Martingales and some generalizations arising from the supersymmetric hyperbolic sigma model

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Abstract

We introduce a family of real random variables \((\beta, \theta)\) arising from the supersymmetric nonlinear sigma model and containing the family \(\beta\) introduced by Sabot, Tarrès, and Zeng \cite{STZ17} in the context of the vertex-reinforced jump process. Using this family we construct an exponential martingale generalizing the one considered in \cite{DMR17}. Moreover, using the full supersymmetric nonlinear sigma model we also construct a generalization of the exponential martingale involving Grassmann variables.

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1 Introduction and main results

The nonlinear supersymmetric hyperbolic sigma \((H^2|2)\) model was introduced by Zirnbauer in \cite{Zir91} as a toy model for quantum diffusion. The corresponding measure can be better analyzed after passing to horospherical coordinates \((u, s)\) as in \cite{SZ04} (for the non-supersymmetric version) and \((u, s, \bar{\psi}, \psi)\) as in \cite{DSZ10} (cf. details below). In particular a phase transition in dimension \(d \geq 3\) was proved, see \cite{DSZ10} and \cite{DS10}.

The \(H^2|2\) model has an interpretation as a random Schrödinger operator \cite{DS10} and unexpectedly also as mixing measure for the vertex-reinforced jump process \cite{ST15}. This process was conceived by Werner and first developed by Davis and Volkov \cite{DV02} \cite{DV04}.

More recently Sabot, Tarrès, and Zeng developed further the random Schrödinger operator interpretation \cite{STZ17} \cite{SZ15}. In particular they derived the explicit law for the random potential, and constructed two families of martingales in discrete time. One of them is the key ingredient to derive a characterization of recurrence/transience behavior of the vertex-reinforced jump process. In \cite{SZ17} Sabot and Zeng connected these families to certain continuous time martingales.

The above two families of discrete time martingales are only the first instances of an infinite hierarchy of martingales described in \cite{DMR17}. All these martingales involve only

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the $u$ components of the $H^{2|2}$ model. In this paper we extend these martingales to even larger families involving all the variables $(u, s, \psi, \overline{\psi})$.

**How this article is organized.** In Sections 1 and 2 we consider only the marginal $\mu^W(du \, ds)$ of the full $H^{2|2}$ model obtained by integrating out the Grassmann variables $(\psi, \overline{\psi})$. It is introduced in Section 1.1. The random variables $u$ encode the asymptotics of local times for a time changed vertex reinforced jump process while the random variables $s$ describe the corresponding fluctuations. For details see [MRT16].

In Section 1.2 we introduce a scaling transformation $S$ for the variables $(u, s)$. The effect of this scaling on the measure $\mu^W$ is formulated in Theorem 1.1. This theorem is in turn the key ingredient to prove the martingale property, which extends Theorem 2.6 and Corollary 2.7 from [DMR17] to test functions depending on $(u, s)$ variables. Note that when the test function depends only on the $u$ variable, we recover the martingales derived in [DMR17]. The martingale property on an infinite graph for the marginal $\mu^W$ is stated in Section 1.4, while Section 1.3 contains some preliminary results in finite volume. All these results are proved in Section 2.

In Section 3 we extend the results of Sections 1 and 2 to the full $H^{2|2}$ super-measure studied in [DSZ10], where Grassmann variables are included. This requires also a generalization of the notion of martingale to a 'susy martingale', not to be confused with the notion of supermartingale in standard probability. Here the test functions may depend on Grassmann variables too. In particular when the test function depends only on the real variables $u, s$ but not on the Grassmann variables, we recover the martingales described in Theorem 1.3 and Corollary 1.4.

### 1.1 The nonlinear supersymmetric hyperbolic sigma model

Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a finite connected graph with vertex set $\tilde{V}$ and set of undirected edges $\tilde{E}$. We assume that $\tilde{G}$ has no direct loops and no parallel edges. We write $i \sim j$ if there is an edge between $i$ and $j$. Let $\delta \in \tilde{V}$ be a distinguished vertex and set $V = \tilde{V} \setminus \{\delta\}$. Every edge $(i \sim j) \in \tilde{E}$ gets a weight $W_{ij} = W_{ji} > 0$. For convenience of notation, we set $W_{ij} = 0$ for all $i, j \in \tilde{V}$ with $i \not\sim j$. The euclidean scalar product is denoted by $\langle a, b \rangle = \sum_{i \in I} a_i b_i$, where $I = V$ or $I = \tilde{V}$, depending on the type of $a$ and $b$.

We define the matrix $A^W(u) \in \mathbb{R}^{\tilde{V} \times \tilde{V}}$ by

$$A^W_{ij}(u) = \begin{cases} -W_{ij}e^{u_i + u_j} & \text{for } i \neq j, \\ \sum_{k \in V} W_{ik}e^{u_i + u_k} & \text{for } i = j. \end{cases} \tag{1.2}$$
Let \( A^W_{VV}(u) \) denote its restriction to \( V \times V \). We define \( \rho^W : \Omega_V \to [0, \infty) \) by

\[
\rho^W(u, s) = e^{-\frac{1}{2}(s, A^W(u)s)} \det A^W_{VV}(u) \prod_{(i \sim j) \in \tilde{E}} e^{-W_{ij}[\cosh(u_i - u_j) - 1]}
\]

(1.3)

\[
= \det A^W_{VV}(u) \prod_{(i \sim j) \in \tilde{E}} e^{-W_{ij}[\cosh(u_i - u_j) - 1] + \frac{1}{2}(s_i - s_j)^2 e^{u_i + u_j}}.
\]

(1.4)

Using the reference measure

\[
\zeta(du_i ds_i) = e^{-u_i} du_i ds_i
\]

(1.5)
on \( \mathbb{R}^2 \), the supersymmetric sigma model is described by the following probability measure on \( \Omega_V \):

\[
\mu^W(du ds) = \rho^W(u, s) \prod_{i \in V} \frac{1}{2\pi} \zeta(du_i ds_i),
\]

(1.6)

where we drop the Dirac measure located at \((u_o, s_o) = (0, 0)\) in the notation. We denote the expectation with respect to \( \mu^W \) by \( \mathbb{E}_{\mu^W} \).

**Notation.** In the following, operations are frequently to be read componentwise, like

\[
a^2 + b^2 = (a_i^2 + b_i^2)_{i \in \tilde{V}}, \quad e^{-u/b/a} = (e^{-u_i b_i/a_i})_{i \in \tilde{V}}, \quad \log a = (\log a_i)_{i \in \tilde{V}}.
\]

**1.2 Scaling**

In this section, we introduce the scaling transformation \( \mathcal{S} \). It arises naturally as action of a group \( \mathcal{G}_V \). The reader may skip this subsection except from the definitions given in formulae (1.18) and (1.22).

**The group \( \mathcal{G} \).** For \((a, b) \in \mathbb{R}^+ \times \mathbb{R}\), we abbreviate

\[
[a, b] := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.
\]

(1.7)

The set of matrices

\[
\mathcal{G} := \{[a, b] : a > 0, b \in \mathbb{R}\}
\]

(1.8)

endowed with matrix multiplication forms a non-Abelian group. Its group operation can be written in the following form:

\[
[a'', b''] = [a, b] \cdot [a', b'] = [aa', b + ab'].
\]

(1.9)

The group \( \mathcal{G} \) has the neutral element \([1, 0]\); the inverse is given by

\[
[a, b]^{-1} = [1/a, -b/a].
\]

(1.10)
We endow \( G \) with the Lebesgue measure in the \((a,b)\)-coordinates \( \lambda(da \, db) = da \, db \). We introduce coordinates \((u,s) \in \mathbb{R}^2\) of \( G \) by
\[
a = e^{-u} \quad \text{and} \quad b = s. \tag{1.11}
\]
In these coordinates the Lebesgue measure \( da \, db \) takes the form of the measure \( \zeta \) from formula (1.5):
\[
da \, db = \zeta(du \, ds). \tag{1.12}
\]

**Right operation on \( G \).** Note that this measure \( \lambda \) is not a Haar measure on \( G \). We define the right operations \( R_{v'} : G \to G, v \mapsto v'' = v \cdot v' \) for \( v' \in G \).

Under \( R_{v'} \), using the notation \( v'' = [a'',b''] = [e^{-u''},s''] \), the measure \( \lambda \) scales as follows:
\[
R_{v'}[\lambda](da'' \, db'') = \frac{1}{a'} da'' \, db'' = \frac{1}{a'} \zeta(du'' \, ds''). \tag{1.14}
\]

**Cartesian power of \( G \).** We define the cartesian power of the group \( G \) with one component pinned to the neutral element:
\[
G_V := \{ [a,b] := ([a_i,b_i])_{i \in \tilde{V}} \in G^{\tilde{V}} : [a_\delta,b_\delta] = [1,0] \}. \tag{1.15}
\]
In particular, the group operation \( \cdot : G_V \times G_V \to G_V \) is understood componentwise. The set \( G_V \) can be identified with the set \( \Omega_V \), defined in (1.1), via the componentwise coordinate change to \((u,s)\)-coordinates
\[
\iota : G_V \to \Omega_V, \quad [a,b] \mapsto (-\log a, b). \tag{1.16}
\]

**\( \mathcal{I} \)-operation as right operation.** Using the identification \( \iota \), the \( \mathcal{I} \)-operation is defined as right operation with inverse elements \([a,b] \in G_V\):
\[
\mathcal{I}_{[a,b]} : \Omega_V \to \Omega_V, \quad \mathcal{I}_{[a,b]} = \iota \circ R_{[a,b]^{-1}} \circ \iota^{-1} = \iota \circ R_{[1/a,-b/a]} \circ \iota^{-1}, \tag{1.17}
\]
i.e. for \((u,s) \in \Omega_V\), using the specifications (1.9) and (1.10) of the group operations,
\[
\mathcal{I}_{[a,b]}(u,s) = \iota((u_{i} + \log a_{i}, s_{i} - e^{-u_{i}} \frac{b_{i}}{a_{i}})_{i \in \tilde{V}}. \tag{1.18}
\]
Note that \([a_\delta,b_\delta] = [1,0] \) implies \( \mathcal{I}_{[a,b]}(u,s) \in \Omega_V \). The map \( \mathcal{I} : G_V \times \Omega_V \to \Omega_V, \mathcal{I}([a,b],(u,s)) = \mathcal{I}_{[a,b]}(u,s) \), is a group action. Indeed, for \( v_1,v_2,v \in G_V \) it holds
\[
\mathcal{I}_{v_1} \circ \mathcal{I}_{v_2}(\iota(v)) = \iota((v'_{2} \cdot v_{1}^{-1}) \cdot v_{1}^{-1}) = \iota(v \cdot (v_1 \cdot v_2)^{-1}) = \mathcal{I}_{v_1 v_2}(\iota(v)). \tag{1.19}
\]
Moreover, for the neutral element \([1, 0] \in G_V\) the map \(\mathcal{J}_{[1, 0]}\) is the identity. Consequently, \(\mathcal{J}_{[a,b]}^{-1}\) is invertible for \([a, b] \in G_V\) with the following inverse:

\[
\mathcal{J}_{[a,b]}^{-1} = \mathcal{J}_{[a,b]}^{-1} : \Omega_V \rightarrow \Omega_V,
(u, s) \mapsto (\bar{u}, \bar{s}) = \iota^{-1}(u, s, [a, b]) = (u - \log a, s + e^{-u}b).
\] (1.20)

Let

\[
\mathcal{W} = \{W = (W_{ij})_{i,j \in \tilde{V}} : W_{ij} = W_{ji}, W_{ij} > 0 \text{ if and only if } i \sim j\} \tag{1.21}
\]
denote the set of possible edge weights. We consider the following group action on \(\mathcal{W}\):

\[
G_V \times \mathcal{W} \rightarrow \mathcal{W}, ([a, b], W) \mapsto W^a = (W_{ij}^a := a_i a_j W_{ij})_{i,j \in \tilde{V}}.
\] (1.22)

### 1.3 Results in finite volume

Denote by \(x_V\) the restriction of a vector \(x \in \mathbb{R}^V\) to \(\mathbb{R}^\tilde{V}\). Let

\[
e_{V,V}^{-u} = \text{diag}(e^{-u_i}, i \in V) \tag{1.23}
\]
denote the diagonal matrix in \(\mathbb{R}^{V \times V}\) with entries \(e^{-u_i}\) on the diagonal. We consider the variables \(\theta^{V,W}(u, s) = (\theta^{V,W}_i(u, s))_{i \in V}\) defined by

\[
\theta^{V,W}(u, s) = e_{V,V}^{-u} A^{W}_{V,V}(u) s_V.
\] (1.24)

Componentwise, we have for \(i \in V\)

\[
\theta^{V,W}_i(u, s) = \sum_{j \in V} W_{ij} e^{u_j}(s_i - s_j). \tag{1.25}
\]

We need the random variables \(\tilde{\beta}^{V,W} = (\tilde{\beta}^{V,W}_i)_{i \in \tilde{V}}\) and their restriction \(\beta^{V,W}\) to \(V\) defined by

\[
\tilde{\beta}^{V,W}_i(u) = \frac{1}{2} \sum_{j \in V} W_{ij} e^{u_j - u_i}, \quad \beta^{V,W} = (\tilde{\beta}^{V,W}_i)_{i \in V}. \tag{1.26}
\]

These variables were introduced in \[STZ17\]. We drop the dependence on \(V\), \(W\), or both if there is no risk of confusion.

The following theorem describes the behavior of the supersymmetric sigma model \(\mu^W\) with respect to the scaling transformation \(\mathcal{J}_{[a,b]}\) and is a fundamental ingredient in this paper. Its extension to Grassmann variables is given in Theorem 3.3.

**Theorem 1.1** Let \([a, b] \in G_V\). The image of \(\mu^W\) under the map \(\mathcal{J}_{[a,b]}\) is absolutely continuous with respect to \(\mu^W\) with the following Radon-Nikodym derivative on \(\Omega_V:\)

\[
\frac{d(\mathcal{J}_{[a,b]} \mu^W)}{d\mu^W}(u, s) = L^W(a, b)^{-1} e^{-((a^2 + b^2 - 1)_{\beta^{W}}(u), (b_{V}, \theta^{W}(u, s)))} \tag{1.27}
\]

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with the constant
\[
\mathcal{L}^W(a, b) := \prod_{(i \sim j) \in E} e^{-W_{ij}(a_i a_j + b_i b_j - 1)} \prod_{j \in V} \frac{1}{a_j}.
\]
(1.28)

In other words, for any measurable function \( f : \Omega_V \to \mathbb{R}_+ \), one has
\[
\mathbb{E}_{\mu^W} \left[ f(u, s) e^{-(a^2 + b^2 - 1)_{V_n, \beta W(u)} - (b_{V_n, \theta W(u, s))}} \right] = \mathcal{L}^W(a, b) \mathbb{E}_{\mu^W} \left[ f \circ \mathcal{R}_{[a, b]} \right].
\]
(1.29)

In particular, \( \mathcal{L}^W \) describes the joint Laplace transform of \( \beta^W \) and \( \theta^W \):
\[
\mathcal{L}^W(a, b) = \mathbb{E}_{\mu^W} \left[ e^{-(a^2 + b^2 - 1)_{V_n, \beta W(u)} - (b_{V_n, \theta W(u, s))}} \right].
\]
(1.30)

The special case \( b = 0 \) was proven as Theorem 3.1 in [DMR17]. For \( a = \sqrt{1 + \lambda} \) and \( b = 0 \) the Laplace transform \( \mathcal{L}^W(a, b) \) in (1.30) equals the Laplace transform \( \mathcal{L}^W(\lambda) \) given by formula (2.9) in [DMR17].

