DYNAMICS OF A STAGE STRUCTURE PREY-PREDATOR MODEL WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE AND ANTI-PREDATOR BEHAVIOR OF ADULT PREY

PRABIR PANJA*

Department of Applied Science
Haldia Institute of Technology, Haldia-721657, W. B., India

SOOVOOJEET JANA

Department of Mathematics
Ramsaday College, Amta, Howrah, India

SHYAMAL KUMAR MONDAL

Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University, Midnapore -721102, W. B., India

(Communicated by Bülent Karasözen)

Abstract. In this paper, a predator-prey interaction model among juvenile prey, adult prey and predator has been developed where stage structure is considered on prey species. The functional responses has been considered as ratio dependent. It is assumed that that the adult prey is strong enough such that it has an anti-predator characteristic. Global dynamics of the co-existing equilibrium point has been discussed with the help of the geometric approach. Furthermore, it is established that the proposed system undergoes through a Hopf bifurcation with respect to some important parameters. Finally, some numerical simulations have been done to test our theoretical results.

1. Introduction. Of late mathematical ecology has become a demanding area of research to the scientists and researchers of different specializations including mathematics, biology, economics etc. They are engaged to find out the solution of various ecological problems which evolve with prey-predator interactions, inter and intra-specific competitions, biological conversations and bio-diversity, extinction and co-existence of populations of different communities in different environment etc. In our present work, we intend to study a mathematical model on prey-predator system with some anti-predator behavior of the adult prey population. This type of system can be observed in red colobus monkeys exhibit mobbing when threatened by chimpanzees. The systematic mathematical analysis can lead to better understanding of such type of interactions. Since the work of Lotka [16], various kinds of mathematical models on prey-predator interaction [17, 23, 12, 11, 18] have been explored to explain the relationship between prey and predator.

2010 Mathematics Subject Classification. Primary: 92B05, 92D25; Secondary: 92D40.
Key words and phrases. Juvenile prey, Adult prey, Ratio dependent functional response, Anti-predator behavior, Global stability, Hopf bifurcation.

* Corresponding author: Prabir Panja, prabirpanja@gmail.com.
It is natural that two or more species living in a common habitant are often attach to one another by interacting in several ways. Choice of suitable growth rate is an important aspect in studying the interactions between a prey and its predator populations. However, in the existing literature, several mathematicians and theoretical ecologists have contributed their different conceptual notions about the growth rate of predator population. Normally, the rate of prey consumption by an average predator is known as functional response which can be classified as (i) prey dependent (ii) predator dependent and (iii) multi species dependent. In prey dependent, the functional response is affected by only prey population, in case of predator dependent, functional response can be determined by considering both predator and prey populations and in multi species dependent the species other than the focal predator and its prey influence the functional response. Traditionally in predator-prey mathematical models, the functional response has been considered depending upon density of prey population only. In 1989, Arditi and Ginzburg [2] suggested a ratio dependent functional response which is a particular type of predator dependence. Here, the response only depends on the ratio of prey population size to predator population size and it is quite better than prey dependent functional response. Although the mathematical form of the ratio dependent functional responses are more complex than the other types, there are very few number of existing literature in prey-predator models in which these type of responses have been considered. In 2004, the ratio dependent functional response was considered by Fan and Li [9]. After that, Banerjee [5] developed a prey predator model considering the ratio dependent functional response in 2010. There exists also some research paper [10, 14, 4] where ratio dependent functional response have been considered. Although biologists routinely label the animals as predators or prey, the ecological role of individuals is often far from clear. There are many examples [8, 13, 19, 21, 1] of role reversals in predators and prey, where an adult prey attacks vulnerable young predators. This implies that a juvenile prey that escapes from predation and become adult and then it can kill vulnerable predators. The juvenile prey to adult predators results in behavioral changes later in life: after becoming adult, these prey kill vulnerable predators at a faster rate than prey that had not been exposed. Anti-predator adaptations are mechanisms developed through evolution that assist prey organisms in their constant struggle against predators. Throughout the animal kingdom, adaptations have evolved for every stage of this struggle. There are very few mathematical model (for example. [22]) in which anti-predator behaviors have been considered to analyze the nonlinear predator-prey system.

The rest of the paper is organized as follows: in next section, we form our mathematical model and in section 3, we investigate the model from the point of view of uniform boundedness of the solutions. In section 4, we investigate, the existence of all equilibria of our formulated model and their local as well as global stability criteria. In section 5 we study the Hopf bifurcation analysis of the original system around the interior equilibrium. Numerical simulation works are presented in section 6. Finally, in last section we give some of the main outcomes of our work.

