On the dualization in distributive lattices and related problems

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Abstract

In this paper, we study the dualization in distributive lattices, a generalization of the well known hypergraph dualization problem. We give a characterization of the complexity of the problem under various combined restrictions on graph classes and posets, including bipartite, split and co-bipartite graphs, and variants of neighborhood inclusion posets. In particular, we show that while the enumeration of minimal dominating sets is possible with linear delay in split graphs, the problem gets as hard as for general graphs in distributive lattices. More surprisingly, this result holds even when the poset coding the lattice is only comparing vertices of included neighborhoods in the graph. If both the poset and the graph are sufficiently restricted, we show that the dualization becomes tractable relying on existing algorithms from the literature.

Keywords: distributive lattice dualization, ideal enumeration, neighborhood inclusion posets, dominating sets, hypergraph transversals.

1 Introduction

The dualization in Boolean lattices is a central problem in algorithmic enumeration as it is equivalent to the enumeration of minimal transversals of a hypergraph, minimal dominating sets of a graph, and many other problems [EMG08]. It is also a problem of practical interest in database theory, logic, artificial intelligence and pattern mining [KPS93, EG95, GMKT97, EGM03, Elb02, NP12]. In the following, we say that an enumeration algorithm is running in output-polynomial time if its running time is bounded by a polynomial depending on the sizes of both the input and the output. If the running time between two outputs is bounded by a polynomial depending on the size of the input alone, then the algorithm is said to be running with polynomial delay; see [JYP88, CKP+17]. To date, it is still open whether the dualization in Boolean lattices is possible in output-polynomial time. The best known algorithm is due to Fredman and Khachiyan and runs in output quasi-polynomial time [FK96]. When generalized to arbitrary lattices, it was recently proved in [BK17] that the dualization is impossible in output-polynomial time unless P=NP. In [DN19], it was shown that this result holds even when the premises in the implicational base (coding the lattice) are of size at most two. In the case of premises of size one – when the lattice is distributive – the problem is still open. The best known algorithm is due to Babin and Kuznetsov and runs in output sub-exponential time [BK17].

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are called respectively denote by of . Let be a graph. We say that is a minimal dominating set if and only if is a minimal set of private neighbors of . It is easy to see that for every , two vertices are adjacent. An is a set of vertices such that every two vertices of are adjacent. An independent set in a graph is a set of vertices such that no two vertices of are adjacent. The subgraph of induced by , denoted by , is the graph . A clique in a graph is a set of vertices such that every two vertices of are adjacent. In this paper, we give generalizations of the two problems of enumerating minimal transversals and minimal dominating sets to the dualization in distributive lattices. In Section 5, we exhibit tractable cases of the problem. We discuss future work in Section 6.

2 Preliminaries

We refer to Die05 for graph terminology not defined below; all graphs considered in this paper are undirected, finite and simple. A graph is a pair where is the set of vertices and is the set of edges. Edges are denoted by or instead of . A clique in a graph is a set of vertices such that every two vertices of are adjacent. An independent set in a graph is a set of vertices such that no two vertices of are adjacent. The subgraph induced by , denoted by , is the graph . We note the set of neighbors of defined by . We note the set of closed neighbors of defined by . If it is not clear from the context, we add the subscript to denote the neighborhood in , as in . Two vertices are called twin if and , and false twin if . For a given , we respectively denote by and the two sets defined by and for any .

Let be a graph. We say that is bipartite (resp. co-bipartite) if can be partitioned into two independent sets (resp. two cliques). If can be partitioned into one independent set and one clique, then is called split.

A dominating set in a graph is a set of vertices such that every vertex of is either in or is connected to some vertex of . It is said to be minimal if it does not contain any dominating set as a proper subset. Let be a dominating set of and be a vertex of . We say that has a private neighbor in if if and only if . The set of private neighbors of in is denoted by . It is easy to see that is a minimal dominating set if and only if dominates and for every . In the following, we denote by the set of minimal dominating sets of and by the problem of enumerating . Sometimes during the paper, we only need to dominate subsets of vertices of . Accordingly, we say that is a minimal dominating set of subset of if and only if for any with .

Output quasi-polynomial time algorithms are known for several subclasses, including distributive lattices coded by products of chains, or those coded by the ideals of an interval order. In this paper, we give generalizations of the two problems of enumerating minimal transversals of a hypergraph, and minimal dominating sets of a graph, toward distributive lattices. In this framework, a partial order on vertices is given in addition to the input (hyper)graph. Then, the task is of enumerating minimal ideals of the poset with the desired property, i.e., transversality and domination. We show that the obtained problems are equivalent to the dualization in distributive lattices, even when considering various combined restrictions on graph classes and posets, including bipartite, split and co-bipartite graphs, and variant of neighborhood inclusion posets; see theorems 3.3 and 4.1. For the restricted cases left by our theorem, we show that the problem is tractable relying on existing algorithms from the literature; see theorems 5.6 and 5.13. A summary of the obtained complexities is given in Figure 8.

The rest of the paper is organized as follows. In Section 2 we introduce necessary concepts and definitions. In Sections 3 and 4 we generalize the two problems of enumerating minimal transversals and minimal dominating sets to the dualization in distributive lattices. In Section 5 we exhibit tractable cases of the problem. We discuss future work in Section 6.
Figure 1: The bipartite incidence graph $I(H)$ of bipartition $X = V(H)$ and $Y = \{y_e \mid e \in H\}$ for the hypergraph $H = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{x_1, x_2, x_5\}$, $e_2 = \{x_1, x_2, x_3\}$, $e_3 = \{x_3, x_4, x_5\}$ and $e_4 = \{x_5, x_6\}$. Then $xy_e \in E(I(H))$ if and only if $x \in e$.

The set of minimal dominating sets of subset $W$ in $G$ is denoted by $D_G(W)$.

We refer to [Ber84] for hypergraph terminology not defined below. A hypergraph $H$ is a pair $(V(H), E(H))$ where $V(H)$ is the set of vertices (or groundset) and $E(H)$ is the set of non-empty subsets of $V(H)$ called edges or hyperedges. In this paper, we will sometimes describe a hypergraph by its set of edges only, and will note $e \in H$ in place of $e \in E(H)$. If $x$ is a vertex of $H$, we denote by $E(x)$ the set of edges incident to $x$ defined by $E(x) = \{e \in E(H) \mid x \in e\}$. A transversal in a hypergraph $H$ is a set of vertices $T$ that intersects every edge of $H$. It is called minimal if it does not contain any other transversal as a proper subset. The set of all minimal transversals of $H$ is denoted by $Tr(H)$ and the problem of enumerating $Tr(H)$ given $H$ is denoted by Trans-Enum. A hypergraph $H$ is called simple (or Sperner) if $e \not\subseteq e'$ for any two distinct $e, e' \in H$. It is well known that hypergraphs can be considered simple for the enumeration of minimal transversals. In the following, we denote by $\mathcal{N}(G)$ the simple hypergraph of closed neighborhoods of $G$ defined by $\mathcal{N}(G) = \text{Min}_{\subseteq} \{N[x] \mid x \in V(G)\}$. Then, it is not hard to see that Dom-Enum is a particular case of Trans-Enum, where the minimal dominating sets of $G$ are exactly the minimal transversals of $\mathcal{N}(G)$. Recently in [KLMN14], it was shown that the two problems are equivalent, even when restricted to co-bipartite graphs. For any hypergraph $H$ we denote by $I(H)$ the bipartite incidence graph of $H$ with bipartition $X = V(H)$ and $Y = \{y_e \mid e \in E(H)\}$, and where there is an edge between $x \in X$ and $y_e \in Y$ if $x$ belongs to $e$ in $H$. The construction of a bipartite incidence graph is given in Figure 1 and will be used in Section 3 of this paper.

