Infinite (continuous) spin fields
in the frame-like formalism

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Abstract

In this paper we elaborate on the gauge invariant frame-like Lagrangian description for the wide class of the so-called infinite (or continuous) spin representations of Poincaré group. We use our previous results on the gauge invariant formalism for the massive mixed symmetry fields corresponding to the Young tableau with two rows $Y(k, l)$ ($Y(k+1/2, l+1/2)$ for the fermionic case). We have shown that the corresponding infinite spin solutions can be constructed as a limit where $k$ goes to infinity, while $l$ remain to be fixed and label different representations. Moreover, our gauge invariant formalism provides a natural generalization to (Anti) de Sitter spaces as well. As in the completely symmetric case considered earlier by Metsaev we have found that there are no unitary solutions in de Sitter space, while there exists a rather wide spectrum of Anti de Sitter ones. In this, the question what representations of the Anti de Sitter group such solutions correspond to remains to be open.

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### Introduction

Besides the very well known finite component massless and massive representations of the Poincaré algebra there exists a number of rather exotic so-called infinite (or continuous) spin ones (see e.g. [1, 2]). In dimensions $d \geq 4$ they have an infinite number of physical degrees of freedom and so may be of some interest for the higher spin theory. Indeed, they attracted some attention last times [3, 4, 5, 6, 7, 8] (see also recent review [9]). One of the reason is that such representation being massless nevertheless are characterized by some dimensionful parameter $\mu$, which may play important role in the possibility to construct interactions with such fields.

It has been already noted several times in different contexts that such infinite spin representations may be considered as a limit of massive higher spin ones where spin goes to
infinity, mass goes to zero while their product $ms = \mu$ remains to be fixed and provides this dimensionful parameter that characterizes the representation. A very nice Lagrangian realization for this idea has been given recently by Metsaev in [10] for the bosonic case and in [11] for the fermionic one (see also [12]). Namely, he has shown that the very same gauge invariant formalism that was previously used for the description of massive higher spin bosonic [13] and fermionic [13] particles can be used to provide Lagrangian gauge invariant formulation for the infinite spin cases. It is interesting that even in flat Minkowski space there exist unitary models that resemble the so-called partially massless models for the finite spin particles in de Sitter space. Recall that one of the nice features of such gauge invariant formulation for the massive higher spins is that it perfectly works not only in flat Minkowski space but in (Anti) de Sitter spaces as well, giving a possibility to investigate all possible massless and partially massless limits. The same holds for the infinite spin cases as well and it appeared that while there are no unitary infinite spin models in de Sitter space there exists quite a lot of different unitary solutions in Anti de Sitter space though till now it is not clear what representations of the Anti de Sitter group they correspond to.

Naturally, the most important open physical question is the possibility to have consistent interacting theories containing such infinite spin fields. A very important step in this direction was recently made by Metsaev [15], who using light-cone formalism provided a classification of cubic interaction vertices for one massless infinite spin field with two massive finite spin ones as well as for two massless infinite spin fields with one massive finite spin one (see also [16] for the interaction of two massive scalars with one massless infinite spin field).

In general, the classification of the infinite spin representations is provided by the representations of the so-called short little group $SO(d-3)$ [11, 12]. It is clear that in $d = 3$ and $d = 4$ dimensions this short little group is trivial so we have just one bosonic and one fermionic infinite spin representation whose description is based on the completely symmetric (spin-)tensors. But in dimensions higher than 4 we face a huge number of such representations. For example, in $d = 5, 6$ such representations are labeled by the parameter $l$ that can take integer or half-integer values for the bosonic and fermionic cases respectively. It seemed natural to assume that the Lagrangian formalism for such representations can be obtained starting with the gauge invariant description for the massive mixed symmetry fields corresponding to the Young tableau with two rows $Y(k, l)$ ($Y(k + 1/2, l + 1/2)$ for the fermionic case) if one takes a limit where $k$ goes to infinity, while $l$ being fixed and labels different representations. In this paper using our previous results [17, 18, 19, 20] we show that it appears to be the case. As in the completely symmetric case our gauge invariant formalism provides an extension to (Anti) de Sitter space and we also found that there are no unitary solutions in de Sitter space while there exists a number of unitary ones in Anti de Sitter space. Let us stress ones again that what representations of the Anti de Sitter group such solutions correspond to is still an open question.

Our paper is organized as follows. We work in the gauge invariant frame-like formalism. In Section 1 we provide all necessary information for the frame-like description of the massless finite component mixed symmetry bosonic fields that serve as the natural building blocks for the infinite spin ones. In section 2 we begin with the completely symmetric case providing frame-like generalization of Metsaev’s results [10]. Then we consider a simplest

\footnote{See [21] for the three-dimensional case}
mixed symmetry bosonic example based on the so-called long hooks $Y(k, 1)$ and then we elaborate on the general mixed symmetry case $Y(k, l)$, $k \to \infty$. Similarly, in Section 3 we provide all necessary information for the frame-like description of the massless finite component mixed symmetry fermionic fields while Section 4 contains three subsections analogous to the ones in Section 2.

**Notations and conventions** We work in the frame-like formalism where the world indices are denoted by the Greek letters, while the local indices — by the Latin ones. To simplify formulas we use the so-called condensed notations for the local indices so that

$$\Phi^{a(k), b(l)}(k_1a_2...a_k, b_1b_2...b_l)$$

where round brackets denote symmetrization which uses the minimum number of terms necessary without any normalization factor. For the local indices denoted by the same letter and placed on the same level we always assume symmetrization, e.g.

$$e^a\Phi^{a(k), b(l)} = e^{a_1a_2...a_{k+1}, b(l)}$$

Working with the fermions we heavily use the completely antisymmetric products of $\gamma$-matrices defined as follows

$$\Gamma^a[n] = \frac{1}{n!} \gamma^a \gamma^b \gamma^c \cdots$$

with a couple of useful relations

$$\gamma^a \Gamma^b[n] = \gamma^b \Gamma^{b[n-1]} + \Gamma^{ab[n]}$$

$$(d - n) \Gamma^a[n] = \Gamma^a[n]$$

(Anti) de Sitter space is described by the (non-dynamic) background frame $e_\mu^a$ and the covariant derivative $D_\mu$. All our expressions for the Lagrangians and gauge transformations are completely antisymmetric on the world indices so the covariant derivatives always act on the local indices only (including an implicit spinor index). For the bosonic and fermionic objects the commutator of the covariant derivatives are normalized as follows

$$[D_\mu, D_\nu] \xi^{a(k)} = -\kappa [e_\mu, e_\nu] \xi^{a(k-1)} + \frac{2\Lambda}{(d - 1)(d - 2)} \kappa = \frac{2\Lambda}{(d - 1)(d - 2)}$$

$$[D_\mu, D_\nu] \xi^{a(k)} = -\kappa [e_\mu, e_\nu] \xi^{a(k-1)} + \frac{1}{2} \Gamma_{\mu\nu} \xi^{a(k)}$$

In what follows we also use the completely antisymmetric products of the inverse frames:

$$\{^\mu _\nu _a \} = e^a \epsilon^\mu \epsilon^\nu, \quad \{^\mu _\nu _a _b _c \} = e^a \epsilon^\mu \epsilon^\nu \epsilon^b \epsilon^c$$

and so on.

## 1 Massless bosonic fields

In this section we provide all necessary information on the massless (finite component) symmetric and mixed symmetry bosonic fields which will serve as the building blocks for our construction of the infinite component cases. In what follows $Y(k, l)$ denote mixed symmetry tensor having two rows with length $k$ and $l$ respectively.
1.1 Completely symmetric tensor \( Y(k + 1, 0) \)

Frame-like description for the completely symmetric tensor \([22, 23]\) requires a physical one-form \( h_\mu a(k) \), symmetric and traceless on its local indices and an auxiliary one-form \( \omega_\mu a(k),b \), symmetric on its first \( k \) local indices, traceless on all local indices and satisfying \( \omega_\mu a(k),a = 0 \).

In flat Minkowski space the free Lagrangian looks like:

\[
(-1)^k \mathcal{L}_0(h_\mu a(k)) = - \left\{ \frac{\mu \nu}{ab} \right\} [\omega_\mu a(k-1),c, \omega_\nu b(k-1),c] + \frac{1}{k} \omega_\mu e(k),a \omega_\mu e(k),b \\
- 2 \left\{ \frac{\mu \nu \alpha}{abc} \right\} \omega_\mu a(k-1),b \partial_\nu h_{\alpha c}(k-1)
\]

This Lagrangian is invariant under the following gauge transformations:

\[
\delta_0 \omega_\mu a(k),b = \partial_\mu \chi a(k),b, \quad \delta_0 h_\mu a(k) = \partial_\mu \zeta a(k),a + \chi a(k), _\mu
\]

where parameters \( \chi a(k),b \) and \( \zeta a(k),a \) have the same properties on their local indices as the gauge fields \( \omega_\mu a(k),b \) and \( h_\mu a(k) \) respectively.

After the replacement of the ordinary partial derivatives by the \( \text{AdS} \) covariant ones the Lagrangian ceases to be invariant:

\[
\delta_0 \mathcal{L}_0 = (-1)^k \frac{2(k + 1)(d + k - 3)}{k} \kappa [\omega_\mu a(k),a \zeta a(k) - \chi a(k),a h_\mu a(k)]
\]

This non-invariance can be compensated with the introduction of the mass-like terms (recall that in \( (A)dS \) space the presence of such terms does not necessarily mean that the field is massive):

\[
(-1)^k \mathcal{L}_2(h_\mu a(k)) = b_k \left\{ \frac{\mu \nu}{ab} \right\} h_\mu a(k-1),b h_\nu a(k-1)
\]

and corresponding corrections to the gauge transformations:

\[
\delta_2 \omega_\mu a(k),b = \frac{b_k}{(k + 1)(d - 2)} [ke_\mu b \zeta a(k) - e_\mu a \zeta a(k-1)b] \\
- \frac{1}{(d + k - 3)} ((k - 1)g^{ba} \zeta a(k-1) - 2g^{a(2)} \zeta a(k-2)b)]
\]

provided

\[
b_k = (k + 1)(d + k - 3)\kappa
\]

1.2 Mixed symmetry tensor \( Y(k + 1, 1) \)

For the frame-like description of the massless mixed symmetry fields in the flat Minkowski space we follow \([24, 25, 26]\).

The frame-like description for the mixed symmetry tensor \( Y(k + 1, 1) \) (so-called long hook) appears to be special and deserves separate consideration. For this case we use a physical two-form \( \Phi_{\mu \nu} a(k) \), completely symmetric and traceless on its local indices and an auxiliary one-form \( \Omega_{\mu \nu} a(k),bc \), with local indices corresponding to the Young tableau \( Y(k, 1, 1) \),

\[\text{For the description of mixed symmetry fields in } AdS \text{ space see e.g. } [27, 28, 29].\]
i.e. symmetric on its first $k$ indices, antisymmetric on the last two ones, traceless on all indices and satisfying the relation $\Omega_\mu{}^{a(k),ab} = 0$. In the flat case the free Lagrangian has the form:

$$(-1)^k L_0 = \left\{ \mu \nu \right\} [\Omega_\mu{}^{ae(k-1),cd} \Omega_\nu{}^{be(k-1),cd} + \frac{2}{k} \Omega_\mu{}^{e(k),ac} \Omega_\nu{}^{e(k),bc}]$$

$$- \left\{ \mu \nu \alpha \beta \right\} \Omega_\mu{}^{ae(k-1),bc} \partial_\nu \Phi_{a\beta}{}^{de(k-1)}$$

(5)

This Lagrangian is invariant under the following gauge transformations:

$$\delta_0 \Omega_\mu{}^{a(k),bc} = \partial_\mu \eta^{a(k),bc}, \quad \delta_0 \Phi_{\mu \nu}{}^{a(k)} = \partial_{[\mu} \xi_{\nu]}{}^{a(k)} + \eta^{a(k)}{}_{\mu \nu}$$

(6)

where the gauge parameters $\eta^{a(k),bc}$ and $\xi^{a(k)}$ have the same properties on their local indices as the corresponding gauge fields $\Omega_\mu{}^{a(k),bc}$ and $\Phi_{\mu \nu}{}^{a(k)}$.

The replacement of the ordinary partial derivatives by the $AdS$ covariant ones spoils the invariance of the Lagrangian:

$$(-1)^k \delta L_0 = \frac{(k + 2)(d + k - 4)}{k} \left\{ \mu \nu \right\} [\eta_\mu{}^{e(k),ab} \Phi_{\mu \nu}{}^{e(k)} - 2 \Omega_\mu{}^{e(k),ab} \xi_{\mu \nu}{}^{e(k)}]$$

(7)

In this special case there are no mass-like terms that can be added to the Lagrangian and the invariance of the Lagrangian cannot be restored without introduction of some additional fields (see the next section). The reason is the essential difference in the spectrum of the mixed symmetry representations of the Poincaré and the $(A)dS$ groups (see [30]).