2. Model Formulation. In our present work we are going to discuss a prey-predator mathematical model with stage structured in prey populations. In this model, it is assumed that recruitment rate of juvenile prey is proportional to the density of existing adult prey population. But, the study of anti-predator behavior is very important in ecology due to morphological changes and attack of adult
prey. Now, according to the model developed by Tang and Xiao [22], it is seen that the growth rate of predator has been decreased by an anti-predator behavioral term ($\eta xy$) involving the densities of all prey populations. But from the literature survey [8, 13, 19], it is seen that only the adult prey can save itself from the attack of predator due to its morphological changes. In communal defense, adult prey groups actively defend themselves by attacking or mobbing a predator. Mobbing is the harassing of a predator by many prey animals. For example, red colobus monkeys exhibit mobbing when threatened by chimpanzees, a common predator [21, 1]. So in our proposed model, only the adult prey has been considered to reduce the growth rate of predators. Due to this reason, we consider a stage structured prey population with juvenile prey or immature prey and adult prey or matured prey with biomass densities $x(t)$ and $y(t)$ at any time $t$ respectively. Here, $z(t)$ be the biomass density of predator population at time $t$. As we have assumed that, adult prey population is strong enough to bear an anti-predator characteristics, therefore, the anti-predator behavior is taken as a bilinear form of both the state variables $y$ and $z$ with $\eta$ as the per capita rate of anti-predator behavior of adult prey to the predator population. In population ecology intra-specific competition is an interaction between the members of the same species compete for limited resource. The effects of density dependent mortality on predator prey model has been investigated by Ruan et al. [20]. So, in this work $\alpha$ is taken as a death rate due of intra-specific competition between the adult prey species.

Again, in population dynamics, the functional response is very important to describe the actual nature of both prey and predator population. Now, for the traditional predator-prey model, the functional response depends upon only density of prey population. But according to Berrymen [6], the predator per capita growth rate should decline with its density also. Therefore, to satisfy the above both criteria a functional response should be a function of prey and predator both [2, 5]. In this regard, the following functional responses of predator for consuming juvenile prey and adult prey should be considered as

$$\beta_1xz \quad \text{and} \quad \beta_2yz$$

respectively. Hence, considering above realistic criteria, a predator-prey model has been developed in this paper which is as follows:

$$\begin{align*}
\frac{dx}{dt} &= \gamma y - \beta x - d_1 x - \frac{\beta_1xz}{z + k_1x + k_2y} \\
\frac{dy}{dt} &= \beta x - d_2 y - \alpha y^2 - \frac{\beta_2yz}{z + k_1x + k_2y} \\
\frac{dz}{dt} &= \frac{\mu \beta_1xz}{z + k_1x + k_2y} + \frac{\mu_1 \beta_2yz}{z + k_1x + k_2y} - d_3 z - \eta y z
\end{align*}$$

with nonnegative initial conditions $x(0) \geq 0, y(0) \geq 0$ and $z(0) \geq 0$.

Here, the other parameters involved in the proposed model are described as follows:

- $\gamma$: recruitment rate of juvenile prey.
- $\beta$: portion of juvenile prey who becomes adult.
- $\beta_1$: predation rate of predator to the juvenile prey.
- $\beta_2$: predation rate of predator to the adult prey.
- $\mu$: conservation rate of juvenile prey to the predator.
- $\mu_1$: conservation rate of adult prey to the predator.
- $\eta$: rate of anti-predator behavior of adult prey to the predator.
- $\alpha$: death rate of adult prey due to intra specific competition.
• $d_1, d_2, d_3$: natural death rate of juvenile prey, adult prey and predator respectively.
• $k_1, k_2$: constants for the functional responses.

3. Boundedness of Solutions. In this section, uniform boundedness of the solutions of our proposed system have been discussed.

**Theorem 3.1.** All solutions of the system (1) will be uniformly bounded in $R^3_+$ if $\mu \leq 1$, $\mu_1 \leq 1$ and $\delta = \min\{d_1, d_3\}$.

**Proof.** Now, we construct a function

$$ W = x + y + z $$

Taking time derivative of $W$ and putting the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$, we have

$$ \frac{dW}{dt} = \gamma y - \beta x - d_1 x - \frac{\beta_1 x z}{z + k_1 x + k_2 y} + \frac{\beta_2 y z}{z + k_1 x + k_2 y} - \beta_1 x z + \mu_1 \frac{\beta_2 y z}{z + k_1 x + k_2 y} - d_3 z - \eta y z $$

i.e., $\frac{dW}{dt} = \gamma y - d_1 x - d_2 y - \alpha y^2 - d_3 z - \beta_1 x z (1 - \mu) - \beta_2 y z (1 - \mu_1) - \eta y z$

i.e., $\frac{dW}{dt} \leq \gamma y - d_1 x - d_2 y - \alpha y^2 - d_3 z$, if $\mu \leq 1$ and $\mu_1 \leq 1$

Let us introduce a positive real number $\delta$ then multiplying $\delta$ with $W$ and adding with the above equation and applying the theory stated in [7], we have

$$ \frac{dW}{dt} + \delta W \leq y(\gamma - d_2 - \alpha y + \delta) - (d_1 - \delta)x - (d_3 - \delta)z. $$

i.e., $\frac{dW}{dt} + \delta W \leq y(\gamma - d_2 - \alpha y + \delta)$, taking $\delta = \min\{d_1, d_3\}$

i.e., $\frac{dW}{dt} + \delta W \leq \frac{(\delta + \gamma - d_2)^2}{4\alpha} = Q$. (say)

i.e., $W \leq \frac{Q}{\delta}(1 - e^{-\delta t}) + W(0)e^{-\delta t}$.

For $t \to \infty$, we have $W \leq \frac{Q}{\delta}$.

