

MATRIX RESOLVENT AND THE DISCRETE KDV HIERARCHY

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ABSTRACT. Based on the matrix-resolvent approach, for an arbitrary solution to the discrete KdV hierarchy, we define the tau-function of the solution, and compare it with another tau-function of the solution defined via reduction of the Toda lattice hierarchy. Explicit formulae for generating series of logarithmic derivatives of the tau-functions are then obtained, and applications to enumeration of ribbon graphs with even valencies and to the special cubic Hodge integrals are under consideration.

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1. INTRODUCTION

The discrete KdV equation (aka the Volterra lattice equation) is an integrable Hamiltonian equation in (1+1) dimensions, i.e., one discrete space variable and one continuous time variable, which extends to a commuting system of Hamiltonian equations, called the discrete KdV integrable hierarchy. This integrable hierarchy has important applications in algebraic geometry and symplectic geometry (in particular in the theory of Riemann surfaces) (see e.g. [18]). Significance of the discrete KdV hierarchy was further pointed out by E. Witten [34] in the study of the GUE partition function with even couplings — the “matrix gravity”, and was recently addressed also in the study of the special cubic Hodge partition function [12, 15, 16] — the topological gravity in the sense of [12, 16]. The explicit relationship between the two gravities, called the Hodge–GUE correspondence, has been established in [12, 16]. In this paper, by using the matrix-resolvent (MR) approach recently introduced and developed in [1, 2, 3, 11] we study the tau-structure for the discrete KdV hierarchy, and apply it to studying the above mentioned enumerative problems.

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1.1. **An equivalent description of the discrete KdV hierarchy.** Let \( P(n) \) be the following difference operator
\[
P(n) := \Lambda + w_n \Lambda^{-1},
\]
where \( \Lambda \) denotes the shift operator \( \Lambda : f_n \mapsto f_{n+1} \). Introduce
\[
A_\ell := (P^{\ell+1})_+, \quad \ell \geq 0.
\]
Here, for an operator \( Q \) of the form \( Q = \sum_{k \in \mathbb{Z}} Q_k \Lambda^k \), the positive part \( Q_+ := \sum_{k \geq 0} Q_k \Lambda^k \).

The discrete KdV hierarchy is defined as the following system of commuting flows:
\[
\frac{\partial P}{\partial s_j} = [A_{2j-1}, P], \quad j \geq 1.
\]
For example, the \( s_1 \)-flow reads
\[
\frac{\partial w_n}{\partial s_1} = w_n (w_{n+1} - w_{n-1}),
\]
which is the discrete KdV equation. The commutativity implies that equations (3) for all \( j \geq 1 \) can be solved together, yielding solutions of the form \( w_n = w_n(s) \), \( s := (s_1, s_2, s_3, \ldots) \).

Let us introduce
\[
L := P^2 = \Lambda^2 + w_n + w_n \Lambda^{-1}.
\]
Then \( A_{2j-1} = (P^{2j})_+ = (L^j)_+ \).

**Proposition 1.** The discrete KdV hierarchy (3) can be equivalently written as
\[
\frac{\partial L}{\partial s_j} = [A_{2j-1}, L], \quad j \geq 1.
\]
The proof will be given in Section 2. For the particular case \( j = 1 \), we have
\[
\frac{\partial (w_{n+1} + w_n)}{\partial s_1} = w_{n+2} w_{n+1} - w_n w_{n-1},
\]
\[
\frac{\partial (w_n w_{n-1})}{\partial s_1} = (w_{n+1} + w_n - w_{n-1} - w_{n-2}) w_n w_{n-1}.
\]
For this case it is clear that equations (7)–(8) are equivalent to equation (4).

Observe that equations (6) are the compatibility conditions of the following scalar Lax pairs:
\[
L \psi_n = \lambda \psi_n, \quad \text{i.e.,} \quad \psi_{n+2} + (w_{n+1} + w_n - \lambda) \psi_n + w_n w_{n-1} \psi_{n-2} = 0,
\]
\[
\frac{\partial \psi_n}{\partial s_j} = A_{2j-1} \psi.
\]
We want to write the spectral problem (9) into a matrix form. The scalar Lax operator \( L \), defined in (5), could be viewed as a reduction of
\[
\tilde{L} = \Lambda^2 + a_1(n) \Lambda + a_2(n) + a_3(n) \Lambda^{-1} + a_4(n) \Lambda^{-2},
\]
which is the Lax operator of a bigraded Toda hierarchy. However, observe that \( L \) contains \( \Lambda^{\text{even}} \) only (with even = -2, 0, 2). So, instead of considering a \( 4 \times 4 \) matrix-valued Lax operator, a \( 2 \times 2 \) matrix-valued operator will be sufficient. Indeed, introduce
\[
\mathcal{L} := \begin{pmatrix} \Lambda^2 & 0 \\ 0 & \Lambda^2 \end{pmatrix} + U_n, \quad U_n := \begin{pmatrix} w_{n+1} + w_n - \lambda & w_n w_{n-1} \\ -1 & 0 \end{pmatrix}.
\]
Then the spectral problem (9) reads
\[ \mathcal{L} \left( \begin{array}{c} \psi_n \\ \psi_{n-2} \end{array} \right) = 0. \] (12)

1.2. The MR approach to defining tau-functions. Denote by \( \mathbb{Z}[w] \) the ring of polynomials with integer coefficients in the variables \( w := (w_n), n \in \mathbb{Z} \).

**Definition 1.** An element \( R_n \in \text{Mat}(2, \mathbb{Z}[w]((\lambda^{-1}))) \) is called a matrix resolvent of \( \mathcal{L} \) if
\[ R_{n+2} U_n - U_n R_n = 0. \] (13)

**Definition 2.** The basic (matrix) resolvent \( R_n \) is defined as the matrix resolvent of \( \mathcal{L} \) satisfying
\[ R_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^{-1}), \] (14)
\[ \text{tr} R_n = 1, \quad \det R_n = 0. \] (15)

The basic resolvent \( R_n \) exists and is unique. See in Section 3 for the proof. Write
\[ R_n(\lambda) = \begin{pmatrix} 1 + \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}. \] (16)

Then Definition 2 for \( R_n(\lambda) \) is equivalent to the following set of equations
\[ \beta_n = -w_n w_{n-1} \gamma_{n+2} \] (17)
\[ \alpha_{n+2} + \alpha_n + 1 = (\lambda - w_{n+1} - w_n) \gamma_{n+2} \] (18)
\[ (\lambda - w_{n+1} - w_n)(\alpha_n - \alpha_{n+2}) = w_n w_{n-1} \gamma_n - w_{n+2} w_{n+1} \gamma_{n+4} \] (19)
\[ \alpha_n + \alpha_n^2 + \beta_n \gamma_n = 0 \] (20)

together with equation (14). These equations give recursive relations and initial values for the coefficients of \( \alpha_n, \beta_n, \gamma_n \) (see (57)–(59) below), which will be called the MR recursive relations.

**Proposition 2.** For an arbitrary solution \( w_n(s) \) to the discrete KdV hierarchy, let \( R_n(\lambda) \) denote the basic resolvent of \( \mathcal{L} \) evaluated at \( w_n = w_n(s) \). There exists a function \( \tau_n^\text{dKdV}(s) \) satisfying
\[ \sum_{i,j \geq 1} \frac{\partial^2}{\partial s_i \partial s_j} \tau_n^\text{dKdV}(s) \lambda^{-i-1} \mu^{-j-1} = \text{tr} \left( \frac{R_n(\lambda) R_n(\mu)}{(\lambda - \mu)^2} \right) - 1, \] (21)
\[ \frac{1}{\lambda} + \sum_{i \geq 1} \frac{1}{\lambda^{i+1}} \frac{\partial}{\partial s_i} \log \tau_n^\text{dKdV} = \left[ R_n(\lambda) \right]_{21}, \] (22)
\[ \frac{\tau_n^\text{dKdV} \tau_{n+2}^\text{dKdV}}{\tau_n^\text{dKdV} \tau_n^\text{dKdV}} = w_n. \] (23)

Moreover, the function \( \tau_n^\text{dKdV}(s) \) is uniquely determined by \( w_n(s) \) up to a factor of the form
\[ e^{\alpha_n + \beta_0 + \sum_{k \geq 1} \beta_k s_k}, \]
where \( \alpha, \beta_0, \beta_1, \beta_2, \ldots \) are arbitrary constants.

We call \( \tau_n^\text{dKdV}(s) \) the tau-function of the solution \( w_n = w_n(s) \) to the discrete KdV hierarchy.
1.3. Main Theorem. Based the MR approach, in [11] we have given definition of tau-function for the Toda lattice. Observe that the discrete KdV hierarchy [3] is a reduction of the Toda lattice hierarchy. Therefore, for the arbitrary solution \( w_n(s) \) to the discrete KdV hierarchy, we can also associate another tau-function \( \tau_n(s) \) of the solution \( w_n(s) \) obtained via the reduction (see Section 4.2 for the precise definition). In particular, this tau-function satisfies that

\[
\frac{\tau_{n+1}(s) \tau_{n-1}(s)}{\tau_n^2(s)}.
\]

Theorem 1 (Main Theorem). There exist constants \( \alpha, \beta, \beta_1, \beta_2, \ldots \) such that

\[
\tau_n(s) = e^{\alpha n + \beta_0 + \sum \beta_k s_k} \tau_n \text{dKdV}(s) \tau_{n+1} \text{dKdV}(s).
\]  

(24)

1.4. Application I. Enumeration of ribbon graphs with even valencies. A ribbon graph is a multi-graph embedded onto a Riemann surface. The genus of the ribbon graph is defined as the smallest possible genus of the embedding. Given \( k \geq 1 \) and \( j_1, \ldots, j_k \geq 1 \), denote

\[
\langle \text{tr } M^{2j_1} \cdots \text{tr } M^{2j_k} \rangle_c := k! \sum_{0 \leq \ell \leq \frac{k}{2}} n^{2-2g-k+\ell} a_g(2j_1, \ldots, 2j_k),
\]  

(25)

\[
a_g(2j_1, \ldots, 2j_k) := \frac{1}{\# \text{Sym } \Gamma}.
\]  

(26)

Here, \( |j| = j_1 + \cdots + j_k \), and \( \sum_{\Gamma} \) denotes summation over connected ribbon graphs \( \Gamma \) with labelled half edges and unlabelled vertices of genus \( g \) with \( k \) vertices of valencies \( 2j_1, \ldots, 2j_k \), and \( \# \text{Sym } \Gamma \) is the order of the symmetry group of \( \Gamma \) generated by permuting the vertices. The notation \( \langle \text{tr } M^{2j_1} \cdots \text{tr } M^{2j_k} \rangle_c \) is borrowed from the literature of random matrices [25, 20, 29, 22, 8, 7], where it is often called a connected Gaussian Unitary Ensemble (GUE) correlator.

