Dynamically Driven Renormalization Group Applied to Sandpile Models

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The general framework for the renormalization group analysis of self-organized critical sandpile models is formulated. The usual real space renormalization scheme for lattice models when applied to nonequilibrium dynamical models must be supplemented by feedback relations coming from the stationarity conditions. On the basis of these ideas the Dynamically Driven Renormalization Group is applied to describe the boundary and bulk critical behavior of sandpile models. A detailed description of the branching nature of sandpile avalanches is given in terms of the generating functions of the underlying branching process.

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I. INTRODUCTION

In the last decade it has been recognized that a fair amount of physical phenomena are characterized by strong fluctuations and long-range correlation functions. According to the theory of equilibrium statistical physics, we expect scale invariance only in presence of certain symmetries or at critical points[1]. We are therefore lead to seek for the origin of the scale invariance in nature in the rich domain of nonequilibrium systems[2,3,5,6]. One might hope, in fact, that there are classes of nonequilibrium systems that generate scale invariance for a wide (and arbitrary) range of physical parameter, providing an explanation for the commonly observed scaling laws.

Pursuing this aim, Bak, Tang and Wiesenfeld proposed the concept of Self-Organized Criticality (SOC)[7] as a unifying framework to describe a vast class of dynamically driven systems which evolve spontaneously in a stationary state with a broad power law distribution of discrete energy dissipating events. To illustrate the basic ideas of SOC, they introduced a cellular automaton model of sandpiles. In this model, criticality seems to emerge automatically if the system is driven at an infinitesimal rate[8,9]. Because of the enormous conceptual potentiality, SOC ideas have reverberated rapidly throughout the sciences, from geophysics to economics and biology, as a prototype mechanism to understand the manifestation of scale invariance and complexity in natural phenomena.

The major source of difficulties in the study of sandpiles models is the absence of a general criterion, like the use of the Gibbs distribution in equilibrium systems, to assign an ensemble statistical measure to a particular configuration of the system. This problem is common to many nonequilibrium systems whose theoretical understanding lies far behind the equilibrium theory. In particular, many relations among sandpiles automata and systems with nonequilibrium absorbing critical point have been recently enlightened[10].

In the past years, we developed a renormalization group (RG) strategy for sandpile models[11] which has also been applied[12] to forest-fire models[13]. This approach deals with the critical properties of the system by introducing in the renormalization equations a dynamical steady state condition which provides nonequilibrium stationary statistical weights to be used in the calculation. This scheme, named the Dynamically Driven Renormalization Group (DDRG) [14], has been successively generalized as a renormalization framework for systems with a non-equilibrium critical steady-state. Recently, the DDRG approach has been improved including higher order proliferations through a general scheme[15]. The method has also been applied to one dimensional sandpiles[16] and other non-equilibrium systems[17].

Here we discuss the application of the DDRG to sandpile models, deriving systematically the previous RG schemes[11,13] and presenting new results. We will introduce the general strategy of the DDRG for the critical height sandpile models, and its practical implementation for increasingly complex proliferation schemes. In order to treat such a high level of calculation complexity, we introduce a generating function for the basic recursion relations. The scheme is then extended by exploiting the analogy with a particular chain chemical reaction. Finally its application to the calculation of the boundary critical behavior is shown.

The paper is organized as follows: in Sec.2 we introduce the class of sandpile automata and its mapping into a general nonequilibrium cellular automaton (CA). Sec.3 presents the Dynamically Driven Renormalization Group general scheme. Sec.4 shows the explicit application of the DDRG to the sandpile in its simple scheme. In Sec.5 we present the actual calculations of the renormalization equations and their generating function and results obtained. Sec.6 describes the extended chemical reaction
scheme and its results. Sec. 7 is devoted to the renormalization analysis of the boundary critical behavior. Sec. 8 presents the summary and conclusions.

II. THE SANDPILE MODEL

The prototype example for SOC is provided by sandpiles: sand is added grain by grain until unstable sand (too large local slope of the pile) slides off. In this way the pile reaches a steady-state, in which additional sand grains fall off the pile by avalanche events. The steady-state is critical since avalanches of any size are observed. This class of models can be used to describe a generic avalanche phenomenon, interpreting the sand as energy, mechanical stress or heat memory.

Sandpile models are cellular automata \[17\] defined in a \(d\)-dimensional lattice. A discrete or continuous variable \(E(i)\), that we denote by energy, is associated with each lattice site \(i\). At each time step an input energy \(\delta E\) is added to a randomly chosen site. When the energy on a site reaches a threshold value \(E_c\), the site relaxes transferring energy to the neighboring sites

\[
E(i) \to E(i) - \sum_e \Delta E(e)
\]

where \(e\) represents the unit vectors on the lattice. A typical choice for the parameters is, for example, \(E_c = 4\) and \(\Delta E(e) = \delta E = 1\), but other possibilities have also been considered. The relaxation of the first site can induce a series of relaxations generating an avalanche. Note that the energy is added to the system only when the configuration is stable (i.e. all the sites are below the threshold). The boundary conditions are usually chosen to be open so that energy can leave the system. In these conditions the system organizes itself into a stationary state characterized by avalanches of all length scales. In particular, the distribution for avalanches sizes \(s\) decays as a power law \(P(s) \sim s^{-\gamma}\), and the linear size of the avalanche scales with time \(r \sim t^\tau\). This model has been extensively studied in the past by means of numerical simulations \[18\, 21\] and several exact results have been derived for abelian sandpiles models (ASM) \[22\, 26\].

Given the above definition of sandpiles we can rephrase them in the language of discrete nonequilibrium probabilistic CA. To each site \(i\) is associated a variable \(s_i\) that can assume \(q\) different values (\(s_i = 1, 2, 3, \ldots, q\)). For instance, each state might correspond to an allowed energy level. The subscript \(i\) labels the lattice site. A complete set \(s\) of lattice variables specifies a configuration of the system. We define \(s\) as a configuration \(s_0\) to a configuration \(s\) in a time step \(\tau\) as a function of a set of parameters \(\mu = \{\mu_i\}\). SOC automata are usually defined by a transition probability given by the product

\[
\langle s \mid T \mid s_0 \rangle = \prod_{i=1}^{N} p(s_i \mid s_0, \{s_{i+\epsilon}^0\})
\]

where \(N = L^d\) is the number of sites, and \(\epsilon\) specifies the nearest neighbor (n.n.) vector. The dynamics is therefore expressed as a product of one-site transition probabilities depending upon the sites and its nearest-neighbors states at the previous time step.

As we said, the common characteristic of SOC systems is the presence of a nonequilibrium critical steady-state, which we can analyze using the DDRG formalism. However, it is worth remarking that SOC systems reach true criticality just in the limit of an infinite slow driving condition. This means that the perturbing time scale is much larger than the dynamical activity one. SOC systems relax far more rapidly than they are perturbed. In practice, this implies that no new grain of sand is dropped until the avalanches started by the previous grain has finished. In this way avalanches cannot overlap, and their dynamics is well defined with respect to the external field. A complete RG analysis should take into account also the driving field. However, since we are interested in the critical point, we will study the system in the limit of slow driving. A more detailed discussion of the complete sandpile automaton phase diagram is provided in ref. \[17\].

