Fermi-Walker coordinates in 2+1 dimensional gravity

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Abstract

It is shown that in 2+1 dimensions the Fermi-Walker gauge allows the general solution of the problem of determining the metric from the sources in terms of simple quadratures. This technique is used to solve the problem of the occurrence of closed time like curves (CTC’s) in stationary solutions. In fact the Fermi-Walker gauge, due to its physical nature, allows to exploit the weak energy condition and in this connection it is proved that, both for open and closed universes with axial symmetry, the energy condition imply the total absence of closed time like curves. The extension of this theorem to the general stationary problem, in absence of axial symmetry is considered and at present the proof of such generalization is subject to some assumptions on the behavior of the determinant of the dreibeins in this gauge.

In the first order formalism the Fermi-Walker gauge is defined by

\[ \sum_i \xi^j \Gamma^a_{bi} = 0 ; \quad \sum_i \xi^i e^a_i = \sum_i \xi^i \delta^a_i. \]  

(1)

Regularity conditions at the space origin \( \xi^i = 0 \) imply

\[ \Gamma^a_{bi}(0, t) = 0, \quad e^a_i(0, t) = \delta^a_i. \]  

(2)

Eqs.(1) are solved by the following formulas

\[ \Gamma^a_{bi}(\xi) = \xi^j \int_0^1 R^a_{bji}(\lambda \xi, t) \lambda d\lambda ; \quad \Gamma^a_{b0}(\xi) = \Gamma^a_{b0}(0, t) + \xi^i \int_0^1 R^a_{b0i}(\lambda \xi, t) d\lambda, \]  

(3)

\[ e^a_0(\xi) = \delta^a_0 + \xi^i \int_0^1 \Gamma^a_{i0j}(\lambda \xi, t) d\lambda ; \quad e^a_i(\xi) = \delta^a_i + \xi^j \int_0^1 \Gamma^a_{ij0}(\lambda \xi, t) \lambda d\lambda, \]  

(4)

being \( R^a_{bji} \) the curvature.

In 2+1 dimensions a simplifying feature occurs because the Riemann tensor, being a linear function of the Ricci tensor, can be written directly in terms of the energy-momentum tensor

\[ \varepsilon_{abc} R_{ab} = -2\kappa T_c. \]  

(5)

Thus also in the time dependent case eqs.(3,4) provide a solution by quadrature of Einstein’s equations. Nevertheless one has to keep in mind that the solving formulas are true only in the Fermi-Walker gauge, in which the energy momentum tensor is not...
an arbitrary function of the coordinates but it is subject to the covariant conservation law and symmetry property that are summarized by the equations

\[ \mathcal{D} T^a = 0 \quad \text{and} \quad \varepsilon_{abc} T^b \wedge e^c = 0. \]  

(6)

It will be useful as done in ref. [2] to introduce the cotangent vectors \( T_\mu = \frac{\partial \xi^0}{\partial \xi^\mu} \), \( P_\mu = \frac{\partial \rho}{\partial \xi^\mu} \) and \( \Theta_\mu = \rho \frac{\partial \theta}{\partial \xi^\mu} \) where \( \rho \) and \( \theta \) are the polar variables in the \((\xi^1, \xi^2)\) plane.

In addition we notice that in \(2+1\) dimensions the most general form of a connection satisfying eq. (6) is

\[ \Gamma^a_{\mu \nu}(\xi) = \varepsilon^{abc} \varepsilon_{\mu \rho \sigma} P^\rho A^c_{\nu}(\xi). \]  

(7)

The covariant conservation equation is already satisfied while writing \( A^c_{\nu}(\xi) = T_c \left[ \Theta^\rho \beta_1 + T^\rho (\frac{\beta_2 - 1}{\rho}) \right] + \Theta_c \left[ \Theta^\rho \alpha_1 + T^\rho (\frac{\alpha_2}{\rho}) \right] + P_c \left[ \Theta^\rho \gamma_1 + T^\rho (\frac{\gamma_2}{\rho}) \right] \),

the symmetry constraint gives

\[ A_1 \alpha_2 - A_2 \alpha_1 + B_2 \beta_1 - B_1 \beta_2 = 0 \]  

(9)

\[ A_2 \gamma_1 - A_1 \gamma_2 + \frac{\partial B_1}{\partial \theta} - \frac{\partial B_2}{\partial t} = 0 \]  

(10)

\[ B_2 \gamma_1 - B_1 \gamma_2 + \frac{\partial A_1}{\partial \theta} - \frac{\partial A_2}{\partial t} = 0, \]  

(11)

where \( A_1 + 1, B_1, A_2, B_2 \) are the primitives of \( \alpha_1, \beta_1, \alpha_2, \beta_1 \). The previous equations can be solved by means of simple quadratures in the time dependent case in presence of rotational symmetry or in the stationary case, in both cases with complete control of the support of the energy momentum tensor [2]. Moreover in the stationary case the projection technique due to Geroch [3] allows to deduce from the completeness of such a projection the completeness of the Fermi-Walker coordinate system.

In the stationary case for example, the explicit solution by quadrature given the \( \alpha_1, \beta_1, \gamma_1 \) is [3]

\[ \alpha_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{B_1} \right) + 2 \alpha_1 I \]  

(12)

\[ \beta_2 = \frac{A_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \theta} \left( \frac{N}{A_1} \right) + 2 \beta_1 I \]  

(13)

\[ \gamma_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \theta} \left( \frac{A_1}{B_1} \right) + 2 \gamma_1 I, \]  

(14)
where

\[ I = \int_0^\rho d\rho \frac{N(A_1^2 B_1 - B_1 \alpha_1)}{(B_1^2 - A_1^2)^2}; \quad N = \frac{1}{2\gamma_1} \frac{\partial}{\partial \theta}(A_1^2 - B_1^2) \]  \hspace{1cm} (15)

while the support conditions for the energy momentum tensor are

\[ \beta_1^2 - \alpha_1^2 - \gamma_1^2 = \text{const.} \]  \hspace{1cm} (16)

\[ \alpha_1 B_1 - \beta_1 A_1 = \text{const.} \]  \hspace{1cm} (17)

outside the sources. Such support equations are related to some Lorentz and Poincaré holonomies [4].

The techniques outlined above turns out to be a powerful tool in investigating the problem of the occurrence of CTCs in 2+1 dimensions [1, 2, 4, 5]. In ref. [1] the following “Kerr” solution in 2+1 dimensions was derived

\[ ds^2 = -(dt + 4GJd\theta)^2 + dr^2 + (1 - 4GM)^2 r^2 d\theta^2, \]  \hspace{1cm} (18)

which has the embarassing feature of having CTCs near the source. The Fermi-Walker gauge due to its physical meaning allows to exploit the weak energy conditions (WEC), to show that the functions

\[ E^{(\pm)}(\rho) \equiv (B_2 \pm A_2)(\alpha_1 \pm \beta_1) - (\alpha_2 \pm \beta_2)(B_1 \pm A_1) \]  \hspace{1cm} (19)

are non increasing in \( \rho \). From this result the following theorem follows [2]:

For a stationary open universe with axial symmetry if the matter sources satisfy the WEC and there are no CTC at space infinity, then there are no CTC at all. Thus the “singular source” related to (18) does not satisfy the WEC.

With the same techniques the theorem on the absence of CTC’s can also be proved for all closed stationary universes with axial symmetry. With regard to the extension to non axially-symmetric stationary universes at present the proof goes through provided \( \det(e) \) in the Fermi-Walker gauge never vanishes.
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