DIMENSION-FREE BOUNDS FOR SUM OF DEPENDENT MATRICES AND OPERATORS WITH HEAVY-TAILED DISTRIBUTION

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\textbf{Abstract.} We study the deviation inequality for a sum of high-dimensional random matrices and operators with dependence and arbitrary heavy tails. There is an increase in the importance of the problem of estimating high-dimensional matrices, and dependence and heavy-tail properties of data are among the most critical topics currently. In this paper, we derive a dimension-free upper bound on the deviation, that is, the bound does not depend explicitly on the dimension of matrices, but depends on their effective rank. Our result is a generalization of several existing studies on the deviation of the sum of matrices. Our proof is based on two techniques: (i) a variational approximation of the dual of moment generating functions, and (ii) robustification through truncation of eigenvalues of matrices. We show that our results are applicable to several problems such as covariance matrix estimation, hidden Markov models, and overparameterized linear regression models.

1. Introduction

We study a non-asymptotic upper bound of deviations of the average of multiple random matrices (or operators) from its expectation. Suppose that we observe a sequence of $n$ random matrices $M_1, \ldots, M_n$ that is potentially high-dimensional, dependent, and heavy-tailed. We are interested in evaluating a deviation of its empirical mean from its expectation $\Sigma := \mathbb{E}[M_1]$ using an operator norm $\| \cdot \|$ for matrices, that is, we study the upper bound.
on the following value for each $n$:

$$\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\|.$$

This problem is foundational and important; moreover, it has a variety of applications, with the most typical example being the estimation of covariance matrices. Let $Y_1, \ldots, Y_n$ be a sequence of random vectors; then, we can estimate its covariance matrix $\Sigma = \mathbb{E}[Y_1 Y_1^\top]$ using the empirical mean $n^{-1} \sum_{\ell=1}^{n} M_\ell$ by defining $M_\ell = Y_\ell Y_\ell^\top$. This setup can easily be applied for estimating Fisher information matrices, for example. Other applications include estimation of adjacency matrices of random graphs (Oliveira, 2009), a signal recovery of the compressed sensing (Donoho, 2006), and linear regression of overparameterization (Bartlett et al., 2020). With the growing importance of data in the modern data science, it is increasingly becoming important to study the upper bound with various settings, such as high-dimensional and more complex distributions and dependencies.

This problem has been actively investigated in several settings. Rudelson (1999) first studied the upper bound on an operator norm of the deviation. A representative topic is the case in which a matrix $M_\ell$ is high-dimensional. Bunea and Xiao (2015); Mendelson and Paouris (2014); Srivastava and Vershynin (2013); Koltchinskii and Lounici (2017) derived an upper bound that does not depend on the dimensionality, referred to as a dimension-free upper bound, using the notion of the effective rank of matrices. Especially, Giulini (2018) studied an infinite-dimensional version of the problem. This allows for the estimation of high-dimensional matrices without assuming sparsity (Cai et al., 2010) or the shape of the distribution (Adamczak et al., 2010; Guédon and Rudelson, 2007; Zhivotovskiy, 2021). Han (2022) studied an asymptotically exact risk of the problem. Lopes et al. (2023) developed a bootstrap method for high-dimensional operators, and also studied a dimension-free bound in this setting. For the case where the distribution of a matrix $M_\ell$ has a heavy-tail, Liaw et al. (2017) derived a dimension-free upper bound to clarify how the tail property affects the bound. Vershynin (2018); Jeong et al. (2022) further improved the tightness of the upper bound.
for heavy-tailed distributions. For a dependent setting of a sequence of matrices, Han and Li (2020) derived a bound on the expected value of the norm of this perturbation, by following Han (2017).

1.1. Focus and Result. Our goal is to derive a dimension-free upper bound for deviations of the empirical mean of random matrices that are dependent and heavy-tailed. This setting is a generalization of that considered in many previous studies. To handle the setup, we introduce the notions discussed below. Let $H$ be a Hilbert space, which we first consider $H = \mathbb{R}^p$ and then extend to the case with the infinite-dimensional $H$.

To study the dimension-free property, we consider $M$ as a symmetric linear operator between $H$. Then, we define its effective rank:

**Definition 1** (Effective Rank). For an operator $M : H \to H$ that is positive semi-definite, the effective rank is defined as

$$r(M) := \frac{\text{Tr}(M)}{\|M\|},$$

where $\text{Tr}(M)$ denotes a trace of $M$ and $\|M\|$ is the operator norm.

This notion has been used in several studies such as Koltchinskii and Lounici (2017), and it allows us to measure the statistical dimension of matrices in terms of the largest eigenvalue and traces, without using the dimension of $M$.

To address the dependent property of a sequence $M_1, ..., M_n$, we consider a coefficient $\Gamma_{\ell,n}$ for $\ell = 1, ..., n$ that bounds the following value:

$$|\mathbb{E}[g(M_{\ell+1}, \ldots, M_n) \mid \mathcal{F}_\ell] - \mathbb{E}[g(M_{\ell+1}, \ldots, M_n)]| \leq \Gamma_{\ell,n},$$

where $g(\cdot)$ is an arbitrary Lipschitz-continuous function that will be rigorously defined, and $\mathcal{F}_\ell = \sigma(M_1, \ldots, M_\ell)$ is the sigma-algebra generated by a subset of the sequence. This definition is used in Rio (2000), Dedecker et al. (2007) and others, and general enough because it includes many dependent processes, such as causal Bernoulli shifts and chains with infinite memory.

We also introduce a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ to uniformly handle the heavy-tail property of a distribution of $M_\ell$, $\ell = 1, ..., n$ such that

$$\max_{1 \leq \ell \leq n} \mathbb{P}(\|M_\ell\| \geq t) \leq F(t), \forall t > 0.$$
This definition of $F$ is a general representation of the tail probability of $\|M_\ell\|$ that includes both polynomial and exponential decay.

As a main result, we derive the probabilistic upper bound on the deviation of the empirical mean of the matrices. That is, for any $t, \tau > 0$, the following inequality holds with probability at least $1 - e^{-t} - n F(\tau)$:

$$\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| \left(2\tau + \max_{\ell=1,\ldots,n} \Gamma_{\ell,n}\right) \sqrt{\frac{4r(\Sigma) + t}{n}} + V\sqrt{F(\tau)},$$

where $V^2 \geq \mathbb{E}[\|M_\ell\|^2]$ denote an upper bound of the moment. The results suggest that (i) we can obtain a dimension-free upper bound even with the general notion of heavy-tail and dependence; (ii) the dependent property affects the constant part of the bound through $\Gamma_{\ell,n}$; and (iii) the heavy-tail property may affect the convergence rate of the bound with respect to a number of observations $n$ by adjusting $\tau$.

From a technical perspective, this study makes two contributions. The first is the evaluation of a moment-generating function using variational inequality, following [Catoni and Giulini, 2017]. That is, we utilize a dual representation to reform the moment-generating function using an arbitrary distribution and then use the form to derive an upper bound for the deviation of the matrices. This approach was employed by [Zhivotovskiy, 2021] and others, and we extend the approach to our setting using random matrices with dependence. The second is the truncation technique used to ensure that the estimator is robust. We apply the technique for this problem by truncating eigenvalues of random matrices. By decomposing the deviation of the empirical mean into deviations from a truncated estimator, we can analyze the effect of the heavy-tail property.

1.2. **Organization.** Section 2 presents our setting of the problem with an average of random matrices, and also provides assumptions and its examples. Section 3 presents the results. We begin with the result for bounded dependent matrices in Theorem 4. We then extend this result to unbounded matrices in Corollary 5. Section 4 describes several applications in which the deviation bound is used. Section 5 contains the proofs of the main results. Section 6 concludes the paper. Appendix provides the rest of the proofs for applications.
1.3. Notations. Let $\mathbb{H}$ be a Hilbert space equipped with the scalar product $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ be the corresponding norm. Let $S$ be the set of symmetric linear continuous operators $\mathbb{H} \to \mathbb{H}$, that is, for any $(u, v) \in \mathbb{H}^2$, for any $M \in S$, $\langle Mu, v \rangle = \langle u, Mv \rangle < \infty$. In the special case $\mathbb{H} = \mathbb{R}^p$, $S$ is simply the set of symmetric matrices. For any $M \in S$ we let $\| M \|$ denote its operator norm $\| M \| = \sup_{\| u \|=1} \| Mu \|$. For a matrix $A$, $\| A \|_F$ is the Frobenius norm of $A$. Throughout this paper, $\mathbf{M} = (M_\ell)_{\ell=1, \ldots, n}$ is a finite random sequence of elements of $S$, whose expectation is constant with $\Sigma = \mathbb{E}[M_\ell]$. The objective of this paper is to study the estimators of $\Sigma$.

2. Assumption and Example

2.1. Assumptions. We present two assumptions on $\mathbf{M}$: an assumption on the dependence between the $M_\ell$'s by a coefficient of weak dependence and an assumption on the tail of the random variable $\| M_\ell \|$.

2.1.1. Dependence. We introduce a coefficient to measure the weak dependence of the process of operators/matrices, which is essentially the one from Rio (2000) and is also presented in Dedecker et al. (2007). In preparation, we define a set of Lipschitz functions on $\ell$ operators/matrices for $\ell = 1, \ldots, n$.

**Definition 2** (Lipschitz function on $\ell$ elements). Let $E$ be a space equipped with the norm $\| \cdot \|_E$. For any $\ell \in \mathbb{N}$ and $L > 0$, we let $\text{Lip}_\ell(E, L)$ denote the set of all functions $h : E^\ell \to \mathbb{R}$ such that for any $(a_1, \ldots, a_\ell, b_1, \ldots, b_\ell) \in E^{2\ell}$,

$$|h(a_1, \ldots, a_\ell) - h(b_1, \ldots, b_\ell)| \leq L \sum_{i=1}^\ell \|a_i - b_i\|_E.$$ 

Using this coefficient, we make the following assumption on weak dependency:

**Assumption 1** (dependent processes). There exist real numbers $(\Gamma_{\ell,n})_{1 \leq \ell \leq n}$ such that for any $\ell \in \{1, \ldots, n\}$ and for any function $g \in \text{Lip}_{n-\ell}(S, 1)$,

$$|\mathbb{E}[g(M_{\ell+1}, \ldots, M_n)|\mathcal{F}_\ell] - \mathbb{E}[g(M_{\ell+1}, \ldots, M_n)]| \leq \Gamma_{\ell,n} \quad (1)$$
almost surely. We set $\Gamma_n = \max_{1 \leq \ell \leq n} \Gamma_{\ell,n}$.

This assumption has several noteworthy points: (i) The coefficient used in the assumption is a generalization of the uniform mixing coefficient for bounded processes; (ii) it can represent the efficiency of the estimation by the volume of the coefficient; and (iii) it is not comparable with the $\alpha/\beta$-mixing property because it can represent certain non-mixing processes. This assumption can cover linear auto-regressive moving-average (ARMA) processes and a causal Bernoulli shift (CBS), which is described later. For more details, refer to Rio (2000); Dedecker and Prieur (2005); Dedecker et al. (2007); Alquier and Wintenberger (2012).