1.4 Results in infinite volume

Let \( G_\infty = (V_\infty, E_\infty) \) be an infinite graph with edge weights \( W_{ij} \). We approximate \( G_\infty \) by finite graphs with wired boundary conditions \( G_n = (\tilde{V}_n, \tilde{E}_n) \), where \( \tilde{V}_n = V_n \cup \{ \delta_n \} \), \( V_n \uparrow V_\infty \), and
\[
\tilde{E}_n = E_n \cup \{(i \sim \delta_n) : i \in V_n \text{ and } \exists j \in V_\infty \setminus V_n \text{ such that } (i \sim j) \in E_\infty \}.
\]
(1.31)

We endow the edges of \( \tilde{G}_n \) with the weights
\[
W_{ij}^{(n)} = W_{ij} \quad \text{if } i \in V_n \text{ and } j \in V_n,
\]
(1.32)
\[
W_{\delta_i \delta_j}^{(n)} = \sum_{j \in V_\infty \setminus V_n} W_{ij} \quad \text{for } i \in V_n, \quad \text{and } \quad W_{\delta_i \delta_n}^{(n)} = 0.
\]
(1.33)

Let \( \mu_n^W \) denote the \( H^{2/2} \) measure defined in (1.6) for the graph \( \tilde{G}_n \) with the weights \( W_{ij}^{(n)} \).

Lemma 1.2 (Kolmogorov consistency) For \( n \in \mathbb{N} \), the joint Laplace transform
\[
\mathcal{L}_n^W(a, b) = \mathbb{E}_{\mu_n^W} \left[ e^{-(a^2 + b^2 - 1)_{V_n, \beta V_n} - (b_{V_n, \theta V_n})} \right]
\]
(1.34)
of \( \beta_{V_n} = (\beta_i)_{i \in V_n} \) and \( \theta_{V_n} = (\theta_i)_{i \in V_n} \) satisfies the consistency relation
\[
\mathcal{L}_n^W(a_{V_n}, b_{V_n}) = \mathcal{L}_{n+1}^W(a, b),
\]
(1.35)
for all \([a, b] \in \mathcal{G}_{V_{n+1}} \) with \([a_i, b_i] = [1, 0] \) for all \( i \in \tilde{V}_{n+1} \setminus V_n \). In particular, the law of \((\beta_{V_n}, \theta_{V_n})\) with respect to \( \mu_n^W \) agrees with the law of \((\beta_{V_{n+1}}, \theta_{V_{n+1}})|_{V_n} \) with respect to \( \mu_{n+1}^W \).
Consistency of the law of $\beta$ was first observed by Sabot and Zeng in [SZ15]; see also Lemma 2.4 in [DMR17].

By Kolmogorov’s consistency theorem, there is a probability space $(\Omega, \mathcal{F}, \mu^W)$ with random variables $\beta_i, \theta_i : \Omega \rightarrow \mathbb{R}$, $i \in V_\infty$, such that for all $n \in \mathbb{N}$ the law of

$$(\beta^{(n)} = (\beta_i)_{i \in V_n}, \theta^{(n)} = (\theta_i)_{i \in V_n})$$

(1.36)

with respect to $\mu^W$ agrees with the law of $(\beta^{V_n}, \theta^{V_n}) : \Omega_{V_n} \rightarrow \mathbb{R}^{V_n} \times \mathbb{R}^{V_n}$ with respect to $\mu_n^W$. Moreover, by Lemma 2.3 in [DMR17], for any finite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V \cup \{\delta\}$, there is a measurable function $f^W_V : \mathbb{R}^V \rightarrow \mathbb{R}^\tilde{V}$ such that

$$(u_i)_{i \in \tilde{V}} = f^W_V(\beta^V) .$$

(1.37)

Using the definition (1.24) of $\theta^W$, we have $s_V = A^W_{VV}(u)\exp u_V \theta^V(u, s)$. Hence,

$$(s_i)_{i \in \tilde{V}} = g^W_V(\beta^V, \theta^V)$$

(1.38)

with the measurable function $g^W_V : \mathbb{R}^V \times \mathbb{R}^V \rightarrow \mathbb{R}^\tilde{V}$, $(\beta, \theta) \mapsto s = (s_i)_{i \in \tilde{V}}$ defined by $s_\delta = 0$ and $s_V = A^W_{VV}(f^W_V(\beta)^{-1}e^W_{VV}(\beta) \theta^V$. This allows us to couple the $u$ and $s$-variables. We define

$u^{(n)} = (u^{(n)}_i)_{i \in \tilde{V}_n} = f^W_{V_n}(\beta^{(n)}),$

(1.39)

$s^{(n)} = (s^{(n)}_i)_{i \in \tilde{V}_n} = g^W_{V_n}(\beta^{(n)}), 

\theta^{(n)} = \sum_{j \in V_\infty \setminus V_n} \alpha_j.$

(1.40)

(1.41)

We consider the following set of parameters

$(-\infty, 0]^{(V_\infty)} = \{\alpha \in (-\infty, 0]^{V_\infty} : \alpha_i \neq 0 \text{ for only finitely many } i \in V_\infty\}.$

(1.42)

For $\alpha \in (-\infty, 0]^{(V_\infty)}$ and $n \in \mathbb{N}$, we define $\alpha^{(n)} = (\alpha^{(n)}_i)_{i \in \tilde{V}_n}$ by

$\alpha^{(n)}_i = \alpha_i$ for $i \in V_n$ and $\alpha^{(n)}_\delta = \sum_{j \in V_\infty \setminus V_n} \alpha_j.$

(1.43)

**Theorem 1.3** For all $\alpha \in (-\infty, 0]^{(V_\infty)}$, the sequence $(M^{(n)}_\alpha)_{n \in \mathbb{N}}$, defined by

$M^{(n)}_\alpha : (u^{(n)}, s^{(n)}) \mapsto \exp \left( \sum_{j \in V_n} \alpha^{(n)}_j e^{u^{(n)}_j} (1 + i s^{(n)}_j) \right),$

(1.44)

is a $\mathbb{C}$-valued martingale with respect to the filtration $(\mathcal{F}_n = \sigma(\beta^{(n)}), \theta^{(n)}))_{n \in \mathbb{N}}$.

Taking derivatives of the martingale $(M^{(n)}_\alpha)_{n \in \mathbb{N}}$ at $\alpha = 0$, we obtain the following hierarchy of martingales.
Corollary 1.4 For all \( k \in \mathbb{N} \) and \( j_1, \ldots, j_k \in V_\infty \),
\[
M^{(n)}_{j_1, \ldots, j_k} = \prod_{l=1}^{k} e^{u^{(n)}_{j_l}} (1 + is^{(n)}_{j_l}), \quad n \in \mathbb{N},
\]
its real and imaginary part are martingales with respect to \( (\mathcal{F}_n = \sigma(\beta^{(n)}, \theta^{(n)}))_{n \in \mathbb{N}} \).

In [DMR17], we showed that \((E_{u^{(n)}_W}[M^{(n)}_{\alpha} | \sigma(u^{(n)})])_{n \in \mathbb{N}}\) and \((E_{\mu^{(n)}_W}[M^{(n)}_{j_1, \ldots, j_k} | \sigma(u^{(n)})])_{n \in \mathbb{N}}\) are martingales. These facts are also immediate consequences of Theorem 1.3 and Corollary 1.4.

## 2 The marginal \( \mu^W(du \, ds) \)

We start with two different proofs for Theorem 1.1.

- The first proof is based on Lemma 2.2 below which describes the ratio between the original and \( \mathcal{S} \)-transformed probability density of two supersymmetric sigma models with different parameters. Also for this lemma two different proofs are given.
  - The first proof is based on explicit computations on the quadratic form associated to the matrix \( A^W \) defined in equation (1.2).
  - The second proof uses the description of the density of the supersymmetric sigma model in terms of \( 2 \times 2 \) determinants connected to the linear algebra of Weyl spinors.

Both these proofs are self-contained.

- The second proof of Theorem 1.1 uses conditioning on the \( u \) variables and a result from [DMR17].

Finally Section 2.4 contains the proof of the martingale property.

### 2.1 First proof of Theorem 1.1

The group \( \mathcal{G}_V \) introduced in formula (1.15) acts on the set \( \mathcal{M} \) of measures on \( \Omega_V \):
\[
\mathcal{G}_V \times \mathcal{M} \to \mathcal{M}, \quad (v, \mu) \mapsto \mathcal{S}_v \mu, \tag{2.1}
\]
where \( \mathcal{S}_v \mu \) denotes the image measure of \( \mu \) with respect to the map \( \mathcal{S}_v : \Omega_V \to \Omega_V \). In particular, using the measure \( \zeta \) introduced in formula (1.5), we consider the product
\[
\zeta_V := \zeta^V \times \delta_{(0,0)} \tag{2.2}
\]
composed of factors \( \zeta \) indexed by \( V \) and one Dirac measure located at \( (0, 0) \in \mathbb{R}^2 \) indexed by the special vertex \( \delta \).
Lemma 2.1 For \([a, b] \in \mathcal{G}_V\), one has

\[ \mathcal{S}_{[a,b]} \zeta_V = \left( \prod_{i \in V} a_i \right) \zeta_V \quad (2.3) \]

**Proof.** This is an immediate consequence of formulas (1.17) and (1.14).  

Lemma 2.2 (Ratio of densities) For \([a, b] \in \mathcal{G}_V\) and \((u, s) \in \Omega_V\), one has

\[ \frac{\rho^W(a, s)}{\rho^W(u, s)} = \prod_{(i,j) \in \tilde{E}} e^{W_{ij}(a_i a_j + b_i b_j - 1)} \prod_{i \in V} \exp[-(a_i^2 + b_i^2 - 1) \beta_i^W(u) - b_i \theta_i^W(u,s)]. \quad (2.4) \]

This lemma is proven in Section 2.2 below.

**First proof of Theorem 1.1.** We abbreviate \(c = (2\pi)^{-|V|}\). From (1.6), we know \(d\mu^W = c \rho^W d\zeta_V\). Substituting \(W\) by \(W^a\), this gives \(d\mu^W = c \rho^W d\zeta_V\). We take now the image measure with respect to \(\mathcal{S}_{[a,b]}\). The following calculation uses the description of \(\mathcal{S}_{[a,b]} \zeta_V\) from Lemma 2.1 and in the last step the ratio of densities given in Lemma 2.2 together with the definition (1.28) of the constant \(L^W(a,b)\).

\[ d(\mathcal{S}_{[a,b]} \mu^W) = c(\rho^W \circ \mathcal{S}_{[a,b]}^{-1}) d(\mathcal{S}_{[a,b]} \zeta_V) = c \frac{\rho^W \circ \mathcal{S}_{[a,b]}^{-1}}{\rho^W} \prod_{i \in V} a_i d\zeta_V = \rho^W \prod_{i \in V} a_i d\mu^W = L^W(a,b)^{-1} e^{-\langle (a^2 + b^2 - 1) V, \beta^W \rangle - \langle b V, \theta^W \rangle} d\mu^W \quad (2.5) \]

This implies the claim (1.27), which is written in (1.29) in a different notation. Taking the test function \(f = 1\), (1.30) is a special case of (1.29).  

### 2.2 Two proofs of Lemma 2.2

The two proofs of Lemma 2.2 given in this subsection are based on two different representations of \(\rho^W\). The first representation, being based on the quadratic form associated to \(A^W\), is given in Lemma 2.3, while the second one, being based on \(2 \times 2\) determinants, is described in Lemma 2.5 below.

#### 2.2.1 First proof

We define the matrix \(H^W_{\beta(t)} \in \mathbb{R}^{\tilde{V} \times \tilde{V}}\) by

\[ (H^W_{\beta(t)})_{ij} = 2 \hat{\beta}_i(t) \delta_{ij} - W_{ij} \quad \text{for } i, j \in \tilde{V}, \quad (2.6) \]
Note that for all \(i, j \in \tilde{V}\), one has

\[
(H^W_{\beta(u)})_{ij} = \begin{cases} 
- W_{ij} & \text{if } i \neq j, \\
2\tilde{\beta}_i(u) = \sum_{k \in \tilde{V}} W_{ik}e^{u_k - u_i} & \text{if } i = j
\end{cases}
\]

\(= e^{-u_i - u_j} A^W_{ij}(u) = (e^{-u} A^W(u)e^{-u})_{ij}; \) (2.7)

recall that the graph \(\tilde{G}\) has no direct loops and hence \(W_{ii} = 0\) by the definition of the weights. Here and in the following, when calculating with matrices, we abbreviate \(e^{\pm u} = \text{diag}(e^{\pm u_i}, i \in \tilde{V})\). Thus, expressions like \(e^{-u}\)'s can be read in two equivalent ways, componentwise or as a matrix multiplication, both meaning the same object \((e^{-u_i}s_i)_{i \in \tilde{V}}\). In contrast to this, we write \(e^{-u} = (e^{-u_i})_{i \in \tilde{V}}\) for the column vector. We denote by \(H^W_{\beta(u)} := (H^W_{\beta(u)})_{VV}\) the restriction to \(V \times V\), i.e. \((H^W_{\beta(u)})_{ij} = 2\beta_i(u)\delta_{ij} - W_{ij}\) for \(i, j \in V\), cf. (1.20).

