Grand Ensembles of deterministic operators. II. Localization for generic ”haarsh” potentials

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Abstract: We consider a class of families of deterministic random lattice Schrödinger operators with potentials depending upon an infinite number of parameters on an auxiliary measurable space. We prove Anderson localization for generic families in the strong disorder regime, using a variant of the Multi-Scale Analysis. In our model, the potential is generated by a function on a torus which is discontinuous (”haarsh”) and constructed with the help of an expansion which reminds Haar’s wavelet expansions (but is not orthogonal), so we call such potentials ”haarsh”. A different approach, also using a parameter exclusion technique, has been used by Chan [Chan07] for one-dimensional lattice Schrödinger operators with quasi-periodic, single-frequency potential which was assumed to be of class $C^3(S^1)$.

1. Introduction. Formulation of the results.

In this paper, we study spectral properties of finite-difference operators, usually called lattice Schrödinger operator (LSO), of the form

$$(Hf)(x) = \sum_{y : \|y-x\|=1} f(y) + V(x)f(x), \quad x, y \in \mathbb{Z}^d.$$ 

Such an operator is obviously bounded whenever the function $V : \mathbb{Z}^d \to \mathbb{R}$ (usually referred to as the potential) is bounded. From both physical and purely mathematical point of view, it makes sense to study not an individual operator, but rather an entire family of operators $H(\omega)$ labeled by the points of the phase space of a dynamical system on some probability space. Moreover, it is convenient to assume ergodicity of the dynamical system in question. In this particular case, in order to define an ergodic family of operators, we need:

(i) a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
(ii) an ergodic dynamical system $T$ with discrete time $\mathbb{Z}^d$, $d \geq 1$, i.e. a representation $T : \mathbb{Z}^d \times \Omega \to \Omega$ of the additive group $\mathbb{Z}^d$ into the group of isomorphisms of $(\Omega, \mathcal{F}, \mathbb{P})$, 

$$T^{x+y} = T^x \circ T^y, \quad T^x, T^y \in \text{Aut}(\Omega, \mathcal{F}, \mathbb{P}),$$

such that any $T$-invariant measurable function on $\Omega$ is a.e. constant;

(iii) a measurable mapping $H$ of the space $\Omega$ into the algebra of bounded operators acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{Z}^d)$ verifying for every $x \in \mathbb{Z}^d$:

$$H(T^x(\omega)) = U^{-x}H(\omega)U^x,$$

where $(U^xf)(y) = f(y-x)$ are the conventional unitary shift operators. A conventional lattice Schrödinger operator is obtained by setting

$$H(\omega) = \Delta + V(x; \omega),$$

where $\Delta$ is the nearest-neighbor discrete Laplacian and $V(x; \omega)$ is the operator of multiplication by the function

$$V(x; \omega) = v(T^x \omega),$$

with some function $v : \Omega \to \mathbb{R}$, which we will call the hull of the potential $V$.

An interesting class of quasi-periodic potentials, e.g., in one dimension, is obtained when $\Omega$ is a torus $\mathbb{T}^r$ of dimension $r \geq 1$ endowed with the Haar measure $\mathbb{P}$ and the dynamical system on $\Omega$ is given by

$$T^x : \omega \mapsto \omega + x\alpha \in \mathbb{T}^r.$$

As is well-known, this dynamical system is ergodic whenever the frequency vector $\alpha$ has incommensurable (rationally independent) coordinates. Taking a function $v : \mathbb{T}^r \to \mathbb{R}$, we can define an ergodic family of quasi-periodic potentials $V : \mathbb{Z} \to \mathbb{R}$ by $V(x; \omega) := v(T^x \omega)$. Multi-dimensional quasi-periodic potentials on $\mathbb{Z}^n$ can be constructed in a similar way (with the help of $n$ incommensurate frequency vectors $\alpha^j \in \mathbb{R}^r, j = 1, \ldots, n$).

In this paper, we do not intend to give an extensive review of earlier works on localization phenomena for quasi-periodic operators. Among the first mathematically rigorous results on the localization phenomenon for a one-dimensional discrete Schrödinger equation with the single-frequency quasi-periodic potential of the form $\cos(\alpha x)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, (also known as Almost Mathieu equation and Harper’s equation) we refer to the papers by Sinai [Sin87] and Fröhlich, Spencer and Wittwer [FSW87]. The case of several basic frequencies was considered by the author and Sinai [CSin91], and later in a cycle of papers by Bourgain, Goldstein and Schlag, for various dynamical systems on a torus $\Omega = \mathbb{T}^r$, where the "hull" $v(\omega)$ was assumed analytic; see, e.g., [BG00], [BGS01], [BS00]. Recently, Chan [Chau7] used some parameter exclusion technique (different from ours) to establish the localization for quasi-periodic operators with $v(\omega)$ of class $C^3$.

One general lesson of several works by Bourgain, Goldstein and Schlag is that certain techniques can be successfully applied, with appropriate modification, to various underlying dynamical systems $T^x$ generating a deterministic disordered potential $V(x; \omega) = v(T^x \omega)$. Below we encapsulate the requirements for the dynamical system in one mild condition - that of "uniformly slow" returns of any trajectory $\{T^x \omega, x \in \mathbb{Z}^d\}$ to smaller and smaller neighborhoods of its starting point $\omega \in \Omega$. Cf. subsection 1.1 below. The uniform low bound on the minimal spacings of finite trajectories $\{T^x \omega, x \in A \subset \mathbb{Z}^d\}$, card $A < \infty$, can be essentially relaxed. We plan to address a more general case in a separate paper.
1.1. Requirement for the dynamical system.

We assume that the underlying dynamical system $T$ on the phase space $\Omega$, endowed with a distance $\text{dist}_T(\cdot, \cdot)$, satisfies the following condition of Uniformly Slow Return (USR, in short):

\begin{equation}
\text{USR: } \exists A, C \in (0, \infty) \ \forall \omega \in \Omega \ \forall x, y \in \mathbb{Z}^\nu \text{ such that } x \neq y \\
\text{dist}_T(T^x\omega, T^y\omega) \geq C\|x - y\|^{-A}, \tag{1.1}
\end{equation}

Actually, this condition can be further relaxed so as to admit the lower bound of the form $Ce^{-\|x-y\|^\beta}$, with some $\beta \in (0, 1)$ and $C > 0$.

