Staggered Chiral Perturbation Theory

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We discuss how to formulate a staggered chiral perturbation theory (SXPT). This amounts to a generalization of the Lee-Sharpe Lagrangian to include more than one flavor (i.e., multiple staggered fields), which turns out to be nontrivial. One loop corrections to pion and kaon masses and decay constants are computed as examples in three cases: the quenched, partially quenched, and full (unquenched) case. The results for one loop mass and decay constant corrections have already been presented in Ref. \textsuperscript{[1]}

In order to reproduce the chiral behavior of staggered fermion simulation data as \(a^2 \to 0\), one must account for the systematic effects arising from \(O(a^2)\) taste violations. (Taste refers to the staggered degrees of freedom resulting from doubling, while flavor refers to true quark flavor.) These taste violations are not negligible at current lattice spacings (\(a \approx 0.9 - 0.12\) fm \textsuperscript{[2]}). Lee and Sharpe have formulated \textsuperscript{[3]} such a staggered chiral perturbation theory (SXPT) for one staggered field. Here we describe the generalization to the case of \(n\) flavors.

We follow a three-step procedure:

\begin{itemize}
  \item First, we generalize the Lee-Sharpe Lagrangian to incorporate multiple flavors. This is the “4+4+...” theory—\(n\) flavors with 4 tastes per flavor.
  \item Next we calculate various meson properties; specifically we will calculate the Goldstone pion mass and decay constant for \(n = 3\).
  \item Finally, we adjust this result by hand to keep only one taste per flavor, thus accounting for the \(\sqrt{\text{Det}}\) in simulations \textsuperscript{[2]}.
\end{itemize}

This can be done for any of the following cases: partially quenched (\(m_{\text{valence}} \neq m_{\text{sea}}\)), “full QCD” (\(m_{\text{valence}} = m_{\text{sea}}\)) and quenched (\(m_{\text{valence}} \neq m_{\text{sea}}\), \(m_{\text{sea}} \to \infty\)); here we will show results for the partially quenched case. Complete results have been reported in Ref. \textsuperscript{[1]} and preliminary fits to simulation data are shown in Ref. \textsuperscript{[4]}.

We begin here by formulating SXPT for \(n\) flavors, stating only the differences between the \(n = 1\) \textsuperscript{[3]} and \(n > 1\) cases. We collect the Goldstone bosons arising from the spontaneous break-
Because of this Fierz transformation, generalizing to multiple flavors is tricky. Returning to the quark level, we recall that these operators come from four-quark operators with a net momentum change of $O(\pi)$, which changes quark taste. This gluon exchange can also change color, but not flavor, so all four-quark operators must be of the flavor-unmixed form: $\bar{q}_i (\gamma_5 \otimes \xi_i) q_j (\gamma_5 \otimes \xi_j') q_j$, where $(\gamma_5 \otimes \xi_i)$ is the (spin$\otimes$flavor) matrix. In the naive theory, each bilinear is separately chirally invariant—these are the “odd” bilinears in the staggered theory. Thus, only the odd-odd four-quark operators are relevant here.

Keeping all of the four-quark operators in the flavor-unmixed form, we then see that, using the standard spurion analysis, the taste matrices $\xi_i$ are singlets under the flavor $SU(n_f)$ symmetry [7]. This means we can make the replacement $\xi_{ij} \rightarrow (\xi_{ij})^n$, where $i$ and $j$ are flavor indices. This must be done before the Fierz transformation performed by Lee and Sharpe to put the operators into single-trace form. Only then do the chiral operators follow from the flavor-unmixed four-quark operators.

For the operators $O_1, O_3, O_4$ and $O_6$ (which we combine into $U'$), we can just replace $\xi_{ij} \rightarrow (\xi_{ij})^n$. Instead of the operators $O_2$ and $O_5$ we have four operators, $O_{2A}, O_{2B}, O_{5A}$ and $O_{5B}$ ("$U''$"), which are not in single-trace form. For example: $O_{2A} = \frac{1}{2} [\text{Tr}(\xi_{ij}^n \Sigma) \text{Tr}(\xi_{ij}^n \Sigma) + h.c.]$ and $O_{2B} = \frac{1}{2}[\text{Tr}(\xi_{ij}^n \Sigma) \text{Tr}(\xi_{ij}^n \Sigma) + h.c.]$. The full potential is then $\mathcal{V} = U + U'$.

One of the consequences of the two-trace form of the terms in $U'$ is the appearance of quark-level hairpin terms similar to $\mathcal{L}_{\text{singlet}}$. These terms are of the form $e^{\lambda'_{ij}} (U_t + D_t + S_t + \cdots)^2$, where $\lambda'_{ij}$ depends on the taste channel we’re discussing ($t = V, A, I$): $\lambda'_{ij} = 4m_0^2 / 3$ for the singlet case, and for the vector and axial-vector cases $\lambda'_{V(A)} = a^2 \delta_{V(A)}$, with $\delta_{V(A)}$ a linear combination of the coefficients in $U'$. These hairpins allow mixing among the flavor-neutral mesons (shown in Fig. 1), which we can resum to give a non-diagonal propagator:

$$G_{MN} = \frac{\delta_{MN}}{q^2 + m_M^2} - \frac{\lambda'}{(q^2 + m_M^2)(q^2 + m_N^2)}$$

Figure 1. The $U - U$ propagator at the chiral (a) and quark (b) level. Each $\times$ corresponds to an insertion of $\lambda'$, and the intermediate meson could be $U, D$ or $S$ (for $n = 3$).

$$\times \left( \frac{(q^2 + m_0^2)(q^2 + m_D^2) \cdots}{(q^2 + m_0^2)(q^2 + m_D^2) \cdots} \right),$$

where the second term we denote by $\mathcal{D}_{MN}$.

