Comonads and Galois corings

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Introduction

The notion of a coring was introduced by M. E. Sweedler in [20] with the objective of formulating and proving a predual to the Jacobson-Bourbaki theorem for extensions of division rings. A fundamental argument in [20] is the following: given division rings $E \subseteq A$, each coideal $J$ of the $A$–coring $A \otimes_E A$ gives rise to a factor coring $\mathfrak{C} = A \otimes_E A/J$. If $g \in \mathfrak{C}$ denotes the group-like element $1 \otimes_E 1 + J$, then $D = \{ a \in A : ag = ga \}$ is an intermediate division ring $E \subseteq D \subseteq A$. Moreover, we have the canonical homomorphism of $A$–corings $\zeta : A \otimes_D A \to \mathfrak{C}$ which sends $1 \otimes_D 1$ onto $g$. It follows from [20, 2.2 Fundamental Lemma] that $\zeta$ is an isomorphism of $A$–corings. This fact is basic to establish the bijective correspondence between coideals of $A \otimes_E A$ and intermediate extensions $E \subseteq D \subseteq A$ [20 Fundamental Theorem]. Sweedler’s fundamental Lemma just referred can be replaced by the fact that the coring $A \otimes_D A$ turns out to be simple cosemisimple [12 Theorem 4.4], [11 Theorem 3.2, Theorem 4.3], [6 28.21] or, alternatively, that $A$ is, as a right $\mathfrak{C}$–comodule, a simple generator of the category of all right $\mathfrak{C}$–comodules. Thus, we see that what is behind [20, 2.1 Fundamental Theorem] can be expressed in categorical terms. In fact, this idea has been recently exploited to state a generalization of Sweedler’s theory for simple artinian rings [8]. In this work we show that the idea of obtaining an isomorphism of corings from categorical properties can be ultimately formulated in terms of comonads.

Each group-like element $g$ of a coring $\mathfrak{C}$ over a ring with unit $A$ gives a canonical homomorphism of $A$–corings $\text{can} : A \otimes_B A \to \mathfrak{C}$, which sends $1 \otimes_B 1$ onto $g$, where $B$ is the subring of $A$ consisting of all $g$–coinvariant elements, and $A \otimes_B A$ is the Sweedler canonical coring [20]. The group-like element $g$ also provides a pair of adjoint functors between the category $\text{Mod}_B$ of right $B$–modules and the category $\text{Comod}_\mathfrak{C}$ of right $\mathfrak{C}$–comodules [3]. The left adjoint is defined as a tensor product functor $\otimes_B A : \text{Mod}_B \to \text{Comod}_\mathfrak{C}$, using

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the right $\mathcal{C}$-comodule structure defined on $A$ by $g$. The right adjoint may be interpreted as the functor $\text{Hom}_A(A, -) : \text{Comod}_A \to \text{Mod}_B$. T. Brzeziński proves [3, Theorem 5.6] that, with $\mathcal{C}$ flat as a left $A$–module, this adjunction is an equivalence of categories if, and only if, $\mathcal{C}$ is Galois (i.e., $\text{can}$ is an isomorphism) and $A$ is faithfully flat as a left $B$–module. The canonical map $\text{can}$ can be interpreted as a comonad in the following way: The $A$–coring $\mathcal{C}$ gives rise to a comonad on $\text{Mod}_A$ built over the functor $- \otimes_A \mathcal{C}$ [6, 18.28]. On the other hand, the adjunction associated to the ring extension $B \subseteq A$ determines [1, Section 3.1] another comonad on $\text{Mod}_A$. In this way, the canonical map $\text{can}$ leads to an homomorphism of comonads $- \otimes_B A : \text{Mod}_B \to \text{Comod}_A$. If $(\mathcal{C}, g)$ is a Galois coring, then these comonads are isomorphic and the functor $- \otimes_B A : \text{Mod}_B \to \text{Comod}_A$ is, up to natural isomorphisms, the Eilenberg-Moore comparison functor [1, Section 3.2]. Thus, one of the implications of [3, Theorem 5.6] may be obtained as a consequence of Beck’s Theorem [1, Section 3.3]. This seems not to be the case of the reciprocal implication. In fact, to deduce that the canonical map $\text{can}$ is an isomorphism from the fact that $- \otimes_B A : \text{Mod}_B \to \text{Comod}_A$ is an equivalence one needs an independent argument, which rests on precise relationship between the canonical map and the counit of the adjunction given by the functors $- \otimes_B A$ and $\text{Hom}_A(A, -)$. More precisely, to deduce that $\mathcal{C}$ is Galois, it is enough to assume that the counit is an isomorphism [6, 18.26], [11, Lemma 3.1], arising the condition of being Galois as part of a characterization of the full and faithful character of the functor $\text{Hom}_A(A, -)$ [6, 18.27], [7, Theorem 3.8], [11, Remark 3.7]. In fact, the referred results are proved in the more general framework of the comatrix corings, introduced in [11], where the role of the group-like element $g$ is played by a right $\mathcal{C}$–comodule $\Sigma$ which is finitely generated and projective as a right module over $A$, and $B = \text{End}(\Sigma \mathcal{C})$. The generalization of [3, Theorem 5.6] in this framework was proved in [11, Theorem 3.2]. Our aim in this paper is to investigate which aspects of these results admit a formulation in terms of comonads. This approach hopefully will permit of focusing in what is specific in each particular future situation, having some relevant general results for granted.

Starting from a comonad $G$ on a category $\mathcal{A}$, and a functor $L : \mathcal{B} \to \mathcal{A}$ with a right adjoint $R : \mathcal{A} \to \mathcal{B}$, there is a bijective correspondence between functors $K : \mathcal{B} \to \mathcal{A}_G$ that factorize throughout $L$ and homomorphisms of comonads $\varphi : LR \to G$ [9, Theorem II.1.1]. The notation $K_\varphi$ will be used to refer to this dependence. Moreover, it can be deduced from [9, Theorem A.1] that $K_\varphi : \mathcal{B} \to \mathcal{A}_G$ admits a right adjoint $D_\varphi : \mathcal{A}_G \to \mathcal{B}$ under mild conditions. We indicate elementary proofs of these facts, with the aim of making them more accessible to non specialists in Category Theory (see Proposition [11, Theorem 1.2] and Proposition [11, Proposition 1.3]). We will prove that $D_\varphi$ is full and faithful if and only if $\varphi$ is an isomorphism and $L$ preserves some equalizers (Theorem [11, Theorem 1.6]), and we will conclude our general results by characterizing when $K_\varphi$ establishes an equivalence between the categories $\mathcal{B}$ and $\mathcal{A}_G$ (Theorem [11, Theorem 1.7]). Obviously, the functors characterized in this way are, a fortiori, comonadic but, in contrast with the approach of Beck’s Theorem, the comonad $G$ is here given beforehand, and each functor $K_\varphi$ corresponds to a “representation” of $G$. Beck’s theorem deals with the situation where $\varphi$ is the identity, that is, $G = LR$ is the comonad associated to the adjunction.

Our point of view is motivated in part by the situation where an entwining structure
between an algebra and a coalgebra is given, together with an entwined module \[5\]. The comonad \(G\) is then given by the coring associated to the entwining structure \[3\] Section 2], and the functor \(K_{ϕ}\) is defined by an entwined module, which is noting but a comodule over the aforementioned coring. It seems natural to study the relationship between the category of entwined modules and the category of modules over the subring of coinvariants defined by the entwined module and, in particular, what the structure of the comonad \(G\) is when these categories are equivalent.

