Approximately invariant solutions of creeping flow equations

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Abstract

In this paper, the steady creeping flow equations of a second grade fluid in cartesian coordinates are considered; the equations involve a small parameter related to the dimensionless non–Newtonian coefficient. According to a recently introduced approach, the first order approximate Lie symmetries of the equations are computed, some classes of approximately invariant solutions are explicitly determined, and a boundary value problem is analyzed. The main aim of the paper is methodological, and the considered mechanical model is used to test the reliability of the procedure in a physically important application.

Keywords. Creeping flow equations; Second grade fluid; Approximate Lie symmetries; Approximately invariant solutions

1 Introduction

Lie theory of continuous transformations provides a unified and powerful approach for handling differential equations [1 2 3 4 5 6 7 8]. It is known that the knowledge of the Lie point symmetries admitted by ordinary differential
equations allows for their order lowering and, possibly, reducing them to quadrature, whereas, in the case of partial differential equations, symmetries can be used for the research of special (invariant) solutions of initial and boundary value problems. Also, the Lie point symmetries are important ingredients in the derivation of conserved quantities, or in the algorithmic construction of invertible point transformations linking different differential equations that turn out to be equivalent [8,9,10,11,12].

Unfortunately, any small perturbation of an equation usually destroys some important symmetries, and this limits the applicability of Lie group methods to concrete problems where equations involving terms of different orders of magnitude may occur. On the other hand, differential equations containing small terms are commonly and successfully investigated by means of perturbative techniques. To fill the gap, in the last decades, some approximate symmetry theories have been proposed and widely applied to concrete models. The first approach to approximate Lie symmetries is due to Baikov, Gazizov and Ibragimov [13], who proposed to expand in a perturbation series the Lie generator in order to have an approximate generator. They developed an elegant theory since all the useful properties of exact Lie symmetries are adapted in the approximate sense. Since its introduction, this approach has been applied to many physical models [14,15,16,17,18,19,20]. Nevertheless, the expanded generator is not consistent with the principles of perturbative analysis [21] because the dependent variables are not expanded. This implies that in several examples the approximately invariant solutions obtained by this method are not the most general ones.

Fushchich and Shtelen [22] proposed a different approach. The dependent variables are expanded in a series as done in usual perturbation theory; terms are then separated at each order of approximation, and a system of equations to be solved in a hierarchy is obtained. This resulting system is assumed to be coupled, and the approximate symmetries of the original equations are defined as the exact symmetries of the equations obtained from separation. This approach has an obvious simple and coherent basis. Per contra, a lot of algebra (especially for higher order perturbations) is needed; moreover, the basic assumption of a fully coupled system is too strong, since the equations at a level are not influenced by those at higher levels. In addition, there is no possibility to work in a hierarchy: for instance, if one computes first order approximate symmetries, and then searches for second order approximate symmetries, all the work must be done from the very beginning. Applications of this method to various equations can be found, for instance, in the papers [18,23,24,25,26]. Moreover, some variants [16,27] of
the Fushchich–Shtelen method have been proposed with the aim of reducing the amount of computations.

In particular, in [16] these two approaches have been compared, and a third method, which is a variant of Fushchich–Shtelen one, has been proposed, by removing the assumption of a fully coupled system. The involved algebra is much less than that required by Fushchich–Shtelen method, and it is possible to work in a hierarchy; nevertheless, the method is not general. An application of this approach, as well as a comparison of the results provided by the other approaches, has been given in [16] by considering the creeping flow equations of a second grade fluid. The conclusion of the authors is that their method is to be preferred either to the Baikov–Gazizov–Ibragimov approach or the Fushchich–Shtelen one.

In a recent paper [28], a new approach to approximate symmetries has been proposed; the method is consistent with perturbative analysis, and inherits the relevant properties of exact Lie symmetries of differential equations. The main aim of this paper is to show that this new method provides reliable results, and avoids much of the weaknesses of the existing approaches. Therefore, the goal we want to pursue is concerned more with the methodological aspects of the used approach than with the solution of specific mechanical problems [29]. In the method we apply, the dependent variables are expanded in power series of the small parameter as done in classical perturbative analysis; then, instead of considering the approximate symmetries as the exact symmetries of the approximate system (as done in Fushchich–Shtelen method), the consequent expansion of the Lie generator is constructed, and the approximate invariance with respect to the approximate Lie generator is introduced, as in Baikov–Gazizov–Ibragimov method. Of course, the method requires more computations than that required for determining exact Lie symmetries; nevertheless, a general Reduce [30] package (ReLie, [31]), doing automatically all the needed work, is available.

This consistent theory allows to extend all the relevant features of Lie group analysis to an approximate context, i.e., it can be used to lower the order of ordinary differential equations as well as to compute approximately invariant solutions of partial differential equations [28].

In this paper, we apply this consistent approach to the creeping flow equations of a second grade fluid; the choice is motivated by the fact that this model has been analyzed in [16] by means of the different known approaches to approximate Lie symmetries whose results have been compared. Our aim is to show that the new consistent approach can effectively be applied in a concrete situation producing reliable results; the method proposed in [28] combines the ideas underlying the various approaches thus allowing to take into account the principles of invariance
under a continuous group of transformations and perturbative analysis, so that it seems the more adequate one to deal with differential equations containing small terms.

After determining the approximate Lie symmetries, by considering two different subalgebras of the admitted Lie algebra of approximate symmetries, we determine explicitly the corresponding approximately invariant solutions. Also, we specialize one of these solutions by considering a boundary value problem for the model of a mud flow over a porous surface. It can be immediately seen that the approximate solution recovered for this boundary value problem turns out to be more general than the one determined in [32], where the unperturbed model has been discussed.

The plan of the paper is as follows. In Section 2, an overview of the new approach to approximate Lie symmetries of differential equations is presented. In Section 3, the equations for the creeping flow of a second grade fluid are presented and briefly discussed: they involve small terms, the small parameter being the dimensionless non–Newtonian coefficient. Then the approximate Lie symmetries are computed. In Section 4, four classes of approximately invariant solutions are determined, and a boundary value problem discussed. Finally, Section 5 contains our conclusions.

## 2 Approximate symmetry theories

In this Section, a brief sketch of the approach to approximate Lie symmetries of differential equations developed in [28] is given.

Let

\[ \Delta(x, u, u^{(r)}; \varepsilon) = 0 \]  

be a differential equation of order \( r \), where \( u^{(r)} \) denotes the set of all derivatives of the dependent variables \( u \in U \subseteq \mathbb{R}^m \) with respect to the independent variables \( x \in X \subseteq \mathbb{R}^n \) up to the order \( r \), involving a small parameter \( \varepsilon \).

If one looks for classical Lie point symmetries, in general it is not guaranteed that the infinitesimal generators depend on the parameter \( \varepsilon \). Nevertheless, the occurrence of terms involving \( \varepsilon \) has dramatic effects since one loses some symmetries admitted by the unperturbed equation

\[ \Delta(x, u, u^{(r)}; 0) = 0. \]  

In perturbation theory [21], a differential equation involving small terms is often
studied by looking for solutions in the form

\[ u(x, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k u_{(k)}(x) + O(\varepsilon^{p+1}), \tag{3} \]

whereupon the differential equation writes as

\[ \Delta \approx \sum_{k=0}^{p} \varepsilon^k \tilde{\Delta}_{(k)}(x, u_{(0)}^{(r)}, \ldots, u_{(k)}^{(r)}), \tag{4} \]

Now, let us consider a Lie generator

\[ \Xi = \sum_{i=1}^{n} \xi_i(x, u; \varepsilon) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(x, u; \varepsilon) \frac{\partial}{\partial u_\alpha}, \tag{5} \]

where we assume that the infinitesimals depend on the small parameter \( \varepsilon \).

