EXISTENCE OF STRICTLY DECREASING POSITIVE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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ABSTRACT. The paper is concerned with a linear neutral differential equation
\[ \dot{y}(t) = -c(t)y(t - \tau(t)) + d(t)\dot{y}(t - \delta(t)) \]
where \( c: [t_0, \infty) \to (0, \infty), d: [t_0, \infty) \to [0, \infty), t_0 \in \mathbb{R} \) and \( \tau, \delta: [t_0, \infty) \to (0, r], r \in \mathbb{R}, r > 0 \) are continuous functions. A new criterion is given for the existence of positive strictly decreasing solutions. The proof is based on the Rybakowski variant of a topological Ważewski principle suitable for differential equations of the delayed type. Unlike in the previous investigations known, this time the progress is achieved by using a special system of initial functions satisfying a so-called sewing condition. The result obtained is extended to more general equations. Comparisons with known results are given as well.

1. Introduction. The aim of the paper is to give a criterion for the existence of positive strictly decreasing solutions to the linear neutral differential equation
\[ \dot{y}(t) = -c(t)y(t - \tau(t)) + d(t)\dot{y}(t - \delta(t)) \]
where \( c: [t_0, \infty) \to (0, \infty), d: [t_0, \infty) \to [0, \infty), t_0 \in \mathbb{R} \) and \( \tau, \delta: [t_0, \infty) \to (0, r], r \in \mathbb{R}, r > 0 \) are continuous functions.

The existence of positive solutions of functional differential equations of delayed type is a classical problem which is satisfactorily solved for various classes of equations in numerous papers and books. We should note, however, that the positivity of solutions to neutral differential equations is investigated to a degree less than that of the positivity of solutions of non-neutral equations with delay.

Some results on the existence of positive solutions for delayed differential equations and their systems are summarized, e.g., in [1, 2, 3, 23, 24, 25].

Let us cite one of the nice classical implicit results on the existence of a positive solution of a linear equation with delay ([39], see also [23, Theorem 2.1.4] and [2, Theorem 2.2.13]), which serves as a source for various explicit sufficient positivity criteria. Consider the equation
\[ \dot{y}(t) + p(t)y(t - \tau(t)) = 0 \]
where \( p, \tau: [t_0, \infty) \to \mathbb{R}_+, \mathbb{R}_+ := [0, \infty) \) are continuous functions, \( \tau(t) \leq t \) and \( \lim_{t \to \infty} (t - \tau(t)) = \infty \). Set \( T_0 = \inf_{t \geq t_0} \{ t - \tau(t) \} \). A function \( y \) is called a solution

2010 Mathematics Subject Classification. Primary: 34K40, 34K25; Secondary: 34K12.
Key words and phrases. Delay; positive solution, neutral equation, sewing condition, retract method.

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of (2) with respect to initial point $t_0$ if $y$ is defined and continuous on $[T_0, \infty)$, differentiable on $[t_0, \infty)$, and satisfies (2) for $t \geq t_0$.

**Theorem 1.1.** Equation (2) has a positive solution with respect to $t_0$ if and only if there exists a continuous function $\lambda(t)$ on $[T_0, \infty)$ such that $\lambda(t) > 0$ for $t \geq t_0$ and

$$\lambda(t) \geq p(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda(s)ds \right), \quad t \geq t_0. \quad (3)$$

The above criterion was generalized for systems of linear and nonlinear differential equations with bounded delay in [9] and for nonlinear systems of differential equations with unbounded delay and with finite memory in [16]. Positive solutions of (2) in the so-called critical case were studied, e.g., in [5, 11, 12, 17, 19, 22, 35] and an overview of some sufficient conditions to equation (2) in the critical case is given in a recent paper [4]. Asymptotic formulas describing two classes of asymptotically different positive solutions are analyzed, e.g., in [13, 14] and [15]. The problem of positive solutions is also investigated in further numerous papers such as [6, 7, 8, 10, 20, 21, 29, 37] and the references therein.

To describe the main result of the paper we should note that, to the best of our knowledge, there is no extension of the implicit-type (with respect to $\lambda$) result given by Theorem 1.1, where the key role is played by inequality (3), to neutral equations of the type (1) if the solutions are understood as continuously differentiable functions (see Definition 1.2) below. In this direction, we will show that in the case of equation (1), inequality (3) can be replaced by

$$\lambda(t) \geq c(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda(s)ds \right) + d(t)\lambda(t-\delta(t)) \exp \left( \int_{t-\delta(t)}^{t} \lambda(s)ds \right), \quad (4)$$

$t \geq t_0$, where $\lambda : [t_0-r, \infty) \to (0, \infty)$. Strictly speaking, Theorem 1.1 for $p > 0$ deals with strictly decreasing positive solutions. Our method gives the same statement in this sense. Namely, inequality (4) is necessary and sufficient for the existence of a positive and strictly decreasing solution of equation (1).

The topological (retract) method of T. Ważewski [38], which was successfully modified to retarded differential equations by K.P. Rybakowski (see, e.g., [33, 34]) serves as a theoretical tool to prove the main result. For a nice overview of topological principle, we also refer to [36]. In [18] the retract principle was modified for neutral functional differential equations. This modification should make it possible to use this in the present paper. Even if [18] contains an illustrative example, showing how this modification works, there is one serious problem restricting the classes of equations suitable for considering by it. Below, we explain the heart of the matter.

We consider a neutral functional differential system of the form

$$\dot{y}(t) = f(t, y_t, \dot{y}_t) \quad (5)$$

where the symbol $\dot{y}$ stands for the derivative (considered, if necessary, as one-sided). Sometimes we use the symbol $y'$ as well (if there is no doubt whether the derivative is one-sided or not).

Let $\mathcal{C}$ be the set of all continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ and $\mathcal{C}^1$ be the set of all continuously differentiable functions $\phi : [-r, 0] \to \mathbb{R}^n$. Assume $t \geq t_0$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$ and $f : E_r \to \mathbb{R}^n$ with $E_r := (t_0, \infty) \times \mathcal{C} \times \mathcal{C}$. 
We pose an initial problem for (5):
\[ y_{t_0} = \phi, \quad \dot{y}_{t_0} = \dot{\phi} \]  
(6)

where \( \phi \in C^1 \). The norm of \( \phi \in C \) is defined as \( \| \phi \|_r := \max_{\theta \in [-r,0]} \| \phi(\theta) \| \) and, if \( \phi \in C^1 \), then
\[ \| \phi \|_r := \max_{\theta \in [-r,0]} \| \phi(\theta) \| + \max_{\theta \in [-r,0]} \| \phi'(\theta) \| \]

where \( \| \cdot \| \) is the Euclidean norm.

In the literature there are various definitions of a solution to neutral differential equations. In the paper, as a solution of (5), (6), we assume a continuously differentiable function within the meaning of the following definition.

**Definition 1.2.** A continuously differentiable function \( y: [t_0 - r, t_0) \rightarrow \mathbb{R}^n \), \( t_0 \in (t_0, \infty] \), is a solution of (5), (6) if \( y_{t_0} = \phi \), \( \dot{y}_{t_0} = \dot{\phi} \) and (5) is satisfied for any \( t \in [t_0, t_0) \).

