Translated points and Rabinowitz Floer homology

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Abstract

We prove that if a contact manifold admits an exact filling then every local contactomorphism isotopic to the identity admits a translated point in the interior of its support, in the sense of Sandon [San11b]. In addition we prove that if the Rabinowitz Floer homology of the filling is non-zero then every contactomorphism isotopic to the identity admits a translated point, and if the Rabinowitz Floer homology of the filling is infinite dimensional then every contactomorphism isotopic to the identity has either infinitely many translated points, or a translated point on a closed leaf. Moreover if the contact manifold has dimension greater than or equal to 3, the latter option generically doesn’t happen. Finally, we prove that a generic contactomorphism on $\mathbb{R}^{2n+1}$ has infinitely many geometrically distinct iterated translated points all of which lie in the interior of its support.

1 Introduction

Let $(\Sigma^{2n-1}, \xi)$ denote a coorientable closed contact manifold, and let $\alpha$ denote a 1-form on $\Sigma$ such that $\xi = \ker \alpha$. Let $R_\alpha$ denote the Reeb vector field of $\alpha$, and let $\phi^\tau_\alpha : \Sigma \to \Sigma$ denote the flow of $R_\alpha$.

Denote by $\text{Cont}(\Sigma, \xi)$ the group of contactomorphisms $\psi : \Sigma \to \Sigma$, and denote by $\text{Cont}_0(\Sigma, \xi) \subseteq \text{Cont}(\Sigma, \xi)$ those contactomorphisms $\psi$ that are contact isotopic to $1$. 

Definition 1.1. Fix $\psi \in \text{Cont}(\Sigma, \xi)$, and write $\psi^* \alpha = \rho \alpha$ for $\rho \in C^\infty(\Sigma, \mathbb{R}^+)$. We say that a point $x \in \Sigma$ is called a translated point for $\psi$ if there exists $\tau \in \mathbb{R}$ such that

$$\psi(x) = \phi^\tau_\alpha(x) \quad \text{and} \quad \rho(x) = 1.$$ 

We say that a point $x \in \Sigma$ is called an iterated translated point for $\psi$ if it is a translated point for some iteration $\psi^n$.

The notion of (iterated) translated points was introduced by Sandon in [San11b] and further explored in [San11a]. We refer to the reader to these papers for a discussion as to why translated points are a worthwhile concept to study.

Let $\xi_{st}$ denote the standard contact structure on $\mathbb{R}^{2n-1}$. Suppose $\sigma : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ is a contactomorphism such that $\mathcal{G}(\sigma) := \text{supp}(\sigma - 1)$ is compact, and suppose that $x : \mathbb{R}^{2n-1} \to U \subseteq \Sigma$ is a Darboux chart onto an open subset $U$ of $\Sigma$. Then we can form a contactomorphism $\psi : \Sigma \to \Sigma$ such that $\psi = \sigma \circ x$ on $U$ and $\psi = 1$ on $\Sigma \setminus U$. We call $\psi$ the local contactomorphism induced from $\sigma$. In this case we are only interested in translated points of $\psi$ in the interior of $\mathcal{G}(\psi)$. Indeed, if $x \in \Sigma \setminus \mathcal{G}(\psi)$ is any periodic point for the Reeb flow $\phi^\tau_\alpha$ then $x$ is vacuously a translated point of $\psi$.

Remark 1.2. Since a ball of arbitrary radius in $\mathbb{R}^{2n-1}$ is contactomorphic to $\mathbb{R}^{2n-1}$ any contactomorphism of $\mathbb{R}^{2n-1}$ gives rise to a local contactomorphism via an appropriate Darboux chart.

Definition 1.3. We say that $(\Sigma, \xi = \ker \alpha)$ admits an exact filling if there exists a compact symplectic manifold $(M, d\lambda_M)$ such that $\Sigma := \partial M$ and such that $\alpha = \lambda_M|_{\Sigma}$.

In this case let us denote by $X := M \cup_{\Sigma} \{ \Sigma \times [1, \infty) \}$ the completion of $M$. Define

$$\lambda := \begin{cases} \lambda_M, & \text{on } M, \\ \rho \alpha, & \text{on } \Sigma \times [1, \infty). \end{cases}$$
Then \((X, d\lambda)\) is an exact symplectic manifold that is convex at infinity, and \(\Sigma \subseteq X\) is a hypersurface of restricted contact type.