By using the expansion (3) of the dependent variables only, we have the following expressions for the infinitesimals:

\[ \xi_i \approx \sum_{k=0}^{p} \varepsilon^k \tilde{\xi}_{(k)i}, \quad \eta_\alpha \approx \sum_{k=0}^{p} \varepsilon^k \tilde{\eta}_{(k)\alpha}, \tag{6} \]

with

\[ \tilde{\xi}_{(0)i} = \xi_{(0)i} = \xi_i(x, u, \varepsilon)|_{\varepsilon=0}, \quad \tilde{\eta}_{(0)\alpha} = \eta_{(0)\alpha} = \eta_\alpha(x, u, \varepsilon)|_{\varepsilon=0}, \tag{7} \]

where the \( \mathcal{R} \) is a recursion operator defined as

\[ \mathcal{R} \left[ \frac{\partial^{|\tau|} f_{(k)}(x, u_{(0)})}{\partial u_{(0)1}^{\tau_1} \cdots \partial u_{(0)m}^{\tau_m}} \right] = \frac{\partial^{|\tau|} f_{(k+1)}(x, u_{(0)})}{\partial u_{(0)1}^{\tau_1} \cdots \partial u_{(0)m}^{\tau_m}} + \sum_{i=1}^{m} \frac{\partial}{\partial u_{(0)i}} \left( \frac{\partial^{|\tau|} f_{(k)}(x, u_{(0)})}{\partial u_{(0)1}^{\tau_1} \cdots \partial u_{(0)m}^{\tau_m}} \right) u_{(1)i}, \tag{8} \]

\[ \mathcal{R}[u_{(k)}] = (k+1)u_{(k+1)}, \]

for \( k \geq 0, j = 1, \ldots, m, |\tau| = \tau_1 + \cdots + \tau_m \). Thence, we have an approximate Lie generator

\[ \Xi \approx \sum_{k=0}^{p} \varepsilon^k \tilde{\Xi}_{(k)}, \tag{9} \]
where
\[ \tilde{\xi}_{(k)} = \sum_{i=1}^{n} \tilde{\xi}^{(k)}_{i}(x, u(0), \ldots, u(k)) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \tilde{\eta}^{(k)}_{\alpha}(x, u(0), \ldots, u(k)) \frac{\partial}{\partial u_{\alpha}}. \] (10)

Since we have to deal with differential equations, we need to prolong the Lie generator to account for the transformation of derivatives. This is done as in classical Lie group analysis of differential equations, i.e., the derivatives are transformed in such a way the contact conditions are preserved. Therefore, we have the prolongations
\[ \Xi^{(0)} = \Xi, \]
\[ \Xi^{(r)} = \Xi^{(r-1)} + \sum_{\alpha=1}^{m} \sum_{i_1=1}^{n} \ldots \sum_{i_r=1}^{n} \eta_{\alpha, i_1 \ldots i_r} \frac{\partial}{\partial x_{i_1} \ldots \partial x_{i_r}}, \quad r > 0, \] (11)

where
\[ \eta_{\alpha, i_1 \ldots i_r} = \frac{D \eta_{\alpha, i_1 \ldots i_{r-1}}}{Dx_{i_r}} - \sum_{k=1}^{n} \frac{D \xi_k}{Dx_{i_r}} \frac{\partial^{r} u_{\alpha}}{\partial x_{i_1} \ldots \partial x_{i_{r-1}} \partial x_k}, \] (12)

along with the Lie derivative defined as
\[ \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \left( \frac{\partial u_{\alpha}}{\partial x_i} \frac{\partial}{\partial u_{\alpha}} + \sum_{j=1}^{n} \frac{\partial^2 u_{\alpha}}{\partial x_j \partial x_i} \frac{\partial}{\partial (u_{\alpha} / \partial x_j)} + \ldots \right). \] (13)

Of course, in the expression of prolongations, we need to take into account the expansions of \( \xi_{i}, \eta_{\alpha}, u_{\alpha} \), and drop the \( O(\varepsilon^{p+1}) \) terms.

**Example 1.** Let \( p = 1 \), and consider the approximate Lie generator
\[ \Xi \approx \sum_{i=1}^{n} \left( \tilde{\xi}^{(0)}_{i} + \varepsilon \left( \tilde{\xi}^{(1)}_{i} + \sum_{\beta=1}^{m} \frac{\partial \xi^{(0)}_{i \beta}}{\partial u^{(0)}_{\beta}} u^{(1)}_{\beta} \right) \right) \frac{\partial}{\partial x_i} \]
\[ + \sum_{\alpha=1}^{m} \left( \eta^{(0)}_{\alpha} + \varepsilon \left( \eta^{(1)}_{\alpha} + \sum_{\beta=1}^{m} \frac{\partial \eta^{(0)}_{\alpha \beta}}{\partial u^{(0)}_{\beta}} u^{(1)}_{\beta} \right) \right) \frac{\partial}{\partial u_{\alpha}} \] (14)

where \( \tilde{\xi}^{(0)}_{i}, \tilde{\xi}^{(1)}_{i}, \eta^{(0)}_{\alpha} \) and \( \eta^{(1)}_{\alpha} \) depend on \( (x, u^{(0)}) \). The first order prolongation is
\[ \Xi^{(1)} \approx \Xi + \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \eta_{\alpha i} \frac{\partial}{\partial u_{\alpha}} \] (15)
where
\[ \eta_{\alpha,i} = \frac{D}{Dx_i} \left( \eta_{(0)\alpha} + \epsilon \left( \eta_{(1)\alpha} + \sum_{\beta=1}^{m} \frac{\partial \eta_{(0)\alpha}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) \]
\[ - \sum_{j=1}^{n} \frac{D}{Dx_i} \left( \xi_{(0)j} + \epsilon \left( \xi_{(1)j} + \sum_{\beta=1}^{m} \frac{\partial \xi_{(0)j}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) \left( \frac{\partial u_{(0)\alpha}}{\partial x_j} + \epsilon \frac{\partial u_{(1)\alpha}}{\partial x_j} \right), \]
(16)

with the Lie derivative now defined as
\[ \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{k=0}^{p} \sum_{\alpha=1}^{m} \left( \frac{\partial u_{(k)\alpha}}{\partial x_i} \frac{\partial}{\partial u_{(k)\alpha}} + \sum_{j=1}^{n} \frac{\partial^2 u_{(k)\alpha}}{\partial x_i \partial x_j} \frac{\partial}{\partial (\partial u_{(k)\alpha}/\partial x_j)} + \cdots \right). \]
(17)

Things go similarly for higher order prolongations.

The approximate (at the order \( p \)) invariance condition of a differential equation reads
\[ \Xi^{(r)} \Delta \bigg|_{\Delta \approx 0} \approx 0. \]
(18)

In the resulting condition we have to insert the expansion of \( u \) in order to obtain the determining equations at the various orders in \( \epsilon \).

The Lie generator \( \tilde{\Xi}_{(0)} \) is always a symmetry of the unperturbed equations \((\epsilon = 0)\); the correction terms \( \sum_{k=1}^{p} \epsilon^k \tilde{\Xi}_{(k)} \) give the deformation of the symmetry due to the terms involving \( \epsilon \).

It is worth of being remarked that not all the symmetries of the unperturbed equations are admitted as the zero–th terms of the approximate symmetries; the symmetries of the unperturbed equations that are the zero–th terms of the approximate symmetries are called stable symmetries [13]. Moreover, if \( \Xi \) is the generator of an approximate Lie point symmetry of a differential equation, then \( \epsilon \Xi \) is a generator of an approximate Lie point symmetry too, but the converse is not true in general.

By the same arguments as in classical Lie theory of differential equations, it is easily ascertained that the approximate Lie point symmetries of a differential equation are the elements of an approximate Lie algebra.

Approximate Lie symmetries of differential equations can be used to determine approximately invariant solutions by appending to the equations at hand the
(approximate) invariant conditions. For example, for first order approximate Lie symmetries, the latter are

\[
\sum_{i=1}^{n} \left( \xi_{(0)i} \frac{\partial u_{(0)} \alpha}{\partial x_i} + \varepsilon \left( \xi_{(1)i} + \sum_{\beta=1}^{m} \frac{\partial \xi_{(0)i}}{\partial u_{(0)} \beta} u_{(1)} \beta + \xi_{(0)i} \frac{\partial u_{(1)} \alpha}{\partial x_i} \right) - \left( \eta_{(0)} \alpha + \varepsilon \left( \eta_{(1)} \alpha + \sum_{\beta=1}^{m} \frac{\partial \eta_{(0)} \alpha}{\partial u_{(0)} \beta} u_{(1)} \beta \right) \right) \right) \approx 0,
\]

where \( \alpha = 1, \ldots, m \).