**Lemma 2.3** For \((u, s) \in \Omega_V\), we have the relations

\[
2 \sum_{(i,j) \in \tilde{E}} W_{ij}[\cosh(u_i - u_j) - 1] = \left\langle e^{-u}_V, A^W(u)e^{-u}_V \right\rangle = \left\langle 1_i, H^W_{\beta(u)} 1_{\tilde{V}} \right\rangle, \tag{2.8}
\]

\[
det A^W_{VV}(u) = \prod_{i \in V} e^{2u_i} \cdot \det H^W_{\beta(u)}, \quad \left\langle s, A^W(u)s \right\rangle = \left\langle e^u s, H^W_{\beta(u)} e^u s \right\rangle. \tag{2.9}
\]

In particular, the density \(\rho^W\) defined in (1.3) can be written as

\[
\rho^W(u, s) = \det A^W_{VV}(u)e^{-\frac{1}{2}\left\langle s, A^W(u)s \right\rangle e^{-\frac{1}{2}\left\langle e^{-u}_V, A^W(u)e^{-u}_V \right\rangle}. \tag{2.10}
\]

**Proof.** The first equation in (2.8) follows directly from

\[
\left\langle e^{-u}_V, A^W(u)e^{-u}_V \right\rangle = \sum_{i,j \in \tilde{V}} e^{-u_i} A^W_{ij}(u)e^{-u_j} = \sum_{i \in \tilde{V}} \sum_{k \in \tilde{V}} W_{ik}e^{u_k - u_i} - 2 \sum_{(i,j) \in \tilde{E}} W_{ij} \tag{2.11}
\]

\[= 2 \sum_{(i,j) \in \tilde{E}} W_{ij}[\cosh(u_i - u_j) - 1],\]

where the first sum on the right-hand side of (2.11) comes from the diagonal terms in \(A^W(u)\) and the second sum from the off-diagonal terms. Using relation (2.7) between \(H^W_{\beta(u)}\) and \(A^W(u)\), the remaining claims in (2.8) and (2.9) follow. The expression (2.10) for \(\rho^W\) is a consequence of formula (2.8). \(\blacksquare\)

**Lemma 2.4** The matrix \(A^W\) is invariant with respect to the \(\mathcal{F}\)-operation in the following sense: For \([a, b] \in G_V\), \((u, s) \in \Omega_V\), and \((\tilde{u}, \tilde{s}) = \mathcal{F}_{[a, b]}^{-1}(u, s) = (u - \log a, s + e^{-u}b)\), the following holds

\[A^{W^a}(\tilde{u}) = A^W(u), \quad \text{i.e. } A^W = A^{W^a} \circ \mathcal{F}_{[a, b]}^{-1}, \tag{2.12}\]
Lemma 2.5. For $i, j \in \tilde{V}$ with $i \neq j$, one has $A_{\tilde{w}}(\tilde{u}) = a_i a_j W_{ij} e^{\tilde{u}_i + \tilde{u}_j} = W_{ij} e^{u_i + u_j} = A_{\tilde{w}}(u)$. Since rows of both matrices $A_{\tilde{w}}(\tilde{u})$ and $A_{\tilde{w}}(u)$ sum up to 0, it follows also $A_{\tilde{w}}(\tilde{u}) = A_{\tilde{w}}(u)$. This proves the claim. □

First proof of Lemma 2.2. Substituting (2.12) into the representation (2.10) for $\rho_{\tilde{w}}$, we obtain

$$
\rho_{\tilde{w}}(\mathcal{S}^{-1}_{[a,b]}(u, s)) = \rho_{\tilde{w}}(\tilde{u}, \tilde{s}) = \det A_{\tilde{V}}(u) e^{-\frac{1}{2} \langle s, A_{\tilde{w}}(u) s \rangle} e^{-\frac{1}{2} \langle e^{-u} A_{\tilde{w}}(u) e^{-u}, W \rangle}. \tag{2.13}
$$

Inserting the definition of $\tilde{u}$ and $\tilde{s}$ in the exponents above and using (2.7), the facts $b_\delta = 0 = s_\delta$ and the definition (1.24) of $\theta_{\tilde{w}}$, we obtain

$$
\langle s, A_{\tilde{w}}(u) s \rangle = \langle s, A_{\tilde{w}}(u) s \rangle + \langle b, e^{-u} A_{\tilde{w}}(u) e^{-u} b \rangle + 2 \langle b, e^{-u} A_{\tilde{w}}(u) s \rangle
$$

$$
= \langle s, A_{\tilde{w}}(u) s \rangle + \langle b, H_{\beta(u)}^W b \rangle + 2 \langle b, \theta_{\tilde{w}}(u, s) \rangle, \tag{2.14}
$$

$$
\langle e^{-u} A_{\tilde{w}}(u) e^{-u}, W \rangle = \langle a, e^{-u} A_{\tilde{w}}(u) e^{-u} a \rangle = \langle a, H_{\beta(u)}^W a \rangle. \tag{2.15}
$$

Using in the second equality (2.10) and (2.8), this implies

$$
\rho_{\tilde{w}}(\mathcal{S}^{-1}_{[a,b]}(u, s)) = \det A_{\tilde{V}}(u) e^{-\frac{1}{2} \left( \langle s, A_{\tilde{w}}(u) s \rangle + \langle b, H_{\beta(u)}^W b \rangle \right) - \langle b, \theta_{\tilde{w}}(u, s) \rangle} e^{-\frac{1}{2} \langle a, H_{\beta(u)}^W a \rangle}
$$

$$
= \rho_{\tilde{w}}(u, s) e^{-\frac{1}{2} \left( \langle a, H_{\beta(u)}^W a \rangle + \langle b, H_{\beta(u)}^W b \rangle - \langle 1_{\tilde{V}}, H_{\beta(u)}^W 1_{\tilde{V}} \rangle \right)} e^{-\langle b, \theta_{\tilde{w}}(u, s) \rangle}. \tag{2.16}
$$

Since $a_\delta^2 + b_\delta^2 - 1 = 0$, the first exponent in the last expression takes the form

$$
- \frac{1}{2} \left( \langle a, H_{\beta(u)}^W a \rangle + \langle b, H_{\beta(u)}^W b \rangle - \langle 1_{\tilde{V}}, H_{\beta(u)}^W 1_{\tilde{V}} \rangle \right)
$$

$$
= \sum_{(i \sim j) \in \tilde{E}} W_{ij} (a_i a_j + b_i b_j - 1) - \sum_{i \in V} (a_i^2 + b_i^2 - 1) \beta_i^W(u). \tag{2.17}
$$

This proves the claim. □

2.2.2 Second proof

We can represent the density $\rho_{\tilde{w}}$ of the supersymmetric sigma model as follows. Recall the bijection $\iota$ introduced in (1.16).

Lemma 2.5 For $(u, s) = \iota(v) \in \Omega_V$ with $v = [a, b] \in \mathcal{G}_V$, the density $\rho_{\tilde{w}}$ defined in (1.13) can be written as follows:

$$
\rho_{\tilde{w}}(u, s) = \det A_{\tilde{V}}(u) \exp \left( \sum_{(i \sim j) \in \tilde{E}} \frac{W_{ij}}{2} \det \left( \frac{v_i v_j}{a_i} - \frac{v_j v_i}{a_j} \right) \right). \tag{2.18}
$$
Proof. Let \((u, s) = \iota(v) \in \Omega_V\). It suffices to prove for all \((i \sim j) \in \tilde{E}\)
\[- \left[ \cosh(u_i - u_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{u_i+u_j} \right] = \frac{1}{2} \det \left( \frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right). \tag{2.19}\]
For \(i \in \tilde{V}\), \(v_i = [e^{-u_i}, s_i] = [a_i, b_i]\), we calculate
\[\frac{v_i v_i^t}{a_i} = e^{u_i} \begin{pmatrix} e^{-u_i} & s_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-u_i} + s_i^2 e^{u_i} & s_i e^{u_i} \\ s_i e^{u_i} & e^{u_i} \end{pmatrix}. \tag{2.20}\]
Consequently, the claim (2.19) follows from
\[\det \left( \frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) = (e^{-u_i} - e^{-u_j} + s_i^2 e^{u_i} - s_j^2 e^{u_j})(e^{u_i} - e^{u_j}) - (s_i e^{u_i} - s_j e^{u_j})^2 = 2 - 2 \cosh(u_i - u_j) - (s_i - s_j)^2 e^{u_i+u_j}. \tag{2.21}\]

To deal with determinants of differences of \(2 \times 2\)-matrices, we need the following elementary lemma, which is motivated by the linear algebra of spinors. Let
\[\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.22}\]

**Lemma 2.6** For all \(v_i = [a_i, b_i], v_j = [a_j, b_j] \in G\), one has
\[\det \left( \frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) = 2 - \frac{\|v_i^t \varepsilon v_j\|^2}{a_i a_j}, \tag{2.23}\]
where \(\| \cdot \|\) means the euclidean norm of \(2 \times 2\)-matrices.

**Proof.** The bilinear form \(\text{trace}(A \varepsilon B^t \varepsilon)\) on \(2 \times 2\)-matrices \(A, B \in \mathbb{R}^{2 \times 2}\) is symmetric. Indeed, using \(\varepsilon^t = -\varepsilon\),
\[\text{trace}(B \varepsilon A^t \varepsilon) = \text{trace}((B \varepsilon A^t \varepsilon)^t) = \text{trace}(\varepsilon A \varepsilon B^t) = \text{trace}(A \varepsilon B^t \varepsilon). \tag{2.24}\]
The corresponding quadratic form is given by
\[\text{trace}(A \varepsilon A^t \varepsilon) = -2 \det A. \tag{2.25}\]
It follows
\[\det(A - B) = \det A + \det B + \text{trace}(A \varepsilon B^t \varepsilon). \tag{2.26}\]
Taking now \(A = a_i^{-1} v_i v_i^t\) and \(B = a_j^{-1} v_j v_j^t = B^t\), which fulfill \(\det A = a_i^{-2} (\det v_i)^2 = 1 = \det B\), we obtain
\[\det \left( \frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) = \det(A - B) = \det A + \det B + \text{trace}(A \varepsilon B^t \varepsilon)
= 2 + \text{trace}(A \varepsilon B \varepsilon) = 2 + \frac{1}{a_i a_j} \text{trace}(v_i v_i^t \varepsilon v_j v_j^t \varepsilon). \tag{2.27}\]
Using $\varepsilon^t = -\varepsilon$ again, we rewrite the last trace as follows:
\[
\text{trace}(v_i v_j' \varepsilon v_j v_j' \varepsilon) = \text{trace}(v_i' \varepsilon v_j v_j' \varepsilon) = -\text{trace}(v_i' \varepsilon v_j (v_j')^t) = -\|v_i' \varepsilon v_j\|^2. \tag{2.28}
\]
Substituting this into (2.27), the claim (2.23) follows. \[\]

**Second proof of Lemma 2.2** We take $v = [a', b]$, $v' = [a', b']$, and $v'' = [a'', b'']$ in $\mathcal{G}_v$ with $v'' = v' \cdot v$ and set $(u, s) = \mathcal{I}(v')$, $(\tilde{u}, \tilde{s}) = \mathcal{I}(v'')$. By (1.20), we have $\mathcal{F}_v^{-1}(u, s) = \mathcal{I}(v'')$. Since $\mathcal{W}_a(\tilde{u}) = \mathcal{W}_a(u)$ as stated in Lemma 2.4, it follows
\[
\det A_{\mathcal{V}V}^\mathcal{W}(\tilde{u}) = \det A_{\mathcal{V}V}^\mathcal{W}(u). \tag{2.29}
\]
Using Lemma 2.5 and this fact, we obtain
\[
\frac{\rho^\mathcal{W}(\mathcal{F}_v^{-1}(u, s))}{\rho^\mathcal{W}(u, s)} = \frac{\rho^\mathcal{W}(\mathcal{I}(a'', b''))}{\rho^\mathcal{W}(\mathcal{I}(a', b'))}
= \exp\left(\sum_{(i, j) \in \tilde{E}} \frac{W_{ij}}{2} \left[ a_i a_j \det \left( \frac{v_i''(v_j'')^t}{a_i''} - \frac{v_j''(v_j'')^t}{a_j''} \right) - \det \left( \frac{v_i'(v_j')^t}{a_i'} - \frac{v_j'(v_j')^t}{a_j'} \right) \right] \right) \tag{2.30}
\]
We apply Lemma 2.6 to $v' = [a', b']$ and $v'' = [a'', b'']$ as follows, using $a''_i = a_i a'_i$ for $i \in \tilde{V}$:
\[
\begin{align*}
& a_i a_j \det \left( \frac{v_i''(v_j'')^t}{a_i''} - \frac{v_j''(v_j'')^t}{a_j''} \right) - \det \left( \frac{v_i'(v_j')^t}{a_i'} - \frac{v_j'(v_j')^t}{a_j'} \right) \\
& = a_i a_j - \frac{a_i a_j}{a_i'' a_j''} ||(v_i'')^t v_j''||^2 - 2 + \frac{||v_i''(v_j'')^t \varepsilon v_j''||^2}{a_i'' a_j''} \\
& = 2(a_i a_j - 1) + \frac{1}{a_i'' a_j''} \left( ||(v_i'')^t v_j''||^2 - ||v_i''(v_j')^t \varepsilon v_j''||^2 \right). \tag{2.31}
\end{align*}
\]
Note that
\[
(v_i')^t v_j' = \begin{pmatrix} 0 & -a_i' \\ a_j' & b_j' - b_i' \end{pmatrix} \quad \text{and} \quad (v_i'')^t v_j'' = \begin{pmatrix} 0 & -a_i'' \\ a_j'' & b_j'' - b_i'' \end{pmatrix} = \begin{pmatrix} 0 & -a_i a_j'' \\ a_j a_j' & b_j'' - b_i'' + a_j' b_j - a_i' b_i \end{pmatrix}. \tag{2.32}
\]
We calculate the last parenthesis in (2.31), writing $\langle \cdot, \cdot \rangle$ for the euclidean scalar product of matrices:
\[
\begin{align*}
||v_i''(v_j')^t \varepsilon v_j''||^2 - ||v_i''(v_j')^t \varepsilon v_j''||^2 &= \langle (v_i'')^t v_j'' \rangle + \langle (v_i')^t v_j + (v_i'')^t v_j'' \rangle - \langle (v_i'')^t v_j'' \rangle \\
& = \langle \begin{pmatrix} 0 & -a_i' \end{pmatrix}(1 + a_i) \begin{pmatrix} a_j' (1 + a_j) & -2(b_i' - b_j') \end{pmatrix} \rangle \begin{pmatrix} a_j'(1 - a_j) & -a_i' (1 - a_i) \end{pmatrix} \\
& = -a_i' a_j' \left( \frac{a_j' (a_i^2 - 1)}{a_j} + \frac{a_i'}{a_j} (a_j^2 - 1) + 2(b_i' - b_j') \left( \frac{b_i}{a_j} - \frac{b_j}{a_i} \right) + a_i' a_j' \left( \frac{b_i}{a_j} - \frac{b_j}{a_i} \right)^2 \right). \tag{2.34}
\end{align*}
\]
This yields

\[
\text{l.h.s. in (2.31)} = 2(a_i a_j + b_i b_j - 1) - \left( \frac{a'_i}{a'_j} (a_i^2 + b_i^2 - 1) + \frac{a'_j}{a'_i} (a_j^2 + b_j^2 - 1) \right) \\
- 2 \frac{b_i}{a'_j} (b'_i - b'_j) - 2 \frac{b_j}{a'_i} (b'_j - b'_i). \tag{2.35}
\]

Multiplying this with \(W_{ij}/2\), summing the result over \((i \sim j) \in \tilde{E}\), and using the symmetry \(W_{ij} = W_{ji}\), we obtain

\[
\sum_{(i \sim j) \in \tilde{E}} \frac{W_{ij}}{2} \left[ a_i a_j \det \left( \frac{v''_i (v''_i)^t}{a''_i} - \frac{v''_j (v''_j)^t}{a''_j} \right) - \det \left( \frac{v'_i (v'_i)^t}{a'_i} - \frac{v'_j (v'_j)^t}{a'_j} \right) \right] \\
= \sum_{(i \sim j) \in \tilde{E}} W_{ij} \left[ a_i a_j + b_i b_j - 1 \right] - \sum_{i \in V} W_{ij} \left[ \frac{1}{2} \frac{a'_i}{a'_j} (a_i^2 + b_i^2 - 1) + \frac{b_i}{a'_j} (b'_i - b'_j) \right]. \tag{2.36}
\]

Next, we rewrite the definitions (1.26) of \(\beta^W_i\) and (1.25) of \(\theta^W_i\), \(i \in V\), in the following form, using \([a', b'] = [e^{-u}, s]\):

\[
\beta^W_i = \frac{1}{2} \sum_{j \in V} W_{ij} e^{u_j - u_i} = \frac{1}{2} \sum_{j \in V} W_{ij} \frac{a'_i}{a'_j}, \tag{2.37}
\]

\[
\theta^W_i = \sum_{j \in V} W_{ij} e^{u_j (s_i - s_j)} = \sum_{j \in V} W_{ij} \frac{1}{a'_j} (b'_i - b'_j). \tag{2.38}
\]

Since \(a_i^2 + b_i^2 - 1 = 0\), \(b_i = 0\), we obtain

\[
\text{l.h.s. in (2.36)} = \sum_{(i \sim j) \in \tilde{E}} W_{ij} \left[ a_i a_j + b_i b_j - 1 \right] - \sum_{i \in V} \left[ (a_i^2 + b_i^2 - 1) \beta^W_i + b_i \theta^W_i \right], \tag{2.39}
\]

Substituting this into (2.30), the claim (2.4) follows. \(\blacksquare\)

### 2.3 Proof of Theorem 1.1 by conditioning

The following lemma describes the conditional distribution of \(\theta^W\) given \(\beta^W\).

