') \succ (K, \delta) \) means \( K' \supseteq K \) and \( \delta' \leq \delta \). For an \( \mathbb{R} \)-valued function \( \varphi \) defined on \( \mathcal{F}(G) \), we define

\[
\limsup_F \varphi(F) := \limsup_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F),
\]

and

\[
\liminf_F \varphi(F) := \liminf_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).
\]

**Remark 2.1.** From the definition of the partial order \( \succ \) it is clear that \( \mathcal{B}(K', \delta') \subseteq \mathcal{B}(K, \delta) \) if \( (K', \delta') \succ (K, \delta) \). Thus it follows that

\[
\limsup_F \varphi(F) = \inf_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F),
\]

\[
\liminf_F \varphi(F) = \sup_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).
\]

Finally, we state some properties of the net limit for the \( \mathbb{R} \)-valued function defined on \( \mathcal{F}(G) \).

**Definition 2.2.** Let \( \psi \) be a real-valued function on the set of all nonempty finite subsets of \( G \). We say that \( \psi(F) \) converges to a limit \( L \) as \( F \) becomes more and more invariant if for every \( \epsilon > 0 \) there are a nonempty finite set \( K \subset G \) and a \( \delta > 0 \) such that \( |\psi(F) - L| < \epsilon \) for every \( F \in \mathcal{B}(K, \delta) \).

**Fact 2.3.** If the limit \( \lim_F \psi(F) \) exists as \( F \) becomes more and more invariant, then

\[
\lim_F \psi(F) = \limsup_F \psi(F) = \liminf_F \psi(F).
\]

### 2.2. Pesin-Pitskel topological pressure on subsets for amenable group actions.

**Definition 2.4.** Let \( F \) be a nonempty finite subset of \( G \), we define a metric \( \rho_F \) on \( X \) by

\[
\rho_F(x, y) = \max_{s \in F} \rho(sx, sy).
\]

For every \( \epsilon > 0 \) we denote by \( B^\rho_F(x, \epsilon) \) the Bowen ball of \( x \) with radius \( \epsilon \) in the metric \( \rho_F \), i.e.,

\[
B^\rho_F(x, \epsilon) = \{ y \in X : \rho_F(x, y) < \epsilon \}.
\]

It is not hard to see that the Bowen ball \( B^\rho_F(x, \epsilon) \) is an open subset of \( (X, \rho) \).

Given a potential \( f \in C(X) \), \( x \in X \) and \( F \) being a nonempty finite subset of \( G \), set

\[
f_F(x) = \sum_{g \in F} f(gx).
\]

Then for any subset \( Z \subseteq X \), \( K \in \mathcal{F}(G) \), \( \delta > 0 \), and \( s \in \mathbb{R} \), we define

\[
P^s_{(K, \delta), \epsilon}(Z, f) = \inf_{\Gamma} \sum_i \exp \left( -s |F_i| + \sup_{y \in B^\rho_F(x_i, \epsilon)} f_F(y) \right),
\]

where the infimum is taken over all finite or countable collections \( \Gamma = \{ B^\rho_F(x_i, \epsilon) \}_{i \in I} \) such that \( F_i \in \mathcal{B}(K, \delta) \), \( x_i \in X \) and \( \bigcup_{i} B^\rho_F(x_i, \epsilon) \subseteq Z \).
The quantities \( P^s_{(K, \delta)}(Z, f) \) do not decrease as \((K, \delta)\) increases in the net \( \Lambda \), hence the following limit exists:

\[
P^s(Z, f) = \lim_{(K, \delta) \in \Lambda} P^s_{(K, \delta)}(Z, f).
\]

**Lemma 2.5.** We have:

1. If \( P^s(Z, f) < +\infty \) and \( t > s \), then \( P^t(Z, f) = 0 \).
2. Also, if \( P^s(Z, f) > 0 \) and \( t < s \), then \( P^s(Z, f) = \infty \).

**Proof.** Let \( \epsilon > 0 \) and \( M = P^s(Z, f) + 1 \).

So there exist a positive real number \( 0 < \delta_0 < 1 \) and a nonempty finite set \( K_0 \in \mathcal{F}(G) \) with \( \frac{2M}{(t-s)|K_0|} < \epsilon \) which satisfy that, for each \((K, \delta) \in \Lambda\) with \((K, \delta) \succ (K_0, \delta_0)\), there is a finite or countable cover \( \Gamma_0 = \{B^p_{F_i}(x, \epsilon_i)\}_{i \in I} \) of \( Z \) with \( F_i \in \mathcal{B}(K, \delta) \) and

\[
\sum_i \exp \left( -s |F_i| + \sup_{y \in B^p_{F_i}(x_i, \epsilon)} f_{F_i}(y) \right) < M.
\]

Note that \((K, \delta) \succ (K_0, \delta_0)\) implies \( \delta < 1 \) and \( \frac{2M}{(t-s)|K|} < \epsilon \).

Then, for all \( i \in I \), one has

\[
|KF_i| - |F_i| \leq |KF_i \setminus F_i| \leq \delta |F_i| < |F_i|.
\]

Thus, it follows that

\[
|K|/2 \leq |KF_i|/2 \leq |F_i| \quad \text{for all } i \in I.
\]

Therefore, for \( s < t \), we have

\[
M > \sum_i \exp \left( -s |F_i| + \sup_{y \in B^p_{F_i}(x_i, \epsilon)} f_{F_i}(y) \right)
= \sum_i \exp \left( -t |F_i| + \sup_{y \in B^p_{F_i}(x_i, \epsilon)} f_{F_i}(y) \right) \cdot \exp(t-s)|F_i|
\geq P^t_{(K, \delta), \epsilon}(Z, f) \cdot \exp \left( \frac{(t-s)|K|}{2} \right)
\geq P^t_{(K, \delta), \epsilon}(Z, f) \frac{(t-s)|K|}{2}.
\]

It follows that

\[
P^t_{(K, \delta), \epsilon}(Z, f) < \frac{2M}{(t-s)|K|} < \epsilon.
\]

The above inequality leads to \( P^t(Z, f) = 0 \).

