Berezinskii-Kosterlitz-Thouless phase transition with Rabi-coupled bosons

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We theoretically investigate the superfluid-normal-state Berezinskii-Kosterlitz-Thouless transition in a binary mixture of bosonic atoms with Rabi coupling under balanced densities. We find the nonmonotonic behavior of the transition temperature with respect to the intercomponent coupling and amplification of the transition temperature for finite values of Rabi coupling, but for small intracomponent couplings. We develop the Nelson-Kosterlitz renormalization-group equations in the two-component Bose mixture and obtain the Nelson-Kosterlitz criterion modified by a fractional parameter, which is responsible for half-integer vortices, and by Rabi coupling. Adopting the renormalization-group approach, we clarify the dependence of the Berezinskii-Kosterlitz-Thouless transition temperature on the Rabi coupling and the intercomponent coupling. Analysis of the first and second sound velocities also reveals the suppression of quasicrossing of the two sound modes with a finite Rabi coupling in the low-temperature regime. Our results for a two-dimensional binary Bose superfluid contribute to the understanding of a broad range of multicomponent quantum systems such as two-dimensional multiband superconductors.

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The Berezinskii-Kosterlitz-Thouless (BKT) transition is one of the most striking phenomena that occur in a two-dimensional (2D) superfluid realized in thin films of $^4$He [1–16], ultracold atoms in a planar geometry [17–38] or in a spherical bubble trap [39–42], and exciton-polariton systems [43–50]. The BKT transition originates from unbindings of vortex-antivortex pairs, and a proliferation of free vortices and antivortices [51–53]. It was first experimentally observed in thin $^4$He films [11] and later also in superconducting films [54–58], ultracold atomic gases [17–19, 22, 23, 28–33, 36], and exciton-polariton systems [48, 49]. A BKT transition to electron-hole superfluidity in 2D atomic double layers has been also predicted and is under current investigation [59, 60]. A stark contrast to three-dimensional (3D) superfluidity is a discontinuous jump of the superfluid density at the BKT transition temperature in a 2D superfluid [53, 61–67]. It also leads to a jump of the second sound velocity, which was experimentally measured recently with a $^{39}$K atomic gas [36]. To theoretically investigate the BKT transition, there are mainly two approaches. One is universal relations which are valid in the vicinity of the BKT transition temperature [22, 23, 68–70]. The other approach is to use the Nelson-Kosterlitz (NK) renormalization group (RG) equations, which are responsible for RG flows of the vortex fugacity and the phase stiffness associated with the superfluid density [53]. An advantage of the RG approach is that it is also valid in the low-temperature regime.

In contrast to a single-component Bose gas, a multicomponent Bose mixture has significant qualitative differences such as the Andreev-Bashkin entrainment effect between different species [38, 71–77], the emergence of fractional circulation of vorticity [78–95], and the modification of the NK criterion [96, 97]. There are also several theoretical analyses of the BKT transition in a bilayer XY model [98, 99], which has similarities to 2D binary Bose mixtures, and a Monte Carlo simulation in a binary Bose mixture with finite Rabi coupling [100]. Finite Rabi coupling makes half-quantized vortices, which are vortices in one of the two components of the Bose atoms, topologically unstable but makes vortex molecules, which consist of two vortices of both components with positive or negative charges, stable. Reference [100] proposed that the topological excitations that induce the BKT transition are also replaced with vortex molecule-antimolecule pairs instead of vortex-antivortex pairs. Renormalization group analysis taking into account these distinct topological excitations is crucial to predict physical quantities such as sound velocities and provide a coherent understanding of multicomponent superfluidity.

In this Letter, we consider a 2D atomic Bose gas confined in a quadratic region of area $L^2$, at temperature $T$, and with a chemical potential $\mu$ across the BKT transition temperature through the RG approach. The bosonic gas is characterized by atoms with two hyperfine components in their energy-level spectrum. In addition to the usual intraspecies ($g = g_{11} = g_{22} > 0$) and interspecies ($g_{12}$) contact interactions, atoms in different hyperfine states interact via an external coherent Rabi coupling of frequency $\omega_{R}(\geq 0)$, which drives an exchange of atoms between the two components. The presence of the Rabi coupling implies that only the total number $N = N_1 + N_2$ of atoms is conserved, with $N_{a} = 1, 2$ being the number of atoms in the $a$th hyperfine component. The existence and stability of the ground state with balanced densities $N_1 = N_2$ were extensively discussed in Refs. [101, 102]. We focus on the balanced and uniform ground state throughout this Letter.

Our two-component Bose-atom systems are a counterpart
to strongly coupled multiband superconductors in which all the partial condensates are close to the Bose-Einstein condensation regime. The Rabi coupling corresponds in multiband superconductors to the Cooper-pair exchange among different bands and even in the case of multiband systems, it is the total number of carriers that is conserved, with redistribution of densities among the bands depending on the parameter configuration and on the renormalization of the chemical potential [103–105]. Hence, the present investigation of Rabi coupled bosons can shed light on the BKT transition and collective modes in 2D multiband superconductors, a growing field of study for their fundamental interest and quantum technology applications [106].

