The maximal principle for properly immersed submanifolds and its applications

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Abstract

In this note we consider the Liouville type theorem for a properly immersed submanifold $M$ in a complete Riemannian manifold $N$. Assume that the sectional curvature $K^N$ of $N$ satisfies $K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)\alpha^2$ for some $L > 0$, $2 > \alpha \geq 0$ and $q_0 \in N$.

(i) If $\Delta |\vec{H}|^{2p-2} \geq k|\vec{H}|^{2p}(p > 1)$ for some constant $k > 0$, then we prove that $M$ is minimal.

(ii) Let $u$ be a smooth nonnegative function on $M$ satisfying $\Delta u \geq ku^a$ for some constant $k > 0$ and $a > 1$. If $|\vec{H}| \leq C(1 + \text{dist}_N(\cdot, q_0)^2)\beta^2$ for some $C > 0$, $0 \leq \beta < 1$, then $u = 0$ on $M$.

As applications we get some nonexistence result for $p$-biharmonic submanifolds.

1 Introduction

In the past several decades harmonic maps play a central role in geometry and analysis. Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a map between Riemannian manifolds $(M, g)$ and $(N, h)$. The energy of $\phi$ is defined by

$$E(\phi) = \int_M \frac{|d\phi|^2}{2} d\nu_g,$$

where $d\nu_g$ is the volume element on $(M, g)$.

The Euler-Lagrange equation of $E$ is

$$\tau(\phi) = \sum_{i=1}^m \{ \tilde{\nabla}_{e_i} d\phi(e_i) - d\phi(\nabla e_i) \} = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on the pullback bundle $\phi^{-1}TN$ and $\{e_i\}$ is a local orthonormal frame field on $M$. In 1983, Eells and Lemaire \cite{Eells-Lemaire1983} proposed
to consider the bienergy functional
\[ E_2(\phi) = \int_M \frac{|\tau(\phi)|^2}{2} d\nu_g, \]
where \( \tau(\phi) \) is the tension field of \( \phi \). Recall that \( \phi \) is harmonic if \( \tau(\phi) = 0 \). The Euler-Lagrange equation for \( E_2 \) is
\[ \tau_2(\phi) = \bar{\Delta}(\tau(\phi)) - \sum_{i=1}^{m} R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0. \]
To further generalize the notion of harmonic maps, Peter and Moser [21] (see also [20]) considered the \( p(p > 1) \)-bienergy functional as follows:
\[ E_p(\phi) = \int_M |\tau(\phi)|^p d\nu_g. \]
The \( p \)-bitension field \( \tau_p(\phi) \) is
\[ \tau_p(\phi) = \bar{\Delta}(|\tau(\phi)|^{p-2}\tau(\phi)) - \sum_{i=1}^{m} \left( R^N(|\tau(\phi)|^{p-2}\tau(\phi), d\phi(e_i))d\phi(e_i) \right). \]
The Euler-Lagrange equation for \( E_p \) is \( \tau_p(\phi) = 0 \) and a map \( u \) satisfying \( \tau_p(\phi) = 0 \) is called \( p \)-biharmonic maps. If \( \phi : (M^m, g) \to (N^{m+t}, h) \) is an isometry immersion, then we call \( u \) \( p \)-biharmonic submanifold and 2-biharmonic submanifolds are called biharmonic submanifolds.

For biharmonic submanifolds, we have the well known Chen’s conjecture [9]:

**conjecture 1.1.** Every biharmonic submanifold in \( E^n \) is minimal.

Chen’s conjecture inspires the research on the nonexistence of biharmonic submanifolds in nonpositively curved manifolds (11, 25, 26, 27, 28, 29, 30, 31, 32, 33, etc.). Motivated by Chen’s conjecture, Yingbo Han [19] proposed the following conjecture:

**conjecture 1.2.** Every complete \( p \)-biharmonic submanifolds in nonpositively curved Riemannian manifold is minimal.

Some partial affirmative answers to conjecture 1.2 were proved in [19] and [6]. In this note we will continue to consider the nonexistence of \( p \)-biharmonic submanifolds in nonpositively curved Riemannian manifold. Before mentioning our main result, we define the following notion (see [27]).

