Nontautological Bielliptic Cycles

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Let $[\mathcal{B}_{2,0,20}]$ and $[\mathcal{B}_{2,0,20}]$ be the classes of the loci of stable resp. smooth bielliptic curves with 20 marked points where the bielliptic involution acts on the marked points as the permutation $(1\ 2)...(19\ 20)$. Graber and Pandharipande proved in [GP03] that these classes are nontautological. In this note we show that their result can be extended to prove that $[\mathcal{B}_g]$ is nontautological for $g \geq 12$ and that $[\mathcal{B}_{12}]$ is nontautological.

1 Introduction

The system of tautological rings $\{R^*(\mathcal{M}_{g,n})\}$ is defined to be the minimal system of $\mathbb{Q}$-subalgebras of the Chow rings $A^*(\mathcal{M}_{g,n})$ closed under pushforward (and hence pullback) along the natural gluing and forgetful morphisms

$$\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} \to \mathcal{M}_{g_1+g_2,n_1+n_2},$$
$$\mathcal{M}_{g,n+2} \to \mathcal{M}_{g+1,n},$$
$$\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}.$$

The tautological ring $R^*(\mathcal{M}_{g,n})$ of the moduli space of smooth curves is the image of $R^*(\mathcal{M}_{g,n})$ under the localization morphism $A^*(\mathcal{M}_{g,n}) \to A^*(\mathcal{M}_{g,n})$. We will denote by $RH^{2*}(\mathcal{M}_{g,n})$ the image of $R^*(\mathcal{M}_{g,n})$ under the cycle map $A^*(\mathcal{M}_{g,n}) \to H^{2*}(\mathcal{M}_{g,n})$ and define $RH^{2*}(\mathcal{M}_{g,n})$ accordingly. We say a cohomology class is tautological if it lies in the tautological subring of its cohomology ring, otherwise we say it is nontautological. In this note we will work over $\mathbb{C}$ and all Chow and cohomology rings are assumed to be taken with rational coefficients.

These tautological rings are relatively well understood. An additive set of generators for the groups $R^*(\mathcal{M}_{g,n})$ is given by decorated boundary strata and there exists an algorithm for computing the intersection product (see [GP03]). The class of many “geometrically defined” loci can be shown to be tautological, for example this is the case for the class of the locus $\mathcal{H}_g$ of hyperelliptic curves in $\mathcal{M}_g$ (see [FP03], Theorem 1)).

Any odd cohomology class of $\mathcal{M}_{g,n}$ is nontautological by definition. Deligne proved that $H^{11}(\mathcal{M}_{1,11}) \neq 0$, thus providing a first example of the existence of nontautological classes. In fact it is known that $H^*(\mathcal{M}_{0,n}) = RH^*(\mathcal{M}_{0,n})$ (see [Kee92]) and that $H^{2*}(\mathcal{M}_{1,n}) = RH^{2*}(\mathcal{M}_{1,n})$ for all $n$ (see [Pet14], Corollary 1.2).

Examples of geometrically defined loci which can be proven to be nontautological are still relatively scarce. In [GP03] Graber and Pandharipande hunt for algebraic classes in $H^{2*}(\mathcal{M}_{g,n})$ and $H^{2*}(\mathcal{M}_{g,n})$ which are nontautological. In particular they show that the classes of the loci $\mathcal{B}_{2,0,20}$ and $\mathcal{B}_{2,0,20}$ of stable resp. smooth bielliptic curves of genus 2 with 20 marked points where the bielliptic involution acts on the set of marked points as the permutation $(1\ 2)...(19\ 20)$...
are nontautological. They also show that for sufficiently high odd genus $h$ the class of the
locus of stable curves of genus $2h$ admitting a map to a curve of genus $h$ is nontautological in
$H^{2\bullet}(\overline{M}_{2h})$. Their result relies on the existence of odd cohomology in $H^{\bullet}(\overline{M}_{h,1})$ which has been
proven to exist in [Pik95] for all $h \geq 8069$. A recent survey of different methods of obtaining
nontautological classes can be found in [FP13].

In this note we prove the following two new results.

**Theorem 1.** The cohomology class $[B_{g,n,2m}]$ is nontautological for all $g + m \geq 12$, $0 \leq n \leq 2g - 2$ and $g \geq 2$.