We first examine the two branches of elementary excitations, which are related to Rabi coupling and intercomponent coupling. To consider the BKT transition, we develop NK RG equations in the two-component Bose gas. We point out that the NK criterion that provides the BKT transition temperature is modified due to the fractional parameter. The fractional parameter is also responsible for the half circulation of vorticity in a population-balanced binary Bose mixture. With finite Rabi coupling, on the other hand, the NK criterion reduces to the one in the single-component case related to the formation of vortex molecule-antimolecule pairs. This modification of the NK criterion is also consistent with previous theoretical predictions based on Monte Carlo analysis under balanced densities [100]. We investigate the dependence of the BKT transition temperature on Rabi coupling and intercomponent coupling. It shows a nonmonotonic behavior with respect to the intercomponent coupling and amplifies the maximum transition temperature for each value of Rabi coupling. Finally, we determine the first and second sound velocities across the BKT transition temperature. We confirm the jump of the second sound velocity at the BKT transition temperature. At low temperatures, in particular, finite Rabi coupling is found to hinder quasiscrossing behavior due to the presence of a gapped mode, in contrast to the single-component superfluids [65, 107–111].

The Bogoliubov spectrum of elementary excitations in a uniform system has two branches given by [101, 102]

\[
E_k^\pm = \sqrt{\epsilon_k [\epsilon_k + 2(\mu + \hbar \omega_R)]},
\]

\[
E_k^{(\pm)} = \sqrt{\epsilon_k (\epsilon_k + 2A) + B},
\]

with \(\epsilon_k = \hbar^2 k^2/(2m)\) and \(m\) being the atomic mass. We set \(\eta = g_{12}/g\), and the two parameters appearing in Eq. (2) are

\[
A = \frac{1 - \eta}{1 + \eta} (\mu + \hbar \omega_R) + 2 \hbar \omega_R,
\]

\[
B = 4 \hbar \omega_R \left[ \frac{1 - \eta}{1 + \eta} (\mu + \hbar \omega_R) + \hbar \omega_R \right].
\]

At the mean-field level, for the uniform ground state with balanced densities, the chemical potential \(\mu\) reads [101, 102]

\[
\mu = \frac{1 + \eta}{2} gn - \hbar \omega_R,
\]

where \(n = N/L^2\) is the 2D total number density of bosons. The uniform ground state with balanced densities, characterized by \(n_1 = n_2 = n/2\), is stable under the conditions \(g + g_{12} > 0\) and \((g - g_{12}) n + 2 \hbar \omega_R > 0\) [101, 102], namely, \(-1 < \eta < 1 + 2 \hbar \omega_R/(gn)\) with \(g > 0\). By using Eq. (5), parameters \(A\) and \(B\) become \(A = gn(1 - \eta)/2 + 2 \hbar \omega_R\) and \(B = 4 \hbar \omega_R [gn(1 - \eta)/2 + \hbar \omega_R]\). For small wavenumbers, the elementary excitations in Eqs. (1) and (2) read \(E_k^{(-)} = c_B \hbar k\) and \(E_k^{(\pm)} = \sqrt{\hbar^2 + \epsilon_k A}/\sqrt{\hbar^2 + \epsilon_k B}\), showing explicitly that the mode \(E_k^{(-)}\) is gapless while the mode \(E_k^{(\pm)}\) is gapped (if \(\omega_R \neq 0\)). Notice that \(c_B = [gn(1 + \eta)/(2m)]^{1/2}\) is the Bogoliubov speed of sound for the uniform system. For \(\eta = 1\), one recovers the familiar expression \(c_B = \sqrt{\hbar n/m}\).

By adopting Landau’s approach [112], at finite temperature \(T\), the superfluid density of the system is given by

\[
n_s^{(0)}(T) = n_0 - n_s^{(-)}(T) - n_s^{(+)}(T),
\]

where

\[
n_s^{(\pm)}(T) = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{\hbar^2 k^2}{2m} f_T(E_k^{(\pm)}),
\]

is the thermally activated normal density due to the elementary excitations. In the formula, \(f_T(E)\) is the derivative with respect to \(E\) of the Bose distribution function \(f_T(E) = 1/[e^{E/(k_BT)} - 1]\), with \(k_B\) being the Boltzmann constant.

