NORMALITY AND UNIQUENESS OF LAGRANGE MULTIPLIERS

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ABSTRACT. In this paper we study, for certain problems in the calculus of variations and optimal control, two different questions related to uniqueness of multipliers appearing in first order necessary conditions. One deals with conditions under which a given multiplier associated with an extremal of a fixed function is unique, a property which, in nonlinear programming, is known to be equivalent to the strict Mangasarian-Fromovitz constraint qualification. We show that, for isoperimetric problems in the calculus of variations, a similar characterization holds, but not in optimal control where the corresponding condition is only sufficient for the uniqueness of the multiplier. The other question is related to the set of multipliers associated with all functions for which a solution to the constrained problem is given. We prove that, for both types of problems, this set is a singleton if and only if a strong normality assumption holds.

Who wants to be normal when you can be unique?
Helena Bonham Carter

1. Introduction. In this paper we study, for certain classes of constrained optimization problems, uniqueness of Lagrange multipliers satisfying first order necessary conditions. The issue arises, for example, in sensitivity analysis of optimization problems (see [16, 17, 21] and references therein), in multi-objective optimization [6], or in convergence analysis of optimization algorithms [7].

The monograph on optimization theory by Giorgi, Guerraggio and Thierfelder [12], which treats both the smooth and nonsmooth case, provides an excellent study of optimality conditions for nonlinear programming problems. There, for the problem of minimizing a given real-valued function on \( \mathbb{R}^n \) subject to inequality and equality constraints, one can find one result on uniqueness of Lagrange multipliers (Theorem 3.10.4). It is inserted in Chapter 3, Section 10, as part of a theorem where second order optimality conditions are derived.

To explain the relation between these two aspects (uniqueness of multipliers and second order conditions) let us first mention that, according to [12], Bazaraa, Sherali and Shetty [2] and Ben-Tal [5] provide a false result on second order necessary conditions (a counterexample is given) and a correct result was derived by Kyparisis [15]. Briefly, following [12], the authors in [2, 5] claim that, assuming the linear independence constraint qualification, second order conditions hold on the set of tangential constraints relative to the original set of inequality and equality constraints. The result given in [15], on the other hand, imposes a weaker assumption
called in that paper the “strict Mangasarian-Fromovitz constraint qualification” (introduced, according to [15], by Fujiwara, Han and Mangasarian in [10]) and the second order conditions then hold on the set of tangential constraints relative to a subset of the previous one which takes into account the sign of the corresponding Lagrange multipliers. Moreover, as shown first in [15] (by using theorems of alternative) and later, similarly, in [12], uniqueness of the Lagrange multipliers turns out to be equivalent precisely to that strict constraint qualification.

Now, the definition of this strict version of the Mangasarian-Fromovitz constraint qualification requires the existence of Lagrange multipliers given beforehand. The set of tangential constraints where the second order conditions hold includes inequalities whenever these multipliers vanish, and equalities if they are positive. However, according to [12], it could not properly be considered a constraint qualification since the set of active indices which are positive is not known before the validation of the first order necessary conditions. A similar statement is made by Wachsmuth in [22], where uniqueness of the multipliers is studied in terms of (non-strict) constraint qualifications. There, it is shown that the linear independence constraint qualification is the weakest constraint qualification which ensures the existence and uniqueness of Lagrange multipliers.

In this paper we study similar questions for certain problems in the calculus of variations and optimal control. For these problems, we are interested in characterizing uniqueness of the multipliers appearing in first order conditions. In particular, we show that the results of [15] and [22] can be extended to problems in the calculus of variations involving isoperimetric inequality constraints. Also, we show how the results of [22] can be generalized, in terms of the corresponding linear independence constraint qualification, to optimal control problems with inequality and equality constraints in the control functions. However, for this kind of problems, we provide some examples showing that the characterization given in [15] of the uniqueness of Lagrange multipliers in nonlinear programming, in terms of the strict Mangasarian-Fromovitz constraint qualification, may not hold. In general, as we shall see, the corresponding constraint qualification implies uniqueness of the multipliers but the converse, contrary to the result of [15] in the finite dimensional case, is not necessarily true.

2. The finite dimensional case. In this section we shall elaborate with more detail on some of the ideas mentioned above. The main object is to summarize (and explain with some detail) the results on uniqueness of Lagrange multipliers for the finite dimensional case established in [15] and [22]. We begin with a classical approach, based on the notion of “constraint qualifications,” yielding the main results of those two references. We then provide a second approach, based on the notion of “regularity” as presented in [13, 14], which allows us to explain the main ideas stated before in (what we believe to be) a clearer and succinct way.

The nonlinear programming problem we shall deal with, which we label (N), is that of minimizing \( f \) on the set \( S \), where \( f, g_i : \mathbb{R}^n \to \mathbb{R} \) (\( i \in A \cup B \)) are given functions, \( A = \{1, \ldots, p\} \), \( B = \{p + 1, \ldots, m\} \), and

\[
S := \{x \in \mathbb{R}^n \mid g_\alpha(x) \leq 0 (\alpha \in A), \ g_\beta(x) = 0 (\beta \in B)\}.
\]

- A classical approach

For this approach we shall assume, as in [5, 15, 22], that the functions defining the problem are continuously differentiable and, when second derivatives occur,
they are twice continuously differentiable (weaker assumptions can be found, for example, in [13, 14, 19]).

**Definition 2.1.** Denote by \( \Lambda(f, x_0) \) the set of all \( \lambda \in \mathbb{R}^m \) satisfying the Karush-Kuhn-Tucker (KKT) conditions (or first order Lagrange multiplier rule)

i. \( \lambda_\alpha \geq 0 \) and \( \lambda_\alpha g_\alpha(x_0) = 0 \) (\( \alpha \in A \)).

ii. If \( F(x) := f(x) + \langle \lambda, g(x) \rangle \) then \( F'(x_0) = 0 \).

The function \( F \) is the standard Lagrangian, \( g \) is the function mapping \( \mathbb{R}^n \) to \( \mathbb{R}^m \) whose components are \( g_1, \ldots, g_m \), and \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^n \) so that \( \langle \lambda, g(x) \rangle = \sum_1^m \lambda_i g_i(x) \). The real numbers \( \lambda_1, \ldots, \lambda_m \) are called the Kuhn-Tucker or Lagrange multipliers.

In general, if \( x_0 \) affords a local minimum to \( f \) on \( S \), the (KKT) conditions may not hold at \( x_0 \), and some additional assumptions should be imposed to guarantee that \( \Lambda(f, x_0) \neq \emptyset \). Assumptions of this nature are usually referred to as constraint qualifications (see [12]) since they involve only the constraints and are independent of the geometric structure of the feasible set \( S \) (a broader definition, in terms of critical directions, is given in [5]). Equivalently, they correspond to conditions which assure the positiveness of the cost multiplier \( \lambda_0 \) in the Fritz John necessary optimality condition, which can be stated as follows.

**Theorem 2.2.** If \( x_0 \) solves (N) locally, then there exist \( \lambda_0 \geq 0 \) and \( \lambda \in \mathbb{R}^m \), not both zero, such that

i. \( \lambda_0 \geq 0 \) and \( \lambda_\alpha g_\alpha(x_0) = 0 \) (\( \alpha \in A \)).

ii. If \( F_0(x) := \lambda_0 f(x) + \langle \lambda, g(x) \rangle \) then \( F_0'(x_0) = 0 \).

Based on the theory of augmentability, a simple proof of this result is provided by McShane [18]. In [12], the proof given uses Motzkin theorem of the alternative. There are many well-known constraint qualifications. A detailed explanation of some of them is given in [12] which includes, to mention a few, those of Slater (1950), Karlin (1959), Arrow-Hurwicz-Uzawa (1961), Nondegeneracy or linear independence (1961), Cottle-Dragomirescu (1967), Mangasarian-Fromovitz (1969), Zangwill (1969), Kuhn-Tucker (1951), Abadie (1967), and Guignard-Gould-Tolle (1971).

For all \( x \in S \), denote by \( I(x) := \{ \alpha \in A \mid g_\alpha(x) = 0 \} \) the set of active (or effective or binding) indices at \( x \). In [15], two well-known constraint qualifications are mentioned, and they can be stated as follows.

**(MF) Mangasarian-Fromovitz** at \( x_0 \). The set \( \{g'_\beta(x_0) \mid \beta \in B \} \) is linearly independent and \( \exists h \) such that

\[
\begin{align*}
g'_\alpha(x_0; h) &< 0 \ (\alpha \in I(x_0)), \\
g'_\beta(x_0; h) &= 0 \ (\beta \in B).
\end{align*}
\]

**(LI) Linear independence condition** at \( x_0 \). The set \( \{g'_i(x_0) \mid i \in I(x_0) \cup B \} \) is linearly independent.

A third condition (“a new condition introduced in [10]”) which is more restrictive than (MF) but less restrictive than (LI) corresponds to the following.

**(SMF) Strict Mangasarian-Fromovitz** at \( x_0 \). Given \( \lambda \in \Lambda(f, x_0) \), if \( \Gamma = \{ \alpha \in A \mid \lambda_\alpha > 0 \} \), then the set \( \{g'_\beta(x_0) \mid \beta \in \Gamma \cup B \} \) is linearly independent and \( \exists h \) such that

\[
\begin{align*}
g'_\alpha(x_0; h) &< 0 \ (\alpha \in I(x_0), \ \lambda_\alpha = 0), \\
g'_\beta(x_0; h) &= 0 \ (\beta \in \Gamma \cup B).
\end{align*}
\]
The characterization of uniqueness of Lagrange multipliers given in [15] corresponds to the following result.