3 The model and the admitted approximate Lie symmetries

It is well known that the theory of Newtonian fluids can result inadequate in predicting the behavior of some fluids, so that constitutive relations for non–Newtonian fluid mechanics need to be considered. Several models have been proposed and examined to explain the nonlinear relationship between the stress and the velocity gradient; among them, a model which has gained much support both on theoretical and experimental reasons is that of second grade fluid. In this model, the constitutive equation for the stress tensor \( T \) of an incompressible fluid reads

\[
T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,
\]

where \( p \) is the pressure, \( \mu \) the viscosity, \( \alpha_1 \) and \( \alpha_2 \) material coefficients which may depend on the temperature, \( A_1 \) and \( A_2 \) are the first two Rivlin–Ericksen tensors [33]. The model of second grade fluids is compatible with thermodynamics and allows for stable solutions if [34]:

\[
\alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0, \quad \mu \geq 0.
\]

Under these hypotheses, the dimensionless field equations for incompressible second order fluids in vectorial form have been derived [34, 35].

To avoid the difficulties arising when the boundary has corners or concave regions, a special orthogonal coordinate system [36], generated by the potential flow corresponding to an inviscid fluid, in which the streamlines and velocity potential lines are chosen as coordinate curves in the plane, can be used. In such a formulation, the equations of motion and the boundary conditions become independent of the shape of the body immersed into the flow.
Another hypothesis, useful in some applications, is that of slow motion; in the case of Newtonian fluids we have the Navier–Stokes equations that can be linearized, and exact solutions can be recovered under suitable boundary conditions. On the contrary, in the case of a second grade fluid, written in a suitable curvilinear coordinate system, we are led to the creeping flow equations, which are highly nonlinear. Either the Newtonian or the second grade creeping flow solutions, together with their respective theoretical features, have been discussed in several papers [37, 38, 39, 40].

In this Section, we consider the creeping flow equations of a second grade fluid, and compute the first order approximate Lie symmetries according to the new approach described in the previous Section. This model has been analyzed in [32], where the exact Lie symmetries have been determined, and in [16], where the approximate Lie symmetries have been computed with the different approaches available in the literature.

By considering the equations of motion for incompressible second order fluids for a special curvilinear coordinate system \((\phi, \psi)\) \([34, 35]\) that we relabel as \((x, y)\), the steady plane creeping flow equations in cartesian coordinates read \([32]\):

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial p}{\partial x} - \frac{1}{\text{Re}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \varepsilon \left( 5 \frac{\partial^2 u^2}{\partial x \partial y^2} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} + u \frac{\partial^3 u}{\partial x^2 \partial y} + v \frac{\partial^3 u}{\partial x \partial y^2} ight) &= 0, \\
\frac{\partial p}{\partial y} - \frac{1}{\text{Re}} \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right) - \varepsilon \left( 5 \frac{\partial^2 u}{\partial x \partial y^2} - \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^3 u}{\partial x \partial y^2} ight) &= 0,
\end{align*}
\]

(22)

where \(u(x, y)\) and \(v(x, y)\) are the velocity components in the \(x\) and \(y\) directions, respectively, \(p(x, y)\) is the pressure, \(\text{Re}\) is the Reynolds number, and \(\varepsilon\) \((\varepsilon \ll 1/\text{Re})\) is the dimensionless non–Newtonian coefficient, selected as the perturbation parameter. The resulting equations are non–dimensionalized and the coefficients entering the above equations are defined as follows:

\[
\frac{1}{\text{Re}} = \frac{\mu}{\rho UL}, \quad \varepsilon = \frac{\alpha_1}{\rho L^2},
\]

(23)
where $L$ and $U$ are reference length and velocity, respectively, and the thermodynamic compatibility conditions (21) have been used (the interested reader may refer also to [41] for a detailed critical analysis of thermodynamical compatibility conditions of fluids of differential type). It is worth of being remarked that for creeping flows the inertial terms have been neglected, and for $\varepsilon = 0$ the classical form of the Navier–Stokes equations, in this coordinate system, can be recovered.

By expanding the dependent variables, say

$$u(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + O(\varepsilon^2),$$
$$v(x,y) = v_0(x,y) + \varepsilon v_1(x,y) + O(\varepsilon^2),$$
$$p(x,y) = p_0(x,y) + \varepsilon p_1(x,y) + O(\varepsilon^2),$$

and using the consistent approach to approximate Lie symmetries [28], we obtain that equations (22) are approximately (at first order) invariant with respect to the approximate Lie groups of point transformations generated by the following vector fields:

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial y}, \quad \Xi_3 = \frac{\partial}{\partial p}, \quad \Xi_4 = \varepsilon \frac{\partial}{\partial x}, \quad \Xi_5 = \varepsilon \frac{\partial}{\partial y},$$
$$\Xi_6 = \varepsilon \left( u_0 \frac{\partial}{\partial u} + v_0 \frac{\partial}{\partial v} + p_0 \frac{\partial}{\partial p} \right), \quad \Xi_7 = \varepsilon \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - p_0 \frac{\partial}{\partial p} \right),$$
$$\Xi_8 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + (u_0 + \varepsilon u_1) \frac{\partial}{\partial u} + (v_0 + \varepsilon v_1) \frac{\partial}{\partial v},$$
$$\Xi_9 = \varepsilon \left( \frac{\partial^2 f_1}{\partial y^2} \frac{\partial}{\partial u} - \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial}{\partial v} + \left( f_2 - \frac{1}{Re} \left( \frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} \right) \right) \frac{\partial}{\partial p} \right),$$

where $f_1 = f_1(x,y)$ and $f_2 = f_2(x)$ are arbitrary functions of the indicated arguments, along with the constraint

$$\frac{df_2}{dx} - \frac{1}{Re} \left( \frac{\partial^4 f_1}{\partial x^4} + 2 \frac{\partial^4 f_1}{\partial x^2 \partial y^2} + \frac{\partial^4 f_1}{\partial y^4} \right) = 0. \quad (26)$$

In order to determine approximately invariant solutions, in what follows we will solve the constraint (26) by choosing the ansatz

$$f_1(x,y) = F(x)G(y) + H(x), \quad (27)$$

whereupon three different cases may be considered:
Case (i)

\[ F(x) = a_3 x^3 + a_4 x^2 + a_5 x + a_6, \]
\[ G(y) = a_1 y + a_2, \]
\[ H'''(x) = \text{Ref}_2(x) + a_7; \]  

(28)

Case (ii)

\[ F(x) = (a_3 + a_5 x) \cos(bx) + (a_4 + a_6 x) \sin(bx), \]
\[ G(y) = a_1 \exp(by) + a_2 \exp(-by), \]
\[ H'''(x) = \text{Ref}_2(x) + a_7; \]  

(29)

Case (iii)

\[ F(x) = (a_3 + a_5 x) \exp(bx) + (a_4 + a_6 x) \exp(-bx), \]
\[ G(y) = a_1 \cos(by) + a_2 \sin(by), \]
\[ H'''(x) = \text{Ref}_2(x) + a_7, \]  

(30)

where \( a_i \) \((i = 1, \ldots, 7)\) and \( b \) are arbitrary constants.

