Learning How to Reach, Swim, Walk and Fly in One Trial: Control of Unknown Systems with Scarce Data and Side Information

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Abstract

We develop a learning-based control algorithm for unknown dynamical systems under very severe data limitations. Specifically, the algorithm has access to streaming and noisy data only from a single and ongoing trial. It accomplishes such performance by effectively leveraging various forms of side information on the dynamics to reduce the sample complexity. Such side information typically comes from elementary laws of physics and qualitative properties of the system. More precisely, the algorithm approximately solves an optimal control problem encoding the system’s desired behavior. To this end, it constructs and iteratively refines a data-driven differential inclusion that contains the unknown vector field of the dynamics. The differential inclusion, used in an interval Taylor-based method, enables to over-approximate the set of states the system may reach. Theoretically, we establish a bound on the suboptimality of the approximate solution with respect to the optimal control with known dynamics. We show that the longer the trial or the more side information is available, the tighter the bound. Empirically, experiments in a high-fidelity F-16 aircraft simulator and MuJoCo’s environments illustrate that, despite the scarcity of data, the algorithm can provide performance comparable to reinforcement learning algorithms trained over millions of environment interactions. Besides, we show that the algorithm outperforms existing techniques combining system identification and model predictive control.

Keywords: Physics-informed learning; data-driven control; system identification; reachable sets.

1. Introduction

Learning how to achieve a complex task has found numerous applications ranging from robotics (Lillicrap et al., 2015; Schulman et al., 2015a; Deisenroth et al., 2013) to fluid dynamics (Kutz, 2017). However, learning algorithms generally suffer from high sample complexity, often requiring millions of samples to achieve the desired performance (Nagabandi et al., 2018; Schulman et al., 2015a). Such data requirements limit the practicability of learning algorithms in real-world scenarios where an excessive number of trials cannot be performed on a physical system. A rather extreme example of such a scenario is an aircraft trying to retain a certain degree of control after abrupt changes in its dynamics, e.g., due to the loss of an engine. In such a scenario, there is a need to learn the dynamics after the abrupt changes using data from only the current trajectory.

We develop a learning-based control algorithm that utilizes data from a single trial and leverages side information on the unknown dynamics to reduce the sample complexity. The data include finitely many noisy samples of the states, the states’ derivatives, and the control signals applied. Under such a severe limitation on the amount of available data, learning can be performed efficiently only by incorporating already known invariant properties of the dynamical system. We refer to such extra knowledge as side information. The side information, typically derived from elementary laws...
Figure 1: The developed learning-based algorithm can achieve near-optimal control of simulated robots and an F-16 aircraft using streaming data obtained from the systems’ ongoing trajectory and side information derived from laws of physics. From left to right, we have the Reacher, Swimmer, Cheetah, and F-16 aircraft simulator environments.

of physics, may be a priori knowledge of the regularity of the dynamics, monotonicity or bounds on the vector field, algebraic constraints on the states, or knowledge of parts of the vector field.

The developed algorithm, using the data and side information available to it, computes an over-approximation of the set of states the system may reach. Then, it incorporates such an over-approximation into a constrained short-horizon optimal control problem, which is solved on the fly.

Specifically, it leverages a data-driven differential inclusion to compute over-approximations of the reachable sets of the system. It first constructs a differential inclusion that contains the unknown vector field. Next, it builds on set contractor programming (Chabert and Jaulin, 2009) to refine the differential inclusion as more data become available. Then, it computes over-approximations of the reachable sets of all dynamics described by the differential inclusion through an interval Taylor-based method (Berz and Makino, 1998; Nedialkov et al., 1999) that can enforce constraints from the side information to reduce the width of the over-approximations.

The obtained over-approximations enable to formulate the data-driven optimal control problem as a nonconvex and uncertain optimization problem. Specifically, we encode the control task as a sequential optimization of a cost function over a time horizon. Even for convex cost functions, the control problem is typically nonconvex. Besides, the predictions of the states’ values at future times cannot be computed due to the unknown dynamics. The developed algorithm leverages the obtained over-approximations to optimize the nonconvex problem under the uncertain states’ predictions.

The algorithm computes approximate solutions to the nonconvex optimization problem through convex relaxations. We develop a sequential convex optimization scheme (Mao et al., 2019) that uses the obtained over-approximation and iteratively linearizes its nonconvex constraint around the previous iteration solution. Thus, each iteration solves a convex optimization problem, and we leverage trust regions to account for the potential errors due to the linearization.

Theoretically, we establish a bound on the suboptimality of the approximate solution with respect to the optimal control solution in the case where the dynamics were known. The bound is proportional to the width of the obtained over-approximations. We show that the longer the trial or the more the side information available, the tighter the over-approximations. Thus, the algorithm achieves near-optimal control as more data streams or more side information is available.

Empirically, through a series of simulation examples, we show that the algorithm can provide performance comparable to reinforcement learning (RL) algorithms, such as D4PG (Barth-Maron et al., 2018) and SAC (Haarnoja et al., 2018), while outperforming the system identification technique with model predictive control SINDyC (Brunton et al., 2016; Kaiser et al., 2018). We train SAC and D4PG over millions of environment interactions before comparing to our approach. We emphasize that if we had made fair comparisons, i.e., the baselines RL algorithms were also trained using streaming data from only the ongoing and single episode, the developed algorithm would have
achieved significantly higher performance than any of these baselines since they cannot learn with such constraints on the amount of data. Specifically, in several control tasks from MuJoCo (Todorov et al., 2012; Tassa et al., 2018), we provide promising and comparative results to D4PG and SAC. Further, in a ground collision avoidance scenario of an F-16 aircraft (Heidlauf et al., 2018), we show that the algorithm outperforms SINDyC and the tuned F-16’s linear-quadratic regulator controller.

Related Work. In our prior work (Djeumou et al., 2021, 2020), we described a data-driven algorithm similar to the algorithm developed in this paper. However, the algorithm (Djeumou et al., 2021, 2020) works only for control-affine dynamics. Further, most of the considered side information is not tailored for robotics systems, and only one-step optimal control problems were investigated. In contrast, the algorithm in this paper is applicable for a more general class of dynamics with polynomial dependency in control. We also evaluate the developed algorithm on highly-complex systems and consider a larger set of side information, e.g., algebraic constraints on states and unknown terms. Besides, we investigate short-horizon rather than one-step optimal control problems.

Several approaches for data-driven control combine model predictive control with system identification or data-driven reachable set estimation. These approaches achieve system identification through sparse regression over a library of nonlinear functions (Kaiser et al., 2018), regression over the set of polynomials of fixed degree with physics-based side information (Ahmadi and El Khadir, 2020), spectral properties of the collected data (Proctor et al., 2016), Koopman theory (Korda and Mezić, 2018), or Gaussian processes (Krause and Ong, 2011; Gahlawat et al., 2020). The approaches (Devonport and Arcak, 2020; Haesaert et al., 2017; Chakrabarty et al., 2018) achieve data-driven estimation of the reachable sets of partially unknown dynamics using either supervised learning or Gaussian processes. They provide only probabilistic guarantees of the correctness of the computed reachable sets while our algorithm computes correct over-approximations.

Recent work (Berberich et al., 2020a,b; Markovsky and Dörfler, 2021; van Waarde et al., 2020; Van Waarde et al., 2020; Coulson et al., 2019) have proposed data-driven control techniques based on the behavioral systems theory foundation (Willems et al., 2004), which bypass the system identification step. These techniques mostly assume linear time-invariant dynamical systems and are extremely performant in such a setting. Except for Ahmadi and El Khadir (2020) that considers limited side information and builds on computationally expensive semidefinite programs solvers, none of the above approaches (in their current form) can exploit the side information in this paper. Besides, through extensive comparisons with SINDyC, DeePC, and Gaussian-based approaches, Djeumou et al. (2021, 2020) empirically demonstrates that: (a) For simple systems such as a unicycle, these techniques achieve significant lower performance (computation time and control suboptimality) than an approach that can exploit side information; (b) These techniques struggle to learn on high-dimensional and complex systems (e.g., quadrotor). Thus, this paper compares against RL techniques even though they work in a drastically different regime of data.

