Global well-posedness and inviscid limit for the modified Korteweg-de Vries-Burgers equation

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Abstract: Considering the Cauchy problem for the modified Korteweg-de Vries-Burgers equation

\[ u_t + u_{xxx} + \epsilon |\partial_x|^{2\alpha} u = 2(u^3)_x, \ u(0) = \phi, \]

where \(0 < \epsilon, \alpha \leq 1\) and \(u\) is a real-valued function, we show that it is uniformly globally well-posed in \(H^s\) (\(s \geq 1\)) for all \(\epsilon \in (0,1]\). Moreover, we prove that for any \(s \geq 1\) and \(T > 0\), its solution converges in \(C([0,T]; H^s)\) to that of the MKdV equation if \(\epsilon\) tends to 0.

Keywords: MKdV-Burgers equation, uniform global well-posedness, inviscid limit behavior

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1 Introduction

In this paper, we study the Cauchy problem for the modified Korteweg-de Vries-Burgers (MKdV-B) equation with fractional dissipation

\[ u_t + u_{xxx} + \epsilon |\partial_x|^{2\alpha} u = 2(u^3)_x, \ u(0) = \phi, \] (1.1)

where \(0 < \epsilon, \alpha \leq 1\), \(u\) is a real-valued function of \((x,t) \in \mathbb{R} \times \mathbb{R}_+\). The equation with quadratic nonlinearity

\[ u_t + u_{xxx} + \epsilon |\partial_x|^{2\alpha} u = 2(u^2)_x, \ u(0) = \phi, \] (1.2)

has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur (see [12]). On the other hand, the cubic nonlinearity is also of much interest.

The Cauchy problems (1.1) and (1.2) has been studied by many authors (see [2, 8, 10, 11, 5, 6] and the reference therein). In [10] Molinet and Ribaud studied Eq. (1.2) in the case \(\alpha = 1\) and showed that (1.2) is globally well-posed in \(H^s\) (\(s > -1\)) by using an \(X^{s,b}\)-type space which contains the dissipative structure. Their result is sharp in the sense that the solution map of (1.2) fails to be \(C^2\) smooth at origin if \(s < -1\). Their result is generalized to the case \(0 < \alpha \leq 1\) by Vento [16], also by Guo and Wang [5] and found a critical wellposedness regularity

\[ s_\alpha = \begin{cases} -3/4, & 0 < \alpha \leq 1/2, \\ -3/(5 - 2\alpha), & 1/2 < \alpha \leq 1. \end{cases} \] (1.3)
In [5], Guo and Wang also proved a uniform global wellposedness in $H^s$ ($s > -3/4$) and that the solution converges in $C([0,T];H^s)$ to that of the KdV equation for any $T > 0$ when $\epsilon$ tends to zero, by using a $l^1$ - variant $X^{s,b}$ space and I-method. For the Eq. (1.1), following the methods in [10], Chen and Li [2] showed global wellposedness in $H^s$, $s > -1/4$ and Chen, Li and Miao [3] obtained in $H^s(s > 1/4 - \alpha/4)$ for the case $0 < \alpha \leq 1$.

Following the ideas in [5], we consider the inviscid limit of Eq. (1.4) as $\epsilon$ tends to zero. Formally, if $\epsilon = 0$ then (1.1) reduces to the mKdV equation

$$u_t + u_{xxx} = 6u^2u_x, \quad u(0) = \phi.$$  

The optimal result on local well-posedness of (1.4) in $H^s$ was obtained by Kenig, Ponce and Vega [9]. They obtained that (1.4) is locally well-posed for $s \geq 1/4$. The result on global well-posedness of (1.4) in $H^s$ was obtained in [4] where it was shown that (1.4) is globally well-posed in $H^s$ for $s > 1/4$ and a kind of modified energy method, so called I-method, is introduced. It is natural to conjecture that the solution of Eq. (1.1) will converge to that of Eq. (1.4) if $\epsilon$ tends to zero. To prove that, we prove first the uniform global well-posedness of Eq. (1.4). Then we need to study the difference equation between (1.1) and (1.4). We first treat the dissipative term as perturbation and then use the uniform Lipschitz continuity property of the solution map. Similar ideas can be found in [17] for the inviscid limit of the complex Ginzburg-Landau equation. For $T > 0$, we denote $S_T^\epsilon, S_T$ the solution map of (1.1), (1.4) respectively. The notation $A \lesssim B$ denotes that there exists a constant $C$, such that $A \leq CB$. Now we state our main results.

**Theorem 1.1.** Assume $0 < \alpha \leq 1$ and $s \geq 1$. Let $\phi \in H^s(\mathbb{R})$. Then for any $T > 0$, the solution map $S_T^\epsilon$ satisfies for all $0 < \epsilon \leq 1$

$$\|S_T^\epsilon \phi\|_{F^s(T)} \lesssim C(T, \|u\|_{H^s})$$

where $F^s(T) \subset C([0,T];H^s)$ which will be defined later and $C(\cdot, \cdot)$ is a continuous function with $C(\cdot, 0) = 0$, and also satisfies that for all $0 < \epsilon \leq 1$

$$\|S_T^\epsilon(\phi_1) - S_T^\epsilon(\phi_2)\|_{C([0,T];H^s)} \leq C(T, \|\phi_1\|_{H^s}, \|\phi_2\|_{H^s})\|\phi_1 - \phi_2\|_{H^s}.$$  

We also have the uniform persistence of regularity, following the standard argument. For local well-posedness we actually prove that complex-valued Eq. (1.1) is uniformly locally well-posed in $H^s(s \geq 1/4)$. For the limit behavior, we have

**Theorem 1.2.** Assume $0 < \alpha \leq 1$. Let $\phi \in H^s(\mathbb{R})$, $s \geq 1$. For any $T > 0$, then

$$\lim_{\epsilon \to 0^+} \|S_T^\epsilon(\phi) - S_T(\phi)\|_{C([0,T];H^s)} = 0.$$  

**Remark 1.3.** We are only concerned with the limit in the same regularity space. There seems no convergence rate. This can be seen from the linear solution,

$$\|e^{-t\partial_x^2} - e^{-t\epsilon^2\partial_x^2}\phi\|_{C([0,T];H^s)} \to 0, \quad \text{as} \ \epsilon \to 0,$$  

but without any convergence rate. We believe that there is a convergence rate if we assume the initial data has higher regularity than the limit space. For example, we prove that

$$\|S_T^\epsilon(\phi_1) - S_T(\phi_2)\|_{C([0,T];H^s)} \lesssim \|\phi_1 - \phi_2\|_{H^s} + \epsilon^{1/2}C(T, \|\phi_1\|_{H^s}, \|\phi_2\|_{H^s})$$

The rest of the paper is organized as following. We present some notations and Banach function spaces in Section 2. We give a symmetric estimate in Section 3. We prove the trilinear estimate in Section 4. We present uniform LWP in Section 5 and prove Theorem 1.1 in Section 6. Theorem 1.2 is proved in Section 7.
2 Notation and Definitions

For \( x, y \in \mathbb{R}, x \sim y \) means that there exist \( C_1, C_2 > 0 \) such that \( C_1|x| \leq |y| \leq C_2|x| \). For \( f \in S' \) we denote by \( \widehat{f} \) or \( \mathcal{F}(f) \) the Fourier transform of \( f \) for both spatial and time variables,

\[
\widehat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x,t)dxdt.
\]

We denote by \( \mathcal{F}_x \) the Fourier transform on spatial variable and if there is no confusion, we still write \( \mathcal{F} = \mathcal{F}_x \). Let \( \mathbb{Z} \) and \( \mathbb{N} \) be the sets of integers and natural numbers, respectively. For the simplicity, we let \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). For \( k \in \mathbb{Z} \), let \( I_k = \{ \xi : |\xi| \in [2^{k-1},2^{k+1}] \} \). For \( k \in \mathbb{Z}_+ \) let \( \bar{I}_k = [-2,2] \) if \( k = 0 \) and \( I_k = I_k \) if \( k \geq 1 \). For \( k \in \mathbb{Z}_+ \) and \( j \geq 0 \) let \( D_{k,j} = \{(\xi,\tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in \bar{I}_j \} \). For \( k \in \mathbb{Z} \) and \( j \geq 0 \) let \( D_{k,j} = \{(\xi,\tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in \bar{I}_j \} \).