A partial order on a set $X$ (or poset) is a binary relation $\leq$ on $X$ which is reflexive, anti-symmetric and transitive, denoted by $P = (X, \leq)$. Two elements $x, y$ of $X$ are said to be comparable if $x \leq y$ or $y \leq x$, otherwise they are said to be incomparable. If there is no element $z$ such that $x < z < y$ then we say that $y$ covers $x$. The Hasse diagram of a poset is the graph on vertex set the elements of the poset, and where there is an edge that goes upward from $x$ to $y$ whenever $y$ covers $x$. A subset of a poset in which every pair of elements is comparable is called a chain. A subset of a poset in which no two distinct elements are comparable is called an antichain. A poset is an antichain poset if the set of its elements is an antichain. A set $I \subseteq X$ is called ideal of $P$ if $x \in I$ and $y \leq x$ imply $y \in I$. If $x \in I$ and $x \leq y$ imply $y \in I$, then $I$ is called filter of $P$. Note that the complementary of an ideal is a filter, and vice versa. For every $x \in P$ we associate the principal ideal of $x$ (or simply ideal of $x$) denoted by $\downarrow x$ and defined by $\downarrow x = \{y \in X \mid y \leq x\}$. The principal filter of $x \in X$ is the dual $\uparrow x = \{y \in X \mid x \leq y\}$. The set of all subsets of $X$ is denoted by $2^X$. The set of all ideals of $P$ is denoted by $\mathcal{I}(P)$. Clearly, $\mathcal{I}(P) \subseteq 2^X$ and $\mathcal{I}(P) = 2^X$ whenever $P$ is an antichain poset. If $S$ is a subset of $X$, we respectively denote by $\downarrow S$ and $\uparrow S$ the two sets defined by $\downarrow S = \bigcup_{x \in S} \downarrow x$ and $\uparrow S = \bigcup_{x \in S} \uparrow x$. We respectively denote by Min$(S)$ and Max$(S)$ the sets of minimal and maximal elements of $S$ with respect to $\leq$.

The next definition is central in this paper.
A poset \(P = (X, \leq)\) that codes the distributive lattice \(\mathcal{L}(P) = (\mathcal{I}(P), \subseteq)\), and the border (curved line) formed by the two dual antichains \(\mathcal{B}^+ = \{\{x_1, x_2\}, \{x_2, x_4\}\}\) and \(\mathcal{B}^- = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}\}\) of \(\mathcal{L}(P)\). For better readability, ideals are denoted by the indexes of their elements in the lattice, i.e., 123 stands for \(\{x_1, x_2, x_3\}\).

**Definition 2.1.** Let \(P = (X, \leq)\) be a partial order and \(\mathcal{B}^+, \mathcal{B}^-\) be two antichains of \(P\). We say that \(\mathcal{B}^+\) and \(\mathcal{B}^-\) are dual in \(P\) whenever \(\downarrow \mathcal{B}^+ \cup \uparrow \mathcal{B}^- = X\) and \(\downarrow \mathcal{B}^+ \cap \uparrow \mathcal{B}^- = \emptyset\).

Note that deciding whether two antichains \(\mathcal{B}^+\) and \(\mathcal{B}^-\) of \(P\) are dual can be done in polynomial time in the size of \(P\) by checking whether \(\mathcal{B}^- = \text{Min}(P \setminus \downarrow \mathcal{B}^+).\) The task becomes difficult when the poset is not fully given, but only an implicit coding (of possibly logarithmic size in the size of \(P\)) is given. This is usually the case when considering dualization problems in lattices.

A lattice is a poset in which every two elements have a supremum (also called join) and an infimum (also called meet); see [Bir04, Gra11]. In this paper however, only the next two characterizations from [Bir04, Gra11] will suffice. We denote by Boolean lattice any poset isomorphic to \((2^X, \subseteq)\) for some set \(X\); such a lattice is also called hypercube. We denote by distributive lattice any poset isomorphic to \((\mathcal{I}(P), \subseteq)\) for some partially ordered set \(P = (X, \leq)\). Then, \(X\) and \(P\) are called implicit coding of the lattice and we denote by \(\mathcal{L}(X)\) and \(\mathcal{L}(P)\) the two lattices coded by \(X\) and \(P\). Clearly, every Boolean lattice is a distributive lattice where \(P\) is an antichain poset, as \(\mathcal{I}(P) = 2^X\) for such \(P\). In fact, it can be seen that each comparability \(x \leq y\) in \(P\) removes from \((2^X, \subseteq)\) the Boolean lattice given by the interval \([y, X \setminus \{x\}]\). At last, observe that \(\mathcal{L}(P)\) may be of exponential size in the size of \(P\); it is in particular the case when the lattice is Boolean. An example of a distributive lattice coded by the ideals of a poset is given in Figure 2.

In this paper, we are concerned with the following problem.

**Dualization in distributive lattices given by the ideals of a poset (DDUALIZATION)**

**Input:** A poset \(P = (X, \leq)\) and an antichain \(\mathcal{B}^+\) of \(\mathcal{L}(P)\).

**Output:** The dual antichain \(\mathcal{B}^-\) of \(\mathcal{B}^+\) in \(\mathcal{L}(P)\).

Note that this problem can be reformulated without any mention of the lattice, namely as the enumeration of all inclusion-wise minimal ideals of a poset \(P\) that are not a subset of any element of a family of ideals \(\mathcal{B}^+\) of \(P\), i.e., the set

\[\mathcal{B}^- = \text{Min}_{\subseteq}\{I \in \mathcal{I}(P) \mid I \nsubseteq B \text{ for any } B \in \mathcal{B}^+\}\].

Then computing a first solution to this problem is easy, as we start from \(I = X\) as an ideal, and remove its maximal elements until it is a minimal ideal such that \(I \nsubseteq B\) for any \(B \in \mathcal{B}^+\). However, it is still open whether this problem can be solved in output-quasi polynomial time. To date, the best known algorithm runs in output sub-exponential time.
$2^{O(n^{0.67} \log^3 N)}$ where $N = |\mathcal{B}^+| + |\mathcal{B}^-|$ and $P$ is given as a $n \times n$ matrix \cite{BK17}. Output quasi-polynomial time algorithms running in time $\text{poly}(N, n) + N^{O(\log N)}$ are known for several subclasses, including distributive lattices coded by products of chains \cite{EMG09}, or distributive lattices coded by the ideals of an interval order \cite{DN19}.

If the poset is an antichain, \textit{i.e.}, if the lattice is Boolean, then this problem calls for enumerating every inclusion-wise minimal subset of $X$ that is not a subset of any $B \in \mathcal{B}^+$, or equivalently, that intersects every set in $\mathcal{H} = \{X \setminus B \mid B \in \mathcal{B}^+\}$. Under this formulation, it is easy to see that the restricted problem is equivalent to \textsc{Trans-Enum} (hence to \textsc{Dom-Enum}), where $Tr(\mathcal{H}) = \mathcal{B}^-$; see \cite{EMG08, NP14}. In this case, the best known algorithm runs in output quasi-polynomial time $N^{o(\log N)}$ where $N = |\mathcal{B}^+| + |\mathcal{B}^-|$, and the existence of an output-polynomial time algorithm remains open after decades of research \cite{EG95, FK95, EMG08}. However, the problem has been precised under various restrictions on graph classes and parameters. Among these results, output-polynomial algorithms were given for line \cite{KLMN12, GHKV15}, split \cite{KLMN14}, chordal \cite{KLM+15}, triangle-free graphs \cite{BDHR18}, graphs of bounded clique-width \cite{Cona09}, LMIM-width \cite{GHK+18}, etc. Other classes of graphs remain open, including co-bipartite (as it is equivalent to \textsc{Trans-Enum}, hence to general graphs \cite{KLMN14}), unit disk graphs \cite{KN08, GHK+16}, comparability graphs, etc.

The aim of this paper is to generalize \textsc{Trans-Enum} and \textsc{Dom-Enum} to the dualization in distributive lattices, in order to obtain finer characterizations on the difficulty of the problem.

### 3 Transversal ideals

We give a generalization of \textsc{Trans-Enum} to the enumeration of minimal ideals of a poset with the transversal property. We show that the obtained problem is equivalent to the dualization in distributive lattices.

Let $\mathcal{H}$ be a hypergraph and $P_\mathcal{H}$ be a partial order on vertices of $\mathcal{H}$. Let $I$ be a subset of vertices of $\mathcal{H}$. We say $I$ is a transversal-ideal of $\mathcal{H}$ w.r.t. $P_\mathcal{H}$ if it is an ideal of $P_\mathcal{H}$, and a transversal of $\mathcal{H}$. It is called minimal if it does not contain any transversal-ideal as a proper subset. We denote by $ITr(\mathcal{H}, P_\mathcal{H})$ the set of minimal transversal-ideals of $\mathcal{H}$ w.r.t. $P_\mathcal{H}$, and define the problem of generating $ITr(\mathcal{H}, P_\mathcal{H})$ as follows.

