The solution of 4-dimensional Schrodinger equation for polynomial inverse potential and its partner potential by using ansatz wavefunction method

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Abstract. The radial part of 4-dimensional Schrodinger equation for Polynomial Inverse potential and its partner potential was solved by using the Ansatz Wavefunction method. By applying of the properties and operators of the Supersymmetric Quantum Mechanic method the partner potential was constructed. The eigenvalues and eigenfunctions of the Polynomial Inverse potential and the eigenvalues and eigenfunctions of its partner potential were different. The partner potential of Polynomial Inverse potential for $n > 1$ has eigenvalue greater than Polynomial Inverse potential as its original potential.

1. Introduction
The D-dimensional Schrodinger equation is the wave equations for higher dimensions in quantum mechanics system [1-2]. The solution of D-dimensional Schrodinger equation for some potentials are exactly solved by methods such as Ansatz Wavefunction method [1,2], Nikiforov-Uvarov (NU) method [3-6], Supersymmetric Quantum Mechanic (SUSY QM) [4,8,16], Asymptotic Iteration Method (AIM) [9], Factorization methods [10] and Romanovski polynomial method [11]. The Ansatz Wavefunction method was a method for the analyze the part radial of D-dimensional Schrodinger equation [1-2]. By using and applying the operators of SUSY QM, the properties of SUSY QM and the ground state wave functions of the original potential, the partner potential was constructed [4].

This study aims to construct partner potential and determine eigenvalue and eigenfunction of the generalized Polynomial Inverse potential. In this paper, the solution of radial D-dimensional Schrodinger equation for Polynomial Inverse potential and its partner potential was studied using Ansatz Wavefunction method. This paper is organized as follows. Introduction is presented in Section 1. The Ansatz Wavefunction method and SUSY QM method is presented in Section 2. The results and discussion are presented in Section 3 and conclusion in Section 4.

2. Method
In this section, we review the solution of D-dimensional Schrodinger equation with Ansatz Wavefunction method and Supersymmetry Quantum Mechanics method. The Generally, Schrodinger equation in D-dimensional [1,12] given as

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi(r,\Omega) + V(r,\Omega)\psi(r,\Omega) = E\psi(r,\Omega)$$  \hspace{1cm} (1)

with

$$\psi(r,\Omega) = \frac{1}{r^{D-1}} U(r) Y_{\ell_1\ell_2\ell_3}(\hat{r})$$ \hspace{1cm} ; \hat{r} = \theta_1, \theta_2, \cdots, \theta_{D-1}$$  \hspace{1cm} (2)

The radial part and angular part of 4-dimensional Schrodinger equation obtained by inserted Eq. (2) into Eq. (1) and solve of using variable separation method with

$$Y_{\ell_1\ell_2\ell_3}(\theta_1,\theta_2,\theta_3) = P(\theta_1)P(\theta_2)P(\theta_3)$$ is given as [13]:

$$\frac{d^2 U}{dr^2} + \left(\lambda_0 + \frac{3}{4}\right) \frac{1}{r^2} U + \frac{2\mu}{\hbar^2} [E - V(r)]U = 0$$  \hspace{1cm} (3)
\[
\frac{\partial^2 P(\theta)}{\partial \theta^2} - V(\theta) P(\theta) + \lambda_2 P(\theta) = 0
\]
(4)

\[
\frac{1}{P(\theta_2)} \left( \frac{1}{\sin \theta_2} \left( \frac{\partial}{\partial \theta_2} \sin \theta_2 \frac{\partial P(\theta_2)}{\partial \theta_2} \right) \right) - V(\theta_2) - \frac{\lambda_2}{\sin^2 \theta_2} = 0
\]
(5)

\[
\frac{1}{P(\theta_3)} \left( \frac{1}{\sin^2 \theta_3} \left( \frac{\partial}{\partial \theta_3} \sin^2 \theta_3 \frac{\partial P(\theta_3)}{\partial \theta_3} \right) \right) - V(\theta_3) + \lambda_3 - \frac{\lambda_2}{\sin^2 \theta_3} = 0
\]
(6)

where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) is variable separation constant.

