Abstract—This article proposes a composite adaptive control architecture using dual adaptation scheme for dynamical systems comprising time-varying uncertain parameters. While majority of the adaptive control schemes in literature address the case of constant parameters, recent research has conceptualized improved adaptive control techniques for time-varying systems with rigorous stability proofs. The proposed work is an effort toward a similar direction, where a novel dual adaptation mechanism is introduced to efficiently tackle the time-varying nature of the parameters. Projection and σ-modification algorithms are strategically combined using congelation of variables to claim a global result for the tracking error space. While the classical adaptive systems demand a restrictive condition of persistence of excitation (PE) for accurate parameter estimation, the proposed work relies on a milder condition, called initial excitation (IE) for the same. A rigorous Lyapunov stability analysis is carried out to establish boundedness of the closed-loop system with a tighter ultimate bound compared to existing results. Further, it is analytically shown that the proposed work can recover the performance of previously designed IE-based adaptive controller in case of time invariant systems.

Index Terms—Adaptive systems, composite adaptive control, initial excitation (IE), persistence of excitation (PE), time-varying system.

I. INTRODUCTION

Adaptive control is a powerful nonlinear dynamic control technique, which can tackle parameteric uncertainty in real-time [1], [2]. Adaptive controllers ensure closed-loop stability of the extended error dynamics involving both tracking error and parameter estimation error. Majority of the developments in adaptive control literature consider constant unknown parameters to establish well-behaved closed-loop error dynamics. Compared to the mammoth parameter estimation literature for constant parameters, the time-varying parameter estimation literature shies away. The stability analysis for the case of unknown time-varying parameters is challenging due to the appearance of an undesirable parameter derivative term in the Lyapunov analysis (for details see [3, Introduction]). One approach for bounding the undesirable term is to use robust adaptive control techniques like σ-modification (σ-mod) [4], projection [5] etc., especially for slow-varying parameters. However, naively using these methods results in poor performance. Recent literature has provided promising results in adaptive control for time-varying systems [3], [6], where robust damping and/or sliding-mode like mechanisms are strategically utilized to ensure improved performance. Lastly, the work in [7] also deals with time-varying parameter, while using barrier Lyapunov function to invoke safety bounds on the tracking error.

Most of the aforementioned techniques have only proved tracking error convergence by invoking Barbalat’s lemma, while parameter estimation error convergence requires the restrictive persistence of excitation (PE) condition on the state/reference input [1]. It has been well-established in literature that the PE condition has difficulty to verify and/or satisfy in practical problems. Composite adaptive control techniques [8], [9] provide a way to improve the parameter estimation algorithm by incorporating prediction error (partial information about parameter estimation error) in addition to tracking error. However, these techniques still require the PE condition for parameter convergence. Research efforts are made in recent past to relax the PE condition in various ways, such as data-driven [10], [11], [12] and filter-based [13], [14] methods. The works in [14], [15], [16], and [17] have devised a condition, called initial excitation (IE), which is shown to be sufficient for parameter convergence using a two-tier filter-based adaptive controller. The IE condition is milder than the classical PE condition since it requires the excitation to sustain only in initial time-window as compared to PE demanding the excitation to sustain for all time.

Unlike the aforementioned IE-based algorithms, which are proven to be efficient for systems with constant unknown parameters, the work in [18] has devised a novel adaptive controller for time-varying systems while ensuring parameter convergence (to an ultimate-bound) under IE condition. However, this algorithm cannot ensure restoring the asymptotic tracking performance if the time-varying parameter becomes constant.

Taking inspiration from the recent works [6], [19], this article utilizes the concept of congelation of variables, which splits the unknown parameter into a nominal component (constant) and a perturbation component (time-varying). A composite adaptive control architecture for time-varying systems is developed using a dual adaptation scheme. The dual adaptation mechanism comprises 1) a primary estimator for estimating the total unknown parameter and 2) a secondary estimator dedicated for the constant nominal component as shown in Fig. 1 red block for primary and blue block for secondary estimator. Projection and σ-mod algorithms are strategically combined in the adaptation mechanism to

Fig. 1. Block diagram for proposed algorithm where the novelty is shown in dotted black lines labeled Dual Parameter Estimators. The red block shows parameter estimator and blue block shows nominal parameter estimator. Red and blue lines show input to the parameter estimator and nominal parameter estimator blocks, respectively.
claim a global result for the tracking error space. The secondary parameter estimator utilizes the notion of IE condition for efficient learning of the nominal component of the parameter. A rigorous Lyapunov stability analysis is carried out to establish uniformly ultimately bounded (UUB) stability of the closed-loop error dynamics. It is analytically proved that the proposed algorithm can recover the performance of previously designed IE-based adaptive controllers [14, 16] in case the plant has constant parameters or time-variation smoothly decays with time.

