Squares of White Noise, $SL(2, \mathbb{C})$ and Kubo – Martin – Schwinger States

D.V. Prokhorenko *

February 5, 2008

Abstract

We investigate the structure of Kubo — Martin — Schwinger (KMS) states on some extension of the universal enveloping algebra of $SL(2, \mathbb{C})$. We find that there exists a one-to-one correspondence between the set of all covariant KMS states on this algebra and the set of all probability measures $d\mu$ on the real half-line $[0, +\infty)$, which decrease faster than any inverse polynomial. This problem is connected to the problem of KMS states on square of white noise algebra.

*Institute of Spectroscopy, RAS 142190 Moskow Region, Troitsk
1 Introduction.

The basis object in quantum field theory is $S$-matrix [1,2], which describes the scattering problem at infinite times. But in a number of cases one is interested in the behaviour of quantum systems at large, but finite intervals of time. The general method for studying dynamical problem in quantum field theory is the method of stochastic limit developed by L. Accardy, I.V. Volovich and others [3]. This method leads to quantum stochastic equation.

In [4] the authors have studied quantum stochastic equations of the form

$$i \frac{d}{d\tau} U_\tau = (a((b^+_{\tau})^2 + b^2_{\tau}) + c b^+_{\tau} b_{\tau}) U_\tau,$$

where $a, b$ are real numbers and $\{b^+_{\tau}, b_{\tau}\}$ — quantum white noise, i.e. the pair of operator-valued distributions satisfying the canonical commutation relations:

$$[b_{\tau}, b_{\tau'}] = [b^+_{\tau}, b^+_{\tau'}] = 0,$$

$$[b_{\tau}, b^+_{\tau'}] = \delta(\tau - \tau').$$

These equations contains the squares of white noise. After the renormalization suggested in [4] these squares generate so-called square of white noise (SWN) algebra [5,6]. KMS states was introduced in [7]. KMS states on SWN algebra were considered in [8], where it was found the example of KMS state on SWN algebra. Our main goal is to clarify the structure of KMS states on SWN algebra. After discretising suggested in section 2 our problem will reduce to analogous problem for $U(\mathfrak{sl}(2, \mathbb{C}))$, the universal enveloping algebra.
of $\mathfrak{sl}(2, \mathbb{C})$, where $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of $SL(2, \mathbb{C})$. In the present paper we give complete description of KMS states on some extension of $\mathfrak{sl}(2, \mathbb{C})$. We find that there exists a one-to-one correspondence between the set of all KMS states on this algebra and the set of all probability measures $d\mu$ on the real half-line $[0, +\infty)$, which decrease faster than any inverse polynomial. Our main result contained in Theorem 1.

2 Problem setup.

In this section the necessary notion are introduced and the main result (Theorem 1) is formulated. Let $\Gamma$ be a space of piecewise continuous functions on $[0, 1]$. $\Gamma$ is a Hilbert algebra with respect to the complex conjugation and the scalar product of the form

$$\langle f | g \rangle = \int_0^1 f^*(x)g(x)dx.$$  \hfill (3)

Let $\mathcal{B}$ be a $\ast$-algebra generated by

$$B(f), B^+(f), N(f), \quad f \in \Gamma$$

with the following relations:

a) $B(f)$ is an antilinear functional of $f$, $B^+(f)$ is a linear functional of $f$, $N(f)$ is a linear functional of $f$.

b)

$$[B(f), B^+(g)] = 2N(f^*g),$$

$$[B(f), N(g)] = 2B(f^*g),$$

$$[B^+(f), N(g)] = 2B(f^*g),$$
The involution is defined by the formulas

\[
(N(f))^* = N(f^*),
\]
\[
(B(f))^* = B^+(f).
\]  

(5)

Algebra $\mathcal{B}$ is called square of white noise algebra [5,6].

Let $\mathcal{E}$ be a $\star$-algebra with a unit. The state $\tau$ on $\mathcal{E}$ is a positive linear functional satisfying the following condition: $\tau(1) = 1$.

Let $\mathcal{E}$ be a $\star$-algebra, $\beta$ be a real positive number and $V_t$ ($t \in \mathbb{R}$) be an one-parameter group of its automorphisms. We say, that the linear functional $\tau$ on $\mathcal{E}$ is a KMS-functional with respect the pair \{\beta, V_t\} if $\forall A, B \in \mathcal{E}$ there exists continuous function $F_{AB} : S_\beta \to \mathbb{C}$ which is holomorphic inside the strip $S_\beta = \{z \in \mathbb{C} \mid 0 \leq \text{Im}z \leq \beta\}$ such that for any real $t$

\[
F_{AB}(t) = \tau(AV_t(B))
\]  

(6)

and

\[
\tau(V_t(A)B) = F_{BA}(t + i\beta).
\]  

(7)

Let $U_t$, $t \in \mathbb{C}$ be an one-parameter group of automorphisms of $\mathcal{B}$, defined by the following relations:

\[
U_t(B^+(f)) = B^+(fe^{it\omega}),
\]
\[
U_t(B(f)) = B(fe^{it\omega}),
\]
\[
U_t(N(f)) = N(f).
\]  

(8)
Here $\omega(x)$ is a real-valued positive continuous function on $[0, 1]$.

Our aim is to classify all KMS states on $\mathcal{B}$ with respect the pair $\{\beta, U_t\}$.

The discrete variant of our problem is to classify all KMS states on the algebra $\mathcal{C}^m$, generated by generators $B_i, B_i^+, N_i$, satisfying the following relations:

$$
[B_i, B_j] = [B_i^+, B_j^+] = [N_i, N_j] = 0,

[B_i, B_j^+] = 2\delta_{i,j}N_i,

[B_i, N_j] = 2\delta_{i,j}B_i,

i, j = 1, ..., m.
$$

(9)

The KMS condition has the form:

$$
\tau(AB) = \tau(BU_t\beta(A)), \ A, B \in \mathcal{C}^m,
$$

(10)

here $U_t$ acts on generators as

$$
U_t(B_i^+) = B_i^+ e^{it\omega_i},

U_t(B_i) = B_i e^{-it\omega_i},

U_t(N_i) = N_i
$$

$$
\omega_i > 0, \ i, j = 1, ..., m.
$$

(11)

In the first instance we consider the case of the algebra $\mathcal{C}^m$ for $m = 1$.

Remind that $\mathfrak{sl}(2, \mathbb{C})$, the Lee algebra of $SL(2, \mathbb{C})$ is generated by generators $X, Y, H$ with the following relations:

$$
[X, Y] = H,

[X, H] = -2X,

[Y, H] = 2Y.
$$

(12)
Using substitution
\[
\frac{1}{\sqrt{2}} B_i = Y_i,
\]
\[
\frac{1}{\sqrt{2}} B_i^+ = -X_i,
\]
\[H_i = N_i\] (13)

we see that the algebra \( C^1 \) coincide with the universal enveloping algebra of \( \mathfrak{sl}(2, \mathbb{C}) \) with the involution of the form
\[
H^* = H, \\
X^* = -Y. \tag{14}
\]

Denote by \( P \) the set of all continuous complex-valued function on \( \mathbb{R} \) which increase slowly than some polynomial at infinity. Let \( a \) be a real number. Denote by \( T_a \) an operator acting in the space \( P \) as follows
\[
T_a : f(x) \mapsto (T_a f)(x) = f(x - a). \tag{15}
\]

**Definition.** Denote by \( \mathcal{A} \) a \(*\)-algebra, generated by generators \( X, Y, N_F, F \in P \) which satisfy the following relations:
\[
N_{\lambda F + \mu G} = \lambda N_F + \mu N_G, \quad \lambda, \mu \in \mathbb{C}, \\
N_{FG} = N_F N_G. \tag{16}
\]

and
\[
[X, Y] = N_x, \\
X N_F = N_{T_a F} X, \\
Y N_F = N_{T_{-a} F} Y. \tag{17}
\]
An involution in $\mathcal{A}$ is defined by the following rules:

$$N_F^* = N_{F^*},$$

$$X^* = -Y.$$  \hspace{1cm} (18)

We can embed $U(\mathfrak{sl}(2, \mathbb{C}))$, the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$ into $\mathcal{A}$ if we identify each element $X^nY^mH^k$ from $U(\mathfrak{sl}(2, \mathbb{C}))$ with the element $X^nY^mN_x^k$ from $\mathcal{A}$.

