The Pauli–Łubanski vector, complex electrodynamics, and photon helicity

Sergey I Kryuchkov¹, Nathan A Lanfear¹ and Sergei K Suslov²,³

¹ School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287-1804, USA
² School of Mathematical and Statistical Sciences & Mathematical, Computational and Modeling Sciences Center, Arizona State University, Tempe, AZ 85287-1804, USA

E-mail: sergeykryuchkov@yahoo.com, nlanfear@asu.edu and sks@asu.edu

Received 13 May 2015
Accepted for publication 18 May 2015
Published 1 July 2015

Abstract
We critically analyze the concept of photon helicity and its connection with the Pauli–Łubanski vector from the viewpoint of the complex electromagnetic field, \( \mathbf{E} + i\mathbf{H} \), sometimes attributed to Riemann but studied by Weber, Silberstein, and Minkowski. To this end, a complex covariant form of Maxwell’s equations is used. Weyl’s two-component wave equation for massless neutrinos is also briefly discussed.

Keywords: Poincaré group, Pauli–Łubanski vector

1. Introduction

All physically interesting unitary ray representations of the proper orthochronous inhomogeneous Lorentz group (known nowadays as the Poincaré group) were classified by Wigner [75] and, since then, this approach has been utilized for the mathematical description of mass and spin of an elementary particle. By definition, the Pauli–Łubanski pseudo-vector is given by

\[
\mathbf{w} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} p^\alpha M^{\beta\gamma}, \quad p^\mu w_{\mu} = 0, \tag{1.1}
\]

where \( p_\mu \) is the relativistic linear momentum operator and \( M^{\beta\gamma} \) are the corresponding angular momentum operators. The mass and spin of a particle are defined in terms of two quadratic invariants (Casimir operators of the Poincaré group) as follows

\[
p^2 = p_\mu p^\mu = m^2, \quad w^2 = w_\mu w^\mu = -m^2(s + 1), \quad m > 0, \tag{1.2}
\]

(see, for example, [3, 4, 15, 37, 40, 43–45, 58–61] and the references therein; we use the standard notations that are introduced in the body of the article).

For the massless fields, when \( m = 0 \), one gets \( w^2 = p^2 = pw = 0 \), and the Pauli–Łubanski vector should be proportional to \( p \):

\[
w_\mu = \lambda p_\mu \tag{1.3}
\]

(acting on the corresponding eigenstates [53, 59]). The number \( \lambda \) is called the helicity of the representation and the value \( |\lambda| \) is sometimes called the spin of a particle with zero mass [15, 59–61]. One of the goals of this article is to show that, in the case of the electromagnetic field, the constant \( \lambda \) in the latter equation is fixed, otherwise violating the classical Maxwell equations.

As a result, instead of being given by the constant of proportionality in relation (1.3), the helicity of the photon should be defined, as it is traditionally done in particle physics, by \( \lambda = \mathbf{k} \cdot \mathbf{M}/k_0 \), where \( k = (k_0, \mathbf{k}) \) and \( \mathbf{M} \) is the photon angular momentum \((k^2 = k_0^2 - k^2 = 0)\). But one needs a proper realization of the action of these operators on the photon field tensor in covariant form [48, 49]; or, in 3D-form,

\[4\] This assumption was made by Bargmann and Wigner [4] for the massless limit of the spinor wave equation for particles with an arbitrary integer or half-integer spin proposed by Dirac [17] (see also [22, 23, 32, 54, 74] and the references therein). The pseudo-vector (1.1) was introduced, in a slightly different notation, by equations (4.a)–(4.b) of [4].

³ Author to whom any correspondence should be addressed.
on the complex electromagnetic field vector \( \mathbf{F} = \mathbf{E} + i \mathbf{H} \) discussed in [5, 6, 11, 14, 38, 39, 48, 58, 62, 64–66, 72]. The sign of the constant \( \lambda = \pm 1 \) is fixed then by a ‘continuity’, in view of the invariance of the upper(lower) light cone under a proper Lorentz transformation (see, for example, [15, 37] and [69]).

2. Maxwell’s equations in complex and covariant form

We would like to introduce two complex vector fields

\[
\mathbf{F} = \mathbf{E} + i \mathbf{H}, \quad \mathbf{G} = \mathbf{D} + i \mathbf{B}
\]

(\( \mathbf{E} \) is the electric field, \( \mathbf{D} \) the electric displacement field, \( \mathbf{H} \) the magnetic field, and \( \mathbf{B} \) the magnetic induction field) and present the phenomenological Maxwell equations in a compact form

\[
\frac{i}{c} \left( \frac{\partial \mathbf{G}}{\partial t} + 4\pi \mathbf{j} \right) = \text{curl} \mathbf{F}, \quad \mathbf{j} = \mathbf{j}^*, \quad \text{div} \mathbf{G} = 4\pi \rho, \quad \rho = \rho^*,
\]

(2.2)

where the asterisk stands for complex conjugation. (We are working in Gaussian units; complex-valued \( \rho \) and \( \mathbf{j} \) may be related to the free magnetic charge and current, which have not been observed yet in nature [13, 62].)

With the help of the complex fields \( \mathbf{F} = \mathbf{E} + i \mathbf{H} \) and \( \mathbf{G} = \mathbf{D} + i \mathbf{B} \), we introduce the following anti-symmetric four-tensor

\[
Q_{\mu \nu} = -Q_{\nu \mu} = \begin{pmatrix}
0 & -G_1 & -G_2 & -G_3 \\
G_1 & 0 & iF_3 & -iF_2 \\
G_2 & -iF_3 & 0 & iF_1 \\
G_3 & iF_2 & -iF_1 & 0
\end{pmatrix}
\]

(2.4)

and use the standard four-vector notation, \( x^\mu = (ct, \mathbf{r}) \) for contravariant coordinates and current, respectively.

