Non-Abelian Bosonization and Higher Spin Symmetries

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Abstract. The higher spin properties of the non-abelian bosonization in the classical theory are investigated. Both the symmetry transformation algebra and the classical current algebra for the non-abelian free fermionic model are linear Gel’fand-Dickey type algebras. However, for the corresponding WZNW model these algebras are different. There exist symmetry transformations which algebra remains the linear Gel’fand-Dickey algebra while in the corresponding current algebra nonlinear terms arised. Moreover, this algebra is closed (in Casimir form) only in an extended current space in which nonlinear currents are included. In the affine sector, it is necessary to be included higher isotopic spin current too. As result we have a triple extended algebra.
1. Introduction

The non-abelian free fermionic model and the WZNW model are not distinguished usually from some authors because as was shown by Witten [1] the $SO(N)$ non-abelian free fermionic model and the corresponding WZNW model are equivalent on a classical as well as a quantum level (see also [2]). This equivalence is understood as the presence of the same ($SO(N) \otimes SO(N)$, conformal and affine Kac-Moody) symmetry. This symmetry involves coinciding current algebras on a classical level as well on a quantum level and moreover one have identical anomalies for both models in the quantum case. Recently the Witten bosonization procedure was checked more directly deriving the WZNW model from the gauged free fermionic model [3], [4] in the path-integral approach.

The currents for both the free fermionic model and WZNW model are chiral, i.e.

$$\partial_z J = \partial_{\bar{z}} \bar{J} = 0,$$  \hspace{1cm} (1.1)

which shows that, besides the conformal and affine Kac-Moody symmetries, these models admit higher spin extensions known as $W$–symmetry. Usually, the Drinfeld-Sokolov reduction procedure is applied [3] – [10] to investigate of the $W$–symmetry in the models with chiral currents (for complete list of references see [11] and [12]). Despite the fact that the higher spin currents obtained by this method are independent, they are connected to the choice of the subgroup for reduction. Hence, these currents do not possess for completeness in general. Moreover, some of the characteristic properties of the model under consideration can be lost, because we start from the affine Kac-Moody algebra and the fields transformation laws usually are not analized.\footnote{One attempt to relate the Drinfeld–Sokolov procedure to the Lagrangian approach is discussed in the preprint [13] appearing after completion of the present paper.} The properties of the examined model become more transparent when the hidden symmetry method is applied [13]. This method consists in searching for additional symmetries of the action which generate higher spin conserved quantities [14], [15]. By means of this method in the papers [16] and [17] some difference between the symmetry properties of the classical free fermionic model and the classical WZNW model with respect to the higher spin transformations was obtained. Indeed, although the action integrals of both models are invariant with respect to the transformations satisfying isomorphic extended symmetry algebras; the $DOP(S^1)$ transformations [18] – [20], the corresponding higher spin currents for both models have different properties [21], [16] and [17]. In the paper [19], [4], [21] it is shown that the higher
spin fermionic currents satisfied a closed current algebra, which coincides with the algebra of the corresponding field transformations [16]. Really, applying the Witten non-abelian bosonization procedure [1] the corresponding bosonic currents and the field transformations generating these currents were obtained in [17]. Moreover, it was shown that these infinitesimal bosonic field transformations being the symmetry of the WZNW action satisfied the same algebra as the corresponding fermionic field transformations. However, the transformation laws for the higher spin bosonic currents essentially differ from the corresponding fermionic currents law. This difference consists in that the higher spin fermionic currents form invariant space, while the corresponding bosonic currents do not span an invariant space. To obtain an invariant bosonic currents space the higher spin Noether currents were complemented by additional nonlinear higher spin currents [22].

The present paper is devoted to the more systematic investigation of the properties of the Witten non-abelian bosonization procedure with respect to the higher spin symmetry within the hidden symmetry approach. In the second section a brief review of the results of the papers [16] is given. The explicit form of the infinitesimal fermionic fields transformation Lie algebra and the corresponding classical current transformation laws, i.e. current algebra are presented in the form convenient for comparison with those obtained for the WZNW model. These higher spin $SO(N)$ algebras are closed only if they are completed to $GL(N)$ algebras. It is shown also that the correspondence between the symmetry transformations and the currents is one to one up to the terms containing derivatives from lower spin quantities.

In the third section the corresponding higher spin bosonized currents and the corresponding field transformations are considered. The infinitesimal transformations being on-shell symmetry of the classical WZNW action satisfied the same linear Gel’fand-Dickey algebra [23] as the fermionic field transformations. However, in this case the correspondence between the field transformations and the conserved currents is not one to one. For instance, from the bilinear higher spin currents which correspond to the linear $DOP(S^1)$ field transformations applying the Poisson brackets we derive the transformations which differ from the initial transformations by nonlinear terms. These nonlinear transformations generate nonlinear higher spin currents involving transformations containing higher nonlinearity and so one. For instance, for any current with spin $s = m$ we can obtain $m - 1$ new nonlinear currents with the same spin.

In the fourth section the classical current algebra for $O(N)$ WZNW model is obtained. The higher spin $U(1)$ current algebra ($W_\infty$ algebra) is closed (in terms of Casimir currents)
only if to the spin $s = m$ bilinear current a new $m - 1$ additional nonlinear currents with the same spin are included. In the case of $SO(N)$ isotopic currents with higher conformal spin we have more complicated situation. In this case to have closed current algebra the initial currents must be completed with $GL(N)$ isotopic tensor currents too. As a result a triple extended affine current algebra is obtained. For the abelian case, which also is considered, the obtained algebra can be considered as nonlinear $\hat{W}_{1+\infty}$ algebra too [24]. However, for the non-abelian case in general we have nonlinear closed algebra (without adding new terms) only in terms of matrix valued currents [24, 25].

In the fifth section some concluding remarks are given.

In the Appendix the explicit form of the unconstrained higher spin currents is given for the $SL(2,R)$ WZNW model in terms of the Gauss decomposition components field.

2. Higher Spin Field Transformations for Free Fermionic Model

As is well known, always when we are dealing with a Lagrangian theory whose action is invariant with respect to some $N$-parametric rigid (gauge) group transformations $G$ with Lie algebra $\mathcal{G}$ we know from the Noether theorem that (for the application of the Noether theorem to the conformal theories see [26]):

i) We have $N$ conserved quantities,

ii) These quantities act as generators of the group $G$,

iii) The corresponding current algebra coincides with the Lie algebra $\mathcal{G}$ of the group $G$.

It is evident that the assertion iii) is a consequence of ii). These assertions take place also for some infinite-parametric group such as conformal group in two-dimensions, affine Kac-Moody transformations and for the higher spin extended transformations as the area-preserving transformations $w_{\infty}$, extended conformal and affine Kac-Moody transformations in the case of free fermionic model, e.t.. However, as was shown in [17] (see also [22]) a class transformations exist for the case of the WZNW model in which terms violating the assertions ii) and iii) appear. The latter means that the current$\leftrightarrow$symmetry correspondence is not invertible in this case. Consequently, applying the Poisson bracket we do not return to the same symmetry transformation and the corresponding current algebra does not coincide with the Lie algebra of the initial infinitesimal fields transformations. In the present paper we consider two examples for comparison: the non-abelian free fermionic
model for which $i) - iii)$ are satisfied for higher spin extended symmetry transformations and the WZNW model for which $ii)$ and $iii)$ are violated on the higher spin level.