Summary of Contribution:
1) A novel dual adaptation mechanism utilizing congelation of variables for uncertain time-varying systems.
2) Strategically combining σ-mod and projection to ensure global tracking (in the tracking error space), unlike a semiglobal result in [20], and without encountering forgetting behavior as typical in σ-mod.
3) Extension of the IE-based adaptation strategy for time-varying system, while ensuring UUB result with the ultimate-bound expression being proportional to parameter perturbation and rate of change of parameter perturbation, unlike σ-mod or projection, which depend on norm of true parameter. This leads to attaining performance recovery for the time-invariant case, unlike [18].

II. PRELIMINARIES

Some of the notations and definitions used throughout the article are stated. For a vector $a$, $||a||$ denotes the Euclidean norm. For a $n \times n$ matrix $A$, $||A||_F$ denotes the Frobenius norm and $Tr(A)$ denotes the trace. $I_n$, $O_n \in \mathbb{R}^{n \times n}$ are column vectors having all entries as 1 and 0, respectively; $I_n$ is the identity matrix of dimension $n \times n$. $O_{n \times 1}, I_{n \times n} \in \mathbb{R}^{n \times m}$ are matrices with all entries as 0 and 1, respectively. $\Sigma_n^{+}$ is the set of symmetric positive-definite matrices of size $n \times n$. The operator vec maps matrix to vector: $\mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{nm}$ by stacking the columns.

III. SYSTEM DESCRIPTION

Consider the following dynamical system from [21]:

$$\dot{x}(t) = Ax(t) + B(u(t) + W^T(t)\phi(x(t)))$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the system matrices, $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $W(t) \in \mathbb{R}^{n_w \times n_u}$ is the unknown time-varying parameter, and $\phi(x(t)) \in \mathbb{R}^{n_w}$ is a known regressor and function of the state.

To characterize the desired response a reference model is designed below:

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t)$$

where $x_m(t) \in \mathbb{R}^{n}$ is the reference model state, $A_m \in \mathbb{R}^{n \times n}$ and $B_m \in \mathbb{R}^{n \times m}$ are reference model matrices. The matrix $A_m$ is designed to be Hurwitz to ensure bounded-input-bounded-output stability of (2) with respect to the piecewise-continuous external reference input $r(t) \in \mathbb{R}^n$.

IV. CONTROL OBJECTIVE AND ASSUMPTIONS

The objective is to design a control law $u(t)$ and parameter update law $\dot{W}(t)$ such that the closed-loop error dynamics including tracking error $\epsilon(t) \equiv x(t) - x_m(t)$ and parameter estimation error $\dot{W}(t) \equiv W(t) - W(t)$ remain UUB.

The following assumptions are made to facilitate the design.

Assumption I: System matrix $A$ and $B$ have the following matching conditions [1], [22]: $A = A_m - BK_z$ and $B = BK_y$, where $K_z \in \mathbb{R}^{n \times n_u}$ and $K_y \in \mathbb{R}^{n \times n_w}$ are called controller parameters.

The above is a standard assumption in MRAC literature [1], which ensures structural similarity between the plant and the reference model. Further based on congelation of variables method in [19], the time-varying parameter $W(t)$ can be decomposed as:

$$W(t) = W^* + \delta W(t)$$

where $W^*$ is the constant nominal component of the parameter and $\delta W(t)$ is the parameteric perturbation component (deviation around the nominal).

Assumption 2: The time-varying parameter can be norm-bounded by constants as $||W^*|| \leq \overline{W}$, $||\delta W(t)||_F \leq \delta_W$, and $||\delta W(t)||_F \leq \delta_W$.  

V. ADAPTIVE CONTROLLER DESIGN

A. Control Input

The control input is designed as follows:

$$u(t) = K_x^T x(t) + K_y^T r(t) - \frac{\dot{W}(t)\phi(x(t))}{\mu_0(t)}$$

where $\mu_0(t)$ is the adaptive component of the controller. We assume known $K_x$ and $K_y$, as estimation and convergence of these parameters are not the focus of current article and can be handled along the lines of [14]. We thus emphasize only on the estimation of the unknown time-varying parameter ($W(t)$).

Using (1), (2), (4), and Assumption 1, the closed-loop error dynamics is given as follows:

$$\dot{\epsilon}(t) = A_m \epsilon(t) - B \dot{W}(t)\phi(x(t)).$$

The parameter update law is subsequently proposed using a dual adaptation mechanism as follows:

1) A primary estimate ($\hat{W}(t) \in \mathbb{R}^{n_w \times n_u}$) for the total time varying parameter ($W(t)$);
2) A secondary estimate ($\hat{W}^*(t) \in \mathbb{R}^{n_w \times n_u}$) for the nominal component of the parameter ($W^*$).

