GROUPS GENERATED BY TWO ELLIPTIC ELEMENTS IN PU(2, 1)

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Abstract. Let $f$ and $g$ be two elliptic elements in $\text{PU}(2, 1)$ of order $m$ and $n$ respectively, where $m \geq n > 2$. We prove that if the distance $\delta(f, g)$ between the complex lines or points fixed by $f$ and $g$ is larger than a certain number, then the group $\langle f, g \rangle$ is discrete nonelementary and isomorphic to the free product $\mathbb{Z}_m \ast \mathbb{Z}_n$.

1. Introduction

A subgroup of Fuchsian groups or Kleinian groups generated by two elements was studied by many authors. An interesting question is to explore the conditions for two elements in Fuchsian groups or Kleinian groups to generate a discrete free group. In [8], Knapp found necessary and sufficient conditions for two elliptic transformations to generate a discontinuous subgroup of $Lf(2, \mathbb{R})$, the group of linear fractional transformations. Lyndon and Ullman [15] gave conditions for two hyperbolic transformations whose fixed points separate each other to generate a discrete free group of rank 2. In general, Purzitsky [12] found necessary and sufficient conditions for the subgroups generated by any pair $A, B \in Lf(2, \mathbb{R})$ to be the discrete free product of the cyclic groups $\langle A \rangle$ and $\langle B \rangle$.

The following Theorem 1 is well known in real hyperbolic geometry. It is essentially contained in [8].

**Theorem 1.** Suppose that $f$ and $g$ are elliptic elements of $\text{PSL}(2, \mathbb{R})$ of order $m$ and $n$. Let $\delta(f, g)$ be the distance between the fixed points of $f$ and $g$. If

$$\cosh \delta(f, g) > \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

then $\langle f, g \rangle$ is discrete and isomorphic to the free product $\langle f \rangle \ast \langle g \rangle$.

F. W Gehring, C. Maclachlan and G. J. Martin proved a similar result in the case of Kleinian groups.

**Theorem 2.** [2] Suppose that $f$ and $g$ are elliptic elements of $\text{PSL}(2, \mathbb{C})$ of order $m$ and $n$. Let $\delta(f, g)$ be the distance between the axes of $f$ and $g$. If

$$\cosh \delta(f, g) > \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

then $\langle f, g \rangle$ is discrete and isomorphic to the free product $\langle f \rangle \ast \langle g \rangle$.

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In this paper, the principal problem we wish to consider is that of giving condition in terms of transformations in complex hyperbolic 2-space for the free product of a cyclic group of order $p$ and a cyclic group of order $q$. We prove the following result.

**Theorem 3.** Let $f, g \in \text{PU}(2,1)$ be elliptic elements of order $m$ and $n$. Let $f$ and $g$ be in one of the following cases

1. $f$ and $g$ are reflections in complex lines;
2. $f$ is reflection in a complex line and $g$ is reflection in a point;
3. $f$ and $g$ are reflections in points.

Let $\delta(f, g)$ be the distance between the complex lines or points fixed by $f$ and $g$. Then

$$\cosh \delta(f, g) > \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}}$$

will imply that $(f, g)$ is discrete and isomorphic to the free product $(f) \ast (g)$.

The pattern of this result is very similar to the analogous results in real hyperbolic space. A possible application of Theorem 3 is in the study of complex hyperbolic triangle groups, see for example A. Pratoussevitch [11], R. E. Schwartz [14].

