BRIESKORN SUBMANIFOLDS, LOCAL MOVES ON KNOTS, AND KNOT PRODUCTS

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Abstract. Let $2p + 1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ be a closed oriented $(2p + 1)$-dimensional submanifold of $S^{2p+3}$. Then $K$ is a Brieskorn submanifold if and only if $K$ is connected, $(p-1)$-connected, simple and has a $(p+1)$-Seifert matrix associated with a simple Seifert hypersurface that is $(-1)^p$-S-equivalent to a Kauffman-Neumann-type, or a $KN$-type (See the body of the paper for a definition.) This result follows from a stronger one: Let $2p + 1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ and $J$ be closed, oriented, $(2p + 1)$-dimensional connected, $(p-1)$-connected, simple submanifolds of $S^{2p+3}$. Then $K$ is equivalent to $J$ if and only if a Seifert matrix associated with a simple Seifert hypersurface for $K$ is $(-1)^p$-S-equivalent to that for $J$. (We also discuss the $2p + 1 = 3$ case in both cases.)

This result implies the following: Let $\mu \in \mathbb{N}$. A 1-link $A$ is pass-move equivalent to a 1-link $B$ if and only if $A \otimes \mu$ Hopf is $(2\mu + 1, 2\mu + 1)$-pass-move equivalent to $B \otimes \mu$ Hopf.

1. Introduction

Our main results are Theorems 3.3 and 3.5 in §3 and Theorem 6.3 in §6. We state them after some preliminaries.
2. Review of simple submanifolds and Brieskorn submanifolds

We work in the smooth category.

Let $J$ (resp. $K$) be a (not necessarily spherical) closed oriented $n$-dimensional sub-
manifold of $S^{n+2}$ ($n \in \mathbb{N}$). (If $J$ is homeomorphic to the standard sphere, $J$ is said to
be spherical.) If there is an orientation preserving diffeomorphism map $f : S^{n+2} \to S^{n+2}$
such that $f(J) = K$ and such that $f|_J$ is regarded an orientation preserving diffeomor-
phism map $J \to K$, we say that $J$ is equivalent to $K$. By obstruction theory there
is a connected compact oriented $(n+1)$-dimensional submanifold $W$ of $S^{n+2}$ such that
$\partial W = J$ (See e.g. Theorem 3 in P.50 of [11]). We call $W$ a Seifert hypersurface
for $J$.

Suppose that a topological space $A$ is a sub-topological space of a topological space
$B$. Let $\overline{A}$ denote the closure of $A$ in $B$. We do not say what $B$ is if there is no danger of
confusion.

Let $M$ be a manifold-with-boundary. Let $\text{Int}M$ denote $M - \partial M$.

Let $P$ be a (not necessarily closed) $n$-dimensional oriented submanifold of a (not neces-
sarily closed) $m$-dimensional oriented manifold-with-boundary $Q$. Let $N(P)$ denote the
tubular neighborhood of $P$ in $Q$.

Let $K$ be a (not necessarily spherical) connected, closed, oriented, $(2p+1)$-dimensional
submanifold of $S^{2p+3}$ ($p \in \mathbb{N}$). $K$ is said to be simple if $K$ satisfies that
$\pi_1(S^{2p+3} - N(K)) \cong \mathbb{Z}$ and $\pi_i(S^{2p+3} - N(K)) \cong 0$ ($2 \leq i \leq p$). Let $V$ be a Seifert
hypersurface for $K$. $V$ is said to be simple if $\pi_i(V)$ is trivial ($1 \leq i \leq p$). See Theorem
7.1.

Let $V$ be a Seifert hypersurface for a closed oriented $(2p+1)$-dimensional submanifold $K \subset S^{n+2}$. Let $x_1, \ldots, x_\mu$ be $(p+1)$-cycles in $V$ which compose an ordered basis of
$H_{p+1}(V; \mathbb{Z})/\text{Tor}$. Recall that the orientation of $V$ is compatible with that of $K$. Push $x_i$
in the positive direction of the normal bundle of $V$. Call it $x_i^+$. Push $x_i$ in the negative
direction of the normal bundle of $V$. Call it $x_i^-$. A $(p+1)$- (positive) Seifert matrix for
$K$ associated with $V$ represented by the ordered basis, $\{x_1, \ldots, x_\mu\}$, is a $(\mu \times \mu)$-matrix
$S = (s_{ij}) = (\text{lk}(x_i, y_j^+))$.

A $(p+1)$- (negative) Seifert matrix for $K$ associated with $V$ represented by the ordered
basis, $\{x_1, \ldots, x_\mu\}$, is a $(\mu \times \mu)$-matrix
$N = (n_{ij}) = (\text{lk}(x_i, y_j^-))$. 

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We sometimes omit to write \( K, V, \) and \( \{x_i\} \) when they are clear from the context. See e.g. [9] for \((p, n + 1 - p)\)-Seifert matrices for \( n \)-knots \((p, n \in \mathbb{N})\).

Let \( A \) be an \( r \times r \)-matrix. Let \( P \) be a unimodular \( r \times r \)-matrix. We say that \( A \) is equivalent to \( A' \) if \( A' = P A P \), where \( t \) denotes an operation of making a transposed matrix.

**Proposition 2.1.** (Well-known.) Let \( S \) (resp. \( S' \)) be a \((p + 1)\)-positive Seifert matrix for the above \( K \) associated with the above \( V \) represented by an ordered basis \( \{x_i\} \) (resp. \( \{x'_i\} \)) of \((p + 1)\)-cycles. Then \( S \) is equivalent to \( S' \).

If a \((p + 1)\)-positive Seifert matrix \( P \) for \( K \) and a \((p + 1)\)-negative Seifert matrix \( N \) for \( K \) are defined by the same ordered basis \( \{x_i\} \), we say that \( P \) and \( N \) are related and that the pair \((P, N)\) is a pair of \((p + 1)\)-related Seifert matrices for \( K \).

**Proposition 2.2.** (Well-known.) Let \( X \) be a \((p + 1)\)-positive Seifert matrix for a \((2p + 1)\)-dimensional closed oriented submanifold \( K \subset S^{2p+3} \) associated with a Seifert hypersurface \( V \). Then \((-1)^p X \) represents the \((p + 1)\)-negative Seifert matrix related to \( X \) for \( K \). Furthermore \( X - (-1)^p X \) represents the intersection product on \( H_{p+1}(V; \mathbb{Z})/\text{Tor} \).

Let \( A \) be an \( r \times r \)-matrix \((r \in \mathbb{N} \cup \{0\})\). We say that \( A \) is \((-1)^p S\)-equivalent to

\[
\begin{pmatrix}
A & \alpha & 0 \\
(-1)^p \alpha & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

(resp. \[
\begin{pmatrix}
A & \beta & 0 \\
(-1)^p \beta & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\])

where \( \alpha \) and \( \beta \) are column vectors. We say that \( A \) is \((-1)^p S\)-equivalent to \( A' \) if \( A \) is equivalent to \( A' \). We say that \( A \) is \((-1)^p S\)-equivalent to \( C \) if \( A \) is \((-1)^p S\)-equivalent to \( B \) and if \( B \) is \((-1)^p S\)-equivalent to \( C \).

Let \( K \) be a closed oriented \((2p + 1)\)-dimensional submanifold of \( S^{2p+3} \) \((p \in \mathbb{N})\). If \( A \) is a \((p + 1)\)-Seifert matrix (resp. negative \((p + 1)\)-Seifert matrix) associated with a simple Seifert hypersurface \( V \) for \( K \), we call \( A \) a simple Seifert matrix (resp. negative simple Seifert matrix) for \( K \). We can define a pair of related simple Seifert matrices for \( K \).

**Note.** If we say that \( K \) has a simple Seifert matrix \( A \), then it means that \( K \) has a simple Seifert hypersurface.

Let \( a_*, \) be an integer \( \geq 2(\ast = 1, ..., q) \), where \( q \in \mathbb{N} \). The submanifold

\[
\{(z_1, ..., z_q) \mid |z_1|^2 + ... + |z_q|^2 = 1, z_\ast \in \mathbb{C} \} \cap \{(z_1, ..., z_q) \mid z_1^{a_1} + ... + z_q^{a_q} = 0, z_\ast \in \mathbb{C} \}
\]

is called the Brieskorn submanifold \( \Sigma(a_1, ..., a_q) \). The oriented diffeomorphism type of the Brieskorn submanifold \( \Sigma(a_1, ..., a_q) \) is called the Brieskorn manifold \( \Sigma(a_1, ..., a_q) \).

Let \( a_\ast \) be integers \( \geq 2 \). Let \( A_a = (\zeta_{i,j}) \) be an \((a - 1) \times (a - 1)\) matrix such that
Let $\Lambda_{a_1,\ldots,a_q} = (-1)^{(q-1)/2} \Lambda_{a_1} \otimes \cdots \otimes \Lambda_{a_q}$. It is called a Kauffman-Neumann-type, or a KN-type. See the last few paragraphs of §6 in [8] for KN-types.

The Brieskorn submanifold $\Sigma(a,b) \subset S^3$ is the torus $(a,b)$ knot (See [17]). It is well-known that they are classified. We say that the Brieskorn submanifold $\Sigma(a) \subset S^1$ is the empty knot of degree $a$ (See [8]).

**Theorem 2.3.** ([6, 8].) Let $a_*$ be integers $\geq 2$. Let $q \geq 3$. Any Brieskorn submanifold $\Sigma(a_1,\ldots,a_q)$ is a (not necessarily spherical) $(2q-3)$-dimensional connected, $(q-3)$-connected, simple submanifold of $S^{2q-1}$ such that $\Lambda_{a_1,\ldots,a_q}$ is a simple Seifert matrix.

We prove that the converse of the $q \geq 4$ case of Theorem 2.3 holds and that that of the $q = 3$ case does not hold in general. See Theorem 3.3, Note 3.4, and Theorem 3.5.

3. Theorems on simple submanifolds and Brieskorn submanifolds

**Theorem 3.1.** Let $p \in \mathbb{N}$. Let $K$ be a closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold of $S^{2p+3}$. Let $P$ be a simple Seifert matrix for $K$. Then the following two conditions are equivalent.