Maxwell’s equations then take the covariant form [34, 39]:

\[
\frac{\partial}{\partial x^\tau} Q_{\mu \nu} = -\frac{\partial}{\partial x^\tau} Q_{\nu \mu} = -\frac{4\pi}{c} j^\mu
\]

(2.5)

with summation over two repeated indices. Indeed, in block form, we have

\[
\frac{\partial Q_{\mu \nu}}{\partial x^\tau} = \frac{\partial}{\partial x^\tau} \begin{pmatrix}
0 & -G_q \\
G_p & ie_{pqr} F_r
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\text{div} \mathbf{G} = -4\pi \rho \\
1 \frac{\partial \mathbf{G}}{\partial t} + i \text{curl} \mathbf{F} = -\frac{4\pi}{c} \mathbf{j}
\end{pmatrix},
\]

(2.6)

which verifies this fact. (Here, \( e_{pqr} \) is the totally anti-symmetric Levi-Civita symbol with \( e_{123} = +1 \).)

The continuity equation

\[
0 \equiv \frac{\partial^2 Q_{\mu \nu}}{\partial x^\mu \partial x^\nu} = -\frac{4\pi}{c} \frac{\partial j^\mu}{\partial x^\nu},
\]

(2.7)

or, in the 3D-form

\[
\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0,
\]

(2.8)

describes conservation of the electrical charge.

3. Dual complex field tensors

Two dual anti-symmetric field tensors of the complex fields, \( \mathbf{F} = \mathbf{E} + i \mathbf{H} \) and \( \mathbf{G} = \mathbf{D} + i \mathbf{B} \), are given by

\[
Q_{\mu \nu} = \begin{pmatrix}
0 & -G_1 & -G_2 & -G_3 \\
G_1 & 0 & iF_3 & -iF_2 \\
G_2 & -iF_3 & 0 & iF_1 \\
G_3 & iF_2 & -iF_1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & -D_1 & -D_2 & -D_3 \\
D_1 & 0 & -H_3 & H_2 \\
D_2 & H_3 & 0 & -H_1 \\
D_3 & -H_2 & H_1 & 0
\end{pmatrix} + i \begin{pmatrix}
0 & -B_1 & -B_2 & -B_3 \\
B_1 & 0 & E_3 & -E_2 \\
B_2 & -E_3 & 0 & E_1 \\
B_3 & E_2 & -E_1 & 0
\end{pmatrix},
\]

(3.1)

and

\[
P_{\mu \nu} = \begin{pmatrix}
0 & F_1 & F_2 & F_3 \\
-F_1 & 0 & iG_3 & -iG_2 \\
-F_2 & -iG_3 & 0 & iG_1 \\
-F_3 & iG_2 & -iG_1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & -B_2 & B_1 & 0
\end{pmatrix} + i \begin{pmatrix}
0 & H_1 & H_2 & H_3 \\
-H_1 & 0 & D_3 & -D_2 \\
-H_2 & -D_3 & 0 & D_1 \\
-H_3 & D_2 & -D_1 & 0
\end{pmatrix}.
\]

(3.2)

The real part of the latter represents the standard electromagnetic field tensor in a medium [5, 55, 68]. As for the imaginary part of (3.1), which, ironically, Pauli called an ‘artificiality’ in view of its non-standard behavior under spatial inversion [55], the use of complex conjugation restores this important symmetry for our complex field tensors.
In the complex case under consideration, the dual tensor identities are given by
\[ e_{\mu \nu} P^{\mu \nu} = 2i P_{\mu \nu}, \quad 2i Q_{\mu \nu} = e^{\mu \nu \sigma \tau} P_{\sigma \tau}. \] (3.3)
Here, \( e_{\mu \nu} = -e_{\nu \mu} \) and \( e_{0123} = +1 \) is the Levi-Civita symbol [26]. Then
\[ 6i \frac{\partial Q_{\mu \nu}}{\partial x^\nu} = e_{\mu \nu \sigma \tau} \left( \frac{\partial P_{\sigma \tau}}{\partial x^\nu} + \frac{\partial P_{\sigma \nu}}{\partial x^\tau} + \frac{\partial P_{\tau \nu}}{\partial x^\sigma} \right) \] (3.4)
and both pairs of Maxwell’s equations can also be presented in the form [34]:
\[ \frac{\partial P_{\mu \nu}}{\partial x^\nu} + \frac{\partial P_{\nu \mu}}{\partial x^\nu} + \frac{\partial P_{\mu \sigma}}{\partial x^\nu} = -\frac{4\pi i}{c} e_{\mu \nu \rho \sigma} j^\rho \] (3.5)
in addition to the one given above
\[ \frac{\partial Q_{\mu \nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\nu. \] (3.6)
(The real part of the first equation traditionally represents the first (homogeneous) pair of Maxwell’s equations and the real part of the second one gives the second pair. In our approach both pairs of Maxwell’s equations appear together; see also [5, 13, 14, 39], and [66] for the case in vacuum. Moreover, a generalization to complex-valued four-current may naturally represent free magnetic charge and magnetic current not yet observed in nature [62].)

Another important property is a cofactor matrix identity
\[ P_{\mu \nu} Q^{\mu \nu} = (F \cdot G) \delta^\lambda_\mu, \] (3.7)
which was originally established, in a general form, by Minkowski [48]. Once again, the dual tensors are given by
\[ P_{\mu \nu} = \begin{pmatrix} \frac{F_\mu}{e_{pqr}} G_r & 0 \\ -F_\nu & i e_{pqr} F_r \end{pmatrix}, \quad Q^{\mu \nu} = \begin{pmatrix} 0 & -G_\mu \\ G_\nu & i e_{pqr} F_r \end{pmatrix}, \] (3.8)
in block form. The covariant field energy-momentum tensor in a medium and the corresponding differential balance equation
\[ \frac{\partial}{\partial x^\nu} \left[ \frac{1}{16\pi} \left( P_{\mu \nu} Q^{\mu \nu} + P_{\mu \nu} Q^{\nu \mu} \right) \right] = \frac{1}{32\pi} \left( P_{\sigma \tau} \frac{\partial Q^{\sigma \nu}}{\partial x^\mu} + P_{\sigma \tau} \frac{\partial Q^{\nu \sigma}}{\partial x^\mu} \right) + \frac{j \cdot E}{e/c} + j \times B/c, \] (3.9)
are derived in terms of these tensors in [34]. (In the rest of the article, we will be dealing with the case of vacuum only, when \( G = F \), but it’s convenient to use both vectors in our calculations anyway in order to emphasize where they are coming from.)