We will see below that $\mathcal{A}$ has enough much representation, so $\mathcal{N}$ is isomorphic to $\mathcal{P}$. Denote by $H$ the element $N_x$, $H = N_x$.

There exists one-parameter group of automorphism $U_t, t \in \mathbb{C}$ of $\mathcal{A}$ acts on generators as follows

$$U_t(X) = e^{it}X, \ U_t(Y) = e^{-it}Y, \ U_t(N_F) = N_F.$$  \hspace{1cm} (19)

The $\star$-subalgebra of $\mathcal{A}$ generated by all elements of the form $N_F$ where $F \in \mathcal{P}$, is called the Cartan subalgebra and is denoted by $\mathcal{N}$.

**Proposition.** The following expression

$$\rho(N_F) = \int_{-\infty}^{+\infty} d\sigma(x)F(x)$$  \hspace{1cm} (20)

define a state $\rho$ on $\mathcal{N}$. Here $d\sigma$ is an arbitrary probability measure on real line which decrease faster than any inverse polynomial. Conversely, for any state $\rho$ on $\mathcal{N}$ there exists probability measure $d\sigma$ on line, which decrease faster than any inverse polynomial at infinity such that for all $F \in \mathcal{P}$ (20) holds.

**Proof.** Let $\varphi$ be a positive functional on the space $C(\mathbb{R})$ of bounded
continuous functions on $\mathbb{R}$. It follows from the Riesz — Markov theorem, that there exists a nonnegative measure $d\mu$, such that

\begin{align*}
a) & \int_{-\infty}^{+\infty} d\mu < \infty, \\
b) & \varphi(g) = \int_{-\infty}^{+\infty} g(x) d\mu(x)
\end{align*}

for all continuous functions $g(x)$ on $\mathbb{R}$ such that $g(x) \to 0$ as $x \to \pm \infty$.

Let us consider functionals $\psi$ on $C(\mathbb{R})$, $n = 1, 2, 3..., $ defined as follows

$$
\psi(F) = \rho(N(1+x^2)_n N_F). \tag{21}
$$

So there exist the nonnegative measures $d\mu_n$, $\int d\mu_n < \infty$ such that

$$
\rho((1 + H^2)^n N_F) = \int_{-\infty}^{+\infty} F(x) d\mu_n(x) \tag{22}
$$

for all continuous function $F(x)$ such that $F(x) \to 0$ as $x \to \pm \infty$.

So for any functions $F(x)$ such that

$$
|F(x)| \leq C(1 + x^2)^{n-1} \tag{23}
$$

for some constant $C$, we have

$$
\rho(N_F) = \int_{-\infty}^{+\infty} d\sigma_n(x) F(x), \tag{24}
$$

where

$$
d\sigma_n = \frac{d\mu_n}{(1 + x^2)^n}. \tag{25}
$$

It is easy to see that $d\sigma_n$ does not depend of $n$. It follows from the representation (25) that $d\sigma_n$ tends to zero faster then any inverse polynomial at infinity. The proposition is proved.
The proof uses the Riesz – Markov theorem. Note that each positive linear functional on the space of continuous function on compact Hausdorff space is continuous.

Let us define characteristic functional of $\rho \chi_{\rho}(t)$ by the following formula

$$\chi_{\rho}(t) = \rho(N_{e^{itx}}).$$

(26)

**Theorem 1.** A state $\rho$ on $\mathcal{N}$ extends to a KMS state on $\mathcal{A}$ with respect the pair $\{\beta, U_t\}$ ($\beta > 0$) if and only if its characteristic functional $\chi_{\rho}(t)$ has the form

$$\chi_{\rho}(t) = m_1 + m_2 \frac{1}{1 - e^{-\beta + 2it}} \int_{0}^{+\infty} d\sigma(\lambda) e^{it\lambda}$$

(27)

for some probability measure $\sigma$ on $(0, +\infty)$ which decrease faster than any inverse polynomial. Here $m_1, m_2$ are arbitrary real numbers such that $m_1 \geq 0, m_2 \geq 0, m_1 + m_2 = 1$. If an extension exists then it is unique.

3 Beginning of the proof.

Let us show that the part ”if” of the theorem holds. In order to construct $\rho$ we will investigate irreducible representations of Lee algebra $\mathfrak{sl}(2, \mathbb{C})$, or more precisely modules over $\mathcal{A}$. All irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ with involution (14) have classified in [6] (see also [8]). Unitary representations of the Lee group $SL(2, \mathbb{C})$ have studied in [9]. We extend this construction to the case of the algebra $\mathcal{A}$

**Definition.** Let $\lambda$ be a real positive number. $V_{\lambda}$ is a module over $\mathcal{A}$
spanned on vectors $|\lambda, n\rangle$, $n = 0, 1, ...$ defined by the following representation of generators $X, Y, N_F$ on $\{|\lambda, n\rangle\}$

\[
\begin{align*}
\hat{Y}|\lambda, p + 1\rangle &= -(\lambda + p)|\lambda, p\rangle, \\
\hat{X}|\lambda, p\rangle &= (p + 1)|\lambda, p + 1\rangle, \\
\hat{Y}|\lambda, 0\rangle &= 0, \\
\hat{N}_F|\lambda, p\rangle &= F(\lambda + 2p)|\lambda, p\rangle, \\
p &= 0, 1, ...
\end{align*}
\]

(28)

**Lemma 1.** For each $\lambda > 0$ there exists an unique scalar product on $V_{\lambda}$ (defined up to arbitrary positive multiplier) such that $\hat{X} = -\hat{Y}^*, \hat{N}_F^* = \hat{N}_F$ with respect to this scalar product.

Proof of this lemma is standard, see[6]

**Definition.** $V_0$ is a module over $A$ spanned on vector $|0, 0\rangle$ such that

\[
\langle 0, 0|0, 0\rangle = 1,
\]

(29)

defined by the following representation of generators $X, Y, N_F$ on $|0, 0\rangle$

\[
\hat{X}|0, 0\rangle = \hat{Y}|0, 0\rangle = \hat{N}_F|0, 0\rangle = 0.
\]

(30)

**KMS states $\rho_\lambda$.** Let $\lambda \in \mathbb{R}$, $\lambda > 0$. Consider the completion $\tilde{V}_\lambda$ of module $V_\lambda$ with respect to a scalar product, defined in the previous Lemma. Consider the trace class operator $\rho_\lambda = e^{-\beta \tilde{H}}$ acting in $\tilde{V}_\lambda$, where $\tilde{H}$ is an unique self-adjoint extension of $H$ from $V_\lambda$ and $Z = \text{tr}\{e^{-\beta \tilde{H}}\}$. Operator $H$ is essentially self-adjoint because $V_\lambda$ contains the basis of eigenvectors of $H$. 

10
see for example [10]

Define the state \( \rho_{\lambda} \) on \( \mathcal{A} \) by the following formula

\[
\rho_{\lambda}(a) = \lim_{\mu \to +\infty} \text{tr}\{\hat{a} E_\mu \rho\},
\]

(31)

where \( \{E_\mu\} \) is a spectral family of \( \dot{H} \). It is easy to prove, that this expression is well defined.

**Lemma 2.** The following equality holds

\[
\rho_{\lambda}(N_{e^{it\lambda}}) = e^{it\lambda} \frac{1 - e^{-\beta}}{1 - e^{(2it - \beta)}}.
\]

(32)

**Proof.** Direct calculation.

Consider a state \( \rho_0 \) on \( \mathcal{A} \) defined by the formula,

\[
\rho_0(X^nY^nN_F) = \delta_{m,0}\delta_{n,0}F(0),
\]

\[
n, m = 0, 1, ..., \quad (33)
\]

and define a state \( \rho \) on \( \mathcal{A} \) of the form

\[
\rho(a) = m_1 \rho_0(a) + m_2 \int_{0}^{+\infty} d\sigma(\lambda) \rho_{\lambda}(a),
\]

\[
m_1 \geq 0, m_2 \geq 0,
\]

\[
m_1 + m_2 = 1,
\]

(34)

where \( d\sigma \) is a probability measure, which decreases faster than any inverse polynomial. By using the definition of \( V_{\lambda} \) and the scalar product on it one can see that \( \forall a \in \mathcal{A} \) \( \rho_{\lambda}(a) \) increase slowly than some polynomial at infinity. So the integral in the right hand side exists. It follows from lemma 2 that the characteristic functional of the restriction of \( \rho \) on \( \mathcal{N} \) has a needed form
So the part "if" is proved.

