Exact solutions of the two-mode model of multicomponent Bose-Einstein condensates

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We find the explicit solution of a two-mode model used to explain vortex dynamics in multicomponent Bose-Einstein condensates. We prove that all the solutions are constants or periodic functions and give explicit formulae for the time evolution of the populations of the two atomic species present in the condensate.

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I. INTRODUCTION

One of the most remarkable achievements of contemporary physics has been the realization of Bose-Einstein condensation with ultracold atomic gases. In those experiments bosonic neutral gases where cooled down below the critical temperature and a collective coherent behavior was observed in the gas cloud.

One later interesting results in this field were the achievement of simultaneous condensation of several atomic species using sympathetic cooling and the generation of multiple condensates using r.f. transitions between different hyperfine levels of Rb. These two species are usually denoted by [1] and [2]. Despite its complexity, this system may be described in the mean field limit using two macroscopic wavefunctions describing each of the atomic species. After these achievements many theoretical and experimental works on multicomponent systems followed.

The first generation of vortices in Bose-Einstein condensates reported in Ref. has been one of the most remarkable goals which have been achieved using multicomponent systems. From Ref. we know that while one may build two possible configurations with a unit charge vortex only one of them is stable. The stable configuration corresponds to the vortex placed on the [1] state, namely the one with the largest scattering length. The configuration with the vortex placed in the [2] state, on the other hand, leads to some kind of instability.

In recent works numerical simulations have been used to show that the dynamics of the unstable configuration can be understood within the framework of mean field theories for the double-condensate system. In a later work, it was shown that a simple two-mode model is enough to describe accurately the dynamics of the multiple-condensate system in two dimensional setups (e.g. oblate condensates where the z-coordinate may be neglected). The model was able to describe the most relevant features found in experiment and also in the mean field theories of the system.

In this paper we concentrate on the analysis of the simple model reported in Ref. proving that the system is integrable and all the solutions are periodic or constants (equilibria). This proves that the tendency of the system to exhibit periodic exchange of vortices between both atomic species is fundamental and not the consequence of a fortunate choice of parameters. We also study the period of the vortex transfer mechanism as a function of the physical parameters of the problem, in particular we find a linear dependence on the total number of particles N of the system.

II. TWO MODE MODEL FOR THE MULTIPLE SPECIES CONDENSATE.

A. Gross-Pitaevskii equations for the two-species system.

In this work we will use the zero temperature approximation, in which collisions between the condensed and non condensed atomic clouds are neglected. In the two species case this leads to a pair of coupled Gross-Pitaevskii equations (GPE) for the condensate wavefunctions of each species

\[ i\hbar \frac{\partial}{\partial t} \psi_1 = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_1 + U_{11} |\psi_1|^2 + U_{12} |\psi_2|^2 \right] \psi_1, \]  

\[ i\hbar \frac{\partial}{\partial t} \psi_2 = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_2 + U_{21} |\psi_1|^2 + U_{22} |\psi_2|^2 \right] \psi_2, \]

where \( U_{ij} = 4\pi \hbar^2 a_{ij}/m \) are constants controlling the nonlinear behavior, which are proportional to the s-wave scattering lengths of 1-1, (a11), 2-2 (a22), and 1-2 (a12) binary collisions.

To simplify the formalism we assume that both trapping potentials are concentric and spherically symmetric, \( V_i(\vec{r}) = V_i(\vec{r}) = \frac{1}{2} m \omega^2 \rho^2 \), just like in the experiment. The consideration of the differences between \( V_1 \) and \( V_2 \) does not add new physics to the model as discussed in Ref.

Next we change to a new set of units based on the trap characteristic length, \( a_0 = \sqrt{\hbar/m\omega} \), and period, \( \tau = 1/\omega \) defined as \( x \rightarrow x/a_0, t \rightarrow t/\tau, \psi_{ij} = 4\pi N_a a_{ij}/a_0 \) and \( \psi_j(x) = N_j \psi_j(x) \). Equations conserve the number of particles on each hyperfine level and so we may choose

\[ \int |\psi_1(\vec{r})|^2 = \int |\psi_2(\vec{r})|^2 \equiv 1. \]

This choice implies that the particle number of each species appears in the nonlinear coefficients \( u_{ij} \).