In the present paper we consider transformations which are on-shell symmetry too, i.e. transformations leaving the action unchanged only on the solutions of the equations of motions. We recall that such kinds of symmetries usually appear in the supersymmetric theories. According to the modified Noether theorem given by Ibraghimov [27], [28] to any continuous one parametric on-shell symmetry transformation there also corresponds one conserved quantity, i.e. the assertion $i)$. As we will see below the assertions $ii)$ and $iii)$ can be violated on the higher spin symmetry level.

First we reconsider the classical free fermionic model whose higher spin conserved currents are given by:

$$\mathcal{V}^n = \bar{\psi} \partial_z^{n+1} \psi,$$

$$\mathcal{J}^n_a = \bar{\psi} t_a \partial_z^n \psi,$$  \hspace{1cm} (2.1a)

being a higher spin extension of the holomorphic components of the stress-energy tensor, $\mathcal{V}^0 = T_{zz}$ and of the $SO(N)$ or $SU(N)(GL(N))$ (for definiteness) isotopic current $\mathcal{J}^0_a = J_a$ respectively. In the formulas (2.1) $z$ denotes a light-cone variable $x_+ = (x_0 + x_1)/2$ or the corresponding complex variable, $\psi$ is a holomorphic component of the free spinor field and $t_a \in \mathcal{G}$ are the generators of the group $G$ in fundamental representation. We restrict our considerations only to the holomorphic components $\partial_z \mathcal{V} = 0$, having in mind that the anti-holomorphic components have a similar form and we are concerned with the $SO(N)$ model for which $\psi$ is a holomorphic component of a anticommuting Majorana spinor, i.e. $\bar{\psi} = \psi$. Notice, that in (2.1) and what follows the currents normalization constants are omitted.

When we are dealing with the complex spinor field all the currents (2.1) are independent, however, for the Majorana spinor field this is not the case. Indeed, using the formula [16]:

$$\partial^m \phi \Omega \chi = \sum_{p \geq 0} \binom{m}{p} \partial^p (\phi \Omega \partial^{m-p} \chi),$$

where $\Omega$ is a constant matrix one can get:

$$\mathcal{V}^1 = \partial \mathcal{V}^0,$$

$$\mathcal{V}^3 = 2 \partial \mathcal{V}^2 - \partial^3 \mathcal{V}^0,$$

$$\mathcal{V}^5 = 3 \partial \mathcal{V}^4 - 5 \partial^3 \mathcal{V}^2 + 3 \partial^5 \mathcal{V}^0,$$

$$\vdots$$

$$\vdots$$

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for symmetric matrix $\Omega$ and

$$
\mathcal{J}_a^1 = \frac{1}{2} \partial \mathcal{J}_a^0,
$$

$$
\mathcal{J}_a^3 = \frac{3}{2} \partial \mathcal{J}_a^2 - \frac{1}{4} \partial^3 \mathcal{J}_a^0,
$$

$$
\mathcal{J}_a^5 = \frac{5}{2} (\partial \mathcal{J}_a^4 - \partial^3 \mathcal{J}_a^2) + \frac{1}{2} \partial^5 \mathcal{J}_a^0,
$$

(2.4)

for skew symmetric matrix $\Omega$. For arbitrary odd spin $U(1)$ current (2.1a) and even spin isotopic vector current (2.1b) we obtain respectively:

$$
\mathcal{V}_{2m+1} = \sum_{p=0}^{m} C_{m}^{p} \partial^{2p+1}\mathcal{V}_{2(m-p)},
$$

(2.5a)

$$
\mathcal{J}_a^{2m+1} = \sum_{p=0}^{m} \tilde{C}_{m}^{p} \partial^{2p+1}\mathcal{J}_a^{2(m-p)},
$$

(2.5b)

where $C_{m}^{p}$ are constants which could be derived from (2.2). The formulas (2.3) and (2.5) show us that by a suitable choice of the constants $R_p^n$ in formula

$$
W^n \to \hat{W}^n = \sum_{p=0}^{n} R_p^n \partial^p W^{n-p}
$$

(2.6)

we can obtain a basis in which $\mathcal{V}_{2m+1} \equiv 0$ and $\mathcal{J}_a^{2m+1} \equiv 0$. For convenience we use the basis (2.1) in which the depending currents are not excluded, hence the relations (2.3) must be kept in mind.

The currents (2.1) are Noetherian, i.e. they follow from the Noether theorem as a consequence of the on-shell symmetry of the free fermionic action with respect to the infinitesimal transformations:

$$
\delta^m \psi(z) = k_m(z) \partial^{m+1} \psi,
$$

(2.7a)

$$
\hat{\delta}^m \psi(z) = \alpha^a_m(z) t_a \partial^m \psi,
$$

(2.7b)

where $k_m(z)$ and $\alpha^a_m(z)$ are arbitrary holomorphic functions. It is easy to check that the transformations (2.7) satisfied the following Lie algebra:

$$
[\delta^m(k), \delta^n(h)]\psi(z) = \sum_{r \geq 0} \delta^{m+n-r+1}(h_{n+1}, k_{m+1}) \psi(z),
$$

(2.8a)
$$[\delta^m(k), \hat{\beta}^n(\hat{\beta})]\psi(z) = \sum_{r \geq 0} \hat{\delta}^{m+n-r+1}(\beta_n, k_{m+1})^r \psi(z), \quad (2.8b)$$

$$\left[\hat{\delta}^m(\hat{\alpha}), \hat{\beta}^n(\hat{\beta})\right]\psi(z) = \sum_{r \geq 0} \left(\delta^m_n - n \delta^m_n a \hat{t}_c a + \hat{\delta}^m - n \hat{t}_c a \right) \psi(z), \quad (2.8c)$$

where $t_{ab} = \{t_a, t_b\}/2,$

$$2[\beta_n, \alpha_m]^{\pm} = \binom{n}{r} \beta_n \partial^r \alpha_n + \binom{m}{r} \alpha_m \partial^r \beta_n \quad (2.9)$$

and it is taken into account that the binomial coefficients $\binom{m}{r} = 0$ for $r > m.$ This algebra contains as subalgebras the Virasoro algebra, the $W_\infty$ algebra, semi-direct product of the Virasoro and the affine Kac-Moody algebras.

**Observation:** If $m \neq n$ in the r.h.s. of (2.8d) the anticommutators of the matrix generators of $SO(N)$ are also included. The origin of these anticommutators is the following: When $m \neq n$ in the commutator of two transformations (2.7d) the matrix products $t_at_b$ and $t_bt_a$ appear by different coefficients, so that they do not form the Lie product only. Indeed, taking into account that the product $t_at_b$ can always be represented as

$$t_at_b = \frac{1}{2}[t_a, t_b] + \frac{1}{2}\{t_a, t_b\}$$

and that $\{t_a, t_b\} \in U(N) (GL(N))$ when $t_a \in SU(N) (SL(N))$ then $t_at_b \in U(N) (GL(N))$. The latter allows us to conclude that the extended affine $SU(N)$ Kac-Moody algebra is not closed alone, because the affine $U(1)$ transformations coincides with the $W-$transformations. However, if $t_a \in SO(N)$ then the anticommutator $\{t_a, t_b\} \notin SO(N)$ for any nontrivial representation of $SO(N).$ The latter is a consequence of the incompleteness of the space of matrices $t_a$ in any representation of $SO(N).$ In this case $t_at_b \in GL(N).$ Indeed the $N(N-1)/2$ skew symmetric $N \times N$ matrix together with $N(N+1)/2$ symmetric matrices $\{t_a, t_b\}$ form a complete $N \times N$ real matrix space.

