A Geometric Proof of Mordell’s Conjecture for Function Fields

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Abstract. Let $C, C'$ be curves over a base scheme $S$ with $g(C) \geq 2$. Then the functor $T \mapsto \{\text{generically smooth } T\text{-morphisms } T \times_S C' \to T \times_S C\}$ from $((S\text{-schemes}))$ to $((\text{sets}))$ is represented by a quasi-finite unramified $S$-scheme. From this one can deduce that for any two integers $g \geq 2$ and $g'$, there is an integer $M(g, g')$ such that for any two curves $C, C'$ over any field $k$ with $g(C) = g$, $g(C') = g'$, there are at most $M(g, g')$ separable $k$-morphisms $C' \to C$. It is conjectured that the arithmetic function $M(g, g')$ is bounded by a linear function of $g'$.

0. Introduction

We recall the works of Y. Manin, H. Grauert and P. Samuel on Mordell’s conjecture for function fields (see [G] and [S]). A main part of the conjecture can be stated as follows.

Theorem 1. Let $C, C'$ be two smooth projective curves over a field $k$, where $C$ had genus $g \geq 2$. Then there are at most a finite number of finite separable $k$-morphisms from $C'$ to $C$.

From this one can deduce that

Theorem 2. Let $C$ be a smooth projective curve over a field $k$ with genus $g \geq 2$. Then for any finitely generated field extension $K \supset k$, $C$ has at most a finite number of smooth $K$-points over $k$.

Here a smooth $K$-point over $k$ means a smooth $k$-morphism $\text{Spec}(K) \to C$, and this is equivalent to a $k$-algebra homomorphism $\phi : K(C) \to K$ (where $K(C)$ is the function field of $C$) such that $K \supset \text{im}(\phi)$ is a separably generated extension.

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Theorem 2 can be restated as

**Theorem 3.** Let $C$ be a smooth projective curve over a field $k$ with genus $g \geq 2$, and $X$ be a variety over $k$. Then there are at most a finite number of generically smooth $k$-morphisms $X \rightarrow C$.

(See also [B], [Hir], [Hr], [N], [Vol], [Voj] for some recent developments along this line.)

In this paper we will give a geometric proof for the above facts. Our main result is

**Theorem 4.** Let $S$ be a noetherian scheme and $C, C'$ be curves over $S$ (i.e. smooth projective morphisms $C \rightarrow S$, $C' \rightarrow S$ of relative dimension 1 with geometrically integral fibers). Suppose the fibers of $C$ over $S$ all have genus $g \geq 2$. Then there is a quasi-finite unramified $S$-scheme $T$ representing the following functor

$$
((S\text{-schemes})) \rightarrow ((\text{sets}))

T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C' \rightarrow T \times_S C\}
$$

where “generically smooth” means there is an open subscheme $U \subset T \times_S C'$, faithfully flat over $T$, such that $f|_U$ is smooth. In particular, there are at most a finite number of generically smooth $S$-morphisms from $C'$ to $C$.

Some main ideas of the proof of Theorem 4 come from [L1] and [L2].

A special case of Theorem 4 is

**Theorem 5.** Let $C, C'$ be smooth projective curves over a field $k$ with genera $g(C) = g \geq 2, g(C') = g'$. Then there is a finite étale $k$-scheme $T$ representing the following functor

$$
((k\text{-schemes})) \rightarrow ((\text{sets}))

T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_k C' \rightarrow T \times_k C\}
$$

This is slightly stronger than Theorem 1.

Using Theorem 4 and moduli theory on curves (cf. [M1] or [M2]), we can get

**Theorem 6.** For any two positive integers $g \geq 2$ and $g'$, there is an integer $M(g, g')$ such that for any two curves $C, C'$ over any field $k$ with $g(C) = g, g(C') = g'$, it holds that

$$
\#\{\text{finite separable } k\text{-morphisms } C' \rightarrow C\} \leq M(g, g').
$$

Furthermore, for any finitely generated field extension $K \supset k$, there is an integer
$M(g, K/k)$ such that for any curve $C$ over $k$ with genus $g$, it holds that

$$\#\{\text{smooth } K\text{-points of } C \text{ over } k\} \leq M(g, K/k).$$

and for any variety $X$ over $k$, there is an integer $M(g, X/k)$ such that for any curve $C$ over $k$ with genus $g$,

$$\#\{\text{generically smooth } k\text{-morphisms } X \to C\} \leq M(g, X/k).$$

This strengthens Theorem 1, Theorem 2 and Theorem 3.

From Theorem 4 and Theorem 6, one can also deduce that

**Theorem 7.** Let $S$ be a noetherian scheme and $C_1, ..., C_m, C'_1, ..., C'_n$ be curves over $S$, such that the fibers of $C_i$ over $S$ all have genus $g_i \geq 2$ ($1 \leq i \leq m$), and the fibers of $C'_j$ over $S$ all have genus $g'_j$ ($1 \leq j \leq n$). Let $X = C_1 \times_S \cdots \times_S C_m$, $Y = C'_1 \times_S \cdots \times_S C'_n$.