With the previous rescaling, the GPE for the multicomponent system read

\[ i\hbar \frac{\partial}{\partial t} \psi_1 = \left[ -\frac{1}{2} \frac{\hbar^2 \nabla^2}{2m} + u_{11} |\psi_1|^2 + u_{12} |\psi_2|^2 \right] \psi_1, \]

\[ i\hbar \frac{\partial}{\partial t} \psi_2 = \left[ -\frac{1}{2} \frac{\hbar^2 \nabla^2}{2m} + u_{21} |\psi_1|^2 + u_{22} |\psi_2|^2 \right] \psi_2. \]

The coefficients of the matrix of nonlinear coefficients satisfy the relations \( u_{11}/u_{12} = a_{11}N_1/a_{12}N_2 \), \( u_{21}/u_{22} = a_{12}N_1/a_{22}N_2 \), which means that except for the particular case...
in which \( N_1 = N_2 = N \), this matrix is nonsymmetric. In terms of the population imbalance \( \beta = N_2/N_1 \), and for a fixed total number of particles the values are \( u_{11} = ga_{11}/(1+\beta) \), \( u_{12} = ga_{12}/(1+\beta) \), \( u_{21} = ga_{12}/(1+\beta) \), \( u_{22} = ga_{22}/(1+\beta) \), where \( g = 4\pi N/a_0 \).

**B. Derivation of a two-mode model**

For completeness we will derive here the two-mode model following a different formalism than that of Ref. Ref. [8]. The idea is to use a limited expansion

\[
\psi_n(t, \vec{r}) = \sum_{k=1}^{2} a_{nk}(t) \phi_k(\vec{r}),
\]

where we take \( \phi_k(\vec{r}) \) to be the lowest energy solutions of the linear problem

\[
\left[ -\frac{1}{2} \Delta + V(\vec{r}) \right] \phi_k(\vec{r}) = E_k \phi_k(\vec{r}).
\]

Although we have chosen a harmonic oscillator basis, any suitable truncated basis including a mode resembling the ground state and another one corresponding to a state with a vortex could be used in this expansion. Despite its simplicity and crudeness (only two modes of the full expansion are kept), this approximation has been shown to retain many qualitative features of the dynamics of the system [8]. Inserting Eq. [3] with \( \phi \) given by [3] into Eq. [3] we get

\[
-\int d^3\vec{r} \psi_n(t, \vec{r}) \partial_t |a_{nk}|^2 = -2u_{nk} C_{12} |a_{n1212}a_{1222}| \sin \Phi_{nk},
\]

where \( C_{jk} = \langle \phi_j| \phi_k^2 \rangle \) and \( \Phi \equiv \Phi_{11} = -\Phi_{12} = -\Phi_{21} = \Phi_{22} = \arg(a_{11}a_{22}/a_{12}a_{21}) \). The equations for the phases are

\[
\partial_t \arg a_{nk} = -E_k - \Re \sum_{k'=1}^{2} \frac{\alpha_{nk'}}{a_{nk}} \langle \phi_k U_n \phi_{k'} \rangle.
\]

where \( \langle A \rangle = \int d^3\vec{r} A \). Calculating \( \langle \phi_k U_n \phi_{k'} \rangle \) for the two cases \( k' = k \) and \( k' = \bar{k} \), where \( \bar{k} \) is the complementary value of the index, \( k, k' = 1, 2 \)

\[
\langle \phi_k U_n \phi_{k} \rangle = \sum_{m=1}^{2} u_{nm} \left\{ C_{kk} |a_{mk}|^2 + C_{12} |a_{mk}|^2 \right\},
\]

\[
\langle \phi_k U_n \phi_{\bar{k}} \rangle = C_{12} \sum_{m=1}^{2} u_{nm} a_{mk} \bar{a}_{mk},
\]

we get the following set of equations for our problem

\[
\partial_t \Phi = -\Gamma - C_{12} \rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi \times
\]

\[
\left[ u_{12} \left( \frac{1}{\rho_{11}^2} - \frac{1}{\rho_{12}^2} \right) + u_{21} \left( \frac{1}{\rho_{21}^2} - \frac{1}{\rho_{22}^2} \right) \right],
\]

where

\[
a_{nk} = \rho_{nk} \exp(i\theta_{nk}),
\]

and

\[
\Gamma = \sum_{nk} \gamma_{nk} \rho_{nk}^2.
\]

