A PAIRING BETWEEN GRAPHS AND TREES

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In this paper we develop a canonical pairing between trees and graphs, which passes to their quotients by Jacobi and Arnold identities. Our first main result is that on these quotients the pairing is perfect, which makes it an effective and simple tool for understanding the Lie and Poisson operads, providing canonical duals. Passing from the operads to free algebras over them, we get canonical models for cofree Lie coalgebras. The functionals on free Lie algebras which result are defined without reference to the embedding of free Lie algebras in tensor algebras. In the course of establishing our main results we reprove standard facts about the modules $\mathcal{L}ie(n)$. We apply the pairing to develop product, coproduct and (co)operad structures, defining notions such as a partition of forests which may be useful elsewhere. Remarkably, we find the cooperad which dual to the Poisson operad more manageable than the Poisson operad itself.

This pairing arises as the pairing between canonical bases for homology and cohomology of configurations in Euclidean space. We elaborate on this topology in the expository paper [8]. A variant of this pairing first appears in work of Melancon and Reutenaur on odd-graded free Lie algebras [5]; see Section 4. The pairing was independently developed and applied by Tourtchine [12, 13]; see further commentary at the end of Section 1. We give a unified, explicit, and fully self-contained account here to be built on in a number of different directions in future work, which will include fundamental new results on Lie coalgebras in algebra and topology [9].

1. The (even) Lie configuration pairing

Definition 1.1. (1) A Tree is an isotopy class of acyclic graph whose vertices are either trivalent or univalent, with a distinguished univalent vertex called the root, embedded in the upper half-plane with the root at the origin. Univalent non-root vertices are called leaves, and they are labeled by some set $L$, usually $n = \{1, \ldots, n\}$. Trivalent vertices are also called internal vertices.

(2) The height of a vertex in a Tree is the number of edges between that vertex and the root.

(3) Define the nadir of a path in a Tree to be the vertex of lowest height which it traverses.

(4) A Graph is a connected oriented graph with vertices labeled by some set $L$, taken to be the appropriate $n$ unless otherwise noted.

(5) Given a Tree $T$ and a Graph $G$ labeled by the same set, define

$$\beta_{G,T} : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$$

by sending an edge $e$ connecting vertices labeled by $i$ and $j$ to the nadir of the shortest path $p_T(e)$ between the leaves of $T$ labeled $i$ and $j$, which we call an edge path. Let $\langle G, T \rangle_{(2)}$ to be one if $\beta_{G,T}$ is a bijection and zero otherwise.
(6) In the definition of \( \beta_{G,T} \), let \( \tau_{G,T} = (-1)^N \) where \( N \) is the number of edges \( e \) in \( G \) for which \( p_T(e) \) travels from left to right (according to the half-planar embedding) at its nadir. Define the configuration pairing \( \langle G, T \rangle \) as \( \tau_{G,T}(G,T)(2) \).

See Figure 1 for an illustration. Note that \( \langle G, T \rangle_{(2)} \) is defined without reference to the orientation data of the Graph \( G \) or the planar embedding of the Tree \( T \). We may alternately view a Tree through its set of vertices ordered by \( v \leq w \) if \( v \) is in the shortest path between \( w \) and the root. In this language, \( \beta_{G,T} \) sends an edge in \( G \) to the greatest lower bound of the two leaves in \( T \) labeled by the endpoints of the edge.

\[
T =
\]

\[
G =
\]

**Figure 1.** Two examples of the configuration pairing, involving one underlying tree but two different labelings of its leaves.

The pairing may be defined for non-trivalent trees, but we have yet to find an application of such generality. We extend the pairing to free modules over a fixed ground ring generated by Trees and Graphs, setting notation as follows.

**Definition 1.2.** Fix a ground ring and let \( \Theta_n \) be the free module generated by Trees with leaves labeled by \( n \). Let \( \Gamma_n \) be the free module generated by Graphs with vertices labeled by \( n \). Extend the configuration pairing \( \langle \cdot \rangle \) to one between \( \Theta_n \) and \( \Gamma_n \) by linearity.

We next show that this pairing factors through canonical quotients of \( \Theta_n \) and \( \Gamma_n \) by Jacobi and Arnold identities. Recall that Trees coincide with elements of free non-associative algebras. For example, the first tree from Figure 1 is identified with \([x_2, x_1], x_3\]. In this language, if we replace a tree \( T \) with a bracket expression \( B \), our pairing can be defined by the map \( \beta_{G,B} \) which sends the edge between \( i \) and \( j \) to the innermost pair of brackets which contains \( x_i \) and \( x_j \).

While it is traditional to define the Jacobi identity in the language of brackets, namely that \([[[A,B], C] + [[B,C], A] + [[C,A], B] = 0 \) for any expressions \( A, B \) and \( C \), we use the language of trees as follows.

**Definition 1.3.** (1) A subtree of a Tree consists of a vertex and all edges and vertices whose shortest path to the root goes through that vertex (that is, all edges and vertices over that vertex).
(2) A fusion of a Tree $T$ with another $S$ is the Tree obtained by identifying the root edge of $S$ with a leaf edge of $T$, embedding $S$ through a standard diffeomorphism of the upper-half plane with a boundary-punctured disk disjoint from the rest $T$.

(3) A Jacobi combination in $\Theta_n$ is a sum of three Trees obtained by taking the tree $T$ from Figure 1 which has three leaves, and fusing a tree $D$ to its root along with three trees $A$, $B$ and $C$ to its leaves in three cyclically-related orders. See Figure 2.

(4) A symmetry combination is the sum of two Trees which are isomorphic as graphs and have the same planar ordering of input edges at each internal vertex but exactly one.

(5) Let $J_n \subset \Theta_n$ be the submodule generated by Jacobi combinations and symmetry combinations.

**Proposition 1.4.** The pairing $\langle \beta, \alpha \rangle$ vanishes whenever $\alpha \in J_n$.

*Proof.* Vanishing on the symmetry combinations in $J_n$ is immediate by our sign convention.

To check vanishing on a Jacobi combination, consider three Trees as in Figure 2. In order for a Graph $\gamma$ to pair non-trivially with one of these trees (otherwise, the relation holds vacuously) the vertices $v_i$ and $w_i$ must be nadirs of edge paths. There must then be precisely two distinct edge paths which begin at a leaf of one of $A$, $B$ and $C$ and end in another. Without loss of generality, we may assume that both of these edge paths begin or end at leaves in $A$.

