HOMOGENEOUS SASAKIAN AND 3-SASAKIAN STRUCTURES FROM THE SPINORIAL VIEWPOINT

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ABSTRACT. We give a spinorial construction of Sasakian and 3-Sasakian structures in arbitrary dimension, generalizing previously known results in dimensions 5 and 7 [FK88, FK89, FK90]. Moreover, we obtain a complete description of the space of invariant spinors on a homogeneous 3-Sasakian space, and show that it is spanned by the Clifford products of invariant differential forms with a certain invariant Killing spinor. Finally, we give a basis for the space of Riemannian Killing spinors on a homogeneous 3-Sasakian space, and determine which of these induce the homogeneous 3-Sasakian structure.

Keywords: Homogeneous spaces; invariant spinors; invariant differential forms; Killing spinors; Sasakian manifolds; 3-Sasakian manifolds.

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1. Introduction

The existence of special spinor fields on a Riemannian manifold efficiently encodes a great deal of geometric information, and is relevant, for example, to the study of immersion theory, Einstein metrics, holonomy theory, and $G$-structures, among others [Fri98, Wan89, Bar93, Fri80, ACFH15]. The most extensively investigated special spinors are the real Riemannian Killing spinors, i.e. those satisfying the differential equation $\nabla_X\psi = \pm \frac{1}{2}X \cdot \psi$ for any vector field $X$, whose existence places strong constraints
on the geometry of the underlying manifold. Indeed, it was shown by Friedrich in [Fri80] that Killing spinors are eigenspinors realizing the lower bound for eigenvalues of the Dirac operator, and that any manifold carrying such spinors is Einstein with scalar curvature \( R = n(n - 1) \). The classification of complete simply connected manifolds with real Killing spinors was subsequently accomplished by Bär in [Bä93], where it is was shown that they correspond to parallel spinors (or equivalently, to a reduction of holonomy) on the metric cone. Comparing with Wang’s classification of geometries carrying parallel spinors [Wan89], this shows that Killing spinors are in fact somewhat rare and, beyond round spheres and isolated cases in dimensions 6 and 7, are only carried by (3-)Sasakian manifolds, i.e. Riemannian manifolds \( (M^{2n+1}, g) \) such that the holonomy of the metric cone \( (M \times \mathbb{R}, g + r^2 \, dr) \) reduces to a subgroup of \( SU(n) \) or \( Sp(n/2) \); for this reason (3-)Sasakian manifolds play an outsized role in spin geometry. Importantly, 3-Sasakian manifolds may be considered in full generality from the perspective of spin geometry due to Kuo’s result that they admit a reduction of the structure group of the tangent bundle to the simply-connected subgroup \( \{1\} \times Sp(n - 1) \) of \( SO(4n - 4) \) and hence are necessarily spin [Kuo70]. In the present paper we shed new light on the correspondence between (3-)Sasakian structures and Killing spinors by giving an explicit construction of the former in terms of the latter:

**Theorem.** Let \( (M,g) \) be a Riemannian spin manifold carrying a pair \( \psi_1, \psi_2 \) of Killing spinors (resp. four Killing spinors \( \psi_1, \psi_2, \psi_3, \psi_4 \)). If the vector field \( \xi_{\psi_1, \psi_2} \), defined by the equation

\[
g(\xi_{\psi_1, \psi_2}, X) := \Re<\psi_1, X \cdot \psi_2> \quad \forall X \in TM
\]

is unit length (resp. if the vector fields \( \xi_{\psi_1, \psi_2}, \xi_{\psi_1, \psi_4} \) are orthonormal), then this vector field determines a Sasakian structure on \( M \) (resp. these two vector fields determine a 3-Sasakian structure on \( M \)). Conversely, any Einstein-Sasakian (resp. 3-Sasakian) structure on a connected simply-connected manifold \( M \) arises by this construction.

Using a new technique valid in all dimensions, this theorem generalizes previous results of Friedrich and Kath in dimensions 5 and 7 [FK88] [FK89] [FK90], which were proved by employing certain special spinorial properties occurring in these dimensions only.

In the latter sections of this paper we concern ourselves mainly with 3-Sasakian manifolds, which initially appeared over fifty years ago in [Kuo70] [Udr69], among others. Notable milestones in the subject include Konishi’s construction of 3-Sasakian structures on certain principal \( SO(3) \)-bundles over quaternionic Kähler manifolds of positive scalar curvature [Kon75], and the result of Boyer et al. that the leaf space of the foliation induced by the Reeb vector fields is a quaternionic Kähler orbifold [BGM94]. Indeed, these results show that 3-Sasakian spaces lie between quaternionic Kähler geometries below and hyperKähler geometries above, emphasizing the fact that they provide natural examples of interesting odd dimensional quaternionic geometries.

Previous work on 3-Sasakian manifolds from the spinorial perspective includes e.g. [FK90] [AF10], which give a thorough and elegant accounting of the situation in dimension 7, however until now very little is known about these spaces in higher dimensions. In Sections 4 and 5 of this paper we provide, in arbitrary dimension, a detailed spinorial picture of the homogeneous 3-Sasakian spaces, which were classified by Boyer, Galicki, and Mann in [BGM94]. Using Kuo’s \( Sp(n - 1) \)-reduction together with the description of these spaces by Draper et al. in terms of 3-Sasakian data [DOP19], we apply our new invariant theoretic approach developed in Section 4 of [AHL22] to give a classification of the invariant spinors:

**Theorem.** For a homogeneous 3-Sasakian manifold \( (M^{4n-1} = G/H, g, \xi, \eta, \varphi_1) \), the space of \( G \)-invariant spinors forms an algebra under the wedge product, and is isomorphic to the algebra of invariant \( \varphi_1 \)-anti-holomorphic differential forms,

\[
\Sigma_{inv} \cong \Lambda^V_{inv}(T^*M).
\]

Furthermore, this algebra is generated by the forms \( y_1 := \frac{1}{\sqrt{2}}(\xi_2 + i\xi_3) \) and \( \omega := -\frac{1}{2}(\Phi_2|_H + i\Phi_3|_H) \), where \( \Phi_i := g(-, \varphi_i(-)) \).
Finally, we use this to give a complete description of the Killing spinors in the homogeneous case. Indeed, a partial construction of the Killing spinors on 3-Sasakian manifolds is given in [FK90] as sections of certain rank two subbundles of the spinor bundle, however this description can produce at most six linearly independent Killing spinors and is thus incomplete for spaces of dimension > 19. In Section 6 we resolve this issue in the homogeneous case, obtaining the following full classification:

**Theorem.** Any Killing spinor on a homogeneous 3-Sasakian manifold $(M^{4n-1} = G/H, g, \xi, \eta, \varphi_i)$ is $G$-invariant. If $n \geq 2$ then the space of Killing spinors has a basis given by

$$
\psi_k := \omega^{k+1} - i(k+1) y_1 \wedge \omega^k, \\
-1 \leq k \leq n - 1,
$$

where we use the conventions $\omega^{-1} = 0$ and $\omega^0 = 1$. If $n = 1$ then the space of Killing spinors has a basis given by $1, y_1$.