In this paper the potential energy of Polynomial Inverse potential generally given as [14]:

\[
V(r) = \frac{a}{r^6} + \frac{b}{r^4} + \frac{c}{r^2} + \frac{d}{r^2} + \frac{e}{r} + \frac{f}{r}
\]
(7)

with \( a, b, c, d, e \) and \( f \) is constant coefficient.

### 2.1 Review of Ansatz wave function method

The wavefunction that can not be completed exactly can be solved by the Ansatz wavefunction method. Ansatz wavefunction method is a method used specifically to determine the solution of the radial part [1]. Schrodinger equation The D-dimensional of the radial part of Equation (3) is reduced to two-order differential equation (in atomic units \( \hbar = 1; m = 1 \)) [1,12] is

\[
\left\{ \frac{d^2}{dr^2} - \frac{l_{D-1}(l_{D-1} + D - 2) + (D - 1)(D - 3)/4}{r^2} \right\} U(r) = [-E + V(r)] U(r)
\]
(8)

where \( l_{D-1} = l_{D-1} + D - 2 \), \( l_{D-1} \) is angular momentum, \( D \) is the spatial dimension and \( \lambda_{D-1} \) is the variable separation constant [1]. Based on Equation (8), the Schrodinger equation of the radial part for \( D = 4 \) is

\[
\left\{ \frac{d^2}{dr^2} - \frac{1}{r^2} \left( \lambda_3 + \frac{3}{4} \right) \right\} U(r) = [-E + V(r)] U(r)
\]
(9)

where

\[
\lambda_3 = l_3(l_3 + 2)
\]
(10)

The solution of Eq. (9) as a radial wave function \( U(r) \) of the Ansatz method [1,2] is expressed as:

\[
U(r) = f_n(r) \exp[g(r)]
\]
(11)

where

\[
f_n(r) = \begin{cases} 
1 & \text{for } n = 0, \\
\prod_{j=1}^{n} (r - \sigma_j^{(n)}) & \text{for } n = 1, 2, 3, ...
\end{cases}
\]
(12)

and \( g(r) \) is the wave function in exponential form. Substitution Eq. (9) into Eq. (11), we obtained

\[
-E_n + V(r) + \frac{\lambda_3 + 3/4}{r^2} = g''(r) + (g'(r))^2 + \frac{2f_n''(r)}{f_n(r)} g'(r)
\]
(13)

Equation (13) is a general equation that will be used in the solution of radial part the 4-dimensional Schrodinger equation to determine the energy and wave function by using the ansatz wavefunction method.

### 2.2 Review of SUSY QM
Supersymmetry Quantum Mechanics (SUSY QM) is the simplest model of SUSY field theory developed based on Witten's proposal [13]. Witten defined two charge operators are commute with Hamiltonian \( H_{ss} \) of the supersymmetry quantum system [4,8,15] is given as

\[
H_{ss} = \begin{pmatrix}
\frac{d^2}{dx^2} + \frac{d\varphi(x)}{dx} + \varphi^2(x) & 0 \\
0 & -\frac{d^2}{dx^2} - \frac{d\varphi(x)}{dx} + \varphi^2(x)
\end{pmatrix}
\]

with partner Hamiltonian \( H_{-} = H_{1} \) and \( H_{+} = H_{2} \). By setting the new SUSY operators for raising operator \( A^{+} = -\frac{d}{dx} + \varphi(x) \) and lowering operator \( A = \frac{d}{dx} + \varphi(x) \). The SUSY Hamiltonians of equation (14) are

\[
H_{-}(x) = H_{1} = A^{+}A, \text{ and } H_{+}(x) = H_{2} = AA^{+}
\]

and

\[
A\psi_{0}(x) = A\psi_{0} = 0
\]

