HOLOMORPHIC FUNCTIONAL CALCULUS ON UPPER TRIANGULAR FORMS IN FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. The decompositions of an element of a finite von Neumann algebra into the sum of a normal operator plus an s.o.t.-quasinilpotent operator, obtained using the Haagerup–Schultz hyperinvariant projections, behave well with respect to holomorphic functional calculus.

1. Introduction and description of results

This note concerns the decomposition theorem for elements of a finite von Neumann algebra, recently proved in [2]. In that paper, given a von Neumann algebra \( \mathcal{M} \) with a normal, faithful, tracial state \( \tau \), by using the hyperinvariant subspaces found by Haagerup and Schultz [3] and their behavior with respect to Brown measure, for every element \( T \in \mathcal{M} \) we constructed a decomposition \( T = N + Q \) where \( N \in \mathcal{M} \) is a normal operator whose Brown measure agrees with that of \( T \) and where \( Q \) is an s.o.t.-quasinilpotent operator. An element \( Q \in \mathcal{M} \) is said to be s.o.t.-quasinilpotent if \( ((Q^*)^nQ^n)^{1/n} \) converges in the strong operator topology to the zero operator — by Corollary 2.7 in [3], this is equivalent to the Brown measure of \( Q \) being concentrated at 0. In fact, \( N \) is obtained as the conditional expectation of \( T \) onto the (abelian) subalgebra generated by an increasing family of Haagerup–Schultz projections.

The Brown measure [1] of an element \( T \) of a finite von Neumann algebra is a sort of spectral distribution measure, whose support is contained in the spectrum \( \sigma(T) \) of \( T \). We will use \( \nu_T \) to denote the Brown measure of \( T \). The Brown measure behaves well under holomorphic (or Riesz) functional calculus. Indeed, Brown proved (Theorem 4.1 of [1]) that if \( h \) is holomorphic on a neighborhood of the spectrum of \( T \), then \( \nu_h(T) = \nu_T \circ h^{-1} \) (the push-forward measure by the function \( h \)).

In this note, we prove the following:

Theorem 1. Let \( T \) be an element of a finite von Neumann algebra \( \mathcal{M} \) (with fixed normal, faithful tracial state \( \tau \)) and let \( T = N + Q \) be a decomposition from [2], with \( N \) normal, \( \nu_N = \nu_T \) and \( Q \) s.o.t.-quasinilpotent.

(i) Let \( h \) be a complex-valued function that is holomorphic on a neighborhood of the spectrum of \( T \). Then

\[ h(T) = h(N) + Q_h, \]

where \( Q_h \) is s.o.t.-quasinilpotent.

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(ii) If $0 \notin \text{supp} \nu_T$ (so that $N$ is invertible), then
$$T = N(I + N^{-1}Q)$$
and $N^{-1}Q$ is s.o.t.-quasinilpotent.

The key result for the proof is Lemma 22 of [2], which allows us to reduce to the case when $N$ and $Q$ commute. Before using this, we require a few easy results about s.o.t.-quasinilpotent operators on Hilbert space.

Lemma 2. Let $\mathfrak{A}$ be a unital algebra and let $N, Q \in \mathfrak{A}$, $T = N + Q$ and suppose that both $N$ and $T$ are invertible. Then
$$T^{-1} = N^{-1} - T^{-1}QN^{-1}.$$  
Proof. We have
$$T^{-1} - N^{-1} = T^{-1}(N - T)N^{-1} = -T^{-1}QN^{-1}.$$  
\hfill \Box

Lemma 3. Let $A$ and $Q$ be bounded operators on a Hilbert space $\mathcal{H}$ such that $AQ = QA$ and suppose $Q$ is s.o.t.-quasinilpotent. Then $AQ$ is s.o.t.-quasinilpotent.

Proof. We have $(AQ)^n = A^nQ^n$ and
$$(A^*)^n(AQ)^n = (Q^*)^n(A^*)^nA^nQ^n \leq \|A\|^2n(Q^*)^nA^nQ^n.$$  
By Loewner’s Theorem, for $n \geq 2$ the function $t \mapsto t^{2/n}$ is operator monotone and we have
$$((A^*)^n(AQ)^n)^{2/n} \leq \|A\|^4((Q^*)^nA^nQ^n)^{2/n}.$$  
Thus, for $\xi \in \mathcal{H}$, we have
$$\|((A^*)^n(AQ)^n)^{1/n}\|_2 = \langle (((A^*)^n(AQ)^n)^{2/n}\xi, \xi \rangle \leq \|A\|^4\|((Q^*)^nA^nQ^n)^{2/n}\xi, \xi | \| = \|A\|^4\|((Q^*)^nA^nQ^n)^{1/n}\|_2.$$  
Since $Q$ is s.o.t.-quasinilpotent, this tends to zero as $n \to \infty.$  
\hfill \Box

Proposition 4. Let $N$ and $Q$ be bounded operators on a Hilbert space $\mathcal{H}$ and suppose $NQ = QN$ and $Q$ is s.o.t.-quasinilpotent. Let $T = N + Q$. Let $h$ be a function that is holomorphic on a neighborhood of the union $\sigma(T) \cup \sigma(N)$ of the spectra of $T$ and $N$. Then $h(T)$ and $h(N)$ commute, and $h(T) - h(N)$ is s.o.t.-quasinilpotent.