B. Primary Parameter Estimator $\hat{W}(t)$

The parameter estimate for $W(t)$ is designed using gamma-projection operation [5, Def. 7 and 11], which is denoted as $\text{Proj}_{W}(W(t), y(t), f(t)) = [\text{Proj}_{W}(W(t), y(t), f_i(t)) \forall i \in \{1, \ldots, n_u\}]$, where

$$f_i(t) \equiv f(W_i(t)) = \frac{\dot{W}(t)W_i(t) - \alpha^2}{2\alpha + e^2}$$

$$y(t) = \phi(x(t))e^T(t)PB - \sigma(W - W^*)$$

Here, $f_i(t) \in \mathbb{R}$ is a convex function and $f(t) = [f_1(t), \ldots, f_{n_u}(t)]$. $W_i(t)$ is $i$th column of $W(t)$, $\alpha^2 \equiv \overline{W} + \delta_W$ is a known upper-bound of $W(t)$ from Assumption 2, and $e \in \mathbb{R}_{>0}$ is a controllability parameter ensuring Lipschitz continuity. $\Gamma_W \in \mathbb{S}_{n_u}^+$ is the adaptation gain matrix and $\sigma \in \mathbb{S}_{n_u}$ is a scalar tune similar to $\sigma$-mod [4]; $y(t) \in \mathbb{R}^{n_w \times n_u}$, $y_i(t) \in \mathbb{R}^{n_w}$ is $i$th column of $y(t)$ and $P \in \mathbb{R}^{n \times n}$ is a positive definite solution of the Lyapunov equation.

$$A_m^T P + P A_m = -Q_m$$

where $Q_m \in \mathbb{S}_n^+$ is a chosen positive definite matrix. Further $\hat{W}^*(t)$ is the nominal parameter estimate designed subsequently in Section V-C.

1Note that ($\overline{W} + \delta_W$) is required to be known as revealed subsequently.
Hence
\[
\tilde{W}_r(t) = \begin{cases} \\
\Gamma y_i(t) - \Gamma \left(\frac{\Upsilon_l}{\Delta f_i(t)} \Gamma y_i(t) \right)^T y_i(t) \\
\Gamma y_i(t) 
\end{cases}
\]
(9a)
\[
\tilde{W}_r(t) = \text{Proj}_r(\tilde{W}_r(t), y_i(t), f_i(t))
\]
(9b)
where, (9a) occurs if \( f_i(t) > 0 \wedge y_i^2(t) \Gamma (\nabla f_i(t)) > 0 \), otherwise (9b) occurs.

Remark 1: The design in (7) is a novel concept, where the second term includes \( \sigma \)-mod, which pulls the primary estimate \( \hat{W} \) toward the secondary estimate \( \hat{W}^* \). Hence, this update law can obviate the drawback of unlearning that exists in traditional \( \sigma \)-mod, provided that the secondary estimate approaches the nominal parameter \( W^* \).

C. Secondary Parameter Estimator \( \hat{W}^* \)

The design of update law for \( \hat{W}^* \) is another novel contribution of the article. The design is motivated from the recent literature on IE-based adaptive control [14, 16], which builds on two-tier filter architecture while ensuring parameter convergence without restrictive PE condition on the regressor. The two-tier filtering scheme is subsequently adopted for the secondary estimate while suitably modifying the scheme in the context of time-varying parameter setting.

The secondary parameter estimator error is defined as follows:
\[
\hat{W}^*(t) = \hat{W}^* - W^*.
\]
(11)

1) First-Layer Filtering: Exploiting the idea of congelation of variables [19], we define a series of filters to extract information about the nominal component of the unknown parameter as follows:
\[
\dot{g}(t) = -p_f g(t) + \dot{e}(t), \quad g(t_0) = \mathbf{0}_n
\]
(12)
\[
\dot{e}(t) = -p_f e(t) + e(t), \quad e(t_0) = \mathbf{0}_n
\]
(13)
\[
\dot{u}_f(t) = -p_f u_f(t) + u_{af}(t), \quad u_f(t_0) = \mathbf{0}_{n_u}
\]
(14)
\[
\dot{f}(t) = -p_f \dot{f}(t) + \phi(x(t)), \quad \dot{f}(x(t_0)) = \mathbf{0}_{n_w}
\]
(15)
where \( p_f \in \mathbb{R}_{>0} \) determines the weight given to past trajectories, \( g(t) \in \mathbb{R}^n \) is the filtered-tracking error derivative, \( e(t) \in \mathbb{R}^n, e(t) \in \mathbb{R}^{n_u} \), and \( \hat{f}(t) \in \mathbb{R}^{n_w} \) are the filtered-tracking error, filtered-control input, and filtered-regressor, respectively. Note that the time derivative of trajectory error \( \dot{e}(t) \) is not available. Therefore, \( g(t) \) is calculated using integration by-parts where no information of \( e(t) \) is required and is given as follows:
\[
g(t) = e(t) - \exp(-p_f (t - t_0)) e(t_0) - p_f \int_{t_0}^t e(s) ds.
\]
(16)

