Higher genus polylogarithms on families of algebraic curves

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Abstract: We describe polylogarithms on families of pointed curves of any genus using the theory of the universal Mumford curve which is the family of all pointed Mumford curves. As an application of the result, we show that the polylogarithms are expressed by rational polylogarithms and multiple zeta values.

MSC-Class: 11G20, 11G55, 11M32, 14H10, 14H15, 14D15, 32G20

1. Introduction

Polylogarithm functions, or polylogarithms for short, describe unipotent periods giving isomorphisms between the Betti and de Rham unipotent fundamental groups which are defined for families of algebraic varieties. Especially, the study of polylogarithms on pointed (algebraic) curves which are rational polylogarithms in the genus 0 case becomes an important subject in quantum field theory and arithmetic geometry. A good reference on this subject is [16]. The aim of this paper is to give a computable theory of polylogarithms on pointed curves of any genus based on Teichmüller’s lego game of Grothendieck [18] describing moduli of pointed curves which was already applied to conformal field theory (cf. [3, 4, 22, 27]). Our technical tool is the theory of the universal Mumford curve by which one can express higher genus polylogarithms in terms of the elliptic associators by Enriquez [12]. By results of this paper, polylogarithms on families of pointed curves are seen to be computable functions of the associated deformation parameters which are expressed mainly by rational polylogarithms and multiple zeta values. The $p$-adic version of our results can be shown using results of Furusho [13, 14] which will be written elsewhere, and we hope that the results are extended to a variation theory of motivic fundamental groups of pointed curves.

The organization of this paper is as follows.

In Sections 2 and 3, we construct the universal Mumford curve (of fixed genus) as the family of all pointed Mumford curves over the deformation space of pointed degenerate curves by gluing generalized Tate curves with the computable comparison of their parameters. The construction in this paper is a modified version of results of [23], and is more suitable for studying polylogarithms. There is another approach by Poineau-Turchetti [31] to the universal Mumford curve using Berkovich geometry over $\mathbb{Z}$ [29, 30]. Our theory has the advantage over results of [31] in the sense that it is directly concerned with degeneration theory of curves.
In Sections 4 and 5, we construct a polylogarithm sheaf on the universal Mumford curve which is associated with unipotent periods. Using the KZ (Knizhnik-Zamolodchikov) associator [11] and its elliptic extension known as the elliptic associators [12], we give a van-Kampen type construction of poly logarithm sheaves on generalized Tate curves deforming maximally degenerate curves. This construction is similar to the method of solving the higher genus Kashiwara-Verne problem by Alekseev-Kawazumi-Kuno-Naef [1, 2], and actually their solutions are regarded as regularized unipotent periods. Then by analytic continuation preserving arithmeticity, this polylogarithm sheaf can be extended on the whole universal Mumford curve, and the associated polylogarithms are computable functions of its deformation parameters. In particular, we show that the polylogarithms are expressed mainly by rational polylogarithms and multiple zeta values which extends results of Deligne [9], Brown [6] and Banks-Panzer-Pym [5] in the genus 0 case, and of Hain [19] in the genus 1 case.

2. Generalized Tate curve

2.1. Schottky uniformization. A Schottky group $\Gamma$ of rank $g$ is defined as a free group with generators $\gamma_i \in PGL_2(\mathbb{C})$ ($i = 1, ..., g$) which map Jordan curves $C_i \subset \mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{\infty\}$ to other Jordan curves $C_{-i} \subset \mathbb{P}^1_\mathbb{C}$ with orientation reversed, where $C_{\pm 1}, ..., C_{\pm g}$ with their interiors are mutually disjoint. Each element $\gamma \in \Gamma - \{1\}$ is conjugate to an element of $PGL_2(\mathbb{C})$ sending $z$ to $\beta \gamma z$ for some $\beta \gamma \in \mathbb{C} \times$ with $|\beta \gamma| < 1$ which is called the multiplier of $\gamma$. Therefore, one has

$$\gamma(z) - \alpha \gamma = \beta \gamma(z) - \alpha' \gamma$$

for some element $\alpha, \alpha' \gamma$ of $\mathbb{P}^1_\mathbb{C}$ called the attractive, repulsive fixed points of $\gamma$ respectively. Then the discontinuity set $\Omega \subset \mathbb{P}^1_\mathbb{C}$ under the action of $\Gamma$ has a fundamental domain $D_\Gamma$ which is given by the complement of the union of the interiors of $C_{\pm 1}, ..., C_{\pm g}$. The quotient space $R_\Gamma = \Omega / \Gamma$ is a (compact) Riemann surface of genus $g$ which is called Schottky uniformized by $\Gamma$ (cf. [32]). Furthermore, by a result of Koebe, every Riemann surface of genus $g$ can be represented in this manner.

2.2. Generalized Tate curve. A (pointed) curve is called degenerate if it is a stable (pointed) curve and the normalization of its irreducible components are all projective (pointed) lines. Then the dual graph $\Delta = (V, E, T)$ of a stable pointed curve is a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \to V, \ b : E \to (V \cup \{unordered \ pairs \ of \ elements \ of \ V\})$$

such that the geometric realization of $\Delta$ is connected and that $\Delta$ is stable, namely its each vertex has at least 3 branches. The number of elements of a finite set $X$ is denoted by $\#X$, and a (connected) stable graph $\Delta = (V, E, T)$ is called of $(g, n)$-type if $\text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z}) = g, \#T = n$. Then under fixing a bijection $\nu : T \to \{1, ..., n\}$, which we
call a numbering of $T$, $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $n$-pointed curve of genus $g$ such that each tail $h \in T$ corresponds to the $\nu(h)$th marked point. In particular, a stable graph without tail is the dual graph of a degenerate (unpointed) curve by this correspondence. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $\sharp T$-pointed curve with dual graph $\Delta$ is maximally degenerate. An orientation of a stable graph $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of $\Delta$, denote by $\pm E = \{e, -e \mid e \in E\}$ the set of oriented edges, and by $v_h$ the terminal vertex of $h \in \pm E$ (resp. the boundary vertex of $h \in T$). For each $h \in \pm E$, denote by let $\{|h| \in E$ be the edge $h$ without orientation.