It is important to stress that the superfluid density obtained in Eq. (6) does not take into account the formation of quantized vortices. The bare superfluid density \(n_s^{(0)}(T)\) goes to zero at a critical temperature that is larger than \(T_c\), the critical temperature of the BKT phase transition induced by the unbinding of vortex-antivortex pairs and the proliferation of free quantized vortices described by NK RG equations [51, 52]. In a single-component 2D Bose gas, the NK RG equations are given by [53, 113–115]

\[
\partial_l K(l)^{-1} = 4 \pi^3 g(y)^2, \quad \partial_l y(l) = [2 - \pi K(l)] y(l),
\]

with \(K(l) \equiv \hbar^2 n_s^{(0)}(T)/(m k_B T) = J(l)/(k_B T), J(l) = \hbar^2 n_s^{(0)}(T)/m\) being the phase stiffness, and \(y(l) \equiv \exp[-\mu_c(l)/(k_B T)]\), where \(\mu_c(l)\) is the vortex chemical potential at the dimensionless scale \(l\). The BKT critical temperature \(T_c^{(0)}\) can be obtained by using the NK criterion which provides a fixed point of Eqs. (8) [53]. According to this criterion, \(T_c^{(0)}\) is given by the implicit formula

\[
k_B T_c^{(0)} = \frac{\pi \hbar^2}{2m} n_s(T_c^{(0)}).
\]

In a binary Bose mixture with balanced densities \(\alpha_{a=1,2} = n_a/n = 1/2\); in contrast, we can obtain the following set of NK RG equations [100, 113–115]

\[
\partial_l K(l)^{-1} = 4 \pi^3 \Theta(\omega_R) y(l)^2,
\]

\[
\partial_l y(l) = [2 - \pi \Theta(\omega_R) K(l)] y(l),
\]
where $\Theta(x)$ is the Heaviside step function with $\Theta(0) = 1/2$. It can be derived from the microscopic Lagrangian as in the single-component case. For the details of the derivation, see the Supplemental Material [115]. The RG equations (10) give the modified NK criterion

$$k_{B}T_{c} = \frac{\pi \hbar^{2}}{2m} \Theta(\omega_{R}) n_{s}(T_{c})$$

(11)

at the BKT critical temperature $T_{c}$. This NK criterion (11) is consistent with the Monte Carlo analysis in Ref. [100]. To calculate the RG flow, we use the initial conditions $K(0) = \hbar^{2} n_{s}^{(0)}(T)/(m k_{B} T)$ and $\mu_{c}(0) = \pi^{2} \Theta(\omega_{R}) J(0)/4$ [116–119], where $n_{s}^{(0)}(T)$ is calculated using Eq. (6) with Eqs. (1), (2), and (7). The maximum value of the RG scale is related to the system size as $L_{\text{max}} = \ln (L/\xi)$, with $\xi = h/\sqrt{2 m g (n/2)}$ being the vortex core size. Here, we note that the higher-order derivative terms in the XY model can lead to corrections in the initial conditions for the RG flow. Indeed, it has been pointed out that the higher-order corrections are important for quantitatively accurate predictions of the BKT transition in XY models in particular for a small vortex chemical potential [120]. In our model of a binary Bose mixture, such a higher-order term of the superfluid velocity can arise and determine a quantitative change in our results with a small vortex chemical potential as well. In this Letter, however, since they are expected to produce moderate quantitative changes, we do not consider the effects of the spin-wave excitations on the vortex excitations, which will be the subject of a future investigation including the functional RG analysis [120, 121].

The modification of the NK criterion in the absence of Rabi coupling reflects the half circulation of vorticity. Indeed, the circulation of vorticity is given by [100]

$$\kappa = \oint ds \cdot v_{s} = \frac{h}{m} \oint ds \cdot \frac{(|\psi_{1}|^{2} \nabla \theta_{1} + |\psi_{2}|^{2} \nabla \theta_{2})}{|\psi_{1}|^{2} + |\psi_{2}|^{2}}$$

(12)

with $\psi_{a=1,2}$ being the $a$th complex bosonic field, where $v_{s}$ is the superfluid velocity associated with the superfluid phase $\theta_{a=1,2}$, and $s$ is the vector along the closed path enclosing vortices. With fractional parameters $\alpha_{a=1,2} = n_{a}/n$, for instance, each of the circulations for vortices $(\psi_{1}, \psi_{2}) \sim (\sqrt{n_{1}} e^{i \theta_{0}}, \sqrt{n_{2}})$, with $\theta_{0} = \arctan(y/x)$, is given by $\kappa_{1} = \pm 2 \pi \alpha_{1} h/m$ [100]. For a population-balanced system $n_{1} = n_{2} = n/2$; in particular, $\alpha_{1,2} = 1/2$ gives rise to half vortices. In the presence of Rabi coupling, on the other hand, topological defects that lead to a BKT transition are replaced with vortex molecule-antimolecule pairs instead of vortex-antivortex pairs [78, 81, 100]. The formation of vortex molecule pairs modifies the RG equations as in Eqs. (10), which recover the ones for the single-component case in Eqs. (8).

Figure 1 shows the renormalized superfluid fraction computed with Eqs. (10) for $\bar{g} = mg/h^{2} = 0.1$ and $\eta = 0$ with $L/\xi = 200$. Figure 1(a) displays the results with $\omega_{R} = h \omega_{R}/(\hbar^{2}/m) = 0, 0.1, 1.0$. The horizontal axis is the dimensionless temperature $k_{B} T/(\hbar^{2}/m) = 2 \pi/(n \lambda_{T}^{2})$.