Hence all the solutions of the system (1) are bounded in the region

$$ \Sigma = \{(x, y, z) \in R^3_+ : W = \frac{Q}{\delta} + \epsilon, \text{ for any } \epsilon > 0\}. $$

**Note:** If the conservation rate of juvenile prey and adult prey to the predator are less than one. Then the solutions of the system (1) will be uniformly bounded.

4. Equilibria and Stability Analysis. In this section, we find all the possible equilibria and discuss their stability analysis.
4.1. Equilibria. The above model has three possible equilibria such as:

(i) The trivial equilibria $E_0(0, 0, 0)$.

(ii) The predator free equilibria $E_1(x_1, y_1, 0)$ exists if $\beta \gamma > d_2(\beta + d_1)$

where $x_1 = \frac{\gamma [\beta \gamma - d_2(\beta + d_1)]}{\alpha(\beta + d_1)^2}$ and $y_1 = \frac{[\beta \gamma - d_2(\beta + d_1)]}{\alpha(\beta + d_1)}$.

(iii) The positive interior equilibria $E^*(x^*, y^*, z^*)$, where

$x^* = \frac{x^*(\mu_{\beta - 1} - d_3 k_1) + y^*(\mu_{\beta - 2} - d_3 k_2) - \eta k_1 x^* y^* - \eta k_2 y^*}{(d_3 + \eta y^*)}$,

$y^* = \frac{\eta}{\eta k_1 + \eta k_2} - \frac{\eta k_1 x^*}{d_3}$,

$z^* = \frac{\eta k_2}{d_3}$.

$A_1y^* + A_2y^* + A_3 = 0$, i.e., $y^* = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$, where $A_1 = (\alpha \mu_1 - \eta k_2)$, $A_2 = \mu_1 d_2 + \mu_2 - \mu_1 d_3 k_2 - \eta k_1 x^*$, $A_3 = x^*(\mu_2 + d_1 + \mu_1 - \mu_1 d_3 k_1)$.

Case 1: If $A_1 > 0$, $A_3 < 0$ then only one positive value of $y^*$ can be obtained

as $y^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$.

Case 2: If $A_1 > 0$, $A_2 < 0$ and $A_3 > 0$ then two positive value of $y^*$ can be found

if $A_2^2 > 4A_1A_3$.

4.2. Stability Analysis. In this section, we have investigated the local stability conditions of different existing equilibrium points.

**Theorem 4.1.** The trivial equilibria $E_0$ will be locally asymptotically stable if $\beta \gamma < d_2(\beta + d_1)$.

**Proof.** To prove this theorem see the method developed by Arino et al. [3].

**Theorem 4.2.** Predator free equilibria $E_1$ is locally asymptotically stable if

$\eta > \left[ \frac{\mu \beta \gamma + \mu_1 \beta_2 (\beta + d_1)}{k_1 \gamma + k_2 (\beta + d_1)} - d_3 \right] \frac{\alpha(\beta + d_1)}{[\beta \gamma - d_2(\beta + d_1)]}$

and $\beta \gamma > d_2(\beta + d_1)$.

**Proof.** The characteristic equation of the jacobian matrix at $E_1(x_1, y_1, 0)$ is

$\frac{1}{(k_1 x_1 + k_2 y_1)}[\mu \beta_1 x_1 + \mu_1 \beta_2 y_1] - d_3 - \eta y_1 - \lambda$

$= \left( \lambda^2 + \lambda(\beta + d_1 + d_2 + 2\alpha y_1) + \beta \gamma - d_2(\beta + d_1) \right) = 0$.

Hence, $E_1$ is locally asymptotically stable if

$\eta > \left[ \frac{\mu \beta \gamma + \mu_1 \beta_2 (\beta + d_1)}{k_1 \gamma + k_2 (\beta + d_1)} - d_3 \right] \frac{\alpha(\beta + d_1)}{[\beta \gamma - d_2(\beta + d_1)]}$

and $\beta \gamma > d_2(\beta + d_1)$. 

**Note:** When the anti-predator behavior of adult prey is grater than the difference of predation of juvenile prey, adult prey by predator and death rate of predator population, divided by the equilibrium biomass of adult prey. Then the predator free equilibrium will be locally asymptotically stable.

**Observation 1:** Suppose that

$\eta > \left[ \frac{\mu \beta \gamma + \mu_1 \beta_2 (\beta + d_1)}{k_1 \gamma + k_2 (\beta + d_1)} - d_3 \right] \frac{\alpha(\beta + d_1)}{[\beta \gamma - d_2(\beta + d_1)]}$
holds. Then for the parametric condition \( \gamma = d_2(\beta + d_1)/\beta \) the model system (1) undergoes through a transcritical bifurcation around the trivial equilibrium point \( E_0 \) as when \( \gamma < d_2(\beta + d_1)/\beta \) then the trivial equilibrium becomes asymptotically stable whereas as for \( \gamma > d_2(\beta + d_1)/\beta \) not only the trivial equilibrium \( E_0 \) becomes unstable but also a new (predator free) equilibrium forms and becomes locally asymptotically stable.

