We address here the simplest fundamental questions concerning QIP in disordered or complex systems: (i) Can one generate entanglement in such systems that would survive quenched averaging over long times? (ii) Can one realize quantum gates with reasonable fidelity? Here we answer both questions affirmatively considering both short and long range disordered systems.

First, we consider a short range disorder Ising Hamiltonian, the so-called Edwards-Anderson (E-A) model of spin glasses which can be straightforwardly implemented using atomic Bose-Fermi, or Bose-Bose mixtures in optical lattices [8, 15]. We address the generation and evolution of nearest neighbor (nn) entanglement in this model. In the short range Ising model without disorder, it is possible to create cluster and graph states (i.e., entanglement) starting from an appropriate initial product state [16]. Here we show that, while the disorder averaged density matrix of two neighboring spins remains always separable, the disorder averaged entanglement (quantified by logarithmic negativity [17]) converges with time to a finite value. The generation of entanglement [16] as well as its evolution for arbitrary times in an Ising model without disorder but with long-range interactions, has also been addressed in Ref. [15]. There it was suggested the possibility of applying similar ideas to disordered systems. We show also that the quantum single-qubit Hadamard gate, can be realized in such system with significant (disorder averaged) fidelity.

Secondly, we consider complex systems with long range (1/r, 1/r^2) interactions, that can be realized for instance, in linear ion traps, using either local magnetic fields, as proposed by Wunderlich and coworkers [18], or by appropriately designed laser excitations [9]. The corresponding Hamiltonian can be mapped into an Ising Neural Network (NN) model with weighted patterns [13]. Those patterns can be used as qubit systems, with the information distributed over the chain. One can also include external parallel, or transverse fields in the model. We show that in such system, it is possible to generate long range bipartite entanglement that undergoes a series of collapses and revivals [20], whose times are found analytically. Finally we study also bipartite and tripartite entanglement dynamics in an infinite range Ising model...
without disorder.

Let us start with the Edwards-Anderson spin glass model described by

$$H_{E-A} = -\frac{1}{4} \sum_{i,j} J_{ij} \sigma_i^x \sigma_j^x.$$  \hfill (1)

Here $\sigma_k^x$ denotes the Pauli operator at the $k$th site, and $J_{ij}$'s describe nn couplings for an arbitrary lattice. In the E-A model these couplings are given by independent Gaussian variables with mean $J$ and variance $\sigma^2$. Starting from a pure product state of the form $|\Psi\rangle = \prod_i |\pm\rangle_i$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, we evaluate the entanglement after a finite time, where the density matrix is given by $\rho(t,\{J_{ij}\}) = \exp\{-iH_{E-A}t\} |\Psi\rangle \langle \Psi | \exp\{+iH_{E-A}t\}$.

The reduced density matrix for a nn pair is obtained by tracing over all other sites. For instance, the reduced density matrix for a 2D square lattice is given by

$$\varrho_{12}(t,\{J_{ij}\}) = \frac{1}{4} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \left[ e^{iJ_{12}t/2} \right.$$

$$\left. \cos(J_{24}t/2) \cos(J_{26}t/2) \cos(J_{28}t/2)|00\rangle \langle 01| + \cos(J_{13}t/2) \cos(J_{15}t/2) \cos(J_{17}t/2)|00\rangle \langle 10| \right] + e^{-iJ_{12}t/2} \left. \cos(J_{13}t/2) \cos(J_{15}t/2) \cos(J_{17}t/2)|01\rangle \langle 01| + \cos(J_{24}t/2) \cos(J_{26}t/2) \cos(J_{28}t/2)|10\rangle \langle 11| \right] + e^{-iJ_{13}t/2} \left. \cos(J_{15}t/2) \cos(J_{17}t/2) \cos(J_{28}t/2)|10\rangle \langle 00| \right\} \mathbb{1} + \text{h.c.},$$

where $\mathbb{1}$ is the identity operator and the indices $3 \ldots 8$ enumerate the six neighbors of 1 and 2. A similar expression can be obtained for the 1D lattice. In both cases, the averaging of the reduced state over $J_{ij}$'s (equivalent to reducing the average $\varrho_{12}(t,\{J_{ij}\})$) is separable. Note, however, that as always in physics of disordered systems, if we are interested in typical values of physical quantities such as free energy, entanglement, etc., we are obliged to perform a "quenched" average, i.e. first calculate the quantity of interest and then average \[ \varrho_{12}(t,\{J_{ij}\}) \] (see also [21,22]).

