Support estimation in high-dimensional heteroscedastic mean regression

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Abstract

A current strand of research in high-dimensional statistics deals with robustifying the available methodology with respect to deviations from the pervasive light-tail assumptions. In this paper we consider a linear mean regression model with random design and potentially heteroscedastic, heavy-tailed errors, and investigate support estimation in this framework. We use a strictly convex, smooth variant of the Huber loss function with tuning parameter depending on the parameters of the problem, as well as the adaptive LASSO penalty for computational efficiency. For the resulting estimator we show sign-consistency and optimal rates of convergence in the $\ell_\infty$ norm as in the homoscedastic, light-tailed setting. In our analysis, we have to deal with the issue that the support of the target parameter in the linear mean regression model and its robustified version may differ substantially even for small values of the tuning parameter of the Huber loss function. Simulations illustrate the favorable numerical performance of the proposed methodology.

Keywords. convergence rates, pseudo Huber loss function, robust high-dimensional regression, support estimation, variable selection

1 Introduction

Data sets with a large number of features, often of the same order as or even of larger order than the number of observational repetitions, have become ever more common in applications such as microarray data analysis, functional magnetic resonance imaging or consumer data analysis. In consequence, much methodological research has been done in the area of high-dimensional statistics, and the field has developed rapidly. State of the art expositions are provided in Wainwright (2019), Hastie et al. (2015) and Giraud (2014).

A current strand of research deals with robustifying the available methodology with respect to deviations from light-tail assumptions in particular on the errors, and sometimes also on the predictors. One common approach is to replace the squared loss function by some other, fixed, robust loss function.

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such as the check function from quantile regression and in particular absolute deviation for the median (Belloni et al., 2011; Fan et al., 2014; Loh, 2017). However, doing so generally changes the target parameter away from the mean, particularly in the random design regression models with potentially heteroscedastic, asymmetric errors that we shall focus on.

More specifically, consider the random design linear regression model

\[ Y_j = X_j^\top \beta^* + \varepsilon_j, \quad j = 1, \ldots, n, \tag{1} \]

in which the real-valued responses \( Y_j \) and the \( p \)-variate covariates \( X_j \in \mathbb{R}^p \) are observed, and \( \beta^* \in \mathbb{R}^p \) is the unknown parameter vector. We allow for a random design with heteroscedastic errors, and assume that \( (X_1, \varepsilon_1), \ldots, (X_n, \varepsilon_n) \) are independent and identically distributed with \( \mathbb{E}[\varepsilon_j | X_j] = 0 \), so that \( \mathbb{E}[Y_j | X_j] = X_j^\top \beta^* \) is the identified conditional mean. We shall focus on the high-dimensional case, where \( p \) is at least of the order \( n \), and consider heavy tailed, non-sub-Gaussian errors \( \varepsilon_j \) but restrict ourselves to light-tailed regressors \( X_j \). Indeed, results in Lederer and Vogt (2020) imply that for uniformly bounded covariates, if the errors have slightly more than a finite fourth moment the ordinary least squares LASSO estimator retains the rates of convergence known from the sub-Gaussian case.

For high-dimensional mean regression under still weaker assumptions, Fan et al. (2017) and Sun et al. (2020) considered LASSO estimates with the Huber loss function (Huber, 1964) with parameter \( \alpha > 0 \) defined by

\[ l_\alpha(x) = \begin{cases} 2\alpha^{-1}|x| - \alpha^{-2} & \text{if } |x| > \alpha^{-1} \\ \alpha^2 & \text{if } |x| \leq \alpha^{-1} \end{cases} \]

To deal with the resulting bias, they let the parameter \( \alpha \) depend in a suitable way on sample size and dimension. A result from Sun et al. (2020) is that if the errors have a finite second moment and the covariates are sub-Gaussian, then for \( \alpha \approx (\log(p)/n)^{1/4} \) the estimator has the same rates of convergence in \( \ell_1 \) and \( \ell_2 \) norms as in the light-tailed case.

We shall study support estimation and rates in \( \ell_\infty \) norm in this framework. Previously, in robust high-dimensional regression Fan et al. (2014) addressed support estimation for quantile regression, and Loh (2017) considered homoscedastic models with independent covariates \( X_j \) and errors \( \varepsilon_j \) in scenarios where a fixed robust loss function gave the desired results. In a seminal paper, Wainwright (2009) studied support estimation with the LASSO and introduced the primal-dual witness proof method, see also Zhao and Yu (2006). A line of research, followed e.g. in Wainwright (2009) then investigated minimal conditions under which for certain design matrices, consisting e.g. of independent and identically distributed Gaussian entries, support recovery is possible for distinct constellations of \( p, n \), the order of sparsity and the minimal non-zero entry of \( \beta^* \). Comprehensive results in this direction, which even include non-Gaussian, heavy-tailed errors, are provided in Ndaoud and Tsybakov (2020). Another line of investigation to which we contribute here aims at providing results on support estimation for general design matrices, and in particular on getting rid of the mutual incoherence condition required by the LASSO. Prominent approaches are the use of nonconvex penalties (Fan and Li, 2001) as well as the adaptive LASSO (Zou, 2006). A high-dimensional analysis of these methods is provided in Loh and Wainwright (2017) and in Zhou et al. (2009); van de Geer et al. (2011).

In this paper, in the random design, heteroscedastic regression model (1) we show sign-consistency and optimal rates of convergence in the \( \ell_\infty \) norm for a computationally feasible estimator with the adaptive LASSO penalty and the following variant of the Huber loss function, sometimes called pseudo Huber loss,

\[ l_\alpha(x) = 2\alpha^{-2}\left(\sqrt{1 + \alpha^2 x^2} - 1\right), \tag{2} \]

as proposed by Charbonnier et al. (1994). In contrast to the Huber loss, \( l_\alpha \) is smooth and strictly convex. We require only slightly more than second moments for the errors. In our proofs we combine and extend methods from Fan et al. (2017), Sun et al. (2020), Loh and Wainwright (2017) and Zhou et al. (2009).
The paper is organized as follows. In Section 2 we introduce the estimator and set up some notation to precisely formulate the problem. Section 3 has the main result of the paper in a qualitative form, where we focus on the orders and discard exact constants. After reporting on the results of numerical experiments in Section 4, we present more precise versions of our results together with the main steps of the proofs in Section 5. Section 6 concludes, while technical proofs are deferred to the supplement in Section 7.

2 The adaptive LASSO with pseudo Huber loss function

We consider an estimator based on minimizing the pseudo Huber loss function with a weighted LASSO penalty given by

$$\hat{\beta}^{\text{WLH}}_n \in \arg \min_{\beta \in \mathbb{R}^p, \|\beta\|_2 \leq C_\beta} \left( L^H_{n,\alpha_n}(\beta) + \lambda_n \sum_{k=1}^p w_k |\beta_k| \right)$$

with regularization parameter $\lambda_n$, robustification parameter $\alpha_n > 0$ and weights $w_k > 0$ for $k \in \{1, \ldots, p\}$, and where the parameter $C_\beta > 0$ (or rather $C_\beta/2$, see Assumption 1, (iv)) is some given a-priori bound on the $\ell_2$ norm of the true parameter $\beta^\ast$. In (3), the empirical loss function $L^H_{n,\alpha}$ associated with the pseudo Huber loss is defined by

$$L^H_{n,\alpha}(\beta) := \frac{1}{n} \sum_{i=1}^n l_\alpha(Y_i - X_i^\top \beta),$$

and $l_\alpha$ is as in (2). We shall call $\hat{\beta}^{\text{WLH}}_n$ the weighted LASSO Huber estimator (WLHE). It estimates the parameter

$$\beta^\ast_{\alpha_n} := \arg \min_{\beta \in \mathbb{R}^p, \|\beta\|_2 \leq C_\beta} \mathbb{E}\left[l_{\alpha_n}(Y_1 - X_1^\top \beta)\right],$$

which coincides with $\beta^\ast$ in the particular case of a symmetric conditional distribution of $\varepsilon_1$ given $X_1$, but differs from $\beta^\ast$ in general. Later on we assume $\mathbb{E}[X_1X_1^\top]$ to be positive definite, hence $\beta^\ast_{\alpha_n}$ is unique by the strict convexity of $l_{\alpha_n}$. For a suitable initial estimator $\hat{\beta}^{\text{init}}_n = (\hat{\beta}_1^{\text{init}}, \ldots, \hat{\beta}_p^{\text{init}})^\top$ of $\beta^\ast$ such as the LASSO Huber estimator from Fan et al. (2017), choosing the (random) weights

$$w_k = \max \left\{ 1/|\hat{\beta}_{n,k}^{\text{init}}|, 1 \right\}, \quad k = 1, \ldots, p,$$

leads to the adaptive LASSO Huber estimator (ALHE) $\hat{\beta}^{\text{ALH}}_n$ which we shall focus on. Here, if $|\hat{\beta}_{n,k}^{\text{init}}| = 0$ so that formally $w_k = \infty$, we require that $\beta_k = 0$.

We shall investigate the sign-consistency as well as the rate of convergence in the $\ell_\infty$ distance of $\hat{\beta}^{\text{ALH}}_n$. To this end, let us set up some notation used in the following. Denote the support of the coefficient vector $\beta^\ast$ and its regularized version $\beta^\ast_{\alpha_n}$ in (5) by

$$S := \text{supp}(\beta^\ast) = \{k \in \{1, \ldots, p\} \mid \beta^\ast_k \neq 0\}, \quad s := |S|,$n

$$S_{\alpha_n} := \text{supp}(\beta^\ast_{\alpha_n}) = \{k \in \{1, \ldots, p\} \mid \beta^\ast_{\alpha_n,k} \neq 0\}, \quad s_{\alpha_n} := |S_{\alpha_n}|,$$

where $|S|$ is the cardinality of $S$. A major additional issue in our investigation will be that the support $S$ of $\beta^\ast$, the object of interest, differs from the support $S_{\alpha_n}$ of $\beta^\ast_{\alpha_n}$, the parameter which is actually estimated. Indeed, even if $\beta^\ast$ is sparse in the sense that $S$ is of small cardinality, this need not be the case for $\beta^\ast_{\alpha_n}$. However, our analysis will show that the adaptive LASSO penalty reliably sets the small superfluous entries of $\beta^\ast_{\alpha_n}$ to zero.
Results on support recovery are well-known to depend, in terms of so-called beta-min conditions, on the smallest absolute value of the entries of $\beta^*$ on its support $S$, which we denote by
\[
\beta_{\text{min}}^* := \min_{k \in S} |\beta_k^*|.
\] (7)

We shall further employ the following notations. For a subset $A \subseteq \{1, \ldots, p\}$, $\beta_A$ denotes the vector $(\beta_A)_i = \beta_i \mathbb{1}\{i \in A\}$, $i \in \{1, \ldots, p\}$, and sometimes also the vector $(\beta_i)_{i \in A} \in \mathbb{R}^{|A|}$, where $|A|$ is the cardinality of $A$. $A^c = \{1, \ldots, p\} \setminus A$ is the complement of $A$. If $Q \in \mathbb{R}^{p \times p}$ is a $p \times p$-matrix, and $B \subseteq \{1, \ldots, p\}$ is a further subset, $Q_{AB} \in \mathbb{R}^{|A| \times |B|}$ has entries according to row indices in $A$ and column indices in $B$. The $\ell_1$, $\ell_2$ and $\ell_\infty$ norms of a vector $x$ are denoted by $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$, and the corresponding matrix operator norms by $\|M\|_{M, \infty}$ for the $\ell_\infty$ norm, and $\|M\|_{M, 2}$ for the $\ell_2$ norm, also called spectral norm. We have for $M \in \mathbb{R}^{p \times q}$ that
\[
\|M\|_{M, \infty} = \max_{x \in \mathbb{R}^p, \|x\|_\infty \leq 1} \|Mx\|_\infty = \max_{1 \leq i \leq p} \sum_{j=1}^q |M_{i,j}|.
\]
The symbol $a \lesssim b$, where $a$ and $b$ will depend on $n$, $p$ and $|S|$, means that $a$ is smaller than $b$ up to constants not depending on $n$, $p$ and $|S|$, and $a \simeq b$ means that $a$ and $b$ are of the same order. Finally, $\text{sign}(t)$ denotes the sign of a number $t$, that is, $\text{sign}(t) = \mathbb{1}\{t > 0\} - \mathbb{1}\{t < 0\}$, and $\text{sign}$ is applied coordinate wise to a vector.

3 Sign-consistency and rate of convergence in $\ell_\infty$ norm

In this section we state our main results on sign-consistency and convergence rates in the $\ell_\infty$ norm of the adaptive LASSO Huber estimator in our setting with heteroscedastic, heavy-tailed and potentially asymmetric errors. Below we give a qualitative version of this result when discarding the constants and focusing on the orders. More precise formulations are provided in the proofs section in Lemma 14 combined with Lemmas 8 and 9.

To derive our results we adopt the following assumptions from Fan et al. (2017).

Assumption 1.

(i) For $m = 2$ or $m = 3$ and $q > 1$ we have that $\mathbb{E}[\mathbb{E}[|\epsilon_1|^{m}|X_1|^q]] \leq C_{r,m} < \infty$, where $C_{r,m} > 0$ is a positive constant.

(ii) For constants $0 < c_{X,1} < c_{X,u}$ we have that $0 < c_{X,1} \leq \lambda_{\min}(\mathbb{E}[X_1X_1^\top]) \leq \lambda_{\max}(\mathbb{E}[X_1X_1^\top]) \leq c_{X,u} < \infty$, where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalues of a symmetric matrix $A$.

(iii) For any $v \in \mathbb{R} \setminus \{0_p\}$ the variable $v^\top X_1$ is sub-Gaussian with variance proxy at most $c_{X,\text{sub}}^2 \|v\|_2^2$, $c_{X,\text{sub}} > 0$, that is $\mathbb{P}(|v^\top X_1| \geq t) \leq 2 \exp (-t^2/(2 c_{X,\text{sub}}^2 \|v\|_2^2))$ for all $t \geq 0$.

(iv) We have the a-priori upper bound $\|\beta^*\|_2 \leq C_{\beta}/2$, where $C_{\beta} \geq 1/8$ is numerical constant.

The assumptions are essentially those from Fan et al. (2017). In (i), we use a weaker moment assumption, but slightly more than the finite second moment as required in Sun et al. (2020). Further, as in Fan et al. (2017, Section 5) for the loss function from Catoni (2012), in (i) we can only make use of moments up to order 3 when estimating the approximation error. The normalization $C_{\beta}/2$ is made for later mathematical convenience.
We shall assume that the initial estimator $\hat{\beta}_n^{\text{init}}$ in the adaptive LASSO achieves the following rates in the $\ell_2$ and $\ell_1$ norms
\[
\left\| \hat{\beta}_n^{\text{init}} - \beta^* \right\|_2 \leq C_{\text{init}} \lambda_n^{\text{init}} \sqrt{\delta}, \quad \left\| \hat{\beta}_n^{\text{init}} - \beta^* \right\|_1 \leq C_{\text{init}} \lambda_n^{\text{init}} s, \quad \text{with} \quad \lambda_n^{\text{init}} \simeq \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \tag{8}
\]
for a positive constant $C_{\text{init}} \geq 1$. The notation suggests that the estimator is based on the regularization parameter $\lambda_n^{\text{init}}$ which then determines its rates. Indeed, under Assumption 1 the original LASSO Huber estimator given as a solution of (3) with Huber loss $\tilde{\ell}$, weights $w_k = 1, k = 1, \ldots, p$, satisfies (8) for $n \gtrsim s \log(p)$ under the scaling $\alpha_n \simeq \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}}$ of the robustification parameter and the choice of the regularization parameter as in (8), with probability at least $1 - 3/p$, see Sun et al. (2020, Theorem 8). From our results in Section 5.1 it follows that the same is true when using the pseudo Huber loss function $l_\alpha$ instead.

In the following result, the constants in the order symbols $\simeq$ and $\lesssim$ have to be chosen appropriately to achieve the estimate with the desired probability (12), see Lemmas 8, 9 and 14 in the proof Section 5 for details.

**Theorem 1** (Sign-consistency and rate in the $\ell_\infty$ norm). In model (1) under Assumption 1, consider the adaptive LASSO estimator $\hat{\beta}_n^{\text{ALH}}$ with initial estimator $\hat{\beta}_n^{\text{init}}$ assumed to satisfy (8). Further, suppose that
\[
\left\| \left( \mathbb{E}[X_1X_1^T] \right)_{SS} \right\|_{M,\infty}^{-1} \leq C_{S,X}, \tag{9}
\]
where $C_{S,X} > 0$ is a positive constant, is also satisfied. Assume that the robustification parameter $\alpha_n$ is chosen of the order
\[
\alpha_n \simeq \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}}, \tag{10}
\]
and that the regularization parameter $\lambda_n$ is chosen of order
\[
\lambda_n \simeq \lambda_n^{\text{init}} \left( \frac{\sqrt{\delta} \log(p)}{n} \right)^{\frac{1}{2}}, \quad \text{where} \quad \delta = \left\{ k \in \{1, \ldots, p\} \mid |\hat{\beta}_n^{\text{init}}| > \lambda_n^{\text{init}} \right\}, \tag{11}
\]
and $\lambda_n^{\text{init}} \simeq (\log(p)/n)^{\frac{1}{2}}$ is as in (8). If $n \gtrsim s^2 \log(p)$ and if $\beta^*$ satisfies a beta-min condition of order $\beta_{\text{min}}^{\beta^*} \gtrsim s \lambda_n^{\text{init}}$, then with probability at least
\[
1 - c_1 \exp(-c_2n) - \frac{c_3}{p^2}, \tag{12}
\]
where $c_1, c_2, c_3 > 0$ are suitable constants, the adaptive LASSO Huber estimator $\hat{\beta}_n^{\text{ALH}}$ as a solution to (3) with weights (6) is unique and satisfies
\[
\text{sign}(\hat{\beta}_n^{\text{ALH}}) = \text{sign}(\beta^*) \quad \text{and} \quad \left\| \hat{\beta}_n^{\text{ALH}} - \beta^* \right\|_{\infty} \lesssim \lambda_n^{\text{init}}. \tag{13}
\]
If we drop assumption (9) but instead have $s \leq \log(p)$, then we retain the sign-consistency in (13) but only obtain a $\ell_\infty$-rate of order
\[
\left\| \hat{\beta}_n^{\text{ALH}} - \beta^* \right\|_{\infty} \lesssim \sqrt{\delta} \lambda_n^{\text{init}}.
\]

**Remark 1.** The order in the beta-min condition $\beta_{\text{min}}^{\beta^*} \gtrsim s (\log(p)/n)^{\frac{1}{2}}$ as required e.g. in our result is the same as in Zhou et al. (2009, equation (4.10)), and quite stronger than the order $\beta_{\text{min}}^{\beta^*} \gtrsim (\log(p)/n)^{\frac{1}{2}}$. 

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required in Loh and Wainwright (2017, Corollary 1, Corollary 3). Potentially, this might be weakened in our setting as well by working with nonconvex regularizers. However, here we preferred to accept this restriction but to have the computationally more efficient adaptive LASSO. The requirement \( n \gtrsim s^2 \log(p) \) however, while being stronger than the \( n \gtrsim \log(p) \) for ordinary least squares in Loh and Wainwright (2017, Corollary 1), is however weaker than e.g. the \( n \gtrsim s^3 \log(p) \) required in Loh and Wainwright (2017, Corollary 3) for logistic regression. The rate in (13) under the additional assumption (9) is optimal, while the final bound without this condition is as in Zhou et al. (2009).

Somewhat unfortunately, this result requires that \( s \leq \log(p) \) and hence is only useful in high dimensions, however, at this stage we were not able to get rid of this assumption. Also, note that the order \( \lambda_n \approx \sqrt{s} \log(p)/n \) of the regularization parameter is smaller than that of the ordinary LASSO. Finally, the bound (13) together with the sign consistency implies that

\[
\left\| \hat{\beta}_{n}^{\text{ALH}} - \beta^* \right\|_2 \lesssim \sqrt{s} \lambda_n^{\text{init}} \quad \text{and} \quad \left\| \hat{\beta}_{n}^{\text{ALH}} - \beta^\star \right\|_1 \lesssim s \lambda_n^{\text{init}},
\]

as for the ordinary LASSO Huber estimator. Our results, in particular Lemmas 6 and 14 in Section 5 imply that this remains true under the weaker set of assumptions in Theorem 1, when dropping (9).