In this paper, we consider mainly the case where $\Omega = \mathbb{T}^\nu$, $\nu \geq 1$, and it is technically convenient to define the distance $\text{dist}_T[\omega', \omega''] \equiv \text{dist}_T(\cdot, \cdot)$ as follows:

\[
\text{dist}_T[\omega_1', \ldots, \omega_\nu'], (\omega_1'', \ldots, \omega_\nu'')] := \max_{1 \leq i \leq \nu} \text{dist}_T[\omega_i', \omega_i''],
\]

where $\text{dist}_T$ is the conventional distance on the unit circle $\mathbb{T}^1$. With this definition, the diameter of a cube of side length $r$ in $\mathbb{T}^\nu$ equals $r$, for any dimension $\nu \geq 1$. The reason for the choice of the phase space $\Omega = \mathbb{T}^\nu$ is that the parametric families of ensembles of potentials $V(x; \omega; \theta)$ are fairly explicit in this case.

For ergodic rotations of the torus $\mathbb{T}^\nu$,

\[ T^x\omega = \omega + x_1\alpha_1 + \cdots + \omega + x_d\alpha_d, \ x \in \mathbb{Z}^d, \ \alpha_j \in \mathbb{T}^\nu, \ 1 \leq j \leq d, \]

the USR property reads as a Diophantine condition for the frequency vectors $\alpha_j$, which we always assume below.

1.2. A general form of Randelette Expansions.

In [C01, C07] we have introduced parametric families of ergodic ensembles of operators $\{H(\omega; \theta), \omega \in \Omega\}$ depending upon a parameter $\theta \in \Theta$ in an auxiliary space $\Theta$. We have shown that it is convenient to endow $\Theta$ with the structure of a probability space, $(\Theta, \mathcal{B}, \mathbb{P}(\theta))$ in such a way that $\theta$ be, in fact, an infinite family of IID random variables on $\Theta$, providing an infinite number of auxiliary parameters allowing to vary the hull $v(\omega; \theta)$ locally in the phase space $\Omega$. We called such parametric families Grand Ensembles.

The above description is yet too general. In the framework of lattice Schrödinger operators, we gave in [C01, C07] a more specific construction where $H(\omega; \theta) = H_0 + V(\cdot; \omega; \theta)$, with $V(x; \omega; \theta) = V(T^x\omega; \theta)$ and

\[ v(\omega; \theta) = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega), \tag{1.2} \]

where the family of random variables $\theta := (\theta_{n,k}, n \geq 1, 1 \leq k \leq K_n)$ on $\Theta$ is IID, and $\varphi_{n,k} := (\varphi_{n,k}), n \geq 1, 1 \leq k \leq K_n < \infty$ are some functions on the phase space $\Omega$ of the underlying dynamical system $T^x$. Representations of the form (1.2) were called randelette expansions.

Further, for the purposes of the MSA, it is convenient to assume that

- $\theta_{n,k}$ have a probability density; e.g. $\theta_{n,k}$ are uniformly distributed in $[-1, 1]$;
- the "amplitudes" $a_n$ of "generations" $(\theta_{n,k}, 1 \leq k \leq K_n)$ satisfy
\(\diamond\) an upper bound, to ensure the convergence of the randelette expansion
\(\diamond\) an appropriate lower bound, to ensure that the contribution of the \(n\)-th generation of \(\theta_{n,k}\) is sufficient to wriggle the values of the potential \(V(T_x^\omega; \theta)\) via the randelettes \(\theta_{n,k}\varphi_{n,k}\) and thus to avoid possible "degeneracies";
• the supports of \(\varphi_{n,k}\) have a diameter rapidly decaying as \(n \to \infty\).

Putting the amplitude of the \(\varphi_{n,k}\) in the coefficient \(a_n\), it is natural to assume that \(|\varphi_{n,k}(\omega)|\) is bounded. Further, in order to control the potential \(V(T_x^\omega; \theta)\) at any lattice site \(x \in \mathbb{Z}^d\) or, equivalently, at every point \(\omega \in \Omega\), it is natural to require that for every \(n \geq 1\), \(\Omega\) be covered by the union of the sets where at least one function \(\varphi_{n,k}\) is nonzero (and, preferably, not too small).

Notice that the dynamics \(T^x\) leaves \(\theta\) invariant.

1.3. Description of haarsh randelette expansions.

A very particular, yet interesting case is where randelettes are piecewise constant functions used in the construction of Haar wavelets\(^1\). For example, if \(\Omega = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}\), we set

\[\varphi_{n,k}(\omega) = 1_{C_{n,k}}(\omega), \quad C_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad n \geq 1, \quad 1 \leq k \leq K_n = 2^n.\]

On a torus of higher dimension, one has to replace intervals of length \(2^{-n}\) by cubes of sidelength \(2^{-n}\). Specifically, given an integer \(n \geq 1\), for each integer vector \((r_1, \ldots, r_\nu)\) with \(1 \leq r_j \leq 2^n\), consider the cube

\[\left[\frac{r_1 - 1}{2^n}, \frac{r_1}{2^n}\right) \times \cdots \times \left[\frac{r_\nu - 1}{2^n}, \frac{r_\nu}{2^n}\right) \subset \mathbb{T}^\nu.\]

These cubes can be numbered, e.g., in the lexicographical order of vectors \((r_1, \ldots, r_\nu)\), and their total number equals \(K_n = 2^{nd}\). We will denote these cubes by \(C_{n,k}, k = 1, \ldots, K_n\).

Next, introduce a countable family of functions on the torus,

\[\varphi_{n,k}(\omega) = 1_{C_{n,k}}(\omega), \quad n \geq 1, \quad k = 1, \ldots, K_n,\]

and a countable family of IID random variables \(\theta_{n,k}\) on an auxiliary probability space \(\Theta, \mathcal{B}, \mathbb{P}(\theta)\), uniformly distributed in \([-1, 1]\).

Finally, pick a positive number \(b > 2d\) and set

\[a_n = 2^{-2bn^2}, \quad n \geq 1. \quad (1.3)\]

Now define a function \(v(\omega; \theta)\) on \(\Omega \times \Theta\),

\[v(\omega; \theta) = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega), \quad (1.4)\]

\(^1\) In fact, the main results of this paper remain true for expansions over the orthogonal Haar wavelets, but we would like to stress that the orthogonality is not relevant here.
which can be viewed as a family of functions $v(\cdot; \theta)$ on the torus, parametrized by $\theta \in \Theta$, or as a particular case of a "random" series of functions, expanded over a given system of functions $\varphi_{n,k}$ with "random" coefficients.

We will call such expansions "haarsh", making reference to Haar’s (Haarsche, in German) wavelets and to the "harsh" nature of the resulting potentials. Constructing a potential out of flat pieces is rather unusual in the framework of the localization theory, where, starting from the pioneering mathematical works by Goldsheid, Molchanov and Pastur, all efforts were usually made so as to avoid flatness of the potential. Yet, with an infinite number of flat components $\theta_{n,k} \varphi_{n,k}(\omega)$, each modulated by its own parameter $\theta_{n,k}$, we proved in [C01, C07] an analog of Wegner bound for the respective Grand Ensembles $H(\omega; \theta)$. This was the first indication that such parametric ensembles may feature the phenomenon of Anderson localization.