i) There is a quasi-finite unramified $S$-scheme $T$ representing the following functor

$$((S\text{-schemes})) \to ((\text{sets})) \quad T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S Y \to T \times_S X\}$$

ii) If $S = \text{Spec}(k)$ for a field $k$, then any generically smooth $k$-morphism $f : Y \to X$ can be factored to a product of morphisms of curves. To be precise, there is an injective map $\lambda : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ and $m$ finite separable morphisms $f_i : C'_\lambda(i) \to C_i$ over $k$, such that $f(x_1, ..., x_n) = (f_1(x_{\lambda(1)}), ..., f_m(x_{\lambda(m)})).$

iii) Let $I$ be the set of all injective maps from $\{1, 2, ..., m\}$ to $\{1, 2, ..., n\}$. If $S = \text{Spec}(k)$ for a field $k$, then

$$\#\{\text{generically smooth } k\text{-morphisms } Y \to X\} \leq \sum_{\lambda \in I} \prod_{i=1}^m M(g_i, g'_\lambda(i)).$$

iv) Suppose $S = \text{Spec}(k)$ for a field $k$. Let $K = K(Y)$ (the function field of $Y$). Then

$$\#\{\text{smooth } K\text{-points of } X \text{ over } k\} \leq \sum_{\lambda \in I} \prod_{i=1}^m M(g_i, g'_\lambda(i)).$$

We conjecture that $M(g, g')$ is bounded by a linear function of $g'$. This holds when $g = g'$.

**Acknowledgement.** I wish to thank Kejian Xu for his stimulating discussions with me on this topic. This paper is related to a recent work of his (see [X]).
1. The main theorem

We first fix some terminologies. Let $S$ be a scheme. By a curve over $S$ we will mean a smooth projective morphisms $\pi : C \to S$ of relative dimension 1 with geometrically integral fibers; in this case if all of the fibers of $\pi$ have the same genus $g$, we say $C$ (or $\pi$) has genus $g$. Let $X, Y$ be schemes over $S$. An $S$-morphism $f : X \to Y$ is called generically smooth if there is an open subscheme $U \subset X$, faithfully flat over $S$, such that $f|_U : U \to Y$ is smooth. If $K$ is a field and $\text{Spec}(K)$ is an $S$-scheme, then an $S$-morphism $f : \text{Spec}(K) \to X$ is called a $K$-point of $X$ over $S$; in this case we say the $K$-point is smooth if $f$ is smooth. If $X$ is a variety over a field $k$, we denote by $K(X)$ the function field of $X$.

Note that when $X$ is a variety over a field $k$ and $K$ is a finitely generated field extension of $k$, a smooth $K$-point of $X$ over $k$ is equivalent to a $k$-algebra homomorphism $\phi : K(X) \to K$ such that $K \supset \text{im}(\phi)$ is a separably generated extension.

Lemma 1. Let $S$ be a noetherian scheme and $X, Y$ be flat projective $S$-schemes. Then there is a locally quasi-projective $S$-scheme $T$ representing the following functor

$$\mathfrak{Mor}_{gs} : ((S \text{-schemes})) \to ((\text{sets}))$$

$$T \mapsto \{\text{generically smooth } T\text{-morphisms } T \times_S Y \to T \times_S X\}$$

Proof. By moduli theory, there is a locally quasi-projective $S$-scheme $T'$ representing the following functor

$$((S \text{-schemes})) \to ((\text{sets}))$$

$$T \mapsto \{T\text{-morphisms } T \times_S Y \to T \times_S X\}$$

Here $T'$ is a locally closed subscheme of the Hilbert scheme $\text{Hilb}_{Y \times_S X/S}$, which represents all of the flat closed subschemes of $Y \times_S X$ over $S$. Let $f : T' \times_S Y \to T' \times_S X$ be the universal morphism over $T'$. Then there is a largest open subscheme $U \subset T' \times_S Y$ such that $f|_U$ is smooth. Let $T$ be the image of $U$ under the projection $\text{pr}_1 : T' \times_S Y \to T'$ (this makes sense because $\text{pr}_1$ is flat, hence is an open map). It is easy to see that $T$ represents $\mathfrak{Mor}_{gs}$. Q.E.D.

Lemma 2. Let $C, C'$ be two curves over a field $k$ with $g(C) = g \geq 2$, $g(C') = g'$. Let $f : C' \to C$ be a separable morphism over $k$. Then $d = \deg(f) \leq g' - 1$. Let $p' \in C'$, $p \in C$ be $k$-points. Let $X = C' \times_k C$, and take the ample invertible sheaf of $X$ to be $O_X(D)$ for the divisor $D = p' \times_k C + C' \times_k p$. Then the graph $\Gamma_f \subset X$ of $f$ has Hilbert polynomial $\chi(x) = (d + 1)x + 1 - g'$. 
Proof. By Hurwitz’s theorem, we have $2g' - 2 \geq d(2g - 2) \geq 2d$, hence $d \leq g' - 1$. Since $\Gamma_f \cong C'$, by Riemann-Roch theorem we have $h^0(D') - h^1(D') = \deg(D') + 1 - g'$ for any divisor $D'$ on $\Gamma_f$. Take $D' = D \cap \Gamma_f$, it is easy to see that $\deg(D') = d + 1$. Hence $\chi(n) = h^0(nD') - h^1(nD') = \deg(nD') + 1 - g' = (d + 1)n + 1 - g'$. This shows that $\chi(x) = (d + 1)x + 1 - g'$. Q.E.D.

**Proposition 1.** Let $S$ be a noetherian scheme and $C, C'$ be curves over $S$ of genera $g \geq 2$ and $g'$ respectively. Let $T$ be the locally quasi-projective $S$-scheme representing the following functor

$$
\begin{align*}
((\text{S-schemes})) &\rightarrow ((\text{sets})) \\
T &\mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C' \rightarrow T \times_S C\}
\end{align*}
$$

(as in Lemma 1). Then $T$ is quasi-finite and unramified over $S$.

