Form factors $f_{B \to \pi}^+(0)$ and $f_{D \to \pi}^+(0)$ in QCD and determination of $|V_{ub}|$ and $|V_{cd}|$

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Abstract

We present a QCD study on $B, D \to \pi$ semileptonic transitions at zero momentum transfer and an estimate of magnitudes of the associated CKM matrix elements. Light cone sum rules (LCSRs) with chiral correlator are applied to calculate the form factors $f_{B \to \pi}^+(0)$ and $f_{D \to \pi}^+(0)$. We show that there is no twist-3 and-5 component involved in the light-cone expansions such that the resulting sum rules have a good convergence and offer an understanding of these form factors at twist-5 level. A detailed $\mathcal{O}(\alpha_s)$ computation is carried out in leading twist-2 approximation and the $\overline{MS}$ masses are employed for the underlying heavy quarks. With the updated inputs and experimental data, we have $f_{B \to \pi}^+(0) = 0.28^{+0.05}_{-0.02}$ and $|V_{ub}| = (3.4^{+0.2}_{-0.6} \pm 0.1 \pm 0.1) \times 10^{-3}$; $f_{D \to \pi}^+(0) = 0.62 \pm 0.03$ and $|V_{cd}| = 0.244 \pm 0.005 \pm 0.003 \pm 0.008$. As a by-product, a numerical estimate for the decay constant $f_D$ is yielded as $f_D = 190^{+12}_{-11}$MeV.

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I. INTRODUCTION

An intensive study on the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements remains a cornerstone of high energy physics programme, in testing the standard model (SM) and exploring new physics. The unitarity of the CKM matrix must be put to a test by phenomenological research on the so called unitarity triangle. Opposite to the side of the triangle whose length depends on, in addition to the CKM parameters $|V_{cb}|$ and $|V_{ud}|$, the elements involving heavy-light quark mixing $|V_{ub}|$ and $|V_{cd}|$, the angle $\beta$ is presently well-measured. So precision determination of them is central to the unitarity testing. Exclusive processes offer an indispensable avenue to understand these parameters. The decays of heavy mesons into a light pseudoscalar meson plus an electron and its antineutrino can proceed at electro-weak tree level and are much less sensitive to new physics, and accordingly they could serve as preferred exclusive channels to probe both elements that we take interest in, namely, $|V_{ub}|$ and $|V_{cd}|$. Then we are confronted with calculation of the hadronic matrix elements, say, that for the $B^0 \rightarrow \pi^+$ transition parameterized usually as

$$\langle \pi(p) | \bar{u} \gamma_\mu b | B(p+q) \rangle = 2f_+^{B\rightarrow\pi}(q^2) p_\mu + (f_+^{B\rightarrow\pi}(q^2) + f_-^{B\rightarrow\pi}(q^2)) q_\mu,$$

(1)

with the momentum assignment specified in brackets, and $f_+^{B\rightarrow\pi}(q^2)$ and $f_-^{B\rightarrow\pi}(q^2)$ being the form factors describing QCD dynamics in the decay, of which only the former is related if the small electron mass is neglected. Combining the partial rates measured in some $q^2$ bins with the form factor predictions of different QCD approaches, one could achieve the values for related $|V_{ij}|$. Another approach is to fit the experimental observations using the various form factor parameterizations. In such way, a strong constraint is imposed on $q^2$ distributions of the form factors such that one may obtain a precise estimate of the products $f_+(0)|V_{ij}|$, in which case theoretical task boils down to estimating the form factors $f_+(0)$ at $q^2 = 0$. Requiring a good knowledge of the form factors, the exclusive avenues to $|V_{ij}|$ are theoretically more challenging than inclusive approaches. The continual data updates have aroused one’s enthusiasm for exploring heavy-to-light transitions to approach an understanding of the CKM parameters. In the wake of the recent accurate measurements of the semileptonic processes by the BaBar \cite{1, 2} and CLEO \cite{3, 4} collaborations, new progress has been achieved in this respect. Some extent of tension, however, still holds between inclusive and exclusive extractions of $|V_{ub}|$. A global data-fitting from CKMFitter \cite{5} and UTfit \cite{6} is in favor of a smaller $|V_{ub}|$ than inclusive determinations. One can be referred to \cite{7} for a comprehensive overview of the current status of the CKM matrix elements.

Developed from QCD sum rule technique, light cone sum rules (LCSRs) \cite{8, 9} have become a powerful competitor in making predictions for heavy to light transitions. Complementary to lattice QCD (LQCD) simulations, this approach is successfully applied to study $B$ decays \cite{9–16}: whereas the former are available for the high $q^2$, LCSR calculation is applicable for the low and intermediate $q^2$. Utilizing the LCSR predictions for $f_+^{B\rightarrow\pi}(q^2)$, one has
launched a painstaking investigation into $|V_{ub}|^{[11–14]}$, with a consistent result with those using LQCD. The same approach has also been taken to understand $D \to \pi, K$ decays in $^{[17, 18]}$, the resulting sum rules $^{[18]}$ being employed to extract $|V_{cd}|$ and $|V_{cs}|$.

The uncertainties in the light meson distribution amplitudes (DAs) involved in the sum rules, however, would have different degrees of impacts on the results. To gain enlightenment on how to further improve accuracy of the LCSR calculations, it is essential to look into the role played by each of the higher twist DAs. A systematic numerical analysis shows that whereas the twist-4 effects account for only a few percent of the total sum rule results, the chirally enhanced twist-3 contributions are numerically large enough to be comparable with the twist-2 ones in the $B$ meson cases, and even about twice as large as the latter for $D$ decays. As a result, there are a few problems left unsolved. To start with, one might doubt whether the potential twist-5 effects are negligible in particular while assessing $D$ decays. Secondly, the sum rule pollution by twist-3 would be serious on account of the combined uncertainties of the DAs and chiral enhancement factor. Finally, since there is an extremely different sensitivity to twist-2 between the sum rules for $f_B^{B\to\pi}(0)$ and $f_B^{D\to\pi}(0)$, a successful LCSR application to the latter does not necessarily assure, with the same inputs, a reliable LCSR prediction for the former. For the moment, these issues are difficult to essentially settle within the LCSR framework. The trick suggested in $^{[9, 15]}$ is available as a temporary scenario to approach them.

Focusing on $f_B^{B\to\pi}(0)$ and $f_B^{D\to\pi}(0)$, in this work we intend to reconsider heavy to light transitions in the revised LCSR version so as to provide a calculation independent of the traditional LCSR ones and further a determination of the associated CKM parameters. We will expound that this approach does not involve the twist-3 and-5 DAs, which enables us to get an understanding of the form factors to twist-5 precision only resorting to the known twist-2 and -4 DAs and to perform a cross-check between the resulting LCSR for $f_B^{B\to\pi}(0)$ and $f_D^{D\to\pi}(0)$. This paper is organized as follows. In the following Section we put forward our derivation of the sum rules in question, including a detailed next-to-leading order (NLO) QCD calculation in twist-2 approximation, and elaborate on the key technical points. The modifications and improvements made are also addressed in comparison with the previous calculations $^{[15, 16]}$. In Section 3, after discussing assignment of the parameters for which updated and consistent findings are selected as inputs, we shift into numerical computation with a systematic error discussion included, by means of up-to-date experimental data, and present our LCSR results for $f_B^{B\to\pi}(0)$ and $f_D^{D\to\pi}(0)$ and the determination of $|V_{ub}|$ and $|V_{cd}|$. Too we report on an estimate of the decay constant $f_D$, as a by-product. The final Section is devoted to a summary.
II. QCD CALCULATION OF $f_B^{D\to\pi}(0)$ AND $f^{D\to\pi}(0)$