So the statement of (1) is obtained.

This contradiction gives \( P^t(Z, f) = \infty \) whenever \( P^s(Z, f) > 0 \) and \( t < s \). Hence (2) is proved.

**Lemma 2.5** implies that there exists a critical value of \( s \), at which \( P^s(Z, f) \) jumps from \( \infty \) to 0, and we write
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$$P_\epsilon(Z, f) = \inf \{ s : P_\epsilon^s(Z, f) = 0 \}$$
$$= \sup \{ s : P_\epsilon^s(Z, f) = \infty \}.$$

We call
$$P(Z, f) = \lim_{\epsilon \to 0} P_\epsilon(Z, f)$$
the Bowen Pesin-Pitskel pressure of $Z$.

(Note that the quantities $P_\epsilon(Z, f)$ are NOT monotone with respect to $\epsilon$.)

Next we define another quantity of topological pressure.

Set
$$\tilde{P}^s_{(K, \delta), \epsilon}(Z, f) = \inf \sum_i \exp \left( -s|F_i| + \inf_{y \in B_{F_i}^\epsilon(x_i, \epsilon)} f_{F_i}(y) \right),$$
where the infimum is taken over all finite or countable collections $\Gamma = \{ B_{F_i}^\epsilon(x_i, \epsilon) \}_{i \in I}$ with $F_i \in \mathcal{B}(K, \delta)$, $x_i \in X$ and $\bigcup_{i \in I} B_{F_i}^\epsilon(x_i, \epsilon) \supseteq Z$. It is not hard to see from the definition of Bowen Pesin-Pitskel pressure that the following fact is true.

**Fact 2.6.** Let $G \acts X$ be a continuous action and $Z = \bigcup_{n=1}^\infty Z_n$. Then
$$\tilde{P}^s_{(K, \delta), \epsilon}(Z, f) \leq \sum_{n=1}^\infty \tilde{P}^s_{(K, \delta), \epsilon}(Z_n, f).$$

We define
$$\tilde{P}^s_{(K, \delta), \epsilon}(Z, f) := \lim_{(K, \delta) \in \Lambda} \tilde{P}^s_{(K, \delta), \epsilon}(Z, f),$$
$$\tilde{P}^s_{\epsilon}(Z, f) := \inf \{ s : \tilde{P}^s_{\epsilon}(Z, f) = 0 \}$$
$$= \sup \{ s : \tilde{P}^s_{\epsilon}(Z, f) = \infty \}.$$

Given $0 < \epsilon_1 < \epsilon_2$. If a family of Bowen balls with radius $\epsilon_1$ covers $Z$, then the family of Bowen balls with the same centers which have radius $\epsilon_2$ also covers $Z$. It is clear that
$$\inf_{y \in B_{F_i}^\epsilon(x_i, \epsilon_1)} f_{F_i}(y) \geq \inf_{y \in B_{F_i}^\epsilon(x_i, \epsilon_2)} f_{F_i}(y).$$
Thus $\tilde{P}^s_{(K, \delta), \epsilon_1}(Z, f) \geq \tilde{P}^s_{(K, \delta), \epsilon_2}(Z, f)$. Furthermore, one has
$$\tilde{P}^s_{\epsilon_1}(Z, f) \geq \tilde{P}^s_{\epsilon_2}(Z, f) \quad \text{and} \quad \tilde{P}^s_{\epsilon_1}(Z, f) \geq \tilde{P}^s_{\epsilon_2}(Z, f).$$

Now, we define
$$\tilde{P}(Z, f) := \lim_{\epsilon \to 0} \tilde{P}^s_{\epsilon}(Z, f).$$
and it is clear that
$$\tilde{P}(Z, f) = \sup_{\epsilon > 0} \tilde{P}^s_{\epsilon}(Z, f). \quad (4)$$

A non-trivial fact is that topological pressures defined in the two ways above actually coincide.

**Theorem 2.7.** For the given data above, one has
$$P(Z, f) = \tilde{P}(Z, f).$$

**Proof.** It is clear that $\tilde{P}(Z, f) \leq P(Z, f)$. Thus it suffices to prove
$$P(Z, f) \leq \tilde{P}(Z, f).$$
We may assume that $\tilde{P}(Z, f) < \infty$. Let $\kappa > \tilde{P}(Z, f)$ be any real number. Due to the equation (4), one has
\[ \tilde{P}_{\epsilon'}(Z, f) < \kappa \text{ for all } \epsilon' > 0. \tag{5} \]
Furthermore, one has
\[ \tilde{P}_{\epsilon}^\kappa(Z, f) = 0. \tag{6} \]
Let $\gamma > 0$. Since $f$ is uniformly continuous on a compact metric space $(X, \rho)$, there is a $\theta > 0$ such that
\[ |f(x) - f(y)| < \gamma \text{ if } \rho(x, y) < \theta. \tag{7} \]
**Claim.** Take $0 < \epsilon < \theta$, then one has
\[ P_{(K, \delta), \epsilon}^{\kappa + \gamma}(Z, f) \leq \tilde{P}_{(K, \delta), \epsilon}^\kappa(Z, f). \]

Given $(K, \delta) \in \Lambda$. Let $\Gamma = \{B_{F_i}^\rho(x_i, \epsilon')\}_{i \in I}$ be a finite or countable collection of Bowen balls such that $\bigcup_{i \in I} B_{F_i}^\rho(x_i, \epsilon') \supseteq Z$ with $F_i \in \mathcal{B}(K, \delta)$ and $x_i \in X$.

