Consensus in Networks Prone to Link Failures

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Abstract

We consider deterministic distributed algorithms solving Consensus in synchronous networks of arbitrary topologies. Links are prone to failures. Agreement is understood as holding in each connected component of a network obtained by removing faulty links. We introduce the concept of stretch, which is a function of the number of connected components of a network and their respective diameters. Fast and early-stopping algorithms solving Consensus are defined by referring to stretch resulting in removing faulty links. We develop algorithms that rely only on nodes knowing their own names and the ability to associate communication with local ports. A network has $n$ nodes and it starts with $m$ functional links. We give a general algorithm operating in time $n$ that uses messages of $O(\log n)$ bits. If we additionally restrict executions to be subject to a bound $\Lambda$ on stretch, then there is a fast algorithm solving Consensus in time $O(\Lambda)$ using messages of $O(\log n)$ bits. Let $\lambda$ be an unknown stretch occurring in an execution; we give an algorithm working in time $(\lambda + 2)^3$ and using messages of $O(n \log n)$ bits. We show that Consensus can be solved in the optimal $O(\lambda)$ time, but at the cost of increasing message size to $O(m \log n)$. We also demonstrate how to solve Consensus by an algorithm that uses only $O(n)$ non-faulty links and works in time $O(nm)$, while nodes start with their ports mapped to neighbors and messages carry $O(m \log n)$ bits. We prove lower bounds on performance of Consensus solutions that refer to parameters of evolving network topologies and the knowledge available to nodes.

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1 Introduction

We consider distributed algorithms reaching agreement in synchronous networks. Nodes represent processing units and links model bi-directional communication channels between pairs of nodes. Links are prone to failures but nodes stay operational at all times. A link failing at a round does not convey the transmitted message. A link that has failed at least once is considered unreliable. The functionality of an unreliable link in the future is unpredictable, in that it may either deliver a transmitted message or fail to do it. We model a network with such link failures as evolving through its subnetworks obtained by removing unreliable links.

The Consensus problem is about nodes reaching agreement on a common value. Each node starts with an initial input and it eventually decides. Consensus is usually specified by the three requirements of termination, validity, and agreement, which constrain the outcome of deciding. These conditions may depend on the nature of faults in a distributed system. We address solving Consensus in a model of dynamic networks that evolve through link failures. Removing unreliable links may decompose a network into disjoint connected components. The Consensus problem needs to be suitably reformulated to reflect the possibility of a network becoming disconnected. We assume a version of agreement allowing nodes in different connected components, of a network obtained by removing unreliable links, to decide on different values, as long as nodes within a connected component decide on the same value.

Our approach to specifying Consensus subsumes the models of link crashes, which is stricter in that a failed link stays faulty forever. Node crashes may also decompose a network into separate connected components, and our approach subsumes this model as well. To see this for node crashes, observe that a node’s crash could be simulated by crashing all links incident to the node at the round of crash. If a node’s crash occurs in a simulation then the node stays operational but it constitutes a single-node connected component, so any decision by the node on some initial input value satisfies agreement and validity.

Let $n$ denote the number of nodes and $m$ the number of links in an initial network. Nodes are equipped with names, which are unique integers represented by $O(\log n)$ bits. Initial input values are assumed to be encodable with $O(\log n)$ bits each. A message is considered “short” if it carries $O(\log n)$ bits, and it is “linear” if it carries $O(n \log n)$ bits. A network operates with “minimal knowledge” if each node initially knows its individual name, the initial input value, and it can distinguish among ports as the sources of incoming and outlets for outgoing communication.

We use a network’s dynamic attribute called “stretch,” which is an integer reflecting the number of connected components and their diameters. The purpose of using stretch is to consider scalability of Consensus solutions to networks evolving through link failures and their removal from the network. In particular, we define “fast” and “early-stopping” algorithms solving Consensus by referring to stretch. Along with time performance, we consider the amount of communication, measured as an upper bound on the number of bits in each individual message. We also study performance of algorithms measured by “link use” defined as the number of reliable links through which nodes transmit messages.

A summary of the results. We study how nodes can reach agreement in a model of dynamic networks in which links are prone to failures. The problem is formulated as a variant of Consensus in which agreement needs to hold independently in each connected component of the network that emerges after removal of unreliable links. We propose a parameter of networks called stretch, which
Table 1: A summary of Consensus algorithms and their respective performance bounds. Notation $n$ denotes the number of nodes and $m$ the number of links in an initial network; none of these numbers is ever assumed to be known. Parameter $\Lambda$ denotes an upper bound on stretch. Notation $\lambda$ denotes the stretch actually occurring in an execution by the time all nodes halt. The dagger symbol $\dagger$ indicates asymptotic optimality of the respective upper bound.

| Algorithm / Section | Time | Message Size | # Links | Knowledge | Lower Bound |
|---------------------|------|--------------|---------|-----------|-------------|
| Fast-Consensus ($\Lambda$) / 3 | $\Omega(\log n)$ | $O(m)$ | $\Lambda$ known | $\lambda \leq \Lambda$ |
| SM-Consensus / 4 | $n$ | $O(\log n)$ | $O(m)$ | minimal | time $\lambda$ |
| LM-Consensus / 5 | $(\lambda + 2)^3$ | $O(n \log n)$ | $O(m)$ | minimal | time $\lambda$ |
| ES-Consensus / 6 | $\lambda + 2$ | $O(m \log n)$ | $O(m)$ | minimal | time $\lambda$ |
| OL-Consensus / 7 | $O(nm)$ | $O(m \log n)$ | $2n$ | neighbors known | $\# \text{ links } \Omega(n)$ |

reflects the number of connected components and their diameters. We reformulate the concepts of fast and early-stopping solutions to Consensus, to adopt them to the studied model of dynamic networks, by relating running time to stretch.

An upper bound on stretch, denoted $\Lambda$, could be given to all nodes, with an understanding that faults occurring in an execution are restricted such that stretch never surpasses $\Lambda$. An algorithm solving Consensus with a known upper bound on stretch $\Lambda$ is fast if it runs in time $O(\Lambda)$. A fast solution to Consensus is discussed in Section 3. We also show a lower bound which demonstrates that, for each natural number $\lambda$ and an algorithm solving Consensus in networks prone to link failures, there exists a network that has stretch $\lambda$ and such that each execution of the algorithm on this network takes at least $\lambda$ rounds. In Section 4 we show how to solve Consensus in time $n$ with short messages in networks where nodes have minimal knowledge. We give an algorithm relying on minimal knowledge working in time $(\lambda + 2)^3$ and using linear messages of $O(n \log n)$ bits, where $\lambda$ is an unknown stretch occurring in an execution; this algorithm is presented in Section 5. A Consensus solution is early-stopping if it operates in time proportional to an unknown stretch actually occurring in an execution. In Section 6, we develop an early-stopping solution to Consensus relying on minimal knowledge that employs messages of $O(m \log n)$ bits. We propose to count the number of reliable links used at the same time by a communication algorithm as its performance metric. To make this performance measure meaningful, nodes start knowing their neighbors, in having a correct mapping of communication ports to neighbors. Optimizing link use in order to achieve an algorithmic goal could be interpreted as relying on a network backbone to accomplish the task. In Section 7 we give a solution to Consensus that uses at most an asymptotically optimum number $2n$ of reliable links and works in $O(nm)$ rounds, without knowing the size $n$ of the network. We also show, in Section 7, that if nodes start with ports not mapped on neighbors, then any Consensus solution has to use $\Omega(m)$ links in some networks of $\Theta(m)$ links, for all $n$ and $m$ such that $n \leq m \leq n^2$. A summary of algorithms developed in this paper, with their performance bounds, is given in Table 1.

**Previous work on Consensus in networks.** Dolev [22] studied Byzantine Consensus in networks with faulty nodes and gave connectivity conditions sufficient and necessary for a solution to exist; see also Fischer et al. [25], and Hadzilacos [28]. Khan et al. [31] considered a related problem
| algorithm          | time   | message size | # links | knowledge | comments                      |
|--------------------|--------|--------------|---------|-----------|-------------------------------|
| Kuhn et. al. [32]  | $O(n)$ | $O(\log n)$ | $O(m)$  | minimal   | for $T = \Omega(n)$           |
| Biely et. al [7], alg. 2 | $2D + 2E$ | $O(m \log n)$ | $O(m)$  | $D$ known | directed links                |
| Biely et. al [7], alg. 4 | $3D + 3$ | $O(nD \log n)$ | $O(m)$  | $D$ known | k-set agreement               |
| Kuhn et. al [33]  | $O(n)$ | $O(m^2 \log n)$ | $O(m)$  | minimal   | $\Delta$-coordinated        |

Table 2: A summary of the previous Consensus results that are most relevant to the model studied in this paper and their respective performance bounds. Notation $n$ denotes the number of nodes and $m$ the number of links in an initial network; none of these numbers are ever assumed to be known. Letter $T$ denotes a number such that in every interval of $T$ rounds there exists a stable connected spanning subgraph, see [32]. Letter $D$ denotes a dynamic source diameter, and $E$ denotes a dynamic graph depth, see [7]. The stretch and parameters $D$ and $E$ in bidirectional networks with unreliable links are related as follows: $D = E = \Lambda \geq \lambda$. Parameter $\Delta$ is an upper bound on the difference in the respective times of termination for any two nodes, see [33].

in the model with restricted Byzantine faults, in particular, in the model requiring a node to broadcast identical messages to all neighbors at a round. Tseng and Vaidya [48] presented necessary and sufficient conditions for the solvability of Consensus in directed graphs under the models of crash and Byzantine failures. In a related work, Choudhury et al. [18] provided a time-optimal algorithm to solve Consensus in directed graphs with node crashes. Castañeda et al. [8] considered networks with nodes prone to crashes with an upper bound $t$ on the number of crashes and showed that, as long as the network remained $(t + 1)$-connected, Consensus was solvable in a number of rounds determined by how conducive the network was to broadcasting. Chlebus et al. [15] studied networks with Byzantine nodes such that the removal of faulty nodes leaves a network that is sufficiently connected; they gave fast solutions to Consensus and showed a separation of Consensus with Byzantine nodes from Consensus with Byzantine nodes using message authentication, with respect to asymptotic time performance in suitably connected networks. Tseng [47] and Tseng and Vaidya [49] surveyed work on reaching agreement in networks subject to node faults. Winkler and Schmidt [52] gave a survey of recent work on Consensus in directed dynamic networks.