4 Decomposition of the state $\rho$ into the direct integral.

Now we begin to prove the part "only if". Let $\rho$ be an KMS functional on $\mathcal{A}$.

Let us make the GNS construction for the state $\rho$. We get a Hilbert space $\mathcal{H}$, the dense subspace $\mathcal{D}$, the representation of $\mathcal{A}$ by means operators, acting from $\mathcal{D}$ to $\mathcal{D}$, cyclic vector $|\Omega\rangle \in \mathcal{D}$ i.e. the vector such that $\hat{A}|\Omega\rangle = \mathcal{D}$. We get also $\rho(a) = \langle \Omega|\hat{a}|\Omega\rangle$. For each $a \in \mathcal{D}$ by $|a\rangle$ denote the vector $\hat{a}|\Omega\rangle$.

**Lemma 3.** There exists an unique projector-valued measure $dE$ in $\mathcal{H}$ such that for all $|f\rangle, |g\rangle \in \mathcal{D}$ and a continuous function $F(\lambda)$, which increase slowly than some polynomial at infinity.

$$\langle f|\hat{N}_F|g\rangle = \int_{-\infty}^{+\infty} F(\lambda)\langle f|dE(\lambda)|g\rangle,$$  \hspace{1cm} (35)

and $\langle f|dE(\lambda)|g\rangle$ decrease faster than any inverse polynomial.

**Proof.** The functional $\rho$ is positive. So for all $f \in \mathcal{D}$ the functional $F \to \langle f|\hat{N}_F|f\rangle$ is positive. Therefore there exists a Borelian measure $d\mu_{f,f}$ which decrease faster then any inverse polynomial such that

$$\langle f|\hat{N}_F|f\rangle = \int_{-\infty}^{+\infty} F(\lambda)d\mu_{f,f}(\lambda).$$  \hspace{1cm} (36)
Using polarization identity we can find the measure \( d\mu_{f,g} \), such that

\[
\langle f|\hat{N}_F|g \rangle = \int_{-\infty}^{+\infty} F(\lambda)d\mu_{f,g}(\lambda).
\] (37)

The measure \( d\mu_{f,g} \) is a linear functional of \( g \) and an antilinear functional of \( f \).

Now, for each bounded Borelian function \( F \) define the following sesquilinear form

\[
\mathcal{N}_F(f,g) := \int_{-\infty}^{+\infty} F(\lambda)d\mu_{f,g}.
\] (38)

It follows from this representation that for each bounded Borelian function \( F \) there exists bounded operator in \( \mathcal{H} \) which we denote by \( \hat{N}_F \) such that

\[
\mathcal{N}(f,g) = \langle f|\hat{N}_F|g \rangle
\] (39)

and

\[
\|\hat{N}_F\| \leq 4 \sup_{x \in \mathbb{R}} |F(x)|.
\] (40)

Now for each \( f, g \in \mathcal{H} \) (not necessary in \( \mathcal{D} \)) we can define the measure \( d\mu_{f,g} \) by the following formula

\[
\mu_{f,g}(B) = \langle f|\hat{N}_{\chi_B}|g \rangle;
\] (41)

where \( \chi_B \) is an indicator of Borelian set \( B \).

Let us prove that the measure \( d\mu_{f,g} \) is \( \sigma \)-additive measure. Let \( B_n, n = 1, 2, ... \) the sequence of Borelian sets, such that

\[
B_1 \subset B_2 \subset ...
\] (42)
Let \( B = \bigcup_{n} B_n \). \( \forall \varepsilon > 0, f, g \in \mathcal{H} \) there exist \( N > 0, \tilde{f}, \tilde{g} \in \mathcal{D} \) such that

\[
|\langle f | \hat{N} \chi_{B_n} | g \rangle - \langle \tilde{f} | \hat{N} \chi_{B_n} | \tilde{g} \rangle| < \varepsilon,
\]

\[
|\langle \tilde{f} | \hat{N} \chi_{B_n} | \tilde{g} \rangle - \langle f | \hat{N} \chi_{B_n} | g \rangle| < \varepsilon,
\]

\[
|\langle \tilde{f} | \hat{N} \chi_{B_n} | \tilde{g} \rangle - \langle \tilde{f} | \hat{N} \chi_{B_n} | g \rangle| < \varepsilon,
\]

if \( n > N \). We have \(|\langle f | \hat{N} \chi_{B_n} | g \rangle - \langle \tilde{f} | \hat{N} \chi_{B_n} | g \rangle| < 3\varepsilon\), therefore \( \mu_{f,g}(B_n) \to \mu_{f,g}(B) \), and \( \mu \) is \( \sigma \)-additive. Now using approximation of Borelian function \( F \) by simple function we can prove that (37) is valid for all \( f, g \in \mathcal{H} \) and bounded Borelian function \( F \).

Let \( F, G \) — be continuous functions with compact support. We have

\[
\langle f | \hat{N}_F \hat{N}_G | g \rangle = \langle f | \hat{N}_{FG} | g \rangle.
\]  

(44)

We have proved the representation (37) for all \( f, g \in \mathcal{H} \) and bounded Borelian function \( F \). From this fact it follows that formula (44) is valid for all bounded Borelian functions \( F, G \). So we have constructed projector-valued measure \( E(B) = \hat{N}_{B} \) such that for all \( f, g \in \mathcal{D} \) and \( F(\lambda) \in \mathcal{P} \)

\[
\langle f | \hat{N}_F | g \rangle = \int_{-\infty}^{+\infty} F(\lambda) \langle f | dE(\lambda) | g \rangle.
\]  

(45)

The lemma is proved

**Remark.** For any function \( F \) which increase slowly than some polynomial the following spectral decomposition \( F(\hat{H}) = \int_{-\infty}^{+\infty} F(\lambda) dE(\lambda) \) define a normal operator in \( \mathcal{H} \), which extends \( \hat{N}_F \).

**Lemma 4.** \( \text{Sp}(\hat{H}) \), the spectrum of \( \hat{H} \) lies in \([0, +\infty)\).
**Proof.** Let $f$ be a continuous function with a compact support. Let us compute $-\rho(XN_fY)$, using the KMS property of the state $\rho$.

\[ -\rho(XN_fY) = -\rho(\hat{N}_fYU_{i\beta}(X)). \]  
(46)

But $U_{i\beta}(X) = e^{-\beta X}$, therefore

\[ -\rho(XN_fY) = e^{-\beta}\rho(N_fN_x) - e^{-\beta}\rho(XN_{T_{-2}f}Y), \]  
(47)

or

\[ -\rho(XN_{T_{-2}f}Y) = e^{\beta}\rho(XN_fY) - \rho(N_fN_x). \]  
(48)

Substituting $f(x)$ for $f(x + 2k)$ in previous equality we get

\[ -\rho(XN_{T_{-2}2k}fY) = -e^{\beta}\rho(XN_{T_{-2}2k}fY) - \rho(N_{T_{-2}2k}fN_x), \]  
\[ k = 0, 1, 2... \]  
(49)

Let $f$ be a continuous function with a compact support $\text{supp} f \subset (-\infty, 0)$. Suppose that $\rho(N_f) = 0$ (this fact will be proven below). Then

\[ \forall |a\rangle, |b\rangle \in D\langle a|\hat{N}_f|b\rangle = 0. \]  
(50)

Indeed

\[ \langle a|\hat{N}_f|b\rangle = \rho(a^*\hat{N}_f b) = \rho(U_{-i\beta}(b)a^*\hat{N}_f). \]  
(51)

Using Schwarz inequality

\[ |\langle a|\hat{N}_f|b\rangle| \leq |\rho((U_{-i\beta}(b))a^*aU_{-i\beta}(b))|^{\frac{1}{2}}\rho(N_f^*N_f)^{\frac{1}{2}} = 0. \]  
(52)
It follows from (35) that $\text{Sp}(\tilde{H}) \subset [0, +\infty)$. So suppose that there exists a segment $[a, b] \subset (-\infty, 0)$ such that $\langle \Omega | E([a, b]) | \Omega \rangle \rangle \neq 0$. Thus there exists a continuous function $f$ with compact support $\text{supp} f \subset (-\infty, 0)$, $f \geq 0$ such that

$$\rho(N_f N_x) < 0.$$  \hspace{1cm} (53)

It follows from positivity of $f$ that:

$$\rho(N_{T_{-2k}f} N_x) \leq 0, \quad k = 1, 2, 3...$$

Moreover, it follows from positivity $f$ that:

$$- \rho(X N_{T_{-2k}f} Y) \geq 0, \quad k = 1, 2, 3...$$

So, we get

$$- \rho(X N_{T_{-2k-2}f} Y) \geq -e^\beta \rho(X N_{T_{-2k}f} Y),$$

$$k = 1, 2...$$

$$-\rho(X N_{T_{-2}f} Y)) > 0.$$ \hspace{1cm} (54)

From (39) we get that $-\rho(X N_{T_{-2k}f} Y)$ tends exponentially to infinity then $k \to +\infty$, but $- \sum_{k=1}^{+\infty} \rho(X N_{T_{-2k}f} Y) < +\infty$. This contradiction concludes the proof.

