Perturbation schemes for systems of nucleons and pions: The relationship of covariant perturbation theory, the convolution integral and time-ordered perturbation theory.

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Abstract

This paper is the first in a series of three which attempt to resolve the difficulties that have plagued the $NN - \pi NN$ problem for the past ten years. The problems may be summarized by saying that the current $NN - \pi NN$ equations cannot fit the experimental data. Various theoretical inconsistencies in the current formulation have been pointed out and this work aims to eliminate these inconsistencies and so, we hope, produce agreement with experiment. This is to be done by using covariant perturbation theory, in which these inconsistencies are not present. The covariant perturbation theory is developed starting from a model Lagrangian, in order to fix notation and phases. (The approach is actually Lagrangian independent but we use a specific Lagrangian for concreteness.) It is shown that both old-fashioned "time-ordered" perturbation theory and the convolution integral of Kvinikhidze and Blankleider may be recovered from the covariant perturbation theory when certain approximations are made. The connection of these results with the work of Klein, Lévy, Macke and Kadyshevsky is discussed. Two forthcoming papers will pursue this covariant calculation in the $NN - \pi NN$ system using the model and perturbation scheme developed in this paper and derive fully covariant $NN - \pi NN$ equations without the double counting problems present in previous covariant equations.
I. INTRODUCTION

Models in nuclear and particle physics are of two main types: (i) Those based directly on Quantum Chromodynamics (QCD), in which quarks are the basic degrees of freedom, e.g. Chiral Perturbation Theory [1,2] or the Nambu-Jona-Lasinio model [3,4]; and (ii) Those in which mesons and baryons are the basic degrees of freedom, e.g. the Walecka models of the nuclear many-body system, which are collectively known as Quantum Hadrodynamics (QHD) [5,6]. In principle these two types of models are connected, since the Lagrangian for QHD should be obtained from that of QCD upon integration of the quark-gluon degrees of freedom. However, as yet this has not been achieved, and one has to resort to models of QCD such as chiral bag models [7] in order to explain the connection between the quark and the hadronic degrees of freedom. Since the hadronic picture can in this way be justified from the quark picture one would expect that the hadronic models would be adequate approximations to the true physical situation. Whether this is the case or not can only be determined by comparison of the predictions of the hadronic models with experiment. A disagreement between theory and experiment can then be due either to some approximation in the calculation using the model, or to the need for explicit quark-gluon degrees of freedom in the model. With the current state of computer technology, light nuclear systems are the only nuclear systems in which we may perform the calculation of experimentally measured quantities while retaining some control over the approximations used. Consequently, only in light nuclear systems may we discover whether the models of the second type, in which mesons and baryons are the basic degrees of freedom, can accurately describe the properties of nuclei. If they cannot, it will be necessary to resort to models of the first type, with explicit quark degrees of freedom, in order to obtain an accurate description of nuclear properties. Therefore, the interaction of pions and photons with light nuclear systems is a testing ground for models in particle and nuclear physics.

The $NN - \pi NN$ system is one example of a light nuclear system in which this program for the testing of models based on meson-baryon degrees of freedom may be pursued. During the last twenty years disagreements between theory and experiment have led to an improvement in the calculations based on meson-baryon degrees of freedom. The physics content of these models has gradually increased as they attempt to reproduce the experimental data ever more closely. To illustrate how experimental results and the need for consistency in the theory have together influenced the development of these $NN - \pi NN$ models, we will briefly review the history of the $NN - \pi NN$ equations. This will also allow us to focus on some of the problems which led to the current investigation. A detailed analysis of the theoretical methods and a comparison of their predictions with experiment may be found in the recent book by Garcilazo and Mizutani [8].

The suggestion that the one pion exchange potential could be treated within the framework of the Faddeev equations was first made by Varma who calculated $NN$ scattering using a one pion exchange potential and the Faddeev equations [9]. In this model, the absorption and production of the pion is the result of the $\pi N$ amplitude in the $P_{11}$ channel having a bound state pole with binding energy equal to the pion mass. This model was later extended by Afnan and Thomas to include pion production in $NN$ scattering, pion absorption in $\pi d$ scattering and $\pi d$ elastic scattering [10]. In this way a unified formulation of the three reactions:
\[
N + N \rightarrow N + N, \\
N + N \leftrightarrow \pi + d, \\
\pi + d \rightarrow \pi + d
\]

was achieved for the first time. The problem with this model was that the nucleons were not identical: one of the nucleons (labeled the \(N'\)) was treated as a \(\pi N\) bound state and so could emit a pion, while the other was an elementary nucleon, and so could not emit a pion.

However, the success of the model in predicting the \(NN\) phase shifts for large angular momenta, and the ability of the model to describe \(\pi d\) scattering by not only summing the multiple scattering series, but also including the effects of real absorption, motivated several groups to reformulate the model in such a way that both nucleons could emit a pion. All of these formulations were based on a field theory allowing the possibility of an arbitrary number of pion emissions and absorptions. However, the constraint of unitarity allowed all the groups to truncate the field theory. Some groups performed this truncation using Feshbach projection operators \[13,14\], while others did it through the classification of diagrams technique of Taylor \[15\]. All groups truncated the theory by neglecting or approximating processes involving four or more particles—the so-called one-pion approximation—and so obtained a set of linear integral equations, known as the \(NN - \pi NN\) equations, which are three-dimensional in momentum space \[16–25\]. The equations were then solved and a detailed comparison made with the extensive experimental data generated in the 1980s. Early attempts at extending the Hilbert space to include the \(\pi NN\) states, i.e. truncating the field theory at the next stage, led to a more complicated set of equations which included contributions from the diagram in Fig. \[1\] \[26–28\]. These equations simplify to the \(NN - \pi NN\) equations when the diagram in Fig. \[1\] is excluded \[29\].

Although these Faddeev-type \(NN - \pi NN\) equations resolved the problem of non-identical nucleons by dressing both fermions, the dressing of the two-nucleon propagator was still not complete. The truncation of the Hilbert space and the use of time-ordered perturbation theory meant that both nucleons could not be dressed at the same time. I.e., in the one-pion approximation, the diagram on the right of Fig. \[2\] was included as part of the dressing for the \(NN\) propagator, but the diagram on the left of the figure was not—in spite of the fact that the two diagrams are of the same order in the coupling constant, and are just different relative time-orders of the same process. This incompleteness of the dressing, first pointed out by Sauer et al. \[30\], gives rise to a weakening of the \(\pi NN\) coupling constant as the number of nucleons in the system increases. In fact, Sauer et al. showed that in models which, like the \(NN - \pi NN\) equations, use an incomplete dressing of the multi-nucleon propagator, the omission of some of the dressing means that the \(\pi NN\) coupling constant goes to zero in the nuclear matter limit. Furthermore, the omission of this dressing also leads to a severe underestimation of the \(pp \leftrightarrow \pi d\) cross-section in theoretical calculations—especially in theories which treat the \(\pi N\) amplitude in the \(P_{11}\) channel properly, i.e. as the sum of a pole and non-pole term. The only way this underestimation can be remedied is to dress both nucleons fully. Recently, Kvinikhidze and Blankleider have addressed the question of how to do this and have elegantly demonstrated how the complete dressing can be achieved with the use of a convolution integral representation for the \(NN\) propagator \[31\].

A second associated problem has plagued all calculations based on the \(NN - \pi NN\) equations which use a \(P_{11}\) \(\pi N\) amplitude that is the sum of a pole and non-pole contribution.
None of these calculations has been able to accurately predict the tensor polarization, $T_{20}$, for $\pi - d$ scattering. In 1988 Jennings suggested that this discrepancy between theory and experiment is a result of the $NN - \pi NN$ equations not including the diagram in Fig. 1, which has since become known as the Jennings mechanism [32,33]. Indeed, the mechanism depicted in Fig. 1 is merely a different time-order of a diagram which is included in the $NN - \pi NN$ equations, shown in Fig. 3. Jennings pointed out that the excluded diagram may well cancel part of the effect of the one included in the $NN - \pi NN$ equations, thus producing the correct result for the tensor polarization in $\pi - d$ scattering.

Since the Jennings mechanism and the extra dressing of the $NN$ propagator are merely different time-orders of diagrams already included in the $NN - \pi NN$ equations, logically they too should be included in any calculation of the $NN - \pi NN$ system. Otherwise, the processes which are occurring on one nucleon restrict the processes which may occur on the other nucleon. Further, the failure of the $NN - \pi NN$ equations to correctly describe the experimental data suggests not only that these two omitted mechanisms should now be included, but that they must now be included.

The omission of these two diagrams from the present theory is a result of using a truncation based on time-ordered perturbation theory and unitarity. One might think that the problem could be resolved by truncating the Hilbert Space at two-pion states, since then the two diagrams discussed above would both be included. As mentioned above, an example of a theory which takes this approach is the model of Stingl and Stelbovics [26–28]. However, the equations obtained from this model are computationally difficult to solve. The model also suffers from similar problems to those of the current $NN - \pi NN$ equations, but at the three-pion level. One example of this is that diagrams such as those on the left of Figure 4 are included in the model, whereas diagrams such as those on the right are not included, because they involve a three-pion intermediate state. Again, this occurs even though the two diagrams are just different time-orders of the same process.

Clearly then the problem rests not so much with the choice of truncation point as with a failure to sum over all time-orders when doing the truncation. Therefore, if we wish to include all the relevant physics in our model we must use a perturbation scheme in which all time-orders are included automatically. This paper presents such a perturbation scheme as follows.

In Section II the field operators we will be using are introduced and the Lagrangian for the theory stated. In Section III we explain how to calculate the single-nucleon and single-pion Green’s functions with this Lagrangian. At this point we introduce an approximation common in intermediate-energy nuclear physics—we ignore anti-nucleonic degrees of freedom. Once we make this approximation, the theory simplifies considerably. We will show that, apart from this approximation, the perturbation scheme developed is a fully covariant theory of nucleons and pions as given in Bjorken and Drell [34]. In Section IV we turn our attention to the two-nucleon Green’s function. It is explained how to evaluate $G_{NN}(p_1', p_2'; p_1, p_2)$ and the Fourier Transform of the equal-time Green’s function $\overline{G}_{NN}(t', \vec{x}_1', \vec{x}_2'; t, \vec{x}_1, \vec{x}_2)$ is calculated. We find that:

$$\overline{G}_{NN}(E', \vec{p}_1', \vec{p}_2'; E, \vec{p}_1, \vec{p}_2) = \int \frac{dz dz'}{(2\pi)^2} G_{NN}(E' - z', \vec{p}_1', \vec{p}_2'; E - z, \vec{p}_1, z, \vec{p}_2), \quad (1.2)$$

where $G_{NN}(E' - z', \vec{p}_1', \vec{p}_2'; E - z, \vec{p}_1, z, \vec{p}_2)$ is calculated using the Feynman Rules for $G_{NN}(p_1', p_2'; p_1, p_2)$. In Section V it is shown that similar results hold for the $m \rightarrow m'$
particle Green’s function, a result of particular relevance to the $\pi NN$ system. This use of a convolution integral to calculate multi-particle Green’s functions bears a strong resemblance to the work of Kvinikhidze and Blankleider (KB) \cite{31,35} and in Section \textit{VI} we explain how to derive KB’s result from our field theory. Since KB’s work uses a convolution integral in order to sum a set of time-ordered (or old-fashioned) perturbation theory diagrams it is necessary to first derive time-ordered perturbation theory from our covariant perturbation scheme. KB’s result then becomes a consequence of Equation (1.2) and the relationship of time-ordered perturbation theory to the covariant perturbation scheme. Once this result is established, the approximations used by KB in order to derive equations for the $NN - \pi NN$ system \cite{35} are critically examined, and are found to have considerable inconsistencies. In Section \textit{VII} we consider how to calculate amplitudes from the Green’s function $G_{NN}(E' - z', \vec{p}_1', z', \vec{p}_2'; E - z, \vec{p}_1, z, \vec{p}_2)$ using Lehmann-Symanzik-Zimmermann (LSZ) reduction \cite{36}. We find that the amplitude generated by a sum of all relative time-orders of time-ordered perturbation theory diagrams may be calculated by evaluating the amplitude given by the Feynman rules in Sections \textit{II} and \textit{III}. However, as mentioned above, the perturbation scheme that generates these Feynman rules is merely an equal-time, no-anti-nucleon approximation to the full covariant perturbation theory. Therefore, if we are to include the missing diagrams in our theory we need to use some sort of covariant perturbation theory in order to derive new four-dimensional $NN - \pi NN$ equations, even if at certain stages in the calculation we make approximations.

The results described in this article are similar to those obtained by Klein, Lévy, Macke and Kadyshevsky (KLMK) \cite{37–48}. The essential difference between KLMK’s work and ours is that the KLMK method is driven by the desire to obtain a three-dimensional integral equation, whereas one of the main theses presented in this article is that four-dimensionality is an essential feature of any theory which is not going to treat different time-orders of the same physical process differently. We conclude our arguments for this thesis in Section \textit{VIII} by briefly describing the work of KLMK in the context of the calculations performed in the previous sections. In particular, we highlight the pieces of physics which are missing from their work, due to their use of a three-dimensional equation rather than a four-dimensional one.

Haberzettl has also recently published some work on the the use of a convolution integral in $N$-body scattering, and has derived four-dimensional integral equations for certain scattering problems \cite{49–51}. While his work appears similar to that discussed here it is approached from a rather different perspective, being based on cluster dynamics rather than field theory.

In summary then, the inability of the current $NN - \pi NN$ equations to correctly describe the experimental data forces the introduction of new physics into the theory of the $NN - \pi NN$ system: namely, the consideration of different time-orders of diagrams already included in the theory. The only natural way to include this missing physics is to use the full covariant perturbation theory developed in this paper. The use of such a perturbation theory will, of course, result in four-dimensional integral equations. Any attempt to reduce the dimensionality of these equations results in a loss of physics, as in the work of KB and KLMK. Therefore, in later papers we pursue the use of the full covariant perturbation theory and so derive covariant equations for the $NN - \pi NN$ system. In this paper we wish only to concentrate on the perturbation scheme we shall be using, and to establish the
approximations in which it reduces to the convolution integral approach, the time-ordered perturbation theory and the work of KB and KLMK.

II. THE LAGRANGIAN

In this section we define the field theory we shall be using in order to fix the notation and normalization used throughout the paper. We define nucleon and meson creation and annihilation operators, state the commutation relations they obey and write down the Lagrangian and hence the Hamiltonian. The Lagrangian we use is similar to that used in Quantum Hadrodynamics calculations [7,8]. Note that although we use a specific Hamiltonian the approach pursued in this paper could equally well be applied to any meson-baryon Hamiltonian with vertices which allow the baryons to emit mesons.

Consider second-quantized nucleon field operators $\psi(\vec{x})$ and $\bar{\psi}(\vec{x})$. We can, of course, Fourier decompose these Schrödinger representation operators, as done by, for example, Itzykson and Zuber [52], and when we do we obtain:

$$\psi(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \frac{m}{E_N(q)} \sum_{\alpha} \left[ b_{\alpha}(q) u_{\alpha}(q) e^{i\vec{q} \cdot \vec{x}} + d^\dagger_{\alpha}(q) v_{\alpha}(q) e^{-i\vec{q} \cdot \vec{x}} \right],$$

$$\bar{\psi}(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \frac{m}{E_N(q)} \sum_{\alpha} \left[ b^\dagger_{\alpha}(q) \bar{u}_{\alpha}(q) e^{-i\vec{q} \cdot \vec{x}} + d_{\alpha}(q) \bar{v}_{\alpha}(q) e^{i\vec{q} \cdot \vec{x}} \right],$$

(2.1)

where $m$ is the nucleon mass; $E_N(q)$ is the energy of a nucleon of momentum $\vec{q}$ and $\alpha$ is a collective label for spin and isospin. Here $b^\dagger(\vec{q})$ and $b(\vec{q})$ are the creation and annihilation operators for a nucleon of momentum $\vec{q}$ while $d^\dagger$ and $d$ are the corresponding anti-nucleon operators and $u(\vec{q})$ and $v(\vec{q})$ are the positive and negative energy spinors corresponding to momentum $\vec{q}$, normalized as in Itzykson and Zuber [52], i.e. such that:

$$\bar{u}_{\alpha}(q) u_{\alpha'}(q) = \delta_{\alpha\alpha'}.$$  

(2.3)

Note that $E_N(q)$ can be chosen to give non-relativistic, semi-relativistic or fully relativistic kinematics. E.g., for full relativistic kinematics choose:

$$E_N(q) = \sqrt{m^2 + q^2}.$$  

(2.4)

Since the nucleons are fermions their operators obey the standard anti-commutation relations:

$$\{ b_{\alpha}(\vec{q}), b^\dagger_{\alpha'}(\vec{q'}) \} = (2\pi)^3 \frac{m}{E_N(q)} \delta^{(3)}(q-q') \delta_{\alpha\alpha'},$$

(2.5)

$$\{ d_{\alpha}(\vec{q}), d^\dagger_{\alpha'}(\vec{q'}) \} = (2\pi)^3 \frac{m}{E_N(q)} \delta^{(3)}(q-q') \delta_{\alpha\alpha'},$$

(2.6)

with all other anti-commutators vanishing.