FIG. 1. Renormalized superfluid fraction calculated with Eqs. (10) for $\bar{g} = mg/h^{2} = 0.1$ and $\eta = 0$. (a) displays the results with $L/\xi = 200$ and $\omega_{R} = h \omega_{R}/(\hbar^{2}/m) = 0, 0.1, 1.0$. The horizontal axis is the dimensionless temperature $2 \pi/(n \lambda_{T}^{2}) = k_{B} T/(\hbar^{2}/m)$. The gray dashed curve stands for the superfluid fraction in a single-component Bose gas with $\bar{g} = 0.1$ calculated with Eqs. (8). The thin dotted curves represent the bare superfluid fraction given by Eq. (6). The thin solid line and thin dotted line stand for $k_{B} T = \pi h^{2} n_{s}(T)/(4 m)$ and $k_{B} T = \pi h^{2} n_{s}(T)/(2 m)$, respectively. (b) shows the 3D plot of the superfluid fraction as a function of the temperature and Rabi coupling.

with $\lambda_{T} = [2 \pi h^{2}/(m k_{B} T)]^{1/2}$ being the thermal wavelength. The thin dotted curves stand for the bare superfluid fraction given by Eq. (6). Due to the finite size, the discontinuity of the renormalized superfluid fraction in the thermodynamic limit $L \to \infty$ is smeared and altered to a continuous drop [115]. In the single-component case plotted by the dashed curve, the superfluid fraction intersects with the thin dotted line for $k_{B} T = \pi h^{2} n_{s}/(4 m)$ at the BKT transition temperature as in Eq. (9) in the thermodynamic limit. In contrast, in a population-balanced binary Bose mixture, the superfluid fraction should intersect with the thin solid line for $k_{B} T = \pi h^{2} n_{s}/(4 m)$ in the absence of Rabi coupling at the BKT transition temperature as in Eq. (11) in the thermodynamic limit. With finite Rabi coupling, on the other hand, the superfluid fraction intersects with the thin dotted line for $k_{B} T = \pi h^{2} n_{s}/(2 m)$ at the BKT transition temperature in the thermodynamic limit as in the single-component Bose gas. A larger value of Rabi coupling shifts the transition temperature to a higher one. Figure 1(b) shows a 3D plot of the renormalized superfluid fraction as a function of the Rabi coupling and the temperature.

Figure 2 shows the phase diagram and the BKT transition temperature. In Fig. 2(a), the curves represent the $\eta$ dependence of the BKT transition temperature in the thermodynamic limit with $\bar{g} = 0.1$ and $\omega_{R} = 0, 0.1, 0.5$. The shaded region below the transition temperature is the superfluid phase with a finite superfluid density for each of the values of Rabi coupling, while the system is in the normal phase above that temperature. We can observe that, as $\eta$ increases from $-1$, the transition temperature first increases. Near $\eta = 1 + 2 h \omega_{R}/(g n)$, it reaches a maximum for each $\omega_{R}$ and changes to a gradual decrease. In particular, at $\omega_{R} = 0$, as displayed in Fig. 2(a), the BKT transition temperature is symmetric with respect to $\eta$ and reaches its maximum at $\eta = 0$. This is a natural consequence of the two symmetric excitations.
scales as \( \xi \approx T^{-1/2} \). The upper branch is the first sound velocity \( c_1 \), and the lower branch is identified as the second sound velocity \( c_2 \), which survives as long as the superfluid fraction is finite. Finite Rabi coupling increases the critical temperature \( T_c^{(0)} \) in the single-component Bose gas faster than \( T_c^{(0)} \) by increasing \( \tilde{g} \).

The propagation of sound waves occurs in a fluid due to density fluctuations, and the sound velocity is determined by thermodynamic properties. In a superfluid, in addition to the density wave, there is a collective mode associated with the entropy fluctuations originating from the no-entropy flow in superfluids.

The collective mode of the entropy wave is called the second sound \([65, 111, 122–124]\). The first and second sound velocities \( c_{1,2} \) are the roots of Landau’s two-fluid equation \( c^4 - (v_T^2 + v_L^2) c^2 + v_L^2 v_T^2 = 0 \), where \( v_T \), \( v_L \), and \( v_s \) are the isothermal, adiabatic, and Landau velocities, respectively, calculated from the free energy \([65, 115]\).

Figure 3 illustrates the first and second sound velocities for \( \tilde{g} = 0.1 \) and \( \eta = 0.5 \) with \( \omega_R = 0.0, 1.0 \). The upper branch is the first sound velocity \( c_1 \), and the lower branch is identified as the second sound velocity \( c_2 \), which survives as long as the superfluid fraction is finite. Finite Rabi coupling increases the critical temperature \( T_c^{(0)} \) in the single-component Bose gas faster than \( T_c^{(0)} \) by increasing \( \tilde{g} \).