The partners potential in the SUSY QM method is \( V_{-} = V_{1} \) and \( V_{+} = V_{2} \) are

\[
V_{-}(x) = V_{1} = \varphi^{2}(x) - \varphi'(x) \quad \text{and} \quad V_{+}(x) = V_{2} = \varphi^{2}(x) + \varphi'(x)
\]

where \( \varphi \) is superpotential. The partner potential \( V_{1} \) is determined as

\[
V_{-}(x; a_{0}) = V_{1}(x; a_{0}) = V_{eff}(x) - E_{0}
\]

with \( V_{eff} \) is effective potential and \( E_{0} \) is the ground state energy.

By manipulating equation (17) and by applying equation (11) and equation (18), we construct the new partner potential \( V_{2} \) as

\[
V_{2}(x) = V_{eff}(x) - E_{0} - 2\frac{d}{dx} \left( \frac{d\psi_{0}/dx}{\psi_{0}} \right)
\]

where \( \psi_{0} \) is the ground state wave function [4,8]. The eigenvalue and the eigenfunction of partner potential is also determined by using ansatz wavefunction method.

3. Results and Discussion

In this section, the solution of the Schrodinger equation in 4-dimensional by using ansatz wavefunction method and Supersymmetry Quantum Mechanics method. The exponential wave function \( g(r) \) of the Polynomial Inverse potential obtained by applying the supersymmetry method is

\[
g(r) = \frac{1}{2} \frac{A}{r^{2}} + \frac{B}{r} + Cr + D \ln r
\]

The solution for \( n = 0 \) then \( f_{0}(r) = 1 \). The Equation (7) and Equation (20) substituted into Equation (13) are obtained

\[
-E_{0} + \frac{a}{r^{5}} + \frac{b}{r^{4}} + \frac{c}{r^{3}} + \frac{d}{r^{2}} + \frac{e}{r} + \frac{f}{r} = \left[ \frac{A^{2}}{r^{5}} + \frac{2AB}{r^{4}} + \frac{B^{2} - 2AD + 3A}{r^{3}} + \frac{2B(1 - D) - 2AC}{r^{2}} + \frac{2BC - \left( \lambda_{3} + \frac{3}{4} \right)}{r} + \frac{2CD + C^{2}}{r} \right]
\]

By comparing the two sections in Eq. (21), we get
\[ a = A^2, \quad b = 2AB, \quad c = B^2 - 2AD + 3A, \quad d = 2B(1-D) - 2AC, \]
\[ e = D(D-1) - 2BC - \left( \lambda_3 + \frac{3}{4} \right), \quad f = 2CD, \]
\[ -E = C^2 \]

Furthermore, from Equation (22) we get the parameter value
\[ A = \pm \sqrt{a} \]  

(25)

The value of parameter \( A \) used in equation (25) is negative. Equation (25) is substituted into Eq. (22) the equation is obtained
\[ B = -\frac{b}{2\sqrt{a}} \]  

(26)

Then Equation (25) and Equation (26) are substituted into Eq. (22) we get
\[ D = \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \]  

(27)

And then Equations (25) to Eq. (27) are substituted into Equation (23)
\[ \frac{b}{\sqrt{a}} C^2 - \left( e + \lambda_3 + \frac{9}{4} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right) C + \frac{1}{2} \left( \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right) f = 0 \]  

(28)

The value of \( C \) is determined from Equation (28) by using algebra and we obtained
\[ C = \left[ e + \lambda_3 + \frac{9}{4} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right] \pm \sqrt{\left( e + \lambda_3 + \frac{9}{4} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right)^2 - 4 \left( \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right) b} \]  

(29)

Furthermore, Equation (29) is substituted into Eq. (24), so that an eigenvalue we get:
\[ E_0 = -\frac{a}{4b^2} \left[ e + \lambda_3 + \frac{9}{4} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right] \pm \sqrt{\left( e + \lambda_3 + \frac{9}{4} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right)^2 - 4 \left( \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right) b} \]  