**Theorem 2.** The cohomology class $[B_{g,0,2m}]$ is nontautological when $g + m = 12$ and $g \geq 2$.

With Theorem 1 we improve the genus for which algebraic nontautological classes on $\overline{M}_g$ are
known to exists from 16138 to 12. As far as the author is aware, Theorem 2 provides the first
example of a nontautological algebraic class on $\overline{M}_g$.

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2 Preliminaries

Admissible double covers were introduced to compactify moduli spaces of double covers of
smooth curves, let us recall the definition:

**Definition 3.** Let $(S, x_1, ..., x_k, y_1, ..., y_{2m})$ be a stable pointed curve of arithmetic genus $g$. An
admissible double cover is the data of a stable pointed curve $(T, x'_1, ..., x'_k, y'_1, ..., y'_{2m})$ of arithmetic
 genus $g'$ and a 2-to-1 map $f : S \to T$ satisfying the following conditions:
- the restriction to the smooth locus $f^{\text{sm}}: S^{\text{sm}} \to T^{\text{sm}}$ is branched exactly at the points
  $x'_1, ..., x'_k$ and the inverse image of $x'_i$ is $x_i$ for all $i = 1, ..., k$,
- the inverse image of $y'_i$ under $f$ is $\{y_{2i}, y_{2i+1}\}$,
- the image under $f$ of each node is a node.

We call $S$ the source curve and $T$ the target curve of the admissible cover. An admissible
hyperelliptic structure on $S$ is an admissible cover where $g' = 0$ and an admissible bielliptic
structure on $S$ is an admissible cover with $g' = 1$. Note that the admissible double cover $S \to T$
induces an involution on $S$ fixing the points $x_1, ..., x_k$ and permuting the points $y_1, ..., y_{2m}$
pairwise.

One can define families of admissible double covers and isomorphisms between them (see
[ACV03, Section 4]). By using the Riemann-Hurwitz formula and by induction on the number
of nodes we can deduce that the number $k$ in the above definition equals $2g + 2 - 4g'$. We
denote the moduli stack of admissible bielliptic covers with $2m$ marked points switched by the
involution by $\overline{B}_{g,2m}^{\text{Adm}}$. When $m = 0$ we simply write $\overline{B}_g^{\text{Adm}}$.

A natural target map and source map from each moduli space of admissible double covers can
be defined as follows. The target map is a finite surjective map which sends each admissible cover
to the target stable pointed curve $(T, x'_1, ..., x'_k, y'_1, ..., y'_{2m}) \in \overline{M}_{g',k+m}$. From the properness of
we deduce that the space of such admissible covers is proper. The dimension of the space of such admissible double covers equals $2g - g' + 2m - 1$. In the bielliptic case we get
\[
\dim \mathcal{B}_{g,2m}^{\text{Adm}} = 2g - 2 + m.
\]
The source map forgets all the structure of an admissible double cover except for
\[(S, x_1, \ldots, x_k, y_1, \ldots, y_{2m}) \in \overline{\mathcal{M}}_{g,k+2m}.
\]
In the bielliptic case this gives a map $\mathcal{B}_{g,2m}^{\text{Adm}} \to \overline{\mathcal{M}}_{g,2g-2+2m}$. We can compose this map with a composition of forgetful maps $\overline{\mathcal{M}}_{g,2g-2+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ which forgets the first $2g - 2 - n$ points (which therefore correspond to the first $2g - 2 - n$ ramification points of the admissible bielliptic covers) and stabilizes. We denote by $\mathcal{B}_{g,n,2m}$ the image substack of $\mathcal{B}_{g,2m}^{\text{Adm}}$ in $\overline{\mathcal{M}}_{g,n+2m}$. The above discussion can be summarized in the following diagram:

\[
\begin{array}{ccc}
\mathcal{B}_{g,2m}^{\text{Adm}} & \to & \mathcal{B}_{g,n,2m} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_{1,2g-2+m} & \hookrightarrow & \overline{\mathcal{M}}_{g,n+2m}
\end{array}
\]
The moduli stack $\mathcal{B}_{g,2m}^{\text{Adm}}$ is the open dense substack of $\mathcal{B}_{g,2m}^{\text{Adm}}$ of admissible bielliptic covers of smooth curves and we denote its image stack in $\overline{\mathcal{M}}_{g,n+2m}$ by $\mathcal{B}_{g,n,2m}$. We have well defined Chow classes

\[
\begin{align*}
[\mathcal{B}_{g,n,2m}] & \in A^{g-1+n+m}(\overline{\mathcal{M}}_{g,n+2m}) \\
[\mathcal{B}_{g,n,2m}] & \in A^{g-1+n+m}(\overline{\mathcal{M}}_{g,n+2m}).
\end{align*}
\]

