Universal relation between thermal entropy and entanglement entropy in CFT

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Inspired by the holographic computation of large interval entanglement entropy of two dimensional conformal field theory at high temperature, it was proposed that the thermal entropy is related to the entanglement entropy as $S_A = \lim_{l \to 0}(S_{EE}(R-l) - S_{EE}(l))$. In this letter, we prove this relation for 2D CFT with discrete spectrum in two different ways. Moreover we show that this relation could break down for 2D noncompact free scalar, which is a CFT with continuous spectrum.

\section*{INTRODUCTION}

The entanglement is a uniquely quantum mechanical property and plays an important role in understanding the quantum many body systems. To measure the entanglement in a bi-partite system, one may define the entanglement entropy as the von Neumann entropy of reduced density matrix of subsystem $A$ \cite{1}

\begin{equation}
S_A = -\text{Tr}_A \rho_A \log \rho_A.
\end{equation}

Here the reduced density matrix is obtained by smearing over the degrees of freedom of subsystem $B$ complement to $A$. If the system is in a pure state, one has $S_A = S_B$. However, if the system is at finite temperature, due to thermal effect there is $S_A \neq S_B$. One may define the quantity $\delta S = S_A - S_B$ to measure the deviation from the purity, which is bounded by the Araki-Lieb inequality \cite{2}

\begin{equation}
|\delta S| = |S_A - S_B| \leq S_{A \cup B}.
\end{equation}

Obviously if $A$ is the whole system, then the above inequality saturates and the entanglement entropy reproduces exactly the thermal entropy of the system. The study of entanglement at finite temperature sheds light on the interplay between quantum nature of the system and its thermodynamics.

Among various studies on the entanglement entropy in many-body systems (see a nice review \cite{3}), the one in quantum field theory is of particular interest. As the quantum field encodes infinite number of degrees of freedom, its vacuum is highly entangled. In this case, the entanglement entropy is called geometric entropy as its leading contribution satisfies an area law \cite{4}.

Quite recently the entanglement entropy opens a new window to study AdS/CFT correspondence. In \cite{5,6}, it was proposed that the entanglement entropy of submanifold $A$ in a conformal field theory (CFT) could be holographically given by the area of a minimal surface in the bulk, which is homogeneous to $A$. In the context of AdS$_3$/CFT$_2$ correspondence, the minimal surface in the bulk is just the geodesic connecting two endpoints of the spacial interval and the geodesic length gives the entanglement entropy of the interval in two dimensional (2D) CFT in the large central charge limit. This picture has been proved in \cite{7,8}.

One interesting implication is from the holographic computation of the single interval entanglement entropy for a 2D CFT on a circle at high temperature \cite{11}. When the interval is short, the entropy could be read from the geodesics in the Banados-Teitelboim-Zanelli (BTZ) black hole background. However, when the interval is large, there could be two possibility. One is the usual geodesic length, while the other one could be the sum of the BTZ black hole horizon length and the geodesic length of a very short interval compliment to the original one. In the large interval limit, the latter one dominate the contribution. This inspired the authors in \cite{11} to propose a universal relation between the thermal entropy and the entanglement entropy

\begin{equation}
S_{th} = \lim_{l \to 0}(S_{EE}(R-l) - S_{EE}(l)).
\end{equation}

Actually, from holographic computation, it was pointed out in \cite{12} that if $l$ is just below a critical value, the Araki-Lieb inequality is saturated. However, this could only be true in the large central charge limit. The relation has been checked in the cases of free fermion \cite{11} and noncompact free boson \cite{13}. It would be interesting to check this relation for general CFT.

In this letter, we prove the relation (3) for the CFTs with discrete spectrum. In applying the replica trick to compute the entanglement entropy, we have to calculate the partition function of CFT on a higher genus Riemann surface coming from pasting the tori along the intervals. Here we present two proofs of the relation (3), one using the complete basis of normal sector states from multi-replica field theory, the other relying on the complete basis from the twist sector states in orbifold CFT. In the latter case, the one-to-one correspondence between the twist sector states and normal sector states in an orbifold CFT allows us to prove (3). Moreover, we present a counter-example to this relation.
**FIRST PROOF**

To compute the entanglement entropy, it is convenient to use the so-called Rényi entropy, which is defined to be

$$S^{(n)}_A = -\frac{1}{n-1} \log \text{Tr} A^n. \quad (4)$$

It gives the entanglement entropy $S_A = \lim_{n \to 1} S^{(n)}_A$, if the analytic continuation $n \to 1$ limit is well-defined. By the replica trick\[^9\], the Rényi entropy in two dimensional quantum field theory can be transformed into calculating partition function on a higher genus Riemann surface\[^10\].

$$S^n = -\frac{1}{n-1} \log \frac{Z_n}{Z_1}, \quad (5)$$

where $Z_n$ is the partition function for $n$ tori connecting along the branch cut. To calculate the partition function, we can cut the torus along a cycle and insert a complete basis there.

