NONCOMMUTATIVE REPRODUCING KERNEL HILBERT SPACES

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Abstract. The theory of positive kernels and associated reproducing kernel Hilbert spaces, especially in the setting of holomorphic functions, has been an important tool for the last several decades in a number of areas of complex analysis and operator theory. An interesting generalization of holomorphic functions, namely free noncommutative functions (e.g., functions of square-matrix arguments of arbitrary size satisfying additional natural compatibility conditions), is now an active area of research, with motivation and applications from a variety of areas (e.g., noncommutative functional calculus, free probability, and optimization theory in linear systems engineering). The purpose of this article is to develop a theory of positive kernels and associated reproducing kernel Hilbert spaces for the setting of free noncommutative function theory.

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1. Introduction

The goal of the present paper is to incorporate the classical theory of positive kernels and reproducing kernel Hilbert spaces (see [4,1]) into the new setting of free noncommutative function theory (see [31]).

We use the following operator-valued adaptation of the notion of positive kernel developed in some depth by Aronszajn in [4]. Let \( \Omega \) be a point set and \( K \) a function from the Cartesian product set \( \Omega \times \Omega \) into the space \( \mathcal{L}(\mathcal{Y}) \) of bounded linear operators on a Hilbert spaces \( \mathcal{Y} \). We say that \( K \) is a positive kernel if

\[
\sum_{i,j=1}^{N} \langle K(\omega_i, \omega_j)y_j, y_i \rangle_E \geq 0 \quad (1.1)
\]

for all \( \omega_1, \ldots, \omega_N \in \Omega, \ y_1, \ldots, y_N \in \mathcal{Y}, \ N = 1, 2, \ldots \). Equivalent conditions are:

- There is a Hilbert space \( \mathcal{H}(K) \) consisting of \( \mathcal{Y} \)-valued functions on \( \Omega \) such that \( K \) has the following reproducing kernel property with respect to \( \mathcal{H}(K) \):
  1. for any \( \omega \in \Omega \) and \( y \in \mathcal{Y} \) the function \( K_{\omega,y} \) given by \( K_{\omega,y}(\omega') = K(\omega', \omega)y \) belongs to \( \mathcal{H}(K) \), and
  2. for all \( f \in \mathcal{H}(K) \) and \( y \in \mathcal{Y} \), the reproducing property

\[
\langle f, K_{\omega,y} \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}} \quad (1.2)
\]

holds.

- There is a Hilbert space \( \mathcal{X} \) and a function \( H: \Omega \to \mathcal{L}(\mathcal{H}(K), \mathcal{Y}) \) so that the following Kolmogorov decomposition holds:

\[
K(\omega', \omega) = H(\omega')H(\omega)^* \quad (1.3)
\]

Moreover, in case \( \Omega \) is a domain in \( \mathbb{C} \) (or more generally in \( \mathbb{C}^d \)), then analyticity of \( K \) in its first argument leads to analyticity of the elements \( f \) of \( \mathcal{H}(K) \) and vice versa. Such kernels and associated reproducing kernel Hilbert spaces actually have origins from the early part of the twentieth century (we mention in particular the original works [32,51] as well as the introduction in [4] and the survey paper [46] for an overview of the history); in particular, one can argue that the Kolmogorov decomposition property (1.3) is actually due to Moore [32] but we follow what has become standard terminology in operator theory circles. Over the following decades, reproducing kernel Hilbert spaces have served as a powerful tool of analysis in a number of areas (in particular, complex analysis and operator theory)—the recent book [1] gives a glimpse of the operator-theory side of this activity.

Much more recent is the axiomatization of what is now being called free noncommutative function theory (see [31]). One version of this theory can be viewed as an extension of the theory of holomorphic functions of several complex variables \( z = (z_1, \ldots, z_d) \) to a theory of functions of matrix tuples \( Z = (Z_1, \ldots, Z_d) \), where now the tuple \( Z = (Z_1, \ldots, Z_d) \) consists of freely noncommuting \( n \times n \) matrices with size \( n \in \mathbb{N} \) also allowed to be
arbitrary. In brief, a noncommutative function is a mapping from \( n \times n \) matrices over the domain set to \( n \times n \) matrices over the range set which respects direct sums and similarities—see Section 2 for precise definitions and the monograph [31] for a complete treatment. This axiomatic framework has appeared in the work of Taylor [47] (see also [48]) as a setting for the study of a functional calculus for tuples of freely noncommuting matrix variables, and independently in the work of Voiculescu and collaborators (see [49, 50]) in the context of the needs of a free-probability theory, as well as in the work of Helton-Klep-McCullough and collaborators (see [25, 26, 27, 29, 11]) on free noncommutative Linear-Matrix-Inequality relaxations and connections with stability analysis and optimization problems in linear systems engineering. It is also closely connected with the functional calculus arising from plugging in freely noncommuting operators for the free indeterminates in a formal power series, making use of the tensor products for the operator multiplication in each term of the series (see [10]). Such formal power series appeared much earlier in connection with the theory of automata and formal languages in the work of Fliess, Kleene, and Schützenberger in the middle part of the last century (see [17] for an overview), and have also occurred in Multidimensional System Theory as the transfer functions for input/state/output linear systems with evolution along a tree [9]. The recent monograph [31] of Kaliuzhnyi-Verbovetskyi and the current third author completes the work of Taylor by developing the theory of free noncommutative functions from first principles. In particular, there it is shown how the “respects direct sums” and “respects similarities” properties combined with some mild additional assumptions (“local boundedness”) leads to Taylor-Taylor series developments (at least locally) for a general noncommutative function, as is obtained globally when one starts with a formal power series and plugs in the components of a freely noncommutative operator tuple as arguments. One can also view Free Noncommutative Function theory as a nonlinear extension of Operator Space theory (see [18, 23, 40, 42])—we explore some aspects of these connections in Section 3.3 below.

In the present paper we extend the theory of noncommutative functions in [31] to a theory of noncommutative kernels. The affine version of such kernels already appears in [31] in connection with the first order difference-differential calculus for noncommutative functions. To develop a theory of kernels generalizing that of Aronszajn for the classical case, what is actually needed is a modification of these kernels (called here simply nc kernels) which treats the second variable as a conjugate variable (see condition (2.9) below). In analogy with the definition of noncommutative function, noncommutative kernels are required to satisfy a “respect direct sums” and “respect similarities” condition, now with respect to both variables. In addition, noncommutative (affine or general) kernels have values in \( \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y})) \) rather than just in \( \mathcal{L}(\mathcal{Y}) \), where \( \mathcal{A} \) is the vector space of point variations in the case of affine kernels, and is required to carry some additional order structure coming from another relevant \( C^* \)-algebra, or more generally,
from an operator system $\mathcal{A}$ (closed selfadjoint subspace of $\mathcal{L}(\mathcal{E})$ for some Hilbert space $\mathcal{E}$). It turns out to be useful to introduce also the notion of global function and global kernel, where one discards the “respects similarities” constraint and demands only the “respects direct sums” condition. Then the analogue of an Aronszajn positive kernel is a completely positive noncommutative (or global) kernel for which one insists that the map $K(Z,Z): \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ be a positive map from $\mathcal{A}$ into $\mathcal{L}(\mathcal{Y})$ for each fixed $Z$ in the underlying noncommutative/global point set. Such positive kernels can be seen as more elaborate versions of the notion of completely positive map as occurs in the operator algebra literature (see \cite{40}) as well as of the notion of completely positive kernel as introduced and developed by Barreto-Bhat-Liebscher-Skeida \cite{16} (see Section 3.3 below for details).

Specifically, given a completely positive global or noncommutative kernel, we obtain a complete analogue of conditions (1.1), (1.2), (1.3) and their mutual equivalences (see Theorem 3.1 below). In particular, the new condition (2) involves the existence of a Hilbert space whose elements now consist of noncommutative functions from the $C^*$-algebra $\mathcal{A}$ to the Hilbert coefficient space $\mathcal{Y}$ which is also equipped with a canonical unitary $*$-representation $\sigma: \mathcal{A} \to \mathcal{H}(K)$. The kernel elements $K_{\omega,y}$ in condition (1.2) are now more elaborate and reproduce not only the functional values of an element $f$ of the space $\mathcal{H}(K)$ but also the functional value of the result of the action of an element $a \in \mathcal{A}$ via the representation acting on $f (\sigma(a)f)$. We also obtain an equivalent “lifted-norm” characterization of such a spaces, where the factor $H$ in the Kolmogorov decomposition (1.3) has a prominent role (see Section 3.2).

We also develop a number of additional structure properties of a space $\mathcal{H}(K)$. Section 3.3 develops automatic complete contractivity and complete boundedness properties for the maps $K(Z,Z)$ and $f(Z)$ (for each $f \in \mathcal{H}(K)$) and relates these results with similar such results in the operator-algebra literature (corresponding to the case where the point set $\Omega$ is the noncommutative envelope of a single point set $\Omega_1 = \{s_0\}$). Section 3.4 builds on results from \cite{31} on smoothness properties for noncommutative functions to develop the correspondence between smoothness properties of the kernel function $K$ and smoothness properties for elements $f$ of $\mathcal{H}(K)$. Section 3.5 makes the identification between the notion of formal reproducing kernel Hilbert space introduced earlier by two of the present authors in \cite{14} (see also \cite{15}) and a noncommutative reproducing kernel Hilbert space introduced here.

Historically, an important notion in applications of reproducing kernel Hilbert spaces to operator theory has been that of a multiplier between two reproducing kernel Hilbert spaces, i.e., an operator-valued function $S$ so that the operator $M_S: f(z) \mapsto S(z)f(z)$ maps the reproducing kernel Hilbert space $\mathcal{H}(K')$ to the reproducing kernel Hilbert space $\mathcal{H}(K)$. Our final Section 4 develops the analogue of this notion for the free noncommutative setting. In particular, there is a free noncommutative analogue of the
de Branges-Rovnyak kernel
\[ K_S(z, w) = K(z, w) - S(z)K'(z, w)S(w)^*, \]
complete positivity of which gives a characterization of the noncommutative function \( S \) being a contractive multiplier from \( \mathcal{H}(K') \) into \( \mathcal{H}(K) \). We also obtain the free noncommutative analogue of the theory of de Branges-Rovnyak spaces and the de Branges-Rovnyak theory of minimal decompositions and Brangesian complementary spaces. In our followup paper [13] we use various parts of the material in the present paper to develop further the work of Agler-McCarthy [2, 3] on noncommutative Pick interpolation and transfer-function realization and introduce a free noncommutative version of “complete Pick kernel” (see [1] for the classical case), as well as develop the interpolation and transfer-function realization theory for the more general noncommutative Schur-Agler class over the noncommutative domain \( \mathbb{D}_Q \) associated with any noncommutative defining function \( Q \).

2. Global and noncommutative function theory

2.1. Noncommutative sets. Let \( S \) be a set. We define \( S_{nc} \) to be the disjoint union of \( n \times n \) matrices over \( S \):
\[ S_{nc} = \bigoplus_{n=1}^{\infty} S^{n \times n}. \]
We let \( S_n \) denote the intersection \( S_n = S_{nc} \cap S^{n \times n} \). A subset \( \Omega \) of \( S_{nc} \) is said to be a noncommutative (nc) set if \( \Omega \) is closed under direct sums:
\[ Z \in \Omega_n, W \in \Omega_m \Rightarrow Z \oplus W = \begin{bmatrix} Z & 0 \\
0 & W \end{bmatrix} \in \Omega_{n+m}. \]

If \( \Omega \) is a subset of \( S_{nc} \) which is not already a nc subset of \( S_{nc} \), it is convenient to introduce the notation \([\Omega]_{nc} \) for the nc envelope of \( \Omega \), i.e., the smallest nc subset of \( S_{nc} \) which contains \( \Omega \). More precisely (see [13, Proposition 2.9]), a point \( Z \in S_{nc} \) is in \([\Omega]_{nc} \) exactly when it has a representation as
\[ Z = \begin{bmatrix} Z^{(1)} & \ldots & Z^{(N)} \end{bmatrix} \]
where each \( Z^{(j)} \in \Omega \) \( (j = 1, \ldots, N) \), or equivalently,
\([\Omega]_{nc} \) is the intersection of all noncommutative subsets of \( S_{nc} \) containing \( \Omega \).

We shall from time to time assume that the set \( S \) carries some additional structure. Examples are as follows:

- \( S \) is a vector space \( V \) over the complex numbers \( \mathbb{C} \): One can use the module structure of \( V \) over \( \mathbb{C} \) to make sense of a matrix multiplication of the form \( \alpha \cdot X \cdot \beta \) where \( \alpha \in \mathbb{C}^{n \times m}, X \in V^{m \times k}, \beta \in \mathbb{C}^{k \times n} \). Note that more generally one can replace \( V \) with a module \( \mathcal{M} \) over a unital abelian ring \( \mathcal{R} \) and then replace the role of \( \mathbb{C} \) with the ring \( \mathcal{R} \) (as is done in [31]), but we shall be mainly interested in the case where \( V \) is equipped with some additional structure which forces a complex vector-space structure on \( V \).
- \( V \) is a concrete operator space, i.e., there is a Hilbert space \( \mathcal{H} \) such that \( V \) is equal to a linear subspace of \( \mathcal{L}(\mathcal{H}) \) (our notation for the space of bounded linear operators on \( \mathcal{H} \)). In this case \( V \) inherits
a norm from \( \mathcal{L}(\mathcal{H}) \). Moreover matrices over \( \mathcal{L}(\mathcal{H}) \), i.e., \( \mathcal{L}(\mathcal{H})^{n \times m} \), can be viewed as the space of bounded linear operators from the Hilbert-space direct sum \( \mathcal{H}^m = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) (\( m \)-fold direct sum) into \( \mathcal{H}^n \) and hence inherits a canonical operator norm from its canonical identification with \( \mathcal{L}(\mathcal{H}^m, \mathcal{H}^n) \). If \( \mathcal{V} \) is a subspace of \( \mathcal{L}(\mathcal{H}) \), then there is a canonical norm on the space \( \mathcal{V}^{n \times m} \) of \( n \times m \) matrices over \( \mathcal{V} \), namely, the norm inherited from it being a subspace of \( \mathcal{L}(\mathcal{H}^m, \mathcal{H}^n) \). Thus any concrete operator space \( \mathcal{V} \) comes equipped not only with a Banach-space norm but also the space of matrices \( \mathcal{V}^{n \times m} \) comes equipped with its own norm in this way. These norms satisfy the Ruan axioms. Conversely, any Banach space \( \mathcal{V} \) such that the each space of matrices \( \mathcal{V}^{n \times m} \) over \( \mathcal{V} \) is equipped with a norm \( \| \cdot \|_{n,m} \) with the collection of such norms satisfying the Ruan axioms (such an object is called an (abstract) operator space) is completely isometrically isomorphic to a concrete operator space (the analogue of the Gelfand-Naimark theorem for operator spaces) by a theorem of Ruan—see e.g. [23, Theorem 2.3.5] or [40, Theorem 13.4]. Here a linear map \( \varphi \) between operator spaces \( \mathcal{V} \) and \( \mathcal{V}_0 \) is said to be completely isometric if not only \( \varphi : \mathcal{V} \to \mathcal{V}_0 \) is isometric but also \( \varphi^{(n,n)} : \mathcal{V}^{n \times n} \to \mathcal{V}_0^{n \times n} \) is isometric for all \( n \in \mathbb{N} \), where in general we set

\[
\varphi^{(n,m)} = \text{id}_{\mathbb{C}^{n \times m}} \otimes \varphi : [v_{ij}]_{i=1,...,n}^{j=1,...,m} \mapsto [\varphi(v_{ij})]_{i=1,...,n}^{j=1,...,m} \tag{2.1}
\]

for any natural numbers \( n, m \).

As a particular case of this setup as well as that in the previous bullet, consider the case where \( \mathcal{V} = \mathbb{C}^d \) (i.e., \( d \)-tuples \( z = (z_1, \ldots, z_d) \) of complex numbers). Then it is convenient to make the identification \( (\mathbb{C}^d)^{n \times n} \) of \( n \times n \) matrices over \( \mathbb{C}^d \) with the space \( (\mathbb{C}^{n \times n})^d \) consisting of \( d \)-tuples of \( n \times n \) complex matrices. Note that \( \mathcal{V} \) can be considered as a subspace of \( \mathcal{L}(\mathbb{C}^d) \) by the simple device of embedding the \( d \)-tuple of complex numbers as the first row of a \( d \times d \) matrix with the other rows set equal to zero. This embedding gives the canonical identification of \( \mathbb{C}^d \) with an operator space. Note that one could equally well identify \( \mathbb{C}^d \) with the set of \( d \times d \) matrices having all but the first column equal to zero. One of the quirks of the theory is that, while these two operator spaces are isometric at level 1, they fail to be isometric to each other at higher levels, i.e., there is no completely isometric map of \( \mathbb{C}_\text{row}^d \) onto \( \mathbb{C}_\text{col}^d \) (see [23] for details).

- \( \mathcal{V} \) is a **concrete operator system**, i.e., a selfadjoint linear subspace of \( \mathcal{L}(\mathcal{H}) \) containing the identity operator \( I_{\mathcal{H}} \) on \( \mathcal{H} \). Any concrete operator system \( \mathcal{V} \), as well as the space \( \mathcal{V}^{n \times n} \) consisting of square matrices over \( \mathcal{V} \) of any size \( n \times n \), is equipped with a special cone, the cone of positive semidefinite elements, inducing an ordering on
\(V^{n \times n}\) satisfying certain compatibility conditions. The Choi-Effros Theorem (see [40, Theorem 13.1]) says that any such abstract operator system is then completely order isomorphic to a concrete operator system.

- \(V\) is a concrete operator algebra, i.e., a (not necessarily self-adjoint) norm-closed subalgebra of \(L(H)\) for some Hilbert space \(H\). Again the space of matrices \(V^{n \times n}\) over \(V\) comes equipped with norms satisfying certain compatibility conditions which also respect the algebra structure. The converse result, that any such abstract operator algebra is completely isometrically isomorphic to a concrete operator algebra, is due to Blecher-Ruan-Sinclair (see [40, Corollary 16.7] or the original [19]).

2.2. Global and noncommutative functions. We suppose that \(\mathcal{S}\) and \(\mathcal{S}_0\) are two sets and that \(\Omega\) is a nc subset of \(\mathcal{S}_{nc}\). We say that a function \(f\) from the nc set \(\Omega_{nc}\subset \mathcal{S}_{nc}\) into \(\mathcal{V}_{0,nc}\) is a global function if

- \(f\) is graded: \(f:\Omega_{n} = \Omega \cap \mathcal{S}^{n \times n} \to \mathcal{V}_{0,n} := (\mathcal{S}_0)^{n \times n}\), and
- \(f\) respects direct sums:
  \[
f\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(W) \end{bmatrix}
  \text{ for all } Z \in \Omega_n, W \in \Omega_m, n, m \in \mathbb{N}.
\]

For the next definition, we assume that \(V\) and \(V_0\) are vector spaces over \(\mathbb{C}\) and we let \(\Omega\) be a nc subset of \(\mathcal{V}_{nc}\). Then, as explained above in Subsection 2.1 for \(\alpha \in \mathbb{C}^{n \times m}, V \in \mathcal{V}_0^{m \times k}\) and \(\beta \in \mathbb{C}^{k \times \ell}\), we can use the module structure of \(\mathcal{V}_0\) over \(\mathbb{C}\) to make sense of the matrix multiplication \(\alpha \cdot V \cdot \beta \in \mathcal{V}_0^{m \times \ell}\) and similarly \(\alpha Z \beta\) makes sense as an element of \(\mathcal{V}^{m \times k}\). Given a function \(f : \Omega \to \mathcal{V}_{0,nc}\), we say that \(f\) is a noncommutative (nc) function if

- \(f\) is graded, i.e., \(f : \Omega_{n} \to \mathcal{V}_{0,n}\), and
- \(f\) respects direct sums:
  \[
f\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(W) \end{bmatrix}
  \text{ for all } Z \in \Omega_n, W \in \Omega_m, n, m \in \mathbb{N}.
\]

It can be shown (see [31, Section I.2.3]) that equivalently: \(f\) is a nc function if and only if

- \(f\) is a global function, i.e., \(f\) is graded and \(f\) respects direct sums, and
- \(f\) respects similarities, i.e.: whenever \(Z, \tilde{Z} \in \Omega_n, \alpha \in \mathbb{C}^{n \times n}\) with \(\alpha Z = \tilde{Z}\alpha\Rightarrow \alpha f(Z) = f(\tilde{Z})\alpha\).

In the concrete case where \(V = \mathbb{C}^d\) for some positive integer \(d\) and we make the identification \((\mathbb{C}^d)^{n \times n} \cong (\mathbb{C}^{n \times n})^d\) as discussed in Subsection 2.1 above, then we view a nc function \(f : \Omega \to \mathcal{V}_{0,nc}\) as a function on \(d\) matrix arguments \(Z = (Z_1, \ldots, Z_d)\), where each \(Z_j \in \mathbb{C}^{n \times n}\) and \(n\) is free to vary over all natural numbers.

The class of nc functions from \(\Omega_{nc}\) to \(\mathcal{V}_{0,nc}\) is one of the main objects of study in the book [31] where the notation \(\mathcal{T}(\Omega; \mathcal{V}_{0,nc})\) is used for the
2.3. Global and noncommutative kernels. Let \( S \) be a set as in the previous subsection with associated noncommutative set \( S_{\text{nc}} \) and let \( \Omega \) be a \( \text{nc} \) subset of \( S_{\text{nc}} \) as above. Let \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \) be two vector spaces. Suppose that \( K \) is a function from \( \Omega \times \Omega \) to \( \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\text{nc}} \) (where \( \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0) \) is the space of linear operators from \( \mathcal{V}_1 \) to \( \mathcal{V}_0 \)). We say that \( K \) is a global kernel if

- \( K \) is graded in the sense that
  \[
  Z \in \Omega_n, \ W \in \Omega_m \Rightarrow K(Z, W) \in \mathcal{L}(\mathcal{V}_1^{n \times m}, \mathcal{V}_0^{n \times m}),
  \]  
  (2.3)

and

- \( K \) respects direct sums in the sense that
  \[
  K \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \right) = \begin{bmatrix} K(Z, W)(P_{11}) & K(Z, W)(P_{12}) \\ K(Z, W)(P_{21}) & K(Z, W)(P_{22}) \end{bmatrix}
  \]  
  (2.4)

We note that the direct sum condition (2.4) can be iterated to arrive at the more general form: \( K \) respects direct sums if and only if: whenever \( Z_i \in \Omega_{n_i}, \ W_j \in \Omega_{m_j} \) and \( P_{ij} \in \mathcal{V}_{ij}^{n_i \times m_j} \) for \( i = 1, \ldots, N, \ j = 1, \ldots, M \)

and we set \( Z = \begin{bmatrix} Z^{(1)} \\ \vdots \\ Z^{(N)} \end{bmatrix} \in \Omega_n, \ W = \begin{bmatrix} W^{(1)} \\ \vdots \\ W^{(M)} \end{bmatrix} \in \Omega_m \) and

\[
K(Z, W)(P) = \begin{bmatrix} K(Z^{(i)}, W^{(j)})(P_{ij}) \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq M}.
\]  
(2.5)

Example 2.1. One way to generate a global kernel is as follows (see Remark 3.6 page 41 for a more general setting). Let \( \mathcal{V}, \mathcal{V}_0, \mathcal{X} \) be vector spaces, let \( H \) be a global function from \( \Omega \) to \( \mathcal{L}(\mathcal{X}, \mathcal{V}_0)_{\text{nc}} \), let \( G \) be a global function from \( \Omega \) to \( \mathcal{L}(\mathcal{V}_0, \mathcal{X})_{\text{nc}} \), and suppose that \( \sigma: \mathcal{V}_1 \rightarrow \mathcal{X} \) is a linear map. Define \( K \) by

\[
K(Z, W)(P) = H(Z)(\text{id}_{\mathcal{V}_0^{n \times m}} \otimes \sigma)(P)G(W)
\]  
(2.6)

for \( Z \in \Omega_n, \ W \in \Omega_m, \ P \in (\mathcal{V}_1)^{n \times m} \) where we use the notation \( \text{nc} \). Then one can check that \( K \) so defined is a global kernel from \( \Omega \times \Omega \) to \( \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\text{nc}} \).

For completeness we mention the following notion of what we shall call affine noncommutative kernels defined as follows. Assume that the underlying set \( S \) is replaced by a vector space \( \mathcal{V} \) with \( \Omega \) is a \( \text{nc} \) subset of \( \mathcal{V}_{\text{nc}} \), and let \( \mathcal{V}_1 \) and \( \mathcal{X}_0 \) be vector spaces. We say that the function \( K \) from \( \Omega \times \Omega \) to \( \mathcal{L}(\mathcal{V}_1, \mathcal{L}(\mathcal{X}_0))_{\text{nc}} \) is a \( \text{nc affine kernel} \) if

- \( K \) is a graded kernel (see (2.3)), and
We denote the class of all such nc kernels by  

\[ \Omega \]  

which we shall call simply noncommutative (nc) kernels \( K \).  

An equivalent set of conditions is:

- **K respects intertwining** in the sense that  
  
  \[ Z \in \Omega_n, \; \tilde{Z} \in \Omega_{\tilde{n}}, \; \alpha \in \mathbb{C}^{\tilde{n} \times n} \text{ such that } \alpha Z = \tilde{Z} \alpha, \] 
  
  \[ W \in \Omega_m, \; \tilde{W} \in \Omega_{\tilde{m}}, \; \beta \in \mathbb{C}^{m \times \tilde{m}} \text{ such that } W \beta = \beta \tilde{W}, \] 
  
  \[ P \in \mathcal{V}_1^{n \times m} \Rightarrow \alpha K(Z,W)(P) \beta = K(\tilde{Z}, \tilde{W})(\alpha P \beta). \tag{2.7} \]

- **K respects similarities** in the following sense:  
  
  \[ Z, \tilde{Z} \in \Omega_n, \; \alpha \in \mathbb{C}^{n \times n} \text{ invertible with } \tilde{Z} = \alpha Z \alpha^{-1}, \] 
  
  \[ W, \tilde{W} \in \Omega_m, \; \beta \in \mathbb{C}^{m \times m} \text{ invertible with } \tilde{W} = \beta^{-1} W \beta, \] 
  
  \[ P \in \mathcal{V}_1^{n \times m} \Rightarrow K(\tilde{Z}, \tilde{W})(P) = \alpha K(Z,W)(\alpha^{-1} P \beta^{-1}) \beta. \tag{2.8} \]

One can check that the formula (2.6) gives rise to an affine kernel in the case where \( H \) and \( G \) are nc functions (rather than just global functions). These kernels arise in a natural way in the noncommutative differential-difference calculus worked out in \[31\].

Our main interest however will be in the following variant of affine kernels which we shall call simply noncommutative (nc) kernels. We again assume that \( \Omega \) is a nc subset of \( \mathcal{V}_{nc} \) for a vector space \( \mathcal{V} \), and that \( \mathcal{V}_1 \) and \( \mathcal{V}_0 \) are vector spaces. We then say that a function \( K: \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{nc} \) is a nc kernel if the following conditions hold:

- **K is graded** (see (2.3)), and

- **K respects intertwining** in the following sense:
  
  \[ Z, \tilde{Z} \in \Omega_n, \; \alpha \in \mathbb{C}^{\tilde{n} \times n} \text{ such that } \alpha Z = \tilde{Z} \alpha, \] 
  
  \[ W, \tilde{W} \in \Omega_m, \; \beta \in \mathbb{C}^{m \times \tilde{m}} \text{ such that } \beta W = \tilde{W} \beta, \] 
  
  \[ P \in \mathcal{V}_1^{n \times m} \Rightarrow \alpha K(Z,W)(P) \beta^* = K(\tilde{Z}, \tilde{W})(\alpha P \beta^*). \tag{2.9} \]

An equivalent set of conditions is:

- **K is graded**, i.e., satisfies (2.3),

- **K respects direct sums** in the following sense:
  
  \[ Z, \tilde{Z} \in \Omega_n, \; W, \; \tilde{W} \in \Omega_m, \; P = [P_{11} \; P_{12}] \in \mathcal{V}_1^{(n+m) \times (\tilde{n}+\tilde{m})} \Rightarrow K\left( \begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix} , \begin{bmatrix} W & 0 \\ 0 & \tilde{W} \end{bmatrix} \right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\tilde{W})(P_{12}) \\ K(\tilde{Z},W)(P_{21}) & K(\tilde{Z},\tilde{W})(P_{22}) \end{bmatrix}. \tag{2.10} \]

- **K respects similarities**:
  
  \[ Z, \tilde{Z} \in \Omega_n, \; \alpha \in \mathbb{C}^{n \times n} \text{ invertible with } \tilde{Z} = \alpha Z \alpha^{-1}, \]
  
  \[ W, \tilde{W} \in \Omega_m, \; \beta \in \mathbb{C}^{m \times m} \text{ invertible with } \tilde{W} = \beta W \beta^{-1}, \]
  
  \[ P \in \mathcal{V}_1^{n \times m} \Rightarrow K(\tilde{Z}, \tilde{W})(P) = \alpha K(Z,W)(\alpha^{-1} P \beta^{-1}) \beta^*. \tag{2.11} \]

We denote the class of all such nc kernels by \( \tilde{T}_1(\Omega; \mathcal{V}_{0,nc}, \mathcal{V}_{1,nc}) \).
2.4. Completely positive global/nc kernels. We now assume that the vector spaces $\mathcal{V}_1$ and $\mathcal{V}_0$ are \textit{operator systems} carrying an associated conjugate-linear involution $P \mapsto P^*$. Then if $P$ is a square matrix over either $\mathcal{V}_1$ or $\mathcal{V}_0$ there is a well-defined notion of positivity $P \succeq 0$ arising from the operator-space structure (see Section 2.1 for some discussion).

We say that a nc kernel $K \in T^1(\Omega; \mathcal{V}_{0,nc}, \mathcal{V}_{1,nc})$ is a \textbf{completely positive noncommutative (cp nc) kernel} if in addition, for all $n \in \mathbb{N}$ we have

$$Z \in \Omega_n, \; P \succeq 0 \text{ in } \mathcal{V}_1^{n \times n} \Rightarrow K(Z,Z)(P) \succeq 0 \text{ in } \mathcal{V}_0^{n \times n}. \tag{2.12}$$

In case $\Omega \subset \mathcal{S}_{nc}$ where $\mathcal{S}$ does not necessarily carry any vector space structure and $K: \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1,\mathcal{V}_0)_{nc}$ is a global kernel, we say that $K$ is a \textbf{completely positive global (cp global) kernel} if $K$ respects direct sums (2.11) and $K$ satisfies (2.12).

In case $\mathcal{V}_1 = \mathcal{A}$ and $\mathcal{V}_0 = \mathcal{B}$ are $C^*$-algebras, we have the following equivalent formulation of the complete positivity condition.

\textbf{Proposition 2.2.} The global kernel $K: \Omega \times \Omega \to \mathcal{L}(\mathcal{A},\mathcal{B})_{nc}$ is cp if and only if: for any $Z^{(j)} \in \Omega_{n_j}$, $P_j \in \mathcal{A}^{N \times n_j}$, $b_j \in \mathcal{B}^{n_j}$ for $n_j \in \mathbb{N}$, $j = 1,2,\ldots,N,$ and $N = 1,2,\ldots$, it holds that

$$\sum_{i,j=1}^N b_i^* K(Z^{(i)},Z^{(j)})(P_i^* P_j) b_j \succeq 0. \tag{2.13}$$

\textit{Proof.} For sufficiency, simply take $N = 1$ in condition (2.13) to recover condition (2.12).