As a corollary, we are subsequently able to deduce the precise Killing spinors which recover the homogeneous 3-Sasakian structure via the preceding construction.

The paper is organized as follows: Section 2 reviews the necessary definitions and background that will be used throughout the paper. Section 3 gives a spinorial construction of Sasakian and 3-Sasakian structures and shows that this construction recovers all such structures. Section 4 examines the behaviour of this construction for homogeneous 3-Sasakian spaces. Section 5 gives a classification of the invariant spinors on homogeneous 3-Sasakian spaces and discusses the relationship with invariant differential forms. Section 6 gives explicit bases for the space of Riemannian Killing spinors carried by these spaces, and determines which of these spinors induce the homogeneous 3-Sasakian structure via the construction in Section 3.

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## 2. Preliminaries

In this section we give basic definitions and background related to the spin representation, spinors on homogeneous spaces and (homogeneous) Sasakian and 3-Sasakian structures. For a thorough introduction to these topics, among others, we recommend [LM89, BFGK91, BGM94, BG99, Fri00, BG08, DOP19].

### 2.1. The Spin Representation via Exterior Forms

Throughout this paper we shall make use of the realization of the spin representation in terms of exterior forms. This realization is well-known in the context of representation theory, however its application to spin geometry and the study of spinors has not yet been widely adopted outside of [Wan89, AHL22]. For a detailed description of this construction we refer the reader to Chapter 6.1.2 in [GW09] (beware their different convention for the Clifford relation), and for its application to spinorial calculations on homogeneous spaces see [AHL22]. We briefly recall here the basic definitions and properties insofar as they relate to this work.

Let $(V = \mathbb{R}^{2n-1}, g)$ with the Euclidean inner product and the standard $g$-orthonormal basis $\{e_1, \ldots, e_{2n-1}\}$. Letting $\varphi: V \to V$ denote the almost complex structure on $(\mathbb{R}e_1)^{\perp}$ given by

$$
\varphi(e_{2j}) = e_{2j+1}, \quad \varphi(e_{2j+1}) = -e_{2j},
$$

the complexification of $V$ can be written as a direct sum

$$
V^C = \mathbb{C}_0 \oplus L \oplus L',
$$

(1)
where \( \mathbb{C}_0 := \mathbb{C}e_1 \) and \( L \) (resp. \( L' \)) denotes the space of \( \varphi \)-holomorphic (resp. \( \varphi \)-anti-holomorphic) vectors. Explicitly, these spaces are given by

\[
L := \text{span}_\mathbb{C}\{x_j := \frac{1}{\sqrt{2}}(e_{2j} - ie_{2j+1})\}_{j=1}^{n-1}, \quad L' := \text{span}_\mathbb{C}\{y_j := \frac{1}{\sqrt{2}}(e_{2j} + ie_{2j+1})\}_{j=1}^{n-1}.
\]

Letting \( u_0 := ie_1 \), we define an action of \( V \) on the algebra \( \Sigma := \Lambda^*L' \) of \( \varphi \)-anti-holomorphic forms via

\[
u_0 := -\text{Id}|_{\Sigma^{\text{even}}} + \text{Id}|_{\Sigma^{\text{odd}}}, \quad x_j \cdot \eta := i\sqrt{2} x_j \eta, \quad y_j \cdot \eta := i\sqrt{2} y_j \wedge \eta,
\]

where \( \Sigma^{\text{even}} \) and \( \Sigma^{\text{odd}} \) denote the even and odd graded parts of \( \Sigma = \Lambda^*L' \). Recalling the definition of the complex Clifford algebra,

\[
\mathcal{C}(V^c, g^c) := T(V^c)/(v \otimes w + w \otimes v = -2g^c(v, w)1),
\]

one easily verifies using the identities (5.43) in [GW09] that the action (3) descends to a representation of \( \mathcal{C}(V^c, g^c) \) on \( \Sigma \). Solving for the real orthonormal basis vectors \( e_1, \ldots, e_{2n-1} \) in (2) gives

\[
e_1 = -iu_0, \quad e_{2j} = \frac{1}{\sqrt{2}}(x_j + y_j), \quad e_{2j+1} = -\frac{i}{\sqrt{2}}(y_j - x_j), \quad \forall \ j = 1, \ldots, n-1,
\]

and the corresponding action on \( \Sigma \) (i.e. the Clifford multiplication) is given by

\[
e_1 = i\text{Id}|_{\Sigma^{\text{even}}} - i\text{Id}|_{\Sigma^{\text{odd}}}, \quad e_{2j} \cdot \eta = i(x_j \cdot \eta + y_j \wedge \eta), \quad e_{2j+1} \cdot \eta = (y_j \wedge \eta - x_j \cdot \eta),
\]

for all \( \eta \in \Sigma \).

**Remark 2.1.** It is possible to similarly define the spin representation for even dimensional spaces by deleting the \( \mathbb{C}_0 \) factor in the decomposition (1) and the corresponding operators \( u_0, e_1 \) in (3) and (4).

2.2. **Spinors on Homogeneous Spaces.** Let \( M = G/H \) be a reductive homogeneous space for a semisimple group \( G \), and fix a reductive decomposition \( g = \mathfrak{h} \oplus \mathfrak{m} \) which is orthogonal with respect to the Killing form on \( g \). In this section we review the construction of some geometrically relevant bundles as homogeneous bundles associated to the projection \( G \to G/H \). For a more detailed introduction to reductive homogeneous spaces we refer to [Arv03], and for worked examples illustrating the process of finding invariant spinors we recommend [AHL22] and Chapters 4.5, 5.4 of [BFGK91].

Letting \( \pi: G \to G/H \) denote the projection map, the tangent space \( T_o M \) at the origin \( o := eH \) is naturally identified with \( \mathfrak{m} \cong \ker d_o \pi \). The other tangent spaces are therefore obtained by displacements of \( \mathfrak{m} \) under the isometries in \( G \), and there is a corresponding description of the tangent bundle as a homogeneous bundle:

\[
TM = G \times_{\text{Ad}|_H} \mathfrak{m},
\]

where \( \pi: G \to G/H \) is viewed as a principal \( H \)-bundle. The isotropy representation, i.e. the action of \( H \) on \( T_o M \), also has an elegant description in terms of the Lie theoretic data; isomorphic to the restricted adjoint representation \( \text{Ad}|_H: \mathfrak{m} \to \mathfrak{m} \). In other words, the isotropy representation is identified with the map \( \text{Ad}|_H: H \to \text{GL}(\mathfrak{m}) \), and an invariant Riemannian metric on \( M \) corresponds to a inner product \( g: \mathfrak{m} \times \mathfrak{m} \to \mathbb{R} \) with the property that the image of \( \text{Ad}|_H \) is contained in the subgroup \( \text{SO}(\mathfrak{m}, g) \) of \( \text{GL}(\mathfrak{m}) \). The oriented frame bundle with respect to an invariant metric is then given as a homogeneous bundle by

\[
P_{\text{SO}} = G \times_{\text{Ad}|_H} \text{SO}(\mathfrak{m}, g).
\]

Suppose now that there exists a lift \( \text{Ad}|_H \) of the isotropy representation to the spin group:

\[
\begin{array}{ccc}
\text{Spin}(\mathfrak{m}, g) & \xrightarrow{2:1} & \text{SO}(\mathfrak{m}, g) \\
\text{Ad}|_H & \xrightarrow{2:1} & \text{SO}(\mathfrak{m}, g)
\end{array}
\]
Such a map induces a spin structure and spinor bundle as the homogeneous bundles
\[ P_{\text{Spin}} := G \times \overline{\text{Ad}}_H \text{Spin}(m, g), \quad \Sigma M := P_{\text{Spin}} \times \sigma \Sigma = G \times \overline{\text{Ad}}_H \Sigma, \]
where \( \sigma : \text{Spin}(m) \to \text{Aut}(\Sigma) \) denotes the spin representation. Furthermore, for a connected isotropy group \( H \), this was shown to be the unique \( G \)-invariant spin structure on \( M = G/H \) by Daura Serrano, Kohn, and Lawn in [DKL21].