Let $\Delta = (V, E, T)$ be a stable graph. Fix an orientation of $\Delta$, and take a subset $E$ of $\pm E \cup T$ whose complement $E_\infty$ satisfies the condition that

$$\pm E \cap E_\infty \cap \{-h \mid h \in E_\infty\} = \emptyset,$$

and that $v_h \neq v_{h'}$ for any distinct $h, h' \in E_\infty$. We attach variables $x_h$ for $h \in E$ and $y_e = y_{-e}$ for $e \in E$. Let $R_\Delta$ be the $\mathbb{Z}$-algebra generated by $x_h$ ($h \in E$), $1/(x_e - x_{-e})$ ($e, -e \in E$) and $1/(x_h - x_{h'})$ ($h, h' \in E$ with $h \neq h'$ and $v_h = v_{h'}$), and let

$$A_\Delta = R_\Delta[[y_e (e \in E)], B_\Delta = A_\Delta \left[ \prod_{e \in E} y_e^{-1} \right].$$

In a similar way to [20, Section 2], we construct the universal Schottky group $\Gamma_\Delta$ associated with oriented $\Delta$ and $E$ as follows. For $h \in \pm E$, denote by $\phi_h$ an element of $GL_2(B_\Delta)$ and hence of $PGL_2(B_\Delta)/B_\Delta^\times$ defined as

$$\xi_h(\phi_h(z)) \xi_{-h}(z) = y_h \quad (z \in \mathbb{P}^1),$$

where $\xi_h$ denotes the local coordinate on $P_{v(h)} = \mathbb{P}^1$ given by

$$\xi_h(z) = \begin{cases} z - x_h & \text{if } h \not\in E_\infty, \\ 1/z & \text{if } h \in E_\infty. \end{cases}$$

For any reduced path $\rho = h(1) \cdot h(2) \cdots h(l)$ which is the product of oriented edges $h(1), ..., h(l)$ such that $v_{h(i)} = v_{-h(i+1)}$, one can associate an element $\rho^*$ of $PGL_2(B_\Delta)$ having reduced expression $\phi_{h(l)} \phi_{h(l-1)} \cdots \phi_{h(1)}$. Fix a base vertex $v_b$ of $V$, and consider the fundamental group $\pi_1(\Delta; v_b)$ which is a free group of rank $g = \text{rank}_2 H_1(\Delta, \mathbb{Z})$. Then the correspondence $\rho \mapsto \rho^*$ gives an injective anti-homomorphism $\pi_1(\Delta; v_b) \to PGL_2(B_\Delta)$ whose image is denoted by $\Gamma_\Delta$.

**Proposition 2.1.** The above element $\phi = \phi_{h(l)} \cdots \phi_{h(1)}$ of $PGL_2(B_\Delta)$ has the attractive (resp. repulsive) fixed points $\alpha_\phi$ (resp. $\alpha'_\phi$) and the multiplier $\beta_\phi$ as elements of $B_\Delta$ such that $\xi_{h(i)}(\alpha_\phi)$, $\xi_{-h(i)}(\alpha'_\phi)$ and $\beta_\phi$ belong to the ideal $I_\Delta$ of $A_\Delta$ generated by $y_e$ ($e \in E$). Furthermore, these elements of $I_\Delta$ are computable which means that all their coefficients as power series in $y_e$ ($e \in E$) can be computed.
Proof. The matrix corresponding to $\phi_h \ (h \in E)$ is
\[
\begin{pmatrix}
x_h & -x_h x_{-h} + y_h \\
1 & -x_{-h}
\end{pmatrix} \equiv \begin{pmatrix}
x_h & -x_h x_{-h} \\
1 & -x_{-h}
\end{pmatrix} \mod (I_{\Delta}),
\]
and hence the matrix $M$ corresponding to $\phi$ satisfies
\[
M \equiv \tau \begin{pmatrix}
x_h(l) & -x_h(l) x_{-h(1)} \\
1 & -x_{-h(1)}
\end{pmatrix} \mod (I_{\Delta}),
\]
where
\[
\tau = \prod_{s=1}^{l-1} (x_h(s) - x_{-h(s+1)}) \in A_{\Delta}^r.
\]
Since $\nu = \det(M/\tau)/\text{tr}(M/\tau)^2 \in I_{\Delta}$, the solutions of the characteristic polynomial of $M/\tau$ are given by $x = u \cdot \text{tr}(M/\tau)$ and $x' = \nu u^{-1} \cdot \text{tr}(M/\tau)$ for some element $u$ of $1 + I_{\Delta}$. Therefore,
\[
\alpha_\phi \equiv x + x_{-h(1)}, \quad \alpha'_\phi \equiv x' + x_{-h(1)} \mod (I_{\Delta})
\]
and $\beta_\phi = \nu u^{-2}$ satisfy
\[
\xi_{h(l)} (\alpha_\phi), \quad \xi_{-h(1)} (\alpha'_\phi), \quad \beta_\phi \in I_{\Delta},
\]
and these elements of $I_{\Delta}$ are computable by the proof. □