For necessity, note that that the left hand side of (2.13) can be rewritten as

$$b^* K(Z,Z)(P^* P)b \tag{2.14}$$

if we set

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}, \quad Z = \begin{bmatrix} Z^{(1)} \\ \vdots \\ Z^{(N)} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & \cdots & P_N \end{bmatrix}$$

as a consequence of (2.13). The arbitrariness of $b \in \mathcal{B}^n$ ($n = \sum_{j=1}^N n_j$) then implies that $K(Z,Z)(P^* P) \succeq 0$. \hfill \Box

\textbf{Remark 2.3.} Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be operator systems. Recall (see e.g. [40]) that a map $\varphi: \mathcal{V}_1 \to \mathcal{V}_0$ between operator systems $\mathcal{V}_1$ and $\mathcal{V}_0$ is said to be \textbf{completely positive} (cp) if, for every $N \in \mathbb{N}$ (with notation as in (2.1))

$$(\text{id}_{\mathcal{C}^{N \times N}} \otimes \varphi)(P) \succeq 0 \text{ in } (\mathcal{V}_0)^{N \times N}$$

whenever $P \succeq 0$ in $(\mathcal{V}_1)^{N \times N}$. \tag{2.15}

Similarly, if $\varphi: \mathcal{O}_1 \to \mathcal{O}_0$ is a linear map between operator spaces $\mathcal{O}_1$ and $\mathcal{O}_0$, $\varphi$ is said to be \textbf{completely bounded} if there is a constant $M < \infty$ so that

$$\|(\text{id}_{\mathcal{C}^{N \times N}} \otimes \varphi)(P)\|_{\mathcal{O}_0^{N \times N}} \leq M \|P\|_{\mathcal{O}_1^{N \times N}} \text{ for all } P \in \mathcal{O}_1^{N \times N}. \tag{2.16}$$
The smallest such constant $M$ is called the \textbf{completely bounded norm} of the map $\varphi$. It is well known (see e.g. [30]) that a completely positive map $\varphi : V_1 \to V_0$ between operator systems is automatically completely bounded with completely bounded norm equal to $\|\varphi(1_{V_1})\|$: $\|\varphi\| = \|\varphi\|_{cb} = \|\varphi(1_{V_1})\|$ for all $A \in \mathcal{O}_1^{N \times N}$. \hspace{1cm} (2.17)

If $K$ is a cp global/nc kernel on the nc set $\Omega$, $n \in \mathbb{N}$ and $Z$ is any point in $\Omega_n$, the identity

$$[K(Z,Z)(P_{ij})]_{i,j=1,...,N} = K\left(\bigoplus_{1}^{N} Z, \bigoplus_{1}^{N} Z\right) ([P_{ij}]_{i,j=1,...,N}),$$

a consequence of the “respects direct sums” condition (2.5) satisfied by $K$, shows that the map $K(Z,Z)$ is completely positive. By the general fact (2.17), it follows that $K(Z,Z)$ is also completely bounded with completely bounded norm equal to its norm equal to $\|K(Z,Z)(I)\|$: $\|K(Z,Z)\| = \|K(Z,Z)\|_{cb} = \|K(Z,Z)(I)\|$. \hspace{1cm} (2.18)

\textbf{Remark 2.4.} One can get cp global kernels and cp nc kernels by specializing Example 2.1 as follows. We assume that $V_1$ and $V_0$ are operator systems. By definition (or by the Choi-Effros theorem if one views $V_0$ as an abstract operator system), there is no loss of generality in viewing $V_0$ as a subspace of a $C^*$-algebra $\mathcal{L}(\mathcal{Y})$ for some Hilbert space $\mathcal{Y}$. Let us assume that $\mathcal{X}$ is another Hilbert space, that $H$ is a global/nc function from $\Omega$ to $\mathcal{L}(\mathcal{X}, \mathcal{Y})_{nc}$, and that $\sigma$ is a cp map (as defined in (2.15)) from $V_1$ into $\mathcal{L}(\mathcal{X})$. Then one can check that the function $K$ defined by

$$K(Z,W)(P) = H(Z)(\text{id}_{C_{m \times m}} \otimes \sigma)(P) H(W)^* \hspace{1cm} (2.19)$$

for $Z \in \Omega_n, W \in \Omega_m, P \in V_1^{n \times m}$ is a cp global/nc kernel $K$ from $\Omega \times \Omega$ to $\mathcal{L}(V_1, \mathcal{L}(\mathcal{Y}))_{nc}$. In case $V_1$ is also a $C^*$-algebra, then a consequence of Theorem 3.1 below (specifically, of the equivalence (1) $\iff$ (3) there) is that (2.19) is the form for any cp global/nc kernel.

\textbf{Remark 2.5.} Suppose that $V_0$ and $V_1$ are operator systems and that the function $K$ from $\Omega \times \Omega$ into $\mathcal{L}(V_1, V_0)_{nc}$ is a cp global/nc kernel. As was mentioned in the previous remark, there is no loss of generality in viewing $V_0$ as a subspace of a full $C^*$-algebra $\mathcal{L}(\mathcal{Y})$, so there is no loss of generality in assuming that $V_0 = \mathcal{L}(\mathcal{Y})$ in the definition of cp global/nc kernel. An interesting open question is: \textit{given a cp global/nc kernel from} $\Omega \times \Omega$ to $\mathcal{L}(V_1, \mathcal{L}(\mathcal{Y}))_{nc}$ \textit{where} $V_1 \subset \mathcal{L}(\mathcal{E})$ \textit{for some Hilbert space} $\mathcal{E}$, \textit{is there a cp global/nc kernel} $\overline{K}$ \textit{from} $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{L}(\mathcal{E}), \mathcal{L}(\mathcal{Y}))_{nc}$ \textit{so that} $\overline{K}(Z,W)(P) = K(Z,W)(P)$ \textit{for all} $Z \in \Omega_n, W \in \Omega_m, P \in V_1^{n \times m}$? If this question also has an affirmative answer, then there is no loss of generality in assuming that $V_1 = \mathcal{L}(\mathcal{E})$ is also a full $C^*$-algebra in the definition of cp global/nc kernel. The special case of this question where one takes $\Omega$ to be the nc envelope of a singleton set $\{\omega_0\}$ has a positive answer by the Arveson-Wittstock Hahn-Banach extension theorem (see [24, 40]). A positive answer to this question
for the general case would in turn guarantee that (2.19) is the form for a
general cp global/nc kernel, even without the assumption that \( V_1 = \mathcal{A} \) is a
\( C^* \)-algebra.

3. Global/noncommutative reproducing kernel Hilbert spaces

3.1. Main result. To formulate our main result, we assume without loss of
generality (see Remark 2.5) that the operator system \( V_0 \) is presented to us
in concrete form as the set of bounded linear operators \( \mathcal{L}(\mathcal{Y}) \) on a Hilbert
space \( \mathcal{Y} \) (a full \( C^* \)-algebra). Then we have the following characterization of
cp global kernels and of cp nc kernels. Part of the result is that the analogue
of representation (2.6) for cp kernels gives a complete char acterization of cp
global/nc kernels (see statement (3) in the statement of The orem 3.1).

**Theorem 3.1.** Suppose that \( \Omega \) is a nc subset of \( S_{nc} \) for some set \( S \), \( \mathcal{Y} \) is a
Hilbert space, \( \mathcal{A} \) is a \( C^* \)-algebra and \( K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{nc} \) is a given
function. Then the following are equivalent.

1. \( K \) is a cp global kernel.
2. There is a Hilbert space \( \mathcal{H}(K) \) whose elements are global functions
   \( f : \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{Y})_{nc} \) such that:
   a. For each \( W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, \) and \( y \in \mathcal{Y}_m, \) the function
      \[ K_{W,v,y} : \Omega_n \to \mathcal{L}(\mathcal{A}, \mathcal{Y})^{n \times n} \cong \mathcal{L}(\mathcal{A}^n, \mathcal{Y}^n) \]
      defined by
      \[ K_{W,v,y}(Z)u = K(Z, W)(uv)y \] (3.1)
      for \( Z \in \Omega_n, u \in \mathcal{A}^n \) belongs to \( \mathcal{H}(K). \)
   b. The kernel elements \( K_{W,v,y} \) as in (3.1) have the reproducing
      property: for \( f \in \mathcal{H}(K), W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, y \in \mathcal{Y}_m, \)
      \[ \langle f(W)(v^*), y \rangle_{\mathcal{Y}_m} = \langle f, K_{W,v,y} \rangle_{\mathcal{H}(K)}. \] (3.2)
   c. \( \mathcal{H}(K) \) is equipped with a unital \( * \)-representation \( \sigma \) mapping \( \mathcal{A} \)
to \( \mathcal{L}(\mathcal{H}(K)) \) such that
      \[ (\sigma(a)f)(W)(v^*) = f(W)(v^*a) \] (3.3)
      for \( a \in \mathcal{A}, W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, \) with action on kernel elements
      \( K_{W,v,y} \) given by
      \[ \sigma(a) : K_{W,v,y} = K_{W,av,y}. \] (3.4)
3. \( K \) has a Kolmogorov decomposition: there is a Hilbert space \( \mathcal{X} \) equipped with a unital \( * \)-representation \( \sigma : \mathcal{A} \to \mathcal{L}(\mathcal{X}) \) with a
global function \( H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{Y})_{nc} \) so that
      \[ K(Z, W)(P) = H(Z)(\text{id}_{\mathcal{X}^{n \times m}} \otimes \sigma)(P)H(W)^* \] (3.5)
      for all \( Z \in \Omega_n, W \in \Omega_m, P \in \mathcal{A}^{n \times m}. \)
Furthermore, statements (1), (2), and (3) remain equivalent if it is assumed that \( \Omega \subset \mathcal{V}_{nc} \) for a vector spaces \( \mathcal{V} \), \( K \) is taken to be a cp nc kernel in statement (1), the elements \( f \) of \( \mathcal{H}(K) \) are taken to be nc functions from \( \Omega \) to \( \mathcal{L}(\mathcal{A}, \mathcal{Y})_{nc} \) in statement (2), and \( \mathcal{H} \) is taken to be a nc function from \( \Omega \) to \( \mathcal{L}(\mathcal{X}, \mathcal{Y})_{nc} \) in statement (3).

We prove \((1) \Rightarrow (2), (2) \Rightarrow (3), \) and \((3) \Rightarrow (1)\).

**Proof of \((1) \Rightarrow (2)\).** Assume first that \( K \) is a cp global kernel and that we have chosen a \( Z \in \Omega_n, W \in \Omega_m, u \in \mathcal{A}^n, v \in \mathcal{A}^{1 \times m} \) and \( y \in \mathcal{Y}^m \). The estimates

\[
\|K(Z,W)(uv)y\|_{\mathcal{Y}^m} \leq \|K(Z,W)(uv)\|_{\mathcal{L}(\mathcal{Y}^m,\mathcal{Y}^m)}\|y\|_{\mathcal{Y}^m} \\
\leq \|K(Z,W)\|_{\mathcal{L}(\mathcal{A}^{n \times m},\mathcal{L}(\mathcal{Y}^m,\mathcal{Y}^m))}\|uv\|_{\mathcal{A}^n \times \mathcal{A}^{1 \times m}}\|y\|_{\mathcal{Y}^m} \\
\leq (\|K(Z,W)\|_{\mathcal{L}(\mathcal{A}^{n \times m},\mathcal{L}(\mathcal{Y}^m,\mathcal{Y}^m))}\|v\|_{\mathcal{A}^{1 \times m}}\|y\|_{\mathcal{Y}^m})\|u\|_{\mathcal{A}^n}
\]

show that the formula (3.1) defines a bounded operator \( K_{W,v,y}(Z) \) from \( \mathcal{A}^n \) to \( \mathcal{Y}^m \).

One can then use the assumption that \( K \) is a global kernel to check that each \( K_{W,v,y} \) is a global function as follows. Given a \( K_{W,v,y} \) \((W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, y \in \mathcal{Y}^m)\), for \( Z_1 \in \Omega_{n_1}, Z_2 \in \Omega_{n_2}, u_1 \in \mathcal{A}^{n_1}, u_2 \in \mathcal{A}^{n_2} \), we have

\[
K_{W,v,y} \left( \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = K \left( \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, W \right) \begin{bmatrix} u_1 v \\ u_2 v \end{bmatrix} \\
= \begin{bmatrix} K(Z_1, W)(u_1 v) \\ K(Z_2, W)(u_2 v) \end{bmatrix} \\
= \begin{bmatrix} K_{W,v,y}(Z_1) \\ K_{W,v,y}(Z_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},
\]

and we conclude that \( K_{W,v,y} \) respects direct sums.

We define a linear space \( \mathcal{H}^\circ(K) \) as the span of such kernel elements

\[
\mathcal{H}^\circ(K) = \text{span}\{K_{W,v,y} : W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, y \in \mathcal{Y}^m, m = 1, 2, \ldots \}
\]

with inner product of two kernel elements given by

\[
\langle K_{W,v,y}, K_{W',v',y'} \rangle_{\mathcal{H}^\circ(K)} = \langle K(W', W)(v'^* v)y, y' \rangle_{\mathcal{Y}^{m'}} \tag{3.6}
\]

and then extend to any two elements of \( \mathcal{H}^\circ(K) \) by sesquilinearity. The fact that \( K \) satisfies the cp condition (expressed in the form (2.13)) implies that the inner product is positive semidefinite. We may then take the completion to arrive at a pseudo-Hilbert space \( \mathcal{H}(K) \). Elements of the completion \( f \) can again be viewed as global functions from \( \Omega \) to \( \mathcal{L}(\mathcal{A}, \mathcal{Y}) \) via the reproducing formula (3.2); indeed one checks directly that (3.2) holds for \( f = K_{W',v',y'} \).
We claim that $\sigma$ is the zero element of the space. As

$$
\langle f, K_{W,v,y} \rangle_{H(K)} = \langle K_{W',v',y'}, K_{W,v,y} \rangle_{H(K)} = \langle K(W,W')(v^*v'), y, y' \rangle_{YM}
$$

$$
= \langle K_{W',v',y'}(W)(v^*), y, y' \rangle_{YM}
$$

$$
= \langle f(W)(v^*), y, y' \rangle_{YM}.
$$

We then justify the validity of the formula for the case that $f$ is a finite linear combination of kernel elements by linearity; when $f$ is the $H(K)$-limit of a sequence of finite linear combination of kernel elements, we simply use the formula (3.2) to define $f(W)$. We note that the identification $f \mapsto (W \mapsto f(W))$ of $f \in H(K)$ with the function $W \mapsto f(W)$ defined by (3.7) is well-defined: if $\langle f, f \rangle_{H(K)} = 0$, then a consequence of the Cauchy-Schwarz inequality is that $f$ is orthogonal to $K_{W,v,y}$ for all $W, v, y$; consequently, for all $W \in \Omega_m$ and $y \in Y_m$ we have

$$
\langle f(W)(v^*), y \rangle_{YM} = \langle f, K_{W,v,y} \rangle_{H(K)} = 0
$$

so that the associated function $W \mapsto f(W)$ is zero. Similarly, the correspondence $f \mapsto (W \mapsto f(W))$ is injective: if $f \in H(K)$ and $f(W) = 0$ for all $W$, then it follows that $\langle f(W)v^*, y \rangle_{YM} = 0$ for all $W \in \Omega_m, v^* \in A^{1 \times m}, y \in Y_m$ for $m = 1, 2, \ldots$ whence it follows from (3.2) that $f$ is orthogonal to all kernel elements $K_{W,v,y}$. As such kernel elements have dense span in $H(K)$ by construction, it follows that $f$ is the zero element of $H(K)$. With this identification, it follows that in fact $H(K)$ is a Hilbert space, i.e., any element with zero self inner-product is the zero element of the space. As we have already seen that kernel elements are global functions, it follows by linearity and taking limits that each element $f$ of $H(K)$ is also a global function from $\Omega$ to $L(A, Y)_{nc}$.

For $a \in A$, we define an action $\sigma(a)$ on kernel elements $K_{W,v,y}$ by

$$
\sigma(a): K_{W,v,y} \mapsto K_{W,av,y}.
$$

It is easily checked that $\sigma$ is additive, multiplicative and unital:

$$
\sigma(a_1 + a_2) = \sigma(a_1) + \sigma(a_2), \quad \sigma(a_1 a_2) = \sigma(a_1) \circ \sigma(a_2), \quad \sigma(1_A) = I_{H(K)}.
$$

We check that $\sigma$ respects adjoints:

$$
\langle \sigma(a)K_{W,v,y}, K_{W',v',y'} \rangle_{H(K)} = \langle K_{W,av,y}, K_{W',v',y'} \rangle_{H(K)} = \langle K(W',W)(v^*av)y, y' \rangle_{YM'}
$$

$$
= \langle K(W',W)((a^*v')^*v)y, y' \rangle_{YM'}
$$

$$
= K_{W,v,y}, K_{W',av,y'} \rangle_{H(K)} = \langle K_{W,v,y}, \sigma(a^*)K_{W',v',y'} \rangle_{H(K)}
$$

and it follows that $\sigma(a)^* = \sigma(a^*)$.

Given $a \in A$, we extend the action $\sigma(a)$ to the span $H^c(K)$ by linearity. We claim that $\sigma(a)$ is bounded and therefore extends to a bounded operator
on all of $\mathcal{H}(K)$. Indeed, we note that, for $f = \sum_{j=1}^{N} K_{W_j,v_j,y_j} \in \mathcal{H}^0(K)$ where say $W_j \in \Omega_{m_j}, v_j \in \mathcal{A}^{1 \times m_j}$,

$$\|a\|^2 \|f\|_{\mathcal{H}(K)}^2 - \|\sigma(a)f\|_{\mathcal{H}(K)}^2 = \|a\|^2 \left( \sum_{j=1}^{N} K_{W_j,v_j,y_j} \right)^2 - \left( \sum_{j=1}^{N} K_{W_j,av_j,y_j} \right)_{\mathcal{H}(K)} = \sum_{i,j=1}^{N} (K(W_i,W_j)(v_i^*(\|a\|^21_{\mathcal{A}} - a^*a)v_j)y_j, y_i)_{\mathcal{Y}^{m_i}} \geq 0$$

since $\|a\|^21_{\mathcal{A}} - a^*a$ is a positive element of $\mathcal{A}$ and the kernel $K$ satisfies the cp condition (2.13). We conclude that $\|\sigma(a)\| \leq \|a\|$ and hence $\sigma$ extends to a unital $*$-representation $\sigma : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}(K))$.

We next compute

$$\langle (\sigma(a)f)(W)(v^*), y \rangle_{\mathcal{Y}^m} = \langle (\sigma(a)f), K_{W,v,y} \rangle_{\mathcal{H}(K)} \text{ (by (3.2))} = \langle f, K_{W,a^*v,y} \rangle_{\mathcal{H}(K)} = \langle f(W)(v^*a), y \rangle_{\mathcal{Y}^m}$$

and the formula (3.3) follows.

In case $\Omega \subset \mathcal{V}_{nc}$ for a vector space $\mathcal{V}$ and $K$ is a cp nc kernel, we verify that the kernel elements $K_{W,v,y}$ are actually nc functions as follows. Suppose first that $f = K_{W,v,y}$ (with $W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, y \in \mathcal{Y}^m$) is a kernel element, and suppose that we are given $W' \in \Omega_n, W' \in \Omega_{\tilde{n}},$ and an $\alpha \in \mathbb{C}^{\tilde{n} \times n}$ with $\alpha W' = W'\alpha$. We then use the assumed intertwining property for $K$ (2.7) to deduce the intertwining property (2.2) for $f$: for $y' \in \mathcal{Y}_{\tilde{n}}$, we have

$$\langle \alpha f(W')v'^*, y' \rangle_{\mathcal{Y}_{\tilde{n}}} = \langle \alpha K_{W,v,y}(W')(v'^*v), y' \rangle_{\mathcal{Y}_{\tilde{n}}} = \langle \alpha K(W',W')(v'^*v)y, y' \rangle_{\mathcal{Y}_{\tilde{n}}} = \langle K(W',W')(\alpha v'^*v)y, y' \rangle_{\mathcal{Y}_{\tilde{n}}} = \langle K_{W,y}(W')(\alpha v'^*), y' \rangle_{\mathcal{Y}_{\tilde{n}}} = \langle f(W')(\alpha v'^*), y' \rangle_{\mathcal{Y}_{\tilde{n}}}

$$

(3.8)

The general case now follows by linearity and taking limits.

This completes the verification of statement (2) in the Theorem. □

Proof of (2) $\Rightarrow$ (3). Given a function $K : \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{V}))_{nc}$ together with a Hilbert space of global functions $\mathcal{H}(K)$ for which properties (a), (b), (c) in statement (2) of the Theorem hold, we must construct a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$ together with a global function $H : \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})_{nc}$ so that we recover the given kernel $K$ from the Kolmogorov decomposition formula (3.3). To this end it is natural to choose $\mathcal{X} = \mathcal{H}(K)$ which is already equipped with the unital $*$-representation $\sigma$ given by (3.3) or (3.4). It remains to construct $H$. 

For this purpose it is convenient introduce some general notation. For $W \in \Omega_m$ and $u \in A^m$, we define the \textbf{directional point-evaluation operator} $ev_{W,u}: H(K) \to Y^m$ by

$$\textup{ev}_{W,u}(f) = f(W)u. \quad \text{(3.9)}$$

More generally, for $W \in \Omega_m$ and $U = [u_1 \cdots u_N] \in A^{m \times N}$, we define $ev_{W,U}: H(K)^N \to Y^m$ by

$$\textup{ev}_{W,U} = [ev_{W,u_1} \cdots ev_{W,u_N}], \quad \text{(3.10)}$$

i.e., for $f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \in H(K)^N$, $W \in \Omega_m$, and $U = [u_1 \cdots u_N] \in A^{m \times N}$ we define

$$\textup{ev}_{W,U}(f) = \sum_{i=1}^{N} \textup{ev}_{W,u_i}f_i = \sum_{i=1}^{N} f_i(W)u_i.$$  

For $W \in \Omega_m$ and $V = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \in A^{N \times m}$ (so each $v_j \in A^{1 \times m}$ for $j = 1, \ldots, N$), and $y \in Y^m$, let us define $K_{W,V,y} \in H(K)^N$ by

$$K_{W,V,y} = \begin{bmatrix} K_{W,v_1,y} \\ \vdots \\ K_{W,v_N,y} \end{bmatrix}.$$  

Then, for $f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \in H(K)^N$, $V = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \in A^{N \times m}$, $W \in \Omega_m$, and $y \in Y^m$, we compute

$$\langle \textup{ev}_{W,V,y}, y \rangle_{Y^m} = \sum_{i=1}^{N} \langle f_i(W)v_i^*, y \rangle_{Y^m} = \sum_{i=1}^{N} \langle f_i, K_{W,v_i,y} \rangle_{H(K)} = \langle f, K_{W,V,y} \rangle_{H(K)^N}$$

and we conclude that

$$\left(\textup{ev}_{W,V}^*\right)^* y = K_{W,V,y} \quad \text{(3.11)}$$
Thus, for $Z \in \Omega_n$, $U = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathcal{A}^{N \times n}$, $y' \in \mathcal{Y}^m$, $W \in \Omega_m$, $V = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} \in \mathcal{A}^{N \times m}$, and $y \in \mathcal{Y}^m$, we have

\[
\langle K_{W,V,y}, K_{Z,U,y'} \rangle_{\mathcal{H}(K)^N} = \langle \text{ev}_{Z,U}^*(K_{W,V,y}), y' \rangle_{\mathcal{Y}^m}
\]

\[
= \sum_{i=1}^N \langle \text{ev}_{Z,u_i}^*(K_{W,V,y}), y' \rangle_{\mathcal{Y}^m}
\]

\[
= \sum_{i=1}^N \langle K(Z,W)(u_i y), y' \rangle_{\mathcal{Y}^m}
\]

\[
= \langle K(Z,W)(\sum_{i=1}^N u_i v_i), y' \rangle_{\mathcal{Y}^m}
\]

\[
= \langle K(Z,W)(U^*V)y, y' \rangle_{\mathcal{Y}^m},
\]

the higher-rank generalization of the formula (3.6).

Next, let us note how the formula (3.4) for the action of $\sigma$ extends to the higher-rank case: for $W \in \Omega_m$, $V \in \mathcal{A}^{N \times m}$, $y \in \mathcal{C}^m$, $P = [P_{ij}]_{1 \leq i \leq N', 1 \leq j \leq N} \in \mathcal{A}^{N' \times N}$, and $y \in \mathcal{Y}^m$, we have

\[
(id_{\mathcal{C}^{N' \times N}} \otimes \sigma)(P)K_{W,V,y} = K_{W,PV,y}.
\]

(3.13)

For $Z \in \Omega_n$ and $f \in \mathcal{H}(K)^n$, we define $H(Z) : \mathcal{H}(K)^n \rightarrow \mathcal{Y}^m$ simply as

\[
H(Z) = \text{ev}_{Z,1_{A^{n \times n}}} : \mathcal{H}(K)^n \rightarrow \mathcal{Y}^m.
\]

(3.14)

Then, for $Z \in \Omega_n$, $P \in \mathcal{A}^{n \times m}$, and $W \in \Omega_m$, we compute

\[
H(Z) \ (id_{\mathcal{C}^{n \times m}} \otimes \sigma)(P)H(W)^*y = H(Z) \ (id_{\mathcal{C}^{n \times m}} \otimes \sigma)(P)K_{W,1_{A^{n \times m}}y}
\]

(by (3.14) and (3.11))

\[
= H(Z)K_{W,P,y} \ (by \ (3.13))
\]

\[
= K(Z,W)(1_{A^{n \times n}} \cdot P)y \ (by \ (3.14) \ and \ (3.12))
\]

\[
= K(Z,W)(P)y
\]

(3.15)

and it follows that $H$ defined as in (3.14) provides the sought-after Kolmogorov decomposition (3.5) for $K$.

It still remains to check that $H$ is a global function given that each $f$ in $\mathcal{H}(K)$ is a global function. In general let us use the notation

\[
E_j^{(n)} = j\text{-th column of } I_n
\]

(3.16)
where $I_n$ is the $n \times n$ identity matrix over $\mathbb{C}$. For $Z \in \Omega_n, \tilde{Z} \in \Omega_{\tilde{n}}, f \in \mathcal{H}(K)^n$, and $\tilde{f} \in \mathcal{H}(K)^{\tilde{n}}$, we compute

$$H \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right) \begin{bmatrix} f \\ \tilde{f} \end{bmatrix} = \left( \text{ev} \left[ Z \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}, 1_A(n+\tilde{n}) \right] \right) \begin{bmatrix} f \\ \tilde{f} \end{bmatrix}$$

$$= \sum_{i=1}^{n} f_i \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right) \left( E_i^{(n+\tilde{n})} \otimes 1_A \right) + \sum_{j=1}^{\tilde{n}} \tilde{f}_j \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right) \left( E_{n+j}^{(n+\tilde{n})} \otimes 1_A \right)$$

$$= \begin{bmatrix} \sum_{i=1}^{n} f_i(Z)(E_i^{(n)}) \otimes 1_A) \\ \sum_{j=1}^{\tilde{n}} \tilde{f}_j(\tilde{Z})(E_j^{(\tilde{n})} \otimes 1_A) \end{bmatrix} = \begin{bmatrix} H(Z) & 0 \\ 0 & H(\tilde{Z}) \end{bmatrix} \begin{bmatrix} f \\ \tilde{f} \end{bmatrix} \quad (3.17)$$

where we used that each $f_i$ and $\tilde{f}_j$ are global functions in the third line of the computation. It follows that $H$ is a global function as wanted.

Finally, we now add the assumption that $\Omega \subset V_{nc}$ where $V$ is a vector space and assume that each $f \in \mathcal{H}(K)$ is a nc function. The goal is to show that then $H$ is a nc function. Toward this end, we suppose that $Z \in \Omega_N, \tilde{Z} \in \Omega_M$ and $\alpha \in \mathbb{C}^{M \times N}$ are such that $\alpha Z = \tilde{Z} \alpha$, and that $f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \in \mathcal{H}(K)^N$.

The following computation verifies the desired result:

$$\alpha H(Z)(f) = \alpha \left( \sum_{j=1}^{N} f_j(Z)(E_j^{(N)}) \otimes 1_A \right) = \sum_{j=1}^{N} f_j(\tilde{Z})(\alpha E_j^{(N)}) \otimes 1_A$$

$$= \sum_{j=1}^{N} f_j(\tilde{Z}) \left( \sum_{i=1}^{M} \alpha_{ij} E_i^{(M)} \otimes 1_A \right) = \sum_{i=1}^{M} \sum_{j=1}^{N} f_j(\tilde{Z})(\alpha_{ij} E_i^{(M)}) \otimes 1_A$$

$$= \sum_{i=1}^{M} \left( \sum_{j=1}^{N} \alpha_{ij} f_j(\tilde{Z})(E_i^{(M)}) \otimes 1_A \right) = H(\tilde{Z})(\alpha f). \quad (3.18)$$

This completes the proof of $(2) \Rightarrow (3)$ in Theorem 3.1.

Proof of $(3) \Rightarrow (1)$. We suppose that the kernel $K(Z, W)$ has a Kolmogorov decomposition (3.3). For $P \succeq 0$ in $\mathcal{A}^{n \times n}$, write $P$ in factored form as $P = R^* R$ for some $R \in \mathcal{A}^{n \times n}$. Then the computation

$$K(Z, Z)(P) = H(Z)(\text{id}_{\mathbb{C}^{n \times n}} \otimes \sigma)(R^* R)H(Z)^*$$

$$= H(Z) \left( ((\text{id}_{\mathbb{C}^{n \times n}} \otimes \sigma)(R))^* \right) (\text{id}_{\mathbb{C}^{n \times n}} \otimes \sigma)(R)H(Z)^*$$

shows that $K(Z, Z)(P)$ is positive.
We check that $K$ is a global kernel if $H$ is a global function. Indeed, for $Z^{(i)} \in \Omega_{n_{i}}, W^{(j)} \in \Omega_{m_{j}}$ and $P_{ij} \in \mathbb{A}^{n_{i} \times m_{j}}$ for $i, j = 1, 2$, we have

$$K \left( \begin{bmatrix} Z^{(1)} & 0 \\ 0 & Z^{(2)} \end{bmatrix} ; \begin{bmatrix} W^{(1)} & 0 \\ 0 & W^{(2)} \end{bmatrix} \right) \left( \begin{bmatrix} P_{11} \cdots P_{12} \\ P_{21} \cdots P_{22} \end{bmatrix} \right)$$

$$= H \left( \begin{bmatrix} Z^{(1)} & 0 \\ 0 & Z^{(2)} \end{bmatrix} \right) \left( \begin{bmatrix} P_{11} \cdots P_{12} \\ P_{21} \cdots P_{22} \end{bmatrix} \right)$$

$$= H(Z^{(1)}) \begin{bmatrix} (id \otimes \sigma)(P_{11}) & (id \otimes \sigma)(P_{12}) \\ (id \otimes \sigma)(P_{21}) & (id \otimes \sigma)(P_{22}) \end{bmatrix} H(W^{(1)})^* 0 \begin{bmatrix} 0 & H(W^{(2)})^* \end{bmatrix}$$

$$= \begin{bmatrix} K(Z^{(1)}, W^{(1)})(P_{11}) & K(Z^{(1)}, W^{(2)})(P_{12}) \\ K(Z^{(2)}, W^{(1)})(P_{21}) & K(Z^{(2)}, W^{(2)})(P_{22}) \end{bmatrix}$$

as required.