4 Approximately invariant solutions

Here, in order to construct approximately invariant solutions to (22), we consider two different one–dimensional subalgebras of the admitted approximate Lie symmetries, \( \Xi_A \)

\[ \Xi_A = \kappa_1 \Xi_3 + \kappa_2 \Xi_6 + \kappa_3 \Xi_7 + \Xi_8 + \Xi_9, \]  

(31)

and

\[ \Xi_B = \Xi_1 + \kappa_1 \Xi_2 + \kappa_2 \Xi_3 + \kappa_3 \Xi_4 + \kappa_4 \Xi_5 + \Xi_9, \]  

(32)

where \( \kappa_1, \kappa_2, \kappa_3 \) and \( \kappa_4 \) are arbitrary constants. The first subalgebra essentially involves a stretching group of the independent and dependent variables, whereas the second one consists of the translation of the independent variables and a non–uniform translation of the dependent variables.
4.1 Approximately invariant solutions with respect to $\Xi_A$

Let us consider the following approximate generator

$$\Xi_A = x(1 + \varepsilon \kappa_3) \frac{\partial}{\partial x} + y(1 + \varepsilon \kappa_3) \frac{\partial}{\partial y}$$

$$+ \left( u_0 + \varepsilon \left( u_1 + \kappa_2 u_0 + \frac{\partial^2 f_1}{\partial y^2} \right) \right) \frac{\partial}{\partial u}$$

$$+ \left( v_0 + \varepsilon \left( v_1 + \kappa_2 v_0 - \frac{\partial^2 f_1}{\partial x \partial y} \right) \right) \frac{\partial}{\partial v}$$

$$+ \left( \kappa_1 + \varepsilon \left( (\kappa_2 - \kappa_3) p_0 + f_2 - \frac{1}{\text{Re}} \left( \frac{\partial^3 f_1}{\partial x^3} + \frac{\partial^3 f_1}{\partial x^3} \right) \right) \right) \frac{\partial}{\partial p}.$$ (33)

The corresponding approximately invariant solutions are such that

$$x(1 + \varepsilon \kappa_3) \frac{\partial u}{\partial x} + y(1 + \varepsilon \kappa_3) \frac{\partial u}{\partial y} = u_0 + \varepsilon \left( u_1 + \kappa_2 u_0 + \frac{\partial^2 f_1}{\partial y^2} \right),$$

$$x(1 + \varepsilon \kappa_3) \frac{\partial v}{\partial x} + y(1 + \varepsilon \kappa_3) \frac{\partial v}{\partial y} = v_0 + \varepsilon \left( v_1 + \kappa_2 v_0 - \frac{\partial^2 f_1}{\partial x \partial y} \right),$$

$$x(1 + \varepsilon \kappa_3) \frac{\partial p}{\partial x} + y(1 + \varepsilon \kappa_3) \frac{\partial p}{\partial y} = \kappa_1 + \varepsilon \left( (\kappa_2 - \kappa_3) p_0ight.$$ $$+ f_2 - \frac{1}{\text{Re}} \left( \frac{\partial^3 f_1}{\partial x^3} + \frac{\partial^3 f_1}{\partial x^3} \right) \right),$$ (34)
whereupon, insertion of (24), and separation of the coefficients of different powers of \( \varepsilon \), provide the system

\[
\begin{align*}
\frac{\partial u_0}{\partial x} + y \frac{\partial u_0}{\partial y} &= u_0, \\
\frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} + \kappa_3 \left( \frac{\partial u_0}{\partial x} + y \frac{\partial u_0}{\partial y} \right) &= u_1 + \kappa_2 u_0 + \frac{\partial^2 f_1}{\partial y^2}, \\
\frac{\partial v_0}{\partial x} + y \frac{\partial v_0}{\partial y} &= v_0, \\
\frac{\partial v_1}{\partial x} + y \frac{\partial v_1}{\partial y} + \kappa_3 \left( \frac{\partial v_0}{\partial x} + y \frac{\partial v_0}{\partial y} \right) &= v_1 + \kappa_2 v_0 - \frac{\partial^2 f_1}{\partial x \partial y}, \\
\frac{\partial p_0}{\partial x} + y \frac{\partial p_0}{\partial y} &= \kappa_1, \\
\frac{\partial p_1}{\partial x} + y \frac{\partial p_1}{\partial y} + \kappa_3 \left( \frac{\partial p_0}{\partial x} + y \frac{\partial p_0}{\partial y} \right) &= (\kappa_2 - \kappa_3) p_0 - a_1 \left( 3a_3 x^2 + 2a_4 x \log(x) - a_5 \right), \\
\frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} &= f_2 - \frac{1}{\Re} \left( \frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} \right).
\end{align*}
\] (35)

By considering the case (i), and using (28), the integration of system (35) yields

\[
\begin{align*}
u_0(x,y) &= x U_0(\omega), \\
v_0(x,y) &= x V_0(\omega), \\
p_0(x,y) &= \kappa_1 \log(x) + P_0(\omega), \\
u_1(x,y) &= x ((\kappa_2 - \kappa_3) U_0(\omega) \log(x) + U_1(\omega)), \\
v_1(x,y) &= x ((\kappa_2 - \kappa_3) V_0(\omega) \log(x) + V_1(\omega)) - a_1 \left( 3a_3 x^2 + 2a_4 x \log(x) - a_5 \right), \\
p_1(x,y) &= P_1(\omega) - 6a_1 a_3 \frac{\Re}{y} + \frac{\kappa_1}{2} (\kappa_2 - \kappa_3) \log^2(x) + \left( (\kappa_2 - \kappa_3) P_0(\omega) - (\kappa_1 \kappa_3 + 6 \frac{a_2 a_3}{\Re} + a_7) \right) \log(x),
\end{align*}
\] (36)

where \( \omega = y/x \), and \( U_0(\omega), V_0(\omega), P_0(\omega), U_1(\omega), V_1(\omega), P_1(\omega) \) are functions to be determined.

Substituting relations (36) into system (22), and separating at the various orders of \( \varepsilon \), the following reduced system of ordinary differential equations is pro-
vided:

\[
\begin{aligned}
V_0' - \omega U_0' + U_0 &= 0, \\
(\omega^2 + 1)U_0'' + \Re(\omega P_0' - \kappa_1) &= 0, \\
\omega(U_0'' + \omega V_0'') - \Re P_0' &= 0, \\
V_1' - \omega U_1' + U_1 + (\kappa_2 - \kappa_3)U_0 &= 0, \\
(\omega^2 + 1)
\left((\omega U_0 - V_0)U_0''' - \frac{U_0'''}{\Re}\right)
+ \omega^2(2(\omega V_0' - V_0)U_0'' \\
- \omega(2(\omega U_0 + V_0) - (5\omega^2 + 2)U_0)U_0'' \\
+ (\kappa_2 - \kappa_3)
\left(P_0 - \frac{1}{\Re}(U_0 - 2\omega U_0')\right) - \kappa_1 \kappa_3 \\
- 6\frac{a_2 a_3}{\Re} - a_7 - \omega P_1' &= 0,
\end{aligned}
\]  

(37)

the prime ' denoting the differentiation with respect to \(\omega\).

By inserting the solution of system (37) into equations (36), and then into the perturbation expansions (24), we get the following approximately invariant solution to system (22):

\[
u(x, y) = \frac{\kappa_1 \Re}{2} y \arctan(y/x) + c_2 x + c_1 y
\]

+ \(\epsilon\left(\left(\frac{\kappa_2 - \kappa_3}{2}\right) \left(\frac{\kappa_1 \left(\log((y/x)^2 + 1) + \log(x)\right) + c_4}{2}\right)\Re y
\]

+ (c_1 + c_3)x - 2c_2 y - a_1 a_4 x

- \left(3a_2 a_3 + \frac{\Re}{2}(\kappa_1 \kappa_3 + a_7)\right) y \arctan(y/x)

+ (\kappa_2 - \kappa_3)(c_2 x + c_1 y) \left(\frac{\log((y/x)^2 + 1) + \log(x)}{2}\right)