Since \(X\) is an exact symplectic manifold that is convex at infinity and \(\Sigma\) is a hypersurface of restricted contact type, the **Rabinowitz Floer homology** of the pair \((\Sigma, X)\) is a well defined \(\mathbb{Z}_2\)-vector space. Rabinowitz Floer homology was discovered by Cieliebak and Frauenfelder in [CF09], and has since generated many applications in symplectic topology (we refer to the survey article [AF10b] for more information on Rabinowitz Floer homology).

We can now state our main result.

**Theorem 1.4.** Suppose \((\Sigma, \xi)\) is a closed contact manifold admitting an exact filling \((M, d\lambda_M)\). Then:

1. If \(\psi \in \text{Cont}_0(\Sigma, \xi) \setminus \{\mathbb{I}\}\) is a local contactomorphism then \(\psi\) has translated point \(x \in \text{int}(S(\psi))\).
2. If the Rabinowitz Floer homology \(\text{RFH}(\Sigma, X)\) does not vanish then every \(\psi \in \text{Cont}_0(\Sigma, \xi)\) has a translated point.
3. If \(\text{RFH}(\Sigma, X)\) is infinite dimensional then for \(\psi \in \text{Cont}_0(\Sigma, \xi)\) either \(\psi\) has infinitely many translated points or \(\psi\) has a translated point lying on a closed leaf of \(R_\alpha\).
4. If \(\dim \Sigma \geq 3\) then a generic \(\psi \in \text{Cont}_0(\Sigma, \xi)\) has no translated point lying on a closed leaf of \(R_\alpha\).
5. For a generic \(\psi \in \text{Cont}_0(\Sigma, \xi)\) the following holds. If \(x \in \Sigma\) is a translated point for \(\psi^n\), \(n \in \mathbb{N}\), then \(x\) is not a translated point for \(\psi, \psi^2, \ldots, \psi^{n-1}\).

**Remark 1.5.** Property 5. in Theorem 1.4 holds in fact for leafwise intersections as the proof will show.

The following corollary is well-known and follows from Chekanov’s work [Che96].

**Corollary 1.6.** Any \(\psi \in \text{Cont}_0(\mathbb{R}^{2n-1}, \xi_{st}) \setminus \{\mathbb{I}\}\) admits a translated point \(x \in \text{int}(S(\psi))\).

**Proof.** This follows from Theorem 1.4 together with Remark 1.2.

**Corollary 1.7.** A generic \(\psi \in \text{Cont}_0(\mathbb{R}^{2n-1}, \xi_{st}) \setminus \{\mathbb{I}\}\) admits infinitely many geometrically distinct iterated translated points all of which lie in \(\text{int}(S(\psi))\).

**Proof.** By the previous corollary every \(\psi^n\) admits a translated point \(x_n \in \text{int}(S(\psi^n))\). By property 5. in Theorem 1.4 the set \(\{x_n : n \in \mathbb{N}\}\) cannot be finite for a generic \(\psi\).

**Remark 1.8.** Sandon proved in [San11a] that any positive \(\psi\) admits infinitely many geometrically distinct translated points.

In order to explain the idea behind the proof of Theorem 1.4 we need to introduce a few more definitions. Recall that from \((\Sigma, \xi)\) we can build the **symplectization** of \(\Sigma\), which is the exact symplectic manifold \((S\Sigma, d(r\alpha))\), where

\[
S\Sigma := \Sigma \times \mathbb{R}^+,
\]

and \(r\) is the coordinate on \(\mathbb{R}^+ := (0, \infty)\). Suppose \(\psi \in \text{Cont}(\Sigma, \xi)\). There exists a unique positive smooth function \(\rho \in C^\infty(\Sigma, \mathbb{R}^+)\) such that \(\psi^*\alpha = \rho \alpha\). We define the **symplectization** of \(\psi\) to be the symplectomorphism \(\varphi : S\Sigma \to S\Sigma\) defined by

\[
\varphi(x, r) = (\psi(x), r \rho(x)^{-1}).
\]

Let us now go back to the completion \(X\) of \(M\). Let \(Y_M\) denote the Liouville vector field of \(\lambda_M\) (defined by \(iy_M d\lambda_M = \lambda_M\)). The entire symplectization \(S\Sigma\) embeds into \(X\) via the flow of \(Y_M\), and under this embedding the vector field \(Y\) on \(X\) defined by

\[
Y := \begin{cases} Y_M, & \text{on } M, \\ r \partial_r, & \text{on } S\Sigma \end{cases}
\]
satisfies \(i_\gamma d\lambda = \lambda\) on all of \(X\). Note that under this embedding \(S\Sigma \hookrightarrow X\), the hypersurface \(\Sigma \times \{1\}\) in \(\Sigma\) is identified with \(\Sigma\) in \(X\).