4. Transformation laws and generators

Under the Lorentz transformation [15, 48, 50, 58]
\[ U(\Lambda) Q^{\mu \nu} (x') = Q^{\mu \nu} (\Lambda^*_\nu x') = \Lambda^*_\mu \Lambda^*_\nu Q^{\mu \nu} (x'), \] (4.1)
where the summation is assumed over any two repeated indices\(^5\). We shall use the following six \( 4 \times 4 \) matrices \((\alpha, \beta = 0, 1, 2, 3)\) are fixed:
\[ (m^{\mu \eta}) = g^{\mu \beta} \delta^\eta_\beta - g^{\eta \beta} \delta^\mu_\beta, \] (4.2)
for the corresponding one-parameter subgroups of the proper Lorentz group [15, 50, 61] with the standard metric, \( g_{\mu \nu} = g^{\mu \nu} = \text{diag} (1, -1, -1, -1) \), in the Minkowski space-time. The four-angular momentum operators
\[ M^{\alpha \beta} = x^\beta \partial^\alpha - x^\alpha \partial^\beta, \quad \partial^\alpha = g^{\alpha \beta} \partial_\beta, \] (4.3)
can be derived as follows
\[ M^{\alpha \beta} Q^{\mu \nu} = \left[ -\frac{d}{d \theta^{\alpha \beta}} Q^{\mu \nu} \left( \Lambda^*_\nu \left( \theta^{\alpha \beta} x' \right) \right) \right] \bigg|_{\theta^{\alpha \beta} = 0} = (m^{\alpha \beta})^\mu \delta^\nu_\eta + Q^{\mu \nu} (m^{\alpha \beta})_\eta^\nu, \] (4.4)
with
\[ (m^{\alpha \beta})^\mu = -\frac{d \Lambda^*_\nu \left( \theta^{\alpha \beta} \right)}{d \theta^{\alpha \beta}} \bigg|_{\theta^{\alpha \beta} = 0} = g^{\alpha \beta} \delta^\mu_\beta - g^{\eta \beta} \delta^\mu_\eta. \] (4.5)

In matrix form
\[ M^{\alpha \beta} Q = m^{\alpha \beta} Q + Q m^{\alpha \beta}^T, \] (4.6)
where \( Q = Q^{\mu \nu} \) and \( m^T \) is the transposed matrix. The latter equations (4.4)–(4.6) define the action of the infinitesimal operators on the complex field tensor
\[ M^{\alpha \beta} Q^{\mu \nu} = g^{\alpha \alpha} Q^{\mu \nu} - g^{\alpha \beta} Q^{\mu \sigma} + g^{\mu \sigma} Q^{\alpha \beta} - g^{\mu \beta} Q^{\alpha \sigma}, \] (4.7)
in the form that is required in equation (5.1) below.

In a similar fashion, for the products of the generators
\[ M^{\alpha \beta} M^{\gamma \delta} Q^{\mu \nu} = \left( m^{\alpha \beta} \right)^\gamma_\epsilon \left( m^{\gamma \delta} \right)^\epsilon_\mu Q^{\nu \sigma} + \left( m^{\epsilon \gamma} \right)^\nu_\epsilon \left( m^{\gamma \delta} \right)^\epsilon_\mu Q^{\nu \sigma} + \left( m^{\epsilon \gamma} \right)^\nu_\epsilon \left( m^{\epsilon \gamma} \right)^\epsilon_\mu Q^{\nu \sigma}, \] (4.8)
or, in matrix form
\[ M^{\alpha \beta} M^{\gamma \delta} Q = \left( m^{\alpha \beta} m^{\gamma \delta} \right) Q - \left( \left( m^{\gamma \delta} m^{\alpha \beta} \right) Q \right)^T + m^{\alpha \beta} Q \left( m^{\gamma \delta} \right)^T - \left( m^{\gamma \delta} Q \left( m^{\alpha \beta} \right)^T \right). \] (4.9)

\(^5\) Although Minkowski considered the transformation of electric and magnetic fields in a complex 3D vector form, see equations (8), (9) and (15) in [48] (or equations (23.5)–(23.6) in [38]), he seems never to have combined the corresponding four-tensors into the complex forms (3.1)–(3.2). In the second article [9], Max Born, who used Minkowski’s notes, did not mention the complex fields. As a result, the complex field tensor seems only to have appeared, for the first time, in [39] (see also [66]).
As a result
\[
\begin{align*}
\left[ M^{\alpha\beta}, \ M^{\gamma\delta} \right] & = M^{\alpha\delta}M^{\gamma\beta} - M^{\alpha\beta}M^{\gamma\delta} \\
& = g^{\alpha\gamma}M^{\beta\delta} - g^{\alpha\delta}M^{\beta\gamma} + g^{\beta\gamma}M^{\alpha\delta} - g^{\beta\delta}M^{\alpha\gamma},
\end{align*}
\]
which follows from (4.4)–(4.5) and can be verified, once again, by using (4.3).

Finally, introducing the infinitesimal operators 
\[ M = (M^{21}, M^{31}, M^{12}) \] and 
\[ N = (M^{01}, M^{02}, M^{03}) \]
for the rotations and boosts, respectively, one can get
\[
N^2 Q = -M^2 Q = 2Q, \quad (M \cdot N) Q = -2iQ.
\] (4.11)
The Casimir operators of the proper Lorentz group are given by 
\[ (M + iN)^2/4 = 0 \] and 
\[ (M-iN)^2/4 = -2 \] in the space of complex anti-symmetric tensors under consideration. In view of 
\[ M^2 = -s(s+1) = -2, \]
we may say that the spin of the photon is equal to one. (Here, we have chosen real-valued generators; see also [2, 4, 27, 58, 61] and [73] for more details on the Lorentz group representations.)