### 1.3 General mixed symmetry tensor $Y(k + 1, m + 1)$

In this case we need the two-forms $\Psi_{\mu \nu}{}^{a(k),b(m)}$ and $\Omega_{\mu \nu}{}^{a(k),b(m),c}$ with their local indices corresponding to the Young tableau $Y(k, m)$ and $Y(k, m, 1)$. The free Lagrangian in the flat case looks like:

$$(-1)^{k+m} L_0 = -\frac{1}{2} \left\{ \mu \nu \alpha \beta \right\} [\Omega_{\mu \nu}{}^{ae(k-1),b(f(m-1)),g} \Omega_{\alpha \beta}{}^{ce(k-1),d(f(m-1)),g}$$

$$+ \frac{1}{k} \Omega_{\mu \nu}{}^{e(k),a(f(m-1)),b} \Omega_{\alpha \beta}{}^{e(k),c(f(m-1)),d}$$

$$+ \frac{1}{m} \Omega_{\mu \nu}{}^{a(e(k-1),f(m)),b} \Omega_{\alpha \beta}{}^{e(k-1),f(m),d}]$$

$$+ \left\{ \mu \nu \alpha \beta \gamma \right\} \Omega_{\mu \nu}{}^{a(f(k-1),b(g(m-1)),c) \partial_\alpha \Psi_{\beta \gamma}{}^{d(f(k-1)),e(g(m-1))}$$

(8)

It is invariant under the following gauge transformations:

$$\delta_0 \Omega_{\mu \nu}{}^{a(k),b(m),c} = \partial_{[\mu} \eta_{\nu]}{}^{a(k),b(m),c},$$

$$\delta_0 \Psi_{\mu \nu}{}^{a(k),b(m)} = \partial_{[\mu} \xi_{\nu]}{}^{a(k),b(m)} + \eta_{[\mu}{}^{a(k),b(m),\nu]}$$

(9)

where the one-form gauge parameters $\eta_{\mu}{}^{a(k),b(m),c}$ and $\xi_{\mu}{}^{a(k),b(m)}$ have the same properties on their local indices as the corresponding gauge fields.
With the replacement of the ordinary partial derivatives by the AdS covariant ones we obtain:

\[ (-1)^{k+m} \delta \mathcal{L}_0 = 2 \kappa \left\{ \frac{\mu \nu \alpha \beta}{abcd} \right\} \left[ \frac{(k+2)(d+k-4) + (d+m-6)}{k} \Omega_{\mu \nu} d^{(k),ae(m-1),b,\xi \alpha} d^{(k),c,e(m-1)} \right. \\
+ \frac{(m+1)(d+m-6)}{m} \Omega_{\mu \nu} \eta_{\mu} d^{(k),ae(m-1),b,\Psi_{\nu \alpha} d^{(k),c,e(m-1)}} \\
+ \frac{(k+2)(d+k-4) + (d+m-6)}{k} \eta_{\mu} d^{(k),ae(m-1),b,\Psi_{\nu \alpha} cd^{(k),c,e(m-1)}} \right. \\
\left. + \frac{(m+1)(d+m-6)}{m} \eta_{\mu} a^{(k),ae(m-1),b,\Psi_{\nu \alpha} cd^{(k),c,e(m-1)}} \right] \tag{10} \]

One can try to compensate for this non-invariance by introducing the mass-like terms

\[ (-1)^{k+m} \mathcal{L}_2 = a_{k,m} \left\{ \frac{\mu \nu \alpha \beta}{abcd} \right\} \eta_{\mu} a^{(k),ae(m-1),b,f(m-1)} \eta_{\mu} a^{(k),c,e(m-1)} \tag{11} \]

as well as the corresponding corrections to the gauge transformations:

\[ \delta_2 \Omega_{\mu \nu} a^{(k),b(m),c} = - \frac{2 a_{k,m}}{(k+2)(m+1)(d-4)} \left[ (k+1) m e^{[\mu} c e_{,\nu]} a^{(k),b(m)} - m e^{[\mu} a e_{,\nu]} a^{(k-1)c,b(m)} \right. \\
\left. + e^{[\mu} a c_{,\nu]} a^{(k-1)b,b(m-1)c} - (k+1) e^{[\mu} a c_{,\nu]} a^{(k),b(m-1)c} - \text{Tr} \right] \tag{12} \]

They produce additional variations:

\[ -4 (-1)^{k+m} a_{k,m} \left\{ \frac{\mu \nu \alpha}{abcd} \right\} \left[ \frac{1}{k} \eta_{\mu} d^{(k),ae(m-1),b,\xi \alpha} d^{(k),c,e(m-1)} + \frac{1}{m} \eta_{\mu} a^{(k),ae(m-1),b,\Psi_{\nu \alpha} cd^{(k),c,e(m-1)}} \right. \\
\left. + \frac{1}{k} \Omega_{\mu \nu} d^{(k),ae(m-1),b,\xi \alpha} d^{(k),c,e(m-1)} + \frac{1}{m} \Omega_{\mu \nu} a^{(k),ae(m-1),b,\xi \alpha cd^{(k),c,e(m-1)}} \right] \]

As it can be easily seen it is not possible to restore the invariance of the Lagrangian by adjusting the only parameter \( a_{k,m} \). Similarly to the previous case the invariance can be restored with the help of additional fields only (see the next section).

### 1.4 Special case — tensor \( Y(l + 1, l + 1) \)

This case corresponds to the rectangular Young tableau and also deserves separate consideration. We need the two-forms \( R_{\mu \nu} a^{(l),b(l)} \) and \( \Omega_{\mu \nu} a^{(l),b(l)} \) with local indices corresponding to the Young tableau \( Y(l, l) \) and \( Y(l, l, 1) \). The free Lagrangian in the flat case has the form:

\[ \mathcal{L}_0 = - \frac{1}{2} \left\{ \frac{\mu \nu \alpha \beta}{abcd} \right\} \left[ \Omega_{\mu \nu} a^{(l-1),b,l} \eta_{\mu \nu} a^{(l-1),c,g \Omega_{\alpha \beta} c^{(l-1),d} g_{(l-1),g}} \right. \\
\left. + \frac{2}{l} \Omega_{\mu \nu} c^{(l),a,l} \eta_{\mu \nu} a^{(l),b,l} \right] \tag{13} \]

It is invariant under the following gauge transformations:

\[ \delta_0 \Omega_{\mu \nu} a^{(l),b,l} = \partial_{[\mu} \eta_{\nu]} a^{(l),b,l} c \]
\[ \delta_0 R_{\mu \nu} a^{(l),b,l} = \partial_{[\mu} \xi_{\nu]} a^{(l),b,l} + \eta_{[\mu} a^{(l),b,l} , \nu] \tag{14} \]
The replacement of the ordinary partial derivatives by the $AdS$ covariant ones produces
\[
\delta L_0 = \frac{4(l+2)(d+l-5)}{l} \kappa \left\{ \mu_{ab}^{\alpha} \left[ \Omega_{\mu\nu}^{(l),ae(l-1),b} \xi_{\alpha}^{d(l),ce(l-1)} + \eta_{\mu}^{d(l),ae(l-1),b} R_{\nu\alpha}^{(d(l),ce(l-1))} \right] \right\}
\]
But in this case the invariance of the Lagrangian can be restored with the help of introduction of mass-like terms:
\[
L_2 = a_{l,l} \left\{ \mu_{abcd}^{\alpha} \right\} R_{\mu\nu}^{ae(l-1),bf(l-1)} R_{\alpha\beta}^{ce(l-1),df(l-1)}
\]
as well as the corresponding corrections to the gauge transformations:
\[
\delta_2 \Omega_{\mu\nu}^{a(l),b(l),c} = -\frac{2a_{l,l}}{(l+2)(d-4)} \left[ l e_{[\mu}^{c} \xi_{\nu]}^{a(l),b(l)} - e_{[\mu}^{a} \xi_{\nu]}^{a(l-1)c,b(l)} - e_{[\mu}^{b} \xi_{\nu]}^{a(l),b(l-1)c} - T r \right]
\]
provided
\[
a_{l,l} = \frac{(l+2)(d+l-5)}{2} \kappa
\]

2 Infinite spin bosonic fields

In this section we consider the explicit Lagrangian descriptions for the number of bosonic infinite spin cases. We begin with case based on the completely symmetric tensors reproducing the results of Metsaev but in the frame-like formalism. Then we present the simplest example for the mixed symmetry tensors based on the so-called long hooks. After that we turn to the case of the general mixed symmetry tensors with two rows.

2.1 Completely symmetric case — $Y(k+1,0), \ k \to \infty$

We follow the results of Metsaev who has shown that the same gauge invariant description for the massive higher spin bosonic fields (for the metric formulation and for the frame-like one) is perfectly capable of describing infinite spin massless and tachyonic cases provided one abandons the restriction on the number of field components. Let us recall that the main idea of such gauge invariant description is to begin with the appropriate set of massless (finite component) fields and then introduce the cross terms into Lagrangian as well as the appropriate corrections to the gauge transformations gluing all them together in such a way as to keep all the gauge symmetries intact and as a result the number of physical degrees of freedom unchanged.

For the completely symmetric case we introduce a set of one-forms ($\omega_{\mu}^{a(k),b}, \ h_{\mu}^{a(k)}$), $1 \leq k < \infty$, as well as one-form $A_\mu$ and zero-forms $B^{ab}$, $\pi^a$ and $\varphi$ for the spin 1 and spin 0 components. We begin with sum of kinetic and mass-like terms for all fields:
\[
\mathcal{L} = \sum_{k=1}^{\infty} L_0(h_{\mu}^{a(k)}) + \frac{1}{2} B_{ab}^{2} - \left\{ \mu_{ab}^{\alpha} \right\} \omega^{ab} D_{\mu} h_{\nu} - \frac{1}{2} \pi_{a}^{2} + \pi^{\mu} D_{\mu} \varphi + \sum_{k=1}^{\infty} L_2(h_{\mu}^{a(k)}) + b_{0} h_{\varphi} + c_{0} \varphi^{2}
\]
where $\mathcal{L}_0(h_{\mu}^{a(k)})$ and $\mathcal{L}_2(h_{\mu}^{a(k)})$ are defined in (1) and (3), and with the set of initial gauge transformations defined in (2) and (4) supplemented with

\[
\delta_0 h_{\mu} = D_{\mu} \zeta, \quad \delta_2 \pi^a = b_0 \zeta^a
\]

Now we add a set of cross-terms gluing all these fields together (the meaning of the coefficients $e_k$ is illustrated by Figure 1):

\[
\mathcal{L}_1 = \sum_{k=2}^{\infty} (-1)^k e_k \left\{ \mu \nu \right\}^{ab} \left[ \omega_{\mu}^{a(c(k-1))} h_{\nu,a(k-1)} - h_{\mu}^{a(c(k-1))} \omega_{\nu,c(k-1)} \right]
\]

\[- e_1 \left\{ \mu \nu \right\}^{ab} \omega_{\mu}^{a,b} h_{\nu} + e_1 \omega^{\mu a} h_{\mu,a} - e_0 \pi^\mu h_{\mu}
\]

as well as the corresponding corrections to the gauge transformations:

\[
\delta_1 \omega^{(k),b} = \frac{e_{k+1}}{2(k+1)}((k+1)\chi^{a(k),b} + \chi^{a(k)b, \mu}) + \frac{e_k}{2(d+k-2)}(e_{\mu}^{a} \chi^{a(k-1),b} - Tr)
\]

\[
\delta_1 h^{a(k)} = \frac{ke_{k+1}}{2(k+1)} \zeta^{a(k)} + \frac{e_k}{2(d+k-3)}(e_{\mu}^{a} \zeta^{a(k-1)} - \frac{2}{(d+2k-4)}g^{a(2)} \zeta^{a(k-2)})
\]

\[
\delta_1 \omega^{ab} = e_1 \chi^{ab}, \quad \delta_1 h_{\mu} = \frac{e_1}{2} \zeta_{\mu}, \quad \delta_1 \varphi = e_0 \zeta
\]

The requirement that the whole Lagrangian be invariant under all gauge transformations produces a number of relations on the parameters:

\[(k+1)(d+k-1)b_{k+1} = k(d+k-2)b_k \]

\[0 = \frac{(d+2k)(d+k-3)}{2(d+2k-2)(d+k-2)} e_{k+1}^2 - \frac{(k^2-1)}{2k^2} e_k^2 - \frac{2b_k}{k} + \frac{2(k+1)(d+k-3)}{k} \kappa \]

\[0 = \frac{(d+2)(d-2)}{2d(d-1)} e_2^2 - e_1^2 - 2b_1 + 4(d-2) \kappa \]

\[e_1^2 = \frac{2(d-2)}{d} e_0, \quad b_1 = -\frac{(d-2)}{2(d-1)} e_0^2, \quad b_0 = -\frac{e_1 e_0}{2} \]

The general solution of these equations has two free parameters and their choice depends on the case we are interested in. As we have already mentioned, our system is capable of describing both massive finite spin case (including all its massless and partially massless limits) as well as the infinite spin (not necessarily massless) cases. For the massive finite spin solution it is natural to choose the spin and the mass as these two parameters where the spin $s$ is defined by the requirement $e_s = 0$ (so that our chain of fields is restricted from
the above, see Figure 1), while we use the relation $m^2 = (s-2)e_{s-1}^2/4$ as our definition of mass. In these terms the solution looks like:

$$b_k = -\frac{s(d+s-3)}{k(d+k-2)}[m^2 - (s-1)(d+s-4)\kappa]$$

$$e_k^2 = 4(s-k)(d+s+k-3)\frac{(k-1)(d+2k-2)}{[m^2 - (s-k-1)(d+s+k-4)\kappa]}$$

$$e_1^2 = 4(s-1)(d+s-2)\frac{d}{[m^2 - (s-2)(d+s-3)\kappa]}$$

$$e_0^2 = \frac{2s(d+s-3)}{(d-2)}[m^2 - (s-1)(d+s-4)\kappa]$$

Let us briefly recall the main properties of this solution. First of all, from the expressions above it is clear that the massless limit $m \to 0$ is possible in flat $\kappa = 0$ and Anti de Sitter $\kappa < 0$ spaces only. In flat space massive spin-$s$ particle decomposes into the whole set of massless fields with spins $s, s-1, \ldots, 0$, while in $AdS$ space it decomposes into massless spin-$s$ field and massive spin-($s-1$) one. In de Sitter space $\kappa > 0$ we have a so-called unitary forbidden region $m^2 < (s-1)(d+s-4)\kappa$. At the boundary of this region lives the first partially massless case where the spin-0 component decouples. Inside the forbidden region we have a number of partially massless cases where one of the $e_l = 0$. In this case the whole system decomposes into two disconnected ones. The first one with the fields with $l \leq k \leq s-1$ describes unitary partially massless particle, while the remaining fields gives non-unitary massive one.

Now let us turn to the main subject of our current work — infinite spin cases. Here we choose $c_0$ and $e_0$ as our two main parameters. Then for all other parameters we obtain:

$$b_k = -\frac{(d-2)}{2k(d+k-2)}c_0^2$$

$$e_k^2 = \frac{2d}{(k-1)(d+2k-2)}[k(d+k-3)c_0 - (k-1)(d+k-2)e_0^2 - 2k(k-1)(d+k-2)(d+k-3)\kappa]$$

$$e_1^2 = \frac{2(d-2)}{d}c_0$$

As in the metric-like formulation by Metsaev [10], it is convenient to introduce auxiliary variables $x_k = k(d+k-3)$. Then we have

$$e_k^2 \sim -2\kappa x_k^2 + [c_0 - e_0^2 + 2(d-2)\kappa]x_k + (d-2)e_0^2$$

Now we consider three spaces in turn.

**De Sitter space** From the expression above it is clear the $e_k^2$ will always become negative at some sufficiently large $k$ so that in general we obtain non-unitary theory. The

---

As is well known, there is no strict definition of mass and masslessness in (Anti) de Sitter spaces. For the completely symmetric (spin-)tensors it is natural to define the massless limit as the one where $e_{s-1} \to 0$, i.e. the limit where the main gauge field completely decouples from all the Stueckelberg ones. As for the concrete normalization of mass, we choose it so that it coincides with the usual definition of mass in flat Minkowski space.
only exceptions appear each time when the parameters are adjusted so that some \( e_s = 0 \). These cases just reproduce massive finite spin particles described above.