Note that in previous studies such as Rio (2000), Assumption 1 is used for bounded processes; that is, $\|M_\ell\|$ is bounded. To address unbounded processes, it is essential to utilize the form of the CBS processes described in the next section. A fact that we will often use in this paper is that when $M = (M_1, \ldots, M_n)$ satisfies Assumption 1, then so does $(f(M_1), \ldots, f(M_n))$ where $f : E \rightarrow E$ is an adequate truncation function. This result generally holds as soon as $f$ is $1$-Lipschitz.

**Proposition 1.** Assume that $M = (M_1, \ldots, M_n)$ satisfies Assumption 1 and that $f : E \rightarrow E$ is $1$-Lipschitz. Then $(f(M_1), \ldots, f(M_n))$ also satisfies Assumption 1.

2.1.2. Heavy-Tail. We introduce an assumption regarding the heavy-tail property of random matrices/operators.

**Assumption 2.** We have $\mathbb{E}[\|M_\ell\|^2] \leq V^2$ for some $V < +\infty$. We know a non-increasing function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $F(t) \rightarrow 0$ for $t \rightarrow \infty$ and such that for any $t \in \mathbb{R}_+$,

$$\max_{1 \leq \ell \leq n} \mathbb{P}(\|M_\ell\| > t) \leq F(t).$$

If $F(t)$ is an exponentially decaying function, the $k$-th moment of $\|M_\ell\|$ exists for all $k \in \mathbb{N}$. In particular, $F(t)$ has the form $\exp(-t^2)$ or $\exp(-t)$, which corresponds to the sub-Gaussian or sub-exponential property of the distributions (Vershynin (2018)). If $F(t)$ is a polynomially decaying function $F(t) = -t^b$ for some $b > 2$, the $k$-th moment does not exist for some $k > 2$, and the form represents the heavier-tail property of $\|M_\ell\|$.
Note that it is possible to let \( F(\cdot) \) depend on \( n \) such as by setting \( F(\cdot) = F_n(\cdot) \) and \( F(\cdot) \) varies as \( n \) increases. Despite the flexibility of this setting, we fixed \( F(\cdot) \) regardless of \( n \) for simplicity.

2.2. Examples. We now provide examples in which Assumptions 1 and 2 are satisfied. A recurring case of interest occurs when we have a stationary \( H \)-valued stochastic process \( (Y_\ell)_{\ell \in \mathbb{Z}} \) and \( M_\ell = Y_\ell Y_\ell^T \) or \( M_\ell = Y_\ell Y_{\ell+h}^T \). Then, the estimation of \( \Sigma = \mathbb{E}[M_\ell] \) respectively corresponds to the estimation of the covariance matrix of \( (Y_\ell)_{\ell \in \mathbb{Z}} \) and to its cross-covariances.

2.2.1. Causal Bernoulli Shift. An important example is a causal Bernoulli shift (CBS), which includes several general processes and also a base for generating unbounded processes.

**Example 1** (Causal Bernoulli shifts, CBS). Let \( \Xi = (\xi_\ell)_{\ell \in \mathbb{Z}} \) be a sequence of bounded i.i.d. \( H \)-valued random variables: \( \|\xi_\ell\| \leq B_\xi \) almost surely. Let \( C : \mathbb{H}^\infty \to \mathbb{H} \) with \( C(0, 0, \ldots) = 0 \). Assume that, for any \( (a_1, b_1, a_2, b_2, \ldots) \in \mathbb{H}^\infty \) we have

\[
\|C(a_1, a_2, \ldots) - C(b_1, b_2, \ldots)\| \leq \sum_{\ell=1}^{\infty} a_\ell \|a_\ell - b_\ell\| \text{ and } A := \sum_{\ell=1}^{\infty} a_\ell < \infty.
\]

Then, we have the following stationary process \( (X_\ell)_{\ell \in \mathbb{Z}} \) given by

\[
X_\ell = C(\xi_\ell, \xi_{\ell-1}, \xi_{\ell-2}, \ldots).
\]

This process \( (X_\ell)_{\ell \in \mathbb{Z}} \) is called a CBS. Note that \( \|X_\ell\| \leq B := AB_\xi \) almost surely.

The form of CBSs is a generalization of known stationary and ergodic processes, such as the causal ARMA process. Then, we provide the following result.

**Proposition 2.** We define that \( (Y_\ell)_{\ell \in \mathbb{Z}} \) is a CBS and \( M_\ell = Y_\ell Y_\ell^T \) for any \( \ell \in \mathbb{Z}^2 \). Then, \( M = (M_\ell)_{\ell = 1, \ldots, n} \) satisfies Assumption 1 with \( \Gamma_{t,n} = 4BB_\xi \sum_{i=t+1}^{\infty} \min(i, n)\alpha_i \) and Assumption 2 with \( F(i) = 1_{\{i \leq 4B^2\}} \).

In this result, we do not have to consider the heavy-tail property, because CBSs are bounded processes. The proof of Proposition 2 is included in that of Proposition 3 which provides a more general setting for unbounded processes.
2.2.2. **Application to Unbounded Process.** We apply the CBS class and generate a process that satisfies the assumptions regarding the choice of the coefficient $\Gamma_{\ell,n}$ and the function $F(\cdot)$. Here, the generated process can be unbounded.

**Proposition 3.** We assume that $(X_\ell)_{\ell \in \mathbb{Z}}$ is a CBS. Let $\mathcal{E} = (\varepsilon_\ell)_{\ell \in \mathbb{Z}}$ be a sequence of centered i.i.d. $\mathbb{H}$-valued random variables with $F_\varepsilon(t) := \mathbb{P}(\|\varepsilon_\ell\|^2 \geq t)$, all independent from $(X_\ell)_{\ell \in \mathbb{Z}}$. We define a process $(Y_\ell)_{\ell \in \mathbb{Z}}$ as

$$Y_\ell = X_\ell + \varepsilon_\ell,$$

and $M_\ell = Y_\ell Y_\ell^\top$ for any $\ell \in \mathbb{Z}^2$. Then, $M = (M_\ell)_{\ell = 1,\ldots,n}$ satisfies Assumption 2 with $\Gamma_{\ell,n} = 4BB_\xi \sum_{i=\ell+1}^{\infty} \min(i,n)\alpha_i$ and Assumption 2 with $F(t) = \mathbf{1}_{\{t \leq 4B_\xi^2\}} + F_\varepsilon(t/4)$.

Furthermore, if $\Gamma := 4BB_\xi \sum_{i=2}^{\infty} i\alpha_i < \infty$, then $\max_{1 \leq \ell \leq n} \Gamma_{\ell,n} \leq \Gamma$ holds.

It is important to note that although CBSs are bounded, the generated process $Y$ here can be unbounded depending upon the choice of $\varepsilon_\ell$. To clarify this point, we show certain possible choices for $F_\varepsilon(t/4)$: (i) **bounded noise**: if $\|\varepsilon_\ell\| \leq \lambda$ almost surely, then $F_\varepsilon(t/4) \leq \mathbf{1}_{\{t \leq 4\lambda^2\}}$, (ii) **noise with moments**: if $k, \lambda$ such that $\mathbb{E}(\|\varepsilon_\ell\|^k/\lambda) \leq 1$ then $F_\varepsilon(t/4) \leq 2^k \lambda t^{-k/2}$, and (iii) **noise with exponential moments**: if $k, \lambda > 0$ such that $\mathbb{E}[\exp(\|\varepsilon_\ell\|^k/\lambda)] \leq 1$, then $F_\varepsilon(t/4) \leq \exp(-t^{k/2}/(2^k \lambda))$.

2.2.3. **Application to Chain with Infinite Memory.** We introduce a chain with infinite memory, which turns out to be a special case of the CBS process.

**Example 2 (Chain with Infinite Memory, CIM).** Let $\Xi = (\xi_\ell)_{\ell \in \mathbb{Z}}$ be a sequence of bounded i.i.d. $\mathbb{H}$-valued random variables: $\|\xi_\ell\| \leq B_\xi$ almost surely. Let $D : \mathbb{H}^\infty \rightarrow \mathbb{H}$ with $D(0,0,\ldots) = 0$. Assume that, for any $(a_0, b_0, a_1, b_1, a_2, b_2, \ldots) \in \mathbb{H}^\infty$ we have

$$\|D(a_0, a_1, a_2, \ldots) - D(b_0, b_1, b_2, \ldots)\| \leq \sum_{\ell=0}^{\infty} \beta_\ell \|a_\ell - b_\ell\| \quad \text{and} \quad \mathcal{B} := \sum_{\ell=1}^{\infty} \beta_\ell < 1.$$

Then, there is a stationary solution $(X_\ell)_{\ell \in \mathbb{Z}}$ to the equation (Doukhan and Wintenberger, 2008):

$$X_\ell = D(\xi_\ell, X_{\ell-1}, X_{\ell-2}, X_{\ell-3}, \ldots).$$

The process $(X_\ell)_{\ell \in \mathbb{Z}}$ is called a chain with infinite memory (CIM).
We now discuss the connection between CBSs and CIMs. Using Proposition 4.1 of [Alquier and Wintenberger (2012)], a CIM can be rewritten as a CBS as

\[ X_\ell = C(\xi_\ell, \xi_{\ell-1}, \xi_{\ell-2}, \ldots) \text{ with } \alpha_\ell = \beta_0 B^{\ell-1}. \]

**Remark 1.** We briefly discuss our framework and its relation to the vector auto-regression (VAR) process. Since the VAR with bounded noise terms is obviously CIM, Assumption 1 and 2 are satisfied by a process with heavy-tailed noise added to the VAR, by using Proposition 3. On the other hand, when the noise in the VAR has a heavy-tail, like \( Y_\ell = A Y_{\ell-1} + \varepsilon_\ell \) with \( A \in \mathbb{R}^p \otimes \mathbb{R}^p \) and the heavy-tailed noise \( \varepsilon_\ell \), we need to apply another technique to handle it. That is, we have to approximate the VAR by a finite-order moving-average (MA) process and show it satisfies Assumption 1 and 2. However, since this approximation error vanishes as \( n \to \infty \), we can apply our result without any problems.

3. **Main Results**

We present our upper bound as the main result in stages. First, we consider the case where \( M_\ell \) is a \( p \times p \) matrix with the bounded property, and then we extend it to the unbounded and heavy-tailed cases. Then, we consider the case where \( M_\ell \) is an operator between infinite-dimensional spaces.

3.1. **Result on \( p \)-Dimensional Matrix.**

3.1.1. **Bounded Case.** We first consider the case \( \mathbb{H} = \mathbb{R}^p \) with \( p \in \mathbb{N} \) and the case in which the matrices \( \|M_\ell\| \) are bounded for all \( \ell = 1, \ldots, n \). Obviously, \( M_\ell \) is not heavy-tailed in this case; hence, we handle the dependence property of process \( M \).