Model-free (Mnih et al., 2015; Oh et al., 2016; Lillicrap et al., 2015; Mnih et al., 2016; Schulman et al., 2015b) and model-based (Nagabandi et al., 2018; Deisenroth and Rasmussen, 2011; Gu et al., 2016; Boedecker et al., 2014; Levine and Abbeel, 2014; Ko and Fox, 2009) RL algorithms have been widely used for data-driven control of complex systems. Model-free algorithms can achieve high performance at the expense of high sample complexity (Schulman et al., 2015b) while model-based algorithms are more data-efficient but are conservative and generally achieve lower performance than model free approaches. In contrast, our algorithm can work with data from only the system’s current trajectory, and increases the data efficiency through side information on the dynamics.
2. Background

**Notation.** We denote an interval by \([a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}\) for some \(a, b \in \mathbb{R}\) such that \(a \leq b\), the set \(\{i, \ldots, j\}\) by \(\mathbb{N}[i, j]\) for \(i, j \in \mathbb{N}\) with \(i \leq j\), the \(k^{th}\) component of a vector \(x\) and the \((k, j)\) component of a matrix \(X\) by \(x_k\) and \(X_{k,j}\), respectively, the weighted norm of a vector \(x \in \mathbb{R}^n\) by \(\|x\|_w = \sqrt{\sum_{i=1}^n (w_ix_i)^2}\) for some \(w \in \mathbb{R}^n\), and the Lipschitz constant of \(f : \mathcal{X} \to \mathbb{R}\) by \(L^w_f = \sup\{L \in \mathbb{R} : |f(x) - f(y)| \leq L\|x - y\|_w, x, y \in \mathcal{X}, x \neq y\}\) for \(\mathcal{X} \subseteq \mathbb{R}^n\).

**Interval Analysis.** We denote the set of intervals on \(\mathbb{R}\) by \(\mathbb{I}\) by \(\mathbb{I} = [\mathbb{A}, \overline{\mathbb{A}}] | \mathbb{A}, \overline{\mathbb{A}} \in \mathbb{R}, \mathbb{A} \leq \overline{\mathbb{A}}\), the set of \(n\)-dimensional interval vectors by \(\mathbb{I}^n\), and the set of \(n \times m\)-dimensional interval matrices by \(\mathbb{I}^{n \times m}\). We carry forward the definitions (Moore, 1966) of arithmetic operations, set inclusion, and intersections of intervals to interval vectors and matrices by applying them componentwise. We use the term interval to specify an interval vector or interval matrix when it is clear from the context.

**Interval-Based Contractor.** Interval-based contractor programming is a mathematical framework to solve constraints involving interval variables. Given an initial over-estimation of the constraint’s solutions, a contractor filters such variable domains, i.e., reduces the interval of each variable, without loss of solutions of the constraints. Consider the constraint \(h(\cdot) \leq 0\). Assume that \(\mathcal{A} = [\mathcal{A}_1, \ldots, \mathcal{A}_n] \in \mathbb{I}^n\) is a set containing the solutions. Then, the contractor operator computes \(C_h^\mathcal{A} = [C_{h_{\mathcal{A}_1}}, \ldots, C_{h_{\mathcal{A}_n}}] \subseteq \mathbb{I}^n\) such that \(C_{h_{\mathcal{A}_i}} \subseteq \mathcal{A}_i, \forall i \in \mathbb{N}[1,n]\) and \(h(x) > 0\) for all \(x \in \mathcal{A} \setminus C_{h_{\mathcal{A}_i}}\).

Several polynomial-time algorithms (Benhamou et al., 1999; Van Hentenryck et al., 1997; Trombettoni et al., 2010) have been developed to compute contractors associated with a given constraint. For example, HC4-Revise (Benhamou et al., 1999) is a linear-time algorithm that provides optimal contractors when each variable appears only once in the constraint. In the following, we use \(C_{h_{\mathcal{A}}}^\mathcal{A}\) to refer to the contracted interval resulting from any of these algorithms.

3. Problem Formulation

This paper considers nonlinear dynamics with polynomial dependency in the control inputs as

\[
\dot{x} = f(x) + \sum_{p=1}^d g_p(x)u[\alpha^p],
\]

where \(d \in \mathbb{N}, \alpha^p \subseteq \mathbb{R}^m\) is known, the state \(x : \mathbb{R}_+ \mapsto \mathcal{X}\) is a continuous-time signal evolving in \(\mathcal{X} \subseteq \mathbb{R}^n\), \(u[\alpha^p] = u_1^{\alpha_1^p} \cdots u_m^{\alpha_m^p}\) is a monomial with variables from the control signal \(u : \mathbb{R}_+ \mapsto \mathcal{U}\) where \(\mathcal{U} \subseteq \mathbb{R}^m\). The vector-valued functions \(f = [f_k] : \mathbb{R}^n \mapsto \mathbb{R}^n\) and \(g_p = [g_{p,k}] : \mathbb{R}^n \mapsto \mathbb{R}^n\) are considered to be nonlinear and unknown. Note that even if the dynamics are not in the class above, Taylor expansion provides a tight approximation of the dynamics that lies in such a class.

**Assumption 1 (LIPSCHITZ SYSTEMS)** Given a set \(\mathcal{A} \subseteq \mathbb{R}^n\), \(f_k\) and \(g_{p,k}\) admit local Lipschitz constants \(L_{f_k}^w, L_{g_{p,k}}^w > 0\) on \(\mathcal{A}\), for some \(w \in \mathbb{R}^n_+\), and for all \(k \in \mathbb{N}[1,n], p \in \mathbb{N}[1,d]\).

Assumption 1 is common in the framework of optimal control. We emphasize that even though we use the weighted norm to define the Lipschitz constants, the results of this paper can be straightforwardly extended to general modulus of continuity assumption on \(f\) and \(g_p\). The weighted norm has the advantage of providing information on the relative importance of each variable in the function.
Besides, the domain $\mathcal{X} \in \mathbb{R}^n$ is bounded. Thus, by Assumption 1, $f_k$ and $g_{p,k}$ admit global Lipschitz constants on $\mathcal{X}$. We exploit such a knowledge by assuming known upper bounds on the Lipschitz constants. That is, we have access to $L_{f_k} \in \mathbb{R}_+$ and $L_{g_{p,k}} \in \mathbb{R}_+$ as known upper bounds on the Lipschitz constants $L_{f_k}$ and $L_{g_{p,k}}$, respectively, for $k \in \mathbb{N}_{[1,n]}$ and $p \in \mathbb{N}_{[1,d]}$. We emphasize that the Lipschitz bounds can be directly estimated from data at the expense of weakening some of the guarantees in this paper. Our numerical experiments use Lipschitz bounds estimated from data.

In a discrete-time setting, we denote the initial time by $t_1 \geq 0$ and the current time by $t_j > t_1$ for some $j > 1$. Let $\mathcal{F}_j = \{(\tilde{x}^i, \tilde{x}^i, u^i)\}_{i=1}^{j-1}$ be the finite-length set of observations obtained between $t_1$ and $t_j$. The dataset $\mathcal{F}_j$ contains $j - 1$ noisy samples of the exact state $x^i = x(t_i)$, the derivative $\dot{x}_i = \dot{x}(t_i)$ of the state, and the applied input $u^i = u(t_i)$. We build on the widely-used bounded noise assumption and consider that $|x(t) - \tilde{x}(t)| \leq \eta, |\dot{x}(t) - \tilde{x}(t)| \leq \tilde{\eta}$ for all $t \in \mathbb{R}_+$ and for some vector values $\eta, \tilde{\eta} \in \mathbb{R}_n$. Here the absolute value and the comparison are conducted elementwise.

We seek to control the unknown dynamical system (1) by finding $u^1, \ldots, u^{j+N} \in \mathcal{U}$ that are solutions of the $N$-step optimal control problem

$$\min_{u^1, \ldots, u^{j+N} \in \mathcal{U}} \sum_{q=j}^{j+N} c(x^q, u^q, x^{q+1} = x(t_{q+1}; x_q, u_q)),$$

where $N$ is the planning horizon, $c$ is a known cost function, $x^j = x(t_j)$ is the known current state of the system, $t_q = t_j + (q - j)\Delta t$, $\Delta t$ is a constant time step, and $x^{q+1} = x(t_{q+1}; x^q, u^q)$ is the state at $t_{q+1}$, i.e., a solution of the differential equation (1) at $t_{q+1}$ when $x^q$ is the initial state and $u^q$ is the constant control applied between $[t_q, t_{q+1}]$. The optimization problem (2) is generally nonconvex since the state at $t_{q+1}$ is nonconvex due to the nonlinear dynamics. Besides, $x^{q+1}$ cannot be computed due to the unknown dynamics.

**Problem 1** Given the dataset $\mathcal{F}_j$, the current state $\tilde{x}^j$, compute an approximate solution to the $N$-step optimal control problem (2) and characterize the suboptimality of such approximation.

