Para-linearity as the Nonassociative Counterpart of Linearity

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Abstract

In an octonionic Hilbert space $H$, the octonionic inner product induces maps which fail to be octonionic linear. This fact motivates us to introduce a new notion of the octonionic para-linearity instead. To do this we encounter an insurmountable obstacle. That is, the axiom

$$\langle pu, u \rangle = p \langle u, u \rangle$$

for any octonion $p$ and element $u \in H$ introduced by Goldstine and Horwitz in 1964 can not be interpreted as a property to be obeyed by the octonionic para-linear maps. In this article, we solve this critical problem by showing that this axiom is in fact non-independent from others. This enables us to initiate the study of octonionic para-linear maps. We can thus establish the octonionic Riesz representation theorem which, up to isomorphism, identifies two octonionic Hilbert spaces, one of which is the dual of the other. The dual space consists of continuous left para-linear functionals and it becomes a right octonionic module under the multiplication defined in terms of the second associators which measure the failure of octonionic linearity. This right multiplication has an alternative expression

$$(f \odot p)(x) = pf(p^{-1}x)p,$$

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which is a generalized Moufang identity. Remarkably, the multiplication is compatible with the canonical norm, i.e.,

$$||f \odot p|| = ||f|| \cdot ||p||.$$

Our final conclusion is that para-linearity is the nonassociative counterpart of linearity.

**Keywords** Octonionic Hilbert space · Para-linear · Riesz representation theorem

**Mathematics Subject Classification** Primary: 17A35 · 46S10

## 1 Introduction

Linearity is an important notion in geometry and analysis in the associative setting. The linear process turns Lie groups into Lie algebras locally [11]. Linear functionals can be used to generalize the theory of holomorphic functions to cases where the target domains are of finite or even infinite dimensions based on the Riesz representation theorem [21].

A natural question arises what is the nonassociative counterpart of linearity in the nonassociative realm. This question has no answer even in octonionic functional analysis since there still lacks the octonionic Riesz representation theorem which depends on the maps abstracted from the inner products. In order to explore the reason for this lack, we start with the octonionic Hilbert theory.

The theory of octonionic Hilbert spaces was initiated by Goldstine and Horwitz [5] in 1964 and has many applications in the spectral theory [17], operator theory [16], and physics [3, 8, 9, 20]. But compared with the quaternionic Hilbert space theory [12, 18, 19, 27, 28], the development of the octonionic Hilbert space theory is lagging behind. Nonetheless, the existing research on the octonionic Hilbert space theory has emerged some effective methods. For example, Goldstine and Horwitz [5] attribute the study of the octonionic Hilbert space to the associative setting; see [6, 23] for the further development. Ludkovsky and Sprössig [16, 17] find the theory can get well developed if the $\mathbb{O}$-Hilbert space admits the tensor decomposition of a real Hilbert space with the algebra of octonions $\mathbb{O}$.

Now, we focus on the definition of the octonionic Hilbert space.

**Definition 1.1** [5] An $\mathbb{O}$-Hilbert space $H$ is a left $\mathbb{O}$-module with an $\mathbb{O}$-inner product

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{O}$$

such that $(H, \text{Re} \langle \cdot, \cdot \rangle)$ is a real Hilbert space. Here, the $\mathbb{O}$-inner product satisfies the following axioms for all $u, v \in H$ and $p \in \mathbb{O}$:

(a) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;
(b) $\langle u, v \rangle = \overline{\langle v, u \rangle}$;
(c) $\langle u, u \rangle \in \mathbb{R}^+$; and $\langle u, u \rangle = 0$ if and only if $u = 0$;
(d) $\langle tu, v \rangle = t \langle u, v \rangle$ for $t \in \mathbb{R}$;
(e) \( \text{Re} \langle pu, v \rangle = \text{Re} (p \langle u, v \rangle) \);
(f) \( \langle pu, u \rangle = p \langle u, u \rangle \).

The functionals that need to be studied in the octonionic Hilbert space are naturally induced by the octonionic inner product

\[
f(x) = \langle x, v \rangle
\]

for any given element \( v \in H \). However, these real linear maps in (1.1) are generally not octonionic linear maps since there may hold

\[
f(px) \neq pf(x)
\]

for some \( p \in \mathbb{O} \) and \( x \in H \).

It is still an unsolved problem so far how to define an appropriate functional in octonionic functional analysis. This is because in Definition 1.1 Axiom (f), as a property of \( \mathbb{O} \)-inner-product, cannot be abstractly interpreted into the property of a functional on a general \( \mathbb{O} \)-normed space.

Fortunately, we can overcome this obstacle in a roundabout way in this article. We find that Axiom (f) is actually non-independent. This motivates us to introduce a kind of maps beyond \( \mathbb{O} \)-linearity, called \( \mathbb{O} \)-para-linear maps, defined in any \( \mathbb{O} \)-modules.

An \( \mathbb{O} \)-para-linear map in a left \( \mathbb{O} \)-module \( M \) is defined to be a real linear map \( f \in \text{Hom}_\mathbb{R}(M, \mathbb{O}) \) subject to the condition

\[
\text{Re} [p, x, f] = 0
\]

for all \( p \in \mathbb{O} \) and \( x \in M \), where the element

\[
[p, x, f] := f(px) - pf(x)
\]

is called the second associator related to \( f \). Condition (1.2) becomes Axiom (e) in Definition 1.1 in the specific case where \( f \) is a map induced by the octonionic inner product as in (1.1).

The set of para-linear functions in a left \( \mathbb{O} \)-module constitutes a right \( \mathbb{O} \)-module under the multiplication

\[
(f \odot p)(x) := f(x)p - [p, x, f].
\]

Similarly, \( \mathbb{K} \)-para-linearity can also be defined for any real normed division algebra

\[
\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O},
\]

where \( \mathbb{H} \) is the algebra of quaternions. Since \( \mathbb{K} \)-para-linearity degenerates to linearity in the associative setting, this shows para-linearity is a nonassociative counterpart of linearity.
**K**-para-linearity can be used to define **K**-Hilbert spaces. In the specific case where **K** = **O**, this definition is belonging to Goldstine and Horwitz [5] except that a non-independent axiom is removed.

**Definition 1.2** Suppose that \( H \) is a real Hilbert space under the real inner product \( \langle \cdot, \cdot \rangle_\mathbb{R} \) and also a left \( \mathbb{K} \)-module. We call \( H \) a \( \mathbb{K} \)-**Hilbert space** if there exists an \( \mathbb{R} \)-bilinear map

\[
\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}
\]

such that

\[
Re \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\mathbb{R}
\]

and satisfying

(i) (\( \mathbb{K} \)-para-linearity) \( \langle \cdot, u \rangle \) is left \( \mathbb{K} \)-para-linear for all \( u \in H \).

(ii) (\( \mathbb{K} \)-hermiticity) \( \langle u, v \rangle = \overline{\langle v, u \rangle} \) for all \( u, v \in H \).

(iii) (Positivity) \( \langle u, u \rangle \in \mathbb{R}^+ \); and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \).

Now, we can use \( \mathbb{O} \)-para-linearity to study the Riesz representation theorem in \( \mathbb{O} \)-Hilbert spaces \( H \). The theorem should be expressed in the following form

\[
H^* \cong H. \tag{1.4}
\]

Here \( H^* \) is the dual space consisting of \( \mathbb{O} \)-para-linear functions and the bijection in (1.4) between two spaces is both an isometric isomorphism as real Banach spaces and a conjugate isomorphism as \( \mathbb{O} \)-modules.

Of the most difficult is to show the compatibility between the module structure of \( H^* \) and its canonical norm, i.e.,

\[
||f \odot p|| = ||f|| ||p||
\]

for all \( f \in H^* \) and \( p \in \mathbb{O} \), where \( f \odot p \) is the right multiplication defined in (1.3).

Overcoming this difficulty requires another expression of the right multiplication

\[
(f \odot p)(x) = pf(p^{-1}x)p,
\]

which is a generalized Moufang identity [24].

The octonionic Riesz representation theorem in (1.4) can be restated as

\[
(H^*)^- \cong H. \tag{1.5}
\]

Here \( (H^*)^- \) denotes the set \( H^* \) endowed with a left \( \mathbb{O} \)-module defined by

\[
p \cdot x := x\overline{p}.
\]
The point is that the bijection in (1.5) is an isomorphism of two left \( \mathbb{O} \)-modules in contrast to the conjugate isomorphism in (1.4).