It is shown in [20, Section 3] and [21, 1.4] (see also [24, Section 2] when $\Delta$ is trivalent and has no loop) that for any stable graph $\Delta$, there exists a stable pointed curve $C_\Delta$ of genus $g$ over $A_{\Delta}$ which satisfies the following properties:

(P1) The closed fiber $C_\Delta \otimes_{A_{\Delta}} R_{\Delta}$ of $C_\Delta$ obtained by substituting $y_e = 0 \ (e \in E)$ becomes the degenerate pointed curve over $R_{\Delta}$ with dual graph $\Delta$. More precisely, this curve is obtained from the collection of $P_v := \mathbb{P}^1_{R_{\Delta}} \ (v \in V)$ by identifying the points $x_e \in P_{v_e}$ and $x_{-e} \in P_{v_{-e}} \ (e \in E)$, where the variables $x_h \ (h \in E)$ are identified with the corresponding points on $P_{v_h}$ and $x_h = \infty \in P_{v_h} \ (h \in E_{\infty})$.

(P2) $C_\Delta$ gives rise to a universal deformation of degenerate pointed curves with dual graph $\Delta$. More precisely, $C_\Delta$ satisfies the following: For a noetherian and normal complete local ring $R$ with residue field $k$, let $C$ be a pointed Mumford curve over $R$, namely a stable pointed curve over $R$ with nonsingular generic fiber such that the closed fiber $C \otimes_R k$ is a degenerate pointed curve with dual graph $\Delta$, in which all double points and marked points are $k$-rational. Then there exists a ring homomorphism $A_{\Delta} \rightarrow R$ giving $C_\Delta \otimes_{A_{\Delta}} R \cong C$.

(P3) $C_\Delta \otimes_{A_{\Delta}} B_{\Delta}$ is smooth over $B_{\Delta}$ and is Mumford uniformized (cf. [28]) by $\Gamma_{\Delta}$.
(P4) Take $x_h$ ($h \in \mathcal{E}$) as complex numbers such that $x_e \neq x_{-e}$ and that $x_h \neq x_{h'}$ if $h \neq h'$ and $v_h = v_{h'}$, and take $y_e$ ($e \in E$) as sufficiently small nonzero complex numbers. Then $C_{\Delta}$ becomes a pointed Riemann surface which is Schottky uniformized by the Schottky group over $\mathbb{C}$ obtained from $\Gamma_{\Delta}$.

3. Universal Mumford curve

3.1. Comparison of deformations. Let $\Delta_1 = (V_1, E_1, T_1)$ be a stable graph which is not trivalent. Then there exists a vertex $v_0 \in V_1$ which has at least 4 branches. Take two elements $h_1, h_2$ of $\pm E_1 \cup T_1$ such that $h_1 \neq h_2$ and $v_{h_1} = v_{h_2} = v_0$, and let $\Delta_2 = (V_2, E_2, T_2)$ be a stable graph obtained from $\Delta_1$ by replacing $v_0$ with an oriented (nonloop) edge $h_0$ such that $v_{h_1} = v_{h_2} = v_{h_0}$ and that $v_h = v_{-h_0}$ for any $h \in \pm E_1 \cup T_1 - \{h_1, h_2\}$ with $v_h = v_0$. Put $e_i = |h_i|$ for $i = 0, 1, 2$. Then we have the following identifications:

$$V_1 = V_2 - \{v_{-h_0}\} \quad \text{(in which } v_0 = v_{h_0})\), \quad E_1 = E_2 - \{e_0\}, \quad T_1 = T_2.$$  

Theorem 3.1.

(1) The generalized Tate curves $C_{\Delta_1}$ and $C_{\Delta_2}$ associated with $\Delta_1$ and $\Delta_2$ respectively are isomorphic over $R_{\Delta_2}(s_{e_0})[[s_e \ (e \in E_1)]]$, where

$$\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \quad \frac{y_e}{s_{e_0} s_{e_1}} \quad (i = 1, 2 \ \text{with} \ h_i \notin T_1), \quad \frac{y_e}{s_e} \quad (e \in E_1 - \{e_1, e_2\})$$

belong to $(A_{\Delta_2})^\times$ if $h_1 \neq -h_2$, and

$$\frac{x_{h_1} - x_{h_2}}{s_{e_0}}, \quad \frac{y_e}{s_e} \quad (e \in E_1)$$

belong to $(A_{\Delta_2})^\times$ if $h_1 = -h_2$. Furthermore, these elements of $(A_{\Delta_2})^\times$ are computable.

(2) The assertion (1) holds in the category of complex geometry when $x_{h_1} - x_{h_2}, y_e$ and $s_e$ are taken to be sufficiently small complex numbers.