Similarly, we define meson field operators $\phi_i(\vec{x})$, with $i$ an isospin label, which we expand via:

$$\phi_i(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_\pi(k)} \left[ a_i(k) e^{+i\vec{k} \cdot \vec{x}} + a_i^\dagger(k) e^{-i\vec{k} \cdot \vec{x}} \right],$$

(2.7)
where $\omega_\pi(\vec{k})$ is the energy of a pion of momentum $\vec{k}$, which in the relativistic kinematics is:

$$\omega_\pi(\vec{k}) = \sqrt{m_\pi^2 + \vec{k}^2};$$  \hspace{1cm} (2.8)$$

and $a_i(\vec{k})$ ($a_\dagger_i(\vec{k})$) destroys (creates) a pion with isospin label $i$ and momentum $\vec{k}$. Again, this is in accordance with the definitions used by Itzykson and Zuber [52]. The physical pion operators are then defined by:

$$\phi_\pm = \mp \frac{1}{\sqrt{2}}(\phi_1 \pm i \phi_2),$$  \hspace{1cm} (2.9)$$

$$\phi_0 = \phi_3.$$  \hspace{1cm} (2.10)$$

Note that while the operators $\phi_i$ are self-adjoint the operators $\phi_\pm$ are not self-adjoint. Since pions are bosons the operators $a_i$ and $a_\dagger_i$ obey the commutation relations:

$$[a_i(\vec{k}), a_\dagger_{i'}(\vec{k}')] = 2\omega_\pi(\vec{k})(2\pi)^3 \delta^{(3)}(k - k')\delta_{ii'},$$  \hspace{1cm} (2.11)$$

$$a_{i'}(\vec{k}) a_i(\vec{k}) = 0, \quad a_\dagger_{i'}(\vec{k}) a_i(\vec{k}) = 0, \quad a_{i'}(\vec{k}) a_\dagger_i(\vec{k}) = \delta_{ii'},$$  \hspace{1cm} (2.12)$$

with all other commutators being zero.

These commutation relations are written for the Schrödinger operators. Naturally, they remain the same if we change to some other representation. The two other representations used in this paper are the Heisenberg representation, which we shall denote by a tilde above the operator in question, e.g. $\tilde{\psi}, \tilde{\phi}$; and the interaction representation, which we denote by a superscript $I$, e.g. $\psi^I, \phi^I$. Schrödinger representation operators will remain unadorned. These three different representations are connected by the standard transformations involving the Hamiltonian of the system.

We define this Hamiltonian via the Lagrangian density, $\mathcal{L}$. We choose:

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_\phi + \mathcal{L}_{int},$$  \hspace{1cm} (2.13)$$

where:

$$\mathcal{L}_D(x) = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x),$$  \hspace{1cm} (2.14)$$

$$\mathcal{L}_\phi(x) = \frac{1}{2}((\partial_\mu \bar{\phi}(x) \cdot \partial^\mu \phi(x)) - m_\pi^2 \bar{\phi}(x) \cdot \phi(x)),$$  \hspace{1cm} (2.15)$$

$$\mathcal{L}_{int}(x) = -ig \int d^4x_N d^4x_N' d^4x_\pi \bar{\psi}(x_N') \gamma_5 \bar{\tau}\psi(x_N) \cdot \bar{\phi}(x_\pi) \Gamma(x - x_N', x - x_N, x - x_\pi).$$  \hspace{1cm} (2.16)$$

Here we have chosen pseudo-scalar coupling for the $\pi - N$ interaction Lagrangian density $\mathcal{L}_I(x)$, and $\Gamma(x - x_N', x - x_N, x - x_\pi)$ is the form factor for this interaction. We introduce this form factor in order to model the finite size of all the particles involved. It will also allow us to remove the divergences from the theory. (See Figure 5 for a depiction of this vertex.)

This Lagrangian is that used in the Walecka model (or Quantum Hadrodynamics-1) [7,8] except for two differences:
1. There are no counter-terms to do the renormalization with. Instead, we include a form factor in the pseudo-scalar interaction Lagrangian, thus allowing us to introduce a cut-off and so remove the infinities from the theory.

2. In the QHD Lagrangian there are fields corresponding to one family of vector mesons and one neutral scalar meson. The above Lagrangian has only one family of pseudo-scalar mesons, the pions. Note, however, that this Lagrangian could easily be extended to include vector mesons.

The use to which this Lagrangian is put will also be different. The QHD Lagrangian is used for doing nuclear structure calculations and consequently is always solved as a bound-state problem, often in the mean-field approximation. Conversely, we will be using our Lagrangian to do scattering theory—in particular, the scattering theory of the $NN - \pi NN$ system.

From this Lagrangian, we may define the Hamiltonian density, $\mathcal{H}$, by a Legendre transformation:

$$\mathcal{H}(x) = \frac{\delta L}{\delta \partial_0 \psi} \partial_0 \psi + \frac{\delta L}{\delta \partial_0 \phi} \cdot \partial_0 \phi - \mathcal{L} \quad (2.17)$$

and hence obtain the Hamiltonian:

$$H = \int d^3x \mathcal{H}(x) = H_K + H_{int}, \quad (2.18)$$

where:

$$H_K = \int d^3x \overline{\psi}(x) T_N(x) \psi(x) + \int d^3x \overline{\phi}(x) T_{\pi}(x) \cdot \phi(x), \quad (2.19)$$

$$H_{int} = +ig \int d^3x \left( \int d^4x_N d^4x_{\pi} \overline{\psi}(x_N) \gamma_5 \overline{\pi} \psi(x_N) \cdot \phi(x_{\pi}) \Gamma(x - x_N, x - x_{\pi}, x - x_{\pi}) \right). \quad (2.20)$$

Note that we may define an interaction Hamiltonian density

$$\mathcal{H}_{int}(x) = ig \int d^4x_N d^4x_{\pi} \overline{\psi}(x_N) \gamma_5 \overline{\pi} \psi(x_N) \cdot \phi(x_{\pi}) \Gamma(x - x_N, x - x_{\pi}, x - x_{\pi}). \quad (2.21)$$

The connection between the position space operators of different representations is now:

$$\tilde{\psi}(x) = e^{iHx^0} \psi(x) e^{-iHx^0} \quad (2.22)$$

$$\psi^I(x) = e^{iH_K x^0} \psi(x) e^{-iH_K x^0}. \quad (2.23)$$

From these we may deduce the results for $\tilde{\psi}$. Furthermore, it is clear that the same equations will hold if we replace $\psi$ by any component of $\phi$.

1The canonical momenta, $\pi$, for the field $\phi$, may not be given by the usual relation, $\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi}$, due to the presence of the dissipative force in the interaction Lagrangian. If this is the case it means that the relation between the Lagrangian and the Hamiltonian is more complicated than that given in Equation (2.17). We ignore this difficulty for the present, since we are only really interested in the overall final form of the Hamiltonian, which we may assume is given by Eqs. (2.18), (2.19) and (2.20).
III. THE SINGLE-NUCLEON AND SINGLE-PION GREEN’S FUNCTIONS

In this section we first give a brief explanation of how to obtain a diagrammatic expansion for the single-particle Green’s functions, and how to find the Feynman Rules in coordinate and momentum space for the calculation of these Green’s functions as a perturbation series of Feynman diagrams. This is done primarily to fix normalizations and phases. We then look at the form taken by the free Green’s functions in the low-energy limit, when anti-nucleonic contributions to the Green’s function can be ignored.

We define the one-particle Green’s functions for nucleons and pions by:

\[ G_N(x', x) = \langle 0 | T(\overline{\psi}(x')\psi(x)) | 0 \rangle, \]  
\[ G_{\pi ji}(x', x) = \langle 0 | T(\overline{\phi}_j(x')\phi_i(x)) | 0 \rangle, \]  

respectively, where \( T \) denotes the usual time-ordering operator.

Consider either of the one-particle Green’s functions. By following the standard procedure as outlined in e.g. Itzykson and Zuber \[52\], perturbation series for \( G_N(x', x) \) and \( G_{\pi ji}(x', x) \) may be derived. We may then use Wick’s theorem to rewrite each term in these series as a sum of products of the free single-particle Green’s functions:

\[ G^{(0)}_N(x', x) = \langle 0 | T(\psi^I(x')\psi^I(x)) | 0 \rangle, \]  
\[ G^{(0)}_{\pi ji}(x', x) = \langle 0 | T(\phi^I_j(x')\phi^I_i(x)) | 0 \rangle. \]  

Consequently we are led to the standard interpretation of the perturbation series as a series of Feynman diagrams. Each Feynman diagram corresponds to an analytic expression given by the Feynman Rules for the theory. The Feynman Rules in coordinate space for the theory of nucleons and pions obtained from the Hamiltonian of the previous Section are the standard ones given in Bjorken and Drell, pp.224-5 \[34\], with two changes: (i) A form factor \( \Gamma \) for pion absorption and emission must be added, and (ii) Rule 4 must be replaced by the following rule:

“4. Assign coordinates \( x \) and \( x' \) to the external points. If dealing with \( G_{\pi ji}(x', x) \) also assign isospin indices \( j \) and \( i \) to the external points. Then, if a nucleon line joins \( x \) (or \( x' \)) to an internal point \( y \), associate with it a factor \( G^{(0)}_N(y, x) \) \( (G^{(0)}_N(x', y)) \). If a pion line joins \( x \) (\( x' \)), with isospin label \( i(j) \), to a vertex \( y \), with isospin index \( k \), then include a factor \( G^{(0)}_{\pi ki}(y, x) \) \( (G^{(0)}_{\pi kj}(x', y)) \) for that line.”

Note that the change comes about because we are dealing with Green’s functions whereas Bjorken and Drell dealt with amplitudes.

Now we may define the Green’s functions in momentum space, via:

\[ G_H(p', p) = \int d^4x d^4x' e^{ip'x'}G_H(x', x)e^{-ipx}, \]  

where \( H \) is either a nucleon or a pion. By translation invariance:

\[ G_H(x', x) = G_H(x' - x) \]  

and so it can be shown that:

\[ G_H(p', p) = (2\pi)^4\delta^{(4)}(p' - p)G_H(p), \]
where:

\[ G_H(p) = \int d^4x_R e^{ipx_R} G_H(x_R). \]  

where \( G_H(x_R) = G_H(x' - x) \) is the coordinate space Green’s function, the rules for which are given in Bjorken and Drell [34].

Consequently we may find the momentum space Green’s function by first using Wick’s theorem to get an expansion for \( G_H(x' - x) \) in terms of the free nucleon and free pion Green’s functions, and then inserting the Fourier representations for the free Green’s functions \( G_N^{(0)}(y' - y) \) and \( G_{\pi ji}^{(0)}(y' - y) \) and the form factor \( \Gamma(y', y, y_\pi) \). For the free Green’s functions we have:

\[ G_N^{(0)}(y' - y) = \int \frac{d^4\tilde{q}}{(2\pi)^4} e^{-i\tilde{q}(y' - y)} G_N^{(0)}(\tilde{q}), \]  

\[ G_{\pi ji}^{(0)}(y' - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(y' - y)} G_{\pi ji}^{(0)}(k). \]  

For the form factor we define:

\[ \Gamma(q', q, k) = \int d^4yd^4y'd^4y_\pi e^{-iq'y'}e^{iky_\pi}e^{iqy}\Gamma(y', y, y_\pi), \]  

and the inverse of this Fourier Transformation may be used to find the Fourier Representation of \( \Gamma(y', y, y_\pi) \).

The formulae thus obtained are substituted into the expression for a particular Feynman diagram, the coordinate space integrations are performed and finally the Fourier transform of the result is taken using Eq.(3.8). (For more detail on this procedure see Itzykson and Zuber pp.267-8 [52].) When this is done the following Feynman Rules for the construction of \( G_H(p) \) are obtained.

**Feynman Rules for the single-particle Green’s function in momentum space**

1. Draw all topologically distinct, connected diagrams with one incoming leg and one outgoing leg.

In each diagram:

2. Assign to the external legs four-momentum \( p \), and to the internal nucleon and pion legs four-momenta \( k_1, k_2, \ldots, k_I \), where \( I \) is the total number of internal lines. To each \( \pi NN \) vertex assign an isospin label \( l_1, l_2, \ldots, l_n \). If the external legs are pion legs assign to the incoming (outgoing) pion leg’s endpoint an isospin label \( i \) (\( j \)).

3. To the nucleon line with four-momentum \( q \), where \( q \) equals some \( k_j, j = 1, \ldots, I \), assign a factor \( G_N^{(0)}(q) \).

4. To the pion line with four-momentum \( k \), connecting vertices with isospin labels \( l_e, l_a \), assign a factor \( G_{\pi l_earlabel}^{(0)}(k) \).
5. To each pion emission vertex assign a factor:

\[ (-ig)(i\gamma_5\tau_l)(2\pi)^4\Gamma(q',q,-k)\delta^{(4)}(q' + k - q); \]

to each pion absorption vertex assign a factor:

\[ (-ig)(i\gamma_5\tau_l)(2\pi)^4\Gamma(q',q,k)\delta^{(4)}(q' - q - k); \]

to each \(N\bar{N}\) annihilation vertex assign a factor:

\[ (-ig)(i\gamma_5\tau_l)(2\pi)^4\Gamma(-q',q,-k)\delta^{(4)}(k - q - q'); \]

and to each \(N\bar{N}\) creation vertex assign a factor:

\[ (-ig)(i\gamma_5\tau_l)(2\pi)^4\Gamma(q',-q,k)\delta^{(4)}(q' + q - k). \]

In each case \(q (q')\) is the nucleon four-momentum before (after) the vertex (where applicable), \(k\) is the pion four-momentum and \(l\) is the isospin index assigned to the vertex.

6. Include a factor of (-1) for each closed fermion loop.

7. Integrate over all internal momenta:

\[
\frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \cdots \frac{d^4k_I}{(2\pi)^4}
\]

and sum over all repeated isospin indices.

It is these rules we shall always work with in the calculations. But, in order to calculate with these rules we need to know what the free single-particle propagators \(G_N^{(0)}(q)\) and \(G_\pi^{(0)}(q)\) are. It follows from Eqs.(2.1), (2.2), (3.1) and (3.8) for the free single-nucleon Green’s function and Eqs.(2.7), (3.2), (3.8) for the free single-pion Green’s function that:

\[
G_N^{(0)}(q) = \frac{i}{q - m}, \tag{3.12}
\]

\[
G_\pi^{(0)}_{ji}(k) = \frac{i\delta_{ij}}{k^2 - m_{\pi}^2}, \tag{3.13}
\]

i.e. the free single-particle Green’s functions are the full covariant propagators. (See Appendix A for the details of this calculation.)

The no-anti-nucleon approximation

However, the use of these full covariant propagators results, in most cases, in a very difficult calculation. In order to simplify the calculation somewhat, a restriction is placed upon Green’s functions in this work. We make the approximation that the contribution made by anti-nucleons to the Green’s functions is negligible, an approximation which should be true in the low energy limit. Making this approximation means that we force:
We expect this to be true at the energies we are interested in. It immediately implies that:

\[ G_N^{(0)}(x', x) = \langle 0 | \psi^I(x') \bar{\psi}^I(x) | 0 \rangle \theta(x'_0 - x_0), \]  

(3.15)

as one would expect in the absence of anti-nucleons. It is also found that, as a direct consequence of the derivation of the single-nucleon Green’s function in momentum space, we have:

\[ G_N^{(0)}(q) = \frac{im}{E_N(q)} \sum_\alpha u_\alpha(q) \bar{u}_\alpha(q) q^+ - E_N(q), \]  

(3.16)

in this approximation. (See Appendix A for the justification of these two statements.)