(i) $P'$ is a simple Seifert matrix for $K$.

(ii) $P'$ is $(-1)^p$-S-equivalent to $P$.

Note that we have the following proposition.

**Proposition 3.2.** Let $p \in \mathbb{N}$. There is a closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold $K$ of $S^{2p+3}$ with the following property: There is a simple Seifert matrix $P$ and a Seifert matrix $R$ for $K$ such that $P$ is not $(-1)^p$-S-equivalent to $R$.

Note. In Proposition 3.2 $R$ is not a simple Seifert matrix. If $R$ is associated with a Seifert hypersurface $W$ for $K$, then $W$ is not a simple Seifert hypersurface.

In [15] it was proved that if $K$ and $J$ are spherical, the following Theorem 3.3 is true. We generalize his result and prove a stronger theorem, which is Theorem 3.3.

**Theorem 3.3.** (1) Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ and $J$ be closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifolds of $S^{2p+3}$. Then (i) is equivalent to (ii).

(i) $K$ is equivalent to $J$. 


(ii) A simple Seifert matrix $P_K$ for $K$ is $(-1)^p$-$S$-equivalent to a simple Seifert matrix $P_J$ for $J$.

(2) Let $K$ and $J$ be closed oriented 3-dimensional simple submanifolds of $S^5$. Suppose that $K$ is diffeomorphic to $J$. Then (i) is equivalent to (ii).

(i) $K$ is equivalent to $J$.

(ii) A simple Seifert matrix $P_K$ for $K$ is $(-1)$-$S$-equivalent to a simple Seifert matrix $P_J$ for $J$.

Note 3.4. In Theorem 3.3(2) we cannot remove the condition that $K$ is diffeomorphic to $J$. It is because there is a closed oriented simple 3-dimensional submanifold $E \subset S^5$ with the following properties:

(i) There is a simple Seifert matrix $P$ for $E$ which is $(-1)$-$S$-equivalent to the empty matrix.

(ii) $E$ is not homeomorphic to the 3-sphere.

See Note to Proof of Theorem 7.2(1) in [9].

By Theorem 3.3 the converse of the $q \geq 4$ case of Theorem 2.3 holds. See the following (1). Note $q = p + 2$

**Theorem 3.5.** (1) Let $2p + 1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ be a closed oriented $(2p + 1)$-dimensional submanifold of $S^{2p+3}$. Then (i) is equivalent to (ii).

(i) $K$ is a Brieskorn submanifold.

(ii) $K$ is connected, $(p - 1)$-connected, simple, and has a simple Seifert matrix $P$ that is $(-1)^p$-$S$-equivalent to a $KN$-type.

(2) Let $a, b$, and $c$ be integers $\geq 2$. Let $K$ be a closed oriented 3-dimensional submanifold of $S^5$. Then (i) is equivalent to (ii).

(i) $K$ is the Brieskorn submanifold $\Sigma(a,b,c)$.

(ii) $K$ is diffeomorphic to the Brieskorn manifold $\Sigma(a,b,c)$. $K$ is simple and has a simple Seifert matrix $P$ that is $(-1)$-$S$-equivalent to $\Lambda_{a,b,c}$.

**Note.** We cannot remove the condition ‘$K$ is diffeomorphic to the Brieskorn manifold $\Sigma(a,b,c)$.’ in (2) of Theorem 3.5. See Note 3.4

By Theorem 3.3 we have the following.

**Theorem 3.6.** (1) Let $2p + 1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ be a closed oriented $(2p + 1)$-dimensional submanifold of $S^{2p+3}$. Let $A$ be a 1-link. Let $a_*$ be an integer $\geq 2$ ($* = 1, \ldots, p$). Then (i) is equivalent to (ii).

(i) $K = A \otimes [a_1] \otimes \ldots \otimes [a_p]$. 

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(ii) $K$ is connected, $(p - 1)$-connected, simple. There is a simple Seifert matrix $P$ for $K$ and a Seifert matrix $P'$ for $A$ such that $P$ is $(-1)^p$-$S$-equivalent to $P' \otimes \Lambda_{a_1, \ldots, a_p}$.

(2) Let $K$ be a closed oriented 3-dimensional submanifold of $S^5$. Let $A$ be a 1-link. Let $a$ be an integer $\geq 2$. Then (i) is equivalent to (ii).

(i) $K = A \otimes [a]$.

(ii) $K$ is diffeomorphic to $A \otimes [a]$. $K$ is simple. There is a simple Seifert matrix $P$ for $K$ and a Seifert matrix $P'$ for $A$ such that $P$ is $(-1)$-$S$-equivalent to $P' \otimes \Lambda_a$.

(3) Let $2p + 1 \geq 5$ and $p \in \mathbb{N}$. Let $K$ be a closed oriented $(2p + 1)$-dimensional submanifold of $S^{2p+3}$. Let $p > q$. Let $2q + 1 \geq 3$ and $q \in \mathbb{N}$. Let $A$ be a simple $(2q + 1)$-submanifold. Let $a_*$ be an integer $\geq 2$ ($* = 1, \ldots, p - q$). Then (i) is equivalent to (ii).

(i) $K = A \otimes [a_1] \otimes \ldots \otimes [a_{p-q}]$.

(ii) $K$ is connected, $(p - 1)$-connected simple. There is a simple Seifert matrix $P$ for $K$ and a simple Seifert matrix $P'$ for $A$ such that $P$ is $(-1)^p$-$S$-equivalent to $(-1)^{\#(p-q)}P' \otimes \Lambda_{a_1, \ldots, a_{p-q}}$.

**Note 3.7.** In the above Theorem 3.6 (1)(ii), (2)(ii), and (3)(ii) we use the following fact, which is proved in §6 of [8]: Let $X$ (resp. $Y$) be a $(2x + 1)$-(resp. $(2y + 1)$-)dimensional closed oriented simple submanifold $\subset S^{2x+3}$ (resp. $S^{2y+3}$), where $x, y \in \mathbb{N} \cup \{-1, 0\}$. Here, we regard $X$ (resp. $Y$) as a 1-link if $x$ (resp. $y$) is 0, and as the empty knot if $x$ (resp. $y$) is $-1$. Let $S_X$ (resp. $S_Y$) be an $(x + 1)$-(resp. $(y + 1)$-)positive Seifert matrix for $X$ (resp. $Y$). Let $S_{X \otimes Y} = (-1)^{xy}S_X \otimes S_Y$. Then $S_{X \otimes Y}$ is an $(x + y + 3)$-positive Seifert matrix for $X \otimes Y$. Furthermore it holds that $X \otimes Y$ is a $(2x + 2y + 5)$-dimensional closed oriented simple submanifold $\subset S^{2x+2y+7}$ if $2x + 2y + 5 \geq 3$.

Let $M$ be a closed oriented $m$-dimensional manifold which we can embed in $S^{m+2}$. Let $K$ be a closed oriented $m$-dimensional submanifold $\subset S^{m+2}$ which is diffeomorphic to $M$. Take a tubular neighborhood of $K$ in $S^{m+2}$. Note that it is diffeomorphic to $M \times D^2$. Of course $\partial (M \times D^2) = M \times \partial D^2 = M \times S^1$.

**Question 3.8.** Let $f : M \times S^1 \to M \times S^1$ be a diffeomorphism map which is represented by $(g, \theta)$, where $g$ is a diffeomorphism map of $M$ and $\theta \in S^1$ . Attach $M \times D^2$ with $S^{m+2} - N(K)$ by such a diffeomorphism $f$. Suppose that $(M \times D^2) \cup_f S^{m+2} - N(K)$ is diffeomorphic to $S^{m+2}$. Thus we obtain a submanifold $M \times \{0\} \subset S^{m+2}$. Call this submanifold $K_f$. Then is $K_f$ equivalent to $K$?

**Question 3.9.** In Question 3.8 replace $f$ with any diffeomorphism of $M \times S^1$. What is the answer?
In Question 3.8 (resp. Question 3.9), if $M$ is the standard sphere, there is such an $f$ such that $K_f$ is nonequivalent to $K$ by [2]. We have the positive answer under some conditions on $K$ if $M$ is the standard sphere by [3,15].

We have a partial solution to Question 3.8 by Theorem 3.3. That is, we have the following:

**Corollary 3.10.** Let $p \in \mathbb{N}$. In Question 3.8, if $K$ is a closed oriented $(2p + 1)$-dimensional connected, $(p - 1)$-connected, simple submanifold of $S^{2p+3}$, we have the positive answer.

**Note.** In [15] it was proved that if $K$ is spherical, Corollary 3.10 is true. Corollary 3.10 is its generalization.

4. Review of knot products

Knot products, or products of knots, were defined and have been researched in [6,8,9].

Let $f : C^n \to C$ be a (complex) polynomial mapping with an isolated singularity at the origin of $C^n$. That is, $f(0) = 0$ and the complex gradient $df$ has an isolated zero at the origin. The link of this singularity is defined by the formula $L(f) = V(f) \cap S^{2n-1}$.

Here the symbol $V(f)$ denotes the variety of $f$, and $S^{2n-1}$ is a sufficiently small sphere about the origin of $C^n$. Given another polynomial $g : C^n \to C$, form $f + g$ with domain $C^{n+m} = C^n \times C^m$ and consider $L(f + g) \subset S^{2n+2m+1}$. We use a topological construction for $L(f + g) \subset S^{2n+2m+1}$ in terms of $L(f) \subset S^{2n+1}$ and $L(g) \subset S^{2m+1}$. The construction generalizes the algebraic situation. Given nice codimension-two embeddings $K \subset S^n$ and $L \subset S^m$, we form a product $K \otimes L \subset S^{n+m+1}$. Then $L(f) \otimes L(-g) \simeq L(f + g)$. See [17].

