Faithful realizations of semiclassical truncations

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Abstract

Realizations of algebras in terms of canonical or bosonic variables can often be
used to simplify calculations and to exhibit underlying properties. There is a long
history of using such methods in order to study symmetry groups related to collective
motion, for instance in nuclear shell models. Here, related questions are addressed
for algebras obtained by turning the quantum commutator into a Poisson bracket
on moments of a quantum state, truncated to a given order. In this application,
canonical realizations allow one to express the quantum back-reaction of moments
on basic expectation values by means of effective potentials. In order to match de-
grees of freedom, faithfulness of the realization is important, which requires that, at
least locally, the space of moments as a Poisson manifold is realized by a complete
set of Casimir–Darboux coordinates in local charts. A systematic method to derive
such variables is presented and applied to certain sets of moments which are impor-
tant for physical applications. If only second-order moments are considered, their
Poisson-bracket relations are isomorphic to the Lie bracket of sp(2N, R), providing
an interesting link with realizations of nuclear shell models.

1 Introduction

Semiclassical truncations approximate quantum dynamics by dynamical systems in which
expectation values are coupled to moments of a state. The classical phase space is thereby
extended to an enlarged manifold with a Poisson bracket of expectation values and mo-
ments derived from the commutator of basic operators. These canonical effective methods
have been used in various contexts, such as quantum chemistry [1] and quantum cosmology [2],
and they reproduce well-known results including tunneling phenomena [3], the
low-energy effective action [4, 5], or the Coleman–Weinberg potential [6]. However, the
enlargement of the classical phase space tends to complicate qualitative interpretations as
well as computations, in particular because moments, unlike expectation values, do not
form canonically conjugate pairs. In this paper, we therefore analyze the problem of con-
structing canonical realizations of Poisson systems, or their Casimir–Darboux coordinates.
To second moment order for a single pair of classical degrees of freedom, an interesting

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canonical realization has been known for some time \cite{7, 11}. Our main goal is to extend these results to multiple degrees of freedom and to higher orders in a semiclassical expansion.

At leading order, semiclassical truncations turn out to be closely related to the Lie algebras \( \text{sp}(2N, \mathbb{R}) \). Our methods and examples can therefore be extended directly to finding canonical realizations for these algebras. Moreover, once a canonical realization is found, one automatically obtains a bosonic realization using the standard Poisson structure on the complex numbers. (Canonical pairs are thereby replaced by classical analogs of annihilation and creation operators.)

We put special emphasis on the construction of faithful realizations, in which the number of independent variables is equal to the dimension of the original system, and the co-rank of the Poisson tensor agrees with the number of Casimir functions. Canonical and bosonic realizations of systems of the type studied here have been used for several decades, but achieving faithfulness often presented a problem. Bosonic realizations go back to theoretical work on magnetic systems \([8]\). Interest in particular in bosonic realizations of \( \text{sp}(6, \mathbb{R}) \) grew after the introduction of a symplectic model of nuclear shells and vibrations \([9]\). Non-faithful bosonic realizations have been used in several papers mainly to compute matrix elements in irreducible representations \([10, 11, 12, 13, 14]\). Some of these studies noted difficulties in finding faithful realizations, starting with \( \text{sp}(4, \mathbb{R}) \) \([13, 14]\). Bosonic and canonical realizations of Lie algebras other than \( \text{sp}(2N, \mathbb{R}) \) have been analyzed and formalized in \([15, 16, 17, 18, 19]\), which in most cases were not faithful.

Our results lead to an extension of some of the results of \([13]\) to a faithful bosonic realization, but we expect the main applications of our methods to be in semiclassical discussions of quantum mechanics. Even though we address quantum systems, the use of semiclassical truncations means that we are interested here in classical realizations of a system with Poisson brackets. We do not consider the more complicated question of constructing bosonic realizations of operator algebras — the main topic of \([13]\) — in which factor ordering questions are relevant.

## 2 Canonical Effective Methods

Canonical effective equations \([4, 5]\) describe quantum effects through interactions between expectation values and moments of a state with respect to a fixed set of basic observables. The commutator of operators induces a Poisson bracket on the space of expectation values and moments, leading to an infinite-dimensional extension of the classical phase space. In semiclassical approximations of varying degrees, finite-dimensional truncations are used for each canonical pair. The Hamiltonian operator then implies an effective Hamiltonian on the extended phase space for each of its finite-dimensions, and quantum dynamics can be analyzed much like a classical dynamical system. Mathematically, canonical effective methods replace partial differential equations for wave functions by a system of coupled ordinary differential equations for an enlarged set of variables.

We assume that the unital \(*\)-algebra \( \mathcal{A} \) of observables defining the quantum system is canonical, that is, generated by the unit operator together with a finite set of self-adjoint
position and momentum operators \( Q_j \) and \( \Pi_k \), \( 1 \leq j, k \leq N \), with canonical commutation relations
\[
[Q_j, \Pi_k] = i\hbar \delta_{jk}. \tag{1}
\]

States are positive linear functionals \( \omega \) from the algebra to the complex numbers, such that \( \omega(a^*a) \geq 0 \) for all \( a \in A \). They may (but need not) be obtained from wave functions or density matrices in or acting on a Hilbert space \( \mathcal{H} \) on which \( A \) may be represented by \( a \mapsto \hat{a} \). In such a case, every \( \psi \in \mathcal{H} \) defines a state \( \omega_\psi : a \mapsto \langle \hat{a} \rangle_\psi \), and every density matrix \( \hat{\rho} \) defines a state \( \omega_\rho : a \mapsto \text{tr}(\hat{a} \hat{\rho}) \). To be specific, and for easier comparison with the physics literature on the subject, we will use the notation \( \langle \hat{a} \rangle \) to denote \( \omega_\psi(a) \), but expectation values could as well be defined using mixed states or algebraic states.

We introduce a set of basic variables taking real values:

**Definition 1**

Given a state on a canonical algebra \( A \) generated by self-adjoint \( Q_j \) and \( \Pi_k \), in addition to the unit, the basic expectation values are \( q_j = \langle \hat{Q}_j \rangle \in \mathbb{R} \) and \( \pi_k = \langle \hat{\Pi}_k \rangle \in \mathbb{R} \). For positive integers \( k_i \) and \( l_i \) such that \( \sum_{i=1}^{N}(k_i + l_i) \geq 2 \), the moments of the state are given by
\[
\Delta \left( q_1^{k_1} \cdots q_N^{k_N} \pi_1^{l_1} \cdots \pi_N^{l_N} \right) = \langle (\hat{Q}_1 - q_1)^{k_1} \cdots (\hat{Q}_N - q_N)^{k_N} (\hat{\Pi}_1 - \pi_1)^{l_1} \cdots (\hat{\Pi}_N - \pi_N)^{l_N} \rangle_{\text{Weyl}}, \tag{2}
\]
where the product of operators is Weyl (totally symmetrically) ordered.

If the state is a Gaussian wave function in the standard Hilbert space on which \( A \) can be represented, the moments obey the hierarchy
\[
\Delta \left( q_1^{k_1} \cdots q_N^{k_N} \pi_1^{l_1} \cdots \pi_N^{l_N} \right) = O \left( \hbar^{\frac{1}{2}} \sum_{n}(l_n + k_n) \right). \tag{3}
\]
This property motivates

**Definition 2**

A state on a canonical algebra \( A \) is semiclassical if its moments obey the hierarchy \( (3) \).

A semiclassical state is much more general than the Gaussian family, which has two free parameters per canonical pair of degrees of freedom. A general semiclassical state, by contrast, allows for infinitely many free parameters per canonical pair of degrees of freedom.

We will use the semiclassical hierarchy mainly in order to truncate the infinite-dimensional space of expectation values and moments:

**Definition 3**

The semiclassical truncation of order \( s \geq 2 \) of a quantum system with canonical algebra \( A \) is a finite-dimensional manifold \( P_s \) with boundary, determined by global coordinates \( q_j, \pi_k \) and all moments \( (2) \) such that \( \sum_{n}(l_n + k_n) \leq s \). Its boundary components are obtained from the Cauchy–Schwarz inequality.
A semiclassical truncation of order $s$ therefore includes variables up to order $\frac{1}{2}s$ in $\hbar$ when evaluated on a Gaussian state. Well-known components of the boundary are given by Heisenberg’s uncertainty principle

$$\Delta(q_j^2)\Delta(\pi_k^2) - \Delta(q_j\pi_k)^2 \geq \frac{\hbar^2}{4}\delta_{jk},$$  \hspace{1cm} (4)

but there are higher-order versions relevant for $s > 2$.

Basic expectation values and moments are equipped with a Poisson bracket defined by

$$\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} = \frac{1}{i\hbar}\langle [\hat{A}, \hat{B}] \rangle,$$  \hspace{1cm} (5)

extended to all moments by using linearity and the Leibniz rule. The Poisson bracket turns any semiclassical truncation into a phase space by ignoring in $\{\Delta_1, \Delta_2\}$ all terms of order higher than $s$ in moments. This condition includes the convention that the product of a moment of order $s_1$ and a moment of order $s_2$ is of semiclassical order $s_1 + s_2$. Moreover, the product of a moment of order $s_1$ with $\hbar s_2$ is of order $s_1 + 2s_2$. The consistency of this notion of order and the resulting truncation has been shown in [21].