This no-anti-nucleon approximation not only affects the free single-nucleon propagator, it imposes a definite time-ordering on any process contributing to the single-nucleon Green’s function. This occurs because the theta-functions in the free single-nucleon propagators define the order of any two pion emission and absorption times \( y_0 \) and \( y'_0 \). I.e., the absence of anti-nucleons determines which of the two times in any free-pion Green’s function, \( G_\pi^{(0)}(y', y) \), comes first and which second. So, we have to replace \( G_\pi^{(0)}(y', y) \) by \( G_\pi^{(0)}(y', y) \theta(y'_0 - y_0) \), where \( y'_0 \) is the later of the two times. In momentum space this has the effect of replacing Eq.(3.13) by:

\[ G_{\pi ji}^{(0)}(k) = \frac{1}{2\omega_\pi(k)} \frac{i \delta_{ij}}{k^+ - \omega_\pi(k)}. \]  

(3.17)

(Again, a justification of this fact is to be found in Appendix A.)

Furthermore, in the approximation in which anti-nucleons cannot be created, it follows from the conservation of nucleon number that no processes other than free pion propagation can contribute to the single-pion Green’s function. Therefore:

\[ G_\pi^{(0)}(k) = G_\pi^{(0)}(k). \]  

(3.18)

By contrast, if we attempt to express \( G_N \) in terms of \( G_N^{(0)} \) we find that \( G_N \) obeys a Schwinger-Dyson equation.

So, the no-anti-nucleon approximation reduces the momentum space propagators from full covariant propagators to those given in Eqs.(3.16) and (3.17) and prevents any process other than free propagation from contributing to the single-pion Green’s function.

IV. THE TWO-NUCLEON GREEN’S FUNCTION

In the previous section we explained how to obtain a diagrammatic expansion and Feynman Rules in both coordinate and momentum space for the single-nucleon and single-pion Green’s functions. Since our aim is to develop a theory of the \( NN - \pi NN \) system we should next consider the \( NN \) Green’s function and its relation to \( G_N \) and \( G_{\pi ji} \).

The two-nucleon Green’s function is defined by:
The procedure for obtaining the Feynman rules for this Green's function is exactly the same as that used for the single-particle Green's functions in the previous section. Upon implementing that procedure we find that the rules obtained in coordinate space for the two-nucleon Green's function are the same as those for the one-nucleon Green's function, with two modifications: (i) In the two-nucleon case diagrams must obviously have two incoming and two outgoing external nucleon lines, and (ii) Disconnected diagrams are now permitted, provided they contain no sub-diagrams which are not ultimately joined to the external lines. (For examples of the type of diagrams which are and are not allowed see Figure 6.)

Consider the two-nucleon Green's function $G_{NN}(x_1',x_2';x_1,x_2)$. Suppose that we restrict the times the particles begin and end their propagation, so that the two nucleons begin their propagation at equal times, i.e.:

$$x_1^0 = x_2^0 \equiv t, \quad (4.2)$$

and end it at equal times, i.e.:

$$x_1'^0 = x_2'^0 \equiv t'. \quad (4.3)$$

Then we denote the Green's function by:

$$\mathcal{G}_{NN}(t',\vec{x}_1',\vec{x}_2';t,\vec{x}_1,\vec{x}_2). \quad (4.4)$$

Now consider the Fourier Transform of this Green's function. Because there is only one initial and one final time, when the Fourier Transform with respect to energy is taken, instead of transforming with respect to the individual particle variables $p_1^0$, $p_2^0$, $p_1'^0$ and $p_2'^0$, which are conjugate to $x_1^0$, $x_2^0$, $x_1'^0$ and $x_2'^0$, we must transform with respect to variables $E$ and $E'$ which are conjugate to $t$ and $t'$. Therefore,

$$\mathcal{G}_{NN}(E',\vec{p}_1',\vec{p}_2';E,\vec{p}_1,\vec{p}_2)$$

$$= \int dt \, dt' \, d^3x_1'd^3x_2'd^3x_1''d^3x_2''e^{i(E'E'-\vec{p}_1'\cdot\vec{x}_1'-\vec{p}_2'\cdot\vec{x}_2')}G_{NN}(t',\vec{x}_1',\vec{x}_2',t,\vec{x}_1,\vec{x}_2)e^{-i(Et-\vec{p}_1\cdot\vec{x}_1-\vec{p}_2\cdot\vec{x}_2)} \quad (4.5)$$

$$= \int d^4x_1 \, d^4x_2 \, d^4x_1' \, d^4x_2' e^{i(E'E'-\vec{p}_1'\cdot\vec{x}_1'-\vec{p}_2'\cdot\vec{x}_2')} \delta(x_1'^0 - x_2'^0)G_{NN}(x_1',x_2';x_1,x_2)\delta(x_1^0 - x_2^0)$$

$$\times e^{-i((Ex_1 - \vec{p}_1)\cdot\vec{x}_1 - (Ex_2 - \vec{p}_2)\cdot\vec{x}_2),} \quad (4.6)$$

where we have recalled that $x_1^0 \equiv t$ and $x_1'^0 \equiv t'$ and then used the two-time Green's function $G_{NN}(x_1',x_2';x_1,x_2)$ in place of the equal-time Green's function. Inserting:

$$\delta(a-b) = \int \frac{dz}{2\pi} e^{iz(a-b)}, \quad (4.7)$$
and then performing the Fourier transforms over \(x_1, x_2, x'_1\) and \(x'_2\) gives:

\[
\bar{G}_{NN}(E', \vec{p}'_1, \vec{p}'_2; E, \vec{p}_1, \vec{p}_2) = \int \frac{dz \, dz'}{(2\pi)^2} G_{NN}(E' - z', \vec{p}'_1, \vec{p}'_2; E - z, \vec{p}_1, z, \vec{p}_2),
\]

where \(G_{NN}(p'_1, \vec{p}'_1, p'_2, \vec{p}'_2; p_1, \vec{p}_1, p_2, \vec{p}_2)\) is the Fourier transform of \(G_{NN}(x'_1, x'_2; x_1, x_2)\). At this stage it becomes clear that \(E\) and \(E'\) are the initial and final total energy of the system, respectively:

\[
E = p_1^0 + p_2^0,
\]

\[
E' = p'_1 + p'_2.
\]

It can also be seen that the integral over \(z\) and \(z'\) corresponds to an integral over all possible initial and final "relative" energies of the two-nucleon system.2

Now the translational invariance of \(G_{NN}(x'_1, x'_2; x_1, x_2)\) implies momentum conservation in \(G_{NN}(p'_1, p'_2; p_1, p_2)\), i.e.

\[
G_{NN}(p'_1, p'_2; p_1, p_2) = (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)G_{NN}(p'_1, p'_2; p_1, p_2).
\]

This suggests that:

\[
G_{NN}(E' - z', \vec{p}'_1, \vec{p}'_2; E - z, \vec{p}_1, z, \vec{p}_2) = (2\pi)^4 \delta(E' - E)\delta(z')G_{NN}(E - z', \vec{p}'_1, z', \vec{p}'_2; E - z, \vec{p}_1, z, \vec{p}_2). \tag{4.12}
\]

Further, translational invariance of \(\bar{G}_{NN}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2)\) implies:

\[
\bar{G}_{NN}(E', \vec{p}'_1, \vec{p}'_2; E, \vec{p}_1, \vec{p}_2) = (2\pi)^4 \delta(E' - E)\delta(z')G_{NN}(E, \vec{p}'_1, \vec{p}'_2; \vec{p}_1, \vec{p}_2).
\]

These last two results suggest that Eq. (4.8) may be rewritten as:

\[
\bar{G}_{NN}(E, \vec{p}'_1, \vec{p}'_2; E, \vec{p}_1, \vec{p}_2) = \int \frac{dz \, dz'}{(2\pi)^2} G_{NN}(E - z', \vec{p}'_1, z', \vec{p}'_2; E - z, \vec{p}_1, z, \vec{p}_2). \tag{4.15}
\]

Therefore the problem of calculating \(G_{NN}(E, \vec{p}'_1, \vec{p}'_2; \vec{p}_1, \vec{p}_2)\) is reduced to that of calculating \(G_{NN}(p'_1, p'_2; p_1, p_2)\). But \(G_{NN}(p'_1, p'_2; p_1, p_2)\) is to be calculated using the Feynman Rules in momentum space for the two-nucleon Green’s function. To obtain these Feynman rules from those in coordinate space we merely proceed as we did above for the one-nucleon Green’s function. This leads to Feynman Rules for the calculation of \(G_{NN}(E - z', \vec{p}'_1, \vec{p}'_2; E - z, \vec{p}_1, z, \vec{p}_2)\) which are exactly the same as those for the single-particle Green’s function, except that Rules 1 and 2 become:

\[\text{Note that the definition of "relative" energy used here differs from the standard } k^0 = p_1^0 - p_2^0. \]

Instead, when the term relative energy is used in this paper it refers to the energy \(z = p_2^0 - E - p_1^0\).
1. Draw all topologically distinct diagrams with two incoming and two outgoing legs, excluding diagrams in which any sub-diagram is not ultimately connected to an external line.

For each diagram:

2. Assign to the incoming (outgoing) external nucleon legs energies $E - z$ and $z$ ($E - z'$ and $z'$) and momenta $\vec{p}_1$ and $\vec{p}_2$ ($\vec{p}_1'$ and $\vec{p}_2'$). Assign to the internal nucleon and pion legs momenta $k_1, \ldots, k_I$, where $I$ is the number of internal lines. Note that once the momenta and energy of the particles in the initial state are chosen there are two ways of choosing the momenta and energy of the particles in the final state. Both possible assignments must be included as separate diagrams. The two resulting diagrams differ by a factor of (-1), since the nucleons are fermions.

This is the only change which needs to be made to the full covariant theory in which anti-nucleons are included. However, if we are using the no-anti-nucleon approximation another change is needed in order to maintain consistency in our work, as follows. Recall that, in the calculation of the single-nucleon Green’s function we had to replace $G_{\pi ji}(0)(y' - y)$ by $G_{\pi ji}(0)(y' - y)\theta(y_0 - y_0)$, because the no-anti-nucleon approximation imposed a definite time-order on pion propagation. However, in the two-nucleon Green’s function there is nothing which specifies the relative time-order of processes taking place on different nucleons. Therefore, pions emitted on one nucleon and absorbed on the other do not have the time-order of their emission and absorption specified, whereas, as described above, pions emitted and absorbed on the same nucleon do have that time-order specified. So, in order to treat transmitted pions and pions that are reabsorbed by the same nucleon which emits them equivalently, we split the transmitted pion’s Green’s function $G_{\pi ji}(0)(y_2, y_1)$ into two possibilities, according to whether $y_1^0 < y_2^0$ or $y_2^0 < y_1^0$, and insert appropriate theta functions. Therefore, for transmitted pions one diagram in which we have $G_{\pi ji}(0)(y', y)$ is replaced by two diagrams, in one of which we have $G_{\pi ji}(0)(y', y)\theta(y_0 - y_0)$, and in the other of which we have $G_{\pi ji}(0)(y', y)\theta(y_0 - y_0)$. This rule ensures that the relative time-order of a particular pion’s emission and absorption is always determined. It is important to note that, other than this restriction, the relative time-orders of processes on different nucleons remains undetermined.

If we use the no-anti-nucleon approximation and make this change in order to maintain consistency then we still have the same rules for the two-nucleon Green’s function as in the full covariant theory, but with the simplified propagators given in Eqs.(3.16) and (3.17).

V. THE MULTI-PARTICLE GREEN’S FUNCTION

The argument for the $m \rightarrow m'$ particle Green’s function is constructed exactly as are the above arguments for the one and two-nucleon Green’s functions. The only change necessary to the Feynman rules for the two-nucleon Green’s function in coordinate space is that we must now draw all possible topologically distinct diagrams with $m$ incoming and $m'$ outgoing legs.

Now we examine the equal-time Green’s function, in order to derive a result analogous to Equation (4.8). The argument used is similar to that by which Eq.(4.8) was derived.
The Fourier transform of the equal-time $m \to m'$ particle Green’s function:

$$\overline{G}(t', \vec{x}', \vec{x}'_2, \ldots, \vec{x}'_m; t, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m),$$

(5.1)

which we denote by $\overline{G}(E', \vec{p}_1', \vec{p}_2', \ldots, \vec{p}_m'; E, \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m)$, is given by the formula:

$$\overline{G}(E', \vec{p}_1', \vec{p}_2', \ldots, \vec{p}_m'; E, \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m) = \int \frac{dz^{(2)}dz^{(3)} \ldots dz^{(m)}dz^{(2)'}dz^{(3)'} \ldots dz^{(m)'}{(2\pi)^{m+m'-2}}}{(2\pi)^{m+m'-2}}$$

$$\times G(E' - \sum_{i=2}^{m'} z^{(i)'}p_{1i}', \vec{z}^{(2)'}\vec{p}_{2i}', \ldots, \vec{z}^{(m)'}p_{mi}'; E - \sum_{i=2}^{m} z^{(i)}\vec{p}_i, \vec{z}^{(2)}\vec{p}_2, \ldots, \vec{z}^{(m)}\vec{p}_m),$$

where $G(p_1', p_2', \ldots, p_{m'}'; p_1, p_2, \ldots, p_m)$ is to be calculated using the Feynman rules in momentum space for the $m \to m'$ particle Green’s function. These are the Feynman rules for the two-nucleon Green’s function, with changes to Rules 1 and 2 to accommodate the different numbers of particles in the initial and final states. In Rule 1 we replace ”two incoming and two outgoing” by ”$m$ incoming and $m'$ outgoing” and Rule 2 becomes:

”2. Assign to the incoming external particle legs four-momenta $p_1, p_2, \ldots, p_m$; assign to the outgoing legs four-momenta $p_1', p_2', \ldots, p_{m'}$. Assign to the internal nucleon and pion legs momenta $k_1, \ldots, k_I$, where $I$ is the number of internal lines. Note that once the four-momenta of the particles in the initial state are chosen there are $m!$ ways of choosing the four-momenta of the particles in the final state. All possible assignments must be included, as separate diagrams, with appropriate relative signs. If, for one such diagram, we make an assignment of the fermion four-momenta $(p_1', \ldots, p_{n_F}')$, where $n_F$ is the number of fermions in the final state, then the relative sign of the other $m! - 1$ diagrams generated from it by permutation of the fermion four-momenta is $(+1)$ for an even permutation of $(p_1', \ldots, p_{n_F}')$ and $(-1)$ for an odd permutation.”

Therefore, the equal-time $m \to m'$ single-energy Green’s function:

$$\overline{G}(E, \vec{p}_1, \vec{p}_2', \ldots, \vec{p}_m', \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m)$$

may be found by calculating

$$G(E' - \sum_{i=2}^{m'} z^{(i)'}\vec{p}_i, \vec{z}^{(2)'}\vec{p}_2, \ldots, \vec{z}^{(m)'}\vec{p}_m'; E - \sum_{i=2}^{m} z^{(i)}\vec{p}_i, \vec{z}^{(2)}\vec{p}_2, \ldots, \vec{z}^{(m)}\vec{p}_m)$$

according to these rules and integrating over $z^{(2)}, z^{(3)}, \ldots, z^{(m)}, z^{(2)'}', \ldots, z^{(m)'}'$. Again, this result is true regardless of whether or not the anti-nucleon approximation is used or not.

VI. DERIVATION OF OLD-FASHIONED TIME-ORDERED PERTURBATION THEORY AND THE RESULT OF KVINKIHKIDZE AND BLANKLEIDER FROM THIS PERTURBATION SCHEME

In this section we will show how the above formalism allows the summation of all relative time-orders in an old-fashioned time-ordered perturbation theory diagram, and so derive the result of Kvinikhidze and Blankleider (KB) [31]. In the light of this calculation we will then comment on certain approximations made by KB in their work on the $NN - \pi NN$ problem.
In order to achieve these goals it is necessary to first examine how to derive old-fashioned perturbation theory rules for the two-nucleon Green’s function from the above Feynman rules. It will be clear from the structure of this derivation that we could generalize this argument to the $m \rightarrow m'$ Green’s function.