The approach to failing links that we work with falls within the scope of modeling dynamic evolution of networks. Next, we review previous work on solving Consensus in networks undergoing topology changes, malfunctioning links and transmission failures. Perry and Toueg [11], Santoro and Widmayer [43, 44], and Charron-Bost and Schiper [11] studied agreement problems in complete networks in the presence of dynamic transmission failures. Kuhn et al. [33] considered $\Delta$-coordinated Binary Consensus in undirected graphs, whose topology could change arbitrarily from round to round, as long it stayed connected; here $\Delta$ is a parameter that bounds from above the difference in times of termination for any two nodes. Paper [33] showed how to solve $\Delta$-coordinated Binary Consensus in $O(\frac{nD}{D+\Delta} + \Delta)$ time using message of $O(m^2 \log n)$ size without a prior knowledge of the network’s diameter $D$. In this paper, in the case of $\Delta = \Theta(n)$, we relate the running time of Consensus solutions to the stretch actually occurring in an execution and show that it is possible to solve this version of Consensus in $O(\lambda + 2)$ rounds with messages of $O(m \log n)$ size. Augustine et al. [4] studied dynamic networks with adversarial churn and gave randomized algorithm to reach almost-everywhere agreement with high probability in poly-logarithmic time. Charron-Bost et al. [10]...
considered approximate Consensus in dynamic networks and provided connectivity restrictions on network evolution that make approximate Consensus solvable. Coulouma et al. [21] characterized oblivious message adversaries for which Consensus is solvable. Biely et al. [7] considered reaching agreement and $k$-set agreement in networks when communication is modeled by directed-graph topologies controlled by adversaries, with the goal to identify constraints on adversaries to make the considered problems solvable. Paper [7] solved $k$-set agreement in time $O(3D + H)$ and using messages of $O(nD \log n)$ size, where $D$ denotes the dynamic source diameter and $H$ denotes the dynamic graph depth, and the code of algorithm includes $D$. Relating the algorithms in [7] to the methodology used in this paper, the parameters $D$ and $H$ are equivalent to an upper bound $\Lambda$ on stretch when it is a part of code. One of the results in this paper, that corresponds to that in [7] for bi-connected topologies, is an algorithm that works in $O(\lambda)$ time and does not know the stretch $\lambda$ occurring in an execution. Kuhn et al. [32] considered dynamic networks in which the network topology changes from round to round such that in every $T \geq 1$ consecutive rounds there exists a stable connected spanning subgraph, where $T$ is a parameter. Paper [32] gave an algorithm that implements any computable function of the initial inputs, working in $O(n + n^2/T)$ time with messages of $O(\log n + d)$ size, where $d$ denotes the size of input values. Schmid et al. [46] showed impossibility results and lower bounds for the number of processes and rounds for synchronous agreement under transient link failures. Winkler et al. [53] investigated solving Consensus in dynamic networks with links controlled by adversaries who only eventually provide a desired behavior of networks. A summary of the previous results, that are most closely related to the model of this paper, is given in Table 2.

Next, we review representative work on other algorithmic problems in dynamic networks. Cornejo et al. [20] considered the aggregation problem in dynamic networks, where the goal is to collect tokens, originally distributed among the nodes, at a minimum number of nodes. Haeupler and Kuhn [29] developed lower bounds on information dissemination in adversarial dynamic networks. Sarma et al. [45] investigated algorithms interpreted as random walks in dynamic networks. Michail et al. [38] studied naming and counting in anonymous dynamic networks. Michail et al. [39] studied propagation of influence in dynamic networks that may be disconnected at any round. General frameworks and models of distributed computing in networks evolving in time were discussed in [9, 32, 34, 37].

**Previous work on scalability of Consensus solutions.** We review the previous work on Consensus solutions that scale well with their performance metrics, including communication and time. Let letter $t$ denote an upper bound on the number of node failures that is known to all nodes, and letter $f$ be an actual number of failures occurring in an execution, which is not assumed to be known. We may call a Consensus solution “fast” if it operates in time $O(t+1)$ and “early stopping” if its running time is $O(f + 1)$. Scaling communication performance can be understood either “globally,” which conservatively means $O(n \text{ polylog } n)$ total communication, or “locally,” which conservatively means $O(\text{polylog } n)$ communication generated by each communicating agent.

Networks of degrees uniformly bounded by a constant provide an ultimate local scaling of communication. Upfal [50] showed that an almost-everywhere agreement can be solved in networks of constant degree with a linear number of crashes allowed to occur. Dolev et al. [24] gave an early stopping solution for Consensus with arbitrary process failures and a lower bound $\min\{t + 1, f + 2\}$ on the number of rounds. Berman et al. [6] developed an early-stopping Consensus solution with arbitrary process failures that was simultaneously optimal with respect to time performance $\min\{t + 1, f + 2\}$ and the number of processes $n > 3t$. Galil et al. [26] developed a crash-resilient Consensus
algorithm using $O(n)$ messages, thus showing that this number of messages is optimal, but their algorithm runs in over-linear time $O(n^{1+\varepsilon})$, for any $0 < \varepsilon < 1$; they also gave an early-stopping algorithm with $O(n + fn^\varepsilon)$ message complexity, for any $0 < \varepsilon < 1$. Garay and Moses [27] presented a fast Consensus solution for arbitrary processor faults that runs on the optimum $n > 3t$ number of processors while using the amount of communication that is polynomial in $n$; they also gave an early-stopping variant of this algorithm. Chlebus and Kowalski [12] developed a gossiping algorithm coping with crashes and applied it to develop a Consensus solution that is fast, by running in $O(t+1)$ time, while sending $O(n \log^2 t)$ messages, provided that $n - t = \Omega(n)$. Chlebus and Kowalski [13] developed a deterministic early-stopping algorithm that scales communication globally by sending $O(n \log^5 n)$ messages. Coan [19] gave an early-stopping Consensus solution that uses messages of size that is logarithmic in the size of the range of input values; see also Bar-Noy et al. [5] and Berman et al. [6]. Chlebus at al. [16] gave a fast deterministic algorithm for Consensus which has nodes send $O(n \log^4 n)$ bits, and showed that no deterministic algorithm can be locally scalable with respect to message complexity. Chlebus and Kowalski [14] gave a randomized Consensus solution terminating in the expected $O(\log n)$ time, while the expected number of bits that each process sends and receives against oblivious adversaries is $O(\log n)$, assuming that a bound $t$ on the number of crashes is a constant fraction of the number $n$ of nodes. Chlebus et al. [17] gave a scalable quantum algorithm to solve binary Consensus, in a system of $n$ crash-prone quantum processes, which works in $O(\text{polylog } n)$ time sending $O(n \text{ polylog } n)$ qubits against the adaptive adversary. Dolev and Lenzen [23] showed that any crash-resilient Consensus algorithm deciding in $f + 1$ rounds has worst-case message complexity $\Omega(n^2 f)$. Alistarh et al. [1] developed a randomized algorithm for asynchronous message passing that scales well to the number of crashes, in that it sends expected $O(nt + t^2 \log^2 t)$ total messages.

**General perspective.** Consensus is among the central algorithmic problems in distributed systems and communication networks; it was introduced by Pease at al. [40] and Lamport et al. [35]. Books by Attiya and Welch [3], Herlihy and Shavit [30], Lynch [36], and Raynal [42] provide general expositions of formal models of distributed systems as frameworks to develop distributed algorithms in systems prone to failures of components. Attiya and Ellen [2] review techniques to show impossibilities and lower bounds for problems in distributed computing. Wattenhofer [51] presents applications of distributed solutions to Consensus in distributed ledger technologies.

## 2 Preliminaries

We model distributed systems as collections of nodes that communicate through a wired communication network. Executions of distributed algorithms are synchronous, in that they are partitioned into global rounds coordinated across the whole network. There are $n$ nodes in a network. Each node has a unique $name$ used to determine its identity; a name can be encoded by $O(\log n)$ bits. Nodes start with a read-only variable $name$ initialized to their names; the private copy of this variable at a node $p$ is denoted by $name_p$.

A node has distinctly labeled ports that act as interfaces to links connecting the node to its neighbors. We say that a node knows neighbors if it can associate with each port the name of a node that receives communication sent through this port and whose communication is received through this port. Links connecting pairs of nodes serve as bi-directional communication channels. Messages are scaled to channel capacity, in that precisely one message can be transmitted at a
round through a link in each direction. The size of a message denotes the number of bits used to encode it. A message transmitted at a round is delivered within the same round. If at least one message is transmitted by a link in an execution then this link is used and otherwise it is unused in this execution.

A network begins with all links fully operational. A link may fail to deliver a message transmitted through it at a round; once this happens for a link, it is considered unreliable. The functionality of an unreliable link is unpredictable, in that it may either deliver a transmitted message or fail to do it. A link that has never failed by a given round is reliable at this round. A path in the network is reliable at a round if it consists only of links that are reliable at this round.

Nodes and links of a network can be interpreted as a simple graph, with nodes serving as vertices and links as undirected edges. A network at the start of an execution is represented by some initial graph $G$, which is simple and connected. An edge representing an unreliable link is removed from the graph $G$ at the first round it fails to deliver a transmitted message. A graph representing the network evolves through a sequence of its subgraphs and may become partitioned into multiple connected components. Once an algorithm’s execution halts, we stop this evolution of the initial graph $G$. An evolving network, and its graph representation $G$, at the first round after all nodes have halted in an execution is called final for this execution and denoted by $G_F$.

Nodes start an execution of a distributed algorithm simultaneously. At one round, a node sends messages through some of its ports, collects messages sent by neighbors, and performs local computation. A node can send messages to any subset of its neighbors and collect messages coming from all neighbors at a round.

Consensus with unreliable links. Each node $p$ starts with an initial value $\text{input}_p$. We assume two properties of such input values. One is that each of them can be represented by $O(\log n)$ bits. The other is that input values can be compared, in the sense of belonging to a domain with a total order. In particular, finitely many initial input values contain a maximum one. We say that a node decides when it sets a dedicated variable to a decision value. The operation of deciding is irrevocable. An algorithm solves Consensus in networks with unreliable links if the following three properties hold in all executions:

Termination: every node eventually decides.

Validity: each decision value is among the initial input values across the whole network.

Agreement: when a node $p$ decides then its decision value is the same as these of the nodes that have already decided and to which $p$ is connected by a reliable path at the round of deciding.

If a message sent by a node executing a Consensus solution carries a constant number of node names and a constant number of input values then the size of such a message is $O(\log n)$, due to our assumptions about encoding names and input values. Messages of size $O(\log n)$ are called short. If a message carries $O(n)$ node names and $O(n)$ initial input values then the size of such a message is $O(n \log n)$. Messages of size $O(n \log n)$ are called linear.

Stretch. Let $H$ be a simple graph. If $H$ is connected then diam($H$) denotes the diameter of $H$. Suppose $H$ has $k$ connected components $C_1, \ldots, C_k$, where $k \geq 1$, and let $d_i = \text{diam}(C_i)$ be the diameter of component $C_i$. The stretch of $H$ is defined as $k - 1 + \sum_{i=1}^{k} d_i$. The stretch of a connected graph equals its diameter, because then $k = 1$. The stretch of $H$ can be interpreted as
the maximum diameter of a graph obtained from $H$ by adding $k - 1$ edges such that the obtained graph is connected. The maximum stretch of a graph with $n$ vertices is $n - 1$, which occurs when all vertices are isolated.

**Knowledge.** A property of distributed communication environments or executions is known if it can be used in codes of algorithms. We say that an algorithm relies on minimal knowledge if each node knows its unique name and can identify a port through which a message arrives and can assign a port for a message to be transmitted through. The number of nodes in a network $n$ is never assumed to be known in this paper.

Neighbors can be discovered at one round of communication by all nodes sending their names to the neighbors: incoming messages allow to assign the sender’s name to a port. This operation requires transmitting through every link in the network. If a goal is to minimize the number of used links then we assume that each node knows its neighbors prior to the beginning of an execution, in having the neighbors correctly mapped on ports.

**Performance metrics.** We categorize performance of algorithms by adapting the concepts of fast and early stopping algorithms used in the model of node crashes. These concepts are usually understood as follows. Let $t$ denote an upper bound on the number of node crashes, which is known to all nodes, and $f$ be an actual number of node crashes occurring in an execution, which is not assumed to be known. A Consensus solution is considered as fast if it operates in time $O(t + 1)$ and early stopping if its running time is $O(f + 1)$. The early-stopping quality in this context means scaling the time performance to the actual number of crashes occurring in an execution.

The notation $\Lambda$ will denote an upper bound on the stretch of a communication network in the course of an execution. If $\Lambda$ is used in considerations then this means that $\Lambda$ is known to all nodes. A Consensus algorithm in a synchronous network with links prone to failures is defined to be fast if it runs in a number of rounds proportional to $\Lambda$.

An initially connected graph $G$ evolves through a nested sequence of its original subgraphs, by removing edges representing faulty links. Let the notation $\lambda_k$ denote a stretch of a network $G$ at a round $k$ of an execution and $\lambda$ be a stretch at the round of termination.