Critical temperature \( T_c^{(0)} \) in the single-component Bose gas can be explained by the behavior of the energy gap in \( E_k^{(+)} \) due to the Rabi coupling. As one increases the Rabi coupling, the gap size also increases, and the normal density \( n_k^{(+)} \) in Eq. (7) decreases, while \( n_k^{(-)} \) is unaffected. This results in an increase of the superfluid density in Eq. (6), thereby leading to an enhancement of the BKT transition temperature according to Eq. (11) by replacing the renormalized superfluid density with the bare one, which is a good approximation at low temperatures as illustrated in Fig. 1(a). The maximum value of the transition temperature scaled by the one in the single-component case is shown in Fig. 2(c) with varying Rabi coupling. It monotonically increases by increasing \( \tilde{g} \).

Figure 2(c) also reveals that the ratio \( T_c^{\text{max}} / T_c^{(0)} \) is prominently enhanced as one decreases the intra-coupling strength \( \tilde{g} \). This behavior comes from monotonically increasing the critical temperature \( T_c^{(0)} \) in the single-component Bose gas faster than \( T_c^{(0)} \) by increasing \( \tilde{g} \).

The propagation of sound waves occurs in a fluid due to density fluctuations, and the sound velocity is determined by thermodynamic properties. In a superfluid, in addition to the density wave, there is another collective mode associated with the entropy fluctuations originating from the no-entropy flow in superfluids. The collective mode of the entropy wave is called the second sound \([65, 111, 122–124]\). The first and second sound velocities \( c_{1,2} \) are the roots of Landau’s two-fluid equation \( c^4 - (v_T^2 + v_L^2) c^2 + v_L^2 v_T^2 = 0 \), where \( v_T \), \( v_L \), and \( v_s \) are the isothermal, adiabatic, and Landau velocities, respectively, calculated from the free energy \([65, 115]\).
and entropy mode, respectively, as illustrated by the dashed curves in Fig. 3(d). As one increases the temperature, the two branches exhibit a quasicrossing at which the density mode and entropy mode start to mix as in the case of the single-component 2D Bose gas plotted with the thin dotted curves in Fig. 3 or a 3D Bose gas [65, 111]. In contrast, the solid curves in Fig. 3 imply that finite Rabi coupling suppresses the quasicrossing, as shown in Fig. 3(d), which is distinct from a single-component 2D Bose gas. This behavior can be understood by the presence of a gapped mode. With finite Rabi coupling, $E_k^{(+)\uparrow}$ is gapped out, as shown in the insets in Figs. 3(c) and 3(d), and most thermally excited bosons occupy only the gapless mode $E_k^{(-)\downarrow} \approx c_B\hbar k$. Then, the major difference from the single-component case is only the additional prefactor $1/2$ in Eqs. (7) which affects the Landau velocity. Consequently, the Landau velocity is found to be identical to the Bogoliubov velocity, which also coincides with the adiabatic velocity at zero temperature [115]. It results in the suppression of quasicrossing at a low temperature. The temperature at which the quasicrossing occurs characterizes the temperature above which the second sound can be detected by a density probe [64, 110, 125, 126]. From an experimental point of view, the suppression of quasicrossing at finite temperature implies that the second sound mode is sensitive to a density probe even in the low-temperature regime, which can be tested with ultracold-atom experiments [76, 125].

In summary, we investigated BKT transition in a Rabi-coupled binary Bose mixture under balanced densities. We have derived the NK RG equations for a binary Bose mixture and pointed out that the NK criterion is subject to change due to the fractional parameter and the Rabi coupling, consistent with the Monte Carlo simulation [100]. Based on the obtained RG equations, we clarified the whole behavior of the BKT transition temperature with respect to the Rabi coupling and intercomponent coupling. We found a nonmonotonic behavior of the transition temperature in terms of the intercomponent coupling and showed the maximum transition temperature for each value of Rabi coupling finding regimes of parameters resulting in an amplification of the transition temperature. Finally, we have studied the first and second sound velocities in this binary Bose mixture. We confirmed the jump in the second sound velocity as well as the superfluid density at the BKT transition temperature and elucidated the quasicrossing behavior of the two sound modes in the low-temperature regime. Our obtained NK criterion is consistent with the prediction based on Monte Carlo analysis for the population-balanced case [100]. On the other hand, Monte Carlo analysis has also predicted a double-step structure of the superfluid density in the population-imbalanced case [100, 101, 127–129]. A challenging open problem is to obtain a consistent result through the RG analysis in this population-imbalanced Bose mixture, extending the approach investigated in this work [121].

