Consensus formation times in fully connected societies

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Abstract

We developed a statistical mechanics approach to the problem of opinion formation in interacting agents, constrained by a set of social rules, \( B \). To provide the agents with an adaptive quality, we represented both the social agents and the social rule by perceptrons. For fully connected societies we find that if the agents’ interaction is weak, all agents adapt to the social rule \( B \), with which they form a consensus; but if the interaction is sufficiently strong a consensus is built against the established status quo. This behavior is observed for all temperatures \( T \) and for all values of the agents’ interaction parameter \( H_0 \), except in the limit \( T \to \infty \) or when the interaction reaches the critical value \( H_0 = 1 \), where no consensus is formed. The agents follow a path where, after a time \( \alpha_c \), they disregard their peers’ opinions on socially neutral issues and reach a full consensus at time \( \alpha_d > \alpha_c \). The measure of time \( \alpha \) is proportional to the volume of information provided to the agents.
In this letter we propose a statistical mechanics approach to study the emergence of consensus in a fully connected society of adaptive agents, in the presence of a social field \( B \). The term \textit{consensus} is understood as the level of agreement amongst the agents in favor or against the predetermined socially accepted position delivered by \( B \). \( B \) represents the set of rules resulting from previous consensus-forming processes, typically observed in any functioning society [2, 3]. Agents form their opinions on social issues based on partial information received regularly during the process. The volume of information increases over time and, the agents being adaptive, they update their opinions accordingly. At the end of the process the rate of agreement between agents and \( B \) is measured to determine whether a consensus is formed supporting or rebutting the social order.

There is sufficient evidence in support of modeling opinions (on \textit{important} issues) with binary variables [4]. We will represent the opinion of agent \( a \) on an issue \( \xi \in \{\pm 1\}^N \) (represented by a binary string of length \( N \)) by \( \sigma_a(\xi) \in \{\pm 1\} \). In mathematical terms \( B \) is a classifier that assigns a binary label \( \sigma_B(\xi) \) to the issue \( \xi \). According to [5], representing \( B \) with a perceptron (with a constant synaptic vector \( B \in \mathbb{R}^N \)) ensures the analytical tractability of the model. In this manner the socially accepted position on \( \xi \) is \( \sigma_B(\xi) = \text{sgn}(B \cdot \xi) \) where \( \text{sgn}(x) = 1 \) if \( x > 0 \), \(-1\) if \( x < 0 \) and 0 otherwise and \( B \cdot \xi = \sum_{j=1}^{N} B_j \xi_j \). For consistency sake we associate each agent \( a \) to a perceptron with an adaptive synaptic vector \( J_a \), such that \( \sigma_a(\xi) = \text{sgn}(J_a \cdot \xi) \).

There is a body of evidence supporting the effect of social influence on opinion formation processes [6]; in consequence, to model the agents’ interactions, we follow the social impact theory [7, 8]. To give a topological structure to the system we consider a society with \( M \) agents \( 1 \leq a \leq M \) linked by a set of social strengths \( \mathcal{S} \equiv \{\eta_{a,c} | 0 \leq \eta_{a,c} \in \mathbb{R}\} \), where \( \eta_{a,c} \) represents the influence agent \( c \) has on the opinion of agent \( a \). We define the neighborhood of \( a \) by \( N_a = \{c | c \neq a \text{ and } \eta_{a,c} > 0\} \) which is the set of agents connected to \( a \). The opinion formation process itself is modeled by an on-line learning scenario [9], where a set of social issues \( \mathcal{L}_P \equiv \{(\xi_{\mu}, \sigma_B(\xi_{\mu})) ; \mu = 1, \ldots, P\} \) is used to define the energy of the society:

\[
E(\{J_a\}; \mathcal{L}_P, \mathcal{S}) \equiv \sum_{\mu=1}^{P} \sum_{a=1}^{M} \Theta(-\sigma_a(\xi_{\mu})\sigma_B(\xi_{\mu})) \\
\left[ 1 - \sum_{c \in N_a} \eta_{a,c} \Theta(-\sigma_c(\xi_{\mu})\sigma_B(\xi_{\mu})) \right] \tag{1}
\]

where \( \Theta(x) = 1 \) if \( x > 0 \) and 0 otherwise. Observe that for independent agents (\( \forall a, c ; \eta_{a,c} = 0 \)) the energy (1) is minimized with a consensus in favor of \( B \). If the social strengths \( \{\eta_{a,c}\} \) are sufficiently large, the energy is minimized with a consensus against \( B \).