**Proof.** First note that $\mathcal{Hilb}_{C' \times_S C/S}$ is a disjoint union of projective $S$-schemes $\mathcal{H}(\chi)$ indexed by (infinitely many) Hilbert polynomials $\chi$. By Lemma 2, we see that $T \subset \mathcal{Hilb}_{C' \times_S C/S}$ is contained in a union of a finite number of $\mathcal{H}(\chi)$s for $\chi(x) = (d + 1)x + 1 - g'$, $1 \leq d < g'$. Hence $T$ is quasi-projective. Therefore it is enough to show that the projection $q : T \rightarrow S$ is unramified.

Denote by $\Phi : T \times_S C' \rightarrow T \times_S C$ the universal morphism over $T$, and $\rho = \text{pr}_2 \circ \Phi : T \times_S C' \rightarrow C$.

Case 1: $S = \text{Spec}(k)$ for an algebraically closed field $k$. Let $t \in T$ be a $k$-point, and let $f : C' \rightarrow C$ be the corresponding separable $k$-morphism. Denote by $\zeta : \{t\} \cong \text{Spec}(k) \rightarrow T$ the inclusion. Let

$$
\alpha = (\rho, \rho \circ ((\zeta \circ \text{pr}_1) \times_k \text{id}_{C'}) : T \times_S C' \rightarrow C \times_k C
$$

(1)
i.e. $\alpha(t', x) = (\rho(t', x), f(x))$ $(\forall t' \in T)$. We have a commutative diagram

$$
\begin{align*}
\xymatrix{C' \cong \text{Spec}(k) \times_k C' \ar[d]_{\zeta \times_k \text{id}_{C'}} & \ar[r]^f & C \ar[d]_{\Delta} \\
T \times_S C' & \ar[l]_{\alpha} C \times_k C
}\end{align*}
$$

(2)

Let $\mathcal{I}, \mathcal{J}$ and $\mathcal{J}_0$ be the ideal sheaves of the closed immersions $\Delta$, $\zeta \times_k \text{id}_{C'}$, and $\zeta$ respectively. Clearly $\mathcal{J} \cong \text{pr}_1^* \mathcal{J}_0$. By (2), $\alpha$ induces a morphism $\alpha^* \mathcal{I} \rightarrow \mathcal{J}$. Applying $(\zeta \times_k \text{id}_{C'})^*$ we get a homomorphism of coherent sheaves on $C'$:

$$
\begin{align*}
\eta : (\zeta \times_k \text{id}_{C'})^*(\alpha^* \mathcal{I}) &\cong f^*(\Delta^* \mathcal{I}) \cong f^*\Omega^1_{C'/k} \\
(\zeta \times_k \text{id}_{C'})^* \mathcal{J} &\cong (\zeta^* \mathcal{J}_0) \otimes_k \mathcal{O}_{C'} \cong \Omega^0_{C'}. \tag{3}
\end{align*}
$$
where \( n = \dim_k(\zeta^*\mathcal{J}_0) \).

We claim that \( \eta = 0 \). Indeed, if \( \eta \neq 0 \), then there would be a non-zero homomorphism \( \eta' : f^*\Omega^1_{C/k} \to \mathcal{O}_{C'} \). Since \( f^*\Omega^1_{C/k} \) is an invertible sheaf, \( \eta' \) would be a monomorphism. Therefore we would have a monomorphism \( H^0(\Omega^1_{C/k}) \hookrightarrow H^0(f^*\Omega^1_{C/k}) \hookrightarrow H^0(\mathcal{O}_{C'}) \cong k \), contrary to \( \dim_k(H^0(\Omega^1_{C/k})) = g \geq 2 \).

Let \( T \subset \mathcal{T} \) be the closed subscheme defined by the ideal sheaf \( \mathcal{J}_0^2 \), and \( V \subset C \times_k \mathcal{C} \) be the closed subscheme defined by the ideal sheaf \( \mathcal{T}^2 \). Then \( \alpha \) induces a morphism \( \alpha_1 : T \to V \). By \( \eta = 0 \) we see that \( \alpha_1^*(\mathcal{I}\mathcal{O}_V) = 0 \), hence \( \alpha_1 \) factors through \( \Delta(\mathcal{C}) \), i.e. \( \rho|_{T \times_k \mathcal{C}} = f \circ \text{pr}_2 : T \times_k \mathcal{C}' \to \mathcal{C} \). By the universality of \( \mathcal{T} \), we have a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{q} & \text{Spec}(k) \\
\downarrow \text{inclusion} & & \downarrow \zeta \\
\mathcal{T} & \xrightarrow{\text{id}} & \mathcal{T}
\end{array}
\]

(4)

where \( q \) is the projection. Hence \( \mathcal{J}_0\mathcal{O}_T = 0 \), i.e. \( \mathcal{J}_0 = \mathcal{J}_0^2 \). This means that

\[
\langle \Omega^1_{\mathcal{T}/k} \rangle_t \cong \zeta^*\mathcal{J}_0 = 0
\]

(5)

where \( \langle \Omega^1_{\mathcal{T}/k} \rangle_t \) is the fiber of \( \Omega^1_{\mathcal{T}/k} \) at \( t \), and \( \zeta^*\mathcal{J}_0 \) can be viewed as \( \mathcal{J}_0/\mathcal{J}_0^2 \) restricted to \( \{t\} \). Since \( t \) is an arbitrary closed point of \( \mathcal{T} \), we have \( \Omega^1_{\mathcal{T}/k} = 0 \), i.e. \( \mathcal{T} \) is unramified over \( k \).