The derivation of this result starts with the variational inequality: with a random parameter \( \theta \) following a measure \( \mu \) and a random variable \( X \), it holds that with probability at least \( 1 - e^{-t} \):

\[ \mathbb{E}_\rho[h(X, \theta)] \leq \mathbb{E}_\rho[\log \mathbb{E}_X[\exp(h(X, \theta))]] + KL(\rho\|\mu) + t. \]

for all probability measures \( \rho \ll \mu \) and measurable function \( h \). This result follows [Catoni and Giulini (2017)] and [Zhivotovskiy (2021)]. In our setting
with the dependent property, the idea is to apply the inequality by Rio (2000) for dependent matrices and obtain

$$\mathbb{E} \left[ \exp (\lambda h(M) - \lambda \mathbb{E}[h(M)]) \right] \leq \exp \left( \frac{\lambda^2 L^2 \sum_{\ell=1}^{n} (2\kappa + \Gamma_{\ell,n})^2}{8n^2} \right).$$

Then, we obtain the below result. We state the concentration bound for the estimation of $\Sigma$ using the empirical mean of $M$.

**Theorem 4.** Assume that $M$ is a sequence of positive semi-definite symmetric random $p \times p$ matrices with $\mathbb{E}[M_{\ell}] = \Sigma$ and $\|M_{\ell}\| \leq \kappa^2$ almost surely for some $\kappa > 0$ and all $\ell = 1, \ldots, n$. Let us assume that Assumption 1 is satisfied. Then for all $t > 0$, it holds that

$$\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell} - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| \left(2\kappa^2 + \Gamma_n\right) \sqrt{\frac{4r(\Sigma) + t}{n}},$$

with probability at least $1 - e^{-t}$.

This upper bound has the following implications: First, it is a dimension-free bound in which $p$ does not appear, and this statistical dimension is described by the effective rank $r(\Sigma)$ except for the effect of $t$. This is identical to the statistical dimension of the independent case of Koltchinskii and Lounici (2017) and others. Second, the effect of this dependence appears as $\Gamma_n$ in the coefficient part $(2\kappa^2 + \Gamma_n)$ of the upper bound. If $M$ is independent, we have $\Gamma_n = 0$. Then, our upper bound corresponds to the upper bound by Koltchinskii and Lounici (2017) up to the universal constants.

3.1.2. **Heavy-Tailed Case.** We now extend Theorem 4 to unbounded heavy-tailed matrices $M_\ell$ and provide an updated upper bound.

The general idea is to apply Theorem 4 to a sequence of transformed matrices $\{f(M_1), \ldots, f(M_n)\}$ where $f : S \rightarrow S$ is a bounded transform, such that $\sup_{M \in S} \|f(M)\| \leq \tau$. This application yields a bound on $\frac{1}{n} \sum_{\ell=1}^{n} f(M_{\ell}) - \mathbb{E}[f(M_{\ell})]$. Then, we handle the effect of $f$, that is, $\frac{1}{n} \sum_{\ell=1}^{n} f(M_{\ell}) - \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell}$ and $\|\mathbb{E}[f(M_{\ell})] - \Sigma\|$, and develop an upper bound on $\frac{1}{n} \sum_{\ell=1}^{n} M_{\ell} - \Sigma$. This results in the introduction of an additional term depending on $\tau$ in the upper bound. This technique leads to the following results:
Corollary 5. Assume that \( M \) is a sequence of positive semi-definite symmetric random \( p \times p \) matrices with \( \mathbb{E} [M_\ell] = \Sigma \), which satisfies Assumptions\(^7\) and\(^2\). For any \( \tau > 0 \) and for all \( t > 0 \), it holds that
\[
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r (\Sigma + t)}{n}} + V \sqrt{F(\tau)}.
\]
with probability at least \( 1 - e^{-t} - nF(\tau) \).

The effect of the heavy-tailed property appears additively in the second term of the derived upper bound. The tightness of the bound will, of course, depend on how we deviate from the boundedness assumption, that is, on the function \( F \). We provide the following examples:

**Bounded Case**: Suppose that \( \|M_\ell\| \leq \kappa \) almost surely for some \( \kappa > 0 \), then Assumption\(^2\) is satisfied with \( F(\tau) = 0 \) for \( \tau \geq \kappa \). Thus, we can take \( \tau = \kappa \) and recover exactly Theorem\(^4\).

**Exponential-Tail Case**: Suppose that \( F(\cdot) \) has an exponential decay, that is, \( F(t) = \exp(-at^b) \) with some \( a, b > 0 \). Then, Corollary\(^5\) states that with probability at least \( 1 - e^{-t} - n \exp(-at^b) \),
\[
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r (\Sigma + t)}{n}} + V \exp \left( -\frac{a}{2} \tau^b \right).
\]

We select \( \tau \geq \left( \frac{\log n + t}{a} \right)^{1/b} \) which implies that \( nF(\tau) \leq \exp(-t) \), and we set \( t = \log(\delta^{-1}) \). Subsequently, for every \( \delta \in (0, 1) \) with probability at least \( 1 - \delta \), we have that
\[
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| \left(2 \left( \frac{\log n + \log(\delta^{-1})}{a} \right)^{1/b} + \Gamma_n \right) \sqrt{\frac{4r (\Sigma + \log(\delta^{-1}))}{n}} + V \sqrt{\frac{\log(\delta^{-1})}{n}}.
\]

In this upper bound, the effect of the heavy-tail appears in the second term \( V \sqrt{\log(\delta^{-1})}/n \), which is relatively less significant. In contrast, the main difference is that the constant part of the first term \( 2(\frac{\log n + \log(\delta^{-1})}{a})^{1/b} + \Gamma_n \) increases in \((\log n)^{1/b}\). Except for the logarithmic increase in \( n \), the
Here, we chose the bound in Theorem 4. The effects of the statistical dimension and the dependent property are identical to the bound in Theorem 4.

**Polynomial-Tail Case**: Suppose that $F(\cdot)$ has a polynomial decay $F(\tau) = a\tau^{-b}$ with $a > 0$ and $b > 2$, then the bound is, with probability at least $1 - e^{-t} - na\tau^{-b}$,

$$
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell} - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r(\Sigma) + t}{n}} + V\sqrt{a\tau^{-b}}.
$$

Here, we chose $\tau$ such that $na\tau^{-b} = \exp(-t)$ to make $\tau = (an)^{1/b} \exp(t/b)$, and also set $t = \log(\delta^{-1})$. Then, for any $\delta \in (0, 1)$, we obtain that with probability at least $1 - \delta$,

$$
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell} - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| (2a^{1/b} + \Gamma_n) \sqrt{\frac{4r(\Sigma) + \log(\delta^{-1})}{n}} + V\sqrt{\frac{\delta}{n}}.
$$

Given that $b > 2$, the second term $V\sqrt{\delta/n}$ decreases in $n$ toward 0, but its convergence rate is slower than that in the previous cases, depending on the selection of $\delta$. The statistical properties of the first term remain mostly unchanged, except for the change in the constant part.

### 3.2. Result on Infinite-Dimensional Operator

Here, we consider the case of an infinite-dimensional separable Hilbert space $\mathbb{H}$, which has also been actively studied (Koltchinskii and Lounici, 2017; Giulini, 2018). As in Giulini (2018), we extend our result for the $p$-dimensional setting to the case.

The approach here is to find the finite-dimensional approximation of the spectral norm of operators on an orthonormal basis. Let $(e_j)_{j \in \mathbb{N}}$ be the orthonormal basis of $\mathbb{H}$ and $\mathbb{H}_k := \text{span} \{e_1, \ldots, e_k\}$. Let $(M_{\ell}^{(j_1,j_2)})_{j_1,j_2=1}^k$ be a sequence of real-valued random variables such that $M_{\ell}^{(j_1,j_2)} := \langle M_{\ell} e_{j_1}, e_{j_2} \rangle$. We see that

$$
\sup_{u_k \in \mathbb{H}_k, \|u_k\|=1} \left| \left( \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell} - \Sigma \right) u_k, u_k \right| = \sup_{\sum_{j=1}^{k} (u_k^{(j)})^2 = 1} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j_1=1}^{k} \sum_{j_2=1}^{k} u_k^{(j_1)} u_k^{(j_2)} (M_{\ell}^{(j_1,j_2)} - \mathbb{E} [M_{\ell}^{(j_1,j_2)}]) \right|.
$$
Then, the right-hand side is clearly a spectral norm of the difference between the sampled matrix and the population one, to which Theorem 4 and Corollary 5 are applicable. Based on this approach of considering a limit $k \to \infty$, we obtain the following result:

**Theorem 6.** Assume that $M$ is a sequence of positive semi-definite symmetric $\mathcal{H} \otimes \mathcal{H}$-valued random operators with $\mathbb{E} [M] = \Sigma$, and also satisfies Assumptions 1 and 2. For any $\tau > 0$ and for all $\ell > 0$, it holds that

$$\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r(\Sigma) + t}{n}} + V \sqrt{F(\tau)}.$$

with probability at least $1 - e^{-t} - nF(\tau)$.

This result shows that the obtained upper bound remains the same, even for infinite dimensions. Note that it is impossible to consider an infinite-dimensional case from the beginning without considering the matrix case. This is because our proof by variational equalities depends on a density function of $p$-dimensional Gaussian vector; therefore, we cannot directly handle this proof in the infinite-dimensional case.

4. **Applications**

4.1. **Covariance Operator Estimation.** We consider the problem of covariance operator estimation using dependent samples with heavy tails. Suppose there exists a CBS $(X_\ell)_{\ell \in \mathbb{N}}$ and a sequence of $\mathcal{H}$-valued random variables $(Y_\ell)_{\ell \in \mathbb{Z}}$, which is a strongly stationary process that follows

$$Y_\ell = X_\ell + \varepsilon_\ell,$$

as the process in Proposition 3. In addition, suppose that $\mathbb{E}[X_1] = 0$ and its covariance operator is $\Sigma \in \mathcal{S}$; that is, $\Sigma$ is defined as $\Sigma u = \mathbb{E}[\langle Y_1, u \rangle Y_1]$ for any $u \in \mathcal{H}$.

Suppose we have $n$ observations $Y = (Y_\ell)_{\ell=1,\ldots,n}$ that follow the process $(Y_\ell)_{\ell \in \mathbb{Z}}$. Then, we define an empirical covariance operator:

$$M_\ell u := \langle Y_\ell, u \rangle Y_\ell,$$
for any $u \in \mathbb{H}$. Using this notion, we obtain $n$ operators $M$ from $Y$ and then obtain the empirical covariance operator as

$$\hat{\Sigma} := \frac{1}{n} \sum_{\ell=1}^{n} M_{\ell}. \quad (2)$$

We connect this setup to our result in Corollary 5 and obtain the following result without proof.

**Proposition 7.** Suppose that the sequence $M$ satisfies Assumptions [7] and [2]. Consider the empirical covariance operator defined in (2). Then, for any $\tau > 0$ and $t > 0$, the following inequality holds with probability at least $1 - e^{-t} - nF(\tau)$:

$$\|\hat{\Sigma} - \Sigma\| \leq 2\sqrt{2} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r(\Sigma) + t}{n}} + V\sqrt{F(\tau)},$$

where $F(\tau) = 1_{\{\tau \leq 4B^2\}} + F_\varepsilon(\tau/4)$.

Examples of specific processes, $(Y_\ell)_{\ell \in \mathbb{Z}}$ include a CBS and its extension to the unbounded processes given in Section 2.2.