Suppose \(\varphi \in \Symp(X,\omega)\). A point \(x \in \Sigma\) is called a leaf-wise intersection point for \((\Sigma, \varphi)\) if there exists \(\tau \in \mathbb{R}\) such that \(\varphi(x) = \phi_\tau^\circ(x)\). The definition still makes sense if \(\tau\) is only defined on \(S\Sigma \subseteq X\) rather than on all of \(X\). In this case it is more convenient to phrase the definition using the hypersurface \(\Sigma \times \{1\}\) inside \(S\Sigma\). Thus if \(\varphi \in \Symp(S\Sigma, d(\rho\alpha))\) then a point \(x \in \Sigma\) is called a leaf-wise intersection point for \((\Sigma, \varphi)\) if there exists \(\tau \in \mathbb{R}\) such that

\[
\varphi(x,1) = (\phi_\tau^\circ(x), 1).
\]

Our starting point is the following observation of Sandon [San13].

**Lemma 1.9.** Fix \(\psi \in \Cont(\Sigma, \xi)\) and let \(\varphi \in \Symp(S\Sigma, d(\rho\alpha))\) denote the symplectization of \(\psi\). Then a point \(x \in \Sigma\) is a translated point for \(\psi\) if and only if \((x,1)\) is a leaf-wise intersection point for \(\varphi\).

This reduces the existence problem for translated points of \(\psi\) to the existence problem of leaf-wise intersections for \(\varphi\). The first and second authors, developed in [AF10a] a variational characterization for the leaf-wise intersection problem using the Rabinowitz action functional, and it is precisely this characterization that we will exploit.

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## 2 Proofs

Fix once and for all a 1-form \(\alpha \in \Omega^1(\Sigma)\) such that \(\xi = \ker \alpha\), and fix an exact symplectic filling \((M, d\lambda_M)\). As usual denote the completion of \((M, d\lambda_M)\) by \((X, d\lambda)\). We denote by

\[
\varphi(\Sigma, \alpha) > 0
\]

(2.1)

the minimal period of an orbit of \(R_\alpha\) that is contractible in \(X\).

We denote by \(\mathcal{C}(\Sigma, \xi)\) the set of contact isotopies \(\{\psi_t\}\) for \(t \in \mathbb{R}\) which satisfy \(\psi_0 = 1\) and \(\psi_{t+1} = \psi_t \circ \psi_1\). The universal cover \(\Cont_0(\Sigma, \xi)\) of \(\Cont_0(\Sigma, \xi)\) consists of equivalence classes of members of \(\mathcal{C}(\Sigma, \xi)\), where two paths \(\{\psi_t\}\) and \(\{\psi'_t\}\) are equivalent if there exists a smooth family \(\{\psi_{s,t}\}\) for \((s, t) \in [0,1] \times X\) such that \(\psi_{0,t} = \psi_t\) and \(\psi_{1,t} = \psi'_t\) with \(\{\psi_{s,t}\}\) \(\in \mathcal{C}(\Sigma, \xi)\) for each \(s \in [0,1]\).

The infinitesimal generator \(W\) of a contact isotopy \(\{\psi_t\} \in \mathcal{C}(\Sigma, \xi)\) is defined by

\[
W(x) := \frac{\partial}{\partial t} \bigg|_{t=0} \psi_t(x),
\]

and we say that \(\{\psi_t\}\) is generated by the function \(h : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}\) defined by

\[
h(t, x) := \alpha_{\psi_t(x)}(W(\psi_t(x))
\]

(2.2)

(\(h\) is 1-periodic because \(\psi_{t+1} = \psi_t \circ \psi_1\)).

