The method of parameterization in the quadratic optimal control problem

V A Srochko, V G Antonik and E V Aksenushkina

1 Department of Computational Mathematics and Optimization, Irkutsk State University, 1 K. Marks St., 664003 Irkutsk, Russia
2 Mathematics and Computer Science Department, Baikal State University, 11 Lenin St., 664003 Irkutsk, Russia
E-mail: srochko@math.isu.ru

Abstract. In the framework of control parameterization method the optimization problem with respect to linear phase system with quadratic functional is considered. Approximation of the control is obtained in the class of piecewise constant functions. It is formed as a linear combination of a special set of support functions. Coefficients of this combination are variables of the finite-dimensional problem. To effectively solve this problem explicit expressions for the functional with respect to parameters of approximations are obtained. As a result the quadratic mathematical programming problem is formulated.

1. Introduction

Different transforming techniques of optimal control problems to finite-dimensional problems are well-known to specialists in the area of numerical methods of optimal control [1, 2, 3, 4].

In the last years new discretization (parameterization) methods have appeared. These methods are competitive in comparison with traditional algorithms of solving of optimal control problems (see review [5]).

Modern methods of finite-dimensional optimization together with powerful software surpass the level of corresponding methods for optimal control problems especially in the framework of nonconvex problems. For example effective methods of solving of linear optimal control problems use discretization (parameterization) of control and transition to special linear programming problems [6]. In nonlinear optimal control problems with phase constraints is widely used full discretization with respect to control and state. It leads to nonlinear programming problems [2].

Nevertheless in the framework of discrete approach procedures of parameterization for only control functions are preferential [7, 8]. At the last time such kind of parameterization is presented with the help of linear combination some support functions [9, 3].

In this paper the optimal control problem with quadratic cost functional and linear phase system is considered. Approximation of control is performed in the class of piecewise constant functions. It has a form of linear combination of some support functions with coefficients – parameters. These coefficients become variables in finite-dimensional problem.

Explicit expressions for the cost functional are obtained. They depend on approximation parameters. As a result quadratic programming problems are formed. Such kind of parameterization keep the convexity property of considered optimal control problem.
At last it is formulated relation between optimality conditions in variational and finite-dimensional problems. It appears that differential extremum condition in the finite-dimensional problem is locally equivalent (for small parameterization step) to the maximum principle for the variational problem.

2. Problem formulation. Control parameterization procedure

Let variables \( t \in [t_0, T] \) (a time), \( u(t) \in R \) (a control), \( x(t) \in R^n \) (phase state) are connected by linear system

\[
\dot{x} = A(t)x + b(t)u, \quad x(t_0) = x^0
\]

with continuous functions \( A(t) \in R^{n \times n}, b(t) \in R^n, [t_0, T] \) and given initial state \( x^0 \).

The set \( V \) of admissible controls consists from piecewise continuous functions \( u(t) \) with the following constraint at each moment of a time

\[
u(t) \in [u_-, u_+], \quad t \in [t_0, T].
\]

We define the quadratic functional

\[
\Phi(u) = \frac{1}{2} \langle x(T), x(T) \rangle.
\]

We consider the problem of extremum searching (minimum or maximum) for this functional at the set of admissible controls with some additional constraints.

Our goal is to reduce this problem to the finite-dimensional optimization problem with the help of parameterization procedure formed as a linear combination of some support functions.

We introduce the uniform mesh \( \Delta \) of points \( t_i = t_0 + ih, \quad i = 0, m \) with the step \( h = \frac{T-t_0}{m} \) \((t_m = T)\). Let \( T_j = (t_{j-1}, t_j), \quad j = 1, m \) be the cells of this mesh. Let us define the corresponding characteristic functions

\[
\chi_j(t) = \begin{cases} 1, & t \in T_j, \\ 0, & t \in [t_0, T] \setminus T_j. \end{cases}
\]

We can note that for \( j \neq k \)

\[
\chi_j(t)\chi_k(t) = 0, \quad t \in [t_0, T].
\]

Let \( y = (y_1, ..., y_m) \) be a set of parameters (coefficients) of linear combination. We define the controls

\[
u(t, y) = \sum_{j=1}^{m} y_j \chi_j(t), \quad t \in [t_0, T].
\]

These controls are piecewise constant functions with values \( y_j \) in the framework of the mesh \( \Delta \):

\[
u(t, y) = y_j, \quad t \in T_j, \quad j = 1, m.\]

We define the set of such controls satisfying the constraint (2)

\[
W = \{ u(\cdot, y) : y_j \in [u_-, u_+], \quad j = 1, m \}.\]

It is evident that \( W \) is the subset of \( V \): \( W \subset V \).