**2. Complex Hyperbolic Space**

First, we recall some terminology. More details can be found in [1, 3, 4, 6, 7]. Let $\mathbb{C}^{2,1}$ denote the complex vector space of dimension 3, equipped with a non-degenerate Hermitian form of signature (2,1). There are several such forms. We use the following form, called the second Hermitian form

$$\langle z, w \rangle = w^* J z$$

where $z, w$ are column vectors in $\mathbb{C}^{2,1}$, the Hermitian transpose is denote by $*$ and $J$ is the Hermitian matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Consider the following subsets of $\mathbb{C}^{2,1}$

$$V_+ = \{v \in \mathbb{C}^{2,1} | \langle v, v \rangle > 0 \},$$

$$V_- = \{v \in \mathbb{C}^{2,1} | \langle v, v \rangle < 0 \},$$

$$V_0 = \{v \in \mathbb{C}^{2,1} | \langle v, v \rangle = 0 \}.$$

Let $P : \mathbb{C}^{2,1} - \{0\} \rightarrow \mathbb{CP}^{2,1}$ be the canonical projection onto complex projective space. Then $H^2_C = P(V_-)$ associated with the Bergman metric is complex hyperbolic space. The biholomorphic isometry group of $H^2_C$ is $\text{PU}(2,1)$ acting by linear projective transformations. Here $\text{PU}(2,1)$ is the projective unitary group with respect to the Hermitian form defining on $\mathbb{C}^{2,1}$. In other words, for all $z$ and $w$ in $\mathbb{C}^{2,1}$ we have

$$w^* J z = \langle z, w \rangle = \langle Bz, Bw \rangle = w^* B^* J Bz.$$
Let \( z \) and \( w \) vary over a basis for \( \mathbb{C}^{2,1} \), we see that \( B^{-1} = JB^*J \). This means that the inverse of \( B \in \text{PU}(2, 1) \) has the following form:

\[
B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{j}{\pi} & \frac{f}{\pi} & \frac{c}{\pi} \\ \frac{h}{\pi} & \frac{e}{\pi} & \frac{b}{\pi} \\ \frac{g}{\pi} & \frac{d}{\pi} & \frac{a}{\pi} \end{bmatrix}.
\]

We define the Siegel domain model of the complex hyperbolic 2-space, \( \mathcal{H}^2_\mathbb{C} \) as follows. We identify points of \( \mathcal{H}^2_\mathbb{C} \) with their horospherical coordinate, \( z = (\xi, \nu, \mu) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ = \mathcal{H}^2_\mathbb{C} \). Similarly, points in \( \partial\mathcal{H}^2_\mathbb{C} = \mathbb{C} \times \mathbb{R} \times \{\infty\} \) are either \( z = (\xi, \nu, 0) \in \mathbb{C} \times \mathbb{R} \times \{0\} \) or a point at infinity, denoted \( q_\infty \). Define the map \( \psi : \mathcal{H}^2_\mathbb{C} \rightarrow \text{PC}^{2,1} \) by

\[
\psi : (\xi, \nu, \mu) \mapsto \begin{bmatrix} -|\xi|^2 - \mu + i\nu \\ \sqrt{2}\xi \\ 1 \end{bmatrix} \text{ for } (\xi, \nu, \mu) \in \mathcal{H}^2_\mathbb{C} - q_\infty,
\]

and

\[
\psi : q_\infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

The map \( \psi \) is a homeomorphism from \( \mathcal{H}^2_\mathbb{C} \) to the set of points \( z \) in \( \text{PC}^{2,1} \) with \( \langle z, z \rangle < 0 \). Also \( \psi \) is a homeomorphism from \( \partial\mathcal{H}^2_\mathbb{C} \) to the set of points \( z \) with \( \langle z, z \rangle = 0 \). Let \( L \) be a complex line intersecting \( \mathcal{H}^2_\mathbb{C} \). Then \( \psi(L) \) is a two-dimensional complex linear subspace of \( \mathbb{C}^{2,1} \). The orthogonal complement of this subspace is a one (complex)-dimensional subspace of \( \mathbb{C}^{2,1} \) spanned by a vector \( p \) with \( \langle p, p \rangle > 0 \). Without loss of generality, we take \( \langle p, p \rangle = 1 \) and call \( p \) the polar vector corresponding to the complex line \( L \) (see page 75 of [3]). The Bergman metric on \( \mathcal{H}^2_\mathbb{C} \) is defined by the following formula for distance \( \rho \) between points \( z \) and \( w \) of \( \mathbb{C}^{2,1} \)

\[
cosh \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle \psi(z), \psi(w) \rangle \langle \psi(w), \psi(z) \rangle}{\langle \psi(z), \psi(z) \rangle \langle \psi(w), \psi(w) \rangle}.
\]

