A NONCOMMUTATIVE NORMAL SUBGROUP THEOREM FOR LATTICES OF SEMISIMPLE LIE GROUPS

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Abstract. In this note we prove a noncommutative version of Margulis’ Normal Subgroup Theorem (NST) for irreducible lattices $\Gamma$ in the product of higher rank simple Lie groups. More precisely, we prove a “co-finiteness” result for every non-trivial $\Gamma$-invariant unital $C^*$-subalgebra $A$ of the reduced $C^*$-algebra of $\Gamma$.

Furthermore, we show that the statement of our result is sharper than the commutative NST, by presenting examples of just-infinite groups for which the conclusion of our theorem fails.

Introduction and the statement of the main result

A discrete group $\Gamma$ is called just-infinite if every non-trivial normal subgroup of $\Gamma$ has finite index in $\Gamma$. This notion has been studied extensively in various contexts, such as combinatorial group theory, ergodic theory, dynamics, etc.

The celebrated Normal Subgroup Theorem (NST) of Margulis [Mar91] states that irreducible lattices in centerless higher rank semisimple Lie groups are just-infinite. This theorem and its proof have been source of inspiration for many deep work in the past few decades.

To mention a few, Stuck–Zimmer’s rigidity theorem [SZ94] is a stronger version of Margulis’ NST, which (under additional assumptions on the ambient group) entails that any ergodic probability measure preserving action of $\Gamma$ on the standard Lebesgue space is essentially free; Bader–Shalom’s NST [BS06] extends the result to irreducible lattices in product of just-non-compact locally compact groups, proving their just-infiniteness. Character rigidity is yet another strengthening of the above properties. Bekka [Bek07] proved that the special linear groups $SL_n(\mathbb{Z})$ are character rigid for every $n \geq 3$. This was generalized to other higher rank lattices by Peterson [Pet15], Boutonnet–Houdayer [BH], and Bader–Boutonnet–Houdayer–Peterson [BBHP20].

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Character rigidity is equivalent to operator algebraic superrigidity, so the latter series of results are essentially noncommutative results. In fact, with the exception of Bekka’s work which was based on representation theoretical tools, the subsequent character rigidity results used mostly operator algebraic techniques.

On a different front, recently, a natural noncommutative notion of just-infiniteness was studied by Grigorchuk–Musat–Rørdam in [GMR18]. More precisely, they define a just-infinite $C^*$-algebra to be an infinite dimensional $C^*$-algebra $A$ with the property that every proper quotient of $A$ is finite dimensional.

However, in the non-amenable case specially, their notion does not naturally relate to just-infiniteness of groups through their $C^*$-algebras. The reduced group $C^*$-algebra $C^*_{\lambda_{\Gamma}}(\Gamma)$, i.e. the $C^*$-algebra generated by the left regular representation $\lambda_{\Gamma}$ of $\Gamma$ in $B(\ell^2(\Gamma))$, is just-infinite for any $C^*$-simple group $\Gamma$, yet the free group $\mathbb{F}_2$ for instance, which is $C^*$-simple, is very far from being just-infinite as a group. Also, the full group $C^*$-algebra $C^*(\Gamma)$ is never just-infinite for any non-amenable group $\Gamma$. On the other extreme, the integer group $\mathbb{Z}$ is just-infinite, but $C^*(\mathbb{Z}) = C(\mathbb{T})$ is far from being a just-infinite $C^*$-algebra. The issue is of course that, in general, there is no natural connection between normal subgroups of $\Gamma$ and ideals of $C^*_{\lambda_{\Gamma}}(\Gamma)$. (Certain amenable groups, however, have just-infinite $C^*$-algebras [BGS17].)

In all the above-mentioned generalizations of just-infiniteness and the NST to the noncommutative setting, normal subgroups are effectively viewed as kernels of homomorphisms. For instance, one concludes the NST from Stuck–Zimmer’s result or character rigidity, by applying the result to the quotient group $\Gamma/N$. We are instead taking a rather dual perspective, and view normal subgroups as invariant elements under the conjugation action of $\Gamma$ on the set of its subgroups.

Then, to generalize to the noncommutative setting, we observe that a subgroup $N$ of $\Gamma$ is normal in $\Gamma$ if and only if $C^*_{\lambda_{\Gamma}}(N)$ is a $\Gamma$-invariant $C^*$-subalgebra of $C^*_{\lambda_{\Gamma}}(\Gamma)$, equivalently, if $C^*_{\pi}(N)$ is a $\Gamma$-invariant $C^*$-subalgebra of $C^*_{\pi}(\Gamma)$ for every unitary representation $\pi$ of $\Gamma$, where $\Gamma$ acts on $C^*_{\pi}(\Gamma)$ and $B(H_\pi)$ by inner automorphisms.