Let $K \subset S^n$ and $L \subset S^m$ be (not necessarily connected,) codimension two, oriented, closed submanifolds. Take smooth maps 

$$
\begin{align*}
&f : D^{n+1} \to D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \\
g : D^{m+1} \to D^2
\end{align*}
$$

such that 

$$
\begin{align*}
&f^{-1}((0, 0)) \cap \partial D^{n+1} \\
g^{-1}((0, 0)) \cap \partial D^{m+1}
\end{align*}
$$

in 

$$
\begin{align*}
&\partial D^{n+1} \\
&\partial D^{m+1}
\end{align*}
$$

are 

$$
\begin{align*}
&K \subset S^n \\
&L \subset S^m.
\end{align*}
$$

Let $f + g$ be a smooth map

$$
D^{n+1} \times D^{m+1} \to D^2
$$

$$(x, y) \mapsto f(x) - g(y).
$$

We define $K \otimes L$ to be $(f + g)^{-1}((0, 0)) \cap \partial(D^{n+1} \times D^{m+1})$ in $\partial(D^{n+1} \times D^{m+1})$. If $S^n - K$ or $S^m - L$ is the total space of a $S^1$ fiber bundle such that each fiber is the interior of a Seifert hypersurface as in [6,8], $K \otimes L$ is a smooth codimension two closed submanifold $\subset S^{n+m+1}$. 

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5. Review of local moves on high dimensional knots

Local moves on high dimensional knots were defined and have been researched in \[9, 18, 19, 20, 21, 22, 23, 24\].

We review pass-moves on 1-links. See \[7\] for detail. Two 1-links are pass-move-equivalent if one is obtained from the other by a sequence of pass-moves. See Figure 5.1 for an illustration of the pass-move. Each of four arcs in the 3-ball may belong to different components of the 1-link. If \(K\) and \(J\) are pass-move-equivalent and if \(K\) and \(K'\) are equivalent, then we also say that \(K'\) and \(J\) are pass-move-equivalent.

We review \((p, q)\)-pass-moves on \(n\)-knots \((p, q \in \mathbb{N}, p + q = n + 1)\). Figure 5.2, which consists of the three figures (1), (2) and (3), is a diagram of a \((p, q)\)-pass-move-triple. Confirm that, if \((p, q) = (1, 1)\), the \((p, q)\)-pass-move is the pass-move on 1-links.

Let \(K_+, K_-, K_0\) be \(n\)-dimensional closed oriented submanifolds \(\subset S^{n+2}\) \((n \in \mathbb{N})\). Let \(B\) be an \((n + 2)\)-ball trivially embedded in \(S^{n+2}\). Suppose that \(K_+\) coincides with \(K_-\), \(K_0\) in \(S^{n+2} - B\).

Take an \((n + 1)\)-dimensional \(p\)-handle \(h^p_*(*) = +, -\) and an \((n + 1)\)-dimensional \((n + 1 - p)\)-handle \(h^{n+1-p}\) in \(B\) with the following properties.

1. \(h^p_* \cap \partial B\) is the attaching part of \(h^p_*\), \(h^{n+1-p} \cap \partial B\) is the attaching part of \(h^{n+1-p}\).
2. \(h^p_*\) (resp. \(h^{n+1-p}\)) is embedded trivially in \(B\).
3. \(h^p_* \cap h^{n+1-p} = \phi\).
4. The attaching part of \(h^p_+\) coincides with that of \(h^p_-\). The linking number (in \(B\)) of 
   \[h^p_+ \cup (h^p_-)\] and \([h^{n+1-p}\) whose attaching part is fixed in \(\partial B\)] is one if an orientation is given.

Let \(K_*(*) = +, -\) satisfy that \(K_* \cap \text{Int}B = (\partial h^p_* - \partial B) \cup (\partial h^{n+1-p} - \partial B)\). Note the following. When we define \(K_+\), \(h_+\) exists in \(B\) and \(h_-\) does not exist in \(B\). When we define \(K_-\), \(h_-\) exists in \(B\) and \(h_+\) does not exist in \(B\).
This cube is $B = D^{n+2} = D^1 \times D^p \times D^{n+1-p}$

$B \cap K_+$

Figure 5.2.(1): A $(p, n + 1 - p)$-pass-move-triple

$B \cap K_-$

Figure 5.2.(2): A $(p, n + 1 - p)$-pass-move-triple
Let

\[ P = K_+ \cap (S^{n+2} - \text{Int}B) \]
\[ Q = h^p_+ \cap \partial B \]
\[ R = h^{n+1-p} \cap \partial B \]
\[ T = P \cup Q \cup R. \]

Then \( T \) is an \( n \)-dimensional oriented closed submanifold \( \subset (S^{n+2} - \text{Int}B) \subset S^{n+2} \). Let \( K_0 \) be \( T \subset S^{n+2} \). Then we say that \((K_+, K_-, K_0)\) is related by a single \((p, n + 1 - p)\)-pass-move in \( B \). We also say that \((K_+, K_-, K_0)\) is a \((p, n + 1 - p)\)-pass-move-triple. We say that \( K_+ \) and \( K_- \) differ by a single \((p, n + 1 - p)\)-pass-move in \( B \).

If \((K_+, K_-, K_0)\) is a \((p, n + 1 - p)\)-pass-move-triple, then we also say that \((K_-, K_+, K_0)\) is a \((p, n + 1 - p)\)-pass-move-triple. If \( K_+ \) and \( K_- \) differ by a single \((p, n + 1 - p)\)-pass-move in \( B \), then we also say that \( K_- \) and \( K_+ \) differ by a single \((p, n + 1 - p)\)-pass-move in \( B \).
Let \((K_+, K_-, K_0)\) be related by a single \((p, n + 1 - p)\)-pass-move in \(B\). Then there is a Seifert hypersurface \(V_+\) for \(K_+\) \((* = +, -, 0)\) with the following properties.

\[
\begin{align*}
V_2 &= V_0 \cup h_4^p \cup h^{n+1-p}(\sharp = +, -).
V_2 \cap B &= h_4^p \cup h^{n+1-p}.
V_0 \cap \text{Int}B &= \phi.
\end{align*}
\]

\(V_0 \cap \partial B\) is the attaching part of \(h_4^p \cup h^{n+1-p}\). (The idea of the proof is the Thom-Pontrjagin construction.)

Then the ordered set \((V_+, V_-, V_0)\) is called a \((p, n + 1 - p)\)-pass-move-triple of Seifert hypersurfaces for \((K_+, K_-, K_0)\). We say that an ordered set \((V_+, V_-, V_0)\) is related by a single \((p, n + 1 - p)\)-pass-move in \(B\). We say that \(V_-\) (resp. \(V_+\)) is obtained from \(V_+\) (resp. \(V_-\)) by a single \((p, n + 1 - p)\)-pass-move in \(B\).

We review twist-moves on high dimensional knots. (Note: In \([22]\) the twist-move is called the XXII-move.) Figure 5.3, which consists of the three figures (1), (2) and (3), is a diagram of a twist-move-triple. Confirm that if \(p = 0\), the twist-move is the crossing change on 1-links.

Let \(K_+, K_-, K_0\) be \((2p + 1)\)-dimensional closed oriented submanifold \(\subset S^{2p+3}\) \((p \in \mathbb{N}\cup\{0\})\). Let \(B\) be a \((2p+3)\)-ball trivially embedded in \(S^{2p+3}\). Suppose that \(K_+\) coincides with \(K_-, K_0\) in \(S^{2p+3} - B\). Take a single \((2p + 2)\)-dimensional \((p + 1)\)-handle \(h_+\) (resp. \(h_-\)) embedded in \(B\) such that \([\text{the handle}] \cap \partial B\) is the attaching part of the handle.

**Note.** \([4, 5, 27, 28]\) etc. imply that the core of \(h_+\) (resp. \(h_-\)) is trivially embedded in \(B\) under the above condition.

Suppose that \((h_+\text{'s attaching part}) \cap (h_-\text{'s attaching part}) = \phi\). Suppose that their attaching parts coincide. Thus we can suppose that we regard \(h_+ \cup h_-\) as an oriented \((2p + 2)\)-submanifold \(\subset S^{2p+1}\) if we give the opposite orientation to \(h_-\). Then we can define a \((p + 1)\)-Seifert matrix for the \((2p + 2)\)-submanifold \(h_+ \cup h_-\). We can suppose that the Seifert matrix is a \(1 \times 1\)-matrix (1).

Let \(K_*(\# = +, -)\) satisfy that \(K_* \cap \text{Int}B = (\partial h_* - \partial B)\). Note the following. When we define \(K_+\), \(h_+\) exists in \(B\) and \(h_-\) does not exist in \(B\). When we define \(K_-\), \(h_-\) exists in \(B\) and \(h_+\) does not exist in \(B\). Let \(P = K_+ \cap (S^{2p+3} - \text{Int}B)\). Let \(Q = h_+ \cap \partial B\). Let \(T = P \cup Q\). Then \(T\) is an \((2p + 1)\)-dimensional oriented closed submanifold in \(S^{2p+3} - \text{Int}B\). Let \(K_0\) be \(T\) in \(S^{2p+3}\). Then we say that an ordered set \((K_+, K_-, K_0)\) is related by a single twist-move. \((K_+, K_-, K_0)\) is called a twist-move-triple. We say that \(K_+\) and \(K_-\) differ by a single twist-move in \(B\). If \((K_+, K_-, K_0)\) is a twist-move-triple,
This cube is $D^{2p+3} = B$.

$B \cap K_+$

Figure 5.3.(1): A twist-move-triple

$B \cap K_-$

Figure 5.3.(2): A twist-move-triple

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then we also say that \((K_-, K_+, K_0)\) is a twist-move-triple. If \(K_+\) and \(K_-\) differ by a single twist-move in \(B\), we also say that \(K_-\) and \(K_+\) differ by a single twist-move in \(B\).

**Note.** Suppose that \(p\) is an odd natural number, put \(p = 2k + 1\). The twist-move for \((4k + 3)\)-submanifolds \(\subset S^{4k+5}\) \((4k + 3 \in \mathbb{N}, \ k \in \mathbb{N} \cup \{0\})\) has the following property: Suppose that \(K_+\) is made into \(K_-\) by the twist-move. Then \(K_-\) is a nonspherical knot in general even if \(K_+\) is a spherical knot. Furthermore the \(H_*(K_-; \mathbb{Z})\) is not congruent to \(H_*(K_+; \mathbb{Z})\) in general. Example: A Seifert hypersurface \(V_*\) for a 3-knot \(K_*\) \((* = +, -)\); Framed link representation of \(V_*\) is the Hopf link such that the framing of one component is zero and that that of the other is two. Framed link representation of \(V_-\) is the Hopf link such that the framing of each component is two.