Equation (5) is rewritten as follows:
\[
W^T \dot{\phi}(x(t)) = \left( B^T B \right)^{-1} B \left( \dot{e}(t) - A_m e(t) \right)
\]
\[
- \delta_W^T(t) \phi(x(t)) + \dot{W}(t) \phi(x(t))
\]
(17)
Substituting (13)–(16) in (17), we get the following:
\[
W^T \dot{\phi}(t) + \Delta_f(t) = \left( B g(t) - A_m e(t) \right) + u_f(t)
\]
(18)
where the term corresponding to time-varying component of the unknown parameter satisfies the following dynamics:
\[
\dot{\Delta}_f(t) = -p_f \Delta_f(t) + \delta_W^T(t) \phi(x(t)), \quad \Delta_f(t_0) = \mathbf{0}_{n_u}.
\]
(19)
Here, \( \Delta_f(t) \in \mathbb{R}^{n_u} \) is the filtered effect of the parameteric perturbation \( \delta u(t) \), which is unknown and is simply shown for analysis purpose.

It can be observed that the first-layer filtering provides an algebraic relation (18) involving \( W^* \) as compared to the differential relation (5) having unmeasurable quantity \( \dot{e}(t) \). Hence, relation (18) can be utilized to design a composite adaptive controller. However, we further define another layer of filter to exploit the benefit of the IE condition [15].

2) Second-Layer Filtering: Consider the following filter-dynamics, which take outer-product of first-layer filter outputs as inputs:
\[
\dot{\phi}_{ff}(t) = -p_{ff} \dot{\phi}_{ff}(t) + \dot{\phi}(t) \phi_T^T(t)
\]
\[
\phi_{ff}(t_0) = \mathbf{0}_{n \times n}
\]
(20)
\[
\dot{u}_{ff}(t) = -p_{ff} u_{ff}(t) + (h(t) + u_f(t)) \phi_T^T(t)
\]
\[
u_{ff}(t_0) = \mathbf{0}_{n_u \times n_w}
\]
(21)
where \( \phi_{ff}(t) \in \mathbb{R}^{n_u \times n_w} \) and \( u_{ff}(t) \in \mathbb{R}^{n_u \times n_w} \) are the double filtered regressor and double filtered control input, respectively.

Using (18), (20), and (21), it can be shown that
\[
u_{ff}(t) = W^T \phi_{ff}(t) + \Delta_f(t)
\]
(22)
where
\[
\dot{\Delta}_f(t) = -p_f \Delta_f(t) + \Delta_f(t) \phi_T^T(t)
\]
\[
\Delta_f(t_0) = \mathbf{0}_{n_u \times n_w}.
\]
(23)
Here, \( \Delta_f(t) \in \mathbb{R}^{n_u \times n_w} \) is the double-filtered effect of the parameteric perturbation \( \delta u(t) \).

3) Initial Excitation: Consider the following IE assumption on the filtered-regressor:

Assumption 3: The filtered-regressor \( \phi_i(x(t)) \) is uniformly initially exciting (u-IE) with respect to dynamics in (1), filters in (13), (14), (15), (20), (21), and (38) with time-window \( T_{IE} \) and degree of excitation \( \gamma_{IE} \), i.e., \( \exists \gamma_{IE} > 0, T_{IE} > 0 \) such that
\[
\int_{t_0}^{t_0 + T_{IE}} \phi_i(\tau) \phi_i^T(\tau) d\tau \geq \gamma_{IE} I_{n_w}
\]
(24)
where \( I_{n_w} \) is the identity matrix of dimension \( n_w \).

Remark 2: The definition of IE condition [16, 17] has a crucial difference with the definition of PE condition. In PE condition, a similar integral inequality has to be satisfied for \( \{t, t + T_{PE} \} \forall t \in [t_0, \infty) \), i.e., the excitation has to persist for all future time. Unlike PE, the IE condition demands the integral inequality only for the initial time-window \( [t_0, t_0 + T_{IE}] \). The IE condition is milder than PE since there is no need for the excitation to persist beyond initial time-window.

Update law of \( W^*(t) \) is designed using the gamma-projection operation similar to (10), while incorporating an IE-based component.
\[
\hat{W}^* = \text{Proj}_{\hat{W}}(\hat{W}^*, y_i^*(t), f_i^*(t))
\]
(25)
where
\[
f_i^*(t) = f_i(\hat{W}^*) = \frac{W^T(t) \hat{W}^*(t) - \alpha^2}{2 \alpha e^T + e^T}
\]
(26)
\[
y_i^*(t) = \gamma_1 C_i(t) + \gamma_2 C_{il}(t) + \gamma_3 s(t) C_{ile}(t)
\]
(27)
where \( \alpha \) is from (6) and \( e \in \mathbb{R}_{>0} \) is a continuity parameter ensuring Lipschitz condition. \( \Gamma_{W} \in \mathbb{R}_{>0} \) is the adaptation gain matrix and \( \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}_{>0} \) are parameter tuning scalars for individual filter terms. \( y_i^*(t) \) can be extracted using (27) and
\[
C_i(t) = -\phi_f(x(t)) \left( W^T(t) \phi_f(x(t)) - (h + u_f) \right)^T
\]
(28)
\[
C_{il}(t) = -\left( W^T(t) \phi_f(x(t)) - u_{ff} \right)^T
\]
(29)
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\[ C_W(t) = -\left( \tilde{W}^T(t) \phi(t) - u(t) \right)^T \]  