When paired with $\gamma$ the third term of this sum is zero since the two edge paths share a common nadir, namely $w_3$. In the first and second terms, the edge path between $A$ and $C$ has reversed its planar orientation. On the other hand, the edge path between $A$ and $B$ carries the same orientation in these two trees, as do all other edge paths of $\gamma$, which are either internal to $A$, $B$, $C$ or $D$ or which have one end in $D$. Thus the signs of $\gamma$ paired with the first two trees will be opposite, so that the sum of the three pairings is zero.

\[\square\]

We next consider a relation on Graphs which we will see as dual to the Jacobi identity.

**Definition 1.5.**

(1) An Arnold combination in $\Gamma_n$ is the sum of three Graphs which differ only on the subgraphs pictured in Figure 3.

(2) A symmetry combination of Graphs is the sum of two graphs which differ in the orientation of exactly one edge.

(3) Let $I_n$ be the submodule of $\Gamma_n$ generated by Arnold and symmetry combinations, as well as Graphs with two edges which have the same vertices.
Arnold combinations of graphs first occurred, to our knowledge, in the computation of the cohomology of braid groups \( [1] \). They also appear in various forms of graph homology, for example in Vassiliev’s original work on knot theory \( [11] \). We will see the Arnold combination of Graphs as dual to the Jacobi combination of Trees, and thus in some contexts use the term Jacobi combinations to refer to both.

**Proposition 1.6.** The pairing \( \langle \beta, \alpha \rangle \) vanishes whenever \( \beta \in I_n \).

**Proof.** First, \( \langle G, T \rangle \) vanishes whenever \( G \) has two edges with the same vertices since in this case \( \beta_{G,T} \) cannot be a bijection. From our definition of \( \tau \) it is immediate that the pairing of a tree \( T \) with a symmetry combination of Graphs is zero.

To show that the pairing of an Arnold combination with a Tree \( T \) vanishes, consider the nadir \( v_{ij} \) of an edge path we now call \( p_{ij} \) between leaves \( i \) and \( j \), as well as the nadirs \( v_{jk} \) and \( v_{ki} \). Two of these three nadirs must agree. Without loss of generality say \( v_{ij} \) and \( v_{ki} \) agree, in which case the pairing of \( T \) with the third Graph in Figure 3 is zero. The pairings with the first two Graphs in Figure 3 differ by multiplication by \(-1\), since the \( p_{jk} \) appears with the same orientation in both cases but \( p_{ij} \) and \( p_{ki} \) have the same nadir but different orientations. The sum of these pairings is thus zero. \( \square \)

**Definition 1.7.** Let \( \text{Lie}(n) = \Theta_n/J_n \), and let \( \mathcal{E}il(n) = \Gamma_n/I_n \).

Propositions 1.4 and 1.6 imply that the pairing \( \langle \cdot, \cdot \rangle \) passes to a pairing between \( \text{Lie}(n) \) and \( \mathcal{E}il(n) \), which by abuse we give the same name. In \( \mathcal{E}il(n) \) any Graph which has a cycle is zero, since we may use the Arnold identity reduce to linear combinations of Graphs with shorter cycles, ultimately until there are two edges which share the same vertices. The modules \( \text{Lie}(n) \) occur in many contexts, and in particular are the entries of the \( \text{Lie} \) operad.

**Theorem 1.8.** The pairing \( \langle \cdot, \cdot \rangle \) between \( \text{Lie}(n) \) and \( \mathcal{E}il(n) \) is perfect.

Our proof uses reductions of these modules to particular bases.

**Definition 1.9.**

1. The tall generators of \( \text{Lie}(n) \) are represented by Trees for which the right branch of any vertex is a leaf, and the leaf labeled by 1 is leftmost.
2. The long generators of \( \mathcal{E}il(n) \) are represented by linear Graphs (all vertices but two are endpoints of exactly two edges) with aligned orientations, with vertex 1 as the initial endpoint.
3. Let \( i_k \) be the label of the \( k \)th vertex from the left in a tall Tree or, respectively, in a long Graph, as in Figure 4. Given a permutation \( \sigma \in \Sigma_n \) such that \( \sigma(1) = 1 \) let \( tT_\sigma \) and \( lG_\sigma \) denote the tall Tree and long Graph, respectively, with \( i_k = \sigma(k) \).

**Lemma 1.10.** The tall generators span \( \text{Lie}(n) \). The long generators span \( \mathcal{E}il(n) \).
Proof. Up to anti-symmetry identities, the tall generators of \( \text{Lie}(n) \) are exactly those for which the leaf 1 has the maximum height possible, namely \( n - 1 \). We may reduce to such Trees by inductively applying the Jacobi, at each step getting a sum of two Trees each of which has height of the leaf 1 increased by one.

In a similar spirit, we may start with any Graph generating \( \text{Eil}(n) \) and first reduce to a linear combination of Graphs all of which have a single edge from 1 to another vertex and no other edges with vertex 1. Indeed, if there is an edge between 1 and \( i \) and between 1 and \( j \), we may after a change of orientation use the Arnold identity to express it as a linear combination of Graphs each with an edge instead between \( i \) and \( j \). This reduction decreases the number of edges with endpoint 1. Given a graph with a single vertex from 1 to another vertex \( i_2 \), we repeat the above procedure with \( i_2 \) in the place of 1 to reduce until there is only one edge from \( i_2 \) to some vertex \( i_3 \), in addition to the one from 1 to \( i_2 \). Repeating the procedure inductively reduces to linear graphs with 1 as an endpoint, and up to a sign we may change orientations to align them. □

Proof of Theorem 1.8. By the previous lemma, it suffices to show that the pairing is perfect on the tall and long generators of \( \text{Lie}(n) \) and \( \text{Eil}(n) \) respectively. By direct computation, \( \langle lG_\sigma, tT_\tau \rangle \) is one if \( \sigma = \tau \) and zero otherwise, a perfect pairing. □

In light of Theorem 1.8, we may view Jacobi, Arnold and anti-symmetry identities as arising as precisely the kernel of our pairing between Graphs and Trees.

We may also deduce and extend Witt’s classic calculation of a basis for \( \text{Lie}(n) \), without reference to the embedding of the free Lie algebra in the corresponding free associative algebra.

Corollary 1.11. The tall Trees and long Graphs form bases for \( \text{Lie}(n) \) and \( \text{Eil}(n) \) respectively, which are free of rank \( (n - 1)! \).

Moreover, we have a reduction method to these bases.