We will apply our general theory to the study of comodules over corings over firm rings. This illustrates how the results on comonads given in the Section \[1\] significantly simplify the treatment of some relevant aspects of the comatrix corings and the Galois comodules investigated in \[11\], \[15\] or \[22\]. We show how the notion of Galois comodule without finiteness conditions as introduced in \[22\] fits perfectly in the general categorical setting. We derive, in particular, some new results on full and faithful functors and equivalences between categories of modules and comodules given by Galois comodules without assuming a priori finiteness conditions (Theorem \[2.3\], Theorem \[2.5\]).

## 1 Functors with values in coalgebras and homomorphisms of comonads

Let \((G, Δ, ε)\) be a comonad (or cotriple) on a category \(A\), that is, a functor \(G : A → A\) together with two natural transformations \(Δ : G → G^2\) and \(ε : G → id_A\) such that the diagrams

\[
\begin{array}{ccc}
G & \xrightarrow{Δ} & G^2 \\
\downarrow{Δ} & & \downarrow{GΔ} \\
G^2 & \xrightarrow{ΔG} & G^3 \\
\end{array}
\quad
\begin{array}{ccc}
G \xleftarrow{Gε} & G^2 \xrightarrow{Gε} & G \\
\downarrow & \downarrow & \downarrow \\
& G & \\
\end{array}
\]

are commutative \[10\], \[1\] Chapter 3\]. We will follow as much as possible \[1\], understanding each statement on monads (triples) automatically as its version for comonads. Consider a functor \(L : B → A\) with a right adjoint \(R : A → B\). If \(η : id_B → RL\) is the unit of the adjunction, and \(ε : LR → id_A\) is its counit, then we have \[1\] Proposition 3.1.2\] the comonad \((LR, δ, ε)\) on \(A\), where \(δ = LηR\). Let us recall the category \(A_G\) of \(G\)-coalgebras \[1\] Section 3.1\], whose objects are pairs \((X, x)\) consisting of an object \(X\) of \(A\) and a morphism \(x : X → GX\) such that

\[Gx \circ x = Δ_X \circ x, \quad ε_X \circ x = id_X\]

Given \(G\)-coalgebras \((X, x), (X', x')\), the morphisms \(f : X → X'\) in \(A\) such that \(Gf \circ x = x' \circ f\) form the set of homomorphisms \(\text{Hom}_{A_G}(X, X')\) in \(A_G\) from \((X, x)\) to \((X', x')\).

Let \(U : A_G → A\) denote the forgetful functor. We will consider those functors \(K : B →\)
\( \mathcal{A}_G \) such that the diagram

\[
\begin{array}{c}
\mathcal{B} \\
\downarrow K \\
\mathcal{A}_G \\
\downarrow L \\
\mathcal{A}
\end{array}
\]

is commutative. We are specially interested in the case where \( K \) provides an equivalence of categories between \( \mathcal{B} \) and \( \mathcal{A}_G \). We start by giving an elementary proof of the bijective correspondence, formulated in [9, Theorem II.1.1], between the functors \( K \) that make commute the diagram (1) and the homomorphisms of comonads \( \varphi : LR \to G \). This correspondence will be a consequence of the following Proposition 1.1 which, in addition, establish some technical facts that will be needed later. Recall [1, Section 3.6] that a homomorphism of comonads from \( LR \) to \( G \) is a natural transformation \( \varphi : LR \to G \) such that \( \Delta \varphi = \varphi^2 \delta \) and \( \varepsilon \varphi = \varepsilon \). The one-to-one correspondence between the mathematical objects described in the statements (A) and (C) of Proposition 1.1 can be deduced from [9, Proposition II.1.4].

**Proposition 1.1.** There exist bijective correspondences between:

(A) Homomorphisms of comonads from \((LR, L\eta R, \varepsilon)\) to \((G, \Delta, \varepsilon)\).

(B) Natural transformations \( R \xrightarrow{\alpha} RG \) such that the following diagrams commute:

\[
\begin{array}{c}
R \\
\downarrow \alpha \\
RG \\
\downarrow \alpha G \\
RG^2 \\
R
\end{array}
\]

and

(C) Natural transformations \( L \xrightarrow{\beta} GL \) such that the following diagrams commute:

\[
\begin{array}{c}
L \\
\downarrow \beta \\
GL \\
\downarrow G\beta \\
G^2L \\
L
\end{array}
\]

**Proof.** We first prove the one-to-one correspondence between the natural transformations described in (A) and (B). Given a homomorphism of comonads \( \varphi : LR \to G \), define the natural transformation

\[
R \xrightarrow{\eta R} RLR \xrightarrow{R \varphi} RG
\]
To check that the first diagram in (2) commutes, consider an object \( X \) of \( \mathcal{A} \) and compute:

\[
R\Delta_X \circ \alpha_X = R\Delta_X \circ R\varphi_X \circ \eta_{RX} \\
= R\varphi_X^2 \circ RL\eta_{RX} \circ \eta_{RX} \quad (\varphi \text{ is of comonads}) \\
= R\varphi_{GX} \circ RLR\varphi_X \circ RL\eta_{RX} \circ \eta_{RX} \quad (\varphi_X^2 = \varphi_{GX} \circ LR\varphi_X) \\
= R\varphi_{GX} \circ RLR\varphi_X \circ \eta_{RLRX} \circ \eta_{RX} \quad (\eta \text{ is natural}) \\
= R\varphi_{GX} \circ \eta_{RGX} \circ R\varphi_X \circ \eta_{RX} \quad (\eta \text{ is natural}) \\
= \alpha_{GX} \circ \alpha_X
\]

For the second diagram in (2), we have:

\[
R\epsilon_X \circ \alpha_X = R\epsilon_X \circ R\varphi_X \circ \eta_{RX} \\
= R\epsilon_X \circ \eta_{RX} \quad (\varphi \text{ is of comonads}) \\
= \text{id}_{RX} \quad (\text{by the adjunction})
\]

Conversely, given a natural transformation \( \alpha : R \to RG \) satisfying (2), we define the natural transformation

\[
LR \xrightarrow{L\alpha} LRG \xrightarrow{\varphi} G
\]

To show that \( \varphi \), defined in (5), is a homomorphism of comonads, we need to use that, for each object \( X \) of \( \mathcal{A} \), one has the identities

\[
\varphi_X^2 = \epsilon_{G^2X} \circ L\alpha_{GX} \circ LR\epsilon_{GX} \circ LRL\alpha_X
\]

and

\[
\varphi_X^2 = G\epsilon_{GX} \circ G\alpha_X \circ \epsilon_{GLRX} \circ L\alpha_{RLX},
\]

by definition of \( \varphi^2 \). Do the following computation:

\[
\varphi_X^2 \circ L\eta_{RX} = \epsilon_{G^2X} \circ L\alpha_{GX} \circ LR\epsilon_{GX} \circ LRL\alpha_X \circ L\eta_{RX} \quad (\text{by } (4)) \\
= \epsilon_{G^2X} \circ L\alpha_{GX} \circ LR\epsilon_{GX} \circ L\eta_{RGX} \circ L\alpha_X \quad (\eta \text{ is natural}) \\
= \epsilon_{G^2X} \circ L\alpha_{GX} \circ L\alpha_X \quad (R\epsilon_{GX} \circ \eta_{RGX} = \text{id}_{RGX}) \\
= \epsilon_{G^2X} \circ LR\Delta_X \circ L\alpha_X \quad (\alpha_{GX} \circ \alpha_X = R\Delta_X \circ \alpha_X) \\
= \Delta_X \circ \epsilon_{GX} \circ L\alpha_X \quad (\epsilon \text{ is natural}) \\
= \Delta_X \circ \varphi_X
\]

This proves that the first condition for \( \varphi \) to be a homomorphism of comonads holds. For the second condition, we compute as follows:

\[
\epsilon_X \circ \varphi_X = \epsilon_X \circ \epsilon_{GX} \circ L\alpha_X \\
= \epsilon_X \circ LR\epsilon_X \circ L\alpha_X \quad (\epsilon \text{ is natural}) \\
= \epsilon_X \quad (R\epsilon_X \circ \alpha_X = \text{id}_{RX})
\]