**Theorem 2.3.** Let \( x_0 \in S \) and suppose \( \lambda \in \Lambda(f, x_0) \). Then (SMF) holds at \( x_0 \) \( \iff \) \( \Lambda(f, x_0) = \{ \lambda \} \).

A second result given in [15], connecting the strict Mangasarian-Fromovitz constraint qualification with second order necessary conditions, is stated as follows.

**Theorem 2.4.** Suppose \( x_0 \) solves (N) locally and \( \lambda \in \Lambda(f, x_0) \). If (SMF) holds at \( x_0 \) (or, equivalently, if \( \lambda \) is unique) then \( F''(x_0; h) \geq 0 \) for all \( h \) satisfying

\[
g'_\alpha(x_0; h) \leq 0 \quad (\alpha \in I(x_0), \ \lambda_\alpha = 0), \quad g'_\beta(x_0; h) = 0 \quad (\beta \in \Gamma \cup B).
\]

The proof of this result, given in [15], says: “The proof parallels exactly the proof of Theorem 3.3 in Ben-Tal [5] which in turn follows directly from a more general Theorem 3.2 in [5].” Let us mention that Theorem 3.3, as stated in [5], corresponds precisely to the false result mentioned in [12] and the correct result mentioned in [12] is precisely the one given in [15], that is, Theorem 2.4 above.

Explicitly, Theorem 3.3 in [5] states that, if \( x_0 \) affords a local minimum to \( f \) on \( S \) and (LI) holds, then \( \exists \lambda \in \Lambda(f, x_0) \) such that \( F''(x_0; h) \geq 0 \) for all \( h \) satisfying

\[
g'_\alpha(x_0; h) \leq 0 \quad (\alpha \in I(x_0)), \quad g'_\beta(x_0; h) = 0 \quad (\beta \in B).
\]

A simple counterexample to this result is given in [12].

Let us turn now to the results given in [22] treating also, though from a different viewpoint, the question of uniqueness of Lagrange multipliers. To do so, let us begin with the concept of tangent cone.

When choosing a specific definition of this notion (also for modified cone approximations, including Clarke’s tangent cone) it feels, as Aubin and Frankowska put it in [1], “like opening the door of a menagerie of tangents, and facing the choice of a favorite pet!” For our purposes, we shall find convenient to choose the definition given by Hestenes [13] which, as shown in [12], is equivalent to the one introduced by Bouligand (1932), also known as the contingent cone to \( S \) at \( x_0 \). Other authors such as Bazaraa, Goode, Nashed, Varaiya, Kurcyusz, Rockafellar, Saks, Rogak, Scott-Thomas, Elster, Thierfelder, and many more, have given various equivalent definitions of such a cone.

**Definition 2.5.** We shall say that a sequence \( \{x_q\} \subset R^n \) converges to \( x_0 \) in the direction \( h \) if \( h \) is a unit vector, \( x_q \neq x_0 \), and

\[
\lim_{q \to \infty} |x_q - x_0| = 0, \quad \lim_{q \to \infty} \frac{x_q - x_0}{|x_q - x_0|} = h.
\]

The tangent cone of \( S \) at \( x_0 \), denoted by \( T_S(x_0) \), is the (closed) cone determined by the unit vectors \( h \) for which there exists a sequence \( \{x_q\} \) in \( S \) converging to \( x_0 \) in the direction \( h \). Equivalently (see [14]), \( T_S(x_0) \) is the set of all \( h \in R^n \) for which there exist a sequence \( \{x_q\} \) in \( S \) and a sequence \( \{t_q\} \) of positive numbers such that

\[
\lim_{q \to \infty} t_q = 0, \quad \lim_{q \to \infty} \frac{x_q - x_0}{t_q} = h.
\]
Let us just briefly mention that the latter is the definition of tangent cone chosen by Wachsmuth in [22] except that the requirement that the sequence \( \{ x_q \} \) should belong to \( S \) is omitted. This, however, crucial in the definition of tangent cone of a specific set \( S \) at a point.

Now, clearly, if \( \{ x_q \} \) converges to \( x_0 \) in the direction \( h \) and \( f \) has a differential at \( x_0 \), then
\[
\lim_{q \to \infty} \frac{f(x_q) - f(x_0)}{|x_q - x_0|} = f'(x_0; h).
\]
If \( f \) has a second differential at \( x_0 \), then
\[
\lim_{q \to \infty} \frac{f(x_q) - f(x_0) - f'(x_0; x_q - x_0)}{|x_q - x_0|^2} = \frac{1}{2} f''(x_0; h).
\]
From these facts, first and second order necessary conditions follow straightforwardly.

**Theorem 2.6.** Suppose \( x_0 \) solves \((N)\) locally. Then \( f'(x_0; h) \geq 0 \) for all \( h \in T_S(x_0) \).

**Proof.** Let \( h \in T_S(x_0) \) be a unit vector and \( \{ x_q \} \subset S \) a sequence converging to \( x_0 \) in the direction \( h \). For large values of \( q \) we have \( f(x_q) \geq f(x_0) \) and, therefore,
\[
0 \leq \lim_{q \to \infty} \frac{f(x_q) - f(x_0)}{|x_q - x_0|} = f'(x_0; h).
\]
If also \( f'(x_0) = 0 \), then
\[
0 \leq \lim_{q \to \infty} \frac{f(x_q) - f(x_0)}{|x_q - x_0|^2} = \frac{1}{2} f''(x_0; h).
\]

Now, recall that, for any \( B \subset \mathbb{R}^n \), the set
\[
B^* = \{ z \in \mathbb{R}^n \mid \langle y, z \rangle \leq 0 \text{ for all } y \in B \}
\]
is a closed convex cone, called the dual or polar cone of \( B \). The dual cone \( T_S^*(x_0) \) of the tangent cone of \( S \) at \( x_0 \) is called the normal cone of \( S \) at \( x_0 \). By the first part of Theorem 2.6, if \( x_0 \) is a local minimum point of a \( C^1 \) function \( f \) on a set \( S \) (actually, merely differentiability at \( x_0 \) is required) then the negative gradient \(-f'(x_0)\) is an outer normal of \( S \) at \( x_0 \), that is, \(-f'(x_0) \in T_S^*(x_0)\), i.e., \( f'(x_0; h) \geq 0 \) for all \( h \in T_S(x_0) \).

From the theory of convex cones (see, for example, [13, 14]) or using the Farkas-Minkowski theorem of the alternative (see [12]), it follows that
\[
\Lambda(f, x_0) \neq \emptyset \iff f'(x_0; h) \geq 0 \text{ for all } h \in R_S(x_0),
\]
that is, \(-f'(x_0) \in R_S^*(x_0)\), where
\[
R_S(x_0) := \{ h \in \mathbb{R}^n \mid g'_\alpha(x_0; h) \leq 0 (\alpha \in I(x_0)), \ g'_{\beta}(x_0; h) = 0 (\beta \in B) \}
\]
is the set of vectors satisfying the tangential constraints at \( x_0 \) (see [13, 14]), also called the linearized tangent cone or the cone of locally constrained directions (see [12]). Note that \( T_S(x_0) \subset R_S(x_0) \) and hence \( R_S^*(x_0) \subset T_S^*(x_0) \).
In [22], the term constraint qualifications corresponds to assumptions on the constraints which ensure that the condition \(- f'(x_0) \in R_S(x_0)\) is a necessary optimality condition for our problem. In view of the remarks given above, this coincides with our previous definition of constraint qualification.

Now, as pointed out in [22], the constraint qualifications are independent of the objective function \(f\). Hence, if a constraint qualification implies a certain property for the multipliers satisfying (KKT), this property would hold for all objective functions (for which \(x_0\) affords a local minimum). With this in mind, define

\[ F(x_0) := \{ f \in C^1(\mathbb{R}^n, \mathbb{R}) \mid x_0 \text{ affords a local minimum to } f \text{ on } S \}. \]

The result on uniqueness of Lagrange multipliers given in [22] is the following.

**Theorem 2.7.** Let \(x_0 \in S\). Then \(\Lambda(f, x_0)\) is a singleton for all \(f \in F(x_0) \iff (LI)\) is satisfied.

Note that this result and that of Kyparisis [15], Theorem 2.3, are quite different. In contrast with the former, the latter states that, given \(f \in C^1(\mathbb{R}^n, \mathbb{R})\), \(x_0 \in S\) and \(\lambda \in \Lambda(f, x_0)\), then \(\Lambda(f, x_0) = \{\lambda\} \iff (SMF)\) holds at \(x_0\). Since the (SMF) condition relies on the existence of Lagrange multipliers and depends (indirectly) on the objective \(f\), Wachsmuth [22] refrains from calling this a constraint qualification and points out that (LI) is indeed a constraint qualification which ensures the existence and uniqueness of Lagrange multipliers.

**A regularity approach**

We shall now give a different approach which shows in a clear way how some of the constraint qualifications have emerged and how second order conditions can be easily established. It is based on the notions of regularity, normality and properness, and we refer to [13, 14] for a full account of these ideas.

Recall that \(T_S(x_0) \subset R_S(x_0)\). We shall say that \(x_0 \in S\) is a regular point of \(S\) if \(T_S(x_0) = R_S(x_0)\). This condition is also known as Abadie’s constraint qualification (see [12]). The first order Lagrange multiplier rule is a consequence of Theorem 2.6 and the following auxiliary result on linear functionals derived in [13, 14] through the theory of convex cones.