**Theorem 4.3.** The interior equilibria \( E^* \) of the system (1) is locally asymptotically stable if \( \sigma_1 > 0, \sigma_3 > 0 \) and \( \sigma_1\sigma_2 - \sigma_3 > 0 \) holds where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are given within the proof.

**Proof.** The characteristic equation of the jacobian matrix at \( E^*(x^*, y^*, z^*) \) is

\[
\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0.
\]

where

\[
\begin{align*}
\sigma_1 &= -(M_{11} + M_{22} + M_{33}), \\
\sigma_2 &= (M_{11}M_{22} - M_{12}M_{21}) + (M_{11}M_{33} - M_{13}M_{31}) \\
&
+ (M_{22}M_{33} - M_{23}M_{32}), \\
\sigma_3 &= M_{11}(M_{22}M_{33} - M_{22}M_{33}) + M_{12}(M_{21}M_{33} - M_{23}M_{31}) \\
&
+ M_{13}(M_{31}M_{22} - M_{21}M_{32}), \\
\sigma_1\sigma_2 - \sigma_3 &= -M_{11}^2(M_{22}^2 + M_{33}^2) - M_{22}^2(M_{11}^2 + M_{33}^2) - M_{33}^2(M_{11}^2 + M_{22}^2) \\
&
+ M_{12}(M_{12}M_{21} + M_{13}M_{31} + M_{22}(M_{12}M_{21} + M_{23}M_{32}) \\
&
+ M_{33}(M_{13}M_{31} - 2M_{11}M_{22}), \\
M_{11} &= -\beta - d_1 - \beta_1z^*(z^* + k_2y^*) \\
&
\left(\frac{\beta_1k_2x^*z^*}{(z^* + k_1x^* + k_2y^*)^2}\right), \\
M_{12} &= \gamma + \frac{\beta_1x^*(k_2x^* + k_2y^*)}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{13} &= \frac{\beta_3x^*z^*}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{21} &= \beta + \frac{\beta_2k_1y^*z^*}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{22} &= -d_2 - 2\alpha y^* - \frac{\beta_2z^*(z^* + k_1x^*)}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{23} &= -\frac{\beta_2y^*(k_1x^* + k_2y^*)}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{31} &= \frac{\mu \beta_1z^*(z^* + k_2y^*) - k_1\mu_1\beta_2y^*z^*}{(z^* + k_1x^* + k_2y^*)^2}, \\
M_{32} &= \frac{\mu_1\beta_2z^*(z^* + k_1x^*) - \mu_1\beta_1k_2z^*}{(z^* + k_1x^* + k_2y^*)^2} - \eta z^*, \\
M_{33} &= \frac{k_1x^* + k_2y^*}{(z^* + k_1x^* + k_2y^*)^2} (\mu_3x^* + \mu_1\beta_2y^*) - d_3 - \eta y^*.
\end{align*}
\]

By using the Routh-Hurwitz criteria, it is observed that the interior equilibrium point will be locally asymptotically stable if \( \sigma_1 > 0, \sigma_3 > 0 \) and \( \sigma_1\sigma_2 - \sigma_3 > 0 \) holds. \( \square \)
4.3. Global Stability. In this section, the global stability properties of interior equilibria has been discussed with the help of geometric approach developed by Li and Muldowney [15].

**Theorem 4.4.** The system (1) is globally asymptotically stable around its interior equilibria if \( \eta < \frac{\{\mu\beta_1 + \mu_1\beta_2\}}{(1 + k_1 + k_2)} - d_3 - \mu_5 \) where the expression of \( \mu_2 \) and \( \mu_5 \) are given within the proof of this theorem.

**Proof.** The system (1) can be expressed as

\[
\frac{dX}{dt} = f(X),
\]

where \( f(X) = \begin{pmatrix} \gamma y - \beta x - d_1 x - \beta_1 x z \\ \beta x - d_2 y - \alpha y^2 - \frac{\beta_1 x z}{z + k_1 x + k_2 y} \\ \mu_1 \beta_2 y z + \frac{\beta_1 x z}{z + k_1 x + k_2 y} - d_3 z - \eta y z \end{pmatrix} \) and \( X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \).

Then the Jacobian matrix \( J \) of the system (1) is

\[
J = \begin{pmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{pmatrix}
\]

\[
J_{11} = -\beta - d_1 - \frac{\beta_1 z (z + k_2 y)}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{12} = \gamma + \frac{\beta_1 k x z}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{13} = \frac{\beta z (z + k_1 x)}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{21} = \beta + \frac{\beta_2 k_1 y z}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{22} = -d_2 - 2\alpha y - \frac{\beta_2 z (z + k_1 x)}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{23} = -\frac{\beta_2 y (k_1 x + k_2 y)}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{31} = \frac{(\mu\beta_1 z (z + k_2 y) - \mu_1 \beta_2 y z)}{(z + k_1 x + k_2 y)^2},
\]

\[
J_{32} = \frac{(\mu_1 \beta_2 z (z + k_1 x) - \mu_1 \beta_2 x z)}{(z + k_1 x + k_2 y)^2} - \eta z,
\]

\[
J_{33} = \frac{k_1 x + k_2 y}{(z + k_1 x + k_2 y)^2} (\mu_1 \beta_2 y x + \mu_1 \beta_2 y z) - d_3 - \eta y.
\]

If \( J^{[2]} \) be the second additive compound matrix jacobian matrix of the \( J \), then

\[
J^{[2]} = \begin{pmatrix}
J_{11} + J_{22} & J_{23} & -J_{13} \\
J_{52} & J_{11} + J_{33} & -J_{12} \\
-J_{31} & J_{21} & J_{22} + J_{33}
\end{pmatrix}
\]
Next, we consider $P(X)$ in $C^1(D)$ in such a way that $P = \text{diag}\left\{ \frac{x}{z}, \frac{x}{z}, \frac{x}{z} \right\}$ and then we have $P^{-1} = \text{diag}\left\{ \frac{z}{x}, \frac{z}{x}, \frac{z}{x} \right\}$. 

