Supersymmetric Calogero-Moser-Sutherland models: superintegrability structure and eigenfunctions

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Abstract

We first review the construction of the supersymmetric extension of the quantum Calogero-Moser-Sutherland (CMS) models. We stress the remarkable fact that this extension is completely captured by the insertion of a fermionic exchange operator in the Hamiltonian: sCMS models (s for supersymmetric) are nothing but special exchange-type CMS models. Under the appropriate projection, the conserved charges can thus be formulated in terms of the standard Dunkl operators. This is illustrated in the rational case, where the explicit form of the $4N$ ($N$ being the number of bosonic variables) conserved charges is presented, together with their full algebra. The existence of $2N$ commuting bosonic charges settles the question of the integrability of the srCMS model. We then prove its superintegrability by displaying $2N - 2$ extra independent charges commuting with the Hamiltonian. In the second part, we consider the supersymmetric version of the trigonometric case (stCMS model) and review the construction of its eigenfunctions, the Jack superpolynomials. This leads to closed-form expressions, as determinants of determinants involving supermonomial symmetric functions. Here we focus on the main ideas and the generic aspects of the construction: those applicable to all models whether supersymmetric or not. Finally, the possible Lie superalgebraic structure underlying the stCMS model and its eigenfunctions is briefly considered.¹

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1 Introduction

This presentation has two main objectives. The first is to stress the interpretation of the supersymmetric extension of the quantum Calogero-Moser-Sutherland (CMS) models \[1\] as special exchange-type CMS models \[2\]. This provides a direct path for establishing their integrability \[3\] and also, as is first shown here, their superintegrability. For this analysis, which is the content of section 2, we focus on the rational case. Our second goal is to provide a simple quasi-qualitative presentation of the main ideas underlying our recent construction of the supersymmetric trigonometric CMS (stCMS) eigenfunctions, that is, of the Jack superpolynomials \[3, 4, 5\]. This is the content of section 3. Most technicalities are avoided. We stress that our method is a general program for constructing explicitly the eigenfunctions of a CMS-type Hamiltonian, supersymmetric or not. Some subsections are thus formulated in rather general terms.

2 Integrability and superintegrability of the srCMS model

2.1 Supersymmetric quantum mechanics and fermionic-exchange operators

Consider a quantum model, whose Hamiltonian is denoted \( \mathcal{H} \), that contains bosonic and fermionic variables. That is, that contains in addition to the 2\( N \) bosonic variables \((x_i, p_i)\), the 2\( N \) fermionic variables \((\theta_i, \theta_j^\dagger)\) (e.g., \( \theta_i \theta_j = - \theta_j \theta_i \), so that \( \theta_i^2 = 0 \)). The bosonic and fermionic variables satisfy respectively a Heisenberg and a Clifford algebra:

\[
[x_j, p_k] = i \delta_{jk}, \quad \{\theta_j, \theta_k^\dagger\} = \delta_{jk}, \tag{1}
\]

where \( \{ , \} \) stands for the anticommutator. All other commutators or anticommutators are equal to zero. We usually work with a differential realization of these algebras:

\[
p_j = -i \frac{\partial}{\partial x_j}, \quad \theta_j^\dagger = \frac{\partial}{\partial \theta_j}. \tag{2}
\]

To construct a supersymmetric Hamiltonian, we will first construct two supersymmetric charges, denoted \( Q \) and \( Q^\dagger \), and define the Hamiltonian as their anticommutator:

\[
\mathcal{H} = \frac{1}{2} \{ Q, Q^\dagger \}. \tag{3}
\]

The Hamiltonian is invariant under a supersymmetric transformation if:

\[
Q^2 = (Q^\dagger)^2 = 0. \tag{4}
\]

With the charges written under the form

\[
Q = \sum_{i=1}^N \theta_i^\dagger A_i(x, p), \quad Q^\dagger = \sum_{i=1}^N \theta_i A_i^\dagger(x, p), \tag{5}
\]

eq. \(4\) requires:

\[
[A_i, A_j] = 0 = [A_i^\dagger, A_j^\dagger], \quad \forall i, j. \tag{6}
\]

We use the qualitative ‘Calogero-Moser-Sutherland’ to describe the generic class of models that includes the models studied by Calogero and Sutherland in the quantum case and by Moser in the classical case. In this work, however, we only treat the quantum case. In this context the name of Moser is often omitted. The rationale for its inclusion is due to the fundamental importance of the Lax formulation in the quantum case, which is a direct extension of the classical one that he introduced.
The generic supersymmetric Hamiltonian thus reads:

\[ H = \frac{1}{2} \left( \sum_i A_i^† A_i + \sum_{i,j} \theta_i^† \theta_j [A_i, A_j^†] \right). \] (7)

For non-relativistic models, the Hamiltonian is proportional to the square of the particles’ momenta. This forces \( A_i \) to be a linear function of the momentum \( p_i \), that is:

\[ Q = \sum_j \theta_j^† (p_j - i\Phi_j(x)), \quad Q^† = \sum_j \theta_j (p_j + i\Phi_j(x)). \] (8)

Condition (6) imposes that the potential \( \Phi_j(x) \) be of the form

\[ \Phi_j(x) = \partial_{x_j} W(x), \] (9)

where \( W(x) \) (called the prepotential), is an arbitrary function of the variables \( x_1, \ldots, x_N \). The supersymmetric Hamiltonian now takes the form [6]:

\[ H = \frac{1}{2} \sum_i (p_i^2 + (\partial_{x_i} W)^2 + \partial_{x_i}^2 W) - \sum_{i,j} \theta_i^† \theta_j^† \partial_{x_i} \partial_{x_j} W. \] (10)

This Hamiltonian is an extension of the purely bosonic model whose potential is \( \sum_i (\partial_{x_i} W)^2 + \partial_{x_i}^2 W \). We stress that this construction fixes uniquely the supersymmetric extension of the model to be considered. Specializing to a symmetric prepotential that can be broken up into a sum of two-body interactions:

\[ W(x) = \sum_{i<j} w_{ij}(x_{ij}) \quad \text{with} \quad w'(x_{ij}) = X_{ij}, \] (11)

where \( X_{ij} \) stands for an antisymmetric function of \( x_{ij} = x_i - x_j \), our Hamiltonian reads:

\[ H = \frac{1}{2} \sum_i p_i^2 + \sum_{i<j} \left[ X_{ij}^2 + X_{ij}' (1 - \theta_i^† \theta_j^†) \right] + \sum_{i<j<k} Y_{ijk}, \] (12)

with

\[ Y_{ijk} = X_{ij} X_{ik} + X_{ji} X_{jk} + X_{ki} X_{kj}. \] (13)

This is the supersymmetric Hamiltonian we were looking for.