**Lemma 2.8.** Suppose \(L, L_i (i \in A \cup B, A = \{1, \ldots, p\}, B = \{p + 1, \ldots, m\})\) are linear functionals on a real vector space \(X\) and

\[ R = \{ x \in X \mid L_\alpha(x) \leq 0 (\alpha \in A), L_\beta(x) = 0 (\beta \in B) \}. \]

If \(L(x) \geq 0\) for all \(x \in R\), then there exists \(\{\lambda_i\}_1^n\) such that \(\lambda_\alpha \geq 0 (\alpha \in A)\) and \(L(x) + \sum_1^n \lambda_i L_i(x) = 0 (x \in X)\). If \(\{L_i\}_1^n\) is linearly independent then \(\{\lambda_i\}_1^n\) is unique.

**Theorem 2.9.** If \(x_0\) solves (N) locally and is a regular point of \(S\), then \(\Lambda(f, x_0) \neq \emptyset\).

**Proof.** By Theorem 2.6 and regularity of \(x_0\) relative to \(S\), \(f'(x_0; h) \geq 0\) for all \(h \in R_S(x_0)\). The result then follows by Lemma 2.8.

The second order Lagrange multiplier rule is also a straightforward consequence of Theorem 2.6.
Theorem 2.10. Suppose \( x_0 \in S \) and \( \lambda \in \Lambda(f, x_0) \). If \( x_0 \) solves (N) locally, then \( F^m(x_0; h) \geq 0 \) for all \( h \in T_S(x_0) \) where \( S_1 := \{ x \in S \mid F(x) = f(x) \} \). In particular, if \( x_0 \) is a regular point of \( S_1 \), then \( F^m(x_0; h) \geq 0 \) for all \( h \in R_S(x_0) \).

Proof. Since \( x_0 \) minimizes \( F \) on \( S_1 \) and \( F'(x_0) = 0 \), the result follows by Theorem 2.6. \( \square \)

Note that \( S_1 \) defined above depends on the Lagrange multiplier \( \lambda \in \mathbb{R}^m \) and, if we set \( \Gamma = \{ \alpha \in A \mid \lambda_\alpha > 0 \} \) as before, then

\[
S_1 = S_1(\lambda) = \{ x \in \mathbb{R}^n \mid g_\alpha(x) \leq 0 \ (\alpha \in A, \ \lambda_\alpha = 0), \ g_\beta(x) = 0 \ (\beta \in \Gamma \cup B) \} = \{ x \in S \mid g_\alpha(x) = 0 \ (\alpha \in \Gamma) \}.
\]

Therefore, by definition of tangential constraints, we have

\[
R_S(x_0) = \{ h \in \mathbb{R}^n \mid g'_\alpha(x_0; h) \leq 0 \ (\alpha \in I(x_0), \ \lambda_\alpha = 0), \ g'_\beta(x_0; h) = 0 \ (\beta \in \Gamma \cup B) \} = \{ h \in R_S(x_0) \mid g'_\alpha(x_0; h) = 0 \ (\alpha \in \Gamma) \} = \{ h \in R_S(x_0) \mid f'(x_0; h) = 0 \}.
\]

Observe that, in the definition of regularity, sequential tangent vectors (elements of \( T_S(x_0) \)) are used. If we replace them with curvilinear tangent vectors, that is, elements of

\[
C_S(x_0) := \{ h \in \mathbb{R}^n \mid \exists \delta > 0, x: [0, \delta) \to S \text{ such that } x(0) = x_0, \ x'(0) = h \},
\]

we obtain a modified regularity condition known as the Kuhn-Tucker constraint qualification. Note that \( C_S(x_0) \subset T_S(x_0) \subset R_S(x_0) \) and the constraint qualification corresponds to the condition \( R_S(x_0) \subset C_S(x_0) \). An even weaker condition introduced in [14] is that of quasiregularity relative to \( S \) in case the outer normals of \( S \) at \( x_0 \), that is, the elements \( u \) of \( T^*_S(x_0) \), are expressible as \( u = \sum_1^n \lambda_\alpha g'_\alpha(x_0) \) where \( \lambda_\alpha \geq 0 \) and \( \lambda_\alpha g_\alpha(x_0) = 0 \ (\alpha \in A) \). Thus if \( x_0 \) is a quasiregular minimum point of \( f \) on \( S \), then \( \Lambda(f, x_0) \neq \emptyset \).

In general, it may be difficult to test for regularity and one usually requires some criteria that implies that condition. A simple criterion is that of normality. As pointed out by Hestenes [13], “it is customary in the calculus of variations to call a condition on the gradients \( g'_1(x_0), \ldots, g'_n(x_0) \) a normality condition if it implies regularity at \( x_0 \).” In [14], normality with respect to \( S \) is defined as follows.

Definition 2.11. A point \( x_0 \in S \) is normal relative to \( S \) if \( \lambda = 0 \) is the only solution to

i. \( \lambda_\alpha \geq 0 \) and \( \lambda_\alpha g_\alpha(x_0) = 0 \ (\alpha \in A) \).

ii. \( \sum_1^n \lambda_\alpha g'_\alpha(x_0) = 0 \).

Let us remark that the extended multiplier rule stated in Theorem 2.2 (Fritz John necessary optimality condition) yields in a natural way this definition of normality since in that theorem, if \( x_0 \) is also a normal point of \( S \), then \( \lambda_0 > 0 \) and the multipliers can be chosen so that \( \lambda_0 = 1 \), implying the nonemptiness of \( \Lambda(f, x_0) \).

Now, as shown in [14], the desired relation between normality and regularity does indeed hold. It is the basic result relating the notions of regularity and normality.

Theorem 2.12. If \( x_0 \) is a normal point of \( S \) then \( x_0 \) is a regular point of \( S \).
The proof of this result, given in [14], relies strongly on the following characterization of normality.

**Definition 2.13.** A point \( x_0 \in S \) is proper relative to \( S \) if the set \( \{ g'_\beta(x_0) \mid \beta \in B \} \) is linearly independent and, if \( p > 0 \), \( \exists h \) such that

\[
\sum_{\alpha \in A} \mu_\alpha g'_\alpha(x_0) = 0 \quad \text{and} \quad g'_\beta(x_0; h) < 0 (\beta \in B).
\]

A proof of the fact that normality and properness are equivalent is given in [14].

In [12] and the classical literature, both are known as constraint qualifications, the former due to Cottle-Dragonirescu and the latter to Mangasarian-Fromovitz. Let us mention that the proof of Theorem 2.3 given in [12, 15] is based on this equivalence which is proved in both references by using theorems of alternative (see Motzkin in [12, Theorem 2.4.19]).

Now, the notion of normality relative to \( S \) can certainly be applied to the subset \( S_1 \) of \( S \). It yields the following condition.

**Note 2.14.** Given \( \lambda \in \mathbb{R}^m \) with \( \lambda_\alpha \geq 0 (\alpha \in A) \), \( x_0 \) is a normal point of \( S_1(\lambda) \) if \( \mu = 0 \) is the only solution to

i. \( \mu_\alpha \geq 0 \) and \( \mu_\alpha g_\alpha(x_0) = 0 (\alpha \in A, \lambda_\alpha = 0) \).

ii. \( \sum_1^m \mu_\alpha g'_\alpha(x_0) = 0 \).

The following fundamental result on second order necessary conditions is a consequence of Theorems 2.10 and 2.12.

**Theorem 2.15.** Suppose \( x_0 \in S \) and \( \lambda \in \Lambda(f, x_0) \). If \( x_0 \) solves (N) locally and is a normal point of \( S_1(\lambda) \) if \( F''(x_0; h) \geq 0 \) for all \( h \in \mathcal{R}_{S_1}(x_0) \).

Note that this result and Theorem 2.4 are the same since the condition (SMF) is no other than properness relative to \( S_1(\lambda) \) which is equivalent to normality relative to \( S_1(\lambda) \). Thus, the correct result by Kyparisis [15] (mentioned in [12]) had been previously established by Hestenes in [14, Theorem 7.5, p 227 and Theorem 10.4 p 241]. Of course, also Theorem 2.3 can be stated in terms of the set \( S_1(\lambda) \).

**Theorem 2.16.** Suppose \( x_0 \in S \) and \( \lambda \in \Lambda(f, x_0) \). Then \( x_0 \) is normal relative to \( S_1(\lambda) \) if \( \lambda = 0 \) is the only solution to

\[
\sum_{i \in I(x_0) \cup B} \lambda_i g'_i(x) = 0,
\]

that is, if \( \lambda = 0 \) is the only solution to

i. \( \lambda_\alpha g_\alpha(x_0) = 0 \) (\( \alpha \in A \)).
ii. $\sum_{i=1}^{m} \lambda_{i} g_i'(x) = 0$.

Thus $x_0$ is normal relative to $S_0(x_0)$ if $[\lambda_{\alpha} g_{\alpha}(x_0) = 0 (\alpha \in A)$ and $\lambda' g'(x_0) = 0] \Rightarrow \lambda = 0$ (here ‘$\ast$’ denotes transpose). This is equivalent to the condition that the linear equations $g_i'(x_0; h) = 0 (i \in I(x_0) \cup B)$ in $h$ be linearly independent, which is precisely the (LI) constraint qualification. And this is the way normality is defined in [14].