## 4 Simulations

In this section we numerically compare the performance of the LASSO Huber estimator (LH denotes the LASSO with Huber loss and LPH the LASSO with pseudo Huber loss) and the adaptive LASSO Huber estimator (ALH with Huber loss and ALPH with pseudo Huber loss) with the well-known classic LASSO (L) and adaptive LASSO (AL) with quadratic loss function in a simulation setting which is similar to that in Fan et al. (2017). We consider the high-dimensional linear regression model (1) with \( p = 400 \) normally distributed covariates \( X_1, \ldots, X_n \sim \mathcal{N}_p(0_p, I_p) \) and \( n = 200 \) observations, and a parameter vector given by

\[
\beta^* = (3, \ldots, 3, 0, \ldots, 0)^	op
\]

with \( S = \text{supp}(\beta^*) = \{1, \ldots, 20\} \) and \( s = |S| = 20 \). In the following we discuss different types of errors (light/ heavy tails, symmetric/ asymmetric, homo-/ heteroscedastic). In the homoscedastic case we assume \( \varepsilon_i = \tilde{\varepsilon}_i \) with \( \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n \) independent and identically distributed with \( E[\tilde{\varepsilon}_1] = 0 \) and independent of the covariates \( X_1, \ldots, X_n \), while in the heteroscedastic case the errors are

\[
\varepsilon_i = \frac{1}{\sqrt{3} \|\beta^*\|_2} \left( X_1^\top \beta^* \right)^2 \tilde{\varepsilon}_i.
\]

Evidently, \((X_1, \varepsilon_1), \ldots, (X_n, \varepsilon_n)\) are independent and identically distributed and \( E[\varepsilon_i | X_i] = 0 \). Furthermore, the factor \( 1/(\sqrt{3} \|\beta^*\|_2) \) implies

\[
E[\varepsilon_i^2] = \frac{1}{3 \|\beta^*\|_2^2} E\left[ (X_1^\top \beta^*)^4 \right] E[\tilde{\varepsilon}_i^2] = \frac{1}{3 \|\beta^*\|_2^2} 3 \|\beta^*\|_2^3 E[\tilde{\varepsilon}_i^2] = E[\varepsilon_i^2]
\]

since \( X_1^\top \beta^* \sim \mathcal{N}(0, \|\beta^*\|_2^2) \). Hence the homo- and heteroscedastic errors have the same variance in our simulations.

To compute the estimators in the simulation we use the functions of the packages \texttt{glmnet} (classic LASSO and adaptive LASSO) and \texttt{hqreg} (LASSO with Huber loss and adaptive LASSO with Huber loss). They have a factor of 1/2 in the quadratic loss. Further, the definition of the Huber loss includes an additional scaling of \( \alpha/2 \) in the package \texttt{hqreg}, see Yi and Huang (2017). As a consequence, for
the Huber loss the regularization parameter $\lambda$ of the (adaptive) LASSO includes this scaling factor of $\alpha$ as well, therefore we actually displayed $\lambda/\alpha$ for the Huber loss, which needs to be compared to $\lambda$ for the ordinary LASSO and the pseudo Huber loss. To compute the estimator for the pseudo Huber loss, we modified the functions of the package `hqreg` which were provided on GitHub by Yi and Huang (2017). This package uses a semismooth Newton coordinate descent algorithm, in contrast to the classical coordinate descent algorithm in `glmnet` or the iterative local adaptive majorize-minimization (I-LAMM) algorithm in Fan et al. (2018).

The parameters $\alpha$ and $\lambda$ of the estimators are chosen such that the $\ell_2$ distance of the respective estimation error is minimal. For this purpose we use 100 independent repetitions, where the errors have a specified distribution, and run through a one- or two-dimensional grid for the parameters in each set. In the adaptive versions of the estimators the parameters of the initial estimators are fixed (and equal to the optimal choices for the LASSO), so that we do not require a four-dimensional grid search for the adaptive LASSO. The resulting choices of the robustification parameter $\alpha$ and the regularization parameter $\lambda$ are displayed in the subsequent tables. Somewhat surprisingly, the tuning parameter for the adaptive version of the estimators differs quite strongly between the LASSO and the estimators based on (pseudo) Huber loss, even for homoscedastic, normally distributed errors.

Next we use these values of the parameters $\lambda$ and $\alpha$ in a Monte-Carlo-Simulation with 1000 iterations. In addition to the $\ell_2$ and $\ell_\infty$ distance, we also compute the average percentage of false positives (FP, noise covariates that are selected) and false negatives (FN, signal covariates that are not selected).

The following tables list the results. Overall we have the following main findings. First, for all methods, the version with adaptive weights is superior to that with ordinary weights for both $\ell_2$ and $\ell_\infty$ estimation error, as well as for the proportion of false positives (FPs). Second, estimators based on Huber and pseudo Huber loss function perform very similarly. Third, in particular for heteroscedastic errors these estimators have a much better performance than the ordinary LASSO, both in terms of estimation error as well as - in the adaptive versions - for their variable selection properties. Of course, the price to pay is that the additional tuning parameter $\alpha$ has to be chosen.

(a) **Symmetric errors with light tails.**

In this scenario we consider normally distributed errors $\tilde{\varepsilon}_i \sim \mathcal{N}(0, 4)$ with variance equal to 4.

|      | L  | AL | LH  | LPH | ALH (LH) | ALPH (LH) | ALPH (LPH) |
|------|----|----|-----|------|----------|-----------|------------|
| $\lambda$ | 0.154 | 0.695 | 0.157 | 0.150 | 0.066 | 0.067 | 0.069 |
| $\alpha$  | 0.115 | 0.061 | 0.153 | 0.061 | 0.153 | 0.050 | 0.050 |
| $\ell_2$ norm | 1.66 | 0.93 | 1.67 | 1.67 | 0.83 | 0.83 | 0.83 |
| $\ell_\infty$ norm | 0.60 | 0.41 | 0.61 | 0.61 | 0.38 | 0.38 | 0.38 |
| FP in %  | 16.14 | 1.83 | 15.76 | 16.32 | 1.06 | 0.99 | 0.97 |
| FN in %  | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Table 1: homoscedastic normally distributed errors.
### Table 2: heteroscedastic normally distributed errors.

| L     | AL   | LH   | ALH (LH) | ALPH (LH) |
|-------|------|------|----------|-----------|
| \( \lambda \) | 0.150 | 0.715 | 0.018    | 0.0003    | 0.0003    |
| \( \alpha \)  |      | 3.476 | 57.068   | 55.474    |
| \( \ell_2 \text{ norm} \) | 1.65  | 0.98  | 1.12     | 0.23      | 0.22      |
| \( \ell_\infty \text{ norm} \) | 0.59  | 0.41  | 0.37     | 0.10      | 0.09      |
| FP in % | 15.81 | 1.91  | 21.47    | 0.96      | 1.08      |
| FN in % | 0.00  | 0.00  | 0.00     | 0.00      | 0.00      |

(b) **Symmetric errors with heavy tails.**

Here we consider \( \tilde{\varepsilon}_i = 2Q_i \) with \( Q_i \sim t_3 \) t-distributed with 3 degrees of freedom.

### Table 3: homoscedastic t-distributed errors.

| L     | AL     | LH     | LPH    | ALH (LH) | ALPH (LH) | ALPH (LPH) |
|-------|--------|--------|--------|----------|-----------|------------|
| \( \lambda \) | 0.262  | 0.901  | 0.142  | 0.080    | 0.059     | 0.040      | 0.033      |
| \( \alpha \)  |        | 0.429  | 0.742  | 0.563    | 0.769     | 0.974      |
| \( \ell_2 \text{ norm} \) | 2.85   | 1.89   | 2.34   | 2.35     | 1.17      | 1.18       | 1.19       |
| \( \ell_\infty \text{ norm} \) | 1.03   | 0.76   | 0.85   | 0.85     | 0.53      | 0.53       | 0.53       |
| FP in % | 15.64  | 2.74   | 16.66  | 17.59    | 1.38      | 1.39       | 1.51       |
| FN in % | 0.03   | 0.05   | 0.00   | 0.00     | 0.00      | 0.00       | 0.00       |

### Table 4: heteroscedastic t-distributed errors.

| L     | AL     | LH     | ALH (LH) | ALPH (LH) |
|-------|--------|--------|----------|-----------|
| \( \lambda \) | 0.226  | 0.849  | 0.019    | 0.0005    | 0.0006    |
| \( \alpha \)  |        | 3.574  | 33.854   | 29.368    |
| \( \ell_2 \text{ norm} \) | 2.71   | 1.87   | 1.37     | 0.28      | 0.28      |
| \( \ell_\infty \text{ norm} \) | 0.94   | 0.72   | 0.46     | 0.12      | 0.11      |
| FP in % | 16.36  | 3.05   | 20.95    | 1.13      | 1.18      |
| FN in % | 0.11   | 0.16   | 0.00     | 0.00      | 0.00      |

(c) **Asymmetric errors with heavy tails.**

Finally we consider \( \tilde{\varepsilon}_i = Q_i - E[Q_i] \) with \( Q_i \sim St(0, 1, 0.6, 3) \) skew t-distributed with location parameter 0, scale parameter 1, skew parameter 0.6 and 3 degrees of freedom. An exact definition can be found in Azzalini and Capitanio (2003) and it is \( E[Q_i] = (0.6/\sqrt{1.36}) \sqrt{3/\pi} / \Gamma(3/2) \).
\[ \begin{array}{lcccccc}
L & AL & LH & LPH & ALPH (LH) & ALPH (LH) & ALPH (LPH) \\
\lambda & 0.118 & 0.709 & 0.070 & 0.058 & 0.019 & 0.011 & 0.010 \\
\alpha & 0.863 & 0.871 & 1.124 & 1.842 & 0.47 & 0.46 & 0.47 \\
\ell_2 \text{ norm} & 1.33 & 0.74 & 1.08 & 1.12 & 0.47 & 0.46 & 0.47 \\
\ell_\infty \text{ norm} & 0.48 & 0.32 & 0.39 & 0.40 & 0.22 & 0.22 & 0.23 \\
FP \text{ in %} & 16.37 & 1.56 & 16.48 & 16.49 & 0.52 & 0.63 & 0.53 \\
FN \text{ in %} & 0.01 & 0.01 & 0.00 & 0.00 & 0.02 & 0.00 & 0.01 \\
\end{array} \]

Table 5: homoscedastic skew t-distributed errors.

\[ \begin{array}{lcccc}
L & AL & LH & LPH & ALPH (LH) \\
\lambda & 0.110 & 0.649 & 0.009 & 0.0003 \\
\alpha & 7.00 & 33.898 & 50.684 \\
\ell_2 \text{ norm} & 1.28 & 0.77 & 0.64 & 0.11 \\
\ell_\infty \text{ norm} & 0.45 & 0.32 & 0.22 & 0.05 \\
FP \text{ in %} & 16.00 & 1.80 & 21.18 & 0.43 \\
FN \text{ in %} & 0.02 & 0.02 & 0.00 & 0.00 \\
\end{array} \]

Table 6: heteroscedastic skew t-distributed errors.

5 Proofs: Main steps

In this section we present the results in more technical form together with the main steps of the proofs. Various technical details are deferred to the supplement, Section 7. Let us give an overview of our approach. In Section 5.1 we start with various technical preparations, including a bound on the approximation bias and the restricted strong convexity condition for the pseudo Huber loss, similar to Fan et al. (2017, Section 5) for the Catoni loss function. Section 5.2 details how to implement the primal-dual witness approach from Wainwright (2009) in our setting. Compared to Loh and Wainwright (2017) and Zhou et al. (2009), the main additional issue is that \( \beta^*_\alpha \) as defined in (5) does not have support \( S \) and, indeed, need not to be sparse. Lemmas 8 and 9 take care of technical expressions, in particular the inverse of the Hessian of empirical loss function restricted to \( S \) and of a term involving the gradient when checking strict dual feasibility. In Section 5.3, we deduce a result for the general weighted LASSO Huber estimator (3), which still involves a mutual incoherence condition. Finally, in Sections 5.4 and 5.5 this is specialized for the adaptive LASSO, first for an initial estimator satisfying general rate assumptions, and then for one which is assumed to satisfy (8), for which we can get rid of mutual incoherence.

We shall use the following additional notation. \( X_n = (X_1, \ldots, X_n)^\top \in \mathbb{R}^{n \times p} \) is the design matrix, where \( X_i \in \mathbb{R}^p \) is the covariate vector in model (1). \( w = (w_1, \ldots, w_p)^\top \) denotes the vector of weights from (3), and we set \( w_{\max}(S) = \max_{i \in S} w_i \) and \( w_{\min}(S^c) = \min_{i \in S^c} w_i \). We denote by \( \nabla \) the gradient of a smooth function, and by \( \partial \) the subgradient of a convex function. For vectors \( x, y \) of same dimension we denote by \( x \odot y \) their Hadamard product (that is coordinate wise product). Inequality signs such as \( x < y \) are understood component wise. A diagonal matrix with real entries \( d_1, \ldots, d_n \) is denoted by \( \text{diag}(d_1, \ldots, d_n) \).
5.1 Technical preparations

We start with some technical preparations, where we extend results from Fan et al. (2017) to the pseudo Huber loss function \( l_\alpha \). See also Fan et al. (2017, Section 5) for similar extensions to the Cantoni loss function (Catoni, 2012). The proofs of the lemmas in this section are provided in the supplement, Section 7.1. To start, straightforward differentiation gives

\[
l'_\alpha(x) = \frac{2x}{\sqrt{1 + \alpha^2 x^2}} \quad \text{so that} \quad |l'_\alpha(x)| \leq \frac{2|x|}{\alpha^2 x^2} = 2\alpha^{-1}, \tag{14}
\]

and

\[
l''_\alpha(x) = \frac{2\alpha^{-3}}{(\alpha^{-2} + x^2)^{3/2}} \quad \text{so that} \quad 0 < l''_\alpha(x) \leq \frac{2\alpha^{-3}}{(\alpha^{-2})^{3/2}} = 2. \tag{15}
\]

In particular \( l_\alpha \) is strictly convex. Also note that \( \lim_{\alpha \to 0} l_\alpha(x) = x^2 \) for all \( x \in \mathbb{R} \). For the empirical loss function in (4) this gives

\[
\nabla \mathcal{L}^H_{n,\alpha}(\beta) = -\frac{1}{n} \sum_{i=1}^n l'_\alpha(Y_i - X_i^T \beta) X_i, \quad \nabla^2 \mathcal{L}^H_{n,\alpha}(\beta) = \frac{1}{n} \sum_{i=1}^n l''_\alpha(Y_i - X_i^T \beta) X_i X_i^T. \tag{16}
\]

The following result is similar to Fan et al. (2017, Theorem 1 and Theorem 6), however, we work with a weaker moment assumption.

**Lemma 1.** Under Assumption 1 we have for \( \beta^*_n \) in (5) that

\[
\| \beta^*_n - \beta^* \|_2 \leq C_{\text{apx}} \alpha_n^{m-1}, \tag{17}
\]

where

\[
C_{\text{apx}} = \frac{2^m c_{X,\text{sub}}}{c_{X,1}} \left[ \left( \frac{q}{q-1} \Gamma \left( \frac{q}{2(q-1)} \right) \right)^{\frac{1}{q}} 10000 \left( 2(2C^2 \gamma^2 c_{X,\text{sub}})^m (2m)! \Gamma(m) \right)^{\frac{1}{2}} \right]
\]

and \( \Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) \, dt, \, x > 0, \) is the gamma function.

**Remark 2.** The above result leads to \( \| \beta^*_n - \beta^* \|_2 < C_\beta/2 \) for an (appropriate) choice of \( \alpha_n \). Together with the assumption \( \| \beta^* \|_2 \leq C_\beta/2 \) this will imply that \( \beta^*_n \) is strictly feasible for (5), that is,

\[
\| \beta^*_n \|_2 < C_\beta, \tag{18}
\]

which we will assume from now on.

Next we show along the lines of Fan et al. (2017, Lemmas 2 and 4) that restricted strong convexity (cf. Loh and Wainwright (2017)) is satisfied by the pseudo Huber loss function.

**Lemma 2.** Under Assumption 1 there exist \( c_{\alpha} > 0 \) (depending on \( c_{X,1}, c_{X,a}, c_{X,\text{sub}} \) and \( C_\beta \)) and \( c_1^2, c_2^2 > 0 \) (depending on \( c_{X,1} \) and \( c_{X,\text{sub}} \)) such that for all \( \| \beta \|_2 \leq 4C_\beta \), \( \| \Delta \|_2 \leq 8C_\beta \) and \( \alpha \leq c_\alpha \) with probability at least \( 1 - c_1^2 \exp(-c_2^2 n) \) the empirical pseudo Huber loss function \( \mathcal{L}^H_{n,\alpha} \) satisfies the restricted strong convexity condition

\[
\langle \nabla \mathcal{L}^H_{n,\alpha}(\beta + \Delta) - \nabla \mathcal{L}^H_{n,\alpha}(\beta), \Delta \rangle \geq c_1^{\text{RSC}} \| \Delta \|_2^2 - c_2^{\text{RSC}} \frac{\log(p)}{n} \| \Delta \|_1^2 \tag{19}
\]

with

\[
c_1^{\text{RSC}} = \frac{c_{X,1}}{16}, \quad c_2^{\text{RSC}} = \frac{1600 c_{X,\text{sub}}^2 (\max \left\{ 4 c_{X,\text{sub}} \sqrt{\log(12 c_{X,\text{sub}} / c_{X,1})}, 1 \right\} )^4}{c_{X,1}}.
\]
Restricted strong convexity in particular implies ordinary strong convexity locally on the support $S$ of $\beta^*$.

**Lemma 3.** Under Assumption 1, if $\alpha \leq c_\alpha$ and $n \geq c_3^{\text{RSC}} \log(p)$ with $c_3^{\text{RSC}} = 2c_2^{\text{RSC}} / c_1^{\text{RSC}}$ we have with probability at least $1 - c^2 n \exp(-c_2^2 n)$ for $\beta \in \mathbb{R}^p$ with $\|\beta\|_2 \leq 4C_\beta$ that

$$\lambda_{\min} \left( \left( \nabla^2 L_{n, \alpha}^H (\beta) \right)_{SS} \right) \geq \frac{c_1^{\text{RSC}}}{2} = \frac{c_X}{32}. \tag{20}$$

The following result gives a bound on the gradient of the empirical loss function, and is analogous to Fan et al. (2017, Lemma 1).

**Lemma 4.** Under Assumption 1 there exist $c_{\text{Grad}}^{1}, c_{\text{Grad}}^{2} > 0$ (depending on $q, C, c_{\text{X, sub}}$ and $C_\beta$) such that for all $\alpha \geq c_1^{\text{Grad}} \left( \log(p) \right)^{\frac{1}{2}}$ with probability at least $1 - 2/p^2$ the $\ell_\infty$ norm of the gradient of the empirical pseudo Huber loss function at $\beta^*_n$ is bounded by

$$\| \nabla L_{n, \alpha}^H (\beta^*_n) \|_\infty \leq c_2^{\text{Grad}} \left( \log(p) \right)^{\frac{1}{2}}. \tag{21}$$

### 5.2 Primal-dual witness approach

The proof of Theorem 1 is based on the primal-dual witness (PDW) approach as originally introduced in Wainwright (2009). Following Loh and Wainwright (2017) we summarize the three main steps as follows. The main results in this section for implementing this approach in our setting is Lemma 7, together with Lemmas 8 and 9.

(i) Optimize the restricted program

$$\hat{\beta}_{n, \text{PDW}} \in \arg \min_{\beta \in \mathbb{R}^p, \text{supp}(\beta) \subseteq S, \|\beta\|_2 \leq C_\beta} \left( L_{n, \alpha}^H (\beta) + \lambda_n \sum_{k \in S} w_k |\beta_k| \right), \tag{22}$$

where we enforce the constraint that $\text{supp}(\hat{\beta}_{n, \text{PDW}}) \subseteq S$, and show that all solutions have norm $< C_\beta$.

(ii) Choose $\hat{\gamma}_n \in \mathbb{R}^p$ such that (a.) $\hat{\gamma}_S \in \partial \|\hat{\beta}_{n, \text{PDW}}\|_1$, (b.) it satisfies the zero-subgradient condition

$$\nabla L_{n, \alpha}^H (\hat{\beta}_{n, \text{PDW}}) + \lambda_n (w \odot \hat{\gamma}) = 0, \tag{23}$$

and (c.) such that $\hat{\gamma}_S^\perp$ satisfies the strict dual feasibility condition $\|\hat{\gamma}_{S^\perp}\|_\infty < 1$.

(iii) Show that $\hat{\beta}_{n, \text{PDW}}$ is also a minimum of the full program (3),

$$\arg \min_{\beta \in \mathbb{R}^p, \|\beta\|_2 \leq C_\beta} \left( L_{n, \alpha}^H (\beta) + \lambda_n \sum_{k=1}^p w_k |\beta_k| \right),$$

and moreover the uniqueness of the minimizer of this program.