Definition 3. For every \( k \geq 1 \), define

\[
E_k(n; \lambda_1, \ldots, \lambda_k) := \sum_{j_1, \ldots, j_k = 1}^{\infty} \frac{\langle \text{tr } M^{2j_1} \cdots \text{tr } M^{2j_k} \rangle_c}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}}.
\]  

(27)

Definition 4. Define a \( 2 \times 2 \) matrix-valued series \( R_n(\lambda) \in \text{Mat}(2, \mathbb{Z}[n][[\lambda^{-1}]] \) by

\[
R_n(\lambda) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{(2j-1)!!}{\lambda^{j+1}} \begin{pmatrix} (2j+1)A_{n,j} - (n-1)B_{n,j} & (n-n^2)B_{n+2,j} \\ B_{n,j} & (n-1)B_{n,j} - (2j+1)A_{n,j} \end{pmatrix},
\]  

(28)

with

\[
A_{n,j} := (n-1)_{2} {F}_{1}(-j, 2-n; 2; 2),
\]  

(29)

\[
B_{n,j} := (n-1){F}_{1}(1-j, 2-n; 2; 2) + (n-n^2){F}_{1}(1-j, 3-n, 2; 2).
\]  

(30)

The number \( a_g(2j_1, \ldots, 2j_k) \) has the alternative expression

\[
a_g(2j_1, \ldots, 2j_k) = \sum_G \frac{\prod_{i=1}^{k} a_g(2j_i)}{\# \text{Sym } G},
\]

where \( \sum_G \) denotes summation over connected ribbon graphs \( G \) with unlabelled half-edges and unlabelled vertices of genus \( g \) with \( k \) vertices of valencies \( 2j_1, \ldots, 2j_k \).
Theorem 2. The following formulae hold true

\[ E_1(n; \lambda) = n \sum_{j \geq 1} \frac{(2j-1)!!}{\lambda^{2j+1}} \left( 2F_1(-j, -n; 2; 1) - j \frac{2}{2F_1(1-j, 1-n; 3; 2)} \right), \]

\[ E_2(n; \lambda_1, \lambda_2) = \frac{(1 + \Lambda)}{(\lambda_1 - \lambda_2)^2} \left[ \text{tr} \left( R_n(\lambda_1) R_n(\lambda_2) \right) \right] - \frac{2}{(\lambda_1 - \lambda_2)^2}, \]

\[ E_k(n; \lambda_1, \ldots, \lambda_k) = \frac{1}{k} \sum_{\lambda \in S_k} \frac{(1 + \Lambda)}{\prod_{\ell=1}^{k} (\lambda_{\ell} - \lambda_{\ell+1})} \left( 1 + \lambda \right) \left( \sum_{j=1}^{\infty} \left( \begin{array}{c} j \\ i \end{array} \right) \left( \begin{array}{c} n \\ i + 1 \end{array} \right) \right), \]

where \( R_n(\lambda) \) is defined in Definition 4, and it is understood that \( \sigma_{k+1} = \sigma_1 \).

In the above formulae

\[ 2F_1(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} z^j = 1 + \frac{a}{c} \frac{z}{1} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2} + \ldots \]

is the Gauss hypergeometric function. Recall that it truncates to a polynomial if \( a \) or \( b \) are non-positive integers. In particular,

\[ n \cdot 2F_1(-j, 1-n; 2; 1) = \sum_{i=0}^{j} 2^i \left( \begin{array}{c} j \\ i \end{array} \right) \left( \begin{array}{c} n \\ i + 1 \end{array} \right). \]

1.5. Application II. Identities for special cubic Hodge integrals. Denote by \( \overline{M}_{g,k} \) the Deligne–Mumford moduli space of stable algebraic curves of genus \( g \) with \( k \) distinct marked points, by \( L_i \) the \( i^{th} \) tautological line bundle on \( \overline{M}_{g,k} \), and \( E_{g,k} \) the Hodge bundle. Denote

\[ \psi_i := c_1(L_i), \quad i = 1, \ldots, k, \]

\[ \lambda_j := c_j(E_{g,k}), \quad j = 0, \ldots, g. \]

The Hodge integrals are some rational numbers defined by

\[ \int_{\overline{M}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} \lambda_1^{j_1} \cdots \lambda_g^{j_g} = \langle \lambda_1^{j_1} \cdots \lambda_g^{j_g} \tau_{i_1} \cdots \tau_{i_k} \rangle_{g,k}, \quad i_1, \ldots, i_k, j_1, \ldots, j_g \geq 0. \]

These numbers are zero unless the degree-dimension matching is satisfied

\[ 3g - 3 + k = \sum_{i=1}^{k} i \sigma_i + \sum_{\ell=1}^{g} \ell j_{\ell}. \]

We are particularly interested in the following special cubic Hodge integrals:

\[ \langle \Omega_g \tau_{i_1} \cdots \tau_{i_k} \rangle_{g,k}, \quad \text{with} \quad \Omega_g := \Lambda_g(-1) \Lambda_g(-1) \Lambda_g(\frac{1}{2}), \]

where \( \Lambda_g(z) := \sum_{j=0}^{g} \lambda_j z^j \) denotes the Chern polynomial of the Hodge bundle \( E_{g,k} \).

Notations. \( \mathcal{Y} \) denotes the set of partitions. For a partition \( \lambda \), denote by \( \ell(\lambda) \) the length of \( \lambda \) and by \( |\lambda| \) the weight of \( \lambda \). Denote \( m(\lambda) := \prod_{i=1}^{\ell(\lambda)} m_i(\lambda) \) with \( m_i(\lambda) \) being the multiplicity of \( i \) in \( \lambda \).
Definition 5. For given $g, k \geq 0$ and an arbitrary set of integers $i_1, \ldots, i_k \geq 0$, define
\[
H_{g,i_1,\ldots,i_k} = 2^{g-1} \sum_{\lambda \in \mathcal{Y}} \frac{(-1)^{\ell(\lambda)}}{m(\lambda)!} \left( H_{g,\tau_1+1} \right)_{g,\ell(\lambda)+k},
\]
where $|i| := i_1 + \cdots + i_k$, $\tau_i := \tau_{i_1} \cdots \tau_{i_k}$, and $\tau_{\lambda+1} := \tau_{\lambda_1+1} \cdots \tau_{\ell(\lambda)+1}$.

It should be noted that according to (34), \( \sum_{\lambda \in \mathcal{Y}} \) in (35) is a finite sum.

Lemma 1. The number $H_{g,i_1,\ldots,i_k}$ vanishes unless $|i| \leq 3g - 3 + k$.

Theorem 3. The numbers $H_{g,i_1,\ldots,i_k}$ satisfy
(i) For $k = 0$,
\[
\sum_{g \geq 0} \epsilon^{2g-2} H_{g,0} = -\frac{3}{8} \epsilon^2 - \frac{1}{8} \epsilon + \frac{1}{2} \zeta'(-1) + \frac{1}{32} \left( \frac{1}{4 \epsilon^2} - \frac{5}{48} \right) \log(1 + \epsilon) + \sum_{p \geq 1} \frac{(p-1)!((-\epsilon)^p)}{(1+\epsilon)^p} \sum_{0 \leq g \leq (p+2)/2} (2g-1) (1 - 2^{p+3-2g}) \frac{B_{p+3-2g} B_{2g}}{(p+3-2g)! (2g)!}.
\]

(ii) For $k = 1$,
\[
\binom{2j}{j} \sum_{g \geq 0} 2^{2g-2} \sum_{i=0}^{3g+3-k} j^{i+1} H_{g,i} - \frac{1}{2\epsilon^2} \left[ \frac{j}{1+j} \binom{2j}{j} - (1 - \frac{\epsilon}{2}) \binom{2j}{j} \right] = \sum_{g,h \geq 0} \sum_{\ell=0}^{h} \epsilon^{2g-2+\ell} \frac{B_{\ell+1}}{\ell+1} \binom{h}{\ell} (1 - 2^{\ell+1}) (1 + \frac{\epsilon}{2})^{h-\ell} a_g(2j).
\]

(iii) For $k \geq 2$,
\[
\epsilon^k \sum_{j_1,\ldots,j_k \geq 1} \prod_{\ell=1}^{k} \frac{(2j_\ell)}{j_\ell} \sum_{g \geq 0} \epsilon^{2g-2} \prod_{i_1,\ldots,i_k \geq 0}^{i_1,\ldots,i_k \geq 0} \prod_{|i| \leq 3g+3-k} H_{g,i_1,\ldots,i_k}
\]
\[
= \frac{1}{k} \sum_{\sigma \in S_k} \frac{\left( R_{j_1+1} \left( \frac{\lambda_{j_1+1}}{\epsilon} \right) \cdots R_{j_k+1} \left( \frac{\lambda_{j_k+1}}{\epsilon} \right) \right)}{\prod_{\ell=1}^{k} (\lambda_{\sigma\ell} - \lambda_{\sigma\ell+1})} - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2} - \frac{\delta_{k,2}}{\prod_{j_1 \neq j_2 \geq 1} \frac{j_1 j_2 \frac{2j_1}{j_1} \frac{2j_2}{j_2}}{1 + j_1 j_2 \lambda_{12} \lambda_{12}+1}},
\]
where $R_n(\lambda)$ is defined the same as in (28).

1.6. Organization of the paper. In Section 2 we derive several useful formulae, and prove Proposition 1. In Section 3 we study MR, and use it to describe the discrete KdV flows and the tau-structure. Section 4 is devoted to the proof of the Main Theorem. Applications to ribbon graph enumerations and Hodge integrals are given in Section 5.

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2. Basic formulation

In this section we do preparations for the later sections, and prove Proposition 1.

2.1. Some useful identities. Recall that \( P(n) := \Lambda + w_n \Lambda^{-1} \), \( L = P^2 \). Denote

\[
P(n)^{\ell+1} =: \sum_{k \in \mathbb{Z}} A_{\ell,k}(n) \Lambda^k, \quad \ell \geq -1, \tag{39}
\]

\[
L(n)^j =: \sum_{k \in \mathbb{Z}} m_{j,k}(n) \Lambda^k, \quad j \geq 0. \tag{40}
\]

It is easy to see that if \( k \) is odd, or if \( |k| > 2j \), then \( m_{j,k} \equiv 0 \). It is also clear that

\[
m_{j,k} = A_{2j-1,k}. \tag{41}
\]

Lemma 2. The coefficients \( A_{\ell,k}(n) \) and \( m_{j,k}(n) \) for any \( k \in \mathbb{Z} \) live in the ring \( \mathbb{Z}[w] \).

**Proof** Follows easily from the definitions (39)–(40).

Lemma 3. The following identities hold true

\[
m_{j,-2}(n) = w_n w_{n-1} m_{j,2}(n-2), \tag{42}
\]

\[
m_{j,0}(n) = m_{j-1,-2}(n) + m_{j-1,-2}(n + 2) + (w_{n+1} + w_n) m_{j-1,0}(n), \tag{43}
\]

\[
m_{j,-2}(n) - m_{j,-2}(n - 2) - (w_{n+1} + w_{n-2}) (m_{j-1,-2}(n) - m_{j-1,-2}(n - 2)) + w_{n-2} w_{n-3} m_{j-1,0}(n - 4) - w_n w_{n-1} m_{j-1,0}(n) = 0. \tag{44}
\]

**Proof** Comparing the constant terms of the identity

\[
L^j = L^j^{-1} L = L L^{-1} \tag{45}
\]

we obtain that

\[
m_{j,0}(n) = m_{j-1,-2}(n) + (w_{n+1} + w_n) m_{j-1,0}(n) + w_{n+2} w_{n+1} m_{j-1,2}(n) = m_{j-1,-2}(n + 2) + (w_{n+1} + w_n) m_{j-1,0}(n) + w_n w_{n-1} m_{j-1,2}(n - 2).
\]

This proves (42)–(43). Similarly, comparing the coefficients of \( \Lambda^{-2} \) of (45) we obtain

\[
m_{j,-2}(n) = m_{j-1,-4}(n) + (w_{n-1} + w_{n-2}) m_{j-1,-2}(n) + w_n w_{n-1} m_{j-1,0}(n) = m_{j-1,-4}(n + 2) + (w_{n+1} + w_n) m_{j-1,-2}(n) + w_n w_{n-1} m_{j-1,0}(n - 2),
\]

which implies identity (44). The lemma is proved.