Space \( (H^*)^- \) can be endowed with an \( \mathbb{O} \)-inner product via isomorphism (1.5) so that it becomes an \( \mathbb{O} \)-Hilbert space. The octonionic Riesz representation theorem shows that the two \( \mathbb{O} \)-Hilbert spaces in (1.5) coincide up to isomorphism. This result is true if \( \mathbb{O} \) is replaced by any normed division algebras \( \mathbb{K} \).

**Theorem 1.3** (Riesz representation theorem) Let \( \mathbb{K} \) be a normed division algebra. Let \( H \) be a \( \mathbb{K} \)-Hilbert space and \( H^* \) its dual consisting of all continuous \( \mathbb{K} \)-para-linear functionals. Then, we can identify the two \( \mathbb{K} \)-Hilbert spaces \( H \) and \( (H^*)^- \) up to isomorphism.

Opposed to the associative setting, we have two distinct spaces

\[
H^{*\mathbb{O}} \subsetneq H^*.
\]

Here \( H^{*\mathbb{O}} \) consists of all continuous \( \mathbb{O} \)-linear functionals of \( H \). Under isomorphism (1.4), \( H^{*\mathbb{O}} \) is corresponding to the nucleus of \( H \) consisting of associative elements. Symbolically,

\[
H^{*\mathbb{O}} \cong \mathcal{A}(H).
\]

This shows that the class of \( \mathbb{O} \)-linear functionals is too small to represent every element in an \( \mathbb{O} \)-Hilbert space.

We remark that there are profound relations between the associative and nonassociative theories. For example, the algebra of octonions is closely related to the Clifford algebra of 6, 7 dimensions [10], and the category of left \( \mathbb{O} \)-modules is shown to be isomorphic to the category of left \( C\ell_7 \)-modules [13]. But the octonionic theory admits its intrinsic objects such as the \( \mathbb{O} \)-para-linear functionals as demonstrated in the octonionic Riesz representation theorem.

When para-linearity established itself as a substitute for linearity in the nonassociative realm, geometry and analysis can be extended to the setting of para-linearity.

For instance, we have established the octonionic Riesz representation theorem which is the most important result in the theory of octonionic Hilbert spaces. It should have important applications in octonionic functional analysis such as the para-linear operators, weak topology related to para-linear functionals, and octonionic \( C^* \) algebras.

A typical example of para-linear operators is the octonionic Dirac operator, defined by

\[
Df = \sum_{i=0}^{7} (\nabla e_i f)e_i
\]

for any octonion-valued function \( f \); see [2, 15] for the classical theory of Dirac operators.
One may also study the para-linear Levi-Civita connection. As shown in [7] by Grigorian, an octonion bundle emerges with a fiberwise non-associative product on a $G_2$-manifold. Notice that (see [7, Proposition 6.1])

$$\nabla_X(AB) = (\nabla_X A)B + A(\nabla_X B) - [T_X, A, B], \quad (1.6)$$

where $A, B$ are sections of the given octonion bundle of a $G_2$-manifold $M$, $X$ is a tangent vector, $\nabla$ is the Levi-Civita connection and $T$ is the full torsion tensor of $M$, $T_X = (0, X \wedge T)$. In particular, when $A$ is a parallel section, we have

$$\text{Re} \left( \nabla_X(AB) - A(\nabla_X B) \right) = 0.$$

Therefore, one can interpret the Levi-Civita connection in $G_2$ manifolds as a para-linear operator provided the notion of para-linearity in $\mathcal{O}$-modules is extended further to the setting of modules over parallel sections. We remark that Spin(7) geometry is closely related to $G_2$ geometry based on the fibration [22]

$$G_2 \hookrightarrow \text{Spin}(7) \longrightarrow S^7.$$

Since the Dixon algebra [4]

$$\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$$

underlie the Standard Model of particle physics, para-linearity has potential applications in this field.

## 2 Preliminaries

In this section, we review some notations and basic properties about octonions $\mathcal{O}$ and left $\mathcal{O}$-modules. We refer to [1, 13] for more details.

### 2.1 Octonions

The set of octonions $\mathcal{O}$ is a nonassociative, noncommutative, normed division algebra over $\mathbb{R}$. Let

$$e_0 = 1, \quad e_1, \ldots, e_7$$

be a basis of $\mathcal{O}$ as a real vector space subject to the multiplication rule

$$e_i e_j = \sum_{k=1}^{7} e_{ijk} e_k - \delta_{ij}. \quad (2.1)$$
for any $i, j = 1, \ldots, 7$. Here, $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ijk}$ is completely anti-symmetric with value 1 precisely when

$$ijk = 123, 145, 176, 246, 257, 347, 365.$$  

Let $x = x_0 + \sum_{i=1}^{7} x_i e_i$ be an octonion with all $x_i \in \mathbb{R}$. We define its conjugate

$$\bar{x} := x_0 - \sum_{i=1}^{7} x_i e_i,$$

its norm

$$|x| = \sqrt{x \bar{x}},$$

and its real part

$$\text{Re } x = x_0 = \frac{1}{2} (x + \bar{x}).$$

### 2.2 $\mathbb{O}$-modules

We recall some basic notations and results on $\mathbb{O}$-modules in this subsection. We refer to [13, 14, 25, 26] for octonionic modules and even for modules of alternative algebras.

A left $\mathbb{O}$-module $M$ is a real vector space $M$ endowed with an $\mathbb{O}$-scalar multiplication such that the left associator is left alternative, i.e., for all $p, q \in \mathbb{O}$ and $m \in M$ we have

$$[p, q, m] = -[q, p, m].$$  

Condition (2.2) is equivalent to the requirement

$$r(rm) = r^2 m$$

for all $r \in \mathbb{O}$ and all $m \in M$.

Of the most important identity involving associators in a left $\mathbb{O}$-module $M$ is the five-term identity

$$[p, q, r]m + p[q, r, m] = [pq, r, m] - [p, qr, m] + [p, q, rm]$$  

for all $p, q, r \in \mathbb{O}$ and $m \in M$. This even holds when $M$ is a nonassociative algebra [24].
In the theory of left $\mathcal{O}$-modules, associative and conjugate associative elements play key roles [13]. Let $M$ be a left $\mathcal{O}$-module. An element $m \in M$ is called an associative element if

$$(pq)m = p(qm)$$

and called a conjugate associative element if

$$(pq)m = q(pm).$$

The set of all associative elements is called the nucleus of $M$ and denoted by $\mathcal{A}(M)$, i.e.,

$$\mathcal{A}(M) := \{ m \in M \mid [p, q, m] = 0 \text{ for all } p, q \in \mathcal{O} \}.$$ 

The set of all conjugate associative elements of $M$ is denoted by $\mathcal{A}^{-}(M)$, i.e.,

$$\mathcal{A}^{-}(M) := \{ m \in M \mid (pq)m = q(pm) \text{ for all } p, q \in \mathcal{O} \}.$$ 

**Theorem 2.1** [13] Let $M$ be a left $\mathcal{O}$-module. Then

$$M \cong \mathcal{O}\mathcal{A}(M) \oplus \mathcal{O}\mathcal{A}^{-}(M).$$ 

3 $\mathcal{O}$-para-Linear Functions

The notion of the second associator is used to describe the extent to which a real linear map is far away from the octonionic linear map. This notion also allows us to define the para-linear map.

**Definition 3.1** Let $M$ be a left $\mathcal{O}$-module and let $\text{Hom}_\mathbb{R}(M, \mathcal{O})$ be the set of all real linear functions $f : M \to \mathcal{O}$. We introduce a real trilinear map

$$[\cdot, \cdot, \cdot] : \mathcal{O} \times M \times \text{Hom}_\mathbb{R}(M, \mathcal{O}) \to \mathcal{O},$$

defined by

$$[p, x, f] := f(px) - pf(x).$$  \hfill (3.1)

We call this map the second associator in $M$, which measures the failure of $\mathcal{O}$-linearity.

