Multiuser quantum repeater for quantum communication network based on quantum dots embedded in optical microcavities

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We propose an efficient multi-user quantum repeater based on the interface between the circularly-polarized photon and the quantum dot embedded in a double-sided optical microcavity. The faithful entanglement distribution is accomplished by the time-bin encoding and the polarization noise on the photons is totally converted into the ambiguity of the time when the arriving photons are directed into the cavities. The nonlocal maximally entangled electron-spin state can in principle be constructed deterministically by converting the time-bin entanglement. The extension of the entangled quantum channel is performed with an efficient parity-check detector and it works with a nearly unity success probability. We also give an efficient scheme for the entanglement purification on nonlocal electron-spin systems to depress the influence of asymmetric noise from optical-fiber channels on different time bins of photons, and discuss the influence from the practical circularly birefringence on time-bin entanglement. Our protocol extends the limit on the coherence time of the memory unit which tightly constrains the distance of the legitimate users in quantum communication network.

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I. INTRODUCTION

The reliable transmission of quantum states over noisy channels is important in quantum communication [1–11]. However, serious problems occur when long-distance quantum communication is considered [12]. Due to the inevitable exponential scaling photon loss in the transmission channel, for a 1000 km of standard telecommunications optical fiber, the success probability of direct transmission for photons is of order 10−20. Apart from the photon loss, even though the photon can arrive at the receiver, the fidelity of the polarized states of the photons transmitted also decreases largely, due to the random birefringence arising from thermal fluctuations, vibrations, and imperfections of the fiber itself. To establish a long-distance entanglement channel, a quantum repeater protocol was originally proposed by Briegel et al. [13] in 1998 to reduce the photon loss rate and suppress the decoherence of entangled photon pairs. Some interesting proposals for the implementations of quantum repeaters have been proposed in various physical systems, such as the quantum repeater with nitrogen vacancy (NV) centers in diamond [14,15], atomic ensembles [16], and single trapped ions [17]. Quantum repeater schemes based on electron spins in quantum dots (QD) [18,21] have also been proposed recently due to their perceived scalability and integrability with current semiconductor technologies, and their long coherence and relaxation time [22].

The spin of the excess electron in a single-electron quantum dot for quantum computation was proposed by Loss and Divincenzo in 1998 [23]. Considering the long electron-spin coherence time (µs is achieved in both a QD ensemble and a single QD), fast manipulation, and easy scalability, QD is one of the good candidates for local storage and processing of quantum information. Single semiconductor QD excitons coupling to a microcavity are extensively researched [24–45]. The giant circular birefringence originated from the spin selective dipole coupling for such spin-cavity systems is utilized to generate photon-photon or spin-photon entanglement [25–28], perform hyper-parallel quantum computing [29,31], build universal quantum gates [32–35], complete entanglement purification [36–38], and realize the long-distance quantum communication [19,21,39,42]. The creation of entanglement between two spatially separated QDs in the work by Waks and Vuchovic [18] is performed with cavity-waveguide systems using weak coherent beams. With some high efficiency photon-number resolving detectors, they implement a complete Bell state analyzer for the QD system [12,15]. At the same time, Simon et al. [14] proposed a scheme for entangling two remote spins based on two-photon coincidence detection. By exploiting the dipole-dipole interaction between trions in neighboring dots, they constitute a controlled-phase gate between two local spins, and it makes the quantum entanglement swapping possible which leads to the realization of a quantum repeater.

Since the seminal work of Franson [46], the time-bin degree of freedom of photons attracted much attention. The two-photon time-bin entanglement source for quantum communication was demonstrated in Ref. [47]. With the encoded time-bin qubits, Kalamidas [18] proposed a single-photon quantum error rejection transmission pro-
tocol in 2005, in which a probabilistic transmission is completed with two Pockels cells (PC) and the deterministic error-free transmission is performed with four PCs. In 2007, Li et al. proposed a faithful qubit-transmission scheme against collective noise without ancillary qubits, resorting to the time-bin degree of freedom of a single photon itself. Recently, the distribution of time-bin entangled qubits over an optical fiber at the scale of 300 km is demonstrated and the two-photon interference fringes observed exhibit a visibility of 84%. A time-bin qubit can also be used to perform quantum computing, and only a single optical path rather than multiple paths is used to complete single-qubit operations and heralded controlled-phase gates. An ultrafast measurement technique for time-bin qubits are implemented which makes time-bin qubits more useful.

In this paper, we show that a multi-user quantum repeater based on the QD-microcavity systems can be constructed deterministically with the help of the time-bin encoder. By using the giant circular birefringence effect for the singly charged QD inside a double-sided microcavity, and the multi-photon coincident detection, the time-bin entanglement can be converted into the deterministic entanglement between the electron-spin system in a heralded way with or without additional single-qubit operations. In our protocol, the entanglement distribution can be performed with unity when none of the photons are lost during the transmission process and only single photon detectors other than the high efficiency photon-number resolving detectors are used to detect the photon state. Besides, an efficient parity-check detector is proposed to extend the quantum channel and perform the entanglement purification.

This paper is organized as follows: In Sec. II A, we introduce the interface between a circularly polarized light and a QD-cavity system. The entanglement distribution for three-user quantum network and the N-user entanglement distribution are given in Secs. II B and II C respectively. In Sec. II D we discuss how to complete entanglement extension with a parity-check detector. The influence of asymmetric noise from optical-fiber channels on different time bins and quantum entanglement purification are given in Sec. II A and the influence from the practical circularly birefringence on time-bin entanglement is given in Sec. II B. A discussion and a summary are given in Sec. IV.

II. MULTI-USER QUANTUM REPEATER

A. The interface between a circularly polarized light and a QD-cavity system

Let us consider a singly charged QD (e.g., for a self-assembled InAs/GaAs quantum dot) embedded inside a resonant double-sided microcavity. Both the top and bottom mirrors of the cavity are partially reflective, shown in Fig. 1(a). The optical properties of a singly charged QD embedded inside a micropillar cavity are dominated by the optical transitions of the negatively charged trion ($X^-$) that consists of two electrons bounded to one hole, shown in Fig. 1(b). When the quantization axis for angular momentum is the z axis for the quantum dot geometry, the single electron states have the spin $J_z = \pm \frac{1}{2}$ (labeled as $|\uparrow\rangle$ and $|\downarrow\rangle$), and the hole $J_z = \pm \frac{3}{2}$ (labeled as $|\uparrow\rangle$ and $|\downarrow\rangle$).

In our protocol, the entanglement distribution can be performed with unity when none of the photons are lost during the transmission process and only single photon detectors other than the high efficiency photon-number resolving detectors are used to detect the photon state. Besides, an efficient parity-check detector is proposed to extend the quantum channel and perform the entanglement purification.

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FIG. 1: The spin-dependent transitions for negatively charged exciton $X^-$. (a) A singly charged QD inside a double-sided optical microcavity. (b) The spin selection rules for optical transition of negatively charged exciton. The symbols $\uparrow$ and $\downarrow$ represent the excess electron-spin projections $|+\frac{1}{2}\rangle$ and $|\pm \frac{1}{2}\rangle$ along the quantization axis (z-direction), respectively. The symbols $\dagger$ and $\ddagger$ represent the spin projections of the hole $|+\frac{1}{2}\rangle$ and $|\pm \frac{1}{2}\rangle$, respectively. $R^\dagger (L^\dagger)$ denotes a right-circularly (a left-circularly) polarized photon propagating along (against) the quantization axis.