To study entanglement, we use the logarithmic negativity (LN) \[ \ln. \] The LN of a bipartite state $\rho_{AB}$ is defined as $E_{LN}(\rho_{AB}) = \log_2 \|\rho^A_{AB}\|_1$, where $\|\cdot\|_1$ is the trace norm, and $\rho^A_{AB}$ denotes the partial transpose of $\rho_{AB}$ with respect to the A-part \[ \text{[23]}. \] Note that $\rho_{ij}(t)$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Consequently, a positive value of the LN implies that the state is entangled and distillable \[ \text{[23,24]}. \] while $E_{LN} = 0$ implies separability \[ \text{[24]}. \]

The entanglement in the spin glass model turns out to be an even function of the couplings. The temporal behavior of $E_{LN}(t)$ in a 2D square lattice is shown in Fig. 1 for two different cases of disorder: with frustration and without it. For $J = 0$, $\sigma^2 = 1$, the system has randomly ferro- ($J > 0$) and antiferro-magnetic ($J < 0$) interactions and is strongly frustrated; $E_{LN}(t)$ is rapidly damped to a constant, and does not show any oscillations. This behaviour differs from the non-frustrated case $J = 5$, $\sigma^2 = 1$, when $E_{LN}(t)$ exhibits oscillations with frequencies $\sim 1/\tau$. For short range interactions, the next-nearest neighbor entanglement vanishes, even before the averaging, for both 1D and 2D. To understand why entanglement converges in time to the same finite value in both the frustrated and non-frustrated cases, notice that as long as the distributions $J_{ij}$'s are sufficiently well-behaved, $J_{ij}/2$ corresponds to a uniform distribution over $[0,2\pi]$ for large enough $t$.

![FIG. 1: Temporal behavior of nn averaged entanglement in a 2D spin glass model, starting from $\Pi_i |+\rangle_i$. For a model with frustration ($J = 0$), $E_{LN}(t)$ converges quickly to a constant value (red curve). For a non-frustrated case ($J = +5$), $E_{LN}(t)$ exhibits damped oscillations (blue curve), converging to the same value $\approx 0.0154$, as reached in the frustrated case. Standard deviation for $t \rightarrow \infty$ is $\approx 0.0704$. It is interesting to note that the dynamical behaviour of $E_{LN}$ depends on $J$, although at large times, they all converge to the same value. The same behavior is encountered in the 1D case, even though there is no frustration in that case.](image-url)
where $S_d = 2\pi^d / \Gamma(d/2)$. Due to the periodicity involved implicitly in $\rho_{23}(t)$, there are $2^d - 1$ such hyperspheres. Considering all states in this volume to have unit entanglement, the average entanglement at long times is $\mathcal{E}_d = V_d (2^d - 1) / (2\pi)^d$. As an example, for the case of the 2D lattice (for which $d = 6$), at long times, the actual entanglement is $\approx 0.0154$, while $\mathcal{E}_d \approx 0.0221$. Although the bipartite entanglement vanishes with increasing number of neighbors, one can expect the multipartite entanglement to be non vanishing due to the fact that the volume of separable states is “super-doubly-exponentially small” with increasing number of parties. 

We show now that spin glasses allows also to implement quantum gates. We focus on the Hadamard gate, which transforms the computational basis into a complementary basis: \(|0\rangle \rightarrow |+\rangle\) and \(|1\rangle \rightarrow |−\rangle\). To implement the Hadamard gate, assume that the computation is performed in a spin lattice, and the particles 1 and 2 are a part of it. We assume that at a certain time, particle 1 is in an arbitrary state $\rho_{12}(t)$ where $|a|^2 + |b|^2 = 1$, and we let system evolve according to the Hamiltonian $H_{E-A}$ for a suitable duration of time, before performing measurement on particle 1 (in a suitable basis). For $J = 5$, $\sigma^2 = 1$, particle 2 attains the Hadamard rotated state $|+\rangle + |−\rangle$, with quenched averaged fidelity greater than 0.85. One can increase such fidelity by increasing the number of spins, and employing assisted measurements. Note, that if we try to prepare the Hadamard rotated state using the classical information obtained only from the measurement of particle 1, the fidelity is only $2/3$. 

