Penrose limits and Green-Schwarz strings

Shun’ya Mizoguchi†

High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan

Takeshi Mogami‡ and Yuji Satoh§

Institute of Physics, University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan

Abstract

We discuss the Green-Schwarz action for type IIB strings in general plane-wave backgrounds obtained as Penrose limits from any IIB supergravity solutions with vanishing background fermions. Using the normal-coordinate expansion in superspace, we prove that the light-cone action is necessarily quadratic in the fermionic coordinates. This proof is valid for more general pp-wave backgrounds under certain conditions. We also write down the complete quadratic action for general bosonic on-shell backgrounds in a form in which its geometrical meaning is manifest both in the Einstein and string frames. When the dilaton and 1-form field strength are vanishing, and the other field strengths are constant, our string-frame action reduces, up to conventions, to the one which has been written down using the supercovariant derivative.
1 Introduction

In general relativity, any solution leads to a plane-wave solution in the Penrose limit [1]. This is also generalized to any supergravity theories in ten and eleven dimensions [2]. The maximally supersymmetric solution supported by a constant RR flux [3] is a particular example in IIB supergravity, which is realized as a Penrose limit of $AdS_5 \times S_5$ [4]. Remarkably, the string theory on this background is exactly solvable in the Green-Schwarz formalism [5]. Moreover, it provides a model for analyzing the AdS/CFT correspondence in the stringy regime [6]. The string theory on the plane wave thus opened up a way to study important questions in string theory such as strings in curved spacetime, strings in non-trivial RR backgrounds and stringy aspects of the AdS/CFT correspondence.

After the works of [3, 5, 6], string models on other plane-wave backgrounds have also been studied. For a partial list, see [7]-[12]. In analyzing those models, it is important to know whether or not the Green-Schwarz string action in light-cone gauge is quadratic in the fermionic coordinates. In the original case [5], this was directly shown by the supercoset method [13], which uses symmetry superalgebras associated with backgrounds. When the plane wave is obtained as a Penrose limit (and dimensional reduction) from $AdS \times M$ spaces in ten or eleven dimensions, this special property is also shown [14, 7, 11] by using the Green-Schwarz string or membrane actions for the corresponding $AdS \times M$ spaces [13]-[17]. However, there is no proof for general plane-wave backgrounds.

For general bosonic curved backgrounds which satisfy supergravity equations of motion, the Green-Schwarz actions are known up to quadratic order in the fermionic coordinates: In the type IIA and IIB cases, they were derived in [18] by starting from the corresponding membrane action in eleven dimensions. In the heterotic case, it was also derived in [19] by starting from the $\kappa$-symmetric action written in superfields [20, 21] and using the ‘$\theta$-expansion’ [19, 22], which is a generalization of the ordinary normal-coordinate expansion for bosonic non-linear $\sigma$-models. The $\theta$-expansion was also applied to the IIB case in [23, 24]. For constant axion, 3- and 5-form field strengths and vanishing dilaton, the IIB action was written down in a concise form in terms of the supercovariant derivative [14]. Once a Green-Schwarz action in a background is assured to terminate at quadratic order in fermions, one can use those results. What is a little worrying here is that there are issues on signs, factors and normalizations [9]. Thus, it would be of some use to provide an explicit derivation of the quadratic actions in an independent manner, in which their geometrical meaning also becomes clear.

In this paper, we discuss the Green-Schwarz action for type IIB strings in general plane-wave backgrounds obtained as Penrose limits of any IIB supergravity solutions with vanishing background fermions. (These backgrounds always possess Killing spinors, at least locally.) We start from the $\kappa$-symmetric action for general on-shell supergravity backgrounds written in terms of superfields [25], and use the $\theta$-expansion. This method has an advantage that it
can be applied to any backgrounds. On the other hand, it also has a disadvantage that, in general, it is difficult to carry out the expansion and express the result in terms of physical fields to higher orders. Because of this, the known complete IIB Green-Schwarz actions in non-trivial backgrounds have been derived by the supercoset method, which relies on high symmetries of backgrounds [5], [13]-[16], or by taking the Penrose limit for these known actions [6, 14, 7] (see, also [10, 12]). In spite of that, we show that the $\theta$-expansion is effective enough, for the plane-wave solutions we consider, to prove that the Green-Schwarz action in light-cone gauge is quadratic in the fermionic coordinates. The proof is also valid under certain conditions for more general pp-wave backgrounds, of which the plane-wave solutions we consider are a subclass. By using this systematic expansion method, we also write down the complete quadratic action for general bosonic on-shell backgrounds in a form in which its geometrical meaning becomes clear both in the Einstein and string frames. We confirm that the quadratic action in the string frame is expressed by the supercovariant derivative. We note that any of the bosonic background fields do not have to be constant in our discussion. When the dilaton and 1-form field strength are vanishing, and the other field strengths are constant, we find an agreement with the action in [14] after taking into account the differences of conventions.

The organization of this paper is as follows. In section 2, we give a brief summary about the Penrose limit, the superspace formulation of IIB supergravity, the IIB Green-Schwarz action for general on-shell backgrounds and the $\theta$-expansion. In section 3, we prove that the light-cone action which we consider is quadratic in the fermionic coordinates. In section 4, we write down the quadratic action both in the Einstein and string frames. We conclude with a discussion in section 5. Our notation and conventions are summarized in Appendix A. Some components of the superfields, which are needed in our discussion, are listed in Appendix B.

Note added

After this work was essentially completed, a preprint [26] appeared in which it is proved that the IIB Green-Schwarz action is quadratic in the fermionic coordinates for a general class of pp-wave solutions which have non-constant 3- and 5-form field strengths, and vanishing dilaton, 1-form field strength and background fermions. The method, however, is different: In [26], the general possible terms in the Green-Schwarz action are listed by examining their tensor structure, while in our $\theta$-expansion each term is automatically obtained explicitly with precise coefficients. The $\theta$-expansion method can also be applied to more general backgrounds other than pp-waves, to show that the corresponding Green-Schwarz action terminates at some order, and obtain its explicit form. In addition, a revised version of [18] appeared in the electronic archive, in which typos are corrected and explanations of their conventions are added.
2 Green-Schwarz action in general curved backgrounds

2.1 Penrose limits

Plane-wave solutions obtained from any other solutions in the Penrose limit have a universal form. In the IIB case, which we are interested in, the bosonic fields in the harmonic coordinate system are given by \[ ds^2 = dx^+ dx^- - h_{\tilde{m} \tilde{n}}(x^+) x^{\tilde{m}} x^{\tilde{n}} (dx^+)^2 - \delta_{\tilde{m} \tilde{n}} dx^{\tilde{m}} dx^{\tilde{n}}, \]
\[ \tau = \tau(x^+), \]
\[ F_p = \frac{1}{(p-1)!} \tilde{F}_{+\tilde{m}_1...\tilde{m}_{p-1}}(x^+) dx^+ \wedge dx^{\tilde{m}_1} \wedge ... \wedge dx^{\tilde{m}_{p-1}}. \]

