ABELIAN VARIETIES NOT ISOGENCEOUS TO JACOBIANS OVER GLOBAL FIELDS

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Abstract. We prove the existence of abelian varieties not isogeneous to Jacobians over characteristic \( p \) function fields. Our methods involve studying the action of degree \( p \) Hecke operators on hypersymmetric points, as well as their effect on the formal neighborhoods using Serre Tate co-ordinates. We moreover use our methods to provide another proof over number fields, as well as proving a version of this result over finite fields.

1. Introduction

This paper concerns the following question: Given an algebraically closed field \( k \) and \( g \geq 4 \), Does there exist an Abelian variety over \( k \) of dimension \( g \) which is not isogeneous to the Jacobian of a stable curve? The history of the question, as far as we know, is that it was first asked by Nicholas Katz and Frans Oort in the setting of \( k = \mathbb{Q} \), and then posed more generally by Ching-Li Chai and Oort in [4]. We mention that an observation of Poonen (unpublished, but credited by Chai and Oort) which helped clarify the situation, was to formulate the question more generally for an arbitrary subvariety of \( A_g \) in place of the Torelli locus. Of course, one can also generalize from \( A_g \) to an arbitrary Shimura variety.

In the case of \( k = \mathbb{Q} \) the question has been answered in the affirmative: Chai-Oort [4] proved it conditional on the Andre-Oort conjecture for \( A_g \), and then the second-named author made their proof unconditional [13]. Since then the Andre-Oort conjecture itself has been proven [14] in this case, removing the need for [13]. Recently, Masser-Zannier [11] proved a stronger result, producing Abelian varieties satisfying the condition which are defined over low-degree fields, and in fact showing that ‘most’ such Abelian varieties satisfy the condition, in a precise sense.

We now turn our attention to the case of finite characteristic, about which much less is known. Over \( \mathbb{F}_p \), the authors [12] have a previous work in which they make conjectures based on arithmetic statistics which suggests that the statement should not have affirmative answer for all \( g \geq 4 \). In fact, [12] suggests that if \( D \subseteq A_g \) is a generically ordinary divisor defined over \( \mathbb{F}_p \) (with \( g > 1 \)), then every \( x \in A_g(\mathbb{F}_p) \) should be isogenous to some \( y \in D(\mathbb{F}_p) \).

In the way of unconditional results, there is only a result of Chai-Oort [4, §4] which deals with the analogous situation of a curve inside \( X(1)^2 \) (which produces non-ordinary points) and a result of the authors [12, Thm 4.1] for a specific hypersurface in \( X(1)^N \) for \( N \geq 270 \), using additive combinatorics. Relatedly, work of the first author, Asvin G. and Q. He [12] shows that if \( C_1, C_2 \) are two generically ordinary curves contained in suitable mod \( p \) Hilbert modular surfaces, then there are infinitely many closed points on \( C_1 \) isogenous to some point on \( C_2 \).

The most analogous setting to number fields is function fields in one variable, and thus the natural analogue to \( \mathbb{Q} \) is \( \mathbb{F}_p(T) \). The main goal of this paper is to deal with precisely that case.

Our main theorem is as follows:

**Theorem 1.1.** Let \( k \) be a finite field, let \( F = \overline{k(t)} \), and \( g > 1 \). Let \( D \subseteq A_g/F \) be a divisor. There exists an \( F \)-valued point of \( A_g \) which is not isogenous to any \( F \)-points of \( D \).

As a stepping stone to our main theorem, we first prove the following:
Theorem 1.2. Let $k$ be a finite field, and $g > 1$. Let $D \subset A_g/k$ be a divisor. Let $F = \overline{k(t)}$. There exists an $F$-valued point of $A_g$ which is not isogenous to any $F$-points of $D$.

The heuristics offered in [12] suggest that Theorem 1.2 is strongest possible theorem that is true in positive characteristic; namely, that $\overline{k(t)}$ is the smallest algebraically closed field over which one can expect the existence of an abelian variety not isogenous to any point on $D$.

Of course, one may formulate the above theorem instead in terms of curves contained in $A_g$ by using the familiar translation between curves and function fields.

1.1. Other results. Theorem 1.2 yields yet another proof of the number field version of Theorem 1.2 (namely, the existence of abelian varieties over number fields not isogenous to Jacobians). Our method also yields the following result over finite fields:

Theorem 1.3. Let $D \subset A_g$ denote a divisor over $\overline{\mathbb{F}}_p$. Then, there exists an ordinary $x \in A_g(\overline{\mathbb{F}}_p)$ such that $T(x) \not\subset D$ for every prime-to-$p$ Hecke correspondence $T$.

Theorem 1.3 acts as a substitute over $\overline{\mathbb{F}}_p$, because as stated before we expect that every ordinary $x \in A_g(\overline{\mathbb{F}}_p)$ is isogenous to some point in $D(\overline{\mathbb{F}}_p)$.

We remark also that Theorem 1.3 also highlights one of the differences/difficulties in handling the $\overline{\mathbb{F}}_p$ case: The Galois orbits are just much smaller, so that $T(x)$ cannot be a single orbit, unlike the case of number fields or function fields.

1.2. Ideas of Proofs.

1.2.1. Proof of Theorem 1.2. Fix a divisor $D \subset (A_g)^\tau$. Our first observation is that in the function field case one can impose very strong ‘local’ conditions on our $F$-point, which rules out being contained in $D$. Thinking of $F$-valued point as curves $C \subset A_g$, this amounts to making a curve $C$ which is highly singular at a given point, such that for example its formal neighborhood contains much of the formal neighborhood of $A_g$. There are some difficulties with implementing this strategy, namely:

1. While the local structure of prime-to-$p$ Hecke operators is very well understood, $p$-power Hecke operators behave poorly in characteristic $p$. These are not etale (or even finite) and strongly distort the local structure.

2. It is a-priori unclear how to insist that every irreducible Hecke translate of $C$ also has this property. A-priori, different branches of $C$ through a point $x$ could separate upon applying a Hecke operator.

We overcome the second difficulty by insisting that the curve has maximal monodromy by using Lefschetz theorems in conjunction with Bertini-style theorems due to Poonen and Charles-Poonen. The first difficulty is the main one, and our idea in overcoming it is to use hypersymmetric abelian varieties (see Definition 3.3 for the definition). The $p$-power Hecke orbit of a hypersymmetric point consists of points defined over the same field as the original point (similar in spirit to the prime-to-$p$ Hecke orbit of a supersingular point). In particular, the orbit of a hypersymmetric point under $p$-power isogenies is finite.

This prevents ‘escape to infinity’ and allows us to impose certain local conditions on this whole finite set at once, which we can then control. To impose our local conditions, we use the bilinear structure on the tangent space (and in fact the entire formal neighborhood) inherited from the Serre-Tate co-ordinates to isolate certain favored directions which behave stably under $p$-power isogenies.