**Definition 3.2** Let $M$ be a left $\mathcal{O}$-module. A real linear map $f : M \to \mathcal{O}$ is called an $\mathcal{O}$-para-linear function if it satisfies

$$\text{Re} [p, x, f] = 0$$

for any $p \in \mathcal{O}$ and $x \in M$. 

\[ Springer \]
We denote by $\text{Hom}_{O}(M, O)$ the set of all $O$-para-linear functions. By definition, we have

$$\text{Hom}_{O}(M, O) = \{ f \in \text{Hom}_{R}(M, O) : \text{Re}[p, x, f] = 0 \text{ for all } p \in O \text{ and } x \in M \}.$$ 

**Example 3.3** The right multiplication on $O$ provides a typical example of $O$-para-linear function, i.e.,

$$R_q \in \text{Hom}_{O}(O, O)$$

for any $q \in O$. Here $R_q$ denotes the right multiplication defined by

$$R_q(x) = xq.$$

In this typical example, we find that the associator and the second associator coincide, i.e.,

$$[p, x, R_q] = [p, x, q] \quad (3.2)$$

for any $p, x, q \in O$. Indeed, by definition we have

$$[p, x, R_q] = R_q(px) - pR_q(x) = (px)q - p(xq) = [p, x, q].$$

Identity (3.2) also implies

$$\text{Re}[p, x, R_q] = 0$$

so that $R_q$ is $O$-para-linear.

If $f \in \text{Hom}_{R}(M, O)$, then it can be written as

$$f(x) = f_R(x) + \sum_{i=1}^{7} e_i f_i(x), \quad (3.3)$$

where

$$f_R(x) := \text{Re} f(x)$$

and $f_i(x) \in \mathbb{R}$ for $i = 1, \ldots, 7$.

Now we give a criterion for determining whether a real linear function $f \in \text{Hom}_{R}(M, O)$ is an $O$-para-linear map.
Theorem 3.4 Let $f \in \text{Hom}_\mathbb{R}(M, \mathbb{O})$ with expression as in (3.3). Then $f$ is $\mathbb{O}$-para-linear if and only if
\[ f_i(x) = f_{\mathbb{R}}(\overline{e_i}x) \] (3.4)
for all $i = 0, \ldots, 7$ and all $x \in \mathbb{O}$.

Proof We assume $f$ is $\mathbb{O}$-para-linear. By definition, we have
\[
  f_{\mathbb{R}}(\overline{e_i}x) = \text{Re} f(\overline{e_i}x) = \text{Re} (\overline{e_i} f(x)) = \text{Re} \left( \overline{e_i} \sum_{j=0}^{7} e_j f_j(x) \right) = f_i(x).
\]

Conversely, under assumptions (3.4), for any $i = 1, \ldots, 7$ we claim
\[ e_i f(x) = e_i f_{\mathbb{R}}(x) + \sum_{j,k=1}^{7} \epsilon_{ijk} e_k f_j(x) - f_i(x) \] (3.5)

Indeed, by calculation we have
\[
  e_i f(x) = e_i(f_{\mathbb{R}}(x) + \sum_{j=1}^{7} e_j f_j(x)) = e_i f_{\mathbb{R}}(x) + \sum_{j=1}^{7} e_i e_j f_j(x)
\]
\[
= e_i f_{\mathbb{R}}(x) + \sum_{j,k=1}^{7} \epsilon_{ijk} e_k f_j(x) - f_i(x)
\]
and the claim holds. From the claim we have
\[
\text{Re} [e_i, x, f] = \text{Re}(f(e_i x) - e_i f(x)) = f_{\mathbb{R}}(e_i x) + f_i(x) = 0.
\]

This finishes the proof.

The second associator of an $\mathbb{O}$-para-linear map in $\text{Hom}_{L,\mathbb{O}}(M, \mathbb{O})$ has a close relation with the associator in $M$. 

\[ \square \]
**Theorem 3.5** Let $f \in \text{Hom}_\mathbb{R}(M, \mathbb{O})$ with expression as in (3.3). Then $f$ is $\mathbb{O}$-para-linear if and only if

$$[p, x, f] = \sum_{i=1}^{7} e_i f_{\mathbb{R}}([e_i, p, x])$$

(3.6)

for all $i = 0, \ldots, 7$ and all $p \in \mathbb{O}$ and $x \in M$.

**Proof** To prove the necessity, we might as well set $p = e_i$ with $i = 1, \ldots, 7$. Since $f$ is $\mathbb{O}$-para-linear, Theorem 3.4 shows

$$f(x) = f_{\mathbb{R}}(x) - \sum_{j=1}^{7} e_j f_{\mathbb{R}}(e_j x).$$

(3.7)

Notice that

$$e_j(e_i x) = (e_j e_i) x - [e_j, e_i, x] = \sum_{k=1}^{7} \epsilon_{ijk} e_k x - \delta_{ij} x - [e_j, e_i, x].$$

(3.8)

Now we replace $x$ by $e_i x$ in (3.7) and then apply (3.8). Due to the real linearity of $f_{\mathbb{R}}$ we have

$$f(e_i x) = f_{\mathbb{R}}(e_i x) - \sum_{j,k=1}^{7} \epsilon_{ijk} e_j f_{\mathbb{R}}(e_k x) + e_i f_{\mathbb{R}}(x) + \sum_{j=1}^{7} e_j f_{\mathbb{R}}([e_j, e_i, x]).$$

(3.9)

On the other hand we have shown in (3.5) that

$$e_i f(x) = e_i f_{\mathbb{R}}(x) + \sum_{j,k=1}^{7} \epsilon_{ijk} e_k f_j(x) - f_i(x).$$

(3.10)

We subtract both sides in (3.9) and (3.10). Since $\epsilon_{ijk}$ is completely antisymmetric, we apply the fact that

$$f_i(x) = -f_{\mathbb{R}}(e_i x)$$

and identity (3.4) to conclude

$$[e_i, x, f] = f(e_i x) - e_i f(x) = \sum_{j=1}^{7} e_j f_{\mathbb{R}}([e_j, p, x]).$$

Conversely, if conditions (3.6) hold, then

$$\text{Re} [p, x, f] = 0$$
so that \( f \) is \( \mathbb{O} \)-para-linear by definition. \( \square \)

**Remark 3.6** Theorems 3.4 and 3.5 hold if \( \mathbb{O} \) is replaced by any Cayley–Dickson algebra \( \mathbb{K} \) (see [10]) under the similar notion of \( \mathbb{K} \)-para-linear functions. \( \mathbb{K} \)-para-linearity becomes \( \mathbb{K} \)-linearity when \( \mathbb{K} \) is an associative normed division algebra because the associators in (3.6) vanish in this setting. As a result, the notion of para-linearity is a canonical generalization of the classical linearity from the associative to the nonassociative realm.

As direct consequences of Theorems 3.4 and 3.5, the second associator vanishes for associative element and an \( \mathbb{O} \)-para-linear map is uniquely determined by its real part.

**Corollary 3.7** Assume that \( f : M \to \mathbb{O} \) is \( \mathbb{O} \)-para-linear.

(i) For each associative element \( x \in \mathcal{A}(M) \), we have

\[
[p, x, f] = 0
\]

and hence for all \( p \in \mathbb{O} \)

\[
f(px) = pf(x).
\]

(ii) \( f_{\mathbb{R}} = 0 \) if and only if \( f = 0 \).

To study the \( \mathbb{O} \)-module structure of \( \text{Hom}_{L_{\mathbb{O}}}(M, \mathbb{O}) \), we need some useful identities about the second associators.