(30)

where \( T = \bar{t}_0 + T_{th} \) in (30); the switching signal \( s(t) = 0 \) if \( t \in [0, \bar{t}_0 + T_{th}] \), otherwise 1 and \( y(t) \in \mathbb{R}^{n_u \times n_y} \).

Lemma 1: \( ||\tilde{W}(t)|| \in \mathcal{L}_\infty \forall t \) with an upper bound \( \tilde{W} \).  

Proof: For proof, refer to Lemma 9 and Example 10 in [5].

Remark 3: It is proved in [16] and [17] that the IE condition can be verified online by checking the minimum eigenvalue of \( \phi(t) \). Hence, the above designed parameter estimator is online implementable.

VI. STABILITY ANALYSIS

A. Ultimate Boundedness of \( \epsilon(t) \) and \( \bar{W}(t) \)

We define the joint tracking and parameter estimation error as \( \zeta(t) = [\epsilon(t) \, \text{vec}(\tilde{W}(t))]^T \) with the system dynamics \( \dot{\zeta}(t) \triangleq f(z, t) = A_{m}(s(t)) - \tilde{B}
\]

Remark 4: Note that the dependence of \( f(z, t) \) on \( t \) is explicitly shown to capture the effect of the time-varying quantities other than \( \epsilon(t) \) and \( \tilde{W}(t) \). Hence, the above representation facilitates to invoke [23, Th. 4.18] to claim uniform ultimate boundedness as follows.

Theorem 2: Using the system model in (1), control design in (4), parameter update laws in (10) and (25), and Assumptions 1 and 2, the trajectory \( \tilde{z}(t) \) is UUB.

Proof: The Lyapunov candidate is defined as follows:

\[ V(t) = \zeta(t)^T \begin{bmatrix} P & 0_{n_y \times n_y} \\ 0_{n_y \times n_y} & \Gamma_W \otimes I_n \end{bmatrix} \zeta(t) \]  

(31)

where \( \otimes \) is kronecker product and \( P \) is from (8).

Taking time derivative along system trajectories, substituting from \( \dot{\zeta}(t) \), above, using \( \text{vec}(\tilde{W})^T (\Gamma_W \otimes I_n) \text{vec}(\tilde{W}) = \text{Tr}(\tilde{W}^T \Gamma_W^{-1} \tilde{W}) \) and assuming the case (9a) as scenario (9b) is subsumed in this we get the following:

\[ \hat{V} = -\epsilon(t)^T Q_m \epsilon(t) - 2\sigma \text{Tr} \left( \tilde{W}^T(t) \phi(t) \delta_W(t) \right) \]

\[ + 2\sigma \text{Tr} \left( \tilde{W}^T(t) \right) \begin{bmatrix} P & 0_{n_y \times n_y} \\ 0_{n_y \times n_y} & \Gamma_W \otimes I_n \end{bmatrix} \zeta(t) \]  

(32)

After canceling like terms, bounding projection term, using \( \delta_W(t) \) from (3), and substituting from (8), we get the following:

\[ \hat{V} \leq -\epsilon(t)^T Q_m \epsilon(t) - 2\sigma \text{Tr} \left( \tilde{W}^T(t) \phi(t) \delta_W(t) \right) \]

\[ - 2\text{Tr} \left( \tilde{W}^T(t) \Gamma_W^{-1} \delta_W(t) \right) \]  

(33)

Adding and subtracting \( 2\sigma \text{Tr}(\tilde{W}(t) \phi(t)) \),

\[ \hat{V} = \epsilon(t)^T Q_m \epsilon(t) - 2\sigma \text{Tr} \left( \tilde{W}^T(t) \phi(t) \right) \]

\[ + 2\sigma \text{Tr} \left( \delta_W(t) \left( \tilde{W}^T(t) - \phi(t) \right) \right) \]  

(34)

Using Young’s inequality

\[ \hat{V} \leq -\epsilon(t)^T Q_m \epsilon(t) - 2\sigma \text{Tr} \left( \tilde{W}^T(t) \phi(t) \delta_W(t) \right) \]

\[ + 2\sigma \text{Tr} \left( \delta_W(t) \right) \]  

(35)

B. Ultimate Boundedness of \( \tilde{W}(t) \)

We rewrite (27) as the following \( \forall t \in [t_0 + T_{th}, \infty) \), \( s(t) = 1 \).

\[ y(t) = \gamma_1 C_1(t) + \gamma_2 C_2(t) + \gamma_3 C_{th}(t) \]  

(36)

Theorem 3: Based on the nominal parameter estimation update law in (25) in the presence of Assumption 3, \( \tilde{W}(t) \) is UUB.

