Higher curvature counter terms cause the bounce in loop cosmology

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Abstract. In the loop approach to the quantisation of gravity, one uses a Hilbert space which is too singular for some operators to be realised as derivatives. This is usually addressed by instead using finite difference operators at the Planck scale, a process known as “polymerisation”. In the symmetry reduced example of loop cosmology, we study an ambiguity in the regularisation which we relate to the ambiguity of fixing the coefficients of infinitely many higher curvature counter terms augmenting the Einstein-Hilbert action. Thus the situation is comparable to he one in a naive perturbative treatment of quantum gravity with a cut-off where the necessary presence of infinitely many higher derivative terms compromises predictability. As a by-product, we demonstrate in an appendix that it is possible to have higher curvature actions for gravity which still lead to first order equations of motion like in the Friedmann case.

Preprint LMU-ASC 58/09

1. Introduction

Isotropic and homogeneous space-times not only model the universe at the largest scales and are thus immediately relevant in cosmology but provide an important truncation of any — quantum or classical — theory of gravity. They are mini-superspace models in the sense that the symmetry reduces the infinity of degrees of freedom of the gravitational field to a finite number. As quantum theories, these are quantum mechanical models which allow to ignore the additional conceptual and technical complications of quantum field theories. Thus they provide ideal testing grounds to compare various approaches to quantum gravity in a well controlled environment with as little of notational clutter as possible.

Classically, these space-times of Friedmann-Robertson-Walker type start with a big bang singularity and a traditional canonical quantisation à la Wheeler and deWitt does not change this picture.
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Recently, however, methods from loop quantum gravity have been applied to this setting (see, for example [1, 2, 3, 4, 5, 6]). The surprising result was that the big bang singularity is resolved into a big bounce. Looking backwards in time, the universe only contracts to a maximum density and then re-expands. In the case of a closed cosmology this leads to an infinite oscillation of the universe between this bounce and the maximum expansion when the gravitational pull stops the expansion and turns it around.

In this letter, we will analyse the origin of the radically different behaviour. We will find that it is due to a regularisation that is characteristic of the loop approach: The kinematical Hilbert space is based on an invariant state. This state, however, is so singular, it does not allow canonical momenta to be represented by derivative operators: The limit of infinitesimal translations does not exist. For a discussion of alternatives of covariant states avoiding this singular behaviour, see [7, 8, 9].

The usual way to avoid this problem is to assume a minimal scale and replace derivatives by finite difference operators of the order of the Planck length $\ell_p$. In the full, non-reduced theory, this is encountered as the fact that the spatial curvature is too singular to be represented by a quantum operator but is re-expressed in terms of holonomies around loops of Planck size which do have a quantum representation. Here, in the cosmological setting, it means that the canonical momentum is not a differential operator but the difference between two translation operators.

In [10], see also [11, 12], it was shown that one can do the corresponding replacement already in the classical Hamilton function. The resulting “polymerised” equations of motion show the same bouncing behaviour as in the loop quantised cosmology. Therefore, strictly speaking, the bounce is not due to the quantisation but to the modification of the Hamiltonian (although that modification is motivated by quantum theory considerations).

There are, however, many different finite difference operators that approximate the derivative in the limit $\ell_p \to 0$. Thus, there is no canonical choice and indeed a lot of arbitrariness [13, 14]. All these different finite difference operators differ by higher derivatives times positive powers of $\ell_p$. In this note, we will investigate the consequences of this arbitrariness. In conventional loop cosmology, the momentum $p$ is replaced by $\sin(\ell_p p)/\ell_p$ but as we will argue, any function $f(\ell_p p)/\ell_p$ with $f(0) = 0$ and $f'(0) = 1$ will do. We will show that as long as $f(p)$ has a local maximum at some $p_{\text{max}} > 0$ there will be a bounce exactly when $p = p_{\text{max}}$.

By Taylor expansion, different choices for $f$ differ by higher derivative operators at 0. We will translate this ambiguity in a covariant language and show that every choice of $f$ can be related to a modification of the Einstein-Hilbert action by higher powers of the curvature tensor. The loop cosmology choice of $f = \sin$ thus corresponds to a particular, infinite collection of higher curvature counter terms in the action. But as there is no physical principle suggesting this particular choice of finite difference operator as replacement for the momentum, any other choice of higher derivative counter terms is as good as this.