**Lemma 2.7** Conditioned on \(\beta^W\), the random vector \(\theta^W \in \mathbb{R}^V\) is normally distributed with mean 0 and covariance matrix

\[
H^W_{\beta(u)} = e^{-u} A^W_{VV}(u) e^{-u}. \tag{2.40}
\]

**Proof.** By definition, conditioned on \(u\), the vector \(s_V\) is centered Gaussian with covariance matrix \(A^{-1}\), where \(A := A^W_{VV}(u)\). Since \(u\) is a function of \(\beta^W\) by Lemma 2.3 of [DMR17],

\[14\]
we have conditioned on $β^W$ that $θ^W = e^W_{V}A_sV$ is also centered Gaussian with covariance matrix $(e^W_{V} A) A^{-1} (e^W_{V} A)^t = e^W_{V} A e^W_{V}$. The representation \( (2.40) \) follows from \( (2.7) \).

Our second proof of Theorem 1.1 uses the known transformation behavior of $μ^{Wa}(du \, ds)$ with respect to $\mathcal{A}_{[a, b]}$ from [DMR17] and the fact that conditionally on $u$, the $s$-variables are jointly Gaussian.

**Proof of Theorem 1.1 by conditioning.** To prove (1.29), it suffices to consider test functions of the form $f(u, s) = g(u) h(s)$ with measurable functions $g, h : \mathbb{R} \rightarrow \mathbb{R}_0^+$. We calculate

$$
E_{μ^{Wa}} [f \circ \mathcal{A}_{[a, b]}] = E_{μ^{Wa}} [g(u + \log a) h(s - e^{-(u + \log a)} b)] \tag{2.41}
$$

The expectation in (2.42) is given by

$$
E_{μ^{Wa}} [g(u + \log a) h(s - e^{-(u + \log a)} b)] = L^W(a, 0)^{-1} E_{μ^W} \left[ g(u) h(s - e^{-b} e^{-\langle a^2 - 1 \rangle V, β^W(u)}) \right]
$$

$$
= L^W(a, 0)^{-1} E_{μ^W} \left[ g(u) E_{μ^W} [h(s - e^{-b}) | u] e^{-\langle a^2 - 1 \rangle V, β^W(u)} \right] \tag{2.42}
$$

with the constant $L^W(a, 0)$ given in (1.28); recall that $β^W$ is a function of $u$. By the definition of the supersymmetric sigma model, cf. (1.6) and (1.3), conditioned on $u$ the vector $s_V$ is centered Gaussian with covariance matrix $A^W_{V}(u)^{-1}$ and $s_{\delta} = 0$. Consequently, abbreviating $c = (2π)^{-|V|/2} \sqrt{\det A^W_{V}(u)}$ and $σ(ds) = δ_0(ds_\delta) \prod_{i \in V} ds_i$, the conditional expectation in (2.42) is given by

$$
E_{μ^W} [h(s - e^{-b}) | u] = c \int_{\mathbb{R}^V} h(s - e^{-b}) e^{-\frac{1}{2} \langle (s, A^W(u) s) \rangle} \sigma(ds)
$$

$$
= c \int_{\mathbb{R}^V} h(s) e^{-\frac{1}{2} \langle (s, A^W(u) s) \rangle - \frac{1}{2} \langle (e^{-b} A^W(u) e^{-b}) \rangle} \sigma(ds)
$$

$$
= c \int_{\mathbb{R}^V} h(s) e^{-\frac{1}{2} \langle (s, A^W(u) s) \rangle - \frac{1}{2} \langle (e^{-b} A^W(u) e^{-b}) \rangle - \langle e^{-b} A^W(u) s \rangle} \sigma(ds). \tag{2.43}
$$

Using $b_\delta = 0$ and (2.7), we obtain

$$
\langle e^{-b} A^W(u) e^{-b} \rangle = \langle b_V, H^W_{β^W} b_V \rangle = 2 \langle b_V, β^W(u) \rangle - \sum_{i,j \in V} W_{ij} b_i b_j. \tag{2.44}
$$

Similarly, using the definition (1.24) of $θ^W$, we obtain

$$
\langle e^{-b} A^W(u) s \rangle = \langle b_V, e^{-W_{V}} A^W_{V}(u) s_V \rangle = \langle b_V, θ^W(u, s) \rangle. \tag{2.45}
$$

Inserting (2.44) and (2.45) into (2.43) yields

$$
E_{μ^W} [h(s - e^{-b}) | u] = \prod_{i,j \in V} e^{\frac{1}{2} W_{ij} b_i b_j} \cdot e^{-\langle b_V, β^W(u) \rangle} c \int_{\mathbb{R}^V} h(s) e^{-\frac{1}{2} \langle (s, A^W(u) s) \rangle - \langle b_V, θ^W(u, s) \rangle} \sigma(ds)
$$

$$
= L^W(a, 0) L^W(a, b)^{-1} \cdot e^{-\langle b_V, β^W(u) \rangle} E_{μ^W} \left[ h(s) e^{-\langle b_V, θ^W(u, s) \rangle} \right]. \tag{2.46}
$$
We conclude that the products over edge sets in (2.47) and (2.48) agree. The claim (1.35) follows from (1.29) applied to the function \( f(u, s) = 1 \). □

2.4 Martingales

Proof of Kolmogorov consistency (Lemma 1.2). By Theorem 1.1, one has

\[
L_n^W (a_V, b_V) = \prod_{(i \sim j) \in E_n} e^{-W_{ij} (a_i a_j + b_i b_j - 1)} \cdot \prod_{j \in V_n} \frac{1}{a_j},
\]

(2.47)

\[
L_{n+1}^W (a, b) = \prod_{(i \sim j) \in \tilde{E}_{n+1}} e^{-W_{ij} (a_i a_j + b_i b_j - 1)} \cdot \prod_{j \in \tilde{V}_{n+1}} \frac{1}{a_j} \tag{2.48}
\]

Since \( a_j = 1 \) for \( j \in V_{n+1} \setminus V_n \), one has

\[
\prod_{j \in V_n} \frac{1}{a_j} = \prod_{j \in V_{n+1}} \frac{1}{a_j} \tag{2.49}
\]

Consider \((i \sim j) \in \tilde{E}_{n+1}\).

Case \(i, j \in V_n\): Then \((i \sim j) \in \tilde{E}_n\) and \(W_{ij}^{(n)} = W_{ij}^{(n+1)}\). Consequently, \(W_{ij}^{(n)} (a_i a_j + b_i b_j - 1) = W_{ij}^{(n+1)} (a_i a_j + b_i b_j - 1)\).

Case \(i, j \in \tilde{V}_{n+1} \setminus V_n\): Then \([a_i, b_i] = [1, 0] = [a_j, b_j]\) and hence \(a_i a_j + b_i b_j - 1 = 0\).

Case \(i \in V_n\) and \(j \in \tilde{V}_{n+1} \setminus V_n\): Then \([a_i, b_i] = [1, 0]\). For the given \(i \in V_n\), we calculate

\[
\sum_{j \in \tilde{V}_{n+1} \setminus V_n \atop (i \sim j) \in \tilde{E}_{n+1}} W_{ij}^{(n+1)} (a_i a_j + b_i b_j - 1) = W_{id_{n+1}}^{(n+1)} + \sum_{j \in \tilde{V}_{n+1} \setminus V_n} W_{ij} (a_i - 1) \tag{2.50}
\]

\[
= \sum_{j \in V_n \setminus V_n} W_{ij} (a_i - 1) = W_{id_n}^{(n)} (a_i - 1) = W_{id_n}^{(n)} (a_i a_{\delta_n} + b_i b_{\delta_n} - 1).
\]

We conclude that the products over edge sets in (2.47) and (2.48) agree. The claim (1.35) follows. This identity holds in particular for \((a^2 + b^2 - 1, b)\) in a neighborhood of the origin. As a consequence, by analytic continuation, the characteristic function of \((\beta V_n, \theta V_n)\) with respect to \(\mu_n^W\) agrees with the characteristic function of \((\beta V_{n+1}, \theta V_{n+1})|_{V_n}\) with respect to \(\mu_{n+1}^W\). The claim follows. □

Proof of Theorem 1.3 (Generating martingale). By the definitions (1.39) and (1.40) of \(u^{(n)}\) and \(s^{(n)}\), it follows that \(M^{(n)}_\alpha\) is \(F_n\)-measurable. For \([a, b] \in G_{V_{n+1}}\) with \([a_i, b_i] = [1, 0]\) for all \(i \in \tilde{V}_{n+1} \setminus V_n\), we show

\[
\mathbb{E}_{\mu_n^W} \left[ M^{(n+1)}_\alpha \prod_{j \in V_n} e^{-(a_j^2 + b_j^2 - 1)\beta_j - b_j \theta_j} \right] = \mathbb{E}_{\mu_{\infty}^W} \left[ M^{(n)}_\alpha \prod_{j \in V_n} e^{-(a_j^2 + b_j^2 - 1)\beta_j - b_j \theta_j} \right]. \tag{2.51}
\]
Note that for \( j \in V_n \), one has \( a_j > 0 \) and \( b_j \in \mathbb{R} \). So in particular, we prove the identity (2.51) for \( a_j^2 + b_j^2 - 1 \) and \( b_j \) belonging to a neighborhood of the origin, which implies the martingale property for \( M^{(n)}_\alpha \).

We rewrite the claim in terms of expectations with respect to the supersymmetric sigma model on finite graphs. Let

\[
\tilde{M}^{(n)}_\alpha : \Omega_{V_n} \rightarrow \mathbb{R}, \quad (u, s) \mapsto e^{(a^{(n)}, e^u(1+is))}.
\]  

Using the definition of the variables \( \beta \) and \( \theta \), the identity (2.51) is equivalent to

\[
\mathbb{E}_{w_{n+1}} \left[ \tilde{M}^{(n+1)}_\alpha \prod_{j \in V_n} e^{-(a_j^2 + b_j^2 - 1)\beta_j^{V_n+1}(u) - b_j \theta_j^{V_n+1}(u, s)} \right]
= \mathbb{E}_{w_n} \left[ \tilde{M}^{(n)}_\alpha \prod_{j \in V_n} e^{-(a_j^2 + b_j^2 - 1)\beta_j^{V_n}(u) - b_j \theta_j^{V_n}(u, s)} \right].
\]  

(2.53)

Since \( a_j^2 + b_j^2 - 1 = 0 = b_j \) for \( j \in \tilde{V}_{n+1} \setminus V_n \), we rewrite the left-hand side of (2.53) using Theorem 1.1 as follows:

\[
\text{lhs}(2.53) = \mathbb{E}_{w_{n+1}} \left[ \tilde{M}^{(n+1)}_\alpha \prod_{j \in V_n} e^{-(a_j^2 + b_j^2 - 1)\beta_j^{V_n+1}(u) - b_j \theta_j^{V_n+1}(u, s)} \right]
= \mathcal{L}_{n+1}^W(a, b) \mathbb{E}_{w_a} \left[ M^{(n+1)}_\alpha \circ J_{a,b} \right],
\]  

(2.54)

where the last expectation is taken with respect to the supersymmetric sigma model on the graph \( \tilde{G}_{n+1} \) with the rescaled weights \( a, a_j W_{ij}^{(n+1)} \). We calculate

\[
\tilde{M}^{(n+1)}_\alpha \circ J_{a,b} = \exp \left( \langle a^{(n+1)}, e^{u+\log_a (1 + i(s - e^{-u-\log a}b))} \rangle \right)
= e^{\langle a^{(n+1)}, e^u(1+is) \rangle} e^{\langle a^{(n+1)}, ib \rangle}.
\]  

(2.55)

Note that \( \langle a^{(n+1)}, ib \rangle \) does not depend on \( u \) or \( s \). Consequently, inserting the last expression into (2.54), we obtain

\[
\text{lhs}(2.53) = \mathcal{L}_{n+1}^W(a, b) e^{\langle a^{(n+1)}, ib \rangle} \mathbb{E}_{w_a} \left[ e^{\langle a^{(n+1)}, e^u(1+is) \rangle} \right].
\]  

(2.56)

By Corollary 5.3 in [DMR17],

\[
\mathbb{E}_{w_a} \left[ e^{\langle a^{(n+1)}, e^u(1+is) \rangle} \right] = e^{\langle a^{(n+1)}, 1 \psi \rangle} = e^{\langle a^{(n+1)}, a \rangle}.
\]  

(2.57)

We conclude

\[
\text{lhs}(2.53) = \mathcal{L}_{n+1}^W(a, b) e^{\langle a^{(n+1)}, a - ib \rangle}.
\]  

(2.58)

The right-hand side of (2.53) can be obtained from the last expression by replacing \( n + 1 \) by \( n \). Thus, the claim (2.53) can be written as follows

\[
\mathcal{L}_{n+1}^W(a, b) e^{\langle a^{(n+1)}, a - ib \rangle} = \mathcal{L}_n^W(a V_n, b V_n) e^{\langle a^{(n)}, a - ib \rangle}.
\]  

(2.59)
By Lemma 1.2 \( \mathcal{L}^W_n(a_{V_n}, b_{V_n}) = \mathcal{L}^W_{n+1}(a, b) \). Furthermore, using \([a_{\delta_{n+1}}, b_{\delta_{n+1}}] = [1, 0] \), we obtain

\[
\langle \alpha^{(n+1)}, a - ib \rangle = \sum_{j \in V_{n+1}} \alpha_j (a_j - ib_j) + \alpha^{(n+1)} = \sum_{j \in V_{n+1}} \alpha_j (a_j - ib_j) + \sum_{j \in V_{\infty} \setminus V_{n+1}} \alpha_j
\]

\[
= \sum_{j \in V_n} \alpha_j (a_j - ib_j) + \sum_{j \in V_{\infty} \setminus V_n} \alpha_j = \langle \alpha^{(n)}, a - ib \rangle. \quad (2.60)
\]

This shows that (2.59) holds and finishes the proof of the martingale property. ■

**Proof of Corollary 1.4.** By Theorem 1.3 \((M^{(n)}_\alpha)_{n \in \mathbb{N}} \) is a martingale for all \( \alpha \in (-\infty, 0]^{(V_\infty)} \). The martingale property is equivalent to

\[
\mathbb{E}_{\mu^W_n}[M^{(n+1)}_\alpha 1_A] = \mathbb{E}_{\mu^W_n}[M^{(n)}_\alpha 1_A] \quad (2.61)
\]

for all \( n \in \mathbb{N}_0 \) and all events \( A \in \mathcal{F}_n \). Taking left-sided derivatives at \( \alpha = 0 \), we get

\[
\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M^{(n)}_\alpha = \partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} e^{\langle \alpha^{(n)}, e^{u^{(n)}}(1+is^{(n)}) \rangle} = M^{(n)}_{j_1, \ldots, j_k} M^{(n)}_\alpha,
\]

\[
\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M^{(n)}_\alpha |_{\alpha = 0} = M^{(n)}_{j_1, \ldots, j_k}. \quad (2.62)
\]

Since \( |\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M^{(n)}_\alpha| \leq |M^{(n)}_{j_1, \ldots, j_k}| \) for all \( \alpha \in (-\infty, 0]^{(V_\infty)} \), we can interchange expectation and differentiation at \( \alpha = 0 \) in (2.61). This yields the martingale property for \( M^{(n)}_{j_1, \ldots, j_k}. \) ■

The following are special cases of Corollary 1.4:

- Since \( M^{(n)}_j = e^{u^{(n)}_j} \left( 1 + is^{(n)}_j \right) \), we know that
  
  \[
  \left( s^{(n)}_j e^{u^{(n)}_j} \right)_{n \in \mathbb{N}} \quad (2.63)
  \]

  is a martingale.

