Conformal Killing graphs with prescribed mean curvature

M. Dajczer* and J. H. de Lira†

Abstract

We prove the existence and uniqueness of graphs with prescribed mean curvature function in a large class of Riemannian manifolds which comprises spaces endowed with a conformal Killing vector field.

1 Introduction

Our aim in this paper is to continue the research theme elaborated in [2] where the existence of a Killing vector field in a manifold permitted to formulate and solve Dirichlet problems associated to the mean curvature equation. Here, we expand our scope by dealing with conformal Killing vector fields and the corresponding notion of graph. Our main goal is to provide a significant improvement of the results in [1] for graphs of constant mean curvature and trivial initial condition. This is achieved by means of a completely different methodology that allows us to generalize, under an unifying perspective, several previous results.

The pure analytic method used here enable us to discard or weaken several assumptions in [1]. For instance, we allow prescribed mean curvature and nontrivial boundary data. Moreover, we remove the restriction on the conformal Killing field to be closed. We also remove the restriction on the Ricci curvature to be minimal in the direction of the Killing field and weaken other requirements on the Ricci tensor. Furthermore, we ruled out some restrictions on the metric of the ambient space so that, in particular, our result applies successfully to a class of product spaces.

To explain the framework of this paper we first fix some terminology. Let $\tilde{M}^{n+1}$ denote a Riemannian manifold endowed with a conformal Killing vector field $Y$ whose orthogonal distribution $\mathcal{D}$ is integrable. Thus, there exists $\rho \in C^\infty(\tilde{M})$ such that $\mathcal{L}_Y\tilde{g} = 2\rho\tilde{g}$, where $\tilde{g}$ is the metric in $\tilde{M}$. It results that the integral leaves of $\mathcal{D}$ are totally umbilical hypersurfaces. If in addition $Y$ is closed, then they are spherical, i.e., have constant mean curvature.

*Partially supported by CNPq and Faperj.
†Partially supported by CNPq and Funcap.
We denote by $\Phi : \mathbb{I} \times M^n \to \bar{M}^{n+1}$ the flow generated by $Y$, where $\mathbb{I} = (-\infty, a)$ is an interval with $a > 0$ and $M^n$ is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as $t = 0$. It may happen that $a = +\infty$, i.e., the vector field $Y$ is complete. For instance, this occurs when the trajectories of $Y$ are circles and we pass to the universal cover in an appropriate manner. Since $\Phi_t = \Phi(t, \cdot)$ is a conformal map for any fixed $t \in \mathbb{I}$, there exists a positive function $\lambda \in C^\infty(\mathbb{I})$ such that $\lambda(0) = 1$ and $\Phi_t^* \bar{g} = \lambda^2(t) \bar{g}$.

Given a bounded domain $\Omega$ in $M$, the conformal Killing graph $\Sigma = \Sigma(z)$ of a function $z$ on $\bar{\Omega}$ is the hypersurface $\Sigma = \{ \bar{u} = \Phi(z(u), u) : u \in \bar{\Omega} \}$.

Proving the existence of a conformal Killing graph with prescribed mean curvature and boundary data requires establishing apriori estimates. This is accomplished by the use of Killing cylinders as barriers. The Killing cylinder $K$ over $\Gamma = \partial \Omega$ is the hypersurface ruled by the flow lines of $Y$ through $\Gamma$, that is, $K = \{ \bar{u} = \Phi(t, u) : u \in \Gamma \}$.

Let $\Omega_0$ denote the largest open subset of points of $\Omega$ that can be joined to $\Gamma$ by a unique minimizing geodesic. At points of $\Omega_0$, we denote

$$\text{Ric}^{\text{rad}}_M(x) = \text{Ric}_M(\eta, \eta)$$

where $\text{Ric}_M$ is the ambient Ricci tensor and $\eta \in T_xM$ is a unit vector tangent to the unique minimizing geodesic from $x \in \Omega_0$ to $\Gamma$.

The following result assures the existence of conformal Killing graphs with prescribed mean curvature $H$ and boundary data $\phi$. Here, the functions $H$ and $\phi$ are defined on $\bar{\Omega}$ and $\Gamma$, respectively. Moreover, $H_K$ denotes the mean curvature of $K$ when calculated pointing inwards.

**Theorem 1.** Let $\Omega \subset M^n$ be a $C^{2,\alpha}$ bounded domain so that $\text{Ric}^{\text{rad}}_M \geq -n \inf_\Gamma H^2_K$. Assume $\lambda_t \geq 0$ and $(\lambda_t/\lambda)_t \geq 0$. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be such that $\inf_\Gamma H_K > H \geq 0$ and $\phi \leq 0$. Then, there exists a unique function $z \in C^{2,\alpha}(\Omega)$ whose conformal Killing graph has mean curvature function $H$ and boundary data $\phi$.

Proposition 6 below implies that Theorem 1 holds under weaker but somewhat more technical assumptions on the ambient Ricci tensor. We also point out that we can prove with minor modifications an existence result for functions $H$ depending also on $t$ by imposing the condition $(\lambda H)_t \geq 0$ instead of $\lambda_t \geq 0$. In the case comprised in Theorem 1 the condition $\lambda_t \geq 0$ says that the mean curvatures of the leaves and of the graph have opposite signs.

It is worth to mention that $\bar{M}$ is conformal to a Riemannian product manifold $\mathbb{I} \times \tilde{M}$ where $\tilde{M}$ is conformal to $M$. A quite remarkable fact is that the mean curvature equation for a general conformal metric to a product metric like this does not satisfy, in general, the maximum principle. In fact, in the final remark of the
paper we see that the class of metrics which we deal in this paper stands as a
borderline for the validity of the elliptic techniques in the treatment of the mean
curvature equation.

Assume now that $Y$ is closed or just a Killing field. In each case, a short addi-
tional argument to the proof of Theorem [1] yields the same conclusion under weaker
assumptions. For instance, in the case of a Killing field we prove a generalization of
the main result in [2] under a weaker assumption on the Ricci curvature.

**Corollary 2.** Assume that $Y$ is a Killing field. Let $\Omega \subset M^n$ be a $C^{2,\alpha}$ bounded
domain such that $\text{Ric}^{rad}_M \geq -n \inf_\Gamma H^2_K$. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be such
that $\inf_\Gamma H_K \geq H \geq 0$. Then, there exists a unique function $z \in C^{2,\alpha}(\Omega)$ whose
conformal Killing graph has mean curvature function $H$ and boundary data $\phi$.

The case of closed conformal Killing fields encompasses a broad range of ex-
amples, namely, product and warped ambient spaces, that have been extensively
considered in the recent pertinent literature. In this case, we can state our result in
terms of the Ricci tensor of $M^n$.

**Corollary 3.** Assume that $Y$ is a closed conformal Killing field. Let $\Omega \subset M^n$ be a $C^{2,\alpha}$ bounded
domain such that $n \text{Ric}^{rad}_M \geq -(n-1)^2 \inf_\Gamma H^2_K$. Assume $\lambda_t \geq 0$ and
$(\lambda_t/\lambda)_t \geq 0$. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be such that $\inf_\Gamma H_K > H \geq 0$ and
$\phi \leq 0$. Then, there exists a unique function $z \in C^{2,\alpha}(\Omega)$ whose conformal Killing
graph has mean curvature function $H$ and boundary data $\phi$.