In the present paper, we make the next step and prove the Anderson localization for generic deterministic (e.g., quasi-periodic potentials) of sufficiently large amplitude, constructed with the help of randelette expansions of the form (1.2), under the assumption that the dynamical system obeys the condition of Uniformly Slow Returns (1.1). We use a variant of the Multi-Scale Analysis and study first the spectral properties of finite-volume approximants of the operator $H(\omega; \theta)$ obtained by its restriction on lattice cubes $A_{L_j}(u) = \{ x \in \mathbb{Z}^d : \|x - u\| \leq L_j \}$, with Dirichlet boundary conditions on the "external boundary" $\partial^+ A_{L_j}(u) = \{ x \in \mathbb{Z}^d : \|x - u\| = L_j + 1 \}$. Here and below, we use the max-norm for vectors $x \in \mathbb{R}^d$: $\|x\| = \max_{1 \leq i \leq d} |x_i|$.

Namely, we prove in Section 5 the following result.

**Theorem 1.** Consider a family of lattice Schrödinger operators in $\ell^2(\mathbb{Z}^d)$,

$$H(\omega; \theta) = \Delta + gV(x; \omega; \theta)$$

where $V(x; \omega; \theta) = v(T^x \omega; \theta)$, with $v(\omega; \theta)$ given by the expansion (1.4), and the dynamical system $T^x$ satisfies the USR condition (1.1) for some $A,C < \infty$.

For sufficiently large $|g| \geq g_0(C,A)$, there exists a subset $\Theta^{(\infty)}(g) \subset \Theta$ of measure $\mathbb{P}(\Theta) \{ \Theta^{(\infty)}(g) \} \geq 1 - c(C,A) g^{-1}$ with the following property: if $\theta \in \Theta^{(\infty)}$, then for any $\omega \in \Omega$ the operator $H(\omega; \theta)$ has pure point spectrum with exponentially decaying eigenfunctions $\psi_{j}(\cdot; \omega; \theta)$:

$$\forall x \in \mathbb{Z}^d |\psi_{j}(x; \omega; \theta)| \leq C_{j}(\omega; \theta) e^{-m\|x\|}, m = m(g,C,A) > 0.$$

Its proof is essentially based on an inductive Lemma 4.3 given in Section 4.

**2. Randelettes, partitions and separation bounds for the potential**

For every $n \geq 1$, the supports $C_{n,k} = \text{supp} \varphi_{n,k}$, $1 \leq k \leq K_n$ naturally define a partition of the phase space $\Omega$:

$$C_n = \{ C_{n,k}, 1 \leq k \leq K_n \}.$$

These partitions form a monotone sequence: $C_{n+1} \subset C_n$, i.e., each element of $C_n$ is a union of some elements of the partition $C_{n+1}$. In the probabilistic language,
the (finite) sigma-algebras $B_n$ canonically generated by (the elements of) the partitions $C_n$ form a monotone family: $B_n \subset B_{n+1}$.

To each element $C_{n,k}$ of the partition $C_n$ corresponds a unique finite sequence of indices $\kappa(n,k) = (k_1, \ldots, k_n = k)$ labeling $n$ elements $C_{i,k_i} \supset C_{n,k}$, $1 \leq i \leq n$, of partitions preceding or equal to $C_n$. Further, we associate with the element $C_{n,k}$ a random variable $\xi_{n,k} = \xi_{n,k}(\theta)$ relative to the probability space $\Theta$.

Next, introduce the approximants of the “hull” $v(\omega; \theta)$ given by (1.2):

$$v_n(\omega; \theta) = \sum_{i=1}^{n} a_i \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}, \ n = 1, 2, \ldots .$$

The random variables $\xi_{n,k}(\theta)$ with different $k$ are strongly correlated via the values $\theta_{n'}$ with $n' < n$. Nevertheless, the variables $\theta_{n,k}(\theta)$, independent for different $k$, bring enough “innovation” and allow to mimick, albeit weakly, various properties of “truly random” potentials $V(x; \omega)$ with IID values.

In this paper, we consider only functions $\varphi_{n,k}(\omega)$ which are indicators of their respective supports, i.e. indicators of the respective partition elements $C_{n,k}$.

Therefore, an approximant $v_n(\omega; \theta)$ can be expressed as follows:

$$v_n(\omega; \theta) = \sum_{k=1}^{K_n} \xi_{n,k}(\theta) \varphi_{n,k}(\omega) = \sum_{k=1}^{K_n} \xi_{n,k}(\theta) 1_{C_{n,k}}(\omega).$$

Further, for any $N \geq 1$, if $b \geq 1$, then we have

$$\sum_{n=N+1}^{\infty} a_n = \sum_{n=N+1}^{\infty} 2^{-bn^2} = 2^{-b(2N+1)} 2^{-bN^2} \sum_{i=0}^{\infty} 2^{-b(N+i)^2+b(N+1)^2} \leq 2^{-b(2N+1)} a_N \sum_{i=0}^{\infty} 2^{-i} = 2^{-2bN} a_N,$$

so the norm $\|v - v_N\|_\infty \equiv \|v - v_N\|_{L^\infty(\Omega \times \Theta)}$ can be bounded as follows:

$$\|v - v_N\|_\infty \leq r_N := 2^{-2bN} a_N.$$ 

Notice that, for $N$ large, the RHS is much smaller than the width $2a_N$ of the distribution of random coefficients $a_N \theta_{N,k}$, $1 \leq k \leq K_N$ (recall that $\theta_{N,k} \sim Unif[-1,1]$). This fact plays an important role in our analysis. Observe also that, since $b > 2d \geq 2$, we have, for all $N \geq 1$,

$$2^{-bN} a_N \geq 4 a_N 2^{-2bN}.$$ \hspace{1cm} (2.2)

Set

$$\tilde{n}(L) = \tilde{n}(L, A, C) := 1 + \frac{2A \ln L - \ln C}{\ln 2}.$$ \hspace{1cm} (2.3)
Then for any \( u \in \mathbb{Z}^d \) and any \( \omega \in \Omega \), all points of the trajectory \( \{ T^x \omega, x \in A_{L^2}(u) \} \) are separated by elements of the partition \( C_n(L) \), since by (1.1), we have
\[
\frac{1}{2} \text{dist}_{\Omega}(T^x \omega, T^y \omega) \geq \frac{1}{2} cL^{-2A} \geq 2^{-\bar{n}(L)}.
\]

Further, consider the events
\[
B_{N,k,k'} = \{ \theta : |\xi_{N,k} - \xi_{N,k'}| \leq g^{-1}2^{-bN}a_N \}, \ k, k' \in [1, K_N], k \neq k',
\]
and their unions \( B_N = \bigcup_{k \neq k'} B_{N,k,k'}, \ \mathcal{B} = B(g) = \bigcup_{N \geq 1} B_N. \)