The reflection and transmission coefficients of this spin-cavity system can be obtained by solving the Heisenberg equations of motion for the cavity field operator $\hat{a}$.
and the trion dipole operator $\sigma_-$ with the input-output
relations [56].

$$
\frac{d\hat{a}}{dt} = -\left[i(\omega_c - \omega) + \kappa + \frac{\kappa s}{2}\right]\hat{a} - g\sigma_-
\quad - \sqrt{\kappa}\hat{a}_i - \sqrt{\kappa}\hat{a}_i',
\frac{d\sigma_-}{dt} = -\left[i(\omega_X - \omega) + \frac{\gamma}{2}\right]\sigma_- - g\sigma_-\hat{a},
\hat{a}_r = \hat{a}_i + \sqrt{\kappa}\hat{a},
\hat{a}_t = \hat{a}_i' + \sqrt{\kappa}\hat{a}.
(1)
$$

Here $\hat{a}_i$ and $\hat{a}_i'$ are the two input field operators, shown in Fig.1(a). $\hat{a}_r$ and $\hat{a}_t$ are the two output field operators. $\omega$, $\omega_c$, and $\omega_X$ are the frequencies of the input photon, cavity mode, and $X^-$ transition, respectively. $\kappa$ and $\kappa s/2$ are the cavity-field decay rate and the side-leakage rate, respectively. $g$ is the coupling strength between $X^-$ and the cavity mode. $\gamma/2$ is the dipole decay rate. In the limit of weak incoming field, the charged QD is predominantly in the ground state in the whole process, that is, $< J_z > \approx -\frac{1}{2}$. The spin-cavity system behaves like a linear beam splitter whose reflection and transmission are given by [26, 57].

$$
r(\omega) = 1 + t(\omega),
t(\omega) = \frac{\sqrt{\kappa s}}{\kappa s}\left[\left(i(\omega_X - \omega) + \frac{\gamma}{2}\kappa\right) + \frac{\gamma}{2\kappa} + 1 + \left(\frac{\gamma}{\kappa}\right)^2\right].
(2)
$$

For the simplification of calculation, we take the case that the dipole is tuned into the cavity mode ($\omega = \omega_X = \omega + \Delta$). When the probe field is uncoupled with the dipole transition (i.e., $g = 0$), Eq.(2) can be simplified as

$$
r_0(\omega) = \frac{\frac{\gamma}{\kappa} + \frac{\gamma}{2\kappa} + 1}{\frac{\gamma}{\kappa} + \frac{\gamma}{2\kappa} + 1},
t_0(\omega) = \frac{-1}{\frac{\gamma}{\kappa} + \frac{\gamma}{2\kappa} + 1}.
(3)
$$

For the condition $\Delta = 0$ and $\frac{\gamma}{\kappa} \ll 1$, both the reflection coefficient $|r(\omega)|$ and transmission coefficient $|t_0(\omega)|$ can approach 1. To be exact, when the circularly polarized photon directed into the spin-cavity system is in the state $S_z = +1$ (i.e., $|L\uparrow\rangle$ or $|R\uparrow\rangle$), the excess electron in the state $\downarrow\uparrow$ will interact with the input photon, provide a new cavity situation, and eventually make the photon be reflected. Upon reflection, both the polarization and the propagation direction of the photon will be flipped. However, if the photon input is in the state $|R\uparrow\rangle$ or $|L\uparrow\rangle$ ($S_z = -1$), it will be transmitted through the cavity and acquires an extra $\pi$ phase, leaving the electron spin state unaffected. The whole process can be summarized into the following transformations:

$$
|R\uparrow\rangle \rightarrow |L\downarrow\rangle, \quad |R\uparrow\rangle \rightarrow -|L\downarrow\rangle,
|L\uparrow\rangle \rightarrow |R\downarrow\rangle, \quad |L\uparrow\rangle \rightarrow -|R\downarrow\rangle.
(4)
$$

When the extra electron is in the state $|\downarrow\rangle$, the evolution can be described as:

$$
|R\uparrow\rangle \rightarrow -|R\downarrow\rangle, \quad |R\downarrow\rangle \rightarrow |L\uparrow\rangle,
|L\uparrow\rangle \rightarrow |R\downarrow\rangle, \quad |L\downarrow\rangle \rightarrow -|L\downarrow\rangle.
(5)
$$

Combining the rules above, we can accomplish the entanglement tranfer from the photon systems into the spin systems and the entanglement purification of the spin systems and the entanglement link for a multi-user quantum repeater.

FIG. 2: (Color online) Schematic architecture of the elementary link among the three locations A, B, and C (i.e., the three nodes A, B, and C). (a) Quantum entanglement distribution among the three nodes with the help of the spatial DOF of photons. (b) The decoder for each quantum node. Here, the hexagon denotes the three-photon GHZ source and the orange rounded rectangles denoted NODE-A, NODE-B, and NODE-C represent the quantum nodes owned by the three users Alice, Bob, and Charlie, respectively. PBS$_i$ ($i = 1, 2, \cdots$) is the polarizing beam splitter that transmit the $|H\rangle$ polarized photon and reflect the $|V\rangle$ polarized photon, respectively. QWP$_j$ ($j = 1, 2$) represents a quarter wave plate which is used to accomplish the transformations $|H\rangle \leftrightarrow |R\rangle$ and $|V\rangle \leftrightarrow |L\rangle$. CPBS$_j$ represents the circularly polarizing beam splitter that transmit the $|R\rangle$ polarized photon and reflect the $|L\rangle$ polarized photon, respectively. PC$_1$ is a fast Pockels cell.
B. Entanglement distribution for three-user quantum network

To show how the spin-cavity system works for the deterministic entanglement distribution and entanglement purification of quantum systems in a Greenberger-Horne-Zeilinger (GHZ) state explicitly, we first take the three-photon GHZ state as an example, and then generalize it to the case with an N-photon GHZ state.

Suppose that there is a three-photon entangled source in some a point among the memory nodes belonged to the users, say Alice, Bob, and Charlie, shown in Fig. 2 (a). The photons abc produced are entangled in a GHZ state in the polarization degree of freedom (DOF), i.e.,

\[ |\Phi^+_{3}\rangle = \frac{1}{\sqrt{2}} (|H\rangle_a |H\rangle_b |H\rangle_c + |V\rangle_a |V\rangle_b |V\rangle_c), \]

where \( |H\rangle \) and \( |V\rangle \) represent the horizontal and vertical polarization modes of photons, respectively. The subscripts \( a, b, \) and \( c \) represent the photons transmitted to Alice, Bob, and Charlie, respectively. Before entering the noise channels, the photonic polarization entanglement is converted into the time-bin entanglement of the three noise channels, the photonic polarization entanglement Alice, Bob, and Charlie, respectively. Before entering the state in the polarization degree of freedom (DOF), i.e.,

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where \( |H\rangle \) and \( |V\rangle \) represent the horizontal and vertical polarization modes of photons, respectively. The subscripts \( a, b, \) and \( c \) represent the photons transmitted to Alice, Bob, and Charlie, respectively. Before entering the noise channels, the photonic polarization entanglement is converted into the time-bin entanglement of the three photons abc by passing the three photons through the encoders placed in their respective paths. All the encoders are made of two polarizing beam splitters (PBS) and a fast Pockels cell (PC). A PBS transmits the \( |H\rangle \) polarization photon and reflects the \( |V\rangle \) polarization photon, respectively. A relative time delay \( \Delta t \) of the nanoseconds scale can be obtained for the \( |V\rangle \) component of the photon when the users appropriately preset the difference between the long optical path length \( l \) of the \( |V\rangle \) polarization photon and the short optical path length \( s \) of the \( |H\rangle \) polarization photon (i.e., the time interval of the unbalanced Mach-Zehnder interferometers). The parties turn on PC only when the \( s \)-path component appears and it is used to implement the bit-flip operation \( |V\rangle \leftrightarrow |H\rangle \). After the encoders, the state of the system composed of the three photons abc is changed into the time-bin entanglement with the polarization states of the three photons all being \( |H\rangle \) state,

\[ |\Phi^+_{3}\rangle_{t_0} = \frac{1}{\sqrt{2}} (|H\rangle_a |H\rangle_b |H\rangle_c (|s\rangle_s |s\rangle_s + |l\rangle_l |l\rangle_l))_{abc}. \]

Here \( |s\rangle \) and \( |l\rangle \) denote the early and late time bins with which the photon passes through the optical short \( s \) and long \( l \) paths, respectively.