We finally point out that concerning constant mean curvature Euclidean radial
graphs over spherical domains, the above result extends the theorems for minimal
radial graphs due to Rado and Tausch as well as for constant mean curvature by
Serrin and Lopez (see [5] and references therein).

## 2 Preliminaries

Let $(\tilde{M}^{n+1}, \tilde{g})$ be a Riemannian manifold endowed with a conformal Killing vector
field $Y$ whose orthogonal distribution $\mathcal{D}$ is integrable. Let $\tilde{\nabla}$ denote the Riemannian
connection in $\tilde{M}^{n+1}$ and

$$\langle X, Z \rangle = \tilde{g}(X, Z).$$

From $\mathcal{L}_Y \tilde{g} = 2\rho \tilde{g}$ we deduce the conformal Killing equation

$$\langle \tilde{\nabla}_X Y, Z \rangle + \langle \tilde{\nabla}_Z Y, X \rangle = 2\rho \langle X, Z \rangle, \tag{1}$$

where $X, Z \in T\tilde{M}$. It is a standard fact (cf. [6]) that the conformal factor $\lambda \in C^\infty(\mathcal{I})$ and
$\rho \in C^\infty(\mathcal{I})$ are related by

$$\rho = \lambda_t/\lambda. \tag{2}$$
Denote
\[ |Y(t,u)|^2 = 1/\tilde{\gamma}(t,u) \quad \text{and} \quad \gamma(u) = \tilde{\gamma}(0,u). \]

It follows from (1) and (2) that
\[ \tilde{\gamma}(t,u) = \gamma(u)/\lambda^2(t). \]  

We have from (1) and the integrability of D that
\[ \langle \nabla_X Y, Z \rangle = \rho \langle X, Z \rangle \]  

for any \( X, Z \in D \). Thus, the leaves \( M_t^n = \Phi_t(M) \) are totally umbilical and the mean curvature \( k = k(t,u) \) of \( M_t \) with respect to the unit normal vector field \( Y/|Y| \) is
\[ k = -\rho /|Y| = -\lambda t \sqrt{\gamma}/\lambda^2. \]  

We assign coordinates \( x^0 = t, x^1, \ldots, x^n \) to points in \( \tilde{M} \) of the form \( \tilde{u} = \Phi(t,u) \) where \( x^1, \ldots, x^n \) are local coordinates in \( M \). Then, the coordinate vector fields are
\[ \partial_0|\tilde{u} = Y(\tilde{u}) \quad \text{and} \quad \partial_i|\tilde{u} = \Phi_t \ast \partial_i|u \quad \text{for} \quad 1 \leq i \leq n. \]

The components of the ambient line element \( ds^2 \) in terms of these coordinates are
\[ \tilde{\sigma}_{00} = \langle \partial_0, \partial_0 \rangle = |Y|^2 = \lambda^2(t)/\gamma(u), \quad \tilde{\sigma}_{0i} = \langle \partial_0, \partial_i \rangle = 0, \quad \tilde{\sigma}_{ij}|\tilde{u} = \lambda^2(t)\sigma_{ij}|u, \]
where \( \sigma_{ij} \) are the local components of the metric \( d\sigma^2 \) in \( M^n \). Setting,
\[ \psi^2(u) = 1/\gamma(u) \]
we have that \( \tilde{M}^{n+1} \) is conformal to the Riemannian warped product manifold \( M^n \times_{\psi} \mathbb{I} \) with conformal factor \( \lambda \), i.e.,
\[ ds^2 = \lambda^2(t)(\psi^2(u)dt^2 + d\sigma^2). \]  

Finally, after the change of variable
\[ r = r(t) = \int_0^t \lambda(\tau)d\tau, \]
we have that (6) takes the form of a Riemannian twisted product
\[ ds^2 = \psi^2(u)dr^2 + \theta^2(r)d\sigma^2 \]
where \( \theta(r) = \lambda(t(r)) \).

The above change of variable is essential to avoid the coefficients of the terms of second order in the quasilinear elliptic mean curvature equation for \( \Sigma(z) \) to depend on the function \( z \) itself.

We conclude this sections with a few examples to illustrate this change of variable. Observe that all Riemannian manifolds in the examples below fit the assumptions in Theorem 1. Moreover, these examples comprise Euclidean and hyperbolic space forms what implies that Theorem 1 assures existence for radial graphs with prescribed mean curvature in these spaces.
Examples 4. Let $\phi \in C^\infty(M)$ be a positive function.

(a) By means of the change of variable $e^t = r$, we have that
\[
\bar{M} = \mathbb{R}_+ \times M, \quad ds^2 = \phi^2(u) \, dr^2 + r^2 \, d\sigma^2
\]
is isometric to
\[
\tilde{M} = \mathbb{R} \times M, \quad d\tilde{s}^2 = e^{2t} (\phi^2(u) \, dt^2 + d\sigma^2).
\]

(b) By means of the change of variable $t = 1 - e^{-r}$, we have that
\[
\bar{M} = \mathbb{R} \times M, \quad ds^2 = \phi^2(u) \, dr^2 + e^{2r} \, d\sigma^2
\]
is isometric to
\[
\tilde{M} = (-\infty, 1) \times M, \quad d\tilde{s}^2 = \frac{1}{(1-t)^2} (\phi^2(u) \, dt^2 + d\sigma^2).
\]

(c) By means of the change of variable $t = c + \log(b \tanh (r/2))$, where $c > 0$ and $b^{-1} = \tanh (c/2)$, we have that
\[
\bar{M} = \mathbb{R}_+ \times M, \quad ds^2 = \phi^2(u) \, dr^2 + (\sinh r)^2 \, d\sigma^2
\]
is isometric to
\[
\tilde{M} = (-\infty, c + \log b) \times M, \quad d\tilde{s}^2 = (\sinh(2 \arctanh b^{-1} e^{t-c}))^2 (\phi^2(u) \, dt^2 + d\sigma^2).
\]

In the particular case when $Y$ is closed conformal Killing field, that is, when
\[
\langle \bar{\nabla}_XY, Z \rangle = \rho \langle X, Z \rangle,
\]
we have that $\gamma$ is constant. Thus, in this case, $\bar{M}^{n+1}$ has a warped product structure and is conformal with conformal factor $\lambda$ to a Riemannian product manifold $I \times M^n$. Observe that now the leaves of $\mathcal{D}$ are spherical, that is, totally umbilical with constant mean curvature $k = k(t)$.