### 4. Reachable Set Over-Approximation via Data-Based Differential Inclusions

In this section, we first construct a differential inclusion $\dot{x} \in f(x) + \sum_{p=1}^{d} g_p(x)u^p$ that contains the unknown vector field. Then, we adapt an interval Taylor-based method to over-approximate the reachability set of dynamics described by the constructed differential inclusion. Finally, we show how additional side information constrains the Taylor expansion to provide tighter over-approximations.

**Lemma 1 (OVER-APPROXIMATION OF $f$ AND $g_p$)** Let the set $\mathcal{E}_j = \{(\tilde{x}^i, C_{\mathcal{F}_i}, C_{G_i})\}_{i=0}^{j-1}$ be such that $C_{\mathcal{F}_i} = [C_{\mathcal{F}_i}] \in \mathbb{R}^n$ and $C_{G_i} = [C_{G_{p,k}}] \in \mathbb{R}^{d \times n}$ satisfy $f_k(\tilde{x}^i) \in C_{\mathcal{F}_k}$ and $g_{p,k}(\tilde{x}^i) \in C_{G_{p,k}}$ for all $p \in \mathbb{N}_{[1,d]}$ and $k \in \mathbb{N}_{[1,n]}$. Then, the interval-valued functions $f = [f_k] : \mathbb{R}^n \to \mathbb{R}^n$ and $g_p = [g_{p,k}] : \mathbb{R}^n \to \mathbb{R}^n$, defined by $f_k(A) = \bigcap_{(\tilde{x}, C_{\mathcal{F}_i}, C_{G_i}) \in \mathcal{E}_j} C_{\mathcal{F}_k} + [-1, 1][\mathcal{F}_k, \eta^w(A - \tilde{x}^i)]$ and $g_{p,k}(A) = \bigcap_{(\tilde{x}^i, C_{\mathcal{F}_i}, C_{G_i}) \in \mathcal{E}_j} C_{G_{p,k}} + [-1, 1][\mathcal{F}_k, \eta^w(A - \tilde{x}^i)]$, are such that $\mathcal{R}(f_k, A) \subseteq f_k(A)$ and $\mathcal{R}(g_{p,k}, A) \subseteq g_{p,k}(A)$ for all $A \subseteq \mathcal{X}$. Furthermore, the function $\eta^w : \mathbb{R}^n \to \mathbb{R}$ can be any straightforward interval extension of the weighted norm $\| \cdot \|_w$.

We provide a proof of the lemma and an expression for $\eta^w$ in the extended version of the paper (Djeumou and Topcu, 2021). Intuitively, Lemma 1 states that if a set $\mathcal{E}_j = \{(\tilde{x}^i, C_{\mathcal{F}_i}, C_{G_i})\}_{i=0}^{j-1}$ is known, it is possible to obtain an analytic formula to over-approximate the unknown $f$ and $g_p$ via the Lipschitz bounds. Lemma 2 enables to compute the set $\mathcal{E}_j$ based on the data $\mathcal{F}_j$.  

5
Lemma 2 (Refinement via Contractor) Given a data point \((\tilde{x}^i, \tilde{x}^j, u^i) \in \mathcal{T}_j\), an interval \(\mathcal{F}^i = [\mathcal{F}^i_k] \subseteq \mathbb{IR}^n\) such that \(f(\tilde{x}^i) \in \mathcal{F}^i\), and an interval \(\mathcal{G}^i = [\mathcal{G}^i_p, k] \subseteq \mathbb{IR}^d \times \mathbb{N}\) such that \(g_{p,k}(\tilde{x}^i) \in \mathcal{G}^i_{p,k}\) for all \(p \in \mathbb{N}_{[1,d]}, k \in \mathbb{N}_{[1,n]}\). Let the intervals \(C_{\mathcal{F}^i} \subseteq \mathbb{IR}^n\) and \(C_{\mathcal{G}^i} \subseteq \mathbb{IR}^d \times \mathbb{N}\) defined by

\[
C_{\mathcal{F}^i_k} = \mathcal{F}^i_k \cap \left\{ \tilde{N}^i_k - \sum_{p=1}^d \mathcal{G}^i_{p,k} u^i[\alpha^p] \right\}, C_{\mathcal{G}^i_p,k} = \left\{ \left\{ \left( S_{p-1,k} - \sum_{l=p+1}^d \mathcal{G}^i_{l,k} u^i[\alpha^l] \right) \cap \mathcal{G}^i_{p,k} \right\} u^i[\alpha^p] \right\}^{-1},
\]

if \(u^i[\alpha^p] \neq 0\),

\[
S_{0,k} = \left\{ \tilde{N}^i_k - C_{\mathcal{F}^i} \right\} \cap \left\{ \sum_{p=1}^d \mathcal{G}^i_{p,k} u^i[\alpha^p] \right\}, S_{p,k} = \left\{ S_{p-1,k} - C_{\mathcal{G}^i_p,k} \right\} \cap \left\{ \sum_{l=p+1}^d \mathcal{G}^i_{l,k} u^i[\alpha^l] \right\},
\]

for successive values of \(k \in \mathbb{N}_{[1,n]}\) and for all \(p \in \mathbb{N}_{[1,d]}\) with \(\tilde{N}^i_k = [\tilde{x}^i - \tilde{\eta}, \tilde{x}^i + \tilde{\eta}]\). Then, \(C_{\mathcal{F}^i}\) and \(C_{\mathcal{G}^i}\) are the smallest intervals enclosing \(f(\tilde{x}^i)\) and \(g_{p,k}(\tilde{x}^i)\), given only the data \((\tilde{x}^i, \tilde{x}^j, u^i, u^j)\), \(\mathcal{F}^i\), \(\mathcal{G}^i\).

Algorithm 1 Construct: Compute \(\mathcal{E}_j\) required to over-approximate \(f\) and \(g_p\) at each data point of a given trajectory.

Input: Dataset \(\mathcal{T}_j\) and a parameter \(M > 0\).

Output: \(\mathcal{E}_j = \{ (\tilde{x}^i, f(\tilde{x}^i), \mathcal{G}^i) \}_{i=0}^{j-1}\).

1. \(A \leftarrow X^j, R_{\mathcal{F}^i} R_{\mathcal{G}^i} = [-M, M]^n\)
2. Define \(\tilde{x}^0 \in A, C_{\mathcal{F}^0} \leftarrow R_{\mathcal{F}^0}, C_{\mathcal{G}^0} \leftarrow R_{\mathcal{G}^0}\)
3. for \(i \in \mathbb{N}_{[1,j]} \land (\tilde{x}^i, \tilde{x}^j, u^j) \in \mathcal{T}_j\) do
4. \(\mathcal{E}_i \leftarrow \text{Refine}(f(\tilde{x}^i), \tilde{x}^j, u^j, \mathcal{E}_{i-1}, \mathcal{T}_i)\)
5. end for
6. return \(\mathcal{E}_j\)

The proof of the lemma is provided in the extended version of the paper (Djeumou and Topcu, 2021). Lemma 2 provides tighter sets \(C_{\mathcal{F}^i} \subseteq \mathcal{F}^i\) and \(C_{\mathcal{G}^i} \subseteq \mathcal{G}^i\) that prune out from \(\mathcal{F}^i\) and \(\mathcal{G}^i\) some values \(f(\tilde{x}^i)\) and \(g_{p,k}(\tilde{x}^i)\) that do not satisfy the dynamics constraint \(\dot{x}^i = f(x^i) + \sum_{p=1}^d g_{p,k}(x^i) u[\alpha^p]\).

Theorem 1 (Data-driven Differential Inclusion) Given a dataset \(\mathcal{T}_j\), the bounds \(\mathcal{F}^i_{p,k}\) and \(\mathcal{G}^i_{p,k}\), it holds that the unknown vector field of the dynamics (1) satisfies

\[
\dot{x} \in h(x,u) \triangleq f(x) + \sum_{p=1}^d g^p(x) u[\alpha^p],
\]

where \(f\) and \(g_p\) are obtained from Lemma 1 with \(\mathcal{E}_j\) taken as the output of Algorithm 1.

Remark 1 (Persistent Excitation) The quality of the differential inclusion (3) depends on how much information on \(f\) and \(g_p\) can be obtained from the dataset \(\mathcal{T}_j\). This is the classical observability problem, sometimes referred to as persistent excitation. Thus, the learning algorithm should sometimes take suboptimal actions through persistent excitations of the system.

Finally, we compute over-approximations of the reachable sets of all dynamics described by the differential inclusion (3). Theorem 2 provides a closed-form expression for such a set.