We shall always assume that

$$\alpha_n \leq c_\alpha$$

holds, where $c_\alpha$ is given in Lemma 2. Later, $\alpha_n$ is chosen of an order tending to zero, so that this is automatically satisfied. The following lemma lists some technical properties of the derivatives of the empirical loss function.
Lemma 5. We may write

\[ \hat{Q} := \int_0^1 \nabla^2 \mathcal{L}_{n,\alpha_n}^H (\beta_{\alpha_n}^* + t (\hat{\beta}_n^{PDW} - \beta_{\alpha_n}^*)) dt = \frac{2}{n} \sum_{i=1}^n d_i \mathbf{X}_i \mathbf{X}_i^T = \frac{2}{n} \mathbf{X}_n^T D \mathbf{X}_n, \]

(24)

where \( D = \text{diag}(d_1, \ldots, d_n) \) with

\[ d_i = \frac{1}{2} \int_0^1 l''_{\alpha_n} \left( \mathbf{Y}_i - \mathbf{X}_i \mathbf{X}_i^T (\beta_{\alpha_n}^* + t (\hat{\beta}_n^{PDW} - \beta_{\alpha_n}^*)) \right) dt \in (0, 1]. \]

Furthermore, under Assumption 1, if \( n \geq c_3^{RSC} \log(p) \) with probability at least \( 1 - c_1^p \exp(-c_2^p n) \) the submatrix \( \hat{Q}_{SS} \) is invertible with minimal eigenvalue bounded below by \( c_{X,1}/32 \) and we have the bound

\[ \left\| (\hat{Q}_{SS})^{-1} \right\|_{M, \infty} \leq \frac{32 \sqrt{s}}{c_{X,1}}. \]

(25)

Proof of Lemma 5. (24) follows from straightforward calculation, see (16). Moreover, every point \( \beta \in \mathbb{R}^p \) between \( \beta_{\alpha_n}^* \) and \( \hat{\beta}_n^{PDW} \) has \( \ell_2 \) norm smaller than or equal to \( C_{\beta} \) because \( \| \beta_{\alpha_n}^* \|_2, \| \hat{\beta}_n^{PDW} \|_2 \leq C_{\beta} \).

Hence (20) implies the invertibility of \( Q_{SS} \) and (25) follows from

\[ \left\| (\hat{Q}_{SS})^{-1} \right\|_{M, \infty} \leq \sqrt{s} \left\| (\hat{Q}_{SS})^{-1} \right\|_{M, 2} \leq \frac{32 \sqrt{s}}{c_{X,1}}. \]

\[ \square \]

In the following lemma we show that \( \hat{\beta}_n^{PDW} \) is strictly feasible, meaning \( \left\| \hat{\beta}_n^{PDW} \right\|_2 < C_{\beta} \) holds, for an appropriate choice of \( \lambda_n \) and \( \alpha_n \).

Lemma 6. Under Assumption 1 we have for \( \hat{\beta}_n^{PDW} \) in (22) with \( \alpha_n \geq c_1^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \) that

\[ \left\| \hat{\beta}_n^{PDW} - \beta^* \right\|_2 \leq \left( c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} + w_{\max}(S) \lambda_n + 2 C_{\beta} c_3^{RSC} \sqrt{s} \frac{\log(p)}{n} \right) \frac{\sqrt{s}}{c_1^{RSC}} + C_{apx} \alpha_n^{m-1} \]

(26)

with probability at least \( 1 - c_1^p \exp(-c_2^p n) - 2/p^2 \).

Proof of Lemma 6. Let

\[ \beta_{\alpha_n,\text{supp}}^* = \arg \min_{\beta \in \mathbb{R}^p, \text{supp}(\beta) \subseteq S, \| \beta \|_2 \leq C_{\beta}} \mathbb{E} \left[ l_{\alpha_n} (\mathbf{Y}_1 - \mathbf{X}_1^T \beta) \right] \quad \text{and} \quad \Delta_{n,PDW}^\alpha = \hat{\beta}_n^{PDW} - \beta_{\alpha_n,\text{supp}}^*, \]

(27)

then \( \hat{\beta}_n^{PDW} \) in (22) is a M-estimator of \( \beta_{\alpha_n,\text{supp}}^* \). Following the proof of Lemma 1 leads on the one hand to

\[ \left\| \beta_{\alpha_n,\text{supp}}^* - \beta^* \right\|_2 \leq C_{apx} \alpha_n^{m-1}. \]

In doing so note that

\[ \mathbb{E} [l_{\alpha_n} (\mathbf{Y}_1 - \mathbf{X}_1^T \beta_{\alpha_n,\text{supp}})] \leq \mathbb{E} [l_{\alpha_n} (\mathbf{Y}_1 - \mathbf{X}_1^T \beta^*)] \quad \text{and} \quad \left\| \beta_{\alpha_n,\text{supp}}^* \right\|_2 \leq C_{\beta} \]

because of (27), \( \text{supp}(\beta^*) = S \) and \( \| \beta^* \|_2 \leq C_{\beta} \) by (iv) of Assumption 1. Further, \( \hat{\beta}_n^{PDW} \) has to satisfy the first-order necessary condition of a convex constrained optimization problem over a convex set to be a minimum of (22), cf. Ruszczynski (2006, Theorem 3.33), that is, there exists \( \mathbf{\gamma} \in \partial \left\| \hat{\beta}_n^{PDW} \right\|_1 \) so that

\[ \left\langle \nabla \mathcal{L}_{n,\alpha_n}^H (\hat{\beta}_n^{PDW}) + \lambda_n (w \odot \mathbf{\gamma}), \beta - \hat{\beta}_n^{PDW} \right\rangle \geq 0 \quad \text{for all feasible } \beta \in \mathbb{R}^p. \]

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Hence by the restricted strong convexity of the empirical pseudo Huber loss in Lemma 2 and the first-order necessary condition it follows that

\[ c_1^{\text{RSC}} \| \Delta_n^{\text{PDW}} \|_2^2 - c_2^{\text{RSC}} \log(p) \left( \frac{\Delta_n^{\text{PDW}}}{n} \right) \leq \left\langle \nabla L_{n,\alpha_n}^H(\hat{\beta}_n) - \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*, \Delta_n^{\text{PDW}}) \right\rangle \\
\leq \left\langle -\nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*, \Delta_n^{\text{PDW}}) \right\rangle \\
\leq \| \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*, \Delta_n^{\text{PDW}}) \|_{\infty} \| \Delta_n^{\text{PDW}} \|_1 + \max(S) \lambda_n \| \Delta_n^{\text{PDW}} \|_1 \]

with probability at least \( 1 - c_1^2 \exp(-c_2^2 n) \). Here the last inequality follows since \( \beta_{\alpha_n}^* \) and \( \hat{\beta}_n \) both have support (contained in) \( S \). Rearranging leads to

\[ c_1^{\text{RSC}} \| \Delta_n^{\text{PDW}} \|_2^2 \leq \left( \| \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*, \Delta_n^{\text{PDW}}) \|_{\infty} + \max(S) \lambda_n + c_2^{\text{RSC}} \log(p) \left( \frac{\Delta_n^{\text{PDW}}}{n} \right) \right) \| \Delta_n^{\text{PDW}} \|_1. \]

We obtain \( \| \Delta_n^{\text{PDW}} \|_1 \leq \sqrt{s} \| \Delta_n^{\text{PDW}} \|_2 \) and \( \| \Delta_n^{\text{PDW}} \|_2 \leq 2C_\beta \) because of (22) and (27). In addition by following the proof of Lemma 4 we get \( \| \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*, \Delta_n^{\text{PDW}}) \|_{\infty} \leq c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \) with probability at least \( 1 - p^2/2 \). Hence it follows that

\[ \| \hat{\beta}_n - \beta_{\alpha_n}^* \|_2 \leq \left( c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} + \max(S) \lambda_n + 2C_\beta c_2^{\text{RSC}} (c_1^2)^{\frac{1}{2}} \right) \frac{\sqrt{s}}{c_1^{\text{RSC}}} \]

and in total the assertion of the lemma.

\[ \Box \]

**Remark 3.** The results below will imply that with (appropriate) choices of \( \lambda \) and \( \alpha_n \),

\[ \| \hat{\beta}_n^{\text{PDW}} - \beta^* \|_2 = O \left( \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \right) \]

with high probability, so that in particular \( \| \hat{\beta}_n^{\text{PDW}} - \beta^* \|_2 < C_\beta/2 \). Together with the assumption \( \| \beta^* \|_2 \leq C_\beta/2 \) this will imply that \( \hat{\beta}_n^{\text{PDW}} \) is strictly feasible for (22), that is,

\[ \| \hat{\beta}_n^{\text{PDW}} \|_2 < C_\beta, \quad \text{(28)} \]

which we will assume from now on.

**Lemma 7** (Solving the PDW construction). Suppose that Assumption 1 holds and that \( \beta^* \) satisfies the beta-min condition

\[ \beta_{\min}^* > C_{\text{apx}} \alpha_n^{m-1} \quad \text{(29)} \]

and that \( n \geq c_3^{\text{RSC}} \cdot \log(p) \). Let \( \hat{\beta}_n^{\text{PDW}} \) be as in the PDW construction, and suppose that \( \hat{\gamma}_S \in \partial \| \hat{\beta}_n^{\text{PDW}} \|_1 \) and the zero-subgradient condition (23). Then, with probability at least \( 1 - c_1^2 \exp(-c_2^2 n) \) the strict dual feasibility condition \( \| \hat{\gamma}_S \|_{\infty} < 1 \) is equivalent to the condition

\[ \left[ \hat{Q}_S(\hat{Q}_S)^{-1} \left( \lambda_n (w_S \otimes \hat{\gamma}_S) + \left( \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*) \right)_S - \left( \nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*) \right)_S \right) \right]_{S^c} \\
+ \left( \hat{Q}_S(S_n \setminus S) - \hat{Q}_S(\hat{Q}_S)^{-1} \hat{Q}_S(S_n \setminus S) \right) \beta_{\alpha_n, S \setminus S_n}^* < w_S \lambda_n. \quad \text{(30)} \]

Furthermore, if (30) is satisfied we have that the minimizer \( \hat{\beta}_n^{\text{WLH}} \) in (3) is unique and given by \( \hat{\beta}_n^{\text{WLH}} = \hat{\beta}_n^{\text{PDW}} \), so that in particular supp(\( \hat{\beta}_n^{\text{WLH}} \)) \( \subseteq \) supp(\( \beta^* \)), and furthermore that

\[ \| \hat{\beta}_n^{\text{WLH}} - \beta^* \|_\infty \leq \phi_{n, \infty}, \quad \text{(31)} \]
where
\[
\phi_{n,\infty} = \left\| (\hat{Q}_{SS})^{-1} \right\|_{M,\infty} \left\| (\nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*))_S \right\|_{\infty} + w_{\max}(S) \lambda_n \left\| (\hat{Q}_{SS})^{-1} \right\|_{M,\infty} + \left\| \beta_{a_n,s}^* - \beta_{a_n,s}^* \right\|_{\infty} + \left\| (\hat{Q}_{SS})^{-1} \right\|_{M,\infty} \left\| (\hat{Q}_{S(S_n\setminus S)} \beta_{a_n,S_n\setminus S}^*)_S \right\|_{\infty}.
\]

(32)

Furthermore, if we have in addition the beta-min condition of the same order
\[
\beta_{a_n}^* > \phi_{n,\infty},
\]
then we have the sign-recovery property \( \text{sign}(\hat{\beta}_{n,\text{WLH}}^*) = \text{sign}(\beta^*) \).

The beta-min condition (29) is required so that the approximation error \( \|\beta_{a_n}^* - \beta^*\|_{\infty} \) is small, which implies that the support \( S_{\alpha_n} \) of \( \beta_{a_n}^* \) contains the support \( S \) of \( \beta^* \). Later, we shall choose \( \alpha_n \) and achieve a rate \( \phi_{n,\infty} \), which in any case also includes an approximation term \( \|\beta_{a_n,s}^* - \beta_S^*\|_{\infty} \) such that (33) implies (29).

**Proof of Lemma 7.** We start by showing that under assumption (29) we have \( S \subseteq S_{\alpha_n} \). To this end, we estimate
\[
|\beta_{a_n,k}^*| = |\beta_{a_n,k}^* - \beta_k^* + \beta_k^*| \geq |\beta_{a_n,k}^* - \beta_k^*| \geq \beta_{a_n,s}^* - \beta_{a_n,\infty}^* + \|\beta_{a_n,\infty}^* - \beta^*\|_2
\]
\[
\geq \beta_{a_n,s}^* - C_{\text{apx}} \alpha_n^{m-1} > 0,
\]

for \( k \in S \) where the first inequality in the last line follows from (17) and the final inequality from (29).

Now, using
\[
\hat{Q}(\beta_{n,PDW}^* - \beta_{a_n}^*) = \nabla L^H_{\alpha_n,\alpha_n}(\beta_{n,PDW}^*) - \nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*),
\]

see (24) for the definition of \( \hat{Q} \), we may rewrite the subgradient condition (23), which holds since \( \beta_{n,PDW}^* \) is strictly feasible as in (28), as
\[
\hat{Q}(\beta_{n,PDW}^* - \beta_{a_n}^*) + \nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*) + \lambda_n(\omega \odot \hat{\gamma}) = 0_p
\]
or in block-form
\[
\begin{bmatrix}
\hat{Q}_{SS} & \hat{Q}_{S(S_n\setminus S)} & \hat{Q}_{SS(S_n\setminus S)} \\
\hat{Q}_{S^cS} & \hat{Q}_{S^c(S_n\setminus S)} & \hat{Q}_{SSc}
\end{bmatrix}
\left[
\begin{bmatrix}
\beta_{a_n}^* - \beta_{a_n,s}^* \\
-\beta_{a_n,S_n\setminus S}^* \\
0_{|S_n\setminus S|}
\end{bmatrix}
\right] + \left[
\begin{bmatrix}
\nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*)_S \\
\nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*)_{S^c}
\end{bmatrix}
\right] + \lambda_n(\omega_S \odot \hat{\gamma}_S - \omega_{S^c} \odot \hat{\gamma}_{S^c}) = 0_p,
\]

where we used that \( \hat{\beta}_{n,PDW}^* = 0_{p_{-a}} \) by the primal-dual witness construction. By invertibility of \( \hat{Q}_{SS} \), see Lemma 5, this leads to
\[
\beta_{n,PDW}^* - \beta_{a_n,\infty} = \hat{Q}_{SS}^{-1} \left( -\lambda_n(\omega_S \odot \hat{\gamma}_S - \nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*))_S + \hat{Q}_{S(S_n\setminus S)} \beta_{a_n,S_n\setminus S}^* \right)
\]

and
\[
\lambda_n(\omega_{S^c} \odot \hat{\gamma}_{S^c}) = \hat{Q}_{S^cS}(\hat{Q}_{SS})^{-1} \left( \lambda_n(\omega_S \odot \hat{\gamma}_S) + \left( \nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*)_S \right) - \left( \nabla L^H_{\alpha_n,\alpha_n}(\beta_{a_n}^*)_{S^c} \right) \right)
\]
\[
+ \left( \hat{Q}_{S^c(S_n\setminus S)} - \hat{Q}_{S^cS}(\hat{Q}_{SS})^{-1} \hat{Q}_{S(S_n\setminus S)} \right) \beta_{a_n,S_n\setminus S}^*.
\]
The second equation shows the equivalence of the strict dual feasibility condition $\|\hat{\gamma}_{S^c}\|_\infty < 1$ and (30). Now, if this holds then we obtain that $\hat{\gamma} \in \partial \|\hat{\beta}_n^{\text{PDW}}\|_1$, and since the loss function $L_n^H$ is convex (and obviously also the weighted $\ell_1$ norm) we obtain by (23) that $\hat{\beta}_n^{\text{PDW}}$ is also a solution of (3), cf. Ruszczynski (2006, Theorem 3.33) and recall from (28) that $\hat{\beta}_n^{\text{PDW}}$ is (assumed to be) strictly feasible. To conclude $\hat{\beta}_n^{\text{WLH}} = \hat{\beta}_n^{\text{PDW}}$ we need to show that this solution is unique. Then apparently $\text{supp}(\hat{\beta}_n^{\text{WLH}}) \subseteq \text{supp}(\beta^*)$ and (32) follows from (34). If the beta-min condition (33) holds, then for $k \in S$

$$\left|\hat{\beta}_{n,k}^{\text{WLH}} - \beta_k^*\right| \leq \left\|\hat{\beta}_n^{\text{WLH}} - \beta^*\right\|_\infty < \beta_{\min}^* \leq |\beta_k^*|,$$

which implies $\text{sign}(\hat{\beta}_{n,k}^{\text{WLH}}) = \text{sign}(\beta_k^*)$ and hence the sign-consistency of $\hat{\beta}_n^{\text{WLH}}$.

It remains to show uniqueness of the solution of the program (3). To this end we show that all stationary points $\hat{\beta}$, that is points satisfying $\nabla L_{n,\alpha}^H(\hat{\beta}) = -\lambda_n (w \odot \hat{\gamma})$ with $\hat{\gamma} \in \partial \|\hat{\beta}\|_1$ have support $S$ (cf. Loh and Wainwright (2017, Lemma 3) or Tibshirani (2013, Section 2.3)). Then strict convexity of the loss function restricted to vectors with support $S$, as implied by (20), concludes the proof.

From the form (16) of the gradient of the loss function we see that uniqueness of the fitted values $\lambda_n \hat{\beta}$ for all stationary points implies uniqueness of the subgradient $\hat{\gamma}$, that is $\hat{\gamma} = \gamma$. The strict dual feasibility condition $\|\hat{\gamma}_{S^c}\|_\infty < 1$ for $\hat{\gamma}$ (cf. Tibshirani (2013, Section 2.3) or Wainwright (2009, Lemma 1 (b))) then implies that $\beta$ must also have support in $S$. Now, uniqueness of the fitted values follows from the strict convexity of the pseudo Huber loss by using Lemma 1 (ii) in Tibshirani (2013). This concludes the proof of the lemma.

In the next two technical results, we show how to take care of the terms involving the gradient of the loss in the strict dual feasibility assumption (30), and how to obtain a sharper bound on the inverse of $\hat{Q}_{SS}$ then (25) under (9).

**Lemma 8** (Strict dual feasibility and norm bound I). Suppose that Assumption 1 and (9) are satisfied and assume that the robustification parameter $\alpha_n$ is chosen as in (10). If $n \geq C_3 s^2 \log(p)$ for a sufficiently large positive constant $C_3 > 0$ then there exist constants $C_1, C_2, C_{Q,S} > 0$ and $C_{Q,L} \geq 1$ such that

$$\left\|\left(\hat{Q}_{SS}\right)^{-1}\right\|_{M,\infty} \leq C_{Q,S}$$

is satisfied with probability at least $1 - C_1/p^2 - 6/p^{5s}$, and

$$\left\|\hat{Q}_{SS}^{-1}(\hat{Q}_{SS})^{-1} - \left(\nabla L_{n,\alpha}^H(\beta^*_{\alpha_n})\right)_{S^c}\right\|_\infty \leq C_{Q,L} c_2^{\text{Grad}} \left(\frac{\log(p)}{n}\right)^{s}$$

with probability at least $1 - (4 + C_1 + C_2)/p^2 - 6/p^{5s}$, where $c_2^{\text{Grad}}$ is as in Lemma 4.

The proof is deferred to the supplement, Section 7.2. If we drop the requirement (9) we still obtain a bound of the form (36) under the somewhat restrictive scaling $s \leq \log(p)$. The bound (35) is no longer valid and needs to be replaced by (25).

**Lemma 9** (Strict dual feasibility and norm bound II). Suppose that Assumption 1 holds and assume that the robustification parameter satisfies $\alpha_n \geq \sqrt{4/3} c_1^{\text{Grad}} \left(\frac{\log(p)}{n}\right)^{s}$, where $c_1^{\text{Grad}}$ is as in Lemma 4. Then for $s \leq \log(p)$ and $n \geq \max\{\gamma_{\text{RSC}} s \log(p), 6 \log(p)\}$ we still have (36) with probability at least $1 - c_0^s \exp(-c_2^s n) - 6/p^2$.

The proof is provided in the supplement, Section 7.3.
5.3 A general result for the weighted LASSO Huber estimator

In the next lemma we consider support recovery and bounds for a generic form of the weighted LASSO Huber estimator. This is similar to Zhou et al. (2009, Lemma 8.2). For clarity of formulation we shall impose (36), and (35) in the second part, as high-level conditions. These are taken care of in the preceding lemmas.

Lemma 10. Consider model (1) under Assumption 1. Suppose that (36) holds true, and that the weights satisfy the mutual incoherence condition, that is for some $\eta \in (0, 1)$ we have that

$$\left| \hat{Q}_{SS}^{-1} (\hat{Q}_{SS})^{-1} (w_S \odot \hat{\gamma}_S) \right| \leq w_S (1 - \eta).$$

For the regularization parameter $\lambda_n$, assume that

$$w_{\min} (S^c) \lambda_n > \frac{4 C_{Q, L} c_{Grad}^2}{\eta} \left( \frac{\log (p)}{n} \right)^{\frac{1}{2}}.$$

Furthermore, suppose that the robustification parameter $\alpha_n$ is chosen in the range

$$c_{\text{Grad}} \left( \frac{\log (p)}{n} \right)^{\frac{1}{2}} \leq \alpha_n \leq \left( \frac{c_{\text{Grad}}^2}{80 C_{\text{apx}} c_{\text{Grad}}^2} \left( \frac{\log (p)}{n} \right)^{\frac{1}{2}} \right)^{-1}.$$

and

$$\beta^*_n > \phi_{n, \infty, s}, \quad \text{where} \quad \phi_{n, \infty, s} = \frac{128}{CX} \max \left\{ c_{\text{Grad}}^2 \left( \frac{s \log (p)}{n} \right)^{\frac{1}{2}}, w_{\max} (S) \lambda_n \sqrt{s} \right\}.$$

Then for $n \geq \max \{ c_{\text{RSC}}^3 \log (p), 6 \log (p) \}$ with probability at least

$$1 - c_p^p \exp (-c_p^p n) - 2 \exp (-2n) - \frac{4}{p^2},$$

the weighted LASSO Huber estimator, as a solution to the program (3), is unique and given by $\hat{\beta}_n^{\text{WLH}} = \hat{\beta}_n^{\text{PDW}}$ and satisfies

$$\text{sign}(\hat{\beta}_n^{\text{WLH}}) = \text{sign}(\beta^*) \quad \text{and} \quad \| \hat{\beta}_n^{\text{WLH}} - \beta^* \|_\infty \leq \phi_{n, \infty, s}.$$

with $\phi_{n, \infty, s}$ in (40).