Let us now move to a long-range interactions spin Ising model, described by the Hamiltonian $H_{irr} = \frac{1}{N} \sum_{i,j} J_{ij} \sigma_i^x \sigma_j^x$, where $N$ is the total number of spins. Such models can be realized with trapped ions [11], where $J_{ij} = \sum_i \sigma_i^x \sigma_j^x / \lambda^i$ (where $\lambda$ describes the phonon eigen-modes (eigen-frequencies). Here we consider two extreme cases. First, we take $\lambda_1 = 1$, $\xi_1 = \text{constant} \forall i$, $\lambda_{\mu} \rightarrow \infty$ for $\mu \geq 2$, so that the interactions are ordered, and the Hamiltonian is $H_{irr} = \frac{1}{N} S^2$, where $S = \sum_{i=1}^{N} \sigma_i^z$. Secondly, we consider the case when $\lambda_{\mu} = 1$ for all $\mu$, when the Hamiltonian becomes $H_{NN} = \frac{1}{N} \sum_{i,j=1}^{N} \xi_{\mu}^{(i)} \xi_{\mu}^{(j)} \sigma_i^x \sigma_j^x$. This is the Hopfield model of a neural network with Hebbian couplings [12]. Here $p$ is the number of “patterns” of the neural network, and the patterns are described by random variables $\xi_{\mu}^{(i)} = \pm 1$, each with probability $\frac{1}{2}$. As in the case of short-range interactions, we take the initial state of the evolution as $\rho = \Pi_{i=1}^{N} |+\rangle_i$, and study the dynamics of entanglement for ordered and disordered Hamiltonians. We provide an efficient method to analytically compute the evolved state of any number of patterns and any number of spins.

Consider first the case of the Hamiltonian $H_{irr}$. We can write the evolution operator $\exp(-iS^2t/N)$ as $\int d\omega (\exp(\omega \sum_{i=1}^{N} |0\rangle_i \langle 0|) + \exp(\omega \sum_{i=1}^{N} |1\rangle_i \langle 1|) + h.c.)$.

As depicted in Fig. 2 there are large ranges of time, for which the bipartite state is separable. Interestingly, this range of separability can be reduced, considering entanglement of the tripartite entangled state $\rho_{123}^{lro}(t)$ in a bipartite cut. Although the interactions in $H_{lro}$ are long-range, they are ordered, so that $\rho_{12}^{lro}(t)$ and $\rho_{13}^{lro}(t)$ takes a relatively simple form. Amazingly, the same method applies for $H_{NN}$, where the interactions are both long-range and disordered. Despite its increased complexity, we can still use the technique for the evolution operator $\exp(-iH_{NN}t)$, that was used in the case of $H_{irr}$. Specifically, we replace in $\exp(-iH_{NN}t)$, the operator $\exp(-iS^2t/N)$ by $\int d\omega \exp(\omega \sum_{i=1}^{N} |0\rangle_i \langle 0|) + S_{\mu} \sqrt{1 - \frac{2}{N}} \omega_{\mu}$, for every $\mu$, where $S_{\mu} = \sum_{i=1}^{N} |\xi_{\mu}^{(i)} \rangle \langle \xi_{\mu}^{(i)}|$. Applying this operator to our initial state, we find that the $N$-particle state at time $t$ is 