- One has \( M^{(n)}_{j, l} = e^{u^{(n)}_j + u^{(n)}_l} \left( 1 - s^{(n)}_j s^{(n)}_l + i \left( s^{(n)}_l + s^{(n)}_j \right) \right) \). Hence,

  \[
  \left( e^{u^{(n)}_j + u^{(n)}_l} \left( 1 - s^{(n)}_j s^{(n)}_l \right) \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left( e^{u^{(n)}_j + u^{(n)}_l} \left( s^{(n)}_l + s^{(n)}_j \right) \right)_{n \in \mathbb{N}} \quad (2.64)
  \]

  are martingales. For \( j = l \), this yields the martingales

  \[
  \left( e^{2u^{(n)}_l} \left( 1 - s^{(n)}_l \right)^2 \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left( 2s^{(n)}_l e^{2u^{(n)}_l} \right)_{n \in \mathbb{N}}. \quad (2.65)
  \]

- One has

  \[
  M^{(n)}_{j, l, m} = e^{u^{(n)}_j + u^{(n)}_l + u^{(n)}_m} \left( 1 - s^{(n)}_j s^{(n)}_l - s^{(n)}_j s^{(n)}_m - s^{(n)}_l s^{(n)}_m \right.
  
  \]

  \[
  + i \left( s^{(n)}_j + s^{(n)}_l + s^{(n)}_m - s^{(n)}_j s^{(n)}_l - s^{(n)}_j s^{(n)}_m \right) \left( s^{(n)}_j s^{(n)}_l \right) \left( s^{(n)}_j s^{(n)}_m \right) \left( s^{(n)}_l s^{(n)}_m \right). \quad (2.66)
  \]
Hence, the following are martingales:

\[
\begin{align*}
&\left( e^{u_j(n)} + u_i(n) + u_{lm}(n) \left( 1 - s_j(n) s_l(n) - s_j(n) s_m(n) \right) \right)_{n \in \mathbb{N}}, \\
&\left( e^{u_j(n)} + u_i(n) + u_{lm}(n) \left( s_j(n) + s_l(n) + s_m(n) - s_j(n) s_l(n) s_m(n) \right) \right)_{n \in \mathbb{N}}, \\
&\left( e^{3u_j(n)} \left( 1 - 3s_j(n)^2 \right) \right)_{n \in \mathbb{N}}, \\
&\left( e^{3u_j(n)} \left( 3s_j(n) - (s_j(n))^3 \right) \right)_{n \in \mathbb{N}}.
\end{align*}
\]

\( (2.67) \) \( (2.68) \) \( (2.69) \)

3 Extension to Grassmann variables

We consider now the full supersymmetric \( H^{2|2} \) model, studied in [DSZ10], including Grassmann variables. We start with some preliminaries in Sections 3.1 and 3.2. In the remaining part, we extend the scaling transformation, the Laplace transform, and the martingales introduced in the previous sections to include Grassmann variables.

3.1 Grassmann algebras

Let \( \mathcal{V} \) be a finite dimensional \( \mathbb{R} \)-vector space. Let

\[
\Lambda \mathcal{V} := \bigoplus_{n=0}^{\dim \mathcal{V}} \Lambda^n \mathcal{V}, \quad \Lambda \mathcal{V}_{\text{even}} := \bigoplus_{0 \leq n \leq \dim \mathcal{V}} \Lambda^n \mathcal{V}, \quad \Lambda \mathcal{V}_{\text{odd}} := \bigoplus_{0 \leq n \leq \dim \mathcal{V}} \Lambda^n \mathcal{V}
\]

be the Grassmann algebra generated by it, its even and its odd subspace, respectively. In particular, \( \mathbb{R} = \Lambda^0 \mathcal{V} \subset \Lambda \mathcal{V} \) and \( \mathcal{V} = \Lambda^1 \mathcal{V} \subset \Lambda \mathcal{V} \). The Grassmann product is bilinear and associative. Moreover, for all \( w, w' \in \Lambda \mathcal{V}_{\text{odd}} \) it is anticommutative: \( ww' = -w'w \). In particular, \( w^2 = 0 \). Let body : \( \Lambda \mathcal{V} \to \Lambda^0 \mathcal{V} = \mathbb{R} \) be the projection to the 0th component and soul : \( \Lambda \mathcal{V} \to \bigoplus_{n=1}^{\dim \mathcal{V}} \Lambda^n \mathcal{V} \), soul\( (w) = w - \text{body}(w) \), denote the projection to the nilpotent part. The subset of positive even elements is defined by

\[
\Lambda \mathcal{V}^{+}_{\text{even}} = \{ a \in \Lambda \mathcal{V}_{\text{even}} : \text{body}(a) > 0 \}.
\]

As a generalization of (1.7), for \( a \in \Lambda \mathcal{V}^{+}_{\text{even}}, b \in \Lambda \mathcal{V}_{\text{even}}, \overline{w}, w \in \Lambda \mathcal{V}_{\text{odd}} \), we set

\[
[a, b, \overline{w}, w] := \begin{pmatrix}
a & b & \overline{w} & w \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The set of matrices, cf. (1.8),

\[
\mathcal{G}(\mathcal{V}) := \{ [a, b, \overline{w}, w] : a \in \Lambda \mathcal{V}^{+}_{\text{even}}, b \in \Lambda \mathcal{V}_{\text{even}}, \overline{w}, w \in \Lambda \mathcal{V}_{\text{odd}} \}
\]

(3.4)
endowed with matrix multiplication forms a group, non-Abelian except in trivial cases, with the neutral element \([1,0,0,0]\). In other words,

\[
[a, b, \overline{w}, w] \cdot [a', b', \overline{w}', w'] = [aa', b + ab', \overline{w} + a\overline{w}', w + aw'],
\]

\[
[a, b, \overline{w}, w]^{-1} = [a^{-1}, -ba^{-1}, -\overline{w}a^{-1}, -wa^{-1}];
\]  

(3.5)

(3.6)

cf. (1.9) and (1.10). Note that \(a^{-1}\) is well-defined because \(\text{body}(a) > 0\).

We take again a finite graph \(\tilde{G} = (\tilde{V}, \tilde{E})\) with \(\tilde{V} = V \cup \{\delta\}\) as in Subsection 1.1. We define the cartesian power of the group \(\mathcal{G}(V)\) with one component pinned to the neutral element:

\[
\mathcal{G}(V)_V := \{[a, b, \overline{w}, w] := ([a_i, b_i, \overline{w}_i, w_i])_{i \in \tilde{V}} \in \mathcal{G}(V)^\tilde{V} : [a_\delta, b_\delta, \overline{w}_\delta, w_\delta] = [1, 0, 0, 0]\}.
\]  

(3.7)

3.2 Superfunctions and superexpectation

Let

\[
\mathcal{A}(V) = \mathcal{A}_V(V) = C^\infty(\Omega_V, \Lambda V) = C^\infty(\Omega_V, \mathbb{R}) \otimes \Lambda V
\]  

(3.8)

be the Grassmann algebra over \(V\) with coefficients being smooth real-valued functions \(f \in C^\infty(\Omega_V, \mathbb{R}), (u, s) \mapsto f(u, s)\). Elements of \(\mathcal{A}(V)\) are called superfunctions.

Assume that the vector space \(V\) has a basis \((\psi_i, \psi_i')_{i \in V}\). Moreover, we set \(\psi_\delta = \psi_\delta' = 0\).

Then, \(\overline{\psi}_i, \psi_i \in V \subseteq \Lambda V_{\text{odd}}, i \in \tilde{V}\), implies \(\psi_i \overline{\psi}_j = -\overline{\psi}_j \psi_i, \psi_i \psi_j = -\psi_j \psi_i, \text{ and } \overline{\psi}_i \overline{\psi}_j = -\overline{\psi}_j \overline{\psi}_i\) for all \(i, j \in \tilde{V}\). To describe a superfunction in \(\mathcal{A}(V)\), the following abbreviations are useful:

\[
\mathcal{I}_V = \{(i_1, \ldots, i_n) \in V^n : n \in \mathbb{N}_0, i_1 \ldots i_n\}
\]  

(3.10)

with respect to some fixed linear order \(<\) of the vertex set \(V\). For \(I = (i_1, \ldots, i_n) \in \mathcal{I}_V\), we set

\[
\overline{\psi}_I = \psi_{i_1} \cdots \psi_{i_n}
\]  

(3.11)

and similarly for \(\psi_I\). By convention, \(\overline{\psi}_\emptyset = \psi_\emptyset = 1\). Thus, a superfunction \(f \in \mathcal{A}(V)\) can be uniquely written as

\[
f(u, s, \overline{\psi}, \psi) = \sum_{I, J \in \mathcal{I}_V} f_{IJ}(u, s)\overline{\psi}_I \psi_J
\]  

(3.12)

with coefficients \(f_{IJ} \in C^\infty(\Omega_V, \mathbb{R})\). Here \(f_{\emptyset\emptyset}\) is the body of \(f\) and \(f - f_{\emptyset\emptyset}\) its nilpotent part. An element \(f \in \mathcal{A}(V)\) is even if \(f_{IJ} = 0\) whenever \(|I| + |J|\) is odd; \(f\) is odd if \(f_{IJ} = 0\) whenever \(|I| + |J|\) is even. Let \(\mathcal{A}(V)_{\text{even}} = C^\infty(\Omega_V, \Lambda V_{\text{even}})\) and \(\mathcal{A}(V)_{\text{odd}} = \)
$C^\infty(\Omega_V, \Lambda \nu_{\text{odd}})$ denote the set of even and odd elements of $A(\nu)$, respectively, and let $A(\nu)^+_{\text{even}} = \{f \in A(\nu)_{\text{even}} : \text{body}(f) > 0\}$. Smooth functions (like $\exp$) of elements in $A(\nu)_{\text{even}}$ are understood as power series in the nilpotent part.

In analogy to the parameter dependent $W^\alpha$ in formula \eqref{1.22} we will consider a further generalization of the supersymmetric sigma model $H^{2|2}$ from [DSZ10] involving parameters that depend on Grassmann variables. Our parameters belong to another Grassmann algebra $\Lambda \nu'$ with another finite-dimensional $\mathbb{R}$-vector space $\nu'$. Both vector spaces $\nu$ and $\nu'$ are viewed as subspaces of their direct sum $\nu'' = \nu \oplus \nu'$. The corresponding Grassmann algebras are related by $\Lambda \nu'' = \Lambda \nu \otimes \Lambda \nu'$ with the Grassmann product extended to be anticommuting. In particular, $\Lambda \nu = \Lambda \nu \otimes \mathbb{R} \subseteq \Lambda \nu''$ and $\Lambda \nu' = \mathbb{R} \otimes \Lambda \nu' \subseteq \Lambda \nu''$.

We will consider superfunctions $f \in A(\nu'')$. Each such function can be represented as in \eqref{3.12} with coefficients $f_{i,j} \in A(\nu')$. In the following, we consider coupling constants $W_{ij} \in \Lambda \nu'_{\text{even}}$ for all $(i \sim j) \in \tilde{E}$ and $W_{ij} = 0$ whenever $(i \sim j) \notin \tilde{E}$. We define the superdensity $\rho^W \in A(\nu'')^+_{\text{even}}$ by

$$
\rho^W(u, s, \bar{\psi}, \psi) = e^{-\frac{V}{2}(s, A^W(u)s)} e^{-\langle \bar{\psi}, A^W(u)\psi \rangle} \prod_{(i \sim j) \in \tilde{E}} e^{-W_{ij}[\cosh(u_i-u_j)-1]}
= e^{-\langle \bar{\psi}, A^W(u)\psi \rangle} \frac{\det A^W_{ij}(u)}{\rho^W(u, s)}
$$

with the matrix $A^W(u) \in \mathbb{R}^{\tilde{V} \times \tilde{V}}$ defined in \eqref{1.22} and the density $\rho^W$ defined in \eqref{1.3}. As Lemma \ref{3.1} below shows, $\rho^W$ is the marginal of $\rho^W$. Therefore we use the same symbol writing the supersymmetric variant with the corresponding bold symbol. This convention will also be used below for other quantities like $\zeta$, $\mu^W$, and $L^W$. Let

$$
d\zeta = d\zeta[u, s, \bar{\psi}, \psi] := \prod_{i \in \nu} \frac{1}{2\pi} \zeta(du_i ds_i) \partial_{\bar{\psi}_i} \partial_{\psi_i} = \prod_{i \in \nu} e^{-u_i} du_i ds_i \partial_{\bar{\psi}_i} \partial_{\psi_i}
$$

be the supersymmetric reference measure, where we suppress again the Dirac measure $\delta_{(0,0)}(du ds)$ in the notation. With these notions the supersymmetric sigma model is given by

$$
\mu^W(du ds \partial_{\bar{\psi}} \partial_{\psi}) := d\zeta[u, s, \bar{\psi}, \psi] \circ \rho^W(u, s, \bar{\psi}, \psi),
$$

where the symbol $\circ$ means that the partial derivatives $\partial_{\bar{\psi}}$ and $\partial_{\psi}$ act not only on the superdensity $\rho^W(u, s, \bar{\psi}, \psi)$, but also on the test function as follows:

$$
\int_{\Omega_\nu} d\mu^W f = \int_{\Omega_\nu} d\zeta(\rho^W f)
$$

for any $f \in A(\nu'')$ for which the integral is defined. Here we use the convention

$$
\partial_{\bar{\psi}_i} \bar{\psi}_i = \partial_{\psi_i} \psi_i = 1.
$$

Note that the superintegral $\int d\mu^W f$ with integrable arguments $f \in A(\nu'')$ takes values in $\Lambda \nu'$. 

