Axion instability and non-linear electromagnetic effect

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We investigate the instability due to dynamical axion field near the topological phase transition of insulators. We first point out that the amplitude of dynamical axion field is bounded for magnetic insulators in general, which suppresses the axion instability. Near the topological phase transition, however, the axion field may have a large fluctuation, which decreases the critical electric field for the instability and increases the axion induced magnetic flux density. Using two different model Hamiltonians, we report the electromagnetic response of the axion field in details.

I. INTRODUCTION

Instability is a key to explore new phenomena in physics. After instability, quantum states are rearranged and a new state of matters appears. A well-known example is the Cooper instability, in which paired electrons condensate and the system hosts superconductivity[1].

In the context of topological phases of matters, roles of the topological $\theta$-term in condensed matter physics have been discussed recently[2–31]. Such a topological term provides nontrivial phenomena which have not been observed in ordinary materials. Modifying the Maxwell equation, the $\theta$-term in topological insulators gives the quantized Kerr effect[16–21] and the quantized topological magnetoelastic effects[2–12]. Furthermore, magnetic monopoles can be realized as mirror images of electrons due to the Witten effect[3].

A new instability arises due to the $\theta$-term when it fluctuates. When $\theta$ is dynamical, which we call axion, it couples to electromagnetic fields, changing the behaviors of electromagnetic propagating modes drastically[22]. Using an analogy between particle physics and condensed-matter one, the axion fluctuation is shown to induce an instability when an applied electric field exceeds a critical value.

A subtlety of the above analysis is that it uses $\theta$ itself as a dynamical variable. Being different from particle physics, no direct axion field exists in condensed-matter systems. As was shown in Ref.[22], an antiferromagnetic fluctuation can induce fluctuations of $\theta$, but the correspondence between them is not exact. In actual condensed-matter physics, the antiferromagnetic field, not the axion field, is primary. Therefore, to understand the instability microscopically, the analysis should be re-examined in terms of the primary antiferromagnetic field.

In this paper, we examine the axion instability in a microscopic point of view. We introduce an antiferromagnetic field, instead of an axion field, and analyze the instability caused by the antiferromagnetic field. In contrast to the naive expectation, it is found that fluctuations of the antiferromagnetic field are insufficient to induce a detectable axion instability. We reveal that $\theta$ is bounded above as a function of the antiferromagnetic field, and thus its effect on electromagnetic fields is strongly suppressed. In order to enhance the axion field, fluctuations other than the antiferromagnetic field is necessary. Analysis of quantum anomaly implies that the necessary fluctuation is related to topological quantum phase transition. Only when fluctuations of the antiferromagnetic order and the topological quantum phase transition coexists, the induced axion field causes significant effects.

This paper is organized as follows. In Sec. II we start from reviewing the axion instability discussed by Ooguri and Oshikawa[23], where the axion field is described in terms of the antiferromagnetic order. We then qualitatively discuss how the non-magnetic fluctuations, which are related to the topological order, affect the axion instability and resulting electromagnetic fields. The detailed calculations including the non-magnetic fluctuations are given in Secs. III–VI. Section III derives the low energy effective theory of the axion electrodynamics from a microscopic model of a topological insulator. Section IV gives general forms of static solutions for the induced fields in a certain setup under an applied electric field. The critical electric field for the instability is also given in this section. Section V analyzes two model cases with and without the non-magnetic fluctuations. Section VI discusses the case when a magnetic field, instead of an electric field, is applied. Section VII summarizes the paper.

II. AXION INSTABILITY DUE TO MAGNETIC AND NON-MAGNETIC FLUCTUATIONS

A. Review of Ooguri-Oshikawa’s theory

We first briefly review the axion instability discussed by Ooguri and Oshikawa[23]. They start from the axionic electrodynamics in an insulator. The effective La-
The Lagrangian densities of the electromagnetic fields $E$ and $B$, the axion field $\theta = \theta_0 + \delta\theta$, and the interaction between them, respectively, with $\epsilon$ and $\mu$ being the dielectric constant and the magnetic permeability, respectively, and $\alpha = e^2/\hbar c$ the fine-structure constant. Here, $\mathcal{L}_\alpha$ originates from the Lagrangian density of the spin-wave mode in the insulator: In a linear approximation, the fluctuation of the axion field $\delta\theta$ is related to that of the antiferromagnetic order $\delta\phi_5$ as

$$\delta\theta = \delta\phi_5 / g,$$

with $g$ being a constant, and $J$, $\nu_4$ and $m$ are the stiffness, velocity and mass of the spin-wave mode, respectively. By solving the equation of motion for $\delta\theta$ under an external uniform electric field $E_0$, they found that the system is unstable when the electric field exceeds the critical value given by

$$E_0^{\text{crit}} = \frac{m\epsilon_0}{\alpha\epsilon_0} \sqrt{\frac{(2\pi)^3 g^2 J}{\mu}}, \quad (5)$$

with $\epsilon_0$ being the dielectric constant of the external material. They further showed by solving the modified Maxwell equation derived from the effective Lagrangian $\mathcal{L}$ that the instability leads to screening of the excess electric field above $E_0^{\text{crit}}$ and induction of a magnetic flux density inside the insulator. (See Appendix A for the details.) In such a situation, however, the axion field becomes much larger than unity and Eq. (4) no longer holds. We therefore need to re-examine the relation between the axion field and the antiferromagnetic order beyond the linear approximation.

**B. Effect of the non-magnetic fluctuations**

Although the fluctuations of the antiferromagnetic order are necessary for the axion instability as discussed in Refs. [22, 23], we note that the antiferromagnetic order is not the only order relevant to the axion field. In general, an axion field $\theta$ can be regarded as a phase of a complex field $\phi = e^{i\theta} \rho$ ($\rho = |\phi|$). Denoting the real (imaginary) part of $\phi$ as $\phi_4$ ($\phi_5$), i.e., $\phi = \phi_4 + i\phi_5$, we find that $\phi_4$ and $\phi_5$ transform as

$$\phi_4 \to \phi_4, \quad \phi_5 \to -\phi_5, \quad (6)$$

under time-reversal and inversion operations, because $\theta$ transforms as $\theta \to -\theta$ under each of these transformations. [Note that the symmetry of $\theta$ is determined so that the $\theta$-term ($\mathcal{L}_\theta \propto \theta\mathbf{E} \cdot \mathbf{B}$) is invariant under these transformations.] Since $\phi_4$ and $\phi_5$ have different symmetry properties, they generally correspond to different orders in an insulator. Indeed, while $\phi_5$ represents an antiferromagnetic order that breaks both the time-reversal and inversion symmetries, as we expected, $\phi_4$ corresponds to a non-magnetic order that preserves both these symmetries.