Decomposition of the state $\rho$ into the direct integral.

Let $C'$ be the ring of all finite linear combination of elements of the form $e^{n\beta} e^{i\pi r}, \quad r \in Q, \quad n \in Z$ with rational coefficients. It is obvious that this ring contains only countable number of elements.

**Definition.** Let $A = \{\eta_i\}, \quad i \in Z$ some countable set of continuous
function with compact support. Let $C$ be a set of functions, which consist of all elements of the form

$$
\prod_{i=1}^{n} \eta_i(x - 2l_1) \ldots \eta_i(x - 2l_n),
$$

$$
n = 0, 1, 2, \ldots, l_i \in \mathbb{Z}.
$$

(55)

It is obvious that $C$ is countable. Let us consider the set of all elements of the form:

$$
P(X, Y, H) N_f N_{e^{i\pi r z}}, \ r \in \mathbb{Q}, f(x) \in C,
$$

(56)

where $P(X, Y, H)$ is a polynomial on its arguments with coefficient from $C'$. This set is countable. The algebra $A^*$ over $C'$, by definition, consists of all linear combination of elements of the form (41) with rational coefficients.

Let $U := N_{e^{i\pi z}}$ be an unitary element from $A^*$. It is clear that this element is a central element of $A^*$. We get that $\tilde{U} = e^{i\pi \tilde{H}}$ be an unitary operator, which acts in $\mathcal{H}$. Let

$$
\tilde{U} = \int_0^{2\pi} e^{i\varphi} dP_{\varphi}
$$

(57)

be its spectral decomposition. \forall a \in A^* we put by definition $d\eta(a) = \langle \Omega | dP_{\varphi} \hat{a} | \Omega \rangle$, and $d\mu = \langle \Omega | dP_{\varphi} | \Omega \rangle$.

**Lemma 5.** For almost all $\varphi \in [0, 2\pi)$ there exists a unique KMS-state $\tau_{\varphi}$ on $A^*$ such that

$$
d\eta(a) = \tau_{\varphi}(a)d\mu(a).
$$

(58)

**Proof.** Obviously we have

$$
P_{[\varphi_1, \varphi_2]} = \sum_{n=-\infty}^{n=+\infty} E_{[\frac{\varphi_1+2n}{2}, \frac{\varphi_1+2n}{2}]}.
$$

(59)
Let us consider the following measures
\[
d\eta(a) = \langle \Omega | (dP_{\varphi}) \hat{a} | \Omega \rangle,
\]
\[
d\mu = \langle \Omega | (dP_{\varphi}) | \Omega \rangle.
\]  
(60)

Let us prove that the measure \(d\eta(a)\) is absolutely continuous measure with respect the measure \(d\mu\). Indeed let \(B\) be a Borelian set such that \(\mu(B) = 0\).

We have
\[
|\eta(a)(B)| = |\langle \Omega | P(B)\hat{a} | \Omega \rangle| \leq \langle \Omega | (\hat{a}\hat{a}^*) | \Omega \rangle^{\frac{1}{2}} \langle \Omega | (P(B)P(B)) | \Omega \rangle^{\frac{1}{2}} = \rho(\hat{a}\hat{a}^*)^{\frac{1}{2}} \langle \Omega | P(B) | \Omega \rangle^{\frac{1}{2}} \rho(\hat{a}\hat{a}^*)^{\frac{1}{2}} = 0.
\]  
(61)

So, by using the Radon—Nickodym theorem we see that there exists the function \(\tau(a)(\varphi)\) such that
\[
\tau(a)(\varphi) d\mu(\varphi) = d\eta(a).
\]  
(62)

We have
\[
\tau(\lambda a_1 + \mu a_2)(\varphi) = \lambda \tau(a_1)(\varphi) + \mu \tau(a_2)(\varphi).
\]  
(63)

The last equality is valid almost everywhere.

Let us prove that \(\tau(a^*a)(\varphi) \geq 0\) for almost all \(\varphi\) and all \(a \in A^*\). Let \(P(e^{i\varphi})\) be an arbitrary positive trigonometric polynomial. According to the Riesz theorem we find that there exists a trigonometric polynomial \(Q(e^{i\varphi})\) such that
\[
P(e^{i\varphi}) = Q^*(e^{i\varphi})Q(e^{i\varphi}), P(N_{e^{i\varphi}}) = Q^*(N_{e^{i\varphi}})Q(N_{e^{i\varphi}}).
\]  
(64)
Using the spectral decomposition of $e^{i\pi \tilde{H}}$ we see
\[ \int_0^{2\pi} \tau(aa^*)(\varphi)P(e^{i\varphi})d\mu(\varphi) = \int_0^{2\pi} \langle \Omega|a\tilde{a}^*dP(\varphi)|\Omega\rangle P(e^{i\varphi}) = \rho(aa^*P(N_{e^{i\pi x}})) \geq 0. \] (65)

So for almost all $\varphi$ and all $a \in A^*$ $\tau(a^*a)(\varphi) \geq 0$. Note that we use the fact that $A^*$ contains only countable number of elements.

Let us prove that $\tau(\varphi)$ is a KMS-functional. Note that $\rho(P(N_{e^{i\pi x}}))$ is a KMS-functional for an arbitrary trigonometric polynomial $P(N_{e^{i\pi x}})$ i.e.
\[ \rho(ABP(N_{e^{i\pi x}})) = \rho(BU_{i\beta}(A)P(N_{e^{i\pi x}})). \] (66)

Using spectral decomposition for $e^{i\pi \tilde{H}}$ we find:
\[ \int_0^{2\pi} \tau(AB)(\varphi)P(e^{i\varphi})d\mu(\varphi) = \int_0^{2\pi} \tau(BU_{i\beta}(A))(\varphi)P(e^{i\varphi})d\mu(\varphi). \] (67)

$P(e^{i\varphi})$ is an arbitrary trigonometric polynomial. So for almost all $\varphi$ and all $A, B \in A^*$ we find:
\[ \tau(AB)(\varphi) = \tau(BU_{i\beta}(A))(\varphi). \] (68)

The lemma is proved.

The proof of this lemma is like to the proof of the von Neumann spectral theorem [11].

Let us make now for all $\varphi$ from the previous lemma the GNS construction for $\tau_\varphi$. We get:

a) The Hilbert space $\mathcal{H}_\varphi$,

b) The dense subspace $\mathcal{D}_\varphi$ over the ring $C'$.

c) The representation $\hat{\cdot}$ of $A^*$ in $\mathcal{H}_\varphi$ by means $C'$-linear operator, acting from

19
$\mathcal{D}_\varphi$ to $\mathcal{D}_\varphi$.

d) The vector $|\Omega_\varphi\rangle \in \mathcal{D}_\varphi$ such that $\mathcal{A}^*|\Omega_\varphi\rangle = \mathcal{D}_\varphi$.