In general, the Poisson tensor on a semiclassical truncation is not invertible, such that there is no natural symplectic structure on a semiclassical phase space. For instance, for $N = 1$ the phase space of a semiclassical truncation of order $s = 1$ is five-dimensional with coordinates $(q, \pi, \Delta(q^2), \Delta(q\pi), \Delta(\pi^2))$, and cannot be symplectic. The non-zero basic brackets are

$$\{q, \pi\} = 1$$  \hspace{1cm} (6)

and

$$\{\Delta(q^2), \Delta(q\pi)\} = 2\Delta(q^2), \quad \{\Delta(q\pi), \Delta(\pi^2)\} = 2\Delta(\pi^2), \quad \{\Delta(q^2), \Delta(\pi^2)\} = 4\Delta(q\pi).$$  \hspace{1cm} (7)

Quantum dynamics is determined by a Hamiltonian element $H \in A$. We assume that the Hamiltonian element is given by a sum of Weyl-ordered products of the canonical generators. It defines the quantum Hamiltonian $H_Q(\langle \cdot \rangle, \Delta) = \langle \hat{H} \rangle(\langle \cdot \rangle, \Delta)$, identified as a function of basic expectation values and moments through the state used in $\langle \hat{H} \rangle$. On a semiclassical truncation of order $s$, the quantum Hamiltonian leads to the effective Hamiltonian of order $s$,

$$H_{\text{eff},s} = \langle H(\hat{Q}_j + (\hat{Q}_j - q_j), \hat{P}_k + (\hat{P}_k - \pi_k)) \rangle$$  \hspace{1cm} (8) = H(q, \pi) + \sum_{s, \sum_n (j_n + k_n) = 2}^{s} \frac{\partial^n H(q, \pi)}{\partial q_1^{j_1} \ldots \partial q_N^{j_N} \partial \pi_1^{k_1} \ldots \partial \pi_N^{k_N}} \frac{\Delta(q_1^{j_1} \ldots q_N^{j_N} \pi_1^{k_1} \ldots \pi_N^{k_N})}{j_1! \ldots j_N! k_1! \ldots k_N!},$$

obtained by a formal Taylor expansion in $\hat{Q}_j - q_j$ and $\hat{P}_k - \pi_k$, where $H(q, \pi)$ is the classical Hamiltonian corresponding to $H \in A$. If the Hamiltonian is a polynomial in basic operators, the expansion in [8] is a finite sum and exact, and merely rearranges the
monomial contributions to $\hat{H}$ in terms of central moments. By definition of the Poisson bracket from the commutator, Hamiltonian equations of motion

$$\dot{f}(\langle \cdot \rangle, \Delta) = \{f(\langle \cdot \rangle, \Delta), H_{\text{eff},s}\}$$

(9)
generated by an effective Hamiltonian are truncations of Heisenberg’s equations of motion evaluated in a state.

3 Faithful realizations of semiclassical truncations

While the Poisson brackets $\{q_j, \pi_k\} = 1$, $\{q_j, \Delta\} = 0 = \{\pi_k, \Delta\}$ involving basic expectation values are simple, the brackets between two moments are non-canonical and, in general, non-linear \[4, 22\]:

$$\{\Delta(q^a \pi^c), \Delta(q^d \pi^e)\} = a d \Delta(q^b \pi^{a-1})\Delta(q^{d-1} \pi^c) - b c \Delta(q^{b-1} \pi^a)\Delta(q^d \pi^{e-1})$$

$$+ \sum_{\text{odd } n=1}^{M} \left(\frac{i\hbar}{2}\right)^{n-1} K_{abcd}^n \Delta(q^{b+d-n} \pi^{a+c-n})(10)$$

with $M = \min(a + c, b + d, a + b, c + d)$ and

$$K_{abcd}^n = \sum_{m=0}^{n} (-1)^m m!(n - m)! \left(\begin{array}{cccc} a & b & c & d \\ m & n - m & n - m & m \end{array}\right).$$

(11)

Since only odd $n$ are included in the sum in (10), all coefficients are real. Whenever a term $\Delta(q)$ or $\Delta(\pi)$ appears on the right, it is understood to be zero, which is consistent with an extension of (2) to $\sum (k_i + l_i) = 1$ because $\langle \hat{a} - a \rangle = 0$ for any operator $\hat{a}$. The brackets (10) are therefore linear in moments if and only if $a + b = 2$ or $c + d = 2$.

We will look for mappings of the moments to new variables such that the Poisson brackets can be simplified. In particular, we will derive canonical realizations of semiclassical truncations.

**Definition 4** A canonical realization of an algebra $(C^\infty(M), \{\cdot, \cdot\})$ on an open submanifold $U \subset M$ is a homomorphism $(C^\infty(U), \{\cdot, \cdot\}) \to (C^\infty(\mathbb{R}^{2p} \times \mathbb{R}^I), \{\cdot, \cdot\}_{\text{can}})$ to the algebra of functions on the Poisson manifold $\mathbb{R}^{2p + I}$ equipped with the canonical Poisson bracket on $\mathbb{R}^{2p}$, while $\{f, C\}_{\text{can}} = 0$ for all $f \in C^\infty(\mathbb{R}^{2p} \times \mathbb{R}^I)$ and $C \in \mathbb{R}^I$.

A canonical realization of $(C^\infty(M), \{\cdot, \cdot\})$ is faithful if $\dim M = 2p + I$ and $2p$ is equal to the rank of the Poisson tensor on $M$.

Our examples of $M$ will be given by open submanifolds of the phase space of a given semiclassical truncation. A closely related concept is that of a bosonic realization:

**Definition 5** A bosonic realization of an algebra $(C^\infty(M), \{\cdot, \cdot\})$ on an open submanifold $U \subset M$ is a homomorphism $(C^\infty(U), \{\cdot, \cdot\}) \to (C^\infty(C^p \times \mathbb{R}^I), \{\cdot, \cdot\}_{\text{bos}})$ to the algebra of
functions on the Poisson manifold $\mathbb{C}^p \times \mathbb{R}^I$, where $\mathbb{C}$ is equipped with the Poisson bracket \[ \{z^*, z\}_{\text{bos}} = i, \text{ while } \{f, C\}_{\text{bos}} = 0 \text{ for all } f \in C^\infty(\mathbb{C}^p \times \mathbb{R}^I) \text{ and } C \in \mathbb{R}^I. \]

A bosonic realization of $(C^\infty(M), \{\cdot, \cdot\})$ is faithful if $\dim M = 2p + I$ and $2p$ is equal to the rank of the Poisson tensor on $M$.

Pullbacks by the local symplectomorphisms 
\[ \Phi: \mathbb{R}^{2p} \to \mathbb{C}^p, (q_j, p_k) \mapsto \left( \frac{1}{\sqrt{2}} (q_l + ip_l) \right) \]  
(12)
define a bijection between canonical realizations and bosonic realizations which preserves faithfulness.

We note that the definitions impose reality conditions on the canonical or bosonic variables. In particular, all $q_j$ and $p_k$ must be real, and a bosonic pair $(z, z')$ with $\{z', z\} = i$ must be such that $z' = z^*$.

### 3.1 Poisson structure of semiclassical truncations

Since basic expectation values have canonical Poisson brackets with one another and zero Poisson brackets with any moment, the non-trivial task is to construct a canonical realization of the space of moments for a given semiclassical truncation, at fixed basic expectation values.

A canonical realization of a semiclassical truncation of order $s$ induces a map 
\[ \mathcal{X}^{(s)}: \mathcal{U} \subset \mathcal{P}_s \to \mathbb{R}^{2p} \times \mathbb{R}^I, (\Delta) \mapsto (s_\alpha, p_\beta, U_\gamma) \]  
(13)
such that the variables $(s_\alpha, p_\beta)$, $\{s_\alpha, p_\beta\} = \delta_{\alpha\beta}$, can be used as coordinates on symplectic leaves defined by constant $U_\gamma$. The coordinates $U_\gamma$ are therefore local expressions of Casimir functions of the Poisson manifold \[23\].

A faithful realization requires a bijective map between the moments and canonical variables. For a single degree of freedom and a semiclassical truncation of order $s$, the dimension $D$ of the phase space is the number of moments up to order $s$, or
\[ D = \sum_{j=2}^{s} (j + 1) = \frac{1}{2} (s^2 + 3s - 4) . \]  
(14)

Note again that this dimension $D$ may be even or odd, depending on $s$. Even if $D$ is even, the Poisson tensor is not guaranteed to be invertible.

Every function on a Poisson manifold we are considering can be expressed as a function of finitely many moments $\Delta_i$ in some ordering. We introduce the Poisson tensor 
\[ P^{(s)}_{ij}(\Delta) = \{\Delta_i, \Delta_j\} , \]  
(15)
such that the Poisson brackets of the set of coordinates $\mathcal{X}^{(s)}(\Delta)$ are 
\[ \{\mathcal{X}^{(s)}_\alpha(\Delta), \mathcal{X}^{(s)}_\beta(\Delta)\} = \sum_{i,j=1}^{D} \frac{\partial \mathcal{X}^{(s)}_\alpha(\Delta)}{\partial \Delta_i} P^{(s)}_{ij}(\Delta) \frac{\partial \mathcal{X}^{(s)}_\beta(\Delta)}{\partial \Delta_j} . \]  
(16)
The dimension of the nullspace of the Poisson tensor is equal to the number of Casimir functions in a neighborhood of a given set of $\Delta_i$.

If the co-rank of the Poisson tensor is equal to $I$, at each point of phase space there exist $I$ linearly independent vectors $w_k$, $k = 1, \ldots, I$ with components $w^i_k$, $i = 1, \ldots, D$, such that
\[ \sum_{j=1}^D \mathbb{P}^{(s)}_{ij} w^j_k = 0, \quad k = 1, \ldots, I. \] (17)

The vectors $w_k = (w^i_k)$ are the eigenvectors of the Poisson tensor with zero eigenvalue. Since this eigenspace has $I$-fold degeneracy, the $w_k$ are not unique if $I > 1$. They can be rearranged in linear combinations with coefficients depending on $\Delta_i$.