**Definition 1.1.** For a complete manifold \( (N, h) \) and \( \alpha \geq 0 \), if the sectional curvature \( K^N \) of \( N \) satisfies
\[ K^N \geq -L(1 + dist_N(\cdot, q_0)^2)^\frac{\alpha}{2}, \]
for some $L > 0$ and $q_0 \in M$, then we say that $K^N$ has a polynomial growth bound of order $\alpha$ from below.

We have

**Theorem 1.1.** Let $(M, g)$ be a properly immersed submanifold in a complete Riemannian manifold $(N, h)$ whose sectional curvature $K^N$ has a polynomial growth bound of order less than 2 from below. Assume that there exists a positive constant $k > 0$ such that $(p > 1)$

$$\Delta |\vec{H}|^{2p-2} \geq k |\vec{H}|^{2p} \text{ on } M. \quad (1.2)$$

Then $M$ is minimal.

**Remark 1.1.** When $p = 2$, theorem 1.1 was proved by Maeta (see [27]). Our proof follows his argument by using the second derivatives’ test to our new test functions. Maeta’s argument was developed by Cheng and Yau in the 1970s (see [12], [13], [14] etc.).

Theorem 1.1 implies the following nonexistence result of $p$-biharmonic submanifolds.

**Theorem 1.2.** Let $(M, g)$ be a properly immersed $p$-biharmonic submanifold in a complete nonpositively curved Riemannian manifold $(N, h)$ whose sectional curvature $K^N$ has a polynomial growth bound of order less than 2 from below, then $M$ is minimal.

Using the same argument, we also have the following Liouville type theorem.

**Theorem 1.3.** Let $(M, g)$ be a properly immersed submanifold in a complete Riemannian manifold $(N, h)$ whose sectional curvature $K^N$ has a polynomial growth bound of order less than 2 from below. Assume that $u$ is a smooth nonnegative function on $M$ satisfying

$$\Delta u \geq ku^a \text{ on } M, \quad (1.3)$$

where $k > 0, a > 1$ are constants. If $|\vec{H}| \leq C(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\beta}{2}}$ for some $C > 0, 0 \leq \beta < 1$ and $q_0 \in N$, then $u = 0$ on $M$. Here $\vec{H}$ is the mean curvature vector field of $M$ in $N$.

This Liouville type theorem was first found by Maeta. In [27] he proved the case of $a = 2$.

The rest of this paper is organized as follows: In section 2 we will briefly recall the theory of $p$-biharmonic submanifolds and submanifold theory. Our main theorems are proved in section 3.
2 Preliminaries

In this section we give more details on the definitions of harmonic maps, biharmonic maps, $p$-biharmonic maps and $p$-biharmonic submanifolds.

Let $u : (M^m, g) \to (N^{m+t}, h)$ be a map from an $m$-dimensional Riemannian manifold $(M, g)$ to an $m + t$-dimensional Riemannian manifold $(N, h)$. The energy of $u$ is defined by

$$E(u) = \int_M |du|^2 \, d\nu_g.$$  

The Euler-Lagrange equation of $E$ is

$$\tau(u) = \sum_{i=1}^m \{ \tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \} = 0,$$

where we denote $\nabla$ the Levi-Civita connection on $(M, g)$, and $\tilde{\nabla}$ the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. A map $u : (M^m, g) \to (N^{m+t}, h)$ is called a harmonic map if $\tau(u) = 0$. To generalize the notion of harmonic maps, Eells and Lemaire [17] proposed to consider the bienergy functional

$$E_2(u) = \int_M |\tau(u)|^2 \, d\nu_g.$$  

The Euler-Lagrange equation for $E_2$ is (see [24])

$$\tau_2(u) = \tilde{\Delta}(|\tau(u)|^2) - \sum_{i=1}^m R^N(|\tau(u)|^2, du(e_i)) du(e_i) = 0.$$  

To further generalize the notion of harmonic maps, Han and Feng [20] (see also [21]) introduced the $F$-bienergy functional

$$E_F(u) = \int_M F\left(\frac{|\tau(u)|^2}{2}\right) d\nu_g,$$

where $F : [0, +\infty)$ and $F'(x) > 0$ if $x > 0$.