Note that, given $x_0 \in S$ and $\lambda \in \mathbb{R}^{\alpha}$ with $\lambda_{\alpha} \geq 0 (\alpha \in A)$, we have $R_{S_{0}}(x_0) \subset R_{S_{1}}(x_0) \subset R_{S}(x_0)$. Also, if $x_0$ is a normal point of $S_{0}$, then it is a normal point of $S_{1}$, and hence a normal point of $S$. Moreover, as mentioned before, normality relative to $S$ implies regularity relative to $S$.

This definition has several implications in the theory of necessary optimality conditions. One of them, as explained in [5], is that in most textbooks (see a list of well-known references in [5]) a result weaker than Theorem 2.15 is cited. Namely, in that theorem, the assumption of normality relative to $S_{1}(\lambda)$ is replaced by (the stronger assumption of) normality relative to $S_{0}$, and the set of tangential constraints $R_{S_{1}}(x_0)$ is replaced by (the, in general, smaller set of tangential constraints)

$$R_{S_{0}}(x_0) = \{ h \in \mathbb{R}^{\alpha} | g_i'(x_0; h) = 0 (i \in I(x_0) \cup B) \}.$$ 

Explicitly, this rather “well-worn result” (as Ben-Tal puts it [5]) is the following.

**Theorem 2.17.** Suppose $x_0 \in S$ and $\lambda \in \Lambda(f, x_0)$. If $x_0$ solves (N) locally and is a normal point of $S_{0}$, then $F''(x_0; h) \geq 0$ for all $h \in R_{S_{0}}(x_0)$.

As pointed out in [5], “The source of this weaker result can be attributed to the traditional way of treating the active inequality constraints as equality constraints.”

Let us also explain a second implication of this condition related to first order conditions and uniqueness of multipliers. As mentioned before, one is interested in obtaining criteria for regularity and a simple one is that of normality. That is, regularity relative to $S$ and normality relative to $S$. However, by a simple application of the implicit function theorem, one can easily prove that, given $x_0 \in S$, normality relative to $S_0$ implies regularity relative to $S$ (see [13, Lemma 10.1, p 35]). By Theorem 2.9, if $x_0$ affords a local minimum to $f$ on $S$ and $x_0$ is normal relative to $S_{0}$, then $\Lambda(f, x_0) \neq \emptyset$. Moreover, as one readily verifies, $\lambda \in \Lambda(f, x_0)$ is unique (see [13, Theorem 10.1, p 36]). On the other hand, if normality is assumed relative to $S$, then there exists $\lambda \in \Lambda(f, x_0)$, but $\lambda$ may not be unique.

Let us end this approach with the main results of [15] and [22]. Let $x_0 \in S$. Theorem 2.3 states that, given $f \in C^{1}(\mathbb{R}^{n}, \mathbb{R})$ and $\lambda \in \Lambda(f, x_0)$, $x_0$ is normal relative to $S_{1}(\lambda)$ if and only if $\Lambda(f, x_0) = \{ \lambda \}$. Theorem 2.7 states that $\Lambda(f, x_0)$ is a singleton for all $f \in C^{1}(\mathbb{R}^{n}, \mathbb{R})$ such that $x_0$ affords a local minimum to $f$ on $S$ if and only if $x_0$ is normal relative to $S_{0}(x_0)$.

**3. Isoperimetric inequality constraints.** In this section we shall deal with a fixed endpoint problem of Lagrange in the calculus of variations posed over piecewise smooth arcs and involving inequality and equality isoperimetric constraints. We pose, as before, the question of uniqueness of Lagrange multipliers appearing in first order necessary conditions. As we shall see, there is a striking likeness between the main results stated in the previous section and those for this problem.

To state the problem, suppose we are given an interval $T := [t_0, t_1]$ in $\mathbb{R}$, two points $\xi_0, \xi_1$ in $\mathbb{R}^{n}$, functions $L$ and $L_{\gamma}$ mapping $T \times \mathbb{R}^{n} \times \mathbb{R}^{n}$ to $\mathbb{R}$ and scalars $b_{\gamma}$.
in \( \mathbb{R} (\gamma = 1, \ldots, q) \). Denote by \( X \) the space of piecewise \( C^1 \) functions mapping \( T \) to \( \mathbb{R}^n \) and let
\[
X_e := \{ x \in X \mid x(t_0) = \xi_0, \ x(t_1) = \xi_1 \},
\]
\[
S := \{ x \in X_e \mid I_\alpha(x) \leq 0 (\alpha \in R), \ I_\beta(x) = 0 (\beta \in Q) \}
\]
where \( R = \{1, \ldots, r \}, Q = \{r + 1, \ldots, q \}, and
\[
I_\gamma(x) = b_\gamma + \int_{t_0}^{t_1} L_\gamma(t, x(t), \dot{x}(t)) dt \quad (x \in X).
\]
Consider the problem, which we label \((V)\), of minimizing \( I \) on \( S \), where
\[
I(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t)) dt \quad (x \in X).
\]

Elements of \( X \) will be called arcs, of \( S \) admissible arcs, and an admissible arc \( x \) is a (local) solution to the problem \((V)\) if (upon shrinking \( T \times \mathbb{R}^n \) if necessary) any other admissible arc \( y \) satisfies \( I(x) \leq I(y) \). Given \( x \in X \) we use the notation \((\dot{x}(t))\) to represent \((t, x(t), \dot{x}(t))\). Also, we assume that the functions \( L, L_\gamma \) are \( C^1 \) and, when second derivatives occur, they are \( C^2 \).

For all \( x \in X \) consider the first variation of \( I \) along \( x \) given by
\[
I'(x; y) := \int_{t_0}^{t_1} \{ L_x(\ddot{x}(t))y(t) + L_\dot{x}(\ddot{x}(t))\dot{y}(t) \} dt \quad (y \in X)
\]
and the second variation of \( I \) along \( x \) given by
\[
I''(x; y) := \int_{t_0}^{t_1} 2\Omega(t, y(t), \dot{y}(t)) dt \quad (y \in X)
\]
where, for all \((t, y, \dot{y}) \in T \times \mathbb{R}^n \times \mathbb{R}^n,
\[
2\Omega(t, y, \dot{y}) := \langle y, L_{xx}(\ddot{x}(t))y \rangle + 2\langle y, L_{x\dot{x}}(\ddot{x}(t))\dot{y} \rangle + \langle \dot{y}, L_{\dot{x}\dot{x}}(\ddot{x}(t))\dot{y} \rangle.
\]
The first and second variations of other integrals such as \( I_\alpha \) are defined in a similar way. Define the set of admissible variations as \( Y := \{ y \in X \mid y(t_0) = y(t_1) = 0 \} \).

The following result provides well-known first order necessary conditions (see, for example, [13]). It is the analogous of Theorem 2.2, the Fritz John necessary optimality condition for problem \((N)\).

**Theorem 3.1.** Suppose \( x_0 \) solves \((V)\). Then there exist \( \lambda_0 \geq 0 \) and \( \lambda \in \mathbb{R}^q \), not both zero, such that
i. \( \lambda_0 \geq 0 \) and \( \lambda_\alpha I_\alpha(x_0) = 0 (\alpha \in R) \).
ii. If \( J_0(x) := \lambda_0 I(x) + \sum_1^q \lambda_\alpha I_\alpha(x) \), then \( J_0'(x_0; y) = 0 \) for all \( y \in Y \).

Based on this result, we define normality relative to \( S \) (compare with Definition 2.11) or weak normality, as follows.

**Definition 3.2.** An arc \( x_0 \) will be said to be normal relative to \( S \) or weakly normal if \( \lambda = 0 \) is the only solution to
i. \( \lambda_0 \geq 0 \) and \( \lambda_\alpha I_\alpha(x_0) = 0 (\alpha \in R) \).
ii. $\sum_j^q \lambda_j I'_j(x_0; y) = 0$ for all $y \in Y$.

Clearly, if $x_0$ solves (V) and is weakly normal, then $\lambda_0 > 0$ in Theorem 3.1 and the multipliers can be chosen so that $\lambda_0 = 1$. In this event, the couple $(x_0, \lambda) \in S \times \mathbb{R}^q$ will be called an extremal and we denote by $E$ the set of all extremals.

**Definition 3.3.** Denote by $\Lambda(L, x_0)$ the set of all $\lambda \in \mathbb{R}^q$ such that $(x_0, \lambda) \in E$, that is,

i. $\lambda_0 \geq 0$ and $\lambda_\alpha I_\alpha(x_0) = 0$ $(\alpha \in R)$.

ii. If $J(x) := I(x) + \sum_j^q \lambda_j I'_j(x)$, then $J'(x_0; y) = 0$ for all $y \in Y$.

Condition (ii), as it is well-known, is equivalent to the existence of $c \in \mathbb{R}^n$ such that

$$F_x(\tilde{x}(t)) = \int_{t_0}^t F_x(\tilde{x}(s))ds + c \quad (t \in T)$$

where $F := L + \sum_j^q \lambda_j L_j$ (see [13]).

By Theorem 3.1, we have the following first order necessary condition.

**Theorem 3.4.** If $x_0$ solves (V) and is normal relative to $S$, then $\Lambda(L, x_0) \neq \emptyset$.

Normality of a local solution to the problem relative to $S$ implies nonemptiness of the set of Lagrange of multipliers but not uniqueness. The same occurs with the nonlinear programming problem. For problem (V), a stronger assumption which implies uniqueness of multipliers as well as second order necessary conditions is usually imposed.

