Coset Construction and Character Sumrules for the Doubly Extended N=4 Superconformal Algebras

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\textbf{ABSTRACT}

Character sumrules associated with the realization of the $N = 4$ superconformal algebra $\tilde{A}_\gamma$ on manifolds corresponding to the group cosets $SU(3)_k/U(1)$ are derived and developed as an important tool in obtaining the modular properties of $\tilde{A}_\gamma$ characters as well as information on certain extensions of that algebra. Their structure strongly suggests the existence of rational conformal field theories with central charges in the range $1 \leq c \leq 4$. The corresponding characters appear in the massive sector of the sumrules and are completely specified in terms of the characters for the parafermionic theory $SU(3)/(SU(2) \times U(1))$ and in terms of the branching functions of massless $\tilde{A}_\gamma$ characters into $SU(2)_k \times SU(2)_1$ characters.
1 Introduction

The doubly extended $N = 4$ superconformal algebras $\mathcal{A}_\gamma$ are a one-parameter family ($\gamma \in \left[\frac{1}{2}, \infty]\right)$ of linear superconformal algebras, i.e. superconformal algebras containing a finite dimensional superalgebra. They are characterized by their finite dimensional superalgebra being the non simple $(D(2|1; \frac{1}{1-\gamma}) \oplus u(1))$, and they contain $N = 4$ supersymmetries, which is a maximum for linear superconformal algebras [1]. $\mathcal{A}_\gamma$ furthermore contains all linear conventional superconformal algebras [2, 3] and provides a general framework to study properties of these subalgebras. Moreover, it has the challenging feature of having no unitary representations falling in a minimal series (for $k^- > 1$, see below). However, non trivial physical applications where conformal invariance plays a central role, such as superstring compactifications, are only well understood when the theory is rational. The Landau-Ginzburg technique developed for describing $N = 2$, $c = 9$ superstrings compactified to 4 dimensions is a famous example of this. It is therefore natural to seek simple extensions of the chiral algebra $\mathcal{A}_\gamma$ which give rise to finite dimensional representations of the modular group. More precisely, we study extensions of the (non-linear $N = 4$) algebra $\tilde{\mathcal{A}}_\gamma$, whose direct sum with the algebra of four free fermions and one free boson $A^{Q,U}$ coincides with $\mathcal{A}_\gamma$. Hints for the existence of such extensions are found when analyzing the character sumrules associated with the realization of $\tilde{\mathcal{A}}_\gamma$ via coset constructions based on quaternionic symmetric spaces, which were discovered a few years ago [4, 5]. It was suggested in [6] that extensions of $\tilde{\mathcal{A}}_\gamma$ yield finite dimensional representations of the modular group. However, the full information on the character sumrules, which is needed to completely identify the particular rational extensions of $\tilde{\mathcal{A}}_\gamma$ involved, was not available then. In this paper, we will consider the coset $SU(3)_{k+}/U(1)$ which together with 4 free (Wolf space) fermions [7] provides a realization of $\tilde{\mathcal{A}}_\gamma$ for $\gamma = 2/(\tilde{k}+3)$ and derive the corresponding character sumrules with particular emphasis on the massive sector. In Section 2, we argue that the functions $F^{A}_{2\ell+,n}(q)$ which couple to the $\tilde{\mathcal{A}}_\gamma$ massive characters in the sumrules are related both to the branching functions of $\tilde{\mathcal{A}}_\gamma$ into its Kac-Moody subalgebra $SU(2)_{k+} \times SU(2)_1$ and to the parafermions for the $SU(3)/(SU(2) \times U(1))$ theory. They are argued to be labelled by a rational conformal theory at central charge $c = c_\phi = 1+3(\tilde{k}^- - 1)^2/(\tilde{k}^+ + 1)(\tilde{k}^++3)$. As a concrete example, the cases $\gamma = 1/2$ and $\gamma = 2/5$ are discussed in detail in Appendix A. The analytic structure of the character sumrules associated to the coset constructions realizing $\tilde{\mathcal{A}}_\gamma$ is best understood when highest weight states of the $\tilde{\mathcal{A}}_\gamma$ and of a certain rational Gaussian model are explicitly constructed in terms of coset operators. This is explained in Section 3, which indeed provides proofs of some of the results alluded to in [3].

The character sumrules also lead to the derivation of the modular properties for the massless characters of the $\tilde{\mathcal{A}}_\gamma$ algebra and for the functions $F^{A}_{2\ell+,n}(q)$, provided one assumes the decoupling of the massless and massive sectors of the sumrules under modular transformations. This hypothesis is justified in Section 4. The massless characters and their modular transformations were presented in [6], but a detailed proof of these transformation properties is given in Section 4 and in Appendix C.
2 Analytic structure of the character sumrules

The $N = 4$ superconformal algebra $\mathcal{A}_\gamma$, whose affine subalgebra is $SU(2)_{k^+} \times SU(2)_{k^-} \times U(1)$, has been studied by various groups over the last few years \cite{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. It consists of the dimension 2 Virasoro generator $L(z)$, seven dimension 1 currents $T^{\pm i}(z)$ ($i = 1, 2, 3$) and $U(z)$, as well as four dimension 3/2 supercurrents $G^a(z)$ and four dimension 1/2 currents (free fermions) $Q^a(z)$ which transform as doublets under the two $SU(2)$ algebras. The commutation relations can be found, for instance, in \cite{12, 13}. The algebra is characterized by the levels $k^+, k^-$ (positive integers in the unitary representations to which we shall restrict our attention) of the two commuting $SU(2)$ subalgebras, or equivalently, by the parameter $\gamma = k^- / k$ and the central charge $c = 6k^+ k^- / k$, where $k = k^+_ \pm k^-$. In fact, the algebra $\mathcal{A}_\gamma$, itself is non-simple, but is the direct sum of the subalgebras, $\mathcal{A}_\gamma$ and $\mathcal{A}_{\gamma U}$: the algebra of free fermions and the affine $U(1)$. The generators of $\mathcal{A}_\gamma$, which is a non-linear superconformal algebra, are constructed from those of $\mathcal{A}_\gamma$ as explicitly shown in \cite{9} (see also \cite{10}). Together with a dimension 2 Virasoro generator $\hat{L}(z)$, $\mathcal{A}_\gamma$ contains six currents $\hat{T}^{\pm i}(z)(i = 1, 2, 3)$ which are dimension 1 primaries wrt $\hat{L}(z)$ and generate two $SU(2)$ algebras at level $\hat{k}^\pm = k^\pm - 1$, and four dimension 3/2 supercurrents $\hat{G}^a(z)$. The non-linearity of the algebra is encoded in the (anti)commutation relations of the dimension 3/2 currents \cite{10}.

The representation theory and corresponding characters were given in \cite{2, 3, 9, 10} for unitary highest weight state representations. These are labelled by the two isospin quantum numbers $\ell^+, \ell^-$, and have conformal dimension $h$, whose lower bound $\bar{h}_0$ is a function of $\ell^+, \ell^-, k^+$ and $\bar{k}^-$. An irreducible representation with conformal dimension $\bar{h}_0$ is called massless or chiral (it corresponds in the Ramond (R) sector to a representation with non-zero Witten index), while any representation with conformal weight $\bar{h} > \bar{h}_0$ is called massive. For fixed $\bar{k}^+, \bar{k}^-$, there is a finite number of massless and an infinite number of massive representations and corresponding characters. In the Neveu Schwarz (NS) sector for instance, the massless characters of the $\mathcal{A}_\gamma$ algebra are labelled as

$$Ch_{\gamma,NS}(\bar{k}^+, \bar{k}^-, \ell^+, \ell^-, \bar{h}_0^{NS}; q, z_+, z_-)$$

(2.1)

with $\ell^\pm = 0, \frac{1}{2}, 1, \ldots, \frac{\bar{k}^\pm}{2}$ and

$$\bar{h}_0^{NS} = \frac{1}{\bar{k}}[(\ell^+ - \ell^-)^2 + k^+ \ell^- + k^- \ell^+]$$

(2.2)

while the massive characters are denoted by

$$Ch_{m,NS}(\bar{k}^+, \bar{k}^-, \ell^+, \ell^-, \bar{h}_0^{NS}; q, z_+, z_-)$$

(2.3)

with $\ell^\pm = 0, \frac{1}{2}, \ldots, \frac{\bar{k}^\pm - 1}{2}$ and $\bar{h}_0^{NS} > \bar{h}_0^{NS}$. Often, for simplicity we shall suppress several of the variables.

Realizations of $\mathcal{A}_\gamma$ on manifolds based on group cosets of the form $SU(\bar{k}^- + 2)/(SU(\bar{k}^-) \times U(1))$ together with 4 $\bar{k}^-$ free (Wolf Space) fermions do exist \cite{2, 3, 4}. The factor $U(1)$ in the denominator precisely corresponds to the $U(1)$ current which decouples from $\mathcal{A}_\gamma$ when considering the W-algebra $\mathcal{A}_\gamma$, and the underlying quaternionic symmetric space (or Wolf space) is $W = SU(\bar{k}^- + 2)/(SU(\bar{k}^-) \times SU(2) \times U(1))$. The case $\bar{k}^- = 0 (\bar{k}^+ \in N^*)$ is very
Hilbert space provides representations for the rational torus algebra \( SU(2) \) currents and their associated Sugawara form \( \hat{L}(z) \). The \( \hat{A}_\gamma \) characters, which in this case are bound to be massless, are affine \( SU(2) \) characters and the theory is rational.

The study of character sumrules corresponding to the above coset realizations in the generic case \( \tilde{k}^{-} > 0 \) should shed some light on the structure of some of the possible rational extensions of \( \hat{A}_\gamma \). For much of this paper we shall restrict our analysis to the value \( \tilde{k}^{-} = 1 \), for which the group coset reduces to \( SU(3)_{\tilde{k}^+}/U(1) \). The cases \( \tilde{k}^{-} > 1 \) are associated with cosets \( SU(\tilde{k}^{-} + 2)/(SU(\tilde{k}^{-}) \times U(1)) \) and certainly merit further investigation [14].

For a given value of \( \tilde{k}^+ \), the realization is based on a Hilbert space written as the direct product of the representation space \( \mathcal{H}_{A}^{SU(3)} \) for the affine Lie algebra \( SU(3)_{\tilde{k}^+} \) and corresponding highest weight \( \Lambda = (\vec{a}, \hat{k}^+, 0) \) where \( \vec{a} = (a_1, a_2) \) is a highest weight in the Dynkin basis of an \( SU(3) \) representation (while the last entry denotes the grade, an index we shall not use too much) [14], and the Fock space \( \mathcal{H}_{WS}^{SU(3)} \) for the four free fermion fields associated with the Wolf space \( SU(3)/(SU(2) \times U(1)) \). In addition to representations for \( \hat{A}_\gamma \), the above Hilbert space provides representations for the rational torus algebra \( A_{3\hat{k}} \) (the extension of a \( U(1) \) algebra by a dimension \( 3\hat{k} \) operator) [13] as mentioned briefly in [1] and proven in detail in sect. 3 (\( k \equiv \tilde{k}^+ + \tilde{k}^- = \tilde{k}^+ + \tilde{k}^- + 2 \)). There it is shown how the “decoupling” \( U(1) \) generator of the algebra \( A_{3\hat{k}} \) emerges as the direct sum of the \( U(1) \) hypercharge of \( SU(3)_{\tilde{k}^+} \) and the \( U(1) \) generator of \( SO(4) \). The above information is encoded in the following character sumrules, written here for the NS sector,

\[
\chi_{\Lambda}^{WS,NS} \cdot \chi_{\Lambda} = \{ \chi_{\Lambda}^{WS,NS} \cdot \chi_{\Lambda} \}_0 + \{ \chi_{\Lambda}^{WS,NS} \cdot \chi_{\Lambda} \}_m, \tag{2.4}
\]

where the character for the WS fermions is related to a reducible \( SO(4) \) representation,

\[
\chi_{\Lambda}^{WS,NS} = \theta_3(q, z_{-} z_{y}) \cdot \eta(q) \eta(q) = \prod_{n=1}^{\infty} (1 + z_{-} z_{y} q^{n-\frac{1}{2}})(1 + z_{-} z_{y}^{-1} q^{n-\frac{1}{2}})(1 + z_{-} z_{y}^{-1} q^{n-\frac{1}{2}})(1 + z_{-} z_{y} q^{n-\frac{1}{2}}), \tag{2.5}
\]

and we denote the character for the unitary representation of \( SU(3)_{\tilde{k}^+} \) with highest weight \( \Lambda = (\vec{a}, \tilde{k}^+, 0) \), by,

\[
\chi_{\Lambda}^{SU(3)_{\tilde{k}^+}}(q, z_+, z_y). \tag{2.6}
\]

The variable \( z_y \) is associated with the “hypercharges”, \( Y_{SU(3)}^{SU(3)}, Y_{WS}^{SU(3)} \) (see next section). Unitarity requires the \( SU(3) \) representations to be integrable,

\[
\tilde{k}^- < \langle \vec{a}, \psi \rangle \geq 0 \tag{2.7}
\]

with \( \psi \) the highest root in \( SU(3) \), or \( a_i \) non-negative integers such that \( 0 \leq a_1 + a_2 \leq \tilde{k}^+ \).

\[\text{Note that for } \tilde{k}^- = 1, \text{ the choice } \tilde{k}^+ = 5 \text{ leads to a contribution of } 9 \text{ from the central charge of } A_\gamma, \text{ which could be relevant in some new compactification scheme.}\]
The RHS of (2.4) contains terms involving massless $\tilde{\mathcal{A}}_{\gamma}$ characters and terms involving massive $\tilde{\mathcal{A}}_{\gamma}$ characters. However, the massive representations reduce into two massless ones as $\tilde{h}^{NS}$ reaches the lower bound (2.2) according to the formula [10],

$$Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}(\ell^+,0) + Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}\left(\ell^+ + \frac{1}{2}, \frac{1}{2}\right) = Ch_m^{\tilde{\mathcal{A}}_{\gamma},NS}(\ell^+,\ell^- = 0) \equiv Ch_m^{\tilde{\mathcal{A}}_{\gamma},NS}(\ell^+).$$

We recall that for $k^- = 1, 2\ell^- = 0, 1$ in massless characters and $2\ell^- = 0$ in massive ones. The splitting into massless and massive contributions in (2.4) may therefore be considered non-unique. For our purposes, the combinations of massless characters:

$$Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}(L = 0) \equiv -Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}\left(0, \frac{1}{2}\right),$$

$$Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}(L = 1, \ldots, k - 3) \equiv \frac{1}{2} \left[ Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}\left(\frac{1}{2}, (L - 1), 0\right) - Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}\left(\frac{1}{2}, L, \frac{1}{2}\right)\right],$$

$$Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}(L = k - 2) \equiv Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}\left(\frac{1}{2}(k - 3), 0\right),$$

introduced in [3] play a central role. In fact the hypothesis for any $\tilde{k}^+$, that the massless and massive sectors defined according to these combinations decouple from each other under modular transformations, is extremely non-trivial, but is in fact consistent with the modular properties of $SU(3)_{\tilde{k}^+}$ and $\mathcal{A}_{3k}$ characters. In particular, the massless combinations will be shown to transform as $SU(2)_{\tilde{k}^+ + 1}$ characters under modular transformations in Section 4.