According to the second part of the Noether theorem (assertion ii) applying the Poisson bracket we derive the following field transformations laws:

$$\delta^{l+1}\psi(z) = \int dk_l(x) \{\mathcal{V}^{l+1}(x), \psi(z)\}_{PB} =$$

$$= -2 \sum_{s \geq 0} \delta_{l+1} \delta_{k_l} \partial^{l+1}\psi + (-)^{l+1} \sum_{r \geq 1} \binom{l+1}{r} \partial^r k_l \partial^{l-r+1}\psi(z), \quad (2.10a)$$
\[
\begin{align*}
\delta^l \psi(z) &= \int dx \alpha^a_l(x) \{ J^l_a(x), \psi(z) \}_{PB} = \\
&= -2 \sum_{s \geq 0} \delta_{l+1,2s} \alpha^a_l \partial^s t_a \psi + (-)^l \sum_{r \geq 1} \binom{l}{r} \partial^r \alpha^a_l \partial^{l-r} t_a \psi(z), \quad (2.10b) \\
\delta^{l+1} \psi(z) &= \int dx \alpha^{ab}_l(x) \{ J^{l+1}_{ab}(x), \psi(z) \}_{PB} \\
&= -2 \sum_{s \geq 0} \delta_{l,2s} \alpha^{ab}_l \partial^{l+1} t_{ab} \psi \\
&\quad + (-)^l+1 \sum_{r \geq 1} \binom{l+1}{r} \partial^r \alpha^{ab}_l \partial^{l-r+1} t_{ab} \psi(z), \quad (2.10c)
\end{align*}
\]

where
\[
J^l_{ab} = \frac{1}{2} \psi \{ t_a, t_b \} \partial^l \psi. \quad (2.11)
\]

Comparing the transformations (2.10) with (2.7) we can conclude that they differ by the terms \(\partial^{m-r} (\delta^r \psi)\), \((r = 0, \ldots, m - 1)\) and moreover that the terms \(h_{l+1} \partial^{l+1} \psi\) are absent for odd \(l\) in (2.10a,c) and for even \(l\) in (2.10b). In both cases the corresponding to these transformations currents are not independent which agree with (2.3). This means that applying both parts of the Noether theorem we return to the same (up to redefinition) transformation. The absence of some terms in formulas (2.10) included in (2.7) is a property of the basis under considerations. The independent Noether currents corresponding to the transformations (2.10) also coincide up to redefinition to the currents (2.1). Another important difference, however, is that the transformations (2.10) are off shell-symmetry of the action, i.e. they leave the free fermionic action up to surface terms for arbitrary fields configurations.

It is straightforward to check that the transformations (2.10) satisfy the Lie algebra which differs from (2.8) only by the factor \((-)^r\).

Using (2.7a) we derive the currents transformation law
\[
\delta^m \chi^m = \sum_{p \geq 1} \binom{n+1}{p} \partial^p k_m \chi^m + n - p + 1 \\
- k_m \sum_{q \geq 0} (-)^{m+q} \binom{m+1}{q} \partial^q \chi^m + n - q + 1, \quad (2.12)
\]

which shows that they span an invariant space with respect to the transformations (2.7a). Moreover, it is easy to check that all the currents (2.1) form invariant space with respect to the both laws (2.7) and (2.10), hence, they satisfied a closed current algebra.
When \( m = 0 \) the law (2.12) becomes:

\[
\delta^0 \mathcal{V}^n = (n + 1) \partial k_0 \mathcal{V}^n + k_0 \partial \mathcal{V}^n + \sum_{p \geq 2} \binom{n + 1}{p} \partial^p k_0 \mathcal{V}^{n-p+1}
\]

(2.13)

hence \( \mathcal{V}^n \) is a quasi-primary field.

3. Higher Spin Symmetry in WZNW Model

First we consider the abelian case in order to gain intuition and to make the choice of higher spin bosonic currents more transparent.

3.1. Abelian Bosonization

According to the abelian bosonization procedure we should pair a bosonic currents to any fermionic current (2.1a)

\[
\mathcal{V}^n(z) = \mathcal{U}^{-1} \partial^{n+2} \mathcal{U}(z) = e^{-\varphi(z)} \partial^n e^{\varphi(z)},
\]

(3.1)

where \( \varphi \) is dimensionless scalar field. We recall that from (3.1) it follows:

\[
\mathcal{V}^n(z) = P^{n+2}(J) = \left( J(z) + \partial \right)^{n+2}.1,
\]

(3.2)

where \( P(J) \) are the Faà di Bruno polynomials [29], [30] and \( J = \mathcal{U}^{-1} \partial \mathcal{U} = \partial \varphi \).

It is easy to show that the currents (3.1) are generated by the transformations

\[
\delta^m \mathcal{U}(z) = k_m(x) \partial^{m+1} \mathcal{U}(z),
\]

(3.3)

being an on-shell symmetry of the action

\[
S = \frac{1}{2} \int d^2 z \partial z \mathcal{U}^{-1} \partial \mathcal{U}.
\]

(3.4)

Here we consider the \( U(1) \) group element \( \mathcal{U} \) as independent field variables, i.e. we deal with the \( U(1) \) chiral model. Notice, that the transformations (3.3) coincide with corresponding fermionic ones (2.7a) by the form, hence they satisfy the same \( W_{1+\infty} \) (linear Gel’fand-Dickey) algebra (2.8a).

The canonical currents corresponding to the transformations (3.3) are given by

\[
\mathcal{V}_{can}^{n} = \partial \mathcal{U}^{-1} \partial^{n+1} \mathcal{U}.
\]

(3.5)
However, taking into account the identity

$$V^{n+1} = (J + \partial)V^n$$  \hspace{1cm} (3.6)$$

which follows from (3.2), we obtain

$$V^n = \partial V^{n-1} - V^n_{can}.$$  \hspace{1cm} (3.7)$$

Consequently, the currents (3.1) can be considered as improved ones. Indeed, setting $n = 0$ in (3.7) we get the improved stress-energy tensor

$$T = -V^0_{can} + \partial j = \frac{1}{2} j^2 + \partial j.$$ 

According to the second part of the Noether theorem we have:

$$\delta^n U = \int dy k_n(y)\{V^n(y), U(z)\}_{PB} \sum_{p \geq 1} (-)^{p-1} \binom{n}{p} \partial^{p-1} k_n \partial^{n-p} U$$

$$= U(z) \sum_{p,r \geq 1} (-)^{p-1} C_{p,r}^{n+2-p-1} \partial^{p-r-1} k_n(z) \partial^r \left( U^{-1} \partial^{n-p+2} U(z) \right),$$  \hspace{1cm} (3.8)$$

where

$$C_{m,n}^{p,q} = \binom{m}{p} \binom{n}{q}$$  \hspace{1cm} (3.9)$$

and it is taken into account that $C_{p,q}^{m,n} = 0$ if $p > m$ or $q > n$.

**Observation:** In the transformation law (3.8) we see also a nonlinear part (with respect to $U$ in contrast to the corresponding fermionic case (2.10), and the bosonic transformations (3.3)) where nonlinear terms are absent. The presence of these nonlinear terms in (3.8) is a consequence of the nonlinearity of the bosonization procedure (3.1) and they appear when higher order derivatives are included. Another consequence of this nonlinearity is the noninvariance of the current space (3.1) with respect to the linear transformations (3.3) (see [17], [22], [21]).