**Minimal transversal-ideals enumeration (\textsc{ITrans-Enum})**

**Input:** A hypergraph $\mathcal{H}$ and a partial order $P_\mathcal{H}$ on vertices of $\mathcal{H}$.

**Output:** The set $ITr(\mathcal{H}, P_\mathcal{H}) = \min \subseteq \{I \in \mathcal{T}(P_\mathcal{H}) \mid I \text{ is a transversal of } \mathcal{H}\}$.

An instance of this problem is given in Figure 3. Observe that as for \textsc{DDualization}, computing a first solution to \textsc{ITrans-Enum} is easy as we start with $I = V(\mathcal{H})$ as a transversal-ideal, and greedily reduce it until it is minimal. It is worth pointing out that in the case where $P_\mathcal{H}$ is an antichain poset, then minimal transversal-ideals of $\mathcal{H}$ w.r.t. $P_\mathcal{H}$ are exactly minimal transversals of $\mathcal{H}$, and that the two problems \textsc{ITrans-Enum} and \textsc{Trans-Enum} are equivalent. In the general case, however, to a single minimal transversal-ideal of $\mathcal{H}$ w.r.t. $P_\mathcal{H}$ can correspond several minimal transversals of $\mathcal{H}$; see Figure 3 for an example. If $P_\mathcal{H}$ is a total (linear) order, then $\mathcal{H}$ admits a unique minimal transversal-ideal no matter the number of minimal transversals. Consequently, the size of $Tr(\mathcal{H})$ may be exponential in the size of $Tr(\mathcal{H}, P_\mathcal{H})$.

By the problem definitions, \textsc{DDualization} clearly appears as a particular case of \textsc{ITrans-Enum} where hyperedges of $\mathcal{H}$ are filters of $P_\mathcal{H}$, and the complementary of $\mathcal{H}$ defines an antichain of $L(P_\mathcal{H})$, as $I$ is a transversal of $\mathcal{H}$ if and only if $I \not\subseteq B$ for any
Figure 3: A hypergraph $\mathcal{H} = \{e_1, e_2, e_3, e_4\}$ and a partial order $P_\mathcal{H}$ on vertices of $\mathcal{H}$, where $e_1 = \{x_1, x_2, x_5\}$, $e_2 = \{x_1, x_2, x_3\}$, $e_3 = \{x_3, x_4, x_5\}$ and $e_4 = \{x_5, x_6\}$. The minimal transversal-ideals for this instance are $I_1 = \{x_2, x_3, x_5\}$ and $I_2 = \{x_2, x_3, x_6\}$. Note that $I_1$ is the ideal of two minimal transversals $T_1 = \{x_2, x_5\}$ and $T_2 = \{x_3, x_5\}$.

$B = X \setminus e$, $e \in \mathcal{H}$. However, not every hypergraph satisfies this property. Nevertheless, we show that the two problems are equivalent by showing that hypergraphs that do not share this property can be closed in the poset with no impact on the solutions to enumerate.

For any hypergraph $\mathcal{H}$ and poset $P_\mathcal{H}$, we denote by $\uparrow \mathcal{H}$ the filter-closed hypergraph of $\mathcal{H}$ with respect to $P_\mathcal{H}$ defined by $V(\uparrow \mathcal{H}) = V(\mathcal{H})$ and $E(\uparrow \mathcal{H}) = \text{Min}_\subseteq \{\uparrow e \mid e \in E(\mathcal{H})\}$. Observe that $|\uparrow \mathcal{H}| \leq |\mathcal{H}|$.

**Lemma 3.1.** Let $I$ be an ideal of $P_\mathcal{H}$. Then $I$ is a transversal-ideal of $\mathcal{H}$ if and only if it is a transversal-ideal of $\uparrow \mathcal{H}$. In particular, $\text{ITr}(\mathcal{H}, P_\mathcal{H}) = \text{ITr}(\uparrow \mathcal{H}, P_\mathcal{H})$.

**Proof.** Let $I$ be an ideal of $P_\mathcal{H}$ and $e$ be an edge of $\mathcal{H}$. We show that $I$ intersects $e$ if and only if it intersects $\uparrow e$. Clearly, if $I$ intersects $e$ then it intersects $\uparrow e$ as $e \subseteq \uparrow e$. Let us assume that $I$ intersects $\uparrow e$ and let $x \in I \cap \uparrow e$. Then there exists $y \in e$ such that $y \leq x$. Since $I$ is an ideal, $y \in I$. Thus $I \cap e \neq \emptyset$. Hence $\text{ITr}(\mathcal{H}, P_\mathcal{H}) = \text{ITr}(\uparrow \mathcal{H}, P_\mathcal{H})$.

**Lemma 3.2.** If $\uparrow \mathcal{H} = \mathcal{H}$ then $x \leq y$ implies $E_x \subseteq E_y$ for all $x, y \in V(\mathcal{H})$.

**Proof.** Let $\mathcal{H}$ such that $\uparrow \mathcal{H} = \mathcal{H}$, $x, y \in V(\mathcal{H})$ such that $x \leq y$, and $E \in E_x$. Since $E = \uparrow E$ and $x \leq y$, $y \in E$. Hence the desired result.

In what follows, we say that $P_\mathcal{H}$ is a poset of incident edge inclusion of $\mathcal{H}$ if $x \leq y$ implies $E_x \subseteq E_y$. By Lemma 3.2, every partial order $P_\mathcal{H}$ is a poset of incident edge inclusion of $\uparrow \mathcal{H}$. We conclude with the following result.

**Theorem 3.3.** $\text{DDUALIZATION}$ and $\text{ITRANS-ENUM}$ are equivalent, even when restricted to posets of incident edge inclusion.

**Proof.** It follows from the the equivalence $I \not\subseteq B$ for any $B \in B^+$ if and only if $I \cap (X \setminus B) \neq \emptyset$ for all $B \in B^+$, that $\text{DDUALIZATION}$ is a particular case of $\text{ITRANS-ENUM}$, where $\mathcal{H} = \{X \setminus B \mid B \in B^+\}$ and $\text{ITr}(\mathcal{H}, P_\mathcal{H}) = B^-$. We show that $\text{ITRANS-ENUM}$ reduces to $\text{DDUALIZATION}$. Let $(\mathcal{H}, P_\mathcal{H})$ be an instance of the first problem and $\mathcal{G} = \uparrow \mathcal{H}$ be the filter-closed hypergraph of $\mathcal{H}$ with respect to $P_\mathcal{H}$. Clearly, $\mathcal{G}$ can be computed in polynomial time in the sizes of $\mathcal{H}$ and $P_\mathcal{H}$, and $B^+ = \{X \setminus e \mid e \in E(\mathcal{G})\}$ defines an antichain of $\mathcal{L}(P_\mathcal{H})$. By Lemma 3.1, $\text{ITr}(\mathcal{G}, P_\mathcal{H}) = \text{ITr}(\mathcal{G}, P_\mathcal{H})$. As $\text{ITr}(\mathcal{G}, P_\mathcal{H}) = \{I \in I(P_\mathcal{H}) \mid I \not\subseteq B \text{ for any } B \in B^+\}$, we deduce that $B^- = \text{ITr}(\mathcal{G}, P_\mathcal{H})$ where $B^-$ is the dual antichain of $B^+$ in $\mathcal{L}(P_\mathcal{H})$. Hence that $\text{ITRANS-ENUM}$ can be solved using an algorithm for $\text{DDUALIZATION}$ on $P_\mathcal{H}$ and $B^+$.
Figure 4: A graph $G$ and a partial order $P_G$ on vertices of $G$, where $\mathcal{N}(G) = \mathcal{H}$ and $\mathcal{H}$ is the hypergraph from Figure 3. The minimal dominating-ideals for this instance are $I_1 = \{x_2, x_3, x_5\}$ and $I_2 = \{x_2, x_3, x_6\}$.

4 Dominating ideals

We define a similar generalization for Dom-Enum toward distributive lattices.