Since all the three photons in both \( s \) and \( l \) time bins launched into the noisy channels in the \( |H\rangle \) polarization, the influences of the collective noise on the photon in different time bins can be taken to be the same one [56–57]. In other words, the noise of each optical-fiber channel is steady in the nanosecond scale that is just the time separation between the \( |s\rangle \) and \( |l\rangle \) time bins, and it can be expressed by a unitary transformation \( U_i \),

\[ U_i |H\rangle_i = \delta_i |H\rangle_i + \eta_i |V\rangle_i. \]

Here \( |\delta_i|^2 + |\eta_i|^2 = 1 \). \( i = (a, b, c) \) describes the noise on the photon \( i \). The state of the three-photon system \( abc \) arriving at the three nodes can be written as

\[ \begin{align*}
|\Phi^+_{3}\rangle_{t_3} &= \frac{1}{\sqrt{2}} (|\delta_a| |H\rangle_a + |\eta_a| |V\rangle_a)(|\delta_b| |H\rangle_b + |\eta_b| |V\rangle_b) \\
&\quad \otimes (|\delta_c| |H\rangle_c + |\eta_c| |V\rangle_c)(|s\rangle_s |s\rangle_s + |l\rangle_l |l\rangle_l),
\end{align*} \]

which is still a three-photon time-bin entanglement but the polarization state of the three-photon system ambiguous since the unitary transformation \( U_i \) \( i = (a, b, c) \) on the photon \( i \) is arbitrary and unknown.

The nodes NODE-A, NODE-B, and NODE-C represent the three parties, Alice, Bob, and Charlie, respectively. They have the same device for their decoders, shown in Fig. 2 (b). After the photons pass through the unbalanced Mach-Zehnder interferometers composed of PBS\(_a\) and PBS\(_b\), the state of the three-photon system \( abc \) evolves into

\[ \begin{align*}
|\Phi^+_{3}\rangle_{t_2} &= \frac{1}{\sqrt{2}} (|\delta_a| |H\rangle_a (|s\rangle_s + |l\rangle_l)) \\
&\quad + |\eta_a| |V\rangle_a (|s\rangle_s + |l\rangle_l)) \otimes (|\delta_b| |H\rangle_b (|s\rangle_s + |l\rangle_l)) \\
&\quad + |\eta_b| |V\rangle_b (|s\rangle_s + |l\rangle_l)) \otimes (|\delta_c| |H\rangle_c (|s\rangle_s + |l\rangle_l)).
\end{align*} \]

Here \( |ij\rangle = |i\rangle |j\rangle \) and the notations \( |s, l\rangle \) \( (|s, l\rangle \) denote the time bins created in the encoder (decoder). PC\(_a\)s at each node is supposed to be active only when the components of \( |s\rangle \) or \( |l\rangle \) time bins appear and implement the bit-flip \( |H\rangle \leftrightarrow |V\rangle \) for the components \( |s\rangle \) and \( |l\rangle \). PBS\(_b\)s transmit the \( |H\rangle \) components and reflect the \( |V\rangle \) components, respectively. A relative time delay with the same amount to that in the encoder is exerted on the \( |V\rangle \) components by setting a longer optical path for the \( |V\rangle \) components. The evolution of state \( |\Phi^+_{3}\rangle_{t_2} \) can be described by

\[ \begin{align*}
|\Phi^+_{3}\rangle_{t_2} \rightarrow &\frac{1}{\sqrt{2}} ((|\delta_a| |V\rangle_a |s\rangle_s + |\eta_a| |H\rangle_a |l\rangle_l) \\
&\quad + (|\eta_a| |V\rangle_a |s\rangle_s + |\delta_a| |H\rangle_a |l\rangle_l)) \\
&\quad \otimes (|\delta_b| |V\rangle_b |s\rangle_s + |\eta_b| |H\rangle_b |l\rangle_l)) \\
&\quad + (|\eta_b| |V\rangle_b |s\rangle_s + |\delta_b| |H\rangle_b |l\rangle_l)) \\
&\quad \otimes (|\delta_c| |V\rangle_c |s\rangle_s + |\eta_c| |H\rangle_c |l\rangle_l)) \\
&\quad + (|\eta_c| |V\rangle_c |s\rangle_s + |\delta_c| |H\rangle_c |l\rangle_l)).
\end{align*} \]

Here the superscripts \( \downarrow \) and \( \uparrow \) are used to describe the different outputs of PBS\(_b\)s and \( \uparrow \) is coincident with the relative orientation of the quantization axis of the
QD-confined spin. \(|s\rangle\) (= |ssl\>, |ssl\>, or |sss\>) denotes the time-bin component with only one delay interval. \(|l\rangle\) (= |sll\>, |ssl\>, or |ssl\>) denotes the time-bin component with two delay intervals. One can easily find that no matter what time bins the three photons are in, they are maximally entangled in the polarization DOF. Now, we only discuss the case \(|\Phi\rangle\rangle_{a}\rangle_{c} = \frac{1}{\sqrt{2}}(1|H\rangle_{a}1|H\rangle_{c} + |V\rangle_{a}|V\rangle_{c}) \otimes |s\rangle_{a}|s\rangle_{b}|s\rangle_{c}

for the entanglement creation of the nonlocal three-electron-spin system shared by Alice, Bob, and Charlie, and the other cases can be discussed in the same way.

To complete the entanglement distribution and entangle the three QD-confined electron spins \(e_{a}\), \(e_{b}\), and \(e_{c}\) owned by the three parties Alice, Bob, and Charlie, respectively, we would like to map the linearly polarized photon into the circularly polarized one \(|H\rangle \leftrightarrow |R\rangle\) and \(|V\rangle \leftrightarrow |L\rangle\). The quarter-wave plates (QWPs) near the photons, can be written as follows:

\[\Phi_{t} = \frac{1}{4}\left[(R^{T}R^{T}R^{T})_{abc} \otimes (|\uparrow\uparrow\uparrow\rangle)_{a,b,c} - (R^{T}R^{T}L^{T})_{abc} \otimes (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle)_{a,b,c} + (L^{T}R^{T}L^{T})_{abc} \otimes (|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle)_{a,b,c} - (L^{T}L^{T}L^{T})_{abc} \otimes (|\downarrow\downarrow\downarrow\rangle - |\uparrow\uparrow\uparrow\rangle)_{a,b,c}\right].\]  

(12)

It can be viewed as a high-dimensional entanglement between the three-photon subsystem and three-QD subsystem. After the three parties measure their photons, they can share three-electron-spin entanglement. For instance, if the outcome of the measurement on the three photons \(abc\) is \((R^{T}R^{T}R^{T})_{abc}\) or \((L^{T}L^{T}L^{T})_{abc}\), the parties can get the state of the three-QD subsystem as

\[\Phi_{3}^{T}_{a} = \frac{1}{2}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)_{a,b,c}.\]  

(13)

With a phase-flip operation \(\sigma_{z}\) = \(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\) on the electron \(e_{c}\), the three parties can transform the state \(|\Phi_{3}^{T}\rangle_{a}\rangle_{c}\) into the desired entangled state \(|\Phi_{3}^{T}\rangle_{a}\rangle_{c}\rangle_{0}\). Here

\[\Phi_{3}^{T}_{a} = \frac{1}{2}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)_{a,b,c}.\]  

(14)

As for the other six outcomes of the measurements on the photons \(abc\), Alice, Bob, and Charlie can also obtain the three-QD entangled state \(|\Phi_{3}^{T}\rangle_{a}\rangle_{c}\rangle_{0}\) with a single-qubit operation on one of the three electrons.