**Minkowski space** Let us begin with the \( c_0^2 > 0 \). If \( c_0 < \epsilon_0^2 \) we again face the situation where \( e_k^2 \) become negative at sufficiently large \( k \) so that in general we obtain non-unitary theory. And again, the only exceptions appear when we adjust parameters so that some \( e_s = 0 \) and these cases reproduce massive finite spin particles. At \( c_0 = \epsilon_0^2 \) (that corresponds to the \( \mu_0 = 0 \) in the Metsaev paper [10]) we obtain unitary massless infinite spin theory, while for the \( c_0 > \epsilon_0^2 \) we obtain unitary tachyonic infinite spin one. A whole set of rather exotic solutions arises if we assume that \( e_0^2 < 0 \) and again adjust the parameters so that some \( e_s = 0 \):

\[
c_0 = \frac{(s - 1)(d + s - 2)}{s(d + s - 3)} \epsilon_0^2
\]

These solutions resemble the massive finite spin case described above. Indeed, the whole set of the fields decomposes into two disconnected subsystems. But this time, the infinite chain with \( s \leq k < \infty \) describes unitary theory (which presumably corresponds to the tachyonic infinite dimensional representation of the Poincaré group), while the remaining fields correspond to the non-unitary massive theory.

**Anti de Sitter space** In this case (see Appendix A.2 and general discussion in Appendix A.1) it is clear that \( e_k^2 \) will always become positive at some sufficiently large \( k \). So there are two general possibilities. The first one is when all \( e_k^2 > 0 \) and we obtain unitary infinite spin theory containing all fields with \( 0 \leq k < \infty \). Besides, similarly to the flat case, there exist a number of discrete solutions where some \( e_s = 0 \) and the unitary part contains fields with \( s \leq k < \infty \) only.

### 2.2 Mixed symmetry example — \( Y(k + 1, 1) \), \( k \to \infty \)

In this section we present a simplest example for the mixed symmetry tensors based on the so-called long hooks. We follow our previous work [18] for the massive finite component case but without any restrictions on the number of field components. So we introduce the following fields: \( \Omega_{\mu a}^{(k), bc} \), \( \Psi_{\mu \nu a}^{(k)} \), \( \omega_{\mu a}^{(k), b} \) and \( h_{\mu a}^{(k)} \), \( 1 \leq k < \infty \) and \( \Omega^{abc} \), \( \Phi_{\mu \nu} \), \( \omega^{ab} \) and \( h_{\mu} \).

The sum of the kinetic terms (there are no possible mass-like terms in this particular case):

\[
\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}_0(\Phi_{\mu \nu}^{a(k)}) - \Omega_{abc} \frac{e_{a(k)}}{2} + \{\mu \nu \alpha\} \Omega^{abc} D_{\mu} \Phi_{\nu \alpha} \\
+ \sum_{k=1}^{\infty} \mathcal{L}_0(h_{\mu}^{a(k)}) + \frac{1}{2} \omega^{ab} - \{\mu \nu\} \omega^{ab} D_{\mu} h_{\nu}
\]

where Lagrangians \( \mathcal{L}_0(\Phi_{\mu \nu}^{a(k)}) \) and \( \mathcal{L}_0(h_{\mu}^{a(k)}) \) are defined in [5] and [1], with the initial set of the gauge transformations given in [6] and [2] supplemented with

\[
\delta_0 \Phi_{\mu \nu} = D[\mu \xi_{\nu}], \quad \delta_0 h_{\mu} = D_{\mu} \zeta
\]

There are three types of the possible cross-terms (see Figure 2).

\( Y(k + 1, 1) \Leftrightarrow Y(k, 1) \). In this case the cross-terms have the form:

\[
\mathcal{L}_{11} = \sum_{k=2}^{\infty} (-1)^k c_k \{\mu \nu \alpha\} \left[ \Omega_{\mu a}^{e(k-1), bc} \Phi_{\nu \alpha, e(k-1)} + \Omega_{\mu}^{e(k-1), ab} \Phi_{\nu \alpha, e(k-1)} \right]
\]
with the corresponding corrections to the gauge transformations:

\[ \delta_{11} \Omega_\mu^{a(k),bc} = -c_{k+1}[\eta_\mu^{a(k),bc} - \frac{1}{(k+2)} \eta^{a(k)[b,c]}_\mu] - \frac{c_k}{(d+k-2)} (e_\mu^a \eta^{a(k-1),bc} - Tr) \]

\[ \delta_{11} \Phi_{\mu\nu}^{a(k)} = \frac{k c_{k+1}}{(k+2)} \xi_{[\mu,\nu]}^{a(k)} - \frac{c_k}{(d+k-4)} (e_{[\mu}^a \xi^{a(k-1)}_{\nu]} - Tr) \]

\[ \delta_{11} \Phi_{\mu}^a = -\frac{3 c_1}{(d-3)} e_\mu^a \xi, \quad \delta_{11} \Omega^{abc} = -3 c_1 \eta^{abc}, \quad \delta_{11} \Phi_{\mu
u} = c_1 \xi_{[\mu,\nu]} \]

\[ Y(k+1, 1) \Leftrightarrow Y(k+1, 0). \] In this case we introduce

\[ \mathcal{L}_{12} = \sum_{k=1}^{\infty} (-1)^k d_k \left[ \frac{1}{k} \{ \mu^\nu_{ab} \} \Omega_\mu e^{(k),ab}_{\nu,e(k)} - \{ \mu^\nu_{abc} \} \omega_\mu^{ae(k-1),b} \Phi_{\rho\alpha,e(k-1)c} \right] \]

\[ + d_0 \{ \mu^\nu_{ab} \} \omega^{ab} \Phi_{\mu\nu} \]

as well as the following corrections to the gauge transformations:

\[ \delta_{12} \Omega_\mu^{a(k),bc} = \frac{d_k}{2(k+2)(d-3)} [(k+1) \chi^{a(k)[b}_\mu e^{c]} + e_\mu^a \chi^{a(k-1)[b,c]} - Tr] \]

\[ \delta_{12} \Phi_{\mu\nu}^{a(k)} = \frac{d_k}{2(k+2)(d+k-4)} e_{[\mu}^a \xi^{a(k-1)}_{\nu]} \]

\[ \delta_{12} \omega_\mu^{a(k),b} = -\frac{d_k}{2} \eta^{a(k),b} \mu, \quad \delta_{12} h_\mu^{a(k)} = -d_k \xi^{a(k)}_\mu, \quad \delta_{12} h_\mu = 2d_0 \xi_\mu \]

\[ Y(k+1, 0) \Leftrightarrow Y(k, 0). \] This case has already been considered in the previous subsection, so we do not repeat it here.

Now we require that the whole Lagrangian be invariant under all gauge transformations. This gives us a number of relations on the parameters:

\[ 2(k+1)c_{k+1}d_k + (k+2)d_{k+1}e_{k+1} = 0 \]

\[ d_k e_{k+1} + \frac{2(d+k-2)}{(d+k-3)} c_{k+1} d_{k+1} = 0 \]
we set some there exists a number of non-unitary partially massless limits which appear each time when symmetric tensor. In turn, in Anti de Sitter space we may set c as follows: 

\[ \Psi_{\mu a}, \Phi_{\mu} \]

mixed tensor \( Y \) and \( \Phi_{\mu} \)

Then for the other parameters we obtain:

\[ Y_{\mu a} = 0 \]. Then the whole system decomposes into two disconnected ones with the fields \( \Psi_{\mu a} \) and \( \Phi_{\mu} \) respectively. The first one provides an example of partially massless theory for the mixed tensor \( Y(s, 1) \), while the second one is the partially massless theory for the completely symmetric tensor. In turn, in Anti de Sitter space we may set \( c_{s-1} = 0 \). In this limit the main pair of fields \( (\Psi_{\mu a}^{(s-1)}, \Phi_{\mu}^{(s-1)}) \) decouples providing one more example of partially massless theory, while the remaining fields correspond to the massive \( Y(s-1, 1) \) theory. Also there exists a number of non-unitary partially massless limits which appear each time when we set some \( c_{k} = 0 \) (and hence \( e_{k} = 0 \)).

For the infinite spin case we choose \( c_{1} \) and \( d_{0} \) as our main parameters (see Figure 2). Then for the other parameters we obtain:

\[ c_{k}^{2} = \frac{2k(d + k - 4)(d + 2k)}{(d + k - 3)(d + 2k - 2)} c_{k+1}^{2} = \frac{2k(d + k - 4)(d + 2k)}{(d + k - 3)(d + 2k - 2)} c_{k+1}^{2} = \frac{2k(d + k - 4)(d + 2k)}{(d + k - 3)(d + 2k - 2)} c_{k+1}^{2} = \]

\[ \frac{2(k + 2)(k - 1)}{k + 1} c_{k}^{2} = 2(k + 2)(d + k - 4) \]

\[ \frac{2(d + 2)(d - 3)}{d(d - 2)} e_{2}^{2} = 18c_{1}^{2} - d_{1}^{2} - 6(d - 3) \]

\[ \frac{k(d + k - 3)(d + 2k)}{2(d + k - 2)(d + 2k - 2)} e_{k+1}^{2} = \frac{(k + 1)(d + k - 3)}{2k} e_{k}^{2} = \frac{(k + 1)(d + k - 3)}{2k} e_{k}^{2} = \frac{(k + 1)(d + k - 3)}{2k} e_{k}^{2} = \]

\[ 4c_{1}d_{0} = d_{1}e_{1}, \quad 3(d - 2)c_{1}d_{1} = (d - 3)d_{0}e_{1} \]

From the first two relations we obtain:

\[ (k + 2)(d + k - 2)d_{k+1}^{2} = (k + 1)(d + k - 3)d_{k}^{2} \]

\[ e_{k}^{2} = \frac{4k(d + k - 3)}{(k + 1)(d + k - 4)} c_{k}^{2} \]

The general solution again depends on the two free parameters. For the massive finite component \( Y(s, 1) \) case we choose \( s \) and \( m^{2} = \frac{sd_{s-1}^{2}}{2s+1} \) as our main ones. Then we obtain:

\[ d_{k}^{2} = \frac{2(s + 1)(d + s - 4)}{(k + 1)(d + k - 3)} m^{2} \]

\[ c_{k}^{2} = \frac{(s - k)(d + k - 4)}{(k - 1)(d + 2k - 2)} [m^{2} + (k + 1)(d + k - 4) \kappa] \]

(30)

\[ c_{1}^{2} = \frac{(s - 1)(d + s - 2)}{6d} [m^{2} + 2(d - 3) \kappa] \]

Let us briefly describe the main features of such theory. This time the massless limit is possible in flat Minkowski space only. In de Sitter space we may set \( m = 0 \) (and hence all \( d_{k} = 0 \)). Then the whole system decomposes into two disconnected ones with the fields \( \Psi_{\mu \nu} \) and \( \Phi_{\mu} \) respectively. The first one provides an example of partially massless theory for the mixed tensor \( Y(s, 1) \), while the second one is the partially massless theory for the completely symmetric tensor. In turn, in Anti de Sitter space we may set \( c_{s-1} = 0 \). In this limit the main pair of fields \( (\Psi_{\mu a}^{(s-1)}, \Phi_{\mu}^{(s-1)}) \) decouples providing one more example of partially massless theory, while the remaining fields correspond to the massive \( Y(s-1, 1) \) theory. Also there exists a number of non-unitary partially massless limits which appear each time when we set some \( c_{k} = 0 \) (and hence \( e_{k} = 0 \)).

For the infinite spin case we choose \( c_{1} \) and \( d_{0} \) as our main parameters (see Figure 2). Then for the other parameters we obtain:

\[ c_{k}^{2} = \frac{1}{(k - 1)(d + 2k - 2)} [(k + 1)(d + k - 4)C - (k - 1)(d + k - 2)D] \]

4This time our main gauge field \( \Psi_{\mu \nu}^{(s-1)} \) is connected with the two Stueckelberg ones — \( \Psi_{\mu \nu}^{(s-2)} \) and \( \Phi_{\mu}^{(s-1)} \) with the coefficients \( c_{s-1} \) and \( d_{s-1} \) correspondingly (see Figure 2). These coefficients are related as follows:

\[ 2(s + 1)(s - 2)c_{s-1}^{2} - sd_{s-1}^{2} = 2s(s + 1)(d + s - 5) \kappa \]

Thus for \( \kappa \neq 0 \) it is impossible to set both these two parameters to 0 and obtain the massless limit. So our \( m \) here is not the mass, but simply a convenient parameter.
unitary massless infinite spin theory, while for $C > D > Y$ (spin one. Note, that in the second case there is a limit $D \to \infty$ in de Sitter space. Indeed, all possible to take a limit $D \to \infty$ into two disconnected ones with the field $\Psi_{\mu\nu}$. In this case it is also possible to take a limit $D \to \infty$ so that the whole system decomposes into two disconnected ones with the field $\Psi_{\mu\nu}$ and $\Phi_{\mu}$ respectively.

The sum of kinetic and mass-like terms:

$$\mathcal{L} = \sum_{k=l}^{\infty} \left[ \sum_{m=1}^{l} \mathcal{L}_0(\Psi_{\mu\nu}^{(k)}, b^{(m)}) + \mathcal{L}_0(\Phi_{\mu\nu}^{(k)}) + \mathcal{L}_0(h_{\mu}^{(k)}) \right] + \sum_{k=l}^{\infty} \left[ \sum_{m=1}^{l} \mathcal{L}_2(\Psi_{\mu\nu}^{(k)}, b^{(m)}) + b_{k,1} \{ \mu \nu \alpha \} \Psi_{\mu\nu}^{(k-1),bc} h_{\alpha}^{(k-1)} + \mathcal{L}_2(h_{\mu}^{(k)}) \right]$$

where the kinetic terms were defined in (8) (13 for the special case $k = m = l$), (5) and (1), while the mass-like terms in (11) (10 for $k = m = l$) and (3). Also as the initial set of gauge transformations we use expressions given at (2), (4), (6), (9), (12), (14), (17).