**Proposition 1** Stretches $\lambda_k$ make a monotonously increasing sequence: $\lambda_i \leq \lambda_j$, for $i < j$.

**Proof:** Consider an edge $e$ belonging to a connected component $C$. If $e$ gets removed from the graph and the connected component $C$ stays intact then its diameter may only increase. Suppose the edge $e$ is a bridge of $C$, and after its removal the connected component $C$ breaks into two connected components $C_1$ and $C_2$. The stretch of the graph after removal of edge $e$ is the stretch before removal of this edge minus the diameter of $C$ plus the diameters of $C_1$ and $C_2$, and this number incremented by $1$. So this new stretch cannot be smaller than the original one, because the diameter of $C$ is at most a sum of the diameters of $C_1$ and $C_2$ incremented by one.

We consider bounds on performance of algorithms with respect to stretch $\lambda$ at a round in which all nodes halt. Let $f : \mathbb{N} \to \mathbb{N}$ be a monotonously increasing function. For a communication algorithm to run in $O(f(\lambda))$ time, it is sufficient and necessary that the running time is $O(f(\lambda_k))$, for any round number $k$ prior to halting, by Proposition 1. We say that an algorithm for link failures is early stopping if it runs in $O(\lambda)$ time.
Fast Consensus

Knowing a bound on stretch makes structuring a Consensus solution particularly straightforward, which we demonstrate next. We also show that stretch is a meaningful yardstick to compare the running time of Consensus algorithms to, since each algorithm may have to be executed in some networks for a number of rounds at least as large as stretch.

We present a fast algorithm solving Consensus in networks with link failures, assuming that a bound on stretch is known. The algorithm is called Fast-Consensus; its pseudocode is presented in Figure 1. Every node strives to eventually decide on as large a value as possible, so a node remembers only the largest value it has witnessed so far. Every node iterates, through \( \Lambda \) rounds, the operation to send a message to all neighbors as soon as an input value has been learned that is greater than the greatest remembered before. The messages in the first round carry the respective senders’ input values. All nodes halt after \( \Lambda \) rounds, and every node decides on the greatest value it has learned about.

**Theorem 1** If the stretch never grows beyond \( \Lambda \) in an execution of algorithm Fast-Consensus (\( \Lambda \)) then the algorithm solves Consensus in \( \Lambda \) rounds using messages of \( O(\log n) \) bit size.

**Proof:** The pseudocode in Figure 1 is structured as a loop of \( \Lambda \) iterations, each iteration taking one round. All nodes decide at the end, which gives termination and the time performance.

Each node \( p \) initializes its private variable \( \text{candidate}_p \) to the initial value \( \text{input}_p \). An iteration of the repeat loop maintains the invariant that the variables \( \text{candidate} \) store only values taken from among the original input values, because only the values in this variable are sent and received. Ultimately, each node \( p \) decides on a value in its \( \text{candidate}_p \) variable, which gives validity.

Next, we show agreement. Consider an execution of the algorithm, and let \( G_F \) denote the final graph representing the network just after halting. Suppose there are \( k \) connected components of the graph \( G_F \) at the round of halting. Let \( C_1, \ldots, C_k \) denote these \( k \) connected components, and \( d_1, \ldots, d_k \) be their respective diameters. The assumption gives that the stretch of \( G_F \), which is \( k - 1 + \sum_{i=1}^{k} d_i \), is at most \( \Lambda \).

Suppose, to arrive as a contradiction, that there are two nodes \( p_1 \) and \( p_2 \) that decide on different values, say, \( p_1 \) decides on \( v_1 \) and \( p_2 \) decides on \( v_2 \), while \( p_1 \) and \( p_2 \) are in the same connected
component of $G_F$. We may assume that $p_1$ and $p_2$ both belong to the component $C_k$ and that $v_1 < v_2$. This means that node $p_1$ has never learned about the value $v_2$, as otherwise it would have decided on a value at least as large as $v_2$. Once the value $v_2$ reaches a component $C_i$, among $C_1, \ldots, C_k$, it takes at most $d_i$ rounds for all nodes in the component to learn about a value as large as $v_2$. The value $v_2$ might have hopped between components, for a total of $k - 1$ rounds of such hops, including a hop to $C_k$. This means that $v_2$ reached $C_k$ within $k - 1 + \sum_{i=1}^{k-1} d_i$ initial rounds of the execution. Then it took at most an additional $d_k$ rounds to reach node $p_1$, for a total of $k - 1 + \sum_{i=1}^{k} d_i$ rounds, which is the stretch of $G_F$. This is a contradiction with node $p_1$ deciding on $v_1$, since the assumption about an upper bound on stretch gives $k - 1 + \sum_{i=1}^{k} d_i \leq \Lambda$.

A message carries a candidate value, which is among the input values. By the assumption on properties of input values, each such a value can be encoded with $O(\log n)$ bits.

We show a lower bound on the amount of time required to solve Consensus in networks with link failures. This lower bound is equal to the stretch of the final graph of the network. We consider specific network topologies with the property that with no link failures the time needed to reach agreement is at least a diameter of the network. If a final graph is connected, then its diameter is the same as stretch.

**Lemma 1** For any algorithm $A$ solving Consensus in networks prone to link failures, and for positive integers $D$ and $n \geq 2D$, there exists a network $G$ with $n$ nodes and with diameter $D$ such that some execution of $A$ on $G$ takes at least $D$ rounds with no link failures.

**Proof:** We define graphs $G(x, D)$, where $x > 0$ is an integer. Start with a cycle of length $2D$, and some three consecutive vertices $u$, $v$, and $w$ on the cycle, where $v$ is connected by edges with $u$ and $w$. Replace vertex $v$ by $x$ copies of $v$, each connected precisely to the neighbors $u$ and $w$ of $v$. This construction is depicted in Figure 2 in which $M_x$ denotes $x$ copies of some vertex $v$. Observe that $G(x, D)$ has $2D + x - 1$ vertices and diameter $D$. Take $G = G(x, D)$ such that $n = 2D + x - 1$.

We consider executions of algorithm $A$ on a network modeled by $G$ in which no link ever fails. Each input value of a node in the network will be either 0 or 1. Start with the initial configuration $C_0$ in which all input values are 0, so decision is on 0 by validity. We proceed through a sequence of initial configurations $C_0, C_1, C_2, \ldots, C_n$, such that $C_i$ has $i$ nodes start with initial value 1 and $n - i$ nodes start with value 0. In particular, in configuration $C_n$ all nodes start with input value 1, so a decision has to be on 1, by validity. There exist two configurations $C_i$ and $C_{i+1}$ such that for $C_i$ the decision is on 0 and for $C_{i+1}$ the decision is on 1. These two configurations differ only in some vertex $p$ having input 0 in $C_i$ and input 1 in $C_{i+1}$.

Consider two executions of algorithm $A$, one starting in configuration $C_i$ and the other in $C_{i+1}$. Each node $q$ of distance $D$ from $p$ sends and receives the same messages in the first $D - 1$ rounds in both executions. Take as $q$ a node that is of distance $D$ from $p$; such a node $q$ exists by the specification of graph $G$. Node $q$ needs to wait with decision until at least the round $D$, because its state transitions in the first $D - 1$ rounds are the same in both executions, while the decisions are different. \hfill \Box

**Corollary 1** For any algorithm $A$ solving Consensus in networks prone to link failures, and for any even positive integer $n$, there exists a network $G$ with $n$ nodes such that some execution of $A$ on $G$ takes at least $n/2$ rounds with no link failures.
Proof: We use a network modeled by graph $G(x, D)$ used in the proof of Lemma 1, in which $x = 1$ and $n = 2D$. We have that $D = n/2$ rounds are necessary to reach agreement in some executions of $A$, by Lemma 1.

Theorem 2 For any natural number $\lambda$ and an algorithm $A$ solving Consensus in networks prone to link failures there exists a network $G$ that has stretch $\lambda$ and such that each execution of $A$ on $G$ takes at least $\lambda$ rounds.

Proof: Let us take a network $G$ with $n \geq 2\lambda$ nodes and with diameter $\lambda$ such that some execution of algorithm $A$ on $G$ takes at least $\lambda$ rounds with no link failures. Such a connected network exists by Lemma 1. The stretch of a connected network equals its diameter.

Theorem 2 shows that algorithm Fast-Consensus($\Lambda$) is asymptotically time-optimal on networks prone to link failures, because the actual stretch $\lambda$ could be as large as an upper bound $\Lambda$ on stretch.

4 Consensus with Short Messages

We present a Consensus algorithm using short messages of $O(\log n)$ bit size. Algorithm Fast-Consensus presented in Section 3 which also employs messages of $O(\log n)$ bit size, relies on an upper bound on stretch $\Lambda$ that is a part of code, and if the actual stretch in an execution goes beyond $\Lambda$ then algorithm Fast-Consensus may not be correct. We assume in this section that nodes rely on minimal knowledge only and the given algorithm is correct for arbitrary link-failure patterns and the resulting stretches. The algorithm terminates in at most $n$ rounds, while the number of nodes $n$ is not known; we assume throughout this Section that $n > 1$. The running time is asymptotically optimal, by Corollary 1 in Section 3.

The algorithm is called SM-Consensus, its pseudocode is in Figure 3. Each node $p$ has its name stored as $\text{name}_p$ and initial input value stored at $\text{input}_p$. A node $p$ forms a pair $(\text{name}_p, \text{input}_p)$, which we call an input pair, and sends it through each port as the first communication in an execution; simultaneously node $p$ receives input pairs from its neighbors. As an execution continues, nodes exchange input pairs among themselves.

A node maintains a list $\text{Inputs}$ of all input pairs that the node has ever learned about. For each port $\alpha$, a node maintains a set $\text{Channel}[\alpha]$ storing all input pairs that have either been received
algorithm SM-Consensus

1. initialize: Inputs to empty list; round ← 1; append (name\(_p\), input\(_p\)) to Inputs
2. for each port α do
   initialize set Channel[α] to empty; send (name\(_p\), input\(_p\)) through α
3. for each port α do
   if a pair (name\(_q\), input\(_q\)) received through α then
      add (name\(_q\), input\(_q\)) to Channel[α]; append (name\(_q\), input\(_q\)) to Inputs
4. repeat
   (a) for each port α do
      if some item in Inputs is not in Channel[α] then
         let x be the first such an item; send x through α; add x to Channel[α]
   (b) for each port α do
      if a pair (name\(_q\), input\(_q\)) was just received through α then
         add (name\(_q\), input\(_q\)) to Channel[α]; append (name\(_q\), input\(_q\)) to Inputs
   (c) round ← round + 1
   until round > |Inputs|
5. decide on the maximum input value in Inputs

Figure 3: A pseudocode for a node \(p\). The operation of adding an item to a set is void if the item is already in the set. The operation of appending an item to a list is void if the item is already in the list. The notation |Inputs| means the number of items in list Inputs.

through α or sent through α. At every round, a node verifies for each port α if there is a message to be sent through the port. This is determined by examining the list Inputs to check if there is an input pair in the list that does not belong to the set Channel[α]: if this is the case then the first such a pair is transmitted through α. At every round, a node may receive a message through a port α: if this occurs, the node adds the received input pair to set Channel[α], unless the pair is already there, and appends the pair to the list Inputs, unless the pair is already in the list. A node maintains a counter of rounds called round, which is incremented in each round. An execution ends when this counter surpasses the size of the list Inputs.

Lemma 2 If nodes \(p\) and \(q\) are connected by a resilient path at a round when node \(q\) decides, and the list Inputs\(_p\) includes an input pair (name\(_r\), input\(_r\)) at that round, then the list Inputs\(_q\) includes this pair (name\(_r\), input\(_r\)) at the round when \(q\) decides.