The critical points of the $F$-bienergy functional with $F(x) = (2x)^\frac{p}{2}(p > 1)$ are called $p$-biharmonic maps and isometric $p$-biharmonic maps are called $p$-biharmonic submanifolds.

The $p$-bitension field $\tau_p(u)$ is

$$\tau_p(u) = \tilde{\Delta}(\tau(u)) - \sum_{i=1}^m R^N(|\tau(u)|^p, du(e_i) du(e_i)).$$  

A $p$-biharmonic map satisfies $\tau_p(u) = 0$. 

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Now we briefly recall the submanifold theory. Let \( u : (M, g) \to (N, h) \) be an isometric immersion from an \( m \)-dimensional Riemannian manifold into an \( m + t \)-dimensional Riemannian manifold. The second fundamental form \( B : TM \times TM \to T^\perp(M) \) is defined by:

\[
\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \Gamma(TM),
\]

(2.2)

where \( \bar{\nabla} \) is the Levi-Civita connection on \( N \) and \( \nabla \) is the Levi-Civita connection on \( M \). The Weingarten formula is given by

\[
\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X, \xi \in \Gamma(TM),
\]

(2.3)

where \( A_\xi \) is called the Weingarten map w.r.t. \( \xi \), and \( \nabla^\perp \) denotes the normal connection on the normal bundle of \( M \) in \( N \). For any \( x \in M \), the mean curvature vector field \( \vec{H} \) of \( M \) at \( x \) is

\[
\vec{H} = \frac{1}{m} \sum_{i=1}^{m} B(e_i, e_i).
\]

If \( u \) is an isometry immersion, we see that \( \{du(e_i)\} \) is a local orthonormal frame of \( M \). In addition, for any \( X, Y \in \Gamma(TM) \),

\[
\nabla du(X, Y) = \tilde{\nabla}_X (du(Y)) - du(\nabla^\perp_X Y) = B(X, Y),
\]

(2.4)

where \( \tilde{\nabla} \) is the connection on the pull back bundle \( u^{-1}TN \), whose fiber at a point \( x \in M \) is \( T_{u(x)}N = T^\perp M \bigoplus TM \). Therefore if \( u \) is an isometric immersion,

\[
\tau(u) = tr \nabla du = tr B = m \vec{H},
\]

and a \( p \)-biharmonic submanifold satisfies the following equatuion:

\[
\tau_p(u) = \tilde{\Delta}(|\vec{H}|^{p-2} \vec{H}) - \sum_{i=1}^{m} \left( R^N(|\vec{H}|^{p-2} \vec{H}, e_i)e_i \right)
\]

(2.5)

where \( \tilde{\Delta} = \sum_{i=1}^{m} (\tilde{\nabla}_e_i \tilde{\nabla}_e_i - \tilde{\nabla}_{\nabla^\perp e_i} e_i) \), \( \tilde{\nabla} \) is the connection on the pullback bundle, and \( R^N \) is the Riemannian curvature tensor on \( N \).

From (2.3), we get for any vector field \( \xi \in \Gamma(T^\perp M) \):

\[
\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi = \tilde{\nabla}_{e_i} (\nabla^\perp_{e_i} \xi - A_\xi e_i)
\]

\[
= \nabla^\perp_{e_i} \nabla^\perp_{e_i} \xi - \tilde{\nabla}_{e_i} A_\xi e_i - A_\xi \nabla^\perp_{e_i} e_i
\]

\[
= \nabla^\perp_{e_i} \nabla^\perp_{e_i} \xi - \nabla_{e_i} A_\xi e_i - B(e_i, A_\xi e_i) + A_\xi \nabla^\perp_{e_i} (e_i),
\]

and

\[
\tilde{\nabla}_{\nabla^\perp e_i} e_i \xi
\]

\[
= \tilde{\nabla}_{\nabla^\perp e_i} e_i \xi - A_\xi (\nabla_{e_i} e_i).
\]
Combining the above two identities, we get
\[ \tilde{\Delta} \xi = \nabla^\perp \xi - \nabla e_i A \xi e_i - B(e_i, A \xi e_i) + A \nabla^\perp \xi (e_i) \]
\[ + \nabla_{e_i} \xi - A \xi (\nabla e_i e_i) \]
\[ = \Delta^\perp \xi - \nabla e_i A \xi e_i - A \xi (\nabla e_i e_i) - B(e_i, A \xi e_i) + A \nabla^\perp \xi (e_i). \]