Finally, we suppose that $\Omega \subseteq \mathcal{V}_{nc}$ for a vector space $\mathcal{V}$ and that $H$ is a nc function. The following calculation shows that then $K$ is a nc kernel. We suppose that we are given $Z \in \Omega_{n}, \tilde{Z} \in \Omega_{\tilde{n}}$ and $\alpha \in \mathbb{C}^{\tilde{n} \times n}$ such that $\alpha Z = \tilde{Z} \alpha$, along with $W \in \Omega_{m}, \tilde{W} \in \Omega_{\tilde{m}}$ and $\beta \in \mathbb{C}^{\tilde{m} \times m}$ with $\beta \tilde{W} = \tilde{W} \beta$. We then use the Kolmogorov decomposition to compute

$$\alpha K(Z, W)(P) \beta^* = \alpha H(Z)(id_{\mathbb{C}^{n \times m}} \otimes \sigma)(P) H(W)^* \beta^*$$

$$= H(\tilde{Z}) \alpha (id_{\mathbb{C}^{n \times m}} \otimes \sigma)(P) \beta^* H(\tilde{W})^*$$

$$= H(\tilde{Z})(id_{\mathbb{C}^{\tilde{n} \times \tilde{n}}} \otimes \sigma)(\alpha P \beta^*) H(\tilde{W})^*$$

$$= K(\tilde{Z}, \tilde{W})(\alpha P \beta^*).$$

This completes the proof of (3) ⇒ (1) in Theorem 3.1 \(\square\)

**Remark 3.2.** The proof of (2) ⇒ (3) in Theorem 3.1 implies that we may take a canonical form for the nc Kolmogorov decomposition (3.5), namely: we may take the state space $\mathcal{X}$ in (3.5) to be the nc reproducing kernel Hilbert space $\mathcal{H}(K)$, the representation $\sigma_{\mathcal{X}}$ to be the canonical representation $\sigma$ (3.3) on $\mathcal{H}(K)$, and the nc function $H: \Omega \rightarrow \mathcal{L}(\mathcal{H}(K), \mathcal{Y})_{nc}$ to have the concrete form (3.14).

To complement the understanding of global/nc kernels, we present the following converse to statement (2) in Theorem 3.1.

**Theorem 3.3.** Suppose that $\mathcal{H}$ is a Hilbert space whose elements consist of global functions $f$ from the nc set $\Omega \subseteq \mathcal{S}_{nc}$ to $\mathcal{L}(\mathcal{A}, \mathcal{Y})$ (where $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{Y}$ is a coefficient Hilbert space) such that

1. for each $W \in \Omega_{m}$, the map $f \mapsto f(W)$ is bounded as an operator from $\mathcal{H}$ to $\mathcal{L}(\mathcal{A}, \mathcal{Y})^{m \times m} \cong \mathcal{L}(\mathcal{A}^{m}, \mathcal{Y}^{m})$, and
2. the mapping $\sigma: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ given by

$$\sigma(a)f(W)(u) = f(W)(ua)$$

(3.19)

(for $f \in \mathcal{H}, W \in \Omega_{m}, u \in \mathcal{A}^{m}$) defines a unital $*$-representation of $\mathcal{A}$. 
Then there is a cp global kernel $K$ so that $\mathcal{H}$ is isometrically equal to the global reproducing kernel Hilbert space $\mathcal{H}(K)$ defined as in statement (2) of Theorem 3.1. Furthermore, if $\Omega \subset \mathcal{V}_{nc}$ for a vector space $\mathcal{V}$ and the elements of $\mathcal{H}$ are nc functions, then $K$ is a cp nc kernel.

**Remark 3.4.** Part of the assumption in condition (2) in Theorem 3.3 is that the space $\mathcal{H}$ is invariant under the $A$ action $f \mapsto \sigma(a)f$ given by formula (3.19). If one assumes only this invariance condition, one gets as a consequence of the closed graph theorem that each $\sigma(a)$ is a bounded linear operator on $\mathcal{H}$. It is also immediate that $\sigma$ is multiplicative: $\sigma(a_1a_2) = \sigma(a_1)\sigma(a_2)$. However in general it need not be the case that $\sigma$ preserves adjoints ($\sigma(a)^* = \sigma(a^*)$). Part of the content of the hypothesis in statement (2) in Theorem 3.3 is that this is the case.

**Proof of Theorem 3.3.** Choose $W \in \Omega_m$, $v \in A^{1 \times m}$, and $y \in \mathcal{Y}^m$. Then $f(W) \in \mathcal{L} (\mathcal{A}, \mathcal{Y})^{m \times m} \cong \mathcal{L}(A^m, \mathcal{Y}^m)$ so $f(W)(v^*) \in \mathcal{Y}^m$. As $f \to f(W)$ is bounded as a linear operator from $\mathcal{H}$ to $\mathcal{L}(A^m, \mathcal{Y}^m)$, it follows that $f \mapsto \langle f(W)v^*, y \rangle_{\mathcal{H}}$ is a bounded linear functional on $\mathcal{H}$. By the Riesz-Frechet theorem there therefore is a $K_{W,v,y} \in \mathcal{H}$ so that

$$
\langle f(W)v^*, y \rangle_{\mathcal{Y}^m} = \langle f, K_{W,v,y} \rangle_{\mathcal{H}}.
$$

(3.20)

We call these special functions $K_{W,v,y}$ in $\mathcal{H}$ kernel elements. It is now a simple matter to compute the action of $\sigma(a)$ on a kernel element (for any $a \in A$):

$$
\langle \sigma(a)f, K_{W,v,y} \rangle_{\mathcal{H}} = \langle (\sigma(a)f)(W)(v^*), y \rangle_{\mathcal{Y}^m} = \langle f(W)(v^*a), y \rangle_{\mathcal{Y}^m} = \langle f, K_{W,a^*v,y} \rangle_{\mathcal{H}}
$$

from which we conclude that

$$
\sigma(a)^*: K_{W,v,y} \mapsto K_{W,a^*v,y}.
$$

(3.21)

By hypothesis, $\sigma(a)^* = \sigma(a^*)$. By replacing $a$ by $a^*$ we see that

$$
\sigma(a): K_{W,v,y} \mapsto K_{W,av,y}.
$$

(3.22)

For $V = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix} \in A^{N \times m}$, $W \in \Omega_m$ and $y \in \mathcal{Y}^m$, we define $K_{W,V,y} \in \mathcal{H}^N = \bigoplus_{j=1}^N \mathcal{H}$ by

$$
K_{W,V,y} = \begin{bmatrix}
K_{W,v_1,y} \\
\vdots \\
K_{W,v_N,y}
\end{bmatrix}.
$$

(3.23)

We extend the representation $\sigma$ to elements of $A^{M \times N}$ in the entrywise way:

$$
(id_{C^{M \times N}} \otimes \sigma)(P) := [\sigma(P_{ij})] \in \mathcal{L}(\mathcal{H})^{M \times N}.
$$

Then the formula extends to the matricial form

$$
\sigma(U)K_{W,V,y} = K_{W,UV,y} \in \mathcal{H}^N.
$$

(3.24)
for $W \in \Omega_{m}$, $V \in \mathcal{A}^{N \times m}$, $y \in \mathcal{Y}^{m}$, $U \in \mathcal{A}^{N \times N}$. Furthermore, non-square versions of the unital $*$-representation properties are preserved:

$$(\text{id}_{\mathcal{C}_{MN}} \otimes \sigma)(P) = (\text{id}_{\mathcal{C}_{MN}} \otimes \sigma)(P^*)$$

$$(\text{id}_{\mathcal{C}_{MN}} \otimes \sigma)(P_1 P_2) = (\text{id}_{\mathcal{C}_{MK}} \otimes \sigma)(P_1)(\text{id}_{\mathcal{C}_{MN}} \otimes \sigma)(P_2)$$

if $P_1 \in \mathcal{A}^{M \times K}$, $P_2 \in \mathcal{A}^{K \times N}$,

$$(\text{id}_{\mathcal{C}_{MN}} \otimes \sigma)(I_N \otimes 1_A) = I_{\mathcal{H}N}.$$  

For

$W \in \Omega_{m}$, $v \in \mathcal{A}^{1 \times m}$, $y \in \mathcal{Y}^{m}$, $W' \in \Omega_{m'}$, $v \in \mathcal{A}^{1 \times m'}$, $y \in \mathcal{Y}^{m'}$,

we can now compute the inner product of the associated kernel elements of $\mathcal{H}$ by using these matricial versions of the representation $\sigma$ as follows:

$$\langle K_{W,v,y}, K_{W',v',y'} \rangle_{\mathcal{H}} = \langle \sigma(v)K_{W,I_m \otimes 1_{A,y}}, \sigma(v')K_{W',I_{m'} \otimes 1_{A,y'}} \rangle_{\mathcal{H}}$$

$$= \langle \sigma(v')^* \sigma(v)K_{W,I_m \otimes 1_{A,y}}, K_{W',I_{m'} \otimes 1_{A,y'}} \rangle_{\mathcal{H}^{m'}}$$

$$= \langle \sigma(v'^*v)K_{W,I_m \otimes 1_{A,y}}, K_{W',I_{m'} \otimes 1_{A,y'}} \rangle_{\mathcal{H}^{m'}}. \quad (3.25)$$

We conclude that the result depends on $v$ and $v'$ only through the product $v'^*v$.

For $W \in \Omega_{m}$ and $u \in \mathcal{A}^{m}$, let us define directional point-evaluation operator by the same formula as used for the case where $\mathcal{H} = \mathcal{H}(K)$, namely define $\text{ev}_{W,u} \in \mathcal{L}(\mathcal{H}, \mathcal{Y}^{m})$ by

$$\text{ev}_{W,u} : f \mapsto f(W)(u). \quad (3.26)$$

From the formula (3.20) we see that $K_{W,v,y} = (\text{ev}_{W,v^*})^* y$ and hence

$$\langle K_{W,v,y}, K_{W',v',y'} \rangle_{\mathcal{H}} := \langle (\text{ev}_{W,v^*})^* y, (\text{ev}_{W',v'^*})^* y' \rangle_{\mathcal{H}}$$

$$= \langle (\text{ev}_{W',v'^*})(\text{ev}_{W,v^*})^* y, y' \rangle_{\mathcal{Y}^{m'}}.$$  

As we have already observed from (3.25) that the dependence of the inner product $\langle K_{W,v,y}, K_{W',v',y'} \rangle_{\mathcal{H}}$ on $v, v'$ is only through the product $v'^*v \in \mathcal{A}^{m' \times m}$, we deduce that there is an $\mathcal{L}(\mathcal{Y}^{m}, \mathcal{Y}^{m'})$-valued function $K^\circ$ of three arguments $(W', W, v'^*v)$ defined by

$$K^\circ(W', W, v'^*v) = (\text{ev}_{W',v'^*})(\text{ev}_{W,v^*})^* \in \mathcal{L}(\mathcal{Y}^{m}, \mathcal{Y}^{m'}) \cong \mathcal{L}(\mathcal{Y})^{m' \times m}. \quad (3.27)$$

For a given $W' \in \Omega_{m'}$ and $W \in \Omega_{m}$, the function $K^\circ(W', W, P)$ is defined only for $P \in \mathcal{A}^{m' \times m}$ having a column-row vector factorization $P = v'^*v$ ($v' \in \mathcal{A}^{1 \times m'}$ and $v \in \mathcal{A}^{1 \times m}$). We extend $K(W', W, \cdot)$ so as to be defined on all of $\mathcal{A}^{m' \times m}$ by making use of the higher-rank kernel elements $K_{W,V,y}$ with $V \in \mathcal{A}^{N \times m}$ (3.22). Given two such kernel elements $K_{W,V,y}$ and $K_{W',V',y'}$ (where $V \in \mathcal{A}^{N \times m}$ and $V' \in \mathcal{A}^{N \times m'}$), the $\mathcal{H}^N$-inner product works out to
allow the third argument to have any factorization $V$ dependence only as a function of the matrix product $V$ and $V'$ the result that the identity (3.28) holds.

We conclude that the inner product $\langle K_{W,V,y}, K_{W',V',y'} \rangle_{\mathcal{H}^N}$ has $(V, V')$-dependence only as a function of the matrix product $V'^*V$. We then use linearity to extend the function $K^*$ (still denoted as $K^*$) defined above to allow the third argument to have any factorization $V'^*V$ with $V \in \mathcal{A}^{N \times m}$ and $V' \in \mathcal{A}^{N \times m'}$ for some common $N \in \mathbb{N}$ if $W \in \Omega_m$ and $W' \in \Omega_{m'}$ with the result that the identity (3.28) holds.

We wish to check next that $K^*$ is linear in its third argument $P = V'^*V \in \mathcal{A}^{m' \times m}$. Toward this end, fix $W \in \Omega_m$ and $W' \in \Omega_{m'}$ and suppose that we have $\mathcal{A}$-matrices

$$
V_1 \in \mathcal{A}^{N_1 \times m}, \quad V'_1 \in \mathcal{A}^{N_1 \times m'}, \quad V_2 \in \mathcal{A}^{N_2 \times m}, \quad V'_2 \in \mathcal{A}^{N_2 \times m'}.
$$

We then compute, for $y \in \mathcal{Y}^m$ and $y' \in \mathcal{Y}^{m'}$,

$$
\begin{align*}
\langle (K(W', W_1'^*V_1 + V_2'^*V_2)y, y') \rangle_{\mathcal{Y}^m} & = \langle K(W', W_1'^*V_1 + V_2'^*V_2)y, y' \rangle_{\mathcal{Y}^m} \\
& = \langle K(W', W_1'^*V_1)y, y' \rangle_{\mathcal{Y}^m} + \langle K(W', V_2'^*V_2)y, y' \rangle_{\mathcal{Y}^m} \\
& = \langle (K(W', W_1'^*V_1) + K(W', V_2'^*V_2))y, y' \rangle_{\mathcal{Y}^m}.
\end{align*}
$$

thereby proving the additivity property

$$
K(W', W, P_1 + P_2) = K(W', W, P_1) + K(W', W, P_2) \quad \text{for } P_1, P_2 \in \mathcal{A}^{m' \times m}.
$$

We therefore write

$$
K(W', W)(P) = K^*(W', W, P) \quad \text{where } K(W', W) \in \mathcal{L}(\mathcal{A}^{m' \times m}, \mathcal{L}(\mathcal{Y}^{m' \times m}))
$$

in case $W' \in \Omega_{m'}$, $W \in \Omega_m$, the generalization of (3.27) to the case where $P$ has higher rank over $\mathcal{A}$. One can check that $K(W', W)$ is indeed a bounded operator by using the identity (3.28) and the fact that the map $f \in \mathcal{H}$ to $f(W) \in \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m)$ is bounded (for $W \in \Omega_m$). We now rewrite the formula (3.28) as

$$
\langle K_{W,V,y}, K_{W',V',y'} \rangle_{\mathcal{H}^N} = \langle K(W', W)(V'^*V)y, y' \rangle_{\mathcal{Y}^{m'}}
$$

for $W \in \Omega_m$, $V \in \mathcal{A}^{N \times m}$, $y \in \mathcal{Y}^m$, $W' \in \Omega_{m'}$, $V' \in \mathcal{A}^{N \times m'}$, $y' \in \mathcal{Y}^{m'}$. If we introduce the higher-rank version of the directional point-evaluation...
operator $\text{ev}_{W,u}$ defined in (3.20), namely, the operator $\text{ev}_{W,U} \in \mathcal{L}(\mathcal{H}^N, \mathcal{Y}^m)$ given by

$$
\text{ev}_{W,U} = [\text{ev}_{W,u_1}, \ldots, \text{ev}_{W,u_N}] : f \mapsto \sum_{i=1}^N f_i(W)(u_i) \text{ if } U = [u_1 \ldots u_n]
$$

(3.31)

then we have a succinct formula for the kernel $K$:

$$
K(W', W)(V^*V) = (\text{ev}_{W', V^*})(\text{ev}_{W, V^*})^*.
$$

(3.32)

From this formula we immediately see that $K$ is Hermitian, i.e.

$$
K(Z, W)(P^*) = K(W, Z)(P^*).
$$

(3.33)

To verify that $K$ is a cp kernel, simply note that

$$
\langle K(W, W)(V^*V)y, y \rangle_{\mathcal{Y}^m} = \|K_{W, V, y}\|_{\mathcal{H}^N}^2 \geq 0
$$

for $W \in \Omega_m$, $V \in \mathcal{A}^{N \times m}$, $y \in \mathcal{Y}^m$. It is also possible to verify the expanded cp condition (2.13) by considering the norm-squared of a linear combination of kernel elements in $\mathcal{H}^N$, but, as explained in Proposition 2.2, this follows automatically once we verify that $K$ is a global kernel.

The graded property of $K$ (property (2.3)) was already noted above (see (3.29)). Therefore to check that $K$ is a global kernel, it remains only to check the “respects direct sums” condition (2.4). Since we have already noted that $K$ is Hermitian (see (3.33)), it suffices to check that $K$ respects direct sums in the first argument, i.e.:

$$
K\left(\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}, W\right) \begin{bmatrix} P \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} K(Z, W)(P) \\ K(\tilde{Z}, W)(\tilde{P}) \end{bmatrix}.
$$

(3.34)

for $Z \in \Omega_n$, $\tilde{Z} \in \Omega_{\tilde{n}}$, $W \in \Omega_m$, $P \in \mathcal{A}^{n \times m}$, $\tilde{P} \in \mathcal{A}^{\tilde{n} \times m}$.

Toward this end, we choose any factorization

$$
\begin{bmatrix} P \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} U^* \\ \tilde{U}^* \end{bmatrix} V
$$

with $U \in \mathcal{A}^{N \times n}$, $\tilde{U} \in \mathcal{A}^{\tilde{N} \times \tilde{n}}$, $V \in \mathcal{A}^{N \times m}$. Making use of the formula (3.32), we see that the desired identity (3.34) comes down to

$$
(\text{ev}_{\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}, \begin{bmatrix} U^* \\ \tilde{U}^* \end{bmatrix}}) (\text{ev}_{W, V^*})^* = \begin{bmatrix} (\text{ev}_{Z, U^*}) (\text{ev}_{W, V^*})^* \\ (\text{ev}_{\tilde{Z}, \tilde{U}^*}) (\text{ev}_{W, V^*})^* \end{bmatrix} = \begin{bmatrix} \text{ev}_{Z, U^*} \\ \text{ev}_{\tilde{Z}, \tilde{U}^*} \end{bmatrix} (\text{ev}_{W, V^*})^*.
$$

Canceling off the common right factor leaves us with

$$
\text{ev}_{\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}, \begin{bmatrix} U^* \\ \tilde{U}^* \end{bmatrix}} = \begin{bmatrix} \text{ev}_{Z, U^*} \\ \text{ev}_{\tilde{Z}, \tilde{U}^*} \end{bmatrix}.
$$

(3.35)

To verify (3.35), it suffices to show that it holds when applied to a generic element $f$ of $\mathcal{H}^N$, namely, we wish to verify

$$
\text{ev}_{\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}, \begin{bmatrix} U^* \\ \tilde{U}^* \end{bmatrix}} f = \begin{bmatrix} \text{ev}_{Z, U^*} \\ \text{ev}_{\tilde{Z}, \tilde{U}^*} \end{bmatrix} f
$$

(3.36)
for all \( f \in \mathcal{H}^N \). Let us write out the rows of \( U^* \) and of \( \tilde{U}^* \) as \( u_1^*, \ldots, u_n^* \) and \( \tilde{u}_1^*, \ldots, \tilde{u}_{\tilde{n}}^* \) so that
\[
U^* = \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix}, \quad \tilde{U}^* = \begin{bmatrix} \tilde{u}_1^* \\ \vdots \\ \tilde{u}_{\tilde{n}}^* \end{bmatrix}.
\]

Then verification of (3.36) amounts to a mild generalization of the computation (3.17) where \( 1_A(\tilde{n}+\tilde{n}) \times (\tilde{n}+\tilde{n}) \) is replaced by \( U^* \tilde{U}^* \) as well as \( E_{ij}^{(\tilde{n})} \otimes 1_A \) replaced by \( u_i^* \) and \( E_{ij}^{(\tilde{n})} \otimes 1_A \) replaced by \( \tilde{u}_j^* \).

Let us now assume that each element \( f \) of \( \mathcal{H} \) respects intertwinings (2.2). We claim that the kernel \( K \) respects intertwinings in the first argument, i.e.,
\[
Z \in \Omega_n, \quad \tilde{Z} \in \Omega_{\tilde{n}}, \quad \alpha \in \mathbb{C}^{\tilde{n} \times n} \text{ such that } \alpha Z = \tilde{Z} \alpha, \ W \in \Omega_m, \ P \in A^{n \times m} \Rightarrow \alpha K(Z, W)(P) = K(\tilde{Z}, W)(\alpha P).
\]

From the formula (3.32), we see that the conclusion of (3.37) is equivalent to
\[
\alpha (ev_{Z, V^*}) (ev_{W, V^*})^* = \left( ev_{\tilde{Z}, \alpha V^*} \right) (ev_{W, V^*})^*.
\]

Canceling off the common right factor converts this to
\[
\alpha (ev_{Z, V^*}) f = \left( ev_{\tilde{Z}, \alpha V^*} \right) f
\]
for a general \( f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \in \mathcal{H}^N \) (\( N \) equal to the number of rows in \( V^* \)).

From the definition (3.31) of \( ev_{Z, V^*} \), we see that (3.38) amounts to
\[
\alpha \sum_{i=1}^N f_i(Z)v_i^* = \sum_{i=1}^N f_i(\tilde{Z})\alpha v_i^*.
\]

We now recall the hypothesis in (3.37), namely, that \( Z, \tilde{Z} \) are in \( \Omega \) with \( \alpha Z = \tilde{Z} \alpha \). As each \( f_i \in \mathcal{H} \) as a nc function, each \( f_i \) in particular respects intertwinings (2.2). Hence
\[
\alpha \sum_{i=1}^N f_i(Z)v_i^* = \sum_{i=1}^N (\alpha f_i(Z)) v_i^* = \sum_{i=1}^N \left( f_i(\tilde{Z})\alpha \right) v_i^* = \sum_{i=1}^N f_i(\tilde{Z})(\alpha v_i^*)
\]
and (3.39) (and then also (3.37)) follows as claimed. The Hermitian property (3.33) of \( K \) then implies that \( K \) has the full kernel “respects intertwining” property (2.9), and hence \( K \) is a nc kernel.

From the formulas (3.30), (3.20), (3.21), we see that \( K \) meets all the conditions in part (2) of Theorem 3.1 to serve as the reproducing kernel for the functional Hilbert space \( \mathcal{H} \), so \( \mathcal{H} = \mathcal{H}(K) \) identically and isometrically. \( \square \)
Theorem 3.3 gives the existence of a reproducing kernel for a space satisfying the hypotheses of the theorem but does not give much information on how to actually compute $K$. The next two results fill in this gap.

In the next result we use the following convention. If $y$ is a vector in $\mathcal{Y}^m$, we can view $y$ as an operator from $\mathcal{C}$ to $\mathcal{Y}^m$ (the column operator space structure for the Hilbert space $\mathcal{Y}$). The adjoint operator $y^* : \mathcal{Y}^m \to \mathcal{C}$ is then given by

$$y^* : y \mapsto \langle y, y \rangle_{\mathcal{Y}^m}. \quad (3.40)$$

We apply this notion in particular to the case where $y = f_i(W)(v_j^*)$ where $f_i$ is a global/nc function on $\Omega$, $W \in \Omega_m$, and $v_j^*$ is a vector in $\mathcal{A}^m$ (so $f_i(W) \in \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m)$ and $f_i(W)(v_j^*) \in \mathcal{Y}^m$). Finally if $V = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}$ is an $N \times m$ matrix over $\mathcal{A}$, we let $f_i(W)(V^*)$ denote the block row matrix

$$f_i(W)(V^*) = [f_i(W)(v_1^*) \cdots f_i(W)(v_N^*)] \in \mathcal{Y}^{m \times N}.$$

**Theorem 3.5.** Suppose that $\mathcal{H}$ is a Hilbert space of global/nc functions from $\Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{Y})_{\text{nc}}$ equipped with a unital $*$-representation $\sigma$ (3.19) as in Theorem 3.3. Let $\{f_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}$. Then the kernel function $K(Z, W)(P)$ for $\mathcal{H}$ is given by

$$K(W', W)(V^*V) = \sum_{i \in I} (f_i(W')(V^*)) (f_i(W)(V^*))^* \in \mathcal{L}(\mathcal{Y}^m, \mathcal{Y}^{m'}) \quad (3.41)$$

for $W' \in \Omega_{m'}$, $W \in \Omega_m$, $V' \in \mathcal{A}^{N \times m'}$, $V \in \mathcal{A}^{N \times m}$, with the series converging in the weak operator topology.

**Proof.** As we have seen in (3.28), for $y \in \mathcal{Y}^m$, $W \in \Omega_m$, $V \in \mathcal{A}^{N \times m}$, $y' \in \mathcal{Y}^{m'}$, $W' \in \Omega_{m'}$, and $V' \in \mathcal{A}^{N \times m'}$ we have

$$\langle K(W', W)(V^*V)y, y' \rangle_{\mathcal{Y}^{m'}} = \langle K(W, W')(V^*V)y, y' \rangle_{\mathcal{H}(K)^N} \quad (3.42)$$

where $K_{W, V, y}$ is the element of $\mathcal{H}(K)^N$ with the reproducing property

$$\left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, K_{W, V, y} \right\rangle_{\mathcal{H}(K)^N} = \sum_{j=1}^N \langle h_j(W)v_j^*, y \rangle_{\mathcal{Y}^m}. \quad (3.43)$$

if we write out $V \in \mathcal{A}^{N \times m}$ in terms of its rows as $V = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}$. Recalling the notation (3.16), we see that $\{f_i \otimes E_j^{(N)} : i \in I, 1 \leq j \leq N\}$ is an orthonormal basis for $\mathcal{H}(K)^N$. By the Parseval equality,

$$\langle K_{W, V, y}, K_{W', V', y'} \rangle_{\mathcal{H}(K)^N} = \sum_{i \in I} \sum_{j=1}^N \langle f_i \otimes E_j^{(N)}, K_{W, V', y'} \rangle \cdot \langle K_{W, V, y}, f_i \otimes E_j^{(N)} \rangle.$$
By the reproducing property (3.43),
\[ \langle f_i \otimes E_j^{(N)}, K_{W',V',y'} \rangle_{\mathcal{H}(K)^N} = \langle f_i, K_{W',V',y'} \rangle_{\mathcal{H}(K)} = \langle f_i(W'), v_j^*, y' \rangle_{\mathcal{Y}^m}. \]
Similarly
\[ \langle K_{W,V,y}, f_i \otimes E_j^{(N)} \rangle_{\mathcal{H}(K)^N} = \langle y, f_i(W) v_j^* \rangle_{\mathcal{Y}^m}. \]
Hence, making use of the convention (3.40) applied to the vectors \( f_i(W) v_j^* \), we have
\[ \langle K_{W,V,y}, K_{W',V',y'} \rangle_{\mathcal{H}(K)^N} = \sum_{i \in I} \sum_{j=1}^N \langle f_i(W') v_j^*, y' \rangle_{\mathcal{Y}^m} \langle y, f_i(W) v_j^* \rangle_{\mathcal{Y}^m}. \]
Recalling the formula (3.42) now leads to the expression (3.41) for the kernel function \( K(W',W)(V'^*V) \). The arbitrariness of the vectors \( y \) and \( y' \) in the preceding analysis leads to the conclusion that the series in (3.41) converges in the weak topology.

In case \( \mathcal{H} \) is finite-dimensional, one can get explicit formulas for the kernel function from an arbitrary basis (not necessarily orthonormal).

**Theorem 3.6.** Let \( \mathcal{H} \) be a finite-dimensional Hilbert space consisting of \( \mathcal{L}(\mathcal{A},\mathcal{Y}) \)-valued global/nc functions and equipped with a unital \(*\)-representation \( \sigma \) as in (3.19), and let \( \{f_1, \ldots, f_S\} \) be a basis (not necessarily orthonormal or orthogonal) for \( \mathcal{H} \). Let us introduce the gramian matrix
\[ G = [\langle f_j,f_i \rangle]_{i,j=1,\ldots,S}, \]
for the basis \( \{f_i\}_{i=1,\ldots,S} \). Then, with the same conventions as used in the statement of Theorem 3.5, the kernel function \( K \) for \( \mathcal{H} \) (existence of which is guaranteed by Theorem 3.3) is given by
\[ K(W',W)(V'^*V) = \sum_{i,j=1}^S \langle f_i(W')(V'^*), (G^{-1})_{ij} (f_j(W)(V^*))^\ast \rangle. \]  

**Proof.** Any \( f \in \mathcal{H} \) has an expansion \( f = \sum_{i=1}^S \alpha_i f_i \) in terms of the basis \( \{f_i\}_{i=1}^S \). We set up a system of equations in order to solve for the coefficients \( \{\alpha_i \in \mathbb{C} : i = 1, \ldots, S\} \):
\[ \langle f,f_i \rangle_{\mathcal{H}} = \sum_{j=1}^S \alpha_j \langle f_j,f_i \rangle_{\mathcal{H}} = \sum_{j=1}^S G_{ij} \alpha_j. \]
Solving for the $\alpha_j$'s gives

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_S \end{bmatrix} = G^{-1} \begin{bmatrix} \langle f, f_1 \rangle \\ \vdots \\ \langle f, f_S \rangle \end{bmatrix},$$

or

$$\alpha_i = \sum_{j=1}^{S} (G^{-1})_{ij} \langle f, f_j \rangle.$$

Thus $f = \sum_{i=1}^{S} \alpha_i f_i$ is given by

$$f = \sum_{i=1}^{S} \alpha_i f_i = \sum_{i,j=1}^{S} (G^{-1})_{ij} \langle f, f_j \rangle_H f_i.$$

Hence, for $W \in \Omega_m$ and $v \in \mathcal{A}^{1 \times m}$,

$$\langle f(W)v^*, y \rangle_{Y_m} = \sum_{i,j=1}^{S} (G^{-1})_{ij} \langle f, f_j \rangle_H \langle f_i(W)v^*, y \rangle_{Y_m}$$

$$= \langle f, \sum_{i,j=1}^{S} (G^{-1})_{ji} \langle y, f_i(W)(v^*) \rangle_{Y_m} f_j \rangle_H$$

$$= \langle f, K_{W,v,y} \rangle_H$$

with $K_{W,v,y}$ given by

$$K_{W,v,y} = \sum_{i,j=1}^{S} (G^{-1})_{ji} f_j (f_i(W)(v^*))^* y$$

where we again make use of the convention (3.40). Then

$$\langle K(W', W)(v^*v)y, y' \rangle_{C_m} = \langle K_{W,u,y}, K_{W',v',y'} \rangle_{H(K)}$$

$$= \left\langle \sum_{i,j=1}^{S} (G^{-1})_{ji} f_j (f_i(W)(v^*))^* y, \sum_{i',j'=1}^{S} (G^{-1})_{j'i'} f_{j'} (f_{i'}(W')(v'^*))^* y' \right\rangle$$

$$= \left\langle \sum_{i,j=1}^{S} (G^{-1})_{ji} G_{j'i'} (G^{-1})_{j'i'} (f_{i'}(W')(v'^*)) (f_{i}(W)(v^*))^* y, y' \right\rangle$$

$$= \left\langle \sum_{i',j'=1}^{S} (G^{-1})_{j'i'} f_{i'} (f_{i}(W)(v^*))^* (f_{i}(W)(v'))^* y, y' \right\rangle$$

which after some minor rearrangement agrees with (3.44).
3.2. Lifted norm spaces. In this section we present a different way of viewing our global/nc reproducing kernel Hilbert spaces. The ingredients for the construction are as follows:

- \( \Omega \) is a nc subset of \( S_{nc} \) where \( S \) is a set,
- \( A \) is a \( C^* \)-algebra,
- \( \mathcal{H} \) is a Hilbert space equipped with a unital \( * \)-representation \( \sigma_{\mathcal{H}} \) mapping \( A \) to \( L(H) \),
- \( \mathcal{Y} \) is a coefficient Hilbert space,
- \( H: \Omega \to L(H, \mathcal{Y})_{nc} \) is a global function. In case \( S \) is taken to be a vector space \( V \), we shall also consider the case where \( H \) is a nc function.