Indeed, applying the linear transformations (3.3) we obtain:

$$\delta^m V^n = -k_m V^{m-1} V^n + \sum_{p \geq 0} \binom{n+2}{p} \partial^p k_m V^{m+n-p+1}.$$  \hspace{1cm} (3.10)$$

If $m, n \neq 0$ the first term in the r.h.s. is essentially nonlinear with respect to the currents $V^n$ and it is not present in the fermionic current transformation laws (2.12). For $m = 0$ the formula (3.10) becomes

$$\delta^0 V^n = (n+2) \partial k_0 V^n - k_0 \partial V^n + \sum_{p \geq 2} \binom{n+2}{p} \partial^p k_m V^{m+n-p+1},$$  \hspace{1cm} (3.11)$$
i.e. the higher spin bosonic currents (3.1) are quasi-primary fields too. However, with respect to the nonlinear transformations (3.8) only the stress-energy tensor \( V^0 \) is a primary field. All this is an indication that we have nonlinear current algebra.

The currents which can be obtained from the nonlinear transformations (3.8) contain nonlinear terms \( V^m V^l, (l = 0, \ldots m) \). Consequently, by multiple application of the Noether theorem we obtain the terms with an increasing nonlinearity. It is easy to check that the product \( V^m V^n \) is impossible (if \( m, n > -1 \)) to be represented linearly in terms of the currents \( V^k \) and their derivatives, i.e.

\[
V^m V^n \neq \sum_{k=0}^{m+n} A_k^{m,n} \delta^k V^{m+n-k}, \tag{3.12}
\]

where \( A_k^{m,n} \) are constant coefficients. Taking into account the formula (3.2) we can represent \( V^n \) also as polynomials of the first current \( V^{-1} = J \) and its derivatives. However, these polynomials do not form a complete system.

For more clarification we turn to the \( U(1) \) group parameter \( \varphi(z) \) in terms of which the transformations laws (3.3) take the form:

\[
\delta^n \varphi = U^{-1} \delta^n U = \sum_{p,r \geq 1} (-)^{p-1} \tau_{p,r}^{n+1} \partial^{p-r-1} \kappa_n(x) \partial^r \mathcal{P}^{n-p}(\partial \varphi), \tag{3.13}
\]

hence, the field \( \varphi \) is always transformed nonlinearly if \( n > 0 \).

Rescaling the field \( \varphi \to h^{-1} \varphi \) and taking the limit \( h \to 0 \) we derive from (3.13) the area-preserving transformations law:

\[
\tilde{\delta}^n \varphi = \lim_{q \to 0} \frac{q^{n+1}}{n+1} \delta^n (q^{-1} \varphi) = k_n(z)(\partial \varphi)^{n+1} \tag{3.14}
\]

which satisfy the \( w_\infty \) Lie algebra:

\[
[\tilde{\delta}^m(k), \tilde{\delta}^n(h)] \varphi = \tilde{\delta}^{m+n} (h_m \partial k_n - k_m \partial h_n) \varphi, \tag{3.15}
\]

i.e. we have a contraction of the \( W_{1+\infty} \)-algebra to the \( w_\infty \)-algebra. We recall that the transformations (3.8) and (3.13) are only on-shell symmetry of the free scalar field action while the \( w_\infty \) transformations (3.14) are off-shell symmetry.

The Noether currents corresponding to the transformations (3.14) are

\[
v^n(z) = \frac{1}{n+2} (\partial \varphi)^{n+2}, \tag{3.16}
\]
If we turn to the group parameters as independent field variables and assume the transformation law
\[ \delta^n \varphi = k_n(x) \partial^n \varphi, \]  
being on-shell symmetry of (3.4) we obtain a bosonic realization of linear Gel’fand-Dikey algebra. The corresponding conserved currents are given by
\[ \tilde{V}^n = \partial \varphi \partial^{n+1} \varphi, \]  
These currents are generators of the same (up to redefinition) field transformations (3.17). Hence, the higher spin symmetry properties of the initial fermionic fields can be reproduced if we consider the group parameters as independent field variables.

### 3.2. Non-abelian Bosonization

Similarly to the foregoing cases we choose the higher spin conserved quantities for the WZNW in the form:
\[ V^n = tr(\partial g^{-1} \partial^{n+1} g), \]  
\[ J^n_a = tr(\partial g^{-1} \partial^n g t_a), \]  
where \( g \) takes its values on the group \( G \), i.e. \( g \in G \). The level parameter as well as the normalization constants are omitted in (3.19) and what follows. For definiteness we suppose \( G = SO(N) \), however our considerations are applicable for any other group \( G \). In the present subsection we consider the canonical conserved quantities only. In paper [17] it was shown that these currents can be represented by the Faá di Bruno polynomials too, which show us that (3.19) is an appropriate bosonization of the higher spin fermionic currents (2.1). Indeed the currents (3.19) can be represented in the form:
\[ V^n = -tr(j(z) U^{n+1}(z)), \]  
\[ J^n_a = -tr(j(z) U^n_a(z)), \]  
where the notations
\[ j(z) = U^1(z) = g^{-1} \partial g(z), \quad U^n(z) = g^{-1} \partial^n g(z), \quad U^n_a(z) = g^{-1} \partial^n g t_a = U^n(z)t_a \]  
are introduced. This coincide with the Sugawara construction for \( n = 0 \), hence for \( n > 0 \), the formulas (3.20) can be considered as extended Sugawara constructions.
It is easy to verify that the following recursive relations hold:

\[ U^{n+1} = (j + \partial)U^n = DU^n, \quad (3.22) \]

where \( D = J + \partial \) is a matrix covariant derivative. Starting from \( U^0 = I \), where \( I \) is the identity matrix and \( U_a^0 = t_a \), we obtain:

\[ U^n = D^n I = P^n(j), \quad U^n_a = D^n t_a = P^n(j)t_a, \quad (3.23) \]

which are the matrix generalizations of the Faà di Bruno polynomials. Inserting (3.23) into (3.21) we get:

\[ V^n(z) = -tr\left(j(z)D^{n+1}(z)\right) = -tr\left(jP^{n+1}(j)\right), \quad J^n_a(z) = -tr\left(j(z)D^n t_a\right) = -tr\left(jP^n(j)t_a\right). \quad (3.24) \]

Rescaling the current \( j \rightarrow q^{-1}j \) in the limiting case \( q \rightarrow 0 \) we obtain:

\[ v^n = \lim_{q \rightarrow 0} \frac{q^{n+2}}{n+2} V^n(q^{-1}j) = \frac{1}{n+2} tr(j)^{n+2}, \quad (3.25a) \]

\[ J^n_a = \lim_{q \rightarrow 0} \frac{q^{n+1}}{n+1} J^n_a(q^{-1}j) = \frac{1}{n+1} tr\left((j)^{n+1}t_a\right), \quad (3.25b) \]

The currents (3.25a) coincide with the area-preserving conserved currents. In order to assure that the currents (3.25b) have a Noether character we should introduce higher rank isotopic tensor currents (see [17] and [22]).