C. N-user entanglement distribution

The principle of our deterministic entanglement creation for three legitimate participants can be extended to the N-participant case directly. Assume the original local N-photon GHZ state in the polarization DOF can be described as

\[\Phi_{N}^{T}_{a} = \frac{1}{\sqrt{2}}(|H\rangle_{a}|H\rangle_{b} \cdots |H\rangle_{z} + |V\rangle_{a}|V\rangle_{b} \cdots |V\rangle_{z}),\]  

(15)

where the subscripts \(a, b, \ldots, z\) represent the photons directed to Alice, Bob, \ldots, and Zach, respectively. With a similar encoder to that in the three-photon case on each of the N photons, the state of the system composed of the N photons \(ab \cdots z\) launched into the noisy quantum channels becomes

\[\Phi_{N}^{T}_{a} = \frac{1}{\sqrt{2}}(|H\rangle_{a}|H\rangle_{b} \cdots |H\rangle_{z} \otimes (|s\rangle_{a}|s\rangle_{b} \cdots |s\rangle_{z} + |l\rangle_{a}|l\rangle_{b} \cdots |l\rangle_{z}).\]  

(16)

It is an N-qubit time-bin entanglement. Here \(|s\rangle_{i}\) and \(|l\rangle_{i}\) (\(i = a, b, \ldots, z\)) denote the components of the photon \(i\) which pass though the short path and the long path of the encoder shown in Fig. 2 (a), respectively. To address the influences of the collective-noise channels on the N photons, we can introduce N unknown unitary operators \(U_{i} = \delta_{i}|H\rangle_{i} + \eta_{i}|V\rangle_{i}\) and they act on the photons \(a, b, \ldots, z\), and respectively. Since the time separation between \(|s\rangle_{i}\) and \(|l\rangle_{i}\) time bins are of the nanosecond scale and taken to be much less than the time of fluctuation of the noise of the channels, the influence on the \(|s\rangle_{i}\) components is identical to that on the \(|l\rangle_{i}\) component. After passing through the noisy channels, the state of the system composed of the N photons evolves into

\[\Phi_{N}^{T}_{a} = \frac{1}{\sqrt{2}}(|H\rangle_{a}|H\rangle_{b} \cdots |s\rangle_{z} + |l\rangle_{a}|l\rangle_{b} \cdots |l\rangle_{z}) \otimes (\delta_{a}|H\rangle_{a} + \eta_{a}|V\rangle_{a})(\delta_{b}|H\rangle_{b} + \eta_{b}|V\rangle_{b}) \otimes \cdots \otimes (\delta_{z}|H\rangle_{z} + \eta_{z}|V\rangle_{z}).\]  

(17)

With the decoder shown in Fig. 2 (b), the parties Alice, Bob, \ldots, and Zach let the photons \(a, b, \ldots, z\) pass through their unbalanced polarization interferometer that is followed by a PC. PBS\(_{a}\) followed with a time delay \(\Delta t\) on the \(|V\rangle\) components is used to separate the \(|H\rangle\) and \(|V\rangle\) components of the photon and let them pass through QWP\(_{1}\) and QWP\(_{2}\), respectively. After the photons successively pass through the optical elements described above, the N-photon state evolves into

\[\Phi_{N}^{T}_{a} = \frac{1}{\sqrt{2}}(|R^{T}\rangle_{a}|R^{T}\rangle_{b} \cdots |R^{T}\rangle_{z} + |L^{T}\rangle_{a}|L^{T}\rangle_{b} \cdots |L^{T}\rangle_{z}) \otimes (\delta_{a}|s\rangle_{a} + \eta_{a}|l\rangle_{a})(\delta_{b}|s\rangle_{b} + \eta_{b}|l\rangle_{b}) \otimes \cdots \otimes (\delta_{z}|s\rangle_{z} + \eta_{z}|l\rangle_{z}).\]  

(18)

Here \(|s\rangle\equiv |sll\rangle\), \(|s\rangle|s\rangle\), or \(|l\rangle\equiv |sls\rangle\), \(|l\rangle|l\rangle\), or \(|l\rangle|s\rangle\).
If all the spins are initialized to be a superposition state of the form $\ket{\Phi}_e = \frac{1}{\sqrt{2}}(\ket{\uparrow} + \ket{\downarrow})$, and the photons are in the state $\ket{\Phi}_N^\uparrow = \frac{1}{\sqrt{2}}(\ket{R^1}_a \ket{R^1}_b \cdots \ket{R^1}_z + \ket{L^1}_a \ket{L^1}_b \cdots \ket{L^1}_z)\ket{S^0}_a \ket{S^0}_b \cdots \ket{S^0}_z$, the state of the hybrid system composed of the $N$ photons and the $N$ electron spins after their interaction assisted by QD-cavity systems can be described as

$$\ket{\Phi_H} = \frac{1}{\sqrt{2^N+1}} \left\{ \left[ (\ket{R^1}_a \ket{\uparrow})_e - (\ket{L^1}_a \ket{\downarrow})_e \right] \otimes \left[ (\ket{R^1}_b \ket{\uparrow})_e - (\ket{L^1}_b \ket{\downarrow})_e \right] \otimes \cdots \otimes \left[ (\ket{R^1}_N \ket{\uparrow})_e - (\ket{L^1}_N \ket{\downarrow})_e \right] \right\}$$

$$+ \left[ (\ket{L^1}_a \ket{\uparrow})_e - (\ket{R^1}_a \ket{\downarrow})_e \right] \otimes (\ket{L^1}_b \ket{\uparrow})_e - (\ket{R^1}_b \ket{\downarrow})_e \right)$$

$$\otimes \cdots \otimes (\ket{L^1}_N \ket{\uparrow})_e - (\ket{R^1}_N \ket{\downarrow})_e \right) \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}$$

(19)

For clearingness, one can rewrite the state $\ket{\Phi_H}$ as that in the three-party case. One has

$$\ket{\Phi_H} = \frac{1}{\sqrt{2^N+1}} \left\{ \sum_{i=1}^{N} \sum_{\alpha_i=0}^{1} \sum_{\beta_i=0}^{1} \left\{ \prod_{i=1}^{N} (-\sigma_{x_i})^{\alpha_i} \prod_{i=1}^{N} (-\sigma_{z_i})^{\beta_i} \right\} \right\}$$

$$\times \left\{ \prod_{i=1}^{N} (\ket{R^1}_i \ket{\uparrow})_a \ket{L^1}_i \ket{\downarrow})_a \right\} \otimes \left\{ \prod_{i=1}^{N} (\ket{L^1}_i \ket{\uparrow})_a \ket{R^1}_i \ket{\downarrow})_a \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}$$

(20)

where the single-qubit operators $\sigma_{x_i} = \ket{R^1}_i \ket{L^1}_i \ket{\uparrow}_i \ket{\downarrow}_i$ and $\sigma_{z_i} = \ket{\uparrow}_i \ket{\downarrow}_i \ket{\uparrow}_i \ket{\downarrow}_i$ are used to complete the bit-flip operations on the $i$-th photon and the $i$-th electron spin, respectively, and $\alpha_i = 1 - \alpha_i$. Alice, Bob, ..., and Zach measure the photons $a, b, \ldots, z$, respectively, in the $\{\ket{R}, \ket{L}\}$ basis, the electron-spin subsystem will be collapsed into the state $\ket{\Phi_H} = \frac{1}{\sqrt{2}}(\ket{\uparrow \uparrow \cdots \uparrow} + \ket{\downarrow \downarrow \cdots \downarrow}) \ket{e_a \cdots e_Z}$. If the number $N$ of the parties is even, no additional operation is required to obtain the target entangled GHZ state $\ket{\Phi_H} = \frac{1}{\sqrt{2}}(\ket{\uparrow \uparrow \cdots \uparrow} + \ket{\uparrow \downarrow \cdots \downarrow} \ket{e_a \cdots e_Z}$; otherwise, Alice performs a phase-flip operation $\sigma_\varphi = \ket{\uparrow \uparrow \cdots \uparrow} \ket{-}\ket{-} \ket{\uparrow \downarrow \cdots \downarrow} \ket{-}$ on the spin $e_a$ to get the target entangled GHZ state $\ket{\Phi_H}$. 

During the entanglement distribution process, the quantum noise on the polarization mode of the photons in our protocol is general. If the giant circular birefringence induced by the single electron spin is reliable, one can complete the entanglement distribution process and get the $N$ remotely separated QD-confined electron spins entangled in the GHZ state $\ket{\Phi_H} = \ket{\Phi_H}$ in a heralded way conditional on the detecting of one photon in each node. The photon loss during the entanglement distribution process does not affect the fidelity of the entangled $N$-QD-electron-spin states.