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Lemma 3.1 The probability measure $\mu^W$ defined in (1.6) is the marginal of the supermeasure $\mu^W$ defined in (3.15) in the following sense. In the special case when the weights $W_{ij}$ are real-valued and the superfunction $f$ is an ordinary function $f = f(u, s)$, i.e. does not depend on any Grassmann variables, we have the real-valued integral

$$\int d\mu^W f = \int d\mu^W f. \quad (3.18)$$

Proof. Since $f$ is an ordinary function, the Grassmann part in $\int d\mu^W f$ is reduced to

$$\prod_{i \in V} \partial_{\bar{\psi}_i} \partial_{\psi_i} e^{-\langle \overline{\psi}, A^W(u) \psi \rangle} = \det A^W_{VV}(u). \quad (3.19)$$

Therefore, the definition (3.13) of $\rho^W$ yields

$$\prod_{i \in V} \partial_{\bar{\psi}_i} \partial_{\psi_i} \rho^W(u, s, \bar{\psi}, \psi) = \rho^W(u, s). \quad (3.20)$$

The result follows. \[\blacksquare\]

3.3 Super scaling transformation

We generalize now the definition (1.17) of the scaling transformation $S_{[a,b]} : \Omega_V \to \Omega_V$ to the present setup involving Grassmann parameters. Take a superparameter $[a, b, \overline{\chi}, \chi] \in G(V)$; recall that $[a_\delta, b_\delta, \overline{\chi}_\delta, \chi_\delta] = [1, 0, 0, 0]$ by (3.7). In order to find an analogue to equation (1.29), we consider a generalization of the pull-back $S^*_{[a,b]} f := f \circ S_{[a,b]}$, $f : \Omega_V \to \mathbb{R}$ (3.21) to a supertransformation $S^*_{[a,b,\overline{\chi},\chi]} : \mathcal{A}(\mathcal{V}') \to \mathcal{A}(\mathcal{V}')$ defined as follows. Take a general element

$$f(u, s, \overline{\psi}, \psi) = \sum_{I,J \in I_V} f_{IJ}(u, s) \overline{\psi}_K \psi_L \in \mathcal{A}({\mathcal{V}')}, \quad (3.22)$$

with coefficients $f_{IJ} \in \mathcal{A}(\mathcal{V}')$. In the following, for any even $u', s'$, we interpret $f_{IJ}(u', s')$ again as power series in the nilpotent part of $u'$ and $s'$. We set

$$(S^*_{[a,b,\overline{\chi},\chi]} f)(u, s, \overline{\psi}, \psi) = \sum_{I,J \in I_V} f_{IJ}(u', s') \overline{\psi}_K' \psi_L' \in \mathcal{A}(\mathcal{V}'), \quad (3.23)$$

where the expressions for $u' = u'(u), s' = s'(u, s), \overline{\psi}' = \overline{\psi}'(u, \overline{\psi}), \psi' = \psi'(u, \psi)$ are given by the following formula, to be read componentwise

$$[e^{-u'}, s', \overline{\psi}', \psi'] = [e^{-u}, s, \overline{\psi}, \psi] \cdot [a, b, \overline{\chi}, \chi]^{-1}. \quad (3.24)$$
This means that the explicit expressions for \( u', s', \overline{\psi}', \) and \( \psi' \) are given by
\[
u'_i = u_i + \log a_i, \quad s'_i = s_i - e^{-u_i} b_i a_i^{-1}, \tag{3.25}
\]
\[
\overline{\psi}'_i = \overline{\psi}_i - e^{-u_i} \overline{x}_i a_i^{-1}, \quad \psi'_i = \psi_i - e^{-u_i} \chi_i a_i^{-1}
\]
for all \( i \in V \). Note that \([e^{-u'_i}, s'_i, \overline{\psi}'_i, \psi'_i] = [1, 0, 0, 0]\), and that \( u'_i \) and \( s'_i \) are even superfunctions in \( \mathcal{A}(V') \).

Note that \( \mathcal{S}_v^* \) is a group operation, i.e. for all \( v, v' \in \mathcal{G}(V') \),
\[
\mathcal{S}_v^{[1,0,0,0]} = \text{id}, \quad \mathcal{S}_v^* v' = \mathcal{S}_v^* \mathcal{S}_v^{v'}, \quad \mathcal{S}_v^{v^{-1}} = (\mathcal{S}_v^*)^{-1}. \tag{3.26}
\]
We will need the following transformation formula for the supermeasure \( d\zeta_V \) with respect to \( \mathcal{S}_v^* \).

**Lemma 3.2** For \( v = [a,b, \overline{x}, \chi] \in \mathcal{G}(V') \) and for any compactly supported (or sufficiently fast decaying) test superfunction \( f \in \mathcal{A}(V') \), one has
\[
\int d\zeta_V \mathcal{S}_v^* f = \prod_{j \in V} a_j \int d\zeta_V f. \tag{3.27}
\]

**Proof.** Using \((\mathcal{S}_v^*)^{-1} (e^{-u}) = e^{-(u_{-\log a_i})}\) and using the supertransformation formula described in Lemma [A.1](#) in the appendix, we calculate
\[
\int d\zeta_V \mathcal{S}_v^* f = (2\pi)^{-|V|} \prod_{i \in V} du_i ds_i \partial_{\overline{\psi}} \partial_{\psi} \left( \mathcal{S}_v^* f(u, s, \overline{\psi}, \psi) \prod_{i \in V} e^{-u_i} \right)
\]
\[
= (2\pi)^{-|V|} \prod_{i \in V} du_i ds_i \partial_{\overline{\psi}} \partial_{\psi} \left( f(u, s, \overline{\psi}, \psi) \prod_{i \in V} e^{-(u_i - \log a_i)} \right)
\]
\[
= (2\pi)^{-|V|} \prod_{i \in V} du_i ds_i \partial_{\overline{\psi}} \partial_{\psi} f(u, s, \overline{\psi}, \psi) \prod_{i \in V} e^{-u_i - \log a_i}. \tag{3.28}
\]
The claim follows. \( \blacksquare \)

### 3.4 Grassmann-Laplace transform

In analogy to the definition \([1.24]\) of \( \theta^{V,W} \), we define odd superfunctions \( \overline{\phi}^{V,W}(u, \overline{\psi}) \) and \( \phi^{V,W}(u, \psi) \) by
\[
\overline{\phi}^{V,W}(u, \overline{\psi}) = e^{-u} A^{W}_{V,V}(u) \overline{\psi}, \quad \phi^{V,W}(u, \psi) = e^{-u} A^{W}_{V,V}(u) \psi, \tag{3.29}
\]
cf. \([1.25]\). Here, the restriction \( \psi_V = (\psi_i)_{i \in V} \) should not be confused with the product \( \psi_I, I \in \mathcal{I}_V \), defined in \([3.11]\). Componentwise, we have for \( i \in V \)
\[
\overline{\phi}^{V,W}_i(u, \overline{\psi}) = \sum_{j \in V} W_{ij} e^{u_j}(\overline{\psi}_i - \overline{\psi}_j), \quad \phi^{V,W}_i(u, \psi) = \sum_{j \in V} W_{ij} e^{u_j}(\psi_i - \psi_j). \tag{3.30}
\]
As for $\beta$ and $\theta$, we will drop the dependence on $V, W$, or both if there is no risk of confusion.

Our goal is to derive a generalization of Theorem 1.1 including Grassmann variables. In the following, we abbreviate for $[a, b, \chi, \chi] \in \mathcal{G}(\mathcal{V})$.

\[
\varpi^{V} = \varpi^{V, W} = (\beta^{V}, \theta^{V}, \phi^{V}), \quad \pi^{V}_{[a, b, \chi, \chi]} = (a^{2} + b^{2} + 2\chi\chi - 1, b, \chi, \chi)_{V},
\]

which fulfill $\varpi^{V}, \pi^{V}_{[a, b, \chi, \chi]} \in (\mathcal{A}(\mathcal{V}^{\prime})_{\text{even}} \times \mathcal{A}(\mathcal{V}^{\prime})_{\text{even}} \times \mathcal{A}(\mathcal{V}^{\prime})_{\text{odd}} \times \mathcal{A}(\mathcal{V}^{\prime})_{\text{odd}})^{V}$. We use the following generalization of the Euclidean scalar product:

\[
\langle \pi^{V}_{[a, b, \chi, \chi]}; \varpi^{V} \rangle = \langle (a^{2} + b^{2} + 2\chi\chi - 1)V, \beta^{W} \rangle + \langle bV, \theta^{W} \rangle + \langle \chi V, \phi^{W} \rangle + \langle \phi^{W}, \chi V \rangle.
\]

Note the reversed order of factors in the last product, which causes a sign change due to anticommutativity.

**Theorem 3.3** For $[a, b, \chi, \chi] \in \mathcal{G}(\mathcal{V})$, the joint Grassmann-Laplace transform of $\beta^{W}, \theta^{W}, \phi^{W},$ and $\varpi^{W}$ is given by

\[
\int d\mu^{W} e^{-\langle \pi^{V}_{[a, b, \chi, \chi]}; \varpi^{V} \rangle} = L^{W}(a, b, \chi, \chi)
\]

with the constant

\[
L^{W}(a, b, \chi, \chi) = \prod_{i \sim j \in E} e^{-W_{ij}(a_{i}a_{j} + b_{i}b_{j} + \chi_{i}\chi_{j} + \chi_{j}\chi_{i} - 1)} \cdot \prod_{j \in V} \frac{1}{a_{j}} \in \Lambda \mathcal{V}_{\text{even}}.
\]

Moreover, for every compactly supported (or sufficiently fast decaying\footnote{A sufficient condition is given above \ref{footnote:fast-decaying}.}) in $u$ and $s$ test superfunction $f \in \mathcal{A}(\mathcal{V}^{\prime})$ it holds

\[
\int d\mu^{W} f e^{-\langle \pi^{V}_{[a, b, \chi, \chi]}; \varpi^{V} \rangle} = L^{W}(a, b, \chi, \chi) \int d\mu^{W_{a}} \mathcal{S}^{*}_{[a, b, \chi, \chi]} f,
\]

where $W^{a} = (W^{a}_{ij} := a_{i}a_{j}W_{ij})_{i, j \in V}$ with $W^{a}_{ij} \in \Lambda \mathcal{V}_{\text{even}}^{\prime}$. Note that equation \ref{eq:laplace-transform} is the analogue of (1.29). We remark that in the special case $b = 0, \chi = 0 = \chi$, which was already treated in Theorem 2.1 in [DMR17], $a^{2} + b^{2} + 2\chi\chi - 1$ just reduces to $a^{2} - 1$, which was called $\lambda$ in the citation. If we want the Laplace parameters $a^{2} + b^{2} + 2\chi\chi - 1$ and $b$ to be real-valued, this enforces the parameters $a$ not to be real-valued but to take values in the even part of a Grassmann algebra. This is why we have to allow Grassmann algebra-valued weights $W^{a}_{ij} \in \Lambda \mathcal{V}_{\text{even}}^{\prime}$ rather than only real-valued weights.
Proof of Theorem 3.3  We abbreviate again $v = [a, b, \chi, \lambda]$. Using Lemma 3.2 we obtain
\[
\int d\mu^{W^a} \mathcal{F}_v f = \int d\zeta_V (\rho^{W^a} \mathcal{F}_v^* f) = \int d\zeta_V (\mathcal{F}_v^* ((\mathcal{F}_v^* \rho^{W^a}) f))
= \prod_{j \in V} a_j \int d\zeta_V ((\mathcal{F}_v^* \rho^{W^a}) f).
\] (3.36)

Note that
\[
(\mathcal{F}_v^* f)(u, s, \overline{\psi}, \psi) = f(u - \log a, s + e^{-u}b, \overline{\psi} + e^{-u} \chi, \psi + e^{-u} \chi).
\] (3.37)

By Lemma 2.4 one has $A^{W^a}(u - \log a) = A^W(u)$ for $a = (a_i)_{i \in V} \in \mathbb{R}^V$ with $a_i = 1$. Since the entries of the matrix $A^W(u)$ are smooth functions of $W_{ij}e^{u_i+u_j}$, this identity remains true if we replace $a_i, i \in V$, by even elements of the Grassmann algebra $\Lambda V$ with body$(a_i) > 0$. Consequently (cf. (2.12)),
\[
\mathcal{F}_v^* A^{W^a} = A^W.
\] (3.38)

The relation (2.10) allows us to rewrite $\rho^{W^a}$ as follows:
\[
\rho^{W^a}(u, s, \overline{\psi}, \psi) = e^{-\frac{1}{2} (s, A^{W^a}(u)s)} e^{-\frac{1}{2} (\overline{\psi}, A^{W^a}(u)\overline{\psi})} e^{-\frac{1}{2} (e^{-u} A^{W^a}(u)e^{-u})}.
\] (3.39)

Using (3.38) and the expression (2.7) for $H^W_{\beta(U)}$, we calculate
\[
\mathcal{F}_v^* = \left(\overline{\psi}, A^W(u)\psi\right) = \left(\overline{\psi} + e^{-u} \chi, A^W(u)\left(\psi + e^{-u} \chi\right)\right)
= \left(\overline{\psi}, A^W(u)\psi\right) + \left(\overline{\phi}^W(u, \overline{\psi}), \chi_V\right) + \left(\chi_V, \phi^W(u, \psi)\right) + \left(\chi, H^W_{\beta(U)}\chi\right).
\] (3.40)

As in (2.14) and (2.15), we obtain
\[
\mathcal{F}_v^* \left(s, A^{W^a}(u)s\right) = \left(s, A^W(u)s\right) + \left(b, H^W_{\beta(U)}b\right) + 2 \left(b_V, \theta^W(u, s)\right),
\]
\[
\mathcal{F}_v^* \left(e^{-u} A^{W^a}(u)e^{-u}\right) = \left(a, H^W_{\beta(U)}a\right).
\] (3.41) (3.42)

Combining the above identities and relation (2.8), we find
\[
\mathcal{F}_v^* \rho^{W^a}(u, s, \overline{\psi}, \psi) = \rho^{W}(u, s, \overline{\psi}, \psi) e^{-\frac{1}{2} \left(\left(a, H^W_{\beta(U)}a\right) + \left(b, H^W_{\beta(U)}b\right) + 2 \left(\chi, H^W_{\beta(U)}\chi\right) - \left(1, H^W_{\beta(U)}1\right)\right)}
\cdot e^{-\left(\theta^W(u, s), \chi_V\right)} e^{-\left(\phi^W(u, \overline{\psi}), \chi_V\right)} e^{-\left(\chi, H^W_{\beta(U)}\chi\right)}.
\] (3.43)

Using $a_i^2 + b_i^2 + 2 \overline{x}_i \chi_i - 1 = 0$, we rewrite the first exponent in the last expression as follows
\[
- \frac{1}{2} \left(\left(a, H^W_{\beta(U)}a\right) + \left(b, H^W_{\beta(U)}b\right) + 2 \left(\chi, H^W_{\beta(U)}\chi\right) - \left(1, H^W_{\beta(U)}1\right)\right)
= \sum_{(i,j) \in E} W_{ij}(a_i a_j + b_i b_j + \overline{x}_i \chi_j + \overline{x}_j \chi_i - 1) - \sum_{i \in V} (a_i^2 + b_i^2 + 2 \overline{x}_i \chi_i - 1) \beta^W_i.
\] (3.44)

Substituting this in (3.43) and the result in (3.36), claim (3.35) follows. Formula (3.33) is the special case of (3.35) for $f$ being the constant 1. ■
3.5 Ward identities

To use symmetries of the supersymmetric sigma model, we consider cartesian coordinates
\[ x = (x_i)_{i \in \tilde{V}}, \quad y = (y_i)_{i \in \tilde{V}}, \quad z = (z_i)_{i \in \tilde{V}}, \quad \xi = (\xi_i)_{i \in \tilde{V}}, \quad \eta = (\eta_i)_{i \in \tilde{V}} \]
defined by
\[ x_i = \sinh u_i - \left( \frac{1}{2} s_i^2 + \frac{1}{2} \bar{\psi}_i \psi_i \right) e^{u_i}, \quad y_i = s_i e^{u_i}, \quad \xi_i = e^{u_i} \bar{\psi}_i, \quad \eta_i = e^{u_i} \psi_i, \]  
\[ z_i = \sqrt{1 + x_i^2 + y_i^2 + 2 \xi_i \eta_i} = \cosh u_i + \left( \frac{1}{2} s_i^2 + \frac{1}{2} \bar{\psi}_i \psi_i \right) e^{u_i}. \]

In particular, \( x_\delta = y_\delta = \xi_\delta = \eta_\delta = 0 \) and \( z_\delta = 1 \). Let
\[ S_{\text{cart}}(x, y, \xi, \eta) = - \sum_{(i \sim j) \in \tilde{E}} W_{ij} (x_i x_j - y_i y_j + z_i z_j - \xi_i \eta_j + \eta_i \xi_j) \]
and define
\[ \int d\mu_W^W f := \int \prod_{i \in \tilde{V}} \frac{dx_i dy_i}{2\pi} \partial_{\xi_i} \partial_{\eta_i} \left( \prod_{i \in \tilde{V}} \frac{1}{z_i} e^{S_{\text{cart}}(x, y, \xi, \eta)} f(x, y, \xi, \eta) \right) \]
for any compactly supported or sufficiently fast decaying test function \( f \).