A. Derivation of Feynman rules for time-ordered perturbation theory

Consider the equal-time Green’s function in the no-anti-nucleon approximation:

$$G_{NN}(t', \vec{x}', \vec{x}; t, \vec{x}_1, \vec{x}_2).$$

Define equal-time two-nucleon annihilation and creation operators in the Heisenberg representation via:

$$\tilde{\psi}_{NN}(t, \vec{x}_1, \vec{x}_2) = \tilde{\psi}(t, \vec{x}_1)\tilde{\psi}(t, \vec{x}_2),$$

$$\tilde{\psi}_{NN}(t, \vec{x}_1, \vec{x}_2) = \tilde{\psi}(t, \vec{x}_2)\tilde{\psi}(t, \vec{x}_1).$$

The interaction and Schrödinger representation equal-time two-nucleon operators may then be found in the usual way.

Currently, the perturbation expansion for $G_{NN}(t', \vec{x}', \vec{x}; t, \vec{x}_1, \vec{x}_2)$ is written in terms of individual particle operators, i.e. $\psi^{I_s}$, $\overline{\psi}^{I_s}$ and $\phi^{I_i}$. The first step in our derivation of time-ordered perturbation theory is to rewrite this perturbation expansion in terms of the two-nucleon operator, $\psi^{I}_{NN}$, and the pionic operators, $\phi^{I}_{i}$. The only change which needs to be made in order to rewrite the perturbation expansion in this way is to prove that the $\pi NN$ vertex:

$$H_{\text{int}}(x) = ig \int d^4x_N d^4x'_N d^4x_\pi \psi(x_N)\gamma_5 \tau \psi(x_N) \cdot \bar{\phi}(x_\pi) \Gamma(x - x'_N, x - x_N, x - x_\pi),$$

which we currently use, may be replaced by a vertex:

$$H_{\text{int}(2)}(x) = ig \int d^3x' \int d^4x_N d^4x_\pi \tilde{\psi}(x') \gamma_5 \tau \psi(x_N) \cdot \bar{\phi}(x_\pi) \Gamma(x - x'_N, x - x_N, x - x_\pi),$$

in which not a single nucleon, but a nucleon pair, is destroyed and recreated. (See Figure 4 for a diagrammatic representation of this new vertex.) The proof that, in the absence of anti-nucleons, rewriting the vertex in this way does not change the value of the Green’s function is outlined in Appendix B. This result then allows us to rewrite the expansion for $G_{NN}(t', \vec{x}', \vec{x}; t, \vec{x}_1, \vec{x}_2)$ obtained from Wick’s theorem in terms of the free Green’s functions:

$$G_{NN}^{(0)}(t', \vec{x}', \vec{x}; t, \vec{x}_1, \vec{x}_2) = \langle 0 | T(\psi_{NN}^{I}(t', \vec{x}', \vec{x}) \bar{\psi}_{NN}(t, \vec{x}_1, \vec{x}_2)) | 0 \rangle,$$

$$G_{\pi j}^{(0)}(y', y) = \langle 0 | T(\phi_{i}^{I}(y') \phi_{i}^{I}(y)) | 0 \rangle.$$
\( \mathcal{G}_{NN}^{(0)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2) = 0 \) if \( t > t' \).

(6.8)

Since in the no-anti-nucleon approximation this Green’s function must appear between any two vertices it follows that the time-order of all vertices is now determined. This is to be contrasted with the perturbation scheme used in the previous section where the relative time-order of most of the events occurring on different nucleons was not determined. In that perturbation scheme absorption and emission of exchanged pions were the only events occurring on different nucleons whose relative time-order was specified. In this perturbation scheme the relative time-order of all events is determined.

Other than this change, however, the two perturbation schemes result in the same Green’s function, since the rewriting of the equal-time two-nucleon Green’s function in terms of free two-nucleon Green’s functions does not change its value. Suppose then that we call the result of a particular diagram evaluated according to the rules of the previous Section:

\[ \mathcal{G}_{NN}^{(1)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2), \]

(6.9)

where the subscript \((1)\) indicates the use of one-nucleon Green’s functions in the calculation. Suppose also that we call the result constructed by the rules obtained below for the theory with a two-nucleon vertex:

\[ \mathcal{G}_{NN}^{(2)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2), \]

(6.10)

where the subscript \((2)\) indicates the use of two-nucleon Green’s functions in this calculation. Then it follows that:

\[ \sum \text{All relative TOs} \mathcal{G}_{NN}^{(2)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2) = \mathcal{G}_{NN}^{(1)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2), \]

(6.11)

where ”All relative TOs” indicates that the sum is over all possible relative time-orders of events on different nucleons, with the exception of the emission and absorption of exchanged pions, whose relative time-order is fixed as soon as the diagram to be considered is chosen. This result may also be deduced from Equation (B15) which was derived in Appendix B. Note that both Green’s functions must be evaluated in the no-anti-nucleon approximation if this equation is to hold. We shall return to Eq.(6.11) shortly.

First, however, we must complete our development of time-ordered perturbation theory, for we have not yet explained how the rules usually used in the calculation of \( \mathcal{G}_{NN}^{(2)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2) \) come about.

Take the Fourier Transform of the Green’s function:

\[ \mathcal{G}_{NN}^{(2)}(t', \vec{x}'_1, \vec{x}'_2; t, \vec{x}_1, \vec{x}_2), \]

in order to obtain:

\[ \mathcal{G}_{NN}^{(2)}(E', \vec{p}'_1, \vec{p}'_2; E, \vec{p}_1, \vec{p}_2). \]

Just as was done for \( \mathcal{G}_{NN}^{(1)} \) above, in Eq.(4.14), energy-momentum conservation in this Green’s function may be expressed via:

\[ \mathcal{G}_{NN}^{(2)}(E', \vec{p}'_1, \vec{p}'_2; E, \vec{p}_1, \vec{p}_2) = (2\pi)^4 \delta(E' - E) \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \mathcal{G}_{NN}^{(2)}(E, \vec{p}'_1, \vec{p}'_2, \vec{p}_1, \vec{p}_2). \]

(6.12)

The rules for calculating \( \mathcal{G}_{NN}^{(2)}(E, \vec{p}'_1, \vec{p}'_2, \vec{p}_1, \vec{p}_2) \) are found from the rules in co-ordinate space by the standard procedure explained in Section III. The rules thus obtained are as follows:
Feynman Rules for the two-nucleon Green’s function in terms of the free two-nucleon Green’s function, in the no-anti-nucleon approximation.

1. Draw all those topologically distinct, connected diagrams which have two incoming and two outgoing legs. Remember that different time-orders now contribute to different diagrams.

In each diagram:

2. Label the pion legs with momenta \( k_1, \ldots, k_{I'} \), label the external nucleon pairs with combined energy-momenta \((E, \vec{p}_1, \vec{p}_2)\) and \((E', \vec{p}_1', \vec{p}_2')\). There are two possible ways of assigning the momenta \( \vec{p}_1' \) and \( \vec{p}_2' \) in the final state, and both possibilities should be included as separate diagrams, with a relative minus sign. Label the internal nucleon pairs with combined energy-momenta:

\[
(E^{(1)}, \vec{q}_1^{(1)}, \vec{q}_2^{(1)}), (E^{(2)}, \vec{q}_1^{(2)}, \vec{q}_2^{(2)}), \ldots, (E^{(n-1)}, \vec{q}_1^{(n-1)}, \vec{q}_2^{(n-1)}).
\]

Assign isospin labels \( l_1, l_2, \ldots, l_n \) to the \( n \) vertices. Note that because of our rewriting of \( H_{\text{int}} \) above, a new nucleon pair is regarded as being created whenever a vertex occurs on either nucleon.

3. To the nucleon pair with combined energy-momentum \((E, \vec{q}_1, \vec{q}_2)\) assign a factor \( G_{NN}^{(0)}(E, \vec{q}_1, \vec{q}_2) \).

4. To the pion line with four-momentum \( k \), joining vertices with isospin labels \( l_e \) and \( l_a \), assign a factor \( G_{\pi l_a l_e}^{(0)}(k) \).

5. For each pion absorption vertex occurring on the nucleon with momentum \( \vec{q}_i \) insert a factor:

\[
(-ig)(2\pi)^7\delta(E' - k_0 - E)\delta^{(3)}(q_i' - k - q_i)\delta^{(3)}(q_i' - q_i)(i\gamma_5\tau_i)
\times \Gamma(E' - E_N(q_i'), q_i', E - E_N(q_i), q_i, k_0, \vec{k}).
\]

For each emission vertex occurring under the same assumptions insert a similar factor, but with \( (k_0, \vec{k}) \) replaced by \( (-k_0, -\vec{k}) \).

Here

\[
\tilde{i} = \begin{cases} 
2 & \text{if } i = 1 \\
1 & \text{if } i = 2.
\end{cases}
\]

Also, \( (E, q_i, \vec{q}_i) \) is the combined energy-momentum of the nucleon-pair before the pion emission or absorption; \( (E', q_i', \vec{q}_i') \) is the combined energy-momentum of the nucleon-pair afterwards, \( (k_0, \vec{k}) \) is the four-momentum of the pion and \( l \) is the isospin label assigned to the vertex.

6. Integrate over all internal energies and momenta:

\[
\frac{dE^{(1)} d^3q^{(1)}_1 d^3q^{(1)}_2}{(2\pi)^7} \frac{dE^{(2)} d^3q^{(2)}_1 d^3q^{(2)}_2}{(2\pi)^7} \cdots \frac{dE^{(n-1)} d^3q^{(n-1)}_1 d^3q^{(n-1)}_2}{(2\pi)^7} \frac{d^4k_1 d^4k_2 \cdots d^4k_{I'}}{(2\pi)^4 (2\pi)^4 \cdots (2\pi)^4}
\]

and sum over all repeated isospin indices.
In these rules \( \overline{G}^{(0)}_{NN}(E, \bar{q}_1, \bar{q}_2) \) is defined by:

\[
(2\pi)^7 \delta(E' - E) \delta^{(3)}(q'_1 - q_1) \delta^{(3)}(q'_2 - q_2) \overline{G}^{(0)}_{NN}(E, \bar{q}_1, \bar{q}_2) = \int dt \ dt' \ d^{3}x_1 \ d^{3}x_2 \ d^{3}x'_1 \ d^{3}x'_2 \times e^{i(E't' - \tilde{p}'_1 \cdot \tilde{x}'_1 - \tilde{p}'_2 \cdot \tilde{x}'_2)} \overline{G}^{(0)}_{NN}(t', \bar{x}'_1, \bar{x}'_2, t, \bar{x}_1, \bar{x}_2) \theta(t' - t) e^{-i(Et - \tilde{p}_1 \cdot \tilde{x}_1 - \tilde{p}_2 \cdot \tilde{x}_2)}. \tag{6.14}
\]

When this Fourier transform is evaluated we find:

\[
\overline{G}^{(0)}_{NN}(E, \bar{q}_1, \bar{q}_2) = \frac{im^2}{E_N(\bar{q}_1)E_N(\bar{q}_2)} \sum_{\alpha_1, \alpha_2} u_{\alpha_1}(\bar{q}_1) \bar{\pi}_{\alpha_1}(\bar{q}_1) u_{\alpha_2}(\bar{q}_2) \bar{\pi}_{\alpha_2}(\bar{q}_2). \tag{6.15}
\]

At the same time \( G^{(0)}_{\pi j}(k) \) is defined exactly as in previous sections and so, in this no-anti-nucleon approximation, \( G^{(0)}_{\pi j}(k) \) is given by Eq.\((3.17)\).

Note that all the arguments in this section assume that the vertex \( \Gamma \) has no analytic structure in its energy variables. This assumption is justified since the presence of analytic structure would indicate that intermediate states in \( \Gamma \) could be exposed. Because \( \Gamma \) is the bare vertex for the theory it cannot have such intermediate states, and so it must have no analytic structure in its energy variables.

This represents one way of formulating the Feynman Rules for the two-nucleon Green’s function in this “time-ordered” or “old-fashioned” perturbation theory.

There is another more usual way of writing these rules. Instead of representing a \( j \)-pion intermediate state using one Green’s function for the two nucleons and another \( j \) for the pions, we may write a single Green’s function for all of the \( j + 2 \) particles in this state. This change can be accomplished either at the co-ordinate space level (in a similar but more complex way to the above change from an expression in terms of one-nucleon operators to one in terms of two-nucleon operators), or it can be done at the momentum-space level, as follows.

Consider any diagram containing a \( j \)-pion intermediate state. Suppose that this diagram is evaluated using the above Feynman rules. We know from these rules that, if the isospin indices which are irrelevant to our argument are suppressed, the Green’s function for the diagram may be written:

\[
\overline{G}^j(E', \tilde{p}'_1, \tilde{p}'_2; E, \tilde{p}_1, \tilde{p}_2) = \int \frac{dE d^3 \tilde{p}_1 d^3 \tilde{p}_2 \ d^4 k^{(1)} d^4 k^{(2)} \ldots d^4 k^{(j)}}{(2\pi)^7} \ F(E', \tilde{p}'_1, \tilde{p}'_2; \tilde{E}, \tilde{p}_1, \tilde{p}_2, k^{(1)}, k^{(2)}, \ldots, k^{(j)}) G^{(0)}_{NN}(\tilde{E}, \tilde{p}_1, \tilde{p}_2) \times G^{(0)}_{\pi}(k^{(1)}) G^{(0)}_{\pi}(k^{(2)}) \ldots G^{(0)}_{\pi}(k^{(j)}) F^\dagger(\tilde{E}, \tilde{p}_1, \tilde{p}_2, k^{(1)}, k^{(2)}, \ldots, k^{(j)}; E, \tilde{p}_1, \tilde{p}_2). \tag{6.16}
\]

(See Figure 8) The Feynman Rules also ensure that energy is conserved in \( F \) and \( F^\dagger \). In particular, we may extract a factor \((2\pi)\delta(E - \tilde{E} - \sum_{m=1}^{j} k^{(m)}_0)\) from \( F^\dagger \) and a factor \((2\pi)\delta(\tilde{E} + \sum_{m=1}^{j} k^{(m)}_0 - E')\) from \( F \). Furthermore, \( F \) and \( F^\dagger \) have no poles in any of their \( j \) \( k_0 \)-variables, because the propagator for the \( m \)th pion, which is the only part of the expression to contain analytic structure in the variable \( k^{(m)}_0 \), appears explicitly in our equation for \( \overline{G}^j \) and so cannot form part of \( F \) and \( F^\dagger \).

Using this energy-conservation result and substituting for \( \overline{G}^{(0)}_{NN} \) and \( G^{(0)}_{\pi} \) gives:
\[ \mathcal{G}^j(E', \vec{p}_1', \vec{p}_2'; E, \vec{p}_1, \vec{p}_2) = (2\pi)\delta(E' - E) \int \frac{d^3\vec{p}_1' d^3\vec{p}_2' d^3k^{(1)} d^3k^{(2)} \ldots d^3k^{(j)}}{(2\pi)^{3j+6}} \frac{d\omega_0^{(1)} d\omega_0^{(2)} \ldots d\omega_0^{(j)}}{(2\pi)^j} \]

\[ F(E', \vec{p}_1', \vec{p}_2'; E - \sum_{m=1}^{j} k_0^{(m)}; \vec{p}_1, \vec{p}_2, k^{(1)}, k^{(2)}, \ldots, k^{(j)}) \]

\[ \frac{im^2}{E_N(\vec{p}_1) E_N(\vec{p}_2)} \sum_{\alpha_1, \alpha_2} u_{\alpha_1}(\vec{p}_1) \bar{\nu}_{\alpha_1}(\vec{p}_1) u_{\alpha_2}(\vec{p}_2) \bar{\nu}_{\alpha_2}(\vec{p}_2) \prod_{m=1}^{j} \frac{1}{2\omega_\pi(k^{(m)})} \prod_{m=1}^{j} i \frac{1}{k_0^{(m)^+} - \omega_\pi(k^{(m)})} \]

\[ F^\dagger(E - \sum_{m=1}^{j} k_0^{(m)}; \vec{p}_1', \vec{p}_2', k^{(1)}, k^{(2)}, \ldots, k^{(j)}; E, \vec{p}_1, \vec{p}_2). \] (6.17)

Performing the \( k_0 \) integrations then gives:

\[ \mathcal{G}^j(E', \vec{p}_1', \vec{p}_2'; E, \vec{p}_1, \vec{p}_2) = (2\pi)\delta(E' - E) \int \frac{d^3\vec{p}_1' d^3\vec{p}_2' d^3k^{(1)} d^3k^{(2)} \ldots d^3k^{(j)}}{(2\pi)^{3j+6}} \]

\[ F(E', \vec{p}_1', \vec{p}_2'; E - \sum_{m=1}^{j} \omega_\pi(k^{(m)}); \vec{p}_1, \vec{p}_2, \omega_\pi(k^{(1)}), k^{(1)}, \omega_\pi(k^{(2)}), k^{(2)}, \ldots, \omega_\pi(k^{(j)}), k^{(j)}) \]

\[ \frac{im^2}{E_N(\vec{p}_1) E_N(\vec{p}_2)} \prod_{m=1}^{j} \frac{1}{2\omega_\pi(k^{(m)})} \sum_{\alpha_1, \alpha_2} u_{\alpha_1}(\vec{p}_1) \bar{\nu}_{\alpha_1}(\vec{p}_1) u_{\alpha_2}(\vec{p}_2) \bar{\nu}_{\alpha_2}(\vec{p}_2) \prod_{m=1}^{j} \frac{1}{\omega_\pi(k^{(m)}) - E_N(\vec{p}_1) - E_N(\vec{p}_2)} \]

\[ F^\dagger(E - \sum_{m=1}^{j} \omega_\pi(k^{(m)}); \vec{p}_1, \vec{p}_2; \omega_\pi(k^{(1)}), k^{(1)}, \omega_\pi(k^{(2)}), k^{(2)}, \ldots, \omega_\pi(k^{(j)}), k^{(j)}; E, \vec{p}_1, \vec{p}_2). \] (6.18)

Since this argument is valid for any \( j \)-pion intermediate state the above set of Feynman rules may be replaced by the following set of Feynman rules, which are those usually used for calculating the Green’s function in old-fashioned time-ordered perturbation theory.