We strongly believe that is conclusion not only holds in the symmetry reduced
cosmological theory but is characteristic for the loop approach and thus also holds in
the full four-dimensional field theory: Different regularization of the spatial curvature
in terms of holonomies of loops of finite size again by higher derivatives upon Taylor
expansion. If the full theory is fully covariant, the terms in this Taylor expansion are
expressible in terms of covariant derivatives of the curvature as those are the only tensors
available. Thus, one should once more expect that changes in the regulator induce higher
curvature terms in the action.

Our conclusion is therefore, that the situation regarding the uniqueness (or rather
non-uniqueness) of quantisation of gravity in the loop framework is not much better
than in a naive perturbative field theory quantisation: There, finding Feynman loop
integrals diverging one can introduce a regularisation for example via a cut-off (a role
playing by the minimal scale $\ell_P$ in loop quantum gravity) but then one has to face
the problem of an effective theory that is to determine an infinite choice of numerical
coefficients for all possible higher curvature counter terms. This freedom to choose a
counter term is paralleled in the choice of finite difference operator or an expression for
the spatial curvature in terms of holonomies in the loop approach.

This note ends in an appendix which demonstrates that at every order of the
curvature tensor in the action, one can find a linear combination of index contractions
that leads to an action which contains only first time derivatives of the scale factor and
thus does not lead to higher order equations of motion than in Einstein theory, a result
with some interest independent of the discussion of loop cosmology.

Note added: After publication of the first version of this note, I was made aware of
[15] which has significant overlap with the work presented here.

2. Classical solution

In the loop quantum gravity (LQG) literature, it is common to use an ADM-type
canonical 3+1 decomposition of the gravitational degrees of freedom. Then, in the
cosmological situation that we are interested in as a toy mini-superspace model, one
performs a symmetry reduction. Here, however, for simplicity, we will start from a
Lagrangian formulation in terms of a metric of the FRW form with flat spatial slices

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).$$

With the scale factor $a$, this contains a single degree of freedom which only depends on
the time coordinate $t$ that in our choice of gauge measures proper time. This formalism
leads to the identical Hamiltonian treatment as the ADM plus symmetry reduction
approach but is more convenient when we later want to trace the influence of higher
curvature corrections to the Einstein-Hilbert action.

The only remains of coordinate invariance are shifts $t \mapsto t + c$ which are generated by
the Hamiltonian (constraint) which will play a central role later on. These shifts however
make $a(t)$ not a good observable since $t$ has no invariant meaning. To nevertheless have
a (local in time) observable degree of freedom one can use a relational approach: One
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introduces a further scalar “clock” field $\phi$ to the theory (which for simplicity we assume to be massless and free). Then, in addition to $a(t)$ there is as well $\phi(t)$ which turns out to be a monotonic function and thus can be inverted to eliminate $t$. As a result, we obtain a true observable $a(\phi)$.

We will start from the Einstein-Hilbert action coupled to the scalar suppressing all constants and using a natural system of units

$$ S = \int d^4x \sqrt{-g}(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = \int dt(-6\dot{a}\dot{a}^2 + a^3\dot{\phi}^2). \quad (2) $$

It is important to realise that it would be wrong at this stage to vary with respect to $a(t)$ and $\phi(t)$: This would lead to second order equations of motion with a conserved energy. Doing so ignores the Hamiltonian constraint implementing the time shifts. In fact, the conserved energy is the Hamiltonian but rather finding it to be conserved one should impose the condition that it vanishes. This correct procedure leads to the Friedmann equation which is first order in time while the second order equation above is the time derivative of this equation.

This procedure is equivalent to varying the action before using the ansatz to obtain the Einstein equations and only then to plug in the metric ansatz. The Friedmann equation is then the 00-component of Einstein’s equation (the constraint) while the weaker second order equations arise from the space-space components.