Given these ingredients we define a Hilbert space \( H_\ell = H_\ell(H, \sigma_{\mathcal{H}}) \) (the lifted norm space associated with \( H \) and \( \sigma_{\mathcal{H}} \)) by

\[
H_\ell = \{ f: \Omega \to L(A, \mathcal{Y})_{nc}: f = f_h \text{ for some } h \in \mathcal{H} \} \tag{3.45}
\]

where the function \( f_h: \Omega \to L(A, \mathcal{Y})_{nc} \) is specified as follows: given \( Z \in \Omega_n, u \in A^n \),

\[
f_h(Z)u = H(Z) (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(u) h \tag{3.46}
\]

with \( H_\ell \)-norm given by

\[
\|f\|^2_{H_\ell} = \min\{\|h\|^2: h \in \mathcal{H} \text{ with } f = f_h\}.
\]

Then we have the following result.

**Theorem 3.7.** Suppose that the Hilbert space \( H_\ell \) is defined as in (3.45). Then \( H_\ell \) is a global reproducing kernel Hilbert space with reproducing kernel \( K \) given by

\[
K(Z, W)(P) = H(Z) (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(P) H(W)^* \tag{3.47}
\]

for \( Z \in \Omega_n, W \in \Omega_m, P \in A^{n \times m} \). If \( S \) is taken to be a vector space \( \mathcal{V} \) and \( H \) is assumed to be a nc function, then \( K \) given by (3.47) is a cp nc kernel.

**Proof.** We first verify that any function of the form \( f_h \) as in (3.46) is a global function from \( \Omega \) to \( L(A, \mathcal{Y})_{nc} \) given that \( H \) is a global function from \( \Omega \) to \( L(\mathcal{H}, \mathcal{Y})_{nc} \). Indeed, from (3.46) we read off

\[
f_h \left( \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = H \left( \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \right) \begin{bmatrix} (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(u_1) \\ (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(u_2) \end{bmatrix} h
\]

\[
= \begin{bmatrix} H(Z_1) & 0 \\ 0 & H(Z_2) \end{bmatrix} \begin{bmatrix} (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(u_1)h \\ (\text{id}_{\mathbb{C}^n} \otimes \sigma_{\mathcal{H}})(u_2)h \end{bmatrix} = \begin{bmatrix} f_h(Z_1) & 0 \\ 0 & f_h(Z_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Similarly, if \( S \) is a vector space \( \mathcal{V} \) and \( H \) is a nc function, we check that each \( f_h \) is a nc function as follows. Suppose that \( Z \in \Omega_n, \bar{Z} \in \Omega_m, \alpha \in \mathbb{C}^{m \times n} \)
are such that \( \alpha Z = \bar{Z} \alpha \). Then the computation
\[
\alpha f_h(Z)(u) = \alpha H(Z)(\text{id}_{\mathbb{C}^n} \otimes \sigma_H)(u)h = H(\bar{Z})\alpha(\text{id}_{\mathbb{C}^n} \otimes \sigma_H)(u)h
\]
\[
= H(\bar{Z})\alpha \begin{bmatrix}
\sigma_H(u_1) \\
\vdots \\
\sigma_H(u_n)
\end{bmatrix} h = H(\bar{Z}) \begin{bmatrix}
\sigma_H(\sum_{j=1}^n \alpha_j u_j) \\
\vdots \\
\sigma_H(\sum_{j=1}^n \alpha_{nj} u_j)
\end{bmatrix} h
\]
\[
= H(\bar{Z}) (\text{id}_{\mathbb{C}^n} \otimes \sigma_H) (\alpha u)h = f_h(\bar{Z})(\alpha u)
\]
verifies that \( f_h \) is a nc function.

Let us check next that, for each \( Z \in \Omega_n \), the map \( f \mapsto f(Z) \) is bounded from \( \mathcal{H}_\ell \) to \( L(\mathcal{A}^n, \mathcal{Y}^n) \). Indeed, if \( h \in \mathcal{H} \) is any choice of vector in \( \mathcal{H} \) such that \( f = f_h \) and if \( u \in \mathcal{A}^n \), then
\[
\|f(Z)u\|_{\mathcal{Y}^n} = \|H(Z)(\text{id}_{\mathbb{C}^n} \otimes \sigma_H)(u)h\| \leq \|H(Z)\|_{L(\mathcal{H}^n, \mathcal{Y}^n)}\|u\|_{\mathcal{A}^n}\|h\|_{\mathcal{H}}.
\]
Minimizing over all \( h \in \mathcal{H} \) for which \( f = f_h \) then gives
\[
\|f(Z)u\|_{\mathcal{Y}^n} \leq \|H(Z)\|_{L(\mathcal{H}^n, \mathcal{Y}^n)}\|u\|_{\mathcal{A}^n}\|f\|_{\mathcal{H}_\ell}
\]
and the boundedness claim follows as wanted. We shall see below a more explicit verification of the boundedness of these point evaluations.

For \( a \in \mathcal{A} \) and \( f \in \mathcal{H} \), let us consider the new function \( \sigma(a)f \) given by the formula (3.47) in Theorem 3.3 for \( Z \in \Omega_n \) and \( u \in \mathcal{A}^n \) define \( \sigma(a)f : \Omega_n \to L(\mathcal{A}^n, \mathcal{Y}^n) \) so that
\[
(\sigma(a)f)(Z)(u) = f(Z)(ua).
\]
Choose \( h \in \mathcal{H} \) so that \( f = f_h \). Then we compute
\[
(\sigma(a)f_h)(Z)(u) = f_h(Z)(ua) = H(Z)(\text{id}_{\mathbb{C}^n} \otimes \sigma_H)(ua)h
\]
\[
= H(Z)(\text{id}_{\mathbb{C}^n} \otimes \sigma_H)(u)\sigma_H(a)h = f_{\sigma_H(a)h}(Z)(u)
\]
i.e., we have verified
\[
\sigma(a)f_h = f_{\sigma_H(a)h}.
\]
In particular \( \mathcal{H}_\ell \) is invariant under the action of \( \mathcal{A} \) defined by \( \sigma \). By Remark 3.4 we see that \( \sigma : \mathcal{A} \to L(\mathcal{H}_\ell) \) is a unital representation of \( \mathcal{A} \) on \( \mathcal{H}_\ell \). We shall see below (after a little more work) that in fact \( \sigma \) is a \(*\)-representation of \( \mathcal{A} \). One could then apply Theorem 3.3 to see that there is a global/nc kernel \( K \) so that \( \mathcal{H}_\ell = \mathcal{H}(K) \). Rather than applying the existence result from Theorem 3.3 we shall show directly that the kernel \( K \) given by (3.47) satisfies all the properties required to be the reproducing kernel for the space \( \mathcal{H}_\ell \).

By definition the map \( h \mapsto f_h \) is a coisometry of \( \mathcal{H} \) onto \( \mathcal{H}_\ell \). Moreover, given \( f \in \mathcal{H}_\ell \), there is always a unique \( h_0 \in \mathcal{H} \) so that \( f = f_{h_0} \) and we have the equality of norms: \( \|f\|_{\mathcal{H}_\ell} = \|h_0\|_{\mathcal{H}} \); simply take \( h_0 = P_{\mathcal{N}^\perp}h_1 \) where \( h_1 \) is any choice of vector in \( \mathcal{H} \) with \( f = f_{h_1} \) and where \( \mathcal{N} = \{h \in \mathcal{H} : f_h = 0\} \). The space \( \mathcal{N} \) can be characterized explicitly as
\[
\mathcal{N} = (\text{span}\{(\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)\text{Ran} H(W)^* : W \in \Omega_m, v \in \mathcal{A}^{1 \times m}, m \in \mathbb{N}\})^\perp.
\]
(3.50)
Given \( W \in \Omega_m \), \( v \in A^{1 \times m} \), and \( y \in \mathcal{Y}^m \), we have
\[
\langle f_h(W)(v^*), y \rangle_{\mathcal{Y}^m} = \langle H(W)(\text{id}_{\mathbb{C}^m} \otimes \sigma_H)(v^*)h, y \rangle_{\mathcal{Y}^m} = \langle h, (\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)H(W)^*y \rangle_{\mathcal{H}}.
\]

From the characterization of the space \( \mathcal{N} \), one can see that the vector \( (\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)H(W)^*y \) is in the initial space of the coisometry \( h \mapsto f_h \). Alternatively we can always choose \( h \) in the initial space of \( h \mapsto f_h \) so that we still have \( f = f_h \). In any case we may apply the map \( h \mapsto f_h \) to each vector in the inner product in the last expression above and preserve the value of the inner product. Thus we get
\[
\langle f_h(W)(v^*), y \rangle_{\mathcal{Y}^m} = \langle f_h, f(\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)H(W)^*y \rangle_{\mathcal{H}_\ell}
= \langle f_h, K_{W,v,y} \rangle_{\mathcal{H}_\ell}
\]
where we have set
\[
K_{W,v,y} = f(\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)H(W)^*y.
\]

As \( K_{W,v,y} \) is clearly an element of \( \mathcal{H}_\ell \), the formula exhibits the fact that the point evaluation \( f_h \mapsto f_h(W) \) is bounded as an operator from \( \mathcal{H}_\ell \) to \( \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m) \), i.e., we have arrived at the promised second proof of this fact. Using the rule \( (3.46) \), we can calculate
\[
K_{W,v,y}(Z)u = H(Z)(\text{id}_{\mathbb{C}^m} \otimes \sigma_H)(u)(\text{id}_{\mathbb{C}^1 \times m} \otimes \sigma_H)(v)H(W)^*y
= H(Z)(\text{id}_{\mathbb{C}^m \times m} \otimes \sigma_H)(uv)H(W)^*y
\]
If \( K_{W',v',y'}(W' \in \Omega_{m'}, v' \in A^{1 \times m'}, y' \in \mathcal{Y}^{m'}) \) is a second kernel element as in \( (3.52) \) and we apply the formula \( (3.51) \) with \( K_{W',v',y'} \) in place of \( f_h \), as a consequence of the evaluation formula \( (3.53) \) we see that
\[
\langle K_{W,v,y}, K_{W',v',y'} \rangle_{\mathcal{H}_\ell} = \langle K_{W,v,y}(W')v^*, y' \rangle_{\mathcal{Y}^{m'}}
= \langle H(W')(\text{id}_{\mathbb{C}^{m' \times m}} \otimes \sigma_H)(v'^*v)y, y' \rangle_{\mathcal{Y}^{m'}}
= \langle K(W', W)(v'^*v)y, y' \rangle_{\mathcal{Y}^{m'}}
\]
where we have
\[
K(Z, W)(P) = H(Z)(\text{id}_{\mathbb{C}^m \times m} \otimes \sigma_H)(P)H(W)^*
\]
for in general \( Z \in \Omega_n \), \( W \in \Omega_m \) and \( P \in A^{n \times m} \). The definition \( (3.55) \) of \( K \) exhibits a Kolmogorov decomposition for \( K \). Hence, as a consequence of \( (3) \) \( \Rightarrow (1) \) in Theorem \( 3.1 \) we see that \( K \) is completely positive. Furthermore, \( H \) being a global/nc function implies that \( K \) is a global/nc kernel, as was verified as part of the proof of \( (3) \) \( \Rightarrow (1) \) in Theorem \( 3.1 \).

From the reproducing property \( (3.51) \), one can read off as in the proof of Theorem \( 3.1 \) or \( 3.3 \) the action of the representation \( \sigma(a)^* \) on a kernel element \( K_{W,v,y} \):
\[
\sigma(a)^*: K_{W,v,y} \mapsto K_{W,a^*v,y}.
\]
Then one can use the fact that the inner product in (3.54) as a function of the pair \((v, v')\) depends only on the product \(v^* v\) to see that \(\sigma(a)^* = \sigma(a^*)\), i.e., indeed \(\sigma : A \to \mathcal{L}(\mathcal{H}_\ell)\) is a unital \(*\)-representation.

By uniqueness in the Riesz-Frechet theorem, it follows that the space \(\mathcal{H}_\ell\) is isometrically identical to the global/nc reproducing kernel spaces \(\mathcal{H}(K)\) with kernel \(K\) given by (3.55). □

Remark 3.8. One can also view Theorem 3.7 as an alternative direct path to the proof of (3) \(\Rightarrow\) (2) in Theorem 3.1; one constructs the reproducing kernel Hilbert space \(\mathcal{H}(K)\) directly from the function \(H\) and a unital \(*\)-representation \(\sigma_H\) in the Kolmogorov decomposition for \(K\) rather than from \(K\) itself as in (1) \(\Rightarrow\) (2).

In the case where \(\mathcal{H} = \mathcal{H}(K)\) and the function \(H : \Omega \to \mathcal{L}(\mathcal{H}(K), Y)\) is given concretely by (3.14) with unital \(*\)-representation \(\sigma_H = \sigma\) given by (3.3), an instructive exercise is to verify directly that indeed we recover \(\mathcal{H}(K)\) as \(\mathcal{H}(K) = \mathcal{H}_\ell(H, \sigma)\).

Let \(Z \in \Omega_n, H = \mathcal{H}(K)\). For \(h \in \mathcal{H}(K), h(Z) \in \mathcal{L}(A^n, Y_n)\). We use (3.14) to define \(H(Z)\):

\[
H(Z) = h_1(Z)(E_1^{(n)} \otimes 1_A) + \cdots + h_n(Z)(E_n^{(n)} \otimes 1_A).
\]

For \(u = \begin{bmatrix} u_1 \\
\vdots \\
u_n \end{bmatrix} \in A^n\) and \(h \in \mathcal{H} = \mathcal{H}(K)\), we use (3.46) to define \(f_h\):

\[
f_h(Z)(u) = H(Z) \begin{bmatrix}
\sigma(u_1)(h) \\
\vdots \\
\sigma(u_n)(h)
\end{bmatrix}
\]

\[
= (\sigma(u_1)h)(Z)(E_1^{(n)} \otimes 1_A) + \cdots + (\sigma(u_n)h)(E_n^{(n)} \otimes 1_A)
\]

\[
= h(Z) \begin{bmatrix} u_1 \\
\vdots \\
0 \\
0 \end{bmatrix} + \cdots + h(Z) \begin{bmatrix} 0 \\
\vdots \\
0 \\
u_n \end{bmatrix}
\]

i.e., the identification map \(h \mapsto f_h\) between \(\mathcal{H} = \mathcal{H}(K)\) and \(\mathcal{H}_\ell\) in this case is the identity:

\[
f_h(Z) = h(Z) \quad \text{for all } Z \in \Omega.
\]

Furthermore, the relation (3.49) identifies the representation \(\sigma = \sigma_{\mathcal{H}(K)}\) already specified on \(\mathcal{H} = \mathcal{H}(K)\) with the representation \(\sigma\) on \(\mathcal{H}_\ell\) specified by (3.48). Furthermore, in the proof of (2) \(\Rightarrow\) (3) in Theorem 3.1 we identified the Kolmogorov decomposition (3.5) with \(H\) being given by (3.14). On the other hand, we have seen that this same expression gives the reproducing kernel for the space \(\mathcal{H}_\ell\) (see (3.55)). We conclude that \(\mathcal{H}(K) = \mathcal{H}_\ell(H, \sigma)\) identically and isometrically.

3.3. Special cases and examples.
3.3.1. The special case $S = \{s_0\}$. Let us consider the special case where $S = \{s_0\}$, $\Omega = \{s_0 \otimes I_n\}$, $A = C^*$-algebra. Let $K : \Omega \times \Omega \to \mathcal{L}(A, \mathcal{L}(\mathcal{Y}))_n$ be a cp global kernel. Then there is only one choice of $Z \in \Omega_n$, namely $Z = s_0 \otimes I_n$; let $\varphi^{(n,m)} = K(s_0 \otimes I_n, s_0 \otimes I_m) \in \mathcal{L}(A^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})$. The content of the “respects direct sums” property \((2.4)\) for the kernel $K$ is that

$$K(s_0 \otimes I_n, s_0 \otimes I_m) = \text{id}_{\mathcal{C}^{n \times m}} \otimes K(s_0, s_0),$$

i.e.,

$$K(s_0 \otimes I_n, s_0 \otimes I_m) = \varphi^{(n,m)}$$

where we set $\varphi = K(s_0, s_0)$ and define $\psi^{(n,m)} = \text{id}_{\mathcal{C}^{n \times m}} \otimes \psi$ as in \((2.15)\) and \((2.16)\):

$$\psi^{(n,m)} = \text{id}_{\mathcal{C}^{n \times m}} \otimes \varphi : [P_{ij}] \mapsto [\varphi(P_{ij})].$$

Thus, for this case where $S = \{s_0\}$ is a singleton set, $K$ being a completely positive kernel is the same as $K(s_0, s_0) = \varphi : A \to \mathcal{L}(\mathcal{Y})$ being a cp map and conversely, $\varphi : A \to \mathcal{L}(\mathcal{Y})$ being a cp map is equivalent to $\varphi$ having a unique extension to a cp global kernel $K$ on the nc envelope $\{s_0\}_n$ of the singleton set $\{s_0\}$, where we set $K(s_0, s_0) = \varphi$. Moreover the Kolmogorov decomposition \((3.3)\) for this case, when restricted to level 1, becomes just a formulation of the Stinespring dilation theorem \((3.5)\) (see also \((3.1)\) Theorem 3.1): given a completely positive map $\varphi : A \to \mathcal{L}(\mathcal{Y})$, there is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma : A \to \mathcal{L}(\mathcal{X})$ and an operator $H(s_0) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ so that $\varphi(a) = H(s_0) \sigma(a) H(s_0)^*$.

Our general results concerning cp global kernels gives the following finer structure for the Stinespring dilation space. We note that the following result is simply the specialization of Theorem 3.1 to the case where $\Omega$ is the nc envelope of the singleton set $\Omega_1 = \{s_0\}$.

**Theorem 3.9.** Let $A$ be a $C^*$-algebra and let $\mathcal{Y}$ be a Hilbert space. Suppose that $\varphi : A \to \mathcal{L}(\mathcal{Y})$ is a given linear map. Then the following are equivalent.

1. $\varphi$ is completely positive, i.e., $\text{id}_{\mathcal{C}^{n \times n}} \otimes \varphi : A^{n \times n} \to \mathcal{L}(\mathcal{Y})^{n \times n}$ maps positive elements to positive elements for each $n \in \mathbb{N}$.

2. There is a Hilbert space $\mathcal{H}(\varphi)$ whose elements $f$ are in the space $\mathcal{L}(\mathcal{A}, \mathcal{Y})$ of bounded linear operators from $\mathcal{A}$ to $\mathcal{Y}$ such that:

   (a) for $v \in \mathcal{A}$ and $y \in \mathcal{Y}$, the kernel element $K_{v,y}$, identified as an element of $\mathcal{L}(\mathcal{A}, \mathcal{Y})$ via the formula

   $$K_{v,y}(u) = \varphi(vu)y$$

   for $u \in \mathcal{A}$, belongs to $\mathcal{H}(\varphi)$.

   (b) The kernel elements $K_{v,y}$ have the reproducing property: for $f \in \mathcal{H}(\varphi)$, $v \in \mathcal{A}$ and $y \in \mathcal{Y}$,

   $$\langle f(v^*), y \rangle_{\mathcal{Y}} = \langle f, K_{v,y} \rangle_{\mathcal{H}(\varphi)}.$$

   (c) $\mathcal{H}(\varphi)$ is equipped with a unital $*$-representation $\sigma$ mapping $\mathcal{A}$ to $\mathcal{L}(\mathcal{H}(\varphi))$ such that

   $$(\sigma(a)f)(u) = f(ua)$$
for \( a \in A, \ u \in A, \) with action on kernel elements \( K_{v,y} \) \( (v \in A, \ y \in Y) \) given by

\[
\sigma(a) : K_{v,y} \mapsto K_{av,y}.
\]

(3) \( \varphi \) has a Stinespring dilation: there is a Hilbert space \( X \) equipped with a unital \(*\)-representation \( \sigma_X : A \to \mathcal{L}(X) \) and a bounded linear operator \( H(s_0) \in \mathcal{L}(X,Y) \) so that \( \varphi(a) = H(s_0)\sigma_X(a)H(s_0)^* \).

**Remark 3.10.** As a consequence of Remark 3.2 specialized to the setting here, we see that we may take the Stinespring dilation space \( X \) in part (3) of Theorem 3.9 to be the reproducing kernel space \( \mathcal{H}(\varphi) \) with associated map \( H(s_0) \in \mathcal{L}(\mathcal{H}(\varphi),Y) \) given by

\[
H(s_0) : f \mapsto f(1_A)
\]

with adjoint \( H(s_0)^* \) given by

\[
H(s_0)^* : y \mapsto K_{1_A,y}.
\]

This gives a new geometric picture for the Stinespring dilation space \( \mathcal{A} \); we refer to the paper of Muhly-Solel [33] for an alternative geometric picture.

Also already noted in the original construction of Stinespring [45], the space \( \mathcal{X} = \mathcal{H}(\varphi) \) can itself be considered as an Aronszajn reproducing kernel Hilbert space with the algebra \( A \) taken to be the point set, with kernel \( K_{\varphi} : A \times A \to \mathcal{L}(Y) \) given by

\[
K_{\varphi}(a,b) = \varphi(b^*a).
\]

At first impression it appears that part (2) of Theorem 3.9 is an oversimplification of part (2) of Theorem 3.1 since part (2) of Theorem 3.9 mentions only kernel elements of the form \( K_{v,y} \) with \( v \in A \) and \( y \in Y \) but not with \( v \in A^{1 \times m} \) and \( y \in Y^m \). The explanation is that the additional kernel elements \( K_{V,Y} \) (say with \( V = [v_1 \ldots v_m] \in A^{1 \times m} \) and \( y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in Y^m \) are present, but do not add any information since in this case we have the linearity relations

\[
K_{V,Y} = \sum_{i=1}^m K_{v_i,y_i}.
\]

This is a consequence of the fact that the level-\( m \) point-set \( \Omega_m \) consists only of the single diagonal point \( s_0 \otimes I_m \). Note first that \( K_{V,Y} \) is really \( K_{s_0 \otimes I_m,V,Y} \) in the notation of Theorem 3.1. Identifying \( f \) with \( f(s_0) \) when convenient (the meaning should be clear from the context) and using that \( f(s_0 \otimes I_m) = f(s_0) \otimes I_m \) (since \( f \in \mathcal{H}(\varphi) \) respects direct sums), we see from
the general reproducing property [3.2] that
\[ \langle f, K_{V,Y} \rangle_{\mathcal{H}(\varphi)} = \langle f(s_0 \otimes I_m)(V^*), y \rangle_{\mathcal{Y}^m} \]
\[ = \left\langle \begin{bmatrix} f(s_0) \\ \vdots \\ f(s_0) \end{bmatrix}, \begin{bmatrix} v_1^* \\ \vdots \\ v_m^* \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \right\rangle_{\mathcal{Y}^m} \]
\[ = \sum_{i=1}^{m} \langle f, K_{V_i,Y_i} \rangle_{\mathcal{H}(K)} \]
\[ = \left\langle f, \sum_{i=1}^{m} K_{V_i,Y_i} \right\rangle_{\mathcal{H}(K)} \]
whence we conclude that \( K_{V,Y} = \sum_{i=1}^{m} K_{V_i,Y_i} \).

A notion closely related to complete positivity is that of complete boundedness which can be formulated more generally for maps between operator spaces \( V_1 \) and \( V_0 \) (recall the definitions from Subsection 2.1). Given operator spaces \( V_1 \) and \( V_0 \) and a linear map \( \varphi: V_1 \to V_0 \), we say that \( \varphi \) is **completely bounded** (cb) if there is a constant \( M < \infty \) so that \( \|\varphi^n\| \leq M \) for all \( n \in \mathbb{N} \); the smallest such \( M \) is defined to be the **completely bounded norm** of \( \varphi \), denoted as \( \|\varphi\|_{cb} \). As yet another piece of useful terminology, let us say that any Hilbert space \( \mathcal{H} \) whose elements consist of global functions \( f: \Omega \to \mathcal{L}(A, \mathcal{Y})_{nc} \) as in the hypotheses of Theorem 3.3 is a **nc functional Hilbert space equipped with the \( * \)-representation** \( \sigma_\mathcal{H}: A \to \mathcal{L}(\mathcal{H}) \) given by formula (3.11). In the setup of Theorem 3.3 it was only assumed that the point evaluation maps \( \text{ev}_Z: f \mapsto f(Z) \) were bounded. In fact it turns out that each \( \text{ev}_Z: \mathcal{H} \to \mathcal{Y} \) as well as the factor \( H(Z): \mathcal{H} \to \mathcal{Y}^m \) (for each \( Z \in \Omega_n \)) are completely bounded, as summed up in the following result. Here the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{Y} \) are given their natural column-space operator structure, i.e., we identify \( \mathcal{H} \) with \( \mathcal{L}(\mathcal{C}, \mathcal{H}) \) and \( \mathcal{Y} \) with \( \mathcal{L}(\mathcal{C}, \mathcal{Y}) \) in the natural way:

\[ h \in \mathcal{H} \equiv h: c \in \mathbb{C} \mapsto ch \in \mathcal{H}. \]

**Proposition 3.11.**  (1) Suppose that \( K \) is a cp global kernel and that \( Z \in \Omega_n \) and \( W \in \Omega_m \). Then \( K(Z,W) \) is cb with
\[ \|K(Z,W)\|_{cb} \leq \|K\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)(1_{\mathcal{A}^{(n+m)\times(n+m)}})\| \]
\[ = \max\{|\|K(Z,Z)(1_{\mathcal{A}^{n\times n}})\|, \|K(W,W)(1_{\mathcal{A}^{m\times m}})\|\}. \tag{3.56} \]
Moreover,
\[ \|K\left(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right)\|_{cb} = \|K(Z,Z)(1_{\mathcal{A}^{n\times n}})\|. \tag{3.57} \]

(2) Suppose that \( \mathcal{H} = \mathcal{H}(K) \) is a nc functional Hilbert space equipped with canonical \( * \)-representation \( \sigma_\mathcal{H} \) as defined above and let \( W \in \Omega_m \). Then \( f(W) \in \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m) \) is completely bounded with cb-norm satisfying
\[ \|f(W)\|_{\mathcal{L}_{cb}(\mathcal{A}^m, \mathcal{Y}^m)} \leq \|f\|_{\mathcal{H}(K)} \|K(W,W)(1_{\mathcal{A}^{m\times m}})\|^{1/2}. \tag{3.58} \]
Moreover, if \( \text{ev}^{(r,s)}_W : \mathcal{H}(K) \to \mathcal{L}((A^m)^{r \times s}, (Y^n)^{r \times s}) \) is the point-evaluation operator given by

\[
\text{ev}^{(r,s)}_W : f \mapsto \text{id}_{C^{r \times s}} \otimes f(W),
\]

then

\[
\|\text{ev}^{(r,s)}_W\|_{\mathcal{L}(\mathcal{H}, \mathcal{L}((A^m)^{r \times s}, (Y^n)^{r \times s}))} = \sup_{V, c} \|K(W, W')(V c c^* V^*)\|^{1/2} \quad (3.59)
\]

where the supremum is taken over \( V \in (A^m)^{r \times s} \) of norm at most 1 and over \( c \in \mathbb{C}^s \) of norm at most 1. In particular, we have the estimate

\[
\|\text{ev}^{(r,s)}_W\|_{\mathcal{L}(\mathcal{H}, \mathcal{L}((A^m)^{r \times s}, (Y^n)^{r \times s}))} \leq \|K(W, W)(1_{A^m \otimes m})\|^{1/2} \quad (3.60)
\]

and, for the case \( m = 1 \), we have the equalities

\[
\|\text{ev}_W\|_{\mathcal{L}(\mathcal{H}, \mathcal{L}(A, \mathcal{Y}))} = \|\text{ev}_W\|_{\mathcal{L}_{cb}(\mathcal{H}, \mathcal{L}(A, \mathcal{Y}))} = \|K(W, W)(1_A)\|^{1/2} \quad (3.61)
\]

where we have set \( \text{ev}_W = \text{ev}^{(1,1)}_W \).