**Definition.** The algebra $\mathcal{A}^{**}$ is an algebra generated by all elements of the form

$$P(X,Y,N_x)N_fN_{ei\pi r x},$$

$$r \in Q.$$  \hspace{1cm} (69)

Here $P(X,Y,N_x)$ is a polynomial, and $f$ is an element of $C$ of the form

$$f(x) = \prod_{i=1}^{m} \eta_i(x - 2k_i)$$

$$m = 1,2...$$  \hspace{1cm} (70)

Let $\mathcal{D}_\varphi'$ by definition be subspace of $\mathcal{D}_\varphi$ of the form $\mathcal{D}_\varphi' = \hat{\mathcal{A}}^*|\Omega_\varphi\rangle$.

**Lemma 6.** We can chose the set $A$ such that $\mathcal{D}_\varphi'$ is a dense subspace of $\mathcal{D}_\varphi$ (for almost all $\varphi$).

**Proof.** Let $a \in \mathcal{A}^*$ and be a $\eta_n(x) \in B$ sequence of real-valued continuous function such that $\text{supp}\eta_n(x) \in [-2n,2n]$ and $0 \leq \eta_n(x) \leq 1$, $\eta_n(x)|_{[-n,n]} = 1$. Let us prove that $\mathcal{D}_\varphi'$ is a dense subset in $\mathcal{D}_\varphi$. We have:

$$\|(\hat{a}\hat{N}_{\eta_n}(x) - \hat{a})|\Omega_\varphi\rangle\| = \tau(a^*a(N_{\eta_n}(x) - 1)^2)(\varphi) \leq$$

$$\leq \tau((a^*a)^2)^{1/2}\tau((N_{\eta_n}(x) - 1)^4)^{1/2}(\varphi).$$  \hspace{1cm} (71)

But $\rho((N_{\eta_n}(x) - 1)^4)^{1/2} \to 0$. So there exists subsequence $\eta_{n_k}$ of the sequence $\eta_n(x)$ such that

$$\rho((N_{\eta_{n_k}}(x) - 1)^4) \leq \frac{1}{2^n}.$$  \hspace{1cm} (72)
Therefore the following series
\[ \sum_{n=1}^{\infty} \int \tau((N_{\eta_n}(x) - 1)^4) \varphi \, d\mu(\varphi) \] (73)
converges. So by using B. Levi theorem we find that \( \tau((\eta_n'(H) - 1)^4)^{1/2}(\varphi) \to 0 \) for almost all \( \varphi \). This fact and inequality (71) implies that \( \|(a N_{\eta_n}(x) - a) |\Omega_\varphi|\| \to 0 \). The lemma is proved.

The following lemma holds.

**Lemma 7.** For all \( \varphi \) from lemma 5 there exists the spectral family
\( F_{\pi^{-1}\varphi+2n} \) in \( \mathcal{H}_\varphi \):
\[
F_{\pi^{-1}\varphi+2n} F_{\pi^{-1}\varphi+2m} = F_{\pi^{-1}\varphi+2n} \delta_{n,m},
\]
\[
\sum_{n=0}^{+\infty} F_{\pi^{-1}\varphi+2n} = I, \quad n, m = 0, 1, 2, \ldots
\] (74)
such that
\[
\langle f | \hat{N}_G | g \rangle = \sum_{n=0}^{+\infty} G(\pi^{-1}\varphi + 2n) \langle f | F_{\pi^{-1}\varphi+2n} | g \rangle. \] (75)
For all \( |f\rangle, |g\rangle \in \mathcal{D}'_\varphi \). Moreover, self-adjoint operator, which acts in \( \mathcal{H}_\varphi \) defined by its spectral decomposition
\[
\hat{H}_\varphi = \sum_{n=0}^{+\infty} (\pi^{-1}\varphi + 2n) F_{\pi^{-1}\varphi+2n} \] (76)
is positive.

**Proof.** Let \( K \) be a smooth function with compact support such that \( N_K \in \mathcal{A}^{**} \). Let \( N \in \mathbb{Z}^+ \) be a number such that \( \text{supp}K \subset [-N, N] \).
Note that the measure

$$\rho(f^*dE_{\pi^{-1}\varphi+2ng}),$$

(77)

where $f, g \in \mathcal{A}^{**}$ is an absolutely continuous measure with respect the measure $d\mu$ because

$$dP_\varphi = \sum_{n=-\infty}^{n=\infty} dE_{\varphi}.$$  

(78)

So there exists the functions $\psi_n(\varphi)[f, g] \in L_1(d\mu)$ such that

$$\rho(f^*dE_{\pi^{-1}\varphi+2ng}) = \psi_n(\varphi)[f, g]d\mu(\varphi).$$

(79)

Note that for almost all $\varphi \psi_n(\varphi)[f, g]$ is a positive sesquilinear form and

$$\tau(\varphi)(f^*g) = \sum_{n=-\infty}^{n=\infty} \psi_n(\varphi)[f, g].$$

(80)

Let $G(e^{i\pi\varphi})$ be an arbitrary trigonometric polynomial. We have

$$\rho(f^*N_KN_{G(e^{i\pi\varphi})}g) = \sum_{n=-N}^{n=N} \int K(\pi^{-1}\varphi + 2n)\psi_n(\varphi)[f, g]G(e^{i\pi\varphi})d\mu(\varphi) =$$

$$= \int \tau(f^*N_Kg)(\varphi)G(e^{i\pi\varphi})d\varphi. \quad (81)$$

So we have

$$\tau(f^*N_Kg)(\varphi) = \sum_{n=-N}^{n=N} K(\pi^{-1}\varphi + 2n)\psi_n(\varphi)[f, g].$$

(82)

Suppose that the function $K_n$ has a support in a small neighborhood of the point $\pi^{-1}\varphi + 2n$, and $K_n(\pi^{-1}\varphi + 2n) = 1$. We have

$$\tau(f^*N_{K_n}g)(\varphi) = \psi_n(f, g).$$

(83)
It follows from this identity that \( \hat{N}_K \) is self-adjoint bounded operator in \( \mathcal{H}_\varphi \).

It follows from (83) that

\[
\tau(f^* N_{K_n} N_{K_n} g)(\varphi) = \psi_n(f, g).
\] (84)

It follows from (84) that \( \hat{N}_{K_n} \) is a projector in \( \mathcal{H}_\varphi \). Let \( F_{\pi^{-1} \varphi + 2n} = \hat{N}_{K_n} \).

One can easily proof using (82) that \( F_{\pi^{-1} \varphi + 2n} F_{\pi^{-1} \varphi + 2m} = 0 \) if \( n \neq m \). The fact, that \( F_{\pi^{-1} \varphi + 2n} = 0 \) if \( n < 0 \) follows from the positivity of \( \bar{H} \) in \( \mathcal{H} \). It follows from (80)

\[
\sum_{n=-\infty}^{n=+\infty} F_{\pi^{-1} \varphi + 2n} = 1.
\] (85)

5 Decomposition \( \mathcal{H}_\varphi \) into the sum of irreducible components.

The following lemma holds.

**Lemma 8.** \( \mathcal{H}_\varphi \) can be decomposed into the direct sum of subspaces \( \mathcal{H}_\varphi^K \), \( k = 0, 1, 2, ... \)

\[
\mathcal{H}_\varphi = \bigoplus_{k=0}^{\infty} \mathcal{H}_\varphi^K
\] (86)

such that

a) For all \( m = 0, 1, 2, ... \) operators \( \hat{X}, \hat{Y}, \hat{N}_F \) (\( F \in A^{**} \)) extends by continuity to bounded operators from \( \text{Ran} F_{\pi^{-1} \varphi + 2m} \) to \( \mathcal{H}_\varphi \). These extensions we will also denote by \( \hat{X}, \hat{Y}, \hat{N}_F \).

c) The following subspaces

\[
\bigcup_m \mathcal{H}_\varphi^m \cap \{\text{Ran} F_{\pi^{-1} \varphi} \oplus ... \oplus \text{Ran} F_{\pi^{-1} \varphi + 2m}\}
\] (87)
are invariant under the action of the operators \( \hat{X}, \hat{Y}, \hat{N}_F \).

d)

\[ \mathcal{H}^n \cap \text{Ran} F_{\pi^{-1} \varphi + 2k} = \emptyset \text{ if } k < n, \]

\[ \hat{X}'(\mathcal{H}^n \cap \text{Ran} F_{\pi^{-1} \varphi + 2k}) = \mathcal{H}^n \cap \text{Ran} F_{\pi^{-1} \varphi + 2k + 2l}, \quad l > 0, \text{ if } k \geq n. \quad (88) \]

\textbf{Proof} We have proved that \( \tau(a)(\varphi) \) is a KMS state on \( \mathcal{A}^{\ast \ast} \) for all \( \varphi \) from \( [0, 2\pi) \setminus A \), where \( \mu(A) = 0 \). Let \( \varphi \) be an element from \( [0, 2\pi) \setminus A \). Let \( n_0 \) be a minimal integer number such that \( F_{\pi^{-1} \varphi + 2n_0} \neq 0 \).