Here, \( \tilde{m}, \tilde{n} = 1, 2, ..., 8 \), \( x^\pm = x^0 \pm x^9 \) and \( p = 3, 5 \). We reserve un-tilted \( m, n \) to denote all spacetime directions; \( m, n = 0, 1, ..., 9 \). \( h_{\tilde{m} \tilde{n}} \) are some functions, \( \tau \) is the complex scalar, and \( \tilde{F}_3 \) and \( \tilde{F}_5 \) are the complex 3-form and self-dual 5-form field strengths, respectively. The corresponding \( (p-1) \)-form potentials \( A_{p-1} \) can have non-trivial components \( A_{\tilde{m}_1...\tilde{m}_{p-1}} \) and \( A_{+\tilde{m}_1...\tilde{m}_{p-2}} \). The non-zero components of the Riemann tensor \( R_{klnm} \) and the Ricci tensor \( R'_{mn} \) are only \( R_{+\tilde{m}+\tilde{n}} \) and \( R'_{++} \), up to symmetry of the tensors. (We put the prime for the Ricci tensor to avoid confusion with the curvature 2-form appearing later.) In addition, the non-vanishing Christoffel symbols \( \Gamma^l_{mn} \) are \( \Gamma^\tilde{m}_{++}, \Gamma^{\tilde{m}+\tilde{n}}, \Gamma_{++}^{\tilde{n}} \). Since all the above fields depend only on \( x^+ \), the covariant derivatives of the tensors, e.g., \( \nabla_k R'_{mn} \), are non-vanishing only for \( \nabla_+ \).

Similar properties hold also for tensors with tangent-space indices. This is confirmed by noting that the spacetime indices are converted to the tangent-space indices through the vielbein, which are read off from
\[ e^+ = dx^+ , \quad e^- = dx^- - h_{\tilde{m} \tilde{n}} x^{\tilde{m}} x^{\tilde{n}} dx^+ , \quad e^\tilde{a} = dx^{\tilde{m}} \delta^\tilde{a}_{\tilde{m}} . \]

Here, we denote the tangent-space indices by \( a, b = 0, 1, ..., 9 \), and \( \tilde{a}, \tilde{b} = 1, ..., 8 \). The non-vanishing components of the spin connection \( \omega^b_{m, a} \) are only \( \omega^\tilde{a}_{+, \tilde{a}} \). We also denote the covariant derivative for tensors with the tangent-space indices by \( D_a \).

From these observations, we find that

\((I)\) The non-vanishing components of the tensors (tensor densities) have neither lower ‘\(-\)’ indices nor upper ‘\(+\)’ indices, except for \( g_{mn}, \epsilon_{m_1,...,m_{10}}, \eta_{ab}, \epsilon_{a_1,...,a_{10}} \).

\((II)\) The tensors have lower ‘\(+\)’ indices at least as many as their mass dimension when all the indices are lowered.

\((III)\) The number of the lower ‘\(+\)’ plus upper ‘\(-\)’ is preserved when the indices are raised, lowered, or contracted with indices of other tensors.
Here, for example, $R_{klmn}$ has dimension two, and $\tilde{F}_\mu$ has one.

### 2.2 IIB Green-Schwarz action in curved backgrounds

The $\kappa$-symmetric IIB Green-Schwarz superstring action was constructed in [25] for general curved backgrounds satisfying the supergravity constraints and, hence, the supergravity equations of motion. To write it down, we need the superspace formulation of type IIB supergravity [27]. In the following, we follow the notation and conventions in [27] unless otherwise stated.

The superspace coordinates are then denoted by $z^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}})$. The field content of the IIB theory is as follows: the frame 1-forms $E^A = (E^a, E^\alpha, \bar{E}^{\bar{\alpha}}) = dz^ME^A_M$, the $SO(1,9) \times U(1)$ connection 1-form $\hat{\Omega}^A_B$; the complex 2-form potential $A$ and the real 4-form potential $B$. From these fields, the torsion 2-form $T^A$, the curvature 2-from $R_A^B$, the complex 3-form field strength $F$, and the 5-form field strength $Z$ are constructed. In addition, there appears a scalar superfield, which is an element of $SU(1,1)$,

$$V = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad u\bar{u} - v\bar{v} = 1. \quad (2.3)$$

From this, one obtains the 1-form,

$$V^{-1}dV = \begin{pmatrix} 2iQ & P \\ P & -2iQ \end{pmatrix}, \quad Q = \bar{Q}. \quad (2.4)$$

The real 1-form $Q$ is identified with the $U(1)$ part of $\hat{\Omega}^A_B$, whereas the complex 1-form $P$ is expanded as $P = E^aP_a + E^\alpha P_\alpha - E^{\bar{\alpha}}P_{\bar{\alpha}}$ with

$$P_\alpha = -2\Lambda_\alpha, \quad P_{\bar{\alpha}} = 0. \quad (2.5)$$

$\Lambda_\alpha$ contains the physical spin 1/2 field of the theory. Also, the scalar field $V$ and the complex 3-form $\mathcal{F}$ combine to form an $SU(1,1)$ invariant 3-form field strength $\tilde{F}$:

$$(\bar{\mathcal{F}}, \mathcal{F})V = (\bar{\mathcal{F}}, \mathcal{F}). \quad (2.6)$$

These fields in the IIB theory satisfy superspace constraints and Bianchi identities. We list their relevant components in Appendix B.

By making use of the above formulation, the IIB Green-Schwarz action for general on-shell curved backgrounds is given by

$$I = \frac{1}{2} \int d^2\xi \left[ \sqrt{-g}g^{ij}\Phi E_i^a E_j^b \eta_{ab} + e^{ij} E_i^B E_j^A \mathcal{B}_{AB} \right]. \quad (2.7)$$


Here, \( g^{ij} \) is the world-sheet metric, \( \epsilon^{ij} \) is the anti-symmetric tensor density and \( E^A_i \equiv \partial_i z^M E^A_M \). \( B_{AB} \) is related to the real closed 3-form \( \mathcal{H} \) by
\[
\mathcal{H} \equiv \mathcal{F} + \bar{\mathcal{F}} = dB.
\] (2.8)

\( \Phi \) is given by the components in \( \mathcal{V} \):
\[
\Phi = w = \bar{w}, \quad w = u - \bar{v},
\] (2.9)

where we have taken a specific local \( U(1) \) gauge so that \( w \) becomes real. In this gauge, one can derive
\[
D_A \Phi = -\frac{1}{2} \Phi (P_A + \bar{P}_A),
\]
\[
Q_A = \frac{i}{4} (P_A - \bar{P}_A),
\]
\[
\mathcal{H} = \Phi (F + \bar{F}).
\] (2.10)