**Lemma 2.1.** For any \( N \geq 1 \) and \( k \neq k' \),
\[
\mathbb{P}(B_{N,k,k'}) \leq \frac{2^{-bN}a_N}{2a_N g} = \frac{1}{2g} 2^{-bN}.
\]
(2.4)

Therefore,
\[
\mathbb{P}(B_N) \leq \frac{1}{4g} 2^{-(b-2d)N}
\]
(2.5)

and
\[
\mathbb{P}(B(g)) \leq \frac{g^{-1}}{2^{b-2d+1} - 2}.
\]
(2.6)

**Proof.** To prove (2.4), notice that
\[
\mathbb{P}(B_{N,k,k'}) = \mathbb{E}(\theta) \left[ \mathbb{P}(B_{N,k,k'} \mid B_{N-1}) \right] \leq \frac{2^{-bN}a_N}{2a_N g} = \frac{1}{2g} 2^{-bN},
\]
since, conditional on \( B_{N-1} \), random variables \( \xi_{N,k}, \xi_{N,k'} \) are independent and uniformly distributed in \([-a_N, a_N] \). The assertion (2.5) follows easily from (2.4) and the inequality
\[
\text{card} \{ (k, k') \mid 1 \leq k \neq k' \leq K_N \} < K_N^2 = 1/2K_N^2 = 1/2 2^{2dN}.
\]

Let us analyze the implications of the above bounds. Set \( \Theta^{(\infty)}(g) = \Theta \setminus B(g) \) and let \( \theta \in \Theta^{(\infty)}(g) \). We see that for every \( n \geq 1 \) and every pair of distinct partition elements \( C_{N,k}, C_{N,k'} \), we have, by virtue of Eqn (2.2):
\[
|g \xi_{N,k} - g \xi_{N,k'}| > g 2^{-bN}a_N \geq 4g 2^{-bN}a_N,
\]
and at the same time,
\[
\|g v - g v_N\|_\infty \leq 2^{-bN}a_N g.
\]

So, for all \( \omega \in C_{N,k}, \ \omega' \in C_{N,k'}, \ k \neq k' \), the condition \( \theta \in \Theta^{(\infty)}(g) \) guarantees that
\[
|g v(\omega; \theta) - g v(\omega'; \theta)| \geq 2^{-bN}a_N g - 2^{-bN}a_N g \geq 2^{-bN^2} g.
\]

The above lower bound on the spacings for the "hull" \( v(\omega; \theta) \) can be interpreted in terms of the potential \( gV(x; \omega; \theta) \). Namely, consider an arbitrary box \( A_L(u) \) of size \( L \). Since \( \text{sep}(A_L(u)) \geq L^{-A} \) by the USR condition (1.1), all points of the
finite trajectory \( \{ T^x \omega, x \in A_L(u) \} \) are separated by elements of the partition \( C_\nu(L) \), so that if \( \theta \in \mathcal{G}_{(\omega)}(g) \), then

\[
\min_{x,y \in A_L(u) : x \neq y} |gV(x; \omega) - gV(y; \omega)| \geq 2^{-b n^2(L)} g.
\]

A simple calculation shows that

\[
g2^{-b n^2(L)} > \exp\left(-L^{1/2}\right)
\]

for large \( L \) or large \( g \). Indeed,

\[
\ln \left[ \exp\left(L^{1/2}\right) 2^{-b n^2(L)} \right] = L^{1/2} - n^2(L)2b \ln 2 = L^{1/2} - Const \ln L \xrightarrow{L \to \infty} +\infty,
\]

while for any fixed \( L \) the required inequality is obtained by taking \( g > 0 \) large enough. So, for \( g \) large enough and all \( L > 1 \) we have

\[
\min_{x,y \in A_L(u) : x \neq y} |gV(x; \omega) - gV(y; \omega)| \geq \exp\left(-L^{1/2}\right). \tag{2.7}
\]

In order to understand the principal mechanisms of the proofs of the main results of this paper, it is important to keep in mind that the above mentioned separation of the potential values in \( A_L(u) \) can be guaranteed even under conditioning on \( \theta_{n', k'} \) with all \( n' < n \) and all \( k' \in [1, K_j] \).

3. Wegner-type bounds and separation of finite-volume spectra

Consider a finite box \( A = A_L(u) \subset \mathbb{Z}^d \) and the Hamiltonian \( H_A = \Delta + gV \). If \( g \) is large enough, then the values of the potential \( \{ V(x), x \in A \} \) can be considered an accurate approximation to the eigenvalues \( E_j^A \) of operator \( H_A \), by virtue of the min-max principle. In particular, if all values of the potential in \( A \) are distinct (and fixed), and \( g \) is large enough, then all spectral spacings \( |E_i^A - E_j^A| \) of \( H_A \) are of order of \( O(g) \). Moreover, for a given pair of disjoint boxes \( A_L(u), A_L(v) \) one can also guarantee that

\[
|E_i^{A_L(u)} - E_j^{A_L(v)}| \geq Const(V)g > 0,
\]

so that the Hausdorff distance between the spectra in these two volumes admits a lower bound

\[
\text{dist} \left[ \Sigma(H_{A_L(u)}), \Sigma(H_{A_L(v)}) \right] \equiv \min_{i \neq j} |E_i^{A_L(u)} - E_j^{A_L(v)}| \geq Const(V)g > 0,
\]

provided that the sample of potential values \( \{ V(x), x \in A_L(u) \cup A_L(v) \} \) has all elements distinct and \( g \) is large enough. Naturally, a similar lower bound holds for all pairs of disjoint boxes \( A_L(u), A_L(v) \subset A_L(w) \) inside a larger box \( A_L(w) \), provided that all potential values \( \{ V(x), x \in A_L(w) \} \) are distinct and \( g \) is large enough. Namely, if \( A_L(u), A_L(v) \subset A_L(w) \) and \( A_L(u) \cap A_L(v) = \emptyset \), then

\[
\text{dist} \left[ \Sigma(H_{A_L(u)}), \Sigma(H_{A_L(v)}) \right] = O(g) > 0.
\]
However, such a simple control of spectral spacings and distances between spectra is impossible at a large scale, once $g$ is fixed.

In the traditional Multi-Scale Analysis of random operators, spectral spacings are controlled in a probabilistic way, using the well-know Wegner bound or its variants. The main *raison d'être* of the auxiliary measurable space $\Theta$ in the framework of the Grand Ensembles is precisely to mimick to a certain extent Wegner-type bounds used in the theory of "truly random" media and to ensure some lower bounds on the spectral spacing for generic "hulls" $v: \Omega \rightarrow \mathbb{R}$ generating a deterministic potential $V(x; \omega; \theta) = v(T^x \omega; \theta)$ for a given underlying dynamical system $\{T^x\}$.