Observe that the model described by (1) possesses two sources of disorder, one introduced through the set of issues \( \mathcal{L}_P \), and the second through the topology imposed by \( \mathcal{S} \). In this letter we present a study on the emergence of consensus in homogeneous, fully connected graphs (i.e. for all index \( a \), \( N_a = \{1, 2, \ldots, a - 1, a + 1, \ldots, M\} \) and \( \eta_{a,c} = \eta_0 \) for all pairs \( (a, c) \)).
We apply the replica trick [10] in order to compute the expectation of the logarithm of the partition function $\log Z = \lim_{n \to 0} n^{-1} (Z^n - 1)$. The average of the replicated partition function is

$$Z^n(\beta, \eta_0) \equiv \mathbb{E} \left[ \exp \left(-\beta \sum_{\tau=1}^n E(\{J_a\}; \{\xi_\mu\}, \eta_0)\right) \right]$$

(2)

where the expectation $\mathbb{E}[\cdot]$ is taken over the issues $\xi$, the social rule $B$ and the agents’ synaptic vectors $J_a$, with probabilities $\mathcal{P}(\xi) \equiv 2^{-N} \prod_{k=1}^N (\delta_{\xi_k,1} + \delta_{\xi_k,-1})$, $d\mathcal{B} \mathcal{P}(B) = \prod_k dB_k \delta(B_k - 1)$ and $dJ \mathcal{P}(J) \equiv \prod_{k=1}^N dJ_k \delta \left(\sum_{k=1}^N J_k^2 - N\right) / \sqrt{2\pi n}$ respectively.

By defining the order one parameters $R_a^\gamma \equiv J_a^\gamma \cdot B / N$, $q_a^\gamma \rho \equiv J_a^\rho / N$, $W_{a,b}^\gamma \equiv J_a^\gamma \cdot J_b^\gamma / N$, and $t_{a,b}^\gamma \rho \equiv J_a^\gamma \cdot J_b^\rho / N$ and imposing the replica symmetric Ansatz, i.e. $R_a^\gamma \equiv R$, $q_a^\gamma \rho \equiv q$, $W_{a,b}^\gamma \equiv W$, and $t_{a,b}^\gamma \rho \equiv t$ with the assumption that the overlaps $W$ and $t$ satisfy the scaling $\tau \equiv M(W - t) \sim O(1)$ (see reference [11], equation (3)), it is possible to demonstrate that the logarithm of $Z^n$ can be decomposed in two terms, an entropic contribution:

$$\mathcal{G}(P) \equiv \frac{1}{2} \left( \ln(1 - q) + \frac{q - W}{1 - q} + \frac{W - R^2}{1 - q + \tau} \right)$$

(3)

with $P = (R, q, W, \tau)$ and an energetic contribution:

$$\mathcal{F}_{n,M}(P; \beta, \eta_0) \equiv \frac{1}{nM} \log \left[ 2 \int_0^\infty Dx \int Dw \prod_a Dw \left( \int Ds \prod_a (B(x, \beta) \mathcal{H}(y_a) + \mathcal{H}(-y_a)) \right)^n \right]$$

(4)

where $Dx \equiv dx e^{-x^2/2} / \sqrt{2\pi}$ is the Gaussian measure, $\mathcal{H}(u) = \int_u^\infty Dx$ the Gardner error function, $B(x, \beta) \equiv \exp \left( \sqrt{2\beta} \eta_0 x - \beta \right)$ and

$$y_a \equiv Ru + \sqrt{W - \frac{\tau}{M} - R^2 w + \left( q - W + \frac{\tau}{M} w_a \right)^2 + \frac{\tau}{M} s}.$$  