To introduce this stronger assumption, denote the set of active indices at an admissible arc $x_0$ by $I_\alpha(x_0) = \{\alpha \in R \mid I_\alpha(x_0) = 0\}$ and, as in the previous section, consider the set

$$S_0 := S_0(x_0) = \{x \in X_e \mid I_\gamma(x) = 0 \ (\gamma \in I_\alpha(x_0) \cup Q)\}.$$

Note that, by definition, $x_0$ is normal relative to $S_0$ if $\lambda = 0$ is the only solution to

i. $\lambda_\alpha I_\alpha(x_0) = 0$ $(\alpha \in R)$.

ii. $\sum_j^q \lambda_j I'_j(x_0; y) = 0$ for all $y \in Y$.

This condition is clearly equivalent to the linearly independence on $Y$ of the first variations $I'_\gamma(x_0; y)$ $(\gamma \in I_\alpha(x_0) \cup Q)$ of $I_\gamma$ along $x_0$, which is equivalent to the existence of $y_\gamma \in Y$ $(\gamma \in I_\alpha(x_0) \cup Q)$ such that

$$|I'_\beta(x_0; y_\gamma)| \neq 0 \quad (\beta, \gamma \in I_\alpha(x_0) \cup Q).$$

We refer to [3] for these and other characterizations of normality relative to $S_0$, a property which we shall call strong normality.

The way first and second order necessary conditions for problem (V) are usually established (see, for example, [13]) can be stated as follows. Both require the assumption of strong normality.

**Theorem 3.5.** If $x_0$ solves (V) and is strongly normal then $\Lambda(L, x_0)$ is a singleton, that is, there exists a unique $\lambda \in \mathbb{R}^q$ such that $(x_0, \lambda) \in E$.

**Theorem 3.6.** Suppose $\lambda \in \Lambda(L, x_0)$. If $x_0$ solves (V) and is strongly normal then $J''(x_0; y) \geq 0$ for all $y \in Y$ satisfying
a. \( I'_a(x_0; y) \leq 0 \) for all \( y \in X_e \) if \( \lambda_\alpha = 0 \);
b. \( I'_{\beta}(x_0; y) = 0 \) for all \( \beta \in R \) with \( \lambda_\beta > 0 \), or \( \beta \in Q \).

In a recent paper (see [4]) the same second order condition of Theorem 3.6 is derived but under a weaker assumption. As in Theorem 2.15, it is expressed in terms of the set \( S_1(\lambda) = \{ x \in S \mid J(x) = I(x) \} \) where a multiplier \( \lambda \in R^q \) with \( \lambda_\alpha \geq 0 \) is given and the function \( J \) is as in Definition 3.3(ii), that is, \( J(x) = I(x) + \sum_1^q \lambda_j I_j(x) \). Clearly we have

\[
S_1 := S_1(\lambda) = \{ x \in X_e \mid I_\alpha(x) = 0 \ (\alpha \in R, \ \lambda_\alpha = 0),
I_{\beta}(x) = 0 \ (\beta \in R \text{ with } \lambda_\beta > 0, \text{ or } \beta \in Q) \}
\]

and \( x_0 \) is normal relative to \( S_1(\lambda) \) if \( \mu = 0 \) is the only solution to

i. \( \mu_\alpha \geq 0 \) and \( \mu_\alpha I_\alpha(x_0) = 0 \) for all \( \alpha \in R \), \( \lambda_\alpha = 0 \).

ii. \( \sum_1^q \mu_j I_j(x_0; y) = 0 \) for all \( y \in Y \).

If we define the set of tangential constraints at \( x_0 \in S_1 \) as the set of those \( y \in Y \) satisfying (a) and (b) of Theorem 3.6, that is,

\[
R_{S_1}(x_0) = \{ y \in Y \mid I'_\alpha(x_0; y) < 0 \ (\alpha \in I_\alpha(x_0)) \text{ and } I'_{\beta}(x_0; y) = 0 \ (\beta \in Q) \}
\]

then the result obtained in [4, Theorem 1.5] can be stated as follows.

**Theorem 3.7.** Suppose \( \lambda \in \Lambda(L, x_0) \). If \( x_0 \) solves (V) and is normal relative to \( S_1(\lambda) \) then \( J''(x_0; y) \geq 0 \) for all \( y \in R_{S_1}(x_0) \).

The proof of this result relies strongly on a characterization of normality, given in [4], in terms of the notion of properness (see below), similar to the one given in the previous section. It corresponds to a Mangasarian-Fromovitz type condition. In [4, Proposition 2.3], normality relative to \( S \) and properness relative to \( S \) are shown to be equivalent.

**Definition 3.8.** We call \( x_0 \in S \text{ proper relative to } S \) if

a. \( \{ I'_{\beta}(x_0; y) \mid \beta \in Q \} \) is linearly independent on \( Y \).
b. There exists \( y \in Y \) such that \( I'_\alpha(x_0; y) < 0 \) for all \( \alpha \in I_\alpha(x_0) \) and \( I'_{\beta}(x_0; y) = 0 \) for all \( \beta \in Q \).

We are now in a position to state and prove the results of [15] and [22] corresponding to our isoperimetric problem. Note that, by [4], we can replace the condition 3.9(a) below with properness of \( x_0 \) relative to \( S_1(\lambda) \).

**Theorem 3.9.** Suppose \( \lambda \in \Lambda(L, x_0) \). Then the following are equivalent:

a. \( x_0 \) is normal relative to \( S_1(\lambda) \).

b. \( \lambda \) is unique in \( \Lambda(L, x_0) \).

**Proof.** (a) \( \Rightarrow \) (b): Let \( \tilde{\lambda} \in \Lambda(L, x_0) \) and set \( \mu := \tilde{\lambda} - \lambda \). We have

\[
\mu_\alpha = \tilde{\lambda}_\alpha \geq 0 \quad \text{and} \quad \mu_\alpha I_\alpha(x_0) = \tilde{\lambda}_\alpha I_\alpha(x_0) = 0 \quad (\alpha \in R, \ \lambda_\alpha = 0)
\]

and, if \( \tilde{J}(x) := I(x) + \sum_1^q \tilde{\lambda}_j I_j(x) \) then 0 = \( \tilde{J}(x_0; y) - J'(x_0; y) = \sum_1^q \mu_j I_j(x; y) \) for all \( y \in Y \). By (a), \( \mu = 0 \) and so \( \lambda = \tilde{\lambda} \).
(b) \(\Rightarrow\) (a): \(\neg(a) \Rightarrow \neg(b)\): Assume \(x_0\) is not a normal point of \(S_1(\lambda)\). Then there exists \(\mu \in \mathbb{R}^1\), \(\mu \neq 0\), satisfying
i. \(\mu_0 \geq 0\) and \(\mu_0 I_\alpha(x_0) = 0\) \((\alpha \in R, \lambda_\alpha = 0)\);
ii. \(\sum_1^q \mu_\gamma I'_\gamma(x_0; y) = 0\) for all \(y \in Y\),
and such that
\[
\max\{|\mu_\alpha| : \alpha \in K\} < \min\{\lambda_\alpha : \alpha \in K\}
\]
where \(K = \{\alpha \in I_\alpha(x_0) | \lambda_\alpha > 0\}\). Let \(\hat{\lambda} := \lambda + \mu\). Let us prove that \(\hat{\lambda} \in \Lambda(L, x_0)\) implying \(\neg(b)\) since \(\lambda \neq \hat{\lambda}\). Indeed, if \(\hat{J}(x) := I(x) + \sum_1^q \lambda_\gamma I_\gamma(x)\), then
\[
\hat{J}'(x_0; y) = J'(x_0; y) + \sum_1^q \mu_\gamma I'_\gamma(x_0; y) = 0 \quad (y \in Y)
\]
and so 3.3(ii) holds. To prove 3.3(i), let \(\alpha \in R\). If \(I_\alpha(x_0) = 0\) then \(\hat{\lambda}_\alpha I_\alpha(x_0) = 0\). If \(I_\alpha(x_0) < 0\) then, by 3.3(i) with respect to \(\hat{\lambda}\), we have \(\lambda_\alpha = 0\) and so, by (i) above, \(0 = \mu_0 I_\alpha(x_0) = \hat{\lambda}_\alpha I_\alpha(x_0)\). Finally, if \(\lambda_\alpha = 0\), then \(\hat{\lambda}_\alpha = \mu_\alpha \geq 0\). If \(\lambda_\alpha > 0\) then
\[
\hat{\lambda}_\alpha = \lambda_\alpha + \mu_\alpha \geq \min_{i \in K} \lambda_i + \mu_\alpha > \max_{i \in K} |\mu_i| + \mu_\alpha \geq 0.
\]
This shows that \(\hat{\lambda} \in \Lambda(L, x_0)\) and the proof is complete. \(\square\)

This result and Theorem 3.7 yield the following second order necessary condition.

**Theorem 3.10.** Suppose \(\lambda \in \Lambda(L, x_0)\). If \(x_0\) solves (V) and \(\lambda\) is unique in \(\Lambda(L, x_0)\), then \(J''(x_0; y) \geq 0\) for all \(y \in R_{S_1}(x_0)\).