III. THE DDRG

The probability distribution of CA such as those shown in the previous section obeys the following master equation (ME)

\[
P(s, t_0 + t) = \sum_{\{s'\}} \langle s \mid T(\mu) \mid s_0 \rangle P(s_0, t_0).
\]

The explicit solution of the master equation is not in general available but we can extract the critical properties of the model by a renormalization group analysis. We coarse grain the system by rescaling lengths and time according to the transformation \(x \to b^{-1}x\) and \(t \to b^{-2}t\). The renormalization transformation is constructed through the operator \(R(S, s)\) that introduces a set of coarse grained variables \(S \equiv \{S_i\}\) and rescales the lengths of the system \[22\]. In general, \(R\) is a projection operator with the properties \(R(S, s) \geq 0\) for any \(\{S_i\}, \{s_i\}\), and \(\sum\{S\} R(S, s) = 1\). These properties preserve the normalization condition of the renormalized distribution. The explicit form of the operator \(R\) is defined case by case in various applications of the method. Usually, it corresponds to a block transformation in which lattice sites are grouped together in a super-site that defines the renormalized variables \(S_i\) by means of a majority or a spanning rule.

We subdivide the time step in intervals of the unitary time scale \((t_0 = 0)\) obtaining the coarse graining of the system as follows:
\[ P'(S, t') = \sum_{\{s\}} R(S, s) \sum_{\{s^0\}} \langle s \mid T^{b^*} (\mu) \mid s^0 \rangle P(s^0, 0) \]  

where we have included the application of the operator \( R \) and \( t' = b^* t \). The meaning of \( \langle s \mid T^{b^*} (\mu) \mid s^0 \rangle \) has to be defined explicitly: the simplest possibility is \( b^* = N \) where \( N \) is an integer number, and \( T^N \) denotes the application of the dynamical operator \( N \) times. In general, since we are dealing with a discrete time evolution, we have to consider \( T^{b^*} \) as a convolution over different paths, chosen by an appropriate condition. The detailed definition of the effective operator \( T^{b^*} \) for the sandpile is reported in the next section. By multiplying and dividing each term of Eq. (5) by \( P'(s^0, 0) = \sum_{\{s^0\}} R(S^0, s^0) P(s^0, 0) \) and using the properties of the operator \( R \), after some algebra we get

\[ P'(S, t') = \sum_{\{s^0\}} \left( \frac{\sum_{\{s\}} \sum_{\{s^0\}} R(S^0, s^0) R(S, s) \langle s \mid T^{b^*} (\mu) \mid s^0 \rangle P(s^0, 0)}{\sum_{\{s^0\}} R(S^0, s^0) P(s^0, 0)} \right) P'(s^0, 0) \]  

which finally identifies the renormalized dynamical operator \( \langle S \mid T^{\prime} \mid S^0 \rangle \). In other words, the new dynamical operator \( T' \) is the sum over all the dynamical paths of \( b^* \) steps that from a starting configuration \( \{ s_i^0 \} \) lead to a configuration \( \{ s_i \} \) which renormalizes respectively in \( \{ S_i^0 \} \) and \( \{ S_i \} \). The sum is weighted by the normalized statistical distribution of each configuration. The scheme discussed so far is a general formulation valid for each system which exhibits a stationary state, and its application presupposes the knowledge of the explicit form of the steady-state distribution \( W(s) = P(s, t \to \infty) \). For instance, in equilibrium phenomena \( W(s) \) is given by the Gibbs distribution. In this case it is possible to apply several methods such as cumulant expansions and exact or approximate decimation to obtain the form of the recursion relations. For non-equilibrium dynamical systems, in general we do not know the form of the steady-state distribution. We will therefore develop an approximate method to evaluate the stationary distribution to be used in the calculation of the renormalized master equation.

The steady-state distribution can in general be split into two parts

\[ W(s) = W^{(i)}(s) + W^{(c)}(s) \]  

where \( W^{(i)}(s) \) and \( W^{(c)}(s) \) are, respectively, the incoherent and coherent part of the distribution. The incoherent part of the distribution does not include correlations among variables and expresses a mean field approximation for the system. The coherent part \( W^{(c)}(s) \) can be subdivided in parts describing different kinds of correlations: nearest-neighbors, next-nearest-neighbors, etc. The incoherent part is a factorized distribution that, for systems characterized by a q-state variables (see sect.2), has the form

\[ W^{(i)}(s) = \prod_i \langle \rho_{s_i} \rangle \]  

where \( \langle \rho_{s_i} \rangle \) is the average density of sites in the \( \kappa \)-state. In this way, we have approximated the probability of each configuration \( \{ s_i \} \) as a product measure of the mean field probability to have a state \( s_i \) in each corresponding site. The incoherent part contribution to the renormalization equation can be obtained by stationarity conditions for the system \( \mathcal{S}_\mu (\{ \langle \rho_{s_i} \rangle \}) = 0 \) to evaluate the densities \( \langle \rho_{s_i} \rangle \). These conditions are derived from dynamical mean field equations which describe the driving of the system to the nonequilibrium steady state by means of balance constraints. The operator \( \mathcal{S}_\mu \) depends upon the same dynamical parameters of the operator \( T \), and by solving the stationary condition equation, the average densities of the \( \kappa \)-states for the coarse grained system are obtained as a function of \( \mu \) at the corresponding iteration of the RG equations. By inserting the approximate distribution in Eq. (6), we thus get the following set of renormalization equations

\[ \langle S \mid T^{\prime} (\mu) \mid S^0 \rangle = \frac{\sum_{\{s^0\}} \sum_{\{s\}} R(S^0, s^0) R(S, s) \langle s \mid T^{b^*} (\mu) \mid s^0 \rangle \prod_i \langle \rho_{s_i} \rangle}{\sum_{\{s^0\}} R(S^0, s^0) \prod_i \langle \rho_{s_i} \rangle} \]  

where the second equation denotes the dynamical steady state condition that allows evaluation of the approximate stationary distribution at each coarse graining scale. We call Eq. (10) the driving condition, since it drives the RG equations acting as a feedback on the scale transformation. Eqs (5) and (10) are the basic renormalization equations from which the desired recursion relations
are derived. Imposing that the renormalized operator $T'$ has the same functional form of the operator $T$, i.e. $T'(\mu) = T(\mu')$, we obtain the rescaled parameter set $\mu' = f(\mu)$. This implies that the renormalized single time distribution $P'(S,t')$ has the same functional form of the original distribution $P(s,t)$. The critical behavior of the model is obtained by studying the fixed points $\mu^* = f(\mu^*)$. Since we are dealing with discrete evolution operators $T$, we define the time scaling factor $b^z$ as the average number of steps and apply the operator $T$ in order to obtain that $T'(\mu) = T(\mu')$ for the coarse grained system. In this way we obtain a time recursion relation $t' = g(\mu)t$, or equivalently $b^z = g(\mu)$, from which it is possible to calculate the dynamical critical exponent $z = \log g(\mu^*)/\log b$. In this form of the DDRG, we take into account only the uncorrelated part of the steady-state probability distribution. The results obtained are not trivial because correlations in the systems are considered in the dynamical renormalization of the operator $T$, that given a starting configuration traces all the possible paths leading to the renormalized final configuration. Moreover, geometrical correlations are treated by the operator $\mathcal{R}$ that maps the system by means of spanning conditions or majority rules. The renormalized uncorrelated part of the stationary distribution is evaluated from the stationary condition with renormalized parameters, thus providing an effective treatment of correlations. One can then improve the results by including higher order contributions to the unknown stationary distribution $W(s)$ using cluster variation methods [31].