From the above observation, we can expect that the non-magnetic order $\phi_4$ affects the critical electric field for the axion instability. Figure 1 illustrates the relation between the axion field, $\theta$, and the non-magnetic and magnetic orders, $\phi_4$ and $\phi_5$. For a system with a fixed $\phi_4 > 0$ and $\phi_5 = 0$, a fluctuation in the magnetic order $\delta\phi_5$ induces a fluctuation of the axion field $\delta\theta = \delta\phi_5 / \phi_4$, which is inversely proportional to $\phi_4$. Thus, $\theta$ is very sensitive to a small change of $\phi_5$ when $\phi_4$ is close to zero, which makes the axion instability easily occur. This fact is already included in Eq. (5): $E_0^{\text{crit}}$ becomes smaller for smaller $g$ which corresponds to $\phi_4$ [see Eq. (4)]. In the next section, we shall see that $\phi_4$ is related to the order that characterizes a topological phase transition. A topological phase transition takes place at $\phi_4 = 0$ when $\phi_5 = 0$.

We further stress that as long as the system remains to be gapful and $\theta$ is well-defined, a large magnetic order of $\phi_5$ is not sufficient to obtain a large axion field $\theta$. In order to have a large $\theta$ that exceeds $2\pi$, we also need to flip the sign of $\phi_4$ according to $\phi_4 = \rho \cos \theta$. This means that such a large $\theta$ is most likely to arise near the topological phase transition point at $\phi_4 = \phi_5 = 0$. In the following discussions, we therefore assume that the system is near the topological phase transition point and that $\phi_4$, as well as $\phi_5$, is a dynamical quantity which varies depending on applied electromagnetic fields.

**III. LOW ENERGY EFFECTIVE THEORY OF THE AXION ELECTRODYNAMICS**

As discussed in Sec. II, the non-magnetic order $\phi_4$ is also relevant to the axion instability. In this section, taking into account $\phi_4$ as well as $\phi_5$, we derive a low energy effective theory of the axion electrodynamics in insulating materials.
FIG. 1. Relation between the axion field, $\theta$, and the non-magnetic and magnetic orders, $\phi_4$ and $\phi_5$. In the model described by Eq. (7), $\phi_4$ corresponds to the order that characterizes the topological quantum phase transition, and $\phi_5$ corresponds to the antiferromagnetic order. In particular, a system preserving the time-reversal symmetry is located on the $\phi_4$ axis (i.e., $\phi_5 = 0$), which is classified as a normal insulator (NI) for $\phi_4 > 0$ or a topological insulator (TI) for $\phi_4 < 0$.

A. Three-dimensional topological insulator as a platform of the axion field

We start with the low-energy Hamiltonian of a non-interacting three-dimensional topological insulator:

$$H_{TI} = \sum_k \psi_k^\dagger H_{TI}(k) \psi_k,$$

where $\psi_k^\dagger$ ($\psi_k$) is a creation (annihilation) operator of an electron with quasi-momentum $k$ which has four (spin and orbital) internal degrees of freedom, and

$$H_{TI}(k) = \sum_{i=1,2,3} \hbar v_i k_i \gamma_i + \phi_4 \gamma_4.$$

Here, $\gamma_{i=1,2,3,4}$ are the $4 \times 4$ Hermitian gamma matrices that satisfy the anti-commutation relation:

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}. \quad (9)$$

$v_i=1,2,3$ is the electron velocity, and $\phi_4$ is the band-gap energy. Equation (8) is obtained as the $k \cdot p$ Hamiltonian of inversion symmetric topological insulators.\[23\]. Indeed, by adding a higher-order regularization term $c k^2 \gamma_4$ ($c > 0$) to $H_{TI}(k)$, one can confirm that the three dimensional $Z_2$ topological number is given by the sign of $\phi_4$\[35\]. When $\phi_4$ is negative (positive), the system is topologically non-trivial (trivial).

We note that $H_{TI}(k)$ has the time-reversal symmetry:

$$T H_{TI}(k) T^{-1} = H_{TI}(-k), \quad (10)$$

where $T$ is an antunitary time-reversal operator which anti-commutes with $\gamma_{i=1,2,3}$ and commutes with $\gamma_4$. For simplicity, we assume that $H_{TI}(k)$ also preserves the inversion symmetry:

$$P H_{TI}(k) P^{-1} = H_{TI}(-k). \quad (11)$$

Namely, the inversion operator $P$ anti-commutes with $\gamma_{i=1,2,3}$ and commutes with $\gamma_4$.

From the symmetry properties of $H_{TI}(k)$ and $\gamma_4$, $\phi_4$ is invariant under both time-reversal and inversion. Hence, $\phi_4$ in Eq. (8) can be the non-magnetic order $\phi_4$ introduced in Sec. II B. On the other hand, there is no magnetic order ($\phi_5 = 0$), since the Hamiltonian preserves both the time-reversal and inversion symmetries. The system described by the Hamiltonian (8) is located on the $\phi_4$ axis in Fig. 1 and the axion field can take either $0$ ($\phi_4 > 0$) or $\pi$ ($\phi_4 < 0$) mod $2\pi$, which corresponds to a normal insulator and a topological insulator, respectively. The origin of the $\phi_4$-$\phi_5$-plane ($\rho \equiv \sqrt{\phi_4^2 + \phi_5^2} = 0$) is a quantum critical point of the topological phase transition between a normal insulator and a topological one.