Let us turn now to the result of [22]. Denote by \(F(x_0)\) the set of all \(C^1\) functions \(L\) mapping \(T \times \mathbb{R}^n \times \mathbb{R}^n\) to \(\mathbb{R}\) such that \(x_0\) solves the problem \(V(L)\) of minimizing \(I(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt\) over \(S\). The theorem on uniqueness of multipliers given in [22, Theorem 2] corresponds, in the context of isoperimetric constraints, to the following result.

**Theorem 3.11.** Let \(x_0 \in S\). Then the following are equivalent:

a. \(x_0\) is strongly normal.

b. \(\Lambda(L, x_0)\) is a singleton for all \(L \in F(x_0)\).

**Proof.** (a) \(\Rightarrow\) (b): Let \(L \in F(x_0)\). Since \(x_0\) solves \(V(L)\), we have by (a) that \(\Lambda(L, x_0) \neq \emptyset\). Suppose \(\lambda\) and \(\hat{\lambda}\) belong to \(\Lambda(L, x_0)\) and let \(\mu := \lambda - \hat{\lambda}\). By Definition 3.3,

i. \(\mu_\alpha I_\alpha(x_0) = 0\) \((\alpha \in R)\);

ii. \(\sum_1^q \mu_\gamma I'_\gamma(x_0; y) = 0\) for all \(y \in Y\)

implying, by (a), that \(\mu = 0\).

(b) \(\Rightarrow\) (a): For all \(i \in R \cup Q\) let \(\mu_\gamma = 1\) if \(i \in I_\alpha(x_0)\) and \(\mu_\gamma = 0\) otherwise, and define
\[
L(t, x, \dot{x}) := -\sum_1^q \mu_\gamma [b_\gamma + L_\gamma(t, x, \dot{x})].
\]

Note that
\[
I(x_0) = \int_{t_0}^{t_1} L(t, x_0(t), \dot{x}_0(t))dt = 0 \leq \int_{t_0}^{t_1} L(t, x(t), u(t))dt = I(x) \quad (x \in S)
\]
and so \( L \in \mathcal{F}(x_0) \). Clearly, \((x_0, \mu) \in \mathcal{E}\) and therefore \(\mu \in \Lambda(L, x_0)\). Now, let \( \nu \in \mathbb{R}^q \) satisfy

i. \( \nu_0 I_\alpha(x_0) = 0 \) (\( \alpha \in \mathbb{R} \)).

ii. \( \sum_1^n \nu_0 I_\alpha(x_0; y) = 0 \) for all \( y \in \mathcal{Y} \).

Condition (a) will follow if we show that \( \nu = 0 \). To prove it, define, for all \( i \in \mathbb{R} \cup Q \),

\[
\mu_i := \mu_i + (\nu_i) / \beta \quad \text{where} \quad \beta = 1 + \max \{ |\nu_\alpha| : \alpha \in I_\alpha(x_0) \}.
\]

We claim that \((x_0, \hat{\mu}) \in \mathcal{E}\). Indeed, 3.3(i) holds since, if \( I_\alpha(x_0) = 0 \), then

\[
\hat{\mu}_\alpha = \frac{\beta + \nu_\alpha}{\beta} \geq 0
\]

and, if \( I_\alpha(x_0) < 0 \), then \( \hat{\mu}_\alpha I_\alpha(x_0) = 0 \). For 3.3(ii) observe that, if \( I(x) = I(x) + \sum_1^m \mu_i I_i(x) \), since \((x_0, \mu) \in \mathcal{E}\) and \( \nu \) satisfies (ii) above, then clearly \( J'(x_0; y) = 0 \) for all \( y \in \mathcal{Y} \). This proves the claim and so \( \hat{\mu} \in \Lambda(L, x_0) \). By (b), \( \hat{\mu} = \mu \) and hence \( \nu = 0 \).

\[ \square \]

4. Optimal control. In this section we shall deal with an optimal control problem posed over piecewise \( C^1 \) trajectories and piecewise continuous controls, and involving inequalities and equalities in the control functions.

We shall encounter crucial differences between this and the previous optimization problems, mainly due to the fact that the constraints are no longer constant but depend explicitly on the time interval under consideration.

To state the problem, suppose we are given an interval \( T := [t_0, t_1] \) in \( \mathbb{R} \), two points \( \xi_0, \xi_1 \) in \( \mathbb{R}^n \), and functions \( L \) and \( f \) mapping \( T \times \mathbb{R}^n \times \mathbb{R}^m \) to \( \mathbb{R} \) and \( \mathbb{R}^n \) respectively, and \( \varphi = (\varphi_1, \ldots, \varphi_q) \) mapping \( \mathbb{R}^m \) to \( \mathbb{R}^q \) (\( q \leq m \)). Denote by \( X \) the space of piecewise \( C^1 \) functions mapping \( T \) to \( \mathbb{R}^n \), and by \( \mathcal{U}_k \) the space of piecewise continuous functions mapping \( T \) to \( \mathbb{R}^k \) (\( k \in \mathbb{N} \)). Let \( Z := X \times \mathcal{U}_m \), and consider the following two sets:

\[
D := \{(x, u) \in Z \mid \dot{x}(t) = f(t, x(t), u(t)) \ (t \in T), \ x(t_0) = \xi_0, \ x(t_1) = \xi_1 \},
\]

\[
S := \{(x, u) \in D \mid \varphi_\alpha(u(t)) \leq 0, \ \varphi_\beta(u(t)) = 0 \ (\alpha \in \mathbb{R}, \ \beta \in Q, \ t \in T) \}
\]

where \( R = \{1, \ldots, r\}, \ Q = \{r + 1, \ldots, q\} \). The problem we shall deal with, which we label \((P)\), is that of minimizing the functional

\[
I(x, u) := \int_{t_0}^{t_1} L(t, x(t), u(t))dt \quad ((x, u) \in Z)
\]

over \( S \). Elements of \( Z \) will be called processes, of \( S \) admissible processes, and a process \((x, u)\) solves \((P)\) (locally) if \((x, u)\) is admissible and \( (x, u) \) as in the previous section, upon shrinking \( T \times \mathbb{R}^n \) if necessary) we have \( I(x, u) \leq I(y, v) \) for all admissible process \((y, v)\). Given \((x, u) \in Z \) we shall use the notation \( (\dot{x}(t)) \) to represent \((t, x(t), u(t))\), and ‘*’ denotes transpose.

With respect to the functions delimiting the problem, we assume that, if \( F := (L, f) \), then \( F(t, \cdot, \cdot) \) is \( C^1 \) for all \( t \in T \) and \( \varphi \) is \( C^1 \): \( F(\cdot, x, u), F_x(\cdot, x, u) \) and \( F_u(\cdot, x, u) \) are piecewise continuous for all \((x, u) \in \mathbb{R}^n \times \mathbb{R}^m \); and there exists an integrable function \( \alpha: T \to \mathbb{R} \) such that, at any point \((t, x, u) \in T \times \mathbb{R}^n \times \mathbb{R}^m \),

\[
|F(t, x, u)| + |F_x(t, x, u)| + |F_u(t, x, u)| \leq \alpha(t).
\]
These assumptions are standard for the derivation of first order necessary conditions (see, for example, [19, 20]). Also, we assume that the $q \times (m+r)$-dimensional matrix
\[
\left( \frac{\partial \varphi_i}{\partial u_k} \delta_{\alpha \alpha} \right) \quad (i = 1, \ldots, q; \alpha = 1, \ldots, r; k = 1, \ldots, m)
\]
has rank $q$ on $U$ (here $\delta_{\alpha \alpha} = 1, \delta_{\alpha \beta} = 0 (\alpha \neq \beta)$), where
\[
U := \{ u \in \mathbb{R}^m \mid \varphi_\alpha(u) \leq 0 (\alpha \in R), \varphi_\beta(u) = 0 (\beta \in Q) \}.
\]
This condition is equivalent to the condition that, at each point $u \in U$, the matrix
\[
\left( \frac{\partial \varphi_i}{\partial u_k} \right) \quad (i = i_1, \ldots, i_p; k = 1, \ldots, m)
\]
has rank $p$, where $i_1, \ldots, i_p$ are the indices $i \in \{1, \ldots, q\}$ such that $\varphi_i(u) = 0$.

For all $(t, x, u, p, \mu, \lambda)$ in $T \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}$ let
\[
H(t, x, u, p, \mu, \lambda) := \langle p, f(t, x, u) \rangle - \lambda L(t, x, u) - \langle \mu, \varphi(u) \rangle.
\]
First order necessary conditions are well established and one version can be stated as follows (see [13, 19]).

**Theorem 4.1.** If $(x_0, u_0)$ solves (P), there exist $\lambda_0 \geq 0$, $p \in X$, and $\mu \in U_q$, not vanishing simultaneously on $T$, such that

a. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t) \varphi_\alpha(u_0(t)) = 0$ ($\alpha \in R$, $t \in T$);

b. $\check{p}(t) = -H^*_t(\check{x}_0(t), p(t), \mu(t), \lambda_0)$ on every interval of continuity of $u_0$.

c. $H_\alpha(\check{x}_0(t), p(t), \mu(t), \lambda_0) = 0$ ($t \in T$).

Based on this result, we define normality relative to $S$ as we did in the two previous sections, that is, if the cost multiplier vanishes in the above system, the only solution is the null one.