In this paper we shall mainly be concerned with 3-Sasakian spaces, which are necessarily spin due to Kuo’s reduction of the structure group of the tangent bundle to the (simply-connected) symplectic group of the horizontal distribution (Theorem 5 in [Kuo70]). For a homogeneous 3-Sasakian space, denote by \( m = m_V \oplus m_H \) the splitting into vertical and horizontal distributions. Invariance of the structure tensors implies that the image of \( H \) under the isotropy representation is contained in the above reduction, i.e.
\[ \text{Ad}_H(H) \subseteq \{1\} \times \text{Sp}(m_H) \subseteq \text{SO}(m), \]
and one therefore obtains a lifting of the isotropy representation and the associated (unique) \( G \)-invariant spin structure as above. Throughout the paper we will always use this spin structure when considering homogeneous 3-Sasakian spaces.

In light of the associated bundle construction of the spinor bundle in (5), spinors are identified with \( H \)-equivariant maps \( \varphi : G \to \Sigma \), i.e. maps satisfying
\[ \varphi(gh) = \Delta \circ \overline{\text{Ad}}_H(h^{-1}) \cdot \varphi(g) \quad \forall g \in G, h \in H. \]
The \( G \)-invariant spinors correspond precisely to the constant \( H \)-equivariant maps \( \varphi : G \to \Sigma \), and we denote by \( \Sigma_{\text{inv}} \subseteq \Sigma M \) the subbundle of such spinors. Equivalently, it follows from (6) that invariant spinors correspond to trivial subrepresentations of \( \sigma \circ \overline{\text{Ad}}_H : H \to \text{GL}(\Sigma) \).

One may similarly realize the bundles of \( k \)-tensors and differential \( k \)-forms on \( M \) as homogeneous bundles via
\[ \otimes^k TM = G \times (\overline{\text{Ad}}_H)^{\otimes k} m^{\otimes k}, \quad \Lambda^k TM = G \times \Lambda^k(\overline{\text{Ad}}_H) \Lambda^k m, \]
and invariant sections then correspond to trivial \( H \)-subrepresentations of \( m^{\otimes k} \) and \( \Lambda^k m \) respectively. The representation theoretic problem of finding trivial subrepresentations is approached in this paper using results from classical invariant theory, together with computer calculations in LiE (HTTP://WWW.MATHELABO.UNIV-POTIERS.FR/MAAVL/LiE/) for certain cases involving the exceptional Lie groups.

2.3. Sasakian and 3-Sasakian Structures. Let us briefly define Sasakian and 3-Sasakian structures and discuss some of their properties, following the exposition in [BGM94].

**Definition 2.2.** A Sasakian structure on a Riemannian manifold \( (M, g) \) is a unit length Killing vector field \( \xi \) such that the endomorphism field \( \varphi := -\nabla^g \xi \) satisfies
\[ (\nabla^g_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X \]
for all \( X, Y \in TM \) (where \( \nabla^g \) denotes the Levi-Civita connection). It is customary to denote a Sasakian structure by \( (M, g, \xi, \eta, \varphi) \), where \( \eta := \xi^\flat \). The vertical and horizontal distributions are defined by
\[ \mathcal{V} := \mathbb{R}\xi, \quad \mathcal{H} := \ker \eta, \]
and the vector field \( \xi \) is called the Reeb vector field. The fundamental 2-form is defined by
\[ \Phi(X, Y) := g(X, \varphi(Y)) \quad \forall X, Y \in TM. \]

Here we collect several basic properties of Sasakian manifolds:
Proposition 2.3. (Based on Proposition 2.2 in [BCM94]). If \((M, g, \xi, \eta, \varphi)\) is a Sasakian manifold, then
\[
\varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \text{Im}(\varphi) \subseteq \mathcal{H}, \quad d\eta = 2\Phi.
\]
\[
0 = N_\varphi(X, Y) := [\varphi(X), \varphi(Y)] + \varphi^2[\varphi(X), \varphi(Y)] - \varphi[X, \varphi(Y)] - \varphi[X, \varphi(Y)] + d\eta(X, Y)\xi,
\]
\[
0 = g(\varphi(X), Y) + g(X, \varphi(Y)), \quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y),
\]
for all \(X, Y \in TM\).

In a similar spirit, we have the notion of a 3-Sasakian structure, which consists of three orthogonal Sasakian structures whose Reeb vector fields satisfy the relations of the imaginary quaternions under the Lie bracket:

Definition 2.4. A 3-Sasakian structure on a Riemannian manifold \((M, g)\) consists of three Sasakian structures \((g, \xi_i, \eta_i, \varphi_i)\), \(i = 1, 2, 3\) such that the Reeb vector fields \(\xi_i\), \(i = 1, 2, 3\) are orthogonal and satisfy
\[
[\xi_i, \xi_j] = 2\xi_k
\]
for any even permutation \((i, j, k)\) of \((1, 2, 3)\). It is customary to denote a 3-Sasakian structure by \((M, g, \xi_i, \eta_i, \varphi_i)\), omitting the “\(i = 1, 2, 3\)”. The vertical and horizontal distributions are defined by
\[
\mathcal{V} := \text{span}_\mathbb{R}\{\xi_i\}_{i=1}^3, \quad \mathcal{H} := \cap_{i=1}^3 \ker(\eta_i),
\]
and the vector fields \(\xi_i\), \(i = 1, 2, 3\) are called the Reeb vector fields. The fundamental 2-forms are defined by
\[
\Phi_i(X, Y) := g(X, \varphi_i(Y)) \quad \forall X, Y \in TM.
\]

In addition to the identities in Proposition 2.3, the tensors defining a 3-Sasakian structure satisfy certain “pseudo-quaternionic” compatibility relations:

Proposition 2.5. (Based on Equation (2.4) in [BCM94] and Equation (1.5) in [AD20]). If \((M, g, \xi_i, \eta_i, \varphi_i)\) is a 3-Sasakian manifold, then
\[
\varphi_i = \varphi_j \circ \varphi_k - \eta_k \otimes \xi_j = -\varphi_k \circ \varphi_j + \eta_j \otimes \xi_k
\]
\[
\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \quad \eta_i = \eta_j \circ \varphi_k = -\eta_k \circ \varphi_j,
\]
for any even permutation \((i, j, k)\) of \((1, 2, 3)\).