**Feynman rules for old-fashioned time-ordered perturbation theory**

1. As for Rule 1 above.

In each diagram:

2. Label the incoming (outgoing) nucleon pairs with momenta \((\vec{p}_1, \vec{p}_2)\) \((\vec{p}_1', \vec{p}_2')\) and internal nucleon pairs with momenta \((\vec{q}_1^{(1)}, \vec{q}_2^{(1)})\), \((\vec{q}_1^{(2)}, \vec{q}_2^{(2)})\), \ldots, \((\vec{q}_1^{(n-1)}, \vec{q}_2^{(n-1)})\), including separate diagrams with a relative minus sign for the two possible momentum assignments in the final state. Label internal pion lines with momenta \(\vec{k}^{(1)}, \vec{k}^{(2)}, \ldots, \vec{k}^{(j')}\). Assign isospin labels \(l_1, l_2, \ldots, l_n\) to the vertices. Recall that a new nucleon pair is regarded as being created whenever a vertex occurs on either nucleon.

3. If \( j \) pions are present at the same time as the nucleon pair, and they have momenta \(\vec{k}^{(1)}, \ldots, \vec{k}^{(j)}\) while the nucleon pair has momenta \(\vec{q}_1\) and \(\vec{q}_2\), then associate, with all the particles present at that time, a Green’s function:

\[ \frac{im^2}{E_N(\vec{q}_1) E_N(\vec{q}_2)} \prod_{m=1}^{j} \frac{1}{2\omega_\pi(k^{(m)})} \sum_{\alpha_1, \alpha_2} u_{\alpha_1}(\vec{q}_1) \bar{\nu}_{\alpha_1}(\vec{q}_1) u_{\alpha_2}(\vec{q}_2) \bar{\nu}_{\alpha_2}(\vec{q}_2) \prod_{m=1}^{j} \delta_{\epsilon l a}, \] (6.19)
where $l_e$ and $l_a$ are the isospin indices of the vertex at which the $l$th pion is emitted and absorbed.

Here "$j$ pions present at the same time" means that a vertical line drawn through the two nucleon lines intersects $j$ pion lines.

4. For each pion absorption vertex occurring on the nucleon with momentum $\vec{q}_i$ at the same time as $j$ spectator pions with momenta $\vec{k}_1, \ldots, \vec{k}_j$ are present, insert a factor:

$$(-ig)(2\pi)^3\delta^{(3)}(q'_i - k - q_i)(2\pi)^3\delta^{(3)}(q'_i - q_i)(2\pi)^3 \prod_{m=1}^{j} \delta^{(3)}(k'_m - k_m)(i\gamma_5\tau_l)$$

$$\times \Gamma(E - \sum_{m=1}^{j} \omega_\pi(\vec{k}_j)) - E_N(\vec{q}_i), \vec{q}_i', E - E_N(\vec{q}_i), \omega_\pi(\vec{k}), \vec{k}).$$

Here $l$ is the isospin label assigned to the vertex.

For an emission vertex occurring under the same conditions we insert a similar factor, but with $\vec{k}$ replaced by $-\vec{k}$ and $\omega_\pi(\vec{k})$ replaced by $-\omega_\pi(\vec{k})$.

5. Integrate over all internal momenta:

$$\int d^3k_1 d^3k_2 \ldots d^3k'_1 d^3q_1^{(1)} d^3q_2^{(1)} d^3q_1^{(2)} d^3q_2^{(2)} \ldots d^3q_{(n-1)} d^3q_{(n-1)}$$

and sum over all repeated isospin indices.

\section*{B. The result of Kvinikhidze and Blankleider}

Having obtained the usual time-ordered perturbation theory rules for calculating $G_{NN}(2)(E', \vec{p}_1', \vec{p}_2'; E, \vec{p}_1, \vec{p}_2)$ we may use Eq.(6.11) to derive some of the results proven by KB using different means [31,35,53].

Consider any time-ordered perturbation theory diagram, and suppose that the Green’s function for the diagram is $G_{NN}(2)$. Define a Green’s function $G^{\Sigma}_{NN}(2)$ which is the sum of all Green’s functions representing diagrams which differ from the diagram under consideration only in the relative time-order of processes occurring on different nucleons, but excluding those diagrams in which the order of emission and absorption of transmitted pions is different. I.e. define $G^{\Sigma}_{NN}(2)$ to be:

$$G^{\Sigma}_{NN(2)} = \sum_{\text{All relative time-orders}} G_{NN(2)}.$$ 

(6.20)

Once this definition is made we may take the Fourier transform of Eq.(6.11) and use Eq.(1.8) to obtain:

$$G^{\Sigma}_{NN(2)}(E', \vec{p}_1', \vec{p}_2'; E, \vec{p}_1, \vec{p}_2) =$$

$$\int \frac{dzdz'}{(2\pi)^2} G_{NN(1)}(E' - z', \vec{p}_1', \vec{z}', \vec{p}_2'; E - z, \vec{p}_1, \vec{z}, \vec{p}_2),$$ 

(6.21)
where $G_{NN(1)}$ must be evaluated in the no-anti-nucleon approximation. Or, if we remove the energy delta-function on both sides:

$$G_{NN(2)}^{Σ}(E, \vec{p}_1', \vec{p}_2; \vec{p}_1, \vec{p}_2) = \int \frac{dz dz'}{(2\pi)^2} G_{NN(1)}(E - z', \vec{p}_1', \vec{p}_2'; E - z, \vec{p}_1, z, \vec{p}_2). \quad (6.22)$$

These are two of the crucial results of this paper. They state that the Green’s function for a sum of time-ordered perturbation theory diagrams may be expressed as the integral over all initial and final relative energies of another Green’s function, which is simply the covariant Green’s function in the no-anti-nucleon approximation. This is the general result, now let us look at two specific examples. Firstly, we will consider an arbitrary disconnected diagram and show how Eq.(6.21) reduces to the original KB convolution integral result for this case. Secondly, we will examine one-pion exchange as an example of what happens when we apply Eq.(6.22) to a connected diagram.

1. Disconnected diagrams

Consider an arbitrary disconnected diagram. By definition, no pion is transmitted from one nucleon to the other. Suppose that, in this diagram, processes $P1$ occur on one nucleon and processes $P2$ occur on the other. Applying the Feynman Rules in Section IV to this situation we see that:

$$G_{NN1}^{(d)}(E' - z', \vec{p}_1', \vec{p}_2; E - z, \vec{p}_1, z, \vec{p}_2) = (2\pi)^8 \delta(z' - z) \delta(E' - E) \times \delta^3(p_1' - p_1) \delta^3(p_2' - p_2) G_{P1}^{P1}(E - z, \vec{p}_1) G_{P2}^{P2}(z, \vec{p}_2), \quad (6.23)$$

where $G_{P1}^{P1}(E - z, \vec{p}_1)$ and $G_{P2}^{P2}(z, \vec{p}_2)$ are the single-nucleon Green’s functions for the processes $P1$ and $P2$ respectively, and are to be calculated from the rules in Section III, applied in the no-anti-nucleon approximation.

Compare this with the form of $G_{NN(2)}^{Σ}$ for the same diagram:

$$G_{NN(2)}^{Σ(d)}(E', \vec{p}_1', \vec{p}_2; E, \vec{p}_1, \vec{p}_2) = (2\pi)^7 \delta(E' - E) \delta^3(p_1' - p_1) \delta^3(p_2' - p_2) G_{NN2}^{Σ}^{(d)}(E, \vec{p}_1, \vec{p}_2). \quad (6.24)$$

Applying Eq.(6.22) to this specific case consequently yields:

$$G_{NN(2)}^{Σ(d)}(E, \vec{p}_1, \vec{p}_2) = \int \frac{dz}{2\pi} G_{P1}^{P1}(E - z, \vec{p}_1) G_{P2}^{P2}(z, \vec{p}_2), \quad (6.25)$$

which is KB’s result up to a factor of $i$ [31]. This factor is missing because we are using propagators which differ from KB’s propagators by a factor of $i$. If we replace all our propagators by KB propagators then, for an $n$th order diagram, a factor $i^{n+1}$ appears on the left-hand side of Eq.(6.25) and a factor $i^{n+2}$ appears on the right-hand side. Therefore, we find that, for any diagram:

$$G_{NN(2)}^{Σ(d)}(E, \vec{p}_1, \vec{p}_2) = -\frac{1}{2\pi i} \int dz G_{P1}^{P1}(E - z, \vec{p}_1) G_{P2}^{P2}(z, \vec{p}_2), \quad (6.26)$$

where the propagators now agree with KB’s propagators.
Now we consider one-pion exchange. Suppose a pion is transmitted from one nucleon, which we label \(N_2\), to the other, which we label \(N_1\). Suppose that before the pion is emitted from \(N_2\) (absorbed on \(N_1\)) certain processes \(P_1\) (\(P_2\)) take place on \(N_1\) (\(N_2\)), and after the pion is emitted on \(N_2\) (absorbed on \(N_1\)) certain processes \(P'_1\) (\(P'_2\)) take place on \(N_1\) (\(N_2\)). Then the Green’s function constructed according to the rules in Section \(\text{V}\) is:

\[
G_{NN(1)}(E - z', \vec{p}_1', \vec{z}', \vec{p}_2, E - z, \vec{p}_1, z, \vec{p}_2) = (2\pi)^3 \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \times G_N^{P_1'}(E - z', \vec{p}_1')(-ig)\Gamma(E - z', \vec{p}_1', E - z, \vec{p}_1, z - z', \vec{p}_1' - \vec{p}_1)G_N^{P_1}(E - z, \vec{p}_1) \times G_\pi(z - z', \vec{p}_2 - \vec{p}_2')G_N^{P_2'}(z', \vec{p}_2')(-ig)\Gamma(z', \vec{p}_2', z, \vec{p}_2, z' - z, \vec{p}_2' - \vec{p}_2)G_N^{P_2}(z, \vec{p}_2),
\]

(6.27)

where all single-particle propagators \(G_N(p_0, \vec{p})\) are calculated using the rules giving the single-particle Green’s function for a process \(P\), given in Section \(\text{III}\), applied in the no-anti-nucleon approximation, and we have suppressed all the spin and isospin structure of the result.

Substituting this into Eq.(6.22) leads to:

\[
\overline{G}_{NN(2)}^{E}(E, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = (2\pi)^3 \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2) \int \frac{dz \, dz'}{(2\pi)^2} \times G_N^{P_1'}(E - z', \vec{p}_1')(-ig)\Gamma(E - z', \vec{p}_1', E - z, \vec{p}_1, z - z', \vec{p}_1' - \vec{p}_1)G_N^{P_1}(E - z, \vec{p}_1) \times G_\pi(z - z', \vec{p}_2 - \vec{p}_2')G_N^{P_2'}(z', \vec{p}_2')(-ig)\Gamma(z', \vec{p}_2', z, \vec{p}_2, z' - z, \vec{p}_2' - \vec{p}_2)G_N^{P_2}(z, \vec{p}_2).
\]

(6.28)

If we again remove the relative factor of \(i\) between our propagators and those of KB we obtain, for an \(n\)th order diagram, a factor \(i^{n+2}\) on the right-hand side and \(i^{n}\) on the left-hand side. Therefore, for any order diagram the result is:

\[
\overline{G}_{NN(2)}^{E}(E, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2)(-\frac{1}{2\pi i})^2 \int dz \, dz' \times G_N^{P_1'}(E - z', \vec{p}_1')(-ig)\Gamma(E - z', \vec{p}_1', E - z, \vec{p}_1, z - z', \vec{p}_1' - \vec{p}_1)G_N^{P_1}(E - z, \vec{p}_1) \times G_\pi(z - z', \vec{p}_2 - \vec{p}_2')G_N^{P_2'}(z', \vec{p}_2')(-ig)\Gamma(z', \vec{p}_2', z, \vec{p}_2, z' - z, \vec{p}_2' - \vec{p}_2)G_N^{P_2}(z, \vec{p}_2),
\]

(6.29)

where, once again, the propagators now agree with KB’s propagators. If one now sums this result over all possible processes \(P_1, P_2, P_1'\) and \(P_2'\) one obtains the sum of all time-orders of one-pion exchange from \(N_2\) to \(N_1\) with fully dressed propagators.

\[
\overline{G}_{NN(2)}^{E}(E, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = \delta^{(3)}(p'_1 + p'_2 - p_1 - p_2)(-\frac{1}{2\pi i})^2 \int dz \, dz' \times G_N(E - z', \vec{p}_1')(-ig)\Gamma(E - z', \vec{p}_1', E - z, \vec{p}_1, z - z', \vec{p}_1' - \vec{p}_1)G_N(E - z, \vec{p}_1) \times G_\pi(z - z', \vec{p}_2 - \vec{p}_2')G_N(z', \vec{p}_2')(-ig)\Gamma(z', \vec{p}_2', z, \vec{p}_2, z' - z, \vec{p}_2' - \vec{p}_2)G_N(z, \vec{p}_2),
\]

(6.30)

where all of the propagators \(G_N\) are fully dressed. This is another result also obtained by KB, as an extension of their formula (6.26) [53]. As we shall see in Section \(\text{VIII}\) this formula for the one-pion exchange potential was also obtained by Klein and McCormick [13]. (See Figure 4 for a pictorial representation of this expression and the energy and momentum assignments involved.)
Clearly, this approach can be extended to two pion exchange, three pion exchange etc. The advantage of such a method of calculating Green’s functions is that it allows us to sum all relative time-orders in time-ordered perturbation theory by merely doing a convolution integral of the result obtained from the rules given in Section [IV] applied in the no-anti-nucleon approximation.

C. The approximation of Kvinikhidze and Blankleider

Equation (6.30) is the exact result for the sum of all relative time-orders for one-pion exchange from \(N2\) to \(N1\), with fully dressed propagators. However, in their work on the \(NN - \pi NN\) problem KB use only an approximation to this result [35]. They represent the one-pion exchange Green’s function by a product of two convolution integrals separated by an inverse \(\pi NN\) propagator:

\[
G_{\Sigma NN}(2)(E, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) \approx \delta(3)(p_1' + p_2' - p_1 - p_2)[\left(-\frac{1}{2\pi i}\right)^2 \int dz' dz'' \\
\times G_N(E - z', \vec{p}_1')(\vec{p}_1, E - z', \vec{p}_1, E - z'', \vec{p}_1', z'' - z', \vec{p}_2 - \vec{p}_1') \\
\times G_N(E - z'', \vec{p}_1)G_{\pi}(z'' - z', \vec{p}_1' - \vec{p}_1)G_N(z', \vec{p}_2')[G_{\pi NN}^{-1}(E)][\left(-\frac{1}{2\pi i}\right)^2 \int dz'' dz \\
\times G_N(z''', \vec{p}_2')G_{\pi}(z - z''', \vec{p}_2 - \vec{p}_2')(\vec{p}_2, z, \vec{p}_2, z''' - z, \vec{p}_2' - \vec{p}_2') \\
\times G_N(E - z, \vec{p}_1)G_N(z, \vec{p}_2)],
\]

(6.31)

where, once again, all propagators are fully dressed. KB have performed calculations which suggest that, at least for this case, the error in making such an approximation is small [33]. Be that as it may, we believe that such an approximation is physically flawed on at least two grounds, and consequently we expect it to produce invalid results, if not in the calculation of one-pion exchange, then in some other calculation.

