Conformal properties of harmonic spinors and lightlike geodesics in signature (1, 1)

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Abstract

We are studying the harmonic and twistor equation on Lorentzian surfaces, that is a two dimensional orientable manifold with a metric of signature (1, 1). We will investigate the properties of the solutions of these equations and try to relate the conformal invariant dimension of the space of harmonic and twistor spinors to the natural conformal invariants given by the Lorentzian metric. We will introduce the notion of semi-conformally flat surfaces and establish a complete classification of the possible dimensions for this family.

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1 Introduction

On every Spin manifold we can canonically construct two first order differential operators, the so-called Dirac and twistor operator. It is known that the dimension of their resp. kernels – the space of harmonic resp. twistor spinors – is a conformal invariant, that is invariant under multiplication of the metric used to define these operators with a smooth and strictly positive function. A natural question that arises is to know how these dimensions can be expressed in terms of conformal invariants given by the (pseudo–)Riemannian metric. This question is particularly interesting in the case of Riemannian and Lorentzian surfaces since they carry a natural conformal structure induced by the isothermal charts.
On Riemannian surfaces, harmonic spinors were studied in [Hi74] and [BaSc92]. The dimension of the space of harmonic spinors depends essentially – unlike the dimension of the space of harmonic forms – on the conformal class of the metric and the Spin structure used to define the Dirac operator. Furthermore, the dimension is bounded. The purpose of this paper is to study the dimension of the space of harmonic resp. twistor spinors in the case of an indefinite, that is a Lorentzian metric, and to relate them to lightlike vector fields and lightlike geodesics. We will show that:

- harmonic and twistor spinors can be described in the same way
- harmonic and twistor spinors are linked to the global behaviour of the lightlike geodesics and the given Spin structure
- nowhere vanishing harmonic resp. twistor spinors cause conformal flatness.

Though the Lorentzian theory of surfaces looks similar to its Riemannian analogue, various new phenomena occur. For instance, we have uncountably many conformal classes of simply connected Lorentzian surfaces (see [We96]). On the other hand, the torus is the only compact surface allowed to carry a Lorentzian metric since the Euler number has to vanish. However, a Lorentzian torus – unlike its Riemannian counterpart – need not be conformally flat as we have no regularity properties for solutions of hyperbolic partial differential equations. Further difficulties are caused by the non–compactness of the isometry group and the indefiniteness of the scalar product on the spinor bundle. In order to by–pass these problems, different approaches are carried out: For Lorentzian tori provided with a left–invariant metric, we can explicitly compute the kernels by tools developed in [Ba81]. Non–conformally flat examples of Lorentzian tori are given in section 4.2.2 where we consider a particular class of metrics for which the resulting partial differential equations with respect to the trivial Spin structure can be explicitly solved. We will generalize these examples by the observation that harmonic and twistor half spinors might be interpreted as parallel spinors along one family of lightlike geodesics. We shall introduce semi-conformally flat Lorentzian surfaces which are a particular class of time–orientable, non–conformally flat Lorentzian surfaces (see 4.35) for which a classification of the possible dimensions in dependence on the Spin structure and the global properties of the lightlike geodesics is achieved. These surfaces can by characterized by the existence of a divergence–free lightlike vector field (cf. 1.38). Furthermore, we will rederive some geometric properties of Lorentzian tori shown in [Sa97].

We now want to state our main result. The lightcones in the tangent space induce two one-dimensional lightlike distributions which according to section 2 may be labelled unambiguously by $\mathcal{X}$ and $\mathcal{Y}$ provided the surface is orientable (which we always tacitly assume). Furthermore, a lightlike
vector field is a section either of the $\mathcal{X}$- or of the $\mathcal{Y}$-distribution. It makes therefore sense to speak of an $\mathcal{X}$- or $\mathcal{Y}$-flow if the corresponding vector fields lie in the $\mathcal{X}$- or $\mathcal{Y}$-distribution, of $\mathcal{X}$- or $\mathcal{Y}$-geodesics or $\mathcal{X}$- or of $\mathcal{Y}$-conformal flatness (depending on the divergence-free lightlike vector field to be $\mathcal{X}$ or $\mathcal{Y}$) etc.. By the classical Poincaré-Bendixson theory for ordinary differential equations on a torus we know that such lightlike $\mathcal{X}$- and $\mathcal{Y}$-integral curves or “lines” are either closed, asymptotic of a closed curve or dense. These global properties of the lightlike integral curve give raise to the notion of “resonant” and “non-resonant” cylinders and tori. On the other hand the existence of harmonic spinors imposes some extra conditions on the holonomy of the principal Spin-bundle: If there is a closed $\mathcal{X}$- and $\mathcal{Y}$-line, then the lift of this line to the Spin bundle has to be closed as well. This is what we mean by “$\mathcal{X}$”- resp. “$\mathcal{Y}$-triviality”. Let $\delta_{\pm} = \dim \ker(D_{\pm} : \Gamma(S_{\pm}) \to \Gamma(S_{\mp}))$ resp. $\tau_{\pm} = \dim \ker(P_{\pm} : \Gamma(S_{\pm}) \to \Gamma(S_{\mp}))$ denote the dimensions of the spaces of positive/negative harmonic resp. twistor half-spinors. Then we assert the following to be true (cf. [143]):

**Theorem.** Let $(M^{1+1}, g)$ be a compact $\mathcal{X}$–conformally flat Lorentzian surface. Then $\delta_+ = \tau_-$ and the only possible dimensions for $\delta_+$ are $0, 1$ and $+\infty$. These cases are characterized as follows:

(i) $\delta_+ \leq 1$ if and only if either

- there exists a dense $\mathcal{X}$–line in which case we have $\delta_+ = 0$ for the non–trivial Spin structures,
- or $M^{1+1}$ is non-resonant or
- there exists no $\mathcal{X}$–trivial resonant cylinder on $M^{1+1}$.

Furthermore, we have $\delta_+ = 1$ for the trivial Spin structure.

(ii) $\delta_+ = +\infty$ if and only if there exists an $\mathcal{X}$–trivial resonant cylinder on $M^{1+1}$. In this case we have $\delta_+ = +\infty$ for every Spin structure.

The same conclusion holds for $\mathcal{Y}$ and $\delta_-$ instead of $\mathcal{X}$ and $\delta_+$, and an analogous assertion can be stated for twistor spinors.

The question to what extent this result carries over to general Lorentzian surfaces remains to be settled.

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2 Lorentzian surfaces

We will give a brief introduction to the theory of Lorentzian surfaces. For details see [We96].

A Lorentzian surface \((M^{1+1}, g)\) is given by a smooth and orientable two-dimensional manifold provided with an indefinite metric, that is \(TM^{1+1}\) splits into the direct sum of a timelike bundle \(\xi\) and a spacelike bundle \(\eta\). Furthermore, the lightcone defined by \(g\) is built out of two locally integrable lightlike (or isotropic) distributions. We call these distributions \(\mathcal{X}\) and \(\mathcal{Y}\) according to the following convention: A vector \(v \in TM^{1+1}\) lies in \(\mathcal{X}\) if and only if there exists a further lightlike vector \(w\) such that \((v, w)\) is an oriented basis and \(v + w\) is spacelike. This convention is well-defined, and reversing the orientation interchanges \(\mathcal{X}\) with \(\mathcal{Y}\). Therefore, we can assign to any lightlike object the \(\mathcal{X}\)- resp. \(\mathcal{Y}\)-type and speak of \(\mathcal{X}\)- resp. \(\mathcal{Y}\)-vector fields, curves, geodesics, etc.. We remark that lightlike vector fields need not exist globally. In fact, their global existence is equivalent to the existence of a global orthonormal basis, so that the orthonormal frame bundle over \(M^{1+1}\) is isomorphic to \(M^{1+1} \times SO_+(1, 1)\) where \(SO_+(1, 1)\) denotes the identity component of the isometry group \(O(1, 1)\). Equivalently, we may assume the existence of a non-vanishing timelike vector field. Lorentzian surfaces which admit such vector fields are said to be time-orientable, since they induce an orientation in a timelike subbundle. Simply connected Lorentzian surfaces are always time-orientable. For further reference, we introduce the following notation: If we fix an orthonormal basis \(s = (s_1, s_2)\), let \(X_s = s_1 + s_2\) and \(Y_s = -s_1 + s_2\) which are \(\mathcal{X}\)- resp. \(\mathcal{Y}\)-vector fields.

By a suitable change of the orthonormal basis, every \(\mathcal{X}\)- resp. \(\mathcal{Y}\)-vector field can be written in this form.

The local integrability of the lightlike distributions guarantees the existence of isotropic resp. isothermal coordinates \((x, y)\), so we may locally write \(g = \lambda^2 dx dy\) resp. \(g = \lambda^2(-dx^2 + dy^2)\) for a smooth \(\lambda \neq 0\). In particular, Lorentzian surfaces are locally conformally flat. Recall that two metrics \(g_1\) and \(g_2\) on a manifold \(M\) are said to be conformally equivalent if and only if there is a smooth \(\lambda > 0\) such that \(g_2 = \lambda g_1\). In the case where \(g_2\) is flat, we say that \(g_1\) is conformally flat. An atlas consisting of isotropic or isothermal charts defines – as in the Riemannian case – a conformal structure on the surface. These correspond bijectively to conformal classes of Lorentzian metrics. It should be noted, however, that the corresponding transition functions have no regularity properties. The two isotropic
distributions $\mathcal{X}$ and $\mathcal{Y}$ are conformal invariants of the Lorentzian surface $(M^{1+1}, g)$. In fact, they determine the conformal class $[g]$. The maximal integral curves of the $\mathcal{X}$- and $\mathcal{Y}$-vector fields which we call $\mathcal{X}$- and $\mathcal{Y}$-lines or null lines for short, are further conformal invariants. The $\mathcal{X}$- and $\mathcal{Y}$-lines through $x$ will be denoted by $l_x$ and $m_x$. As we can locally straighten out the null lines by choosing isotropic null coordinates, only the global properties of the null lines encode conformal information.

In the case of a simply connected surface, it can be shown that there are no closed null lines, that two different null lines intersect in at most one point and that every null line is properly embedded in $M^{1+1}$.

Since the existence of a Lorentzian metric on a compact surface is equivalent to $\chi(M) = 0$ where $\chi$ denotes the Euler characteristic, every compact Lorentzian surface is diffeomorphic to a torus. According to the Poincaré–Bendixson theory for ordinary differential equations on the torus, a null line on a compact Lorentzian surface is either dense, a closed curve homeomorphic to $S^1$ which cannot be contracted to a point, or an asymptotic of a closed null line of the same type.

In [Så97], the explicit behaviour of the null lines and further properties are discussed for metrics of the form

$$g(x_1, x_2) = E(x_1)dx_1^2 + 2F(x_1)dx_1dx_2 - G(x_1)dx_2^2.$$ 

Up to a finite covering, all Lorentzian tori with non–trivial isometry group are of that type. If $G \equiv 0$ resp. $|G| > 0$, then $g$ is flat resp. conformally flat. We consider the family of metrics $G'$ where $G(0) = 0$ and $G$ has only isolated zeros in $p_0 = 0, p_1, \ldots, p_{n-1} \in (0, 1)$, $p_{n+k} = p_k + 1$ for all integer $k$. Then $(\mathbb{R}^{1+1}, g)$ is incomplete in the three causal senses for all $g \in G'$, and so is $(T^{1+1}, g)$.