$$
\rho^{NN}(t) = \int (\Pi_{\mu} dr_{\mu} \rho_{\mu}^{s_{\mu}^{(i)}} e^{\sum_{\mu} r_{\mu} s_{\mu}^{(i)} / N}) \\
\Pi_{\mu=1}^{N} \left[ e^{-2i\sqrt{\sum_{\mu} \xi_{\mu}^{(i)} r_{\mu}^{(i)}} / N} |0\rangle_i \langle 0|_i + e^{2i\sqrt{\sum_{\mu} 2 \xi_{\mu}^{(i)} r_{\mu}^{(i)}} / N} |1\rangle_i \langle 1|_i \\
+ \left\{ e^{-2i\sqrt{\sum_{\mu} \xi_{\mu}^{(i)} r_{\mu}^{(i)}} / N} |0\rangle_i \langle 1|_i + h.c. \right\} \right], \quad (3)
$$

where $r_{\mu} = \omega_{\mu} + \omega_{\mu}^{\prime}$, $s_{\mu} = \omega_{\mu} - \omega_{\mu}^{\prime}$, with $\mu = 1, \ldots, p$. 

![FIG. 2: Generation of entanglement of bipartite states $\rho_{12}^{lro}(t)$ with respect to time and number of spins. Collapses and revivals of the entanglement are clearly depicted.](image-url)
After tracing out all except particles 1 and 2 we obtain:

\[ \rho_{12}^{N}(t) = 1/4 \left\{ \langle 00\rangle \langle 00 \rangle + \langle 01\rangle \langle 01 \rangle + \langle 10\rangle \langle 10 \rangle + \langle 11\rangle \langle 11 \rangle \right\} \]

\[ + \left[ e^{-4t \sum \xi_{\mu}^{(1)} \xi_{\mu}^{(2)}/N} \left( \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(2)}/N) \langle 00\rangle \langle 01 \rangle \right) \right. \]

\[ + \left. \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} (\xi_{\mu}^{(1)} + \xi_{\mu}^{(2)})/N) \langle 00\rangle \langle 11 \rangle \right\} \]

\[ + \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} (\xi_{\mu}^{(1)} - \xi_{\mu}^{(2)})/N) \langle 01\rangle \langle 10 \rangle \]

\[ + \left[ e^{4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(2)}/N} \left( \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(1)}/N) \langle 10\rangle \langle 11 \rangle \right) \right. \]

\[ + \left. \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(2)}/N) \langle 10\rangle \langle 11 \rangle \right\} + \text{h.c.} \right\}. \]

(4)

For \( N \) large, and \( t/N \) small, the above expression can be simplified using the fact that \( \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(2)}/N) \) self-averages to the value \( \exp\left[-(8t^2/N^2) \sum_{i \neq 1,2} \sum_{i=1,2} (N \sum_{\mu} x_{\mu}^i)^2 \right] \), where for all \( i, x_{\mu}^i = +1 \) or \(-1\) with probability \(1/2\) each. Therefore, for large \( N \) and small \( t/N \), we have that \( \Pi_{i \neq 1,2} \cos(4t \sum \xi_{\mu}^{(i)} \xi_{\mu}^{(2)}/N) \) self-averages to the value \( \exp\left[-(8t^2/N)p\right] \), so that after time \( t \sim \sqrt{N/p} \), all the off-diagonal elements of the state \( \rho_{12}^{NN}(t) \) become vanishingly small. Therefore, for the first time, nearest neighbor entanglement in the evolved state appears and persists for times of order \( \tau_{C} \sim \sqrt{N/p} \). However, there are repeated revivals in entanglement, with the period being \( \tau_{R} \sim \pi N/2 \) for odd \( p \), and \( \tau_{R} \sim \pi N \) for even \( p \). Note, that the period of revivals is independent of the number of patterns in the model (cf. [24]).

Summarizing, we have studied disordered and complex spin systems with short-range and long range interactions that can be realized with trapped atoms or ions. We have shown that in both cases it is possible to generate quenched averaged entanglement over long times. In the case of short range interactions, we considered Edwards-Anderson model in 1D and 2D square lattice. We have shown that in such disordered system, it is possible to implement also distinctly quantum single-qubit gates with high fidelity. We have also demonstrated that it is possible to generate entanglement in the spin system with long range interactions, corresponding to the Hopfield neural network model. We have shown that in such case, entanglement exhibits a sequence of collapses and revivals.

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