Firstly, to try and express one-pion exchange as in Eq.(6.31) violates conservation of energy in the following sense. If the energy arguments of the Green’s functions are treated as off-shell energies of the particles involved, then the energy and momentum assignments in Eq.(6.31) correspond to the energies and momenta indicated in Figure 10. These momenta are to be compared to those in the exact expression for the sum of time-ordered perturbation theory diagrams, depicted in Figure 9. It is clear that in the exact expression the energy and momentum of each individual particle is conserved throughout the diagram, except at the vertices, where a pion interacts with a nucleon, thus modifying the energy and momentum of the nucleon with which it interacts. However, the \(\pi NN\) vertices are constructed in such a way that energy and momentum are conserved there too; therefore, one can say that the energy and momentum of the individual particles is conserved in this exact expression for the sum of time-ordered perturbation theory diagrams. In the approximate expression, the momentum of each individual particle is still conserved, in the same sense. However, an inspection of the energy assignments reveals that, although the total energy is the same at all times during the pion exchange process, the energy of the three individual particles jumps suddenly and for no physical reason at the point where the two disconnected pieces are joined together to form a connected diagram. Since the relative energy of the particles is integrated over this jump in energy is possibly quite large, and so this violation of energy
conservation is bound to have an effect somewhere in any calculation performed using such an approximation.

A second associated flaw in such an approximation lies in the diagrams it omits from the Green's function. The types of diagrams omitted are represented in Figure 11. The justification for ignoring such diagrams is that, as mentioned above, KB have performed a calculation which shows their effect to be small [53]. This appears reasonable, until one carefully examines what physics is omitted and what is included in such an approximation. Observe that diagrams such as Figures 12 and 13 are still included in KB's calculation. Breaking down these two included diagrams and the omitted diagram into smaller pieces implies that KB's approximation amounts to always including the process on the left of Figure 14 but only sometimes including the process on the right. The process on the right of Figure 14 is excluded in some diagrams, such as Figure 11, but included in others, such as Figure 13. In other words, in certain diagrams KB ignore the Jennings mechanism [32,33] when a spectator pion is present. I.e., diagrams which contain the same physical processes are treated differently, with some time-orders included and some excluded.

These two problems reflect a fundamental flaw in the KB approach. KB use a convolution integral in order to sum all relative time-orders in a disconnected diagram, an approach, which, as we have seen, leads to the calculation of a Green's function which is a no-anti-nucleon approximation to the full covariant propagator. In other words, in KB's model, disconnected pieces are calculated in an approach which is an approximation to covariant perturbation theory. However, when they come to construct convolution integral expressions for connected diagrams they do so by joining disconnected pieces, written as convolution integrals, together using an inverse $\pi NN$ propagator, as in Equation (6.31). This method of joining disconnected diagrams together is consistent with time-ordered perturbation theory, not with covariant perturbation theory. Consequently, their technique for forming connected diagrams from disconnected pieces is inconsistent with their calculation of disconnected diagrams. This leads, firstly, to violations of energy conservation, since although the total energy $E$ is conserved the relative energy is not constrained to be conserved, and secondly, to the unequal treatment of certain diagrams which only differ in the relative time-order of processes on different nucleons, which is, ironically, the problem the convolution integral was introduced to fix.

If one wishes to join disconnected pieces together in a fashion consistent with the use of a convolution integral for disconnected diagrams one needs to reexamine the roots of the convolution integral technique, as we have done here. This leads to expressions such as Eq.(6.29) which are the exact sums of all relative time-orders in a set of time-ordered perturbation theory diagrams. But, such a result is only an equal-time no-anti-nucleon approximation to the full covariant perturbation theory expression. Therefore, we intend to persist with the covariant calculation in which all the physics is naturally included, rather than pursuing approximations with questionable physical motivation, such as those used by Kvinikhidze and Blankleider.

VII. AMPLITUDES

In this section we obtain the amplitudes which correspond to the two-nucleon Green's functions we have used so far. This is done via the standard procedure for obtaining ampli-
tudes from Green’s functions: LSZ reduction \cite{30}, in which the external legs of the Green’s function are amputated in order to obtain the corresponding amplitude. Again, it will be clear that the work of this section can be extended to any \(m \rightarrow m'\) particle Green’s function.

For the two-nucleon system the S-matrix is defined to be:

\[
\langle p_1', p_2'|T_{NN}|p_1, p_2 \rangle = G_N^{-1}(p_1')G_N^{-1}(p_2')G_{NN}(p_1', p_2'; p_1, p_2)G_N^{-1}(p_1)G_N^{-1}(p_2),
\]

and we have ignored spin and isospin indices. Note the use of dressed nucleon propagators \(G_N\) here.\footnote{Since we are using dressed nucleon propagators we should include factors of \(Z^{-\frac{1}{2}}\), where \(Z\) is the wave function renormalization, if we wish to get the LSZ reduction formula exactly correct. However, the inclusion of such factors makes no difference to the overall result and so we omit them in order to make the argument as clear as possible.} Firstly, we apply this formula to the general two-nucleon Green’s function obtained from the rules in Section IV:

\[
G_{NN}(E - z', \vec{p}_1', \vec{z}', \vec{p}_2'; E - z, \vec{p}_1, \vec{z}, \vec{p}_2).
\]

Note that we have defined a total energy \(E = p_{10}' + p_{20}' = p_{10} + p_{20}\) and relative energies \(z' = p_{20}'\), \(z = p_{10}\) in order to facilitate the derivation of a relation between the amplitude obtained from this Green’s function and the one obtained from the time-ordered perturbation theory Green’s function. When the free one-nucleon Green’s function is substituted in, this equation becomes, in the no-anti-nucleon approximation:

\[
G_{NN}(E - z', \vec{p}_1', \vec{z}', \vec{p}_2'; E - z, \vec{p}_1, \vec{z}, \vec{p}_2) = i \frac{E + z' - E_N(\vec{p}_1')}{E + z - E_N(\vec{p}_1)} \frac{z'^+ - E_N(\vec{p}_2')}{z^+ - E_N(\vec{p}_2)},
\]

where \(E_N\) may contain terms representing the dressing of the nucleon. (The positive-energy projection operators may be omitted here, and throughout the rest of this section, since they have no effect, due to our use of the no-anti-nucleon approximation.) Secondly, consider any time-ordered perturbation theory Green’s function:

\[
\overline{G}_{NN}(E, \vec{p}_1', \vec{p}_2'; \vec{p}_1, \vec{p}_2)
\]

LSZ reduction applied to \(\overline{G}_{NN}(E, \vec{p}_1', \vec{p}_2'; \vec{p}_1, \vec{p}_2)\) implies:

\[
\overline{G}_{NN}(E, \vec{p}_1', \vec{p}_2'; \vec{p}_1, \vec{p}_2) = i \frac{E + z' - E_N(\vec{p}_1')}{E + z - E_N(\vec{p}_1)} \frac{z'^+ - E_N(\vec{p}_2')}{z^+ - E_N(\vec{p}_2)}\overline{T}_{NN}(E, \vec{p}_1', \vec{p}_2'; \vec{p}_1, \vec{p}_2),
\]

(7.6)
where, once again, $E_N$ may contain terms representing the dressing of the nucleons. Above we defined $\Sigma_{NN}^{\Sigma(2)}$ to be the sum of all Green's functions differing only in the relative time-order of processes which occur on different nucleons, excluding transmitted pion emission and absorption. Now we also define:

$$T_{NN(2)} = \sum_{\text{All time-orders}} T_{NN(2)}.$$

Since Eq.(7.6) holds between each term in this sum over all time-orders, it must relate $\Sigma_{NN}^{\Sigma(2)}$ to $T_{NN(2)}$ as well. Therefore, Eq.(7.6) for $G_{NN(2)}^{\Sigma}$ and Eq.(7.4) for $G_{NN(1)}$ may be substituted into Eq.(6.22) in order to obtain the relationship between $T_{NN(2)}$ and $T_{NN(1)}$.

When this is done we find:

$$T_{NN(2)}^{\Sigma}(E, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = \int \frac{dzdz'}{(-2\pi i)^2} (\frac{1}{E^+ - E^-(\vec{p}_1') + \frac{1}{z'} - E_N(\vec{p}_2')}\times T_{NN(1)}(E - z', \vec{p}_1', \vec{p}_2', E - z, \vec{p}_1, \vec{p}_2) (\frac{1}{E^+ - z - E_N(\vec{p}_1)} + \frac{1}{z^+ - E_N(\vec{p}_2')}) (7.8)$$

If we wanted to calculate a physical quantity from a sum of time-ordered perturbation theory diagrams in which all relative time-orders are included, we would need to calculate $T_{NN(2)}$ on-shell. However, it may well be easier to calculate $T_{NN(1)}$ and then use the relation between $T_{NN(1)}$ and $T_{NN(2)}$ to obtain $T_{NN(2)}$ on-shell.

In order to find out if this is feasible we set $E = E_{on}$ with

$$E_{on} = E_N(\vec{p}_1) + E_N(\vec{p}_2) = E_N(\vec{p}_1') + E_N(\vec{p}_2').$$

For this value of $E$ Eq.(7.8) reads:

$$T_{NN(2)}^{\Sigma}(E_{on}, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = \int \frac{dzdz'}{(-2\pi i)^2} \text{disc}(\frac{1}{z' - E_N(\vec{p}_2')})\times T_{NN(1)}(E_{on} - z', \vec{p}_1', \vec{p}_2', E_{on} - z, \vec{p}_1, \vec{p}_2) \text{disc}(\frac{1}{z - E_N(\vec{p}_2')}). (7.10)$$

But:

$$\text{disc}(\frac{1}{z - w}) = -2\pi i \delta(z - w). (7.11)$$

Therefore,

$$T_{NN(2)}^{\Sigma}(E_{on}, \vec{p}_1', \vec{p}_2', \vec{p}_1, \vec{p}_2) = T_{NN(1)}(E_N(\vec{p}_1'), \vec{p}_1, E_N(\vec{p}_2'), \vec{p}_2; E_N(\vec{p}_1), \vec{p}_1, E_N(\vec{p}_2), \vec{p}_2). (7.12)$$

This is a more general version of a result obtained by Kvinikhidze and Blankleider by different means [35]. It states that, on-shell, the amplitude obtained from the no-anti-nucleon two-nucleon Green's function is equal to the sum over all time-orders of the amplitudes obtained from time-ordered perturbation theory.

\footnote{Note that the two-nucleon propagator used in this equation does not contain the full dressing on both nucleons, since certain time-orders are excluded from it. As discussed in Section 4, this inadequate dressing was the original reason for KB's suggesting the use of a convolution integral for the free two-nucleon Green's function [31].}
VIII. THE METHOD OF KLEIN, LÉVY, MACKE AND KADYSHEVSKY (KLMK)

In the 1950s and 60s there was considerable interest in techniques for reducing the Bethe-Salpeter (BS) equation to a three-dimensional integral equation. In this section we explain how this work is connected to our work above and in so doing reveal that the KLMK method suffers from similar inconsistencies to those of the KB method discussed above.

Initial work by Salpeter showed that if an instantaneous interaction was used in the Bethe-Salpeter equation a three-dimensional integral equation for the positive-energy component of the wave function was obtained [37]. This work was then extended by Lévy and Klein, who showed that, in the ladder approximation, the BS equation could be approximated by the following integral equation for the wave function

\[ \phi^{++}(\vec{p}) = \Lambda^{(1)}(\vec{p})\Lambda^{(2)}(-\vec{p}) \int d^3k V_{OPE}(E,\vec{p},\vec{p} - \vec{k}) \phi^{++}(\vec{p} - \vec{k}), \]  

(8.1)

where the wave function \( \phi^{++}(\vec{p}) \) is defined by:

\[ \phi^{++}(\vec{p}) = (-2\pi i) \frac{1}{2} E + p_0 - E_N(\vec{p}) \frac{1}{2} E - p_0 - E_N(-\vec{p}) \Lambda^{(1)}(\vec{p}) \Lambda^{(2)}(-\vec{p}) \psi(p), \]  

(8.2)

with \( p \) the relative four-momentum and \( E \) the total energy in the center-of-mass frame. The kernel \( V_{OPE}(E,\vec{p},\vec{p} - \vec{k}) \) is given by:

\[
V_{OPE}(E,\vec{p},\vec{p} - \vec{k}) = -\lambda(E - 2E_N(\vec{p})) \left[ \frac{1}{2\pi i} \right]^2 \int d^3k dk_0 dp_0 \frac{1}{\frac{1}{2} E + p_0 - E_N(\vec{p})} \frac{1}{\frac{1}{2} E - p_0 - E_N(-\vec{p})} \Gamma^{(1)}(p,p - k) \frac{1}{k^2 + \mu^2} \Gamma^{(2)}(-p,-p + k) \times \frac{1}{\frac{1}{2} E + p_0 - k_0 - E_N(\vec{p} - \vec{k})} \frac{1}{\frac{1}{2} E - p_0 + k_0 - E_N(\vec{k} - \vec{p})} \right] (E - 2E_N(\vec{p} - \vec{k})).
\]  

(8.3)

If we regard \( E_N \) as containing a self-energy term for the nucleon then this kernel contains both fully dressed vertices and fully dressed propagators and we may write:

\[
V_{OPE}(E,\vec{p},\vec{p} - \vec{k}) = (E - 2E_N(\vec{p})) \mathcal{G}^{\Sigma}_{NN(2)}(E,\vec{p},\vec{p} - \vec{k},\vec{k} - \vec{p})(E - 2E_N(\vec{p} - \vec{k})),
\]  

(8.4)

where \( \mathcal{G}^{\Sigma}_{NN(2)}(E,\vec{p},\vec{p} - \vec{k},\vec{k} - \vec{p}) \) is the sum of the convolution integral Green’s functions for pion exchange from \( N1 \) to \( N2 \) and \( N2 \) to \( N1 \), which are given respectively, by Eq.(6.30) as written, and by the same equation with \( N1 \) and \( N2 \) swapped. Note that in order to establish this fact we need to perform the following substitutions, which merely correspond to changing the frame of reference to the centre-of-mass frame for the two-nucleon system:

\[
\vec{p} = \vec{p}', \quad \vec{p}_2 = -\vec{p}_2';
\]  

(8.5)

\[
p_0 = \frac{1}{2} E - z';
\]  

(8.6)

\[
\vec{k} = \vec{p}_2 - \vec{p}_2' = \vec{p}_1 - \vec{p}_1';
\]  

(8.7)

\[
k_0 = z - z'.
\]  

(8.8)
Using Eq. (7.6) we see that this $V_{OPE}$ is the amplitude for the sum of all the time-ordered perturbation theory diagrams which represent the different time-orders of fully-dressed one-pion exchange.