5. The Pauli–Lubański vector and Maxwell’s equations in vacuum

As follows from the representation theory of the Poincaré group [4, 15] and the geometry of the Minkowski space–time [47, 53], for the case of massless particles, the Pauli–Lubański vector should be collinear to the operator of the four-linear momentum. For a classical electromagnetic field, this relation takes the form
\[
\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \partial^\alpha \left( M^{\mu\nu} Q^{\alpha\beta} \right) = -i \partial_\mu Q^{\mu\beta},
\] (5.1)
and by (4.7), we find that
\[
g^{\alpha\mu} \varepsilon_{\mu\nu\alpha\beta} \partial^\alpha Q^{\nu\beta} - g^{\beta\mu} \varepsilon_{\mu\nu\beta\alpha} \partial^\alpha Q^{\nu\alpha} = -i \partial_\mu Q^{\mu\beta}
\]
(5.2)
\[
(\alpha, \beta = 0, 1, 2, 3 \text{ are fixed; no summation is assumed over these indices}).\]
By a direct, but rather tedious evaluation, one can verify that the latter equation, which is written in terms of a third rank four-tensor, is equivalent to the original system of Maxwell equations in vacuum, \( \partial_\mu Q^{\mu\nu} = 0 \). As a result, the helicity of the photon, or a harmonic circular classical electromagnetic wave, cannot be defined as an undetermined sign, or an extra \pm 1 \text{ factor}, in the right hand side of equation (5.1) as it is stated in standard textbooks on the quantum field theory [4, 15, 59–61]. (This misconception has been one of our main motivations for writing this article.)

In view of (5.1), for the rotations and boosts, 
\[ M = (M^{21}, M^{31}, M^{12}) \] and 
\[ N = (M^{01}, M^{02}, M^{03}) \], respectively, the following standard equations hold
\[
(\nabla \cdot M)Q = i \partial_0 Q, \quad \partial_0 = \frac{1}{c} \partial_t \]
\[
\partial_0 M Q + (\nabla \times N) Q = i \nabla Q,
\] (5.4)
where \( Q = Q^{\mu\nu} = -Q^{\nu\mu} \) is the complex field tensor and the actions of operators \( M \) and \( N \) on this tensor are explicitly defined by (4.7).

\textbf{Note.} In vacuum, when \( G = F \) and \( \rho = 0, \ j = 0 \), two different covariant forms of Maxwell’s equations are given by
\[
\partial_\mu Q^{\mu\nu} = 0, \quad \partial^\nu p_{\mu\nu} = 0,
\] (5.5)
where \( \partial^\nu = g^{\nu\rho} \partial_\rho = g^{\nu\rho} \partial/\partial x^\rho \). The second equation follows from (5.2), when one takes \( \mu = \beta \) and sums over \( \beta = 0, 1, 2, 3 \) with the help of (3.3). As another useful consequence of our equation (5.2), one can directly show that the d’Alembert operator annihilates any component of the complex field tensor in vacuum
\[
\partial^\nu g^{\mu\rho} \partial_\rho Q^{\nu\alpha} = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) Q^{\mu\nu} = \Box Q^{\mu\nu} = 0,
\] (5.6)
thus de-coupling the system. It is worth noting that, in covariant form, our derivation does not require any formula of 3D-vector calculus. (The general theory of relativistic-invariant equations is studied in classical accounts [4, 9, 16, 17, 22, 23, 27, 32, 46, 54, 56]; see also [13, 28, 43–45, 74] and the references therein.)

6. Examples

In a matrix form, equation (5.3) can be rewritten as follows
\[
\begin{pmatrix}
0 & -\partial_2 G_3 + \partial_3 G_2 \\
\partial_2 G_3 - \partial_3 G_2 & 0 \\
-\partial_1 G_3 + \partial_3 G_1 & -i(\partial_1 F_2 - \partial_2 F_1) \\
\partial_1 G_3 - \partial_3 G_1 & -\partial_1 G_3 - \partial_2 G_1 \\
i(\partial_2 F_3 - \partial_3 F_1) & i(\partial_2 F_3 - \partial_3 F_1) \\
0 & i(\partial_2 F_3 - \partial_3 F_1) \\
-\partial_2 F_3 - \partial_3 F_2 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
0 & G_1 & G_1 & 0 & 0 & 0 \\
G_1 & G_1 & iF_3 & -iF_2 & 0 & 0 \\
iF_3 & -iF_2 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
(6.1)
or
\[
\begin{pmatrix}
0 & -\left( \text{curl } G \right)_q \\
\left( \text{curl } G \right)_p & -i e_{pqr} \left( \text{curl } F \right)_r \\
\end{pmatrix}
= \frac{i}{c} \frac{\partial}{\partial t} \begin{pmatrix}
0 & -G_q \\
G_p & i e_{pqr} F_r \\
\end{pmatrix},
\] (6.2)
in a more compact block form. In vacuum, \( G = F \) and this matrix relation implies the single complex Maxwell equation, \( \text{curl } F = (i/c) \partial F/\partial t. \)

---

\( ^6 \) Multiple meanings of the word ‘photon’ are analyzed in [31].
In a similar fashion, for the first component of (5.4), namely, \( \partial_0 M_1 Q + \left( \partial_2 N_3 - \partial_3 N_2 \right) Q = i \partial_0 Q \), we obtain,

\[
\begin{pmatrix}
0 & 0 & G_3 - G_2 \\
0 & i f_2 & i f_3 \\
G_2 & -i f_3 & 0 \\
\end{pmatrix}
\]

\[+ \begin{pmatrix}
0 \\
-i (\partial_2 F_2 + \partial_3 F_3) & 0 \\
i \partial_2 F_1 & \partial_3 G_1 \\
i \partial_2 F_1 & -\partial_3 G_1 \\
G_2 & -i f_3 & 0 \\
G_3 & i f_2 & -i f_1 \\
\end{pmatrix}
\]

(6.3)

Once again, in vacuum, \( \mathbf{G} = \mathbf{F} \) and this matrix relation is satisfied in view of the pair of complex Maxwell equations, \( \text{curl } \mathbf{F} = (i/c) \partial \mathbf{F} / \partial t \) and \( \partial \mathbf{F} = 0 \). (Cyclic permutations of the spatial indices cover the two remaining components.)