When the background satisfies the supergravity constraints, the action (2.7) has the \( \kappa \) symmetry given by
\[
\delta E^a_a \equiv \delta z^M E^a_M = 0,
\]
\[
\delta E^a = \delta z^M E^a_M = 2E^a_i (\sigma_a)^{\alpha\beta} g^{ij} \eta_{j\beta},
\]
\[
\delta E^c = \frac{1}{2} \delta E^\alpha \Lambda^\alpha - \delta \bar{E}^\alpha \bar{\Lambda}^\alpha,
\]
\[
\delta \Phi = \Phi (\delta E^\alpha \Lambda^\alpha - \delta \bar{E}^\alpha \bar{\Lambda}^\alpha),
\]
\[
\delta (\sqrt{-g} g^{ij}) = 4i (g^{ik} \epsilon^{jl} + \epsilon^{ik} g^{jl}) (E^a_k \eta_{a\alpha} + \bar{E}^a_k \bar{\eta}_{a\alpha}) + 2 (g^{ij} \epsilon^{kl} - 2 \epsilon^{kj} g^{il}) E^c_k (\sigma_c)^{\alpha\beta} \eta_{a\alpha} \Lambda^\beta - \eta_{a\alpha} \bar{\Lambda}^\beta.
\] (2.11)

Here, \( (\sigma_a)^{\alpha\beta} \) (and \( (\sigma_a)_{\alpha\beta} \)) are the 16 \( \times \) 16 \( \gamma \)-matrices. \( \eta_{j\beta} \) satisfies \( \sqrt{-g} g^{ij} \eta_{j\beta} = -\epsilon^{ij} \bar{\eta}_{j\beta} \). Note that the action in (2.7) is expressed in the Einstein frame.

### 2.3 Expansion by fermionic coordinates

In order to analyze a model described by the Green-Schwarz action in the previous subsection, it is desirable to express it in terms of the superspace coordinates \( z^M \). A systematic method for such a purpose was developed in the heterotic case (\( \theta \)-expansion) [19], following the algorithm in [22]. This is a sophisticated version of the ordinary normal-coordinate expansion for bosonic non-linear \( \sigma \)-models. It is also straightforward to apply this method to the IIB case [23, 24].

Following this method, the action \( I \) in (2.7) is covariantly expanded as a functional of \( z^M \). Let \( y^M \) be a tangent vector of a geodesic through \( z^M \), which satisfies \( y^B D_B y^A = 0 \) with \( y^A = y^M E^A_M \). We then have
\[
I(z') = e^{\Delta(z,y)} I(z),
\]
\[
\Delta(z, y) = \int d^2 \xi \ y^A(\xi) D_A(\xi).
\] (2.12)
Here, \( z'^M = z^M + y^M \) when the normal coordinates are taken. \( D_A(\xi) \) is the functional covariant derivative. For example, for a supervector \( X^B(\xi) \),

\[
D_A(\xi)X^B(\xi') = E_A^M (z(\xi)) \frac{\delta X^B(\xi')}{\delta z^M(\xi)} + \delta(\xi, \xi')(-1)^{[C][A]}X^C(\xi)\hat{\Omega}_{AC}^B(\xi),
\]

where \( \delta(\xi, \xi') \) is the delta function and \([A]\) takes 0 for bosonic indices and 1 for fermionic indices. Since we are considering the IIB case, the covariant derivative includes the \( U(1) \) connection as well as the ordinary spin connection. From the definition of \( \Delta \), it follows that

\[
\Delta X^{BC\ldots}_{DE\ldots} = y^A D_A X^{BC\ldots}_{DE\ldots},
\]

\[
\Delta y^A = 0,
\]

\[
\Delta E_i^A = D_i y^A + E_i^C y^B T_{BC}^A,
\]

\[
\Delta(D_i y^A) = y^B E_i^D y^C R_{CDB}^A,
\]

with \( X^{BC\ldots}_{DE\ldots} \) an arbitrary tensor and \( D_i \equiv E_i^A D_A \).

Since we would like to obtain the expansion with respect to the fermionic variables only, at the final stage of the expansion, we set

\[
z^M = z^M_0 \equiv (x^m, 0, 0), \quad y^M = y^M_0 \equiv (0, y^\mu_0, y^\bar{\mu}_0),
\]

At \( z^M = z^M_0 \), the superfields retain their lowest components in the fermionic coordinates:

\[
E_m^a = e_m^a(x), \quad E_m^\alpha = \psi_m^\alpha(x), \quad E_\mu^\alpha = \bar{\psi}_\mu^\alpha(x), \quad E_\mu^\bar{\alpha} = \delta_\mu^\bar{\alpha}, \quad E_\mu^\bar{\bar{\alpha}} = -\bar{\delta}_\mu^\bar{\bar{\alpha}},
\]

\[
E_\mu^a = E_\mu^\bar{\bar{\alpha}} = E_\mu^\bar{\bar{\bar{\alpha}}} = 0,
\]

\[
Q_m = q_m(x),
\]

\[
\hat{\Omega}_{m,ab}^\beta = \omega_{m,ab}^\beta(x), \quad \hat{\Omega}_{m,\alpha}^\beta = \frac{1}{4}(\sigma^{ab})_\alpha^\beta \omega_{m,ab}(x) + i\delta_\alpha^\beta e q_m(x),
\]

\[
\hat{\Omega}_{\mu,AB}^\beta = \hat{\Omega}_{\mu,AB}^\beta = 0.
\]

Here, \( e \) represents the charge of the fields, e.g., \( e = +1 \) for \( \theta^\alpha \). Thus, for \( y^A_0 \equiv y^M_0 E^A_M(z_0) \), one has

\[
y^a_0 = 0, \quad y^\alpha_0 = y^\alpha_0 E^\alpha_\mu = \theta^\alpha, \quad y^\bar{\alpha}_0 = -y^\bar{\bar{\alpha}}_0 E^\bar{\bar{\alpha}}_\mu = \bar{\theta}^\alpha.
\]

Since the terms dropped in setting \( z^M = z^M_0 \) may give contributions at higher orders by further acting with \( \Delta \), (2.15)-(2.17) cannot be substituted at intermediate stages of the expansion. However, there is an exception: Let us recall the rules in (2.14) and the fact that tensors with the Lorentz indices such as \( R_{AB} \) are non-vanishing only when both of \( A, B \) are bosonic or fermionic. One then finds that acting with \( \Delta \) on \( y^a \) yields no contribution even at higher orders and, thus, \( y^a = 0 \) can be set at any stage. We also note that, in the case we
are interested in, the background fermions are set to be zero at the final stage. We express this by an arrow: \( W \rightarrow W' \) means \( W \) (background fermions = 0) = \( W' \).

In this way, one can in principle derive the IIB Green-Schwarz action for general on-shell backgrounds explicitly in terms of \((x^m, \theta^\alpha, \bar{\theta}^\bar{\alpha})\). The computation at low orders is straightforward. For example, the zero-th order term in the expansion, \( I^{(0)} \), is obtained simply by setting \( z^M = z^M_0 \) in \( I(z) \). At the first order \( I^{(1)} = \Delta I \), we need to compute \( \Delta E^a_i \) and \( \Delta \Phi \).