Klein went on to generalize this result, showing how the BS equation:

$$M = I + I GM, \tag{8.9}$$

where $M$ and $I$ are regarded as operators in the Hilbert space of energy-momentum states $|p\rangle$ could always be written as a three-dimensional equation of the form:

$$T(E) = V(E) + V(E)G_{NN}^{++}(E)T(E), \tag{8.10}$$

where

$$G_{NN}^{++}(E) = \Lambda_+^{(1)} \Lambda_+^{(2)} G_{NN}(E), \tag{8.11}$$

with the t-matrix $T(E)$ defined by:

$$T(E) = (\frac{1}{2\pi i})^2 G_{NN}^{-1}(E)[\int dp'_0 dp_0 G_N^{(+)}(\frac{1}{2} E - p'_0)G_N^{(+)}(\frac{1}{2} E + p_0)$$

$$\times M(p'_0, p_0; E)G_N^{(+)}(\frac{1}{2} E - p_0)G_N^{(+)}(\frac{1}{2} E + p_0)]G_{NN}^{-1}(E), \tag{8.12}$$

and the potential $V(E)$ given by:

$$V(E) = V_1(E) + V_2(E) - V_1(E)G_{NN}^{++}(E)V_1(E) + \ldots, \tag{8.13}$$

where:

$$V_1(E) = (\frac{1}{2\pi i})^2 G_{NN}^{-1}(E)[\int dp'_0 dp_0 G_N^{(+)}(\frac{1}{2} E - p'_0)G_N^{(+)}(\frac{1}{2} E + p_0)$$

$$\times I(p'_0, p_0; E)G_N^{(+)}(\frac{1}{2} E - p_0)G_N^{(+)}(\frac{1}{2} E + p_0)]G_{NN}^{-1}(E), \tag{8.14}$$

$$V_2(E) = (\frac{1}{2\pi i})^2 G_{NN}^{-1}(E)[\int dp'_0 dk_0 dp_0 G_N^{(+)}(\frac{1}{2} E - p'_0)G_N^{(+)}(\frac{1}{2} E + p_0)I(p'_0, k_0; E)$$

$$\times G(k_0, E)I(k_0, p_0; E)G_N^{(+)}(\frac{1}{2} E - p_0)G_N^{(+)}(\frac{1}{2} E + p_0)]G_{NN}^{-1}(E) \ldots. \tag{8.15}$$

All operators in these formulae are now considered to be operators in three-dimensional momentum space.

Note that the propagator $G_{NN}(E)$ is the time-ordered perturbation theory propagator:

$$G_{NN}(E) = \frac{1}{E^+ - E_N - E_N^+}. \tag{8.16}$$

If dressed particles are involved the use of this time-ordered perturbation theory propagator is open to question à la KB [31]. But, we ignore this difficulty for the present and assume that this $G_{NN}(E)$ is the correct propagator to use, since any such problem may be fixed by making a minor modification to the approach outlined here.

Now if the interaction kernel $I$ contains only the single pion exchange that gives the ladder BS equation then:
\[
\langle \vec{p}'|V_1(E)|\vec{p}\rangle = V_{OPE}(E, \vec{p}', \vec{p}).
\] (8.17)

This result was also obtained by Kadyshevsky and his collaborators [47,48], by proceeding from the Lippman-Schwinger equation for the amplitude. Klein’s early work focused on deriving this result for the wave function BS equation, although he later also derived it for the BS equation for amplitudes [46].

Klein also explained how to obtain a time-ordered perturbation theory expression for \(V(E)\) from the results given here [41]. This explanation indicated that the potential \(V(E)\) could be regarded as the sum of all possible time-ordered perturbation theory diagrams for the process under consideration. That is to say, given any BS equation interaction kernel \(I\), there are two equivalent ways of obtaining the three-dimensional integral equation (8.10) corresponding to the BS equation (8.9):

1. Use Eq. (8.13) in order to derive the equivalent energy-dependent potential \(V(E)\);
2. Sum all two-particle irreducible time-ordered perturbation theory diagrams allowed for this \(I\); the result is \(V(E)\).

That these two methods are both equivalent to the BS equation is a manifestation of the result derived above in Sections IV–VI: both the convolution integral and time-ordered perturbation theory may be obtained from field theory in the equal-time no-anti-nucleon approximation.

Although in theory both of these two methods provide three-dimensional integral equations containing exactly the same information as the BS equation, in practice it is not feasible to obtain the full \(V(E)\) which is necessary in order to make the equivalence exact. Consequently approximations to the potential, such as that defined by Equation (8.3), must be made. Sometimes additional modifications to the propagator are also made in order to further facilitate the calculation, see e.g. [48]. However, here we merely wish to examine the quality of the approximation:

\[
V(E) = V_1(E),
\] (8.18)

which is obtained by taking the first term in the series (8.13) for \(V\) when the BS equation is being used in the ladder approximation, and so \(I\) consists only of one-pion exchange. If dressed propagators are being used the time-ordered perturbation theory diagrams which are omitted from this approximation but included in the ladder BS equation include those shown in Figure 13. These omitted diagrams are to be contrasted with those in Figure 16, which are included in the calculation using Eqs. (8.10) and (8.18). It is clear that, as is the case in the KB method and the current \(NN \rightarrow \pi NN\) equations, different time-orders of the same physical process are treated differently.

Connected to this problem is the problem of relative energy non-conservation, which occurs here in a similar way to that in which it occurred in the KB approximation discussed in Section VIII C. If we examine the energy assignments for the two-pion exchange diagram generated by Equations (8.13) and (8.18) we find that they correspond to those drawn in Figure 17. Although the energy assignments for one-pion exchange are now correct (compare with Figures 8 and 10) there is still a sudden jump in energy where one one-pion exchange diagram, calculated using a certain pair of relative energies, is joined to another one-pion
exchange diagram, which is calculated using different relative energies. Again, as discussed above for the KB approximation, this jump, could, in principle, be infinite, since the relative energy variables $k_0$ and $k'_0$ are integrated over. Consequently, an examination of KLMK’s techniques shows that although their work contains many of the same elements as those pursued in this paper, ultimately it has a different purpose, since it places a higher value on obtaining a tractable three-dimensional integral equation than on including all possible time-orders of the relevant processes. In this sense it is more akin to the work of KB than to the approach espoused here, since both KB and KLMK restrict themselves to considering only three-dimensional integral equations and using energy-integrations to sum certain pieces in their essentially three-dimensional theories. They pay for this restriction however, since both approaches violate relative energy conservation and neither includes all time-orders of physically relevant processes. If these two problems are to be eliminated a fully four-dimensional calculation must be pursued.

**IX. CONCLUSION**

Current models of the $NN - \pi NN$ system are unsatisfactory in two ways. Firstly, they fail to correctly predict the experimental data. Secondly, because all calculations are done, in some sense or another, in a time-ordered framework, diagrams which are merely different time-orders of the same process are treated unequally—the most famous example of this being the non-inclusion of the Jennings diagram in the current $NN - \pi NN$ equations \[32,33\]. It is to be hoped that a resolution of the second problem will, by producing a theoretically consistent model, lead to a diminution of the first problem. Therefore, the question is how to consistently include all time-orders in a calculation.

In this paper we displayed a scheme for doing just this. Starting from a Lagrangian closely related to those of Quantum Hadrodynamics \[7,8\] we explained how to calculate fully covariant one-nucleon, one-pion, two-nucleon and $m \rightarrow m'$-particle Green’s functions, either with or without anti-nucleonic degrees of freedom. It was then shown that, in the absence of anti-nucleonic degrees of freedom, any covariant perturbation theory diagram may be expressed as the sum of a set of time-ordered perturbation theory diagrams, provided we restrict all particles to have the same time in the initial and final states. This result made it clear that Kvinikhidze and Blankleider’s original convolution integral formula for disconnected diagrams \[31\] is merely a manifestation of the more general relationship between covariant perturbation theory and time-ordered perturbation theory. This fact allowed us to point out the inconsistency in the KB approach to the $NN - \pi NN$ problem \[35\], which leads to a violation of relative energy conservation. To complete our derivation of the properties of convolution integral theory we explained how LSZ reduction \[36\] could be performed on convolution integral Green’s functions. We then observed that KLMK’s work \[38–48\], which also used a convolution integral to calculate the one-pion exchange potential, has problems similar to those which occur in the KB approximation. These problems occur because any approximate three-dimensional integral equation, which involves an energy integration for the propagators, as in the work of KB, or for one-pion exchange, as in the work of KLMK, results in relative energy non-conservation. This is connected to the fact that such three-dimensional integral equations always omit certain time-orders of some physical process for which other time-orders are included. And, if there is one thing the failure of
the current $NN - \pi NN$ equations teaches us, it is that we treat different time-orders of the same physical process differently at our peril. As stressed above, the only way to circumvent this difficulty and consistently include all time-orders of the relevant physical processes in a natural way is to use time-dependent perturbation theory and derive four-dimensional integral equations.

Having developed these points in this work the next task is the application of the perturbation theory developed in the first five sections of this paper in order to derive fully covariant scattering equations for systems of nucleons and pions. The technique chosen to do this is the classification-of-diagrams method of Taylor [15]. Because there are some subtleties in the application of Taylor’s method to perturbation schemes other than time-ordered perturbation theory we will explain Taylor’s technique fully and elucidate these subtleties in a forthcoming paper [54]. In a third paper the results developed in this second paper will be applied to the $NN - \pi NN$ system [55].

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APPENDIX A: CALCULATION OF THE ONE-PARTICLE FREE GREEN’S FUNCTIONS

In this appendix we calculate the one-nucleon and one-pion free Green’s functions. We then apply the no-anti-nucleon approximation to these Green’s functions, and hence obtain the form of the free Green’s functions in the no-anti-nucleon approximation.

1. Calculation of the full covariant Green’s functions

We begin by noting that the Green’s functions in question are defined by:

$$G_N^{(0)}(x', x) = \langle 0 | T(\psi^I(x') \overline{\psi}^I(x)) | 0 \rangle,$$

(A1)

$$G_{\pi j}^{(0)}(x', x) = \langle 0 | T(\phi^I_j(x') \phi^I_j(x)) | 0 \rangle.$$

(A2)

Firstly, consider the one-nucleon Green’s function. Using the definition of the time-ordering operator $T$, the expansions of the Schrödinger representation field operators $\psi(\vec{x})$, $\overline{\psi}(\vec{x})$:

$$\psi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_N(k)} \sum_\alpha [b_\alpha(k) u_\alpha(k) e^{i\vec{k} \cdot \vec{x}} + d_\alpha^\dagger(k) v_\alpha(k) e^{-i\vec{k} \cdot \vec{x}}],$$

(A3)

$$\overline{\psi}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_N(k)} \sum_\alpha [b_\alpha^\dagger(k) \overline{u}_\alpha(k) e^{-i\vec{k} \cdot \vec{x}} + d_\alpha(k) \overline{v}_\alpha(k) e^{i\vec{k} \cdot \vec{x}}].$$

(A4)
and noting that
\[ G_N^{(0)}(x', x) = G_N^{(0)+}(x', x)\theta(x'_0 - x_0) + G_N^{(0)-}(x', x)\theta(x_0 - x'_0); \tag{A5} \]
we have simplified \(G_N^{(0)+}\) and \(G_N^{(0)-}\) by eliminating those terms which are zero due to:
\[ b_\alpha(\vec{p})|0\rangle = 0, \quad A8 \]
\[ d_\alpha(\vec{p})|0\rangle = 0. \tag{A9} \]

Now consider \(G_N^{(0)}(p)\), defined by:
\[ G_N^{(0)}(p', p) = (2\pi)^4\delta(4)(p - p')G_N^{(0)}(p) \tag{A10} \]
where:
\[ G_N^{(0)}(p', p) = \int d^4x d^4x' e^{ip'x'} G_N^{(0)}(x', x)e^{-ipx} = G_N^{(0)+}(p', p) + G_N^{(0)-}(p', p). \tag{A11} \]

Here \(G_N^{(0)+}(p', p)\) and \(G_N^{(0)-}(p', p)\) are the Fourier transforms of \(G_N^{(0)+}(x', x)\theta(x'_0 - x_0)\) and \(G_N^{(0)-}(x', x)\theta(x_0 - x'_0)\) respectively. Evaluation of \(G_N^{(0)+}(p', p)\) gives:
\[ G_N^{(0)+}(p', p) = \int dx_0 dx'_0 \frac{m^2\sum_{\alpha\alpha'} u_{\alpha'}(\vec{p'})\pi_{\alpha}(\vec{p})}{E_N(\vec{p})E_N(\vec{p'})} \langle 0|b_{\alpha'}(\vec{p'})e^{-iH_K(x'_0 - x_0)}b_{\alpha}(\vec{p})|0\rangle e^{i(p'_0x'_0 - p_0x_0)}. \tag{A12} \]

Transforming to relative and average times:
\[ T = \frac{x_0 + x'_0}{2}, \quad \tau = x'_0 - x_0; \tag{A13} \]
and noting that \(dx_0 dx'_0 = dT d\tau\) gives:
\[ G_N^{(0)+}(p', p) \]
\[ = \int dT e^{i(p'_0 - p_0)T} \int_0^\infty d\tau \frac{m^2\sum_{\alpha\alpha'} u_{\alpha'}(\vec{p'})\pi_{\alpha}(\vec{p})}{E_N(\vec{p})E_N(\vec{p'})} \langle 0|a_{\alpha}(\vec{p})e^{i(p'_0 - p_0)H_K + \tau H_K}a_{\alpha}(\vec{p})|0\rangle \tag{A14} \]
\[ = 2\pi\delta(p'_0 - p_0) \frac{m^2\sum_{\alpha\alpha'} u_{\alpha'}(\vec{p'})\pi_{\alpha}(\vec{p})}{E_N(\vec{p})E_N(\vec{p'})} \lim_{\epsilon \to 0^+} \int_0^\infty d\tau \langle 0|a_{\alpha}(\vec{p})e^{i(p'_0 - p_0 + i\epsilon - H_K)\tau}a_{\alpha}(\vec{p})|0\rangle \tag{A15} \]
\[ = 2\pi\delta(p'_0 - p_0) \frac{m^2}{E_N(\vec{p})E_N(\vec{p'})} \sum_{\alpha\alpha'} u_{\alpha}(\vec{p'})\pi_{\alpha}(\vec{p}) \lim_{\epsilon \to 0^+} \frac{i}{p_0 + i\epsilon - E_N(\vec{p'})} (\vec{p'}\alpha'|\vec{p}\alpha), \tag{A16} \]
where $H_K|\tilde{p}'\alpha'\rangle = E_N(\tilde{p}')|\tilde{p}'\alpha'\rangle$. Now the Fock space states $|\tilde{p}\alpha\rangle$ are normalized such that:

$$\langle\tilde{p}'\alpha'|\tilde{p}\alpha\rangle = \delta_{\alpha\alpha'}(2\pi)^3 \frac{E_N(\tilde{p})}{m} \delta^{(3)}(p' - p),$$  \hspace{1cm} (A17)

hence:

$$G_N^{(0)^+}(p', p) = (2\pi)^4 \delta^{(4)}(p' - p) \frac{im}{E_N(\tilde{p})} \sum_{\alpha} u_{\alpha}(\tilde{p})\pi_{\alpha}(\tilde{p}).$$ \hspace{1cm} (A18)

Therefore,

$$G_N^{(0)^+}(p) = \frac{im}{E_N(\tilde{p})} \sum_{\alpha} u_{\alpha}(\tilde{p})\pi_{\alpha}(\tilde{p}),$$ \hspace{1cm} (A19)

in agreement with Garcilazo up to a factor of $i$. This result is also in agreement with van Faassen, again up to a factor of $i$. van Faassen absorbs the factor of $\frac{m}{E_N(\tilde{p})}$ into his spinor normalization \[57\].

A similar calculation for $G_N^{(0)^-}(p)$ yields:

$$G_N^{(0)^-}(p) = \frac{im}{E_N(-\tilde{p})} \sum_{\alpha} v_{\alpha}(-\tilde{p})\pi_{\alpha}(-\tilde{p}),$$ \hspace{1cm} (A20)

again in agreement with Garcilazo. Noting that $E_N(-\tilde{p}) = E_N(\tilde{p})$ these two results may be combined in order to produce:

$$G_N^{(0)}(p) = \frac{im}{E_N(\tilde{p})} \left[ \frac{\sum_{\alpha} u_{\alpha}(\tilde{p})\pi_{\alpha}(\tilde{p})}{p_0^+ - E_N(\tilde{p})} + \frac{\sum_{\alpha} v_{\alpha}(-\tilde{p})\pi_{\alpha}(-\tilde{p})}{p_0^+ + E_N(\tilde{p})} \right],$$ \hspace{1cm} (A21)

which may then be simplified to yield:

$$G_N^{(0)}(p) = \frac{i}{\tilde{p} - m},$$ \hspace{1cm} (A22)

in accordance with Eq.(3.12).