Take any $u, v \in B_{F_i}^\rho(x_i, \epsilon)$ and $g \in F_i$, it is clear that $\rho(gu, gv) < \epsilon < \theta$. Then one has
\[ \frac{1}{|F_i|} \sum_{g \in F_i} |f(gu) - f(gv)| < \gamma. \]
Furthermore,
\[ \sup_{u \in B_{F_i}^\rho(x_i, \epsilon)} \sum_{g \in F_i} f(gu) - \inf_{u \in B_{F_i}^\rho(x_i, \epsilon)} \sum_{g \in F_i} f(gv) \]
\[ \leq \left\{ \sum_{g \in F_i} f(gu) - \sum_{g \in F_i} f(gv) \right\}^\prime : u, v \in B_{F_i}^\rho(x_i, \epsilon) \]
\[ \leq \left\{ \sum_{g \in F_i} |f(gu) - f(gv)| : u, v \in B_{F_i}^\rho(x_i, \epsilon) \right\} \]
\[ \leq |F_i| \gamma. \]
Therefore,
\[ -(\kappa + \gamma)|F_i| + \sup_{y \in B_{F_i}^\rho(x_i, \epsilon)} \sum_{g \in F_i} f(gy) \leq -\kappa|F_i| + \inf_{x \in B_{F_i}^\rho(x_i, \epsilon)} \sum_{g \in F_i} f(gx). \]
Then it follows that
\[ P_{(K, \delta), \epsilon}^{\kappa + \gamma}(Z, f) \leq \sum_{i \in I} \exp \left( -(\kappa + \gamma)|F_i| + \sup_{y \in B_{F_i}^\rho(x_i, \epsilon)} \sum_{g \in F_i} f(gy) \right) \]
\[ \leq \sum_{i \in I} \exp \left( -\kappa|F_i| + \inf_{y \in B_{F_i}^\rho(x_i, \epsilon)} f_{F_i}(y) \right). \]
The arbitrariness of the collection $\Gamma$ implies that
\[ P_{(K, \delta), \epsilon}^{\kappa + \gamma}(Z, f) \leq \tilde{P}_{(K, \delta), \epsilon}^\kappa(Z, f). \]
Then the claim is verified by taking the net limit $(K, \delta) \in \Lambda$.

Combining with (6), one gets
\[ P_{\epsilon}^{\kappa + \gamma}(Z, f) = 0, \]
which implies that
\[ P_{c}(Z, f) \leq \kappa + \gamma. \]

Then by taking limsup as \( \epsilon \to 0 \), one gets \( P(Z, f) \leq \kappa + \gamma. \)

2.3. Weighted Pesin-Pitskel topological pressure on subsets.

**Definition 2.8.** For any given potential \( f \in C(X) \), any bounded function \( \varphi : X \to \mathbb{R} \), \( \forall (K, \delta) \in \Lambda \) and \( \forall \epsilon > 0 \), define
\[
W_{\epsilon}^{s}(K, \delta), f, \epsilon(\varphi) = \inf_{\Gamma} \sum_{j} \exp \left( -s|F_{j}| + \sup_{y \in B_{\epsilon}^{f}(x, \epsilon)} f_{F_{j}}(y) \right),
\]
where the infimum is taken over all finite or countable collections \( \Gamma = \{ (B_{\epsilon}^{f}(x_{j}, \epsilon), c_{j}) \}_{j \in J} \) such that \( 0 < c_{j} < \infty, x_{j} \in X, F_{j} \in \mathcal{B}(K, \delta) \) and \( \sum_{j \in J} c_{j} 1_{B_{\epsilon}^{f}(x_{j}, \epsilon)} \geq \varphi \).

For \( Z \subseteq X \) and \( \varphi = 1_{Z} \) we set
\[
W_{\epsilon}^{s}(K, \delta), f, \epsilon(Z) := W_{\epsilon}^{s}(K, \delta), f, \epsilon(1_{Z}).
\]
The quantities \( W_{\epsilon}^{s}(K, \delta), f, \epsilon(Z) \) do not increase as \( (K, \delta) \) increases, hence the following limit exists:
\[
W_{f, \epsilon}(Z) = \lim_{(K, \delta) \in \Lambda} W_{\epsilon}^{s}(K, \delta), f, \epsilon(Z).
\]
Clearly, there exists a critical value of the parameter \( s \), at which \( W_{f, \epsilon}(Z) \) jumps from \( \infty \) to \( 0 \), and we define
\[
PW_{\epsilon}(Z, f) := \inf \{ s : W_{f, \epsilon}(Z) = 0 \} = \sup \{ s : W_{f, \epsilon}(Z) = \infty \}.
\]

We define
\[
PW(Z, f) := \limsup_{\epsilon \to 0} PW_{\epsilon}(Z, f).
\]
(Note that the quantities \( PW_{\epsilon}(Z, f) \) are NOT monotone with respect to \( \epsilon \).)

**Lemma 2.9** (See [15], Theorem 2.1). Let \( (X, \rho) \) be a compact metric space and \( \mathcal{B} = \{ B(x_{i}, r_{i}) \}_{i \in I} \) be a family of closed (or open) balls in \( X \). Then there exists a finite or countable subfamily \( \mathcal{B}' = \{ B(x_{i}, r_{i}) \}_{i \in I'} \) of pairwise disjoint balls in \( \mathcal{B} \) such that
\[
\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{i \in I'} B(x_{i}, 5r_{i}).
\]

The next proposition demonstrate the control relation between the (local) Pesin-Pitskel topological pressure and the (local) weighted Pesin-Pitskel topological pressure on the same subset \( Z \).