If in addition (35) is satisfied, we may replace $\phi_{n, \infty, s}$ in the beta-min condition (40) and in the $\ell_\infty$ bound in (42) by

$$\phi_{n, \infty, f} = 4 C_{Q, S} \max \left\{ c_{\text{Grad}}^2 \left( \frac{\log (p)}{n} \right)^{\frac{1}{2}}, w_{\max} (S) \lambda_n \right\}.$$

Proof of Lemma 10. We shall apply Lemma 7. Using the mutual incoherence condition (37), in order to show strict dual feasibility as in (30) it suffices to prove that

$$\left\| \hat{Q}_{SS}^{-1} (\hat{Q}_{SS})^{-1} \left( \nabla \mathcal{L}_{n, \alpha_n}^H (\beta^*_n) \right) \right\|_{\infty}$$

$$+ \left\| \left( \hat{Q}_{SS} (S_n \setminus S) \right) \beta_{\alpha_n, S_n \setminus S} \right\|_{\infty} < \frac{w_{\min} (S^c) \eta}{2} \lambda_n.$$
The first term is bounded by (36) (which is satisfied by assumption). We prove in the supplement, Section 7.4 that
\[
\left\| \left( \tilde{Q}^{S \setminus (S_n \setminus S)} - \tilde{Q}^{S \setminus S} \right)^{-1} \tilde{Q}^{S(S_n \setminus S)} \right\|_{\infty} \leq 80 \, c_{\text{apx}}^2 \, c_{\text{sub}} \, \alpha_n^{m-1}
\]
with probability at least \(1 - 2 \exp(-2n) - 2/p^2\). Then the choices of \(\lambda_n\) and \(\alpha_n\) in (38) and in (39) imply (44). Since the first beta-min condition in Lemma 7 is also satisfied in both cases by choice of \(\alpha_n\) in (39), the first part of that lemma up to (32) applies. Here we assumed that \(\sqrt{s} \geq c_{X,1}/(2560 \, c_{\text{sub}}^2)\) for (40) and \(320 \, c_{\text{sub}}^2 \, C_{Q,S} \geq 1\) for (43), which can be arranged by choosing the constants appropriately. Now we show that \(\phi_{n,\infty}\) in (32) is bounded by \(\phi_{n,\infty,s}\) and, under the additional condition (35) is even bounded by \(\phi_{n,\infty,f}\). Then (40) (or the analogous condition with \(\phi_{n,\infty,f}\)) implies the beta-min condition (33) in Lemma 7, which concludes the proof.

To this end, note that \(\phi_{n,\infty}\) is bounded by four times the maximum of the summands in (32). In addition (25) leads to
\[
4 \, w_{\max}(S) \, \lambda_n \left\| (\tilde{Q}^{SS})^{-1} \right\|_{M,\infty} \leq \frac{128 \, w_{\max}(S) \, \lambda_n \, \sqrt{s}}{c_{X,1}}
\]
and together with Lemma 4 and the lower bound of \(\alpha_n\) in (39) leads to
\[
4 \left\| (\tilde{Q}^{SS})^{-1} \right\|_{M,\infty} \left\| \left( \nabla L_{n,\alpha_n}^H (\beta^{*}_{\alpha_n}) \right)_S \right\|_{\infty} \leq \frac{128 \, c_{\text{Grad}}^2 \, c_{X,1}}{c_{\text{sub}}} \left( \frac{s \log(p)}{n} \right)^{1/2}
\]
with probability at least \(1 - c_{v}^2 \exp(-c_{v}^2 n) - 2/p^2\). Further, Lemma 1 implies
\[
4 \left\| \beta_{\alpha_n,S} - \beta^{*}_S \right\|_{\infty} \leq 4 \left\| \beta^{*}_{\alpha_n} - \beta^{*}_S \right\|_{2} \leq 4 \, C_{\text{apx}} \, \alpha_n^{m-1} \leq \frac{128 \, c_{\text{Grad}}^2 \, c_{X,1}}{c_{\text{sub}}} \left( \frac{s \log(p)}{n} \right)^{1/2}
\]
with the choice of \(\alpha_n\) in (39). Finally, in the supplement, Section 7.4, we also show that
\[
\left\| \tilde{Q}^{S(S_n \setminus S)} \beta^{*}_{\alpha_n,S_n \setminus S} \right\|_{\infty} \leq 80 \, c_{\text{apx}}^2 \, c_{\text{sub}} \, \alpha_n^{m-1}
\]
with high probability. Together with (25) these imply
\[
4 \left\| (\tilde{Q}^{SS})^{-1} \right\|_{M,\infty} \left\| \tilde{Q}^{S(S_n \setminus S)} \beta^{*}_{\alpha_n,S_n \setminus S} \right\|_{\infty} \leq \frac{10240 \, c_{\text{apx}}^2 \, c_{\text{sub}}^2 \, \sqrt{s} \, \alpha_n^{m-1}}{c_{X,1}} \leq \frac{128 \, c_{\text{Grad}}^2 \, c_{X,1}}{c_{\text{sub}}} \left( \frac{s \log(p)}{n} \right)^{1/2}
\]
by the choice of \(\alpha_n\), which concludes the proof of \(\phi_{n,\infty} \leq \phi_{n,\infty,s}\). To show \(\phi_{n,\infty} \leq \phi_{n,\infty,f}\) under the assumption (35), after arranging \(80 \, c_{\text{sub}}^2 \, C_{Q,S} \geq 1\) we proceed analogously (and use the estimate (35) instead of (25) in the previous inequalities). This concludes the proof of the lemma.

\[\square\]

5.4 The adaptive LASSO with generic first-stage estimator

The next step is to provide a result on the estimation error of the adaptive LASSO Huber estimator \(\hat{\beta}_{n,\text{ALH}}^{\text{init}}\) in (3) with weights in (6) for a generic initial estimator, similar to Zhou et al. (2009, Theorem 4.3)

**Lemma 11.** Consider model (1) under Assumption 1. Suppose that (36) holds and that \(\alpha_n\) is chosen according to (39). For the estimation error
\[
\Delta_n := \hat{\beta}_{n,\text{ALH}}^{\text{init}} - \beta^{
\]

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of the initial estimator $\hat{\beta}_{n,0}^{\text{init}}$, assume that we have upper bounds of the form

$$\|\Delta_{n,S}\|_\infty \leq a_n < 1, \quad \|\Delta_{n,S'}\|_\infty \leq b_n < 1,$$

(48)

with sequences $(a_n)$ and $(b_n)$ tending to zero. Furthermore, assume that for some $\eta \in (0,1)$ and $C_\lambda > 4/\eta$ the regularization parameter is chosen from the range

$$\frac{4 C_{Q,C} e^{\text{Grad} b_n}}{\eta} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} < \lambda_n \leq C_\lambda C_{Q,C} e^{\text{Grad} b_n} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}},$$

(49)

and in addition suppose that there is a sequence $q_n \leq (1 - \eta)/b_n$, which may grow if $b_n \downarrow 0$, such that

$$\left\| \hat{Q}_{S'\bar{S}} (\hat{Q}_{\bar{S}S})^{-1} \right\|_{M,\infty} \leq q_n.$$  

(50)

Finally, setting

$$\phi_{n,\infty,s,1} = \frac{128}{c_{X,1}} \max \left\{ C_\text{Grad} \left( \frac{s \log(p)}{n} \right)^{\frac{1}{2}}, \frac{\lambda_n}{\sqrt{n}} \right\},$$

suppose that the beta-min assumption

$$\beta_{\text{min}}^* > \max \left\{ 2 a_n, \phi_{n,\infty,s,1}, 2 \max \left\{ \frac{q_n}{1 - \eta}, C_\lambda C_{Q,C} b_n \right\} \right\}$$

(51)

is satisfied. Then for $n \geq \max \left\{ c_3^{\text{SC}} s \log(p), 6 \log(p) \right\}$ with probability at least equal to (41), the adaptive LASSO Huber estimator, given as a solution to the program (3) with weights in (6), is unique and given by $\hat{\beta}_{n,\text{ALH}} = \hat{\beta}_{n,\text{PDW}}$ and satisfies

$$\text{sign}(\hat{\beta}_{n,\text{ALH}}) = \text{sign}(\beta^*) \quad \text{and} \quad \left\| \hat{\beta}_{n,\text{ALH}} - \beta^* \right\|_\infty \leq \phi_{n,\infty,s,1}.$$  

(52)

If in addition (35) is also assumed, we can replace $\phi_{n,\infty,s,1}$ in the beta-min condition (51) and in the upper bound of the $\ell_\infty$ distance of the estimation error by

$$\phi_{n,\infty,f,1} = 4 C_{Q,S} \max \left\{ c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}}, \lambda_n \right\}.$$  

(53)

**Proof of Lemma 11.** We shall apply Lemma 10. To this end, we start by checking the mutual incoherence condition (37) and the condition (38) on the regularization parameter $\lambda_n$. For (38), since $|\hat{\beta}_{n,k}^{\text{init}}| = |\beta_k^* + \Delta_{n,k}| = |\Delta_{n,k}| \leq \|\Delta_{n,S'}\|_\infty$ for $k \in S^c$, from (48) we obtain that

$$w_{\text{min}}(S^c) = \min_{k \in S^c} \left\{ \max \left\{ \left\| \hat{\beta}_{n,k}^{\text{init}} \right\|^{-1}, 1 \right\} \right\} \geq \left\| \Delta_{n,S'} \right\|^{-1}_\infty \geq \frac{1}{b_n},$$

(54)

and hence $w_{\text{min}}(S^c) \lambda_n \geq \lambda_n/b_n$, which together with the assumption (49) on $\lambda_n$ gives (38). Next, we turn to the mutual incoherence condition (37), for which it suffices to prove

$$\left\| \hat{Q}_{S'\bar{S}} (\hat{Q}_{\bar{S}S})^{-1} \right\|_{M,\infty} \leq \frac{w_{\text{min}}(S^c)}{w_{\text{max}}(S)} (1 - \eta).$$  

(55)

From the beta-min condition (51) and the bounds in (48) we have in particular that $\beta_{\text{min}}^*/2 > a_n \geq \|\Delta_{n,S}\|_\infty \geq |\Delta_{n,k}|$ and hence that

$$\left| \hat{\beta}_{n,k}^{\text{init}} \right| = |\beta_k^* + \Delta_{n,k}| \geq |\beta_k^*| - |\Delta_{n,k}| > \beta_{\text{min}}^* - \frac{\beta_{\text{min}}}{2} = \frac{\beta_{\text{min}}^*}{2}.$$
for \( k \in S \). This together with the definition of the weights implies

\[
\frac{w_{\text{max}}(S)}{c_{\chi,1}} \leq 128 \\sup_{k \in S} \left\{ \max \left\{ \left( \frac{\hat{\beta}_{n,k}^{\text{init}}}{\lambda_n} \right)^{-1} \right\} \right\} \leq \max \{ 2/\beta_{\text{min}}^*, 1 \}.
\]

(56)

In order to conclude (55) we now consider two cases. If \( \beta_{\text{min}}^* \leq 2 \), then we have \( w_{\text{max}}(S) \leq 2/\beta_{\text{min}}^* \) because of (56) and hence with (54) and the last term in the beta-min condition (51) we obtain

\[
\frac{w_{\text{min}}(S)}{w_{\text{max}}(S)}(1 - \eta) > \frac{\beta_{\text{min}}^*(1 - \eta)}{2b_n} > \frac{\eta}{q_n} \geq \| \hat{Q}_{S'}S(\hat{Q}_{S'}S)^{-1} \|_{M,\infty}
\]

by (50). If \( \beta_{\text{min}}^* > 2 \), then \( w_{\text{max}}(S) \leq 1 \) and by (50), (54) and the choice of \( q_n \) it follows that

\[
\frac{w_{\text{min}}(S)}{w_{\text{max}}(S)}(1 - \eta) \geq \frac{1 - \eta}{b_n} \geq q_n = \| \hat{Q}_{S'}S(\hat{Q}_{S'}S)^{-1} \|_{M,\infty},
\]

so that (55) is satisfied in both cases.

Next we show that \( \phi_{n,\infty,s} \leq \phi_{n,\infty,s,1} \), then the beta-min condition (51) directly implies (40). Comparing \( \phi_{n,\infty,s,1} \) and \( \phi_{n,\infty,s} \), it remains to show that

\[
\frac{128 w_{\text{max}}(S) \lambda_n \sqrt{s}}{c_{\chi,1}} \leq \frac{128}{c_{\chi,1}} \max \left\{ \frac{c_2 \text{Grad} \left( \frac{s \log(p)}{n} \right)}{n} \lambda_n \sqrt{s} \right\}.
\]

(57)

To this end, note that the last lower bound in the inequality (51) implies

\[
\frac{128 c_{2\text{Grad}} \left( \frac{s \log(p)}{n} \right)}{c_{\chi,1}} \lambda_n \sqrt{s} > \frac{256 \lambda_n \sqrt{s}}{\beta_{\text{min}}^*}
\]

by the choice of the regularization parameter \( \lambda_n \) in (49). This together with (56) implies (57). So Lemma 10 applies and we conclude that the \( \ell_{\infty} \) bound in (42) can be reduced to (52).

For the sharper bound \( \phi_{n,\infty,f} \leq \phi_{n,\infty,f,1} \), under assumption (35) one argues similarly. This concludes the proof.

\[\square\]

5.5 The adaptive LASSO with the LASSO in the first stage

We start with the following lemma, which is analogous to Zhou et al. (2009, Lemma 4.2) and which gives a superset \( \overline{S} \) of the support \( S \), the cardinality of which is of the same order \( s \). This is used to determine the order of regularization in the adaptive LASSO Huber estimator in the theorem to follow.

**Lemma 12** (Thresholding procedure). If the initial estimator \( \hat{\beta}_n^{\text{init}} \) satisfies (8), and if the following beta-min condition

\[
\beta_{\text{min}}^* > 2 C_{\text{init}} \lambda_n^{\text{init}} \sqrt{s}
\]

(58)

holds, then the set \( \overline{S} = \{ k \in \{1, \ldots, p\} \mid \left| \hat{\beta}_{n,k}^{\text{init}} \right| > \lambda_n^{\text{init}} \} \) satisfies

\[
S \subseteq \overline{S} \quad \text{and} \quad s \leq |\overline{S}| \leq 2 C_{\text{init}} s.
\]

(59)

**Proof of Lemma 12.**
Let \( \Delta_n = \hat{\beta}_n^{\text{init}} - \beta^* \). Then from (8) it follows that

\[
\| \Delta_{n,S} \|_{\infty} \leq \| \Delta_n \|_{\infty} \leq \| \Delta_n \|_2 \leq C_{\text{init}} \lambda_n^{\text{init}} \sqrt{s}
\]

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and hence for all $k \in S$ that
\[
|\hat{\beta}_{n,k}^{\text{init}}| = |\beta^*_k + \Delta_{n,k}| \geq |\beta^*_k| - |\Delta_{n,k}| \geq \beta^{*\text{min}} - \|\Delta_{n,S}\|_\infty > C_{n} \lambda_n^{\text{init}} \sqrt{s}
\]
because of inequality (58). In consequence, the definition of the set $\overline{S}$ implies the membership $S \subseteq \overline{S}$. Furthermore, for $k \in S^c$ (since $\beta^*_k = 0$) it is
\[
|\hat{\beta}_{n,k}^{\text{init}}| = |\beta^*_k + \Delta_{n,k}| = |\Delta_{n,k}|
\]
and the upper bound of the $\ell_1$ norm of the estimation error in (8) leads to
\[
\|\hat{\beta}_{n,S^c}^{\text{init}}\|_1 \leq \|\Delta_{n,S^c}\|_1 \leq C_{n} \lambda_n^{\text{init}} s.
\]
Hence we include at most $C_{n} \lambda_n^{\text{init}} s$ more entries from $S^c$ in $\overline{S}$, thus
\[
s \leq |\overline{S}| \leq s + C_{n} \lambda_n^{\text{init}} s \leq 2 C_{n} \lambda_n^{\text{init}} s,
\]
which completes the proof. \hfill \square

**Lemma 13.** Suppose Assumption 1 and $n \geq \max \{c^{\text{RSC}} \log(p), 6 \log(p)\}$ hold. Then
\[
\max_{k \in \{1, \ldots, p-s\}} \left\| \left( e^T_{k} \hat{Q}_{S^cS} \hat{Q}_{SS}^{-1} \right)^T \right\|_2 \leq \frac{33 c_{X,\text{sub}}}{\sqrt{\lambda_n^{\text{init}}}} \log(p) \quad \text{and} \quad \|\hat{Q}_{S^cS} \hat{Q}_{SS}^{-1}\|_{L,\infty} \leq \frac{33 c_{X,\text{sub}} \sqrt{s}}{\sqrt{\lambda_n^{\text{init}}}} \quad (60)
\]
with probability at least $1 - c^P \exp(-c^P n) - 2/p^2$.

The technical proof of this lemma is deferred to Section 7.3.

For clarity of formulation in the following result we shall again impose (36), and (35) in the second part, as high-level conditions. Theorem 1 then follows from the following Lemma 14 together with Lemmas 8 and 9.

**Lemma 14** (Adaptive LASSO Huber with LASSO in the first stage). Consider model (1) under Assumption 1. Suppose that (36) holds true, and that the robustification parameter $\alpha_n$ is chosen according to (39). Suppose that the initial estimator $\hat{\beta}_{n}^{\text{init}}$ satisfies (8) with $\lambda_n^{\text{init}} = C_{n} \lambda_n^{\text{init}}$, for some constant $C_{\lambda,\text{init}} \geq 16 c^{\text{Grad}} / c_{X,1}$, and that for suitable $\eta \in (0, 1)$ and $C_{\lambda,1} > 2 (2 C_{n})^{1/2} / \eta$ the regularization parameter is chosen from the range
\[
\frac{4 C_{Q,1} c^{\text{Grad}}_{\lambda,1} \lambda_n^{\text{init}}}{\eta} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} < \lambda_n \leq C_{\lambda,1} C_{Q,1} c_{\text{grad}}^{\text{Grad}} \lambda_n^{\text{init}} \left( \frac{4 \log(p)}{2 n} \right)^{\frac{1}{2}} \quad (61)
\]
with $\overline{S} = \{k \in \{1, \ldots, p\} \mid |\hat{\beta}_{n,k}^{\text{init}}| > \lambda_n^{\text{init}}\}$ as above. In addition suppose that the sample size satisfies
\[
n \geq \max \left\{ \left( \frac{33 c_{X,\text{sub}} C_{n} \lambda_n^{\text{init}}}{(1 - \eta) \sqrt{\lambda_n^{\text{init}}}} \right)^2 s^2 \log(p), \max \left\{ c^R_{3}, \left( \frac{64 c^{\text{Grad}} / c_{X,1}}{2} \right)^2 s \log(p), 6 \log(p) \right\} \right\}, \quad (62)
\]
and that we have the beta-min condition
\[
\beta^{*\text{min}} > 2 \max \left\{ \frac{33 c_{X,\text{sub}} \sqrt{s}}{\sqrt{\lambda_n^{\text{init}}}} \lambda_n \sqrt{s}, C_{\lambda,1} c_{Q,1} C_{n} \right\} \lambda_n^{\text{init}} \sqrt{s} \cdot \quad (63)
\]
Then with probability at least
\[
1 - c^P \exp(-c^P n) - 2 \exp(-2n) - \frac{4}{p^2}
\]
the adaptive LASSO Huber estimator, given as a solution to the program (3) with weights in (6), is unique and given by $\hat{\beta}_{n}^{ALH} = \hat{\beta}_{PDW}^{P}$ and satisfies

$$\text{sign}(\hat{\beta}_{n}^{ALH}) = \text{sign}(\beta^{*})$$

and

$$\left\| \hat{\beta}_{n}^{ALH} - \beta^{*} \right\|_{\infty} \leq 2 C_{\lambda,L} C_{Q,L} C_{\text{init}} \lambda^{\text{init}}_{n} \sqrt{s}. \quad (64)$$

If in addition (35) is also assumed, the upper bound of the $\ell_{\infty}$ distance of the estimation error reduces to

$$\left\| \hat{\beta}_{n}^{ALH} - \beta^{*} \right\|_{\infty} \leq \max \left\{ \frac{4 C_{Q,L} C_{2}^{\text{grad}}}{C_{\lambda,\text{init}}} \left( \frac{\sqrt{s} \log(p)}{n} \right)^{\frac{1}{2}}, \frac{C_{\lambda,L} C_{Q,L} C_{\text{init}}}{16} \frac{c}{C_{X,1}} \right\} \lambda^{\text{init}}_{n}. \quad (65)$$