In particular, let us consider the two subfamilies

$$(2.1) \quad G_1 = \{ g \in G' \mid G'(0,1) > 0 \}$$

$$(2.2) \quad G_2 = \{ g \in G' \mid G'(p_i) \neq 0, F(p_i)F(0) > 0, 0 \leq i \leq n - 1 \}.$$ 

We have the following proposition (see [Så97]):

**2.1 Proposition.** Let $g \in G_1 \cup G_2$.

(i) Let $\eta_0 = \text{sgn} F(0)$. Then

$$X_1 = G\partial_{x_1} + \left( F + \eta_0 \sqrt{EG + F^2} \right) \partial_{x_2}$$

$$X_2 = \partial_{x_1} + \left( F - \eta_0 \sqrt{EG + F^2} \over G \right) \partial_{x_2}$$

are two linearly independent isotropic vector fields for $g$ (the choice of $\eta_0$ guaranteeing the existence of the limit in $p_i$).
(ii) The inextendible null geodesics of \(X_2\) are complete. Hence there exists incomplete \(X_1\)-geodesics since \((T^{1+1}, g)\) is lightlike incomplete.

As we remarked above, a Lorentzian torus need not be conformally flat. In fact, we have the following characterization of conformal flatness (see [Sa97]):

2.2 Proposition. Let \((M^{1+1}, g)\) be a Lorentzian surface.

(i) If there is a nowhere vanishing time- or spacelike conformal vector field, then \((M^{1+1}, g)\) is conformally flat. The converse is true if in addition \(M\) is compact.

(ii) Every conformally flat compact Lorentzian surface is complete.

We recall that a vector field \(K\) is called conformal if \(\mathcal{L}_K g = \sigma g\) for a smooth function \(\sigma\) (where \(\mathcal{L}\) denotes the Lie derivative).

3 Pseudo–Riemannian Spin geometry

We will give a brief survey of the relevant Spin geometric features we use in the fourth section. We focus mainly on the signature \((1, 1)\). A general reference is [Ba81].

Let \((\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle_{p,q})\) be the standard pseudo-euclidean vector space of signature \((p, q)\) where \(p\) is the dimension of a maximal timelike subspace. We shall always assume that the \(p\) first vectors of an orthonormal basis are timelike. For \(p + q = 2m\) we can identify the associated Clifford algebra \(\text{Cliff}(\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle_{p,q})\) with \(\text{End}(\Delta_{p,q}) = \mathbb{C}(2^m)\) obtaining thereby an action \(\mu\) of \(\text{Cl}^{p,q}_{\text{p,q}}\) on \(\Delta_{p,q} = \mathbb{C}^{2^m}\). This action will be denoted by \(\cdot\), that is \(\mu(x, v) = x \cdot v\). An explicit isomorphism in signature \((1, 1)\) is given by extension of the mapping

\[
(3.3) \quad e_1 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

Next, we define the groups \(\text{Spin}(p, q)\) and \(\text{Spin}_+(p, q)\). Let \(S_{p,q} = \{ x \in \mathbb{R}^{p+q} \mid \langle x, x \rangle_{p,q} = 1 \}\) and \(H_{p,q} = \{ x \in \mathbb{R}^{p+q} \mid \langle x, x \rangle_{p,q} = -1 \}\).

3.1 Definition.

\[
\text{Spin}(p, q) = \{ x_1 \ldots x_{2l} \mid x_i \in H_{p,q} \cup S_{p,q} \} \\
\text{Spin}_+(p, q) = \{ x_1 \ldots x_{2l} \mid x_i \in H_{p,q} \cup S_{p,q} \text{ with an even number of timelike factors} \}
\]
In order to deal simultaneously with the pairs \( SO(p, q)/\text{Spin}(p, q) \) and \( SO_+(p, q)/\text{Spin}_+(p, q) \) we write \( G(p, q) \) and \( \tilde{G}(p, q) \), where \( \tilde{G}(p, q) = \text{Spin}(p, q) \) if \( G(p, q) = SO(p, q) \) and \( \tilde{G}(p, q) = \text{Spin}_+(p, q) \) if \( G(p, q) = SO_+(p, q) \). If we use \( \text{Spin}(1, 1) \) to represent \( \text{Spin}(1, 1) \), we get the following:

### 3.2 Lemma

\[
\text{Spin}(1, 1) = \left\{ g_a = \begin{pmatrix} a & 0 \\ 0 & \pm \frac{1}{a} \end{pmatrix} \mid a \in \mathbb{R} \setminus \{0\} \right\}
\]

The volume element \( \omega = e_1 \cdot e_2 \) of \( \text{Cl}_{1,1} \) defines – viewed as an endomorphism of \( \Delta_{1,1} \) – a splitting of \( \Delta_{1,1} \) into the direct sum of \( \Delta^+_{1,1} = \text{Eigenspace of } \omega \text{ for } -1 = (z_1) \) and \( \Delta^-_{1,1} = \text{Eigenspace of } \omega \text{ for } 1 = (iz_2) \) where \( (z_1, z_2) \) denotes the standard basis of \( \mathbb{C}^2 \) (the sign convention follows [Ba81] and is motivated by the higher dimensional case). Reversing the orientation interchanges \( \Delta^+_{1,1} \) with \( \Delta^-_{1,1} \).

Next, we consider Spin structures of \( (M^{p+q}, g) \), that is reductions \( (Q, f) \) of the \( G(p, q) \)-frame bundle \( P \) to a \( \tilde{G}(p, q) \)-bundle \( Q \). We have the following criterion for the existence of such reductions:

### 3.3 Proposition

Let \( (M^{p+q}, g) \) be a connected pseudo–Riemannian manifold and \( TM = \xi^p \oplus \eta^q \) a splitting into a time- and spacelike bundle resp. of maximal rank.

- (i) \( (M^{p+q}, g) \) is Spin if and only if \( w_2(TM) = w_2^2(\eta) \) where \( w_i \in H^i(M, \mathbb{Z}_2) \) denotes the \( i \)-th Stiefel–Whitney–class.

- (ii) If \( (M^{p+q}, g) \) is time–oriented, then the mapping \( \text{Spin}(M^{p+q}, g) \to \pi_1(P, x), (Q, f) \mapsto f_*\pi_1(Q, y) \) for \( y \in f^{-1}(x) \) is injective. In particular, two Spin structures which are isomorphic as a twofold covering of \( P \) are isomorphic as Spin structures.

- (iii) If \( \text{Spin}(M^{p+q}, g) \neq \emptyset \), then \( \text{card}(\text{Spin}(M^{p+q}, g)) = \text{card}(H^1(M^{p+q}, \mathbb{Z}_2)) \).

For a proof of (i) see [Ba81], for (ii) and (iii) see [Ba81].

In particular, every time–orientable Lorentzian surface admits a Spin structure. One is explicitly given by \( Q_0 = M^{1+1} \times \text{Spin}_+(1, 1) \) and \( f_0(x, a) = (x, \Lambda(a)) \). This Spin structure will be referred to as the trivial one; it is unique up to isomorphism if \( M \) is simply connected. On the other hand, if \( M \) is time–orientable and compact, then \( (M^{1+1}, g) \) carries four non–isomorphic Spin structures since \( H^1(T^{1+1}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

If \( \tilde{g} = \kappa^2 g \) is conformally equivalent to \( g \), we can canonically associate a Spin structure \((\tilde{Q}, \tilde{f})\) with every Spin structure \((Q, f)\) on \((M^{p+q}, g)\): If \( \Phi_\kappa : P_g \to P_{\tilde{g}} \) is the isomorphism defined by \( s = (s_1, \ldots, s_{p+q}) \mapsto \frac{1}{\kappa} s = (\frac{1}{\kappa} s_1, \ldots, \frac{1}{\kappa} s_{p+q}) \), then the subgroup \( (\Phi_\kappa \circ f)_# \pi_1(Q, q) \) in \( \pi_1(P_g, \Phi_\kappa(f(q))) \) distinguishes by a Spin structure \((\tilde{Q}, \tilde{f})\) which can be shown to be isomorphic with \((Q, f)\).
Now, we fix a Spin structure \((Q, f)\) over \((M^{p+q}, g)\). The associated fiber bundle

\[
S = Q \times_{\hat{G}(p,q)} \Delta_{p,q}
\]

is a complex vector bundle of rank \(2^{\lfloor \frac{p+q}{2} \rfloor}\) which is called the spinor bundle associated with \((Q, f)\).

The set of smooth sections of \(S\) is denoted by \(\Gamma(S)\): its elements are called spinor fields or spinors for short. A spinor \(\varphi\) may be represented by a \(\hat{G}(p,q)\)-equivariant function \(\hat{\varphi} \in C^\infty(Q, \Delta_{p,q})\hat{G}(p,q)\), that is, \(\hat{\varphi}(gq) = g^{-1}\hat{\varphi}(q)\) for all \(q \in Q\) and \(g \in \hat{G}(p,q)\). One may also think of a spinor as a collection of local sections \(\tilde{s} : U \to Q\) covering \(M\) together with a family of local trivializations \(\varphi_\tilde{s} \in C^\infty(U, \Delta_{p,q})\) verifying \(\varphi_\tilde{s}g = g^{-1}\varphi_\tilde{s}\). Furthermore, we can consider the sections of \(S^\pm = Q \times_{\hat{G}(1,1)} \Delta^\pm_{1,1}\), where the fiberwise splitting is induced by the volume element of \(\text{Cliff}(T_xM^{1+1}, g_x)\), the Clifford algebra generated by \((T_xM^{1+1}, g_x)\). The corresponding sections in \(\Gamma(S^\pm)\) are called half spinors. To emphasize the sign, we also speak of positive or negative (half-)spinors. In the subsequent chapters, we will also use the following representation of half spinors: If \(\varphi \in \Gamma(S^\pm)\) is a positive resp. negative spinor, we can write \(\tilde{\varphi}(g) = \tilde{f}^\pm(q)u_{\pm}\) for \(\tilde{f}^\pm \in C^\infty(Q, \mathbb{C})\). Using the representation of \(\text{Spin}(1,1)\) of \(\mathbb{H}\), the transformation law of \(\tilde{f}^+\) is given by \(\tilde{f}^+(gg_a) = \frac{1}{a}\tilde{f}^+(q)\) and \(\tilde{f}^-(gg_a) = a\tilde{f}^-(q)\). The same holds if one considers the complex valued functions \(f_\tilde{s}\) given by the local trivializations \(\varphi_\tilde{s} = f_\tilde{s}u_{\pm}\).