1It is unclear to us whether one can impose analogously strong local conditions over number fields.
1.2.2. Proof of Theorem 1.2. It turns out that the hardest case of this theorem is Theorem 1.2. Our approach is essentially to notice that \( \mathbb{k}(T) \) contains many distinct subfields - in fact, copies of itself: \( \mathbb{k}(P(T)) \). One can show that if \( D \) is defined over some function field \( E \) of \( F \), then we may find a smaller subfield \( E' \subset E \) such that the intersection of \( D \) with its own Galois conjugates over \( E' \) is defined over \( \mathbb{k} \). But if \( C \) is a curve defined over \( E' \), then \( C \subset D \) implies that \( C \) is contained in the intersection of \( D \) with all its galois conjugates over \( E' \). One must be a bit careful, and we work with curves instead of function fields, but this idea can essentially be pushed through to reduce to Theorem 1.2.

1.2.3. Proofs of Additional Results. This same idea can be carried out (with a lot less difficulty, as we no longer face any characteristic \( p \) issues) to find a curve \( C \subset \mathbb{A}_g \) none of whose Hecke translates are contained in \( D \), where \( C \) is now a curve defined over a Number Field. We use the existence of such a curve, along with a "big monodromy" result due to Zywina, to provide an intersection-theoretic proof of the existence of Number Field-valued points of \( \mathbb{A}_g \) that are not isogenous to any point of \( D \). This method also goes through to prove Theorem 1.3.

1.3. Structure of Paper. In section 2 we show how to construct Abelian varieties over curves with prescribed local and monodromy conditions. In section 3 we review the theory of Serre-Tate coordinates, introduce the notion of primitive Serre-Tate directions and study how they behave under isogenies. This is the technical heart of the paper. We then go on to prove Theorem 1.2, and then upgrade it to Theorem 1.1. Finally, in Section 4 we give a couple of other results: we explain how our methods can give a rather short proof of the same result over number fields (although relying on a strong Monodromy theorem of Zywina) and we prove Theorem 1.3.

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2. Constructing families of Abelian varieties with specified local conditions

In what follows, for an irreducible curve \( C \) when referring to the geometric monodromy of \( C \), we mean \( \pi_1(C) \) where \( \eta \) is the generic point.

We fix a prime \( r > 6p \) which is distinct from the characteristic \( p \). We fix a smooth compactification of \( \mathbb{A}_g[r] \), with boundary divisor \( D \). Note that over \( \mathbb{A}_g[r] \) we have the \( n \)-torsion local systems \( \mathcal{L}_n \) for \( n \) prime to \( p \), as well as the \( p^m \)-torsion local systems \( \mathcal{L}_p^m \) over the ordinary locus \( \mathbb{A}_g^{\text{ord}}[r] \), defined as the dual of the connected component of the \( p^m \)-torsion of the universal Abelian scheme.

We shall need the following result:

**Theorem 2.1.** Fix \( g > 1 \), and an auxiliary prime \( r > 6p \). For any finite set of \( \text{Spec} \mathbb{F}_p[[t]] \) points \( x_1, \ldots, x_n \) whose differential is injective in \( \mathbb{A}_g^{\text{ord}}[r] \), and positive integers \( m_1, \ldots, m_n \) there exists a curve \( C \subset \mathbb{A}_g^{\text{ord}}[r] \) such that

1. \( C \) admits \( \text{Spec} \mathbb{F}_p[[t]] \) points specializing to all of the \( x_i \) mod the \( t^{m_i} \), and
2. The geometric monodromy of \( C \) surjects onto \( \Gamma_g(\mathbb{Z}_r) \times \prod_{t \neq p, r} \text{P}(\text{Sp}_{2g}(\mathbb{Z}_t)) \times \text{GL}_g(\mathbb{Z}_p) \)

where

\[
\text{P}(\text{Sp}_{2g}(\mathbb{Z}_t)) := \text{Sp}_{2g}(\mathbb{Z}_t)/\mathbb{Z}_t^	imes
\]

The idea for the proof is twofold: For the prime-to-\( p \) part we use tame Lefschetz theorems (see [6, Theorem 7.3] and [7]), and for the \( p \)-part we simply pick enough \( \mathbb{F}_q \) points whose Frobenius images generate.

\(^2\)The purpose of \( r \)-torsion is simply to rigidify the moduli problem.
Lemma 2.2. For a positive integer \( n \) prime to \( p \), the \( n \)-torsion local system \( \mathcal{L}_n \) over \( \mathcal{A}_g[r] \) is tamely ramified at \( D \).

Proof. The image of inertia at the boundary in \( \text{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}) \) is well-known to be unipotent, and therefore has order prime to \( p \). \hfill \square

Next, let \( k \) be a finite field over which the union of all the \( x_i \) is defined. We pick finitely many points in \( \mathcal{A}_g[r](k) \) whose Frobenius images contain every conjugacy class in \( \text{GL}_g(\mathbb{Z}/p^2\mathbb{Z}) \). Call the union of all those \( R \). It follows that every curve defined over \( k \) containing \( R \) as a smooth divisor has fundamental group surjecting onto \( \text{GL}_g(\mathbb{Z}/p^2\mathbb{Z}) \) and hence onto all of \( \text{GL}_g(\mathbb{Z}_p) \). We moreover claim that such a curve \( C \) has geometric fundamental group surjecting onto \( \text{GL}_g(\mathbb{Z}/p^2\mathbb{Z}) \). Indeed, pick a point \( Q \in R \subset C \) such that the Frobenius at \( Q \) maps trivially to \( \text{GL}_g(\mathbb{Z}/p^2\mathbb{Z}) \). Then from the sequence

\[
1 \to \pi_1(C, \overline{Q}) \to \pi_1(C, Q) \to \pi_1(k) \to 1
\]

the claim follows.

Next, as we will be applying Goursat’s lemma to join the \( p \)-part and prime-to-\( p \) part of the Monodromy, we shall need to consider quotients of \( \Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \mathbb{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \) which can occur as quotients of \( \text{GL}_g(\mathbb{Z}_p) \). Since the Chevalley groups are simple (with finitely many possible exceptions) most factors don’t contribute, and so it is easy to see that all such quotients factor through a single finite quotient group \( H \) of \( \Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \mathbb{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \) and \( H' \) of \( \text{GL}_g(\mathbb{Z}_p) \).

Now, by possibly increasing \( k \) we increase \( R \) to contain points in \( \mathcal{A}_g^{\text{ord}}[r](k) \) whose Frobenius image contain every conjugacy class in \( H \times H' \). As above, every curve defined over \( k \) and containing \( R \) as a smooth divisor has geometric fundamental group surjecting onto \( H \times H' \). We moreover insist that \( R \) is distinct from the closed points in the image of the \( x_i \). We are now ready to complete the proof.