(30)

where \( E_0 \) is ground state eigenvalue. Equation (12), Eq. (20) and Eq. (25) to Eq. (27) are substituted into Eq. (11), we obtained
\[ U_0(r) = N_0 r^\left( \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}} \right) \exp \left[ -\frac{\sqrt{a}}{2} \frac{1}{r} - \frac{b}{2\sqrt{a}} \frac{1}{r} + Cr \right] \]  

(31)

where \( U_0(r) \) is ground state eigenfunction. Equation (30) and Equation (31) are used to determine the potential partner radial part.

The solution for \( n = 1 \), then \( f_1(r) = r - \sigma_1^{(1)} \). The solution of this case is done with the same steps as for \( n = 0 \) and then we obtained
\[ -E_1 + \frac{a}{r^5} + \frac{b}{r^3} + \frac{c}{r} + \frac{d}{r^2} + \frac{e}{r} \left( r - \sigma_1^{(1)} \right) = \]
\[ \left( \frac{\lambda^2}{r^5} + \frac{2AB}{r^3} + \frac{B^2 - 2AD + 3A}{r^2} + \frac{2B(1-D) - 2AC}{r^3} \right) + \frac{D(D-1) - 2BC - \left( \lambda_3 + \frac{3}{4} \right)}{r^2} + \frac{2CD}{r} = \]
\[ \left( r - \sigma_1^{(1)} \right) - \frac{2A}{r^3} - \frac{2B}{r^2} + 2C + \frac{2D}{r} \]  

(32)

4
By comparing the two sections in Eq. (32) we get
\[ -a\sigma^{(1)}_1 = -A^2\sigma^{(1)}_1 \]  
\[ a - b\sigma^{(1)}_1 = A^2 - 2AB\sigma^{(1)}_1 \]  
\[ b - c\sigma^{(1)}_1 = 2AB - [B^2 - 2AD + 3A]\sigma^{(1)}_1 \]  
\[ c - d\sigma^{(1)}_1 = B^2 - 2AD + A - (2B(1 - D) - 2AC)\sigma^{(1)}_1 \]  
\[ d - e\sigma^{(1)}_1 = -2BD - 2AC - (D(D - 1) - 2BC - (\lambda_3 + 3/4))\sigma^{(1)}_1 \]  
\[ e - f\sigma^{(1)}_1 = D(D - 1) - 2BC + 2D - (\lambda_3 + 3/4) - 2CD\sigma^{(1)}_1 \]  
\[ f + E\sigma^{(1)}_1 = 2C(D + 1) - C^2\sigma^{(1)}_1 \]  
\[ -E_i = C^2 \]  

From Equation (34) to Eq. (39) we get parameter values
\[ A = \pm \sqrt{a}, \quad B = -\frac{b}{2\sqrt{a}}, \quad D = \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}, \]  
\[ C = \left( e + \lambda_3 + \frac{3}{4} \right) \pm \sqrt{\left( e + \lambda_3 + \frac{3}{4} \right)^2 - 4\left(\sigma^{(0)}_1 + \frac{b}{2\sqrt{a}}\right)\left(\frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}\right)f} \]  
\[ 4\left(\sigma^{(0)}_1 + \frac{b}{2\sqrt{a}}\right) \]  

Furthermore, Equation (42) is substituted into Eq. (40), then obtained
\[ E_i = \left( e + \lambda_3 + \frac{3}{4} \right) \pm \sqrt{\left( e + \lambda_3 + \frac{3}{4} \right)^2 - 4\left(\sigma^{(0)}_1 + \frac{b}{2\sqrt{a}}\right)\left(\frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}\right)f} \]  
\[ 4\left(\sigma^{(0)}_1 + \frac{b}{2\sqrt{a}}\right) \]  

where \( E_i \) is the first level excited energy. Equation (12), Equation (20) and Equation (41) are substituted into Eq. (11) we get
\[ U_i(r) = N_i(r - \sigma^{(1)}_1)r^{\frac{3}{2} - \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}} \exp\left[ -\frac{\sqrt{a}}{2} \frac{1}{r^2} - \frac{b}{2\sqrt{a}} \frac{1}{r} + Cr \right] \]  

where \( U_i(r) \) is the eigenfunction of first level.