It follows from (12)-(15) and (17) that

\[
\partial_t \Gamma = 2\gamma \omega C_{02} \rho_{11} \rho_{12} \rho_{22} \rho_{24} \sin \Phi,
\]

where

\[
\gamma = u_{12} (\gamma_{12} - \gamma_{11}) + u_{21} (\gamma_{11} - \gamma_{12}).
\]

The following relations are then evident

\[
\rho_{11}^2 = \rho_{11}^2 - \frac{u_{12}}{\gamma} (\Gamma - \Gamma)
\]

\[
\rho_{21}^2 = \rho_{21}^2 + \frac{u_{12}}{\gamma} (\Gamma - \Gamma)
\]

\[
\rho_{22}^2 = \rho_{22}^2 - \frac{u_{12}}{\gamma} (\Gamma - \Gamma).
\]
with $\tilde{\Gamma} = \sum_{nk} \gamma_{nk} \tilde{\rho}_{nk}$. These equations imply that all the relevant densities may be obtained from a single quantity, $\Gamma(t)$. Our goal now will be to find an equation ruling its dynamics. Let us notice that Eq. (11) can be rewritten as

$$\partial_t \rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi = \frac{1}{2C_{12} \gamma^*} \Gamma \partial_t \Gamma$$

(32) and solved:

$$\rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi - \rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi = \frac{1}{4C_{12} \gamma^*} (\Gamma^2 - \tilde{\Gamma}^2)$$

(33)

Using the last equation and (28)-(31), Eq. (26) for $\Gamma$, can be presented as

$$\Gamma^2 = P_4(\Gamma)$$

(34)

where

$$P_4(\Gamma) = -\frac{1}{4} [\Gamma^2 - \tilde{\Gamma}^2 + 4 \gamma^* C_{12} \rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi]^2 +$$

$$4 C_{12}^2 \frac{u_1^2 u_2^2}{\gamma^*} (\Gamma - \Gamma_{11}) (\Gamma - \Gamma_{12}) (\Gamma - \Gamma_{21}) (\Gamma - \Gamma_{22})$$

(35)

and the constants $\Gamma_{nk}$: $n,k = 1,2$ are defined as $\Gamma_{11} = \Gamma + \gamma_{12}/u_1 u_2$, $\Gamma_{12} = \Gamma - \gamma_{12}/u_1 u_2$, $\Gamma_{21} = \Gamma + \gamma_{21}/u_1 u_2$, $\Gamma_{22} = \Gamma - \gamma_{21}/u_1 u_2$. Thus, the solution of our equations can be obtained from a single equation for $\Gamma$, which, using the previous equations, allows to obtain the expressions for the densities $\rho_{jk}(t)$. In particular, Eq. (34) leads to

$$t - t_0 = \int_{\tilde{\Gamma}}^{\Gamma(t)} \frac{d\Gamma}{\sqrt{P_4(\Gamma)}}$$

(36)

Let us rewrite the polynomial (33) in terms of its roots as follows

$$P_4(\Gamma) = P_* (\Gamma - P_1) (\Gamma - P_2) (\Gamma - Q_1) (\Gamma - Q_2)$$

(37)

where $P_* = 4 C_{12}^2 u_1^2 u_2^2 / \gamma^*$ and $P_1, P_2, Q_1, Q_2$ are the roots of $P_4$. There are at least two real roots $P_1, P_2$ of $P_4$ satisfying that $\tilde{\Gamma}$ is always located between them, i.e. $P_1 < \tilde{\Gamma} < P_2$.

To prove the last affirmation let us first compute

$$P_4(\Gamma) = 4 C_{12}^2 \gamma^* \sin^2 \Phi > 0.$$  

Next we evaluate

$$P_4(\Gamma_{11}) = -\frac{1}{4} [\Gamma_{11} - \tilde{\Gamma}^2 + 4 \gamma^* C_{12} \frac{u_1^2 u_2^2}{\gamma^*} \sin^2 \Phi]^2 < 0,$$

$$P_4(\Gamma_{12}) = -\frac{1}{4} [\Gamma_{12} - \tilde{\Gamma}^2 + 4 \gamma^* C_{12} \frac{u_1^2 u_2^2}{\gamma^*} \sin^2 \Phi]^2 < 0.$$  

This means that there is at least one real root located in each of the intervals $P_1 \in [\tilde{\Gamma}, \Gamma_{11}]$ and $P_2 \in [\tilde{\Gamma}, \Gamma_{12}]$.