It is interesting to note that the symmetries of the multiple-flavor theory are only slightly modified from those of the single-flavor theory. For example:

- The spontaneously broken symmetry ($m = 0$ and $a = 0$): $SU(4n)_L \times SU(4n)_R \rightarrow SU(4n)_\text{vec}$, for any $n \geq 1$.
- The residual chiral symmetry ($m = 0$ and $a \neq 0$): $U(n)_c \times U(n)_c$. We use $t$ and $r$ (not $L$ and $R$) to denote the left and right symmetries to remind us that they are mixtures of chiral spin and taste.
- Fermion number ($m \neq 0$ for $n > 1$, we assume all $n$ masses are nondegenerate) and $a \neq 0$: $U(1)_{\text{VEC}}$ for one flavor, and this becomes $(U(1)_{\text{vec}})^n$ for more than one flavor.

We show here the $n = 3$ partially quenched result for the pion mass and decay constant. Here a “pion” is any flavor-charged meson, and $n$ refers to the number of sea quarks. We add two quenched valence quarks, $x$ and $y$, and calculate the properties of the Goldstone “pion” $P^+_5 = x\bar{y}$. $X$ and $Y$ will refer to the flavor-neutral mesons composed of these quarks.

For the pion mass, we calculate the pion self energy evaluated at $p^2 = -m_{P^+}^2$. All connected (at the quark level) terms cancel and we are left only with terms involving the disconnected terms. To one loop, we have:

$$\frac{m_{P^+}^2}{(m_x + m_y)} = \mu \left\{ 1 + \frac{1}{16\pi^2 f^2} \int \frac{d^4q}{q^2} \left( 2D^0 \right) + \mathcal{D} \right\} + [\text{a.t.}],$$

In this expression and below we leave off the analytic terms (denoted by [a.t.]) for simplicity. Note
also that the pion mass vanishes in the chiral limit \((m_x, m_y \to 0)\) as it must, since this is the true Goldstone boson.

For the pion decay constant, we calculate the matrix element \(\langle 0 | j_{a5} | P_5^+ (p) \rangle = -i f_{P_5^+} p_{\mu} \), where \(j_{a5}\) is the axial current for \(P_5^+\). To one loop we find

\[
f_{P_5^+} = f \left( 1 + \frac{1}{16\pi^2} \delta f_{P_5^+} + \text{[a.t.]} \right), \quad (6)
\]

\[
\delta f_{P_5^+} = -\frac{1}{8} \int \frac{d^4 q}{\pi^2} \left[ \sum_{q,i} \left( \frac{1}{q^2 + m_i^2} \right) - 2D^I_{XY} + 4D^V_{XY} + (4D^V_{XX} + 8D^V_{XY} + 4D^V_{YY}) + (V \to A) \right]. \quad (7)
\]

The sum over \(Q\) is over mesons with one sea and one valence quark, and \(t\) runs over the 16 tastes.

These expressions are quite general, and any relevant result can be found by taking limits before performing the momentum integrals. For the “full QCD” case, set \(m_x = m_u\) and \(m_y = m_d\) \((m_s)\) for the true pion (kaon); for the quenched case, take \(m_u, d, s \to \infty\).

Once one takes the desired limits, one must perform the integrals in Eqs. 6 and 7. The integrands are ratios of products of terms of the form \((q^2 + m^2)^{\alpha}\). These can be expanded as sums of poles times their residues. Performing the integrals, we keep only the chiral logarithms (the analytic pieces are absorbed into “[a.t.]”, and we leave off finite volume effects here). For a single pole, we get a term \(\ell (m^2) \equiv m^2 \ln (m^2 / \Lambda^2)\), while for a double pole, we get \(\ell (m^2) \equiv - (\ln (m^2 / \Lambda^2) + 1)\) (with \(\Lambda\) the chiral scale). The residues multiplying these logarithms are complicated, and are given in full detail along with the analytic terms in Ref. [1]. Also, see Ref. [4] for explicit expressions for \(m_{\pi}^2\) and \(f_{\pi}\).

The last step before having the final result for the mass and decay constant is to adjust from four to one tastes per flavor. This is done using the “quark flow technique,” in Ref. [1], where we determine where quark loops arise, and multiply each corresponding loop by 1/4. However, from that approach it is not clear that this works at all orders in perturbation theory. Further, even if it does work, the application of the quark flow technique could be quite complex.

One can use the replica method to automate this: Take \(n_4\) the quarks of each flavor \(i\). Then calculate quantities of interest to a given number of loops as analytic functions of the \(n_i\); in the end, set each \(n_i = 1/4\). This should take into account the transition from 4 \(\to 1\) tastes per flavor at all orders automatically.

Another interesting possibility that may arise in SxPT is that of an unusual phase. If \(a^2 \Delta_A = m_{\pi}^2 - m_{\pi_{\text{true}}}^2\)

\[
\delta'_{A, \text{crit}} = -4 \Delta_A \frac{1 + a^2 \Delta_A / m_{\pi_{\text{true}}}^2}{2 + 3a^2 \Delta_A / m_{\pi_{\text{true}}}^2}.
\]

\(m_{\pi_{\text{true}}}^2\) could become negative for small, but non-zero \(m_u = m_d\). From fits, this does not appear likely for the physical case of QCD. A corresponding condition for \(m_u = m_d = m_s\) may be satisfied, although such a phase would disappear in the continuum limit. The possibility of an unusual phase requires further study [8].

Calculations for \(m_{P_5^+}^2\) and \(f_{P_5^+}\) are complete for the partially quenched, full and quenched cases. Preliminary fits are shown in Ref. [4], and appear promising. The next goal is to include heavy quarks so as to calculate heavy-light decay constants, and an extension to baryons [8].

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