**Definition 4.2.** An admissible process $(x_0, u_0) \in S$ is normal relative to $S$ if, given $(p, \mu) \in X \times U_q$ satisfying

i. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t) \varphi_\alpha(u_0(t)) = 0$ ($\alpha \in R$, $t \in T$);

ii. $\check{p}(t) = -f^*_t(\check{x}_0(t))p(t) \quad [\check{H}^*_t(\check{x}_0(t), p(t), \mu(t), 0)] (t \in T)$;

iii. $0 = f^*_t(\check{x}_0(t))p(t) - \varphi^\alpha(u_0(t))\mu(t) \quad [\check{H}_\alpha(\check{x}_0(t), p(t), \mu(t), 0)] (t \in T)$, then $p \equiv 0$. Note that, in this event, also $\mu \equiv 0$.

**Definition 4.3.** Let $E$ the set of all $(x, u, p, \mu) \in X \times U_q$ such that

a. $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t) \varphi_\alpha(u(t)) = 0$ ($\alpha \in R$, $t \in T$);

b. $\check{p}(t) = -f^*_t(\check{x}(t))p(t) + L^*_t(\check{x}(t)) (t \in T)$;

c. $f^*_t(\check{x}(t))p(t) = L^*_t(\check{x}(t)) + \varphi^\alpha(u(t))\mu(t)$ ($t \in T$).

The elements of $E$ in the above definition will be called extremals. For all $(x_0, u_0) \in S$, denote by $\Lambda(L, x_0, u_0)$ the set of all $(p, \mu) \in X \times U_q$ such that $(x_0, u_0, p, \mu) \in E$. From the above definitions it follows that, in Theorem 4.1, if $(x_0, u_0)$ is a normal process of $S$, then $\Lambda(L, x_0, u_0) \neq \emptyset$, that is, there exists $(p, \mu) \in X \times U_q$ such that $(x_0, u_0, p, \mu)$ is an extremal.
Let us now introduce the corresponding sets $S_0(u_0)$ and $S_1(\mu)$. Denote the set of active indices at $u \in \mathbb{R}^m$ by $I_u(u) := \{ \alpha \in R \mid \varphi_\alpha(u) = 0 \}$. Given $u_0 \in \mathcal{U}_m$, let

$$S_0 := S_0(u_0) = \{(x, u) \in D \mid \varphi_i(u(t)) = 0 \ (i \in I_u(u_0(t)) \cup Q, \ t \in T)\}.$$ 

For $\mu \in \mathcal{U}_q$ with $\mu_\alpha(t) \geq 0 \ (\alpha \in R, \ t \in T)$, define

$$S_1 := S_1(\mu) = \{(x, u) \in D \mid \varphi_\alpha(u(t)) \leq 0 \ (\alpha \in R, \ \mu_\alpha(t) = 0, \ t \in T), \ \varphi_\beta(u(t)) = 0 \ (\beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q, \ t \in T)\}.$$ 

Note that $S_1 = \{(x, u) \in S \mid \varphi_\alpha(u(t)) = 0 \ (\alpha \in R, \ \mu_\alpha(t) > 0, \ t \in T)\}$. Applying to these two sets of constraints the definition of normality, given in Definition 4.2, we obtain the following.

**Note 4.4.** An admissible process $(x_0, u_0)$ is normal relative to $S_0(u_0)$ if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0 \ (\alpha \in R, \ t \in T)$;

ii. $\dot{p}(t) = -f^*_\alpha(x_0(t))p(t) \ (t \in T)$;

iii. $f^*_\alpha(x_0(t))q(t) = \varphi^*_\alpha(u_0(t))\mu(t) \ (t \in T)$, 

then $p \equiv 0$. In this event, we have $\mu \equiv 0$.

**Note 4.5.** An admissible process $(x_0, u_0)$ is normal relative to $S_1(\mu)$ if, given $(q, \nu) \in X \times \mathcal{U}_q$ satisfying

i. $\nu_\alpha(t) \geq 0$ and $\nu_\alpha(t)\varphi_\alpha(u_0(t)) = 0 \ (\alpha \in R, \ \mu_\alpha(t) = 0, \ t \in T)$;

ii. $\dot{q}(t) = -f^*_\alpha(x_0(t))q(t) \ (t \in T)$;

iii. $f^*_\alpha(x_0(t))q(t) = \varphi^*_\alpha(u_0(t))\nu(t) \ (t \in T)$, 

then $q \equiv 0$. In this event, we also have $\nu \equiv 0$.

Now, for our optimal control problem, uniqueness of the pair $(p, \mu)$ such that $(x_0, u_0, p, \mu)$ is an extremal follows under the assumption of normality relative to $S_0(u_0)$. This assumption, however, can be weakened and the result follows if the admissible process is normal relative to $S_1(\mu)$ (we refer to [8, 9, 12] for a connection between this assumption and second order necessary conditions). Let us prove this result.

**Theorem 4.6.** Let $(x_0, u_0) \in S$ and suppose there exists $(p, \mu) \in X \times \mathcal{U}_q$ such that $(x_0, u_0, p, \mu) \in S$. If $(x_0, u_0)$ is normal relative to $S_1(\mu)$, then $\Lambda(L, x_0, u_0) = \{(p, \mu)\}$.

**Proof.** Suppose $(\bar{p}, \bar{\mu}) \in X \times \mathcal{U}_q$ is such that $(x_0, u_0, \bar{p}, \bar{\mu}) \in S$ and set $(q, \nu) := (\bar{p} - p, \bar{\mu} - \mu)$. Let us show that $(q, \nu)$ satisfies the conditions of Note 4.5. If $\alpha \in R$ with $\mu_\alpha(t) = 0$, then

$$\nu_\alpha(t) = \bar{\mu}_\alpha(t) \geq 0 \quad \text{and} \quad \nu_\alpha(t)\varphi_\alpha(u_0(t)) = \bar{\mu}_\alpha(t)\varphi_\alpha(u_0(t)) = 0$$

and so 4.5(i) holds. The other two conditions 4.5(ii) and 4.5(iii) hold since

$$\dot{q}(t) = \bar{p}(t) - \dot{p}(t) = -f^*_\alpha(x_0(t))q(t) \ (t \in T),$$

$$f^*_\alpha(x_0(t))q(t) = f^*_\alpha(x_0(t))(\bar{p}(t) - p(t)) = \varphi^*_\alpha(u_0(t))\nu(t) \ (t \in T).$$

Since $(x_0, u_0)$ is normal relative to $S_1(\mu)$, $(q, \nu) \equiv (0, 0)$.

$\square$
Thus, normality relative to $S_1(\mu)$ implies uniqueness of the multiplier $(p, \mu)$ in $\Lambda(L, x_0, u_0)$. This result corresponds, in the two previous problems, to the sufficiency result on uniqueness of Theorems 2.16 and 3.9. The converse, however, may not hold for the optimal control problem. The reason can be easily explained. If we argue by contradiction, as in the previous proofs, that is, if $(x_0, u_0)$ is not normal relative to $S_1(\mu)$, then there exists $(q, \nu) \neq (0, 0)$ satisfying 4.5(i)-(iii) but, contrary to the previous cases, we cannot assure that $\nu$ can be chosen so that

$$\max \{|\nu_\alpha(t)| : \alpha \in K(t)\} < \min \{\mu_\alpha(t) : \alpha \in K(t)\} \quad (t \in T)$$

where $K(t) = \{\alpha \in I_\alpha(u_0(t)) \mid \mu_\alpha(t) > 0\}$. If this were the case, and we let $(\hat{p}, \hat{\mu}) := (p + q, \mu + \nu)$, then one could actually prove that $(\hat{p}, \hat{\mu})$ satisfies 4.3(a)-(c) with respect to $(x_0, u_0)$ showing that, since $(\hat{p}, \hat{\mu}) \neq (p, \mu)$, $(p, \mu)$ is not unique. Indeed, note first that

$$\dot{\hat{p}}(t) = \hat{p}(t) + \dot{q}(t) = -f^*_x(\tilde{x}_0(t))\hat{p}(t) + L^*_x(x_0(t)),$$

$$f^*_x(\tilde{x}_0(t))\dot{\hat{p}}(t) = f^*_x(\tilde{x}_0(t))(p(t) + q(t)) = L^*_x(\tilde{x}_0(t)) + \varphi''(u_0(t))\hat{p}(t)$$

and so 4.3(b) and (c) hold. To see that also 4.3(a) holds we have to show that, for all $\alpha \in R$ and $t \in T$, $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0$. Let $\alpha \in R$ and $t \in T$. If $\mu_\alpha(t) = 0$ then $\mu_\alpha(t) = \nu_\alpha(t) \geq 0$. If $\mu_\alpha(t) > 0$ then

$$\dot{\mu}_\alpha(t) = \mu_\alpha(t) + \nu_\alpha(t) \geq \min_{\alpha \in K(t)} \mu_\alpha(t) + \nu_\alpha(t) > \max_{\alpha \in K(t)} |\nu_\alpha(t)| + \nu_\alpha(t) \geq 0.$$ 

This proves the first relation in 4.3(a). For the second note that, if $\varphi_\alpha(u_0(t)) = 0$ then $\mu_\alpha(t)\varphi_\alpha(u_0(t)) = 0$ and, if $\varphi_\alpha(u_0(t)) < 0$ then, by 4.3(a), $\mu_\alpha(t) = 0$ and so, by 4.5(i),

$$0 = \nu_\alpha(t)\varphi_\alpha(u_0(t)) = \mu_\alpha(t)\varphi_\alpha(u_0(t)).$$

This would complete the proof. However, as we said before, this implication may not be true and we can, in fact, provide a counterexample.