Let \((K_+, K_-, K_0)\) be related by a single twist-move in \(B\). Then there is a Seifert hypersurface \(V_*\) for \(K_*\) \((* = +, -, 0)\) with the following properties.

1. \(V_* = V_0 \cup h_\sharp\) \((\sharp = +, -)\). \(V_* \cap B = h_*\).
2. \(V_0 \cap \text{Int } B = \phi\).

\(V_0 \cap \partial B\) is the attaching part of \(h_*\).

(The idea of the proof is the Thom-Pontrjagin construction.)

The ordered set \((V_+, V_-, V_0)\) is called a twist-move-triple of Seifert hypersurfaces for \((K_+, K_-, K_0)\). We say that \(V_-\) (resp. \(V_+\)) is obtained from \(V_+\) (resp. \(V_-\)) by a single twist-move in \(B\).
6. Theorems on local moves on knots and products of knots

In [9, 10] we obtained many results on relations between knot products and local moves on knots. In this paper we prove the following results.

**Theorem 6.1.** Let \( 2p + 1 \geq 5 \) and \( p \in \mathbb{N} \). Let \( J \) be a \((2p+1)\)-dimensional smooth submanifold \( \subset S^{2p+5} \). Suppose that \( J \) and \( K \) differ by a single twist-move and are nonequivalent. Suppose that \( J \) is equivalent to \( A \otimes [2] \) for a \((2p−1)\)-dimensional connected, \((p−2)\)-connected, simple submanifold \( A \subset S^{2p+1} \). Then there is a \((2p−1)\)-dimensional connected, \((p−2)\)-connected, simple submanifold \( B \subset S^{2p+1} \) with the following properties:

(i) \( K \) is equivalent to \( B \otimes [2] \).
(ii) \( A \) and \( B \) differ by a single twist-move and are nonequivalent.
(iii) The equivalence class of such \( B \) is unique.

Note. Compare the above Theorem 6.1 with Theorem 7.3 of [9].

**Theorem 6.2.** Let \( A \) be a 1-link. Let \( \mu \in \mathbb{N} \). Let \( J = A \otimes \mu \text{Hopf} \). Let \( K \) be obtained from \( J \) by one \((2\mu + 1, 2\mu + 1)\)-pass-move. Then there is a 1-link \( B \) such that \( K = B \otimes \mu \text{Hopf} \) and such that \( A \) is pass-move equivalent to \( B \).

Note. We abbreviate \( A \otimes \text{(the Hopf link)} \) to \( A \otimes \text{Hopf} \).

In Theorem 8.1 of [9] we proved the following: If a 1-knot \( A \) is obtained from a 1-knot \( B \) by one pass-move, then \( A \otimes \mu \text{Hopf} \) is obtained from \( B \otimes \mu \text{Hopf} \) by one \((2\mu + 1, 2\mu + 1)\)-pass-move.

In Theorem 8.10 of [9] we proved the case where \( A \) is a knot in Theorem 6.2.

These two theorems imply the following: A 1-knot \( A \) is pass-move equivalent to a 1-knot \( B \) if and only if \( A \otimes \mu \text{Hopf} \) is \((2\mu + 1, 2\mu + 1)\)-pass-move equivalent to \( B \otimes \mu \text{Hopf} \).

In Theorem 4.1 of [10] we proved the following: If a 1-link \( A \) is obtained from a 1-link \( B \) by one pass-move, then \( A \otimes \mu \text{Hopf} \) is obtained from \( B \otimes \mu \text{Hopf} \) by one \((2\mu + 1, 2\mu + 1)\)-pass-move. By this fact and Theorem 6.2 we have the following.

**Theorem 6.3.** Let \( \mu \in \mathbb{N} \). A 1-link \( A \) is pass-move equivalent to a 1-link \( B \) if and only if \( A \otimes \mu \text{Hopf} \) is \((2\mu + 1, 2\mu + 1)\)-pass-move equivalent to \( B \otimes \mu \text{Hopf} \).

7. Handle decompositions of Seifert hypersurfaces

We use Theorem 7.1 in order to prove our other theorems in this paper. A special case of Theorem 7.1 is proved in [12, 15], which includes the case where \( K \) is PL homeomorphic to the standard sphere. Theorem 7.1 is stronger than this special case.

We review handle decompositions. See [11, 25, 26] for detail.

Let \( W \) be a \( w \)-dimensional compact manifold \((w \in \mathbb{N})\). Take a handle decomposition \((B \times [0, 1]) \cup (0\text{-handles}) \cup \ldots \cup (w\text{-handles}) \cup (T \times [0, 1])\), where there may not be an \( i \)-handle \((0 \leq i \leq w)\). Note that \( B \) (resp. \( T \)) is a compact \((w − 1)\)-dimensional submanifold of
∂W. Note that ∂B is diffeomorphic to ∂T. Note that ∂W is diffeomorphic to $B \cup_\alpha T$, where $\alpha$ is a diffeomorphism map from $\partial B \to \partial T$. If a handle is attached to $B \times [0, 1]$, its attaching part is embedded in $B \times \{1\}$. No handle is attached to $T \times [0, 1]$ although $(a \text{ handle}) \cap (T \times [0, 1]) \neq \emptyset$ may hold. $B$ (resp. $T$) may be the empty set. We do not suppose whether $B$ (resp. $T$) is connected or not. We do not suppose whether $B$ (resp. $T$) is closed or not. We say that $B$ is the bottom of this handle decomposition and that $T$ is the top of this handle decomposition.

For a handle decomposition, regard the core (resp. cocore) of each handle as the cocore (resp. core) of it and replace the top (resp. the bottom) with the bottom (resp. the top), then we obtain a new handle decomposition. It is called the dual handle decomposition of the handle decomposition. If a handle $h$ in a handle decomposition is changed into a handle $\bar{h}$ in its dual handle decomposition, $\bar{h}$ is called the dual handle of $h$. Note that we have the following: If a dual handle is attached to $T \times [0, 1]$, its attaching part is embedded in $T \times \{0\}$. No dual handle is attached to $B \times [0, 1]$ although $(a \text{ handle}) \cap (B \times [0, 1]) \neq \emptyset$ may hold.

Let $V$ be a compact $(n+1)$-dimensional manifold ($n+1 \in \mathbb{N}$). For the convenience of the application (see Theorem 7.1), we suppose that the dimension is $n+1$. If a handle decomposition of $V$ satisfies the following conditions, we say that the handle decomposition is a special handle decomposition of $V$.

1. The top $T$ is connected or empty. The bottom $B$ is connected or empty.
2. It has only one (resp. no) $(n+1)$-dimensional 0-handle if $B = \emptyset$ (resp. $B \neq \emptyset$). It has no $(n+1)$-dimensional $i$-handle ($1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$).
3. The dual handle decomposition has only one (resp. no) $(n+1)$-dimensional 0-handle if $T = \emptyset$ (resp. $T \neq \emptyset$). The dual handle decomposition has no $(n+1)$-dimensional $i$-handle ($1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$).

**Example.** If the above $V$ is 6-dimensional and has a special decomposition with $B = \emptyset$ and $T = \partial V$, then $V$ has a handle decomposition $(\text{one 0-handle}) \cup (3\text{-handles})$, where there may be no 3-handle. If the above $V$ is 7-dimensional and has a special decomposition $B = \emptyset$ and $T = \partial V$, then $V$ has a handle decomposition $(\text{one 0-handle}) \cup (3\text{-handles}) \cup (4\text{-handles})$, where there may be no 3-handle, or where there may be no 4-handle.

We define ‘surgeries by using embedded handles’ as follows: Let $X$ be an $x$-dimensional ‘submanifold-with-boundary’ of an $m$-dimensional ‘manifold-with-boundary’ $M$ ($x, m \in \mathbb{N}, x < m$). Suppose that we can embed $X \times [0, 1]$ in $M$ so that $X \times \{0\} = X$. Suppose that an $(x+1)$-dimensional $p$-handle $h^p$ is embedded in $M$ and is attached to $X \times [0, 1]$ ($p \in \mathbb{N} \cup \{0\}, 0 \leq p \leq x$). Suppose that the attaching part of $h^p$ is embedded in $X \times \{1\}$. Suppose that $h^p \cap (X \times [0, 1])$ is only the attaching part of $h^p$. Let $X' = \partial (h^p \cup (X \times [0, 1])) - (X \times \{0\})$. Note that there are two cases, $\partial X = \emptyset$ and
$\partial X \neq \phi$. Note that $X$ is the bottom and $X'$ the top of this handle decomposition. Then we say that $X'$ is obtained from $X$ by the surgery by using the embedded handle $h^p$. We do not say that we use $X \times [0, 1]$ if there is no danger of confusion.

Let $n \in \mathbb{N} \cup \{0\}$. Let $K$ be an $n$-dimensional oriented closed submanifold of a (not necessarily closed) $(n + 2)$-dimensional oriented compact manifold-with-boundary $Q$. We suppose that $K$ satisfies the following condition $(\ast)$: $(K \cap \text{Int}Q) = K - \partial Q$ is connected and is an $n$-dimensional open submanifold of $K$. $K - \partial Q$ is transverse to $\partial Q$ and is an $n$-dimensional compact submanifold of $Q$. $K \cap \partial Q$ is a (not necessarily connected) $n$-dimensional compact submanifold of $\partial Q$.

If $K \cap \partial Q \neq \phi$, we define the tubular neighborhood $N(K)$ of $K$ in $Q$ as follows: Take the tubular neighborhood of $K - \partial Q$ in $Q$, and say $X$. Take the tubular neighborhood of $K - X$ in $\partial Q - X$, and say $Y$. Let $N(K) - X$ (resp. $N(K) - X$) be the total space of the restriction of “the collar neighborhood of $\partial Q$ in $Q$” to $Y - X$ (resp. $Y$) as a fiber bundle. Let $N(K)$ be diffeomorphic to $K \times D^2$.