We will abuse notation and also denote the image of these classes in the respective cohomology rings by $[\mathcal{B}_{g,n,2m}]$ and $[\mathcal{B}_{g,n,2m}]$. In a completely analogous way, we can define spaces of admissible hyperelliptic covers $\overline{\mathcal{H}}_{g,2m}$ and the loci $\overline{\mathcal{H}}_{g,n,2m}$ and $\mathcal{H}_{g,n,2m}$ in $\overline{\mathcal{M}}_{g,n+2m}$ and $\mathcal{M}_{g,n+2m}$ for all $0 \leq n \leq 2g + 2$.

**Notation 4.** We will denote by $\overline{\mathcal{M}}_{g,n}^D$ (resp. $\mathcal{M}_{g,n}^D$) the moduli stack parameterizing trivial étale double covers

\[f : (C_1: y_1,1, \ldots, y_{n1}) \cup (C_2: y_{12}, \ldots, y_{n2}) \to (C: y_1, \ldots, y_n)\]
mapping two isomorphic stable (resp. smooth) curves $(C_1: y_1,1, \ldots, y_{n1}) \simeq (C_2: y_{12}, \ldots, y_{n2})$ to a curve $(C: y_1, \ldots, y_n)$ such that $f^{-1}(y_i) = (y_{i1}, y_{i2})$.

Our proof of Theorem 4 relies on the following result for pullbacks along gluing morphisms.

**Proposition 5 (GP03 Proposition 1).** Let $\xi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ be the gluing morphism and let $\gamma \in RH^{*}(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2})$, then

\[\xi^{*}(\gamma) \in RH^{*}((\overline{\mathcal{M}}_{g_1,n_1+1})) \otimes RH^{*}(\overline{\mathcal{M}}_{g_2,n_2+1}).\]

We say that a cycle $\lambda \in H^{*}(\overline{\mathcal{M}}_{g_1,n_1}) \otimes H^{*}(\overline{\mathcal{M}}_{g_2,n_2})$ admits a tautological Künneth decomposition if $\lambda \in RH^{*}(\overline{\mathcal{M}}_{g_1,n_1}) \otimes RH^{*}(\overline{\mathcal{M}}_{g_2,n_2})$. 

\[\text{dim} \mathcal{B}_{g,2m}^{\text{Adm}} = 2g - 2 + m.\]
3 Proof of Theorem 1 and 2

We are now ready to prove Theorem 1. We start by proving the following weaker result.

Proposition 6. We have
\[ [B_{g,0,2m}] \not\in RH^\bullet(M_{g,2m}) \]
for \( g + m = 12 \) and \( g \geq 2 \).

Proof. Let \( \iota_1 : M_{1,11} \times M_{1,11} \to \overline{M}_{1,11} \times \overline{M}_{1,11} \) be the inclusion and \( \iota_2 : \overline{M}_{1,11} \times \overline{M}_{1,11} \to \overline{M}_{g,2m} \) the gluing morphism which glues the corresponding first \( g - 1 \) points of the two factors and orders the remaining points by sending the \( k \)'th marked point of the first curve to \( 2k - 1 \) and the \( k \)'th marked point of the second curve to \( 2k \). Let \( \iota \) be the composition \( \iota_2 \circ \iota_1 \) and let \( \Delta \) resp. \( \Delta_o \) be the diagonal of \( \overline{M}_{1,11} \times \overline{M}_{1,11} \) resp. \( M_{1,11} \times M_{1,11} \) so that \( \iota^*_1(\Delta) = [\Delta_o] \). In Lemma 7 we will prove that \( \iota^*([B_{g,0,2m}]) = \alpha[\Delta_o] \) for some \( \alpha \in \mathbb{Q}_{>0} \). Let \( \partial(M_{1,11} \times M_{1,11}) := ((\partial \overline{M}_{1,11}) \times \overline{M}_{1,11}) \cup (\overline{M}_{1,11} \times (\partial \overline{M}_{1,11})) \). Since the sequence
\[ A^{10}(\partial(M_{1,11} \times M_{1,11})) \longrightarrow A^{11}((\overline{M}_{1,11}) \times M_{1,11}) \xrightarrow{\iota^*_1} A^{11}(M_{1,11} \times M_{1,11}) \longrightarrow 0 \]
is exact there exists a class \( B \in A^{10}(\partial(M_{1,11} \times M_{1,11})) \) such that \( \iota^*_2([B_{g,0,2m}]) = \alpha[\Delta] + B \).