+ c_6 x + (c_5 - c_1 (\kappa_2 - \kappa_3)) y\),

(38)
\( v(x,y) = -\frac{\kappa_1 \text{Re} x}{2} \arctan(y/x) + \left( \frac{\kappa_1 \text{Re}}{2} - c_2 \right) y + c_3 x \)  

\[ (39) \]

\[ + \varepsilon \left( \left( -\frac{\kappa_2 - \kappa_3}{2} \right) \left( \frac{\kappa_1 \left( \log((y/x)^2 + 1) \right)}{2} + \log(x) \right) + c_4 \right) \text{Re} x \]

\[ + 2c_2 x + (c_1 + c_3) y + a_1 a_4 y \]

\[ + \left( 3a_2 a_3 + \frac{\text{Re}}{2} (\kappa_1 \kappa_2 + a_7) \right) x \arctan(y/x) \]

\[ + \left( (\kappa_2 - \kappa_3) \left( c_3 x + \left( \frac{\kappa_1 \text{Re}}{2} - c_2 \right) y \right) \right. \]

\[ \left. - 2a_1 a_4 x \right) \left( \frac{\log((y/x)^2 + 1)}{2} + \log(x) \right) - 3a_1 a_3 x^2 + c_7 x \]

\[ - \left( c_6 - \frac{\text{Re}}{2} (c_4 (\kappa_2 - \kappa_3) - \kappa_1 \kappa_2 - a_7) + 3a_2 a_3 \right) y + a_1 a_5 \right), \]

\[ p(x,y) = \kappa_1 \left( \frac{\log((y/x)^2 + 1)}{2} + \log(x) \right) + c_4 \]

\[ (40) \]

\[ + \varepsilon \left( -\frac{\kappa_1}{2} (\kappa_2 - \kappa_3) \arctan^2(y/x) + \frac{\kappa_1}{8} (\kappa_2 - \kappa_3) \log^2((y/x)^2 + 1) \right) \]

\[ + \frac{\left( \kappa_2 - \kappa_3 \right) (c_3 - c_1) - 2a_1 a_4}{\text{Re}} \arctan(y/x) \]

\[ + \left( \left( \frac{\kappa_2 - \kappa_3}{2} \left( \kappa_1 \log(x) + c_4 \right) - \frac{\kappa_1 \kappa_3}{2} \right) \right. \]

\[ \left. - 3a_2 a_3 - \frac{a_7}{2} \right) \log((y/x)^2 + 1) + \frac{\kappa_1}{2} (\kappa_2 - \kappa_3) \log^2(x) \]

\[ + \left( c_4 (\kappa_2 - \kappa_3) - \kappa_1 \kappa_3 - 6a_2 a_3 \right) \left( \frac{a_1 a_4}{\text{Re}} - a_7 \right) \log(x) \]

\[ + \kappa_1 \text{Re} \frac{x}{x^2 + y^2} \left( (4c_2 - \kappa_1 \text{Re}) x + 2(c_1 + c_3) y - 6a_1 a_3 \right) \text{Re} y + c_8 \right), \]

where \( c_i \ (i = 1, \ldots, 8) \) are arbitrary constants.
4.2 Approximately invariant solutions with respect to $\Xi_B$

Let us now consider the approximate generator

$$\Xi_B = (1 + \varepsilon \kappa_3) \frac{\partial}{\partial x} + (\kappa_1 + \varepsilon \kappa_4) \frac{\partial}{\partial y} + \varepsilon \frac{\partial^2 f_1}{\partial u} - \varepsilon \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial}{\partial y}$$

$$+ \left( \kappa_2 + \varepsilon \left( f_2 - \frac{1}{\text{Re}} \left( \frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} \right) \right) \right) \frac{\partial}{\partial p}. \quad (41)$$

The corresponding approximately invariant solutions are such that

$$\begin{align*}
(1 + \varepsilon \kappa_3) \frac{\partial u}{\partial x} + (\kappa_1 + \varepsilon \kappa_4) \frac{\partial u}{\partial y} &= \varepsilon \frac{\partial^2 f_1}{\partial y^2}, \\
(1 + \varepsilon \kappa_3) \frac{\partial v}{\partial x} + (\kappa_1 + \varepsilon \kappa_4) \frac{\partial v}{\partial y} &= -\varepsilon \frac{\partial^2 f_1}{\partial x \partial y}, \\
(1 + \varepsilon \kappa_3) \frac{\partial p}{\partial x} + (\kappa_1 + \varepsilon \kappa_4) \frac{\partial p}{\partial y} &= \kappa_2 + \varepsilon \left( f_2 - \frac{1}{\text{Re}} \left( \frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} \right) \right),
\end{align*} \quad (42)$$

whereupon, insertion of (24), and separation of the coefficients of different powers of $\varepsilon$, provide the system

$$\begin{cases}
\frac{\partial u_0}{\partial x} + \kappa_1 \frac{\partial u_0}{\partial y} = 0, \\
\frac{\partial u_1}{\partial x} + \kappa_1 \frac{\partial u_1}{\partial y} + \kappa_3 \frac{\partial u_0}{\partial x} + \kappa_4 \frac{\partial u_0}{\partial y} = \frac{\partial^2 f_1}{\partial y^2}, \\
\frac{\partial v_0}{\partial x} + \kappa_1 \frac{\partial v_0}{\partial y} = 0, \\
\frac{\partial v_1}{\partial x} + \kappa_1 \frac{\partial v_1}{\partial y} + \kappa_3 \frac{\partial v_0}{\partial x} + \kappa_4 \frac{\partial v_0}{\partial y} = -\frac{\partial^2 f_1}{\partial x \partial y}, \\
\frac{\partial p_0}{\partial x} + \kappa_1 \frac{\partial p_0}{\partial y} = \kappa_2, \\
\frac{\partial p_1}{\partial x} + \kappa_1 \frac{\partial p_1}{\partial y} + \kappa_3 \frac{\partial p_0}{\partial x} + \kappa_4 \frac{\partial p_0}{\partial y} = f_2 - \frac{1}{\text{Re}} \left( \frac{\partial^3 f_1}{\partial x \partial y^2} + \frac{\partial^3 f_1}{\partial x^3} \right). \quad (43)
\end{cases}$$

We are able to explicitly determine approximately invariant solutions in all the cases (i), (ii) and (iii).
4.2.1 Case (i)

In this case, the solution to system (43) is

\[
\begin{align*}
  u_0(x, y) &= U_0(\omega), \\
  v_0(x, y) &= V_0(\omega), \\
  p_0(x, y) &= \kappa_2 x + P_0(\omega), \\
  u_1(x, y) &= (\kappa_1 \kappa_3 - \kappa_4) x U_0(\omega) + U_1(\omega), \\
  v_1(x, y) &= ((\kappa_1 \kappa_3 - \kappa_4) V_0(\omega) - a_1 (a_3 x^2 + a_4 x + a_5)) x + V_1(\omega), \\
  p_1(x, y) &= \left( (\kappa_1 \kappa_3 - \kappa_4) P_0(\omega) - 3 \frac{a_3}{\text{Re}} (2(a_1 \omega + a_2) + \kappa_1 a_1 x) \\
  &\quad - \kappa_2 \kappa_3 - \kappa_4 \right) x + P_1(\omega),
\end{align*}
\]  

(44)

the prime \( ' \) denoting the differentiation with respect to \( \omega \), where \( \omega = y - \kappa_1 x \), and \( U_0(\omega), V_0(\omega), P_0(\omega), U_1(\omega), V_1(\omega), P_1(\omega) \) satisfy the following reduced system of ordinary differential equations:

\[
\begin{align*}
  V_0' - \kappa_1 U_0' &= 0, \\
  (\kappa_1^2 + 1) U_0'' + \text{Re}(\kappa_1 P_0' - \kappa_2) &= 0, \\
  \kappa_1 (U_0'' + \kappa_1 V_0'') - \text{Re} P_0'' &= 0, \\
  V_1' - \kappa_1 U_1' + (\kappa_1 \kappa_3 - \kappa_4) U_0' &= 0, \\
  (\kappa_1^2 + 1) \left( (\kappa_1 U_0 - V_0) U_0''' - \frac{U_1''}{\text{Re}} \right) + \kappa_1^2 (2 \kappa_1 V_0' - U_0') V_0'' \\
  + \kappa_1 \left( (5 \kappa_1^2 + 2) U_0'' + \frac{2(\kappa_1 \kappa_3 - \kappa_4)}{\text{Re}} \right) U_0'' - \kappa_1 P_1' \\
  + (\kappa_1 \kappa_3 - \kappa_4) P_0' - \kappa_2 \kappa_3 - \kappa_4 - \kappa_1 - \alpha &= 0, \\
  \kappa_1 (U_1 - (\kappa_1^2 + 1) V_0) U_1''' + \kappa_3^3 U_0 V_0''' - \frac{\kappa_1}{\text{Re}} (U_1'' + \kappa_1 V_1'') \\
  - \left( \kappa_1 (\kappa_1^2 - 1) V_0' + (5 \kappa_1^2 + 2) U_0' - \frac{\kappa_1 \kappa_3 - \kappa_4}{\text{Re}} \right) U_0'' \\
  - \kappa_1 \left( \kappa_1^2 U_0' - \frac{2}{\text{Re}} \kappa_1 \kappa_3 - \kappa_4 \right) V_0'' + P_1' + \beta &= 0,
\end{align*}
\]

(45)

where \( \alpha = -6 \frac{a_3}{\text{Re}} (a_1 \omega + a_2) \) and \( \beta = 2 \frac{a_1 a_4}{\text{Re}} \).