Quite naturally, some "hulls" labeled by $\theta \in \Theta$ have to be excluded, essentially for the same reasons that some samples of IID random potentials have to be excluded, if we aim to prove localization: for example, an identically zero sample of potential gives rise to an operator $H = \Delta$ with a purely a.c. spectrum. Similarly, setting all $\theta_{n,k} = 0$, we get $V(x; \omega; \theta) \equiv 0$.

In our earlier works [C01, C07], we proved Wegner-type estimates for generic deterministic potentials constructed with the help of suitable randelette expansions. For the reader’s convenience, we summarize below the main results of the papers works [C01, C07], adapted to our model and notations.

**Lemma 3.1.**

$$\mathbb{P}^{(\theta)} \left\{ \text{dist} \left[ \Sigma(H_{\lambda_{1}(u)}, \Sigma(H_{\lambda_{2}(v)}) \right] \leq \epsilon \right\} \leq \epsilon a_{n}^{-1} e^{-2b\eta(L)}.$$ 

Now we will show that the above probabilistic lower bound on the distance between two finite-volume spectra can be improved and expressed in terms of the parameter $\theta$ only, like it was done in the previous section for a given (and then fixed) spatial scale $L$.

### 3.1. Simplicity of spectra and upper bounds on the resolvents.

Consider a box $A_{L_0}(u)$ and set $\delta_0 = 2$, $g_0 = e^{mL_0} + 4d$, with some $m > 0$, so that $g_0 \geq e^{mL_0} + 2\| \Delta_{A_{L_0}(u)} \|$. By virtue of the min-max principle, if $\text{sep}(V, A_{L_0}) \geq \delta_0$, then all eigenvalues $\{E_{\alpha}^{A_{L_0}(u)}, \alpha = 1, \ldots, |A_{L_0}(u)| \}$ of $H_{A_{L_0}(u)} = \Delta + gV$ are distinct, and the minimal distance between them obeys

$$\min_{j \neq j'} |E_{j}^{A_{L_0}(u)} - E_{j'}^{A_{L_0}(u)}| \geq g_0 \delta_0 - 2\| \Delta_{A_{L_0}(u)} \| \geq 2e^{mL_0} + 4d > 0.$$ 

Indeed, each eigenvalue $E_{j}^{A_{L_0}(u)}$ of $H_{A_{L_0}(u)}$ is a "perturbation" of one of the eigenvalues of the multiplication operator $V$, i.e. there exists $x = x(j) \in A$ such that $|E_{j}^{A_{L_0}(u)} - V_{x(j)}| \leq \| \Delta_{A_{L_0}(u)} \|$. Furthermore, for any complex $\zeta$ such that

$$\min_{x \in A} \text{dist}(\zeta, gV_x) \geq e^{mL_0} + 2d$$

we have $\text{dist}(\zeta, \Sigma(H_{A_{L_0}(u)})) \geq e^{mL_0}$, yielding the bounds

$$\| (H_{A} - \zeta)^{-1} \| \leq e^{-mL_0}.$$
and
\[ \max_{x,y \in A_{L_0}(u)} |G_{A_{L_0}(u)}(x,y;\zeta)| \leq e^{-mL_0}. \]

In particular, the above upper bound on the Green functions holds true on the contour \( I_\zeta = \{ \zeta \in \mathbb{C} : |\zeta - V(x;\omega;\theta)| = g_0 \} \). Using a standard perturbation theory for non-degenerate spectra of self-adjoint operators, one can easily obtain localization bounds on eigenfunctions of \( H_{A_{L_0}(0)}(\omega;\theta) \), for any \( \omega \), and, by virtue of the covariance relation \( H_{A_{L_0}(u)}(\omega;\theta) = H_{A_{L_0}(0)}(T^u\omega;\theta) \), for any box of size \( L_0 \) in \( \mathbb{Z}^d \).

We come, therefore, to the following

**Lemma 3.2.** Fix a real number \( m > 0 \) and an integer \( L_0 > 1 \), and set \( \delta_0 = 2, g_0 = e^{2mL_0 + 4d} \). Then for any \( g > g_0 \) there exists a subset \( \Theta^{(0)} \subset \Theta \) of measure \( \mathbb{P}(\Theta^{(0)}) \geq 1 - Cg^{-1} \) such that if \( \theta \in \Theta^{(0)} \), then for any \( \omega \in \Omega \), any \( u \in \mathbb{Z}^d \) and any \( \zeta \in \mathbb{C} \), either
\[ \min_{x \in A_{L_0}(u)} |\zeta - gV(x;\omega;\theta)| \leq g_0 \delta_0 \]

or
\[ \max_{x,y \in A_{L_0}(u)} |G_{A_{L_0}(u)}(x,y;\zeta)| \leq e^{-2mL_0}. \quad (3.1) \]

**Definition 3.1.** A box \( \Lambda_L(u) \) will be called \((\zeta,m)\)-nonsingular (or \((\zeta,m)\)-NS) if the following bound holds:
\[ \max_{y \in \partial - \Lambda_L(u)} |G_{\Lambda_L(u)}(u,y;\zeta)| \leq e^{-\gamma(m,\ell)} \]
where
\[ \gamma(m,\ell) = m\ell(1 + \ell^{-1/8}). \]
Otherwise, it will be called \((\zeta,m)\)-singular (or \((\zeta,m)\)-S).

In view of the above definition, observe that the inequality (3.1) implies
\[ \max_{x,y \in A_{L_0}(u)} |G_{A_{L_0}(u)}(x,y;\zeta)| \leq e^{-m(1+L_0^{-1/8})L_0}, \]
which reads as the condition of \((\zeta,m)\)-non-singularity of the box \( A_{L_0}(u) \).

A violation of the \((\zeta,m)\)-non-singularity property for a box \( A_L(u) \) is considered, in the context of the MSA, as an unwanted event. In the case where it occurs, it is important to know how "bad" is the singular box. Introduce the following

**Definition 3.2.** Let \( \zeta \in \mathbb{C} \). A box \( \Lambda_L(u) \) will be called \((\zeta)\)-nonresonant (or \((\zeta)\)-NR, in short) if the following bound holds:
\[ \text{dist}(\zeta, \Sigma(H_{\Lambda_L(u)})) \geq e^{-\ell^{1/4}}. \]
Otherwise, it will be called \((\zeta)\)-resonant (or \((\zeta)\)-R).

We will also need a modified notion of resonance.
Definition 3.3. Let \( \zeta \in \mathbb{C} \). A box \( A_l(u) \) will be called \((\zeta, L)\)-nonresonant (or \((\zeta, L)\)-NR, in short) if the following bound holds:

\[
\text{dist}(\zeta, \Sigma(H_{A_l(u)})) \geq e^{-L^{1/4}}.
\]

Otherwise, it will be called \((\zeta, L)\)-resonant (or \((\zeta, L)\)-R, in short).