V. Kolmanovskii and A. Myshkis [28] considered the initial-value problem for neutral differential equations (5), (6). Although this problem should be expected to have a continuously differentiable solution on an interval \( [t_0, t_\phi) \), in general, this is not true. Even if the functional \( f \) and the initial function \( \phi \) are arbitrarily smooth, and the initial problem can be solved by the method of steps, the continuous solution may, generally speaking, have jumps of the derivative for arbitrarily large \( t \). Such jumps will be absent if the initial function \( \phi \) satisfies the sewing condition

\[ \dot{\phi}(0) = f(t_0, \phi, \dot{\phi}). \]  
(7)

**Theorem 1.3.** [28, p.107] Let \( f: E_r \rightarrow \mathbb{R}^n \) be a continuous functional satisfying, in some neighborhood of any point of \( E_r \), the Lipschitz condition
\[ \| f(t, \psi_1, \chi_1) - f(t, \psi_2, \chi_2) \| \leq L_1 \| \psi_1 - \psi_2 \| + L_2 \| \chi_1 - \chi_2 \| \]

with constants \( L_i \in [0, \infty) \), \( i = 1, 2 \). Assume also \( \phi \in C^1 \) and the sewing condition (7) being fulfilled. Then, there exists a \( t_\phi \in (t_0, \infty] \) such that:

a) There exists a solution \( y \) of (5), (6) on \( [t_0 - r, t_\phi) \).

b) On any interval \( [t_0 - r, t_1) \subset [t_0 - r, t_\phi), t_1 > t_0 \), this solution is unique.

c) If \( t_\phi < \infty \), then \( \dot{x}(t) \) has not a finite limit as \( t \rightarrow t_\phi \).

d) The solution \( y \) and \( \dot{y} \) depend continuously on \( \phi \).

For a particular case of system (5) given by
\[ \dot{y}(t) = f(t, y(t), \dot{y}(t)) \]
\[ := f(t, y(t - h_1(t)), \ldots, y(t - h_o(t)), \dot{y}(t - g_1(t)), \ldots, \dot{y}(t - g_\ell(t))) \]

where indices \( o \geq 0 \) and \( \ell \geq 1 \), a more general result can be proved easily by the method of steps (compare [28, pages 111, 96, and 15]).

**Theorem 1.4.** Let
\[ f: [t_0, \infty) \times \mathbb{R}^{o+\ell} \rightarrow \mathbb{R}^n \]
\[ h_i: [t_0, \infty) \rightarrow (0, r], \quad i = 1, \ldots, o \quad \text{and} \quad g_j: [t_0, \infty) \rightarrow (0, r], \quad j = 1, \ldots, \ell \]
be continuous functions. Assume also \( \phi \in C^1 \) and the sewing condition (7), in the case considered having the form
\[ \dot{\phi}(0) = f(t_0, \phi(-h_1(t_0)), \ldots, \phi(-h_o(t_0)), \dot{\phi}(-g_1(t_0)), \ldots, \dot{\phi}(-g_\ell(t_0))) \]  
(8)

being fulfilled. Then:
a) There exists a solution \( y \) of (5), (6) on \( [t_0 - r, \infty) \).
b) On any interval \( [t_0 - r, t_1] \subset [t_0 - r, \infty) \), \( t_1 > t_0 \), this solution is unique.
c) The solution \( y \) and \( \dot{y} \) depend continuously on \( \phi \).

To succeed in applying Theorem 1.3 (or Theorem 1.4) to prove the existence and uniqueness of a continuously differentiable (by Definition 1.2) solution, the sewing condition (7) (or (8)) must be fulfilled. If not, then, generally speaking, a solution has no continuous derivative and certainly, it has no two-sided derivative for \( t = t_0 \). To define an initial function that satisfies the sewing condition is usually not an easy task. The above weighty circumstance when applying the retract principle to neutral functional differential equations, follows from the necessity to satisfy the sewing condition. When the retract principle is used, it is necessary to construct not only one initial function but a set of functions, called the set of initial functions, satisfying several assumptions. One of the assumption is that every function of this set must satisfy a sewing condition. So, from above it follows that, technically, is not easy to construct such a set. In the present paper, we perform, for the case of linear neutral differential equation (1), the relevant construction of a set of initial functions when dealing with a criterion for a solution to be positive. This is an important progress as, eventually, we are able to prove that such a positive solution is continuously differentiable (in the meaning of Definition 1.2).

The rest of the paper is structured as follows. In Part 2 we give a generalization of the retract principle to neutral functional differential equations, previously developed in [18]. The main result (a criterion for the existence of a positive strictly decreasing and continuously differentiable solution of neutral differential equation (1)) is given in Part 3 where a special construction of a system of initial functions satisfying the sewing condition is also developed. For a more general equation than (1), a criterion for the existence of a positive strictly decreasing and continuously differentiable solution is formulated in Part 4. Some open questions, corollaries and remarks as well as comparisons with some of the previous results are listed in Part 5.

2. Retract method. This part provides necessary background. It is mainly taken from papers [18] and [34]. Note that the underlying ideas are based, in addition to the paper of the founder T. Ważewski [38], on the so-called Razumikhin condition in the theory of stability, e.g., [30, 31, 32], and on Razumikhin’s type of extension of Ważewski’s principle by K.P. Rybakowski [33, 34]). Mentioned are the necessary changes of the original versions, making it possible to prove a criterion for the existence of positive solutions to equation (1).

If a set \( A \subset \mathbb{R} \times \mathbb{R}^n \) is given, then int \( A \), \( \overline{A} \) and \( \partial A \) denote, as usual, the interior, the closure, and the boundary of \( A \), respectively.

Definition 2.1. (compare [18, 34]) Let \( \Lambda \) be a topological space, let a subset \( \tilde{\Omega} \subset \mathbb{R} \times \Lambda \) be open in \( \mathbb{R} \times \Lambda \), and let \( x \) be a mapping associating with every \( (\delta, \lambda) \in \tilde{\Omega} \) a function \( x(\delta, \lambda) : D_{\delta, \lambda} \to \mathbb{R}^n \) where \( D_{\delta, \lambda} \) is an interval in \( \mathbb{R} \). Assume (1)–(3):

1. \( \delta \in D_{\delta, \lambda} \).
2. If \( t \in \text{int} D_{\delta, \lambda} \), then there is an open neighbourhood \( O(\delta, \lambda) \) of \( (\delta, \lambda) \) in \( \tilde{\Omega} \) such that \( t \in D_{\delta', \lambda'} \) holds for all \( (\delta', \lambda') \in O(\delta, \lambda) \).
3. If \( (\delta', \lambda'), (\delta, \lambda) \in \tilde{\Omega} \), and \( t' \in D_{\delta', \lambda'} \), \( t \in D_{\delta, \lambda} \), then
\[
\lim_{(\delta', \lambda', t') \to (\delta, \lambda, t)} x(\delta', \lambda')(t') = x(\delta, \lambda)(t).
\]
Then, \((\Lambda, \tilde{\Omega}, x)\) is called a system of curves in \(\mathbb{R}^n\).