IV. RENORMALIZATION SCHEME FOR SANDPILE MODELS

Here, we show in detail the DDRG scheme for sandpile models. For the sake of clarity, we start by considering the minimum proliferation scheme. A more refined scheme is discussed in the following sections.

To simplify the description of sandpile models as much as possible, we can reduce the number of states of each site in the following way. At any scale, we divide the sites in critical ($s_i = 1$) and stable ($s_i = 0$). Stable sites do not relax when energy is added to them. On the other hand, critical sites relax when they receive an energy grain $\delta E_{in}$. In this formalism we define $\langle \rho \rangle$ as the density of critical sites. For convenience, we also define unstable sites ($s_i = 2$), as those that are relaxing, even though they are not present in the static configurations of the system. These definitions can be extended to a generic scale $b$. For instance, a cell at scale $b$ is considered critical if the addition of energy $\delta E_{in}(b)$ induces a relaxation of the size of the cell (i.e. the avalanche spans the cell).

In a relaxation event at the minimal scale energy is equally distributed in the four directions. This is no longer the case at a coarse grained level where different possibilities arise: the energy in principle can be distributed to one, two, three or four neighbors. It is also worth remarking that in a certain case unstable sites at the coarse grained scale do not dissipate energy to nearest neighbors, representing just intra-site energy rearrangements. These processes define the probability that relaxation events take place on the renormalized scale without energy transfer. All these events occur with probabilities

$$\vec{P} = (p_0, p_1, p_2, p_3, p_4)$$

In terms of the matrix element $\langle 0|T|2 \rangle$ the vector $\vec{P}$ represents the probabilities

$$p_n = \langle 0|T|2 \rangle_n$$ (11)

where $\langle 0|T|2 \rangle_n$ is the probability that a relaxing site becomes stable and transfers energy to $n$ neighbors. In this way, we have obtained the set of parameters that describes the dynamics. Of course, the choice of the parameter space is not uniquely determined; one encounters proliferation problems typical of real space RG methods. For instance, higher order proliferations are due to multiple relaxations of the same site and sites becoming critical during the dynamical process (i.e.: $\langle 1|T|2 \rangle$). In the following, the practical implementation of the method considers just the minimal proliferation we have reported above. In the next sections, a more refined scheme will be treated.

First of all, let us show how the driving condition is obtained by imposing the stationarity of the process. The average energy of a site evolves according to the following equation written in the continuum notation:

$$\frac{dE(t)}{dt} = \delta E_{in} - \delta E_{out}$$ (12)

where $\delta E_{in}$ is the average energy entering into the site either because of relaxation in a neighboring site or because of the external perturbations, and $\delta E_{out}$ is the average energy dissipated by the site. The stationary state is characterized by the balance between the energy that goes in and the energy that goes out of the system. We assume that energy is transferred in “quanta” $\delta E = \delta E_{in}$ in each direction and we on average obtain

$$\delta E = \langle \rho \rangle \delta E \sum_n np_n$$ (13)

which implies

$$\langle \rho \rangle = \frac{1}{\sum_n np_n}$$ (14)

This relation gives the average density of critical sites in the steady-state, allowing us to evaluate the approximate stationary distribution at each scale.

The renormalized matrix element is then obtained by considering all the renormalized processes that span the cell and transfer energy outside
\[ p'_n = (S_i = 0 \mid T' \mid S_i^0 = 2)_n. \] (15)

We proceed in defining explicitly a renormalization procedure for the dynamics by considering a finite truncation on four-sites cells. This corresponds to a cell-to-site transformation on a square lattice, in which each cell at the coarser scale is formed by four sub-cells at the finer scale: the length scaling factor is \( b = 2 \). In this case, the operator \( \mathcal{R} \) can be written in the following way:

\[ \mathcal{R}(S, s) = \prod_J \mathcal{R}(S_J, \{s_i\}_J) \] (16)

where each term is acting on a specific cell \( J \) and \( \{s_i\}_J \) denotes the configurations of sites belonging to that cell. A cell is renormalized as a relaxing one if it contains a relaxing sub-cell which transfers energy to a critical sub-cell. In this way, we ensure that the occurring relaxation process is extending over the size of the renormalized length scale independently of the successive avalanche evolution. A critical cell is therefore defined by a cell which can be spanned by a path of relaxation events. The scheme considers only connected paths that span the cell from left to right or top to bottom. This spanning rule implies that only paths extending over the size of the resulting length scale contribute to the renormalized dynamics, and it ensures the connectivity properties of the avalanche in the renormalization procedure.

Every cell at the coarser scale can be characterized by an index \( \alpha \) that indicates the configuration of sub-cells, and we have that \( \sum_{\{s_i\}} \rightarrow \sum_{\alpha} \). The approximated stationary distribution (Eq. (8)) for each of these configurations is given by

\[ W_{\alpha}(\rho) = n_{\alpha} \prod_{i=1}^{4}(\rho_{s_i}) \] (17)

where \( n_{\alpha} \) is a factor due to the multiplicity of each configuration.

By using this scheme and replacing sums over configurations with sums over the index \( \alpha \), the recursion relations can then be rewritten in the simpler form

\[ p'_n = \frac{1}{N} \sum_{\alpha} W_{\alpha}(\rho) \sum_{\alpha'} \langle \alpha' \mid T^b(p_n') \mid \alpha \rangle_n \] (18)

where \( \langle \alpha \rangle, \langle \alpha' \rangle \) denotes the four site configurations which renormalize in \( \mid S_i^0 = 2 \) and \( \mid S_i = 0 \), respectively. In the above expression the denominator of eq. (18) is adsorbed in the normalization factor \( \mathcal{N} \).

The effective operator \( T^{b^2} \) contains all the dynamical processes that contribute to the definition of a meaningful renormalized dynamics. We define the following transformation

\[ \langle s \mid T^{b^2}(\mu) \mid s^0 \rangle = \sum_N \mathcal{D}_N \langle s \mid T^N(\mu) \mid s^0 \rangle \] (19)

where \( \mathcal{D}_N \) is the renormalization operator for the dynamical evolution of the system: it is a projection operator that samples only the paths of \( N \) time steps which have to be considered in the definition of the effective operator \( T^{b^2} \).