3 Killing cylinders

Let $\Omega \subset M^n$ be a bounded domain with regular boundary $\Gamma$. The Killing cylinder $K$ over $\Gamma$ determined by the conformal Killing field $Y$ is the hypersurface defined by
\[
K = \{ \Phi(t, u) : t \in \mathbb{R}, \ u \in \Gamma \}.
\]

Let $t^1, \ldots, t^{n-1}$ be local coordinates for $\Gamma$. We denote by $(\tau_{ij})$ the components of the metric in $\Gamma$ with respect to these coordinates. It results that $t, t^1, \ldots, t^{n-1}$ are local coordinates for $K$. Let $\eta$ be the inward unit normal vector along $\Gamma$ as a submanifold of $M$. Then
\[
\tilde{\eta} = \frac{1}{\lambda} \Phi_* \eta
\]
is an unit normal vector field to \( K \). Thus,
\[
\langle \bar{\eta}, \partial_t \rangle = 0 = \langle \bar{\eta}, \partial_{t^i} \rangle
\]
where \( \partial_t = \partial/\partial t \) and \( \partial_{t^i} = \partial/\partial t^i \). We deduce from (1) that
\[
\langle \nabla_{\partial_{t^i}} \partial_t, \bar{\eta} \rangle = \rho \langle \partial_{t^i}, \bar{\eta} \rangle = 0.
\]
Hence \( \partial_t \) is a principal direction of \( K \) with corresponding principal curvature
\[
\kappa = \bar{\gamma} \langle \nabla_Y Y, \bar{\eta} \rangle.
\]  
It follows from (1) and (3) that
\[
\kappa = -\frac{1}{2} \bar{\gamma} \eta (\bar{\gamma} - 1) = \frac{1}{2} \bar{\gamma} \eta (\gamma) = \frac{1}{\lambda} \eta (\log \sqrt{\bar{\gamma}}).
\]
Lemma 5. The mean curvature \( H_K \) of the Killing cylinder \( K \) is given by
\[
nH_K(t, u) = \kappa(t, u) + \frac{n-1}{\lambda(t)} H_{\Gamma}(u).
\]  
Proof: We have that
\[
nH_K = \kappa + t^{ij} \langle \nabla_{\partial_{t^i}} \partial_{t^j}, \bar{\eta} \rangle |_{(t, u)} + \lambda^{-2} \tau^{ij} \langle \Phi_{t^i} \nabla_{\partial_{t^j}} \partial_{t^i} \rangle |_{(t, u)} + \lambda^{-1} \Phi_{t^i} \eta |_{(t, u)}
\]
and the proof follows. \( \blacksquare \)

We denote by \( \Gamma_\epsilon \) and \( K_\epsilon \) the level sets \( d = \epsilon \) in \( M \) and \( \tilde{M} \), respectively. By \( H_{K_\epsilon} \) we denote the mean curvature of the Killing cylinder \( K_\epsilon \) over \( \Gamma_\epsilon \).

Proposition 6. Assume that the Ricci curvature tensor of \( \tilde{M} \) satisfies
\[
\inf_{\Omega_0} \{ \text{Ric}^\text{rad}_M + (nk^2 - \sqrt{\bar{\gamma}} k_i) \} |_{t=0} \geq -n \inf_{\Gamma} H^2_K.
\]  
Then, \( H_{K_\epsilon} |_{x_0} \geq H_K |_{y_0} \) if \( y_0 \in \Gamma \) is the closest point to a given point \( x_0 \in \Gamma_\epsilon \subset \Omega_0 \).
**Proof:** The coordinate $d$-curve in $\Phi(t, u)$ is the image by $\Phi_t$ of the coordinate $d$-curve passing through $u \in M$. Thus,

$$\bar{\eta}|_{\Phi_t(u)} = \frac{1}{\lambda} \Phi_t*(u)\partial_t|_u = \frac{1}{\lambda} \partial_t|_{\Phi_t(u)} = \frac{1}{\lambda} \partial_t^*|_{\Phi(t,u)}.$$

Extend $\bar{\eta}$ near $K$ as the velocity vector field of the geodesics in $M_t$ departing orthogonally from $K \cap M$. We obtain for $1 \leq i, j \leq n - 1$ that

$$\lambda^2 \bar{\eta}\left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \bar{\partial}_j\right> = \partial_v \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\partial}_v, \bar{\eta}\right> = \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\nabla}_{\bar{\partial}_v} \bar{\partial}_v, \bar{\partial}_j\right> + \left<\bar{R}(\partial_v, \eta)\partial_v, \partial_i\right> + \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\partial}_v, \bar{\nabla}_{\bar{\partial}_j} \bar{\partial}_v\right> = -\lambda^2 \left<\bar{\nabla}_{\bar{\eta}} \bar{\eta}, Y\right> + \left<\bar{R}(\partial_v, \eta)\eta, Y\right> + \left<\bar{\nabla}_{\bar{\eta}} \eta, Y\right> - \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \bar{\nabla}_{\bar{\partial}_j} \bar{\eta}\right>.$$  

Using (11) and (13), we have

$$\bar{\eta}\left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \bar{\partial}_j\right> = -k^2 t_{ij} - \left<\bar{R}(\partial_v, \eta)\eta, \bar{\partial}_j\right> + \left<A^2 \partial_v, \bar{\partial}_j\right> = (11)$$

where $A_\epsilon$ denotes the Weingarten map of $K_\epsilon$ relative to $\bar{\eta}$.

For the remaining case $i = j = 0$, we have

$$\bar{\eta}\left<\bar{\nabla}_{\bar{\partial}_0} \bar{\eta}, \partial_0\right> = \left<\bar{\nabla}_Y \bar{\nabla}_{\eta}, Y\right> + \left<\bar{R}(\eta, Y)\eta, Y\right> + \left<\bar{\nabla}_{[\eta, Y]} \eta, Y\right> + \left<\bar{\nabla}_Y \eta, \bar{\nabla}_Y Y\right>.$$  

However,

$$[\bar{\eta}, Y] = -[Y, \bar{\eta}] = -[\partial_v, \lambda^{-1} \partial_v] = \frac{\lambda t}{\lambda^2} \partial_v = \rho \bar{\eta}. = (12)$$

Thus,

$$\left<\bar{\nabla}_{[\eta, Y]} \bar{\eta}, Y\right> = \rho \left<\bar{\nabla}_Y \bar{\eta}, Y\right> = -\rho^2.$$

Using (12) we have

$$\left<\bar{\nabla}_Y \eta, \bar{\nabla}_Y Y\right> = \left<\bar{\nabla}_Y \eta, \bar{\nabla}_Y Y\right> + \left<\nabla_Y \bar{\eta}, [\eta, Y]\right> = \left<A^2 Y, Y\right>$$

and

$$\left<\nabla_Y \bar{\nabla}_Y \eta, Y\right> = Y \left<\nabla_Y \eta, Y\right> - \gamma \left<\nabla \eta, Y\right> \left<\nabla_Y Y, Y\right> = -Y(\rho) + \rho^2.$$

It follows that

$$\bar{\eta}\left<\bar{\nabla}_{\bar{\partial}_0} \bar{\eta}, \partial_0\right> = -Y(\rho) - \left<\bar{R}(Y, \eta)\eta, Y\right> + \left<A^2 Y, Y\right>.$$  

(13)