Here we denote by $C^*_{\pi}(\Gamma)$ the $C^*$-algebra generated by the unitaries $\pi(g), g \in \Gamma$. We denote $\pi \prec \lambda_{\Gamma}$ if $\pi$ is weakly contained in $\lambda_{\Gamma}$, that is, if the map $\lambda_{\Gamma}(g) \mapsto \pi(g), g \in \Gamma$, extends to a representation of $C^*_{\lambda_{\Gamma}}(\Gamma)$, which we still denote by $\pi : C^*_{\lambda_{\Gamma}}(\Gamma) \to C^*_{\pi}(\Gamma) \subset B(H_\pi)$. 
Definition 1. Let $\pi$ be a unitary representation of $\Gamma$, and let $A$ be an invariant unital $C^*$-subalgebra of $C^*_\pi(\Gamma)$. We say $A$ is co-finite in $C^*_\pi(\Gamma)$ if the commutant $A' \subset B(H_\pi)$ admits a normal $\Gamma$-invariant state.

A discrete group $\Gamma$ is said to be nc-just-infinite (for noncommutative just-infinite) if it is infinite and for every non-trivial invariant unital $C^*$-subalgebra $A$ of $C^*_{\lambda\Gamma}(\Gamma)$ and every $\pi \prec \lambda\Gamma$, $\pi(A)$ is co-finite in $C^*_\pi(\Gamma)$.

We note that if $\Gamma$ is nc-just-infinite, then $\Gamma$ is a just-infinite group. Indeed, let $N$ be a non-trivial normal subgroup of $\Gamma$. Then, $C^*_{\lambda\Gamma}(N)$ is a non-trivial $\Gamma$-invariant unital $C^*$-subalgebra of $C^*_{\lambda\Gamma}(\Gamma)$, and we have $\ell^\infty(\Gamma/N) \subset C^*_{\lambda\Gamma}(N)'$. Thus, if $C^*_{\lambda\Gamma}(N)$ is co-finite in $C^*_{\lambda\Gamma}(\Gamma)$, then the coset space $\Gamma/N$ admits a $\Gamma$-invariant probability measure, which implies that $\Gamma/N$ is finite.

The main result of this note is the following.

Theorem 2. Assume $G$ is a connected semisimple Lie group with trivial center, no non-trivial compact factors, and such that all its simple factors have real rank at least two. Let $\Gamma < G$ be an irreducible lattice. Then $\Gamma$ is nc-just-infinite.

The key ingredient of the proof is a recent deep result of Boutonnet–Houdayer [BH, Theorem B], which is a noncommutative generalization of a structure theorem of Nevo–Zimmer [NZ02] for stationary actions of lattices $\Gamma$ as in the statement of Theorem 2. This presents a limit to generalize our proof for more general lattices $\Gamma$. However, considering the commutative picture [Mar91, BS06], we conjecture that if $\Gamma$ is an irreducible lattice in $G$, where $G$ is either a centerless connected semisimple Lie group with real rank at least 2, or the product of two locally compact, non-discrete, compactly generated just-non-compact groups, not both isomorphic to $\mathbb{R}$, then $\Gamma$ is nc-just-infinite.

The results mentioned at the beginning of the section, including Margulis’ NST, are considered rigidity results for $\Gamma$. However, there are just-infinite groups that should not really be considered as rigid objects in the same spirit as higher rank lattices. For instance, any simple group is just-infinite. Perhaps a more extreme case is the integer group $\mathbb{Z}$ which is just-infinite (and in fact, satisfies the stronger conclusion of Stuck–Zimmer’s result).

We see below that nc-just-infinite is indeed stronger than the commutative property, hence in particular, our Theorem 2 is sharper than the commutative NST for the lattices $\Gamma$ as in the statement.
Proposition 3. Let $\Gamma$ be a nc-just-infinite group. Then

(i) $\Gamma$ is either amenable or $C^*$-simple.

(ii) $\Gamma$ is icc.