Definition 2.6. Let \((M, g, \xi, \eta, \varphi)\) be a 3-Sasakian manifold. A local frame \(e_1, \ldots, e_{4n-1}\) of \(TM\) is called adapted if
\[
e_i = \xi_i, \quad e_{4p+i} = \varphi_i(e_{4p}), \quad \text{for all } i = 1, 2, 3, \ p = 1, \ldots, n - 1.
\]

Finally, for 3-Sasakian manifolds we recall that there is a particularly useful choice of metric connection adapted to the geometry: the so-called canonical connection (see Section 4 of [AD20], where this connection is introduced for the more general class of 3-(\(\alpha, \delta\))-Sasaki manifolds). We review its main properties in the following proposition:

Proposition 2.7. (Based on Section 4 of [AD20]). For a 3-Sasakian manifold \((M, g, \varphi, \xi_i, \eta_i)\), the canonical connection \(\nabla\) is the unique metric connection with skew torsion such that
\[
\nabla_X\varphi_i = -2(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \text{for all } X \in TM.
\]
The derivatives of the other structure tensors are
\[
\nabla_X\xi_i = -2(\eta_k(X)\xi_j - \eta_j(X)\xi_k), \quad \nabla_X\eta_i = -2(\eta_k(X)\eta_j - \eta_j(X)\eta_k),
\]
and the torsion 3-form is given by
\[
T = \sum_{i=1}^3 \eta_i \wedge d\eta_i.
\]
2.4. **Homogeneous 3-Sasakian Spaces.** First, we recall Boyer, Galicki, and Mann’s classification of homogeneous 3-Sasakian spaces:

**Theorem 2.8.** (based on Theorem C in [BGM94]). The homogeneous 3-Sasakian spaces \( M^{4n-1} = G/H, g \) are precisely

\[
\begin{align*}
\mathbb{S}^{4n-1} &\cong \frac{\text{Sp}(n)}{\text{Sp}(n-1)}, \\
\mathbb{R}^{4n-1} &\cong \frac{\text{Sp}(n)}{\text{Sp}(n-1) \times \mathbb{Z}_2}, \\
\text{SO}(n+3) &\cong \frac{G_2}{\text{Sp}(n-1) \times \text{Sp}(1)}, \\
\text{SU}(n+1) &\cong \frac{\text{Sp}(n-1) \times \mathbb{Z}_2}{\text{Sp}(n-3) + \text{Sp}(3)} + \text{SU}(6), \\
\text{SU}(n+1) &\cong \frac{E_7}{\text{Spin}(12), E_8}.
\end{align*}
\]

where the permissible values of \( n \) are as follows:

| \( n \) | \( n \geq 1 \) | \( n \geq 2 \) | \( n \geq 4 \) |

This classification was obtained by proving that any homogeneous 3-Sasakian space fibers over a Wolf space with a finite list of possibilities for the fiber, and then using the classification of Wolf spaces in [Wol65]. Recently, Goertsches et al. obtained a new proof of the classification using root systems of complex simple Lie algebras to construct homogeneous 3-Sasakian spaces [GRS21]. Previously, and also from the algebraic point of view, the invariant connections on homogeneous 3-Sasakian spaces were studied in detail by Draper et al. in [DOP19]. Importantly, they gave a characterization of these spaces in terms of purely Lie theoretic data called 3-Sasakian data:

**Theorem 2.9.** (Based on Definition 4.1 and Theorem 4.2 in [DOP19]). Let \( M = G/H \) be a homogeneous space with connected isotropy group \( H \), satisfying the following properties:

(i) There is a \( \mathbb{Z}_2 \)-graded decomposition \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) such that \( \mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h} \);

(ii) There exists an \( \mathfrak{h}^\mathbb{C} \)-module \( U \) such that \( \mathfrak{g}_1^\mathbb{C} \cong \mathbb{C}^2 \otimes U \) as a module for \( \mathfrak{g}_0^\mathbb{C} \cong \mathfrak{sp}(1)^\mathbb{C} \oplus \mathfrak{h}^\mathbb{C} \), where \( \mathbb{C}^2 \) is the standard representation of \( \mathfrak{sp}(1)^\mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}) \).

Then there is a 3-Sasakian structure \( (\xi_1, \eta, \varphi) \) on \( M = G/H \) determined by the tensors

\[
\begin{align*}
\xi_1 &:= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
\xi_2 &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\xi_3 &:= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\
\varphi_i &:= \frac{1}{2} \text{ad}(\xi_i)|_{\mathfrak{sp}(1)} + \text{ad}(\xi_i)|_{\mathfrak{g}_1},
\end{align*}
\]

\[
g := -\frac{1}{4(n+1)} \kappa|_{\mathfrak{sp}(1) \times \mathfrak{sp}(1)} - \frac{1}{8(n+1)} \kappa|_{\mathfrak{g}_0 \times \mathfrak{g}_1},
\]

where \( \kappa \) denotes the Killing form of \( \mathfrak{g} \). Furthermore, the Nomizu map of the Levi-Civita connection \( \nabla^g \) is given by

\[
A^g(X)Y = \begin{cases} 
\frac{1}{2}[X,Y]_m & \text{if } X, Y \in \mathfrak{sp}(1) \text{ or } X, Y \in \mathfrak{g}_1, \\
0 & \text{if } X \in \mathfrak{sp}(1), Y \in \mathfrak{g}_1, \\
[X,Y]_m & \text{if } X \in \mathfrak{g}_1, Y \in \mathfrak{sp}(1),
\end{cases}
\]

where subscript \( m \) denotes projection onto the reductive complement \( \mathfrak{m} := \mathfrak{sp}(1) \oplus \mathfrak{g}_1 \).

Indeed, they proved that all simply-connected homogeneous 3-Sasakian spaces can be constructed from 3-Sasakian data, and they gave an explicit description of the data in each case which will be extremely useful for our purposes.

Finally, we recall the result of Agricola, Dileo, and Stecker that, in the homogeneous case, the Nomizu map of the canonical connection (see Proposition 2.7) takes a simple form:
Proposition 2.10. (Based on Proposition 4.2.1 in [ADS21].) For a homogeneous 3-Sasakian space, the Nomizu map $\Lambda$ of the canonical connection $\nabla$ is given by
\[
\Lambda(X)Y = \begin{cases} 
-[X,Y] & \text{if } X \in \mathcal{V}, \\
0 & \text{if } X \in \mathcal{H},
\end{cases}
\]
for all $X, Y \in \mathfrak{m}$.

3. The (3-)Sasakian Structures Induced by Killing Spinors

In this section we explain how to construct Sasakian and 3-Sasakian structures from Riemannian Killing spinors, generalizing the constructions of Friedrich and Kath in dimensions 5 and 7 [FK88, FK89, FK90]. We also show that all Sasakian and 3-Sasakian structures on connected, simply-connected Einstein-Sasakian or 3-Sasakian manifolds arise from this construction by considering certain subbundles of the spinor bundle defined in [FK90].