We use \( f * g \) will stand for the convolution on time and spatial variables, i.e.,

\[
(f * g)(t,x) = \int_{\mathbb{R}^2} f(t-s,x-y)g(s,y)dsdy.
\]

Let \( \eta_0 : \mathbb{R} \to [0,1] \) denote an even smooth function supported in \([-8/5,8/5]\) and equal to 1 in \([-5/4,5/4]\). For \( k \in \mathbb{N} \) let \( \eta_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}) \) and \( \eta_{\leq k} = \sum_{k'=0}^k \eta_{k'} \). For \( k \in \mathbb{Z} \) let \( \chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1}) \). Roughly speaking, \( \{\chi_k\}_{k \in \mathbb{Z}} \) is the homogeneous decomposition function sequence and \( \{\eta_k\}_{k \in \mathbb{Z}_+} \) is the non-homogeneous decomposition function sequence to the frequency space. For \( k \in \mathbb{Z}_+ \) let \( P_k \) denote the operator on \( L^2(\mathbb{R}) \) defined by

\[
\widehat{P_k}u(\xi) = \chi_k(\xi)\widehat{u}(\xi)
\]

By a slight abuse of notation we also define the operator \( P_k \) on \( L^2(\mathbb{R} \times \mathbb{R}) \) by the formula \( \mathcal{F}(P_k u)(\xi,\tau) = \chi_k(\xi)\mathcal{F}(u)(\xi,\tau) \). For \( l \in \mathbb{Z} \) let

\[
P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.
\]

We define the Lebesgue spaces \( L^p_T L^q_x \) and \( L^p_x L^q_T \) by the norms

\[
\|f\|_{L^p_T L^q_x} = \left( \int_{(0,T]} \|f\|^p_{L^q_x} \right)^{1/p}, \quad \|f\|_{L^p_x L^q_T} = \left( \int_{(0,T]} \|f\|^q_T \right)^{1/q}.
\]  \hspace{1cm} (2.1)

We denote by \( W_0 \) the semigroup associated with Airy-equation

\[
\mathcal{F}_x(W_0(t)\phi)(\xi) = \exp[i\xi^3 t] \widehat{\phi}(\xi), \quad \forall \ t \in \mathbb{R}, \ \phi \in \mathcal{S}'.
\]

For \( 0 < \epsilon \leq 1 \) and \( 0 < \alpha \leq 1 \), we denote by \( W_{\epsilon}^\alpha \) the semigroup associated with the free evolution of \( (1.1) \),

\[
\mathcal{F}_x(W_{\epsilon}^\alpha(t)\phi)(\xi) = \exp[-\epsilon|\xi|^{2\alpha} t + i\xi^3 t] \widehat{\phi}(\xi), \quad \forall \ t \geq 0, \ \phi \in \mathcal{S}',
\]

and we extend \( W_{\epsilon}^\alpha \) to a linear operator defined on the whole real axis by setting

\[
\mathcal{F}_x(W_{\epsilon}^\alpha(t)\phi)(\xi) = \exp[-\epsilon|\xi|^{2\alpha} |t| + i\xi^3 t] \widehat{\phi}(\xi), \quad \forall \ t \in \mathbb{R}, \ \phi \in \mathcal{S}'.
\]

Let

\[
L(f)(x,t) = 2W_0(t) \int_{\mathbb{R}^2} e^{ix\xi} e^{it\tau'} - e^{-\epsilon|\xi|^2\alpha \over i\tau'} + \epsilon|\xi|^{2\alpha} \mathcal{F}(W_0(-t) f)(\xi,\tau')d\xi d\tau'.
\]  \hspace{1cm} (2.2)

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To study the low regularity of \((1.2)\), Molinet and Ribaud introduced the variant version of Bourgain’s spaces with dissipation
\[
\|u\|_{X^{b,s,\alpha}} = \|\langle i(\tau - \xi^3) + |\xi|^{2\alpha}\rangle^b \langle \xi \rangle^s \tilde{u}\|_{L^2(\mathbb{R}^2)},
\]
where \(\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}\). The standard \(X^{b,s}\) space for \((1.1)\) used by Bourgain, Kenig, Ponce and Vega (see [1], [9]) is defined by
\[
\|u\|_{X^{b,s}} = \|\langle \tau - \xi^3 \rangle^b \langle \xi \rangle^s \tilde{u}\|_{L^2(\mathbb{R}^2)}.
\]
We introduce the Banach spaces used in [5]. For \(k \in \mathbb{Z}_+\) we define the dyadic \(X^{b,s}\)-type normed spaces \(X_k = X_k(\mathbb{R}^2)\),
\[
X_k = \{ f \in L^2(\mathbb{R}^2) : f(\xi,\tau) \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \xi^3) \cdot f\|_{L^2} \}.
\]
This kind of spaces were introduced, for instance, in [8], [15] and [7] for the BO equation. From the definition of \(X_k\), we see that for any \(l \in \mathbb{Z}_+\) and \(f_k \in X_k\) (see also [8]),
\[
\sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \xi^3) \int |f_k(\xi,\tau')|2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4}d\tau'|_{L^2} \lesssim \|f_k\|_{X_k}.
\]
Hence for any \(l \in \mathbb{Z}_+, t_0 \in \mathbb{R}, f_k \in X_k\) and \(\gamma \in \mathcal{S}(\mathbb{R}), \text{ then}
\[
\|\mathcal{F}[\gamma(2^l(t-t_0)) \cdot \mathcal{F}^{-1} f_k]\|_{X_k} \lesssim \|f_k\|_{X_k}. \tag{2.5}
\]
For \(s \geq 0\), we define the following spaces:
\[
F^s = \{ u \in \mathcal{S}'(\mathbb{R}^2) : \|u\|_{F^s}^2 = \sum_{k \in \mathbb{Z}_+} 2^{2sk} \|\eta_k(\xi)\mathcal{F}(u)\|_{X_k}^2 < \infty \}, \tag{2.6}
\]
\[
N^s = \{ u \in \mathcal{S}'(\mathbb{R}^2) : \|u\|_{N^s}^2 = \sum_{k \in \mathbb{Z}_+} 2^{2sk} \|\langle i + \tau - \xi^3 \rangle^{-1} \eta_k(\xi)\mathcal{F}(u)\|_{X_k}^2 < \infty \}. \tag{2.7}
\]
For \(T \geq 0\), we define the time-localized spaces \(X^{b,s,\alpha}_T, X^{b,s}_T, F^s(T)\) and \(N^s(T)\)
\[
\|u\|_{X^{b,s,\alpha}_T} = \inf_{w \in X^{b,s,\alpha}_T} \{ \|w\|_{X^{b,s,\alpha}_T}, w(t) = u(t) \text{ on } [0,T] \};
\]
\[
\|u\|_{X^{b,s}_T} = \inf_{w \in X^{b,s}_T} \{ \|w\|_{X^{b,s}_T}, w(t) = u(t) \text{ on } [0,T] \};
\]
\[
\|u\|_{F^s(T)} = \inf_{w \in F^s} \{ \|w\|_{F^s}, w(t) = u(t) \text{ on } [0,T] \};
\]
\[
\|u\|_{N^s(T)} = \inf_{w \in N^s} \{ \|w\|_{N^s}, w(t) = u(t) \text{ on } [0,T] \}. \tag{2.8}
\]
As a conclusion to this section, we recall a result in [5].

**Proposition 2.1** (Proposition 2.1, [5]). Let \(Y\) be a Banach space of functions on \(\mathbb{R} \times \mathbb{R}\) with the property that
\[
\|e^{it\tau_0} e^{-it\theta^3} f\|_Y \lesssim \|f\|_{H^s(\mathbb{R})}
\]
holds for all \(f \in H^s(\mathbb{R}) \text{ and } \tau_0 \in \mathbb{R}\). Then we have the embedding
\[
\left( \sum_{k \in \mathbb{Z}_+} \|P_k u\|_{F^s}^2 \right)^{1/2} \lesssim \|u\|_{F^s}. \tag{2.9}
\]
3 A symmetric estimate

According to the standard fixed point argument, we will need the following trilinear estimate.

Lemma 3.1. If \( s \geq \frac{1}{4} \), then exists \( C > 0 \), such that for any \( u, v, w \in F^s \)

\[
\|\partial_x(uvw)\|_{N^s} \leq C(\|u\|_{F^s} \|v\|_{F^s} \|w\|_{F^s} + \|v\|_{F^s} \|u\|_{F^s} \|w\|_{F^s} + \|w\|_{F^s} \|v\|_{F^s} \|u\|_{F^s})
\] (3.1)

Now we prove a symmetric estimate which will be used to prove the trilinear estimate, closely following the methods in [6]. Similar ideas for the bilinear estimates can be found in [7]. For \( \xi_1, \xi_2, \xi_3 \in \mathbb{R} \) and \( \omega : \mathbb{R} \to \mathbb{R} \) defined as \( \omega(\xi) = \xi^3 \). Let

\[
\Omega(\xi_1, \xi_2, \xi_3) = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3).
\] (3.2)

This is the resonance function that plays a crucial role in the trilinear estimate of the \( X^{s,b} \)-type space, see [13] for a perspective discussion. For compactly supported functions \( f, g, h, u \in L^2(\mathbb{R} \times \mathbb{R}) \). Let

\[
J(f, g, h, u) = \int_{\mathbb{R}^6} f(\xi_1, \mu_1) g(\xi_2, \mu_2) h(\xi_3, \mu_3)
\]
\[
u(\xi_1 + \xi_2 + \xi_3, \mu_1 + \mu_2 + \mu_3 + \Omega(\xi_1, \xi_2, \xi_3)) d\xi_1 d\xi_2 d\xi_3 d\mu_1 d\mu_2 d\mu_3.
\]

Lemma 3.2. Assume \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \), \( k_1 \leq k_2 \leq k_3 \leq k_4 \), \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \) and \( f_{k,j} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in \( I_{k_i} \times I_{j_i} \), \( i = 1, 2, 3, 4 \). For simplicity we write \( J = |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \).