In particular, the above proposition implies:

(a) There are simple non-amenable groups $\Gamma$ that are not nc-just-infinite. Indeed, the existence of non-$C^*$-simple groups with trivial amenable radicals was proven by Le Boudec in [LB17]. In the remarks after [LB17, Theorem C], it is asserted that those examples include groups admitting simple subgroups of index two. Obviously any such subgroup is also non-$C^*$-simple, and non-amenable, and therefore also non nc-just-infinite by Proposition 3.

(b) There are groups that satisfy the conclusion of Stuck–Zimmer’s result, but are not nc-just-infinite. Indeed, the integer group $\mathbb{Z}$ is such an example.

Remark. (1) Note that for a given unitary representation $\pi$ of $\Gamma$ and an invariant subalgebra $A \subset C^*_\pi(\Gamma)$, co-finiteness of $A$ is effectively a von Neumann algebraic property: it is equivalent to the same property for the bi-commutant $A''$. The reason we consider all representations of $C^*_{\lambda_F}(\Gamma)$, is to have a $C^*$-algebraic version of the property. For instance, if $\Gamma$ is an icc group and $I \subset C^*_{\lambda_F}(\Gamma)$ is a non-trivial ideal, then since $L(\Gamma)$ is a factor, $I'' = L(\Gamma)$. Therefore, if we let $A = C 1 + I$, then $A' = R(\Gamma)$, and so the $\Gamma$-invariant $C^*$-subalgebra $A$ is co-finite in $C^*_{\lambda_F}(\Gamma)$. But, if $\Gamma$ is non-amenable and non nc-just-infinite (see item (a) above), then there exists $\pi \prec \lambda_{\Gamma}$ such that $\pi(A)$ is not co-finite in $C^*_{\pi}(\Gamma)$.

(2) We observe that if $N$ is a finite-index normal subgroup $\Gamma$, then for any unitary representation $\pi$ of $\Gamma$, $C^*_\pi(N)$ is co-finite in $C^*_{\lambda_F}(\Gamma)$. Indeed, in this case, the $\Gamma$-action on $M := C^*_\pi(N)'$ factors through the finite group $\Gamma/N$, hence $M$ admits a $\Gamma$-invariant normal state.

Proofs of Theorem 2 and Proposition 3

The proof of Theorem 2 is structured similarly to the proof of the commutative NST, consisting of two parts: the Amenability half, and the Property (T) half.

Proof of Theorem 2. Let $A \subset C^*_{\lambda_F}(\Gamma)$ be a non-trivial $\Gamma$-invariant unital $C^*$-subalgebra, and let $\pi$ be a unitary representation of $\Gamma$ which is weakly contained in $\lambda_{\Gamma}$. 
Amenability half. We prove that $\pi(A)'$ admits a $\Gamma$-invariant state.

Let $P$ be a minimal parabolic subgroup, and $K$ a maximal compact subgroup of $G$ such that $G = KP$. Let $\nu_P$ be the unique $K$-invariant probability on $G/P$, and $\mu$ a fully supported probability on $\Gamma$ such that $(G/P, \nu_P)$ is the Poisson boundary of the $(\Gamma, \mu)$-random walk \cite{Fur67}.

The action $G \curvearrowright G/P$ is faithful; indeed, since $\ker(G \curvearrowright G/P)$ is contained in $P$, it is a normal amenable subgroup of $G$, which is also a subproduct of $G$ by semisimplicity. Since each factor of $G$ is nonamenable, it follows that $\ker(G \curvearrowright G/P)$ is trivial. In particular, $\Gamma$ also acts faithfully on $G/P$. Thus, since the action $\Gamma \curvearrowright G/P$ is strongly proximal \cite{Fur67}, it follows $\Gamma$ has trivial amenable radical \cite[Proposition 7]{Fur03}. This implies $\Gamma$ is $C^*$-simple by \cite[Theorem 1]{BCdH94} (cf. \cite[Corollary 6.10]{BKKO17}). Hence, $\pi$ is weakly equivalent to $\lambda_{\Gamma}$, and therefore the map $\pi(g) \mapsto \lambda_{\Gamma}(g)$ extends to a $\Gamma$-equivariant ucp map $\psi: B(H_\mu) \to B(\ell^2(\Gamma))$. Since $C_\lambda^*(\Gamma)$ lies in the multiplicative domain of $\psi$, it follows that $\psi(\pi(A)') \subset A'$. Therefore, it suffices to show that $A'$ admits a $\Gamma$-invariant state.