We also have

$$\bar{\eta}\left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \partial_i\right> = -\left<\bar{\nabla}_Y A_\epsilon \partial_i, \bar{\partial}_i\right> - \left<A_i \partial_i, \bar{\nabla}_Y \eta\right> = -\left<\left(\bar{\nabla}_Y A_\epsilon\right) \partial_i, \partial_i\right> + 2 \left<A^2 \partial_i, \partial_i\right>$$

for $0 \leq i, j \leq n - 1$. Taking traces with respect to the metric $(t_{ij})$ in $K$ with $t^{00} = \dot{\gamma}$ and $t^{00} = 0$, and using (11) and (13) gives

$$\text{tr} \bar{\nabla}_Y A_\epsilon = t^{ij} \left<\left(\bar{\nabla}_Y A_\epsilon\right) \partial_i, \partial_j\right> = -t^{ij} \eta \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \partial_j\right> + 2 t^{ij} \left<A^2 \partial_i, \partial_j\right>$$

$$= -\gamma \eta \left<\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \partial_j\right> + 2 \gamma \left<A^2 \partial_i, \partial_j\right> - \sum_{i,j=1}^{n-1} t^{ij} \left<\left(\bar{\nabla}_{\bar{\partial}_i} \bar{\eta}, \partial_j\right) - 2 \left<A^2 \partial_i, \partial_j\right>\right>$$

$$= \gamma Y(\rho) + (n - 1)k^2 + \text{Ric}_{\bar{\xi}}(\eta, \eta) + \text{tr} A^2.$$
However,
\[ \sqrt{\gamma} \kappa_t = \frac{Y(k)}{|Y|} = -\frac{1}{|Y|^2} Y(\rho) + kY\left(\frac{1}{|Y|}\right) = -\dot{\gamma}Y(\rho) + k^2. \] (14)

We conclude that
\[ \text{tr} \tilde{\nabla}_\eta A_\epsilon = \text{tr} A_\epsilon^2 + \text{Ric}_{\tilde{M}}(\bar{\eta}, \bar{\eta}) + nk^2 - \sqrt{\gamma} k_t. \]

Since \( n\dot{H}_{K_\epsilon} = \tilde{\nabla}_\eta \text{tr} A_\epsilon = \text{tr} \tilde{\nabla}_\eta A_\epsilon \), at \( d = \epsilon \), it follows that
\[ n\dot{H}_{K_\epsilon} \geq nH_{K_\epsilon}^2 + \text{Ric}_{\tilde{M}}(\bar{\eta}, \bar{\eta}) + nk^2 - \sqrt{\gamma} k_t. \]

Therefore, at \( t = 0 \) and by the assumption on \( \text{Ric}^{rad}_{\tilde{M}} \) there exist constants \( c > 0 \) and \( d_0 > 0 \) such that
\[ \dot{H}_{K_d} \geq c(H_{K_d} - \inf \Gamma H_K) \]
for \( d \in [0, d_0] \). Hence, \( H_{K_d} \) does not decrease with increasing \( d \).

4 Conformal Killing graphs

The conformal Killing graph \( \Sigma^n \) of a function \( z: \bar{\Omega} \subset M^n \to \mathbb{I} \) is the hypersurface in \( M^{n+1} \) given by
\[ \Sigma^n = \{ X(u) = \Phi(z(u), u) : u \in \bar{\Omega} \}. \]

We show next that the partial differential equation for a prescribed mean curvature function \( H \) in \( \bar{\Omega} \) is the quasilinear elliptic equation of divergence form
\[ \text{div} \left( \frac{\nabla z}{\sqrt{\gamma + |\nabla z|^2}} \right) - \frac{1}{\sqrt{\gamma + |\nabla z|^2}} \left( \frac{\langle \nabla \gamma, \nabla z \rangle}{2\gamma} + n\gamma \rho \right) - n\lambda H = 0. \] (15)

Recall that \( \rho = \lambda_t/\lambda \). Here, the gradient \( \nabla \) and the divergence \( \text{div} \) are differential operators in the leaf \( M^n \) endowed with the metric \( d\sigma^2 \).

A sufficient condition to have a maximum principle for (15) (see Theorem 10.1 in [3]) is
\[ (\lambda_t/\lambda)_t = \rho_t \geq 0 \quad \text{and} \quad \lambda_t H \geq 0. \] (16)

The graph \( \Sigma \) is parametrized in terms of local coordinates by
\[ X(u) \in \Sigma \mapsto (z(x^1, \ldots, x^n), x^1, \ldots, x^n). \]

Thus, the tangent space to \( \Sigma \) at \( X(u) \) is spanned by the vectors
\[ X_i = z_i \partial_0|_{X(u)} + \partial_i|_{X(u)}, \] (17)
where \( z_i = \partial z/\partial x^i \). Hence, the metric induced on \( \Sigma \) is
\[ g_{ij}|_{X(u)} = \lambda^2(z(u)) \left( \sigma_{ij}(u) + \frac{z_i z_j}{\gamma(u)} \right). \]
The inverse is
\[ g^{ij} |_{X(u)} = \frac{1}{\lambda^2(z(u))} \left( \sigma^{ij}(u) - \frac{z^i z^j}{\gamma(u) + |\nabla z|^2} \right) \]
where \( z^i = \sigma^{ik} z_k \) and \( |\nabla z|^2 = z^i z_j \) as usual.

Fix the orientation on \( \Sigma \) given by the unit normal vector field
\[ N|_{X(u)} = \frac{1}{\lambda^2 W} \left( \gamma(u) \partial_t |_{X(u)} - \Phi_{z(u)}^* \nabla z(u) \right), \]
where
\[ \lambda^2 W^2 = \gamma + |\nabla z|^2. \]
Notice that \( \langle N, Y \rangle > 0 \). We compute the second fundamental form
\[ a_{ij} = \langle \nabla X_i, X_j, N \rangle \]
of \( \Sigma \). From (17) and (18) we obtain
\[
\begin{align*}
\lambda^2 W a_{ij} &= \gamma z_{ij} \langle \partial_t, \partial_0 \rangle + \gamma z_i z_j \langle \nabla_{\partial_0} \partial_t, \partial_0 \rangle + \gamma z_j \langle \nabla_{\partial_0} \partial_t, \partial_0 \rangle + \gamma z_i \langle \nabla_{\partial_0} \partial_t, \partial_0 \rangle \\
&+ \gamma \langle \nabla_{\partial_0} \partial_j, \partial_t \rangle - z_i z_j \langle \nabla_{\partial_0} \partial_0, \Phi_{z(u)}^* \nabla z \rangle - z_j \langle \nabla_{\partial_0} \partial_0, \Phi_{z(u)}^* \nabla z \rangle \\
&- z_i \langle \nabla_{\partial_0} \partial_t, \Phi_{z(u)}^* \nabla z \rangle - \langle \nabla_{\partial_0} \partial_j, \Phi_{z(u)}^* \nabla z \rangle.
\end{align*}
\]