The starting point of LCSR calculation is to consider a correlation function with $T$ product of currents sandwiched between the vacuum and a light meson state $L$. In the coordinate space and for large and negative virtuality of the current operators, the correlation function can be in form expanded, in the small light cone distance $x^2 \approx 0$, as,

$$ \text{correlation function} \sim \sum_m C_m(x) \langle L(p)|O_m(x,0)|0 \rangle, $$

(2)

where $C_m(x)$ are the Wilson coefficients, $O_m(x,0)$ the nonlocal operators built out of quark and/or gluon fields, and the matrix elements $\langle L(p)|O_m(x,0)|0 \rangle$ have an expansion form in term of the light cone DAs $\Psi^{(n)}$ with increasing twist $n$. The power series $\sum C_n(x \cdot p)^n (x \cdot p \sim 1$ for a large external momentum $p$) appearing in the expansion process are summed up effectively, which works out some of the problems with the expansion in the small distance $x \approx 0$. Switching (2) to momentum space, we have

$$ \text{correlation function} \sim \sum_n T^{(n)}_H \otimes \Psi^{(n)}, $$

(3)

a factorized form with the hard kernel $T^{(n)}_H$ being convoluted with $\Psi^{(n)}$. Whereas the process-independent $\Psi^{(n)}$ parameterize the long distance effects below a factorization scale $\mu$, the process-dependent amplitudes $T^{(n)}_H$ describe the hard-scattering dynamics above $\mu$, which are perturbatively calculable and have the following expansions in $\alpha_s$:

$$ T^{(n)}_H = T^{(n)}_0 + \frac{\alpha_s C_F}{4\pi} T^{(n)}_1 + \cdots. $$

(4)

If calculation is restricted to $O(\alpha_s)$ accuracy, we need just to estimate the leading order (LO) contributions $T^{(n)}_0$ and NLO corrections $T^{(n)}_1$. Then the remaining procedure is standard.

Now let us take up our LCSR calculations of $f_B^{D\to\pi}(0)$ and $f^{D\to\pi}(0)$. Allowing for the similarity of the two situations, for definiteness we would like to concentrate on the former. Moreover, throughout the paper the chiral limit $m_\pi = 0$ is taken. We follow [9, 15] and adopt the following correlation function:

$$ \Pi_\mu(p,q) = i \int d^4x e^{iqx} \langle \pi(p)|T\{J^{V+A}_\mu(x),J^{P+S}_B(0)\}|0 \rangle = F((p+q)^2)p_\mu + \tilde{F}((p+q)^2)q_\mu. $$

(5)

Here we substitute the chiral currents $J^{V+A}_\mu(x) = \bar{u}(x)\gamma_\mu(1 + \gamma_5)b(x)$ and $J^{P+S}_B = m_b\bar{b}(0)i(1 + \gamma_5)d(0)$, respectively, for the operators adopted usually $J_\mu(x) = \bar{u}(x)\gamma_\mu b(x)$ and $J_B = m_b\bar{b}(0)i\gamma_5d(0)$. The operator replacements do not violate renormalization group invariance of the correlation function, for both $J^{V+A}_\mu$ and $J^{P+S}_B$, like the latter two, have an anomalous dimension of zero, and however make the correlation function receive an additional contribution from the set of scalar $B$ mesons. In view of that the mass of the lowest
scalar $B$ meson is slightly below the one of the first excited state of the pseudoscalar $B$ mesons, we could safely isolate the pole term of the pseudoscalar ground state from the contributions of higher resonances and continuum states.

For the present purpose, it is sufficient to consider the part proportional to $p_\mu$ in (5), that is, the invariant function $F((p + q)^2)$. It has the pole term of interest to us,

$$F_{\text{pole}}((p + q)^2) = \frac{2m_B^2 f_{B \to \pi}^B(0)}{m_B^2 - (p + q)^2},$$

where $m_B$ and $f_B$ indicate, respectively, the $B$ meson mass and decay constant defined as

$$\langle B|\bar{b}i\gamma_5 d|0\rangle = \frac{m_B^2 f_B}{m_b}.$$  

The spectral function $\rho^H(s)$ is introduced to reckon in the higher state contributions in a dispersion integral starting with the threshold $s_B^0$, which should be assigned near the squared mass of the lowest scalar $B$ meson. At this point, what remains to be done is the light cone expansion calculation on $F((p + q)^2)$, from which the corresponding QCD spectral function $\rho^{QCD}(s)$ is extracted in order to get the sum rule for $f_{B \to \pi}^B(0)$ by matching the Borel improved theoretical and phenomenological forms with the duality assumption $\rho^H(s) = \rho^{QCD}(s)\Theta(s - s_B^0)$.

The light cone expansion of (5) goes effectively in the large space-like momentum region $(p + q)^2 - m_b^2 << 0$ for the $b\bar{d}$ channel. At tree-level and to NLO in the light-cone expansion of the $b$ quark propagator, it can be illustrated by the two Feynman diagrams as depicted in Fig.1. In comparison with Fig.1(a), which corresponds to the leading term in the quark propagator and illustrates the two-particle contribution, Fig.1(b) portrays the three-particle Fock state effect due to the soft-emission correction to the free quark propagator, which is expressed as

$$-ig_s \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int_0^1 dv \left[ \frac{1}{2} \frac{k + m_b}{(m_b^2 - k^2)^2} G^{\mu\nu}(vx)\sigma_{\mu\nu} + \frac{1}{m_b^2 - k^2} v x_\mu G^{\mu\nu}(vx)\gamma_\nu \right].$$
The contribution of Fig.1(a) to $F((p + q)^2)$ is easy to estimate, using the definition of the pionic two-particle DAs:

\[
\langle \pi(p)|\bar{u}_\alpha(x)d_\beta(0)|0\rangle_{x^2\rightarrow 0} = i\frac{f_\pi}{4}\int_0^1 du \ e^{i\mu p x}\left[(\gamma_5)_{\beta\alpha}\varphi_\pi(u)
\right.
\]
\[
- (\gamma_5)_{\beta\alpha}\mu_\pi\phi_{3\pi}^p(u) + \frac{1}{6}(\sigma_{xyz}\gamma_5)_{\beta\alpha}p^z x^\eta\mu_\pi\phi_{3\pi}^g(u)
\]
\[
+ \frac{1}{16}(\gamma_5)_{\beta\alpha}x^2\phi_{4\pi}(u) - i\frac{1}{2}(\gamma_5)_{\beta\alpha}\int_0^u \psi_{4\pi}(v)dv\right],
\]  

(9)

where $u$ is the fraction of the light cone momentum $p_0 + p_3$ of the pion carried by the constituent $u$ quark. While $\varphi_\pi(u)$ denotes the twist-2 DA, both $\phi_{3\pi}^p(u)$ and $\phi_{3\pi}^g(u)$, which are accompanied by the chiral enhancement factor $\mu_\pi$, have twist-3, and the other two functions are both of twist-4. From the following trace form, which emerges obviously as one works in the momentum space,

\[
Tr\left\{[d(\bar{u}p)\bar{u}(up)]\gamma_\mu(1 + \gamma_5)(\not{q} + up\not{u} + m_b)(1 + \gamma_5)\right\},
\]