As in real hyperbolic geometry, \( A \) holomorphic complex hyperbolic isometry \( g \) is said to be:

(i) loxodromic if it fixes no point in \( \mathcal{H}^2_\mathbb{C} \) but exactly two points of \( \partial\mathcal{H}^2_\mathbb{C} \);
(ii) parabolic if it fixes no point in \( \mathcal{H}^2_\mathbb{C} \) but exactly one point of \( \partial\mathcal{H}^2_\mathbb{C} \);
(iii) elliptic if it fixes at least one point of \( \mathcal{H}^2_\mathbb{C} \).

The matrices corresponding to a loxodromic element and a parabolic element can be found in [10]. We will only give some matrices corresponding to the elliptic elements in this paper. If \( A \) is an elliptic element, then there are now three cases. First, suppose that \( A \) has a repeated eigenvalue with a two dimensional eigenspace containing both positive and negative vectors. This eigenspace corresponds to a complex line \( L \) on which \( A \) acts as the identity. In particular, there are points of \( \partial\mathcal{H}^2_\mathbb{C} \) fixed by \( A \) and so \( A \) is called boundary elliptic. As \( A \) fixes \( L \) and rotates \( \mathcal{H}^2_\mathbb{C} \) around \( L \), it is complex reflection in the line \( L \). If \( A \) is not boundary elliptic, then it has an eigenspace spanned by a negative vector \( w \). This corresponds to a fixed point \( w \in \mathcal{H}^2_\mathbb{C} \). In this case \( A \) is called regular elliptic. There are two possibilities. Either \( A \) has a repeated eigenvalue with an eigenspace spanned by two positive vectors. In this case \( A \) is complex reflection in the point \( w \). Otherwise, \( A \) has three distinct eigenvalues.
Proposition 2.1. (1) Every boundary elliptic element in $H^2_C$ is conjugate to
\[
\begin{bmatrix}
    u^{-1/3} & 0 & 0 \\
    0 & u^{2/3} & 0 \\
    0 & 0 & u^{-1/3}
\end{bmatrix},
\]
where $u = e^{i\theta}$.

(2) Every regular elliptic element in $H^2_C$ is conjugate to
\[
\begin{bmatrix}
    (u + w)/2 & (u - w)/2 & 0 \\
    0 & v & 0 \\
    (u - w)/2 & 0 & (u + w)/2
\end{bmatrix},
\]
where $w = e^{2i\theta/3}$ and $u = v = e^{-i\theta/3}$.

Suppose that $A \in SU(2,1)$ is an elliptic element. We define the order of $A$ as
\[
\text{order}(A) = \inf \{ m > 0, A^m = I \}.
\]

As in the case of real hyperbolic geometry, a discrete subgroup of $SU(2,1)$ can not contain elliptic elements of infinite order.

3. The Heisenberg group

Just as the boundary of real hyperbolic space may be identified with the one point compactification of Euclidean space, so the boundary of complex hyperbolic space may be identified with one point compactification of the Heisenberg group. We now collect some of the basic facts about the Heisenberg group that will be used later.

Consider the 3 dimensional Heisenberg group $\mathfrak{H}$ which is the set $\mathbb{C} \times \mathbb{R}$ (with coordinates $(\xi, \nu)$) endowed with the multiplication law
\[
(\xi_1, \nu_1) \circ (\xi_2, \nu_2) = (\xi_1 + \xi_2, \nu_1 + \nu_2 + 2\mathfrak{I}\langle\xi_1, \xi_2\rangle),
\]
where $\langle\cdot, \cdot\rangle$ is the standard positive definite Hermitian form on $\mathbb{C}$. The Heisenberg norm assigns to $(\xi, \nu)$ the non-negative real number
\[
|\langle\xi, \nu\rangle|_0 = (||\xi||^4 + \nu^2)^{1/2} = ||\xi||^2 - i\nu\frac{1}{2}
\]
where $||\xi||^2 = \langle\xi, \xi\rangle = \sum |\xi_i|^2$. This enables us to define the Cygan metric on the Heisenberg group:
\[
\rho_0((\xi_1, \nu_1), (\xi_2, \nu_2)) = |\langle\xi_1 - \xi_2, \nu_1 - \nu_2 + 2\mathfrak{I}\langle\xi_1, \xi_2\rangle\rangle|_0 = |\langle\xi_1, \nu_1\rangle^{-1} \circ (\xi_2, \nu_2)|_0.
\]