As for Riemannian Spin bundles the covariant derivative \(\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)\) is induced by the lift to \(Q\) of the Levi–Civita–connection \(Z\) in \(P\). Fixing a local orthonormal basis \(s = (s_1, s_2)\), we get

\[
\nabla^S_\tilde{V}\varphi = [\tilde{s}, V(\varphi_\tilde{s})] - \frac{1}{2}g(\nabla^SC_{s_1, s_2}e_1 \cdot e_2 \cdot \varphi_\tilde{s}].
\]

We also verify the product rule

\[
\nabla^S_\tilde{V}(W \cdot \varphi) = (\nabla^C_{\tilde{V}}W) \cdot \varphi + W \cdot \nabla^S_\tilde{V}\varphi.
\]

The main difficulty in the pseudo-Riemannian setup is to define a suitable scalar product on \(S\). This can be done as follows: Assume \((M^{1+1}, g)\) to be time-orientable. Let \(TM^{1+1} = \xi^1 \oplus \eta^1\) be a splitting into a (now trivial) time- resp. spacelike vector bundle. Fix orientations in \(\xi\) and \(\eta\). \(P\) can be reduced to the structure group \(K = SO(1) \times SO(1)\), which is maximal compact in \(SO_+(1,1)\). The reduced bundle is given by \(P_\xi = \{(s_1, s_2) | s_1\ \text{positively oriented in } \xi, s_2\ \text{positively oriented in } \eta\}\). Then \(\hat{Q}_\xi = f^{-1}(P_\xi)\) is the reduction of \(Q\) to \(\hat{K} = (\text{Spin}_+(1) \times \text{Spin}_+(1))/\mathbb{Z}_2\) which is maximal compact in \(\text{Spin}_+(1,1)\). We have \(S = \hat{Q}_\xi \times_{\hat{K}} \Delta_{1,1}\) and \(TM = P_\xi \times_{\hat{K}} \mathbb{R}^{1+1}\). Let \((\cdot, \cdot)_{\Delta_{1,1}}\) denote the standard hermitian product on \(\Delta_{1,1}\) which is \(\hat{K}\)-invariant, but not \(\text{Spin}_+(1,1)\)-invariant. We can extend this scalar product to a fiberwise defined scalar product \((\cdot, \cdot)_\xi\) on \(S\). Now let \(J_\xi : S \to S, J_\xi ([g, v]) =
\]

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\([\tilde{q}, e_1 \cdot v]\) for \(\tilde{q} \in \tilde{Q}_\xi\) and the unit vector field \(e_1 \in \xi\). We define

\[
< \varphi, \psi >_x = (J_\xi \varphi, \psi)_{\xi_x} = (e_1 \cdot v, w)_{\Delta_1},
\]

where \(\varphi(x) = [\tilde{q}, v]\) and \(\psi(x) = [\tilde{q}, w]\). This is an indefinite, \(\text{Spin}_+(1,1)\)-invariant scalar product on \(S\). Then the formulae

\[
V(< \varphi, \psi >) = < \nabla^S V \varphi, \psi > + < \varphi, \nabla^S V \psi >
\]

and

\[
< X \cdot \varphi, \psi > = < \varphi, X \cdot \psi >
\]

hold.

Given \((S, \nabla^S)\) over \((M^{1+1}, g)\), we can canonically define two first order differential operators, namely the Dirac operator \(D\) and the twistor operator \(P\). In terms of a local orthonormal basis \(s = (s_1, s_2)\), they are given by

\[
D\varphi = -s_1 \cdot \nabla^S_{s_1} \varphi + s_2 \cdot \nabla^S_{s_2} \varphi \quad \text{and} \quad P\varphi = -s_1 \otimes \left( \nabla^S_{s_1} \varphi + \frac{1}{2} s_1 \cdot D\varphi \right) + s_2 \otimes \left( \nabla^S_{s_2} \varphi + \frac{1}{2} s_2 \cdot D\varphi \right).
\]

In the fourth section, we will deal with the equations \(D\varphi = 0\) and \(P\psi = 0\) called the harmonic resp. the twistor equation, the latter being equivalent to \(\nabla^S_V \varphi = -\frac{1}{2} V \cdot D\varphi\) for every \(V \in \mathfrak{X}(M^{1+1})\). The solutions are referred to as harmonic resp. twistor spinors. The vector spaces of harmonic resp. twistor spinors will be denoted by \(\mathcal{H}_\pm\) and \(\mathcal{T}_\pm\). Furthermore, we will consider \(\mathcal{H}_\pm = \Gamma(S^\pm) \cap \mathcal{H}\) and analogously \(\mathcal{T}_\pm = \Gamma(S^\pm) \cap \mathcal{T}\). The superscript \(0\) denotes the space of harmonic and twistor spinors respectively. Their dimension with respect to the trivial Spin structure. We are mainly interested in the numbers \(\delta(\pm) = \dim(\mathcal{H}_\pm)\) and \(\tau(\pm) = \dim(\mathcal{T}_\pm)\), since they have the following well-known property (see [Ba99] and [BFGK91]):

3.4 Proposition. Let \(\tilde{g} = \lambda g\) and \(\tilde{\mathcal{H}}_\pm\) resp. \(\tilde{\mathcal{T}}_\pm\) the space of harmonic resp. twistor \((\text{half–})\) spinors with respect to \(\tilde{g}\). Then the maps

\[
\begin{align*}
(i) \quad \tilde{\varphi} \in \tilde{\mathcal{H}}_\pm & \mapsto \lambda^{\frac{1}{4}} \tilde{\varphi} \in \mathcal{H}_\pm \\
(ii) \quad \tilde{\psi} \in \tilde{\mathcal{T}}_\pm & \mapsto \lambda^{-\frac{1}{4}} \tilde{\psi} \in \mathcal{T}_\pm
\end{align*}
\]

are isomorphisms. In particular, \(\delta(\pm)\) and \(\tau(\pm)\) are conformal invariants.
4 Spinor field equations and lightlike geodesics in signature (1, 1)

4.1 Harmonic and twistor spinors

4.1 Proposition.

(i) Let $\varphi$ be in $\Gamma (S^+) \text{ resp. } \Gamma (S^-)$. Then $\varphi$ is harmonic if and only if

$$\nabla^S_X \varphi \equiv 0 \text{ resp. } \nabla^S_Y \varphi \equiv 0$$

holds for all $\mathcal{X}$–vector fields $X$ resp. $\mathcal{Y}$–vector fields $Y$.

(ii) Let $\varphi$ be in $\Gamma (S^+) \text{ resp. } \Gamma (S^-)$. Then $\varphi$ is twistor if and only if

$$\nabla^S_Y \varphi \equiv 0 \text{ resp. } \nabla^S_X \varphi \equiv 0$$

holds for all $\mathcal{Y}$–vector fields $Y$ resp. $\mathcal{X}$–vector fields $X$.

Proof. We prove the assertion only for positive spinors, the remaining cases being showed in the same way.

Let $\varphi \in \Gamma (S^+)$ and let $X$ and $Y$ be a $\mathcal{X}$– resp. $\mathcal{Y}$–vector field which we write $X = X_s = (s_1 + s_2)$ and $Y = Y_s = (-s_1 + s_2)$.

(i) Using the local expression of $D$, we see that $\varphi$ is harmonic if and only if

$$s_1 \cdot \nabla^S_{s_1} \varphi = s_2 \cdot \nabla^S_{s_2} \varphi$$

which is equivalent to

$$\nabla^S_{s_1} \varphi = \omega \cdot \nabla^S_{s_2} \varphi = -\nabla^S_{s_1} \varphi$$

(with the volume element $\omega = s_1 \cdot s_2$). Hence $\varphi$ is harmonic if and only if

$$\nabla^S_{s_1 + s_2} \varphi = 0.$$  \[\Box\]

(ii) $\nabla^S_{s_i} \varphi = -\frac{1}{2} s_i \cdot D \varphi$ for $i = 1, 2$ is equivalent to

$$\nabla^S_{s_1} \varphi = -\omega \cdot \nabla^S_{s_2} \varphi$$

and

$$\nabla^S_{s_2} = -\omega \cdot \nabla^S_{s_1} \varphi,$$

hence to

$$\nabla^S_{s_1 + s_2} \varphi = 0.$$  \[\Box\]

Since $\nabla^S_{s_i} \varphi (x) = [q_x, V^* (\tilde{\varphi} (q_x))]$ for all $q_x \in \pi_Q^{-1} (x)$ and $V \in \mathcal{X} (M)$ (where $V^*$ denotes the horizontal lift of $V$ to $Q$), proposition 4.1 may be restated as follows:

4.2 Corollary 1.

(i) $\mathcal{H}_+ = \{ \tilde{\varphi} \in C^\infty (Q, \Delta^+_{1,1})^{\tilde{G}(1,1)} | \tilde{\varphi} \text{ is constant along the horizontal lifts of } \mathcal{X} - \text{curves} \}$

(ii) $\mathcal{H}_- = \{ \tilde{\varphi} \in C^\infty (Q, \Delta^-_{1,1})^{\tilde{G}(1,1)} | \tilde{\varphi} \text{ is constant along the horizontal lifts of } \mathcal{Y} - \text{curves} \}$

(iii) $\mathcal{I}_+ = \{ \tilde{\varphi} \in C^\infty (Q, \Delta^+_{1,1})^{\tilde{G}(1,1)} | \tilde{\varphi} \text{ is constant along the horizontal lifts of } \mathcal{Y} - \text{curves} \}$

(iv) $\mathcal{I}_- = \{ \tilde{\varphi} \in C^\infty (Q, \Delta^-_{1,1})^{\tilde{G}(1,1)} | \tilde{\varphi} \text{ is constant along the horizontal lifts of } \mathcal{X} - \text{curves} \}$
A further characterization is given by the formula \( \nabla^S_{t}(t)\varphi = \frac{d}{ds} \mathcal{P}^Q_{\gamma,t+s-t} \varphi(\alpha(t+s)) |_{s=0} \) for any smooth curve \( \gamma \), where \( \mathcal{P}^Q_{\gamma,t+s-t} \) denotes the parallel transport of \( Q \) along \( \gamma \) between the fibers \( \pi_Q^{-1}(\gamma(t+s)) \) and \( \pi_Q^{-1}(\gamma(t)) \):

4.3 Corollary 2. Let \( \varphi \in \Gamma(S^+) \). Then \( \varphi \) is a positive harmonic spinor if and only if for any \( \mathcal{X} \)-curve \( \alpha \) joining two points \( x \) and \( y \) in \( M^{1+1} \), we have \( \varphi(y) = [\mathcal{P}^Q_{\alpha,x \to y} Q, v] \) for \( \varphi(x) = [q, v] \). Analogous statements hold for \( \mathcal{S}_- \), \( \Xi_+ \) and \( \Xi_- \).

As a first application, we note the following:

4.4 Proposition. There is a bijective correspondence between the sets \( \{ \varphi \in \mathcal{S}_+ \mid \varphi(x) \neq 0 \text{ for all } x \} \) and \( \{ \varphi \in \mathcal{S}_- \mid \psi(x) \neq 0 \text{ for all } x \} \).

Proof. If \( \varphi \in \Gamma(S^+) \) is given by \( \tilde{f}^+ u_1 \) for \( \tilde{f}^+ \in C^\infty(Q, \mathbb{C}) \), we can define a twistor spinor \( \psi_{\tilde{f}^+} \in \Gamma(S^-) \) by \( \psi_{\tilde{f}^+}(qg_a) = \frac{1}{\tilde{f}^+ (qg_a)} a \frac{1}{\tilde{f}^+ (q)} a \psi_{\tilde{f}^+} (q) = g_a^{-1} \psi_{\tilde{f}^+} (q) \), so \( \frac{1}{\tilde{f}^+} u_1 \) defines indeed a \( \tilde{G}(1,1) \)-invariant function. Because of \( X^* (\frac{1}{\tilde{f}^+}) = 0 \), \( \psi = [q, \frac{1}{\tilde{f}^+ (q)} u_1] \) defines a negative twistor spinor.

4.5 Proposition. Let \( \varphi \in \mathcal{S}_+ \). If \( \varphi(x) = 0 \), then \( \varphi |_{l_x} \equiv 0 \). In particular, we have \( \delta_+ \leq 1 \) for any Spin structure if there exists a dense null line on \( (M^{1+1}, g) \). Analogous statements hold for \( \mathcal{S}_-, \Xi_+ \) and \( \Xi_- \).

Proof. The first assertion is a consequence of the above corollaries. Assume that there is a \( x \in M^{1+1} \) with \( l_x \) is dense in \( M^{1+1} \). Let \( \varphi_1, \varphi_2 \in \mathcal{S}_+ \) with \( \varphi_1 \neq 0 \). Pick \( c \in \mathbb{C} \) such that \( \varphi_2(x) = c \varphi_1(x) \). Hence \( (\varphi_2 - c \varphi_1) |_{l_x} \equiv 0 \), that is \( \varphi_2 \equiv c \varphi_1 \) for continuity reasons.

4.2 Examples

We now apply the preceding results to compute some explicit examples. For the sake of simplicity, we will only deal with positive harmonic spinors, but all examples extend to the remaining cases in an obvious way.