(10)

we see readily that the twist-3 components make a vanishing contribution to the light cone expansion, because of the corresponding Dirac wavefunctions. In fact, the same happens to the three-particle situation, as shown from a straightforward computation with (8) and the decomposition:

\[
\langle \pi(p)|\bar{u}_\alpha(x)g_s G_{\mu\nu}(vx)d_\beta(0)|0\rangle_{x^2\rightarrow 0} = \frac{1}{4}\int D\alpha_i e^{ip\cdot x(\alpha_1 + \alpha_3)}\left[i f_3\pi(\sigma^{\mu\nu}\gamma_5)_{\beta\alpha}
\right.
\]
\[
\times(p_\mu p_\rho g_{\nu\lambda} - p_\nu p_\rho g_{\mu\lambda})\Phi_{3\pi}(\alpha_i) - f_\pi(\gamma^\mu\gamma_5)_{\beta\alpha}\left\{p_\nu g_{\mu\rho} - p_\mu g_{\nu\rho}\right\}\Psi_{4\pi}(\alpha_i)
\]
\[
+ \frac{p_\rho(p_\mu x_\nu - p_\nu x_\mu)}{p \cdot x}\left(\Phi_{4\pi}(\alpha_i) + \Psi_{4\pi}(\alpha_i)\right)\right\} - i\frac{f_\pi}{2}\epsilon_{\mu\nu\rho\sigma}(\gamma_5)_{\beta\alpha}
\]
\[
\times\left\{(p^\lambda g^{\delta\rho} - p^\delta g^{\lambda\rho})\bar{\Psi}_{4\pi}(\alpha_i) + \frac{p^\mu(p^\delta x^\lambda - p^\lambda x^\delta)}{p \cdot x}\left(\bar{\Phi}_{4\pi}(\alpha_i) + \bar{\Psi}_{4\pi}(\alpha_i)\right)\right\}\right],
\]  

(11)

where $G_{\mu\nu}$ is the gluonic field strength tensor and $D\alpha_i = d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)$; $\Phi_{3\pi}(\alpha_i)$ indicates the twist-3 component of the three-particle DAs, and the remaining functions are all of twist-4. In the usual LCSR application to $D$ decays, the chirally enhanced twist-3 terms provide a leading contribution, which engenders much negative influence as aforementioned.

At present, the two-particle contribution $F_0^{(2p)}((p + q)^2)$ can be written down in a form that the DAs are convoluted with the corresponding LO hard scattering amplitudes,

\[
F_0^{(2p)}((p + q)^2) = - f_\pi\int_0^1 du \ T_0^{(2)}((p + q)^2, u)\varphi_\pi(u)
\]
\[
- T_0^{(4)}((p + q)^2, u)\int_0^u \psi_{4\pi}(v)dv - \tilde{T}_0^{(4)}((p + q)^2, u)\phi_4(u),
\]  

(12)
with

\[ T_0^{(2)}((p+q)^2, u) = -2 \frac{m_b^2}{m_b^2 - u(p+q)^2}, \]  
\[ T_0^{(4)}((p+q)^2, u) = 2 \frac{u}{(m_b^2 - u(p+q)^2)^2} \left( ud\frac{d}{du} + 1 \right), \]  
\[ \widetilde{T}_0^{(4)}((p+q)^2, u) = -2 \frac{u^2}{2(m_b^2 - u(p+q)^2)^2} du^2. \]  

The three-particle contribution is of the following convolution

\[ F_0^{(3p)}((p+q)^2) = -f_\pi \int_0^1 du T_0^{(4)}((p+q)^2, u) I_{4\pi}(u), \]  
with

\[ T_0^{(4)}((p+q)^2, u) = 2 \frac{u}{(m_b^2 - u(p+q)^2)^2} d\frac{d}{du}, \]  
\[ I_{4\pi}(u) = \int_0^u d\alpha_1 \int_{(u-\alpha_1)/(1-\alpha_1)}^1 \frac{dv}{v} \left[ 2\Psi_{4\pi}(\alpha_i) + 2\tilde{\Psi}_{4\pi}(\alpha_i) \right] - \Phi_{4\pi}(\alpha_i) - \tilde{\Phi}_{4\pi}(\alpha_i) \bigg|_{\alpha_2=1-\alpha_1-\alpha_3}^{\alpha_3=(u-\alpha_1)/v}. \]  

Then we can attain the imaginary part of \( F_0^{QCD}((p+q)^2) = F_0^{(2p)}((p+q)^2) + F_0^{(3p)}((p+q)^2) \) via estimating the ones of the hard kernels in (13–15) and (17), and further the desired QCD spectral function \( \rho_{0}^{QCD}(s) \). The result is as follows:

\[ \rho_{0}^{QCD}(s) = 2f_\pi \int_0^1 du \delta(1 - u) \frac{s}{m_b^2} \left[ \varphi_{4\pi}(u) + \frac{u}{m_b^2} \left( ud\frac{d}{du} + 1 \right) \right] \int_0^u \psi_{4\pi}(v) dv \]  
\[ - \frac{u^2}{4m_b^2} d^2 \phi_{4\pi}(u) - \frac{u}{m_b^2} dI_{4\pi}(u). \]  

At twist-4 level, we have provided a complete LO light cone QCD representation for \( F((p+q)^2) \). In its present form, the ensuing continuum substraction could be enforced systematically for the twist-4 as well as twist-2 parts, with the known QCD spectral function. This improves explicitly the previous treatment \([15, 16]\) in which the twist-4 contribution is written down in a form not suitable for continuum substraction.

Our main task is to evaluate the gluon emission effect on the LCSR for \( f_B^{B \to \pi}(0) \) at one loop level. It should be sufficient for this purpose to calculate the NLO parts of the leading twist-2 and the chirally enhanced twist-3 contributions. To be specific, we are about to compute the six Feynman diagrams plotted in Fig.2, to the accuracy in question. Fig.2(a) depicts diagrammatically the hard-exchange correction between the outgoing and spectator
quarks in the $B \rightarrow \pi$ transition. From the nature of the correlation function, we deduce easily that there is no UV divergence in Fig.2(a) or it could not be canceled out. Of the other figures, Figs.2(b, e) and Figs.2(c, f) involve, respectively, the partial one-loop contributions to the $1 + \gamma_5$ vertex and to the $\gamma_\mu(1 + \gamma_5)$ one, while Fig.2(d) does the remaining loop contribution to both operators. It is conceivable that each of these five includes both UV and IR divergences, except Fig.2(d) which is merely UV divergent because obviously if any IR divergence arises it can not be reasonably absorbed into a pionic DA.

It is found that the twist-3 components still produce no effect at one-loop level, for the same reason as in the tree-level case. Hence the NLO computation is reduced to a calculation of the $\mathcal{O}(\alpha_s)$ correction to the LO twist-2 contribution $T_0^{(2)}$ (for brevity, hereafter we indicate the LO twist-2 contribution by the symbol $T_0$ instead of $T_0^{(2)}$ and the corresponding NLO correction by $T_1$, up to a prefactor $\alpha_s/4\pi$). We work in the Feynman gauge. In addition, we use the dimensional regularization and $\overline{MS}$ scheme to deal with the ultraviolet (UV)
and infrared (IR) divergences appearing in the calculation, such that the LO evolution kernel of \( \varphi_\pi(u) \) achieved early in the same prescription is available for a proof of QCD factorization for the resulting twist-2 contribution to \( F((p+q)^2) \) as we attempt to segregate the long distance contribution from the perturbative kernel. The calculation is tedious and complicated. Here we present, for the first time, some details of the diagram calculation. We summarize the divergence contribution to \( T_1 \) from each of the diagrams in Fig.2 as follows,

\[
T_{1(a)}^{\text{div}}(u, r) = 4 \frac{1}{ur^2} \left[ (1 - r) \ln(1 - r) - \left( \frac{1}{u} \right) \ln(1 - ur) \right] \Delta_{IR},
\]

\[
T_{1(b)}^{\text{div}}(u, r) = 4 \frac{1}{1 - ur} \left[ \left( \frac{r - 1}{ur} \right) \ln(1 - ur) + 1 \right] \Delta_{IR} - 2 \Delta_{UV},
\]

\[
T_{1(c)}^{\text{div}}(u, r) = 4 \frac{1}{1 - ur} \left[ \left( \frac{1}{ur} \right) \ln(1 - ur) + 1 \right] \Delta_{IR} - \frac{1}{2} \Delta_{UV},
\]

\[
T_{1(d)}^{\text{div}}(u, r) = 4 \frac{2 + ur}{(1 - ur)^2} \Delta_{UV},
\]

\[
T_{1(e+f)}^{\text{div}}(u, r) = - \frac{2}{1 - ur} \Delta_{IR} + \frac{2}{1 - ur} \Delta_{UV}.
\]

