SINGULAR POINTS ON PRODUCT OF CERTAIN HOMOGENEOUS SPACES

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Abstract. In this paper, we extend the dimension formula for singular vectors on products of certain homogeneous spaces.

1. Introduction

The roots of the theory of Diophantine approximation lie in Dirichlet’s Theorem. Given a matrix of real numbers \( \theta \in M_{m \times n}(\mathbb{R}) \), it asserts that the system of inequalities

\[
\begin{align*}
\| \theta q - p \| &< Q^{-n/m} \\
0 &< \| q \| \leq Q
\end{align*}
\]

has an integer solution \((p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n\) for all real numbers \(Q > 1\). Here \(\| \cdot \|\) denotes the supremum norm. One of the central topics in Diophantine approximation is to study the subsets of matrices for which one can go beyond (D). In this paper, we focus on singular matrices.

A matrix \( \theta \in M_{m \times n}(\mathbb{R}) \) is called singular if for every \( \epsilon > 0 \), the system of inequalities

\[
\begin{align*}
\| \theta q - p \| &< \epsilon Q^{-n/m} \\
0 &< \| q \| \leq Q
\end{align*}
\]

has an integer solution \((p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n\) for all sufficiently large \(Q\). We denote by \( \text{Sing}_{m,n} \) the set of all singular matrices in \( M_{m \times n}(\mathbb{R}) \). This notion was introduced by Khintchine [10] in 1937 in the setting of simultaneous approximation of vectors, and was generalized to the matrix setting later in [11]. See Moshchevitin’s survey [13] on the topic. Khintchine showed that the Lebesgue measure of \( \text{Sing}_{m,n} \) is 0. A natural question is

**Question 1.1.** What is \( \dim \text{Sing}_{m,n} \)?

Here and throught the paper, “dim” refers to the Hausdorff dimension. It is well-known that, when \((m, n) = (1, 1)\), the set \( \text{Sing}_{1,1} \) coincides with the set of rational numbers \(\mathbb{Q}\). For \((m, n) \neq (1, 1)\), Question 1.1 is a challenge, that saw breakthroughs only recently. First, in 2011, Cheung [3] proved that the Hausdorff dimension of singular pairs \( \text{Sing}_{2,1} \) is 4/3. This was extended in [4] by Cheung and Chevallier to the set of singular vectors of dimension \(n\), namely they showed that \( \dim \text{Sing}_{n,1} = n^2/(n+1) \) for all \( n \geq 2 \). Note that by Khintchine’s transference principle \( \text{Sing}_{m,n} \) and \( \text{Sing}_{n,m} \) have the same dimension. In general, the sharp upper bound of \( \dim \text{Sing}_{m,n} \) was obtained by Kadyrov, Kleinbock, Lindenstrauss and Margulis in [9] using the contraction property of the height function. The sharp...
lower bound of $\dim \text{Sing}_{m,n}$ was obtained recently by Das, Fishman, Simmons and Urbaniśki [7]. So Question 1.1 was answered completely:

**Theorem 1.2.** [3][4][9][7] For any $(m, n) \in \mathbb{N}^2$ with $(m, n) \neq (1, 1)$,

$$\dim \text{Sing}_{m,n} = mn - \frac{mn}{m+n}.$$  

Thanks to Dani’s correspondence [5], it is now well-known that many Diophantine properties of $\theta$ can be reformulated dynamically. Let $m, n \in \mathbb{N}$, $G = \text{SL}(m + n, \mathbb{R})$, $\Gamma = \text{SL}(m + n, \mathbb{Z})$, $Y_{m+n} = G/\Gamma$ and

$$F_{m,n}^+ = \{g_t^{(m,n)} : t \geq 0\} \quad \text{where} \quad g_t^{(m,n)} = \begin{pmatrix} e^{mt} I_m & e^{-mt} I_n \\ 0 & 1 \end{pmatrix} \in G. \quad (1.1)$$

For $\theta \in M_{m \times n}(\mathbb{R})$, set

$$u_\theta = \begin{pmatrix} I_m & \theta \\ 0 & I_n \end{pmatrix} \in G \quad \text{and} \quad x_\theta = u_\theta z^{m+n} \in Y_{m+n}. \quad (1.2)$$

Then $\theta \in M_{m \times n}(\mathbb{R})$ is singular if and only if the trajectory $F_{m,n}^+ x_\theta$ is divergent, i.e. it eventually leaves every compact subset of $Y_{m+n}$.

In general, let $G$ be a semisimple Lie group, $\Gamma \subset G$ be a lattice and $F^+ = \{g_t : t \geq 0\} \subset G$ be a one-parameter subsemigroup. A point $x \in G/\Gamma$ is called $F^+$-singular if the corresponding orbit $F^+ x$ is divergent on $G/\Gamma$. The set of singular points is denoted by $D(F^+, G/\Gamma)$, and was extensively studied in recent years, see [8]. A related notion was introduced by Kadyrov, Kleinbock, Lindenstrauss and Margulis [9]. Let $\delta \in (0, 1]$, a point $x \in G/\Gamma$ is called $(F^+, \delta)$-singular if for any compact subset $K$ of $G/\Gamma$ one has

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_K(g_t x) \, dt \leq 1 - \delta,$$

where $1_K$ denotes the characteristic function of $K$. The set of $(F^+, \delta)$-singular points is denoted by $D_\delta(F^+, G/\Gamma)$. We have the following natural question that extends Question 1.1.

**Question 1.3.** What is $\dim D(F^+, G/\Gamma)$ and $\dim D_\delta(F^+, G/\Gamma)$?

As a direct consequence of Theorem 1.2, we have for any $(m, n) \in \mathbb{N}^2$ with $(m, n) \neq (1, 1)$,

$$\dim D(F_{m,n}^+, Y_{m+n}) = \dim Y_{m+n} - \frac{mn}{m+n}. \quad (1.3)$$

Moreover, the following theorem was proved by Das, Fishman, Simmons and Urbaniśki in [7].

**Theorem 1.4.** Let $(m, n) \in \mathbb{N}^2$ and $\delta \in (0, 1]$. Then

$$\dim D_\delta(F_{m,n}^+, Y_{m+n}) = \dim Y_{m+n} - \delta \frac{mn}{m+n}. \quad (1.4)$$

The upper bound was established previously by Kadyrov, Kleinbock, Lindenstrauss and Margulis [9] using the contraction property of the height function.
In this paper, we consider certain special cases of Question 1.3, namely, when the homogeneous system \((F^+, G/\Gamma)\) is a product of unweighted homogeneous systems. More precisely, let

\[
G = \prod_{1 \leq i \leq s} G_i, \quad \Gamma = \prod_{1 \leq i \leq s} \Gamma_i, \quad X_i = G_i/\Gamma_i, \quad X = G/\Gamma = \prod_{1 \leq i \leq s} X_i, \tag{1.5}
\]

where

\[
G_i = \text{SL}(m_i + n_i, \mathbb{R}) \quad \text{and} \quad \Gamma_i = \text{SL}(m_i + n_i, \mathbb{Z}),
\]

with \(s \geq 2\) and \((m_i, n_i) \in \mathbb{N}^2\). Let

\[
A^+ = \prod_{1 \leq i \leq s} F^+_i, \quad \text{where} \quad F^+_i = F^+_{m_i,n_i},
\]

where we use the notation in (1.1). Let \(F^+\) be a one-parameter subsemigroup of \(A^+\) that projects non-trivially to each component. The homogeneous system \((F^+, G/\Gamma)\) is the main object of our study. For any such \(F^+\), there exists \(a = (a_1, \ldots, a_s) \in \mathbb{R}_+^s\), where \(\mathbb{R}_+ = (0, \infty)\), such that

\[
F^+_a = \left\{ g_t = \left( g_{a_i t}^{(m_i, n_i)} \right) : t \geq 0 \right\}. \tag{1.6}
\]

We say \(F^+_a\) is the one-parameter subsemigroup of \(A^+\) associated to the weight vector \(a\). We have \(F^+_a = F^+_r\) if and only if \(a = ra'\) for some positive constant \(r\).

Note that, \(x = (x_1, \ldots, x_s) \in X\) is \(F^+_a\)-singular (resp. \((F^+_a, \delta)\)-singular) if for some \(1 \leq i \leq s\), \(x_i\) is \(F^+_i\)-singular (resp. \((F^+_i, \delta)\)-singular). This motivates the following definition. We say \(x \in X\) is essentially \(F^+_a\)-singular (resp. essentially \((F^+_a, \delta)\)-singular) if \(x\) is \(F^+_a\)-singular (resp. \((F^+_a, \delta)\)-singular) but none of \(x_j\) is \(F^+_i\)-singular (resp. \((F^+_i, \delta)\)-singular). The set of essentially \(F^+_a\)-singular (resp. essentially \((F^+_a, \delta)\)-singular) points is denoted as \(D^e(F^+_a, X)\) (resp. \(D^e_\delta(F^+_a, X)\)).