**Definition 2.2.** If \(A \subset A^*\) are any two sets of a topological space and \(\pi: A^* \to A\) is a continuous mapping from \(A^*\) onto \(A\) such that \(\pi(p) = p\) for every \(p \in A\), then \(\pi\) is said to be a *retraction* of \(A^*\) onto \(A\). If there exists a retraction of \(A^*\) onto \(A\), \(A^*\) is called a *retract* of \(A^*\).

**Lemma 2.3.** *(Compare [18, 34]*) Let \((\Lambda, \tilde{\Omega}, x)\) be a system of curves in \(\mathbb{R}^n\). Let \(\tilde{\omega}, W, Z\) be sets. Assume the below conditions \((1)–(4)\):

1. \(\tilde{\omega} \subset [t_0 - r, t_\ast] \times \mathbb{R}^n, t_\ast > t_0\), the cross-section \(\{(t, y) \in \tilde{\omega}\}\) is an open simply connected set for every \(t \in [t_0 - r, t_\ast)\), and \(W \subset \partial \tilde{\omega}\).
2. \(Z \subset \tilde{\omega} \cup W, Z \cap W\) is a retract of \(W\), but not a retract of \(Z\).
3. There is a continuous map \(q: B \to \Lambda\) where \(B = \tilde{\Omega} \cap (Z \cup W)\) such that, for any \(z = (\delta, y) \in B, (\delta, q(z)) \in \tilde{\Omega}\), and, if also \(z \in W\), then \(x(\delta, q(z))(\bar{\delta}) = y\).
4. Let \(A\) be the set of all \(z = (\delta, y) \in Z \cap \tilde{\omega}\) such that, for fixed \((\delta, y) \in A,\) there is a \(t > \delta, t \in D_{\delta,q(z)}\) and \((t, x(\delta, q(z)))(t) \notin \tilde{\omega}\).

Assume that, for every \(z = (\delta, y) \in A\), there is a \(t(z), t(z) > \delta\), such that:

a) \(t(z) \in D_{\delta,q(z)}\) and, for all \(t, \delta \leq t < t(z), (t, x(\delta, q(z)))(t) \in \tilde{\omega}\),

b) \((t(z), x(\delta, q(z)))(t(z)) \in W\),

c) For any \(\sigma > 0\), there is a \(t, t(z) < t \leq t(z) + \sigma\) such that \(t \in D_{\delta,q(z)}\) and \((t, x(\delta, q(z)))(t) \notin \tilde{\omega}\).

(4) For any \(z = (\delta, y) \in W \cap B\) and all \(\sigma > 0\), there is a \(t, \delta \leq t = \delta + \sigma\) such that \(t \in D_{\delta,q(z)}\) and \((t, x(\delta, q(z)))(t) \notin \tilde{\omega}\).

Then, there is a \(z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}\) such that, for every \(t \in D_{\delta_0,q(z_0)}\),

\[ (t, x(\delta_0, q(z_0)))(t) \in \tilde{\omega}. \]

**Remark 1.** Let

\[ \Lambda = \mathcal{C}^1, \quad \tilde{\Omega} \subset \{(t, \lambda) \in [t_0, \infty) \times \mathcal{C}^1 \text{ such that } \dot{\lambda}(0) = f(t_0, \lambda, \dot{\lambda})\} \]

and function \(f\) satisfies all the assumptions of Theorem 1.3. In this case, through each \((t_0, \lambda) \in \tilde{\Omega}\), there exists a unique solution \(y(t_0, \lambda, \dot{\lambda})\) of (5) defined on its maximal interval \([t_0 - r, a_\lambda]\). Let \(D_{t_0, \lambda} = [t_0 - r, a_\lambda]\) where \(a_\lambda > t_0\). Then, \((\Lambda, \tilde{\Omega}, y)\) is a system of curves in \(\mathbb{R}^n\) within the meaning of Definition 2.1. A similar remark holds when all the assumptions of Theorem 1.4 are satisfied.

Usually, when applying Lemma 2.3 to prove the existence of a solution of a given system with the graph staying in a prescribed domain \(\tilde{\omega}\), the form of \(\tilde{\omega}\) should be specified. As a standard shape of such a domain, used in numerous investigations, serves the so-called polyfacial set defined below.

**Definition 2.4.** Let \(p\) and \(s\) be nonnegative integers, \(p + s > 0, t_\ast > t_0\), and let

\[ l_i: [t_0 - r, t_\ast] \times \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \ldots, p, \]

\[ m_j: [t_0 - r, t_\ast] \times \mathbb{R}^n \to \mathbb{R}, \quad j = 1, \ldots, s \]

be continuously differentiable functions. The set

\[ \omega := \{(t, y) \in [t_0 - r, t_\ast] \times \mathbb{R}^n, \quad l_i(t, y) < 0, \quad m_j(t, y) < 0, \quad \text{for all } i, j \} \]

is called a *polyfacial set* provided that the cross-section

\[ \omega \cap \{(t, y): t = t^*, y \in \mathbb{R}^n\} \]

is an open and simply connected set for every fixed \(t^* \in [t_0 - r, t_\ast]\).
When \( p = 0 \) in Definition 2.4, the functions \( l_i, i = 1, \ldots, p \) are not defined. Similarly, if \( s = 0 \), the functions \( m_j, j = 1, \ldots, s \) are omitted. In order to prove the existence of a solution of (5) satisfying the property (9), a polyfacial set \( \omega \) should meet some additional requirements. We can characterize such requirements as properties guaranteeing the properties of solutions of system (5), formulated for the system of curves \((\Lambda, \tilde{\Omega}, x)\) in Lemma 2.3. Because of the neutrality of the equations, we need to be able to foresee the properties of the derivatives of solutions as described by auxiliary inequalities.

**Definition 2.5.** (compare [18]) Let \( q \) be a nonnegative integer, \( t_* > t_0 \), and let
\[
c_k : [t_0 - r, t_*) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad k = 1, \ldots, q,
\]
be continuous functions. A polyfacial set \( \omega \) is called regular with respect to Eq. (5) and auxiliary inequalities
\[
c_k(t, y, x) \leq 0, \quad k = 1, \ldots, q
\]
if \( \alpha - \delta \) below hold:

\( \alpha \) If \((t, \phi) \in \mathbb{R} \times C^1 \) and \((t + \theta, \phi(\theta)) \in \omega \) for \( \theta \in [-r, 0) \), then \((t, \phi, \dot{\phi}) \in E_r.