(a) For any \( k_1 \leq k_2 \leq k_3 \leq k_4 \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \),

\[
J \leq C 2^{(j_{\min}+j_{\text{thd}})/2} 2^{(k_{\min}+k_{\text{thd}})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (3.3)

(b) If \( k_2 \leq k_3 - 5 \) and \( j_2 \neq j_{\max} \),

\[
J \leq C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{\max}/2} 2^{-k_{\max}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2};
\] (3.4)

if \( k_2 \leq k_3 - 5 \) and \( j_2 = j_{\max} \),

\[
J \leq C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{\max}/2} 2^{-k_{\max}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (3.5)

(c) For any \( k_1, k_2, k_3, k_4 \in \mathbb{N} \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \),

\[
J \leq C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{\max}/2} 2^{-(k_1+k_2+k_3)/6} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (3.6)

(d) If \( k_{\min} \leq k_{\max} - 10 \), then

\[
J \leq C 2^{(j_1+j_2+j_3+j_4)/2} 2^{-3k_{\max}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (3.7)

Here we use \( k_{\max}, k_{\sec}, k_{\text{thd}} \) and \( k_{\min} \) denote the maximum, the second maximum, the third maximum number and the minimum of numbers \( k_1, k_2, k_3 \) and \( k_4 \). The notations \( j_{\max}, j_{\sec}, j_{\text{thd}} \) and \( j_{\min} \) are similar.
Proof. Let \( A_{k_i}(\xi) = \int \rho_{k_i,j_i}(\xi,\mu)^2 d\mu \)^{1/2}, i = 1, 2, 3, 4, then \( \|A_{k_i}\|_{L^2}\) equals \( \|f_{k_i,j_i}\|_{L^2(\xi,\mu)}\). Using the Cauchy-Schwarz inequality and the support properties of the functions \( f_{k_i,j_i} \),

\[
|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \\
\leq C 2^{(j_{\min} + j_{\text{thrd}})/2} \int_{\mathbb{R}^3} A_{k_1}(\xi_1)A_{k_2}(\xi_1)A_{k_3}(\xi_1)A_{k_4}(\xi_1 + \xi_2 + \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
\leq C 2^{(k_{\min} + k_{\text{thrd}})/2} 2^{(j_{\min} + j_{\text{thrd}})/2} 2 \prod_{i=1}^4 \|A_{k_i,j_i}\|_{L^2},
\]

which is part (a), as desired.

For part (b), by examining the supports of the functions, \( J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) \equiv 0 \) unless
\[
k_4 \leq k_3 + 5. \tag{3.8}
\]
Simple changes of variables in the integration and the observation that the function \( \omega \) is odd show that
\[
|J(f, g, h, u)| = |J(g, f, h, u)| = |J(f, h, g, u)| = |J(\tilde{f}, \tilde{g}, u, h)|, \tag{3.9}
\]
where \( \tilde{f}(\xi, \mu) = f(-\xi, -\mu), \tilde{g}(\xi, \mu) = g(-\xi, -\mu) \). We assume first that \( j_2 \neq j_{\text{max}} \) and \( j_4 = j_{\text{max}} \), then we will prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) nonnegative functions supported in \( I_{k_i}, i = 1, 2, 3 \) and \( g : \mathbb{R}^2 \to \mathbb{R}^+ \) is an \( L^2 \) function supported in \( I_{k_4} \times \tilde{I}_{j_4} \), then
\[
\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3)) d\xi_1 d\xi_2 d\xi_3 \\
\lesssim 2^{(j_1 + j_2 + j_3)} 2^{-k_{\text{max}}/2} \|g_1\| L^2 \|g_2\| L^2 \|g_3\| L^2 \|g\| L^2. \tag{3.10}
\]
This suffices for (3.4).

To prove (3.10), we first observe that since \( k_2 \leq k_3 - 5 \) then \( |\xi_3 + \xi_2| \sim |\xi_3| \). By change of variables \( \xi'_1 = \xi_1, \xi'_2 = \xi_2, \xi'_3 = \xi_2 + \xi_3 \), we get that the left side of (3.10) is dominated by
\[
\int_{[\xi'_1 \sim 2^{k_1}, \xi'_2 \sim 2^{k_2}, \xi'_3 \sim 2^{k_3}]} g_1(\xi'_1)g_2(\xi'_2) \\
g_3(\xi'_3 - \xi'_2)g(\xi'_1 + \xi'_2, \Omega(\xi'_1, \xi'_2, \xi'_3 - \xi'_2)) d\xi'_1 d\xi'_2 d\xi'_3. \tag{3.11}
\]
Note that in the integration area we have
\[
\left| \frac{\partial}{\partial \xi'_2} \left[ \Omega(\xi'_1, \xi'_2, \xi'_3 - \xi'_2) \right] \right| = |\omega'(\xi'_2) - \omega'(\xi'_3 - \xi'_2)| \sim 2^{2k_3},
\]
where we use the fact \( \omega'(\xi) \sim |\xi|^2 \) and \( k_2 \leq k_3 - 5 \). So we have \( \|g(\xi'_1 + \xi'_2, \Omega(\xi'_1, \xi'_2, \xi'_3 - \xi'_2))\|_{L^2(\xi'_2)} = 2^{-k_3}\|g(\xi'_1 + \xi'_2, \mu_2)\|_{L^2(\mu_2)} \). By change of variable \( \mu_2 = \Omega(\xi'_1, \xi'_2, \xi'_3 - \xi'_2) \), we get that (3.11) is dominated by
\[
2^{-k_3} \int_{[\xi'_1 \sim 2^{k_1}]} g_1(\xi'_1) \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2} d\xi'_1 \\
\lesssim 2^{-k_{\text{max}}/2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}. \tag{3.12}
\]
If \( j_3 = j_{\text{max}} \), this case is identical to the case \( j_4 = j_{\text{max}} \) in view of (3.9). If \( j_1 = j_{\text{max}} \), similar to (3.10), it suffices to prove \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) nonnegative functions supported in
$I_{k_1}, i = 2, 3, 4$ and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an $L^2$ nonnegative function supported in $I_{k_1} \times I_{j_1}$, then

$$
\int_{\mathbb{R}^3} g_2(\xi_2) g_3(\xi_3) g_4(\xi_4) g(\xi_2 + \xi_3 + \xi_4, \Omega(\xi_2, \xi_3, \xi_4)) d\xi_2 d\xi_3 d\xi_4 \\
\lesssim 2^{-k_{\text{max}}} 2^{k_{\text{min}}/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}.
$$

(3.13)

Indeed, by change of variables $\xi' = \xi_2, \xi_3 = \xi_3, \xi'_4 = \xi_2 + \xi_3 + \xi_4$ and the observation that in the area $|\xi'_2| \sim 2^{k_2}, |\xi'_3| \sim 2^{k_3}, |\xi'_4| \sim 2^{k_1},$

$$
|\frac{\partial}{\partial \xi'_2} [\Omega(\xi'_2, \xi'_3, \xi'_4 - \xi'_2 - \xi'_3)]| = |\omega'(\xi'_2) - \omega'(\xi'_4 - \xi'_2 - \xi'_3)| \sim 2^{2k_3},
$$

we get from Cauchy-Schwartz inequality that

$$
\int_{\mathbb{R}^3} g_2(\xi_2) g_3(\xi_3) g_4(\xi_4) g(\xi_2 + \xi_3 + \xi_4, \Omega(\xi_2, \xi_3, \xi_4)) d\xi_2 d\xi_3 d\xi_4 \\
\lesssim \int_{|\xi'_2| \sim 2^{k_2}, |\xi'_3| \sim 2^{k_3}, |\xi'_4| \sim 2^{k_1}} g_2(\xi'_2) g_3(\xi'_3) \\
\cdot g_4(\xi'_4 - \xi'_2 - \xi'_3) g(\xi'_4, \Omega(\xi'_2, \xi'_3, \xi'_4 - \xi'_2 - \xi'_3)) d\xi'_2 d\xi'_3 d\xi'_4 \\
\lesssim 2^{-k_3} \int_{|\xi'_2| \sim 2^{k_2}, |\xi'_3| \sim 2^{k_1}} g_3(\xi'_3) \|g_2(\xi'_2) g_4(\xi'_4 - \xi'_2 - \xi'_3)\|_{L^{2}_{\xi'_2}} \|g(\xi'_4, \cdot)\|_{L^{2}_{\xi'_2}} d\xi'_3 d\xi'_4 \\
\lesssim 2^{-k_{\text{max}}} 2^{k_{\text{min}}/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}.
$$

(3.14)

We assume now that $j_2 = j_{\text{max}}$. This case is identical to the case $j_1 = j_{\text{max}}$ in view of (3.9). We note that we actually prove that if $k_2 \leq k_3 - 5$ then

$$
J \leq C 2^{(j_1 + j_3 + j_4)/2} 2^{-k_{\text{max}}} 2^{k_{\text{min}}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
$$

(3.15)

Therefore, we complete the proof of part (b).