Here \( \bar{u} = 1 - u, r = (p + q)^2/m_b^2 \) and

\[
\Delta_{IR}(\Delta_{UV}) = \frac{1}{\varepsilon_{IR}} \left( \frac{1}{\varepsilon_{UV}} \right) - \gamma_E + \ln 4\pi
\]

with the \( \varepsilon_{UV} \) and \( \varepsilon_{IR} \) introduced to regularize the UV and IR divergences, respectively. Obviously, the yielded results are as expected.

Adding all the divergent and finite terms together, we have the NLO correction

\[
T_1(u, r) = 2 \left\{ \frac{1}{1 - ur} \left[ 3 - 2 \ln(1 - r) \frac{1 - r - ur}{ur^2} - 2 \ln \left( \frac{1 - ur}{1 - r} \right) \frac{1 - r - urm^2}{ur^2} \right] \Delta_{IR}
+ \frac{6ur}{(1 - ur)^2} \Delta_{UV} + \frac{1 + ur}{(1 - ur)^2} \left[ 3 - 3 \ln \frac{m_b^2}{\mu^2} + \frac{1}{ur} \right]
+ 2 \left[ \frac{1}{ur} - \frac{1}{r(1 - ur)} - \left( \frac{1}{ur^2} - \frac{1}{1 - ur} \right) \ln \frac{m_b^2}{\mu^2} \right] \ln(1 - r)
+ 2 \left( \frac{1}{1 - ur} - \frac{1}{ur^2} \right) \left( \ln^2(1 - r) + \text{Li}_2(r) \right)
+ \left[ \frac{4}{1 - ur} + \frac{ur + u^2r + \pi}{\pi(1 - ur)^2} - 2 \left( \frac{2}{1 - ur} - \frac{1 - ur}{ur^2} \right) \ln \frac{m_b^2}{\mu^2} \right] \ln(1 - ur)
- 2 \left( \frac{2}{1 - ur} - \frac{1 - ur}{ur^2} \right) \ln^2(1 - ur) \right\},
\]

with the dilogarithm \( \text{Li}_2(x) = - \int_0^x dt \frac{\ln(1-t)}{t} \).

Keep in mind that up to now the quark mass has been treated as a bare quantity. A mass renormalization must be performed in the \( \overline{MS} \) scheme, in order to have a UV renormalized
hard-scattering amplitude $T$ via adding $T_1$ to $T_0$. It can be done by making the parameter replacement $m_b \rightarrow Z_m m_b$ in the related expressions, with the renormalization constant $Z_m = 1 - 3 \Delta_{UV} \frac{\alpha_s C_F}{4\pi}$. As a result, the tree level expression (13), to the accuracy required, is modified to the form

$$T_0(u, r) = \frac{2}{ur - 1} - \frac{\alpha_s C_F}{4\pi} \frac{12ur}{(1 - ur)^2} \Delta_{UV},$$

but the NLO term $T_1(u, r)$ keeps its form unchanged. Here $m_b$ entering $r$ should be understood as the $\overline{MS}$ mass. The additional UV divergent contribution in (28), as it should be, precisely cancels out the one of (27). Then a complete UV renormalized result is obtained as

$$T(u, r) = T_0(u, r) + \frac{\alpha_s C_F}{4\pi} T_1(u, r)$$

$$= 2 \left\{ \frac{1}{1 - ur} \left[ 3 - 2 \ln(1 - r) \frac{1 - r - ur}{ur^2} - 2 \ln \left( \frac{1 - ur}{1 - r} \right) \frac{1 - r - u\bar{u}r^2}{u\bar{u}r^2} \right] \Delta_{IR} + \frac{1 + ur}{(1 - ur)^2} \left( 3 - 3 \ln m_b^2 \mu^2 + \frac{1}{ur} \right) \right. $$

$$+ 2 \left[ \frac{1}{ur} - \frac{1}{r(1 - ur)} - \left( \frac{1}{ur^2} - \frac{1}{1 - ur} \right) \ln \frac{m_b^2}{\mu^2} \right] \ln(1 - r)$$

$$+ 2 \left( \frac{1}{1 - ur} - \frac{1}{ur^2} \right) \left( \ln^2(1 - r) + \text{Li}_2(r) \right)$$

$$+ \left[ \frac{4}{1 - ur} + \frac{ur + u^2r + \bar{u}}{\bar{u}(ur)^2} - 2 \left( \frac{2}{1 - ur} - \frac{1 - \bar{u}r}{u\bar{u}r^2} \right) \ln \frac{m_b^2}{\mu^2} \right] \ln(1 - ur)$$

$$- 2 \left( \frac{2}{1 - ur} - \frac{1 - \bar{u}r}{u\bar{u}r^2} \right) \left( \ln^2(1 - ur) + \text{Li}_2(ur) \right) \right\}.$$  

(29)

We need to add that superior to use of the pole mass for the b quark [15, 16], employing the $\overline{MS}$ mass could render not only the calculation free from some element of uncertainty but the physical meaning more obvious even when the calculation is performed at QCD tree level, as shown in (28).

To proceed, we embark on handling the IR divergence term,

$$T_{IR}(u, r) = \frac{2\Delta_{IR}}{1 - ur} \left[ 3 - 2\ln(1 - r) \frac{1 - r - ur}{ur^2} - 2\ln \left( \frac{1 - ur}{1 - r} \right) \frac{1 - r - u\bar{u}r^2}{u\bar{u}r^2} \right].$$

If we try to subtract the divergent part from the UV renormalized hard amplitude to represent the invariant function $F((p + q)^2)$ in the form of QCD factorization, it has to abide by the form

$$T_{IR}(u, r) = -\Delta_{IR} \int_0^1 dv V_0(v, u) T_0(v, r),$$

(30)

where $V_0(v, u)$ is the kernel of the evolution equation of the pionic twist-2 DA [19]. As checked readily, this is indeed the case. We can therefore eliminate the divergence by defining a scale
dependent DA as
\[
\varphi_\pi(u, \mu) = \varphi_\pi(u) - \Delta_{\text{IR}} \frac{\alpha_s C_F}{4\pi} \int_0^1 dv \, V_0(u, v) \, \varphi_\pi(v),
\]
which is convoluted with the perturbative kernel \( T^H(u, r, \mu) = T(u, r) |_{\Delta_{\text{IR}}=0} \). As a result, the twist-2 contribution to \( F((p + q)^2) \) observes, at NLO, the following QCD factorization:
\[
F^{\text{QCD}}((p + q)^2) = -f_\pi \int_0^1 dv \, T^H(u, r, \mu) \, \varphi_\pi(u, \mu).
\]
Up to higher order corrections in \( \alpha_s, \mu \) dependence of \( T^H(u, r, \mu) \) compensates that of \( \varphi_\pi(u, \mu) \). It should be understood that in the above operations the factorization and renormalization scales have been set identical for simplicity.