D. Entanglement extension with parity-check detector

FIG. 3: (Color online) Schematic diagram of the parity-check detector for the two QDs in the centering node owned by Zach. $H_1$ and $H_2$ are two half-wave plates and each is used to complete the Hadamard rotation on the circular polarized photons.

After the successful generation of the $N$-electron GHZ state $\ket{\Phi_H} = \frac{1}{\sqrt{2}}(\ket{\uparrow \uparrow \cdots \uparrow} + \ket{\downarrow \downarrow \cdots \downarrow})$ for two electron spins confined, respectively, in distant cavities separated within the attenuation length, one can extend the length of the quantum channel by local entanglement swapping, which can be performed efficiently with a parity-check detector shown in Fig. 3. Instead of subsequently inputting one probe photon into two target cavities $25, 26, 36, 42, 45$, one can split the incident photon into two spatial modes with a 50/50 beam splitter (BS), and then send each spatial mode into one cavity, respectively. In other words, only one effective input-output is involved in our scheme, which makes the parity-check detector more efficient, especially in the lower coupling regime.

Suppose the two stationary qubits $QD_1$ and $QD_2$ are in arbitrary superposition states $\ket{\Phi}_e = \alpha_1 \ket{\uparrow} + \beta_1 \ket{\downarrow}$ and $\ket{\Phi}_e = \alpha_2 \ket{\uparrow} + \beta_2 \ket{\downarrow}$, respectively. Here $|\alpha_1|^2 + |\beta_1|^2 = 1$. A polarized photon $a$ in the state $\ket{\Phi}_p = \frac{1}{\sqrt{2}}(\ket{R} + \ket{L})$ is incident into the $a_{in}$ input of the parity-check detector (shown in Fig. 3). After passing through the BS, it is changed into the state

$$\ket{\Phi}_p = \frac{1}{2}(\ket{R} + \ket{L}) \otimes (\ket{a_1} + \ket{a_2})$$

(21)
where \(|a_1\rangle\) and \(|a_2\rangle\) are two spatial modes of the photon \(a\) that send to \(QD_1\) and \(QD_2\), respectively. The photon in the states \(|R\rangle\) and \(|L\rangle\) is separated by CPBSs, and then enters the cavities. When leaving the cavities, the system composed of the photon a \(QD_1\), and \(QD_2\) evolves into the state \(|\Phi_{pe2}\rangle\).

\[
|\Phi_{pe2}\rangle = \frac{1}{2}\left( (|R\rangle + |L\rangle) \otimes (|a_1\rangle \otimes (|\alpha| \uparrow - |\beta| \downarrow) \langle \uparrow \rangle) \otimes |\Phi_{e_2}\rangle 
+ |a_2\rangle \otimes (|\alpha| \uparrow - |\beta| \downarrow) \otimes |\Phi_{e_1}\rangle \right),
\]

(22)

where the superscripts \(\uparrow\) and \(\downarrow\) are used to discriminate the propagating directions of the photon with respect to the spin of the QDs, as previously described in Sec. II A.

Here, the subscripts \(a_1\) and \(a_2\) represent spatial modes of photon \(a\) sending to the left analyzer and the right analyzer, respectively. One can project the state of the two remaining QDs nondestructively into the state

\[
|\Phi_0\rangle_{e_2} = \frac{1}{\sqrt{2}}(\langle \alpha | \alpha_2 \rangle \uparrow, \uparrow - |\beta| \beta_2 \downarrow, \downarrow)
\]

(24)

when a \(|R\rangle\) polarized photon is detected in one of the two analyzers; otherwise the state of the two remaining QDs will collapse into

\[
|\Phi_1\rangle_{e_2} = \frac{1}{\sqrt{2}}(\langle \alpha | \beta_2 \rangle \uparrow, \uparrow - |\beta| \alpha_2 \downarrow, \uparrow)).
\]

(25)

That is to say, the click of the photon detector \(D_1\) or \(D_3\) announces the even parity of the two QDs and the click of \(D_2\) or \(D_4\) heralds the odd parity of the two QDs.

Considering \(N + 1\) communication nodes, say, Alice (\(e_a\)), Bob (\(e_b\)), . . . , Zach (\(e_z\)) and Dean (\(e_d\)), the original \(N\)-electron GHZ state shared by Alice, Bob, . . . , Zach is \(|\Phi_N\rangle_{\tilde{e}}\), and the Bell state shared by Zach (\(e_z\)) and Dean is \(|\Phi_2\rangle_{\tilde{e}}\). The total spin state of the \(N' = N + 2\) electrons can be written as

\[
|\Phi_{N'}\rangle_{\tilde{e}} = |\Phi^+_N\rangle_{\tilde{e}} \otimes |\Phi^+_2\rangle_{\tilde{e}}
\]

\[
= \frac{1}{2}(|\uparrow, \uparrow, \uparrow, \uparrow, \uparrow + |\downarrow, \downarrow, \downarrow, \downarrow)_{e_\tilde{e}}
\]

(26)

If Zach applies the parity-check detector on the two QDs \(e_z\) and \(e_{z'}\), the system composed of the \(N + 2\) electrons will evolves into \(|\Phi_{pe1}\rangle\) before the click of photon detectors in the parity-check detector shown in Fig. 3. Here

\[
|\Phi_{pe1}\rangle = \frac{1}{2\sqrt{2}}(|R_{a_1}| \otimes (|\uparrow, \uparrow, \uparrow, \downarrow \rangle_{e_a \cdots e_{z} e_d e_z} 
- |\downarrow, \downarrow, \downarrow, \downarrow \rangle_{e_a \cdots e_{z} e_d e_z}) 
+ |L_{a_2}| \otimes (|\uparrow, \uparrow, \uparrow, \downarrow \rangle_{e_a \cdots e_{z} e_d e_z} 
- |\downarrow, \downarrow, \downarrow, \downarrow \rangle_{e_a \cdots e_{z} e_d e_z})
\]

(27)

where the stationary \(N + 2\) combined QDs are divided into the even parity case \(|\Phi_{N'}(e^0)\rangle\) and the odd one \(|\Phi_{N'}(e^1)\rangle\) conditioned on the detection of \(|R\rangle\) and \(|L\rangle\) photon, respectively, i.e.,

\[
|\Phi_{N'}(e^0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow, \uparrow, \uparrow, \uparrow \rangle_{e_a \cdots e_{z} e_d e_z})
\]

\[
|\Phi_{N'}(e^1)\rangle = \frac{1}{\sqrt{2}}(|\uparrow, \uparrow, \uparrow, \downarrow \rangle_{e_a \cdots e_{z} e_d e_z}).
\]

(28)

With these \(N\)-spin GHZ states, Zack performs a Hadamard operation on the two QDs \(e_z\) and \(e_{z'}\), and then he measures the state of \(e_z\) and \(e_{z'}\) with the basis \(|\uparrow\rangle, |\downarrow\rangle\}, which will project the remaining \(N\) QDs \(e_a, e_b, \cdots, e_d\) into the desired GHZ state of the form \(|\Phi_{N'}\rangle\), up to a local operation on the QD \(e_d\). The span of the GHZ quantum channel is eventually further extended, and the parties in quantum communication network can arbitrarily extend their communication distance with the same quantum entanglement swapping process described above in principle.

III. INFLUENCE ON QUANTUM REPEATER FROM ASYMMETRIC NOISE AND THE PRACTICAL CIRCULARLY BIREFRINGENCE

In the previous section, we detailed the principle and the process of our multi-user quantum repeater protocol. The influences of the noisy channels exerted on the early time bin and the late time bin of the photons are considered to be the same ones and the fluctuation of the noise at the nanosecond scale is neglected. The spin-selection rule is taken to be perfect and the coupling strength \(g \gg (\kappa, \gamma)\), meanwhile the resonant condition \(|\Delta| \approx 0\) is satisfied. However, when it comes to the practical situation, the two kinds of requirements above will be slightly modified. First, the channel may be very noisy and the fiber parameters have local fast variations, the influences of the noisy channels can change at the different time bins. The unitary transformation \(U_1^d\) at the late time bin \(|l\rangle\) is different from that \(U_1^s\) at the early time bin \(|s\rangle\). Second, the heavy-light hole mixing can reduce the
fidelity of the optical selection rules \cite{62}, and it can be improved for charged excitons due to the quenched exchanged interaction \cite{63}. Meanwhile, the finite linewidth of the input light pulse will inevitably make the resonant condition diffusion. The side leakage of the cavity $\kappa$, and the limited coupling strength $g$ will also lead to the imperfect birefringent propagation of the input photons \cite{26,12}. For instance, when the electron spin is in spin-up state of the three-photon system after passing through the cavity and the $|L^i\rangle$ or $|R^i\rangle$ supposes to be totally transmitted has a probability $\nu_0$ to be reflected. Then the final state of three-electron subsystem created will be less entangled.