4.2.1 Simply connected surfaces

As observed in section 2, two different null lines intersect at most once and closed null lines cannot exist. Furthermore, the frame bundle \( P \) is trivial since \( (M^{1+1}, g) \) is time-orientable, and the resulting trivial Spin structure is unique up to isomorphism.
4.6 Proposition. If $M^{1+1}$ is simply connected, then $\delta_+ = +\infty$.

Proof. Let $\beta : [0, 1] \to M^{1+1}$ be a $\mathcal{V}$-curve with lift $\hat{\beta}$ to $Q$, and let $f_n : [0, 1] \to \mathbb{C}$ be a family of linearly independent smooth functions whose support is strictly contained in $[0, 1]$. Define a Spin$_+$(1, 1)-equivariant function $\hat{\varphi}_n : Q \to \Delta^+_1$ by extending $\hat{\varphi}_n(\hat{\beta}(t)) = f_n(t)u_1$ first to $Q_{\beta}$ by the transitive action of Spin$_+$(1, 1) on the fibers, and secondly to $M^{1+1}$ by parallel transport on $A = \bigcup_{x \in \beta} l_x$ and $\hat{\varphi}_n \equiv 0$ on $A^c$.

This construction depends crucially on the fact that for simply connected surfaces, the local and the global behaviour of the null lines are the same. Therefore we can extend local solutions to global ones. This observation is the key for the construction of harmonic and twistor spinors in 4.3. We will link the global behaviour of the null lines to the spinors; by studying the null lines in the large, we will be able to extend local solutions or to find obstructions for doing so.

4.2.2 Diagonal metrics on Lorentzian tori

We consider Lorentzian tori $(T^{1+1}, g, \lambda)$ whose metric is given by a diagonal metric

$$g_\lambda(x_1, x_2) = -\lambda_1^2(x_1, x_2)dx_1^2 + \lambda_2^2(x_1, x_2)dx_2^2$$

for $\lambda_1, \lambda_2 \neq 0$ in $C^\infty(\mathbb{R}^2)^2$.

Left-invariant metrics

First we consider the case where $\lambda_1, \lambda_2$ are constant. Thus $(T^{1+1}, g_\lambda)$ may be seen as a Lie group provided with a left–invariant metric. Since $\pi_1(T^{1+1})$ has no two–torsions, we may treat the harmonic and the twistor equation for all four Spin structures simultaneously by tools developed in [Ba81] which we will briefly sketch:

The problem is to compare two non–isomorphic Spin structures $(Q_1, f_1)$ and $(Q_2, f_2)$ and to find conditions for $\Gamma(S_1)$ and $\Gamma(S_2)$ to be isomorphic. Let $(M^{p+q}, g)$ be a pseudo–Riemannian Spin manifold of signature $(p, q)$. Let $R' = \{(q_1, q_2) \in Q_1 \times Q_2 : f_1(q_1) = f_2(q_2)\}$. $Z_2$ acts naturally on each fiber of $Q_1$, hence on $R'$. The pair $(R, \mu)$, where $\theta : R \to P \colon [q_1, q_2] \mapsto f_1(q_1)$, is called the deformation of $(Q_1, f_1)$ and $(Q_2, f_2)$. If $(Q_1, f_1)$ and $(Q_2, f_2)$ are isomorphic, then $R$ is isomorphic to $P \times Z_2$. $G(p, q) \times Z_2$ acts on $R$ by $[q_1, q_2], (A, m) = [q_1a, q_2am]$, where $a \in \lambda^{-1}(A)$. This action is well defined, therefore providing $R$ with the structure of a $G(p, q) \times Z_2$–fiber bundle.

Next we define the vector bundle $E = R/G(p, q) \times Z_2 \mathbb{R}$ over $M$. Its complexification $E^C$ is given by $E^C = R/G(p, q) \times Z_2 \mathbb{C}$. Let $\tilde{s}_1 : U \to Q_1$ be two local sections and let $[\tilde{s}] = [(\tilde{s}_1, \tilde{s}_2)] : U \to R$ where $[\ ]$ denotes the equivalence classes in $R$. Let $e \in E^C$. Then $e$ can be represented in the form $e = [\tilde{e}], z$ ($\{\}$ denoting the equivalence classes in $R/G(p, q)$).
4.7 Proposition. The map \( \beta : S_1 \otimes E^C \to S_2 \) defined by \( \beta ([\tilde{s}_1, v]_x \otimes [(\tilde{s}), z]_x) = [\tilde{s}_2, zv]_x \) is a vector bundle isomorphism.

Hence, in the case where \( E^C \) is trivial, the spinor bundles \( S_1 \) and \( S_2 \) are isomorphic. For instance, this happens if \( (Q_1, f_1) \) and \( (Q_2, f_2) \) are isomorphic, for \( E^C \) is then isomorphic to \( M^{p+q} \times \mathbb{Z}^2 \). Thus equivalent Spin structures induce isomorphic spinor bundles. Furthermore, we yield the following

4.8 Corollary. On a surface \( M \) each two spinor bundles are isomorphic.

Proof. For the first Chern class of the complexification \( E^C \) of the real line bundle \( E \) holds \( 2c_1(E^C) = 0 \). Since \( H^2(M, \mathbb{Z}) = 0 \) or \( \mathbb{Z} \) depending on whether or not \( M \) is compact, \( E^C \) must be trivial. ■

Next we want to know how the spinor derivative transforms under this isomorphism:

Let \( \nabla_{E^C} \) be the connection induced by the lift to \( R \) of the Levi–Civita connection of \( P \). Then one shows that \( \nabla_{E^C} \) is flat, hence for \( \eta = [(\tilde{s}), z] \in \Gamma(E^C_U) \) with \( z : U \to \mathbb{C} \), we have \( \nabla_{E^C} \eta = [(\tilde{s}), V(\eta)] \).

Therefore, the following diagram commutes for every \( V \in \mathcal{X}(M^{p+q}) \):

\[
\begin{array}{ccc}
\Gamma(S_1 \otimes E^C) & \xrightarrow{\beta} & \Gamma(S_2) \\
\nabla_{V}^{S_1 \otimes E^C} & \circlearrowleft & \nabla_{V}^{S_2} \\
\Gamma(S_1 \otimes E^C) & \xrightarrow{\beta} & \Gamma(S_2)
\end{array}
\]

where \( \nabla_{V}^{S_1 \otimes E^C} (\varphi \otimes \eta) = \nabla_{V}^{S_1} \varphi \otimes \eta + \varphi \otimes \nabla_{V}^{E^C} \eta \).

If we assume \( E^C \) to be trivial, then we can choose a nowhere vanishing section \( e : M^{p+q} \to E^C \).

We define the complex-valued form \( \omega_e \) by the equation \( \nabla_{V}^{E^C} e = \omega_e(V)e \) and \( \alpha_e : \Gamma(S_1) \to \Gamma(S_1 \otimes E^C) \) by \( \alpha_e(\varphi)(x) = \varphi_x \otimes e_x \). Then the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma(S_1) & \xrightarrow{\alpha_e} & \Gamma(S_1 \otimes E^C) \\
\nabla_{V}^{S_1} + \omega_e(V) & \circlearrowleft & \nabla_{V}^{S_2} \\
\Gamma(S_1) & \xrightarrow{\alpha_e} & \Gamma(S_1 \otimes E^C)
\end{array}
\]

Let us now consider the special case of a connected Lie group \( G \) provided with a left–invariant metric \( g \). Let \( p : \tilde{G} \to G \) be the universal cover of \( G \). \( \pi_1(G) \) acts as a group of deck transformations. Since \( g \) is left–invariant, we can trivialize \( P \) by choosing \( n \) left–invariant vector fields on \( G \), i.e.
$P = G \times SO_+ (p,q)$. Therefore, $Q_0 = G \times Spin_+(p,q)$ with $f_0 = id \times \lambda$ defines the trivial Spin structure on $G$. The lifts $\tilde{X}_i$ of the vector fields $X_i$ to $\tilde{G}$ are globally left– and $\pi_1(G)$–invariant vector fields on $\tilde{G}$, hence $\tilde{P} = \tilde{G} \times SO_+ (p,q)$ and $\tilde{Q}_0 = \tilde{G} \times Spin_+(p,q)$, $\tilde{f}_0 = id \times \lambda$. We know that $Spin(G,g) \cong Hom(\pi_1(G), \mathbb{Z}_2)$. Let $\chi \in Hom(\pi_1(G), \mathbb{Z}_2)$. $\pi_1(G,g)$ acts on $\tilde{G} \times Spin_+(p,q)$ through $\chi$ by $\omega.(\tilde{g},a) = (\omega \tilde{g}, \chi(\omega) a)$, where $\omega \in \pi_1(G,e)$. Let $Q_\chi = \tilde{G} \times \mathbb{Z}_2$ Spin$(p,q)$ and $f_\chi : Q_\chi \to P$, $[\tilde{g},a] \mapsto [p(\tilde{g}), \lambda(a)]$. Then $(Q_\chi, f_\chi)$ defines a Spin structure and the following proposition holds (cf. [Ba81]):

4.9 PROPOSITION.

(i) $Spin(G,g) \cong \{(Q_\chi, f_\chi) \mid \chi \in Hom(\pi_1(G), \mathbb{Z}_2)\}$

(ii) The spinor bundle $S_\chi = Q_\chi \times Spin_+(p,q) \Delta_{p,q}$ associated with the Spin structure $(Q_\chi, f_\chi)$ is given by $S_\chi = \tilde{G} \times \chi \Delta_{p,q}$.

(iii) The deformation of $(Q_0, f_0)$ and $(Q_\chi, f_\chi)$ is given by $R_\chi = \left(\tilde{G}/\ker(\chi)\right) \times SO_+(p,q)$. Furthermore, $E_\chi^C = \left(\tilde{G}/\ker(\chi)\right) \times \mathbb{Z}_2$.

Assume that we have a nowhere vanishing section $e_\chi \in \Gamma(E^C)$. Such a section is given by a map $e_\chi : \tilde{G} \to \mathbb{C}$ without zeros such that $\epsilon(\omega \tilde{g}) = \chi(\omega)e(\tilde{g})$. If for $\chi \equiv 1$ we have $e_1 \equiv 1$, we can identify $S_1$ with the trivial Spin structure and $\Gamma(S_1)$ with $C^\infty(G, \Delta_{p,q})$. Let $(g, Id)$ be a global section of $P \cong G \times SO_+(p,q)$ and let $[\tilde{g}, 1] : U \to Q_\chi$ be a local lift of this section. Then $\gamma(x) = [\{\tilde{g}(x)\}, 1] \in \Gamma(E^C_U)$ corresponds to this section and $e_\chi(x) = \left[\{\tilde{g}(x)\}, e_\chi(\tilde{g}(x))\right] = e_\chi(\tilde{g}(x))\gamma(x)$. Since $\nabla^E \gamma = 0$, we get

$$\nabla^E_V e_\chi = (de_\chi)(V^*)\gamma = e_\chi^{-1}(de_\chi)(V^*)e_\chi,$$

hence $\omega_\chi(V) = e_\chi^{-1}V^*(e_\chi)$, where $V^*$ denotes the lift of $V \in \mathfrak{X}(G)$ to $Q_\chi$.

Identifying $\Gamma(S_\chi)$ with $\Gamma(S_1) = C^\infty(G, \Delta_{p,q})$ yields $\nabla^S_V \varphi = \nabla^S_V \varphi + e_\chi^{-1}V^*(e_\chi)\varphi$ for $\varphi \in C^\infty(G, \Delta_{p,q})$.