Now, let \( \varphi : LR \to G \) be a homomorphism of comonads and consider the natural transformation \( \alpha : R \to RG \) defined in (4). If we consider the homomorphism of comonads, say
ϕ’, defined, from this α, in (5), then we will see that it coincides with the original ϕ. For this, we compute, for an object X of A:

\[
\varphi_X' = \epsilon_{GX} \circ L\alpha_X \\
= \epsilon_{GX} \circ LR\varphi_X \circ L\eta_{RX} \\
= \varphi_X \circ \epsilon_{LRX} \circ L\eta_{RX} \quad (\epsilon \text{ is natural)} \\
= \varphi_X
\]

Finally, we have to see that, given a natural transformation α : R → RG subject to (2), and defining first ϕ according to (5), and the new natural transformation, say α’, by (4), we reobtain the original α. This follows from the following computation, for X an object of A:

\[
\alpha_X' = R\varphi_X \circ \eta_{RX} \\
= R\epsilon_{GX} \circ R\alpha_X \circ \eta_{RX} \quad (\eta \text{ is natural)} \\
= R\epsilon_{GX} \circ \eta_{RGX} \circ \alpha_X \\
= \alpha_X
\]

The correspondence between the natural transformations described in (A) and (C) can be proved in a similar way (details can be found in [14, Proposición 1.1]). The assignment goes as follows: starting from a homomorphism of comonads ϕ : LR → G, we define the natural transformation

\[
\begin{array}{c}
L \\
\downarrow L\eta \\
LRL \xrightarrow{\varphi_L} GL \\
\downarrow \beta \\
G
\end{array}
\quad (7)
\]

which turns out to make commute the diagrams (3). Conversely, to each natural transformation β : L → GL making commute the diagrams in (3), we assign the homomorphism of comonads ϕ given by

\[
\begin{array}{c}
LR \xrightarrow{\beta_R} GLR \xrightarrow{G\epsilon} G \\
\downarrow \varphi \\
G
\end{array}
\quad (8)
\]

Given a homomorphism of comonads ϕ : LR → G, and the corresponding natural transformation β given by (7), we can define the functor \( K_\varphi : B \to A_G \) defined as

\[
K_\varphi Y = (LY, \beta_Y) = (LY, \varphi_{LY} \circ L\eta_Y)
\]
on objects Y of B, and as \( K_\varphi f = Lf \) on morphisms f of B. The diagrams (3) show that \((LY, \beta_Y)\) is a G–coalgebra and the naturalness of β implies that Lf is a homomorphism of G–coalgebras. This functor \( K_\varphi \) satisfies that \( UK_\varphi = L \). Conversely, given a functor \( K : B \to A_G \) making commute the diagram (11), define, for each object Y of B, \( \beta_Y : LY \to GLY \) as the structure morphism of the G–coalgebra KY. It is easy to check that this assignment defines a natural transformation \( \beta : L \to GL \) which makes commute (3). We thus deduce from Proposition (11)
Theorem 1.2. [9, Theorem II.1.1]. Given a comonad \( G \) on a category \( A \), and a functor \( L : B \to A \), if \( L \) has a right adjoint \( R : A \to B \), then there exists a bijective correspondence between functors \( K : B \to A_G \) making commute (11) and homomorphisms of comonads \( \varphi : LR \to G \).

The situation described in Proposition 1.1 requires a suitable terminology. Thus, given a homomorphism of comonads \( \varphi : LR \to G \), we will refer to the natural transformations \( \alpha : R \to RG \) and \( \beta : L \to GL \) as the co-induced representation (resp. induced representation) of \( \varphi \). When the starting datum is either a natural transformation \( \alpha \) subject to the conditions (2), or a natural transformation \( \beta \) making commute the diagrams in (3), then the corresponding homomorphism of comonads \( \varphi : LR \to G \) will be called canonical transformation associated to \( \alpha \) (resp. \( \beta \)).

We next study when \( K \varphi \) has a right adjoint. The following Proposition 1.3 can be deduced from [9, Theorem A.1]. We give an elementary proof in our case.

Proposition 1.3. Assume that for every \( G \)-coalgebra \((X,x)\) there exists in \( B \) the equalizer of the pair of morphisms \( \alpha_X, Rx : RX \to RGX \). Then the functor \( K \varphi : B \to A_G \) has a right adjoint \( D \varphi : A_G \to B \), whose value at \((X,x)\) is the equalizer

\[
D \varphi X \xrightarrow{eq_X} RX \xrightarrow{\alpha_X} RGX
\]  

(9)

Proof. Given objects \( X \) of \( A \) and \( Y \) of \( B \), denote by \( \Phi : \text{Hom}_A(LY,X) \to \text{Hom}_B(Y,RX) \) the isomorphism of the adjunction, which, in terms of the counity \( \eta \), is defined by \( \Phi(h) = Rh \circ \eta_Y \) for \( h : LY \to X \). Consider the following commutative diagram of maps between sets:

\[
\begin{array}{ccc}
\text{Hom}_{A_G}(K \varphi Y, X) & \xrightarrow{- \circ \beta_Y} & \text{Hom}_B(Y, D \varphi X) \\
\text{Hom}_A(LY,X) \xrightarrow{\Phi} \text{Hom}_B(Y,RX) \downarrow \downarrow \\
\text{Hom}_A(LY,GX) \xrightarrow{\Phi} \text{Hom}_B(Y, RGX),
\end{array}
\]

where \( \Upsilon \) is given by the restriction of \( \Phi \) to \( \text{Hom}_{A_G}(K \varphi Y, X) \). In fact, the bottom square commutes serially, in the sense of [11 page 112] (a detailed verification can be found in [14 Proposición 2.1]). Now, the vertical edges are equalizers, the left one by definition of homomorphism of \( G \)-coalgebras, and the right edge by the universal property of the equalizer (12). Hence, the natural isomorphism \( \Phi : \text{Hom}_A(LY,X) \to \text{Hom}_B(Y,RX) \) induces, by restriction, a natural isomorphism \( \Upsilon : \text{Hom}_{A_G}(K \varphi Y, X) \to \text{Hom}_B(Y, D \varphi X) \).

Remark 1.4. If we put in the diagram (10) \( X = K \varphi Y \), for an object \( Y \) of \( B \), since \( id_{K \varphi Y} \) is a homomorphism of \( G \)-coalgebras, we get that \( \eta_Y = \Phi(id_{K \varphi Y}) \) factorizes throughout \( D \varphi K \varphi Y \). In this way, the unit of the adjunction \( K \varphi \dashv D \varphi \), denoted by \( \tilde{\eta} \), is uniquely
determined at \( Y \) by the universal property of an equalizer, according to the following diagram:

\[
\begin{array}{ccc}
D_\varphi K_\varphi Y & \xrightarrow{\alpha_{LY}} & RLY \\
\downarrow \varrho_Y & & \downarrow R\beta_Y \\
Y & \xrightarrow{\eta_Y} & Y
\end{array}
\]

To make explicit the counit \( \hat{\epsilon} \) at an object \((X, x)\) of \( \mathcal{A}_G \), take \( Y = D_\varphi X \) in diagram (10). Then we obtain the morphism

\[
\hat{\epsilon}_X = \Upsilon^{-1}(id_{D_\varphi X}) = \Phi^{-1}(eq_X) = \epsilon_X \circ Leq_X
\]

which is of \( G \)-coalgebras in view of (10). This situation is resumed by the diagram in \( \mathcal{A} \):

\[
\begin{array}{ccc}
K_\varphi D_\varphi X & \xrightarrow{Leq_X} & LRX \\
\downarrow \hat{\epsilon}_X & & \downarrow \epsilon_X \\
X & \xrightarrow{\phi_X} & GX
\end{array}
\]

well understood that the underlying object in \( \mathcal{A} \) of \( K_\varphi D_\varphi X \) is, by definition, \( LD_\varphi X \).