(3) Suppose that \( K : \Omega \times \Omega \to \mathcal{L}(A, \mathcal{L}(Y)) \) is a cp global kernel with nc function \( H \) in its Kolmogorov decomposition \((3.5)\) and suppose that \( Z \) be a point in \( \Omega_n \). Then the map \( H(Z) : X^n \to Y^n \) is completely bounded with cb norm given by

\[
\|H(Z)\|_{cb} = \|H(Z)\| = \|K(Z, Z)(1_{A^{n \times n}})\|^{1/2}. \quad (3.62)
\]

Proof. To prove (1), suppose that \( K \) is a cp global kernel and we fix \( Z \in \Omega_n \) and \( W \in \Omega_m \). A general fact is that a cp map \( \varphi : A \to \mathcal{L}(Y) \) is automatically cb with \( \|\varphi\|_{cb} = \|\varphi(1_A)\| \). In particular, we conclude that, for each fixed \( Z \in \Omega_n \) and \( W \in \Omega_m \), \( K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}]) : A^{(n+m) \times (n+m)} \to \mathcal{L}(Y)^{(n+m) \times (n+m)} \) is cb with \( \|K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}])\|_{cb} = \|K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}]) (I_{n+m} \otimes 1_A)\| \). But a consequence of the “respects direct sums” property is that

\[
K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}]) ([\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}]) = \begin{bmatrix} 0 & K(Z, W)(P) \\ 0 & 0 \end{bmatrix},
\]

from which we read off that

\[
\|\text{id}_{C^{n \times n}} \otimes K(Z, W)\| \leq \|\text{id}_{C^{n \times n}} \otimes K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}])\|
\]

and hence

\[
\|K(Z, W)\|_{cb} \leq \|K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}])\|_{cb} = \|K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}]) (1_{A^{n \times n}})\|
\]

\((N = n + m)\) and it follows that \( K(Z, W) \) is cb with cb-bound as in \((3.56)\). The equality in \((3.59)\) follows from the identity

\[
K([\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}], [\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}]) (1_{A^{n \times n}}) = \begin{bmatrix} K(Z, Z)(1_{A^{n \times n}}) & 0 \\ 0 & K(W, W)(1_{A^{m \times m}}) \end{bmatrix},
\]
a consequence of the “respects direct sums” property (2.4). Similarly, the cb-norm of $K \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right)$ is $\| K \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right) \|_{(A_{2n} \times 2n),}$ Then the formula (3.57) follows from the identity

$$K \left( \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \right) = \begin{bmatrix} K(Z, Z)(A_n) \\ 0 \\ 0 \\ K(Z, Z)(A_n) \end{bmatrix}.$$

To prove (2), we assume that $\mathcal{H} = \mathcal{H}(K)$ for the cp global kernel $K$ and $f \in \mathcal{H}$. We wish to estimate the norm of

$$\text{id}_{C^{r \times s}} \otimes f(W) : [v_{ij}] \mapsto \left[ f(W)(v_{ij}) \right] \in (Y^m)^{r \times s} \cong \mathcal{L}(C^s, (Y^m)^r)$$

where $V = [v_{ij}] \in (A^m)^{r \times s}$. Let us use the notation $v_i = [v_{i1} \cdots v_{is}]$ for the $i$-th row of $V$. Choose $c = \left[ \begin{smallmatrix} c_1 \\ \vdots \\ c_s \end{smallmatrix} \right] \in C^s$. Viewing $(\text{id}_{C^{r \times s}} \otimes f(W))(V)$ as an operator from $C^s$ to $(Y^m)^r$, we have

$$(\text{id}_{C^{r \times s}} \otimes f(W))(V) : c \mapsto \text{col}_{1 \leq i \leq r} \left[ \sum_{j=1}^s f(W)(v_{ij})c_j \right]$$

$$= \text{col}_{1 \leq i \leq r} \left[ \sum_{j=1}^s f(W)(v_{ij})c_j \right] = \text{col}_{1 \leq i \leq r} \left[ f(W)(v_i c) \right].$$

If $y = \left[ \begin{smallmatrix} y_1 \\ \vdots \\ y_r \end{smallmatrix} \right]$ is an arbitrary element of $(Y^m)^r$, we then have

$$(\text{id}_{C^{r \times s}} \otimes f(W))(V)c, y)_{(Y^m)^r} = \sum_{i=1}^r \langle f(W)(v_i c), y_i \rangle_{Y^m}$$

$$= \sum_{i=1}^r \langle f, K_{W,c}v_i^* y_i \rangle_{\mathcal{H}(K)}$$

$$= \left\langle f, \sum_{i=1}^r K_{W,c}v_i^* y_i \right\rangle_{\mathcal{H}(K)}.$$ 

(3.63)

This leads to the estimate

$$|\langle (\text{id}_{C^{r \times s}} \otimes f(W))(V)c, y \rangle_{(Y^m)^r} | \leq \| f \|_{\mathcal{H}(K)} \left\| \sum_{i=1}^r K_{W,c}v_i^* y_i \right\|_{\mathcal{H}(K)}.$$ 

(3.64)
We next estimate
\[
\left\| \sum_{i=1}^{r} K_{W,e^{*}v_{i}^{*};y_{i}} \right\|_{\mathcal{L}(\mathcal{H}(K))}^{2} = \sum_{i,j=1}^{r} \langle K(W,W) (v_{i}c^{*}v_{j}^{*}) y_{j}, y_{i} \rangle \|_{\mathcal{Y}^{m}}^{2}
\]
\[
= \left\langle K \left( \bigoplus_{1}^{r} W, \bigoplus_{1}^{r} W \right) (Vc^{*}V^{*}) y_{j}, y_{j} \right\rangle \|_{\mathcal{Y}^{m}}^{r}
\]
\[
\leq \left\| K \left( \bigoplus_{1}^{r} W, \bigoplus_{1}^{r} W \right) \right\|_{\mathcal{L}(\mathcal{A}_{mr}^{m}, \mathcal{L}(\mathcal{Y}^{m}))} \| Vc^{*}V^{*} \|_{\mathcal{A}_{mr}^{m}} \| y \|_{\mathcal{Y}^{m}}^{2} \|_{(\mathcal{Y}^{m})^{r}}
\]
(3.65)

By (3.57) and an easy induction,
\[
\left\| K \left( \bigoplus_{1}^{r} W, \bigoplus_{1}^{r} W \right) \right\|_{\mathcal{L}(\mathcal{A}_{mr}^{m}, \mathcal{L}(\mathcal{Y}^{m}))} = \| K(W,W)(1_{\mathcal{A}_{m}^{m}}) \|_{\mathcal{L}(\mathcal{Y}^{m})}.
\]
(3.66)

Note next that \( Vc^{*}V^{*} \leq \| c \|_{2}^{2} V \) in \( \mathcal{A}_{mr}^{m} \) and hence
\[
\| Vc^{*}V^{*} \|_{\mathcal{A}_{mr}^{m}} \leq \| c \|_{2}^{2} \| V \|_{\mathcal{A}_{mr}^{m}}^{2}.
\]
(3.67)

Putting all the pieces (3.64), (3.65), (3.65), (3.66) together, we arrive at
\[
\| (id_{\mathcal{Y}^{m} \times s} \otimes f(W))(V)c \| \leq \| f \|_{\mathcal{H}(K)} \| K(W,W)(1_{\mathcal{A}_{m}^{m}}) \|_{1/2} \| V \| \| c \| \quad (3.68)
\]
from which the estimate (3.58) follows.

We use the generalized reproducing property (3.63) to compute the norm of \( ev_{W}^{(r,s)} \) as follows:
\[
\| ev_{W}^{(r,s)}(f) \| = \sup_{\| f \| \leq 1, \| V \| \leq 1, \| c \| \leq 1} \| (id_{\mathcal{Y}^{m} \times s} \otimes f(W))(V)c \|
\]
\[
= \sup_{\| f \| \leq 1, \| V \| \leq 1, \| c \| \leq 1, \| y \| \leq 1} \| \langle (id_{\mathcal{Y}^{m} \times s} \otimes f(W))(V)c, y \rangle \|
\]
\[
= \sup_{\| f \| \leq 1, \| V \| \leq 1, \| c \| \leq 1, \| y \| \leq 1} \left\| \sum_{i=1}^{r} K_{W,e^{*}v_{i}^{*};y_{i}} \right\|_{\mathcal{H}(K)} \quad \text{(by (3.63))}
\]
\[
= \sup_{\| V \| \leq 1, \| c \| \leq 1, \| y \| \leq 1} \left\| \sum_{i=1}^{r} K_{W,e^{*}v_{i}^{*};y_{i}} \right\|^{1/2}
\]
\[
= \sup_{\| V \| \leq 1, \| c \| \leq 1, \| y \| \leq 1} \left\langle K \left( \bigoplus_{1}^{r} W, \bigoplus_{1}^{r} W \right) (Vc^{*}V^{*}) y, y \right\rangle \quad \text{(by (3.65))}
\]
where the last step is by the first part of the calculation (3.65). This completes the verification of (3.59).
From the rest of the calculation [3.65] we arrive at the uniform estimate (independent of \(r\) and \(s\))

\[
\| \text{ev}_W^{(r,s)} \| \leq \| K(W,W)(1_{A^{m \times m}}) \|^{1/2}.
\]

For the case \(m = 1\) and \(r = s = 1\), we may specialize \(V\) to \(1_A\) and \(c\) to \(1 \in \mathbb{C}\) to get

\[
\| \text{ev}_W \| = \sup_{\| V \| \leq 1, \| c \| \leq 1} \langle K(W,W)(Vc^*c^*V^*), y, y \rangle^{1/2}
\]
\[
\geq \sup_{\| y \| \leq 1} \langle K(W,W)(1_A), y, y \rangle^{1/2}
\]
\[
= \| K(W,W)(1_A) \|^{1/2}
\]

We therefore have the squeeze play

\[
\| K(W,W)(1_A) \|^{1/2} = \| \text{ev}_W \|_{\mathcal{L}(H_0, L_0)} 
\]
\[
\leq \| \text{ev}_W \|_{\mathcal{L}(H_0, L_0)} \leq \| K(W,W)(1_A) \|^{1/2}
\]

from which the string of equalities (3.61) for the case \(r = s = 1\) and \(m = 1\) follows.

To prove (3), we use the formula (3.14) for \(H(Z)\). For \(Z \in \Omega_n\) and \(F = [F_{ij}] \in (\mathcal{H}(K)^n)^{r \times s}\), we have

\[
(\text{id}_{C^{r \times s}} \otimes H(Z))(F) \in (\mathcal{Y}^n)^{r \times s} \cong \mathcal{L}(C^s, \mathcal{Y}^{nr}).
\]

For \(c = [c_j]_{j=1}^s \in C^s\), we therefore wish to estimate the norm of

\[
(\text{id}_{C^{r \times s}} \otimes H(Z))(F) \cdot c = [H(Z)(F_{ij})] \cdot [c_j] = \left[ \sum_{j=1}^s c_j H(Z)(F_{ij}) \right]_{i=1}^r
\]
\[
= \left[ \sum_{j=1}^s c_j F_{ij}(Z)(1_{A^{n \times n}}) \right]_{i=1}^r.
\]

For \(y = [y_i]_{i=1}^r \in \mathcal{Y}^{nr}\), we compute

\[
\langle [H(Z)(F_{ij})] \cdot [c_j], y \rangle_{\mathcal{Y}^{nr}} = \left\langle \left[ \sum_{j=1}^s c_j F_{ij}(Z)(1_{A^{n \times n}}) \right]_{i=1}^r, [y_i]_{i=1}^r \right\rangle_{\mathcal{Y}^{nr}}
\]
\[
= \sum_{i=1}^r \sum_{j=1}^s c_j \langle F_{ij}, KZ(1_{A^{n \times n}}), y_i \rangle_{\mathcal{H}(K)^n}
\]
\[
= \left\langle \left[ \sum_{j=1}^s F_{ij} c_j \right]_{i=1}^r, K_{\otimes Z}(1_{A^{nr \times nr}}), y \right\rangle_{\mathcal{H}(K)^{nr}}.
\]
We then estimate

\[ |\langle [H(Z)(F_{ij}) \cdot [c_j], y]\rangle_{Y^m}| \leq \left\| \sum_{j=1}^{s} F_{ij} c_j \right\|_{\mathcal{H}(K)^n} \cdot \left\| K(\oplus_1^r Z, \oplus_1^s Z) (1_{\mathcal{A}^{n \times n}}) \right\|^{1/2} \]

and it follows that

\[ \left\| \text{id}_{C^{s \times s} \otimes H(Z)} \right\| \leq \left\| K(Z, Z)(1_{\mathcal{A}^{n \times n}}) \right\|^{1/2}. \]

for all \( r, s \in \mathbb{N} \). In particular we get the estimate

\[ \|H(Z)\|_{\text{cb}} \leq \left\| K(Z, Z)(1_{\mathcal{A}^{n \times n}}) \right\|^{1/2}. \]

On the other hand, from (3.14) and (3.11) we read off that

\[ \|H(Z)\| = \sup_{\|v\| \leq 1} \|K_Z(1_{\mathcal{A}^{n \times n}}, v)\|_{\mathcal{H}(K)^n} = \left\| K(Z, Z)(1_{\mathcal{A}^{n \times n}}) \right\|^{1/2}. \]

We again get a squeeze play

\[ \left\| K(Z, Z)(1_{\mathcal{A}^{n \times n}}) \right\|^{1/2} = \|H(Z)\| \leq \|H(Z)\|_{\text{cb}} \leq \left\| K(Z, Z)(1_{\mathcal{A}^{n \times n}}) \right\|^{1/2} \]

from which the string of equalities (3.62) follows.

**Remark 3.12.** Boundedness vs. Complete Boundedness in general.

We have several comments exploring the connections of Proposition 3.11 with the operator algebra literature.

1. We note that statement (3) in Proposition 3.11 assures us that the bounded operator \( H(Z) \) between Hilbert spaces \( \mathcal{H}(K)^n \) and \( Y^m \) is in fact completely bounded with \( \text{cb} \) norm equal to its operator norm in \( \mathcal{L}(\mathcal{H}(K)^n, Y^m) \). This in fact is a general phenomenon for Hilbert space operators: in our notation, for Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), a linear operator \( T \) from \( \mathcal{H} \) to \( \mathcal{K} \) is bounded if and only if it is completely bounded (as an operator between the operator spaces \( \mathcal{H}_{\text{col}} \) and \( \mathcal{K}_{\text{col}} \)) and moreover the identity map from \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) to \( \mathcal{L}_{\text{cb}}(\mathcal{H}_{\text{col}}, \mathcal{K}_{\text{col}}) \) is a complete isometry (see the result of Effros-Ruan [23, Theorem 4.1]).

2. Similarly, in case \( \mathcal{A} = \mathbb{C}, W \in \Omega_1 \) and \( f \in \mathcal{H}(K) \), \( f(W) \in \mathcal{L}(\mathbb{C}, \mathcal{Y}) \) is a Hilbert space operator and hence \( f(W) \) bounded as an element of \( \mathcal{Y} \cong \mathcal{L}(\mathbb{C}, \mathcal{Y}) \) implies that \( f(W) \) is completely bounded with \( \text{cb} \) norm equal to its operator norm. However, in case \( \mathcal{A} \neq \mathbb{C} \) and/or \( W \in \Omega_m \) with \( m > 1 \), it would appear that statement (2) in Proposition 3.11 is not automatic from more general considerations. However, according to another result of Effros-Ruan [23] as reformulated by Pisier (see [41], page 3986), an element \( u \) of \( \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m) \) has \( \text{cb} \) norm at most 1 if and only if, for any finite sequence \( x_1, \ldots, x_N \) of elements from \( \mathcal{A} \), it happens that

\[ \sum_{i=1}^{N} \|u(x_i)\|_{Y^m}^2 \leq \sum_{i=1}^{N} x_i^* x_i \|A^m\|. \]
After a rescaling, we get: \( u \in \mathcal{L}(\mathcal{A}^m, \mathcal{Y}^m) \) has cb norm at most \( M \) if and only if, for any finite sequence \( x_1, \ldots, x_n \) of elements from \( \mathcal{A}^m \),

\[
\sum_{i=1}^{N} \| u(x_i) \|_{\mathcal{Y}^m}^2 \leq M^2 \| \sum_{i=1}^{N} x_i^* x_i \|_{\mathcal{A}^m}. \quad (3.69)
\]

Let us apply this criterion for the case where \( u = f(W) \) for an \( f \in \mathcal{H}(K) \), \( W \in \Omega_m \) and \( x_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{im} \end{bmatrix} \in \mathcal{A}^m \) for \( i = 1, \ldots, N \). Making use of the canonical *-action \( \sigma_{\mathcal{H}(K)} = \sigma \) of \( \mathcal{A} \) on \( \mathcal{H}(K) \) given by (3.3), we note that

\[
f(W)(x_i) = f(W) \left( \sum_{j=1}^{m} E_j^{(m)} \otimes x_{ij} \right) \\
= \sum_{j=1}^{m} (\sigma(x_{ij}) f) (W) \left( E_j^{(m)} \otimes 1_{\mathcal{A}} \right).
\]

For \( y \in \mathcal{Y}^m \) we then compute

\[
\langle f(W)(x_i), y \rangle_{\mathcal{Y}^m} = \left\langle \sum_{j=1}^{m} (\sigma(x_{ij}) f) (W) \left( E_j^{(m)} \otimes 1_{\mathcal{A}} \right), y \right\rangle_{\mathcal{Y}^m} \\
= \sum_{j=1}^{m} \left\langle \sigma(x_{ij}) f, K_{W,E_j^{(m)}} \otimes 1_{\mathcal{A}} y \right\rangle_{\mathcal{Y}^m}.
\]

Hence

\[
|\langle f(W)(x_i), y \rangle_{\mathcal{Y}^m}| = \left| \left\langle \sum_{j=1}^{m} \sigma(x_{ij}) f, K_{W,E_j^{(m)}} \otimes 1_{\mathcal{A}} y \right\rangle_{\mathcal{H}(K)} \right| \\
\leq \left\| \sum_{j=1}^{m} \sigma(x_{ij}) f \right\| \left\langle K(W,W) \left( E_j^{(m)} E_j^{(m)*} \otimes 1_{\mathcal{A}} \right) y, y \right\rangle^{1/2} \\
\leq \left\| \sum_{j=1}^{m} \sigma(x_{ij}) f \right\| \| K(W,W)(1_{\mathcal{A}^m \times m}) \|^{1/2} \| y \|
\]

and we conclude that

\[
\| f(W)(x_i) \|^2 \leq \left\| \sum_{j=1}^{m} \sigma(x_{ij}) f \right\|^2 \| K(W,W)(1_{\mathcal{A}^m \times m}) \|.
\]
Let us set $M := \|K(W,W)(1_{A_m \times m})\|^{1/2}$ and now sum over $i$ to get

$$
\sum_{i=1}^{N} ||f(W)x_i||^2 \leq M^2 \sum_{i=1}^{N} \left\langle \sum_{j,\ell=1}^{m} \sigma(x_{ij})f, \sigma(x_{i\ell})f \right\rangle_{H(K)}
$$

$$
= M^2 \sum_{i=1}^{N} \left\langle \sigma \left( \sum_{j,\ell=1}^{m} x_{i\ell}^* x_{ij} \right) f, f \right\rangle_{H(K)}
$$

$$
\leq M^2 \left\| \sum_{i=1}^{N} x_i^* x_i \right\|_{A_m} \|f\|_{H(K)}^2.
$$

As a consequence of the Effros-Ruan–Pisier criterion (3.69), we conclude that $f(W)$ is a cb map from $A^m$ to $Y^m$ with cb norm at most $\|f\|_{H(K)} \cdot \|K(W,W)(1_{A_m \times m})\|$, thereby giving an alternate proof of statement (2) in Proposition 3.11.

3. An equivalent formulation of the Effros-Ruan criterion is (see [11], page 3986): $u$ in $L(A, Y)$ has cb norm at most 1 if and only if there is a state $\varphi$ on $A$ such that

$$
\text{for all } x \in A, \|u(x)\|_Y^2 \leq \varphi(x^* x).
$$

We can use our theory of global Reproducing Kernel Hilbert Spaces to prove the sufficiency side (presumably the easy side) of this criterion as follows. Assume that there is a state $\varphi$ on $A$ so that (3.70) holds. Use the state $\varphi$ to define an inner product on $A$:

$$
\langle a, b \rangle_{H^o} = \varphi(b^* a).
$$

View the elements of $A$ as elements of $L(A, Y)$ according to the formula

$$
a \equiv f_a : x \mapsto u(xa).
$$

A consequence of the assumption (3.70) is that

$$
\|f_a(x)\|_Y^2 = \|u(xa)\|_Y^2 \leq \varphi(a^* x^* xa) \leq \|x\|_A^2 \varphi(a^* a) = \|x\|_A^2 \|a\|_{H^o}^2,
$$

for each $x \in A$. Hence when we let $H$ be the completion of $H^o$ in the $H^o$ inner product, the elements $f$ of the completion can still be identified as elements of $L(A, Y)$ with the property that $\|f\|_{L(A, Y)} \leq \|f\|_H$.

For $v$ a fixed element of $A$, one can check that the map $\sigma(v) : f_a \mapsto f_{va}$ is a $*$-representation of $A$ on $L(H^o)$ and extends to a $*$-representation of $A$ on $L(H)$ of the functional form:

$$
(\sigma(v)f)(x) = f(xv).
$$

We use this $*$-representation to show that, for each $f \in H$, the map

$$
id_{C^n} \otimes f : \begin{bmatrix} a_1 \\
\vdots \\
a_n \end{bmatrix} \mapsto \begin{bmatrix} f(a_1) \\
\vdots \\
f(a_n) \end{bmatrix}
$$
has $\mathcal{L}(A^n, Y^n)$-norm bounded by $\|f\|_H$ as well. Indeed, note that
\[
\left\| \begin{bmatrix} f(a_1) \\ \vdots \\ f(a_n) \end{bmatrix} \right\|_{Y^n}^2 = \sum_{i=1}^{n} \|f(a_i)\|_Y^2
\]
\[
= \sum_{i=1}^{n} \|\sigma(a_i)f(1_A)\|_H^2
\]
\[
\leq \sum_{i=1}^{n} \|\sigma(a_i)f\|_H^2 \|1_A\|_A^2
\]
\[
= \sum_{i=1}^{n} \langle \sigma(a_i^*a_i)f, f \rangle_H
\]
\[
= \left\langle \sigma \left( \sum_{i=1}^{n} a_i^*a_i \right), f \right\rangle \mathcal{L}(A)
\]
\[
\leq \left\| \sum_{i=1}^{n} a_i^*a_i \right\|_{\mathcal{L}(A)} \|f\|_H^2
\]
\[
= \left\| \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\|_{A^n}^2 \|f\|_H^2.
\]

We have now verified that $\mathcal{H}$ satisfies all the hypotheses of Theorem 3.3 specialized to the case where $\Omega = \bigoplus_{n=0}^{\infty} \{s_0 \otimes I_n\}$. We conclude that $\mathcal{H} = \mathcal{H}(\phi)$ is a nc reproducing kernel Hilbert space over the nc envelope $\bigoplus_{n=1}^{\infty} \{s_0 \otimes I_n\}$ of a singleton-point set $\{s_0\}$ for some completely positive map $\phi: A \to \mathcal{L}(Y)$. By statement (2) in Proposition 3.11 it follows that each $f \in \mathcal{H}$ is actually completely bounded:

\[
\|f\|_{\mathcal{L}(A,Y)} \leq \|f\|_{\mathcal{L}(A,Y)} \leq \|f\|_H.
\]
as an element of $\mathcal{L}(A,Y)$. In particular, $u = f_{1_A}$ is completely bounded, and the sufficiency direction of the second Effros-Ruan criterion (3.70) follows.

3.3.2. Reproducing kernel Hilbert spaces in the sense of Aronszajn. Let $\Omega$ be a set of points (not necessarily having any noncommutative structure), let $Y$ be a Hilbert space and suppose that $K$ is a function from $\Omega \times \Omega$ to $\mathcal{L}(Y)$. We say that $K$ is a positive kernel (in the sense of Aronszajn [4] who worked out much of the theory for the case $Y = \mathbb{C}$) if

\[
\sum_{i,j=1}^{N} \langle K(z_i, z_j)y_j, y_i \rangle_Y \geq 0
\]
for all choices of points $z_1, \ldots, z_N$ and vectors $y_1, \ldots, y_n \in Y$, $N = 1, 2, \ldots$; in other words, the matrix $[K(w_i, w_j)]_{i,j=1}^{N}$ is positive in $\mathcal{L}(Y)^{N \times N}$ for all choices of $z_1, \ldots, z_N \in \Omega$, for any $N = 1, 2, \ldots$. Equivalently, if we let
and we define \( \tilde{\Omega} = [\Omega]_{nc} \) be the nc envelope of \( \Omega \) (see Section 2.1), i.e., the nc set \( \tilde{\Omega} \) with
\[
\tilde{\Omega}_n = \left\{ \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} : z_1, \ldots, z_n \in \Omega \right\}
\]
and we define \( \tilde{K} : \tilde{\Omega} \times \tilde{\Omega} \to \mathcal{L}(\mathcal{C}, \mathcal{L}(\mathcal{Y}))_{nc} \) by
\[
\tilde{K}(Z, W)(P) = [K(z_i, w_j)]_{i,j}
\]
if \( Z = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}, W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \), and \( P = [p_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m} \), then \( \tilde{K} \) is a cp global kernel as defined in Section 3.1 and in fact is the unique extension of \( K \) (defined on \( \Omega \times \Omega \) to a cp global kernel on \( [\Omega]_{nc} \times [\Omega]_{nc} \). Conversely, if \( \mathcal{A} = \mathbb{C} \) and if \( \tilde{K} \) is any global kernel on \( \tilde{\Omega} = [\Omega]_{nc} \), then \( K(z, w) := \tilde{K}(z, w)(1) \) is a positive kernel in the sense of Aronszajn on \( \Omega \). The versions of Theorems 3.1 and 3.3 for this special case have been mainstays in the operator theory literature for many decades now (see e.g. [II]).

Conversely, there are a couple of ways to associate a positive kernel in the sense of Aronszajn to a general cp global kernel which we now discuss.

1. **Fix the \( \mathcal{A} \)-argument.** Let \( K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{nc} \) be a cp global kernel and let \( P \) be a fixed positive element of \( \mathcal{A}^{n \times n} \) for some \( n \in \mathbb{N} \). Then it is easily seen that the kernel \( K_P : \Omega_n \times \Omega_n \to \mathcal{L}(\mathcal{Y}^n) \) defined by
\[
K_P(Z, W) = K(Z, W)(P)
\]
is a positive kernel in the sense of Aronszajn. There is a resulting reproducing kernel Hilbert space \( \mathcal{H}(K) \) whose elements are \( \mathcal{Y}^n \)-valued functions defined on the set of points \( \Omega_n \). Any Kolmogorov decomposition (3.5) for the cp global kernel \( K \) induces a standard Aronszajn-Kolmogorov decomposition for \( K_P \): if \( P \succeq 0 \) in \( \mathcal{A}^{n \times n} \) and we factor \( P \) as \( P = UU^* \) with \( U \in \mathcal{A}^{n \times k} \), then
\[
K_P(Z, W) = H_P(Z)H_P(W)^* \quad \text{where} \quad H_P(Z) := H(Z)(I_{\mathcal{C}^{n \times k}} \otimes \sigma)(U).
\]

In the special case where \( \mathcal{A} = \mathbb{C} \), the next result shows that complete positivity of the nc kernel \( K \) can often be checked just by looking at various kinds of positivity for the kernels \( K_{I_n} \) for each \( n = 1, 2, \ldots \).

**Theorem 3.13.** Assume that \( \mathcal{Y} \) is an operator space and that \( \Omega \) is a nc subset of \( \mathcal{Y} \). Suppose that \( K : \Omega \times \Omega \to \mathcal{L}(\mathcal{C}, \mathcal{Y})_{nc} \) is a nc kernel. Assume either:

(a) (i) the underlying vector space \( \mathcal{Y} \) is \( \mathbb{C}^d \) (see the second bullet in the discussion in Subsection 2.7) and the nc set \( \Omega \subset \mathbb{C}_{nc}^d \cong \prod_{n=1}^{\infty} (\mathbb{C}^{n \times n})^d \) has the special form \( \Omega = \prod_{n=0}^{\infty} N(0; \epsilon)_n \) where \( N(0; \epsilon)_n \) is the \( \epsilon \)-ball in \( (\mathbb{C}^{n \times n})^d \) centered at the origin of radius \( \epsilon \) for some \( \epsilon > 0 \), and (ii) \( K_{I_n} : \Omega_n \times \Omega_n \to \mathcal{L}(\mathcal{Y}^n) \) given by \( K_{I_n}(Z, W) = K(Z, W)(I_n) \) is a positive kernel in the sense of Aronszajn on \( \Omega_n = N(0; \epsilon)_n \) for each \( n \in \mathbb{N} \), or
(b) (i) \( \Omega \) is similarity-invariant: given \( W \in \Omega_n \) and \( S \in \mathbb{C}^{n \times n} \) with \( S \) invertible, it holds that \( SWS^{-1} \in \Omega_n \), and (ii) \( K_n(Z, Z) := K(Z, Z)(I_n) \) is positive semidefinite for all \( Z \in \Omega_n \) for all \( n \in \mathbb{N} \).

Then \( K \) is a cp nc kernel.

Proof. The proof of case (a) which we have makes use of the connections between nc RKHSs studied here and the formal nc RKHSs studied in [14]; we therefore postpone the proof of case (a) to Subsection 3.5 where formal nc RKHSs are reviewed.

It remains to show that hypothesis (b) \( \Rightarrow K \) is cp. We therefore assume (b), so \( K(Z, Z)(I_n) \geq 0 \) for each \( n \in \mathbb{N} \). We must show that then \( K(Z, Z)(P) \geq 0 \) for any \( P \geq 0 \) in \( \mathbb{C}^{n \times n} \). Assume first that \( P \) is invertible. Then \( P \) factors as \( P = SS^* \) with \( S \in \mathbb{C}^{n \times n} \) invertible. By assumption \( \tilde{Z} = S^{-1}ZS \) is again in \( \Omega_n \). By similarity-invariance of the nc kernel \( K \), we have

\[
K(Z, Z)(P) = K(Z, Z)(SS^*) = SK(\tilde{Z}, \tilde{Z})(I) S^* \geq 0.
\]

As any positive \( P \) can be approximated by strictly positive definite operators, it follows that \( K \) is cp. \( \square \)

As an immediate corollary of case (a) of Theorem 3.13, we obtain the following result.

Corollary 3.14. Suppose that the nc subset \( \Omega \) of \( \mathbb{C}^d_{\text{nc}} \approx \prod_{n=1}^{\infty} (\mathbb{C}^{n \times n})^d \) is taken to be \((\text{Nilp})^d = \prod_{n=1}^{\infty} (\text{Nilp})^d_n \) where \((\text{Nilp})^d_n \) consists of \( d \)-tuples \( Z = (Z_1, \ldots, Z_d) \) of \( n \times n \) matrices which are jointly nilpotent in the sense that \( Z^\alpha = 0 \) as soon as the length \( |\alpha| \) of the word \( \alpha \) is sufficiently large. Suppose that \( K \) is a nc kernel on \((\text{Nilp})^d \) with the property that \( K(Z, Z)(I_n) \geq 0 \) for all \( Z \in (\text{Nilp})^d_n \) for all \( n \in \mathbb{N} \). Then \( K \) is cp.