Proof. If $\lambda$ is outside of $\sigma(T) \cup \sigma(N)$, then by Lemma 2
$$(T - \lambda)^{-1} = (N - \lambda)^{-1} - (T - \lambda)^{-1}Q(N - \lambda)^{-1}. \quad (1)$$  
Let $C$ be a contour in the domain of the complement $\sigma(T) \cup \sigma(N)$, with winding number 1 around each point in $\sigma(T) \cup \sigma(N)$. Then
$$h(T) = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - T)^{-1} d\lambda$$  
$$h(N) = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - N)^{-1} d\lambda.$$  

For any complex numbers \( \lambda_1 \) and \( \lambda_2 \) outside of \( \sigma(T) \cup \sigma(N) \), the operators \((\lambda_1 - T)^{-1}\), \((\lambda_2 - N)^{-1}\) and \(Q\) commute; thus, \(h(T)\) and \(h(N)\) commute with each other. Using (1), we have
\[
h(T) - h(N) = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - T)^{-1}Q(\lambda - N)^{-1} d\lambda = AQ,
\]
where
\[
A = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - T)^{-1}(\lambda - N)^{-1} d\lambda.
\]
We have \(AQ = QA\). By Lemma 3, \(AQ\) is s.o.t.-quasinilpotent.

For the remainder of this note, \(\mathcal{M}\) will be a finite von Neumann algebra with specified normal, faithful, tracial state \(\tau\).

**Lemma 5.** Let \(T \in \mathcal{M}\). Suppose \(p \in \mathcal{M}\) is a \(T\)-invariant projection with \(p \notin \{0, 1\}\).

(i) If \(T\) is invertible, then \(p\) is \(T^{-1}\)-invariant. Moreover, we have
\[
T^{-1}p = (pTp)^{-1},
\]
\[
(1 - p)T^{-1} = ((1 - p)T(1 - p))^{-1},
\]
where the inverses on the right-hand-sides are in \(p\mathcal{M}p\) and \((1 - p)\mathcal{M}(1 - p)\), respectively.

(ii) The union of the spectra of \(pTp\) and \((1 - p)T(1 - p)\) (in \(p\mathcal{M}p\) and \((1 - p)\mathcal{M}(1 - p)\), respectively) equals the spectrum of \(T\).

(iii) If \(h\) is a function that is holomorphic on a neighborhood of \(\sigma(T)\), then \(p\) is \(h(T)\)-invariant. Moreover, \(h(T)p = h(pTp)\).

**Proof.** For (i), a key fact is that one-sided invertible elements of \(\mathcal{M}\) are always invertible. Thus, writing \(T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\) with respect to the projections \(p\) and \((1 - p)\) (so that \(a = pTp\), \(b = pT(1 - p)\) and \(c = (1 - p)T(1 - p)\)) writing \(T^{-1} = \begin{pmatrix} x & y \\ w & z \end{pmatrix}\) and multiplying, we easily see that \(a\) and \(c\) must be invertible and
\[
T^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{pmatrix}.
\]
Thus, \(p\) is \(T^{-1}\)-invariant.

For (ii) we use (i) and the fact that the formula (2) shows that \(T\) is invertible whenever \(pTp\) and \((1 - p)T(1 - p)\) are invertible.

For (iii), writing
\[
h(T) = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - T)^{-1} d\lambda
\]
for a suitable contour \(C\), where this is a Riemann integral that converges in norm, the result follows by applying part (i). \(\square\)

For a von Neumann subalgebra \(\mathcal{D}\) of \(\mathcal{M}\), let \(\text{Exp}_\mathcal{D}\) and \(\text{Exp}_{\mathcal{D}'}\), respectively denote the \(\tau\)-preserving conditional expectations onto \(\mathcal{D}\) and, respectively, the relative commutant of \(\mathcal{D}\) in \(\mathcal{M}\).

**Lemma 6.** Let \(T \in \mathcal{M}\).
Lemma 7. Let \( T \in \mathcal{M} \) and let \( p_i \) and \( D \) be as in either part (i) or part (ii) of Lemma 6. Suppose a function \( h \) is holomorphic on a neighborhood of the spectrum of \( T \). Then \( \text{Exp}_{D'}(h(T)) = h(\text{Exp}_{D'}(T)) \).