Our next aim is to prove that \( D_\varphi \) is a full and faithful functor if and only if \( \varphi \) is an isomorphism of comonads and the functor \( L \) preserves the equalizers (9).

**Lemma 1.5.** Assume that the equalizer (9) exists for every \( G \)-coalgebra \((X, x)\). Then the following diagram

\[
\begin{array}{ccc}
LD_\varphi X & \xrightarrow{Leq_X} & LRX \\
\downarrow \hat{\epsilon}_X & & \downarrow \epsilon_X \\
X & \xrightarrow{\phi_X} & GX
\end{array}
\]

is (serially) commutative.

**Proof.** That the right square commutes serially is an easy consequence of the naturalness of \( \varphi \) and of the fact that it is a homomorphism of comonads. For the commutativity of the left square, we do the following computation:

\[
x \circ \hat{\epsilon}_X = x \circ \epsilon_X \circ Leq_X \\
= \epsilon_{GX} \circ LRx \circ Leq_X \\
= \epsilon_{GX} \circ L\alpha_X \circ Leq_X \\
= \epsilon_{GX} \circ LR\varphi_X \circ L\eta_{RX} \circ Leq_X \\
= \varphi_X \circ \epsilon_{LRX} \circ L\eta_{RX} \circ Leq_X \\
= \varphi_X \circ Leq_X
\]

(\( \epsilon \) is natural)

(\( eq_X \) equalizes \((Rx, \alpha_X)\))

(\( \epsilon \) is natural)
We are now ready to state the main result in this section. Recall that a right adjoint functor is faithful and full if and only if the counit is an isomorphism [2 Proposition 3.4.1]

**Theorem 1.6.** Let $L : \mathcal{B} \to \mathcal{A}$ be a functor admitting a right adjoint $R : \mathcal{A} \to \mathcal{B}$, and let $G : \mathcal{A} \to \mathcal{A}$ be a comonad. Consider a functor $K_\varphi : \mathcal{B} \to \mathcal{A}_G$ that makes commute (1) with corresponding homomorphism of comonads $\varphi : LR \to G$, and let $\alpha : R \to RG$, $\beta : L \to GL$ be its representations. Assume that for every $G$–coalgebra $(X,x)$, there exists in $\mathcal{B}$ the equalizer of $\alpha_X, Rx$. Then the right adjoint $D_\varphi : \mathcal{A}_G \to \mathcal{B}$ to the functor $K_\varphi$ is faithful and full if and only if $L$ preserves the equalizers of the form (9) and $\varphi$ is an isomorphism of comonads.

**Proof.** Let $X$ be any object of $\mathcal{A}$. An easy computation shows that

$$ RX \xrightarrow{\alpha_X} RGX \xrightarrow{\alpha_{GX}} RG^2X $$

(14)

is a contractible equalizer in the sense of [1 Section 3.3]. The equalizer (14) shows that $eq_{GX} = \alpha_X$ and $D_\varphi GX = RX$. From this, by applying the functor $L$ to (14), and taking (12) into account, we get that $\hat{\epsilon_{GX}} = \epsilon_{GX} \circ L\alpha_X = \varphi_X$. Thus, if $D_\varphi$ is full and faithful, then $\hat{\epsilon}_{GX} = \varphi_X$ is an isomorphism for every $G$–coalgebra $(X,x)$. Moreover, since the bottom row in the commutative diagram (13) is an equalizer [1 Proposition 3.3.4], we get that its top row is an equalizer as well. This means that $L$ preserves the equalizers of the form (9). Conversely, if $\varphi$ is an isomorphism and $L$ preserves all equalizers of the form (9), then, by Lemma 1.5, $\hat{\epsilon}_X$ is an isomorphism for every $G$–coalgebra $(X,x)$, whence $D_\varphi$ is full and faithful. 

Any isomorphism of comonads $\varphi : LR \to G$ induces [1 Theorem 3.3] an equivalence of categories $\mathcal{A}_{LR} \cong \mathcal{A}_G$. By combining this fact with Theorem 1.6 and Beck’s Theorem [1 Theorem 3.10], we may obtain Theorem 1.7. We prefer to include here an explicit proof.

**Theorem 1.7.** Let $L : \mathcal{B} \to \mathcal{A}$ be a functor with a right adjoint $R : \mathcal{A} \to \mathcal{B}$, and $G : \mathcal{A} \to \mathcal{A}$ any comonad. Consider a functor $K_\varphi : \mathcal{B} \to \mathcal{A}_G$ that makes commute (1) with the corresponding homomorphism of comonads $\varphi : LR \to G$, and let $\alpha : R \to RG$, $\beta : L \to GL$ be its representations. Assume that for every $G$–coalgebra $(X,x)$, there exists in $\mathcal{B}$ the equalizer of $\alpha_X, Rx$. Then the functor $K_\varphi$ is an equivalence of categories between $\mathcal{B}$ and $\mathcal{A}_G$ if and only if $L$ preserves the equalizers of the form (9), reflects isomorphisms, and $\varphi$ is an isomorphism of comonads.

**Proof.** We first observe that for every object $Y$ of $\mathcal{B}$, the unit $\hat{\eta}_Y$ of the adjunction $K_\varphi \dashv D_\varphi$ stated in Proposition 1.3 is given, according to the Remark 1.4 by the equalizer in the
horizontal row in the commutative diagram

If we apply the functor $L$ to the commutative diagram (15) we obtain the diagram (16)

commutative as well. Here, the morphisms $\epsilon_{LRLY}$ and $\epsilon_{LY}$ make the diagonal row a contractible equalizer. If $K_\varphi$ is an equivalence of categories, then its right adjoint $D_\varphi$ is obviously faithful and full and, by Theorem 1.6 $\varphi$ is a natural isomorphism and $L$ preserves the equalizers of the form (15). Since the forgetful functor $U_G : A_G \to A$ reflects isomorphisms [11 Proposition 3.3.1], we get from $L = U_G \circ K_\varphi$ that $L$ reflects isomorphisms. Conversely, if $\varphi$ is a natural isomorphism and $L$ preserves isomorphisms, then, by Theorem 1.6 the counit of the adjunction $K_\varphi \dashv D_\varphi$ is an isomorphism. From the diagram (16) we deduce that $L\eta_Y$ is an isomorphism and, since $L$ reflects isomorphisms, $\hat{\eta}_Y$ must be an isomorphism. We have thus proved that the unit of the adjunction $K_\varphi \dashv D_\varphi$ is also a natural isomorphism. Therefore, $K_\varphi$ is an equivalence of categories.

2 Corings over firm rings

Let $A$ be a ring, which is not assumed to have a unit. By $\text{Mod}_A$ we denote the category of all right $A$–modules. We may also consider left modules or bimodules over different rings. The tensor product over $A$ will be denoted by $- \otimes_A -$ . A right $A$–module $M$ is said to be firm [19] if the “multiplication” map $\varpi^+_M : M \otimes_A A \to M$ is bijective. Its inverse will be denoted by $d^+_M : M \to M \otimes_A A$. Left firm $A$–modules are defined analogously, with notations $\varpi^-_M$ and $d^-_M$ for the “multiplication” map and its inverse, respectively. Obviously, $\varpi^+_A = \varpi^-_A$, which implies, in case of being bijective, that $d^+_A = d^-_A$. We will say then that $A$ is a firm ring. If $A$ is firm, then the full subcategory $\text{Mod}_A$ of $\text{Mod}_A$ whose objects are all firm modules is abelian [19 (4.6)], [16 Corollary 1.3], [17 Proposition 2.7], even thought,
in general, that equalizers cannot be computed in Abelian Groups, due essentially to the lack of exactness of the functor $- \otimes_A A$.