By using (2.14), (2.10), and components in (2.5) and Appendix B, the \( \Delta \Phi \) and \( \Delta E^a_i \) are found to be

\[
\Delta \Phi = \Phi(y^\alpha \Lambda_\alpha - \bar{y}\bar{\alpha} \bar{\Lambda}_{\bar{\alpha}}), \\
\Delta E^a_i = -i[y^\alpha (\sigma_\alpha)_{\alpha\beta} \bar{E}^\beta_j + \bar{y}_{\bar{\alpha}} (\sigma_{\bar{\alpha}})_{\alpha\beta} E^\beta_i].
\]

(2.18)

We then have

\[
L^{(1)} = \frac{1}{2} \left[ \sqrt{-g} g^{ij} \left( \Delta \Phi E^a_i E^b_j + 2\Phi E^a_i \Delta E^b_j \right) \eta^{ab} + \epsilon^{ij} y^c E^b_i E^a_j \mathcal{H}_{ABC} \right]
\]

\[
= \frac{1}{2} \Phi E^a_i \left[ \sqrt{-g} g^{ij} y^a - \epsilon^{ij} \bar{y}^\alpha \right] (\sigma_{\alpha})_{\alpha\beta} \left[ (\sigma_{\beta})^{\beta\gamma} \Lambda_\gamma E^b_j - 2i \bar{E}^\beta_j \right] + \text{h.c.},
\]

(2.19)

where \( I^{(n)} = \int d^2 \xi L^{(n)} \). h.c. stands for hermitian conjugates.

After some amount of algebras, the expression at the second order is also obtained. We discuss this in section 4. However, as mentioned in Introduction, at higher orders it is becoming more and more difficult to carry out the expansion and express the results in terms of physical fields.

### 3 Light-cone action is quadratic in fermionic coordinates

In this section, we prove that the light-cone action in the plane-wave backgrounds summarized in subsection 2.1 is quadratic in the fermionic coordinates \( \theta, \bar{\theta} \). We also see that the proof is valid for more general pp-wave backgrounds under certain conditions.

Before starting our discussion, we would like to make comments on the works in [19] and [24]. In [19], it was argued that the light-cone Green-Schwarz action for the heterotic theory in a certain class of curved backgrounds is quadratic in the fermionic coordinates. The argument was based on the claim, \( O^{-\hat{a}\hat{b}} O^{-\hat{c}\hat{d}} = 0 \) for a Majorana-Weyl spinor in light-cone gauge \( (\sigma^+)_{\alpha\beta} \vartheta^\beta = 0 \), where \( O^{-\hat{a}\hat{b}} = \vartheta^\alpha (\sigma^{-\hat{a}\hat{b}})_{\alpha\beta} \bar{\vartheta}^\beta \). However, this cannot be true: If this was true, one could show that any term of the form \( \vartheta^\alpha \vartheta^\beta \bar{\vartheta}^\gamma \bar{\vartheta}^\delta \) would be vanishing by making use of the Fierz transformations. In fact, we have checked that the numbers of the independent non-vanishing terms of \( \vartheta^\alpha \vartheta^\beta \bar{\vartheta}^\gamma \bar{\vartheta}^\delta \) and those of \( O^{-\hat{a}\hat{b}} O^{-\hat{c}\hat{d}} \) are the same. Similar confusions are found in [24] in the argument that some quartic terms in fermions are vanishing. In the following discussion, we do not use this kind of argument.

As a preliminary to our proof, let us note several facts. First, in light-cone gauge for the IIB case,

\[
(\sigma^+)_{\alpha\beta} \vartheta^\beta = (\sigma^+)_{\alpha\beta} \bar{\vartheta}^\beta = 0.
\]

(3.1)
Furthermore, at $z^M = z_0^M$, one has (2.16) and

$$D_a \theta^\alpha = \partial_a \theta^\alpha - \frac{1}{4} \omega_{a, bc}(\sigma^{bc})^\alpha_\beta \theta^\beta + i q_a \theta^\alpha. \quad (3.2)$$

From these and (I) listed in section 2, it follows that

$$(\sigma^+)_{\alpha\beta} D_i \theta^\beta = (\sigma^+)_{\alpha\beta} D_i \bar{\theta}^\beta = 0. \quad (3.3)$$

In comparing the connections in (3.2) and (2.16), we note that $(\sigma^{bc})^\alpha_\beta = - (\sigma^{bc})^\alpha_\beta$. (The expression of the spin connection in terms of the vielbein is read off from that of the torsion in [27].)

Second, since we are considering the case in which background fermions are vanishing, background fields with odd number of spinor indices are vanishing. From the representation theory of the Lorentz group, background fields with even number of spinor indices should be expressed by terms of the form: ($\gamma$-matrices)$ \times$ (background fields only with bosonic indices). Thus, the spinors $\theta, \bar{\theta}$ have to be contracted with the $\gamma$-matrices in order to obtain non-vanishing terms including them. Since $\theta, \bar{\theta}$ has a definite chirality, the possible contractions in light-cone gauge give only the following bilinears,

$$\bar{\theta} \sigma^{-\tilde{a}\tilde{b}} \theta, \quad \theta \sigma^{-\tilde{a}\tilde{b}} \theta, \quad \bar{\theta} \sigma^{-\tilde{a}\tilde{b}} \theta, \quad \bar{\theta} \sigma^{-\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \theta. \quad (3.4)$$

Third, recalling (2.19) and (2.14), one finds that the fields appearing in $I^{(n)} (n \geq 1)$ are:

(1) $\Phi, H_{ABC}, T_{AB}^C, R_{ABC}^P$, (their covariant derivatives);  
(2) $\eta_{ab}, \epsilon_{a_1, \ldots, a_{10}}$;  
(3) $E_i^A, y^{\dot{\alpha}}, D_i y^{\dot{\alpha}}$ with $\dot{\alpha} = (\alpha, \bar{\alpha})$. The 2-form potential, for example, does not appear in the expansion except at the zero-th order. It could appear only through the decomposition of tensors with two spinor indices of the form $X_{\alpha\beta}$. The decomposition, however, is carried out algebraically by using the constraints and Bianchi identities, but they do not include the 2-form potential. After setting $z^M = z_0^M$ and background fermions to be zero, the background fields in set (1) are expressed by the bosonic physical fields described in subsection 2.1. Otherwise, the superspace would contain degrees of freedom which are redundant for the IIB theory. Thus, the fields which appear in the expansion satisfy the properties (I)-(III) listed in section 2. In fact, for the fields coming from (1), the property (II) becomes more precise:

(II') They have lower ‘+’ indices as many as their mass dimension when all the indices are lowered.

Now, let $W^{(2n;p)}_{a_1, \ldots, a_k} (x, \theta, \bar{\theta})$ with mass dimension $p \geq 0$ be a term which is made of $2n$-spinors and the background bosonic fields in set (1) and (2) above. Then, the following statement holds:

**Lemma:** In light-cone gauge, $W^{(2n;p)}_{a_1, \ldots, a_k} \neq 0$ only when $m_+ = m_- + 2n + p$, where $m_+(m_-)$ is the number of $+(\ldots)$ indices among $(a_1, \ldots, a_k)$. 