Theorem 2 (Data-driven Reachable Set Over-approximation) Given the dataset \(\mathcal{T}_j\), a constant control signal \(u : t \rightarrow u^q\) on the interval \([t_q, t_{q+1}]\) with \(u^q \in U\), and the uncertain set
In this section, we develop an algorithm that computes approximate solutions to the optimal control problem. Specifically, this constraint can be formulated as the new constraint \( w(f(x), [g_p, k(x)], u, x) \geq 0 \). In some cases, another constraint \( z(f(x), [g_p, k(x)], \frac{\partial f}{\partial x}(x), [\frac{\partial g_p, k}{\partial x}(x)], u, x) \geq 0 \) can be derived by differentiating \( w \). The new constraints \( w \) and \( z \) can be incorporated in the computation of \( R^{q+1} \) through contractors. More specifically, the refinement algorithm and the interval extensions of the Jacobian can be improved by additionally contracting with respect to the constraints \( w \) and \( z \). Thus, such side information enables to obtain a tighter \( R^{q+1} \). We develop on more side information in the extended paper.

5. Approximate Optimal Control

In this section, we develop an algorithm that computes approximate solutions to the optimal control problem (2) using over-approximations of the reachable sets. Further, we characterize the suboptimality of the approximate solutions with respect to the case of known dynamics.
The nonconvexity in the optimal control problem (2) is due to the possibly nonconvex cost function \( c \) and the nonconvex constraint \( x^{q+1} = x(t_{q+1}; x^q, u^q) \). We replace such an expression by \( x^{q+1} = \hat{h}^\theta(x^q, u^q) \in \mathcal{R}^{q+1} \), where the function \( \hat{h}^\theta \), parameterized with \( \theta \in \mathbb{R}^n \), is a trajectory picked inside \( \mathcal{R}^{q+1} \). For example, a straightforward choice can be \( \hat{h}^\theta(x^q, u^q) = \theta \mathcal{R}^{q+1} + (1 - \theta)\mathcal{R}^{q+1} \), for \( \theta \in [0, 1]^n \). Then, we solve the nonconvex problem by sequentially linearizing \( x^{q+1} \) and the cost function \( c \) around the solution of the \( s \)th iteration. This results into a convex subproblem that is solved to full optimality. The obtained solutions are then used at the \((s + 1)\)th iteration.

**Linearization.** Let \( x = [x^j; \ldots; x^{j+N}] \in \mathbb{R}^{nN} \) and \( u = [u^j; \ldots; u^{j+N}] \in \mathbb{R}^{mN} \). We denote the solutions of the \( s \)th iteration by \( x^s = [x^{j+1}; \ldots; x^{j+N+s}] \) and \( u^s = [u^{j+1}; \ldots; x^{j+N+s}] \). Then, we can approximate the gradient of \( h^\theta \) (or \( x(t_{q+1}; x^q, u^q) \)) around \( x^s, u^s \) as follows:

\[
\begin{align*}
A^{q,s} &= \frac{\partial h^\theta(x^q, u^q)}{\partial x^q} \bigg|_{x^{q,s}, u^{q,s}} \in \mathbb{I} + \left( \mathcal{J}^f(x^q,s) + \sum_{p=1}^d \mathcal{J}^{g_p}(x^q,s,u^p) \right) \Delta t = A^{q,s}, \\
B^{q,s} &= \frac{\partial h^\theta(x^q, u^q)}{\partial u^q} \bigg|_{x^{q,s}, u^{q,s}} \in \mathbb{I} + \left( \Delta t \mathcal{G}_{p,k}(x^q,s) \frac{\partial u^p}{\partial u^q} \bigg|_{u^{q,s}} \right) = B^{q,s},
\end{align*}
\]

where \( \mathbb{I} \) is the identity matrix of appropriate dimensions. The Jacobians \( \mathcal{J}^f(x^q,s), \mathcal{J}^{g_p}(x^q,s) \) are exactly \( \mathcal{J}^f \) and \( \mathcal{J}^{g_p} \) when no extra side information are given. With side information, the matrices are computed through chain rules as described in side information 1. Note that since we neglect the term in \( \Delta t^2 \), \( A^{q,s} \) and \( B^{q,s} \) are approximations of the actual range of the gradients of \( h^\theta \).

Next, we define the variables \( \Delta x = x - x^s, \Delta x^q = x^q - x^{q,s}, \Delta u = u - u^s, \) and \( \Delta u^q = u^q - u^{q,s} \) in terms of the unknown solutions of the current iteration \( x \) and \( u \). Thus, at the \((s + 1)\)th iteration, the first-order approximation of \( x^{q+1} = \hat{h}^\theta(x^q, u^q) \) around the previous solution \( x^{q,s}, u^{q,s} \) is

\[
x^{q+1,s} + \Delta x^{q+1} = \hat{h}^\theta(x^{q,s}, u^{q,s}) + A^{q,s} \Delta x^q + B^{q,s} \Delta u^q + v^q,
\]

where \( v = [v^j; \ldots; x^{j+N}] \) are penalty variables that enable the linearization to be always feasible. Further, to ensure that the variable \( v^q \) is used only when necessary, we augment the cost function with the sufficiently large penalization weight \( \lambda > 0 \). Thus, the solution for the \((s + 1)\)th iteration, optimizes the penalized and linearized cost given by \( L^s(\Delta x, \Delta u) = \sum_{q=1}^{j+N} \left( c(x^{q,s}, u^{q,s}, x^{q+1,s}) + \nabla c(x^{q,s}, u^{q,s}, x^{q+1,s}) (\Delta x; \Delta u) + \lambda \sum_{q=1}^{j+N} \|v^q\| \right) \), where we also linearize the possibly nonconvex function \( c \) given that \( \nabla c \) is its gradient, and \( \| \cdot \| \) can be either the infinity norm or 1-norm. In order to verify the linearization accuracy, we also define the nonlinear realized cost \( J(x, u) = \sum_{q=1}^{j+N} c(x^q, u^q, x^{q+1}) + \lambda \sum_{q=1}^{j+N} \| x^{q+1} - h^\theta(x^q, u^q) \| \).

**Trust region constraints and linearized problem.** We impose the trust region constraint \( \| \Delta u \| \leq r^s \) to ensure that \( u \) does not deviate significantly from the control input \( u^s \) obtained in the previous iteration, where \( r^s \) will be updated at each iteration so that the \( x \) remains close to \( x^s \). This update rule enables to keep the solutions within the region where the linearization is accurate. As a consequence, each iteration of our algorithm solves the following linear optimization problem:

\[
\begin{align*}
\text{minimize} & \quad L^s(\Delta x, \Delta u) \\
\text{subject to} & \quad (5), \| \Delta u \| \leq r^s, u^s + \Delta u \in \mathcal{U}^N, x^s + \Delta x \in \mathcal{X}^N.
\end{align*}
\]

The optimal solution of the linearized problem is either accepted and used in the next iteration or rejected until convergence. When the linearization is considered accurate, i.e., the realized cost \( J \) and linearized cost \( L^s \) are similar, the solution is accepted and the trust region is expanded. Otherwise, the solution is rejected and the trust region is contracted.
Theorem 3 (Suboptimality Bound) Assume that $L_c$ with the 2-norm is the Lipschitz constant of the cost $c$ on $X \times U \times X$. Let $C_j^*$ and $C_j$ be the optimal costs of the $N$-step control problem (2) when the dynamics are known, e.g. $x^{q+1} = x(\cdot, x^q, u^q)$ is known, and the dynamics are unknown, e.g., $x^{q+1} = h^0(x^q, u^q) \in \mathcal{R}^{q+1}$. Then, $|C_j^* - C_j| \leq L_c \left( \|\text{wd}(\mathcal{R}_{j+1}^{q+1})\|_2 + \sum_{q=j+1}^{q=N} \|\text{wd}(\mathcal{R}_q^j)\|_2 \right)$ holds with $\text{wd}(\mathcal{A}) = \overline{A} - \underline{A}$ being the width of the interval $\mathcal{A}$. The interval $\mathcal{R}_q^j$ is the over-approximation of the reachable set at time index $t_q$ from the initial uncertain set $\mathcal{R}_q$ (with $\mathcal{R}_q = \hat{x}_j$) and for all $u^q \in U$.

Theorem 3 provides that the suboptimality bound is proportional to the width of the over-approximation of the reachable set. Thus, our algorithm achieves near-optimal control with more data along the trajectory and more side information, as the over-approximations become tighter.