The equations determining these modular transformations are tremendously overdetermined, and the fact that the above decoupling assumption leads to a consistent solution is at present our strongest evidence in favour of that assumption. With this definition, the massless part of the sumrule (2.4) is given by

$$\{\chi^{WS,NS}(q, z_y, z_-) \cdot \chi_{\Lambda}(q, z_y, z_+)\}_{0} \equiv \sum_{L=0}^{k-2} M_A^L(q, z_y) Ch_0^{\tilde{\mathcal{A}}_{\gamma},NS}(L, q, z_+, z_-),$$

(2.10)

where the matrix $M_A^L$ (given in [3] without proof) will be proven in the next section to be

$$M_A^L(q, z_y) =$$

$$-\delta_{L,0} \left\{ \delta_{a_1,0} \chi_{-a_2+3(1+a_2)}^3(q, z_y) + \delta_{a_2,0} \chi_{a_1-3(1+a_1)}^3(q, z_y) + \delta_{a_1+a_2-k-3} \chi_{a_1-2+3k}(q, z_y) \right\}$$

$$+\delta_{L,k-2} \left\{ \delta_{a_1,0} \chi_{-a_2-3(k-1-a_2)}^3(q, z_y) + \delta_{a_2,0} \chi_{a_1+2(k-1-a_1)}^3(q, z_y) + \delta_{a_1+a_2-k-3} \chi_{a_1-2}(q, z_y) \right\}$$

$$(1 - \delta_{L,0}) (1 - \delta_{L,k-2}) \left\{ \delta_{L,k-2} \chi_{-a_2-3(k-1-a_2)}^3(q, z_y) \right\}$$

$$+\delta_{L,k-2-a_1} \chi_{a_1+2(1+a_2)}^3(q, z_y) + \delta_{L,a_1+a_2+1} \chi_{a_1-a_2}^3(q, z_y) - \delta_{L,a_1} \chi_{a_1-2+3(a_2+2)}^3(q, z_y)$$

$$-\delta_{L,a_2} \chi_{a_1-2+3(a_1+1)}^3(q, z_y)$$

(2.11)

and the characters $\chi_{3k}^3(q, z_y)$ appearing in the above matrix are those of the rational torus algebra $\mathcal{A}_{3k}$. They are related to generalised theta functions in the following way,

$$\chi_{3k}^3(q, z_y) = \frac{1}{\eta(q)} \theta_{m,3k}(q, z_y^{2/3}) \equiv \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z} + \frac{m}{3k}} q^{3kn^2} z_y^{2kn}.$$
Concerning the massive part of the sumrule we note that the massive characters (in the NS sector here) may be written quite generally as \[\text{[9, 10]},\]

\[\text{Ch}_{m}^{\tilde{h}_{m},NS}(\tilde{k}^{+}, \tilde{k}^{-}, \ell^{+}, \ell^{-}, \tilde{h}; q, z_{+}, z_{-}) = q^{\tilde{h}_{m}} \text{Ch}_{m}^{\tilde{h}_{m},NS}(\tilde{k}^{+}, \tilde{k}^{-}, \ell^{+}, \ell^{-}, \tilde{h}_{m}; q, z_{+}, z_{-}) = \chi_{c_{\phi}, h_{\phi}}(q) \chi_{F}^{NS}(q, z_{+}, z_{-}) \chi_{2\ell+1}^{k_{+}}(q, z_{+}) \chi_{2\ell-1}^{k_{-}}(q, z_{-}).\]

(2.13)

Here, \(\chi_{F}^{NS}(q, z_{+}, z_{-})\) is a character for 4 free fermions (two complex NS ones), \(\chi_{2\ell}^{k}\) is the \(SU(2)_{k}\) characters for isospin \(\ell\) given by

\[\chi_{2\ell}(q, z) = q^{-\frac{1}{2}} z^{-1} \prod_{n=1}^{\infty} (1 - q^{n})^{-1} (1 - q^{n} z^{2})^{-1} \prod_{m \in \mathbb{Z}^{+}} \frac{q^{(k+2)m^{2}} [z^{2(k+2)m} - z^{-2(k+2)m}]}{1 - q^{m}},\]

and finally

\[\chi_{c_{\phi}, h_{\phi}}(q) = \frac{q^{h_{\phi} - (c_{\phi} - 1)/24}}{\eta(q)}\]

is the Virasoro character corresponding to

\[h_{\phi} = \tilde{h} - \tilde{h}_{0} + \delta \tilde{h},\]
\[\delta \tilde{h} \equiv \frac{k^{+}k^{-}}{4k} \left( \frac{2\ell^{+}}{k^{+}} + \frac{2\ell^{-}}{k^{-}} \right) \left( \frac{\tilde{k}^{+} - 1 - 2\ell^{+}}{k^{+}} + \frac{\tilde{k}^{-} - 1 - 2\ell^{-}}{k^{-}} \right) \geq 0;\]
\[c_{\phi} = 1 + 6 \left( \sqrt{\frac{k^{+}k^{-}}{k}} - \sqrt{\frac{k}{k+k^{-}}} \right)^{2}\]

(2.16)

at conformal dimension \(h_{\phi}\) and central charge \(c_{\phi}\) with

\[\tilde{h} - \tilde{h}_{m} \equiv h_{\phi} - \frac{1}{24} (c_{\phi} - 1).\]

Thus

\[\tilde{h}_{m} \equiv \tilde{h}_{0} + \frac{k^{+}2\ell^{+} + 1}{2k^{+}} - \frac{1}{2}\]

(2.17)

(for \(k^{-} = \tilde{k}^{-} + 1 = 2\)). The interpretation of this structure in terms of free fields was recently given in ref. \[\text{[12]}\]. Then also

\[\text{Ch}_{m}^{\tilde{h}_{m},NS}(\tilde{k}^{+}, \tilde{k}^{-}, \ell^{+}, \ell^{-}, \tilde{h}; q, z_{+}, z_{-}) = \chi_{c_{\phi}, h_{\phi}}(q) \cdot \eta(q) \cdot \text{Ch}_{m}^{\tilde{h}_{m},NS}(\tilde{k}^{+}, \tilde{k}^{-}, \ell^{+}, \ell^{-}, \tilde{h}_{m}; q, z_{+}, z_{-}).\]

(2.18)

For \(\tilde{k}^{-} = 1\) only \(\ell^{-} = 0\) occurs in the massive case. We may then write the massive part of the sumrule as

\[\{\chi_{WS,NS}(q, z_{y}, z_{-}) \cdot \chi_{\Lambda}(q, z_{y}, z_{+})\}_{m} = \sum_{2\ell^{+} = 0}^{\tilde{k}^{+}-1} \sum_{n \in \mathbb{Z}_{k}} \tilde{M}_{2\ell^{+}, n}(q, z_{+}, z_{-}, z_{y}) F_{2\ell^{+}, n}(q)\]

(2.19)
where
\[ \tilde{M}^{\Lambda}_{2\ell^+,n}(q, z_+, z_-, z_y) = C h_{m}^{\tilde{\Lambda}, NS}(\tilde{h}_{m}, \ell^+, 0; q, z_+, z_-) \chi^{3k}_{-2(a_1-a_2)+6\ell^++6n}(q, z_y). \]  

This result will be proven in the next section. We see that the functions \( F^\Lambda_{2\ell^+,n}(q) \) may be interpreted as an infinite sum of Virasoro characters, multiplied by \( \eta \). Together with the \( \tilde{\Lambda} \) character at the particular value of conformal dimension given by \( \tilde{h}_{m} \), they provide the contribution from an infinite sum of massive characters. It follows from the structure of the sum rule that these functions have the form of a certain fractional power of \( q \) multiplied by an infinite power series in \( q \) with positive integer coefficients, which we have analyzed by algebraic manipulation programs for \( \tilde{\omega} \) and \( \tilde{\nu} \).

Under the first flow, the \( \tilde{\Lambda} \) algebra presumably in fact giving rise to a rational conformal field theory at \( \tilde{k}^+ = 1, 2 \). From the strongly motivated assumption (see sect. 4) that the massless and massive parts of the sumrule decouple from each other under modular transformations, it follows that the functions \( F^\Lambda_{2\ell^+,n}(q) \) in fact also carry a finite dimensional representation of the modular group. Furthermore, from the above discussion it then further appears plausible that they are related to characters of some extended algebra presumably in fact giving rise to a rational conformal field theory at \( c = c_\phi \), perhaps corresponding to some \( W \)-extension of the Virasoro algebra or some coset construction.

The modular forms \( F^\Lambda_{2\ell^+,n}(q) \) satisfy the following symmetry relations:
\[
F^\Lambda_{2\ell^+,n}(q) = F^\Lambda_{2\ell^+,n+k}(q), \\
F^\Lambda_{2\ell^+,n}(q) = F^{\Lambda_C}_{2\ell^+-n-2\ell^+}(q),
\]  

as is obtained by letting \( z_y \rightarrow z_y^{-1} \) in (2.4), a shift which relates the sumrule for an integrable \( SU(3)_{\tilde{k}^+} \) representation \( \tilde{a} = (a_1, a_2) \) to the sumrule for its conjugate \( \tilde{a}_C = (a_2, a_1) \). This can be seen from the following properties of the \( z_y \)-dependent functions in the sumrule,
\[
\chi^{SU(3)_{\tilde{k}^+}}_\Lambda(q, z_+, z_y^{-1}) = \chi^{SU(3)_{\tilde{k}^+}}_{\Lambda_C}(q, z_+, z_y), \\
\chi^{WS,NS}_{\Lambda}(q, z_-, z_y^{-1}) = \chi^{WS,NS}_{\Lambda}(q, z_-, z_y)
\]  

and
\[
\chi^{3k}_m(q, z_y^{-1}) = \chi^{-3k}_{-m}(q, z_y),
\]  

with \( \Lambda_C \equiv (\tilde{\omega}_C, \tilde{k}^+, 0) \). Two other symmetries follow from allowing the sumrule (2.4) to “flow” as
\[
z_\pm \rightarrow q^{1/2}z_\pm, \quad z_y \rightarrow q^{1/2}z_y,
\]  

or
\[
z_\pm \rightarrow z_\pm, \quad z_y \rightarrow q^{1/2}z_y.
\]  

Under the first flow, the \( SU(3)_{\tilde{k}^+} \) and the 4 fermion characters respectively transform as
\[
\chi^{SU(3)_{\tilde{k}^+}}_{\Lambda}(q, q^{1/2}z_+, \sqrt{2}z_y) = q^{-\tilde{k}^+}(z_y/z_+) \chi^{SU(3)_{\tilde{k}^+}}_{\Lambda}(q, z_+, z_y), \\
\chi^{WS,NS}_{\Lambda}(q, q^{1/2}z_-, \sqrt{2}z_y) = q^{-\tilde{k}^+}z_y^{-3/2} \chi^{WS,NS}_{\Lambda}(q, z_-, z_y).
\]  

(2.26)
On the other hand, considerations of spectral flow for $\tilde{A}_3$ characters [10, 11] lead to

$$C h_{k_+}^{\tilde{A}_3,NS}(L, q, q^{-\frac{3}{2}} z_+, q^{\frac{3}{2}} z_-) = -q^{-\frac{k_+}{2}} z_+^{1-\frac{1}{2}} z_- C h_{k_+}^{\tilde{A}_3,NS}(0, q, z_+, z_-),$$

$$C h_{m}^{\tilde{A}_3,NS}(\ell^+, \ell^- = 0, q, q^{-\frac{1}{2}} z_+, q^{\frac{1}{2}} z_-) = q^{-\frac{k_+}{2}} z_+^{1-\frac{1}{2}} z_- C h_{m}^{\tilde{A}_3,NS}(\frac{k_+ - 1}{2} - \ell^+, 0, q, z_+, z_-),$$