### 4.2. Lagged Covariance Matrix Estimation

We consider an estimation problem for a lagged covariance matrix, which is also called a cross-covariance matrix. Consider the same process $(Y_\ell)_{\ell \in \mathbb{Z}}$ as in Section 4.1. The objective here is to estimate

$$\Sigma_1 := \mathbb{E}[Y_\ell Y^T_{\ell+1}],$$

from $n$ observations $Y = (Y_\ell)_{\ell=1,\ldots,n}$. This problem is obviously extended to $\Sigma_h := \mathbb{E}[Y_\ell Y^T_{\ell+h}]$ for $h \geq 2$. Note that $\Sigma_1$ is not symmetric; hence, our main results cannot be directly applied to a naive estimator, $\hat{\Sigma}_1 := (n - 1)^{-1} \sum_{\ell=1}^{n-1} Y_\ell Y^T_{\ell+1}$. We also consider a covariance matrix $\Sigma = \mathbb{E}[Y_\ell Y^T_\ell]$, which can be regarded as $\Sigma_0$, and its empirical estimator $\hat{\Sigma} := (n - 1)^{-1} \sum_{\ell=1}^{n-1} Y_\ell Y^T_\ell$.

To estimate $\Sigma_1$, we define an augmented process and estimator for the covariance matrix of the process. We define a Hilbert space $\mathbb{H}_2^2$ equipped with the scalar product $\langle (y_1, y_2), (y_1', y_2') \rangle = \langle y_1, y_1' \rangle + \langle y_2, y_2' \rangle$ for $(y_1, y_2), (y_1', y_2') \in \mathbb{H}^2$. Let $\tilde{Y}_\ell = (Y_\ell, Y_{\ell+1})^T$ be an $\mathbb{H}_2^2$-valued augmented process generated by $(Y_\ell)_{\ell \in \mathbb{Z}}$, whose covariance function is written as

$$\Sigma_{0:1} := \mathbb{E}\left[\tilde{Y}_\ell \tilde{Y}_\ell^T\right] = \begin{pmatrix} \mathbb{E}[Y_\ell Y^T_\ell] & \mathbb{E}[Y_\ell Y^T_{\ell+1}] \\ \mathbb{E}[Y_{\ell+1} Y^T_\ell] & \mathbb{E}[Y_{\ell+1} Y^T_{\ell+1}] \end{pmatrix} = \begin{pmatrix} \Sigma_0 & \Sigma_1 \\ \Sigma_1^T & \Sigma_0 \end{pmatrix}. \quad (3)$$
In the following, we estimate $\Sigma_{0:1}$ and apply the result to the estimation for $\Sigma_1$.

Using observations $Y$, we generate $\tilde{Y}_1, \ldots, \tilde{Y}_{n-1}$ and its sample-wise product matrices $M_1, \ldots, M_{n-1}$ as

$$M_\ell := \tilde{Y}_\ell \tilde{Y}_\ell^\top = \begin{pmatrix} Y_\ell Y_\ell^\top & Y_\ell Y_{\ell+1}^\top \\ Y_{\ell+1} Y_\ell^\top & Y_{\ell+1} Y_{\ell+1}^\top \end{pmatrix}.$$ 

We then construct an empirical estimator

$$\hat{\Sigma}_{0:1} := \frac{1}{n-1} \sum_{\ell=1}^{n-1} M_\ell = \begin{pmatrix} \hat{\Sigma} & \hat{\Sigma}_1 \\ \hat{\Sigma}_1^\top & \hat{\Sigma} \end{pmatrix}. \quad (4)$$

We show a concentration inequality for $\hat{\Sigma}_{0:1}$ and then additionally show the convergence of $\hat{\Sigma}$ and $\hat{\Sigma}_1$.

**Proposition 8.** Assume that $Y$ is a CBS, as in Example 7. Consider matrix (3) and estimator (4). Then, for any $\tau > 0$ and $t > 0$, the following inequality holds with probability at least $1 - e^{-t} - nF(\tau)$:

$$\|\hat{\Sigma}_{0:1} - \Sigma_{0:1}\| \leq 4\sqrt{2} (\|\Sigma_{0:1}\|) \left(32B^2 + \Gamma_n\right) \sqrt{\frac{4r(\Sigma_{0:1}) + t}{n-1}} + V\sqrt{F(\tau)},$$

where $F(\tau) = 41_{\{\tau \leq 4B^2\}} + 4F_\varepsilon(t/2)$. Furthermore, with the same probability, we obtain

$$\max\{\|\hat{\Sigma} - \Sigma\|, \|\hat{\Sigma}_1 - \Sigma_1\|\} \leq 4\sqrt{2} (\|\Sigma_1\| + \|\Sigma\|) \left(32B^2 + \Gamma_n\right) \sqrt{\frac{4r(\Sigma_{0:1}) + t}{n-1}} + V\sqrt{F(\tau)}.$$

The first statement follows Theorem 4 for estimator $\hat{\Sigma}_{0:1}$ for a symmetric matrix $\Sigma_{0:1}$. The second statement simply follows the first statement and the facts $\|\Sigma_{0:1}\| \leq \|\Sigma_1\| + \|\Sigma\|$. By using the relation $\text{tr}(\Sigma_{0:1}) = 2\text{tr}(\Sigma) = 2\|\Sigma\|_r(\Sigma)$, we can update the upper bound.

### 4.3. Linear Hidden Markov Model

We consider a linear hidden Markov model (HMM) and study its estimator. Specifically, we consider a HMM model with a lag order 1 and set $\mathbb{H} = \mathbb{R}^p$. Suppose that we observe a sequence of $p$-dimensional random vectors $Y = (Y_\ell)_{\ell=0}^n$ which follows the following equations for $\ell \in \mathbb{Z}$:

$$Y_\ell = X_\ell + \varepsilon_\ell,$$  \quad (5)
where $A \in \mathbb{R}^{p \times p}$ is an unknown parameter matrix such that $\|A\| \in (0, 1)$, and $(X_\ell)_{\ell \in \mathbb{Z}}$ is a latent process such that $\|X_\ell\| \leq B$ almost surely. Here, $(\varepsilon_\ell)_{\ell \in \mathbb{Z}}$ is a sequence of i.i.d. $p$-dimensional noise variable with zero mean and finite variance, and $\xi_\ell$ is a sequence of i.i.d. $p$-dimensional bounded noise variable with zero mean such that $\|\xi_\ell\| \leq B_\xi$ almost surely. Under condition $\|A\| \in (0, 1)$ and the bounded property of $\xi_\ell$, the upper bound $B$ is guaranteed to be finite. For brevity, we assume that $\mathbb{E}[\varepsilon_\ell \varepsilon_\ell^T] = \mathbb{E}[\xi_\ell \xi_\ell^T] = I$.

We also define a covariance matrix $\Sigma := \mathbb{E}[Y_\ell Y_\ell^T]$ and lagged covariance matrix $\Sigma_1 := \mathbb{E}[Y_{\ell+1} Y_\ell^T]$. The goal of this problem is to estimate the unknown parameter matrix $A$.

We consider a convenient form of the HMM model. Let us define noise matrices $E = (\varepsilon_0, \ldots, \varepsilon_{n-1}) \in \mathbb{R}^{p \times n}$, $E_+ = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^{p \times n}$, and $Z = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{p \times n}$ and also define matrices $Y = (Y_0, \ldots, Y_{n-1}) \in \mathbb{R}^{p \times n}$ and $Y_+ = (Y_1, \ldots, Y_n) \in \mathbb{R}^{p \times n}$. Then, we rewrite models (5) and (6) as

$$(Y_+ - E_+) = A(Y - E) + Z.$$

Multiplying $(Y - U)^T$ on both sides from the right and taking an expectation yields

$$A = (\mathbb{E}[(Y_+ - E_+)(Y - E)^T] - \mathbb{E}[Z(Y - E)^T])\mathbb{E}[(Y - E)(Y - E)^T]^{-1}$$
$$= \Sigma_1(\Sigma + I)^{-1}.$$

Here, we utilize the independent properties of the noise, and $\mathbb{E}[\varepsilon_\ell \varepsilon_\ell^T] = I$.

We now consider the estimator for $A$. Using empirical estimators $\hat{\Sigma} := YY^T/n = n^{-1} \sum_{\ell=0}^{n-1} Y_\ell Y_\ell^T$ and $\hat{\Sigma}_1 := Y_+ Y^T/n = n^{-1} \sum_{\ell=0}^{n-1} Y_{\ell+1} Y_\ell^T$, we define the following estimator:

$$\hat{A} := \hat{\Sigma}_1(\hat{\Sigma} + I)^{-1}. \quad (7)$$

Then, we obtain the following result:

**Proposition 9.** Consider the HMM model (5)-(6) and estimator (7) for the parameters in the model. Then, for any $t > 2B^2$ with probability at least $1 - \exp(-t) - n F_\xi(t/2)$, the following inequality holds:

$$\|\hat{A} - A\|$$
\begin{align*}
\leq 4\sqrt{2} \left( \|\Sigma\| + \|\Sigma_1\| \right) \left( \|\Sigma_1\| + 1 \right) \left( 32B^2 + \Gamma_n \right) \frac{\sqrt{4r(\Sigma) + t}}{n} + 2V(1 + \|\Sigma_1\|) \sqrt{F_p(t/2)}.
\end{align*}

This result is obtained by bounding the estimation error \( \|\widehat{A} - A\| \) with the estimation errors of the covariance matrix \( \Sigma \) and the lagged covariance matrix \( \Sigma_1 \), as described in Proposition 4.2. Note that it is possible to extend the number of lags in this HMM model to larger than 1.

4.4. Overparameterized Linear Regression. Herein, we study a linear regression problem with dependent and heavy-tailed covariates following the overparameterization framework developed by Bartlett et al. (2020).

Let \((X_\ell)_{\ell \in \mathbb{Z}}\) be a CBS as a \( \mathbb{H} \)-valued latent process and \((Y_\ell)_{\ell \in \mathbb{Z}}\) be a generated process as a \( \mathbb{H} \)-valued covariate such that

\[ Y_\ell = X_\ell + \varepsilon_\ell, \tag{8} \]

where \( \varepsilon_\ell \) is an i.i.d. \( \mathbb{H} \)-valued noise variable with a mean value of zero. In addition, we define \( \theta^* \in \mathbb{H} \) as a true unknown parameter and a covariance operator \( \Sigma = \mathbb{E}[Y_\ell Y_\ell^\top] \). For \( \ell \in \mathbb{Z} \), we consider an \( \mathbb{R} \)-valued random variable \( Z_\ell \) to be a response that follows the below model:

\[ Z_\ell = \langle \theta^*, Y_\ell \rangle + U_\ell, \tag{9} \]

where \( U_\ell \) is an \( \mathbb{R} \)-valued independent random variable with a mean of zero and a variance of \( \sigma^2 > 0 \).

The goal of the regression problem is to estimate \( \theta^* \) from the observations \((Z_1, Y_1), \ldots, (Z_n, Y_n)\). We introduce a design matrix and operator as \( Z = (Z_1, \ldots, Z_n)^\top \in \mathbb{R}^n \) and \( Y : \mathbb{H} \to \mathbb{R}^n \) such that \( Y^\top \theta = (Y_1^\top \theta, \ldots, Y_n^\top \theta)^\top \in \mathbb{R}^n \) for \( \theta \in \mathbb{H} \). Define \( \Pi_Y := Y^\top(YY^\top)^{-1}Y \) and \( \hat{\Sigma} = Y^\top Y/n \).