Now, $P_f = \frac{dP}{dX} = \text{diag}\left\{ \frac{\dot{x}}{z} - \frac{x}{z^2}, \frac{\dot{z}}{z} - \frac{x}{z^2}, \frac{\dot{z}}{z} - \frac{x}{z^2} \right\}$.

Next, we consider $8P$. PANJA, S. JANA AND S.K. MONDAL

Then, using the above equation (2), we have $P_fP^{-1} = \text{diag}\left\{ \frac{\dot{x}}{z} - \frac{\dot{z}}{z}, \frac{\dot{z}}{z} - \frac{\dot{z}}{z}, \frac{\dot{z}}{z} - \frac{\dot{z}}{z} \right\}$ and $PJ^{[2]}P^{-1} = J^{[2]}$.

Also, we have

\[ B = P_fP^{-1} + PJ^{[2]}P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \]

where $B_{11} = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + J_{11} + J_{22}$, $B_{12} = \begin{pmatrix} J_{23} & -J_{13} \end{pmatrix}$, $B_{21} = \begin{pmatrix} J_{32} & -J_{31} \end{pmatrix}^t$ and $B_{22} = \begin{pmatrix} J_{11} + J_{33} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$.

Let us introduce the following vector norm in $R^4$ of the form

\[ |(u, v, w)| = \max\{|u|, |v| + |w|\}. \]

where $(u, v, w)$ is the vector in $R^4$ and Lozinskii measure with respect to this norm is denoted by $\Gamma$.

So, $\Gamma(B) \leq \sup\{p_1, p_2\}$ where $p_1 = \Gamma_1(B_{11}) + |B_{12}|$ and $p_2 = \Gamma_1(B_{22}) + |B_{21}|$, where $|B_{12}|, |B_{21}|$ are the matrix norms with respect to the $L^1$ vector norm and $\Gamma_1$ is the Lozinskii measure with respect to that norm. Then the required values can be obtained as

\[
\Gamma_1(B_{11}) = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + J_{11} + J_{22}, \\
|B_{12}| = \max\{|J_{23}|, | - J_{13}|\}, |B_{21}| = \max\{|J_{32}|, | - J_{31}|\}.
\]

\[
\Gamma_1(B_{22}) = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + \max\{J_{11} + J_{33} + J_{21} + J_{12} + J_{22}\}.
\]

From the third equation of system (1) we have

\[
\frac{\dot{z}}{z} = \frac{\mu_1 \beta_1 x}{z + k_1 x + k_2 y} + \frac{\mu_1 \beta_2 y}{z + k_1 x + k_2 y} - d_3 - \eta y. \tag{2}
\]

Then, using the above equation (2), we have

\[
p_1 = \frac{\dot{x}}{x} - \frac{\mu_1 \beta_1 x}{z + k_1 x + k_2 y} - \frac{\mu_1 \beta_2 y}{z + k_1 x + k_2 y} + d_3 + \eta y + J_{11} + J_{22} \\
\quad + \max\{|J_{23}|, | - J_{13}|\}.
\]

\[
p_2 = \frac{\dot{x}}{x} - \frac{\mu_1 \beta_1 x}{z + k_1 x + k_2 y} - \frac{\mu_1 \beta_2 y}{z + k_1 x + k_2 y} + d_3 + \eta y \\
\quad + \max\{J_{11} + J_{33} + J_{21} + J_{12} + J_{22}\} + \max\{|J_{32}|, | - J_{31}|\}.
\]

Therefore, from the above we have

\[
\Gamma(B) \leq \frac{\dot{x}}{x} - \frac{\mu_1 \beta_1 x}{z + k_1 x + k_2 y} - \frac{\mu_1 \beta_2 y}{z + k_1 x + k_2 y} + d_3 + \eta y \\
\quad + \max\{|J_{11} + J_{22} + |J_{23}|, J_{11} + J_{22} + | - J_{13}|\}, \tag{3}
\]

\[
\max\{J_{11} + J_{33} + J_{21} + J_{32}, J_{12} + J_{22} + | - J_{31}|\}.
\]
Hence, the system (1) will be globally asymptotically stable around the interior predator free equilibria

\[ E \]