Proof. Using that the Riemann integral converges in norm, that \( \text{Exp}_{D'} \) is norm continuous and applying Lemma 6, we get

\[
\text{Exp}_{D'}(h(T)) = \frac{1}{2\pi i} \int_C h(\lambda)\text{Exp}_{D'}((\lambda - T)^{-1}) \, d\lambda
\]

\[
= \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - \text{Exp}_{D'}(T))^{-1} \, d\lambda = h(\text{Exp}_{D'}(T)).
\]

For convenience, here is the statement of Lemma 22 of [2] and an immediate consequence.

Lemma 8. Let \( T \in \mathcal{M} \). For any increasing, right-continuous family of \( T \)-invariant projections \((q_t)_{0 \leq t \leq 1}\) with \( q_0 = 0 \) and \( q_1 = 1 \), letting \( D \) be the von Neumann algebra generated by the set of all the \( q_t \), the Fuglede–Kadison determinants of \( T \) and \( \text{Exp}_{D'}(T) \) agree. Since the same is true for \( T - \lambda \) and \( \text{Exp}_{D'}(T) - \lambda \) for all complex numbers \( \lambda \), we have that the Brown measures of \( T \) and \( \text{Exp}_{D'}(T) \) agree.

Now we have all the ingredients to prove our main result.

Proof of Theorem In Theorem 6 of [2] the decomposition \( T = N + Q \) is constructed by considering an increasing, right-continuous family \((p_t)_{0 \leq t \leq 1}\) of Haagerup–Schultz projections, with \( p_0 = 0 \) and \( p_1 = 1 \), that are \( T \)-invariant, letting \( D \) be the von...
Neumann algebra generated by the set of projections in this family and taking \( N = \text{Exp}_D(T) \). In particular, each \( p_t \) is also \( Q \)-invariant.

For (i), we need to show that the Brown measure of \( h(T) - h(N) \) is the Dirac mass at 0. By Lemma 5(iii), each \( p_t \) is \( h(T) \)-invariant. So by Lemma 8 the Brown measures of \( h(T) - h(N) \) and \( \text{Exp}_{D'}(h(T) - h(N)) \) agree. Since \( h(N) \in D \), we have \( \text{Exp}_{D'}(h(N)) = h(N) \) and by Lemma 7 we have \( \text{Exp}_{D'}(h(T)) = h(\text{Exp}_{D'}(T)) \). Combining these facts we get
\[
\nu_{h(T) - h(N)} = \nu_{h(\text{Exp}_{D'}(T)) - h(N)}. \tag{4}
\]
We have
\[
\text{Exp}_{D'}(T) = N + \text{Exp}_{D'}(Q)
\]
and \( \text{Exp}_{D'}(Q) \) is s.o.t.-quasinilpotent. This last statement follows formally from Lemma 8 and the fact that \( Q \) is s.o.t.-quasinilpotent. However, we should mention that the fact that \( \text{Exp}_{D'}(Q) \) is s.o.t.-quasinilpotent was actually proved directly in [2] as a step in the proof that \( Q \) is s.o.t.-quasinilpotent. In any case, since \( N \) and \( \text{Exp}_{D'}(T) \) commute and \( \text{Exp}_{D'}(Q) \) is s.o.t.-quasinilpotent, by Proposition 4 it follows that \( h(\text{Exp}_{D'}(T)) - h(N) \) is s.o.t.-quasinilpotent. Using (4), we get that \( h(T) - h(N) \) is s.o.t.-quasinilpotent, as desired.

For (ii), the projections \( p_t \) form a right-continuous family, each of which is invariant under \( N^{-1}Q \). By Lemma 8 the Brown measure of \( N^{-1}Q \) equals the Brown measure of
\[
\text{Exp}_{D'}(N^{-1}Q) = N^{-1}\text{Exp}_{D'}(Q). \tag{5}
\]
But since \( N^{-1} \) and \( \text{Exp}_{D'}(Q) \) commute and since the latter is s.o.t.-quasinilpotent, by Lemma 3 their product (5) is s.o.t.-quasinilpotent. \( \square \)

References

[1] L. G. Brown, Lidskii’s theorem in the type II case, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 1–35.

[2] K. Dykema, F. Sukochev, and D. Zanin, A decomposition theorem in II_1–factors, J. reine angew. Math., to appear, available at [http://arxiv.org/abs/1302.1114](http://arxiv.org/abs/1302.1114).

[3] U. Haagerup and H. Schultz, Invariant subspaces for operators in a general II_1-factor, Publ. Math. Inst. Hautes Études Sci. 109 (2009), 19-111.

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