The main observation at this point is the following: the term

\[ \kappa_{ij} = 1 - \theta_i^† \theta_j^† = 1 - (\theta_i - \theta_j)(\partial_{\theta_i} - \partial_{\theta_j}). \] (14)

which captures the whole dependence of the Hamiltonian upon the fermionic variables, is a fermionic-exchange operator [7]. In other words, its action on an arbitrary function \( g \) of the fermionic variables \( \theta_i \) and \( \theta_i^† \) reads

\[ \kappa_{ij} g(\theta_i, \theta_j, \theta_i^†, \theta_j^†) = g(\theta_j, \theta_i, \theta_j^†, \theta_i^†) \kappa_{ij}. \] (15)

In addition, it satisfies all the properties of an exchange operator:

\[ \kappa_{ij} = \kappa_{ji}, \quad \kappa_{ij}^† = \kappa_{ij}, \quad \kappa_{ij} \kappa_{jk} = \kappa_{ik} \kappa_{ij} = \kappa_{jk} \kappa_{ki}, \quad \kappa_{ij}^2 = 1. \] (16)

That the supersymmetric extension is fully captured by the introduction of a fermionic exchange operator is thus a generic feature of supersymmetric many-body problems whose interaction is decomposable into a sum of two-body interactions. This is also a key technical tool in our subsequent analysis, as we will shortly explain. But we would like to make the observation that, up this point, the discussion is quite general and not restricted to integrable problems.
Having an Hamiltonian expressed in terms of a fermionic exchange operator implies that under the right projection, we can trade \( \kappa_{ij} \) for the ordinary exchange operator, \( K_{ij} \), that exchanges the variables \( x_i \) and \( x_j \):

\[
K_{ij} f(x_i, x_j) = f(x_j, x_i) K_{ij},
\]

with \( f(x_i, x_j) \) standing for a function or an operator. The suitable projection is in our case the one on the space \( P_{SN} \) of functions invariant under the simultaneous exchange of the bosonic and the fermionic variables, that is, invariant under the action of

\[
K_{ij} = K_{ij} \kappa_{ij}.
\]

On this space, the supersymmetric Hamiltonian reduces to an ordinary exchange-type Hamiltonian \[3\]. Stated differently: acting on the proper space, we can study many features of the supersymmetric model without even introducing fermionic degrees of freedom!

### 2.2 The integrability of the srCMS model

To work with a concrete example, consider the simple srCMS model \[6\]:

\[
H(r) = -\frac{1}{2} \sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < k \leq N} \frac{\beta(\beta - K_{jk})}{(x_j - x_k)^2}.
\]

We will establish the integrability of this model using the Dunkl-operator formalism, thus relying heavily on the projection trick.

Using the rational Dunkl operators,

\[
D_j = \partial_{x_j} - \beta \sum_{k \neq j} \frac{1}{x_{jk}} K_{jk},
\]

we find

\[
-\frac{1}{2} \sum_{j=1}^{N} D_j^2 = -\frac{1}{2} \sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < k \leq N} \frac{\beta(\beta - K_{jk})}{(x_j - x_k)^2}.
\]

The Dunkl operators are commuting and covariant:

\[
[D_j, D_i] = 0, \quad K_{ij} D_j = D_i K_{ij}.
\]

Therefore, the conservation laws of the rCMS model (with exchange terms) are simply \( \sum_j D_j^n \), for \( n = 1, \ldots, N \). The conserved charges of the rCMS model without exchange terms are obtained by simply projecting these expressions onto the space of symmetric functions, which amounts to replacing every factor \( K_{ij} \) by 1 once pushed to the right. On the other hand, the conserved charges of the srCMS model are obtained as follows:

\[
\mathcal{H}_n = \sum_j D_j^n \bigg|_{P_{SN}}, \quad n = 1, \ldots, N.
\]

In particular, \( \mathcal{H}_2 = -2H(r) \). The explicit dependence of the srCMS conserved charges \( \mathcal{H}_n \) upon the fermionic variables can be obtained by implementing the projection, that is, replacing every factor \( \kappa_{ij} \) by \( K_{ij} \) once shifted to the right. The proof of the commutativity of these charges leans on a simple property of the projection:

\[
[A \bigg|_{P_{SN}}, B \bigg|_{P_{SN}}] = [A, B] \bigg|_{P_{SN}} \quad \text{if} \quad [\kappa_{ij}, A] = [\kappa_{ij}, B] = 0.
\]

The operators \( \mathcal{H}_n \) meet this requirement since \( [\kappa_{ij}, (\sum_k D_k^n)] = 0 \). Therefore, the commutator can be evaluated before doing the projection and its vanishing is an immediate consequence of the commutativity of the Dunkl operators.
We have thus found the supersymmetric extension of all the usual rCMS charges and proved their commutativity. This however does not imply the integrability of the srCMS model because there are more degrees of freedom in the supersymmetric case (these are the $2N$ extra Grassmannian variables). Consequently, one should expect more conserved charges. It is actually rather easy to construct $3N$ extra charges involving explicitly (i.e., even before the projection) some fermionic variables [3]:

\[
Q_{(n)} = \sum_i \theta_i D_i^n \bigg|_{p^{\text{SN}}} , \quad n = 0, 1, \ldots, N - 1 ,
\]

\[
Q_{(n)}^\dagger = \sum_i \theta_i D_i^n \bigg|_{p^{\text{SN}}} , \quad n = 0, 1, \ldots, N - 1 ,
\]

\[
I_{(n)} = \sum_i \theta_i \theta_i^\dagger D_i^n \bigg|_{p^{\text{SN}}} , \quad n = 0, 1, \ldots, N - 1 .
\]

These charges satisfy the following algebra:

\[
\{ Q_{(n)}, Q_{(m)} \} = \{ Q_{(n)}^\dagger, Q_{(m)}^\dagger \} = \{ I_{(n)}, I_{(m)} \} = 0 ,
\]

\[
[ Q_{(n)}, H_{(m)} ] = [ Q_{(n)}^\dagger, H_{(m)} ] = [ I_{(n)}, H_{(m)} ] = 0 .
\]

(26)

Together with

\[
\{ Q_{(n)}^\dagger, Q_{(m)}^\dagger \} = H_{(n+m)} ,
\]

\[
[ Q_{(n)}^\dagger, I_{(m)} ] = \mathcal{Q}_{(n+m)}^\dagger , \quad [ Q_{(n)}, I_{(m)} ] = - \mathcal{Q}_{(n+m)} .
\]

(27)

There are thus $4N$ conserved charges, $2N$ bosonic and $2N$ fermionic. Only the $2N$ bosonic ones are mutually commuting and independent.