The replicated partition function is, in the limit of large $N$:

$$Z^n(\beta, \eta_0) = \operatorname{extr}_P \left\{ \exp \left[ nNM \left( \mathcal{G}(P) + \alpha \mathcal{F}_{n,M}(P; \beta, \eta_0) \right) \right] \right\},$$

where $\alpha \equiv P / N$ is a measure of the volume of information presented to the agents. Let us define the arithmetic average $\bar{y} \equiv M^{-1} \sum_a y_a$ and by using the approximation $(\frac{M}{\ell})^{-1} \sum_{\ell=1}^\ell \mathcal{H}(\bar{y}) \approx \mathcal{H}(\bar{y})$ we can approach the replicated product in equation (4) by

$$\prod_a \left( B(\mathcal{H}(y_a) + \mathcal{H}(-y_a)) \approx [B \mathcal{H}(\bar{y}) + \mathcal{H}(\bar{y})]^M .$$

(5)

To ensure the extensivity of the energy (11) we impose the scaling $M\eta_0 = H_0 \sim O(1)$. Thus, by applying a Gaussian approximation to the RHS of (5) we have that, in leading order in $M$, the energetic contribution can be expressed as:

$$\mathcal{F}_{n,M}(P; \beta, \eta_0) \approx -2 \sqrt{1 - q \over W} \int dz \sqrt{2\pi} \exp \left( -\frac{1 - q z^2}{2W} \right) \mathcal{H} \left( -\frac{1 - q}{\sqrt{W(W - R^2)}} Rz \right) \Phi(z; \tau; q; \beta, H_0)$$

(6)
where $\Phi(z; \tau, q; \beta, H_0)$ is the minimum over $u \in (0, 1)$ and $\sigma \in \mathbb{R}$ of the function
\[
\Omega(u, \sigma; z; \tau, q; \beta, H_0) = \frac{1 - q (\sigma - z)^2}{\tau} + \frac{[u - \mathcal{H}(\sigma)]^2}{2\mathcal{H}(\sigma)\mathcal{H}(-\sigma)} - u^2\beta H_0 + u\beta + O \left( \frac{\log M}{M} \right). \tag{7}
\]

For small values of $\tau$ we have that $\Phi(z; \tau, q; \beta, H_0) = \Phi(z; \beta, H_0) + \frac{\tau - \hat{\Phi}(z; \beta, H_0)}{1 - q} + O(\tau^2)$ for suitable functions $\Phi(z; \beta, H_0)$ and $\hat{\Phi}(z; \beta, H_0)$. By defining the quantities:
\[
\begin{align*}
a_1 &\equiv \Theta(2H_0 - 1) \max \left\{ 0, \frac{\beta(2H_0 - 1) - 1}{\beta(2H_0 - 1)} \right\} \tag{8} \\
a_2 &\equiv \min \left\{ 1, \frac{1}{\beta} \right\} \tag{9} \\
a_3 &\equiv \frac{1}{2} - \frac{\sqrt{\beta^2(1 - H_0)^2 + 1} - 1}{2\beta(1 - H_0)} \tag{10} \\
b_0 &\equiv \Theta(a_2 - a_1)a_2 + \Theta(a_1 - a_2)a_3 \tag{11} \\
b_1 &\equiv \Theta(a_2 - a_1)a_1 + \Theta(a_1 - a_2)a_3 \tag{12}
\end{align*}
\]