Therefore by decomposing the $p$-biharmonic submanifold equation into its normal and tangential parts respectively we get [19]:
\[ \Delta^\perp (|\tilde{H}|^p)^2 - \sum_{i=1}^{m} B(A|\tilde{H}|^p, e_i) + \sum_{i=1}^{m} (R^N(|\tilde{H}|^p, e_i) e_i)^\perp = 0, \quad (2.6) \]
\[ Tr_g(\nabla A|\tilde{H}|^p) + Tr_g[A \nabla^\perp |\tilde{H}|^p, (.)] - \sum_{i=1}^{m} (R^N(|\tilde{H}|^p, e_i) e_i)^\perp = 0. \quad (2.7) \]

3 Proof of theorems

In this section, we will need the following Hessian comparison theorem (see [4]).

**Lemma 3.1.** Let $(N, h)$ be a complete Riemannian manifold with $\text{sect} \geq K (K < 0)$. For any point $q \in M$ the distance function $r(x) = d(x, q)$ satisfies
\[ D^2 r \leq \sqrt{|K| \coth(\sqrt{|K|} r)} h, \]
at all points where $r$ is smooth (i.e. away from $q$ and the cut loss). Here $D^2 r$ denotes the Hessian of $r$.

3.1 Proof of Theorem 1.1

**Proof.** If $M$ is compact we see that $\tilde{H} = 0$ follows from the standard maximal principle. Therefore we assume that $M$ is noncompact. We will prove the theorem by a contradiction argument. Here we follow Maeta’s [27] argument by choosing new test functions.

Suppose that $\tilde{H}(x_0) \neq 0$ for some $x_0 \in M$. Set $u(x) = |\tilde{H}(x)|^{2p-2}$ for $x \in M$. For each $\rho > 0$ let
\[ F(x) = F_\rho(x) = (\rho^2 - r^2(\phi(x)))^{2p-2} u(x), \]
for $x \in M \cap X^{-1}(B_\rho)$, where $\phi : M \to \mathbb{R}^n$ is the isometric immersion, $B_\rho$ is the standard ball in $\mathbb{R}^n$ with radius $\rho$ and $r(\phi(x)) = \text{dist}_N(\phi(x), q_0)$ for some $q_0 \in N$. 

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Assume that $x_0 \in X^{-1}(B_{r_0})$. For each $\rho \geq \rho_0$, $F = F_\rho$ is a nonnegative function which is not identically zero on $M \cap X^{-1}(B_\rho)$ and equals zero on the boundary. Assume that $q \in M \cap X^{-1}(B_\rho)$ is the maximum point of $F(q)$ exists because $\phi$ is properly immersed.

(i) $\phi(q)$ is not on the cut loss of $q_0$. Then $\nabla F(q) = 0$ and hence we get at $q$

$$\frac{\nabla u}{u} = \frac{(2p - 2)\nabla r^2(\phi(x))}{\rho^2 - r^2(\phi(x))}.$$  \hspace{1cm} (3.1)

In addition at $q$

$$0 \geq \Delta F(x) = (2p - 2)(2p - 3)(\rho^2 - r^2(\phi(x)))^{2p-4}|\nabla r^2(\phi(x))|^2 u(x)$$
\begin{align*}
&\quad - (2p - 2)(\rho^2 - r^2(\phi(x)))^{2p-3} \Delta r^2(\phi(x)) u(x) \\
&\quad - 2(2p - 2)(\rho^2 - r^2(\phi(x)))^{2p-3} \langle \nabla r^2(\phi(x)), \nabla u \rangle_g \\
&\quad + (\rho^2 - r^2(\phi(x)))^{2p-2} \Delta u. \quad \hspace{1cm} (3.2)
\end{align*}