**Proposition 2.1.** Let there be given a countable amenable group \( G \). Suppose \( G \) acts continuously on a compact metric space \( X \). For any given \( f \in C(X) \), any \( Z \subseteq X \), any \( s \geq 0 \) and any \( \epsilon, \kappa > 0 \), one has
\[
\overline{P}_{\epsilon}^{s+\kappa}(Z, f) \leq W_{\epsilon}^{s}(Z, f) \leq P_{\epsilon}^{s}(Z, f).
\]

**Proof.** Let \( Z \subseteq X \), \( s \geq 0 \), \( \epsilon, \kappa > 0 \). Take \( \varphi = 1_{Z} \) and \( c_{i} \equiv 1 \) in the definition (8), we see that \( W_{\epsilon}^{s}(K, \delta), f, \epsilon(Z) \leq P_{\epsilon}^{s}(K, \delta), f, \epsilon(Z, f) \) for each \( (K, \delta) \in \Lambda \). So \( W_{\epsilon}^{s}(Z, f) \leq P_{\epsilon}^{s}(Z, f) \). To complete the proof, we only need to prove \( \overline{P}_{\epsilon}^{s+\kappa}(Z, f) \leq W_{\epsilon}^{s}(Z, f) \).
Take $K_0 \subset G$ a nonempty set with $|K_0| > 2/\kappa$. Let $(K, \delta) \in \Lambda$ with $(K, \delta) > (K_0, 1)$. It is clear that

$$\frac{2}{\kappa} < |K_0| \leq |K| \leq |KF| \leq 2|F|$$

(9)

since $|KF| - |F| \leq |KF \setminus F| < \delta|F|$, $K \supseteq K_0$ and $\delta \leq 1$.

Let $\{(B_{F_i}^\rho(x_i, \epsilon), c_i)\}_{i=1}^\infty$ be a finite or countable family of Bowen open balls such that $x_i \in X$, $0 < c_i < \infty$, $F_i \in \mathcal{B}(K, \delta)$, and $\sum_{i=1}^\infty c_i 1_{B_{F_i}^\rho(x_i, \epsilon)} \geq 1_Z$ (if the family is finite then all Bowen balls $B_{F_i}^\rho(x_i, \epsilon)$ ($i \geq N$) are emptyset for some $N \in \mathbb{N}$).

Since the group $G$ is countable, the cardinality of $\mathscr{F}(G)$ is also countable. We list the members of $\mathcal{B}(K, \delta)$ as follows:

$$\mathcal{B}(K, \delta) = \{H_1, \cdots, H_n, \cdots\},$$

where $H_n \neq H_m$ for $n \neq m$. Let $n \in \mathbb{N}$ and $H_n \in \mathcal{B}(K, \delta)$. Set

$$\mathcal{I}_n := \{i \in \mathbb{N} : F_i = H_n\} \quad \text{and} \quad \mathcal{I}_{n,k} := \{i \in \mathcal{I}_n : i \leq k\}$$

for $k \in \mathbb{N}$. (Note that $F_i$ maybe equal to $F_j$ for some $i \neq j$ but the Bowen balls $B_{F_i}^\rho(x_i, \epsilon)$ and $B_{F_j}^\rho(x_j, \epsilon)$ are different.) Denote by $B_i = B_{F_i}^\rho(x_i, \epsilon)$ and $5B_i = B_{F_i}^\rho(x_i, 5\epsilon)$. Obviously, we may assume $B_i \neq B_j$ if $i \neq j$. For $t > 0$, we set

$$Z_{n,t} = \left\{x \in Z : \sum_{i \in \mathcal{I}_n} c_i 1_{B_i}(x) > t\right\},$$

and

$$Z_{n,k,t} = \left\{x \in Z : \sum_{i \in \mathcal{I}_{n,k}} c_i 1_{B_i}(x) > t\right\}.$$

Next we will show the following inequality in three steps:

$$\bar{P}_{(K, \delta), \epsilon}(Z, f) \leq \sum_{i=1}^\infty c_i \exp \left(-s|F_i| + \sup_{y \in B_{F_i}^\rho(x_i, \epsilon)} f_{F_i}(y)\right),$$

(10)

where $f_{F_i}(y) = \sum_{g \in F_i} f(gy)$. Then the conclusion follows from this.

**Step 1.** In this step we will show that for each $n, k \in \mathbb{N}$ and $t > 0$, there exists a finite set $\mathcal{J}_{n,k,t} \subset \mathcal{I}_{n,k}$ such that $\{B_i : i \in \mathcal{J}_{n,k,t}\}$ are mutually disjoint, $Z_{n,k,t} \subset \bigcup_{i \in \mathcal{J}_{n,k,t}} 5B_i$ and

$$\sum_{i \in \mathcal{J}_{n,k,t}} \exp \left(-s|F_i| + \sup_{y \in B_i} f_{F_i}(y)\right) \leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i \exp \left(-s|F_i| + \sup_{y \in B_i} f_{F_i}(y)\right).$$

Let $\emptyset \neq W \subset \mathcal{I}_{n,k}$. We write

$$Z_W = \left\{x \in Z_{n,k,t} : x \in \bigcap_{i \in W} B_i \text{ and } x \notin B_{i'} \text{ for all } i' \in \mathcal{I}_{n,k}\setminus W\right\}.$$

If $Z_W \neq \emptyset$, then for any point $\bar{x} \in Z_W$, one has

$$\sum_{i \in \mathcal{I}_{n,k}} c_i 1_{B_i}(\bar{x}) = \sum_{i \in W} c_i > t.$$

(11)

Set

$$\mathcal{G} = \{W \subset \mathcal{I}_{n,k} : Z_W \neq \emptyset\}.$$
It is clear that the set $G$ is finite, and $|G| \leq 2^{|n, k|}$. Meanwhile, it is easy to see that
\[ Z_{n, k, t} = \bigcup_{W \in G} Z_W. \]  \hfill (12)

Let $W \in G$ and $x_0$ be any point in $Z_W$. Then
\[ \sum_{i \in I_{n, k}} c_i 1_{B_i}(x_0) = \sum_{i \in W} c_i > t. \]