Lemma 4. The following identities hold true

\[
A_{\ell,-1}(n) = w_n A_{\ell,1}(n - 1), \tag{46}
\]

\[
A_{\ell,0}(n) = w_{n+1} A_{\ell-1,1}(n) + w_n A_{\ell-1,1}(n - 1), \tag{47}
\]

\[
w_n A_{\ell,1}(n - 1) - w_{n+1} A_{\ell,1}(n) + w_{n+1} A_{\ell-1,0}(n - 1) - w_n A_{\ell-1,0}(n - 1) = 0, \tag{48}
\]

\[
A_{\ell,0}(n + 1) - A_{\ell,0}(n) = w_{n+2} A_{\ell,2}(n) - w_n A_{\ell,2}(n - 1). \tag{49}
\]
Proof. Identities \((46) - (48)\) are contained in the Lemma 2.2.1 of \cite{11} (see the proof therein). Identity \((49)\) follows from comparing coefficients of \(\Lambda\) on the both sides of the following identity:

\[
P^{\ell+1}P = PP^{\ell+1}.
\]

The lemma is proved. \(\square\)

Taking \(\ell = 2j - 1\) in identity \((59)\) and using \((11)\) we obtain

\[
m_{j,0}(n + 1) - m_{j,0}(n) = w_{n+2}m_{j,2}(n) - w_n m_{j,2}(n - 1).
\]  

Identity \((51)\) implies that \(w_{n+1}\) satisfies \((3)\), then it satisfies \((6)\); vice versa.

2.2. Proof of Proposition \(1\) Note that this proposition means the following: if \(w_n = w_n(s)\) satisfies \((3)\), then it satisfies \((6)\); vice versa.

Firstly, let \(w_n = w_n(s)\) be an arbitrary solution to \((3)\), i.e.

\[
\frac{\partial P}{\partial s_j} = [A_2j - 1, P]
\]

for all \(j \geq 1\). Since \(L = P^2\) we have

\[
\frac{\partial L}{\partial s_j} = P\frac{\partial P}{\partial s_j} + \frac{\partial P}{\partial s_j} = P [A_2j - 1, P] + [A_2j - 1, P] P = [A_2j - 1, L].
\]

Secondly, let \(w_n = w_n(s)\) be an arbitrary solution to \((6)\), namely, it satisfies that

\[
\frac{\partial w_{n+1} + w_n}{\partial s_j} = w_{n+2} w_{n+1} m_{j,2}(n) - w_n w_{n-1} m_{j,2}(n - 2),
\]  

\[
\frac{\partial (w_n w_{n-1})}{\partial s_j} = w_n w_{n-1} (m_{j,0}(n) - m_{j,0}(n - 2)).
\]

Identity \((51)\) implies that

\[
(\Lambda + 1)\frac{\partial w_n}{\partial s_j} = w_{n+2} w_{n+1} m_{j,2}(n) - w_{n+1} w_n m_{j,2}(n - 1)
\]

\[
+ w_{n+1} w_n m_{j,2}(n - 1) - w_n w_{n-1} m_{j,2}(n - 2)
\]

\[
= w_{n+1} (m_{j,0}(n + 1) - m_{j,0}(n)) + w_n (m_{j,0}(n) - m_{j,0}(n - 1)),
\]

where we have used identity \((50)\). Identity \((52)\) implies that

\[
w_n \frac{\partial w_{n+1}}{\partial s_j} + w_n \frac{\partial w_n}{\partial s_j} = w_{n+1} w_n (m_{j,0}(n + 1) - m_{j,0}(n)) + w_{n+1} w_n (m_{j,0}(n) - m_{j,0}(n - 1)).
\]

Combining the above two identities and assuming that \(w_n \neq w_{n+1}\) yields

\[
\frac{\partial w_n}{\partial s_j} = w_n (m_{j,0}(n) - m_{j,0}(n - 1)) = \text{Coef}_{\Lambda}^{-1} [A_2j, P].
\]

(It is easy to see from \((51)\) that solutions satisfying \(w_n \equiv w_{n+1}\) are independent of \(s\). Therefore these trivial solutions also satisfy \((3)\).) The proposition is proved. \(\square\)
2.3. Lax pairs in the matrix form. Let us write the scalar Lax pairs (9)–(10) into the matrix form.

Lemma 5. The wave function $\psi_n$ satisfies that

$$
\frac{\partial \psi_n}{\partial s_j} = \lambda^j \psi_n + \sum_{i=1}^{j} \lambda^{j-i} \left( m_{i-1,-2} \psi_n - w_n w_{n-1} m_{i-1,0} \psi_{n-2} \right), \quad j \geq 1.
$$

Proof. We have for any $j \geq 1$

$$
(L_j^j) = (L_j^{j-1} L_j^L) = (L_j^{j-1})_+ L_+ + \left( (L_j^{j-1})_+ - (L_j^{j-1})_- \right)_+ + \left( (L_j^{j-1})_- + (L_j^{j-1})_+ \right)_+
$$

$$
= (L_j^{j-1})_+ L - \left( (L_j^{j-1})_+ - (L_j^{j-1})_- \right)_+ + \left( (L_j^{j-1})_- + (L_j^{j-1})_+ \right)_+
$$

$$
= (L_j^{j-1})_+ L + m_{j-1,-2} - w_n w_{n-1} m_{j-1,0} \Lambda^{-2}.
$$

In the above derivations it is understood that $L = L(n)$ and $m_{j,k} = m_{j,k}(n)$. Therefore,

$$
A_{2j-1} = (L_j^j)_+ = L^j + \sum_{i=1}^{j} \left( m_{i-1,-2} - w_n w_{n-1} m_{i-1,0} \Lambda^{-2} \right) L^{j-i}, \quad \forall j \geq 0.
$$

The lemma is proved.

Lemma 6. The vector-valued wave function $\Psi_n = (\psi_n, \psi_{n-2})^T$ satisfies that

$$
\frac{\partial \Psi_n}{\partial s_j} = V_j(n) \Psi_n, \quad j \geq 1,
$$

where $V_j(n)$ are the following $2 \times 2$ matrices

$$
V_j(n) := \begin{pmatrix}
\lambda^j + \sum_{i=1}^{j} \lambda^{j-i} m_{i-1,-2}(n) & -w_n w_{n-1} \sum_{i=1}^{j} \lambda^{j-i} m_{i-1,0}(n) \\
\sum_{i=1}^{j} \lambda^{j-i} m_{i-1,0}(n-2) & m_{j,0}(n-2) - \sum_{i=1}^{j} \lambda^{j-i} m_{i-1,-2}(n)
\end{pmatrix}.
$$

Proof. Equation (55) follows straightforwardly from (54) and (9).

We arrive at

Proposition 3. The discrete KdV hierarchy are the compatibility conditions of (12) and (55):

$$
\frac{\partial U_n}{\partial s_j} = V_j(n+2) U_n - U_n V_j(n), \quad j = 1, 2, 3, \ldots.
$$

3. Tau-structure

3.1. The MR recursive relations. Write

$$
\alpha_n = \sum_{j \geq 0} \frac{a_{n,j}}{\lambda^{j+1}}, \quad \gamma_n = \sum_{j \geq 0} \frac{c_{n,j}}{\lambda^{j+1}}.
$$
Then we find that $a_{n,j}, c_{n,j}$ satisfy

\begin{align}
    c_{n,j+1} &= (w_{n-1} + w_{n-2}) c_{n,j} + a_{n,j} + a_{n-2,j}, \\
    a_{n,j+1} &= a_{n+1,j+1} + (w_{n+1} + w_n)(a_{n+2} - a_{n,j}) + w_{n+1} w_n c_{n+4,j} - w_n w_{n-1} c_{n,j} = 0, \\
    a_{n,j} &= \sum_{i=0}^{j-1} \left( w_n w_{n-1} c_{n,i} c_{n,j-1-i} - a_{n,i} a_{n,j-1-i} \right)
\end{align}

as well as

\begin{align}
    a_{n,0} &= 0, \quad c_{n,0} = 1.
\end{align}

**Lemma 7.** The basic resolvent of $L$ exists and is unique.

**Proof** Observe that multiplying (18) and (19) gives (20). This proves existence of $R_n$. Uniqueness follows directly from the MR recursive relations (57)–(59), as we can solve $a_{n,j}, c_{n,j}$ uniquely in an algebraic way for all $j \geq 1$. The lemma is proved.

For the reader’s convenience we give in below the first few terms of $R_n$:

\[
    R_n(\lambda) = \left( \begin{array}{c}
    1 + \frac{w_n}{\lambda} + \cdots - \frac{w_{n-1} w_n}{\lambda^2} - \frac{w_{n-1} (w_n + w_{n+1}) w_n}{\lambda^2} + \cdots \\
    \frac{1}{\lambda} + \frac{w_n - 2 + w_{n-1}}{\lambda^2} + \cdots - \frac{w_{n-1} w_n}{\lambda^2} + \cdots
    \end{array} \right).
\]

### 3.2. MR and the discrete KdV flows.

Let $R_n$ be the basic matrix resolvent of $L$.

**Lemma 8.** The following formulae hold true

\begin{align}
    c_{n,j} &= m_{j,0}(n-2), \\
    a_{n,j} &= m_{j,-2}(n).
\end{align}

**Proof** By identifying the recursive relations as well as the initial values of the recursions.

**Lemma 9.** The matrices $V_j(n)$ have the following expressions

\[
    V_j(n) = (\lambda^j R_n)_+ + \begin{pmatrix} 0 & 0 \\ 0 & c_{n,j} \end{pmatrix},
\]

where “$+$” means taking the polynomial part in $\lambda$ as $\lambda \to \infty$.

### 3.3. Loop operator.

Introduce a linear operator $\nabla(\lambda)$ by

\[
    \nabla(\lambda) := \sum_{j \geq 1} \frac{1}{\lambda^{j+1}} \frac{\partial}{\partial s_j}.
\]

It readily follows from equation (63) that

\[
    \nabla(\mu) \Psi_n(\lambda) = \left[ \frac{R_n(\mu)}{\mu - \lambda} + Q_n(\mu) \right] \Psi_n(\lambda),
\]

where

\[
    Q_n(\mu) := -\frac{I}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma_n(\mu) \end{pmatrix}.
\]
Lemma 10. The following formula holds true
\[ \nabla(\mu) R_n(\lambda) = \frac{1}{\mu - \lambda} \left[ R_n(\mu), R_n(\lambda) \right] + \left[ Q_n(\mu), R_n(\lambda) \right]. \] (65)

3.4. From MR to tau-function. The MR allows us to define tau-function of an arbitrary solution of the discrete KdV hierarchy.