In order to discuss axion instability, we need to take into account a term that breaks both the time-reversal and inversion symmetries. The simplest form of such a term is $\phi_5 \gamma_5$ with $\gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4$: Under an operation $S = T$ or $P$, $\gamma_5$ is transformed to $S \gamma_5 S^{-1} = -\gamma_5$, and hence $\phi_5$ is odd under both time-reversal and inversion. This $\phi_5$ is the antiferromagnetic order $\phi_5$ introduced in Sec. II B. In the following discussions, we consider the minimal model that describes the axion instability, i.e., the Hamiltonian (7) with replacing $H_{TI}(k)$ with

$$H(k) = \sum_{i=1,2,3} \hbar v_i k_i \gamma_i + \phi_4 \gamma_4 + \phi_5 \gamma_5. \quad (12)$$

Although $\phi_4$ and $\phi_5$ are material parameters, they should be determined self-consistently in the presence of external electromagnetic fields so as to minimize the total energy including the electromagnetic ones (see Sec. IV).

B. Effective Lagrangian for the axion field

We use Fujikawa’s method\[36, 37\] to relate $\phi_4$ and $\phi_5$ to the axion field $\theta$ and derive the effective Lagrangian. In the Lagrangian formalism, the Hamiltonian (7) with Eq. (12) is rewritten in terms of the Lagrangian density as

$$\mathcal{L}_{\text{el}} = \bar{\psi} \left( i \hbar \sum_{\mu=0,1,2,3} v_\mu \Gamma_\mu D_\mu - \phi_4 \right) \psi - i \phi_5 \bar{\psi} \Gamma_5 \psi, \quad (13)$$

where $\psi(x) = \sum_k \psi_k e^{ik \cdot x}$, $\Gamma_0 = \gamma_4$, $\Gamma_{i=1,2,3} = \gamma_4 \gamma_i$, $\Gamma_5 = -i \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3$, $v_0 = 1$ and $\psi = \psi^\dagger \Gamma_0$. Here we have replaced the partial derivative $\partial_\mu$ with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, where $A_\mu$ is the gauge field. In the absence of electromagnetic fields, an insulator has its pristine values of $\phi_4$ and $\phi_5$, which are denoted by $\phi_4^{(0)}$ and $\phi_5^{(0)}$, respectively. This means, there is an effective potential energy $V_\phi(\phi_4, \phi_5)$ which has a minimum at
\( \phi(t, \psi, \theta) \) in the \( \psi_4, \phi_5 \)-plane. By taking into account the contribution of this potential energy \( V_a(\phi_4, \phi_5) \), as well as the Lagrangian density \( \mathcal{L}_{\text{em}} \) of the electromagnetic fields, the total Lagrangian density \( \mathcal{L} \) is given by
\[
\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_\theta - V_a(\phi_4, \phi_5),
\]
from which, the partition function is given in the path integral formalism as
\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ i \int d^4x \mathcal{L} \right].
\]
We then perform the gauge transformation:
\[
\psi' = e^{-i\Gamma_\theta/2} \psi, \quad \bar{\psi}' = \bar{\psi} e^{-i\Gamma_\theta/2},
\]
with \( \theta \) defined by
\[
\phi_4 = \rho \cos \theta, \quad \phi_5 = \rho \sin \theta,
\]
so as to eliminate the time-reversal breaking term, \( i\bar{\psi} \Gamma_5 \psi \), in \( \mathcal{L}_\theta \). This procedure, however, produces the \( \theta \)-term \( \mathcal{L}_\theta \) as the Jacobian of the path-integral measure, resulting in
\[
Z = \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' \exp \left[ i \int d^4x \left( \mathcal{L}' + \mathcal{L}_\theta \right) \right],
\]
where
\[
\mathcal{L}' = \mathcal{L}_{\text{em}} + \bar{\psi}' \left( i\hbar \nu_\mu \Gamma_\mu D'_\mu - \rho \right) \psi' - V_a(\phi_4, \phi_5),
\]
\[
D'_\mu = \partial_\mu + i e A_\mu + \frac{i}{2} \Gamma_5 \partial_\mu \theta,
\]
and \( \theta_0 \) in \( \mathcal{L}_\theta \) is given by \( \theta_0 = \text{arg}(\phi_4(0) + i \phi_5(0)) \). By integrating with respect to \( \psi' \) and \( \bar{\psi}' \), the following effective Lagrangian density \( \mathcal{L}_b \) for the bosonic fields \( \phi_4 \) and \( \phi_5 \) and the electromagnetic fields \( E \) and \( B \) is obtained:
\[
\mathcal{L}_b = \mathcal{L}_{\text{em}} + \mathcal{L}_\theta - V_a(\phi_4, \phi_5)
\]
\[
-\hbar \text{in det} \left[ i \hbar \sum_{\mu=0,1,2,3} \nu_\mu \Gamma_\mu D'_\mu - \rho \right].
\]
Here, the last term of the right hand side is the contribution from the integration of \( \psi' \) and \( \bar{\psi}' \), which can be evaluated by using the derivative expansion in the low energy limit. Since this term does not depend on \( \theta \), it does not provide any correction of the \( \theta \)-term, although it induces the kinetic term of \( \theta \) and \( \rho \), and the coupling between \( A_\mu \) and the current \( J_\mu \) of electron, and renormalizes \( e, \mu \) and \( V_a(\phi_4, \phi_5) \). Thus, using the same notation of \( e, \mu \) and \( V_a(\phi_4, \phi_5) \) to represent the renormalized ones, we eventually have the following low energy Lagrangian:
\[
\mathcal{L}_b = \mathcal{L}_{\text{em}} + \mathcal{L}_\theta - V_a(\phi_4, \phi_5),
\]
where we have omitted the kinetic term of \( \theta \) and \( \rho \) since we only discuss the static property in this paper. Note that Eq. (22) differs from the Lagrangian density introduced in Sec. II A in the potential energy of the axion field: \( V_a(\phi_4, \phi_5) \) in Eq. (22) is a function of \( \phi_4 \) as well as \( \phi_5 \) and can include higher-order terms of \( \delta \theta \).