Now we consider possible cross-terms (see Figure 3). $Y(k+1,m+1) \leftrightarrow Y(k,m+1)$. In this case they have the form:

$$(-1)^{k+m} \mathcal{L}_1 = c_{k,m} \{ \mu \nu \alpha \beta \} \left[ \Omega_{\mu\nu}^{(k-1),bf(m-1),c} \Psi_{\alpha\beta}^{(k-1),e} + \Omega_{\mu\nu}^{(k-1),af(m-1),b} \Psi_{\alpha\beta}^{(k-1),ce(k-1),df(m-1)} \right]$$

De Sitter space There are no unitary models except for the massive finite component $Y(s,1)$ ones described above.

Flat Minkowski space There are two general possibilities. For $C = D > 0$ we obtain unitary massless infinite spin theory, while for $C > D > 0$ we obtain unitary tahyonic infinite spin one. Note, that in the second case there is a limit $D \to 0$ that resembles the partially massless limit in de Sitter space. Indeed, all $d_k \to 0$ and the whole system decomposes into two disconnected ones with the field $\Psi_{\mu\nu}$ and $\Phi_{\mu}$ respectively.

Anti de Sitter space First of all there are solutions where all $c_k^2 > 0$ so that we obtain unitary infinite spin theory containing all our fields (see Appendix A.3). Moreover, it is possible to take a limit $D \to 0$ (so that all $d_k \to 0$) when the whole system decomposes into two subsystems with the fields $\Psi_{\mu\nu}$ and $\Phi_{\mu}$ respectively. The second class of solutions appears when some $c_s = 0$ (and hence $e_s = 0$), while all $c_k^2 > 0$ for $k > s$. In this case the unitary part of the theory contains fields $\Psi_{\mu\nu}^{(k)}$ and $\Phi_{\mu}^{(k)}$ with $k > s$. Note that in this case it is also possible to take a limit $D \to 0$ so that the whole system decomposes into four independent ones where the subsystem where the fields $\Psi_{\mu\nu}^{(k)}$, $s \leq k < \infty$ provides one more non-trivial example of the unitary infinite spin theory.

2.3 General case $Y(k+1,l+1)$, $k \to \infty$

In this case we also follow our previous work on the massive mixed symmetry tensors [20] but without a restriction on the number of field components. Thus we introduce: $(\Omega_{\mu\nu}^{(k),bf(m),c})$, $\Psi_{\mu\nu}^{(k),bf(m)}$, $l \leq k < \infty$, $1 \leq m \leq l$, $(\Omega_{\mu\nu}^{(k),bc})$, $\Phi_{\mu\nu}^{(k)}$ and $(\omega_{\mu\nu}^{(k),b})$, $h_{\mu}^{(k)}$, $l \leq k < \infty$. The sum of kinetic and mass-like terms:

$$d_k^2 = \frac{8(d-3)}{3(k+1)(d+k-3)}d_0^2$$

where

$$C = \frac{3dc_1^2}{(d-3)}, \quad D = \frac{2d_0^2}{3}$$

To analyze this solution it is convenient to introduce $x_k = (k+1)(d+k-4)$. Then we have:

$$c_k^2 \sim -kx_k^2 + [C - D + 2(d-3)]x_k + 2(d-3)D, \quad d_k^2 \sim D$$

De Sitter space There are no unitary models except for the massive finite component $Y(s,1)$ ones described above.
with the corresponding corrections for the gauge transformations

\[
\delta_1 \Omega_{\mu\nu}^{a(k),b(m),c} = -\frac{c_{k,m}}{(d + k - 3)} [e_{[\mu} a_{\eta_{\nu}]} a^{(k-1),b(m),c} - Tr] \\
+ \frac{k c_{k+1,m}}{(k + 1)} [\eta_{[\mu,\nu]} a^{(k),b(m),c} + \frac{1}{(k - m + 2)} \eta_{[\mu} a^{(k),b(m-1)\nu]} \\
+ \frac{1}{(k + 2)} \eta_{[\mu} a^{(k),b(m)} + \frac{1}{(k + 2)(k - m + 2)} \eta_{[\mu} a^{(k),b(m-1)c\nu]}]
\]

\[
\delta_1 \Phi_{\mu\nu}^{a(k),b(m)} = -\frac{c_{k,m}}{(d + k - 4)} [e_{[\mu} a_{\eta_{\nu}]} a^{(k-1),b(m)} - Tr] \\
+ \frac{k c_{k+1,m}}{(k + 2)} [\xi_{[\mu,\nu]} a^{(k),b(m)} + \frac{1}{(k - m + 2)} \xi_{[\mu} a^{(k),b(m-1)c\nu]}]
\]

Note that the two special cases \(Y(k + 1, 1) \leftrightarrow Y(k, 1)\) and \(Y(k + 1, 0) \leftrightarrow Y(k, 0)\) have already been considered in the previous subsections.

Figure 3: General mixed symmetry case

\[Y(k + 1, m + 1) \leftrightarrow Y(k + 1, m).\] here we introduce:

\[
(-1)^{k+m} \mathcal{L}_1 = d_{k,m} \left\{_{\mu\nu^a\beta}^{abcd} \right\} [\Omega_{\mu\nu}^{ae(k-1),bf(m-1),c} \Psi_{\alpha\beta}^{de(k-1),f(m-1)} \\
+ \Omega_{\mu\nu}^{ae(k-1),fb(m-1),b} \Psi_{\alpha\beta}^{ce(k-1),df(m-1)}]
\]
together with the following corrections:

$$
\delta_1 \Omega_{\mu \nu} a(k), b(m), c = \frac{d_{k,m}}{(k - m + 2)(d + m - 4)} [(k - m + 1)e_{[\mu} b_{\nu]} a(k), b(m-1), c - T r]
- e_{[\mu} a_{c}] a(k-1)b, b(m-1)c - T r]
- d_{k,m+1}[\eta_{\mu} a(k), b(m) c + \frac{1}{(m+1)} \eta_{\mu} a(k), b(m) c]
$$

$$
\delta_1 \Psi_{\mu \nu} a(k), b(m) = - \frac{d_{k,m}}{(k - m + 2)(d + m - 5)} [(k - m + 1)\xi_{[\mu} a(k), b(m-1) e_{\nu]} b - e_{[\mu} a_{b}] a(k-1)b, b(m-1) - T r]
- \frac{m}{(m+1)} d_{k,m+1}\xi_{[\mu} a(k), b(m) e_{\nu]}
$$

Here there is also a special case $Y(k + 1, 1) \Leftrightarrow Y(k + 1, 0)$ that has already been considered in the previous subsection.

The requirement that the whole Lagrangian be invariant under the all appropriately corrected gauge transformations produces a great number of relations on the parameters. First of all we obtain the following important relations:

$$
a_{k,m} = \frac{l(l + 1)(d + l - 3)(d + l - 4)}{m(k + 1)(d + k - 3)(d + m - 4)} a_{l,l}
$$

$$
d_{k,m}^2 = \frac{(l - m + 1)(d + l + m - 3)}{(k - m + 1)(d + k + m - 3)} d_{l,m}^2
$$

$$
c_{k,m}^2 = \frac{(k - l)(d + k + l - 3)}{(k - m)(d + k + m - 3)} c_{k,l}^2
$$

So all the parameters in the Lagrangian and gauge transformations are expressed in terms of the main ones: $a_{l,l}$, $d_{l,m}$, $m \leq l$ (corresponding to the left-most column in Figure 3) and $c_{k,l}$, $k > l$ (corresponding to the upper row).

For these main parameters we obtain:

$$
0 = \frac{2k(d + k)(d + k + m - 3)(d + k - 4) - 1}{k(d + k - 3)(d + 2k - 2)(d + k + m - 3)} c_{k+1,m}^2
- \frac{2(k - 1)(k + 2)(k - m + 1) - 1}{k(k + 1)(k - m + 1)} c_{k,m}^2
- \frac{2m(d + 2m - 2)((d + k + m - 3)(d + 5) + (k - m + 1)(m + 2) - 1)}{(k - m + 1)(d + 2m - 4)(d + m - 4)(d + k + m - 3)} d_{k,m+1}^2
- \frac{2(m - 1)}{km} d_{k,m}^2 - \frac{4}{k} a_{k,m} + \frac{2(k + 2)(d + k - 4) + (d + m - 6)}{k} r
$$

$$
0 = - \frac{2k(d + k)(m + 1)}{m(d + k - 3)(d + 2k - 2)(d + k + m - 3)} c_{k+1,m}^2
+ \frac{2(k - 1)(m + 1)}{m(k + 1)(k - m + 1)} c_{k,m}^2
$$

15
The relation on these parameters shows that for $\kappa$ number of infinite spin cases.

In de Sitter space we have an unitary forbidden region $M^2 < l(d + l - 5)\kappa$. At the boundary of this region we find the only unitary partially massless limit where $d_{l,0} = 0$ (and hence all $d_{k,0} = 0$) and all the fields in the lowest row on Figure 3 decouple from the rest ones. Inside the forbidden region there is a number of non-unitary partially massless limits which appear each time when some $d_{l,m} = 0$ (and hence all $d_{k,m} = 0$). In this case the Figure 3 splits vertically into two disconnected parts.

In Anti de Sitter space we also have an unitary forbidden region $M^2 < -(s - l)(d + s + l - 5)\kappa$. At the boundary of this region we again find the only unitary partially massless limit where $c_{s-1,l} = 0$ (and hence all $c_{s-1,m} = 0$) so that the right-most column in Figure 3 decouples. Inside the forbidden region there exists a number of non-unitary partially massless limit which appear each time when some $c_{k,l} = 0$ (and hence all $c_{k,m} = 0$) when the Figure 3 splits horizontally.

For the infinite spin case we choose $c_{l+1,l}$ and $d_{l,l}$ as our main parameters (see Figure 3). Then we obtain:

\[
a_{l,l} = \frac{(d + l - 5)(l + 2)}{4(d + l - 3)} C - \frac{(l + 2)(d + l - 5)}{4l} D + \frac{(l + 2)(d + l - 5)}{2} \kappa
\]

\[
c_{k,l}^2 = \frac{1}{2(k - 1)(d + 2k - 2)} [(k - l + 1)(d + k + l - 4)C]
\]

Once again the general solution of these equations depends on the two arbitrary parameters and it is capable of describing both the massive finite component case as well as a number of infinite spin cases.

For the massive case defined with $c_{s,l} = 0$ (and hence all $c_{s,m} = 0, 0 \leq m \leq l$) we obtain:

\[
a_{l,l} = -\frac{(s + 1)(d + s - 4)}{2l(d + l - 3)} [M^2 - l(d + l - 5)\kappa]
\]

\[
c_{k,l}^2 = \frac{(s - k)(d + s + k + 1)}{(k - 1)(d + 2k - 2)} [M^2 + (k - l + 1)(d + k + l - 4)\kappa]
\]

\[
d_{l,m}^2 = \frac{(s - m + 1)(d + l + m - 4)(d + s + m - 4)}{(m - 1)(d + 2m - 4)(d + l + m - 3)} [M^2 - (l - m)(d + l + m - 5)\kappa]
\]

\[
d_{l,1}^2 = \frac{s(d + l - 3)(d + s - 3)}{(d - 2)(d + l - 2)} [M^2 - (l - 1)(d + l - 4)\kappa]
\]

\[
d_{l,0}^2 = \frac{2(s + 1)(d + l - 4)(d + s - 4)}{(d - 4)(d + l - 3)} [M^2 - l(d + l - 5)\kappa]
\]

In the previous subsection our main gauge field $\Psi_{\mu\nu}^{a(s-1),b(l)}$ is connected with the two Stueckelberg ones $\Psi_{\mu\nu}^{a(s-2),b(l)}$ and $\Psi_{\mu\nu}^{a(s-1),b(l-1)}$ with the parameters $c_{s-1,l}$ and $d_{s-1,l}$ respectively (see Figure 3). The relation on these parameters

\[
c_{s-1,l}^2 - d_{s-1,l}^2 \sim \kappa
\]

shows that for $\kappa \neq 0$ it is impossible to set both these parameters to 0 and obtain the massless limit. Thus our $M$ here is not the mass, but just a convenient parameter.


\[
d_{l,k}^2 = \frac{-(k - l - 1)(d + k + l - 2)D}{2(k - 1)(d + 2k - 4)(d + l + k - 3)} - \frac{2(\kappa)(d + k + l - 4)\kappa}{(l - k)(d + k + l - 5)C} \\
(\text{denominator}) \\
+ (l - k + 2)(d + l + k - 3)D \\
- \frac{2(l - k)(l - k + 2)(d + l + k - 3)(d + l + k - 5)\kappa}{(l - k + 2)(d + l + k - 3)} D \\
\]

where

\[
C = \frac{l(d + 2l)}{(d + 2l - 3)c_{l+1,l}^2}, \quad D = (l - 1)d_{l,l}^2
\]

Let us introduce the variables \(x_k = k(d + k - 5)\) and the function

\[
F(x_k) = \quad -2\kappa x_k^2 + [C - D + (l(d + l - 5) + (l + 2)(d + l - 3))\kappa] x_k \\
+ (l + 2)(d + l - 3)D - l(d + l - 5)C - 2l(l + 2)(d + l - 3)(d + l - 5)\kappa
\]

Then we find:

\[
\begin{align*}
c_{k-1,l}^2 & \sim F(x_k), \quad l + 2 < k < \infty \\
d_{l,k}^2 & \sim F(x_k), \quad 0 \leq k \leq l \\
d_{l,l} & \sim F(x_0)
\end{align*}
\]

In flat Minkowski space we obtain a region of the parameters

\[
C \geq D > \frac{l(d + l - 5)}{(l + 2)(d + l - 3)} C
\]

corresponding to the unitary infinite spin theory containing all our fields. Beyond this region there exists a number of partially massless limits where one of the \(d_{l,k} = 0\) (and hence all \(d_{m,k} = 0\), \(m \geq l\)): \(D = \frac{(\kappa)(d + l + k - 5)}{(l - k + 2)(d + l + k - 3)} C, \quad C > 0\)

and the Figure 3 splits vertically into two disconnected parts.

In Anti de Sitter space there exists a whole region of parameters (see Appendix A.4) which corresponds to the unitary infinite spin theory containing all our fields. Beyond this region there are two types of discrete solutions. The first one corresponds to the cases where some \(c_{s,l} = 0\) (and hence all \(c_{s,m} = 0\), \(0 \leq m \leq l\)) so that Figure 3 splits horizontally. The second one corresponds to the cases where some \(d_{l,m} = 0\) (and hence all \(d_{k,m} = 0\), \(l \leq k < \infty\)) when Figure 3 splits vertically. Moreover, these two sets have common points when simultaneously some \(c_{s,l} = 0\) and \(d_{l,l} = 0\) so that Figure 3 splits into the four disconnected parts.