Definition 6. A family of elements \( \Omega_{pq}(n) \in \mathbb{Z}[w] \), \( p, q \geq 1 \) are called a tau-structure of the discrete KdV hierarchy if
\[ \Omega_{pq}(n) = \Omega_{qp}(n), \quad \forall \ p, q \geq 1 \] (66)
and for an arbitrary solution \( w_n = w_n(s) \) of the discrete KdV hierarchy
\[ \frac{\partial \Omega_{pq}(n)}{\partial s_r} = \frac{\partial \Omega_{pr}(n)}{\partial s_q}, \quad \forall \ p, q, r \geq 1. \] (67)

Definition 7. Define \( \Omega_{ij}(n), \ i, j \geq 1 \) via the generating series
\[ \sum_{i,j \geq 1} \Omega_{ij}(n) \lambda^{i-1} \mu^{j-1} = \frac{\text{tr} \left( R_n(\lambda) R_n(\mu) \right) - 1}{(\lambda - \mu)^2}. \] (68)

Lemma 11. \( \Omega_{ij}(n), \ i, j \geq 1 \) (68) are well-defined, and live in \( \mathbb{Z}[w] \). Moreover, they form a tau-structure of the discrete KdV hierarchy.

Proof The proof is almost identical with the one for the Toda lattice hierarchy \[11\] (or the one for the Drinfeld–Sokolov hierarchies \[3\]); details are omitted here.

Proof of Proposition 2 By Lemma 11 it suffices to prove the compatibility between (21) – (23).

Firstly, on one hand,
\[ \sum_{i,j \geq 1} \lambda^{i-1} \mu^{j-1} \left[ \Omega_{ij}(n + 2) - \Omega_{ij}(n) \right] \]
\[ = \frac{\text{tr} \left( R_{n+2}(\lambda) R_{n+2}(\mu) \right) - \text{tr} \left( R_n(\lambda) R_n(\mu) \right)}{(\lambda - \mu)^2} \]
\[ = \frac{(1 + 2\alpha_n(\lambda)) \gamma_{n+2}(\mu) - (1 + 2\alpha_n(\mu)) \gamma_{n+2}(\lambda)}{\lambda - \mu} - \gamma_{n+2}(\lambda) \gamma_{n+2}(\mu). \]

On the other hand,
\[ \nabla(\mu) \left[ R_{n+2}(\lambda) \right]_{21} = \frac{(1 + 2\alpha_{n+2}(\mu)) \gamma_{n+2}(\lambda) - (1 + 2\alpha_{n+2}(\lambda)) \gamma_{n+2}(\mu)}{\lambda - \mu} + \gamma_{n+2}(\lambda) \gamma_{n+2}(\mu). \]

Hence by using (68) we find that
\[ \sum_{i,j \geq 1} \lambda^{i-1} \mu^{j-1} \left[ \Omega_{ij}(n + 2) - \Omega_{ij}(n) \right] = \nabla(\mu) \left[ R_{n+2}(\lambda) \right]_{21}. \] (69)

This proves the compatibility between (21) and (22).
Secondly, on one hand, 
\[
\sum_{i,j \geq 1} \lambda^{-i-1} \mu^{-j-1} \left[ \Omega_{ij}(n+2) + \Omega_{ij}(n-1) - \Omega_{ij}(n+1) - \Omega_{ij}(n) \right]
\]
\[= \sum_{i,j \geq 1} \lambda^{-i-1} \mu^{-j-1} \left[ \Omega_{ij}(n+2) - \Omega_{ij}(n) \right] - \sum_{i,j \geq 1} \lambda^{-i-1} \mu^{-j-1} \left[ \Omega_{ij}(n+1) - \Omega_{ij}(n-1) \right].
\]
On the other hand, 
\[
\nabla(\mu) \nabla(\lambda) \log w_n = \nabla(\mu) \left[ \gamma_{n+2}(\lambda) - \gamma_{n+1}(\lambda) \right] = \nabla(\mu) \gamma_{n+2}(\lambda) - \nabla(\mu) \gamma_{n+1}(\lambda).
\]
Using (69) we find 
\[
\sum_{i,j \geq 1} \lambda^{-i-1} \mu^{-j-1} \left[ \Omega_{ij}(n+2) + \Omega_{ij}(n-1) - \Omega_{ij}(n+1) - \Omega_{ij}(n) \right] = \nabla(\mu) \nabla(\lambda) \log w_n.
\]
This proves compatibility between (21) and (23). Thirdly, the following identity 
\[
\nabla(\lambda) \log w_n = \gamma_{n+2}(\lambda) - \gamma_{n+1}(\lambda)
\]
shows the compatibility between (22) and (23). The proposition is proved.

3.5. Generating series of multi-point correlations functions.

**Definition 8.** For an arbitrary solution \( w_n = w_n(s) \) to the discrete KdV hierarchy, let \( \tau_n^{dKdV} = \tau_n^{dKdV}(s) \) denote the tau-function of this solution. We call 
\[
\partial^k \log \tau_n^{dKdV}(s) \quad j_1, \ldots, j_k \geq 1, \quad k \geq 1
\]
the \( k \)-point correlation functions of the solution \( w_n = w_n(s) \).

**Theorem 4.** \( \forall k \geq 3 \), the generating series of \( k \)-point correlation functions has the following expression: 
\[
\sum_{j_1, \ldots, j_k \geq 1} \frac{1}{\lambda_1^{j_1} \cdots \lambda_k^{j_k}} \partial^k \log \tau_n^{dKdV}(s) = - \frac{1}{k} \sum_{\sigma \in S_k} \text{tr} \left( R_n(\lambda_{\sigma_1}) \cdots R_n(\lambda_{\sigma_k}) \right) \prod_{i=1}^k (\lambda_{\sigma_i} - \lambda_{\sigma_i+1}),
\]
where it is understood that \( \sigma_{k+1} = \sigma_1 \).

The proof can be achieved by the mathematical induction, as in [1]; we hence omit the details.

We see from Theorem 4 the easy fact that the multi-logarithmic derivatives \( \frac{\partial^k \log \tau_n^{dKdV}(s)}{\partial s_{j_1} \cdots \partial s_{j_k}} \), \( k \geq 2 \) all live in \( \mathbb{Z}[w] \), as their generating series are expressed by MR via algebraic manipulations.

4. Main Theorem

4.1. Matrix resolvent approach to the Toda lattice hierarchy. Denote 
\[
\mathcal{P} := \Lambda + v_n^{Toda} + w_n^{Toda} \Lambda^{-1}, \quad \mathcal{A}_\ell := (\mathcal{P}^{\ell+1})_+ , \quad \ell \geq 0.
\]
The Toda lattice hierarchy is defined as the following system of commuting flows 
\[
\frac{\partial \mathcal{P}}{\partial \ell} = \left[ \mathcal{A}_\ell, \mathcal{P} \right], \quad \ell \geq 0.
\]
Let us briefly review part of the results of \cite{[11]}. Introduce $U_n = \begin{pmatrix} v_n^{\text{Toda}} - \lambda & w_n^{\text{Toda}} \\ -1 & 0 \end{pmatrix}$. The basic resolvent $R_n$ associated to $\mathcal{P}^M := \Lambda + U_n$ is defined as the unique solution in $\text{Mat}(2, \mathbb{Z}[v^{\text{Toda}}, w^{\text{Toda}}][[\lambda^{-1}]]$ to the problem:
\begin{align*}
R_{n+1} U_n - U_n R_n &= 0, \\
R_n &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^{-1}), \\
\text{tr} R_n &= 1, \quad \det R_n = 0.
\end{align*}

Write
\begin{equation}
R_n(\lambda) = \begin{pmatrix} 1 + A_n(\lambda) & B_n(\lambda) \\ G_n(\lambda) & -A_n(\lambda) \end{pmatrix}, \quad A_n, B_n, G_n \in \mathbb{Z}[v^{\text{Toda}}, w^{\text{Toda}}][[\lambda^{-1}]].
\end{equation}

Then $A_n, B_n, G_n$ satisfy that
\begin{align*}
B_n &= -w_n^{\text{Toda}} G_{n+1} \\
A_{n+1} + A_n + 1 &= G_{n+1} (\lambda - v_n^{\text{Toda}}) \\
(\lambda - v_n^{\text{Toda}})(A_n - A_{n+1}) &= w_n^{\text{Toda}} G_n - w_{n+1}^{\text{Toda}} G_{n+2} \\
A_n + A_n^2 &= B_n G_n.
\end{align*}

The following lemma was proven in \cite{[11]}.

**Lemma 12** (\cite{[11]}). For an arbitrary solution $v_n^{\text{Toda}} = v_n^{\text{Toda}}(t)$, $w_n^{\text{Toda}} = w_n^{\text{Toda}}(t)$ to the Toda lattice hierarchy there exists a function $\tau_n^{\text{Toda}}(t)$ such that
\begin{equation}
\sum_{i, j \geq 0} \frac{1}{\lambda^{i+2} \mu^{j+2}} \frac{\partial^2 \log \tau_n^{\text{Toda}}(t)}{\partial t_i \partial t_j} = \frac{\text{tr} R_n(t, \lambda) R_n(t, \mu) - 1}{(\lambda - \mu)^2},
\end{equation}
\begin{equation}
\frac{1}{\lambda} + \sum_{i \geq 0} \frac{1}{\lambda^{i+2}} \frac{\partial}{\partial t_i} \log \frac{\tau_{n+1}^{\text{Toda}}(t)}{\tau_n^{\text{Toda}}(t)} = [R_{n+1}(t, \lambda)]_{21}
\end{equation}
\begin{equation}
\tau_{n+1}^{\text{Toda}}(t) \tau_{n-1}^{\text{Toda}}(t) = w_n.
\end{equation}

The function $\tau_n^{\text{Toda}}(t)$ is uniquely determined by the solution $v_n^{\text{Toda}}(t)$, $w_n^{\text{Toda}}(t)$ up to $\tau_n^{\text{Toda}}(t) \mapsto e^{a_0 + a_1 n + \sum_{i \geq 0} b_i t_i} \tau_n^{\text{Toda}}(t)$ for some constants $a_0, a_1, b_0, b_1, b_2, \ldots$.

**Definition 9** (\cite{[11]}). $\tau_n^{\text{Toda}}(t)$ is called the tau-function of the solution $v_n^{\text{Toda}}(t)$, $w_n^{\text{Toda}}(t)$.