Let $A$ be a firm ring. An $A$–coring is a coalgebra in the monoidal category of all firm $A$–bimodules (see, e.g., [6, 38.33]). Explicitly, an $A$–coring is a firm $A$–bimodule $C$ endowed with two homomorphisms of $A$–bimodules $\Delta_C : C \to C \otimes_A C$, and $\varepsilon_C : C \to A$ that satisfy the equations:

$$ (C \otimes_A \Delta_C) \circ \Delta_C = (\Delta_C \otimes_A C) \circ \Delta_C \tag{17} $$

$$ (\varepsilon_C \otimes_A C) \circ \Delta_C = d_C^-, \quad (C \otimes_A \varepsilon_C) \circ \Delta_C = d_C^+ \tag{18} $$

In (17) we have considered the canonical isomorphism $C \otimes_A (C \otimes_A C) \cong (C \otimes_A C) \otimes_A C$ as one equality, denoting the “common value” by $C \otimes_A C \otimes_A C$. When $A$ is unital, we recover the original definition from [20]. A direct computation, using (17) and (18), shows that each $A$–coring $(C, \Delta_C, \varepsilon_C)$ determines a comonad on $\text{Mod}_A$ defined by the functor

$$ - \otimes_A C : \text{Mod}_A \longrightarrow \text{Mod}_A $$

and the natural transformations

$$ - \otimes_A C \xleftarrow{-\otimes_A \Delta_C} - \otimes_A C \otimes_A C $$

$$ - \otimes_A C \xleftarrow{-\otimes_A \varepsilon_C} - \otimes_A A \cong id $$

The category of coalgebras for this comonad is the category $\text{Comod}_C$ of all right $C$–comodules.

Let $B$ be a firm ring, and a firm $B \to A$–bimodule $\Sigma$. Arguing as in [15], we have a pair of adjoint functors

$$ \text{Mod}_B \xleftarrow{-\otimes_B \Sigma} \text{Mod}_A, \quad - \otimes_B \Sigma \dashv \text{Hom}_A(\Sigma, -) \otimes_B B \tag{19} $$

which is obtained by composing the adjunctions

$$ \text{Mod}_B \xleftarrow{-\otimes_B \Sigma} \text{Mod}_A, \quad - \otimes_B \Sigma \dashv \text{Hom}_A(\Sigma, -) $$

and

$$ \text{Mod}_B \xleftarrow{J} \text{Mod}_B, \quad J \dashv - \otimes_B B $$

where $J : \text{Mod}_B \to \text{Mod}_B$ is the inclusion functor. It will be useful to give explicitly the unit and the counit of the adjunction (19). To do this, given $y \in Y$ for $Y$ a right firm $B$–module, we will use the notation $d^+_C(y) = y^b \otimes_B b \in Y \otimes_B B$ (sum understood). Of
course, this element of the tensor product is determined by the condition \(y^b = y\). The counit of the adjunction is

\[
\eta_Y : Y \longrightarrow \text{Hom}_A(\Sigma, Y \otimes_B \Sigma) \otimes_B B,
\]

\[
\eta_Y(y) = (y^b \otimes_B -) \otimes_B b,
\]  

(20)
and the unit

\[
\epsilon_X : \text{Hom}_A(\Sigma, X) \otimes_B B \otimes_B \Sigma \longrightarrow X,
\]

\[
\epsilon_X(f \otimes_B b \otimes_B u) = f(bu).
\]  

(21)
We have then the associated comonad

\[
(\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma, \eta_{\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma}, \epsilon)
\]
Assume we have a structure of \(B - C\)-bicomodule over \(\Sigma\), that is, a homomorphism of \(B - A\)-bimodules \(\varrho_\Sigma : \Sigma \rightarrow \Sigma \otimes_A C\) such that

\[
(\varrho_\Sigma \otimes_A C) \circ \varrho_\Sigma = (\Sigma \otimes_A \Delta_C) \circ \varrho_\Sigma,
\]

\[
(\Sigma \otimes_A \varepsilon_C) \circ \varrho_\Sigma = d_\Sigma^C
\]  

(22)
It follows from (22) that the natural transformation

\[
\beta : - \otimes_B \Sigma \longrightarrow - \otimes_B \Sigma \otimes_A C
\]
(23)
satisfies the conditions of the statement (C) of the Proposition \[\text{[1]}\] and, therefore, it gives rise to a canonical homomorphism of comonads \text{can} defined by the composite

\[
\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \longrightarrow \text{can} \longrightarrow \text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \otimes_A C
\]

or, using a simplified version of Heynemann-Sweedler’s notation, we have, for each right \(B\)-module \(X\):

\[
\text{Hom}_A(\Sigma, X) \otimes_B B \otimes_B \Sigma \longrightarrow \text{can}_X \longrightarrow X \otimes_A C
\]

\[
f \otimes_B b \otimes_B u \longrightarrow f(bu_{[0]} \otimes_A u_{[1]}
\]
where \(\varrho_\Sigma(u) = u_{[0]} \otimes_A u_{[1]}\) (sum understood).

We can now apply Proposition \[\text{[1.3]}\] and Theorem \[\text{[1.6]}\] to obtain:

**Theorem 2.1.** The functor \(- \otimes_B \Sigma : \text{Mod}_B \rightarrow \text{Comod}_C\) has a right adjoint

\[
\text{Hom}_C(\Sigma, -) \otimes_B B : \text{Comod}_C \longrightarrow \text{Mod}_B
\]
This functor is faithful and full if and only if \text{can} is a natural isomorphism and \(- \otimes_B \Sigma : \text{Mod}_B \rightarrow \text{Mod}_A\) preserves the equalizer

\[
\text{Hom}_C(\Sigma, X) \otimes_B B \longrightarrow \text{Hom}_A(\Sigma, X) \otimes_B B \longrightarrow \text{can}_X \longrightarrow \text{Hom}_A(\Sigma, X \otimes_A C) \otimes_B B
\]

\[
f \otimes_B b \otimes_B u \longrightarrow f(bu_{[0]} \otimes_A u_{[1]}
\]
for every right \(C\)-comodule \((X, \varrho_X)\), where \(\alpha_X(f \otimes_B b) = [(f \otimes_A C) \circ \varrho_\Sigma] \otimes_B b\) for \(f \otimes_B b \in \text{Hom}_A(\Sigma, X) \otimes_B B\).
Proof. It is enough to check that, in the present situation, $\alpha_X$, as defined in (4), is given as in the statement of the theorem, and that $- \otimes_B B : \text{Mod}_B \to \text{Mod}_B$ is left exact since it is right adjoint to the inclusion functor $J : \text{Mod}_B \to \text{Mod}_B$. \hfill \square

Theorem 1.7 has the following consequence:

**Theorem 2.2.** Let $\Sigma$ be a $B - C$-bicomodule. The functor $- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C$ is an equivalence of categories if and only if $- \otimes_B \Sigma : \text{Mod}_B \to \text{Mod}_A$ preserves the equalizers of the form (24), reflects isomorphisms, and the canonical transformation can is an isomorphism.