Hence
\[ \sum_{i \in W} c_i > t \text{ for all } W \in G. \]

The number of above inequalities is finite since $|G| \leq 2^{|n, k|}$. So we can choose the positive rational number $t' > t$ and $c'_i < c_i$ for each $i \in I_{n, k}$ such that
\[ \sum_{i \in W} c'_i > t' \text{ for all } W \in G. \]

Let $M$ be a common denominator of all $c'_i$ and $t'$. Multiplying by $M$ from both sides above, one has that, if $x_0 \in Z_W$, then
\[ \sum_{i \in I_{n, k}} Mc'_i 1_{B_i}(x_0) = \sum_{i \in W} Mc'_i > Mt' \text{ for all } W \in G. \]

(Note that $Mc_i$ and $Mt'$ are positive integers.) Then it follows that
\[ Z_W \subseteq \{ y \in Z : \sum_{i \in I_{n, k}} Mc'_i 1_{B_i}(y) > Mt' \}. \]

Combining with (12), one has
\[ Z_{n, k, t} \subseteq \{ y \in Z : \sum_{i \in I_{n, k}} Mc'_i 1_{B_i}(y) > Mt' \}. \]

Set
\[ Z'_{n, k, t} = \{ y \in Z : \sum_{i \in I_{n, k}} Mc'_i 1_{B_i}(y) > Mt' \}. \]

Let $B = \{ B_i : i \in I_{n, k} \}$ and define $v_0 : B \to Z$ by $u(B_i) = Mc'_i$. By Lemma 2.9 (use the metric $\rho_{H_n}$ instead of $\rho$), there exists a mutually disjoint subfamily $B_1$ of $B$ such that
\[ \bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{B \in \mathcal{B}_1} 5B. \]

It is clear that $\mathcal{B}_1 \subseteq \mathcal{B} = \{ B \in \mathcal{B} : v_0(B) \geq 1 \}$. We define
\[ v_1(B) = \begin{cases} v_0(B) - 1 & \text{if } B \in \mathcal{B}_1, \\ v_0(B) & \text{if } B \in \mathcal{B} \setminus \mathcal{B}_1. \end{cases} \]

Claim.
\[ Z'_{n, k, t} \subseteq \left\{ x \in Z : \sum_{B \in \mathcal{B}} v_1(B) > Mt' - 1 \right\}. \]

Let $x' \in Z'_{n, k, t}$. It is clear that
\[ \sum_{i \in I_{n, k}} v_0(B_i) 1_{B_i}(x') = \sum_{B \in \mathcal{B}} v_0(B) > Mt'. \]  \hfill (13)
On the other hand, one has
\[
\sum_{B \in \mathcal{B}} \sum_{x' \in B} v_0(B) = \sum_{B \in \mathcal{B}} v_0(B) + \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_0(B) \\
= \sum_{B \in \mathcal{B}} (v_1(B) + 1) + \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_0(B).
\]

(14)

Since \(\mathcal{B}_1\) is mutually disjoint, there exists at most one member \(B^* \in \mathcal{B}_1\) containing the point \(x'\), i.e., \(x' \in B^*\). Thus, combing with (13) and (14), one has
\[
M_{t'} < \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_1(B) + \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_0(B) \\
= 1 + v_1(B^*) + \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_1(B) \\
= 1 + \sum_{B \in \mathcal{B} \setminus \mathcal{B}_1} v_1(B).
\]

Hence the claim is verified.

In a similar way as above, we can continue this process. Suppose we have found the family \(\mathcal{B}_{j-1}\) (\(j \leq M_{t'}\)) being mutually disjoint such that
\[
Z'_{n,k,t} \subseteq \left\{ x \in \mathcal{Z} : \sum_{B \in \mathcal{B}} v_{j-1}(B) > M_{t'} - (j - 1) \right\}.
\]

The above relationship implies that
\[
Z'_{n,k,t} \subseteq \bigcup\{B : B \in \mathcal{B}, v_{j-1} \geq 1\}.
\]

Hence, by Lemma 2.9 (use the metric \(\rho_{\mathcal{H}_n}\) instead of \(\rho\)), there exists a mutually disjoint subfamily \(\mathcal{B}_j\) of \(\{B : B \in \mathcal{B}, v_{j-1}(B) \geq 1\}\) such that
\[
Z'_{n,k,t} \subseteq \bigcup\{B : B \in \mathcal{B}, v_{j-1} \geq 1\} \subseteq \bigcup_{B : B \in \mathcal{B}_j} 5B.
\]

Thus we define the function \(v_j\) as follows:
\[
v_j(B) = \begin{cases} 
  v_{j-1}(B) - 1 & \text{if } B \in \mathcal{B}_j, \\
  v_{j-1}(B) & \text{if } B \in \mathcal{B} \setminus \mathcal{B}_j.
\end{cases}
\]

With a similar argument as in the proof of the Claim, one gets
\[
Z'_{n,k,t} \subseteq \left\{ x \in \mathcal{Z} : \sum_{B \in \mathcal{B}} v_j(B) > M_{t'} - j \right\}.
\]

Hence we obtain mutually disjoint subfamilies of \(\mathcal{B}\), say \(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{M_{t'}}\), such that
\[
Z'_{n,k,t} \subseteq \bigcup_{B \in \mathcal{B}_j} 5B.
\]

By a calculation and note that \(F_i = H_n\) for \(i \in \mathcal{I}_n\), one has
\[
\sum_{j=1}^{M_{t'}} \sum_{B \in \mathcal{B}_j} \exp \left(-s|H_n| + \sup_{y \in B} f_{H_n}(y)\right)
\]
which tells us that

In this step we show that for each finite nonempty subset

Step 2. in Step 1, then

with respect to the Hausdorff distance, there exists a subsequence

all nonempty compact subsets of the compact metric space ($X, \rho_{H_n}$). Now one has that

Recall that $\nu_{H_n}$ is the smallest. Due to

Choose $j_0 \in \{1, \cdots, M't\}$ such that

is the smallest. Due to $t \leq t'$, one has

Recall that $t' > t$ and $c'_i < c_i$. Thus

which tells us that $J_{n,k,t} = \{i \in I_{n,k} : B_i \in \mathcal{B}_i\}$ fit in our purpose.