\( \beta \) If \((t, \phi) \in \mathbb{R} \times C^1 \), \((t + \theta, \phi(\theta)) \in \omega \) for \( \theta \in [-r, 0) \) and, moreover,
\[
c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \quad \theta \in [-r, 0), \quad k = 1, \ldots, q,
\]
then also
\[
c_k(t + \theta, \phi(\theta), f(t, \phi, \dot{\phi})) \leq 0, \quad k = 1, \ldots, q.
\]

\( \gamma \) For all \( i = 1, \ldots, p \), all \((t, y) \in \partial \omega \) for which \( l_i(t, y) = 0 \) and for all \( \phi \in C^1 \) for which \( \phi(0) = y \), \((t + \theta, \phi(\theta)) \in \omega \), \( \theta \in [-r, 0) \) and
\[
c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \quad \theta \in [-r, 0), \quad k = 1, \ldots, q,
\]
it follows that:
\[
Dl_i(t, y) = \frac{\partial l_i}{\partial t}(t, y) + \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t, y) \cdot f_r(t, \phi, \dot{\phi}) > 0.
\]

\( \delta \) For all \( j = 1, \ldots, s \), all \((t, y) \in \partial \omega \) for which \( m_j(t, y) = 0 \) and for all \( \phi \in C^1 \) for which \( \phi(0) = y \), \((t + \theta, \phi(\theta)) \in \omega \), \( \theta \in [-r, 0) \) and
\[
c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \quad \theta \in [-r, 0), \quad k = 1, \ldots, q,
\]
it follows that:
\[
Dm_j(t, y) = \frac{\partial m_j}{\partial t}(t, y) + \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t, y) \cdot f_r(t, \phi, \dot{\phi}) < 0.
\]
If \( \omega \) is a polyfacial set, then define the set \( W \) used in Lemma 2.3 as
\[
W := \{(t, y) \in \partial \omega : m_j(t, y) < 0, j = 1, \ldots, s\}.
\]
Moreover, we need to specify the properties of the mapping \( q \) in Lemma 2.3. The following definition describes the admissible behavior of functions with respect to \( \omega \). A fixed set of functions generated by this mapping and satisfying properties gathered in the following definition is called a set of initial functions.

**Definition 2.6** (Set of initial functions). Let \( Z \) be a subset of \( \omega \cup W \) and let the mapping
\[
q : B \to C^1, \quad B := \overline{Z} \cap (Z \cup W)
\]
be continuous. We assume that, if \( z = (\delta, y) \in B \), then \((\delta, q(z)) \in \tilde{\Omega}\). If moreover

\[
\text{...}
\]
1) For \( z \in Z \cap \omega \), we have \((\delta + \theta, q(z)(\theta)) \in \omega \) for \( \theta \in [-r, 0] \).

2) For \( z \in W \cap B \), we have \((\delta, q(z)(\delta)) = z \) and either

\[ 2a) \quad (\delta + \theta, q(z)(\theta)) \in \omega \quad \text{for} \quad \theta \in [-r, 0] \]

or

\[ 2b) \quad (\delta + \theta, q(z)(\theta)) \in \overline{\omega} \quad \text{for} \quad \theta \in [-r, 0] \]

and, for all \( \sigma > 0 \), there is a \( t = t(\sigma, z) \), \( \delta < t \leq \delta + \sigma \) such that \( t \) is within the domain of definition of solution \( x(\delta, q(z)) \) of (5) and \((t, x(\delta, q(z))(t)) \notin \overline{\omega} \),

then such a set of functions is called a set of initial functions for (5) with respect to \( \omega \) and \( Z \).

Finally, we will formulate the below theorem as an application of Lemma 2.3 for a system of neutral equations (5). Therefore, its proof is omitted.

**Theorem 2.7.** Let \( \omega \) be a nonempty polyfacial set, regular with respect to (5) and inequalities (10). Assume \( \phi \in C^1 \) and the sewing condition (7) being fulfilled. Let a fixed \( t_* \in (t_0, \infty) \) exist such that:

a) There exists a solution \( y \) of (5), (6) on \([t_0 - r, t_*]\).

b) On any interval \([t_0 - r, t_1] \subset [t_0 - h, t_*] \), \( t_1 > t_0 \), this solution is unique.

c) If \( t_* < \infty \), then \( \dot{y}(t) \) has not a finite limit as \( t \to t_*^- \).

d) The solution \( y \) and \( \dot{y} \) depend continuously on \( \phi \).

Assume that \( q \) defines a set of initial functions for (5) with respect to \( \omega \) and \( Z \) and that the derivative of every solution \( x(\delta, q(z))(t) \) of (5) defined by any \( z = (\delta, x) \in B \) has a finite left limit at every point \( t \) provided that

\[ (t, x(\delta, q(z))(t)) \in \overline{\omega}. \]

Let, moreover, \( Z \cap W \) be a retract of \( W \), but not a retract of \( Z \). Then, there exists at least one point \( z_0 = (\delta_0, x_0) \in Z \cap \omega \) such that a solution \( x(\delta_0, q(z_0))(t) \) exists on \([t_0 - r, t_*]\) and

\[ (t, x(\delta_0, q(z_0))(t)) \in \omega \]

holds for all \( t \in [t_0 - r, t_*] \).

3. **Main result.** In this section we give a criterion (sufficient and necessary conditions) for the existence of a positive and strictly decreasing solution of the equation (1).

Equation (1) is a particular case of equation (5) if the functional \( f \) in the right-hand side of (5) is specified as

\[ f(t, \phi, \dot{\phi}) := -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)). \]

Such a functional \( f \) is used in the remaining part of the paper.

**Theorem 3.1.** For the existence of a positive strictly decreasing solution of (1) on \([t_0 - r, \infty)\), a necessary and sufficient condition is that there exists a continuous function \( \lambda: [t_0 - r, \infty) \to (0, \infty) \) such that inequality (4) holds for \( t \geq t_0 \).

**Proof. Necessity.** Let a continuously differentiable positive strictly decreasing solution \( y = y(t) \) of (1) be given on \([t_0 - r, \infty)\). From (1) we conclude \( \dot{y}(t) < 0 \) for every \( t \in [t_0, \infty) \). We show that \( y(t) \) can be expressed in the form

\[ y(t) = \exp \left( - \int_{t_0}^{t} \lambda(s) \, ds \right), \quad t \geq t_0 - r \tag{15} \]
where \( \lambda \) satisfies all conditions formulated in the theorem. Taking the derivative of \( y \), we get
\[
\dot{y}(t) = -\lambda(t) \exp \left( -\int_{t_0}^{t} \lambda(s) \, ds \right), \quad t \geq t_0 - r
\] (16)
and, therefore,
\[
\lambda(t) := -\frac{\dot{y}(t)}{y(t)}, \quad t \geq t_0 - r.
\] (17)
It can be seen from (15)-(17) that \( \lambda(t) > 0 \) if \( t \geq t_0 - r \). Substitute (15) into (1), assuming \( t \geq t_0 \), and divide the equation obtained by \( \exp \left( -\int_{t_0}^{t} \lambda(s) \, ds \right) \). We get
\[
\lambda(t) = c(t) \exp \left( \int_{t_0}^{t} \lambda(s) \, ds \right) + d(t)(t - \delta(t)) \exp \left( \int_{t_0}^{t} \lambda(s) \, ds \right)
\]
where \( t \geq t_0 \). This means that inequality (4) holds.