**Proposition 3.8** Suppose that \( f \in \text{Hom}_{L_{\mathbb{O}}}(M, \mathbb{O}) \). Then for any \( p, q \in \mathbb{O} \) and \( x \in M \) we have

\[
p[p, x, f] = [p, x, f]p = [p, px, f]; \quad (3.11)
\]

\[
\text{Re} (f([p, q, x])) = \text{Re} ([p, x, f]q); \quad (3.12)
\]

\[
\text{Re} ([p, x, f]q) = -\text{Re} ([q, x, f]p). \quad (3.13)
\]

**Proof** As usual, we write

\[
f(x) = f_{\mathbb{R}}(x) + \sum_{i=1}^{7} e_i f_i(x).
\]

To prove (3.11), without loss of generality we may assume that \( p \in \text{Im}(\mathbb{O}) \). Let \( p = \sum_{i=1}^{7} p_i e_i \) with \( p_i \in \mathbb{R} \). Then by Theorem 3.5, we have

\[
p[p, x, f] = \sum_{i, j=1}^{7} p_i p_j e_i [e_j, x, f]
\]
\[
\sum_{i,j,k=1}^{7} p_i p_j e_i e_k f_\mathbb{R}([e_k, e_j, x]).
\]

Similarly, we get
\[
[p, x, f] \bar{p} = \sum_{i,j=1}^{7} p_i p_j [e_j, x, f] \bar{e}_i = - \sum_{i,j,k=1}^{7} p_i p_j e_k e_i f_\mathbb{R}([e_k, e_j, x]).
\]

Since
\[
e_i e_k + e_k e_i = -2\delta_{ik},
\]

it follows that
\[
p[p, x, f] - [p, x, f] \bar{p} = -2 \sum_{i,j=1}^{7} p_i p_j f_\mathbb{R}([e_i, e_j, x]).
\]

Because of the skew symmetricity of the associator, we get
\[
p[p, x, f] = [p, x, f] \bar{p}.
\]

By direct calculations, we get
\[
[p, \bar{p} x, f] = f(|p|^2 x) - pf(\bar{p} x)
\]
\[
= |p|^2 f(x) - pf(\bar{p} x)
\]
\[
= p(\bar{p} f(x)) - pf(\bar{p} x)
\]
\[
= -p[\bar{p}, x, f]
\]
\[
= p[p, x, f].
\]

Now, we come to prove (3.12). without loss of generality we may assume \( q = e_i \) with \( i = 1, \ldots, 7 \). By direct calculations, we have
\[
\text{Re} ([p, x, f]q) = \text{Re} \left( \sum_{j=1}^{7} (e_j f_\mathbb{R}([e_j, p, x])) e_i \right)
\]
\[
= \text{Re} \left( \sum_{j=1}^{7} f_\mathbb{R}([e_j, p, x]) e_j e_i \right)
\]
\[
= -f_\mathbb{R}([e_i, p, x])
\]
\[
= \text{Re} \left( f([p, q, x]) \right).
\]

Identity (3.13) follows from (3.12) directly.

\[\square\]
Remark 3.9 Identity (3.11) generalizes the third identity of Lemma 3.9 in [7].

Utilizing Proposition 3.8, we can endow $\text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$ with a canonical right $\mathcal{O}$-module structure. Let $r \in \mathcal{O}$, $f \in \text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$, and $x \in M$. We define the right scalar multiplication

$$(f \circ r)(x) := f(x)r - [r, x, f]. \quad (3.14)$$

Theorem 3.10 If $M$ be a left $\mathcal{O}$-module, then $\text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$ is a right $\mathcal{O}$-module.

Proof First we need to show that for any $f \in \text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$ and $r \in \mathcal{O}$ we have $f \circ r \in \text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$.

That is to prove

$$\text{Re} [p, x, f \circ r] = 0. \quad (3.15)$$

By direct calculations, we have

$$[p, x, f \circ r] = (f \circ r)(px) - p((f \circ r)(x))$$

$$= f(px)r - [r, px, f] - p(f(x)r - [r, x, f])$$

$$= (pf(x) + [p, x, f])r - [r, px, f] - p(f(x)r - [r, x, f])$$

$$= [p, f(x), r] + [p, x, f]r - [r, px, f] + pr, x, f). \quad (3.16)$$

Since

$$\text{Re} [p, f(x), r] = 0, \quad \text{Re} [r, px, f] = 0,$$

(3.16) tells us that we have to show

$$\text{Re} ([p, x, f]r) = -\text{Re} (pr, x, f)).$$

This can be easily deduced by applying identity (3.13) and the fact that

$$\text{Re} (pq) = \text{Re} (qp)$$

for all $p, q \in \mathcal{O}$.

Next we come to show that $\text{Hom}_{L\mathcal{O}}(M, \mathcal{O})$ is a right $\mathcal{O}$-module. It suffices to prove

$$(f \circ r) \circ r = f \circ (r^2)$$

for any $r \in \mathcal{O}$. Due to the uniqueness as shown in assertion (ii) of Corollary 3.7, we need to prove

$$\text{Re} ((f \circ r) \circ r - f \circ (r^2))(x) = 0.$$
By definition (3.14) and identity (3.11), we have

\[
((f \odot r) \odot r)(x) = ((f \odot r)(x))r - [r, x, f \odot r] = f(x)r^2 - [r, x, f]r - [r, x, f \odot r] \tag{3.17}
\]

and

\[
(f \odot (r^2))(x) = f(x)r^2 - [r^2, x, f]. \tag{3.18}
\]

Notice that \(\text{Re} [r, x, f \odot r] = 0\), \(\text{Re} [r^2, x, f] = 0\), and

\[
[r, x, f] = -[\overline{r}, x, f]. \tag{3.19}
\]

We thus conclude from (3.17) and (3.18) that

\[
\text{Re} \left( (f \odot r) \odot r - f \odot (r^2) \right)(x) = \text{Re} ([\overline{r}, x, f]r) = \text{Re} ([\overline{r}, rx, f]) = 0.
\]

The last but one step used (3.11). This completes the proof. \(\square\)

4 \(\mathbb{O}\)-Hilbert Spaces

We adopt the definition of \(\mathbb{O}\)-Hilbert spaces by Goldstine and Horwitz [5] except that we describe it in terms of \(\mathbb{O}\)-para-linearity. In addition, we find a non-independent axiom given by Goldstine and Horwitz [5].

4.1 Definition of Hilbert \(\mathbb{O}\)-modules

We first introduce a succinct definition of \(\mathbb{O}\)-Hilbert spaces. This definition also suits all normed division algebras.

**Definition 4.1** A left \(\mathbb{O}\)-module \(H\) is called a **pre-Hilbert left \(\mathbb{O}\)**-module if there exists an \(\mathbb{R}\)-bilinear map \(\langle \cdot, \cdot \rangle : H \times H \to \mathbb{O}\), referred to as an \(\mathbb{O}\)-inner product, satisfying:

(i) **(\(\mathbb{O}\)-para-linearity)** \(\langle \cdot, u \rangle\) is left \(\mathbb{O}\)-para-linear for all \(u \in H\).

(ii) **(Octonion hermiticity)** \(\langle u, v \rangle = \langle v, u \rangle^*\) for all \(u, v \in H\).

(iii) **(Positivity)** \(\langle u, u \rangle \in \mathbb{R}^+\); and \(\langle u, u \rangle = 0\) if and only if \(u = 0\).

Under axioms in Definition 4.1, we can deduce in Lemma 4.6 below that

\[
\langle pu, u \rangle = p \langle u, u \rangle \tag{4.1}
\]

for all \(p \in \mathbb{O}\) and all \(u \in H\). This equation is one of the axioms of Goldstine and Horwitz as part \((f)\) of [5, Postulate 2].
We introduce a real trilinear map
\[ [\cdot, \cdot, \cdot] : \mathbb{O} \times M \times M \rightarrow \mathbb{O}, \]
defined by
\[ [p, u, v] := \langle pu, v \rangle - p \langle u, v \rangle \]
for \( u, v \in H \) and \( p \in \mathbb{O} \). We call this map the second associator in \( H \).

Identity (4.1) can be restated as
\[ [p, u, u] = 0. \]

**Remark 4.2** The second associator has been introduced in the space \( \text{Hom}_{L\mathbb{O}}(H, \mathbb{O}) \) in Sect. 3. These two notions in \( H \) and \( H^\ast \) coincide if we identify \( H \) with \( H^\ast \) (see Theorem 5.6). In fact, for any \( p \in \mathbb{O} \) and \( u, v \in H \) we have
\[ [p, u, v] = [p, u, v'], \tag{4.2} \]
where \( v' \in \text{Hom}_{L\mathbb{O}}(H, \mathbb{O}) \) is a map associated to \( v \) via the inner product, i.e.,
\[ v' = \langle \cdot, v \rangle. \]

Indeed, by definition we have
\[
[p, u, v'] = v'(pu) - pv'(u) \\
= \langle pu, v \rangle - p \langle u, v \rangle \\
= [p, u, v].
\]

**Example 4.3** If \( \mathbb{O} \) is regarded as an \( \mathbb{O} \)-Hilbert space with the inner product
\[ \langle x, y \rangle := x \overline{y} \]
then its second associator of \( p, u, v \) is given by
\[ [p, u, v] = \langle pu, v \rangle - p \langle u, v \rangle \\
= (pu)\overline{v} - p(u\overline{v}), \]
which is exactly the associator of \( p, u, \overline{v} \) in the algebra \( \mathbb{O} \).