Proof of Theorem 2.1. First, consider a series of blowups at points of \( \mathcal{A}_g[r] \) to obtain a smooth variety \( B \) in which all the \( x_i \) separate. Now we apply Poonens result [15, Thm 1.3] to obtain a smooth curve \( C_0 \) which is obtained from \( B \) by repeatedly intersecting with the ample class, such that \( C_0 \) is smooth, intersects \( D \) transversally, is defined over \( k \), and contains the lifts of the \( x_i \) mod \( t^{m_i} \). Note that by [3, Thm 1.1] we may insist that \( C_0 \) is moreover irreducible. We will define \( C \) to be the image of \( C_0 \), so this already implies that condition (1) is satisfied.

Next, we will moreover ensure (using the same result) that \( C_0 \) contains the points above \( R \) as smooth divisors, which will imply that the geometric monodromy of \( C_0 \) surjects onto \( \text{GL}_g(\mathbb{Z}_p) \). Separately, the Lefschetz theorem proven by Esnault-Kindler together with lemma 2.2 implies that the fundamental group of \( C_0 \) surjects onto \( \Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \mathbb{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \).

Finally, Goursat’s lemma together with the fact that the geometric monodromy of \( C_0 \) surjects onto \( H \times H' \) implies that the fundamental group of \( C_0 \) surjects onto all of \( \Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \mathbb{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \times \text{GL}_g(\mathbb{Z}_p) \) as desired. \hfill \square

3. Hecke translates in characteristic \( p \)

3.1. Background on Serre-Tate coordinates. We will briefly recall Serre-Tate coordinates and describe the local action of \( p \)-power hecke operators in terms of these coordinates. For a thorough treatment of Serre-Tate coordinates, see [10], or [12].

Let \( \mathcal{G} \) denote an ordinary \( p \)-divisible group over an algebraically closed field \( k \) with dimension \( g \) and height \( 2g \). We have that \( \mathcal{G} \cong \mathcal{G}^{\text{mult}} \times \mathcal{G}^{\text{et}} \), where \( \mathcal{G}^{\text{mult}} \cong \mu_{p^{\infty}}^g \) and \( \mathcal{G}^{\text{et}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^g \). The \( p \)-divisible groups \( \mathcal{G}^{\text{mult}} \) and \( \mathcal{G}^{\text{et}} \) are rigid, and the functoriality of the connected-étalé exact sequence
implies that deformations of of $\mathcal{G}$ to an Artin local $k$-algebra $R$ are in bijection with extensions of $\mathcal{G}^\mathrm{et} \times \text{Spec } R$ by $\mathcal{G}^\text{mult} \times \text{Spec } R$. In particular, the deformation space of $\mathcal{G}$ has a natural group structure.

Let $T_p(\mathcal{G})$ denote the Tate-module of $\mathcal{G}^\mathrm{et}$, and let $T_p(\mathcal{G}^\vee)$ denote the Tate-module of $(\mathcal{G}^\text{mult})^\vee$, where $\vee$ denotes taking the Cartier dual. Then, the formal deformation space of $\mathcal{G}$, which we have already seen has the structure of a group, actually has the structure of a formal torus, and is canonically isomorphic to $\text{Hom}(T_p(\mathcal{G}) \otimes T_p(\mathcal{G}^\vee), \hat{\mathbb{G}}_m)$. It will later also be convenient to use the canonical identification of this torus with $(T_p(\mathcal{G})^* \otimes T_p(\mathcal{G}^\vee)^*) \otimes \hat{\mathbb{G}}_m$. Here, $^*$ denotes taking the linear dual of a $\mathbb{Z}_p$-module.

Specifically, let $R$ denote any Artin local $k$-algebra, with maximal ideal $\mathfrak{m}$. Then, the deformations of $\mathcal{G}$ to $R$ are in bijection with bilinear maps $q : T_p(\mathcal{G}) \times T_p(\mathcal{G}^\vee) \to 1 + \mathfrak{m}$. Therefore, a $\mathbb{Z}_p$-basis $e_i = \{e_1, \ldots, e_g\}$ of $T_p(\mathcal{G})$ and $m_j = \{m_1, \ldots, m_g\}$ of $T_p(\mathcal{G}^\vee)$ yield coordinates $Q = \{q_{ij}\}$ on the deformation space, i.e. the characteristic $p$ deformation space is isomorphic to $\text{Spf } k[[q_{ij} - 1]]$, and an $R$-valued point of the deformation space corresponds to a choice of $g^2$ elements in $1 + \mathfrak{m}$, which is the same data as a bilinear map $q : T_p(\mathcal{G}) \times T_p(\mathcal{G}^\vee) \to 1 + \mathfrak{m}$. Changing the coordinates $e_i$ by $A \in \text{GL}_g(\mathbb{Z}_p)$ and $m_j$ by $B \in \text{GL}_g(\mathbb{Z}_p)$ yields a new set of coordinates $Q' = \{q'_{ij}\}$, with $Q' = A^{-1}QB^T$. The same analysis holds when $R$ is a complete local $k$-algebra. Here we think of $Q$ as an element of $M_g((1 + \mathfrak{m})^\times)$, which inherits the structure of a $M_g(\mathbb{Z}_p)$ bi-module from the $\mathbb{Z}_p$-module structure of $(1 + \mathfrak{m})^\times$.

Let $\mathcal{H}$ denote another $p$-divisible group and let $\phi : \mathcal{G} \to \mathcal{H}$ be an isogeny. We pick bases $\{e'_i, m'_j\}$ for $T_p(\mathcal{H}), T_p(\mathcal{H}^\vee)$, and let $X$ and $Y$ denote the matrices (necessarily with non-zero determinant) of $\phi : T_p(\mathcal{G}) \to T_p(\mathcal{H})$ and $\phi^\vee : T_p(\mathcal{H}^\vee) \to T_p(\mathcal{G}^\vee)$ in these coordinates. Let $\mathcal{H}/k[[t]]$ deform $\mathcal{G}$, and let $Q = \{q_{ij}(t)\}$ denote its Serre-Tate coordinates. The following result is well known (see for example [10] Theorem 2.1 or [8] Proposition 2.23).

**Proposition 3.1.** Notation as above, and consider the matrix $Q' = X^{-1}QY$. Note that a-priori the entries of $Q'$ are valued only in $k((t^{1/\infty}))$.

1. Suppose that the entries of $Q'$ are valued in $1 + tk[[t]]$. Then, $\phi$ lifts uniquely to an isogeny $\tilde{\mathcal{G}} \to \tilde{\mathcal{H}}$ over $k[[t]]$ (where $\tilde{\mathcal{H}}$ is necessarily a deformation of $\mathcal{H}'$) and the Serre-Tate coordinates (with respect to $\{e'_i, m'_j\}$) of $\mathcal{H}'$ are $Q'$.