Actually, it would seem at first sight that there are $(N+1)$ $I_{(n)}$-type charges and thus a total of $2N+1$ mutually commuting conserved charges. However, it can be checked that say $I_{(N)}$ can be expressed in terms of the lower order $I_{(n)}$ as well as the $H_{(n)}$’s. Let us illustrate this in the simple context of two particles ($N = 2$):

\[
H_1 = D_1 + D_2 ,
\]

\[
H_2 = D_1^2 + D_2^2 ,
\]

\[
I_0 = \theta_1 \theta_1^\dagger + \theta_2 \theta_2^\dagger ,
\]

\[
I_1 = \theta_1 \theta_1^\dagger D_1 + \theta_2 \theta_2^\dagger D_2 .
\]

(28)

The operator $I_2$ depends of the preceding four conserved quantities. Indeed, it is easily checked that

\[
I_2 = I_1 H_1 - \frac{1}{2} I_0 \left( H_1^2 - H_2 \right) .
\]

(29)

It is clear that there can be no more than $2N$ mutually commuting conserved charges since this is the maximal number of commuting operators in the free case ($\beta = 0$), in which case these quantities are $\{ p_i, \theta_i \theta_i^\dagger \}$.

The srCMS model is thus seen to be integrable in the usual sense. \(^3\)

### 2.3 The superintegrability of the rsCMS model

Consider the following $2N$ quantities:

\[
\mathcal{L}_n = \sum_j x_j D_j^{n+1} \bigg|_{p^{\text{SN}}} , \quad n = -1, \ldots, N - 2 ,
\]

\[
\mathcal{M}_n = \sum_j x_j \theta_j \theta_j^\dagger D_j^{n+1} \bigg|_{p^{\text{SN}}} , \quad n = -1, \ldots, N - 2 .
\]

(30)

\(^3\)The integrability can also be established via the Lax formalism [3]. The form of the Lax operator is actually the same as in the non-supersymmetric case, except that every factor $X_{ij}$ (differentiated or not) is replaced by $X_{ij} \kappa_{ij}$. 

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5
They satisfy the algebra
\[
[L_n, L_m] = (n - m)L_{n+m}, \quad [\mathcal{H}_n, L_m] = n\mathcal{H}_{n+m}, \quad [\mathcal{H}_n, \mathcal{M}_m] = n\mathcal{I}_{n+m}.
\] (31)
These relations are easily obtained using:
\[
[D^n_l, x_j] = \delta_{ij} \left( nD^{n-1}_l - \beta \sum_{k \neq l} \frac{D^n_l - D^n_k}{D_l - D_k} K_{ij} \right) - \beta (1 - \delta_{ij}) \frac{D^n_l - D^n_j}{D_l - D_j} K_{ij},
\] (32)
for \( n = 0, 1, 2 \ldots \) and:
\[
[x_i D^n_l, x_j D^m_j] = \delta_{ij} \left[ (n - m)D^{n+m-1}_l + \beta x_i \sum_{k \neq l} D^n_l D^m_k - D^m_k D^n_k K_{ij} \right]
- \beta (1 - \delta_{ij}) \left[ \frac{x_i D^{n+m}_l + x_j D^{n+m}_j}{D_l - D_j} K_{ij} - (x_i + x_j) \frac{D^m_j D^n_l}{D_l - D_j} K_{ij} \right],
\] (33)
for \( n, m = 1, 2, 3 \ldots \).

We can construct from these, \( 2N - 2 \) new and independent conserved charges (i.e., commuting with \( \mathcal{H}_2 \)):
\[
J_n = \mathcal{H}_{n+1} - \mathcal{L}_{n-1} - \mathcal{H}_1, \quad K_n = \mathcal{I}_{n+1} - \mathcal{M}_{n-1} - \mathcal{I}_1.
\] (34)
This a direct supersymmetric extension of the argument given in [8]. The srCMS model is thus not only super and integrable, but also superintegrable.

3 The stCMS model: construction of the eigenfunctions

3.1 The stCMS Hamiltonian

We now move to the second part of this work and discuss the construction of the eigenfunctions of the stCMS model. If the integrability structure of the (s)CMS models is most simply analyzed in the rational case, the study of the eigenfunctions is most naturally done in the trigonometric case. The Hamiltonian of the stCMS model reads [7, 3]:
\[
\mathcal{H}_{(t)} = -\frac{1}{2} \sum_{i=1}^{N} x_i^2 + \left( \frac{\pi}{L} \right)^2 \sum_{i<j} \frac{\beta (\beta - \kappa_{ij})}{\sin^2 (\pi x_{ij} / L)} - \left( \frac{\pi \beta}{L} \right)^2 \frac{N(N^2 - 1)}{6},
\] (35)
where \( L \) is the circumference of the circle on which the particles are confined. Removing the contribution of the ground-state wave function,
\[
\psi_0(x) = \Delta^\beta (x) \equiv \prod_{j<k} \sin^\beta \left( \frac{\pi x_{jk}}{L} \right),
\] (36)
\footnotesize
\begin{itemize}
  \item The expression of \([x_i D^n_l, x_j D^m_j]\) contains many misprints in [8].
  \item Furthermore, the algebra [9] obviously implies the algebraic linearization of the 4N equations of motion:
\end{itemize}
\[
\frac{d\mathcal{H}_n}{dt} = 0, \quad \frac{d\mathcal{L}_n}{dt} = 0, \quad \frac{d\mathcal{M}_n}{dt} = -i\mathcal{H}_{n+2}, \quad \frac{d\mathcal{I}_n}{dt} = -i\mathcal{L}_{n+2}.
\] Such a linearization is also possible for the stCMS model. It follows from a direct generalization of the approach of [10] (using the Lax formalism). On the other hand, note that the superintegrability of the classical rCMS model was proved in [9].

\footnotesize
\begin{itemize}
  \item Of course there are no bound states in the rational case. The presence of bound states necessitates the introduction of an harmonic confinement. But then the eigenfunctions turn out (somewhat surprisingly) to be expressible in terms of the eigenfunctions of the trigonometric case, the Jack polynomials [13].
\end{itemize}
the transformed Hamiltonian then becomes
\[
\hat{\mathcal{H}} = \frac{1}{2} \left( \frac{L}{\pi} \right)^2 \Delta^{-\beta} \mathcal{H}(\ell) \Delta^{\beta}.
\] (37)

Expressed in terms of the new bosonic variables \( z_j = e^{2\pi i x_j/L} \), it finally reads
\[
\hat{\mathcal{H}} = \sum_i (z_i \partial_i)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j) - 2\beta \sum_{i<j} \frac{z_i z_j}{z_{ij}} (1 - \kappa_{ij}).
\] (38)

### 3.2 The \( \hat{\mathcal{H}} \) eigenfunctions as symmetric superpolynomials

The eigenfunctions of the Hamiltonian are superpolynomials, namely polynomials of the bosonic variables \( z_i \) and the fermionic ones \( \theta_i \). Looking for the proper generalization of the Jack polynomials, we are interested in eigenfunctions that are symmetric with respect to the simultaneous interchange of both types of variables, i.e., invariant under the action of \( \mathcal{K}_{ij} \).