The operator \( \mathcal{D}_N \) is chosen on the basis of physical considerations: spanning conditions etc. In addition, \( \mathcal{D}_N \) should satisfy some general properties in order to preserve the symmetry or the internal space of the dynamical variables. For instance, we have to ensure the normalization of the effective dynamical operator by the property

\[ \sum_s \sum_N \mathcal{D}_N \langle s \mid T^N(\mu) \mid s^0 \rangle = 1 \] (20)

Moreover, \( \mathcal{D}_N \) must be consistent with the definition of the renormalization operator \( \mathcal{R} \): it should describe dynamical processes among renormalized variables of the same type of those given by the operator \( \mathcal{R} \). Finally, \( \mathcal{D}_N \) has to preserve the form of the dynamical operator \( T \) at each scale. This condition imposes that the time scaling is consistent with the length scaling used in \( \mathcal{R} \). In this way, it is possible to map the renormalized system in the old one with renormalized variables. The operator \( \mathcal{D}_N \) is therefore defined explicitly as an operator acting on the paths internal to four site cells. It selects for each \( N \) just relaxation paths which consist of \( N \) connected non-contemporary relaxation events that leave the cell without unstable sites. In a mathematical form it reads as

\[ \mathcal{D}_N = \prod_{i \in \{\alpha'\}} (1 - \delta_{2,s_i}) \prod_{J=0}^{N-1} \prod_{m=1}^{4} \delta(m - \sum_{i \in \{\alpha_J\}} \delta_{2,s_i}) \] (21)

where \( \alpha_J \)'s are the intermediate cell configurations during the dynamical evolution and \( \sum_{i \in \{\alpha_J\}} \) denotes the sum over all the sites in the cells. In the above expression, each delta function acts on a different intermediate cell eliminating those paths which do not have activity at each dynamical step. Furthermore, the operator ensures that in the cell \( \alpha' \) (Nth step) no activity is present; i.e. the process has stopped. Finally, we have to write the equation that gives the time scaling factor from the total average over contributing processes to the renormalized matrix element \( \langle 0 \mid T^t \mid 2 \rangle \)

\[ g(p_n) = \frac{1}{N} \sum_{\alpha} W_{\alpha}(\rho) \sum_{\alpha'} \sum_{N} N \mathcal{D}_N \langle \alpha' \mid T^N(p_n') \mid \alpha \rangle \] (22)

where we used the DDRG scheme to explicitly get the stationary weights, and \( \mathcal{N} \) is an opportune normalization factor.

The above relations will provide the consistent rescaling of time by imposing that \( b^2 = g(p_n^* \rho) \) from which it is possible to calculate the dynamical critical exponent.
V. MINIMAL PROLIFERATION
CALCULATIONS AND RESULTS

The explicit evaluation of the recursion relations depends on the choice of the spanning condition. In the following, the scheme used considers only connected paths that span the cell from left to right or from top to bottom. This spanning rule implies that only paths extending over the size of coarse grained length scale contribute to the renormalized dynamics, and it ensures the connectivity of the avalanche in the renormalization procedure.

An example of such a path is shown in Fig. (1). In this case \( \alpha = 2 \) and the path shown refers to the probability that the unstable sub-cell relaxes towards the other critical sub-cell [Fig. 1(b)]. This occurs with probability \((1/4)p_1\). At this point we consider the probability that the next relaxation event at the fine scale involves two neighboring sites, one inside and the other outside the original cell of size \( b = 2 \) [Fig.1(c)]. This occurs with probability \((2/3)p_2\). This series of relaxation processes contributes to the term \( D_2(\alpha' | T^2(p_{n'})) \alpha = 2_1 \) that characterizes the relaxation processes at the coarse grained scale. By summing over all the paths that lead to a \( p'_1 \) process one obtains for \( \alpha = 2 \)

\[
\sum_{\alpha'} \sum_{N} D_N(\alpha' | T^N(p_{n'}) | \alpha = 2)_1 = \\
\left( \frac{p_1}{6} + \frac{p_2}{3} \right) \left( \frac{2p_1}{3} + \frac{2p_2}{9} \right) + \left( \frac{p_2}{6} + \frac{p_3}{3} \right) \left( \frac{2p_1}{3} + \frac{3p_2}{9} \right). 
\] (23)

In a similar way one can also write the expression for the complete recursion relations. Proliferation effects due to multiple relaxations of the same site and sites becoming critical during the dynamical process are not considered in this scheme. However, the complete polynomials for \( p'_n \) involve more than two hundred terms, and we developed a generating function that allows their systematic calculation.

The generating function allows us to find the form of the renormalized operator \( T' \), without writing down the explicit form of all relaxation paths in the coarse grained cell. The basic idea of this method is to renormalize the function which describes all relaxation paths at once, rather then the probabilities of separate relaxation paths, by using the branching structure of avalanches.

To describe in detail the branching process underlying the large scale behavior of the sandpile model, we consider the generating function

\[
\sigma(N, E, S, W) = p_0 + \frac{p_1}{4}(N + E + S + W) \\
+ \frac{p_2}{6}(NE + NS + NW + ES + EW + SW) \\
+ \frac{p_3}{4}(NES + NEW + NSW + ESW) + p_4 NESW,
\] (24)

where symbols \( N, E, S \) and \( W \) correspond to the north, east, south and west directions on the square lattice, respectively. The coefficient in each term of this polynomial gives the probability for the process to go in the corresponding directions. The generating function takes into account all possible relaxation processes in the cell. It is easy to check directly that this function has the following properties:

1. If the argument corresponding to any direction is replaced by zero, the function counts the relaxation processes that do not send energy to this direction (Fig 2(b)).

2. If the argument corresponding to any direction is replaced by unit, the function counts the relaxation processes whether or not the energy is transferred in that direction (Fig 2(c)).

3. The generating function is normalized so that \( \sigma(1, 1, 1) = 1 \).

If there is a critical cell near the relaxing one, the outgoing energy can initiate the relaxation of the cell. It is easy to see that we can replace the argument corresponding to this direction by another generating function corresponding to the relaxation of the second cell. Finally, we obtain the generating function of this two-step relaxation process. For example, the function \( \sigma(N_1, \sigma(N_2, E_2, S_2, 1), S_1, W_1) \) describes the processes where the cell 1 relaxes first. Then, if the energy goes to the direction \( N_1 \) it initiates the relaxation of the cell 2 (Fig 3(d)). The symbols \( N_i, E_i, S_i, W_i \) denote the directions outgoing from the cell \( i \).