C. Modified Maxwell equations and Hamiltonian density including the dynamical axion field

By taking the functional derivative of Eq. (22) with respect to \( A_{\mu=0,1,2,3} \) and \( \delta \theta \), we obtain the modified Maxwell equations:
\[
\nabla \cdot B = 0,
\]
\[
\frac{1}{c} \partial_t B + \nabla \times E = 0,
\]
\[
\nabla \cdot D = 0,
\]
\[
-\frac{1}{c} \partial_t D + \nabla \times H = 0,
\]
\[
-\frac{\alpha}{4\pi^2} E \cdot B + \frac{\partial V_a}{\partial \theta} = 0,
\]
with the constitutive relations
\[
D = \varepsilon E + 4\pi P_\theta,
\]
\[
H = \frac{1}{\mu} B - 4\pi M_\theta,
\]
\[
P_\theta = \frac{\alpha}{4\pi^2} \theta B,
\]
\[
M_\theta = \frac{\alpha}{4\pi^2} \theta E.
\]
The Hamiltonian density corresponding to Eq. (22) is given by
\[
\mathcal{H}_b = \mathcal{H}_{\text{em}} + V_a(\phi_4, \phi_5)
\]
where \( \mathcal{H}_{\text{em}} \) is the Hamiltonian density of the electromagnetic fields given by
\[
\mathcal{H}_{\text{em}} = \frac{1}{8\pi} (E \cdot D + B \cdot H) = \frac{1}{8\pi} \left( \varepsilon E^2 + \frac{1}{\mu} B^2 \right).
\]
It should be noted that the term that corresponds to \( \mathcal{L}_\theta \) vanishes in the Hamiltonian formalism because it is topological and does not contribute to the energy.

IV. RESPONSE TO AN APPLIED ELECTRIC FIELD

A. Setup

Following Ref. [23], we consider an interface between two insulators described by Eq. (22), and apply an electric field \( E_0 \) perpendicular to the interface (Fig. 2). We assume that both the insulators have \( \theta_0 = 0 \). We further assume that the potential energy \( V_a(\phi_4, \phi_5) \) for the bottom (top) insulator is steep (shallow) so that the axion field in the bottom insulator is fixed to \( \theta = 0 \) even in the presence of applied fields whereas a nonzero axion field \( \theta = \delta \theta \) can be induced in the top insulator in response to applied fields. In this paper, we refer to the bottom (top) insulator as a normal (an axion) insulator. The normal insulator can be a vacuum.
Let \( \epsilon_0 \) and \( \mu_0 \) (\( \epsilon \) and \( \mu \)) be the dielectric constant and the magnetic permeability of the normal (axion) insulator, respectively. The boundary condition at the interface is obtained from Eqs. (25) and (23) as

\[
\epsilon E + \frac{\alpha}{\pi} \theta B = \epsilon_0 E_0, \quad (34)
\]
\[
B = B_0 \quad (35)
\]
where \( E \) and \( B(B_0) \) are the electric field and the magnetic flux density normal to the interface in the axion(normal) insulator, respectively.

Hence, at the minimum of \( \mathcal{H}_{em} \), \( B \) and \( E \) are given as functions of \( \theta \) as

\[
B = \tilde{B}(\theta), \quad (40)
\]
\[
E = \tilde{E}(\theta) = \frac{\epsilon_0 E_0}{\epsilon_{eff}(\theta)}, \quad (41)
\]
where Eq. (41) is obtained by substituting Eq. (40) in Eq. (34). Equation (41) indicates that \( \epsilon_{eff}(\theta) \) can be regarded as an effective dielectric constant modified by the axion field.

Figure 2 illustrates the \( \theta \) dependence of \( \tilde{B}(\theta) \) and \( \tilde{E}(\theta) \), which shows that the nonzero axion field induces a magnetic flux density and screens the electric field instead. This result can also be explained from the constituent equations (28)-(31) and the boundary condition (34) as follows: When an external electric field \( E_0 \) is applied, an electric field \( E \) is generated to satisfy the boundary condition, which induces \( M_0 \) via Eq. (31); this \( M_0 \) works as a magnetic flux density \( B \) and induces \( P_0 \) via Eq. (30).

Next, we minimize the Hamiltonian density \( \mathcal{H}_{em} \) with respect to \( \rho \) and \( \theta \). With the optimized \( E \) and \( \rho \) given by Eqs. (40) and (41), respectively, the Hamiltonian density (32) can be written as a function of \( \rho \) and \( \theta \) as

![Figure 2](image-url)

**FIG. 2.** Schematic of the setup. We consider an interface between a normal insulator (bottom) and an axion insulator (top): The axion field is fixed to zero in the former, whereas nonzero axion field can be induced in the latter in response to applied fields. Electric field \( E_0 \) is applied perpendicular to the interface. The electric field \( E \) and the magnetic flux density \( B \), as well as the axion field \( \theta \), in the axion insulator are determined so as to minimize the energy of the system. The magnetic flux density in the normal insulator is the same as that in the axion insulator due to Eq. (23).

**B. Static solutions**

In the following, we assume that energy dissipation of the system is large enough so that the system relaxes to a stationary state within a finite time after applying an electric field. Static solutions for the induced fields are obtained by minimizing Eq. (32) as a function of \( E, B, \rho, \) and \( \theta \) under the boundary condition (34).

First, we minimize the Hamiltonian density (33) with respect to \( E \) and \( B \). Substituting Eq. (34) in Eq. (33), we obtain

\[
\mathcal{H}_{em}(B, \theta) = \frac{\epsilon_{eff}(\theta)}{8\pi\epsilon\mu} \left( B - \tilde{B}(\theta) \right)^2 + \frac{(\epsilon_0 E_0)^2}{8\pi\epsilon_{eff}(\theta)}, \quad (36)
\]

with

\[
\tilde{B}(\theta) = \frac{\theta/\Theta_0}{1 + (\theta/\Theta_0)^2} \frac{\epsilon_0 E_0}{\sqrt{\epsilon/\mu}}, \quad (37)
\]
\[
\epsilon_{eff}(\theta) = \epsilon \left[ 1 + (\theta/\Theta_0)^2 \right]^2, \quad (38)
\]
\[
\Theta_0 = \frac{\pi}{\alpha} \sqrt{\epsilon/\mu} = 4.3 \times 10^2 \sqrt{\epsilon/\mu}. \quad (39)
\]

where \( \epsilon_{eff}(\theta) \) works as an effective dielectric constant. The solution that converged by repeating this process is Eqs. (40) and (41).