8
We show this by induction with respect to \( n \). For \( n = 0 \), \( W^{(0;p)}_{a_{1},...,a_{k}} \) contains only bosonic fields in (1) and (2). Since only \( \eta_{-+}, \epsilon_{-+}, ... \) have lower indices, each lower \( \) is accompanied by one lower \( + \). Also, \( W^{(0;p)}_{a_{1},...,a_{k}} \) has dimension \( p \) and, hence, has to have \( p \) lower \( + \) in addition to \( + \)’s in \( \eta_{-+}, \epsilon_{-+}, ... \). Thus, one can make \( W^{(0;p)}_{a_{1},...,a_{k}} \) only when \( m_{+} = m_{-} + p \). We then suppose that the above statement holds for \( W^{(2(n-1);p)}_{a_{1},...,a_{k}} \). We also denote bilinears in (3.4) by \( O^{(r)} \) with \( \{ -r \} \) expressing their index structures: \( \{ -r \} = -,-[\hat{a}\hat{b}],[\hat{a}\hat{b}\hat{c}\hat{d}] \). The brackets stand for anti-symmetrization. Extracting one bilinear, \( W^{(2n;p)}_{a_{1},...,a_{k}} \) takes the form \( W^{(2n;p)}_{a_{1},...,a_{k}} = \sum O^{(r)} W^{(2(n-1);p+1)}_{\{ -r \} a_{1},...,a_{k}} \). The right-hand side is non-vanishing only when \( m_{+} = (m_{-} + 1) + 2(n - 1) + (p + 1) = m_{-} + 2n + p \), and so is \( W^{(2n;p)}_{a_{1},...,a_{k}} \).

Using this result, we can analyze the possible order in \( \theta, \bar{\theta} \) of the action \( I \). Taking into account (2.19) and the fact that we eventually set the background fermions to be zero, the possibly non-vanishing terms in the \( \theta \)-expansion are of the forms,

\[
(\Delta^{2k}\Phi)(\Delta^{2l}E^{a}_{i})(\Delta^{2m}E^{b}_{j})[\theta(\sigma_{ab} + \eta_{ab})(\Delta^{2n-1}\Lambda)],
\]

\[
(\Delta^{2k}\Phi)(\Delta^{2l}E^{a}_{i})\theta^{\alpha}(\sigma_{a\alpha})(\Delta^{2m-1}E^{b}_{j}),
\]

(3.5)

their hermitian conjugates and similar expressions obtained by replacing \( \theta \) with \( \bar{\theta} \).

\( \Delta^{2n}\Phi \) has dimension zero and \( 2n \) spinors. Thus, the lemma shows that in light-cone gauge this is non-vanishing only when \( n = 0 \) after setting \( z^{M} = z_{0}^{M} \) and the background fermions to be zero. Similarly, \( \theta(\sigma_{ab} + \eta_{ab})(\Delta^{2n-1}\Lambda) \) has dimension 0 and \( 2n \) \( (n \geq 1) \) spinors and, thus, this is vanishing.

To know the possible order of the remaining terms of the forms \( \Delta^{2n}E^{a}_{i} \) and \( \Delta^{2n-1}E^{\alpha}_{i} \), we need a little more analysis, because these have an extra world-sheet index. Similarly to the heterotic case discussed in [19], the general form of \( \Delta^{2n}E^{a}_{i} \) follows from (2.14) to be given by

\[
\Delta^{2n}E^{a}_{ia} \rightarrow W^{ia}_{ab}E^{b}_{i} + W^{i\alpha}_{a\alpha}D_{\alpha}\theta^{\beta},
\]

(3.6)

where \( \theta^{\beta} = \bar{\theta^{\beta}} \). In our setting, \( W^{ia}_{a\beta} \) should be

\[
W^{ia}_{a\beta} \rightarrow \theta^{\hat{\beta}}W^{ia}_{\hat{a}\hat{\beta}},
\]

(3.7)

and \( W^{i\alpha\beta} \) takes the form for \( (\hat{\alpha}, \hat{\beta}) = (\alpha, \beta) \)

\[
S_{ab}(\sigma^{b})_{\alpha\beta} + M_{abcd}(\sigma^{bcd})_{\alpha\beta} + N_{abcdef}(\sigma^{abcdef})_{\alpha\beta},
\]

(3.8)

and similarly for \( (\hat{\alpha}, \hat{\beta}) = (\alpha, \tilde{\beta}), (\tilde{\alpha}, \beta), (\tilde{\alpha}, \tilde{\beta}) \). Consequently, we have

\[
\Delta^{2n}E^{a}_{ia} \rightarrow W^{ia}_{ab}E^{b}_{i} + S_{a-}(\theta\sigma^{-}D_{i}\theta),
\]

\[
+ M_{a-\hat{c}d}(\theta\sigma^{-}\tilde{d}D_{i}\theta) + N_{a-\hat{c}d\hat{e}f}(\theta\sigma^{-}\tilde{d}\tilde{e}fD_{i}\theta) + \cdots,
\]

(3.9)

where the ellipses stand for similar terms obtained by replacing \( \theta \) with \( \bar{\theta} \). Extracting the world-sheet derivative, we can apply the lemma again. Since \( W^{ia}_{ab}, S_{a-}, M_{a-\hat{c}d}, N_{a-\hat{c}d\hat{e}f} \) have
dimension 0, we find that $W_{++} \rightarrow \mathcal{O}(\theta^2)$; $W_{--}, W_{a\tilde{b}}, S_{+-}, M_{+-\tilde{c}\tilde{d}}, N_{+-\tilde{c}\tilde{d}\tilde{f}} \rightarrow \mathcal{O}(\theta^0)$; and others are vanishing. Thus, the only non-vanishing term among $\Delta^{2n} E_{ia}$ ($n \geq 1$) is $\Delta^{2} E_{i+} \sim \mathcal{O}(\theta^2)$.

Similarly, for $\Delta^{2n-1} E_{i\alpha}$, we find that

$$\Delta^{2n-1} E_{i\alpha} \rightarrow W_a E_i^a \theta^\alpha + \text{SD}_i \theta^\alpha + \cdots.$$ (3.10)

Since $W_a$ has dimension 1 and $S$ has 0, $W_+, S \rightarrow \mathcal{O}(\theta^0)$ and $W_-, W_a \rightarrow 0$. Thus, $\Delta^{2n-1} E_{i\alpha}$ is non-vanishing only for $n = 1$ with $\mathcal{O}(\theta^1)$.

Summarizing, the terms of the forms in (3.5) are at most of order two. Therefore, we conclude that the light-cone Green-Schwarz action in the plane-wave geometries with vanishing background fermions is quadratic in the fermionic coordinates $\theta$ and $\bar{\theta}$.

We also note that, in the proof, we have used only the properties (I), (II') and (III). Thus, the proof holds for more general pp-wave backgrounds as long as these properties are satisfied.