Let $G$ be a graph and $P_G$ be a partial order on vertices of $G$. Let $D$ be a subset of vertices of $G$. We say that $D$ is a dominating-ideal of $G$ w.r.t. $P_G$ if it is an ideal of $P_G$ and a dominating set of $G$. It is called minimal if it does not contain any dominating-ideal as a proper subset. Note that a dominating-ideal $I$ is minimal if and only if $\text{Priv}(I, x) \neq \emptyset$ for all $x \in \text{Max}(I)$. We denote by $\mathcal{ID}(G, P_G)$ the set of minimal dominating-ideals of $G$ w.r.t. $P_G$, and define the problem of generating $\mathcal{ID}(G, P_G)$ as follows.

Minimal dominating-ideals enumeration (IDOM-Enum)

**Input:** A graph $G$ and a partial order $P_G$ on vertices of $G$.

**Output:** The set $\mathcal{ID}(G, P_G) = \text{Min}_{\subseteq} \{I \in \mathcal{I}(P_G) \mid I \text{ dominates } G\}$.

An instance of this problem is given in Figure 4. Observe that as for the classical case, when $P$ is an antichain, IDOM-Enum is a particular case of ITRANS-Enum where $\mathcal{ID}(G, P_G) = TTr(\mathcal{N}(G), P_G)$. The rest of this section is devoted to the proof of their equivalence.

In the following, we say that $P_G$ is a (closed) neighborhood inclusion poset of $G$ if $x \leq y$ implies $N[x] \subseteq N[y]$, and that $P_G$ is a weak (closed) neighborhood inclusion poset of $G$ if $x \leq y$ implies either $N[x] \subseteq N[y]$, or $N[x] \supseteq N[y]$. Clearly, every neighborhood inclusion poset is a weak neighborhood inclusion poset. It can be seen that the first restriction is closely related to the one of Lemma 3.2, as to every neighborhood inclusion poset of a graph corresponds an incident edge inclusion poset in $\mathcal{N}(G)$. Hence, neighborhood inclusion posets naturally appear when considering dualization problems in distributive lattices.

**Theorem 4.1.** ITRANS-Enum and IDOM-Enum are equivalent, even when restricted to:
1. bipartite graphs;
2. split graphs and weak neighborhood inclusion posets; and
3. co-bipartite graphs and neighborhood inclusion posets.

**Proof.** Clearly, IDOM-Enum is a particular case of ITRANS-Enum where $\mathcal{ID}(G, P_G) = TTr(\mathcal{N}(G), P_G)$. We show that ITRANS-Enum reduces to IDOM-Enum. Let $(H, P_H)$ be a non-trivial (such that $H \neq \emptyset$) instance of ITRANS-Enum. Note that by Lemma 3.1, we can restrict ourself to the case where $H = I^H$. Hence by Lemma 3.2, $x \leq y$ in $P_H$ implies $E_x \subseteq E_y$. Let $I(H)$ be the bipartite incidence graph of $H$ of bipartition $X = V(H)$ and $Y = \{y_e \mid e \in E(H)\}$, and where $xy_e \in E(I(H))$ if and only if $x \in X$, $y_e \in Y$ and $x \in e$; see Section 2 and Figure 1. A first observation is the following:
Observation 4.2. Let \( x, y \in P_H \). Then \( x \leq y \) implies \( N(x) \subseteq N(y) \) in \( I(H) \).

The remainder of the proof is separated into three parts: we will adapt the construction of the bipartite incidence graph according to each item of the theorem.

Let us first consider Item 1. Let \( G \) be the graph obtained from \( I(H) \) by adding a single vertex \( v \) connected to every vertex of \( X \). Then \( G \) is bipartite with bipartition \( X \) and \( Y \cup \{v\} \). Let \( P_G \) be the poset obtained from \( P_H \) by making every \( y \in Y \) greater than every \( x \in X \), and \( v \) incomparable with every other vertex, i.e., \( P_G = P_H \cup \{x < y \mid x \in X, y \in Y\} \). We prove the following.

Claim 4.3. Let \( I \subsetneq V(H) \). Then \( I \in ITG(H, P_H) \) if and only if \( I \cup \{v\} \in ITD(G, P_G) \).

Proof of the claim. Let \( I \subsetneq V(H) \) such that \( I \in ITG(H, P_H) \). As \( H \) is non-empty, \( I \neq \emptyset \). By construction, \( I \) is an ideal of \( P_G \) and it is a minimal dominating-ideal of \( X \), i.e., \( Y \subseteq N[I] \) and \( Y \not\subseteq N[I \setminus \{x\}] \) for any \( x \in \max(I) \). By hypothesis \( I \neq V(H) \), hence \( I \) does not dominate \( G \), and \( I \cup \{v\} \) does; \( v \) is here to dominate elements of \( X \) that are not in the transversal. Since \( v \) is not adjacent to \( Y \), it does not still private neighbors to vertices in \( I \). Hence \( I \cup \{v\} \) is a minimal dominating-ideal of \( G \). Let \( I \subsetneq V(H) \) such that \( I \in ITD(G, P_G) \). Note that \( y \notin I \) for any \( y \in Y \) as \( X \subseteq y \) and \( X \) dominates \( G \). As \( H \) is non-empty, \( I \cap X \neq \emptyset \). By hypothesis, \( I \neq X \). Thus \( v \) has a private neighbor in \( X \) and \( \text{Priv}(I, x) \subseteq Y \) for all \( x \in I \). Hence \( I \) is a minimal transversal-ideal of \( H \).

Let us now consider Item 2. Let \( G \) be the graph obtained from \( I(H) \) by completing \( X \) into a clique, and by adding a single vertex \( v \) connected to every vertex of the graph, i.e., \( v \) is universal. Then \( G \) is split with clique \( X \cup \{v\} \) and independent set \( Y \). Let \( P_G \) be the poset obtained from \( P_H \) by making every \( y \in Y \) greater than \( v \), i.e., \( P_G = P_H \cup \{v < y \mid y \in Y\} \). We prove the following two claims.

Claim 4.4. \( P_G \) is a weak neighborhood inclusion poset on \( G \).

Proof of the claim. Clearly, \( x \leq y \) either implies \( x, y \in X \), or both \( x = v \) and \( y \in Y \).

In the first case, it follows from Observation 4.2 that \( N[x] \subseteq N[y] \) as \( X \cup v \) induces a clique. In the other case, \( N[v] \supseteq N[y] \) as \( v \) is universal. Hence \( P_G \) is a weak neighborhood inclusion poset.

Claim 4.5. Let \( I \subseteq V(H) \). Then \( I \in ITG(H, P_H) \) if and only if \( I \in ITD(G, P_G) \), \( I \neq \{v\} \).

Proof of the claim. Let \( I \subseteq V(H) \) such that \( I \in ITG(H, P_H) \). As \( H \) is non-empty, \( I \neq \emptyset \). By construction, \( I \) is an ideal of \( P_G \), \( I \neq \{v\} \), and it is a minimal dominating-ideal of \( Y \).

As \( I \) dominates \( X \cup \{v\} \), it is a minimal dominating-ideal of \( G \). Let \( I \subseteq V(H) \) such that \( I \in ITD(G, P_G) \) and \( I \neq \{v\} \). Note that \( y \notin I \) for any \( y \in Y \) as \( v \in y \) and \( v \) dominates \( G \). Since \( I \neq \{v\} \), \( I \subseteq X \). Since \( X \) induces a clique, \( \text{Priv}(I, x) \subseteq Y \) for all \( x \in I \). Hence \( I \) is a minimal transversal-ideal of \( H \).

We now consider Item 3. Let \( G \) be the graph obtained from \( I(H) \) by adding a single vertex \( v \) connected to every vertex of \( X \), and by completing both \( X \) and \( Y \) into a clique. Then \( G \) is co-bipartite with cliques \( X \cup \{v\} \) and \( Y \). Let \( P_G = P_H \). We prove the following two claims.

Claim 4.6. \( P_G \) is a neighborhood inclusion poset on \( G \).

Proof of the claim. Let \( x, y \in P_G \) such that \( x \leq y \). It follows from Observation 4.2 that \( N[x] \subseteq N[y] \) as \( X \cup v \) induces a clique.
Claim 4.7. Let $I \subseteq V(\mathcal{H})$. Then $I \in \mathcal{ITr}(\mathcal{H}, P_{\mathcal{H}})$ if and only if $I \in \mathcal{ID}(G, P_G)$ and $I \notin \{(x, y) \mid x \in X \cup \{v\}, y \in Y\}$.