Before we do so, we will change variables: Since in loop quantum gravity, it is the area which is naturally quantised in Planck units we will instead of $a$ use $v = a^3$. In terms of this, the Lagrangian reads

$$ L = -2\frac{\dot{v}^2}{3v} + v\phi'^2. \quad (3) $$

To proceed to a Hamiltonian formulation, we need the canonical momenta

$$ \beta = \frac{\partial L}{\partial \ddot{v}} = -4\frac{\dot{v}}{3v} \quad p = \frac{\partial L}{\partial \dot{\phi}} = 2v\dot{\phi} \quad (4) $$

and obtain the Hamiltonian

$$ H = -\frac{3}{8}v\beta^2 + \frac{1}{4}p^2 = \frac{1}{v}\left(\sqrt{\frac{3}{8}v\beta + \frac{1}{2}p}\right)\left(-\sqrt{\frac{3}{8}v\beta + \frac{1}{2}p}\right). \quad (5) $$

This, as explained above, leads to the constraint $\sqrt{3/8v\beta} \approx \pm p/2$ (as usual equations with $\approx$ hold on the constraint surface). We have written the Hamiltonian in this factorised form as $v$ is always positive and thus depending on the signs of $\beta$ and $p$ only one of the factors can vanish. Swapping the two factors corresponds then to time reversal. This factorisation is useful since if $H = H_1H_2$ and in fact $H_1 \approx 0$, one can express the time evolution of an phase space function $A$ as $\{H,A\} \approx \{H_1,A\}H_2$. As long as one is only interested in ratios of time derivatives of phase space functions one can thus use $H_1$ instead of $H$ as we will do repeatedly below.

The Hamilton equations of motion

$$ \dot{p} = 0 \quad \dot{\phi} = \frac{p}{2v} \quad \dot{v} = -\frac{3}{v}\beta \approx \pm \sqrt{3/8}p \quad (6) $$
are solved by \( v = |\sqrt{3/8}p(t-t_0)| \) and \( \phi = \sqrt{2/3} \ln |t-t_0| \). This reproduces the expected big bang/big crunch singularity \( v = 0 \) at finite proper time \( t = t_0 \).

The evolution of the volume as a function of the clock field \( v = \exp(\pm \sqrt{3/2}(\phi - \phi_0)) \) has the singularity at \( \phi = \pm \infty \). It is either found algebraically from the solutions \( v(t) \) and \( \phi(t) \) or directly by solving the gauge invariant equation of motion

\[
\frac{dv}{d\phi} = \frac{\dot{v}}{\dot{\phi}} = \frac{\partial H/\partial \beta}{\partial H/\partial p} = -\frac{3v^2}{2p} \approx \pm \sqrt{3/4}v. \tag{7}
\]

This concludes our discussion of the classical system (2).

3. The LQG modification

The Hamiltonian (5) is simple enough to be quantised canonically by considering wave functions \( \Psi(v,\phi) \) and replacing \( v \) and \( \phi \) by multiplication operators and the momenta by derivatives \( \hat{\beta} = -i\partial/\partial v \) and \( \hat{p} = -i\partial/\partial \phi \). This Wheeler-deWitt equation can then be turned into a wave equation by the substitution \( x = \ln(v) \).

This simple quantisation is, however, not available in the context of loop quantum gravity. There, the kinematical Hilbert space is constructed as the GNS representation built upon an invariant state (for an extended discussion, see for example [7, 8, 9]. The invariance (as opposed to covariance as in the Wheeler-deWitt theory) implies a discontinuous representation of at least one of the unitary operators which can be thought of the exponentials of the canonical variables \( v \) and \( \beta \).

As a result, the derivative \(-i\partial/\partial v\) is too singular to be defined in this “polymer Hilbert space”. In the full, covariant theory this is encountered as the problem that the spatial curvature is too singular to be defined as an operator (as it would be a derivative in a conventional treatment).

The solution found in the loop literature is to observe that although the curvature does not exist as a quantum operator, the holonomies do and the curvature can be approximated by holonomies around small loops. In the reduced theory discussed here, this translates to the observation that although the momentum \( \beta \) cannot be defined as a quantum operator in the polymer space, the translation operator for a finite distance in \( v \) can via \( U(\ell)\Psi(v,\phi) = \Psi(v+\ell,\phi) \). This operator can be viewed as the quantisation of the Weyl operator \( \exp(i\ell\beta) \) which does not suffer the problems of the quantisation of \( \beta \).