Proof. It suffices to observe that \((\text{Nilp})^d \) is similarity-invariant for each \( n \in \mathbb{N} \) and then quote case (b) of Theorem 3.13. \( \square \)

To make precise the connections between \( \mathcal{H}(K_P) \) and \( \mathcal{H}(K) \) in general, we use the model Kolmogorov decomposition for \( K \) as in (3.14) and (3.15). Fix a factorization \( P = UU^* \) of \( P \) where, say, \( U \in \mathcal{A}^{n \times k} \). It then follows that the space \( \mathcal{H}(K_P) \) can be identified as the lifted norm space

\[
\mathcal{H}(K_P) = \{ H(\cdot)\sigma(U)f : f \in \mathcal{H}(K)^k \}
\]

where \( H \) is given by (3.14), i.e., as the space of functions

\[
Z \mapsto (\text{id}_{\mathcal{C}^{n \times k}} \otimes \sigma)(U) \begin{bmatrix} f_1 \\ \vdots \\ f_k \\ \hat{f}_1 \\ \vdots \\ \hat{f}_k \end{bmatrix} (Z) (1_{\mathcal{A}^{n \times n}}) =: \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} (Z)(U)
\]

with norm so that the map

\[
\Gamma_U : f \mapsto \begin{bmatrix} f_1 \\ \vdots \\ \hat{f}_k \end{bmatrix} (Z)(U)
\]
is a coisometry: 
\[ \| \Gamma_U f \|_{\mathcal{H}(K_P)} = \| P_{(\text{Ker} \Gamma_U)^\perp} f \|_{\mathcal{H}(K)^k}. \]

Note that 
\[ \text{Ker} \Gamma_U = \left\{ \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \in \mathcal{H}(K)^k : \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} (Z)(U) = 0 \text{ for all } Z \in \Omega_n \right\}, \]

\[ (\text{Ker} \Gamma_U)^\perp = \text{span} \{ K_{W,U^*}, y : W \in \Omega_n, y \in \mathcal{Y} \}. \]

Let us now assume that \( \mathcal{A} \) is a von Neumann algebra (e.g., \( \mathcal{A} = \mathcal{L}(K) \) for a Hilbert space \( K \)). Then the Douglas lemma (see [22]) holds inside \( \mathcal{A} : U \in \mathcal{A}^{n \times k'}, U \in \mathcal{A}^{n \times k} \) and \( U^* U \leq U U^* \Rightarrow \) there is a \( C \in \mathcal{A}^{k \times k'} \) with \( C C^* \leq 1_{\mathcal{A}^{k \times k}} \) and \( UC = U' \). If \( P' = U^* U \leq P = U U^* \) in \( \mathcal{A}^{n \times n} \), then \( K(\cdot, \cdot)(P') \leq K(\cdot, \cdot)(P) \) as \( \mathcal{L}(\mathcal{Y}) \)-valued kernels on \( \Omega_n \) and it follows that \( \mathcal{H}(K_{P'}) \) is contained contractively in \( \mathcal{H}(K_P) \). Using the factorization \( P' = U^* U \) and \( P = U U^* \), we then have 
\[ \mathcal{H}(K_{P'}) = \left\{ Z \mapsto \begin{bmatrix} f_1 \\ \vdots \\ f_{k'} \end{bmatrix} (Z)(U') = \left( (\text{id}_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C) \begin{bmatrix} f_1 \\ \vdots \\ f_{k'} \end{bmatrix} \right) (Z)(U) : \begin{bmatrix} f_1 \\ \vdots \\ f_{k'} \end{bmatrix} \in \mathcal{H}(K)^{k'} \right\}, \]

\[ \mathcal{H}(K_P) = \left\{ Z \mapsto \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} (Z)(U) : \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \in \mathcal{H}(K)^k \right\}. \]

and we see that \( (\text{id}_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C) : \mathcal{H}(K)^{k'} \to \mathcal{H}(K)^k \) satisfies \( \sigma(C)(\text{Ker} \Gamma_U) = \) Ker \( \Gamma_U \) and hence defines a mapping from \( \mathcal{H}(K_{P'}) \) to \( \mathcal{H}(K_P) \). However it is not the case that \( (\text{id}_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C) \) maps \( (\text{Ker} \Gamma_U)^\perp \) into \( (\text{Ker} \Gamma_U)^\perp \) so, after the identifications 
\[ \mathcal{H}(K_P) \cong \mathcal{H}(K)^k \otimes \text{Ker} \Gamma_U, \quad \mathcal{H}(K_{P'}) \cong \mathcal{H}(K)^k \otimes \text{Ker} \Gamma_U, \]
the map \( (I_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C) : \mathcal{H}(K_{P'}) \to \mathcal{H}(K_P) \) becomes 
\[ P_{(\text{Ker} \Gamma_U)^\perp} (I_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C) |_{(\text{Ker} \Gamma_U)^\perp}. \]

On the other hand the map \( (\text{id}_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C)^* : K_{W,U^*,y} \mapsto K_{W,U^*,y} \) with \( U^* U = C^* C \) is given explicitly on kernel elements by 
\[ (\text{id}_{\mathcal{C}^{k \times k'}} \otimes \sigma)(C)^* : K_{W,U^*,y} \mapsto K_{W,U^*,y} \text{ with } U^* U = C^* C. \]

However this calculus of spaces \( \mathcal{H}(K_P) \) has awkward aspects when one tries to apply it to get at more detailed structure of the original space \( \mathcal{H}(K) \). Unresolved issues are:

- Analyze in terms of \( P \) and \( P' \) when \( \mathcal{H}(K_P) \cap \mathcal{H}(K_{P'}) = \{0\} \) and when \( \mathcal{H}_{P'} \subset \mathcal{H}_P \).
- Analyze how to recover \( \mathcal{H}(K) \) from all the auxiliary Aronszajn reproducing kernel Hilbert spaces \( \{\mathcal{H}(K_P) : P \in \mathcal{A}^{n \times n} \text{ with } P \geq 0\}. \)
2. **Use the $A$-argument to enlarge the set of points.** In this approach, we view the set of points as $\Omega \times A$ rather than as $\Omega$; in case $\Omega$ is the nc envelope $\{s_0\}_{nc}$ of a singleton set $\{s_0\}$, this idea appears in statement (2) of Theorem 3.9 above. Given a cp global kernel $K$ from $\Omega \times \Omega$ to $L(A, L(Y))_{nc}$ and $n \in \mathbb{N}$, we define an Aronszajn-type kernel $K_n$ from $(\Omega_n \times A_n) \times (\Omega \times A^n)$ to $L(Y)$ by

$$\tilde{K}_n((Z,u),(W,v)) = K(Z,W)(uv^*)$$

Then $K_n$ has Aronszajn-Kolmogorov decomposition

$$K_n((Z,u),(W,v)) = H_n(Z,u)H_n(W,v)^*$$

with norm so that $\Gamma_n$ is a coisometry. Note that

$$(\ker \Gamma_n)^\perp = \text{span}\{K_{W,v^*,y} : W \in \Omega_n, v \in A^n, y \in Y^n\}$$

This collection of subspaces forms an increasing sequence of subspaces with dense union in $\mathcal{H}(K)$, so the corresponding orthogonal projection $P_n$ of $\mathcal{H}(K)$ onto $(\ker \Gamma_n)^\perp$ converges strongly to the $I_{\mathcal{H}(K)}$. We therefore have, for all $f \in \mathcal{H}(K)$,

$$\|f\|_{\mathcal{H}(K)} = \lim_{n \to \infty} \|P_n f\|_{\mathcal{H}(K)} = \lim \|\Gamma_n f\|_{\mathcal{H}(K_n)}.$$ 

It is also possible to consider instead functions on $\Omega_n \times A_{n \times N}$ for any $N \in \mathbb{N}$ in the Aronszajn-type reproducing kernel Hilbert space determined by the kernel $K_{n,N}$ defined by

$$K_{n,N}((Z,U),(W,V)) = K(Z,W)(UV^*)$$

and then obtain results analogous to those mentioned in the previous paragraph, but with $\mathcal{H}(K)^N$ replacing $\mathcal{H}(K)$. In this way we arrive at a picture of the reproducing kernel Hilbert space $\mathcal{H}(K)$ associated with a cp global kernel as an asymptotic limit of Aronszajn-type reproducing kernel Hilbert spaces $\mathcal{H}(K_n)$, or more generally, $\mathcal{H}(K)^N$ as the asymptotic limit of the family $\mathcal{H}(K_{n,N})$.

3.3.3. **Completely positive kernels in the sense of Barreto-Bhat-Liebscher-Skeide.** A unified setting for completely positive maps between $C^*$-algebras and Aronszajn-type positive kernels already appears in the work of Barreto-Bhat-Liebscher-Skeide [16, Section 3]. Given a point set $\Omega_1$, a $C^*$-algebra $\mathcal{A}$ and a Hilbert space $\mathcal{Y}$, we say that the function $K$ from $\Omega_1 \times \Omega_1$ to $L(\mathcal{A}, L(\mathcal{Y}))$ is a **completely positive kernel** (in the sense of Barreto-Bhat-Liebscher-Skeide) if (one among several equivalent definitions), for all...
choices of $z_1, \ldots, z_N \in \Omega_1$, $a_1, \ldots, a_N \in \mathcal{A}$, and $y_1, \ldots, y_N \in \mathcal{Y}$ for any $N \in \mathbb{N}$,
\begin{equation}
\sum_{i=1}^{N} \langle K(z_i, z_j)(a_i^* a_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0, \tag{3.72}
\end{equation}
or equivalently, the map from $\mathcal{A}$ to $\mathcal{L}(\mathcal{Y})^{N \times N}$ given by
\begin{equation}
a \mapsto [K(z_i, z_j)(a)]_{i,j=1,\ldots,N} \tag{3.73}
\end{equation}
is a completely positive map for any choice of $z_1, \ldots, z_N \in \Omega_1$ for any $N = 1, 2, \ldots$. Indeed, in case $\mathcal{A} = \mathbb{C}$ and we define $\tilde{K}$ from $\Omega_1 \times \Omega_1$ into $\mathcal{L}(\mathcal{Y})$ by $\tilde{K}(z, w) = K(z, w)(1)$, then $\tilde{K}$ is an Aronszajn-type kernel and any Aronszajn-type kernel arises in this way from a BBLS-completely positive kernel with $\mathcal{A} = \mathbb{C}$. Similarly, if $\Omega_1$ consists of a single point, say $z_0$, and we define $\varphi: \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ by $\varphi(a) = K(z_0, z_0)(a)$, then $\varphi$ is a (linear) completely positive map and any completely positive map arises in this way from a BBLS-positive kernel over the 1-point set $\Omega_1 = \{z_0\}$ (this last observation already appears in Stinespring’s paper [45, proof of Theorem 1]).

To put the ideas into our framework of cp global kernels, we proceed as follows. Let $\Omega$ be the nc envelope $[\Omega]_{nc}$ of $\Omega_1$:
\begin{equation}
\Omega = \Pi_{n=1}^{\infty} \left\{ \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} : z_j \in \Omega_1 \text{ for } 1 \leq j \leq n \right\}. \tag{3.74}
\end{equation}
Given a function $K$ from $\Omega_1 \times \Omega_1$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$, let $\tilde{K}$ be the unique extension of $K$ to a global kernel on $\Omega$ (i.e., $\tilde{K}$ is the unique extension of $K$ to a function mapping $\Omega_n \times \Omega_m$ into $\mathcal{L}(\mathcal{A}^{n \times m}, \mathcal{L}(\mathcal{Y})^{n \times m})$ for all $n, m \in \mathbb{N}$ which satisfies the “respects direct sums” property (2.4)). We leave it to the reader to verify: Then $\tilde{K}$ is a cp global kernel. Conversely, for the special case where $\Omega$ is the nc envelope of $\Omega_1$, any cp global kernel arises in this way from a BBLS-completely positive kernel on $\Omega_1$. Thus, from the point of view here, the notion of BBLS-completely positive kernel corresponds to the special case of a general cp global kernel where the set of points $\Omega$ is commutative (i.e., $\Omega_n$ consists only of commuting diagonal matrices).

We can use the BBLS-complete positivity condition to define a notion of completely positive global/nc kernel on subsets $\Omega$ which are not necessarily nc subsets of the ambient nc set $\mathcal{S}_{nc}$ and or $\mathcal{V}_{nc}$. Suppose first that $\Omega$ is any subset of $\mathcal{S}_{nc}$ and that $K: \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{nc}$ is a graded kernel, i.e., $K$ satisfies condition (2.3) with $\mathcal{V}_1 = \mathcal{A}$ and $\mathcal{V}_0 = \mathcal{L}(\mathcal{Y})$. Let us say that $K$ is a cp global kernel on $\Omega$ if

1. $K$ respects direct sums, i.e., $K$ satisfies condition (2.3) for any choice of $Z, \tilde{Z}, W, \tilde{W} \in \Omega$ for which it is the case that $\begin{bmatrix} Z & 0 \\ 0 & \tilde{Z} \end{bmatrix}$ and $\begin{bmatrix} W & 0 \\ 0 & \tilde{W} \end{bmatrix}$ are also in $\Omega$, and
(2) $K$ satisfies the BBLS-complete positivity condition (3.72) or equivalently (3.73), i.e.,
\[
\sum_{i=1}^{N} \langle K(Z^{(i)}, Z^{(j)})(a_i^*a_j)y_j, y_i \rangle_Y \geq 0,
\]
for all $Z^{(1)} \in \Omega_{n_1}, \ldots, Z^{(N)} \in \Omega_{n_N}$, $a_1 \in A^{\kappa \times n_1}, \ldots, a_N \in A^{\kappa \times n_N}$ for any $N, \kappa \in \mathbb{N}$, or equivalently, the map from $A^{N \times N}$ to $L(Y)^{N \times N}$ given by
\[
[a_{ij}]_{i,j=1,...,N} \mapsto [K(Z^{(i)}, Z^{(j)})(a_{ij})]_{i,j=1,...,N}
\]
is completely positive for any choice of $Z^{(1)} \in \Omega_{n_1}, \ldots, Z^{(N)} \in \Omega_{n_N}$ where we set $N = \sum_{i=1}^{N} n_i$ (where $[a_{ij}]_{i,j=1,...,N}$ with $a_{ij} \in A^{n_i \times n_j}$ is a generic element of $A^{N \times N}$).

If $\Omega$ is not already a nc subset, we extend $K$ to $\Omega_{nc} \times \Omega_{nc}$ (where $\Omega_{nc}$ is the nc envelope of $\Omega$ consisting of all possible finite direct sums of elements of $\Omega$) by
\[
\tilde{K}
\begin{bmatrix}
Z^{(1)} \\
\vdots \\
Z^{(N)}
\end{bmatrix},
\begin{bmatrix}
W^{(1)} \\
\vdots \\
W^{(M)}
\end{bmatrix}
\rangle = [K(Z^{(i)}, W^{(j)})([a_{ij}])]_{i=1,...,N;j=1,...,M}
\]
for any $Z^{(1)}, \ldots, Z^{(N)}, W^{(1)}, \ldots, W^{(N)} \in \Omega$. If it happens that $Z^{(1)}, \ldots, Z^{(N)}, W^{(1)}, \ldots, W^{(N)} \in \Omega$, then the formula (3.77) is consistent with how $K$ is already defined by the “respects direct sums” condition. Then $\tilde{K}$ is a cp global kernel on $\Omega_{nc} \times \Omega_{nc}$ which when restricted to $\Omega \times \Omega$ agrees with $K$.

Similarly, if $\Omega$ is a (not necessarily nc) subset of $\mathcal{V}_{nc}$ for a complex vector space $\mathcal{V}$ and if $K$ is a graded function from $\Omega \times \Omega$ into $L(A, L(Y))_{nc}$ (so (2.3) is satisfied, we say that $K$ is a cp nc kernel on $\Omega$ if

(1) $K$ respects intertwinings, i.e., condition (2.7) holds, and
(2) $K$ satisfies the BBLS-complete positivity condition (3.75) or equivalently (3.76) on $\Omega$.

Under these conditions, $K$ extends to a well-defined cp nc kernel $\tilde{K}$ on the nc envelope $\Omega_{nc}$ of $\Omega$.

We mention that the reproducing kernel Hilbert space associated with a BBLS-completely positive kernel was studied in the paper of Ball-Biswas-Fang-ter Horst [6] with the $A$-argument taken as part of the point set as in approach (2) in Subsection 3.3.2 above, and the BBLS-completely positive kernels appear prominently in the characterization of the generalized Schur classes and associated generalized Nevanlinna-Pick interpolation theory of Muhly-Solel (see [34, 35, 36]).
3.4. Smoothness properties. We now explore the extent to which smoothness on the kernel leads to smoothness for the functions in the associated reproducing kernel Hilbert space \( \mathcal{H}(K) \) and in the factor \( H(Z) \) in the Kolmogorov decomposition for \( K \).

Throughout this subsection we assume that \( \Omega \) is a nc subset of \( \mathcal{V}_{nc} \) for a vector space \( \mathcal{V} \). We consider three possible topologies for \( \Omega \), the finite topology, the disjoint union topology, and the uniform topology (see \([31]\)), described as follows.

- We say that \( \Omega \) is open in the finite topology if, given \( W \in \Omega_m \) and \( H_W \in \mathcal{V}_m \), there exists \( \epsilon > 0 \) so that \( W + tH_W \in \Omega_m \) for all \( t \in \mathbb{C} \) with \( |t| < \epsilon \).
- Assume that \( \mathcal{V} \) is a Banach space equipped with an admissible family of matrix norms (a generalization of operator space defined in Section 7.1 of \([31]\)) and that \( \Omega \) is an open subset of \( \mathcal{V}_{nc} \). We say that \( \Omega \) is open in the disjoint union topology if, given \( W \in \Omega_m \) there exists \( \epsilon > 0 \) so that \( W + H_W \in \Omega_m \) as long as \( H_W \in \mathcal{V}_m \) has \( \|H_W\| < \epsilon \).
- Assume that \( \mathcal{V} \) and \( \Omega \) are as in the immediately preceding definition. We say that \( \Omega \) is open in the uniform topology if, given \( W \in \Omega_m \) there is \( \epsilon > 0 \) so that, for all \( N \in \mathbb{N} \) and \( D_W^{(N)} \in \mathcal{V}^{mN \times mN} \) with \( \|D_W^{(N)}\| < \epsilon \), it holds that \( \bigoplus_N W + D_W^{(N)} \in \Omega_{nN} \).

Suppose now that \( K \) is a nc cp kernel in \( \tilde{T}(\Omega; \mathcal{A}_{nc}, \mathcal{L}(\mathcal{V})_{nc}) \). We have three notions of local boundedness for \( K \) (with fixed \( \mathcal{A} \)-argument \( P \)) depending on the choice of topology on \( \Omega \).

- Assume that \( \Omega \) is finitely open. We say that the nc cp kernel \( K \) is P-locally bounded along slices if, for each choice of points \( Z \in \Omega_n \) and \( W \in \Omega_m \), \( \mathcal{A} \)-matrix \( P \in \mathcal{A}^{n \times m} \) and direction vectors \( D_Z \in \mathcal{V}^{m \times n} \) and \( D_W \in \mathcal{V}^{m \times m} \), there is an \( \epsilon > 0 \) so that, for \( t \in \mathbb{C} \) with \( |t| < \epsilon \) we have not only \( Z + tD_Z \in \Omega_n \) and \( W + tD_W \in \Omega_m \) but also \( \|K(Z + tD_Z, W + tD_W)(P)\| \) is bounded for \( t \in \mathbb{C} \) with \( |t| < \epsilon \).
- Assume that \( \Omega \) is open in the disjoint union topology. We say that the nc cp kernel \( K \) is P-locally bounded if, given points \( Z \in \Omega_n \) and \( W \in \Omega_m \) and \( \mathcal{A} \)-matrix \( P \in \mathcal{A}^{n \times m} \), there is an \( \epsilon > 0 \) so that \( \|K(Z + D_Z, W + D_W)(P)\| \) is uniformly bounded over all direction vectors \( D_Z \in \mathcal{V}^{m \times n} \) and \( D_W \in \mathcal{V}^{m \times m} \) satisfying \( \|D_Z\| < \epsilon \) and \( \|D_W\| < \epsilon \).
- Assume that \( \Omega \) is open in the uniform topology. We say that the nc cp kernel \( K \) is uniformly P-locally bounded if, given points \( Z \in \Omega_n \) and \( W \in \Omega_m \) and \( \mathcal{A} \)-matrix \( P \in \mathcal{A}^{n \times m} \), there is an \( \epsilon > 0 \) so that not only are \( \bigoplus_N Z + D_Z^{(N)} \in \Omega_{N_n} \) and \( \bigoplus_N W + D_W^{(N)} \in \Omega_{N_m} \) but also \( \|K(\bigoplus_N Z + D_Z^{(N)}, \bigoplus_N W + D_W^{(N)})(\bigoplus_N P)\| \) is uniformly bounded (independently of the choice of \( N \in \mathbb{N} \)) over all direction vectors \( D_Z^{(N)} \in \mathcal{V}^{N_n \times N_n} \) and \( D_W^{(N)} \in \mathcal{V}^{N_m \times N_m} \) satisfying \( \|D_Z^{(N)}\| < \epsilon \) and \( \|D_W^{(N)}\| < \epsilon \).
Similar definitions apply to nc functions. If \( f \) is a nc function on nc set \( \Omega \) which is open in the finite topology, we say that \( f \) is **locally bounded on slices** if, given a point \( Z \in \Omega_n \) and direction vector \( D_W \in \mathcal{V}^{m \times n} \), there is an \( \epsilon > 0 \) so that not only is \( Z + tD_Z \in \Omega_n \) but also \( \|f(Z + tD_Z)\| \) is bounded for \( t \in \mathbb{C} \) with \( |t| < \epsilon \). Similarly, if \( \Omega \) is open in the disjoint union topology and \( f \) is a nc function on \( \Omega \), we say that \( f \) is **locally bounded** if, given a point \( Z \in \Omega_n \) there is a \( \epsilon > 0 \) so that not only is \( Z + D_Z \in \Omega_n \) but also \( \|f(Z + D_Z)\| < \epsilon \) is bounded over all direction vectors \( D_Z \in \mathcal{V}^{m \times n} \) satisfying \( \|D_Z\| < \epsilon \). Finally, if \( \Omega \) is open in the uniform topology and \( f \) is a nc function on \( \Omega \), we say that \( f \) is **uniformly locally bounded** if, given any \( Z \in \Omega_n \), there is an \( \epsilon > 0 \) so that not only is \( \bigoplus_1^N Z + D_Z^{(N)} \in \Omega_{N_n} \) but also \( \|f(\bigoplus_1^N Z + D_Z^{(N)})\| < \epsilon \) is bounded (independently of the choice of \( N \in \mathbb{N} \)) over all direction vectors \( D_Z^{(N)} \in \mathcal{V}^{mN \times nN} \) satisfying \( \|D_Z^{(N)}\| < \epsilon \).

The significance of these various local boundedness conditions is that they imply corresponding analyticity properties for the nc function \( f \): (1) if \( f \) is locally bounded along slices, then \( f \) is Gâteaux differentiable at each point \( Z \in \Omega \) (see Theorem 7.2 in [31]), (2) if \( f \) is locally bounded, then \( f \) is Fréchet differentiable at each \( Z \in \Omega \) (see Theorem 7.4 in [31]), and (3) if \( f \) is uniformly locally bounded, then \( f \) is what is called “uniformly analytic” which in turn implies particularly nice convergence properties for its local Taylor-Taylor series (see Theorem 7.21 in [31]).

Assume that \( \Omega \) is finitely open and that the cp nc kernel \( K \) is \( P \)-locally bounded along slices. Then one can use the property of invariance with respect to direct sums to see that it suffices to assume that the \( P \)-local boundedness property holds with \( Z = W \in \Omega_n \); indeed note that

\[
K \left( \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \right) \left( \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \right) = \begin{bmatrix} K(Z,Z)(P_{11}) & K(Z,W)(P_{12}) \\ K(W,Z)(P_{21}) & K(W,W)(P_{22}) \end{bmatrix}
\]

If \( K \) is a cp nc kernel, then \( K \left( \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \right) \) is a completely positive map from \( \mathcal{A}^{(n+m) \times (n+m)} \) to \( \mathcal{L}(\mathcal{V}^{m+n}) \). This implies that \( K(Z,W) \) is even a completely bounded map. Thus, in working with the \( P \)-locally bounded condition, we may restrict to the case where \( Z = W \). It also suffices to work with \( P \) equal to the identity matrix \( P = I_{A^{n \times n}} \), since \( \|K(Z,Z)\| = \|K(Z,Z)(1_{A^{n \times n}})\| \). An analogous comment applies to the case where \( K \) is locally \( P \)-bounded or uniformly \( P \)-locally bounded.

**Theorem 3.15.** Let \( \Omega \subset \mathcal{V}_{nc} \) be a finitely open nc set.

1. Let \( K \) be a cp nc kernel on \( \Omega \) with values in \( \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{V}))_{nc} \) that is I-locally bounded on slices. Let \( (\mathcal{H}(K), \sigma) \) be the corresponding nc reproducing kernel Hilbert space and let \( H \) be the factor in the corresponding minimal Kolmogorov decomposition. Then each \( f \in \mathcal{H}(K) \) as well as \( H \) are locally bounded on slices, even when the completely bounded norm is used in the target domain. More precisely, let \( n \in \mathbb{N}, Z \in \Omega_n, D_Z \in \mathcal{V}^{m \times n} \) and assume that \( K(Z + tD_Z, Z + tD_Z)(1_{A^{n \times n}}) \) is bounded for \( |t| < \epsilon \); then for each \( f \in \mathcal{H}(K) \),
\[ \|f(Z + tD_Z)\| \text{ and } \|f(Z + tD_Z)\|_{cb} \text{ as well as } \|H(Z + tD_Z)\| \text{ and } \\
\|H(Z + tD_Z)\|_{cb} \text{ are bounded for } |t| < \epsilon. \]

(2) Conversely, let \( \mathcal{H} \) be a nc functional Hilbert space on \( \Omega \) with values in \( L(\mathcal{A},\mathcal{Y})_{nc} \) with an \( \mathcal{A} \)-action. Assume that for all \( n \in \mathbb{N}, Z \in \Omega_n, D_Z \in \mathcal{V}^{n \times n} \) there exists an \( \epsilon > 0 \) such that \( \|f(Z + tD_Z)\| \) is bounded for \( |t| < \epsilon \) for all \( f \in \mathcal{H} \) (in this case we say that the nc functional Hilbert space \( \mathcal{H} \) is \textbf{locally bounded on slices}). Then the corresponding cp nc kernel \( K \) is \( P \)-locally bounded on slices, even when the completely bounded norm is used in the target domain; more precisely, in the notation of the previous sentence, \( \|K(Z + tH_Z, Z + tH_Z)_{A^n \times n}\| = \|K(Z + tD_Z, Z + tD_Z)\|_{cb} \) is bounded for \( |t| < \epsilon. \)

\textit{Proof.} Assume first that \( K \) is a cp nc kernel on \( \Omega \) and \( f \in \mathcal{H}(K) \). For a point \( W \in \Omega_m \), direction vector \( D_W \in \mathcal{V}^{m \times m} \), row \( \mathcal{A} \)-tuple \( v \in \mathcal{A}^m \), and vector \( y \in \mathcal{Y}^m \), by Proposition 3.11 (specifically the estimate (3.58)), we have

\[ \|f(W + tD_W)\|_{L(\mathcal{A}^m,\mathcal{Y}^m)} = \|f(W + tD_W)\|_{L_{cb}(\mathcal{A}^m,\mathcal{Y}^m)} \leq \|f\|_{\mathcal{H}(K)} \|K(W + tD_W, W + tD_W)(1_{A^n \times m})\|^{1/2}_{L_{cb}(\mathcal{Y}^m)}. \]

The assumption that \( K \) is \( P \)-locally bounded then leads to the conclusion that \( f \) is locally bounded in cb-norm with radius of tolerance \( \epsilon \) equal to the radius of tolerance for \( K \) independent of the choice of \( f \in \mathcal{H}(K) \).

We next analyze the boundedness along slices for the minimal factor \( H \) in the Kolmogorov decomposition for \( K \). Again by Proposition 3.11 (specifically for this case the equality (3.62)), we have

\[ \|H(W + tD_W)\| = \|H(W + tD_W)\|_{cb} = \|K(W + tD_W, W + tD_W)(1_{A^n \times m})\|^{1/2}. \]

We conclude that \( H \) is locally bounded in cb-norm along slices as long as \( K \) is \( P \)-locally bounded along slices with radius of tolerance \( \epsilon \) the same as for \( K \). This completes the proof of statement (1) in Theorem 3.15.

We next suppose that \( \mathcal{H} \) is a nc functional Hilbert space with \( \mathcal{A} \)-action (as in Theorem 3.3) which furthermore is bounded along slices. More precisely, this means that for each \( n \in \mathbb{N}, Z \in \Omega_n, D_Z \in \mathcal{V}^{n \times n}, \) there exists \( \epsilon > 0 \) such that \( \|f(Z + tD_Z)\| \) is bounded for \( |t| < \epsilon \). We emphasize that the assumption here is that the radius of tolerance \( \epsilon \) depends on \( Z \) and \( D_Z \) but not on the choice of \( f \) in \( \mathcal{H}(K) \). Consider the family of linear operators

\[ \{H(Z + tD_Z): \mathcal{H}(K)^n \to \mathcal{Y}^n: t \in \mathbb{C} \text{ with } |t| < \epsilon\}. \]
Note that when we apply any operator in this family to a vector \( f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \) in \( \mathcal{H}(K)^n \) we get
\[
H(Z + tD_Z) \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} (Z + tD_Z)(1_{\mathcal{A}^{n\times n}})
= f_1(Z + tD_Z)(E_1^{(n)} \otimes 1_{\mathcal{A}}) + \cdots + f_n(Z + tD_Z)(E_n^{(n)} \otimes 1_{\mathcal{A}}).
\]
As each term in the final expression is bounded for \(|t| < \epsilon\), it follows that
\[\|H(Z + tD_Z)f\| \text{ is bounded for } |t| < \epsilon \text{ for each } f \in \mathcal{H}(K)^n.\]
It is now a consequence of the Principle of Uniform Boundedness (see e.g. [43]) that in fact \(\|H(Z + tD_Z)\|\) (and hence also \(\|H(Z + tD_Z)\|_{cb}\) by (3.62)) is bounded for \(|t| < \epsilon\). This completes the proof of statement (2) in the Theorem. \(\square\)

There are results analogous to that of Theorem 3.15 when we replace the hypothesis that \(\Omega\) is open in the finite topology by the hypothesis that \(\Omega\) is open in the disjoint union or in the uniform topology, and replace the “locally bounded on slices” condition by “locally bounded” or by “uniformly locally bounded” respectively. For the record we state these results.

**Theorem 3.16.** Let \(\Omega \subset \mathcal{V}_{nc}\) be a nc set which is open in the disjoint union topology.

1. Let \(K\) be a cp nc kernel on \(\Omega\) with values in \(\mathcal{L}(\mathcal{A},\mathcal{L}(\mathcal{Y}))_{nc}\) that is \(P\)-locally bounded. Let \((\mathcal{H}(K),\sigma)\) be the corresponding nc reproducing kernel Hilbert space and let \(H\) be the factor in the corresponding minimal Kolmogorov decomposition. Then each \(f \in \mathcal{H}(K)\) as well as \(H\) are both locally bounded and locally completely bounded. More precisely, let \(n \in \mathbb{N}\), \(Z \in \Omega_n\), and assume that \(K(Z + D_Z, Z + D_Z)(1_{\mathcal{A}^{n\times n}})\) is defined and bounded for all \(D_Z \in \mathcal{V}^{n\times n}\) with \(\|D_Z\| < \epsilon\); then for each \(f \in \mathcal{H}(K)\), \(\|f(Z + D_Z)\|\) and \(\|f(Z + D_Z)\|_{cb}\) as well as \(\|H(Z + D_Z)\|\) and \(\|H(Z + D_Z)\|_{cb}\) are bounded for \(\|D_Z\| < \epsilon\).

2. Conversely, let \(\mathcal{H}\) be a nc functional Hilbert space on \(\Omega\) with values in \(\mathcal{L}(\mathcal{A},\mathcal{Y})_{nc}\) with an \(\mathcal{A}\)-action. Assume that for all \(n \in \mathbb{N}\), \(Z \in \Omega_n\), there exists an \(\epsilon > 0\) such that \(\|f(Z + D_Z)\|\) is defined and bounded for all \(D_Z \in \mathcal{V}^{n\times n}\) such that \(\|D_Z\| < \epsilon\) and for all \(f \in \mathcal{H}\) (in this case we say that the nc functional Hilbert space \(H\) is **locally bounded**). Then the corresponding cp nc kernel \(K\) associated with \(\mathcal{H}\) by Theorem 3.15 is \(P\)-locally bounded and \(P\)-locally completely bounded; more precisely, \(\|K(Z + D_Z, Z + D_Z)(1_{\mathcal{A}^{n\times n}})\| = \|K(Z + D_Z, Z + D_Z)(1_{\mathcal{A}^{n\times n}})\|_{cb}\) is defined and bounded for all \(D_Z \in \mathcal{V}^{n\times n}\) with \(\|D_Z\| < \epsilon\).