Proof. Since the assertion (2) follows from (1) and the property 2.2 (P4) of generalized Tate curves, we prove (1) when $\Delta_1$ has no tail from which this assertion in general case follows. Over a certain open subset of $\{ (x_h, y_e) \in \mathbb{C}^{\pm E_1} \times \mathbb{C}^{E_1} \}$ with sufficiently small absolute values $|x_{h_1} - x_{h_2}|$ and $|y_e|$, $C_{\Delta_1}$ gives a deformation of the degenerate curve with dual graph $\Delta_2$. Hence by the universality of generalized Tate curves, there exists an injective homomorphism

$$(*): (R_{\Delta_1} \otimes \mathbb{C})[[y_e \ (e \in E_1)]] \hookrightarrow (R_{\Delta_2} \otimes \mathbb{C})((s_0))[[s_e \ (e \in E_1)]]$$

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which induces an isomorphism $C_{\Delta_1} \cong C_{\Delta_2}$ such that the degenerations of $C_{\Delta_1}$ given by $x_{h_1} - x_{h_2} \to 0$ and $y_e \to 0$ ($e \in E_1$) correspond to those of $C_{\Delta_2}$ given by $s_{e_0} \to 0$ and $s_e \to 0$ respectively. Since these two curves are Mumford uniformized, a result of Mumford (cf. [28, Corollary 4.11]) implies that the uniformizing groups $\Gamma_{\Delta_1}, \Gamma_{\Delta_2}$ of $C_{\Delta_1}, C_{\Delta_2}$ respectively are conjugate over the quotient field of $(R_{\Delta_2} \otimes \mathbb{C})((s_0))[[s_e]]$. Denote by $\iota : \Gamma_{\Delta_1} \to \Gamma_{\Delta_2}$ the isomorphism defined by this conjugation. Since eigenvalues are invariant under conjugation and the cross ratio

$$[a, b; c, d] = \frac{(a - c)(b - d)}{(a - d)(b - c)}$$

of 4 points $a, b, c, d$ is invariant under linear fractional transformation, one can see the following:

(A) For any $\gamma \in \Gamma_{\Delta_1}$, the multiplier of $\gamma$ is equal to that of $\iota(\gamma)$ via the injection ($\ast$).

(B) For any $\gamma_i \in \Gamma_{\Delta_1}$ ($i = 1, \ldots, 4$), the cross ratio $[a_1, a_2; a_3, a_4]$ of the attractive fixed points $a_i$ of $\gamma_i$ is equal to that of $\iota(\gamma_i)$ via the injection ($\ast$).

We consider the case that $h_1 \neq -h_2$. Put

$$A_1 = R_{\Delta_1} \left[ \left[ x_{h_1} - x_{h_2}, \frac{y_e}{x_{h_1} - x_{h_2}} (i = 1, 2), \ y_e (e \in E_1 - \{e_1, e_2\}) \right] \right]$$

whose quotient field is denoted by $\Omega_1$, and let $I_1$ be the kernel of the natural surjection $A_1 \to R_{\Delta_1}$. Then from (A) and (B) as above, by using Proposition 2.1 and results in [20, Section 1], we will show that the isomorphism ($\ast$) descends to $A_1 \cong A_{\Delta_2}$, where the variables are related as in the first case of the assertion. For $\gamma \in \Gamma_{\Delta_1}$ with reduced expression $\phi_{h(l)} \cdots \phi_{h(1)}$, by Proposition 2.1 and its proof, the attractive fixed point $a$ of $\gamma$ satisfies $\xi_{h(l)}(a) \in I_1$. For each $v \in V_1$, fix a path $\rho_v$ in $\Delta_1$ from the base point $v_0$ to $v$. By the assumption on $v_0$, one can take distinct oriented edges $h_3, h_4 \in \pm E_1 - \{h_1, h_2\}$ with terminal vertex $v_0$, and $\rho_i \in \pi_1(\Delta_1; v_0)$ ($i = 1, \ldots, 4$) with reduced expression $\cdots h_i$. Then the attractive fixed points $a_i$ of $\gamma_i = (\rho_v^*)^{-1} \cdots \rho_1^* \cdot \rho_0^*$ satisfy that $[a_1, a_3; a_2, a_4] \in (x_{h_1} - x_{h_2}) \cdot (A_1)^x$. Furthermore, by [20, Proposition 1.4 and Theorem 1.5], the attractive fixed points $a_i'$ of $\iota(\gamma_i)$ satisfy that $[a'_1, a'_3; a'_2, a'_4] \in s_0 \cdot (A_{\Delta_2})^x$, and hence from (B), we have the comparison of $x_{h_1} - x_{h_2}$ and $s_0$. Since $h_1$ is not a loop by the assumption, the comparison of $y_1/(x_{h_1} - x_{h_2})$ and $s_1$ follows from applying (B) to $\gamma_i = (\rho_v^*)^{-1} \cdots \rho_i^* \cdot \rho_0^*$ ($i = 1, \ldots, 4$), where $\rho_i \in \pi_1(\Delta_1; v_0)$ has reduced expression

$$\begin{align*}
\rho_1 &= \cdots h_2, \\
\rho_2 &= \cdots h_3, \\
\rho_3 &= \cdots h_5 \cdot h_1, \\
\rho_4 &= \cdots h_6 \cdot h_1,
\end{align*}$$

for distinct oriented edges $h_5, h_6$ with terminal vertex $v_{-h_1}$. Similarly, we have the comparison of $y_2/(x_{h_1} - x_{h_2})$ (resp. $y_e (e \in E_1 - \{\text{loops}\})$) and $s_2$ (resp. $s_e$), and
further if $e \in E_1$ is a loop, then the comparison of $y_e$ and $s_e$ follows from applying (A) to $\gamma = (\rho^{s_e})^{-1} \cdot \phi_e \cdot \rho^{s_e}$. Therefore, the injection gives rise to $A_1 \cong A_{\Delta_1}$ under which we have
\[
\frac{x_{h_1} - x_{h_2}}{s_{e_0}} \cdot \frac{y_{e_1}}{s_{e_0} s_{e_1}} \cdot \frac{y_e}{s_e} \quad (i = 1, 2), \quad \frac{y_e}{s_e} \quad (e \in E_1 - \{e_1, e_2\}) \in (A_{\Delta_2})^x.\]
Furthermore, by the proof of Proposition 2.1 and [20, Proposition 1.4 and Theorem 1.5], the above argument implies that these elements of $(A_{\Delta_2})^x$ are computable.