Second associators in an \( \mathbb{O} \)-Hilbert space can be studied with the aid of the theory of \( \mathbb{O} \)-para-linear functionals based on relation (4.2).
Lemma 4.4  Let $H$ be a pre-Hilbert (left) $\mathbb{O}$-module, $p, q \in \mathbb{O}$ and $v, u \in H$. Then, we have

$$[p, v, u] = \sum_{i=1}^7 e_i \Re \langle [e_i, p, v], u \rangle; \quad (4.3)$$

$$[pq, v, u] = \langle [p, q, v], u \rangle - [p, q, \langle v, u \rangle] + p[q, v, u] + [p, qv, u]; \quad (4.4)$$

$$\Re \langle [p, q, v], u \rangle = -\Re \langle p[q, v, u] \rangle. \quad (4.5)$$

Proof  We fix $u \in H$ and take $f := u' \in \text{Hom}_{\mathbb{L}\mathbb{O}}(H, \mathbb{O})$ in Theorem 3.5. Since

$$f(x) = u'(x) = \langle x, u \rangle$$

and

$$f_{\mathbb{R}}(x) = \Re \langle x, u \rangle,$$

we thus conclude that identity (4.3) follows from Theorem 3.5.

Identity (4.4) can be proved in the manner similar to the proof of identity (2.3). Identity (4.5) follows from (4.4) by taking the real part on both sides. $\square$

If $H$ is a pre-Hilbert (left) $\mathbb{O}$-module, then $(H, \langle \cdot, \cdot \rangle_{\mathbb{R}})$ is a real Hilbert space, where

$$\langle x, y \rangle_{\mathbb{R}} := \Re \langle x, y \rangle.$$

Conversely, it is quite natural to ask if a real Hilbert space with a structure of left $\mathbb{O}$-module admits an $\mathbb{O}$-inner product.

Theorem 4.5  Let $H$ be a left $\mathbb{O}$-module equipped with a real valued inner product $\langle \cdot, \cdot \rangle_0$. Then, $H$ admits an $\mathbb{O}$-inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle \cdot, \cdot \rangle_{\mathbb{R}} = \langle \cdot, \cdot \rangle_0 \quad (4.6)$$

if and only if

$$\langle px, y \rangle_0 = \langle x, \overline{p} y \rangle_0 \quad (4.7)$$

holds for all $p \in \mathbb{O}$ and all $x, y \in H$.

Proof  Suppose the $\mathbb{O}$-inner product $\langle \cdot, \cdot \rangle$ in $H$ satisfies (4.6). To prove (4.7), we need to show

$$\Re \langle px, y \rangle = \Re \langle x, \overline{p} y \rangle$$

for all $p \in \mathbb{O}$ and $x, y \in H$. By direct calculation, we have

$$\Re \langle px, y \rangle = \Re (p \langle x, y \rangle + [p, x, y])$$
Conversely, suppose that (4.7) holds. We claim that the following inner product satisfies all conditions:

$$\langle x, y \rangle_{\mathbb{O}} := \langle x, y \rangle_0 - \sum_{i=1}^{7} (e_i x, y)_0 e_i.$$  

(4.8)

According to the construction, equation (4.6) obviously holds. It remains to show that $\langle \cdot, \cdot \rangle_{\mathbb{O}}$ is an $\mathbb{O}$-inner product.

Evidently, $\langle \cdot, \cdot \rangle_{\mathbb{O}}$ is $\mathbb{R}$-bilinear. We come to prove the para-linearity. Define

$$f(x) := \langle x, y \rangle_{\mathbb{O}}$$  

(4.9)

for any given $y \in H$. Then $f$ is clearly a real linear map. We write

$$f(x) = f_{\mathbb{R}}(x) + \sum_{i=1}^{7} f_i(x) e_i,$$  

(4.10)

where $f_{\mathbb{R}}(x), f_i(x) \in \mathbb{R}$. Combining (4.8) with (4.9), we obtain another expression of $f$ which demonstrates that

$$f_i(x) = - (e_i x, y)_0 = - \text{Re} \langle e_i x, y \rangle = - \text{Re} f(e_i x) = - f_{\mathbb{R}}(e_i x)$$

for $i = 1, \ldots, 7$. Then Theorem 3.4 implies that $f$ is left para-linear. This proves the para-linearity of $\langle \cdot, \cdot \rangle_{\mathbb{O}}$.

The octonionic hermiticity follows from condition (4.7). Indeed,

$$\overline{\langle x, y \rangle_{\mathbb{O}}} = \langle x, y \rangle_0 + \sum_{i=1}^{7} (e_i x, y)_0 e_i$$

$$= (y, x)_0 + \sum_{i=1}^{7} \langle x, e_i y \rangle_0 e_i$$

$$= (y, x)_0 - \sum_{i=1}^{7} (e_i y, x)_0 e_i$$

$$= (y, x)_{\mathbb{O}}.$$  

$\square$ Springer
Finally, we show the positivity of \(\langle \cdot, \cdot \rangle_\mathbb{O} \). For each \(i = 1, \ldots, 7\), we have
\[
\langle e_i x, x \rangle_0 = \langle x, e_i x \rangle_0 = -\langle x, e_i x \rangle_0 = -\langle e_i x, x \rangle_0
\]
so that \(\langle e_i x, x \rangle_0 = 0\). This means
\[
\langle x, x \rangle_\mathbb{O} = \langle x, x \rangle_0,
\]
which implies the positivity.

The associator \([p, q, u]\) in the \(\mathbb{O}\)-Hilbert space \(H\) is a skew-adjoint map as a function of \(u\), while the second associator \([p, u, v]\) in the \(\mathbb{O}\)-Hilbert space \(H\) is an alternating real bilinear map of \(u\) and \(v\).

**Lemma 4.6** Let \(H\) be a pre-Hilbert left \(\mathbb{O}\)-module. For any \(u, v \in H\) and \(p, q \in \mathbb{O}\), we have
\[
\langle ([p, q, u], v) \rangle_\mathbb{R} = -\langle u, [p, q, v] \rangle_\mathbb{R} ; \tag{4.11}
\]
\[
[p, u, v] = -[p, v, u]. \tag{4.12}
\]

**Proof** Identity (4.11) follows from Theorem 4.5. Indeed,
\[
\langle ([p, q, u], v) \rangle_\mathbb{R} = \langle (pq)u - p(qu), v \rangle_\mathbb{R} = \langle u, (pq)v - q(pv) \rangle_\mathbb{R} = \langle u, [\overline{q}, \overline{p}, v] \rangle_\mathbb{R} = \langle u, [q, p, v] \rangle_\mathbb{R} = -\langle u, [p, q, v] \rangle_\mathbb{R}.
\]
Due to identities (4.3) and (4.11) we obtain
\[
[p, u, v] = \sum_{i=1}^{7} e_i \langle [e_i, p, u], v \rangle_\mathbb{R} = -\sum_{i=1}^{7} e_i \langle [e_i, p, v], u \rangle_\mathbb{R} = -[p, v, u].
\]
This proves (4.12). \(\square\)

As a direct consequence of Lemma 4.6, we have for any \(p \in \mathbb{O}\) and \(u \in H\)
\[
[p, u, u] = 0.
\]

The next lemma tells us how it works when factors of octonions pass through the \(\mathbb{O}\)-inner product.
Lemma 4.7 Let $H$ be a pre-Hilbert left $\mathbb{O}$-module. For any $u, v \in H$ and $p, q \in \mathbb{O}$, the following identities hold:

\begin{align*}
\langle u, pv \rangle &= \langle u, v \rangle \overline{p} + \overline{[p, u, v]}; \\
\langle pu, qv \rangle &= p(\langle u, v \rangle \overline{q}) + p[q, u, v] + [p, u, qv]; \\
\langle pu, qv \rangle &= (p \langle u, v \rangle \overline{q}) + [pq, u, v] + \langle [p, q, v], u \rangle.
\end{align*}