Manifestly, \( \mathcal{H} \) leaves invariant the space of polynomials of a given degree in \( z \) and a given degree in \( \theta \), being homogeneous in both sets of variables. We consider eigenfunctions of the form:
\[
A^{(m)}(z; \theta; \beta) = \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq N} \theta_{i_1} \cdots \theta_{i_m} A^{(i_1 \ldots i_m)}(z; \beta), \quad m = 0, 1, 2, 3, \ldots,
\] (39)
where \( A^{(i_1 \ldots i_m)} \) is a homogeneous polynomial in \( z \). Due to the presence of \( m \) fermionic variables in its expansion, \( A^{(m)} \) is said to belong to the \( m \)-fermion sector. We stress that the simple dependence upon the fermionic variables, which factorizes in monomial prefactors, is a consequence of their anticommuting nature.

We now clarify the symmetry properties, with respect to the \( z \) variables, of the eigenfunctions \( A^{(m)} \), assumed to be invariant under the action of the exchange operators \( \mathcal{K}_{ij} \). Given that the \( \theta \) products are antisymmetric, i.e.,
\[
\kappa_{i_a i_b} \theta_{i_1} \cdots \theta_{i_m} = -\theta_{i_1} \cdots \theta_{i_m} \quad \text{if} \quad i_a, i_b \in \{i_1, \ldots, i_m\},
\] (40)
the superpolynomials \( A^{i_1 \ldots i_m} \) must be partially antisymmetric to ensure the complete symmetry of \( A^{(m)} \). More precisely, the functions \( A^{i_1 \ldots i_m} \) must satisfy the following relations:\footnote{Note that the case \( m = 1 \), with \( A^{(1)} = \sum_i \theta_i A^{(1;1/\beta)} \), is special in that regard: \( K_{ij} A^k = A^k \) if and only if \( i, j \neq k \).}
\[
K_{ij} A^{i_1 \ldots i_m}(z; \beta) = -A^{i_1 \ldots i_m}(z; \beta) \quad \forall \quad i \text{ and } j \in \{i_1 \ldots i_m\},
\]
\[
K_{ij} A^{i_1 \ldots i_m}(z; \beta) = -A^{i_1 \ldots i_m}(z; 1/\beta) \quad \forall \quad i \text{ and } j \notin \{i_1 \ldots i_m\}. \] (41)

We have thus established that any symmetric eigenfunction of the stCMS model contains terms of \textit{mixed symmetry} in \( z \): each polynomial \( A^{i_1 \ldots i_m} \) is completely antisymmetric in the variables \( \{z_{i_1}, \ldots, z_{i_m}\} \), and totally symmetric in the remaining variables \( z/\{z_{i_1}, \ldots, z_{i_m}\} \).

To proceed further, we need to address the following points: how to label the eigenfunctions; how to define a natural basis for the space of superpolynomials; how to define the eigenfunctions via a triangular expansion in that basis. These points are considered successively in the following subsections.

### 3.3 Superpartitions

Symmetric polynomials are indexed by partitions. In the same manner, symmetric superpolynomials, i.e., polynomials in \( \mathcal{P}^{S_N} \), can be indexed by \textit{superpartitions}. A superpartition in the \( m \)-fermion sector is a sequence of non-negative integers that generates two partitions separated by a semicolon \footnote{Note the case \( m = 1 \), with \( A^{(1)} = \sum_i \theta_i A^{(1;1/\beta)} \), is special in that regard: \( K_{ij} A^k = A^k \) if and only if \( i, j \neq k \).}:
\[
\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N) = (\lambda^a; \lambda^s),
\] (42)
the first one being associated to an antisymmetric function of the variables \( \{ z_i, \ldots, z_{i_m} \} \)

\[
\lambda^a = (\Lambda_1, \ldots, \Lambda_m), \quad \Lambda_i > \Lambda_{i+1} \quad i = 1, \ldots m - 1,
\]

and the second one, to a symmetric function of the variables \( \{ z_{i_m+1}, \ldots, z_{i_N} \} \)

\[
\lambda^s = (\Lambda_{m+1}, \ldots, \Lambda_N), \quad \Lambda_i \geq \Lambda_{i+1} \quad i = m + 1, \ldots N.
\]

In the zero-fermion sector \((m = 0)\), the semicolon disappears and we recover the partition \(\lambda^s\).

The weight (or degree) of a superpartition is simply the sum of its parts. For instance, the only possible superpartitions of weight 3 in the 1-fermion sector are:

\[
(3;0), \quad (2;1), \quad (1;2), \quad (1;1,1), \quad (0;3), \quad (0;2,1), \quad (0;1,1,1),
\]

while in the 2-fermion sector, they are:

\[
(3,0;0) \quad (2,1;0) \quad (2,0;1) \quad (1,0;2) \quad (1,0;1,1).
\]

### 3.4 The monomial symmetric superpolynomials basis

Given our goal of constructing the superextension of the Jack polynomials, which themselves decompose triangularly in the symmetric monomial basis, the superextension of the latter will provide our natural expansion basis. The monomial symmetric superpolynomials (supermonomials for short) are defined as:

\[
m_{\alpha}(z, \theta) = m(\lambda_1, \ldots, \lambda_m; \lambda_{m+1}, \ldots, \lambda_N)(z, \theta) = \sum_{\sigma \in S_N} \sum_{\xi \in \Lambda} \theta^{\sigma(1) \cdot \cdots \cdot \sigma(m)} z^{\sigma(\lambda)},
\]

where the prime indicates that the summation is restricted to distinct terms, and where

\[
|z|^{\sigma(\lambda)} = z_1^{\lambda_1} \cdots z_m^{\lambda_m} \cdot z_{m+1}^{\lambda_{m+1}} \cdots z_N^{\lambda_N}
\]

and

\[
\theta^{\sigma(1) \cdot \cdots \cdot \sigma(m)} = \theta_{\sigma(1)} \cdots \theta_{\sigma(m)}.
\]

Clearly, in the zero-fermion sector, a supermonomial reduces to an ordinary symmetric monomial.