In the full theory, it is then appealed to foaminess of space-time in a theory of quantum gravity and any occurrence of the spatial curvature is replaced by an expression in terms of holonomies around loops of Planck size which classically converge to the curvature in the limit of zero loop size. As in the quantum theory the limit does not exist one is satisfied with finite size loops of Planck length.

To parallel this procedure, in loop quantum cosmology, the occurrence of the momentum \( \beta \) in the Hamiltonian is replaced by the operator

\[
\tilde{\beta} = \frac{U(\ell) - U(-\ell)}{2i\ell}. \tag{8}
\]
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This replacement is known as "polymerisation", see for example [16]. Note well, that this is the quantisation of the classical $\sin(\ell \beta)/\ell$.

The distance $\ell$ is of course kept fixed as the limit $\ell \to 0$ does not exist in the quantum theory. This is motivated by observing that in the classical theory, $\lim_{\ell \to 0}(\exp(i\ell \beta) - \exp(-i\ell \beta))/2i\ell = \beta$. As in the numerical solution of differential equations, the derivative $\hat{\beta} = i\partial/\partial v$ is replaced by the finite difference operator $\tilde{\beta}$.

At this point, the change of variables from $a$ to $v = a^3$ is important: The operator $\tilde{\beta}$ relates the wave function at two different points of constant difference $\Delta v = \ell$ rather than at constant $\Delta a$. We see that the regularisation explicitly breaks coordinate independence and singles out the coordinate $v$.

The regularisation is, however, far from unique. In a section below, we will study the consequences of this arbitrariness.

In this LQC modified quantum theory, it is then found [1, 2, 3, 4, 5, 6] that the time evolution is radically different compared to the classical theory investigated above: The contraction (looking back in time) is is stopped at a finite value for $v$ and $\beta$ and, instead of a big bang or big crunch, there is a bounce and the universe re-expands.

As explained in [10], see also [11, 12], one does not have to (possibly numerically) solve the quantum time evolution equation (for effective actions used in loop cosmology, see also [17, 18, 19]). It is sufficient to replace $\beta$ by $\sin(\ell \beta)/\ell$ in the classical Hamiltonian, viz

$$\tilde{H} = -\frac{3}{8} \frac{v \sin(\ell \beta)^2}{\ell^2} + \frac{p^2}{4v} = \frac{1}{v} \left( \sqrt{\frac{3}{8}} \frac{v \sin(\ell \beta)}{\ell} + \frac{p}{2} \right) \left( -\sqrt{\frac{3}{8}} \frac{v \sin(\ell \beta)}{\ell} + \frac{p}{2} \right)$$

(9)

and solve the resulting equations of motion. In particular, for the gauge invariant evolution, one now finds

$$\frac{dv}{d\phi} = \frac{\dot{v}}{\phi} = \frac{\partial H}{\partial \beta} = \sqrt{\frac{3}{2}} \frac{v \cos(\ell \beta)}{p} \approx \pm \sqrt{\frac{3}{2}} v^2 - (\ell p)^2.$$  

(10)

This is solved by $v = \sqrt{3/2\ell|p| \cosh(\sqrt{3/2}(\phi - \phi_0))}$. Obviously, this solution is a bouncing universe with a minimum at $v_{\text{min}} = \sqrt{3/2\ell|p|}$. Note well that the momentum (curvature) at the minimum is $\beta_{\text{max}} = \pi/2\ell$ independent of initial conditions. Going back to the equation of motion, we see that the universe bounces when $\dot{v} = \partial H/\partial \beta = 0$.