Every right $C$-comodule $\Sigma$, is a $T - C$-bicomodule, where $T = \text{End}(\Sigma_C)$. Of course, in this case we have a natural isomorphism $\text{Hom}_A(\Sigma, -) \otimes_T T \cong \text{Hom}_A(\Sigma, -)$. Following [22], we will say that $\Sigma$ is a Galois $C$-comodule when $\text{can} : \text{Hom}_A(\Sigma, -) \otimes A \Sigma \to - \otimes_A \mathcal{C}$ is an isomorphism. More generally, if $\Sigma$ is a $B - \mathcal{C}$-bicomodule firm as a $B - A$-bimodule, then we have a homomorphism of rings $\lambda : B \to T$. Assume that $B$ is a left ideal of $T$ (that is, that the image under $\lambda$ of $B$ is a left ideal or $T$). By [15] Lemma 4.10, Lemma 4.11, we have a commutative diagram of natural transformations:

\[
\begin{array}{ccc}
\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma & \xrightarrow{\cong} & \text{Hom}_A(\Sigma, -) \otimes_T \Sigma \\
\downarrow \text{can} & & \downarrow \text{can} \\
- \otimes_A \mathcal{C} & & - \otimes_A \mathcal{C}
\end{array}
\]  

(25)

Therefore, $\Sigma$ is a Galois right $\mathcal{C}$-comodule in the sense of [22] if and only if

\[\text{can} : \text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \to - \otimes_A \mathcal{C}\]

is an isomorphism. Obviously, Theorems 2.1 and 2.2 can be formulated in the case where $B = T = \text{End}(\Sigma_C)$, characterizing then when $\text{Hom}_C(\Sigma, -) : \text{Comod}_C \to \text{Mod}_T$ is full and faithful or it gives an equivalence of categories. This evokes Gabriel-Popescu’s Theorem [13].

A left module $M$ over a firm ring $R$ will be said to be flat if the functor $- \otimes_R M : \text{Mod}_R \to \text{Ab}$ is exact, where $\text{Ab}$ denotes the category of abelian groups.

The following theorem gives a general version, in the sense that no finiteness condition is assumed a priori on the comodule $\Sigma$, of [4, Theorem 2.1.(1)], [7, Theorem 3.8], [6, 18.27], [15] Theorem 4.9.

**Theorem 2.3.** Let $\Sigma$ be a $B - \mathcal{C}$-bicomodule, where $A$ is unital and $\mathcal{C}$ is an $A$-coring, flat as a left $A$-module. Assume $B$ to be firm, and that $B\Sigma$ is a firm module. If $B$ is a left ideal of $T = \text{End}(\Sigma_C)$, then the following statements are equivalent

(i) The functor $\text{Hom}_C(\Sigma, -) : \text{Comod}_C \to \text{Mod}_T$ is full and faithful;

(ii) $\Sigma$ is a generator of the category $\text{Comod}_C$;
(iii) \( \text{can} : \text{Hom}_C(\Sigma, -) \otimes_T \Sigma \to - \otimes_A C \) is an isomorphism, and \( T\Sigma \) is flat;

(iv) \( \text{can} : \text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \to - \otimes_A C \) is an isomorphism of comonads on \( \text{Mod}_A \) and \( B\Sigma \) is flat;

(v) the functor \( \text{Hom}_C(\Sigma, -) \otimes_B C : \text{Comod}_C \to \text{Mod}_B \) is full and faithful.

Proof. By [12, Proposition 1.2], \( A\Sigma \) is flat if and only if \( \text{Comod}_C \) is a Grothendieck category and the forgetful functor \( U : \text{Comod}_C \to \text{Mod}_A \) is exact.

(i) \( \iff \) (ii) is a consequence of Gabriel-Popescu’s Theorem [18, Cap. III, Teoremă 9.1.(2)].

(ii) \( \implies \) (iii) By Theorem 2.1, with \( B = T \), \( \Sigma \) is a Galois \( C \)-comodule. By Gabriel-Popescu’s Theorem [18, Cap. III, Teoremă 9.1.(3)] the functor \(- \otimes_T \Sigma : \text{Mod}_T \to \text{Comod}_C\) is exact. Since \( U : \text{Comod}_C \to \text{Mod}_A \) is exact, we get that \( T\Sigma \) is flat.

(iii) \( \implies \) (i) By Theorem 2.1 with \( B = T \).

(iii) \( \implies \) (iv) In view of diagram (25), \( \text{can} : \text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \to - \otimes_A C \) is an isomorphism. We have proved that (iii) is equivalent to (ii). Therefore, we deduce from [15, Proposition 4.13] that \( B\Sigma \) is flat.

(iv) \( \implies \) (v) By Theorem 2.1.

(v) \( \implies \) (ii) Follows by the standard argument: For any right \( C \)-comodule \( X \), take a free presentation \( T(I) \to \text{Hom}_C(\Sigma, X) \to 0 \). After tensoring on the right by \( B \otimes_B \Sigma \), and taking that \( T \otimes_B B \sim B \) [15, Lemma 4.10] into account, we get a presentation \( \Sigma(I) \to X \to 0 \).

Lemma 2.4. Let \( \Sigma \) be a \( B\)-\( C \)-bicomodule, where \( B \) is a firm ring. Assume that \( \Sigma \) is firm as a left \( B \)-module. If the counit of the adjunction \(- \otimes_B \Sigma \dashv \text{Hom}_C(\Sigma, -) \otimes_B \Sigma\) evaluated at \( B \) is surjective, then \( B \) is a left ideal of \( T = \text{End}(\Sigma_C) \).

Proof. The proof of the implication (iv) \( \implies \) (v) of [15, Theorem 4.15] runs here: In view of (20) and Remark 1.3, the hypothesis says that the map

\[ \eta_B : B \to \text{Hom}_C(\Sigma, B \otimes_B \Sigma) \otimes_B B, \quad \eta_B(b) = (b \otimes_B -) \otimes_B c \]

is surjective. By composing \( \eta_B \) with the isomorphism \( \text{Hom}_C(\Sigma, B \otimes_B \Sigma) \cong \text{Hom}_C(\Sigma, \Sigma) \otimes_B B = T \otimes_B B \), we obtain that every element of \( T \otimes_B B \) is a finite sum of elements of the form \( b \cdot \otimes_B c \). Thus, \( B \) is a left ideal of \( T \).

Theorem 2.5. Let \( \Sigma \) be a \( B\)-\( C \)-bicomodule, where \( A \) is unital and \( C \) is an \( A \)-coring, flat as a left \( A \)-module. Assume \( B \) to be firm, and that \( B\Sigma \) is a firm module. The following statements are equivalent:

(i) The functor \(- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C\) is an equivalence of categories;

(ii) \( \text{can} : \text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma \to - \otimes_A C \) is an isomorphism, and \( B\Sigma \) is faithfully flat;

(iii) \( \Sigma \) is a generator of \( \text{Comod}_C \) such that the functor \(- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C\) is full and faithful;
(iv) $\Sigma$ is a generator of $\text{Comod}_C$ such that the functor $- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C$ is faithful and $B$ is a left ideal of $T$.

Proof. $\text{(i)} \Rightarrow \text{(ii)}$ can be an isomorphism by Theorem 2.2. By [12, Proposition 1.2], the forgetful functor $\text{Comod}_C \to \text{Mod}_A$ is faithful and exact. Therefore, the functor $- \otimes_B \Sigma : \text{Mod}_B \to \text{Mod}_A$ is faithful and exact.

$\text{(ii)} \Rightarrow \text{(i)}$ By Theorem 2.2.

$\text{(i)} \Rightarrow \text{(iii)}$ Since $B$ is a generator of $\text{Mod}_B$, $\Sigma \cong B \otimes_B \Sigma$ is a generator of $\text{Comod}_C$.

$\text{(iii)} \Rightarrow \text{(iv)}$ By Lemma 2.4.

$\text{(iv)} \Rightarrow \text{(ii)}$ The forgetful functor $\text{Comod}_C \to \text{Mod}_A$ is faithful. Therefore, $- \otimes_B \Sigma : \text{Mod}_B \to \text{Mod}_A$ is faithful. By Theorem 2.3, $B \Sigma$ is flat and can is an isomorphism.

Remark 2.6. Theorem 2.5 shows, in conjunction with Theorem 2.3, that if $- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C$ is an equivalence of categories, then $\Sigma$ is a Galois comodule in the sense of [22].