**Theorem 3.17.** Let \(\Omega \subset \mathcal{V}_{nc}\) be a nc set which is open in the uniform topology.

1. Let \(K\) be a cp nc kernel on \(\Omega\) with values in \(\mathcal{L}(\mathcal{A},\mathcal{L}(\mathcal{Y}))_{nc}\) that is uniformly \(P\)-locally bounded. Let \((\mathcal{H}(K),\sigma)\) be the corresponding nc
reproducing kernel Hilbert space and let $H$ be the factor in the corresponding minimal Kolmogorov decomposition. Then each $f \in \mathcal{H}(K)$ as well as $H$ are uniformly locally bounded. More precisely, let $Z \in \Omega_n$, and assume that $K(\bigoplus_1^N Z + D_Z^{(N)}, \bigoplus_1^N Z + D_Z^{(N)}) (1_{\mathbb{A}^{n \times n}})$ is defined and bounded for all $D_Z^{(N)} \in \mathcal{V}^{nN \times nN}$ with $\|D_Z\| < \epsilon$ for all $N \in \mathbb{N}$; then for each $f \in \mathcal{H}(K)$, $\|f(\bigoplus_1^N Z + D_Z^{(N)})\|$ as well as $\|H(\bigoplus_1^N Z + D_Z^{(N)})\|$ are bounded for all $D_Z \in \mathcal{V}^{nN \times nN}$ with $\|D_Z\| < \epsilon$ for all $N \in \mathbb{N}$.

(2) Conversely, let $\mathcal{H}$ be a nc functional Hilbert space on $\Omega$ with values in $\mathcal{L}(\mathcal{A}, \mathcal{Y})_{\text{nc}}$ with an $\mathcal{A}$-action. Assume that for each $Z \in \Omega_n$, there exists an $\epsilon > 0$ such that $\|f(\bigoplus_1^N Z + D_Z^{(N)})\|$ is bounded for all $D_Z \in \mathcal{V}^{n \times n}$ such that $\|D_Z\| < \epsilon$ and for all $f \in \mathcal{H}_K$ (in this case we say that the nc functional Hilbert space $\mathcal{H}$ is uniformly locally bounded). Then the corresponding cp nc kernel $K$ associated with $\mathcal{H}$ as in Theorem 3.3 is uniformly $P$-locally bounded; more precisely, $\|K(\bigoplus_1^N Z + D_Z^{(N)}, \bigoplus_1^N Z + D_Z^{(N)}) (1_{\mathbb{A}^{n \times n}})\|$ is defined and bounded for all $D_Z \in \mathcal{V}^{nN \times nN}$ with $\|D_Z\| < \epsilon$ for all $N \in \mathbb{N}$.

**Proof of Theorems 3.16 and 3.17** As the proofs of both theorems parallel closely the proof of Theorem 3.15, we only sketch the main ideas. The hypothesis and conclusion of Theorem 3.15, for both statements (1) and (2), involves the boundedness of a family of operators parametrized by $t \in \mathbb{C}$ with $|t| < \epsilon$. The hypothesis and conclusion of Theorems 3.16 and 3.17 are the same, but with the modification that the family of operators is parametrized by increment vectors $D_Z \in \mathcal{V}^{n \times n}$ with $\|D_Z\| < \epsilon$, or by increment vectors $D_Z^{(N)} \in \mathcal{V}^{nN \times nN}$ with $N \in \mathbb{N}$ arbitrary with $\|D_Z^{(N)}\| < \epsilon$. With this modification of the set of operators whose boundedness is of interest, the proofs of Theorems 3.16 and 3.17 go through in exactly the same way as in the proof of Theorem 3.15.

3.5. **Functional versus formal noncommutative reproducing kernel Hilbert spaces.** The goal of this subsection is to establish a dictionary between the global/nc functional reproducing kernel Hilbert spaces being discussed here and the notion of formal nc reproducing kernel Hilbert spaces introduced by two of the present authors in [14]. Toward this end, we first need to review the setup from [14].

We let $\mathbb{F}_{d}^d$ be the monoid on $d$ generators $\{1, \ldots, d\}$ (also known as the unital free semigroup with $d$ generators). Elements of $\mathbb{F}_{d}^d$ are written as words $a = i_N \cdots i_1$ with letters $i_j$ from the alphabet consisting of the first $d$ natural numbers $\{1, \ldots, d\}$. Multiplication is by concatenation:

$$a \cdot b = i_N \cdots i_1 j_M \cdots j_1 \text{ if } a = i_N \cdots i_1 \text{ and } b = j_M \cdots j_1.$$
The empty word, denoted as $\emptyset$, serves as the unit element for $\mathbb{F}_d^+$. For $a = i_N \cdots i_1$ an element of $\mathbb{F}_d^+$, we let $a^\top = i_1 \cdots i_N$ denote the transpose of $a$ and let $|a| = N$ denote the length of (or number of letters in) $a$.

Given a collection of freely noncommuting indeterminates $z = (z_1, \ldots, z_d)$ and given a word $a = i_N \cdots i_1$ in $\mathbb{F}_d^+$, we let $z^a$ denote the noncommutative monomial $z^a = z_{i_N} \cdots z_{i_1}$, where we take $z^\emptyset = 1$. Given also a linear space $X$, we let $X\langle\langle z\rangle\rangle$ denote the space of all formal power series $f(z) = \sum_{a \in \mathbb{F}_d^+} f_a z^a$ where the coefficients $f_a$ are in $X$. Suppose that $X'$ is an algebra such that $X$ is a left module over $X'$. Given

$$F(z) = \sum_{a \in \mathbb{F}_d^+} F_a z^a \in X'\langle\langle z\rangle\rangle, \quad f(z) = \sum_{b \in \mathbb{F}_d^+} f_b z^b \in X\langle\langle z\rangle\rangle,$$

we define the (noncommutative convolution) product $F \cdot f(z) \in X'\langle\langle z\rangle\rangle$ by

$$(F \cdot f)(z) = \sum_{\gamma \in \mathbb{F}_d^+} \left( \sum_{a,b \in \mathbb{F}_d^+ : \gamma = ab} F_a f_b \right) z^\gamma.$$

We now suppose that we are given a Hilbert space $\mathcal{H}$ whose elements $f(z)$ are formal power series $f(z) = \sum_{a \in \mathbb{F}_d^+} f_a z^a \in \mathcal{Y}\langle\langle z\rangle\rangle$ for a coefficient Hilbert space $\mathcal{Y}$. We say that $\mathcal{H}$ is a NFRKHS (noncommutative formal reproducing kernel Hilbert space) if, for each $a \in \mathbb{F}_d^+$, the linear operator $\text{ev}_a : f \mapsto f_a$ mapping $f$ to its $a$-th formal power series coefficient in $\mathcal{Y}$ is continuous. As any such power series is completely determined by the list of its coefficients $\{f_a : a \in \mathbb{F}_d^+\}$, equivalently we can view an element $f(z)$ as a function $a \mapsto f_a$ on $\mathbb{F}_d^+$. Hence, by the Aronszajn theory of reproducing kernel Hilbert spaces (see Subsection 3.3.2 above), there is an Aronszajn-type positive kernel $K : \mathbb{F}_d^+ \times \mathbb{F}_d^+ \to L(\mathcal{Y})$ so that $\mathcal{H}$ is the reproducing kernel Hilbert space associated with $K$. To spell this out for this context, we denote the value of $K$ at the pair of words $(a, b)$ by $K_{a,b}$. Since we view an element $f \in \mathcal{H}$ as a formal power series $\sum_{a \in \mathbb{F}_d^+} f_a z^a$ rather than as a function $a \mapsto f_a$, we write, for a given $b \in \mathbb{F}_d^+$ and $y \in \mathcal{Y}$, the element $\text{ev}_b^* y \in \mathcal{H}$ as

$$(\text{ev}_b^* y)(z) = K_b(z)y = \sum_{a \in \mathbb{F}_d^+} K_{a,b} y z^a.$$

Then the reproducing kernel property assumes the form

$$\langle f, K_b(\cdot)y \rangle_{\mathcal{H}} = \langle f_b, y \rangle_{\mathcal{Y}}. \quad (3.78)$$

Following [14], we make the notation more suggestive of the classical case as follows. We let $w = (w_1, \ldots, w_d)$ be a second $d$-tuple of freely noncommuting indeterminates. For suggestive formal reasons which will become clear below, we also introduce the conjugate $d$-tuple of freely noncommuting indeterminates

$$\overline{w} = (\overline{w}_1, \ldots, \overline{w}_d)$$
In general, given a coefficient Hilbert space $\mathcal{C}$, we can use the $\mathcal{C}$-inner product to define a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}(\langle w \rangle)} : \mathcal{C} \times \mathcal{C}(\langle w \rangle) \to \mathbb{C}(\langle w \rangle)$$

by

$$\langle c, \sum_{a \in F^+_d} f_a w^a \rangle_{\mathcal{C} \times \mathcal{C}(\langle w \rangle)} = \sum_{a \in F^+_d} \langle c, f_a \rangle_{\mathcal{C}} w^{a^\top}.$$

We shall also have use of the reverse-order version of the pairing:

$$\langle \sum_{a \in F^+_d} f_a w^a, c \rangle_{\mathcal{C}(\langle z \rangle) \times \mathcal{C}} = \sum_{a \in F^+_d} \langle f_a, c \rangle_{\mathcal{C}} w^a.$$

Then the reproducing kernel property (3.78) can be written more suggestively as

$$\langle f, K(\cdot, w) y \rangle_{\mathcal{H} \times \mathcal{H}(\langle w \rangle)} = \langle f(w), y(\langle w \rangle) \rangle_{\mathcal{Y} \times \mathcal{Y}}. \quad (3.79)$$

Here $K(z, w)$ has the property that, for each $y \in \mathcal{Y}$, the formal power series in $w$ given by

$$K(z, w) y = \sum_{a \in F^+_d} (K_{a,b} y) z^a w^b \gamma^0$$

can be viewed as an element of $\mathcal{H}(\langle w \rangle)$. Whenever $\mathcal{H}$ is a Hilbert space of formal power series with the structure as laid out above, we shall say that $\mathcal{H}$ is the noncommutative formal reproducing kernel Hilbert space (NFRKHS) with reproducing kernel $K \in \mathcal{L}(\mathcal{Y}(\langle z, w \rangle))$.

The following result amounts to Theorem 3.1 from [14].

**Theorem 3.18.** Let

$$K(z, w) = \sum_{a,b \in F^+_d} K_{a,b} z^a w^b \gamma^0 \in \mathcal{L}(\mathcal{Y}(\langle z, w \rangle))$$

be a given element of $\mathcal{L}(\mathcal{Y}(\langle z, w \rangle))$ where $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$ are $d$-tuples of freely noncommuting indeterminates. Then the following conditions are equivalent:

$\textit{a}$ These can be seen as special cases of the more general pairing (which we shall not need in the sequel)

$$\langle \sum_{a \in F^+_d} f_a w^a, \sum_{b \in F^+_d} g_b w^b \rangle_{\mathcal{C}(\langle w \rangle) \times \mathcal{C}(\langle w \rangle)} = \sum_{\gamma \in F^+_d} \sum_{a, b : \gamma = b^\top a} \langle f_a, g_b \rangle_{\mathcal{C}} w^\gamma,$$

or equivalently

$$\langle f(w), g(w) \rangle_{\mathcal{C}(\langle w \rangle) \times \mathcal{C}(\langle w \rangle)} = \langle g(w)^* f(w) \rangle_{\mathcal{C}(\langle w \rangle)}$$

where we set $(\sum_a g_a w^a)^* = \sum_a g_a^* w^{a^\top} \in \mathcal{L}(\mathcal{C}) \langle \langle w \rangle \rangle$ where $g_a^* \in \mathcal{L}(\mathcal{C}, \mathbb{C})$ is given by $g_a^* : c \mapsto \langle c, g_a \rangle_{\mathcal{C}}$. Using this formalism one can easily check the identity

$$\langle S(w) f(w), g(w) \rangle_{\mathcal{Y}(\langle w \rangle) \times \mathcal{Y}(\langle w \rangle)} = \langle f(w), S(w)^* g(w) \rangle_{\mathcal{U}(\langle w \rangle) \times \mathcal{U}(\langle w \rangle)}$$

for $S(w) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})(\langle w \rangle)$, $f(w) \in \mathcal{U}(\langle w \rangle)$ and $g(w) \in \mathcal{Y}(\langle w \rangle)$. 
(1) \( K(z, w) \) is a positive formal kernel in the sense that
\[
\sum_{a, b \in \mathbb{F}_d^+} \langle K_{a, b} y_a, y_b \rangle y \geq 0
\]
for all finitely supported \( Y \)-valued functions \( a \mapsto y_a \) on \( \mathbb{F}_d^+ \).

(2) \( K \) is the reproducing kernel for a uniquely determined NFRKHS \( \mathcal{H}(K) \) of formal power series in the set of noncommuting indeterminates \( z = (z_1, \ldots, z_d) \).

(3) There is an auxiliary Hilbert space \( X \) and a noncommutative formal power series \( H(z) \in L(X, \mathcal{Y} \langle \langle z \rangle \rangle) \) such that
\[
K(z, w) = H(z) H(w)^* \tag{3.80}
\]
where we use the convention \((w^b)^* = \overline{w^b}^T\) so that
\[
H(w)^* = \sum_{b \in \mathbb{F}_d^+} (H^b)^T \overline{w^b} \text{ if } H(z) = \sum_{a \in \mathbb{F}_d^+} H_a z^a.
\]

Moreover, in this case the NFRKHS \( \mathcal{H}(K) \) can be defined directly in terms of the formal power series \( H(z) \) appearing in condition (2) by
\[
\mathcal{H}(K) = \{ H(z) x : x \in X \}
\]
with norm taken to be the “lifted norm”
\[
\|H(z)x\|_{\mathcal{H}(K)} = \|Qx\|_\mathcal{H} \tag{3.81}
\]
where \( Q \) is the orthogonal projection of \( X \) onto the orthogonal complement of the kernel of the map \( M_H : X \mapsto \mathcal{Y} \langle \langle z \rangle \rangle \) given by \( M_H : x \mapsto H(z)x \).

We now suppose that we are given a formal positive kernel \( K \) and the associated NFRKHS \( \mathcal{H}(K) \) as in Theorem 3.18. We wish to make the connection with nc reproducing kernel Hilbert spaces by evaluating formal power series \( f(z) \in \mathcal{H}(K) \) at matrix tuple points \( Z = (Z_1, \ldots, Z_d) \in (\mathbb{C}^{n \times n})^d \cong (\mathbb{C}^d)^{n \times n} \).

We therefore introduce the nc set \( \Omega \subset (\mathbb{C}^d)_{nc} \) by
\[
\Omega = \{ Z \in (\mathbb{C}^d)_{nc} : \sum_{\ell=0}^\infty \sum_{a \in \mathbb{F}_d^+: |a| = \ell} Z^a \otimes f_a \text{ converges} \}
\]
for all \( f(z) = \sum_{a \in \mathbb{F}_d^+} f_a z^a \in \mathcal{H}(K) \}. \tag{3.82}

Here the convergence is taken in the weak topology of \( \mathcal{Y}^{m \times m} \) if \( Z \in (\mathbb{C}^d)^{m \times m} \). From the lifted norm characterization (3.81) of \( \mathcal{H}(K) \), it is clear that \( \Omega \) can alternatively be characterized as
\[
\Omega = \{ Z \in (\mathbb{C}^d)_{nc} : \sum_{\ell=0}^\infty \sum_{a \in \mathbb{F}_d^+: |a| = \ell} Z^a \otimes H_a \text{ converges} \} \tag{3.83}
\]
where, if \( Z \in (\mathbb{C}^d)^{m \times m} \), the convergence is in \( L(X^m, \mathcal{Y}^m) \) with the weak operator topology.
The next result is the main tool for arriving at a nc reproducing kernel Hilbert space from a NFRKHS.

**Proposition 3.19.** Suppose that $K$ is a formal positive kernel with associated NFRKHS $\mathcal{H}(K)$. Define the nc set $\Omega$ as in either (3.82) or (3.83), and suppose that $Z \in \Omega_n$. Then the iterated sum

$$\sum_{s=0}^{\infty} \sum_{a \in F_d : |a|=s} \left( \sum_{t=0}^{\infty} \sum_{b \in F_d : |b|=t} Z^a P W^s b^T \otimes K_{ab} \right)$$

(3.84)

converges for all $W \in \Omega_m$ and $P \in \mathbb{C}^{n \times m}$ for all $m = 1, 2, \ldots$. The same result holds if the order of iteration is reversed.

**Proof.** Given $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathbb{C}^{n \times m}$, from the second characterization (3.83) of $\Omega$ we see that $\sum_{s=0}^{\infty} \sum_{a \in F_d : |a|=s} Z^a \otimes H_a$ converges and that

$$\left( \sum_{t=0}^{\infty} \sum_{b \in F_d : |b|=t} W^b \otimes H_b \right)^* = \sum_{t=0}^{\infty} \sum_{b \in F_d : |b|=t} W^s t^T \otimes H_b$$

converges weakly, from which it follows that the iterated sum

$$\sum_{s=0}^{\infty} \sum_{a \in F_d : |a|=s} \left( \sum_{t=0}^{\infty} \sum_{b \in F_d : |b|=t} Z^a P W^s b^T \otimes K_{ab} \right)$$

$$= \sum_{s=0}^{\infty} \sum_{a \in F_d : |a|=s} (Z^a \otimes H_a) P \left( \sum_{t=0}^{\infty} \sum_{b \in F_d : |b|=t} W^s b^T \otimes H_b \right)$$

converges in either order. \qed

Given the result of Proposition 3.19, we can associate a cp nc kernel $H$ with a given formal positive kernel $K$ as follows. Suppose that $K$ has formal Kolmogorov decomposition $K(z, w) = H(z)H(w)^*$ as in (3.80). For $Z \in \Omega_n$, we may use the convergent series in (3.83) to define a function $Z \mapsto H(Z)$:

$$H(Z) = \sum_{\ell=0}^{\infty} \left[ \sum_{a \in F_d : |a|=\ell} H_a \otimes Z^a \right] \in \mathcal{L}(\mathcal{X}^n,\mathcal{Y}^m).$$

We then define a kernel function $K$ from $\Omega_n \times \Omega_m$ to $\mathcal{L}(\mathbb{C}^{n \times m}, \mathcal{L}(\mathcal{Y}^n \times m))$ by

$$K(Z, W)(P) = H(Z)(P \otimes \text{id}_{\mathcal{L}(\mathcal{X})})H(W)^*.$$  

(3.85)

As $H$ is given in terms of a convergent tensor-calculus power series, it follows that $H$ is a nc function from $\Omega$ to $\mathcal{L}(\mathcal{X}, \mathcal{Y})_{\text{nc}}$ (see [31]). Then $K$ is given
in terms of a Kolmogorov decomposition (with \( A = \mathbb{C} \) and \( \sigma \) the trivial representation of \( \mathbb{C} \) on \( \mathbb{C} \)), and hence is a cp nc kernel.

The following result establishes the precise correspondence between the nc RKHS \( \mathcal{H}(K) \) and the formal nc RKHS \( \mathcal{H}(\tilde{K}) \) when \( K \) and \( \tilde{K} \) are related as in (3.85).

**Theorem 3.20.** Suppose that \( K \) is a formal positive kernel with formal Kolmogorov decomposition as in (3.80), and let \( K \) be the function from \( \Omega_n \times \Omega_m \) to \( \mathcal{L}(\mathbb{C}^n \otimes \mathcal{L}(\mathcal{Y}^m)) \) given by (3.85). Then \( K \) is a cp nc kernel. Furthermore \( \mathcal{H}(K) \) is the isometric image of \( \mathcal{H}(\tilde{K}) \) under the map

\[
\sum_{a \in F_d^+} f_a z^a \mapsto \left( Z \in \Omega_n \mapsto \left( u \in \mathbb{C}^n \mapsto \sum_{a \in F_d^+} Z^a u \otimes f_a \right) \right).
\]

Conversely, suppose that \( K \) from \( \Omega_n \times \Omega_m \) to \( \mathcal{L}(\mathbb{C}^n \otimes \mathcal{L}(\mathcal{Y}^m)) \) \((n, m \in \mathbb{N} \) arbitrary\) is a cp nc kernel defined on either (a) the nc set \( \Omega \) taken to be \( \text{Nilp}_d^* = \bigcap_{n=1}^{\infty} \text{Nilp}_n^d \) where \( \text{Nilp}_n^d \) is defined to be the space of jointly nilpotent \( d \)-tuples \( Z = (Z_1, \ldots, Z_d) \) of \( n \times n \) matrices \( (\text{so } Z^a = 0 \text{ once } |a| \text{ is sufficiently large}) \), or (b) on the nc set \( \Omega \subset (\mathbb{C}^d)_{\text{nc}} \) where \( \Omega \) is a finitely open set containing \( 0 \) on which \( K \) is \( I \)-locally bounded along slices. Suppose that \( K \) has nc Kolmogorov decomposition \( K(Z, W)(P) = H(Z)(P \otimes \text{id}_{\mathcal{L}(\mathcal{Y})})H(W)^* \) for a nc function \( H : \Omega \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})_{\text{nc}} \). (By the results of (31), \( H(Z) \) has a Taylor-Taylor series representation \( H(Z) = \sum_{a \in F_d^+} Z^a \otimes H_a \) either on all of \( \text{Nilp}_d \) in case (a), or on an appropriate \( \tilde{\Omega} \subset \Omega \) in case (b)). Define a formal power series \( H(z) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})(\langle z \rangle) \) and a formal kernel \( K \in \mathcal{Y}(\langle z, \overline{w} \rangle) \) by

\[
H(z) = \sum_{a \in F_d^+} H_a z^a, \quad K(z, w) = H(z)^* H(w) = \sum_{a, b \in F_d^+} (H_a H^*_{ab}) z^a \overline{w}^b.
\]

Then \( K \) is a formal positive kernel with convergence set containing \( \text{Nilp}_d^* \) \((\text{in case (a)}) \) or \( \tilde{\Omega} \) \((\text{in case (b)}) \), and we recover \( \mathcal{H}(K) \) from \( \mathcal{H}(\tilde{K}) \) via the inverse of the map (3.86).

**Proof.** As is seen from part (3) of Theorem (3.18) the space \( \mathcal{H}(K) \) can be presented directly in terms of the formal Kolmogorov decomposition (3.80) of the formal positive kernel \( K \) as the space

\[
\mathcal{H}(K) = \{ f_x(z) : x \in \mathcal{X} \}
\]

where we set \( f_x(z) = H(z)x, \) with norm given by

\[
\|f\|_{\mathcal{H}(K)}^2 = \inf\{\|x\|^2 : x \in \mathcal{X} \text{ such that } f = f_x\}.
\]

On the other hand, by Theorem (3.7) the nc RKHS \( \mathcal{H}(K) \) has a similar lifted norm description in terms of its nc Kolmogorov decomposition (3.5), namely:

\[
\mathcal{H}(K) = \{ f_x : x \in \mathcal{X} \}.
\]
where we set \( f_x(Z)u = H(Z)(\text{id}_{C^n} \otimes \sigma_x)(u)x \) with norm given by
\[
\|f\|_{\mathcal{H}(K)}^2 = \inf\{\|x\|^2 : f = f_x\}.
\]
For the case at hand here where the cp nc kernel \( K \) is derived from a formal positive kernel \( K \) as in (3.85), the \( C^* \)-algebra \( \mathcal{A} \) is just \( \mathbb{C} \) and the representation \( \sigma_x \) is the trivial representation. Then the function \( Z \mapsto f_x(Z) \), acting on a vector \( u \in \mathbb{C}^n \) for \( Z \in \Omega_n \) can be written more concretely as
\[
f_x(Z)u = H(Z)(u \otimes x) = \sum_{a \in F_d^+} Z^a u \otimes H_a x.
\]
In terms of these respective parametrizations of the spaces \( \mathcal{H}(K) \) and \( \mathcal{H}(K) \) via the state space \( \mathcal{X} \) for the Kolmogorov decompositions, we see that the map (3.86) is given by
\[
\phi: \sum_{a \in F_d^+} (H_a x)z^a \mapsto (Z \in \Omega_n \mapsto (u \in \mathbb{C}^n \mapsto \sum_{a \in F_d^+} Z^a u \otimes H_a x)). \tag{3.87}
\]
Given the lifted-norm-space characterizations of \( \mathcal{H}(K) \) and \( \mathcal{H}(K) \), we see that indeed the map \( \phi \) (3.87) maps \( \mathcal{H}(K) \) onto \( \mathcal{H}(K) \). To see that \( \phi \) is well-defined, note that \( \sum_{a \in F_d^+} (H_a x)z^a = 0 \) in \( \mathcal{H}(K) \) means that \( H_a x = 0 \) for all \( a \in F_d^+ \). It then follows that \( \sum_{a \in F_d^+} Z^a u \otimes H_a x = 0 \) for all \( Z \in \Omega_n \) and all \( u \in \mathbb{C}^n \), i.e., the right-hand side of (3.87) is the zero element of \( \mathcal{H}(K) \). Conversely, suppose that \( \sum_{a \in F_d^+} Z^a u \otimes H_a x = 0 \) for all \( Z \in \Omega_n \) and all \( u \in \mathbb{C}^n \). We identify the map \( u \mapsto \sum_{a \in F_d^+} Z^a u \otimes H_a x \) with an element of \( \mathbb{C}^{n \times n} \otimes \mathcal{Y} \), namely, \( \sum_{a \in F_d^+} Z^a \otimes H_a x \). As a function of \( Z \), this can be identified as a nc function \( g: \Omega \to \mathcal{Y}_{nc} \):
\[
g(Z) = \sum_{a \in F_d^+} Z^a \otimes H_a x. \tag{3.88}
\]
By results from [31] on noncommutative Taylor series, we have the identification
\[
Z^a \otimes H_a x = Z^a \Delta_R^{a^\top} g(0, \ldots, 0) \quad \text{for } |a| + 1 \text{ times}. \tag{3.89}
\]
Whether we are in the case where \( \Omega = \text{Nilp}_d \) or where \( \Omega \) is a nc ball around the origin where all the series \( f(Z) = \sum_{a \in F_d^+} Z^a \otimes f_a \) converge, the function \( g \) is also given as the sum of its Taylor series (finite in the nilpotent case)
\[
g(Z) = \sum_{a \in F_d^+} Z^a \Delta_R^{a^\top} g(0, \ldots, 0).
\]
Moreover, the higher-order nc derivatives \( \Delta_R^{a^\top} g \) are uniquely determined by the associated nc function \( g \). Hence the function \( Z \mapsto g(Z) \) in (3.88) being identically equal to zero forces all the higher order derivatives \( \Delta_R^{a^\top} g(0, \ldots, 0) \) to be zero, which in turn, due to the identity (3.89), forces \( H_a x = 0 \) for all
a ∈ F^+_d. Hence \( \sum_{a \in F^+_d} (H_a x) z^a = 0 \) as an element of \( \mathcal{Y} \langle \langle z \rangle \rangle \). In this way we see that the map \( \phi \) is injective as well.

Finally to show that \( \phi \) is an isometry from \( \mathcal{H}(K) \) onto \( \mathcal{H}(K) \), it suffices to show the set identity

\[
\{ x \in X : f_x = f \} = \{ x \in X : f_x = f \}
\]

whenever \( f \in H(Z) \) and \( f \in H(K) \) are related as in (3.86). This is a consequence of the fact that \( f_x = 0 \) in \( H(Z) \) exactly when \( f_x = 0 \) in \( H(K) \).

This in turn is a consequence of the analysis done in the previous paragraph.

Conversely, let \( K \) be a cp nc kernel either on \( \Omega = \text{Nilp}^d \) (case (a)) or on a finitely open set \( \Omega \subset \mathbb{C}^d \) containing the origin where \( K \) is \( I \)-locally bounded (case (b)). Let \( K(Z,W)(P) = H(Z)(P \otimes \text{id}_{\mathcal{Y}})H(W)^* \) be a minimal Kolmogorov decomposition (i.e. span \( \{ \text{Ran} H(W)^* : W \in \Omega \} = X \) from which it follows that span \( \{ \text{Ran} (P \otimes \text{id}_{\mathcal{Y}})H(W)^* : P \in \mathbb{C}^{n \times n}, W \in \Omega_n \} = X_n \) for each \( n \in \mathbb{N} \), where, in case (b), \( H \) is locally bounded along slices. By results from [31], \( H(Z) = \sum_{a \in F^+_d} Z^a \otimes H_a \) either on all of \( \text{Nilp}^d \) (case (a)) or on an appropriate \( \tilde{\Omega} \subset \Omega \) (in case (b)). Let \( H(z) = \sum_{a \in F^+_d} H_a z^a \),

\[
K(z,w) = \sum_{a,b \in F^+_d} K_{a,b} z^a w^b
\]

where \( K_{a,b} = H_a H_b^* \). Then \( K \) is a formal positive nc kernel with convergence set containing either \( \text{Nilp}^d \) or \( \tilde{\Omega} \). □

Remark 3.21. We here note some results from the work of Kaliuzhnyi-Verbovetskyi–Vinnikov [30] which may be viewed as formal analogues of various parts of Theorem 3.13.

1. Formal version of Corollary 3.14

Theorem 3.22. The formal power series

\[
K(z,w) = \sum_{a,b} K_{a,b} z^a w^b \in \mathcal{L}(\mathcal{Y} \langle \langle z,w \rangle \rangle)
\]

is a formal positive kernel if and only if, for every \( n \in \mathbb{N} \) and \( Z \in \text{Nilp}_n^d \),

\[
K_{I_n}(Z,Z) = \sum_{a,b \in F^+_d} K_{a,b} \otimes Z^a Z^b \in \mathcal{L}(\mathcal{Y} \otimes \mathbb{C}^n)
\]

(note that the a priori infinite series is actually finite since \( Z \in \text{Nilp}_n^d \) is a positive semidefinite operator.

Using the connection between formal positive kernels and cp nc kernels given by Theorem 3.20, we see that Theorem 3.22 may be considered as equivalent to Corollary 3.14. A direct proof of Theorem 3.22 can be found in [30, Theorem 3].

2. Formal version of Theorem 3.13 part (a): The following formal version of Theorem 3.13 part (a) appears as Theorem 2 in [30].

Theorem 3.23. Suppose that \( K(z,w) \in \mathcal{L}(\mathcal{Y} \langle \langle z,w \rangle \rangle) \) is a formal power series which is uniformly convergent when a \( (n \times n) \)-matrix d-tuple pair
(Z, W) is substituted for the formal indeterminates (z, w), as long as Z, W are in some norm-open ball around the origin $U_n$ in $\mathbb{C}^{n \times n}$ for each $n \in \mathbb{N}$. Suppose also that the associated kernel $(Z, W) \mapsto K(Z, W)$ obtained by this substitution is a positive kernel in the sense of Aronszajn on each open ball $U_n$ for all $n \in \mathbb{N}$. Then it follows that $K$ is a positive kernel.

We are now ready to complete the proof of Theorem 3.13 part (a) by using its formal analogue Theorem 3.23 as the basic ingredient.