To estimate \( \theta^* \), we define a minimum norm estimator as:

\[ \hat{\theta} = \arg\min_{\theta \in \mathbb{H}} \{ \|\theta\|^2 : Y^\top Y \theta = Y^\top Z \} = Y^\top (YY^\top)^{\dagger}Z, \tag{10} \]

where \( \dagger \) denotes the pseudo-inverse of operators. The excess risk of estimator \( \hat{\theta} \) is measured using

\[ R(\hat{\theta}) := \mathbb{E}(Z_*Y_*)[(Z_* - \langle Y_*, \hat{\theta} \rangle)^2 - (Z_* - \langle Y_*, \theta^* \rangle)^2], \tag{11} \]
where \((Z_s, Y_s)\) is an i.i.d. copy of \((Z_1, Y_1)\) from the regression model (9) and \(E_{(Z_s, Y_s)}[-]\) is the expectation with respect to \((Z_s, Y_s)\).

We present a technical assumption that specializes in the overparameterization setting. Let \(\Pi_{\Sigma}^\perp\) be a projection operator onto a linear space spanned by vectors orthogonal to any eigenvector of \(\Sigma\).

**Assumption 3.** \(\text{dim}(\Pi_{\Sigma}^\perp(Y)) > n\) holds almost surely.

This assumption is identical to Assumption 1 in Bartlett et al. (2020) and is intended to deal with cases where there are no degeneracies, such as perfect dependence among the variables.

With this setting, we bound the risk of the estimator for the overparameterized linear regression model.

**Proposition 10.** Consider the linear regression model (9) with the process (8) by a CBS \((X_t)_{t \in \mathbb{Z}}\). Suppose that Assumption 3 holds. Consider the estimator (10) and its excess risk (11). Then, for any \(t \in (B^2, n/c)\) with some \(c > 1\), at least \(1 - \exp(-t) - nF_\varepsilon(t/4)\), we have

\[
R(\hat{\theta}) \leq 2\sqrt{2c} ||\theta^*||^2 \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r(\Sigma) + t}{n}} + V\sqrt{F_\varepsilon(t/4)} + c\sigma^2 \text{tr}(C),
\]

where \(C = (YY^T)^{-1}Y\Sigma Y^T (YY^T)^{-1}\).

This result indicates that we can bound the bias term of the risk of the overparameterized linear regression estimator, even in the dependent and heavy-tailed setting. Note that the last term \(c\sigma^2 \text{tr}(C)\) represents the variance of the risk, which converges to zero by removing correlations and controlling for them using different techniques. This is outside our interest, but see Lemma 11 in Bartlett et al. (2020) for further details.

5. **Proofs for Main Results in Section 3**

5.1. **Outline.** We first state two lemmas at the core of our proofs in Section 5.2. Lemma 11 appears in many forms in the proofs of the PAC-Bayes bounds (Catoni, 2007; Alquier, 2021). For convenience, we use here the version stated in (Catoni and Giulini, 2017; Zhivotovskiy, 2021). Lemma 12 is Rio’s version of Hoeffding’s inequality (Rio, 2000) for weakly dependent random variables:
Then, we prove Theorem 4 in Section 5.3. To do so, we essentially follow the techniques developed in (Catoni and Giulini, 2017; Zhivotovskiy, 2021). However, both these studies rely on exponential inequalities for independent random variables. Instead, we use Rio’s inequality, which requires the boundedness assumption.

Then, in Section 5.4, we introduce a truncation function that transforms unbounded matrices into bounded ones. We thus apply Theorem 4 to the truncated matrices. Then, we control the effect of the truncation function to prove Corollary 5.

5.2. Preliminary results.

**Lemma 11** (Catoni and Giulini (2017)). Assume that \( X \) is a random variable defined in a measurable space \((X, \mathcal{A})\), and \((\Theta, \mathcal{F})\) is a measurable parameter space. Let \( \mu \) be a random measure on \((\Theta, \mathcal{F})\) and \( h : X \times \Theta \rightarrow \mathbb{R} \) be a real-valued \( \mathcal{A} \otimes \mathcal{F} / \mathcal{B}(\mathbb{R}) \)-measurable function such that \( \mathbb{E}_X[\exp(h(X, \theta))] < \infty \) for \( \mu \)-almost all \( \theta \). It holds that with probability at least \( 1 - e^{-t} \), for all the probability measures \( \rho \ll \mu \) simultaneously,

\[
\mathbb{E}_{\rho}[h(X, \theta)] \leq \mathbb{E}_{\rho}[\log \mathbb{E}_X[\exp(h(X, \theta))] + \text{KL}(\rho \| \mu) + t.
\]

**Proof.** The proof is merely a descendant of the duality relationship.

\[
\mathbb{E}_X\left[\exp\left(\sup_{\rho \ll \mu} \left(\mathbb{E}_{\rho}[h(X, \theta)] - \log \mathbb{E}_X[\exp(h(X, \theta))]\right)\right) - \text{KL}(\rho \| \mu)\right]
= \mathbb{E}_X\mathbb{E}_\mu\left[\exp(h(X, \theta))\right] - \mathbb{E}_X[\exp(h(X, \theta))]
= \mathbb{E}_\mu\mathbb{E}_X\left[\frac{\exp(h(X, \theta))}{\mathbb{E}_X[\exp(h(X, \theta))]}\right]
= 1.
\]

We use Tonelli’s theorem to exchange the order of expectations. Then Markov’s inequality leads to the inequality that holds with probability at least \( 1 - e^{-t} \);

\[
\sup_{\rho \ll \mu} \left(\mathbb{E}_{\rho}[h(X, \theta)] - \log \mathbb{E}_X[\exp(h(X, \theta))]\right) - \text{KL}(\rho \| \mu) < t.
\]

This completes the proof. □
Lemma 12 (Hoeffding’ inequality applied to matrices (Rio, 2000)). Let \( \{M_1, \ldots, M_n\} \) be a sequence of positive semi-definite symmetric random matrices with \( \max_{\ell=1,\ldots,n} \|M_{\ell}\| \leq \kappa^2 \) almost surely for some \( \kappa > 0 \). Let us assume that Assumption 1 is satisfied. Then, for any function \( h \in \text{Lip}_n(E, L) \) and for any \( \lambda > 0 \) we have

\[
\mathbb{E} \left[ \exp (\lambda h(M_1, \ldots, M_n)) - \lambda \mathbb{E}[h(M_1, \ldots, M_n)] \right] \\
\leq \exp \left( \frac{\lambda^2 L^2 \sum_{\ell=1}^n (2\kappa + \Gamma_{\ell,n})^2}{8n^2} \right).
\]

5.3. Bounded Case (Theorem 4).

Proof of Theorem 4. The proof consists of truncation of \( \rho = \rho_{u,v} \) given by Zhivotovskiy (2021) and the lemma above obtained using duality.

Step 1: Let us assume that \( \Sigma \) is invertible. Otherwise, we only need to consider a lower-dimensional subspace, and the proof is indifferent to the case with invertible \( \Sigma \). Let \( \mu \) denote a \( 2p \)-dimensional product measure of two \( p \)-dimensional Gaussian measures with a zero mean and covariance \( (2r(\Sigma))^{-1} \Sigma \). We define \( \mathbb{S}^{p-1} \) as a unit ball in \( \mathbb{R}^p \). Let us set \( u, v \in \Sigma^{1/2} \mathbb{S}^{p-1} \) and define \( f_u, f_v \) as probability density functions with respect to the Lebesgue measure such that

\[
f_u(x) = \frac{\exp (-r(\Sigma)(x-u)^T \Sigma^{-1}(x-u)) \mathbf{1}_{\{\|x-u\| \leq \sqrt{\|\Sigma\|}\}}}{\int \exp (-r(\Sigma)(x'-u)^T \Sigma^{-1}(x'-u)) \mathbf{1}_{\{\|x'-u\| \leq \sqrt{\|\Sigma\|}\}} dx'},
\]

\[
f_v(x) = \frac{\exp (-r(\Sigma)(x-v)^T \Sigma^{-1}(x-v)) \mathbf{1}_{\{\|x-v\| \leq \sqrt{\|\Sigma\|}\}}}{\int \exp (-r(\Sigma)(x'-v)^T \Sigma^{-1}(x'-v)) \mathbf{1}_{\{\|x'-v\| \leq \sqrt{\|\Sigma\|}\}} dx'}.
\]

Here, \( \mathbf{1}_{\{\mathcal{E}\}} \) is an indicator function which is 1 if an event \( \mathcal{E} \) is true and 0 otherwise. Assume that the independent random vectors \( \theta, \eta \) have densities \( f_u \) and \( f_v \), then \( \mathbb{E}[(\theta, \eta)] = (u, v) \) by the symmetricity of \( f_u, f_v \), and \( \max \{\|\theta\|, \|\eta\|\} \leq 2\sqrt{\|\Sigma\|} \) almost surely. Let \( \rho_{u,v} \) be a probability measure of \( \theta, \eta \) given by \( \rho_{u,v}(dx, dy) = f_u(x) f_v(y) dx dy, x, y \in \mathbb{R}^d \). In the proof of Theorem 1 of Zhivotovskiy (2021), it is shown that

\[
\text{KL} (\rho_{u,v}||\mu) \leq 2 \log 2 + 2r(\Sigma).
\]
(Step 2) Let \( f(A, \theta, \eta) := \theta^\top A \eta \) for any \( A \in \mathbb{R}^p \otimes \mathbb{R}^p \) and \( \theta, \eta \in \mathbb{R}^p \). Lemma \( \text{[12]} \) with \( h(M_1, \ldots, M_n) = \sum_{\ell=1}^n f(M_\ell, \theta, \eta) \) gives that for any \( \lambda > 0 \),

\[
\mathbb{E}_M \left[ \exp \left( \lambda \sum_{\ell=1}^n f(M_\ell, \theta, \eta) \right) \right] = \mathbb{E}_M \left[ \exp \left( \lambda \sum_{\ell=1}^n \theta^\top M_\ell \eta \right) \right] \leq \exp \left( n \lambda \theta^\top \Sigma \eta + \frac{n^2 \lambda^2 \|\theta\|^2 \|\eta\|^2 (2\kappa^2 + \Gamma_n)^2}{8n} \right),
\]

because for any \( A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{R}^p \otimes \mathbb{R}^p \),

\[
|h(A_1, \ldots, A_n) - h(B_1, \ldots, B_n)| = \left| \theta^\top \left( \sum_{\ell=1}^n (A_\ell - B_\ell) \right) \right| \eta \leq \|\theta\| \|\eta\| \sum_{\ell=1}^n \|A_\ell - B_\ell\|.
\]