Step 2. In this step we show that for each finite nonempty subset $K \subset G$, $1 > \delta > 0$ and $\kappa, t > 0$, one has

Let us assume $Z_{n,t} \neq \emptyset$, otherwise there is nothing to prove. Since $Z_{n,t} = \bigcup_{k \in \mathbb{N}} Z_{n,k,t}$, $Z_{n,k,t} \neq \emptyset$ for $k$ large enough. Take $J_{n,k,t}$ to be the sets constructed in Step 1, then $J_{n,k,t} \neq \emptyset$ for $k$ large enough.

Define $\mathcal{E}_{n,k,t} := \{x_i : i \in J_{n,k,t}\}$, where $x_i$ is the center of the Bowen ball $B_i = B_{\rho_{H_n}}(x_i, \epsilon)$. Now one has that $F_i = H_n$ when $i \in I_n$. Since the space of all nonempty compact subsets of the compact metric space $(X, \rho_{H_n})$ is compact with respect to the Hausdorff distance, there exists a subsequence $\{k_j\}_{j \geq 1}$ of $\mathbb{N}$ and nonempty compact sets $\mathcal{E}_{n,j,t} \subset X$ such that $\mathcal{E}_{n,k_j,t} \to \mathcal{E}_{n,t}$ for $j \to \infty$. Thus $\mathcal{E}_{n,t}$ is a finite set and there is a $N' \in \mathbb{N}$ such that $|\mathcal{E}_{n,t}| = |\mathcal{E}_{n,k_j,t}|$ for $j \geq N'$. Hence

$$
\bigcup_{x \in \mathcal{E}_{n,t}} B_{\rho_{H_n}}(x, 5.5\epsilon) \supseteq \bigcup_{x \in \mathcal{E}_{n,k_j,t}} B_{\rho_{H_n}}(x, 5\epsilon) = \bigcup_{i \in J_{n,k_j,t}} 5B_i \supseteq Z_{n,k_j,t}
$$
for all \( j \geq N' \). Thus

\[
\bigcup_{x \in \mathcal{E}_{n,t}} B^\rho_{H_n}(x, 6\epsilon) \supseteq Z_{n,t}.
\]  

(15)

Let

\[
\theta = \min_{u, v \in \mathcal{E}_{n,t}} \rho(u, v) \quad \text{and} \quad \eta = \min(\theta/3, \epsilon).
\]

Since \( \mathcal{E}_{n,k_j,t} \rightarrow \mathcal{E}_{n,t} \) with respect the metric \( \rho_{H_n} \), there is a positive integer \( N'_1 \geq N' \) such that the Hausdorff distance between \( \mathcal{E}_{n,k_j,t} \) and \( \mathcal{E}_{n,t} \) is less than \( \eta \) for all \( j \geq N'_1 \). So for each point \( z \in \mathcal{E}_{n,t} \), there exists only one point \( x \in \mathcal{E}_{n,k_j,t} \) satisfying \( \rho_{H_n}(x, z) < \epsilon \) for all \( j \geq N'_1 \). In other words, for \( j \geq N'_1 \), there is a 1-1 map \( \psi_j \) from \( \mathcal{E}_{n,t} \) onto \( \mathcal{E}_{n,k_j,t} \) satisfying \( \rho_{H_n}(z, \psi_j(z)) < \epsilon \) for \( j \geq N'_1 \) and \( z \in \mathcal{E}_{n,t} \). Note that \( z \in B^\rho_{H_n}(\psi_j(z), \epsilon) \), thus

\[
\sum_{z \in \mathcal{E}_{n,t}} \exp \left( -s|H_n| + \inf_{y \in B^\rho_{H_n}(z, 6\epsilon)} f_{H_n}(y) \right) \leq \sum_{z \in \mathcal{E}_{n,t}} \exp \left( -s|H_n| + f_{H_n}(z) \right)
\]

\[
\leq \sum_{z \in \mathcal{E}_{n,t}} \left( -s|H_n| + \sup_{y \in B^\rho_{H_n}(z, \epsilon)} f_{H_n}(y) \right)
\]

\[
= \sum_{x \in \mathcal{E}_{n,k_j,t}} \left( -s|H_n| + \sup_{y \in B^\rho_{H_n}(x, \epsilon)} f_{H_n}(y) \right)
\]

\[
= \sum_{i \in J_{n,k_j,t}} \left( -s|F_i| + \sup_{y \in B^\rho_{H_n}(x, \epsilon)} f_{F_i}(y) \right).
\]

Since the inequality (9) and \( H_n \in \mathcal{B}(K, \delta) \subseteq \mathcal{B}(K_0, 1) \) for each \( n \in \mathbb{N} \), one has \( |K| \leq 2|H_n| \) and \( \kappa|K_0| > 2 \). Then by (15), Step 1 and the inequalities above, one has

\[
\bar{P}_{(K, \delta), 6\epsilon}(Z_{n,t}, f) \leq \sum_{z \in \mathcal{E}_{n,t}} \exp \left( -(s + \kappa)|H_n| + \inf_{y \in B^\rho_{H_n}(z, 6\epsilon)} f_{H_n}(y) \right)
\]

\[
\leq \frac{1}{\kappa|H_n|} \sum_{z \in \mathcal{E}_{n,t}} \exp \left( -s|F_n| + \inf_{y \in B^\rho_{H_n}(z, 6\epsilon)} f_{F_n}(y) \right)
\]

\[
\leq \frac{2}{\kappa|K_0|} \sum_{i \in J_{n,k_j,t}} \left( -s|F_n| + \sup_{y \in B^\rho_{H_n}(x, \epsilon)} f_{F_n}(y) \right)
\]

\[
\leq \frac{1}{t} \sum_{i \in I_n} c_i \exp \left( -s|F_i| + \sup_{y \in B_i} f_{F_i}(y) \right).
\]