For part (c), setting $f^k_{i,j_i}(\xi, \tau) = f_{k_i,j_i}(\xi, \tau - \omega(\xi)), i = 1, 2, 3, 4$, then we get

$$
|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \\
= \left| \int_{\mathbb{R}^6} \prod_{i=1}^{3} f_{k_i,j_i}^k(\xi_i, \tau_i) f_{k_4,j_4}(\xi_1 + \xi_2 + \xi_3, \tau_1 + \tau_2 + \tau_3) d\xi_1 d\xi_2 d\xi_3 d\tau_1 d\tau_2 d\tau_3 \right|
$$

Making variables change $\xi = \xi_1 + \xi_2 + \xi_3, \tau = \tau_1 + \tau_2 + \tau_3$, we have

$$
|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \\
= \left| \int_{\mathbb{R}^6} f_{k_1,j_1}^* f_{k_2,j_2}^* f_{k_3,j_3}^* \cdot f_{k_4,j_4}^* (\xi, \tau) d\xi d\mu \right| \\
\lesssim \|f_{k_1,j_1}^* \|_{L^2} \|f_{k_2,j_2}^* \|_{L^2} \|f_{k_3,j_3}^* \|_{L^2} \|f_{k_4,j_4}^* \|_{L^2} \\
\lesssim \prod_{i=1}^{3} \|\mathcal{F}^{-1} f_{k_i,j_i}^* \|_{L^2_{\xi_i}} \|f_{k_4,j_4}^* \|_{L^2_{\xi_3}}
$$

On the other hand

$$
\mathcal{F}^{-1}(f^k_{k_i,j_i}) = \int_{\mathbb{R}^2} f_{k_i,j_i}(\xi, \tau - \omega(\xi)) e^{ix\xi} e^{it\tau} d\xi d\tau \\
= \int_{\mathbb{R}^2} f_{k_i,j_i}(\xi, \tau) e^{ix\xi} e^{it\omega(\xi)} d\xi d\tau,
$$
If we can show that
\[ \|F^{-1} f_{k,i,j}^t \|_{L^2_{t,x}} \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} f_{k,i,j}^t(\xi,\tau)e^{ix\xi}e^{it\omega(\xi)}d\xi\|_{L^2_{t,x}} d\tau \]
\[ \lesssim 2^{j/2}2^{-k/6}\|f_{k,i,j}\|_{L^2} \]  \hspace{1cm} (3.16)
then part (c) follows by symmetry.

Now we give the proof of (3.16) by using a result of Wang [18].

**Lemma 3.3.** Let \( U_m = e^{it(-\Delta)^{m/2}} \), \( m \geq 2 \), define
\[ m^* = \begin{cases} \infty, & n \leq m, \\ 2n/(n-m), & n > m, \end{cases} \]  \hspace{1cm} (3.17)
and
\[ \frac{2}{\gamma(\cdot)} = n\left(\frac{1}{2} - \frac{1}{\gamma(\cdot)}\right), \]
\[ 2\sigma(m,\cdot) = n(2m)(\frac{1}{2} - \frac{1}{\gamma(\cdot)}). \]  \hspace{1cm} (3.18)

For \( 2 \leq r, p < 2^* \), we have
\[ \|U_m(t)\phi\|_{L^{\gamma(p)}(I,\dot{B}^s_{p,2})} \lesssim \|\phi\|_{\dot{H}^s} \]
\[ \|A U_m(t)f\|_{L^{\gamma(p)}(I,\dot{B}^s_{p,2})} \lesssim \|f\|_{L^{\gamma(r)}(I,\dot{B}^{s_{\cdot}}_{r',2})} \]  \hspace{1cm} (3.20)

Here \( I \subset \mathbb{R} \) is a any interval, \( A U_m := \int_0^t U_m(t-\tau) \cdot d\tau \).

Choosing \( m = 3, n = 1, s = 0, p = 6 \) and \( r = 2 \) in Lemma 3.3, we get (3.10).

For part (d), we need to consider two cases: \( \xi_1 \cdot \xi_2 > 0 \) or \( \xi_1 \cdot \xi_2 < 0 \). Observing that if we let \( \xi_4 = -(\xi_1 + \xi_2 + \xi_3) \), then \( \Omega(\xi_1, \xi_2, \xi_3) = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4) \). The former case is easier to handle. When \( \xi_1 \cdot \xi_2 > 0 \), because of \( k_1 \leq k_4 - 10 \), we have \( \Omega(\xi_1, \xi_2, \xi_3) \geq 2^{k_2}2^{k_3}2^{k_4} \sim 2^{k_2 + 2k_3} \).

If \( k_2 \leq k_3 - 5 \), notice that \( j_{\text{max}} \geq k_2 + 2k_3 - 20 \), part (d) holds by part (b).

If \( k_2 \geq k_3 - 5 \), observing \( \frac{-k_1-k_3+k_4}{6} \sim \frac{-k_1+2k_3}{6} \) and owning to \( j_{\text{max}} \geq k_2 + 2k_3 - 20 \), part (d) holds by part (c).

We assume now \( \xi_1 \cdot \xi_2 < 0 \). Now we should consider serval cases according to \( j_i = j_{\text{max}} \). If \( j_1 = j_{\text{max}} \), it suffices to prove that if \( A_i \) is \( L^2 \) nonnegative functions supported in \( I_{k_i} \), \( i = 1, 2, 3 \) and \( B \) is a \( L^2 \) nonnegative function supported in \( I_{k_4} \times \tilde{I}_{j_4} \), then
\[ \int_{\mathbb{R}(\xi_1, \xi_2 < 0)} A_1(\xi_1)A_2(\xi_2)A_3(\xi_3)B(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3 \]
\[ \lesssim 2^{j/2}2^{-2k}2^{\gamma/2-2k}2^\gamma2^\gamma2^\gamma2^\gamma \]  \hspace{1cm} (3.22)
By localizing \( |\xi_1 + \xi_2| \sim 2^l \) for \( l \in \mathbb{Z} \), we get that the right-hand side of (3.22) is dominated by
\[ \sum_l \int_{\mathbb{R}^3} \chi_l(\xi_1 + \xi_2)A_1(\xi_1)A_2(\xi_2)A_3(\xi_3)B(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3. \]  \hspace{1cm} (3.23)

From the support properties of the functions \( A_i, B \) and the fact that in the integration area
\[ |\Omega(\xi_1, \xi_2, \xi_3)| = |3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)| \sim 2^{j+2k_3}, \]
We get that
\[ j_{\text{max}} \geq l + 2k_3 - 20. \]  
(3.24)

By change of variables \( \xi_1' = \xi_1 + \xi_2, \xi_2' = \xi_2, \xi_3' = \xi_1 + \xi_3 \), we obtain that (3.24) is dominated by

\[
\sum_l \int_{|\xi_1'| \sim 2^l, |\xi_2'| \sim 2^{k_2}, |\xi_3'| \sim 2^{k_3}} \chi_l(\xi_1')A_1(\xi_1' - \xi_2')A_2(\xi_2')A_3(\xi_2 + \xi_3' - \xi_1')
\]
\[
\quad \cdot B(\xi_2' + \xi_3', \Omega(\xi_1' - \xi_2', \xi_2', \xi_2 + \xi_3' - \xi_1'))d\xi_1'd\xi_2'd\xi_3'.
\]  
(3.25)

Since in the integration area
\[
\left| \frac{\partial}{\partial \xi_1'}[\Omega(\xi_1' - \xi_2', \xi_2', \xi_2 + \xi_3' - \xi_1')] \right| = |\omega'(\xi_1' - \xi_2') - \omega'(\xi_2' + \xi_3' - \xi_1')| \sim 2^{2k_3},
\]  
(3.26)

then we get from (3.26) that (3.25) is dominated by

\[
\sum_l \int_{|\xi_1'| \sim 2^l} \chi_l(\xi_1') \| A_1 \|_{L^2} \| A_2 \|_{L^2} \| A_3 \|_{L^2}
\]
\[
\quad \cdot \| A_2(\xi_2')B(\xi_2' + \xi_3', \Omega(\xi_1' - \xi_2', \xi_2', \xi_2 + \xi_3' - \xi_1')) \|_{L^2_{\xi_2', \xi_3'}}d\xi_1'
\]  
(3.27)

where we used (3.21) in the last inequality. From symmetry we know the case \( j_3 = j_{\text{max}} \) is identical to the case \( j_4 = j_{\text{max}} \); the case \( j_1 = j_{\text{max}} \) is identical to the case \( j_2 = j_{\text{max}} \). Thus it reduces to prove the case \( j_2 = j_{\text{max}} \). It suffices to prove that if \( A_i \) is \( L^2 \) nonnegative functions supported in \( I_{k_2}, i = 1, 3, 4 \) and \( B \) is a \( L^2 \) nonnegative function supported in \( \tilde{I}_{j_2} \), then

\[
\int_{\mathbb{R}^3 \cap \{\xi_1, \xi_2 < 0\}} A_1(\xi_1)A_3(\xi_3)A_4(\xi_4)B(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1d\xi_3d\xi_4
\]  
(3.28)