Corollary 1.12. Given \( \alpha \in \text{Lie}(n) \), \( \alpha = \sum_{\sigma \in \Sigma_n} \langle lG_\sigma, \alpha \rangle tT_\sigma \). Similarly, given \( \beta \in \text{Eil}(n) \), \( \beta = \sum_{\sigma \in \Sigma_n} \langle \beta, tT_\sigma \rangle lG_\sigma \).

For example, by looking at the tree \( T \) which represents \( \alpha = [x_2, x_3], [x_1, x_4] \), we see that any long Graph which pairs with it non-trivially must start out with an edge from one to four, so the
two possibilities are $1 \rightarrow 4 \rightarrow 3 \rightarrow 2$ and $1 \rightarrow 4 \rightarrow 2 \rightarrow 3$. By computing these pairings we have 
\[ \alpha = [[[x_1, x_4], x_3], x_2] - [[[x_1, x_4], x_2], x_3]. \]

We reiterate that this duality between Lie trees and graphs modulo Arnold identities was first noticed by Tourtchine. Using a recursive definition, it is developed in [12]. It is applied in a form close to ours in [13]. Indeed, in that paper Section 2 gives various bases for the modules $\mathcal{E}il(n)$ (which are denoted $T_M^+$) and Section 5, in particular Statement 5.4, is devoted to the duality between $\mathcal{L}ie(n)$ (denoted $B_M^+$) and $\mathcal{E}il(n)$.

2. The (even) Poisson configuration pairing

We next make the straightforward passage to disconnected trees and graphs, which pertain to Poisson algebras and configuration spaces.

**Definition 2.1.**

1. Let $\Phi_n$ be the free module spanned by unordered collections of Trees, which we call Forests, with leaves labeled by $n$.
2. To a Forest $F$ associate a partition $\rho(F)$ of $n$ by setting $i \sim j$ if $i$ and $j$ are leaves in the same tree. Let $\Phi^n_P$ be the submodule spanned by all $F$ with $\rho(F) = P$. We have $\Phi^n_P \cong \bigotimes_{S_i \in P} \Theta_{#S_i}$.
3. Let $\Phi^n_k$ be the submodule spanned by all $F$ with a total of $k$ internal vertices. We have that
   \[ \Phi_n = \bigoplus_k \Phi^n_k = \bigoplus_P \Phi^n_P, \quad \text{with} \quad \Phi^n_k = \bigoplus_P \Phi^n_P | \Sigma(#S_i-1) = k. \]

Thus $\Phi_n^{-1}$ is isomorphic to $\Theta_n$, as is $\Phi^n_P$ where $P$ is the trivial partition.

**Definition 2.2.**

1. Let $\Delta_n$ be the free module spanned by unordered collections of Graphs, which we call Diagrams, with vertices collectively labeled by $n$. Equivalently, $\Delta_n$ is the free module spanned by possibly disconnected oriented graphs.
2. If $D$ is a Diagram, let $\rho(D)$ be the partition of $n$ according to the connected components of $D$. Let $\Delta^n_P$ be the submodule of $\Delta_n$ spanned by all Diagrams $D$ with $\rho(D) = P$. We have that $\Delta^n_P \cong \bigotimes_{S_i \in P} \Gamma_{#S_i}$.
3. Let $\Delta^n_k$ be the submodule spanned by all $D$ with a total of $k$ edges. We have that
   \[ \Delta_n = \bigoplus_k \Delta^n_k = \bigoplus_P \Delta^n_P. \]

**Definition 2.3.**

1. Extend the pairing $\langle \cdot, \cdot \rangle$ to $\Phi_n$ and $\Delta_n$ by setting $\langle D, F \rangle$ to be zero unless $\rho(D) = \rho(F)$ and pairing $\Phi^n_P \cong \bigotimes_{S_i \in P} \Theta_{#S_i}$ with $\Delta^n_P \cong \bigotimes_{S_i \in P} \Gamma_{#S_i}$ through the tensor product of the pairings between $\Theta_{#S_i}$ and $\Gamma_{#S_i}$.
2. Define a Jacobi combination of Forests to be the sum of three forests whose component Trees are identical but for one component Tree of each, which together constitute a Jacobi combination of Trees. Extend all definitions of Jacobi, Arnold, and symmetry combinations from Trees and Graphs to Forests and Diagrams in similar fashion. By abuse of notation, let $J_n$ be the submodule of $\Phi_n$ generated by Jacobi and symmetry combinations, and let $I_n$ be the submodule of $\Delta_n$ generated by Arnold and symmetry combinations, as well as Diagrams with two edges which have the same vertices.
(3) Let \( P_{ois}(n) = \Phi_n/J_n \), and let \( Siop(n) = \Delta_n/I_n \). The submodules \( J_n \) and \( I_n \) are generated by homogeneous elements, so these quotients decompose as

\[
P_{ois}(n) = \bigoplus_k P_{ois_k}(n) = \bigoplus_k P_{ios_P}(n) \quad \text{and} \quad Siop(n) = \bigoplus_k Siop_k(n) = \bigoplus P_{ios_P}(n),
\]

with \( P_{ios_P}(n) \cong \otimes_{S \in P} \text{Lie}(\#S_i) \) and with \( Siop_P(n) \cong \otimes_{S \in P} \mathcal{E} \mathcal{I} 1(\#S_i) \).

Thus, for example \( P_{ois_{n-1}}(n) \cong \text{Lie}(n) \).

This definition of \( P_{ois}(n) \) is isomorphic to the more customary one using bracket expressions.

**Proposition 2.4.** \( P_{ois}(n) \) is isomorphic to the free module generated expressions with two multiplications \([,] \) and \( \cdot \) in the variables \( x_1, \ldots, x_n \), using each variable once, quotiented by anti-symmetry and the Jacobi identity in \([,] \), by commutativity in \( \cdot \) and by the Leibniz rule that \([,] \) defines a derivation with respect to \( \cdot \).

**Proof.** We may use the Leibniz rule to reduce to expressions where there are no \( \cdot \) multiplications which appear inside any bracket. The resulting \( \cdot \) products of pure bracket expressions have no relations defined by the Leibniz rule, and are naturally represented by forests. The anti-symmetry, Jacobi and commutativity relations then translate exactly between these products of brackets and forests.

The definition of the configuration pairing extends naturally to all bracket expressions.

**Definition 2.5.** If \( D \) is a Diagram and \( B \) is a bracket expression define the map \( \beta_{D,B} \) when possible by sending the edge between \( i \) and \( j \) to the innermost pair of brackets which contain \( x_i \) and \( x_j \). If either no such pair of brackets exist or when \( x_i \) and \( x_j \) are multiplied by \( \cdot \), internal to any brackets, we say that \( \beta_{D,B} \) is not defined.