The main result of this paper is as follows.

**Theorem 1.5.** Let \(X\) be a product of \(s\) homogeneous spaces given by (1.5) with \(s \geq 2\) and \((m_i, n_i) \in \mathbb{N}^2\). Let \(F^+_a\) be the one-parameter semigroup in (1.6) associated to \(a \in \mathbb{R}_+^s\). Then

\[
\dim D^e(F^+_a, X) = \dim X - \min_{1 \leq i \leq s} \frac{m_i n_i}{m_i + n_i}, \tag{1.7}
\]

and

\[
\dim D^e_\delta(F^+_a, X) = \dim X - \min_{1 \leq i \leq s} \delta \frac{m_i n_i}{m_i + n_i}. \tag{1.8}
\]

Note that 1-singular does not imply singular (see, for example, Remark 4.13). Hence (1.8) does not imply (1.7). It follows from the definition that,

\[
D(F^+_a, X) = D^e(F^+_a, X) \cup \bigcup_{i=1}^s \{ (x_1, \ldots, x_s) \in X : x_i \text{ is } F^+_i\text{-singular} \}.
\]

\[
D^e_\delta(F^+_a, X) = D^e_\delta(F^+_a, X) \cup \bigcup_{i=1}^s \{ (x_1, \ldots, x_s) \in X : x_i \text{ is } (F^+_i, \delta)\text{-singular} \}.
\]

Hence, in view of (1.3) and (1.4), we have the following corollary of Theorem 1.5.
Corollary 1.6. Let the notation be as in Theorem 1.5. Then
\begin{equation}
\dim D(F_{a}^{+}, X) = \dim X - \min_{1 \leq i \leq s} \frac{m_{i}n_{i}}{m_{i} + n_{i}}, \tag{1.9}
\end{equation}
and
\begin{equation}
\dim D_{\delta}(F_{a}^{+}, X) = \dim X - \min_{1 \leq i \leq s} \delta \frac{m_{i}n_{i}}{m_{i} + n_{i}}, \tag{1.10}
\end{equation}

When \((m_{i}, n_{i}) = (1, 1)\) for all \(1 \leq i \leq s\), the formula (1.9) was conjectured by Y. Cheung via private communication with the fourth named author. Cheung’s motivation is his result in [2] where he established (1.9) in the case where \((m_{i}, n_{i}) = (1, 1)\) and \(a = (1, \ldots, 1)\).

It will be clear from the proof that the dimension formulas in Theorem 1.5 and Corollary 1.6 are local. This means that for any non-empty open subset \(U\) of \(X\), the intersection of the singular set with \(U\) has the same dimension as itself. Our method also implies that the Hausdorff dimensions of \(\bigcup_{a \in \mathbb{R}^{s}_{+}} D(F_{a}^{+}, X)\) and \(\bigcup_{a \in \mathbb{R}^{s}_{+}} D_{\delta}(F_{a}^{+}, X)\) are equal to the right hand side of (1.9) and (1.10), respectively. We will give the proof of this stronger upper bound at the end of Section 3.1.

In Section 2, we reduce the proof of Theorem 1.5 to Proposition 2.1 which gives the dimension formula of singular points on an unstable horospherical leaf. The proof of Proposition 2.1 is the main body of the paper and is given in two independent parts.

In Section 3, we give the estimate from above using the Eskin-Margulis-Mozes height function along the lines of [9]. When the dynamical system \((F_{a}^{+}, X)\) has only one positive Lyapunov exponent, the optimal upper bound follows rather directly from the arguments in [9]. Otherwise, essential new ideas are needed. We construct a universal covering of the set of singular points independent of the weight \(a\). The key step is Lemma 3.5, which forms the main innovative part of Section 3.

In Section 4, we give the estimate from below using the variational principle in parametric geometry of numbers introduced in [7]. The variational principle enables us to study a very large family of Diophantine sets, namely, that can be described using templates, which are certain piecewise linear functions. In particular, the Hausdorff dimension of a Diophantine set that is associated to certain template can be computed using only the information of the template. By carefully choosing templates for different \((m_{i}, n_{i})\), we manage to construct certain product sets of matrices that give the optimal lower bound.

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2. A FIRST REDUCTION

In this section, we state the proposition we are going to prove in this paper and deduce Theorem 1.5 from it. Let us fix \(s \geq 2\) and \((m_{i}, n_{i}) \in \mathbb{N}^{2}\) for all \(1 \leq i \leq s\) from now on. Set
\[ M_{i} = M_{m_{i} \times n_{i}}(\mathbb{R}) \quad \text{and} \quad M = \prod_{i=1}^{s} M_{i}. \]
For $\Theta = (\theta_1, \ldots, \theta_s) \in \mathbb{M}$, let

$$u_\Theta = (u_{\theta_1}, \ldots, u_{\theta_s}) \in G \quad \text{and} \quad x_\Theta = (x_{\theta_1}, \ldots, x_{\theta_s}) \in X,$$

where $u_\theta$ and $x_\theta$ are as in (1.2). It is easily checked that the set $U := \{u_\Theta : \Theta \in \mathbb{M}\}$ is the expanding horospherical subgroup of $G$ with respect to $F_a^+$ for any $a \in \mathbb{R}^+_*$. Set

$$D^e(F_a^+, G) = \{\Theta \in \mathbb{M} : x_\Theta \text{ is essentially } F_a^+\text{-singular}\}$$

and

$$D^e_\delta(F_a^+, G) = \{\Theta \in \mathbb{M} : x_\Theta \text{ is essentially } (F_a^+, \delta)\text{-singular}\}.$$

Then Theorem 1.5 can be deduced from the following proposition.

**Proposition 2.1.** Let notation be as in Theorem 1.5. Then

$$\dim D^e(F_a^+, \mathbb{M}) = \sum_{i=1}^s m_i n_i - \min_{1 \leq i \leq s} \frac{m_i n_i}{m_i + n_i}, \tag{2.1}$$

and

$$\dim D^e_\delta(F_a^+, \mathbb{M}) = \sum_{i=1}^s m_i n_i - \min_{1 \leq i \leq s} \delta \frac{m_i n_i}{m_i + n_i}. \tag{2.2}$$

**Proof of Theorem 1.5 modulo Proposition 2.1.** Let $P$ be the weakly contracting subgroup of $G$ with respect to $F_a^+$, i.e.,

$$P = \{h \in G : \text{the set } \{ghg^{-1} : g \in F_a^+\} \text{ is bounded}\}.$$

The multiplication map from $P \times U$ into $G$ is everywhere regular with image $PU := \{pu : p \in P, u \in U\}$, see [12, Lemma 6.44]. Therefore, $PU$ is Zariski open in $G$. On the other hand, by Borel’s density theorem [1], $\Gamma$ is Zariski dense in $G$. So we have $\pi(P \times U) = X$, where

$$\pi : P \times \mathbb{M} \to G/\Gamma, \quad (p, \Theta) \mapsto pu(\Theta)\Gamma.$$

Note that for any $p \in P$ and $x \in X$, $px$ is essentially $F_a^+$-singular (resp. essentially $(F_a^+, \delta)$-singular) if and only if $x$ is essentially $F_a^+$-singular (resp. essentially $(F_a^+, \delta)$-singular). Hence we have

$$\pi^{-1}(D^e(F_a^+, X)) = P \times D^e(F_a^+, \mathbb{M}), \quad \pi^{-1}(D^e_\delta(F_a^+, X)) = P \times D^e_\delta(F_a^+, \mathbb{M}).$$

Since the multiplication map $P \times U \to PU$ is a diffeomorphism, locally the map $\pi$ is a diffeomorphism. Thus

$$\dim D^e(F_a^+, X) = \dim \pi^{-1}(D^e(F_a^+, X)), \quad \dim D^e_\delta(F_a^+, X) = \dim \pi^{-1}(D^e_\delta(F_a^+, X)).$$

Note that for any subset $Y$ of $\mathbb{M}$, $\dim P \times Y = \dim P + \dim Y$. So Theorem 1.5 follows from Proposition 2.1 and the fact that $\dim \mathbb{M} = \sum_{i=1}^s m_i n_i$. \hfill \Box

### 3. The upper bound

This section is devoted to the proof of the following proposition.

**Proposition 3.1.** Let $\delta \in (0, 1]$ and $a \in \mathbb{R}^+_*$, then

$$\dim D_\delta(F_a^+, \mathbb{M}) \leq \sum_{i=1}^s m_i n_i - \min_{1 \leq i \leq s} \delta \frac{m_i n_i}{m_i + n_i}.$$
Clearly, we have
\[ \mathcal{D}_\delta^c(F^+_a, M) \subset \mathcal{D}_\delta(F^+_a, M) \text{ and } \mathcal{D}_\delta^c(F^+_a, M) \subset D_1(F^+_a, M). \] (3.1)
Hence Proposition 3.1 gives the sharp upper bound of Proposition 2.1.