Suppose one of the eigenvectors, $w_k$, can be expressed as
\[ w^i_k = \frac{\partial C_k(\Delta)}{\partial \Delta_i}. \] (18)

Then $C_k(\Delta)$ is a Casimir function which commutes with any function on the Poisson manifold. At a given point, each 1-form $dC_k$ defines a smooth submanifold of codimension one in the Poisson manifold through $dC_k = 0$. As the eigenvectors $w_k$, and therefore the $dC_k$, are linearly independent, the intersections of all $I$ $(D-1)$-dimensional submanifolds is a $(D-I)$-dimensional submanifold, called a symplectic leaf. If we choose local coordinates $(v_1, \ldots, v_{D-I})$ on a symplectic leaf, we have $(v_1, \ldots, v_{2n}, C_1, \ldots, C_I)$ as a coordinate system on phase space, where $n = \frac{1}{2}(D-I)$. The Poisson tensor in these coordinates takes the form
\[ \mathbb{P}^{(s)}_{ij} = \left( \begin{array}{c|c} \mathbb{P}^{(s)}_{\alpha\beta} & 0 \\ \hline 0 & 0 \end{array} \right), \] (19)

where $\mathbb{P}^{(s)}_{\alpha\beta} = \{v_\alpha, v_\beta\}$ and $\det(\mathbb{P}^{(s)}_{\alpha\beta}) \neq 0$. A faithful canonical realization provides a map
\[ (v_1, \ldots, v_{2n}, C_1, \ldots, C_I) \rightarrow (s_1, \ldots, s_n, p_1, \ldots, p_n, U_1, \ldots, U_I) \] (20)
of the local coordinates. After applying this map, the Poisson tensor has the form with
\[ \tilde{\mathbb{P}}^{(s)}_{\alpha\beta} = \left( \begin{array}{cc} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{array} \right). \] (21)

Darboux' theorem shows that local canonical coordinates $s_\alpha$ and $p_\beta$ exist.

As $\dot{C}_I = \{C_I, H\} = 0$ for any Hamiltonian $H$, motion is always confined to a symplectic leaf $C_I = \text{const}$. Moreover, the existence of a Casimir function implies that the Hamiltonian is not unique because $\{f, H\} = \{f, H + \lambda C_I\}$ for any phase-space function $f$ and $\lambda \in \mathbb{R}$.

### 3.2 Algebraic structure of second-order semiclassical truncations

For a system with $N$ classical degrees of freedom, we collectively refer to $q_j$ and $p_k$ as $x_i$, $i = 1, \ldots, 2N$. As can be seen from or directly from commutators, the Poisson tensor takes the form
\[ \tilde{\mathbb{P}}^{(s)}_{\alpha\beta} = \left( \begin{array}{cc} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{array} \right). \] (21)
brackets of second-order semiclassical truncations are then of the form
\[
\{ \Delta(x_i x_j), \Delta(x_k x_l) \} = \sum_{m \leq n} f_{ij,kl}^{mn} \Delta(x_m x_n) .
\]  
(22)
The \( \Delta(x_i x_j) \) form an independent set of moments if we require that \( i \leq j \).

The brackets are linear and form a Lie algebra with structure constants
\[
f_{ij,kl}^{mn} = \tau_{ik} \delta_j^m \delta_l^n + \tau_{il} \delta_j^m \delta_k^n + \tau_{jk} \delta_i^m \delta_l^n + \tau_{jl} \delta_i^m \delta_k^n + \tau_{ik} \delta_j^n \delta_l^m + \tau_{il} \delta_k^n \delta_j^m - \tau_{jk} \delta_i^n \delta_l^m - \tau_{jl} \delta_i^n \delta_k^m ,
\]  
(23)
using \( \tau_{ij} = \{ x_i, x_j \} \). For \( \tau_{ij} \), we have the identity
\[
\sum_j \tau_{ij} \tau_{jk} = \sum_j \{ x_i, x_j \} \{ x_j, x_k \} = -\delta_{ik}
\]  
(24)
because both brackets are non-zero if and only if \( x_j \) is canonically conjugate to both \( x_i \) and \( x_k \), which implies \( x_i = x_k \) for basic variables. We note that the \( f_{ij,kl}^{mn} \) are manifestly symmetric in the index pairs \((i,j)\) and \((k,l)\), but not in \((m,n)\).

Instead of summing over restricted double indices, it is more convenient to symmetrize all of them explicitly, in particular
\[
f_{ij,kl}^{(mn)} = \frac{1}{2} \left( \tau_{ik} \delta_j^m \delta_l^n + \tau_{il} \delta_j^m \delta_k^n + \tau_{jk} \delta_i^m \delta_l^n + \tau_{jl} \delta_i^m \delta_k^n + \tau_{ik} \delta_j^n \delta_l^m + \tau_{il} \delta_k^n \delta_l^m - \tau_{jk} \delta_i^n \delta_l^m - \tau_{jl} \delta_k^n \delta_l^m \right) ,
\]  
(25)
and include all \( \Delta(x_m x_n) \) in (22) using \( \Delta(x_m x_n) = \Delta(x_n x_m) \). Summations over restricted double indices \((m,n)\) such that \( m \leq n \) can then be replaced by two full summations over \( m \) and \( n \). For instance,
\[
\{ \Delta(x_i x_j), \Delta(x_k x_l) \} = \sum_{m \leq n} f_{ij,kl}^{mn} \Delta(x_m x_n) = \sum_{m,n} f_{ij,kl}^{(mn)} \Delta(x_m x_n) .
\]  
(26)

3.2.1 Cartan metric and root vectors

We compute the Cartan metric
\[
g_{ij,kl} = \sum_{m,n,op} f_{ij,mn}^{(op)} f_{kl,op}^{(mn)} = 4(N + 1) \left( \tau_{il} \tau_{kj} + \tau_{ik} \tau_{lj} \right) .
\]  
(27)

Lemma 1 The Cartan metric \((27)\) is non-degenerate.

Proof: The metric acts on objects of the form \( V = \sum_{i,j} V_{ij} \Delta(x_i x_j) \) via
\[
g(V_1, V_2) = \sum_{i,j,k,l} g_{ij,kl} V_{ij}^{V_{kl}} .
\]  
(28)
For $V$ to be non-zero we need $\text{Sym}(V_{ij}) = \frac{1}{2}(V_{ij} + V_{ji}) \neq 0$ because $\Delta(x_ix_j) = \Delta(x_jx_i)$. Suppose there is a non zero object $V$ in the null space of $g$, such that $g(V, \cdot) = 0$ or $\sum_{i,j} V_{ij} g_{ij;kl} = 0$. Using (27) and rearranging, we find

$$0 = 8(N + 1) \sum_{i,j} \tau_{li} \text{Sym}(V_{ij}) \tau_{jk}. \quad (29)$$

Because $\tau$ is invertible, (29) implies that $V_{ij}$ is antisymmetric, but then $V = 0$. We conclude that $g$ is non-degenerate.

The algebra of second-order moments is therefore a semi-simple Lie algebra. We can show that it is actually simple, and identify it, by examining its Dynkin diagram. We should first find the Cartan subalgebra.

**Lemma 2** The adjoint action of any moment of the form $\Delta(q_i q_j)$, $\Delta(\pi_i \pi_j)$, or $\Delta(q_k \pi_l)$ with $k \neq l$ is nilpotent.

**Proof:** The claim is easy to see for $\Delta(q_i q_j)$ and $\Delta(\pi_i \pi_j)$: The adjoint action of $\Delta(q_i q_j)$ on a moment $\Delta$ is a sum of moments in which any $\pi_k$ that may appear in $\Delta$ is replaced by $q_k$, if $k = i$ or $k = j$. After applying this action twice, no $\pi_k$ is left and the third application gives zero. Analogous arguments hold for $\Delta(\pi_i \pi_j)$.

For $\Delta(q_k \pi_l)$ with $k \neq l$, the adjoint action is non-zero only on moments of the form $\Delta(x\pi_k)$ or $\Delta(yq_l)$, where $x$ and $y$ can be any position or momentum component. In the first case, we compute

$$
\{\Delta(q_k \pi_l), \Delta(x\pi_k)\} = \Delta(x\pi_l) + \{\pi_l, x\} \Delta(q_k \pi_k)
= \begin{cases} 
\Delta(q_k \pi_l) - \Delta(q_k \pi_k) & \text{if } x = q_l \\
\Delta(x\pi_l) & \text{if } x \neq q_l
\end{cases}
$$

Therefore,

$$
\{\Delta(q_k \pi_l), \Delta(q_k \pi_l), \Delta(x\pi_k)\} = \begin{cases} 
-\Delta(q_k \pi_l) & \text{if } x = q_l \\
\{q_k, x\} \Delta(\pi_l^2) & \text{if } x \neq q_l
\end{cases}
$$

The next adjoint action of $\Delta(q_k \pi_l)$ gives zero, and similarly on $\Delta(yq_l)$. Therefore, $\Delta(q_i \pi_i)$ gives zero, and similarly on $\Delta(yq_i)$.

Since nilpotent actions are non-diagonalizable, we construct the Cartan subalgebra from moments of the form $\Delta(q_i \pi_i)$. Since they Poisson commute with one another, they span the Cartan subalgebra

$$H = \langle \Delta(q_i \pi_i) \rangle_{1 \leq i \leq N} \quad (30)$$

The moments $\Delta(q_i \pi_i)$ are orthogonal to one another and have the same norm with respect to the Cartan metric.
Figure 1: The Dynkin diagram for a second-order semiclassical truncation. We adopt the convention that the filled circles correspond to shorter roots and the empty circles correspond to longer roots.