For instance, consider the Lorentzian torus $(T^{1+1}, g_0)$. The universal cover is given by $p : \mathbb{R}^2 \to T^{1+1}$, $p(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2})$. Then $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{R}^2$ by $(z_1, z_2). (x_1, x_2) = (x_1 + z_1, x_2 + z_2)$. On the other hand, $Hom(\pi_1(T), \mathbb{Z}_2)$ can be identified with $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(a_1, a_2) \mid a_i \in \{\pm 1\}\}$ by $\chi_1 = \chi(0 \oplus 1) = e^{i\frac{\pi}{2}(1-a_1)}$ and $\chi_2 = \chi(0 \oplus 1) = e^{i\frac{\pi}{2}(1-a_2)}$. We define

$$\epsilon_a(x_1, x_2) = e^{i\frac{\pi}{2}(x_1(1-a_1) + x_2(1-a_2))}.$$

Then $\epsilon_{(1,1)} = 1$ and $\epsilon_\chi((z_1, z_2), (x_1, x_2)) = \chi_1^{z_1} \chi_2^{z_2} \epsilon_\chi(x_1, x_2) = \chi(z_1, z_2) \epsilon_\chi(x_1, x_2)$. Now $\omega_\chi(X_s)(x) = e_\chi^{-1}((s_1 + s_2)^*)(\epsilon_\chi(x))$, hence

$$\omega_\chi(X_s)(x) = \frac{i\pi}{2} \left( -\frac{1-a_1}{\lambda_1} + \frac{1-a_2}{\lambda_2} \right).$$
Consequently, for a spinor \( \tilde{f}u_1 \in \Gamma(S^+(a_1, a_2)) \cong C^\infty(T^{1+1}, \mathbb{C}u_1) \) to be harmonic, we yield the following equation:
\[
\nabla_X^S \tilde{f} + \omega(X) \tilde{f} = \frac{1}{\lambda_1} \partial_{x_1} \tilde{f} + \frac{1}{\lambda_2} \partial_{x_2} \tilde{f} + \frac{i\pi}{2} \left( \frac{1 - a_1}{\lambda_1} + \frac{1 - a_2}{\lambda_2} \right) \tilde{f} = 0.
\]
Let the development of \( \tilde{f} \) into a Fourier series be given by \( \tilde{f}(x_1, x_2) = \sum_{k,l \in \mathbb{Z}} \tilde{f}_{kl} e^{2\pi i(kx_1 + lx_2)} \). Then we get the equation
\[
\sum_{k,l \in \mathbb{Z}} \tilde{f}_{kl} \left( \frac{4k - a_1 + 1}{\lambda_1} + \frac{4l - a_2 + 1}{\lambda_2} \right) e^{2\pi i(kx_1 + lx_2)} = 0.
\]
Thus \( \tilde{f} \in C^\infty(T^{1+1}, \mathbb{C}) \) defines a harmonic spinor if and only if
\[
\tilde{f}_{kl} = 0 \text{ or } 4k - a_1 + 1 = -\frac{\lambda_1}{\lambda_2} (4l - a_2 + 1).
\]
Since constants are solutions for the trivial Spin structure and \( 4k - a_1 + 1 \) and \( 4l - a_2 + 1 \) are in \( \mathbb{Z} \), we finally find:

| \( S(a_1, a_2) \) | \( \delta_+ \) for \( \lambda_{a_1} \) | \( \lambda_{a_2} \) in \( \mathbb{Q} \) | \( \lambda_{a_2} \) not in \( \mathbb{Q} \) |
|-------------------|-----------------|-----------------|-----------------|
| \((+1, +1)\)      | \(+\infty\)     | 1               |                 |
| \((+1, -1)\)      | \(+\infty\)     | 0               |                 |
| \((-1, +1)\)      | \(+\infty\)     | 0               |                 |
| \((-1, -1)\)      | \(+\infty\)     | 0               |                 |

**Closed metrics**

Next, let \( \lambda_i \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}^2} \) be two periodic functions satisfying the additional condition:
\[
\partial_{x_2} \lambda_1 + \partial_{x_1} \lambda_2 = 0,
\]
that is the form \( \lambda = \lambda_1 dx_1 - \lambda_2 dx_2 \) is closed. Therefore we refer to this type of metrics as *closed*. Fix the orthonormal frame \( s = \left( \frac{1}{\lambda_1} \partial_{x_1}, \frac{1}{\lambda_2} \partial_{x_2} \right) \). Then \( \text{div}(X_s) = 0 \) – a fact we will reconsider later. In terms of Fourier coefficients, the closedness condition may be restated as
\[
(4.5) \quad I\lambda_{kl} = k\lambda_{2kl},
\]
where \( \lambda_{kl} \) denotes the kl-th Fourier coefficient of \( \lambda_i \). In particular, we have \( \lambda_{10} = \lambda_{20} = 0 \) for \( l \) and \( k \) different from 0. These formulae will prove useful for the subsequent computations. We fix the trivial Spin structure and trivialize with respect to the basis \( s \). Using (3.4), we may rephrase the equation
\[
\nabla_X^S \tilde{f}u_1 = 0
\]
as
\[
-\lambda_2 \partial_{x_1} \tilde{f} = \lambda_1 \partial_{x_2} \tilde{f} + \frac{1}{2} (\partial_{x_2} \lambda_1 + \partial_{x_1} \lambda_2),
\]
that

\[ -\lambda_2 \partial_{x_1} \tilde{f} = \lambda_1 \partial_{x_2} \tilde{f}. \]

We remark that the constant spinor \( u_1 \) defines a solution that projects onto the torus. Furthermore, if there exists a dense null line, we already know by \( \text{[4.10]} \) that there cannot exist any further linearly independent solutions.

In order to find non-trivial solutions we define the function

\[ \tilde{f}_\alpha(x_1, x_2) = \exp \left( i \pi \alpha \int_0^{x_1} (\lambda_1(s, x_2) + \lambda_1(s, 0)) ds - \int_0^{x_2} (\lambda_2(x_1, t) + \lambda_2(0, t)) dt \right) \]

for \( 0 \neq \alpha \in \mathbb{R} \). Then \( \tilde{f}_\alpha \) defines a solution for \( \text{[4.6]} \) on \( \mathbb{R}^2 \). In order to be inducible on the torus \( T \), we must impose the double-periodicity of \( \tilde{f}_\alpha \), that is \( \tilde{f}_\alpha(x_1 + n, x_2 + m) = \tilde{f}_\alpha(x_1, x_2) \) for \( n, m \in \mathbb{Z} \). If \( l_i = \int_{T_{1+i}} \lambda_i dt \) denotes the 0-th Fourier coefficient of \( \lambda_i \), we get the following criterion:

**4.10 Lemma.** \( \tilde{f}_\alpha \in C^\infty(\mathbb{R}^2, \mathbb{C})^{\mathbb{Z}^2} \) if and only if \( \alpha l_i \in \mathbb{Z} \) for \( i = 1, 2 \).

**Proof.** We have \( \tilde{f}_\alpha(x_1 + n, x_2 + m) = \tilde{f}_\alpha(x_1, x_2) \) if and only if

\[ \alpha \left( n \int_0^{1} (\lambda_1(s, x_2) + \lambda_1(s, 0)) ds - m \int_0^{1} (\lambda_2(x_1, t) + \lambda_2(0, t)) dt \right) \in 2\mathbb{Z}. \]

Now, \( \int_0^{1} \lambda_1(s, x_2) ds = l_1 \) and \( \int_0^{1} \lambda_2(x_1, t) ds = l_2 \) (use \( \text{[4.10]} \)), so \( \tilde{f}_\alpha \in C^\infty(\mathbb{R}^2, \mathbb{C})^{\mathbb{Z}^2} \) if and only if \( \alpha(nl_1 - ml_2) \in \mathbb{Z} \) for every \( n \) and \( m \) in \( \mathbb{Z} \).

In particular, if \( \frac{n}{l_2} = \frac{p}{q} \in \mathbb{Q} \) (where \( p \) and \( q \) have no common divisor), then \( \tilde{f}_{\frac{p}{q}} \) defines a solution. Since the set \( \{ \tilde{f}_{nm} u_1 \mid m \in \mathbb{Z} \} \) is linearly independent in \( \mathcal{H}_0^0 \), we have \( \delta_0^+ = +\infty \).

**4.11 Example.**

Let \( c > 0 \) be a rational number and \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function without zeros with period 1 such that \( -f(2x) \neq c \) for every \( x \in \mathbb{R} \) and \( f(2p_1) = -\frac{c}{2} \) for \( p_0 = 0, p_1, \ldots, p_n = 1 \in [0,1] \), but \( f'(2p_1) \neq 0 \).

For instance, we could choose \( c = 2 \) and \( f(x) = \frac{1}{10} \cos(2\pi(x + \frac{1}{4})) - 1 \). Then the diagonal metric defined by \( \lambda_1(x_1, x_2) = -f(x_1 - x_2) \) and \( \lambda_2 = -f(x_1 - x_2) - c \) is closed. Furthermore, \( \frac{1}{l_2} = \frac{1}{2c} \in \mathbb{Q} \), hence \( \delta_+^0 = +\infty \). If we express \( g_\lambda \) in the new coordinates \( (x, y) \) given by \( x = \frac{x_2 - x_1}{2} \) and \( y = \frac{x_1 + x_2}{2} \), we get \( g_\lambda(x, y) = (2c f(2x) + c^2) dx^2 - 4(f^2(2x) + 2f(2x) + 2) dxdy + (2c f(2x) + c^2)dy^2 \). Because of our assumptions \( g_\lambda \in \mathcal{G}_2 \) (see \( \text{[2.4]} \)), so \( g_\lambda \) provides an example of a non-conformally flat diagonal metric since it is not complete.
4.12 Lemma. There are no dense $\mathcal{X}$–null lines if and only if $\frac{1}{\lambda_2} \notin \mathbb{Q}$.

Proof. As the global properties of the null lines such as denseness are independent of the parametrization, we can consider the flow of any $\mathcal{X}$–vector field. For instance, we may choose $X = \lambda_2 \partial_{x_1} + \lambda_1 \partial_{x_2}$, where we assume that $\lambda_1, \lambda_2 > 0$. Let the flow of $X$ be given by $(x_1(t), x_2(t))$. We establish the assertion by computing the rotation number of this flow (see [Ha69] for details). $\lambda_2 > 0$ implies $\lambda_1 dx_1 - \lambda_2 dx_2 = 0$. Since the form $\lambda = \lambda_1 dx_1 - \lambda_2 dx_2$ is closed, we yield an exact ordinary differential equation on $\mathbb{R}^2$. Hence we have to find a $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\partial_{x_1} F = \lambda_1$ and $\partial_{x_2} F = -\lambda_2$. The initial condition $x_2(0)$ is determined by $F(0, x_2(0)) = c$ for a constant $c$.

Integration of $\partial_{x_1} F = \lambda_1$ yields $F(x_1, x_2) = \int_0^{x_1} \lambda_1(s, x_2)ds + f(x_2)$, where $f'(x_2) = -\lambda_2(x_1, x_2) - \int_0^{x_1} (\partial_2 \lambda_1)(s, x_2)ds = -\lambda_2(x_1, x_2) + \int_0^{x_1} (\partial_1 \lambda_2)(s, x_2)ds = -\lambda_2(0, x_2)$. A possible $F$ is given by $F(x_1, x_2) = \int_0^{x_1} \lambda_1(s, x_2)ds - \int_0^{x_2} \lambda_2(0, s)ds$. We choose $c = 0$ and use the Fourier series of $\lambda_1$ and $\lambda_2$ to get the following equation:

$$l_1 x_1 + \sum_{k \neq 0} \lambda_{1k} \frac{e^{2\pi ikx_1} - 1}{2\pi ik} \cdot e^{2\pi ikx_2} - l_2 x_2 - \sum_{k \neq 0} \lambda_{2k} \frac{e^{2\pi ikx_2} - 1}{2\pi ik} = 0.$$

Evaluating in $x_1 = n \in \mathbb{Z}$ yields $l_1 n - l_2 x_2(n) = \sum_{k \neq 0} \lambda_{2k} \frac{e^{2\pi ikx_2(n)} - 1}{2\pi ik} = 0$, hence

$$\frac{x_2(n)}{n} = \frac{l_1}{l_2} - \frac{1}{n} \cdot \frac{l_2}{l_1} \sum_{k \neq 0} \lambda_{2k} \frac{e^{2\pi ikx_2(n)} - 1}{2\pi ik} = 0.$$

Since the Fourier series of smooth functions are absolutely convergent, we have $\rho = \lim_{n \to +\infty} \frac{x_2(n)}{n} = \frac{l_1}{l_2}$, whence the assertion.