Let \( x(t, y), \quad t \in [t_0, T] \) be the solution of the phase system (1), that corresponds to the control \( u(t, y) \). The following expression is true:

\[
x(t, y) = x(t_0) + \sum_{j=1}^{m} y_j x^j(t), \quad t \in [t_0, T].
\]

Here \( x^j(t) \) is the solution of the phase Cauchy problem with the support function \( \chi_j(t) \) and the null initial condition

\[
\dot{x} = A(t)x + b(t)\chi_j(t), \quad x(t_0) = 0.
\]
We note that $x^j(t) = 0, \ t \in [t_0, t_j-1], \ j = 2, m$.

If we will apply the parameterization procedure to the approximate solving of the original problem then we will obtain finite-dimensional optimization problems regarding variables $y$ or $z$. To effectively solve these problems we need to obtain explicit expression of the cost function and her derivatives regarding variables $y, z$.

3. Expressions for the cost functional

Using the formula (4) we have

$$
\Phi(y) = \frac{1}{2} \langle x(T, 0), x(T, 0) \rangle + \sum_{j=1}^{m} y_j \langle x(T, 0), x^j(T) \rangle + \\
+ \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} y_j y_k \langle x^j(T), x^k(T) \rangle.
$$

We designate:

$m$-vector $d$ with coordinates $\langle x(T, 0), x^i(T) \rangle, \ j = 1, m$,

$(n \times m)$-matrix $X$ with columns $x^j(T), \ j = 1, m$.

As a result we obtain the first formula for the functional $\Phi$ at the set $W$

$$
\Phi(y) = \Phi(0) + \langle d, y \rangle + \frac{1}{2} \langle Xy, Xy \rangle.
$$

Coordinates of the vector $d$ and columns of the matrix $X$ are calculated through Cauchy problems for the phase system (1). These problems should be solved at the segment $[t_j-1, T], \ j = 1, m$.

Now we have to obtain the second formula for the functional $\Phi$ with the help of the conjugate system

$$
\dot{\psi} = -A(t)^T \psi, \ t \in [t_0, T].
$$

(6)

Let $\psi^i(t), \ i = 1, n$ be the solution of this system with initial condition $\psi(T) = e^i$, where $e^i \in R^n$ is the unit vector. We define functions

$$
s_i(t) = \langle \psi^i(t), b(t) \rangle, \ t \in [t_0, T], \ i = 1, n.
$$

Then we have the expression

$$
x_i(T, u) = \langle \psi^i(t_0), x(t_0) \rangle + \int_{t_0}^{T} s_i(t) u(t) dt, \ i = 1, n,
$$

(7)

that gives the opportunity to obtain explicit formula for the functional $\Phi(u)$ depending on control

$$
\Phi(u) = \frac{1}{2} \sum_{i=1}^{n} x_i^2(T, u) = \Phi(0) + \\
+ \sum_{i=1}^{n} \langle \psi^i(t_0), x(t_0) \rangle \int_{t_0}^{T} s_i(t) u(t) dt + \frac{1}{2} \sum_{i=1}^{n} (\int_{t_0}^{T} s_i(t) u(t) dt)^2.
$$

Further we apply the formula (3) for the control:

$$
\int_{t_0}^{T} s_i(t) u(t) dt = \sum_{j=1}^{m} y_j \int_{t_j}^{T} s_i(t) dt
$$
and designate
\[ c_{ij} = \int_{T_j} s_i(t)dt, \quad c_j = \sum_{i=1}^{n} \langle \psi^i(t_0), x(t_0) \rangle c_{ij}. \] (8)

As a result we have the second formula for the functional \( \Phi \) at set \( W \)
\[ \Phi(y) = \Phi(0) + \sum_{j=1}^{m} c_j y_j + \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} c_{ij} y_j \right)^2. \]

The realization of this formula is connected to solutions of the conjugate system (6) together with the integration of the functions \( s_i(t) \) with respect to cells \( T_j \).

Vector-matrix presentation of this formula has the form
\[ \Phi(y) = \Phi(0) + \langle c, y \rangle + \frac{1}{2} \langle Cy, Cy \rangle \]
with \( m \)-vector \( c = \{c_j\} \) and \( (n \times m) \)-matrix \( C = \{c_{ij}\} \).

4. Optimality conditions
Let us consider the following problem
\[ \frac{1}{2} \langle x(T), x(T) \rangle \rightarrow \text{ext}, \quad u(t) \in [u_-, u_+], \quad t \in [t_0, T] \]
in the linear control system.

We formulate the corresponding quadratic programming problems with the convex cost function and the simplest constraints with respect to variables:
\[ \langle d, y \rangle + \frac{1}{2} \langle Xy, Xy \rangle \rightarrow \text{ext}, \quad \langle c, y \rangle + \frac{1}{2} \langle Cy, Cy \rangle \rightarrow \text{ext}, \quad y \in [u_-, u_+]. \]

We may note that for the minimum operation (\( \text{ext} = \min \)) we have the convex minimization problems that could be solved by finite number of iterations \([10, 11]\). And for the maximum operation we have the convex maximization problems. Global solution of such problems could be searched by methods from \([4, 12]\).