We now turn to the free pion Green’s function:

$$G_{\pi ji}^{(0)}(x', x) = \langle 0|T(\phi_{ji}^+(x')\phi_{ji}^+(x))|0\rangle.$$ \hspace{1cm} (A23)

Using the expansion of the Schrödinger operator:

$$\phi_{i}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_\pi(k)}[a_i(k)e^{i\vec{k}\cdot\vec{x}} + a_i^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}]$$ \hspace{1cm} (A24)

and the relation of the interaction and Schrödinger operators gives, by a similar argument to that for the free one-nucleon Green’s function:

$$G_{\pi ji}^{(0)}(x', x) = G_{\pi ji}^{(0)^+}(x', x)\theta(x'_0 - x_0) + G_{\pi ji}^{(0)^-}(x', x)\theta(x_0 - x'_0),$$ \hspace{1cm} (A25)

with:
\[ G_{\pi ji}^{(0)^+}(x', x) = \int \frac{d^3k \, d^3k'}{(2\pi)^6} \frac{1}{2\omega_\pi(k)2\omega_\pi(k')} (0) a_j(\vec{k}') e^{-iH_\pi(x_0' - x_0)} a_i^\dagger(\vec{k}) |0\rangle e^{i(\vec{k}' - \vec{x}' - \vec{k} - \vec{x})}, \quad (A26) \]

\[ G_{\pi ji}^{(0)^-}(x', x) = \int \frac{d^3k \, d^3k'}{(2\pi)^6} \frac{1}{2\omega_\pi(k)2\omega_\pi(k')} (0) a_i(\vec{k}) e^{-iH_\pi(x_0 - x_0')} a_j^\dagger(\vec{k}'') |0\rangle e^{i(\vec{k} - \vec{x} - \vec{k}'')}. \quad (A27) \]

If we define \( G_{\pi ji}^{(0)^+}(k', k) \) and \( G_{\pi ji}^{(0)^-}(k', k) \) to be the Fourier transforms of \( G_{\pi ji}^{(0)^+}(x', x) \theta(x_0' - x_0) \) and \( G_{\pi ji}^{(0)^-}(x', x) \theta(x_0 - x_0') \), we may obtain, by a similar procedure to that used for the free one-nucleon Green’s function above:

\[ G_{\pi ji}^{(0)^+}(k', k) = (2\pi)^4 \delta^{(4)}(k' - k) \frac{i}{2\omega_\pi(k')} \frac{\delta_{ij}}{k_0^+ - \omega_\pi(k)}, \quad (A28) \]

\[ G_{\pi ji}^{(0)^-}(k', k) = - (2\pi)^4 \delta^{(4)}(k' - k) \frac{i}{2\omega_\pi(-k')} \frac{\delta_{ij}}{k_0^- + \omega_\pi(-k)}. \quad (A29) \]

Note that once again, these formulae agree with Garcilazo up to a factor of \( i \). Note also that:

\[ G_{\pi ji}^{(0)^-}(k', k) = G_{\pi ji}^{(0)^+}(-k, -k'). \quad (A30) \]

If we use the fact \( \omega_\pi(k) = \omega_\pi(-k) \), these may be combined to give the total Green’s function:

\[ G_{\pi ji}^{(0)}(k', k) = (2\pi)^4 \delta^{(4)}(k' - k)G_{\pi ji}^{(0)}(k). \quad (A31) \]

with:

\[ G_{\pi ji}^{(0)}(k) = \frac{i\delta_{ij}}{k^2 - m_\pi^2}, \quad (A32) \]

in agreement with Eq.\( (3.13) \).

2. The no-anti-nucleon approximation

So far we have not made any approximations in the calculation of these Green’s functions. In this section we examine what happens if we make the no-anti-nucleon approximation, i.e. take the limit \( v(\vec{p}) \to 0 \) and \( \pi(\vec{p}) \to 0 \). Examining Eq.\( (A7) \) reveals, in this limit, we obtain:

\[ G_N^{(0)^-}(x', x) = 0, \quad (A33) \]

and so,

\[ G_N^{(0)}(x', x) = G_N^{(0)^+}(x', x) \theta(x_0' - x_0), \quad (A34) \]

in the no-anti-nucleon approximation. This proves Eq.\( (3.13) \).

Furthermore, since:

\[ G_N^{(0)}(p', p) = G_N^{(0)^+}(p', p) + G_N^{(0)^-}(p', p), \quad (A35) \]
and \( G_N^{(0)}(p,p') \) is zero in this approximation, it follows that:

\[
G_N^{(0)}(p', p) = G_N^{(0)+}(p', p).
\]

Consequently, Eq.(A19) implies that, in the no-anti-nucleon approximation:

\[
G_N^{(0)}(p) = \text{im} \frac{E_N(p)}{p_0^+ - E_N(p)} \sum_\alpha u_\alpha(p) \pi_\alpha(p),
\]

as claimed in Eq.(3.16).

**APPENDIX B: REWRITING THE PERTURBATION EXPANSION OF THE TWO-NUCLEON GREEN’S FUNCTION IN ORDER TO OBTAIN OLD-FASHIONED PERTURBATION THEORY**

In this Appendix we establish that the vertex containing single-particle operators, i.e.:

\[
H_{\text{int}}(x) = ig \int d^4x_N d^4x_N' \bar{\psi}(x_N') \gamma_5 \tau \psi(x_N) \cdot \bar{\phi}(x_\pi) \Gamma(x - x_N', x - x_N, x - x_\pi),
\]

may be re-expressed in terms of two-nucleon operators, which in the Schrödinger representation are defined by:

\[
\psi_{NN}(\vec{x}_1, \vec{x}_2) = \psi(\vec{x}_1) \psi(\vec{x}_2),
\]

\[
\bar{\psi}_{NN}(\vec{x}_1, \vec{x}_2) = \bar{\psi}(\vec{x}_2) \bar{\psi}(\vec{x}_1).
\]

It immediately follows that the equal-time two-nucleon Green’s function may be rewritten in terms of these two-particle operators without changing its value, as was claimed in Section VII.

**Claim:** Suppose that the state \(|S\rangle^I\) contains two nucleons, and both these nucleons in \(|S\rangle^I\) are created at the same time. Define \(H_{\text{int}}^I(y)\) by:

\[
H_{\text{int}}^I(y_0, \vec{y}, \vec{y}') = \bar{\psi}^I(y_0, \vec{y}') \psi^I(y_0, \vec{y})
\]

\[
= ig \int d^4y_N d^4y_N' d^4y_\pi \bar{\psi}_{NN}(y_N, \vec{y}_N, \vec{y}_N') \gamma_5 \tau
\]

\[
\times \psi_{NN}(y_N, \vec{y}_N, \vec{y}_N') \cdot \bar{\phi}_I(y_\pi) \Gamma(y - y_N', y - y_N, y - y_\pi).
\]

Then:

\[
H_{\text{int}}^I(y) |S\rangle^I = \int d^3y \cdot \bar{H}_{\text{int}}^I(y_0, \vec{y}, \vec{y}') |S\rangle^I.
\]

(Naturally these statements, if true in one representation, are true in any representation. We have chosen the interaction representation, because, as will be seen in Eq.(B14), it is operators in that representation which the Green’s function is expressed in terms of.)

**Proof:**

Consider:
where \(|S\rangle^I\) is a state in the interaction representation containing \(n\), as yet, unspecified number of nucleons, all of which were created at the same time. Firstly we note that, using the anti-commutation relations of the \(\bar{\psi}\)s and \(\psi\)s, and the definition of \(\mathcal{H}^I_{int}\), we may obtain:

\[
\mathcal{H}_{int}(y_0, \bar{y}, \bar{y}') = \mathcal{H}^I_{int}(y_0, \bar{y})e^{iH_Ky_0}\bar{\psi}(\bar{y}')\psi(\bar{y}')e^{-iH_Ky_0} - \mathcal{H}^I_{int}(y_0, \bar{y})\delta^{(3)}(y - y').
\]  

(B8)

Now if the state \(|S\rangle^I\) consists of nucleons which are all created at the same time \(t\) and pions created at unspecified times it may be written, in the interaction representation, as:

\[
|S\rangle^I = \prod_{k=1}^{n_N^S} \phi^I(z_k) \prod_{j=1}^{n_S^N} e^{iH_Kt}\bar{\psi}(\bar{x}_j)|0\rangle,
\]

(B9)

where \(n_N^S\) and \(n_S^N\) are the numbers of nucleons and pions in the state \(|S\rangle\) and \((t, \bar{x}_1), \ldots, (t, \bar{x}_{n_N^S})\) and \(z_1, \ldots, z_{n_S^N}\) are the space-time points at which these nucleons and pions are created. Clearly, the operator \((\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}'))\) commutes with all the pionic operators, and, it may also be shown to commute with \(H_K\), leaving us with:

\[
\int d^3y'\mathcal{H}_{int}(y_0, \bar{y}, \bar{y}')|S\rangle = \mathcal{H}^I_{int}(y_0, \bar{y}) \prod_{k=1}^{n_N^S} \phi^I(z_k)e^{iH_Kt}\left(\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}')\right) \prod_{j=1}^{n_S^N} \bar{\psi}(\bar{x}_j)|0\rangle - \mathcal{H}^I_{int}(y_0, \bar{y})|S\rangle.
\]

(B10)

Furthermore, from the anti-commutation relations of the \(\psi\)s and \(\bar{\psi}\)s we find that:

\[
(\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}'))\bar{\psi}(\bar{x}) = \bar{\psi}(\bar{x}) + \bar{\psi}(\bar{x})(\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}')).
\]

(B11)

This implies that:

\[
(\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}')) \prod_{j=1}^{n_S^N} \bar{\psi}(\bar{x}_j) = n_S^N \prod_{j=1}^{n_S^N} \bar{\psi}(\bar{x}_j) + \prod_{j=1}^{n_S^N} \bar{\psi}(\bar{x}_j)(\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}')),
\]

(B12)

suggesting that \((\int d^3y'\bar{\psi}(\bar{y}')\psi(\bar{y}'))\) may be interpreted as the fermionic number operator, as we would expect. Substituting this in Eq.(B11) and simplifying the result gives:

\[
\int d^3y'\mathcal{H}^I_{int}(y_0, \bar{y}, \bar{y}')|S\rangle^I = (n_S^N - 1)\mathcal{H}^I_{int}(y_0, \bar{y})|S\rangle^I,
\]

(B13)

which, if \(n_S^N = 2\) proves Eq.(B6), thus proving the claim.

Now consider the definition of \(G_{NN}(x_1', x_2'; x_1, x_2)\). By methods covered in most text books on field theory, the following expression for \(G_{NN}(x_1, x_2; x_1', x_2')\) may be derived:

\[
G_{NN}(x_1', x_2'; x_1, x_2) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4y_1d^4y_2 \cdots d^4y_n \times \langle 0 | T(\psi^I(x_1')\psi^I(x_2')\mathcal{H}^I_{int}(y_0) \cdots \mathcal{H}^I_{int}(y_1)\bar{\psi}^I(x_1)\bar{\psi}^I(x_2)) | 0 \rangle_{con},
\]

(B14)
where the \textit{con} indicates that those contributions to the matrix element which contain vacuum-vacuum subdiagrams are to be ignored. Suppose that we make the no-anti-nucleon approximation. Suppose that we also consider only the equal-time Green’s function, i.e. we set $x_1^0 = x_2^0 \equiv t$ and $x_1'^0 = x_2'^0 \equiv t'$. We begin by using the definition of $T$ in order to rewrite this Green’s function as a sum over all possible time-orders of the field operators in the vacuum expectation value. Each time-order is written as a separate term in the sum, with the integration over $y_1^0, y_2^0, \ldots, y_n^0$ being suitably restricted.

We wish to prove that, if we consider any term in the sum over all time-orders, replacing $H_{\text{int}}(y)$ by $\int d^3 y' H_{\text{int}(2)}(y_0, \vec{y}, \vec{y}')$ does not change the value of that particular term. So consider any one term in the sum over all time-orders. Firstly note that if the presence of anti-nucleons is forbidden then the time $t$ must be the earliest time, and consequently the operators $\psi^I(x_1)$ and $\psi^I(x_2)$ must be the first to act. Now the state $\psi^I(x_1)\psi^I(x_2)|0\rangle$ has a nucleon number of two, and since nucleon number is conserved in this theory, it follows that the number of nucleons in the state on which $H_{\text{int}}$ is acting is always two. Furthermore, if all vertices $H_{\text{int}}(y)$ up to the $j$th one have been replaced by vertices $\int d^3 y' H_{\text{int}(2)}(y_0, \vec{y}, \vec{y}')$ then it is clear that the $j$th interaction Hamiltonian $H_{\text{int}}(y_j)$ acts on a state in which two nucleons, created at the same time, are present. Consequently, $H_{\text{int}}(y_j)$ may also be replaced by a vertex $\int d^3 y_j' H_{\text{int}(2)}(y_j^0, \vec{y}_j, \vec{y}'_j)$. Since the first $H_{\text{int}}(y)$ acts directly on the state $\psi^I(x_1)\psi^I(x_2)|0\rangle$ which clearly contains two nucleons, which, by assumption, are created at the same time, it follows, by induction, that replacing all vertices $H_{\text{int}}(y)$ by vertices $\int d^3 y' H_{\text{int}(2)}(y_0, \vec{y}, \vec{y}')$ does not change the value of this term in the Green’s function. Consequently, Eq.(B14) may be rewritten:

$$G_{NN}(x_1', x_2'; x_1, x_2) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum \text{All TOs} \int_{\Omega} dy_1^0 dy_1^1 dy_1^2 dy_2^0 dy_2^1 dy_2^2 \ldots dy_n^0 dy_n^1 dy_n^2 \times \langle 0| \psi_{NN}^I(t, \vec{x}_1', \vec{x}_2') H_{\text{int}(2)}^{I}(y_n^0, \vec{y}_n, \vec{y}'_n) \ldots H_{\text{int}(2)}^{I}(y_1^0, \vec{y}_1, \vec{y}'_1) \overline{\psi}_{NN}^I(x_1^0, \vec{x}_1, \vec{x}_2)|0\rangle_{\text{con}},$$  

where $\Omega$ is a region of integration appropriate for the particular time-order under consideration. Thus, we have proved that, in the absence of anti-nucleons, the equal-time two-nucleon Green’s function may be rewritten in terms of the two-particle Green’s functions without changing its value.
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FIGURES

FIG. 1. The so-called Jennings term, which is not included in the $NN - \pi NN$ equations.

FIG. 2. The diagram on the left is not included in the $NN$ propagator used in the $NN - \pi NN$ equations, while the diagram on the right is.

FIG. 3. This diagram is currently included in the $NN - \pi NN$ equations, even though it is merely a different time-order of the excluded Jennings term.

FIG. 4. The diagram on the right is not included in the Stingl and Stelbovics model, since it involves a three-pion intermediate state. This occurs even though it is merely a different time order of the left-hand diagram, which is included in the model.
FIG. 5. The $\pi NN$ vertex we will use in this work.

FIG. 6. The diagram on the left contains a vacuum-vacuum subdiagram and consequently is not included in the perturbation expansion for the two-nucleon Green’s function. However, the diagram on the right, although it is disconnected like the left-hand diagram, contains no vacuum-vacuum subdiagram and consequently is included in the perturbation expansion.

FIG. 7. A diagrammatic representation of the two-nucleon vertex, in which one nucleon is a spectator. To be compared to the vertex in Figure 5 which is only a one-nucleon vertex.

FIG. 8. A diagrammatic representation of the Green’s function for a process with a $j$-pion intermediate state.
FIG. 9. The energy and momentum assignments in the exact convolution integral result for one-pion exchange.

FIG. 10. The energy and momentum assignments in the Kvinikhidze and Blankleider (KB) approximation to the exact convolution integral expression for one-pion exchange, which is shown in Figure 9.

FIG. 11. An example of the diagrams left out of the calculation in KB’s approximation to one-pion exchange.

FIG. 12. An example of the diagrams included in the calculation in KB’s approximation to one-pion exchange.
FIG. 13. A diagram containing the Jennings mechanism with spectator pion which is included in KB’s calculation of one-pion exchange.

FIG. 14. The process on the left is always included in KB’s approximation to the exact convolution integral one-pion exchange formula, while the process on the right is not always included; it is forbidden in certain diagrams, e.g. Figure 11, and allowed in others, e.g. Figure 13.

FIG. 15. These diagrams are an example of those omitted from the calculation of two-pion exchange in the KLMK method. Note that they are merely a different time-order of other included diagrams, see e.g. Figure 16.
FIG. 16. These diagrams are included in the calculation of two-pion exchange in the KLMK method.

FIG. 17. The energy assignments in the KLMK method calculation of two-pion exchange.