Using these properties we can write down the generating function \( \Sigma_\alpha \), counting the relaxation processes in the block that consists of four cells for different \( \alpha \). To this end, we must take into account only the processes which match the spanning condition. Therefore, it is necessary to eliminate all processes, in which only one cell relaxes. As the coefficients of the polynomial \( \Sigma_\alpha \) have the meaning of probabilities, they should finally be normalized by the condition

\[
\Sigma_\alpha(1, ..., 1) = 1. 
\] (25)

The generating function corresponding to the relaxation processes inside the block with \( \alpha = 2 \) is

\[
\Sigma_2 = \\
\{ \sigma(\sigma(N_2, 1, 1, W_2), 1, S_1, W_1) - \sigma(0, 1, S_1, W_1) \} + \\
\{ \sigma(N_2, 1, \sigma(1, 1, S_1, W_1), W_2) - \sigma(N_2, 1, 0, W_2) \} + (26)
\]
where \( Z_2 \) is a normalization factor chosen so that \( \Sigma_2(1, ..., 1) = 1 \). To write this function we start from the left down cell and define the arguments of the \( \sigma \)-function corresponding to the toppling of this cell. For the process definitely spans the block, the left up cell should topple and we write another \( \sigma \)-function instead of the symbol \( N_1 \). By going eastward the process will terminate inside the block and this branch of the toppling process cannot affect the neighboring blocks. Hence, we should write the number 1 instead of the symbol \( E_1 \). The other symbols \( S_1 \) and \( W_1 \) correspond to the branches of the relaxation process that goes immediately out of the initial block of cells. To consider only the processes that span the block, we must subtract the \( \sigma \)-function describing processes that do not send the energy from the first critical cell to the second one. Then, we add analogous \( \sigma \)-functions for the processes starting from the relaxation of the left upper cell. The term \( c.p. \) denotes all possible cyclic permutations of the critical cells inside the block. Analogously, we can write \( \sigma \)-functions of all other types of blocks.

To obtain complete generating function \( \Sigma \) for the block of four cells, we should sum up all \( \Sigma_\alpha \)-functions with the weights of blocks and normalize the result

\[
\Sigma = \frac{1}{Z} \sum_\alpha W_\alpha \Sigma_\alpha. \tag{27}
\]

Now, to transform the \( \Sigma \)-function from the block of four cells at the scale \( b^k \) to larger cell at the next scale \( b^{k+1} \), we replace the directions \( N_1, N_2, ... \) outgoing from the initial block by the new arguments corresponding to the directions \( N, ... \) outgoing from the new renormalized cell. In other words, two bonds that connect the neighboring blocks are coupled to the only bond on the lattice at the next scale, as is shown in fig. 3. Eventually, we obtain the following generating function:

\[
\Sigma(N, E, S, W) = \frac{W_2 \Sigma_2 + W_3 \Sigma_3 + W_4 \Sigma_4}{Z} \tag{28}
\]

where

\[
\Sigma_2 = (\sigma(\sigma(N, 1, 1, W), 1, S, W) - \sigma(0, 1, S, W)) + (\sigma(N, 1, \sigma(1, 1, S, W), W) - \sigma(N, 1, 0, W)) + c.p./Z_2.
\]

\[
\Sigma_3(N, E, S, W) = (\sigma(\sigma(N, \sigma(N, E, 1, 1), 1, W), 1, S, W) - \sigma(0, 1, S, W) + \sigma(\sigma(N, 1, 1, W), \sigma(1, E, S, 1), S, W) - \sigma(0, 0, S, W) + \sigma(1, \sigma(\sigma(N, E, 1, 1), 1, E, S, 1), S, W) - \sigma(1, 0, S, W) + c.p.)/Z_3.
\]

\[
\Sigma_4(N, E, S, W) = (\sigma(\sigma(N, \sigma(N, E, 1, 1), 1, W), 0, S, W) + \sigma(0, \sigma(N, E, 1, \sigma(N, 1, 1, W)), E, S, 1, S, W) - 2\sigma(0, 0, S, W) + (\sigma(0, 1, W) - \sigma(1, S, W) - \sigma(0, 1, S, W) + \sigma(0, 0, S, W))
\]

\[
\sigma(N, 0, 1, W)\sigma(0, E, S, 1) + \sigma(N, E, 1, 1)
\]

\[
\sigma(N, 0, 1, W)(\sigma(1, E, S, 1) - \sigma(0, E, S, 1)) + \sigma(0, E, S, 1)(\sigma(N, 1, 1, W) - \sigma(N, 0, 1, W)) + \sigma(1, E, S, 1) - \sigma(0, E, S, 1))
\]

\[
\sigma(N, 1, 1, W) - \sigma(N, 0, 1, W))) + c.p.)/Z_4.
\]

Here, \( Z_i \) and \( Z \) are the normalization factors and \( c.p. \) denote the expressions obtained from the previous polynomial by all possible cyclic permutations of its arguments. This generating function is the polynomial that contains only the first and second powers of its arguments. The last terms correspond to the processes when two energy portions are transferred from the initial block to the neighboring block by the two paths. However, according to the RG strategy, these processes should be considered as the transfer of the coarse grained energy portion at the larger scale. Therefore, all second powers of the arguments should be replaced by the first ones. The result obtained is the generating function describing the relaxation of renormalized cells. It depends on the same products of its arguments as the generating function for initial cells, but the coefficients of this polynomial are different and equal to the probabilities of relaxation processes that send energy to the given direction at the new scale. Taking these coefficients, we obtain the sought recursion relations which link the parameters of the cell at the scale \( b^k \) with the same parameters at the scale \( b^{k+1} \)

\[
\vec{P}(k + 1) = \vec{f}(\vec{P}(k), \rho(k)) \tag{30}
\]

The above set of RG equations supplemented with the driving condition

\[
\langle \rho^{k+1} \rangle = \frac{1}{\sum_n \rho_n b_{n+1}} \tag{31}
\]

define completely the DDRG recursion relations for sandpile models. Given this scheme the flow diagram and the relative fixed point in the parameters space \( \langle \rho, \vec{P} \rangle \) can be studied. We consider here the calculation scheme implemented with \( \vec{P} = (p_0, p_1, p_2, p_3, p_4) \). Despite the enlargement of the phase space by including the proliferation characterized by the probability \( p_0 \) the flow in the phase space is very similar to those obtained in ref. [1], where this parameter was not considered. A single attractive fixed point is obtained and the numerical value of this fixed point is very close to that obtained in the approach of Ref. [1]. The complete attractiveness of the fixed point corresponds to the lack of relevant scaling field, i.e., the control parameter. This must be the case, because we implement our RG scheme in the infinite time scale separation limit. This implies the relevant scaling field are constrained to their critical values. In other words we are
restricting the study of the system on its critical surface. In Tab. I we report the results obtained within this calculation scheme. The single fixed point is the signature of a single universality for all the non-directed sandpile models. This is questioned from numerical simulations which show evidences this could not be the case [32]. It appears, in fact, that BTW model and Manna model belong to different universality classes. This distinction does not appear in the present proliferation scheme. The present approach, however, can not provide a definitive settling of this issue. Mainly, it depends on the fact that we still are neglecting some proliferations such as the possibility of multiple topplings, that could be relevant for the identification of different universality classes.