As a corollary we get that $P_4(\Gamma) > 0$ for $P_1 < \Gamma < P_2$. Concerning the other two roots $Q_1, Q_2$ ($Q_2 > Q_1$ if real), nothing may be said in general. However, for the case $P_* > 0$ it is clear that $P_4(\pm \infty) > 0$ and then $Q_{1,2}$ are real numbers. When $P_* < 0$ it happens that $Q_1, Q_2$ could be (and, in fact, they are) complex conjugate roots for certain parameter ranges. In any case, the boundedness of $\Gamma \in [P_1, P_2]$ ensures the periodicity of the solutions.

Let us now proceed to find an explicit form for the solutions. First, to present polynomial $P_4(\Gamma)$ in the canonical form we make the transform:

$$\Gamma(t) = \alpha Y + \beta$$

(38)

where

$$\alpha = \frac{1}{4} (\delta + 1) P_2 - \frac{1}{4} (\delta - 1) P_1,$$

$$\beta = \frac{1}{4} (\delta + 1) P_2 + \frac{1}{4} (\delta - 1) P_1,$$

$$\delta = \sqrt{|P_1 - Q_1||P_1 - Q_2|} + \sqrt{|P_2 - Q_1||P_2 - Q_2|}$$

(39)

(40)

(41)

Equivalently we may also write

$$\Gamma(t) = P_0 + \frac{\Delta}{\delta} \sqrt{\frac{Y}{Y + \delta}},$$

(42)

with the quantities $\Delta$ and $P_0$ being given by

$$\Delta = \alpha \delta - \beta = \frac{1}{4} (\delta^2 - 1) (P_2 - P_1),$$

(43)

$$P_0 = \frac{\beta}{\delta} = \frac{1}{28} [\delta (\delta + 1) P_2 + (\delta - 1) P_1],$$

(44)

It is clear that $P_1 < P_0 < P_2$ and thus $P(\Gamma) > 0$. In terms of $Y$ the polynomial $P_4(\Gamma)$ can be written as

$$P_4(\Gamma) = \delta^4 P_0 \left(1 - Y^2 \right) \left(1 - k^2 Y^2 \right)$$

(45)

Thus, the function $Y(t)$ satisfies the equation

$$Y(t) = \tilde{\Omega} \left(1 - t \tilde{t}, k \right),$$

(46)

where

$$\tilde{\Omega} = \frac{\delta^2}{\Delta} \sqrt{P_0}$$

$$= \frac{1}{2 \sqrt{|P_1| \left| \sqrt{|P_1 - Q_1||P_2 - Q_2|} + \sqrt{|P_1 - Q_2||P_2 - Q_1|} \right|}},$$

(47)

Equation (34) can be solved as follows:

$$Y(t) = \sin \left( \tilde{\Omega} \left( t - \tilde{t} \right), k \right),$$

(48)

where

$$k = \frac{\sqrt{|P_1 - Q_1||P_2 - Q_2|} - \sqrt{|P_1 - Q_2||P_2 - Q_1|} \left| \sqrt{|P_1 - Q_1||P_2 - Q_2|} + \sqrt{|P_1 - Q_2||P_2 - Q_1|} \right|}{\sqrt{|P_1 - Q_1||P_2 - Q_2|} + \sqrt{|P_1 - Q_2||P_2 - Q_1|}}$$

(49)

and the constant $\tilde{t}$ may be determined from the initial conditions by solving the equations

$$\sin \left( \tilde{\Omega} \left( t_0 - \hat{t} \right), k \right) = \frac{\Delta}{\alpha - \beta} - \delta.$$  

(50)

This allows us to get a explicit solution for our problem. To do so let us first use Eq. (13) to find

$$\Gamma(t) = \frac{\alpha \sin \left( \tilde{\Omega} \left( t - \tilde{t} \right), k \right) + \beta}{\sin \left( \tilde{\Omega} \left( t - \tilde{t} \right), k \right) + \delta}.$$  

(51)
Thus, inserting this explicit expression for $\Gamma(t)$ and Eqs. (28)-(31), one may get explicit expressions for the populations $\rho_{jk}(t), j, k = 1, 2$. Typical forms of the solutions for different parameter values are shown in Fig. 1.