One can clearly see that there is no chance of changing the sign \( + \) into \( - \) in the right hand side without a violation of Maxwell’s equations. Indeed, let us pick just one of the matrix elements from both sides, say, \( \partial_2 F_2 + \partial_3 F_3 = -\partial_1 G_1 \), which indicates also that the left and right hand sides are coming from the different pairs of Maxwell’s equations (2.2)–(2.3).

Note. In the case of Weyl’s two-component wave equation for massless neutrinos, one can choose

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2 \]

and, in block form

\[ M^{\phi\phi} = -M^{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & \pm i \sigma_3 \\ \pm i \sigma_2 & e_{pq} \sigma_q \end{pmatrix} \]

(6.5)

where \( \sigma_1, \sigma_2, \sigma_3 \) are the standard 2 \( \times \) 2 Pauli matrices with the products given by \( \sigma_p \sigma_q = i e_{pq} \sigma_3 + \delta_{pq} \) (see, for example, [61]). As a result, equations (1.3) are satisfied provided that

\[ \partial_0 \phi = \pm \mathbf{\sigma} \cdot \mathbf{V} \phi, \quad \lambda = \pm \frac{1}{2}. \]

(6.6)

respectively (details are left to the reader). Thus, the relativistic Weyl equation for a massless particle with the spin 1/2 can be derived from the representation theory of the Poincaré group.

### 7. Helicity

In particle physics [8, 58, 61, 69], the helicity is defined as the projection of the angular momentum \( \mathbf{M} \) on the direction of motion \( \mathbf{p} \):

\[ \lambda = \frac{\mathbf{p} \cdot \mathbf{M}}{||\mathbf{p}||} = -\frac{\mathbf{p}_0}{||\mathbf{p}||} \]

(7.1)

The helicity states are eigenstates of the operator:

\[ \lambda \langle \mathbf{p}, \lambda \rangle = \frac{\mathbf{p} \cdot \mathbf{M}}{||\mathbf{p}||} \langle \mathbf{p}, \lambda \rangle = \lambda \langle \mathbf{p}, \lambda \rangle . \]

(7.2)

For massless particles one can define the spin as \( s = |\lambda| \) and, if the parity is conserved, the particle will have only two independent helicity eigenstates \( \langle \mathbf{p}, \lambda = s \rangle \) and \( \langle \mathbf{p}, \lambda = -s \rangle \).

In the case of the classical electromagnetic field, equations (5.3) and/or (6.2) show that the helicity operator is proportional to the ‘energy operator’:

\[ \lambda = \frac{i}{c} \frac{\partial}{\partial t}. \]

(7.3)

As a result, these two operators have common eigenstates, \( \langle \mathbf{k}, \lambda \rangle = Q^\lambda, \) in the space of complex anti-symmetric four-tensors of the second rank. (The simplest covariant helicity states will be constructed in the next section.)

On the other hand, in 3D-complex electrodynamics, one can take the complex vector field \( \langle \mathbf{k}, \lambda \rangle = \mathbf{E} + i \mathbf{H} \) and choose the following real-valued spin matrices [71]:

\[
s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0, \\ -1 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(7.4)

such that the standard relations hold \( [s_p, s_q] = s_p s_q - s_q s_p = e_{pq} s_r \) and \( s_1^2 + s_2^2 + s_3^2 = -2. \) Then

\[ (\nabla \cdot \mathbf{s}) \mathbf{F} = \partial_1 (s_1 \mathbf{F}) + \partial_2 (s_2 \mathbf{F}) + \partial_3 (s_3 \mathbf{F}) \]

\[ = \partial_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + \partial_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \]

\[+ \partial_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \text{curl } \mathbf{F}. \]

(7.5)

Once again, our representation (7.3) for the helicity operator holds in view of the Maxwell equation in vacuum, \( \text{curl } \mathbf{F} = (i/c) \partial \mathbf{F} / \partial t. \)
Note. In view of (7.3), the traditional definition of helicity (7.2) is related to separation of variables in Maxwell’s equations. Letting \( Q^{\mu\nu} = q(t)Z^{\mu\nu}(r) \), one gets
\[
0 = \frac{\partial Q^{\mu\nu}}{\partial x^\nu} = \frac{1}{c} \frac{\partial Q^{\mu\nu}}{\partial t} + \frac{\partial Q^{\mu\nu}}{\partial x_\rho} \frac{\partial Z^{\mu\nu}(r)}{\partial x_\rho},
\]
(7.6)
or
\[
-\frac{q}{cq} Z^{\mu\nu} = \frac{\partial Z^{\mu\nu}(r)}{\partial x_\rho}, \quad q = e^{-i\omega t},
\]
(7.7)
where \( \omega \) must be a real-valued constant of the separation of variables in order to have bounded solutions. As a result
\[
\frac{\partial Z^{\mu\nu}}{\partial x_\rho} = \frac{\omega}{c} Z^{\mu\nu}(r),
\]
(7.8)
thus giving a covariant form of the corresponding eigenvalue problems in different curvilinear coordinates [67, 71].