The class \( B \) admits a tautological Künneth decomposition by Lemma 5. Given a basis \( \{e_i\}_{i \in I} \) for \( H^\bullet(M_{1,11}) \) with dual basis \( \{\hat{e}_i\}_{i \in I} \) the cohomology class of the diagonal can be written as
\[ [\Delta] = \sum_{i \in I} (-1)^{\deg e_i} e_i \otimes \hat{e}_i. \]

In particular since \( H^{11}(M_{1,11}) \neq 0 \) the diagonal \( [\Delta] \) does not admit a tautological Künneth decomposition. Since the pullback of a tautological class along a (composition of) gluing morphisms admits a tautological Künneth decomposition by Proposition 5 this shows that \( [B_{g,0,2m}] \) is nontautological.

Lemma 7. Consider the composition of gluing morphisms \( \iota : M_{1,11} \times M_{1,11} \to \overline{M}_{g,2m} \) defined above. We have \( \iota^*([B_{g,2m}]) = \alpha[\Delta_o] \) for some \( \alpha \in \mathbb{Q}_{>0} \).

Proof. Consider the fiber diagram
\[
\begin{array}{ccc}
F & \longrightarrow & B_{g,2m}^{Adm} \\
\downarrow & & \downarrow \phi \\
M_{1,11} \times M_{1,11} & \xrightarrow{\iota} & \overline{M}_{g,2m}
\end{array}
\]
We will describe the fiber product \( F \), or rather the push forward of its class to \( M_{1,11} \times M_{1,11} \).

Consider the moduli stack \( M_{1,11}^{D} \), there is a closed embedding \( M_{1,11}^D \to M_{1,11} \times M_{1,11} \), \((C_1 \cup C_2 \to C) \to (C_1, C_2)\) with image the diagonal \( \Delta_o \). We define a map \( \eta : M_{1,11}^D \to \overline{M}_{g,2m}^{Adm} \) as follows: on the source curves \( y \) attatches rational bridges \( R_i \) between the corresponding marked points \( y_{i,1} \) of \( C_1 \) and \( y_{i,2} \) of \( C_2 \) for all \( 1 \leq i \leq g - 1 \) and on the target curve it attaches a rational curve \( R_i' \) with two marked points to the corresponding marked point \( y_i \) of \( C \). The trivial double cover \( C_1 \cup C_2 \to C \) then induces an admissible double cover
\[
\left( C_1 \cup C_2 \cup \bigcup_{i=1}^{g-1} R_i ; y_{g,1}, y_{g,2}, \ldots, y_{11,1}, y_{11,2} \right) \to \left( C \cup \bigcup_{i=1}^{g-1} R_i' ; y_g, \ldots, y_1 \right),
\]
branched at the marked points of each \(R_i\), which maps each pair of marked points \(y_{1,i}, y_{2,i}\) of \(C_1 \cup C_2 \cup \bigcup_{i=1}^{g-1} R_i\) to the corresponding marked point \(y_i\) of \(C \cup \bigcup_{i=1}^{g-1} R_i\).

By the universal property of fiber products we get a map \(\mathcal{M}_{1,11}^D \to F\). We claim that the composition \(\mathcal{M}_{1,11}^D \to F \to F^{\text{red}}\) is a finite surjective morphism. The map \(F \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}\) is proper since properness is stable under base extension, the map \(\mathcal{M}_{1,11}^D \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}\) is proper because \(\overline{\mathcal{M}_{g,2m}}\) is proper. It follows that \(\mathcal{M}_{1,11}^D \to F\) is proper. Since the map \(\mathcal{M}_{1,11}^D \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}\) is quasi-finite so is \(\mathcal{M}_{1,11}^D \to F\). Since \(\mathcal{M}_{1,11}^D \to F^{\text{red}}\) is proper and quasi-finite and \(F^{\text{red}}\) is of finite type (and reduced) it remains to check that this map induces a surjection on closed points.