For this set of parametric values Figure 1 has been drawn which shows that the system is stable. Therefore, from above equation (3), we have

\[ \mu_3 = \max \{ -\beta - d_1 - \frac{\beta_1(1 + k_2)}{(1 + k_1 + k_2)^2} - d_2 - 2\alpha \mu_2 - \frac{\beta_2(1 + k_1)}{(1 + k_1 + k_2)^2} \] 

\[ + \frac{\beta_2(k_1 + k_2)}{(1 + k_1 + k_2)^2} - \beta - d_1 - \frac{\beta_1(1 + k_2)}{(1 + k_1 + k_2)^2} - d_2 - 2\alpha \mu_2 \] 

\[ - \frac{\beta_2(1 + k_1)}{(1 + k_1 + k_2)^2} + \frac{\beta_1(k_1 + k_2)}{(1 + k_1 + k_2)^2} \} \]

\[ \mu_4 = \max \{ -\beta - d_1 - \frac{\beta_1(1 + k_2)}{(1 + k_1 + k_2)^2} + \frac{k_1 + k_2}{(1 + k_1 + k_2)^2} (\mu \beta_1 + \mu_1 \beta_2) - d_3 \] 

\[ - \eta \mu_2 + \beta + \frac{\beta_2 k_1}{(1 + k_1 + k_2)^2} + |(\mu \beta_1(1 + k_1) - \mu_1 \beta_2 k_1)| - \eta \mu_2 |, \]

\[ \gamma + \frac{\beta_1 k_2}{(1 + k_1 + k_2)^2} - d_2 - 2\alpha \mu_2 - \frac{\beta_2(1 + k_1)}{(1 + k_1 + k_2)^2} \] 

\[ + |(\mu \beta_1(1 + k_2) - k_1 \mu_1 \beta_2)| \} \] 

\[ \mu_5 = \max \{ \mu_3, \mu_4 \} \]

Therefore, from above equation (3), we have

\[ \Gamma (B) \leq \frac{\dot{x}}{x} - \frac{\mu \beta_1}{1 + k_1 + k_2} - \frac{\mu_1 \beta_2}{1 + k_1 + k_2} + d_3 + \eta \mu_2 + \mu_5 \]

\[ i.e., \Gamma (B) \leq \frac{\dot{x}}{x} - \left( \frac{\mu \beta_1}{1 + k_1 + k_2} + \frac{\mu_1 \beta_2}{1 + k_1 + k_2} - d_3 - \eta \mu_2 - \mu_5 \right) \]

Now, integrating the above equation in \([0, t]\) we have

\[ \int_{0}^{t} \Gamma (B) ds \leq \log \frac{x(t)}{x(0)} - \left( \frac{\mu \beta_1}{1 + k_1 + k_2} + \frac{\mu_1 \beta_2}{1 + k_1 + k_2} - d_3 - \eta \mu_2 - \mu_5 \right) t. \]

\[ \frac{1}{t} \int_{0}^{t} \Gamma (B) ds \leq \frac{1}{t} \log \frac{x(t)}{x(0)} - \left( \frac{\mu \beta_1}{1 + k_1 + k_2} + \frac{\mu_1 \beta_2}{1 + k_1 + k_2} - d_3 - \eta \mu_2 - \mu_5 \right). \]

\[ \lim_{t \to \infty} \sup \frac{1}{t} \int_{0}^{t} \Gamma (B) ds < - \left( \frac{\mu \beta_1}{1 + k_1 + k_2} + \frac{\mu_1 \beta_2}{1 + k_1 + k_2} - d_3 - \eta \mu_2 - \mu_5 \right) < 0. \]

Hence, the system (1) will be globally asymptotically stable around the interior equilibria \( E^* \) if \( \eta < \left\{ \frac{(\mu \beta_1 + \mu_1 \beta_2)}{(1 + k_1 + k_2)} - d_3 - \mu_5 \right\} / \mu_2 \).

5. Numerical Simulations. In this section, the dynamical behavior of the proposed model (1) has been discussed numerically using MATLAB. Let us consider a set of parametric values as: \( \gamma = 1.5, \beta = 0.3, d_1 = 0.25, \beta_1 = 0.1, k_1 = 0.1, k_2 = 0.5, d_2 = 0.13, \alpha = 0.14, \beta_2 = 0.2, \mu = 0.3, \mu_1 = 0.4, d_3 = 0.2, \eta = 0.2 \). For this set of parametric values Figure 1 has been drawn which shows that the predator free equilibria \( E_1(13.41, 4.911, 0) \) is locally asymptotically stable.
Again, we consider another set of parametric values as $\gamma = 1.5$, $\beta = 0.29$, $d_1 = 0.25$, $\beta_1 = 0.1$, $k_1 = 0.01$, $k_2 = 0.02$, $d_2 = 0.13$, $\alpha = 0.14$, $\beta_2 = 0.2$, $\mu = 0.75$, $\mu_1 = 0.8$, $d_3 = 0.25$, $\eta = 0.01$. Using the above set of parametric values Figure 2 has been drawn. From this figure, it is seen that the interior equilibria $E^*(6.076, 2.579, 3.034)$ is locally asymptotically stable.

