A NOTE ON $l^2$ NORMS OF WEIGHTED MEAN MATRICES

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Abstract. We give a proof of Cartlidge’s result on the $l^p$ operator norms of weighted mean matrices for $p = 2$ on interpreting the norms as eigenvalues of certain matrices.

1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p} + \frac{1}{q} = 1$. Let $l^p$ be the Banach space of all complex sequences $a = (a_n)_{n \geq 1}$ with norm

$$||a|| := \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty.$$ 

The celebrated Hardy’s inequality ([7, Theorem 326]) asserts that for $p > 1$,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^p \leq \left( \frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} |a_k|^p.$$ 

Hardy’s inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |c_{j,k} a_k|^{p} \leq \sum_{k=1}^{\infty} |a_k|^p,$$

in which $C = (c_{j,k})$ and the parameter $p$ are assumed fixed ($p > 1$), and the estimate is to hold for all complex sequences $a$. The $l^p$ operator norm of $C$ is then defined as the $p$-th root of the smallest value of the constant $U$:

$$||C||_{p,p} = U^{\frac{1}{p}}.$$

Hardy’s inequality thus asserts that the Cesàro matrix operator $C$, given by $c_{j,k} = 1/j, k \leq j$ and 0 otherwise, is bounded on $l^p$ and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix $A$ is a summability matrix if its entries satisfy: $a_{j,k} \geq 0, a_{j,k} = 0$ for $k > j$ and $\sum_{k=1}^{j} a_{j,k} = 1$. We say a summability matrix $A$ is a weighted mean matrix if its entries satisfy:

$$a_{j,k} = \frac{\lambda_k}{\Lambda_j}, 1 \leq k \leq j; \Lambda_j = \sum_{i=1}^{j} \lambda_i, \lambda_i \geq 0, \lambda_1 > 0.$$

Hardy’s inequality (1.1) now motivates one to determine the $l^p$ operator norm of an arbitrary summability matrix $A$. In an unpublished dissertation [4], Cartlidge studied weighted mean matrices as operators on $l^p$ and obtained the following result (see also [2, p. 416, Theorem C]).

Theorem 1.1. Let $1 < p < \infty$ be fixed. Let $A$ be a weighted mean matrix given by (1.2). If

$$L = \sup_{n} \left( \frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_{n}}{\lambda_{n}} \right) < p,$$

then $||A||_{p,p} \leq p/(p - L)$.
We note here there are several published proofs of Cartlidge’s result. Borwein [3] proved a far more general result than Theorem [1.1] on the $l^p$ norms of generalized Hausdorff matrices. Rhoades [11] Theorem 1] obtained a slightly general result than Theorem 1.1, using a modification of the proof of Cartlidge. Recently, the author [6] also gave a simple proof of Theorem 1.1.

It is our goal in this note to give another proof of Theorem 1.1 for the case $p = 2$, following an approach of Wang and Yuan in [12], which interprets the left-hand side of (1.1) when $p = 2$ as a quadratic form so that Hardy’s inequality follows from estimations of the eigenvalues of the corresponding matrix associated to the quadratic form. We will show in the next section that the same idea also works for the case of weighted mean matrices.

2. Proof of Theorem 1.1 for $p = 2$

We may assume $a_n$ being real without loss of generality and it suffices to prove the theorem for any finite summation from $n = 1$ to $N$ with $N \geq 1$. We also note that it follows from our assumption on $L$ that $\lambda_n > 0$. Now consider

$$\sum_{n=1}^{N} \left( \sum_{i=1}^{n} \frac{\lambda_i a_i}{\Lambda_n} \right)^2 = \sum_{n=1}^{N} \left( \sum_{i,j=1}^{n} \frac{\lambda_i \lambda_j a_i a_j}{\Lambda_n^2} \right) = \sum_{n=1}^{N} \alpha_{i,j} a_i a_j, \quad \alpha_{i,j} = \sum_{k=\max(i,j)}^{N} \frac{\lambda_i \lambda_j}{\Lambda_k^2}.$$  

We view the above as a quadratic form and define the associated matrix $A$ to be

$$A = (\alpha_{i,j})_{1 \leq i,j \leq N}.$$  

We note that the matrix $A$ here is certainly positive definite, being equal to $B^t B$ with $B$ a lower-triangular matrix,

$$B = \left( b_{i,j} \right)_{1 \leq i,j \leq N}, \quad b_{i,j} = \lambda_j / \Lambda_i, \quad 1 \leq j \leq i; \quad b_{i,j} = 0, \quad j > i.$$  

It is easy to check that the entries of $B^{-1}$ are given by

$$(B^{-1})_{i,i} = \frac{\Lambda_i}{\Lambda_i}, \quad (B^{-1})_{i+1,i} = -\frac{\Lambda_i}{\Lambda_{i+1}}, \quad (B^{-1})_{i,j} = 0 \quad \text{otherwise}.$$  