### 3 Massless fermionic fields

In this section we provide all necessary information on the massless (finite component) symmetric and mixed symmetry fermionic fields which will serve as the building blocks for our construction of the infinite component cases. In what follows \(Y(k + 1/2, l + 1/2)\) denote mixed symmetry spin-tensor having two rows of tensor indices with length \(k\) and \(l\) respectively as well as implicit spinor index.
3.1 Completely symmetric spin-tensor \( Y(k + 3/2, 1/2) \)

The frame-like description for such spin-tensor \([32]\) uses the one-form \( \Phi_{\mu}^{a(k)} \), completely symmetric on its local indices and \( \gamma \)-transverse, i.e. \( \gamma_b \Phi_{\mu}^{bat(k-1)} = 0 \). The free Lagrangian in the flat case has the form:

\[
(-1)^k \mathcal{L}_0(\Phi_{\mu}^{a(k)}) = -i \{ \frac{\mu \nu \alpha \beta \gamma}{abc} \} \left[ \Phi_{\mu}^{d(k)} \Gamma^{abc} \partial_{\nu} \Phi_{\alpha}^{d(k)} - 6k \Phi_{\mu}^{ad(k-1)} \gamma^b \partial_{\nu} \Phi_{\alpha}^{cd(k-1)} \right]
\]

(40)

It is invariant under the following gauge transformations:

\[
\delta_0 \Phi_{\mu}^{a(k)} = \partial_{\mu} \zeta^{a(k)} + \eta^{a(k), \mu}
\]

(41)

where parameters \( \zeta^{a(k)} \) and \( \eta^{a(k), b} \) are such that:

\[
(\gamma \zeta)^{a(k-1)} = 0, \quad \eta^{a(k), a} = 0, \quad (\gamma \eta)^{a(k-1), b} = \gamma_b \eta^{a(k), b} = 0
\]

If we replace the ordinary partial derivatives by the \( AdS \) covariant ones, the Lagrangian ceases to be gauge invariant:

\[
\delta \mathcal{L}_0 = i(-1)^k \frac{3(d + 2k - 1)(d + 2k - 2)}{2} \kappa \epsilon_{\mu \nu \alpha \beta \gamma} \Phi_{\mu}^{b(k)} \gamma^a \zeta^{a(k-1)}
\]

But the gauge invariance of the Lagrangian can be restored by adding the following mass-like terms to the Lagrangian:

\[
\mathcal{L}_1 = (-1)^{k+1} b_k \{ \frac{\mu \nu}{ab} \} \left[ \Phi_{\mu}^{c(k)} \Gamma^{ab} \Phi_{\nu}^{c(k)} + 2k \Phi_{\mu}^{ac(k-1)} \Phi_{\nu}^{bc(k-1)} \right]
\]

(42)

as well as the corresponding corrections to the gauge transformations

\[
\delta_1 \Phi_{\mu}^{a(k)} = -i b_k \frac{2}{3(d-2)\kappa} \left[ \gamma_{\mu} \zeta^{a(k)} - \frac{2}{(d + 2k - 2)} \gamma^a \zeta^{\mu} \right]
\]

(43)

provided

\[
b_k^2 = -\frac{9}{4}(d + 2k - 2)^2 \kappa
\]

As usual for the fermions it requires \( \kappa < 0 \), i.e \( AdS \) space.

3.2 Mixed symmetry spin-tensor \( Y(k + 3/2, 3/2) \)

In this and the following subsections we use the results of \([19]\) (see also \([33, 34]\)) which provided the generalization of the bosonic formalism \([25]\) to the fermionic case.

Let us consider the simplest example of the mixed symmetric spin-tensors which appears to be special and has to be considered separately. We need the two-form \( \Psi_{\mu \nu}^{a(k)} \), completely symmetric and \( \gamma \)-transverse on its local indices. The free Lagrangian in the flat space can be written as follows:

\[
(-1)^k \mathcal{L}_0(\Psi_{\mu \nu}^{a(k)}) = -i \{ \frac{\mu \nu \alpha \beta \gamma}{ab \cdots e} \} \left[ \Psi_{\mu \nu}^{f(k)} \Gamma^{ab \cdots e} \partial_{\alpha} \Psi_{\beta \gamma}^{f(k)} - 10k \Psi_{\mu \nu}^{af(k-1)} \Gamma^{bc \cdots e} \partial_{\alpha} \Psi_{\beta \gamma}^{ef(k-1)} \right]
\]

(44)
It is invariant under the following gauge transformations:

$$\delta_0 \Psi_{\mu\nu}^{a(k)} = \partial_\mu \xi_\nu^{a(k)} + \eta^{a(k),\mu\nu}$$

(45)

As usual, the replacement of the ordinary partial derivatives spoils the invariance of the Lagrangian that we can try to restore by introducing the mass-like terms to the Lagrangian

$$\mathcal{L}_1 = (-1)^k a_k \left\{ \frac{\mu \alpha \beta}{a b c d} \right\} \left[ \bar{\Psi}_{\mu \nu} e^{(k)} \Gamma^{a b c d} \Psi_{\alpha \beta} e^{(k)} + 6 k \bar{\Psi}_{\mu \nu} a e^{(k-1)} \Gamma^{b c} \Psi_{\alpha \beta} d e^{(k-1)} \right]$$

(46)

as well as the corresponding corrections to the gauge transformations:

$$i \delta_1 \Psi_{\mu\nu}^{a(k)} = \frac{i a_k}{5(d-4)} [\gamma_\mu \xi_\nu^{a(k)}] + \frac{2}{(d + 2k - 2)} \gamma^a \xi_{[\mu,\nu]}^{a(k-1)}$$

(47)

Collecting together all variations we obtain

$$i (-1)^k \left[ \frac{8(d-3)}{5(d-4)} a_k, 0^2 \right] - 10(d-3)(d-4) \kappa \left\{ \frac{\mu \alpha \beta}{a b c} \right\} \left[ \bar{\Psi}_{\mu \nu} (k) \Gamma^{a b c} \xi_\alpha^{(k)} - 3 k \bar{\Psi}_{\mu \nu} a^{(k-1)} \gamma^b \xi_\alpha^{(k-1)} \right]$$

$$i (-1)^k \left[ \frac{8 a_k, 0^2}{5(d-4)(d+2k-2)} - 10 k (2d+2k-5) \kappa \left\{ \frac{\mu \alpha \beta}{a b c} \right\} \left[ \bar{\Psi}_{\mu \nu} (k) \Gamma^{a b c} \xi_\alpha^{(k)} + 6 \bar{\Psi}_{\mu \nu} a^{(k-1)} \gamma^b \xi_\alpha^{(k-1)} \right]$$

As it is easy to see it is impossible to restore the gauge invariance by adjusting the only parameter $a_{k,0}$.

### 3.3 General case — spin-tensor $Y(k+3/2, m + 3/2)$

The frame-like formalism requires the two-form $\Psi_{\mu \nu}^{a(k), b(m)}$, with the local indices corresponding to the Young tableau $Y(k, m)$. The free Lagrangian in the flat space looks as follows:

$$(-1)^{k+m} \mathcal{L}_0 (\Psi_{\mu \nu}^{a(k), b(m)}) = -i \left\{ \frac{\mu \alpha \beta}{a b c d e} \right\} \left[ \bar{\Psi}_{\mu \nu} \Gamma^{a b c d e} \right] \partial_\alpha \Psi_{\beta \gamma} f(k), h(m)$$

$$-10 k \bar{\Psi}_{\mu \nu} a f(k-1), h(m) \Gamma^{b c d e} \partial_\alpha \Psi_{\beta \gamma} e f(k-1), h(m)$$

$$-10 m \bar{\Psi}_{\mu \nu} f(k), a h(m-1) \Gamma^{b c d e} \partial_\alpha \Psi_{\beta \gamma} f(k), e h(m-1)$$

$$-60 k m \bar{\Psi}_{\mu \nu} a f(k-1), b h(m-1) \gamma^c \partial_\alpha \Psi_{\beta \gamma} d f(k-1), e h(m-1)$$

(48)

It is invariant under the following gauge transformations:

$$\delta_0 \Psi_{\mu \nu}^{a(k), b(m)} = \partial_\mu \xi_\nu^{a(k), b(m)} + \eta^{a(k), b(m), \nu}$$

(49)

The replacement of the ordinary partial derivatives by the AdS covariant ones produces:

$$\delta \mathcal{L}_0 = -10 i k (-1)^{k+m} \left\{ \frac{\mu \alpha}{a b c} \right\}$$

$$\left[ [(d + 2k - 3)(d + 2m - 4) + (k - m)(2k - 2m + 3)] \bar{\Psi}_{\mu \nu} d(k), e(m) \Gamma^{a b c} \xi_\alpha d^{(k), e(m)} \right.$$  

$$-3 k [(d + 2m - 5)(d + 2m - 6) + 3 m - 4 k - 6] \bar{\Psi}_{\mu \nu} a d(k-1), e(m) \gamma^b \xi_\alpha c d(k-1), e(m)$$  

$$-3 m (d + 2k - 1)(d + 2k - 2) \bar{\Psi}_{\mu \nu} d(k), a c(m-1) \gamma^b \xi_\alpha d^{(k), e(m-1)}$$

(50)
At the same time if we introduce the mass-like term into the Lagrangian:

\[ (-1)^{k+m} \mathcal{L}_1 = a_{k,m} \{ \frac{\mu \nu}{abcd} \} [\bar{\Psi}_{\mu \nu} f(k),h(m)] \Gamma^{abcd} \Psi_{\alpha \beta} f(k),h(m) \]

\[ + 6k \bar{\Psi}_{\mu \nu} a f(k-1),h(m) \Gamma^{bc} \Psi_{\alpha \beta} d f(k-1),h(m) \]

\[ + 6m \bar{\Psi}_{\mu \nu} f(k),ah(m-1) \Gamma^{bc} \Psi_{\alpha \beta} f(k),dh(m-1) \]

\[ - 12km \bar{\Psi}_{\mu \nu} a f(k-1),bh(m-1) \Psi_{\alpha \beta} c f(k-1),dh(m-1) \]  

(51)

as well as the corresponding corrections to the gauge transformations:

\[ \delta_1 \Psi_{\mu \nu} a(k),b(m) = \frac{i a_{k,m}}{5(d-4)} \gamma_{[\mu \xi_{\nu]} a(k),b(m)} + \frac{2}{(d+2k-2)} \gamma^{a \xi_{[\mu,\nu]} a(k-1),b(m)} \]

\[ + \frac{2}{(d+2m-2)} \gamma^{b \xi_{[\mu} a(k),b(m-1)]_{\nu]} \]

\[ - \frac{4}{(d+2k-2)(d+2m-2)} \gamma^{c \xi_{[\mu} a(k-1),b(m-1)]_{\nu]} \]  

(52)

we obtain:

\[ \delta_1 \mathcal{L}_1 = -i (-1)^{k+m} \frac{8a_{k,m}^2}{5(d+2k-2)(d+2m-4)} \{ \frac{\mu \nu}{abcd} \} \]

\[ \left[ [(d+2k-1)(d+2m-4)+k-m+2] \bar{\Psi}_{\mu \nu} d^{(k),e(m)} \Gamma^{d(k),e(m)} \xi_{a(k),b(m)} \right. \]

\[ -3k[(d+2k-1)(d+2m-4)+2k-2m+4] \bar{\Psi}_{\mu \nu} a d^{(k-1),e(m)} \gamma^{b \xi_{a(k),d(k),e(m)} \gamma^{d(k),e(m)}} \]

\[ -3m(d+2k-1)(d+2m-4) \bar{\Psi}_{\mu \nu} d^{(k),ae(m-1)} \gamma^{b \xi_{a(k),d(k),e(m-1)}} \]  

(53)

Thus in this case it is also impossible to restore the gauge invariance by adjusting the parameter \( a_{k,m} \).

3.4 Special case \( Y(l+3/2, l+3/2) \)

This time we introduce a two-form \( R_{\mu \nu}^{a(l),b(l)} \) with the free Lagrangian:

\[ \mathcal{L}_0(R_{\mu \nu}^{a(l),b(l)}) = -i \left\{ \frac{\mu \nu}{abcd} \right\} [\bar{R}_{\mu \nu} f(l),h(l)] \Gamma^{abcd} \partial_{\alpha} R_{\beta \gamma} f(l),h(l) \]

\[ -20l \bar{R}_{\mu \nu} f(l),ah(l-1) \Gamma^{abcd} \partial_{\alpha} R_{\beta \gamma} f(l),eh(l-1) \]

\[ -60l^2 \bar{R}_{\mu \nu} a f(l-1),b(h(l-1)) \gamma^{c} \partial_{\alpha} R_{\beta \gamma} f(l-1),eh(l-1) \]  

(54)

which is invariant under the gauge transformations

\[ \delta_0 R_{\mu \nu}^{a(l),b(l)} = \partial_{[\mu} \xi_{\nu]} a(l),b(l) + \eta_{[\mu} a(l),b(l),_{\nu]} \]

(55)

Switching to the AdS covariant derivatives we spoil the invariance of the Lagrangian:

\[ \delta \mathcal{L}_0 = -10i(d+2l-3)(d+2l-4)k \left\{ \frac{\mu \nu}{abcd} \right\} \]

\[ [\bar{R}_{\mu \nu} d^{l,e(l)} \Gamma^{abcd} \xi_{a(l),e(l)} - 6l \bar{R}_{\mu \nu} d^{l,l-1} \gamma^{b \xi_{a(l),e(l-1)}} \]  

(56)
In this section we present an analogous construction for the fermionic case. As in the bosonic case we begin with the completely symmetric spin-tensors, then we give a simple example for the mixed symmetry spin-tensors based on the long hooks and at last we consider general case based on the mixed symmetry spin-tensors with two rows.

4.1 Symmetric case — $Y(k + 3/2, 1/2), \ k \rightarrow \infty$

In the recent paper [11] Metsaev has shown that the very same gauge invariant formalism that was previously used for the description of massive higher spin fermionic fields (see [14] for the metric like formulation and [17] for the frame-like one) is also capable of describing a number of infinite spin cases. In this subsection we reproduce the results of [11] but in the frame-like formalism. We follow our previous results in [17] but without a restriction on the number of field components. Thus we introduce a set of spinor one-forms $\Phi^a_{\mu \nu}$, $0 \leq k < \infty$ as well as spinor zero-form $\phi$. We begin with the sum of kinetic and mass-like terms for all these fields:

$$\mathcal{L} = \sum_{k=0}^{\infty} \left[ \mathcal{L}_0(\Phi^a_{\mu \nu}^{(k)}) + \mathcal{L}_2(\Phi^a_{\mu \nu}^{(k)}) \right] + i e^\mu_a \bar{\phi} \gamma^a D_\mu \phi + c_0 \bar{\phi} \phi \tag{58}$$

where Lagrangians $\mathcal{L}_0(\Phi^a_{\mu \nu}^{(k)})$ and $\mathcal{L}_2(\Phi^a_{\mu \nu}^{(k)})$ were defined in (40) and (42), with the initial set of gauge transformations defined in (11) and (12).