The entire set of moments forms a Cartan–Weyl basis. For any \( \Delta(q_i \pi_i) \), the set of basic moments \( \Delta(x_k x_l) \) with \( k \leq l \) is an eigenbasis of the adjoint action with eigenvalues

- 2 if \( x_k = x_l = \pi_i \),
- 1 if \( x_l = \pi_i \) and \( q_i \neq x_k \neq \pi_i \),
- -1 if \( x_k = q_i \) and \( q_i \neq x_l \neq \pi_i \),
- -2 if \( x_k = x_l = q_i \), and zero otherwise.

The eigenvectors with eigenvalues \( \pm 2 \) have eigenvalue 0 with any other \( \Delta(q_i \pi_i) \), while the eigenvectors with eigenvalues \( \pm 1 \) are shared by two moments of the form \( \Delta(q_i \pi_i) \). The root system is therefore given by all vectors with only two non-zero components of opposite sign and absolute value one, and vectors with a single non-zero component equal to \( \pm 2 \). A suitable subset of eigenmoments with the smallest possible positive eigenvalues for the adjoint action of all \( \Delta(q_i \pi_i) \) gives the simple root vectors

\[
\{ \Delta(q_2 \pi_1), \Delta(q_3 \pi_2), \ldots, \Delta(q_N \pi_{N-1}), \Delta(\pi_N^2) \}\,.
\]

with simple roots

\[
\begin{pmatrix}
1 \\
-1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
-1 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix},
\ldots,
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
\vdots \\
0 \\
-2
\end{pmatrix}.
\]

The resulting Dynkin diagram, shown in Fig. [1] belongs to \( \text{sp}(2N, \mathbb{R}) \).

The Casimir functions of \( \text{sp}(2N, \mathbb{R}) \) can therefore be thought of as approximate constants of motion in quantum mechanics: At the second semiclassical order, the Hamiltonian is a function of basic expectation values and second-order moments, and the \( \text{sp}(2N, \mathbb{R}) \) Casimir functions commute with any such function. These constants of motion can be written as

\[
U_{2m} \propto \text{tr} \left[ (\tau \Delta)^{2m} \right], \quad m \leq N
\]

where \( \Delta \) is a matrix with components \( \Delta_{ij} = \Delta(x_i x_j) \), and \( \tau_{ij} = \{x_i, x_j\} \) as before. There is one approximate constant of motion per classical degree of freedom.

10
3.2.2 Example of $\text{sp}(4, \mathbb{R})$

For two classical degrees of freedom, we show the Cartan metric ordering the moments as

$$\{ \Delta(\pi_1^2), \Delta(\pi_1 q_1), \Delta(q_1^2), \Delta(\pi_2^2), \Delta(\pi_2 q_2), \Delta(\pi_2 q_1), \Delta(\pi_1 q_2), \Delta(q_1 q_2) \}.$$ 

The result,

$$g = \begin{pmatrix} 0 & 0 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(35)

is easily seen to be non-degenerate. The Cartan subalgebra is

$$H = \{ \Delta(q_1 \pi_1), \Delta(q_2 \pi_2) \},$$

(36)

and the simple root vectors

$$\{ \Delta(q_2 \pi_1), \Delta(\pi_2^2) \}$$

(37)

imply simple roots

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

(38)

corresponding to the Cartan matrix

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

(39)

defining $\text{sp}(4, \mathbb{R})$ (or $C_2$).

3.3 Examples

We present standard examples of faithful realizations before we proceed with the general theory.

3.3.1 The Lie algebra $\text{su}(2)$

The Poisson bracket for $\text{su}(2)$ with generators $S_i, i = 1, 2, 3,$ is given by

$$\{S_i, S_j\} = \sum_{k=1}^{3} \epsilon_{ijk} S_k.$$ 

(40)
It is well known that $S^2 = \sum_{i=1}^{3} S_i^2$ is a Casimir function of this algebra. The task is to find a pair of functions of the generators that are canonically conjugate with respect to the original Poisson tensor. These variables can be defined implicitly by

$$S_x = \sqrt{S^2 - S_z^2} \cos(\phi) \quad , \quad S_y = \sqrt{S^2 - S_z^2} \sin(\phi),$$

such that $\{\phi, S_z\} = 1$. Solving for $\phi$ and inserting it into the Poisson bracket, we indeed have

$$\{\phi, S_z\} = \{\arctan(S_y/S_x), S_z\} = \partial \arctan(S_y/S_x)/\partial S_x \{S_x, S_z\} + \partial \arctan(S_y/S_x)/\partial S_y \{S_y, S_z\} = 1.$$  

(42)

### 3.3.2 The Lie algebra su(1,1)

The Lie algebra su(1,1) is defined by the relations

$$[K_0, K_1] = -K_2 \quad , \quad [K_1, K_2] = K_0 \quad , \quad [K_0, K_2] = K_1.$$  

(43)

For this bracket, a faithful canonical realization is given by

$$K_0 = k + \frac{1}{2}(s^2 + p_s^2) \quad , \quad K_1 = \frac{s}{2} \sqrt{4k + s^2 + p_s^2} \quad , \quad K_2 = \frac{ps}{2} \sqrt{4k + s^2 + p_s^2},$$

(44)

where $K_1^2 + K_2^2 - K_0^2 = -k^2$ is the Casimir function and $s$ and $p_s$ are canonically conjugate variables.

### 3.3.3 The Lie algebra sp(2, $\mathbb{R}$)

The Lie algebra sp(2, $\mathbb{R}$) can be expressed as the set of matrices of the form $\begin{pmatrix} c & a \\ b & -c \end{pmatrix}$, with generators

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(45)

and relations

$$[A, B] = C \quad , \quad [A, C] = -2A \quad , \quad [B, C] = 2B.$$  

(46)

Over the complex numbers, this Lie algebra is isomorphic to su(1, 1) via

$$A = K_2 + iK_1 \quad , \quad B = K_2 - iK_1 \quad , \quad C = 2iK_0.$$  

(47)

The canonical realization (44) can therefore be mapped to this case:

$$A = \frac{1}{2}(p_s + is)\sqrt{4k + s^2 + p_s^2} \quad , \quad B = \frac{1}{2}(p_s - is)\sqrt{4k + s^2 + p_s^2} \quad , \quad C = i(2k + s^2 + p_s^2).$$

(48)
However, because $\text{sp}(2, \mathbb{R})$ and $\text{su}(1, 1)$ are different real forms, these generators are not real. The generators (48) therefore do not present a suitable canonical realization for our purposes.

Similarly, using $b = 2^{-1/2}(s + ip_s)$, we obtain generators

$$A = ib^* \sqrt{b^* b + 2k}, \quad B = -ib \sqrt{b^* b + 2k}, \quad C = 2i(b^* b + k)$$

(49)
of Holstein–Primakoff type [8] in which $A$ and $B = A^*$ can be quantized to raising and lowering operators. However, these generators are not real either, and do not present a suitable bosonic realization.

### 3.3.4 Second-order semiclassical truncation for a single pair of classical degrees of freedom

The constructions used in [7, 1] can be interpreted as a faithful canonical realization

$$\Delta(q^2) = s^2, \quad \Delta(q\pi) = sp_s, \quad \Delta(\pi^2) = p_s^2 + \frac{U}{s^2}$$

(50)
of a semiclassical truncation with $N = 1$, $s = 2$, and Casimir function $U$.

The mapping

$$A = -\frac{1}{2} \Delta(\pi^2), \quad B = \frac{1}{2} \Delta(q^2), \quad C = \Delta(q\pi)$$

(51)
generates an isomorphism to $\text{sp}(2, \mathbb{R})$, giving a simple example of the results of Section 3.2, and a corresponding faithful canonical realization of $\text{sp}(2, \mathbb{R})$. If we use the canonical realization (44) of $\text{su}(1, 1)$, on the other hand, we obtain complex expressions for the moments and therefore violate the reality conditions imposed on faithful canonical realizations.

Using (51), the canonical realization (50) can be related to (49) if we define

$$b' = -\frac{\sqrt{2}iA}{\sqrt{-iC + 2k}} = \frac{i}{\sqrt{2}} \frac{p_s^2 + U/s^2}{\sqrt{U - isp_s}}, \quad b = \frac{\sqrt{2}iB}{\sqrt{-iC + 2k}} = \frac{i}{\sqrt{2}} \frac{s^2}{\sqrt{U - isp_s}}$$

(52)

with $U = 4k^2$, such that $\{b', b\} = i$. However, reality conditions are again violated because $b' \neq b^*$.

### 3.3.5 Non-faithful bosonic realization of $\text{sp}(2N, \mathbb{R})$

The Lie algebra $\text{sp}(2N, \mathbb{R})$ can be written with $N(2N + 1)$ generators $A_{ij}$ ($i \leq j$), $B_{ij}$ ($i \leq j$) and $C_{ij}$ where $i, j = 1, \ldots, N$ and relations [11]

$$[A_{ij}, A_{i'j'}] = 0 = [B_{ij}, B_{i'j'}]$$

(53)

$$[B_{ij}, A_{i'j'}] = C_{j'j} \delta_{ii'} + C_{i'i} \delta_{jj'} + C_{i'i} \delta_{ji'} + C_{ii'} \delta_{jj'}$$

(54)

$$[C_{ij}, A_{i'j'}] = A_{ij} \delta_{ii'} + A_{i'i} \delta_{jj'}$$

(55)

$$[C_{ij}, B_{i'j'}] = -B_{j'j} \delta_{ii'} - B_{j'i} \delta_{jj'}$$

(56)

$$[C_{ij}, C_{i'j'}] = C_{i'j'} \delta_{ij'} - C_{ij'} \delta_{i'j}$$

(57)
It has a bosonic realization \[24\, 11\, 13\, 14\]

\[A_{ij} = \sum_{\alpha=1}^{n} b^*_i \alpha b_j \alpha, \quad B_{ij} = \sum_{\alpha=1}^{n} b_i \alpha b_j \alpha, \quad C_{ij} = \frac{1}{2} \sum_{\alpha=1}^{n} (b^*_i \alpha b_j \alpha + b_j \alpha b^*_i \alpha) \quad (58)\]

for every integer \(n \geq 1\), with \(nN\) boson variables \(b_{i\alpha}\) (implying \(2nN\) degrees of freedom).