Define the pairing \( \langle D, B \rangle \) as \( 0 \) if \( \beta_{D,B} \) is not a bijection between the set of edges of \( D \) and the set of bracket pairs of \( B \) (in particular, if it is not defined), or \( (-1)^k \), where \( k \) is the number of edges \( i \to j \) for which the corresponding \( x_j \) is to the left of \( x_i \).

It is immediate that this definition of the pairing agrees with that of Definition 2.3 on bracket expressions which correspond to Forests. Moreover, just as respecting Jacobi, Arnold and anti-symmetries was intrinsic in the Lie setting, respecting the Leibniz rule is intrinsic in this setting.

**Proposition 2.6.** \( \langle D, B \rangle = \langle D, B' + B'' \rangle \), where \( B \) is a bracket expression, \( B' \) is obtained from \( B \) by substituting \( Y \cdot [X, Z] \) for \( [X, Y \cdot Z] \), and \( B'' \) is obtained by substituting \( [X, Y] \cdot Z \), for some sub-expressions \( X, Y, Z \).

**Proof.** The map \( \beta_{D,B} \) will be a bijection if and only if exactly one of \( \beta_{D,B'} \) or \( \beta_{D,B''} \) is a bijection, in which case the signs of the pairing will also agree.

Finally, from Theorem 1.8 and the decompositions of \( P_{ois}(n) \) and \( Siop(n) \) into (sums of) tensor products of \( \text{Lie}(\#S_i) \) and \( \mathcal{E} \mathcal{I} 1(\#S_i) \), we immediately have the following.

**Theorem 2.7.** The configuration pairing \( \langle \cdot \rangle \) between \( \Phi_n \) and \( \Delta_n \) descends to a perfect pairing between \( P_{ois_k}(n) \) and \( Siop_k(n) \).
3. The Odd Lie and Poisson Configuration Pairings

There is a closely related pairing between graphs and trees whose sign is determined not by orientation of edges but by ordering. Recall Definition 1.1 as we make the following.

**Definition 3.1.**  
(1) An eoGraph is a connected graph with ordered edges and labeled vertices.  
(2) By abuse define $\beta_{G,T}$ and $\langle G, T \rangle$ for $G$ an eoGraph and $T$ a Tree as in Definition 1.1.  
(3) Order the internal vertices of a Tree $T$ from left to right in accordance with its embedding in the upper half-plane, so that if a vertex $v$ sits over the left branch of another vertex $w$ then $v < w$, and if $v$ sits over the right branch of $w$ then $w < v$.  
(4) Let $\sigma_{G,T}$ be the sign of the permutation defined through $\beta_{G,T}$ on the orderings of the edges of $G$ and the internal vertices of $T$. Define the pairing $\langle G, T \rangle$ as $\sigma_{G,T} \langle G, T \rangle$.  
(5) Let $\Theta_n$ be the free module generated by eoGraphs with $n$ vertices, and extend $\langle , \rangle$ to a pairing between $\Theta_n$ and $\Theta_n$ by linearity.

The Jacobi and symmetry combinations of trees on which this pairing will vanish are the ones for graded Lie algebras generated in odd degrees.

**Definition 3.2.**  
(1) An odd Jacobi combination is a linear combination of three Trees as in Definition 1.3 but with each Tree having a coefficient as indicated in Figure 5, where $\#T$ is the number of leaves in a Tree $T$.  
(2) An odd symmetry combination of two Trees is the sum of which differ only by switching left and right branches of a single vertex, as in part 4 of Definition 1.3 but with a coefficient of $(-1)^{|A||B|}$, where $A$ and $B$ are the subtrees emanating from this vertex, for one of the resulting Trees.  
(3) Let $\Theta_n \subset \Theta_n$ be the submodule generated by odd Jacobi combinations and odd symmetry combinations.

(4) An Arnold combination of eoGraphs is the sum of three eoGraphs which differ only on the subgraphs pictured in Figure 6. The two edges in each of these subgraphs are ordered consecutively and in the same position in all three eoGraphs; their ordering with respect to each other is indicated by the labels of ‘I’ and ‘II’.

**Figure 5. An odd Jacobi combination of Trees.**

**Figure 6. An Arnold combination of eoGraphs.**
(5) A symmetry combination of eoGraphs is a linear combination of two graphs which differ only in the ordering of their edges, where one of the two eoGraphs has a coefficient given by the sign of the permutation relating these orderings.

(6) Let $oJ_n$ be the submodule of $O\Gamma_n$ generated by Arnold and symmetry combinations, as well as Graphs with two edges which have the same vertices.

(7) Let $\text{Lie}^o(n)$ be $\Theta_n$/oJn, and let $\text{Eil}^o(n)$ be $\Omega_n$/oIn.

**Theorem 3.3.** The configuration pairing $\langle,\rangle$ passes to a perfect pairing between $\text{Lie}^o(n)$ and $\text{Eil}^o(n)$.

**Proof.** Once we have shown that the configuration pairing vanishes on $oJ_n$ and $oI_n$, the arguments from the proof of Theorem 1.8 apply to show the pairing is perfect. Indeed, the tall generators of $\text{Lie}^o(n)$ share their definition with those of $\text{Lie}(n)$, and we define the long generators of $O\Gamma_n$ to be the linear graphs with vertex 1 as an endpoint where the edges are ordered according to their linear position. There are still $(n-1)!$ tall generators of $\text{Lie}^o(n)$ and long generators of $\text{Eil}^o(n)$, which by inspection pair perfectly. The process of Lemma 1.10 applies almost verbatim, with only the coefficients changing in the reduction process.

We show vanishing on odd Jacobi and symmetry combinations of Trees and eoGraphs by straightforward computation. In pairing an eoGraph $\gamma$ with the two Trees in an odd symmetry combination, only the signs of the two pairings might differ since the Trees are isomorphic as graphs. The order of edges of $\gamma$ is fixed, so we consider the order of internal vertices of the two Trees, which differ only around the vertex $v$ whose branches are the subtrees $A$ and $B$ of Definition 3.2 part (2). The transposition of the vertices in $A$ and $B$ is a composite of $|A|-1|B|+|B|-1=|A||B|-1$ transpositions. When the resulting sign of $(-1)^{|A||B|-1}$ accounting for the difference between the two pairings is multiplied by the $(-1)^{|A||B|}$ of Definition 3.2 part (2), we see these two terms differ by their sign and thus cancel.