In the case when \( g \) is defined over the orthogonal group, i.e. \( g \in SO(N) \), hence \( g^{-1} = g^T \) which allows us to write

\[ (1 - (-)^m)V^m(x) = \sum_{p \geq 1} (-)^{m-p} \binom{m}{p} \partial^p V^{m-p}, \quad (3.26a) \]

\[ (1 - (-)^m)J^m_a(x) = \sum_{p \geq 1} (-)^{m-p} \binom{m}{p} \partial^p J^{m-p}_a, \quad (3.26b) \]

Notice, that the equality (3.26a) takes place only for \( (t_a)^T = -t_a \), while the second rank symmetric tensor currents \( J^{n}_{ab} \) satisfy the Eq. (3.26a). From (3.26) we obtain

\[ V^{2m+1} = \sum_{p=0}^{m} B_{2p+1}^{2m+1} \partial^{2p+1} V^{2(m-p)}, \quad (3.27a) \]

\[ J^{2m+1}_a = \sum_{p=0}^{m} \tilde{B}_{2p+1}^{2m+1} \partial^{2p+1} J^{2(m-p)}_a, \quad (3.27b) \]
where \( B^m_p \) are coefficients which can be determined from (3.26). The explicit form of (3.27) for the currents with lowest spin reads:

\[
\begin{align*}
V^{-1} &= tr g^T \partial g \equiv 0, \\
V^1 &= \frac{1}{2} \partial V^0, \\
V^3 &= \frac{3}{2} \partial V^2 - \frac{1}{4} \partial^3 V^0, \\
V^5 &= \frac{1}{2}(-5 \partial V^4 + 5 \partial^3 V^2 - \partial^5 V^0), \\
&\vdots \\
J^1_a &= tr g^T t_a \partial g \equiv 0, \\
J^3_a &= \partial J^2_a, \\
J^5_a &= 2 \partial J^4_a - \partial^3 J^2_a. \\
&\vdots
\end{align*}
\]

(3.28)

Notice that the coefficients in these relations for the currents \( V \) and \( J_a \) with coinciding index differ because the current \( V^n \) has a conformal spin \( s = n + 2 \) while the conformal spin of the current \( J \) is \( s = n + 1 \).

The formulas (3.27) ((3.28)) show us that the odd spin \( U(1) \) currents \( V^{2m+1} \) are represented as derivatives from the underlying even spin currents, while the even spin currents \( J_a^{2m+1} \) (the current \( J_a^0 \) has conformal spin equal to one) are represented in terms of derivatives from the underlying odd spin currents \( J_a^{2l} \). Consequently, in the case of the \( SO(N) \) WZNW model the \( U(1) \) currents with even spin are independent, while only the isotopic vector currents with odd spin are independent which corresponds to the \( SO(N) \) free fermionic model. By suitable redefinition of the currents (2.10) we can pass to a basis in which all dependent currents vanish identically.

Notice, that in the current algebra for the \( SO(N) \) currents (3.19) as well as in the algebra of the corresponding field transformation laws, which would be considered below, there also arise the symmetric matrix generators \( t_{ab} = \{t_a, t_b\}/2 \) which as in the fermionic case complement the \( SO(N) \) algebra to the \( GL(N) \) algebra.

We recall that the currents (3.19) follow from the Noether theorem as a consequence of the on-shell symmetry of the WZNW action with respect to the linear transformations (see Ref. [17]):

\[
\begin{align*}
\delta^m g(z) &= k_m(z) \partial^{m+1} g(z), \\
\hat{\delta}^m g(z) &= \alpha_m^a(z) t_a \partial^{m+1} g(z),
\end{align*}
\]

(3.29a) (3.29b)
where \( k_m, \alpha_m \) are arbitrary holomorphic functions. These transformations satisfy the same Lie algebra (2.8) as the corresponding fermionic transformations (2.7). Taking into account that \( \delta^m g^{-1} = -g^{-1} \delta^m g g^{-1} \) it is easy to verify that these transformations satisfy the same Lie algebra as the transformations for \( \delta^m g \). Notice, that due to the dimensionlessness of the scalar field \( g \) the transformations (2.10a) can be considered twofold: ones as space-time conformal transformations and others as \( U(1) \) gauge transformations. From this property the Sugawara construction for the stress-energy tensor follows. In the present paper the transformations (2.10a) are considered as \( U(1) \) gauge transformations. However, as in the abelian case the currents (3.19) do not form invariant space with respect to the transformations (3.29) (see Refs. [17], [22]). Applying the equal-time Poisson brackets we derive the fields transformations generating from the currents (3.19):

\[
\delta^m g(x) = \int dy k_m(y) \{ V^m(y), g(x) \}_P = \frac{4\pi}{N} \left\{ -k_m(x) \partial^{m+1} g(x) + g(x) \sum_{p \geq 1, q \geq 0} (-)^{p-1} C^{m+1,p-1}_{p,q} \partial^q k_m \partial^{p-q-1} \left( \partial g^{-1} \partial^{m-p+1} g \right) \right\},
\]

\[
\hat{\delta}^m g(x) = \int dy \alpha^a_m(y) \{ J^m_a(y), g(x) \}_P = \frac{4\pi}{N} \left\{ -\alpha^a_m(x) t_a \partial^m g(x) + g(x) \sum_{p \geq 1, q \geq 0} C^{m,p-1}_{p,q} \partial^q \alpha^a_m \partial^{p-q-1} \left( \partial g^{-1} t_a \partial^{m-p+1} g \right) \right\},
\]

We use the Poisson bracket defined in [1] in which the higher spin current variation is inserted by

\[
\delta J^m_A = tr \left( \partial \delta g^{-1} \partial^m g A + \partial g^{-1} \partial^m \delta g A \right) = tr \left( -\partial^m g A g^{-1} \partial (\delta g g^{-1}) + \sum_{p \geq 1} \binom{m}{p} \partial^{m-p} g A \partial g^{-1} \partial^p (\delta g g^{-1}) \right)
\]

and

\[
(\delta gg^{-1})(x)(\delta gg^{-1})(y) = \frac{4\pi}{N} I \otimes I \epsilon(x - y).
\]

Notice, that the anti-holomorphic coordinate \( \bar{z} \) appears as evolution (time) variable.

**Observation:** The variations (3.30) for \( m = 0 \) are linear and coincide with the ordinary conformal and gauge (affine Kac-Moody) transformations, while for \( m > 0 \) nonlinear terms arise. The appearance of these nonlinear terms is in contrast with the initial transformations (3.29) and the corresponding spinor transformations (2.10).
We note, that the transformations (3.30) derived by means of the Poisson bracket are on-shell symmetry of the WZNW action too, which allow us to apply the Noether theorem again. The nonlinear terms in (3.30) generate nonlinear currents of type $tr(jU^mU^n)$ and $tr(jt_aU^mt_bU^n)$. By repeatedly applying the Noether theorem we obtain set of nonlinear currents:

$$V^{n_1,n_2,...,n_l} = tr(U^{n_1}U^{n_2}...U^{n_l}), \quad (l = 1, 2, ...). \quad (3.33)$$

Taking into account that $U^0 = I$ and $U^1 = j = g^{-1}\partial g$ from (3.33) we obtain $V^{n_1,...,n_r,0,...,0} = V^{n_1,...,n_r}$ if ($r = 2, 3, ...$) and $V^n = V^{1,n+1,0,...,0}$ if $r = 1$. It is straightforward to verify that the extended currents space \{V^{n_1,n_2,...,n_l}\} is invariant with respect to the transformations (3.30). For instance, the transformation law for the currents (3.33) with respect to the linear transformations (3.29) reads

$$\delta^m V^{n_1,n_2,...,n_l} = \sum_{q=0}^{l-1} \left\{-k_m V^{n_1,...,n_q,m,n_{q+1},...,n_l} \right\} + \sum_{p \geq 0} \left( \begin{array}{c} n_q \hfill p \end{array} \right) \partial^p k_m V^{n_1,...,n_{q-1}n_q+m-p+1,n_{q+1},...,n_l} \right\}. \quad (3.34)$$

To have an invariant current space with respect to the extended affine transformations too, we include symmetric isotopic tensor currents $J^{n_1,n_2,...,n_l}_{a_1,...,a_p}$, $(l, p = 1, 2, ...; p < l)$. For example the explicit form of the rank 2 tensor currents is:

$$J^{n_1,...,n_l}_{a,b} = \sum_{q,p=0,q\neq p}^{l} tr(U^{n_1}...U^{n_r}t_a...U^{n_q}t_b...U^{n_l}). \quad (3.35)$$

It is straightforward to check that the set of all tensor currents form an invariant space.