We consider a large interval on a circle of radius $R$. The interval length is $L = R - l$ with $l$ being very small $l/R < 1$. Without losing generality, we set the branch points at $u_1 = \frac{l}{2}$, $u_2 = -\frac{l}{2}$ such that the interval extends from $u_1$ to $u_2$ winding around the spacial cycle. In other words, the interval is the union $[-\frac{L}{2}, \frac{L}{2}] \bigcup [\frac{L}{2}, \frac{3L}{2}]$. The CFT is at a finite temperature, with thermal radius being $\beta = 1/T$. We assume that the CFT has a discrete spectrum. As we are working with Euclideanized field theory, we are allowed to quantize the theory along the thermal direction or along the spacial direction. Let us first consider the quantization along the thermal direction. In this case, the thermal density matrix is of the form

$$\rho_{th} = e^{-\beta H} \quad (6)$$

with the Hamiltonian being

$$H = \frac{2\pi}{R}(L_0 + \tilde{L}_0 - \frac{c}{12}) \quad (7)$$

Then we have the thermal partition function

$$Z_1 = \text{Tr} e^{-\beta H} = e^{\frac{2\pi n c}{K} \beta} \sum_i e^{-\frac{2\pi n c}{K} \Delta_i}, \quad (8)$$

where the summation is over all the excited states with conformal dimension $\Delta_i$. To compute the partition function $Z_n$ on $n$-sheeted Riemann surface, we should cut the cylinder along the spacial cycle and insert a set of complete basis of $n$-replica CFT\[^15\] [^16]. In this way, we find that

$$Z_n = \sum_{j_1,j_2,\cdots,j_n} \langle j_1,j_2,\cdots,j_n \mid T^+(u_1)T^-(u_2) \mid j_1,j_2,\cdots,j_n \rangle \cdot \exp \left(-\frac{2\pi c}{L} \sum_{j_i} \Delta_{j_i} + \frac{2\pi n c}{12} \right),$$

where the summation is over all the excitations of CFT in every replica and $\Delta_i$ is the conformal dimension of the excitation in $j_i$-th replica. Here $T^-(u_1)$ and $T^+(u_2)$ are two twist operators inserting at the branch points of the interval. The above correlation function is defined on a $n$-sheeted cylinder pasting along the branch cut shown in Fig. 1a. As the interval is very large, these two twist operators are very close to each other. Actually, we may modify the branch cut in such a way that it extends the whole spatial cycle, being subtracted by the compliment small interval, as shown in Fig. 1b. As a result, we can rewrite $Z_n$ as

$$Z_n = \sum_{j_1,j_2,\cdots,j_n} \langle j_1,j_2,\cdots,j_n \mid T^-(u_1)T^+(u_2) \mid j_2,\cdots,j_n,j_1 \rangle \cdot \exp \left(-\frac{2\pi c}{L} \sum_{j_i} \Delta_{j_i} + \frac{2\pi n c}{12} \right).$$

Here we have taken into account of the effect of the interval over the whole spacial cycle, which transform the states into next replica. In large interval limit, we may use the operator product expansion (OPE)

$$T^-(\frac{L}{2})T^+(\frac{L}{2}) \sim c_R l^{-\frac{c}{12}}(1 + O(l)), \quad (9)$$

where we only keep the first term such that the correlation function of the twist operator reduces to the inner product of the complete basis, which is non-vanishing only when the excitations on different replica are the same. Consequently

$$Z_n = e^{\frac{2\pi n c}{K} \beta} c_R l^{-\frac{c}{12}}(\sum_i e^{-\frac{2\pi n c}{K} \Delta_i} + O(l)), \quad (10)$$

from which we have

$$S_n = -\frac{1}{n-1} \left( \log (c_R l^{-\frac{c}{12}}) + \log \left( \sum_i e^{-\frac{2\pi n c}{K} \Delta_i} \right) \right) + O(l).$$