The Heisenberg group acts on itself by Heisenberg translation. For $(\xi_0, \nu_0) \in \mathfrak{H}$, this is
\[
T_{\xi_0, \nu_0} : (\xi, \nu) \mapsto (\xi + \xi_0, \nu + \nu_0 + 2\mathfrak{I}\langle\xi_0, \xi\rangle) = (\xi_0, \nu_0) \circ (\xi, \nu).
\]

Heisenberg group translation by $(0', \nu_0)$ where $0'$ is origin in $\mathbb{C}$ and $\nu_0 \in \mathbb{R}$ are called vertical translations.
4. The Ford isometric spheres

In [3] Goldman had extended the definition of isometric spheres of Möbius transformations acting on the upper half space to the Ford isometric spheres of complex hyperbolic transformations of the Siegel domain. These spheres and their associated geometric properties have been extensively used in [3, 5, 9, 10].

Let $q_\infty = (1, 0, 0) \in \mathbb{C}^2$.

**Definition 4.1.** [9] Let $X \in \text{PU}(2, 1)$. Suppose that $X$ does not fix $q_\infty$. Then the isometric sphere of $X$ is the hypersurface

$$I_X = \{ z \in \mathbb{H}_C^2 : |\langle Z, q_\infty \rangle| = |\langle Z, X^{-1}(q_\infty) \rangle| \}.$$ 

for any $Z \in \mathbb{C}^3$ which maps onto $z$ projectively.

As in real case, $X$ maps $I_X$ to $I_X^{-1}$ and $X$ maps the component of $\overline{\mathbb{H}}_C^2 \setminus I_X$ containing $q_\infty$ to the component of $\overline{\mathbb{H}}_C^2 \setminus I_X^{-1}$ not containing $q_\infty$.

**Proposition 4.1.** [9] Let $X \in \text{PU}(2, 1)$ such that $X(q_\infty) \neq q_\infty$. Then the isometric sphere is the sphere for Cygan metric $\rho_0$ with center at $X^{-1}(q_\infty)$ and radius $r_X = \sqrt{1/|g|}$.

5. Proof of the Theorem 3

In this section, we prove Theorem 3. The basic structure of this proof resembles the original proof of [2]. On the first hand, we give a proof of part one of Theorem 3.

Suppose that boundary elliptic element $A \in \text{PU}(2, 1)$ fixes 0 and $\infty$. This means that complex line $L_A$ fixed by $A$ is spanned by 0 and $\infty$. In other words

$$p_A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$ 

Assume that $m \geq n$, let $f$ and $g$ be boundary elliptic elements in $\text{PU}(2, 1)$ of order $m$ and $n$ respectively, that is, $f$ and $g$ are reflections in complex lines and set

$$\delta = \delta(f, g)$$

and $\omega^2 = e^{\delta+i\phi}$, where $\delta$ is the distance between the complex lines fixed by $f$ and $g$ respectively.

The statement is invariant with respect to conjugation by elements in $\text{PU}(2, 1)$. Thus by means of conjugation we may choose some matrices form of $f$ and $g$ for the convenience of our calculations.