In order to establish our assertion, it suffices to show that the maximum eigenvalue of $A$ is less than $4/(2 - L)^2$ or the minimum eigenvalue of its inverse $A^{-1}$ is greater than $(2 - L)^2/4$ which is equivalent to proving that the matrix $A^{-1} - \lambda I_N$ is positive definite, where $\lambda = (2 - L)^2/4$ and $I_N$ is the $N \times N$ identity matrix. Using the expression $A^{-1} = B^{-1} (B^{-1})^t$, we see that this is equivalent to showing that for any integer $N \geq 1$ and any real sequence $a = (a_n)_{1 \leq n \leq N}$,

$$\sum_{n=1}^{N-1} \left( \frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1} \right)^2 + \frac{\Lambda_N^2}{\lambda_N^2} a_N^2 \geq \frac{(2 - L)^2}{4} \sum_{n=1}^{N} a_n^2.$$  

(2.1)

For any integer $n \geq 1$ and fixed constants $\alpha, \beta, a_{n+1}, \mu_n$ (here $\alpha, \beta$ may depend on $n$), we consider the following function:

$$f(a_n) := (\alpha a_n - \beta a_{n+1})^2 - \mu_n a_n^2.$$  

When $\mu_n > \alpha^2$, it is easy to see that

$$f(a_n) \leq f \left( \frac{\alpha \beta a_n + \alpha a_{n+1}}{\alpha^2 - \mu_n} \right) = \frac{\beta^2 \mu_n a_n^2}{\alpha^2 - \mu_n},$$  

(2.2)

with the above inequality reversed when $\mu_n < \alpha^2$.

On taking $\alpha = \Lambda_n / \lambda_n, \beta = \Lambda_n / \lambda_{n+1}$ here, we obtain that for any $0 < \mu_n < \Lambda_n^2 / \lambda_n^2$,

$$\left( \frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1} \right)^2 - \mu_n a_n^2 \geq - \frac{\Lambda_n^2 / \lambda_{n+1}}{\lambda_n^2 / \lambda_{n+1} - \mu_n} \mu_n a_n^2.$$  

(2.3)
Summing the above inequality for \( n = 1, \ldots, N - 1 \) yields:

\[
(2.3) \quad \sum_{n=1}^{N-1} \left( \frac{\Lambda_2}{\Lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1} \right)^2 + \frac{\Lambda_2^2}{\lambda_N^2} a_N^2 \\
\geq \mu_1 a_1^2 + \sum_{n=1}^{N-2} \left( \mu_{n+1} - \frac{\Lambda_2^2/\lambda_{n+1}}{\Lambda_n^2/\lambda_n^2 - \mu_n} \right) a_{n+1}^2 + \left( \frac{\Lambda_2^2}{\lambda_N^2} - \frac{\Lambda_{N-1}^2/\lambda_{N-1}^2}{\Lambda_{N-1}^2/\lambda_{N-1}^2 - \mu_{N-1}} \right) a_N^2.
\]

We now want to find a number \( \mu_n \) satisfying \( 0 < \mu_n < \Lambda_n^2/\lambda_n^2 \) for any integer \( n \geq 1 \), such that the following inequality holds for \( 1 \leq n \leq N - 1 \):

\[
(2.4) \quad \mu_{n+1} - \frac{\Lambda_2^2/\lambda_{n+1}}{\Lambda_n^2/\lambda_n^2 - \mu_n} \mu_n \geq \frac{(2 - L)^2}{4}.
\]

For this purpose, we set

\[
\mu_n = (k + c) \frac{\Lambda_n}{\lambda_n} - c, \quad k = (2 - L)^2/4,
\]

with \( c \) a constant to be specified later. For the so chosen \( \mu_n \)'s, inequality (2.4) can be seen to be equivalent to

\[
(2.5) \quad (k + c)x^2 + (k + c)x - ((k + c)x - c)y - (k + c)((k + c)x - c) - c \geq 0.
\]

where \( x = \Lambda_n/\lambda_n \) and \( y = \Lambda_{n+1}/\lambda_{n+1} \). Note that by the assumption (1.3), it follows that \( y \leq x + L \) and the case \( n = 1 \) of (1.3) implies \( L > 0 \). Note also that \( x \geq 1 \) so that the left-hand side expression of inequality (2.5) is a decreasing function of \( y \) for fixed \( x \). Hence we can replace \( y \) there and conclude that (2.5) follows from the following inequality:

\[
(k + c)x^2 + (k + c)x - ((k + c)x - c)(L + x) - (k + c)((k + c)x - c) - c \geq 0.
\]

Equivalently, we can recast the above inequality as:

\[
(2.6) \quad \left(k + 2c - L(k + c) - (k + c)^2 \right) x + cL + c(k + c) - c \geq 0.
\]