Obviously, a \((\zeta, m)\)-nonsingular box is automatically \(\zeta\)-nonresonant, and a \((\zeta, L)\)-resonant box, with \( L \geq \ell \), is also \((\zeta, \ell)\)-resonant.

In what follows, we will use a sequence of integers (scales) \( L_j, j \geq 0 \), defined recursively for a given \( L_0 \geq 4 \):

\[
L_{j+1} := \left[ L_j^{1/2} \right] L_j, \quad j = 1, 2, \ldots
\]

(3.2)

where \( \lfloor \cdot \rfloor \) stands for the integer part, so that \( L_{j+1} \) is a multiple of \( L_j \) with

\[
L_j^{3/2} \leq L_{j+1} < L_j^{3/2} + L_j.
\]

Further, set \( \delta_0 = 2 \) and for all \( j \geq 1 \),

\[
\delta_j = 2^{-\tilde{n}(L_j)} a_{\tilde{n}(L_j)} ,
\]

(3.3)

with \( \tilde{n}(L) = 1 + (\ln 2)^{-1} (2A \ln L - \ln C) \), as in Eqn (2.3). Observe that

\[
\delta_j = 2^{-\tilde{n}(L_j)} a_{\tilde{n}(L_j)} < \text{Const} L_j^{-4A^2 b \ln L_j} < \text{Const} L_0^{-4A^2 b \ln L_0},
\]

so that \( \sum_{j \geq 0} \delta_j \) converges, and its sum is small for large \( L_0 \):

\[
\sum_{j \geq 0} \delta_j \leq \text{Const} \sum_{j \geq 0} L_0^{-4A^2 b \ln L_j} \leq \text{Const} L_0^{-4A^2 b \ln L_0}.
\]

3.2. "Good" \( \theta \)-sets for uniform separation of spectra: the initial scale.

Lemma 3.3. Assume that the dynamical system \( \{ T^x, x \in \mathbb{Z}^d \} \) satisfies the USR condition (1.1) with given values of constants \( A, C > 0 \). Fix an integer \( L_0 \geq 4 \), a real number \( m > 0 \) and set \( g_0 = 4e^{mL_0} + 4d \). Then for all \( g \geq g_0 \), there exists a subset \( \Theta^{(1)} \subset \Theta \) of measure

\[
\mu(\Theta^{(1)}) \geq 1 - \text{Const} g^{-1}
\]

such that for any \( \theta \in \Theta^{(1)} \) and any \( \omega \in \Omega \):

(A) \( \forall \ u \in \mathbb{Z}^d, \text{sep}(g V(\cdot; \omega; \theta), A_{L_0}(u)) \geq g \delta_0 \geq g_0 \delta_0 \); 

(B) for any pair of disjoint boxes \( A_{L_0}(u_1), A_{L_0}(u_2) \) with \( \| u_1 - u_2 \| \leq L_1^{4/3}, \)

\[
\text{dist}(\Sigma(H_{A_{L_0}(u_1)}), \Sigma(H_{A_{L_0}(u_2)})) \geq g \delta_0;
\]

(C) for any disjoint pair \( A_{L_0}(u_1), A_{L_0}(u_2) \) such that \( \| u_1 - u_2 \| \leq L_1^{4/3}, \)

\[
\forall \ \zeta \in \mathbb{C} \text{ if } A_{L_0}(u_1) \text{ is } \zeta\text{-NR, then } A_{L_0}(u_2) \text{ is } (\zeta, m)\text{-NS.}
\]

(3.5)

In particular,

\[
\forall \ \zeta \in \mathbb{C} \text{ either } A_{L_0}(u_1) \text{ or } A_{L_0}(u_2) \text{ is } (\zeta, m)\text{-NS.}
\]

(3.6)
Remark. Observe that \( L_1^{4/3} = O(L_0^2) \); the equality would be exact without rounding in the definition of the scales \( L_{j+1} \), \( j \geq 1 \). It is technically convenient to have the above separation bounds in a box slightly larger than \( L_1 \) (resp., larger than \( L_{j+1} \), in Lemmas 3.4 and 3.5 below). This is used in the proof of spectral localization in Section 5.

It is worth mentioning that, in the Multi-Scale Analysis of random operators, the assertion (C) of the above theorem is usually established with high probability relative to the space \( \Omega \). Here, it is \"deterministic\" in \( \omega \), but holds only with \"high probability\" in the auxiliary parameter space \( \Theta \).

3.3. \"Good\" \( \theta \)-sets for uniform separation of spectra: an arbitrary scale.

Introduce the following statement which should be considered as a property of the variables \( \theta_{n,k} \) or as an event relative to the probability space \( \Theta \):

\[
\text{Sep}(j) := \min_{1 \leq k \neq k' \leq K_{j+1}} |\xi_{\tilde{n}(L_j), k} - \xi_{\tilde{n}(L_j), k'}| \geq \delta_j .
\]

Lemma 3.4. Let \( j \geq 1 \) and consider the scales \( L_j, L_{j+1} \) defined in Eqn (3.2). Assume that a set \( \Theta^{(j)} \subset \Theta \) satisfying the following condition \( \text{Sep}(j) \). Define a subset \( \Theta^{(j+1)} \subset \Theta^{(j)} \) satisfying \( \text{Sep}(j+1) \):

\[
\Theta^{(j+1)} = \left\{ \theta \in \Theta^{(j)} : \min_{1 \leq k \neq k' \leq K_{j+1}} |\xi_{\tilde{n}(L_{j+1}), k} - \xi_{\tilde{n}(L_{j+1}), k'}| \geq \delta_{j+1} \right\}
\]

Then the following properties hold true:

(A) \( \mathbb{P}(\Theta^{(j+1)}) \leq \frac{1}{2^j} 2^{-(b-2)\tilde{n}(L_j)} \).