**Sufficiency.** In this part we make use of Theorem 2.7. The proof is divided into five steps.

**Step 1. Definition of the polyfacial set \( \omega \).** We set \( n = p = 1, s = 0, t_* = \infty \) and define a function (see Definition 2.4)
\[
l_1(t, y) := y \left( y - \nu \exp \left( -\int_{t_0}^{t} \lambda(s) \, ds \right) \right)
\]
where \( y \in \mathbb{R}, \nu > 1 \) is a constant and \( \lambda \) satisfies inequality (4). Then, the set
\[
\omega := \{(t, y) \in [t_0 - r, \infty) \times \mathbb{R}; l_1(t, y) < 0\}
\] (18)
is a polyfacial set within the meaning of Definition 2.4 since, for every fixed \( t^* \in [t_0 - r, \infty) \), the set
\[
\omega \cap \{(t, y) : t = t^*, y \in \mathbb{R}\} = \left\{(t, y) : t = t^*, 0 < y < \nu \exp \left( -\int_{t_0}^{t} \lambda(s) \, ds \right) \right\}
\]
is open and simply connected.

**Step 2. Regularity of \( \omega \).** Set \( q = 1 \). Define a function (see Definition 2.5)
\[
c_1(t, y, x) := x \left( x + \nu \lambda(t) \exp \left( -\int_{t_0}^{t} \lambda(s) \, ds \right) \right).
\] (19)
In general, the function \( c_1 \) depends on three variables. Nevertheless, in the construction below it is not necessary to use the variable \( y \).

We show that the set \( \omega \) defined by (18) is regular with respect to equation (1) and auxiliary inequality \( c_1(t, y, x) \leq 0 \) by Definition 2.5. Therefore, we will verify all its assumptions \( \alpha - \delta \) (denoted below as \( \alpha^* - \delta^* \)).

\( \alpha^* \) If \((t, \phi) \in \mathbb{R} \times C^1 \) and \((t + \theta, \phi(\theta)) \in \omega \) for \( \theta \in [-r, 0) \), then the functional \( f \) is defined at \((t, \phi, \dot{\phi})\). Thus, point \( \alpha \) of Definition 2.5 holds.

\( \beta^* \) Let \((t, \phi) \in \mathbb{R} \times C^1 \), \((t + \theta, \phi(\theta)) \in \omega \) for \( \theta \in [-r, 0) \) and
\[
c_1(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \quad \theta \in [-r, 0).
\] (20)
From (19) and (20) we get
\[
-\nu \lambda(t + \theta) \exp \left( -\int_{t_0}^{t+\theta} \lambda(s) \, ds \right) \leq \dot{\phi}(\theta) \leq 0, \quad \theta \in [-r, 0).
\] (21)
In addition, we have
\[ f(t, \phi, \dot{\phi}) = -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)) < 0 \] (22)
since \( c(t) > 0 \) and \( \phi(-\tau(t)) > 0 \). Now using the definition of \( \omega \) (18) and inequalities (21), (4), we get
\[ f(t, \phi, \dot{\phi}) = -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)) \geq -\nu c(t) \exp \left( -\int_{t_0}^t \lambda(s)ds \right) \]
\[ -\nu d(t)\lambda(t - \delta(t)) \exp \left( -\int_{t_0}^{t - \delta(t)} \lambda(s)ds \right) \]
\[ = \nu \exp \left( -\int_{t_0}^t \lambda(s)ds \right) \left( -c(t) \exp \left( \int_{t - \tau(t)}^t \lambda(s)ds \right) \right. \]
\[ - d(t)\lambda(t - \delta(t)) \exp \left( \int_{t - \delta(t)}^{t - \delta(t)} \lambda(s)ds \right) \]
\[ \geq -\nu \lambda(t) \exp \left( -\int_{t_0}^t \lambda(s)ds \right). \] (23)

Combining (22) and (23), we obtain
\[ -\nu \lambda(t) \exp \left( -\int_{t_0}^t \lambda(s)ds \right) \leq f(t, \phi, \dot{\phi}) < 0. \] (24)

A consequence of (24) is the inequality
\[ c_1(t + \theta, \phi(\theta), f(t, \phi, \dot{\phi})) \]
\[ = f(t, \phi, \dot{\phi}) \left( f(t, \phi, \dot{\phi}) + \nu \lambda(t) \exp \left( -\int_{t_0}^t \lambda(s)ds \right) \right) \leq 0. \]

A consequence of (24) is the inequality
\[ c_1(t + \theta, \phi(\theta), f(t, \phi, \dot{\phi})) \]
\[ = f(t, \phi, \dot{\phi}) \left( f(t, \phi, \dot{\phi}) + \nu \lambda(t) \exp \left( -\int_{t_0}^t \lambda(s)ds \right) \right) \leq 0. \]

Thus, point \( \beta \) of Definition 2.5 holds.

\( \gamma^* \) Let \( \phi \in C^1([-r, 0], \mathbb{R}) \) be such that \( (t + \theta, \phi(\theta)) \in \omega \) for \( \theta \in [-r, 0) \) and \( (t, \phi(0)) \in \partial \omega \). Then, either
\[ \phi(0) = 0 \] (25)
or
\[ \phi(0) = \nu \exp \left( -\int_{t_0}^t \lambda(s)ds \right). \] (26)

Moreover, we assume that (13) holds, i.e.,
\[ c_1(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \]
\[ = \dot{\phi}(\theta) \left( \dot{\phi}(\theta) + \nu \lambda(t + \theta) \exp \left( -\int_{t_0}^{t + \theta} \lambda(s)ds \right) \right) \leq 0, \ \theta \in [-r, 0). \] (27)

Let (25) be true. We will use the properties \( \phi(-\tau(t)) > 0 \) (it follows from definition (18) of the set \( \omega \)) and \( \dot{\phi}(-\delta(t)) \leq 0 \) (it is a consequence of (27)) to get
\[ DL_1(t, y) = DL_1(t, 0) = \frac{\partial l_1}{\partial t} (t, 0) + \frac{\partial l_1}{\partial y} (t, 0) \cdot f(t, \phi, \dot{\phi}) \]
Let (26) be true. We will use the properties
\[
\phi(-\tau(t)) < \nu \exp \left(-\int_{t_0}^{t-\tau(t)} \lambda(s) ds \right)
\]
(it follows from definition (18) of the set \(\omega\)) and
\[
\dot{\phi}(-\delta(t)) \geq \nu \lambda(t - \delta(t)) \exp \left(-\int_{t_0}^{t-\delta(t)} \lambda(s) ds \right)
\]
(it is a consequence of (27)).

Then,
\[
Dl_1(t, y) = Dl_1 \left(t, \nu \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) \right)
\]
\[
= \frac{\partial l_1}{\partial t} \left(t, \nu \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) \right) + \frac{\partial l_1}{\partial y} \left(t, \nu \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) \right) f(t, \phi, \dot{\phi})
\]
\[
= \nu \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right)
\]
\[
\cdot \left( \nu \lambda(t) \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) - c(t) \phi(-\tau(t)) + d(t) \dot{\phi}(-\delta(t)) \right)
\]
\[
> \nu \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) \left( \nu \lambda(t) \exp \left(-\int_{t_0}^{t} \lambda(s) ds \right) \right)
\]
\[
- \nu c(t) \exp \left(-\int_{t_0}^{t-\tau(t)} \lambda(s) ds \right) - \nu d(t) \lambda(t - \delta(t)) \exp \left(-\int_{t_0}^{t-\delta(t)} \lambda(s) ds \right)
\]
\[
\geq \nu^2 \exp \left(-2 \int_{t_0}^{t} \lambda(s) ds \right) \left( \lambda(t) - c(t) \exp \left(\int_{t-\tau(t)}^{t} \lambda(s) ds \right) \right)
\]
\[
- d(t) \lambda(t - \delta) \exp \left(\int_{t-\delta(t)}^{t} \lambda(s) ds \right) \right) \geq | \text{ by (4) } | \geq 0.
\]

Thus, point \(\gamma\) of Definition 2.5 holds.