**Theorem 7.1.** Let $n \in \mathbb{N}$ and $n \geq 3$. Let $K$ be a closed oriented $n$-dimensional connected, $([n^2] - 1)$-connected submanifold of $Q$. Let $Q$ be an $(n + 2)$-dimensional manifold that is $S^{n+2}$, or $B^{n+2}$, or $S^{n+1} \times [0, 1]$. We suppose that $K$ satisfies the above condition $(\ast)$. We do not suppose whether $K \cap \partial Q = \phi$ or $K \cap \partial Q \neq \phi$. Let $N(K)$ be the tubular neighborhood of $K$ in $Q$. Then the following three conditions are equivalent:

1. $\pi_i(Q - N(K)) = \begin{cases} \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil. \end{cases}$

2. There is a $\left\lceil \frac{n-1}{2} \right\rceil$-connected Seifert hypersurface $V$ for $K$. (Note: $V$ is $(n + 1)$-dimensional.)

3. There is a Seifert hypersurface $V$ for $K$ that has a special handle decomposition whose bottom is the empty set and whose top is $\partial V$.

**Proof of Theorem 7.1.** Since $[K] = 0 \in H_n(Q; \mathbb{Z})$, there is a Seifert hypersurface $V$ for $K$ even if $K \cap \partial Q \neq \phi$. By using isotopy of $V$, we can suppose that $\text{Int} V \subset \text{Int} Q$ and that $V$ is transverse to $\partial Q$. Let $N(V)$ be the tubular neighborhood of $V$ in $Q$. Note that $Q - N(K) - N(V)$ is isotopic to $Q - N(V)$ in $Q$. We prove the following proposition.

**Proposition 7.2.** (iii) $\implies$ (i).

**Proof of Proposition 7.2.** By van Kampen theorem $\pi_1(Q - N(V)) \cong 1$. Reason: Note that $Q = N(V) \cup Q - N(V)$ and that $N(V) \cap Q - N(V)$ is a disjoint union of two copies of $V$. 

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By Mayer-Vietoris exact sequence on $N(V)$ and $\overline{Q - N(V)}$, we have $H_i(\overline{Q - N(V)}) \cong 0$ for $2 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$.

Consider the infinite cyclic covering space $\overline{Q - N(K)}$ of $\overline{Q - N(K)}$. Note that $\overline{Q - N(K)}$ is a union of the lift of $N(V)$ and that of $\overline{Q - N(V)}$. By Mayer-Vietoris exact sequence on them, $H_i(\overline{Q - N(K)}) \cong 0$ if $1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$. By van Kampen theorem $\pi_1(\overline{Q - N(K)}) \cong 1$.

By Hurewicz theorem $\pi_i(\overline{Q - N(V)}) \cong 0$ for $2 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$. By well-known facts on covering spaces, $\pi_i(\overline{Q - N(V)}) \cong \pi_i(\overline{Q - N(V)})$ for $2 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$. This completes the proof of Proposition 7.2. \hfill \Box

**Proposition 7.3.** (i) $\implies$ (ii).

**Proof of Proposition 7.3.** We prove the following claim.

**Claim 7.4.** There is a simply-connected Seifert hypersurface for $K$

**Proof of Claim 7.4.** Take a Seifert hypersurface $V$ for $K$. Let $\{g_1, ..., g_n\}$ be a set of generators of $\pi_1 V$. Let $g_i$ also denote a circle that represents the element $g_i \in \pi_1 V$. Note that the dimension of $V$ is $n + 1$ and that $n + 1 \geq 4$. Hence we can suppose that $g_i$ is embedded in $V$.

Since $g_i$ is embedded in $V$, the intersection product of $[g_i]$ and $[V, K]$ in $H_5(\overline{Q - N(K)}, \partial \overline{N(K)})$ is zero. Furthermore recall $\pi_1(\overline{Q - N(K)}) = \mathbb{Z}$. Hence we can take a continuous map $f_i : D^2_2 \to \overline{Q - N(K)}$ such that $f_i(\partial D^2_2) = g_i$.

Note that the dimension of $\overline{Q - N(K)}$ is $n + 2$, and that $n + 2 \geq 5$. Hence we can suppose that $f_i$ is an embedding.

We need the following lemma.

**Lemma 7.5.** Let $C$ be an embedded circle in $V$. Suppose that there is an embedded 2-disc $D$ in $\overline{Q - N(K)}$ such that $\partial D = C$. Note that $D$ may intersect $V$. Let $N(C)$ be the tubular neighborhood of $C$ in $\overline{Q - N(K)}$. Then we can suppose that $D \cap V \cap N(C)$ is $C$. (Recall that $n + 2 \geq 5$).

**Proof of Lemma 7.5.** Since $V$ is orientable, $N(C) = C \times D^{n+1}$. Hence $\partial N(C)$ is the trivial $S^n$-bundle over $C$. Since $n \geq 2$, all sections of this trivial $S^n$-bundle over $C$ are homotopic. There is a section perpendicular to $V$ at $C$. Another section is defined by $D \cap N(C)$. Both sections are homotopic. Hence $D \cap V \cap N(C) = C$. This completes the proof of Lemma 7.5. \hfill \Box

Let $N(g_i)$ be the tubular neighborhood of $g_i$ in $\overline{Q - N(K)}$. By Lemma 7.5 $f_i(D^2_2) \cap V \cap N(g_i) = g_i$. Hence we can suppose that $f_i(D^2_2)$ intersects $V$ transversely, that $(f_i(D^2_2) \cap V) - g_i$ is a disjoint union of some circles, and that $f^{-1}(\text{the circles})$ is in...
the interior of $D^2_t$. Take an innermost circle of the circles $f_i(D^2_t) \cap V \subset D^2_t$. It bounds a disc in $D^2_t$. Note that (the disc) $\cap$ (the other circles)$= \phi$. Note that $f$(the disc) is embedded in $Q - N(K)$ and that $f$(the disc) $\cap V = f$(the innermost circle). Take an embedded $(n + 2)$-dimensional 2-handle whose core is $f$(the disc) and whose attaching part is embedded in $V$. Carry out a surgery on $V$ by using this 2-handle. (Note that we remove the interior of $g_i \times D^n$ and add $D^2 \times S^{n-1}$.) Repeating this procedure on such circles in each $D_i$, we obtain a new $V$. By van Kampen theorem this new $V$ is simply-connected. This completes the proof of Claim 7.4 \hfill $\square$

It is trivial that Proposition 7.3 follows from the following Claims 7.6 and 7.7.

**Claim 7.6.** Let $N(V)$ be the tubular neighborhood of $V$ in $Q - N(K)$. Note that $N(V) = V \times [-1, 1]$. Let $r \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that there is an $(r - 1)$-connected Seifert hypersurface. Then there is an $(r - 1)$-connected Seifert hypersurface $V$ with the following condition: For $t = 1,-1$, the homomorphism $\iota : \pi_r(V \times \{ t \}) \to \pi_r(Q - N(K) - N(V))$ that is induced by the natural inclusion map is injective.

**Proof of Claim 7.6.** The $r = 1$ case follows from Claim 7.4.

We prove the $r \geq 2$ case. Take an $(r - 1)$-connected Seifert hypersurface $V$ for $K$. Suppose that $\alpha \in \pi_r(V \times \{ t \})$ satisfies the condition $\iota(\alpha) = 0$ ($t \in \{1, -1\}$).

Note that the dimension of $V$ is $n + 1$, and that $2r \leq n + 1$. Hence $\alpha$ is represented by an embedded $r$-sphere in $V$.

Let $\alpha$ also denote this $r$-sphere. Then there is a continuous map $f : D^{r+1} \to Q - N(K)$ such that $f(\partial D^{r+1}) = \alpha$. Since $\iota(\alpha) = 0$, $f$(Int$D^{r+1}$) $\cap V = \phi$.

Note that the dimension of $Q - N(K)$ is $n + 2$, that the dimension of $D^{r+1}$ is $r + 1$, that $r + 1 \geq 3$, and that $2(r + 1) \leq n + 2$. Hence we can suppose that $f$ is an embedding map.

Take an embedded $(n + 2)$-dimensional $(r + 1)$-handle whose core is $f(D^{r+1})$ and whose attaching part is embedded in $V$. Carry out a surgery on $V$ by using this handle. (Note that we remove the interior of $S^r \times D^{n+1-r}$ from $V$ and attach $D^{r+1} \times S^{n-r}$.) We obtain a new $V$. Repeating this procedure. Since $\pi_r V$ is finitely generated, $\iota : \pi_r(V \times \{ t \}) \to \pi_r(Q - N(K) - N(V))$ becomes injective for $t = 1,-1$ after finite times of this procedure. This completes the proof of Claim 7.6 \hfill $\square$

**Claim 7.7.** Let $r \leq \lfloor \frac{n-1}{2} \rfloor$. Let $V$ be an $(r - 1)$-connected Seifert hypersurface for $K$. Suppose that $\pi_r(V \times \{ t \}) \to \pi_r(Q - N(K) - N(V))$ is injective for $t = 1,-1$. Then $\pi_r V = 0$.

**Proof of Claim 7.7.** Take any element $\alpha \in \pi_r V$. Note that the dimension of $V$ is $n + 1$ and that $2r \leq n + 1$. Hence $\alpha$ is represented by an embedded $r$-sphere in $V$. 18
Let \( \alpha \) also denote this \( r \)-sphere. Since \( \pi_r(\overline{Q - N(K)}) = 0 \), there is a continuous map \( f : D^{r+1} \to \overline{Q - N(K)} \) such that \( f(\partial D^{r+1}) = \alpha \).

Note that the dimension of \( \overline{Q - N(K)} \) is \( n + 2 \), that the dimension of \( D^{r+1} \) is \( r + 1 \), that \( r + 1 \geq 3 \), and that \( 2(r + 1) \leq n + 2 \). Hence we can suppose that \( f \) is an embedding map.

We prove the following:

**Claim 7.8.** Let \( N(\alpha) \) be the tubular neighborhood of \( \alpha \) in \( \overline{Q - N(K)} \). Then \( f(D^{p+1}) \cap V \cap N(\alpha) = \alpha \).