The quantity we are interested in is

$$\lim_{l \to 0} (S_{EE}(R - l) - S_{EE}(l)) = -\lim_{n \to 1} \frac{1}{n-1} \left( \log Z^n \frac{c_R}{R} - n \log Z^n \frac{\beta}{R} \right)$$

$$= \log Z^n \frac{\beta}{R} - \frac{R Z^n \frac{\beta}{R}}{Z^n \frac{\beta}{R}}, \quad (11)$$

with

$$Z(x) \equiv \sum_i e^{-\frac{2\pi x \Delta_i}. \quad (12)$$

Note that the terms proportional to $c_n$ have been cancelled.
then the field satisfies the twist boundary condition of twist sector states. This requires us to insert a complete basis of the interval, the field satisfy nontrivial monodromy a complete set of basis there. However if we cut the

FIG. 1: Riemann surface for finite temperature large interval entanglement entropy. (a): The dashed line is the branch cut, and the arrow indicates the field transforming from $j$-th to $(j + 1)$-th replica. (b): The modified branch cut.

On the other hand, the thermal entropy could be obtained by

$$S_{th} = - \frac{\partial F}{\partial T} = \log Z[\beta] - \frac{R}{\beta} Z'[\beta],$$  \hspace{1cm} (13)$$

which is the same as \((11)\). Therefore we prove the relation \((3)\).

We would like to emphasize that the above proof is valid for any temperature. From holographic point of view, the absence of the term linear in $c$ in the entanglement entropy seems indicate that this discussion is only true in the low temperature limit. However, this is just an illusion. Our proof above is obviously independent of the temperature. One may worry that at the high temperature one should quantize the theory along the spacial direction rather than the thermal direction. This worry is not necessary as we know that the partition function is modular invariant. Actually it is possible to discuss the problem from the quantization along the spacial direction, as we show below.

SECOND PROOF

If we quantize the theory along the spacial direction, the density matrix is of the form

$$\rho_s = e^{-RH_s} = e^{-\frac{2\pi R}{\beta}}(L_0 + \tilde{L}_0 - \frac{c}{12}).$$  \hspace{1cm} (14)$$

In this case, we need to cut the thermal cycle and insert a complete set of basis there. However if we cut the thermal cycle across the branch cut, due to the presence of the interval, the field satisfy nontrivial monodromy condition. This requires us to insert a complete basis of twist sector states.

Suppose that one of the branch points is at the origin, then the field satisfies the twist boundary condition

$$\phi^{(j)}(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \phi^{(j+1)}(z, \bar{z}), \hspace{1cm} (15)$$

with $j = 1, \cdots, n$ labelling the sheets. And $\phi$ can be any field in a CFT. We can redefine other $n$ fields as

$$\phi^{(t,k)}(z, \bar{z}) = \sum_{j=1}^{n} e^{\frac{2\pi R}{\beta}kj} \phi^{(j)}(z, \bar{z})$$  \hspace{1cm} (16)$$

with the monodromy condition

$$\phi^{(t,k)}(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \phi^{(t,k)}(z, \bar{z})e^{-\frac{2\pi R}{\beta}k}. \hspace{1cm} (17)$$

The mode expansion of the field in the twist sector is

$$\phi^{(t,k)} = \frac{1}{n^{h+\frac{c}{24}(1-\frac{1}{12})}} e^{\frac{2\pi R}{\beta}k} \sum_{m=-h, \bar{m}=-\bar{h} \atop m-\bar{m}=k+n} \Phi_m \Phi_{\bar{m}} e^{\frac{2\pi R}{\beta}h} e^{\frac{2\pi R}{\beta}z}.$$

The lowest state in the twist sector are

$$\phi_{-h\Phi_{-\bar{h}} | t \rangle}, \hspace{1cm} (18)$$

with the conformal dimension \((\frac{h}{n} + \frac{c}{24}(1-\frac{1}{12}) \atop \frac{h}{n} + \frac{c}{24}(1-\frac{1}{12}))\). Here \(| t \rangle\) is the twist vacuum with conformal dimension $h = \frac{c}{24} n (1 - \frac{1}{12})$. And the higher conformal dimension states can be built by acting creation operators on this state. There is an one-to-one correspondence between the twist sector states and normal sector states, with their energies being related by \((14)\)

$$H = \frac{2\pi R}{\beta}(L_{\text{twist}} + \tilde{L}_{\text{twist}} - \frac{nc}{12}) = \frac{H_{\text{normal}}}{n}. \hspace{1cm} (19)$$

With this correspondence, let us discuss the relation \((3)\) again. The partition function $Z_1$ is now

$$Z_1 = e^{\frac{2\pi R}{\beta} \frac{n}{12}} \sum_t e^{-\frac{2\pi R}{\beta} \Delta_i}. \hspace{1cm} (20)$$