Case 2: \( S = \text{Spec}(k) \) for an arbitrary field \( k \). Let \( \overline{k} \) be the algebraic closure of \( k \), and denote \( \overline{C} = C \otimes_k \overline{k}, \overline{C}' = C' \otimes_k \overline{k} \). Then \( \mathcal{T} \otimes_k \overline{k} \) represents

\[
((\overline{k}\text{-schemes})) \to (\text{(sets)})
\]

\[
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_k \overline{C}' \to T \times_k \overline{C}\}
\]

Hence \( \mathcal{T} \otimes_k \overline{k} \) is unramified over \( \overline{k} \) by Case 1. This shows that \( \mathcal{T} \) is unramified over \( k \).

Case 3: general case. We need to show that \( \Omega^1_{\mathcal{T}/S} = 0 \), for this it is enough to show \( \langle \Omega^1_{\mathcal{T}/S} \rangle_t = 0 \) for any closed point \( t \in \mathcal{T} \). Let \( s = q(t) \in S \), and let \( k \) be the residue field at \( s \) (i.e. \( \{s\} \) can be viewed as a morphism \( \text{Spec}(k) \to S \)). Denote by \( \mathcal{C}_s, \mathcal{C}'_s \) and \( \mathcal{T}_s \) the fibers of \( \mathcal{C}, \mathcal{C}' \) and \( \mathcal{T} \) over \( s \) respectively. Then \( \mathcal{T}_s \) represents

\[
((k\text{-schemes})) \to (\text{(sets)})
\]

\[
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_k \mathcal{C}'_s \to T \times_k \mathcal{C}_s\}
\]

Hence \( \mathcal{T}_s \) is unramified over \( k \) by Case 2, i.e. \( \Omega^1_{\mathcal{T}_s/k} = 0 \). Note that \( \Omega^1_{\mathcal{T}_s/k} \cong \Omega^1_{\mathcal{T}/S}|_{\mathcal{T}_s} \), we have \( \Omega^1_{\mathcal{T}/S} = 0 \). Q.E.D.
Lemma 3. Let \( \pi : X \to S \) be an unramified separated morphism of noetherian schemes. Then \( \pi \) has at most a finite number of sections \( \zeta : S \to X \).

Proof. Let \( \zeta : S \to X \) be a section of \( \pi \). It is easy to see the following diagram is cartesian:

\[
\begin{array}{ccc}
S & \xrightarrow{\zeta} & X \\
\downarrow{\zeta} & & \downarrow{\Delta} \\
X & \xrightarrow{\beta} & X \times_S X
\end{array}
\]

where \( \beta = (\zeta \circ \pi, \text{id}_X) \) (i.e. \( \beta(x) = (\zeta(\pi(x)), x) \)). Hence \( \zeta \) is a closed immersion because \( \Delta \) is a closed immersion. Let \( J \) be the ideal sheaf of \( \zeta(S) \subset X \), and \( I \) be the ideal sheaf of \( \Delta(X) \subset X \times_S X \). Then \( \beta^* I \cong J \) because (6) is cartesian. Since \( \pi \) is unramified, we have \( I = I^2 \). Hence \( J = J^2 \). This shows that \( X \) is a disjoint union of \( \zeta(S) \) with another closed subscheme, hence each connected component of \( S \) maps isomorphically to a connected component of \( X \) under \( \zeta \). Let \( \mathcal{X} \) be the set of connected components of \( X \) and \( \mathcal{G} \) be the set of connected components of \( S \) (both are finite sets). Then any \( \zeta \) is uniquely determined by an injective map from \( \mathcal{X} \) to \( \mathcal{G} \). Hence \( \pi \) has at most a finite number of sections. Q.E.D.

Theorem 1. Let \( S \) be a noetherian scheme and \( C, C' \) be curves over \( S \), where \( C \to S \) has genus \( g \geq 2 \). Then

i) There is a quasi-finite unramified \( S \)-scheme \( T \) representing the following functor

\[
((S\text{-schemes})) \to (\text{(sets)})
\]

\[
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C' \to T \times_S C\}
\]

ii) There are at most a finite number of generically smooth \( S \)-morphisms from \( C' \) to \( C \).

iii) In particular, if \( S = \text{Spec}(k) \) for a field \( k \), then there is a finite étale \( k \)-scheme \( \mathcal{T} \) representing the following functor

\[
((k\text{-schemes})) \to (\text{(sets)})
\]

\[
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_k C' \to T \times_k C\}
\]

and there are at most a finite number of finite separable \( k \)-morphisms from \( C' \) to \( C \).

Proof. i) For any connected component \( U \subset S \), the fibers of \( C' \) over \( U \) all have same genus. Let \( S_1, \ldots, S_n \) be the connected components of \( S \), \( C_i = C \times_S S_i \),
\[
C_i' = C' \times_S S_i \quad (1 \leq i \leq n).
\]
By Proposition 1, there is a quasi-finite unramified \(S_i\)-scheme \(T_i\) representing the following functor

\[
((S_i\text{-schemes})) \to ((\text{sets}))
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C_i' \to T \times_S C_i\}
\]

(1 \(\leq i \leq n\)). Clearly \(T = \biguplus_{i=1}^n T_i\) represents

\[
\{\text{S-schemes}\} \to ((\text{sets}))
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C' \to T \times_S C\}
\]

ii) Let \(q : T \to S\) be the projection, which is quasi-finite, hence separated. By i), a generically smooth \(S\)-morphism \(C' \to C\) is equivalent to a section \(\zeta : S \to T\) of \(q\). By Lemma 3, \(q\) has at most a finite number of sections, hence there are at most a finite number of generically smooth \(S\)-morphisms \(C' \to C\).

iii) When \(S = \text{Spec}(k)\) for a field \(k\), a \(k\)-scheme is quasi-finite unramified iff it is finite étale, and a \(k\)-morphism \(C' \to C\) is generically smooth iff it is finite separable. Hence the statements hold by i) and ii). Q.E.D.