3.1. Auxiliary sets. In this section we cover \( \mathcal{D}_\delta(F^+_a, M) \) by sets whose dimensions are easier to estimate from above.

For any \( 1 \leq i \leq s \), we choose and fix a right invariant Riemannian metric \( \text{dist}_i(\cdot, \cdot) \) on \( G_i \), which naturally induces a metric on \( X_i = G_i/\Gamma_i \), also denoted by “\( \text{dist}_i \)”, as follows:
\[ \text{dist}_i(g\Gamma_i, h\Gamma_i) = \inf_{\gamma \in \Gamma_i} \text{dist}(g\gamma, h), \] where \( g, h \in G_i \).

Set “\( \text{dist} \)” to be the metric on \( X \) given by
\[ \text{dist} = \max_{1 \leq i \leq s} \text{dist}_i. \] (3.2)

For \( R > 0 \), let
\[ B^X_R = \{ x \in X : \text{dist}(x, [e]) \leq R \} \text{ and } E^X_R = X \setminus B^X_R, \]
where \( e \) denotes the identity element. For \( R, T > 0 \) and \( 0 < \delta \leq 1 \), let
\[ \tilde{\mathcal{D}}_\delta(F^+_a, R, T) = \left\{ \Theta \in M : \frac{1}{T} \int_0^T \mathbf{1}_{E^X_R}(g_t x\Theta) \, dt \geq \delta \right\}. \] (3.3)
The value \( \frac{1}{T} \int_0^T \mathbf{1}_{E^X_R}(g_t x\Theta) \, dt \) measures the proportion of the time up to \( T \) that the orbit \( F^+_a x\Theta \) spends in the set \( E^X_R \). Thus, the set \( \tilde{\mathcal{D}}_\delta(F^+_a, R, T) \) can be thought of as an approximation to the set \( \mathcal{D}_\delta(F^+_a, M) \). Their precise relation can be stated as follows: for any \( 0 < \delta' < \delta \leq 1 \) and \( R > 0 \), we have
\[ \mathcal{D}_\delta(F^+_a, M) \subset \liminf_{T \to \infty} \tilde{\mathcal{D}}_{\delta'}(F^+_a, R, T) := \bigcup_{T_1>0} \bigcap_{T>T_1} \tilde{\mathcal{D}}_{\delta'}(F^+_a, R, T). \] (3.4)
This gives our first enlargement of \( \mathcal{D}_\delta(F^+_a, M) \).

Next we cover each \( \tilde{\mathcal{D}}_{\delta'}(F^+_a, R, T) \) by a set defined using the data on each component of \( X = \prod_{i=1}^s X_i \). For \( 1 \leq i \leq s \) and \( R > 0 \), we set
\[ B^X_{Ri} = \{ x \in X_i : \text{dist}_i(x, [e]) \leq R \} \text{ and } E^X_{Ri} = X_i \setminus B^X_{Ri}. \]
We write \( g_{i,t} = g_t^{(m_i, n_i)} \) to simplify the notation. For \( R, T > 0 \) and \( \Theta \in M_i \), set
\[ \mathcal{A}_i(R, T, \Theta) = \frac{1}{T} \int_0^T \mathbf{1}_{E^X_{Ri}}(g_{i,t} x\Theta) \, dt. \]
Since \( \text{dist} \) is defined as the maximum of all the \( \text{dist}_i \), we have
\[ \frac{1}{T} \int_0^T \mathbf{1}_{E^X_R}(g_t x\Theta) \, dt = \frac{1}{T} \int_0^T \max_{1 \leq i \leq s} \mathbf{1}_{E^X_{Ri}}(g_{i,t} x\Theta) \, dt \leq \mathcal{A}(F^+_a, R, T, \Theta), \]
where
\[ \mathcal{A}(F^+_a, R, T, \Theta) = \sum_{i=1}^s \mathcal{A}_i(R, a_i T, \Theta_i). \]
This together with (3.3) implies
\[ \tilde{\mathcal{D}}_{\delta'}(F^+_a, R, T) \subset \mathcal{D}_{\delta'}(F^+_a, R, T) := \left\{ \Theta \in M : \mathcal{A}(F^+_a, R, T, \Theta) \geq \delta' \right\}. \] (3.5)
Combining (3.4) and (3.5), we get, for any $0 < \delta' < \delta \leq 1$,
\[
D_{\delta}(F^{+}_{a}, M) \subset \liminf_{T \to \infty} D_{\delta'}(F^{+}_{a}, R, T) := \bigcup_{T_1 > 0} \bigcap_{T > T_1} D_{\delta'}(F^{+}_{a}, R, T). \quad (3.6)
\]

We summarize what we have obtained in the following lemma.

**Lemma 3.2.** Suppose $0 < \delta' < \delta \leq 1$, then
\[
D_{\delta}(F^{+}_{a}, M) \subset \liminf_{T \to \infty} D_{\delta'}(F^{+}_{a}, R, T). \quad (3.7)
\]

The key step in our proof of Proposition 3.1 is that the right hand side of (3.7) is contained in the limsup set associated to any weight vector. More precisely, we have the following lemma.

**Lemma 3.3.** Let $0 < \delta' < \delta \leq 1$ and $a, b \in \mathbb{R}^{s}_{+}$, then
\[
\liminf_{T \to \infty} D_{\delta}(F^{+}_{a}, R, T) \subset \limsup_{T \to \infty} D_{\delta'}(F^{+}_{b}, R, T) := \bigcap_{R > 0} \bigcup_{T_1 > 0} \bigcap_{T > T_1} D_{\delta'}(F^{+}_{b}, R, T). \quad (3.8)
\]

The proof of Lemma 3.3 will be given in Section 3.2. The above two lemmas reduce the proof of Proposition 3.1 to estimating the dimension of the right hand side of (3.8) for a convenient weight $b$. The special weight we are using will be
\[
b' = (b'_1, \ldots, b'_s), \text{ where } b'_i = \frac{1}{m_i + n_i}. \quad (3.9)
\]

In this case the dynamical system $(F^{+}_{b'}, X)$ has a single positive Lyapunov exponent. By Lemmas 3.2 and 3.3, for any $0 < \delta' < \delta \leq 1$
\[
D_{\delta}(F^{+}_{a}, M) \subset D_{\delta'} := \bigcap_{R > 0} \bigcup_{T_1 > 0} \bigcap_{T > T_1} D_{\delta'}(F^{+}_{b'}, R, T). \quad (3.10)
\]

So Proposition 3.1 will follow from the following lemma.

**Lemma 3.4.** Let $\delta \in (0, 1]$, then
\[
\dim D_{\delta} \leq \sum_{i=1}^{s} m_in_i - \min_{1 \leq i \leq s} \delta \frac{m_in_i}{m_i + n_i}. \quad (3.11)
\]

The proof of Lemma 3.4 will be given in Section 3.3. Since the right hand side of (3.10) does not depend on $a \in \mathbb{R}^{s}_{+}$, Lemma 3.4 also implies
\[
\dim \left( \bigcup_{a \in \mathbb{R}^{s}_{+}} D_{\delta}(F^{+}_{a}, M) \right) \leq \sum_{i=1}^{s} m_in_i - \min_{1 \leq i \leq s} \delta \frac{m_in_i}{m_i + n_i}.
\]

### 3.2. Proof of Lemma 3.3.

The proof of Lemma 3.3 is based on the following key lemma.

**Lemma 3.5.** Let $s \in \mathbb{N}$, $1 = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_s > 0$ and $f_1, f_2, \ldots, f_s : \mathbb{R}_{+} \to [0, \infty)$ be bounded functions. Then for any $\epsilon > 0$ and $t_0 > 0$, there exists $t \geq t_0$ such that
\[
\sum_{i=1}^{s} f_i(t) \leq \epsilon + \sum_{i=1}^{s} f_i(\sigma_it). \quad (3.12)
\]
Proof. We argue by induction on $s$. For $s = 1$, since $\sigma_1 = 1$, the inequality (3.12) is trivial. Suppose $s \geq 2$ and the lemma holds for $s - 1$. Let $\epsilon > 0, t_0 > 0$. By assumption, the function $f_s$ is bounded, hence there exists $Q \in \mathbb{N}$ such that $f_s(x) \leq Q\epsilon$ for all $x \in \mathbb{R}_+$. Then there exists $T > T_0$ for any $\epsilon > 0, t_0 > 0$. We claim that $g_s(\sigma_s^{-q}t_1) \leq Q\epsilon$ for all $x \in \mathbb{R}_+$. By the induction hypothesis, there exists $t_1 \geq t_0$ such that

$$
\sum_{i=1}^{s-1} g_i(t_1) \leq \epsilon + \sum_{i=1}^{s-1} g_i(\sigma_i t_1). \tag{3.14}
$$

We claim that

$$
\sum_{q=0}^{Q} \left( \epsilon + \sum_{i=1}^{s} f_i(\sigma_i \sigma_s^{-q}t_1) - \sum_{i=1}^{s} f_i(\sigma_s^{-q}t_1) \right) \geq 0. \tag{3.15}
$$

Summing the index $q$ first and using (3.13) for $1 \leq i \leq s - 1$, we have the left hand side of (3.15) is equal to

$$(Q + 1)\epsilon + \sum_{i=1}^{s-1} g_i(\sigma_i t_1) + \sum_{q=0}^{Q} f_s(\sigma_s^{-q+1}t_1) - \sum_{i=1}^{s-1} g_i(t_1) - \sum_{q=0}^{Q} f_s(\sigma_s^{-q}t_1)$$

$$= (Q\epsilon + f_s(\sigma_s t_1) - f_s(\sigma_s^{-Q}t_1)) + \left( \epsilon + \sum_{i=1}^{s-1} g_i(\sigma_i t_1) - \sum_{i=1}^{s-1} g_i(t_1) \right). \tag{3.16}
$$

The first term of (3.16) is nonnegative since $0 \leq f_s(x) \leq Q\epsilon$ for all $x \in \mathbb{R}_+$. The second term of (3.16) is nonnegative by (3.14). Therefore, (3.15) holds.