![Graphs of $\rho_{11}(t)$, $\rho_{12}(t)$, and $\rho_{22}(t)$](image)

**FIG. 1.** Evolution of the densities $\rho_{11}(t)$ (solid) and $\rho_{12}(t)$ (dashed) for two set of combinations of parameters and initial conditions (a) $g = 100$, $a_{11} = 1$, $a_{12} = 0.97$, $a_{22} = 0.94$, $\beta = N_1/N_2 = 1$, $\rho_{11}(t_0) = 0.78$, $\phi_{11}(t_0) = 0$, $\phi_{12}(t_0) = 0.5$, $\phi_{21}(t_0) = 0.5$, $\phi_{22}(t_0) = 0.7$. (b) $g = 100$, $a_{11} = 1$, $a_{12} = 0.1$, $a_{22} = 0.02$, $\beta = N_1/N_2 = 1/4$, $\rho_{11}(t_0) = 0.9$, $\phi_{11}(t_0) = 0$, $\phi_{12}(t_0) = 0$, $\rho_{21}(t_0) = 0.9$, $\phi_{21}(t_0) = 1.5$, $\phi_{22}(t_0) = 0$. We fix $\tilde{t} = 0$ and get the corresponding (irrelevant) value for $t_0$ from Eq. (50).

The period of oscillations of the populations $\rho_{jk}(t)$ is that of the sn function, i.e.

$$T = \frac{4K(k)}{\Omega} = \frac{4\Delta K(k)}{\delta^2 \sqrt{P_0} \sqrt{P_*}} = \frac{8K(k)}{\sqrt{|P_1 - P_2|}} \times \frac{1}{\sqrt{|P_1 - Q_1| |P_2 - Q_2|}}$$  \hspace{1cm} (52)

where the quantity $K(k) = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2 \theta}$ is the complete elliptic integral of first kind [10].

Concerning the period, the dependence on the physical parameter $N$ is easy to obtain. Let us define the new constants (not dependent on $N$) $\gamma_{ij} = \gamma_{ij}/N$, $\bar{u}_{ij} = u_{ij}/N$. Thus, the period scales as $T(k, N) = N^{-1} T(k, 1)$.

Concerning the instability mechanisms described in Refs. [9], they are now completely understood within the framework of Eq. (51). For instance in Fig. 2 we plot typical solutions in the regime of vortex transfer for parameter values typical of the experiments [1]. Both the stability of the vortex when it is placed in [1] and its instability when put on [2] are well described by the exact solutions of the two-mode model.

![Graphs of $\rho_{12}(t)$ and $\rho_{22}(t)$](image)

**FIG. 2.** Evolution of the densities $\rho_{12}(t)$ and $\rho_{22}(t)$ (containing the vorticity of each component) for two set of combinations of initial conditions corresponding respectively to (a) a stable vortex in $|1>$ and (b) an unstable vortex in $|2>$.

Parameter values are $g = 100$, $\beta = N_1/N_2 = 1$, $a_{11} = 1$, $a_{12} = 0.97$, $a_{22} = 0.94$. Initial conditions are (a) $\rho_{11}(t_0) = 0.001$, $\phi_{11}(t_0) = 0$, $\phi_{12}(t_0) = 0.1$, $\rho_{21}(t_0) = 0.999$, $\phi_{21}(t_0) = 0$, $\phi_{22}(0) = 0.2$. (b) $\rho_{11}(t_0) = 0.999$, $\phi_{11}(t_0) = 0$, $\phi_{12}(t_0) = 0.1$, $\rho_{21}(t_0) = 0.001$, $\phi_{21}(t_0) = 0$, $\phi_{22}(t_0) = 0.2$. We fix $\tilde{t} = 0$ and get the corresponding (irrelevant) value for $t_0$ from Eq. (50).
IV. CONCLUSIONS

In this paper we have integrated the two-mode model developed for the explanation of the vortex transfer mechanisms described in Refs. [5,6,8]. The most remarkable result is that all the solutions are periodic functions and thus the transfer mechanisms of at least part of the vorticity are natural within the range of validity of the model, which was previously found to be at least that of pancake traps and some regimes of fully three-dimensional traps. It has been shown that the frequency of the oscillations depends linearly on the total number of particles of the condensate while the dependence on the other parameters (relative populations and scattering lengths) is nontrivial and given by our explicit formulae.

We hope that the technique described here will be useful to analyze the similar problem which arises when a Josephson coupling between both species is incorporated into the experimental setup by means of an off-resonant laser [11].

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