This gives Theorem 0.4, Theorem 0.5 and Theorem 0.1.

**Corollary 1.** Let \(C\) be a curve of genus \(g \geq 2\) over a field \(k\).

i) For any finitely generated field extension \(K \supseteq k\), there are at most a finite number of \(k\)-algebra homomorphisms \(\phi : K(C) \to K\) such that \(K \supseteq \text{im}(\phi)\) is a separably generated extension. In other words, \(C\) has at most a finite number of smooth \(K\)-points over \(k\).

ii) For any variety \(X\) over \(k\), there are at most a finite number of generically smooth \(k\)-morphisms from \(X\) to \(C\).

**Proof.** i) Note that there is a one to one correspondence

\[
\{\text{smooth } K\text{-points of } C\text{ over } k\} \leftrightarrow \{\text{separably generated } k\text{-extensions } K(C) \hookrightarrow K\}
\]

Let \(k'\) be the algebraic closure of \(k\) in \(K\) (i.e. \(k' = \{a \in K | a\text{ is algebraic over } k\}\)), then \(k' \supseteq k\) is a finite extension. Let \(C' = C \otimes_k k'\), then \(K(C') \cong K(C)[k']\). Let \(m = |\text{Gal}(k'/k)|\), then every field extension \(K(C) \hookrightarrow K\) over \(k\) induces \(m\) field extensions \(K(C') \hookrightarrow K\). Therefore it is enough to show that there are at most a finite number of separably generated \(k'\)-extensions \(K(C') \hookrightarrow K\). Thus we may assume \(k = k'\), i.e. \(k\) is algebraically closed in \(K\).

Let \(n = \text{tr.deg}(K/k)\). Then we can take \(n\) subfields \(L_1, ..., L_n \subset K\) containing \(k\) with \(\text{tr.deg}(L_i/k) = n - 1 \quad (1 \leq i \leq n)\), such that for each \(i\), \(L_i\) is algebraically
closed in $K$, $L_i \subset K$ is separably generated, and $\bigcap_{i=1}^{n} L_i = k$. Let $C_i = C \otimes_k L_i$ 
$(1 \leq i \leq n)$. Then $g(C_i) = g(C) = g \geq 2$. Since $\text{tr.deg}(K/L_i) = 1$, there is an $L_i$-curve $C_i'$ such that $K(C_i') \cong K$.

For any $k$-homomorphism $\phi : K(C) \to K$ such that $K$ is separably generated over $\text{im}(\phi)$, there is at least one $i$ such that $\text{im}(\phi) \not\subset L_i$ (hence $\text{im}(\phi) \cap L_i = k$), and $K(C) \otimes_k L_i \to K$ is smooth, hence induces an $L_i$-homomorphism $\phi_i : K(C_i) \to K$ such that $K \supset \text{im}(\phi_i)$ is a separable extension, which is equivalent to a finite separable $L_i$-morphism $C_i' \to C_i$. By Theorem 1.iii), there are at most a finite number of finite separable $L_i$-morphisms from $C_i'$ to $C_i$. Note that $\phi_i$ uniquely determines $\phi$, hence there is a monomorphism

\[
\{\text{smooth } K\text{-points of } C \text{ over } k\} \hookrightarrow \bigcup_{i=1}^{n} \{\text{finite separable } L_i\text{-morphisms } C_i' \to C_i\}
\]

This shows $\{\text{smooth } K\text{-points of } C \text{ over } k\}$ is a finite set.

ii) Let $K = K(X)$, then any generically smooth $k$-morphism from $f : X \to C$ gives a smooth $K$-point $f' : \text{Spec}(K) \to C$, and $f$ is uniquely determined by $f'$. This gives a monomorphism

\[
\{\text{generically smooth } k\text{-morphisms } X \to C\} \hookrightarrow \{\text{smooth } K\text{-points of } C \text{ over } k\}
\]

Hence $\#\{\text{generically smooth } k\text{-morphisms } X \to C\} < \infty$. Q.E.D.

This gives Theorem 0.2 and Theorem 0.3.

### 2. Some consequences

In this section we will prove Theorem 0.6 and Theorem 0.7. First we generalize Theorem 1.1.

**Theorem 1.** Let $S$ be a noetherian scheme and $C_1, \ldots, C_m, C_1', \ldots, C_n'$ be curves over $S$, such that for each $i$ $(1 \leq i \leq m)$, $C_i \to S$ has genus $g_i \geq 2$. Let $T_{i,j}$ $(1 \leq i \leq m, 1 \leq j \leq n)$ be the $S$-scheme representing the following functor

\[
((S\text{-schemes})) \to ((\text{sets}))
\]

\[
T \mapsto \{\text{generically smooth } T\text{-morphisms } f : T \times_S C_j' \to T \times_S C_i\}
\]

as in Theorem 1.1, and let $f_{i,j} : T_{i,j} \times_S C_j' \to T_{i,j} \times_S C_i$ be the universal morphism. Let $X = C_1 \times_S \cdots \times_S C_m$ and $Y = C_1' \times_S \cdots \times_S C_n'$. Then there is a quasi-finite unramified $S$-scheme $T$ representing the following functor

\[
\text{Mor}_{gs} : ((S\text{-schemes})) \to ((\text{sets}))
\]
$T \mapsto \{\text{generically smooth } T\text{-morphisms } T \times_S Y \to T \times_S X\}$