8. Covariant harmonic wave solutions

In vacuum, \( \partial_\nu Q^{\mu\nu} = 0 \), where
\[
Q^{\mu\nu} = \begin{pmatrix} 0 & -F_{\nu} \\ F_{\nu} & i e_{\rho\sigma} F_{\sigma} \end{pmatrix}, \quad F = fe^{i(\omega t - k \cdot r)} = E + iH.
\]
(8.1)
Here, \( f \) = constant is a complex polarization vector to be determined and
\[
x^\mu = (ct, r), \quad k_\mu = (\omega/c, -k), \quad kx = k_\mu x^\mu = \omega t - k \cdot r.
\]
(8.2)
In a compact form, \( Q^{\mu\nu} = A^{\mu\nu} e^{ikx} \) and \( A^{\mu\nu} k_\nu k_\rho = 0, \) where
\[
A^{\mu\nu} = \begin{pmatrix} 0 & -f_\nu \\ f_\nu & i e_{\rho\sigma} f_\sigma \end{pmatrix} = \text{constant}.
\]
(8.3)
This tensor is an eigenfunction of the four-gradient, \( i^{-1} \partial_\nu Q^{\mu\nu} = k_\nu Q^{\mu\nu}. \)

As a result
\[
\begin{pmatrix} 0 & -f_1 & -f_2 & -f_3 \\ f_1 & 0 & i f_3 & -i f_2 \\ f_2 & -i f_3 & 0 & -k_1 \\ f_3 & i f_2 & -i f_1 & 0 \end{pmatrix} \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
(8.4)
and \( \det A = -(f \cdot f) = 0 \) (Lorentz invariant by Minkowski [48]). The complex invariant, \( B^2 = (E + iH)^2 = 0, \) results in \( E^2 = H^2 \) and \( E \cdot H = 0, \) as required [38].

In 3D-form, the latter system of linear equations gives an eigenvalue problem:
\[
if \times f = \frac{\omega}{c} f, \quad f \cdot f = 0.
\]
(8.5)
The eigenvalues are
\[
-\omega/c, \quad -ik_1, \quad ik_2
\]
\[
= \frac{-\omega}{c} \left(k_1^2 + k_2^2 + k_3^2 - \frac{\omega^2}{c^2}\right) = 0.
\]
(8.6)
The case \( \omega = 0, \) when \( f = k, \) does not satisfy the second condition \( f^2 = 0 \) unless \( k = 0. \)

Therefore, there are only two eigenvectors \( \{f, f^*\}, \) corresponding to \( \omega/c = \pm k = \pm \sqrt{k_1^2 + k_2^2 + k_3^2}. \)
\[
f = k \times (l \times k) + ik(k \times l),
\]
\[
f^* = f_{k \rightarrow -k},
\]
(8.7)
respectively [14]. Here, \( l \) is an arbitrary real vector that is not collinear to \( k (k \neq \text{constant} \ L) \) and \( f \cdot f^* = 1. \) (A similar eigenvalue problem occurs in the mean magnetic field generation, called \( \alpha \Omega - \text{dynamo, in cosmic astrophysics [24].} \))

Example. Let \( \{e_1, e_2\} \) be an orthonormal basis in \( R^3. \) One can choose \( l = e_1 \) and \( k = ke_3. \) Then
\[
f = e_1 + ie_2, \quad f^* = \frac{e_1 - ie_2}{\sqrt{2}}
\]
(8.8)
(see [14, 38], and [67] for more details).

9. Discrete transformations and polarization

The complex Maxwell equations in vacuum
\[
\frac{i}{c} \frac{\partial F}{\partial t} = \text{curl } F, \quad \text{div } F = 0,
\]
(9.1)
are invariant under the following discrete transformations: spatial inversion \( P: F(r,t) \rightarrow F^*(r,-t); \) time reversal \( T: F(r,t) \rightarrow F^*(r,-t); \) and space–time inversion \( PT: F(r,t) \rightarrow F(-r,-t). \) They, together with the identity transformation, correspond to the four connected components of the Poincaré group. (These transformations form the so-called Klein group \{Identity, P, T, PT, PT\}.)

The action of this group generates the following four circularly polarized waves \( (\omega = +c k)\)
\[
F_1 = fe^{i(kr - \omega t)},
\]
\[
F_2 = f^* e^{-i(kr + \omega t)} = F_1^* \big|_{t \rightarrow -t} = TF_1
\]
(9.2)
and
\[
F_3 = f e^{i(kr - \omega t)} = F_1 \big|_{t \rightarrow -t, r \rightarrow -r} = (PT)F_1,
\]
\[
F_4 = f^* e^{-i(kr + \omega t)} = F_3^* \big|_{t \rightarrow -t} = TF_3 = PF_1.
\]
(9.3)
They represent right- and left-handed circularly polarized waves moving along the vector \( k \) in opposite directions. One can easily verify that the solutions \( \{F_1, F_2\} \) correspond to
\[ \lambda = +1 \text{ and } \{ F_1, F_2 \} \text{ have } \lambda = -1. \] Also, \( F_1 \cdot F_1 = F_2^2 = 0 \) and \( F_2 \cdot F_3 = (F^2)_w = 0. \) These four solutions are linearly independent.

**Example.** The standard circular, elliptic, and linear polarizations of the classical electromagnetic waves occur as a result of superposition of the complex solutions under consideration. With the help of the polarization vectors (8.8), one gets

\[
F = \frac{F_1 + F_3}{2} + \frac{c_2 - F_1 - F_3}{2} = E + iH \\
= c_1 e_1 \cos (k \cdot r - \omega t) - c_2 e_2 \sin (k \cdot r - \omega t) \\
+ i \left[ c_2 e_1 \cos (k \cdot r - \omega t - \frac{\pi}{2}) - c_1 e_2 \sin (k \cdot r - \omega t - \frac{\pi}{2}) \right] \\
(9.4)
\]

where \( c_1 = c_1^* \) and \( c_2 = c_2^* \). For the elliptic polarization, we choose \( |c_1| > |c_2| \text{ or } |c_1| < |c_2| \); the linear polarization arises, for instance, if \( c_1 \neq 0 \) and \( c_2 = 0 \) (see [38] and [67], problems 2.128–2.134, for more details).