(2.27)

for $L = 0, ..., k_+ + k_-$ (with $k_- = 1$ here). Here both sides are evaluated for the value of conformal dimension equal to $h_m$ (eq. 2.17). Finally,

$$\chi^3_{m}(q, q^{3/2} z_y) = q^{-3k/4} z_y^{-k} \chi^{3k}_{m+3k}(q, z_y).$$

(2.28)

It is now straightforward to conclude that the modular forms $F_{2\ell^+, n}(q)$ obey the following symmetry relation,

$$F_{2\ell^+, n}(q) = F_{k_+ - 1 - 2\ell^+, n + 2\ell^+ + 2}(q).$$

(2.29)

The second flow (2.23) relates the character sumrules for $SU(3)_{\tilde{k}+}$ representations pertaining to the same orbit under the order 3 transformation

$$\phi(a_1, a_2) = (a_2, \tilde{k}^+ - (a_1 + a_2)),$$

(2.30)

$$\chi^{SU(3)_{\tilde{k}+}}_{\Lambda}(q, z_+, q^\ell z_y) = q^{-\frac{k^+}{2}} z_{y}^{\frac{2k^+}{2}} \chi^{SU(3)_{\tilde{k}+}}_{\Lambda}(q, z_+, z_y).$$

(2.31)

$(\phi(\Lambda) \equiv (\phi^e(\tilde{a}), \tilde{k}^+, 0))$. Furthermore, the 4-fermion character and the theta functions transform as $(\epsilon = \pm 1)$

$$\chi^{WS,NS}(q, z_-, q^\ell z_y) = q^{-1} z_y^{-2\epsilon} \chi^{WS,NS}(q, z_-, z_y),$$

$$\chi^3_{m}(q, q^{\ell} z_y) = q^{-3\epsilon} z_y^{-3} \chi^{3k}_{m+2\ell k}(q, z_y).$$

(2.32)

These properties lead to the third relation between the modular forms $F_{2\ell^+, n}(q)$,

$$F_{2\ell^+, n}(q) = F^{\phi^e(\Lambda)}_{2\ell^+, n + \epsilon + 3(\epsilon - 1)a_1 + 3(\epsilon + 1)a_2}(q).$$

(2.33)

The three sets of symmetries described so far allow for a complete determination of the functions $F_{2\ell^+, n}(q)$ in the very particular case where $\tilde{k}^+ = 1$. It is shown in Appendix A that the massive sector of the sumrules in this case depends on the two functions

$$2 F_{0,0}^{(0,0),1,0}(q) = \theta_{0,1}(q),$$

$$2 F_{0,1}^{(0,0),1,0}(q) = \theta_{1,1}(q),$$

(2.34)

which, when multiplied by $1/\eta$, provide the two characters for the $SU(2)_1$ conformal field theory with central charge 1, in total agreement with the expected value of $c_\phi$ (2.16). When $\tilde{k}^+ \geq 2$, some further information on the functions $F_{2\ell^+, n}(q)$ can be obtained by deriving their modular properties as motivated in section 4, and by relating them to the branching functions in the reduction of $\tilde{A}_3$ representations according to their $SU(2)_{\tilde{k}+} \times SU(2)_{\tilde{k}-}$ content. To this end it is useful to separate the variables $z_+, z_-$ and $z_y$ in the sumrule (2.4). The character
corresponding to four free fermions may always be written in terms of $SU(2)_1$ characters in the following way (see also \[17\]),
\[
\chi^{WS,NS}(q, z_-, z_y) = \chi^1_0(q, z_-)\chi^1_0(q, z_y) + \chi^1_1(q, z_-)\chi^1_1(q, z_y).
\]

Next, the decomposition of the $SU(3)_{k+}$ characters in $SU(2)_{k^+}$ characters for a regular embedding of $SU(2)$ in $SU(3)$, is given by the following general structure,
\[
\chi^{SU(3)_{k+}}_\Lambda(q, z_+, z_y) = \sum_{2\ell^+ = 0}^{\tilde{k}^+} \sum_{n \in \mathbb{Z}_{k^+}} P^\Lambda_{2\ell^+, n}(q) \chi^{3\tilde{k}^+}_{4(a_1 - a_2) + 6(n + \ell^+)}(q, z_y) \chi^{\tilde{k}^+}_{2\ell^+}(q, z_+),
\]
where the functions $P^\Lambda_{2\ell^+, n}(q)$ satisfy the periodicity condition $P^\Lambda_{2\ell^+, n + k^+}(q) = P^\Lambda_{2\ell^+, n}(q)$, and obey the symmetry relations
\[
\begin{align*}
P^\Lambda_{2\ell^+, n}(q) &= P^\Lambda_{2\ell^+, n - 2\ell^+}(q), \\
P^\Lambda_{2\ell^+, n + k^+}(q) &= P^\Lambda_{2\ell^+, n + 2\ell^+}(q), \\
P^\Lambda_{2\ell^+, n}(q) &= P^{\Lambda'}_{2\ell^+, n - (\ell+1)a_1 - (\ell+1)a_2}(q),
\end{align*}
\]
which immediately follow from the relations \[2.22\], \[2.26\] and \[2.31\].

As discussed in \[18\], the functions $P^\Lambda_{2\ell^+, n}(q)$ actually provide the characters of the parafermionic theory $SU(3)/(SU(2) \times U(1))$. We illustrate in Appendix A how to derive expressions for these characters for the cases $\tilde{k}^+ = 1$ and $\tilde{k}^+ = 2$ in terms of familiar modular forms. For instance, in the simplest case where $\tilde{k}^+ = 1$, one has, for all $\vec{a}, \vec{a}'$ corresponding to integrable representations \[2.7\],
\[
P^{(\vec{a}, 1, 0)}_{0, 0}(q) = P^{(\vec{a}', 1, 0)}_{1, 0}(q) = 1.
\]
The LHS of the sumrule \[2.4\] can now be rewritten as
\[
\chi^{WS,NS}(q, z_-, z_y)\chi^{SU(3)_{k+}}_\Lambda(q, z_+, z_y)
= \sum_{2\ell^+ = 0}^{\tilde{k}^+} \sum_{n = 0}^{\tilde{k}^+-1} P^\Lambda_{2\ell^+, n}(q) \chi^{3\tilde{k}^+}_{4(a_1 - a_2) + 6(n + \ell^+)}(q, z_y) \chi^{\tilde{k}^+}_{2\ell^+}(q, z_+)
= \sum_{2\ell^+ = 0}^{\tilde{k}^+} \sum_{n = 0}^{\tilde{k}^+-1} P^\Lambda_{2\ell^+, n}(q)
\times \prod_{r = 0}^{k-1} \frac{1}{\eta(q)} \theta_{2(\ell^* + r)\tilde{k}^* - 4(a_1 - a_2) - 6(n + \ell^*) + (\ell^* + r) + 4(a_1 - a_2) + 6(n + \ell^*)}(q, z_y).
\]
The last equality is based on the fact that the $SU(2)_1$ characters are related to theta functions at level 1
\[
\chi^1_{2\ell^*}(q, z_y) \equiv \frac{1}{\eta(q)} \theta_{2\ell^* + 1}(q, z_y^2),
\]
and on another remarkable identity between theta functions, proven in Appendix B,
\[
\theta_{a, 1}(q, z_y^2)\theta_{b, 3\tilde{k}^*}(q, z_y^{2/3}) = \sum_{r = 0}^{k-1} \theta_{(a+2r)\tilde{k}^* - b, k\tilde{k}^*}(q) \theta_{6r + 3a + b, 3k}(q, z_y^{2/3}).
\]
It is now clear from (2.40) that the $z_y$ dependence of the character sumrule is completely encoded in the characters for the $A_{3k}$ algebra which were given above (2.12). Indeed it follows that the realization of $\tilde{\mathcal{A}}_\gamma$ involves that algebra as will be shown more explicitly in the next section. Furthermore, the dependence on the variables $z_+, z_-$ appears through the product of two $SU(2)$ characters, at levels $\tilde{k}^+$ and 1. In order to compare with the RHS of the sumrule (2.4), one is naturally led to express the $\tilde{\mathcal{A}}_\gamma$ characters in terms of the characters for the $SU(2)_{\tilde{k}^+} \times SU(2)_1$ Kac-Moody subalgebra. This is a relatively easy task for the massive $\tilde{\mathcal{A}}_\gamma$ characters. Indeed, as explained above, the latter factorize in the product containing two affine $SU(2)$ characters at level $\tilde{k}^+ - 1$ and $\tilde{k}^- - 1$ [9]. From this expression and from the GKO formula [19]

\[
\chi_{2\ell^+}(q, z) = \sum_{\ell \equiv 2\ell^+ \mod 2}^{\tilde{k}^+} \chi_{2\ell^+}(q, z) \chi_{2\ell^++1, 2\ell^++1}(q) \tag{2.42}
\]

where $\chi_{\rho, q}^{Vir, (m)}$ are the unitary minimal model Virasoro characters for $c = 1 - 6/m(m + 1)$, one easily works out that

\[
\eta(q) Ch_{m, NS}^{\tilde{\mathcal{A}}_\gamma, NS}(\tilde{k}^+, \tilde{k}^-, \tilde{k}^NS, \ell^+, \ell^-; q, z_+, z_-) = \sum_{\ell^+ = 0}^{\tilde{k}^+} \sum_{\ell^- = 0}^{\tilde{k}^-} \chi_{2\ell^+}(q, z_+) \chi_{2\ell^-}(q, z_-) \chi_{\ell^+, \ell^-}^{Vir,(\tilde{k}^+)}(q) \chi_{\ell^+, \ell^-}^{Vir,(\tilde{k}^-)}(q), \tag{2.43}
\]

where $\chi_{\ell, \ell'}^{Vir,(k^\pm)}(q)$ are Virasoro characters at level $k^\pm$, and are understood to be 1 whenever $k^+$ or $k^-$ are integers smaller than 3.

Similar expressions hold in the Ramond sector and can be obtained by spectral flow from (2.43).

For the massless characters it is not possible to provide similar explicit expressions for the corresponding branching functions. However, these branching functions are related in a very interesting way to the modular forms associated with the massive sumrule above. Let us introduce the branching functions $Y^{(\tilde{k}^+ + 2)}(q)$ for the NS $\tilde{\mathcal{A}}_\gamma$ massless characters when $\tilde{k}^- = 1$, so that the combinations (2.9) are written as

\[
Ch_{0, NS}^{\tilde{\mathcal{A}}_\gamma, NS}(L, q, z_+, z_-) = \sum_{\ell^+ = 0}^{\tilde{k}^+} \sum_{\ell^- = 0}^{1} \frac{(-1)^{2\ell^++L-1}}{2\ell^++2\ell^- \equiv L-1 \mod 2} Y_{\ell^+, \ell^-, q}^{(\tilde{k}^+ + 2)}(q) \chi_{2\ell^+}(q, z_+) \chi_{2\ell^-}(q, z_-). \tag{2.44}
\]

These branching functions seem to have an intriguing relation to minimal unitary Virasoro characters [20]. There are as many functions $Y^{(\tilde{k}^+ + 2)}(q)$ at fixed value of $\tilde{k}^+$ as there are Virasoro characters at level $\tilde{k}^+ + 2$, they possess the same symmetries as the corresponding Virasoro characters, and they transform in a similar way under modular transformations [20].

Let us finish this section by summarizing the analytic structure of the character sumrules
in the NS sector when \( \tilde{k}^- = 1 \),

\[
\sum_{2\ell^- = 0}^{1} \sum_{2\ell^+ = 0}^{\tilde{k}^+} \chi_{2\ell^+}^+(q, z_+ \chi_{2\ell^-}^1(q, z_-)
\]

\[
\times \sum_{n=0}^{\tilde{k}^++1} P_{2\ell^+}^\Lambda(n, q) \prod_{r=0}^{k-1} \eta(q) \theta_{2(\ell^- + r)}(q) \chi_{2\ell^+}^+(q, z_+) \chi_{2\ell^-}^1(q, z_-)
\]

\[
= \sum_{2\ell^- = 0}^{1} \sum_{2\ell^+ = 0}^{\tilde{k}^+} \chi_{2\ell^+}^+(q, z_+ \chi_{2\ell^-}^1(q, z_-)
\]

\[
\times \sum_{L=0}^{L+1 = 2\ell^+ + 2\ell^- \mod 2} \{ (-1)^{2\ell^+ + L + 1} M^L_\Lambda(q, z_y) Y_{2\ell^+ + 1, L+1}(q)
\]

\[
+ (1 - \delta_{L,0})(1 - \delta_{L,2-}) \sum_{n \in \mathbb{Z}_k} \chi_{-2(a_1 - a_2) + 6n + 3(L-1)}^k(q, z_y) F^A_{L-1, n}(q) \eta(q)_{-1} \chi_{L, 2\ell^+ + 1}^{Vir, (k^+)}(q) \}.
\]

(2.45)

The information encoded in the above expression for the sumrules and the symmetry properties (2.21), (2.29), (2.33), (2.37) allow for a complete determination of the functions \( P_{2\ell^+, n}^\Lambda(q) \), and for expressing the functions \( F_{2\ell^+, n}^\Lambda(q) \) in terms of the branching functions \( Y_{(k^+ + 2)}^+(q) \), as derived in Appendix A for \( \tilde{k}^+ = 1, 2 \). The latter are not yet known for \( \tilde{k}^+ > 1 \), and will be discussed further elsewhere [20]. Let us however mention that the results obtained by Kazama and Susuki [21] may provide an indirect route to the determination of these branching functions. Indeed, these authors constructed \( N = 2 \) superconformal theories by considering cosets of the form

\[
SU(n + m) \times SO(2nm) \quad \frac{SU(n) \times SU(m) \times U(1)}{SU(n) \times SO(2nm) \times U(1)}
\]

(2.46)

involving hermitian symmetric spaces. When \( n = 1 \) and \( m = 2 \), one has the following schematic structure

\[
SU(3)_{\tilde{k}^+} \times SO(4)_1 \sim [N = 2]_{KS} \times SU(2)_{\tilde{k}^+ + 1} \times U(1).
\]

(2.47)

On the other hand, our analysis gives

\[
SU(3)_{\tilde{k}^+} \times SO(4)_1 \sim [\hat{\mathcal{A}}_\gamma] \times U(1),
\]

(2.48)

where we have found that both the \( U(1) \) and the \([\hat{\mathcal{A}}_\gamma]\) occur with extended algebras. This result underlines once more [3] the intimate connection between the doubly extended \( N = 4 \) superconformal algebras and these Kazama-Suzuki \( N = 2 \) theories.