By inserting the solution of system (45) into equations (44) and then into the perturbation expansions (24), the approximately invariant solution of system (22) is given:

\[
  u(x, y) = \frac{\kappa_2 \text{Re}}{2(\kappa_1^2 + 1)^2} (y - \kappa_1 x)^2 + c_1 (y - \kappa_1 x) + c_2 \quad (46)
\]
In this case, the representation of the approximately invariant solution is

\[ v(x,y) = \frac{\kappa_1 \kappa_2 \text{Re}}{2(\kappa_1^2 + 1)^2} (y - \kappa_1 x)^2 + \kappa_1 c_1 (y - \kappa_1 x) + c_3 \]

\[ p(x,y) = \frac{\kappa_1 \kappa_2}{\kappa_1^2 + 1} (y - \kappa_1 x) + \kappa_2 x + c_4 \]

where \( c_i \) \((i = 1, \ldots, 8)\) are arbitrary constants.

4.2.2 Case (ii)

In this case, the representation of the approximately invariant solution is

\[ u_0(x,y) = U_0(\omega), \quad v_0(x,y) = V_0(\omega), \quad p_0(x,y) = \kappa_2 x + P_0(\omega), \]
\[
u_1(x,y) = \left( \frac{a_2 \exp(-by)(a_5 - a_6 \kappa_1) + a_1 \exp(by)(a_5 + a_6 \kappa_1)}{\kappa_1^2 + 1} \right) \sin(bx) \\
+ a_2 \exp(-by) \frac{b(a_3 - a_4 \kappa_1)(\kappa_1^2 + 1) - a_6(\kappa_1^2 - 1) + 2a_5 \kappa_1}{(\kappa_1^2 + 1)^2} \\
+ a_1 \exp(by) \frac{b(a_3 + a_4 \kappa_1)(\kappa_1^2 + 1) - a_6(\kappa_1^2 - 1) - 2a_5 \kappa_1}{(\kappa_1^2 + 1)^2} \sin(bx) \\
- \left( \frac{a_2 \exp(-by)(a_6 + a_5 \kappa_1) + a_1 \exp(by)(a_6 - a_5 \kappa_1)}{\kappa_1^2 + 1} \right) \cos(bx) \\
+ a_2 \exp(-by) \frac{b(a_4 + a_3 \kappa_1)(\kappa_1^2 + 1) + a_5(\kappa_1^2 - 1) + 2a_6 \kappa_1}{(\kappa_1^2 + 1)^2} \\
+ a_1 \exp(by) \frac{b(a_4 - a_3 \kappa_1)(\kappa_1^2 + 1) + a_5(\kappa_1^2 - 1) - 2a_6 \kappa_1}{(\kappa_1^2 + 1)^2} \cos(bx) \\
+ (\kappa_1 \kappa_3 - \kappa_4) x U_0'(\omega) + U_1(\omega),
\]
\[
v_1(x,y) = \left( \frac{a_2 \exp(-by)(a_6 + a_5 \kappa_1) - a_1 \exp(by)(a_6 - a_5 \kappa_1)}{\kappa_1^2 + 1} \right) \sin(bx) \\
+ a_2 \exp(-by) \frac{b(a_4 + a_3 \kappa_1)(\kappa_1^2 + 1) - a_6 \kappa_1(\kappa_1^2 - 1) + 2a_5 \kappa_1^2}{(\kappa_1^2 + 1)^2} \\
- a_1 \exp(by) \frac{b(a_4 - a_3 \kappa_1)(\kappa_1^2 + 1) + a_6 \kappa_1(\kappa_1^2 - 1) + 2a_5 \kappa_1^2}{(\kappa_1^2 + 1)^2} \sin(bx) \\
+ \left( \frac{a_2 \exp(-by)(a_5 - a_6 \kappa_1) - a_1 \exp(by)(a_5 + a_6 \kappa_1)}{\kappa_1^2 + 1} \right) \cos(bx) \\
+ a_2 \exp(-by) \frac{b(a_3 - a_4 \kappa_1)(\kappa_1^2 + 1) - a_5 \kappa_1(\kappa_1^2 - 1) - 2a_6 \kappa_1^2}{(\kappa_1^2 + 1)^2} \\
- a_1 \exp(by) \frac{b(a_3 + a_4 \kappa_1)(\kappa_1^2 + 1) + a_5 \kappa_1(\kappa_1^2 - 1) - 2a_6 \kappa_1^2}{(\kappa_1^2 + 1)^2} \cos(bx) \\
+ (\kappa_1 \kappa_3 - \kappa_4) x V_0'(\omega) + V_1(\omega),
\]
\[
p_1(x,y) = \frac{2b(a_2 \exp(-by)(a_5 - a_6 \kappa_1) + a_1 \exp(by)(a_5 + a_6 \kappa_1))}{\text{Re}(\kappa_1^2 + 1)} \sin(bx) \\
- \frac{2b(a_2 \exp(-by)(a_6 + a_5 \kappa_1) + a_1 \exp(by)(a_6 - a_5 \kappa_1))}{\text{Re}(\kappa_1^2 + 1)} \cos(bx) \\
+ (\kappa_1 \kappa_3 - \kappa_4) P'_0(\omega) - \kappa_2 \kappa_3 - a_7 \right) x + P_1(\omega),
\]
the prime $'$ denoting the differentiation with respect to $\omega$, where $\omega = y - \kappa_1 x$, and $U_0(\omega), V_0(\omega), P_0(\omega), U_1(\omega), V_1(\omega), P_1(\omega)$ satisfy the reduced system (45) with $\alpha = \beta = 0$.

By solving the reduced system, we finally obtain the following approximately invariant solution:

$$u(x,y) = \frac{\kappa_2 Re}{2(\kappa_1^2 + 1)^2}(y - \kappa_1 x)^2 + c_1 (y - \kappa_1 x) + c_2$$

$$+ \varepsilon \left( \frac{Re}{2(\kappa_1^2 + 1)^3} - a_7(\kappa_1^2 + 1) \right)(y - \kappa_1 x)^2$$

$$+ \left( \frac{\kappa_2 Re(\kappa_1 - \kappa_4)}{\kappa_1^2 + 1}x + c_5 \right)(y - \kappa_1 x)$$

$$+ \left( a_2 \exp(-by)(a_5 - a_6 \kappa_1) + a_1 \exp(by)(a_5 + a_6 \kappa_1) \right) bx$$

$$+ a_2 \exp(-by) \frac{b(a_3 - a_4 \kappa_1)(\kappa_1^2 + 1) - a_6(\kappa_1^2 - 1) + 2a_5 \kappa_1}{(\kappa_1^2 + 1)^2}$$

$$+ a_1 \exp(by) \frac{b(a_3 + a_4 \kappa_1)(\kappa_1^2 + 1) - a_6(\kappa_1^2 - 1) - 2a_5 \kappa_1}{(\kappa_1^2 + 1)^2} \sin(bx)$$

$$- \left( a_2 \exp(-by)(a_6 + a_5 \kappa_1) + a_1 \exp(by)(a_6 - a_5 \kappa_1) \right) bx$$

$$+ a_2 \exp(-by) \frac{b(a_4 + a_3 \kappa_1)(\kappa_1^2 + 1) + a_5(\kappa_1^2 - 1) + 2a_6 \kappa_1}{(\kappa_1^2 + 1)^2}$$

$$+ a_1 \exp(by) \frac{b(a_4 - a_3 \kappa_1)(\kappa_1^2 + 1) + a_5(\kappa_1^2 - 1) - 2a_6 \kappa_1}{(\kappa_1^2 + 1)^2} \cos(bx)$$

$$+ c_1 (\kappa_1 \kappa_3 - \kappa_4)x + c_6$$.  