**Definition 10.** The logarithmic derivatives of $\tau_n^{\text{Toda}}(t)$
\begin{equation}
\frac{\partial^k \log \tau_n^{\text{Toda}}(t)}{\partial t_{i_1} \cdots \partial t_{i_k}}, \quad i_1, \ldots, i_k \geq 0, k \geq 1
\end{equation}
are called $k$-point correlations functions of the Toda lattice hierarchy. Define
\begin{equation}
C_k(\lambda_1, \ldots, \lambda_k; n; t) := \sum_{i_1, \ldots, i_k \geq 0} \frac{1}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} \frac{\partial^k \log \tau_n^{\text{Toda}}(t)}{\partial t_{i_1} \cdots \partial t_{i_k}}.
\end{equation}
4.2. Reduction to the discrete KdV hierarchy. We consider solutions of Toda lattice hierarchy in the ring $\mathbb{C}[[t_0,t_1,\ldots]] \otimes \mathcal{V}$, where $\mathcal{V}$ is any ring of functions of $n$, closed under $\Lambda$ and $\Lambda^{-1}$. These solutions can be specified by (i.e. in 1-1 correspondence to) the initial value:

$$f(n) = v_{n}^{\text{Toda}}(t = 0), \quad g(n) = w_{n}^{\text{Toda}}(t = 0).$$

Let us explain how a subset of solutions of the Toda lattice hierarchy be reduced to solutions of the discrete KdV hierarchy. On one hand, let $v_{n}^{\text{Toda}} = v_{n}^{\text{Toda}}(t)$, $w_{n}^{\text{Toda}} = w_{n}^{\text{Toda}}(t)$ be an arbitrary solution in $\mathbb{C}[[t_0,t_1,\ldots]] \otimes \mathcal{V}$ of the Toda lattice hierarchy satisfying the following type of initial conditions

$$f(n) \equiv 0.$$  

It follows that

$$v_{n}^{\text{Toda}}|_{t_0=t_2=t_4=\cdots=0} \equiv 0, \quad (\forall n, t_1, t_3, t_5, \cdots).$$

This further implies that the commuting flows $\partial_{w_{n}^{\text{Toda}}(t)}|_{t_0-t_2=t_4=\cdots=0} (j \geq 1)$ are decoupled, namely, there are no $v_{n}^{\text{Toda}}$-dependence in these flows (of course when restricting to $t_0 = t_2 = t_4 = \cdots = 0$). Moreover, these flows coincide with the discrete KdV hierarchy (3). Therefore if we define

$$w_{n}(s) := w_{n}^{\text{Toda}}(t)|_{t_{2s-1}=s, \, t_{2s-2}=0, \, i \geq 1},$$

then $w_{n} = w_{n}(s)$ is a solution to the discrete KdV hierarchy. On the other hand, let $w_{n} = w_{n}(s)$ be an arbitrary solution to the discrete KdV hierarchy in the ring $\mathbb{C}[s_1, s_2, \ldots] \otimes \mathcal{V}$. Let $g(n)$ denote its initial value, i.e. $g(n) := g(n) = w_{n}(s = 0)$. Define $v_{n}^{\text{Toda}}(t), \, w_{n}^{\text{Toda}}(t)$ as the unique solution in $\mathbb{C}[[t_0,t_1,\ldots]] \otimes \mathcal{V}$ to the Toda lattice hierarchy with $(f(n) \equiv 0, g(n))$ as the initial value. Then $w_{n}^{\text{Toda}}(t)|_{t_{2s-1}=s, \, t_{2s-2}=0, \, i \geq 1} = w_{n}(s)$.

Hence the correspondence between solutions of the discrete KdV hierarchy and the subset of solutions of the Toda Lattice hierarchy has been established.

For a solution $(v_{n}^{\text{Toda}}(t), w_{n}^{\text{Toda}}(t))$ to the Toda lattice hierarchy satisfying $v_{n}^{\text{Toda}}(0) \equiv 0 (\forall n)$, let $\tau_{n}^{\text{Toda}}(t)$ denote the tau-function of this solution. Define $w_{n}(s)$ as in (85), and

$$\tau_{n}(s) := \tau_{n}^{\text{Toda}}(t_0 = 0, t_1 = s_1, t_2 = 0, t_3 = s_2, \ldots).$$

Then we know that the function $w_{n} = w_{n}(s)$ satisfies the discrete KdV hierarchy (3), and that

$$w_{n}(s) = \frac{\tau_{n+1}(s) \tau_{n-1}(s)}{\tau_{n}^{2}(s)}.$$  

As indicated above, all solutions of the discrete KdV hierarchy can be obtained from this way.

**Definition 11.** We call $\tau_{n}(s)$ the tau-function reduced from the Toda lattice hierarchy of the solution $w_{n} = w_{n}(s)$ to the discrete KdV hierarchy.

Introduce the notations:

$$A_n(\lambda) := A_n(\lambda)|_{v_{n}^{\text{Toda}} \equiv 0, w_{n}^{\text{Toda}} \equiv w_{n}},$$

$$B_n(\lambda) := B_n(\lambda)|_{v_{n}^{\text{Toda}} \equiv 0, w_{n}^{\text{Toda}} \equiv w_{n}},$$

$$G_n(\lambda) := G_n(\lambda)|_{v_{n}^{\text{Toda}} \equiv 0, w_{n}^{\text{Toda}} \equiv w_{n}}.$$

Clearly, $A_n, \, B_n, \, G_n$ belong to $\mathbb{Z}[w][[\lambda^{-1}]]$. Note that definitions of $A_n(\lambda), \, B_n(\lambda), \, G_n(\lambda)$ are in the absolute sense, namely, they do not depend on whether $w_{n}$ is a solution or not.
Lemma 13. The function $A_n(\lambda)$ satisfies
\[ w_{n+1}(A_{n+2}(\lambda) + A_{n+1}(\lambda) + 1) - w_n(A_n(\lambda) + A_{n-1}(\lambda) + 1) = \lambda^2 (A_{n+1}(\lambda) - A_n(\lambda)). \] (88)

Proof Following from (79) and (80) with $v_n^{\text{Toda}} \equiv 0$.

4.3. Proof of the Main Theorem. Firstly, on one hand, it follows from the Lemma 1.2.3 of [11] that
\[ m_{j,0}(n;s) = \frac{\partial}{\partial s_j} \log \frac{\tau_{n+1}(s)}{\tau_n(s)}, \quad j \geq 1. \] (89)

On the other hand, from (22) and (61) we find
\[ m_{j,0}(n;s) = \frac{\partial}{\partial s_j} \log \frac{\tau_{n+2}^{\text{dKdV}}(s)}{\tau_n^{\text{dKdV}}(s)}, \quad j \geq 1. \] (90)

Comparing the above two expressions we find
\[ \log \frac{\tau_{n+1}(s)}{\tau_n(s)} - \log \frac{\tau_{n+2}^{\text{dKdV}}(s)}{\tau_n^{\text{dKdV}}(s)} = S(n), \] (91)

where $S(n)$ is some function depending only on $n$. Equation (91) implies that
\[ \log \tau_n(s) - (\Lambda + 1) \log \tau_n^{\text{dKdV}}(s) = \tilde{S}(n) + f(s), \] (92)

where $\tilde{S}(n)$ is some function depending only on $n$, and $f(s)$ is some function depending only on $s$.

Secondly, it follows easily from (23) and (87) that
\[ \frac{\tau_{n+1}(s) \tau_{n-1}(s)}{\tau_n^2(s)} = \frac{\tau_{n+2}^{\text{dKdV}}(s) \tau_{n-1}^{\text{dKdV}}(s)}{\tau_{n+1}^{\text{dKdV}}(s) \tau_n^{\text{dKdV}}(s)}. \] (93)

Substituting (92) in (93) we find that
\[ \log \tau_n(s) - (\Lambda + 1) \log \tau_n^{\text{dKdV}}(s) = \alpha n + f(s), \] (94)

where $\alpha$ is some constant.

Thirdly, on one hand, using (21) we find
\[ \sum_{i,j \geq 1} \frac{\partial^2 \log \tau_n^{\text{dKdV}}(s)}{\partial s_i \partial s_j} \frac{1}{\lambda^{i+1} \mu^{j+1}} \]
\[ = \frac{\alpha_n(\lambda) + \alpha_n(\mu) + 2\alpha_n(\lambda)\alpha_n(\mu) - w_n w_{n-1} (\gamma_n(\lambda) \gamma_{n+2}(\mu) + \gamma_n(\mu) \gamma_{n+2}(\lambda))}{(\lambda - \mu)^2}. \] (95)

Therefore,
\[ \sum_{i,j \geq 1} \frac{\partial^2 \log \tau_n^{\text{dKdV}}(s)}{\partial s_i \partial s_j} \frac{1}{\lambda^{2i+1} \mu^{2j+1}} \]
\[ = \lambda \mu \frac{\alpha_n(\lambda^2) + \alpha_n(\mu^2) + 2\alpha_n(\lambda^2)\alpha_n(\mu^2) - w_n w_{n-1} (\gamma_n(\lambda^2) \gamma_{n+2}(\mu^2) + \gamma_n(\mu^2) \gamma_{n+2}(\lambda^2))}{(\lambda^2 - \mu^2)^2} \]
\[ := W_2(\lambda, \mu; n, s). \] (96)
So
\[
\sum_{i,j \geq 1} \left( \frac{\partial^2 \log \tau^{dKdV}(s)}{\partial s_i \partial s_j} + \frac{\partial^2 \log \tau^{dKdV}(s)}{\partial s_i \partial s_j} \right) \frac{1}{\lambda^{2i+1} \mu^{2j+1}} = W_2(\lambda, \mu; n, s) + W_2(\lambda, \mu; n+1, s).
\] (97)

On the other hand, denote
\[
P(n)^{\ell+1} =: \sum_{k \in \mathbb{Z}} P_{\ell,k}(n) \Lambda^k, \quad \ell = -1, 0, 1, 2, \ldots.
\]

Clearly,
\[
m_{j,k}(n) = P_{2j-1,k}(n), \quad j \geq 0.
\]

Using the Definition 1.2.4 and the Lemma 1.2.3 of [11] we have
\[
C_2(\lambda, \mu; n; t) = \sum_{i,j \geq 0} \frac{\partial^2 \log \tau^{Toda}(t)}{\partial t_i \partial t_j} \frac{1}{\lambda^{i+2} \mu^{j+2}} = A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) - w_n(\gamma_{n+1}(\lambda)\gamma_n(\mu) + \gamma_{n+1}(\mu)\gamma_n(\lambda))
\]
\[
(\lambda - \mu)^2
\]
where
\[
A_n(\lambda) = \sum_{\ell \geq 0} \frac{P_{\ell-1,1}(n)}{\lambda^{\ell+1}}, \quad G_n(\lambda) = \sum_{\ell \geq 0} \frac{P_{\ell-1,0}(n-1)}{\lambda^{\ell+1}}.
\]

Taking
\[
t_{2i-2} = 0, \quad t_{2i-1} = s_i \quad (i \geq 1)
\]
we have
\[
G_n(\lambda) = \sum_{j \geq 0} \frac{P_{2j-1,0}(n-1)}{\lambda^{2j+1}} = \sum_{j \geq 0} \frac{m_{j,0}(n-1)}{\lambda^{2j+1}} = \sum_{j \geq 0} \frac{c_{n+1,j}}{\lambda^{2j+1}} = \lambda \gamma_{n+1}(\lambda^2). \quad (98)
\]

It follows that
\[
A_n(\lambda) = (\Lambda + 1)^{-1} \left( \lambda^2 \gamma_{n+2}(\lambda^2) - 1 \right) = \lambda^2 (\Lambda + 1)^{-1} \gamma_{n+2}(\lambda^2) - \frac{1}{2}, \quad (99)
\]
\[
\alpha_n(\lambda) = (\Lambda^2 + 1)^{-1} \left( (\lambda - w_{n+1} - w_n) \gamma_{n+2}(\lambda) \right) - \frac{1}{2}. \quad (100)
\]

**Lemma 14.** The following identities hold true:
\[
\gamma_n(\lambda^2) = \frac{A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1}{\lambda^2}, \quad (101)
\]
\[
G_n(\lambda) = \frac{A_n(\lambda) + A_{n-1}(\lambda) + 1}{\lambda}, \quad (102)
\]
\[
\alpha_n(\lambda^2) = \frac{A_{n-1}(\lambda) - w_{n-1}}{\lambda^2} \left( A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1 \right), \quad (103)
\]

**Proof** Identities (101), (102) are easy consequences of (99), (98).
Note that identity \( [50] \) implies that
\[
\alpha_n(\lambda^2) = \frac{1}{2} \left( w_{n-1} \gamma_{n+1}(\lambda^2) - w_{n-2} \gamma_{n-1}(\lambda^2) + (\lambda^2 - 2w_{n-1}) \gamma_n(\lambda^2) - 1 \right)
\]
\[
= \frac{1}{2} \left( w_{n-1} \frac{A_n(\lambda) + A_{n-1}(\lambda) + 1}{\lambda^2} - w_{n-2} \frac{A_{n-2}(\lambda) + A_{n-3}(\lambda) + 1}{\lambda^2} \right.
\]
\[
+ (\lambda^2 - 2w_{n-1}) \frac{A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1}{\lambda^2} - 1 \right).
\]

Applying Lemma \([13]\) in this identity yields
\[
\alpha_n(\lambda^2) = \frac{1}{2\lambda^2} \left( \lambda^2 (A_{n-1}(\lambda) - A_{n-2}(\lambda)) + (\lambda^2 - 2w_{n-1}) (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) - \lambda^2 \right)
\]
\[
= \frac{1}{\lambda^2} \left( \lambda^2 A_{n-1}(\lambda) - w_{n-1} (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) \right).
\]

The lemma is proved.