Suppose \(\{\psi_t\} \in \mathcal{C}(\Sigma, \xi)\). In this case if \(\psi := \psi_1\) then the symplectization \(\varphi\) of \(\psi\) belongs to \(\Ham(S\Sigma, d(\rho\alpha))\). Indeed, if we define the contact Hamiltonian \(H : \mathbb{R}/\mathbb{Z} \times S\Sigma \to \mathbb{R}\) associated to \(\{\psi_t\}\) by

\[
H(t, x, r) := rh(t, x),
\]

where \(h\) is the function from (2.2), and we denote by \(\varphi_t\) the Hamiltonian flow of \(H\), then it can be shown (see for instance [AF11] Proposition 2.3) that

\[
\varphi_t(x, r) = (\psi_t(x), r\rho_t(x)^{-1}).
\]
Thus the symplectization $\varphi$ of $\psi$ is simply $\varphi_1$. We define

$$F_0 : S\Sigma \to \mathbb{R}$$

by

$$F_0(x, r) := f(r),$$

where

$$f(r) := \frac{1}{2}(r^2 - 1) \text{ on } (1/2, \infty),$$

$$f''(r) \geq 0 \text{ for all } r \in \mathbb{R}^+,$$

$$\lim_{r \to 0} f(r) = -\frac{1}{2} + \varepsilon$$  \hspace{1cm} (2.3)

for some small $\varepsilon > 0$. Note that the Hamiltonian vector field $X_{F_0}$ is given by $X_{F_0}(x, r) = f'(r)R_\alpha(x)$; in particular $X_{F_0}|_{\Sigma \times \{1\}} = R_\alpha$.

Let $\chi \in C^\infty(S^1, [0, \infty))$ denote a smooth function such that if $\bar{\chi}(t) := \hat{\chi}(0) \chi(t) F_0(v)$ then there exists $t_0 \in (0, 1/2)$ such that $\bar{\chi}(t) \equiv 1$ on $[t_0, 1]$, and such that on $[0, t_0]$ the function $\bar{\chi}$ is strictly increasing.

Finally fix a smooth function $\vartheta : [0, 1] \to [0, 1]$ such that $\vartheta(t) = 0$ for $t \in [0, 1/2]$, and such that $\vartheta(1) = 1$ with $0 \leq \dot{\vartheta}(t) \leq 4$ for all $t \in [0, 1]$. Denote by $\mathcal{L}(S\Sigma) := C^\infty(S^1, S\Sigma)$. We now define the Rabinowitz action functional we will work with.

**Definition 2.1.** We define the Rabinowitz action functional

$$\mathcal{A} : \mathcal{L}(S\Sigma) \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}(v, \eta) := \int_0^1 v^* \lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 \dot{\vartheta}(t) H(\vartheta(t), v) dt.$$

A simple calculation tells us that if $(v, \eta) \in \text{Crit}(\mathcal{A})$ then if we write $v(t) = (x(t), r(t))$ we have

$$\begin{cases}
\dot{v}(t) = \eta \chi(t) X_{F_0}(v) + \dot{\vartheta}(t) X_H(\vartheta(t), v), \\
\int_0^1 \chi(t) F_0(v) dt = 0.
\end{cases}$$

The following lemma appears in [AF10a, Proposition 2.4], and explains the connection between the Rabinowitz action functional and leaf-wise intersection points (and hence translated points, via Lemma 1.9).

**Lemma 2.2.** Define

$$e : \text{Crit}(\mathcal{A}) \to \Sigma$$

by

$$e(v, \eta) := x(1/2)$$

where $v = (x, r)$. Then $e$ is a surjection onto the set of translated points for $\psi$. If $\psi$ has no translated points lying on closed leaves of $R_\alpha$ then $e$ is a bijection. Moreover

$$\mathcal{A}(v, \eta) = \eta$$

for $(v, \eta) \in \text{Crit}(\mathcal{A})$. 

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Proof. Suppose \((v, \eta) \in \text{Crit}(\mathcal{A})\). Write \(v(t) = (x(t), r(t)) \in \Sigma \times \mathbb{R}^+\). For \(t \in [0, 1/2]\) one has
\[ v(t) = \eta \chi(t) X_{F_0}(v) \]
Since \(F_0 = \chi f\) with \(f\) autonomous, \(F_0\) is constant on its flow lines and thus \(t \mapsto F_0(v(t)) = r(t)\) is constant for \(t \in [0, 1/2]\). The second condition tells this constant is 1, and hence \(v(1/2) = (\phi_{\varphi}^t(x(0)), 1)\).

Next, for \(t \in [1/2, 1]\) we have \(\dot{v}(t) = \dot{\varphi}(t) X_H(\varphi(t), v)\). Thus \(v(t) = \varphi_{\varphi(t)}(v(1/2))\) for \(t \in [1/2, 1]\). In particular, \(\varphi(v(1/2), 1) = (\phi_\varphi^c(v(1/2)), 1)\), and thus \(v(1/2)\) is a leaf-wise intersection point of \(\varphi\).