\(\delta^*\) Since \(s = 0\), there is no function of the type \(m_j, j = 1, \ldots, s\) (see Definition 2.4) in the definition (18) of polyfacial set \(\omega\).

We conclude that the set \(\omega\) defined by (18) is regular by Definition 2.5 with respect to equation (1) and auxiliary inequality \(c(t, y, x) \leq 0\).

**Step 3. Using Theorem 2.7 - sets \(W\) and \(Z\).** To apply Theorem 2.7, we define the set \(W\) in accordance with (14) as
\[
W := \{(t, y) \in \partial \omega : m_j(t, y) < 0, j = 1, \ldots, s\} = \{(t, y) \in \partial \omega\}
\]
since no function of the type \(m_j, j = 1, \ldots, s\) is used. Moreover, define
\[
Z := \{(t, y) \in \omega \cup W : t = t_0\} = \{(t_0, y) : y \in [0, 1]\}.
\]
Obviously, $Z \cap W$ is a retract of $W$, but not a retract of $Z$.

**Step 4. Using Theorem 2.7 - Initial functions for (1).** Now we will construct a set of initial functions for (1) with respect to $\omega$ and $Z$ such that every initial function $\phi$ satisfies the sewing condition (7), i.e.

$$S(t_0, \phi) = 0$$

where

$$S(t_0, \phi) := f(t_0, \phi, \dot{\phi}) - \dot{\phi}(0) = -c(t_0)\phi(-\tau(t_0)) + d(t_0)\phi(-\delta(t_0)) - \dot{\phi}(0).$$

Define, for any $z = (t_0, y) \in Z$ (recall that $y \in [0, 1]$) two initial functions $\varphi_{y}^{\text{max}} \in C([t_0, \theta])$, $\varphi_{y}^{\text{min}} \in C([t_0, \theta])$:

$$\varphi_{y}^{\text{max}}(s) := \nu \exp \left( - \int_{t_0}^{t_0+s} \lambda(u) \, du \right) - \nu + y,$$

$$\varphi_{y}^{\text{min}}(s) := \frac{1}{2} ks^2 + y$$

where, for a constant $\varepsilon \in (0, 1)$,

$$k := \frac{\varepsilon}{r} \cdot \min_{-r \leq \theta \leq 0} \lambda(t_0 + \theta) \exp \left( - \int_{t_0}^{t_0+\theta} \lambda(u) \, du \right) > 0.$$ 

Obviously, $\varphi_{y}^{\text{max}}(0) = y$, $\varphi_{y}^{\text{min}}(0) = y$. For $s \in [-r, 0)$, we prove

$$0 < \varphi_{y}^{\text{min}}(s) < \varphi_{y}^{\text{max}}(s) < \nu \exp \left( - \int_{t_0}^{t_0+s} \lambda(u) \, du \right).$$

The left-hand inequality in (29) holds since $y \in [0, 1]$ and $k > 0$. The right-hand inequality in (29) holds since $-\nu + y < 0$. To prove the middle inequality in (29), we define a function

$$\Psi(s) := \varphi_{y}^{\text{min}}(s) - \varphi_{y}^{\text{max}}(s), \ s \in [-r, 0].$$

Then, for $s \in [-r, 0)$,

$$\Psi'(s) = \frac{\varepsilon}{r} s \min_{-r \leq \theta \leq 0} \lambda(t_0 + \theta) \exp \left( - \int_{t_0}^{t_0+\theta} \lambda(u) \, du \right)$$

$$+ \nu \lambda(t_0 + s) \exp \left( - \int_{t_0}^{t_0+s} \lambda(u) \, du \right)$$

$$\geq -\varepsilon \min_{-r \leq \theta \leq 0} \lambda(t_0 + \theta) \exp \left( - \int_{t_0}^{t_0+\theta} \lambda(u) \, du \right)$$

$$+ \nu \lambda(t_0 + s) \exp \left( - \int_{t_0}^{t_0+s} \lambda(u) \, du \right) > 0.$$ 

Therefore,

$$\varphi_{y}^{\text{min}}(s) - \varphi_{y}^{\text{max}}(s) = \Psi(s) < \Psi(0) = 0, \ s \in [-r, 0)$$

and the middle inequality in (29) is proved.

Moreover, the following chain of inequalities obviously hold
We show that the values $S(t_0, \varphi_y^{\text{max}}), S(t_0, \varphi_y^{\text{min}})$ take opposite signs. Using (4), we get

$$
S(t_0, \varphi_y^{\text{max}}) = -c(t_0) \left( \nu \exp \left( -\int_{t_0}^{t_0-\tau(t_0)} \lambda(s) \, ds \right) - \nu + y \right)
- \nu d(t_0) \lambda(t_0 - \delta(t_0)) \exp \left( -\int_{t_0}^{t_0-\delta(t_0)} \lambda(s) \, ds \right) + \nu \lambda(t_0)
= -\nu c(t_0) \exp \left( \int_{t_0-\tau(t_0)}^{t_0} \lambda(s) \, ds \right)
- \nu d(t_0) \lambda(t_0 - \delta(t_0)) \exp \left( \int_{t_0-\delta(t_0)}^{t_0} \lambda(s) \, ds \right)
+ \nu \lambda(t_0) + c(t_0)(\nu - y) > 0,
$$

and

$$
S(t_0, \varphi_y^{\text{min}}) = -c(t_0) \left( \frac{k}{2} (-\tau(t_0))^2 + y \right) - d(t_0) k \delta(t_0) < 0.
$$

Define a one-parameter family of functions $\varphi_y^\alpha$, depending on a parameter $\alpha \in [0, 1]$ as

$$
\varphi_y^\alpha(s) := \alpha \varphi_y^{\text{max}}(s) + (1 - \alpha) \varphi_y^{\text{min}}(s), \ s \in [-r, 0].
$$

Then, by (31) and (32),

$$
S(t_0, \varphi_y^0)S(t_0, \varphi_y^1) = S(t_0, \varphi_y^{\text{min}})S(t_0, \varphi_y^{\text{max}}) < 0.
$$