Also, another set of parametric values have been considered as $\gamma = 0.5$, $\beta = 2.5$, $d_1 = 0.25$, $\beta_1 = 0.5$, $k_1 = 1.0$, $k_2 = 1.0$, $d_2 = 0.23$, $\alpha = 0.1$, $\beta_2 = 0.29$, $\mu = 0.2$, $\mu_1 = 0.7$, $d_3 = 0.11$, $\eta = 0.0001$. For this set of parametric values Figure 3 has been drawn with different initial conditions such as $(1.2, 3)$, $(2.3, 4)$, $(2.5, 1)$ and $(2.6, 1)$. From this figure, it is observed that the interior equilibria is globally asymptotically stable.

To discuss the dynamical behavior of system (1), the parametric values are considered as follows: $\gamma = 0.5$, $\beta = 2.5$, $d_1 = 0.25$, $\beta_1 = 0.5$, $k_1 = 1.0$, $k_2 = 1.0$, $d_2 =$
0.23, \alpha = 0.09, \beta_2 = 0.29, \mu = 0.2, \mu_1 = 0.98, d_3 = 0.11. Using this set of parametric values Figure 4 has been drawn with respect to change of the parameter \eta \in (0,0.01). From this figure, it is observed that the proposed system becomes unstable within the interval 0 \leq \eta < 0.0048 and the system remains stable in \eta > 0.0048. So, it can be concluded from this figure that the anti-predator behavior rate (\eta) has an important role in the stability of the system (1).

Using the same set of parametric values used in Figure 4 except \gamma = 1.2, d_1 = 0.19, \beta_1 = 0.9, d_2 = 0.24, \alpha = 0.089, \eta = 0.01, the Figure 5 has been drawn with respect to the parameter \beta \in (0.4,0.5). From this figure, it is seen that the system remains unstable in 0.4 \leq \beta \leq 0.411 and becomes stable in \beta > 0.411. Then, it can be concluded that the transmission rate from juvenile to adult prey (\beta) has a major role to the stability of the system (1).

Taking the same set of parametric values used in Figure 4 except \gamma = 1.1, \beta = 0.5, d_1 = 0.2, d_2 = 0.23, \alpha = 0.09, \eta = 0.1, the Figure 6 has been drawn with respect to the parameter \beta_1 \in (0.95,0.98). From this figure, it is observed that as the value of \beta_1 increases then the system remains stable in the range 0.95 \leq \beta_1 < 0.9623 and
unstable if $\beta_1 > 0.9623$. So, it can be concluded that the predation rate of juvenile prey to the predator ($\beta_1$) has vital impact on the stability of system (1).

Using the same set of parametric values except $\gamma = 1.2, \beta = 0.6, d_1 = 0.2, \beta_1 = 0.9, d_2 = 0.24, \alpha = 0.089, \beta_2 = 0.37, \eta = 0.01$, the Figure 7 has been drawn with respect to the parameter $\beta_2 \in (0.37, 0.38)$. From this figure, it is seen that as the value of $\beta_2$ increases then the system remains stable within the range $0.37 \leq \beta_2 \leq 0.3774$ and the system becomes unstable when $\beta_2 > 0.3774$. It can be concluded from this figure that predation rate of adult prey to the predator ($\beta_2$) has the ability to change the behavior of the proposed system (1).

Again, we consider the same set of parametric values except $\beta = 0.9, d_1 = 0.0017, \beta_1 = 0.9, d_2 = 0.19, \alpha = 0.09, \beta_2 = 0.28, \eta = 0.01$, the Figure 8 has been drawn with respect to the parameter $\gamma \in (0.6, 0.7)$. Then, the system remains unstable within the range $0.6 \leq \gamma < 0.608$ and stable if $\gamma > 0.608$. From this figure,
it can be concluded that the recruitment rate of juvenile prey ($\gamma$) has an important role on the stability of the system (1).

From the numerical simulation, it is seen that system becomes unstable within the interval $0 \leq \eta < 0.0048$ and the system remains stable in $\eta > 0.0048$. Also, it is observed that the system remains unstable in the interval $0.4 \leq \beta \leq 0.411$ and becomes stable in $\beta > 0.411$. Again, it is seen that as the value of $\beta_1$ increases then the system remains stable in the range $0.95 \leq \beta_1 < 0.9623$ and unstable if $\beta_1 > 0.9623$. It is observed that as the value of $\beta_2$ increases then the system remains stable within the range $0.37 \leq \beta_2 \leq 0.3774$ and the system becomes unstable when $\beta_2 > 0.3774$. Again, it is seen that the system remains unstable within the range $0.6 \leq \gamma < 0.608$ and stable if $\gamma > 0.608$. So, from the numerical simulation results
it can be concluded that the parameters $\eta, \beta, \beta_1, \beta_2$ and $\gamma$ have an significant effects on the stability of the proposed system.

6. Conclusion. In our present work, we thoroughly described the dynamical behavior of our proposed model through both theoretical and numerical experiments. Our investigation ensures the existence of three possible equilibria including vanishing or trivial, predator free and the co-existence one. The asymptotic local stability analysis of the equilibria shows that all the equilibria are not only conditionally asymptotically stable but also there is a possibility of occurring transcritical bifurcation around the trivial equilibrium. Thus, it can be concluded that the recruitment rate of juvenile prey populations ($\gamma$) plays an important role regarding the existence of entire prey species in the system as a comparative less value of $\gamma$ would tend to the system asymptotically stable around trivial equilibrium (see theorem 2) whereas a comparative higher value of $\gamma$ assures the system existence of entire prey species (see theorem 3).