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Supplemental Material for "Berezinskii-Kosterlitz-Thouless phase transition with Rabi-coupled bosons"

Derivation of Nelson-Kosterlitz renormalization group equations

A single-component weakly-interacting Bose gas is described by the Lagrangian density

$$\mathcal{L} = i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} |\nabla \psi|^2 - \frac{g}{2} |\psi|^4,$$  \hspace{1cm} (S1)

with $\psi(r, t)$ the complex bosonic field, $m$ the atomic mass, and $g$ the interaction strength. To address the BKT physics, we employ Popov’s treatment: $\psi(r, t) = \tilde{\psi}(r, t)e^{i\theta(r)}$ assuming a time-independent phase $\theta(r)$ as a slowly varying field and $\bar{\psi}(r, t)$ as a fast field of the superfluid phase [S1, S2]. By integrating out the fast variable $\bar{\psi}$, the Euclidean action associated with the formation of vortices is given by

$$S[\theta] = \hbar \int d^2r \left( \frac{K}{2} |\nabla \theta|^2 \right),$$  \hspace{1cm} (S2)

where $K = h^2 n_s/(mk_B T) = J/(k_B T)$ with $J = h^2 n_s/m$ the phase stiffness and $n_s$ the superfluid density [S2]. Practically, Eq. (S2) can be obtained just by regarding the quasi-condensate density as a uniform superfluid density $n = |\psi|^2 = n_s$ and inserting it into Eq. (S1). In the presence of vortices, the XY model in Eq. (S2), which is equivalent to a Coulomb gas apart from the analytic spin-wave contribution, can be mapped to the sine-Gordon model described by [S3, S4]

$$S_{SG}[\phi] = \frac{\hbar}{2\pi^2 K} \int d^2r \left( \nabla \phi \right)^2 - \frac{2\hbar y}{\alpha^2} \int d^2r \cos(2\phi),$$  \hspace{1cm} (S3)

with $\phi(r)$ the analytic real field for the Coulomb gas, $y \equiv \exp[-\mu_v/(k_B T)]$ the dimensionless parameter characterizing the strength of the cosine potential corresponding to the vortex fugacity where $\mu_v$ is the vortex chemical potential, and $\alpha$ the short-range cutoff [S3, S4, S5]. To develop RG equations, we consider a correlation function [S3]

$$R(r_1 - r_2) = \left\langle e^{i\phi(r_1)}e^{-i\phi(r_2)} \right\rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{i\phi(r_1)}e^{-i\phi(r_2)}e^{-S_{SG}[\phi]/\hbar},$$  \hspace{1cm} (S4)

where $Z = \int \mathcal{D}\phi \exp[-S_{SG}[\phi]/\hbar]$ is the partition function. Neglecting the cosine potential in Eq. (S3), we get $R_0(r_1 - r_2) = \exp[-\pi K F(r_1 - r_2)/2]$ with $F(r) = \ln(|r|/\alpha)$. Perturbative expansion in terms of $y$ up to $O(y^2)$ results in

$$R(r_1 - r_2) = R_0(r_1 - r_2) \left[ 1 + \frac{y^2}{\alpha^4} \sum_{\sigma = \pm 1} \int d^2r' \int d^2r'' e^{-2\pi K F(r' - r'')} \left[ e^{\pi \sigma K G(r_1, r_2; r', r'')} - 1 \right] \right],$$  \hspace{1cm} (S5)

with $G(r_1, r_2; r', r'') = F(r_1 - r') - F(r_1 - r'') + F(r_2 - r'') - F(r_2 - r')$. Assuming $|r'| = |r' - r''| \ll |r' + r''|/2$, Eq. (S5) reduces to

$$R(r_1 - r_2) = R_0(r_1 - r_2) \left[ 1 + \frac{y^2}{\alpha^4} \int d^2r' e^{-2\pi K F(r')} \right].$$  \hspace{1cm} (S6)

Defining the effective strength $K_{eff}$ by $R(r) = \exp[-\pi K_{eff} F(r)/2]$, it is given by

$$K_{eff}^{-1} = K^{-1} + 4\pi^3 y^2 \int_0^{\infty} dx x^{3-2\pi K},$$  \hspace{1cm} (S7)

with a dimensionless length scale $x = r/\alpha$ up to $O(y^2)$. Splitting the spatial integral at a boundary $b = e^{dl} = 1 + dl$ and introducing

$$\tilde{K}^{-1} = K^{-1} + 4\pi^3 \tilde{y}^2 \int_1^{\infty} dx x^{3-2\pi \tilde{K}}, \hspace{1cm} \tilde{y} = y b^{2-\pi K},$$  \hspace{1cm} (S8)

one obtains

$$K_{eff}^{-1} = \tilde{K}^{-1} + 4\pi^3 \tilde{y}^2 \int_1^{\infty} dx x^{3-4\pi \tilde{K}},$$  \hspace{1cm} (S9)
which is equivalent to Eq. (S7) after rescaling $x \to x/b$. The set of equations (S8) leads to the NK RG equations (8) with $K(l) = \hbar^2 n_2(l) / (mk_B T)$ and $y(l) = \exp[-\mu_v(l)/k_B T]$ where $\mu_v(l)$ is the vortex chemical potential at the dimensionless scale $l$ [S3, S4, S5]. The BKT critical temperature $T_c^{(0)}$ can be obtained by the NK criterion (9) which provides a fixed point of Eqs. (8) [S5, S6].