### 3 Embedding of \( \hat{\mathcal{A}}_\gamma \oplus \mathcal{A}_{3k} \) in an affine \( \hat{SU}(3)_{\tilde{k}^+} \) Module \( \otimes \) Fermion Fock Space.

The reduction of this module according to the rational torus algebra \( \mathcal{A}_{3k} \) and \( \hat{\mathcal{A}}_\gamma \) was qualitatively described above. It plays a crucial role for the derivation of the results we present. In this section we indicate some details of the pertinent proofs.
3.1 Proof that $\tilde{A}_γ$ highest weight states come as multiplets forming representations of $A_{3k}$

For more details concerning the construction of $\tilde{A}_γ$ in terms of Wolf space fermions and affine $SU(3)_{k+}$ we refer to ref. [7]. Here we briefly introduce the notation. The construction is based on two complex fermions, $ψ^+(z), ψ_+(z)$ and $ψ^-(z), ψ_-(z)$ pairwise conjugate, as well as on the affine currents of $SU_3(3)_{k+}$, denoted $V^i(z), V_i(z)$ with $i \in \{+,-,θ,m\}$. The $SU(2)_{k+}$ subalgebra of $\tilde{A}_γ$ is then generated by $V^θ = T^+, V_θ = T^-\ θ$ and their OPE, involving $T^{+3}$ given as a combination of the Cartan generator currents, $V^m, V_m$. Let us denote by $Y_{SU(3)}$ the $U(1)$ current in $SU(3)_{k+}$ corresponding to the usual $SU(3)$ “hypercharge”. It is proportional to the combination of $V^m, V_m$ orthogonal to $T^{+3}$. Then $(V^+, V_-)$ form an $ℓ^+ = \frac{1}{2}$ doublet with $Y_{SU(3)} = +1$ and $(V^−, V_+)$ form an $ℓ^- = \frac{1}{2}$ doublet with $Y_{SU(3)} = -1$.

The WS fermions build the $SU(2)_{1}$ generators as follows

$$T^{−} = −2iψ_+ψ_−, \quad T^{−} = −2iψ^+ψ^−, \quad T^{−3} = ψ^+ψ_+ + ψ^-ψ_−.$$  \hfill (3.1)

So $(ψ_−, ψ^+)$ form an $ℓ^- = \frac{1}{2}$ doublet with $Y_{WS} = +1$ whereas $(ψ_+, ψ^-)$ form an $ℓ^- = \frac{1}{2}$ doublet with $Y_{WS} = -1$. Here the $U(1)$ current corresponding to “Wolf space hypercharge” is expressed by

$$Y_{WS} ≡ (ψ^+ψ_+ − ψ^-ψ_−).$$

The $U(1)$ current, $U(z)$ of $A_γ$, which decouples from $\tilde{A}_γ$ is then given by

$$U = i\sqrt{3} Y = i\sqrt{3} (Y_{SU(3)} + Y_{WS}).$$ \hfill (3.2)

The four supersymmetry currents of $\tilde{A}_γ$ are given by

$$G_+ = \frac{1}{\sqrt{2}} (V^+ψ_+ + V^-ψ_−), \quad G_- = \frac{1}{\sqrt{2}} (V_+ψ^+ + V_−ψ^-),$$

$$G_{+K} = \frac{i}{\sqrt{2}} (V^+ψ^- − V^-ψ^+), \quad G_{−K} = −\frac{i}{\sqrt{2}} (V_+ψ_− − V_−ψ_+).$$ \hfill (3.3)

Finally the Virasoro generator of $\tilde{A}_γ$ may be expressed as (normal ordering implied)

$$L = −\frac{1}{8} (V^iV_i + V_iV^i) − (∂ψ_+ψ^+ + ∂ψ^+ψ_+ + ∂ψ_−ψ^- + ∂ψ^-ψ_−) + \frac{1}{k} UU.$$ \hfill (3.4)

The rational torus model character, (2.12), corresponds to an irreducible representation of $A_{3k}$ labelled by the integer, $m$, which splits into infinitely many representations of $Y(z) = \frac{2}{i\sqrt{3}} U(z)$ with zero modes $Y ≡ \frac{2}{\sqrt{3}} u$ each of which has

$$h^Y = \frac{3}{4k} Y^2 = \frac{u^2}{k}, \quad Y = \frac{1}{3} (m + 6kn), \quad n \in \mathbb{Z}.$$ \hfill (3.5)

In ref. [7] a finite set of $\tilde{A}_γ$ highest weight states were explicitly constructed in terms of the WS fermions and the $SU(3)_{k+}$ generators. Massless as well as massive examples were
presented. Our present assertion is that these were merely examples of what are in fact infinite families of \( \tilde{A}_\gamma \) highest weight states in the space

\[
\mathcal{H}(\Lambda, \tilde{k}^+) = \mathcal{H}^{WS} \otimes \mathcal{H}_{\Lambda}^{SU(3)_{\tilde{k}^+}},
\]

the tensor product of the Fock space for the WS fermions and a module for \( SU(3)_{\tilde{k}^+} \) in the representation labelled, \( \Lambda = (\tilde{a}, \tilde{k}^+, 0) \), \( \tilde{a} \) being the weight of the corresponding \( SU(3) \) representation.

In fact we shall now present an operator, \( K(p) \) for all \( p \in \mathbb{Z} \) with the property that if \( |hws\rangle \) is a certain \( \tilde{A}_\gamma \) highest weight state in \( \mathcal{H}(\Lambda, \tilde{k}^+) \), then the set of states

\[
\{|hws, p\rangle \equiv K(p)|hws\rangle |p \in \mathbb{Z}\}
\]

have the following properties,

1. they are \( \tilde{A}_\gamma \) highest weight states;
2. they are \( Y \) highest weight states;
3. they carry an irreducible representation of \( A_{3k} \) labelled by an integer, \( m \) easily computable in terms of \( |hws\rangle \) (see below).

For simplicity let us first consider the simplest possible \( \tilde{A}_\gamma \) highest weight state; the WS vacuum tensored with the \( SU(3)_{\tilde{k}^+} \) singlet highest weight state. Denote that combined state by \(|0\rangle\). We define the action of \( K(p) \) by the state \(|p\rangle\) obtained by

\[
|p\rangle \equiv K(p)|0\rangle \equiv \left\{ \begin{array}{ll}
\{(V^+ V^-)_{\tilde{k}^+} (\psi^+ \psi_-)^3\}^p & p \geq 0 \\
\{(V^+ V^-)_{\tilde{k}^+} (\psi^+ \psi_-)^3\}^{|p|} & p < 0.
\end{array} \right.
\]

(3.6)

Here we have introduced the following condensed notation: Let \( \{A_i(z)\} \) be any set of primary field operators with some standard mode expansion. By the state

\[
A_{i_1} A_{i_2} \ldots A_{i_n} |0\rangle,
\]

we mean to first consider the state

\[
A_{i_1}(z_1) A_{i_2}(z_2) \ldots A_{i_n}(z_n) |0\rangle
\]

followed by taking successively the limits, \( z_n \to 0 \), then \( z_{n-1} \to 0 \), etc. and finally extracting the leading term in that limit. As an example, with this prescription (we consider the NS sector, only for simplicity)

\[
(\psi^+ \psi_-)^3 |0\rangle \equiv \psi^+_{-5/2} (\psi_-)_{-5/2} \psi^+_{-3/2} (\psi_-)_{-3/2} \psi^+_{-1/2} (\psi_-)_{-1/2} |0\rangle.
\]

This notation implies that we have defined the action of the operator, \( K(p) \) for all states. It is easy to verify that the state \(|p\rangle\) has the following properties

\[
\psi^+_r |p\rangle = (\psi_-)_r |p\rangle = 0, \quad r \geq -3p + \frac{1}{2}
\]

\[
(\psi^+_r |p\rangle = \psi^-_r |p\rangle = 0, \quad r \geq +3p + \frac{1}{2}
\]

\[
V^+_N |p\rangle = (V^-)_N |p\rangle = 0, \quad N \geq -3p
\]

\[
(V^+_N |p\rangle = V^-_N |p\rangle = 0, \quad N \geq +3p
\]

\[
(V^-_{-3p-1})^n |p\rangle = [(V^-)_{-3p-1}]^n |p\rangle = 0, \quad n > \tilde{k}^+.
\]

(3.7)
With this it is easy to prove that the states, \(|p⟩\) are all massless \(\tilde{A}_γ\) highest weight states. Further they are highest weight states of the \(U(1)\) algebra of \(Y(z)\), and finally they have the quantum numbers

\[
\ell^+ = \ell^- = 0 \quad Y = 2pk = 2u/\sqrt{3} \quad h = 3p^2k \quad h^Y = \frac{u^2}{k} = 3p^2k = h,
\]

so that the conformal dimension \(\tilde{h}\) pertaining to \(\tilde{A}_γ\) is \(\tilde{h} = h - h^Y = 0\) independent of \(p\). Hence, indeed these states furnish a representation of \(A_{3k}\) with \(m = 0\).

It is simple to see that this technique in fact works also when \(K(p)\) acts on other \(\tilde{A}_γ\) highest weight states: the resulting state is (i) also an \(\tilde{A}_γ\) highest weight state with the same \(\tilde{A}_γ\) quantum numbers, (ii) it is a highest weight state for \(Y(z)\), and (iii) the set of all these for \(p ∈ \mathbb{Z}\) provides a representation of \(A_{3k}\) with

\[
m ≡ 3Y \mod 6k
\]

where \(Y\) is the total “hypercharge” of any of the states, \(K(p)|\text{hws}\rangle\).

This technique implies that in the following we only have to understand one \(\tilde{A}_γ\) highest weight state for each set of \(\tilde{A}_γ\) quantum numbers and \(m\). The operator \(K(p)\) will furnish the infinite tower corresponding to the representation of \(A_{3k}\).

### 3.2 Massless \(\tilde{A}_γ\) states

We now provide a complete list of all massless \(\tilde{A}_γ\) highest weight states in \(\mathcal{H}(\tilde{k}^+, \Lambda)\). For \(\bar{a}\) an \(SU(3)\) weight corresponding to an integrable representation (for the given integer value of \(\tilde{k}^+\)), i.e. \(a_i\) non zero integers such that,

\[
0 ≤ a_1 + a_2 ≤ \tilde{k}^+,
\]

let,

\[
|\bar{a}; p⟩ = K(p)|\bar{a}; 0⟩,
\]

the last state being,

\[
|\bar{a}; 0⟩ ≡ |\bar{a}\rangle ⊗ |0⟩,
\]

the tensor product of the \(SU(3)_{\tilde{k}^+}\) highest weight state, \(|\bar{a}\rangle\) and the WS Fock space vacuum, \(|0⟩\). In fact, according to the above we only need consider the \(p = 0\) case, the rest being trivially obtained by acting with \(K(p)\).

The complete list of massless states for

\[
\Lambda = (\bar{a}, \tilde{k}^+, 0)
\]
The states considered in ref. \cite{7} were however here we must find all massless \( \tilde{\psi} \). These were enough to exhaust the quantum number possibilities for massless \( \tilde{\psi} \). To notice a particularly relevant property of these states when presented in terms of the space of the quarks which is a flavour singlet. We shall understand the relevant projection to the vacuum of the quarks and the WS fermions. The state, \(|\tilde{\psi}_i;0\rangle\) may then be thought of as the following (or rather a combination of such states to make a flavour singlet)

\[
|\tilde{\psi}_i;0\rangle = (q_{f_1}^{i_1})_{-\frac{1}{2}} \cdots (q_{f_{a_1}}^{i_{a_1}})_{-\frac{1}{2}} (\bar{q}_{f_2}^{i_2})_{-\frac{1}{2}} \cdots (\bar{q}_{f_{a_2}}^{i_{a_2}})_{-\frac{1}{2}} |0\rangle
\]

The quantum numbers of these states are as follows,

\[
\begin{align*}
2\ell_i^- &= 0, \quad i = 1, 2, 3 \\
2\ell_i^+ &= 1, \quad i = 4, 5, 6 \\
2\ell_i^+(\tilde{\alpha}) &= \tilde{k}^+ - a_1, \quad m_1(\tilde{\alpha}) = a_1 - a_2 - 3(\tilde{k}^+ - a_2) \\
2\ell_i^+(\tilde{\alpha}) &= \tilde{k}^+ - a_2, \quad m_2(\tilde{\alpha}) = a_1 - a_2 + 3(\tilde{k}^+ - a_1) \\
2\ell_i^+(\tilde{\alpha}) &= a_1 + a_2, \quad m_3(\tilde{\alpha}) = a_1 - a_2 \\
2\ell_i^+(\tilde{\alpha}) &= \tilde{k}^+ - 2\ell_i^+(\tilde{\alpha}), \quad m_{i+3}(\tilde{\alpha}) = m_i(\tilde{\alpha}) + 3k, \quad i = 1, 2, 3.
\end{align*}
\]

The states considered in ref. \cite{7} were

\[|\overline{3}(\tilde{\alpha};0), \quad |(4)\tilde{\alpha};0), \quad |(5)\tilde{\alpha};0).\]

These were enough to exhaust the quantum number possibilities for massless \( \tilde{\psi} \). states, however here we must find all massless \( \tilde{\psi} \) states in \( \mathcal{H}(A, \tilde{k}^+) \). That the above list is exhaustive is partly conjectural. In this connection as well as in general, however, it is interesting to notice a particularly relevant property of these states when represented in terms of the standard fermionic “quark model” of the \( SU(3)_{\tilde{k}^+} \) current algebra.