Proof of Lemma 14. We shall apply Lemma 11. To check the assumptions, for (48) using (8) we get $\|\Delta_{n}\|_{\infty} \leq \|\Delta_{n}\|_{2} \leq C_{\text{init}} \lambda^{\text{init}}_{n} \sqrt{s} = a_{n} = b_{n}$. For the lower bound in (49), using (61), Lemma 12 and the choice of $b_{n}$ we estimate

$$\lambda_{n} > \frac{4 C_{Q,L} C_{2}^{\text{grad}} C_{\text{init}}}{\eta} \lambda^{\text{init}}_{n} \left( \frac{\sqrt{s} \log(p)}{n} \right)^{\frac{1}{2}} \geq \frac{4 C_{Q,L} C_{2}^{\text{grad}} C_{\text{init}}}{\eta} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \lambda^{\text{init}}_{n},$$

and similarly for the upper bound

$$\lambda_{n} \leq C_{\lambda,L} C_{Q,L} C_{2}^{\text{grad}} C_{\text{init}} \lambda^{\text{init}}_{n} \left( \frac{s \log(p)}{n} \right)^{\frac{1}{2}} \leq C_{\lambda,L} C_{Q,L} C_{2}^{\text{grad}} b_{n} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}},$$

with $C_{\lambda,L} > 4/\eta$. Next, (50) follows from Lemma 13 with $q_{n} = 33 c_{X,\text{sub}} \sqrt{s}/\sqrt{\xi_{1}}$ with high probability. In addition the choice of $b_{n}$ and the lower bound (62) of the sample size implies

$$q_{n} \leq \frac{33 c_{X,\text{sub}}}{33 c_{X,\text{sub}} C_{\text{init}} C_{\lambda,\text{init}} (s \log(p)/n)^{\frac{1}{2}}} = \frac{(1 - \eta) \sqrt{\xi_{1}}}{b_{n}} = 1 - \frac{\eta}{b_{n}}.$$

So finally we have to check the beta-min condition in (51), which concludes the proof of the lemma in this setting. The last term in the maximum is given by (63) and the choice of $b_{n}$ and $q_{n}$, and $\beta^{\text{min}}_{n} \geq 2 a_{n}$ is clear because of the choice of $a_{n}$ and (63). Hence for applying Lemma 11 it remains to show that

$$\phi_{n,\infty,s,1} \leq 2 C_{\lambda,L} C_{Q,L} C_{\text{init}} \lambda_{n}^{\text{init}} \sqrt{s}.$$ 

This bound implies then also (64) because of (52). It is

$$\frac{128 C_{2}^{\text{grad}}}{c_{X,1}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \leq \frac{128 C_{2}^{\text{grad}}}{c_{X,1} C_{\lambda,\text{init}}} \lambda^{\text{init}}_{n} \sqrt{s} \leq \frac{8 C_{Q,L} C_{\text{init}} \sqrt{s}}{\lambda^{\text{init}}_{n}} \leq 2 C_{\lambda,L} C_{Q,L} C_{\text{init}} \lambda_{n}^{\text{init}} \sqrt{s},$$

since $16 C_{2}^{\text{grad}} \leq c_{X,1} C_{\lambda,\text{init}} C_{Q,L} C_{\text{init}}$. Moreover, (61) and (62) together with Lemma 12 lead to

$$\frac{128 C_{Q,L} C_{2}^{\text{grad}}}{c_{X,1}} \left( \frac{\sqrt{s} \log(p)}{2 n} \right)^{\frac{1}{2}} \leq \frac{128 C_{Q,L} C_{2}^{\text{grad}}}{c_{X,1}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \leq 2 C_{\lambda,L} C_{Q,L} C_{\text{init}} \lambda_{n}^{\text{init}} \sqrt{s}.$$ 

Under the stronger assumption (35) we show

$$\phi_{n,\infty,f,1} \leq \max \left\{ \frac{4 C_{Q,L} C_{2}^{\text{grad}}}{C_{\lambda,\text{init}}}, \frac{C_{\lambda,L} C_{Q,L} C_{\text{init}}}{16} \right\} \lambda^{\text{init}}_{n},$$

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which implies (65) because of (53). Note that the upper bound is obviously also smaller than the right term in (63). It is easy to see that

\[ 4 C_{\mathcal{Q}, \mathcal{S}} \frac{\mu}{C_{\lambda, \text{init}}} \frac{\log(p)}{n} \leq 4 C_{\mathcal{Q}, \mathcal{S}} \frac{\mu}{C_{\lambda, \text{init}}} \frac{\log(p)}{n} \]

and

\[ 4 C_{\mathcal{Q}, \mathcal{S}} \lambda_n \leq 4 C_{\lambda, L} C_{\mathcal{Q}, \mathcal{S}} C_{\mathcal{Q}, \mathcal{L}} C_{\text{init}} \frac{\log(p)}{n} \leq 4 C_{\lambda, L} C_{\mathcal{Q}, \mathcal{S}} C_{\mathcal{Q}, \mathcal{L}} C_{\text{init}} \frac{\log(p)}{n} \]

by (61), (62) and Lemma 12, which concludes the proof.

6 Conclusions

In their recent paper, Sun et al. (2020) extended the analysis from Fan et al. (2017) to fixed designs, as well as to conditional moments of \( \varepsilon_1 \) of order strictly smaller than 2, in which case they showed that the rates of convergence deteriorate. Results on support estimation, rates of convergence in the \( \ell_\infty \) norm together with a data-driven choice of the robustification parameter would be of some interest in this setting as well. Another possible extension or modification of our method would be the use of nonconvex penalty functions such as SCAD as in Loh and Wainwright (2017), with the methodological aim to achieve milder beta-min conditions.

The paper was partially motivated by the problem of selecting the random or correlated coefficients in a linear random coefficient regression model

\[ Y_j = X_j^\top \beta_j + \varepsilon_j, \quad j = 1, \ldots, n, \quad (66) \]

where \( \beta_j \) are also independent and identically distributed random vectors. Versions of this model have been studied quite intensely - mainly in a nonparametric framework - in the recent econometrics literature (Hoderlein et al., 2010; Dunker et al., 2019).

Writing \( \bar{X}_j = (1, X_j^\top) \) and \( \theta_j = (\varepsilon_j, \beta_j^\top) \) we may consider the heteroscedastic regression model

\[ Y_j = \bar{X}_j^\top \mathbb{E}[\theta_j] + \bar{X}_j^\top (\theta_j - \mathbb{E}[\theta_j]) \]

as model for the first moments of the \( \beta_j \) and \( \varepsilon_j \). A similar, but more complicated heteroscedastic mean regression model - involving products and squares of entries of \( \bar{X}_j \) - can be designed for the entries of the covariance matrix of the \( \theta_j \), where the response \( (Y_j - \mathbb{E}[Y_j])^2 \) also involves the estimation error from the first stage mean regression. Support estimation then selects those coefficients with non-zero variances as well as the correlated pairs of coefficients.

Another extension of some interest would be to robustify asymmetric versions of least squares regression (Newey and Powell, 1987; Gu and Zou, 2016), that is, high-dimensional expectile regression.

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7 Supplement: Further technical proofs

At first we introduce further notations. For a random variable \( Y \in \mathbb{R} \) we write \( Y \sim \text{subG}(\tau) \) with \( \tau > 0 \) if \( P(|Y| \geq t) \leq 2 \exp \left( -\frac{t^2}{2\tau^2} \right) \) for all \( t \geq 0 \), and for a random vector \( \mathbf{Y} \in \mathbb{R}^d \) we write \( \mathbf{Y} \sim \text{subG}_d(\tau) \) if \( P(|v^\top \mathbf{Y}| \geq t) \leq 2 \exp \left( -\frac{t^2}{2\tau^2\|v\|^2} \right) \) for all \( v \in \mathbb{R}^d \setminus \{0_d\} \) and \( t > 0 \). In addition a random variable \( Y \sim \text{subE}(\tau, b) \) is called sub-Exponential with \( \tau, b > 0 \) if \( E[Y] = 0 \) and \( E[\exp(tY)] \leq \exp(\frac{t^2\tau^2}{2}) \) for all \( |t| < 1/b \). Furthermore, we denote by \( \tilde{X}_1, \ldots, \tilde{X}_p \in \mathbb{R}^n \) the columns of \( \mathbf{X}_n \) and the rows are \( \mathbf{X}_1 = (X_{i,1}, \ldots, X_{i,p})^\top \). Finally, \( e_k \) is the \( k \)th unit vector, with \( k \)th coordinate equal to 1, and zero entries otherwise. The dimension of \( e_k \) will depend on and be clear from the context.

7.1 Proofs for Section 5.1

Proof of Lemma 1. Let \( l(x) = x^2 \), then by (ii) of Assumption 1 we get

\[
E \left[ (Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) - l(Y_1 - \mathbf{X}_1^\top \beta^*) \right] = (\beta_{a_n}^* - \beta^*)^\top E \left[ \mathbf{X}_1 \mathbf{X}_1^\top \right] (\beta_{a_n}^* - \beta^*) \\
\geq c_1 \| \beta_{a_n}^* - \beta^* \|^2_2.
\]  

(67)

Let \( g_{a_n}(x) = l(x) - l_{a_n}(x) = x^2 - 2a_n^{-2} \left( \frac{x}{2} + \frac{a_n^2}{2} \right), \) then

\[
E \left[ l(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) - l(Y_1 - \mathbf{X}_1^\top \beta^*) \right] = E \left[ l(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) - l_{a_n}(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) \right] \sum_{k=1}^{p} E \left[ g_{a_n}(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) - g_{a_n}(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) \right]
\]

(68)

because \( \beta_{a_n}^* \) minimizes \( E[l_{a_n}(Y_1 - \mathbf{X}_1^\top \beta)] \) over \( \| \beta \|_2 \leq C_\beta \) and \( \| \beta^* \|_2 \leq C_\beta \) by (iv) of Assumption 1. Furthermore, the mean value theorem implies

\[
E \left[ g_{a_n}(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) - g_{a_n}(Y_1 - \mathbf{X}_1^\top \beta_{a_n}^*) \right] = E \left[ g'_{a_n}(Z)(\mathbf{X}_1^\top(\beta^* - \beta_{a_n}^*)) \right] \\
\leq E \left[ g'_{a_n}(Z) \| \mathbf{X}_1^\top(\beta^* - \beta_{a_n}^*) \| \|Z\| \geq a_n^{-1} \right] + E \left[ g'_{a_n}(Z) \| \mathbf{X}_1^\top(\beta^* - \beta_{a_n}^*) \| \{ |Z| < a_n^{-1} \} \right]
\]

(69)

with \( Z = Y_1 - \mathbf{X}_1^\top \tilde{\beta} \) and \( \tilde{\beta} \) between \( \beta^* \) and \( \beta_{a_n}^* \). Note that \( \tilde{\beta} \) is also a random vector. For the first summand we obtain from (14) that

\[
E \left[ g'_{a_n}(Z) \| \mathbf{X}_1^\top(\beta^* - \beta_{a_n}^*) \| \|Z\| \geq a_n^{-1} \right] \\
\leq 2 E \left[ |Z| \left( 1 - \frac{1}{\sqrt{1 + a_n^2 Z^2}} \right) \| \mathbf{X}_1^\top(\beta^* - \beta_{a_n}^*) \| \{ |Z| \geq a_n^{-1} \} \right]
\]

Let \( P(x) \) be distribution of \( e_1 \) conditional on \( \mathbf{X}_1 \) and \( E_c \) the corresponding conditional expectation. Then we get the inequality

\[
E_c \left[ |Z| \left( 1 - \frac{1}{\alpha_n \epsilon \sqrt{Z^2 + 1}} \right) \|Z\| \geq a_n^{-1} \right] \leq E_c \left[ |Z| \|Z\| \geq a_n^{-1} \right] \\
= \int_0^{\infty} \|P_c[|Z| \geq a_n^{-1}, |Z| > t] \right) dt
\]

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Moreover, for the first term in the brackets we obtain by Hölder’s inequality and (i) of Assumption 1 implies that

The mean on the right hand side can be upper bounded by

Now we analyze the second term in (69). Taking the derivative in the series expansion

because

where \( m \in \{2, 3\} \) is given in Assumption 1, and in consequence

Now we analyze the second term in (69). Taking the derivative in the series expansion

implies that

and hence that

because \( \alpha_n |Z| < 1 \). Moreover, it is

and in consequence

So in total we obtain by (67) - (72) the inequality

The mean on the right hand side can be upper bounded by

Moreover, for the first term in the brackets we obtain by Hölder’s inequality and (i) of Assumption 1

\[
E \left[ \| \beta^{*} - \beta^{*} \|_2^2 \leq \frac{5}{C_{\alpha} \mathcal{I}} E \left[ \| Z \|^m \| X_1^T (\beta^{*} - \beta^{*}_n) \| X_1 \right] \alpha_n^{-1} \right].
\]
In addition note that $X_i^T(\beta^* - \beta_{\alpha}^*) \sim \text{subG}(c_{\text{sub}} \| \beta^* - \beta_{\alpha}^* \|_2)$ by (iii) of Assumption 1, and that the moments of a sub-Gaussian random variable $Q \sim \text{subG}(\tau)$ with $\tau > 0$ are bounded by

$$E[|Q|^r] \leq (2\tau^r)^{\frac{r}{r-1}} \Gamma\left(\frac{r}{2}\right), \quad E[|Q|^r]^{\frac{r}{2}} \leq \sqrt{2} \left(\frac{r}{\Gamma\left(\frac{r}{2}\right)}\right)^{\frac{r}{2}} \tau$$

(75)

for $r > 1$. This can be proven analog to Rigollet and Hütter (2019, Lemma 1.4). Hence

$$E[|\varepsilon_i^{\text{sub}}|^m |X_i^T(\beta^* - \beta_{\alpha}^*)|] \leq \sqrt{2} (C_{\text{m}, 1})^{\frac{r}{2}} \left(\frac{q}{q - 1} \Gamma\left(\frac{q}{2(q - 1)}\right)\right)^{\frac{r}{2}} c_{\text{sub}} \| \beta^* - \beta_{\alpha}^* \|_2.$$ 

(76)

For the second term in the brackets in (74) the Cauchy-Schwarz inequality implies

$$E\left[|X_i^T(\beta^* - \tilde{\beta})|^m |X_i^T(\beta^* - \beta_{\alpha}^*)|\right] \leq \left(E\left[|X_i^T(\beta^* - \tilde{\beta})|^{2m}\right] E\left[|X_i^T(\beta^* - \beta_{\alpha}^*)|^2\right]\right)^{\frac{1}{2}} \leq 2 E\left[|X_i^T(\beta^* - \tilde{\beta})|^{2m}\right]^{\frac{1}{2}} c_{\text{sub}} \| \beta^* - \beta_{\alpha}^* \|_2.$$ 

(77)

To give a upper bound for the remaining expected value we consider at first a tail bound for the appropriate random variable. Let $L$ be the line between $\beta^*$ and $\beta_{\alpha}^*$. Moreover, $X_i^T \beta^*$ and $X_i^T \beta_{\alpha}$ are sub-Gaussian with variance proxy $C_{\beta}^2 c_{\text{sub}}^2$ by (iii) and (iv) of Assumption 1 and $\| \beta_{\alpha}^* \|_2 \leq C_{\beta}$ by (5). Hence Rigollet and Hütter (2019, Theorem 1.16) leads to

$$\mathbb{P}\left(\| \beta^* - \tilde{\beta} \|^T X_1 > x\right) \leq \mathbb{P}\left(\max_{u \in L} |u^T X_1| > x\right) \leq 4 \exp \left(-\frac{x^2}{2 C_{\beta}^2 c_{\text{sub}}^2}\right).$$

In addition Rigollet and Hütter (2019, Lemma 1.4) and the corresponding proof imply

$$E\left[|X_i^T(\beta^* - \tilde{\beta})|^{2m}\right] \leq 2 \left(2 C_{\beta}^2 c_{\text{sub}}^2\right)^m (2m)! \Gamma(m).$$ 

(78)

In total (73) - (78) leads to

$$\| \beta_{\alpha}^* - \beta^* \|_2 \leq C_{\text{apx}} \alpha_{\alpha}^{m-1}$$

with

$$C_{\text{apx}} = \frac{52^m c_{\text{sub}}}{c_{\text{m}, 1}} \left((C_{\text{m}, 1})^{\frac{r}{2}} \left(\frac{q}{q - 1} \Gamma\left(\frac{q}{2(q - 1)}\right)\right)^{\frac{r}{2}} \right)^{\frac{r}{q}} + \left(2 \left(2 C_{\beta}^2 c_{\text{sub}}^2\right)^m (2m)! \Gamma(m)\right)^{\frac{1}{2}}.$$

Proof of Lemma 2. We obtain

$$\langle \nabla L_n^H(\beta + \Delta) - \nabla L_n^H(\beta), \Delta \rangle = \frac{1}{n} \sum_{i=1}^n \left(l'_n(Y_i - X_i^T \beta) - l'_n(Y_i - X_i^T (\beta + \Delta))\right) X_i^T \Delta$$

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for $\beta, \Delta \in \mathbb{R}^p$ by (16). Firstly we show that

$$\left\langle \nabla \mathcal{L}_{n,\alpha}^H(\beta + \Delta) - \nabla \mathcal{L}_{n,\alpha}^H(\beta), \Delta \right\rangle \geq \frac{1}{2n} \sum_{i=1}^{n} \varphi_{\tau\|\Delta\|_2} \left( X_i^T \Delta \mathbb{I} \{ |Y_i - X_i^T \beta| \leq T \} \right)$$

(79)

for all $\alpha \leq 1/(T + 8 \tau C_\beta)$ and $(\beta, \Delta) \in A := \{(\beta, \Delta) : \|\beta\|_2 \leq 4C_\beta \text{ and } \|\Delta\|_2 \leq 8C_\beta \}$, where

$$\varphi_t(u) = u^2 \mathbb{I} \{ |u| \leq t/2 \} + (t - |u|)^2 \mathbb{I} \{ t/2 < |u| \leq t \}$$

and

$$T = 96 \frac{c_{X,\text{sub}} \sqrt{c_{X,\text{sub}} C_\beta}}{c_X}, \quad \tau = \max \left\{ 4c_{X,\text{sub}} \sqrt{\log(12c_{X,\text{sub}}/c_X)}, 1 \right\}.$$

The function $\varphi_t$ satisfies obviously $\varphi_t(u) \leq u^2 \mathbb{I} \{ |u| \leq t \}$. Let $i \in \{1, \ldots, n\}$ be fixed, then we get on the one hand

$$\varphi_{\tau\|\Delta\|_2} \left( X_i^T \Delta \mathbb{I} \{ |Y_i - X_i^T \beta| \leq T \} \right) = 0$$

if $|X_i^T \Delta| > \tau\|\Delta\|_2$ or $|Y_i - X_i^T \beta| > T$. In addition we have always

$$\left( l'_\alpha(Y_i - X_i^T \beta) - l'_\alpha(Y_i - X_i^T (\beta + \Delta)) \right) X_i^T \Delta \geq 0$$

because of the convexity of $g(\beta) = l_\alpha(Y_i - X_i^T \beta)$. On the other hand, if $|X_i^T \Delta| \leq \tau\|\Delta\|_2$ and $|Y_i - X_i^T \beta| \leq T$ we get

$$|Y_i - X_i^T \beta| \leq T \leq \alpha^{-1}$$

and

$$|Y_i - X_i^T (\beta + \Delta)| \leq |Y_i - X_i^T \beta| + |X_i^T \Delta| \leq T + \tau\|\Delta\|_2 \leq T + 8\tau C_\beta \leq \alpha^{-1}$$

because $(\beta, \Delta) \in A$ and the choice of $\alpha$. In addition the mean value theorem implies

$$l''_\alpha(Y_i - X_i^T \beta) - l''_\alpha(Y_i - X_i^T (\beta + \Delta)) = l''_\alpha(c) \left( Y_i - X_i^T \beta - Y_i + X_i^T (\beta + \Delta) \right)$$

with $c \in (Y_i - X_i^T \beta, Y_i - X_i^T (\beta + \Delta))$ since the pseudo Huber loss $l_\alpha$ is twice differentiable. The above conditions lead to $|c| \leq \alpha^{-1}$ as well. Moreover, note that

$$l''_\alpha(c) = \frac{2\alpha^{-3}}{(\alpha^{-2} + c^2)^{3/2}} \geq \frac{2\alpha^{-3}}{(2\alpha^{-2})^{3/2}} = \frac{2}{2^{3/2}} \geq \frac{1}{2}$$

for all $|c| \leq \alpha^{-1}$. Hence it follows that

$$\left( l'_\alpha(Y_i - X_i^T \beta) - l'_\alpha(Y_i - X_i^T (\beta + \Delta)) \right) X_i^T \Delta = l''_\alpha(c) (X_i^T \Delta)^2 \geq \frac{1}{2} (X_i^T \Delta)^2$$

if $|X_i^T \Delta| \leq \tau\|\Delta\|_2$ and $|Y_i - X_i^T \beta| \leq T$. So in total inequality (79) is satisfied for all $(\beta, \Delta) \in A$ and $\alpha \leq 1/(T + 8\tau C_\beta)$. Furthermore, the condition of $\alpha$ reduces to $\alpha \leq c_\alpha$ where $c_\alpha$ is a positive constant depending on $c_{X,\text{sub}}, c_{X,\text{sub}}$, and $C_\beta$, because of the choice of $T$ and $\tau$. The proof of Fan et al. (2017, Lemma 2) provides

$$\frac{1}{n} \sum_{i=1}^{n} \varphi_{\tau\|\Delta\|_2} \left( X_i^T \Delta \mathbb{I} \{ |Y_i - X_i^T \beta| \leq T \} \right) \geq c_1 \|\Delta\|_2 \left( \|\Delta\|_2 - c_2 \left( \log(p) \right)^{\frac{3}{2}} \|\Delta\|_1 \right)$$

(80)
with \( c_1 = c_{X1}/4 \) and \( c_2 = 160 \tau^2 c_{X, sub}/c_{X1} \). Additionally the proof of Fan et al. (2017, Lemma 4) leads to

\[
c_1 \| \Delta \|^2 \left( \| \Delta \| - c_2 \left( \frac{\log(p)}{n} \right) \right) \geq \frac{c_1}{2} \| \Delta \|^2 - \frac{c_1 c_2^2 \log(p)}{2n} \| \Delta \|^2. \tag{81}
\]

All in all the inequalities (79) - (81) imply the assertion of Lemma 2.