A binary Bose mixture is described by

$$L = \sum_{a=1,2} \left[ i\hbar \psi_a^* \partial_t \psi_a - \frac{\hbar^2}{2m} \left| \nabla \psi_a \right|^2 - \frac{g}{2} \left| \psi_a \right|^4 \right] - g_{12} \left| \psi_1 \right|^2 \left| \psi_2 \right|^2 + \hbar \omega_R \left[ \psi_1^* \psi_2 + \psi_2^* \psi_1 \right].$$

(S10)

With the transformation $\psi_a(r,t) = \tilde{\psi}_a(r,t)e^{i\omega_a(r)}$, by integrating out $\tilde{\psi}_a$, the Euclidean action of a binary Bose mixture relevant to the formation of vortices is given by

$$S[\theta_1, \theta_2] = \hbar \int d^2 r \left[ \frac{K_1}{2} \left| \nabla \theta_1 \right|^2 + \frac{K_2}{2} \left| \nabla \theta_2 \right|^2 \right],$$

(S11)

in the absence of Rabi coupling $\omega_R = 0$ with $K_a = 1, 2 = \alpha_a K = J_a / (k_B T)$ where $J_a = \alpha_a J$ is the phase stiffness of each component with $\alpha_a = n_a / n$ the fractional parameter. As in the single-component XY model, the binary XY model (S11) can be mapped to the binary sine-Gordon model

$$S_{\text{SG}}[\phi_1, \phi_2] = \frac{\hbar}{2\pi^2} \int d^2 r \left[ \frac{K_1}{2} \left| \nabla \phi_1 \right|^2 + \frac{1}{K_2} \left| \nabla \phi_2 \right|^2 \right] + \frac{2\hbar}{\alpha^2} \int d^2 r \left[ y_1 \cos (2\phi_1) + y_2 \cos (2\phi_2) \right].$$

(S12)

We can follow a similar manner in the single-component case to derive the RG equations. Up to $O(y_1^2, y_2^2)$, for the symmetric case $\alpha_1 = \alpha_2 = 1/2$ with $y_1 = y_2 = y$, it provides a set of NK RG equations [S3, S4, S7]

$$\partial_l K(l)^{-1} = 2\pi^2 y^2_a, \quad \partial_l y(l) = \left[ 2 - \frac{\pi^2}{2} K(l) \right] y(l),$$

(S13)

which are Eqs. (10) for $\omega_R = 0$.

In the presence of Rabi coupling, the two components are coupled as

$$L_\theta = -\frac{J_1}{2} \left| \nabla \theta_1 \right|^2 - \frac{J_2}{2} \left| \nabla \theta_2 \right|^2 + 2\hbar \omega_R \sqrt{n_1 n_2} \cos (\theta_1 - \theta_2).$$

(S14)

It provides the equations of motion

$$J_1 \nabla^2 \theta_1 = 2\hbar \omega_R \sin (\theta_1 - \theta_2), \quad J_2 \nabla^2 \theta_2 = -2\hbar \omega_R \sin (\theta_1 - \theta_2).$$

(S15)

This set of equations of motion leads to domain wall solutions which are metastable states with a finite Rabi coupling [S8]. Inserting the equations of motion (S15) into the Lagrangian (S14), we obtain the optimized Lagrangian

$$L_\theta^{\text{opt}} = -\frac{2J_1}{2} \left| \nabla \theta_1 \right|^2 + \text{const.} + O(\nabla^4),$$

(S16)

for $n_1 = n_2 = n/2$. The action associated with the optimized Lagrangian (S16) is equivalent to Eq. (S2) in the single-component case because $2J_1 = J$. Note that the effective phase stiffness as a coefficient of $-(\nabla \theta_1)^2/2$ in Eq. (S16) is replaced with $J_1$ instead of $2J_1$ in the absence of Rabi coupling $\omega_R = 0$ because the two phases are no longer coupled. This is consistent with Eqs. (S13). As a result, neglecting the higher-order derivatives of $O(\nabla^4)$ and following the procedure mentioned above, we restore the RG equations identical to the ones in the single-component case. Taking into account Eqs. (S13) in the absence of Rabi coupling, we can write the RG equations as in Eqs. (10).