**Step 3.** Finally we show the aimed inequality: for any \( (K, \delta) \succ (K_0, 1) \), we have

\[
\bar{P}_{(K, \delta), 6\epsilon}(Z, f) \leq \sum_{i \in I} c_i \exp \left( -s|F_i| + \sup_{y \in B_i} f_{F_i}(y) \right).
\]

Let \( 0 < t < 1 \). Note that \( Z \subseteq \bigcup_{n=1}^\infty Z_{n,t} \). Then, by Fact 2.6, we get

\[
\bar{P}_{(K, \delta), 6\epsilon}(Z, f) \leq \sum_{n=1}^\infty \bar{P}_{(K, \delta), 6\epsilon}(Z_{n,t}, f)
\]
Let $t \to 1$, the inequality can be obtained and we complete the three steps.

From this inequality, take the net limit $(K, \delta) \in \Lambda$, and by the arbitrariness of the family $\{(B^c_{p,}(x, \epsilon), c_i)\}_{i=1}^{\infty}$, one has $\bar{P}_{6e}^{c+\kappa}(Z, f) \leq W_{f, \epsilon}(Z)$. $\square$

The following theorem is a crucial ingredient in the proof of the variational principle.

**Theorem 2.10.** $P(Z, f) = PW(Z, f)$.

**Proof.** From Proposition 2.1, we know that

$$\limsup_{\epsilon \to 0} \bar{P}_{6e}(Z, f) \leq \limsup_{\epsilon \to 0} PW_\epsilon(Z, f) \leq \limsup_{\epsilon \to 0} P_\epsilon(Z, f).$$

The arbitrariness of $\kappa$ implies that

$$\bar{P}(Z, f) \leq PW(Z, f) \leq P(Z, f),$$

combing with Theorem 2.7, one gets $P(Z, f) = PW(Z, f)$. $\square$

**Proposition 2.2.** For any $s \geq 0$, $(K, \delta) \in \Lambda$, $\epsilon > 0$, and any given nonempty compact subset $K \subseteq X$ and $c = W_{(K, \delta), \epsilon}(K, f) > 0$, there is a Borel probability measure $\mu$ on $X$ such that $\mu(K) = 1$ and

$$\mu(B^c_{p,}(x, \epsilon)) \leq \frac{1}{c} \exp\left(-s|F| + \sup_{y \in B^c_{p,}(x, \epsilon)} f_F(y)\right), \quad \forall x \in X, F \in \mathfrak{B}(K, \delta).$$

**Proof.** We define a function $p$ on $C(X)$ by

$$p(\varphi) = (1/c)W_{(K, \delta), \epsilon}(\chi_K \cdot \varphi).$$

Then it is easy to verify that

1. $p(\varphi + \psi) \leq p(\varphi) + p(\psi)$ for all $\varphi, \psi \in C(X)$;
2. $p(t\varphi) = tp(\varphi)$ for any $t \geq 0$ and $\varphi \in C(X)$;
3. $p(1) = 1$, $0 \leq p(\varphi) \leq \|\varphi\|_{\infty}$ for all $\varphi \in C(X)$;
4. $p(\varphi) = 0$ for $\varphi \in C(X)$ with $\varphi \leq 0$.

By the Hahn-Banach theorem, one can extend the linear functional $t \to tp(1)$ on the subspace of constant functions to a linear functional $L : C(X) \to \mathbb{R}$ satisfying

$$L(1) = p(1) \text{ and } -p(\varphi) \leq L(\varphi) \leq p(\varphi), \forall \varphi \in C(X).$$

If $\varphi \in C(X)$ with $\varphi \geq 0$, then $p(-\varphi) = 0$, so $L(\varphi) \geq 0$. By the Riesz representation theorem, there exists a Borel probability measure $\mu$ on $X$ such that

$$L(\varphi) = \int_X \varphi \, d\mu, \forall \varphi \in C(X).$$

Now we show $\mu(K) = 1$. To see this, for any compact set $E \subseteq X \setminus K$, by the Uryson’s lemma, there is a $\varphi' \in C(X)$ such that $0 \leq \varphi' \leq 1$, $\varphi'(x) = 1$ for any $x \in E$, and $\varphi'(x) = 0$ for any $x \in K$. Then $\chi_K \cdot \varphi' \equiv 0$ and thus $p(\varphi') = 0$. Hence $\mu(E) \leq L(\varphi') \leq p(\varphi') = 0$, which implies $\mu(X \setminus K) = 0$, and so $\mu(K) = 1$. 
Next we show the inequality. For any compact set $E \subset B^\rho_F(x, \epsilon)$, by the Uryson’s lemma, there exists a $\varphi^* \in C(X)$ such that $0 \leq \varphi^* \leq 1$, $\varphi^*(y) = 1$ for $y \in E$, and $\varphi^*(y) = 0$ for $y \in X \setminus E$. Then $\mu(E) \leq L(\varphi^*) \leq p(\varphi^*)$.