Here is an example with \(N = 4\) that neatly illustrates the mixed symmetry of each component:

\[
m_{(1,0,1,1)} = \theta_1 \theta_2(z_1 - z_2)(z_3 z_4) + \theta_1 \theta_3(z_1 - z_3)(z_2 z_4) + \theta_2 \theta_3(z_2 - z_3)(z_1 z_4) + \theta_2 \theta_4(z_2 - z_4)(z_1 z_3) + \theta_3 \theta_4(z_3 - z_4)(z_1 z_2)
\]

Two other examples (with \(N = 3\)) will clarify the different role of a zero entry in each sector:

\[
m_{(3;0)} = \theta_1 z_1^3 + \theta_2 z_2^3 + \theta_3 z_3^3
\]

\[
m_{(0;3)} = \theta_1 (z_2^3 + z_3^3) + \theta_2 (z_1^3 + z_3^3) + \theta_3 (z_1^3 + z_2^3)
\]

There are other simple bases for the space of symmetric superpolynomials. For instance, one can combine the \(2N\) algebraically independent symmetric power sums,

\[
p_n = \sum_i z_i^n = m_{(n)}, \quad q_{n-1} = \sum_i \theta_i z_i^{n-1} = m_{(n,0)}, \quad n = 1, \ldots, N,
\]

to form a new basis:

\[
p_{\lambda} = q_{\alpha} p_{\lambda} = q_{\lambda_1} \cdots q_{\lambda_m} p_{\lambda_{m+1}} \cdots p_{\lambda_N}.
\]

One can also generate the whole space with the elementary symmetric superfunctions,

\[
e_n = \sum_{1 \leq i_1 < \cdots < i_n \leq N} z_{i_1} \cdots z_{i_n}, \quad f_{n-1} = \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq N} \theta_j z_{i_1} \cdots z_{i_{n-1}} = m_{(0,1n-1)}.
\]

or with the complete symmetric superfunctions,

\[
h_n = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq N} z_{i_1} \cdots z_{i_n} = \sum_{|\lambda| = n} m_{\lambda}, \quad j_{n-1} = \sum_{|\lambda| = n-1} (\Lambda_1 + 1) m_{\lambda},
\]

where \(n = 1, \ldots, N\) while \(|\lambda| = \sum_i \lambda_i\) and \(m\) stand for the degrees in \(z\) and \(\theta\) respectively.
3.5 Jack superpolynomials: a first definition

We now define our first candidates for the role of Jack superpolynomials (in the $m$-fermion sector) as the unique eigenfunctions of the supersymmetric Hamiltonian $\hat{H}$,

$$\hat{H} J_{\Lambda}(z, \theta; \beta) = \varepsilon_{\Lambda} J_{\Lambda}(z, \theta; \beta),$$

that can be decomposed triangularly in terms of monomial superfunctions:

$$J_{\Lambda}(z, \theta; \beta) = m_{\Lambda}(z, \theta) + \sum_{\Omega: \Omega < \Lambda} c_{\Lambda, \Omega}(\beta)m_{\Omega}(z, \theta).$$

Clearly, for this definition to be complete, we need to specify the ordering ($<$) underlying the triangular decomposition. The simplest and most natural choice at this point is to formulate the ordering in terms of the partitions $\Lambda^*$ and $\Omega^*$ associated respectively to the superpartitions $\Lambda$ and $\Omega$, by rearranging their parts in decreasing order. (For instance, the rearrangement of $\Lambda = (7, 4, 3, 1; 8, 6, 5, 3, 1)$ is the partition $\Lambda^* = (8, 7, 6, 5, 4, 3, 3, 3, 1, 1)$.) We thus say that $\Omega < \Lambda$ if $\Omega^* < \Lambda^*$ with respect to the dominance ordering on partitions:

$$\Lambda^* \geq \Omega^* \iff \Lambda_i^* + \Lambda_2^* + \ldots + \Lambda_i^* \geq \Omega_1^* + \Omega_2^* + \ldots + \Omega_i^*, \quad \forall i,$$

with $\Lambda_i^* = (\Lambda^*)_i$. This turns out to be sufficient to calculate explicitly the Jack superpolynomials $J_{\Lambda}$. Here is an example of such a polynomial:

$$J_{(2;2)} = m_{(2;2)} + \frac{2\beta}{1 + \beta} m_{(2;1^2)} + \frac{\beta}{1 + \beta} m_{(1;2^1)} + \frac{6\beta^2}{(1 + \beta)(1 + 2\beta)} m_{(1^3)}.$$

Although complete, the above characterization is not quite precise. Indeed, not all supermonomials $m_{\Omega}$ labeled by a superpartition $\Omega$ dominated by $\Lambda$ in the above way do appear in the decomposition. For instance, the terms $m_{(0;2;1,1)}$ and $m_{(0;1^4)}$, although allowed by the ordering condition, are not present in the expression of $J_{(2;2)}$. This poses the following natural question: can we characterize precisely the terms that appear in the supermonomial triangular decomposition of $J_{\Lambda}$, that is, can we pinpoint the precise ordering at work? A second (and more ambitious) related question is the following: can we write down an explicit formula for the coefficients $c_{\Lambda, \Omega}$?

The answer to both of these questions is yes. The clue to obtain the answer lies in the following observation: the action of $\hat{H}$ on the supermonomial basis is triangular and this triangularity determines precisely the ordering entering in the definition of the eigenfunctions. Moreover, the action of $\hat{H}$ can be computed exactly, essentially because it can be reduced to a two-body computation. This turns out to provide all the data required for evaluating the coefficients $c_{\Lambda, \Omega}$. These conclusions are completely general and apply to any CMS model, supersymmetric or not. In the next subsections, we indicate the key steps in reaching these conclusions, keeping the presentation rather general. Technical details can be found in [4].