Combining inequalities (3.1) and (3.2) we have at $q$

$$\frac{\Delta u(x)}{u(x)} \leq \frac{(2p - 2)(2p - 1)|\nabla r^2(\phi(x))|^2}{(\rho^2 - r^2(\phi(x)))^2} + \frac{(2p - 2)|\Delta r^2(\phi(x))|}{\rho^2 - r^2(\phi(x))}. \quad \hspace{1cm} (3.3)$$

By a direct computation we see that

$$|\nabla r^2(\phi(x))|^2 \leq 4mr^2(\phi(x)),$$

and

$$\Delta r^2(\phi(x)) = 2 \sum_{i=1}^{m} ((\nabla r)(\phi(x)), d\phi(e_i))^2$$
\begin{align*}
&\quad + 2r(\phi(x)) \sum_{i=1}^{m} (D^2 r)(\phi(x)) (d\phi(e_i), d\phi(e_i)) + 2r(\phi(x)) ((\nabla r)(\phi(x)), \tau(\phi)(x)) \\
&\quad \leq 2m + 2r(\phi(x)) \sum_{i=1}^{m} (D^2 r)(\phi(x)) (d\phi(e_i), d\phi(e_i)) + 2mr(\phi(x)) |\vec{H}(x)|. \quad \hspace{1cm} (3.4)
\end{align*}

where $m = \dim M$, $\nabla$ is the gradient on $(N, h)$ and $D^2r$ denotes the Hessian of $r$. Since the sectional curvature $K^N$ of $N$ satisfies $K^N \geq -L(1 + r^2)\tilde{\tau}$, by the Hessian comparison theorem see lemma 3.1 we get

$$\sum_{i=1}^{m} (D^2 r)(\phi(x)) (d\phi(e_i), d\phi(e_i)) \leq m \sqrt{L(1 + r^2)\tilde{\tau}} \coth \left( \sqrt{L(1 + r^2)\tilde{\tau}} r(\phi(x)) \right). \quad \hspace{1cm} (3.5)$$

Combining the last two inequalities we obtain

$$\Delta r^2(\phi(x)) \leq 2m + 2mr(\phi(x)) |\vec{H}(x)|. \quad \hspace{1cm} (3.6)$$

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Recall that $\Delta |\vec{H}|^{2p-2} \geq k |\vec{H}|^{2p}$, i.e. $\Delta u \geq ku^{2p-2}$, thus from inequalities (3.3), (3.6) we obtain

$$ku(q)^{\frac{1}{p-1}} \leq \frac{4m(2p-2)(2p-1)r^2(\phi(q))}{(\rho^2 - r^2(\phi(q)))^2} + \frac{(2p-2)2m \sqrt{L(1 + r^2)^2 r(\phi(q))} \coth \left( \sqrt{L(1 + r^2)^2 r(\phi(q))} \right)}{\rho^2 - r^2(\phi(q))} + \frac{(2p-2)2mr(\phi(q))|\vec{H}(q)|}{\rho^2 - r^2(\phi(q))}.$$  

(3.7)

From the last inequality one gets

$$u(q) \leq C(p, k, m)\left[\frac{r^{2p-2}(\phi(q))}{(\rho^2 - r^2(\phi(q)))^{2p-2}} + \frac{1}{(\rho^2 - r^2(\phi(q)))^{p-1}} \right]$$

(3.8)

where $C(p, k, m)$ is a constant depends only on $p, k, m$. Therefore

$$F(q) \leq C(p, k, m)\left[\frac{r^{2p-2}(\phi(q))}{(\rho^2 - r^2(\phi(q)))^{2p-2}} + \frac{1}{(\rho^2 - r^2(\phi(q)))^{p-1}} \right]$$