Furthermore,

$$\mathcal{T} \cong \prod_{\lambda \in I} \mathcal{T}_{1,\lambda(1)} \times_S \cdots \times_S \mathcal{T}_{m,\lambda(m)}$$  \hspace{1cm} (1)

where $I$ is the set of all injective maps from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$, and the universal morphism over $\mathcal{T}_\lambda = \mathcal{T}_{1,\lambda(1)} \times_S \cdots \times_S \mathcal{T}_{m,\lambda(m)}$ is

$$f_\lambda = (f_{1,\lambda(1)} \times_S \cdots \times_S f_{m,\lambda(m)}) \circ (\text{id}_{\mathcal{T}_\lambda} \times_S q_\lambda) : \mathcal{T}_\lambda \times_S Y \to \mathcal{T}_\lambda \times_S X$$  \hspace{1cm} (2)

where $q_\lambda = \text{pr}_{\lambda(1)} \times_S \cdots \times_S \text{pr}_{\lambda(m)} : Y \to C_{\lambda(1)}' \times_S \cdots \times_S C_{\lambda(m)}'$ (i.e. $(x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(m)})$). In particular, if $S$ is connected, then for any generically smooth $S$-morphism $f : Y \to X$, there is a $\lambda \in I$ and generically smooth $S$-morphisms $\phi_i : C_{\lambda(i)}' \to C_i$ ($1 \leq i \leq m$) such that $f((x_1, \ldots, x_n) = (\phi_1(x_{\lambda(1)}), \ldots, \phi_m(x_{\lambda(m)}))$.

**Proof.** By Lemma 1.1, there is a locally quasi-projective $S$-scheme $\mathcal{T}$ representing $\mathfrak{Mor}_{gs}$. Let $f : \mathcal{T} \times_S Y \to \mathcal{T} \times_S X$ be the universal $\mathcal{T}$-morphism. We now show that the projection $q : \mathcal{T} \to S$ is quasi-finite and unramified.

Case 1: $m = 1$. For each $j$ ($1 \leq j \leq n$), denote by

$$Y_j = C_1' \times_S \cdots \times_S C_{j-1}' \times_S C_{j+1}' \times_S \cdots \times_S C_n'$$  \hspace{1cm} (3)

and $p_j : Y \to Y_j$ the projection. For each $j$ ($1 \leq j \leq n$), $f$ is equivalent to the morphism $\phi_j = p_j \times_S f : \mathcal{T} \times_S Y \to \mathcal{T} \times_S Y_j \times_S X$ over $\mathcal{T} \times_S Y_j$. Let $U_j \subset \mathcal{T}$ be the largest open subscheme over which $\phi_j$ is generically smooth. Then it is easy to see that $\phi_j$ is generically smooth over $U_j$. Hence there is an induced $S$-morphism $q_j : U_j \times_S Y_j \to \mathcal{T}_{1,j}$. Since $\mathcal{T}_{1,j}$ is quasi-finite over $S$, $q_j$ factors through $U_j$. In other words, $f|_{U_j \times_S Y}$ is equal to the composition of the projection $\text{pr}_j : U_j \times_S Y \to U_j \times_S C_j'$ and the pull-back of $f_{1,j}$ via an $S$-morphism $h_j : U_j \to \mathcal{T}_j$. By the universality of $\mathcal{T}$, we see $h_j$ is an isomorphism. Furthermore, for any point $t \in \mathcal{T}$, at least one $\phi_j$ is generically smooth over $t$, hence $\mathcal{T} = \bigcup_{j=1}^n U_j$. The above argument also shows that the $U_j$'s are disjoint to each other. Therefore we have

$$\mathcal{T} \cong \prod_{j=1}^n \mathcal{T}_{1,j}$$  \hspace{1cm} (4)

over $S$, and the universal morphism over $\mathcal{T}_{1,j}$ is

$$f_{1,j} \circ \text{pr}_j : \mathcal{T}_{1,j} \times_S Y \to \mathcal{T}_{1,j} \times_S X$$  \hspace{1cm} (5)
Case 2: general case. Let \( T \) be a connected \( S \)-scheme. Then a \( T \)-morphism \( \phi : T \times_S Y \to T \times_S X \) is equivalent to \( m \) \( T \)-morphisms \( \phi_i = \text{pr}_i \circ \phi : T \times_S Y \to T \times_S C_i \). If \( \phi_i \) is generically smooth, then by Case 1, there is a unique \( j \) such that \( \phi_i = \psi_{ij} \circ \text{pr}_j \), where \( \psi_{ij} : T \times_S C'_j \to T \times_S C_i \) is the pull-back of \( f_{ij} \) via an \( S \)-morphism \( T \to T_{ij} \). Denote \( \lambda(i) = j \). It is easy to see that \( \phi \) is generically smooth iff every \( \phi_i \) is generically smooth and \( \lambda(i) \neq \lambda(i') \) for any \( i \neq i' \). Thus \( \lambda \in I \), and \( \phi \) is equal to the pull-back of \( f_\lambda \) via a unique \( S \)-morphism \( T \to T_\lambda \).

From this we see that \( T \) is isomorphic to a disjoint union of all \( T_\lambda \)'s, hence is quasi-finite and unramified over \( S \). Q.E.D.