Let \( V_{\text{cart}} \) denote the \( \mathbb{R} \)-vector space with basis \( (\xi_i, \eta_i)_{i \in \tilde{V}} \). Let \( S_{\text{susy}}(\Omega_V, \xi, \eta) \) denote the space of superfunctions of the form
\[ f_{\text{cart}} : \Omega_V \to \mathcal{A}(V_{\text{cart}}), \quad (x, y) \mapsto f_{\text{cart}}(x, y, \xi, \eta) = \sum_{I, J \in \tilde{I}_V} f_{IJ}(x, y) \xi_I \eta_J, \]
where the coefficients \( f_{IJ} \) are Schwartz functions and
\[ \xi_I = \prod_{i \in I} \xi_i, \quad \eta_J = \prod_{j \in J} \eta_j. \]

After doing the change of coordinates given in (3.45), we obtain the test function in horospherical coordinates \( f_{\text{hor}} : \Omega_V \to \mathcal{A}(V) \),
\[ (u, s) \mapsto f_{\text{hor}}(u, s, \bar{s}, \bar{\psi}, \psi) = f_{\text{cart}}(x(u, s, \bar{s}, \bar{\psi}, \psi), y(u, s, \bar{s}, \bar{\psi}, \psi), \xi(u, s, \bar{s}, \bar{\psi}, \psi), \eta(u, s, \bar{s}, \bar{\psi}, \psi)). \]

These notions can be directly extended to superfunctions involving parameters that depend on Grassmann variables by considering \( f_{\text{cart}}, S_{\text{cart}} : \Omega_V \to \mathcal{A}(V_{\text{cart}}) \otimes \Lambda V' \). Lemma 5.1 of [DMR17] implies that for any superfunction \( f_{\text{cart}}(x, y, \xi, \eta) \) with the property \( e^{S_{\text{cart}}} f_{\text{cart}} \in S_{\text{susy}}(\Omega_V, \xi, \eta) \otimes \Lambda V' \), one has
\[ \int d\mu_{\text{cart}}^W f_{\text{cart}} = \int d\mu_W f_{\text{hor}}. \]
Lemma 3.4 (Ward identities) Let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function and \( \tau = (\tau_i)_{i \in V} \in (V'_\text{odd})^V \). If \( f((\alpha, x + z + iy) + (\tau, \xi + i\eta)) e^{S_{\text{cart}}} \in S_{\text{susy}}(\Omega_V, \xi, \eta) \otimes \Lambda V' \), then the following identity holds

\[
\int \mu^W_{\text{cart}} f((\alpha, x + z + iy) + (\tau, \xi + i\eta)) = f((\alpha, 1)). \tag{3.53}
\]

Proof. Let \( \varphi \in \mathbb{R} \). We define \( \xi^\varphi = (\xi^\varphi_j)_{j \in \tilde{V}}, \eta^\varphi = (\eta^\varphi_j)_{j \in \tilde{V}} \) by

\[
\begin{pmatrix}
\xi^\varphi_j \\
\eta^\varphi_j
\end{pmatrix} =
\begin{pmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{pmatrix}
\begin{pmatrix}
\xi_j \\
\eta_j
\end{pmatrix}. \tag{3.54}
\]

Note that \( S_{\text{cart}}(x, y, \xi, \eta) = S_{\text{cart}}(x, y, \xi^\varphi, \eta^\varphi) \). Furthermore, the supertransformation \( (x, y, \xi, \eta) \mapsto (x, y, \xi^\varphi, \eta^\varphi) \) has super Jacobian 1 and hence leaves the reference supermeasure \( dx dy \partial_\xi \partial_\eta \) invariant. The assumption \( f((\alpha, x + z + iy) + (\tau, \xi + i\eta)) e^{S_{\text{cart}}} \in S_{\text{susy}}(\Omega_V, \xi, \eta) \otimes \Lambda V' \) assures that all expectations in the following calculations exist and are finite and justifies that we can exchange the order of integration in (3.56), below. It follows

\[
\text{lhs}(3.53) = \int \mu^W_{\text{cart}} f((\alpha, x + z + iy) + (\tau, \xi^\varphi + i\eta^\varphi))
\]

\[
= \int \mu^W_{\text{cart}} f((\alpha, x + z + iy) + e^{-i\varphi} (\tau, \xi + i\eta)). \tag{3.55}
\]

Consequently,

\[
\text{lhs}(3.53) = \frac{1}{2\pi} \int_0^{2\pi} \int \mu^W_{\text{cart}} f((\alpha, x + z + iy) + e^{-i\varphi} (\tau, \xi + i\eta)) d\varphi
\]

\[
= \int \mu^W_{\text{cart}} \frac{1}{2\pi} \int_0^{2\pi} f((\alpha, x + z + iy) + e^{-i\varphi} (\tau, \xi + i\eta)) d\varphi. \tag{3.56}
\]

Note that

\[
g(r) := \frac{1}{2\pi} \int_0^{2\pi} f((\alpha, x + z + iy) + e^{-i\varphi} r) d\varphi - f((\alpha, x + z + iy)) \tag{3.57}
\]

is an analytic superfunction of \( r \in \Lambda V'_\text{even} \), which vanishes for all \( r \in \mathbb{R} \) by the mean value theorem for holomorphic functions. Consequently, using that \( g(r) \) for \( r \in \Lambda V'_\text{even} \) is defined as a Taylor series in the nilpotent part of \( \mathbb{R} \), we obtain \( g(r) = 0 \) for all \( r \in \Lambda V'_\text{even} \). This yields

\[
\text{lhs}(3.53) = \int \mu^W_{\text{cart}} f((\alpha, x + z + iy)). \tag{3.58}
\]

The claim (3.53) follows from Lemma 5.2 of [DMR17].
Corollary 3.5 (Ward identity for exp) For all \( \alpha \in (-\infty, 0] \) and \( \tau = (\tau_i)_{i \in \mathcal{V}} \in L_{\text{odd}}^V \), one has

\[
\int d\mu_W e^{\langle \alpha, e^n(1+i\alpha) \rangle + \langle \tau, e^n(\bar{\psi} + i\psi) \rangle} = e^{\langle \alpha, 1 \rangle},
\]

using the abbreviation \( e^n(1+i) = (e^{u_j}(1 + is_j))_{j \in \mathcal{V}} \).

Proof. We apply Lemma 3.4 to the function \( f = \exp \). Note that since \( \text{body}(x_j + z_j) = \text{body}(e^{u_j}) > 0 \) and \( \alpha_j \leq 0 \) the assumption \( e^{(\alpha,x+z+iy)+\langle \tau, \xi +i\eta \rangle_{\text{cart}} \in \mathbb{R}_{\text{susy}}(\Omega_V, \xi, \eta)} \otimes \Lambda V' \) is satisfied. Using (3.35) and (3.36), we find

\[
\begin{aligned}
\text{body}(\tau) \in \mathbb{R}_{\text{susy}}(\Omega_V, \xi, \eta) \otimes \Lambda V' \quad \text{is satisfied. Using (3.35) and (3.36), we find}
\end{aligned}
\]

For all \( i,j \) and \( \beta, \theta \in V \), we take weights \( W_{ij} = W_{ji} \in (\Lambda V'_\infty)_\text{even} \) such that \( W_{ij} \in (\Lambda V'_\infty)_\text{even} \) whenever \( i \sim j \) is an edge in \( \tilde{G}_n \) for some \( n \) and \( W_{ij} = 0 \) whenever \( i \) and \( j \) are not connected by an edge in the infinite graph \( G_\infty \). The edges of \( G_n \) are given the weights \( W_{ij}^{(n)} \) defined as in (1.32) and (1.33). Let \( \mu_W \) denote the supersymmetric sigma model with Grassmann variables defined in (3.13) for the graph \( \tilde{G}_n \) with weights \( W_{ij}^{(n)} \).

Let \( n \in \mathbb{N} \). Recall the definition (3.31) of \( \bar{\omega}V_n \) and \( \bar{\pi}V_n^{V_n} \) for \( [a, b, \chi, \bar{\chi}] \in G(V'_n)V_n \). We consider the joint Grassmann-Laplace transform

\[
\mathcal{L}_n^W(a, b, \chi, \bar{\chi}) = \int d\mu_W e^{-\langle \bar{\pi}V_n, \bar{\omega}V_n \rangle}.
\]

Test functions. Following the discussion above eq. (3.52) we will consider the space \( T_n \) of test functions \( f \in A(V_n) \otimes \Lambda V'_n \) such that \( e^{\bar{\omega}V_n} f \in \mathbb{R}_{\text{susy}}(\Omega_V, \xi, \eta) \otimes \Lambda V'_n \).

Functions of \( \beta, \theta, \bar{\theta}, \phi \). Let \( U_n \) be a vector space with basis \((\phi_i, \bar{\phi}_i)_{i \in V_n} \). In analogy to the definition (3.8) of \( A(V) \), we denote by \( B_n(U_n) = C^\infty(\mathbb{R}V_n \times \mathbb{R}V_n, \mathcal{U}_n) \) the Grassmann algebra over \( U_n \) where the coefficients are given by smooth real-valued functions \( f_{IJ} \in C^\infty(\mathbb{R}V_n \times \mathbb{R}V_n, \mathbb{R}) \), \( (\beta, \theta) \mapsto f_{IJ}(\beta, \theta) \). If we insert the functions \( \beta = \beta V_n(u) \),
\[ \theta = \theta^V(u, s), \quad \phi = \phi^V(u, \psi), \quad \text{and} \quad \phi = \phi^V(u, \psi), \] cf. formulas [1.26], [1.24], and [3.29], in the representation

\[ f(\beta, \theta, \phi, \phi) = \sum_{I, J \in I(V_n)} f_{I J}(\beta, \theta, \phi, \phi) \in \mathcal{B}_n(U_n), \quad (3.61) \]

the superfunction in horospherical coordinates can be written as

\[ f_{\text{hor}}(u, s, \psi, \psi) = f(\varphi^V(u, s, \psi, \psi)) = \sum_{I, J \in I(V_n)} \tilde{f}_{I J}(u, s) \psi_I \psi_J. \quad (3.62) \]

Again, these definitions extend directly to functions involving Grassmann-dependent parameters \( \mathcal{B}_n(U_n) \otimes \Lambda V'_n \).

**Lemma 3.6 (Consistency)**

For \( n \in \mathbb{N} \) and \([a, b, \bar{\chi}, \chi] \in \mathcal{G}(V_{n+1})V_{n+1}\) with \([a_i, b_i, \bar{\chi}_i, \chi_i] = [1, 0, 0, 0]\) for all \( i \in V_{n+1} \setminus V_n\), one has

\[ \mathcal{L}^W_n(aV_n, bV_n, \bar{\chi}V_n, \chi V_n) = \mathcal{L}^W_{n+1}(a, b, \bar{\chi}, \chi). \quad (3.63) \]

Consequently, for any superfunction \( f \in \mathcal{B}_n(U_n) \otimes \Lambda \mathcal{V}'_n \) such that \( f_{\text{hor}} \in \mathcal{U}_n \) one has

\[ \int d\mu^W_n f(\varphi^V_n) = \int d\mu^W_{n+1} f((\varphi^V_{n+1})|_{V_n}). \quad (3.64) \]

Informally speaking, this means that the (super-)law of \( \varphi^V_n = (\beta^V_n, \theta^V_n, \phi^V_n, \phi^V_n) \) with respect to \( \mu^W_n \) agrees with the (super-)law of \( \varphi^V_{n+1}|_{V_n} = (\beta^V_{n+1}, \theta^V_{n+1}, \phi^V_{n+1}, \phi^V_{n+1})|_{V_n} \) with respect to \( \mu^W_{n+1} \).

**Proof.** Using the expression [3.34] for the Grassmann-Laplace transform, the proof of [3.63] is in complete analogy with the proof of Lemma 1.2 using Theorem 3.3 as the analogue of Theorem 1.1 and replacing expressions of the form \( a_i a_j + b_i b_j - 1 \) originating from formula [1.28] by expressions \( a_i a_j + b_i b_j + \bar{\chi}_i \chi_j + \bar{\chi}_j \chi_i - 1 \), appearing in formula [3.34].

To prove [3.64], we consider first the special case \( f(\varphi^V_n) = e^{-\langle \pi^V_{[a, b, \bar{\chi}, \chi]} \varphi^V_n \rangle} \). We claim \( f_{\text{hor}} \in \mathcal{U}_n \). Indeed note that replacing \( u \) in \( \beta(u) \) with \( u = u(x, y, \xi, \eta) \) we can write (cf. Lemma 2.3)

\[ S_{\text{cart}}(x, y, \xi, \eta) = -\frac{1}{2} \langle 1_{\varphi^V_n}, H_{\beta} 1_{\varphi^V_n} \rangle - \frac{1}{2} \langle y, H_{\beta} y \rangle - \langle \xi, H_{\beta} \eta \rangle, \quad (3.65) \]

\[ -\langle \pi^V_{[a, b, \bar{\chi}, \chi]} \varphi^V_n \rangle = -S_{\text{cart}} + C_W(a, b, \bar{\chi}, \chi) \]

\[ -\frac{1}{2} \langle a, H_{\beta} a \rangle - \frac{1}{2} \langle (y + b), H_{\beta} (y + b) \rangle - \langle (\xi + \bar{\chi}), H_{\beta} (\eta + \chi) \rangle. \quad (3.66) \]
where
\[ C_W(a, b; \chi, \chi) := \sum_{(i,j) \in \mathcal{E}_n} W_{ij} \left[ 1 - a_i a_j - b_i b_j - \chi_i \chi_j - \chi_j \chi_i \right] \] (3.67)
is a constant in \((\mathcal{V}_n')^\text{even}\). Letting \(c := \min\{\text{body}(a^2_j) : j \in \mathcal{V}_n\} > 0\) we have
\[ e^{\mathcal{S}_{\text{cart}} f_{\text{cart}}(x, y, \xi, \eta)} = e^{\mathcal{S}_{\text{cart}}(x,y+b,\xi+\chi,\eta+\chi)} e^{F(x,y,\xi,\eta)} e^{C_W(a,b;\chi,\chi)} \] (3.68)
where \(\text{body}(F(x, y, \xi, \eta)) \leq 0\), and all derivatives of \(F\) of any order in \(x, y, \xi, \eta\) are algebraic functions of these variables without singularities. Hence \(e^{\mathcal{S}_{\text{cart}} f_{\text{cart}}} \in \mathcal{S}_{\text{nasy}}(\Omega_{\mathcal{V}_n}, \xi, \eta) \otimes \Lambda \mathcal{V}_n'\).