The solution for \( n = 2; \ f_2(r) = (r - \sigma^{(2)}_1)(r - \sigma^{(2)}_1) \). The solution of this case is carried out with the same steps as for \( n = 0 \) and \( n = 1 \) and the following equation is obtained
\[ a = A^2, \quad b = 2AB, \quad c = B^2 - 2AD + 3A \]  
\[ e - f(\sigma^{(2)}_1 + \sigma^{(2)}_1) = D(D - 1) - 2BC - (\lambda_3 + 3/4) + 2 + 4D - 2C(D + 1)(\sigma^{(2)}_1 + \sigma^{(2)}_1) \]  
\[ f = 2C(D + 1) \]  
\[ E_2 = -C^2 \]  

By using Eq. (45) - Eq. (47) we obtain the following parameter values
\[ A = \pm \sqrt{a}, \quad B = -\frac{b}{2\sqrt{a}}, \quad D = \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}, \]

\[ C = \left( e + \lambda_{3} + \frac{3}{4} - D - 2 \right) \pm \sqrt{\left( e + \lambda_{3} + \frac{3}{4} - D - 2 \right)^2 - 4 \left( \sigma_{2}^{(2)} + \sigma_{1}^{(2)} \right) Df \}

\[ = 4 \left( \sigma_{2}^{(2)} + \sigma_{1}^{(2)} \right) + \frac{b}{2\sqrt{a}} \] (50)

Furthermore, Equation (50) is substituted into Eq. (48), the eigenvalue is obtained

\[ E_{2} = -\left( e + \lambda_{3} + \frac{3}{4} - D - 2 \right) \pm \sqrt{\left( e + \lambda_{3} + \frac{3}{4} - D - 2 \right)^2 - 4 \left( \sigma_{2}^{(2)} + \sigma_{1}^{(2)} \right) Df \}

\[ = 4 \left( \sigma_{2}^{(2)} + \sigma_{1}^{(2)} \right) + \frac{b}{2\sqrt{a}} \] (51)

where \( E_{2} \) is the second level excited energy. Equation (12), Equation (20) and Equation (49) are substituted into Eq. (11), we get

\[ U_{2}(r) = N_{2} (r - \sigma_{2}^{(2)})(r - \sigma_{1}^{(2)}) r^{\frac{3}{2}} \exp \left[ -\frac{\sqrt{a}}{2r} - \frac{1}{2\sqrt{a}} \right] + Cr \] (52)

Where \( U_{2}(r) \) is the second level eigenfunction.

The energy equations and the wave functions in generally from Equations (30), (43), (51), (31), (44) and Eq. (52) are consecutively

\[ E_{n} = \left[ e + \lambda_{n} + \frac{3}{4} (1-n)D - n(n-1) \right] \pm \sqrt{\left( e + \lambda_{n} + \frac{3}{4} (1-n)D - n(n-1) \right)^2 - 4 \left( \sum_{i=1}^{n} \sigma_{i}^{(n)} + \frac{b}{2\sqrt{a}} \right) Df} \]

\[ = 4 \left( \sum_{i=1}^{n} \sigma_{i}^{(n)} \right) + \frac{b}{2\sqrt{a}} \] (53)

and

\[ U_{n}(r) = N_{n} \left[ \prod_{i=1}^{n} (r - \sigma_{1}^{(n)}) \right] r^{\nu} \exp \left[ \frac{A}{2r^2} + \frac{B}{r} + Cr \right] \] (54)