A. Influence of asymmetric noise on different time bins and entanglement purification

Suppose the influences of the noisy channels at the different time bins are different. The unitary transformations at the early time bin and the late time bin are described as $U^e_i$ and $U^l_j$, respectively,

$$U^e_i |H\rangle = \delta_i |H\rangle + \eta_i |V\rangle,$$

$$U^l_j |H\rangle = \delta_j |H\rangle + \eta'_j |V\rangle,$$  

(30) (31)

where $|\delta_i|^2 + |\eta_i|^2 = |\delta_j|^2 + |\eta'_j|^2 = 1$ ($i = a, b, c$). The state of the three-photon system after passing through the noisy channel can be described as

$$|\Phi^+_1\rangle = \frac{1}{\sqrt{2}} \left( |s\rangle_a |s\rangle_b |s\rangle_c (\delta_a |H\rangle a + \eta_a |V\rangle a) + |l\rangle_a |l\rangle_b |l\rangle_c (\delta^*_b |H\rangle b + \eta^*_b |V\rangle b) \right)$$

$$\otimes (\delta^*_c |H\rangle c + \eta^*_c |V\rangle c) \right)$$

(32)

which is a partially entangled GHZ state when considering the time-bin qubits but the polarization states of the three photons at different time bins are ambiguous and separable. With the same decoder in each node shown in Fig. 2 (b), the early components of the photons $abc$ will be converted into the time-bin qubits with the polarization state be $|V\rangle_a |V\rangle_b |V\rangle_c$, while the late components of $abc$ will be converted into one another kind of time-bin qubits with the polarization state be $|H\rangle_a |H\rangle_b |H\rangle_c$ and it equals to the early ones only when the symmetric noise model is effective. The state of the three photons before entering the cavities can be written as follows

$$|\Phi^+_3\rangle = \frac{1}{\sqrt{2}} |V^i\rangle_a |V^i\rangle_b |V^i\rangle_c \left[ (\delta_a |s\rangle_a + \eta_a |l^i\rangle a) \right.$$ 

$$\otimes (\delta^*_b |s\rangle_b + \eta^*_b |l^i\rangle b) (\delta^*_c |s\rangle_c + \eta^*_c |l^i\rangle c) \} + (|H^i\rangle_a |H^i\rangle_b |H^i\rangle_c \left[ (\delta^*_a |s\rangle_a + \eta^*_a |l^i\rangle a) \right.$$ 

$$\otimes (\delta^*_b |s\rangle_b + \eta^*_b |l^i\rangle b) (\delta^*_c |s\rangle_c + \eta^*_c |l^i\rangle c) \}$$

(33)

which is a partially entangled polarization state at any deterministic time bins. Since the unitary transformations are arbitrary and unknown, one can describe the three-photon state with density matrix $\rho$

$$\rho = \mu |\Phi^+_3\rangle \langle \Phi^+_3| + (1 - \mu) |\Phi^-_3\rangle \langle \Phi^-_3|$$

(34)

which can be viewed as a mixture of $|\Phi^+_3\rangle_0 = \frac{1}{\sqrt{2}} \left( (|H^i\rangle_a |H^i\rangle_b |H^i\rangle_c + |V^i\rangle_a |V^i\rangle_b |V^i\rangle_c \right) \langle \Phi^-_3\rangle_0 = \frac{1}{\sqrt{2}} \left( (|H^i\rangle_a |H^i\rangle_b |H^i\rangle_c - |V^i\rangle_a |V^i\rangle_b |V^i\rangle_c \right)$

with the probabilities $\mu$ and $1 - \mu$, respectively.

We would like to consider the case that the three photons are in state $|\Phi^-_3\rangle_0$ first for purification. With the similar process, we can complete the purification for $|\Phi^+_3\rangle_0$, where the relative phase between different polarization modes of the photons will be mapped into the relative phase between the different spin states of the QD-confined electrons. The state of the hybrid system composed of the three photons $abc$ and the three electron spins $e_a e_b e_c$, after the interactions evolves to $|\Phi^-_3\rangle_0$ shown in Eq. (29),

$$|\Phi^-_3\rangle = ((|R^e\rangle_0 |\uparrow\rangle e_a - |L^e\rangle_0 |\downarrow\rangle e_a) ((|R^e\rangle_0 |\downarrow\rangle e_a - |L^e\rangle_0 |\uparrow\rangle e_a) \right.$$ 

$$\otimes (|R^e\rangle_c |\uparrow\rangle e_c - |L^e\rangle_c |\downarrow\rangle e_c)$$

$$-(|L^e\rangle_0 |\uparrow\rangle e_a - |R^e\rangle_0 |\downarrow\rangle e_a) (|L^e\rangle_0 |\downarrow\rangle e_a - |R^e\rangle_0 |\uparrow\rangle e_a) \right.$$ 

$$\otimes (|L^e\rangle_c |\uparrow\rangle e_c - |R^e\rangle_c |\downarrow\rangle e_c).$$

(35)

Compare the hybrid states Eq. (33) with Eq. (32), one can easily draw a conclusion when the three photons input are in mixed state $\rho$, the detection of the one photon in each node will project the three electron spins $e_a e_b e_c$ into a mixed state of the form $\rho''$.

$$\rho'' = \mu |\Phi^+_0\rangle \langle \Phi^+_0| + (1 - \mu) |\Phi^-_0\rangle \langle \Phi^-_0|.$$

(36)

It is a mixture of the two GHZ states $|\Phi^+_0\rangle_0$ and $|\Phi^-_0\rangle_0$ with the probabilities $\mu$ and $1 - \mu$, respectively. Here

$$|\Phi^+_0\rangle_0 = \frac{1}{2} (|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle),$$

$$|\Phi^-_0\rangle_0 = \frac{1}{2} (|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle).$$

(37)

When the parties in quantum communication network obtain the mixed state $\rho''$, they can use the entanglement purification protocols to increase the fidelity of the entangled channel among Alice, Bob, and Charlie. Since the phase-flip error cannot be purified directly, the parties can perform a Hadamard operation on each QD, and the joint state of $A$, $B$, and $C$ is converted into

$$\rho''_H = \mu |\Phi^{+'}_0\rangle \langle \Phi^{+'}_0| + (1 - \mu) |\Phi^{-'}_0\rangle \langle \Phi^{-'}_0|.$$

(38)

Here

$$|\Phi^{+'}_0\rangle_0 = \frac{1}{2} (|\uparrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle),$$

$$|\Phi^{-'}_0\rangle_0 = \frac{1}{2} (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle).$$

(39)
The phase-flip error is totally mapped into the bit-flip error, and the parties can perform the entanglement purification proposal with our efficient parity-check detector.