2. If the entries of $Q'$ aren’t valued in $1 + tk[[t]]$ (eg. if some entry looks like $(1 + t)^{1/2}$), then $\phi$ doesn’t lift to an isogeny over $k[[t]]$ with source $\tilde{\mathcal{G}}$. However, let $n = n(\mathcal{G}, \phi)$ denote the smallest positive integer such that the entries of $Q'(t^n)$ are valued in $1 + tk[[t]]$. Then, $\phi$ lifts uniquely to an isogeny $\tilde{\mathcal{G}} \times_{\text{Spf } k[[t]]} \text{Spf } k[[s]] \to \tilde{\mathcal{H}}$, where the map $\text{Spf } k[[s]] \to \text{Spf } k[[t]]$ is given by $t \mapsto s^n$, and $\mathcal{H}'$ is a deformation of $\mathcal{H}'$ to $k[[s]]$. The Serre-Tate coordinates of $\tilde{\mathcal{G}} \times_{\text{Spf } k[[t]]} \text{Spf } k[[s]]$ are $Q(t) = Q(s^n)$, and of $\mathcal{H}'$ are $Q'(s) = X^{-1}Q(s^n)Y$.

In either case, we will let $\tilde{\phi}$ denote the isogeny that lifts $\phi$, which is already defined over $\text{Spf } k[[t]]$ in the first case, but which is only defined over $\text{Spf } k[[s]] = \text{Spf } k[[t^{1/\infty}]]$ in the second case.

Suppose now that $\mathcal{G}$ was equipped with a principal polarization, $\lambda$. This yields canonical isomorphisms $T_p(\mathcal{G}) \to T_p(\mathcal{G}^\vee)$ and $T_p(\mathcal{G})^* \to T_p(\mathcal{G}^\vee)^*$, both of which we will also denote by $\lambda$. Suppose that we have chosen coordinates such that $m_i = \lambda(e_i)$ (which yield dual bases $\mu_i = \lambda(e_i)$ of $T_p(\mathcal{G})^*, T_p(\mathcal{G}^\vee)^*$). Then, $\lambda$ lifts to a deformation of $\mathcal{G}$ precisely when $q_{ij} = q_{ji}$ ([3] Corollary 2.2.4). The deformation space of $(\mathcal{G}, \lambda)$ is $\text{Hom}^\text{Sym}(T_p(\mathcal{G}) \otimes T_p(\mathcal{G}^\vee), \hat{\mathbb{G}}_m) \subset \text{Hom}(T_p(\mathcal{G}) \otimes T_p(\mathcal{G}^\vee), \hat{\mathbb{G}}_m)$, the set of all symmetric maps. This space is canonically isomorphic to $(T_p(\mathcal{G})^* \otimes T_p(\mathcal{G}^\vee)^*)^\text{Sym} \otimes \hat{\mathbb{G}}_m$, where $(T_p(\mathcal{G})^* \otimes T_p(\mathcal{G}^\vee)^*)^\text{Sym}$ is the $\mathbb{Z}_p$-span of $\{e_i \otimes \mu_j + e_j \otimes \mu_i, e_i \otimes \mu_i\}$. In coordinates, this space equals $\text{Spf } k[[q_{ij}]]/(\{q_{ij} - q_{ji}\})$. 

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3.2. Description of \( p \)-power Hecke operators on \( A_g \) mod \( p \). We now briefly describe \( p \)-power Hecke operators on \( A_g^{\text{ord}} \) mod \( p \). For further details, the interested reader may consult [2, Chapter VII Section 4].

Let \((L, \lambda), (L', \lambda')\) denote ordinary principally polarized \( p \)-divisible groups as above, and let \( \phi \rightarrow L' \) denote an isogeny such that \( \phi^*(\lambda_1) = p^n \lambda_1 \). Suppose that we choose bases for \( T_p(L), T_p(L'), T_p(L') \) and \( T_p(L',\nu) \) which are compatible with \( \lambda, \lambda' \). The choice of bases induce isomorphisms \( L, L' \rightarrow \left( \mathbb{Q}_p/\mathbb{Z}_p \right)^g \times (\mathbb{Z}_p^\infty)^g \). The isogeny \( \phi \) acts by multiplication by a matrix \( X \in M_g(\mathbb{Z}_p) \) on the first summand, and by \( Y \in M_g(\mathbb{Z}_p) \) on the second summand, and the condition \( \phi^* \lambda' = p^n \lambda \) is equivalent to \( X \cdot Y^T = p^n \text{Id} \). We say that the matrix of \( \phi \) is \((A, B)\).

To that end, let \((n : n_1, n_2, \ldots, n_g)\) denote a sequence of integers such that \( n \geq n_1 \geq n_2 \geq \cdots \geq n_g \geq 0 \). A \( p \)-power Hecke operator \( \tau \) is said to have type \((n : n_1, n_2, \ldots, n_g)\) if it parameterizes all ordinary isogenies \( \{\phi : L \rightarrow L'\} \) such that the matrix \( \phi \) is \((A, B)\) where \( AB^T = p^n \text{Id} \).

\[
\begin{pmatrix}
p^{n_1} & & & \\
& p^{n_2} & & \\
& & \ddots & \\
& & & p^{n_g}
\end{pmatrix} \in \text{GL}_g(\mathbb{Z}_p).
\]

We note that the Hecke correspondence \( \tau \) is Frobenius precisely when it has type \((1 : 0, 0, \ldots, 0)\) and is Verschibung precisely when it has type \((1 : 1, 1, \ldots, 1)\). Further, \( \tau \) factors through Frobenius precisely when \( n > n_1 \), and factors through Verschibung when \( n_2 > 0 \). The Hecke correspondence \( \tau \) induces a closed subvariety \( A_g^{\text{ord}}[\tau] \subset A_g^{\text{ord}} \times A_g^{\text{ord}} \), with the two canonical projections \( \text{Pr}_1(\tau), \text{Pr}_2(\tau) \) to \( A_g \). The maps \( \text{Pr}_i(\tau) \) are usually inseparable.

Now, let \( C \subset A_g \) denote a reduced generically ordinary curve, and let \( C^{\text{ord}} \) be the intersection of \( C \) with the ordinary locus of \( A_g \). We define \( \tau(C^{\text{ord}}) \subset A_g^{\text{ord}} \) to be the unique reduced subscheme corresponding to \( \text{Pr}_1(\tau)_1, \text{Pr}_1(\tau)_2 C^{\text{ord}} \), and we define \( \tau(C) \) to be the Zariski closure of \( \tau(C^{\text{ord}}) \) in \( A_g \). We make the following observation that will be used below: Let \( \tau \) denote a Hecke correspondence which doesn’t factor through Frobenius, and suppose the Serre-Tate coordinates of some branch of the completion of \( C \) at an ordinary point are \( Q = q_{ij} \), where none of the \( q_{ij} \) are \( p \)-th powers. Then the Serre-Tate coordinates of \( \tau \) applied to this branch are calculated using the recipe outlined in Proposition 3.3.1 (2), i.e. the Serre-Tate coordinates of the various branches equal \( A^{-1}Q^{\nu}B^T \), where \( A \) is as in 3.2 and \( B \) satisfies \( AB^T = p^n \text{Id} \).