As the case \( j_4 = j_{\text{max}} \), we get that the right-hand side of (3.28) is dominated by

\[
\sum_l \int_{\mathbb{R}^3} \chi_l(\xi_3 + \xi_4)A_1(\xi_1)A_4(\xi_4)A_3(\xi_4)B(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1d\xi_3d\xi_4.
\]  
(3.29)

From the support properties of the functions \( A_i, B \) and the fact that in the integration area
\[
|\Omega(\xi_1, \xi_2, \xi_3)| = |3(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)| \sim 2^{l+2k_3},
\]

We get that
\[ j_{\text{max}} \geq l + 2k_3 - 20. \]  
(3.30)

By changing variables \( \xi_1' = \xi_1 + \xi_3, \xi_3' = \xi_3 + \xi_4, \xi_4' = \xi_1 + \xi_3 + \xi_4 \), we obtain that (3.29) is dominated by

\[
\sum_l \int_{|\xi_1'| \sim 2^l, |\xi_4'| \sim 2^{k_2}, |\xi_1'| \sim 2^{k_3}} \chi_l(\xi_4')A_1(\xi_1' - \xi_4')A_3(\xi_3' + \xi_4' - \xi_1')
\]
\[
\quad \cdot B(\xi_4', \Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_1' - \xi_1'))d\xi_1'd\xi_3'd\xi_4'.
\]  
(3.31)
Since in the integration area,
\[
\left| \frac{\partial}{\partial \xi_3} \Omega(\xi_4 - \xi_3', \xi_1' + \xi_3 - \xi_4, \xi_3' - \xi_1') \right| = \left| -\omega'(\xi_4' - \xi_3') + \omega'(\xi_1' + \xi_3 - \xi_4') \right| \sim 2^{2k_3},
\]
then we get from (3.32) that (3.31) is dominated by
\[
\sum_I \int_{|\xi|^2} \chi_I(\xi) \|A_1\|_{L^2} \|A_3\|_{L^2} \|A_4(\xi - \xi_3')B(\xi_4', \Omega(\xi_4' - \xi_3') \xi_1' + \xi_3 - \xi_4') \|_{L^2} \|B\|_2 \|d\xi_3\|_2 \leq \sum_I 2^{g/2} 2^{-k_3} \|A_1\|_{L^2} \|A_3\|_{L^2} \|A_4\|_{L^2} \|B\|_2 \leq 2^{j_{\max}/2} 2^{-2k_3} \|A_1\|_{L^2} \|A_3\|_{L^2} \|A_4\|_{L^2} \|B\|_2,
\]
where we used (3.30) in the last inequality. Therefore, we complete the proof of part (d).

We restate Lemma 3.7 in a form that is suitable for the trilinear estimates in the next sections.

**Corollary 3.4.** Assume \(k_1, k_2, k_3, k_4 \in \mathbb{Z}\), \(j_1, j_2, j_3, j_4 \in \mathbb{Z}_+\) and \(f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})\) are functions supported in \(\hat{D}_{k_i,j_i}\), \(i = 1, 2, 3\).

(a) For any \(k_1, k_2, k_3, k_4 \in \mathbb{Z}\) and \(j_1, j_2, j_3, j_4 \in \mathbb{Z}_+\),
\[
\|1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \leq C 2^{(k_{\min} + k_{\text{thd}})/2} 2^{(j_{\min} + j_{\text{thd}})/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}.
\]

(b) For any \(k_1, k_2, k_3, k_4 \in \mathbb{Z}\) with \(k_{\text{thd}} \leq k_{\text{sec}} - 5\) and \(j_1, j_2, j_3, j_4 \in \mathbb{Z}_+\). If for some \(i \in \{1, 2, 3, 4\}\) such that \((k_i, j_i) = (k_{\text{thd}}, j_{\max})\), then
\[
\|1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \leq C 2^{-k_{\max}/2} 2^{-k_{\text{thd}}/2} 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-j_{\max}/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2},
\]
else we have
\[
\|1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \leq C 2^{-k_{\max}/2} 2^{k_{\min}/2} 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-j_{\max}/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}.
\]

(c) For any \(k_1, k_2, k_3, k_4 \in \mathbb{N}\) and \(j_1, j_2, j_3, j_4 \in \mathbb{Z}_+\),
\[
\|1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \leq C 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-j_{\max}/2} 2^{-(k_{\min} + k_{\text{thd}} + k_{\text{sec}})/6} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}.
\]
(d) For any \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \) with \( k_{\min} \leq k_{\max} - 10 \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \),
\[
\|1_{D_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \leq C'2^{(j_1+j_2+j_3+j_4)/2}2^{-3k_{\max}/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}.
\]  

\textbf{Proof.} Clearly, we have
\[
\|1_{D_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})(\xi, \tau)\|_{L^2} = \sup_{\|f\|_{L^2}=1} |\int_{D_{k_4,j_4}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} d\xi d\tau|.
\]  

Let \( f_{k_i,j_i} = 1_{D_{k_i,j_i}} \cdot f \), we have \( f_{k_i,j_i}(\xi, \mu) = f_{k_i,j_i}(\xi, \mu + \omega(\xi)), \) \( i = 1, 2, 3, 4 \). The functions \( f_{k_i,j_i} \) are supported in \( I_{k_i} \times \bigcup_{|m| \leq 3} I_{j_i+m}, \|f_{k_i,j_i}\|_{L^2} = \|f_{k_i,j_i}\|_{L^2} \). Using simple changes of variables, we get
\[
\int_{D_{k_4,j_4}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} d\xi d\tau = J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}).
\]  

Then Corollary 3.4 follows from Lemma 3.2. \( \square \)

### 4 Proof of the trilinear estimate

In this section, we use the following propositions to prove Lemma 3.1. For simplicity, we let
\[
G = 2^{k_1}\|\eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{X_{k_4}}
\]

\textbf{Proposition 4.1.} For \( 0 \leq k_1 \leq k_2 \leq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, \) we have
\[
G \lesssim 2^{k_1/2} \prod_{i=1}^3 \|f_{k_i}\|_{X_{k_i}}.
\]

\textbf{Proof.} According to the definition of \( X_k \)
\[
G \leq 2^{k_1} \sum_{j_4=0}^{\infty} 2^{j_4/2} \|1_{D_{k_4,j_4}}(\xi, \tau)(\tau - \omega(\xi) + i)^{-1}(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}
\]

\[
\lesssim 2^{k_1} \sum_{j_4=0}^{\infty} 2^{j_4/2} 2^{-j_4} \| \sum_{j_4=0}^{\infty} 2^{j_4} \|1_{D_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2} \| \|X_{k_4}\|
\]

\[
\lesssim 2^{k_1} \sum_{j_4=0}^{\infty} 2^{-j_4} 2^{-k_{\max}/2} 2^{(j_1+j_2+j_3+j_4)/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}
\]

\[
\lesssim 2^{k_1} \prod_{i=1}^3 \|f_{k_i}\|_{X_{k_i}}.
\]

Here we use the Corollary 3.4 (d) and \( (\tau - \omega(\xi) + i)^{-1} \sim 2^{-j_4} \). \( \square \)

\textbf{Proposition 4.2.} For \( 0 \leq k_1 \leq k_2 \leq k_3, k_2 \geq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, k_1 \leq k_2 - 10, \) we have
\[
G \lesssim 2^{k_1/2} \prod_{i=1}^3 \|f_{k_i}\|_{X_{k_i}}.
\]
Proof. The proof is similar to Proposition 4.1, we omit it.

Proposition 4.3. For $0 \leq k_1 \leq k_2 \leq k_3, k_1 \geq k_3 - 30, k_3 \geq 110, |k_4 - k_3| \leq 5$, we have $G \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}$.

Proof. According to Corollary 3.4 (c), we have

$$\| 1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2} \lesssim 2^{j_1+j_2+j_3+j_4/2}2^{-j_{\max}/2}2^{-(k_{min}+k_{thd}+k_{sec})/6} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}.$$ 

So we get

$$G \lesssim 2^{k_4} \sum_{j_4=0}^{\infty} 2^{-j_4/2}2^{j_1+j_2+j_3+j_4/2}2^{-j_{\max}/2}2^{-(k_{min}+k_{thd}+k_{sec})/6} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2} \lesssim 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}.$$ 

In the last inequality, we use the fact that $k_{min} \geq k_{\max} - 40$ and $|k_4 - k_3| \leq 5$.

Proposition 4.4. For $0 \leq k_1 \leq k_2 \leq k_3, k_4 \leq k_3 - 10, k_3 \geq 110, |k_2 - k_3| \leq 5$, when $k_1 \leq k_2 - 6$, we have $G \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}$;

when $|k_1 - k_2| \leq 5$, we have $G \lesssim 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}$.

Proof. Firstly, we consider the case $k_1 \leq k_2 - 10$. Using the method in prove lemma 3.2 and making variables change $\xi'_1 = \xi_1, \xi'_2 = \xi_1 + \xi_2, \xi'_3 = \xi_3$, we can easily get

$$\| 1_{\hat{D}_{k_4,j_4}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2} \lesssim 2^{-k_{\max}+k_{1}/2}2^{k_1/2}2^{j_1+j_2+j_3+j_4/2}2^{-j_{\max}/2} \prod_{i=1}^{3} \|f_{k_i}\|_{L^2}.$$ 

By this result, we get

$$G \lesssim 2^{k_4} \sum_{j_4=0}^{\infty} 2^{-j_4/2}2^{-k_{\max}+k_{1}/2}2^{k_1/2}2^{j_1+j_2+j_3+j_4/2}2^{-j_{\max}/2} \prod_{i=1}^{3} \|f_{k_i}\|_{L^2} \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}.$$ 

Secondly, we consider the case $|k_1 - k_2| \leq 5$. Observing in this case, we have $k_{min} \geq k_{\max} - 30$, the result we need follows by Corollary 3.4 (c).