We conclude that the bouncing behaviour characteristic for loop quantum cosmology can already be found in the classical evolution once one applies the LQG substitution $\beta \mapsto \sin(\ell \beta)/\ell$ in the Hamilton function. One can then obtain the LQC wave equation (the finite difference equation) form the Wheeler-de Witt-quantisation of the LQG-modified Hamiltonian as $\exp(\ell \partial_v)$ is in fact the translation operator $\Psi(v) \mapsto \Psi(v + \ell)$. Thus we can view loop quantum cosmology as an ordinary quantum theory with a modified Hamiltonian.
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Figure 1. Evolution of the scale factor: LQC bounce (solid), classical expansion (dashed) and classical contraction (dotted)

4. Counter term ambiguities

As it is well known from numerical evaluation of derivatives, the replacement of the derivative with respect to $v$ by the finite difference operator is far from unique. This was already stressed in [13, 14], lattice refinement for loop cosmology was also addressed in [20]. One could as well use

$$\tilde{\beta}' = \frac{U(2\ell) - U(-2\ell)}{4i\ell}$$ (11)

or for example a linear combination

$$\tilde{\beta}_\alpha = \alpha \frac{U(\ell) - U(-\ell)}{2i\ell} + (1 - \alpha) \frac{U(2\ell) - U(-2\ell)}{4i\ell}$$ (12)

or a linear combination of even more translation operators. All these have in common that the Taylor expansion for small $\beta$ starts with $\beta + O(\ell^2 \beta^3)$. For example, we have $\tilde{\beta}_\alpha = \beta + \frac{3\alpha - 4}{6} \ell^2 \beta^3 + O(\ell^4 \beta^5)$. In general, with the appropriate choice of $n$ translation operators $U$, one pick the first $n$ terms in the Taylor expansion for small $\beta$.

Therefore, with arbitrary precision for small $\beta$ (for example up to a possible maximum at a bounce, see below), one can approximate any function $f(\ell \beta)/\ell$. The only physical condition is the $\beta \to 0$ limit (the “unloopy limit”) in which it should be given by $\beta$, that is $f(0) = 0$ and $f'(0) = 1$.

More generally, one could also use an expression that also depends on the other variables $v$, $\phi$, and $p$. An example would be the above mentioned possibility to consider constant step sizes in $a$ rather than $v$. In this letter, we will for simplicity, however, restrict our attention to functions $f$ which only depend on $\beta$.

For general such $f$, the Hamiltonian reads

$$H_f = \frac{-3}{8} \frac{vf(\ell \beta)}{\ell^2} + \frac{p^2}{4v} = \frac{1}{v} \left( \sqrt{\frac{3}{8} \frac{vf(\ell \beta)}{\ell}} + \frac{p}{2} \right) \left( -\sqrt{\frac{3}{8} \frac{vf(\ell \beta)}{\ell}} + \frac{p}{2} \right)$$ (13)
the gauge invariant evolution equation corresponding to (7) and (10) now reads
\[
\frac{dv}{d\phi} = \frac{\dot{v}}{\dot{\phi}} = \frac{\partial H/\partial \beta}{\partial H/\partial p} = \sqrt{\frac{3}{2}} v f'(\ell \beta).
\]
(14)

We see that there is a bounce when \( f'(\ell \beta) = 0 \) for some \( \beta \) whereas there is a big bang/big crunch when \( f \) is strictly monotonic. For example, the classical case (5) corresponds to \( f(x) = x \) and has an initial singularity whereas the LQC case with \( f = \sin \) bounces.

Coming from a loop perspective, one would restrict attention to functions \( f \) that can be build out of Weyl operators \( U(\cdot) \) and symmetry then restricts one to linear combinations \( f(x) = \sum_n a_n \sin(b_n x) \). Obviously, finite sums always have maxima and thus always bounce. But even in the polymer Hilbert space, one can have infinite sums as long as \( \sum_n |a_n|^2 < \infty \). Amongst those there is for example
\[
f_s(x) = \frac{1}{(32^{1/4} - 1) \zeta(-1/4)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}} \sin(nx),
\]
(15)

which is monotonic between \(-\pi/2 < x < \pi/2\) and has poles at \(|x| = \pi/2\). The time evolution with this polymerisation function is even more singular than in the classical case as it reaches \( v = 0 \) for finite \( \phi \).