For our purposes, this realization violates reality conditions. Moreover, it is not faithful: Since \(2N + 1\) is odd, the number of degrees of freedom cannot match the dimension \(N(2N+1)\) of \(sp(2N, \mathbb{R})\), and since \(sp(2N, \mathbb{R})\) has rank \(N\), it has \(N\) Casimirs. For a faithful bosonic realization, one therefore needs \(N^2\) boson variables \(b_{i\alpha}\) (that is, \(n = N\)) and \(N\) Casimir variables. Finding an explicit realization of this form has proven to be difficult even for \(sp(4, \mathbb{R})\). For instance, possible expressions have been given up to solving complicated partial differential equations \[13\] or diagonalizing large matrices \[14\]. In the next section, we will solve this problem for the analogous question of finding a faithful canonical realization of a second-order semiclassical truncation with two classical degrees of freedom, which is algebraically equivalent to \(sp(4, \mathbb{R})\).

### 4 Constructing Casimir–Darboux coordinates

A partially constructive proof of Darboux’ theorem for symplectic manifolds is presented in \[25\]: Given a symplectic manifold \((M, \omega)\), the following steps demonstrate the existence of Darboux coordinates \((q_j, \pi_k)\) in a neighborhood \(U \subset M\) around a given point \(x \in M\), such that \(\omega = \sum_j dq_j \wedge d\pi_j\). We first choose some function on \(M\), calling it \(q_1\), such that \(dq_1 \neq 0\) at \(x\). Its Hamiltonian vector field \(X_{q_1}\) is then non-zero and generates a non-trivial flow \(F_{q_1}(t) = \exp(tX_{q_1})\) in a neighborhood of \(x\). Choosing a hypersurface transverse to \(X_{q_1}\), we can endow the whole neighborhood with a pair of coordinates given by \(q_1\) and \(\pi_1 = -t\), defined by the parameter \(t\) of the Hamiltonian flow such that \(t = 0\) on the hypersurface. These two coordinates are canonically conjugate because

\[\{q_1, \pi_1\} = X_{q_1}t = \frac{\partial}{\partial t} t = 1. \quad (59)\]

We then move on to the hypersurface defined by \(q_1 = 0 = \pi_1\), apply the previous steps, and iterate until we have the required number of coordinates \(q_j\) and \(\pi_k\) defined on a family of hypersurfaces of decreasing dimension. Starting with the last hypersurface of dimension two, we iteratively transport the coordinates into a neighborhood within the next higher hypersurface by declaring that they take constant values on all lines of the flows \(F_{q_i}(s)F_{\pi_j}(t)\), if \(q_i\) and \(\pi_j\) have already been transported in this way. The proof concludes by showing that the coordinates transported to the neighborhood \(U\) of \(x\) in \(M\) are indeed canonical.

The steps used to prove Darboux’ theorem for symplectic manifolds can be simplified and extended to a systematic procedure to derive Casimir–Darboux coordinates on Poisson manifolds. We keep the first step, but instead of using hypersurfaces of constant canonical
coordinates we construct hypersurfaces which are Poisson orthogonal to the already constructed canonical pairs. This modification eliminates the need to transport coordinates from hypersurfaces to the full manifold. We first illustrate the method for the second-order semiclassical truncation of a single pair of classical degrees of freedom.

4.1 Canonical realization for a single pair of degrees of freedom at second order

The Poisson brackets of our non-canonical coordinates $\Delta(q^2), \Delta(q\pi)$ and $\Delta(\pi^2)$ are given in (7):

$$\{\Delta(q^2), \Delta(q\pi)\} = 2\Delta(q^2) \quad \{\Delta(q\pi), \Delta(\pi^2)\} = 2\Delta(\pi^2) \quad \{\Delta(q^2), \Delta(\pi^2)\} = 4\Delta(q\pi).$$  

As our first canonical coordinate we choose $s = \sqrt{\Delta(q^2)}$. Identifying the (negative) parameter along its Hamiltonian flow with the new momentum $p_s$, we have the differential equations

$$\frac{\partial \Delta(q^2)}{\partial p_s} = -\{\Delta(q^2), \sqrt{\Delta(q^2)}\} = 0 \quad \frac{\partial \Delta(q\pi)}{\partial p_s} = -\{\Delta(q\pi), \sqrt{\Delta(q^2)}\} = \sqrt{\Delta(q^2)} = s \quad \frac{\partial \Delta(\pi^2)}{\partial p_s} = -\{\Delta(\pi^2), \sqrt{\Delta(q^2)}\} = 2\frac{\Delta(q\pi)}{\sqrt{\Delta(q^2)}} = 2\frac{\Delta(q\pi)}{s}. \quad (61)$$

Since $s$ is held constant in these equations, we can first solve (62) by a simple integration,

$$\Delta(q\pi) = sp_s + f_1(s), \quad (64)$$

insert the result in (63) and integrate once more:

$$\Delta(\pi^2) = p_s^2 + 2\frac{f_1(s)}{s}p_s + f_2(s). \quad (65)$$

Computing $\{\Delta(q\pi), \Delta(\pi^2)\}$ using the canonical nature of the variables $s$ and $p_s$, and requiring that it equal $2\Delta(\pi^2)$ implies two equations:

$$\frac{df_1}{ds} = \frac{f_1(s)}{s}, \quad \frac{df_2}{ds} = 2\frac{f_1(s)f_1(s)}{s^2} - 2\frac{f_2(s)}{s}. \quad (66)$$

They are solved by

$$f_1(s) = U_2 s \quad f_2(s) = \frac{U_1}{s^2} + U_2^2 \quad (67)$$

with constants $U_1$ and $U_2$. We can eliminate $U_2$ by a canonical transformation replacing $p_s$ with $p_s + U_2$. The constant $U_1$ is the Casimir coordinate. The resulting moments in terms of Casimir–Darboux variables are

$$\Delta(q^2) = s^2 \quad \Delta(q\pi) = sp_s \quad \Delta(\pi^2) = p_s^2 + \frac{U_1}{s^2}. \quad (68)$$
as in \([50]\) or \([7, 1]\). The Casimir coordinate \(U_1\) can be interpreted as the left-hand side of Heisenberg’s uncertainty relation,

\[
\Delta(q^2)\Delta(\pi^2) - \Delta(q\pi)^2 = U_1 \geq \frac{\hbar^2}{4},
\]

which is a constant of motion at second semiclassical order.

4.2 Poisson tensors of rank greater than two

If we have a Poisson tensor of rank greater than two, we have to iterate the procedure used in our example in order to find additional canonical pairs. In general, it may be difficult to solve some of the differential equations explicitly.

Instead of using general solutions and eliminating surplus parameters through canonical transformations, in practice it is more useful to make suitable choices for functions such as \(f_1\) and \(f_2\) in the preceding example. There are wrong choices in the sense that the procedure may terminate before the required number of coordinates has been found, in which case one obtains a non-faithful canonical realization. Usually, it is not difficult to see which choices lead to a loss of degrees of freedom.

In order to iterate the procedure, we use the following method related to the notion of Dirac observables in canonical relativistic systems \([26, 27, 28]\). Having found a canonical pair \((s, p_s)\) on a (sub)manifold of dimension \(d\), we construct \(d - 2\) independent functions \(f_i\) such that \(\{f_i, s\} = 0 = \{f_i, p_s\}\) for all \(i\). These functions are then Dirac observables with respect to \(s\) and \(p_s\). The construction of Dirac observables is, in general, a very difficult task, and in fact presents one of the main problems of canonical quantum gravity. Here, however, the structure of already-constructed canonical coordinates helps to make the construction of suitable \(f_i\) feasible. In particular, the free functions that remain after constructing \(s\) and \(p_s\), such as \(f_1\) and \(f_2\) in the example, are, by construction, independent of \(s\), and therefore already fulfill \(\{f_i, p_s\} = 0\).

Only a single set of conditions, \(\{f_i, s\} = 0\), then remains to be implemented by suitable combinations of the original \(f_i\), which can be done by eliminating integration parameters in the flow \(F_s(t)\). For instance, had we not already known that \(U_1\) in \((69)\) is a Casimir function, we could have derived it as follows: The flow generated by \(s^2 = \Delta(q^2)\) on the remaining moments is determined by the differential equations

\[
\frac{d\Delta(q\pi)}{dt} = -2\Delta(q^2), \quad \frac{d\Delta(\pi^2)}{dt} = -4\Delta(q\pi).
\]

The first equation implies that \(\Delta(q\pi)[t] = -2\Delta(q^2)t + d\) with \(t\)-independent \(d\). Inserting this solution in the second equation, we find \(\Delta(\pi^2)[t] = 4\Delta(q^2)t^2 - 4dt + e\) with another constant \(e\). We now eliminate \(t\) by inserting \(t = \frac{1}{2}(d - \Delta(q\pi)[t])/s^2\) in \(\Delta(\pi^2)[t]::

\[
\Delta(\pi^2)[t] = \frac{\Delta(q\pi)[t]^2}{\Delta(q^2)} - 3\frac{d^2}{\Delta(q^2)} + e.
\]
Therefore, \( U_1 = \Delta(q^2)\Delta(\pi^2)[t] - \Delta(q\pi)[t] = -3d^2 + es^2 \) is independent of \( t \), which implies \( dU_1/dt = \{U_1, \Delta(q^2)\} = 0 \), and \( U_1 \) is a Dirac observable with respect to \( \Delta(q^2) \) which can be used as a coordinate Poisson orthogonal to \( s \).