(3.9)

which implies that

$$F(q) \leq C(p, k, m, L)\left(1 + \rho^2\right)\frac{1}{(\rho^2 - r^2(\phi(q)))^{p-1}}.$$  

where $C(p, k, m, L)$ is a constant depends only on $p, k, m, L$. Since $q$ is the maximum of $F$, for any $x \in M \cap B_\rho$ we have

$$F(x) \leq F(q) \leq C(p, k, m, L)\left(1 + \rho^2\right)\frac{1}{(\rho^2 - r^2(\phi(q)))^{p-1}}.$$  

Therefore

$$|\vec{H}(x)|^{2p-2} \leq \frac{C(p, k, m, L)\left(1 + \rho^2\right)\frac{1}{(\rho^2 - r^2(\phi(q)))^{2p-2}}}{(\rho^2 - r^2(\phi(q)))^{p-2}},$$  

(3.10)

for any $x \in M \cap B_\rho$ and $\rho \geq \rho_0$.

(ii) If $\phi(q)$ is on the cut loss of $q_0$, then we use a method of Calabi (see [3]). Let $\sigma$ be a minimal geodesic joining $\phi(q)$ and $q_0$. Then for any $q'$ in the
interior of $\sigma$, $q'$ is not conjugate to $q_0$. Fix for such a point $q'$. Let $U_{q'} \subseteq B_{\rho}$ be a conical neighborhood of the geodesic segment of $\sigma$ joining $q'$ and $\phi(q)$ such that for any $\phi(x) \in U_{q'}$, there is at most one minimizing geodesic joining $q'$ and $\phi(x)$. Let $\bar{r}(\phi(x)) = \text{dist}_{U_{q'}}(\phi(x), q')$ in the manifold $U_{q'}$. Then we have $\bar{r}(\phi(x)) \geq \text{dist}_N(\phi(x), q')$, $r(\phi(x)) \leq r(q') + \bar{r}(\phi(x))$, $r(\phi(q)) = r(q') + \bar{r}(\phi(q))$.

We claim that the function

$$F_{\rho, q'}(x) := (\rho^2 - \{r(q') + \bar{r}(\phi(x))\}^2)^{2p-2}u(x) \text{ for } x \in \phi^{-1}(U_{q'})$$

also attains a local maximum at the point $q$. In fact, for any point $x \in \phi^{-1}(U_{q'})$ we have

\[
F_{\rho, q'}(q) = (\rho^2 - \{r(q') + \bar{r}(\phi(q))\}^2)^{2p-2}u(q) \\
= (\rho^2 - r^2(\phi(q)))^{2p-2}u(q) \\
= F_{\rho}(q) \geq F_{\rho}(x) \\
= (\rho^2 - r^2(\phi(x)))^{2p-2}u(x) \\
\geq (\rho^2 - \{r(q') + \bar{r}(\phi(x))\}^2)^{2p-2}u(x) \\
= F_{\rho, q'}(x).
\]

Therefore the claim is proved and we play the second derivative’s test to $F_{\rho, q'}(x)$ at $q$, the same argument as before shows that

\[
F_{\rho, q'}(q) \leq C(p, k, m)\{r^{2p-2}(\phi(q)) + \left[1 + \sqrt{L(1 + r^2)\frac{r}{2}r(\phi(q)))\cosh\left(\sqrt{L(1 + r^2)\frac{r}{2}r(\phi(q))}\right)}\right]^{p-1}(\rho^2 - r^2(\phi(q)))^{p-1} + \sqrt{F_{\rho, q'}(q)r(\phi(q))^{p-1}}],
\]

which implies that

$$F_{\rho, q'}(q) \leq C(p, k, m, L)(1 + \rho^2)^{\sqrt{(a+6)/(p-1)}}.$$

Take $q' \to q_0$ we have $F_{\rho, q'}(q) = F_{\rho}(q)$ and hence

$$F_{\rho}(q) \leq C(p, k, m, L)(1 + \rho^2)^{\sqrt{(a+6)/(p-1)}}.$$

Therefore

$$|\bar{H}(x)|^{2p-2} \leq \frac{C(p, k, m)(1 + \rho^2)^{\sqrt{(a+6)/(p-1)}}}{(\rho^2 - r^2(\phi(x)))^{2p-2}}, \quad (3.11)$$

for any $x \in M \cap B_{\rho}$ and $\rho \geq \rho_0$. Let $x = x_0$ and $\rho \to +\infty$ we get $\bar{H}(x_0) = 0$, a contradiction. Therefore $M$ is minimal.
3.2 Proof of Theorem [1.2]