Thus consider \( 3 \times \tilde{k}^+ \) complex fermion pairs,

\( q_i^f(z), \ \bar{q}_i^f(z) \)

with \( i = 1, 2, 3 \) being \( SU(3) \) triplet indices and \( f = 1, 2, \ldots, \tilde{k}^+ \) being “flavour” indices. Our \( SU(3)_{\tilde{k}^+} \) representation spaces may be thought of as lying in the subspace of the Fock space of the quarks which is a flavour singlet. We shall understand the relevant projection to flavour singlets being performed whenever necessary. By \(|0\rangle\) we now understand the combined vacuum of the quarks and the WS fermions. The state, \(|\tilde{\alpha};0\rangle\) may then be thought of as the following (or rather a combination of such states to make a flavour singlet)

\[
|\tilde{\alpha};0\rangle = (q_{f_1}^{i_1})_{-\frac{1}{2}} \cdots (q_{f_{a_1}}^{i_{a_1}})_{-\frac{1}{2}} (\bar{q}_{f_2}^{i_2})_{-\frac{1}{2}} \cdots (\bar{q}_{f_{a_2}}^{i_{a_2}})_{-\frac{1}{2}} |0\rangle
\]

the sets, \( \{f_1, \ldots, f_{a_1}\} \) and \( \{f'_1, \ldots, f'_{a_2}\} \) being non-overlapping. Similarly the \( SU(3)_{\tilde{k}^+} \) generators entering the supercurrents are

\[
\begin{align*}
V^+ &\sim \bar{q}_3^f q_1^f \\
V_- &\sim \bar{q}_2^f q_3^f \\
V^+ &\sim \bar{q}_3^f q_1^f \\
V_- &\sim \bar{q}_2^f q_3^f
\end{align*}
\]

(3.11)
sum over $f$ implied.

The states, $|p\rangle$ above then have the following further properties

$$
(q^f_1)_r|p\rangle = (q^f_2)_r|p\rangle = 0, \quad r \geq -p + \frac{1}{2},
$$

$$
(q^f_1)_r|p\rangle = (q^f_2)_r|p\rangle = 0, \quad r \geq +p + \frac{1}{2},
$$

$$
(q^f_3)_r|p\rangle = 0, \quad r \geq -2p + \frac{1}{2},
$$

$$
(q^f_3)_r|p\rangle = 0, \quad r \geq +2p + \frac{1}{2}.
$$

(3.12)

These relations together with those in (3.7) show that the states $|p\rangle$ have the nice property of simply representing shifted vacuum levels for all the fermions – WS ones and quarks alike – evenly in all flavours. Likewise, all the massless states given in (3.9) have the property that they represent shifted fermionic vacuum states, however with the vacuum levels shifted by amounts depending on flavour. More precisely, linear combinations of such states corresponding to projections to flavours singlets. Finally we point out that the enumerations of the massless $\tilde{A}_\gamma$ highest weight states in (3.9) immediately allow us to write down the expression for the matrix, $M^{\Lambda L}(q, z_y)$, (2.11), by inspection.

### 3.3 Massive $\tilde{A}_\gamma$ states

From the above discussion several points concerning the embedding of massive highest weight $\tilde{A}_\gamma$ states in $\mathcal{H}(\Lambda, \tilde{k}^+)$ immediately emerge.

First our general argument demonstrates that for given $\tilde{A}_\gamma$ quantum numbers, they too come in infinite multiplets corresponding to a representation of $A_{3k}$. Second, our studies of modular properties of characters as summarized in Section 4, imply that for given $\ell^+$ ($\ell^-$ is necessarily 0 for massive representations in the NS sector and $0 \leq 2\ell^+ \leq \tilde{k}^+ - 1$, cf. ref. [8]), there must be a denumerable infinity of massive $\tilde{A}_\gamma$ highest weight states corresponding to a certain infinite set of $\tilde{h}$ values. This follows because we know that the characters of $WS \otimes SU(3)_{\tilde{k}^+}$ transform according to a finite representation of the modular group whereas that will be possible only if infinitely many $\tilde{A}_\gamma$ characters are involved on the right hand side of the sumrule for $\{\chi^{WS,NS}, \chi_\Lambda\}_m$ in eq. (2.19). It is completely outside the scope of the present paper to provide an understanding as to how these infinitely many massive $\tilde{A}_\gamma$ representations are embedded in detail in $\mathcal{H}(\Lambda, \tilde{k}^+)$. It is interesting to remark, however, that in the quark model description introduced above they will all correspond to certain (solitonic) excitations above the shifted vacua corresponding to the massless states.

Also, for a given $SU(3)_{\tilde{k}^+}$ representation $\Lambda$ containing the $SU(3)$ representation, labelled by $\tilde{a}$, and for given $\ell^+$ it is not too hard to provide the rule to work out what $m$ values will occur in the sum rule as far as the representations of $A_{3k}$ are concerned. To this we now turn.

First notice that if we have imbedded in $\mathcal{H}(\Lambda, \tilde{k}^+)$ a representation of $\tilde{A}_\gamma \oplus A_{3k}$ with quantum numbers $\ell^+$ for $\tilde{A}_\gamma$ ($\ell^- = 0$) and $\overline{m}$ for $A_{3k}$, then we can expect all the ones with the same $\ell^+$ and with $m = \overline{m} + 6\mathbb{Z}_k$ to arise as well. The corresponding states may be obtained from the highest weight of the first one by a suitable operator, and since neither $\ell^+$
nor $\ell^-$ is allowed to change that operator must involve an integer number of pairs $(V^+, V_-)$ and $\psi^+\psi_-$ (or their conjugates), each pair contributing an even increment to $Y$ and thus a multiple of 6 to $m$.

To determine the offset in $m$ away from $6\mathbb{Z}_k$, we notice that the highest weight state in the representation of $\tilde{A}_\gamma \otimes \mathcal{A}_{3k}$ must have $(\ell^+, Y)$ quantum numbers lying in the weight lattice of the $SU(3)$ representation in question. A little consideration shows that this implies that the rule for obtaining the possible $m$ values for given $\Lambda$ and $\ell^+$ is as follows. Let

$$t \equiv a_1 - a_2 \mod 3$$

be the triality of the $SU(3)$ representation, then

$$m = \begin{cases} 
-2t + 6\mathbb{Z}_k & \text{for } 2\ell^+ \text{ even} \\
3 - 2t + 6\mathbb{Z}_k & \text{for } 2\ell^+ \text{ odd.}
\end{cases}$$

(3.13)

This means that we may write the massive part of the sumrule, (2.19) as follows,

$$\{\chi^{WS,NS}(q, z_-, z_y)\chi_{\Lambda}(q, z_+, z_y)\}_m \left[ \sum_{2t^+=0}^{k+1} C h_{m,NS}^{\tilde{A}_\gamma}(\ell^+, q, z_+, z_-) \sum_{n \in \mathbb{Z}_k} \chi_{-2(a_1 - a_2) + 6n + 6\ell^+}(q, z_y) F^{\Lambda}_{2\ell^+, n}(q) \right],$$

(3.14)

where the massive $\tilde{A}_\gamma$ character is taken at the special value of conformal dimension $\tilde{h}_m$ of the previous section, and where the set of massive representations arising is given by the expansion of the modular forms, $F^{\Lambda}_{2\ell^+, n}(q)$ in a power series in $q$ with positive integer coefficients, apart from some overall prefactor involving a fractional power of $q$.

This justifies (2.20).

4 Modular Properties of $\tilde{A}_\gamma$ characters

The very form of the massive $\tilde{A}_\gamma$ characters (2.43) allows for a straightforward derivation of their modular properties in terms of the modular transformations of $S\tilde{U}(2)_{\tilde{k}+} \times S\tilde{U}(2)_{\tilde{k}+}$ and Virasoro characters at level $\tilde{k}+1$. Because the combinations of massless $\tilde{A}_\gamma$ characters $Ch_{0,NS}^{\tilde{A}_\gamma}(L)$ do not have the simple form of the massive ones, their modular transformations are not so easily obtained. If one assumes that the massive and massless sectors of the sumrules (2.4) decouple from each other under S and T, the number of equations for the determination of the unknown modular transformation matrices by far exceeds the number of unknowns, more and more so as $\tilde{k}+ \to \infty$. Thus, finding a consistent solution we take as a strong indication for the validity of the conjecture. A consistent solution is one compatible with the modular transformations of the LHS of the sumrules, which involve $S\tilde{U}(3)_{\tilde{k}+}$. We will show below that it implies the remarkable result, that the combinations of massless characters transform like $S\tilde{U}(2)_{\tilde{k}+1}$ characters. Indeed, as was pointed out in [6], when the angular variables $z_{\pm}$ are suitably correlated, they do indeed reduce to such characters, but for general values of $z_{\pm}$ that is not the case. Furthermore, the decoupling hypothesis implies that the functions $F^{\Lambda}_{2\ell^+, n}(q)$ introduced in (2.19) carry a finite representation of the
modular group. When multiplied by $1/\eta$, these functions have the right properties to be the characters (or linear combinations thereof) of a rational conformal field theory with central charge

$$c_\phi = 1 + 3(\tilde{k}^+ - 1)^2/(\tilde{k}^+ + 1)(\tilde{k}^+ + 3).$$

(4.1)

In the case $\tilde{k}^+ = 1$, as explained in Appendix A, the functions $F^\Lambda_{2\ell,n}(q)$ are completely determined. The decoupling may be verified and the combinations of massless characters actually transform as $SU(2)_2$ characters. We leave for further investigation [20] the general formula for the modular transformations of the functions $F^\Lambda_{2\ell,n}(q)$ and only give here the result for the next theory in the series ($\tilde{k}^+ = 2$). If the functions $F^\Lambda_{2\ell,n}(q)$ are organized in a column vector

$$\frac{1}{\eta} (F_{0,0}^{((0,0),2,0)}, F_{0,1}^{((0,0),2,0)}, F_{0,2}^{((0,0),2,0)}, F_{0,0}^{((1,1),2,0)}, F_{0,1}^{((1,1),2,0)}, F_{0,2}^{((1,1),2,0)})^T,$$

(4.2)

the matrix of their S transform is

$$S = \frac{2}{5} \begin{bmatrix} s_3 & 2s_3 & 2s_3 & s_6 & 2s_6 & 2s_6 \\ s_3 & 2s_3c_6 & -s_6 & s_6 & s_{12} & -2s_6c_3 \\ -s_6 & 2s_3c_6 & s_6 & -2s_6c_3 & s_{12} & s_6 \\ s_6 & 2s_6 & -s_6 & -2s_3 & s_{12} & -2s_3 \\ s_6 & s_{12} & -2s_6c_3 & -s_3 & -2s_3c_6 & s_6 \\ -s_6 & -2s_6c_3 & s_{12} & -s_3 & s_6 & -2s_3c_6 \end{bmatrix},$$

where we have defined $s_n = \sin \frac{n\pi}{15}$ and $c_n = \cos \frac{n\pi}{15}$. The above matrix is not orthogonal such as would be expected for the transformation matrix for a set of characters. However, it seems very likely that these modular functions are instead combinations of characters of some extended algebra, with the reduction of angular variables implying symmetries that have forced us to look at such combinations only. Indeed in the limit $z_y = 1$ the characters of the rational torus model $A_{3k}$ exhibit this very behaviour.

Let us now concentrate on the massless sector and show how the $SU(2)_{\tilde{k}^+ + 1}$ transformation laws emerge as a consistent solution to the modular transformations of the sumrules. The S transform of the massless sector of the sumrule (2.10) for a weight $\vec{a}$ of triality $t$ is given by (we set $z_y = 1$ in the following for simplicity)

$$\{\chi^{W,S,NS}_{\vec{a}} \cdot \sum_{\epsilon = 0, \pm 1} \sum_{\vec{a}' \in (0,0)} \left[S(t, \epsilon)\right]^{\phi^t(\vec{a})} \chi_{\phi^t(\Lambda')} \}(0) = (M(t)^S)^L_{\vec{a}} (Ch_0^{A_{3k},NS})^{S}(L),$$

(4.3)

where we use the S transform of the affine $SU(3)_{\tilde{k}^+}$ characters in the form

$$(\chi^t_\Lambda)^S = \Sigma_{\vec{a}}^{\vec{a}'} \chi^t_{\Lambda'},$$

(4.4)

with $\Lambda' = (\vec{a}', \tilde{k}^+, 0)$ and

$$\phi^t(\Lambda) \equiv \phi^t(\vec{a}), \tilde{k}^+, 0).$$

Further,

$$\Sigma_{\vec{a}}^{\vec{a'}} = \Sigma_{\vec{a}}^{\vec{a'}} + (\Sigma_{\vec{a}}^{\vec{a'}})^*,$$

(4.5)
and
\[ \Sigma_\vec{a}^\epsilon = \frac{i}{k\sqrt{3}} z_k^{-2(\alpha_1 - \alpha_2)(\alpha'_1 - \alpha'_2)} (z_k^{-2\alpha_1\alpha'_2} - 1)(z_k^{-2\alpha_2\alpha'_1} - 1). \] (4.6)

The notations are \( z_k = e^{i\pi \alpha}, \alpha_i = a_i + 1, \) and \( \vec{a} = (a_2, a_1) \) when \( \vec{a} = (a_1, a_2) \). Also we write \( \Lambda_C \equiv (\vec{a}_C, k^+, 0) \).