The Poisson bracket of two Dirac observables is also a Dirac observable. (This property may be useful for calculating further Dirac observables once more than two have been found.) Given a complete set of Dirac observables, they form coordinates on a Poisson manifold, and we can compute their Poisson brackets from their expressions in terms of the original variables. On this new Poisson manifold, we proceed as in the first step, and then iterate. The procedure terminates when we reach the full dimension, in which case the Poisson manifold is symplectic, or when we obtain a complete set of Poisson commuting Dirac observables. The commuting Dirac observables are the Casimir functions. Because all coordinates constructed in this way are functions of the original variables (the moments in our case of interest), there is no need to transport coordinates to successive hypersurfaces.

### 4.3 Second-order canonical realization for two classical degrees of freedom

A non-trivial example of our general procedure is given by the second-order semiclassical truncation of a system with two pairs of classical degrees of freedom, \((q_1, \pi_1)\) and \((q_2, \pi_2)\). We obtain ten moments: two fluctuations and one covariance for each pair, as well as four cross-covariance such as \( \Delta(q_1q_2) \). The rank of the resulting Poisson tensor is eight, so that we should construct four canonical pairs and two Casimir functions.

Since we already discussed the case of a single canonical pair, we can speed up the first step and construct two canonical pairs at the same time by defining \( s = \sqrt{\Delta(q_1^2)} \) and \( s_2 = \sqrt{\Delta(q_2^2)} \). Their canonical momenta can be generated as in the case of a single degree of freedom, but analogs of the functions \( f_i \) could now depend on all the remaining canonical variables: We have

\[
\Delta(q_1\pi_1) = s_1p_1 + f_{q_1\pi_1}, \quad \Delta(\pi_1^2) = p_1^2 + 2\frac{P_1}{s_1}f_{q_1\pi_1} + f_{\pi_1^2} \quad (72)
\]

and

\[
\Delta(q_2\pi_2) = s_2p_2 + f_{q_2\pi_2}, \quad \Delta(\pi_2^2) = p_2^2 + 2\frac{P_2}{s_2}f_{q_2\pi_2} + f_{\pi_2^2} \quad (73)
\]

with four functions \( f_{q_1\pi_1}, f_{\pi_1^2}, f_{q_2\pi_2} \) and \( f_{\pi_2^2} \) independent of \( s_1, p_1, s_2 \) and \( p_2 \).

We now have to find spaces which are Poisson orthogonal to \((s_1, p_1, s_2, p_2)\), or functions of the moments which Poisson commute with all four canonical coordinates. If we choose \( f_{q_1\pi_1} = 0 = f_{q_2\pi_2}, \) this condition is equivalent to having moments which Poisson commute with \( \Delta(q_1^2), \Delta(q_1p_1), \Delta(q_2^2) \) and \( \Delta(q_2p_2) \). Two such functions are

\[
f_{\pi_1^2} = s_1^2\Delta(\pi_1^2) - s_1^2p_1^2 = \Delta(q_1^2)\Delta(\pi_1^2) - \Delta(q_1\pi_1)^2 =: f_1 \quad (74)
\]

and

\[
f_{\pi_2^2} = s_2^2\Delta(\pi_2^2) - s_2^2p_2^2 = \Delta(q_2^2)\Delta(\pi_2^2) - \Delta(q_2\pi_2)^2 =: f_2 \quad (75)
\]
obtained simply by solving (72) and (73) for \( f_{\pi_1^2} \) and \( f_{\pi_2^2} \). After computing the Poisson brackets between all the cross-covariances and \( \Delta(q_1^2) = s_{11}^2 \), \( \Delta(q_1 \pi_1) = s_{1p}^1 \), \( \Delta(q_2^2) = s_{22}^2 \) and \( \Delta(q_2 \pi_2) = s_{2p}^2 \), we can construct a complete set of other Poisson commuting functions by integrating flow equations generated by \( \Delta(q_1^2) \), \( \Delta(q_1 \pi_1) \), \( \Delta(q_2^2) \) and \( \Delta(q_2 \pi_2) \). The resulting combinations are

\[
\begin{align*}
    f_3 &= \Delta(q_1 \pi_2) \Delta(q_2 \pi_1) - \Delta(q_1 q_2) \Delta(\pi_1 \pi_2) \\
    f_4 &= \Delta(q_1^2) \frac{\Delta(q_2 \pi_1)}{\Delta(q_1 q_2)} - \Delta(q_1 \pi_1) \\
    f_5 &= \Delta(q_2^2) \frac{\Delta(q_1 \pi_2)}{\Delta(q_1 q_2)} - \Delta(q_2 \pi_2) \\
    f_6 &= \frac{\Delta(q_1^2) \Delta(q_2^2)}{\Delta(q_1 q_2)^2},
\end{align*}
\]

as can be checked explicitly. The Poisson brackets between these six functions are closed, so that we can iterate the procedure.

We start the next step by defining \( s_3 = f_6 \), which is the inverse of the squared correlation between the two particle positions. Its flow equations impose conditions on derivatives of functions Poisson-commuting with \( p_3 \), which can again be integrated. Solving some of the integrals, we obtain \( p_3 \) as a function of the \( f_i \) and \( s_3 \), explicitly

\[
p_3 = \frac{f_4 + f_5}{4s_3(1 - s_3)}. \tag{80}
\]

Moreover, the four combinations

\[
\begin{align*}
    g_1 &= f_1 + \frac{(f_4 + f_5)^2}{4(1 - f_6)} + \frac{1}{2} \frac{(f_4 + f_5)(f_4 - f_5)}{1 - f_6} \\
    g_2 &= f_2 + \frac{(f_4 + f_5)^2}{4(1 - f_6)} - \frac{1}{2} \frac{(f_4 + f_5)(f_4 - f_5)}{1 - f_6} \\
    g_3 &= f_3 + \frac{(f_4 + f_5)^2}{4(1 - f_6)} \\
    g_4 &= \frac{1}{2} (f_4 - f_5)
\end{align*}
\]

Poisson commute with \( s_3 \) and \( p_3 \), as can again be checked explicitly. It turns out that

\[
g_1 + g_2 - 2g_3 = U_1 \tag{85}
\]

is the quadratic Casimir of the full moment system. Using \( U_1 \), we have three remaining variables, which can conveniently be chosen to be \( g_1 \pm g_2 \) and \( g_4 \). Their mutual Poisson brackets are again closed.
The next step of the procedure leads to the combinations

\[
\begin{align*}
    h_1 &= \frac{g_4}{\sqrt{s_3 - 1}} \\
    h_2 &= (g_1 - g_2)\sqrt{\frac{s_3 - 1}{s_3}} \\
    h_3 &= \frac{(1 - s_3)(g_1 + g_2) + s_3U_1 + 2(1 + s_3)(1 - s_3)^{-1}g_4^2}{\sqrt{s_3}}
\end{align*}
\]  

Poisson-commuting with \(s_3\) and \(p_3\), in addition to \(U_1\). We choose \(p_4 = h_1\) as our final canonical momentum, such that invariance under its flow implies

\[
\begin{align*}
    h_2 &= A(p_4) \cos(s_4) \\
    h_3 &= A(p_4) \sin(s_4)
\end{align*}
\]

with some function \(A(p_4)\). From the remaining Poisson brackets of \(h_i\), it follows that

\[
A(p_4)\frac{dA(p_4)}{dp_4} = -8p_4U_1 + 32p_4^3.
\]

The general solution of this equation is

\[
A(p_4) = \sqrt{U_2 - 8p_4^2U_1 + 16p_4^4}
\]

with a constant of integration \(U_2\) which can be interpreted as the second Casimir. (At this point, it could be any function of the quadratic and quartic Casimirs).

To summarize, we express the original moments in terms of Casimir–Darboux variables. For moments of the first classical pair of degrees of freedom, we find

\[
\begin{align*}
    \Delta(q_1^2) &= s_1^2, \quad \Delta(q_1\pi_1) = s_1p_1 \\
    \Delta(\pi_1^2) &= p_1^2 + \frac{\Phi(s_3, p_3, s_4, p_4)}{s_1^2}
\end{align*}
\]

with

\[
\Phi(s_3, p_3, s_4, p_4) = -\frac{s_3 + 1}{s_3 - 1}p_4^2 - 4s_3\sqrt{s_3 - 1}p_3p_4 + 4s_3^2(s_3 - 1)p_3^2 + \frac{1}{2s_3 - 1}U_1
\]

\[\]  

For moments of the second classical pair of degrees of freedom,

\[
\begin{align*}
    \Delta(q_2^2) &= s_2^2, \quad \Delta(q_2\pi_2) = s_2p_2 \\
    \Delta(\pi_2^2) &= p_2^2 + \frac{\Gamma(s_3, p_3, s_4, p_4)}{s_2^2}
\end{align*}
\]
with
\[
\Gamma(s_3, p_3, s_4, p_4) = -\frac{s_3 + 1}{s_3 - 1} p_4^2 + 4s_3\sqrt{s_3 - 1} p_3 p_4 + 4s_3^2 (s_3 - 1) p_3^2 + \frac{1}{2} \frac{s_3}{s_3 - 1} U_1
\]
\[
-\frac{1}{2} \frac{\sqrt{s_3}}{s_3 - 1} \sqrt{U_2 - 8p_4^2 U_1 + 16p_4^4} \left(-\sqrt{s_3 - 1} \cos (s_4) + \sin (s_4)\right).
\]