Proof: The Lyapunov candidate is defined as follows:

\[ V = \frac{1}{2} \text{Tr} \left( \tilde{W}^T(t) \Gamma_W^{-1} \tilde{W}(t) \right) \]  

(37)

Taking time-derivative along system trajectory \( \forall t \in [t_0 + T_{th}, \infty) \), substituting (25), and upper-bounding the mix-term of \( \tilde{W}(t) \) and \( \text{Proj}_{W}^\dagger \tilde{W}(t, y(t), f(t)) \) similar to (32) using [5, Lemma 8] yields the following:

\[ \hat{V} = -\gamma_1 \text{Tr} \left( \tilde{W}^T(t) \phi(t) \phi(t) \tilde{W}(t) - \phi(t) \Delta_f(t) \right) \]

\[ - 2\gamma_2 \text{Tr} \left( \tilde{W}^T(t) \phi(t) \tilde{W}(t) - \phi(t) \Delta_f(t) \right) \]  

(38)

Using (15), (19), and (23) and bounding \( ||\phi(t)|| \leq \bar{\phi} \) as \( x(t) \in \mathcal{L}_\infty \) from Theorem 2, we get the following bounds on the undesirable terms:

\[ ||\phi(t)|| \leq \bar{\phi} \]  

(39)

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which gives the following:
\[ V^* \leq - \text{Tr} \left( \hat{W}^T(t)(\gamma_2 \phi(T) + \gamma_3 \phi(T)) \hat{W}(t) \right) + \| \hat{W}(t) \|_F \left( \gamma_2 + 3 \gamma_3 \frac{\hat{W}^T(t)}{P_{ff}} \right) \]
\[ + \| \hat{W}(t) \|_F \left( 3 + \gamma_2 \frac{\hat{W}^T(t)}{P_{ff}} \right) \]  
(40)

as the parameter perturbation is full rank from Assumption 3.

From (39), we get as follows:
\[ V^*(t) \leq \frac{\lambda_{\text{max}}(\hat{W}^T)}{2} \| \hat{W}(t) \|_F^2. \]  
(43)

Substituting (43) in (42) and putting \( \beta_2^* = \lambda_{\text{max}}(\hat{W}^T) \), we finally get the following:
\[ V^* \leq - \frac{\beta_1}{\beta_2^*} V^* + \frac{c^2 e^2}{2\beta_1}. \]  
(44)

From [23, Th. 4.18], the solution of the error system \( \tilde{W}(t) \) is UUB.

Remark 6: From (39) and (44) we get \( \frac{\lambda_{\text{min}}(\hat{W}^T)}{2} \| W(t) \|_F^2 \leq V^* \leq \frac{c^2 e^2}{2\beta_1^*} \), which results in \( V^*(t) \in L_\infty \) with an upper-bound given as
\[ \frac{c^2 e^2}{2\beta_1^*} \sqrt{\lambda_{\text{min}}(\hat{W}^T)}. \]

Hence, the IE condition will dictate a smaller ultimate bound of \( \| \hat{W}(t) \|_F \) than the all-time bound in Lemma 1 when
\[ \frac{c^2 e^2}{2\beta_1^*} \sqrt{\lambda_{\text{min}}(\hat{W}^T)} \leq \tilde{W} + n_u(\alpha^* + e^*). \]  
(45)

Therefore, after replacing \( c^* \) and \( \beta_1^* \) from (40) and (41), respectively, in (45), the following upper-bound on \( \delta_W(t) \) would have a guaranteed advantage for IE-based design:
\[ \delta_W \leq \frac{\lambda_{\text{min}}(\hat{W}^T)}{2\beta_2^*} \sqrt{\lambda_{\text{min}}(\hat{W}^T)} + \frac{2\gamma_2 \gamma_3}{\sigma} (\tilde{W} + n_u(\alpha^* + e^*). \]  
(46)

It can be inferred that sufficient degree of excitation \([\gamma_{\text{IE}} \text{ from} \) (24), which directly affects the magnitude of \( \lambda_{\text{min}}(\phi(T)) \), ensures satisfaction of the above inequality. Equation (46) further provides a smaller upper bound on \( W_{\text{est}}(t) \) in Theorem 2 given as follows:
\[ \sigma \left( \frac{c^* \sqrt{\lambda_{\text{min}}(\hat{W}^T)} + \tilde{W}}{\lambda_{\text{min}}(\hat{W}^T)} \right) \leq c_W. \]  
(47)

Remark 7: From (47) the proposed framework shows a bound, which is proportional to parameter perturbation (\( \delta_W \)) and rate of change of parameter perturbation (\( \dot{\delta}_W \)), unlike for the classical \( \sigma \)-mod and projection based parameter estimator which depends on the norm of true parameter (\( \tilde{W} + \delta_W \)).