The same result we find if we consider the algebra of nonlinear infinitesimal transformations (3.30).

4. Bosonic Current Algebra

4.1. Abelian Model

First we consider the abelian model whose currents are given by (3.1). Notice, that for our purposes it is more convenient to consider the $U(1)$ principal chiral model for which
the group element \( U \) are independent field variables. In this case we derive the following current algebra of the improved currents (3.1):

\[
\{V^m(x), V^n(y)\}_\text{PB} = \delta V^m(x) \delta V^n(y) = \begin{array}{c}
4\pi \sum_{p,q,r \geq 1, s \geq 0} C_{p,q}^{m+2,p-1} C_{q,s}^{n+2,q} \partial^{q-s} V^{m-p} \partial^{p-r-1} V^{n-q} \partial^{r+s} \delta(x-y),
\end{array}
\]

where we used

\[
\delta V^m(x) = \sum_{p \geq 1} \left( \begin{array}{c}
m + 2 \\
p \end{array} \right) V^{m-p} \partial^p (U^{-1} \delta U)
\]

and

\[
U^{-1}(x) \delta U(x) U^{-1}(y) \delta U(y) = 4\pi \varepsilon(x-y).
\]

The r.h.s. of (4.1) contains linear as well nonlinear terms which can be seen by rewriting (4.1) in the following equivalent form:

\[
\{V^m(x), V^n(y)\}_\text{PB} = \delta V^m(x) \delta V^n(y)
= 4\pi \left\{ \sum_{p,q,r \geq 1, s \geq 0} C_{p,q}^{m+2,p-1} C_{q,s}^{n+2,q} \right.
\times \partial^{q-s} V^{m-p} \partial^{p-r-1} V^{n-q} \partial^{r+s} \delta(x-y)
\]

\[
+ \sum_{q,r \geq 1} C_{q,r}^{n+2,m+1} \partial^{m-r+1} V^{n-q} \partial^{q+r} \delta(x-y)
\]

\[
+ \sum_{p,s \geq 1} C_{p,s}^{m+2,n+2} \partial^{n-s+2} V^{m-p} \partial^{p+s-1} \delta(x-y)
+ \partial^{m+n+3} \delta(x-y) \left\}
\]

The first term in the r.h.s. of (4.4) gives the nonlinear Gel’fand-Dikey algebra while the second and third terms represent the linear part of the same algebra. Although we are dealing with a classical theory there appear central terms in (4.4) for any \( m,n \). We note, that the algebra (4.4) becomes pure linear only in the case \( m = n = 0 \) i.e. for the Virasoro subalgebra. Indeed, inserting \( m = 0 \) into (4.4) we obtain

\[
\{V^0(x), V^n(y)\}_\text{PB} = 4\pi \sum_{q \geq 1} \left( \begin{array}{c}
q + 2 \\
q \end{array} \right) \left( \begin{array}{c}
m + 2 \\
r \end{array} \right) \partial^{q-s} V^{-1} V^{n-q} \partial^s \delta(x-y)
\]

\[
+ \partial (V^{n-q} \partial^q \delta(x-y))
\]

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which if \( n > 0 \) also contain nonlinear terms. Nonlinear terms disappear only if we set \( n = 0 \):

\[
\{V^0(x), V^0(y)\}_{PB} = \left( 2V^0 \partial + \partial V^0 + 2\partial V^{-1} \partial + 3V^{-1} \partial^2 + \partial^3 \right) \delta(x - y),
\]

(4.6)
i.e. \( V^0 \) is a quasi-primary field. Consequently, after the bosonization the linear current algebra transforms into nonlinear ones with central terms on the classical level. The latter is obvious for the improved quantities.

Here we also give the corresponding algebra for the canonical currents (3.5) which reads:

\[
\{\tilde{V}^m(x), \tilde{V}^n(y)\}_{PB} = \partial \tilde{V}^{m-1} \tilde{V}^{n-1} \delta(x - y) + \tilde{V}^{m-1} \tilde{V}^{n-1} \partial \delta(x - y) - \sum_{q \geq 1, s \geq 0} C_{q,s}^{n+1,q} \partial^{q-s} \tilde{V}^{m-1} \partial^{q} \partial^{s} \delta(x - y)
\]

(4.7)

\[
+ \sum_{p \geq 1, r \geq 0} C_{p,r}^{m+1,p} \tilde{V}^{m-p} \partial^{p-r} \partial^{r} \delta(x - y)
\]

\[
+ \sum_{p,q \geq 1, r,s \geq 0} C_{p,q}^{m+1,p-1} C_{q,s}^{n+1,q} \partial^{q-s} \tilde{V}^{m-p} \partial^{p-r} \partial^{r+s} \delta(x - y),
\]

without a central term contrary to the improved current algebra (4.1).

It is easy to check that the currents (3.16) satisfy the same \( w_\infty \) algebra (3.15) with respect to the Poisson brackets as the corresponding infinitesimal transformations (3.14).

4.2. Non-abelian case

In the non-abelian case we derive the following current algebra:

\[
\{V^m(x), V^n(y)\}_{PB} = tr \left( \partial U^{m+1} U^{n+1} \right) \delta(x - y) + tr \left( U^{m+1} U^{n+1} \right) \partial \delta(x - y)
\]

\[
- \sum_{q \geq 1, s \geq 0} C_{q,s}^{n+1,q} tr \left( \partial^{q-s} U^{m+1} \partial W^{n-q+1} \right) \partial^{s} \delta(x - y)
\]

\[
+ \sum_{p \geq 1, r \geq 0} C_{p,r}^{m+1,p} tr \left( W^{m-p+1} \partial^{p-r} U^{n+1} \right) \partial^{r} \delta(x - y)
\]

\[
+ \sum_{p,q \geq 1, r,s \geq 0} C_{p,q}^{m+1,p-1} C_{q,s}^{n+1,q} tr \left( \partial^{q-s} W^{m-p+1} \partial^{p-r-1} W^{n+1} \right)
\]

\[
\times \partial^{r+s} \delta(x - y),
\]

(4.8a)

\[
\{V^m(x), J_a^n(y)\}_{PB} = tr \left( \partial U^{m+1} J_a^n \right) \delta(x - y) + tr \left( U^{m+1} J_a^n \right) \partial \delta(x - y)
\]
\[
- \sum_{q \geq 1, s \geq 0} C_{q,s}^{n,q} \tr \left( \partial^{q-s} U_{m+1} \partial j_{a}^{n-q} \right) \partial^{s} \delta(x-y)
+ \sum_{p \geq 1, r \geq 0} C_{p,r}^{m+1,p} \tr \left( W_{m-p+1} \partial^{p-r} Y_{a}^{n} \right) \partial^{r} \delta(x-y)
+ \sum_{p,q \geq 1, r,s \geq 0} C_{q,s}^{n+1,p} \tr \left( \partial^{q-s} W_{m-p+1} \partial^{p-r} J_{a}^{n} \right) \times \partial^{r+s} \delta(x-y),
\]
\]

\[
\{ J_{a}^{m}(x), J_{b}^{n}(y) \}_{PB} = \tr \left( \partial Y_{n}^{m} Y_{b}^{n} \right) \delta(x-y) + \tr \left( Y_{b}^{m} Y_{n}^{n} \right) \partial \delta(x-y)
- \sum_{q \geq 1, s \geq 0} C_{q,s}^{n,q} \tr \left( \partial^{q-s} Y_{a}^{m} \partial j_{b}^{n-q} \right) \partial^{s} \delta(x-y)
+ \sum_{p \geq 1, r \geq 0} C_{p,r}^{m+1,p} \tr \left( j_{a}^{m-p} \partial^{p-r} Y_{b}^{n} \right) \partial^{r} \delta(x-y)
+ \sum_{p,q \geq 1, r,s \geq 0} C_{q,s}^{n+1,p} \tr \left( \partial^{q-s} j_{a}^{m-p} \partial^{p-r} J_{b}^{n} \right) \times \partial^{r+s} \delta(x-y),
\]

where the following notations are used

\[
W^{n}(x) = \partial g^{-1} \partial^{n} g(x) = -U^{1} U^{n}(x),
Y_{a}^{n}(x) = g^{-1} \partial^{n} g(x) t_{a},
\]

\[
(4.9)
\]