Definition 3.1. A spinor $\psi \in \Gamma(\Sigma M)$ is called a (Riemannian) Killing spinor if it satisfies
\[
\nabla^g_X \psi = \lambda X \cdot \psi, \quad \lambda = \frac{1}{2} \text{ or } -\frac{1}{2},
\]
for all $X \in TM$. We shall refer interchangeably to Riemannian Killing spinors and Killing spinors, and, unless otherwise stated, we will only consider Killing spinors with $\lambda = \frac{1}{2}$ in this paper.

Generalizing to arbitrary dimension the 1-form and dual vector field considered in [FK88] (also in Section 5 of [FK90] and Chapter 4.4 of [BFGK91]), we make the following definition:

Definition 3.2. Let $(M, g)$ be a spin manifold. Given a pair of spinors $\psi_1, \psi_2$, the associated 1-form $\eta_{\psi_1, \psi_2}$ and its metric dual $\xi_{\psi_1, \psi_2}$ (the associated vector field) are defined by
\[
\eta_{\psi_1, \psi_2}(X) := \Re \langle \psi_1, X \cdot \psi_2 \rangle, \quad \xi_{\psi_1, \psi_2} := \eta_{\psi_1, \psi_2}^\sharp \quad \text{for all } X \in TM,
\]
where $\langle \cdot, \cdot \rangle$ denotes the usual Hermitian metric on the spinor bundle and $\Re$ is the real part. We also define the associated endomorphism field $\varphi_{\psi_1, \psi_2}$ by
\[
\varphi_{\psi_1, \psi_2}(X) := -\frac{1}{2} (X \cdot d\eta_{\psi_1, \psi_2})^\sharp, \quad \text{for all } X \in TM.
\]

Remark 3.3. Observe that the operator $\Re$ in (9) is unnecessary when $\psi_1 \in \Re i \psi_2$, since skew-symmetry of the Clifford multiplication with respect to the Hermitian product $\langle \cdot, \cdot \rangle$ ensures that $\langle i\psi, X \cdot \psi \rangle$ is purely real. Thus, by setting $\psi_1 = i\psi_2$ in (9), our definition recovers the 1-form and vector field considered in Section 4 of [FK89] and Chapter 1.5 of [BFGK91]. We also note that it is possible to choose non-vanishing spinors $\psi_1, \psi_2$ such that the associated vector field $\xi_{\psi_1, \psi_2}$ is identically zero. This is the case, for example, when $\psi_1 = \psi_2$.

As we will frequently encounter the tensors (9) and (10) and their derivatives, let us summarize some relevant facts:

Lemma 3.4. Let $(M, g)$ be a Riemannian spin manifold with spinor bundle $\Sigma M$, and denote by $\langle \cdot, \cdot \rangle$ the usual Hermitian scalar product on the fibers of $\Sigma M$.

(i) Differentiation of the Hermitian product $\langle \cdot, \cdot \rangle$ commutes with $\Re$, i.e.
\[
X \langle \Re \varphi, \psi \rangle = \Re (X \langle \varphi, \psi \rangle)
\]
for all vector fields $X$ and spinors $\varphi, \psi$.

(ii) For any spinor $\psi \in \Gamma(\Sigma M)$ and vector fields $X, Y \in TM$ we have
\[
\Re \langle X \cdot \psi, Y \cdot \psi \rangle = g(X, Y) \|\psi\|^2.
\]
(iii) If \( \psi \) is a Riemannian Killing spinor then \( ||\psi|| \) is constant on each connected component of \( M \).

**Proof.** These can be found elsewhere throughout the literature; in the interest of comprehensiveness we recall the proofs here.

(i) One easily calculates:

\[
X(\Re(\varphi, \psi)) = \frac{1}{2} X((\varphi, \psi) + (\bar{\varphi}, \bar{\psi})) = \frac{1}{2} X((\varphi, \psi) + (\varphi, \varphi)) \\
= \frac{1}{2} \left( (\nabla_X^g \varphi, \psi) + (\varphi, \nabla_X^g \psi) + (\nabla_X^g \psi, \varphi) + (\psi, \nabla_X^g \varphi) \right) \\
= \frac{1}{2} \left( (\nabla_X^g \varphi, \psi) + (\nabla_X^g \varphi, \psi) + (\varphi, \nabla_X^g \psi) + (\varphi, \nabla_X^g \psi) \right) \\
= \Re(\nabla_X^g \varphi, \psi) + \Re(\varphi, \nabla_X^g \psi) = \Re(X(\varphi, \psi)).
\]

(ii) Using the skew-symmetry of the Hermitian product with respect to Clifford multiplication, we calculate:

\[
\Re(X \cdot \psi, Y \cdot \psi) = -\Re(\psi, X \cdot Y \cdot \psi) = \Re(\psi, Y \cdot X \cdot \psi) + \Re(\psi, 2g(X, Y)\psi) \\
= \Re(Y \cdot X \cdot \psi) + 2g(X, Y) ||\psi||^2 = \Re(Y \cdot X \cdot \psi) \Re(\varphi, \nabla_X^g \psi) + 2g(X, Y) ||\psi||^2 \\
= -\Re(X \cdot \psi, Y \cdot \psi) + 2g(X, Y) ||\psi||^2,
\]

and the result follows.

(iii) Differentiating the square of the norm with respect to any \( X \in TM \) and using skew-symmetry of the Hermitian product with respect to Clifford multiplication gives:

\[
X||\psi||^2 = X(\psi, \psi) = \langle \nabla_X^g \psi, \psi \rangle + \langle \psi, \nabla_X^g \psi \rangle = \frac{1}{2}(X \cdot \psi) + \frac{1}{2} \langle \psi, X \cdot \psi \rangle = 0.
\]

Since one of the defining conditions of a Sasakian structure involves the exterior derivative of the Reeb 1-form, we also need an identity relating \( d\eta_{\psi_1, \psi_2} \) to the spinors \( \psi_1, \psi_2 \). Generalizing the identity calculated by Friedrich and Kath in the proof of Theorem 2 in [FK89], we have:

**Lemma 3.5.** If \((M, g)\) is a Riemannian spin manifold carrying a pair of Killing spinors \( \psi_1, \psi_2 \), then the exterior derivative of the 1-form \( \eta_{\psi_1, \psi_2} \) dual to \( \xi_{\psi_1, \psi_2} \) is given by

\[
d\eta(X,Y) = \Re[-\langle \psi_1, X \cdot Y \cdot \psi_2 \rangle + \langle \psi_1, Y \cdot X \cdot \psi_2 \rangle], \quad \text{for all} \ X, Y \in TM.
\]

**Proof.** The result follows by a calculation analogous to that found in the proof of Theorem 2 in [FK89].

We now arrive at the first main result of this section, which gives conditions for the tensors \( \xi_{\psi_1, \psi_2}, \eta_{\psi_1, \psi_2}, \) and \( \varphi_{\psi_1, \psi_2} \) determined by a pair of spinors \( \psi_1, \psi_2 \) via (9) and (10) to constitute a Sasakian structure.