In particular, in the case when \( S = \text{Spec}(k) \) for a field \( k \), we have

**Corollary 1.** Let \( C_1, ..., C_m, C'_1, ..., C'_n \) be curves over a field \( k \), with \( g(C_i) = g_i \geq 2 \) (\( 1 \leq i \leq m \)), \( g(C'_j) = g'_j \) (\( 1 \leq j \leq n \)). Let \( X = C_1 \times_k \cdots \times_k C_m \), \( Y = C'_1 \times_k \cdots \times_k C'_n \). Then for any generically smooth \( k \)-morphism \( f : Y \to X \), there is an injective map \( \lambda : \{1, 2, ..., m\} \to \{1, 2, ..., n\} \) and \( m \) finite separable morphisms \( f_i : C'_{\lambda(i)} \to C_i \) over \( k \), such that \( f(x_1, ..., x_n) = (f_1(x_{\lambda(1)}), ..., f_m(x_{\lambda(m)})) \).

**Lemma 1.** Let \( \pi : X \to S \) be a quasi-finite morphism of noetherian schemes. Then there is an integer \( M \) such that for any point \( s \in S \), the fiber \( X_s \) has degree \( \leq M \) over \( s \).

**Proof.** Since we are only concerned with the fiber degrees, we can assume \( S \) is reduced.

We use noetherian induction on \( S \), when \( X = \emptyset \) there is nothing to prove.

Suppose \( X \neq \emptyset \). Take a generic point \( \xi \in X \) such that \( \zeta = \pi(\xi) \) is not a specialization of any \( \pi(x) \) \( (x \in X) \), hence any point of \( \pi^{-1}(\zeta) \) is a generic point of \( X \) (because \( \pi \) is quasi-finite). Take an open neighborhood \( U' \subset S \) of \( \zeta \) such that the generic points of \( \pi^{-1}(U') \) are all in \( \pi^{-1}(\zeta) \). Let \( V \subset S \) be the closure of \( \{\zeta\} \), with reduced induced scheme structure. Since \( \pi \) is quasi-finite, we can take an irreducible open neighborhood \( U \subset V \cap U' \) of \( \zeta \) such that \( \pi^{-1}(U) \to U \) is finite. Furthermore, noting that \( \pi^{-1}(U) \to U \) is generically flat, we can take \( U \) such that \( \pi^{-1}(U) \to U \) is flat, hence the fibers of \( \pi^{-1}(U) \to U \) all have degree \( d = \deg(\pi^{-1}(\zeta)/\zeta) \).

Take an open subset \( U_1 \subset U' \) such that \( U = U_1 \cap V \). Note that \( \pi^{-1}(U) = \pi^{-1}(U_1) \). Let \( S' = S - U_1 \) with reduced induced scheme structure, and let \( X' = X \times_S S' \). By noetherian induction, there is an integer \( M \) such that for any \( x \in X' \), the fiber degree \( \deg(X_s/s) \leq M \). Hence for any \( x \in X \), we have \( \deg(X_s/s) \leq \max(M, d) \). Q.E.D.
For any $g \geq 0$, there is a “catalog space of curves of genus $g$”, which is a quasi-projective scheme $S_g$ over $\mathbb{Z}$ together with a curve $C_g$ over $S_g$ such that for any curve $C$ of genus $g$ over any field $k$, there is at least one $k$-point $\text{Spec}(k) \to S_g$ over which the fiber of $C_g$ is isomorphic to $C$. (We have many choices of $S_g$, and we don’t use the moduli space $M_g$ of curves of genus $g$ because $M_g$ is not a fine moduli space, i.e. there is no universal curve over $M_g$.) For any $g \geq 2$ and $g' \geq 0$, denote by $S_{g,g'} = S_g \times S_{g'}$. Then over $S = S_{g,g'}$ there are two curves $C = C_g \times S_{g'}$ and $C' = S_g \times C_{g'}$, of genera $g$ and $g'$ respectively. By Theorem 1.1, there is a quasi-finite unramified $S$-scheme $T = T_{g,g'}$ representing

$$\{S\text{-schemes}\} \rightarrow ((\text{sets}))$$

$$T \mapsto \{\text{generically smooth } T \times_S C' \to T \times_S C\}$$

By Lemma 1, there is an integer $M$ such that for any $s \in S$, the fiber $T_s$ has degree $\leq M$ over $s$. For any field $k$ and any two curves $C, C'$ over $k$ with $g(C) = g$, $g(C') = g'$, there is a $k$-point $s : \text{Spec}(k) \to S$ such that the fibers $T_s \cong C, T'_s \cong C'$ over $k$. By $\deg(T_s/s) \leq M$, we see there are at most $M$ generically smooth (i.e. finite separable) $k$-morphisms from $C'$ to $C$. Denote by $M(g,g') = M$, we get

**Proposition 1.** For any two curves $C, C'$ over any field $k$ with $g(C) = g, g(C') = g'$, we have

$$\#\{\text{finite separable } k\text{-morphisms } C' \to C\} \leq M(g,g').$$

**Remark 1.** We can take $M(g,g')$ to be the smallest integer such that Proposition 1 holds. In this way we define an integer-valued function of two integer variables $g \geq 2$ and $g'$. By Hurwitz’s Theorem, it is easy to see that $M(g,g') = 0$ when $g' < g$. For the bound of $M(g,g')$, we have the following conjecture.

**Conjecture.** There are constants $a, b \in \mathbb{R}$ such that $M(g,g') \leq ag' + b$ for any $g \geq 2$ and any $g'$.

The following example gives an evidence of the conjecture.