We take two copies of three-QD systems $e_a e_b e_c$ and $e'_a e'_b e'_c$, for each round of purification and they both are in the state $\rho_0'$. The combined six-QD system is in the state $\rho_0''$ which could be viewed as the mixture of four pure states $|\Phi_0^+\rangle \otimes |\Phi_0^+\rangle$, $|\Phi_0^+\rangle \otimes |\Phi_0^-\rangle$, $|\Phi_0^-\rangle \otimes |\Phi_0^+\rangle$, and $|\Phi_0^+\rangle \otimes |\Phi_0^-\rangle$ with the probabilities of $\mu^2$, $\mu(1-\mu)$, $(1-\mu)^2$, respectively. After the parity-check detection performed by Alice, Bob, and Charlie, if all the outcomes of parity-check detectors are even, the total six-QD system $e_a e_b e_c e'_a e'_b e'_c$ will be projected into the state

$$|\varphi\rangle = \frac{1}{2}(|↑↑↑↑↑↑⟩ + |↑↓↑↑↓↑⟩ + |↓↑↑↓↑↑⟩ + |↓↓↑↑↑↑⟩)$$

(40)

with the probability of $\frac{\mu^2}{4}$ and

$$|\varphi'\rangle = \frac{1}{2}(|↓↓↓↓↓↓⟩ + |↓↑↑↑↑↑⟩ + |↑↓↓↑↑↑⟩ + |↑↑↑↑↑↓⟩)$$

(41)

with the probability of $\frac{\mu^2}{4}$. If only Alice gets an even-parity result, both Bob and Charlie perform a bit-flip operation on their electron spins, which leads to the same projection of the QD system to that when the outcomes of the three parity-check detectors are all even. As for the case with zero or two of the outcomes are even, which originates from the cross state $|\Phi_0^+\rangle \otimes |\Phi_0^-\rangle$ and $|\Phi_0^-\rangle \otimes |\Phi_0^+\rangle$, it leads to the error and should be discarded. In other words, the joint QD system will be projected into

$$\rho_{h_1}'' = \frac{\mu^2}{\mu^2 + (1-\mu)^2} |\varphi\rangle \langle \varphi| + \frac{(1-\mu)^2}{\mu^2 + (1-\mu)^2} |\varphi'\rangle \langle \varphi'|$$

(42)

with the probability of $\mu^2 + (1-\mu)^2$, when one or three of Alice, Bob, and Charlie get the even parity in their parity-check detection processes.

In order to obtain the GHZ state of a three-QD subsystem $e_a e_b e_c$, Alice, Bob, and Charlie perform a Hadamard operation on each of the three electron spins $e_a e_b e_c$, respectively, and then the states $|\varphi\rangle$ and $|\varphi'\rangle$ will evolve as follows

$$|\varphi\rangle \rightarrow \frac{1}{4\sqrt{2}}(|↑↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩)) |e_a e_b e_c\rangle$$

$$+ |↑↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩)) |e_a e_b e_c\rangle$$

$$+ |↓↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩)) |e_a e_b e_c\rangle$$

$$+ |↓↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩)) |e_a e_b e_c\rangle$$

$$|\varphi'\rangle \rightarrow \frac{1}{4\sqrt{2}}(|↑↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↑↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↓↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↓↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↑↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↑↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↓↑↑⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

$$+ |↓↓↓⟩ e'_a e'_b e'_c ((|↑⟩ + |↓⟩) |e_a e_b e_c\rangle$$

(43)

By measuring the spin states of the QDs $e_a$, $e_b$, and $e_c$ with the basis $\{ |↑⟩, |↓⟩ \}$, they can, with or without some phase-flip operations, get the desired three-QD subsystem $e'_a e'_b e'_c$ in the states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ with the probabilities of $\frac{\mu^2}{\mu^2 + (1-\mu)^2}$ and $\frac{(1-\mu)^2}{\mu^2 + (1-\mu)^2}$, respectively. Finally, the Hadamard operations on $e'_a$, $e'_b$, and $e'_c$ will convert the joint states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ back into $|\Phi_0^+\rangle$ and $|\Phi_0^-\rangle$, leaving the whole system in state

$$\rho_f'' = \frac{\mu^2}{\mu^2 + (1-\mu)^2} |\Phi_0^+\rangle e'_{a,b,c}$$

$$+ \frac{(1-\mu)^2}{\mu^2 + (1-\mu)^2} |\Phi_0^-\rangle e'_{a,b,c}.$$  

(44)

It is a mixed entangled state with more entanglement than the original one $\rho''$ when $\mu > 1/2$. To get a higher-fidelity entangled state, the parties can iterate the purification protocol several rounds with the method described above. For instance, if the initial state with the fidelity $\mu > 0.7$ is available, one can achieve the state with the fidelity $f > 0.997$ for only two rounds.
B. Influence on fidelity and efficiency from the practical circularly birefringence

To discuss the influence of the imperfect circular birefringence for the QD-cavity unit on the fidelity of the quantum distribution process, let’s take the entanglement distribution with the symmetric noise model as an example. In this case, the three photons $abc$ input into the cavities are described by Eq. (11), and the QD-confined electron spins $e_i$ ($i = a, b, c$) are all initialized to be a superposition state of the form $|\Phi\rangle_{e_i} = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$. The original transformation relationship described in Eq. (11) and Eq. (12) for an ideal QD-cavity unit should be modified a little. When the electron spin is in the spin-up state $|\uparrow\rangle$, one has the following transformations

$$
|R^t, \uparrow\rangle \rightarrow r|L^t, \uparrow\rangle + t|R^t, \uparrow\rangle,
|L^t, \uparrow\rangle \rightarrow t_0|L^t, \uparrow\rangle + r_0|L^t, \uparrow\rangle,
$$

(46)

where $r$ (t) and $r_0$ ($t_0$) are the reflection (transmission) coefficients shown in Eq. (2) and Eq. (3). In other words, the incident circularly polarized photon $|R^t\rangle$ or $|L^t\rangle$ totally reflected in the ideal case has a probability $t$ to be transmitted through the cavity, and $|L^t\rangle$ or $|R^t\rangle$ supposed to be totally transmitted has a probability $r$ to be reflected. When the extra electron is in the state $|\downarrow\rangle$, the evolution can be described similarly as:

$$
|R^t, \downarrow\rangle \rightarrow t_0|R^t, \downarrow\rangle + r_0|L^t, \downarrow\rangle,
|L^t, \downarrow\rangle \rightarrow r_0|R^t, \downarrow\rangle + t_0|L^t, \downarrow\rangle,
|L^t, \downarrow\rangle \rightarrow |L^t, \downarrow\rangle + t_0|R^t, \downarrow\rangle.
$$

(47)

According to the practical transformations in Eq. (46) and Eq. (47), the total state of the hybrid photon-QD system after the reflection of the photons $abc$, can be written as

$$
|\Phi'\rangle = \frac{1}{2\sqrt{2}} \left[ \prod_{i=a,b,c} \left( (|R^t, \uparrow\rangle + t|L^t, \uparrow\rangle)|\uparrow\rangle_{e_i} + r_0|R^t, \uparrow\rangle + t_0|L^t, \uparrow\rangle)|\uparrow\rangle_{e_i} \right. \\
+ \left. \prod_{j=a,b,c} \left( (|L^t, \uparrow\rangle + t|R^t, \uparrow\rangle)|\uparrow\rangle_{e_j} + r_0|L^t, \uparrow\rangle + t_0|R^t, \uparrow\rangle)|\uparrow\rangle_{e_j} \right). 
$$

(48)

It is not easy for us to discuss the influence of the practical circular birefringence on the QD-cavity unit directly. We would like to rewrite the state $|\Phi'\rangle$ according to the collective state of the photons $abc$ as we do with the ideal QD-cavity unit in Eq. (12), and the modified state of the the combined system can be written as

$$
|\Phi'\rangle = \frac{1}{\sqrt{C}} \left[ \prod_{i,a,b,c} \left( (|R^t, \uparrow\rangle + t|L^t, \uparrow\rangle)|\uparrow\rangle_{e_i} + r_0|R^t, \uparrow\rangle + t_0|L^t, \uparrow\rangle)|\uparrow\rangle_{e_i} \right. \\
+ \left. \prod_{j,a,b,c} \left( (|L^t, \uparrow\rangle + t|R^t, \uparrow\rangle)|\uparrow\rangle_{e_j} + r_0|L^t, \uparrow\rangle + t_0|R^t, \uparrow\rangle)|\uparrow\rangle_{e_j} \right). 
$$

(49)