It holds that

\[
\frac{1}{n} \mathbb{E}_{\rho_{n,v}} \left[ \log \mathbb{E}_M \left[ \exp \left( \lambda \sum_{\ell=1}^n f(M_\ell, \theta, \eta) \right) \right] \right] \leq \frac{1}{n} \mathbb{E}_{\rho_{n,v}} \left[ n \lambda \theta^\top \Sigma \eta + \frac{n^2 \lambda^2 \|\theta\|^2 \|\eta\|^2 (2\kappa^2 + \Gamma_n)^2}{8n} \right] = \mathbb{E}_{\rho_{n,v}} \left[ \lambda \theta^\top \Sigma \eta + \frac{\lambda^2 \|\theta\|^2 \|\eta\|^2 (2\kappa^2 + \Gamma_n)^2}{8} \right] \leq \lambda u^\top \Sigma v + \frac{\lambda^2 (2\sqrt{\|\Sigma\|^4} (2\kappa^2 + \Gamma_n)^2}{8} = \lambda u^\top \Sigma v + 2\lambda^2 \|\Sigma\|^2 (2\kappa^2 + \Gamma_n)^2.
\]

The last inequality follows the fact \( \max \{\|\theta\|, \|\eta\|\} \leq 2\sqrt{\|\Sigma\|} \). Therefore, from Lemma \( \text{[11]} \) with \( h(M_1, \ldots, M_n, \theta, \eta) = \lambda \sum_{\ell=1}^n f(M_\ell, \theta, \eta) \) and the fact that \( \log 2 \leq r(\Sigma) \) for any \( \Sigma \), we obtain

\[
\frac{1}{n} \sum_{\ell=1}^n \lambda u^\top M_\ell v \leq \lambda u^\top \Sigma v + 2\lambda^2 \|\Sigma\|^2 (2\kappa^2 + \Gamma_n)^2 + \frac{4r(\Sigma) + t}{n},
\]
simultaneously for all \( u, v \) with probability at least \( 1 - e^{-t} \). By choosing
\[
\lambda = \sqrt{\frac{4r(\Sigma) + t}{2n||\Sigma||^2(2\kappa^2 + \Gamma_n)^2}},
\]
we obtain a bound such that
\[
\left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \leq 2\sqrt{2}||\Sigma|| \left(2\kappa^2 + \Gamma_n\right) \sqrt{\frac{4r(\Sigma) + t}{n}}.
\]

This is our claim. \( \square \)

5.4. Heavy-Tailed Case (Corollary 5). We firstly present a truncation function, which is necessary to our robustification strategy for heavy-tailed random matrices.

**Definition 3.** For any \( \tau > 0 \), we define the truncation function \( \psi_\tau: \mathbb{R} \to \mathbb{R} \) as follows:
\[
\psi_\tau(x) = \begin{cases} 
-\tau & \text{if } x < \tau, \\
x & \text{if } |x| \leq \tau, \\
\tau & \text{if } x > \tau.
\end{cases}
\]

There is a standard method to extend a real function \( \mathbb{R} \to \mathbb{R} \) to a function of symmetric matrices \( \mathbb{S} \to \mathbb{S} \), by applying the function to the eigenvalues of the matrix. More precisely, given \( A \in \mathbb{S} \), \( A \) can be written as
\[
A = Q \begin{pmatrix} 
\lambda_1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \lambda_p
\end{pmatrix} Q^T,
\]
for some matrix \( Q \) such that \( QQ^T = I \), where \( (\lambda_1, \ldots, \lambda_p) \) are the eigenvalues of \( A \). We then define \( \psi_\tau(A) \) by
\[
\psi_\tau(A) = Q \begin{pmatrix} 
\psi_\tau(\lambda_1) & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \psi_\tau(\lambda_p)
\end{pmatrix} Q^T.
\]

We can now state the first corollary of Theorem 4.

**Corollary 13.** Assume that \( \{M_1, \ldots, M_n\} \) satisfies Assumption 7. Fix \( \tau > 0 \). Then for all \( t > 0 \), it holds that
\[
\left\| \frac{1}{n} \sum_{\ell=1}^{n} (\psi_\tau(M_\ell) - \mathbb{E}[\psi_\tau(M_\ell)]) \right\| \leq 2\sqrt{2}||\Sigma|| \left(2\tau + \Gamma_n\right) \sqrt{\frac{4r(\Sigma) + t}{n}}.
\]
with probability at least $1 - e^{-t}$.

**Proof of Corollary 13** Because $x \mapsto \psi_\tau(x)$ is 1-Lipschitz, the sequence of matrices $\{\psi_\tau(M_1) - \mathbb{E}[\psi_\tau(M_1)], \ldots, \psi_\tau(M_n) - \mathbb{E}[\psi_\tau(M_1)]\}$ satisfies Assumption 1. As they are bounded by $\tau$ and all have the same expectation (zero), so we can apply Theorem 4 to yield the result. □

As stated in the outline of the proof, we now have to understand what is the difference between the expectation of the truncated matrices, and the expectations of the (non-truncated) matrices themselves.

**Proposition 14.** Fix $\tau > 0$. Under Assumption 2 we have

$$\max_{1 \leq \ell \leq n} \|\mathbb{E}[\psi_\tau(M_\ell)] - \mathbb{E}[M_\ell]\| \leq V \sqrt{F(\tau)}.$$

**Proof of Proposition 14** For any $\ell = 1, \ldots, n$, we have

$$\|\mathbb{E}[\psi_\tau(M_\ell)] - \mathbb{E}[M_\ell]\| \leq \mathbb{E}[\|\psi_\tau(M_\ell) - M_\ell\|]$$

$$= \mathbb{E}[\|M_\ell\| - \tau 1_{\|M_\ell\| > \tau}]$$

$$\leq \mathbb{E}[\|M_\ell\| 1_{\|M_\ell\| > \tau}]$$

$$\leq \sqrt{\mathbb{E}[\|M_\ell\|^2]} \mathbb{P}(\|M_\ell\| > \tau)$$

$$\leq V \sqrt{F(\tau)}.$$

□

**Corollary 15.** Assume that $\{M_1, \ldots, M_n\}$ satisfies Assumptions 1 and 2 and $\mathbb{E}[M_\ell] = \Sigma$. Fix $\tau > 0$. For all $t > 0$, it holds that

$$\left\| \frac{1}{n} \sum_{\ell=1}^n \psi_\tau(M_\ell) - \Sigma \right\| \leq 2\sqrt{\Sigma} \|\Sigma\| (2\tau + \Gamma_n) \sqrt{\frac{4r(\Sigma) + t}{n}} + V \sqrt{F(\tau)}$$

with probability at least $1 - e^{-t}$.

**Proof of Corollary 15** First, we decompose the norm as

$$\left\| \frac{1}{n} \sum_{\ell=1}^n \psi_\tau(M_\ell) - \Sigma \right\|$$

$$\leq \left\| \frac{1}{n} \sum_{\ell=1}^n (\psi_\tau(M_\ell) - \mathbb{E}[\psi_\tau(M_\ell)]) \right\| + \left\| \frac{1}{n} \sum_{\ell=1}^n (\mathbb{E}[\psi_\tau(M_\ell)] - \mathbb{E}[M_\ell]) \right\|$$
can upper bound the first term with probability 1 − \(E\) and where we use the triangle inequality in the first line, and Jensen’s inequality and \(E[M_\ell] = \Sigma\) in the second line. As Assumption [1] is satisfied, we can upper bound the first term with probability 1 − \(e^{-t}\) by Corollary [13]. Because Assumption [2] is also satisfied, we can bound the second term, Proposition [14].

Note that Corollary [15] already provides an estimation result for \(\Sigma\) when matrices \(M_\ell\) are unbounded. However, in contrast to Corollary [5], not only does the bound depend on \(\tau\) but the estimator \(\frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell)\) does as well. A mistake in the choice of \(\tau\) can lead to poor estimation in practice.

To control the distance between this estimator \(\frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell)\), and the standard estimator \(\frac{1}{n} \sum_{\ell=1}^{\ell} M_\ell\), we prove the following proposition.

**Proposition 16.** Under Assumption [2] we have

\[
\mathbb{P}\left( \left\| 1 \frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell) - 1 \frac{1}{n} \sum_{\ell=1}^{\ell} M_\ell \right\| \geq 0 \right) \leq nF(\tau).
\]

**Proof of Proposition [16]** We have

\[
\mathbb{P}\left( \left\| 1 \frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell) - 1 \frac{1}{n} \sum_{\ell=1}^{\ell} M_\ell \right\| \geq 0 \right)
\leq \mathbb{P}\left( 1 \frac{1}{n} \sum_{\ell=1}^{\ell} \|\psi_\tau(M_\ell) - M_\ell\| \geq 0 \right)
\leq \mathbb{P}\left( \exists \ell : \|\psi_\tau(M_\ell) - M_\ell\| \geq 0 \right)
\leq n\mathbb{P}\left( \|\psi_\tau(M_\ell) - M_\ell\| \geq 0 \right)
\leq nF(\tau).
\]

We are now in a position to prove Corollary [5].

**Proof of Corollary [5]** Using the triangle inequality,

\[
\left\| 1 \frac{1}{n} \sum_{\ell=1}^{\ell} M_\ell - \Sigma \right\| \leq \left\| 1 \frac{1}{n} \sum_{\ell=1}^{\ell} M_\ell - 1 \frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell) \right\| + \left\| 1 \frac{1}{n} \sum_{\ell=1}^{\ell} \psi_\tau(M_\ell) - \Sigma \right\|.
\]
We remind that the assumptions of Corollary [5] include: \( \{M_1, \ldots, M_n\} \) satisfy Assumptions [1] and [2] and \( \mathbb{E}[M_\ell] = \Sigma \), which allows us to use Corollary [15] to upper bound the second term with probability \( 1 - e^{-t} \). This allows for the use of Proposition [16] to prove that the first term will be null with probability at least \( 1 - nF(\tau) \). \( \square \)

5.5. Infinite-Dimensional Case (Theorem [6]).

Proof of Theorem [6]. For a sequence of \( \mathbb{H} \otimes \mathbb{H} \)-valued positive semi-definite symmetric random operators \( \{M_1, \ldots, M_n\} \) with \( \mathbb{E}[M_\ell] = \Sigma \) and \( \max_{1 \leq \ell \leq n} \|M_\ell\| \leq \kappa^2 \) almost surely for some \( \kappa > 0 \) satisfying Assumption [1],

\[
P \left( \sup_{u_k \in \mathbb{H}_k, \|u_k\| = 1} \left( \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right) u_k, u_k \right) \geq 2 \sqrt{2} \|\Sigma\| \left( 2 \kappa^2 + \Gamma_n \right) \sqrt{\frac{4r(\Sigma) + t}{n}} \leq e^{-t},
\]

because for \( \Sigma_k \) such that \( \Sigma_k^{(j_1,j_2)} = \mathbb{E}[M_\ell^{(j_1,j_2)}] \) and \( \Sigma := \mathbb{E}[M_\ell], \|\Sigma_k\| \leq \|\Sigma\| \) and \( \text{tr} (\Sigma_k) \leq \text{tr} (\Sigma) \), and \( \Gamma_n \) is also uniform for each, as we can see from the proof. Note that for any \( c \geq 0 \) and \( k \in \mathbb{N} \),

\[
\left\{ \sup_{u_k \in \mathbb{H}_k, \|u_k\| = 1} \left( \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right) u_k, u_k \right\} \geq c
\]

\[
\subseteq \left\{ \sup_{u_{k+1} \in \mathbb{H}_{k+1}, \|u_{k+1}\| = 1} \left( \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right) u_{k+1}, u_{k+1} \right\} \geq c
\]

and

\[
\lim_{k \to \infty} \left\{ \sup_{u_k \in \mathbb{H}_k, \|u_k\| = 1} \left( \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right) u_k, u_k \right\} \geq c = \left\{ \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \geq c \right\}.
\]

The continuity of \( P \) leads to

\[
P \left( \left\| \frac{1}{n} \sum_{\ell=1}^{n} M_\ell - \Sigma \right\| \geq 2 \sqrt{2} \|\Sigma\| \left( 2 \kappa^2 + \Gamma_n \right) \sqrt{\frac{4r(\Sigma) + t}{n}} \right) \leq e^{-t}.
\]

Then, using the same approach to extend Theorem [4] to Corollary [5] we obtain the statement. \( \square \)
We study the perturbation of an empirical mean of random matrices from its expected value such that they are dependent and the distribution has a heavy-tail. The upper bound derived here is independent of dimension of the matrices, then is instead mainly described by a trace of the expectation and a tail of the distribution. In addition, a constant part of the upper bound increases with the strength of the dependence. The proof here is based on a variational inequality and robustification by truncation. Our result is applied to the estimation problem of covariance operators/matrices, the parameter estimation in linear hidden Markov models, and linear regression under overparameterization.