**Theorem 3.6.** Let \((M, g)\) be a Riemannian spin manifold carrying a pair \( \psi_1, \psi_2 \) of Killing spinors (not necessarily linearly independent). If the associated vector field \( \xi_{\psi_1, \psi_2} \) is unit length, then the tensors \((g, \xi := \xi_{\psi_1, \psi_2}, \eta := \eta_{\psi_1, \psi_2}, \varphi := \varphi_{\psi_1, \psi_2})\) determine a Sasakian structure on \( M \).

**Proof.** By Proposition 2.1.2 in [BG99], it suffices to show that \( \xi \) is a Killing vector field and the \((1,1)\)-tensor \( \alpha(X) := -\nabla_X^g \xi \) satisfies

\[
(\nabla_X^g \alpha)Y = g(X,Y)\xi - \eta(Y)X.
\]
One easily sees from the calculation on page 30 of [BFGK91] that \( \xi \) is a Killing vector field (actually they prove this only for the case \( \psi_1 = i \psi_2 \), but the same argument works in general by using Lemma 3.4(iii)). Next we claim that \( \alpha = \varphi \). To see this, we use Lemmas 3.4(i) and 3.5 to calculate:

\[
g(\alpha(X),Y) = g(-\nabla_X^\theta \xi, Y) = -X g(\xi,Y) + g(\xi,\nabla_X^\theta Y) = \mathcal{R}[\neg X(\psi_1,Y \cdot \psi_2) + \langle \psi_1,(\nabla_X^\theta Y) \cdot \psi_2 \rangle]
\]

\[
= \mathcal{R}[\neg \langle \nabla_X^\theta \psi_1,Y \cdot \psi_2 \rangle - \langle \psi_1,Y \cdot \nabla_X^\theta \psi_2 \rangle) = \mathcal{R}\left[ -\frac{1}{2}\langle X \cdot \psi_1,Y \cdot \psi_2 \rangle - \frac{1}{2}\langle \psi_1,Y \cdot X \cdot \psi_2 \rangle \right]
\]

\[
= -\frac{1}{2} d\eta(X,Y) = g(Y, \varphi(X)),
\]

proving the claim. It remains only to prove (11), substituting \( \varphi \) in the place of \( \alpha \). Let \( \{e_1, \ldots, e_{2n-1}\} \) be a local orthonormal geodesic frame defined in a neighbourhood of a point \( p \in M \), i.e. a local orthonormal frame such that \( \nabla_{e_i} e_j(p) = 0 \). We calculate at the point \( p \in M \):

\[
(\nabla_{e_i}^g \varphi)e_j = \nabla_{e_i}(\varphi(e_j)) = -\frac{1}{2} \sum_{s=1}^{2n-1} \nabla_{e_i}^g (d\eta(e_j, e_s)e_s) = -\frac{1}{2} \sum_{s=1}^{2n-1} \nabla_{e_i}^g [\mathcal{R}[\neg \langle \psi_1, e_s \cdot e_j \cdot \psi_2 \rangle - \langle \psi_1, e_s \cdot e_j \cdot \nabla_{e_i}^g \psi_2 \rangle]e_s
\]

\[
= -\frac{1}{2} \sum_{s \neq j} \mathcal{R}(\langle e_i \cdot \psi_1, e_s \cdot e_j \cdot \psi_2 \rangle + \langle \psi_1, e_s \cdot e_j \cdot \nabla_{e_i}^g \psi_2 \rangle)e_s = -\frac{1}{2} \sum_{s \neq j} \mathcal{R}(\langle e_i \cdot \psi_1, (e_s \cdot e_j \cdot e_i - e_i \cdot e_s \cdot e_j) \cdot \psi_2 \rangle)e_s
\]

\[
= -\frac{1}{2} \sum_{s \neq j} \mathcal{R}(\langle \psi_1, (-e_s \cdot e_i \cdot e_j - 2\delta_{i,j} e_s - e_i \cdot e_s \cdot e_j) \cdot \psi_2 \rangle)e_s = -\frac{1}{2} \sum_{s \neq j} \mathcal{R}(\langle \psi_1, (2\delta_{i,s} e_j - 2\delta_{i,j} e_s) \cdot \psi_2 \rangle)e_s
\]

\[
= -\sum_{s \neq j} \delta_{i,s} \eta(e_j) - \delta_{i,j} \eta(e_s))e_s = \delta_{i,j}(\xi - \eta(e_j) e_j) - \sum_{s \neq j} \delta_{i,s} \eta(e_j)e_s = \left\{ \begin{array}{ll} -\eta(e_j)e_i & \mbox{if } i \neq j, \\ \xi - \eta(e_j)e_j & \mbox{if } i = j, \end{array} \right.
\]

\[
= g(e_i,e_j)\xi - \eta(e_j)e_i,
\]

completing the proof. \( \square \)

In fact, we shall prove in the next theorem that any Sasakian structure arises from this construction. To that end, we consider the bundles

\[
E_{\pm} := \{ \psi \in \Sigma M : (\pm 2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \psi = 0 \quad \forall X \in TM \}
\]

defined by Friedrich and Kath in [FK90]. They showed that, on a simply-connected Einstein-Sasakian spin manifold, these bundles admit a global basis consisting of Riemannian Killing spinors for the constants \( \pm \frac{1}{2} \) (see the proof of Theorem 1 in [FK90]). For our purposes, it is important to observe that \( \text{rank}(E_{-}) \geq 1 \) (see the proof of Theorem 1 in Chapter 4.2 of [BFGK91], noting that the roles of \( E_{+}, E_{-} \) are reversed here due to our convention for the Clifford multiplication). We have:

**Theorem 3.7.** If \( (M^{2n-1}, g, \xi, \eta, \varphi) \) is a connected, simply-connected Einstein-Sasakian spin manifold, then the Sasakian structure arises from the preceding construction.

**Proof.** Let \( \psi \in \Gamma(E_{-}) \) be a Killing spinor, and assume without loss of generality that \( ||\psi|| = 1 \) (otherwise we may rescale it to unit length using Lemma 3.4(iii))). Defining \( \psi' := -\xi \cdot \psi \), we note that \( \psi' \in \Gamma(E_{-}) \), since \( \xi \) anti-commutes in the Clifford algebra with the operators \( (-2\varphi(X) + \xi \cdot X - X \cdot \xi) \) defining the bundle \( E_{-} \). Furthermore, \( \psi' \) is a Killing spinor due to the calculation

\[
\nabla_X^\theta \psi' = -\nabla_X^\theta (\xi \cdot \psi) = -\nabla_X^\theta (\xi \cdot \psi) = \varphi(X) \cdot \psi = -\frac{1}{2} \xi \cdot X \cdot \psi
\]

\[
= \frac{1}{2}(\xi \cdot X - X \cdot \xi) \cdot \psi - \frac{1}{2} \xi \cdot X \cdot \psi = -\frac{1}{2} X \cdot \xi \cdot \psi = \frac{1}{2} X \cdot \psi',
\]