**Proof of Claim 7.8.** We need the following claim:

**Claim 7.9.** Let \( p \in \mathbb{N} \). Let \( q \geq p + 2 \). Let \( \alpha \) be an \( \mathbb{R}^q \)-bundle over \( S^p \). Let \( \tau \) be the tangent bundle of \( S^p \). Let \( \varepsilon^r \) be the trivial \( \mathbb{R}^r \)-bundle over \( S^p \). If \( \alpha \oplus \tau = \varepsilon^{p+q} \), then \( \alpha = \varepsilon^q \).

**Proof of Claim 7.9.** Since \( q \geq p \), we have \( \alpha = \varepsilon^1 \oplus \beta \), where \( \beta \) is an \( \mathbb{R}^{q-1} \)-bundle. Since \( \tau \oplus \varepsilon^1 = \varepsilon^{p+1} \), \( \alpha \oplus \tau = (\beta \oplus \varepsilon^1) \oplus \tau = (\tau \oplus \varepsilon^1) \oplus \beta = \varepsilon^{p+1} \oplus \beta \). Hence \( \varepsilon^{p+q} = \varepsilon^{p+1} \oplus \beta \).

Recall that \( \pi_i(SO(n)) \cong \pi_i(SO(n + 1)) \) if \( 1 \leq i \leq n - 2 \) and \( n \in \mathbb{N} - \{1\} \). Reason; The exact sequence \( \pi_i(SO(n)) \to \pi_i(SO(n + 1)) \to \pi_i(S^n) \).

Recall that \( \mathbb{R}^r \)-bundles over \( S^p \) are classified by \( \pi_{p-1}SO(r) \). Note that \( p - 1 \leq (q - 1) - 2 \) and that \( \varepsilon^{p+1} \oplus \beta \) is the trivial \( \mathbb{R}^{p+q} \)-bundle. Hence \( \beta \) is the trivial \( \mathbb{R}^{q-1} \)-bundle over \( S^p \).

Hence \( \beta \oplus \varepsilon^1 \) is the trivial \( \mathbb{R}^q \)-bundle over \( S^p \). This completes the proof of Claim 7.9. \( \square \)

Let \( N'(\alpha) \) be the tubular neighborhood of \( \alpha \) in \( V \) (Recall that \( N(\alpha) \) is the tubular neighborhood of \( \alpha \) in \( \overline{Q - N(K)} \)). By Claim 7.9, \( N'(\alpha) = S^r \times D^{n+1-r} \) and \( N(\alpha) = S^r \times D^{n+2-r} \). Hence \( \partial N(\alpha) = S^r \times S^{n+1-r} \) is the trivial \( S^{n+1-r} \)-bundle over \( S^r \).

Since \( n + 2 - r \geq r + 2 \), all sections of this trivial \( S^{n+1-r} \)-bundle over \( S^r \) are homotopic. There is a section that is perpendicular to \( N'(V) \) at \( \alpha \). Another section is defined by \( f(D^{r+1}) \cap N(\alpha) \). Both sections are homotopic. This completes the proof of Claim 7.8. \( \square \)

Recall that \( f \) is an embedding map. By Claim 7.8 we can suppose that \( f(D^{r+1}) \) intersects \( V \) transversely and that \( f(D^{r+1}) \cap V \) is a disjoint union of connected, closed, oriented, \( r \)-dimensional manifolds. Note that each of these \( r \)-manifolds is not an \( r \)-sphere in general. Take an innermost connected \( r \)-manifold \( M \) of these \( r \)-manifolds. Note that \( f^{-1}(M) \) in \( D^{r+1} \) is diffeomorphic to \( M \). There is an \( (r + 1) \)-dimensional compact connected, oriented, manifold \( W \) embedded in \( D^{r+1} \) such that \( M = \partial W \). By the existence of \( W \), \( M \) is a vanishing \( r \)-cycle in \( \overline{Q - N(K)} \). – \( N(V) \).

By Hurewicz theorem \( \pi_r(V) = H_r(V; \mathbb{Z}) \). By Hurewicz theorem, Mayor-Vietoris theorem, and van Kampen theorem, \( \pi_1(\overline{Q - N(K)} \cap N(V)) = 1 \).
\[ \pi_i(Q - N(K) - N(V)) = 0 \text{ for } 2 \leq i \leq r - 1, \quad \pi_r(Q - N(K) - N(V)) = H_r(Q - N(K) - N(V)). \]

Hence there is an \( r \)-sphere embedded in \( V \) that is homologous to \( M \) and the homotopy class \([M] \in \pi_r(Q - N(K) - N(V))\) is zero. Since \( \pi_r V \to \pi_r(Q - N(K) - N(V)) \) is injective, the homotopy class \([M] \in \pi_r V\) is zero. By obstruction theory there is a continuous map \( f : W \to V \) such that \( f|M : M \to V \).

Repeating this procedure, we obtain a new \( f \) such that \( f(D^{p+1} \cap V) = \phi \). Hence \( \alpha \) is null-homotopic in \( Q - N(K) - N(V) \). Since \( \pi_r V \to \pi_r(Q - N(K) - N(V)) \) is injective, \( \alpha \) is null-homotopic in \( V \). Hence \( \pi_r V = 0 \). This completes the proof of Claim 7.7. □

This completes the proof of Proposition 7.3. □

**Proposition 7.10.** (ii) \( \implies \) (iii) if \( n \geq 5 \).

**Proof of Proposition 7.10.** We prove the following proposition that is stronger than Proposition 7.10.

**Proposition 7.11.** Let \( n \geq 5 \). There is a \([n-1]\)-connected Seifert hypersurface \( V \) for \( K \). Let \( B \cup T = K \), where \( B \) (resp. \( T \)) may be the empty set. Let \( B \) and \( T \) be connected, \([\lfloor \frac{n}{2} \rfloor - 1]\)-connected, compact. Then there is a special handle decomposition of \( V \) whose bottom is \( B \) and whose top is \( T \).

**Proof of Proposition 7.11.** \( V \) satisfies the following (\(*\)): There is a handle decomposition of \( V \) with the following conditions;

(i) The bottom is \( B \). The top is \( T \).
(ii) It has only one (resp. no) 0-handle if \( B = \phi \) (resp. if \( B \neq \phi \)).
(iii) It has only one (resp. no) \((n + 1)\)-handle if \( T = \phi \) (resp. if \( T \neq \phi \)).

Reason: Use 1-handles and cancel one or some handles if necessary.

Note that \( V \) is simply-connected and \((n + 1)\)-dimensional, that \( B \) is connected, and that \( n + 1 \geq 6 \). Hence we have the following (see e.g. Lemma 1.21 in §1.3 of [16]).

**Claim 7.12.** There is a Seifert hypersurface \( V \) for the \( n \)-dimensional submanifold \( K \) whose handle decomposition satisfies the above (\(*\)) and has no 1-handle.

**Claim 7.13.** There is a Seifert hypersurface \( V \) for the \( n \)-dimensional submanifold \( K \) whose handle decomposition satisfies Claim 7.12 and that has no 2-handle.

**Proof of Claim 7.13.** Take a handle decomposition of \( V \) that satisfies Claim 7.12. Take a sub-handle-decomposition
\[ T_H = \begin{cases} 
\text{(the only one 0-handle) } \cup \text{(all 2-handles)} & \text{if } B = \phi \\
(B \times [0, 1]) \cup \text{(all 2-handles)} & \text{if } B \neq \phi 
\end{cases} \]

of this handle decomposition.

Note that \( V \) is parallelizable, that \( B \) is simply-connected, that the dimension of \( B \) is \( n \), and that \( n \geq 5 \). Hence we have the following (\( \sharp \))

\[ T_H = \begin{cases} 
\nu(S^2 \times D^{n-1}) & \text{if } B = \phi \\
(B \times [0, 1])\nu(S^2 \times D^{n-1}) & \text{if } B \neq \phi, 
\end{cases} \]

where \( \nu \in \{ 0 \} \cup \mathbb{N} \).

Note that \( V \) is connected, \( \lceil \frac{n-1}{2} \rceil \)-connected, compact and that \( B \) is connected, \( \lceil \frac{n}{2} \rceil - 1 \)-connected, compact. Hence \( H_2(V, B; \mathbb{Z}) = \pi_2(V, B) = 0 \) by using Mayor-Vietoris theorem, the homotopy exact sequence of pair, Hurewicz theorem.

By these facts and the above (\( \sharp \)) we can eliminate all 2-handles. This completes the proof of Claim 7.13. \( \square \)

**Note.** Suppose that two compact connected manifolds \( A^a \) intersect \( B^b \) transversely in a simply-connected, connected, compact manifold \( C^{a+b+1} \). Whitney trick does not work in general when \( a \) or \( b \) is 2 even if \( a + b \geq 5 \). Reason: Let \( a \) (resp. \( b \)) be two. Whitney disc may intersect \( B \) (resp. \( A \)).

**Claim 7.14.** There is a Seifert hypersurface \( V \) for the \( n \)-dimensional submanifold \( K \) whose handle decomposition satisfies Claim 7.13 and has no \( i \)-handle \((1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor)\).

**Proof of Claim 7.14.** Note that \( V \) is connected, \( \lceil \frac{n-1}{2} \rceil \)-connected, compact and that \( B \) is connected, \( \lceil \frac{n}{2} \rceil - 1 \)-connected, compact. Hence \( H_i(V, B; \mathbb{Z}) = \pi_i(V, B) = 0 \) \((1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor)\) by using Mayor-Vietoris theorem, the homotopy exact sequence of pair, Hurewicz theorem. By this fact and \( n + 1 \geq 6 \), we can eliminate all \( i \)-handles \((1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor)\) by using Whitney trick. This completes the proof of Claim 7.14. \( \square \)

Take the dual handle decomposition of the handle decomposition that satisfies Claim 7.14. Eliminate one or some handles if necessary in the same manner as above. This completes the proof of Proposition 7.11. \( \square \)

This completes the proof of Proposition 7.10. \( \square \)

**Proposition 7.15.** (ii) \( \implies \) (iii) if \( n = 3, 4 \).

**Proof of Proposition 7.15.** It is trivial that Proposition 7.15 follows from the following proposition.

**Proposition 7.16.** The \( n = 3, 4 \) case of Proposition 7.11 holds.
Proof of Proposition 7.16. Take a simply-connected Seifert hypersurface $V$ for $K \subset S^{n+2}$. Since $V$ is connected, compact, and $B$ is connected, $V$ satisfies the condition $(\ast)$ in the first paragraph of Proof of Proposition 7.11. Take a 1-handle $h^1$ of the handle decomposition. Since $V$ is oriented, $B \cup h^1 = B_2(S^1 \times D^n)$. Since $\pi_1 V = 1$, there is a continuous map $f : D^2 \to V$ such that $f(\partial D^2) = S^1 \times \{0\}$. Push off $f(\text{Int} D^2)$ from $V$, keeping $f(\partial D^2)$, in the positive direction of the normal bundle of $V$ in $S^{n+2} - N(K)$. Thus we obtain a continuous map $g : D^2 \to S^{n+2} - N(K)$ such that $D^2 \cap V = \partial D^2 = S^1 \times \{0\}$.