For the special case \(f(\pi V_n) = e^{-\langle \pi V_n, V_n \rangle \alpha V_n} \) claim (3.64) reads
\[ \int d\mu_n^W e^{-\langle \pi V_n, V_n \rangle \alpha V_n} = \int d\mu_{n+1}^W e^{-\langle \pi V_n, V_{n+1} \rangle \alpha V_{n+1} V_n} \] (3.69)
This formula is just another way of writing equation (3.63). For the remainder of this proof, we consider \(c := a^2 + b^2 + 2\chi \chi - 1\), \(b, \chi, \chi, \chi\) as our list of independent variables, viewing \(a = \sqrt{c - b^2 - 2\chi \chi + 1}\) as a function of \((c, b, \chi, \chi)\). This makes sense as long as \(\text{body}(c - b^2) > -1\). We take all iterated Grassmann derivatives of the form \(\prod_{k=1}^m \partial_{\chi k} \prod_{k=1}^m \partial_{\chi k} \) with \(i_k, k_k \in \mathcal{V}_n\) in equation (3.69). Afterwards, we set \(\chi = 0\) and \(\chi = 0\). For \(I, J \in \mathcal{I}_{\mathcal{V}_n}\), we obtain
\[ \int d\mu_n^W \phi_I^V_n \phi_J^V_n e^{-\langle c \alpha V_n, V_n \rangle - \langle b \alpha V_n, \theta V_n \rangle} = \int d\mu_{n+1}^W \phi_I^{V_{n+1}} \phi_J^{V_{n+1}} e^{-\langle c \alpha V_n, \theta V_{n+1} \rangle V_n - \langle b \alpha V_{n+1}, \theta V_{n+1} \rangle} \] (3.70)
for any Grassmann monomial \(g\). Note that the identity (3.70) holds in particular for all real \(b, c\) in a neighborhood of the origin.

For a general function assume first the weights \(W_{ij}\) take only real values. Then, \(\beta\) and \(\theta\) take only real values because the integration variables \(u\) and \(s\) take real values. Hence, using the uniqueness theorem for Laplace transforms and the representation (3.61) of the superfunction \(f\), the claim (3.64) follows under our additional assumption \(W_{ij} \in \mathbb{R}\); note that the hypothesis \(f \in \mathcal{B}_{\mathcal{V}_n}(\mathcal{U}_n) \otimes \Lambda \mathcal{V}_n'\) with \(f_{\text{hor}} \in \mathcal{T}_n\) provides the necessary integrability. Because both sides of the claim (3.64) are analytic superfunctions in the weights \(W_{ij}\), the claim follows also in the general case. ■

We remark that in the above proof, it is essential to allow the scaling parameters \(a\) to take values in the even part of a Grassmann algebra rather than taking only real values, because we have written \(a = \sqrt{c - b^2 - 2\chi \chi + 1}\) with real \(c\) and \(b\) and Grassmann variables \(\chi\) and \(\chi\).

For \(\alpha \in (-\infty, 0)^{V_{\infty}}\) we use again the definition of \(\alpha^{(n)}\) given in formula (1.43). On the contrary, given \(\tau = (\tau_i)_{i \in V_{\infty}}\) such that \(\tau_i \in (\Lambda \mathcal{V}_n')^\text{odd}\) for all \(n \in \mathbb{N}\) and \(i \in \mathcal{V}_n\), we denote by \(\tau^{(n)}\) the restriction of \(\tau\) to \(V_n\). Note that \(\Lambda \mathcal{V}_n' \subseteq \Lambda \mathcal{V}_{n+1}'\).

The following theorem is an extension of the martingale property stated in Theorem 1.3.
Theorem 3.7 For $n \in \mathbb{N}$, $\alpha \in (-\infty, 0]$, and $\tau = \tau_i \in V_\infty$ as above, let

$$M_{\alpha, \tau}^{(n)} = M_{\alpha, \tau}^{(n)}(u, s, \overline{\psi}, \psi) = e^{\langle \alpha(n), e^{u(1+is)} \rangle} e^{\langle \tau, e^{u(\overline{\psi}+i\psi)} \rangle}.$$ (3.71)

For any test superfunction $g \in B_{\nu}(U_n) \otimes \mathbb{N}_n$ with $g_{\text{hor}} \in T_n$, one has

$$\int d\mu_{n+1}^W M_{\alpha, \tau}^{(n)} g(\overline{\omega} V_{n+1}|V_n) = \int d\mu_n^W M_{\alpha, \tau}^{(n)} g(\overline{\omega} V_n).$$ (3.72)

Note that in (3.71) we need a definition for $\alpha_{\delta_n}$ because $e^{\delta_n (1+i\delta_n)} = 1$. In contrast to this, $e^{\delta_n (\psi_{\delta_n} + i\psi_{\delta_n})} = 0$, hence no definition of $\tau_{\delta_n}$ is needed.

Proof of Theorem 3.7 The proof is in complete analogy to the proof of Theorem 3.3 with an extended set of variables.

We consider first the special case $g(\overline{\omega} V_n) = e^{-\langle \pi_{[a,b,\overline{\chi},\chi]} W V_n \rangle}$ with $[a, b, \overline{\chi}, \chi] \in \mathcal{G}(V_n)|V_n$. Note that with this choice $g_{\text{hor}} \in T_n$. Now, set $[u, b_i, \overline{\chi}_i, \chi_i] = [1, 0, 0, 0]$ for $i \in \mathbb{N}_n \setminus \mathbb{N}_n$.

The fact $\langle \pi_{[a,b,\overline{\chi},\chi]} W V_{n+1}|V_n \rangle = \langle \pi_{[a,b,\overline{\chi},\chi]} W V_{n+1} \rangle$ and equation (3.55) from Theorem 3.3 yield

$$\int d\mu_{n+1}^W M_{\alpha, \tau}^{(n)} e^{-\langle \pi_{[a,b,\overline{\chi},\chi]} W V_{n+1}|V_n \rangle} = \int d\mu_{n+1}^W e^{-\langle \pi_{[a,b,\overline{\chi},\chi]} W V_{n+1} \rangle}$$

$$= \mathcal{L}_{n+1}^W (a, b, \overline{\chi}, \chi) \int d\mu_{n+1}^W \mathcal{S}^*_{[a,b,\overline{\chi},\chi]} M_{\alpha, \tau}^{(n+1)}.$$ (3.73)

The following calculation is analogous to formula (2.55):

$$\mathcal{S}^*_{[a,b,\overline{\chi},\chi]} M_{\alpha, \tau}^{(n+1)} = \exp \left( \langle \alpha^{(n+1)}, e^{u+\log a} (1 + i(s - e^{-u-\log a})b) \rangle \right) \cdot \exp \left( \langle \tau^{(n+1)}, e^{u+\log a} (\overline{\psi} - e^{-u-\log a} \chi + i(\psi - e^{-u-\log a} \overline{\chi})) \rangle \right)$$

$$= e^{\langle \alpha^{(n+1)}, e^{u(1+is)} \rangle + \langle \tau^{(n+1)}, e^{u(\psi+i\psi)} \rangle} e^{-\langle \alpha^{(n+1)}, ib \rangle - \langle \tau^{(n+1)}, \overline{\chi} + ix \rangle}. \quad (3.74)$$

Inserting this in (3.73) and using the Ward identity from Corollary 3.5 we obtain the following analog of the calculation from formula (2.56) to (2.58):

$$\int d\mu_{n+1}^W e^{-\langle \pi_{[a,b,\overline{\chi},\chi]} W V_{n+1} \rangle}$$

$$= \mathcal{L}_{n+1}^W (a, b, \overline{\chi}, \chi) e^{-\langle \alpha^{(n+1)}, ib \rangle - \langle \tau^{(n+1)}, \overline{\chi} + ix \rangle} \int d\mu_{n+1}^W e^{\langle \alpha^{(n+1)}, e^{u(1+is)} \rangle + \langle \tau^{(n+1)}, e^{u(\psi+i\psi)} \rangle}$$

$$= \mathcal{L}_{n+1}^W (a, b, \overline{\chi}, \chi) e^{-\langle \alpha^{(n+1)}, ib \rangle - \langle \tau^{(n+1)}, \overline{\chi} + ix \rangle} e^{\langle \alpha^{(n+1)}, 1 \rangle}$$

$$= \mathcal{L}_{n+1}^W (a, b, \overline{\chi}, \chi) e^{\langle \alpha^{(n+1)}, a - ib \rangle - \langle \tau^{(n+1)}, \overline{\chi} + ix \rangle}. \quad (3.75)$$

In the same way, replacing $n + 1$ by $n$ yields

$$\int d\mu_{n}^W M_{\alpha, \tau}^{(n)} e^{-\langle \pi_{[a,b,\overline{\chi},\chi]} W V_n \rangle} = \mathcal{L}_{n}^W ((a, b, \overline{\chi}, \chi) e^{\langle \alpha^{(n)}, a - ib \rangle - \langle \tau^{(n)}, \overline{\chi} + ix \rangle}. \quad (3.76)$$
The consistency result from Lemma 3.6 can be written in the form $L_{n+1}^W(a, b, \bar{\chi}, \chi) = L_n^W((a, b, \bar{\chi}, \chi)|_{V_n})$. Identity (2.60) states $\langle \alpha^{(n+1)}, a - ib \rangle = \langle \alpha^{(n)}, a - ib \rangle$. Finally, using $\chi_j = 0$ for all $j \in \tilde{V}_{n+1} \setminus V_n$, we obtain

$$\langle \tau^{(n+1)}, \bar{\chi} + i\chi \rangle = \sum_{j \in V_{n+1}} \tau_j^{(n+1)}(\bar{\chi}_j + i\chi_j) = \sum_{j \in V_n} \tau_j^{(n)}(\bar{\chi}_j + i\chi_j) = \langle \tau^{(n)}, \bar{\chi} + i\chi \rangle. \quad (3.77)$$

It follows that

$$\int d\mu_{n+1}^W M_{\alpha, \tau}^{(n+1)} e^{-\langle \pi_{V_{n+1}}^V, \bar{\omega}_{n+1}^V \rangle} = \int d\mu_n^W M_{\alpha, \tau}^{(n)} e^{-\langle \pi_{V_n}^V, \bar{\omega}_V \rangle}. \quad (3.78)$$

Using the same argument as in the proof of Lemma 3.6, replacing the supermeasure $d\mu_k^W$, $k \in \{n, n+1\}$, by $d\mu_k^W M_{\alpha, \tau}^{(k)}$, the claim (3.72) follows for any superfunction $g \in B_{\mathcal{V}_n}(U_n) \otimes \Lambda \mathcal{V}_n'$ with $g_{\text{hor}} \in \mathcal{T}_n$.

**Corollary 3.8** For $n, k, m \in \mathbb{N}$ and $j_1, \ldots, j_k, l_1, \ldots, l_m \in V_{n+1}$, let

$$M_{j_1, \ldots, j_k, l_1, \ldots, l_m}^{(n)} = \prod_{p=1}^k e^{u_p^{(n)}} (1 + is_p^{(n)}) \prod_{q=1}^m e^{u_q^{(n)}} (\psi_{l_q} + i\psi_{l_q}^{(n)}). \quad (3.79)$$

For any superfunction $g \in B_{\mathcal{V}_n}(U_n) \otimes \Lambda \mathcal{V}_n'$ with $g_{\text{hor}} \in \mathcal{T}_n$, one has

$$\int d\mu_{n+1}^W M_{j_1, \ldots, j_k, l_1, \ldots, l_m}^{(n+1)} g(\bar{\omega}_{n+1}^V|_{V_n}) = \int d\mu_n^W M_{j_1, \ldots, j_k, l_1, \ldots, l_m}^{(n)} g(\bar{\omega}_V). \quad (3.80)$$

The same holds for the real and imaginary part of $M_{j_1, \ldots, j_k, l_1, \ldots, l_m}^{(n)}$.

**Proof.** In analogy to Corollary 1.3, the proof follows directly from the Taylor expansion of formula (3.72) with respect to $\alpha$ and $\tau$.

**A Coordinate transformations for superfunctions**

We abbreviate $x = (u, s, \bar{\psi}, \psi) = (u_i, s_i, \bar{\psi}_i, \psi_i)_{i \in V}$ and $dx = \prod_{i \in V} du_i ds_i \partial_{\bar{\psi}_i} \partial_{\psi_i}$.

**Lemma A.1** For $v \in \mathcal{G}(\mathcal{V})_V$ and any compactly supported (or sufficiently fast decaying) superfunction $f$, one has

$$\int dx (\mathcal{S}_v^* f)(x) = \int dx f(x). \quad (A.1)$$

**Proof.** Consider a supermatrix

$$M = \begin{pmatrix} A & \Sigma \\ \Gamma & B \end{pmatrix} \quad (A.2)$$
where \( A, B \) have even entries, \( \Sigma, \Gamma \) have odd entries, and \( A \) and \( B \) are invertible. Its superdeterminant is defined by

\[
\text{sdet } M = \frac{\det(A - \Sigma B^{-1}\Gamma)}{\det B}.
\]  

(A.3)

It plays an analogous role in Berezin’s supertransformation formula as the ordinary determinant plays in the classical transformation formula; cf. Theorem 2.1 in [Ber87].

For \( v = [a, b, \overline{\chi}, \chi] \), the change of coordinates generating \( \mathcal{S}_v^* \) is given by

\[
x'(x) = (u', s', \overline{\psi}', \psi') = (u + \log a, s - e^{-u}ba^{-1}, \overline{\psi} - e^{-u}a^{-1}, \psi - e^{-u}a^{-1}).
\]  

(A.4)

This transformation has the super Jacobi matrix given by

\[
\frac{\partial x'}{\partial x} = \left( \begin{array}{cccc}
\frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial s} & \frac{\partial u'}{\partial \psi} & \frac{\partial u'}{\partial \overline{\psi}} \\
\frac{\partial s'}{\partial u} & \frac{\partial s'}{\partial s} & \frac{\partial s'}{\partial \psi} & \frac{\partial s'}{\partial \overline{\psi}} \\
\frac{\partial \psi'}{\partial u} & \frac{\partial \psi'}{\partial s} & \frac{\partial \psi'}{\partial \psi} & \frac{\partial \psi'}{\partial \overline{\psi}} \\
\frac{\partial \overline{\psi}'}{\partial u} & \frac{\partial \overline{\psi}'}{\partial s} & \frac{\partial \overline{\psi}'}{\partial \psi} & \frac{\partial \overline{\psi}'}{\partial \overline{\psi}}
\end{array} \right) = \left( \begin{array}{cc}
A & 0 \\
\Gamma & 1
\end{array} \right)
\]  

(A.5)

with

\[
A = \left( \begin{array}{cc}
1 & 0 \\
e^{-u}ba^{-1} & 1
\end{array} \right), \quad 1 = \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right), \quad \Gamma = \left( \begin{array}{cc}
e^{-u}a^{-1} & 0 \\
e^{-u}a^{-1} & 0
\end{array} \right).
\]  

(A.6)

Here \( e^{-u}ba^{-1} \) is the diagonal matrix with the entries \( e^{-u_i}ba^{-1} \). This super Jacobi matrix has the superdeterminant \( \text{sdet} \frac{\partial x'}{\partial x} = 1 \). Consequently, the inverse supertransformation has the superdeterminant \( \text{sdet} \frac{\partial x}{\partial x'} = 1 \), as well. We obtain

\[
\int dx \mathcal{S}_v^*(f)(x) = \int dx f(x'(x)) = \int dx' f(x') \text{sdet} \frac{\partial x}{\partial x'} = \int dx' f(x').
\]  

(A.7)

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