### 4 Explicit form of quadratic action in general backgrounds

In the previous section, we have shown that the action (2.7) becomes quadratic in $\theta$ and $\bar{\theta}$ for the plane- and pp-wave backgrounds which we are consider. Thus, we can obtain the complete action by performing the $\theta$-expansion to second order. In this section, we write down the explicit form of the action up to second order in general bosonic on-shell backgrounds. The action for the backgrounds discussed in the previous section is obtained by substituting corresponding field configurations.

As we mentioned in Introduction, the IIB Green-Schwarz action up to this order has been discussed for general bosonic on-shell backgrounds: It has been derived by starting from the membrane action in eleven dimensions and by using T-duality in [18], or by the $\theta$-expansion in [24] (see also, [23]). For a class of plane-wave geometries with vanishing dilaton and constant axion and other field strengths, it has been written down in [14].

Our motivations for discussing the action to quadratic order, in spite of the existing results, are as follows. First, there are issues on the precise form of the IIB Green-Schwarz action for general backgrounds (see, e.g., [9]). Thus, for fixing the precise form, we think that it is useful to compute it in a coherent manner described so far. Second, it turns out that our results are different from those in [24] in signs. Third, we find that it is possible to write down the action in a form in which its geometrical meaning is manifest both in the Einstein and string frames. In this way, the comparison becomes easier with the action in [14] in corresponding cases, which is used in recent works.

Recalling the first order terms in (2.19), we see that the second order terms in $I^{(2)} = \frac{1}{2} \Delta^2 I$
are given by
\[ L^{(2)} \rightarrow \frac{1}{4} \Phi E_i^a \left( \sqrt{-g} g^{ij} \theta - \epsilon^{ij}\dot{\theta} \right) \sigma_a \left[ \sigma_b (\Delta \Lambda) E_j^b - 2i\Delta \dot{E_j} \right] + \text{h.c.} \]  
(4.1)

By making use of the components of the superfields in Appendix B, we find that
\[ \Delta E_j^a \rightarrow E_j^a (\mathcal{D}_a \theta)\alpha, \quad \Delta \Lambda \alpha \rightarrow (\mathcal{D} \theta)\alpha, \]
\[ (\mathcal{D}_a \theta)\alpha \equiv D_a \theta^\alpha - i Z_{acde} (\sigma^{bcde})^\alpha_\beta \bar{\theta}^\beta - \frac{3}{16} F^{abc} (\sigma^{bc})^\alpha_\beta \bar{\theta}^\beta + \frac{1}{48} F^{abc} (\sigma_{abcd})^\alpha_\beta \bar{\theta}^\beta, \]
\[ (\mathcal{D} \theta)\alpha \equiv \frac{i}{2} P_a (\sigma^a)_{\beta} \bar{\theta}^\beta + \frac{i}{24} F_{abc} (\sigma^{abc})^\alpha_\beta \theta^\beta. \]  
(4.2)

We note that \((\mathcal{D}_a \theta)^\alpha\) and \((\mathcal{D} \theta)_\alpha\) are precisely the supersymmetry transformations in the Einstein frame with parameter \(\theta\) for the gravitino, \(\delta \psi^a_{\alpha}\), and for the spin 1/2 field, \(\delta \lambda_\alpha\), respectively \([27]\). We will see that these terms add up to the supersymmetry transformation of the gravitino in the string frame. This is consistent with the fact that the passage from the Einstein frame to the string frame involves mixing of the gravitino and spin 1/2 field. (In the above, all the superfields are evaluated at \(z^M = z_0^M\). Thus, if we faithfully follow the notation in \([27]\), the fields should be denoted by small letters. However, here and in the following, we often do not distinguish the capital and small letters, since no confusion may occur.)

Substituting (4.2) into (4.1) gives
\[ L^{(2)} \rightarrow \frac{1}{4} \Phi E_i^a E_j^b \left( \sqrt{-g} g^{ij} \theta - \epsilon^{ij}\dot{\theta} \right) \sigma_a \left[ \sigma_b \mathcal{D} \theta - 2i\mathcal{D}_b \theta \right] + \text{h.c.} \]  
(4.3)

The term \((\sigma_b \mathcal{D} \theta - 2i\mathcal{D}_b \theta)\) is also rewritten in a form \(-2i (\mathcal{D}^{E:1}_b \bar{\theta} + \mathcal{D}^{E:2}_b \theta)\) with
\[ \mathcal{D}^{E:1}_b = \partial_b - \frac{1}{4} \omega_{bcd} \sigma^{cd} - \frac{1}{4} (\bar{P}_b + \sigma_{bc} P^c) + i Z_{bcde} \sigma^{cdef}, \]
\[ \mathcal{D}^{E:2}_b = -\frac{1}{48} (F^{cde} - \bar{F}^{cde}) \sigma_{bcde} - \frac{1}{16} (F_{bcd} + 3 \bar{F}_{bcd}) \sigma^{cd}. \]  
(4.4)

Here, the \(U(1)\) connection in the covariant derivative has been expressed by \(P_a\) through (2.10).

In sum, the IIB Green-Schwarz action in the Einstein frame for general bosonic on-shell backgrounds is given by
\[ I = \int d^2 \xi \left( L^{(0)} + L^{(2)} \right), \]
\[ L^{(0)} = \frac{1}{2} \left[ \sqrt{-g} g^{ij} \Phi E_i^a(x) E_j^b(x) \eta_{ab} + \epsilon^{ij} E_i^a(x) E_j^b(x) B_{ab}(x) \right], \]  
(4.5)
\[ L^{(2)} = -\frac{i}{2} \Phi E_i^a E_j^b \left( \sqrt{-g} g^{ij} \theta - \epsilon^{ij}\dot{\theta} \right) \sigma_a (\mathcal{D}^{E:1}_b \bar{\theta} + \mathcal{D}^{E:2}_b \theta) + \text{h.c.}, \]
with \(B_{ab}(x)\) being the field \(B_{ab}(z)\) evaluated at \(z^M = z_0^M\), and \(\mathcal{D}^{E:1(2)}_b\) given in (4.4).
We would also like to obtain the string-frame action. For this purpose, we make a rescaling of the metric by \(\Omega_1^2 = e^{\frac{\phi}{4}}\),

\[
\tilde{c}_m^a = e^{\frac{\phi}{4}} c_m^a; \quad \sigma^a, X^{mn...} : \text{fixed}.
\]  

(4.6)

Under this transformation, for example, \(F_{abc} = e^{\frac{\phi}{4}} \tilde{F}_{abc}\), and

\[
E_j^a \omega_{a, bc} \sigma^{bc} = \tilde{E}_j^a (\tilde{\omega}_{a, bc} + \eta_{ab} \partial_c \ln \Phi) \sigma^{bc}.
\]  

(4.7)

By further rescaling the fermionic coordinates as \(e^{\frac{\phi}{4}} \tilde{\theta}, e^{\frac{\phi}{4}} \tilde{\bar{\theta}}\), and dropping tildes, we then arrive at the string-frame action

\[
I = \int d^2 \xi (L^{(0)}_S + L^{(2)}_S),
\]