The Levi-Civita connections in \( M \) and \( M_t \) are determined by the same Christoffel symbols since \( \tilde{\sigma}_{ij} |_u = \lambda^2 \sigma_{ij} |_u \) if \( u \in M \) and \( \tilde{u} = \Phi_t(u) \in M_t \). We have from (2) and (1) that
\[ \langle \nabla_{\partial_t} \partial_j, \partial_t \rangle = -\rho(z(u)) \langle \partial_t |_{X(u)}, \partial_j |_{X(u)} \rangle = -\lambda \lambda_t(z(u)) \sigma_{ij}(u). \]
The terms involving the flow lines acceleration are
\[
\begin{align*}
\langle \nabla_{\partial_0} \partial_t, \partial_0 \rangle |_{X(u)} &= \frac{1}{2} \partial_t |_{X(u)} = \frac{\lambda^2(z(u))}{\gamma(u)} \\
\langle \nabla_{\partial_0} \partial_t, \partial_0 \rangle |_{X(u)} &= \frac{1}{2} \partial_t |_{X(u)} = -\frac{\lambda^2(z(u)) \gamma_t(u)}{2 \gamma^2(u)}.
\end{align*}
\]
Similarly,
\[
\begin{align*}
\langle \nabla_{\partial_0} \partial_t, \Phi_{z(u)}^* \nabla z(u) \rangle &= \langle \Phi_{z(u)}^* \nabla_{\partial_0} \partial_t |_u, \Phi_{z(u)}^* \nabla z(u) \rangle = \lambda^2(z(u)) \langle \nabla_{\partial_0} \partial_t |_u, \nabla z |_u \rangle \\
\langle \nabla_{\partial_0} \partial_0, \Phi_{z(u)}^* \nabla z \rangle &= \langle \nabla_{\partial_0} \partial_0 |_{X(u)}, z^i \partial_j |_{X(u)} \rangle = z^i \rho(\partial_t |_{X(u)}, \partial_j |_{X(u)}) = \gamma_i \lambda \lambda_t(z(u)).
\end{align*}
\]
Since \( z_{i,j} = z_{ij} - \langle \nabla_{\partial_0} \partial_j, \nabla z \rangle \) are the Hessian components, we have
\[ Wa_{ij} = z_{i,j} - \frac{\lambda_t}{\gamma} \gamma z_j - \frac{\lambda_t}{\lambda} \lambda \gamma \sigma_{ij} - \frac{\gamma_i}{2 \gamma} z_j - \frac{\gamma_j}{2 \gamma} z_i - \frac{\gamma_k}{2 \gamma^2} z_i z_j z^k. \]
We easily obtain
\[
\lambda^4 W^3 a^i_k = \lambda^4 W^3 g^{ij} a_{jk} = (\lambda^2 W^2 \sigma^{ij} - z^iz^j) z_{j;ik} - \frac{1}{2} z^i \gamma_k - \lambda^2 W^2 \left( \frac{\gamma^i}{2\gamma} z_k + \gamma \rho \delta^i_k \right).
\]  
(19)

Taking traces after dividing both sides by \(\lambda^3 W^3\) yields
\[
n\lambda H = \frac{1}{\lambda W} \left( \sigma^{ij} - \frac{z^iz^j}{\lambda^2 W^2} \right) z_{j;i} - \frac{1}{\lambda W} \left( \frac{\gamma^i z^i}{2\gamma} + n\gamma \rho \right).
\]  
(20)

Equivalently, we have
\[
\mathcal{Q}[z] := \left( \frac{z^i}{\sqrt{\gamma + z^k z_k}} \right)_{;i} - \frac{1}{\sqrt{\gamma + z^k z_k}} \left( \frac{\gamma^i z^i}{2\gamma} + n\gamma \rho \right) - n\lambda H = 0
\]  
(21)

as we wished.

## 5 The proof of Theorem 1

Finding a conformal Killing graph \(\Sigma(z)\) with prescribed mean curvature function \(H\) and boundary data \(\phi\) amounts to solve the Dirichlet problem
\[
\begin{cases}
\mathcal{Q}[z] = 0 \\
z|\Gamma = \phi.
\end{cases}
\]  
(22)

**Proof of Theorem 1**: We apply the well-known continuity method to the family parametrized by \(\tau \in [0, 1]\) of Dirichlet problems
\[
\begin{cases}
\mathcal{Q}_\tau[z] = 0, \\
z|\Gamma = \tau \phi
\end{cases}
\]  
(23)

where
\[
\mathcal{Q}_\tau[z] = \text{div} \left( \frac{\nabla z}{\sqrt{\gamma + |\nabla z|^2}} \right) - \frac{\langle \nabla \gamma, \nabla z \rangle}{2\gamma \sqrt{\gamma + |\nabla z|^2}} - \tau \left( \frac{n\gamma \rho}{\sqrt{\gamma + |\nabla z|^2}} + n\lambda H \right).
\]

Let \(I\) be the subset of \([0, 1]\) consisting of values of \(\tau\) for which the Dirichlet problem (23) has a \(C^{2,\alpha}\) solution. Then, the proof reduces to show that \(I = [0, 1]\). First, observe that \(I\) is nonempty since \(z = 0\) is a solution for \(\tau = 0\). Moreover, we have that \(I\) is open in view of (16). The difficult part is to show that \(I\) is closed. This follows from the a priori estimates given below in Propositions 8, 9 and 10 and standard theory of quasilinear elliptic partial differential equations [3].

**Remark 7**. We point out that our existence results still hold if \(\phi\) is only assumed continuous. We may approximate \(\phi\) uniformly by smooth boundary data and use the interior gradient estimate to obtain strong convergence on compact subsets of \(\Omega\). A local barrier argument shows that the limiting solutions achieve the given boundary data.
6 Height estimates

In this section, we obtain an a priori height estimate.

**Proposition 8.** Under the assumptions of Theorem 7 there exists a positive constant $C = C(\Omega, H)$ independent of $\tau$ such that

$$|z_\tau|_0 \leq C + |\phi|_0$$

for any solution $z_\tau$ of the Dirichlet problem (23).

**Proof:** In view of (16) we may apply the Comparison Principle (cf. Theorem 10.1 in [3]) to (23). From

$$Q_\tau[\sup \phi] \leq 0 = Q_\tau[z_\tau] \quad \text{and} \quad z_\tau|\Gamma \leq \tau \sup \phi,$$

we conclude that $z_\tau \leq \tau \sup \phi$. Similarly, for a solution $z$ of (22) we obtain from

$$Q_\tau[z] \geq 0 = Q_\tau[z_\tau] \quad \text{and} \quad z|\Gamma \leq z_\tau|\Gamma$$

that $z \leq z_\tau$.

Next we construct barriers on $\Omega_0$ which are subsolutions to (22) of the form

$$\varphi(u) = \inf_\Gamma \phi + f(d(u))$$

where $d(u) = \text{dist}(u, \Gamma)$ and the real function $f$ will be chosen later. Hence,

$$\varphi_i = f'd_i \quad \text{and} \quad \varphi_{ij} = f''d_id_j + f'd_{ij}. \quad (25)$$

At points in $\Omega_0$, we have

$$|\nabla d| = 1.$$  

(26)

It follows that

$$d^id_{ij} = 0$$

(27)

and

$$2\langle \nabla \partial_i \nabla d, \partial_d \rangle = \partial_d |\nabla d|^2 = 0.$$  

Moreover,

$$d^i_j = \sigma^{ij}d_{ij} = \sigma^{ij} \langle \nabla \partial_i \nabla d, \partial_j \rangle = -(n - 1)H_\Gamma \epsilon,$$  

(28)

where $H_\Gamma \epsilon$ denotes the mean curvature of $\Gamma \epsilon \subset \Omega_0$ with respect to $\eta$.