Finally, we have
\[
\Delta(\pi_1 \pi_2) = \frac{p_1 p_2}{\sqrt{s_3}} + \sqrt{\frac{s_3 - 1}{s_3}} \left(\frac{p_2}{s_1} - \frac{p_1}{s_2}\right) p_4
\]
\[
-2\frac{\sqrt{s_3}}{s_3} (s_3 - 1) \left(\frac{p_1}{s_2} + \frac{p_2}{s_1}\right) p_3 + \frac{(3s_3 - 1)}{s_1 s_2 \sqrt{s_3} (s_3 - 1)} p_4^2
\]
\[
-4 \frac{(s_3 - 1) s_3^{3/2}}{s_1 s_2} p_3^2 \frac{\sqrt{s_3}}{2s_1 s_2 (s_3 - 1)} U_1
\]
\[
+ \frac{s_3}{2s_1 s_2 (s_3 - 1)} \sin (s_4) \sqrt{U_2 - 8p_4^2 U_1 + 16p_4^4}
\]
\[
\Delta(q_1 \pi_2) = \frac{p_2 s_1}{\sqrt{s_3}} - \sqrt{\frac{s_3 - 1}{s_3}} \frac{s_1}{s_2} p_4 - 2 (s_3 - 1) \sqrt{s_3} \frac{s_1}{s_2} p_3
\]
\[
\Delta(q_2 \pi_1) = \frac{p_1 s_2}{\sqrt{s_3}} + \sqrt{\frac{s_3 - 1}{s_3}} \frac{s_2}{s_1} p_4 - 2 (s_3 - 1) \sqrt{s_3} \frac{s_2}{s_1} p_3
\]
\[
\Delta(q_1 q_2) = \frac{s_1 s_2}{\sqrt{s_3}}
\]

for the cross-covariances.

4.4 Third-order semiclassical truncation for single pair of degrees of freedom

Third-order moments are subject to linear Poisson brackets within a third-order truncation. In particular, the Poisson bracket of any pair of third-order moments is zero within this truncation, and we have linear brackets between second-order and third-order moments, such as
\[
\{\Delta(q^2), \Delta(q^2 \pi)\} = 2\Delta(q^3) , \quad \{\Delta(q^2), \Delta(q^2 \pi^2)\} = 4\Delta(q^2 \pi) , \quad \{\Delta(q^2), \Delta(\pi^3)\} = 6\Delta(q^2 \pi^2)
\]
and so on. Thanks to the truncation, the brackets still define a linear Lie algebra, but it is not semisimple because the third-order moments span an Abelian ideal. This seven-dimensional Lie algebra is the semidirect product \(\text{sp}(2, \mathbb{R}) \ltimes \mathbb{R}^4\) where \(\text{sp}(2, \mathbb{R})\), spanned by the second-order moments, acts on \(\mathbb{R}^4\), spanned by the third-order moments, according
to

\[
A = -\frac{1}{2} \Delta(\pi^2) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

(104)

\[
B = \frac{1}{2} \Delta(q^2) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(105)

\[
C = \Delta(q\pi) = \begin{pmatrix}
-3 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix}
\]

(106)

using (51). Computing the Casimir

\[
K = -\frac{1}{2}(AB + BA) - \frac{1}{4}C^2 = -\frac{15}{4}I = -\frac{3}{2} \left( \frac{3}{2} + 1 \right) I,
\]

(107)

this action is recognized as the spin-3/2 representation of \(\mathfrak{sp}(2, \mathbb{R})\).

Guided by our second-order examples, we make the choice

\[
\Delta(q^2) = s_1^2
\]

(108)

\[
\Delta(q\pi) = s_1 p_1
\]

(109)

as the first step in the introduction of canonical coordinates. Suitable variables on the hypersurface Poisson orthogonal to \((s_1, p_1)\) are

\[
f_1 = \Delta(q^2)\Delta(\pi^2) - \Delta(q\pi)^2
\]

\[
f_2 = \Delta(q^2)\frac{\Delta(q^2\pi)}{\Delta(q^2)} - \Delta(q\pi)
\]

\[
f_3 = \frac{\Delta(q^2)^2}{\Delta(q^3)^2} \left( \Delta(q^2\pi)^2 - \Delta(q^2\pi)\Delta(q^3) \right)
\]

\[
f_4 = 2\Delta(q\pi) + \Delta(q^2) \frac{\Delta(q^3)\Delta(\pi^3) - \Delta(q^2\pi)\Delta(q^2\pi)}{\Delta(q^2\pi)^2 - \Delta(q^2\pi)\Delta(q^3)}
\]

The dimension of the Poisson manifold at third order is \(D = 7\), while the rank of the Poisson tensor is six. We therefore expect three degrees of freedom and one Casimir function. One additional coordinate Poisson commuting with \((s_1, p_1)\) is needed to have seven independent variables. Since the Poisson brackets of \(f_i\) are closed, the last variable Poisson commuting with \((s_1, p_1)\) has to be the Casimir function, which by ansatz can be found to be

\[
U_1 = 4 \left( \Delta(q^2)^2 - \Delta(q^2\pi)\Delta(q^3) \right) \left( \Delta(q^2\pi)^2 - \Delta(q^3)\Delta(q^2\pi) \right)
\]

(110)

\[
- \left( \Delta(q^2\pi)\Delta(q^2\pi) - \Delta(q^3)\Delta(\pi^3) \right)^2.
\]

(111)
To initiate the next step, we choose

\[ s_2 = f_3 \]  \hspace{1cm} (112)

and integrate its flow equations. The resulting expressions tell us that

\[ p_2 = \frac{6f_2 + f_4}{16s_2}, \]  \hspace{1cm} (113)

while

\[ g_1 = f_1 + \frac{(6f_2 + f_4)^2}{16}, \quad g_2 = -\frac{1}{2}f_2 - \frac{1}{4}f_4 \]  \hspace{1cm} (114)

Poisson commute with \( s_2 \) but not with \( p_2 \). After a further transformation of variables, we obtain the remaining canonical pair

\[ s_3 = \frac{g_2}{\sqrt{s_2}} \]  \hspace{1cm} (115)
\[ p_3 = -\frac{2g_1 - 7s_2 + 10p_3^2s_2}{6\sqrt{s_2}(-1 + 4p_3^2)}, \]  \hspace{1cm} (116)

as can be checked directly.

The resulting faithful canonical realization is given by the second-order moments

\[ \Delta(\pi^2) = p_1^2 + \frac{f_1(s_2, p_2, s_3, p_3)}{s_1^2} \]  \hspace{1cm} (117)
\[ \Delta(q\pi) = s_1p_1 \]  \hspace{1cm} (118)
\[ \Delta(q^2) = s_1^2 \]  \hspace{1cm} (119)

where

\[ f_1(s_2, p_2, s_3, p_3) = -3\sqrt{s_2} \left(4s_3^2 - 1\right)p_3 + \frac{1}{2} \left(7 - 10s_3^2\right)s_2 - 16s_3^2p_2^2, \]  \hspace{1cm} (120)

and third-order moments

\[ \Delta(\pi^3) = \frac{1}{\sqrt{s_2}s_1^3} \Phi(s_i, p_j) \left(\frac{U_1}{16s_2s_3^2 - 4s_2}\right)^{1/4} \]  \hspace{1cm} (121)
\[ \Delta(q^3) = \frac{1}{s_1\sqrt{s_2}} (p_1s_1 + (s_3 + 1)\sqrt{s_2} + 4s_2p_2) \]  \hspace{1cm} (122)
\[ \times (p_1s_1 + (s_3 + 1)\sqrt{s_2} + 4s_2p_2) \left(\frac{U_1}{16s_2s_3^2 - 4s_2}\right)^{1/4} \]  \hspace{1cm} (123)

\[ \Delta(q^3) = \frac{1}{\sqrt{s_2}} \left(p_1s_1^2 + s_1(p_3\sqrt{s_2} + 4s_2p_2)\right) \left(\frac{U_1}{16s_2s_3^2 - 4s_2}\right)^{1/4} \]  \hspace{1cm} (124)

with

\[ \Phi(s_i, p_j) = p_1^3s_1^3 + 3p_1^2s_1^2\sqrt{s_2}s_3 + 3p_1s_1s_2)s_3^2 + 4s_1p_1p_2 - 1) + 64p_2^3s_2^3 \]
\[ + s_2^{3/2}s_3 \left(s_3^2 + 24s_1p_1p_2 - 7) + 48p_2^2s_2^{5/2} + 12p_2s_2^2 \left(s_3^2 + 4s_1p_1p_2 - 1) \right. \]  \hspace{1cm} (125)
4.5 Momentum dependence

In [1], the moments are quadratic in the new momentum $p_s$. This property is useful because it implies an effective Hamiltonian (8) with standard kinetic term, quadratic in the classical momentum $\pi$ (the expectation value) and the new momentum $p_s$ related to $\Delta(\pi^2)$:

$$
\langle \hat{H} \rangle = \frac{\langle \hat{\pi}^2 \rangle}{2m} + V(\hat{q}) = \frac{\pi^2 + \Delta(\pi^2)}{2m} + V(q) + \frac{1}{2}V''(q)\Delta(q^2) + \cdots
$$

$$
= \frac{\pi^2}{2m} + \frac{p_s^2}{2m} + \frac{U}{2m} + V(q) + \frac{1}{2}V''(q)s^2 + \cdots
$$

(126)

The corresponding property for a bosonic realization implies that generators of a Lie algebra have some terms bilinear in the boson variables. (However, bosonic realizations corresponding to canonical realizations of moment algebras cannot be completely bilinear, owing to Casimir terms such as $U/s^2$.) Our third-order realization for a single classical degree of freedom is similar in that $\Delta(\pi^2)$ is quadratic in the new momenta, although with $s$-dependent coefficients.