$$v(x,y) = \frac{\kappa_1 \kappa_2 Re}{2(\kappa_1^2 + 1)^2}(y - \kappa_1 x)^2 + \kappa_1 c_1 (y - \kappa_1 x) + c_3$$

$$+ \varepsilon \left( \frac{Re}{2(\kappa_1^2 + 1)^3} - a_7 \kappa_1(\kappa_1^2 + 1) \right)(y - \kappa_1 x)^2$$

$$+ \left( \kappa_1 \kappa_3 - \kappa_4 \right) \frac{\kappa_1 \kappa_2 Re}{(\kappa_1^2 + 1)^2}x + c_5 \right)(y - \kappa_1 x)$$

$$+ \left( a_2 \exp(-by)(a_5 - a_6 \kappa_1) + a_1 \exp(by)(a_6 - a_5 \kappa_1) \right) bx$$
In this last case, the representation of the approximately invariant solution is

\[ u_0(x, y) = U_0(\omega), \quad v_0(x, y) = V_0(\omega), \quad p_0(x, y) = \kappa_2 x + P_0(\omega), \]

\[ u_1(x, y) = \left( \frac{a_6 \exp(-bx)(a_2 - a_1 \kappa_1) - a_5 \exp(bx)(a_2 + a_1 \kappa_1)}{(\kappa_1^2 + 1)} \right) b x \]

\[ + \exp(-bx) \frac{b a_4(a_2 - a_1 \kappa_1)(\kappa_1^2 + 1) - a_6(a_2(\kappa_1^2 - 1) + 2a_1 \kappa_1)}{(\kappa_1^2 + 1)^2} \]

\[ + a_2 \exp(-by) \frac{b(a_4 + a_3 \kappa_1)(\kappa_1^2 + 1) - a_6 \kappa_1(\kappa_1^2 - 1) + 2a_5 \kappa_1^2}{(\kappa_1^2 + 1)^2} \]

\[ - a_1 \exp(by) \frac{b(a_4 - a_3 \kappa_1)(\kappa_1^2 + 1) + a_6 \kappa_1(\kappa_1^2 - 1) + 2a_5 \kappa_1^2}{(\kappa_1^2 + 1)^2} \sin(bx) \]

\[ + \left( \frac{a_2 \exp(-by)(a_5 - a_6 \kappa_1) - a_1 \exp(by)(a_5 + a_6 \kappa_1)}{\kappa_1^2 + 1} \right) \sin(bx) \]

\[ + a_2 \exp(-by) \frac{b(a_3 - a_4 \kappa_1)(\kappa_1^2 + 1) - a_5 \kappa_1(\kappa_1^2 - 1) - 2a_6 \kappa_1^2}{(\kappa_1^2 + 1)^2} \cos(bx) \]

\[ - a_1 \exp(by) \frac{b(a_3 + a_4 \kappa_1)(\kappa_1^2 + 1) + a_5 \kappa_1(\kappa_1^2 - 1) - 2a_6 \kappa_1^2}{(\kappa_1^2 + 1)^2} \cos(bx) \]

\[ + \kappa_1 c_1(\kappa_1^2 - \kappa_3^2)x + c_7 \right), \]

\[ p(x, y) = \frac{\kappa_1 \kappa_2}{\kappa_1^2 + 1}(y - \kappa_1 x) + \kappa_2 x + c_4 \]

\[ + \epsilon \left( \frac{\kappa_1^2 \Re^2}{(\kappa_1^2 + 1)^2} (y - \kappa_1 x)^2 + 2\kappa_2 \left( c_1 \Re - \frac{\kappa_1 \kappa_3 - \kappa_4}{(\kappa_1^2 + 1)^2} \right) (y - \kappa_1 x) \right) \]

\[ + \frac{2b(a_2 \exp(-by)(a_5 - a_6 \kappa_1) + a_1 \exp(by)(a_5 + a_6 \kappa_1))}{\Re(\kappa_1^2 + 1)} \sin(bx) \]

\[ - \frac{2b(a_2 \exp(-by)(a_6 + a_5 \kappa_1) + a_1 \exp(by)(a_6 - a_5 \kappa_1))}{\Re(\kappa_1^2 + 1)} \cos(bx) \]

\[ - \frac{\kappa_2 \kappa_3 + a_7}{\kappa_1^2 + 1} x - \frac{\kappa_2 \kappa_4 + a_7 \kappa_1}{\kappa_1^2 + 1} y + c_8 \right), \]  

(52)

where \( c_i \ (i = 1, \ldots, 8) \) are arbitrary constants.

### 4.2.3 Case (iii)

In this last case, the representation of the approximately invariant solution is
\[ v_1(x,y) = \left( \frac{a_6 \exp(-bx)(a_1 + a_2 \kappa_1) + a_5 \exp(bx)(a_1 - a_2 \kappa_1)}{\kappa_1^2 + 1} - \frac{a_6 \exp(-bx)(a_1 + a_2 \kappa_1) - a_5 \exp(bx)(a_1 - a_2 \kappa_1)}{\kappa_1^2 + 1} \right) \sin(by) \\
+ \exp(-bx) \frac{ba_4(a_1 + a_2 \kappa_1)(\kappa_1^2 + 1) - a_6(a_1(\kappa_1^2 - 1) - 2a_2 \kappa_1)}{(\kappa_1^2 + 1)^2} \sin(by) \\
+ \exp(-bx) \frac{ba_4(a_1 + a_2 \kappa_1)(\kappa_1^2 + 1) - a_6(a_1(\kappa_1^2 - 1) - 2a_2 \kappa_1)}{(\kappa_1^2 + 1)^2} \cos(by) \\
+ \frac{2b(a_6 \exp(-bx)(a_2 - a_1 \kappa_1) - a_5 \exp(bx)(a_2 + a_1 \kappa_1))}{\Re(\kappa_1^2 + 1)} \sin(by) \\
+ \frac{2b(a_6 \exp(-bx)(a_1 + a_2 \kappa_1) - a_5 \exp(bx)(a_1 - a_2 \kappa_1))}{\Re(\kappa_1^2 + 1)} \cos(by) \\
+ \left( (\kappa_1 \kappa_3 - \kappa_4) U_0'(\omega) + U_1(\omega) \right) x + P_1(\omega), \] (53)

the prime ' denoting the differentiation with respect to \( \omega \), where \( \omega = y - \kappa_1 x \),

and \( U_0(\omega), V_0(\omega), P_0(\omega), U_1(\omega), V_1(\omega), P_1(\omega) \) satisfy, once again, the reduced system \([45]^{45}\) with \( \alpha = \beta = 0 \).
Finally, we are able to recover the following approximately invariant solution:

\[
\begin{align*}
    u(x, y) &= \frac{k_2 \text{Re}}{2(k_1^2 + 1)^2} (y - \kappa_1 x)^2 + c_1 (y - \kappa_1 x) + c_2 \\
    &+ \varepsilon \left( \frac{\text{Re} k_2 (k_1 (3 \kappa_1 k_3 - 4 \kappa_4) - \kappa_3) - a_7 (k_1^2 + 1)}{2(k_1^2 + 1)^3} (y - \kappa_1 x)^2 \\
    &+ \left( \frac{k_2 \text{Re}(k_1 k_3 - \kappa_4)}{(k_1^2 + 1)^2} x + c_5 \right) (y - \kappa_1 x) \\
    &+ \left( a_6 \exp(-bx)(a_1 + a_2 \kappa_1) - a_5 \exp(bx)(a_1 - a_2 \kappa_1) \right) bx \\
    &+ \exp(-bx) b a_4(a_1 + a_2 \kappa_1)(k_1^2 + 1) - a_6(a_1(k_1^2 + 1) + 2a_1 \kappa_1) \\
    &+ \exp(bx) b a_3(a_1 - a_2 \kappa_1)(k_1^2 + 1) + a_5(a_1(k_1^2 - 1) + 2a_1 \kappa_1) \right) \sin(by) \\
    &+ \left( a_6 \exp(-bx)(a_1 + a_2 \kappa_1) - a_5 \exp(bx)(a_1 - a_2 \kappa_1) \right) bx \\
    &+ \exp(-bx) b a_4(a_1 + a_2 \kappa_1)(k_1^2 + 1) - a_6(a_1(k_1^2 - 1) + 2a_1 \kappa_1) \\
    &+ \exp(bx) b a_3(a_1 - a_2 \kappa_1)(k_1^2 + 1) + a_5(a_1(k_1^2 - 1) + 2a_1 \kappa_1) \right) \cos(by) \\
    &+ c_1(k_1 k_3 - \kappa_4)x + c_3 \right), \\
    v(x, y) &= \frac{k_1 k_2 \text{Re}}{2(k_1^2 + 1)^2} (y - \kappa_1 x)^2 + \kappa_1 c_1 (y - \kappa_1 x) + c_3 \\
    &+ \varepsilon \left( \frac{\text{Re} k_2 (k_1 (2(k_1^2 - 1) k_3 - 3 \kappa_1 k_4)) - a_7 k_1 (k_1^2 + 1)}{2(k_1^2 + 1)^3} (y - \kappa_1 x)^2 \\
    &+ \left( k_1 k_3 - \kappa_4 \frac{k_2 \text{Re}(k_1 k_3 - \kappa_4)}{(k_1^2 + 1)^2} x + c_5 \right) (y - \kappa_1 x) \\
    &+ \left( a_6 \exp(-bx)(a_1 + a_2 \kappa_1) + a_5 \exp(bx)(a_1 - a_2 \kappa_1) \right) bx \\
    &+ \exp(-bx) b a_4(a_1 + a_2 \kappa_1)(k_1^2 + 1) - a_6 k_1(a_2(k_1^2 - 1) + 2a_1 \kappa_1) \\
    &+ \exp(bx) b a_3(a_1 - a_2 \kappa_1)(k_1^2 + 1) + a_5(a_1(k_1^2 - 1) + 2a_1 \kappa_1) \right) \cos(by) \\
    &+ c_1(k_1 k_3 - \kappa_4)x + c_6 \right)
\end{align*}
\]
+ \exp(bx) \frac{ba_3(a_1 - a_2 \kappa_1)(\kappa_1^2 + 1) - a_5 \kappa_1(a_2(\kappa_1^2 - 1) - 2a_1 \kappa_1)}{(\kappa_1^2 + 1)^2} \sin(by)
- \left( \frac{a_6 \exp(-bx)(a_2 - a_1 \kappa_1) + a_5 \exp(bx)(a_2 + a_1 \kappa_1)}{\kappa_1^2 + 1} \right) bx
+ \exp(-bx) \frac{ba_4(a_2 - a_1 \kappa_1)(\kappa_1^2 + 1) + a_6 \kappa_1(a_1(\kappa_1^2 - 1) - 2a_2 \kappa_1)}{(\kappa_1^2 + 1)^2}
+ \exp(bx) \frac{ba_3(a_2 + a_1 \kappa_1)(\kappa_1^2 + 1) + a_5 \kappa_1(a_1(\kappa_1^2 - 1) + 2a_2 \kappa_1)}{(\kappa_1^2 + 1)^2} \cos(by)
+ \kappa_1 c_1(\kappa_1 \kappa_3 - \kappa_4)x + c_7,

p(x,y) = \frac{\kappa_1 \kappa_2}{\kappa_1^2 + 1}(y - \kappa_1 x) + \kappa_2 x + c_4
+ \varepsilon \left( \frac{\kappa_2^2 \Re^2}{(\kappa_1^2 + 1)^2}(y - \kappa_1 x)^2 + 2\kappa_2 \left( c_1 \Re - \frac{\kappa_1 \kappa_3 - \kappa_4}{(\kappa_1^2 + 1)^2} \right)(y - \kappa_1 x) \right)
+ \frac{2b \left( a_6 \exp(-bx)(a_2 - a_1 \kappa_1) - a_5 \exp(bx)(a_2 + a_1 \kappa_1) \right)}{\Re(\kappa_1^2 + 1)} \sin(by)
+ \frac{2b \left( a_6 \exp(-bx)(a_1 + a_2 \kappa_1) - a_5 \exp(bx)(a_1 - a_2 \kappa_1) \right)}{\Re(\kappa_1^2 + 1)} \cos(by)
- \frac{\kappa_2 \kappa_3 + a_7}{\kappa_1^2 + 1} x - \frac{\kappa_2 \kappa_4 + a_7 \kappa_1}{\kappa_1^2 + 1} y + c_8,

where \(c_i (i = 1, \ldots, 8)\) are arbitrary constants.

### 4.3 A boundary value problem

In [32], the physical problem of a mud flow over a porous surface has been considered. According to this model, the porosity and the suction velocity increases over the length. In nature, a mud flow, that occurs especially on the slopes surrounding young, narrow and asymmetric depression basins, may spread on detrital porous sediments starting from less porous sandy parts to more porous gravelly parts of the plain. In this situation, the boundary conditions read

\[
\begin{align*}
    u(x, 0) &= 0, & v(x, 0) &= -v_0 x, & u(-\infty, y) &= u_0 y, \\
    \frac{\partial v}{\partial y}(x, \infty) &= 0, & p(-\infty, y) &= p_0,
\end{align*}
\]
and the solution has been explicitly given in [32].

Here, we want to analyze the same boundary value problem, but in the approximate sense. Let us consider the approximate boundary value problem

\[ u(x,0) = O(\varepsilon), \quad v(x,0) = -v_0 x + O(\varepsilon), \quad u(-\infty,y) = u_0 y + O(\varepsilon), \]

\[ \frac{\partial v}{\partial y}(x,\infty) = O(\varepsilon), \quad p(-\infty,y) = p_0 + O(\varepsilon). \]

By using (58) into the approximately invariant solution given by (38)–(40), we obtain the approximate solution:

\[ u(x,y) = u_0 y + \varepsilon \left( -a_1 a_4 x \arctan \left( \frac{y}{x} \right) + c_5 y \right), \]

\[ v(x,y) = -v_0 x + \varepsilon \left( -3a_1 a_3 x^2 + c_7 x + a_1 a_5 \right. \]

\[ + a_1 a_4 \left( y \arctan \left( \frac{y}{x} \right) - x \log(x^2 + y^2) \right) \left), \right. \]

\[ p(x,y) = p_0 + \varepsilon \left( -2 \frac{a_1 a_4}{\Re} \arctan \left( \frac{y}{x} \right) - 6 \frac{a_1 a_3}{\Re} y + c_8 \right). \]

It is immediately to be verified that the exact solution given in [32] can be recovered in (59) for \( \varepsilon = 0 \).

5 Conclusions

In this paper, we explicitly determine some classes of approximately invariant solutions of the steady creeping flow equations of second grade fluids by using a recently introduced approach [28] to approximate Lie symmetries that is consistent with the principles of perturbative analysis. The same equations have been analyzed in [16], where the results of three different approximate symmetry methods have been compared. In [16], the authors show that some solutions can be obtained with one method (essentially, the Fushchich–Shtelen method and, a fortiori, with their method, that simply shorten the length of the needed computations but is not general, since it is assumed that the differential equations are linear at zeroth–order), but not with the Baikov–Gazizov–Ibragimov method. The same problem is not encountered here; in all cases approximately invariant solutions can be determined and often are more general of the ones characterized in [16].

The method here used, proposed along the lines of the approach by Baikov–Gazizov–Ibragimov [13], but taking into account the expansion of the dependent variables, allows to yield correct terms for the approximate solutions. Further applications of the approach proposed in [28] are currently under investigation,
and aim to show the advantages of the method when analyzing the approximate symmetries of differential equations containing small terms.

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