3.6 Generalities: the triangular action of the Hamiltonian, the induced ordering and a determinantal formula for the eigenfunctions

Consider a generic Hamiltonian $H$ and a generic basis $m_a$. (Typically, the $m_a$ will be monomial functions but another basis could be used.) Suppose that, in a given subspace$^9$ spanned by three basis elements $m_a$, $m_b$, $m_c$ (for some labels $a, b, c$), the action of $H$ takes the following form:

$$H m_a = \epsilon_a m_a + v_{ak} m_k + v_{ac} m_c,$$
$$H m_b = \epsilon_b m_b + v_{bc} m_c,$$
$$H m_c = \epsilon_c m_c.$$

$^9$In the context of the stCMS, this subspace could be fixed by the set of superpartitions with given degree $n$ and given fermion number $m$. However, this could also be a subspace pertaining to a a non-supersymmetric problem, hence characterized by some conditions on ordinary partitions.
where the eigenvalues are all distinct. The action of $H$ is obviously triangular. It is also clear that the triangularity entails an ordering: $a > b > c$. The key point is that this ordering is precisely the one governing the triangular expansion of the eigenfunctions of $H$, denoted $J_a$, in the $m_a$ basis. This follows from the following result: if $H$ acts on the $m_a$'s as in (55), its eigenfunction reads [11]:

$$J_a \propto \begin{vmatrix} m_c & m_b & m_a \\ \epsilon_c - \epsilon_a & v_{bc} & v_{ac} \\ 0 & \epsilon_b - \epsilon_a & v_{ab} \end{vmatrix}. \quad (56)$$

Let us verify that $(H - \epsilon_a)J_a$ does indeed vanish. Since the entries of the determinant are numbers except for the first row, $H$ acts nontrivially only on this row:

$$(H - \epsilon_a)J_a \propto \begin{vmatrix} (\epsilon_c - \epsilon_a)m_c & (\epsilon_b - \epsilon_a)m_b + v_{bc}m_c & (\epsilon_a - \epsilon_a)m_a + v_{ab}m_b + v_{ac}m_c \\ \epsilon_c - \epsilon_a & v_{bc} & v_{ac} \\ 0 & \epsilon_b - \epsilon_a & v_{ab} \end{vmatrix}. \quad (57)$$

The coefficient of $m_a$ is identically zero. The coefficient of $m_b$ is most simply obtained by setting $m_c = 0$. This leads to

$$(H - \epsilon_a)J_a\big|_{m_c=0} \propto \begin{vmatrix} 0 & (\epsilon_b - \epsilon_a)m_b & v_{ab}m_b \\ \epsilon_c - \epsilon_a & v_{bc} & v_{ac} \\ 0 & \epsilon_b - \epsilon_a & v_{ab} \end{vmatrix}. \quad (58)$$

The first row being proportional to the third one, the determinant is zero. Similarly, by setting $m_b = 0$, we see that the first two rows become proportional, which again enforces the vanishing of the determinant. Since the expansion coefficients of $(H - \epsilon_a)J_a$ in the monomial basis are all zero, we can conclude that $(H - \epsilon_a)J_a = 0$.

The eigenfunction $J_a$ is thus given by the determinant (58) up to a multiplicative constant. Note that it is not equal to zero because the eigenvalues being all distinct, the coefficient of $m_a$ cannot vanish. But this shows readily (i.e., by expanding the determinant) that it can be written under the form $J_a \propto (m_a + \cdots)$ where the dots refer to lower order terms in the ordering induced by the action of $H$ in the $m_a$ basis. This is precisely the point we wanted to emphasize.

Note that the eigenfunction can be determined uniquely by simply enforcing its leading expansion coefficient to be one (monic condition).

This simple example shows neatly that the ordering governing the decomposition of $J_a$ in the $m_a$ basis is encoded in the triangularity of $H$ on $\{m_a\}$. However, it is somewhat misleading in its simplicity. It suggests that all terms occurring in the decomposition of $J_a$ can be compared with each others and that a single `chain of ordering’ (which refers to the case where all $m_i$ with $i$ comparable to $a$ do appear in the action of $H m_a$) is always involved. We will thus consider a second slightly more complicated case that captures the generic features.

Consider a five-dimensional subspace spanned by $m_a$, $m_b$, $m_c$, $m_d$, $m_e$ and suppose the following action of Hamiltonian:

$$
\begin{align*}
H m_a &= \epsilon_a m_a + v_{ab}m_b + v_{ac}m_c + v_{ad}m_d \\
H m_b &= \epsilon_b m_b + v_{bc}m_c \\
H m_c &= \epsilon_c m_c + v_{ce}m_e \\
H m_d &= \epsilon_d m_d + v_{de}m_e \\
H m_e &= \epsilon_e m_e.
\end{align*}
\quad (59)
$$

This action implies the following chains of ordering: $a > b > c$ as well as $c > e$ and $d > e$. Hence, we see that the ordering within the subspace is only partial because for instance $c$ and $d$ cannot be
compared (this is also the case for \(b\) and \(d\)). The corresponding eigenfunction is\(^{10}\)

\[
J_a \propto \begin{vmatrix}
m_e & m_d & m_c & m_b & m_a \\
\epsilon_e - \epsilon_a & v_{de} & v_{ce} & 0 & 0 \\
0 & \epsilon_d - \epsilon_a & 0 & 0 & v_{ad} \\
0 & 0 & \epsilon_c - \epsilon_a & v_{bc} & v_{ac} \\
0 & 0 & 0 & \epsilon_c - \epsilon_a & v_{ab}
\end{vmatrix}.
\] (60)

It thus follows that both \(m_c\) and \(m_d\) appear in the expression of \(J_a\), even if \(c\) and \(d\) cannot be compared with each other. The main point is that they are both comparable to \(a\). This example also illustrates our second point: \(e\) is comparable to \(a\) but \(m_e\) does not appear in the expression of \(H m_a\). In other words, \(e\) is compared to \(a\) by a sequence of two chains of orderings: \(a > b > c\) and \(c > e\). We will later relate the number of chains of ordering and the number of applications of a ladder operator on labels.

The construction of determinantal expressions for the eigenfunctions provides a strong motivation for obtaining explicitly the coefficients \(v_{ab}\) appearing in the action of \(H\) on the monomial basis: they are the building blocks of the expansion coefficients \(c_{ab}\) of \(J_a\) in that basis. The knowledge of these coefficients leads thus to `closed-form expressions for the eigenfunctions'! Quite remarkably, these coefficients are indeed computable, as we will now show.

### 3.7 Generalities: the explicit action of \(H\) by the universal dressing of a model-dependent \(N = 2\) computation

As we just noticed, the specification of the ordering requires the determination of the coefficients \(v_{ab}\) that do not vanish in the action of \(H\) on \(m_a\). Actually, we will see that it is not much more complicated to compute precisely all these coefficients \(v_{ab}\). And, as pointed out, this yields directly the eigenfunctions.

The strategy is the following: we first compute the action of \(H\) in the two-particle sector. Such computations are always very easy. For a CMS-type model, whose Hamiltonian is a sum of two-body interactions, this computation turns out to encode the core value of the coefficient \(v_{ab}\). Indeed, to go from \(N = 2\) to a general \(N\) simply amounts to dressing the result by a symmetry factor \([12, 11, 4]\). And another remarkable fact is that, although the \(N = 2\) computation is model dependent, the symmetry dressing appears to be universal, that is, the symmetry factor boils down to a symmetry of the labels (partitions or superpartitions) of the eigenfunctions, hence independent of the root structure, or the rational or trigonometric version of the (s)CMS under consideration. This is certainly so for all the cases we have considered so far (including the rational case with confinement \([14]\)).