Proposition 4.5. For $0 \leq k_1 \leq k_2 \leq k_3, \max(k_3,k_4) \leq 120$, we have

$$G \lesssim 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{X_k_i}.$$
Proof. Using Corollary 3.4 (a), noticing that \( k_i \leq 120, i = 1, 2, 3, 4 \), we have

\[
G \lesssim 2^{(k_{\min} + k_{\text{thd}})/2} \prod_{i=1}^{3} \| f_{k_i} \|_{X_{k_i}} \\
\lesssim 2^{(k_1 + k_2)/4} \prod_{i=1}^{3} \| f_{k_i} \|_{X_{k_i}}
\]

Now we turn to the proof of Lemma 3.1.

**Proof:** In view of definition, we get

\[
\| \partial_x(uvw) \|_{N_s}^2 = \sum_{k_4 = 0}^{\infty} 2^{sk_4} \| \eta_{k_4}(\xi - \omega(\xi) + i)^{-1} F[\partial_x(uvw)] \|_{X_{k_4}}^2
\]

For the simplicity of notation, setting \( f_{k_1} = \eta_{k_1}(\xi) F(u)(\xi, \tau), f_{k_2} = \eta_{k_2}(\xi) F(v)(\xi, \tau), \) and \( f_{k_3} = \eta_{k_3}(\xi) F(w)(\xi, \tau), \) for \( k_1, k_2, k_3 \in \mathbb{Z}_+, \) then we get

\[
\| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} F[\partial_x(uvw)] \|_{X_{k_4}} \\
\lesssim \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} \| \xi \cdot \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}} \\
\lesssim \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}}.
\]

From symmetry, it suffices to bound

\[
\sum_{0 \leq k_1 \leq k_2 \leq k_3} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}}.
\]

Dividing the summation into the several parts, we get

\[
\sum_{k_1 \leq k_2 \leq k_3} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}} \\
\leq \sum_{j=1}^{5} \sum_{(k_1, k_2, k_3, k_4) \in A_j} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}}, \tag{4.1}
\]

where we denote

\[
A_1 = \{ 0 \leq k_1 \leq k_2 \leq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5 \}; \\
A_2 = \{ 0 \leq k_1 \leq k_2 \leq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, k_1 \leq k_2 - 10 \}; \\
A_3 = \{ 0 \leq k_1 \leq k_2 \leq k_3, k_1 \geq k_3 - 30, k_3 \geq 110, |k_4 - k_3| \leq 5 \}; \\
A_4 = \{ 0 \leq k_1 \leq k_2 \leq k_3, k_4 \leq k_3 - 10, k_3 \geq 110, |k_2 - k_3| \leq 5 \}; \\
A_5 = \{ 0 \leq k_1 \leq k_2 \leq k_3, \max(k_3, k_4) \leq 120 \}.
\]

We will apply Proposition 4.14.5 obtained in the beginning of this section to bound the five terms in (4.1). For example, for the first term, from Proposition 4.1, we have

\[
\| 2^{sk_4} \sum_{k_i \in A_1} 2^{k_4} \| \eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{X_{k_4}} \|_{L_{k_4}^2} \\
\lesssim \| 2^{sk_4} \sum_{k_i \in A_1} 2^{(k_1)/2} \| f_{k_1} \|_{X_{k_1}} \| f_{k_2} \|_{X_{k_2}} \| f_{k_3} \|_{X_{k_3}} \| f_{k_4} \|_{k_4}^2 \\
\lesssim \| u \|_{F^{1/4}} \| v \|_{F^{1/4}} \| w \|_{F^s}.
\]
For the other terms we can handle them in the same way. Therefore we complete the proof of the Lemma 3.1.

5 Uniform LWP for MKdV-B equation

In this section we study the uniform local well-posedness for the MKdV-Burgers equation. We will prove a time localized version of Theorem 1.1 where \( T = T(\|\phi\|_{H^s}) \) is small. In [3], the result of three estimates in \( X^{b,s} \) space is depend on \( \alpha, \varepsilon \), so it is not proper in this situation. To get a uniform result about \( \alpha, \varepsilon \), we will use the space \( F^s \). Let us recall that (1.1) is invariant in the following scaling

\[
 u(x, t) \rightarrow \lambda u(\lambda x, \lambda^3 t), \quad \phi(x) \rightarrow \lambda \phi(\lambda x), \quad \epsilon \rightarrow \lambda^{4-2\alpha} \epsilon, \quad \forall \ 0 < \lambda \leq 1.
\]  

This invariance is very important in the proof of Theorem 1.1 and also crucial for the uniform global well-posedness in the next section. We first show that \( F^s(T) \hookrightarrow C([0, T], H^s) \) for \( s \in \mathbb{R} \), \( T \in (0, 1] \) in the following proposition. We state some results of Guo, Proposition 5.1-5.4 can be found in [5].

**Proposition 5.1.** If \( s \in \mathbb{R} \), \( T \in (0, 1] \) and \( u \in F^s(T) \), then

\[
 \sup_{t \in [0, T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)}.
\]

**Proposition 5.2.** If \( s \in \mathbb{R} \) and \( u \in L_t^2 H_x^s \), then

\[
 \|u\|_{N^s} \lesssim \|u\|_{L_t^2 H_x^s}.
\]

We recall the estimate in [5] for the free solution.

**Proposition 5.3.** Let \( s \in \mathbb{R} \). There exists \( C > 0 \) such that for any \( 0 \leq \epsilon \leq 1 \)

\[
 \|\psi(t) W^{\alpha}_\epsilon(t) \phi\|_{F^s} \leq C \|\phi\|_{H^s}, \quad \forall \ \phi \in H^s(\mathbb{R}).
\]

Similarly for the inhomogeneous linear operator we have

**Proposition 5.4.** Let \( s \in \mathbb{R} \). There exists \( C > 0 \) such that for all \( v \in S(\mathbb{R}^2) \) and \( 0 \leq \epsilon \leq 1 \),

\[
 \|\psi(t)L(v)\|_{F^s} \leq C \|v\|_{N^s}.
\]

We next show (1.1) is uniformly (on \( 0 < \epsilon \leq 1 \)) locally well-posed in \( H^s \), \( s \geq 1/4 \). The procedure is quite standard. See [9], for instance. By the scaling (5.1), we see that \( u \) solves (1.1) if and only if \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^3 t) \) solves

\[
 \partial_t u_\lambda + \partial_x^3 u_\lambda + \epsilon \lambda^{4-2\alpha} |\partial_x|^{2\alpha} u_\lambda + 6u_\lambda^2 \phi(\lambda \cdot) = \lambda^2 \phi(\lambda \cdot) = 0, \quad u_\lambda(0) = \lambda^2 \phi(\lambda \cdot).
\]

Since \( s \geq 1/4 \),

\[
 \|\lambda^2 \phi(\lambda x)\|_{H^s} = O(\lambda^{3/2 + s} \|\phi\|_{H^s}) \quad \text{as} \ \lambda \rightarrow 0,
\]

thus we can first restrict ourselves to considering (1.1) with data \( \phi \) satisfying

\[
 \|\phi\|_{H^s} = r \ll 1.
\]
We will mainly work on the integral equation

$$u(t) = W_\varepsilon^\alpha(t)\phi_1 - L(\partial_x(\psi^3 u^3))(x,t),$$  \hspace{1cm} (5.9)

and a truncated form

$$u(t) = \psi(t) [W_\varepsilon^\alpha(t)\phi_1 - L(\partial_x(\psi^3 u^3))(x,t)],$$  \hspace{1cm} (5.10)

where $\psi$ is a smooth time cutoff function satisfying $\psi \in C_0^\infty(\mathbb{R})$, $\text{supp}\psi \subset [-2,2]$, $\psi \equiv 1$ on $[-1,1]$.