In pairing an eoGraph $\gamma$ with the three Trees in an odd Jacobi combination as in Figure 5, we may as in the proof of Proposition 1.8 assume that there are precisely two distinct edges which begin at a leaf of one of $A$, $B$, and $C$ and end in another, and that they both end at leaves in $A$. As before, the pairing of the third term with $\gamma$ is zero. Because the ordering of edges in $\gamma$ is fixed, the difference in sign between the pairings of $\gamma$ with the first and second trees is the sign of the bijection between the ordered sets of vertices in and between $A$, $B$, and $C$. In the first Tree these vertices appear in the following order: vertices in $A$, $v_1$, vertices in $B$, $w_1$, vertices in $C$. In the second Tree these vertices are in the order: vertices in $C$, $v_2$, vertices in $A$, $w_2$, vertices of $B$. Under $\beta_{\gamma,T}$, $v_1$ and $w_2$ will correspond to the same vertex in $\gamma$, as will $v_2$ and $w_1$. Because there are $|T|-1$ vertices in a subtree $T$, the sign of this permutation is $-1$ to the power $|C|-1)(|A|-1)+1+(|B|-1)+1+(|B|-1)+1+(|A|-1)$, which is equal to $(-1)^{|C||A|}$ of the first tree is multiplied by $(-1)^{|A||B|}$ and the second by $(-1)^{|B||C|}$ these two pairings will have opposite signs and thus cancel.

The vanishing of the configuration pairing on odd symmetry combinations of eoGraphs is immediate. To show that the pairing of an odd Arnold combination of graphs with a Tree $T$ vanishes, we may as in Proposition 1.8 let $v_{ij}$ denote the nadir of the edge path between leaves $i$ and $j$ and without loss of generality assume that $v_{ij}$ and $v_{ki}$ agree, in which case the pairing of $T$ with the third Graph in Figure 6 is zero. The pairings with the first two Graphs in Figure 6 differ by multiplication by $-1$, since $v_{ij}=v_{ki}$ is matched with edge I in the first eoGraph and edge II in the second while $v_{jk}$ has the opposite matchings, giving a sum of zero. $\square$
We extend to the Poisson setting as in Section 2. Recall Definition 4.1 of the module of Forests. There are completely straightforward generalizations of these definitions where pre- or super-scripts of $o$ decorate all of the modules. We omit the repeated definitions. Moreover, we may also use the language of expressions in variables $x_1, \ldots, x_n$ using brackets and products, in which case the appropriately defined pairing respects the odd-graded Leibniz rule.

From Theorem 3.3 and the definition of $\langle \cdot \rangle$ on $\mathcal{P}ois^o(n)$ and $\mathcal{S}iop^o(n)$ through their decompositions into (sums of) tensor products of $\text{Lie}^o(#S_i - 1)$ and $\text{Eil}^o(#S_i - 1)$, we immediately have the following.

**Theorem 3.4.** The configuration pairing $\langle \cdot \rangle$ between $\Phi_n$ and $o\Delta_n$ descends to a perfect pairing between $\mathcal{P}ois_k^o(n)$ and $\mathcal{S}iop_k^o(n)$.

4. **Functionals on free Lie algebras**

Working over a field in this section, let $V$ be a vector space and $V^*$ be its dual. Let $L(V)$ (respectively $L^o(V)$) be the (respectively oddly graded) free Lie algebra generated by $V$, and $L_n(V)$ (respectively $L_n^o(V)$) be the $n$th graded summand. We construct functionals on $L(V)$ and $L^o(V)$ using elements of $V^*$. First to set notation we recall that Trees may be used to define elements of $L(V)$ and $L^o(V)$.

**Definition 4.1.** Given $v_1, \ldots, v_n \in V$ and $T \in \Theta_n$, define $\mu_T(v_1, \ldots, v_n)$ to be the product of $v_i$ according to $T$. That is, translate $T$ into a bracket of free variables $x_i$ (as before Definition 1.3) substitute $v_i$ for $x_i$, and take the resulting product in $L(V)$ or $L^o(V)$.

The $\mu_T(v_1, \ldots, v_n)$ span $L_n^o(V)$ (respectively $L^o(V)$).

**Definition 4.2.** Let $E(V)$, respectively $E^o(V)$, be the module spanned by Graphs (respectively eoGraphs), with vertices labeled by elements of $V$, up to linearity in each vertex and Arnold and symmetry relations. Given $w_1, \ldots, w_n \in V^*$ and $G \in \Gamma_n$ let $\gamma_G(w_1, \ldots, w_n)$ denote the element of $E(V)$ (respectively $E^o(V)$) where $w_i$ labels the $i$th vertex of $G$.

**Definition 4.3.** Define pairings between $L(V)$ and $E(V^*)$ (respectively $L^o(V)$ and $E^o(V^*)$) by

$$\langle \gamma_G(w_1, \ldots, w_n), \mu_T(v_1, \ldots, v_n) \rangle = \sum_{\sigma \in \Sigma_n} \langle G, \sigma \cdot T \rangle \prod_i w_{\sigma(i)}(v_i),$$

(and similarly in the odd case) where $\sigma \in \Sigma_n$ acts on $T$ by permuting the labels of its leaves. Extend to all of $L(V)$ linearly.

**Proposition 4.4.** The pairing of Definition 4.3 is well-defined.

*Proof.* Note that $\mu_T(v_1, \ldots, v_n) = \mu_{\tau \cdot T}(v_{\tau(1)}, \ldots, v_{\tau(n)})$, and similarly for $\gamma_G$ for any permutation $\tau$. We compute immediately that the pairing gives the same value on any of these representations. With this established, the only other equalities to check arise from Jacobi and (anti-)symmetry identities in $L(V)$ along with Arnold and symmetry identities in $E(V)$, which with freedom to permute the leaves of $T$ and the $v_i$ all follow from the vanishing of $\langle \cdot, \cdot \rangle$ on the corresponding identities of (o)Graphs and Trees. \( \Box \)

Take for example the free Lie algebra on two letters, so that $V$ is spanned by say $a$ and $b$. Let $T$ be the first tree in Figure 1 and let $v_1 = a, v_2 = b, v_3 = b$, so that $\mu_T(v_1, v_2, v_3) = [[b, a]b]$. Let
G be as in Figure 1 and compute that \( \langle \gamma_G(a^*, b^*, b^*), [[b, a]b] \rangle \) is one, since the only term in the sum of Definition 4.3 which is non-zero is the one with \( \sigma = id \). All other terms vanish either because \( \prod w_{\sigma(i)}(v_i) \) will be zero, since some term such as \( b^*(a) \) will occur, or in the case that \( \sigma \) transposes 2 and 3 because the resulting \( \langle G, \sigma \cdot T \rangle \) is zero, as computed in Figure 1.