Combining

$$\langle \nabla \gamma, \nabla z \rangle = -\frac{2}{|Y|^4} \langle \nabla \nabla z Y, Y \rangle = 2\gamma^2 \langle \nabla Y Y, \nabla z \rangle.$$  

11
with \((20)\) yields

\[
Q[\varphi] = \frac{1}{U} \left( \varphi^i - \frac{\varphi^i \varphi^j \varphi_{ij}}{U^2} \right) - \frac{\gamma}{U^3} (\gamma + U^2) \langle \nabla_Y Y, \nabla \varphi \rangle - \frac{n \gamma \rho}{U} - n \lambda H. \tag{29}
\]

where

\[
U = \lambda W = \sqrt{\gamma + f^2}.
\]

Using \((25)\) and \((28)\) we obtain

\[
Q[\varphi] = \frac{\gamma}{U^3} (f'' - \gamma \langle \nabla_Y Y, \eta \rangle f') - \frac{f'}{U} ((n-1)H_{\Gamma} + \gamma \langle \nabla_Y Y, \eta \rangle) - \frac{n \gamma \rho}{U} - n \lambda H.
\]

We chose in \((24)\) the test function

\[
f = \frac{e^{DB}}{D} (e^{-Dd} - 1)
\]

where \(B > \text{diam}(\Omega)\) and \(D > 0\) is a constant to be chosen later. Then,

\[
f' = -e^{D(B-d)} \quad \text{and} \quad f'' = -Df'.
\]

Using \((14)\) and that \(\rho_t \geq 0\) by assumption, we obtain

\[
nk^2 - \sqrt{\gamma} k_t = (n-1)k^2 + \bar{\gamma} \rho_t \geq 0.
\]

It follows from Proposition \([6]\) that

\[
H_{K_t} \geq H^* > H \geq 0
\]

where \(H^* = \inf_{\Gamma} H_K > 0\). Since \(\lambda(\varphi) \leq 1\), we obtain using Lemma \([5]\) that

\[
Q[\varphi] \geq -\frac{\gamma f'}{U^3} (D + \kappa_\epsilon) - \frac{f'}{U} nH^* - \frac{n \gamma \rho}{U} - nH.
\]

We require \(D > \sup_{\Omega^0} |\kappa|\) and denote \(nD_0 = D + \kappa_\epsilon\). Thus \(Q[\varphi] > 0\) if

\[
HU^3 < -H^* f'U^2 - \gamma \rho U^2 - \gamma D_0 f'. \tag{30}
\]

Since \(f' \to -\infty\) as \(D \to +\infty\), we conclude that for \(D\) sufficiently large the inequality holds. Hence, we have shown that

\[
Q[\varphi] > 0 = Q[z] \quad \text{and} \quad \varphi|_{\Gamma} \leq z|_{\Gamma}
\]

To prove that \(\varphi \leq z\) on \(\bar{\Omega}\) we just follow the reasoning in the proof of Lemma 6 in \([2]\) (see \([3]\), p. 171). We conclude that \(\varphi\) is a continuous subsolution for the Dirichlet problem \((22)\).
7 Boundary gradient estimates

In this section, we establish an a priori gradient estimate along the boundary of the domain.

Proposition 9. Under the assumptions of Theorem 1 there exists a positive constant
\[ C = C(\Omega, H, \phi, |z_0|) \]
 independent of \( \tau \) such that
\[ \sup_{\Gamma} |\nabla z_\tau| \leq C \]
for any solution \( z_\tau \) of the Dirichlet problem (23).

Proof: We argue for \( \tau = 1 \). We use barriers of the form \( w + \phi \) along a tubular neighborhood \( \Omega_\epsilon \) of \( \Gamma \) where we extended \( \phi \) to \( \Omega_\epsilon \) by taking \( \phi(t^i, d) = \phi(t^i) \). We choose \( w = f(d) \) where
\[ f(d) = -\bar{\mu} \ln(1 + \mu d) \]
and \( \mu > 0, \bar{\mu} > 0 \) are constants. Hence,
\[ f' = \frac{-\mu \bar{\mu}}{1 + \mu d} \quad \text{and} \quad f'' = \frac{1}{\mu} f'^2. \]

We choose \( \bar{\mu} = c/ \ln(1 + \mu) \) with \( c > 0 \) to be chosen later. Hence,
\[ f(\epsilon) = \frac{-c \ln(1 + c \mu)}{\ln(1 + \mu)} \to -c \quad \text{as} \quad \mu \to +\infty \quad \text{for} \quad \epsilon > 0, \]
and
\[ f'(0) = \frac{-c \mu}{\ln(1 + \mu)} \to -\infty \quad \text{as} \quad \mu \to +\infty. \] (31)

A simple estimate using (29) gives
\[ Q[w + \phi] = a^{ij}(x, \nabla w + \nabla \phi)(w_i^j + \phi_i^j) + b(x, \nabla w + \nabla \phi) - n\lambda H \]
\[ \geq a^{ij}w_i^j + \Lambda|\phi|_{2,\alpha} + b - n\lambda H. \]

Here \( \Lambda = \gamma/U^3 \) is the lowest eigenvalue of the matrix
\[ a^{ij} = \frac{\delta^{ij}}{U} - \frac{1}{U^3}(w^i + \phi^i)(w^j + \phi^j) \]
and
\[ U^2 = \theta + f'^2 \quad \text{where} \quad \theta = \gamma + |\nabla \phi|^2 \]
from (26). Moreover,
\[ b = -\frac{\gamma}{U^3}(\gamma + U^2)\langle \nabla_Y Y, \nabla w + \nabla \phi \rangle - \frac{n\gamma \rho}{U}. \]
It follows from (26) and (27) that
\[ w^i w^j \xi^i \xi^j = f'^2 d^i d^j f'' d^i d^j = f'^2 f'', \]
\[ w^i \phi^j \xi^i \xi^j = f' f'' \langle \nabla d, \nabla \phi \rangle = 0 \]
and
\[ \phi^i \phi^j \xi^i \xi^j = f' \phi^i \phi^j d^i d^j. \]

Since
\[ \Delta w = f'' + f' \Delta d = f'' - (n - 1) f' H \Gamma, \]
we obtain
\[ a^i w^i_{\phi^j} = -(n - 1) \frac{f'}{U} H \Gamma d + \frac{f''}{U^3} (\gamma + |\nabla \phi|^2) - \frac{f'}{U^3} \phi^i \phi^j d^i d^j. \]

Moreover, from (7) and \( \nabla w = f' \eta \) we have
\[ b = -\frac{f'}{U} \left( \frac{\gamma}{U^2} + 1 \right) \kappa - \frac{\gamma}{U} \left( \frac{\gamma}{U^2} + 1 \right) \langle \nabla Y, \nabla \phi \rangle - \frac{n \gamma \rho}{U}. \]

Using Lemma 5, we obtain
\[ Q[w + \phi]U^3 \geq -n f' H \Gamma d + \lambda H U^2 + \gamma |\phi|_{2,0}^2 - n \gamma \rho U^2 - \gamma \langle \nabla Y, \nabla \phi \rangle U^2 \]
\[ + f'' (\gamma + |\nabla \phi|^2) - f' \gamma \kappa - f' \phi^i \phi^j d^i d^j - \gamma^2 \langle \nabla Y, \nabla \phi \rangle. \]

Since \( \phi \leq 0 \), we have \( \lambda(\phi) \leq 1 \) and \( \rho(\phi) \leq \rho_0 \). At points of \( \Gamma \), we obtain
\[ Q[w + \phi]U^3 \geq -n (f' H \Gamma d + H \sqrt{\theta + f'^2})(\theta + f'^2) \frac{1}{c} \ln(1 + \mu)(\gamma + |\nabla \phi|^2) f'^2 \]
\[ - \gamma (n \rho_0 + \langle \nabla Y, \nabla \phi \rangle)(\theta + f'^2) - (\gamma \kappa + \phi^i \phi^j d^i d^j)f' + \gamma |\phi|_{2,0} - \gamma^2 \langle \nabla Y, \nabla \phi \rangle \]
where \( f' = f'(0) \) satisfies (31). It is easy to see using inf_\Gamma H_K \geq H \geq 0 that choosing \( \mu \) large enough and then \( c \) large enough assures that \( Q[w + \phi] > 0 \) on a small tubular neighborhood \( \Omega_\epsilon \) of \( \Gamma \) and that \( w + \phi \leq z \) on both boundary components. Therefore, \( w + \phi \) is a locally defined lower barrier for the Dirichlet problem (22).