One can show the assertion in the case that $h_1 = -h_2$ similarly. □

3.2. Construction of the universal Mumford curve. For nonnegative integers $g, n$ such that $2g - 2 + n > 0$, denote by $\overline{\mathcal{M}}_{g, n}$ the moduli stack over $\mathbb{Z}$ of stable $n$-pointed curves of genus $g$ (cf. [10, 25, 26]). Then by definition, there exists the universal stable pointed curve $\mathcal{C}_{g, n}$ over $\overline{\mathcal{M}}_{g, n}$.

**Theorem 3.2.** There exists a deformation space $\mathcal{S}_{g, n}$ of all $n$-pointed degenerate curves of genus $g$, and a stable $n$-pointed curve of genus $g$ over $\mathcal{S}_{g, n}$ whose fiber by the canonical morphism $\text{Spec}(A_{\Delta}) \to \mathcal{S}_{g, n}$ becomes the generalized Tate curve $C_{\Delta}$ for each stable graph $\Delta$ of $(g, n)$-type.

**Proof.** Let $\Delta = (V, E, T)$ be a stable graph of $(g, n)$-type, and take a system of coordinates on $P_v = \mathbb{P}^1_{R_{\Delta}} (v \in V)$ such that $x_h = \infty \ (h \in E_{\infty})$ and that $\{0, 1\} \subset P_v$ is contained in the set of points given by $x_h \ (h \in E$ with $v_h = v)$. Under this system of coordinates, one has the generalized Tate curve $C_{\Delta}$ whose closed fiber $C_{\Delta} \otimes_{A_{\Delta}} R_\Delta$ gives a family of degenerate curves over the open subspace of
\[
\mathcal{S}_\Delta = \{ (p_h \in P_{v_h})_{h \in E \cup T} \mid p_h \neq p_{h'} \ (h \neq h', v_h = v_{h'}) \}
\]
defined as $p_e \neq p_{-e}$ for nonloop edges $e \in E$. Therefore, taking another system of coordinates on $P_v$ obtained by mutual changes of $0, 1, \infty$ and comparing the associated generalized Tate curves with the original $C_{\Delta}$ as in Theorem 3.1, $C_{\Delta}$ can be extended over the deformation space of all pointed degenerate curves with dual graph $\Delta$. Since two stable graphs of $(g, n)$-type can be translated by a combination of replacements $\Delta_1 \leftrightarrow \Delta_2$ given in 3.1, one can define a scheme $\mathcal{S}_{g, n}$ obtained by gluing $\text{Spec}(A_{\Delta}) (\Delta$: stable graphs of $(g, n)$-type) along the isomorphism given in Theorem 3.1. Then $\mathcal{S}_{g, n}$ is regarded as the deformation space of all $n$-pointed degenerate curves of genus $g$ over which there exists a stable $n$-pointed curve of genus $g$ obtained by gluing $C_{\Delta}$. □

**Definition 3.3.** We call the above stable $n$-pointed curve of genus $g$ over $\mathcal{S}_{g, n}$ the $n$-pointed universal Mumford curve of genus $g$ which is the fiber of $\mathcal{C}_{g, n}$ by the canonical morphism $\mathcal{S}_{g, n} \to \overline{\mathcal{M}}_{g, n}$. By 2.2 (P2) and (P4), one can see that this universal Mumford curve gives rise to all $n$-pointed Mumford curves of genus $g$, and to all $n$-pointed Riemann surfaces of genus $g$ close to degenerate curves.

**Remark 3.4.** Gerritzen-Herrlich [15] introduced the extended Schottky space $\overline{S}_g$ of genus $g > 1$ as the fine moduli space of stable complex curves of genus $g$ with Schottky
structure. For integers $g, n$ as above, one can consider the extended Schottky space $\mathcal{S}_{g,n}$ for stable $n$-pointed complex curves of genus $g$ with Schottky structure. Then by the result of Koebe referred in 2.1, $\mathcal{S}_{g,n}/\text{Out}(F_g)$ becomes a covering space of the moduli space of stable $n$-pointed complex curves of genus $g$ which is also the complex analytic space $\mathcal{M}_{g,n}$. Furthermore, the $n$-pointed universal Mumford curve of genus $g$ can be analytically extended to the universal family of stable pointed complex curves over $\mathcal{S}_{g,n}/\text{Out}(F_g)$.

4. Polylogarithms and associators

4.1. Polylogarithms for Riemann surfaces. Fix nonnegative integers $g, n$ such that $2g - 2 + n > 0$, let $(\mathcal{R}; p_0, p_1, \ldots, p_n)$ be a family over a complex space $S$ of $(n+1)$-pointed compact Riemann surfaces of genus $g$, and put $\mathcal{R}^s = \mathcal{R} \setminus \{p_1, \ldots, p_n\}$. Then the completed group algebras

$$\mathbb{C}\hat{\pi}_1(\mathcal{R}_s^s; p_0) = \lim_{\leftarrow} \mathbb{C}\pi_1(\mathcal{R}_s^s; p_0)/J^n_s;$$

over $\mathbb{C}$ of the fundamental groups $\pi_1(\mathcal{R}_s^s; p_0)$ for fibers $\mathcal{R}_s^s$ over $s \in S$ give rise to a local system on $S$. Under taking its de Rham realization, we define the unipotent periods as the monodromy representations between (tangential) sections of $\mathcal{R}^s/S$ which are described by multi-valued functions on $S$ called polylogarithms on $(\mathcal{R}; p_1, \ldots, p_n)/S$.