In conclusion, it is worth noting that, here, we have only discussed the classical electromagnetic field in vacuum. Different aspects of the ‘photon paradigm’ are emphasized in [31]. The photon wave functions are dealt with in [1, 8, 10, 11, 21, 25, 61]. For quantization in the complex form, see [10, 12–14] and the references therein. (General quantization procedures are discussed, for example, in [13, 15, 30, 33, 35, 51, 52, 59–61, 63, 67].) Coherent states of light and dynamical invariants are reviewed in [18–20, 30, 63]. The squeezed states of the photons and atoms in a cavity and their relations with so-called ‘missing’ solutions for the harmonic oscillator are analyzed in [36, 41, 42]. Professor Toptygin kindly pointed out an intrinsic importance of the helicity concept from the sub-atomic world (parity violation in beta decay [57, 76]) to cosmic astrophysics (possible amplification of galactic magnetic fields by the turbulent dynamo mechanism [7, 24, 70]). Last but not least, organic compounds appear often in the form of only one of two stereoisomers. As a result, in optically active biological substances, these molecules rotate polarized light to the left [29], thus creating another old unexplained puzzle.

**Acknowledgments**

This research was partially supported by NSF grant DMS #1535822. We are grateful to Prof Dr Patric Muggli for his hospitality at Max-Planck-Institut für Physik, Werner-Heisenberg-Institute in Munich; March 2015. We are indebted to Prof Dr Gerald A Goldin, Prof Dr John Klauder, Prof Dr Margarita A Man’ko, Prof Dr Vladimir I Man’ko, Prof Dr Svetlana Roudenko, and Prof Dr Igor N Toptygin for their valuable comments and encouragement. The third-named author was partially supported by the AFOSR grant FA9550-11-1-0220. Comments from one of the referees, which have helped to improve the presentation, are appreciated.

**References**

[1] Akhiezer A and Berestetskii V B 1965 *Quantum Electrodynamics* (New York: Interscience)

[2] Bargmann V 1947 Irreducible unitary representations of the Lorentz group Ann. Math. 48 568–640

[3] Bargmann V 1954 On unitary ray representations of continuous groups Ann. Math. 59 1–46

[4] Bargmann V and Wigner E 1948 Group theoretical discussion of relativistic wave equations Proc. Natl Acad. Sci. USA 34 211–23

[5] Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover)

[6] Bateman H 1915 *The Mathematical Analysis of Electrical and Optical Wave-Motion* (Cambridge: Cambridge University Press) p 4

[7] Beck R 2009 Galactic and extragalactic magnetic fields—a concise review Astrophys. Space Sci. Trans. 5 43–47

[8] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1971 *Relativistic Quantum Theory* (Oxford, New York: Pergamon)

[9] Bhabha H J 1945 Relativistic wave equations for the elementary particles Rev. Mod. Phys. 17 200–16

[10] Bialynicki-Birula I 1994 On the wave function of the photon *Acta Phys. Pol.* A 86 97–116

[11] Bialynicki-Birula I 1996 Photon wave function *Progress in Optics* vol 36 ed E Wolf (Amsterdam: Elsevier) pp 245–94

[12] Bialynicki-Birula I 2006 Photon as a quantum particle *Acta Phys. Pol.* B 37 935–46

[13] Bialynicki-Birula I and Bialynicki-Birula Z 1975 *Quantum Electrodynamics* (Oxford, New York: Pergamon, PWN–Polish Scientific Publishers)

[14] Bialynicki-Birula I and Bialynicki-Birula Z 2013 The role of Riemann–Silberstein vector in classical and quantum theories of electromagnetism *J. Phys. B: At. Mol. Opt. Phys.* 46 053001

[15] Bogolubov N N, Logunov A A, Oksak A I and Todorov I T 1990 *General Principles of Quantum Field Theory* (Dordrecht: Kluwer)

[16] Dirac P A M 1928 The quantum theory of the electron *Proc. R. Soc. London A* 117 610–24

[17] Dirac P A M 1936 Relativistic wave equations *Proc. R. Soc. London A* 155 447–59

[18] Dodonov V V, Malkin I A and Man’ko V I 1975 Integrals of motion, Green functions, and coherent states of dynamical systems *Int. J. Theor. Phys.* 14 37–54

[19] Dodonov V V and Man’ko V I 1987 Invariants and correlated states of nonstationary quantum systems *Invariants and the Evolution of Nonstationary Quantum Systems (Proc. of Lebedev Physics Institute vol 183)* (Commack: Nova Science) pp 103–261 (Engl. transl. 1989)

[20] Dodonov V V and Man’ko V I (ed) 2003 ‘Nonclassical’ states in quantum optics: brief historical review *Theory of Nonclassical States of Light* (London: Taylor and Francis)

[21] Dutra S M 2005 *Cavity Quantum Electrodynamics: The Strange Theory of Light in a Box* (Hoboken, NJ: Wiley)

[22] Fierz M 1939 Über die relativistische theorie kräftefreier teilchen mit beliebigem spin *Helv. Phys. Acta* 12 3–37

[23] Fierz M 1940 Über den dreihimpuls von teilchen mit ruhmasse null und beliebigem spin *Helv. Phys. Acta* 13 45–60

[24] Fleishman G D and Toptygin I N 2013 *Cosmic Electrodynamics: Electrodynamics and Magnetic Hydrodynamics of Cosmic Plasmas* (New York: Springer)