In order to prove the last statement, we first note that
\[
\lambda(X_{F_0}(x, r)) = f'(r) \alpha_x(R_x(x)) = f'(r),
\]
and hence
\[
\mathcal{A}(v, \eta) = \int_0^{1/2} \lambda(\eta \chi X_{F_0}(v)) dt + \int_{1/2}^1 \left[ \lambda(\dot{\varphi}X_H(\varphi(t), v)) - \dot{\varphi}H(\varphi(t), v) \right] dt
\]
\[ = \eta + 0. \]

Unfortunately, in order to be able to define the Rabinowitz Floer homology, we cannot work with \(\mathcal{A}\) as it is not defined on all of \(\mathcal{L}X \times \mathbb{R}\). In order to rectify this, we extend \(F_0\) and \(H\) to Hamiltonians defined on all of \(X\). Here are the details. Define
\[ F : X \to \mathbb{R} \]
by setting
\[ F|_{X \setminus S\Sigma} := -1/2 + \varepsilon, \]
where \(\varepsilon > 0\) is as in [2,3], and defining \(F = F_0\) on \(S\Sigma\). Next, for \(c > 0\) let \(\beta_c \in C^\infty([0, \infty), [0, 1])\) denote a smooth function such that
\[
\beta_c(r) = \begin{cases} 
1, & r \in [e^{-c}, e^c], \\
0, & r \in [0, e^{-2c}] \cup [e^c + 1, \infty),
\end{cases}
\]
and such that
\[
0 \leq \beta_c'(r) \leq 2e^{2c} \quad \text{for} \quad r \in [e^{-2c}, e^{-c}],
\]
\[
-2 \leq \beta_c(r) \leq 0 \quad \text{for} \quad r \in [e^c, e^c + 1].
\]
Then define \(H_c : [0, 1] \times X \to \mathbb{R}\) by
\[
H_c|_{[0,1] \times (X \setminus S\Sigma)} := 0,
\]
and for \((t, x, r) \in [0, 1] \times S\Sigma\),
\[
H_c(t, x, r) := \beta_c(r) \dot{r} \varphi(t) h(\varphi(t), x).
\]

Remark 2.3. Note that for any \(c > 0\), \(H_c\) is a compactly supported 1-periodic Hamiltonian on \(X\) with the property that \(H_c(t, \cdot, \cdot) = 0\) for \(t \in [0, 1/2]\). Moreover the Hofer norm \(\|H_c\|\) of \(H_c\) satisfies
\[
\|H_c\| \leq 4(e^c + 1)(h_+ + h_-),
\]
where
\[
h_+ := \max_{(t, x) \in \mathbb{R}/\mathbb{Z} \times \Sigma} h(t, x), \quad h_- := \min_{(t, x) \in \mathbb{R}/\mathbb{Z} \times \Sigma} h(t, x).
\]
Remark 2.4. Suppose $\sigma : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ is a contactomorphism such that $\mathcal{G}(\sigma)$ is compact, and suppose that $x : \mathbb{R}^{2n-1} \to U \subseteq \Sigma$ is a Darboux chart onto an open subset $U$ of $\Sigma$. Let $\psi : \Sigma \to \Sigma$ denote the local contactomorphism such that $\psi = \sigma \circ x$ on $U$ and $\psi = \mathbb{I}$ on $\Sigma \setminus U$. Given $R > 0$, let $\tau_R : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ denote the contact rescaling defined by $\tau_R(x,y,z) = (Rx, Ry, R^2z)$ for $(x,y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$. There is a 1-1 correspondence between the translated points of $\sigma$ and the translated points of the conjugation $\sigma_R := \tau_R \circ \sigma \circ \tau_R^{-1}$ as follows: $(x,y,z)$ is a translated point of $\sigma$ if and only if $\tau_R(x,y,z)$ is translated point of $\sigma_R$. Moreover if $\sigma$ is generated by the function $h(t,x)$ then $\sigma_R$ is generated by $R^2h(t, \tau_R^{-1}(x))$.

We denote by $\psi_R$ the local contactomorphism of $\Sigma$ corresponding to $\sigma_R$ and the function $H_{c,R}$ corresponding to $\psi_R$. Then for fixed $c > 0$ we can choose $R$ so small that the Hofer norm of $H_{c,R}$ is smaller than $\varphi(\Sigma, \alpha)$.

We now extend the Rabinowitz action functional $\mathcal{A}$ to a new functional

$$\mathcal{A}_c : \mathcal{L} X \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}_c(v, \eta) := \int_0^1 \dot{v}^* \lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 H_c(t,v) dt.$$  

The following result is the key to the present paper. The proof is similar to (but simpler than) [AF11 Proposition 4.3].