where

\[ A = -\sqrt{a}, \quad B = -\frac{b}{2\sqrt{a}}, \quad D = \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}, \]

\[ C = \left( e + \lambda_{n} + \frac{3}{4} (1-n)D - n(n-1) \right) \pm \sqrt{\left( e + \lambda_{n} + \frac{3}{4} (1-n)D - n(n-1) \right)^2 - 4 \left( \sum_{i=1}^{n} \sigma_{i}^{(n)} + \frac{b}{2\sqrt{a}} \right) Df} \]

\[ = 4 \left( \sum_{i=1}^{n} \sigma_{i}^{(n)} \right) + \frac{b}{2\sqrt{a}} \] (55)

\( E_{n} \) is eigenvalue, \( U_{n}(r) \) adalah eigenfunction of Polynomial Inverse potential. Furthermore, from Equation (53) and Equation (54), the potential partner construction is obtained
with $E_0$ in Eq. (30), parameters $A$, $B$, $D$ and $C$ in Equations (25) to Equation (27). Equation (55) is a potential partner of the radial part of the Inverse Polynomial potential. The potential partner will be solved in the 4-dimension Schrödinger equation using the ansatz wave function method.

The exponential wave function $g(r)$ potential partner of Polynomial Inverse potential obtained by applying the supersymmetry method is

$$g(r) = \frac{1}{2} \frac{A'}{r^2} + \frac{B'}{r} + C' r + D' \ln r$$

(56)

The solution for $n = 0$; $f_0(r) = 1$. Equation (12) and Equation (56) substituted into Equation (13) we obtained

$$-E_0 + \frac{a}{r^6} + \frac{b}{r^5} + \frac{c}{r^4} + \frac{d + 2A}{r^3} + \frac{e + \lambda_3 + \frac{3}{4} + 2B + 2D}{r^2} + \frac{f}{r} - E_0 - 2C = \frac{A'^2}{r^6} + \frac{2A'B'}{r^5} + \frac{B^2 + 2A'D^2 + 3A'}{r^4} + \frac{2B'(1-D') - 2A'C'}{r^3} + \frac{D'(D'-1) - 2B'C' - \left(\lambda_3 + \frac{3}{4}\right)}{r^2} + \frac{2C'D' + C'^2}{r}$$

(57)

By comparing the two sections in Eq. (57), we get

$$a = A'^2, \quad b = 2A'B', \quad c = B^2 - 2A'D^2 + 3A', \quad d + 2A = 2B'(1-D') - 2A'C',$$

(58)

$$e + \lambda_3 + \frac{3}{4} + 2B + 2D = D'(D'-1) - 2B' + C' - \left(\lambda_3 + \frac{3}{4}\right)$$

(59)

$$f = 2C'D'$$

(60)

$$E_0' = C'^2 - E_0 - 2C$$

(61)

From Eq. (58) we obtain

$$A' = \pm \sqrt{a}$$

(62)

$$B' = -\frac{b}{2\sqrt{a}}$$

(63)

$$D' = \frac{3}{2} + \frac{c}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}$$

(64)

and then, from Equation (59), (60) and Eq. (62) through Eq. (64) we get

$$\frac{b}{\sqrt{a}} C'^2 - \left(e + \lambda + \lambda_3 + \frac{3}{2} \frac{b}{\sqrt{a}} + \frac{3}{2} \frac{\lambda}{\sqrt{a}} - \frac{b^2}{2\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}\right) C + \frac{1}{2} \left(\frac{3}{2} + \frac{\lambda}{\sqrt{a}} - \frac{b^2}{8a\sqrt{a}}\right) f = 0$$

(65)