By (3.15), there exists $0 \leq q \leq Q$ such that

$$\epsilon + \sum_{i=1}^{s} f_i(\sigma_i \sigma_s^{-q}t_1) - \sum_{i=1}^{s} f_i(\sigma_s^{-q}t_1) \geq 0.$$

This implies that $t = \sigma_s^{-q}t_1$ satisfies (3.12). Note that $t \geq t_1$, since $\sigma_s \leq 1$. This completes the proof. \qed

Proof of Lemma 3.3. Assume the contrary that, there exists

$$
\Theta \in \bigcap_{T_1 > 0} \bigcap_{T > T_1} D_\delta(F^+_a, R, T) \setminus \bigcup_{T_1 > 0} \bigcup_{T > T_1} D_{\delta'}(F^+_b, R, T).
$$

Then there exists $T_1 > 0$ such that, for any $T \geq T_1$,

$$
\Theta \in D_\delta(F^+_a, R, T) \quad \text{but} \quad \Theta \notin D_{\delta'}(F^+_b, R, T).
$$

In view of the definition of $D_\delta(F^+_a, R, T)$ in (3.5), for any $T \geq T_1$,

$$
\mathcal{A}(F^+_a, R, T, \Theta) \geq \delta \quad \text{and} \quad \mathcal{A}(F^+_b, R, T, \Theta) < \delta'. \tag{3.17}
$$
Note that the right hand side of (3.8) is unchanged if we rescale $b$. So by possibly rescaling $b = (b_1, \ldots, b_s)$ and reordering $1 \leq i \leq s$ if necessary, we may assume that

$$1 = \frac{b_1}{a_1} \geq \frac{b_2}{a_2} \geq \cdots \geq \frac{b_s}{a_s} > 0.$$  

Applying Lemma 3.5 to the functions $f_i(t) = A_i(R, a_i t, \theta_i)$, with

$$\epsilon = \frac{1}{2}(\delta - \delta'), \quad t_0 = T_1 \quad \text{and} \quad \sigma_i = \frac{b_i}{a_i},$$

we know that there exists $T \geq T_1$, such that

$$A(F^+_a, R, T, \Theta) = \sum_{i=1}^s A_i(R, a_i T, \theta_i) \leq \frac{1}{2}(\delta - \delta') + \sum_{i=1}^s A_i(R, b_i T, \theta_i) = \frac{1}{2}(\delta - \delta') + A(F^+_b, R, T, \Theta).$$

This leads to a contradiction to (3.17), hence completes the proof.

3.3. Proof of Lemma 3.4. For simplicity, we write

$$\alpha_i = \frac{m_i n_i}{m_i + n_i} \quad \text{where} \quad 1 \leq i \leq s.$$  

Without loss of generality, we may assume that

$$\alpha_1 = \min_{1 \leq i \leq s} \alpha_i.$$  

To simplify the notation, we write $b = b'$ and $b_i = b'_i = \frac{1}{m_i + n_i}$. Let

$$D_\delta(F^+_i, R, T) = \{ \theta \in M_i : A_i(R, T, \theta) \geq \delta \}.$$  

The following lemma gives a covering of $D_\delta(F^+_b, R, T)$ by finitely many product sets.

**Lemma 3.6.** Let $\delta \in (0, 1]$. For any $\epsilon \in (0, \delta)$, there exists a finite subset $S = S(\epsilon) \subset [0, 1]^s$ satisfying that,

(1) for any $(\delta_1, \ldots, \delta_s) \in S$, $\sum_{i=1}^s \delta_i \geq \delta - \epsilon$.

(2) for any $T > 0$,

$$D_\delta(F^+_b, R, T) \subset \bigcup_{(\delta_i) \in S} \prod_{i=1}^s D_{\delta_i}(F^+_i, R, b_i T). \quad (3.18)$$

**Proof.** We consider the compact subset

$$K = \left\{ (\delta_1, \ldots, \delta_s) \in [0, 1] : \sum_{i=1}^s \delta_i = \delta \right\}.$$  

It follows directly from the definition that

$$D_\delta(F^+_b, R, T) \subset \bigcup_{(\delta_i) \in K} \prod_{i=1}^s D_{\delta_i}(F^+_i, R, b_i T).$$
On the other hand, for any fixed
\[ v = (\delta'_1, \ldots, \delta'_s) \in [0,1]^s \] with \( \sum_{i=1}^s \delta'_i \geq \delta - \epsilon \), (3.19)
the set
\[ N_v := \{ (\delta_1, \ldots, \delta_s) \in K : \delta'_i < \delta_i \text{ for all } 1 \leq i \leq s \} \]
is open. Note that for any \((\delta_1, \ldots, \delta_s) \in N_v\),
\[ D_{\delta_i}(F_i^+, R, b_iT) \subset D_{\delta'_i}(F_i^+, R, b_iT) \quad \text{for all } 1 \leq i \leq s. \]
Moreover, for any \((\delta_1, \ldots, \delta_s) \in K\), there exists \( v \) satisfying (3.19) such that \((\delta_1, \ldots, \delta_s) \in N_v\). In view of the compactness of \( K \), there exists a finite set \( S = S(\epsilon) \subset [0,1]^s \) consisting of \( v \) as in (3.19) such that any \((\delta_1, \ldots, \delta_s) \in K\) is contained in \( N_v \) for some \( v \in S \). This completes the proof.

Let us fix a Euclidean metric \( d_i \) on each \( M_i \) and denote the associated open metric ball\(^1\) of radius \( r \) centered at 0 as \( B_r^M \). Set the metric on \( M \) to be \( d = \max_{1 \leq i \leq s} d_i \) and \( B_r^M = B_{r}^{M_1} \times \cdots \times B_{r}^{M_s} \) to be the associated metric ball of radius \( r \) centered at 0. For simplicity, we write the right hand side of (3.11) as \( \alpha \), that is
\[ \alpha = \left( \sum_{i=1}^s m_i n_i \right) - \delta \alpha_1. \]

**Lemma 3.7.** For any \( r, \epsilon > 0 \), there exists \( T = T(r, \epsilon) > 0 \) and \( R = R(T) > 0 \) such that for any \( \ell \in \mathbb{N} \), the set
\[ D_{\delta}(F^+_{b_i}, R, \ell T) \cap B_r^M \]
can be covered by no more than \( e^{(\alpha + \epsilon)\ell T} \) balls of radius \( e^{-\ell T} \).

**Proof.** Recall that \( \alpha_1 = \min_{1 \leq i \leq s} \alpha_i \). Let \( S(\epsilon/2\alpha_1) \) be given as in Lemma 3.6, then the set (3.20) can be covered by the set
\[ \bigcup_{(\delta_i) \in S(\epsilon/2\alpha_1)} \left( \prod_{i=1}^s D_{\delta_i}(F_i^+, R, b_iT) \cap B_r^M \right). \]

Hence it suffices to study the set \( \prod_{i=1}^s D_{\delta_i}(F_i^+, R, b_iT) \cap B_r^M \).