**Proof of Lemma 3.** For \( v \in B_S := \{ v \in \mathbb{R}^P \mid \text{supp}(v) \subseteq S, \| v \|_2 = 1 \} \) we have that

\[
\langle \nabla^2 \mathcal{L}_{n, \alpha}^H(\beta) \rangle v = \lim_{t \to 0} \frac{\nabla \mathcal{L}_{n, \alpha}^H(\beta + tv) - \nabla \mathcal{L}_{n, \alpha}^H(\beta)}{t}
\]

and hence that

\[
v^T \langle \nabla^2 \mathcal{L}_{n, \alpha}^H(\beta) \rangle v = \lim_{t \to 0} \frac{\langle \nabla \mathcal{L}_{n, \alpha}^H(\beta + tv) - \nabla \mathcal{L}_{n, \alpha}^H(\beta), tv \rangle}{t^2} \tag{82}
\]

The RSC condition (19) implies that for \( t \leq 1 \) and \( v \in B_S \) we obtain

\[
\langle \nabla \mathcal{L}_{n, \alpha}^H(\beta + tv) - \nabla \mathcal{L}_{n, \alpha}^H(\beta), tv \rangle \geq t^2 \left( c_1^{RSC} \| v \|_2^2 - c_2^{RSC} \frac{\log(p)}{n} \| v \|_1^2 \right) \geq t^2 \left( c_1^{RSC} - c_2^{RSC} \frac{s \log(p)}{n} \right)
\]

where we used \( \| v \|_1 \leq \sqrt{s} \| v \|_2 \) since \( \text{supp}(v) \subseteq S \) and \( \| v \|_2 = 1 \). Plugging this into (82) together with the condition \( n \geq c_3^{RSC} s \log(p) \) gives

\[
v^T \langle \nabla^2 \mathcal{L}_{n, \alpha}^H(\beta) \rangle v \geq c_1^{RSC} - \frac{c_1^{RSC}}{2} = c_1^{RSC} \tag{83}
\]

which is equivalent to the estimate (20).

**Proof of Lemma 4.** By (14) and (16) we obtain

\[
\nabla \mathcal{L}_{n, \alpha}^H(\beta_{\alpha_n}^*) = -\frac{1}{n} \sum_{i=1}^n l'_{\alpha_n}(Y_i - X_i^\top \beta_{\alpha_n}^*) X_i
\]

with \( |l'_{\alpha_n}(x)| \leq 2\alpha_n^{-1} \) and \( |l'_{\alpha_n}(x)| \leq 2|x| \) for all \( x \in \mathbb{R} \). Furthermore, by (75) in the proof of Lemma 1 it follows that

\[
\mathbb{E}[|Q|^u]^{\frac{1}{u}} \leq \left( \frac{2\tau^2}{\tau} \frac{ru}{2} \Gamma\left( \frac{ru}{2} \right) \right)^{\frac{1}{ru}} \leq (2\tau^2)^{\frac{1}{u}} \left( ru \right)^{\frac{1}{ru}} \leq (2\tau^2)^{\frac{1}{u}} (u)^{\frac{1}{ru}} \leq (2\tau^2)^{\frac{1}{u}} u^{\frac{1}{ru}} \tag{84}
\]

for \( Q \sim \text{subG(\tau)} \) with \( \tau > 0 \) and \( u, r \in \mathbb{N} \) with \( u \geq 2 \) and \( r/2 \in \mathbb{N} \). In the last inequality we bound the \( r \) largest factors of \( ru \) by \( ru \), then the next \( r \) largest factors by \( r(u - 1) \) and so on. Now we choose \( 1 < q_1 \leq q \), where \( q \) is given in Assumption 1, such that \( r_1 = q_1/(q_1 - 1) \in \mathbb{N} \) and \( r_1 \) is even. Then we obtain

\[
\mathbb{E}\left[ \left( l'_{\alpha_n}(Y_i - X_i^\top \beta_{\alpha_n}^*) X_{i,k} \right)^2 \right] \leq 4 \mathbb{E}\left[ \left( \varepsilon_i - X_i^\top (\beta^* - \beta_{\alpha_n}^*) \right)^2 X_{i,k}^2 \right]
\]

\[
\leq 8 \mathbb{E}\left[ \left( \varepsilon_i^2 + (X_i^\top (\beta^* - \beta_{\alpha_n}^*))^2 \right) X_{i,k}^2 \right]
\]

\[
= 8 \mathbb{E}\left[ \varepsilon_i^2 |X_i| X_{i,k}^2 + (X_i^\top (\beta^* - \beta_{\alpha_n}^*))^2 X_{i,k}^2 \right]
\]

\[
\leq 8 \mathbb{E}\left[ \left( 1 + \mathbb{E}[\varepsilon_i^m |X_i|] \right) X_{i,k}^2 + (X_i^\top (\beta^* - \beta_{\alpha_n}^*))^2 X_{i,k}^2 \right]
\]

\[
\leq c_3^{\text{grad}}
\]

29
with

\[ c_3^{\text{Grad}} = 32 c_{X_{\text{sub}}}^2 \left( 1 + (1 + C_{e,m})^{\frac{1}{m}} r_1^2 + 2^2 64 c_{X_{\text{sub}}}^2 C_3^2 \right) \]

for \( k = 1, \ldots, p \). In the last inequality we used the Hölder and Cauchy-Schwarz inequality, (84) with \( u = 2 \) and \( r \in \{2, r_1\} \), and the fact that \( X_{i,k} \sim \text{subG}(\alpha_{x_{\text{sub}}}) \) and \( X_i^T (\beta^* - \beta_{\alpha_n}^*) \sim \text{subG}(2C_3 \epsilon_{X_{\text{sub}}}) \) by (iii) of Assumption 1. Analog we obtain for higher moments, \( u \geq 3 \), using \( |l_{\alpha_n}^r(x)|^u \leq 4 (2 \alpha_n^{-1})^{u-2} x^2 \), the estimate

\[
\mathbb{E} \left[ |l_{\alpha_n}^r(Y_i - X_i^T \beta_{\alpha_n}^*) X_{i,k}|^u \right] \leq 4 \left( \frac{2}{\alpha_n} \right)^{u-2} \mathbb{E} \left[ (\epsilon_i - X_i^T (\beta^* - \beta_{\alpha_n}^*))^2 |X_{i,k}|^u \right]
\]

\[
\leq 8 \left( \frac{2}{\alpha_n} \right)^{u-2} \mathbb{E} \left[ (1 + (X_i)^m |X_i|) |X_{i,k}|^u + (X_i^T (\beta^* - \beta_{\alpha_n}^*))^2 |X_{i,k}|^u \right]
\]

\[
\leq 8 \left( \frac{2}{\alpha_n} \right)^{u-2} 2 \frac{c_{X_{\text{sub}}}^u}{\alpha_n} \left( 1 + (1 + C_{e,m})^{\frac{1}{m}} r_1^u + 2^u 64 c_{X_{\text{sub}}}^2 C_3^2 \right)
\]

\[
= \frac{u!}{2} \left( \frac{2 c_{X_{\text{sub}}}^u}{\alpha_n} \right)^{u-2} 16 c_{X_{\text{sub}}}^2 \left( 1 + (1 + C_{e,m})^{\frac{1}{m}} r_1^u + 2^u 64 c_{X_{\text{sub}}}^2 C_3^2 \right)
\]

with

\[ c_4^{\text{Grad}} = \sqrt{2} \max(r_1, 2) c_{X_{\text{sub}}} . \]

In addition note that \( \mathbb{E}[l_{\alpha_n}^r(Y_i - X_i^T \beta_{\alpha_n}^*) X_{i,k}] = 0 \) because of (5) and (18). Now Bernstein’s inequality, cf. Massart (2007, Proposition 2.9), leads to

\[
P \left( \left| \frac{1}{n} \sum_{i=1}^n l_{\alpha_n}^r(Y_i - X_i^T \beta_{\alpha_n}^*) X_{i,k} \right| \geq \left( \frac{2 c_{X_{\text{sub}}}^u}{\alpha_n} \frac{c_{\text{Grad}} x}{n} \right)^{\frac{1}{2}} + \frac{c_{X_{\text{sub}}}^2 \log(n)}{2} \right) \leq 2 \exp(-x)
\]

for \( x > 0 \) since the terms of the sum are independent. Let \( x = 3 \log(p) \) and \( c_{\text{Grad}} = \sqrt{96 c_{\text{Grad}}^3 / 4} \), then by the choice of \( \alpha_n \) we get

\[
\frac{2 c_{\text{Grad}}^3}{\alpha_n} \leq \frac{2 c_{\text{Grad}}^3}{c_{\text{Grad}}^3} \left( \frac{c_{\text{Grad}}^3 (\log(p))}{96n} \right)^{\frac{1}{2}} = \left( \frac{c_{\text{Grad}}^3}{n} \right)^{\frac{1}{2}}
\]

and hence

\[
P \left( \left| \frac{1}{n} \sum_{i=1}^n l_{\alpha_n}^r(Y_i - X_i^T \beta_{\alpha_n}^*) X_{i,k} \right| \geq 2 \left( \frac{6 c_{\text{Grad}}^3 \log(p)}{n} \right)^{\frac{1}{2}} \right) \leq 2 \exp \left( -3 \log(p) \right) .
\]

Union bound implies

\[
P \left( \left\| \nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*) \right\|_{\infty} \geq 2 \left( \frac{6 c_{\text{Grad}}^3 \log(p)}{n} \right)^{\frac{1}{2}} \right) \leq 2 \exp \left( -3 \log(p) + \log(p) \right) = \frac{2}{p^2}
\]

and \( c_2^{\text{Grad}} = 2(6c_3^{\text{Grad}})^{\frac{1}{2}} \).

7.2 Proof of Lemma 8

The proof of Lemma 8 relies on the following two technical results.
Lemma 15. Suppose Assumption 1 and \( \alpha_n = C_\alpha \left( \log(n) \right) \frac{1}{\alpha} \) for some positive constant \( C_\alpha > 0 \) hold. If in addition \( n \geq \max \left\{ (576 \log(6) C_\alpha C_2^2 C_{\text{sub}}^2 s^2 \log(p), 16 \log(24) s \log(p) \right\} \), then there exist positive constants \( C_1, C_2, C_3 > 0 \) such that

\[
\left\| \tilde{Q}_{SS} - \mathbb{E}[X_i X_i^\top]_{SS} \right\|_{M2} \leq C_2 \max \left\{ \left( \frac{s}{n} \right)^{\frac{1}{2}}, \frac{s}{n} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}}, \frac{\left( \log(p) \right)^{\frac{1}{2}}}{n}, \alpha_n^\frac{1}{2}, \alpha_n^{\frac{1}{2}}, \alpha_n^{-\frac{1}{2}}, \alpha_n \right\}
\]

\[
\leq \frac{C_3}{\sqrt{s}}
\]

with probability at least \( 1 - C_1/p^2 - 6/p^5 \).

Proof of Lemma 15. The following proof uses elements of the proof of Lemma 1 in Sun et al. (2020). Let \( B_2^* = \{ u \in \mathbb{R}^n \mid \| u \|_2 \leq 1 \} \). Then using (24) in Lemma 5 we have

\[
\left\| \frac{2}{n} \sum_{i=1}^n (X_i)_S (X_i)_S^\top - \tilde{Q}_{SS} \right\|_{M2} = \max_{u \in B_2^*} \sum_{i=1}^n \left( \frac{2}{n} \sum_{i=1}^n (1 - d_i) (X_i)_S (X_i)_S^\top \right) u
\]

\[
= \max_{u \in B_2^*} \left( Z_1(u) + Z_2(u) \right)
\]

\[
\leq \max_{u \in B_2^*} Z_1(u) + \max_{u \in B_2^*} Z_2(u)
\]

(86)

with

\[
Z_1(u) = \frac{1}{n} \sum_{i=1}^n \left( \int_0^1 \left( 2 - l'_n \left( X_i - X_i^\top (\beta_n^* + t (\tilde{\beta}_n^{PDW} - \beta_n^*)) \right) \right) \right.
\]

\[
\cdot \mathbb{1}_{[0, \alpha_n / 2]} \left( \left| X_i - X_i^\top (\beta_n^* + t (\tilde{\beta}_n^{PDW} - \beta_n^*)) \right| \right) \mathbb{1} (u^\top (X_i)_S)^2 dt.
\]

\[
Z_2(u) = \frac{1}{n} \sum_{i=1}^n \left( \int_0^1 \left( 2 - l'_n \left( X_i - X_i^\top (\beta_n^* + t (\tilde{\beta}_n^{PDW} - \beta_n^*)) \right) \right) \right.
\]

\[
\cdot \mathbb{1}_{(\alpha_n / 2, \alpha_n]} \left( \left| X_i - X_i^\top (\beta_n^* + t (\tilde{\beta}_n^{PDW} - \beta_n^*)) \right| \right) \mathbb{1} (u^\top (X_i)_S)^2 dt.
\]

To handle the first sum in (86) we consider the series expansion in (71), which implies

\[
|2 - l'_n (x)| = \left| -2 \sum_{k=2}^\infty \frac{1}{k} (2k - 1) \alpha_n^{2k-2} x^{2k-2} \right| \leq 3 \alpha_n^2 x^2.
\]

if \( \alpha_n^2 x^2 < 1 \). Hence for small \( \alpha_n \) we get

\[
\max_{u \in B_2^*} Z_1(u) \leq \max_{u \in B_2^*} \frac{3 \alpha_n^2}{n} \sum_{i=1}^n \left( u^\top (X_i)_S \right)^2.
\]

Standard spectral norm bounds on the sample covariance matrix (with independent and identically distributed sub-Gaussian rows), cf. Wainwright (2019, Theorem 6.5), and (ii) of Assumption 1 lead to

\[
\max_{u \in B_2^*} \frac{1}{n} \sum_{i=1}^n \left( u^\top (X_i)_S \right)^2 \leq \left\| \frac{1}{n} \sum_{i=1}^n (X_i)_S (X_i)_S^\top \right\|_{M2}
\]

\[
\leq \left\| \mathbb{E}[X_i X_i^\top]_{SS} \right\|_{M2} + \left\| \frac{1}{n} \sum_{i=1}^n (X_i)_S (X_i)_S^\top - \mathbb{E}[X_i X_i^\top]_{SS} \right\|_{M2}
\]

\[
\leq c N + C_4 \left( \frac{s}{n} \right)^{\frac{1}{2}} + \frac{s}{n} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}}
\]

(87)
with probability at least $1 - C_1/p^2$ for some positive constants $C_1, C_4 > 0$. Hence

$$\max_{u \in B_2^n} Z_n^2(u) \leq 3 (c_X u + 3C_4) \alpha_n$$ \hspace{1cm} (88)

with high probability. For the second sum in (86) we firstly estimate

$$\max_{u \in B_2^n} Z_n^2(u) \leq \max_{u \in B_2^n} \frac{2}{n} \sum_{i=1}^{n} \left( \int_0^1 \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( Y_i - X_i^\top (\beta_{\alpha_i}^* + t (\hat{\beta}_{n}^{PDW} - \beta_{\alpha_i}^*)) \right) dt \right) \left( u^\top (X_i)_S \right)^2$$

because of (15). Now we can rearrange the term in the indicator function as

$$\left| Y_i - X_i^\top (\beta_{\alpha_i}^* + t (\hat{\beta}_{n}^{PDW} - \beta_{\alpha_i}^*)) \right| = \left| \varepsilon_i + (1 - t) X_i^\top (\beta^* - \beta_{\alpha_i}^*) + t X_i^\top (\beta^* - \hat{\beta}_{n}^{PDW}) \right|.$$ 

Using the inequality

$$\mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( |Q_1 + Q_2 + Q_3| \right) \leq \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |Q_1| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |Q_2| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |Q_3| \right)$$

for random variables $Q_1, Q_2$ and $Q_3$ leads to

$$\frac{2}{n} \sum_{i=1}^{n} \left( \int_0^1 \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( Y_i - X_i^\top (\beta_{\alpha_i}^* + t (\hat{\beta}_{n}^{PDW} - \beta_{\alpha_i}^*)) \right) dt \right) \left( u^\top (X_i)_S \right)^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \left( \int_0^1 \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( |\varepsilon_i| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |(1 - t) X_i^\top (\beta^* - \beta_{\alpha_i}^*)| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |t X_i^\top (\beta^* - \hat{\beta}_{n}^{PDW})| \right) dt \right) \left( u^\top (X_i)_S \right)^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \left( \int_0^1 \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( |\varepsilon_i| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |X_i^\top (\beta^* - \beta_{\alpha_i}^*)| \right) + \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |X_i^\top (\beta^* - \hat{\beta}_{n}^{PDW})| \right) dt \right) \left( u^\top (X_i)_S \right)^2$$

$$= \frac{2}{n} \sum_{i=1}^{n} \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} \left( |\varepsilon_i| \right) \left( u^\top (X_i)_S \right)^2$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |X_i^\top (\beta^* - \beta_{\alpha_i}^*)| \right) \left( u^\top (X_i)_S \right)^2$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} \left( |X_i^\top (\beta^* - \hat{\beta}_{n}^{PDW})| \right) \left( u^\top (X_i)_S \right)^2. \hspace{1cm} (90)$$