\( D_\varphi \) is a dense subset in \( \text{Ran}(F_{\pi^{-1} \varphi + 2n_0}) \).

Let \( \eta(\lambda) \) be a smooth function such that \( \text{supp}\eta(\lambda) \) is placed at a small neighborhood of the point \( \pi^{-1} \varphi + 2n_0 \), and \( \eta(\pi^{-1} \varphi + 2n_0) = 1 \). So \( \eta(H) = F_{\pi^{-1} \varphi + 2n_0} \). We can think that \( N_\eta \in \mathcal{A}^{\ast \ast} \). Using the Pythagoras theorem we find that \( D_{\pi^{-1} \varphi + 2n_0} := D_\varphi \cap \text{Ran}(F_{\pi^{-1} \varphi + 2n_0}) \) is a dense set in \( \text{Ran}(F_{\pi^{-1} \varphi + 2n_0}) \).

Let us define the following operators \( \tilde{X}, \tilde{Y} \) acting in \( \tilde{\mathcal{A}}^{\ast \ast} D_{\pi^{-1} \varphi + 2n_0} \) according with the following formula:

\[ \tilde{X} = \lim_{n \to \infty} \eta_n(H) \hat{X}, \]

\[ \tilde{Y} = \lim_{n \to \infty} \eta_n(H) \hat{Y}, \]

\[ \tilde{H} = \lim_{n \to \infty} \eta_n(H) \hat{H}. \quad (89) \]

Here \( \eta_n(H) \in B \) — is a sequence of real-valued functions such that

a) \( 0 \leq \eta_n(H) \leq 1, \)

b) \( \text{supp}\eta_n(\lambda) \subset [-2n, 2n], \)

c) \( \eta_n(H)|_{[-n,n]} = 1, \)

d) \( N_{\eta_n} \in \mathcal{A}^{\ast \ast}. \)
To define the limits (89) we need no any topology because \( \eta_n(H) \hat{X}, \eta_n(H) \hat{Y} \) become stabilize on \( \hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} \). Note that the following relation holds

\[
[\hat{X}, \hat{Y}] = \hat{N}_x,
\]

\[
\hat{X} \hat{N}_F = \hat{N}_{T_aF} \hat{X},
\]

\[
\hat{Y} \hat{N}_F = \hat{N}_{T_{-a}F} \hat{Y}.
\] (90)

and

\[
N_F^* = N_{F^*},
\]

\[
X^* = -Y.
\] (91)

Note that \( \forall n \in \mathbb{Z} \) the operators \( \hat{X}, \hat{Y}, \hat{H} \) are the bounded operators from \( F_{\pi^{-1}\varphi+2n} \cap \hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} \) to \( \mathcal{H}_\varphi \). The proof of this fact is similar to derivation of the formula for scalar product on \( V_\lambda \). So we can extend the operators \( \hat{X}, \hat{Y}, \hat{H} \) to the operators acting in \( \hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} \) with invariant domain:

\[
\text{Lin}\{ \bigcup_n \text{Ran} F_{\pi^{-1}\varphi+2n} \cap \hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} \}
\]

It is easy to see that

\[
\hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} = \bigcup_n \hat{X}^n \text{Ran} F_{\pi^{-1}\varphi+2n_0}.
\] (92)

Note that the formulas (89) defines operators \( \hat{X}, \hat{Y}, \hat{H} \) acting in \( \hat{A}^{\ast \ast}\text{Lin}\{ \bigcup_n (\text{Ran} F_{\pi^{-1}\varphi+2n} \cap \mathcal{D}) \} \). Denote by \( \mathcal{H}_\varphi^{2n_0} \) the space \( \hat{A}^{\ast \ast}D_{\pi^{-1}\varphi+2n_0} \).

Let us prove that for all \( n = 1, 2, \ldots \) the operators \( \hat{X}_n \) are the bounded operators from \( \text{Ran} F_{\pi^{-1}\varphi+2n_0+2} \cap \mathcal{D}_\varphi \) to \( \mathcal{H}_\varphi \). Let \( \psi \in \text{Ran} F_{\pi^{-1}\varphi+2n_0+2} \). We
can represent $\psi$ as a sum:

$$\psi = f_1 + f_2,$$

$$f_1 \in D_\varphi \cap \mathcal{H}_\varphi^{n_0} \cap F_{\pi^{-1}\varphi+2n_0+2},$$

$$f_2 \in D_\varphi \cap F_{\pi^{-1}\varphi+2n_0+2}. \quad (93)$$

For all $\varepsilon > 0$ we can find decomposition (93) such that the projection of the vector $f_2$ to the space $\mathcal{H}_\varphi^{n_0} \cap F_{\pi^{-1}\varphi+2n_0+2}$ has a norm which is less than $\varepsilon$. So we can think $\|f_1\| \leq 2\|\psi\|$, $\|f_2\| \leq 2\|\psi\|$. Let us calculate $\langle f_2 | \tilde{Y}^n \tilde{X}^n | f_2 \rangle$.

For all $n \in \mathbb{Z}^+$ we will prove by induction there exists constant $C_n$ such that

$$\langle f_2 | \tilde{Y}^n \tilde{X}^n | f_2 \rangle \leq C_n \| f_2 \|^2. \quad (94)$$

We have

$$\langle f_2 | \tilde{Y}^{n+1} \tilde{X}^{n+1} | f_2 \rangle =$$

$$= \langle f_2 | \tilde{Y}^n \tilde{X}^{n+1} \tilde{Y} | f_2 \rangle + \sum_{i=0}^{n+1} \langle f_2 | \tilde{Y}^n \tilde{X}^i [\tilde{Y}, \tilde{X}] \tilde{X}^{n-i} | f_2 \rangle \quad (95)$$

The second term in the right hand side of last equality is equal to

$$C \langle f_2 | \tilde{Y}^n \tilde{X}^n | f_2 \rangle \quad (96)$$

for some constant $C$ and we must to estimate the first term $\langle f_2 | \tilde{Y}^n \tilde{X}^{n+1} \tilde{Y} | f_2 \rangle$.

Note that $\tilde{Y} | f_2 \rangle \in \text{Ran} \in F_{\pi^{-1}\varphi+2n_0}$ and there exists the constant $C'$ such that $\| \tilde{Y} | f_2 \rangle \| \leq \| f_2 \|$. We have proven that $\forall n = 0, 1, 2...$ the operators $\tilde{X}, \tilde{Y}, \tilde{H}$ are the bounded operators on $\text{Ran} F_{\pi^{-1}\varphi+2n_0} \cap A^{\text{ss}} D_{\pi^{-1}\varphi+2n_0}$, $n = 0, 1, 2...$.

So there exists a constant $C''$ such that $\langle f_2 | \tilde{Y}^n \tilde{X}^{n+1} \tilde{Y} | f_2 \rangle \leq C'' \langle f_2 | f_2 \rangle$. The statement is proved.
So all the powers of $\tilde{X}$ we can extend from $\text{Ran} F_{\pi^{-1}\varphi+2n_0+2} \cap \mathcal{D}_\varphi$ to the $\text{Ran} F_{\pi^{-1}\varphi+2n_0+2}$. It is easy to prove as above that the operators $\tilde{X}, \tilde{Y}, \tilde{H}$ are the bounded operators on $\tilde{X}^n(\text{Ran} F_{\pi^{-1}\varphi+2n_0+2} \cap \mathcal{D}_\varphi)$. So we can extend the operators $\tilde{X}, \tilde{Y}, \tilde{H}$ to the operators which acts in the space

$$\text{Lin}\{\bigcup_n \tilde{X}^n(\text{Ran} F_{\pi^{-1}\varphi+2n_0} \oplus \text{Ran} F_{\pi^{-1}\varphi+2n_0+2})\}.$$  \hspace{1cm} (97)

For all $N = 1, 2, 3...$ the restrictions of this operators to the spaces

$$\text{Lin}\{\bigcup_{n=1}^N \tilde{X}^n(\text{Ran} F_{\pi^{-1}\varphi+2n_0} \oplus \text{Ran} F_{\pi^{-1}\varphi+2n_0+2})\}$$ \hspace{1cm} (98)

are the bounded operators.