By definition an object of \(F\) over \(S\) consists of a curve \(\bar{C} := (\bar{C}_1, \bar{C}_2) \in \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}(\mathbb{C})\), an object \((S \to T) \in \overline{B}_{g,2m}(\mathbb{C})\) and an isomorphism \(\gamma : \iota(\bar{C}) \sim \phi(S \to T)\). To prove the claim we will show that \((\bar{C}, (S \to T), \gamma)\) is isomorphic to an object in the image of \(\mathcal{M}_{1,11}^D(\mathbb{C})\).

Let \(f : \bar{C}_1 \cup \bar{C}_2 \to \iota(\bar{C})\) be the map of curves induced by \(\iota\), set \(C := \iota(\bar{C})\), \(C_1 := f(\bar{C}_1)\) and \(C_2 := f(\bar{C}_2)\), let \(\tau\) be the involution on \(C\) induced by the bielliptic involution of \(S \to T\) and let \(Q_i\) be the node of \(C\) corresponding to the \(i\)’th marking of \(\bar{C}_1\) and \(\bar{C}_2\).

Since \(C_1\) and \(C_2\) are smooth there are two possibilities for the action of \(\tau\) on \(C\): Either it fixes \(C_1\) and \(C_2\) or it switches the whole of \(C_1\) with the whole of \(C_2\). Suppose \(\tau\) fixes \(C_1\) and \(C_2\).

By construction the involution \(\tau\) maps marked points lying on \(C_1\) to marked points lying on \(C_2\) so this is only possible if \(C\) has no marked points at all. In this case \(\tau\) must fix the different strands of \(C\) at each \(Q_i\). If the inverse image of \(Q_i\) in \(S\) were to be a rational bridge \(R_i\) then this rational bridge would have 2 marked ramification points which are not nodes, but this would imply that \(\tau\) switches the nodes on the rational bridge and therefore switches the strands of \(C\) at \(Q_i\). It follows that the inverse image of each \(Q_i\) in \(S\) is a single node \(\hat{Q}_i\). Since \(C_1\) and \(C_2\) are smooth, \(\tau\) induces an involution on the set of nodes \(\{\hat{Q}_1, \ldots, \hat{Q}_{11}\}\). We can thus find distinct \(\hat{Q}_i, \hat{Q}_j \neq \tau(\hat{Q}_i)\) such that \(S \setminus \{\hat{Q}_1, \tau(\hat{Q}_1), \hat{Q}_j, \tau(\hat{Q}_j)\}\) is connected. But this means that there are at least two nodes \(P_i\) and \(P_j\) of \(T\) such that \(T \setminus \{P_i, P_j\}\) is connected. This would imply that the arithmetic genus of \(T\) is at least 2, which is a contradiction.

We can therefore assume \(\tau\) maps \(C_1\) to \(C_2\). Let us first suppose that \(\tau\) does not fix all nodes, so there exist some distinct \(i, j\) such that \(\tau(\hat{Q}_i) = \hat{Q}_j\). Let \(P\) be the image of \(\{\hat{Q}_i, \hat{Q}_j\}\) under the bielliptic map. Like before we see that \(T \setminus \{P\}\) is connected and it therefore has arithmetic genus 0 (since by assumption the arithmetic genus of \(T\) is 1). However the arithmetic genus of \(C_1 \setminus \{\hat{Q}_i, \hat{Q}_j\}\) is 1 and the bielliptic map restricts to an isomorphism \(C_1 \setminus \{\hat{Q}_i, \hat{Q}_j\} \to T \setminus \{P\}\), which is a contradiction.

We have thus proven that \(\tau\) switches the components \(C_1\) and \(C_2\) and fixes the nodes \(Q_i\), which implies that \(((\bar{C}_1, \bar{C}_2), (S \to T), \gamma)\) is isomorphic to an object in the image of \(\overline{\mathcal{M}_{g,2m}}\). This proves that the map \(\mathcal{M}_{1,11}^D \to F^{\text{red}}\) is surjective.