From Eq. (65) by using algebra, we have

$$C' = \frac{\left(\lambda + \frac{3}{2} \frac{\lambda}{\sqrt{a}} - \frac{b^2}{2\sqrt{a}} + \frac{b^2}{8a\sqrt{a}}\right) \pm \left(\lambda + \frac{3}{2} \frac{\lambda}{\sqrt{a}} - \frac{b^2}{2\sqrt{a}} + \frac{b^2}{8a\sqrt{a}}\right)^2 - 4 \left(\frac{3}{2} - \frac{\lambda}{\sqrt{a}} + \frac{\lambda^2}{4a}\right) \frac{b}{\sqrt{a}} f}{2\frac{b}{\sqrt{a}}}$$

(66)

with

$$\lambda = e + \lambda + \lambda_3 + \frac{3}{2} - \frac{b}{\sqrt{a}}$$

(67)
Equation (66) is substituted into Eq. (61) we get

$$E_0 = -\frac{a}{4b^2} \left[ \left( \frac{\lambda + 3}{2} + \frac{c}{2a\sqrt{\alpha} - b^2} \right) \pm \sqrt{\left( \frac{\lambda + 3}{2} + \frac{c}{2a\sqrt{\alpha} - b^2} \right)^2 - 4 \left( \frac{3}{2a\sqrt{\alpha} - b^2} \right)^2} \right] ^2 - E_0 - 2C \quad (68)$$

where $E_0$ is the ground state energy. Equation (12), Equations (56) and Equations (62) through Equation (64) are substituted to Eq. (11) we have

$$U'_0(r) = N_0 r \left( \frac{3}{2} \cdot \frac{c}{2a\sqrt{\alpha} - b^2} \right) \exp \left[ -\frac{\sqrt{a}}{2} \cdot \frac{1}{r^2} - \frac{b}{2\sqrt{\alpha}} \cdot \frac{1}{r} + C' r \right] \quad (69)$$

where $U'_0(r)$ is the ground state eigenfunction.

The solution for $n = 1$: $f_1(r) = r - \sigma_1^{(1)}$. The solution of this case is done with the same steps as for $n = 0$, we obtained

$$E_1 = -\left[ \left( \lambda + 2D' \right) \pm \sqrt{\left( \lambda + 2D' \right)^2 - 4 \left( \sigma_1^{(1)} + \frac{b}{2\sqrt{\alpha}} \right) D' f} \right] ^2 - E_0 - 2C \quad (70)$$

and

$$U'_1(r) = N_1 \left( r - \sigma_1^{(1)} \right) r \left( \frac{3}{2} \cdot \frac{c}{2a\sqrt{\alpha} - b^2} \right) \exp \left[ -\frac{\sqrt{a}}{2} \cdot \frac{1}{r^2} - \frac{b}{2\sqrt{\alpha}} \cdot \frac{1}{r} + C' r \right] \quad (71)$$

where $E_1$ is the first level of excited energy dan $U'_1(r)$ is the first level of eigenfunction.

The solution for $n = 2$: $f_2(r) = \left( r - \sigma_2^{(2)} \right) \left( r - \sigma_1^{(2)} \right)$. The solution of this case is done with the same steps as $n = 0$ and $n = 1$ we have

$$E_2 = -\left[ \left( \lambda + D' - 2 \right) \pm \sqrt{\left( \lambda + D' - 2 \right)^2 - 4 \left( \sigma_2^{(2)} + \sigma_1^{(2)} \right) D' f} \right] ^2 - E_0 - 2C \quad (72)$$

and

$$U'_2(r) = N_2 \left( r - \sigma_2^{(2)} \right) \left( r - \sigma_1^{(2)} \right) r \left( \frac{3}{2} \cdot \frac{c}{2a\sqrt{\alpha} - b^2} \right) \exp \left[ -\frac{\sqrt{a}}{2} \cdot \frac{1}{r^2} - \frac{b}{2\sqrt{\alpha}} \cdot \frac{1}{r} + C r \right] \quad (73)$$

where $E_2$ is the second level of excited energy dan $U'_2(r)$ is a second level eigenfunction.