Now we add a set of cross-terms gluing all these fields together (see Figure 4):

$$\mathcal{L}_1 = \sum_{k=1}^{\infty} (-1)^{k+1} i e_k \left\{ \gamma^\mu_{\ a b c} \left( \Phi^a_{\mu \nu} \gamma^{bc(k-1)} \gamma^b \Psi_{\nu, c(k-1)} - \bar{\Phi}^a_{\mu \nu} \gamma^{bc(k-1)} \right) \right\}$$

$$+ i e_0 e^\mu_a (\bar{\Phi} \gamma^a \phi - \bar{\phi} \gamma^a \Phi) \tag{59}$$

as well as the corresponding corrections to the gauge transformations:

$$\delta_1 \Phi^a_{\mu \nu}^{(k)} = \frac{\epsilon_{k+1}}{6(k+1)} \zeta^a_{\mu \nu}^{(k)} + \frac{\epsilon_k}{6k(d+k-2)} (\epsilon^a_{\mu \gamma} \zeta^{(k-1)} - Tr) \tag{59}$$

$$\delta_1 \phi = \epsilon_0 \zeta \tag{60}$$
Then we require that the whole Lagrangian be gauge invariant. This gives us a number
of recurrent relations on the parameters:

\[ b_{k+1} = \frac{(d+2k)}{(d+2k+2)}b_k, \quad k \geq 0, \quad b_0 = -\frac{3(d-2)}{d}c_0 \]

\[ \frac{(d+2k-2)}{6(k+1)(d+2k)}e_{k+1}^2 = \frac{e_k^2}{6k} - \frac{2(d+2k-1)}{3(d+2k-2)}b_k^2 - \frac{3}{2}(d+2k-1)(d+2k-2)\kappa \]

\[ \frac{(d-2)}{6d}e_1^2 = e_0^2 - \frac{2(d-1)}{3(d-2)}b_0^2 - \frac{3}{2}(d-1)(d-2)\kappa \]

As in the bosonic case the general solution depends on the two arbitrary parameters. For
the description of massive fermionic spin-(s + 1/2) field we obtain:

\[ b_k^2 = \frac{9(d+2s-2)^2}{(d+2k)^2}[m^2 - \frac{1}{4}(d+2s-4)^2\kappa] \]

\[ e_k^2 = \frac{36k(s-k)(d+s+k-2)}{(d+2k-2)}[m^2 - (s-k-1)(d+s+k-3)\kappa] \]

\[ e_0^2 = \frac{6s(d+s-2)}{(d-2)}[m^2 - (s-1)(d+s-3)\kappa] \quad (61) \]

Let us briefly recall the main properties of this theory. The massless limit is possible in flat
and AdS spaces only. In flat space such limit results in the sum of massless fields with spins
s + 1/2, s − 1/2, . . . , 1/2, while in AdS space we get massless spin-(s + 1/2) and massive spin-
(s − 1/2) fields. In de Sitter space we have an unitary forbidden region \( m^2 < \frac{1}{4}(d+2s-4)^2\kappa \). Inside this region lives a number of non-unitary partially massless limits that appears each
time then one of the \( e_k = 0 \).

Let us turn now to the infinite spin solution. Here we choose \( c_0 \) and \( e_0 \) as our main
parameters. We obtain:

\[ b_k = \frac{(d-2)c_0}{(d+2k)} \]

\[ e_k^2 = \frac{6k(d+2k-2)}{(d-2)}e_0^2 - \frac{36k^2(d+k-2)}{(d+2k-2)}c_0^2 - 9k^2(d+k-2)(d+2k-2)\kappa \]

\[ e_0^2 \]

It is convenient to introduce auxiliary variables \( x_k = k(d+k-2) \). Then we get:

\[ e_k^2 \sim -4\kappa x_k^2 + \left[ \frac{8e_0^2}{3(d-2)} - 4c_0 - (d-2)^2\kappa \right] x_k + \frac{2(d-2)e_0^2}{3} \]

\[ (64) \]
The spectrum of the infinite spin solutions is essentially the same as in the bosonic case. There are no such solutions in de Sitter space. In flat Minkowski space for the $2e_0^2 = 3(d - 2) c_0 > 0$ (that again corresponds to $\mu_0 = 0$ in the Metsaev paper [11]) we obtain the massless unitary infinite spin model, while for $2e_0^2 > 3(d - 2) c_0 > 0$ — unitary tahyonic one. Besides, for $e_0^2 < 0$ there exists a discrete spectrum of solutions which appear each time when we adjust the parameters so that some $e_s = 0$ and all $e_k^2 > 0$, $k > s$:

$$c_0^2 = \frac{(d + 2s - 2)^2}{6(d - 2)s(d + s - 2)} e_0^2,$$

$$e_0^2 > \frac{6s(s + 1)(d + s - 2)(d + s - 1)}{(s - 2)} \kappa, \quad s > 0$$

(65)

$$c_0 = 0, \quad e_0^2 > 0, \quad s = 0$$

In Anti de Sitter space we also have two general types of solution. First of all, there exists the whole range of the parameters (see Appendix A.5) such that all $e_k^2 > 0$ and all our fields remain to be connected. Besides, there are discrete solutions where unitary part of the theory contains the fields $\Phi_\mu^{a(k)}$ with $k > s$ only.

### 4.2 Mixed symmetry example $Y(k + 3/2, 3/2) \rightarrow \infty$

In this section we present an example for mixed symmetry fermionic case based on the long hooks. We follow our previous results on the massive finite component case [19], but without a restriction on the number of components. Thus we introduce the spinor two-forms $\Psi_{\mu\nu}^{a(k)}$ and spinor one-forms $\Phi_\mu^{a(k)}$, $0 \leq k < \infty$. We begin with the sum of kinetic and mass-like terms:

$$\mathcal{L} = \sum_{k=0}^{\infty} [\mathcal{L}_0(\Psi_{\mu\nu}^{a(k)}) + \mathcal{L}_0(\Phi_\mu^{a(k)}) + \mathcal{L}_2(\Psi_{\mu\nu}^{a(k)}) + \mathcal{L}_2(\Phi_\mu^{a(k)})]$$

(66)

which were defined in [14], [40], [46] and [42] correspondingly. Also we use initial gauge transformations that were defined in [11], [43], [45] and [47].

Let us consider possible cross-terms (see Figure 5).

$Y(k + 5/2, 3/2) \leftrightarrow Y(k + 3/2, 3/2)$. In this case the cross-terms have the form:

$$-i(-1)^k \mathcal{L}_{11} = c_k \left\{ \mu_{a\beta abcd} \right\} \left[ \overline{\Psi}_{\mu\nu}^{ae(k)} \Gamma_{\alpha\beta}^{bcd} \Psi_{\alpha\beta}^{\epsilon(k)} - \overline{\Psi}_{\mu\nu}^{e(k)} \Gamma_{abc}^{\delta} \Psi_{\alpha\beta}^{de(k)} \right]$$

(67)

while the corresponding corrections to the gauge transformations look like:

$$\delta_{11}\Psi_{\mu\nu}^{a(k)} = \frac{c_k}{10(k + 2)} e_{[\mu, \nu]}^{a(k)} - \frac{c_{k-1}}{10k(d + k - 3)} (e_{[\mu}^{a} \xi_{\nu]}^{a(k-1)} - Tr)$$

(68)

$Y(k + 3/2, 3/2) \leftrightarrow Y(k + 3/2, 1/2)$. Here we introduce the following terms:

$$-i(-1)^k \mathcal{L}_{12} = d_k \left\{ \mu_{a\beta ab} \right\} \left[ \overline{\Phi}_\mu^{d(k)} \Gamma_{abc}^{\alpha} \Phi_\alpha^{d(k)} - 6k \overline{\Phi}_\mu^{ad(k-1)} \gamma_\beta \Phi_\alpha^{cd(k-1)} \right]$$

$$+ d_k \left\{ \mu_{a\beta ab} \right\} \left[ \Phi_\mu^{d(k)} \Gamma_{abc}^{\alpha} \Phi_\alpha^{d(k)} - 6k \Phi_\mu^{ad(k-1)} \gamma_\beta \Phi_\alpha^{cd(k-1)} \right]$$

(69)

as well as the corresponding corrections to the gauge transformations:

$$\delta_{12}\Psi_{\mu\nu}^{a(k)} = \frac{(k + 1)d_k}{10(k + 2)(d - 3)(d - 4)} \Gamma_{\mu\nu}^{a(k)} - \frac{1}{2(k + 1)(d + k - 3)}$$

$$\frac{((d - 3)(d - 4) + 2k + 2) e_{[\mu}^{a(k-1)} - (d + 2k - 1) \gamma^{[a} \xi_{\nu]}^{a(k-1)}]}{(d + 3)(d + 4)}$$

(70)

$$\delta\Phi_\mu^{a(k)} = 2d_k \xi_\mu^{a(k)}$$
when the whole system decomposes into two disconnected ones containing fields from the
Minkowski space only. In de Sitter space there is an unitary forbidden region
inside this region there exists the only partial massless limit

\[ \delta_{13} \Phi_\mu^{a(k)} = \frac{e_k}{6(k+1)} \varepsilon_\mu^{a(k)} + \frac{e_{k-1}}{6(d+k-2)} (e_\mu^{a(k-1)} - Tr) \]  

(72)

Now we require that the total Lagrangian be invariant under the resulting gauge transformations. This gives a number of recurrent relations on the parameters:

\[ (d + 2k + 2)a_{k+1} = (d + 2k)a_k \]

\[ (k + 2)(d + l - 1)d_{k+1}^2 = (k + 1)(d + k - 2)d_k^2 \]

\[ b_k = \frac{3(d - 2)}{5(d - 4)} a_k \]

\[ e_k^2 = \frac{9(k + 1)(d + k - 1)}{25(k + 2)(d + k - 2)} c_k^2 \]

\[ \frac{2(d - 3)(d - 4)}{5(k + 1)(k + 2)(d + k - 2)(d + 2k)} c_k^2 = \frac{4(k + 1)}{(k + 2)} a_k^2 - \frac{8(d - 3)}{5(d - 4)} a_k^2 - 10(d - 3)(d - 4) \kappa \]

For the massive finite spin case where some \( c_s = 0 \) (and hence \( e_s = 0 \)) we obtain the following solution (here \( m \) is not the mass, but just a convenient parameter):

\[ d_k^2 = \frac{(s + 1)(d + s - 2)}{(k + 1)(d + k - 2)} m^2 \]

\[ c_k^2 = \frac{10(k + 1)(d + s + k)(s - k)}{(d + 2k)} \left[ (d - 4)(s + 1) m^2 + 10(k + 2)(d + k - 2) \kappa \right] \]

\[ a_k^2 = \frac{(d + 2s)^2(d - 4)}{4(d + 2k)^2(d - 3)(s + 2)} \left[ 10(s + 1)(d + s - 2)m^2 - 25(d - 3)(d - 4)(s + 2) \kappa \right] \]

The massless limit (that requires \( c_{s-1} \to 0 \) and \( d_s \to 0 \) simultaneously) is possible in flat Minkowski space only. In de Sitter space there is an unitary forbidden region

\[ m^2 < \frac{5(d - 4)(d - 3)(s + 2)}{2(s + 1)(d + s - 2)} \kappa \]

Inside this region there exists the only partial massless limit \( d_s \to 0 \) (and hence all \( d_k \to 0 \)) when the whole system decomposes into two disconnected ones containing fields from the
upper and lower rows in the Figure 5. In Anti de Sitter space we also find an unitary forbidden region

\[ m^2 < -\frac{10(d-3)(s+2)(d+s-3)}{(d-4)} \kappa \]

At the boundary lives the only unitary partially massless limit where \( c_{s-1} = 0 \) (and hence \( e_{s-1} = 0 \)). In this limit two main fields \( \Psi_{\mu\nu}^{a(s)} \) and \( \Phi_{\mu}^{a(s)} \) decouple. Inside the unitary forbidden region there is a number of partially massless limits that appear each time when one of the \( c_l = 0 \) (and hence \( e_l = 0 \)).

For the infinite spin solutions we choose the parameters \( c_0 \) and \( d_0 \) as our main ones (see Figure 5). Then we obtain:

\[
\begin{align*}
  a_k^2 &= \frac{50}{8(d+2k)^2}[-4(d-4)^2C + 4d^2D - d^2(d-4)^2\kappa] \\
  c_k^2 &= \frac{200(k+1)}{2(d+2k)}[(d+k-2)C - k(d+k)D - k(k+2)(d+k)(d+k-2)\kappa] \\
  d_k^2 &= \frac{(d-2)}{(k+1)(d+k-2)}d_0^2
\end{align*}
\]

where

\[
C = \frac{d_0^2}{200(d-2)}, \quad D = \frac{(d-4)d_0^2}{20(d-3)}
\]

If we introduce the auxiliary variables \( x_k = k(d+k) \) and function

\[
F(x_k) = -\kappa x_k^2 + (C - D - d\kappa)x_k + 2dC
\]

then we will find that

\[
\begin{align*}
  c_k^2 &\sim F(x_l), \quad k \geq 0 \\
  d_k^2 &\sim F(x_l), \quad l = -2 \\
  a_k^2 &\sim F(x_l), \quad l = -\frac{d}{2}
\end{align*}
\]

In flat Minkowski space we obtain the unitary infinite spin solution where all our fields enter for the quite restricted range of the parameters:

\[
C \geq D \geq \frac{(d-4)^2}{d^2}C
\]

In Anti de Sitter space there is a region of parameters (see Appendix A.6) when all our fields remain to be connected. At the boundary of this region there is a limit when all \( d_k = 0 \) so that Figure 5 splits vertically into two parts containing fields \( \Psi_{\mu\nu} \) and \( \Phi_{\mu} \) respectively. Besides, there is a discrete set of solutions where some \( c_s = 0 \) (and hence \( e_s = 0 \)) while all \( c_k^2 > 0 \) for \( k > s \) so that Figure 5 splits horizontally with the unitary part containing fields \( \Psi_{\mu\nu}^{a(k)} \) and \( \Phi_{\mu}^{a(k)} \) with \( k > s \) only. Moreover, these solutions admit a limit where all \( d_k = 0 \) and Figure 5 splits into four disconnected parts.
4.3 General case $Y(k + 3/2, l + 3/2)$, $k \to \infty$

We need the following set of fields: $\Psi_{\mu\nu}^{a(k), b(m)}$, $0 \leq m \leq l$, $l \leq k \leq \infty$, and $\Phi_{\mu}^{a(k)}$, $l \leq k \leq \infty$. The sum of their kinetic and mass-like terms is:

$$\mathcal{L} = \sum_{k=1}^{\infty} \sum_{m=0}^{l} \left( \mathcal{L}_0(\Psi_{\mu\nu}^{a(k), b(m)}) + \mathcal{L}_2(\Psi_{\mu\nu}^{a(k), b(m)}) + \mathcal{L}_0(\Phi_{\mu}^{a(k)}) + \mathcal{L}_2(\Phi_{\mu}^{a(k)}) \right)$$

(74)

where all these Lagrangians as well as corresponding gauge transformations were defined in the previous section.