3.3. \( p \)-power Hecke correspondence on Serre-Tate coordinates. We will now describe the action of \( p \)-power Hecke correspondences locally in terms of Serre-Tate coordinates. For a point \( x \in A_g \), we will denote the associated principally polarized abelian variety by \( A_x \) and its associated \( p \)-divisible group by either \( A_x[p^\infty] \) or \( G_x \). We note that the formal neighbourhood \( \mathcal{A}_g \) of \( A_g \) at an ordinary point \( A \) is canonically isomorphic to the polarized deformation space of \( A_x \) (which, by the Serre-Tate lifting theorem, is the same as the polarized deformation space of \( G_x \)). Therefore, this space canonically has the structure of a formal torus, and in coordinates, is isomorphic to \( \text{Hom}_{\text{Sym}}(T_p(G_x) \otimes T_p(G_x^\text{sym}), \mathbb{G}_m) \).

**Definition.** Let \( x \in A_g \) denote an ordinary point, and let \( \ell \subset T_x A_g \) denote a one-dimensional subspace. We say that \( \ell \) is a Serre-Tate direction if it equals the tangent space of a rank-1 formal subtorus of \( \mathcal{A}_g \), i.e. \( L \otimes \mathbb{G}_m \) where \( L \subset (T_p(G)^\bullet \otimes T_p(G)^\text{sym})^{\text{Sym}} \) is a saturated rank-1 \( \mathbb{Z}_p \)-submodule. We say that \( \ell \) is a Primitive Serre-Tate direction if in addition \( L \) is spanned by a primitive tensor in \( (T_p(G)^\bullet \otimes T_p(G)^\text{sym})^{\text{Sym}} \).

Analogously, we define \( T^k_x A_g \) to be the \( k \)-th order neighborhood of \( x \) in \( A_g \). We define a primitive \( k \)-th order Serre-Tate direction to be the base change of a rank-1 formal subtorus of \( A_g \) to this neighborhood.
We note that for any positive $k$, there are only finitely many $k$th order Serre-Tate directions.

**Definition.** An ordinary abelian variety $A/\mathbb{F}_q$ is said to be **hypersymmetric** if $A$ is isogenous to $E^9$, where $E/\mathbb{F}_q$ is an ordinary Elliptic curve.

Let $C \subset \mathcal{A}_g$ denote a generically ordinary reduced irreducible curve, that has the following properties:

1. $C$ passes through every $\mathbb{F}_q$-rational ordinary hypersymmetric point $x$.
2. For any $x \in \mathcal{A}_g(\mathbb{F}_q)$ with $A_x$ as above, $C$ has nodal singularities at $x$, and for every $k$th order primitive Serre-Tate direction $\ell_k$ at $x$, there exists a formally smooth local branch $C_{\ell_k,x}$ of $C$ at $x$ whose tangent space equals $\ell_k$.

We have the following key theorem:

**Theorem 3.2.** Let $\tau$ denote any $p$-power Hecke correspondence, and let $C$ be as above. Then $\tau(C)$ contains every ordinary $x \in \mathcal{A}_g(\mathbb{F}_q)$ which is hypersymmetric. Further, $\tau(C)$ contains every $k$th order primitive Serre-Tate direction in $T_x^k\mathcal{A}_g$.

**Lemma 3.3.** Let $x \in \mathcal{A}_g(\mathbb{F}_q)$ be ordinary and hypersymmetric. Then, $\tau(x) \in \mathcal{A}_g(\mathbb{F}_q)$.

**Proof.** Every subgroup of $A_x[p^\infty]$ has the form $G_{\text{et}} \times G_{\text{mult}}$, where $G_{\text{et}} \subset A_x[p^\infty]_{\text{et}}$, and $G_{\text{mult}} \subset A_x[p^\infty]_{\text{mult}}$. Since $A_x$ is hypersymmetric, the Galois representation on $T_p(\mathcal{A}_g[p^\infty]_{\text{et}})$ acts through a scalar (because this statement is trivially true for $E$). It follows that every subgroup of $A_x[p^\infty]$ is defined over $\mathbb{F}_q$, whence the result follows.

**Proof of Theorem 3.2.** Let $x \in \mathcal{A}_g(\mathbb{F}_q)$ denote some fixed hypersymmetric point, and let $\ell_k$ denote a $k$th order primitive Serre-Tate direction, and let $C_{\ell_k} = \text{Spf}[\mathbb{F}[t]]\cong \mathfrak{n}_k$ denote the formally smooth local branch of $C$ at $x$ that realizes $\ell_k$. We fix polarization-compatible bases $\{\epsilon_i\}$ of $T_p(\mathcal{A}_g)$ and $\{\mu_j\}$, with dual bases $\{\epsilon_i\}$ and $\{\mu_j\}$, such that $\ell_k$ is realized by the sub-torus corresponding to the span of $\epsilon_1 \otimes \mu_1$.

Let $y \in \tau^\vee(x)$, such that the associated isogeny $\phi^\vee : A_x \rightarrow A_y$ is obtained by the quotienting $A$ by the diagonal subgroup with type $(n : n'_1, \ldots, n'_g)$. By Lemma 3.3, $y \in \mathcal{A}_g(\mathbb{F}_q)$, and therefore our hypothesis implies that $C$ passes through $y$, and there exists a formally smooth branch of $C$ through $y$ realizing any fixed $k$th order primitive Serre-Tate direction. Let $e_i'$ and $m_j'$ be defined as $e_i' = \frac{1}{p^n_t} \phi^\vee(e_i)$, and $m_j' \in T_p(\mathcal{A}_g)$ be defined analogously – $\{e_i\}$ and $\{m_j\}$ are bases of $T_p(\mathcal{A}_g)$ and $T_p(\mathcal{A}_g)$ respectively. Let $\{e_i'\}$ and $\{\mu_j\}$ denote dual bases. Consider the branch $C_{y,e_i'}$, where $\ell_k$ is realized by $e_i' \otimes \mu_1$. The Serre-Tate coordinates of $C_{y,e_i'}$ are given by functions $q_{ij}(t)$ where $q_{i1}(t) \equiv 1 + t \mod t^{k+1}$ and $q_{ij}(t) \equiv 1 \mod t^{k+1}$.

Let $\phi : A_y \rightarrow A_x$ denote the dual of $\phi^\vee$. We now lift $\phi$ to an isogeny $\tilde{\phi}$ whose source is the abelian scheme $A_{C_{y,e_i'}}$. By Proposition 3.3, the Serre-Tate coordinates of $\tilde{\phi}(A_{C_{y,e_i'}})$ are given by $Q(s) = X^{-1}Q(s^{\text{et}})Y$, where $X$ and $Y$ are diagonal matrices with $ith$ entry $p^{n_i}$ and $p^{n-i}$ respectively. It follows that the $k$th order tangent direction of $\tilde{\phi}(A_{C_{y,e_i'}})$ is equal to $\ell_k$. As $x, \ell_k$ and $\tau$ were arbitrary, the result follows.

3.4. **Zariski-Density of rank-1 formal branches.**

**Proposition 3.4.** Let $x$ be a hypersymmetric point, and suppose $D \subset \mathcal{A}_g$ is a variety such that $T_xD$ contains all rank-1 primitive subtori. Then $D = \mathcal{A}_g$.