We remark that

\[
V^{n}(x) = \tr W^{n+1},
J_{a}^{n}(x) = \tr j_{a}^{n}(x).
\]

\[
(4.10)
\]

We note that in the l.h.s. of the current algebra (4.8) nonlinear terms (with respect to the matrixvalued currents (4.9)) appear, which in the general case are impossible to be represented (lineally or nonlinearly) in terms of the currents (4.10) only. Consequently, to have a closed algebra additional nonlinear currents of type (3.33) and (3.35) should be included (in both cases linear and nonlinear algebra).

Taking into account

\[
\partial U^{n}(x) = U^{n+1} - U^{1} U^{n}
\]

\[
(4.11)
\]
we derive the following generalized Laibnitz formula:

\[ \partial^l U^m(x) = \sum_{s \geq 0} \binom{l}{s} \sum_{r_1 \geq 0} \binom{s}{r_1} \times \sum_{r_2 \geq 0} \binom{s - r_1}{r_2} \ldots \sum_{r_{k-1} \geq 0} \binom{s - r_1 - \ldots - r_{k-1}}{r_k} \times U^{r_k} U^{r_{k-1}} \ldots U^{r_2} U^{r_1} U^{m+l-s} \delta_{r_1+r_2+\ldots+r_k-s,0}. \]  

(4.12)

Hence, the r.h.s. of (4.12) can be represented in terms of the currents (3.33) and (3.35). As was pointed out above the currents (3.33) and (3.35) are also Noetherian and are generated by the nonlinear part of the transformations (3.30). The nonlinear currents (3.33) satisfy the following current algebra:

\[ \{ V^{\{m\}_k}(x), V^{\{n\}_l}(x) \}_{PB} = \sum_{CP(m)} \sum_{CP(n)} tr\left( U^{m_1}(x) U^{m_2}(x) \ldots U^{m_{k-1}}(x) \right) \]

\[ \times \sum_{p,q \geq 0} \sum_{r,s \geq 0} C_{p,q}^{m_1} C_{q,s}^{n_1} \partial^q - s U^{m_1 - p} \partial^p - r - 1 U^{n_1 - q} \partial^r + s - 1 \delta(x-y), \]  

(4.13)

where \{m\}_k = (m_1, \ldots, m_k) and \(CP(m)\) denotes the cyclic permutations of \(m_1, \ldots, m_k\).

Taking into account (4.12) the r.h.s. of (4.13) can be rewritten only in terms of the currents (3.33) (without derivative terms). This allow us to conclude that currents (3.33) satisfy a closed linear algebra if \(l\) runs to \(\infty\). In the case of isotopic currents (3.35) one obtains closed linear algebra if currents with increasing isotopic indices up to

\[ J^{m_1, \ldots, m_k} a_1, \ldots, a_k = tr\left( U^{m_1} t_{a_1} \ldots U^{m_2} t_{a_2} \ldots U^{m_k} t_{a_k} \right), \quad (k = 1, 2, \ldots) \]  

(4.14)

are included. Consequently, the currents (3.33), (3.35), (4.14) satisfy a closed linear affine algebra if \(t_a \in SL(N)\) also in the case \(g \in SO(N)\).

Notice, that we obtained an algebra which is a triple extension of the semi-direct product of the Virasoro algebra and the affine Kac-Moody algebra. This extension is carried out with respect to: the conformal spin, the degree of nonlinearity and the isotopic spin. Let us recall that there exists possibility for the algebra (4.8) to be considered as a nonlinear algebra for matrix valued currents. We point out, that there are several possibilities to obtain particular extended algebras from the general construction considered above. A simple example of the contraction procedure is presented below.
Inserting (3.23) into (3.33), (3.33), (4.14) and after redefinition \( j \to q^{-1}j \) in the limiting case \( q \to 0 \) we get:

\[
v^n(x) = \lim_{q \to 0} \frac{q^{n+2}}{n+2} v^{n_1,n_2,...,n_k}_{a_1,a_2,...,a_l} = \frac{1}{n+2} tr j^{n+2}, \tag{4.15a}
\]

\[
v^n_a(x) = \lim_{q \to 0} \frac{q^{n+1}}{n+1} j^{n_1,n_2,...,n_k}_{a_1,a_2,...,a_l} = \frac{1}{n+1} tr (ta_j)^{(n+1)}, \tag{4.15b}
\]

\[
v^n_{a_1,a_2,...,a_l}(x) = \lim_{q \to 0} \frac{q^{n+1}}{n+1} j^{n_1,n_2,...,n_k}_{a_1,a_2,...,a_l}, \tag{4.15c}
\]

where \( n = \sum_{r=0}^{k} n_r \) and \( l \leq k \). We note, that if \( 1 < l < k \) there are various possibilities for re-arranging the matrix generators \( t_a \) and the currents \( U^n \) in the formula (4.15a) .

With respect to the Poisson bracket the currents (4.13) satisfy the following algebra:

\[
\{v^m(x), v^n(y)\}_{PB} = (m+1) \partial v^{m+n}(x) \delta(x-y) + (m+n+2) v^{m+n} \partial \delta(x-y), \tag{4.16a}
\]

\[
\{v^m(x), v^b^n(y)\}_{PB} = \frac{1}{n+1} \sum_{p=0}^{n} tr \left( \partial (j^{m+1}) j^p t_b j^{n-p} \right) \delta(x-y) + (m+n+1) v^m_b \partial \delta(x-y), \tag{4.16b}
\]

\[
\{v^a^m(x), v^b^n(y)\}_{PB} = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} tr \left( \partial (j^p t_a j^{m-p}) j^p t_b j^{n-p} \right) \delta(x-y) + tr \left( j^{n+p-q} t_a j^{m+q-p} t_b \right) \partial \delta(x-y). \tag{4.16c}
\]

We note, that the terms of type \( tr (\partial(j^m) A j^n) \) can be represented as a total derivative only if \( A = I \), otherwise, the term \( tr(j^{m-k} \partial j^k t_a j^n) \equiv tr(j^{m-k} U^j t_a j^n) - tr(j^{m+1} t_a j^n) \) also is included. Consequently, it can be concluded that only the currents (4.15a) satisfy a closed linear algebra (4.16a) coinciding with the area-preserving \( w_\infty \) algebra. The r.h.s. of (4.16b,c) shows that the isotopic vector and generally isotopic tensor currents do not form closed algebra if the currents containing the derivatives of \( j \) are not included, i.e. the currents of type (4.14).