An immediate objection could be formulated with regards to this program: although it is clear that the action of \(H\) on \(m_a\) is well-defined in the \(N = 2\) sector, we know that for a general \(N\)-body problem, the action of \(H\) on a particular element \(m_a\), that itself does not vanish upon reduction to \(N = 2\), may contain terms that disappear upon reduction. Are these terms properly taken into account?

To make the above considerations more concrete and precise, we will now return to the stCMS model. This will also allow us to address the potential objections in a definite context.

### 3.8 The transposition of the action of \(\tilde{H}\) on superpartitions: ladder operators

When the action of the Hamiltonian is a sum of two-body terms \(\sum \tilde{H}_{ij}\) (up to a derivative part that acts diagonally), as is the case for the (s)CMS model, the various terms appearing in the decomposition of \(\tilde{H} m_A\) can be characterized by the action of a ladder operator \(R_{ij}(t)\) acting on the superpartitions

\(^{10}\)The validity of this result requires that eigenvalues, corresponding to labels that can be compared, be distinct.
More precisely, $R_{ij}^{(\ell)}$ acts on two parts $\Lambda_i$ and $\Lambda_j$ of a superpartition as follows:

$$
R_{ij}^{(\ell)}(\Lambda_1, \ldots, \Lambda_i, \ldots, \Lambda_j, \ldots, \Lambda_N) = \begin{cases} 
(\Lambda_1, \ldots, \Lambda_i - \ell, \ldots, \Lambda_j + \ell, \ldots, \Lambda_N) & \text{if } \Lambda_i > \Lambda_j, \\
(\Lambda_1, \ldots, \Lambda_i + \ell, \ldots, \Lambda_j - \ell, \ldots, \Lambda_N) & \text{if } \Lambda_j > \Lambda_i.
\end{cases}
$$

(61)

for $i < j$ and $\ell \geq 0$. This action of $R_{ij}^{(\ell)}$ is non-zero only in the following cases:

I : $i, j \in \{1, \ldots, m\}$ and $\left\lfloor \frac{\Lambda_i - \Lambda_j - 1}{2} \right\rfloor \geq \ell,$

II : $i \in \{1, \ldots, m\}, j \in \{m + 1, \ldots, N\}$ and $|\Lambda_i - \Lambda_j| \geq 1 \geq \ell,$

III : $i, j \in \{m + 1, \ldots, N\}$ and $\left\lfloor \frac{\Lambda_i - \Lambda_j}{2} \right\rfloor \geq \ell.$

(62)

Here $\left\lfloor x \right\rfloor$ stands for the largest integer smaller or equal to $x$. This characterization is obtained by a consideration of the action of $\bar{\gamma}$ on the $N = 2$ sector. Three cases must then be distinguished, according to their fermion number: $m = 2, 1, 0$, corresponding respectively to pairs $(i, j)$ of type I, II, III. The results pertaining to ordinary Jack polynomials correspond to case III.

Let us make the induced ordering (denoted $\geq^\gamma$) explicit. Given a sequence

$$
\gamma = (\gamma_1, \ldots, \gamma_m; \gamma_{m+1}, \ldots, \gamma_N),
$$

(63)

we will denote by $\bar{\gamma}$ the superpartition whose antisymmetric part is the rearrangement of $(\gamma_1, \ldots, \gamma_m)$ and whose symmetric part is the rearrangement of $(\gamma_{m+1}, \ldots, \gamma_N)$. For example, we have

$$
(1, 3, 2; 2, 3, 1, 2) = (3, 2, 1; 3, 2, 2, 1).
$$

(64)

We say that

$$
\Lambda \geq^\gamma \Omega \quad \text{iff} \quad \Omega = R_{i_k, j_k}^{(\ell_k)} \cdots R_{i_1, j_1}^{(\ell_1)} \Lambda
$$

(65)

for a given sequence of operators $R_{i_k, j_k}^{(\ell_k)}, \ldots, R_{i_1, j_1}^{(\ell_1)}$. This is a refinement of the ordering introduced previously at the level of the corresponding partitions $\Lambda^*$ and a generalization of the dominance ordering, which is recovered in the zero-fermion sector.

All the supermonomials $m_\Omega$ appearing in the expansion of $\mathcal{H} m_\Lambda$ are precisely those whose labeling superpartition is obtained from $\Lambda$ by one application of the ladder operator $R$, that is, all $\Omega$ that can be written as $\Omega = R_{ij}^{(\ell)} \Lambda$ for some $i < j$, and $\ell > 0$. This implies that the number of non-zero parts of $\Omega$ exceeds that of $\Lambda$ by at most one. On the other hand, the supermonomials $m_\Gamma$ appearing in the expansion of $J_\Lambda$ (which is now understood to represent a determinant expansion) are precisely those whose label $\Gamma$ is obtained from $\Lambda$ by any number of applications of the ladder operator $R$. We can now make the connection with the loose terminology of the previous sections: one application of $R$ corresponds to a single chain of ordering while multiple chains of ordering are described by the action of a sequence of ladder operators.

The expressions for the coefficients $v_{\Lambda, \Omega}$, as well as a detailed description of the symmetry factors, can be found in [4] and will not be repeated here. Let us instead address the potential problem pointed out previously: the action of $\mathcal{H}$ on a two-part superpartition may contain more than two parts, hence lie beyond the $N = 2$ sector, which was claimed to contain all the relevant information. Consider for instance

$$
\mathcal{H} m_{(3, 1)} = (10 + 4 \beta N - 6 \beta) m_{(3, 1)} + 2 \beta m_{(2, 2)} + 8 \beta m_{(2, 1, 1)} + 2 \beta m_{(1, 2, 1)}
$$

(66)

which holds for arbitrary values of $N > 2$. However, if we specialize to $N = 2$, the last two terms disappear. How could these be taken into account by a $N = 2$ computation? The point is that the different terms $m_{\Omega}$ in $\mathcal{H} m_\Lambda$ are not necessarily linked to the same two-body calculation. While the contribution of the first non-diagonal term $m_{(2, 2)}$ can be described by the two-body interaction $\mathcal{H}_{12}$, that is, $(2; 2) = R_{12}^{(1)} (3; 1)$, this is not the proper two-body interaction for the description of the Hamiltonian, there are in addition some terms that act non-diagonally (as in the (s)rCMS model [11]), their action is taken into account by another ladder operator acting on a single entry of a superpartition.
other two terms. Indeed, we see that \((2; 1, 1) = R^{(1)}_{13} (3; 1, 0)\). This shows that the relevant \(N = 2\) computation is rather \(\hat{H}_{13} m_3 (3; 0)\). Similarly, because \((1; 1, 2) = R^{(2)}_{13} (3; 1, 0)\) and \((1; 1, 2) = (1; 2, 1)\), the corresponding two-body problem is again \(\hat{H}_{13} m_3 (3; 0)\).