Remark 8: Our proposed formulation of dual adaptation has structural similarity with recent development in higher order adaptive control formulation [24, 25], where there are two update laws consisting of a surrogate variable and the actual parameter estimator. The surrogate variable updates according to the actual parameter estimator law from traditional MRAC, whereas the actual parameter estimator chases this surrogate variable. In the proposed method, the total estimator \( \tilde{W}(t) \) is analogously chasing the nominal estimator \( W^*(t) \). However, a detailed analytical comparison between these two techniques is yet to be explored and will be considered in future research.

C. Performance Recovery in Disturbance-Free Scenario

This section establishes a special feature of our proposed dual adaptation scheme in a disturbance-free setting (\( \delta_W(t) = 0 \)). It is claimed that the algorithm can recover the performance of previous works on IE-based adaptive control for constant parameter models [14, 16].

In particular, exponential convergence of the tracking and parameter estimation errors can be guaranteed in this special case of time-invariant system, i.e., \( W(t) = W^* \) with no perturbation term \( \delta_W(t) \).

Theorem 4: Using the system dynamics in (1), controller design in (4), the unknown parameter \( W(t) = W^* \), and the update laws in (10) and (25) guarantee exponential convergence in the extended state space \( [e^T(t), \tilde{W}^T(t)]^T \), provided the IE condition in Assumption 3 is satisfied.

Proof: We can directly use (33), with \( \delta_W(t) = 0, \delta_W(t) = 0, \) and replace \( W(t) = W^* \) to get the following:
\[ \dot{V} = -e^T(t)Q_m e(t) - 2\sigma T r \left( \tilde{W}^T(t) \tilde{W}(t) \right) + 2\sigma T r \left( \tilde{W}^T(t) \tilde{W}(t) \right) \]

Using Young’s inequality
\[ \leq - \lambda_{\text{min}}(Q_m)\| e(t) \|_2^2 - \sigma \| \tilde{W}(t) \|_F^2 + \sigma \| \tilde{W}(t) \|_F^2 \]
\[ \leq - \lambda_{\text{min}}(Q_m)\| e(t) \|_2^2 - \frac{2\sigma}{\lambda_{\text{min}}(\hat{W}^T)} V^*. \]
\[ \leq - \frac{\beta_1}{\beta_2} V + c_W V^*. \]  
(48)

where \( \beta_1, \beta_2 \) are same as in (37), \( c_W = \frac{2\sigma}{\lambda_{\text{min}}(\hat{W}^T)} > 0 \) and \( \lambda_{\text{min}}(\hat{W}^T)/2\| W^*(t) \|_F^2 \leq V^*(t) \) is from (39).

Analyzing the convergence rate of \( V^*(t) \) when \( W(t) = W^* \), we remove the time-varying term from (42) \( (c^* \) becomes 0), to get as follows:
\[ V^* \leq - \frac{\beta_1}{\beta_2} V^* \]
\[ \Rightarrow V^*(t) \leq \exp \left( - \frac{\beta_1}{\beta_2} T \right) V^*(T) \forall t \geq T \]  
(49)

where \( \beta_1, \beta_2 > 0 \).

Finally, integrating (48), replacing in (49), and defining \( c_{\Omega} = 2\beta_1/\beta_2, \) we get the following:
\[ V(t) \leq \exp \left( - \frac{\beta_1}{\beta_2} (t - T) \right) V(T) + c_{\Omega} \int_0^T \exp \left( - \frac{\beta_1}{\beta_2} (t - \tau) - c_{\Omega} \right) d\tau V^*(T) \]
\[ = \exp \left( - \frac{\beta_1}{\beta_2} (t - T) \right) V(T) + c_{\Omega} \exp \left( - \frac{\beta_1}{\beta_2} \right) V^*(T) \]
\[ - c_{\Omega} \exp \left( - \frac{\beta_1}{\beta_2} \right) \exp \left( c_{\Omega} T \right) V^*(T) \]  
(50)
Fig. 2. Tracking error, parameter estimation error, and nominal parameter estimation error for the proposed method with unknown time-varying parameter. For parameter error, $\hat{W}(t) = W(t) - W(t)$ and $\hat{W} = \hat{W} - W$. (a) Tracking Error $e(t)$. (b) Parameter estimation error $\hat{W}(t)$. (c) Nominal parameter estimation error $\hat{W}(t)$.

\[ \leq \exp \left\{ \frac{\beta_1}{\beta_2} (t - T) \right\} V(T) + \frac{c_1}{c_2} \exp \left\{ -c_1t \right\} V^*(T) \]  

where, $c_1 = \beta_1 / \beta_2 - c_1$ and $V(t) \to 0$ as $t \to \infty$.