We define the operator

$$\Phi_{\phi}(u) = \psi(t)W_\varepsilon^\alpha(t)\phi - \psi(t)L(\partial_x(\psi^3 u^3)),$$  \hspace{1cm} (5.11)

where $L$ is defined by (2.2). We will prove that $\Phi_{\phi}(\cdot)$ is a contraction mapping from

$$\mathcal{B} = \{w \in F^s : \|w\|_{F^s} \leq 2cr\}$$  \hspace{1cm} (5.12)

into itself. From Propositions 5.2, 5.3 and 5.4 we get if $w \in \mathcal{B}$, then

$$\|\Phi_{\phi}(w)\|_{F^s} \leq c\|\phi\|_{H^s} + 2\|\partial_x(\psi(t)^3 w^3(\cdot, t))\|_{N^s} \leq cr + 2c\|w\|_{F^s}^{\frac{3}{2}} \leq cr + 2c(2cr)^3 \leq 2cr,$$  \hspace{1cm} (5.13)

provided $r$ satisfies $8c^3 r^2 \leq 1/2$. Similarly, for $w, h \in \mathcal{B}$

$$\|\Phi_{\phi}(w) - \Phi_{\phi}(h)\|_{F^s} \leq c\|\partial_x(\psi^3(\tau)(u^3(\tau) - h^3(\tau)))\|_{F^s} \leq c(\|h^2(w-h)\|_{F^s} + \|w(w^2-h^2)\|_{F^s}) \leq c(\|w\|_{F^s}\|w+h\|_{F^s}\|w-h\|_{F^s} + \|h\|_{F^s}\|h\|_{F^s}\|w-h\|_{F^s}) \leq 8c^3 r^2\|w-h\|_{F^s} \leq \frac{1}{2}\|w-h\|_{F^s}.$$  \hspace{1cm} (5.14)

Thus $\Phi_{\phi}(\cdot)$ is a contraction. There exists a unique $u \in \mathcal{B}$ such that

$$u = \psi(t)W_\varepsilon^\alpha(t)\phi - 2\psi(t)L(\partial_x(\psi^3 u^3)).$$  \hspace{1cm} (5.15)

Hence $u$ solves the integral equation (5.9) in the time interval $[0,1]$. Similar to Guo [5], we can show that $u \in X^{1/2,s,\alpha}$. For general $\phi \in H^s$, by using the scaling (5.1) and the uniqueness result in [3], we immediately obtain that Theorem 1.1 holds for a small $T = T(\|\phi\|_{H^s}) > 0$.

### 6 Uniform global well-posedness for KdV-B equation

In this section we will extend the uniform local solution obtained in the last section to a uniform global solution. The standard way is to use conservation law. We can verify that if $v$ was a smooth solution of (1.4), then

$$H_1[v] = \int_{\mathbb{R}} (v_x)^2 - v^4 + v^2 dx$$  \hspace{1cm} (6.1)
is a conservation quantity for (1.4). However, there are less symmetries for (1.1). Let \( u \) be a smooth solution of (1.1), we have

\[
\frac{d}{dt} H_1[u] = \int_\mathbb{R} 2u_x \partial_x u_t - 4u^3 u_t + 2uu_t dx
\]

\[
= \int_\mathbb{R} 2u_x \partial_x (-u_{xxx} - \varepsilon |\partial_x|^{2\alpha} u + 2(u^3)_x) - 4u^3 (-u_{xxx} - \varepsilon |\partial_x|^{2\alpha} u + 2(u^3)_x) + 2u(-u_{xxx} - \varepsilon |\partial_x|^{2\alpha} u + 2(u^3)_x) dx
\]

\[
= \int_\mathbb{R} 2u_x \partial_x (-\varepsilon |\partial_x|^{2\alpha} u) + 4u^3 (\varepsilon |\partial_x|^{2\alpha} u) - 2u(\varepsilon |\partial_x|^{2\alpha} u) dx
\]

\[
= -2\varepsilon \int_\mathbb{R} (\Lambda^{1+\alpha} u)^2 dx + 4\varepsilon \int_\mathbb{R} u^3 \Lambda^{2\alpha} u^2 dx - 2\varepsilon \int_\mathbb{R} |\Lambda^\alpha u|^2 dx
\]

\[
\leq -\varepsilon \int_\mathbb{R} (\Lambda^{2\alpha} u)^2 dx + 4\varepsilon \int_\mathbb{R} u^3 \Lambda^{2\alpha} u^2 dx
\]

(6.2)

where we use the notation \( \Lambda = |\partial_x| \). Using Cauchy-Schwartz inequality, we have

\[
4\varepsilon \int_\mathbb{R} u^3 \Lambda^{2\alpha} u^2 dx \leq 4\varepsilon \|u^3\|_2 \|\Lambda^{2\alpha} u\|_2
\]

\[
\leq 4\varepsilon \|u\|_6^3 \|\Lambda^{2\alpha} u\|_2
\]

\[
\leq 4\varepsilon (\|u\|_6^6 + \frac{1}{8} \|\Lambda^{2\alpha} u\|_2^2)
\]

\[
\leq 8\varepsilon \|u\|_6^6 + \frac{\varepsilon}{2} \|\Lambda^{2\alpha} u\|_2^2
\]

Therefore, we have

\[
\frac{d}{dt} H_1[u] + \frac{\varepsilon}{2} \|\Lambda^{2\alpha} u\|_2^2 \leq \|u\|_6^6
\]

Using Galiardo-Nirenberg inequality

\[
\|u\|_6^6 \leq \|u\|_2^\frac{6}{2} \|u_x\|_2^2, \quad \|u\|_4^4 \leq \|u\|_2^3 \|u_x\|_2
\]

Hence, we get

\[
\sup_{[0,T]} \|u(t)\|_{H^1} + \varepsilon \left( \int_0^T \|\Lambda^{2\alpha} u\|_2^2 d\tau \right)^{\frac{1}{2}} \leq C(T, \|u_0\|_{H^1})
\]

(6.3)

By a standard limit argument, \((6.3)\) holds for \(H^1\)-strong solution. Thus if \( \phi \in H^1 \), then we get that (1.1) is uniformly globally well-posed.

7 Limit Behavior

In this section we prove Theorem 1.2. From the remark 1.3 if we consider the limit behavior in \( H^1 \) we need a \( H^2 \)-conservation quantity. We first give a \( H^2 \)-conservation quantity for (1.4). It is well-known that the following KdV equation

\[
u_t + u_{xxx} = 3(u^2)_x, \quad u(0) = \phi.
\]

(7.1)

is completely integrable and has infinite conservation laws. As a corollary one obtains that if \( v \) was a smooth solution to (7.1), then for any \( k \in \mathbb{Z}_+ \),

\[
\sup_{t \in \mathbb{R}} \|v(t)\|_{H^k} \lesssim \|v_0\|_{H^k}.
\]

(7.2)
Now we use Miura transform $M$ to establish the relation of (7.1) and (1.4). If $v$ was a solution of (1.4), then $Mu = \partial_x v + v^2$ is a solution of (7.1), see (14). Using this fact, we can find a $H^2$-conservation quantity of (1.4). We can easily verify that

$$H_1[u] = \int_R (\partial_x u)^2 + 2u^3dx$$

(7.3)

is a $H^1$-conservation quantity of (7.1). Let $u = \partial_x v + v^2$ in (7.3), we get

$$H_2[u] = \int_R (u_{xx})^2 + 10u^2u_x^2 + 2u^6dx$$

(7.4)

is a $H^2$-conservation quantity of (1.4). Obviously, (1.4) has $L^2$-conservation law, so

$$H'_2[u] = \int_R (u_{xx})^2 + 10u^2u_x^2 + 2u^6 + u^2dx$$

(7.5)

is also a $H^2$-conservation quantity of (1.4).

However, there are less symmetries for (1.1). We can still expect that the $H^k$ norm of the solution remains dominated for a finite time $T > 0$, since the dissipative term behaves well for $t > 0$. We already see that for $k = 1$ from (6.1). Now we prove for $k = 2$ which will suffice for our purpose. We do not pursue for $k \geq 3$. Assume $u$ is a smooth solution to (1.1). By the equation (1.1) and partial integration in (7.5), we have

$$\frac{d}{dt} H'_2[u] = \int_R 2u_{xx}\partial_{xx}(u_t) + 20uu_x^2u_t + 20u^2u_x\partial_x(u_t) + 12u^5u_t + 2uu_tdx$$

$$= \int_R 2u_{xx}\partial_x(-uu_{xx} - \varepsilon|\partial_x|^{2\alpha}u + 2(u^3_x))dx$$

$$+ \int_R 20uu_x\partial_x(-uu_{xx} - \varepsilon|\partial_x|^{2\alpha}u + 2(u^3_x))dx$$

$$+ \int_R (20uu_x^2 + 12u^5 + 2u)(-uu_{xx} - \varepsilon|\partial_x|^{2\alpha}u + 2(u^3_x))dx$$

$$= -2\varepsilon \int_R (|\partial_x|^{\alpha+2}u)^2dx - 20\varepsilon \int_R uu_x^2(|\partial_x|^{2\alpha}u)dx$$

$$-20\varepsilon \int_R u^2u_x|\partial_x|^{2\alpha}udx - 12\varepsilon \int_R u^5|\partial_x|^{2\alpha}udx - 2\varepsilon\|\partial_x|^{\alpha}u\|_2^2$$

$$\leq -2\varepsilon\|\Lambda^{\alpha+2}u\|_2^2 - 2\varepsilon\|\Lambda^\alpha u\|_2^2 - 20\varepsilon \int_R uu_x^2\Lambda^{2\alpha}udx$$

$$-20\varepsilon \int_R u^2u_x\partial_x(\Lambda^{2\alpha}u)dx - 12\varepsilon \int_R u^5\Lambda^{2\alpha}udx$$

$$\lesssim -\varepsilon\|\Lambda^{\alpha+2}u\|_2^2 - \varepsilon\|\Lambda^\alpha u\|_2^2 + \|uu_x^2\|_2^2 + \|u^5\|_2^2 + \|u^2u_x\|_2^2 + \|uu_x\|_2^2 + \|u\|_6^2 + \|u\|_2^2$$

where we use the notation $\Lambda = |\partial_x|$ and Cauchy-Schwarz inequality. Thus we have

$$\frac{d}{dt} H'_2[u] + \frac{\varepsilon}{2}\|\Lambda^{2\alpha+1}u\|_2^2$$

$$\lesssim \|uu_x^2\|_2^2 + \|u^5\|_2^2 + \|u^2u_x\|_2^2 + \|uu_x\|_2^2 + \|u\|_6^2 + \|u\|_2^2$$

$$\lesssim \|u\|_\infty^2\|u_x\|_4^4 + \|u\|_\infty\|u_x\|_2^2 + \|u\|_{10}^{10} + \|u\|_{4}^{4} + \|u\|_{4}^{4} + \|u\|_{3}^{3}\|u_x\|_2^2$$

(7.6)

Using Galiardo-Nirenberg inequality, the right of (7.6) can be dominated by $\|u\|_{H^2}$. 