...
where we have used the identity $\varphi = -\nabla g \xi$ and the defining condition of $E_-$. Using Lemma 3.4(ii) we calculate
\[ \eta_{\psi, \psi'}(X) = g(\xi_{\psi, \psi'}, X) = \Re(\psi, X \cdot \psi') = -\Re(\psi, X \cdot \xi \cdot \psi) = g(X, \xi) ||\psi||^2 = \eta(X) \]
for any $X \in TM$, hence $\eta_{\psi, \psi'} = \eta$ (or equivalently $\xi_{\psi, \psi'} = \xi$). In particular $\xi_{\psi, \psi'}$ is unit length, so it follows from Theorem 3.6 that the pair $\psi, \psi'$ induces a Sasakian structure on $(M, g)$. We have already shown that the Reeb vector field and dual one form of the induced structure agree with $\xi, \eta$, and the result follows by noting that $\varphi_{\psi, \psi'} = -\nabla g \xi_{\psi, \psi'} = -\nabla g \xi = \varphi$. \hfill \Box

As corollaries, we obtain analogous construction and uniqueness results for 3-Sasakian manifolds:

**Theorem 3.8.** Let $(M, g)$ be a Riemannian spin manifold carrying Killing spinors $\psi_1, \psi_2, \psi_3, \psi_4$ (not necessarily linearly independent). If $\xi_{\psi_1, \psi_2}$ and $\xi_{\psi_3, \psi_4}$ are orthogonal unit length vector fields, then the two Sasakian structures induced by Theorem 3.6 determine a 3-Sasakian structure on $(M, g)$.

**Proof.** This follows from Theorem 3.6 and the fact that two Sasakian structures with orthogonal Reeb vector fields uniquely determine a 3-Sasakian structure (see e.g. p.556 of [FK90] for the construction of the structure tensors of the third Sasakian structure in terms of the other two). \hfill \Box

To prove a uniqueness result analogous to Theorem 3.7 for 3-Sasakian manifolds, we consider the natural generalization
\[ E_i^- := \{ \psi \in \Sigma M : (\pm 2\varphi_i(X) + \xi_i \cdot X - X \cdot \xi_i) \cdot \psi = 0 \quad \forall X \in TM \}, \quad i = 1, 2, 3 \]
of Friedrich and Kath’s bundles (12) to the 3-Sasakian setting (see the proof of Theorem 6 in [FK90]). Indeed, performing the argument from the proof of Theorem 3.7 to each of the three Sasakian structures individually yields:

**Theorem 3.9.** If $(M^{4n-1}, g, \xi, \eta, \varphi_i)_{i=1,2,3}$ is a connected, simply-connected 3-Sasakian manifold, then the 3-Sasakian structure arises from the preceding construction.

Let us briefly compare our results with those of Friedrich and Kath in the 5- and 7-dimensional setting. These can be found in [FK90], Sections 5 and 6 of [FK90], and also appeared subsequently in Chapters 4.3 and 4.4 of the book [BFGK91]. In dimension 5, their construction relies on the fact that the spin representation of Spin(5) acts transitively on the unit sphere in the spinor module (which no longer holds in dimension $> 9$ [MS13]). Given a non-zero spinor $\psi$, this allows them to arrange a particular choice of frame in which a unique unit length solution $\xi \in TM$ to the equation $\xi \cdot \psi = i\psi$ is readily apparent. For a non-zero Killing spinor $\psi$, and under an appropriate normalization of the scalar curvature, they then show that this vector field $\xi$ determines a Sasakian structure. Similarly, given an orthonormal pair of Killing spinors $\psi_1, \psi_2$ in dimension 7, Friedrich and Kath use the orthogonal decomposition $\Sigma M^7 = C\psi_1 \oplus_\perp (TM^7 \cdot \psi_1)$ to find a unique unit length vector field $\xi$ satisfying $\xi \cdot \psi_1 = \psi_2$, which goes on to become the Reeb vector field of a Sasakian structure. This decomposition of the spinor bundle occurs as a coincidence in dimension 7, and fails in higher dimensions since the dimension of the spinor module grows much faster than the dimension of the manifold. In both cases, Friedrich and Kath note that their vector field $\xi$ is dual to the 1-form defined essentially by (9), but there is no mention of the fact that this 1-form can be taken as the starting point to perform such a construction in arbitrary dimension, as we have done in this paper.

Aside from the preceding comments about dimension, we also note that our results are slightly different in spirit: Friedrich and Kath prove that a Killing spinor in dimension 5 (resp. two Killing spinors in dimension 7) defines a specific unit length vector field which is in fact the Reeb vector field of a Sasakian structure. On the other hand, our Theorem 3.6 requires the assumption that the vector field $\xi_{\psi_1, \psi_2}$ induced by a pair of Killing spinors is unit length; this is then shown in Theorem 3.7 to be a reasonable assumption in the sense that Sasakian manifolds always carry spinors $\psi_1, \psi_2$ inducing a
unit length vector field, so no cases are “lost” by imposing this. The similarities and differences are much the same when comparing Section 6 of [FK90] to our Theorem 3.8.

4. Invariance of Spinors and their Associated (3-)Sasakian Structures

Given the relationship described above it is natural to ask whether, on a homogeneous manifold, invariance of a (3-)Sasakian structure implies invariance of the associated unit spinor(s) and vice versa. One already sees from Remark 4.42 in [AHL22] that the homogeneous sphere $S^{4n-1} = \frac{Sp(n) \cup U(1)}{Sp(n-1) \cup U(1)}$ with a certain choice of invariant Riemannian metric admits a non-invariant spinor $\psi_1 \in \Gamma(E_1^+) \corresponding to an invariant Sasakian structure. However, it turns out that if the spinor is invariant then so too must be the corresponding Sasakian structure, as we prove in this section. This suggests that an invariant spinor is a more fundamental geometric object than an invariant (3-)Sasakian structure, capturing more of the homogeneity data of the space. To begin, we have the following lemma:

**Lemma 4.1.** For any $X \in \mathbb{R}^k$ and $\theta \in \Lambda^p \mathbb{R}^k$, the identity

$$\theta \cdot X - X \cdot \theta = ((-1)^p + 1) X \wedge \theta$$

holds in the Clifford algebra.

**Proof.** The Clifford algebra identities

$$X \cdot \theta = (X \wedge \theta) - (X \wedge \theta), \quad \theta \cdot X = (-1)^p [X \wedge \theta + X \wedge \theta]$$

appear as Equations (1.4) in Chapter 1.2 of [BFGK91], and the result follows by subtracting them. $\square$

Considering a 2-form $T = \sum_{i<j} T_{ij} e_i \wedge e_j \in \Lambda^2 \mathbb{R}^k \cong so(k)$ and its spin lift $\tilde{T} = \frac{1}{2} \sum_{i<j} T_{ij} e_i \cdot e_j$, we immediately obtain the corollary:

**Corollary 4.2.** Let $T \in so(k)$ be a skew-symmetric linear transformation and $\tilde{T} \in \text{spin}(k)$ its spin lift under the Lie algebra isomorphism $so(k) \cong \text{spin}(k)$. Then, for any $X \in \mathbb{R}^k$, the identity

$$\tilde{T} \cdot X - X \cdot \tilde{T} = T(X)$$

holds in the Clifford algebra, where $T(\theta)$ refers to the standard action of $so(k)$ on differential forms.