Observe that \( C_2(\lambda, \mu; n, s) \) satisfies the parity symmetries
\[
C_2(\lambda, \mu) = C_2(-\lambda, -\mu), \quad C_2(\lambda, -\mu) = C_2(-\lambda, \mu).
\]

So
\[
\sum_{i,j \geq 1} \frac{\partial^2 \log \tau_n(t)}{\partial t_{2i-1} \partial t_{2j-1}} \frac{1}{\lambda^{2i+1} \mu^{2j+1}} = \frac{C_2(\lambda, \mu) - C_2(-\lambda, \mu)}{2}.
\]

Lemma 15. The following identity hold true:
\[
\lambda \mu \frac{\alpha_n(\lambda^2) + \alpha_n(\mu^2) + 2\alpha_n(\lambda^2)\alpha_n(\mu^2) - w_n w_{n-1} (\gamma_n(\lambda^2) \gamma_{n+2}(\mu^2) + \gamma_n(\mu^2) \gamma_{n+2}(\lambda^2))}{(\lambda^2 - \mu^2)^2}
\]
\[
+ \lambda \mu \frac{\alpha_{n+1}(\lambda^2) + \alpha_{n+1}(\mu^2) + 2\alpha_{n+1}(\lambda^2)\alpha_{n+1}(\mu^2) - w_{n+1} w_n (\gamma_{n+1}(\lambda^2) \gamma_{n+3}(\mu^2) + \gamma_{n+1}(\mu^2) \gamma_{n+3}(\lambda^2))}{(\lambda^2 - \mu^2)^2}
\]
\[
= \frac{A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) - w_n (G_{n+1}(\lambda)G_n(\mu) + G_{n+1}(\mu)G_n(\lambda))}{2(\lambda - \mu)^2}
\]
\[
- \frac{A_n(-\lambda) + A_n(\mu) + 2A_n(-\lambda)A_n(\mu) - w_n (G_{n+1}(-\lambda)G_n(\mu) + G_{n+1}(\mu)G_n(-\lambda))}{2(\lambda + \mu)^2}.
\]

Proof Applying \([101] - [103]\) and the parity symmetry
\[
A_n(-\lambda) = A_n(\lambda)
we find that it suffices to prove the following equality

\[- \lambda \mu + 2 \lambda \mu \left[ A_{n-1}(\lambda) - \frac{w_{n-1}}{\lambda^2} (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) + \frac{\lambda^2}{2} \right] \]

\[\left[ A_{n-1}(\mu) - \frac{w_{n-1}}{\mu^2} (A_{n-1}(\mu) + A_{n-2}(\mu) + 1) + \frac{\mu^2}{2} \right] \]

\[- \frac{w_n w_{n-1}}{\lambda \mu} \left[ (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) (A_{n+1}(\mu) + A_n(\mu) + 1) \right. \]

\[\left. + (A_{n-1}(\mu) + A_{n-2}(\mu) + 1) (A_{n+1}(\lambda) + A_n(\lambda) + 1) \right] \]

\[- \frac{w_{n+1} w_n}{\lambda \mu} \left[ (A_n(\lambda) + A_{n-1}(\lambda) + 1) (A_{n+2}(\mu) + A_{n+1}(\mu) + 1) \right. \]

\[\left. + (A_n(\mu) + A_{n-1}(\mu) + 1) (A_{n+2}(\lambda) + A_{n+1}(\lambda) + 1) \right] \]

\[+ 2 \lambda \mu \left[ A_n(\lambda) - \frac{w_n}{\lambda^2} (A_n(\lambda) + A_{n-1}(\lambda) + 1) + \frac{\lambda^2}{2} \right] \]

\[\left[ A_n(\mu) - \frac{w_n}{\mu^2} (A_n(\mu) + A_{n-1}(\mu) + 1) + \frac{\mu^2}{2} \right] \]

\[= \frac{(\lambda + \mu)^2}{2} \left[ A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) \right. \]

\[- \frac{w_n}{\lambda \mu} \left( (A_{n+1}(\lambda) + A_n(\lambda) + 1) (A_n(\mu) + A_{n-1}(\mu) + 1) \right. \]

\[\left. + (A_{n+1}(\mu) + A_n(\mu) + 1) (A_n(\lambda) + A_{n-1}(\lambda) + 1) \right) \]

\[- \frac{(\lambda - \mu)^2}{2} \left[ A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) \right. \]

\[+ \frac{w_n}{\lambda \mu} \left( (A_{n+1}(\lambda) + A_n(\lambda) + 1) (A_n(\mu) + A_{n-1}(\mu) + 1) \right. \]

\[\left. + (A_{n+1}(\mu) + A_n(\mu) + 1) (A_n(\lambda) + A_{n-1}(\lambda) + 1) \right) \right] . \]

Noting that

\[\lambda \mu \cdot \text{lhs} = \]

\[- \lambda^2 \mu^2 + 2 \left[ \lambda^2 A_{n-1}(\lambda) - w_{n-1} (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) + \frac{\lambda^2}{2} \right] \]

\[\left[ \mu^2 A_{n-1}(\mu) - w_{n-1} (A_{n-1}(\mu) + A_{n-2}(\mu) + 1) + \frac{\mu^2}{2} \right] \]

\[- w_{n-1} \left[ (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) \left( \mu^2 (A_n(\mu) - A_{n-1}(\mu)) + w_{n-1} (A_{n-1}(\mu) + A_{n-2}(\mu) + 1) \right) \right. \]

\[\left. + (A_{n-1}(\mu) + A_{n-2}(\mu) + 1) \left( \lambda^2 (A_n(\lambda) - A_{n-1}(\lambda)) + w_{n-1} (A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) \right) \right] \]

\[- w_n \left[ (A_n(\lambda) + A_{n-1}(\lambda) + 1) \left( \mu^2 (A_{n+1}(\mu) - A_n(\mu)) + w_n (A_n(\mu) + A_{n-1}(\mu) + 1) \right) \right. \]

\[\left. + (A_n(\mu) + A_{n-1}(\mu) + 1) \left( \lambda^2 (A_{n+1}(\lambda) - A_n(\lambda)) + w_n (A_n(\lambda) + A_{n-1}(\lambda) + 1) \right) \right] \]

\[+ 2 \left[ \lambda^2 A_n(\lambda) - w_n (A_n(\lambda) + A_{n-1}(\lambda) + 1) + \frac{\lambda^2}{2} \right] \]
\[
\left[ \mu^2 A_n(\mu) - w_n \left( A_n(\mu) + A_{n-1}(\mu) + 1 \right) + \frac{\mu^2}{2} \right]
\]
and that
\[
\lambda \mu \cdot \text{rhs} = 2 \lambda^2 \mu^2 \left( A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) \right) \\
+ w_n \left( \lambda^2 + \mu^2 \right) \left( (A_{n+1}(\lambda) + A_n(\lambda) + 1) (A_n(\mu) + A_{n-1}(\mu) + 1) \\
+ (A_{n+1}(\mu) + A_n(\mu) + 1) (A_n(\lambda) + A_{n-1}(\lambda) + 1) \right),
\]
we find
\[
\lambda \mu \cdot \text{(lhs - rhs)} \\
= \lambda^2 \mu^2 (2A_{n-1}(\lambda)A_{n-1}(\mu) - 2A_n(\lambda)A_n(\mu) + A_{n-1}(\lambda) + A_{n-1}(\mu) - A_n(\lambda) - A_n(\mu)) \\
- \mu^2 (A_{n-1}(\mu) + A_n(\mu) + 1)(w_{n-1}(A_{n-2}(\lambda) + A_{n-1}(\lambda) + 1) - w_n(A_n(\lambda) + A_{n+1}(\lambda) + 1)) \\
- \lambda^2 (A_{n-1}(\lambda) + A_n(\lambda) + 1)(w_{n-1}(A_{n-2}(\mu) + A_{n-1}(\mu) + 1) - w_n(A_n(\mu) + A_{n+1}(\mu) + 1)) \\
= \lambda^2 \mu^2 (2A_{n-1}(\lambda)A_{n-1}(\mu) - 2A_n(\lambda)A_n(\mu) + A_{n-1}(\lambda) + A_{n-1}(\mu) - A_n(\lambda) - A_n(\mu)) \\
+ \lambda^2 \mu^2 (A_{n-1}(\mu) + A_n(\mu) + 1)(A_n(\lambda) - A_{n-1}(\lambda)) \\
+ \lambda^2 \mu^2 (A_{n-1}(\lambda) + A_n(\lambda) + 1)(A_n(\mu) - A_{n-1}(\mu)) = 0.
\]
The lemma is proved. \(\square\)

**End of proof of Theorem** \(\Box\) It follows from Lemma \(\Box\) that
\[
\frac{\partial^2 \log \tau_n(s)}{\partial s_i \partial s_j} = \frac{\partial^2}{\partial s_i \partial s_j} (\Lambda + 1) \log \tau_n^{\text{dKdV}}(s).
\]
Combining with \(\Box\) we find that
\[
f(s) = \beta_0 + \sum_{k \geq 1} \beta_k s_k,
\]
where \(\beta_0, \beta_1, \beta_2, \ldots\) are constants (independent of \(n\)). The theorem is proved. \(\square\)

**Corollary 1.** Fix \(k \geq 2\). Let \(w_n = w_n(s)\) be an arbitrary solution to the discrete KdV hierarchy, and \(\tau_n\) the tau-function reduced from the Toda lattice hierarchy of \(w_n(s)\). The following formula holds true:
\[
\sum_{j_1, \ldots, j_k=1}^{\infty} \frac{1}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}} \frac{\partial^k \log \tau_n(s)}{\partial s_{j_1} \cdots \partial s_{j_k}} = - \frac{1}{k} \sum_{\sigma \in S_k} \frac{(1 + \Lambda) \text{tr} [R_n(\lambda_{\sigma_1}) \cdots R_n(\lambda_{\sigma_k})]}{\prod_{i=1}^{k} (\lambda_{\sigma_i} - \lambda_{\sigma_{i+1}})} - \frac{2}{(\lambda - \mu)^2},
\]
where it is understood that \(\sigma_{k+1} = \sigma_1\).