Having in hand the hard kernel available, we can calculate the QCD spectral function to write \( F^{\text{QCD}}((p + q)^2) \) as a dispersion integral. For \( r = (p + q)^2 / m_b^2 = s / m_b^2 > 1 \), we have
\[
\rho_{\text{QCD}}(s) = -\frac{1}{\pi r} f_\pi \int_0^r d\eta \, \text{Im} T^H(u, r, \mu) \, \varphi_\pi(u, \mu) |_{u=\eta/r},
\]
where we take the operation
\[
\frac{\rho_{\text{QCD}}(\eta)}{1 - \eta} \bigg|_{+} = \frac{F(\eta) - F(1)}{1 - \eta},
\]
to avert the redundant divergences possibly occurring as the integral in (33) is performed over the interval \([0, r]\).
Using (33) and counting the twist-4 contribution covered in (20), we have the final sum rule for the product $f_B f_{B \rightarrow \pi}(0)$

$$f_B f_{B \rightarrow \pi}(0) e^{-\frac{m_B^2}{M^2}} = -\frac{m_B^2 f_\pi}{2\pi m_B^2} \int_{m_B^2}^{s_0^B} ds \ e^{-\frac{m_B^2}{M^2}} \frac{1}{s} \int_0^{s/m_B^2} d\eta \ \text{Im}T \left( \frac{m_B^2}{s \eta}, \frac{s}{m_B^2}, \mu \right) \ \varphi_\pi \left( \frac{m_B^2}{s \eta}, \mu \right)$$

$$+ \frac{f_\pi}{m_B^2} \int_{u_0}^{1} du e^{-\frac{m_B^2}{uM^2}} \left( -\frac{u d^2 \phi_{4\pi}(u)}{4 du^2} + u \psi_{4\pi}(u) + \int_0^{u} dv \psi_{4\pi}(v) - \frac{d}{du} I_{4\pi}(u) \right)$$

$$\equiv K(s_0^B, M^2), \quad (36)$$

with $M^2$ indicating the Borel parameter with respect to $(p + q)^2$ and $u_0 = m_B^2/s_0^B$.

Converting (36) into the corresponding sum rule for $D \rightarrow \pi$ transition by a simple replacement of the parameters, we put an end to our derivation of the LCSRs for $f_{B \rightarrow \pi}(0)$ and $f_{D \rightarrow \pi}(0)$, to $O(\alpha_s)$ precision in twist-2 approximation and at tree-level for twist-4 contributions.

We close this Section with a few remarks. Albeit the LCSR calculations are done at twist-4 level, the results remain valid to twist-5 accuracy. The reason is simple. The twist-5 DAs as well as twist-3 ones play no role in the present context due to the Dirac structures of the related nonlocal operators, of which both $\bar{d}(x)\gamma_5 u(0)$ and $\bar{d}(x)\sigma_{\mu\nu}\gamma_5 u(0)$, as sandwiched between the vacuum and a pion state, bring about a chirally enhanced twist expansion. Apart from helping reduce sum rule pollution by long-distance parameters, the disappearance of twist-3 and -5 components from the light cone expansions guarantees the resulting LCSRs well convergent. We are going to return to this point in the following Section.

III. CHOICE OF THE INPUTS AND NUMERICAL DISCUSSION

Presently, theoretical estimates of $f_{B \rightarrow \pi}(0)$ and $f_{D \rightarrow \pi}(0)$ with twist-5 accuracy are obtainable in the sum rules to have been given and the inputs to properly be selected. On the experimental side, from the measured shapes of the form factors for $B \rightarrow \pi l\bar{\nu}$, the CKM matrix element $|V_{ub}|$ multiplied by $f_{B \rightarrow \pi}(0)$ is numerically inferred as $[1]$

$$f_{B \rightarrow \pi}(0) |V_{ub}| = (9.4 \pm 0.3 \pm 0.3) \times 10^{-4}. \quad (37)$$

For the semileptonic processes $D \rightarrow \pi l\bar{\nu}$, a similar manipulation $[3]$ gives

$$f_{D \rightarrow \pi}(0) |V_{cd}| = 0.150 \pm 0.004 \pm 0.001. \quad (38)$$

Then the yielded theoretical predictions could have $|V_{ub}|$ and $|V_{cd}|$ extracted from these up-to-date data.

Aimed at determining $|V_{ub}|$ and $|V_{cd}|$, we must do our best to enhance reliability of the LCSR assessments for the form factors in question. So special care should be taken when
making our choice of the parameters entering the sum rules. The main sources of uncertainty are, of course, the related DAs, which can merely be understood at a phenomenological level. Based on the conformal symmetry of massless QCD, we can parameterize these DAs by expanding them in terms of matrix elements of conformal operators. The twist-2 DA $\varphi_\pi(u)$ is of the following expansion in the Gegenbauer polynomials:

$$\varphi_\pi(u) = 6u\bar{u} \left( 1 + a_2(\mu)C_2^{3/2}(u - \bar{u}) + a_4(\mu)C_4^{3/2}(u - \bar{u}) + \cdots \right),$$

(39)

with the even moments $a_{2n}(\mu)$ remaining to be determined. The Gegenbauer polynomials of higher-degree (large $n$) are rapidly oscillating and so one neglects usually their effects on the numerical integrals included in the sum rules by retaining only the first few terms of the expansion. Some scenarios have been put forward to examine the higher-moment effects. We are willing to mention the prescriptions suggested in [20] and in [16]. In [20] Ball and Talbot (BT) presume that $a_{2n}$ fall off as powers of $n$, $a_{2n} \propto 1/(n + 1)^p$, in order to build a DA model. In comparison, authors of [16] consider a modified transverse momentum $K_\perp$ dependent Brodsky-Huang-Lepage (BHL) wavefunction,

$$\Psi_\pi(u, K_\perp) = \left[ 1 + B_\pi C_2^{3/2}(2u - 1) + C_\pi C_4^{3/2}(2u - 1) \right] \times \frac{A_\pi}{u(1-u)} \exp \left[ -\beta_\pi^2 \frac{K_\perp^2 + m_q^2}{u(1-u)} \right],$$

(40)

which is integrated over $|K_\perp| \leq \mu$ to give a twist-2 DA. Phenomenological studies with both models are in support of the rationality of using an expansion truncated after $n = 2$. We stick to such disposal. In one-loop approximation taken as default for all the renormalized parameters except QCD coupling, $a_2(\mu)$ and $a_4(\mu)$ respect the renormalization group equations

$$a_2(\mu_2) = [L(\mu_2, \mu_1)]^{25C_F/6\beta_0} a_2(\mu_1),$$

(41)

$$a_4(\mu_2) = [L(\mu_2, \mu_1)]^{91C_F/10\beta_0} a_4(\mu_1),$$

(42)

with $L(\mu_2, \mu_1) = \frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}$, $C_F = 4/3$ and $\beta_0 = 11 - \frac{2n_f}{3}$, $n_f$ being the number of active quark flavors. To our knowledge, all the existing estimates for $a_2(\mu)$ are basically consistent with each other and have the averaged central value of 0.25 at $\mu = 1$ GeV. In the light of the current experimental constraints imposed on LCSR calculations, $a_2(1$ GeV) appears to prefer varying between 0.16 – 0.19 [12–14, 21]. The situation is not optimistic about $a_4(\mu)$. The findings differ among the various studies to a large extent, and even there would be a difference in sign between numerical estimates. Fortunately, the sum rule results depend less sensitively on $a_4(\mu)$ than on $a_2(\mu)$. We would like to use as a consistent input the findings [14], $a_2(\mu = 1$GeV) = 0.17 ± 0.08 and $a_4(\mu = 1$GeV) = 0.06 ± 0.1, from fitting the LCSR calculation of the pionic electromagnetic form factor to the recent experimental
observation. Concerning the twist-4 DAs, the three-particle components are specified by only two parameters to NLO in conformal spin, and are of the following forms:

\[ \Phi_{4\pi}(\alpha_i) = 120\delta^2_\pi\epsilon_\pi(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3, \]  
\[ \Psi_{4\pi}(\alpha_i) = 30\delta^2_\pi(\mu)(\alpha_1 - \alpha_2)\alpha_3^2\left[\frac{1}{3} + 2\epsilon_\pi(1 - 2\alpha_3)\right], \]  
\[ \bar{\Phi}_{4\pi}(\alpha_i) = -120\delta^2_\pi\alpha_1\alpha_2\alpha_3\left[\frac{1}{3} + \epsilon_\pi(1 - 3\alpha_3)\right], \]  
\[ \bar{\Psi}_{4\pi}(\alpha_i) = 30\delta^2_\pi\alpha_3^2(1 - \alpha_3)\left[\frac{1}{3} + 2\epsilon_\pi(1 - 2\alpha_3)\right], \]

where the nonperturbative quantities \( \delta^2_\pi \) and \( \epsilon_\pi \) have the scale dependence

\[ \delta^2_\pi(\mu_2) = [L(\mu_2, \mu_1)]^{\alpha_s^{\text{MS}}(\mu_1)}\delta^2_\pi(\mu_1), \]
\[ (\delta^2_\pi\epsilon_\pi)(\mu_2) = [L(\mu_2, \mu_1)]^{10\beta_0\pi}(\delta^2_\pi\epsilon_\pi)(\mu_1), \]

and the parameter values \[ \delta^2_\pi = (0.18 \pm 0.06)\text{GeV}^2 \] and \( \epsilon_\pi = 0.2 \pm 0.1 \) normalized at 1 GeV, which are to be adopted as inputs. Resorting to equation of motion the two-particle components, without introducing any new parameter, can be understood as

\[ \phi_{4\pi}(u) = \frac{200}{3}\delta^2_\pi u^2\bar{u}^2 + 8\delta^2_\pi\epsilon_\pi\{u\bar{u}(2 + 13u\bar{u}) + 2u^3(10 - 15u + 6u^2)\ln u + 2\bar{u}^3(10 - 15\bar{u} + 6\bar{u}^2)\ln \bar{u}\}, \]
\[ \psi_{4\pi}(u) = \frac{20}{3}\delta^2_\pi C_\pi^4(2u - 1). \]

The \( \overline{\text{MS}} \) quark masses \( m_b \) and \( m_c \) comply with the proverbial LO evolution equations. The bottomonium \[ 23 \] and charmonium \[ 23, 24 \] sum rule results with four-loop precision, \( m_b(m_b) = 4.164 \pm 0.025 \text{ GeV} \) and \( m_c(m_c) = 1.29 \pm 0.03 \text{ GeV} \), are applicable well to the present discussion. As far as QCD coupling goes, we use two-loop running down from \( \alpha_s(M_z) = 0.1176 \pm 0.002 \) \[ 25 \]. Additionally, the factorization scales are assigned, according to the typical virtuality of the heavy quarks, as \( \mu_b = 3 \text{ GeV} \) and \( \mu_c = 1.5 \text{ GeV} \) in the respective cases of \( B \) and \( D \) mesons.

Among the hadronic parameters are the decay constants \( f_B \), \( f_D \), and \( f_\pi \), apart from the heavy meson masses determined experimentally \[ 25 \] as \( m_B = 5.279 \text{ GeV} \) and \( m_D = 1.865 \text{ GeV} \). The value of \( f_\pi \) is measured at \( f_\pi = 130.4 \text{ MeV} \) \[ 25 \], from the exclusive processes \( \pi \to \mu\bar{\nu}_\mu \) and \( \pi \to \mu\bar{\nu}_\mu\gamma \). Recently, an updated measurement of \( f_D \) has already been reported by the CLEO collaboration \[ 4 \], \( f_D = 205.8 \pm 8.9 \text{ MeV} \). However, it is on the basis of combining the experimental data on \( f_D \) multiplied by \( |V_{cd}| \),

\[ f_D|V_{cd}| = 46.4 \pm 2.0 \text{ MeV} \]  
\[ (50) \]
and the assumption $|V_{cd}| = |V_{us}| = 0.2255 \pm 0.0019$, and hence could only serve as an input in the sum rule calculation of $f_{+}^{D \to \pi}(0)$. Instead of a direct estimate of $f_{+}^{D \to \pi}(0)$, we consider the sum rule for the product $f_{D} f_{+}^{D \to \pi}(0)$, which in conjunction with the experimental numbers (38) and (50) allows us to consistently make predictions for the quantities $f_{+}^{D \to \pi}(0)$, $|V_{cd}|$ and $f_{D}$ as well. By contrast, leptonic $B$ decays are made difficult to detect experimentally by higher helicity suppression. To have a measurement analogous to (50), the only opportunity is furnished by $B \to \tau \nu$ well established lately [26]. Nevertheless the results yielded in the SM are less persuasive. The reason is that these modes turn out to be sensitive to possible extensions of the SM such as the two-Higgs doublet models and minimal supersymmetric extensions. We must have recourse to theoretical predictions for $f_{B}$ to make an assessment of $f_{B}^{B \to \pi}(0)$. As a consistent choice, here we make use of the interval $f_{B} = 214^{−5}_{+7}$ MeV [13] from a sum rule with the $\overline{MS}$ quark mass.

The remaining parameters are intrinsic to the sum rules, containing the effective threshold $s_{0}^{B}$ ($s_{0}^{D}$) and Borel variables $M^{2}$. The former can be set at the neighborhood of the squared mass of the lowest scalar $B$ meson ($D$ meson). An alternative manner, which has proven to be more effective, is through use of an auxiliary sum rule obtained, for example, by taking logarithmic derivative of $1/M^{2}$ for (36),

$$m_{B}^{2} = - \frac{\partial}{\partial M^{-2}} \ln K(s_{0}^{B}, M^{2}).$$

(51)

Requiring the measured value of the $B$ meson mass to be reproduced precisely from the above sum rule, we get the effective interval $s_{0}^{B} = (34 \pm 0.5)$ GeV$^{2}$ in accordance with the sum rule estimate in heavy quark effective theory [27]. Similarly, $s_{0}^{D}$ is fixed at $(6.5 \pm 0.25)$ GeV$^{2}$. The Borel intervals could be specified in the standard procedure. We have $M^{2} = (18 \pm 3)$ GeV$^{2}$ and $M^{2} = (6 \pm 3)$ GeV$^{2}$, corresponding to, respectively, the sum rules for $B$ and $D$ mesons. As both inherent parameters vary within their separate ranges allowed, it is demonstrated that the twist-4 effects are kept at a numerical level less than 4%, and also the continuum contributions are highly suppressed, not exceeding 20%.

Using the inputs given above, the numerical discussion can be done. Our sum rule result for $f_{B} f_{+}^{B \to \pi}(0)$ reads

$$f_{B} f_{+}^{B \to \pi}(0) = 59^{+10}_{−4} \text{ MeV},$$

(52)

with the uncertainty achieved by adding in quadrature all the errors caused by variations of the inputs, of which the scale parameter $\mu$ is set to the interval between $(2.5 − 6.0)$ GeV. We address this result is because it is independent of the value for $f_{B}$ and therefore of less uncertainty, and moreover is convenient for a numerical update of the sum rule for $f_{+}^{B \to \pi}(0)$ once the theoretical estimate of $f_{B}$ gets improved in the future. Substituting the parameter value for $f_{B}$ into (52), we obtain

$$f_{+}^{B \to \pi}(0) = 0.28^{+0.05}_{−0.02}.$$  

(53)
FIG. 3: Dependence of the LCSR for $f_{B \pi}^+(0)$ on the Borel parameter $M^2$ (a) and on the factorization scale $\mu$ (b).