(4.8)

with

\[
D^{S;1}_b = \partial_b - \frac{1}{4} \omega_{b, cd} \sigma^{cd} - \frac{1}{4} \left[ \tilde{P}_b + \sigma_{bc} (P^c + \frac{1}{2} \partial^c \phi) \right] + ie^{\phi} Z_{bdef} \sigma^{def},
\]

\[
D^{S;2}_b = -\frac{1}{48} e^{\frac{\phi}{4}} (F_{cde} - F^{cde}) \sigma_{b, cd} - \frac{1}{16} e^{\frac{\phi}{4}} (F_{bcd} + 3 \tilde{F}_{bcd}) \sigma^{cd}.
\]  

(4.9)

The quadratic term \(L^{(2)}_S\) is written also in terms of two Majorana-Weyl spinors \(\vartheta^I (I = 1, 2)\) defined by

\[
\theta = \vartheta^1 + i \vartheta^2, \quad \bar{\theta} = \vartheta^1 - i \vartheta^2.
\]  

(4.10)

To proceed, we note that

\[
\frac{1}{2} (\theta A \bar{\theta} + \bar{\theta} A \theta) = \vartheta^I [\delta_{IJ} A^1 + (\rho_0)_{IJ} A^2] \vartheta^J,
\]  

(4.11)

\[
\frac{1}{2} (\bar{\theta} A \bar{\theta} + \bar{\theta} A \theta) = \bar{\vartheta}^I [(\rho_3)_{IJ} A^1 + (\rho_1)_{IJ} A^2] \bar{\vartheta}^J = \bar{\vartheta} \rho_3 [A^1 + \rho_0 A^2] \vartheta,
\]

where \(A = A^1 + i A^2\) and

\[
\rho_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(4.12)

We have dropped indices \(I, J\) in the last expression in (4.11) in an obvious way. From this, we find that any terms appear always in combinations of the form \((\sqrt{-g} g^{ij} \vartheta - (\rho_3)_{IJ} \vartheta^I \vartheta^J)\). Collecting all the terms, we then have

\[
L^{(2)}_S = -i E_i^a E_j^b \left[ \sqrt{-g} g^{ij} \delta_{IJ} - (\rho_3)_{IJ} \right] \vartheta^I \sigma_a D^{S;1}_b \vartheta^J,
\]  

(4.13)
with

\[ D_b^S = \partial_b - \frac{1}{4} \omega_{b,cd} \sigma^{cd} - \frac{1}{4} \Im \rho^c \rho_0 (\sigma_{bc} - \eta_{bc}) + e^\phi Z_{bcdef} \rho_0 \sigma^{cdef} \]

\[ - \frac{1}{4} e^\frac{\phi}{2} \Re F_{bcdef} \rho_0 \sigma^{cdef} \]

\[ - \frac{1}{8} \eta_{bc} \sigma_{b} \]

\[ + \frac{1}{24} e^\frac{\phi}{2} \Im F_{bcd} \rho_1 \sigma^{cd} + \frac{1}{24} e^\frac{\phi}{2} \Im F_{cde} \rho_1 \sigma^{de} . \]

(4.14)

Here, we have used (2.10).

In order to further express this in terms of the fields in modern string-theory conventions, we compare the equations of motion in [27], whose notation we adopt in this paper, and those in term of the fields in [28] (with typos corrected):

\[ R_{ab} = -2 \tilde{\rho} (\rho_a \rho_b) - \tilde{F} (\rho_a \rho_b) + \frac{1}{12} \eta_{ab} \tilde{F}_{cde} F^{cde} - 384 Z_{a \rho_0} \rho_3 \sigma^{cdef} \]

\[ + \frac{1}{48} \eta_{ab} \left[ e^{-\frac{\phi}{2}} (H^{p})_{abcd} (H^{p})_{b}^{cd} + e^\frac{\phi}{2} (F^{p})_{abcd} (F^{p})_{b}^{cd} \right] \]

\[ + \frac{1}{192} \tilde{F}_{abcd} (H^{p})_{abcd} (H^{p})_{abcd} - \frac{1}{96} (F^{p})_{acdef} (F^{p})_{b}^{acdef} , \]

(4.15)

with \( \tau^P = C^P + i e^{-\phi^P} \). We have put superscripts \( P \) on the fields in [28], and the primes on the Ricci tensors; the sign of the Ricci tensor in [28] has also been flipped so as to conform to the conventions in [27]; the dilaton has been denoted by \( \phi^P \), to avoid confusion with \( \Phi \) in the action (2.7). Comparing the above two equations gives

\[ \phi = \phi^P \], \( P_a = -\frac{1}{2} (\partial_a \phi^P + i e^{\phi^P} F^a_P) \), \( F_{abc} = \frac{1}{2} (e^{-\frac{\phi}{2}} H_{abc} + i e^{\frac{\phi}{2}} F^P_{abc}) \), \( Z_{abcd} = -\frac{1}{192} \tilde{F}^P_{abcd} \).

(4.16)

To fix the signs and phases, we have used (2.10) and other equations of motion. The relative signs of \( F^P_a, H^P_{abc}, F^P_{abc}, \tilde{F}^P_{abc} \) are, however, determined only up to the symmetry of the equations of motion. Using these identifications and dropping superscripts \( P \), we obtain

\[ D_b^S = \partial_b - \left( \frac{1}{4} \omega_{b,cd} + \frac{1}{8} H_{bcd} \rho_3 \right) \sigma^{cd} \]

\[ - e^\phi \left( \frac{1}{8} F_{c} \rho_0 \sigma_{c} + \frac{1}{48} \tilde{F}_{cde} \rho_1 \sigma^{cde} + \frac{1}{480} \tilde{F}_{cdefg} \rho_0 \sigma^{cdefg} \right) \sigma_b . \]

(4.17)

This is in complete agreement with the covariant derivative appearing in the string-frame supersymmetry transformation of the gravitino, e.g., in [29]. Note that different normalizations are also used in the literature: For example, the 5-form field strengths in [9] is different by a factor 4; the RR field strengths in [14] by a factor 2, which is absorbed into the shift of the constant part of dilaton. Thus, when \( \phi = F_a = 0 \) and other field strengths are constant, our quadratic terms in \( L^{(2)} \) agree with those in [14].