6. Numerical Experiments

In this section, we empirically demonstrate that the algorithm, using data from only the current trial and the least amount of side information necessary to learn, can achieve performance comparable to the highly-tuned implementations of D4PG (Hoffman et al., 2020) and SAC (Yarats and Kostrikov, 2020) trained over ten million of interactions with the environments. We emphasize that the comparison is unfair to our algorithm since, at each evaluating episode, it learns from only the thousand data obtained during the episode. Further, we show in an F-16 aircraft simulator, a 13-states and 4-control inputs nonlinear dynamics with polynomial control, that (a) The algorithm outperforms systems identification approaches such as SINDyc (Kaiser et al., 2018) ; (b) The algorithm can meet real-time requirements. We provide further details on the numerical experiments in the extended paper (Djeumou and Topcu, 2021). A video of the simulations is at https://tinyurl.com/hdem8x76, and the code at https://github.com/wuwushrek/datacontrolreach.git.

Experiments in MuJoCo. The equations of motion for multi-joint dynamical systems in the MuJoCo environment are as follows: $M(q)\ddot{q} + b(\dot{q}, q) = h(u) + J^T_c(q)F_c(\dot{q}, q, u)$, where $q$ is the system’s state, $M(q)$ is the inertial matrix, $b(\dot{q}, q)$ contains coriolis, centrifugal, gravitational and passive forces, $J^T_c(q)$ is the contact Jacobian matrix, and $F_c(\dot{q}, q, u)$ is the contact force.

For each environment, the cost function is provided by MuJoCo, and we perform numeric differentiation in order to find its gradient. The Lipschitz bounds are under-estimated using only 1000 data points obtained prior to the on-the-fly control. The Reacher environment does not consider any side information other than the Lipschitz bounds, while Swimmer and Cheetah consider that $M(q)$ is known (Side information 1) in order to start learning. Indeed, without such side information, our algorithm fails to learn to control due to the large over-approximations of reachable sets. $M(q)$ is typically obtained for a robot through Euler-Lagrange formulation that uses the kinetic and potential energy. Further, we reduce the over-approximation of the contact force $F_c$ by considering the Coulomb law of friction. That is, via Side information 2, we impose the constraints $F_c^1 \geq 0$ and $F_c^1 \geq \sqrt{(F_c^2)^2\mu_1 + (F_c^3)^2\mu_2}$ at each contact point, where $F_c^1$ is the normal force value, $F_c^2$ and $F_c^3$ are the tangential forces, and $\mu_1, \mu_2$ are the friction coefficients.

Figure 2 demonstrates that it is possible to learn to control with only data from a single episode by leveraging side information. In Cheetah, more side information can improve our algorithm’s performance. We reduced the time-step value of Reacher to accommodate our algorithm and observed that D4PG was unable to learn the task solely due to such a change, while SAC was not affected.
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**Figure 2:** From left to right, we plot the (average) immediate reward over 100 episodes. The experiments show that our algorithm can yield performance comparable to D4PG and SAC.

**Data-driven control of an F-16 aircraft.** We consider a scenario involving an F-16 aircraft (Heidlauf et al., 2018) diving towards the ground at a low altitude and a high downward pitch angle. We show how our algorithm can prevent a ground collision using only the measurements obtained during the dive and elementary laws of physics as side information. We compare our algorithm with the linear-quadratic regulator (LQR) of the simulator, a pre-trained neural network for the task, and SINDYc achieving sparse system identification from a library of functions.

Our algorithm considers the structural knowledge of rigid-body dynamics while assuming that the aerodynamics forces and moments are completely unknown. In other words, the effect of the control inputs on the aircraft is unknown. For example, from the first principles, the lateral velocity’s derivative is given by $rv - qw - g \sin \theta + \frac{F_u}{m}$, where the structure is generic but the aerodynamic force $F_u$ (specific to the aircraft) is unknown.

We use the library PySINDY (de Silva et al., 2020) for the comparison with system identification. We considered monomials (up to degree 6), sines and cosines of the state, and the products of these functions with the control inputs as the library functions. We provide the noisy measurements of the state and its derivatives to both SINDYc and our algorithm. Our algorithm uses Lipschitz bounds estimated using 1000 data points. Finally, the neural network baseline was trained via policy optimization. Figure 3 empirically demonstrates the effectiveness of the proposed approach.

**Figure 3:** Our algorithm enables the F-16 to avoid the ground collision while the embedded LQR controller and SINDYc fail to avoid the crash. Further, it can be applied in real time since the compute time is less than the control time step enforced by the simulator.

7. **Conclusion**

This paper develops a learning-based, data-efficient control algorithm for unknown systems using streaming data from an ongoing trial and available side information. The experiments demonstrate that it is possible, with data from a single episode and side information, to perform comparably to learning algorithms trained over millions of environment interactions. Further, we empirically show that the algorithm is fast and can be used in a scenario with real-time constraints.
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Learning to Reach, Swim, Walk and Fly in One Trial: 
Control of Unknown Systems with Scarce Data and Side Information
– Supplementary Material –

In this supplementary material, we provide the proofs of the lemma and theorems described in the paper, and additional insights on the numerical experiments.

Proof of Lemma 1
This is a direct result from combining the arithmetic of intervals and the definition of the bounds \( \overline{f}_k \) and \( \overline{g}_{p,l} \) provided by the Lipschitz assumption. Specifically, from the upper bound \( \overline{f}_k \) on the Lipschitz constant of \( f_k \), we have that
\[
|f_k(x) - f_k(y)| \leq \overline{f}_k \|x - y\|_w, \quad \forall x, y \in \mathcal{X}.
\]
Hence, given \((\tilde{x}^i, C_{F_i}, C_{G_i^l}) \in \mathcal{E}_j \) and \( x \in \mathcal{X} \), we can write that \( f_k(x) \in f_k(\tilde{x}^i) + [-1, 1] \overline{f}_k \|x - \tilde{x}^i\|_w \), and therefore \( f_k(x) \in C_{\mathcal{F}_i} + [-1, 1] \overline{f}_k \|x - \tilde{x}^i\|_w \) due to \( f(\tilde{x}^i) \in C_{\mathcal{F}_i} \). Now we want to extend the function \( \| \cdot \|_w \) to a function \( \eta^w(\cdot) \) in the domain of intervals. We have that
\[
\eta^w(S) = \beta_1 \left( \sum_{i=1}^n w_i \beta_2(S_i) \right), \quad \forall S \in \mathbb{R}^n,
\]
where the functions \( \beta_1 : \mathbb{R} \mapsto \mathbb{R} \) and \( \beta_2 : \mathbb{R} \mapsto \mathbb{R} \) are interval extensions of \( \sqrt{\cdot} \) and \((\cdot)^2\), respectively. For any \( S \in \mathbb{R} \), we have that
\[
\beta_1(S) = \begin{cases} 
\sqrt{S}, & \text{if } S \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]
\[
\beta_2(S) = \begin{cases} 
\max\{S^2, S, 0\}, & \text{if } 0 \in S, \\
\max\{S^2, S\}, & \text{otherwise}.
\end{cases}
\]

By monotonicity of \( \sqrt{\cdot} \) and \((\cdot)^2\), it is easy to see that \( \{\sqrt{a} | a \in A\} \subseteq \beta_1(A) \) and \( \{a^2 | a \in A\} \subseteq \beta_2(A) \) for given \( A \in \mathbb{R} \). Hence, it is immediate that \( \eta^w \) is an interval extension of \( \| \cdot \|_w \). Thus, for all \( x \in A \subseteq \mathcal{X} \), we have
\[
f_k(x) \in f_k(\tilde{x}^i) + [-1, 1] \overline{f}_k \|x - \tilde{x}^i\|_w \in f_k(x^i) + [-1, 1] \overline{f}_k \eta^w(A - \tilde{x}^i).
\]
Therefore, \( \mathcal{A}(f_k, A) = \{f_k(x) | x \in A\} \subseteq f_k(\tilde{x}^i) + [-1, 1] \overline{f}_k \eta^w(A - \tilde{x}^i) \). The previous belonging relation is valid for every data \((\tilde{x}^i, C_{F_i}, C_{G_i^l}) \in \mathcal{E}_j \) and, as a result, \( \mathcal{A}(f_k, A) \subseteq f_k(A) \). The same reasoning applied to \( \overline{g}_{p,k} \) enables to show that \( \mathcal{A}(g_{p,k}, A) \subseteq g_{p,k}(A) \).