We finally get:

4.13 Proposition. Let $g$ be conformally equivalent to a closed diagonal metric. Then the following holds: 1 and $+\infty$ are the only possible values for $\delta_+^0$. Furthermore, these dimensions are characterized as follows:

(i) $\delta_+^0 = 1$ iff $\frac{l_1}{l_2} \notin \mathbb{Q}$ iff all $\mathcal{X}$–null lines are dense.

(ii) $\delta_+^0 = +\infty$ iff $\frac{l_1}{l_2} \in \mathbb{Q}$ iff all $\mathcal{X}$–null lines are closed or asymptotic of a closed $\mathcal{X}$–null line.

4.3 $\mu$–surfaces

By a half spinor which is harmonic or twistor may be seen as an object that is constant along the lifts of the corresponding null lines. Unfortunately, we have no a priori control over the parallel
transport in \( Q \), and due to the non-compactness of \( \tilde{G}(1,1) \), the lifts of the null lines may even be unbounded (see \ref{4.17}). Therefore, we focus on cases where a direct link between the null lines on \((M^{1+1},g)\) and the harmonic and twistor spinors can be established, without lifting the null lines to \( Q \).

As we did earlier, we restrict our investigation to the case of positive harmonic spinors. One can prove analogous results by interchanging suitably \( S^+/S^- \) and \( X/Y \), as pointed out above.

4.14 Definition. Let \( \varphi \in \Gamma(S^+) \) be a positive harmonic spinor. A smooth function \( \mu_\varphi : M^{1+1} \to \mathbb{C} \) verifying

\begin{enumerate}[(i)]
  \item \( \mu_\varphi(x) = 0 \) if and only if \( \varphi(x) = 0 \) and
  \item \( \mu_\varphi \) is constant along \( X \)-curves
\end{enumerate}

is said to be a mass functional for \( \varphi \). A \( \mu \)-surface is a Lorentzian surface admitting a mass functional for every \( \varphi \in \Gamma(S^+) \).

We shall give examples of \( \mu \)-surfaces in the following section (see \ref{4.36} and \ref{4.37} in conjunction with \ref{4.47}). The reason for looking for such mass functionals is the following property:

4.15 Lemma. Let \((M^{1+1},g)\) be a \( \mu \)-surface. Let \( \varphi \) be a positive harmonic spinor and \( x \in M \) such that \( \varphi(x) = 0 \). If \( (x_n) \subset l \) for a fixed \( X \)-line \( l \) converges to \( x \), then \( \varphi|_l \equiv 0 \).

The general idea to produce obstructions to the inequality \( \delta_+ \geq 2 \) is to assure that a single zero of a harmonic spinor is propagated along all null lines, therefore forcing the spinor to be zero everywhere. We note the following ”heritage principle”:

4.16 Corollary. Let \((M^{1+1},g)\) be a \( \mu \)-surface. Let \( \varphi \) be a positive harmonic spinor, and \( l_\infty \) a closed \( X \)-line. If \( \varphi|_{l_\infty} \equiv 0 \), then \( \varphi|_l \equiv 0 \) for every asymptotic \( l \) of \( l_\infty \).

Although all Spin structures on a \( \mu \)-surface can be treated simultaneously as we shall see, the following proposition illustrates how the non-trivial Spin structures differ from the trivial one in terms of the parallel transport:

4.17 Proposition. Let \((M^{1+1},g)\) be a \( \mu \)-surface with a dense \( X \)-line \( l \). Then there exists a local section \( \tilde{s} : U \to Q \) and a convergent sequence \( (x_n) \subset U \cap l \) with \( x_n \to x \in U \) such that the following property holds: If \( (a_n) \subset \mathbb{R} \) is defined by \( P^Q_{l: x_0 \to x_n} \tilde{s}(x_0) = \tilde{s}(x_n)g_{a_n} \), then for a subsequence \( (a_{n_i}) \) we have \( a_{n_i} \to \pm \infty \) or \( a_{n_i} \to 0 \), that is \( \{g_{a_n}\} \) is unbounded in \( \tilde{G}(1,1) \).
(Recall that according to 3.2, \( g_{a_n} = \begin{pmatrix} a_n & 0 \\ 0 & \frac{1}{a_n} \end{pmatrix} \).)

Proof. Assume the opposite. Then consider the horizontal lift \( l^* \) of \( l \) to \( Q \). Extend \( l^* \) to a (continuous) section \( \tilde{s} \) of \( l \) to \( Q \). Extend \( l^* \) to a section \( \tilde{s} \) of \( M^{1+1} \) by \( \tilde{s}(x) = \lim_{n \to \infty} l^*(x_n) \) for \( x_n \in l \to x \). This limit exists indeed, since \( l^*(x_n) = \mathcal{P}_{l:x \to x_n} l^*(x_0) \) is bounded in \( Q \) by assumption. Hence \( Q \) would be isomorphic to the trivial Spin structure. \( \blacksquare \)

4.18 Corollary. If there exists a dense \( \mathcal{A} \)-line on a \( \mu \)-surface \( (M^{1+1}, g) \), then \( \delta_+ = 0 \) for every non–trivial Spin structure.

Proof. Using the notation of the preceding proposition, we have

\[
\varphi(x_n) = [\mathcal{P}_{l:x \to x_n} \tilde{s}(x_0), f\tilde{s}(x_0)u_1] = [\tilde{s}(x_n), g_{a_n} f\tilde{s}(x_0)u_1] = [\tilde{s}(x_n), f\tilde{s}(x_0)a_nu_1] \to \varphi(x).
\]

Hence, if \( a_n \to \pm \infty \), then \( f\tilde{s}(x_0) = 0 \). If \( a_n \to 0 \), then \( \varphi(x) = 0 \). In both cases, the spinor \( \varphi \) has a zero, implying \( \varphi \equiv 0 \) by 4.15. \( \blacksquare \)

From now on, we will mostly consider compact \( \mu \)-surfaces, though the techniques and results can be applied to Lorentzian cylinders as well. Due to the “denseness obstruction” 4.5 we can restrict our attention to the case where no dense null lines occur. First, we introduce the subsequent notation:

Let \( x \in M^{1+1} \) and \( l_1 \) and \( l_2 \) be two closed \( \mathcal{A} \)-lines which do not contain \( x \). Since \( M^{1+1} \) is homeomorphic to a torus, the connected components of \( M^{1+1} \setminus (l_1 \cup l_2) \) are open in \( M^{1+1} \) and homeomorphic to a cylinder without boundary. Let \( C_{l_1l_2}(x) \) denote the cylinder which contains \( x \). Its closure is given by \( \overline{C_{l_1l_2}(x)} = C_{l_1l_2}(x) \cup l_1 \cup l_2 \). In the case where \( l_1 = l_2 \) as a set, we have \( \overline{C_{l_1l_2}(x)} = M^{1+1} \), so the whole torus itself may be considered as a closed cylinder. Now let be \( x \) such that \( l_x \) is an asymptotic of the two closed null lines \( l_1 \) and \( l_2 \). Then the cylinder \( C_{l_1l_2}(x) \) will be written \( A_{l_1l_2}(x) \). For further reference, such a cylinder will be called asymptotic. Closed null lines are not allowed to be homotopic to a single point, hence there are no more closed null lines in any asymptotic cylinder. Since every asymptotic tends to \( l_1 \) or \( l_2 \), we get the

4.19 Lemma. Let \( \varphi \) be a positive harmonic spinor on a compact \( \mu \)-surface. Then its mass functional \( \mu_\varphi \) is constant on every closed asymptotic cylinder.

Thus, if the spinor has a zero in an asymptotic cylinder, it must be zero on the whole cylinder. In order to treat the case where the union of closed null lines is dense in \( M^{1+1} \), we introduce a further type of cylinders which does not contain “ribbons” of closed null lines:
4.20 Definition. A cylinder \( C_{l_1 l_2} \) is called non–resonant if for any two arbitrary closed null lines \( \tilde{l}_1, \tilde{l}_2 \) in the closure of \( C_{l_1 l_2} \) there is an asymptotic \( l \) in \( C_{l_1 l_2} \).

4.21 Lemma. Let \( \varphi \) be a positive harmonic spinor, and \( C = C_{l_1 l_2} \) be a non–resonant cylinder. Then \( \mu_\varphi \) is constant on \( C \).

Proof. Consider the set \( A := \{ x \in C \mid l_x \text{ is an asymptotic} \} \). \( A \) is open and dense in \( C \). As the total differential of \( \mu_\varphi \) vanishes on \( A \) as a consequence of 4.19, the denseness of \( A \) implies the result.

4.22 Corollary. Let \((M^{1+1}, g)\) be a non–resonant \( \mu \)–cylinder and let \( \varphi_1, \varphi_2 \) be two positive harmonic spinors. If there exists a \( x \in M^{1+1} \) such that \( \varphi_1(x) = \varphi_2(x) \), then \( \varphi_1 \equiv \varphi_2 \). In particular, every positive harmonic spinor with a zero is identically zero, and \( \delta_+ \leq 1 \).

4.23 Definition. Let \((M^{1+1}, g)\) be a Lorentzian surface and \( l \) a closed \( X \)-line. The Spin bundle \( Q \) is called \( X \)-trivial along \( l \), if the relation \( P^Q_{l: x \rightarrow x} q = q \) holds for every \( x \in l \) and \( q \in (Q|_l)_x \).

4.24 Lemma. Let \( \varphi \) be a positive harmonic spinor that has no zero along a closed \( X \)-line \( l \). Then \( Q \) must be \( X \)-trivial along \( l \).

Proof. By 1.3 we know that \( \varphi(x) = [q, v] = [P^Q_{l: x \rightarrow x} q, v] \). Hence, if \( P^Q_{l: x \rightarrow x} q = qg \) for a uniquely determined \( g \in Spin(1, 1) \), we have \( g^{-1}v = v \). It follows \( g = id \) by 3.2.

So far, we have obtained obstructions to the inequality \( \delta_+ \geq 2 \). In the case where this inequality holds, a third type of cylinder becomes interesting:

4.25 Definition. A closed cylinder \( R_{l_1 l_2} = \overline{C_{l_1 l_2}} \) which does not consist of a single \( X \)-line is said to be resonant if \( l_x \) is closed for every \( x \in R_{l_1 l_2} \).

4.26 Proposition. Let \((M^{1+1}, g)\) be a compact \( \mu \)-surface.

(i) A non–trivial positive harmonic spinor cannot be zero on every resonant cylinder.

(ii) If \( \delta_+ \geq 2 \), then there exists a \( X \)–trivial resonant cylinder \( R \) on \( M^{1+1} \), that is \( Q \) is \( X \)-trivial along every closed \( X \)-line in \( R \).
Proof. (ii) is a consequence of (i). To prove the first assertion, let us assume the opposite. It suffices to show that \( \mu_\varphi \) is locally constant.