Define $\theta_\mu = \sum_{g \in \Gamma} \mu(g)\Ad_{\rho(g)}$, where $\rho: \Gamma \to B(\ell^2(\Gamma))$ is the right regular representation. Then $\theta_\mu$ is a $\Gamma$-equivariant ucp map on $B(\ell^2(\Gamma))$, and we denote by $\mathcal{H}_\mu$ the noncommutative Poisson boundary of $(\Gamma, \mu)$ in the sense of Izumi \cite{Izu04}, i.e. the space $\{a \in B(\ell^2(\Gamma)) : \theta_\mu(a) = a\}$ of fixed points of $\theta_\mu$. We also denote $\mathcal{H}_\mu = \mathcal{H}_\mu \cap \ell^\infty(\Gamma)$ for the space of bounded $\mu$-harmonic functions on $\Gamma$.

Any point-weak* limit $\mathcal{E}_\mu$ of the sequence of maps $\frac{1}{n}\sum_{k=1}^n \theta_{\mu^k}$ defines a $\Gamma$-equivariant ucp idempotent from $B(\ell^2(\Gamma))$ onto $\mathcal{H}_\mu$, and endowed with the Choi–Effros product defined by $\mathcal{E}_\mu$, the space $\mathcal{H}_\mu$ becomes a von Neumann algebras. Note that $\mathcal{H}_\mu$ is also canonically equipped with a $\Gamma$-action since it is a $\Gamma$-invariant subspace of $B(\ell^2(\Gamma))$.

Since $C_{\lambda_{\Gamma}}^*(\Gamma)$ lies in the multiplicative domain of $\mathcal{E}_\mu$, it follows that $\mathcal{E}_\mu(A') \subset A' \cap \mathcal{H}_\mu$. We will show that the latter intersection admits a $\Gamma$-invariant state, hence its composition with $\mathcal{E}_\mu$ yields a $\Gamma$-invariant state on $A'$, and this will conclude the proof of this part.

By \cite[Theorem 4.1]{Izu04}, there is a $\Gamma$-equivariant von Neumann algebra isomorphism $\Phi: \Gamma \rtimes L^\infty(G/P) \to \mathcal{H}_\mu$ that restricts to the identity map on $C_{\lambda_{\Gamma}}^*(\Gamma)$, and to the Poisson transform $\mathcal{P}_{\nu_P}: L^\infty(G/P) \to \mathcal{H}_\mu$ defined by $\mathcal{P}_{\nu_P}(f)(g) = \int_{G/P} f(gx) \, d\nu_P(x)$ for $f \in L^\infty(G/P)$ and $g \in \Gamma$.

Since $\Phi$ is identity on $C_{\lambda_{\Gamma}}^*(\Gamma)$, it maps the relative commutant $B$ of $A$ in the crossed product $\Gamma \rtimes L^\infty(G/P)$ onto $A' \cap \mathcal{H}_\mu$. Thus, it
is now suffices to show that the $\Gamma$-von Neumann algebra $B$ admits a $\Gamma$-invariant state.

Let $\Xi_0 : \Gamma \rtimes L^\infty(G/P) \to L^\infty(G/P)$ be the canonical conditional expectation, which is $\Gamma$-equivariant and faithful. Then, the composition $\varrho(\cdot) = \int_{G/P} \Xi_0(\cdot)(x) d\nu_P(x)$ defines a faithful normal $\mu$-stationary state on $\Gamma \rtimes L^\infty(G/P)$. We will show that the restriction of $\varrho$ to $B$ is $\Gamma$-invariant.

First, we see that $\Gamma \rtimes B$ is ergodic, i.e. the $\Gamma$-invariant subalgebra $\{b \in B : gb = b \forall g \in \Gamma\}$ of $B$ is trivial. One can show this, for instance, by using the freeness of $\Gamma \rtimes G/P$ to conclude a unique stationarity as in \cite[Example 4.13]{HK}, and then apply \cite[Proposition 2.7.(3)]{BBHP}; (alternatively, \cite[Theorem A]{DP} can be used to show the claim). However, it was pointed out to us by Rémi Boutonnet that the ergodicity in the above follows from the following general fact, which we include for the future easy reference; we thank him for providing us with the proof.

**Lemma 4 (Boutonnet).** Let $\Gamma$ be an icc group with $\mu \in \text{Prob}(\Gamma)$ whose support generates $\Gamma$. If $(X, \nu)$ is any ergodic $(\Gamma, \mu)$-space, then the action of $\Gamma$ on the crossed product $\Gamma \rtimes L^\infty(X)$ by conjugation is ergodic.