We begin with the following two elements in $\text{SU}(2, 1)$

$$U_1 = \begin{bmatrix} u_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & u_1^{\frac{1}{2}} & 0 \\ 0 & 0 & u_1^{-\frac{1}{2}} \end{bmatrix}$$

and

$$U_2 = \begin{bmatrix} u_2^{-\frac{1}{2}} & 0 & 0 \\ 0 & u_2^{\frac{1}{2}} & 0 \\ 0 & 0 & u_2^{-\frac{1}{2}} \end{bmatrix}$$
where \( u_1 = e^{\frac{2\pi i}{m}} \), \( u_2 = e^{\frac{2\pi i}{n}} \).

Obviously, \( U_1 \) and \( U_2 \) are boundary elliptic elements of order \( m \) and \( n \). Moreover, \( U_1 \) and \( U_2 \) fix the same complex line with polar vector

\[
p = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Now suppose that \( A \) and \( B \) in \( SU(2, 1) \) have the following forms

\[
A = \begin{bmatrix} \sqrt{\omega/2} & \sqrt{\omega/2} & \sqrt{\omega} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2\sqrt{\omega} & 1/\sqrt{2\omega} & -1/2\sqrt{\omega} \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} -1/2\sqrt{\omega} & 1/\sqrt{2\omega} & 1/2\sqrt{\omega} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ \sqrt{\omega}/2 & \sqrt{\omega}/2 & -\sqrt{\omega}/2 \end{bmatrix}.
\]

We define the matrices representative \( F \) and \( G \) of \( f \) and \( g \) as follows

\[
F = AU_1A^{-1}, \quad G = BU_2B^{-1}.
\]

Elementary calculations show that

\[
F = \begin{bmatrix} \frac{1}{2}(u_1^{-\frac{1}{2}} + u_1^{\frac{1}{2}}) & 0 & \frac{|\omega|}{2}(u_1^{\frac{1}{2}} - u_1^{-\frac{1}{2}}) \\ 0 & u_1^{\frac{1}{2}} & 0 \\ \frac{|\omega|}{2}(u_1^{\frac{1}{2}} - u_1^{-\frac{1}{2}}) & 0 & \frac{1}{2}(u_1^{-\frac{1}{2}} + u_1^{\frac{1}{2}}) \end{bmatrix}
\]

and

\[
G = \begin{bmatrix} \frac{1}{2}(u_2^{-\frac{1}{2}} + u_2^{\frac{1}{2}}) & 0 & \frac{|\omega|}{2}(u_2^{\frac{1}{2}} - u_2^{-\frac{1}{2}}) \\ 0 & u_2^{-\frac{1}{2}} & 0 \\ \frac{|\omega|}{2}(u_2^{\frac{1}{2}} - u_2^{-\frac{1}{2}}) & 0 & \frac{1}{2}(u_2^{-\frac{1}{2}} + u_2^{\frac{1}{2}}) \end{bmatrix}.
\]

We can see that \( f \) and \( g \) are boundary elliptic elements of order \( m \) and \( n \). The complex line fixed by \( f \) has the polar vector \( p_f = (\sqrt{\omega/2}, 0, 1/\sqrt{2\omega})^T \) and the complex line fixed by \( g \) has polar vector \( p_g = (\sqrt{\omega/2}, 0, 1/\sqrt{2\omega})^T \).

**Proposition 5.1.** Let \( f \) and \( g \) be boundary elliptic elements in \( PU(2, 1) \) having the matrices of above. Then the complex lines fixed by \( f \) and \( g \) with polar vectors \( p_f \) and \( p_g \) has distance \( \delta \).

**Proof:** Let the complex line \( C_f \) fixed by \( f \) with polar vector \( p_f \) and the complex line \( C_g \) fixed by \( g \) with polar vector \( p_g \). Then by distance formulas in [13] we have

\[
\text{dist}(C_f, C_g) = 2 \cosh^{-1}(|p_f, p_g|) = 2 \cosh^{-1}(\frac{\sqrt{\omega}}{\sqrt{2}} + \frac{\sqrt{\omega}}{\sqrt{2}}) = 2 \cosh^{-1}(\frac{\sqrt{\omega}}{\sqrt{2}} + \frac{1}{\sqrt{2\omega}}) = 2 \cosh^{-1}(\frac{1}{2}(\omega + 1)) = \delta.
\]
Suppose that $A \in \text{SU}(2,1)$ does not fix $q_{\infty}$, which is equivalent to requiring that $g$ be non-zero when $A$ has the form (1). Then the isometric sphere of $A$ is the sphere in the Cygan metric with center $A^{-1}(\infty)$ and radius $r_A = \frac{1}{\sqrt{|g|}}$. In Heisenberg coordinates