We note that \( \tilde{E}(\theta) \) and \( \tilde{B}(\theta) \) are functions of \( \theta/\Theta_0 \). Depending on \( |\theta|/\Theta_0 \), they have three different behaviors. In region (I) \( |\theta|/\Theta_0 \ll 1, \tilde{E}(\theta) \) keeps almost a constant value \( \epsilon_0 E_0/\epsilon \), and no significant magnetic field \( \tilde{B}(\theta) \) is induced. In region (II) \( |\theta|/\Theta_0 \sim 1, \tilde{E}(\theta) \) begins to screened, and \( \tilde{B}(\theta) \) is induced. Finally, in region (III) \( |\theta|/\Theta_0 \gg 1 \), both \( \tilde{E}(\theta) \) and \( \tilde{B}(\theta) \) are screened. When \( |\theta|/\Theta_0 \), the contribution of \( \mathcal{L}_0 \) becomes large compared to \( \mathcal{L}_{em} \), and therefore, the interaction effect between the axion and electromagnetic fields becomes more significant. Actually, as shown in Fig. 3, \( \tilde{E}(\theta) \) and \( \tilde{B}(\theta) \) largely deviate from their values at \( \theta = 0 \) around \( |\theta| \sim \Theta_0 \). A typical value of \( \Theta_0 \) is in the order of \( 10^3 \sim 10^4 \), which means, a large axion field is required to observe the axion electromagnetism. We also note that \( \tilde{E}(\theta) \) and \( \tilde{B}(\theta) \) are not periodic in \( \theta \) mod \( 2\pi \). Although \( 2\pi \) periodicity in \( \theta \) is imposed in a closed space-time with periodic boundary conditions, this is not the case due to the existence of the interface, as pointed out in Ref. [23].

Next, we minimize the Hamiltonian density \( \mathcal{H}_{b} \) with respect to \( \rho \) and \( \theta \). With the optimized \( E \) and \( \rho \) given by Eqs. (40) and (41), respectively, the Hamiltonian density (32) can be written as a function of \( \rho \) and \( \theta \) as

\[
\mathcal{H}_{b} = \mathcal{H}_{em}(\theta) + V_{a}(\phi_4, \phi_5), \quad (42)
\]
\[
\mathcal{H}_{em}(\theta) = \frac{(\epsilon_0 E_0)^2}{8\pi\epsilon \left[ 1 + (\theta/\Theta_0)^2 \right]^2}, \quad (43)
\]

with \( \phi_4 = \rho \cos \theta \) and \( \phi_5 = \rho \sin \theta \). When both \( \phi_4 \) and \( \phi_5 \) fluctuate, \( \rho \) and \( \theta \) can change independently, and stationary solutions are obtained by solving \( \partial \mathcal{H}_{b}/\partial \rho = 0 \) and \( \partial \mathcal{H}_{b}/\partial \theta = 0 \). The former equation reduces to
\[ M_{\text{eff}}^2 = \left. \frac{\partial^2 \tilde{\mathcal{H}}_{\text{em}}(\theta)}{\partial \theta^2} \right|_{\theta = 0} = M_{\text{em}}^2 + M_a^2 \]  

with

\[ M_{\text{em}}^2 = \left. \frac{\partial^2 \mathcal{H}_{\text{em}}(\theta)}{\partial \theta^2} \right|_{\theta = 0} = -\frac{1}{\Theta_0^2} \left( \epsilon_0 E_0 \right)^2 \leq 0, \]  

\[ M_a^2 = \left. \frac{\partial^2 \tilde{V}_a(\theta)}{\partial \theta^2} \right|_{\theta = 0} > 0. \]

Here, the sign of \( M_a^2 \) is fixed to be positive by definition of \( \tilde{V}_a(\phi_4, \phi_5) \). Note that sign change of the squared mass of the axion \( M_{\text{eff}}^2 \) occurs when the external electric field \( E_0 \) is bigger than a certain threshold. Defining the threshold \( E_{\text{crit}}^0 \) as \( E_0 \) that satisfies \( M_{\text{eff}}^2 = 0 \), we obtain

\[ E_{\text{crit}}^0 = \left. \frac{4\pi \tilde{\mathcal{E}}_0 M_a}{\epsilon_0} \right. \]  

When \( E_0 < E_{\text{crit}}^0 \), \( \tilde{\mathcal{H}}_{\text{ib}}(\theta) \) has positive curvature \( M_{\text{eff}}^2 > 0 \) (bradyonic), so \( \theta = 0 \) remains to be a (at least local) minimum of \( \tilde{\mathcal{H}}_{\text{ib}}(\theta) \). On the other hand when \( E_0 > E_{\text{crit}}^0 \), \( \tilde{\mathcal{H}}_{\text{ib}}(\theta) \) has negative curvature \( M_{\text{eff}}^2 < 0 \) (tachyonic), so \( \theta = 0 \) becomes unstable, i.e., the axion instability occurs.

Although one may think that \( E_{\text{crit}}^0 \) is too large to cause the axion instability in realistic systems, Eq. (49) indicates that \( E_{\text{crit}}^0 \) takes a lower value for smaller \( M_a \). Note that Eq. (48) is rewritten as

\[ M_a^2 = \left. \frac{\partial^2 \tilde{V}_a(\theta)}{\partial \theta^2} \right|_{\theta = 0} > 0. \]  

Therefore, there are two ways to reduce the critical value \( E_{\text{crit}}^0 \): One is to reduce the value of \( M_a \) by going near the quantum critical point of the antiferromagnetic order, as discussed in Ref.\[22\]; the other is to reduce the value of \( \tilde{\mathcal{E}}_0 \) by going near the topological quantum phase transition as discussed in the previous section.