Recall that \( b_i = 1/(m_i + n_i) \). According to [9, Theorem 1.5], there exists \( T_i, C_i > 0 \) such that, for any \( T > T_i \), there exists \( R_i = R_i(T) \) such that for any \( \ell \in \mathbb{N} \) and \( R \geq R_i \), the set \( D_{\delta_i}(F_i, R, b_i\ell T) \cap B_r^{M_i} \) can be covered by no more than \( C T^{3s} e^{(m_i n_i - \delta_i \alpha_i)\ell T} \) balls of radius \( e^{-\ell T} \). Hence for any \( T \geq \max_{1 \leq i \leq s} T_i \) and \( R \geq \max_{1 \leq i \leq s} R_i(T) \) the set \( \prod_{i=1}^s D_{\delta_i}(F_i^+, R, b_iT) \cap B_r^M \) can be covered by no more than
\[ C T^{3s} e^{\left( \sum_{i=1}^s m_i n_i - \sum_{i=1}^s \delta_i \alpha_i \right)\ell T} \]
balls of radius \( e^{-\ell T} \), where \( C = \prod_{i=1}^s C_i \). Take \( T_0 \) large enough, we may assume that for any \( T \geq T_0 \),
\[ \# S(\epsilon/2\alpha_1) \cdot C T^{3s} e^{\frac{T}{T}} \leq e^{\frac{T}{T}}. \]

\(^1\)In this paper all the metric balls are assumed to be open.
On the other hand, according to Lemma 3.6, we have

\[ \sum_{i=1}^{s} m_i n_i - \sum_{i=1}^{s} \delta_i \alpha_i \leq \sum_{i=1}^{s} m_i n_i - \left( \sum_{i=1}^{s} \delta_i \right) \alpha_1 \leq \alpha + \frac{\varepsilon}{2}. \]

In summary, for any \( T \geq \max_{0 \leq i \leq s} T_i \) and \( R \geq \max_{1 \leq i \leq s} R_i(T) \), the set (3.20) can be covered by no more than

\[ 2^{s} S(\varepsilon/2\alpha_1) \cdot \max_{(c_i) \in S(\varepsilon/2\alpha_1)} C T^{3s} e^{(\sum_{i=1}^{s} m_i n_i - \sum_{i=1}^{s} \delta_i \alpha_i)T} \leq e^{(\sigma + \varepsilon)T} \]

balls of radius \( e^{-\ell T} \). This completes the proof. \( \square \)

**Proof of Lemma 3.4.** By the definition of Hausdorff dimension, it suffice to show that for any \( r > 0 \) and \( \sigma > 0 \),

\[ \mathcal{H}^{\sigma}(D_{\delta} \cap B_r^{M}) = 0. \]

Recall that, for a subset \( Z \subset M \)

\[ \mathcal{H}^{\sigma}(Z) = \lim_{\beta \to 0} \mathcal{H}_{\beta}^{\sigma}(Z), \]

where

\[ \mathcal{H}_{\beta}^{\sigma}(Z) = \inf \left\{ \sum_{n} |U_n|^\sigma : Z \subset \bigcup_n U_n, |U_n| \leq \beta \right\}. \]

Hence it suffices to show that for any \( \beta > 0 \),

\[ \mathcal{H}_{\beta}^{\sigma}(D_{\delta} \cap B_r^{M}) = 0. \quad (3.21) \]

We claim that for any \( R > 0 \) and \( T > 0 \)

\[ D_{\delta} \subset \bigcap_{\ell \in N} \bigcup_{\ell \in N, \ell \geq \ell_1} D_{\delta}(F_b^{+}, R, \ell T). \quad (3.22) \]

According to the definition of \( D_{\delta} \) in (3.10), it suffices to prove that there exists \( R_1 > R \) such that for any \( t \in [\ell-1,T] \) if, \( \Theta \in D_{\delta}(F_b^{+}, R_1, t) \), then

\[ \Theta \in D_{\delta}(F_b^{+}, R, (\ell - 1)T) \cup D_{\delta}(F_b^{+}, R, \ell T). \]

If \( \Theta \not\in D_{\delta}(F_b^{+}, R, (\ell - 1)T) \), then \( \Theta \not\in D_{\delta}(F_b^{+}, R_1, (\ell - 1)T) \). The assumption \( \Theta \in D_{\delta}(F_b^{+}, R_1, t) \) implies that there exists \( t_1 \in [(\ell - 1)T, T] \) such that \( g_{t_1} x_{\Theta} \in E_{R_1}^X \).

The claim now follows by taking \( R_1 \) sufficiently large so that, if \( g_{t_1} x_{\Theta} \in E_{R_1}^X \) for some \( t_1 \in [(\ell - 1)T, T] \), then \( g_{t_2} x_{\Theta} \in E_{R}^X \) for all \( t_2 \in [(\ell - 1)T, T] \).

By applying Lemma 3.7 to \( \varepsilon = \frac{1}{2}(\sigma - \alpha) \) and \( r \), we can find \( T > 0 \) and \( R > 0 \) such that for any \( \ell \in N \), the set

\[ D_{\delta}(F_b, R, \ell T) \cap B_r^{M} \]

can be covered by no more than \( e^{(\sigma + \varepsilon)T} \) balls of radius \( e^{-\ell T} \). We choose \( \ell_1 \in N \) with \( 2e^{-\ell_1 T} \leq \beta \). By (3.22),

\[ D_{\delta} \cap B_r^{M} \subset \bigcup_{\ell \in N, \ell \geq \ell_1} D_{\delta}(F_b^{+}, R, \ell T) \cap B_r^{M}. \]
It follows that
\[
H_{\sigma}(D_\delta(F^+_b, M) \cap B_r^M) \leq \sum_{\ell \geq \ell_1} e^{(a+\epsilon)\ell T} e^{-\sigma \ell T} = e^{\frac{1}{2} \sigma (\sigma - \alpha)} \frac{T}{1 - e^{-\frac{1}{2} (\sigma - \alpha) T}}.
\]

By letting $\ell_1$ go to infinity, we have (3.21) holds. This completes the proof. \qed

4. The lower bound

This section is devoted to proving the lower bound in Proposition 2.1.

**Proposition 4.1.** For $a \in \mathbb{R}_+^s$ and $\delta \in (0, 1]$, we have
\[
\dim D^\delta(F^+_a, M) \geq \dim X - \min_{1 \leq i \leq s} \frac{m_i n_i}{m_i + n_i},
\]
and
\[
\dim D^\delta(F^+_a, M) \geq \dim X - \min_{1 \leq i \leq s} \delta \frac{m_i n_i}{m_i + n_i}.
\]

The main tool of the proof is the variational principle in parametric geometry of numbers developed by Das, Fishman, Simmons and Urbani\’ski in [6, 7]. It allows us to construct a set of points with the given Diophantine properties (including singularity), whose Hausdorff dimension is computable. Before we head to the proof of Proposition 4.1, we recall the settings of parametric geometry of numbers.

4.1. **Parametric geometry of numbers and variational principle.** Parametric geometry of numbers originates in a question by Schmidt [15]. It was developed by Schmidt and Summerer [16, 17] and then Roy [14]. Recently, Das, Fishman, Simmons and Urbani\’ski [6, 7] provided a variational principle, that gives a quantitative version of an important theorem by Roy.

Let $m, n \in \mathbb{N}$ and $\theta \in M_{m \times n}(\mathbb{R})$. The main purpose of parametric geometry of numbers is to study the trajectory \( \{g_t^{(m,n)}x_\theta : t \geq 0\} \subset Y_{m+n} \) through the successive minima function
\[
h(t) = h_\theta(t) := (h_1(t), \ldots, h_{m+n}(t)) : \mathbb{R}_+ \to \mathbb{R}^{m+n}
\]
where for $1 \leq k \leq m + n$,
\[
h_k(t) = \log \lambda_k(g_t^{(m,n)}x_\theta),
\]
and $\lambda_k$ denotes the $k$-th successive minima.

It is easy to see that $h(t)$ is piecewise linear with few possible slopes. Minkowski’s first and second convex body theorems give further information. In a landmark paper [14], Roy showed that when $m$ or $n$ is 1, the successive minima functions are precisely approximated by Roy-systems, a relatively simple combinatorial object. In [6, 7], Das, Fishman, Simmons and Urbani\’ski extend this result to arbitrary $m$ and $n$ and quantified the result.
Definition 4.2 (Das, Fishman, Simmons, Urbański). Let \((m, n) \in \mathbb{N}^2\) and \(I \subset \mathbb{R}_+\) be an interval. An \(m \times n\) template on \(I\) is a piecewise linear map \(L = (L_1, \ldots, L_{m+n}) : I \to \mathbb{R}^{m+n}\) with the following properties:

1. \(L_1 \leq L_2 \leq \cdots \leq L_{m+n}\).
2. \(-1/n \leq L'_k \leq 1/m\), for all \(1 \leq k \leq m+n\).
3. For all \(j = 0, \ldots, m+n\) and for any interval \(I' \subset I\) such that \(L_j < L_{j+1}\) on \(I'\), the function \(F_j = \sum_{0<k\leq j} L_i\) is convex and piecewise linear on \(I'\) with slopes in

\[Z(j) := \left\{ \frac{k_1}{m} - \frac{k_2}{n} \mid 0 \leq k_1 \leq m, 0 \leq k_2 \leq n, k_1 + k_2 = j \right\}.\]

Remark 4.3. It is easily checked that, on any interval \(I\), the constant function \(L = 0\) is a template, called the trivial template.