[25] Fock V 1930 La mécanique des photons *Compt. Rend.* 190 1399–401
Fock V A 2004 The mechanics of photons Selected Works: Quantum Mechanics and Quantum Field Theory ed L D Faddeev, L A Khalfin and I V Komarov (London/Boca Raton, FL: Chapman and Hall/CRC) pp 183–5 (Engl. transl.)
[26] Fock V A 1965 The Theory of Space Time and Gravitation (New York: Pergamon)
[27] Gel’fand I M, Minlos R A and Ya Shapiro Z 1963 Representations of the Rotation and Lorentz Groups and Their Applications (New York: Pergamon)
[28] Goldin G A and Shitelen V M 2001 On Galilean invariance and nonlinearity in electrodynamics and quantum mechanics Phys. Lett. A 279 331–26
[29] Hecht E 2002 Optics 4th edn (San Francisco, CA: Addison-Wesley)
[30] Klauder J R and Sudarshan E C G 1968 Fundamentals of Quantum Optics (New York: Benjamin)
[31] Klyshko D N 1994 Quantum optics: quantum, classical, and metaphysical aspects Phys.-Usp. 37 1097–123
[32] Kramers H A, Belinfante F J and Luba
[33] Kryuchkov S I, Lanfear N A and Suslov S K 2015 Complex field quantization Int. J. Theor. Phys. 52 4445–60
[34] Kryuchkov S I, Mahalov A and Suslov S K 2013 On the problem of electromagnetic-field quantization J. Math. Phys. 54 013508
[35] Kryuchkov S I, Lanfear N A and Suslov S K 2015 Complex form of classical and quantum electrodynamics, in preparation.
[36] Kryuchkov S I, Suazo E and Suslov S K 2014 On photon statistics in variable media arXiv:1401.2924v2
[37] Kryuchkov S I, Suslov S K and Vega-Guzmán J M 2013 The minimum-uncertainty squeezed states for atoms and photons in a cavity J. Phys. B: At. Mol. Opt. Phys. 46 104007 (IOP Select and Highlight of 2013)
[38] Kuznetsov G I, Liberman M A, Makarov A A and Smorodinskii Ya A 2006 Selected Works 2nd edn (Moscow: YPCC, Classics of Science) pp 300–11 (in Russian)
[39] Landau L D and Lifshitz E M 1975 The Classical Theory of Fields 4th edn (Oxford: Butterworth-Heinemann)
[40] Laporte O and Uhlenbeck G E 1931 Applications of spinor analysis to the Maxwell and Dirac equations Phys. Rev. 37 1380–97
[41] Lomont J S and Moses H E 1962 Simple realizations of the infinitesimal generators of the proper orthochronous inhomogeneous Lorentz group for mass zero J. Math. Phys. 3 405–8
[42] López R M, Suslov S K and Vega-Guzmán J M 2013 Reconstructing the Schrödinger groups Phys. Scr. 87 038118
[43] López R M, Suslov S K and Vega-Guzmán J M 2013 On a hidden symmetry of quantum harmonic oscillators J. Diff. Equ. 19 543–54
[44] Lubański J K 1941 Sur le spin des particules élémentaires Physica 8 44–52
[45] Lubański J K 1942 Sur la théorie des particules élémentaires de spin quelconque Physica B 9 325–38
[46] Majoran E 1993 Helicity and unitary representations of the Lorentz group Fundamentale Probleme in Quantum Theory (New York: Academic Press) pp 1–69 (Engl. transl. 1992)
[47] Minkowski H 1909 Raum und Zeit Physikalische Zeitschrift 110 104–11; Jahresbericht der Deutschen Mathematiker-Vereinigung pp 75–88
[48] Minkowski H 1908 Die grundlagen für die elektromagnetischen vorgänge in bewegten körpern Nachr. König. Ges. Wiss. Göttingen, math.-phys. Kl. 53–111; The fundamental equations for electromagnetic processes in moving bodies The Principle of Relativity ed M Sahar (Calkutta: University Press) pp 1–69 (Engl. transl. 1920)
[49] Minkowski H 1909 Eine ableitung der grundgleichungen für die elektromagnetischen vorgängen in bewegten körpern vom Standpunkt der elektronentheorie Math. Ann. 68 526–56
[50] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[51] Moskalev A N 2006 Relativistic Field Theory (St. Petersburg: PIYaF RAN) (in Russian)
[52] Mukhanov V and Winitzki S 2007 Introduction to Quantum Effects in Gravity (Cambridge: Cambridge University Press)
[53] Naber G L 2012 The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity 2nd edn (Berlin: Springer)
[54] Pauli W 1941 Relativistic field theories of elementary particles Rev. Mod. Phys. 13 203–32
[55] Pauli W 1938 Theory of Relativity (Oxford: Pergamon)
[56] Proca A 1936 Sur la théorie ondulatoire des électrons positifs et négatifs J. Phys. Radioïum 7 547–53
[57] Roy A 2001 Discovery of parity violation: breakdown of a symmetry principle Resonance 6 32–43
[58] Rumer Yu B and Fet A I 1977 Group Theory and Quantized Fields (Moscow: Nauka) (in Russian)
[59] Ryder L H 1996 Quantum Field Theory 2nd edn (Cambridge: Cambridge University Press)
[60] Scheck F 2007 Quantum Physics (Berlin: Springer)
[61] Schweber S S 1961 An Introduction to Relativistic Quantum Field Theory (Evaston: Row, Peteron and Company)
[62] Schwinger J, DeRood L L Jr, Milton K A and Wu-y Tsai 1998 Classical Electrodynamics (Reading, MA: Perseus Books)
[63] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[64] Silberstein L 1907 Elektromagnetische grundgleichungen in bivectorieller behandlung Ann. Phys. 327 579–86
[65] Silberstein L 1907 Nachtrag zur abhandlung über ‘elektromagnetische grundgleichungen in bivectorieller behandlung’ Ann. Phys. 329 783–4
[66] Taylor N W 1952 A simplified form of the relativistic electromagnetic equations Aust. J. Sci. Res. A 5 423–9
[67] Toptygin I N 2014 The Principle of Relativity ed M Saha (Calcutta: University Press) pp 1–69 (Engl. transl. 1920)
[68] Toptygin I N 2015 The Principle of Relativity ed M Saha (Calcutta: University Press) pp 1–69 (Engl. transl. 1920)
[69] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[70] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[71] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[72] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[73] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[74] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[75] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[76] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[77] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[78] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[79] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[80] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[81] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[82] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[83] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[84] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[85] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[86] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[87] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[88] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[89] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)
[90] Vallée J P 2004 Cosmic magnetic effects in Gravity (Cambridge: Cambridge University Press)