Proof: Let us consider a modification of algorithm SM-Consensus, as presented in Figure 3, in which the repeat loop is iterated indefinitely, rather than stopping after |Inputs| iterations. In an execution of the modified algorithm, eventually lists Inputs and sets Channel[α], for all ports α, stabilize in all nodes, so that messages are no longer sent. By that round, the input pair (name\(_r\), input\(_r\)) will have reached node \(q\), because there is a resilient path from \(p\) to \(q\). We show that this also occurs in an execution of algorithm SM-Consensus without the modification of the condition controlling the repeat loop, for the same link failures in the two executions. To this end, it suffices to demonstrate that node \(q\) withholds deciding long enough.

Let \(γ\) be a path between nodes \(r\) and \(p\) through which pair (name\(_r\), input\(_r\)) arrives to node \(p\) first; suppose \(γ = (s_1,s_2,...,s_k)\), where \(r = s_1\) and \(s_k = p\). Let \(δ\) be a resilient path connecting...
node $p$ to $q$ at a round when node $q$ decides; suppose $\delta = (t_1, t_2, \ldots, t_\ell)$, where $p = t_1$ and $t_\ell = q$. We want to show that all input pairs originating at the nodes on the paths $\gamma$ and $\delta$ reach node $q$ by the round in which $q$ decides. Consider a path $\zeta$ obtained by concatenating path $\gamma$ with path $\delta$ at the node $p = s_k = t_1$. Let us denote the nodes on this path as $\zeta = (u_1, \ldots, u_\ell)$, where $u_1 = r$ and $u_\ell = q$, and denote by $v_i$ a pair of the form $(\text{name}, \text{input})$ originating at $u_i$, for reference.

The simplest scenario, for node $q$ to receive input pair $v_1 = (\text{name}, \text{input}_r)$, occurs when pair $v_1$ gets sent in each consecutive round along the path $\zeta$ until it reaches node $q$. During these transmissions, node $q$ receives consecutive pairs $v_i$, starting from $v_\ell = (\text{name}_q, \text{input}_q)$ originating in $q$, through $v_1 = (\text{name}_r, \text{input}_r)$. In this scenario, a new pair gets added to list $\text{Inputs}_q$ in each of these rounds, which makes node $q$ postpone deciding for a total of at least $\ell$ rounds. We call this the basic scenario. Let us consider conceptual queues $Q_i$ at each of the nodes $u_i$ on $\zeta$, for $i < \ell$, where $Q_i$ stores pairs that still need to be sent to $u_{i+1}$. These queues are manipulated by actions specified in instructions (4a) and (4b) in the pseudocode in Figure 3. In the basic scenario, each queue $Q_i$ is initialized with $u_i$, and then each pair $v_i$ moves through the queues $Q_j$, for $j = i + 1, i + 2, \ldots$ during consecutive rounds, to eventually arrive at $u_\ell = q$.

Next, we consider two possible alternative ways for input pairs to be sent along $\zeta$ to arrive at $q$, while departing from the basic scenario.

One possible departure from the basic scenario occurs when a pair $v_i$ gets delivered to $u_j$, for $j > i$, by a shortcut outside of $\zeta$ rather than through consecutive nodes $u_{i+1}, u_{i+2}, \ldots, u_j$. This may result in pair $v_i$ reaching $q$ earlier than in the basic scenario, while postponing delivery of other pairs by one round. What does not change is that a new pair gets delivered to $q$ at each round, so $q$ keeps waiting, and also $v_i$ gets added to $Q_k$ to be later removed from $Q_k$ exactly once, at each node $u_k$ on the path $\zeta$, for $k \geq i$, while sent down the path towards $q$, by the rules of manipulating lists $\text{Inputs}$ and sets $\text{Channel}$ specified in instructions (4a) and (4b) in the pseudocode in Figure 3. In this scenario, pair $(\text{name}_r, \text{input}_r)$ gets delivered to $q$ no later than in the basic scenario.

Another possible departure occurs when a pair $v$ originating at a node outside $\zeta$ gets delivered to a node on $\zeta$ and then travels along $\zeta$ towards $q$. Each such a pair $v$ simply increases the number of pairs transmitted along $\zeta$, so it may delay $v_1 = (\text{name}_r, \text{input}_r)$ from reaching $q$ by one round. At the same time, such a pair $v$ increments the size of the list $\text{Inputs}$ at $q$ by one when added to it, thereby extending the time period until $q$ decides by one round. This makes node $q$ wait long enough to receive input pair $(\text{name}_r, \text{input}_r)$ by the round it decides.

The two possible departures from the basic scenario discussed above can both occur in an execution, and also there could be multiple instances of pairs creating such scenarios departing from the basic one, without affecting the conclusion that node $q$ waits long enough to receive pair $(\text{name}_r, \text{input}_r)$ prior to deciding. \vspace{-1em}

\begin{theorem}
Algorithm SM-Consensus solves Consensus in $n$ rounds relying on minimal knowledge and using short messages each of $O(\log n)$ bits.
\end{theorem}

\textbf{Proof:} Input pairs stored in the list $\text{Inputs}$ at a node have names of nodes as their first components, so there can be at most $n$ pairs ever added to this list. A node decides immediately when a round number exceeds the size of list $\text{Inputs}$, which occurs by round $n$. This gives termination and the bound on running time. A node decides on an input value in an input pair present in its list $\text{Inputs}$, which stores pairs that originated at nodes of the network; this implies validity. By Lemma 2 all
nodes in a connected component of the final graph $G_F$ have identical lists \textbf{Inputs} at a round of deciding. A decision is on a maximum value in a list, which is uniquely determined by the set of pairs stored in the list; this gives agreement. A message carries one input pair, so it consists of $O(\log n)$ bits, per the assumptions about names and input values. \hfill \Box

5 Consensus with Linear Messages

The algorithm presented in this section uses linear messages of $O(n \log n)$ bit size. The algorithm works in $(\lambda + 2)^3$ time, where $\lambda$ is the stretch occurring at the round of halting. Nodes rely only on the minimal knowledge: each node knows its own name and can distinguish ports by their communication functionality.

\textbf{An overview of the algorithm.} Every node maintains a counter of round numbers, incremented when a round begins. In each round, node $p$ generates a new timestamp $r$ equal to the current value of the round counter, and forms a pair $(\text{name}_p, r)$, which we call a \textit{timestamp pair} of node $p$. Such pairs flow through the network forwarded by all nodes to their neighbors. For the purpose of this forwarding, if node $p$ has a timestamp pair of a node $q$ and $p$ either generates a new timestamp pair for $q$, in case $p = q$, or receives new timestamp pairs for $q$, when $q \neq p$, then $p$ keeps only the pair for $q$ with the greatest timestamp and discards pairs with smaller timestamps.

An execution of the algorithm at a node is partitioned into \textit{epochs}, each being a contiguous interval of rounds. Epochs are not coordinated among nodes, and each node governs its own epochs. The first epoch begins at round zero, and for the following epochs, the last round of an epoch is remembered in order to discern timestamp pairs sent in the following epochs. For the purpose of monitoring progress of discovering the nodes in the connected component during an epoch, each node maintains a separate collection of timestamp pairs, which we call \textit{pairs serving the epoch}. This collection stores only timestamp pairs sent in the current epoch, a pair with the greatest timestamp per node which originally generated the pair.

The \textit{status} of a node $q$ at node $p$ during an epoch can be either absent, valid, or expired; we simply say that node $q$ is absent, valid, or expired at $p$, respectively. If node $p$ does not have a timestamp pair for $q$ serving the epoch then $q$ is \textit{absent}. If at a round of an epoch node $p$ either adds a timestamp pair serving the epoch for an absent node $q$ or replaces a timestamp pair of a node $q$ by a new timestamp pair with a greater timestamp than the previously held one, then $q$ is \textit{valid} at this round. If node $p$ has a timestamp pair for a node $q$ serving the epoch but does not replace it at a round with a different timestamp pair to make it valid, then $q$ is \textit{expired} at this round. If node $p$ at a round $t_1$ receives an epoch-serving timestamp pair $(\text{name}_q, t_2)$ for a node $q$ that replaces a previously stored timestamp for $q$ then the number $t_1 - t_2$ is called the \textit{range} of $q$ at $p$. Ranges are determined only for valid nodes at $p$, since $q$ becomes valid after $p$ receives $(\text{name}_q, t_2)$. A range of $q$ is the length of a path traversed by a timestamp pair of $q$ to reach $p$ in the epoch, at the round the timestamp pair is received, so it is a lower bound on the distance from $q$ to $p$ at this round.

We say that an epoch of node $p$ \textit{stabilizes} at a round if either no new node has its status changed from absent to valid at $p$ or no node gets its range changed at $p$. If an epoch stabilizes at a round, then the epoch ends. During an epoch, a node $p$ builds a set of names of nodes from which it has received timestamp pairs serving this epoch. A similar set produced in the previous epoch is also stored. As an epoch ends, $p$ compares the two sets and decides if they are equal. We want nodes
algorithm LM-Consensus

1. candidate-value\textsubscript{p} ← input\textsubscript{p}, round ← 0, Timestamps ← ∅, Nodes ← ⊥
2. repeat
   (a) epoch ← round, PreviousNodes ← Nodes, EpochTimestamps ← ∅
   (b) repeat
      i. round ← round + 1
      ii. add pair (name\textsubscript{p}, round) to sets Timestamps and EpochTimestamps
      iii. for each port do
         A. send Timestamps and (candidate, candidate-value\textsubscript{p}) through the port
         B. receive messages coming through the port
      iv. for each received pair (candidate, x) do
         if x > candidate-value\textsubscript{p} then assign candidate-value\textsubscript{p} ← x
      v. for each received timestamp pair (node, y) do
         A. add (node, y) to Timestamps if this is a good update
         B. if y > epoch then add (node, y) to EpochTimestamps if this is a good update
   (c) until epoch stabilized at the round
   (d) set Nodes to the set of first coordinates of timestamp pairs in EpochTimestamps
3. until PreviousNodes = Nodes
4. send (decision, candidate-value) through each port
5. decide on candidate-value

Figure 4: A pseudocode for a node p. Each iteration of the main repeat-loop \(2\) makes an epoch. Symbol ⊥ denotes a value different from any actual set of nodes, so the initialization of Nodes to ⊥ in line \(1\) guarantees execution of at least two epochs. A good update of a timestamp pair for a node q either adds a first such a pair for q or replaces a present pair for q with one with a greater timestamp. At each round, p checks to see if a message of the form (decision, z) has been received, and if so then p forwards this message through each port, then decides on z, and halts.

to decide on the maximum input value they have ever learned about.

Implementation of the algorithm. The algorithm is called LM-Consensus, its pseudocode is given in Figure 4. An execution starts with initialization of some variables by instruction \(1\). The main repeat loop follows as instruction \(2\); one iteration of the main repeat-loop makes an epoch. The pseudocode refers to a number of variables; we review them next.

Each node p uses a variable candidate-value\textsubscript{p}, which it initializes to input\textsubscript{p}. Node p creates a pair (candidate, candidate-value\textsubscript{p}), which we call a candidate pair of p. Nodes keep forwarding their candidate pairs to neighbors continually. If a node p receives a candidate pair of some other node with a value x such that x > candidate-value\textsubscript{p} then p sets its candidate-value\textsubscript{p} to x. An execution concludes with deciding by performing instruction \(5\). Just before deciding, a node notifies the neighbors of the decision. Once a notification of a decision is received, the recipient forwards the decision to its neighbors, decides on the same value, and halts.

The variable round is an integer counter of rounds, which is incremented in each iteration of the inner repeat loop by executing instruction \(2(b)\). The round counter is used to generate timestamps. The variable Timestamps stores timestamp pairs that p has received and forwards to its neighbors. The variable EpochTimestamps stores timestamp pairs serving the current epoch.
Each set Timestamps and EpochTimestamps stores at most one timestamp pair per node, the one with the greatest received timestamp. Each iteration of the inner repeat loop (2b) implements one round of sending and collecting messages through each port by executing instruction (2b iii). The inner repeat loop (2b) ends as soon as the epoch stays stable at a round, which is represented by condition (2c).