5. Applications

5.1. Enumeration of ribbon graphs with even valencies. Enumeration of ribbon graphs is closely related to random matrix theory \(\Box\) \(\Box\) \(\Box\) \(\Box\) \(\Box\) \(\Box\) \(\Box\) \(\Box\) (e.g. to the Gaussian Unitary Ensembles (GUE) correlators); the partition function (with coupling constants) in a random matrix theory is often a tau-function of some integrable system.
Theorem\[2\] Define $F_n(s)$ and $Z_n(s)$ by
\[
F_n(s) := \frac{n^2}{2} \left( \log n - \frac{3}{2} \right) - \frac{1}{12} \log n + \zeta'(1 - 1) + \sum_{g \geq 2} \frac{B_{2g}}{4g(1 - 1)n^{2g - 2}}
+ \sum_{k \geq 0} \frac{1}{k!} \sum_{j_1, \ldots, j_k \geq 1} \langle \text{tr} M^{2j_1} \cdots \text{tr} M^{2j_k} \rangle_{e} s_{j_1} \cdots s_{j_k},
\]
\[
Z_n(s) := e^{\sum_{g \geq 2} \frac{B_{2g}}{4g(1 - 1)n^{2g - 2}}}.
\]

Then $Z_n(s)$ is a particular tau-function (of the discrete KdV hierarchy) reduced from the Toda lattice hierarchy [11]. The initial value of $w_n(s) := \frac{Z_{n+1}(s)Z_{n-1}(s)}{Z_n(s)}$ is given by $w_n(s = 0) = n$.

The theorem then follows from Lemma [14], Corollary [11] as well as the Theorem 1.1.1 of [11].

Define a formal series $Z(x, s; \epsilon)$ by
\[
\log Z(x, s; \epsilon) = \frac{x^2}{2\epsilon^2} \left( \log x - \frac{3}{2} \right) - \frac{1}{12} \log x + \zeta'(1 - 1) + \sum_{g \geq 2} \epsilon^{2g-2} \frac{B_{2g}}{4g(1 - 1)x^{2g-2}}
+ \sum_{g \geq 0} \epsilon^{2g-2} \sum_{k \geq 1} \sum_{j_1, \ldots, j_k \geq 1} a_g(2j_1, \ldots, 2j_k) s_{j_1} \cdots s_{j_k} x^{2-2g-k+|j|}.
\]

Clearly,
\[
Z(x, s; \epsilon = 1) = Z_n(s).
\]

Recall that in [11] we had proved that $\forall \epsilon$, $Z(x, s; \epsilon)$ is a tau-function reduced from the Toda lattice of the discrete KdV hierarchy under the identification $n = x/\epsilon$ as well as the flow rescalings $\partial_{s_j} \mapsto \epsilon \partial_{s_j}$. More precisely, define
\[
w(x, s; \epsilon) := \frac{Z(x + \epsilon, s; \epsilon)Z(x - \epsilon, s; \epsilon)}{Z(x, s; \epsilon)^2},
\]
then $w(x, s; \epsilon)$ is a particular solution to the discrete KdV hierarchy:
\[
\epsilon \frac{\partial L}{\partial s_j} = [A_{2j-1}, L]
\]
with $L := \Lambda^2 + w(x + \epsilon) + w(x) + w(x)w(x - \epsilon)\Lambda^{-2}$, $A_{2j-1} := L^j$, $\Lambda := e^{\epsilon \partial_x}$. Validity of these statements can be found in the Appendix of [11]. The initial data of this solution is given by
\[
w(x, 0; \epsilon) \equiv x = n \epsilon.
\]

Let $Z^{dKdV}(x, s; \epsilon)$ be the tau-function of the solution $w(x, s; \epsilon)$. The following corollary follows from the Main Theorem.

**Corollary 2.** There exist constants $\alpha, \beta_0, \beta_1, \beta_2, \cdots$ such that
\[
Z(x, s; \epsilon) = e^{\alpha x + \beta_0 + \sum_{k \geq 1} \beta_k s_j} Z^{dKdV}(x, s; \epsilon) Z^{dKdV}(x + \epsilon, s; \epsilon).
\]

Note that the constants $\alpha, \beta_0, \beta_1, \beta_2, \cdots$ right above now can depend on $\epsilon$. In what follows, we fix the ambiguities simply by requiring $Z^{dKdV}(x, s; \epsilon)$ to be the unique function satisfying
\[
Z(x, s; \epsilon) = Z^{dKdV}(x, s; \epsilon) Z^{dKdV}(x + \epsilon, s; \epsilon).
\]
**Remark.** The following formal series of $s$

$$Z^{dKdV}(x + \frac{\epsilon}{2}, s; \epsilon) =: \tilde{Z}(x, s; \epsilon)$$

was introduced in [16] by Si-Qi Liu, Youjin Zhang and the authors of the present paper, called the modified GUE partition function with even couplings, which plays an important role in a proof of the Hodge–GUE correspondence [16]. Moreover, Liu, Zhang and the authors derived the Dubrovin–Zhang loop equation for $\log \tilde{Z}$ from the corresponding Virasoro constraints, which also provides an algorithm for computing the modified GUE correlators of an arbitrary genus [16]. Very recently, Jian Zhou [36] derived the topological recursion of Chekhov–Eynard–Orantin type for the modified GUE correlators from the Virasoro constraints constructed in [16]; moreover, as a consequence of the topological recursion, an interesting formula between intersection numbers and $k$-point functions of modified GUE correlators was obtained by Zhou [36] (see the Theorem 3 in [36] for the details); it remains an open question to match the formula obtained by Zhou with another interesting formula obtained by Gaëtan Borot and Elba García-Failde [5] (see the Corollary 12.3 in [5]) as a consequence of the Hodge–GUE correspondence (or with some other consequence like (119) below), which may lead to a new proof of the Hodge–GUE correspondence. Last but not least, as an application of Theorem 2 let us give a third algorithm for computing the modified GUE correlators based on the following full genera formulæ:

\[
e^2 \sum_{i_1, \ldots, i_k \geq 0} \frac{\langle \phi_{i_1} \phi_{i_2} \rangle(x; \epsilon)}{\lambda_1^{i_1+1} \lambda_k^{i_k+1}} = \frac{\text{tr} \left( R_{(x+\frac{\epsilon}{2})/\epsilon}(\lambda_1) R_{(x+\frac{\epsilon}{2})/\epsilon}(\lambda_2) \right)}{(\lambda_1 - \lambda_2)^2} - \frac{1}{(\lambda_1 - \lambda_2)^2}, \tag{111}
\]

\[
e^k \sum_{i_1, \ldots, i_k \geq 0} \frac{\langle \phi_{i_1} \cdots \phi_{i_k} \rangle(x; \epsilon)}{\lambda_1^{i_1+1} \cdots \lambda_k^{i_k+1}} = -\frac{1}{k} \sum_{\sigma \in S_k} \frac{\text{tr} \left( R_{(x+\frac{\epsilon}{2})/\epsilon}(\lambda_{\sigma_1}) \cdots R_{(x+\frac{\epsilon}{2})/\epsilon}(\lambda_{\sigma_k}) \right)}{\prod_{\ell=1}^k (\lambda_{\sigma_\ell} - \lambda_{\sigma_{\ell+1}})} \quad (k \geq 3), \tag{112}
\]

where $\langle \phi_{i_1} \cdots \phi_{i_k} \rangle(x; \epsilon)$ denote the modified GUE correlators with even couplings, defined by

$$\langle \phi_{i_1} \cdots \phi_{i_k} \rangle(x; \epsilon) := \frac{\partial^k \log \tilde{Z}}{\partial s_{i_1} \cdots \partial s_{i_k}}(x, s = 0; \epsilon),$$

and $R_n(\lambda)$ is defined in Definition 4. We notice that the reason that one can talk about “genus” for $\log \tilde{Z}$ is because $\log \tilde{Z}$ is even in $\epsilon$ and so are $\langle \phi_{i_1} \cdots \phi_{i_k} \rangle(x; \epsilon)$. A concrete algorithm using the formulæ of the form (111)–(112) for calculating the corresponding correlators including certain large genus asymptotics is given in [13].

Using the definitions of $Z(x, s; \epsilon)$, $Z^{dKdV}(x, s; \epsilon)$ and the following elementary identity

$$\frac{1}{1 + \epsilon^2} = \sum_{\ell \geq 0} \frac{B_{\ell+1}}{(\ell + 1)!} (1 - 2^{\ell+1}) z^\ell,$$

where $B_m$ denotes the $m$th Bernoulli number, we obtain the following proposition.
Proposition 4. The logarithm of $Z^{dKdV}(x; s; \epsilon)$ has the following expression

$$
\log Z^{dKdV}(x; s; \epsilon) = \left(\frac{1}{4} \log x - \frac{3}{8} x^2\right) \frac{x^2}{\epsilon^2} + \frac{1}{4} \left(1 - \log x\right) \frac{x}{\epsilon} + \frac{1}{24} (1 - 12 \zeta'(-1) - \log x) + \sum_{p \geq 1} x^p \frac{\epsilon^p}{p} (-1)^p (p - 1)! \sum_{0 \leq g \leq (p+2)/2} (2g - 1) \left(1 - 2^{p+3-2g}\right) \frac{B_{p+3-2g} B_{2g}}{(p + 3 - 2g)! (2g)!} + \sum_{g, h \geq 0} \sum_{k \geq 1} \sum_{\ell = 0}^h \frac{\epsilon^{2g-2+\ell} B_{\ell+1}}{\ell + 1} \left(\frac{h}{\ell}\right) (1 - 2^{\ell+1}) x^{h-\ell} \sum_{j_1, \ldots, j_k \geq 1} a_g(2j_1, \ldots, 2j_k) s_{j_1} \ldots s_{j_k}. \tag{113}
$$

5.2. Identities for cubic Hodge integrals. The so-called special cubic Hodge free energy is defined as the following generating series of Hodge integrals

$$
\mathcal{H}(t; \epsilon) = \sum_{g \geq 0} \sum_{k \geq 0} \sum_{i_1, \ldots, i_k \geq 0} t_{i_1} \cdots t_{i_k} \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1) \Lambda_g(-1) \Lambda_g\left(\frac{1}{2}\right) \psi_{i_1}^j \cdots \psi_{i_k}^j.
$$

Here, $t = (t_0, t_1, \ldots)$. (Warning: Avoid from confusing with the flow variables $t_\ell, \ell \geq 0$ of the Toda lattice hierarchy used in Section 4.)

Theorem (Hodge–GUE correspondence [16, 12]). We have

$$
\log Z(x; s; \epsilon) + \epsilon^{-2} \left(-\frac{1}{2} \sum_{j_1, j_2 \geq 1} \frac{j_1 j_2}{j_1 + j_2} \tilde{s}_{j_1} \tilde{s}_{j_2} + \sum_{j \geq 1} \frac{j}{1 + j} \tilde{s}_j - x \sum_{j \geq 1} \tilde{s}_j - \frac{x}{4} + x\right) - \zeta'(-1) = \mathcal{H}\left(t(x - \frac{\epsilon}{2}, s); \sqrt{2} \epsilon\right) + \mathcal{H}\left(t(x + \frac{\epsilon}{2}, s); \sqrt{2} \epsilon\right), \tag{114}
$$

where $\tilde{s}_j := \left(\frac{3}{2}j\right)s_j$ and $t_i(x, s) := \sum_{j \geq 1} j^{i+1} \tilde{s}_j - 1 + \delta_{i,1} + x \cdot \delta_{i,0}, i \geq 0$.