TABLE I: The LCSR result for $f_{B \to \pi}^+(0)$ with the uncertainty estimates due to the variation of the input.

| Central value | $M^2$ | $s_0^B$ | $\mu$ | $m_B$ | $f_B$ | $a_2^\pi$ | $a_4^\pi$ |
|---------------|-------|---------|-------|-------|-------|----------|----------|
| $f_{+ \to \pi}^+(0)$ | +0.002 | +0.007 | +0.05 | +0.008 | +0.007 | +0.008 | +0.01 |
| 0.277         | -0.001 | -0.008 | -0.01 | -0.008 | -0.009 | -0.008 | -0.01 |

Illustrating stability of the numerical result, we display the variations of the sum rule for $f_{+ \to \pi}^+(0)$ with the Borel and the scale parameters, respectively, in Figs.3(a) and 3(b). It is distinctly observed that the $M^2$ dependence is considerably weak in the Borel interval required, and there is a moderate $\mu$ dependence. Furthermore, to have an explicit understanding of the role that every source of uncertainty plays in the uncertainty evaluation, we collect in Tab.1 the individual uncertainty contributions estimated by altering each of the inputs within its specified range. Those not listed therein are tiny and included in the total uncertainty. A comparison is drawn among the LCSR predictions for $f_{+ \to \pi}^+(0)$ in Tab.2, there being a result quite close to one another. We can understand it as follows: (1) No matter which of the two correlation functions one adopts for a LCSR estimate of that quantity, the light-cone expansion reveals a good convergence, as will be addressed. (2) All these calculations employ essentially the same inputs for the leading twist-2 DA, along with a $f_B$ consistently determined from the sum rules. It is exceptionally hard to have a LQCD calculation to compare with, since the pionic energy goes beyond the restriction by the lattice spacing. Nonetheless, it is claimed [28] that $f_{B \to \pi}^+(0)$ is estimable in an improved LQCD simulation, with the result $f_{+ \to \pi}^+(0) = 0.27 \pm 0.07 \pm 0.05$.

Now the experimental measurement (37), with the aid of the theoretical prediction (53),
TABLE II: Comparison of theoretical predictions for the form factors $f_{B \rightarrow \pi}^+(0)$ and $f_{D \rightarrow \pi}^+(0)$.

| Approach         | Ref. | $f_{B \rightarrow \pi}^+(0)$       | $f_{D \rightarrow \pi}^+(0)$       |
|------------------|------|-----------------------------------|-----------------------------------|
| LCSR             | [14] | 0.281 ± 0.05                      | 0.63 ± 0.11                       |
|                  | [11] | 0.258 ± 0.331                     |                                   |
|                  | [17] | 0.26^{+0.04}_{-0.03}              | 0.67^{+0.10}_{-0.07}              |
|                  | [13] |                                   | 0.62 ± 0.03                       |
| This work        |      | 0.28^{+0.05}_{-0.02}              |                                   |
| Lattice QCD      | [30] |                                   | 0.57 ± 0.06 ± 0.02                |
|                  | [29] |                                   | 0.64 ± 0.03 ± 0.06                |
|                  | [28] |                                   | 0.74 ± 0.06 ± 0.04                |
|                  | [31] |                                   | 0.666 ± 0.029                     |
|                  | [32] |                                   | 0.65 ± 0.06 ± 0.06                |

allows for extracting the desired CKM matrix element $|V_{ub}|$. We have the interval:

$$|V_{ub}| = (3.4^{+0.2}_{-0.6} ± 0.1 ± 0.1) \times 10^{-3},$$  \hspace{1cm} (54)

where the first error originates from the uncertainty of $f_{B \rightarrow \pi}^+(0)$ and the others do from the corresponding experimental ones. Obviously, an analogous result can be extracted in the other LCSR estimates of $f_{B \rightarrow \pi}^+(0)$ in Tab.2. There is also a similar determination from matching the LCSR calculations and experimental partial rates for $q^2 \leq 12\text{GeV}^2$ [14]. All these are upheld by the findings in LQCD simulations for a high $q^2$ and consistent with the CKM fit upshots [5, 6].

Corresponding to (52), the product $f_D f_{D \rightarrow \pi}^+(0)$ has the numerical value

$$f_D f_{D \rightarrow \pi}^+ = 117^{+8}_{-7} \text{ MeV}. \hspace{1cm} (55)$$

Tab.3 provides a summary of the major uncertainty contributions to the sum rule. As exhibited in Figs.4(a) and 4(b), the stability of the sum rule holds as well as in the B meson situation, as $M^2$ changes in the interval specified and $\mu$ ranges from 1 to 3 GeV. Intriguingly, using the same inputs as ours for most of the parameters this quantity is explored in the LCSR approach [18] and the yielded result $f_D f_{D \rightarrow \pi}^+(0) = 137^{+19}_{-14} \text{ MeV}$ is compatible with our prediction within the errors, but showing a larger central value. We remark on this difference. The twist expansion in $x^2 \approx 0$ is the basic thought of the LCSR approach. For heavy to light transition, such an expansion must match the one in the inverse of heavy quark mass $m_Q$. One shows, indeed, that in the heavy quark expansion the end point behaviors of the higher-twist DAs entering a traditional LCSR might modify, but does not violate the twist hierarchy. For instance, the twist-3 term, which is formally $1/m_Q$ suppressed versus the
FIG. 4: Dependence of the LCSR for $f_D f_D^{\pi}(0)$ on the Borel parameter $M^2$ (a) and on the factorization scale $\mu$ (b).

twist-2 part, behaves the same as the latter in the heavy quark limit. However, an explicit calculation with a finite $m_Q$ demonstrates that whereas the twist expansion works better for $B$ decays, there is a considerable numerical violation of the hierarchy relation in the $D$ meson cases, where the twist-3 components contribute to the sum rules much more than the twist-2 ones due to the chiral enhancement factor $\mu_\pi > 1$. The fact that the sum rule for $f_+^{D\to\pi}(0)$ is poorly convergent implies that the twist-5 effect is not negligible and should be considered, even if we work in twist-4 approximation. Currently nothing is known, however, about the twist-5 DAs except that they provide the sum rule with a term formally $1/m_Q^2$ suppressed with respective to the twist-3 one. To have a sketchy understanding of their influence on the LCSR calculation, authors of [18] suppose that the ratio of the twist-5 to twist-3 parts is identical to the one of the twist-4 and -2 terms, while in [17] the twist-4 term is multiplied by a factor of 3. Anyway, it is still obscure that how much on earth do the twist-5 components, in particular those with the chiral enhancement factor, contribute to the sum rule for $f_+^{D\to\pi}(0)$. We leave it as an open question until a reliable twist-5 model wavefunction is presented. Given that the present scenario ensures, to twist-5 precision, the light-cone expansion to converge well whether for $B$ or $D$ decays, this issue gets, at any rate, settled provisionally.