Theorem 4.4 (Das, Fishman, Simmons, Urbański). Let \((m, n) \in \mathbb{N}^2\).

1. For every \(m \times n\) template \(L\) on \(\mathbb{R}_+\), there exists an \(m \times n\) matrix \(\theta\) such that \(h_\theta(t) - L(t)\) is bounded.
2. For every \(m \times n\) matrix \(\theta\), there exists an \(m \times n\) template \(L\) on \(\mathbb{R}_+\) such that \(h_\theta(t) - L(t)\) is bounded.

When \(m\) or \(n\) equals 1, a template is a Roy-sytem (up to a language translation) and Theorem 4.4 is Roy’s theorem [14].

The variational principle provides a quantitative version of Theorem 4.4. It is expressed in terms of lower average contraction rate of a template, as described below.

For a time \(t\), one can define the local contraction rate \(\delta(L, t)\) of a template \(L\) at \(t\). The definition can be found in [7, Definition 2.5], we omit it here as it is very technical. For our purpose, we only need Lemma 4.11 and Remark 4.5 given below.

Denote the average contraction rate on an interval \([T_1, T_2]\) by

\[\Delta(L, [T_1, T_2]) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \delta(L, t) \, dt.\]

If \(T_1\) is the start of the interval of the domain of \(L\), we omit it and simply write \(\Delta(L, T_2)\). The lower average contraction rate \(\underline{\delta}(L)\) of a template \(L\) is defined by

\[\underline{\delta}(L) = \liminf_{T \to \infty} \Delta(L, T).\]

Remark 4.5. The lower average contraction rate for the trivial \(m \times n\) template is \(mn\).

Given a \(m \times n\) template \(L\) on \(\mathbb{R}_+\), set

\[\mathcal{M}(L) = \{ \theta \in M_{m \times n}(\mathbb{R}) \mid h_\theta - L\text{ is bounded} \}.\]
Namely, $\mathcal{M}(L)$ denotes the set of matrices whose successive minima function is at finite distance from $L$. Given a set $L$ of $m \times n$ templates, we consider

$$\mathcal{M}(L) = \bigcup_{L \in \mathcal{L}} M(L).$$

The variational principle reads as follows.

**Theorem 4.6** (Das, Fishman, Simmons, Urbański). Let $\mathcal{L}$ be a (Borel) collection of templates closed under finite perturbations. Then

$$\dim \mathcal{M}(L) = \sup_{L \in \mathcal{L}} \delta(L).$$  \hfill (4.3)

**4.2. Reformulation of definitions and strategy of proof.** Let us fix $s \in \mathbb{N}$, $(m_i, n_i) \in \mathbb{N}^2$ for each $1 \leq i \leq s$ and a weight vector $a = (a_1, \ldots, a_s) \in \mathbb{R}^s_+$. Without loss of generality, we may suppose that

$$\frac{m_1n_1}{m_1 + n_1} = \min_{1 \leq i \leq s} \frac{m_in_i}{m_i + n_i}.$$

By Mahler’s compactness criterion, we can reformulate various singular properties using the successive minima function.

**Proposition 4.7.** Let $\Theta = (\theta_1, \ldots, \theta_s) \in \mathbb{M}$.

1. $x_\Theta$ is $F^+_a$-singular if and only if
   $$\limsup_{t \to \infty} \min_{1 \leq i \leq s} h_{\theta_i,1}(a_i t) = -\infty.$$

2. $x_\Theta$ is essentially $F^+_a$-singular if and only if it is $F^+_a$-singular and
   $$\limsup_{t \to \infty} h_{\theta_i,1}(t) > -\infty \quad \text{for all } 1 \leq i \leq s.$$

3. $x_\Theta$ is $(F^+_a, \delta)$-singular if and only if
   $$\limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_{[-C,C]}(\min_{1 \leq i \leq s} h_{\theta_i,1}(a_i t)) \, dt \leq 1 - \delta \quad \text{for all } C > 0.$$

4. $x_\Theta$ is essentially $(F^+_a, \delta)$-singular if and only if it is $(F^+_a, \delta)$-singular and
   $$\limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_{[-C,C]}(h_{\theta_i,1}(t)) \, dt > 1 - \delta, \quad \text{for all } 1 \leq i \leq s \text{ and } C > 0.$$

The proof of Proposition 4.1 will be based on Theorem 4.6 and Proposition 4.7. Specifically, we are going to construct certain $s$-tuple of templates $(L^1, \ldots, L^s)$ with $L^i$ a $m_i \times n_i$ template on $\mathbb{R}_+$ for each $1 \leq i \leq s$ such that $\prod_{i=1}^s \mathcal{M}(L^i)$ is a subset of $D^e(F^+_a, \mathcal{M})$ or $D^e_\delta(F^+_a, \mathcal{M})$, using the reformulation of Proposition 4.7. Then a lower bound for the Hausdorff dimension of the sets $D^e(F^+_a, \mathcal{M})$ and $D^e_\delta(F^+_a, \mathcal{M})$ can be deduced using Theorem 4.6. In particular, Proposition 4.1 will be deduced from the following two lemmas.

For the sake of convenience, here and throughout the paper, when saying an $s$-tuple of templates $(L^1, \ldots, L^s)$, we mean an $s$-tuple of templates $(L^1, \ldots, L^s)$ with $L^i$ a $m_i \times n_i$ template on $\mathbb{R}_+$ for each $1 \leq i \leq s$. 

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Lemma 4.8. There exists an $s$-tuple of templates $(L^1, \ldots, L^s)$ satisfying

\[
\begin{align*}
\delta(L^1) &= \frac{m_1n_1}{m_1 + n_1}, \\
\delta(L^i) &= m_in_i \quad \text{for all } 2 \leq i \leq s, \\
\limsup_{t \to \infty} \min_{1 \leq i \leq s} L^i_1(a_it) &= -\infty, \\
\limsup_{t \to \infty} L^i_1(t) &= 0 \quad \text{for all } 1 \leq i \leq s.
\end{align*}
\] (4.4)

(4.5)

Lemma 4.9. Let $\delta \in (0, 1]$ and $\delta_i \in (0, \delta)$ satisfying $\sum_{1 \leq i \leq s} \delta_i \geq \delta$. Then there exists an $s$-tuple of templates $(L^1, \ldots, L^s)$ such that,

\[
\begin{align*}
\delta(L^i) &= m_in_i - \delta \frac{m_in_i}{m_i + n_i} \quad \text{for all } 1 \leq i \leq s, \\
\limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_{[-C,C]}(L^i_1(t)) \, dt &= 1 - \delta_i \quad \text{for all } 1 \leq i \leq s \text{ and } C > 0, \\
\limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_{[-C,C]} \left( \min_{1 \leq i \leq s} L^i_1(t) \right) \, dt &= 1 - \delta \quad \text{for all } C > 0.
\end{align*}
\] (4.8)

(4.9)

Now we are ready to prove Proposition 4.1 modulo these two lemmas.

Proof of Proposition 4.1. According to Proposition 4.7, for any $s$-tuple of templates $(L^1, \ldots, L^s)$ that satisfies (4.6) and (4.7), we have

\[
\prod_{i=1}^s \mathcal{M}(L^i) \subset D^\varepsilon(F^+_a, M).
\]

Then, (4.4) and (4.5) imply that

\[
\dim D^\varepsilon(F^+_a, M) \geq \dim \prod_{i=1}^s \mathcal{M}(L^i)
\]

\[
\geq \sum_{1 \leq i \leq s} \dim \mathcal{M}(L^i)
\]

\[
\geq \sum_{1 \leq i \leq s} m_in_i - \frac{m_1n_1}{m_1 + n_1}.
\]

This completes the proof of (4.1).

The proof of (4.2) is similar. Indeed, according to Proposition 4.7, for any $s$-tuple of templates $(L^1, \ldots, L^s)$ that satisfies (4.9) and (4.10), we have

\[
\prod_{i=1}^s \mathcal{M}(L^i) \subset D^\varepsilon_\delta(F^+_a, M).
\]
Then (4.8) implies that, for any \( \delta_1, \ldots, \delta_s \in (0, \delta) \) satisfying \( \sum_{i=1}^{s} \delta_i \geq \delta \), we have

\[
\dim D_{\delta}(F_{\alpha}, M) \geq \dim \prod_{i=1}^{s} M(L^i) \\
\geq \sum_{1 \leq i \leq s} \dim M(L^i) \\
\geq \sum_{1 \leq i \leq s} m_i n_i - \sum_{1 \leq i \leq s} \delta_i \frac{m_i n_i}{m_1 + n_1}.
\]

Thus, it follows that

\[
\dim D_{\delta}(F_{\alpha}, M) \geq \sum_{1 \leq i \leq s} m_i n_i - \delta \frac{m_1 n_1}{m_1 + n_1}.
\]

This completes the proof. \( \square \)

4.3. Standard templates. In this section, we recall the notion of standard template defined by two points \((t, \varepsilon)\) and \((t', \varepsilon')\) introduced in [7], which will be the building blocks of our construction.