Proof We first prove (4.13). By the octonionic hermiticity, we conclude from (4.12) that

\begin{align*}
\langle u, pv \rangle &= \overline{\langle pv, u \rangle} \\
&= \overline{p \langle v, u \rangle} + \overline{[p, v, u]} \\
&= \langle u, v \rangle \overline{p} - \overline{[p, v, u]} \\
&= \langle u, v \rangle \overline{p} + [p, u, v].
\end{align*}

(4.14) can be deduced from (4.13) directly. Indeed,

\begin{align*}
\langle pu, qv \rangle &= p \langle u, qv \rangle + [p, u, qv] \\
&= p(\langle u, v \rangle \overline{q}) + p[q, u, v] + [p, u, qv].
\end{align*}

It follows from identities (4.4) and (4.12) that

\begin{align*}
p[q, u, v] + [p, u, qv] &= \langle [p, q, v], u \rangle - [p, q, \langle v, u \rangle] + [pq, u, v].
\end{align*}

Combining this with (4.14), we get

\begin{align*}
\langle pu, qv \rangle &= p(\langle u, v \rangle \overline{q}) + \langle [p, q, v], u \rangle - [p, q, \langle v, u \rangle] + [pq, u, v] \\
&= p(\langle u, v \rangle \overline{q}) + [p, u, v, \overline{q}] + \langle [p, q, v], u \rangle + [pq, u, v] \\
&= (p \langle u, v \rangle \overline{q}) + [pq, u, v] + \langle [p, q, v], u \rangle.
\end{align*}

This completes the proof.

Remark 4.8 All discussions in this subsection still hold when $\mathbb{O}$ is replaced by an arbitrary real normed division algebra $\mathbb{K}$. This verifies that the Hilbert space over any normed division algebras can be defined uniformly.

### 4.2 Norm of $\mathbb{O}$-Hilbert Space

In this subsection, we collect some known results about the norm of an $\mathbb{O}$-Hilbert space; see [5]. Our method is suitable for any Hilbert space over any normed division algebras.

Definition 4.9 A normed left $\mathbb{O}$-module is a pair $(X, \| \cdot \|)$ consisting of a left $\mathbb{O}$-module $M$ and a map $\| \cdot \| : X \to \mathbb{R}$ satisfying the following axioms.
(i) \(|x| \geq 0\) for all \(x \in X\) with equality if and only if \(x = 0\).
(ii) \(|px| = |p| |x|\) for all \(p \in \mathbb{O}\) and \(x \in X\).
(iii) \(|x + y| \leq |x| + |y|\) for all \(x, y \in X\).

A normed \(\mathbb{O}\)-module is said to be a Banach \(\mathbb{O}\)-module if it is complete with respect to its canonical norm.

**Definition 4.10** An \(\mathbb{O}\)-Hilbert space is a pre-Hilbert left \(\mathbb{O}\)-module \(H\) which is complete with respect to the norm \(|.| : H \to \mathbb{R}^+\) induced by the \(\mathbb{O}\)-inner product, i.e.,

\[
|u| = \sqrt{\langle u, u \rangle}. \tag{4.16}
\]

It is easy to check that in an \(\mathbb{O}\)-Hilbert space there holds the polarization identity:

\[
\langle u, v \rangle = \frac{1}{4} \sum_{i=0}^{7} e_i \left( |e_i u + v|^2 - |e_i u - v|^2 \right). \tag{4.17}
\]

This means that the \(\mathbb{O}\)-inner-product can be recovered from the norm.

**Theorem 4.11** Let \(H\) be a Hilbert left \(\mathbb{O}\)-module. Then \(H\) is a Banach left \(\mathbb{O}\)-module with respect to the norm induced from the \(\mathbb{O}\)-inner product.

**Proof** Since \((H, |.|)\) is a real Banach space, it remains to show that

\[
|pu| = |p| |u|
\]

for any \(p \in \mathbb{O}\) and \(x \in X\). According to (4.12) and (4.15) we obtain

\[
\langle pu, pu \rangle = (p \langle u, u \rangle) p + [pp, u, u] + \langle [p, p, u], u \rangle = |p|^2 |u|^2.
\]

This proves that \(H\) is a Banach left \(\mathbb{O}\)-module.

The Cauchy–Schwarz inequality has been established in [5]. Here we give an alternative proof based on Lemma 4.7.

**Proposition 4.12** For all \(u, v \in H\), we have the Cauchy–Schwarz inequality

\[
|\langle u, v \rangle| \leq |u| |v| \tag{4.18}
\]

Equality holds in (4.18) if and only if \(u\) and \(v\) are \(\mathbb{O}\)-linearly dependent.

**Proof** To prove the Cauchy–Schwarz inequality, without loss of generality we may assume \(|v| = 1\). Denote \(p = \langle u, v \rangle\). It follows from (4.13) that

\[
\langle u - pv, pv \rangle = \text{Re} \left( \langle u - pv, v \rangle \overline{p} \right) = \text{Re} \left( \langle u, v \rangle - p \langle v, v \rangle \overline{p} \right)
\]
\[ = 0. \]

This implies that
\[ ||u||^2 = ||u - pv||^2 + ||pv||^2 \geq ||pv||^2. \] (4.19)

Since
\[ ||pv|| = |p| \cdot ||v|| = |p|, \]
we can rewrite (4.19) as
\[ ||u||^2 \leq ||u - pv||^2 = ||u|| \cdot ||v||^2. \]

Moreover, in view of (4.19), we conclude that
\[ ||u - pv||^2 = 0, \]
which means \( u, v \) are \( \mathbb{O} \)-linearly dependent. \( \square \)

## 5 Riesz Representation Theorem

Let \( H \) be an \( \mathbb{O} \)-Hilbert space. We consider
\[ H^* := \{ f \in \text{Hom}_{L,\mathbb{O}}(H, \mathbb{O}) \mid f \text{ is continuous} \}. \]

For any \( f \in H^* \), we define the right scalar multiplication by
\[ (f \circ r)(x) := f(x)r - [r, x, f] \] (5.1)

and define the norm
\[ ||f|| = \sup_{||x|| \leq 1} ||f(x)||. \] (5.2)

This makes \( H^* \) a Banach right \( \mathbb{O} \)-module. Indeed, Theorem 3.10 shows that \( H^* \) is a right \( \mathbb{O} \) module. To show it is a normed space, we need to prove
\[ ||f \circ p|| = ||f|| \cdot |p| \]
for all \( f \in H^* \) and \( p \in \mathbb{O} \). This needs an alternative expression of the right multiplication in (5.1) below.

**Lemma 5.1** For any \( f \in H^* \) and \( p \in \mathbb{O} \), we have
\[ (f \circ p)(x) = pf(p^{-1}x)p. \] (5.3)
**Proof** Without loss of generality we can assume $|p| = 1$. This means $p^{-1} = \bar{p}$. By the definition of the second associator in (3.1), we have

$$f(\bar{p}x) = \bar{p}f(x) + [\bar{p}, x, f].$$

Multiplying the left and right above by $p$, we obtain

$$pf(p^{-1}x)p = \left(p(\bar{p}f(x) + [\bar{p}, x, f])\right)p.$$

We thus can apply the identity

$$[\bar{p}, x, f] = -[p, x, f]$$

and identity (3.11)

$$p[p, x, f] = [p, x, f]p$$

to conclude

$$pf(p^{-1}x)p = f(x)p - [p, x, f]. \quad (5.4)$$

By the definition of the right scalar multiplication in (3.14), the right side above is exactly $(f \odot p)(x)$ as desired. 

**Remark 5.2** In our typical Example 3.3, we have $\odot$-para-linear function $R_q \in \text{Hom}_{\mathbb{O}}(\mathbb{O}, \odot)$ and its right multiplication by an octonion $p$ is given by

$$R_q \odot p = R_{qp}.$$  

Indeed,

$$R_q \odot p(x) = R_q(x)p - [p, x, R_q]$$

$$= (xq)p - [p, x, q]$$

$$= (xq)p - [x, q, p]$$

$$= x(qp)$$

$$= R_{qp}(x).$$

In this setting, identity (5.3) becomes

$$(R_q \odot p)(x) = pR_q(p^{-1}x)p.$$ 

That is,

$$x(qp) = p((p^{-1}x)q)p \quad (5.5)$$
for any \( p, q, x \in \mathcal{O} \). This is a variant of the following Moufang identity \[24\]

\[(xy)(ax) = x(ya)x. \tag{5.6}\]

Consequently, (5.3) is a generalized Moufang identity.