Comparing this theorem with Corollary 2 (also cf. [110]) we obtain

Proposition 5. The following formula holds true:

$$
\log Z^{dKdV}(x; s; \epsilon) = \mathcal{H}\left(t(x - \frac{\epsilon}{2}, s); \epsilon\right) + \frac{1}{4} \epsilon^2 \sum_{j_1, j_2 \geq 1} \frac{j_1 j_2}{j_1 + j_2} \tilde{s}_{j_1} \tilde{s}_{j_2} + \frac{x - \epsilon}{2 \epsilon^2} \left(\sum_{j \geq 1} \tilde{s}_j - 1\right) - \frac{1}{2} \epsilon^2 \sum_{j \geq 1} \frac{j}{1 + j} \tilde{s}_j + \frac{1}{2} \zeta'(-1) + \frac{1}{8 \epsilon^2}. \tag{115}
$$

Denote $\Omega_g := \Lambda_g(-1) \Lambda_g(-1) \Lambda_g\left(\frac{1}{2}\right)$ as before, and write

$$
\Omega_g =: \sum_{d \geq 0} \Omega_g^{[d]}, \quad \Omega_g^{[d]} \in H^{2d}(\overline{\mathcal{M}}_{g,k}).
$$

It might be helpful to notice that for $g = 1$, $\deg \Omega_1 \leq 1$; for $g \geq 2$, $\deg \Omega_g \leq 3g - 3$.

Definition 12. For any given $k \geq 0, i_1, \ldots, i_k \geq 0$, define a formal series $H_{i_1, \ldots, i_k}(x; \epsilon)$ by

$$
H_{i_1, \ldots, i_k}(x; \epsilon) := 2^{g-1} \sum_{g = 0}^{\infty} \epsilon^{2g-2} \sum_{d = 0}^{3g} \sum_{\lambda \in \mathbb{Z}} \frac{(-1)^{\ell(\lambda)}}{m(\lambda)!} \left(\Omega_g^{[d]} e^{(x-1)\tau_0} \tau_{\lambda+1} \tau_{i_1} \cdots \tau_{i_k}\right)_g. \tag{116}
$$
Similarly, for \( k \geq 0, j_1, \ldots, j_k \geq 1 \), define

\[
e_{j_1, \ldots, j_k}(x; \epsilon) := \frac{\partial^k \log Z(x, s; \epsilon)}{\partial s_{j_1} \ldots \partial s_{j_k}} \bigg|_{s=0},
\]

\[
b_{j_1, \ldots, j_k}(x; \epsilon) := \frac{\partial^k \log Z^{dKdV}(x, s; \epsilon)}{\partial s_{j_1} \ldots \partial s_{j_k}} \bigg|_{s=0}.
\]

Note that in the notation \( \langle \ldots \rangle_{g,m} \), we omit the index \( m \) from \( \langle \ldots \rangle_{g,m} \). For such an abbreviation, \( m \) should be recovered from counting the number of \( \tau \)'s in “...”. Therefore, for each fixed \( g, d \) and for each monomial in the Taylor expansion \( e^{(x-1)m} = \sum_{r=0}^{\infty} \frac{1}{r!} (x-1)^r \), the above summation over partitions \( \sum_{\lambda \in \mathcal{P}} \) is a finite sum, i.e. the degree-dimension matching \( |\lambda| = 3g - 3 + k + r - d - |\lambda| \) has to be hold. Lemma \( \ref{lemma:1} \) also easily follows from this constrain with \( r = 0 \) taken. The numbers \( H_{g,i_1,\ldots,i_k} \) defined by (35) and the formal series \( H_{i_1,\ldots,i_k}(x; \epsilon) \) are clearly related by

\[
H_{i_1,\ldots,i_k}(x = 1; \epsilon) = \sum_{g=0}^{\infty} \epsilon^{2g-2} H_{g,i_1,\ldots,i_k}.
\]

It is also clear that

\[
e_{j_1,\ldots,j_k}(n; \epsilon = 1) = \langle \text{tr } M^{2j_1} \ldots \text{tr } M^{2j_k} \rangle, \quad k \geq 1,
\]

\[
e_{j_1,\ldots,j_k}(x; \epsilon) = b_{j_1,\ldots,j_k}(x; \epsilon) + b_{j_1,\ldots,j_k}(x + \epsilon; \epsilon), \quad k \geq 0.
\]

For a given non-negative integer \( k \) and a give set of positive integers \( j_1, \ldots, j_k \), the formal series \( e_{j_1,\ldots,j_k}(x; \epsilon), b_{j_1,\ldots,j_k}(x; \epsilon) \) have the following expressions

\[
e_0(x; \epsilon) = \frac{x^2}{2\epsilon^2} \left( \log x - \frac{3}{2} \right) - \frac{1}{12} \log x + \zeta'(-1) + \sum_{g \geq 2} \epsilon^{2g-2} \frac{B_{2g}}{4g(g-1)x^{2g-2}},
\]

\[
e_{j_1,\ldots,j_k}(x; \epsilon) = k! \sum_{0 \leq g \leq 1 + \frac{j_1}{2} + \frac{j_2}{2}} \epsilon^{2g-2} x^{2-2g-k+|j|} a_g(2j_1, \ldots, 2j_k), \quad k \geq 1,
\]

\[
b_0(x; \epsilon) = \left( \frac{1}{4} \log x - \frac{3}{8} \right) \frac{x^2}{\epsilon^2} + \frac{1}{4} \left( 1 - \log x \right) \frac{x}{\epsilon} + \frac{1}{24} \left( 12 \zeta'(-1) - \log x \right) + \sum_{p \geq 1} e^p x^p (-1)^p (p - 1)! \sum_{0 \leq g \leq (p+2)/2} (2g - 1) \left( 1 - 2^{p+3-2g} \right) \frac{B_{p+3-2g} B_{2g}}{(p + 3 - 2g)! (2g)!},
\]

\[
b_{j_1,\ldots,j_k}(x; \epsilon) = k! \sum_{g,h,k \geq 0} \sum_{|j| = h} \epsilon^{2g-2+\ell} \frac{B_{\ell+1}}{\ell+1} \left( \frac{h}{\ell} \right) \left( 1 - 2^{\ell+1} \right) x^{h-\ell} a_g(2j_1, \ldots, 2j_k), \quad k \geq 1.
\]

Proposition 6. The following identity holds true:

\[
e_{j_1,\ldots,j_k}(x; \epsilon) = \prod_{\ell=1}^{k} \left( 2j_{\ell} \right) \sum_{i_1,\ldots,i_k \geq 0} \sum_{i_{j_{\ell}}+1} \left( H_{i_1,\ldots,i_k}(x - \frac{\epsilon}{2}; \epsilon) + H_{i_1,\ldots,i_k}(x + \frac{\epsilon}{2}; \epsilon) \right) + \delta_{k,2} \frac{j_1 j_2}{j_1 + j_2} \left( \frac{2j_1}{j_1} \right) \left( \frac{2j_2}{j_2} \right) - \delta_{k,1} \left( \frac{j_1}{1 + j_1} \left( \frac{2j_1}{j_1} \right) - x \left( \frac{2j_1}{j_1} \right) \right) + \delta_{k,0} \zeta'(-1).
\]
Proposition 7. The following identity holds true:

\[
b_{j_1,\ldots,j_k}(x;\epsilon) = \prod_{\ell=1}^{k} \left(\begin{array}{c} 2j_{\ell} \\ j_{\ell} \end{array}\right) \sum_{i_1,\ldots,i_k \geq 0} \prod_{\ell=1}^{k} j_{\ell}^{i_{\ell}+1} H_{i_1,\ldots,i_k}(x - \frac{\epsilon}{2};\epsilon) + \delta_{k,0} \zeta(-1) \frac{1}{2}
\]

\[
+ \delta_{k,2} \frac{j_1 j_2}{j_1 + j_2} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) \left(\begin{array}{c} 2j_2 \\ j_2 \end{array}\right) - \delta_{k,1} \frac{1}{2} \frac{1}{1 + j_1} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) - (x - \epsilon) \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right). \tag{120}
\]

Corollary 3. For any \(k \geq 0\), \(j_1,\ldots,j_k \geq 1\), the following formulae hold true:

\[
e_{j_1,\ldots,j_k}(1;\epsilon) = \prod_{\ell=1}^{k} \left(\begin{array}{c} 2j_{\ell} \\ j_{\ell} \end{array}\right) \sum_{i_1,\ldots,i_k \geq 0} \prod_{\ell=1}^{k} j_{\ell}^{i_{\ell}+1} \left(\begin{array}{c} H_{i_1,\ldots,i_k}(1 - \frac{\epsilon}{2};\epsilon) + H_{i_1,\ldots,i_k}(1 + \frac{\epsilon}{2};\epsilon) \end{array}\right)
\]

\[
+ \delta_{k,2} \frac{j_1 j_2}{j_1 + j_2} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) \left(\begin{array}{c} 2j_2 \\ j_2 \end{array}\right) - \delta_{k,1} \frac{1}{2} \frac{1}{1 + j_1} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) - (\frac{\epsilon}{2} + 1) \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) + \delta_{k,0} \zeta(-1), \tag{121}
\]

\[
e_{j_1,\ldots,j_k}(\frac{\epsilon}{2} + 1;\epsilon) = \prod_{\ell=1}^{k} \left(\begin{array}{c} 2j_{\ell} \\ j_{\ell} \end{array}\right) \sum_{i_1,\ldots,i_k \geq 0} \prod_{\ell=1}^{k} j_{\ell}^{i_{\ell}+1} \left(\begin{array}{c} H_{i_1,\ldots,i_k}(1;\epsilon) + H_{i_1,\ldots,i_k}(\epsilon;\epsilon) \end{array}\right)
\]

\[
+ \delta_{k,2} \frac{j_1 j_2}{j_1 + j_2} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) \left(\begin{array}{c} 2j_2 \\ j_2 \end{array}\right) - \delta_{k,1} \frac{1}{2} \frac{1}{1 + j_1} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) - (\frac{\epsilon}{2} + 1) \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) + \delta_{k,0} \zeta(-1), \tag{122}
\]

\[
b_{j_1,\ldots,j_k}(\frac{\epsilon}{2} + 1;\epsilon) = \prod_{\ell=1}^{k} \left(\begin{array}{c} 2j_{\ell} \\ j_{\ell} \end{array}\right) \sum_{g \geq 0} \sum_{i_1,\ldots,i_k \geq 0} H_{g,i_1,\ldots,i_k} \prod_{\ell=1}^{k} j_{\ell}^{i_{\ell}+1} + \delta_{k,0} \zeta(-1) \frac{1}{2}
\]

\[
+ \delta_{k,2} \frac{j_1 j_2}{2} \frac{1}{j_1 + j_2} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) \left(\begin{array}{c} 2j_2 \\ j_2 \end{array}\right) - \delta_{k,1} \frac{1}{2} \frac{1}{1 + j_1} \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right) - (1 - \frac{\epsilon}{2}) \left(\begin{array}{c} 2j_1 \\ j_1 \end{array}\right). \tag{123}
\]

Combining (123) of the above corollary and Theorem 4 proves Theorem 3.

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