$$A^{-1}(\infty) = \left(\frac{h}{\sqrt{2g}}, -\frac{3}{g}\right).$$

Similarly the isometric sphere of $A^{-1}$ is the Cygan sphere of radius $\frac{1}{\sqrt{|g|}}$ with center

$$A(\infty) = \left(-\frac{d}{\sqrt{2g}}, \frac{3}{g}\right).$$

The isometric spheres of $f$ and $g$ are easily calculated from their matrices representative $F$ and $G$. Using $u_1 = e^{2\pi i / m}$ we have $u_1^{2/3} - u_1^{-1/3} = 2e^{\pi i / 6m} \sin(\pi / m)$ and $u_1^{2/3} + u_1^{-1/3} = 2e^{\pi i / 6m} \cos(\pi / m)$. The Cygan isometric sphere $I_f$ of $f$ has radius

$$r_f = \frac{1}{|\frac{1}{2|\omega|}(u_1^{\frac{2}{3}} - u_1^{-\frac{1}{3}})|} = \sqrt{\frac{|\omega|}{\sin(\pi / m)}}$$

and center

$$\left(0, -\frac{3}{2|\omega|}(u_1^{\frac{2}{3}} + u_1^{-\frac{1}{3}})\right) = \left(0, \frac{|\omega| \cos(\pi / m)}{\sin(\pi / m)}\right).$$

Similarly the Cygan isometric sphere $I_{f^{-1}}$ of $f^{-1}$ has radius $r_{f^{-1}} = r_f$ and center

$$\left(0, -\frac{|\omega| \cos(\pi / m)}{\sin(\pi / m)}\right).$$

The Cygan isometric spheres $I_g$ and $I_{g^{-1}}$ of $g$ have radius

$$r_g = r_{g^{-1}} = \frac{1}{\sqrt{|\omega|\left(u_2^{\frac{2}{3}} - u_2^{-\frac{1}{3}}\right)}} = \sqrt{\frac{1}{|\omega| \sin(\pi / n)}}.$$ 

and the centers of $I_g$ and $I_{g^{-1}}$ are

$$\left(0, \frac{\cos(\pi / n)}{|\omega| \sin(\pi / n)}\right), \quad \left(0, -\frac{\cos(\pi / n)}{|\omega| \sin(\pi / n)}\right)$$

respectively.

The fundamental domain for the action of $f$ on the Heisenberg group is the exterior of these two spheres $I_f$ and $I_{f^{-1}}$ together with the region bounded by their intersection.

We observe that the fundamental domain of $f$ contains the Heisenberg sphere $S_f^*$ with center $(0,0)$ and radius

$$r_f^* = \sqrt{\frac{|\omega|}{\sin(\pi / m)}(1 - \cos(\pi / m))}.$$
The Cygan isometric spheres $I_g$ and $I_{g^{-1}}$ of $g$ are contained in the Heisenberg sphere $S_g^*$ with center $(0,0)$ and radius

$$r_g^* = \sqrt{\frac{1}{|\omega| \sin(\pi/n)} (1 + \cos(\pi/n))}.$$

The interiors of $I_g$ and $I_{g^{-1}}$ are contained in the interiors of $I_f$ and $I_{f^{-1}}$ if $r_g^* \leq r_f^*$. That is

$$|\omega|^2 = e^\delta \geq \frac{\sin(\pi/m)}{1 - \cos(\pi/m)} \frac{1 + \cos(\pi/n)}{\sin(\pi/n)}.$$