We consider each of the three terms separately. By (iii) of Assumption 1 we get for fixed $u \in B_2^n$ that $u^\top (X_i)_S \sim \text{subG}(cX_{sub})$, and following the proof of Rigollet and Hüttner (2019, Lemma 1.12) together with $(\mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} (|\varepsilon_i|))^2 = \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} (|\varepsilon_i|)$ leads to

$$Q_i(u) = \mathbb{1}_{(\alpha_i^{-1/2}, \infty)} (|\varepsilon_i|) \left( u^\top (X_i)_S \right)^2 - \mathbb{E} \left[ \mathbb{1}_{(\alpha_i^{-1/2}/3, \infty)} (|\varepsilon_i|) \left( u^\top (X_i)_S \right)^2 \right]$$

$$\sim \text{subE}(16c_X^2, 16c_X^2).$$
Bernstein’s inequality, cf. Rigollet and Hütter (2019, Theorem 1.13), implies

$$
P\left(\left|\frac{2}{n}\sum_{i=1}^{n} Q_i(u)\right| > x\right) \leq 2 \max_{u} \left\{ \exp\left(-\frac{x^2 n}{2048 c_{X,\text{sub}}^2}\right), \exp\left(-\frac{x n}{64 c_{X,\text{sub}}^2}\right)\right\}
$$

for $x > 0$ and fixed $u \in B_2^x$. Now we proceed with a covering argument. Consider a $1/8$-cover $A$ of cardinality $N = N(1/8; B_2^x, \|\cdot\|_2) \leq 24^s$ of the unit Euclidean ball of $\mathbb{R}^s$ with respect to the Euclidean distance (cf. Lemma 1.18 in Rigollet and Hütter (2019) or Example 5.8 in Wainwright (2019)). We can argue similarly to the proof in Wainwright (2019, Theorem 6.5) since we consider also a quadratic form, and obtain for $x = 256 \sqrt{\log(24) c_{X,\text{sub}}^2 \left(\frac{2 \log(p)}{n}\right)^{\frac{1}{2}}}$ that

$$
P\left(\max_{u \in B_2^x} \left|\frac{2}{n}\sum_{i=1}^{n} Q_i(u)\right| > x\right) \leq P\left(\max_{u \in A} \left|\frac{2}{n}\sum_{i=1}^{n} Q_i(u)\right| > \frac{x}{2}\right)
$$

$$
\leq 2 |N| \max \left\{ \exp\left(-8 \log(24) s \log(p)\right), \exp\left(-4 \log(24) s \log(p)\right)\right\}
$$

$$
\leq 2 \exp\left(\log(24) s - 8 \log(24) s \log(p)\right)
$$

$$
\leq 2 \exp\left(-4 \log(24) s \log(p)\right)
$$

$$
\leq \frac{2}{p^{5s}}
$$

(91)

since $4 \log(p) \geq 1$ if $p \geq 2$, and by assumption $n \geq 16 \log(24) s \log(p)$. In addition we obtain

$$
\frac{2}{n}\sum_{i=1}^{n} \mathbb{E}\left[\mathbb{I}_{(\alpha^{-1/2}/3, \infty)}(|\varepsilon_i|) \left(u^\top (X_i) S\right)^2\right] = 2 \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}_{(\alpha^{-1/2}/3, \infty)}(|\varepsilon_i|) | X_1\right] \left(u^\top (X_1) S\right)^2\right]
$$

$$
\leq 2 \left(1 + C_{e,m}\right) (9\alpha_n)^{\frac{p}{m}} \mathbb{E}\left[\left(u^\top (X_1) S\right)^2\right]
$$

$$
\leq 2 \left(1 + C_{e,m}\right) c_{X,\text{sub}}^2 (9\alpha_n)^{\frac{p}{m}},
$$

by Assumption 1 and an application of the conditional version of Markov’s inequality,

$$
\mathbb{E}\left[\mathbb{I}_{(\alpha^{-1/2}/3, \infty)}(|\varepsilon_i|) | X_1\right] = P\left(|\varepsilon_i| > \frac{1}{3\alpha_n^2} \left| X_1\right.\right) \leq (9\alpha_n)^{\frac{p}{m}} \mathbb{E}\left[|\varepsilon_i|^m | X_1\right]
$$

$$
\leq (9\alpha_n)^{\frac{p}{m}} \left(1 + \mathbb{E}[|\varepsilon_i|^m | X_1]\right)^{\frac{1}{m}}
$$

$$
\leq \left(1 + C_{e,m}\right) (9\alpha_n)^{\frac{p}{m}}.
$$

By building the maximum of the expected values over $u \in B_2^x$ and collecting terms we find that

$$
\max_{u \in B_2^x} \frac{2}{n}\sum_{i=1}^{n} \mathbb{I}_{(\alpha^{-1/2}/3, \infty)}(|\varepsilon_i|) \left(u^\top (X_i) S\right)^2 \leq 256 \sqrt{\log(24) c_{X,\text{sub}}^2 \left(\frac{s \log(p)}{n}\right)^{\frac{1}{2}}}
$$

$$
+ 2 \left(1 + C_{e,m}\right) c_{X,\text{sub}}^2 (9\alpha_n)^{\frac{p}{m}}
$$

(92)

with probability at least $1 - 2/p^{5s}$. We proceed similar for the second and third sum in (90), hence it is sufficient to consider the rates of the expected values

$$
\mathbb{E}\left[\mathbb{I}_{(\alpha^{-1/2}/3, \infty)} \left| \left(X_1^\top (\beta^* - \beta)\right) \left(u^\top (X_1) S\right)^2\right.\right]
$$

33
with $\beta = \beta_{\alpha_n}^*$ and $\beta = \beta_n^{PDW}$. Obviously it is

$$I_{(\alpha_n^{-1/2}, \infty)} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_{\alpha_n}^*) \right| \right) \leq 3 \left| \mathbf{X}_1^T (\beta^* - \beta_{\alpha_n}^*) \right| \frac{\alpha_n}{4}$$

and hence by Assumption 1, Rigollet and Hütter (2019, Lemma 1.4) and the Cauchy-Schwarz inequality

$$E \left[ I_{(\alpha_n^{-1/2}, \infty)} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_{\alpha_n}^*) \right| \right) \left( u^T (\mathbf{X}_1)_S \right)^2 \right] \leq 3 \alpha_n \sqrt{E \left[ \left| \mathbf{X}_1^T (\beta^* - \beta_{\alpha_n}^*) \right| \left( u^T (\mathbf{X}_1)_S \right)^2 \right]}$$

$$\leq 12 \sqrt{\alpha_n} \| \beta^* - \beta_{\alpha_n}^* \|_2 \frac{\alpha_n}{4}.$$

Lemma 1 implies

$$E \left[ I_{(\alpha_n^{-1/2}, \infty)} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_{\alpha_n}^*) \right| \right) \left( u^T (\mathbf{X}_1)_S \right)^2 \right] \leq 12 C_{apx} c_{X, sub} \alpha_n \frac{m - \frac{1}{2}}{2}.$$  (93)

The vector $\beta_n^{PDW}$ has support $S$ and satisfies $\| \beta_n^{PDW} - \beta^* \|_2 \leq 2C_\beta$ by (22) and (iv) of Assumption 1, hence it follows that

$$E \left[ I_{(\alpha_n^{-1/2}, \infty)} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_n^{PDW}) \right| \right) \left( u^T (\mathbf{X}_1)_S \right)^2 \right] = \mathbb{P} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_n^{PDW}) \right| > \frac{1}{3\alpha_n} \right)$$

$$\leq \mathbb{P} \left( \max_{u \in \mathbb{R}^\ell} \left( u^T (\mathbf{X}_1)_S \right) > \frac{1}{3\alpha_n} \right)$$

$$\leq \mathbb{P} \left( \max_{u \in \mathbb{R}^\ell} \left( u^T (2C_\beta (\mathbf{X}_1)_S) \right) > \frac{1}{3\alpha_n} \right)$$

$$\leq \exp \left( \log(6) s - \frac{1}{288 C_\beta^2 c_{X, sub} \alpha_n} \right)$$

by Rigollet and Hütter (2019, Theorem 1.19) together with Assumption 1. By the choice of $\alpha_n$ and the sample size $n$ we obtain

$$\exp \left( \log(6) s - \frac{1}{288 C_\beta^2 c_{X, sub} \alpha_n} \right) = \exp \left( \log(6) s - \frac{\sqrt{n}}{576 C_\beta^2 c_{X, sub} C_\alpha \sqrt{\log(p)}} \right)$$

$$\leq \exp \left( \log(6) s - \log(6) s \right) \exp \left( \frac{1}{\sqrt{576 C_\beta^2 c_{X, sub} \alpha_n}} \right)$$

$$= 2 \left( \frac{576 C_\beta^2 c_{X, sub} \alpha_n}{\sqrt{576 C_\beta^2 c_{X, sub} \alpha_n}} \right)^2$$

since $\exp(x) \geq x^2/2$ for $x > 0$. Therefore

$$E \left[ I_{(\alpha_n^{-1/2}, \infty)} \left( \left| \mathbf{X}_1^T (\beta^* - \beta_n^{PDW}) \right| \right) \left( u^T (\mathbf{X}_1)_S \right)^2 \right] \leq \sqrt{2} 2304 C_\beta^2 c_{X, sub} \alpha_n$$  (94)

by the Cauchy-Schwarz inequality. So finally the previous considerations in (86) - (94) showed that

$$\left\| \frac{2}{n} \sum_{i=1}^n (\mathbf{X}_i)_S^T (\mathbf{X}_i)_S - \hat{Q}_{SS} \right\|_{m, 2} \leq 768 \sqrt{\log(24) c_{X, sub} \left( \frac{s \log(p)}{n} \right)} + 2 (1 + C_{e.m}) c_{X, sub} (9\alpha_n)^{\frac{3}{2}}$$

$$+ 24 C_{apx} c_{X, sub} \alpha_n^{m - \frac{1}{2}} + 3(c_{X, u} + 3C_4) + \sqrt{2} 4608 C_\beta^2 c_{X, sub} \alpha_n$$

$$\leq C_5 \max \left\{ \left( \frac{s \log(p)}{n} \right)^{\frac{1}{2}}, \alpha_n^{\frac{3}{2}}, \alpha_n^{m - \frac{1}{2}}, \alpha_n \right\}$$
for a positive constant $C_5 > 0$ with probability at least $1 - C_1/p^2 - 6/p^{5s}$. Furthermore, repeated application of the spectral norm bound in (87) leads to

\[
\left\| \mathcal{Q}_{SS} - \mathbb{E}[X_1 X_1^\top]_{SS} \right\|_{M,2} \leq \left\| \mathcal{Q}_{SS} - \frac{2}{n} \sum_{i=1}^{n} (X_i)_{S} (X_i)_{S}^\top \right\|_{M,2} \\
+ 2 \left\| \frac{1}{n} \sum_{i=1}^{n} (X_i)_{S} (X_i)_{S}^\top - \mathbb{E}[X_1 X_1^\top]_{SS} \right\|_{M,2} \\
\leq C_2 \max \left\{ \left( \frac{s}{n} \right)^{\frac{1}{2}}, \left( \frac{s}{n} \right) \left( \frac{n \log(p)}{n} \right)^{\frac{1}{2}}, \left( \frac{s \log(p)}{n} \right) \left( \frac{\alpha_{\omega}}{\alpha_n} \right)^{\frac{1}{2}}, \alpha_n \right\}
\]

for a positive constant $C_2 > 0$. By the choices of $\alpha_n$ and $n \gtrsim s^2 \log(p)$ together with $m \in \{2,3\}$ finally it follows that

\[
\left\| \mathcal{Q}_{SS} - \mathbb{E}[X_1 X_1^\top]_{SS} \right\|_{M,2} \leq C_6 \left( \frac{s \log(p)}{n} \right)^{\frac{1}{2}} \leq \frac{C_3}{\sqrt{s}}
\]

for some positive constants $C_3, C_6 > 0$ with probability at least $1 - C_1/p^2 - 6/p^{5s}$.

Lemma 16. Let $M \in \mathbb{R}^{[A] \times [B]}$ be a matrix with $A, B \subseteq \{1, \ldots, p\}$ and $\max_{k \in \{1, \ldots, |A|\}} \left\| M^\top e_k \right\|_2 \leq C_M$ for some positive constant $C_M > 0$. Suppose Assumption 1 and $\alpha_n \geq \frac{c_1 \log(p)}{n}$ holds, then with probability at least $1 - 2/p^2$ the $\ell_\infty$ norm of $M(\nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*))_B$ is bounded by

\[
\left\| M(\nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*))_B \right\|_\infty \leq C_M \frac{c_2 \log(p)}{n} \leq C_M \frac{c_2 \log(p)}{n}.
\]

Proof of Lemma 16. We follow the proof of Lemma 4. It is

\[
M(\nabla L_{n,\alpha_n}^H(\beta_{\alpha_n}^*))_B = M \left( -\frac{1}{n} \sum_{i=1}^{n} \nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*)(X_i)_B \right) = -\frac{1}{n} \sum_{i=1}^{n} \nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) Z_i
\]

with $Z_i = M(X_i)_B$. The random vectors $\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) Z_i, \ldots, \nu_{\alpha_n}^i (Y_n - X_n^\top \beta_{\alpha_n}^*) Z_n$ are independent and identically distributed because $(X_1, \varepsilon_1), \ldots, (X_n, \varepsilon_n)$ are independent and identically distributed. In addition (iii) of Assumption 1 and $\max_{k \in \{1, \ldots, |A|\}} \left\| M^\top e_k \right\|_2 \leq C_M$ imply that the entries $Z_{i,k} = e_k^\top Z_i$ of $Z_i$ are sub-Gaussian with variance proxy $C_M c_2^{\text{Grad}}$. This leads to

\[
\mathbb{E} \left[ (\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) Z_{i,k})^2 \right] \leq C_M^2 c_3^{\text{Grad}}
\]

for $u \in \mathbb{N}$, $u \geq 3$, where $c_3^{\text{Grad}}$ and $c_5^{\text{Grad}}$ are given in the proof of Lemma 4. Moreover, we obtain

\[
\mathbb{E} \left[ (\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) Z_i)^u \right] \leq \frac{u!}{2} \left( \frac{2 C_M c_3^{\text{Grad}}}{\alpha_n} \right)^{u-2} c_4^{\text{Grad}}
\]

for $u \in \mathbb{N}$, $u \geq 3$, where $c_3^{\text{Grad}}$ and $c_5^{\text{Grad}}$ are given in the proof of Lemma 4. Moreover, we obtain

\[
\mathbb{E} \left[ (\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) Z_i)_1 \right] = M \mathbb{E} \left[ (\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) (X_i)_B \right] = 0_{|A|}
\]

since $\mathbb{E}[(\nu_{\alpha_n}^i (Y_i - X_i^\top \beta_{\alpha_n}^*) X_i)_1] = 0_p$ (see proof of Lemma 4). Arguing as in the proof of Lemma 4 concludes the proof.

\[35\]
Proof of Lemma 8.
For the first part we invoke Lemma 15 and obtain (if $C_3 \geq \max \{ (576 \log(6) C_\alpha C_\beta^2 \ell(X_{\text{sub}})^2, 16 \log(24) \}$ in Lemma 8)

$$\| \hat{Q}_{SS} - E[X_1X_1^\top]_{SS} \|_{M,2} \leq \frac{C_4}{\sqrt{s}}$$

with probability at least $1 - C_1/p^2 - 6/p^5s$ for some positive constants $C_1, C_4 > 0$. Moreover, we have

$$\left\| \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right\|_{M,2} \leq \left\| \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right\|_{M,\infty} \leq C_{S,X} \tag{96}$$

by (9) and the symmetry of the matrix. Hence by Loh and Wainwright (2017, Lemma 11) we conclude that

$$\left\| (\hat{Q}_{SS} - E[X_1X_1^\top]_{SS})^{-1} \right\|_{M,\infty} \leq 2C_{S,X} \left\| \hat{Q}_{SS} - E[X_1X_1^\top]_{SS} \right\|_{M,2} \leq \frac{2C_4C_{S,X}^2}{\sqrt{s}}, \tag{97}$$

with high probability if $\sqrt{s} \geq 2C_4 C_{S,X}$. Finally the triangle inequality and once again (9) lead to

$$\left\| (\hat{Q}_{SS} - E[X_1X_1^\top]_{SS})^{-1} \right\|_{M,\infty} \leq C_{S,X} + \sqrt{s} \left\| (\hat{Q}_{SS} - E[X_1X_1^\top]_{SS})^{-1} \right\|_{M,2} \leq C_{S,X} + 2C_4 C_{S,X}^2$$

with probability at least $1 - C_1/p^2 - 6/p^5s$.

To prove the second part of this lemma we follow the inequalities

$$\left\| \hat{Q}_{SS} (\hat{Q}_{SS})^{-1} \left( \nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*) \right)_S \right\|_\infty \leq \left\| \left( E[X_1X_1^\top]_{SS} \right)^{-1} \left( \nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*) \right)_S \right\|_\infty$$

$$+ \left\| \left( \hat{Q}_{SS} - E[X_1X_1^\top]_{SS} \right)^{-1} \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right\|_{M,\infty} \right) \left( \nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*) \right)_S \right\|_\infty \tag{98}$$

and

$$\left\| \left( \hat{Q}_{SS} (\hat{Q}_{SS})^{-1} - E[X_1X_1^\top]_{SS} \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right) \left( \nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*) \right)_S \right\|_\infty$$

$$\leq \max_{k \in \{1, \ldots, p-s\}} \left\| \left( e_k^\top \left( \hat{Q}_{SS} (\hat{Q}_{SS})^{-1} - E[X_1X_1^\top]_{SS} \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right) \right) e_k \right\|_2 \left\| (\nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*))_S \right\|_2$$

$$\leq \max_{k \in \{1, \ldots, p-s\}} \left\| \left( e_k^\top E[X_1X_1^\top]_{SS} \Delta_1 \right)^\top \right\|_2 \left\| \left( e_k^\top \Delta_2 \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right)^\top \right\|_2$$

$$\cdot \left\| (\nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*))_S \right\|_2$$

$$\leq \max_{k \in \{1, \ldots, p-s\}} \left\| \Delta_1 \right\|_{M,2} \left\| E[X_1X_1^\top]_{SS} \right\| e_k \left\| \left( E[X_1X_1^\top]_{SS} \right)^{-1} \right\|_{M,2} \left\| \Delta_2 e_k \right\|_2 + \left\| \Delta_1 \right\|_{M,2} \left\| \Delta_2 e_k \right\|_2$$

$$\cdot \left\| (\nabla L_{n,\alpha_n}^H (\beta_{\alpha_n}^*))_S \right\|_2 \tag{99}$$
with
\[ \Delta_1 = (\hat{Q}_{SS})^{-1} - \left( E[X_1X_1^\top]_{SS} \right)^{-1} \quad \text{and} \quad \Delta_2 = \hat{Q}_{SS^c} - E[X_1X_1^\top]_{SS^c} \]
in Loh and Wainwright (2017, Corollary 3). Note that (97) implies \( \|\Delta_1\|_{M,2} \leq 2C_4 C_5^3 X/\sqrt{s} \). For the
first term in (98) we shall apply Lemma 16 with \( M = E[X_1X_1^\top]_{SS^c} \left( E[X_1X_1^\top]_{SS^c} \right)^{-1} \). We obtain
\[
\max_{k \in \{1, \ldots, p-s\}} \left\| E[X_1X_1^\top]_{SS^c} e_k \right\|_2 \leq \max_{k \in S} \left\| E[X_1X_1^\top] \right\|_2 \leq \max_{u \in \mathbb{R}^p, \|u\|_2 = 1} \left\| E[X_1X_1^\top] u \right\|_2 \leq c_{X,u}
\]
by (ii) of Assumption 1, and hence together with (96) the estimate
\[
\max_{k \in \{1, \ldots, p-s\}} \left\| E[X_1X_1^\top]_{SS^c} e_k \right\|_2 \leq C_{S,X} c_{X,u}.
\]
Lemma 16 and the choice of \( \alpha_n \) in (10) lead to
\[
\left\| E[X_1X_1^\top]_{SS^c} \right\|_2 \leq C_{S,X} c_{X,u} \epsilon_2 \frac{\text{Grad} \left( \frac{\log(p)}{n} \right) }{\sqrt{s}} \quad \text{(100)}
\]
with probability at least \( 1 - 2/p^2 \). In addition we get by Lemma 4 also
\[
\left\| \left( L_{n,\alpha_n}^H (\beta_{n,\alpha_n}^*) \right) \right\|_2 \leq \sqrt{s} \left\| \left( L_{n,\alpha_n}^H (\beta_{n,\alpha_n}^*) \right) \right\|_\infty \leq c_2 \frac{\text{Grad} \left( \frac{s \log(p)}{n} \right) }{\sqrt{s}} \quad \text{(101)}
\]
with the same probability. The final task is now to study the rate of \( \max_{k \in \{1, \ldots, p-s\}} \| \Delta_2 e_k \|_2 \). First of all it is
\[
\max_{k \in \{1, \ldots, p-s\}} \left\| \left( \hat{Q}_{SS^c} - E[X_1X_1^\top]_{SS^c} \right) e_k \right\|_2 \leq \sqrt{s} \max_{k \in \{1, \ldots, p-s\}} \left\| \hat{Q}_{SS^c} - E[X_1X_1^\top]_{SS^c} \right\|_2 \epsilon_k
\]
\[
= \sqrt{s} \max_{k \in \{1, \ldots, p-s\}} \left\| e_k^\top \left( \hat{Q}_{SS^c} - E[X_1X_1^\top]_{SS^c} \right) e_k \right\|
\]
\[
\leq \sqrt{s} \max_{k \in \{1, \ldots, p-s\}} \left| e_k^\top \left( \hat{Q} - E[X_1X_1^\top] \right) e_k \right|
\]
We proceed similar to the proof of Lemma 15 but here we have only the maximum over \( p^2 \) elements in comparison to the \( 24^s \) elements in the mentioned proof. In addition we use the fact that the centered product of two sub-Gaussian random variables is sub-Exponential, cf. Vershynin (2018, Lemma 2.7.7), and that also the centered product of two sub-Gaussian random variables and a bounded random variable is sub-Exponential. Hence we don’t have the rates depending on \( s \) in (87) and in (91) the factor \( s \) can be dropped. It follows that there exist positive constants \( C_2, C_5, C_6 > 0 \) such that
\[
\max_{k,j \in \{1, \ldots, p\}} \left| e_j^\top \left( \hat{Q} - E[X_1X_1^\top] \right) e_k \right| \leq C_6 \max \left\{ \left( \frac{\log(p)}{n} \right)^{1/2}, \alpha_n^{1/2}, \alpha_n^{1/2}, \alpha_n \right\} \leq \frac{C_6}{\sqrt{s}}
\]
with probability at least \( 1 - C_2/p^2 \) by the choices of \( \alpha_n \) in (10) and \( n \geq s^2 \log(p) \) together with \( m \in \{2,3\} \). Hence
\[
\max_{k \in \{1, \ldots, p-s\}} \| \Delta_2 e_k \|_2 \leq \frac{C_6}{\sqrt{s}} \quad \text{(102)}
\]
37
with high probability and in total we obtain by (98) - (102) the inequality
\[
\|\hat{Q}_{S:S}^{-1}(\hat{Q}_{SS})^{-1}(\nabla L_{n,\alpha_n}(\beta^*_{\alpha_n}))\|_\infty \leq C_{S:X} c_{X,u} c_{\text{Grad}}^{\frac{3}{2}} \left(\frac{\log(p)}{n}\right)^{\frac{1}{2}} + c_{\text{Grad}}^{\frac{3}{2}} \left(\frac{s \log(p)}{n}\right)^{\frac{1}{2}} \cdot \left(\frac{2 C_4 c_{X,u} C_{S:X}^2}{\sqrt{s}} + \frac{C_5 C_{S:X}}{\sqrt{s}} + \frac{2 C_4 C_6 C_{S:X}^2}{s}\right)
\]
\[
\leq C_{\text{Grad}} \left(\frac{\log(p)}{n}\right)^{\frac{1}{2}}
\]
with probability at least \(1 - (4 + C_1 + C_2)/p^2 - 6/p^5\) for some positive constant \(C_7 > 0\). Renewed application of Lemma 4 and the triangular inequality lead to
\[
\|\hat{Q}_{S:S}^{-1}(\hat{Q}_{SS})^{-1}(\nabla L_{n,\alpha_n}(\beta^*_{\alpha_n}))\|_\infty \leq (1 + C_7) c_{\text{Grad}}^{\frac{3}{2}} \left(\frac{\log(p)}{n}\right)^{\frac{1}{2}}.
\]
\[
\square
\]

### 7.3 Proofs of Lemmas 9 and 13

We start with proving Lemma 13. For this purpose we need a technical result concerning the column normalization of the design matrix \(X_n\).