In the above formula, the weights \( \vec{a}' \) correspond to integrable \( SU(3)_{k^+} \) representations and are classified according to the \( \epsilon \) region to which they belong. By definition, a weight \( \vec{a}' = (a'_1, a'_2) \) belongs to the region \( \epsilon = 0 \) if it is the representative of its orbit under the transformation \( \phi (2.30) \) with the lowest \( a'_1 + a'_2 \) value. If two weights within the same orbit have the same \( a'_1 + a'_2 \) value, then the weight with the highest \( a'_1 \) value will belong to the \( \epsilon = 0 \) region. One naturally obtains weights in the \( \epsilon = \pm 1 \) regions by applying \( \phi^{\pm 1} \) to the \( \epsilon = 0 \) weights. For a weight \( \vec{a} \) of triality \( t \), and \( \vec{a}' \) having a value of \( \epsilon \), one may write
\[ S(t, \epsilon)_{\vec{a}}^{\vec{a}'} = y^{-t\epsilon} S(t)_{\vec{a}}^{\vec{a}'}, \] (4.7)

where \( y = e^{2\pi i} \). The unitarity condition then takes the form
\[ \sum_{\epsilon} \sum_{\vec{a}'} [S(t)_{\vec{a}}^{\vec{a}'}(S(t')_{\vec{a}}^{\vec{a}'})^*] y^{\epsilon(t'-t)} = 1. \] (4.8)

It is clear that the above expression vanishes whenever \( t \neq t' \), whereas for \( t = t' \), the sum over \( \epsilon \) merely gives a factor 3. It follows that the reduced matrices \( S(t) \) are normalized so that \( S(t)S(t)^\dagger = 1/3 \) for all \( t \). A complication occurs whenever \( k^+ \) is a multiple of 3 and \( t \neq 0 \) since the matrix \( S(t) \) is not square, having an extra column corresponding to the central representation with Dynkin labels \((\frac{k^+}{3}, \frac{k^+}{3})\). One however can prove that the matrix element of \( S(t \neq 0) \) having this representative as column label vanishes.

If one defines the S transform of combinations of massless \( \hat{A}_\gamma \) characters as
\[ (Ch_0^{\hat{A}_\gamma, NS}(L))^S = S_L^{(4.9)} Ch_0^{\hat{A}_\gamma, NS}(L'), \]
the above considerations allow to rewrite the transformation law \((4.3)\) as the following “master equation”
\[ 3 \sum_{\vec{a}} (S(t)_{\vec{a}}^{\vec{a}'})^*(M(t)^S)^L_{\vec{a}} = \sum_{\epsilon, L'} M_{\phi^L(\Lambda')}(\epsilon) y^{-t\epsilon} S_{L'}^L. \] (4.10)

Here, the modular transformation matrix for the \( SU(3) \) characters is the reduced one pertaining to \( \epsilon = 0 \). The index \( \vec{a} \) runs over triality \( t \) representations only, whereas \( \vec{a}' \) is a representation pertaining to \( \epsilon = 0 \). For \( k^+ \) a multiple of 3 and \( t \neq 0 \), it is understood that the representation \( \vec{a}' = (\frac{k^+}{3}, \frac{k^+}{3}) \) is removed. As explicitly shown in Appendix C, a solution to the above equation is provided by
\[ S_{L'}^L = -\frac{2}{k} \sin \left( \frac{\pi}{k} (L + 1)(L' + 1) \right), \quad L, L' = 0, ..., k - 2, \] (4.11)

which corresponds, up to a sign, to the elements of the S transform matrix for \( SU(2)_{k^+ + 1} \) characters.

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5 Conclusions

In a series of previous publications [9, 10, 11, 6] we provided a rather detailed study of the characters for the unitary highest weight representations of the $N = 4$ doubly extended superconformal algebra $\tilde{A}_\gamma$. In this paper we have presented a number of investigations aimed at understanding field theories based on this algebra. To this end we have made use of previously known coset constructions realizing the algebra $\tilde{A}_\gamma$ [3, 4, 7, 5]. We have concentrated on the family based on the Wolf space $SU(3)/SU(2) \times U(1)$ which corresponds to one of the two central extensions being unity ($\tilde{k}^-$ in our case) but the other, $\tilde{k}^+$, being arbitrary. We have studied the character sumrules induced by this realization, and our main results are as follows.

First it has been possible to obtain information on the modular transformation properties of the massless characters. Remarkably, certain combinations of massless characters transform exactly as $SU(2)$ characters of level $\tilde{k}^+ + \tilde{k}^-$. This result was mentioned in [6] where it emerges from a completely different point of view, and some of the theorems stated there are proven in this paper.

Second, we show in two different ways that the realization gives rise to $\tilde{A}_\gamma \times U(1)$, or more interestingly to the particular rational extension $A_{3k}$ of $U(1)$ having an extra dimension $3k = 3(k^+ + k^-)$ operator in the algebra [15]. This result is analogous to what happens when one studies the branching of affine $SU(3)$ characters into $SU(2)$ ones [18], and in fact provides one way of proving that result. Perhaps more importantly this reduction allows the derivation of the modular properties of the combinations of massless characters.

Third, the massive part of the sumrule is shown to be written as infinite sums of massive $\tilde{A}_\gamma$ characters, which not only involve the $A_{3k}$ characters again, but also certain modular forms $F_\Lambda^A$, which appear to transform according to a finite dimensional representation of the modular group. These forms multiply the massive characters evaluated at a particular conformal dimension, which is chosen in such a way that the massive characters also transform according to a finite dimensional representation of the modular group. This strongly implies that these modular forms are intimately related to a separate rational conformal field theory, a suggestion which is further strengthened by comparison with a recent free field realization of these algebras [12]. We have presented several general properties of these modular forms and we have demonstrated their relation to a variety of branching functions. These studies have revealed a number of remarkable identities. In special cases it has been possible to relate some of the $F_\Lambda^A$ to other known modular forms. It appears that they are also related to coset theories based on certain of the Kazama-Suzuki $N = 2$ theories [21]. This last fact is potentially significant and deserves much further investigation.

In conclusion, we have made progress towards understanding the modular properties of the $\tilde{A}_\gamma$ characters in general and towards an understanding of the rational conformal field theories involved in connection with certain coset realizations in particular. This latter subject contains a rich set of interesting structures relating the conformal field theories with a chiral $\tilde{A}_\gamma$ algebra to several other known but a priori unrelated conformal field theories.
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A Determination of the functions $P_{2\ell^+, n}^\Lambda(q)$ and $F_{2\ell^+, n}^\Lambda(q)$ for $\tilde{k}^+ = 1, 2$ and $\tilde{k}^- = 1$

A.1 The case $\tilde{k}^+ = 1$

When $\tilde{k}^- = \tilde{k}^+ = 1$, i.e. $k = 4$, the three integrable $SU(3)_1$ representations to consider pertain to the single orbit of the transformation $\phi (2.30)$. It is therefore sufficient to consider the sumrule (2.4) for one of them (say the singlet) to determine all functions $P_{2\ell^+, n}^\Lambda(q)$ and $F_{2\ell^+, n}^\Lambda(q)$. The symmetries (2.21), (2.29), (2.33) and (2.37) described in Section 2 allow for a particularly simple sumrule,

\[
\begin{align*}
\sum_{2\ell^+=0}^{1} \sum_{2\ell^-=0}^{1} & \chi^{2\ell^+}(q, z_+)\chi^{2\ell^-}(q, z_-)P_{0, 0}^{(0, 0), 1, 0}(q) \sum_{r=0}^{3} \frac{1}{\eta} \theta_{2(2\ell^+-r)-6\ell^+, 4}(q)\chi^{12}_6(q, z_y) \\
& = Ch_0^{A^+, NS}(0, q, z_+, z_-)[\chi^{12}_6(q, z_y)] + \chi^{12}_6(q, z_y)] \\
& + Ch_0^{A^+, NS}(1, q, z_+, z_-)[\chi^{12}_6(q, z_y)] + \chi^{12}_6(q, z_y)] \\
& + Ch_0^{A^+, NS}(2, q, z_+, z_-)[\chi^{12}_6(q, z_y)] + \chi^{12}_6(q, z_y)] \\
& + Ch_0^{A^+, NS}(0; q, z_+, z_-) \left\{ F_{0,0}^{(0,0), 1, 0}(q)[\chi^{12}_6(q, z_y)] + \chi^{12}_6(q, z_y)] \right\} \\
& + F_{0,1}^{(0,0), 1, 0}(q)[\chi^{12}_6(q, z_y)] + \chi^{12}_6(q, z_y)].\end{align*}
\]

In order to determine the functions $P_{0,0}^{(0,0), 1, 0}(q)$ and $F_{0, n}^{(0,0), 1, 0}(q)$, $n = 0, 1$, it is sufficient to identify the coefficients of the rational torus characters $\chi^{12}_6(q, z_y)$ and $\chi^{12}_6(q, z_y)$ on both sides of the sumrule. Recalling that $\theta_{m,k}(q) = \theta_{-m,k}(q) = \theta_{2k-m,k}(q)$, one gets

\[
\begin{align*}
Ch_0^{A^+, NS}(1, q, z_+, z_-) & = F_{0,0}^{(0,0), 1, 0}(q)Ch_0^{A^+, NS}(0, q, z_+, z_-) \\
= & \frac{1}{\eta} F_{0,0}^{(0,0), 1, 0}(q)[\theta_{0,4}(q)\chi^{1}_6(q, z_+)]\chi^{1}_6(q, z_-) + \theta_{4,4}(q)\chi^{1}_6(q, z_+)\chi^{1}_6(q, z_-)], \quad (A.2)
\end{align*}
\]

and

\[
\begin{align*}
F_{0,1}^{(0,0), 1, 0}(q)Ch_0^{A^+, NS}(0, q, z_+, z_-) = & \frac{1}{\eta} \theta_{2,4}(q) F_{0,0}^{(0,0), 1, 0}(q)[\chi^{1}_6(q, z_+)]\chi^{1}_6(q, z_-) + \chi^{1}_6(q, z_+)\chi^{1}_6(q, z_-)].
\end{align*}
\]

(A.3)
Next, one can use the decomposition of $\tilde{A}_\gamma$ characters in $SU(2)_1 \times SU(2)_1$ characters (2.43), (2.44), namely

$$
Ch_{0}^{\tilde{A}_{\gamma} NS}(0, q, z_+, z_-) = \frac{1}{\eta} [\chi_0^1(q, z_+) \chi_0^1(q, z_-) + \chi_1^1(q, z_+) \chi_1^1(q, z_-)], \quad (A.4)
$$

$$
Ch_{0}^{\tilde{A}_{\gamma} NS}(1, q, z_+, z_-) = Y_{1/16}^{(3)}(q)[\chi_0^1(q, z_+) \chi_0^1(q, z_-) - \chi_1^1(q, z_+) \chi_1^1(q, z_-)], \quad (A.5)
$$

where $Y_{1/16}^{(3)}(q) \equiv Y_{1,2}^{(3)}(q) = Y_{2,2}^{(3)}(q)$. By identifying the coefficients of the $SU(2)_1$ bilinears in the relations above, one obtains the following results,

$$
F_{0,0}^{(0,0),1,0}(q) = \frac{1}{2}[\theta_{0,4}(q) + \theta_{4,4}(q)]P_{0,0}^{(0,0),1,0}(q)
$$

$$
F_{0,1}^{(0,0),1,0}(q) = \theta_{2,4}(q)P_{0,1}^{(0,0),1,0}(q)
$$

$$
Y_{1/16}^{(3)}(q) = \frac{1}{2\eta}P_{0,0}^{(0,0),1,0}(q)[\theta_{0,4}(q) - \theta_{4,4}(q)]. \quad (A.6)
$$

One can see how the functions $P_{0,0}^{(0,0),1,0}(q)$ and $F_{0,0}^{(0,0),1,0}(q)$ are related to the branching function $Y_{1/16}^{(3)}(q)$. The nice feature of the particular case $\tilde{k}^- = \tilde{k}^+ = 1$ is that it is possible to completely determine the branching functions $Y_{1/16}^{(3)}(q)$, as we shall now see. It was noticed in [3] that the Ramond massless $\tilde{A}_\gamma$ characters reduce to $SU(2)$ characters when $z_- = -z_+^{-1}$, a property which, in the NS sector and when $\tilde{k}^- = \tilde{k}^+ = 1$, can be expressed as

$$
Ch_{0}^{\tilde{A}_{\gamma} NS}(L, q, z_+, z_- = -q^{-1/2}z_+^{-1}) = -q^{-1/4}z_+^{-1} \chi_L^2(q, z_+). \quad (A.7)
$$