Proof of Lemma 2
The proof is very similar to the result in Djeumou et al. (2021), proven for the noiseless data setting. Given the knowledge that \( f(\tilde{x}^i) \in \mathcal{F}_i \) and \( g_{p,k}(\tilde{x}^i) \in \mathcal{G}_{p,k}^l \), we seek for tighter intervals \( C_{F_i} \subseteq \mathcal{F}_i \) and \( C_{G_{p,k}^l} \subseteq \mathcal{G}_{p,k}^l \) that prune out some values \( f_k(\tilde{x}^i) \) and \( g_{p,k}(\tilde{x}^i) \) from \( \mathcal{F}_k \) and \( \mathcal{G}_{p,k}^l \) that do not satisfy...
the constraint \( \dot{x}^i = f(\bar{x}) + \sum_{p=1}^{d} g_p(\bar{x}^i)u^i[\alpha_p] \). Note that the proof below can be straightforwardly adapted for general linear constraints in the form \( z^T w = v \) where \( v \) and \( w \) are known and \( z \) is the variable that need to be contracted. We have that \( \dot{x} \in [\bar{x} - \bar{\eta}, \bar{x} + \bar{\eta}] \) by the noise bound assumption. Therefore, we have that

\[
f(\bar{x}) = \dot{x}^i - \sum_{p=1}^{d} g_p(\bar{x}^i)u^i[\alpha_p] \in (\bar{x}^i - \bar{\eta}, \bar{x}^i + \bar{\eta}) - \sum_{p=1}^{d} G_{p,k}^i u^i[\alpha_p]) \cap F^i = C_{F^i}.
\]

Therefore, a similar reasoning using the tighter interval \( C_{F^i} \) and interval arithmetic provides that

\[
\sum_{p=1}^{d} g_p(\bar{x}^i)u^i[\alpha_p] \in (\bar{x}^i - \bar{\eta}, \bar{x}^i + \bar{\eta}) - (\sum_{p=1}^{d} G_{p,k}^i u^i[\alpha_p]) = S_0.
\]

Note that plugging back \( S_0 \) instead of \( \sum_{p=1}^{d} G_{p,k}^i u^i[\alpha_p] \) in the expression of \( C_{F^i} \) will not yield a tighter set. Therefore, \( C_{F^i} \) and \( S_0 \) are optimal. Next, we focus on the term \( \sum_{p=1}^{d} g_p(\bar{x}^i)u^i[\alpha_p] \in S_0 \).

For all \( k \in \mathbb{N}_{[1,n]} \), we have that

\[
g_{1,k}(\bar{x}^i)u^i[\alpha_1] = \sum_{p=1}^{d} g_{p,k}(\bar{x}^i)u^i[\alpha_p] - \sum_{p=1}^{d} g_{p,k}(\bar{x}^i)u^i[\alpha_p] \in (\sum_{k=0}^{d} G_{l,k}^i u^i[\alpha_1]) \cap (\sum_{p=1}^{d} G_{p,k}^i u^i[\alpha_1]),
\]

and we can, in a similar manner, deduce that

\[
\sum_{p=1}^{d} g_{p,k}(\bar{x}^i)u^i[\alpha_p] \in (\sum_{k=0}^{d} G_{l,k}^i u^i[\alpha_1]) \cap (\sum_{p=1}^{d} G_{p,k}^i u^i[\alpha_1]).
\]

Using the same argument as for the optimality of \( S_0 \) and \( C_{F^i} \), we can say that \( S_{1,k} \) and \( C_{G_{1,k}^i} \) are optimal. Finally, we apply the previous step in a sequential manner for \( p = 2, \ldots, d \) to the equality \( g_{p,k}(\bar{x}^i)u^i[\alpha_p] = \sum_{p=1}^{d} g_{l,k}(\bar{x}^i)u^i[\alpha_p] - \sum_{l>p}^{d} g_{l,k}(\bar{x}^i)u^i[\alpha_p] \) in order to obtain optimal intervals \( S_{p,k} \) and \( C_{G_{p,k}^i} \).

**Proof of Theorem 1**

The result is straightforward from Lemma 1 and Lemma 2. First, let \( i \in \mathbb{N}_{[1,j]} \). We show that for all \( (\bar{x}^i, C_{F^i}, C_{G_{1,k}^i}) \in \mathcal{E}_j \) given by Algorithm 1, we have \( f_1(\bar{x}^i) \in C_{F_{1,k}^i} \) and \( g_{p,k}(\bar{x}^i) \in C_{G_{p,k}^i} \) for all \( p, k \in \mathbb{N}_{[1,d]} \times \mathbb{N}_{[1,n]} \). Specifically, as a consequence of line 1 of Algorithm 2 and Lemma 1, we have that \( f_1(\bar{x}^i) \in F_{1,k}^i \) and \( g_{p,k}(\bar{x}^i) \in G_{p,k}^i \). Hence, by line 2 of Algorithm 2 and Lemma 2, we immediately have that \( f_1(\bar{x}^i) \in C_{F_{1,k}^i} \) and \( g_{p,k}(\bar{x}^i) \in C_{G_{p,k}^i} \). Thus, \( \mathcal{E}_j \) can be used in Lemma 1 to conclude that \( f_1(x) \in F_{1,k}^i \) and \( g_{p,k}(x) \in G_{p,k}^i \) for all \( x \in \mathcal{X} \). Therefore, we have \( \dot{x} = f(x) + \sum_{p=1}^{d} g_p(x)u[\alpha_p] \in f(x) + \sum_{p=1}^{d} g_p(x)u[\alpha_p] \) through straightforward interval arithmetic.
Proof of Theorem 2

This proof leverages an interval Taylor-based method to over-approximate the reachable set of dynamics described by differential inclusions.

First, consider the case of known dynamics in the form $\dot{x} = h(x, u)$, where $h : X \times U \mapsto \mathbb{R}^n$ is $C^D_h$, i.e., it admits continuous partial derivatives of order $1, \ldots, D_h$ on $X$. Given $\mathcal{R}^q$ such that $x(t_q) \in \mathcal{R}^n$ and a control signal $u$ that is $\mathcal{C}^D_u$ on the interval $[t_q, t_{q+1}]$, interval Taylor-based methods Berz and Makino (1998); Nedialkov et al. (1999) provide an over-approximation $\mathcal{R}^{q+1}$ of the reachable set at $t_{q+1}$ under the control $v$ as follows:

$$\mathcal{R}^{q+1} = \mathcal{R}^q + \sum_{d=1}^{D-1} (t_{q+1} - t_q) \sum_{k,l} \left[ h^[[d]](\mathcal{R}^q, v)(t_q) + (t_{q+1} - t_q) \partial f_{k,l}^{[[d]]}(\mathcal{P}^q, v) \right] \left[ t_q, t_{q+1} \right]$$

(10)

where $D \leq \min(D_u + 1, D_h)$ is the order of the Taylor expansion, $h$ is an interval extension of $h$, $v$ is an interval extension of $v$, $h^{[[d]]}$ are interval extensions of the Taylor coefficients $h_{k,l}^{[[d]]}$ defined inductively by

$$h_{k,l}^{[1]} = h, \quad h_{k,l}^{[d+1]} = \frac{1}{d+1} \left( \partial h_{k,l}^{[[d]]} \frac{\partial h}{\partial x} + \sum_{l=0}^{d-1} \partial h_{k,l}^{[[d]]} u^{[(l+1)]} \right),$$

(11)

and the set $\mathcal{P}^q \in \mathbb{I}^n$ is an a priori rough enclosure of $\{ x(t_{q+1}; v^q, x^q) \in X | x^q \in \mathcal{R}^q \}$ and is a solution of the fixed-point equation

$$\mathcal{R}^q + [0, t_{q+1} - t_q] \mathcal{R}(h, \mathcal{P}^q \times v([t_q, t_{q+1}])) \subseteq \mathcal{P}^q.$$  

(12)

Then, in the setting of our problem, we have that $h(x, u) = f(x) + \sum_{p=1}^d g_p(x) u^[[\alpha]^p]$ where $f$ and $g_p$ are unknown functions. Further, with only the Lipschitz assumption on $f$ and $g_p$, we are limited to a Taylor expansion of order $D = 2$. By Corollary 1, $h$ defined in (3) is a straightforward interval extension of the unknown $h$. Further, since $\mathcal{R}(h, \mathcal{P}^q \times u([t_q, t_{q+1}])) \subseteq h(\mathcal{P}^q, u([t_q, t_{q+1}]))$, the set $\mathcal{P}^q$ in Theorem 2 is an a priori rough enclosure that satisfies the fixed-point equation (12) when $u([t_q, t_{q+1}]) = u^q$. Thus, we apply the Taylor expansion (10) to obtain

$$\mathcal{R}^{q+1} = \mathcal{R}^q + \Delta t \left( h(\mathcal{R}^q, u^q) \right) + \frac{\Delta t^2}{2} \left( \partial h \frac{\partial h}{\partial x} (\mathcal{P}^q, u^q) + \frac{\Delta t^2}{2} \left( \partial h \frac{\partial h}{\partial u} (\mathcal{P}^q, u^q) \right) \right) [t_q, t_{q+1}]$$

(13)