**Proof.** Denote by $\Xi : \Gamma \rtimes L^\infty(X) \to L^\infty(X)$ the canonical conditional expectation. Let $a \in \Gamma \rtimes L^\infty(X)$ be $\Gamma$-fixed, and set $b = a - \Xi(a)$. Then $b$ is $\Gamma$-fixed and $\Xi(b) = 0$. For $g \in \Gamma$, let $\phi_g := \Xi(b \lambda_{g^{-1}}) \in L^\infty(X)$ be the Fourier coefficient of $b$ at $g$. Since $b$ is $\Gamma$-fixed, the map $g \mapsto |\phi_g|^2$ is $\Gamma$-equivariant with respect to the conjugation action of $\Gamma$ on itself, and so the function $\zeta(g) := \int_X |\phi_g|^2 d\nu$ is $\mu$-harmonic (for the conjugate action). Moreover, $\sum_{g \in \Gamma} \zeta(g) = \int_X \Xi_0(b^*b) d\nu < \infty$, and so $\zeta \in \ell^1(\Gamma)$. Thus, $\zeta$ attains its maximum values on a finite conjugate-invariant subset of $\Gamma$, that is $\{e\}$ by the icc assumption. Since $\zeta \geq 0$ and $\zeta(e) = 0$, it follows $\zeta = 0$. So $b = 0$, and $a = \Xi(a)$. The function $\Xi(a) \in L^\infty(X)$ is $\Gamma$-invariant, hence constant by ergodicity of $\Gamma \rtimes (X, \nu)$. \hfill $\Box$

Returning to the proof of the theorem, we can now invoke \cite[Theorem B]{BH} to conclude that either the restriction $\varrho|_B$ is $\Gamma$-invariant, or that there exists a closed subgroup $P \subset Q \subset G$ and a $\Gamma$-equivariant von Neumann algebra embedding $\vartheta : L^\infty(G/Q, \nu_Q) \to B$, such that $\varrho \circ \vartheta = \nu_Q$, where $\nu_Q$ is the pushforward of $\nu_P$ under the canonical map $G/P \to G/Q$. We will rule out the latter possibility.

For the sake of contradiction, assume otherwise, that such $\vartheta$ exists. Then the composition $\Xi_0 \circ \vartheta$ is a normal $\Gamma$-equivariant von Neumann algebra homomorphism from $L^\infty(G/Q, \nu_Q)$ into $L^\infty(G/P, \nu_P)$. The
canonical embedding is the unique such map (see e.g. [BS06, Theorem 2.14]), so $\Xi_0 \circ \vartheta = \text{id}\big|_{L^\infty(G/Q)}$. Since $\Xi_0$ is a faithful conditional expectation, it follows $\vartheta = \text{id}\big|_{L^\infty(G/Q)}$ by [Ham85, Lemma 3.3]. In particular, $L^\infty(G/Q) \subset B \cap L^\infty(G/P)$, and therefore, $\Phi(L^\infty(G/Q)) \subset A' \cap \mathcal{H}_\mu$. As remarked above, $\Phi$ restricts to the Poisson transform on $L^\infty(G/P)$, so $A \subset \left(\mathcal{P}_\nu\langle L^\infty(G/Q)\rangle\right)'$.

Since $A$ is non-trivial, and $A \cap \ell^\infty(\Gamma) = C1$, there exists $a \in A$ and $g, h \in \Gamma$ with $g \neq h$ such that $\langle a\delta_g, \delta_h \rangle \neq 0$. Denoting by $m_\phi \in B(\ell^2(\Gamma))$ the multiplication operator associated to a function $\phi \in \ell^\infty(\Gamma)$, a commutes with $\rho(g)m_{P_\nu\psi}(f)\rho(g^{-1}) = m_{P_\psi\nu}(f)$ for every $g \in \Gamma$ and $f \in C(G/Q)$. The action $G \curvearrowright G/Q$ is strongly proximal, so given $x \in G/Q$ there exits a net $(g_i)_i$ in $\Gamma$ such that $g_i\nu Q \xrightarrow{\text{weak}*} \delta_x$ in $\text{Prob}(G/Q)$. This implies $m_{P_\psi\nu}(f) \xrightarrow{\text{WOT}} m_{P_\nu}(f)$ in $B(\ell^2(\Gamma))$, and therefore $a$ commutes with $m_{P_\nu}(f)$ for every $f \in C(G/Q)$ and $x \in G/Q$. Hence,