**Example 1.** Let $C, C'$ be curves over a field $k$ with $g(C) = g(C') = g \geq 2$. Then by Hurwitz’s theorem, any finite separable $k$-morphism $C' \to C$ is an isomorphism. Hence $\#\{\text{finite separable } k\text{-morphisms } C' \to C\} \leq |\text{Aut}(C/k)|$. It is well-known that $|\text{Aut}(C/k)| \leq ag + b$ for some constants $a, b$ (this can be shown using Hurwitz’s theorem). In other words, the conjecture holds when $g = g'$.

**Corollary 2.** Let $K \supset k$ be a finitely generated field extension, and $g \geq 2$ be an integer. Then there is an integer $M(g,K/k)$ such that for any curve $C$ over $k$ with
genus $g$,

$$\#\{\text{smooth } K\text{-points of } C \text{ over } k\} \leq M(g, K/k).$$

Therefore for any variety $X$ over $k$, there is an integer $M(g, X/k)$ such that for any curve $C$ over $k$ with genus $g$,

$$\#\{\text{generically smooth } k\text{-morphisms } X \to C\} \leq M(g, X/k).$$

**Proof.** For simplicity we may assume $k$ is algebraically closed. Let $n = \text{tr.deg}(K/k)$.

Look at the proof of Corollary 1.1.i), there can be found $n$ subfields $L_1, \ldots, L_n \subset K$ containing $k$ with $\text{tr.deg}(L_i/k) = n − 1$ ($1 \leq i \leq n$), such that for each $i$, $L_i$ is algebraically closed in $K$, $L_i \subset K$ is separably generated, and $\bigcap_{i=1}^{n} L_i = k$. For each $i$, there is an $L_i$-curve $C_i'$ such that $K(C_i') \cong K$. Let $g_i' = g(C_i')$. For any $k$-curve $C$ of genus $g$, a smooth $K$-point of $C$ over $k$ is equivalent to a finite separable $L_i$-morphism $C_i' \to C \otimes_k L_i$ for some $i$. Hence

$$\#\{\text{smooth } K\text{-points of } C \text{ over } k\} \leq \sum_{i=1}^{n} M(g, g_i').$$

We can take $M(g, K/k) = \sum_{i=1}^{n} M(g, g_i')$.

The last statement can be easily deduced by the first one, as in the proof of Corollary 1.1.ii). Q.E.D.

Proposition 1 and Corollary 2 together give Theorem 0.6.

By Corollary 1 and Proposition 1 we get Theorem 2.

**Theorem 2.** Let $C_1, \ldots, C_m$, $C_1', \ldots, C_n'$ be curves over a field $k$, with $g(C_i) = g_i \geq 2$ ($1 \leq i \leq m$), $g(C_j') = g_j'$ ($1 \leq j \leq n$). Let $X = C_1 \times_k \cdots \times_k C_m$, $Y = C_1' \times_k \cdots \times_k C_n'$. Then

$$\#\{\text{generically smooth } k\text{-morphisms } Y \to X\} \leq \sum_{\lambda \in I} \prod_{i=1}^{m} M(g_i, g_{\lambda(i)}').$$

where $I$ is the set of all injective maps from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$.

**Corollary 3.** Notation as in Theorem 2. For the field $K = K(Y)$ we have

$$\#\{\text{smooth } K\text{-points of } X \text{ over } k\} \leq \sum_{\lambda \in I} \prod_{i=1}^{m} M(g_i, g_{\lambda(i)}').$$

**Proof.** It is enough to show that for each $i$ ($1 \leq i \leq m$), a smooth $k$-morphism $\text{Spec}(K) \to C_i$ is equivalent to a generically smooth $k$-morphism $Y \to C_i$. For
each \( j \) \((1 \leq j \leq n)\), let

\[
Y_j = C'_1 \times_k \cdots \times_k C'_{j-1} \times_k C'_{j+1} \times_k \cdots \times_k C'_n
\]

and let \( K_j = K(Y_j) \), viewed as a subfield of \( K \). For any smooth \( k \)-morphism \( \phi : \text{Spec}(K) \to C_i \), there is a \( j \) such that \( \phi \) is equivalent to a generically smooth \( K_j \)-morphism \( f : C'_j \otimes_k K_i \to C_i \otimes_k K_i \). This is then equivalent to a \( k \)-morphism \( \zeta : \text{Spec}(K_i) \to T_{i,j} \) (notation in Theorem 1). Note that the image of \( \zeta \) is a \( k \)-point, because \( Y_i \) has geometrically integral fibers. This shows that \( f = f_0 \otimes_k K_i \) for a finite separable \( k \)-morphism \( f_0 : C'_j \to C_i \), hence is equivalent to \( f_0 \circ \text{pr}_j : Y \to C_i \).

Q.E.D.

Theorem 1, Corollary 1, Theorem 2 and Corollary 3 together give Theorem 0.7.

Example 2. Let \( C \) be a curve of genus \( g \geq 2 \) over a field \( k \), and let \( X = C \times_k C \). Denote by \( \iota : X \to X \) the morphism by exchanging factors (i.e. \( \iota(x, y) = (y, x) \)). By Theorem 1 and Example 1, we see that any finite separable \( k \)-morphism \( X \to X \) is an isomorphism, and is either equal to \( \sigma \times_k \tau \) for some \( \sigma, \tau \in \text{Aut}(C/k) \), or equal to \( (\sigma \times_k \tau) \circ \iota \) for some \( \sigma, \tau \in \text{Aut}(C/k) \). Hence \( \text{Aut}(X/k) \cong (\mathbb{Z}/2\mathbb{Z}) \rtimes (\text{Aut}(C/k) \times \text{Aut}(C/k)) \). Therefore \( |\text{Aut}(X/k)| = 2|\text{Aut}(C/k)|^2 \leq 2M(g, g)^2 \).

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