The operator

$$
S(t_0, \varphi_y^\alpha) = -c(t_0) \left[ \alpha \varphi_y^{\text{max}}(-\tau(t_0)) + (1 - \alpha) \varphi_y^{\text{min}}(-\tau(t_0)) \right]
+ d(t_0) \left[ \alpha \varphi_y^{\text{max}}(-\delta(t_0)) + (1 - \alpha) \varphi_y^{\text{min}}(-\delta(t_0)) \right] - \varphi_y^\alpha(0),
$$

where

$$
\varphi_y^\alpha(0) = \alpha \varphi_y^{\text{max}}(0) + (1 - \alpha) \varphi_y^{\text{min}}(0) = -\alpha \nu \lambda(t_0),
$$

is strongly monotone with respect to $\alpha$, since, due to (4),

$$
\frac{\partial}{\partial \alpha} S(t_0, \varphi_y^\alpha) = -c(t_0) \left[ \varphi_y^{\text{max}}(-\tau(t_0)) - \varphi_y^{\text{min}}(-\tau(t_0)) \right]
+ d(t_0) \left[ \varphi_y^{\text{max}}(-\delta(t_0)) - \varphi_y^{\text{min}}(-\delta(t_0)) \right] + \nu \lambda(t_0)
= -c(t_0) \left[ \nu \exp \left( -\int_{t_0}^{t_0-\tau(t_0)} \lambda(u) \, du \right) - \nu + y - \frac{k}{2} (-\tau(t_0))^2 - y \right]
+ d(t_0) \left[ -\nu \lambda(t_0 - \delta(t_0)) \exp \left( -\int_{t_0}^{t_0-\delta(t_0)} \lambda(u) \, du \right) - k(-\delta(t_0)) \right] + \nu \lambda(t_0)
$$
\( \alpha = \alpha_y \in [0,1] \) such that \( S(t_0, \varphi_y^{\alpha_y}) = 0 \), i.e., the sewing condition (28) is true. This value, as can be seen in (34), is defined by the formula

\[
\alpha_y = \frac{c(t_0)\phi_y^{\min}(-\tau(t_0)) - d(t_0)\phi_y^{\min}(-\delta(t_0))}{c(t_0)\Psi(-\tau(t_0)) - d(t_0)\Psi(-\delta(t_0)) + \nu\lambda(t_0)}
\]

and depends continuously on \( y \) since \( \phi_y^{\min}, \phi_y^{\max} \) and \( \Psi \) depend continuously on \( y \). Therefore, the function

\[
\varphi_y^{\alpha_y}(s) = \alpha_y\phi_y^{\max}(s) + (1 - \alpha_y)\phi_y^{\min}(s)
\]

is continuous with respect to \( y \) as well.

Applying (30), we see that, for any function \( \varphi_y^{\alpha_y}(s), s \in [-r,0] \), defined by (33), we have:

\[
\dot{\varphi}_y^{\alpha_y}(s) = \alpha_y\dot{\phi}_y^{\max}(s) + (1 - \alpha_y)\dot{\phi}_y^{\min}(s)
\]

\[
\leq \alpha_y\phi_y^{\min}(s) + (1 - \alpha_y)\phi_y^{\min}(s) = \phi_y^{\min}(s), \ s \in [-r,0],
\]

\[
\dot{\varphi}_y^{\alpha_y}(s) = \alpha_y\phi_y^{\max}(s) + (1 - \alpha_y)\phi_y^{\min}(s)
\]

\[
\geq \alpha_y\phi_y^{\max}(s) + (1 - \alpha_y)\phi_y^{\max}(s) = \phi_y^{\max}(s), \ s \in [-r,0].
\]

**Step 5.** Using Theorem 2.7 - Initial functions for (1) and mapping \( q \). By Definition 2.6, we will construct a continuous mapping \( q: B \to C^1 \) where the set \( B \) is defined in Lemma 2.3, point (2) and, in our case, becomes

\( B = \mathbb{Z} \cap (\mathbb{Z} \cup W) = \mathbb{Z} \).

Then, \( q \) maps the set \( Z \) into the space of initial functions satisfying the sewing condition. Define such a mapping \( q: B \to C^1[-r,0] \) for every \( z = (t_0, y) \in B \) by the formula

\[
q(z) = q((t_0, y)) = \varphi_y^{\alpha_y}.
\]

This mapping is continuous and

\[
(t_0 + \theta, q(z)(\theta)) = (t_0 + \theta, \alpha_y\varphi_y^{\max}(\theta) + (1 - \alpha_y)\varphi_y^{\min}(\theta)) \in \omega \quad \text{for} \quad \theta \in [-r,0),
\]

\[
(t_0, q(z)(0)) = (t_0, \alpha_y\varphi_y^{\max}(0) + (1 - \alpha_y)\varphi_y^{\min}(0)) = z.
\]
The mapping \( q \) satisfies conditions 1) and 2a) of Definition 2.6. All assumptions of Theorem 2.7 are now fulfilled. Therefore, there exists at least one point \( z_0 = (t_0, y_0) \in Z \cap \omega \) such that a solution \( x(t_0, q(z_0))(t) \) of (1) exists on \([t_0 - r, \infty)\) and

\[
(t, x(t_0, q(z_0))(t)) \in \omega
\]

holds for all \( t \in [t_0 - r, \infty) \). Because of the shape of \( \omega \), such a solution is positive and, by (4), it is strictly decreasing.

**Remark 2.** Let all assumptions of Theorem 3.1 be true. From its proof (see (36) and the definition (18) of the set \( \omega \)) we deduce that, if (4) holds for \( t \geq t_0 \), then there exist a positive strictly decreasing solution \( y = y(t) \) of (1) on \([t_0 - r, \infty)\) satisfying the inequalities

\[
0 < y(t) < \exp \left(- \int_{t_0}^{t} \lambda(s)ds\right), \quad t \in [t_0 - r, \infty).
\]

Moreover, from formulas (11) and (12) of Definition 2.5, such a solution satisfies

\[
c_1(t, y(t), \dot{y}(t)) \leq 0, \quad t \in [t_0 - r, \infty),
\]

i.e.,

\[
- \lambda(t) \exp \left(- \int_{t_0}^{t} \lambda(s)ds\right) \leq \dot{y}(t) \leq 0, \quad t \in [t_0 - r, \infty).
\]

Due to the linearity of (1), the coefficient \( \nu \) is omitted in (37) and (38).

4. **Generalization.** Consider an equation

\[
\dot{y}(t) = - \sum_{i=1}^{m} c_i(t)y(t - \tau_i(t)) + \sum_{j=1}^{r} d_j(t)\dot{y}(t - \delta_j(t))
\]

(39)

where \( c_i, d_j : [t_0, \infty) \to [0, \infty) \), and \( \tau_i, \delta_j : [t_0, \infty) \to (0, r] \) are continuous functions. Moreover, assume \( \sum_{i=1}^{m} c_i(t) > 0, \quad t \in [t_0, \infty) \). Obviously, equation (39) is more general than equation (1). Now we will formulate a generalization of Theorem 3.1. We omit its proof since it is similar to that of Theorem 3.1. Note that the system of initial functions can be used in the proof without any changes.

**Theorem 4.1.** For the existence of a positive strictly decreasing solution of (39) on \([t_0 - r, \infty)\), a necessary and sufficient condition is that there exists a continuous function \( \lambda : [t_0 - r, \infty) \to (0, \infty) \) such that the inequality

\[
\lambda(t) \geq \sum_{i=1}^{m} c_i(t)\exp \left(\int_{t-\tau_i(t)}^{t} \lambda(s)ds\right) + \sum_{j=1}^{r} d_j(t)\lambda(t - \delta_j(t))\exp \left(\int_{t-\delta_j(t)}^{t} \lambda(s)ds\right)
\]

holds for \( t \geq t_0 \). Moreover, if this inequality holds, then there exists a positive strictly decreasing solution \( y = y(t) \) of (39) on \([t_0 - r, \infty)\) satisfying inequalities (37) and (38).