**Lemma 17.** Let \(X_n = (X_1, \ldots, X_n)\top \in \mathbb{R}^{n \times p}\) be a matrix with independent and identically distributed rows \(X_i \sim \text{subG}(c_{X,\text{sub}})\) with variance proxy \(c_{X,\text{sub}}^2 > 0\). Then for \(n \geq 6 \log(p)\) the columns \(X_k\) of \(X_n\) satisfy with probability at least \(1 - 2/p^2\)
\[
\frac{1}{n} \max_{k \in \{1, \ldots, p\}} \left\|X_k\right\|_2 \leq 17 c_{X,\text{sub}}^2.
\]
\[
(103)
\]

**Proof.** We have \(X_{i,k} = e_i \top X_k \sim \text{subG}(c_{X,\text{sub}})\) for all \(i = 1, \ldots, n\) and \(k = 1, \ldots, p\) by the definition of a sub-Gaussian random vector. Rigollet and Hüttner (2019, Lemma 1.12) implies \(X_{i,k}^2 - \mathbb{E}[X_{i,k}^2] \sim \text{subE}(16 c_{X,\text{sub}}^2, 16 c_{X,\text{sub}}^2)\) and with Bernstein’s inequality, cf. Rigollet and Hüttner (2019, Theorem 1.13), it follows that
\[
\mathbb{P} \left(\max_{1 \leq i \leq n} \left|\frac{1}{n} \sum_{i=1}^{n} (X_{i,k}^2 - \mathbb{E}[X_{i,k}^2])\right| > x\right) \leq 2 \max \left\{\exp \left(-\frac{x^2 n}{512 c_{X,\text{sub}}^4}\right), \exp \left(-\frac{x n}{32 c_{X,\text{sub}}^2}\right)\right\},
\]
for all \(x > 0\) and \(k = 1, \ldots, p\) since \(X_{1,k}, \ldots, X_{n,k}\) are independent and identically distributed. By the union bound and the condition \(n \geq 6 \log(p)\) we obtain
\[
\mathbb{P} \left(\max_{1 \leq i \leq n} \left|\frac{1}{n} \sum_{i=1}^{n} (\mathbb{E}[X_{i,k}^2])\right| > 16 c_{X,\text{sub}}^2\right) \leq 2 p \exp \left(-\frac{n}{2}\right) \leq \frac{2}{p^2}.
\]
Furthermore, we have for all \(k = 1, \ldots, p\) the estimate
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_{i,k}^2] \leq c_{X,\text{sub}}^2
\]
since \(X_{1,k}\) is sub-Gaussian with variance proxy \(c_{X,\text{sub}}^2\), and therefore we get
\[
\max_{1 \leq i \leq n} \frac{1}{n} \left\|\hat{X}_k\right\|_2 \leq \max_{k \in \{1, \ldots, p\}} \frac{1}{n} \sum_{i=1}^{n} \left|X_{i,k}^2 - \mathbb{E}[X_{i,k}^2]\right| + \max_{k \in \{1, \ldots, p\}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_{i,k}^2]
\]
\[
\leq 16 c_{X,\text{sub}}^2 + c_{X,\text{sub}}^2 = 17 c_{X,\text{sub}}^2
\]
with high probability. \[
\square
\]
Proof of Lemma 13. We follow the proof of Zhou et al. (2009, Lemma 10.3). It is
\[
\hat{Q}_{S}^{-1} = \frac{2}{n} X_n^T D X_n \left( \frac{2}{n} X_n^T D X_n \right)^{-1} = X_n^T D X_n (X_n^T D X_n)^{-1},
\]
see Lemma 5. For \( k \in S_c \) let
\[
r_k = \left( X_n^T D X_n \right)^{-1} X_n^T D \tilde{x}_k \in \mathbb{R}^s,
\]
then we have
\[
\left\| \hat{Q}_{S}^{-1} \right\|_{M, \infty} = \max_{k \in S_c} \| r_k \|_1.
\]
Furthermore, on the one hand the column normalization in Lemma 17 under the condition \( n \geq 6 \log(p) \) and the submultiplicativity of the spectral norm lead to
\[
\max_{k \in S_c} \left\| D \hat{X}_n^T r_k \right\|_2 \leq \max_{k \in S_c} \left( \left\| D \hat{X}_n^T \left( X_n^T D X_n \right)^{-1} X_n^T D \hat{X}_k \right\|_M, 2 \right) \left\| D \hat{X}_k \right\|_2 \leq \max_{k \in S_c} \left\| \hat{X}_k \right\|_2 \leq \sqrt{\frac{1}{T} c_{\text{sub}} \sqrt{n}}
\]
with probability at least \( 1 - 2/p^2 \) since \( D \hat{X}_n^T \left( X_n^T D X_n \right)^{-1} X_n^T D \hat{X}_k \) is an orthogonal projection matrix and \( D \) a diagonal matrix with entries smaller than or equal to 1. On the other hand under the condition \( n \geq c_{\text{RSC}}^3 \log(p) \) the smallest eigenvalue of \( \hat{Q}_{SS} = \frac{2}{n} X_n^T D X_n \) is bounded below by \( c_{\text{X}, 3}/32 \) with probability at least \( 1 - c_1 p \exp(-c_1 n) \), see Lemma 5, which implies
\[
\left\| D \hat{X}_n^T r_k \right\|_2^2 = \frac{n}{2} r_k^T \left( \frac{2}{n} X_n^T D X_n \right) r_k \geq \frac{c_{\text{X}, 1}}{64} \| r_k \|_2^2
\]
for all \( k \in S_c \). Hence we obtain by the last inequalities the estimate
\[
\max_{k \in S_c} \| r_k \|_2 \leq \max_{k \in S_c} \left( \frac{64}{c_{\text{X}, 1} n} \right)^{1/2} \left\| D \hat{X}_n^T r_k \right\|_2 \leq \frac{33 c_{\text{X}, \text{sub}}}{\sqrt{c_{\text{X}, 1}}}
\]
and in total
\[
\left\| \hat{Q}_{S}^{-1} \right\|_{M, \infty} = \max_{k \in S_c} \| r_k \|_1 \leq \max_{k \in S_c} \sqrt{s} \| r_k \|_2 \leq \frac{33 c_{\text{X}, \text{sub}} \sqrt{s}}{\sqrt{c_{\text{X}, 1}}}
\]
with high probability.

Lemma 9 immediately follows from the above lemma and Lemma 4.

Lemma 18. Suppose Assumption 1 and \( \alpha_n \geq \sqrt{4/3 c_1^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{1/2}} \) hold. Then for \( s \leq \log(p) \) and \( n \geq \max \{ c_{\text{RSC}}^3 s \log(p), 6 \log(p) \} \) we have that
\[
\left\| \hat{Q}_{S}^{-1} \left( \nabla L^H_{\alpha_n, \beta_n} \right) \right\|_S \leq \frac{\sqrt{166 c_{\text{X}, \text{sub}}^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{1/2}}}{\sqrt{3} c_{\text{X}, 1}} c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{1/2}
\]
with probability at least \( 1 - c_1 p \exp(-c_2 p) - 4/p^2 \).
Proof. Set

\[ T = \left\{ \left\| \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right\|_\infty \leq \frac{\sqrt{3} \, 106 \, c_{X, 1}^2 \log(p)}{\sqrt{3} \, c_{X, 1}} \right\}. \]

Then

\[
\mathbb{P}(T^c) \leq \mathbb{P}(T^c \cap \left\{ \max_{k \in \{1, \ldots, p^s\}} \left\| \left( e_k^T \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \right) \right\|_2 \leq \frac{33 \, c_{X, 1}}{\sqrt{n}} \right\})
\]

\[
+ \mathbb{P}(T^c \cap \left\{ \max_{k \in \{1, \ldots, p^s\}} \left\| \left( e_k^T \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \right) \right\|_2 > \frac{33 \, c_{X, 1}}{\sqrt{n}} \right\})
\]

\[
\leq \mathbb{P}(T^c \cap \left\{ \max_{k \in \{1, \ldots, p^s\}} \left\| \left( e_k^T \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \right) \right\|_2 \leq \frac{33 \, c_{X, 1}}{\sqrt{n}} \right\})
\]

\[
+ c_1 \exp(-c_2^2 n) + 2/p^2
\]

(105)

because of Lemma 13. Further, by definition of the event \( T \),

\[
\mathbb{P}(T^c \cap \left\{ \max_{k \in \{1, \ldots, p^s\}} \left\| \left( e_k^T \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \right) \right\|_2 \leq \frac{33 \, c_{X, 1}}{\sqrt{n}} \right\})
\]

\[
= \mathbb{P}(\left\{ \max_{k \in \{1, \ldots, p^s\}} \left| e_k^T \mathcal{Q}_{S^c S} (\mathcal{Q}_{SS})^{-1} \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right|_S > \frac{\sqrt{3} \, 106 \, c_{X, 1}^2 \log(p)}{\sqrt{3} \, c_{X, 1}} \right\})
\]

\[
\leq \mathbb{P}(\left\{ \max_{u \in \mathbb{R}^p, \|u\|_2 \leq \frac{33 \, c_{X, 1}}{\sqrt{3} \, c_{X, 1}}} \left| u^T \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right|_S \right\})
\]

\[
= \mathbb{P}(\left\{ \max_{u \in \mathbb{R}^p, \|u\|_2 \leq \frac{33 \, c_{X, 1}}{\sqrt{3} \, c_{X, 1}}} \left| u^T \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right|_S \right\})
\]

(106)

In the following we proceed with a covering argument. Let \( A \) denote a 1/2-cover of cardinality \( N = \frac{n}{1/2} = 2^\frac{n}{\theta} \) of the ball \( B_2 = \{ u \in \mathbb{R}^p : \|u\|_2 \leq 1 \} \) of \( \mathbb{R}^p \) with respect to the Euclidean distance (cf. Rigollet and Hütter (2019, Definition 1.17) or Wainwright (2019, Definition 5.1)). Then, as in the proof of Rigollet and Hütter (2019, Theorem 1.19), we obtain

\[
\mathbb{P}(\max_{u \in \mathbb{R}^p} \left| u^T \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right|_S) > \frac{\sqrt{3} \, 106 \, c_{X, 1}^2 \log(p)}{\sqrt{3} \, c_{X, 1}} \right\})
\]

\[
\leq \mathbb{P}(\max_{u \in \mathbb{R}^p} \left| u^T \left( \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) \right|_S) > \frac{\sqrt{3} \, 106 \, c_{X, 1}^2 \log(p)}{\sqrt{3} \, c_{X, 1}} \right\})
\]

(107)

Now we can write for fixed \( u \in A \), analog to the proof of Lemma 16,

\[
u^T \left( \frac{33 \, c_{X, 1}}{\sqrt{3} \, c_{X, 1}} \nabla \mathcal{L}_{n, \alpha_n} (\beta_{\alpha_n}) \right) = \frac{1}{n} \sum_{i=1}^n v_{\alpha_i} (Y_i - X_i^T \beta_{\alpha_n}) Z_i
\]

with \( Z_i = 33 \, c_{X, 1}^2 / \sqrt{3} \, c_{X, 1} u^T (X_i) S \). The random variables \( v_{\alpha_i} (Y_i - X_i^T \beta_{\alpha_n}) Z_i \) are independent and identically distributed and have mean equal to zero, see proof of Lemma 16 for
more details. In addition (iii) of Assumption 1 implies that the random variables \(Z_1, \ldots, Z_n\) are sub-Gaussian with variance proxy \(1089 c^2_{\text{sub}} / c_{x,1}\). This leads to

\[
\mathbb{E} \left[ (l'_{\alpha_n} (Y_i - X_i^T \beta_{\alpha_n}^*) Z_i)^2 \right] \leq \frac{1089 c^2_{\text{sub}} c_{x,4}^{\text{Grad}}}{c_{x,1}}
\]

and

\[
\mathbb{E} \left[ |l'_{\alpha_n} (Y_i - X_i^T \beta_{\alpha_n}^*) Z_i|^u \right] \leq \frac{u!}{2} \left( \frac{233 c_{x,\text{sub}} c_{x,1}^{\text{Grad}}}{\sqrt{c_{x,1}} \alpha_n} \right)^{u-2} c_{x,1}^{\text{Grad}}
\]

for \(u \in \mathbb{N}, u \geq 3\), where \(c_{x,1}^{\text{Grad}}\) and \(c_{x,4}^{\text{Grad}}\) are given in the proof of Lemma 4. Bernstein's inequality and the choice of \(\alpha_n\) leads for fixed \(u \in A\) to

\[
P \left( \frac{1}{n} \sum_{i=1}^n l'_{\alpha_n} (Y_i - X_i^T \beta_{\alpha_n}^*) Z_i \geq \frac{66 c_{x,\text{sub}}}{\sqrt{c_{x,1}}} \left( \frac{8 c_{x,1}^{\text{Grad}} \log(p)}{n} \right)^{\frac{1}{2}} \right) \leq 2 \exp \left( -4 \log(p) \right),
\]

see proof of Lemma 4 for more details. By the union bound and the definition of \(c_2^{\text{Grad}}\) in the proof of Lemma 4 we get

\[
P \left( \max_{u \in A} \left| u^T \left( \frac{33 c_{x,\text{sub}}}{\sqrt{c_{x,1}}} \left( \nabla L_{n,\alpha_n} (\beta_{\alpha_n}^*) \right) \right) \right| > \frac{\sqrt{7} 33 c_{x,\text{sub}}}{\sqrt{3} c_{x,1}} c_2^{\text{Grad}} \left( \frac{\log(p)}{n} \right)^{\frac{1}{2}} \right) \leq 2 N \exp \left( -4 \log(p) \right) \leq 2 \exp \left( -4 \log(p) + s \log(6) \right)
\]

\[
\leq 2 \exp \left( -4 \log(p) + 2 \log(p) \right) = \frac{2}{p^2}
\]

(108)

since the 1/2-covering-number \(N\) can be upper bounded by \(6^s\), cf. Rigollet and Hütter (2019, Lemma 1.18) or Wainwright (2019, Example 5.8), and we assumed \(s \leq \log(p)\). In conclusion the inequalities (105) - (108) imply the assertion of the lemma. \(\square\)

### 7.4 Proof of (45) and of (46)

From (24) in Lemma 5 we obtain

\[
Q_{\beta_{n,\alpha_n} \setminus S} = \hat{Q}_{\mathcal{S}^c} \hat{Q}_{S^c}^{-1} \hat{Q}_{S(S_n \setminus S)} \beta_{\alpha_n, S_n \setminus S}^*
\]

\[
= \left( \frac{2}{n} \nabla \nabla_{n,S} D X_{n,S_n \setminus S}^T D X_{n, S_n \setminus S} (X_{n,S}^T D X_n, S_n \setminus S) \right) \beta_{\alpha_n, S_n \setminus S}^*
\]

\[
= \nabla \nabla_{n,S} \left( \frac{2}{n} D \nabla_{n,S} D X_{n,S_n \setminus S} (X_{n,S}^T D X_n, S_n \setminus S) \beta_{\alpha_n, S_n \setminus S}^* \right)
\]

The matrix in brackets, which we will denote by \(P\), is an orthogonal projection matrix. Therefore using Lemma 17, on an event with probability at least \(1 - 2/p^2\) we obtain

\[
\max_{k \in \{1, \ldots, p\}} \left\| \left( \frac{2}{n} \frac{1}{n} X_{n,S}^T D \hat{P} D \hat{P} \right) \right\| \leq \max_{k \in S^c} \left( \frac{2}{n} \right) \left\| D \hat{P} \right\|_{M,2} \left\| \hat{P} \right\|_{M,2} \left\| D \hat{P} \right\|_{M,2} \left\| X \right\|_{2} \leq \frac{2 \sqrt{\frac{17}{2}} c_{x,\text{sub}}}{n}
\]
since the entries of the diagonal matrix $D$ are smaller than or equal to 1. Setting $Q = X_{n,S_n} \beta_{\alpha_n,S_n}^*$, for $x > 0$ this leads to

$$
P\left( \max_{k \in \{1, \ldots, p\}} \left| \frac{2}{n} X_{n,S_n}^T D^\top P D^\top X_{n,S_n} \beta_{\alpha_n,S_n}^* \right| > x, \quad \frac{1}{n} \left\| \hat{X}_n \right\|_2 \leq \sqrt{\frac{c_{\text{sub}}}{n}} \right) \leq \frac{1}{n} \left\| \left( \frac{2}{n} X_{n,S_n}^T D^\top P D^\top \right)^\top \right\|_2 \leq \frac{2 \sqrt{\frac{c_{\text{sub}}}{n}}}{n} \right)
$$

The vector $Q$ has independent and sub-Gaussian entries

$$(X_i)_{S_n}^T \beta_{\alpha_n,S_n}^* = X_i^T \left( \beta_{\alpha_n,S_n}^*, 0_{(S_n \cap S^c)^c} \right) \sim \text{subG} \left( c_{\text{sub}}, \| \beta_{\alpha_n,S_n}^* \|_2 \right)
$$

by (iii) of Assumption 1, and Rigollet and Hütter (2019, Theorem 1.6) implies

$$2 \sqrt{\frac{c_{\text{sub}}}{n}} Q \sim \text{subG}_n \left( 2 \sqrt{\frac{c_{\text{sub}}}{n}} \| \beta_{\alpha_n,S_n}^* \|_2 \right)
$$

Finally Rigollet and Hütter (2019, Theorem 1.19) with the choice $\delta = \exp(-2n)$ leads to

$$\mathbb{P} \left( \max_{u \in \mathbb{R}^n: \|u\|_2 \leq 1} \left| u^T \left( 2 \sqrt{\frac{c_{\text{sub}}}{n}} Q \right) \right| > 16 \sqrt{\frac{c_{\text{sub}}}{n}} \| \beta_{\alpha_n,S_n}^* \|_2 \right) \leq \exp \left( -2n \right),$$

so that we obtain overall

$$\mathbb{P} \left( \left\| \left( \hat{Q}_{S,S_n} \right) - \hat{Q}_{S,S_n} \right\|_{\infty} > 16 \sqrt{\frac{c_{\text{sub}}}{n}} \| \beta_{\alpha_n,S_n}^* \|_2 \right) \leq \exp \left( -2n \right) + 2/p^2. \quad (109)$$

Similarly, for the vector $\hat{Q}_{S,S_n} \beta_{\alpha_n,S_n}^* = \frac{2}{n} X_{n,S}^T D X_{n,S_n} \beta_{\alpha_n,S_n}^*$, arguing as for (109) we obtain

$$\mathbb{P} \left( \left\| \hat{Q}_{S,S_n} \beta_{\alpha_n,S_n}^* \right\|_{\infty} > 16 \sqrt{\frac{c_{\text{sub}}}{n}} \| \beta_{\alpha_n,S_n}^* \|_2 \right) \leq \exp \left( -2n \right) + 2/p^2.$$

From Lemma 1 we get

$$\left\| \beta_{\alpha_n,S_n}^* \right\|_2 = \left\| \beta_{\alpha_n,S_n}^* - \beta_{S_n}^* \right\|_2 \leq \left\| \beta_{\alpha_n}^* - \beta^* \right\|_2 \leq C_{\text{apx}} \alpha_n^{-m-1}$$

since $\beta^*_l = 0$ for all $l \in S_{\alpha_n} \setminus S$. So in total we have

$$\left\| \hat{Q}_{S,S_n} \beta_{\alpha_n,S_n}^* \right\|_{\infty} \leq 80 C_{\text{apx}} c_{\text{sub}}^2 \alpha_n^{m-1}$$

with probability at least $1 - 2 \exp(-2n) - 2/p^2$, which yields the claimed inequalities in (45) and (46).