$$f(gx)\langle a\delta_g, \delta_h \rangle = \langle am_{P_\nu}(f)\delta_g, \delta_h \rangle = \langle m_{P_\nu}(f)a\delta_g, \delta_h \rangle = \langle a\delta_g, m_{P_\nu}(f)\delta_h \rangle = f(hx)\langle a\delta_g, \delta_h \rangle$$

for all $f \in C(G/Q)$ and $x \in G/Q$, which implies that $gh^{-1}$ lies in $\ker(\Gamma \curvearrowright G/Q)$. But it follows from similar reasoning as in the beginning of the proof that $\Gamma \curvearrowright G/Q$ is faithful. This contradicts $g \neq h$, and therefore completes the proof of the first half.

**Property (T) half.** The assumptions imply that $G$ has property (T) and since $\Gamma$ is a lattice in $G$, $\Gamma$ also has property (T) (see e.g. [BdlHV08, Theorem 1.7.1]). Recall that $\pi$ is a unitary representation of $\Gamma$ weakly contained in $\lambda_G$.

Let $M = \pi(A)\,'$, and let $(M, H_s, J_s, P_s)$ be its standard form in the sense of [Haa73]. By [Haa73, Theorem 3.2], the action $\Gamma \curvearrowright M$ is implemented by a unitary representation $\sigma$ of $\Gamma$ on $H_s$. As usual, we endow $B(H_s)$ with the $\Gamma$-action defined by unitaries $\sigma_g, g \in \Gamma$.

By the Amenability half, there exists a $\Gamma$-invariant state on $M$. Since $M$ has separable predual, a standard argument yields a sequence $(\omega_n)$ of normal states on $M$ such that $\|\omega_n \circ \sigma_g - \omega_n\| = \|g \omega_n - \omega_n\| \xrightarrow{n \to \infty} 0$ for every $g \in \Gamma$.

By [Haa73, Lemma 2.10], the map $\xi \mapsto \omega_\xi|_M$ is a $\Gamma$-equivariant homeomorphism from $P_s$ onto $M_s^+$, where $\omega_\xi$ is the vector state defined by the vector $\xi$. Hence, there exists an almost $\sigma$-invariant sequence in $H_s$, and property (T) implies the existence of a $\sigma(\Gamma)$-invariant unit.
vector $\xi_0$ in $H_\pi$. The vector state $\omega_{\xi_0}$ restricts to a normal $\Gamma$-invariant state on $M$. This completes the proof.

Proof of Proposition 3. Let $\Gamma$ be a nc-just-infinite group.

(i) Assume that $\Gamma$ is not $C^*$-simple. Let $\pi: C^*_\lambda(\Gamma) \to B(H_\pi)$ be a non-faithful representation, and let $I = \ker(\pi)$. In particular, $\pi$ defines a unitary representation of $\Gamma$ which is weakly contained in $\lambda_\Gamma$. Define $A = I + C1$. Then $A$ is a $\Gamma$-invariant unital $C^*$-subalgebra of $C^*_\lambda(\Gamma)$, and $\pi(A) = C1$. Thus, by the assumption, there exists a $\Gamma$-invariant state on $B(H_\pi)$. Since $\pi$ is weakly contained in $\lambda_\Gamma$, there is also a $\Gamma$-invariant state on $B(\ell^2(\Gamma))$ [Bek90, Corollary 5.3]. Restricting to $\ell^\infty(\Gamma)$, this yields an invariant mean on $\ell^\infty(\Gamma)$, and so $\Gamma$ is amenable.

(ii) Assume $\Gamma$ is not icc. Let $p$ be a projection in the center of the group von Neumann algebra $L\Gamma$ such that both $p$ and $p^\perp$ are non-zero. Then, $A = Cp \oplus Cp^\perp$ is a non-trivial $\Gamma$-invariant unital $C^*$-subalgebra of $C^*_\lambda(\Gamma)$, and $A' = pB(\ell^2(\Gamma))p \oplus p^\perp B(\ell^2(\Gamma))p^\perp$. The map $a \mapsto p^*ap + p^\perp ap^\perp$ from $B(\ell^2(\Gamma))$ to $A'$ is normal, $\Gamma$-equivariant, and ucp. Hence, we obtain a $\Gamma$-invariant normal state $B(\ell^2(\Gamma))$, which implies that $\Gamma$ is finite. This is a contradiction as nc-just-infinite groups are by definition assumed to be infinite.

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