5. **Concluding discussions.** From the proof of Theorem 3.1, we conclude that a positive solution (if inequality (4) holds) is generated by a function from a one-parameter family of functions \( \varphi_{y}^{q_{y}} \), defined by formula (35) where the parameter \( y \in [0, 1] \). More specifically, as it follows from points (2) and (3) of Lemma 2.3, we can restrict the values of the parameter \( y \) only to values \( y \in (0, 1) \). In this connection, the following open problem arises.
Open Problem 1. How to compute a value (values) of parameter \( y = y^* \in (0, 1) \) such that the initial function \( \varphi_{y^*} \) determines a positive solution of equation (1) (or (39)) indicated in Theorem 3.1?

A solution to this open problem can have certain importance, e.g., in numerical computations.

Because of the linearity of considered equations and the existence of a positive solution, we conclude that there exists a one-parameter family of linearly dependent positive solutions of equation (1) on interval \([t_0 - r, \infty)\).

It is easy to explain, that there exists a one-parameter family of linearly independent positive solutions of equation (1) on interval \([t_0 - r, \infty)\). Looking again at the proof of Theorem 3.1, we emphasize that the definition of the function \( \varphi_{y}^{\text{min}} \) depends (through the constant \( k \)) on a parameter \( \varepsilon \in (0, 1) \). Therefore, each function in the system of initial functions \( \varphi_{y}^{*} \) where \( y \in (0, 1) \), relevant to a choice of \( \varepsilon \), is linearly independent on an interval \([t_0 - r, t_0]\) of every function in the system of initial functions \( \varphi_{y}^{*} \) constructed for a different choice of \( \varepsilon \). Consequently, positive solutions defined by different initial functions, being linearly independent on interval \([t_0 - r, t_0]\) are linearly independent positive solutions of equation (1) on \([t_0 - r, \infty)\). One cannot, however, conclude that such a type of linear independence on the interval \([t_0 - r, \infty)\) implies the existence of a one-parameter family of linearly independent positive solutions of equation (1) on every interval \([t_1 - r, \infty)\) where \( t_1 \geq t_0 \). This assertion can be wrong due to, e.g., the effect of solution pasting (we refer to [26, Part 3.5]). A similar discussion applies to the function \( \varphi_{y}^{\text{max}} \) and the parameter \( \nu \). Nevertheless, we formulate the following open problem connected with this topic.

Open Problem 2. Indicate sufficient conditions for the existence of at least a one-parameter family of linearly independent positive solutions of equation (1) (or (39)) on every interval \([t_1 - r, \infty)\) where \( t_1 \geq t_0 \).

Obviously, Theorem 3.1 is a generalization of Theorem 1.1 to neutral differential equations. Now, we will restrict our discussion only to equation (1) and its special cases although it is easy to formulate corresponding remarks to more general equation (39) and its special cases.

Let the functions \( c(t), \delta(t) \) and delays \( \tau(t), \delta(t) \) in equation (1) be constant, i.e., \( c(t) \equiv c = \text{const}, \delta(t) \equiv \delta = \text{const}, \tau(t) \equiv \tau = \text{const}, \delta(t) \equiv \delta = \text{const} \) and equation (1) becomes

\[
\dot{y}(t) = -cy(t-\tau) + d\delta(t-\delta). \tag{40}
\]

Then, Theorem 3.1 is formulated as

**Theorem 5.1.** For the existence of a positive strictly decreasing solution of (40) on \([t_0 - r, \infty)\), a necessary and sufficient condition is that there exists a continuous function \( \lambda: [t_0 - r, \infty) \rightarrow (0, \infty) \) such that inequality

\[
\lambda(t) \geq c \exp \left( \int_{t-\tau}^{t} \lambda(s)ds \right) + d\lambda(t-\delta) \exp \left( \int_{t-\delta}^{t} \lambda(s)ds \right) \tag{41}
\]

holds for \( t \geq t_0 \).

From Theorem 5.1 and formula (41) where \( \lambda(t) \equiv \lambda = \text{const} \), we immediately get the following corollaries. These criteria are well-known, we refer, e.g., to [24, Theorem 5.2.10, Corollary 5.2.11], [25, Theorem 6.7.1]. Similar criteria can be found, e.g., in [1, Corollary 6.5], [2, Theorem 3.5.3] and [23, Theorem 3.2.3].
Corollary 1. For the existence of a positive strictly decreasing solution of (40) on \([t_0 - r, \infty)\) it is sufficient the existence of a positive constant \(\lambda\) such that inequality
\[
\lambda \geq ce^{\lambda \tau} + \lambda de^{\lambda \delta}
\]
holds.

For the choice \(\lambda = 1/\tau\) or \(\lambda = 1/\delta\) in (42), we get

Corollary 2. For the existence of a positive strictly decreasing solution of (40) on \([t_0 - r, \infty)\) it is sufficient that either inequality
\[
1 > ce^{\tau} + de^{\delta/\tau}
\]
or inequality
\[
1 > cde^{\tau/\delta} + de
\]
hold.

Corollaries 1, 2 can be improved in view of Remark 2 (formulas (37), (38)) in the sense that if inequalities (42), (43), (44) are valid, then on \([t_0, \infty)\) there exist a positive solution vanishing for \(t \to \infty\) and having negative and vanishing for \(t \to \infty\) continuous derivative.

Remark 3. In the paper we regard solutions of equation (1) as continuously differentiable functions satisfying the given equation everywhere. As noted, e.g., in [28, p. 107] it leads to some complications, since the sewing condition must be valid for continuously differentiable initial functions. In the proof of Theorem 3.1, a modification of the retract principle suitable for neutral differential equations was used. This principle, to be successfully applied, needs not only one initial function, but a whole family of initial functions satisfying the sewing condition. Therefore, the crucial moment of the proof was a special construction of such a family of initial functions.

To compare our results with, e.g., those given in [1, Theorem 6.1] we emphasize that the definition of a solution substantially differs (a solution is defined as an absolutely continuous function satisfying the equation almost everywhere).

In [1, 3, 23, 24, 25] part of the results is devoted to the existence of positive solutions of neutral equations having, e.g., the form
\[
(y(t) + P(t)y(t - \tau))' + Q(t)y(t - \sigma) = 0, \quad t \geq t_0
\]
under various conditions for \(P\) and \(Q\). The substantial difference is that the delays in the equation, unlike those in our investigation, are constant. Thus, the results derived in the cited sources are, in principle, not applicable to equation (1).

Acknowledgments. The authors greatly appreciate the work of the anonymous referee, whose comments and suggestions have helped to improve the paper in many aspects. The first author was supported by the Czech Science Foundation under the project 16-08549S. The research of the second author was carried out under the project CEITEC 2020 (LQ1601) with financial support from the Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme II. The work of the first author was realized in CEITEC - Central European Institute of Technology with research infrastructure supported by the project CZ.1.05/1.1.00/02.0068 financed from European Regional Development Fund.
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Received December 2016; revised April 2017.  

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