Remark 9: The above theorem is a crucial feature of the proposed scheme entailing performance recovery in the disturbance free case. The result indicates that dual adaptation mechanism can ensure parameter convergence ($\hat{W}(t) \to 0$ and $\hat{W}^*(t) \to 0$) under the IE condition when $\delta_W(t) = 0$ and thereby also invoke exponential rate of convergence for tracking error $e(t)$. The proposed algorithm successfully unifies the cases of time-varying and time-invariant parameter using the dual adaptation principle unlike the recent result [18]. Further it can be inferred that even in the time-varying case if $\delta_W(t) \to 0$, asymptotic convergence of the error dynamics can be obtained using the proposed algorithm.

VII. SIMULATION RESULTS

We validate the performance of the proposed algorithm using the following reference system and actual system of a linearized 2-DOF robot model taken from [17], while injecting a time-varying perturbation component $\Delta A(t)$.

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.6 & 2 & 1.2 & -1.4 \\ 0 & 0 & 0 & 1 \\ 2.25 & -1.25 & 3.75 & -2.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sin(\frac{t}{2}) & \sin(t) & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \sin(\frac{t}{2}) & \frac{1}{\sqrt{2}} \cos(\frac{t}{2}) & \frac{1}{\sqrt{2}} \cos(\frac{t}{2}) & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \Delta A(t) \]

\[ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -15 & -11.25 & -1.2 & -1.6 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -2 & -3 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \]
where, $A_m$ is Hurwitz, $A$ and $A_m$ satisfy Assumption 1. We use the following external input $r(t) = [5, 10\sin(t) \exp(-t/1000)]$, which is IE but not PE. To check for the condition mentioned in (46), we need $\lambda_{\min}(\phi_f(T))$, which is extracted online once the IE condition is satisfied. Based on the above model, it can be shown that $\Delta A(t)x = B\delta_W(t)\phi(x)$, where $\phi(x(t))$ is constructed using elements of $x(t)$, and from (34) we can compute $\phi$, which will be used in (46). The matrix $P$ is chosen as follows:

$$P = \begin{bmatrix}
1.415 & 0.0128 & 0.1356 & 0.1542 \\
0.0128 & 0.0623 & -0.0597 & -0.0625 \\
0.1356 & -0.0597 & 1.2775 & 0.2858 \\
0.1542 & -0.0625 & 0.2858 & 0.2953
\end{bmatrix}$$

which gives $Q_m = -I_4$. We choose the following values for the tuning gains of parameter estimator and filters: $\Gamma_V = 10I_4, \Gamma_W = 10I_4, \gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 5, p_V = 2, p_f = 0.5, \delta_W = 21, \epsilon = 3, \epsilon^* = 2,$ and $\sigma = 0.1$. The performance of our algorithm is shown in Fig. 2 with tracking and parameter estimation errors along each state. Note that in general there is no access to the nominal parameter as breaking the time-varying component into fixed parameter and time-varying parameter is unknown, however, for the efficacy of the results we provide error along the fixed parameter component in Fig. 2(c).

We compare our proposed algorithm with (i) classical MRAC with projection, and (ii) a recent work [18]—represented by red and blue, respectively, while the proposed method is represented in black as shown in Figs. 3 and 4. Two following scenarios are considered:

1) unknown time-varying parameter following $A$ matrix defined above with results in Fig. 3;
2) time-varying component of parameter made zero to show performance recovery of proposed method; results are shown in Fig. 4.

The control law for classical MRAC (i) is similar to (4), with the change in parameter estimation law which follows (10) for projection-operator with $y(t) = \phi(x(t))e^T(t)PB$.

From Fig. 3(a) and (b) we see that the proposed algorithm performs significantly better in terms smaller tracking error norm and parameter estimation error norm in steady-state along with faster convergence. Furthermore, our method has relatively small peak controller norm of 12.5 units as compared to 17.5 units for (i) and (ii) as shown in Fig. 3(c). Lastly, Fig. 4(a) and (b) illustrate exponential convergence of the proposed algorithm in terms of tracking error and parameter estimation error when time-varying perturbation component of the parameter is removed.

VIII. CONCLUSION

In this article we design a composite adaptive controller using dual adaptation technique for time-varying dynamical systems. The novel dual adaptation mechanism can efficiently deal with the time-varying nature of the parameters. A combined approach using Projection and $\sigma$-mod algorithms is conceptualized while exploiting congelation of variables to claim a global result for the tracking error space. Unlike the classical adaptive systems requiring the restrictive PE condition for accurate parameter estimation, the proposed work builds on the milder IE condition. A rigorous Lyapunov stability analysis is performed to ensure boundedness of the closed-loop error dynamics. Moreover, the proposed algorithm can recover the performance of previously designed IE-based adaptive controller in case of plants having constant parameters. Simulation results corroborate the efficacy of the design in contrast to state-of-the-art.

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