we can split the real line in three non-intersecting segments $\mathbb{D}_{z_0}, \mathbb{D}_0$ and $\mathbb{D}_1$, such that $\mathbb{D}_{z_0} \equiv \{ x \in \mathbb{R} | b_1 < \mathcal{H}(-x) < b_0 \}$, $\mathbb{D}_0 \equiv \{ x \in \mathbb{R} | b_0 < \mathcal{H}(-x) \}$ and $\mathbb{D}_1 \equiv \{ x \in \mathbb{R} | b_1 > \mathcal{H}(-x) \}$. In the zeroth order of $\tau$ we have that:
\[
\Phi(z; \beta, H_0) \equiv \begin{cases} 
\Phi_{z_0} = \frac{\beta H(z)[1 - H_0 H(z)]}{1 - 2H_0 H(z)H(-z)} & z \in \mathbb{D}_{z_0} \\
\Phi_0 = \frac{\mathcal{H}(z)}{2\mathcal{H}(-z)} & z \in \mathbb{D}_0 \\
\Phi_1 = \frac{\mathcal{H}(-z)}{2\mathcal{H}(z)} + \beta(1 - H_0) & z \in \mathbb{D}_1.
\end{cases} \tag{13}
\]

$\Phi(z; \beta, H_0)$ is continuous in $z$ but not differentiable at the boundaries $z = -\mathcal{H}^{-1}(a_1)$ (between $\Phi_1$ to the left and $\Phi_{z_0}$ to the right) and $z = -\mathcal{H}^{-1}(a_2)$ (between $\Phi_{z_0}$ to the left and $\Phi_0$ to the right) if $a_1 < a_2$ or $z = -\mathcal{H}^{-1}(a_3)$ (between $\Phi_1$ to the left and $\Phi_0$ to the right) otherwise. In the plain defined by the independent parameters $\beta$ and $H_0$ the components $\Phi_{z_0}, \Phi_0$ and $\Phi_1$ cover the areas illustrated in figure [H]. Observe that the component $\Phi_{z_0}$ appears in the sector $\mathcal{S}_{z_0} \equiv \{ (\beta, H_0) | \beta \leq 1 \text{ and } H_0 \geq 0 \} \cup \{ (\beta, H_0) | \beta > 1 \text{ and } 2H_0 < \beta/(\beta - 1) \}$, the component $\Phi_1$ appears in the sector $\mathcal{S}_1 \equiv \{ (\beta, H_0) | \beta \geq 0 \text{ and } 2H_0 > (1 + \beta)/\beta \}$ and the component $\Phi_0$ appears in the sector $\mathcal{S}_0 \equiv \{ (\beta, H_0) | \beta \geq 1 \text{ and } H_0 \geq 0 \}$.

The fragmentation of the function $\Phi(z; \beta, H_0)$ over the plane $(\beta, H_0)$ is the consequence of the interaction between a very large number of agents and the feature responsible for the complex behavior described in the following.

$R$, the overlap between a typical $J$ and $B$, represents the level of agreement with the social rule $B$. $q$, the overlap between synaptic vectors belonging to the same replicated agent, represents the level of variability that remains in the space of compatible synaptic vectors (known as the version space). $W$ is the projection of an agent’s synaptic vector in the direction of another agent’s synaptic vector.
Figure 1: Distribution of the components (13) in the plain $(\beta, H_0)$ (color on-line).

Within the same replicated system. If two agents $a$ and $c$ have the same overlap with $B$, $R_a = R_c = R$ (as in the case considered) the relationship between $R$ and $W_{a,c}$ is $W_{a,c} = R^2 + (1 - R^2) \cos \phi$ where $\phi$ is the angle between $J_{a,\perp} \equiv J_a - RB/\sqrt{N}$ and $J_{c,\perp} \equiv J_c - RB/\sqrt{N}$, which are the components of $J_a$ and $J_c$ perpendicular to $B$ respectively. An interesting effect is observed when we consider opinions on socially neutral issues, which are issues $S_0 \in \{\pm 1\}^N$ such that $S_0 \cdot B = 0$. If $\phi = \pi/2$ then $W_{a,c} = R^2$, which implies that the opinion of an agent $a$ on $S_0$ is independent on the opinion agent $c \neq a$ on $S_0$. Disregarding $\tau$ and by defining $w \equiv (1 - q)^{1/2}W$ and $r \equiv (1 - q)^{-1/2}R$, we have that the free energy of the system is:

$$
\beta f(\alpha, \beta, H_0) = \text{extr}_{\{r, q, w\}} \phi(r, q, w; \beta, H_0),
$$

where

$$
\phi(r, q, w; \beta, H_0) \equiv -\frac{1}{2} \left( \ln(1 - q) + \frac{q}{1 - q} \right) + \frac{r^2}{2} + 2\alpha \int \frac{dz}{\sqrt{2\pi w}} \exp \left( -\frac{z^2}{2w} \right) \mathcal{H}(-\kappa z) \Phi(z),
$$