(B) For any \( \theta \in \Theta^{(j)} \) and any \( \omega \in \Omega \), and for any pair of disjoint cubes \( A_{L_j}(x), A_{L_j}(y) \subset A_{L_{j+1}}^{1/3}(0) \),

\[
\text{dist}(\Sigma(H_{A_{L_j}(x)})), \Sigma(H_{A_{L_j}(y)})) \geq \delta_{j+1} ;
\]

(C) As a consequence, for any \( \theta \in \Theta^{(j)} \), any \( \omega \in \Omega \), any \( u \in \mathbb{Z}^d \) and any pair of disjoint cubes \( A_{L_j}(x), A_{L_j}(y) \subset A_{L_{j+1}}^{1/3}(u) \),

\[
\text{dist}(\Sigma(H_{A_{L_j}(x)})), \Sigma(H_{A_{L_j}(y)})) \geq 4\delta_{j+1} .
\]

The assertion (C) of Lemma 3.4 leads directly to the main result of this section:

Lemma 3.5. Let \( \Theta^{(\infty)} = \bigcap_{j=0}^{\infty} \Theta^{(j)} \). If \( \theta \in \Theta^{\infty} \), then for any \( j \geq 0 \), any \( u \in \mathbb{Z}^d \), all \( \omega \in \Omega \) and any disjoint pair of boxes \( A_{L_j}(x), A_{L_j}(y) \subset A_{L_{j+1}}^{1/3}(u) \),

\[
\text{dist}(\Sigma(H_{A_{L_j}(x)}(\omega; \theta))), \Sigma(H_{A_{L_j}(y)}(\omega; \theta))) \geq 4\delta_{j+1} .
\]
4. Decay of Green functions in finite boxes

In our recent manuscript \cite{C08}, we have introduced the following useful notion of a "subharmonic" function on the lattice which allows to simplify the inductive step of the MSA. For the readers convenience, we summarize below the results of \cite{C08}, adapted to our model and the notations of the present paper.

**Definition 4.1.** Consider a set $\Lambda \subset \mathbb{Z}^d$ (not necessarily finite), a subset $\mathcal{S} \subset \Lambda$ and a bounded function $f : \Lambda \to \mathbb{C}$. Let $L \geq 0$ be an integer and $q > 0$. Function $f$ is called $(\ell, q)$-subharmonic in $\Lambda$ and regular on $\Lambda \setminus \mathcal{S}$ (or, equivalently, $(\ell, q, \mathcal{S})$-subharmonic), if for any $u \notin \mathcal{S}$ with $\text{dist}(u, \partial \Lambda) \geq \ell$, we have

$$|f(u)| \leq q \max_{y : \|y - u\| = \ell} |f(y)|$$

(4.1)

and for any $u \in \mathcal{S}$

$$|f(u)| \leq q \max_{y : \text{dist}(y, \mathcal{S}) \in [1, 2\ell - 1]} |f(y)|.$$  

(4.2)

The motivation for this definition comes from the following observations.

Consider a pair of boxes $\Lambda_u(u) \subset \Lambda_L(x_0)$. If $\Lambda_u(u)$ is $(E, m)$-nonsingular, then the Geometric Resolvent Identity (GRI),

$$G_{\Lambda_L(x_0)}(u, y) = \sum_{(w, w') \in \partial \Lambda_u(u)} G_{\Lambda_u(u)}(u, w) G_{\Lambda_L(x_0)}(w', y), \quad y \notin \Lambda_u(u),$$

implies that function $f(x) = f_y(x) := G_{\Lambda_L(x_0)}(x, y; E)$ satisfies

$$|f(u)| \leq \tilde{q} \cdot \max_{v : \|v - u\| = \ell} |f(v)|,$$

(4.3)

with

$$\tilde{q} = \tilde{q}(d, \ell; E) = 2d\ell^{d-1}e^{-\gamma(m, \ell)}.$$  

(4.4)

Now suppose that $\Lambda_u(u)$ is $(E, m)$-S, but $\Lambda_L(x)$ is $E$-CNR and, in addition, for any $w$ with $\text{dist}(w, \Lambda_u(u)) = \ell + 1$ the box $\Lambda_u(w)$ is $(E, m)$-nonsingular. Then, by GRI applied twice,

$$|G(u, y)| \leq 2d(6\ell)^{d-1}e^{L^A} \max_{w : \|w - u\| = 2\ell} |G(w, y; E)|$$

$$\leq 2d(6\ell)^{d-1}e^{L^A}2d(2\ell)^{d-1}e^{-\gamma(m, \ell)} \max_{v : \|v - u\| \in [\ell + 1, 3\ell + 1]} |G(v, y; E)|.$$  

Therefore, with

$$q(d, \ell, E) := 4d^2(12\ell^2)^{d-1}e^{L^A}e^{-\gamma(m, \ell)} > \tilde{q}(d, \ell, E),$$

(4.5)

we obtain

$$|G(u, y)| \leq q(d, \ell, E) \max_{v : \|v - u\| \in [\ell + 1, 3\ell + 1]} |G(v, y; E)|.$$  

(4.6)

**Lemma 4.1.** Let $f$ be an $(\ell, q, \mathcal{S})$-subharmonic function on $\Lambda = \Lambda_L(x)$. Suppose that $\text{diam}(\mathcal{S}) \leq 2\ell$. Then

$$|f(x)| \leq q^{[(L-2)/\ell]} \mathcal{M}(f, \Lambda).$$

(4.7)
Lemma 4.2. Fix an integer \( j \geq 0 \) and let \( L_j, L_{j+1} \) be defined as in Eqn (3.2). Consider a box \( A_{L_{j+1}}(u) \), \( u \in \mathbb{Z}^d \). Fix \( \theta \in \Theta^{(j+1)}_\delta \), with \( \Theta^{(j+1)}_\delta \) defined in Eqn (3.8), and consider the family of operators \( \mathcal{H}(\omega; \theta), \omega \in \mathcal{O} \). If \( A_{L_{j+1}}(u) \) is \( \zeta \)-nonresonant and \( L_0 \) is large enough (and so are, therefore, all \( L_j \) with \( j > 1 \)), then the box \( A_{L_{j+1}}(u) \) is \( (\zeta, m) \)-nonsingular:

\[
\max_{y \in \partial A_{L_{j+1}}(u)} |G_{A_{L_{j+1}}(u)}(x, y; \zeta)| \leq e^{-\gamma(m, L_{j+1})}.
\]

**Proof.** By virtue of Lemma 3.3, for any \( y \in \partial A_{L_{j+1}}(u) \), we have

\[
|G_{A_{L_{j+1}}(u)}(x, y; \zeta)| \leq q \left[ \frac{L_{j+1}-2L_j}{\delta} \right] e^{L_{j+1}^{1/2}}.
\]

Fix \( m \geq 1 \) and assume that \( L_j \) is large enough. Then we can write that

\[
mL_{j+1} \left[ 1 - \frac{3L_j}{L_{j+1}} \right] \left( 1 + L_j^{-1/4} \right) - L_{j+1}^{-1/2} \geq mL_{j+1}(1 + L_{j+1}^{-1/4}),
\]

provided that \( L_j \) is large enough. Finally,

\[
|G_{A_{L_{j+1}}(u)}(x, y; \zeta)| \leq e^{-mL_{j+1}(1+L_{j+1}^{-1/4})} = e^{-\gamma(m, L_{j+1})}. \quad \square
\]

So, we come to the following important conclusion.