It follows that the pushforward of \(\Delta_o\) to \(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}\) equals the pushforward of \(F\) to \(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}\) up to a scalar. Since

\[
\text{codim}_{\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}} \Delta_o = 11 = \text{codim}_{\overline{B}_{g,2m}} \overline{B}_{g,2m}^{\text{Adm}}
\]

we see that \(\iota^* \overline{B}_{g,2m}^{\text{Adm}} = \alpha \Delta_o\) for some \(\alpha \in \mathbb{Q}_{>0}\) and \(g + m = 12\).

\[\square\]

**Lemma 8.** Every algebraic class of codimension 11 in \(\overline{\mathcal{M}_{1,11}} \times \overline{\mathcal{M}_{1,11}}\) supported on \(\partial(\overline{\mathcal{M}_{1,11}} \times \overline{\mathcal{M}_{1,11}})\) admits a tautological Künneth decomposition.

\[\text{As in} \ [\text{Vis89}, \text{Definition 1.8}].\]
ii Every algebraic class on $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ of complex codimension less than 11 admits a tautological Künneth decomposition.

**Proof.** This is a slightly weaker version of [GP03 Lemma 3], the proof given there required that $RH^*(\mathcal{M}_{1,n}) = H^*(\mathcal{M}_{1,n})$ and $H^k(\mathcal{M}_{1,n}) = 0$ for $n < 11$, for which there was no reference at the time of [GP03]. The first equation is [Pet14, Corollary 1.2]. The second condition follows from Getzler’s computations for $n < 11$ in [Get98].

9. We have now concluded the proof of Proposition 6. To prove Theorem 11 it remains to show that $[\mathcal{B}_{g,n,2m}]$ is nontautological for all $n$, $g$, $m$ with $g + m > 12$.

**Proof of Theorem 7.** We will show in Lemma 10 and 11 that if $[\mathcal{B}_{g,n,2m}]$ is nontautological then so are $[\mathcal{B}_{g,n+1,2m}]$ for $n \leq 2g - 3$, and $[\mathcal{B}_{g,n,2m+2}]$. In Lemma 12 we will show that if $[\mathcal{B}_{g,1,0}]$ is nontautological then so is $[\mathcal{B}_{g+1}]$. Using these statements inductively, with base case the statement of Proposition 6, we conclude that $[\mathcal{B}_{g,n,2m}]$ is nontautological for all $g + m \geq 12$. □

**Lemma 10.** If $[\mathcal{B}_{g,n,2m}]$ is nontautological and $n \leq 2g - 3$ then so is $[\mathcal{B}_{g,n+1,2m}]$.

**Proof.** Let $\pi : \mathcal{M}_{g,n+1,2m} \to \mathcal{M}_{g,n+2m}$ be the morphism which forgets the first point and stabilizes. Since $\pi([\mathcal{B}_{g,n+1,2m}]) = [\mathcal{B}_{g,n,2m}]$ and $\dim [\mathcal{B}_{g,n+1,2m}] = \dim [\mathcal{B}_{g,n,2m}]$ we have $\pi_*[\mathcal{B}_{g,n+1,2m}] = \alpha[\mathcal{B}_{g,n,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Because the push forward of a tautological class by the forgetful morphism is tautological, the result follows. □

**Lemma 11.** If $[\mathcal{B}_{g,n,2m}]$ is nontautological then so is $[\mathcal{B}_{g,n,2m+2}]$.

**Proof.** Suppose $n < 2g - 2$ then by the previous result $[\mathcal{B}_{g,n+1,2m}]$ is nontautological. Consider the gluing morphism

$$\sigma : \mathcal{M}_{g,n+2m+1} \times \mathcal{M}_{0,3} \to \mathcal{M}_{g,n+2m+2}$$

which glues the first points of both curves together, then $\sigma^{-1}([\mathcal{B}_{g,n,2m+2}]) = [\mathcal{B}_{g,n+1,2m}]$.

Since $\operatorname{codim}_{\mathcal{M}_{g,n,2m+2}} [\mathcal{B}_{g,n,2m+2}] = \operatorname{codim}_{\mathcal{M}_{g,n+2m+1}} [\mathcal{B}_{g,n+1,2m}]$ it follows that $\sigma^*([\mathcal{B}_{g,n,2m+2}]) = \alpha[\mathcal{B}_{g,n,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Since $\sigma$ is a gluing morphism and the pullback of a tautological class along $\sigma$ admits tautological Künneth decomposition $[\mathcal{B}_{g,n,2m+2}]$ is nontautological.