Recall that $\dot{u} = 0$ since the control signal $u = u^q$ is constant on $[t_q, t_{q+1}]$. Furthermore, for all $k, l \in \mathbb{N}_{[1,n]}$, we have

$$\frac{\partial h_{k,l}^{[[d]]}}{\partial x_l}(x, u^q) = \frac{\partial f_k}{\partial x_l}(x) + \sum_{p=1}^d \frac{\partial g_{p,k}}{\partial x_l}(x) u^[[\alpha]^p].$$

Thus, by definition of the upper bounds on the Lipschitz constants of $f_k$ and $g_{p,k}$ we have

$$\frac{\partial f_k}{\partial x_l}(x) \in [-1, 1] w_k \bar{f}_k \quad \text{and} \quad \frac{\partial g_{p,k}}{\partial x_l}(x) \in [-1, 1] w_k \bar{g}_{p,k}.$$  

Therefore, $\frac{\partial h}{\partial x}(\mathcal{P}^q, u^q) = J^f + \sum_{p=1}^d J^g_{p} u^[[\alpha]^p]$ is an interval extension of the jacobian of $h$ with respect to $x$. Finally, merging $\frac{\partial h}{\partial x}$ and $\dot{u} = 0$ into (13) provides the over-approximating set (4).
Proof of Theorem 3

Let \( u \) and \( v \) be the optimal control values corresponding to the cost \( C_j^* \) and \( C_j \). For notation simplicity, let \( h(w^q, u^q) \) denotes \( x(t_{q+1}; w^q, u^q) \). Thus, \( w^{q+1} = h(w^q, u^q) \) and \( y^{q+1} = h^\theta(y^q, v^q) \) are completely determined by the control values \( u^q, v^q \) and the current state \( y^j = x^j \). We have

\[
|\hat{C}_j - C_j^*| = \begin{cases} 
\hat{C}_j - \sum_{q=j}^{j+N} c(w^q, u^q, h(w^q, u^q)), & \text{if } \hat{C}_j \geq C_j^* \\
C_j^* - \sum_{q=j}^{j+N} c(y^q, v^q, h^\theta(y^q, v^q)), & \text{otherwise}
\end{cases}
\]

(14)

\[
\leq \begin{cases} 
\sum_{q=j}^{j+N} (c(y^q, u^q, h^\theta(y^q, u^q)) - c(w^q, u^q, h(w^q, u^q)), & \text{if } \hat{C}_j \geq C_j^* \\
\sum_{q=j}^{j+N} (c(w^q, v^q, h(w^q, u^q)) - c(y^q, v^q, h^\theta(y^q, v^q)), & \text{otherwise}
\end{cases}
\]

(15)

\[
\leq \begin{cases} 
\sum_{q=j}^{j+N} L_c(\|h(w^q, u^q) - h^\theta(y^q, u^q)\|_2 + \|u^q - y^q\|_2), & \text{if } \hat{C}_j \geq C_j^* \\
\sum_{q=j}^{j+N} L_c(\|h(w^q, v^q) - h^\theta(y^q, v^q)\|_2 + \|u^q - y^q\|_2), & \text{otherwise}
\end{cases}
\]

(16)

\[
\leq L_c \left( \|\text{wd}(R_{\hat{u}}^{j+N+1})\|_2 + \sum_{q=j+1}^{j+N} 2\|\text{wd}(R_{u}^{q})\|_2 \right).
\]

(17)

The first inequality is obtained by definition of \( \hat{C}_j \) and \( C_j^* \) as optimal solutions of the optimal control problem under the different dynamics \( h \) and \( h^\theta \). That is, for any control other than \( u^q, v^q \), the cost returned when rolling out the unknown dynamics given by \( h \) \((h^\theta)\) is suboptimal. The second inequality uses the definition of the Lipschitz constant of \( c \). Finally, in the last inequality, we use the fact that \( h(w^q, u^q), h(w^q, v^q) \in R_{\hat{u}}^{j+N+1} \) and \( h^\theta(y^q, v^q), h^\theta(y^q, u^q) \in R_{u}^{j+N+1} \) to conclude.

Incorporating More Side Information

We recall that general constraints on the states or its derivatives, in the form of nonlinear mathematical (in)equalities, can be used as side information via the interval contractor programming framework. In the following, we provide others examples of side information that can be incorporated in the proposed learning algorithm.

Side information 3 (DECOUPLING AMONG STATES) The qualitative knowledge that some components of the vector field \( \dot{x} \) do not depend on some components of the state \( x \). In other words, the subset of states for which some components of \( f \) and \( g_p \) are independent is known.

For example, if the state \( x_{l,k}(t) \) does not directly affect \( \dot{x}_{l,k}(t) \) for some \( l, k \in \mathbb{N}_{[1,n]} \) under any control signal in \( U \), we can obtain a tighter over-approximation of the reachable set by setting to zero the intervals \( \mathcal{F}^f_{k,l} \) and \( \mathcal{F}^{g_p}_{k,l} \) for all \( p \in \mathbb{N}_{[1,d]} \).

Side information 4 (GRADIENT BOUNDS) We are given bounds on the gradient of some components of \( f \) and \( g_p \). Such side information may include the monotonicity of \( f \) or \( g_p \).
These bounds can be used to provide tight interval extensions $\mathcal{J}^f$ and $\mathcal{J}^{g_p}$. For example, if the function $f_k$ is known to be non-decreasing with respect to the variable $x_l$ on a set $A \subseteq \mathcal{X}$, then we obtain a tighter $\mathcal{R}^{j+1}$ by the update $\mathcal{J}_{k,l}^f \leftarrow \mathcal{J}_{k,l}^f \cap \mathbb{R}_+$ if $\mathcal{P}^q \subseteq A$.

**Side information 5 (Vector field bounds)** We are given the sets $\mathcal{R}^{f,A} \in \mathbb{R}^n$ and $\mathcal{R}^{g,A} \in \mathbb{R}^{n \times m}$ as supersets of the range of $f$ and $g_p$, respectively, over a given set $A \subseteq \mathcal{X}$.

Given a set $S \subseteq A$, tight extensions of $f$ and $g_p$ over $S$ can be obtained by the update $f(S) \leftarrow f(S) \cap \mathcal{R}^{f,A}$ and $g_p(S) \leftarrow g_p(S) \cap \mathcal{R}^{g,A}$. The tight extensions can be directly used in Corollary 1, the refinement algorithm, and the initialization of the construction algorithm.

### Sequential Convex Programming Algorithm

**Algorithm 3** Sequential convex programming with trust region to find an approximate solution to the $N$-step optimal control problem.

**Input:** Dataset $\mathcal{T}_j$, current system’s state $x_j$, initial trust region $r_1 > 0$, penalty weight $\lambda > 0$, trust region parameters $0 < \rho_0 < \rho_1 < 1$, $\alpha > 1$, and optimality tolerance $\epsilon_{\text{tol}} > 0$.

**Output:** $u, x$

1. Initialize $x_1^0, u^1 \in U^N, s \leftarrow 1$ {\textit{The starting point does not need to be feasible}}
2. while $\text{True}$ do
3.   Solve linearized problem (6) at $x^s, u^s, r^s$ to obtain $\Delta x^{s+1}, \Delta u^{s+1}$
4.   $\Delta J^s \leftarrow J(x^s, u^s) - J(x^s + \Delta x^{s+1}, u^s + \Delta u^{s+1})$ {\textit{Variation of the realized cost}}
5.   $\Delta L^s \leftarrow J(x^s, u^s) - L^s(\Delta x^{s+1}, \Delta u^{s+1})$ {\textit{Variation of the linearized cost}}
6.   if $|\Delta J^s| \leq \epsilon_{\text{tol}}$ then
7.     return $x^s, u^s$ {\textit{Found solution}}
8.   end if
9.   $\rho^s \leftarrow \Delta J^s / \Delta L^s$ {\textit{Encode the quality of linearization}}
10. if $\rho^s < \rho_0$ then
11.     $r^s \leftarrow r^s / \alpha$ {\textit{Contract trust region}}
12. else
13.     $s \leftarrow s + 1$ {\textit{Update estimate}}
14. end if
15. $\alpha^s \leftarrow r^s / \alpha$ if $\rho^s \leq \rho_1$ else $r^s \alpha$ {\textit{Contract or expand trust region}}
16. end while
17. return $x^s, u^s$

Algorithm 3 summarizes the trust-region-based sequential convex optimization scheme to compute approximate (possibly local) solutions to the $N$-step optimal control problem (2). Specifically, the quality of the linear approximation can be understood by inspecting the ratio $\rho^s$. The ratio $\rho^s$ compares the realized reduction $\Delta J^s$ to the predicted cost $\Delta L^s$. When $\rho^s \leq \rho_0$ with $\rho_0$ sufficiently close to 0, the linearization is considered inaccurate. Then, we contract the trust region $r^s$ and restart the iteration. If not, the solutions $\Delta x^{s+1}$ and $\Delta u^{s+1}$ are considered acceptable. Then, we move to
the next iteration and contract or expand the trust region depending on if $\rho^k$ is below or above the threshold $\rho_1$ typically chosen to be close to 1.