Put by definition

$$\tilde{H}_\varphi^{n_0+1} = \text{Lin}\{\bigcup_n \tilde{X}^n(\text{Ran} F_{\pi^{-1}\varphi+2n_0} \oplus \text{Ran} F_{\pi^{-1}\varphi+2n_0+2})\}.$$ \hspace{1cm} (99)

Now the operators $\tilde{X}, \tilde{Y}$ are defined on

$$\text{Lin}\{\bigcup_{n \geq n_0} \tilde{H}_\varphi^{n_0+1} \cap F_{\pi^{-1}\varphi+2n}\}.$$ \hspace{1cm} (100)

Let us consider the space

$$\Pi_\varphi^{n_0+1} := \text{Ran} F_{\pi^{-1}\varphi+2n_0+2} \ominus \tilde{H}_\varphi^{n_0}$$ \hspace{1cm} (101)

and the space $\mathcal{H}_\varphi^{n_0+1} := \text{Lin}\{\bigcup_n \tilde{X}^n\Pi_\varphi^{n_0+1}\}$ It is clear that

$$\tilde{H}_\varphi^{n_0+1} = \mathcal{H}_\varphi^{n_0} \oplus \mathcal{H}_\varphi^{n_0+1}.$$ \hspace{1cm} (102)

Continuing this procedure to infinity we will prove the lemma.

Note that the proof of this lemma is like to the well-known geometric proof of the theorem about Jordan normal form of operator [12].
6 Decomposition of the state $\tau_\varphi$ into the sum of the Gibbs states and the end of the proof.

Let us decompose the vector $|\Omega_\varphi\rangle$ into the following direct sum

$$|\Omega_\varphi\rangle = \sum_{m=0}^{\infty} |\Omega_{\varphi m}\rangle,$$  \hspace{1cm} (103)

where $|\Omega_{\varphi m}\rangle \in \mathcal{H}_\varphi^m$.

It follows from lemma 8 that for all $P(X,Y)N_F \in \mathcal{A}^{**}$

$$\langle \Omega_\varphi|P(\hat{X},\hat{Y})\hat{N}_F|\Omega_\varphi\rangle = \sum_{m=0}^{\infty} \langle \Omega_{\varphi m}|P(\hat{X},\hat{Y})\hat{N}_F|\Omega_{\varphi m}\rangle.$$  \hspace{1cm} (104)

Now we state the following

**Lemma 9.** The following states

$$\tau_{\varphi n}(P(X,Y)N_F) := \frac{1}{\langle \Omega_{\varphi n}|\Omega_{\varphi n}\rangle} \langle \Omega_{\varphi n}|P(\hat{X},\hat{Y})\hat{N}_F|\Omega_{\varphi n}\rangle$$  \hspace{1cm} (105)

are well defined and the KMS states.

**Proof.** Let us show that $\tau_n(P(\tilde{X},\tilde{Y})\tilde{N}_F)(\varphi)$ are the KMS states. Let us introduce, the operators $\tilde{X},\tilde{Y},\tilde{H}$ defined on $\bigcup_{m}\{\text{Ran} F_{\pi^{-1}\varphi+2m+2n_0} \oplus \ldots \oplus \text{Ran} F_{\pi^{-1}\varphi+2m+2n_0}\}$ such that

a) The subspaces $\mathcal{H}_{\varphi}^n$ are invariant under the action of the operators $\tilde{X},\tilde{Y},\tilde{H}$,

b) The restriction of $\tilde{X},\tilde{Y},\tilde{H}$ to $\mathcal{H}_{\varphi}^{n_0}$ coincide with the restriction of $\tilde{X},\tilde{Y},\tilde{H}$ to $\mathcal{H}_{\varphi}^{n_0}$ respectively, and the restriction of $\tilde{X},\tilde{Y},\tilde{H}$ to $\mathcal{H}_{\varphi}^{n_0+m}$ are

28
equal to zero as \( m > 0 \). We will find these operators in the following form:

\[
\tilde{X} = \sum_{m=0}^{\infty} C_1^m \tilde{X}^{m+1} \tilde{Y}^m F_{\pi^{-1}\varphi+2n_0+2m},
\]

\( (106) \)

\[
\tilde{Y} = \sum_{m=0}^{\infty} C_2^m \tilde{X}^m \tilde{Y}^{m+1} F_{\pi^{-1}\varphi+2n_0+2m+2},
\]

\( (107) \)

\[
\tilde{N}_F = \sum_{m=0}^{\infty} D_m \tilde{X}^m \tilde{Y}^m F_{\pi^{-1}\varphi+2n_0+2m},
\]

\( (108) \)

It is easy to find such \( C_1^m, C_2^m, D_m \), such that the restriction of \( \tilde{X}, \tilde{Y}, \tilde{H} \) to \( \mathcal{H}_{\varphi}^{n_0} \) coincide with \( \tilde{X}, \tilde{Y}, \tilde{H} \) respectively. It is clear that these operator are equal to zero on \( \mathcal{H}_{\varphi}^{n_0+m} \) \( m > 0 \). So we have:

\[
\tau_0(P(X, Y)N_F) = \langle \Omega_{\varphi} | P(\tilde{X}, \tilde{Y})\tilde{N}_F | \Omega_{\varphi} \rangle.
\]

\( (109) \)

for all \( N_F \in \mathcal{A}^{**} \). Note that the group of automorphisms \( U_t \) acts on \( \tilde{X}, \tilde{Y}, \tilde{N}_F \) as follows

\[
U_t(\tilde{X}) = e^{it} \tilde{X},
\]

\[
U_t(\tilde{Y}) = e^{-it} \tilde{Y},
\]

\[
U_t(\tilde{N}_F) = \tilde{N}_F.
\]

\( (110) \)

So we have prove the KMS property of the functional \( \tau_0 \). The prove of the KMS property of \( \tau_1, \tau_2, ... \) is analogues to the previous prove.

Then the following lemma holds.

**Lemma 10.** We can chose the set \( A \) such that for all \( a \in \mathcal{A}^{**} \)

\[
\tau_{\varphi n}(a) = \rho_{\pi^{-1}\varphi+2n}(a).
\]

\( (111) \)

**Proof.** Let us prove that \( \tau_{\varphi l}(a) \) \( l = 0, 2, ... \) is defined by the Gibbs formula.

Note that the Hilbert space \( \mathcal{H}_{\varphi}^{n_0+l} \) is isomorphic to \( \Gamma \otimes \tilde{\mathcal{V}}_{\pi^{-1}\varphi+2n_0+2l} \) Here \( \Gamma \)
is a some Hilbert space, and $\otimes$ means the tensor product of Hilbert space. The domain of restriction of operators $\hat{X}, \hat{Y}, \hat{F}(H)$ to $\mathcal{H}_\varphi^{m_0+l}$ consider with $\Gamma \otimes V_{\pi^{-1}\varphi + 2n_0 + 2l}$. Here $\otimes$ means an algebraic tensor product and operators $\hat{X}, \hat{Y}, \hat{F}(H)$ at this representation have the form $1 \otimes \hat{X}_\lambda, 1 \otimes \hat{Y}_\lambda, 1 \otimes (\hat{N}_F)_\lambda$, where $\hat{X}_\lambda, \hat{Y}_\lambda, (\hat{N}_F)_\lambda$ — are the representations of elements $X, Y, N_F$ in $V_\lambda$, where $\lambda = \pi^{-1}\varphi + 2n_0 + 2l$.