The avalanche exponent \( \tau \) can be obtained directly from the fixed point parameters. By using the discrete length scale \( b^{(k)}_R = 2^k \) and the avalanche distribution in the form \( P(r)dr = r(1 - 2r)dr \) we can define the probability that the relaxation process spans the cell of size \( b^{(k)}_R \) and dies at the neighboring cells not extending over the scale \( b^{(k + 1)}_R \)

\[
K = \int_{b^{(k)}_R}^{b^{(k+1)}_R} P(r)dr \int_{b^{(k-1)}_R}^{\infty} P(r)dr = 1 - 2^{(1 - r)} \tag{32}
\]

Asymptotically \( (k \to \infty) \) we can express \( K \) in terms of fixed point parameters \( \rho^* \) and \( p^* \) in the following way:

\[
K = p^*_0 + p^*_1 (1 - \rho^*) + p^*_2 (1 - \rho^*)^2 + p^*_3 (1 - \rho^*)^3 + p^*_4 (1 - \rho^*)^4. \tag{33}
\]

This equation gives the total probability that a relaxation process occurs without triggering other sites, and therefore it does not extend on length scales larger than that of a single cell. Using these two expressions, Eqs. \( \{32,33\} \), the exponent \( \tau \) is given by the formula

\[
\tau = 1 - \frac{1}{2} \ln(1 - K) = 1.262 \tag{34}
\]

in excellent agreement with the proposed value \( \tau = 5/4 \) \([28]\) and large scale numerical simulations \([18,19,21]\). The obtained value is also in good agreement with the value obtained in the calculation of ref \([1]\), showing the robustness of the method with respect to different proliferation scheme. In order to overcome some of the approximations considered so far, we will present in the next section an improved scheme which takes into account a wider set of dynamical parameters. This scheme allows us to study also the critical behavior at the boundary of the system.

VI. EXTENDED KINETIC EQUATION SCHEME

In this section we treat more explicitly the dynamics of the original sandpile model. Considering the evolution of stable cells, we can take into account some of the processes which were neglected in the scheme discussed in the previous section.

To keep the connection with the original formulation of the sandpile model, we will characterize the static properties of a cell by four quantities

\[
\vec{N}(k) = (n_A, n_B, n_C, n_D), \quad n_A + n_B + n_C + n_D = 1, \tag{35}
\]

which are nothing but the probabilities for a cell to behave like a site on the initial lattice with a height 1, 2, 3 or 4, respectively, in the coarse grained dynamics, i.e. the addition of a ”coarse grained particle” to the cell transforms it to the next one in the alphabet. For example, the cell B characterized by the vector \((0,1,0,0)\) will be transformed to the cell C with the vector \((0,0,1,0)\). The last variable \( n_D \) is the probability for the cell to behave like a critical one in a sense that the addition of a ”coarse grained particle” to the cell induces relaxations into some neighboring cells or, in other words, subrelaxation processes on a minimal scale span the cell and transfer energy to some of its neighbors.

As we stressed in the previous sections, independently of the dynamics of the model at the minimal scale, each critical cell is characterized by the vector

\[
\vec{F}(k) = (p_1, p_2, p_3, p_4), \quad p_1 + p_2 + p_3 + p_4 = 1, \tag{36}
\]

that gives the probabilities for the energy to go to 1, 2, 3 or 4 neighboring cells after the relaxation of the critical cell. Here, because we have already enlarged the phase space by introducing the densities \( n_\alpha \), we do not include \( p_0 \) in the calculation scheme.

In this framework the coarse grained dynamics of the sandpile model can be represented as the following branching process on the sublattice \( \mathcal{L}_b \):

\[
\begin{align*}
A + \varphi & \rightarrow B, \\
B + \varphi & \rightarrow C, \\
C + \varphi & \rightarrow D, \\
D + \varphi & \rightarrow \begin{cases} p_1 : D + \varphi \\ p_2 : C + 2\varphi \\ p_3 : B + 3\varphi \\ p_4 : A + 4\varphi. \end{cases}
\end{align*} \tag{37}
\]

Here, \( \varphi \) and \( \varphi \) denote the ”coarse grained particles” obtained by the cell and the particles transferred to the neighboring cells, respectively.

These processes can be formally reinterpreted as an irreversible chemical reaction which takes place at each cell of the sublattice \( \mathcal{L}_b \). Now the coarse grained variables \( n_A, n_B, n_C, n_D \) and \( n_\varphi \) denote the concentrations of the respective species A, B, C, D, and \( \varphi \). Following to standard prescriptions of the chemical physics we can write down kinetic equations corresponding to this scheme of chemical reactions
where \( \bar{p} = p_1 + 2p_2 + 3p_3 + 4p_4 \) is equal to the average number of particles leaving the cell on toppling and \( \mathbf{r} \) is the position vector of the cell in the 2D space. The noise term \( \eta(\mathbf{r}, t) \), being non-negative, mimics the random addition of particles to the system. The diffusion term \( \nabla^2(n_\varphi n_D) \) describes the transfer of particles into the neighboring cells, and the diffusion coefficient \( \nu \) for the discrete Laplacian on the square lattice is equal to 1/4.

The only mobile specie in this scheme of reactions is \( \varphi \) and it is the field \( n_\varphi \) which describes the dynamics of avalanches. When it is equal to zero, all toppling processes die. Then, due to the noise term \( \eta(\mathbf{r}, t) \), particles are added randomly into the system initiating a branching process directed to the open boundary of the system. This process mutates species in the cells it has visited and topples the critical ones. Finally, the system will move to the steady state where the probability that the activity will die is on average balanced by the probability that the activity will branch. Thus, the chain reaction maintains this stationary state and all further avalanches cannot change the concentrations of species A, B, C, and D. Therefore, the steady state is characterized by the conditions that

\[
\dot{n}_A = \dot{n}_B = \dot{n}_C = \dot{n}_D = 0
\]

and Eqs. \((38-41)\) lead to the following relationships between concentrations of species \( \tilde{N}(k) \) at the stationary state and branching probabilities \( \tilde{P}(k) \)

\[
\begin{align*}
    n_A^* &= p_4/\bar{p}, \\
    n_B^* &= (p_3 + p_4)/\bar{p}, \\
    n_C^* &= (p_2 + p_3 + p_4)/\bar{p}, \\
    n_D^* &= (p_1 + p_2 + p_3 + p_4)/\bar{p} = 1/\bar{p}.
\end{align*}
\]

The relation \((47)\) between the probability \( n_A^* \) and branching probabilities \( \tilde{P}(k) \) can also be derived from the assumption that at the stationary state the flow of particles in a cell was on average balanced by the flow of particles out of the cell.

Thus, we have found the driving conditions for the sandpile models. Using them we can link the statistic weights of any static configurations of a cell with the dynamic parameters. Now, we can realize the renormalization procedure described previously. To this end, we must consider all types of blocks of four cells, whose relaxation matches the spanning condition. Such blocks and some of their relaxation schemes are shown in fig. 4.

While the previous scheme deals only with the cells being critical before the relaxation of the block, this one allows us to consider the cells becoming critical during the relaxation(fig. 4c).