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Using Cauchy-Schwartz inequality, we get

$$\sup_{[0,T]} \|u(t)\|_{H^2} + \epsilon^{1/2} \left( \int_0^T \| \Lambda^{2\alpha+1} u(\tau) \|_{L^2}^2 d\tau \right)^{1/2} \leq C(T, \|\phi\|_{H^2}), \quad \forall \ T > 0. \quad (7.7)$$

Assume $u_\epsilon$ is a $H^1$-strong solution to (1.1) obtained in the last section and $v$ is a $H^1$-strong solution to (1.4) in [3], with initial data $\phi_1, \phi_2 \in H^1$ respectively. We still denote by $u_\epsilon, v$ the extension of $u_\epsilon, v$. From the scaling (5.1), we may assume first that $\|\phi_1\|_{H^1}, \|\phi_2\|_{H^1} \ll 1$. Let $w = u_\epsilon - v$, $\phi = \phi_1 - \phi_2$, then $w$ solves

$$\begin{align*}
\left\{ \begin{array}{l}
w_t + w_{xxx} + \epsilon |\partial_x|^{2\alpha} w = 2(w(v^2 + u_\epsilon^2 + vu_\epsilon))_x, \ t \in \mathbb{R}_+, x \in \mathbb{R}, \\
v(0) = \phi.
\end{array} \right.
\end{align*} \quad (7.8)$$

We first view $\epsilon |\partial_x|^{2\alpha} w$ as a perturbation to the difference equation of the MKdV equation. Considering the integral equation of (7.8)

$$w(x,t) = W_0(t)\phi - \int_0^t W_0(t-\tau)[\epsilon |\partial_x|^{2\alpha} w + 2(w(v^2 + u_\epsilon^2 + vu_\epsilon))_x]d\tau, \ t \geq 0. \quad (7.9)$$

Then $w$ solves the following integral equation on $t \in [0,1]$,

$$w(x,t) = \psi(t)W_0(t)\phi - \chi_{\mathbb{R}_+} \int_0^t W_0(t-\tau) \psi(\tau)(\epsilon |\partial_x|^{2\alpha} w(\tau))d\tau - 2\chi_{\mathbb{R}_+} \int_0^t W_0(t-\tau) (w(v^2 + u_\epsilon^2 + vu_\epsilon))_x(\tau)d\tau. \quad (7.10)$$

By Proposition 5.2, 5.3, 5.4 and Lemma 3.1, for $1/4 \leq s \leq 1$, we get

$$\|w\|_{F^s} \lesssim \|\phi\|_{H^1} + \epsilon \|u_\epsilon\|_{L^2_{[0,T]}H^{2\alpha+s}} + \|w\|_{F^s}|u_\epsilon|_{F^s} + \|w\|_{F^s}\|u_\epsilon\|_{F^s} + \|w\|_{F^s}\|v\|_{F^s}. \quad (7.11)$$

Since from Theorem 1.1 we have

$$\|v\|_{F^s} \lesssim \|\phi_2\|_{H^s} \ll 1, \quad \|u_\epsilon\|_{F^s} \lesssim \|\phi_1\|_{H^s} \ll 1,$$

then we get that

$$\|w\|_{F^s} \lesssim \|\phi\|_{H^s} + \epsilon \|u_\epsilon\|_{L^2_{[0,T]}H^{2\alpha+s}}. \quad (7.12)$$

From Proposition 5.1 and (7.7) we get

$$\|u_\epsilon - v\|_{C([0,T],H^s)} \lesssim \|\phi_1 - \phi_2\|_{H^s} + \epsilon^{1/2}C(\|\phi_1\|_{H^2}, \|\phi_2\|_{H^1}). \quad (7.13)$$

For general $\phi_1, \phi_2 \in H^1$, using the scaling (5.1), then we immediately get that there exists $T = T(\|\phi_1\|_{H^1}, \|\phi_2\|_{H^1}) > 0$ such that

$$\|u_\epsilon - v\|_{C([0,T],H^s)} \lesssim \|\phi_1 - \phi_2\|_{H^s} + \epsilon^{1/2}C(T, \|\phi_1\|_{H^2}, \|\phi_2\|_{H^1}). \quad (7.14)$$

Therefore, (7.14) automatically holds for any $T > 0$, due to (6.1) and (7.7).

**Proof of Theorem 1.2** For fixed $T > 0$, we need to prove that $\forall \ \eta > 0$, there exists $\sigma > 0$ such that if $0 < \epsilon < \sigma$ then

$$\|S_T^\epsilon(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} < \eta. \quad (7.15)$$
We denote $\varphi_K = P_{\leq K} \varphi$, then we get
\[
\|S_T^*(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} \\
\leq \|S_T^*(\varphi) - S_T^*(\varphi_K)\|_{C([0,T];H^s)} \\
+ \|S_T^*(\varphi_K) - S_T(\varphi_K)\|_{C([0,T];H^s)} + \|S_T(\varphi_K) - S_T(\varphi)\|_{C([0,T];H^s)}.
\] (7.16)

From Theorem 1.1 and (7.14), we get
\[
\|S_T^*(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} \lesssim \|\varphi_K - \varphi\|_{H^s} + \epsilon^{1/2}C(T, K, \|\varphi\|_{H^s}).
\] (7.17)

We first fix $K$ large enough, then let $\epsilon$ go to zero, therefore (7.15) holds.

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**References**

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II. Geom. Funct. Anal., 3:107-156, 209-262, 1993.

[2] Wengu Chen, Junfeng Li, On the low regularity of the modified Korteweg-de Vries equation with a dissipative term, Journal of Differential Equations 240(2007) 125-144.

[3] Wengu Chen, Junfeng Li and Changxing Miao, The well-posedness of cauchy problem for dissipative modified korteweg deVries equations, Differential and Integral Equations, 20 (2007), 1285-1301.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$. J. Amer. Math. Soc., 16(3):705-749, 2003.

[5] Zihua Guo and Baoxiang Wang, Global well-posedness and inviscid limit for the Korteweg-de Vries-Burgers equation, [arXiv:0803.2450](http://arxiv.org/abs/0803.2450)

[6] Zihua Guo, Local Well-posedness and a priori bounds for the modified Benjamin-Ono equation without using a gauge transformation, [arXiv:0807.3764v1](http://arxiv.org/abs/0807.3764v1)

[7] A. D. Ionescu, C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces, J. Amer. Math. Soc., 20 (2007), no. 3, 753-798.

[8] A. D. Ionescu, C. E. Kenig, D. Tataru, Global well-posedness of KP-I initial-value problem in the energy space, [arXiv:0705.4239](http://arxiv.org/abs/0705.4239v1)

[9] C. E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Communications on Pure and Applied Mathematics 46 (1993), no. 4, 527-620.

[10] L. Molinet and F. Ribaud, Well-posedness results for the generalized Benjamin-Ono equation with arbitrary large initial data, International Mathematics Research Notices 2004 (2004), no. 70, 3757-3795.

[11] L. Molinet and F. Ribaud,The cauchy problem for dissipative korteweg-de Vries equations in Sobolev spaces of negative order,Indiana Univ.math.J.50(2001)1745-1776.

[12] E. Ott, N. Sudan, Damping of solitary waves, J. Phys. Fluids 13 (1970), no. 6, 1432-1434.

[13] T. Tao, Multiplier weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations, Amer. J. Math., 123(5):839-908, 2001. MR 2002k:35283

[14] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics 106.

[15] D. Tataru, Local and global results for wave maps I, Comm. Partial Differential Equations 23 (1998), 1781-1793.

[16] S. Vento, Global well-posedness for dissipative Korteweg-de Vries equations,[arXiv:0706.1730v1](http://arxiv.org/abs/0706.1730v1).

[17] Baoxiang Wang, The Limit Behavior of Solutions for the Cauchy Problem of the Complex Ginzburg-Landau Equation, Communications on Pure and Applied Mathematics, 53 (2002), 0481-0508.

[18] Baoxiang Wang, Interudition to nonlinear evolution equations, preprint.