Proof of the claim. Let $I \subseteq V(\mathcal{H})$ such that $I \in \mathcal{ITr}(\mathcal{H}, P_{\mathcal{H}})$. As $\mathcal{H}$ is non-empty, $I \neq \emptyset$. By construction, $I$ is an ideal of $P_G$, $I \notin \{(x, y) \mid x \in X \cup \{v\}, y \in Y\}$, and it is a minimal dominating-ideal of $Y$. As $I$ dominates $X \cup \{v\}$, it is a minimal dominating-ideal of $G$. Let $I \subseteq V(\mathcal{H})$ such that $I \in \mathcal{ID}(G, P_G)$ and $I \notin \{(x, y) \mid x \in X \cup \{v\}, y \in Y\}$. Note that $y \notin I$ for any $y \in Y$ or else, as $y$ is non adjacent to any vertex in $Y$, $I$ must contain one vertex of $X \cup \{v\}$ to dominate $G$. Then $I$ does not contain any other vertex as it dominates $G$, and $I \in \{(x, y) \mid x \in X \cup \{v\}, y \in Y\}$, which is excluded. Moreover, $v \notin I$ or else, $I$ must contain some $x \in X$ to dominate $Y$ and $N[v] \subseteq N[x]$. Hence $I \subseteq X$. Since $X$ induces a clique, $\text{Priv}(I, x) \subseteq Y$ for all $x \in I$. Hence $I$ is a minimal transversal-ideal of $\mathcal{H}$.

The proof of the theorem follows from claims [4.3] [4.4] [4.5] [4.6] and [4.7] observing that $G = I(\mathcal{H})$ is constructed in polynomial time in the sizes of $\mathcal{H}$ and $P_{\mathcal{H}}$, and that $\mathcal{ITr}(\mathcal{H}, P_{\mathcal{H}})$ can be enumerated with polynomial delay from $\mathcal{ID}(G, P_G)$ on the constructed graph and poset. Indeed, in the case of Item 1 only one extra solution (namely $I = V(\mathcal{H})$) has to be handled separately. In the case of Item 2 only one solution (namely $I = \{v\}$) has to be discarded. In the case of Item 3 at most $|V(G)|^2$ solutions (namely every subsets of $V(G)$ of size two) have to be discarded. Hence the desired result.

5 Tractable cases for dominating-ideals enumeration

In the following, we show that the restricted cases left by Theorem 4.1 are tractable, using existing algorithms and techniques from the literature for the enumeration of minimal dominating sets in split and triangle-free graphs [KLAM14 BDHR18]. Our results rely on the following important property.

Proposition 5.1. Let $G$ be a graph and $P_G$ be a weak neighborhood inclusion poset on $G$. Then, every minimal dominating set of $G$ is an antichain of $P_G$. Hence there is a bijection between minimal dominating sets of $G$ and their ideal in $P_G$. If in addition $P_G$ is a neighborhood inclusion poset, then Max$(D)$ dominates $G$ whenever $D$ does.

Proof. Let $D$ be a dominating set of $G$ and $x, y \in D$ be two comparable elements of $P_G$. If $P_G$ is a weak neighborhood inclusion poset, then either $N[x] \subseteq N[y]$ or $N[x] \supseteq N[y]$. Thus, either $D \setminus \{x\}$ or $D \setminus \{y\}$ dominates $G$ and we deduce that every minimal dominating set of $G$ is an antichain of $P_G$. If $P_G$ is a neighborhood inclusion poset, then $D \setminus \{y\}$ dominates $G$ and we deduce that Max$(D)$ dominates $G$. Since the set of antichains and the set of ideals of a poset are in bijection, we conclude to a bijection between minimal dominating sets of $G$ and their ideal in $P_G$.

A consequence of Proposition 5.1 is the following equality.

$$\mathcal{ID}(G, P_G) = \text{Min}_\subseteq \{\downarrow D \mid D \in \mathcal{D}(G)\}. \tag{5}$$

Note that instances that verify this property are not trivially tractable, as two of the constructed instances in the proof of Theorem 4.1 satisfy Proposition 5.1 despite the fact that the problem on such instances is DDUALIZATION-hard, hence TRANS-ENUM-hard.
5.1 Split graphs and neighborhood inclusion posets

In [KLMN14], the authors give a polynomial delay algorithm to enumerate minimal dominating sets in split graphs. The algorithm relies on the two observations that if $G$ is a split graph of maximal independent set $S$, and clique $C$, then the set of intersections of minimal dominating sets with $C$ is in bijection with $\mathcal{D}(G)$, and it forms an independence system. A pair $(X, S)$ where $S \subseteq 2^X$ is said to be an independence system if $\emptyset \in S$ and if $S \in S$ implies that $S' \in S$ for all $S' \subseteq S$. We show that these observations can be generalized in our case, giving a polynomial delay algorithm to enumerate $\mathcal{ID}(G, P_G)$ whenever $G$ is split and $P_G$ is a neighborhood inclusion poset.

In what follows, we use the same notation as in [KLMN14] to denote the intersection of a dominating set $D$ with some set $W \subseteq V(G)$, namely $D_W = D \cap W$. We extend this notation to the set of minimal dominating sets as follows:

$$\mathcal{D}_W(G) \overset{\text{def}}{=} \{ D_W \mid D \in \mathcal{D}(G) \}.$$  

**Proposition 5.2 ([KLMN14]).** Let $G$ be a split graph with maximal independent set $S$, clique $C$, and let $D$ be a minimal dominating set of $G$. Then $D_S = S \setminus N(D_C)$, i.e., $D$ is uniquely characterized by its intersection with $C$.

Furthermore, $\mathcal{D}_C(G) = \{ A \subseteq C \mid \forall x \in A, \text{Priv}(A, x) \cap S \neq \emptyset \}$ and

1. $\mathcal{D}_C(G)$ and $\mathcal{D}(G)$ are in bijection,
2. $(C, \mathcal{D}_C(G))$ is an independence system.

In the following, we consider a split graph $G$ and a neighborhood inclusion poset $P_G$. As $P_G$ is a neighborhood inclusion poset, Equality 6 applies. The next proposition allows us to consider a decomposition of $G$ into a maximal independent set $S$ and a clique $C$, such that $S \subseteq \text{Min}(P_G)$.

**Proposition 5.3.** Let $G$ be a split graph and $P_G$ be a neighborhood inclusion poset on $G$. Then there exists a decomposition of $G$ into a maximal independent set $S$, and a clique $C$, such that $S \subseteq \text{Min}(P_G)$.

**Proof.** Let $S, C$ be a decomposition of $G$ that maximizes the independent set. If $x \in S$ and $x \notin \text{Min}(P_G)$, then there exists some $y_x \in C$ such that $y_x \leq x$, $N[x] = N[y_x]$, and thus such that $S \setminus \{x\} \cup \{y_x\}$ and $C \setminus \{y_x\} \cup \{x\}$ is still a decomposition of $G$ that maximizes the independent set. \hfill \Box

We now define

$$\mathcal{D}_C(G, P_G) \overset{\text{def}}{=} \{ D_C \mid D \in \mathcal{D}(G) \text{ and } \downarrow D \in \mathcal{D}(G, P_G) \},$$

and show that Proposition 5.2 holds for this set.

**Lemma 5.4.** Let $G$ be a split graph with maximal independent set $S$ and clique $C$, and $P_G$ be a neighborhood inclusion poset on $G$. Then $\mathcal{D}_C(G, P_G)$ and $\mathcal{ID}(G, P_G)$ are in bijection, and $\mathcal{D}_C(G, P_G) \subseteq \mathcal{D}_C(G)$.

**Proof.** The bijection between $\mathcal{D}_C(G, P_G)$ and $\mathcal{ID}(G, P_G)$ follows from propositions 5.1 and 5.2 and Equality 6, where to every $A \in \mathcal{D}_C(G, P_G)$ corresponds a unique $I \in \mathcal{ID}(G, P_G)$ such that $I = \downarrow (A \cup (S \setminus N(A)))$, and to every $I \in \mathcal{ID}(G, P_G)$ corresponds a unique $A \in \mathcal{D}_C(G, P_G)$ such that $A = \text{Max}(I) \cap C$.

The inclusion $\mathcal{D}_C(G, P_G) \subseteq \mathcal{D}_C(G)$ follows from Equality 6 as $A \in \mathcal{D}_C(G, P_G)$ implies $A = D_C$ for some $D \in \mathcal{D}(G)$ such that $\downarrow D \in \mathcal{D}(G, P_G)$.