The limitation of our result is the tightness of the obtained upper bound. It is difficult to achieve lower bounds when random matrices are dependent and heavy-tailed, while some lower bounds are known when they are independent and each element is Gaussian. The lower bound in this case is an interesting subject for future research.

**Appendix A. Proof for Examples**

*Proof of Proposition 7.* Let \( f : E \to E \) be a 1-Lipschitz function and define \( G_\ell = \sigma(f(M_1), \ldots, f(M_\ell)) \). Our objective is to prove that, for any \( g \in \text{Lip}_{n-\ell}(S, 1) \), we have

\[
\|E\left[g(f(M_{\ell+1}), \ldots, f(M_n))|G_\ell\right] - E\left[g(f(M_{\ell+1}, \ldots, f(M_n))\right]\| \leq \Gamma_{\ell, n}. \tag{12}
\]

Let \( h \) be defined by \( h(a_1, \ldots, a_\ell) = g(f(a_1), \ldots, f(a_\ell)) \). Then \( h \in \text{Lip}_{n-\ell}(S, 1) \). Indeed,

\[
|h(a_1, \ldots, a_\ell) - h(b_1, \ldots, b_\ell)| \\
= |g(f(a_1), \ldots, f(a_\ell)) - g(f(b_1), \ldots, f(b_\ell))| \\
\leq L \sum_{i=1}^{\ell} \|f(a_i) - f(b_i)\|_E \\
\leq L \sum_{i=1}^{\ell} \|a_i - b_i\|_E,
\]
satisfies Assumption 1 and \( h \) that we can rewrite as
\[ L_{\infty} \]
to the correct
This is almost (12), however, the conditional expectation is not with respect to the correct \( \sigma \)-algebra. This is easily fixed, because \( G \subseteq F_\ell \). Thus,
\[ \left| \mathbb{E}[g(f(M_{\ell+1}), \ldots, f(M_n)) | \mathcal{F}_\ell] - \mathbb{E}[g(f(M_{\ell+1}, \ldots, f(M_n))] \right| \leq \Gamma_{\ell,n} \]
by using (13).

**Proof of Proposition 3.** We define \((\tilde{\xi}_\ell)_{\ell \in \mathbb{Z}}\) as an independent copy \( \Xi \). We fix \( \ell \in \{1, \ldots, n\} \), we are going to check (1). In order to do so, we define, for \( m > \ell \),
\[ \tilde{X}_m = C(\xi_m, \xi_{m-1}, \ldots, \xi_{\ell+1}, \tilde{\xi}_\ell, \tilde{\xi}_{\ell-1}, \tilde{\xi}_{\ell-2}, \ldots), \]
and \( \tilde{Y}_m = \tilde{X}_m + \varepsilon_m \). We put \( G_\ell = \sigma(\xi_\ell, \xi_{\ell-1}, \tilde{\xi}_{\ell-2}, \ldots; \varepsilon_\ell, \varepsilon_{\ell-1}, \ldots) \). Then, for \( g \in \text{Lip}_{n-\ell}(S, 1) \), we have
\[ \mathbb{E}[g(M_{\ell+1}, \ldots, M_n) | \mathcal{F}_\ell] - \mathbb{E}[g(M_{\ell+1}, \ldots, M_n)] \]
and we will prove an upper bound on \( \mathbb{E}[g(M_{\ell+1}, \ldots, M_n) | \mathcal{G}_\ell] - \mathbb{E}[g(M_{\ell+1}, \ldots, M_n)] \).
Hence, we have
\[ \mathbb{E}[g(M_{\ell+1}, \ldots, M_n) | \mathcal{G}_\ell] - \mathbb{E}[g(M_{\ell+1}, \ldots, M_n)] \]
\[ = \mathbb{E}[g(\tilde{Y}_{\ell+1} Y_{\ell+1}^T, \ldots, \tilde{Y}_n Y_n^T) - g(Y_{\ell+1} Y_{\ell+1}^T, \ldots, Y_n Y_n^T) | \mathcal{G}_\ell] \]
\[ \leq \sum_{m=\ell+1}^n \mathbb{E} \left\| \tilde{Y}_m Y_m^T - Y_m Y_m^T \right\| \]
\[ \begin{align*}
&= \sum_{m=\ell+1}^n \left\| \mathbb{E} \left[ (\bar{X}_m + \varepsilon_m)(\bar{X}_m + \varepsilon_m)^\top - (X_m + \varepsilon_m)(X_m + \varepsilon_m)^\top | \mathcal{G}_\ell \right] \right\| \\
&= \sum_{m=\ell+1}^n \left\| \mathbb{E} \left[ \bar{X}_m \bar{X}_m^\top - X_m X_m^\top | \mathcal{G}_\ell \right] \right\| \\
&= \sum_{m=\ell+1}^n \left\| \mathbb{E} \left[ \bar{X}_m \bar{X}_m^\top - \bar{X}_m X_m^\top + \bar{X}_m X_m^\top - X_m X_m^\top | \mathcal{G}_\ell \right] \right\| \\
&\leq \sum_{m=\ell+1}^n \left( \left\| \mathbb{E} \left[ \bar{X}_m \bar{X}_m^\top - \bar{X}_m X_m^\top | \mathcal{G}_\ell \right] \right\| + \left\| \mathbb{E} \left[ \bar{X}_m X_m^\top - X_m X_m^\top | \mathcal{G}_\ell \right] \right\| \right) \\
&\leq \sum_{m=\ell+1}^n B \left( \left\| \mathbb{E} \left[ \bar{X}_m^\top - X_m^\top | \mathcal{G}_\ell \right] \right\| + \left\| \mathbb{E} \left[ \bar{X}_m - X_m | \mathcal{G}_\ell \right] \right\| \right).
\end{align*} \]

Then, we obtain
\[
\left\| \mathbb{E} \left[ \bar{X}_m - X_m | \mathcal{G}_\ell \right] \right\| = \left\| \mathbb{E} \left[ C(\xi_m, \xi_{m-1}, \ldots, \xi_{\ell+1}, \xi, \xi, \xi_{\ell-1}, \ldots) - C(\xi_m, \xi_{m-1}, \ldots, \xi_{\ell+1}, \xi, \xi_{\ell-1}, \ldots) | \mathcal{G}_\ell \right] \right\| \\
\leq \sum_{i=m-\ell}^\infty \alpha_i \mathbb{E} \left[ \left\| \bar{\xi}_{m-i} - \xi_{m-i} \right\| | \mathcal{G}_\ell \right] \leq 2 \sum_{i=m-\ell}^\infty \alpha_i B_{\xi}
\]

and thus,
\[
\mathbb{E} \left[ g(M_{\ell+1}, \ldots, M_n) | \mathcal{G}_\ell \right] - \mathbb{E} \left[ g(M_{\ell+1}, \ldots, M_n) \right] \leq \sum_{m=\ell+1}^n \left[ 4B \sum_{i=m-\ell}^\infty \alpha_i B_{\xi} \right] \\
\leq 4B \sum_{i=\ell+1}^\infty \min(i, n) \alpha_i B_{\xi}.
\]

Thus, (1) is satisfied with \( \Gamma_{\ell,n} = 4BB_{\xi} \sum_{i=\ell+1}^\infty \min(i, n) \alpha_i \). Let us now check Assumption 2. We have obviously:
\[
\mathbb{P}(\|M_\ell\| \geq t) = \mathbb{P}(\| (X_\ell + \varepsilon_\ell)(X_\ell + \varepsilon_\ell)^\top \| \geq t) \\
= \mathbb{P}(\|X_\ell + \varepsilon_\ell\|^2 \geq t) \\
= \mathbb{P}(\|X_\ell + \varepsilon_\ell\| \geq \sqrt{t}) \\
\leq \mathbb{P}(\|X_\ell\| \geq \sqrt{t}/2) + \mathbb{P}(\|\varepsilon_\ell\| \geq \sqrt{t}/2) \\
\leq 1_{\{i \leq 4B^2\}} + \mathbb{P}(\|\varepsilon_\ell\| \geq \sqrt{t}/2),
\]

which ends the proof. □
Proof of Proposition 8. Since \((X_\ell)_{\ell \in \mathbb{N}}\) is a CBS, we have \(X_\ell = C(\xi_\ell, \xi_{\ell-1}, \xi_{\ell-2}, \ldots)\) with
\[
\|C(a_1, a_2, \ldots) - C(b_1, b_2, \ldots)\| \leq \sum_{\ell=1}^{\infty} a_\ell \|a_\ell - b_\ell\| \quad \text{and} \quad \mathcal{A} := \sum_{\ell=1}^{\infty} a_\ell < \infty.
\]
Using the form, we show that \((\tilde{X}_\ell)_{\ell \in \mathbb{N}} := ((X_\ell, X_{\ell+1})^T)_{\ell \in \mathbb{N}}\) is also a CBS, since we have
\[
\tilde{X}_\ell = (C(\xi_\ell, \xi_{\ell-1}, \xi_{\ell-2}, \ldots), C(\xi_{\ell-1}, \xi_{\ell-2}, \xi_{\ell-3}, \ldots)) = D(\xi_\ell, \xi_{\ell-1}, \xi_{\ell-2}, \ldots)
\]
with some function \(D\) which satisfies
\[
\|D(a_1, a_2, \ldots) - D(b_1, b_2, \ldots)\| \leq \sum_{\ell=1}^{\infty} (a_\ell + a_{\ell+1}) \|a_\ell - b_\ell\|.
\]
Since \((\tilde{X}_\ell)_{\ell \in \mathbb{N}}\) is a CBS, Proposition 3 proves that \((M_1, \ldots, M_{\ell-1})\) satisfies Assumption 1 with \(\Gamma_{\ell,n} = 8BB_\xi \sum_{i=\ell+1}^{n} \min(i, n) a_i\) and \(\Gamma_n := 8BB_\xi \sum_{i=2}^{n} \min(i, n) a_i\).