The variable Nodes stores the names of nodes from which timestamp pairs serving the epoch have been received. The variable Nodes is calculated at the end of an epoch by instruction (2d). The set of nodes in Nodes at the end of an epoch is stored as PreviousNodes at the start of the next epoch. The main repeat loop (2) stops to be iterated as soon as the set of names of nodes stored in Nodes stays the same as the set stored in PreviousNodes, which is checked by condition (3).

We want the set Nodes to be calculated at least twice by at least two consecutive epochs. To guarantee this, PreviousNodes is initialized to a special value denoted ⊥ by instruction (2a) in the pseudocode, when the instruction is executed for the first time. The value ⊥ is defined by the property that it is different from any set of nodes.

Correctness and time performance. Node \( p \) has heard of node \( q \) in an epoch, if node \( p \) received a timestamp pair from \( q \) serving this epoch. We mean an epoch according to the node \( p \), since epochs are not coordinated across the network, and so a different node \( q \) could be in a different epoch by its count of iterations of the main repeat loop (2) in the pseudocode in Figure 4.

Lemma 3 If a node \( p \) has not heard of some node \( q \) by a round of an epoch yet and that node is still connected to \( p \) by a reliable path at the next round, then the epoch continues through the next round.

Proof: The proof is by induction on the round numbers in an epoch. Node \( p \) starts an epoch with an empty set of timestamps pairs serving the epoch and adds at least own timestamp pair to make itself valid at the first round of the epoch. If node \( p \) has neighbors connected to it by reliable links, then, at the first round of the epoch, node \( p \) hears of them and makes valid as well. This means that an epoch never stabilizes at the first round, so it always continues beyond the first round. This provides the base of induction.

Next we consider the inductive step. Let round \( i \) of an epoch executed by node \( p \) be such that node \( p \) has not heard of some node \( q \) in its current connected component up to round \( i - 1 \), and that such a node \( q \) stays connected to \( p \) during round \( i \). If node \( p \) has not heard of node \( q \) by round \( i - 1 \), then \( q \)'s distance from \( p \) is greater than \( i - 1 \) at the beginning of round \( i \), so the distance is at least \( i \). Let \( π = (s_0, s_1, \ldots, s_ℓ) \) be a shortest path from \( p \) to \( q \), that exists at round \( i \), where \( p = s_0 \) and \( s_ℓ = q \), for \( ℓ \geq i \). Node \( s_i \) has its timestamp pair delivered by \( s_1 \) to \( p = s_0 \) at round \( i \), because such a timestamp pair has just completed its traversal of path \( π \) towards \( p \). This arrival establishes the range of \( s_i \) as \( i \), because this is the distance to \( p \) at this round. If node \( p \) has not heard of \( s_i \) before, then \( p \) changes the status of \( s_i \) from absent to valid. If node \( p \) has heard of \( s_i \) before, then the range of \( s_i \) at \( p \) is at most \( i - 1 \) after round \( i - 1 \), so it gets changed to \( i \). The epoch does not stabilize during round \( i \) in either case, so it continues through the next round \( i + 1 \).

Each epoch at a node \( p \) eventually comes to an end. This is because eventually all nodes in the connected component of \( p \) are either valid or expired at \( p \), and their ranges stabilize to ones resulting from link crashes that occurred in the execution up to this point.

Lemma 4 The duration of an epoch at a node is at most \((λ + 2)^2\).
**Proof:** Let us consider a node $p$. At a round $i$ of an epoch, node $p$ has the status of every node of distance at most $i$ at round $i$ set to either valid or expired. Eventually a round $k$ occurs such that the diameter of the connected component of $p$ is $k$, and after this round node $p$ will have heard of every node in its connected component. If a node $q$ is of distance $i$ from $p$ at a round $j$, and this distance stays equal to $i$ for at least $i$ rounds following round $j$, then the range of every node on a shortest path from $p$ to $q$ gets updated to its current value, which happens by round $j + i$. Let $q$ be an arbitrary node. The distance from $q$ to node $p$ can change at most as many times as the diameter of the connected component of $p$ at the round $p$ ends the epoch. After each such a change, it takes up to as many rounds as the distance from $p$ to $q$ to have ranges of nodes on a shortest path from $q$ to $p$ updated to new values.

Let $D$ denote the diameter of the connected component of node $p$ when it ends the epoch. If $D = 0$ then the epoch ends after two rounds. If $D \geq 1$ then the round in which $p$ ends the epoch is at most 

$$D + \sum_{i=1}^{D-1} i^2 = D + \frac{D(D-1)}{2}.$$ 

The obtained bound is at most $D^2$. It follows that an epoch takes at most $(\lambda + 2)^2$ rounds. 

**Theorem 4** Algorithm LM-Consensus solves Consensus in $(\lambda + 2)^3$ rounds relying on minimal knowledge and using messages of $\mathcal{O}(n \log n)$ bit size.

**Proof:** If a node $p$ decides then the decision is on its candidate value stored in the variable $\text{candidate-value}$. Such a decision value is the initial input value of some node, by the instructions (1) and (2(b)iv) in Figure 4. This gives validity.

For agreement, we argue that when a node $p$ decides, then it knows the maximum candidate value of all nodes in its connected component at the round of deciding. The equality of sets Nodes and PreviousNodes, verified as condition [3] in the pseudocode in Figure 4 at the end of the epoch in which $p$ decides, guarantees that the nodes of the connected components at the ends of the current and previous epochs have stayed the same. Consider the maximum candidate value present among the nodes of the connected component of $p$ at the round of its deciding. This candidate value was also maximum among the values held by nodes of the connected component of $p$ in the previous epoch, since no nodes got disconnected from $p$ in the current epoch. Suppose the maximum candidate value was held by a node $q$ in the connected component at the end of the previous epoch. Node $p$ hears from $q$ in the current epoch, by Lemma 3. The candidate value of $q$ travels along the same paths to $p$ as timestamp pairs from $q$. It follows that node $p$ decides on the candidate value of node $q$, and this candidate value is maximum among candidate values of all nodes in the connected component at the round of deciding.

Next, we estimate the running time. A connected component of a node $p$ can evolve through a sequence of contractions, occurring when the connected component shrinks and some nodes get disconnected to make other connected components. The number of epochs for every node is at least two, and it is at most the number of connected components of the final graph plus one, for which the stretch plus one is an upper bound. An epoch at a node $p$ takes at most $(\lambda + 2)^2$ time, by Lemma 4. The number of rounds by halting for a node is a sum of lengths of its epochs, so it is at most $(\lambda + 2)^3$. 

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Finally, we estimate the size of messages. A node sends its variable `Timestamps` and a candidate pair `(candidate, candidate-value)` in a message. The set `Timestamps` includes at most one timestamp pair per node. A node’s name needs $O(\log n)$ bits and a timestamp needs at most $\log n^3 = O(\log n)$ bits, because $\lambda < n$. Each candidate value is some original input of a node, so it also needs $O(\log n)$ bits. □

6 Early Stopping Consensus

We give an early-stopping Consensus algorithm whose time performance $O(\lambda)$ scales optimally to the stretch $\lambda$ occurring in an execution by the time of halting. Nodes rely only on the minimal knowledge, similarly as in algorithms SM-Consensus (in Section 4) and LM-Consensus (in Section 5), but messages carry $O(m \log n)$ bits, which is more than short messages of $O(\log n)$ bits in algorithm SM-Consensus and linear messages of $O(n \log n)$ bits in LM-Consensus.

An overview of the algorithm. A graph whose vertices represent nodes and edges stand for links is like a map of the network at a round. Nodes executing the algorithm keep sending their knowledge of the network’s map to neighbors, while simultaneously receiving similar information from them. The goal for each node is to build an approximation of a map of the network, which we call a snapshot. More precisely, a snapshot of the network at a node $p$ consists of the names of nodes as vertices and edges between vertices representing links, but only these nodes and links that $p$ has heard about. A snapshot at a node $p$ may include the initial input for a vertex, should the node $p$ know the input of a node represented by this vertex. A whole snapshot encoded as a message takes $O(m \log n)$ bits. We explain how nodes manage snapshots next.

The model of minimal knowledge means that initially a node knows only its own name and input, but its ports are not labeled with names of the respective neighbors. To mitigate this, at the first round, every node sends its name through all the communication ports. At this very round, nodes receive the names of their respective neighbors coming through communication ports. This results in every node discovering its neighbors, which they use to assign neighbors’ names to ports. During the first round, nodes do not send their input values. After the first round, a node has a snapshot consisting of its own name, the names of neighbors that submitted their names, and edges connecting the node to its newly discovered neighbors.

Starting from the second round, nodes iterate the following per round: send the current snapshot to each neighbor, receive snapshots from the neighbors, and update the snapshot by incorporating newly acquired knowledge. Links may be marked as unreliable, if they have failed at least once to deliver a transmitted message. Edges representing unreliable links get removed from the snapshot. Such a removal is permanent, in that an edge representing an unreliable link is not restored to the snapshot. A snapshot at a node $p$ determines a part of a connected component to which $p$ belongs, but a snapshot may not be up to date, as links may fail and there could be a delay in $p$ learning about it.

We say that a node $p$ has heard of a node $q$ if $\text{name}_q$ is a vertex in the snapshot at $p$. Each node hears of its neighbors by the end of the first round. A node $q$ that belongs to the same connected component as a node $p$, according to the current snapshot at $p$, and such that $p$ knows the initial input of $q$, is considered as settled by $p$.

A node $p$ executing the algorithm participates in exchanging snapshots with neighbors, as long
Algorithm ES-Consensus

1. Nodes ← \{name_p\}, Inputs ← \{(name_p, input_p)\}, Links ← \emptyset, Unreliable ← \emptyset
2. for each port do
   (a) send name_p through this port
   (b) if name_q received through this port then
        i. assign name_q to the port as a name of the neighbor
        ii. add name_q to Nodes; add edge \{name_p, name_q\} to Links
3. while there exists an unsettled node in p’s connected component in the snapshot do
   for each neighbor q do
       i. send sets Nodes, Links, Unreliable, Inputs to q
       ii. if a message from q was just received then
           update the sets Nodes, Links, Unreliable, Inputs
           by adding new elements included in this message from q
           else add edge \{name_p, name_q\} to Unreliable
4. for each neighbor q do send sets Nodes, Links, Unreliable, Inputs to q
5. decide on the maximum input value at the second coordinate of a pair in Inputs

Figure 5: A pseudocode for a node p. A node q is considered unsettled by p if it is in the same connected component as p, according to the snapshot at p, and there is no pair of the form \(\text{name}_q, \?\) in Inputs_p.

as its connected component contains nodes that have not been settled yet, according to the current snapshot. At the end of the first round, every node considers all its neighbors as pending settling. A soon as a node realizes that all nodes in its connected component according to the snapshot are settled, it sends its snapshot to the neighbors for the last time, finds the maximum input among the nodes in its snapshot, decides on this maximum value, and halts.

Implementation of the algorithm. The algorithm is called ES-Consensus, its pseudocode is given in Figure 5. The pseudocode refers to a number of variables that we introduce next. A set variable Nodes at a node p stores the names of all nodes that node p has ever learned about, and a set variable Links stores links known by p to have transmitted messages successfully at least once, a link is represented as a set of two names of nodes at the endpoints of the link. A set variable Unreliable stores edges representing links known to have failed. Knowledge about failures can be acquired in two ways: either directly, when a neighbor is expected to send a message at a round and no message arrives through the link, or indirectly, contained in a snapshot received from a neighbor. A node stores all known initial input values of nodes q as pairs \(\text{name}_q, \text{input}_q\) in a set variable Inputs. Nodes keep notifying their neighbors of the values of some of their private variables during iterations of the while loop in instruction (3) in Figure 5. A node iterates this loop until all vertices in the connected component of the node are settled, which is sufficient to decide. Once a node is ready to decide, it forwards its snapshot to all neighbors for the last time, decides on the maximum input value in some pair in Inputs, and halts.