It should be heavily stressed that the remarkable fact that the result for general \(N\) can be deduced out of the \(N = 2\) one is true for the action of \(\hat{H}\) in the supermonomial basis; this would not be possible directly for the eigenfunctions \(J_A\).

### 3.9 \(J_A\) as eigenfunctions of the \(\hat{H}_n\) charges

The method sketched in the previous subsections indicate that we can construct closed-form expressions for the eigenfunctions \(J_A\) of the stCMS Hamiltonian \(\hat{H}\), our first candidates for the role of Jack superpolynomials. Actually, we can prove that the \(J_A\)’s diagonalize the whole tower of conserved charges \(\hat{H}_n\) [5] (recall that the bar indicates that the contribution of the ground state wave function has been taken away):

\[
\hat{H}_n J_A = e^{(n)}_A J_A.
\]

Unfortunately these superpolynomials are not orthogonal. In other words, the action of the \(\hat{H}_n\)’s leaves a residual degeneracy. Indeed, we can check that \(e^{(n)}_A = e^{(n)}_\Omega\) for all \(n\) if \(\Lambda^* = \Omega^*\) even when \(\Lambda \neq \Omega\) [5]. However, in retrospect, this is not too surprising: we have not constructed the common eigenfunctions of all the commuting conserved charges. We still need to diagonalize the \(\bar{I}_n\) charges.

### 3.10 Constructing orthogonal eigenfunctions: diagonalization of the \(\bar{I}_n\) charges

As just indicated, the eigenfunctions of the Hamiltonian \(\hat{H} = \hat{H}_2\) happen to be eigenfunctions of all the \(\hat{H}_n\) operators. Similarly, to study of the eigenfunctions of the complete set \(\{\bar{I}_n\}\), it suffices to consider only the first non-trivial charge \(\bar{I}_1\) [4].

The remaining problem is thus to construct linear combinations of the \(J_A\) that are eigenfunctions of \(\bar{I}_1\). The strategy is by now clear: we first compute the action of \(\bar{I}_1\) on the \(J_A\) basis, determine the new ordering induced by the underlying triangularity, compute the expansion coefficients exactly, and then write down the eigenfunctions, now denoted \(J_A\), in determinantal form. The self-adjointness property of the charges \(\{\hat{H}_n\}\) and \(\{\bar{I}_n\}\) ensures the orthogonality of the \(J_A\). These are thus the genuine Jack superpolynomials. The details of this construction can be found in [4].

Let us mention that a more direct way to build the orthogonal Jack superpolynomials is also presented in [5]. It amounts to simply symmetrizing the product of a non-symmetric Jack polynomial with a monomial in the fermionic variables. Even though this approach is more direct, we prefer not to include it since it is not as much in the spirit of this presentation.

### 3.11 A remark on the underlying Lie superalgebraic structure

Let us conclude this section on the stCMS model by addressing the following question: is there a Lie superalgebra structure underlying this problem? Given that the tCMS Hamiltonian is closely linked to the \(su(N)\) root structure, the natural guess is that the stCMS Hamiltonian would be related to the superalgebra \(su(m, N - m)\). The root structure of \(su(m, N - m)\) is (see for instance [15]):

\[
\delta_i - \delta_j, \quad \epsilon_k - \epsilon_\ell, \quad 1 \leq i < j \leq m \quad \text{and} \quad m + 1 \leq k < \ell \leq N - m \quad (68)
\]

with

\[
\delta_i \cdot \delta_j = -\delta_{ij}, \quad \epsilon_k \cdot \epsilon_\ell = \delta_{k\ell}, \quad \delta_i \cdot \epsilon_k = 0 \quad (69)
\]

When \(m = 0\), we recover the \(su(N)\) roots. The roots \(\delta_i - \delta_j\) and \(\epsilon_k - \epsilon_\ell\) are said to be bosonic while the remaining roots \(\delta_i - \epsilon_k\) are called fermionic. However, it should be clear that there is no
genuine notion of statistics at the level of the roots themselves (although the generators do have a definite statistics) since for instance the difference between two fermionic roots could be bosonic (e.g., $\delta_1 - \epsilon_1 - (\delta_2 - \epsilon_1) = \delta_1 - \delta_2$). This, of course, is not a deep observation. However, it readily implies that the Hamiltonian constructed from the root system of a Lie superalgebra, $su(m, N - m)$ for instance, does not contain anticommuting variables (see [16] for example). Hence, the stCMS Hamiltonian does not appear to have an immediate interpretation in terms of the $su(m, N - m)$ algebra.

However, there does exist a link with the $su(m, N - m)$ root structure, but at the level of the action of the ladder operator $R_{ij}^{(l)}$. Referring to equations (61) and (62), we see that the action of $R_{ij}^{(l)}$ in case I and III corresponds to the subtraction of positive bosonic roots, while in case II, it amounts to subtract a positive or a negative fermionic root. That both positive and negative fermionic roots are involved is not surprising given the relativity of the positivity requirement for these roots. At the level of superpartitions, the sign of the fermionic roots should not play any role because it is simply related to our choice of relative ordering for the two partitions $\lambda^e$ and $\lambda^v$ composing the superpartition.

4 Conclusion

The integrability and superintegrability of the sCMS models rely on a fundamental observation: the supersymmetric extension is fully captured by a fermionic exchange operator. This allowed us to infer the (super)integrability by means of a projection argument. Many generalizations of the srCMS model could be shown, in a similar way, to be superintegrable: the srCMS models formulated for general root systems, the extension of the rCMS model with many supersymmetries, the srCMS model with spin degrees of freedom, etc.

On the other hand, we have stressed that the method outlined in section 3 for constructing eigenfunctions is quite general. It has already been extended to the srCMS case with confinement, that is, to the construction of orthogonal generalized Hermite (super)polynomials [14]. Generalized Jacobi and Laguerre polynomials in superspace could also be obtained along this line from the sr/tCMS model with $B$-type roots. The method can be easily extended to models with more supersymmetries or including spin degrees of freedom. The astonishing power of the method gives us hope that this line of attack could provide a breakthrough in the study the eigenfunctions of elliptic models.

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