For the partition function on $n$-sheeted Riemann surface, we have

$$Z_n = e^{\frac{2\pi R}{\beta} \frac{n}{12}} \sum_t (t, i \mid T - \frac{1}{2}) (t', i) e^{-\frac{2\pi R}{\beta} \Delta_i},$$

where the label $t$ denotes the twist sector, and the summation is over all the states in the twisted sector. As the branch points are near each other, we can still use the OPE of the twist operators \((9)\), from which we find

$$S_n = -\frac{1}{n-1} \left( \frac{\pi c R}{6n \beta} + \log (e^{\frac{c}{24} (n - \frac{1}{12})}) + \log (\sum_t e^{-\frac{2\pi R}{\beta} \Delta_i}) \right) - n\left( \frac{\pi c R}{6 \beta} + \log (\sum_t e^{-\frac{2\pi R}{\beta} \Delta_i}) + O(l) \right).$$

Consequently, we have

$$\lim_{l \to 0} (S_{EE}(l) - S_{EE}(R - l)) = \frac{\pi c R}{3 \beta} + \log Z[R] + \frac{R Z'[\beta]}{Z[\beta]} \hspace{1cm} (21)$$
The thermal entropy is now
\[ S_{th} = \frac{1}{3} \pi \varepsilon \frac{R}{\beta} + \log Z[\frac{R}{\beta}] + \frac{R^2}{\varepsilon} Z[\frac{R}{\beta}], \]
which is the same as \[21\]. This complete our second proof of the relation \[3\].

In the large central charge limit, the first term in \[21\] gives the entropy of the BTZ black hole in the bulk, which is dual to the CFT at high temperature. And the remaining terms correspond to the quantum corrections. However, we would like to emphasize again that the relation \[21\] is valid for all temperatures. It equals to the relation \[13\], both of which gives the thermal entropy of the CFT.

**A COUNTER-EXAMPLE**

In the above proof, we have assumed that the CFT has discrete spectrum. It would be interesting to see if the relation \[3\] is still true for the CFT with continuous spectrum. However, we find a counter-example: non-compact free scalar \[17\]. One may compute the large and short interval expansion of the partition function and check the relation directly, as discussed carefully in \[14\]. In the large interval expansion, there appears a term of the form \(\log(|\log l|)\), which could not be cancelled by the short interval terms. The reason for such kind of term could be found in the discussion above. One essential part in the above proofs is to use the OPE of the twist operators. However, in the case of noncompact scalar, the OPE relation \[1\] breaks down. Actually, the relation changes to
\[
\mathcal{T}^{-}(\frac{l}{2}) \mathcal{T}^{-}(\frac{l}{2}) = c_{n} e^{-k(1-\frac{1}{n})} \left\{ V^{n-1} \int \prod_{j=1}^{n} dk_{j} \delta(\sum_{j=1}^{n} k_{j}) \cdot \prod_{j=1}^{n} \frac{(\frac{l}{n})^{2k_{j}^{2}} e^{-ik_{j} \phi^{(i)}(u)}}{|u=0|} \cdot \prod_{1 \leq j < j' \leq n} \left( 2 \sin \frac{\pi}{n} (j_{2} - j_{1})^{2k_{j}k_{j'}} + O(l) \right) \right\},
\]
where \(V\) is the regularized volume in the target space. Different from the small interval case, the operators \(\prod_{j=1}^{n} e^{ik_{j} \phi^{(i)}}\) can have contribution to the partition function in the large interval limit. In the end, we find that
\[
\lim_{l \to 0} S_{EE}(R - l) - S_{EE}(l) = \frac{1}{2} \log(|\log l|) + S_{th} + \text{const}.
\]
This result could be obtained by direct expansion of the \(W\) functions in the partition function as well\[13\]. The log-logarithmic divergence stems from the continuous spectrum of the non-compact scalar. It is remarkable that after removing this divergence, the relation \[3\] still holds after absorbing the constant term by regularization.

**CONCLUSION AND DISCUSSION**

In this letter, we proved the universal relation \[3\] between the thermal entropy and entanglement entropy for the CFTs with discrete spectrum. We also presented a counter-example for this relation. One interesting issue is to see if the relation is generically broken down for the CFT with continuous spectrum. Another interesting question is to check if the relation \[3\] could be true for a generic 2D quantum field theory without conformal symmetry.

In higher dimension, the recent study on the holographic entanglement entropy suggest that the Araki-Lieb inequality could be saturated, which is called entanglement plateau\[12\]. This suggests that there exist some kind of generalization of the relation \[3\] in higher dimension. It would be interesting to study such relation directly in the field theory.

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[17] Our conclusion is different from the one in [13], where the relation (3) was claimed to be confirmed.