The decomposition formula (2.44) when $z_- = -q^{-1/2}z_+^{-1}$ gives the following set of relations,

$$
\chi_0^2(q, z_+) = Y_0^{(3)}(q) \chi_0^1(q, z_+) \chi_0^1(q, z_-) + Y_{\frac{3}{2}}^{(3)}(q) \chi_1^1(q, z_+) \chi_1^1(q, z_-),
$$

$$
\chi_2^2(q, z_+) = 2Y_{1/16}^{(3)}(q) \chi_0^1(q, z_+) \chi_1^1(q, z_-),
$$

$$
\chi_2^2(q, z_+) = Y_{\frac{1}{2}}^{(3)}(q) \chi_0^1(q, z_+) \chi_1^1(q, z_+) + Y_{0}^{(3)}(q) \chi_1^1(q, z_+) \chi_1^1(q, z_-), \quad (A.8)
$$

where we have used the well-known fact,

$$
\chi_{2\ell}^k(q, -q^{-1/2}z_+) = (-1)^{2\ell}q^{-1/4}z_+^{-1}\chi_{k-2\ell}^k(q, z), \quad (A.9)
$$

and introduced the notation $Y_0^{(3)}(q) \equiv Y_{1,1}^{(3)}(q) = Y_{2,3}^{(3)}(q)$ and $Y_{\frac{3}{2}}^{(3)}(q) \equiv Y_{2,1}^{(3)}(q) = Y_{1,3}^{(3)}(q)$. The GKO character sum rules [19] allow to solve for the branching rules with the result,

$$
Y_0^{(3)}(q) = \frac{1}{2V_{ir,(3)}^{Vir}} = \frac{1}{2}\frac{\chi_0^{Vir,(3)}}{[\chi_0^{Vir,(3)}]^2 - [\chi_{\frac{3}{2}}^{Vir,(3)}]^2} = \eta^{-1}\theta_{1,4}(q),
$$

$$
Y_{\frac{3}{2}}^{(3)}(q) = -\frac{1}{2V_{ir,(3)}^{Vir}} = \frac{\chi_{\frac{3}{2}}^{Vir,(3)}}{[\chi_0^{Vir,(3)}]^2 - [\chi_{\frac{3}{2}}^{Vir,(3)}]^2} = -\eta^{-1}\theta_{3,4}(q),
$$

$$
Y_{1/16}^{(3)}(q) = \frac{1}{2V_{ir,(3)}^{Vir}} = \frac{1}{2}\frac{\chi_{1/16}^{Vir,(3)}}{[\chi_0^{Vir,(3)}]^2 - [\chi_{\frac{3}{2}}^{Vir,(3)}]^2} = \eta^{-1}[\theta_{0,4}(q) - \theta_{4,4}(q)]. \quad (A.10)
$$
It is now straightforward to obtain the analytic results summarized in the following,

\[ P_{0,0}^{(1,0),1,0}(q) = P_{1,0}^{(1,0),1,0}(q) = 1, \]
\[ F_{0,0}^{(1,0),1,0}(q) = F_{0,2}^{(1,0),1,0}(q) = \frac{1}{2} [\theta_{0,4}(q) + \theta_{4,4}(q)] = \frac{\eta}{2} \chi^1_0(q), \]
\[ F_{0,1}^{(1,0),1,0}(q) = F_{0,3}^{(1,0),1,0}(q) = \theta_{2,4}(q) = \frac{\eta}{2} \chi^1_1(q). \quad (A.11) \]

The functions corresponding to the non-singlet representations are obtained by implementing the symmetries \( (2.33), (2.37) \). The results above were presented in a different form in \([6]\). In this new derivation, it is remarkable that the functions \( F_{0,n}^{(\tilde{a},1),0}(q) \) and the branching functions \( Y_{r,s}^{(3)}(q) \) span a basis for theta functions at level 4. The occurrence of the latter is obviously deeply rooted in the relation \((2.41)\) which links theta functions at level 1 and 3 to theta functions at level 4 and 12. As we shall see when \( \tilde{k}^+ = 2 \), the theta functions at level 10 enter in the expressions for \( F_{2\tilde{k}^+}^{(\tilde{a},2),0}(q) \) and \( Y_{m,n}^{(4)}(q) \), but in a much more complicated pattern.

A.2 The case \( \tilde{k}^+ = 2 \)

The six integrable \( SU(3)_2 \) representations organize themselves in two orbits of the transformation \( \phi \), and it is sufficient to consider the sumrules corresponding to one representative of each orbit, say the singlet and the octet, for which the sumrules take the following form,

\[
\sum_{2\ell^+ = 0}^{1} \sum_{2\ell^- = 0}^{2} \chi_{2\ell^+}^1(q, z_+) \chi_{2\ell^-}^1(q, z_-)
\times \sum_{n=0}^{1} P_{2\ell^+,n}^{(0,0),2,0}(q) \sum_{r=0}^{4} \frac{1}{\eta} \theta_{4(\ell^-+r)-6(n+\ell^+),10}(q) \chi_{6(\ell^++\ell^-+n+r)}^{15}(q, z_y) = \\
- \text{Ch}_{0}^{\tilde{a},NS}(0, q, z_+, z_-)[\chi_{15}^{15}(q, z_y) + \chi_{-3}^{15}(q, z_y)] + \text{Ch}_{0}^{\tilde{a},NS}(1, q, z_+, z_-) \chi_{10}^{15}(q, z_y)
- \text{Ch}_{0}^{\tilde{a},NS}(2, q, z_+, z_-) \chi_{15}^{15}(q, z_y) + \text{Ch}_{0}^{\tilde{a},NS}(3, q, z_+, z_-)[\chi_{12}^{15}(q, z_y) + \chi_{-12}^{15}(q, z_y)]
+ \text{Ch}_{m}^{\tilde{a},NS}(0; q, z_+, z_-)
\times \left\{ F_{0,0}^{(0,0),2,0}(q) \chi_{15}^{15}(q, z_y) + F_{0,1}^{(0,0),2,0}(q) \chi_{15}^{15}(q, z_y) + \chi_{-9}^{15}(q, z_y) \right\}
+ F_{0,2}^{(0,0),2,0}(q) \left[ \chi_{12}^{15}(q, z_y) + \chi_{-12}^{15}(q, z_y) \right]
+ \text{Ch}_{m}^{\tilde{a},NS} \left( \frac{1}{2}; q, z_+, z_- \right)
\times \left\{ F_{0,0}^{(0,0),2,0}(q) \chi_{15}^{15}(q, z_y) + F_{0,1}^{(0,0),2,0}(q) \chi_{15}^{15}(q, z_y) + \chi_{-9}^{15}(q, z_y) \right\}
+ F_{0,2}^{(0,0),2,0}(q) \left[ \chi_{12}^{15}(q, z_y) + \chi_{-12}^{15}(q, z_y) \right], \quad (A.12) \]
In the above, we have used the symmetries of the modular forms \( F^{(g,2,0)}_{2e,n} \) \((2.21), (2.29), (2.33)\). In order to determine the parafermionic characters and to express the branching functions \( Y_{r,s}^{(4)}(q) \), it is sufficient to identify in both sides of the sumrules the coefficients of \( \chi_0^2(q, z_+) \chi_1^0(q, z_-) \), \( \chi_1^2(q, z_+) \chi_1^1(q, z_-) \) and \( \chi_2^2(q, z_+) \chi_0^1(q, z_-) \). To this end, we write, according to \((2.43),(2.44)\),

\[
\begin{align*}
Ch_{m}^{\tilde{A}^{+},NS}(0; q, z_+, z_-) &= \eta^{-1}[\chi_0^{Vir,(3)}(q)\chi_0^2(q, z_+)\chi_0^1(q, z_-) + \chi_1^{Vir,(3)}(q)\chi_1^2(q, z_+)\chi_1^1(q, z_-) \\
&+ \chi_1^{Vir,(3)}(q)\chi_2^2(q, z_+)\chi_0^1(q, z_-)], \\
Ch_{m}^{\tilde{A}^{+},NS}(1; q, z_+, z_-) &= Y_0^{(4)}(q)\chi_0^1(q, z_+)\chi_0^1(q, z_-) - Y_2^{(4)}(q)\chi_1^2(q, z_+)\chi_1^1(q, z_-) \\
&+ Y_3^{(4)}(q)\chi_2^2(q, z_+)\chi_0^1(q, z_-), \\
Ch_{m}^{\tilde{A}^{+},NS}(3; q, z_+, z_-) &= Y_3^{(4)}(q)\chi_0^1(q, z_+)\chi_0^1(q, z_-) - Y_7^{(4)}(q)\chi_1^2(q, z_+)\chi_1^1(q, z_-) \\
&+ Y_9^{(4)}(q)\chi_2^2(q, z_+)\chi_0^1(q, z_-),
\end{align*}
\]

(14.1)

where we introduced the notations \( Y_0^{(4)} \equiv Y_0^{(4)} = Y_{1,1}^{(4)} = Y_{3,1}^{(4)} \), \( Y_1^{(4)} \equiv Y_1^{(4)} = Y_{1,2}^{(4)} = Y_{3,2}^{(4)} \), \( Y_2^{(4)} \equiv Y_2^{(4)} = Y_{2,1}^{(4)} \), \( Y_3^{(4)} \equiv Y_3^{(4)} = Y_{2,4}^{(4)} \) and \( Y_4^{(4)} \equiv Y_4^{(4)} = Y_{3,8}^{(4)} \). Using exactly the same arguments as in the case \( k^+ = 1 \), one can derive some relations among the branching functions \( Y_{r,s}^{(4)}(q) \) (recall that in the previous case, the property (A.7) combined with the GKO sumrules allowed for a complete determination of \( Y_0^{(3)}(q), Y_1^{(3)}(q) \) and \( Y_2^{(3)}(q) \),

\[
\begin{align*}
Y_{7/16}^{(4)} &= \frac{1}{\chi_1^{1/2}}[\chi_1^{Vir,(4)} - \chi_0^{Vir,(3)} Y_0^{(4)}], \\
Y_{3/2}^{(4)} &= \frac{1}{\chi_1^{1/2}}[\chi_3^{Vir,(4)} - \chi_0^{Vir,(3)} Y_0^{(4)}], \\
Y_{3/8}^{(4)} &= \frac{1}{\chi_1^{1/2}}[\chi_3^{Vir,(4)} + \chi_1^{Vir,(3)} Y_3^{(4)}], \\
Y_{1/10}^{(4)} &= \frac{1}{\chi_1^{1/2}}[\chi_7^{Vir,(4)} + \chi_0^{Vir,(3)} Y_3^{(4)}].
\end{align*}
\]

(15.1)
A tedious but tractable computation provides us with an expression for the parafermionic characters in terms of standard modular functions,

\[
P^{(0,0),2,0}_{0,1} = \chi_{3/80} \left[ \chi_{1/2} \theta_{4,10} - \chi_0 \theta_{6,10} \right] \Delta^{-1},
\]

\[
P^{(0,0),2,0}_{0,0} = \chi_{3/80} \left[ \chi_{1/2} \theta_{4,10} - \chi_0 \theta_{6,10} \right] \Delta^{-1},
\]

\[
P^{(0,0),2,0}_{1,0} = \chi_{3/80} \chi_{1/16} \left[ \theta_{4,10} - \theta_{6,10}^2 \right] \left[ \theta_{1,10} + \theta_{9,10} \right]^{-1} \Delta^{-1},
\]

\[
P^{(1,1),2,0}_{0,1} = -\chi_{7/16} \left[ \chi_{1/2} \theta_{8,10} - \chi_0 \theta_{2,10} \right] \Delta^{-1},
\]

\[
P^{(1,1),2,0}_{0,0} = -\chi_{7/16} \chi_{1/16} \left[ \theta_{8,10} - \theta_{2,10}^2 \right] \left[ \theta_{3,10} + \theta_{7,10} \right]^{-1} \Delta^{-1},
\]

where

\[
\Delta = \left[ (\chi_0 \theta_{Vir}^{(3)})^2 - (\chi_{1/2} \theta_{Vir}^{(3)})^2 \right] \left[ \theta_{2,10} \theta_{4,10} - \theta_{6,10} \theta_{8,10} \right] \eta^{-1}.
\]

Two of the six modular forms \(F_{2e+n}^A(q)\) can also be written in terms of standard modular forms, while the four others are expressed in terms of the branching functions as announced,

\[
F^{(0,0),2,0}_{0,1} (q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( P^{(0,0),2,0}_{0,1}(q) \theta_{4,10}(q) + P^{(0,0),2,0}_{0,0}(q) \theta_{6,10}(q) \right),
\]

\[
F^{(1,1),2,0}_{0,2}(q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( P^{(1,1),2,0}_{0,2}(q) \theta_{8,10}(q) + P^{(1,1),2,0}_{0,0}(q) \theta_{2,10}(q) \right),
\]

\[
F^{(0,0),2,0}_{0,2}(q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( \left[ P^{(0,0),2,0}_{0,1}(q) \theta_{8,10}(q) + P^{(0,0),2,0}_{0,0}(q) \theta_{2,10}(q) \right] - \eta Y_{0}^{(4)}(q) \right),
\]

\[
F^{(0,0),2,0}_{0,0}(q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( \left[ P^{(0,0),2,0}_{0,1}(q) \theta_{10,10}(q) + P^{(0,0),2,0}_{0,0}(q) \theta_{10,10}(q) \right] - \eta Y_{3/5}^{(4)}(q) \right),
\]

\[
F^{(1,1),2,0}_{0,1}(q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( \left[ P^{(1,1),2,0}_{0,1}(q) \theta_{4,10}(q) + P^{(1,1),2,0}_{0,0}(q) \theta_{6,10}(q) \right] + \eta Y_{3/5}^{(4)}(q) \right),
\]

\[
F^{(1,1),2,0}_{0,0}(q) = \frac{1}{\chi_{1/2}^{(3)}(q)} \left( \left[ P^{(1,1),2,0}_{0,1}(q) \theta_{10,10}(q) + P^{(1,1),2,0}_{0,0}(q) \theta_{10,10}(q) \right] - \eta Y_{0}^{(4)}(q) \right).
\]

(A.18)

B A theta function identity.