5. Conclusions

The main results obtained in the present article is that when the bosonization procedure is applied some higher spin symmetry properties are changed. For the free fermionic model it is well known that the higher spin currents and their generating transformations form the same Lie algebra – the \( DOP(S^1) \) algebra (coinciding with the linear part of the
Gel’fand-Dickey algebra) which agree with the Noether theorem. This result take place for the both abelian and non-abelian models. In the corresponding bosonic models (the free single scalar field model and the WZNW model), we start from the same type \((DOP(S^1))\) linear fields transformations and its corresponding bilinear higher spin currents. By cyclic application of the Noether theorem nonlinear terms arise in the field transformations laws, hence in the corresponding currents. The scalar field current algebra contains both the linear and non-linear Gel’fand-Dickey algebras. Moreover, in the general non-abelian case current algebra is closed only if the terms with an increasing degree of nonlinearity are included. These nonlinear terms involve currents with increasing isotopic spin too. As a result we obtain triple extended algebra. This current algebra can be considered as well as a nonlinear extended algebra of type \(\hat{W}_\infty\) if we consider matrix valued currents. All this allows us to conclude that some difference appears between the higher spin symmetry properties of the non-abelian free fermionic model and the corresponding WZNW model. We recall that on the ordinary symmetry level both models are completely equivalent.

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**Appendix A.**

Here as an example is considered the group \(SL(2, R)\). For this group the Gauss decomposition reads:

\[
g(x) = \Psi \Phi \Delta,
\]

where

\[
\Psi = \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}.
\]

(A.2)

Here \(\psi, \phi\) and \(\chi\) are real scalar dimensionless fields.

Inserting (A.1) and (A.2) into (3.21) we obtain

\[
U^m = U^m_+ \sigma_- + U^m_- \sigma_+ + U^m_3 \sigma_3 + U^m_0 \sigma_0,
\]

(A.3)
where

\[ U^m_+ = \partial^m \psi + 2\psi A_m - e^{2\phi} \psi^2 \partial^m \chi \]
\[ + \sum_{p \geq 1} \binom{m}{p} \left( e^{2\phi} \psi^p \partial^p (\partial \phi) - e^{2\phi} \psi^{m-p} \partial^p \chi - e^{2\phi} \psi^2 \partial^p \chi P^{m-p}(\partial \phi) \right) \]
\[ - \sum_{q \geq 1} \binom{m-p}{q} e^{2\phi} \psi^{m-p-q} \partial^q \chi P^q(\partial \phi), \quad (A.4a) \]

\[ U^m_- = e^{2\phi} \partial^m \chi + \sum_{p \geq 1} \binom{m}{p} e^{2\phi} \partial^p \psi P^{m-p}(\partial \phi), \quad (A.4b) \]

\[ U^m_3 = -A_m + e^{2\phi} \psi \partial^m \chi \]
\[ + \frac{1}{2} \sum_{p \geq 1} \binom{m}{p} \left( e^{2\phi} \partial^m \psi \partial^p \chi + 2e^{2\phi} \partial^p \chi P^{m-p}(\partial \phi) \right) \]
\[ + \sum_{q \geq 1} \binom{m-p}{q} e^{2\phi} \partial^{m-p-q} \psi \partial^q \chi P^q(\partial \phi), \quad (A.4c) \]

\[ U^m_0 = B_m + \frac{1}{2} \sum_{p \geq 1} \binom{m}{p} \left( e^{2\phi} \partial^m \psi \partial^p \chi \right) \]
\[ + \sum_{q \geq 1} \binom{m-p}{q} e^{2\phi} \partial^{m-p-q} \psi \partial^q \chi P^q(\partial \phi). \quad (A.4d) \]

Here \( A_m \) and \( B_m \) are defined by

\[ \Phi^{-1} \partial^m \Phi = -A_m \sigma_3 + B_m \sigma_0 \quad (A.5) \]

\( \sigma_0 = I \) and \( \sigma_\pm, \sigma_0 \) are matrix generators of the \( SL(2, R) \) algebra:

\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (A.6) \]

From (A.3) it holds that

\[ A_{m+1} = \partial A_m + \partial \phi B_m, \]
\[ B_{m+1} = \partial B_m + \partial \phi A_m, \quad (A.7) \]

where \( A_0 = 0, B_0 = 1 \). From (A.7) it follows that the sum \( A_m + B_m \) satisfies the condition (3.30), i.e. we have

\[ P^m(\partial \phi) = A_m(\phi) + B_m(\phi), \quad (A.8) \]

where \( P^m \) are the Faà di Bruno polynomials. Taking into account that \( U^1 = j(x) \) from (A.3) we obtain

\[ j(x) = g^{-1} \partial g(x) \]
\[ = (\partial \psi + 2\psi \partial \phi - e^{2\phi} \psi^2 \partial \chi) \sigma_- + e^{2\phi} \partial \chi \sigma_+ + (-\partial \phi + e^{2\phi} \psi \partial \chi) \sigma_3. \quad (A.9) \]
Now, it is easy to compute
\[ j^2 = (\partial \phi)^2 + e^{2\phi} \partial \chi \partial \psi)I = 2TI, \]  
where \( I \) is the identity matrix and \( T \) is the stress-energy tensor. Then, from (A.3) and (A.4) we have:
\[ j^{2k} = 2T^k I, \]  
\[ j^{2k+1} = 2T^k j(x) \quad \text{for} \; k = 0, 1, \ldots \]  
Inserting (A.11) into (4.15) we find:
\[ v^{2k} = T^k, \]  
\[ v^{2k+1} = 0. \]  
Consequently, all the odd spin \( w_\infty \) conserved quantities (4.15) vanish in the case of \( SL(2, R) \) WZWN model, while the even spin quantities are a product of stress-energy tensor.

From (A.3) we get
\[ V^m = \frac{1}{2} tr U^{m+2} = U^m_{0} + 2 \]  
which for \( m > -1 \) do not vanish identically. Consequently, the formulas (A.3), (A.4) and (A.13) show us that the \( W_\infty \) currents with any spins do not vanished identically.

Next we show that the currents \( V^{m,n,...} \) can not be represented in terms of the currents \( V^m \), i.e.
\[ V^{m,n,...} \neq V^m V^n \ldots \]  
Indeed inserting (A.3) in the product \( U^m U^n \) we get
\[ U^m U^n = (U^m_+ U^n_+ - U^m_+ U^n_0 + U^m_0 U^n_+ + U^m_0 U^n_0) \sigma_- 
+ (U^m_3 U^n_0 - U^m_3 U^n_+ + U^m_0 U^n_+ + U^m_0 U^n_0) \sigma_+ 
\left( \frac{1}{2} (U^m_+ U^n_+ - U^m_0 U^n_0) + U^m_0 U^n_+ + U^m_0 U^n_0 \right) \sigma_3 
\left( \frac{1}{2} (U^m_+ U^n_+ + U^m_+ U^n_0) + U^m_0 U^n_0 + U^m_0 U^n_3 \right) \sigma_0. \]  
Consequently, in general \( tr(U^m U^n) \neq tr(U^m)tr(U^n) \) if \( m, n \geq 0 \). This allows us to conclude that we have a closed current algebra only if we consider an extended currents space in which higher spin nonlinear currents are included. This is also the case of WZNW model for which the gauge fixing \( U^m_\pm = 0, U^m_3 = -A^m, U^m_0 = B^m \) is impose. Substituting in (A.15) we obtain:
\[ tr(U^m U^n) = 2(A^m A^n + B^m B^n) \neq cB^m B^n \]  
because according to (A.7) we have \( A^m \neq B^m \).
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