Let \( x \in M^{1+1} \). If \( l_x \) is an asymptotic, then \( \mu_\varphi|_{\mathcal{A}_x(x)} \) is constant by 4.19. Otherwise, \( l_x \) is closed. If for every neighbourhood \( U \) of \( x \) there exists \( x' \in U \) such that \( l_x \) is an asymptotic, then \( x \) is in the closure of a non–resonant cylinder \( C \). If \( x \in \text{int}(C) \), then \( \mu_\varphi \) is constant on a neighbourhood of \( x \) by 4.21. If not, then \( x \in \partial C \cap \partial R \), where \( R \) is a resonant cylinder. Thus \( \mu_\varphi \equiv 0 \) on a neighbourhood of \( x \), since \( \mu_\varphi|_C \equiv \text{const} \) and \( \mu_\varphi|R \equiv 0 \) by assumption.

On the other hand, whenever there exists an \( \mathcal{X} \)–trivial resonant cylinder on \( M^{1+1} \), then we can produce harmonic spinors as in 4.6, since the \( \mathcal{X} \)–triviality guarantees that the spinors constructed in this way are well defined. Hence we arrive at the following proposition, generalizing the left–invariant case:

**4.27 Theorem.** Let \((M^{1+1},g)\) be a compact \( \mu \)–surface. Then the only possible dimensions are \( \delta_+ = 0, 1 \) and \( +\infty \). These cases are characterized as follows:

(i) \( \delta_+ \leq 1 \) if and only if either

- there exists a dense \( \mathcal{X} \)–line in which case we have \( \delta_+ = 0 \) for the non–trivial Spin structures or
- \( M^{1+1} \) is non–resonant or
- there exists no \( \mathcal{X} \)–trivial resonant cylinder on \( M^{1+1} \).

(ii) \( \delta_+ = +\infty \) if and only if there exists a resonant \( \mathcal{X} \)–trivial cylinder on \( M^{1+1} \). In this case, we have \( \delta_+ = +\infty \) for every Spin structure.

As we have already seen for the left–invariant case, the dimensions \( \delta_+ = 0 \) and \( 1 \) can occur and depend on the given Spin structure.

### 4.4 Spinors and conformal flatness

We will now study the relationship between the existence of harmonic and twistor spinors and conformal flatness. In particular, we will consider the geometric implications of \( \mathcal{X} \)–triviality.

As we saw in 2.2, conformal flatness is related to the existence of nowhere vanishing time– resp. spacelike conformal vector fields. With every spinor \( \varphi \in \Gamma(S) \) we can canonically associate a vector field that is conformal in the case of a twistor spinor:
4.28 Definition. Let \((M^{p+q}, g)\) be an orientable and time–orientable pseudo–Riemannian Spin manifold, and let \(\psi \in \Gamma(S)\). We define the associated vector field \(V_\psi\) by the equation
\[
g(V_\psi, W) = i^{p+1} \langle W \cdot \psi, \psi \rangle
\]
for \(W \in \mathfrak{X}(M^{p+q})\).

A direct computation yields the following proposition (see, for instance, [Ba99]):

4.29 Proposition. Let \(\varphi \in \Gamma(S)\) be a twistor spinor. Then \(V_\psi\) is a conformal vector field. More precisely, we have \(\mathcal{L}_{V_\psi} g = \frac{1}{n} \text{Re} \left( i^{p+1} \langle D\psi, \psi \rangle \right) g \).

Next, we determine the associated vector field of a spinor in signature \((1,1)\):

4.30 Lemma. Let \((M^{1+1}, g)\) be time–orientable and \(\psi \in \Gamma(S)\). Let \(s = (s_1, s_2) : U \to P\) be an orthonormal frame with a lift \(\tilde{s}\) to \(\tilde{Q}_\xi\) (cf. section 3). Let \(\tilde{\psi}_s = \psi_1^+ u_1 + \psi_2^- u_{-1} \in C^\infty(U, \Delta_{1,1})\) be the local trivialization of \(\psi\) with respect to \(\tilde{s}\). Then \(V_\psi = |\psi_1^+|^2 X_s - |\psi_2^-|^2 Y_s\). In particular, \(V_\psi\) is a causal vector field which is timelike if the local components \(\psi_s^+\) and \(\psi_s^-\) have no zeros, and lightlike in case of a half–spinor without zeros.

Proof. Let \(w_1\) and \(w_2\) be the local components of \(W \in \mathfrak{X}(M)\) with respect to \(s\), that is \(W = w_1 s_1 + w_2 s_2 = [s, w_1 e_1 + w_2 e_2]\). A direct computation yields \(\langle W \cdot \psi, \psi \rangle = (|\psi_2^-|^2 + |\psi_1^+|^2)w_1 + (|\psi_2^-|^2 - |\psi_1^+|^2)w_2\). If \(V_\psi = V_{\psi_1} s_1 + V_{\psi_2} s_2\), we get \(V_{\psi_1} = \left( |\psi_1^+|^2 + |\psi_2^-|^2 \right)\) and \(V_{\psi_2} = \left( |\psi_1^+|^2 - |\psi_2^-|^2 \right)\). Furthermore, since \(\lambda = g(X_\psi, Y_\psi) > 0\) and \(g(V_\psi, V_\psi) = -2\lambda |\psi_1^+|^2 |\psi_2^-|^2\), the vector field \(V_\psi\) is causal.

4.31 Remark. Let \(X\) be a \(\mathcal{X}\)–vector field and \(\varphi\) a positive harmonic spinor. Then \(X \cdot \varphi = 0\). In particular we get \(V_\varphi \cdot \varphi = 0\).

4.32 Corollary. Let \((M^{1+1}, g)\) and \(\psi^\pm \in \Gamma(S^\pm)\) be two twistor spinors without zeros. Then \((M^{1+1}, g)\) is conformally flat. In particular if \(M\) is compact, then \((M^{1+1}, g)\) is complete.

4.33 Remark. According to 4.4 the same result holds for harmonic instead of twistor spinors.

As we have already seen for \(\mathcal{X}\)–triviality, harmonic resp. twistor spinors without zeros induce certain “flatness” properties. Therefore, we will look more closely to Lorentzian surfaces that admit nowhere vanishing solutions to the harmonic resp. twistor equation.

First, we recall the following statement:
4.34 Proposition. Let $\varphi \in \Gamma(S)$ be a harmonic spinor on $(M^{p+q}, g)$. Then $\text{div}(V_{\varphi}) = 0$.

4.35 Definition. A time–orientable Lorentzian surface $(M^{1+1}, g)$ is said to be $X$– resp. $Y$–conformally flat if there exists a global orthonormal frame $s = (s_1, s_2)$ such that $\text{div}(X_s) = 0$ resp. $\text{div}(Y_s) = 0$. We call a Lorentzian surface $(M^{1+1}, g)$ semi-conformally flat (s.c.f.) if $(M^{1+1}, g)$ is either $X$– or $Y$–conformally flat.

The notion of semi-conformal flatness will be justified in 4.42.

As we did for spinors, we will concentrate on $X$–conformally flat surfaces; analogous statements hold for $Y$–conformally flat ones.

Since every $X$–vector field can be written as $X_s$ with respect to a suitably chosen basis, a time–orientable Lorentzian surface is $X$–conformally flat if and only if there exists a $X$–vector field $X$ such that $\text{div}(X) = 0$. Furthermore, it follows that the notion of semi-conformal flatness is invariant under conformal change of the metric: If there exists a $X$–vector field $X$ on $(M^{1+1}, g)$ with $\text{div}(X) = 0$, then we can find another $X$–vector field $\tilde{X} \in \mathcal{X}(M)$ with $\tilde{\text{div}}(\tilde{X}) = 0$ (where $\tilde{\text{div}}$ denotes the divergence operator associated with the conformally changed metric $\tilde{g} = \lambda g$) as can be seen from the formula $\tilde{\text{div}}(V) = V(\ln(\lambda)) + \text{div}(V)$.

4.36 Examples.

(i) As we remarked in section 4.2.2, closed diagonal metrics admit lightlike divergence-free vector fields and are therefore s.c.f.

(ii) We are going to exhibit further examples by a direct computation of the divergence: Let us consider a Lorentzian torus with standard coordinates $(x_1, x_2)$ and volume form $\omega$. Let $V = k \partial x_1 + l \partial x_2$. Using the formula $d(i_X \omega) = \text{div}(X)\omega$, we get $\text{div}(V) = \partial_1 k + \partial_2 l + \frac{1}{2} V(\ln|\det(g)|)$. In particular, if $\det(g) \equiv 1$, then $\text{div}(V) = \partial_1 k + \partial_2 l$. As semi-conformal flatness is a conformal invariant, we may always assume – by rescaling the metric with the factor $\frac{1}{\det(g)}$ – this assumption to be fulfilled. For instance, if we consider the family of metrics given by (2.1) and (2.2), we get:

4.37 Corollary. Every metric in $G_1 \cup G_2$ defines a s.c.f. surface.

4.38 Proposition. Let $(M^{1+1}, g)$ be a time–orientable Lorentzian surface. Then the following assertions are equivalent:

(i) $(M^{1+1}, g)$ is $X$–conformally flat
(ii) There exists a global section $M^{1+1} \to P$ such that $\text{div}(X_s) = 0$

(iii) There exists a $\mathcal{X}$–vector field $X$ such that $\text{div}(X) = 0$

(iv) $\nabla^L_X s_i = 0$ for $i = 1, 2$

(v) There exists a positive harmonic spinor without zeros

(vi) There exists a negative twistor spinor without zeros

Proof. Only the implications (ii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (ii), (v) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (v) need proof.

(ii) $\Leftrightarrow$ (v): By a direct application of the Koszul formula, we prove the

4.39 Lemma.

$$-\frac{1}{2}g(X_s, [X_s, Y_s]) = \text{div}(X_s) = g(\nabla^L_X s_1, s_2) = -g(\nabla^L_X s_2, s_1)$$

Then (156) implies:

4.40 Corollary. Locally, we have the identity $\nabla^S_{X_s} [\tilde{s}, \varphi_{\tilde{s}}] = [\tilde{s}, X_s(\varphi_{\tilde{s}})] - \frac{1}{2}\text{div}(X_s)e_1 \cdot e_2 \cdot \varphi_{\tilde{s}}$. In particular, if $(M^{1+1}, g)$ is $\mathcal{X}$–conformally flat, we get $\nabla^S_{X_s} [\tilde{s}, \varphi_{\tilde{s}}] = [\tilde{s}, X_s(\varphi_{\tilde{s}})]$.

Furthermore, $g(\nabla^L_X s_i, s_i) = 0$ for $i,j \in \{1, 2\}$. By the lemma, we have $g(\nabla^L_X s_1, s_2) = -g(\nabla^L_X s_2, s_1) = \text{div}(X_s)$, whence the equivalence.

(v) $\Rightarrow$ (iii): Since $\text{div}(e^f X) = e^f (X(f) + \text{div}(X))$, the $\mathcal{X}$–vector field $e^f X$ will be divergence–free if and only if $X(f) = -\text{div}(X)$ holds. By 4.30, we have $V_\varphi = \lambda X_s$ with $\lambda \neq 0$. Application of 4.34 yields $X_s(\ln |\lambda|) = -\text{div}(X_s)$.

(ii) $\Rightarrow$ (v): Let $\tilde{s} : M^{1+1} \to Q_0$ be a global section in the trivial bundle. Then $\varphi_{\tilde{s}} = u_1$ defines a positive harmonic spinor without zeros.