Using

$$\frac{\sin \theta}{1 - \cos \theta} = \frac{1 + \cos \theta}{\sin \theta},$$

This translates into

$$\cosh(\delta) = \frac{|\omega|^2 + |\omega|^{-2}}{2} \geq \frac{\cos \frac{\pi}{n} \cos \frac{\pi}{m} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}}.$$

We have therefore seen that the exterior of a fundamental domain for $(g)$ lies inside a fundamental domain for $(f)$. It follows from the simplest version of Kleinian-Maskit combination theorem that the group $(f,g)$ is discrete and isomorphic to the free product of cyclic groups,

$$(f,g) \cong \langle f \rangle * \langle g \rangle \cong \mathbb{Z}_m * \mathbb{Z}_n.$$

It is straightforward to extend the main result to the case where either or both of $f$ and $g$ are complex reflection in a point. If $f$ and $g$ are complex reflections in a point then the expressions for $F$ and $G$ on the above become

$$F = \begin{bmatrix} \frac{1}{2} (u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}) & 0 & \frac{|\omega|}{2} (u_1^{-\frac{1}{3}} - u_1^{\frac{2}{3}}) \\ 0 & u_1^{\frac{1}{3}} & 0 \\ \frac{1}{2|\omega|} (u_1^{-\frac{1}{3}} - u_1^{\frac{2}{3}}) & 0 & \frac{1}{2} (u_1^{-\frac{1}{3}} + u_1^{\frac{2}{3}}) \end{bmatrix},$$

which fixes $p_f = (\sqrt{\omega}/2, 0, 1/\sqrt{2\omega})^T$ and

$$G = \begin{bmatrix} \frac{1}{2} (u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}) & 0 & -\frac{1}{2|\omega|} (u_2^{\frac{1}{3}} - u_2^{-\frac{2}{3}}) \\ 0 & u_2^{\frac{1}{3}} & 0 \\ \frac{|\omega|}{2} (u_2^{-\frac{1}{3}} - u_2^{\frac{2}{3}}) & 0 & \frac{1}{2} (u_2^{-\frac{1}{3}} + u_2^{\frac{2}{3}}) \end{bmatrix},$$

which fixes $p_g = (-1/\sqrt{2\omega}, 0, \sqrt{\omega}/2)^T$.

The distance between the fixed points or lines may be calculated as in [13]. Namely, when one of $p_f$ and $p_g$ is in $V_+$ and the other in $V_-$ (that is one of $f$ and $g$ is complex reflection in a point and the other is complex reflection in a complex line) then the distance between this point and complex line is $\delta(f,g)$ where

$$\sinh^2 \left( \frac{\delta(f,g)}{2} \right) = \frac{\langle p_f, p_g \rangle \langle p_g, p_f \rangle - \langle p_f, p_f \rangle \langle p_g, p_g \rangle}{\langle p_f, p_f \rangle \langle p_g, p_g \rangle} = |\omega|/2 - 1/2\omega^2.$$

Similarly, when $p_f$ and $p_g$ are both in $V_-$, so $f$ and $g$ each are complex reflection in a point then the distance between these points is $\delta(f,g)$ where
cosh^2 \left( \frac{\delta(f,g)}{2} \right) = \frac{\langle pf, pg \rangle \langle pg, pf \rangle}{\langle pf, pf \rangle \langle pg, pg \rangle} = |\omega/2 + 1/2|^{2}.

In either case

\[ \cosh^2 \left( \frac{\delta(f,g)}{2} \right) = \frac{|\omega|^2 + |\omega|^{-2}}{2}. \]

The same identity holds in the case where \( f \) and \( g \) fix complex lines and \( \delta(f,g) \) denotes the distance between these complex lines.

In each case the isometric spheres and fundamental domains are the same and so the other calculations go through with no changes.

Remark 5.1. The group generated by \( f \) and \( g \) preserves a (unique) complex line \( L \). The restriction of the Bergman metric to \( L \) is just the Poincaré metric and both \( f \) and \( g \) act on \( L \) as elliptic hyperbolic isometries. Theorem 3 is a natural generalisation of the result for real hyperbolic space of dimensions 2 and 3.

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