4.2. KZ connection and associator. For an integer $n \geq 3$, a KZ connection on an $n$-pointed projective line $(\mathbb{P}^1; p_1, \ldots, p_n)$ is defined as a trivial bundle with flat connection which has regular singularities at $p_i$ with residue $X_{p_i}$ $(i = 1, \ldots, n)$, where $X_{p_i}$ are symbols satisfying $\sum_{i=1}^n X_{p_i} = 0$. We take a coordinate $z$ on $\mathbb{P}^1$ such that $p_n = \infty$. Then the connection form is given by the equation

$$d - \sum_{i=1}^{n-1} \frac{X_{p_i}}{z - p_i} dz = 0,$$

and hence the associated monodromy along a path $\gamma \subset \mathbb{P}^1_C$ from $a$ to $b$ is a sum of words of $X_{p_1}, \ldots, X_{p_n}$ whose coefficients are rational polylogarithms given by iterated integrals

$$\int_{\gamma} \frac{dz}{z - p_{i(1)}} \cdots \frac{dz}{z - p_{i(k)}},$$

where $p_{i(j)} \in \{p_1, \ldots, p_{n-1}\}$ (cf. [6, 17]). These polylogarithms are convergent when $a \neq p_{i(1)}$, $b \neq p_{i(k)}$, and are generally described in the shuffle regularization (cf. [5, 2.1.5]) as sums of convergent polylogarithms times

$$\frac{1}{j!} \left( \int_{\gamma} \frac{dz}{z - a} \right)^j \frac{1}{k!} \left( \int_{\gamma} \frac{dz}{z - b} \right)^k.$$
when \( a = p_i(1) \) or \( b = p_i(k) \). Therefore, we have a de Rham realization of
\[
\mathbb{C}^\wedge_{\pi_1}(\mathbb{P}^1 - \{p_1, \ldots, p_n\})
\]
as the ring \( \mathbb{C} \langle \langle X_{p_i} \rangle \rangle \) of noncommutative formal power series over \( \mathbb{C} \) of the symbols \( X_{p_i} \)
\((i = 1, \ldots, n - 1)\).

We assume that \( n = 3, p_1 = 0, p_2 = 1 \). Then there exists a unique solution \( G_0(z) \)
of the KZ connection such that
\[
\lim_{z \to 0} \frac{G_0(z)}{z^{X_0}} = 1; \quad z^{X_0} := \sum_{k=0}^{\infty} \frac{\log(z)^k}{k!} (X_0)^k \quad (0 < z < 1).
\]
Furthermore, \( G_0(z) \cdot z^{-X_0} \) is represented as a noncommutative formal power series in
\( X_0, X_1 \) whose constant term is 1 and coefficients are rational polylogarithm functions
\[
\text{Li}_{k_1, \ldots, k_l}(z) = \int_0^z \prod_{k=1}^{l} w_0 \cdots w_{n_k} w_0 \cdots w_{n_k} w_0 \cdots w_{n_k} w_0 w_1 \cdots w_1 w_0 \cdots w_0 w_1
\]
\[
= \sum_{0 < n_1 < \cdots < n_l} \frac{z^{n_0}}{n_1 \cdots n_l} \in \mathbb{Q}[[z]] \quad (|z| < 1),
\]
where \( w_0 = dz/z, w_1 = dz/(1-z) \). Denote by \( G_1(z) \) the solution of the KZ connection
such that
\[
\lim_{z \to 1} \frac{G_1(z)}{(1-z)^{X_1}} = 1.
\]
Then the connection matrix
\[
\Phi_{\text{KZ}}(X_0, X_1) = G_1(z)^{-1} \cdot G_0(z)
\]
is called the \textit{KZ associator} and is represented as a noncommutative formal power series
in \( X_0, X_1 \) whose coefficients are expressed by \textit{multiple zeta values}
\[
\zeta(k_1, \ldots, k_l) = \sum_{0 < n_1 < \cdots < n_l} \frac{1}{n_1^{k_1} \cdots n_l^{k_l}}
\]
for \( k_l > 1 \).

4.3. \textit{Elliptic associators.} Based on the study by Calaque-Enriquez-Etingof [7] of the
universal KZB (Knizhnik-Zamolodchikov-Bernard) equation, the \textit{elliptic associators}
are introduced by Enriquez [12] for describing motivic (unipotent) fundamental groups of
elliptic curves minus unit elements. The de Rham theoretic aspect of elliptic associators
is described as follows.

Let \( q \) be a variable, and denote by \( E_q \) the Tate (elliptic) curve over \( \mathbb{Z}[[q]] \) which is
formally represented as
\[
(\mathbb{P}^1 - \{0, \infty\})/\langle q \rangle = \mathbb{G}_m/\langle q \rangle,
\]
where $\langle q \rangle = \{ q^a \mid a \in \mathbb{Z} \}$. Then $E_q|_{q=0}$ is identified with the space $\mathbb{P}^1/(0 = \infty)$ obtained from $\mathbb{P}^1$ by identifying $0 = \infty$. Take symbols $T, A$ as a de Rham framing of the first cohomology group of $E_q$ minus the unit element 1, and put $f(T)(x) = f(\text{ad}_T)(x)$ for a noncommutative formal power series over $\mathbb{Q}$ in $T, A$. Then