D. Correspondence with Ooguri-Oshikawa’s theory

For a small fluctuation of \( \theta \approx \delta \theta \), the potential term in Eq. (44) can be approximated up to the second order in \( \delta \theta \):

\[ \tilde{V}_a(\theta) = \frac{M_a^2}{2} (\delta \theta)^2 = \frac{M_a^2}{2} (\delta \phi_5)^2. \]  

Comparing this equation with Eqs. (2) and (4), we obtain the correspondence relation: \( \tilde{\mathcal{E}}_0 \leftrightarrow g \) and \( M_a^2/2 \leftrightarrow g^2 J_n \). Therefore, \( E_{\text{crit}}^0 \) in Eq. (49) coincides with that in Eq. (7).
V. MODEL ANALYSIS

In this section, we analyze the axion instability based on model potentials. Before going to the detailed analysis, we first present a general consideration. In the initial state, the system is an ordinary (non-topological) insulator with time-reversal invariance. It should be noted that $\theta = 0$ is not the global minimum of $\mathcal{H}_b(\theta)$ for $E_0 \neq 0$: because the first term of Eq. (44) is a decreasing function of $|\theta|$ and the second term is periodic in $\theta$, the inequality $\mathcal{H}_b(\theta) > \mathcal{H}_b(\theta + \text{sgn}(\theta)2\pi)$ always holds. Therefore, once the axion instability occurs, a large axion field $\theta$ such that $|\theta|/\Theta_0 \gg 1$ is expected to emerge, unless the potential $\tilde{V}_a(\theta)$ has a singularity. Note that $\tilde{V}_a(\theta)$ scales as a function of $\theta/\Theta_0$. Furthermore, Fig. 3 indicates that such a large $\theta$ leads to almost complete screening of the electric field and induction of a small magnetic field inside the axion insulator. On the other hand when $\tilde{V}_a(\theta)$ diverges at a certain $\theta$, the induced $\theta$ is bounded to be less than $2\pi \ll \Theta_0$. In this case, screening of $E$ and induction of $B$ are small as seen from Fig. 3. Hence, although the critical electric field $E_0^{\text{crit}}$, which is determined by behaviors around $\theta = 0$, agrees with the result by Ooguri and Oshikawa, we find that the resulting behaviors of $E$ and $B$ are totally different. Below, we consider two model potentials with and without non-magnetic fluctuations which correspond to analytic and singular $\tilde{V}_a(\theta)$, respectively.

A. Instability due to coexisting non-magnetic and magnetic fluctuations

First, we consider a model hosting both non-magnetic and magnetic fluctuations. As we will see below, this model corresponds to the region (III) in Fig 3. The coexistence of non-magnetic and magnetic fluctuations makes it possible to induce a huge value of $\theta$, but it suppresses $B$ and $E$ according to Eqs. (40) and (41). As an example of a $2\pi$ periodic potential, consider the following model potential:

\[
\tilde{V}_a(\theta) = M_\alpha^2 (1 - \cos \theta),
\]

which has the minimum value $0$ at $\theta = 2\pi n$ and the maximum value $2M_\alpha^2$ at $\theta = 2\pi (n + 1/2)$ with $n$ being an integer. In the presence of the external electric field $E_0$, the $\theta$ dependence of $\mathcal{H}_b(\theta)$ is given by

\[
\mathcal{H}_b(\theta) = \tilde{\mathcal{H}}_{\text{em}}(0) + M_\alpha^2 \frac{\theta^2}{1 + (\theta/\Theta_0)^2} \left( \frac{E_0}{E_0^{\text{crit}}} \right)^2 + 2(1 - \cos \theta),
\]

In Fig 4, we plot $\tilde{\mathcal{H}}_{\theta}(\theta) - \tilde{\mathcal{H}}_{\theta}(0)$ for $E_0/E_0^{\text{crit}} = 0.9, 1.0$ and $1.1$ as a function of $\theta$. Here, we have assumed $\epsilon = \mu = 1$ and used $\Theta_0 = 4.3 \times 10^2$ [see Eq. (39)]. In the scale of $\theta$ shown in Fig. 4(a), $\theta = 0$ seems to be a maximum of $\tilde{\mathcal{H}}_{\theta}(\theta)$, but actually this point is a local minimum (maximum) for $E_0 < E_0^{\text{crit}}$ ($E_0 > E_0^{\text{crit}}$) as seen in Fig. 4(b). Because of the steep peak shown in Fig. 4(a), which comes from the first term in the square bracket in Eq. (54), when $E_0$ exceeds the critical value $E_0^{\text{crit}}$ and the axion instability occurs, $|\theta|$ becomes much larger than $\Theta_0$ as we mentioned above. The system is expected to end up with the first stationary point $\theta_{\text{min}}$, which is estimated as follows.

From $\partial \mathcal{H}_b(\theta)/\partial \theta \bigg|_{\theta = \theta_{\text{min}}} = 0$, we have

\[
\Theta_0 \left( \frac{E_0}{E_0^{\text{crit}}} \right)^2 f(\theta_{\text{min}}) = \sin \theta_{\text{min}},
\]

with

\[
f(\theta_{\text{min}}) = \frac{(\theta_{\text{min}}/\Theta_0)}{[1 + (\theta_{\text{min}}/\Theta_0)^2]^2}.
\]

Since $\Theta_0$ in Eq. (55) is much larger than the right-hand side in Eq. (39) [a typical value of $\Theta_0$ is in the order of $10^4$], $f(\theta_{\text{min}})$ should be much smaller than 1, which means $|\theta_{\text{min}}|/\Theta_0 \ll 1$ and $f(\theta_{\text{min}}) \simeq (\theta_{\text{min}}/\Theta_0)^{-3}$. The first positive solution of Eq. (55) arises around where the left-hand side of Eq. (55) decreases to unity, i.e.,

\[
\theta_{\text{min}} \simeq \Theta_0^{1/3} \left( \frac{E_0}{E_0^{\text{crit}}} \right)^{2/3}.
\]

With this $\theta_{\text{min}}$, the electromagnetic fields $B$ and $E$ inside the axion insulator are evaluated from Eqs. (40) and (41) as

\[
B = \tilde{B}(\theta_{\text{min}}) \simeq \Theta_0^{-1/2} \left( \frac{E_0}{E_0^{\text{crit}}} \right)^{1/4},
\]

\[
E = \tilde{E}(\theta_{\text{min}}) \simeq \Theta_0^{-1/2} \left( \frac{E_0}{E_0^{\text{crit}}} \right)^{-1/4}.
\]

In Fig. 5, we plot the axion field $\theta$ and the electromagnetic fields $E$ and $B$ induced in the axion insulator as functions of an applied electric field $E_0$. One can see that $\theta$ suddenly increases to a huge value at $E_0 = E_0^{\text{crit}}$, which almost completely screens the electric field $E$. The axion field also induces a magnetic flux density $B$, but its amplitude is quite small since the induced $B$ is in the region (III) of Fig 3. For comparison, the results of Ooguri-Oshikawa’s theory are shown in Fig. 9.