In order for the above inequality to hold for all \( x \geq 1 \), we need to choose \( c \) so that

\[
(2.7) \quad k + 2c - L(k + c) - (k + c)^2 \geq 0.
\]

We now choose the value of \( c \) so that the left-hand side expression above when considered as a function of \( c \) is maximized. It is easy to see that in this case, \( c + k = (2 - L)/2 \) so that

\[
c = \frac{L(2 - L)}{4}.
\]

It is then easy to check that for the so chosen \( c \), the left-hand side expression of inequality (2.7) is reduced to 0 and the left-hand side expression of inequality (2.6) becomes \( cL/2 > 0 \). It follows that inequality (2.4) holds for \( 1 \leq n \leq N - 1 \). Note also that in our case \( \mu_1 = k \) and for \( 1 \leq n \leq N \),

\[
0 < \mu_n = (k + c) \frac{\Lambda_n}{\lambda_n} - c < (k + c) \frac{\Lambda_n}{\lambda_n} = \left( \frac{2 - L}{2} \right) \frac{\Lambda_n}{\lambda_n} < \frac{\Lambda_n}{\lambda_n} \leq \frac{\Lambda_2^2}{\lambda_N^2},
\]

so that our assumption \( 0 < \mu_n < \Lambda_n^2/\lambda_n^2 \) is satisfied. In particular, we have

\[
\frac{\Lambda_2^2}{\lambda_N^2} - \frac{\Lambda_{N-1}^2/\lambda_{N-1}^2}{\Lambda_n^2/\lambda_n^2 - \mu_{N-1}} \mu_{N-1} \geq \mu_N - \frac{\Lambda_2^2/\lambda_{N-1}^2}{\Lambda_{N-1}^2/\lambda_{N-1}^2 - \mu_{N-1}} \mu_{N-1} \geq \frac{(2 - L)^2}{4}.
\]

From this we see that inequality (2.1) follows from inequality (2.3) and this completes the proof.
3. Further Discussions

We point out here inequality (3.1) can be regarded as an analogue to the following discrete inequality of Wirtinger’s type studied by Fan, Taussky and Todd [11, Theorem 8]:

\[
a_1^2 + \sum_{n=1}^{N-1} (a_n - a_{n+1})^2 + a_N^2 \geq 2 \left( 1 - \cos \frac{\pi}{N+1} \right) \sum_{n=1}^{N} a_n^2.
\]

Converses of the above inequality was found by Milovanović and Milovanović [9]:

\[
a_1^2 + \sum_{n=1}^{N-1} (a_n - a_{n+1})^2 + a_N^2 \leq 2 \left( 1 + \cos \frac{\pi}{N+1} \right) \sum_{n=1}^{N} a_n^2.
\]

Simple proofs of inequalities (3.1) and (3.2) were given by Redheffer [10] and Alzer [1], respectively. Our proof of Theorem 1.1 for \( p = 2 \) in the previous section is motivated by the methods used in [10] and [1].

To end this paper, we note the paper [8] contains several generalizations of inequalities of (3.1) and (3.2), one of them can be stated as:

**Theorem 3.1.** For any real sequence \( a = (a_n)_{1 \leq n \leq N} \), and two positive real numbers \( a, b \),

\[
(a^2 + b^2 - 2ab \cos \frac{\pi}{N+1}) \sum_{n=1}^{N} a_n^2 \leq b^2 a_1^2 + \sum_{n=1}^{N-1} (a_n - ba_{n+1})^2 + a_N^2 \leq \left( a^2 + b^2 + 2ab \cos \frac{\pi}{N+1} \right) \sum_{n=1}^{N} a_n^2.
\]

The proof given in [8] to the above theorem is to regard

\[ b^2 a_1^2 + \sum_{n=1}^{N-1} (a_n - ba_{n+1})^2 + a_N^2 \]

as a quadratic form with the associated matrix \( A \) being symmetric tridiagonal with its entries given by

\[ (A)_{i,i} = a^2 + b^2, \quad (A)_{i,i+1} = (A)_{i+1,i} = -ab, \quad (A)_{i,j} = 0 \text{ otherwise.} \]

The eigenvalues of \( A \) are shown in [8] to be \( a^2 + b^2 + 2ab \cos \left( \frac{\pi k}{N+1} \right), 1 \leq k \leq N \), from which Theorem 3.1 follows easily.

We note here one can also give a proof of Theorem 3.1 following the methods in [10] and [1] as one checks readily that the right-hand side inequality of (3.3) follows on taking \( \alpha = a, \beta = b, \mu_n = a^2 + ab \sin(n+1)t/\sin(nt), t = \pi/(N+1) \) in inequality (2.2) and summing for \( n = 1, \ldots, N-1 \). Similarly, the left-hand side inequality of (3.3) follows from on taking \( \alpha = a, \beta = b, \mu_n = a^2 - ab \sin(n+1)t/\sin(nt), t = \pi/(N+1) \) in inequality (2.2) (with inequality reversed there).

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