Proof of Theorem 3.13 part (a): Suppose that the nc set $\Omega$ is a uniform ball $N(0; \epsilon)$ around 0 in $(\mathbb{C}^d)_{\text{nc}}$ and that the kernel $K_{I_n}$ is Aronszajn-positive on $N(0; \epsilon)_n$ for each $n \in \mathbb{N}$. We also assume that $K$ is locally $I$-bounded on slices. Then we use the fact (not proven in [31] and verifiable by results done there for the order-0 case) that $K$ has a Taylor-Taylor series expansion centered at the origin:

$$K(Z, W)(P) = \sum_{a, b} K_{a, b} \otimes Z^a P W^* b^T$$

for some Taylor coefficient moments $K_{a, b} \in L(Y)$. We then may associate a formal kernel $K(z, w) = \sum_{a, b} K_{a, b} z^a w^{* b^T}$. The hypothesis that $K_{I_n}$ is a positive Aronszajn kernel on each $N(0; \epsilon)_n$ implies that $K$ satisfies the hypotheses of Theorem 3.23. We conclude that $K$ is a positive formal kernel, hence has a formal Kolmogorov decomposition

$$K(z, w) = H(z) H(w)^* = \sum_{a, b} H_a H_b^* z^a w^{* b^T}.$$ 

Plugging in matrix pairs $(Z, W)$ for the formal indeterminates $(z, w)$ into this relation gives us

$$K(Z, W)(I_n) = H(Z) H(W)^* = \sum_{a, b} H_a H_b^* \otimes Z^a W^* b^T.$$ 

Our goal is to show that $K$ is a cp nc kernel, i.e., that $K(Z, Z)(P) \succeq 0$ whenever $Z \in N(0; \epsilon)_n$ and $P \succeq 0$ in $\mathbb{C}^{n \times n}$. Fix $Z \in N(0; \epsilon)_n$. While $N(0; \epsilon)_n$ is not similarity-invariant, it is invariant under local similarities, i.e., given $Z \in N(0; \epsilon)$, there is a $\eta > 0$ so that $S \in \mathbb{C}^{n \times n}$ with $\|S\|, \|S^{-1}\|$ both at most $\eta$ implies that $\tilde{Z} = S^{-1} Z S \in N(0; \epsilon)$. If $P$ has the form $SS^*$ with $S$ as above, then the nc kernel properties of $K$ imply that

$$K(Z, Z)(P) = K(Z, Z)(SS^*) = S K(\tilde{Z}, \tilde{Z})(I) S^* \succeq 0.$$ 

On the other hand, we also know that

$$S K(\tilde{Z}, \tilde{Z})(I) S^* = S H(\tilde{Z}) H(\tilde{Z})^* S^* = H(Z) SS^* H(Z)^*$$

since $H$ is a nc function. We conclude that

$$K(Z, Z)(P) = H(Z) P H(Z)^*$$  \hspace{1cm} (3.90)

for $P \succeq 0$ in a sufficiently small neighborhood around $I_n$ (where the neighborhood depends on the fixed point $Z \in N(0; \epsilon)$). But both sides of (3.90)
are linear in $P$, in particular, entire in the matrix entries of $P$. We conclude that the identity (3.90) actually holds for all Hermitian $P \in \mathbb{C}^{n \times n}$, and then, by linearity, for all $P \in \mathbb{C}^{n \times n}$. We have thus exhibited a nc Kolmogorov decomposition for $K$ and we conclude that $K$ is cp as wanted. \hfill $\square$

As is pointed out in [30], the hypothesis that $K_{I_n}$ is a positive kernel in the sense of Aronszajn cannot be weakened to the hypothesis that $K(Z, Z)(I_n) \succeq 0$ for all $Z \in N(0; \epsilon)$; one must use the full force of Aronszajn-positivity of $K_{I_n}$ to deduce that the associated formal kernel $K$ is a positive formal kernel.

3. The polynomial case. Another interesting case of Theorem 3.20 is the case where $K$ is a formal polynomial, i.e., $K_{a, b} = 0$ for all but finitely many $a, b \in \mathbb{F}_d^+$. By Theorem 3.22, it follows that the assumption that $K(Z, Z)(I_n) \succeq 0$ for all $Z \in \text{Nilp}_n^d$ for all $n \in \mathbb{N}$ is enough to imply that $K$ is a positive kernel and hence has a formal Kolmogorov decomposition $K(Z, W) = H(Z)H(W)$ for some $H \in \mathcal{L}(X, Y)(\langle z \rangle)$. The result of Theorem 4 from [30] is that the Kolmogorov factor $H$ can be taken also to be a polynomial; in addition there are estimates on the degree of $H$ in terms of the degree of $K$. By the correspondence between formal positive kernels and cp nc kernels given by Theorem 3.20, it is clear that one can also formulate a non-formal version of this result: if $K(Z, W)(P) = K_{a, b} \otimes Z^a PW^{b^T}$ is a nc kernel such that $K(Z, Z)(I_n) \succeq 0$ for $Z \in \text{Nilp}_n^d$ and $n \in \mathbb{N}$, then $K$ has a nc Kolmogorov decomposition $K(Z, W)(P) = H(Z)PH(W)^*$ where $H(Z) = \sum_{a \in \mathbb{F}_d^+} H_a \otimes Z^a$ is also a polynomial ($H_a = 0$ for all but finitely many words $a \in \mathbb{F}_d^+$).

We note some additional corollaries of Theorem 3.20.

Corollary 3.24. Let $K$ be a nc kernel in either case (a) or case (b) as in the converse side of Theorem 3.20, so $K$ itself has a Taylor-Taylor series (a verifiable fact not proved in [31])

$$K(Z, W)(P) = \sum_{a, b \in \mathbb{F}_d^+} Z^a PW^{b^T} \otimes K_{a, b}.$$ 

Then $K$ is a cp nc kernel if and only if $[K_{a, b}]_{a, b \in \mathbb{F}_d^+} \succeq 0$ in the sense that statement (1) in Theorem 3.18 holds.

Corollary 3.25. Let $H$ be a Hilbert space of nc functions with values in $\mathcal{Y}_{nc}$ with bounded point evaluations on a nc set $\Omega \subset \mathbb{C}_d^d$ containing 0. Let $f(z) = \sum_{a \in \mathbb{F}_d^+} Z^a \otimes f_a$ be the Taylor-Taylor series for $f$ centered at 0. Then the Taylor-Taylor-coefficient maps $f \mapsto f_a$ and $f \mapsto \Delta^a f(0, \ldots, 0)$ are all bounded.

We note that it is also possible to prove Corollary 3.25 directly, a good exercise for those having some facility with the techniques developed in [31].

Remark 3.26. The following two questions remain open.
• We suppose that we are given a formal positive kernel $K$ with associated convergence set $\Omega$ and assume that the point $Z \in (\mathbb{C}^d)^n \cong (\mathbb{C}^n)^d$ is such that the iterated series converges for all $W \in \Omega$. Does it then follow that $Z$ itself is in $\Omega$?

• If $Z \in (\mathbb{C}^n)^d$ is such that $K(Z, Z)$ converges, does it follow that $Z \in \Omega$?

In the corresponding commutative situation, the answer to both questions is positive, as can be seen by using the fact that the domain of convergence of the kernel function is a logarithmically convex complete Reinhard domain combined with the symmetry property $K(Z, W)(P^*) = K(W, Z)(P^*)$ of the kernel function.

4. Multipliers between nc reproducing kernel Hilbert spaces

4.1. Characterization of contractive multipliers. Let us suppose that we are given two cp nc kernels $K'$ and $K$, both defined on the Cartesian product $\Omega \times \Omega$ of a nc set $\Omega \subset S_{nc}$ with values in $\mathcal{L}(A,\mathcal{L}(U))_{nc}$ and $\mathcal{L}(\mathcal{A},\mathcal{L}(\mathcal{Y}))_{nc}$ respectively, where $\mathcal{A}$ is a $C^*$-algebra and $U$ and $\mathcal{Y}$ are two auxiliary Hilbert spaces. Suppose next that $S$ is a global function on $\Omega$ with values in $\mathcal{L}(\mathcal{U},\mathcal{Y})_{nc}$. We say that $S$ is a multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$, written as $S \in \mathcal{M}(K', K)$ if the operator $M_S$ given by

$$(M_S f)(W) = S(W) f(W) \quad \text{for} \quad W \in \Omega_m$$

maps $\mathcal{H}(K')$ boundedly into $\mathcal{H}(K)$. Here are a few preliminary observations concerning such operators $M_S$.

• If $S$ is a global function with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{nc}$ and $f$ is a global function with values in $\mathcal{L}(A, \mathcal{U})_{nc}$, then $W \mapsto S(W) f(W)$ is a global function with values in $\mathcal{L}(\mathcal{A}, \mathcal{Y})_{nc}$. Simply compute, for $Z \in \Omega_n$ and $W \in \Omega_m$,

$$(M_S f) \left( \begin{bmatrix} Z & 0 \\ W & 0 \end{bmatrix} \right) = S \left( \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \right) f \left( \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \right)$$

$$= S(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}) f(\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix})$$

$$= S(Z) f(Z) \begin{bmatrix} 0 & 0 \\ 0 & S(W) f(W) \end{bmatrix} = \begin{bmatrix} (M_S f)(Z) & 0 \\ 0 & (M_S f)(W) \end{bmatrix}.$$

• If $S$ is a vector space $\mathcal{V}$ and we make the stronger assumption that both $S$ and $f$ are nc functions, then $M_S f$ is also a nc function. If $Z \in \Omega_n$, $\tilde{Z} \in \Omega_{\tilde{n}}$ and $\alpha \in \mathbb{C}^{\tilde{n} \times n}$ is such that $\alpha Z = \tilde{Z} \alpha$, then

$$\alpha \cdot (M_S f)(Z) = \alpha S(Z) f(Z) = S(\tilde{Z}) \alpha f(Z) = S(\tilde{Z}) f(\tilde{Z}) \alpha$$

$$= (M_S f)(\tilde{Z}) \cdot \alpha.$$

• Let $\sigma'$ and $\sigma$ denote the canonical $\mathcal{A}$-actions on $\mathcal{H}(K')$ and $\mathcal{H}(K)$ respectively. Suppose that $S$ is a global/nc function on $\Omega$ with
values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\text{nc}}$ and $f$ is a global/nc function on $\Omega$. Then, for all $a \in \mathcal{A}$,

$$\sigma'(a)(M_S f) = M_S(\sigma(a)f),$$

i.e., $M_S$ intertwines $\sigma(a)$ with $\sigma'(a)$ for all $a \in \mathcal{A}$. Indeed, compute

$$(\sigma'(a)(M_S f))(W)(v) = (M_S f)(W)(va) = S(W)f(W)(va) = (M_S \sigma(a)f)(W)(v).$$

- Suppose that $S$ is a global/nc function on $\Omega$ with values $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\text{nc}}$ with the property that $M_S f \in \mathcal{H}(K)$ for each $f \in \mathcal{H}(K')$, i.e., $M_S$ is well defined as an operator from $\mathcal{H}(K')$ into $\mathcal{H}(K)$. Then $S \in \mathcal{M}(K', K)$. By the Closed Graph Theorem (see e.g. [43]), it suffices to check that $M_S$ is closed as an operator from $\mathcal{H}(K')$ into $\mathcal{H}(K)$. We therefore assume that $\{f_n\}$ is a sequence in $\mathcal{H}(K')$ with $\lim_{n \to \infty} f_n = f$ in $\mathcal{H}(K')$ and that $M_S f_n$ converges in $\mathcal{H}(K)$ to the global/nc function $g \in \mathcal{H}(K)$. Then, due to the boundedness of the point-evaluation maps we have, for each $W \in \Omega_m$ and $v \in A^m$,

$$g(W)v = \lim_{n \to \infty} (M_S f_n)(W)v = \lim_{n \to \infty} S(W)f_n(W)v = S(W) \left( \lim_{n \to \infty} f_n(W)v \right)$$

$$(4.1)$$

$$= S(W)f(W)v = (M_S f)(W)v$$

from which it follows that $M_S f = g$ in $\mathcal{H}(K)$.

Given $K'$, $K$, and $S$ as above, we shall say that $S$ is a **contraction multiplier** from $\mathcal{H}(K')$ to $\mathcal{H}(K)$, written as $S \in \mathcal{BM}(K', K)$, if $S \in \mathcal{M}(K', K)$ with operator norm of $M_S$ at most 1: $\|M_S\|_{\mathcal{L}^2(\mathcal{H}(K'), \mathcal{H}(K))} \leq 1$. Our main result concerning contraction multipliers is the following.

**Theorem 4.1.** Given global/nc kernels $K'$ and $K$ from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{U}))_{\text{nc}}$ and $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$ respectively and given a global/nc function $S$ from $\Omega$ to $\mathcal{L}(\mathcal{U}, \mathcal{Y})_{\text{nc}}$, the following are equivalent:

1. $S \in \mathcal{BM}(K', K)$.
2. The kernel $K_S$ from $\Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$ given by

$$K_S(Z, W)(P) = K(Z, W)(P) - S(Z) K'(Z, W)(P) S(W)^*$$

is a cp global/nc kernel.

**Proof.** It is easily verified that if $K$ and $K'$ are global/nc kernels and $S$ is a global/nc function, then $K_S$ is a global/nc kernel.

If $M_S \in \mathcal{M}(K', K)$, we compute the action of $M_S$ on a kernel element $K_{W, v, y} \in \mathcal{H}(K)$ (4.1.1) of $\mathcal{H}(K)$ as follows: for $f \in \mathcal{H}(K')$, $W \in \Omega_m$, $v \in A^{1 \times m}$
we have
\[ \langle f, (M_S)^* K_{W,v,y} \rangle_{\mathcal{H}(K)} = \langle M_S f, K_{W,v,y} \rangle_{\mathcal{H}(K)} \]
\[ = \langle (M_S f)(W)(v^*), y \rangle_{\mathcal{Y}^m} \text{ (by (3.2) for } \mathcal{H}(K)) \]
\[ = \langle S(W)f(W^*)(v^*), y \rangle_{\mathcal{Y}^m} \]
\[ = \langle f(W^*)(v^*), S(W^*)y \rangle_{\mathcal{Y}^m} \]
\[ = \langle f, K_{W,v,S(W^*)y} \rangle_{\mathcal{H}(K')} \text{ (by (3.2) for } \mathcal{H}(K')) \]
and hence
\[ (M_S)^*: K_{W,v,y} \mapsto K_{W,v,S(W^*)y}. \] (4.2)

It follows that
\[ \langle K_S(W,W)(v^*v)\gamma, y \rangle_{\mathcal{Y}^m} \]
\[ = \langle K(W,W)(v^*v)\gamma, y \rangle_{\mathcal{Y}^m} - \langle K''(W,W)(v^*v)S(W^*)y, S(W^*)y \rangle_{\mathcal{Y}^m} \]
\[ = \|K_{W,v,y}\|_{\mathcal{H}(K)}^2 - \|K'_{W,v,S(W^*)y}K_{W,v,S(W^*)y}\|_{\mathcal{H}(K')} \]
\[ = \|K_{W,v,y}\|_{\mathcal{H}(K)}^2 - \|M^*_y K_{W,v,y}\|_{\mathcal{H}(K')}^2 \geq 0 \]
and we conclude that $K_S$ is cp. This completes the proof of necessity (1) $\Rightarrow$ (2).

Conversely, assume only that $K_S$ is cp. As we are still assuming that $K'$ and $K$ are global/nc kernels and that $S$ is a global/nc function, we already know that $K_S$ is a global/nc kernel. The proof of the necessity direction motivates us to define an operator $\Gamma$ on kernel elements $K_{W,v,y}$ of $\mathcal{H}(K)$ by
\[ \Gamma: K_{W,v,y} \mapsto K'_{W,v,S(W^*)y} \]
and then extend to finite linear combinations of kernel elements by linearity. The computation, for $W^{(j)} \in \Omega_{m_j}$ and $v_j \in \mathcal{A}^{1 \times m_j}$,
\[ \left\| \sum_{j=1}^N K_{W^{(j)},v_j,y_j} \right\|_{\mathcal{H}(K)}^2 - \left\| \Gamma \left( \sum_{j=1}^N K_{W^{(j)},v_j,y_j} \right) \right\|_{\mathcal{H}(K')}^2 \]
\[ = \sum_{i,j=1}^N \langle K(W^{(i)},W^{(j)})(v^*_i v_j)\gamma, y_i \rangle_{\mathcal{Y}^{m_i}} \]
\[ - \sum_{i,j=1}^N \langle S((W^{(i)}))K'(W^{(i)},W^{(j)})(v^*_i v_j)S(W^{(j)})^*\gamma, y_i \rangle_{\mathcal{Y}^{m_i}} \]
\[ = \sum_{i,j=1}^N \langle K_S(W^{(i)},W^{(j)})(v^*_i v_j)\gamma, y_i \rangle_{\mathcal{Y}^{m_i}} \geq 0 \text{ (by property (2.13)).} \]

We conclude that $\Gamma$ is well-defined on the span of the kernel elements in $\mathcal{H}(K)$ and extends by continuity to a well-defined contraction operator from all of $\mathcal{H}(K)$ into $\mathcal{H}(K')$. Furthermore, by reading the computation (4.2) backwards, we see that $\Gamma^*: \mathcal{H}(K') \to \mathcal{H}(K)$ is given by
\[ M_S : f(Z) \mapsto S(Z)f(Z). \] Hence \( M_S = \Gamma^* \) is a contraction from \( \mathcal{H}(K') \) to \( \mathcal{H}(K) \), i.e., \( S \in \overline{BM}(K', K) \).

**Remark 4.2.** We note that the formula for \( M_S^z \) on kernel elements gives us a second way to see that \( M_S \) intertwines \( \sigma(a) \) with \( \sigma'(a) \) for each \( a \in \mathcal{A} \), once we recall the action of \( \sigma(a) \) and \( \sigma'(a) \) on kernel elements given by \( (3.4) \):

\[
\sigma'(a)(M_S)^*K_{W,v,y} = (\sigma'(a)K_{W,v,S(W)^y})^y = (M_S)^*K_{W,av,S(W)^y} = (M_S)^*\sigma(a)K_{W,v,y}.
\]

### 4.2. The global/nc reproducing kernel Hilbert spaces associated with a contractive multiplier \( S \).

In the classical setting where \( S \) is a contractive multiplier between the Hardy spaces \( H^2 \otimes \mathcal{U} \) and \( H^2 \otimes \mathcal{V} \) over the unit disk (equal to reproducing kernel Hilbert spaces with Szegő kernel \( k(z, w) = \frac{1}{1 - z\overline{w}} \) tensored with either \( I_\mathcal{U} \) or \( I_\mathcal{V} \)), the associated kernel \( K_S(z, w) = \frac{I_\mathcal{V} - S(z)S(w)^*}{1 - z\overline{w}} \) has become known as a de Branges-Rovnyak kernel and the associated Hilbert space \( \mathcal{H}(K_S) \) as a de Branges-Rovnyak space, due to the fundamental work of de Branges-Rovnyak [20] [21] (see also [7] [12] [37] [38] [39] [44]). Much of the theory associated with these spaces goes through in our more general global/nc setting.

As a starting point for the discussion, we review the general theory of minimal decompositions for Hilbert spaces \( \mathcal{H}' \) which are contractively included in another Hilbert space \( \mathcal{H} \), summarized as follows. The following general formulation comes from [7].

**Proposition 4.3.** Let \( \mathcal{K} \) be a Hilbert space which is contractively included in a larger Hilbert space \( \mathcal{H} \) but with its own possibly distinct norm:

\[ k \in \mathcal{K} \Rightarrow k \in \mathcal{H} \text{ and then } \|k\|_\mathcal{H} \leq \|k\|_\mathcal{K}. \]

Define another Hilbert space \( \mathcal{K}^{\perp_{\text{dBR}}} \) (the **Brangesian complement to \( \mathcal{K} \)**) by

\[
\mathcal{K}^{\perp_{\text{dBR}}} = \{ h \in \mathcal{H} : \|h\|^2_{\mathcal{K}^{\perp_{\text{dBR}}}} := \sup\{\|h + k\|^2_{\mathcal{H}} - \|k\|^2_{\mathcal{K}} : k \in \mathcal{K}\} < \infty \}
\]

Then \( \mathcal{K} \) and \( \mathcal{K}^{\perp_{\text{dBR}}} \) are complementary subspaces of \( \mathcal{H} \) in the sense that each \( h \in \mathcal{H} \) has a decomposition \( h = k + k' \) with \( k \in \mathcal{K} \) and \( k' \in \mathcal{K}^{\perp_{\text{dBR}}} \). Then also the norm of any \( h \in \mathcal{H} \) is given by

\[
\|h\|_{\mathcal{H}}^2 = \inf\{\|k\|_{\mathcal{K}}^2 + \|k'\|^2_{\mathcal{K}^{\perp_{\text{dBR}}}} : k \in \mathcal{K}, k' \in \mathcal{K}^{\perp_{\text{dBR}}} \text{ such that } h = k + k'\}.
\]

(4.3)

Moreover:

1. There is a unique choice of vectors \((k, k') \in \mathcal{K} \times \mathcal{K}^{\perp_{\text{dBR}}} \) for which the infimum in (4.3) is attained, namely: \( k = \iota^*h \) and \( k' = (I_\mathcal{H} - \iota^*)h \) where \( \iota : k \mapsto k \) is the inclusion map considered as a contraction operator from \( \mathcal{K} \) into \( \mathcal{H} \).

2. The space \( \mathcal{K} \) can be characterized as the Brangesian complement \( \mathcal{K}'' := (\mathcal{K}^{\perp_{\text{dBR}}})^{\perp_{\text{dBR}}} \) of \( \mathcal{K}' := \mathcal{K}^{\perp_{\text{dBR}}} \):

\[
\mathcal{K} = \{ h \in \mathcal{H} : \|h\|^2_{\mathcal{K}''} := \sup\{\|h + k\|^2_{\mathcal{H}} - \|k'\|^2_{\mathcal{K}^{\perp_{\text{dBR}}}} : k' \in \mathcal{K}^{\perp_{\text{dBR}}}\} < \infty \}
\]
and then \( \|k\|^2_K = \|k\|^2_{K'} \).

A particular application of Proposition 4.3 is to the case where \( A \) is a contraction operator from a Hilbert space \( K \) to a Hilbert space \( H \) and we define two linear submanifolds of \( H \) by

\[
\mathcal{H}_A = \text{Ran} \left( I - AA^* \right)^{1/2}, \quad \mathcal{M}_A = \text{Ran} A \tag{4.4}
\]

with respective pull-back norms

\[
\|(I - AA^*)^{1/2} g\|_{\mathcal{H}_A} = \|Qg\|_{\mathcal{H}}, \quad \|Ah\|_{\mathcal{M}_A} = \|Q'h\|_{K} \tag{4.5}
\]

where \( Q: \mathcal{H} \to (\text{Ker} (I - AA^*))^{1/2} \) and \( Q': K \to (\text{Ker} A)^{1} \) are the orthogonal projections. Then it is easily verified that both \( \mathcal{H}_A \) and \( \mathcal{M}_A \) are themselves Hilbert spaces (in particular, complete in their respective norms) and are each contractively included in \( H \) with respective contractive adjoint inclusion maps \((\iota_{\mathcal{H}_A})^* : \mathcal{H} \to \mathcal{H}_A \) and \((\iota_{\mathcal{M}_A})^* : \mathcal{H} \to \mathcal{M}_A \) given by

\[
(\iota_{\mathcal{H}_A})^* : h \mapsto (I - AA^*)h, \quad (\iota_{\mathcal{M}_A})^* : h \mapsto AA^*h.
\]

One can then verify that \( \mathcal{H}_A \) and \( \mathcal{M}_A \) are Brangesian complements of each other.

Suppose now that \( K' \) and \( K \) are cp global/nc kernels from \( \Omega \times \Omega \) to respectively \( \mathcal{L}(A_{\text{nc}}, \mathcal{L}(U)_{\text{nc}}) \) and \( \mathcal{L}(A_{\text{nc}}, \mathcal{L}(Y)_{\text{nc}}) \) respectively, and that \( S \in \mathcal{B}M(K', K) \) is a contractive multiplier. Our interest is to apply the discussion of the preceding paragraph to the case where \( A = M_S: \mathcal{H}(K') \to \mathcal{H}(K) \). The ensuing result gives an explicit geometric characterization of the global/nc version of the de Branges-Rovnyak space \( \mathcal{H}(K) \).

**Theorem 4.4.** Suppose that \( K', K \) are two cp global/nc kernels and \( S \in \mathcal{B}M(K', K) \) is a contractive multiplier as above. Then the global/nc reproducing kernel Hilbert space \( \mathcal{H}(K_S) \) associated with the cp global/nc kernel \( K_S \) (4.1) is isometrically equal to the pull-back space \( \mathcal{H}_{M_S} \) defined by (4.4), (4.5) with \( A = M_S \), specifically: \( \mathcal{H}(K_S) = \text{Ran} \left( I - M_SM_S^* \right)^{1/2} \) with

\[
\|(I - M_SM_S^*)^{1/2} h\|_{\mathcal{H}(K_S)} = \|Qh\|_{\mathcal{H}(K)}
\]

where \( Q: \mathcal{H}(K) \to (\text{Ker} (I - M_SM_S^*))^{1/2} \) is the orthogonal projection.

Equivalently, \( \mathcal{H}(K_S) \) is the Brangesian complement of the Hilbert space \( \mathcal{M}_{M_S} \) contractively included in \( \mathcal{H}(K) \) defined by \( \mathcal{M}_S = \text{Ran} M_S \) with pull-back norm given by

\[
\|M_Sg\|_{\mathcal{M}_{M_S}} = \|Q'g\|_{\mathcal{H}(K')} \]

where \( Q': \mathcal{H}(K') \to (\text{Ker} M_S)^{1} \) is the orthogonal projection. The space \( \mathcal{M}_S \) is itself a global/nc reproducing kernel Hilbert space with reproducing kernel \( K_S^0 \) given by

\[
K_S^0(Z,W)(P) := S(Z)K'(Z,W)(P)S(W)^*.
\]

**Proof.** All these results follow from the general discussion preceding the theorem once we verify that \( \mathcal{H}(K_S) \) is isometrically equal to \( \mathcal{H}_{M_S} \) and that
$\mathcal{H}(K_{S}^{2})$ is isometrically equal to $\mathcal{M}_{M_{S}}$. We do only the first case as the second is similar.

The starting point is the following consequence of the formula (4.2) for the action of $M_{S}$ on a kernel element in $\mathcal{H}(K)$: for $W \in \Omega_{m}$, $v \in \mathcal{A}_{1 \times m}$, and $y \in \mathcal{Y}^{m}$ we have

$$(K_{S})_{W,v,y} = (I - M_{S}M_{S}^{*})K_{W,v,y}.$$ 

Hence we can compute the norm-squared of a finite linear combination of kernel elements in $\mathcal{H}(K_{S})$ as follows. If $f = \sum_{j=1}^{N}(K_{S})_{W^{(j)},v_{j},y_{j}}$ for some $W^{(j)} \in \Omega_{m_{j}}$, $v_{j} \in \mathcal{A}_{1 \times m_{j}}$ and $y_{j} \in \mathcal{Y}^{m_{j}}$, then $f$ has the form $f = (I - M_{S}M_{S}^{*})g$ where $g = \sum_{j=1}^{N}K_{W^{(j)},v_{j},y_{j}}$ is the corresponding finite linear combination of kernel elements from $\mathcal{H}(K)$. Furthermore,

$$\|f\|^{2}_{\mathcal{H}(K_{S})} = \|(I - M_{S}M_{S}^{*})g\|^{2}_{\mathcal{H}(K_{S})}$$

$$= \|\sum_{j=1}^{N}(K_{S})_{W^{(j)},v_{j},y_{j}}\|^{2}_{\mathcal{H}(K_{S})} = \sum_{i,j=1}^{N}\langle K_{S}(W^{(i)},W^{(j)})(v_{i}^{*}v_{j}y_{j},y_{i})\rangle_{\mathcal{Y}^{m_{i}}}$$

$$= \sum_{i,j=1}^{N}\langle K(W^{(i)},W^{(j)})(v_{i}^{*}v_{j}y_{j},y_{i})\rangle_{\mathcal{Y}^{m_{i}}}$$

$$- \sum_{i,j=1}^{N}\langle K(W^{(i)},W^{(j)})(v_{i}^{*}v_{j}S(W^{(j)})^{*}y_{j},S(W^{(i)})^{*}y_{i})\rangle_{\mathcal{U}^{m_{i}}}$$

$$= \sum_{i,j=1}^{N}\left(\langle K_{W^{(i)},v_{j},y_{j}},K_{W^{(i)},v_{i},y_{i}}\rangle_{\mathcal{H}(K)} - \langle M_{S}^{*}K_{W^{(i)},v_{j},y_{j}},M_{S}^{*}K_{W^{(i)},v_{i},y_{i}}\rangle_{\mathcal{H}(K')}\right)$$

$$= \left(\langle (I - M_{S}M_{S}^{*})\left(\sum_{j=1}^{N}K_{W^{(j)},v_{j},y_{j}}\right),\sum_{i=1}^{N}K_{W^{(i)},v_{i},y_{i}}\right)_{\mathcal{H}(K)} = \|f\|^{2}_{\mathcal{H}_{M_{S}}}.$$ 

and we conclude that a dense subset of $\mathcal{H}(K_{S})$ is isometrically equal to a dense subset of $\mathcal{H}_{M_{S}}$. By taking limits and using the boundedness of the respective point-evaluations, we get that $\mathcal{H}(K_{S})$ is equal to $\mathcal{H}_{M_{S}}$ isometrically as claimed.

As a special case, consider the situation where $K'$ and $K$ are two cp global/nc kernels from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}_{nc},\mathcal{L}(\mathcal{Y})_{nc})$ (i.e., in this case $\mathcal{U} = \mathcal{Y}$) and suppose that $\mathcal{H}(K')$ is contractively included in $\mathcal{H}(K)$. This is the precise situation where $S = I$ is in the contractive multiplier class $\mathcal{BM}(K',K)$, where we use the notation $I$ for the global/nc function $S(Z) = I_{\mathcal{Y}^{m}}$ if $Z \in \Omega_{n}$. We thus arrive at the following corollary concerning contractive inclusions of global/nc reproducing kernel Hilbert spaces.

**Corollary 4.5.** Suppose that $\mathcal{H}(K')$ and $\mathcal{H}(K)$ are two global/nc reproducing kernel Hilbert spaces of functions with values in $\mathcal{L}(\mathcal{A},\mathcal{Y})_{nc}$. Then $\mathcal{H}(K')$
is contained contractively in $\mathcal{H}(K)$ if and only the global/nc kernel $K''$ given by

$$K''(Z,W)(P) = K(Z,W)(P) - K'(Z,W)(P)$$

is cp. In this case the Brangesian complement $\mathcal{H}(K')_{dBR} = \mathcal{H}(K) \ominus_{dBR} \mathcal{H}(K')$ is also a global/nc reproducing kernel Hilbert space with associated cp kernel equal to $\mathcal{H}(K - K')$:

$$\mathcal{H}(K) \ominus_{dBR} \mathcal{H}(K') = \mathcal{H}(K - K').$$

**Remark 4.6.** Section 3 of the [14] develops many of the results of this section for the setting of formal rather than concrete nc RKHSs. In particular [14, Theorem 3.15] is the formal analogue of Theorem [4.1] for the special case where both $K$ and $K'$ are taken to be the formal nc Szegő kernel and [14, Theorem 3.4] is roughly a formal analogue of Proposition [4.3]. The formal setting of Theorem [4.4] for the special case where $K'$ and $K$ are formal Szegő kernels is worked out in [8, Proposition 4.1]. Additional results for the formal setting can be obtained by using the correspondence between functional and formal RKHSs explained in Subsection [3.5] to transfer results here for the functional setting to the formal setting.

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