4.41 Corollary 1. On a $\mathcal{X}$–conformally flat surface, we have $\delta_+^0 \geq 1$.

4.42 Corollary 2. $(M^{1+1}, g)$ is conformally flat if and only if $(M^{1+1}, g)$ is $\mathcal{X}$– and $\mathcal{Y}$–conformally flat.

Proof. For a flat metric, every constant defines a harmonic resp. twistor spinor with respect to the trivial Spin structure. The implication follows then from 133. We yield the converse from 132.

Next, we will prove some further properties of $\mathcal{X}$–conformally flat surfaces:
4.43 Definition. A Lorentzian surface \((M^{1+1}, g)\) is said to be \(\mathcal{X}\)–resp. \(\mathcal{Y}\)–complete, if every \(\mathcal{X}\)–resp. \(\mathcal{Y}\)–geodesic is complete. A Lorentzian surface that is \(\mathcal{X}\)–resp. \(\mathcal{Y}\)–complete is said to be semi–null complete.

4.44 Proposition. A compact \(\mathcal{X}\)–conformally flat surface is \(\mathcal{X}\)–complete.

Proof. By 4.38 (v), we have \(\nabla_X^C X_s = 0\), so that the \(X_s\)–geodesics are given by the flow of \(X_s\). □

4.45 Examples.

(i) The interdependence of semi-conformal flatness, positive harmonic spinors without zeros and semi-completeness is demonstrated by 4.11: Since the harmonic half–spinors we found have no zeros, the surface must be semi-complete. But we showed that this metric is in \(G_2\), so it is not conformally flat. Hence, from 4.42 follows that \((T^{1+1}, g)\) is complete for one type of isotropic geodesics, and that there must be incomplete geodesics for the other type, in accordance with proposition 2.1.

(ii) Consider the following example taken from [RoS´ a93]: Let \(\tau : [0, 1] \to \mathbb{R}\) be a smooth function with \(\tau(a) = 0\), but \(\tau'(a) \neq 0\), and whose support is strictly contained in \([0, 1]\). Extend \(\tau\) periodically on the whole real line and define \(g^\tau(x,y) = 2dx dy - \tau(x)dy^2\). Then \(g^\tau \in G'\), hence \(g^\tau\) is lightlike incomplete. For instance, \(\gamma(t) = (a, \tau'(a) \ln(t + \frac{1}{\tau(a)})\) is a closed incomplete geodesic which without loss of generality we assume to be \(\mathcal{X}\). Thus any positive harmonic spinor must be zero on \(\gamma\). But as \(\tau|_{[-\epsilon, \epsilon]} \equiv 0\) for \(\epsilon\) sufficiently small, \((T^{1+1}, g)\) contains an \(\mathcal{X}\)-trivial resonant cylinder, and therefore \(\delta_+ = +\infty\). This example shows that there exists not conformally flat tori with \(\mathcal{X}\)-trivial resonant cylinder which are not \(\mathcal{X}\)-conformally flat.

4.46 Proposition. On a \(\mathcal{X}\)–conformally flat surface, we have \(\mathcal{H}_+ \cong \mathcal{I}_-\).

Proof. The maps \(\Phi : \mathcal{H}_+ \to \mathcal{I}_-, \varphi \mapsto \frac{i}{2} Y_s \cdot \varphi\) and \(\Psi : \mathcal{I}_- \to \mathcal{H}_+, \psi \mapsto \frac{i}{2} X_s \cdot \varphi\) are bundle isomorphisms inverse of one another. □

4.47 Proposition. Every s.c.f. surface is a \(\mu\)–surface.

Proof. Let \(s = (s_1, s_2) : M \to P\) be a (global) orthonormal frame with \(\text{div}(X_s) = 0\). We define

\[\mu_\varphi(x) = \langle Y_s \cdot \varphi, \varphi \rangle.\]

Let \(\tilde{s}\) be a local lift of \(s\) to \(\tilde{Q}_{s_1}\). For this section, let \(\varphi = [\tilde{s}, \varphi_{\tilde{s}}]\). Then \(\mu_\varphi(x) = \langle Y_s \cdot \varphi, \varphi \rangle(x) = -2|\varphi(x)|^2\), where \(|\cdot|\) denotes the absolute value function on \(\mathbb{C}\). Thus (i) and (ii) of definition 4.14 hold.

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Since $\varphi$ is a positive harmonic spinor, we have $\nabla^S_X \varphi = 0$. Consequently, we get $X_\mu(\mu_\varphi) = \langle \nabla^C_X Y_s \cdot \varphi, \varphi \rangle$. Since $\nabla^C_X Y_s = -\nabla^C_X s_1 + \nabla^C_X s_2 = 0$ by 4.38 (iv), the assertion follows.

The classification of the possible dimensions of $\delta^+_\tau$ in 4.27 may be restated as follows:

**4.48 Theorem.** Let $(M^{1+1}, g)$ be a compact $\mathcal{X}$–conformally flat Lorentzian surface. Then $\delta^+_\tau = \tau_-$ and the only possible dimensions for $\delta^+_\tau$ are $0, 1$ and $+\infty$. These cases are characterized as follows:

(i) $\delta^+_\tau \leq 1$ if and only if either

- there exists a dense $\mathcal{X}$–line in which case we have $\delta^+_\tau = 0$ for the non–trivial Spin structures,
- or
- $M^{1+1}$ is non-resonant or
- there exists no $\mathcal{X}$–trivial resonant cylinder on $M^{1+1}$.

Furthermore, $\delta^+_0 = 1$.

(ii) $\delta^+_\tau = +\infty$ if and only if there exists a $\mathcal{X}$–trivial resonant cylinder on $M^{1+1}$. In this case we have $\delta^+_\tau = +\infty$ for every Spin structure.

Next we will show that in some sense $\mathcal{X}$–conformal flatness is forced by positive harmonic spinors which have non–zero ”mass”:

**4.49 Definition.** Let $(M^{1+1}, g)$ be a Lorentzian surface and $(P, \pi, M^{1+1}; G)$ a principal fiber bundle over $M^{1+1}$. A (local) section $s : U \to P$ is said to be $\mathcal{X}$– resp. $\mathcal{Y}$–parallel if for every $\mathcal{X}$– resp. $\mathcal{Y}$–curve $\alpha : [a, b] \to U$, we have $P^\alpha_{\mathcal{X} \to \mathcal{Y}} s(\alpha(a)) = s(\alpha(b))$, where $P^\alpha_{\mathcal{X} \to \mathcal{Y}}$ denotes the parallel transport in $P$ along $\alpha$.

**4.50 Lemma.** Let $s = (s_1, s_2) : U \to P$ be a local section of the orthonormal frame bundle $P$. Then $s$ is $\mathcal{X}$-parallel if and only if $\text{div}(X_s) = 0$. Furthermore, if $s$ can be lifted to a section $\tilde{s} : U \to Q$ of $Q$, then $\tilde{s}$ is $\mathcal{X}$-parallel if and only if $\text{div}(X_{\tilde{s}}) = 0$.

**Proof.** Let $\alpha$ be the flow generated by $X_s$ in $U$, and let $P^{\mathcal{X}}_{\alpha}$ be the usual parallel transport in $TM^{1+1}$ along $\alpha$ induced by $P^\mathcal{P}_{\alpha}$. We have $\nabla^{\mathcal{X}}_{X_s} s_j(x) = \frac{d}{dt} P^{\mathcal{X}}_{\alpha \to 0} s_j(\alpha(t)) |_{t=0} = \frac{d}{dt}[s(\alpha(0)), e_j] = 0$, hence $\text{div}(X_s) = 0$ by 4.38 (y).

For the converse, let $Z$ denote the Levi-Civita-connection in $P$. Since $\text{div}(X_s) = -\frac{1}{2} g(\nabla^{\mathcal{X}}_{X_s} s_1, s_2) = 0$, we get $s^* Z(X) = Z(ds(X)) = 0$ for every $\mathcal{X}$-vector field $X$ (cf. 4.40). Hence $s^* Z(\alpha'(t)) = 0$ for
every $\mathcal{X}$-curve $\alpha : [a, b] \to M^{1+1}$, that is $\alpha^*_{s(\alpha(a))} = \text{lift of } \alpha$ starting in $s(\alpha(b))$. It follows that $\mathcal{P}_a^p s = \alpha^*_{s(\alpha(a))} = s(\alpha(b))$.

Since $f \circ \mathcal{P}^Q = \mathcal{P}^P \circ f$ and $\hat{s}^* Z(X_s) = -\text{div}(X_s)\omega = 0$ for the lifts of $s$ and $Z$ to $Q$, we deduce the same result for the Spin bundle $Q$. ■

The notion of $\mathcal{X}$-triviality can then be reformulated as follows:

**4.51 Corollary.** Let $l$ be a closed $\mathcal{X}$-line. Then $Q$ is $\mathcal{X}$-trivial along $l$ if and only if $l$ can be parametrized such that $\text{div}(l') = 0$. In particular, such a parametrization makes $l$ into a geodesic.

**Proof.** Choose an orthonormal frame $s$ such that $l' = s_1 + s_2$ and repeat the reasoning of 4.50. ■

**4.52 Remark.** As in the case of conformal flatness the condition $\text{div}(X) = 0$ can always be locally realized: Indeed, if $\beta : (a, b) \to U$ is a $\mathcal{Y}$-curve, pick a section $s : |\beta| \to P$ and extend this section on $U$ by parallel transport of $s(\beta(t))$ in the $\mathcal{X}$-direction. Hence $s : U \to P$ is $\mathcal{X}$-parallel by construction and well defined if $U$ is conveniently chosen, i.e. $\beta$ intersects every $\mathcal{X}$-line in $U$ only once and there are no closed $\mathcal{X}$-lines (e.g. if $U$ is simply connected). Then $\text{div}(X_s) = 0$ by 4.50.

**4.5 Conclusion**

Like for compact Riemannian surfaces, $\delta_+$ depends both on the conformal class of the metric and on the Spin structure. $\delta_+$ may be unbounded, in contrast to what is known for the Riemannian case, where the dimension is bounded by $[g + 1] \leq \frac{2}{3}$ (see [Hi74]). Furthermore, for Lorentzian surfaces we have a certain symmetry between harmonic and twistor spinors.

In the case of $\mu$–surfaces, the conformal invariants $\delta$ and $\tau$ reflect the global behaviour of the null lines. In some regular cases, where the global and local behaviour is quite similar (e.g. for simply connected surfaces or resonant tori), $\delta$ and $\tau$ are $+\infty$. If a "pathological" behaviour such as dense null lines occurs, then $\delta$ and $\tau$ are less than or equal 2, and we have a kind of "dynamic" dependence on the conformal class. No intermediate values are attained. Although $\delta$ and $\tau$ are weaker conformal invariants than the null lines, in some cases they allow us to distinguish between conformal classes. Furthermore, solutions with "mass", that is solutions without zeros, force conformal flatness. All techniques used – above all the characterization of harmonic and twistor half spinors as a kind of parallel spinors along the lightlike distributions – are genuine for the signature $(1, 1)$. On the other hand the case of a pseudo-Riemannian signature $(p, q)$ with $p + q \geq 3$ is significantly different. For
instance, the dimension of the space of twistor spinors on a connected pseudo-Riemannian manifold is bounded by $2^{\left[\frac{\dim}{2}\right]+1}$ (see [Ba99] and [BFGK91]).

It is not clear altogether to what extent these techniques can be applied to a general Lorentzian surface or which are the geometric obstructions for doing so. The next obvious step would be to investigate the class of asymptotic cylinders. One could try to find counterexamples of the "heritage principle", that is, harmonic or twistor spinors which are zero on the closed null lines, but have no zeros on the asymptotic cylinder itself, or metrics for which $\delta_\pm$ or $\tau_\pm$ may attain values other than 0, 1 or $+\infty$.

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