It is easy to verify that \( w + \tau \phi \) is a lower barrier for solutions of (23). Similarly, just using that \( H_K \geq 0 \) we can see that \( -w + \tau \phi \) is a upper barrier for (23). This concludes the proof for any value of \( \tau \). ■

8 Interior gradient estimates

In this section, we establish an a priori global estimate for the gradient.
Proposition 10. Under the assumptions of Theorem 1 there exists a positive constant \( C = C(\Omega, H, \phi, |\nabla z|_\Gamma_0) \) independent of \( \tau \) such that
\[
\sup_{\Omega} |\nabla z_\tau| \leq C
\]
for any solution \( z_\tau \) of the Dirichlet problem (23).

Proof: The proof follows a similar guideline as in [2]. Consider on \( \Sigma(z) \) the function
\[
\chi = v e^{2Kz},
\]
where \( v = |\nabla z|^2 = z^i z_i \) and \( K > 0 \) is a constant. We already have the desired bound if \( \chi \) achieves its maximum on \( \Gamma \). Thus, we assume that \( \chi \) attains maximum value at an interior point \( \bar{u} \in \Omega \) where \( |\nabla z| > 0 \). This assumption enables us to choose a local normal coordinate system \( x^1, \ldots, x^n \) such that \( \partial_1|\bar{u} = \nabla z/|\nabla z| \) and \( \sigma_{ij}(\bar{u}) = \delta_{ij} \). Hence, at \( \bar{u} \) we have
\[
z_1 = |\nabla z| > 0 \text{ and } z_j = 0 \text{ if } j \geq 2.
\]
Since \( v_j = 2z^i z_{ij} \), we also obtain at \( \bar{u} \) from
\[
\chi_j = 2e^{2Kz} \left( K v z_j + z^i z_{ij} \right) = 0
\]
that \( z^i z_{ij} = -K v z_j \). Therefore, at \( \bar{u} \) we obtain
\[
z_{1,1} = -K v, \quad v_1 = -2K v^{3/2} \quad \text{and} \quad v_j = 0 = z_{1,j} \text{ if } j \geq 2.
\]
Moreover, we may assume after rotating the \( \partial_j \) that \( (z_{i,j}), 2 \leq i, j \leq n, \) is diagonal. We write (20) as
\[
\tilde{a}^{ij} z_{ij} = \tilde{b}, \quad (32)
\]
where
\[
\tilde{a}^{ij} = (\gamma + v) \sigma^{ij} - z^i z^j
\]
and
\[
\tilde{b} = \frac{1}{2} \gamma_i z^i + (\gamma + v) \left( \frac{\gamma_h z^i + n \gamma \rho}{2\gamma} \right) + n \lambda H (\gamma + v)^{3/2}.
\]
(33)
Hence, at \( \bar{u} \) we have
\[
(\gamma + v) z^i_{i,1} = \tilde{b} - K v^2. \quad (34)
\]
Covariant differentiation of (32) followed by contraction with \( \nabla z \) at \( \bar{u} \) gives
\[
\frac{1}{(\gamma + v)} (\gamma_1 v^{1/2} - 2K v^2)(\tilde{b} - K v^2) - 2K^2 v^3 + \tilde{a}^{ij} z^i_{ij} z_{ij} = v^{1/2} \tilde{b}_{i,1}. \quad (35)
\]
The third derivatives of \( z \) satisfy the Ricci identity
\[
z_{i,jl} - z_{i,dl} = R_{ikjl} z^k,
\]
where by $R_{ikjm}$ we denote the coefficients of the curvature tensor of $M$. Hence,

$$\tilde{a}^{ij}z^iz_{i;j} = (\gamma + v)\sigma^{ij}z^i(z_{i;j} + R_{ikjl}z^k) - z^iz^jz^l(z_{i;j} + R_{ikjl}z^k)$$

$$= \tilde{a}^{ij}z^iz_{i;j} + (\gamma + v)R_{kl}z^kz^l - \tilde{a}^{ij}z^iz_{i;j} + R_{11}(\gamma + v)v$$

since $R_{jklm}z^lz^m = 0$.

To estimate $\tilde{a}^{ij}z^iz_{i;j}$ we use the Hessian matrix of $\chi$. Since

$$\chi_{ij} = 2e^{2Kz}(2K^2z_iz_jv + 2Kz_iz_lz_{l;j} + Kz_{i;j}v + 2Kz_jz^lz_{l;i} + z^lz_{l;i} + z^lz_{l;i;j})$$

is nonpositive at $\bar{u}$, it results that $\tilde{a}^{ij}\chi_{ij} \leq 0$. Hence,

$$0 \geq 2K^2\tilde{a}^{ij}z_iz_jv + 4K\tilde{a}^{ij}z_iz_lz_{l;j} + K\tilde{a}^{ij}z_{i;j}v + \tilde{a}^{ij}z^lz_{l;i} + \tilde{a}^{ij}z^lz_{l;i;j}$$

$$= (\gamma + v)(Kv^i + z^iz^l) - 2K^2\gamma v^2 + \tilde{a}^{ij}z^lz_{l;i;j}.$$}

Since $(z_{i,k})$ is diagonal, it follows that

$$z^l_{i;l}z^l_j = (z^l_{i;l})^2 \geq (z^l_{i;1})^2 = K^2v^2.$$}

Using (34) and (36) we obtain

$$\tilde{a}^{ij}z^iz_{i;j} \leq K^2\gamma v^2 - K\tilde{b}v.$$}

Thus,

$$\tilde{a}^{ij}z^iz_{i;j} \leq K^2\gamma v^2 - K\tilde{b}v + R(\gamma + v)v \quad (37)$$

where $R = R_{11}$. Replacing (37) into (35) and multiplying both sides by $(\gamma + v)$ gives

$$K\tilde{b}v(\gamma + 3v) + K^2\gamma v^2(v - \gamma) - Rv(\gamma + v)^2 + \gamma_1v^{1/2}(Kv^2 - \tilde{b}) + \tilde{b}_1v^{1/2}(\gamma + v) \leq 0. \quad (38)$$

Since $\tilde{b} = \tilde{b}(x, z, \nabla z)$, we have at $\bar{u}$ that

$$\tilde{b}_{1;1} = \tilde{b}_{x^1} + \tilde{b}_zv^{1/2} - K\tilde{b}_zv.$$}

We rewrite (38) as

$$K^2\gamma v^2(v - \gamma) - Rv(\gamma + v)^2 + \gamma_1v^{1/2}(Kv^2 - \tilde{b}) + (\gamma + v)\left(\tilde{b}_zv + \tilde{b}_{x^1}v^{1/2}\right) + K\left(\tilde{b}(\gamma + 3v) - \tilde{b}_{x^1}(\gamma + v)v^{1/2}\right) \leq 0. \quad (39)$$

From (33), we obtain

$$\tilde{b}_{x^1} = 2n\gamma_1 + \gamma_1 + n\gamma_1\rho v^{1/2} + \frac{3}{2}n\lambda H\gamma_1(\gamma + v)^{1/2} + (\frac{3}{2}\gamma_1/\gamma_1)\gamma_1v^{1/2} + n\lambda H(\gamma + v)^{3/2}.$$}

Moreover,

$$\tilde{b}_z = n\gamma_1\rho(\gamma + v) + n\lambda H(\gamma + v)^{3/2} \geq 0$$
since $\rho_t \geq 0$ and $\lambda_t H \geq 0$.