13
5 Discussion

In this paper, we have considered the IIB Green-Schwarz action in general plane-wave backgrounds obtained as Penrose limits from any IIB supergravity solutions with vanishing background fermions. Starting from the Green-Schwarz action in terms of superfields for general on-shell backgrounds, and using the $\theta$-expansion, we have shown that the action in light-cone gauge is quadratic in the fermionic coordinates. For this to hold, the bosonic background fields do not have to be constant. As long as the properties (I), (II) and (III) are satisfied, the proof is valid also for more general pp-wave solutions, of which the plane-wave solutions above are a subclass. We have also discussed the explicit form of the IIB Green-Schwarz action for general bosonic on-shell backgrounds up to second order in the fermionic coordinates. We have written it down in a form in which its geometrical meaning becomes clear both in the Einstein and string frames. This quadratic action (not necessarily in light-cone gauge) is valid, up to this order, for any on-shell backgrounds as long as background fermions are vanishing. The complete action for the plane- and pp-waves which we have considered is read off by substituting the corresponding background fields. We have found that, when the dilaton and 1-form field strength are vanishing, and other field strengths are constant, our string-frame action agrees with the one in [14] up to conventions. Our results may be used for further exploring strings in curved spacetime and RR backgrounds, string dualities, and the AdS/CFT correspondence. They would also be of some use to fix the issues on the precise form of the IIB Green-Schwarz action.

The proof for the IIB case may be easily applied to other cases with appropriate modifications. When the Yang-Mill part is set to be zero, the proof that the heterotic Green-Schwarz action is quadratic in the fermionic coordinates is given essentially just by truncating one Majorana-Weyl spinor, e.g., $\vartheta^2$. When the Yang-Mills part is turned on, the 3-form field strength can have non-vanishing components for $H_{\tilde{a}\tilde{b}\tilde{c}}$ because of the Chern-Simons term and the dimensionful coupling. We may need appropriate changes in the analysis in this case. A similar proof for the IIA case may also be given essentially by flipping the chirality of one Majorana-Weyl spinor. Although it is difficult to carry out the $\theta$-expansion to higher orders in a totally general case, the expansion would be under control for special backgrounds such as maximally symmetric spaces.
Acknowledgments

We would like to thank S. Hyun, N. Ishibashi, Y. Kiem, K. Mohri, J. Park, M. Sakaguchi and H. Shin for useful discussions, conversations and correspondences. Y.S. would also like to thank Korea Advanced Institute of Science and Technology, where part of this work was done, for its warm hospitality. The work of S.M. is supported in part by Grant-in-Aid for Scientific Research (C)(2) #14540286 from The Ministry of Education, Culture, Sports, Science and Technology, whereas the work of Y.S. is supported in part by University of Tsukuba Research Projects.

Appendix

A Notation and conventions

We use the following conventions for the indices:

- $m, n, \ldots = 0, 1, \ldots, 9$: vector indices for spacetime
- $\tilde{m}, \tilde{n}, \ldots = 1, \ldots, 8$: vector indices for transverse directions of spacetime
- $\mu, \nu, \ldots; \bar{\mu}, \bar{\nu}, \ldots = 1, \ldots, 16$: spinor indices for spacetime
- $M, N, \ldots$: $M = (m, \mu, \bar{\mu}), \ N = (n, \nu, \bar{\nu}), \ldots$
- $a, b, \ldots = 0, 1, \ldots, 9$: vector indices for tangent space
- $\tilde{a}, \tilde{b}, \ldots = 1, \ldots, 8$: vector indices for transverse directions of tangent space
- $\alpha, \beta, \ldots; \bar{\alpha}, \bar{\beta}, \ldots = 1, \ldots, 16$: spinor indices for tangent space
- $\dot{\alpha}, \dot{\beta}, \ldots$: $\dot{\alpha} = (\alpha, \bar{\alpha}), \ \dot{\beta} = (\beta, \bar{\beta}), \ldots$
- $A, B, \ldots$: $A = (a, \alpha, \bar{\alpha}), \ B = (b, \beta, \bar{\beta}), \ldots$

For the notation and conventions of the spinors, superspace and super differential forms, we follow [27]. For example, $\eta_{ab} = \text{diag}(+1, -1, \ldots, -1)$, the scalar product of supervectors is $U^A V_A = U^a V_a + U^a V_a - U^\bar{a} V_{\bar{a}}$, the 5-form field strength is self-dual with $\epsilon^{01\ldots 9} = +1$, and so on. We use the 16 component notation for the spinors. $\gamma$-matrices are expressed by $16 \times 16$ matrices $(\sigma^a)^{\alpha\beta}$ and $(\sigma^a)_{\alpha\beta}$, which satisfy $\{\sigma^a, \sigma^b\} = 2\eta^{ab}$. The anti-symmetrization of $\sigma^a$’s is, for example, $\sigma^{\alpha\beta} = \frac{1}{2}(\sigma^a \sigma^b - \sigma^b \sigma^a)$. The Weyl spinors with upper indices satisfy $(\sigma^0 \ldots \sigma^9)^{\alpha\beta} \theta^\beta = -\theta^\alpha$. Since the spin connection act on a vector $\varphi^a$ as $D_m \varphi^a = \partial_m \varphi^a + \varphi^b \omega^a_{mb} + \cdots$, the action on the spinors is determined as in (3.2).

We often do not distinguish the notation for the full superfields and that for their lowest components in $\theta$, since no confusion may occur.
B Components of superfields

We list some components of the superfields which are derived via constraints and Biachi identities. These are needed to calculate the quadratic action in section 4 (and check the \( \kappa \)-symmetry).

\( T_{AB}^c \):

\[
T_{ab}^c = T_{a\beta}^c = T_{ab}^\gamma = T_{a\beta}^\gamma = 0,
\]
\[
T_{a\beta}^c = -i(\sigma^c)_{a\beta}, \quad T_{a\beta}^\gamma = -\frac{3}{16}\tilde{F}_{abc}(\sigma^{bc})_\beta^\gamma - \frac{1}{48}\tilde{F}_{abcd}(\sigma_{abcd})_\beta^\gamma, \quad (B.1)
\]
\[
T_{a\beta}^\gamma \rightarrow iZ_{abcd}(\sigma^{cde})_\beta^\gamma, \quad T_{a\beta}^\gamma = -(\tilde{T}_{a\beta}^\gamma), \quad T_{a\beta}^\gamma = -(\tilde{T}_{a\beta}^\gamma)
\]

where \( \rightarrow \) stands for setting background fermions to be zero.

\( \mathcal{H}_{ABC} \) in the gauge (2.9),

\[
\mathcal{H}_{a\beta\gamma} = \mathcal{H}_{a\beta\gamma} = \mathcal{H}_{a\beta\gamma} = \mathcal{H}_{a\beta\gamma} = 0, \quad (B.2)
\]
\[
\mathcal{H}_{a\beta\gamma} = \mathcal{H}_{a\beta\gamma} = -i\Phi(\sigma_a)_{\beta\gamma}, \quad \mathcal{H}_{ab\gamma} = -\Phi(\sigma_{ab})_\gamma^\delta\Lambda^\delta, \quad \mathcal{H}_{ab\gamma} = -\Phi(\sigma_{ab})_\gamma^\delta\Lambda^\delta.
\]

\( D\Lambda_\beta \):

\[
D_{\alpha}\Lambda_\beta = -\frac{i}{24}F_{abc}(\sigma^{abc})_{\alpha\beta}, \quad \bar{D}_{\alpha}\Lambda_\beta = -\frac{i}{2}P_a(\sigma^a)_{\alpha\beta}\quad (B.3)
\]

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