**Implementation Details**

All the experiments in this paper were performed on a computer with an Intel Core i9-9900 CPU 3.1GHz ×16 processors and 31.2 Gb of RAM. We use Gurobi 9.0 [Gurobi Optimization (2021)] to solve each subproblem of the sequential convex optimization (6), and we used the control tasks of DeepMind Control Suite [Tassa et al. (2018)] for comparison with RL algorithms.

**Code.** All the implementations are written and tested in Python 3.8, and we will release the full code at [https://github.com/wuwushrek/datacontrolreach.git](https://github.com/wuwushrek/datacontrolreach.git). We emphasize that this code is still under development and can be significantly improved both on its organization and efficiency.

**Comparisons with SAC and D4PG.** We utilized the implementation of D4PG from [Hoffman et al. (2020)] and SAC from [Yarats and Kostrikov (2020)] for a comparison with the proposed data-driven control algorithm. The default hyper-parameters provided by the implementations of D4PG and SAC were used to train the agents in the Reacher, Swimmer, and Cheetah environment. Such hyper-parameters have been empirically demonstrated to provide high performance in control tasks of MuJoCo. We recall that D4PG and SAC were pre-trained using ten million of iterations with the environment. Then, the testing phase was done for 100 episodes. Our algorithm only learns from the data obtained during the 1000 time steps of each episode.

**Experiments in MuJoCo.** We recall that the Reacher environment model has been slightly modified to accommodate our learning algorithm. Specifically, our algorithm works well with smaller time steps and we modified the original Reacher model to accommodate to that. We provide in the code the new Reacher model description to be used inside the DeepMind Control Suite framework. The others environments were not modified as the time steps were already small enough for our algorithm to succeed in learning the control tasks.

The Lipschitz bounds were estimated using trajectories generated by an excitation-based control of the system. We provide a code to compute under-estimation of the Lipschitz bounds for any MuJoCo environment. In addition, the nonconvex cost functions were linearized as detailed in this paper using finite differentiation and the well-documented API provided by MuJoCo.

**Experiments on the F-16 Aircraft Simulator**

We provide in this section additional details on the F-16 aircraft simulator since the simulator is not publicly available yet.

The F-16 aircraft’s flight control system [Heidlauf et al. (2018)] is described by a hierarchical feedback control loop consisting of an autopilot and a low-level controller. The autopilot performs higher-level maneuvers, such as ground collision avoidance, waypoint tracking, and more. In contrast, the low-level control tracks the references from the autopilot and maintains stability by actuating the flight control surfaces appropriately. The control system uses a closed-loop feedback control to actuate the flight control surfaces, including the thrust, the ailerons, elevators, and rudders, in order to meet the desired flight objectives.
Inside the simulator, the underlying nonlinear dynamics, containing 13-states and 4-control inputs, capture the (6-DOF) movement of an aircraft through the standard aerodynamic equations. The dynamics describe the evolution of the system’s states, namely velocity $v_t$, angle of attack $\alpha$, sideslip $\beta$, altitude $h$, attitude angles: roll $\phi$, pitch $\theta$, yaw $\psi$, and their corresponding rates $p$, $q$, $r$, engine power and two more states for translation along north and east. The plant model is built on several linearly interpolated lookup tables that incorporate wind tunnel data describing the engine model, the various coefficients including damping, force and moment coefficients, and the moments due to the control surfaces. As a consequence, with known look-up tables, the resulting dynamics have polynomial dependency in the control input.
Side Information and F-16 Dynamics. Our algorithm considers as side information the following knowledge of the rigid-body dynamics of a 6-DOF Stevens et al. (2015)

\[
\begin{align*}
\dot{u} &= rv - qw - g \sin \theta + \frac{F_u}{m}, \\
\dot{v} &= -ru + pw + g \sin \phi \cos \theta + \frac{F_v}{m}, \\
\dot{w} &= qu - pv + g \cos \phi \cos \theta + \frac{F_w}{m}, \\
\dot{v}_t &= \frac{u \dot{u} + v \dot{v} + w \dot{w}}{v_t}, \\
\dot{\alpha} &= \frac{u \dot{w} - w \dot{u}}{u^2 + w^2}, \\
\dot{\beta} &= \frac{(v \dot{u} - v \dot{v}_t) \cos \beta}{u^2 + w^2}, \\
\dot{p} &= \frac{J_y - J_z}{J_x}qr + \frac{M_p}{J_x}, \\
\dot{q} &= \frac{J_z - J_x}{J_y}pr + \frac{M_q}{J_y}, \\
\dot{r} &= \frac{J_x - J_y}{J_z}pq + \frac{M_r}{J_z}, \\
\dot{\phi} &= p + \tan \theta(q \sin \phi + r \sin \phi), \\
\dot{\theta} &= q \cos \phi - r \sin \phi, \\
\dot{\psi} &= \frac{q \sin \phi + r \cos \phi}{\cos \theta}, \\
\end{align*}
\]

where \(x = [v_t, \alpha, \beta, p, q, r, \phi, \theta, \psi, \text{power}, h, p_n, p_e]\) is the full state vector of the aircraft, the intermediary variables \(u = v_t \cos \alpha \cos \beta, v = v_t \sin \beta,\) and \(w = v_t \sin \alpha \cos \beta\) represent respectively the axial, lateral, and vertical velocities in the body frame, \(v_t\) is the truth velocity, \(\alpha\) is the angle of attack, \(\beta\) is sideslip, \(p\) is the pitch rate, \(q\) is the roll rate, \(r\) is the yaw rate, \(\phi\) is the roll angle, \(\theta\) is the pitch angle, \(\psi\) is the yaw angle, \(h\) is the altitude, power is the resulting power when applying thrust, \(p_n\) is the position on the north axis, and \(p_e\) is the position on the east axis. The aerodynamics forces and moments are given by \(F_u, F_v, F_w\) and \(M_p, M_q, M_r\), respectively. Such possibly time-varying forces and moments depend on the wing and control surfaces, the states, and the control inputs of the aircraft. That is, we have the dependencies \(F_u(x, u), F_v(x, u), F_w(x, u), M_p(x, u), M_q(x, u), M_r(x, u),\) where \(u = [\text{thrust}, \delta_a, \delta_e, \delta_r]\) is the vector of control inputs. Here, \(\delta_a, \delta_e,\) and \(\delta_r\) are the aileron, elevator, and rudder control inputs, respectively, \(m\) is the mass of the aircraft, \(J_x, J_y,\) and \(J_z\) are the inertia moments, and \(g\) is the gravity constant.

Specifically, even though the form of the dynamics above is known, we consider that the power dynamics, the forces and moments, typically approximated via lookup tables and experiments, are unknown functions of the states and control inputs. In other words, the effect of the control inputs on the aircraft are unknown but we still want to retain some degree of control using streaming data from a single trial. Such unknown functions are nonlinear in the states and polynomial in the control inputs.
Data-Driven Differential Inclusion. We empirically demonstrate in Figure 5 that the data-driven differential inclusion constructed from streaming data from the ongoing trajectory and Lipschitz bounds is indeed tight.

![Graph showing differential inclusion and unknown vector field comparison](image)

Figure 5: From left to right, we show how the constructed data-driven differential inclusion can be indeed tight on the F-16 aircraft example.

Ground Collision Scenario. In the ground collision avoidance scenario, we initialize the simulator such that the plane is diving nose down towards the ground with an extremely high downward pitch angle. The autopilot uses a PID law to compute the references on the system’s states that our algorithm must track to avoid the crash. The initial condition considered is given by $\theta = -85\pi/180$, $v_t = 540$, $h = 3600$, $\phi = \pi/4$, $\psi = -\pi/4$, $\beta = 0$, $\alpha = 2.5\pi/180$, $p = q = r = 0$.

At each time step, the autopilot computes and adjusts the state setpoints required to avoid the ground collision. Thus, the cost function that our algorithm is trying to optimize is given by: $\text{cost}(x, u) = \|x - x_{\text{target}}\|_2$, where $x_{\text{target}}$ is the target provided by the autopilot. We compare the highly-tuned LQR controller of the simulator to our algorithm trying to avoid ground collision from streaming data obtained while diving towards the ground.