(14)

where $\kappa \equiv r/\sqrt{w(w - r^2)}$. The conditions $\partial_r \phi = \partial_q \phi = \partial_w \phi = 0$ imply that $q = 0$ and

$$
r = -\sqrt{2\pi} \alpha \int dz \mathcal{N}(z|0, w - r^2) \Phi'(z)
$$

(15)

$$
r^2 = -2\alpha \int dz \mathcal{N}(z|0, w) z \mathcal{H}(-\kappa z) \Phi'(z),
$$

(16)

where $\mathcal{N}(x|\mu, \sigma^2) \equiv \exp (- (x - \mu)^2 / 2\sigma^2) / \sqrt{2\pi}$ is a Gaussian distribution in $x$, centered at $\mu$, with standard deviation $\sigma$. Clearly, $r^2 \leq w$. If there is an increment of $r^2$ towards $w$, the Gaussian distribution in (15) becomes sharply concentrated at 0. Moreover, if the parameter $r^2$ reaches $w$ for
a finite volume of information \( \alpha_c \) we have that

\[
\alpha_c = -\sqrt{\frac{\pi}{2}} \frac{r}{\Phi'(0)}
\]

(17)

\[
r = \frac{\sqrt{2\pi}}{\Phi'(0)} \int dz \mathcal{N}(z|0, r^2) \Theta(rz) z \Phi'(z),
\]

(18)

where

\[
\Phi'(0) = \sqrt{\frac{2}{\pi}} \text{sgn}(H_0 - 1) \left\{ \begin{array}{ll}
\frac{\beta(1-H_0)}{2-\beta H_0} & \beta < 2 \text{ and } H_0 < \frac{2+2\beta}{2\beta} \\
1 & \text{otherwise}
\end{array} \right.
\]

(19)

By defining

\[
I \equiv -\frac{1}{2} \int dz \mathcal{N}(z|0, r^2) \Theta((1-H_0)z) \Phi''(z)
\]

\[
J \equiv -\frac{r}{2 \Phi'(0)} \int dz \mathcal{N}(z|0, r^2) \Theta((1-H_0)z) \Phi'(z)
\]

we have that the determinant of the Hessian when \( w = r^2 \), and \( \alpha_c \) and \( r \) are given by (17) and (18) respectively is:

\[
|H| = \frac{1}{4} \left[ \frac{1}{r^2} - \frac{3}{2} \frac{\Phi''(0)}{\Phi'(0)} + \left( 1 - \frac{r^2 \Phi''(0)}{\Phi'(0)} \right) (3I + J) \right]
\]

(20)

where

\[
\frac{\Phi''(0)}{\Phi'(0)} = \left\{ \begin{array}{ll}
\frac{1}{\pi} (12-\pi) \frac{\beta H_0 + 2\pi}{2-\beta H_0} & \beta < 2 \text{ and } H_0 < \frac{2+2\beta}{2\beta} \\
-\frac{12-\pi}{\pi} & \text{otherwise}
\end{array} \right.
\]

The determinant (20) is found to be positive for all \( \beta > 0 \) and \( H_0 \neq 1 \). Thus the solution \( w = r^2 \) at \( \alpha_c \) given by (17) with \( r \) given by (18) is stable. A plot of the \( \log(\alpha_c) \) as a function of \( \beta \) and \( H_0 \) is presented in figure 2. From figure 2 we observe that there is a sector of the \((\beta, H_0)\) plane for which the system takes a relatively long time to reach the solution \( r^2 = w \). This is the sector for which \( z = 0 \in D_{z_0} \). In this manner we can construct the phase diagram shown in figure 3. Note that for all values of \( \alpha \) larger than \( \alpha_c \), the equation (15) is no longer satisfied given that the minimum occurs at the border of definition of the parameter \( r \) (either \( w \) or \(-w\)) but the derivative \( \partial_r \phi \) is not zero.
Most of the opinion formation process occurs for $\alpha > \alpha_c$. The effective energy for $\alpha > \alpha_c$ can be defined as