We mainly study the behavior of the system around co-existing equilibrium due to its natural importance. Apart from locally asymptotical stability criteria, we investigate the existence of Hopf bifurcation and global asymptotic stability criteria of the interior equilibrium. A geometric method has been applied to investigate the global asymptotic stability of the proposed system around the interior equilibrium. There are very few papers on prey predator mathematical model with anti-predator behavior. The existing paper considered the anti-predator behavior on the whole prey species [22]. But in reality, it is seen that adults prey attacks their vulnerable predators and save the younger prey from predation. So, in this mathematical model we introduced the anti-predator behavior on the adult prey only. Also, the ratio dependent functional response function has been considered along with the anti-predator behavior of adult prey population which are newly introduced in this paper.

REFERENCES

[1] M. Andersson and C. G. Wiklund, Clumping versus spacing out: Experiments on nest predation in fieldfares (Turdus pilaris), Animal Behavior, 26 (1978), 1207–1212.
[2] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics ratio dependence, Journal of Theoretical Biology, 139 (1989), 311–326.
[3] O. Arino, A. Abdllaoui, J. Mikram and J. Chattopadhyay, Infection on prey population may act as biological control in ratio-dependent predator-prey model, Nonlinearity, 17 (2004), 1101–1116.
[4] M. Bandyopadhyay and J. Chattopadhyay, Ratio-dependent predator-prey model: Effect of environmental fluctuation and stability, Nonlinearity, 18 (2005), 913–936.
[5] M. Banerjee, Self-replication of spatial patterns in a ratio-dependent predator-prey model, Mathematical and Computer Modelling, 51 (2010), 44–52.
[6] A. A. Berryman, The origin and evolution of predator-prey theory, Ecology, 73 (1992), 1530–1535.
[7] G. Birkoff, G. C. Rota, Ordinary Differential Equations, Ginn, Boston, 1982.
[8] Y. Choh, M. Ignacio, M. W. Sabelis and A. Janssen, Predator-prey role reversal, juvenile experience and adult antipredator behavior, Scientific Reports, 2 (2012), 7–28.
[9] Y. H. Fan and W. T. Li, Permanence for a delayed discrete ratio-dependent predator-prey system with Holling type functional response, Journal of Mathematical Analysis and Applications, 299 (2004), 357–374.
[10] H. I. Freedman and A. M. Mathsen, Persistence in predator-prey systems with ratio-dependent predator influence, Bulletin of Mathematical Biology, 55 (1993), 817–827.
[11] S. Gakkhar and K. Negi, A mathematical model for viral infection in toxin producing phytoplankton and zooplankton system, *Applied Mathematics and Computation*, 179 (2006), 301–313.

[12] A. Hugo, E. S. Massawe and O. D. Makinde, An ecoepidemiological mathematical model with treatment and disease infection in both prey and predator population, *Journal of Ecology and The Natural Environment*, 4 (2012), 266–279.

[13] A. Janssen, F. Faraji, T. Van Der Hammen, S. Magalhaes and M. W. Sabelis, Interspecific infanticide deters predators, *Ecology Letters*, 5 (2002), 490–494.

[14] Y. Kuang and E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *Journal of Mathematical Biology*, 43 (2001), 221–290.

[15] M. Y. Li and J. S. Muldowney, A geometric approach global stability problems, *SIAM Journal on Mathematical Analysis*, 27 (1996), 1070–1083.

[16] A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins, Baltimore, 1925.

[17] P. Panja and S. K. Mondal, Stability analysis of coexistence of three species prey-predator model, *Nonlinear Dynamics*, 81 (2015), 373–382.

[18] Y. Pei, L. Chen, Q. Zhang and C. Li, Extinction and permanence of one-preymulti-predators of Holling type II function response system with impulsive biological control, *Journal of Theoretical Biology*, 235 (2005), 495–503.

[19] G. A. Polis, C. A. Myers and R. D. Holt, The ecology and evaluation of intraguild predation-potential competitors that eat each other, *Annual Review of Ecology and Systematics*, 20 (1989), 297–330.

[20] S. Ruan, A. Ardito, P. Ricciardi and D. L. DeAngalis, Coexistence in competition models with density dependent mortality, *Comptes Rendus Biologies*, 330 (2007), 845–854.

[21] C. B. Stanford, The influence of chimpanzee predation on group size and anti-predator behavior in red colobus monkeys, *Animal Behavior*, 49 (1995), 577–587.

[22] B. Tang and Y. Xiao, Bifurcation analysis of a predator-prey model with anti-predator behavior, *Chaos, Solitons & Fractals*, 70 (2015), 58–68.

[23] F. Wei and Q. Fu, Hopf bifurcation and stability for predator-prey systems with Beddington-DeAngelis type functional response and stage structure for prey incorporating refuge, *Applied Mathematical Modelling*, 40 (2016), 126–134.

Received February 2020; 1st revision February 2020; Final revision February 2020.

*E-mail address:* prabirpanja@gmail.com
*E-mail address:* soovoojeet@gmail.com
*E-mail address:* shyamal_260180@yahoo.com