In the particular case where $B$ is unital and $\Sigma_A$ is finitely generated and projective, we have a natural isomorphism

$$\nu : \text{Hom}_A(\Sigma, -) \otimes_B B \cong - \otimes_A \Sigma^* \quad (f \otimes_B b \mapsto f(\text{be}_i) \otimes_A e^*_i)$$

where $\Sigma^* = \text{Hom}_A(\Sigma, A)$, and $\{(e_i, e^*_i)\}$ is a finite dual basis for $\Sigma_A$. We have then the adjoint pair $- \otimes_B \Sigma \dashv - \otimes_A \Sigma^*$ and the associated comonad $- \otimes_A \Sigma^* \otimes_B \Sigma$ on $\text{Mod}_A$. In particular, $\Sigma^* \otimes_B \Sigma$ is an $A$–coring, the comatrix coring associated to the bimodule $B \Sigma_A$ (see [11]). If $\Sigma$ is a $B$–$\mathcal{C}$–bicomodule, then we have a commutative diagram of homomorphisms of comonads

$$\begin{array}{ccc}
\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma & \xrightarrow{\nu \otimes_B \Sigma} & - \otimes_A \Sigma^* \otimes_B \Sigma \\
\text{can} & & \text{can} \\
& \otimes_A \mathcal{C}, & \otimes_A \mathcal{C},
\end{array}$$

where $\text{can}(\phi \otimes_B u) = \phi(u_{[0]})u_{[1]}$ for $\phi \otimes_B u \in \Sigma^* \otimes_B \Sigma$ is a homomorphism of $A$–corings from $\Sigma^* \otimes_B \Sigma$ to $\mathcal{C}$ (see [11]).

Corollary 2.7. [11, Theorem 3.2, Theorem 3.10] Let $\Sigma$ be a $B$–$\mathcal{C}$–bicomodule, where $B$ is a unital ring and $\mathcal{C}$ is a coring over a unital ring $A$. The following statements are equivalent:

(i) $A \mathcal{C}$ is flat and the functor $- \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_\mathcal{C}$ is an equivalence of categories;

(ii) $\Sigma_A$ is finitely generated and projective, the canonical map $\text{can} : \Sigma^* \otimes_B \Sigma \to \mathcal{C}$ is an isomorphism, and $B \Sigma$ is faithfully flat;

(iii) $A \mathcal{C}$ is flat, $\Sigma$ is a finitely generated projective generator of $\text{Comod}_\mathcal{C}$, and $\lambda : B \to T$ is an isomorphism.
Proof. \( \mathfrak{H} \Rightarrow \mathfrak{M} \) The equivalence of categories \( - \otimes_B \Sigma \) preserves finitely generated and projective generators, whence \( \Sigma \cong B \otimes_B \Sigma \) is a finitely generated projective generator of \( \text{Comod}_C \). On the other hand, by Theorem 2.6, \( B \) is a left ideal of \( T \). Since \( B \) is a unital subring of \( T \), we must have \( B = T \).

\( \mathfrak{M} \Rightarrow \mathfrak{H} \) Since the forgetful functor \( U : \text{Comod}_C \to \text{Mod}_A \) has an exact right adjoint \( - \otimes_A \mathfrak{C} \) which preserves colimits, \( U \) preserves finitely generated and projective objects. The rest follows directly from Theorem 2.5 and diagram (26).

\( \mathfrak{H} \Rightarrow \mathfrak{I} \) Since \( \text{can} \) is an isomorphism, and the left \( A \)-module \( \Sigma^* \otimes_B \Sigma \) is flat, we get that \( A \mathfrak{C} \) is flat. That \( - \otimes_B \Sigma \) is an equivalence of categories follows from Theorem 2.5 and diagram (26). \( \square \)

**Remark 2.8.** It follows from Corollary 2.7 that if \( - \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_C \) is an equivalence, and \( B \) is unital, then the coring \( \mathfrak{C} \) is Galois in the sense of \( \mathfrak{I} \) (or \( \Sigma \) is a Galois comodule as in \( \mathfrak{I} \)), that is, \( \text{can} : \Sigma^* \otimes_T \Sigma \to \mathfrak{C} \) is an isomorphism.

Next, we will derive some fundamental results of \( \mathfrak{I} \). Write \( \Sigma^* = \text{Hom}_A(\Sigma, A) \), and assume that \( A \) is a ring with unit. The \( B \)-bimodule \( S = \Sigma \otimes_A \Sigma^* \) has the structure of a \( B \)-ring (without unit, in general), with associative multiplication given by

\[
\mu(x \otimes_A \phi \otimes_B y \otimes_A \psi) = x \phi(y) \otimes_A \psi = x \otimes_A \phi(y) \psi, \quad (x \otimes_A \phi \otimes_B y \otimes_A \psi) \in \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^*.
\]

The map \( S \to \text{End}(\Sigma_A) \) sending \( x \otimes_A \phi \) onto the endomorphism \( y \mapsto x \phi(y) \) is a homomorphism of \( B \)-rings. In particular, \( \Sigma \) is an \( S - A \)-bimodule. Analogously, the homomorphism \( S \to \text{End}(\Sigma^*) \) that maps \( y \otimes_A \psi \) to the endomorphism \( \phi \mapsto \phi(y) \psi \) is a homomorphism of \( B \)-rings, and \( \Sigma^* \) is then an \( A - S \)-bimodule. The situation studied in \( \mathfrak{I} \) begins with a homomorphism of rings \( \iota : R \to S \), where \( R \) is a firm ring. We will use the notation \( \iota(r) = e_r \otimes_R e_r^* \) for \( r \in R \) (sum understood). By restriction of scalars, \( \Sigma \) is then an \( R - A \)-bimodule, with the left action of \( R \) defined explicitly by \( ru = e_r e^*_s(u) \), for \( r \in R \), \( u \in \Sigma \). The \( A - R \)-bimodule structure on \( \Sigma^* \) is given by the right action of \( R \) defined by \( \phi r = \phi(e_r)e^*_s \). Now, \( S = \Sigma \otimes_A \Sigma^* \) becomes an \( R \)-ring, and thus we could take \( B = R \) without loss of generality (this was not formally the case in \( \mathfrak{I} \), where the initial ring \( B \) was assumed to be unital).

If \( \Sigma \) is firm as a left \( R \)-module, then we have, following \( \mathfrak{I} \), a natural isomorphism

\[
\nu : \text{Hom}_A(\Sigma, -) \otimes_R R \cong - \otimes_A \Sigma^* \otimes_R R \quad (h \otimes_R r \mapsto h(e_s) \otimes_A e^*_s \otimes_B r^*) \quad (27)
\]

In view of (27) the adjunction (19) leads, using the notation \( \Sigma^\dagger = \Sigma^* \otimes_R R \), to an adjunction

\[
\begin{array}{ccc}
\text{Mod}_R & \cong \otimes_R \Sigma^\dagger & \text{Mod}_A,
\end{array}
\]

whose counit is

\[
\epsilon_M : M \otimes_A \Sigma^\dagger \otimes_R \Sigma \longrightarrow M \quad (m \otimes_A \phi \otimes_R r \otimes_R x \mapsto m\phi(rx)), \quad (29)
\]
and whose unit is
\[ \eta_N : N \xrightarrow{\cong} N \otimes_R \Sigma \otimes_A \Sigma^! \quad (n \mapsto n^r \otimes_R e^s \otimes_A e^*_s \otimes_R r^s) \]

From the adjoint pair (28) we have the comonad on \( \text{Mod}_A \):
\[ (\otimes_A \Sigma^! \otimes_R \Sigma, \eta_{\otimes_A \Sigma^!} \otimes_R \Sigma, \epsilon) \tag{30} \]

Evaluating at \( A \), we obtain the \( A \)-coring \( A \otimes_A \Sigma^! \otimes_R \Sigma \cong \Sigma^! \otimes_R \Sigma \) with comultiplication
\[ \Delta^! = \eta_{A \otimes_A \Sigma^!} \otimes_R \Sigma \] and counity \( \epsilon_A \). Explicitly,
\[ \Delta^! (\phi \otimes_R r \otimes_R u) = \phi \otimes_R s \otimes_R e^t \otimes_A e^*_t \otimes_R (r^s)^t \otimes_R u \]
\[ \epsilon_A (\phi \otimes_R r \otimes_B u) = \phi(ru) \]

Moreover, the comonad (30) is determined by the commatrix coring \( (\Sigma^! \otimes_R \Sigma, \Delta^!, \epsilon_A) \), since the functor underlying to the comonad is a tensor product over \( A \).