Melancon and Reutenaur give a pairing equivalent to this one for graded Lie algebras generated in odd degrees in [5]. Their main result is the following.

**Theorem 4.5** (Theorem 4.1 of [5]). The pairing between \( L^0(V) \) and \( E^0(V^*) \) is perfect.

Thus \( E(V^*) \) is a model for the free Lie coalgebra on \( V^* \). We extend this result to graded Lie algebras, without using embeddings of free Lie algebras in associative algebras, in [9].

### 5. Product and coproduct structures

The modules of Diagrams \( \Delta_n \) and \( o\Delta_n \) carry a multiplication which is elementary.

**Definition 5.1.** Given Diagrams (respectively oDiagrams) \( D_1 \) and \( D_2 \) their product \( D_1 \cdot D_2 \) is the Diagram \( D \) whose edges are the union of the edges in \( D_1 \) and \( D_2 \), carrying the same orientations (or respectively having the edges of \( D_1 \) occur in their given order before the edges of \( D_2 \) occur in their given order).

Under this product \( \Delta_n \) and \( o\Delta_n \) are commutative (respectively graded commutative, with \( o\Delta_n^k \) in degree \( k \)) rings with unit, namely the Diagram with no edges.

**Proposition 5.2.** The multiplication on \( \Delta_n \) and \( o\Delta_n \) passes to the quotients \( Siop(n) \) and \( Siop^0(n) \).

Just as the product on \( Siop(n) \) and \( Siop^0(n) \) are defined through unions of edges, there is a coproduct on \( Pois(n) \) and \( Pois^0(n) \) defined through partitions of internal vertices.

**Definition 5.3.** A partition of a Forest \( F \) with \( n \) leaves and \( k \) internal vertices is a pair of Forests \( F_1, F_2 \) each with \( n \) leaves and with \( \ell_1 \) and \( \ell_2 \) internal vertices where \( \ell_1 + \ell_2 = k \), defined as follows.

1. Partition the set of internal vertices of \( F \) into two sets, \( S_1 \) and \( S_2 \).
2. For each vertex in \( S_1 \), choose one leaf above each of the two branches of that vertex.
3. Take the smallest subgraph \( G_1 \) of \( F \) containing all of the vertices in \( S_1 \) along with all of the leaves chosen in the previous step.
4. Obtain \( F_1 \) by replacing pairs of edges of \( G_1 \) connected by a bivalent vertex with a single edge, adding a root edge to the lowest vertex of each connected component, and adding a single-edge Tree for each leaf vertex not chosen in step 2.
5. Repeat the previous three steps using \( S_2 \) instead of \( S_1 \) to obtain \( F_2 \).

![Figure 7](image-url) A partition of a Forest \( F \) into \( F_1, F_2 \). The internal vertices chosen for \( F_1 \) are those labeled by \( b, c, e \), and for \( F_2 \) are \( a, d \).
Definition 5.4. Define a coassociative, cocommutative coproduct $c : \Phi_n^k \to \bigoplus_{\ell_1+\ell_2=k} \Phi_{\ell_1}^n \otimes \Phi_{\ell_2}^n$ by $c(F) = \sum_{(F_1,F_2) \in P_F} F_1 \otimes F_2$, where $P_F$ is the set of partitions of $F$.

Theorem 5.5. Let $F \in \Phi_n^k$ and let $G_1$ and $G_2$ in $\Delta_n^{\ell_1}$ and $\Delta_n^{\ell_2}$ (or $\alpha \Delta_n^{\ell_1}$ and $\alpha \Delta_n^{\ell_2}$ respectively) with $\ell_1 + \ell_2 = k$. Then $\langle G_1 \cdot G_2, F \rangle = \langle G_1 \otimes G_2, c(F) \rangle$, where $\langle \cdot, \cdot \rangle$ is the direct sum of tensor products of configuration pairings between $\bigoplus_{\ell_1+\ell_2=n} \Phi_{\ell_1}^n \otimes \Phi_{\ell_2}^n$ and $\bigoplus_{\ell_1+\ell_2=n} \Delta_n^{\ell_1} \otimes \Delta_n^{\ell_2}$ (or respectively $\bigoplus_{\ell_1+\ell_2=n} \alpha \Delta_n^{\ell_1} \otimes \alpha \Delta_n^{\ell_2}$).

Proof. If $\beta_{G_1,G_2,F}$ is a bijection then the nadirs of the paths $p_F(e)$ for $e$ in $G_1$ and the nadirs for $e \in G_2$ partition the internal vertices of $F$ into two sets. If we remember the vertex labels of the edges $e$ we also have a choice of two leaves over each internal vertex of $F$, which gives rise to a unique partition of $F$ into say $(\phi_1,$ $\phi_2)$. From the definition of partition of a Forest we see that $\beta_{G_1,\phi_1}$ and $\beta_{G_2,\phi_2}$ are bijections. Moreover, the sign $\tau_{G_1,G_2,F}$ equals the product $\tau_{G_1,\phi_1} \cdot \tau_{G_2,\phi_2}$ (respectively $\sigma_{G_1,G_2,F} = \sigma_{G_1,\phi_1} \sigma_{G_2,\phi_2}$). Finally, for any other partition of $F$ into some $(F_1,F_2)$ will $\beta_{G_1,F_1}$ and $\beta_{G_2,F_2}$ be bijections. In all such cases there will be some edge in either $G_1$ or $G_2$ whose endpoints correspond to leaves which are in different connected components of $F_1$ or $F_2$. Thus

$$
\langle G_1 \otimes G_2, c(F) \rangle = \langle G_1 \otimes G_2, \sum_{(F_1,F_2) \in P_F} F_1 \otimes F_2 \rangle = \langle G_1 \otimes G_2, \phi_1 \otimes \phi_2 \rangle = \langle G_1 \cdot G_2, F \rangle,
$$

or similarly with $\sigma$’s replacing $\tau$’s in the odd setting.

If $\beta_{G_1,G_2,F}$ is not a bijection, with say an internal vertex $v$ not in its image, then for any partition $(F_1,F_2)$ of $F$, $v$ cannot be in the image of $\beta_{G_1,F_1}$ or $\beta_{G_2,F_2}$, so both $\langle G_1 \cdot G_2, F \rangle$ and $\langle G_1 \otimes G_2, c(F) \rangle$ are zero. \hfill $\square$

We conclude this section with the following.