All possible choices for \( f \) will differ by higher powers of \( \ell \beta \) which in the conventional quantisation corresponds to higher derivatives with respect to \( v \) with coefficients that vanish for \( \ell = 0 \). In the remainder of this section, we will argue that these higher derivative terms can be translated back to the action where they correspond to the addition of higher curvature corrections to the Einstein-Hilbert action. Thus, any choice for a regularising function \( f \) corresponds to a particular choice of higher curvature counter terms in the action.

One should compare this to what one would do in a naive perturbative quantisation of gravity: There, one would expand the Einstein-Hilbert action around some background to obtain Feynman rules and then would find that the Feynman integrals are divergent and the theory is non-renormalisable since Newton’s constant which controls the perturbative expansion has negative mass dimension. All integrals, however, are finite if one uses some regularisation (for example point-splitting where operators cannot get closer than a distance \( \ell \)). But then, one should treat gravity as an effective theory and not only include the Einstein-Hilbert term but all possible scalars that can be built out of powers of covariant derivatives of the curvature. An infinite number of coefficients would have to be determined experimentally and would be subject to change once one changes the regulator.

We see that here the situation is very similar: By picking a regularising function \( f \) which avoids short distance singularities as in a differential quotient representing \( \beta \) in traditional quantum mechanics, one implicitly chooses a particular set of coefficients for the higher curvature counter terms. The ambiguity in \( f \) can be mapped to the well known ambiguity in picking a higher derivative effective action.

To see this, let us first translate the LQG corrected Hamiltonian (13) back to a Lagrangian \( L = \dot{v} \beta + \dot{\phi} p - H_f \). Unfortunately, this cannot be done explicitly for general
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$f$ since one has to invert $\dot{v} = -(3/4)v f(\ell \beta) f'(\ell \beta)/\ell$ to find $\beta$ as a function of $\dot{v}$. Thus, we present it here in the LQC case $f = \sin$. We find $\beta = -\arcsin(8\ell \dot{v}/3v)/2\ell$ and eventually

$$\bar{L} = \frac{3a^3}{16\ell^2} \left(1 - \sqrt{1 - s^2} - s \arcsin(s)\right) + a^3 \dot{\phi}^2,$$

where we defined $s = 8\ell(\dot{a}/a)$ which is related to the square root of the curvature. This Lagrangian contains arbitrary even powers of the derivative $\dot{a}$ and, in the limit $\ell \to 0$, reduces to [3]. It was already discussed in [15].

As a final step, we want to discuss how such a higher derivative Lagrangian can arise from a higher curvature covariant action in four dimensions. There is, however, a minor technical difficulty: For the metric [1], the Ricci scalar is

$$R = 6(\dot{a}/a)^2 - 6\ddot{a}/a,$$

and the Einstein-Hilbert Lagrangian $\mathcal{L} = \sqrt{-g}R$ not only contains $\dot{a}$ but as well the second derivative $\ddot{a}$. So far, this did not cause any trouble when proceeding to the Hamiltonian setting, since one can integrate by parts to get rid of second time derivatives. If one naively adds higher powers of the Ricci scalar $R^n$ to the action, not all second derivatives $\ddot{a}$ can be eliminated by integration by parts anymore and thus [16] can not be written covariantly only in terms of the Ricci scalar. It is well known in the literature, that the generalisation of the Friedmann equation including generic higher order curvature correction ceases to be first order.

This difficulty can be avoided once one includes traces of higher powers of the Ricci tensor $R^{(n)} = R_{\mu_1}^{\mu_2} R_{\mu_2}^{\mu_3} \cdots R_{\mu_n}^{\mu_1}$. In the appendix, it is shown that up to a total derivative, any function $a^3P((\dot{a}/a)^2)$ for analytic $P$ can be written as $\sqrt{-g}$ times a power series in the $R^{(n)}$. In particular, this is true for the LQC Lagrangian $\bar{L}$ above (cf. [16]).

We have therefore shown that the polymerisation of the Hamilton operator by replacing $\beta$ by $f(\ell \beta)/\ell$ is equivalent to a modification of the Einstein Hilbert action by higher order curvature corrections. Different choices for the function $f$ correspond to different choices of higher derivative counter terms.