$$W_0 := \left( \frac{T}{e^T - 1} \right) (A), \quad W_1 := [T, A], \quad W_\infty := \left( \frac{T}{e^{-T} - 1} \right) (A)$$

satisfy $W_0 + W_1 + W_\infty = 0$. We consider the KZ equation

$$d - W_0 \frac{dz}{z} - W_1 \frac{dz}{z-1} = 0$$

on $\mathbb{P}^1 - \{0, 1, \infty\}$ whose fibers are identified with $\mathbb{C} \langle \langle T, A \rangle \rangle$. Since $e^T(W_0) + W_\infty = 0$, there exists the associated vector bundle on $\mathbb{P}^1/(0 = \infty)$ with connection via the left transition operator by $e^T$ at $0 = \infty$. Furthermore, the associated monodromy around 0 with base point $0 \rightarrow 1$ is

$$\Phi_{\text{KZ}}(W_0, W_1) \cdot e^{2\pi \sqrt{-1} W_0} \cdot \Phi_{\text{KZ}}(W_0, W_1)^{-1}$$

and that from $1 \rightarrow \infty$ is

$$e^{\pi \sqrt{-1} W_1} \cdot \Phi_{\text{KZ}}(W_\infty, W_1) \cdot e^T \cdot \Phi_{\text{KZ}}(W_0, W_1)^{-1}$$

which are the formulas of elliptic associators given in [12, Proposition 4.8].

5. Higher genus polylogarithm

5.1. Polylogarithm sheaf on a generalized Tate curve. We construct polylogarithm sheaves by a similar method of Alekseev-Kawazumi-Kuno-Naef [1, Propositions 7 and 8] and [2, Proposition 6.6 and Theorem 6.11] on solving the higher genus Kashiwara-Verne problem. Let $\Delta_0 = (V_0, E_0, T_0)$ be a trivalent graph with orientation of $(g, n)$-type, and fix a maximal subtree $M_0$ of $\Delta_0$. Then $E_0 - M_0$ consists of $g$ elements which we denote by $e_1, \ldots, e_g$. Consider the ring

$$\mathcal{X}_{g,n} = \mathbb{Q} \langle \langle X_t, T_i, A_i \rangle \rangle$$

of noncommutative formal power series over $\mathbb{Q}$ in the symbols $X_h$ ($h \in \pm M_0 \cup T_0$) and $T_i, A_i$ ($i = 1, ..., g$) satisfying the conditions

- For each $h \in \pm M_0$, $X_h + X_{-h} = 0$.
- For each $v \in V$, the sum of $X_h$ for $h \in \pm E_0 \cup T_0$ with $v_h = v$ is equal to 0, where

$$X_{e_i} = \left( \frac{T_i}{e^{T_i} - 1} \right)(A_i),$$

$$X_{-e_i} = \left( \frac{T_i}{e^{-T_i} - 1} \right)(A_i) = -e^{T_i} (X_{e_i}) = -X_{e_i} - [T_i, A_i].$$
Theorem 5.1. There exists a vector bundle with flat connection called a polylogarithm sheaf $\mathcal{V}_{\Delta_0}$ on $C_{\Delta_0}$ with fiber $X_{g,n}$ which is constructed by gluing the KZ connection on $P_{v_0} = \mathbb{P}^1$ with regular singularities at $x_h$ with residue $X_h$ for $h \in \pm E_0 \cup T_0$ such that $v_h = v_0$, and the elliptic associators corresponding to $e_1, \ldots, e_g$.

Proof. We recall that $C_{\Delta_0}$ is the deformation of the union of $P_v \ (v \in V_0)$ by the relations $\xi_h \cdot \xi_{-h} = y_h \ (h \in \pm E_0)$, where $\xi_h$ denotes the local coordinate at $x_h \in P_{v_h}$. For each $v \in V_0$, denote by $(\mathcal{V}_v, \mathcal{F}_v)$ the KZ connection on $P_v$ with regular singularities at $x_h$ with residue $X_h$ for $h \in \pm E_0 \cup T_0$ such that $v_h = v$. Since $\Delta$ is trivalent, the marked projective lines $(P_v; x_h (v_h = v))$ are identified with $(\mathbb{P}^1; 0, 1, \infty)$. If $h \in M_0$, then $X_h \frac{d\xi_h}{\xi_h} = X_{-h} \frac{d\xi_{-h}}{\xi_{-h}}$, and hence one can glue $(\mathcal{V}_{v_h}, \mathcal{F}_{v_h})$ and $(\mathcal{V}_{v_{-h}}, \mathcal{F}_{v_{-h}})$ around $y_h = 0$ in terms of rational polylogarithm functions. If $h = e_i$ for some $i \in \{1, \ldots, g\}$, then $e^T_i (X_{e_i}) \frac{d\xi_{e_i}}{\xi_{e_i}} = X_{-e_i} \frac{d\xi_{-e_i}}{\xi_{-e_i}}$, and hence $(\mathcal{V}_{v_{e_i}}, \mathcal{F}_{v_{e_i}})$ and $(\mathcal{V}_{v_{-e_i}}, \mathcal{F}_{v_{-e_i}})$ are glued around $y_h = 0$ similarly in terms of rational polylogarithm functions and the elliptic associators. Therefore, we obtain a vector bundle with flat connection on

$$\hat{C}_{\Delta_0} = \lim_{\longrightarrow} C_{\Delta_0} \otimes_{A_{\Delta_0}} (A_{\Delta_0}/I_{\Delta_0})^n,$$

where $I_{\Delta_0}$ is the ideal of $A_{\Delta_0}$ generated by $q_i, y_{e_i} \ (i = 1, \ldots, g)$. Then by Grothendieck's existence theorem, there exists the associated vector bundle with flat connection on $C_{\Delta_0}$. □