Lemma 4.3. Assume that \( \theta \in \Theta^{(j)} \). Then, for a sufficiently large number \( g_0 > 0 \) and any \( g > g_0 \), a box \( A_{L_{j+1}}(u) \) can be \( (\zeta, m) \)-S only if it is \( \zeta \)-resonant itself or it contains a box \( A_{L_{j+1}}(u) \) which is \( (\zeta, L_{j+1}) \)-resonant.

**Proof.** Fix an element \( \theta \in \Theta^{(\infty)} \). The assertion of the lemma will be proved by induction on \( j = 0, 1, \ldots \).

For \( j = 0 \), the assertion follows from the Lemma 3.2. Indeed, consider an arbitrary box \( A_{L_0}(v) \) and a complex number \( \zeta \) such that

\[
\text{dist} \left[ \zeta, \Sigma_{H_{A_{L_0}}(\omega; \theta)} \right] \geq \text{dist} [\zeta, \{ gV(x; \omega; \theta) \}] - 2d \geq 2e^{-\gamma(m, L)} + 2d.
\]

Then the box \( A_{L_0}(\omega; \theta) \) must be \( (\zeta, m) \)-nonsingular, since \( \theta \in \Theta^{(\infty)} \subset \Theta^{(0)} \).

Now fix an integer \( j \geq 0 \) and assume that the assertion of the Lemma 4.3 is proven for all \( j' \leq j \). Notice that \( \theta \in \Theta^{(\infty)} \subset \Theta^{(j)} \).

By Lemma 3.4, for any complex \( \zeta \), no box \( A_{L_j}(u) \) can contain two (or more) \( \zeta \)-resonant sub-boxes \( L_j(x) \).

Next, Lemma 4.2 says that a box \( A_{L_j}(v) \) can be \( (\zeta, m) \)-singular only if it is \( \zeta \)-resonant. Therefore, no box \( A_{L_{j+1}}(u) \) can contain two (or more) \( \zeta \)-singular sub-boxes \( L_j(v) \), whatever be \( \zeta \in \mathbb{C} \).
Moreover, Lemma 4.2 guarantees that $A_{L_j+1}(u)$ is $(\zeta, m)$-nonsingular, unless one of the two events occurs: either
• $A_{L_j+1}(u)$ is $\zeta$-resonant,
or
• $A_{L_j+1}(u)$ contains at least one sub-box $A_{L_j}(v)$ which is $(\zeta, L_{j+1})$-resonant.

By induction, the assertion of Lemma 4.3 follows. \(\square\)

5. Proof of the Anderson Localization for generic haarsh potentials

Results of Sections 3 and 4 provide a sufficient input for the usual proof of spectral localization of operators $H(\omega; \theta)$ with $\theta \in \Theta(\infty)$, the idea of which goes back to [FMSS85, DK89]. Indeed, the situation here is even simpler than in the context of IID random potentials, since, by virtue of Lemma 3.5, for any energy $E$, finding a disjoint pair of $(E, m)$-singular cubes $A_{L_j}(v), A_{L_k}(v')$ inside any cube $A_{L_{j+1}}(u)$ is not just unlikely (as it is in the case of IID potentials), but even impossible.

For the reader’s convenience, we give below the derivation of the spectral localization from Lemmas 3.5 and 4.3.

Consider the operator $H = \Delta + V$ with a bounded potential $V$ (the boundedness of $V$ can be relaxed, but in our case it is granted by the construction). Let $\mu$ be its spectral measure (defined with respect to a given normalized function $f \in \ell^2(\mathbb{Z}^d)$). As is well-known, $\mu$-a.e. generalized eigenfunction $\psi = \psi_E$ of $H$, with (generalized) eigenvalue $E$, is polynomially bounded:

$$|\psi_E(x)| \leq c(\psi_E)||x||^C, \ c(\psi_E), C < \infty.$$ 

So, consider such a polynomially bounded (generalized) eigenfunction $\psi$. There exists a point $x_0$ where $\psi(x_0) \neq 0$. The first general observation is that all cubes $A_{L_j}(x_0)$ with sufficiently large $L_j$ must be $(E, m)$-singular, for otherwise we would have

$$|\psi(x_0)| \leq \text{Const} L_j^{d-1} e^{-mL_j} \max_{||y-x_0||=L_j} |\psi(y)| \leq \text{Const} L_j^{C'} e^{-mL_j} \xrightarrow{j \to \infty} 0,$$

which contradicts the assumption that $\psi(x_0) \neq 0$.

Now suppose that, for some $j_0 < \infty$ and every $j \geq j_0$, the cube $A_{L_j}(x_0)$ is $(E, m)$-singular. Fix an arbitrary number $\rho \in (0, 1)$, pick a number $\eta > \frac{1+\rho}{1-\rho}$ and set

$$\tilde{A}_{j+1} = A_{2^n L_{j+1}}(x_0) \setminus A_{2^n 2 L_j}(x_0).$$

Notice that if $x \in \tilde{A}_{j+1}$, then

$$\text{dist}[x, \partial \tilde{A}_{j+1}] \geq \rho||x-x_0||.$$

Pick any point $x \in \tilde{A}_{j+1}$ and consider the cube $A_R(x)$ with $R = \rho||x-x_0||$. Then by Lemma 3.5, all cubes $A_{L_j}(u) \subset A_R(x)$ (which are automatically disjoint from $A_{L_j}(x_0)$) must be $(E, m)$-nonsingular. Further, the Lemma 4.2 applies to the function $\psi(x)$ which is $(L_j, q, \emptyset)$-subharmonic in $A_R(x)$ with $q =$
$\text{Const} L_j^{d-1} e^{-mL_j}$ and has a global upper bound $\mathcal{M}(\psi, A_R(y)) \leq \text{Const} L_j^{3C'/2}$, so that we obtain

$$|\psi(x)| \leq C L_j^{C'/2} q^{-j}$$

$$\leq C L_j^{C'/2} \left( \text{Const} L_j^{d-1} e^{-mL_j} \right)^{\frac{q(x)-q(x_0)}{L_j}}$$

$$\leq e^{-m\rho' \|x\|},$$

where $\rho' \in (0, \rho)$ can be made arbitrarily close to 1 for $j$ large enough by choosing $\rho \in (0, 1)$ sufficiently close to 1. This completes the proof of Theorem 1. □

6. Concluding remarks

The technical results of sections 3 and 4 imply not only spectral localization for “haarsch” Hamiltonians, but also dynamical localization. Our methods can be extended to the potentials generated by “hulls” $v(\omega; \theta)$ of finite smoothness. We prefer to do so in a forthcoming manuscript, while keeping the size of this paper within reasonable limits, and to focus here on the concept of a Grand Ensemble and on the probabilistic techniques which allow to establish the localization phenomenon for generic deterministic random Hamiltonians.

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