Note that the dimension of $S^{n+2} - N(K)$ is $n + 2$, that the dimension of $D^2$ is 2, and that $n + 2 \geq 2 \times 2$. Hence we can suppose that $g$ is an embedding map.

Take an embedded $(n+2)$-dimensional 2-handle whose core is $D^2$ and whose attaching part is embedded in $V$. Carry out a surgery on $V$ by using this 2-handle. (Note $n - 1 \neq 1$. Note that we remove the interior of $S^1 \times D^n$ from $V$ and add $D^2 \times S^{n-1}$.) Then the 1-handle $h^1$ is eliminated and a new $(n - 1)$-handle is obtained. (Note that the dual of an $(n-1)$-handle is a 2-handle not a 1-handle.) We eliminate the 1-handle $h^1$ and obtain a new $V$. Repeating this procedure, we eliminate all 1-handles.

Take its dual handle decomposition. Since $V$ is connected, compact, and $B$ is connected, it also satisfies the condition $(\ast)$ (Of course we replace $B$ (resp. $T$) in the condition $(\ast)$ with $T$ (resp. $B$)). It has no $n$-handle. We can eliminate all 1-handles as above so that we do not obtain a new $n$-handle.

Therefore the new $V$ has a handle decomposition such that the bottom is $B$, that the top is $T$, that it satisfies the condition $(\ast)$, and that it has no 1-handle, no $n$-handle. This completes the proof of Proposition 7.16.

This completes the proof of Proposition 7.15.

This completes the proof of Theorem 7.1.

8. Proof of Theorem 3.1

We prove (ii)$\implies$(i): Attach an embedded $(2p + 2)$-dimensional $(p+1)$-handle to a simple Seifert hypersurface for $K$, carry out a surgery by using this handle, and obtain a new one. Then a simple Seifert matrix associated with the old one is $(-1)^p$-S-equivalent to that associated with the new one. Repeat this procedure.

Claim 8.1. (i) $\implies$ (ii).

Proof of Claim 8.1. Let $(X,Y)$ denote a pair (A manifold, its submanifold ). Let $(X,Y) \times [0,1]$ denote a pair (The manifold $\times [0,1]$, its submanifold $\times [0,1]$) which is a level preserving embedding. Take $(S^{2p+3}, K) \times [0,1]$. Let $V$ (resp. $V'$) be a simple Seifert surface for $K$ whose simple Seifert matrix is $P$ (resp. $P'$). Take a $(2p+2)$-dimensional
submanifold \( V \cup (K \times [0, 1]) \cup V' \) in a \((2p + 4)\)-dimensional manifold \( S^{2p+3} \times [0, 1] \). By Proposition 7.11 and 7.16 we can suppose that a \((2p+3)\)-dimensional Seifert hypersurface \( W \) for the \((2p + 2)\)-dimensional submanifold \( V \cup (K \times [0, 1]) \cup V' \) has a special handle decomposition

\[
(V \times [0, 1]) \cup ((p + 1)\text{-handles}) \cup (p + 2)\text{-handles}) \cup (V' \times [0, 1]).
\]

Let \( \Phi : W \to [0, 1] \) be a height function and a Morse function which gives this handle decomposition. By using isotopy we can suppose the following: Let \( 0 = t_0 \leq t_1 \leq \ldots \leq t_\nu = 1 \) be a partition of \( I \) satisfying

(i) Each \( t_i \) is a regular value of \( \Phi \).

(ii) At most one critical value of \( \Phi \) lies in each interval \((t_i, t_{i+1})\).

Therefore it suffices to prove the case where \( \Phi \) has only one critical point. If we change \([0, 1]\) to \([0, -1]\), then the index \( \xi \) of critical point becomes \( 2p + 3 - \xi \). Hence it suffices to prove the case where \( \Phi \) has only one critical point of index \((p + 1)\). That is, it suffices to prove the case where \( W \) has a handle decomposition

\[
(V \times [0, 1]) \cup (\text{only one } (p + 1)\text{-handle}) \cup (V' \times [0, 1]).
\]

We can suppose that this only one \((p + 1)\)-handle is attached to \( V \) in \( S^{2p+3} \times \{t\} \) for a \( t \) by a famous method in ‘Morse theory with handle bodies’. Hence a simple Seifert matrix for \( V \) is \((-1)^p\)-\( S \)-equivalent to that for \( V' \). This completes the proof of Claim 8.1.

This completes the proof of Theorem 3.1.

9. Proof of Proposition 3.2

We show an example. Embed \( S^3 \times D^3 \) in \( S^7 \) such that \( S^3 \times \{\text{the center}\} \) is the standard 3-sphere trivially embedded in \( S^7 \). Let the submanifold \( \partial(S^3 \times D^3) \) in \( S^7 \) satisfy that the \( S^3 \times D^3 \) is its simple Seifert hypersurface whose simple Seifert matrix is a \( 1 \times 1 \)-matrix \((0)\). This 5-dimensional submanifold is called \( K \). We can suppose that \( D^4 \times S^2 \) bounds \( K \). Hence \( \phi \) and \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) are 3-Seifert matrices for \( K \), where \( \phi \) is the empty matrix. Note that \((0)\) is not \( S \)-equivalent to \( \phi \) or \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \).

We show another example. There is a closed oriented 3-dimensional submanifold \( K \) of \( S^5 \) with the following property. (See §10 of [9] for this submanifold for detail.)

(1) There is a Seifert hypersurface \( V \) for \( K \) whose framed link representation is the \((2, 2a)\) torus link such that the framing of each component is zero \((a \in \mathbb{N} - \{1\})\). \( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \) is a simple Seifert matrix for \( K \).
There is a Seifert hypersurface $W$ for $K$ whose framed link representation is the $(2, 2a)$ torus link such that the framing of one component is zero and the other component is the dot circle. (Carry out a surgery on $V$ by using a 5-dimensional 3-handle embedded in $S^5$ and obtain $W$.) $W$ is a 4-dimensional homology ball. Hence $\phi$ and $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ are Seifert matrices for $K$. Note that $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, where $a \in \mathbb{N} - \{1\}$, is not $(-1)$-$S$-equivalent to $\phi$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. □

10. Proof of Theorem 3.3

Theorem 3.1 implies that (i) $\implies$ (ii) in (1) (resp. (2)).

Lemma 10.1. (ii) $\implies$ (i) in (1).

Proof of Proposition 10.1. It suffices to prove the case where $P_J$ is equivalent to $P_K$. By Theorem 3.1 and ‘Theorem 7.1 and its proof’ we can suppose that there is a simple Seifert hypersurface $V_*$ with a special handle decomposition (the top $\partial V_*$, the bottom $\phi$) whose simple Seifert matrix is $P_*$ ($*=J,K$). By Proposition 7.2, $P_* + (-1)^p P_*(*) = J,K$ is the intersection product on $H_p(V_*;\mathbb{Z})$. Embed $(2p+2)$-dimensional 0-handle $h^0$ in $S^{2p+3}$ trivially. We can take $p$-spheres in $\partial h^0$ and attach embedded $(2p+2)$-dimensional $(p+1)$-handles to $h^0$ along the $p$-spheres so that a simple Seifert matrix associated with the result $h^0 \cup ((p+1)$-handles) is $P_* (*)=J,K$.

Then $V_J$ is diffeomorphic to $V_K$. Reason: The core of the attached part of each $(2p+2)$-dimensional $(p+1)$-handle is a $p$-sphere. The boundary of $h^0$ is a $(2p+1)$-sphere. The core of each $(2p+2)$-dimensional $(p+1)$-handle is $(p+1)$-dimensional. $S^{2p+3}$ is $(2p+3)$-dimensional. Furthermore $p \geq 2$.

Furthermore there is a diffeomorphism map $f:S^{2p+3} \rightarrow S^{2p+3}$ such that $f(V_J) = V_K$. This completes the proof of Proposition 10.1. □

Lemma 10.2. (ii) $\implies$ (i) in (2).

Proof of Proposition 10.2. By Theorem 3.1 and ‘Theorem 7.1 and its proof’ we can suppose that there is a simple Seifert hypersurface $V_*$ with a special handle decomposition (the top $\partial V_*$, the bottom $\phi$) whose simple Seifert matrix $P'_*$ is $(-1)$-$S$-equivalent to $P_*$ ($*=J,K$).

Since $J$ is diffeomorphic to $K$, we can make a closed oriented 4-manifold $M = V_J \cup (-V_K)$ by identifying $J$ with $-K$. It holds that $\sigma(V_*) = \sigma(P'_* + P'_*) = \sigma(P_* + P_*).$ Hence $\sigma(V_J) = \sigma(V_K)$. By Novikov additivity, $\sigma(M) = 0$. Hence there is a compact spin oriented 5-dimensional manifold $W$ such that $\partial W = M$. Surgeries by using 6-dimensional handles let $W$ have a special handle decomposition.
\[ W = V_J \times [0, 1] \cup (5\text{-dimensional 2-handles}) \cup (5\text{-dimensional 3-handles}) \cup V_K \times [0, 1]. \] (See §IX, X of [11].)