An execution of the algorithm starts with each node announcing its name to all neighbors, by executing instruction (2) in Figure 5. This allows every node to discover its neighbors and map its ports to the neighbors’ names. A node does not send its input in the first round of communication. A node sends its snapshot to the neighbors for the first time at the second round, by instruction (3)
in the pseudocode in Figure 5.

Node $p$ has heard of a node $q$ if $\text{name}_q$ is in set $\text{Nodes}_p$. Node $p$ has settled node $q$ once the pair $(\text{name}_q, \text{input}_q)$ is in $\text{Inputs}_p$ and node $q$ belongs to the connected component of $p$ according to its snapshot. We say that node $p$ knows the state of node $q$ at the end of a round $i$ if the following inclusions hold: $\text{Nodes}_q \subseteq \text{Nodes}_p$, $\text{Links}_q \subseteq \text{Links}_p$, $\text{Unreliable}_q \subseteq \text{Unreliable}_p$, and $\text{Inputs}_q \subseteq \text{Inputs}_p$, where $\text{Nodes}_q$, $\text{Links}_q$, $\text{Unreliable}_q$, and $\text{Inputs}_q$ denote the values of these variables at $q$ at the end of round $i$. If node $p$ hears of its neighbor $q$ at the first round, then $p$ knows only the $q$'s name, but does not know either the input or any neighbor of $q$ other than oneself.

Correctness and performance. We show that the algorithm is a correct Consensus solution that is early stopping.

**Lemma 5** Once node $p$ settles a node $q$, then $p$ knows the state of $q$ at the end of the first round.

**Proof:** After the first round, the set $\text{Inputs}$ at $q$ includes the only pair $(\text{name}_q, \text{input}_q)$. This is because of the initialization in instruction (1) of the pseudocode in Figure 5, and since $q$ does not receive the neighbors' inputs at the first round, by instructions (2). Node $q$ learns the names of its neighbors during the first round, which $q$ uses to populate $\text{Nodes}$ and $\text{Links}$. During the first iteration of the while loop in Figure 5, which occurs at the second round, node $q$ sends the pair $(\text{name}_q, \text{input}_q)$ to the neighbors, along with the contents of sets $\text{Nodes}$, $\text{Links}$, and $\text{Unreliable}$, as they were at the end of the first round. Pair $(\text{name}_q, \text{input}_q)$ spreads through the network carried in snapshots sent to neighbors, along with the contents of the sets $\text{Nodes}$, $\text{Links}$, and $\text{Unreliable}$ at node $q$. When a pair $(\text{name}_q, \text{input}_q)$ reaches node $p$ for the first time, the contributions of the sets $\text{Inputs}_q$, $\text{Nodes}_q$, $\text{Links}_q$, and $\text{Unreliable}_q$ to the received snapshot are as from these set variables at the end of the first round at $q$.

Nodes settle their neighbors by the end of the second round, after the first iteration of the while loop in Figure 5 as long as the links to these neighbors have not failed. As the while loop iterates in consecutive rounds, once a node $p$ hears of some node $q$ at a round $i$, then the earliest $p$ is expected to settle node $q$ is at the next round $i + 1$. To see this, observe that a node $p$ hears of its neighbors at the first round and settles them in the second round, unless some links to neighbors failed. A failure of a link to a neighbor may postpone settling this neighbor, if its input value manages to reach $p$ eventually, or $p$ may never settle this neighbor, if it gets disconnected from $p$. The delay of at least one round between hearing of a node and settling this node is maintained through each iteration of the while loop. Once node $p$ settles a node $q$, it learns the state of $q$ at the end of the first round, by Lemma 5. Such a state may include names of nodes in $\text{Nodes}_q$ that are unsettled by $p$ yet; if this is the case then node $p$ continues iterating the while loop.

After each round, a node builds a snapshot as an approximation of the network’s topology. The vertices of this graph are names of nodes from the set $\text{Nodes}$, and these links in the set $\text{Links}$ that are not in $\text{Unreliable}$ make the edges. We say that a node $p$ completes survey of the network by a round if $p$ has settled all nodes in its connected component according to the snapshot of this round. A node keeps communicating with neighbors until it completes survey, and then one more time, by instruction (4) in the pseudocode in Figure 5. This extra round of communication serves the purpose to help neighbors complete their surveys in turn, as they may need the information that has just allowed the sender to complete its survey. Finally, a node decides on the maximum from the set of all input values stored in pairs in $\text{Inputs}$, and halts.
Lemma 6 If node $p$ decides on $\text{input}_r$, another node $q$ also decides, and nodes $p$ and $q$ are connected by a reliable path at a round when they have already decided, then $q$ has node $r$ as settled in its snapshot at this round.

Proof: Suppose $q$ does not have $r$ in its snapshot as settled at the first round in which both $p$ and $q$ have completed surveys, to arrive at a contradiction. Node $p$ has $q$ in its connected component of the snapshot and, similarly, node $q$ has $p$ in its connected component of the snapshot, because they are connected by a reliable path at the first round after completing surveys. Let $\gamma = (s_1, \ldots, s_k)$ be a reliable path connecting $p = s_1$ with $q = s_k$ at the first round in which both $p$ and $q$ have completed surveys and such that $q$ settled $p$ by a snapshot that arrived through this path. Let a node $s_i$ on the path $\gamma$ be such that $i$ is the greatest index $j$ of a node $s_j$ in $\gamma$ such that $s_j$ settled $r$ in its snapshot. An index $i$ with this property exists because node $p = s_1$ is such. Moreover, the inequality $i < k$ holds because $q = s_k$ does not have $r$ settled in its snapshot. There is a reliable path $\delta = (t_1, \ldots, t_m)$ from $r = t_1$ to $s_i = t_m$ through which a snapshot arrived first bringing $\text{input}_r$ to make $s_i$ settle $r$. Consider a path $\zeta$ obtained by concatenating $\delta$ with a part of $\gamma$ starting at $s_i$ and ending at $s_k = q$, where $i < k$. Let us denote the nodes on this path as $(u_1, \ldots, u_\ell) = \zeta$, where $u_1 = r$ and $u_\ell = q$, for reference.

We examine the flow of information along $\zeta$ from $r = u_1$ towards $u_\ell = q$. At the first round, node $u_1 = q$ learns the name of its neighbor $u_{\ell-1}$. At the second round, node $u_\ell$ settles $u_{\ell-1}$ and learns of the node $u_{\ell-2}$, by Lemma 5. In general, a node $u_j$, such that $j > 1$, hears of its neighbor $u_{j-1}$ at the first round and settles it at the second round. At a round a node $u_j$ settles its neighbor $u_{j-1}$, it also hears of the node $u_{j-2}$ as still unsettled. This creates a chain of dependencies such that node $u_j$ heard of a node up the path $\zeta$ towards $r$ that is still unsettled and in the same connected component in its snapshot.

As snapshots with $\text{input}_r$ move along $\zeta$ towards $q$, this chain of dependencies, starting at a node that received $\text{input}_r$ most recently and ending at $q$, stays unbroken. This is because of the following two reasons. First, the part of $\zeta$ taken from $\delta$ provides reliable edges at all times, since $\text{input}_r$ manages to reach $s_i$: the only possibility of this not being the case would be to settle a node on this path via a different shorter path to $s_i$, but this is a shortest path by its choice. Second, the part consisting of $\gamma$ provides reliable edges during the considered rounds, since these edges are still reliable when $q$ completes survey. This makes node $q$ eventually hear of $r$, and receive $\text{input}_r$ at the next round, which is a contradiction.

$\square$

Theorem 5 Algorithm ES-Consensus is an early stopping solution of Consensus that relies on minimal knowledge and uses messages carrying $\mathcal{O}(m \log n)$ bits.

Proof: A node decides on an input value from its snapshot, which gives validity.

We show agreement as follows. Consider two nodes $p$ and $q$ that are connected by a reliable path at the first round when each of these nodes has already decided. Suppose, to arrive at a contradiction, that node $p$ decided on a value that is greater than the value that node $q$ decided on. Let $r$ be the node that provided its $\text{input}_r$ as the decision value for $p$. By Lemma 6, node $q$ has node $r$ settled at the round of deciding, so $q$ decides on a value that is at least as large as $\text{input}_r$, which is a contradiction.

Next, we estimate the number of rounds needed for each node to halt. As an execution proceeds, information flows through the connected components of the network, by iteratively sending
a snapshot to neighbors and updating it at the same round. A value that gets decided on, in a particular connected component, may travel along a path that shares its parts with multiple connected components. If such a path crosses a connected component then its length is upper bounded by the connected component’s diameter. If a link on such a path fails, this may occur after the future decision value got transmitted through this link, and the endpoints of this link may belong to different connected components. This shows that the number of hops a decision value makes on its way between a pair of nodes it at most the stretch. There are only two rounds that cannot be accounted for by this counting: the first round, during which nodes discover their neighbors, and the last round, when a node notifies its neighbors of its snapshot for the last time, so if an execution terminates at round $t$, then the stretch at this round is at least $t - 2$. It follows that the algorithm terminates by round $\lambda + 2$.

7 Optimizing Link Use

We present an algorithm solving Consensus with an optimal $O(n)$ link use. We depart from the model of minimal knowledge used in the previous sections and assume that nodes know their neighbors at the outset, in having names of the corresponding neighbors associated with all ports. We identify links, determined by ports of a node, by names of the respective neighbors of the node, and use the terms ports and incident links interchangeably. The Consensus algorithm we describe next makes nodes halt in $O(nm)$ rounds. We complement the algorithm by showing that optimal link use $O(n)$ is only possible when each node starts with a mapping of ports on its neighbors, because otherwise $\Omega(m)$ is a lower bound on the link use. We also show that no algorithm can simultaneously be early stopping and have the optimal $O(n)$ link use.

An overview of the algorithm. A node categorizes its incident links as either passive, active or unreliable; these are exclusive categories that evolve in time. An active link is used to send messages through it, so a node categorizes an incident link as active once it receives a message through it. Initially, one link incident to a node is considered as active by the node, and all the remaining incident links are considered passive. A link is passive at a round if none of its endpoint nodes has ever attempted a transmission through this link. A node transmits through an active port at every round, unless the node decides and halts. It follows that if a node $p$ considers a link active, which connects it to a neighbor $q$, then $q$ considers the link active as well, possibly with a delay of one round. Similarly, if a node $p$ considers a link passive, which connects it to a neighbor $q$, then $q$ considers the link passive as well, possibly for one round longer than $p$. A node $p$ detects a failure of an active link and begins to consider it unreliable after the link fails to deliver a message to $p$ as it should. For an active link connecting $p$ with $q$, once node $p$ considers the link unreliable then $q$ considers the link unreliable as well, possibly with a delay of one round.

The state of a node $p$ at a round consists of its name, the input value, and a set of its neighbors, with each incident link categorized as either passive, active, or unreliable, representing this categorization of links by node $p$ at the round. States of a node may evolve in time, in that an incident link may change its categorization. Links start as passive, except for one incident link per node initialized as active, then they may become active, and finally they may become unreliable.

A snapshot of the network at a node represents the node’s knowledge of its connected component in the network restricted to active edges and states of its nodes. Formally, a snapshot of network at
A node $p$ at a round is a collection of states of some nodes that $p$ has received and stores. A snapshot allows to create a map of a portion of the network, which is a graph with names of nodes as vertices and edges representing links. This map can include input values of some nodes, should they become known. A part of such a map is a connected component of a node with nodes reachable by active links. Formally, an active connected component of node $p$ at a round is a connected component, of the vertex representing $p$, in a graph that is a map of the network according to the snapshot of $p$ at the round with only active links represented by edges.