**Example 4.7.** Consider the following functions:

$$f(t, x, u) = u, \quad L(t, x, u) = \frac{1}{2} t u(2-u), \quad \varphi(u) = u^2 - 1.$$ 

Let $x_0 \in X$ be arbitrary and set

$$u_0(t) := \begin{cases} 1 & \text{if } t \in [-1, 0] \\ -1 & \text{if } t \in (0, 1]. \end{cases}$$

Suppose $(x_0, u_0, p, \mu) \in \mathcal{E}$. By Definition 4.3 we have

$$\mu(t) \geq 0, \quad \dot{p}(t) = 0, \quad p(t) = t(1 - u_0(t)) + 2u_0(t)\mu(t) \quad (t \in [-1, 1]).$$

Thus $p$ is a constant satisfying

$$p = \begin{cases} 2\mu(t) & \text{if } t \in [-1, 0] \\ 2t - 2\mu(t) & \text{if } t \in (0, 1]. \end{cases}$$
Since $\mu(t) \geq 0$ for all $t \in [-1, 1]$, from the first relation we have $p \geq 0$ and, from the second, $p \leq 2t$ for all $t \in (0, 1]$ and so $p \leq 0$. Thus $p \equiv 0$ and therefore

$$
\mu(t) = \begin{cases} 
0 & \text{if } t \in [-1, 0] \\
t & \text{if } t \in [0, 1].
\end{cases}
$$

This implies that $(p, \mu)$ is the only pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$.

Now, consider the pair $(q, \nu)$ with $q \equiv 2$ and $\nu(t) := u_0(t)$ ($t \in [-1, 1]$). It satisfies Note 4.5(i) since

$$
\nu(t) \geq 0 \text{ and } \nu(t)\varphi(u_0(t)) = 0
$$

for all $t \in [-1, 1]$ such that $\mu(t) = 0$, that is, all $t \in [-1, 0]$. Moreover, $\dot{q}(t) = 0$ and $q(t) = 2u_0(t)v(t) = 2$ for all $t \in [-1, 1]$ and so also 4.5(ii) and (iii) hold. Since $(q, \nu) \neq (0, 0)$, it follows that $(x_0, u_0)$ is not normal relative to $S_1(\mu)$. \qed

A similar, though more general example, is the following.

**Example 4.8.** Consider the following functions:

$$
f(t, x, u) = u, \quad L(t, x, u) = b(t)u, \quad \varphi(u) = \frac{1}{2}(u^2 - 1)
$$

where

$$
b(t) = \begin{cases} 
0 & \text{if } t \in [-1, 0] \\
t^2 & \text{if } t \in [0, 1].
\end{cases}
$$

Let $x_0 \in X$ be arbitrary and set

$$
u_0(t) := \begin{cases} 
1 & \text{if } t \in [-1, 0] \\
-1 & \text{if } t \in (0, 1].
\end{cases}
$$

Proceeding as in the previous example, if $(x_0, u_0, p, \mu)$ is an extremal, then

$$
\mu(t) \geq 0, \quad p(t) = 0, \quad p(t) = b(t) + u_0(t)\mu(t) \quad (t \in [-1, 1]).
$$

Thus $p$ is a constant satisfying

$$
p = \begin{cases} 
\mu(t) & \text{if } t \in [-1, 0] \\
t^2 - \mu(t) & \text{if } t \in (0, 1].
\end{cases}
$$

Since $\mu(t) \geq 0$ for all $t \in [-1, 1]$, from the first relation we have $p \geq 0$ and, from the second, $p \leq t^2$ for all $t \in (0, 1]$ and so $p \leq 0$. Thus $p \equiv 0$ and therefore $\mu \equiv b$. This implies that $(p, \mu)$ is the only pair such that $(x_0, u_0, p, \mu) \in \mathcal{E}$.

Now, consider the pair $(q, \nu)$ with $q \equiv 1$ and $\nu(t) := u_0(t)$ ($t \in [-1, 1]$). It satisfies 4.5(i) since

$$
\nu(t) \geq 0 \text{ and } \nu(t)\varphi(u_0(t)) = 0
$$

for all $t \in [-1, 1]$ such that $\mu(t) = 0$, that is, all $t \in [-1, 0]$. Moreover, $\dot{q}(t) = 0$ and $q(t) = u_0(t)v(t) = 1$ for all $t \in [-1, 1]$ and so also 4.5(ii) and (iii) hold. Since $(q, \nu) \neq (0, 0)$, we conclude that $(x_0, u_0)$ is not normal relative to $S_1(\mu)$. \qed

The main result on uniqueness given in [22], as we show next, has its corresponding counterpart for our problem. Denote by $\mathcal{F}(x_0, u_0)$ the set of all $L$ satisfying the smoothness assumptions given at the beginning of the section and such that
\((x_0, u_0)\) solves the problem \(P(L)\) of minimizing \(I(x, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt\) subject to \((x, u) \in S\).

**Theorem 4.9.** Let \((x_0, u_0) \in S\). Then the following are equivalent:

\(a.\) \((x_0, u_0)\) is normal relative to \(S_0\).
\(b.\) \(\Lambda(L, x_0, u_0)\) is a singleton for all \(L \in F(x_0, u_0)\).

**Proof.** \((a) \Rightarrow (b):\) Let \(L \in \mathcal{F}(x_0, u_0)\). By \((a)\), since \((x_0, u_0)\) solves \(P(L)\), there exists \((p, \mu) \in \Lambda(L, x_0, u_0)\). Suppose \((q, \nu) \in \Lambda(L, x_0, u_0)\). Then

i. \([\mu_\alpha(t) - \nu_\alpha(t)]\varphi_\alpha(u(t)) = 0 (\alpha \in R, t \in T)\);
ii. \([\hat{p}(t) - \hat{q}(t)] = -f_\nu^*(\hat{x}(t))\nu(t) + f_\mu^*(\hat{x}(t))\mu(t) (t \in T)\);
iii. \(f_\nu^*(\hat{x}(t))\nu(t) = \varphi^*(u(t))\mu(t) + \varphi^*(u(t))\nu(t) (t \in T)\),

implying, by \((a)\), that \((p, \mu) \equiv (q, \nu)\).

\((b) \Rightarrow (a):\) For all \(i \in R \cup Q\) and \(t \in T\), let \(\mu_i(t) = 1\) if \(i \in I_\mu(u_0(t))\) and \(\mu_i(t) = 0\) otherwise. Define \(L(t, x, u) := -\langle \mu(t), \varphi(u) \rangle\). Note that, for all \((x, u) \in S\),

\[I(x_0, u_0) = \int_{t_0}^{t_1} L(t, x_0(t), u_0(t))dt = 0 \leq \int_{t_0}^{t_1} L(t, x(t), u(t))dt = I(x, u)\]

and so \(L \in \mathcal{F}(x_0, u_0)\). Clearly \((x_0, u_0, 0, \mu) \in \mathcal{E}\) and therefore \((0, \mu) \in \Lambda(L, x_0, u_0)\).

Note that

\[H(t, x, u, 0, \mu(t), 1) = -L(t, x, u) - \langle \mu(t), \varphi(u) \rangle = 0.\]

Now, let \((q, \nu) \in X \times \mathcal{U}_t\) satisfy

i. \(\nu_\alpha(t)\varphi_\alpha(u_0(t)) = 0 (\alpha \in R, t \in T)\);
ii. \(\hat{q}(t) = -f_\nu^* (\hat{x}_0(t))\nu(t) (t \in T)\);
iii. \(f_\nu^* (\hat{x}_0(t))\nu(t) = \varphi^*(u_0(t))\nu(t) (t \in T)\).

Condition \(a)\) will follow if we show that \((q, \nu) \equiv (0, 0)\). To prove it, define

\[\hat{\mu}_i (t) := \mu_i (t) + \langle \nu_i (t) / \beta \rangle \quad \text{and} \quad \hat{p}(t) := q(t) / \beta \quad (i \in R \cup Q, t \in T)\]

where \(\beta = 1 + \max \{ |\nu_\alpha(t)| : \alpha \in I_\nu(u_0(t)), t \in T \}\).

We claim that \((x_0, u_0, \hat{p}, \hat{\mu}) \in \mathcal{E}\). Indeed, 4.3(a) holds since, if \(\varphi_\alpha(u_0(t)) = 0\), then

\[\hat{\mu}_\alpha(t) = \frac{\beta + \nu_\alpha(t)}{\beta} \geq 0\]

and, if \(\varphi_\alpha(u_0(t)) < 0\), then \(\hat{\mu}_\alpha(t)\varphi_\alpha(u_0(t)) = 0\). For 2.3(b) and (c), observe that

\[\hat{p}(t) = \hat{q}(t) / \beta = -f_\nu^* (\hat{x}_0(t))\nu(t) / \beta = -f_\nu^* (\hat{x}_0(t))\hat{p}(t)\]

and, similarly,

\[f_\nu^* (\hat{x}_0(t))\hat{p}(t) = f_\nu^* (\hat{x}_0(t))\nu(t) / \beta = \varphi^*(u_0(t))\nu(t) / \beta = L_\nu (\hat{x}_0(t)) + \varphi^*(u_0(t)) \hat{\mu}(t).\]

This proves the claim and so \((\hat{p}, \hat{\mu}) \in \Lambda(L, x_0, u_0)\). By \((b)\), \((\hat{p}, \hat{\mu}) \equiv (0, \mu)\) and hence \((q, \nu) \equiv (0, 0)\).

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