We may assume from now on that $\gamma(\bar{u}) \leq v(\bar{u})$. It follows that there exist constants $A_i = A_i(\gamma, \lambda, \rho, H, \nabla \gamma, \nabla H), 1 \leq i \leq 4$, such that

$$ (\gamma + v) \left( \bar{b}_2 v + \bar{b}_4 v^{1/2} \right) \geq \left( (\gamma_1/2\gamma)_1 - 2^{3/2} n \lambda |H_1| \right) v^3 + A_1 v^2 + A_2 v^{3/2} + A_3 v + A_4 v^{1/2}. $$

We also estimate

$$ \gamma_1 v^{1/2} (K^2 - \tilde{b}) \geq K \gamma_1 v^{5/2} + B_1 v^2 + B_2 v^{3/2} + B_3 v + B_4 v^{1/2}, $$

where $B_i = B_i(\gamma, \lambda, \rho, H, \nabla \gamma, \nabla H), 1 \leq i \leq 4$, are constants. Finally, we have

$$ \tilde{b}(\gamma + 3v) - \tilde{b}_2 (\gamma + v) v^{1/2} = \gamma_1 v^{3/2} + n \gamma \rho (\gamma + v)^2 + n \lambda \gamma (\gamma + v)^{3/2} \geq \gamma_1 v^{3/2}. $$

With the above estimates, it follows from (39) that

$$ (K^2 \gamma - R + (\gamma_1/2\gamma)_1 - 2^{3/2} n \lambda |H_1|) v^3 + 2 K \gamma_1 v^{5/2} $$

$$ - (K^2 \gamma^2 + 2 R \gamma - D_1) v^2 + D_2 v^{3/2} - (R \gamma^2 - D_3) v + D_4 v^{1/2} \leq 0 $$

where $D_i = A_i + B_i$. Choosing $K$ large enough so that the coefficient of $v^3$ is positive implies that $v(\bar{u})$ is bounded by a constant $K_0$ which depends only on $\Omega$ and $H$.

Using the fact that $\bar{u}$ is a maximum point for $\chi$, we conclude that

$$ v(u) \leq K_0 e^{2K(z(\bar{u})-z(u))} \leq K_0 e^{4K|z_0|}. $$

Finally, we observe that minor modifications in the calculations above give indeed

$$ |\nabla z_\tau| \leq K_0^{1/2} e^{2K|z_0|} $$

for any $\tau \in [0, 1]$, and this concludes the proof. $\blacksquare$

9 Proof of corollaries

In this section we prove Corollaries 2 and 3 in the Introduction.

Proof of Corollary: We first observe that the proofs of Propositions 9 and 10 still work under the weaker assumption $\inf \gamma H_K \geq H \geq 0$. Thus, it suffices to show that Proposition 8 still works if $H^* \geq H$ when $Y$ is a Killing field. In this situation, it is easy to see that (30) is equivalent to

$$ (H^*)^2 (\gamma + f^2) f^2 - H^2 (\gamma + f^2)^3 + 2 H^* \gamma D_0 f^2 (\gamma + f^2) + \gamma^2 D_0^2 f^2 > 0. $$

Clearly, the last inequality holds for $D$ sufficiently large and the proof follows. $\blacksquare$

Proof of Corollary: Being $Y$ closed we may assume $\gamma = 1$. Thus (8) and (9) yield

$$ n^2 H_k^2 = (n - 1)^2 H_1^2. $$
On the other hand, the relation between the Ricci tensors of $\bar{M}^{n+1}$ and $M^n$ is
\[
\text{Ric}_{\bar{M}}(X, Z) = \text{Ric}_M(X, Z) - (nk^2 - k_t)\langle X, Z \rangle
\]
for any $X, Z \in TM$. Thus (10) is equivalent to
\[
\inf_{\Omega_0} \text{Ric}_M^{rad} \geq -\frac{(n - 1)^2}{n} \inf \Gamma H^2,
\]
and the proof follows. 

Remark 11. We have from (6) that the ambient space $\bar{M}^{n+1}$ is a product manifold endowed with the metric
\[
ds^2 = \lambda^2(t)\psi^2(u)(dt^2 + \psi^{-2}d\sigma^2).
\]
It is thus natural to consider the general situation of an ambient space $\mathbb{R} \times M^n$ endowed with the conformal metric
\[
\tilde{g} = \lambda^2(t, u)g = e^{2\varphi(t, u)}g
\]
where $g$ is the product metric in $\mathbb{R} \times M^n$. In this case, the mean curvature equation for the graph $X = (z(u), u)$ is
\[
\text{div} \left( \frac{\nabla z}{\sqrt{1 + |\nabla z|^2}} \right) - \frac{n}{\sqrt{1 + |\nabla z|^2}} \left( \langle \bar{\nabla} z, \bar{\nabla} \varphi \rangle - \varphi_t \right) - n\lambda H = 0,
\]
where $z(t, u) = z(u)$ and we compute $\langle \bar{\nabla} z, \bar{\nabla} \varphi \rangle$ in the ambient space. We easily conclude that in order to have a maximum principle for the above equation we have to ask $\varphi_t$ to be independent of $t$, that is, the function $\lambda$ has to separate variables. But this is precisely the case we studied in this paper.

References

[1] M. Dajczer and L. J. Alías, Normal geodesic graphs of constant mean curvature. J. Diff. Geometry 75 (2007), 387–401.

[2] M. Dajczer, P. Hinojosa and J. H. de Lira, Killing graphs with prescribed mean curvature. Calc. Var. Partial Diff. Equations 33 (2008), 231–248.

[3] D. Gilbarg and N. Trudinger, “Elliptic partial differential equations of second order”. Springer-Verlag, New York 2001.

[4] Y. Li and L. Nirenberg, Regularity of the distance function to the boundary. Rendiconti Accademia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni 123 (2005), 257–264.
[5] R. Lopez, *A note on radial graphs with constant mean curvature*. Manuscripta Math. **110** (2003), 45–54.

[6] W. Poor, “Differential geometric structures”. Dover, New York 2007.

Marcos Dajczer  
IMPA  
Estrada Dona Castorina, 110  
22460-320 – Rio de Janeiro – Brazil  
marcos@impa.br

Jorge Herbert S. de Lira  
UFC - Departamento de Matematica  
Bloco 914 – Campus do Pici  
60455-760 – Fortaleza – Ceara – Brazil  
jorge.lira@pq.cnpq.br