5.2. Polylogarithm sheaf on the universal Mumford curve. Let $\Delta = (V, E, T)$ be a stable graph of $(g, n)$-type with orientation. Then $\Delta$ is obtained from $\Delta_0$ by repeating alterations $\Delta_1 \leftrightarrow \Delta_2$ given in 3.1 not contracting $e_1, \ldots, e_g$ to points. Then we attach symbols $X^\Delta_h$ with each $h \in \pm E \cup T$ by the following rules:

- If $\Delta = \Delta_0$, then $X^\Delta_h = X_h$.
- For any $v \in V$, the sum of $X^\Delta_h \ (h \in \pm E \cup T \text{ with } v_h = v)$ is 0.
- Let $\Delta_i = (V_i, E_i, T_i) \ (i = 1, 2)$ be above stable graphs which are related as in 3.1. Then $X^\Delta_h = X^\Delta_{h_i}$ for $h \neq h_0$.

Theorem 5.2. The vector bundle with flat connection $(\mathcal{V}_{\Delta_0}, \mathcal{F}_{\Delta_0})$ on $C_{\Delta_0}$ can be analytically continued to a vector bundle with flat connection $(\mathcal{V}, \mathcal{F})$ on the universal Mumford curve $C_{g,n}/\mathcal{S}_{g,n}$ which is called the universal polylogarithm sheaf. For each trivalent graph $\Delta = (V, E, T)$ of $(g, n)$-type, the restriction of $(\mathcal{V}, \mathcal{F})$ to $C_\Delta$ is constructed as in Theorem 5.1.

Proof. The assertion follows from that a gluing process of KZ connections on two projective lines as in the proof of Theorem 5.1 gives rise to a KZ connection on the glued projective line. □
5.3. Higher genus polylogarithms and their formulas.

**Definition 5.3.** For a stable graph \( \Delta \) of \((g, n)\)-type, denote by \((R_\Delta; x_t \ (t \in T))\) the family of Riemann surfaces obtained from \((C_\Delta; x_t \ (t \in T))\) as in 2.2 (P4). Then polylogarithms on \( C_\Delta \) are defined as the coefficients of words of \( X^h_\Delta \) \((h \in \pm E \cup T)\) which give rise to the monodromy between tangential points at \( x_t \in P_v \) of the universal polylogarithm sheaf on \((R_\Delta; x_t \ (t \in T))\).

**Theorem 5.4.** Let \( \Delta \) be a stable graph of \((g, n)\)-type, and denote by \( y_1, \ldots, y_k \) deformation parameters associated with \( E \). Let \( U \) be a small open subset of the complex moduli space of degenerate curves with dual graph \( \Delta \), take a small disc \( D \) in \( \mathbb{C} \) centered at 0 and a contractible neighborhood \( W \subset D - \{0\} \) of a unit tangential point at 0. Then each polylogarithm on \( C_\Delta \) is expressed uniquely as a finite sum

\[
\sum_{j_1, \ldots, j_k} L_{j_1, \ldots, j_k} \left( \frac{\log y_1}{2\pi \sqrt{-1}} \right)^{j_1} \cdots \left( \frac{\log y_k}{2\pi \sqrt{-1}} \right)^{j_k},
\]

on \( U \times W^k \), where \( L_{j_1, \ldots, j_k} \) are holomorphic functions \( L_{j_1, \ldots, j_k} \) on \( U \times D^k \). Furthermore, \( L_{j_1, \ldots, j_k} |_{U \times \{0\}^k} \) are represented as \( \mathbb{Z} \)-linear sums of products of \( \pi \sqrt{-1} \), multiple zeta values and (convergent) rational polylogarithms on the irreducible components of \( C_\Delta \otimes \mathbb{A}_\Delta R_\Delta \). In particular, if \( \Delta \) is trivalent, then \( L_{j_1, \ldots, j_k} |_{U \times \{0\}^k} \) are \( \mathbb{Z} \)-linear sums of products of \( \pi \sqrt{-1} \) and multiple zeta values.

**Proof.** As is stated in the proof of Theorem 5.2, we may assume that \( \Delta = (V, E, T) \) is trivalent. Then \( R_\Delta \) is identified with \( \mathbb{Z} \), and \( C_\Delta \) is obtained by gluing \( P_v = \mathbb{P}^1_{\mathbb{Z}} \) \((v \in V)\) via the set of parameters \( \{y_e \mid e \in E\} = \{y_i \mid i = 1, \ldots, k\} \). By the description in Theorem 5.2 of \((V, F)\) on \( C_\Delta \), the associated higher genus polylogarithms on \( C_\Delta \) are obtained as the products of

\[
\xi_h^X (\xi_h^{X^h_\Delta})^{-1} = \frac{X^h_\Delta}{y_{|h|}} = \sum_{k=0}^\infty \frac{(\log(y_{|h|})^k}{k!} (X^h_\Delta)^k,
\]

of elements of \( A_{g,n} \) with constant term 1 and coefficients \( \text{Li}_{k_1, \ldots, k_l}(\xi_h) \in \mathcal{Q}[[\xi_h]] \) and of the KZ and elliptic associators. Therefore, the assertion follows from that the KZ associator is expressed by the multiple zeta values, and that the higher genus polylogarithms are independent of \( \xi_h \). \( \square \)

**Acknowledgments**

This work is partially supported by the JSPS Grant-in-Aid for Scientific Research No. 20K03516.
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