**Finite size effect on the superfluid density**

Figure S1 shows the renormalized superfluid fraction with varying system size length $L$ in a Rabi-coupled binary Bose mixture. In the thermodynamic limit $L \to \infty$, the renormalized superfluid density is finite below the BKT transition temperature, while it discontinuously drops to zero at the BKT transition temperature, which is determined by the NK criterion (11). As mentioned in the main text, Fig. S1 indicates that a finite system size smears the discontinuous drop.
The isothermal, adiabatic, and Landau velocities are given by

\[ v_T = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_T}, \quad v_s = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_s}, \quad v_L = \sqrt{\frac{n_N T s^2}{n_N c_V}}, \]  

respectively with

\[ P = -\left( \frac{\partial F}{\partial L^2} \right)_{N,T}, \quad s = \frac{1}{m N} \left( \frac{\partial F}{\partial T} \right)_{N,L^2}, \quad c_V = T \left( \frac{\partial s}{\partial T} \right)_{N,L^2}, \]

the pressure, the entropy per mass unit, and the specific heat at constant volume respectively. In a 2D single-component Bose gas, the free energy \( F \) at the mean-field level is given by [S9]

\[ F = \frac{g N^2}{2 L^2} + L^2 k_B T \int \frac{d^2 k}{(2\pi)^2} \ln \left[ 1 - e^{-E_k/(k_B T)} \right], \]

with the Bogoliubov spectrum \( E_k = \sqrt{\varepsilon_k (\varepsilon_k + 2gn)} \). The first term is free energy at zero temperature. The second term involving thermal excitations represents the thermal contribution at a finite temperature. In the phononic regime at low temperatures, the linear dispersion \( E_k = c_B \hbar k \) with \( c_B = \sqrt{gn/m} \) gives analytic expressions of the free energy and the normal density as

\[ \frac{F}{N} = \frac{gn}{2} - \frac{\zeta(3)}{2\pi\hbar^2} \frac{(k_B T)^3}{n c_B^2}, \quad n_n = -\frac{3\zeta(3)}{2\pi\hbar^2} f^{'T}(E_k) = \frac{3\zeta(3)}{2\pi\hbar^2} \frac{(k_B T)^3}{m c_B^4}, \]

with \( \zeta(x) \) the Riemann zeta function [S9, S10]. At zero temperature, Eqs. (S17) with Eqs. (S18) and (S20) lead to

\[ c_1 = v_T = v_s = c_B, \quad c_2 = v_L = \frac{c_B}{\sqrt{2}}. \]  

In a two-component Bose mixture, the free energy \( F \) at the mean-field level is given by

\[ F = \frac{1 + \eta gn}{4} \frac{N^2}{L^2} - \hbar \omega_R N + L^2 k_B T \int \frac{d^2 k}{(2\pi)^2} \ln \left[ 1 - e^{-E_k^{(-)}/(k_B T)} \right] + \ln \left[ 1 - e^{-E_k^{(+)}/(k_B T)} \right]. \]

In the phononic regime at low temperatures without Rabi coupling \( \omega_R = 0 \), \( E_k^{(-)} = c_B \hbar k \) with \( c_B = \sqrt{(1 + \eta gn)/(2m)} \) and \( E_k^{(+)} = c^+ \hbar k \) with \( c^+ = \sqrt{(1 - \eta gn)/(2m)} \) provide

\[ \frac{F}{N} = \frac{1 + \eta gn}{2} \frac{N^2}{2\pi\hbar^2} \frac{\zeta(3)}{n} \frac{(k_B T)^3}{n (c^+_B)^2} + (c^+_B)^2, \quad n_n = n_n^{(+)} + n_n^{(-)} = \frac{3\zeta(3)}{2\pi\hbar^2} \frac{(k_B T)^3}{m (c^+_B)^2} + \frac{(1 + \eta gn)}{2m (c^+_B)^2}. \]
which results in

\[ v_T = v_s = c_B, \quad v_L = \sqrt{\frac{c_1^2 + c_B^2}{c_1^4 + c_B^4}} = \sqrt{\frac{1 - \eta}{1 + \eta^2}} c_B, \]  

(S24)

at zero temperature. For \(-1 < \eta \leq 0\), \(v_L \geq v_s\) leads to \(c_1 = v_L\) and \(c_2 = v_s\). For \(0 < \eta < 1\), \(v_L < v_s\) leads to \(c_1 = v_s\) and \(c_2 = v_L\). With a finite Rabi coupling \(\omega_R > 0\), on the other hand, the thermal contribution associated with the gapped branch \(E_k^{(+)}\) vanishes at zero temperature. Then, the free energy and normal density are given by

\[ F = \frac{1 + \eta}{2} \frac{gn}{2} \frac{\hbar}{2} \frac{\zeta(3)}{2\pi} \frac{(k_B T)^3}{n c_B^4}, \quad n_n = \frac{3\zeta(3)}{2\pi} \frac{(k_B T)^3}{2m c_B^4}. \]  

(S25)

The thermal part of free energy is identical to the one in the single-component case while the normal density is half of that in the single-component case in Eqs. (S20) because of the prefactor \(1/2\) in Eqs. (7). As a result, the sound velocities at zero temperature change to

\[ c_{1,2} = v_T = v_s = v_L = c_B, \]  

(S26)

at any value of inter-component coupling \(\eta\). The difference from the single-component case in Eqs. (S21) is ascribed to the modification of the normal density in Eqs. (S25).

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