Since $\tilde{\mathcal{H}}_{\theta}(\theta)$ is an even function of $\theta$, $\theta = -\theta_{\text{min}}$ is also a solution of Eq. (55). The sign of the axion field is spontaneously determined. In other words, time-reversal symmetry is spontaneously broken at the onset of the instability. Accordingly, the direction of the induced magnetic field is determined.

B. Instability due to magnetic fluctuations

In this subsection, we consider a model with fixed $\phi_4 = \tilde{\rho}(0) > 0$, where only the antiferromagnetic order
FIG. 4. Hamiltonian density $\tilde{\mathcal{H}}_b(\theta)$ [Eq. (54)] in the model with coexisting non-magnetic and magnetic fluctuations as a function of the induced axion field $\theta$. We assume an axion insulator with $\epsilon = \mu = 1$ and use $\Theta_0 = 4.3 \times 10^2$. (a) shows behavior of $\tilde{\mathcal{H}}_b(\theta)$ in a wide range of $\theta$, and (b)–(d) are the enlarged views in the regions marked by boxes in (a). (b) The point $\theta = 0$ changes from a local minimum to a maximum at $E_0 = E_{0}^{\text{crit}}$. Due to the $\cos \theta$ term in Eq. (54), (c) $\tilde{\mathcal{H}}_b(\theta)$ exhibits a wavy curve for $\theta < \theta_{\text{min}}$, and (d) local minima periodically appear for $\theta > \theta_{\text{min}}$, where $\theta_{\text{min}}$ is the $\theta$ for the first local minimum at $E_0 > 0$. We assume that the system under $E_0 > E_{0}^{\text{crit}}$ relax to the first local minimum, $\theta = \theta_{\text{min}}$, and derive the electromagnetic fields [Eqs. (58) and (59)] inside the axion insulator.

$\phi_5$ can fluctuate. As is shown below, only a small $\theta$ can be induced in this case, which corresponds to the region (1) in Fig. 3.

Since $\phi_4 = \rho(\theta) \cos \theta$ is a constant $\rho(0)$, we have $\phi_5 = \tilde{\rho}(\theta) \sin \theta = \tilde{\rho}(0) \tan \theta$. Then, we assume the following quadratic potential for $\phi_5$,

$$\tilde{\mathcal{V}}_a(\theta) = \frac{M^2}{2} \phi_5^2 = \frac{M^2}{2} \tan^2 \theta,$$

which has a minimum at $\theta = 0$. We note that the induced $|\theta|$ is $\pi/2$ at most because there is an infinitely high potential barrier at $|\theta| = \pi/2$.

The energy density is given by $\tilde{\mathcal{H}}_b(\theta)$ in Eq. (44) with the potential (60). Since $|\theta| \ll \Theta_0$ even at $E_0 > E_{0}^{\text{crit}}$, $\tilde{\mathcal{H}}_b(\theta)$ is approximately given by

$$\tilde{\mathcal{H}}_b(\theta) \simeq \tilde{\mathcal{H}}_{\text{em}}(0) + \frac{M^2}{2} \left[ -\theta^2 \left( \frac{E_0}{E_{0}^{\text{crit}}} \right)^2 + \tan^2 \theta \right].$$

In Fig. 5, we show $\tilde{\mathcal{H}}_b(\theta) - \tilde{\mathcal{H}}_b(0)$ for $E_0/E_{0}^{\text{crit}} = 0.9, 1.0$ and 1.1 as a function of $\theta$. When $E_0 > E_{0}^{\text{crit}}$, $\tilde{\mathcal{H}}_b(\theta)$ has double minima at nonzero $\theta$, and thus time-reversal symmetry is spontaneously broken by choosing one of the two minima, resulting in the antiferromagnetic order $\phi_5$.

We numerically find the position of the minima of...
from a minimum to a local maximum at $E_E$ to Eq. (44) can be written as

$$\theta$$

Hence, the induced axion field $\theta$ maximized $B$ in Eq. (35), we obtain the solution $\theta = \pm \pi / 2$, the position of the minima is restricted in $|\theta| < \pi / 2$ and goes to $|\theta| = \pi / 2$ as $E_0 / E_0^{\text{crit}} \to \infty$.

Eq. (61), which is shown in Fig. 6(a). The corresponding $E$ and $B$ are obtained from Eqs. (41) and (40) and shown in Figs. 7(c) and 7(d), respectively. In particular, in the limit of $E_0 / E_0^{\text{crit}} \to \infty$, $\theta$ approaches to $\pi / 2$ or $-\pi / 2$ because of the divergence of the potential term at $\theta = \pm \pi / 2$. In this limit, $B$ and $E$ inside the axion insulator are linear in $E_0$:

$$\lim_{E_0 \to \infty} B = \tilde{B} \left( \frac{\pi}{2} \right) \sim \pm 3.6 \times 10^{-3} \frac{\mu}{\epsilon} \epsilon_0 E_0,$$

$$\lim_{E_0 \to \infty} E = \tilde{E} \left( \frac{\pi}{2} \right) \sim \left( 1 - 1.3 \times 10^{-5} \frac{\mu}{\epsilon} \right) \epsilon_0 E_0 / \epsilon.$$ (63)

Equation (63) clearly shows that the electric field $E$ is only partially screened ($1.3 \times 10^{-5} \mu / \epsilon$). While the massive axion electrodynamics analyzed by Ooguri and Oshikawa shows screening of the excess electric field above $E_0^{\text{crit}}$ (see Appendix A), no such significant screening is seen in this model because large induction of $\theta$ is strongly suppressed in insulators with fixed $\phi_4$.

VI. RESPONSE TO AN APPLIED MAGNETIC FIELD

So far, an electric field is applied to the interface. Now, we comment briefly what happens when a magnetic field $B_0$, instead of $E_0$, is applied.

The boundary condition at the interface is given by Eqs. (32) and (33). Then, minimizing the Hamiltonian density (32) with respect to $E$ under the boundary condition (33), we obtain the solution $E = 0$. With the optimized $B$ and $E$, the Hamiltonian density corresponding to Eq. (44) can be written as

$$\tilde{\mathcal{H}}_b = \frac{B_0^2}{8 \pi \mu} + \tilde{V}_a(\theta).$$ (64)

Hence, the induced axion field $\theta$ is determined by solving

$$\partial \tilde{V}_a / \partial \theta = 0,$$ (65)

and the effective square mass of axion is obtained as

$$M_{\text{eff}}^2 = \left. \frac{\partial^2 \tilde{V}_a(\theta)}{\partial \theta^2} \right|_{\theta=0} > 0.$$ (66)

Therefore, no axion instability happens and no electric field is induced inside the axion insulator.