To obtain the recursion relations by the method presented in the previous section, it is necessary to calculate the statistic weights of all configurations considered. The statistic weight of the block is given by the product of probabilities \( n_i \) for all cells in the block, multiplied by the number of different blocks with the same relaxation schemes. Thus, the following weights must be ascribed to the blocks shown in fig. 4.

\[
\begin{align*}
    W_a &= 4n_D^2(n_A + n_B + n_C)^2, \\
    W_b &= 4n_D^3(n_A + n_B), \\
    W_c &= 4n_D^3n_C, \\
    W_d &= n_d^4.
\end{align*}
\]

Expressing them through the probabilities \( \tilde{P} \), by using driving conditions \((44-47)\), we obtain the complete system of renormalization equations.

\[
\tilde{P}(k + 1) = \tilde{f}(\tilde{P}(k))
\]

Given this set of RG transformations we can study how the system evolves under successive doubling of length scale. The final result is independent upon the initial conditions \( n_i \) and \( p_i \) at the minimal scale. Also in this case the fixed point is attractive in the whole phase space and the system evolves spontaneously toward the fixed point values \( p_i^* \) and \( n_i^* \) shown in Tables II and III. These results can be compared with the exact ones obtained for the sandpile \([24]\). The exact height probabilities for the sandpile are reported in Tab.II and compared with our RG results. In order to calculate the avalanche exponent we can use the eq.\((52)\) by expressing \( K \) in terms of the fixed point parameters in the following way:

\[
K = p_1(1 - n_D^d) + p_2^*(1 - n_D^d)^2 + p_3^*(1 - n_D^d)^3 + p_4^*(1 - n_D^d)^4.
\]

More generally the probability \((53)\) should be represented via \( \sigma \)-function

\[
K = \sigma^*(1 - n_D^d(N), 1 - n_D^d(E), 1 - n_D^d(S), 1 - n_D^d(W))
\]

where \( n_D(N), n_D(E), n_D(S) \) and \( n_D(W) \) are the concentrations of critical cells at the nearest neighbors. By using the fixed point values we finally obtain \( \tau = 1.248 \), that again is in very good agreement with the proposed value \( 5/4 \) \([28]\) and numerical simulations \([18, 19, 21]\). Also in this case the scheme results in a single universality class for the sandpile models. In the extended scheme we include part of the proliferations by allowing different heights, and taking into account that some of the sites becomes critical during the relaxation event. We do not, however, consider multiple relaxation of the same sites during the spanning time, nor we allows a renormalization of the energy transfer (\( \delta E \)). This parameters could
be important in the case of the Manna model as pointed out in ref. [32]. Work is in progress to extend the present DDRG scheme in order to include also these further proliferations.

VII. THE BOUNDARY CRITICAL PROPERTIES

Since the critical properties of the sandpile model are quite similar to those of second order phase transitions, we proceed here along the same lines followed in the study of equilibrium critical phenomena. In particular we determine also the surface critical exponents, which in general differ from the bulk ones. This is of special importance in the two-dimensional case where conformal field theory connects surface and bulk properties of the model [33].

A. Open boundary

The fact that the boundary is open means that after the relaxation of the boundary site the energy can leave the system. We consider the critical energy of sites at the open boundary to be \( E_c = 4 \). It is more convenient to consider the boundary lying along the diagonal of the lattice and construct the renormalized cell in the following way. We consider the block of cells \( 2 \times 2 \) that contains one bulk, two boundary and one external cells as shown in fig. 3. For the critical properties of the model at large scales does not depend on the local structure of the lattice, the results obtained should not depend on the specific choice of the boundary. To describe the boundary cells at an arbitrary scale, we introduce the vectors

\[
\vec{N}^o(k) = (n_A^o, n_B^o, n_C^o, n_D^o),
\]

\[
n_A^o + n_B^o + n_C^o + n_D^o = 1,
\]

\[
\vec{P}^o(k) = (p_1^o, p_2^o, p_3^o, p_4^o),
\]

\[
p_1^o + p_2^o + p_3^o + p_4^o = 1,
\]

(55)

which have the same meaning as in the previous section. The kinetic equations and feedback relations coincide with the bulk ones. Thus, it is only necessary to find the correct form of recursion relations. To write the generating function for the boundary block, we can use again the generating functions describing the relaxations of bulk and boundary cells. Also we have to introduce a special generating function corresponding to the relaxation of the unphysical external cell. Since there are not processes transferring energy from the external half-plane of the lattice to the internal one, we require that the external cell immediately transfers the energy outside the lattice. Thus, we provide conservation of the flow of energy through the boundary in the scale transformation. For the block shown in fig. 3 the generating function of relaxation of the external cell has the simple form

\[
\sigma^{out} = \frac{N + E}{2}.
\]

(57)

Statistical weights of the static boundary configurations are given by the product of the probabilities of one bulk and two boundary cells. Now, the coefficients of the generating function of the renormalized cell gives us the recursion relations

\[
\vec{P}^o(k + 1) = f_\sigma(\vec{P}^o(k), \vec{P}(k)).
\]

(58)

Together with the bulk recursion relations of Eq. (52), Eq. (58) represents the complete system of the renormalization equations for the case of the open boundary. This system also has only one fixed point. The obtained fixed point parameters are given in Tables I and II. The comparison of fixed point height probabilities with the exact values obtained for ASM [25] shows rather good agreement.

To calculate the boundary critical exponent we should use the generating function describing the relaxation of the boundary cell in the fixed-point of RG flow. It is given by eq. (52), where

\[
K = \sigma^*_\sigma(1, 1, 1 - n_B^0, 1 - n_D^0).
\]

(59)

This indicates that the toppling of coarse grained boundary cell should stop in two neighboring internal cells. The result \( \tau_o = 1.486 \) is very close to \( \tau_o = \frac{3}{2} \) calculated exactly in [26] for open boundary of ASM. Such a good agreement is probably to be ascribed to the fact that in the boundary avalanche in ASM each site topples only once [27]. Therefore renormalization scheme that does not take into account multiple topplings gives most realistic results near open boundary.

B. Closed boundary

Let us consider the sandpile model on the half-plane. The edge of this half-plain is the closed boundary directed along one of the lattice axes. Each boundary site has three neighboring sites. Two of them also belong to the boundary. The fact that the boundary is closed means that after the relaxation of a boundary site energy does not leave the lattice, being distributed among the neighboring sites. Since the energy can leave the boundary site only in three directions, it is quite natural to take critical energy of a boundary site \( E_c = 3 \).

In order to follow the RG strategy, we again perform the site-to-cell transformation, replacing the block of four cells by a single cell at the larger scale. The cells at an arbitrary scale can be considered as either boundary or bulk ones. While the former consist only of bulk cells at the smaller scale, the latter include both the bulk and the boundary cells of smaller size. The renormalization of the bulk cell is described by the system of RG equations obtained in the previous section. Let us introduce the
description for the dynamics of boundary cells at an arbitrary scale. The static states of a boundary cell can be represented by three symbols A, B, C, which correspond to energy values \( E = 1, 2, 3 \) of boundary sites on the initial lattice. The addition of energy transforms the cell from the state A into the state B or C, sending the energy to three, two or one neighboring cells, respectively. The probabilities for the cell on the closed boundary to be in one of the three states is given by the vector

\[
\vec{N}^c = (n_A^c, n_B^c, n_C^c), \quad n_A^c + n_B^c + n_C^c = 1.
\] (60)

In the same way, the probabilities for the energy to be transferred in one, two or three directions is given by the vector

\[
\vec{P}^c = (p_1^c, p_2^c, p_3^c), \quad p_1^c + p_2^c + p_3^c = 1.
\] (61)