For Assumption 2, we utilize the fact that the largest eigenvalue of a matrix is no more than a sum of largest eigenvalues of its submatrices and obtain
\[
\mathbb{P}(\|M_\ell\| \geq t) \leq \mathbb{P}(\|(X_\ell + \varepsilon_\ell)(X_\ell + \varepsilon_\ell)^T\| \geq t/4) + \mathbb{P}(\|(X_\ell + \varepsilon_\ell)(X_{\ell+1} + \varepsilon_{\ell+1})^T\| \geq t/2) + \mathbb{P}(\|(X_{\ell+1} + \varepsilon_{\ell+1})(X_{\ell+1} + \varepsilon_{\ell+1})^T\| \geq t/4) \\
= 2\mathbb{P}(\|X_\ell + \varepsilon_\ell\|^2 \geq t/4) + \mathbb{P}(\|(X_\ell + \varepsilon_\ell)(X_{\ell+1} + \varepsilon_{\ell+1})^T\| \geq t/2) \\
\leq 2\mathbb{P}(\|X_\ell + \varepsilon_\ell\| \geq \sqrt{t/2}) + \mathbb{P}(\|X_\ell + \varepsilon_\ell\||X_{\ell+1} + \varepsilon_{\ell+1}| \geq t/2) \\
\leq 2\mathbb{P}(\|X_\ell + \varepsilon_\ell\| \geq \sqrt{t/2}) + 2\mathbb{P}(\|X_\ell + \varepsilon_\ell\| \geq \sqrt{t/2}) \quad \text{and} \quad \mathbb{P}(\|X_\ell + \varepsilon_\ell\| \geq \sqrt{t/2}) \\
\leq 4\mathbb{P}(\|X_\ell\| \geq \sqrt{t/2}) + 4\mathbb{P}(\|\varepsilon_\ell\| \geq \sqrt{t/2}) \\
\leq 41_{\{t \leq 4B^2\}} + 4\mathbb{P}(\|\varepsilon_\ell\| \geq \sqrt{t/2}).
\]
Hence, Assumption 2 holds \(F(\tau) = 41_{\{t \leq 2B^2\}} + 4\mathbb{P}(\|\varepsilon_\ell\| \geq \sqrt{t/2})\). Thus, Corollary 5 shows the first statement.

Finally, the fact \(\|\Sigma_{0:1}\| \leq \|\Sigma_1\| + \|\Sigma\|\) yields the second statement. \(\Box\)
Proof of Proposition \cite{2} First, we confirm that \((X_t)_{t \in \mathbb{Z}}\) is a CBS in Example \cite{1} by its definition. Hence, by Proposition \cite{2} a sequence of matrices generated by \(Y_t Y_t^\top\) satisfies Assumption \cite{1} and \cite{2}.

We show that the estimation error \(\|\hat{A} - A\|\) is bounded by estimation error of \(\hat{\Sigma}\) and \(\Sigma_1\). We bound the error as

\[
\|\hat{A} - A\| = \| (\hat{\Sigma}_1 - \Sigma_1)(\hat{\Sigma} + I)^{-1} + \Sigma_{Y,1}(\hat{\Sigma} + I)^{-1} - (\Sigma + I)^{-1}) \|
\leq \|\hat{\Sigma}_1 - \Sigma_1\| \| (\hat{\Sigma} + I)^{-1}\| + \|\Sigma_1(\Sigma + I)^{-1} - (\Sigma - \hat{\Sigma})(\hat{\Sigma} + I)^{-1}\| \\
\leq \|\hat{\Sigma}_1 - \Sigma_1\| + \|\Sigma - \hat{\Sigma}\|\|\Sigma_1\|.
\]

(14)

Here, we use the facts \(\| (\hat{\Sigma} + I)^{-1}\| \leq 1\) and \(\| (\Sigma + I)^{-1}\| \leq 1\).

We combine the above results. By the same discussion for Proposition \cite{3} in Section \cite{4.2} then we have the bound. Then, for any \(t \geq 4B^2\), Corollary \cite{5} yields

\[
\max\{\|\hat{\Sigma} - \Sigma\|, \|\hat{\Sigma}_1 - \Sigma_1\|\}
\leq 4\sqrt{2}(\|\Sigma_1\| + \|\Sigma\|)(32B^2 + \Gamma_n)\sqrt{\frac{4r(\Sigma_{0:1}) + t}{n}} + \mathbb{P}(\|\varepsilon\| \geq \sqrt{t}/2),
\]

where the definition of \(\Sigma_{0:1}\) follows Section \cite{4.2}. We combine this inequality to the result (14), and we obtain the statement.

Proof of Proposition \cite{70} By Lemma 2 and 19 in \cite{Bartlett et al. 2020}, the risk \(R(\hat{\theta})\) is evaluated as

\[
R(\hat{\theta}) \leq 2(\theta^* + I - \Pi_{\gamma})\Sigma(I - \Pi_{\gamma})\theta^* + \sigma^2\text{tr}((YY^\top)^{-1}Y\Sigma Y^\top(YY^\top)^{-1})
= 2(\theta^*)^\top B\theta^* + c\sigma^2\text{tr}(C),
\]

where \(B = (I - \Pi_{\gamma})\Sigma(I - \Pi_{\gamma})\). We bound the first term as

\[
(\theta^*)^\top B\theta^* = (\theta^*)^\top (I - \Pi_{\gamma})\Sigma(I - \Pi_{\gamma})\theta^*
\leq \|\theta^*\|^2\|I - \Pi_{\gamma}\|\|\Sigma - n^{-1}Y^\top Y\|
\leq \|\theta^*\|^2\|\Sigma - n^{-1}Y^\top Y\|,
\]

where the second equality follows \((I - \Pi_{\gamma})Y^\top = (I - Y^\top(YY^\top)^{-1}Y)Y^\top = Y^\top - Y^\top(YY^\top)^{-1}(YY^\top) = 0\), and the second inequality follows \(\|I - \Pi_{\gamma}\| \leq 1\) from the non-expansive property of projection operators. Recalling that \(n^{-1}Y^\top Y = \hat{\Sigma}\) as in (2), Proposition \cite{2} yields the statement. \qed
Acknowledgement

The first author’s work is partially supported by Japan Science and Technology Agency CREST (grant number JPMJCR21D2). The second author’s work is partially funded by CY Initiative of Excellence (grant “Investissements d’Avenir” ANR-16-IDEX-0008), Project “EcoDep” PSI-AAP2020-0000000013. The third author’s work is partially supported by Japan Society for the Promotion of Science KAKENHI (grant number 21K11780) and Japan Science and Technology Agency FOREST (grant number JP-MJFR2161).

References

Adamczak, R., Litvak, A., Pajor, A. and Tomczak-Jaegermann, N. (2010) Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles, Journal of the American Mathematical Society, 23, 535–561.

Alquier, P. (2021) User-friendly introduction to PAC-Bayes bounds, arXiv preprint arXiv:2110.11216.

Alquier, P. and Wintenberger, O. (2012) Model selection for weakly dependent time series forecasting, Bernoulli, 18, 883–913.

Bartlett, P. L., Long, P. M., Lugosi, G. and Tsigler, A. (2020) Benign overfitting in linear regression, Proceedings of the National Academy of Sciences, 117, 30063–30070.

Bunea, F. and Xiao, L. (2015) On the sample covariance matrix estimator of reduced effective rank population matrices, with applications to fpca, Bernoulli, 21, 1200–1230.

Cai, T. T., Zhang, C.-H. and Zhou, H. H. (2010) Optimal rates of convergence for covariance matrix estimation, The Annals of Statistics, 38, 2118–2144.

Catoni, O. (2007) PAC-Bayesian supervised classification: the thermodynamics of statistical learning, Institute of Mathematical Statistics Lecture Notes – Monograph Series, 56, Institute of Mathematical Statistics, Beachwood, OH, Ohio.

Catoni, O. and Giuliani, I. (2017) Dimension-free pac-bayesian bounds for matrices, vectors, and linear least squares regression, arXiv preprint arXiv:1712.02747.
Dedecker, J., Doukhan, P., Lang, G., José Rafael, L. R., Louhichi, S. and Prieur, C. (2007) Weak dependence, in Weak dependence: With examples and applications, Springer, pp. 9–20.

Dedecker, J. and Prieur, C. (2005) New dependence coefficients. examples and applications to statistics, Probability Theory and Related Fields, 132, 203–236.

Donoho, D. L. (2006) Compressed sensing, IEEE Transactions on information theory, 52, 1289–1306.

Doukhan, P. and Wintenberger, O. (2008) Weakly dependent chains with infinite memory, Stochastic Processes and their Applications, 118, 1997–2013.

Giulini, I. (2018) Robust dimension-free gram operator estimates, Bernoulli, 24, 3864–3923.

Guédon, O. and Rudelson, M. (2007) Lp-moments of random vectors via majorizing measures, Advances in Mathematics, 208, 798–823.

Han, F. and Li, Y. (2020) Moment bounds for large autocovariance matrices under dependence, Journal of Theoretical Probability, 33, 1445–1492.

Han, Q. (2022) Exact spectral norm error of sample covariance, arXiv preprint arXiv:2207.13594.

Handel, R. v. (2017) Structured random matrices, Convexity and concentration, pp. 107–156.

Jeong, H., Li, X., Plan, Y. and Yilmaz, O. (2022) Sub-gaussian matrices on sets: Optimal tail dependence and applications, Communications on Pure and Applied Mathematics, 75, 1713–1754.

Koltchinskii, V. and Lounici, K. (2017) Concentration inequalities and moment bounds for sample covariance operators, Bernoulli, 23, 110–133.

Liaw, C., Mehrabian, A., Plan, Y. and Vershynin, R. (2017) A simple tool for bounding the deviation of random matrices on geometric sets, in Geometric aspects of functional analysis, Springer, New York, pp. 277–299.

Lopes, M. E., Erichson, N. B. and Mahoney, M. W. (2023) Bootstrapping the operator norm in high dimensions: Error estimation for covariance matrices and sketching, Bernoulli, 29, 428–450.

Mendelson, S. and Paouris, G. (2014) On the singular values of random matrices, Journal of the European Mathematical Society, 16, 823–834.
Oliveira, R. I. (2009) Concentration of the adjacency matrix and of the laplacian in random graphs with independent edges, *arXiv preprint arXiv:0911.0600*.

Rio, E. (2000) Inégalités de Hoeffding pour les fonctions lipschitziennes de suites dépendantes, *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 330, 905–908.

Rudelson, M. (1999) Random vectors in the isotropic position, *Journal of Functional Analysis*, 164, 60–72.

Srivastava, N. and Vershynin, R. (2013) Covariance estimation for distributions with $2+\epsilon$ moments, *The Annals of Probability*, 41, 3081–3111.

Vershynin, R. (2018) *High-Dimensional Probability: An Introduction with Applications in Data Science*, Cambridge University Press, Cambridge.

Zhivotovskiy, N. (2021) Dimension-free bounds for sums of independent matrices and simple tensors via the variational principle, *arXiv preprint arXiv:2108.08198*.