The generally, the eigenvalue and eigenfunction from Equation (68), (70), (72), (69), (71) dan Eq. (73) respectively is
\[ E_n = -\frac{(\lambda + (3-n)D^n - n(n-1)) \pm \sqrt{(\lambda + (3-n)D^n - n(n-1))^2 - 4\left(\sum_{i=1}^{n} \sigma_i^{(n)} + \frac{b}{2\sqrt{a}}\right)D'f}}{4\left(\sum_{i=1}^{n} \sigma_i^{(n)} + \frac{b}{2\sqrt{a}}\right)} - E_n - 2C \quad (74) \]

and

\[ U_n(r) = N_n \left( \prod_{i=1}^{n} (r - \sigma_i^{(n)}) \right)^{\alpha} \exp \left[ \frac{1}{2} A^2 + \frac{B^2}{r} + C'r \right] \quad (75) \]

where

\[ A' = -\sqrt{a}, \quad B' = -\frac{b}{2\sqrt{a}}, \quad D' = \frac{3c}{2} + \frac{b^2}{8a\sqrt{a}}, \]

\[ C' = \frac{(\lambda + (3-n)D^n - n(n-1)) \pm \sqrt{(\lambda + (3-n)D^n - n(n-1))^2 - 4\left(\sum_{i=1}^{n} \sigma_i^{(n)} + \frac{b}{2\sqrt{a}}\right)D'f}}{4\left(\sum_{i=1}^{n} \sigma_i^{(n)} + \frac{b}{2\sqrt{a}}\right)}. \]

\( E_n \) is the eigenvalue and \( U_n(r) \) is the eigenfunction of the potential partner of Polynomial Inverse potential. The eigenvalue of Equation (74) has different with the eigenvalue of Equation (53) because the effect of ground state eigenvalue of Equation (30) in Equation (74). The energy spectrum of Equation (53) and Equation (74) is shown in Table 1.

**Table 1.** Energy spectrum for Polynomial Inverse potential and its partner potential in 4-dimensional variation radial quantum number \( n \) with \( \lambda_1 = 24.4614 \) and \( \lambda_1' = 7.0794 \).

| \( N \) | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( E_n \left( \text{fm}^{-1} \right) \) | \( E_n' \left( \text{fm}^{-1} \right) \) |
|------|------|------|------|------|------|------|-----------------|-----------------|
| 0    | -2   | 1    | -3   | -1   | 2    | -3   | 25.4706         | 22.9843         |
| 1    | -2   | 1    | -3   | -1   | 2    | -3   | 22.9957         | 21.6931         |
| 2    | -2   | 1    | -3   | -1   | 2    | -3   | 17.2794         | 19.2794         |
| 3    | -2   | 1    | -3   | -1   | 2    | -3   | 9.6340          | 15.0501         |
| 4    | -2   | 1    | -3   | -1   | 2    | -3   | 2.1214          | 9.6844          |

Table 1 shows that the increase in radial quantum numbers causes the energy spectrum to decrease. Partner potential for \( n > 1 \) has a larger energy spectrum than the energy spectrum of Polynomial Inverse potential as its original potential. The energy spectrum of Polynomial Inverse potential has different with the energy spectrum of its partner potential. This result is consistent with the result of research about the solution of construction of partner potential for Hylleraas potential in Ref. (4).

**4. Conclusion**

In this paper, we have presented the solution of 4-dimensional Schrodinger equation of radial part for Polynomial Inverse potential and its partner potential by using Ansatz wavefunction method and construction of potential by using Supersymmetric quantum mechanics method. We obtained the eigenvalue and eigenfunction from radial part solutions, which the eigenvalue and eigenfunction depend on the parameters of all components of the composed potential. Partner potential was considered to be
new potential. The partner potential of Polynomial Inverse potential for $n > 1$ has eigenvalue greater than Polynomial Inverse potential as its original potential.

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