The relaxation process at the boundary cell can be represented as follows:

\[
\begin{align*}
A + \varphi & \rightarrow B, \\
B + \varphi & \rightarrow C, \\
C + \varphi & \rightarrow \begin{cases} 
\varphi + 1 : C + 1 \varphi, \\
\varphi + 2 : B + 2 \varphi, \\
\varphi + 3 : A + 3 \varphi.
\end{cases}
\end{align*}
\]

where \( \varphi \) and \( \bar{\varphi} \) are the energy obtained by the cell and transferred to the neighboring cell, respectively. Hence, we can write the following kinetic equations for the energy transfer:

\[
\begin{align*}
n_A^c &= n_\varphi (p_3^c n_C^c - n_\lambda^c), \\
n_B^c &= n_\varphi (p_2^c n_C^c + n_\lambda^c - n_B^c), \\
n_C^c &= n_\varphi (p_1^c n_C^c + n_B^c - n_C^c), \\
n_{\bar{\varphi}}^c &= n_\varphi (\bar{\varphi} c n_D^c - 1) + \bar{\varphi} c \nu \Delta^2 (n_{\bar{\varphi}} n_C^c) + \eta(r, t)
\end{align*}
\] (62-65)

Here, the discrete Laplacian \( \Delta \) must be understood with the Neumann boundary conditions. The steady state corresponds to the conditions, \( \vec{N} = 0 \). This leads us to the following driving conditions for the closed boundary:

\[
\begin{align*}
n_A^c &= p_3^c / \bar{\varphi}, \\
n_B^c &= (p_2^c + p_3^c) / \bar{\varphi}^c, \\
n_C^c &= (p_1^c + p_2^c + p_3^c) / \bar{\varphi}^c = 1 / \bar{\varphi}^c
\end{align*}
\] (66-68)

To perform the standard renormalization procedure described above and find the recursion relations, the generating function method can be employed. To this end, we introduce the generating function for the relaxation of a cell on the closed boundary as the following polynomial:

\[
\sigma_c(N, E, W) = \frac{p_c^c}{3} (N + E + W) + \frac{p_c^2}{3} (NE + NW + EW) + p_c^3 NEW.
\] (70)

The general idea of the generating function for the block of four cells is the same as in the bulk case. The difference is that the generating functions for boundary blocks oriented differently with respect to the boundary should be calculated separately and cannot be obtained by simple cyclic permutations of the arguments. Finally, applying the renormalization procedure with the use of the feedback relations \( \{66, 67, 68\} \), we obtain the recursion relations

\[
\vec{F}^c(k + 1) = \vec{F}^c(\vec{P}^c(k), \vec{P}(k)),
\] (71)

where \( \vec{P}(k) \) matches the bulk recursion relations of Eq. (62). The obtained height probabilities (Table [II]) are in good agreement with those calculated exactly in the case of ASM with closed boundary. To calculate the critical exponent \( \tau \) describing the distribution of avalanches near a closed boundary, we use the Eq. (68), expressing \( K \) through the fixed point generating function of the boundary cell:

\[
K = \sigma_c^I (1 - n_C^c, 1 - n_D, 1 - n_C^c).
\] (72)

The critical exponent results to be \( \tau_c = 1.239 \). The correction to the bulk critical exponent due to the half-plane geometry was not presented before. The agreement of the bulk and open boundary exponents with exact values allows us to expect such an accuracy in the case of the closed boundary.

**VIII. DISCUSSION AND CONCLUSIONS**

In this paper we have presented the detailed application of the DDRG to the sandpile model. We have concentrated on the BTW model which we have studied using schemes of increasing complexity. In the simple scheme the sites are subdivided in three states (stable, critical and active) and the RG transformation acts on the energy transfer probabilities \( p_i \). The scheme is then extended in order to treat explicitly the four states probability densities \( n_{\alpha} \), which can be obtained self-consistently. The fixed point values of \( n_{\alpha} \) and \( p_i \) are in good agreement with exact results. In addition, we compute the critical exponent \( \tau \) describing the avalanche size distribution. The result is in good agreement with numerical and analytical estimations and appears to be robust with respect to the different approximations. Finally, we study the boundary scaling of the sandpile model, obtaining results in good agreement with exact results. The present analysis shows that the DDRG is a complete and systematic tool to study nonequilibrium systems in general and sandpile models in particular.
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| $p_0^*$ | $p_1^*$ | $p_2^*$ | $p_3^*$ | $p_4^*$ |
|-------|-------|-------|-------|-------|
| 0.091 | 0.345 | 0.379 | 0.161 | 0.024 |

$\rho^* = 0.595$

**TABLE I.** The fixed point probabilities for the energy transfer from the relaxing cell, including the probability $p_0$.

| $p_1^*$ | $p_2^*$ | $p_3^*$ | $p_4^*$ |
|-------|-------|-------|-------|
| Bulk 0.295 | 0.435 | 0.229 | 0.0414 |
| Open 0.142 | 0.417 | 0.351 | 0.0899 |
| Closed 0.526 | 0.394 | 0.0799 |

**TABLE II.** Relaxation probabilities in the extended RG scheme.

| $n_A$ | $n_B$ | $n_C$ | $n_D$ |
|-------|-------|-------|-------|
| Bulk RG 0.0205 | 0.134 | 0.349 | 0.496 |
| Exact [24] 0.0736 | 0.174 | 0.306 | 0.446 |
| Open RG 0.0377 | 0.184 | 0.359 | 0.419 |
| boundary Exact [24] 0.1144 | 0.217 | 0.316 | 0.363 |
| Closed RG 0.0514 | 0.305 | 0.643 |
| boundary Exact [24] 0.113 | 0.318 | 0.568 |

**TABLE III.** Height probabilities in the stationary state.

| $\tau$ | $\tau_o$ | $\tau_c$ |
|-------|-------|-------|
| RG 1.248 | 1.486 | 1.239 |
| Exact 1.25 [28] | 1.5 [24] | ? |

**TABLE IV.** Critical exponents $\tau$, $\tau_o$ and $\tau_c$ for the avalanche size distribution in the bulk, open and closed boundaries, respectively.
FIG. 1. Example of the renormalization scheme for the relaxation dynamics. For details see the text.
FIG. 2. Samples of application of generating functions.
FIG. 3. Renormalizing the lattice we couple the two bonds connecting neighbouring blocks into the only bond of the new lattice. $N_1, N_2 \rightarrow N$, $E_2, E_3 \rightarrow S$, $S_3, S_4 \rightarrow W$, $W_4, W_1 \rightarrow E$. 
FIG. 4. We show the four different types of the blocks for the cemichal reaction model and some relaxation schemes spanning them. The other schemes can be obtained from these figures by rotations. It is convenient for calculations to subdivide the relaxation processes in the block (d) into three parts shown in the dashed box.
FIG. 5. The procedure of cell-to-site transformation at the open boundary of the lattice. One internal cell (I), two boundary cells (B) and one auxiliary external cell (E) build the boundary cell at the next scale $2b$. 