**Theorem 5.3** \( H^* \) is a Banach right \( \mathcal{O} \)-module.

**Proof** For any \( f \in H^* \) and \( p \in \mathcal{O} \), it follows from identity (5.3) that

\[
||f \circ p|| = \sup_{||x|| \leq 1} \left| pf(p^{-1}x)p \right| \\
\leq \sup_{||x|| \leq 1} |p| ||f|| \left| p^{-1}x \right| |p| \\
\leq |p| ||f||.
\]

Thus \( f \circ p \) is a continuous \( \mathcal{O} \)-para-linear functional, i.e., \( f \circ p \in H^* \). In view of Theorem 3.10, we conclude that \( H^* \) is an \( \mathcal{O} \)-submodule of the right \( \mathcal{O} \)-module \( \text{Hom}_{L, \mathcal{O}}(H, \mathcal{O}) \).

Since \( p \) and \( f \) are arbitrarily fixed, we can replace \( p \) by \( p^{-1} \) and \( f \) by \( f \circ p \) to get

\[
\left| \left| (f \circ p) \circ p^{-1} \right| \right| \leq |p^{-1}| ||f \circ p||
\]

so that

\[
||f|| = \left| \left| (f \circ p) \circ p^{-1} \right| \right| \\
\leq |p^{-1}| ||f \circ p|| \\
\leq |p^{-1}| |p| ||f|| \\
= ||f||.
\]

This implies

\[
||f \circ p|| = |p| ||f||.
\]

Hence \( H^* \) is a normed right \( \mathcal{O} \)-module with respect to norm (5.2).

Finally, we show that \( H^* \) is a Banach right \( \mathcal{O} \)-module. Let \( H^{*R} \) stand for the real dual space of \( H \) regarded as a real Hilbert space, i.e.,

\[ H^{*R} = \{ f : H \rightarrow \mathbb{R} \mid f \text{ is a bounded real linear functional} \}. \]

It is well-known that \( H^{*R} \) is a real Banach space. Therefore, if \( f_n \in H^* \) is a Cauchy sequence, then there exists \( f \in H^{*R} \) such that \( \lim_{n \to \infty} f_n = f \). It is easy to check that the maps

\[ [p, x, \cdot] : H^* \rightarrow \mathcal{O} \]
and

\[ \text{Re} : \mathbb{O} \rightarrow \mathbb{O} \]

are both continuous. Hence

\[ \text{Re} \left[ p, x, f \right] = \text{Re} \left( \left( p, x, \lim_{n \to \infty} f_n \right) \right) = \text{Re} \left( \lim_{n \to \infty} \left[ p, x, f_n \right] \right) = \lim_{n \to \infty} \text{Re} \left( \left[ p, x, f_n \right] \right) = 0. \]

This proves \( f \in H^* \).

The following lemma shows that \( H^* \) as a real Hilbert space is isometric with \( H^{*\mathbb{R}} \) via the real part operator.

**Lemma 5.4** The real part map

\[ \text{Re} : H^* \rightarrow H^{*\mathbb{R}}, \]

defined by \((\text{Re} f)(x) := f_{\mathbb{R}}(x)\), is a norm-preserving map. That is, for any \( f \in H^* \) we have

\[ ||f||_{H^*} = ||f_{\mathbb{R}}||_{H^{*\mathbb{R}}}. \]

**Proof** Let \( f \in H^* \) and \( x \in H \) with \( ||x|| \leq 1 \). Then there exists \( p \in \mathbb{O} \) with \( |p| = 1 \) such that

\[ |f(x)| = pf(x) = \text{Re}(pf(x)). \]

Since \( f \) is para-linear, it follows that

\[ |f(x)| = \text{Re}(pf(x)) = \text{Re}(f(px)) = f_{\mathbb{R}}(px) \leq ||f_{\mathbb{R}}||_{H^{*\mathbb{R}}} ||px|| \leq ||f_{\mathbb{R}}||_{H^{*\mathbb{R}}}. \]

Thus we obtain \( ||f||_{H^*} \leq ||f_{\mathbb{R}}||_{H^{*\mathbb{R}}}. \)

On the other hand, it is easy to see that \( ||f_{\mathbb{R}}||_{H^{*\mathbb{R}}} \leq ||f||_{H^*} \). This completes the proof.

To state the octonionic version of Riesz representation theorem, we need introduce the notion of conjugate \( \mathbb{O} \)-linearity.

**Definition 5.5** Let \( M_1 \) be a left \( \mathbb{O} \)-module and \( M_2 \) a right \( \mathbb{O} \)-module. A map \( f \in \text{Hom}_{\mathbb{R}}(M_1, M_2) \) is called **conjugate \( \mathbb{O} \)-linear** if

\[ f(px) = f(x)\overline{p} \]

for all \( p \in \mathbb{O} \) and all \( x \in M_1 \).
Theorem 5.6 (Riesz representation theorem) Let $H$ be an $\mathcal{O}$-Hilbert space. Then there is an isometric conjugate $\mathcal{O}$-isomorphism

$$H^* \cong H.$$ (5.7)

Proof Let $f \in H^*$ be para-linear. By Theorem 3.4, it can be written as

$$f(x) = f_\mathbb{R}(x) - \sum_{i=1}^2 e_i f_\mathbb{R}(e_i x),$$

where $f : H \to \mathbb{R}$ is a real linear functional. Then the classical Riesz representation theorem shows there exists a unique element $z_f \in H$ such that

$$f_\mathbb{R}(x) = \langle x, z_f \rangle_\mathbb{R}$$

and

$$\|z_f\| = \|f_\mathbb{R}\|.$$ Since both $f$ and $\langle \cdot, z_f \rangle$ are $\mathcal{O}$-para-linear and share the same real part, they must coincide as shown by Corollary 3.7, i.e.,

$$f(x) = \langle x, z_f \rangle.$$

By Lemma 5.4 we have

$$\|z_f\| = \|f\|. $$

Recall that $H^*$ is a Banach right $\mathcal{O}$-module by Theorem 5.3. Hence, the map

$$\sigma : H^* \to H$$

$$f \mapsto z_f$$

is an isometry between $H^*$ and $H$ as real Banach spaces.

We next prove that $\sigma$ is a conjugate $\mathcal{O}$-homomorphism. For any $x \in H$ and any $r \in \mathcal{O}$, by definition (3.14) we have

$$\langle x, z_f \circ r \rangle = (f \circ r)(x)$$

$$= f(x)r - [r, x, f]$$

$$= \langle x, z_f \rangle \bar{r} + [\bar{r}, x, z_f]$$

$$= \langle x, \bar{r}z_f \rangle.$$ This shows that

$$\sigma (f \circ r) = \bar{r} \sigma (f).$$
This implies that $\sigma$ is conjugate $\mathbb{O}$-linear.

We finally show that $\sigma$ is a conjugate $\mathbb{O}$-isomorphism. We define 

$$\tau : H \rightarrow H^*$$

$$y \mapsto y' := \langle \cdot, y \rangle.$$ 

We now prove $\tau$ is also a conjugate $\mathbb{O}$-homomorphism. For any $x \in H$ and any $r \in \mathbb{O}$, it follows from (3.14) and (4.2) that 

$$(y' \odot r)(x) = y'(x)r - [r, x, y']$$

$$= \langle x, y \rangle r - [r, x, y]$$

$$= \langle x, y \rangle \overline{r} + [\overline{r}, x, y]$$

$$= \langle x, \overline{r}y \rangle$$

$$= (\overline{r}y)'(x).$$ 

This shows that 

$$\tau(y) \odot r = \tau(\overline{r}y),$$

i.e., $\tau$ is also a conjugate $\mathbb{O}$-homomorphism.