Let us go back to our numerical calculation. Combining the sum rule prediction (55) with the product of the two experimental numbers (38) and (50), we could yield the square of $|V_{cd}|$ and further the magnitude of $V_{cd}$:

$$|V_{cd}| = 0.244 \pm 0.005 \pm 0.003 \pm 0.008,$$

(56)

where the first and second errors are of an experimental origin and the third is due to the theoretical uncertainty. This result deviates by about 2% from the Wolfenstein approxima-
TABLE III: The LCSR result for $f_D f_+^{D\to\pi}(0)$ with the uncertainty estimates due to the variation of the input.

| Central value | $M^2$ | $s_0^D$ | $\mu$ | $m_c$ | $a_2^\pi$ | $a_4^2$ | $\omega_4^2$ | $\delta_4^2$ |
|---------------|-------|---------|-------|-------|----------|--------|-----------|-----------|
| $f_D f_+^{D\to\pi}(0)$ | +0.0007 | +0.0015 | +0.0062 | -0.0006 | +0.0001 | +0.005 | +0.0017 | +0.0003 | +0.0014 |
| 0.117 | -0.0001 | -0.0016 | -0.0049 | -0.0003 | -0.005 | -0.0016 | -0.0002 | -0.0013 |

The $|V_{cd}| = |V_{us}| = 0.2255 \pm 0.0024$ [25], and is in good keeping with $|V_{cd}| = 0.234 \pm 0.007 \pm 0.002 \pm 0.025$ extracted from (38) by using the LQCD estimate $f_{D\pi}^+(0) = 0.64 \pm 0.03 \pm 0.06$ [29]. Certainly we have a slightly larger central value than achieved in [18], where the same data are combined with the LCSR result $f_D f_+^{D\to\pi}(0) = 137^{+19}_{-14}$ MeV.

To proceed, substitution of (56) in (38) gets

$$f_+^{D\to\pi}(0) = 0.62 \pm 0.03, \quad (57)$$

where all the theoretical and experimental errors are in quadrature covered in the total uncertainty. There are abundant researches on $f_+^{D\to\pi}(0)$, from which we pick just out several typical LCSR and lattice predictions and arrange them, along with the present estimate, into the tabulation in Tab.2. At first sight, there exists a good accordance among all the LCSR results listed. Yet this should not be taken too seriously, for more or less parameters, on which the sum rules have relatively sensitive dependence, are chosen to have different inputs in these calculations. For instance, the obviously different parameter values are employed for both $f_D$ and $a_2$ in [17] and [18]. The LQCD evaluations turn out to have a different extent of deviation from each other in the central values, but without any conflict within the errors. A comprehensive survey shows that $f_+^{D\to\pi}(0)$ prefers taking a value larger than 0.6.

Lastly, we would like to present, as a by-product, our assessment for the decay constant $f_D$. From (50) and (56) follows that

$$f_D = 190^{+12}_{-11} \text{ MeV}, \quad (58)$$

with the same error disposal as in the $f_+^{D\to\pi}(0)$ case. It falls into a somewhat wide interval formed by the existing findings of $f_D$, which can be illuminated by the following examples. The CLEO measures $f_D = 205.8 \pm 8.9$ MeV [4], on the assumption $|V_{cd}| = |V_{us}| = 0.2255 \pm 0.0019$. LQCD simulation predicts the three-flavor results $f_D = 218.9 \pm 11.3$ MeV [34] and $f_D = 213 \pm 4$ MeV [35], and two-flavor one $f_D = 197 \pm 4$ MeV [33]. Compared with all these determinations, QCD sum rules provide, besides the two-loop result $203 \pm 20$ MeV [36], the three-loop ones $f_D = 195 \pm 20$ MeV [37] and $f_D = 177 \pm 21$ MeV [38]. Hence one should step up efforts to improve calculations and promote understanding of that quantity. The present estimate, however, could be accommodated by $f_D \approx 200$ MeV, a result gradually becoming
accepted on the basis of a multitude of phenomenological investigations, and in particular
accords well with those from the two-flavor LQCD [33] as well as three-loop QCD sum rules
[37]. Meanwhile, these consistencies further expand support for the validity of our findings
in (56) and (57).

In the above discussion, a cross check has been made automatically between our LCSR
predictions for \( f_+^{B \rightarrow \pi}(0) \) and \( f_+^{D \rightarrow \pi}(0) \). In contrast, it is out of the question for a traditional
LCSR calculation, since twist-2 and -3 contributions, as emphasized, dominate respectively
in the two sum rules, which consequently show exceedingly different sensitivities to both of
them. In addition, from the observation that the twist-2 part predominates entirely over the
twist-4 one in the present LCSR framework, we can benefit a lot in attempting to acquire
a constraint on \( a_2 \) and \( a_4 \) from the data on \( B \) and \( D \) decays. No doubt, this would enhance
significantly our confidence in the LCSR applications to heavy-to-light transitions.

IV. SUMMARY

We have addressed in some detail a QCD assessment for \( B, D \rightarrow \pi \) transitions at the
zero momentum transfer, in an improved LCSR approach, and presented our determinations
of the form factors \( f_+^{B \rightarrow \pi}(0) \) and \( f_+^{D \rightarrow \pi}(0) \) as well as the CKM matrix elements \( |V_{ub}| \) and
\( |V_{cd}| \). We have also yielded a numerical estimate of the decay constant \( f_D \).

To \( O(\alpha_s) \) accuracy for twist-2 contributions and with the \( \overline{MS} \) masses for the heavy
quarks, the LCSR calculation on \( f_B f_+^{B \rightarrow \pi}(0) \) and \( f_D f_+^{D \rightarrow \pi}(0) \) is carried out and the resulting
sum rules bear the two remarkable characteristics: (1) They receive no contribution from
not only the twist-3 but also the unknown twist-5 components, which are regarded usually
as a serious source of uncertainty in the conventional LCSR applications, among others, to
\( D \) decays, and therefore are available to twist-5 accuracy. (2) The twist-2 parts play a fully
dominant role over the twist-4 ones so that the twist hierarchy required for convergence of
the light cone expansions is preserved well and the higher-twist effects are kept under good
control. The numerical analysis is performed with the updated inputs and experimental
data; the validity and the self-consistency of the sum rule results are checked up and verified
by a numerical comparison with some of typical theoretical predictions. Our findings are
such as below:

\[
\begin{align*}
f_+^{B \rightarrow \pi}(0) &= 0.28^{+0.05}_{-0.02}, \quad |V_{ub}| = (3.4^{+0.2}_{-0.6} \pm 0.1 \pm 0.1) \times 10^{-3}, \\
f_+^{D \rightarrow \pi}(0) &= 0.62 \pm 0.03, \quad |V_{cd}| = 0.244 \pm 0.005 \pm 0.003 \pm 0.008, \\
f_D &= 190^{+12}_{-11} \text{ MeV}.
\end{align*}
\]

The present results can be improved once the related inputs or experimental data become
updated. Albeit unlikely to give help in understanding the existing discrepancy between in-
clusive and exclusive $|V_{ub}|$ determinations, an improvement on the $|V_{ub}|$ determination is expected especially. However, it demands evidently a more decided knowledge of $f_B$, apart from a significant advance in experiment and in theoretical or phenomenological research on the pionic twist-2 DA. A continued and intensive study of $f_B$ helps also in the confirmation whether or not non-SM physics shows an explicitly observable effect in $\tau$-leptonic and corresponding semileptonic $B$ decays, which are expected to be detectable to a high precision in the running LHC or foreseeable super $B$ factor. On the other hand, although so far our discussion on the form factors has been restricted to the largest recoil point $q^2 = 0$, $q^2$ dependence of them is understandable within the kinematical regions allowed by their individual light-cone expansion calculations. Then it is possible to extrapolate the results to the large $q^2$ regions in various ways available so as to have an all-around understanding of their behaviors. Too it is interesting to generalize the present discussion to the decays into $K$ meson. We put off these studies to a future issue.

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