**Definition 4.10.** Given two points \((t', \varepsilon')\) and \((t'', \varepsilon'')\), with \(0 < t' < t''\) and \(\varepsilon', \varepsilon'' \geq 0\). Denote \(\Delta t = t'' - t'\) and \(\Delta \varepsilon = \varepsilon'' - \varepsilon'\). The pair \(((t', \varepsilon'), (t'', \varepsilon''))\) is said to be **admissible** if it satisfies the following conditions:

\[
-\Delta t \leq \Delta \varepsilon \leq \frac{\Delta t}{n}, \quad (4.11)
\]

\[
\Delta \varepsilon \geq -\frac{n-1}{2n} \Delta t \quad \text{if} \quad m = 1 \quad \text{and} \quad \Delta \varepsilon \leq \frac{m-1}{2m} \Delta t \quad \text{if} \quad n = 1, \quad (4.12)
\]

\[
(m-1) \left( \frac{1}{n} \Delta t - \Delta \varepsilon \right) \geq (m+n) \varepsilon' \quad \text{or} \quad (m-1) \left( \frac{1}{n} \Delta t + \Delta \varepsilon \right) \geq (m+n) \varepsilon'' \quad (4.13)
\]

We define the **standard template** \(L((t', \varepsilon'), (t'', \varepsilon''))\) associated to an admissible pair \(((t', \varepsilon'), (t'', \varepsilon''))\) on an interval \([t', t'']\) in the following way.

- Let \(g_1, g_2 : [t', t''] \to \mathbb{R}\) be piecewise linear functions such that
  \[g_1(t') = g_2(t') = -\varepsilon', \quad g_1(t'') = g_2(t'') = -\varepsilon''\]
  and \(g_i\) has two intervals of linearity: one on which \(g' = 1/m\) and the other on which \(g' = -1/n\). For \(i = 1\) the latter interval comes first while for \(i = 2\) the former interval comes first. The existence of such functions \(g_1\) and \(g_2\) is guaranteed by (4.11). Finally, let \(g_3 = \ldots = g_d\) be functions on \([t', t'']\) chosen so that \(g_1(t) + \cdots + g_d(t) = 0\) for all \(t \in [t', t'']\).

- For each \(t \in [t_k, t_{k+1}]\), let \(L(t) = g(t)\) if \(g(t) \leq g_3(t)\); otherwise let \(L_1(t) = g_1(t)\) and let \(L_2(t) = \cdots = L_d(t) = 0\) be chosen so that \(L_1(t) + \cdots + L_d(t) = 0\).

Moreover, a sequence of points \(\{(t_i, \varepsilon_i)\}_{1 \leq i \leq k}\) is called admissible if for all \(1 \leq l \leq k - 1\), the pair \(((t_l, \varepsilon_l), (t_{l+1}, \varepsilon_{l+1}))\) is admissible. We define the standard template \(L\) associated to \(\{(t_i, \varepsilon_i)\}_{1 \leq i \leq k}\) to be the template on the interval \([t_1, t_k]\) that equals \(L((t_i, \varepsilon_i), (t_{i+1}, \varepsilon_{i+1}))\) on \([t_i, t_{i+1}]\).

**Lemma 4.11.** Let \(L\) be the standard template associated to a pair points \((t', \varepsilon')\) and \((t'', \varepsilon'')\). Then
(1) $L_1(t) \leq -\min\{\varepsilon', \varepsilon''\}$ for all $t \in [t', t'']$.

(2) The average contraction rate on the interval $[t', t'']$ is given by

\[\Delta([t', t'']) = mn - \frac{mn}{m + n} - O\left(\frac{\max(\varepsilon', \varepsilon'')}{{t''} - {t'}}\right)\]

Proof. (1) follows directly from the definition. For the proof of (2), see [7, Definition 12.4].

The following simple observation will be useful.

**Lemma 4.12.** Any pair of points $((t', \varepsilon'), (t'', \varepsilon''))$ that satisfies

\[t'' - t' \geq (m + n)^2 \max(\varepsilon', \varepsilon'')\] (4.14)

is admissible.

Proof. Given (4.14), the conditions (4.11), (4.12) and (4.13) are easily checked.

4.4. The construction. The proof of Lemma 4.8 will occupy this and the next section. In this section, we construct the templates we need and in the next section, we verify that they satisfy the conditions of Lemma 4.8.

The core of our construction lies in the first two spaces. For $3 \leq i \leq s$, we simply set $L^i$ to be the trivial template.

Set $T_0 = 1$ and $T_{k+1} = T_k + \sqrt{T_k}$. Set

\[l_k = \lceil\sqrt[3]{T_k}\rceil, \quad \gamma_k = l_k^{-1} \sqrt{T_k},\]

and for $1 \leq l \leq l_k$, set $t_{k,l} = T_k + l\gamma_k$. Clearly, $\gamma_k$ goes to infinity when $k$ goes to infinity. So there exists $k_0 > 0$ such that for any $k \geq k_0$,

\[l_k \geq 4, \quad \text{and} \quad a_i\gamma_k \geq (m_i + n_i)^2 \log \gamma_k \text{ for } i = 1, 2.\] (4.15)

We define $L^1$ as follows (see Figure 1):

- On the interval $[0, a_1T_{k_0}]$, set $L^1$ to be the trivial template.
- On the interval $[a_1T_k, a_1T_{k+1}]$ for $k \geq k_0$, set $L^1$ to be the concatenation of standard templates associated to the sequence of points

\[(a_1T_k, 0), (a_1t_{k,1}, \log \gamma_k), \ldots, (a_1t_{k,l_{k-1}}, \log \gamma_k), (a_1T_{k+1}, 0).\]

According to Lemma 4.12 and (4.15), the sequence of points is admissible, hence the construction is valid.

Also, we define $L^2$ as follows (see Figure 1):

- On the interval $[0, a_2T_{k_0}]$, set $L^2$ to be the trivial template.
- On the interval $[a_2T_{k_0}, a_2T_{k_0+1}]$, set $L^2$ to be the standard template associated to the pair of points

\[(a_2T_{k_0}, 0), (a_2T_{k_0+1}, \log \gamma_k).\]

- Let $k \geq k_0 + 1$. On the subinterval $[a_2T_k, a_2t_{k,2}]$ of $[a_2T_k, a_2T_{k+1}]$, set $L^2$ to be the concatenation of two standard templates associated to the sequence of points

\[(a_2T_k, \log \gamma_k), (a_2t_{k,1}, \log \gamma_k), (a_2t_{k,2}, 0).\]
On the subinterval \([a_2 t_{k,2}, a_2 t_{k,l_k-2}]\), set \(L^2\) to be the trivial template. On
the last subinterval \([a_2 t_{k,l_k-2}, a_2 T_{k+1}]\), set \(L^2\) to be the concatenation of two
standard templates associated to the sequence of points
\((a_2 t_{k,l_k-2}, 0), (a_2 t_{k,l_k-1}, \log \gamma_k), (a_2 T_{k+1}, \log \gamma_k)\).

According to Lemma 4.12 and (4.15), the sequence of points is admissible, hence
the construction is valid.

\[
\begin{array}{ccccccccccc}
T_k & t_{k,1} & t_{k,2} & \cdots & t_{k,l} & t_{k,l+1} & \cdots & t_{k,l_k-2} & t_{k,l_k-1} & T_{k+1} & t \\
\hline
\hline
L^2(a_2 t) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

\[
\begin{array}{ccccccccccc}
T_k & t_{k,1} & t_{k,2} & \cdots & t_{k,l} & t_{k,l+1} & \cdots & t_{k,l_k-2} & t_{k,l_k-1} & T_{k+1} & t \\
\hline
\hline
L^1(a_1 t) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

\textbf{Figure 1.} Templates \(L^1(a_1 t)\) and \(L^2(a_2 t)\) on a generic interval \(t \in [T_k, T_{k+1}], k \geq k_0 + 1.\)

4.5. \textbf{Proof of Lemma 4.8.} By Lemma 4.11(1) and our construction, for \(k \geq k_0 + 1\) we have

\[
L^1_1(a_1 t) \leq -\log(\gamma_k) \text{ on the interval } [t_{k,1}, t_{k,l_k-1}],
\]
\[
L^2_1(a_2 t) \leq -\log(\gamma_k) \text{ on the interval } [T_k, t_{k,1}] \cup [t_{k,l_k-1}, T_{k+1}].
\]

Thus,

\[
\max_{T_k \leq t \leq T_{k+1}} \min_{1 \leq i \leq s} L^i_1(a_i t) \leq -\log(\gamma_k) \rightarrow_{k \rightarrow \infty} \infty,
\]

which proves (4.6). And (4.7) follows from the simple observation that

\[
\sup_{T_k \leq t \leq T_{k+1}} L^i_1(a_i t) = 0 \text{ for all } 1 \leq i \leq s.
\]