In order to prove the following product identity (cf. eq. (2.41))

\[
\theta_{a,1}(q, z^2) \theta_{b,3/2}(q, z^{2/3}) = \sum_{r=0}^{k-1} \theta_{a+2r} \theta_{b,3/2}(q, z^{2/3}),
\]

(B.1)

where \(k = \tilde{k} + 3\), simply rewrite the LHS according to the definition of generalized theta functions, i.e.

\[
\theta_{a,1}(q, z^2) \theta_{b,3/2}(q, z^{2/3}) = \sum_{n,m} \eta^{n^2+am+a+b} \eta^{n^2+3k+m^2+bn} \eta^{\frac{k^2}{12k^2+3}} \eta^{2n+2\tilde{m}^2+3k^2+3m+n},
\]

(B.2)
Rename $2n + 2\hat{k} + m = 2ks + 2r$ with $r = 0, \ldots, k - 1$, and reorganize the double sum on $n, m$ by a sum on $s, m$ and $r$ in such a way that the above expression becomes

$$
\sum_{r=0}^{k-1} \sum_{s,m} q^{k^2 s^2 + \hat{k}(\hat{k} + 3)m^2 + s(k(2r + a - 2\hat{k} + m) + m(-a\hat{k} - b - 2\hat{k} + r) + r^2 + ar + s^2 \frac{a}{12k^+} + s^2 \frac{a^2}{12k^+} z} \delta_{2ks + 2r + a + b,}. \tag{B.3}
$$

Finally replace $m \to -m + s$ in the above sum, and notice that

$$
r^2 + ar + \frac{a^2}{4} + \frac{b^2}{12k^+} = \frac{(ak^+ + 2r\hat{k}^+ - b)^2}{4kk^+} + \frac{(3a + 6r + b)^2}{12k}. \tag{B.4}
$$

## C Solution to the master equation

In order to determine the elements $S_L^L$ for $1 \leq L \leq k - 3$ and $0 \leq L' \leq k - 2$, it is sufficient to consider trility $t = 0$ in the master equation (1.11), and the representations $\bar{\alpha} = (p, 0)$, $\bar{\alpha}' = (p, 0)$ and $\bar{\alpha}' = (0, p)$ with $p$ integer in the range $0 \leq p \leq \frac{k^+}{4}$ (in order to have a representation pertaining to the $\epsilon = 0$ region). The entries $S_L^0$ and $S_L^{k-2}$ are special but can be obtained along similar lines to those described below. As can be seen from the explicit expression (2.11), the non-zero entries of $(M(0)^{k^+_L})_L^L (1 \leq L \leq k - 3)$ occur for

\begin{align*}
\bar{\alpha} &= (k - 2 - L, k - 2 - L - 3q) \quad \text{and} \quad k - 1 - 2L \leq 3q \leq k - 2 - L, \tag{C.1} \\
\bar{\alpha}' &= (k - 2 - L + 3q, k - 2 - L) \quad \text{and} \quad L - (k - 2) \leq 3q \leq 2L - (k - 1), \tag{C.2} \\
\bar{\alpha} &= \left(\frac{1}{2}(L - 1 + 3q), \frac{1}{2}(L - 1 - 3q)\right) \quad \text{and} \quad -L + 1 \leq 3q \leq L - 1, \tag{C.3} \\
\bar{\alpha}' &= (L, L - 3q) \quad \text{and} \quad 2L - (k - 3) \leq 3q \leq L, \tag{C.4} \\
\bar{\alpha} &= (L + 3q, L) \quad \text{and} \quad -L \leq 3q \leq -2L + (k - 3), \tag{C.5} \\
\bar{\alpha}' &= \left(\frac{1}{2}(k - 3 - L + 3q), \frac{1}{2}(k - 3 - L - 3q)\right) \quad \text{and} \quad L - (k - 3) \leq 3q \leq (k - 3) - L, \tag{C.6}
\end{align*}

and $q$ is further restricted by the condition that $a_1, a_2$ are integers. According to (4.3), (4.4), the relevant elements of the S transform matrix of $S U(3)_{k^+}$ characters are given by

$$
S(t = 0)_{\bar{\alpha}}^{(p, p)} = \frac{2}{k\sqrt{3}} \left\{ \sin \left(\frac{2\pi}{k}(p + 1)(a_1 + 1)\right) + \sin \left(\frac{2\pi}{k}(p + 1)(a_2 + 1)\right) - \sin \left(\frac{2\pi}{k}(p + 1)(a_1 + a_2 + 2)\right) \right\}, \tag{C.7}
$$

\begin{align*}
S(t = 0)_{\bar{\alpha}}^{(p, 0)} + S(t = 0)_{\bar{\alpha}}^{(0, p)} &= \frac{2}{k\sqrt{3}} \left\{ \sin \left(\frac{2\pi}{k}(p + 1)(a_1 + 1) - qp\right) + \sin \left(\frac{2\pi}{k}(p + 1)(a_2 + 1) + qp\right) \right. \\
&\quad + \sin \left(\frac{2\pi}{k}(a_1 + 1) + qp\right) + \sin \left(\frac{2\pi}{k}(a_2 + 1) - qp\right) \\
&\quad - \sin \left(\frac{2\pi}{k}(a_1 + 1) + (a_2 + 1)(p + 1) + qp\right) \right\}
\end{align*}
\[
- \sin \left( \frac{2\pi}{k} \left[ (a_2 + 1) + (a_1 + 1)(p + 1) - qp \right] \right),
\]
where use has been made of the fact that the relevant representations \( \vec{a} \) should have triality zero, and therefore one can choose \( a_1 - a_2 = 3q \). When \( \vec{a} \) is restricted according to the above, the LHS of the master equation for the representation \( \vec{a}' = (p, p) \) can be written as
\[
\text{LHS} = -\frac{4}{k} \sqrt{3} \sin \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \sum_{6q-3(L+1) \in I} \chi^{3k;S}_{6q-3(L+1)} \cdot \left\{ \cos \left( \frac{\pi}{k} (p + 1)(L + 1) \right) - \cos \left( \frac{\pi}{k} (p + 1)(6q - 3(L + 1)) \right) \right\},
\]
where \( I \) symbolizes a sum over the intervals
\[
\begin{align*}
I_1 &= I_2 = [-2k - L + 1, -2k + L - 1] \\
I_3 &= [-L + 1, L - 1] \\
I_4 &= I_5 = [-2k + L + 3, -(L + 3)] \\
I_6 &= [-4k + L + 3, -2k - (L + 3)].
\end{align*}
\]
The above expression (C.9) is obtained as a sum of six terms corresponding to the six restrictions on \( \vec{a} \) listed previously, where one has relabelled \( 3q \equiv 6q' - 3(L + 1) \) in item 3 in order for \( \frac{1}{2}(L - 1 \pm 3q) \) to be an integer, \( 3q \equiv 6q' + 3k - 3(L + 1) \) in item 6 in order for \( \frac{1}{2}(k - 3 - L + 3q) \) to be an integer, and \( q \equiv -q \) in item 5. The symmetries of the cosine and of the characters is such that one may replace any interval with minus the same thing (denoted \( I_i \)) so that the sum over \( I = I_1 \cup I_2 \ldots \cup I_6 \) is equivalent to the sum over
\[
I = I_6 \cup I_1 \cup I_4 \cup I_3 \cup T_5 \cup T_2;
\]
which in turn is equivalent to the sum over the interval of length \( 6k \)
\[
I' = [-4k + (L + 3), 2k + (L + 3)].
\]
Therefore, the LHS (C.9) becomes
\[
\text{LHS} = -\frac{4}{k} \sqrt{3} \sin \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \sum_{q=0}^{k-1} \chi^{3k;S}_{6q-3(L+1)} \cdot \left\{ \cos \left( \frac{\pi}{k} (p + 1)(L + 1) \right) - \cos \left( \frac{\pi}{k} (p + 1)(6q - 3(L + 1)) \right) \right\} \]
\[
= -\frac{4}{k} \sqrt{3} \sin \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \cdot \frac{1}{\sqrt{6k}} \cdot \sum_{m' \in \mathbb{Z}_{6k}} \chi^{3k}_{m'} \sum_{q=0}^{k-1} \cos \left( \frac{\pi}{k} m'(2q - (L + 1)) \right) \]
\[
= -\frac{4}{k} \sqrt{3} \sin \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \cdot \frac{1}{\sqrt{6k}} \cdot \sum_{m' \in \mathbb{Z}_{6k}} \sum_{q=0}^{k-1} \cos \left( \frac{\pi}{k} m'(2q - (L + 1)) \right).
\]
\[\begin{align*}
\left\{ \chi_m^{3k} \cos \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \\
- \frac{1}{2} \left( \chi_{m'}^{3k} + \chi_{m'-3(p+1)}^{3k} \right) \sum_{q=0}^{k-1} \cos \left( \frac{\pi}{k} m' (2q - (L + 1)) \right) \right\} \\
= \sqrt{\frac{2}{k}} \sum_{n'=0}^{5} (-)^{n'L} \\
\cdot \left\{ -\chi_{kn'}^{3k} \sin \left( \frac{2\pi}{k} (p + 1)(L + 1) \right) \\
+ (\chi_{kn'+3(p+1)}^{3k} + \chi_{kn'-3(p+1)}^{3k}) \sin \left( \frac{\pi}{k} (p + 1)(L + 1) \right) \right\},
\end{align*}\]

where we used the S transform of the \(A_{3k}\) characters in the following form,

\[
(\chi_m^{3k})^S = \frac{1}{\sqrt{6k}} \sum_{m' \in \mathbb{Z}_{6k}} \exp \left\{ -i m' m \right\} \chi_{m'}^{3k}.
\]

Moreover, the RHS of the master equation when \(t = 0\) and \(\vec{a'} = (p, p)\) is given by

\[
\text{RHS} = \sum_{L'} [(M(\epsilon = 0))_{(p,p)}^{L'} + (M(\epsilon = -1))_{(k-3-2p,p)}^{L'} + (M(\epsilon = 1))_{(p,k-3-2p)}^{L'}] S_{L'}^L \\
= \sum_{i=0}^{2} \left\{ \chi_{2ik}^{3k} S_{2p+1}^L - [\chi_{2ik+3(p+1)}^{3k} + \chi_{2ik-3(p+1)}^{3k}] S_{p}^L - \chi_{(2i+1)k}^{3k} S_{k-3-2p}^L \\
+ [\chi_{(2i+1)k+3(p+1)}^{3k} + \chi_{(2i+1)k-3(p+1)}^{3k}] S_{k-2-p}^L \right\}.
\]

On the other hand, very similar manipulations for the combined contribution of \(\vec{a'} = (p, 0), (0, p)\) lead to the following expression for the LHS of the master equation,

\[
\text{LHS} = \frac{4\sqrt{3}}{k} \sum_{q=0}^{k-1} \chi_{6q-3(L+1)}^{3k,S} \left\{ \sin \left( \frac{\pi}{k} (L + 1)(p + 1) \right) \cos \left( \frac{\pi}{k} (p + 3)(L + 1 - 2q) \right) \\
+ \sin \left( \frac{\pi}{k} (L + 1) \right) \cos \left( \frac{\pi}{k} (2p + 3)(L + 1 - 2q) \right) \\
- \sin \left( \frac{\pi}{k} (L + 1)(p + 2) \right) \cos \left( \frac{\pi}{k} p(L + 1 - 2q) \right) \right\} \\
= \sqrt{\frac{2}{k}} \sum_{n'=0}^{5} (-1)^{n'(L+1)} \left\{ (\chi_{kn'+2p+3}^{3k} + \chi_{kn'-2p-3}^{3k}) \sin \left( \frac{\pi}{k} (L + 1) \right) \\
+ (\chi_{kn'+p+3}^{3k} + \chi_{kn'-p-3}^{3k}) \sin \left( \frac{\pi}{k} (L + 1)(p + 1) \right) \\
- (\chi_{kn'+p}^{3k} + \chi_{kn'-p}^{3k}) \sin \left( \frac{\pi}{k} (L + 1)(p + 2) \right) \right\}.
\]

A tedious but straightforward analysis of the RHS for \(t = 0\) and \(\vec{a'} = (0, p), (p, 0)\) leads to the following result

\[
\text{RHS} = \sum_{i=0}^{2} \left\{ -\chi_{2ik+(2p+3)}^{3k} + \chi_{2ik-(2p+3)}^{3k} \right\} S_0^L + (\chi_{2ik+p}^{3k} + \chi_{2ik-p}^{3k}) S_{p+1}^L
\]
\[
+ \left( \chi_{(2i+1)k+p+3} + \chi_{(2i+1)k-p-3} \right) S_{k-p-2}^L - \left( \chi_{2ik+p+3} + \chi_{2ik-p-3} \right) S_{p}^L
- \left( \chi_{(2i+1)k+p} + \chi_{(2i+1)k-p} \right) S_{k-p-3}^L + \left( \chi_{(2i+1)k+2p+3} + \chi_{(2i+1)k-2p-3} \right) S_{k-2}^L \right). \quad (C.17)
\]

Comparison between (C.13) and (C.15) on the one hand, and between (C.16) and (C.17) on the other allows us to write

\[
S_{L'}^L = -\sqrt{\frac{2}{k}} \sin \left( \frac{\pi}{k} \left( L + 1 \right) \right) \quad (C.18)
\]

for \(1 \leq L \leq k-3\) and \(0 \leq L' \leq k-2\), which corresponds, up to a sign, to the elements of the S transform matrix for \(SU(2)_{k+1}\) characters.
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