Now we introduce all possible cross-terms:

$$Y(k + 3/2, m + 3/2) \leftrightarrow Y(k + 1/2, m + 3/2)$$

$$\mathcal{L}_{11} = \sum_{k=l+1}^{\infty} \sum_{m=1}^{l} \frac{(-1)^{k+m} c_{k,m}}{10(k - m + 2)(k + 2)} \left[ (k - n + 2) \xi_{[\mu,\nu]}^{a(k), b(m)} + \xi_{[\mu}^{a(k), b(m+1), \nu]} \right]$$

(75)

Corrections to the gauge transformations:

$$\delta_{111}^{a(k), b(m)} = -\frac{c_{k+1,m}}{10(k - m + 2)(k + 2)} \left[ (k - n + 2) \xi_{[\mu,\nu]}^{a(k), b(m)} + \xi_{[\mu}^{a(k), b(m+1), \nu]} \right] - \frac{c_{k,m}}{10(k - m + 2)(k + 2)} \left[ \xi_{[\mu}^{a(k), b(m)} + \ldots \right]$$

(76)

$$Y(k + 3/2, m + 3/2) \leftrightarrow Y(k + 1/2, m + 1/2)$$

$$\mathcal{L}_{12} = \sum_{k=l}^{\infty} \sum_{m=1}^{l} \frac{(-1)^{k+m} d_{k,m}}{10(k - m + 2)(k + 2)} \left[ \Psi_{\mu\nu}^{e(k), a(f(m+1))} \right]$$

(77)

Corrections:

$$\delta_{12}^{a(k), b(m)} = -\frac{d_{k,m}}{10(m - k + 2)(m - m + 4)} \left[ (k - m + 1) \xi_{[\mu}^{a(k), b(m), \nu]} \right]$$

(78)

There are three special cases $Y(k + 5/2, 3/2) \leftrightarrow Y(k + 3/2, 3/2)$, $Y(k + 3/2, 3/2) \leftrightarrow Y(k + 3/2, 1/2)$, $Y(k + 5/2, 1/2) \leftrightarrow Y(k + 3/2, 1/2)$ that have been considered in the previous subsection.

---

The meaning of the coefficients is essentially the same as in the bosonic case, see Figure 3.
Now we require that the whole Lagrangian be gauge invariant. First of all, this gives a number of important relations on the parameters:

\[ a_{k,m} = \frac{(d+2l)(d+2l-2)}{(d+2k)(d+2m-2)} a_{l,t} \]
\[ c_{k,m}^2 = \frac{(k-l)(d+k+l-2)}{(k-m)(d+k+m-2)} c_{k,l}^2 \]
\[ d_{k,m}^2 = \frac{(l-m+1)(d+l+m-2)}{(k-m+1)(d+k+m-2)} d_{l,m}^2 \]
\[ e_k^2 = \frac{9(k-l)(d+k+l-2)}{25(k+1)(d+k-3)} e_{k,2}^2 \]

Thus all the parameters are determined by the \(a_{l,t}, c_{k,l}, k \geq l+1\) and \(d_{l,m}, 0 \leq m \leq l\). Further, for these parameters we obtain:

\[ c_{k,l}^2 = \frac{k(d+2k-2)}{(k+1)(d+2k)} c_{k+1,l} + \frac{4k(d+2k-1)}{(d+2k-2)} a_{k,t}^2 \]
\[ + 25k(d+2k-1)(2+2k-2) \]
\[ d_{l,m}^2 = \frac{m(l-m)((d+2m-8)(d+l+m-1)+6)}{(m+1)(l-m+1)(d+2m-2)(d+l+m-2)} d_{l+1,m}^2 \]
\[ + \frac{m(d^2+3dl+dm-7d+2l^2+2lm-8l-6m+12)}{(l+1)(d+2l)(d+l+m-2)} c_{l+1,m}^2 \]
\[ + \frac{8m(d+2l-1)(d+2m-4)+l-m+2}{(d+2l-2)(d+2m-4)} a_{l,t}^2 \]
\[ + 50m((d+2l-3)(d+2m-4)+(l-m)(2l-2m+3)) \kappa \]

For the massive mixed symmetry spin-tensor \(Y(s+3/2, l+3/2)\) we obtain the following solution:

\[ a_{l,t}^2 = \frac{(d+2s)^2}{4(d+2l)^2} [m^2 - 25(d+2l-4)^2 \kappa] \]
\[ c_{k,l}^2 = \frac{k(s-k+1)(d+s+k-1)}{(d+2k-2)} [m^2 + 100(k-l+1)(d+k+l-3) \kappa] \]
\[ d_{l,n}^2 = \frac{n(s-n+2)(d+s+n-2)(d+l+n-3)}{(d+2n-4)(d+l+n-2)} [m^2 - 100(l-n)(d+l+n-4) \kappa] \]

Once again the massless limit is possible in flat Minkowski space only. In de Sitter space we obtain an unitary forbidden region

\[ m^2 < 25(d+2l-4)^2 \kappa \]

Inside this region we find a number of non-unitary partially massless limits where some \(d_{l,k} = 0\). In Anti de Sitter space there is also an unitary forbidden region

\[ m^2 < -100(s-l+1)(d+s+l-3) \kappa \]
At the boundary of this region lives the only unitary partially massless case where \( c_{s,l} = 0 \), while inside this region there exists a number of non-unitary partially massless models where some \( c_{k,l} = 0 \).

For the infinite spin solutions we choose \( c_{l+1,l} \) and \( d_{l,0} \) as our main parameters. Then we obtain:

\[
\begin{align*}
    a_{l,l}^2 &= -\frac{25(d-4)^2}{(d+2l)^2} C + 25D - \frac{25(d-4)^2}{4} \kappa \\
    c_{k,l}^2 &= \frac{100k}{(d+2k-2)^2} [(k+1)(d+k-3)C - (k-l-1)(d+k+l-1)D \\
                 &\quad - (k+1)(d+k-3)(k-l-1)(d+k+l-1)\kappa] \\
    d_{l,k}^2 &= \frac{100k(d+l+k-3)}{(d+2k-4)(d+l+k-2)^2} [k(d+k-4)C + (l-k+2)(d+l+k-2)D \\
                 &\quad + k(l-k+2)(d+k-4)(d+l+k-2)\kappa]
\end{align*}
\]

where

\[
C = \frac{(d+2l)c_{l+1,l}^2}{100(l+1)(l+2)(d+l-2)}, \quad D = \frac{(d-4)d_{l,0}^2}{10(l+2)(d+l-3)}
\]

Introducing variables \( x_k = k(d+k-4) \) and the function

\[
F(x_k) = -\kappa x_k^2 + [C - D + (l+2)(d+l-2)\kappa]x_k - (l+2)(d+l-2)D
\]

we find:

\[
\begin{align*}
    c_{k-1,l}^2 &\sim F(x_k), \quad l + 2 \leq k < \infty \\
    d_{l,k}^2 &\sim F(x_k), \quad 0 \leq k \leq l \\
    a_{l,l}^2 &\sim F(x_k), \quad k = -\frac{(d-4)}{2}
\end{align*}
\]

For flat Minkowski space there exists an unitary region of parameters:

\[
\frac{(d+2l)^2}{(d-4)^2} D \geq C \geq D
\]

where all our fields remain to be connected.

In Anti de Sitter space there exists a range of the parameters (see appendix A.7) where unitary theory contains all our fields. Besides, there exist two sets of discrete solutions. The first one corresponds to the situations when some \( c_{s,l} = 0 \) (and hence all \( c_{s,m} = 0 \) for \( 0 \leq m \leq l \)) so that the whole set of fields decomposes into finite and infinite parts, the infinite part being unitary. The second type of solutions corresponds to the cases when some \( d_{l,s} = 0 \) (and hence all \( d_{k,s} = 0 \) for \( k \geq l \)) so that the whole set of fields splits into two infinite parts, one of them being unitary. Moreover, there are special points where simultaneously \( c_{s,l} = 0 \) and \( d_{l,l} = 0 \) when our system decomposes into four disconnected parts.
A On the unitary solutions in $AdS$ space

A.1 Method of resolving the unitarity conditions

In every case, the unitarity condition of the Lagrangian is given by a system of linear inequalities, where each of them has the form $(A_i)^2(k_i) \geq 0$, where $A_i$ stands for the letter of the coefficient and $k_i$ stands for its ordinal number. The unitarity condition of the partially massless limit is given by a system of linear inequalities and linear equations: $(A_i)^2(k_i) > 0, (B_j)^2(k_j) = 0$. Then, it turns out that the sign of any coefficient is determined by a sign of a square trinomial

$$P(x) = -\kappa x^2 + (B\kappa + C)x + D$$  \hspace{1cm} (80)

at some $x$, wherein $B$ depends only on the dimension of the space-time, while $C$ and $D$ are two independent linear combinations of squares of two coefficients of Lagrangian. Let $A_1$ and $A_2$ be the coefficients. Hence, the unitarity condition is a system of linear inequalities of two variables. The solution of such system is a convex set, on whose border one or two inequalities become equalities. Let us sort the conditions by $x$: $P(x_i) \sim 0$, where ”$\sim$” is either ”$\geq$” or ”$=$”. Then, we will frequently use two following assertions:

1. If a square trinomial $P(x)$ with positive leading coefficient is such that conditions $P(x_0) = 0, P(x_1) \geq 0$ and $x_0 < x_1$ hold, then for any $x > x_1$ the inequality $P(x) > 0$ holds. The same is true for $x < x_1 < x_0$

2. If a square trinomial $P(x)$ with positive leading coefficient is such that condition $P(x_1) = P(x_2) = 0$ holds and $x_1 < x_2$, then $P(x) < 0$ for any $x \in (x_1, x_2)$ and $P(x) > 0$ for any $x < x_1 \lor x > x_2$.

Then in case of $\kappa < 0$ the second assertion implies that all the points determined by expressions $P(x_i) = 0, P(x_{i+1}) = 0$, and only they are the vertices of the region of unitarity. Hence, the boundary of the region of unitarity is a union of segments of lines $P(x_i) = 0$ whose ends are points $P(x_i) = 0, P(x_{i-1}) = 0$ and $P(x_i) = 0, P(x_{i+1}) = 0$. The first and the last conditions, if they exist, give rays instead of segments. The region of unitarity can be defined as a union of regions given by inequalities $A_1 \in [(A_1)_k, (A_1)_{k+1}], A_2 \in [f_k(A_1), g_k(A_1)]$, where $(A_1)_k$ are coordinates of vertices of the region of unitarity, in ascending order, and $f_k, g_k$ are either $\pm \infty$ or linear functions obtained by resolving equations $P(x_i) = 0$ with the respect to $A_2$. The convexity of the region of unitarity implies that $g_k(A_1) \geq f_k(A_1)$ for $A_1 \in [(A_1)_k, (A_1)_{k+1}]$.

As to partially massless limits, each of them is given by one or two equations and a system of inequalities. In case of one equation $P(x_i) = 0$, the assertion 1 implies that all of the inequalities, except $P(x_{i-1}) = 0$ and $P(x_{i+1}) = 0$ if they exist, can be removed without altering the solution. Hence, the unitary partially massless limit is either a segment or a ray, whose ends are points where $P(x_{i-1}) = 0$ and $P(x_{i+1}) = 0$ intersect $P(x_i) = 0$. Similarly to the case of complete Lagrangian, the area can be expressed either in form $A_2 = aA_1 + b, A_1 \in [(A_1)_1, (A_1)_2]$ or in form $A_1 = (A_1)_0, A_2 \in [(A_2)_1, (A_2)_2]$. In case of two equalities the assertion 2 implies that the solution is non-empty if and only if one equality is next to the other in the list. In that case, the solution is a unique point.
An example of different areas is given in Figure 1. Here, grey area is the region of unitarity, given by inequalities $P(x_i) = 0, i \in \{1, 4\}$, red segment is the partially massless limit given by conditions $P(x_1) \geq 0, P(x_2) = 0, P(x_3) \geq 0, P(x_4) \geq 0$, green ray the partially massless limit given by conditions $P(x_3) = 0, P(x_4) \geq 0$, and the blue dot is a double partially massless limit given by conditions $P(x_1) = 0, P(x_3) = 0, P(x_4) \geq 0$.

Figure 6: Example of different unitarity regions in $AdS$ space

In the flat case, all the inequalities become homogeneous. Hence, the unitarity region is an angle with the vertex at the origin, i.e. point $A_1 = A_2 = 0$. The list of inequalities contains an infinite number of conditions for increasing and unbounded sequence of $x_i$. The region of unitarity thus is defined by inequalities $Cx_1 + D \geq 0, C \geq 0$. In contrast to the $AdS$ case, the trivial case $P(x_i) = 0$ is possible in flat space. It corresponds to values $A_1 = A_2 = 0$ and defines totally massless limit. Any non-trivial partially massless limit is given by one equation $P(x_i) = 0$ and an infinite list of inequalities. Such limit is possible if and only if the equation is the first element of the list of conditions. The solution in that case is $P(x_i) = 0, C > 0$.