**Proof.** We will in fact prove the stronger claim that all rank-1 formal tori are dense in $\text{Sym}^2(\mathbb{Z}_p) \otimes \mathbb{G}_m$. Letting $q_{ij}$ be the co-ordinates as before, we consider the element $v \in \mathbb{Z}_p^g$ such that $v_i = p^{N_i}$ where $N$ will be chosen to be a large integer. Then the corresponding primitive rank 1 torus has
co-ordinates \( Q_{ij}(t) = t^{P_{ij}^N + N_j} \). Now let \( f \in k[[q_{ij}]] \) be a non-zero power series which vanishes on all such primitive rank 1 tori. There are finitely many monomials in \( f \) which are dominated by all the rest. Note that \( \prod_{i,j} q_{ij}^{r_{ij}} \) evaluates at \( \bar{Q}(t) \) to \( t^{M_{ij}} \) where \( M_{ij} = \sum_{i,j} r_{ij}P_{ij}^N + N^j \). For large enough \( N \) these exponents are all distinct, which means there is a single term of \( f(\bar{Q}(t)) \) of smallest degree, which is a contradiction.

\[ \square \]

3.5. **Algebraicity of pure Serre-Tate directions.** As in \([13]\) Prop 5.4], for we may consider the completion along the diagonal in \( D \times D \) as a formal scheme \( D^\wedge \) over \( D \), and we may also view this as sitting inside the pull back \( i^*A_g^\wedge \). Now for any integer \( m > 0 \) we may also take the corresponding \( n \)'th order infinitesimal subscheme \( D_m \) sitting inside \( i^*A_g \). Now, if we go up to a level cover \( A_g^{ord} \) where the \( p \)-adic Tate module has been trivialized mod \( p^n \), then the infinitesimal scheme \( (A_g)_n \) can be trivialized as \( \text{Sym}^2(\mathbb{Z}/p^m\mathbb{Z})^g \otimes \mathbb{G}_m \) and there is a closed subscheme consisting of the pure Serre-Tate directions. Hence the same is true in \( A_g^{ord} \). Thus, pulling back, to \( D \) we see that there is a closed subvariety \( R_n \subset i^*A_g^{n} \) whose fibers at closed point consist of the unions of all pure Serre-Tate directions to order \( n \). Moreover, \( R_n \) is etale-locally a product, thus its restriction to any closed subvariety of \( D \) is irreducible. It follows that there is a Zariski-closed subscheme \( D_{st,n} \) of \( D \) consisting of all the points at which \( D \) contains all the pure Serre-tate dicretions to order \( n \).

3.6. **Proof of Theorem 1.2.** Suppose that \( D \subsetneq A_g \) is a subvariety, and let \( x \in A_g(k) \) be a hypersymmetric point. If \( D \) does not contain any point isogenous to \( x \) the result follows simply by picking a curve \( C \) passing through \( x \). Thus, at the cost of increasing the finite field \( k \), we assume \( D \) contains a \( k \)- point isogenous to \( x \). By Lemma 3.4 it follows that \( D \) does not contain all formal subtori passing through \( x \). It follows that there is some positive integer \( M \) such that \( T_x^MD \) does not contain all \( M \)'th order pure Serre-Tate directions. Thus, by the discussion in the previous subsection, \( x \in D_{st,M} \). It follows by Noetherianity that the tower \( D_{st,M} \) stabilizes, and thus for which the above statement is true for all hypersymmetric points.

We now pick a prime \( r > 6p \) and increase \( k \) so that the pre-images of \( x \) in \( A_g[r] \) are defined over \( k \). We now use Theorem 2.1 to produce a curve \( C \subset A_g[r] \) which

1. For every hypersymmetric point isogenous to \( x \), and every \( M \)'th order pure Serre-Tate direction \( \eta \) through \( x \), \( C \) admits a \( \overline{\mathbb{F}}[[t]] \) point specializing to \( \eta \), and
2. The geometric monodromy of \( C \) surjects onto \( \Gamma_g(Z_r) \times \prod_{\ell \neq p,r} \text{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \times \text{GL}_g(\mathbb{Z}_p) \).

By condition 2 above, every Hecke translate \( TC \) of \( C \) is irreducible. Since the isogenous curves to \( C \) correspond to irreducible components of its Hecke translates, it suffices to show that \( TC \not\subset D \) for every hecke operator \( T \). We factor \( T = T_eT_p \) for \( T_e \) a prime-to-\( p \) Hecke correspondence and \( T_p \) a \( p \)-power Hecke correspondence.

By Theorem 3.2 the curve \( T_pC \) also satisfies condition 1 above. Now since \( TC \) is irreducible, it follows that for each point \( y \in T_e x \) the formal neighborhood of \( TC = T_e(T_p C) \) at \( y \) contains the image of the formal neighborhood of \( T_p C \) at \( x \). Since \( T_e \) is etale, it follows that \( TC \) satisfies condition 1 above at \( y \). Therefore, \( TC \) cannot be contained in \( D \).

The result about function field valued points follows immediately by considering the generic point of \( C \).

\[ \square \]

3.7. **Proof of Theorem 1.1.** We now suppose that \( D \subsetneq (A_g)_k \) is a subvariety. Identifying \( k(T) \) with the function field of \( P^1 \) we may realize \( D \) as the generic point of a (strict) subvariety \( D \subset (A_g \times P^1)_k \), no generic points of which are contained in a fiber. We let \( E \subset (A_g)_k \) be the constant
part $D$, so that $E_F \subset D$ and $E$ is the maximum such subvariety. By increasing $k$ we assume $E$ is defined over $k$. We are looking for a curve $C \subset (A_g \times \mathbb{P}^1)_k$ such that no Hecke translate of $C$ is contained inside $D$.

As a first step, we pick $N$ sufficiently large so that $A_N$ is not a sub-quotient of $\prod_{\ell \neq p} \text{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \times \text{GL}_g(\mathbb{Z}_p)$, and let $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ be a degree $1$ map whose Galois group is $A_N$. We shall need the following two lemmas:

**Lemma 3.5.** Let $K \subset L$ be a finite separable extension of fields, and let $G$ be the Galois group of the normal closure of $L$ over $K$. Suppose that $H$ is a profinite group with no irreducible constituent in common with $G$, and let $\phi: \text{Gal}_K \to H$ be a group homomorphism. Then $\phi(\text{Gal}_L) = \phi(\text{Gal}_K)$.

**Proof.** By increasing $L$ we may as well assume $L/K$ is Galois. Then $\text{Gal}_L \subset \text{Gal}_K$ is normal, and so we get a map $G \to \phi(\text{Gal}_K)/\phi(\text{Gal}_L)$, which must be trivial by our assumptions. The statement follows.