VII. SUMMARY

In this paper, we examine axion instability from a microscopic point of view. We introduce an antiferromag-
netic field, instead of an axion field, and analyze axion instability caused by the antiferromagnetic field. From a general argument, it is pointed out that a non-magnetic order describing a topological transition is relevant to the axion dynamics, as well as the antiferromagnetic order. Since an axion field is related to both magnetic and non-magnetic orders, fluctuations of the antiferromagnetic field are insufficient to induce a visible axion instability.

Starting from a microscopic Hamiltonian for a topological insulator with an additional term that breaks both time-reversal and inversion symmetries, we describe an axion field in terms of an antiferromagnetic field and an energy gap, which correspond to the magnetic and non-magnetic orders, respectively. Then we derive an effective Lagrangian for the axion field, $\theta$, and the electromagnetic fields, $E$ and $B$, with a phenomenologically introduced potential for the axion field, which is a $2\pi$-periodic function and includes higher order terms of $\theta$. This potential keeps the system with $\theta = 0$ at a local minimum of the energy density even under electromagnetic fields. When an applied electric field exceeds a critical value, however, $\theta = 0$ becomes an unstable point and the system relaxes to a new local minimum.

To see the effect of the non-magnetic fluctuations, we calculate induced $\theta$, $E$, and $B$ in response to an applied electric field for two model potentials with and without non-magnetic fluctuations. In the case when both magnetic and non-magnetic fields fluctuate, a large amplitude of an axion field is induced above the critical field. As a result, an applied electric field is almost completely screened. Contrarily to this, in the case when only the magnetic order fluctuates, the amplitude of the induced axion field is bounded above by $\pi/2$, which cannot induce a significant screening of an electric field. In both cases the induced magnetic field is small since it becomes significant only at around $\theta = \pi/\alpha \sqrt{\epsilon/\mu}$, where $\epsilon$, $\mu$, and $\alpha$ are the dielectric constant and the magnetic permeability of the insulator and the fine-structure constant, respectively. We also note that no axion instability occurs when a magnetic field, instead of an electric field, is applied. Our result suggests that a system that is close to the quantum critical point of the antiferromagnetic order is appropriate for investigating axion electromagnetism.

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Appendix A: Ooguri-Oshikawa’s theory

For comparison, we revisit the Ooguri-Oshikawa model. They consider the following model potential,

$$\tilde{V}_a(\theta) = \frac{M_a^2}{2} \theta^2,$$  \hspace{1cm} (A1)

which has the minimum value 0 at $\theta = 0$. In the presence of an external electric field $E_0$ normal to the interface between the axion insulator and the normal insulator, $\tilde{\mathcal{H}}_a(\theta)$ is given by

$$\tilde{\mathcal{H}}_a(\theta) = \tilde{\mathcal{H}}_{\text{em}}(0) + \frac{M_a^2 \Theta_0^2}{2} \left[ -\frac{(\theta/\Theta_0)^2}{1 + (\theta/\Theta_0)^2} \left( \frac{E_0}{E_0^{\text{crit}}} \right)^2 + (\theta/\Theta_0)^2 \right].$$  \hspace{1cm} (A2)

The right-hand side of Eq. (A2) is shown in Fig. 8. Then, $\partial \tilde{\mathcal{H}}_a(\theta)/\partial \theta = 0$ reduces to

$$\frac{1}{\theta/\Theta_0} \left[ \frac{1}{(\theta/\Theta_0)^2} - \left( \frac{E_0}{E_0^{\text{crit}}} - 1 \right) \right] = 0.$$  \hspace{1cm} (A3)

Above the critical electric field, $\tilde{\mathcal{H}}_a(\theta)$ has two minima at

$$\theta = \pm \Theta_0 \sqrt{\frac{E_0}{E_0^{\text{crit}}} - 1}.$$  \hspace{1cm} (A4)

Substituting Eq. (A4) in Eqs. (40) and (41), the induced $B$ and $E$ are obtained as

$$B = \pm \frac{\epsilon_0 E_0^{\text{crit}}}{\sqrt{\epsilon/\mu}} \sqrt{\frac{E_0}{E_0^{\text{crit}}} - 1},$$  \hspace{1cm} (A5)

$$E = \frac{\epsilon_0 E_0^{\text{crit}}}{\epsilon}.$$  \hspace{1cm} (A6)

We illustrate the induced $\theta$, $E$, and $B$ in Fig. 3(a)-(d). This model corresponds to the region (II) of Fig. 3.

In comparison with the other models considered in the main text, the Ooguri-Oshikawa model induces a larger magnetic field. However, the justification of their analysis is not obvious. Their model potential takes into account only the squared term of $\theta$, and neglects the higher order terms. However, when a larger magnetic field is induced, $\theta$ becomes $O(1)$ so the higher order terms can not be neglected. Indeed, if $\theta$ originates from only magnetic fluctuations, our analysis in Sec. IVB indicates that the axion instability should be suppressed due to the higher order terms.

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FIG. 8. $\tilde{H}_b(\theta)$ based on Ooguri-Oshikawa’s theory [23] is plotted as a function of $\theta$. 

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FIG. 9. Behaviors of (a) the axion field $\theta$, (b) the electric field $E$, and (c) the magnetic flux density $B$ inside an axion insulator in response to an applied electric field $E_0$ based on Oogri-Oshikawa’s theory\,[23], where the values at $E_0 > E_0^{\text{crit}}$ are given by Eqs. (A4), (A6), and (A5), respectively. The $\epsilon$ and $\mu$ dependences are all included in the scaling factor: $\Theta_0$ for $\theta$, $\epsilon_0 E_0^{\text{crit}}/\epsilon$ for $E$, and $\epsilon_0 E_0^{\text{crit}}/\sqrt{\epsilon/\mu}$ for $B$. Above the critical field, the electric field takes a constant value $\epsilon_0 E_0^{\text{crit}}/\epsilon$, and a significant magnetic flux density is induced.