If $n = 2g - 2$ we can first prove that $[\mathcal{B}_{g,n-1,2m+2}]$ is nontautological in the same way by pulling back through the map $\mathcal{M}_{g,n+2m} \times \mathcal{M}_{0,3} \to \mathcal{M}_{g,n+2m+2}$ and then use Lemma 10. □

**Lemma 12.** If $[\mathcal{B}_{g,1,0}]$ is nontautological then so is $[\mathcal{B}_{g+1}]$.

**Proof.** Let $\epsilon : \mathcal{M}_{g,1} \times \mathcal{M}_{1,1} \to \mathcal{M}_{g+1}$ be the gluing morphism. From the description of the boundary divisors of $\mathcal{B}_{g+1}$ (see [Pag16, Page 1275-1276]) it follows that there exists $\alpha', \beta' \in \mathbb{Q}_{>0}$ such that

$$\epsilon^*[\mathcal{B}_{g+1}] = \alpha[\mathcal{M}_{g,1,0} \times \mathcal{M}_{1,1}] + \beta[\mathcal{M}_{g-1,0,2} \times \mathcal{M}_{1,1}^D] \in H^*(\mathcal{M}_{g,1} \times \mathcal{M}_{1,1})$$

The class $[\mathcal{M}_{g-1,0,2} \times \mathcal{M}_{1,1}^D]$ admits a tautological Künneth decomposition (since the class of the hyperelliptic locus is tautological by [EP05 Theorem 1] and therefore so is its pushforward under a gluing morphism with a tautological class). The class $[\mathcal{M}_{g,1} \times \mathcal{M}_{1,1}]$ does not admit a tautological Künneth decomposition by assumption. It follows by Proposition 6 that $[\mathcal{B}_{g+1}]$ is nontautological. □

13. We will now prove a similar result for the open locus of $\mathcal{M}_{g,2m}$ where $g + m = 12$. 


Proof of Theorem [2]. The case where \( g = 2 \) is treated in [GP03 Section 3]. We use a similar argument to prove the remaining cases. The proof runs by contradiction. Suppose \([B_{g,0,2m}] \in RH^*(\mathcal{M}_{g,2m})\) then there is some collection of cycles \( Z_i \) in \( \mathcal{M}_{g,2m} \), of complex codimension \( 11 \) and supported on \( \partial \mathcal{M}_{g,2m} \) such that \( \sum[Z_i] + [B_{g,0,2m}] \) is a tautological class. Consider again the gluing morphism \( \iota_2: \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \rightarrow \mathcal{M}_{g,2m} \) as above. By assumption the pullback of \( \sum[Z_i] + [B_{g,0,2m}] \) to \( \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \) admits a tautological Künneth decomposition whereby the pullback of \( \sum[Z_i] \) to \( \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \) must be nontautological.

We shall use the usual notation that \( \Delta_j \) is the locus of curves in \( \mathcal{M}_{g,2m} \) consisting of two curves, one of which has genus \( j \), glued together in a single node, and \( \Delta_{irr} \) is the locus that generically parametrizes irreducible singular curves. Since \( \iota_2(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \) does not have a separating node we see that \( \iota_2(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \not\subset \Delta_j \). The intersection

\[
\Delta_j \cap (\mathcal{M}_{1,11} \times \mathcal{M}_{1,11})
\]

therefore lies in \( \partial(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \). It follows by Lemma 8.ii that \( \iota_2^*[Z_i] \) admits a tautological Künneth decomposition if \( \text{Supp} Z_i \subset \Delta_j \).

Consider now the \( Z_i \) with support inside \( \Delta_{irr} \). We can decompose the map \( \iota_2 \) as

\[
\mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \xrightarrow{\iota'_2} \mathcal{M}_{g-1,2m+2} \xrightarrow{\iota'_2} \mathcal{M}_{g,2m}
\]

Then there exist cycles \( Y_i \) in \( \mathcal{M}_{g-1,2m+2} \) such that \( \iota'_2^*[Y_i] = [Z_i] \). Now

\[
\iota_2^*[Z_i] = \iota'_2^* \iota_1^*[Z_i] = \iota'_2^* (c_1(N_{\mathcal{M}_{g-1,2m+2} \mathcal{M}_{g,2m}}) \cap [Y_i]).
\]

We see that \( \iota_2^*[Z_i] \) decomposes as a product of algebraic classes of codimension less than 11, which admit tautological Künneth decomposition by Lemma 8.ii.

We conclude that all the \([Z_i]\) have tautological Künneth decomposition when pulled back to \( \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \). Therefore \( \iota_2^*(\sum[Z_i] + [B_{g,0,2m}]) \) does not admit a tautological Künneth decomposition. It follows by Proposition 5 that \([B_{g,2m}]\) is nontautological. \( \square \)

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