Moreover, it is easy to check that 

$$\sigma \tau = id, \quad \tau \sigma = id.$$ 

Hence $\sigma$ is a conjugate $\mathbb{O}$-isomorphism. This finishes the proof. $\square$

Associated to a right $\mathbb{O}$-module $M$ is an induced left $\mathbb{O}$-module $M^-$ with multiplication defined by 

$$p \cdot x := x \overline{p}$$

for all $x \in M$ and all $p \in \mathbb{O}$. We can thus identify two left modules $H$ and $(H^*)^-$ by the isomorphism $\sigma$ in Theorem 5.6. Moreover, this isomorphism induces on $(H^*)^-$ a canonical $\mathbb{O}$-inner product 

$$\langle f, g \rangle := \langle z_f, z_g \rangle$$

for any $f, g \in (H^*)^-$. By definition, the octonion hermiticity and positivity can be verified directly. It remains to show the para-linearity. By calculation, for any $r \in \mathbb{O}$ and $f, g \in H^*$ we have 

$$\langle r \cdot f, g \rangle = \langle f \odot \overline{r}, g \rangle$$

$$= \langle z_{f \odot r}, z_g \rangle$$

$$= \langle rz_f, z_g \rangle.$$
\[ = r\langle z_f, z_g \rangle + [r, z_f, z_g]. \]

This shows that \( \langle \cdot, g \rangle \) is para-linear as desired. Then \( \sigma \) is an isomorphism between two Hilbert left \( \mathcal{O} \)-modules.

Based on Theorem 5.6, we can characterize the set

\[ H^{*\mathcal{O}} = \{ f : H \rightarrow \mathcal{O} \mid f \text{ is a bounded } \mathcal{O}\text{-linear functional} \} \]

in terms of associative elements of \( H \). To do this, we need a criterion for an element \( x \in H \) to be an associative element.

**Lemma 5.7** Let \( H \) be a Hilbert left \( \mathcal{O} \)-module and \( x \in H \). Then \( x \in \mathcal{A}(H) \) if and only if

\[ [p, x, y] = 0 \quad (5.8) \]

for all \( p \in \mathcal{O} \) and \( y \in H \).

**Proof** If \( x \in \mathcal{A}(H) \), then Corollary 3.7 shows that

\[ [p, x, y] = [p, x, y'] = 0, \]

where \( y' \) is defined as in the proof of Theorem 5.6.

Conversely, if \( x \in H \) satisfies hypothesis (5.6), then identity (4.5) implies that

\[ \langle [q, p, x], y \rangle_{\mathbb{R}} = 0 \]

for any \( p, q \in \mathcal{O} \) and \( y \in H \). Since \( (H, \langle \cdot, \cdot \rangle_{\mathbb{R}}) \) is a real Hilbert space, we thus conclude that

\[ [q, p, x] = 0 \]

for all \( p, q \in \mathcal{O} \), which means that \( x \in \mathcal{A}(H) \). This completes the proof. \( \square \)

**Theorem 5.8** Let \( H \) be an \( \mathcal{O} \)-Hilbert space. Then, there is an isometric isomorphism of real Banach spaces

\[ H^{*\mathcal{O}} \cong \mathcal{A}(H). \quad (5.9) \]

**Proof** According to Theorem 5.6, it suffices to show that

\[ \tau|_{\mathcal{A}(H)} : \mathcal{A}(H) \rightarrow H^{*\mathcal{O}} \]

is surjective.
If \( f \in H^{*\mathbb{O}} \), then Theorem 5.6 ensures the existence of an element \( z_f \in H \) such that

\[
f(x) = \langle x, z_f \rangle.
\]

Obviously,

\[
\tau(z_f) = f.
\]

It remains to show that \( z_f \in \mathcal{A}(H) \). Since \( f \) is \( \mathbb{O} \)-linear, it follows from identity (4.2) that for any \( p \in \mathbb{O} \),

\[
[p, z_f, x] = -(p, x, z_f) = -(p, x, f) = -(f(px) - pf(x)) = 0.
\]

Then we conclude from Lemma 5.7 that \( z_f \in \mathcal{A}(H) \). This completes the proof. \( \square \)

**Remark 5.9** Isomorphisms (5.7) and (5.9) indicate that the objects in octonionic functional analysis are \( \mathbb{O} \)-para-linear maps rather than \( \mathbb{O} \)-linear maps.

**References**

1. Baez, J.C.: The octonions. Bull. Am. Math. Soc. (N.S.) 39(2), 145–205 (2002)
2. Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators, Grundlehren Text Editions. Springer, Berlin (2004). Corrected reprint of the 1992 original
3. De Leo, S., Abdel-Khalek, K.: Octonionic quantum mechanics and complex geometry. Progr. Theoret. Phys. 96(4), 823–831 (1996)
4. Furey, N., Hughes, M.J.: One generation of standard model Weyl representations as a single copy of \( \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \). Phys. Lett. B, 827, Paper No. 136959 (2022)
5. Goldstine, H.H., Horwitz, L.P.: Hilbert space with non-associative scalars. I. Math. Ann. 154, 1–27 (1964)
6. Goldstine, H.H., Horwitz, L.P.: Hilbert space with non-associative scalars. II. Math. Ann. 164, 291–316 (1966)
7. Grigorian, S.: \( G_2 \)-structures and octonion bundles. Adv. Math. 308, 142–207 (2017)
8. Günaydin, M.: Octonionic Hilbert spaces, the Poincaré group and \( SU(3) \). J. Math. Phys. 17(10), 1875–1883 (1976)
9. Günaydin, M., Gürsey, F.: Quark structure and octonions. J. Math. Phys. 14, 1651–1667 (1973)
10. Harvey, F.R.: Spinors and Calibrations. Perspectives in Mathematics, vol. 9. Academic Press Inc, Boston (1990)
11. Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces, Graduate Studies in Mathematics, vol. 34. American Mathematical Society, Providence (2001). Corrected reprint of the 1978 original
12. Horwitz, L.P., Razon, A.: Tensor product of quaternion Hilbert modules. In: Classical and quantum systems (Goslar, 1991), pp. 266–268. World Science Publisher, River Edge (1993)
13. Huo, Q., Li, Y., Ren, G.: Classification of left octonionic modules. Adv. Appl. Clifford Algebr. 31(1):Paper No. 11 (2021)
14. Jacobson, N.: Structure of alternative and Jordan bimodules. Osaka Math. J. 6, 1–71 (1954)
15. Lawson Jr. H.B., Michelsohn M.L.: Spin Geometry, Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton (1989)
16. Ludkovsky, S.V.: Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra. Sovrem. Mat. Prilozh. 35, 98–162 (2005)
17. Ludkovsky, S.V., Sprössig, W.: Spectral representations of operators in Hilbert spaces over quaternions and octonions. Complex Var. Elliptic Equ. 57(12), 1301–1324 (2012)
18. Razon, A., Horwitz, L.P.: Uniqueness of the scalar product in the tensor product of quaternion Hilbert modules. J. Math. Phys. 33(9), 3098–3104 (1992)
19. Razon, A., Horwitz, L.P.: Projection operators and states in the tensor product of quaternion Hilbert modules. Acta Appl. Math. 24(2), 179–194 (1991)
20. Rembieliński, J.: Tensor product of the octonionic Hilbert spaces and colour confinement. J. Phys. A 11(11), 2323–2331 (1978)
21. Rudin, W.: Function theory in the unit ball of $\mathbb{C}^n$, Classics in Mathematics. Springer, Berlin (2008). Reprint of the 1980 edition
22. Salamon, D.A., Walpuski, T.: Notes on the octonions. In: Proceedings of the Gökova Geometry-Topology Conference 2016, pp. 1–85. Gökova Geometry/Topology Conference (GGT), Gökova (2017)
23. Saworotnow, P.P.: A generalized Hilbert space. Duke Math. J. 35, 191–197 (1968)
24. Schafer, R.D.: An Introduction to Nonassociative Algebras. Dover Publications, Inc., New York (1995). Corrected reprint of the 1966 original
25. Shestakov, I., Trushina, M.: Irreducible bimodules over alternative algebras and superalgebras. Trans. Am. Math. Soc. 368(7), 4657–4684 (2016)
26. Slinko, A.M., Sestakov, I.P.: Right representations of algebras. Algebra i Logika, 13(5):544–588, 605–606 (1974)
27. Soffer, A., Horwitz, L.P.: $B^*$-algebra representations in a quaternionic Hilbert module. J. Math. Phys. 24(12), 2780–2782 (1983)
28. Viswanath, K.: Normal operations on quaternionic Hilbert spaces. Trans. Am. Math. Soc. 162, 337–350 (1971)

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