Proof. Recall that the normal part of the $p$-biharmonic submanifolds is

$$\Delta^\perp (|\vec{H}|^{p-2} \vec{H}) - \sum_{i=1}^{m} B(A_i |\vec{H}|^{p-2} \vec{H}, e_i) + \sum_{i=1}^{m} (R^N(\vec{H} |\vec{H}|^{p-2} \vec{H}, e_i) e_i)^\perp = 0.$$ 

Therefore

$$\Delta |\vec{H}|^{2p-2} = 2\langle \Delta^\perp (|\vec{H}|^{p-2} \vec{H}), |\vec{H}|^{p-2} \vec{H} \rangle + 2|\nabla (|\vec{H}|^{p-2} \vec{H})|^2 \geq 2 \sum_{i=1}^{m} \langle B(A_i |\vec{H}|^{p-2} \vec{H}, e_i), |\vec{H}|^{p-2} \vec{H} \rangle,$$

where in the first inequality we used the assumption of nonpositive curvature.

Therefore $M$ is minimal by theorem [1.1].

3.3 Proof of Theorem [1.3]

Proof. Similar to the proof of theorem [1.1] set $F_\rho(x) = (\rho^2 - r^2(\phi(x)))^{2a-2} u(x)$. If $u(x_0) \neq 0$, then using the second derivatives’ test to $F_\rho$ at the maximum point $q$ for $\rho$ big enough such that $x_0 \in B_\rho$, we will get

$$ku(q) = \frac{4m(2a - 2)(2a - 1)r^2(\phi(q))}{(\rho^2 - r^2(\phi(q)))^2} \left\{ \frac{2m + 2m \sqrt{L(1+r^2)\frac{\alpha}{r(\phi(q))} \coth \left( r\sqrt{L(1+r^2)} \phi(q) \right) \frac{r(\phi(q))}{\sqrt{L(1+r^2)\frac{\alpha}{r(\phi(q))}}}}{\rho^2 - r^2(\phi(q))} \right\}^a.$$

Therefore

$$u(q) \leq C(a, k, m) \left[ \frac{r^{2a-2}(\phi(q))}{(\rho^2 - r^2(\phi(q)))^{2a-2}} \right]^{a-1} + \frac{1 + \sqrt{L(1+r^2)\alpha r(\phi(q)) \coth \left( r\sqrt{L(1+r^2)} \phi(q) \right)}}{\rho^2 - r^2(\phi(q))^{a-1}} + |\vec{H}|^{a-1} r(\phi(q))^{a-1} \frac{1}{(\rho^2 - r^2(\phi(q)))^{a-1}},$$

which implies that

$$F_\rho(q) \leq C(a, k, m, L) \max \left\{ (1 + \rho^2)^{\frac{2a-6}{3}}, (1 + \rho^2)^{\frac{2a-3}{3}} \right\}.$$
where we used the assumption that $|\vec{H}| \leq C(1 + dist_N(\cdot, q_0)^2)^{\frac{a}{2}}$. Therefore
\[
(\rho^2 - r^2(\phi(x)))^{2a-2} u(x) \leq F_{\rho}(q) \leq C(a, k, m, L) \max\{(1 + \rho^2)^{\frac{a-2}{2}(a-1)}, (1 + \rho^2)^{\frac{2}{2}(a-1)}\},
\]
which implies that
\[
u(x) \leq C(a, k, m, L) \max\{(1 + \rho^2)^{\frac{a-2}{2}(a-1)}, (1 + \rho^2)^{\frac{2}{2}(a-1)}\}
\frac{(\rho^2 - r^2(\phi(x)))^{2a-2}}{\nu(x)}.\]
Because $\alpha < 2$ and $\beta < 1$, let $x = x_0$ and $\rho \to +\infty$ we obtain $u(x_0) = 0$, a contradiction. Thus $u = 0$ on $M$. \qed

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