\hfill \Box

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Lemma 5.5. Let $G$ be a split graph with maximal independent set $S \subseteq \text{Min}(P_G)$ and clique $C$, and $P_G$ be a neighborhood inclusion poset on $G$. Then $(C, \mathcal{D}_C(G, P_G))$ is an independence system that can be enumerated with polynomial delay given $G$ and $P_G$.

Proof. We first show that $(C, \mathcal{D}_C(G, P_G))$ is an independence system. Let $\emptyset \neq A \subseteq C$ such that $A \in \mathcal{D}_C(G, P_G)$. We show that $A \setminus \{x\} \notin \mathcal{D}_C(G, P_G)$ for all $x \in A$. Let us assume that it is not the case, and let $x \in A$ such that $A \setminus \{x\} \notin \mathcal{D}_C(G, P_G)$. By Proposition 5.4, since $A \in \mathcal{D}_C(G, P_G)$ and since $\mathcal{D}_C(G)$ is an independence system, both $A$ and $A \setminus \{x\}$ belong to $\mathcal{D}_C(G)$. Let $D, D' \in \mathcal{D}(G)$ such that $A = D_C$ and $A \setminus \{x\} = D'_C$. By Proposition 5.2, $D' = D \setminus \{x\} \cup \{s_1, \ldots, s_k\}$ where $\{s_1, \ldots, s_k\} = \text{Priv}(A, x) \cap S$. As by hypothesis $A \setminus \{x\} \notin \mathcal{D}_C(G, P_G)$, there exists $D^* \subseteq D(G)$ such that $D^* \subseteq D'$. Now, note that $\text{Min}(P_G) \cap D' \subseteq D^*$ or else there exists $w \in \text{Min}(P_G) \cap D' \setminus D^*$, hence $D^* \subseteq (D' \setminus \{w\})$, and we deduce that $\downarrow (D' \setminus \{w\})$ dominates $G$, which by Proposition 5.1 implies that $\text{Max}(\downarrow (D' \setminus \{w\})) = D' \setminus \{w\}$ dominates $G$, which contradicts the fact that $D'$ is a minimal dominating set. Therefore $\{s_1, \ldots, s_k\} \subseteq D^*$ and as $\downarrow D^* \subseteq \downarrow D'$, there exist $u \in D^*$ and $v \in D' \setminus \text{Min}(P_G) \setminus D^*$ such that $u < v$. Note that $v \in D$ (as $D' \subseteq D$) and $v \notin x$ (as $x \notin D'$). Let $D^0 = D^* \cup \{x\} \setminus \{s_1, \ldots, s_k\}$. Clearly $D^0$ dominates $G$. As $\downarrow D^0 \subseteq \downarrow D'$, $\downarrow D^0 \subseteq \downarrow D$. Moreover $\downarrow D^0 \subseteq \downarrow D$ as $v \in D$ and $v \notin D^0$. This contradicts the hypothesis that $A \in \mathcal{D}_C(G, P_G)$. Hence $A \cup \{x\} \in \mathcal{D}_C(G, P_G)$.

Now, note that testing whether some arbitrary set $A \subseteq C$ belongs to $\mathcal{D}_C(G, P_G)$ can be done in polynomial time in the sizes of $G$ in $\mathcal{D}(G)$ such that $D_C = A$, using Proposition 5.2 and test whether $\downarrow D \in \mathcal{D}(G)$ by checking if $\text{Priv}(\downarrow D, x) \neq \emptyset$ for every $x \in D$. Hence, $\mathcal{D}_C(G, P_G)$ can be enumerated with polynomial delay by adding vertices of $C$ one by one from the emptyset to maximal elements of $\mathcal{D}_C(G, P_G)$, checking at each step whether the new set belongs to $\mathcal{D}_C(G, P_G)$. Repetitions are avoided with a linear order on vertices of $C$; see [KLMN13] for further details on the enumeration of an independence system.

We deduce a polynomial delay algorithm to enumerate $\mathcal{I}(G, P_G)$ whenever $G$ is a split graph and $P_G$ is a neighborhood inclusion poset on $G$. The algorithm first computes a decomposition $S, C$ that maximizes the independent set, makes $S$ a subset of $\text{Min}(P_G)$ using Proposition 5.3 and enumerates the independence system $(C, \mathcal{D}_C(G, P_G))$ with polynomial delay using Lemma 5.5. For every $A \in \mathcal{D}_C(G, P_G)$, it outputs the unique corresponding $I = \downarrow D$ such that $D_C = A$ using Lemma 5.4. This can clearly be done with polynomial delay per solution. We conclude with the following result.

Theorem 5.6. There is a polynomial delay algorithm for $\text{IDOM-ENUM}$ whenever $G$ is split and $P_G$ is a neighborhood inclusion poset.

5.2 Triangle-free graphs and weak neighborhood inclusion posets

In [BDHRS18], the authors give an output-polynomial algorithm to enumerate minimal dominating sets in triangle-free graphs, i.e., graphs with no induced clique of size three. These graphs include bipartite graphs. We rely on this algorithm to show that $\mathcal{I}(G, P_G)$ can be enumerated in output-polynomial time in the same graph class, whenever $P_G$ is a weak neighborhood inclusion poset on $G$. Our argument is based on the next observation.

Proposition 5.7. Let $G$ be a triangle-free graph and $P_G$ be a weak neighborhood inclusion poset on $G$. Then $P_G$ is of height at most two, and it is partitioned into an antichain $A$ of isolated elements (that are both minimal and maximal in $P_G$), and a family $S$ of $k$ disjoint stars $S_1, \ldots, S_k$ of respective center $u_1, \ldots, u_k$ such that either $S_i = \downarrow u_i$ or $S_i = \uparrow u_i$, for

1$S_i$ induces a star in the Hasse diagram of $P_G$. 11
all $i \in [k]$. Furthermore, vertices in $S_i \setminus u_i$ are of degree one in $G$.

**Proof.** This situation is depicted in Figure 5. We first show that $P_G$ is of height at most two. Suppose that there exist $x, y, z$ such that $x < y < z$. Then $xy, xz, yz \in E(G)$ which contradicts the fact the $G$ is triangle-free.

Let us now prove the rest of the proposition. Let $A = \operatorname{Min}(P_G) \cap \operatorname{Max}(P_G)$, and $B = P_G \setminus A$. Let $S \subseteq B$ be a connected component in the Hasse diagram of $P_G$, and let $x, y \in S$ such that $x < y$. Two symmetric cases arises depending on whether $N[x] \subseteq N[y]$ or $N[x] \supseteq N[y]$. If $N[x] \subseteq N[y]$ then $x$ is of degree one in $G$ (or else the other neighbor of $x$ would be connected to both $x$ and $y$ and would induce a triangle in $G$). Moreover, every other element $z \neq x$ that is comparable with $y$ verifies $N[z] \subseteq N[y]$ (or else it verifies $N[z] \supseteq N[y]$ and $xyz$ induces a triangle in $G$), hence is of degree one (by previous remark).

Also, it verifies $z < y$ as $P_G$ is of height at most two. Hence $S$ induces a star of center $y$ in the Hasse diagram of $P_G$, such that $S = \uparrow y$, and where every vertex in $S \setminus y$ is of degree one in $G$. The other case $N[x] \supseteq N[y]$ leads to the symmetric situation where $S = \downarrow x$ and where every vertex in $S \setminus x$ is of degree one in $G$.