Note that $\chi_K \cdot \varphi^* \leq \chi_{B^\rho_F(x, \epsilon)}$ and $F \in \mathcal{B}(K, \delta)$. Consider the family consisting of one element $\{(B^\rho_F(x, \epsilon), 1)\}$. By definition, one gets

$$\exp \left( -s |F| + \sup_{y \in B^\rho_F(x, \epsilon)} f_F(y) \right) \geq W^s_{K, \delta}, f, \epsilon(\varphi) = cp(\varphi^*) \geq c \mu(E).$$

Since $E$ is arbitrary, one has

$$\mu(B^\rho_F(x, \epsilon)) \leq \frac{1}{c} \exp \left( -s |F| + \sup_{y \in B^\rho_F(x, \epsilon)} f_F(y) \right).$$

3. Local measure theoretic pressure for amenable group actions.

**Definition 3.1.** For any nonempty finite subset $F$ of an amenable group $G$, $f \in C(X)$, $\delta > 0$, $\mu \in \mathcal{M}(X)$ and $x \in X$, we define

$$h_{\mu}(\rho, f, x, \delta, F) := \frac{1}{|F|} \log \left( e^{f_F(x)} \mu(B^\rho_F(x, \delta))^{-1} \right),$$

and

$$h_{\mu}(\rho, f, x, \delta) := \liminf_{F} h_{\mu}(\rho, f, x, \delta, F).$$

The local measure theoretic pressure is defined by

$$h^P_{\mu}(\rho, f, x) := \lim_{\delta \to 0} h_{\mu}(\rho, f, x, \delta).$$

(Note that $h_{\mu}(\rho, f, x, \delta)$ increases as $\delta$ decreases.)

It is not hard to see that the function $h^P_{\mu}(\rho, f, x)$ is Borel measurable (Please see [9] Fact 4.6). The measure theoretic pressure for the action is defined by

$$P_{\mu}(G, f) := \int_X h^P_{\mu}(\rho, f, x) \, d\mu(x).$$

For the relationship between the local measure theoretic pressure and the Bowen Pesin-Pitskel pressure, we have:

**Theorem 3.2** (See [9], Theorem 4.8). Let $G \acts (X, \rho)$ be a continuous action, $f \in C(X)$, and let $\mu$ be a Borel probability measure on $X$, and $Z \subseteq X$ be a Borel subset with $\mu(Z) > 0$. Then for any $s \in \mathbb{R}$, if

$$h^P_{\mu}(\rho, f, x) \geq s$$

for all $x \in Z$,

then $P(Z, f) \geq s$.

4. A proof of the variational principle. Now we proceed to prove Theorem 1.1.

**Proof.** For any Borel probability measure $\mu$ on $X$ with $\mu(K) = 1$, and for any $\delta > 0$, it follows that the set

$$K_\delta = \{ x \in K : h^P_{\mu}(\rho, f, x) \geq P_{\mu}(G, f) - \delta \}$$

has positive measure in $\mu$. Thus, by Theorem 3.2, one has

$$P(K_\delta, f) \geq P_{\mu}(G, f) - \delta.$$
Since \( K_\delta \subset K \), one has \( P(K, f) \geq P_\mu(G, f) - \delta \). The arbitrariness of \( \delta \) and \( \mu \) imply that

\[
P(K, f) \geq \sup \{ P_\mu(G, f) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}.
\]

In what follows, we will show the other direction of the inequality:

\[
P(K, f) \leq \sup \{ P_\mu(G, f) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}.
\]

We may assume that \( P(K, f) > -\infty \), otherwise there is nothing to prove. By Theorem 2.10 one has \( P(K, f) = \PW(K, f) \).

Given any point \( x \in X \) and any \( \kappa > 0 \). Since \( f \) is uniformly continuous, it is easy to see that

\[
\lim_{\epsilon \to 0} \liminf_F \frac{1}{|F|} \left( \sup_{y \in B_F^\epsilon(x,\epsilon)} f_F(y) - f_F(x) \right) = 0.
\]

Note that

\[
\lim_{\epsilon \to 0} \liminf_F \frac{1}{|F|} \left( \sup_{y \in B_F^\epsilon(x,\epsilon)} f_F(y) - f_F(x) \right) = \inf_{\epsilon > 0} \liminf_F \frac{1}{|F|} \left( \sup_{y \in B_F^\epsilon(x,\epsilon)} f_F(y) - f_F(x) \right),
\]

thus

\[
\liminf_F \frac{1}{|F|} \left( \sup_{y \in B_F^\epsilon(x,\epsilon)} f_F(y) - f_F(x) \right) > -\kappa \tag{16}
\]

for all \( \epsilon > 0 \).

Set \( s = \PW(K, f) - \kappa \), by the definition of \( \PW(K, f) \), one can find an \( \epsilon' > 0 \) and a \((K, \delta) \in A\) such that \( W^{s}_{(K,\delta), f,\epsilon'}(K) > 0 \). By Proposition 2.2, there exists a \( \tilde{\mu} \in \mathcal{M}(X) \) with \( \tilde{\mu}(K) = 1 \) such that

\[
\tilde{\mu}(B_F^{\epsilon'}(x,\epsilon')) \leq \frac{1}{e} \exp \left( -s |F| + \sup_{y \in B_F^{\epsilon'}(x,\epsilon')} f_F(y) \right)
\]

for all \( x \in X, F \in \mathcal{B}(K, \delta) \). Therefore,

\[
h_F^U(\rho, f, x) \geq h_{\tilde{\mu}}(\rho, f, x, \epsilon')
\]

\[
= \liminf_F \frac{1}{|F|} \log \left( e^{f_F(x)} \tilde{\mu}(B_F^{\epsilon'}(x,\delta))^{-1} \right)
\]

\[
\geq s + \liminf_F \frac{1}{|F|} \left( f_F(x) - \sup_{y \in B_F^{\epsilon'}(x,\epsilon')} f_F(y) \right)
\]

\[
> \PW(K, f) - 2\kappa.
\]

Since \( \kappa \) is arbitrary, one has

\[
P_{\tilde{\mu}}(G, f) \geq P(K, f),
\]

and so

\[
P(K, f) \leq \sup \{ P_\mu(G, f) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}.
\]

\( \square \)
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