On the other hand, by Remark 4.5, for \(i \geq 3, \delta(L_i) = m_i n_i.\) Hence, to prove (4.4)
and (4.5), we are left to compute \(\delta(L_i)\) for \(i = 1, 2.\) Note that for \(i = 1, 2, k > 0,\)
and \(T \in [a_i T_k, a_i T_{k+1}],\) we have

\[
\Delta(L^i, T) - \Delta(L^i, a_i T_k) = O \left( \frac{T - a_i T_k}{a_i T_k} \right) = O \left( \frac{1}{\sqrt{T_k}} \right) \rightarrow_{k \rightarrow \infty} 0.
\]
Hence it suffices to compute $\Delta(L^i, a_i T_k)$. By definition,

$$\Delta(L^i, a_i T_k) = \sum_{0 \leq j \leq k-1} \frac{T_{j+1} - T_j}{T_k} \Delta(L^i, [a_i T_j, a_i T_{j+1}]).$$

(4.16)

As $T_{j+1} - T_j = \sqrt{T_j}$ goes infinity when $j$ goes to infinity, to complete the proof, it suffices to show that

$$\lim_{k \to \infty} \Delta(L^1, [a_i T_k, a_i T_{k+1}]) = m_1 n_1 - \frac{m_1 n_1}{m_1+ n_1},$$

$$\lim_{k \to \infty} \Delta(L^2, [a_2 T_k, a_2 T_{k+1}]) = m_2 n_2.$$

In view of Lemma 4.11(2), we get

$$\lim_{k \to \infty} \Delta(L^1, [a_i T_k, a_i T_{k+1}]) = \lim_{k \to \infty} \sum_{0 \leq l \leq k-1} \frac{t_{k,l+1} - t_{k,l}}{\sqrt{T_k}} \Delta(L^1, [a_i t_{k,l}, a_i t_{k,l+1}])$$

$$= m_1 n_1 - \frac{m_1 n_1}{m_1 + n_1} + \lim_{k \to \infty} O \left( \frac{\log \gamma_k}{\gamma_k} \right)$$

and

$$\lim_{k \to \infty} \Delta(L^2, [a_2 T_k, a_2 T_{k+1}])$$

$$= \lim_{k \to \infty} \sum_{0 \leq l \leq k-1} \frac{t_{k,l+1} - t_{k,l}}{\sqrt{T_k}} \Delta(L^2, [a_2 t_{k,l}, a_2 t_{k,l+1}])$$

$$= \lim_{k \to \infty} \frac{1}{l_k} \left( (l_k - 4) m_2 n_2 + 4 \left( m_2 n_2 - \frac{m_2 n_2}{m_2 + n_2} + O \left( \frac{\log \gamma_k}{\gamma_k} \right) \right) \right)$$

$$= m_2 n_2.$$

Here we used the fact that both $\gamma_k$ and $l_k$ tend to infinity when $k$ goes to infinity. This completes the proof of Lemma 4.8. \qed

**Remark 4.13.** Note that for the $L^1$ constructed above, any matrix $\theta$ with $h_\theta - L^1$ bounded is 1-singular, but not singular. In particular, it shows that there are lots of matrices that are 1-singular, but not singular.

4.6. **The construction.** This and the next subsection are devoted to the proof of Lemma 4.9. The construction of templates we need is given in this section and their properties are verified in the next subsection.

Set $T_0 = 1$ and $T_{k+1} = T_k + \sqrt{T_k}$. Set

$$l_k = \lfloor \sqrt{T_k} \rfloor, \quad \gamma_k = l_k^{-1} \sqrt{T_k},$$

and for $1 \leq l \leq l_k$, set $t_{k,l} = T_k + l \gamma_k$. For any $\delta_i \in (0, \delta)$ with $\sum_{1 \leq i \leq s} \delta_i \geq \delta$, we choose $\alpha_i, \beta_i \in (0, 1)$ with the following properties:

1. $\beta_i - \alpha_i = 1 - \delta_i$, for all $1 \leq i \leq s$,
2. The length of the interval $\cap_{1 \leq i \leq s} [\alpha_i, \beta_i]$ is $1 - \delta$. 
There exists $k_0 > 0$ such that for any $k \geq k_0$,

$$2\gamma_k < \alpha_i \sqrt{T_k}, \quad 2\gamma_k < (1 - \beta_i) \sqrt{T_k} \quad \text{and} \quad a_i \gamma_k \geq (m_i + n_i)^2 \log \gamma_k \quad \text{for all} \quad 1 \leq i \leq s.$$  \hfill (4.17)

Let

$$p^i_k = \max \left\{ 1 \leq l \leq l_k : l \gamma_k \leq \alpha_i \sqrt{T_k} \right\} \quad \text{and} \quad q^i_k = \min \left\{ 1 \leq l \leq l_k : l \gamma_k \geq \beta_i \sqrt{T_k} \right\}.$$

According to (4.17), we have $p^i_k \geq 2$ and $l_k - q^i_k \geq 2$ for all $1 \leq i \leq s$.

Now we define the template $L^i$ as follows (see Figure 2):

- On the interval $[0, a_i T_{k_0}]$, set $L^i$ to be the trivial template.
- Let $k \geq k_0$. On the subinterval $[a_i T_k, a_i t_{k,p^i_k}]$ of $[a_i T_k, a_i T_{k+1}]$, set $L^i$ to be the concatenation of standard templates associated to the sequence of points

  $$(a_i T_k, 0), (a_i t_{k,1}, \log \gamma_k), \ldots, (a_i t_{k,p^i_k-1}, \log \gamma_k), (a_i t_{k,p^i_k}, 0).$$

On the subinterval $[a_i t_{k,p^i_k}, a_i t_{k,q^i_k}]$, set $L^i$ to be the trivial template. On the last subinterval $[a_i t_{k,q^i_k}, a_i T_{k+1}]$, set $L^i$ to be the concatenation of standard templates associated to the sequence of points

  $$(a_i t_{k,q^i_k}, 0), (a_i t_{k,q^i_k+1}, \log \gamma_k), \ldots, (a_i t_{k,l_k-1}, \log \gamma_k), (a_i T_{k+1}, 0).$$

According to Lemma 4.12 and (4.17), the sequence of points is admissible. Hence the construction is valid.

**Figure 2.** Templates $L^i(a_i t)$ on a generic interval $t \in [T_k, T_{k+1}]$.

### 4.7. Proof of Lemma 4.9

Arguing as in Section 4.5, to prove (4.8), (4.9) and (4.10), it suffices to show that, for all $1 \leq i \leq s$,

$$\lim_{k \to \infty} \Delta(L^i, [a_i T_k, a_i T_{k+1}]) = m_i n_i - \delta_i \frac{m_i n_i}{m_i + n_i},$$  \hfill (4.18)

$$\lim_{k \to \infty} \frac{1}{a_i \sqrt{T_k}} \int_{a_i T_k}^{a_i T_{k+1}} 1_{[\bar{C}, C]}(L^i_1(t)) \, dt = 1 - \delta_i,$$  \hfill (4.19)

$$\lim_{k \to \infty} \frac{1}{a_i \sqrt{T_k}} \int_{a_i T_k}^{a_i T_{k+1}} 1_{[\bar{C}, C]} \left( \min_{1 \leq i \leq s} L^i_1(t) \right) \, dt = 1 - \delta.$$  \hfill (4.20)
In view of Lemma 4.11(2), we get
\[
\lim_{k \to \infty} \Delta(L^i, [a_i T_k, a_i T_{k+1}]) \\
= \lim_{k \to \infty} \sum_{0 \leq l \leq k-1} \frac{t_{k,l+1} - t_{k,l}}{T_j} \Delta(L^i, [a_i t_{k,l}, a_i t_{k,l+1}]) \\
= \lim_{k \to \infty} \frac{1}{l_k} \left( (q_k^i - p_k^i) m_i n_i + (l_k + p_k^i - q_k^i) \left( m_i n_i - \frac{m_i n_i}{m_i + n_i} + O \left( \frac{\log \gamma_k}{\gamma_k} \right) \right) \right) \\
= m_i n_i - \delta_i \frac{m_i n_i}{m_i + n_i}.
\]
This proves (4.18).

Note that $\gamma_k$ goes to infinity as $k$ goes to infinity. So according to Lemma 4.11(1), for any $C > 0$, we have
\[
\lim_{k \to \infty} \frac{1}{a_i T_k} \int_{a_i T_k}^{a_i T_{k+1}} 1_{[-C,C]}(L^i(t)) \, dt = \lim_{k \to \infty} \frac{q_k^i - p_k^i + O(1)}{l_k} = 1 - \delta_i.
\]
This proves (4.19). The proof of (4.20) is similar, hence omitted. This completes the prove of Lemma 4.9.

\[\text{References}\]

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