### A.2 Completely symmetric tensor $Y(k + 1, 0)$

In this case unitarity requires that all $e_k^2 \geq 0, k \in \mathbb{N}$. Applying the general method to the function $\mathcal{F}(\mathcal{F})$ we find that the region of unitarity (such that all our fields enter) is determined by the following set of conditions:

\[
\begin{align*}
    c_0^2 &\geq \frac{2s(s + 1)(s + d - 2)(s + d - 3)|\kappa|}{(d - 2)} - \frac{2(s + 1)(s + 2)(s + d - 2)(s + d - 11)|\kappa|}{(d - 2)}, \\
    c_0 &\geq \frac{s(d + s - 1)[e_0^2 + 2(s + 1)(d + s - 2)|\kappa|]}{(s + 1)(d + s - 2)}, \quad s \in \mathbb{R}^+, +\infty
\end{align*}
\]

(81)
Besides, there exists a discrete set of unitary solutions that appear each time when some \( e_s = 0, s > 0 \), while all \( e_k^2 > 0, k > s \):

\[
\begin{align*}
c_0 &= \frac{(s - 1)(d + s - 2)[e_0^2 + 2s(s + d - 3)k]}{s(s + d - 3)}, \\
e_0^2 &\geq \frac{2s(s + 1)(s + d - 3)(s + d - 2)k}{(d - 2)}
\end{align*}
\]

as well as a special case

\[
c_0 \geq 0, \quad e_0^2 = 0.
\]

### A.3 Mixed symmetry tensor \( Y(k + 1, 1) \)

In this case unitarity requires that all \( e_k^2 \geq 0, d_k^2 \geq 0, k \in [0, +\infty) \). The general analysis applied to the function (33) shows that the unitarity region is determined by the following set of conditions:

\[
C \in \left[ \frac{k(k - 1)(k + d - 2)(k + d - 1)k}{2(d - 3)}, \frac{k(k + 1)(k + d - 1)(k + d)k}{2(d - 3)} \right],
\]

\[
D \in \left[ 0, \frac{(k + 2)(k + d - 3)(C + k(k + d - 1)k)}{k(k + d - 1)} \right], \quad k \in [1, +\infty)
\]

At the boundary of this region corresponding to \( D = 0 \) and \( C > 0 \) all \( d_k^2 = 0 \) so that the whole system decomposes into two subsystems with the fields \( \Psi_{\mu \nu} \) and \( \Phi_\mu \) respectively. There exists also a set of discrete solutions when some \( e_s = 0 \) (and hence \( e_s = 0 \)), while all \( c_k^2 > 0, k > s \):

\[
C = \frac{(k - 1)(k + d - 2)(D - (k + 1)(k + d - 4)k)}{(k + 1)(k + d - 4)},
\]

\[
D \in \left( 0, \frac{(k + 1)(k + 2)(k + d - 4)(k + d - 3)k}{2(d - 3)} \right)
\]

In this case the unitary part of the theory contains fields \( \Psi_{\mu \nu}^{a(k)} \) and \( \Phi_\mu^{a(k)} \) with \( k > s \). Note that in this case it is also possible to take a limit \( D \to 0 \) so that the whole system decomposes into four independent ones where the subsystem with the fields \( \Psi_{\mu \nu}^{a(k)}, s \leq k < \infty \) provides one more non-trivial example of the unitary infinite spin theory.

### A.4 General bosonic case \( Y(k + 1, l + 1) \)

In this case unitarity requires that all \( c_{k,l}^2 \geq 0, k > l \) and \( d_{l,k}^2 \geq 0, 0 \leq k \leq l \). All these parameters are given by the values of the one and the same function (39), so that our general method applicable here as well. We find that unitary region of parameters is determined by the following set of conditions:

\[
C \in \left[ \frac{k(k + 1)(d + 2l + k - 1)(d + 2l + k)k}{d + 2l - 3}, \frac{(k + 1)(k + 2)(d + 2l + k)(d + 2l + k + 1)k}{d + 2l - 3} \right],
\]
\[ D \leq \frac{(k + 3)(d + 2l + k - 2)(2(k + 1)(d + 2l + k)|\kappa| + C)}{(k + 1)(d + 2l + k)}, \quad k \in 0, +\infty, \]

\[ C \in \left[ \frac{3}{5}(d + 2l - 4)|\kappa| \right], \quad D \geq 0 \]

\[ C \in \left[ \frac{(k + 2)(k + 3)(d + 2l - 4)(d + 2l - k - 3)|\kappa|}{(k + 3)(k + 4)(d + 2l - k - 5)(d + 2l - k - 4)|\kappa|} \right], \quad d + 2l - 3 \]

\[ D \geq \frac{(k + 1)(d + 2l - k - 6)(C - 2(k + 3)(d + 2l - k - 4)|\kappa|)}{(k + 3)(d + 2l - k - 4)} \], \quad k \in 0, l - 1, \]

\[ C \in \left[ \frac{(l + 1)(l + 2)(d + l - 3)(d + l - 2)|\kappa|}{d + 2l - 3} \right], +\infty \),

\[ D \geq \frac{l(l + d - 5)(C - 2(l + 2)(d + l - 3)|\kappa|)}{(l + 2)(l + d - 3)} \]

Besides, there are two types of discrete solutions. The first one corresponds to the cases where some \( c_{c,l} = 0 \) (and hence all \( c_{s,m} = 0, 0 \leq m \leq l \)):

\[ C \in \left[ -2(s - l - 2)(s + l + d - 3)|\kappa|, \frac{(s - l - 2)((s - l - 1)(s + l + d - 3)(s + l + d - 2))|\kappa|}{(d + 2l - 3)} \right], \quad (86) \]

\[ D = \frac{(s - l)(s + l + d - 5)(C - (s - l - 2)(s + l + d - 3)|\kappa|)}{(s - l - 2)(s + l + d - 3)} \]

so that Figure 3 splits horizontally. The second one corresponds to the cases where some \( d_{l,m} = 0 \) (and hence all \( d_{k,m} = 0, l \leq k < \infty \)):

\[ C \geq \frac{(l + 2 - s)((l + 1 - s)(s + l + d - 3)(s + l + d - 2))|\kappa|}{(d + 2l - 3)}, \quad (87) \]

\[ D = \frac{(l - s)(s + l + d - 5)(C - (l + 2 - s)(s + l + d - 3)|\kappa|)}{(l + 2 - s)(s + l + d - 3)}, \quad s < l \]

\[ C \geq 0, \quad D = 0, \quad s = l \]

when Figure 3 splits vertically. Moreover, these two sets have common points when simultaneously some \( c_{s,l} = 0 \) and \( d_{l,l} = 0 \):

\[ D = 0, \quad C = -2(s - l - 1)(s + l + d - 2)|\kappa| \]

(88)

so that Figure 3 splits into four disconnected parts.

**A.5 Completely symmetric spin-tensor** \( Y(k + 3/2, 0) \)

In this case unitarity requires that all \( \epsilon_k^2 \geq 0 \) for \( k \geq 0 \). Using our general method for the function (64) we find the the unitarity region is determined by the following set of parameters:

\[ \epsilon_0^2 \in \left[ \frac{6s(s + 1)(s + d - 1)(s + d - 2)|\kappa|}{(d - 2)}, \frac{6(s + 1)(s + 2)(s + d)(s + d - 1)|\kappa|}{(d - 2)} \right]. \]
Besides, there exists a set of discrete solutions where some $e_s = 0$ while all $e_k^2 > 0$ for $k > s$:

$$ c_0 = \left( \frac{d + 2s - 2)^2[2e_0^2 + 3(d - 2)s(d + s - 2)\kappa]}{12(d - 2)s(d + s - 2)} \right), \quad s \in 0, +\infty, \quad (89) $$

with unitary part containing fields $\Phi_\mu^{a(k)}$ with $k > s$ only.

A.6 Mixed symmetry spin-tensor $Y(k + 3/2, 3/2)$

In this case unitarity requires that all $c_0^2 \geq 0$, $d_k^2 \geq 0$ and $a_k^2 \geq 0$ for $k \in 0, +\infty$. All these parameters are related with the values of one and the same function $\kappa$ so that our general method is valid. We obtain for the unitary region the following set of conditions:

$$ C \in \left[ \frac{k(k + 1)(k + d)(k + d + 1)\kappa}{2(d - 2)}, \frac{(k + 1)(k + 2)(k + d + 1)(k + d + 2)\kappa}{2(d - 2)} \right], $$

$$ D \leq \frac{(k + 3)(k + d + 1)(C + (k + 1)(k + d + 1)\kappa)}{(k + 1)(k + d + 1)}, \quad k \in 0, +\infty, $$

$$ C \in \left[ 0, \frac{\kappa d^2}{4} \right], \quad D \geq 0 $$

$$ C \geq \frac{\kappa d^2}{4}, \quad D \geq \frac{(d - 4)^2(4C - \kappa d^2)}{4d^2} $$

At the boundary of this region there is a limit when all $d_k = 0$, namely:

$$ D = 0, \quad \frac{d^2\kappa}{4} \geq C > 0 \quad (91) $$

so that Figure 5 splits vertically into two parts containing fields $\Psi_{\mu\nu}^{a(k)}$ and $\Phi_\mu^{a(k)}$, respectively. Besides, there is a discrete set of solutions where some $c_s = 0$ (and hence $e_s = 0$) while all $c_k^2 > 0$ for $k > s$:

$$ D = \frac{(s + 2)(s + d - 2)(C + s(s + d)\kappa)}{s(s + d)}, $$

$$ C \in \left[ -s(s + d)|\kappa|, \frac{s(s + 1)(s + d)(s + d + 1)\kappa}{2(d - 2)} \right], \quad s > 0 \quad (92) $$

$$ C = 0, \quad D \in [0, -3(d - 1)|\kappa|], \quad s = 0 $$

so that Figure 5 splits horizontally with the unitary part containing fields $\Psi_{\mu\nu}^{a(k)}$ and $\Phi_\mu^{a(k)}$ with $k > s$ only. Moreover, these solutions admit a limit

$$ D = 0, \quad C = -s(s + d)|\kappa| \quad (93) $$

where all $d_k = 0$ and Figure 5 splits into four disconnected parts.
A.7 General fermionic case $Y(k + 3/2, l + 3/2)$

In this case unitarity requires that $c_{k,l}^2 \geq 0$, $d_{l,m}^2 \geq 0$, $a_{l,t}^2 \geq 0$, $k \in l + 1, +\infty$, $n \in 0, l$. All of these parameters are related with the different values of the one and the same function \[79\]. Using our general method we find that the unitarity region is determined by the following set of conditions:

\[
C \in \left[ \frac{s(s + 1)(2l + s + d - 1)(2l + s + d)\kappa}{(l + 2)(l + d - 2)} \right] \cdot \left[ \frac{(s + 2)(2l + s + d + 1)(2l + s + d)\kappa}{(l + 2)(l + d - 2)} \right], \\
D \leq \frac{(s + l + 2)(d + s + l - 2)(\kappa|s(d + 2l + s) + C)}{s(d + 2l + s)}, \quad s \in 0, +\infty,
\]

\[
C \in \left[ \frac{s(s + 1)(2l + s + d - 1)(2l + s + d)\kappa}{(l + 2)(l + d - 2)} \right] \cdot \left[ \frac{(s + 2)(2l + s + d + 1)(2l + s + d)\kappa}{(l + 2)(l + d - 2)} \right], \\
D \geq \frac{(l - 1)(l + d - 5)(3(2l + d - 3)\kappa - C)}{3(2l + d - 3)}, \quad s \in 0, l - 2,
\]

\[
C \in \left[ \frac{(s + 2)(s + 3)(d + 2l - s - 2)(d + 2l - s - 3)\kappa}{(l + 2)(l + d - 2)} \right] \cdot \left[ \frac{(s + 3)(s + 4)(d + 2l - s - 3)(d + 2l - s - 4)\kappa}{(l + 2)(l + d - 2)} \right], \\
D \geq \frac{(l - s - 1)(d + l - s - 5)(\kappa|s(s + 3)(d + 2l - s - 2) - C)}{(s + 3)(d + 2l - s - 2)}, \quad s \in 0, l - 2,
\]

\[
C \in \left[ l + 1)(l + d - 1)\kappa, \frac{(d + 2l)^2|\kappa|}{4} \right], \quad D \geq 0,
\]

\[
C \in \left[ \frac{(d + 2l)^2|\kappa|}{4}, +\infty \right], \quad D \geq \frac{(d - 4)^2[4C + |\kappa|(d + 2l)^2]}{4(d + 2l)^2}
\]

Besides, there exist two sets of discrete solutions. The first one corresponds to the situation when some $c_{s-1,l} = 0$ (and hence all $c_{s-1,m} = 0$ for $0 \leq m \leq l$):

\[
C \in \left[ \frac{s(s + 1)(2l + s + d - 1)(2l + s + d)\kappa}{(l + 2)(d + l - 2)} \right] \cdot \left[ \frac{(s + 2)(2l + s + d + 1)(2l + s + d)\kappa}{(l + 2)(d + l - 2)} \right], \\
D = \frac{(s - l)(s + d - 4)(C - (s - l - 2)(s + l + d - 2)\kappa)}{(s - l - 2)(s + l + d - 2)}, \quad s > l + 2 \quad \text{(94)}
\]

\[
C = 0, \quad D \in \left[ |\kappa|l(l + d - 4), |\kappa|l(l + 3)(l + d - 1) \right], \quad s = l + 2
\]

where the whole set of fields decomposes into the finite and infinite parts, the infinite part being unitary. The second type of solutions corresponds to the cases when some $d_{l,s} = 0$.
(and hence all $d_{k,s} = 0$ for $k \geq l$):

$$D = \frac{s(s + d - 4)((l + 2 - s)(s + l + d - 2)|\kappa| - C)}{(l + 2 - s)(s + l + d - 2)}$$

$$C \in \left[ \frac{(l + 2 - s)(s + l + d - 2)(d + 2l)^2|\kappa|}{4(l + 2)(l + d - 2)}, \frac{|\kappa|(l + 2 - s)(l + 1 - s)(s + l + d - 2)(s + l + d - 1)}{(l + 2)(l + d - 2)} \right], \quad s < l \quad (95)$$

$$D = \frac{l(l + d - 4)(2(2l + d - 2)|\kappa| - C)}{2(2l + d - 2)},$$

$$C \in \left[ 0, \frac{|\kappa|(d + 2l - 2)(d + 2l)^2}{2(2l + d - 2)(l + d - 2)} \right], \quad s = l$$

so that the whole set of fields splits into two infinite parts, one of them being unitary. Moreover, there are special points where simultaneously $c_{s,l} = 0$ and $d_{l,l} = 0$:

$$D = \frac{|\kappa|ls(l + d - 4)(s + d - 4)}{(l + 2)(l + d - 2)},$$

$$C = -\frac{2|\kappa|(2l + d - 2)(s - l - 2)(s + l + d - 2)}{(l + 2)(l + d - 2)} \quad (96)$$

when our system decomposes into four disconnected parts.

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