Since \( \pi_1(V_*) = 1, V_\ast \times [0, 1] \cup (5\text{-dimensional 2-handles}) \) is\
\[ V_\ast \times [0, 1] \cup (S^2 \times D^3)(\zeta \in \mathbb{N} \cup \{0\}). \] Hence
\[ \partial(V_\ast \times [0, 1] \cup (5\text{-dimensional 2-handles})) - \text{Int} V_\ast \times \{0\} \] is diffeomorphic to
\[ V_\ast \cup (S^2 \times S^3)(\zeta \in \mathbb{N} \cup \{0\}). \]

Hence we have the following: By surgeries using 5-dimensional 2-handles embedded in
\( S^{2p+3} \) we obtain a new Seifert hypersurface \( \tilde{V}_\ast \) for \(* \) with the following properties: Its simple Seifert matrix is \( \tilde{P}_\ast \). \( \tilde{V}_J \) is diffeomorphic to \( \tilde{V}_K \). \( \tilde{P}_J \) is equivalent to \( \tilde{P}_K \).

Since \( 2 \leq 5 \), there is a framed link \( L_\ast = (L_{s_1}, ..., L_{s_{d_\ast}}) \) that represent \( \tilde{V}_\ast \) with the following properties (\(* = J, K\)):

1. \( L_J \) is equivalent to \( L_K \).
2. The framing of \( L_{ji} \) is the same as that of \( L_{K_i} \).
3. \( \tilde{V}_\ast \) has a handle decomposition of only one 0-handle and 2-handles \( h_{si}^2 \) that is attached to \( L_{si} \) with the framing. (Note that \( \tilde{V}_J \) is diffeomorphic to \( \tilde{V}_K \)).
4. The 2-handles make a basis of \( H_2(\tilde{P}_J; \mathbb{Z}) \). (Note that \( \tilde{P}_J \) is equivalent to \( \tilde{P}_K \).)

Thus \( \tilde{V}_J \) is equivalent to \( \tilde{V}_K \) as submanifolds in \( S^5 \). This completes the proof of Proposition 10.2 \( \square \)

This completes the proof of Theorem 3.3 \( \square \)

11. PROOF OF THEOREM 6.1

Claim 11.1. There is a simple Seifert matrix \( P_\ast \) for \(* (\ast = J, K) \) with the following property:
Each element of \( P_J \) is the same as that of \( P_K \) except for only one diagonal element. They differ by one.

Proof of Claim 11.1. There is a \((2p + 3)\)-ball \( B \) where the twist-move is carried out. Take a \((2p + 2)\)-dimensional \((p + 1)\)-handle \( h \) associated with the twist move that is embedded in \( B \). Let \( Z \) be a \((2p + 1)\)-dimensional closed oriented submanifold \((J - B) \cup (h \cap \partial B) \subset S^{2p+3} \). Thus \((J, K, Z) \) is a twist-move-triple. Note that \( Z \) is embedded in \( S^{2p+3} - B \). By the construction of \( Z \), \( Z \) is \((p - 1)\)-connected. Furthermore \( S^{2p+3} - N(J) \) is made by attaching one \((2p + 3)\)-dimensional \((p + 2)\)-handle to \( S^{2p+3} - B - N(Z) \). Hence \( \pi_i(S^{2p+3} - N(J)) = \pi_i(S^{2p+3} - N(K)) \) \( (1 \leq i \leq p) \). Hence \( Z \) is a \((2p + 1)\)-dimensional connected, \((p - 1)\)-connected, simple submanifold of \( S^{2p+3} - B \). By Theorem 7.1 there is a \( p \)-connected Seifert hypersurface \( W \) in \( S^{2p+3} - B \) for \( Z \) that has a special handle decomposition (the top \( Z \), the bottom \( \phi \)). Note that the handle \( h \) is attached to \( W \). By using
we obtain a simple Seifert hypersurface $V_J$ (resp. $V_K$) for $J$ (resp. $K$) that has a special handle decomposition. This completes the proof of Claim 11.1.

Claim 11.2. $(-1)^{p-1}P_J$ is a simple Seifert matrix for $A$.

Proof of Claim 11.2. Let $P_A$ be a simple Seifert matrix for $A$. \[ \cdot \cdot \cdot (1) \]

Claim 11.3. $(-1)^{p-1}P_A$ is $(-1)^p$-S-equivalent to $P_J$.

Proof of Claim 11.3. Since $A \otimes [2] = J$, $(-1)^{p-1}P_A$ is a simple Seifert matrix for $J$. (Reason: Note 3.7. Note that a Seifert matrix for $[2]$ is a $1 \times 1$-matrix $\Lambda_2 = (1)$. Recall that $\Lambda_2$ is defined right before Theorem 2.3.) By Theorem 3.3 we have Claim 11.3.

Claim 11.4. $(-1)^{p-1}P_A$ is $(-1)^{p+1}$-S-equivalent to $P_J$.

Proof of Claim 11.4. By Note 3.7, $(-1)^{p}P_J$ is a simple Seifert matrix for $J \otimes [2]$. By Note 3.7 and Claim 11.3 $(-1)^{p}(-1)^{p-1}P_A$ is a simple Seifert matrix for $J \otimes [2]$. By Theorem 3.3 it holds that $(-1)^{p}(-1)^{p-1}P_A$ is $(-1)^{p+1}$-S-equivalent to $(-1)^{p}P_J$. Hence we have Claim 11.4.

Note. Claims 11.3 and 11.4 hold on time. Reason: Recall Proposition 2.2. The way of making the intersection matrix from a pair of related positive Seifert matrix and negative one depends on whether $p$ is odd or even in general.

By Claim 11.4 $P_A$ is $(-1)^{p+1}$-S-equivalent to $(-1)^{p-1}P_J$. This fact, the above (1), and Theorem 3.1 imply Claim 11.2.

By Claims 11.1 and 11.2, we can make a $(2p - 1)$-dimensional connected, $(p - 1)$-connected, simple submanifold $B$ of $S^{2p+1}$ from $A$ by one twist move so that a simple Seifert matrix of $B$ is $(-1)^{p-1}P_K$. By [6, 8] and Theorem 3.3 we have (i) (ii) (iii) in Theorem 6.1.

12. Proof of Theorem 6.2

Claim 12.1. There is a simple Seifert matrix $P_\ast$ for $\ast$ (\( \ast = J, K \)) with the following property: Let $p_{\ast,ij}$ be the $(i,j)$-element of $P_\ast$. Let $P_\ast$ be a $c \times c$-matrix ($c \in \mathbb{N}$). There are natural numbers $a, b \leq c$ such that $a \neq b$ and that

\[
\begin{align*}
p_{\ast,ij} &= p_{K,ij} - 1 & \text{if } (i,j) = (a,b) \\
p_{\ast,ij} &= p_{K,ij} & \text{if } (i,j) \neq (a,b) \text{ and } (i,j) \neq (b,a).
\end{align*}
\]

(Recall that the $(b,a)$-element is the same as the $(a,b)$-element.)
Proof of Claim 12.1. There is a \((4\mu+3)\)-ball \(B\) where the \((2\mu+1,2\mu+1)\)-pass-move is carried out. Take \((4\mu+2)\)-dimensional \((2\mu+1)\)-handles \(h\) and \(h'\) associated with the \((2\mu+1,2\mu+1)\)-pass-move that are embedded in \(B\). Let \(Z\) be a \((4\mu+1)\)-dimensional closed oriented submanifold \((J-B)\cup(h\cap\partial B)\cup(h'\cap\partial B)\subset S^{4\mu+3}\). Thus \((J,K,Z)\) is a \((2\mu+1,2\mu+1)\)-pass-move-triple. Note that \(Z\) is embedded in \(S^{4\mu+3}-B\). By the construction of \(Z\), \(Z\) is \((2\mu-1)\)-connected. Furthermore \(S^{4\mu+3}-N(J)\) (resp. \(S^{4\mu+3}-N(K)\)) is made by attaching two \((4\mu+3)\)-dimensional \((2\mu+2)\)-handles and one \((4\mu+3)\)-dimensional \((4\mu+2)\)-handles to \(S^{4\mu+3}-B\). Hence \(\pi_i(S^{4\mu+3}-N(J))=\pi_i(S^{4\mu+3}-N(K)) (1\leq i\leq 2\mu)\). Hence \(Z\) is a \((4\mu+1)\)-dimensional connected, \((2\mu-1)\)-connected, simple submanifold of \(S^{4\mu+3}-B\). By Theorem 7.1 there is a \(2\mu\)-connected Seifert hypersurface \(W\) in \(S^{4\mu+3}-B\) for \(Z\) that has a special handle decomposition (the top \(Z\), the bottom \(\phi\)). Note that the handles \(h\) and \(h'\) is attached to \(W\). By using this \(W\cup h\cup h'\) we obtain a simple Seifert hypersurface \(V_J\) (resp. \(V_K\)) for \(J\) (resp. \(K\)) that has a special handle decomposition. Hence we have Claim 12.1.

Let \(P_A\) be a Seifert matrix for the 1-knot \(A\). . . (1)

Claim 12.2. \(P_A\) is \(S\)-equivalent to \((-1)^\mu P_J\).

Proof of Claim 12.2. Since \(J = A\otimes^{\mu}\text{Hopf}\), \((-1)^\mu P_A\) is a simple Seifert matrix for \(J\). (Reason: See Note 3.7. Note that a Seifert matrix for the Hopf link is a \(1\times 1\)-matrix \(\Lambda_{2,2} = (-1)\). Recall that \(\Lambda_{2,2}\) is defined right before Theorem 2.3.) By Theorem 3.3 \((-1)^\mu P_A\) is \(S\)-equivalent to \(P_J\). Hence we have Claim 12.2.

By Claim 12.2 we have the following:

Claim 12.3. There is a 1-link \(A'\) whose Seifert matrix is \((-1)^\mu P_J\).

By Claim 12.1 and 12.3, we have the following:

Claim 12.4. We can make a 1-link \(B\) from \(A'\) by one pass-move so that a Seifert matrix of \(B\) is \((-1)^\mu P_K\).

\(A\) is pass-move equivalent to \(A'\) because of the above (1), Claims 12.2 and 12.3. By this and Claim 12.4 \(A\) is pass-move equivalent to \(B\). By Theorem 3.3 and Claim 12.4 \(B\otimes^{\mu}\text{Hopf}\) is equivalent to \(K\). This completes the proof of Theorem 6.2.

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