Corollary 5.6. The coproduct $c$ passes from $\Phi_n$ to its quotients $\mathcal{Pois}(n)$ and $\mathcal{Pois}^\alpha(n)$. Thus the equality of Theorem 5.5 holds for $F \in \mathcal{Pois}(n)$ and $G_i \in \mathcal{Siop}(n)$ or $F \in \mathcal{Pois}^\alpha(n)$ and $G_i \in \mathcal{Siop}^\alpha(n)$.

Proof of Corollary 5.6. Assume that $F$ is in the Jacobi and symmetry submodule $J_n$ (respectively $\alpha J_n$). By Theorem 5.5, $\langle G_1 \otimes G_2, c(F) \rangle = \langle G_1 \cdot G_2, F \rangle = 0$, since the configuration pairing vanishes on $F$. But the pairing $\langle \cdot, \cdot \rangle$ is perfect, so $c(F) = 0$. \hfill $\square$

6. Operad structures

It is well-known that the $\mathcal{Lie}(n)$ and $\mathcal{Pois}(n)$ assemble to form operads. In this section we determine the linearly dual cooperad structure on the $\mathcal{Eil}(n)$ and $\mathcal{Siop}(n)$. Similar ideas were developed in [5] for graph complexes. For simplicity we restrict attention to the non-$\Sigma$ operad structure, giving the following definitions in order to set notation.

Definition 6.1. (1) Rooted planar trees, which we call rp-trees, share much of their definition with Trees, but are unlabeled and not restricted to have only either trivalent or univalent vertices.

(2) Given an rp-tree $\tau$ and a set of edges $E$ the contraction of $\tau$ by $E$ is the rp-tree $\tau'$ obtained by, for each edge $e \in E$, identifying its two vertices (altering the embedding only in a small neighborhood of $e$) and removing $e$ from the set of edges.
(3) Let $\Upsilon$ denote the category of rp-trees, in which there is a unique morphism $f_{\tau,\tau'}$ from $\tau$ to $\tau'$, if $\tau'$ is the contraction of $\tau$ along some set of non-leaf edges $E$. Let $\Upsilon_n$ denote the full subcategory of rp-trees with $n$ leaves.

(4) Each $\Upsilon_n$ has a terminal object, namely the unique tree with one vertex, called the $n$th corolla $\gamma_n$ as in [4]. We allow for the tree $\gamma_0$ which has no leaves, only a root vertex, and is the only element of $\Upsilon_0$.

(5) An edge is called redundant if one of its vertices is bivalent. For a vertex $v$ let $|v|$ denote its valence minus one.

**Definition 6.2.** A non-$\Sigma$ operad is a functor $\mathcal{O}$ from $\Upsilon$ to a symmetric monoidal category $(\mathcal{C}, \otimes)$ which satisfies the following axioms.

1. $\mathcal{O}(\tau) = \otimes_{v \in \tau} \mathcal{O}(\gamma_{|v|})$.
2. $\mathcal{O}(\gamma_1) = 1_{\mathcal{C}} = \mathcal{O}(\gamma_0)$.
3. If $e$ is a redundant edge and $v$ is its terminal vertex then under the decomposition of axiom $\mathcal{O}(e_{\{e\}}) = 1_{\mathcal{C}}$ is the identity map on $\otimes_{v' \neq e} F(\gamma_{|v'|})$ tensored with the isomorphism $(1_{\mathcal{C}} \otimes -)$.

4. If $\mu$ is a subtree of $\tau$, by which we mean a collection of vertices and their branches which is itself a tree, and if $f_{\mu,\mu'}$ and $f_{\tau,\tau'}$ contract the same set of edges, then under the decomposition of axiom $\mathcal{O}(\tau,\tau') = \mathcal{O}(\mu,\mu') \otimes id$.

By axiom (4), the values of $\mathcal{O}$ on morphisms may be computed by composing morphisms on subtrees, so we may identify some subset of basic morphisms through which all morphisms factor. The basic class we consider is that of all morphisms $\tau \rightarrow \gamma_n$ where $\gamma_n$ is a corolla. This class includes the $c_i$ operations and May’s structure maps.

**Definition 6.3.** The module $\Theta = \bigoplus_i \Theta_i$ forms an operad which associates to the morphism $\tau \rightarrow \gamma_n$ in $\Upsilon$ the homomorphism $f_{\tau}$ sending $\bigotimes_{v_j} T_{v_j}$, where $v_i$ ranges over internal vertices in $\tau$ and $T_{v_j} \in \Theta_{|v_j|}$, to the tree $S \in \Theta_n$, called the grafting of the $T_{v_j}$ and obtained as a quotient of them as follows. If $v_i$ in $\tau$, is the other vertex of the $k$th branch (in the planar ordering of edges) of $v_j$, we identify the root edge of $T_{v_i}$ with the $k$th leaf of $T_{v_j}$. Label the vertices of $S$ by elements of $n$ according to the total ordering where leaf $\ell$ is less than leaf $m$ if they both sit over some $T_{v_j}$ and the leaf of $T_{v_i}$ over which $\ell$ sits has a smaller label than that over which $m$ sits.

This operad structure passes immediately to the quotient $\mathcal{L}ie = \bigoplus_i \mathcal{L}ie(i)$, known as the Lie operad. In the odd setting of $\mathcal{L}ie^{\rho} = \bigoplus_i \mathcal{L}ie^{\rho}(i)$, $f_{\tau}$ sends $\bigotimes_{v_j} T_{v_j}$ to $(\text{sign}\rho)S$, where $S$ is as above and $\rho$ is the permutation which relates the order of internal vertices as they occur in $\bigotimes_{v_j} T_{v_j}$ with the order of the corresponding internal vertices of $S$.

Through the configuration pairing duality, we know that $\mathcal{E}il = \bigoplus_i \mathcal{E}il(i)$ form a cooperad, which we understand explicitly as follows.

**Definition 6.4.** Label both the leaves of an rp-tree and the branches of each internal vertex $v$ with elements of $n$ and $|v|$ respectively, from left to right using the orientation in the upper half plane. To an rp-tree $\tau$ with $n$ leaves and two distinct integers $j, k \in n$ let $v$ be the nadir of the shortest path between leaves labelled $i$ and $j$ and define $J_v(j), J_v(k)$ to be the labels of the branches of $v$ over which leaves $j$ and $k$ lie.