In the following, we denote by $\{v_1^i, \ldots, v_i^i\}$ the set of branches of some star $S_i \in S$, $i \in [k]$, and by $u_i$ its center. Then, we denote by $G_{\text{re}}$ and $P_{G_{\text{re}}}$ the reduced graph and poset obtained from $G$ and $P_G$, where every star $S_i \in S$ had its branches $\{v_1^i, \ldots, v_i^i\}$ contracted into a single element $v_i$, and where every edge $u_iu_j$ that connect two distinct stars $S_i, S_j$ in $G$ has been removed. We denote by $B_a$ and $B_v$ the sets $B_a = \{u_1, \ldots, u_k\}$ and $B_v = \{v_1, \ldots, v_k\}$. The resulting graph is detailed below and is given in Figure 6. Observe that $P_{G_{\text{re}}}$ is partitioned into an antichain $A$ of isolated elements (that are both minimal and maximal in $P_{G_{\text{re}}}$, and left untouched by our transformation), and a set $B = B_a \cup B_v = \{u_1, v_1, \ldots, u_k, v_k\}$ of $k$ disjoint chains $u_iv_i$ (such that either $u_i < v_i$ or $v_i < u_i$), $i \in [k]$. The graph $G_{\text{re}}$ is partitioned into one triangle-free graph induced by $A$ (left untouched by our transformation), and an induced matching $\{u_1v_1, \ldots, u_kv_k\}$ ($B_a$ and $B_v$ induce two independent sets), where $u_i$ is disconnected from $A$, and $u_i$ is arbitrarily connected to $A$, for every $i \in [k]$. Clearly, $G_{\text{re}}$ and $P_{G_{\text{re}}}$ can be constructed in polynomial time in the sizes of $G$ and $P_G$. The following property is implicit in [KLMNT14] and can also be found in the Ph.D. thesis of Mary [Mar13].

![Figure 6: The decomposition $(A, B)$ of a reduced triangle-free graph $G_{\text{re}}$.](image-url)
Proposition 5.8 ([KLMN14, Mar13]). Let $G$ be a graph and $uv$ be an edge of $G$. Then $\mathcal{D}(G) = \mathcal{D}(G - uv)$ whenever there exists $u' \neq u$, $v' \neq v$ such that $N_{G - uv}[u'] \subseteq N_{G - uv}[u]$ and $N_{G - uv}[v'] \subseteq N_{G - uv}[v]$. Such an edge $uv$ is called redundant.

Lemma 5.9. There is a bijection between $\mathcal{TD}(G, P_G)$ and $\mathcal{ID}(G_{re}, P_{G_{re}})$.

Proof. Let $S$ be a star of Proposition 5.7 of center $u$ and branches $v^1, \ldots, v^l$. Then, observe that $v^1, \ldots, v^l$ are false twins in $G$, i.e., $N(v^i) = N(v^j) = u$ for all $i, j \in [l]$. It is easy to see that a minimal dominating set contains $v^i$ for some $i$ if and only if contains the whole set $\{v^1, \ldots, v^l\}$ as a subset. Hence, the contraction of all branches $\{v^1, \ldots, v^l\}$ of $S$ into a representative vertex $v$ in both $G$ and $P_G$ has no impact on the complexity of enumerating minimal dominating sets: one can replace $v$ by $\{v^1, \ldots, v^l\}$ for every $D \in \mathcal{D}(G)$ such that $\downarrow D \in \mathcal{TD}(G, P_G)$ and $v \in D$ to obtain solutions of the graph before contraction. As for the deleted edges $u_i u_j$, $i, j \in [k], i \neq j$, they are all redundant as $N_{G - u_i u_j}[v_p] \subseteq N_{G - u_i u_j}[u_p]$ for all $i, j, p \in [k], i \neq j$. By Proposition 5.8 they can be removed from $G$ with no incidence on domination. \hfill \Box

Proposition 5.10. For every minimal dominating set $D$ such that $\downarrow D \in \mathcal{TD}(G_{re}, P_{G_{re}})$, $\text{Min}(P_{G_{re}}) \cap B_u \subseteq D$.

Proof. Let $u \in B_u \cap \text{Min}(P_G)$ and $v \in B_v$ be the unique vertex such that $u < v$. Since $v$ is of degree one in $G$, it must be dominated by either itself, or $u$. Since $u < v$, a dominating-ideal that contains $v$ is not minimal. Hence $\text{Min}(P_{G_{re}}) \cap B_u \subseteq D$ for all minimal dominating set $D$ such that $\downarrow D \in \mathcal{TD}(G_{re}, P_{G_{re}})$.

Let $G$ be a graph and $W, D$ be two subsets of vertices of $G$. Recall that $\mathcal{D}(G)(W)$ denotes the set of minimal dominating sets of subset $W$ in $G$; see Section 2. We now rely on an implicit result from [BDHR18].

Theorem 5.11 ([BDHR18]). There is an algorithm that, given a graph $G$ and a set $W$ such that $G[W]$ is triangle-free, enumerates $\mathcal{D}(G)(W)$ in total time $\text{poly}(|G| \cdot |\mathcal{D}(G)(W)|^2$ and polynomial space.

Let us define the set $B_u = \text{Min}(B) = \{w_1, \ldots, w_k\}$. Note that $w_i = \text{Min}_i \{u_i, v_i\}$ for all $i \in [k]$. We now consider the set

$$A' \overset{\text{def}}{=} A \setminus \bigcup_{i=1}^{k} N[w_i].$$

Clearly, $G_{re}[A']$ is triangle-free. Hence, $\mathcal{D}_{G_{re}}(A')$ can be enumerated in output-polynomial time $\text{poly}(|G_{re}| \cdot |\mathcal{D}_{G_{re}}(A')|^2$ using the algorithm of Theorem 5.11. We now show how to compute $\mathcal{TD}(G_{re}, P_{G_{re}})$ given $\mathcal{D}_{G_{re}}(A')$.

Lemma 5.12. Let $D$ be a minimal dominating set of $G$. Then $\downarrow D \in \mathcal{TD}(G_{re}, P_{G_{re}})$ if and only if $D = D^* \cup \{w_i \mid v_i \notin N[D^*]\}, D^* \in \mathcal{D}_{G_{re}}(A')$.

Proof. The situation of this lemma is depicted in Figure 7. We show the first implication. Let $D \in \mathcal{D}(G)$ such that $\downarrow D \in \mathcal{TD}(G_{re}, P_{G_{re}})$, and let $D^* = D \setminus \{w_i \mid w_i \in D\}$. Clearly, $D^*$ dominates $A'$. Let $t \in D^*$. We show that it has a private neighbor in $A'$. Let $a$ be a private neighbor of $t$ (with respect to $D$) such that $a \notin A'$. If no such $a$ exists, then we proved our claim, as in that case $t$ must have a private neighbor in $A'$. Else, $a$ belongs to $N[w_i]$ for some $i \in [k]$. If $w_i = u_i$ then by Proposition 5.10 $w_i \in D$ which contradicts the fact that $a$ is a private neighbor of $t$. If $w_i = v_i$, then $a \in \{u_i, v_i\}$. Since either $u_i$
or \( v_i \) belongs to \( D \) (as \( v_i \) is of degree one), it must be that either \( t = u_i \) or \( t = v_i \). As \( t \neq w_i = v_i \), we know that \( t = u_i \). In that case, \( t \) has another private neighbor \( a' \neq a \) that is non-adjacent to \( v_i \) (or else \( D \) is not a minimal dominating-ideal as \( t = u_i \) can be replaced by \( v_i \), \( a \in N[v_i] \), and \( v_i < u_i \)). At last, if \( a' \) belongs to \( w_j \) for some \( j \in [k] \), then \( w_j = u_j \) (as \( N[v_i] = \{u_i, v_i\} \) and \( B \) is a induced matching, hence \( a \neq u_j \)) which by Proposition 5.10 is absurd, as \( w_j \in D \). Hence \( a' \in A' \), which proves our claim. Hence \( D^* \) minimally dominates \( A' \), i.e., \( D^* \in D_{G_{re}}(A') \). Now, note that \( w_i \in D \) if and only if \( v_i \notin N[D^*] \). Indeed, if \( v_i \notin N[D^*] \) then \( w_i \in D \) or else \( w_i \notin D \), by Proposition 5.10 \( w_i = v_i \), hence \( u_i \in D \), \( u_i \in D^* \), and \( v_i \notin N[D^*] \) which is absurd. If \( v_i \in N[D^*] \), then \( u_i \in D^* \), \( w_i = v_i \), and \( w_i \notin D \) or else \( \{u_i, v_i\} \subseteq D \) which is absurd since \( D \) is an antichain. Hence \( D = D^* \cup \{w_i \mid v_i \notin N[D^*]\} \) which concludes the first implication.

We show the other implication. Let \( D^* \in D_{G_{re}}(A') \) and \( D = D^* \cup \{w_i \mid v_i \notin N[D^*]\} \). Clearly \( D \) dominates \( G_{re} \) as for all \( i \in [k] \), either \( v_i \in N[D^*] \) and therefore \( u_i \in D^* \) (as \( v_i \) is disconnected from \( A' \)) and \( N[w_i] \) is dominated, or \( v_i \notin N[D^*] \) and \( w_i \) dominates \( N[w_i] \).