**Lemma 3.6.** Let $D \subset (A_g \times \mathbb{P}^1)_k$ be as above. There is an integer $M$ such that for any $M$ distinct points $x_1, \ldots, x_M$ of $\mathbb{P}^1(k)$, we have $\bigcap_{i=1}^{m} D_{x_i} = E$.

**Proof.** We will base change to an uncountable field $k$, and prove the statement there (from which the original statement clearly follows).

For any $M$, consider the set $S_M$ of points $x_1$ for which one can find distinct $x_2, \ldots, x_M$ such that $\bigcap_{i=1}^{m} D_{x_i} \neq E$. This is clearly a constructible set, so is either finite or co-finite. Moreover the set $S_M$ is clearly descending (perhaps non-strictly) with $M$. It follows that if $S_M$ is not eventually empty, then $\bigcap_{m} S_M \neq \emptyset$. Thus we may find a point $x_1$ which is contained in this intersection.

We may now iterate with $x_2$, etc., to produce a countably infinite sequence of points $x_i$ such that $\bigcap_{i=1}^{m} D_{x_i} \neq E$ for any positive integer $m$. By Noetherianity this intersectio eventually stabilizes to a subvariety $E'$ properly containing $E$. This $E'$ is contained in infinitely many fibers of $D$ and hence in all of them. Thus $E'_E \subset D$ contradicting the definition of $E$.

We now construct our curve $C$ as follows. Fix a prime $r > 6p$. Following Theorem 1.2 we next construct an irreducible curve $C_0 \subset A_g[r]$ such that

1. No Hecke translate of $C_0$ is contained in $E$.
2. The geometric monodromy of $C \cap A^\text{ord}_g[r]$ surjects onto $\Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \text{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \times \text{GL}_g(\mathbb{Z}_p)$, where $r > 6p$ is a prime as in the proof of Theorem 1.2.

We do this the same way as in the proof of Theorem 1.1. Next, we pick a map $\psi: C_0 \to \mathbb{P}^1$ arbitrarily, and let $C_1$ be the graph of this map in $A_g[r] \times \mathbb{P}^1$. Finally, we let $C$ be the base change of this map under $\phi \times \text{Id}_{A_g[r]}$, for $N > \deg \psi$. It follows that $C$ is irreducible since $A_N$ does not admit a non-trivial homomorphism to $S_{\deg \psi}$.

By lemma 5.6 the monodromy group of $C$ surjects onto $\Gamma_g(\mathbb{Z}_r) \times \prod_{\ell \neq p,r} \text{P}(\text{Sp}_{2g}(\mathbb{Z}_\ell)) \times \text{GL}_g(\mathbb{Z}_p)$. It follows that for each Hecke operator $T$, that $TC$ is irreducible. Suppose that $TC \subset D$.

For a generic point $s \in \mathbb{P}^1$, the fiber $\phi^{-1}(s)$ consists of $N$ points $x_1, \ldots, x_N$, and so it follows from theorem 3.6 that

$$(TC)_{x_j} = \bigcap_{i=1}^{m} (TC)_{x_i} \subset \bigcap_{i=1}^{m} D_{x_i} = E.$$

\[\text{One may construct } E \text{ by intersecting all base changes of } D \text{ along automorphisms of } F \text{ over } \overline{k}.\]
It follows that \( TC \subset E \). Since \( E \) is constant, by projecting to \( A_g \), this in fact means that \( TC_0 \subset E \) which is a contradiction.

4. ADDITIONAL RESULTS

4.1. The case of number fields. Let \( H \subset A_g \) denote the Hodge line bundle. For any proper curve \( C \subset A_g \), we define \((C.H)\) to denote the degree of this line bundle pulled back to \( C \).

Lemma 4.1. Let \( T \) denote a Hecke operator on \( A_g \). Then, for any proper curve \( C \subset A_g \) we have 
\[
\deg T(C) = \deg(T) \cdot \deg(C).
\]

Proof. This follows directly from the fact that the two different pull-backs of \( H \) to the graph of the Hecke correspondence are equal.

Definition. Let \( X \) be some variety defined over a field \( L \), and let \( K/L \) denote a finite extension. We say that a point \( x \in X(K) \) is primitive if \( K \) is the smallest field of definition of \( x \).

Lemma 4.2. Let \( C \subset A_g \) denote a curve defined over \( \mathbb{Q} \) with maximal monodromy. There exists a constant \( c \in \mathbb{Z} \) such that for any positive integer \( n \in \mathbb{Z} \), there exists a number field \( K/\mathbb{Q} \) which is Galois and has degree \( \geq n \), and a primitive point \( x \in C(K) \) such that the monodromy of \( A_x \) has index bounded by \( c \) in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \).

Proof. Let \( \pi : C \rightarrow \mathbb{P}^1 \) denote a finite map of degree \( d \), and consider \( B = \text{Res}_{C/\mathbb{P}^1}(A) \), which is an abelian scheme over an open subset \( U \subset \mathbb{P}^1 \) of dimension \( gd \). Work of Zywina [16, Theorem 1.1] implies that for any number field \( L \), there exists a constant \( c_L \) such that for a density 1 set of points \( y \in U(L) \), the monodromy of \( B_y \) has index bounded by \( c_L \) in the monodromy of \( B_U \). In Section 1.2 of loc. cit., Zywina explicitly describes the constant \( c_L \). Indeed, if \( L/\mathbb{Q} \) were a Galois extension which has the property that the monodromy of \( B_{UL} \) were the same as the monodromy of \( B_{U\mathbb{Q}} \), then the constant \( c_L \) equals \( c_\mathbb{Q} \). It is easy to see that there exist fields \( L \) with this property with \([L: \mathbb{Q}]\) arbitrarily large; for example, by choosing \( L \) so that \( \text{Gal}(L/\mathbb{Q}) = G(\mathbb{Z}/\ell^n\mathbb{Z}) \), where \( G/\mathbb{Z}_\ell \) is some exceptional simple group and \( \ell \) is any prime.

We now use the above discussion to deduce the existence of number fields \( K \) with arbitrarily large degree and points \( x \in C(K) \) with monodromy as claimed. Let \( y \in U(L) \) satisfy the conclusions of the previous paragraph. By Hilbert irreducibility, we may also assume that that \( \pi^{-1}(y) = \{x_1 \ldots x_d\} \), where \( x_i \in C(K_i) \) where \( K_i/L \) are conjugate degree \( d \) extensions of \( L \) and the points \( x_i \) are conjugate to each other. We have that \( \prod_{i=1}^d A_{x_i} \) descends to \( L \) and equals \( B_y \), and that the monodromy of \( (\prod_{i=1}^d A_{x_i})_{K_1} \) equals the intersections of the monodromies of \( B_C \) and \( B_y \) inside the monodromy of \( B_U \). It then follows that the monodromy of \( (\prod_{i=1}^d A_{x_i})_{K_1} \) has index bounded by \( c = c_\mathbb{Q} \) inside the monodromy of \( B_C \), and projecting onto the first factor yields that the monodromy of \( A_{x_1} \) has index bounded by \( c \) in the monodromy of \( A_C \), which is equal to \( \text{GSp}(\hat{\mathbb{Z}}) \) by our assumption on the curve \( C \).

\[ \square \]

Theorem 4.3. Let \( V \subset A_g \) denote a closed subvariety. Then, there exists an abelian variety over a number field which is not isogenous to any point of \( V \).