Proposition 2.5. There exists $c > 0$ such that if $c > c_0$ then if $(v, \eta) \in \text{Crit}(\mathcal{A}_c)$ then $v(S^1) \subseteq \Sigma \times \mathbb{R}^+$, and moreover if we write $v(t) = (x(t), r(t))$ then $r(S^1) \subseteq (e^{-c/2}, e^{c/2})$.

Proof. We know that $r(t) = 1 \in (e^{-c/2}, e^{c/2})$ for all $t \in [0,1/2]$. Thus if

$$I := \left\{ t \in S^1 : r(t) \in (e^{-c/2}, e^{c/2}) \right\}$$

then $I$ is a non-empty open interval containing the interval $[0,1/2]$. Let $I_0 \subseteq I$ denote the connected component containing 0. We show that $I_0$ is closed, whence $I_0 = I = [0,1]$.

If $v(t) \in \Sigma \times (e^{-c}, e^c)$ and $t \in [1/2,1]$ then $r(t)$ satisfies the equation

$$\dot{r}(t) = -\hat{\theta}(\xi(t)) \frac{\hat{\rho}_\xi(t)(x(t))}{\rho_\xi(t)(x(t))} \cdot r(t).$$

Set

$$C := \max \left\{ \frac{\hat{\rho}(x)}{\rho(x)} : (t,x) \in [0,1] \times \Sigma \right\}.$$ 

Since $0 \leq \hat{\theta} \leq 4$, we see that for $t \in I_0 \cap [1/2,1]$ it holds that

$$e^{-4C} \leq r(t) \leq e^{4C}.$$ 

In particular, provided $c > c_0 := 8C$ then we have that if $v(t) \in \Sigma \times (e^{-c}, e^c)$ then actually $v(t) \in \Sigma \times (e^{-c/2}, e^{c/2})$. This shows that $I_0$ is closed as required. 

As an immediate corollary, we obtain:

Corollary 2.6. For $c > c_0$ the critical point equation and the critical values for critical points of $\mathcal{A}_c$ are independent of $c$. In fact, they agree with those of $\mathcal{A}$.

Remark 2.7. It is important to note that if $\psi$ is a local contactomorphism then the constant $c_0 = c_0(\psi)$ is invariant under the contact rescaling $\tau_R$ from Remark 2.4. More precisely, if $\psi_R$ is the rescaling of $\psi$ as in Remark 2.4 then $c_0(\psi) = c_0(\psi_R)$.
Fix a family \( J = (J_t)_{t \in \mathbb{S}^1} \) of \( \omega \)-compatible almost complex structures on \( X \) such that the restriction \( J_t|_\Sigma \) is of SFT-type (see \[CFO10\]). From \( J \) we obtain an \( L^2 \)-inner product \( \langle \cdot, \cdot \rangle_J \) on \( \mathcal{L}^X \times \mathbb{R} \) by

\[
\langle (\zeta, l), (\zeta', l') \rangle_J := \int_0^1 \omega(J_t \zeta(t), \zeta'(t))dt + ll'.
\]

We denote by \( \nabla_J \alpha_c \) the gradient of \( \alpha_c \) with respect to \( \langle \cdot, \cdot \rangle_J \). Given \( -\infty < a < b < \infty \) we denote by \( \mathcal{M}_J(\alpha_c)_a^b \) the set of smooth maps \( u = (v, \eta) \in C^\infty(\mathbb{R}, \mathcal{L}^X) \) that satisfy

\[
\partial_s u + \nabla_J \alpha_c(u(s)) = 0,
\]

\[
a < \alpha_c(u(s)) < b \quad \text{for all } s \in \mathbb{R}.
\]

The following result is by now standard (see \[AS09, AF10a\]).

Proposition 2.8. Given \( -\infty < a < b < \infty \) and \( J \) as above, if \( c > c_0 \) then there exists a compact set \( K = K(c, J, a, b) \subseteq X \times \mathbb{R} \) such that for all \( u = (v, \eta) \in \mathcal{M}_J(\alpha_c)_a^b \) one has

\[
u(R \times S^1) \subseteq K.
\]

Theorem 1.4 follows from Proposition 2.8 by arguments from \[AF10a\], as we now explain.