This can be compared to a naive field theoretic quantisation of gravity: Even though the Einstein theory is non-renormalisable, one can still expand around a fixed background and compute Feynman loop integrals. Those, of course, will be divergent but can be regularised for example by point splitting. To balance the effect of changing the regulator one then has to add an infinity of counter terms with coefficients to be determined by experiment. Any choice of counter terms is as good as any other as an effective theory.

We find exactly the same situation in loop quantum cosmology: One evades short distance singularities by point splitting in time (replacing derivatives by finite difference operators with step size of order $\ell$) and this induces an infinite series of higher curvature corrections. Once more, the regularisation is far from unique and changing it changes the numerical coefficients of the counter terms.
Thus the finiteness properties of loop cosmology are just the same as in naive field theoretic quantisation with a finite cut-off (or point splitting). Presumably this conclusion extends to the full, symmetry unreduced theory of loop quantum gravity.

5. Conclusions

Perturbative quantum gravity in four dimensions is known to be non-renormalisable. Still, it can be made finite by hand by regulating divergent loop integrals, for example by point splitting. Slight changes in the regulator will induce changes of the coefficients of an infinity of higher curvature terms in the effective action that have positive powers of the cut-off scale as coefficients. This suggests that those higher curvature counter terms should have been included in already at the classical action and their coefficients eventually have to be determined experimentally as they are not predicted by the theory. Such a treatment of quantum gravity is not satisfactory due to its lack of predictability.

Loop quantum gravity is a non-perturbative approach to the quantisation of gravity supposedly avoiding these problems from non-renormalisability by manifest finiteness as at no stage divergent Feynman integrals have to be computed. In this note, we have given evidence that this not the case and in fact the situation in loop quantum gravity is the same as in the naive, perturbative treatment. We have shown how he same problems of ambiguities in higher curvature terms reappear also in the loop approach.

There, the “polymer” Hilbert space does not allow to define operators for the spatial curvature as the derivative with respect to the holonomies that it would be in conventional quantisation is too singular to exist. Therefore, the curvature has to be approximated by holonomies around Planck sized loops. Classically, this approximation would be exact in the limit of shrinking the loop to a point, but in the quantum theory it is kept constant at the Planck scale. This step was argued to be the analog of regularisation and is well known not to be unique.

In this note, we studied the ambiguity of this regularisation: There is an infinity of inequivalent expression in terms of holonomies that approximate the curvature once the loops shrink to points.

We studied this in detail in the symmetry reduced version of loop quantum gravity known as loop cosmology. There, the scale factor $a$ and a scalar field $\phi$ only depend on time and instead of representing the canonical momentum for $a$ (or the volume element $v = a^3$) by a derivative one regulates it by a finite difference operator at lattice spacing proportional to $\ell$.

This “polymerisation” can already be performed at the classical level by replacing the momentum $\beta$ by $\sin(\ell \beta)/\ell$. Once more, this regularisation is far from unique and the ambiguity can be parametrised by a function $f$ with $f(0) = 0$ and $f'(0) = 1$. Any such $f$ leads to a different regularisation.

We then translated this ambiguity back to the level of the action where we demonstrated that different $f$ correspond to the addition of different higher curvature corrections to the Einstein-Hilbert action. Therefore, in the loop treatment, it seems,
there is the same lack of predictability due to an infinite number of possible counter-
terms with unknown coefficients as in the naive, perturbative approach. This has not
been considered so far since only a single polymerisation function \( f = \sin \) has been
considered in the literature. Unfortunately, there is no physical principle that singles
out this particular choice and others should be regarded as equally well motivated as
this.

We used the fact that the regularisation can already be applied to the classical
theory before quantization although the modification is only required after quantization
to give well defined operators in the appropriate (polymer) Hilbert space. This modified
classical theory, however, already shows features that are supposedly quantum in nature
like a bounce avoiding a big bang/big crunch singularity. By restricting our attention
to this modified classical theory we could avoid the question if the full quantised theory
actually exists and if it can be described by a simple effective action as we have been
using it.

Acknowledgments

The author would like to thank John Baez, Giuseppe Policastro, and Ivo Sachs for
discussions and suggestions and Ghanashyam Date for pointing out [15]. This work was
supported by the Elitenetzwerk Bayern.