Proof. Without loss of generality, we suppose that \( V \) is a divisor, and by replacing \( V \) by the union of its Galois conjugates, we may assume that \( V \) is defined over \( \mathbb{Q} \). We may proceed as in the positive characteristic setting to find a curve \( C \subset A_g \) having maximal monodromy which satisfies \( T(C) \not\subset V \) for every Hecke correspondence \( T \). In fact, by using the characteristic-zero Lefschetz theorem detailed in [8, Theorem 1.2] for smooth quasi-projective varieties, we may find a proper curve \( C \subset A_g \) defined over \( \mathbb{Q} \) with maximal monodromy, and which satisfies such that \( T(C) \not\subset V \) for every Hecke correspondence \( T \).

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We first note that for a proper curve $C \subset A_g$, we have
\begin{equation}
(T(C) \cdot V) = \deg(T)(C \cdot V).
\end{equation}
This is true when $V = H$ by Lemma 4.1. In general, we note that $A_g$ has Picard group $\mathbb{Z}$, and hence $(C' \cdot V) = n_V(C' \cdot H)$ for any proper curve $C \subset A_g$, where the divisors $[H], [V]$ satisfy $n_V[H] = V$ in Pic$(A_g)$. By Lemma 4.2 there exists a Galois extension $K/Q$ of degree $\geq n$ and a primitive point $x \in C_Q$ whose monodromy has index bounded by $c$ in $\text{GSp}_{2g}(\hat{\mathbb{Z}})$, where we now choose $n > c(C \cdot V)$. For any fixed Hecke correspondence, we have that $T(x)$ is stabilized by $\text{Gal}(\overline{K}/K)$, and the monodromy assumptions on $A_x$ implies that each orbit has at least $\frac{\deg(T)}{c}$ elements. The same statement holds for $\sigma(x)$ with $\sigma \in \text{Gal}(K/Q)$, and further, we have that the sets $T(x)$ and $T(\sigma(x))$ are in the same $\text{Gal}(\overline{Q}/Q)$ orbit.

Suppose now that $T(x)$ intersected $V$ non-trivially for some Hecke operator $T$. As we have assumed $V$ is defined over $Q$, it follows that $T(x) \cap V$ is stable under $\text{Gal}(\overline{Q}/Q)$. The previous paragraph implies that $T(\sigma(x)) \cap V$ has size $\geq \frac{\deg(T)}{c}$ for every $\sigma \in \text{Gal}(K/Q)$. Further, even if the same point $y$ is contained in $T(\sigma_1(x)) \cap V$ and $T(\sigma_2(x)) \cap V$, this point contributes twice to the intersection $T(C) \cdot V$, because this implies that $T(C)$ must be singular at $y$, with branches corresponding to the branch of $C$ through $\sigma_1(x)$ and through $\sigma_2(x)$. It therefore follows that
\begin{equation}
T(C) \cdot V \geq \frac{\deg(T)}{c} \cdot n.
\end{equation}
But our initial choice of $n$ was such that $n > c \cdot N_V \cdot \deg(C)$, and so Equations (4.1.1) and (4.1.2) cannot both hold simultaneously. It therefore follows that $T(x)$ must be disjoint from $V$ for every Hecke correspondence $T$, and the result follows.

4.2. An analogue over $\mathbb{F}_p$. We will now prove that for $D \subset A_g$ a divisor, there exists an ordinary $x \in A_g(\mathbb{F}_p)$ such that $T(x) \not\subset D$ for every prime-to-$p$ Hecke operator $T$.

Proof of Theorem 4.3. By the main result of this paper, there exists a generically ordinary $C \subset A_g$ that satisfies $T(C) \not\subset A_g$ for every Hecke operator $T$. An identical intersection-theoretic argument as in the proof of Theorem 4.3 yields that for a point $x \in C(\mathbb{F}_p)$, whose minimal field of definition is $\mathbb{F}_q$ where $q$ is a large enough power of $p$, $T(x) \not\subset D$. The result follows.

\begin{flushright}
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\end{flushright}

References

[1] C.L.Chai. Families of ordinary abelian varieties: canonical coordinates, $p$-adic monodromy, Tate-linear subvarieties and Hecke orbits. Preprint.
[2] C.-L.Chai, G.Faltings. Degenerations of abelian varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 22. Springer-Verlag, Berlin, 1990. xii+316 pp.
[3] C-L.Chai, F.Oort. Moduli of abelian varieties and $p$-divisible groups. Arithmetic geometry, 441–536, Clay Math. Proc., 8, Amer. Math. Soc., Providence, RI, 2009.
[4] C-L.Chai, F.Oort. Abelian varieties isogenous to a Jacobian. Ann. of Math. (2) 176 (2012), no. 1, 589–635.
[5] F.Charles, B.Poonen. Bertini irreducibility theorems over finite fields J. Amer. Math. Soc. 29 (2016), 81-94.
[6] H.Esnault. Survey on some aspects of Lefschetz theorems in algebraic geometry. Rev. Mat. Complut. 30 (2017), no. 2, 217–232.
[7] H.Esnault, L.Kindler. Lefschetz theorems for tamely ramified coverings. Proc. Amer. Math. Soc. 144 (2016), no. 12, 5071–5080.
[8] M.Goresky, R.MacPherson. Stratified Morse theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 14. Springer-Verlag, Berlin, 1988.
[9] Anvin,G. Q.He, A.N.Shankar. Just likely intersections in Hilbert modular varieties. In preparation.

$^4$This can happen if $\sigma_1(x)$ is isogenous to $\sigma_2(x)$. 

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[10] N.Katz *Serre-Tate local moduli*. Algebraic surfaces (Orsay, 1976–78), pp. 138–202, Lecture Notes in Math., 868, Springer, Berlin-New York, 1981.

[11] D.Masser, U.Zannier. *Abelian varieties isogenous to no Jacobian*. Ann. of Math. (2) 191 (2020), no. 2, 635–674.

[12] A.N.Shankar, J.Tsimerman *Unlikely intersections in finite characteristic*. Forum Math. Sigma 6 (2018), Paper No. e13.

[13] J.Tsimerman. *The existence of an abelian variety over $\mathbb{Q}$ isogenous to no Jacobian*. Ann. of Math. (2) 176 (2012), no. 1, 637–650.

[14] J.Tsimerman. *The André-Oort conjecture for $\mathbb{A}_g$*. Ann. of Math. (2) 187 (2018), no. 2, 379–390.

[15] B.Poonen. *Bertini theorems over finite fields*. Ann. of Math., 160 (2004), 1099-1127

[16] D.Zywina. *Families of abelian varieties and large Galois images*. [arXiv:1910.14174](https://arxiv.org/abs/1910.14174)