**Proof of Theorem 1.4.**

(1) Suppose \( \psi \) is a local contactomorphism. Fix \( c > c_0 \). After possibly replacing \( \psi \) by \( \psi_R \) for some \( R \) sufficiently large (see Remark 2.3) we may assume \( \|H_c\| < \psi(\Sigma, \alpha) \)

where \( H_c \) is the Hamiltonian corresponding to \( \psi_R \) - note we are implicitly using Remark 2.7 here.

It follows from the proof of Theorem A in \[AF10a\] that there exists a critical point \((v, \eta)\) of \( \alpha_c \) with \( |\eta| \leq \|H_c\| \). Thus the translated point is necessarily a genuine translated point of \( \psi_R \), that is, \( x(0) \in \text{int}(\mathcal{S}(\psi_R)) \). Thus \( \psi_R \), and hence \( \psi \), has a translated point in the interior of its support.

Moreover \( \text{RFH}(\alpha_c)_a^b \) is independent of the choice of \( c > c_0 \). Thus it makes sense to define \( \text{RFH}(\{\psi_t\}, \Sigma, X) \) via

\[
\text{RFH}(\{\psi_t\}, \Sigma, X) := \lim_{a \downarrow -\infty} \lim_{b \uparrow \infty} \text{RFH}_s(\alpha_c)_a^b.
\]

See \[AF10a\] for more information. In fact, by arguing as in \[AF10a\] Theorem 2.16, we have

\[
\text{RFH}(\{\psi_t\}, \Sigma, X) \cong \text{RFH}(\Sigma, X),
\]

where \( \text{RFH}(\Sigma, X) \) denotes the Rabinowitz Floer homology of \( (\Sigma, X) \), as defined in \[CF09\]. The second statement of Theorem 1.4 now follows from Lemma 2.2 and Corollary 2.6 exactly as in \[AF10a\] Theorem C.

(3) The third statement in Theorem 1.4 follows from the Main Theorem in \[Kan10\].

(4) The fact that generically one doesn’t find translated points on closed Reeb orbits when \( \text{dim } \Sigma \geq 3 \) is proved exactly as in \[AF08\] Theorem 3.3] (as in Statement (2) above, the fact that
(5.) Finally, the fifth statement is proved by arguing as follows. Fix $k \in \{2, 3, \ldots, \infty\}$. Denote by $\mathcal{H}^k$ the class of $C^k$ contact Hamiltonians $H$, reparametrized so that $H(t, \cdot) = 0$ for $t \in [0, 1/2]$, which additionally have possibly been cutoff outside of a neighborhood of $\Sigma \times \{1\}$.

Recall given (any) two Hamiltonians $K_1, K_2$, the composition $K_1 \# K_2$ is defined by

$$(K_1 \# K_2)(t, p) := K_1(t, p) + K_2(t, (\phi_t^{K_1})^{-1}(p)).$$

Denote by $K_m^\# := K^\# \ldots \# K$ ($m$ times). Note that if $H \in \mathcal{H}^k$ then $H^m \# \in \mathcal{H}^k$ for all $m \in \mathbb{N}$.

Given $H \in \mathcal{H}^k$, we denote by $\mathcal{A}_H$ the Rabinowitz action functional

$$\mathcal{A}_H(v, \eta) = \int_0^1 v^*\lambda - \eta \int_0^1 \chi(t) F_0(v) dt - \int_0^1 H(t, v) dt$$

(so that the functional $\mathcal{A}_\varepsilon$ would now be written as $\mathcal{A}_{H, \varepsilon}$).

Let $\mathcal{L} = W^{1,2}(S^1, X)$ and let $\mathcal{E}$ denote the Banach bundle over $\mathcal{L}$ with fibre $\mathcal{E}_v := L^2(S^1, v^*TX)$.

Fix $l, m \in \mathbb{N}$. We now define a section

$$\sigma : \mathcal{L} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \to \mathcal{E}^\vee \times \mathbb{R} \times \mathcal{E}_v^\vee \times \mathbb{R}$$

by

$$\sigma(v, \eta, w, \tau, H) := (d\mathcal{A}_{H}((v, \eta), d\mathcal{A}_{H}(w, \tau)).$$

Let $\mathcal{M} := \sigma^{-1}(\text{zero section})$, so that

$$\mathcal{M} = \text{Crit}(\mathcal{A}_{H, \#}) \times \text{Crit}(\mathcal{A}_{H=m}).$$