Unlike the example of a single pair of degrees of freedom, the moments for two pairs of degrees of freedom, given so far, are not quadratic in the new momenta. In fact, we can prove by ansatz that, for a second-order semiclassical truncation for two classical degrees of freedom, there is no faithful representation quadratic in momenta with $s$-independent coefficients. The Poisson tensor has rank eight, so that we are looking for four canonical pairs $(s_j, p_i)$ and two Casimir functions.

We write

$$
\Delta(\pi_1^2) = p_1^2 + p_3^2 + F_1(s_i)p_1 + F_2(s_i)p_3 + F(s_i)
$$

$$
\Delta(\pi_2^2) = p_2^2 + p_4^2 + G_1(s_i)p_2 + G_2(s_i)p_4 + G(s_i)
$$

$$
\Delta(\pi_1\pi_2) = p_1p_2 + p_3p_4 + H_1(s_i)p_1 + H_2(s_i)p_2 + H_3(s_i)p_3 + H_4(s_i)p_4 + H_5(s_i)
$$

(127) (128) (129)

and choose

$$
\Delta(q_1^2) = s_1^2 + s_3^2 , \quad \Delta(q_2^2) = s_2^2 + s_4^2 , \quad \Delta(q_1q_2) = s_1s_2 + s_3s_4 .
$$

(130)

A realization of the entire algebra can be generated by taking Poisson brackets: We can compute

$$
\Delta(\pi_1\pi_2) = \frac{1}{4} \{ \{\Delta(q_1q_2), \Delta(\pi_1^2)\} , \Delta(\pi_2^2) \}
$$

(131)

and, given this moment,

$$
\Delta(q_1\pi_2) = \frac{1}{2} \{\Delta(q_1^2), \Delta(\pi_1\pi_2)\} , \quad \Delta(q_2\pi_1) = \frac{1}{2} \{\Delta(q_2^2), \Delta(\pi_1\pi_2)\} .
$$

(132)

Finally, once we know these three moments, we compute

$$
\Delta(q_1\pi_1) + \Delta(q_2\pi_2) = \{\Delta(q_1q_2), \Delta(\pi_1\pi_2)\} , \quad -\Delta(q_1\pi_1) + \Delta(q_2\pi_2) = \{\Delta(q_1\pi_2), \Delta(q_2\pi_1)\}
$$

(133)
from which $\Delta(q_1\pi_1)$ and $\Delta(q_2\pi_2)$ follow from linear combinations. If $F_1 = F_2 = F_3 = 0$, $G_1 = G_2 = G_3 = 0$, and $H_1 = H_2 = H_3 = H_4 = H_5 = 0$, we have a non-faithful realization because there are no Casimir variables. We therefore have to find suitable functions depending on two additional variables, $U_1$ and $U_2$, such that the required Poisson brackets are realized.

Evaluating all Poisson brackets for consistency conditions, such as $\{\Delta(\pi_1^2), \Delta(\pi_2^2)\} = 0$, we find the following mapping:

$$
\Delta(q_1^2) = s_1^2 + s_3^2 \quad (134)
$$

$$
\Delta(q_2^2) = s_2^2 + s_4^2 \quad (137)
$$

$$
\Delta(q_1\pi_1) = s_1p_1 + s_3p_3 + \frac{1}{2}s_1s_2U_1\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) + \frac{1}{2}s_3s_4U_2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right) \quad (135)
$$

$$
\Delta(q_2\pi_2) = s_2p_2 + s_4p_4 + \frac{1}{2}s_1s_2U_1\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) + \frac{1}{2}s_3s_4U_2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right) \quad (138)
$$

$$
\Delta(\pi_1^2) = p_1^2 + p_3^2 + p_1s_2U_1\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right) + p_3s_4U_2\left(\frac{1}{s_3^2} - \frac{1}{s_4^2}\right) \quad (136)
$$

$$
\Delta(\pi_2^2) = p_2^2 + p_4^2 + p_2s_1U_1\left(\frac{1}{s_2^2} - \frac{1}{s_3^2}\right) + p_4s_3U_2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right) \quad (139)
$$

for the first classical degree of freedom,

$$
\Delta(q_1q_2) = s_1s_2 + s_3s_4 \quad (140)
$$

$$
\Delta(q_1\pi_2) = s_1p_2 + s_3p_4 + \frac{1}{2}s_1^2U_1\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right) + \frac{1}{2}s_3^2U_2\left(\frac{1}{s_3^2} - \frac{1}{s_4^2}\right) \quad (141)
$$

$$
\Delta(q_2\pi_1) = s_2p_1 + s_4p_3 + \frac{1}{2}s_2^2U_1\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) + \frac{1}{2}s_4^2U_2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right) \quad (142)
$$

$$
\Delta(\pi_1\pi_2) = p_1p_2 + p_3p_4 + \frac{1}{2}p_1s_1U_1\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right) + \frac{1}{2}p_2s_2U_1\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) \quad (143)
$$

$$
+ \frac{1}{2}p_3s_3U_2\left(\frac{1}{s_3^2} - \frac{1}{s_4^2}\right) + \frac{1}{2}p_4s_4U_2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right)
$$

$$
- \frac{1}{4}s_1s_2U_1^2\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right)^2 - \frac{1}{4}s_3s_4U_2^2\left(\frac{1}{s_4^2} - \frac{1}{s_3^2}\right)^2
$$

for the cross-covariances.
If the two free parameters $U_1$ and $U_2$ were independent Casimir functions, we would have a faithful canonical realization. However, the rank of the Jacobian of the transformation from $(s_i, p_j, U_I)$ to the moments can be seen to equal seven, and therefore the realization is not faithful. Moreover, the quadratic Casimir of the algebra,

$$C_2 = \text{tr} \left( \left( \tau \Delta \right)^2 \right),$$  \hfill (144)

can be computed explicitly and does not equal a function of $U_1$ and $U_2$ — it depends on the coordinates as well. If the map were faithful, we would have

$$\frac{\partial C_2}{\partial s_i} = \frac{\partial C_2}{\partial p_j} = 0.$$  \hfill (145)

Finally, we note that the canonical transformation

$$P_1 = p_1 + \frac{1}{2} s_2 U_1 \left( \frac{1}{s_2^2} - \frac{1}{s_1^2} \right), \quad P_2 = p_2 + \frac{1}{2} s_1 U_1 \left( \frac{1}{s_1^2} - \frac{1}{s_2^2} \right)$$
$$P_3 = p_3 + \frac{1}{2} s_4 U_2 \left( \frac{1}{s_4^2} - \frac{1}{s_3^2} \right), \quad P_4 = p_4 + \frac{1}{2} s_3 U_2 \left( \frac{1}{s_3^2} - \frac{1}{s_4^2} \right)$$

and $S_i = s_i$ maps our realization to the non-faithful

$$\Delta(q_1^2) = S_1^2 + S_3^2, \quad \Delta(q_1 p_1) = S_1 P_1 + S_3 P_3, \quad \Delta(p_1^2) = P_1^2 + P_3^2$$
$$\Delta(q_2^2) = S_2^2 + S_4^2, \quad \Delta(q_2 p_2) = S_2 P_2 + S_4 P_4, \quad \Delta(p_2^2) = P_2^2 + P_4^2$$
$$\Delta(q_1 q_2) = S_1 S_2 + S_3 S_4, \quad \Delta(q_1 p_2) = S_1 P_2 + S_3 P_4$$
$$\Delta(q_2 p_1) = S_2 P_1 + S_4 P_3, \quad \Delta(p_1 p_2) = P_1 P_2 + P_3 P_4,$$

in which there are no free parameters that could play the role of Casimir functions. The only possibilities are therefore realizations non-quadratic in momenta, or with non-standard, $s$-dependent kinetic terms. None of these options can lead to a bilinear bosonic realization.

### 4.6 Realizations of sp$(2n, \mathbb{R})$

The isomorphism between second-order semiclassical truncations and sp$(2n, \mathbb{R})$ implies that faithful bosonic realizations of sp$(4, \mathbb{R})$ cannot be bilinear in the boson variables. This result underlines some of the difficulties in finding such realizations pointed out in [13, 14]. Given the generators $A_{ij} (i \leq j)$, $B_{ij} (i \leq j)$ and $C_{ij}, i, j = 1, \ldots, N$, of sp$(2N, \mathbb{R})$ with relations (53), it is easy to see that an explicit isomorphism between sp$(2N, \mathbb{R})$ and the second-order semiclassical truncation with $N$ classical degrees of freedom is given by

$$A_{ij} = \Delta(\pi_i \pi_j), \quad B_{ij} = \Delta(q_i q_j), \quad C_{ij} = \Delta(q_i p_j).$$  \hfill (146)

In particular, for sp$(4, \mathbb{R})$ we obtain a realization from (93)–(102) with four bosonic variables $b_1 = \frac{1}{\sqrt{2}} (s_1 + ip_1)$, $b_2 = \frac{1}{\sqrt{2}} (s_2 + ip_2)$, $b_3 = \frac{1}{\sqrt{2}} (s_3 + ip_3)$ and $b_4 = \frac{1}{\sqrt{2}} (s_4 + ip_4)$, in addition to two Casimir variables $U_1$ and $U_2$. We do not reproduce here all generators
obtained by substituting bosonic variables in (93)–(102), but note that the resulting expressions are rather different from the non-faithful form (58). Even the moments that are bilinear in bosonic variables, such as \( B_{11} = s_1^2 = \frac{1}{2}(b_1 + b_1^\ast)^2 \) or \( C_{11} = s_1 p_1 = \frac{i}{2}((b_1^\ast)^2 - b_1^2) \), depend on different combinations of the \( b_i \). These changes are required to maintain the reality conditions implied by a bosonic realization. Moreover, our realization brings in the two Casimir variables \( U_1 \) and \( U_2 \) in a way that requires a non-bilinear realization.

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