Now $\forall a = P(X, Y)N_F \in \mathcal{A}^{**}$

$$
\tau_l(P(X, Y)N_F) = \sum_{n=0}^{\infty} \langle \Omega_\varphi | (P(\hat{X}, \hat{Y})F(H)F_{\varphi + 2n_0 + 2l}|\Omega_\varphi) =
= \sum_{n=0}^{\infty} ((P(\hat{X}_\lambda, \hat{Y}_\lambda)\hat{N}_F)_{n,n} \langle \Omega_\varphi | F_{\varphi + 2n_0 + 2l}|\Omega_\varphi) \rangle.
$$

Here symbol $(*)_n,m$ — means the matrix element between the vectors $|\lambda, n\rangle, |\lambda, m\rangle$. We must prove that $\langle \Omega_\varphi | F_{\varphi + 2n_0 + 2l}|\Omega_\varphi \rangle$ is proportional to the Gibbs weight. It is easy to do by considering the element $\langle \Omega_\varphi | \hat{X}F_{\lambda + 2n}\hat{Y}|\Omega_\varphi \rangle$ and using the KMS property. So our lemma is proved.

So, we see that for all $a \in \mathcal{A}^{**}$

$$
\tau_\varphi(a) = \sum_{i=0}^{\infty} m_i(\varphi)\rho_{\pi^{-1}\varphi + 2i}(a),
$$

where by definition $m_i(\varphi) = \langle \Omega_{\varphi n}|\Omega_{\varphi n}\rangle$. Let us consider the measure $d\sigma(\lambda)$ which coincides with $d\mu(\pi(\lambda - 2k))m_k(\pi(\lambda - 2k))$ on each interval $[k, 2k + 2)$. So our state can be represented as

$$
\rho(a) = \int_{0}^{+\infty} d\sigma(\lambda)\rho_\lambda(a).
$$
Now let $a = N_{e^{itx}} \in A^*$, where $t \in \mathbb{Q}$. Let $\eta_n(x) \in B$ be a sequence of continuous functions such that $\text{supp}\eta_n(x) \in [-2n, 2n]$, $\eta_n(x)|_{[-n,n]} = 1$, $0 \leq \eta_n(x) \leq 1$. We have

$$\int_0^{+\infty} d\sigma(\lambda) \rho_\lambda(N_{\eta_n}N_{e^{itx}}) = \rho(N_{\eta_n}N_{e^{itx}}),$$

(115)

Where $\rho_\lambda$ defined in (32,33). The right hand side of this equality can be represented as follows

$$\rho(N_{\eta_n}N_{e^{itx}}) = \int_{-\infty}^{+\infty} d\mu(x)e^{itx}\eta_n(x)$$

(116)

for some measure $d\mu$ decreasing faster than any inverse polynomial. So the right hand side of (115) tends to $\rho(N_{e^{itx}}$ as $n \to \infty$. The integrand in the left hand side of (115) satisfy $|\rho_\lambda(N_{\eta_n}N_{e^{itx}})| \leq 1$ and $\lim_{n \to \infty} \rho_\lambda(N_{\eta_n}N_{e^{itx}}) = \rho_\lambda(N_{e^{itx}})$. So

$$\int_{-\infty}^{+\infty} d\sigma(\lambda) \rho_\lambda(N_{e^{itx}}) = \rho(N_{e^{itx}})$$

(117)

or

$$\sigma(\{0\}) + \frac{1 - e^{-\beta}}{1 - e^{-\beta+i\tau}} \int_{-\infty}^{+\infty} d\sigma(\lambda)e^{it\lambda} = \int_{-\infty}^{+\infty} d\mu(x)e^{itx}\eta_n(x).$$

(118)

Both sides of equality (118) are continuous on $t$ so (118) is valid for arbitrary $t$. It follows from (118) that $d\sigma$ is a linear combination of $d\mu(\lambda) - \sigma(0)\delta(\lambda)$ and $d\mu(\lambda - 2) - \alpha(0)\delta(\lambda - 2)$ therefore $d\sigma$ decrease faster than any inverse polynomial.

Therefore the part "only if" is proved.
7 Uniqueness

We can prove the uniqueness of $\rho$ by induction on the number $B, B^+$ by using the KMS property. Let $\frac{1}{\sqrt{2}} B := Y, \ -\frac{1}{\sqrt{2}} B^+ := X$. consider the following expression

$$\rho(B^{\pm}, ..., B^{\pm} N_F), \quad (119)$$

where the number of elements $B^{\pm}$ is equal to $n$. The base of induction ($n = 0$) is obvious. Suppose that the statement is proved for $m = n - 1, n - 2, ... 1$. Consider the expression

$$\rho(B^+ AN_F), \quad (120)$$

Here $A$ is a product of $n - 1$ operators $B^{\pm}$. Using the KMS property we have:

$$\rho(B^+ AN_F) = e^{-\beta} \rho(B^+ AN_{T_{2F}}) + e^{-\beta} \rho([A, B^+] N_{T_{2F}}). \quad (121)$$

Iterating this identity we find:

$$\rho(B^+ AN_F) = e^{-\beta k} \rho(B^+ AN_{T_{2kF}}) + \sum_{j=1}^{k} e^{-\beta j} \rho([A, B^+] T_{-2jF}). \quad (122)$$

We can represent $\rho(a N_j b)$ $a, b \in \mathcal{D}$ as an integral by some measure which decrease faster then any inverse polynomial. So the first term tends to zero as $k^M e^{-\beta k}$ as $k \to \infty$ for some integer $M$. The sum in the second term has the limit because $\rho([A, B^+] T_{-2jF})$ tends to zero as $j^M e^{-\beta j}$ as $j \to \infty$ for some integer $M$. So

$$\rho(B^+ AN_F) = \sum_{j=1}^{\infty} e^{-\beta j} \rho([A, B^+] T_{-2jF}). \quad (123)$$
But \([A, B^+]T_{-2j}F\) contains only \(n - 1\) operators \(B^\pm\). We can write analogues for \(\rho(BAN_F)\). These representations prove the uniqueness of \(\rho\).

8 Conclusion.

In the present paper we have investigated the structure of Kubo-Martin-Shwinger states on universal enveloping algebra of \(\mathfrak{sl}(2, \mathbb{C})\). It is interesting to generalize our results to the infinite dimensional case, general Lie algebras and quantum groups.

9 Acknowledgements.

I would like to thank I.V. Volovich, L. Accardy, A.N. Pechen and R.A. Roschin for very useful discussions.

This work was partially supported by the Russian Foundation of Basis Research (project 05-01-008884), the grand of the president of the Russian Federation (project NSh-1542.2003.1) and the program ”Modern problems of theoretical mathematics” of the mathematical Sciences department of the Russian Academy of Sciences.

References

[1] N.N. Bogoliubov, and D.V. Shirkov, *Introduction to the theory of quantization fields*, (New York, Interscience, 1959).
[2] R. Jost, *The General theory of Quantized Fields*, (Providence, American Mathematical Society, 1965).

[3] L. Accardy, Yu.G. Lu, I.V. Volovich, *Quantum Theory and Its Stochastic Limit*, (Springer, Berlin, 2002).

[4] L. Accardy, I. Volovich, *Quantum White Noise with Singular Non-Linear Interaction*, quant-ph/9704029.

[5] L. Accardy, Y.G. Lu, I.V. Volovich, *White noise approach to classical and quantum stochastic calculi*, Centro Vito Volterra, Universita di Roma ”Tor Vergata”, Preprint 375, 1999.

[6] L. Accardi, U. Franz, M. Skeide, *Renormalized squares of white noise and other non-gaussian noises as Levy processes on real Lie algebra*, Centro Vito Volterra, Universita di Roma ”Tor Vergata”, Preprint 423, 2000, Commun. Math. Phys. 228(2002) 123-150.

[7] R. Haag, M.M. Hugenholtz, M. Vinnik, Commun. Math. Phys. 5, 215 (1967)

[8] Luigi Accardi, Grigori Amosov, Uwe Franz, *KMS states on the square of white noise algebra*, quant-ph/0208070.

[9] I.M. Gelfand, M. I. Graev, I.I. Piatetsky-Shapiro, *Teoria predstavleni i avtomorfnye funkchii*, (M. Nauka, Fizmatlit, 1996).

[10] J. Glimm, A. Jaffe, *Quantum Physics A Functional Integral Point of View*, (Springer-Verlag, New York, Heidelberg, Berlin 1981).

[11] K. Maurin, *Methods of Gilbert spaces*, ( Mir, Moscow, 1965.) 1959).
[12] I.M. Gelfand, *Lectures on linear algebra*, (M: Nauka, Fizmatlit, 1966.)