The vertical derivative of $\sigma$,

$$D\sigma(v, \eta, w, \tau, H) := T_v\mathcal{L} \times \mathbb{R} \times T_w\mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \to \mathcal{E}^\vee \times \mathbb{R} \times \mathcal{E}_v^\vee \times \mathbb{R}$$

is given by

$$\left(\dot{v}, \dot{\eta}, \dot{w}, \dot{\tau}, \dot{H}\right) \mapsto \left(\mathbf{H}_{\mathcal{A}_{H}}(v, \eta)((\dot{v}, \dot{\eta}, \dot{H}), \bullet), \int_0^1 \dot{H}^\#(t, v) dt, \left(\mathbf{H}_{\mathcal{A}_{H=m}}(w, \tau)((\dot{w}, \dot{\tau}, \dot{H^m}), \bullet), \int_0^1 \dot{H}^m(t, w) dt\right)\right),$$

where, e.g. $\mathbf{H}_{\mathcal{A}_{H}}(v, \eta)$ denotes the Hessian of $\mathcal{A}_{H}$ at the critical point $(v, \eta)$.

In general one cannot hope that $D\sigma(v, \eta, w, \tau, H)$ is surjective. However if we set

$$\mathcal{R} := \{(v, \eta, w, \tau, H) : v(t) \neq w(t) \text{ for all } t \in [1/2, 1]\}$$

then $D\sigma(v, \eta, w, \tau, H)$ is surjective for $(v, \eta, w, \tau, H) \in \mathcal{M}^* := \mathcal{M} \cap \mathcal{R}$. In fact, if

$$\mathcal{V}(v, \eta, w, \tau, H) \subseteq T_v\mathcal{L} \times \mathbb{R} \times T_w\mathcal{L} \times \mathbb{R} \times \mathcal{H}^k$$

denotes the subspace of quintuples $(\dot{v}, \dot{\eta}, \dot{w}, \dot{\tau}, \dot{H})$ satisfying $\dot{v}(0) = \dot{w}(1/2) = 0$ then the fact that $D\sigma(v, \eta, w, \tau, H)\mathcal{V}(v, \eta, w, \tau, H)$ is surjective for all $(v, \eta, w, \tau, H) \in \mathcal{M}^*$ can be proved exactly as in [AF10a] Proposition A.2. Now define

$$\phi_{\text{eval}} : \mathcal{L} \times \mathbb{R} \times \mathcal{L} \times \mathbb{R} \times \mathcal{H}^k \to \Sigma \times \Sigma$$

by

$$\phi_{\text{eval}}(v, \eta, w, \tau, H) := (v(0), w(1/2)).$$

Then from the claim it follows that $\phi_{\text{eval}}\mid_{\mathcal{M}^*}$ is a submersion for generic $H \in \mathcal{H}^k$ (see for instance [AF08] Lemma 3.5). Thus there exists a generic set $\mathcal{M}_{l,m} \subseteq \mathcal{M}^*$ for which the following property holds: if $(v, \eta)$ is a critical point of $\mathcal{A}_{H^m}$, and $(w, \tau)$ is a critical point of $\mathcal{A}_{H=m}$, which satisfies
Claim: Suppose $H \in \mathcal{H}^*$. Set $\varphi := \phi^H_t$. Then for all pairs $(l, m)$ of positive integers, $\varphi^l$ and $\varphi^m$ do not have any common leaf-wise intersection points.

To prove the claim we argue by contradiction. Without loss of generality assume $l \leq m$, and suppose $x \in \Sigma$ is a common leaf-wise intersection point of $\varphi^l$ and $\varphi^m$. Thus there exists $\eta, \tau \in \mathbb{R}$ such that

$$
\varphi^l(x) = \phi^\eta_\alpha(x), \quad \varphi^m(x) = \phi^\tau_\alpha(x).
$$

Then

$$
\varphi^{m-l}(\varphi^l(x)) = \phi^{\tau-\eta}_{\alpha}(\varphi^l(x)),
$$

so $\varphi^l(x)$ is a leaf-wise intersection point of $\varphi^{m-l}$. Let $(v, -\eta) \in \text{Crit}(\mathcal{H}^l)$ and $(w, -\tau + \eta) \in \text{Crit}(\mathcal{H}^m)$ denote the critical points of $\mathcal{H}^l$ and $\mathcal{H}^m$ corresponding to $x$ and $\varphi^l(x)$ respectively, so that $v(0) = \varphi^l(x)$ and $v(1/2) = x$, and $w(0) = \varphi^m(x)$ and $w(1/2) = \varphi^l(x) = \varphi^l(0)$. By construction $v(t) \neq w(t)$ for all $t \in [1/2, 1]$, and this gives the desired contradiction. 

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