Here the shorthand $(|r\rangle + r_0|\downarrow\rangle)^{\otimes 3} \equiv (|r\rangle + r_0|\downarrow\rangle)^{\otimes 3}$ when the outcomes of the measurement on the three photons $abc$ result in $S = \{|R^t, R^t, R^t\rangle, |L^t, L^t, L^t\rangle\}$ and $AS = \{|R^t, R^t, L^t\rangle, |R^t, L^t, R^t\rangle, |L^t, R^t, L^t\rangle, |R^t, L^t, L^t\rangle, |L^t, L^t, R^t\rangle, |L^t, R^t, R^t\rangle\}$, respectively.

$$
C_1 = |r^3 + t^3|^2 + 3|r|^2|t|^{2} + 3|t|^2|t|^2 + r_0^2 + t_0^2 + r_0^2 + t_0^2.
$$

(50)

$$
C_2 = |r^2 + t^2 + 2|rr\rangle + 2|tt\rangle|\rangle^2 + r_0^2 + t_0^2 + r_0^2 + t_0^2 + 2|rr\rangle + 2|tt\rangle|\rangle^2 + |r_0^2 + t_0^2 + r_0^2 + t_0^2.
$$

(51)

When the outcomes of the measurement on the photons $abc$ result in $S$, Charlie performs $\sigma_{e_i} = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ on the electron spin $e_i$ as the same as he does in the case that the perfect circular birefringence for the QD-cavity unit is effective. The state of the three electron spins $e_a e_b e_c$ is converted into

$$
|\Phi'\rangle = \frac{1}{\sqrt{C_1}} \left[ (|r\rangle + r_0|\downarrow\rangle)^{\otimes 2}(|t\rangle + t_0|\downarrow\rangle) \right. \\
+ \left. (|t\rangle + t_0|\downarrow\rangle)^{\otimes 2}(|t\rangle + t_0|\downarrow\rangle) \right].
$$

(52)

With respect to the ideal state $|\Phi_0\rangle$ shown in Eq. (14), the fidelity $F_1$ of the practically entangled state of the
three remote QDs in Eq. (52) can be written as
\begin{equation}
F_1 = |\langle \Phi_3^+ | \Phi_3^\pm \rangle_0 | = \frac{|r^3 + t^2 - r_0^3 - t_0^2|^2}{C_1}.
\end{equation}

One can easily find that when the state of the three photons belongs to AS, the fidelity $F_2$ of the heralded entanglement between the three electron spins $e_a e_b e_c$ can be detailed as
\begin{equation}
F_2 = \frac{|r^2 t_0 + t^2 r_0 - r_0^2 t - t_0^2 r|^2}{C_2}.
\end{equation}

During the entanglement extension process, we utilize a PCD on two local spin-qubits, and perform the partial measurements on the spins. The success probability of the entanglement extension nearly equals to the efficiency of the PCD
\begin{equation}
\eta = \frac{|r|^2 + |t|^2 + |r_0|^2 + |t_0|^2}{2}.
\end{equation}

Although the strong coupling is experimentally challenging, it has been observed in various QD-cavity systems \cite{64, 65}. For micropillars with the diameter $d_c$ around 1.5 µm, the $X^-$ dipole decay rate $\gamma/2 \approx \mu eV$ when the temperature $T = 2K$. The coupling strength $g = 80\mu eV$ and the cavity quality factor including the side leaks $Q > 4 \times 10^4$ has been experimentally realized with $I_{0.6}G_{0.4}A$ in a similar experiment setup \cite{66}. In other words, $g/(\kappa + \kappa_s) > 2.4$ is achievable. Meanwhile, the coupling strength $g$ depends on the QD exciton oscillator strength and the mode volume $V$, while $\kappa$ is only determined by the cavity quality factor \cite{69}, and they can be controlled independently to achieve a larger $g/(\kappa + \kappa_s)$. Recently, the coupling strength $g = 16\mu eV$ and a cavity spectral width as low as $\kappa = 20.5\mu eV (Q = 65000)$ has been achieved in a 7.3 µm diameter micropillar \cite{68}. And then, the quality factor is improved to $Q = 2.15 \times 10^5 (\kappa = 6.2 \mu eV)$ with lower side leakage \cite{69}.

The fidelities $F_1$ and $F_2$ of the three-photon entanglement distribution process are shown in Fig.4 as the function of the side leakage of the spin-cavity system $\kappa_s/\kappa$ and the coupling strength $g/\kappa$ on the resonant interaction condition, where $\frac{\kappa}{\gamma} = 0.1$, $\omega_c = \omega_{X^=} = \omega_0$. One can see that when the coupling strength $g/\kappa$ increases, both $F_1$ and $F_2$ increase and approach unity when $g/\kappa = 1$ and the side leakage is involved and $\kappa_s/\kappa = 0.2$, $F_1 = 0.978$ and $F_2 = 0.984$. The respective efficiency $\eta$ and the our PCD and that in Refs.\cite{25, 26, 36, 42} are shown in Fig.5 respectively for the cases $\frac{\kappa}{\gamma} = 0.1$ and $\omega_c = \omega_{X^=} = \omega_0$. They are also the functions of $\kappa_s/\kappa$ and $g/\kappa$. When $\kappa_s = 0$, both $\eta$ and $\eta'$ increase with the coupling strength $g/\kappa$ and approach unity when $g/\kappa > 1.2$. When the side leakage $\kappa_s/\kappa = 0.2$, one has $\eta > \eta'$, i.e., $g/\kappa = 2$, $\eta = 0.905$ while $\eta' = 0.817$. That is, our multi-user quantum repeater network is more efficient that than the PCDs in Refs.\cite{25, 26, 36, 42}. Moreover, it is achievable in experiment with current techniques.

\section*{IV. DISCUSSION AND SUMMARY}

By far, we have detailed the process of establishing the quantum entangled channel for quantum communication network. The photons entangled in the spatial DOF are exploited to entangle the remotely separated QD-cavity units. Currently, it is not easy to prepare the spatial entanglement directly, the protocols for generating N-photon GHZ state based on spin-QD-single-side-
cavity unit [28] and the spin-QD-double-side-cavity unit [42] have been proposed, and other systems can also be used to generate the N-photon GHZ states [70–73]. With some optical elements, the polarized GHZ entanglement can be transformed into the spatial entanglement before the transmission over noisy optical-fiber channels, shown in Fig. 2 Along with spatial Bell states, one can perform the topological extension of the entanglement across the quantum network by teleportation, and the multi-party quantum channels can be achieved finally.

In our protocol, the electron spin of the QD functions as a quantum node. Just before the arriving of the incident photon, the users initialize their spins by optical pumping or optical cooling [74,75], followed by single-spin rotations [76]. The time needed for the coherent control of electron spins has been suppressed into the scale of picosecond in semiconductor quantum dot [77]. Meanwhile, the electron-spin coherence time as high as $T_2 \approx 2.6\mu s$ has been experimentally achieved [78], which is quite long compared with all preparation and measurement time (ns scales). Hence, the cavity photon time $\tau = 4.5\mu s$ will be the dominant time interval for the exciton dephasing [42]. The previous fidelity of the final quantum networking will be reduced by amount of $1 - \exp(-\tau/T_2) \approx 0.002$.

The imperfection in our scheme comes from photon loss, which is also an inevitable problem in the previously published schemes [14–16]. The photon loss occurs due to the cavity imperfection, the fiber absorption, and the inefficiency of the photon detector. As the successful generation of the electron-spin GHZ state and the completion of quantum connection are heralded by the detection of photons, the photon loss will only affect the efficiency of our scheme and has no effect on the fidelity of the quantum channel established. During the transmission of the photons, there is no restriction on the electron spins. In other words, if the photon loss is negligible, the distance of the adjacent nodes can be arbitrary long other than limited by the coherent time of the quantum nodes [15,16]. Moreover, with the current significant progress on the source of entangled photon pairs that the repetition rate as high as $10^6/10^7\ s^{-1}$ have been achieved [79,80], we are able to accomplish our schemes with a high efficiency.

In summary, we have proposed an efficient multiparty quantum repeater protocol for spin-photon systems. The users can establish a heralded quantum channel entangled in the GHZ state $|\Phi^+\rangle_e^N$ upon the detection of the photons. It is independent of the noise parameters of the channel, and can be applied directly in the quantum communication network protocols.

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