Returning to the case where \( (\Sigma, \phi_{\Sigma}) \) is an \( R - \mathcal{C} \)-bicomodule, we have that the natural transformation (23) gives rise, by Proposition 1.1, to a homomorphism of comonads \( \text{can}^! \) defined (see (31)) at \( (X, \phi_X) \) by
\[ \text{can}^! _X = (X \otimes_A \epsilon_A \otimes_A \mathcal{C}) \circ (X \otimes_A \Sigma^! \otimes_R \phi_{\Sigma}) \]

Obviously,
\[ \text{can}^! _X = X \otimes_A \text{can}^! \tag{31} \]

where \( \text{can}^! \) is the map
\[ \text{can}^! : \Sigma^! \otimes_R \Sigma \longrightarrow \mathcal{C}, \quad \phi \otimes_B r \otimes_R u \mapsto \phi(ru_{[0]}u_{[1]}) \tag{32} \]

which turns out to be a homomorphism of \( A \)-corings, because \( \text{can}^! \) is a homomorphism of comonads. According to Proposition 1.3, the functor \( \otimes_R \Sigma : \text{Mod}_R \rightarrow \text{Comod}_\mathcal{C} \) has a right adjoint that, over a right \( \mathcal{C} \)-comodule \( (X, \phi_X) \), is defined as the equalizer in \( \text{Mod}_R \) of the pair \( (\alpha_X, \phi_X \otimes_A \Sigma^!) \). An easy computation shows that, in the present case, \( \alpha_X = X \otimes_A \alpha_{\Sigma^!} \), where
\[ \alpha_{\Sigma^!} = (\text{can}^! \otimes_A \Sigma^!) \circ \eta_{\Sigma^!} \tag{33} \]

From the diagrams (2) we get that \( (\Sigma^!, \alpha_{\Sigma^!}) \) is a \( \mathcal{C} - R \)-bicomodule. In this way, the functor defined in Proposition 1.3 becomes a cotensor product, as it is defined by the equalizer
\[ X \square_{\mathcal{C}} \Sigma^! \longrightarrow X \otimes_A \Sigma^! \xrightarrow{\eta_X \otimes_A \alpha_{\Sigma^!}} X \otimes_A \mathcal{C} \otimes_A \Sigma^! \xrightarrow{\phi_X \otimes_A \Sigma^!} X \otimes_A \Sigma^! \tag{34} \]

From Proposition 1.3 and Theorem 1.6 we deduce:
Theorem 2.9. [15, Theorem 4.9] Let \( \mathcal{C} \) be an \( A \)-coring and \( \Sigma \) a right \( \mathcal{C} \)-comodule, which is as well a left \( B \)-module. Let \( \iota : R \to \Sigma \otimes_A \Sigma^* \) be a homomorphism of rings, where \( R \) is a firm ring. If \( \Sigma \) is an \( R \)-\( \mathcal{C} \)-bicomodule such that \( \Sigma \) is firm as a left \( R \)-module, then the functor \( - \otimes_R \Sigma : \text{Mod}_R \to \text{Comod}_\mathcal{C} \) admits as a right adjoint the cotensor product functor \( - \square \Sigma^\dagger : \text{Comod}_\mathcal{C} \to \text{Mod}_R \). This functor is faithful and full if and only if \( \text{can}^\dagger : \Sigma^\dagger \otimes_R \Sigma \to \mathcal{C} \) is an isomorphism of \( A \)-corings and \( - \otimes_R \Sigma : \text{Mod}_R \to \text{Mod}_A \) preserves the equalizers of the form (31).

Theorem 2.10. Let \( \mathcal{C} \) be an \( A \)-coring and \( \Sigma \) a right \( \mathcal{C} \)-comodule, which is as well a left \( B \)-module. Let \( \iota : R \to \Sigma \otimes_A \Sigma^* \) be a homomorphism of rings, where \( R \) is a firm ring. If \( \Sigma \) is an \( R \)-\( \mathcal{C} \)-bicomodule such that \( \Sigma \) is firm as a left \( R \)-module, then the functor \( - \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_\mathcal{C} \) is an equivalence of categories if and only if \( \text{can}^\dagger : \Sigma^\dagger \otimes_B \Sigma \to \mathcal{C} \) is an isomorphism of \( A \)-corings, and \( - \otimes_B \Sigma : \text{Mod}_B \to \text{Mod}_A \) reflects isomorphisms and preserves the equalizers of the form (34).

Remark 2.11. Start from a \( B \)-\( \mathcal{C} \)-bicomodule \( \Sigma \), with \( B \) and \( B \Sigma \) firm, and consider a ring homomorphism \( \iota : B \to S = \Sigma \otimes_A \Sigma^* \) (that is, put \( B = R \) in the foregoing discussion, which essentially does not constitute a restriction, as we discussed before). This is just to say that the left multiplication ring homomorphism \( \lambda : B \to T = \text{End}(\Sigma_e) \) factorizes throughout \( S \), in the sense of the commutative diagram

\[
\begin{array}{ccc}
\Sigma \otimes_A \Sigma^* & \to & \text{End}(\Sigma_A) \\
\downarrow & & \downarrow \\
B & \to & \text{End}(\Sigma_e)
\end{array}
\]

Now, since in this case the functors \( \text{Hom}_e(\Sigma, -) \otimes_B B, - \square_e \Sigma^\dagger : \text{Comod}_\mathcal{C} \to \text{Mod}_B \) are both right adjoint to \( - \otimes_B \Sigma : \text{Mod}_B \to \text{Comod}_\mathcal{C} \), then they are isomorphic. On the other hand, from the natural isomorphism \( \nu \) given in (27), we get the commutative diagram of homomorphisms of comonads

\[
\begin{array}{ccc}
\text{Hom}_A(\Sigma, -) \otimes_B B \otimes_B \Sigma & \to & - \otimes_A \Sigma^\dagger \otimes_B \Sigma \\
\downarrow \text{can} & & \downarrow \text{can} \\
- \otimes_A \mathcal{C},
\end{array}
\]

Therefore, \( \text{can} \) is a natural isomorphism if and only if \( \text{can} \) is an isomorphism [21, Theorem 4.2]. In this way, we can add the following two statements to the list of equivalent conditions in Theorem 2.3, connecting Theorem 2.3 and [15, Theorem 4.9]:

\( (iv') \) \( \text{can} : \Sigma^\dagger \otimes_B \Sigma \to \mathcal{C} \) is an isomorphism of corings and \( B \Sigma \) is flat;

\( (v') \) the functor \( - \square_e \Sigma^\dagger : \text{Comod}_\mathcal{C} \to \text{Mod}_B \) is full and faithful.

Analogously, in Theorem 2.5 we can add the following equivalent condition, connecting Theorem 2.5 and [15, Theorem 4.15]:

\( (ii') \) \( \text{can} : \Sigma^\dagger \otimes_B \Sigma \to \mathcal{C} \) is an isomorphism of corings and \( B \Sigma \) is faithfully flat.
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