The module $\Gamma = \bigoplus_i \Gamma_i$, forms a cooperad which associates to the morphism $\tau \rightarrow \gamma_n$ the homomorphism $g_{\tau}$ sending $G \in \Gamma$ to $\bigotimes_{v_i} G_{v_i}$, as $v_i$ ranges over internal vertices in $\tau$. The graph $G_{v_i} \in \Gamma_{|v_i|}$, is defined by having for each edge in $G$, say between $j$ and $k$, an an edge between
Definition 6.7. (1) Let \( \pi \) be a morphism in \( \mathcal{Y} \) in which \( \tau \) is a tree with one internal vertex over the \( i \)th root edge. Such morphisms give rise to what are known as \( c_\tau \)-operations, which generate an operad structure. Define an operad structure on \( \mathcal{Pois} = \bigoplus \mathcal{Pois}(n) \) by sending \( f \) to the map \( \mathcal{Pois}(n) \otimes \mathcal{Pois}(m) \to \mathcal{Pois}(n + m - 1) \) where \( B_1 \otimes B_2 \) is sent to the bracket expression in which the variables in \( B_2 \) are re-labeled from \( x_i \) to \( x_m \), the variables \( x_j \) in \( B_1 \) with \( j > i \) are re-labeled by \( x_{j+m-1} \), and then \( B_2 \) is substituted for \( x_i \) in \( B_1 \).

(2) Apply Definition 6.4 of a (co)operad structure on the Graph module \( \Gamma \) verbatim to define such a structure on the Diagram module \( \Delta = \bigoplus \Delta_i \).

Finally, we treat the case of Forests and the Poisson operad. Here we use Proposition 2.4 choosing to describe the operad structure in terms of bracket expressions.

Theorem 6.5. Let \( \tau \) be an \( r/p \)-tree and let \( T_{v_i} \in \Theta_{|v_i|} \), with \( v_i \) ranging over the internal vertices of \( \tau \). Let \( G \in \Gamma_n \) or \( o\Gamma_n \), where \( n = \sum |v_i| \). Then \( \langle G, f_\tau(\otimes T_{v_i}) \rangle = \langle g_\tau(G), \otimes T_{v_i} \rangle \), where \( \langle , \rangle \otimes \) denotes the tensor product of (respectively even or odd) configuration pairings.

Proof. A vertex \( w \) in the \( T_v \) subtree of \( T = f_\tau(\otimes T_{v_i}) \) is in the image of \( \beta_{G,T} \) if and only if there is an edge in \( G \) with vertices whose labels coincide with those of one leaf above the left branch of \( w \) and one leaf above the right branch. Such leaves correspond to the leaves of \( \tau \) which lie above the corresponding edges of \( v \). We see that this is also the condition for \( w \) to be in the image of \( \beta_{G,T} \), as well.

To determine the signs in the even setting note that in pairing with both \( T \) and \( \otimes T_{v_i} \), the orientation of an edge path as it passes through \( w \) depends only on the leaves of \( T_v \) connected by that edge, which in turn in both cases only depends on the leaves of \( \tau \) connected by that edge path. When \( G \in o\Gamma_n \), the signs of the permutations \( \rho \) and \( \pi \) in the definition of \( f_\tau \) and \( g_\tau \) relate the signs of \( \langle G, f_\tau(\otimes T_{v_i}) \rangle \) and \( \langle g_\tau(G), \otimes T_{v_i} \rangle \).

In summary \( \beta_{G,T} \) is a bijection if and only if all of the \( \beta_{G,T_v} \) are, and signs agree, establishing the result. \( \square \)

The argument of Corollary 5.6 adapts to this setting to give the following main result.

Corollary 6.6. The (co)operad structure map \( g_\tau \) passes from \( \Gamma_n \) and \( o\Gamma_n \) to their quotients \( \mathcal{E}il(n) \) and \( \mathcal{E}il''(n) \) respectively.

For example, by a small abuse we may consider the second tree of the Forest \( F \) from Figure 7. If \( j, k = 3, 4 \) then \( v \) is the vertex labeled \( d \), and \( J_d(3) = 2 \) while \( J_d(4) = 1 \).

This definition is closely related to the choose-two operad, introduced in Section 2.2 of [7]. This operad structure is illustrated in the more general setting of Forests and Diagrams in Figure 8.

If we consider \( E(V) \) as in Definition 4.2 then the canonical cooperad action induces a Lie coalgebra structure under which a tree \( G \) maps to \( \sum_{e \in G} G^t \otimes G'' - G'' \otimes G' \), where \( G' \) and \( G'' \) are the sub-trees obtained by removing \( e \). We use this Lie coalgebra structure in [9].
Theorem 6.8. Let $\tau$ be an $rp$-tree and $f_\tau$ the morphism from $\tau$ to the corresponding corolla. Let $B_{v_i}$ be bracket expressions in $|v_i|$ variables, where $v_i$ ranges over the internal vertices of $\tau$. Let $D \in \Delta_n$ or $o\Delta_n$, where $n = \sum |v_i|$. Then $\langle D, f_\tau(\bigotimes B_{v_i}) \rangle = \langle g_\tau(D), \bigotimes B_{v_i} \rangle$, where $\langle \cdot \rangle_\otimes$ denotes the tensor product of (respectively even or odd) configuration pairings.

The proof is entirely analogous to that of Theorem 6.5. The proof would be more involved if we had used Forests instead of bracket expressions to define the operad structure, because the Leibniz rule must then be used to compute the final result of a structure map. But by Proposition 2.6 the configuration pairing works perfectly well for bracket expressions, for which the proof is straightforward.

Corollary 6.9. The cooperad structure map $g_\tau$ passes from $\Delta_n$ and $o\Delta_n$ to their quotients $S_{iop}(n)$ and $S_{iop}^o(n)$ respectively.

This cooperad structure is more manageable than the operad structure on $\mathcal{P}ois$, for which the Leibniz rule is needed to reduce to any basis. For example, in $\mathcal{S}$ where we establish the classical result that the homology of the little disks operads are $\mathcal{P}ois$ or $\mathcal{P}ois^o$, it is simpler to work with cohomology and show that the cooperad structure there agrees $S_{iop}$ or $S_{iop}^o$.

We end with a small illustration of Theorem 6.8.

Figure 8. Examples of the operad structure maps $f_\tau$ and $g_\tau$. Both $\langle D, f_\tau(\bigotimes F_{v_i}) \rangle$ and $\langle g_\tau(D), \bigotimes F_{v_i} \rangle_\otimes$ are equal to $-1$.

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