Appendix

In this appendix, we will show that it is possible to write every even power of \( (\dot{a}/a)^{2n} \)
as a linear combination of contractions of \( n \) powers of the Ricci tensor up to a total
time derivative. Thus, every effective Lagrangian that arises by polymerising with some
function \( f \) can be obtained from a covariant four-dimensional Lagrangian.

This result is of independent interest since it shows that it is possible to have
gravity with higher curvature corrections while maintaining the first order nature of the
corrected Friedmann equation.

The problem arises as the Ricci scalar \([17]\) alone not only contains \((\dot{a}/a)^2\) but a
second derivatives \(\ddot{a}/a\) as well. A term that contains only one factor \(\ddot{a}\) can be converted
to a term containing only \(a\) and \(\dot{a}\), but this is not possible for higher powers \(\ddot{a}^k\). We will
show, that at each power \(n\) of the curvature, it is possible to find a linear combination
of contractions of the Ricci tensor (as the metric \([1]\) is conformally flat, the Weyl tensor
vanishes) that contains only \((\dot{a}/a)^{2n}\) as well as \((\ddot{a}/a)(\dot{a}/a)^{2n-2}\) which can be converted
to the wanted term by integration by parts.

It is convenient to go the conformal frame by changing to a new time coordinate \(T\)
defined by \(dT/dt = 1/a\) in which the metric reads

\[
ds^2 = a^2(-dT^2 + dx^2 + dy^2 + dz^2).
\]
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In this metric, the Ricci tensor is

\[ R_{\mu}^{\nu} = \left( 3(a'/a)^2 - 3a''/a^3 \right) - ((a'/a)^2 + a''/a^3) I_3 \]  

(19)

where we denote \( a' = da/dT = a\dot{a} \) and \( a'' = d^2a/dT^2 = a\ddot{a} + a^2\dot{a}. \) As before, we denote traces of powers of the Ricci tensor by \( R^{(n)} = R_{\mu_1}^{\mu_2}R_{\mu_2}^{\mu_3} \cdots R_{\mu_n}^{\mu_1}. \) \( R^{(1)} = -6a''/a^3 \) is the Ricci scalar and does not contain the “wanted term” \((a'/a)^2 = (\dot{a}/a)^2\) without second derivatives.

Let us write \( A_{k,l} \) for \((a'/a^2)^{2k}(a''/a^3)^l\) with \( k, l \geq 0. \) We want to show that for each \( n \) there is a polynomial \( P_n \) in the \( R^{(k)} \) which is a linear combination of only \( A_{n,0} \) and \( A_{n-1,1}. \) For \( n > 1, P_n \) is not a multiple of \( A_{n-1,1}. \)

We prove this by induction on \( n. \) For \( n = 1, \) this is true since \( R^{(1)} = -6A_{0,1}. \) Now assume, we found the \( P_l \) for all \( l < n. \) We start with \( R^{(n)}, \) this contains \( A_{n,0} \) with coefficient \( 3^n + 3(-1)^n \) which does not vanish for \( n > 1. \) In general, it is a linear combination of \( n + 1 \) terms \( A_{n-k,k} \) with \( 0 \leq k \leq n. \) Now observe that, by induction, \( P_k := P_l R^{(1)} = A_{n-k,k} \) contains only \( A_{n-k,k} \) and \( A_{n-1-k,k+1} \) for \( n - 1 \) values \( k = 1, \ldots, n - 1. \) These form a basis of the vector space spanned by \( \{A_{n-2,2}, A_{n-3,3}, \ldots, A_{0,0}\}. \) Thus it is possible to find \( P_n \) as a linear combination of \( R^{(n)} \) and the \( P_k^m. \)

This completes our proof, since \( A_{n,0} = (\dot{a}/a)^{2n} \) and \( A_{n-1,1} = (\dot{a}/a)^{2n} + (\dot{a}/a)^{2n-2}\dot{a}/a \) and since the second term can be converted to \((\dot{a}/a)^{2n}\) as well by an integration by parts up to a total derivative.

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