$$\phi_{\text{eff}}(q, w; \alpha, \beta, H_0) \equiv -\log(1 - q) - \frac{q}{2(1 - q)} - \frac{w}{2} + 2\alpha \int \frac{dz}{\sqrt{2\pi}w} \exp \left( -\frac{z^2}{2w} \right) \Theta ((1 - H_0)z) \Phi(z).$$  \hspace{1cm} (21)

The new saddle point equations are

$$0 = q$$
$$w = 2\alpha \int dz \mathcal{N}(z \mid 0, w) \left( 1 - \frac{z^2}{w} \right) \Theta ((1 - H_0)z) \Phi(z).$$

There is a maximum value of $\alpha = \alpha_d$ such that $w = 1$:

$$\alpha^{-1}_d = 2 \int dz (1 - z^2) \Theta ((1 - H_0)z) \Phi(z).$$  \hspace{1cm} (22)

The determinant of the Hessian at $\alpha_d$ with $q = 0$ and $w = 1$ is:

$$|H| = \frac{3}{8} - \frac{1}{8} \frac{\int Dz (3z^2 - z^4) \Theta ((1 - H_0)z) \Phi(z)}{\int Dz (1 - z^2) \Theta ((1 - H_0)z) \Phi(z)}$$

which is positive for all values of $\beta$ and $H_0$. As it is shown in figure 3, $\alpha_c < \alpha_d$ for all values of $\beta > 0$ and $H_0 \neq 1$.

**Discussion:** We presented a model for the opinion formation process in a society of interacting agents, represented by binary perceptrons, in the presence of a social field $B$. The field is the result of many opinion formation processes prior to the current one and provides the socially acceptable position on which issues’ opinions are formed.

Although we worked in a fully connected graph with non directed links, we observed the asymptotic formation of a consensus for all temperatures and values of the interaction, with the exception of the
lines $\beta = 0$ and $H_0 = 1$. On the line $\beta = 0$ consensus is not achieved due to large energy fluctuations in the system. At $H_0 = 1$ competing attitudes towards following either $B$ or neighboring agents cancel each other and consensus is never reached.

The solution to the saddle point equations for the energy (14) reveal the following behavior: Firstly, the overlap $q = 0$ whatever the value of $\alpha$. This indicates that a maximum of variability is kept in the version space. The first milestone in the opinion formation process is reached at $\alpha_c(\beta, H_0)$, which is the volume of information at which $R^2 = W$. From this point onwards the agents approach consensus disregarding the opinion of their peers on socially neutral issues. The majority of the opinion formation process occurs for volumes $\alpha_c < \alpha < \alpha_d$, where the effective energy of the system is described by (21). At $\alpha_d W = 1$ and consensus is reached. For values of $H_0 < 1 R = \sqrt{W} = 1$ and all the agents follow the status quo imposed by $B$. For $H_0 > 1 R = -\sqrt{W} = -1$ and the consensus is against $B$. In both cases the agents reach a consensus following a path that maximizes the diversity of opinions in the only manner allowed: by developing independent attitudes towards socially neutral issues. $\alpha$ is a time-like parameter, thus the reported $\alpha_c$ and $\alpha_d$ can be considered as characteristic times of the model, which, for a fully connected system, are expected to be shorter than the characteristic times of a system defined on a more realistic graph.

As it is expected from a mean field approximation, phenomena associated to the correlation length of the system (like the presence of clusters reported in [5, 17]), cannot be addressed within this framework. To do so we will need to consider more realistic graph topologies, particularly by introducing non-symmetric interaction (directed graphs) [18] and connectivity dynamics [19, 20] which facilitates the exchange of information between agents [21, 22].
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