A node $p$ sends a summary of its knowledge of the states of nodes in the network to neighbors through all its active links at each round. If $p$ receives a message with such knowledge from a neighbor, then $p$ updates its knowledge and the snapshot by incorporating newly learned information. Such new information may include either a state of a node $q$ such that $p$ has never had a state of $q$, or a subsequent state of node $q$ such that $p$ has already had some state of $q$. At each round, node $p$ determines its active connected component based on the current snapshot. If we refer to an active connected component of $p$ then this means acting connected component according to the current snapshot. We say that a node $p$ has heard of a node $q$ if name $q$ occurs in the snapshot at $p$; node $p$ may either store some $q$’s state or $q$’s name may belong to a state of some other node that $p$ stores. A node $p$ considers another node $q$ settled if $p$ has $q$’s state in its snapshot. A node $p$ considers its active connected component settled if $p$ has settled all nodes in its active connected component.

If node $p$ has heard about another node $q$ such that $q$ does not belong to node $p$’s active connected component, but it is connected to a node $r$ in the active connected component by a passive link, then node $p$ considers the link connecting $q$ to $r$ as outgoing. If there is an outgoing link in $p$’s active connected component then $p$ considers its active connected component extendible, otherwise $p$ considers its active connected component enclosed.

The idea of the algorithm is for all nodes to participate in making some outgoing links active, each time the active connected component is settled and still extendible. All nodes in the active connected component of a node $p$ can choose the same outgoing link to make it active, once the active connected component becomes settled, because each node knows the same set of outgoing links. Once $p$’s active connected component becomes settled and enclosed then $p$ can decide.

Implementation of the algorithm. The algorithm is called OL-Consensus, its pseudocode is in Figure 6. Each node stores links it knows as unreliable in a set Unreliable, initialized to the empty set. Each node stores links it considers active in a set Active, initialized to some incident link. Each node stores passive links in a set Passive, which a node initializes to the set of all incident links except for the one initially activated link.

All nodes maintain a variable round as a counter of rounds. In each round, a node creates a timestamp pair which consists of its current state and the value of round counter used as a timestamp. A node $p$ stores timestamp pairs in a set Timestamps. For each node $q$ different from $p$, a node $p$ stores a timestamp pair for $q$ if such a pair arrived in messages and only one pair with the largest timestamp. These variables are initialized by instruction (1) in Figure 6.

Initialization in an execution is followed by iterating a loop performed by instruction (2) in the pseudocode in Figure 6. The purpose of an iteration is to identify a new settled active connected component; we call an iteration epoch. An epoch is determined by the round in which it started, remembered in the variable epoch by instruction (2a). The knowledge of an active connected component of a node $p$ identified in an epoch is stored in a set Snapshot, which is initialized at the outset of an epoch to just $p$’s state by instruction (2a). This knowledge is represented as a
algorithm OL-Consensus

1. Unreliable ← ∅, Active ← \{\{p, q\}\} where \(q\) is some neighbor, Passive ← set of links to \(p\)'s neighbors, except for the neighbor \(q\) used in Active, state ← (name\(_p\), input\(_p\), Active, Passive, Unreliable), round ← 0, timestamp ← (state, round)

2. repeat
   (a) epoch ← round, Snapshot ← \{state\}
   (b) repeat
      i. round ← round + 1, add timestamp to set Timestamps
      ii. for each incident link \(\alpha\) do
         A. if \(\alpha\) is in Active then send Timestamps through \(\alpha\)
         B. if \(\alpha\) is mature in Active and no message received through \(\alpha\) then move \(\alpha\) to Unreliable
         C. if a message received through \(\alpha\) then place \(\alpha\) in Active
      iii. for each received timestamp pair (state, \(y\)) do
         A. add (state, \(y\)) to Timestamps
         B. if \(y >\) epoch then add state to Snapshot
   (c) until the active connected component is settled
   (d) if the active connected component is extendible then
      i. identify an outgoing edge as a connector
      ii. if the connector is incident to \(p\) then place it in Active
3. until the active connected component is enclosed
4. set decision-value to the maximum input value in Snapshot
5. send pair (decision, decision-value) through each active incident link
6. decide on decision-value

Figure 6: A pseudocode for a node \(p\). We assume that \(p\) has at least one neighbor. In each round, node \(p\) checks to see if a pair of the form \((\text{decision}, z)\) has been received, and if so then \(p\) forwards this pair through each active port, decides on \(z\), and halts.

collection of states of nodes that arrived to \(p\) in timestamp pairs, with timestamps indicating that they were created after the start of the current epoch, as verified by instruction \((2b)iiiB\). The main part of an epoch is implemented as an inner repeat loop \((2b)\). An iteration of this loop implements a round of communication with neighbors through active links and updating the state by instruction \((2b)ii\). An incident link in Active is mature if either it became active because a message arrived through it or \(p\) made it active spontaneously at some round \(i\) and the current round is at least \(i + 2\). If a mature active link fails to deliver a message then \(p\) moves it to Unreliable.

A set variable Timestamps stores timestamp pairs that a node sends in each message and updates after receiving messages at a round. A set variable Snapshot is used to construct an active connected component. Snapshot is rebuilt in each epoch, starting only with the current \(p\)'s state. We separate storing timestamp pairs in a set Timestamps used for communication from storing states in Snapshot to build an active connected component because nodes do not synchronize epochs among themselves, to facilitate proper advancement of epochs in other nodes.

We say that node \(p\) completes survey of the network by a round if \(p\) has settled all nodes in its active connected component according to the snapshot of this round. The inner repeat loop \((2b)ii\).
terminates once \( p \) completes survey of the network, by condition (2c) controlling the loop. If the active connected component is extendible, then \( p \) identifies a connector which is an outgoing edge to be made active. We may identify an outgoing edge that is minimal with respect to lexicographic order among all outgoing links for a settled active connected component to be designated as a connector. If a connector is a link incident to \( p \) then \( p \) moves it to set Active, by instruction (2d).

Once an epoch ends, by condition (2c), and the active connected component is enclosed, then the main repeat loop ends, by condition (3) controlling the loop. At this point, node \( p \) is ready to decide, and the decision is on the maximum input value in a state stored in \textit{Snapshot}, by instruction (4). Node \( p \) notifies each neighbor connected by an active link of the decision value, by instruction (5), and decides, by instruction (6). The pseudocode in Figure 6 omits what pertains to handling messages that could be generated in round (5) by some nodes. Namely, as a first thing at a round, node \( p \) verifies if a pair of the form \((\text{decision}, z)\) has been received in the previous round, and if so then \( p \) forwards this pair through all active ports, decides on this value \( z \), and halts.

**Correctness and performance bounds.** We show that algorithm \textsc{OL-Consensus} is a correct solution to Consensus, and estimate its performance. The algorithm uses a similar paradigm to survey connected components as algorithm \textsc{ES-Consensus}. Nodes first learn of nodes names and later of their input values. This creates a chain of dependencies along paths through connected components, which in the case of algorithm \textsc{OL-Consensus} consist of active links.

**Lemma 7** If two nodes \( p \) and \( q \) are connected by a reliable path just after they both decided then each of them counts the other node in its last active connected component.

**Proof:** Let us consider an arbitrary reliable path from \( p \) to \( q \). Each link on this path is either active or passive. This is because once a passive link becomes active, then a message is sent through it in each round, so a failure is immediately detected. A passive link is never activated only if it already connects two nodes in the same active connected component. An active link makes its endpoints belong to the same active connected component.

The following Lemma 8 is analogous to Lemma 6 about algorithm \textsc{ES-Consensus}, which also uses sufficiently large messages to build a map approximating the network. We need it to show agreement. A proof may be structured similarly to that of Lemma 6, we include a detailed argument for the sake of completeness.

**Lemma 8** If a node \( p \) decides on input \( r \), another node \( q \) also decides, and nodes \( p \) and \( q \) are connected by a reliable path at a round when they have already decided, then \( q \) has node \( r \) as settled in its snapshot at this round.

**Proof:** Suppose \( q \) does not have \( r \) in its snapshot as settled at the first round in which both \( p \) and \( q \) have completed surveys, to arrive at a contradiction. Nodes \( p \) and \( q \) have each other in their active connected components, by Lemma 7. Let \( \gamma = (s_1, \ldots, s_k) \) be a path consisting of active links that connects \( p = s_1 \) with \( q = s_k \) just after both \( p \) and \( q \) have completed surveys and such that \( q \) settled \( p \) by a chain of Timestamps that arrived through this path. Let node \( s_i \) on the path \( \gamma \) be such that \( i \) is the greatest index \( j \) of a node \( s_j \) in \( \gamma \) that settled \( r \) in its snapshot. Such an index \( i \) exists because node \( p = s_1 \) has this property. The inequality \( i < k \) holds because \( q = s_k \) does not have \( r \) settled in its snapshot. There is a path \( \delta = (t_1, \ldots, t_m) \) from \( r = t_1 \) to \( s_i = t_m \) consisting of
active links through which Timestamps arrived first bringing state, to make $s_i$ settle $r$. Consider a path $\zeta$ consisting of active links obtained by concatenating $\delta$ with a part of $\gamma$ starting at $s_i$ and ending at $s_k = q$, where $i < k$. Let us denote the nodes on this path as $(u_1, \ldots, u_\ell) = \zeta$, where $u_1 = r$ and $u_\ell = q$.

At the first round, node $u_\ell = q$ learns a state of its neighbor $u_{\ell-1}$. At the second round, node $u_\ell$ settles $u_{\ell-1}$ and hears of the node $u_{\ell-2}$. In general, a node $u_j$, such that $j > 1$, hears of its neighbor $u_{j-1}$ at the first round and settles it at the second round. If a node $u_j$ settles its neighbor $u_{j-1}$ at a round then it also hears of the node $u_{j-2}$ as still unsettled. This creates a chain of dependencies such that node $u_j$ heard of a node up the path $\zeta$ towards $r$ that is still unsettled and in the same connected component in its snapshot.

As Timestamps with state, move along $\zeta$ towards $q$, this chain of dependencies, starting at a node that received state, most recently and ending at $q$, stays unbroken. This is because of the following two reasons. First, the part of $\zeta$ taken from $\delta$ provides active edges at all times, since state, manages to reach $s_i$: the only possibility of this not being the case would be to settle a node on this path via a different shorter path to $s_i$, but this is a shortest path by its choice. Second, the part consisting of $\gamma$ provides active edges during the considered rounds, since these edges are still reliable when $q$ completes survey. This makes node $q$ to eventually hear of $r$, and receive state, at the next round, which is a contradiction. □

We consider an auxiliary activation process on a graph that models activation of connectors in an execution of algorithm OL-Consensus. The process acts on a given simple graph $H$ that has some $k$ vertices and proceeds through rounds. An edge of $H$ may progress through three states, first passive, then possibly active, and then possibly be deleted. We consider subgraphs determined by active edges, and connected components that we call active connected components. An edge connecting a vertex in a connected component $C$ to a vertex in a different connected component is considered as outgoing from $C$. In the beginning of a round, if an active connected component has an outgoing edge, then one such an edge is made active. At the end of a round, some active edges may be deleted. The process continues until there are no outgoing edges.

**Lemma 9** In the course of the activation process on a graph with $k$ vertices, the number of active edges in the graph at any round is at most $2k - 2$.