Note that if \( t \in D^* \) then it has private neighbors in \( A' \) that are not adjacent to any \( w_i \) (by construction), hence such that no ideal \( I \subseteq \downarrow (D \setminus \{t\}) \) can dominate. If \( t \in D \setminus D^* \) then \( t = w_i \) for some \( i \in [k] \), it has \( v_i \) for private neighbor, and it is minimal in \( P_{G_{re}} \). Hence \( \downarrow D \) is minimal dominating-ideal of \( G \).

We conclude to the existence of an output-polynomial algorithm to enumerate the set \( \mathcal{TD}(G, P_G) \) whenever \( G \) is triangle-free and \( P_G \) is a weak neighborhood inclusion poset. The algorithm first computes \( G_{re} \) and \( P_{G_{re}} \) in polynomial time in the sizes of \( G \) and \( P_G \), and then enumerates \( \mathcal{TD}(G, P_G) \) using lemmas 5.9 and 5.12.

**Theorem 5.13.** There is an algorithm that, given a triangle-free graph \( G \) and a weak neighborhood inclusion poset \( P_G \), enumerates \( \mathcal{TD}(G, P_G) \) in output-polynomial time.

### 6 Conclusion

In this paper, we generalized the two problems of enumerating minimal transversals of a hypergraph, and minimal dominating sets of a graph, to the enumeration of minimal ideals of a poset with the desired property, i.e., transversality and domination. We showed that the obtained problems are equivalent to the dualization in distributive lattices, even when considering various combined restrictions on graph classes and posets, including bipartite, split and co-bipartite graphs, and variant of neighborhood inclusion posets; see theorems 3.3 and 4.1. For the restricted cases left by our theorem, we showed that the problem is tractable relying on existing algorithms from the literature; see theorems 5.6 and 5.11. A summary of the obtained complexities is given in Figure 8.

We leave open the complexity status of distributive lattice dualization in general. Note that the results of theorems 5.6 and 5.11 characterize couples of antichains (coded by a graph) and distributive lattices (coded by a poset) for which the dualization is tractable.
Graph classes | N.I. posets | Weak N.I. posets | Arbitrary posets
--- | --- | --- | ---
Bipartite | OutputP | OutputP | D-hard
Split | PolyD | D-hard | D-hard
Co-bipartite | D-hard | D-hard | D-hard

Figure 8: Summary of the complexity results obtained in theorems 4.1, 5.6 and 5.13. OutputP stands for output-polynomial, and PolyD for polynomial delay. N.I. stands for neighborhood inclusion, and D-hard for D\textsc{Dualization}-hard.

For future work, we would be interested in characterizations that only depend on the poset, in order to obtain classes of lattices for which the dualization is tractable, as in [DN19, Elb09], using graph structures presented in this paper.

References

[BDHR18] Marthe Bonamy, Oscar Defrain, Marc Heinrich, and Jean-Florent Raymond. Enumerating minimal dominating sets in triangle-free graphs. arXiv preprint arXiv:1810.00789, 2018.

[Ber84] Claude Berge. Hypergraphs: combinatorics of finite sets, volume 45. Elsevier, 1984.

[Bir40] Garrett Birkhoff. Lattice theory, volume 25. American Mathematical Soc., 1940.

[BK17] Mikhail A Babin and Sergei O Kuznetsov. Dualization in lattices given by ordered sets of irreducibles. Theoretical Computer Science, 658:316–326, 2017.

[CKP+17] Nadia Creignou, Markus Kröll, Reinhard Pichler, Sebastian Skritek, and Heribert Vollmer. On the complexity of hard enumeration problems. In International Conference on Language and Automata Theory and Applications, pages 183–195. Springer, 2017.

[Cou09] Bruno Courcelle. Linear delay enumeration and monadic second-order logic. Discrete Applied Mathematics, 157(12):2675–2700, 2009.

[Die05] Reinhard Diestel. Graph theory. 2005, volume 101. Grad. Texts in Math, 2005.

[DN19] Oscar Defrain and Lhouari Nourine. Dualization in lattices given by implicative bases. ArXiv e-prints, 2019.

[DP02] Brian A Davey and Hilary A Priestley. Introduction to lattices and order. Cambridge university press, 2002.

[EG95] Thomas Eiter and Georg Gottlob. Identifying the minimal transversals of a hypergraph and related problems. SIAM Journal on Computing, 24(6):1278–1304, 1995.

[EGM03] Thomas Eiter, Georg Gottlob, and Kazuhsa Makino. New results on monotone dualization and generating hypergraph transversals. SIAM Journal on Computing, 32(2):514–537, 2003.
[Elb02] Khaled M Elbassioni. An algorithm for dualization in products of lattices and its applications. In European Symposium on Algorithms, pages 424–435. Springer, 2002.

[Elb09] Khaled M Elbassioni. Algorithms for dualization over products of partially ordered sets. SIAM Journal on Discrete Mathematics, 23(1):487–510, 2009.

[EMG08] Thomas Eiter, Kazuhisa Makino, and Georg Gottlob. Computational aspects of monotone dualization: A brief survey. Discrete Applied Mathematics, 156(11):2035–2049, 2008.

[FK96] Michael L. Fredman and Leonid Khachiyan. On the complexity of dualization of monotone disjunctive normal forms. Journal of Algorithms, 21(3):618–628, 1996.

[GHK+16] Petr A Golovach, Pinar Heggernes, Mamadou M Kanté, Dieter Kratsch, and Yngve Villanger. Enumerating minimal dominating sets in chordal bipartite graphs. Discrete Applied Mathematics, 199:30–36, 2016.

[GHK+18] Petr A. Golovach, Pinar Heggernes, Mamadou Moustapha Kanté, Dieter Kratsch, Sigve H. Sæther, and Yngve Villanger. Output-polynomial enumeration on graphs of bounded (local) linear mim-width. Algorithmica, 80(2):714–741, Feb 2018.

[GHKV15] Petr A Golovach, Pinar Heggernes, Dieter Kratsch, and Yngve Villanger. An incremental polynomial time algorithm to enumerate all minimal edge dominating sets. Algorithmica, 72(3):836–859, 2015.

[GMKT97] Dimitrios Gunopulos, Heikki Mannila, Roni Khardon, and Hannu Toivonen. Data mining, hypergraph transversals, and machine learning. In PODS, pages 209–216. ACM, 1997.

[Grä11] George Grätzer. Lattice theory: foundation. Springer Science & Business Media, 2011.

[JYP88] David S Johnson, Mihalis Yannakakis, and Christos H Papadimitriou. On generating all maximal independent sets. Information Processing Letters, 27(3):119–123, 1988.

[KLM+15] Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine, and Takeaki Uno. A polynomial delay algorithm for enumerating minimal dominating sets in chordal graphs. In International Workshop on Graph-Theoretic Concepts in Computer Science, pages 138–153. Springer, 2015.

[KLMN12] Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, and Lhouari Nourine. On the neighbourhood helly of some graph classes and applications to the enumeration of minimal dominating sets. In International Symposium on Algorithms and Computation, pages 289–298. Springer, 2012.

[KLMN14] Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, and Lhouari Nourine. On the enumeration of minimal dominating sets and related notions. SIAM Journal on Discrete Mathematics, 28(4):1916–1929, 2014.
Mamadou Moustapha Kanté and Lhouari Nourine. Minimal dominating set enumeration. *Encyclopedia of Algorithms*, pages 1–5, 2008.

Dimitris Kavvadias, Christos H Papadimitriou, and Martha Sideri. On horn envelopes and hypergraph transversals. In *International Symposium on Algorithms and Computation*, pages 399–405. Springer, 1993.

Arnaud Mary. *Enumération des dominants minimaux d’un graphe*. PhD thesis, Université Clermont Auvergne, 2013.

Lhouari Nourine and Jean-Marc Petit. Extending set-based dualization: Application to pattern mining. In *Proceedings of the 20th European Conference on Artificial Intelligence*, pages 630–635. IOS Press, 2012.

Lhouari Nourine and Jean Marc Petit. Dualization on partially ordered sets: Preliminary results. In *International Workshop on Information Search, Integration, and Personalization*, pages 23–34. Springer, 2014.