Let us establish the relation between variational and finite-dimensional problems with respect to optimality conditions. We choose the following problems
\[ \Phi(u) = \frac{1}{2} \langle x(T), x(T) \rangle \rightarrow \max, \quad u \in [u_-, u_+], \] (9)
\[ \Phi(y) = \Phi(0) + \sum_{j=1}^{m} c_j y_j + \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} c_{ij} y_j \right)^2 \rightarrow \max, \quad y_j \in [u_-, u_+]. \] (10)

Here coefficients \( c_j, c_{ij} \) are defined by formulas (8).

The transition from (9) to (10) is made with the help of the control \( u(t, y) \). We want to formulate the maximum principle for this control in the framework of the problem (9). Let \( x(t, y) \) be the corresponding phase trajectory, \( \psi(t, y) \) be the solution of the conjugate problem
\[ \dot{\psi} = -A(t)^T \psi, \quad \psi(T) = x(T, y). \]

Since \( \psi^i(T) = e^i, \ i = 1, \ldots, n \), then
\[ \psi(T) = \sum_{i=1}^{n} x_i(T, y)\psi^i(T). \]
Hence, given the formula (7)

\[ \psi(t, y) = \sum_{i=1}^{n} x_i(T, y)\psi^i(t) = \sum_{i=1}^{n} (\psi^i(t_0), x(t_0)) + \sum_{j=1}^{m} c_{ij}y_j(\psi^i(t)). \]

We note that

\[ c_{ij} = \int_{t_{j-1}}^{t_j} s_i(t)\,dt = s_i(t_j)h + o(h). \]  

Then

\[ \langle \psi(t, y), b(t) \rangle = \sum_{i=1}^{n} (\psi^i(t_0), x(t_0))s_i(t) + O(h). \]

As a result, the maximum principle for the control \( u(t, y) \) in the problem (9) has the following form

\[ u(t, y) = \begin{cases} u_-, & \langle \psi(t, y), b(t) \rangle < 0, \\ u_+, & \langle \psi(t, y), b(t) \rangle > 0. \end{cases} \]

Let us consider this condition at the points \( t_j, j = 1, m \). We designate

\[ \delta_j = \sum_{i=1}^{n} (\psi^i(t_0), x(t_0))s_i(t_j) \]

and assume that \( \delta_j \neq 0, j = 1, m \). Then \( \langle \psi(t_j, y), b(t_j) \rangle = \delta_j + O(h), j = 1, m \). It means that for small values \( h : \text{sign} \langle \psi(t_j, y), b(t_j) \rangle = \text{sign} \delta_j \), and the maximum principle for the control \( u(t, y) \) at the points \( t_j \) is written in the following form

\[ u(t_j, y) = \begin{cases} u_-, & \delta_j < 0, \\ u_+, & \delta_j > 0. \end{cases} \]

Let us consider the differential maximum condition in the problem (10). The partial derivatives are presented by the next formulas

\[ \frac{\partial \Phi(y)}{\partial y_j} = c_j + \sum_{i=1}^{n} (\sum_{j=1}^{m} c_{ij}y_j)c_{ij} = \sum_{i=1}^{n} (\langle \psi^i(t_0), x(t_0) \rangle + \sum_{j=1}^{m} c_{ij}y_j)c_{ij}, j = 1, m. \]

We formulate the necessary maximum condition in the problem (10)

\[ y_j = \begin{cases} u_-, & \frac{\partial \Phi(y)}{\partial y_j} < 0, \\ u_+, & \frac{\partial \Phi(y)}{\partial y_j} > 0. \end{cases} \]

Taking into account the formula (11) we obtain

\[ \frac{\partial \Phi(y)}{\partial y_j} = h \sum_{i=1}^{n} (\psi^i(t_0), x(t_0))s_i(t_j) + o(h) = h\delta_j + o(h). \]

Therefore, for small \( h \)

\[ \text{sign} \frac{\partial \Phi(y)}{\partial y_j} = \text{sign} \delta_j, \quad j = 1, m. \]

It means that the maximum condition for the point \( y \) in the problem (10) has the form

\[ y_j = \begin{cases} u_-, & \delta_j < 0, \\ u_+, & \delta_j > 0. \end{cases} \]

This formula coincides with (12).

Thus, for small values of parameterization step \( h > 0 \) the maximum principle for the control \( u(t, y) \) at the points \( t_j \) in the problem (9) coincides with the differential maximum condition for the point \( y \) in the problem (10).
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