**Proof:** We start with no active edges, so each vertex is an active connected component and $k$ is the number of such components. Consider activation of new edges at a round as occurring sequentially. As an edge is activated, it may connect two different active connected components, thus decreasing their number, or it may close a cycle. Suppose an active connected component $C_1$ gets connected to active connected component $C_2$, then $C_2$ to $C_3$, and so on through $C_i$, with the edge activated in $C_i$ connecting $C_i$ to some $C_j$, for $1 \leq j < i$. That last edge from a vertex in $C_i$ to a vertex in $C_j$ is not needed to make all $C_i$, for $1 \leq j \leq i$, into one connected component, so we treat it as an extra edge. A number of such extra edges created at a round is maximized when $i = 2$, because for each decrease of the number of connected components with one activated edge we also activate another edge. A tree minimizes the number of connected components to one, and a tree on $k$ vertices has $k - 1$ edges. We obtain that the number of active edges at each round is at most twice the number of edges in a tree of $k$ vertices, which is $2(k - 1)$. □
**Theorem 6** Algorithm OL-Consensus solves Consensus in $O(nm)$ rounds with less than $2n$ links used at any round and sending messages of $O(m \log n)$ bits.

**Proof:** A node decides on an input value from a state in its snapshot. All states include original input values, which gives validity.

Consider two nodes $p$ and $q$ connected by a reliable path at the first round when each of these nodes has already decided. We want to show that $p$ and $q$ decide on the same value. Suppose, to arrive at a contradiction, that node $p$ decided on a value that is greater than the value that node $q$ decided on. Let $r$ be the node whose state provided its input value as the decision value for $p$. By Lemma 8, node $q$ has node $r$ settled at the round of deciding, so $q$ decides on a value that is at least as large as $\text{input}_r$, which is a contradiction. This gives agreement.

As an execution proceeds, information flows through each active connected component, by iteratively sending timestamp pairs to neighbors via active links and simultaneously updating latest states in timestamp pairs. The length of a path of active links traversed by a timestamp pair may be as long as the number of nodes in an active connected component minus one. This means that the duration of an epoch is less than $n$ rounds. After a new connector is added, it takes the length of an epoch for all nodes to settle on the state of nodes in an active connected component. There may be up to $m$ links added as connectors. This means that every node halts in $O(nm)$ rounds.

The process of making links active is modeled by the activation process on a graph representing the network. By Lemma 9, the number of links that are active at the same time is less than $2n$. □

**Lower bounds for link usage.** We now switch to a situation when the destinations of ports are not initially known to nodes. For any positive integers $n$ and $m$ such that $m = O(n^2)$, we design a graph $\mathcal{G}(n, m)$ with $\Theta(n)$ vertices and $\Theta(m)$ edges, which makes any Consensus solution to use $\Theta(m)$ links even if the nodes know the parameters $n$ and $m$. We drop the parameters $n$ and $m$ from the notation $\mathcal{G}(n, m)$ and simply use $\mathcal{G}$ whenever they are fixed and understood from context.

Consider any positive integers $n$ and $m$ such that $m = O(n^2)$. Let graph $\mathcal{G}$ consist of two identical parts $G_1$ and $G_2$ as subgraphs. The parts are $\lceil \frac{m}{n} \rceil$-regular graphs of $\lceil \frac{n}{2} \rceil$ vertices each. Without loss of generality, we can assume that the number $\lceil \frac{m}{n} \rceil$ is even, to guarantee that such regular graphs exist. Graph $\mathcal{G}$ is obtained by connecting $G_1$ and $G_2$ with $\lceil \frac{n}{2} \rceil$ edges such that each vertex from $G_1$ has exactly one neighbor in $G_2$.

By the construction, graph $\mathcal{G}$ has $2 \lceil \frac{n}{2} \rceil = \Theta(n)$ vertices and $(\lceil \frac{m}{n} \rceil + 1) \lceil \frac{n}{2} \rceil = \Theta(m)$ edges. Let us assume now that the destinations of outgoing links are not initially known to the nodes. This means that ports can be associated with neighbors’s names only after receiving messages through them. The following Theorem 7 holds even if $n$ and $m$ can be a part of code.

**Theorem 7** For any Consensus algorithm $A$ relying on minimal knowledge, and positive integer numbers $n$ and $m$ such that $n \leq m$ and $m \leq n^2$, there exists a network $\mathcal{G}(n, m)$ with $\Theta(n)$ nodes and $\Theta(m)$ links and an execution of algorithm $A$ on $\mathcal{G}(n, m)$ that uses $\Theta(m)$ links.

**Proof:** If node $p$ executing $A$ sends a message by an incident link, through which it has not received nor sent any message yet, then the receiving node could be any original neighbor of node $p$ in the network $\mathcal{G} = \mathcal{G}(n, m)$ which has not communicated with $p$ yet. We refer as uncovered neighbors of $p$ at a round to these among $p$’s neighbors that $p$ has not received a message from nor sent a direct
message to yet. Choosing which uncovered neighbor of node \( p \) is connected by a link, identifiable only by its port at \( p \), is a part of an adversarial strategy of failing links.

Algorithm \( \mathcal{A} \) gets executed on some initial configurations of network \( \mathcal{G} \), as defined above. Let the notation \( \mathcal{C}_{i,j} \) denote an initial configuration in which the nodes from part \( G_1 \) have \( i \in \{0,1\} \) as their input values while the nodes from part \( G_2 \) have \( j \in \{0,1\} \). We call a link between two nodes in \( G_1 \) unused, by a given round, if none of its endpoints has tried to send a message through it yet. Let \( k_1 \) denote the number of nodes in \( G_1 \) that have at least one unused link incident to some other node in \( G_2 \). At the beginning, \( k_1 = |G_1| \) and \( k_1 \) may decrease in the course of an execution.

For any initial configuration \( \mathcal{C}_{i,j} \), where \( i,j \in \{0,1\} \), consider the following adversarial strategy to fail links. Let a node \( p \) be in \( G_1 \) and let \( \ell \) denote a link that \( p \) wants to send a message through at a round. Suppose node \( p \) has at least two unused links but wants to send a message by only one of them. The adversary allows the message to be delivered and the other endpoint of \( \ell \) is chosen to be an arbitrary uncovered neighbor of \( p \) in \( G_1 \). Suppose there is only one unused link incident to \( p \) and node \( p \) wants to send a message through it. Then, if \( k_1 > 1 \), let this link \( \ell \) fail before it delivers a message, and otherwise, if \( k_1 = 1 \) then let link \( \ell \) deliver the message to a remaining unassigned neighbor of \( p \). It follows that in this case the message must go outside \( G_1 \). Each time number \( k_1 \) decreases by one, then all but one reliable links incident to some node of \( G_1 \) have been used and they will not be failed by the adversary in the continuation of the execution. Hence, before the first message is delivered from some node in part \( G_1 \) to some node in part \( G_2 \), at least \( (\lceil \frac{m}{n} \rceil + 1) \cdot (|G_1| - 1) \) reliable links will have been used for communication between nodes in \( G_1 \). A similar reasoning applies to communication within part \( G_2 \) of \( \mathcal{G} \).

Assume now, to arrive at a contradiction, that in all executions starting from the configurations \( \mathcal{C}_{0,0} \), \( \mathcal{C}_{0,1} \), and \( \mathcal{C}_{1,1} \), in which the above strategy has been used, fewer than \( (\lceil \frac{m}{n} \rceil + 1) \cdot (|G_1| - 1) \) non-faulty links have been used. We call these executions \( \mathcal{E}_0 \), \( \mathcal{E}_b \), and \( \mathcal{E}_1 \), respectively. Nodes in part \( G_1 \) do not communicate with nodes in \( G_2 \) in any of these executions. For nodes in part \( G_1 \), an execution starting from \( \mathcal{C}_{0,0} \) is indistinguishable from an execution starting from \( \mathcal{C}_{0,1} \). Similarly, for nodes in part \( G_2 \), an execution starting from \( \mathcal{C}_{0,1} \) is indistinguishable from an execution starting from \( \mathcal{C}_{1,1} \). It follows that each node in part \( G_1 \) decides on the same value in execution \( \mathcal{E}_b \) as in execution \( \mathcal{E}_0 \), and a decision has to be on 0 by validity. Each node in part \( G_2 \) decides on the same value as in execution \( \mathcal{E}_1 \), and a decision is on 1, by validity. This is a contradiction with agreement, as both parts \( G_1 \) and \( G_2 \) are connected in \( \mathcal{E}_b \) and have to decide on the same value. Therefore, in one of the considered executions, more than these many reliable links have to be used: \( (\lceil \frac{m}{n} \rceil + 1) \cdot (|G_1| - 1) = (\lceil \frac{m}{n} \rceil + 1) \cdot (|G_2| - 1), \) which is \( \Theta(m) \).

\begin{proof}
Let \( \mathcal{A} \) be a Consensus algorithm that uses \( O(n) \) reliable links concurrently when executed in networks with \( n \) nodes. For all natural numbers \( n \) and \( \lambda \leq n \), there exists a network \( \mathcal{G} \) with stretch \( \lambda \) on which an execution of algorithm \( \mathcal{A} \) takes \( \Omega(n) \) rounds.

Let \( n \) and \( \lambda \leq n \) be natural numbers; we assume both \( n \) and \( \lambda \) are even to simplify the notations. Let \( H_1 \) be a network with \( \frac{n}{2} \) nodes and diameter \( \frac{3}{2} - 1 \). Let \( H_2 \) be a copy of \( H_1 \). Let us form a network \( \mathcal{G} \) by taking nodes in \( H_1 \) and \( H_2 \) and adding all possible links between nodes in \( H_1 \) and \( H_2 \). The diameter of \( \mathcal{G} \) is at most \( 2 \cdot (\frac{3}{2} - 1) + 1 = \lambda - 1 \).

The following strategy for an adversary to fail links is applied. For a round \( i \), let \( K_i \) be a set of all reliable links between \( H_1 \) and \( H_2 \) by which nodes attempt to send messages at this round. The
adversary fails all links from \( K \) before any message arrives, as long as \( H_1 \) and \( H_2 \) stay connected. Because there are \( \frac{n^2}{4} \) links between nodes in parts \( H_1 \) and \( H_2 \), this guarantees that no two nodes, of which one is in \( H_1 \) and the other in \( H_2 \), exchange a message during at least \( \Omega\left(\frac{n^2}{4}\right) = \Omega(n) \) rounds.

Let us consider the following three initial configurations of \( G \). In the first configuration \( I_1 \), all nodes start with the input value 0. In the second configuration \( I_2 \), all nodes start with inputs 1. In the third configuration \( I_3 \), nodes from \( H_1 \) start with input 0, while nodes from \( H_2 \) start with input 1. We consider executions of algorithm \( A \) starting with each of the initial configurations \( I_k \), for \( k = 1, 2, 3 \), when the adversary applies the strategy described above to fail edges; let \( E_k \) be the respective execution. If all nodes halt before some pair of nodes such that one is from \( H_1 \) and the other is from \( H_2 \) communicate among themselves, then the nodes in \( H_1 \) cannot distinguish execution \( E_1 \) from \( E_3 \) and the nodes in \( H_2 \) cannot distinguish execution \( E_2 \) from \( E_3 \). Let us consider decisions by nodes in execution \( E_3 \). The nodes in \( H_1 \) decide on 0, as they should in \( E_1 \), by validity, while the nodes in \( H_2 \) decide on 1, as they should in \( E_2 \), by validity. This contradicts the requirement of agreement.

We conclude by settling the question if an algorithm can be simultaneously early stopping and optimize link use.

**Corollary 2** If a Consensus algorithm uses \( \mathcal{O}(n) \) reliable links concurrently at any time, when executed in networks of \( n \) nodes, then this algorithm cannot be early stopping.

**Proof:** Let us consider integers \( n \) and \( \lambda_n \leq n \) such that \( \lambda_n = o(n) \). By Theorem 8 the algorithm works in \( \Omega(n) \) rounds. The algorithm cannot be early stopping, because this would mean running in time \( \mathcal{O}(\lambda_n) = o(n) \).

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