These formulas also easily generalize for the commutator of a 2-form with a form of arbitrary degree:

**Lemma 4.3.** Let $T \in so(k)$ be a skew-symmetric linear transformation and $\tilde{T} \in \text{spin}(k)$ its spin lift under the Lie algebra isomorphism $so(k) \cong \text{spin}(k)$. Then, for any $\theta \in \Lambda^p \mathbb{R}^k$, the identity

$$\tilde{T} \cdot \theta - \theta \cdot \tilde{T} = T(\theta),$$

holds in the Clifford algebra, where $T(\theta)$ refers to the standard action of $so(k)$ on differential forms.

**Proof.** It suffices to prove the result for $T = e_i \wedge e_j$ and $\theta = e_i \wedge \cdots \wedge e_{i_p}$. We calculate:

$$\tilde{T} \cdot \theta - \theta \cdot \tilde{T} = \frac{1}{2} (e_i \cdot e_j \cdot e_{k_1} \cdot \cdots \cdot e_{k_p} - e_i \cdot \cdots \cdot e_{k_p} \cdot e_i \cdot e_j) = \begin{cases} 0 & i, j \not\in \{i_1, \ldots, l_p\}, \\ 0 & i, j \in \{i_1, \ldots, l_p\}, \\ e_i \cdot e_j \cdot \theta & \text{otherwise} \end{cases} = T(\theta).$$

$\square$
Finally, we show that the Clifford product of an invariant vector or differential form with an invariant spinor is again invariant.

**Lemma 4.4.** If \((M = G/H, g)\) is a homogeneous spin manifold carrying an invariant spinor \(\psi\), then \(\theta \cdot \psi \in \Sigma\) is invariant for any invariant form \(\theta \in \Lambda^k_{\text{inv}} \mathfrak{m}, k \geq 0\).

**Proof.** For any isotropy operator \(h \in \mathfrak{h} \subseteq \mathfrak{so}(m)\), it follows from (14) that
\[
\tilde{h} \cdot \theta \cdot \psi - \theta \cdot \tilde{h} \cdot \psi = h(\theta) \cdot \psi.
\]
Invariance of \(\psi\) and \(\theta\) gives \(\tilde{h} \cdot \psi = 0 = h(\theta)\), hence \(\tilde{h} \cdot \theta \cdot \psi = 0\) as desired.

With these lemmas, it is easy to prove that invariant Killing spinors induce invariant (3-)Sasakian structures via our construction from Section 4.

**Theorem 4.5.** If \((M^{2n-1} = G/H, g)\) is a Riemannian homogeneous spin manifold carrying invariant Killing spinors \(\psi_1, \psi_2\) such that \(\xi_{\psi_1, \psi_2}\) is unit length, then the induced Sasakian structure \((\xi_{\psi_1, \psi_2}, \eta_{\psi_1, \psi_2}, \varphi_{\psi_1, \psi_2})\) is invariant.

**Proof.** It suffices to show that the structure tensors \(\xi := \xi_{\psi_1, \psi_2}, \eta := \eta_{\psi_1, \psi_2}, \varphi := \varphi_{\psi_1, \psi_2}\) are invariant. Using (14) and invariance of \(g, (\ , \ ), \psi_1\), and \(\psi_2\), we calculate:
\[
g([h, \xi], X) = -g(\xi, [h, X]) = -\Re(\psi_1, [h, X] \cdot \psi_2) = -\Re(\psi_1, (\tilde{h} \cdot X - X \cdot \tilde{h}) \cdot \psi_2)
\]
\[
= -\Re(\psi_1, \tilde{h} \cdot X \cdot \psi_2) = \Re(\tilde{h} \cdot \psi_1, X \cdot \psi_2) = 0
\]
for all \(X \in \mathcal{T}M\), hence \(\xi, \eta\) are invariant. It follows that \(\varphi = -\nabla^g \xi\) is invariant as well.

By the same argument, one also obtains the analogous result in the 3-Sasakian setting:

**Theorem 4.6.** If \((M^{4n-1} = G/H, g)\) is a Riemannian homogeneous spin manifold carrying invariant Killing spinors \(\psi_1, \psi_2, \psi_3, \psi_4\) such that \(\xi_{\psi_1, \psi_2}\) and \(\xi_{\psi_3, \psi_4}\) are orthogonal and unit length, then the induced 3-Sasakian structure is invariant.

5. INVARIANT DIFFERENTIAL FORMS AND SPINORS

Expanding upon the work [DOP19], in this section we describe the invariant \(\varphi_1\)- (anti-)holomorphic differential forms on homogeneous 3-Sasakian spaces. We also describe the invariant spinors carried by these spaces, and the relationship between the forms and spinors. We would like to emphasize that this approach exploits the exterior form viewpoint of the spin representation, which greatly simplifies calculations and allows one to easily prove results for spaces of arbitrary dimension.

**Lemma 5.1.** For any simply connected-homogeneous 3-Sasakian space \((M = G/H, g)\), with a fixed choice of adapted basis for \(m\) and the associated Clifford algebra representation (4), the spinor \(1 \in \Lambda^* \mathcal{L}'\) is \(G\)-invariant.

**Proof.** Comparing with Theorem 2.8, there are only eight possibilities for \(G\). Arguing similarly to the proof of Proposition 4.15 in [AHL22], we have that \(1 \in \Gamma(E_1^-)\). We also note that the action of \(G\) on \(\Sigma\) restricts to a representation on \(E_1^-\) (by a similar calculation as in the proof of Proposition 3.7 in [Kat99]), and this representation will be trivial in most cases due to the fact that \(\dim \mathfrak{c}(E_1^-) = 2\). This leaves only the case \(G = \text{Sp}(1)\) to consider, which has trivial isotropy group \(H = \{1\}\) and hence all spinors are invariant.
In order to prove the first major result of this section, we will make use of the First Fundamental Theorems of Invariant Theory for the classical complex simple Lie groups, which can be found e.g. in [FH91] [Sch08]; we will use the formulations presented in [Sch08] as these are more suited to our purposes. We will also need the description of the exterior powers of the standard representation of $SO(n, \mathbb{C})$ as highest weight modules (see e.g Chapter 5.5.2 of [GW09]). Here we summarize these results:

**Theorem 5.2.** (Based on the First Fundamental Theorems in Section 5 of [Sch08]). Let $SL(n, \mathbb{C})$ act on $\mathbb{C}^n$ by its standard representation and, if $n = 2l$ is even, let $Sp(2l, \mathbb{C})$ also act by its standard representation. Denote by $e_1, \ldots, e_n$ (resp. $e_1^*, \ldots, e_n^*$) the standard basis for $\mathbb{C}^n$ (resp. the dual basis for $(\mathbb{C}^n)^*$), and let

$$T := T(\mathbb{C}^n \oplus (\mathbb{C}^n)^*) = T(\mathbb{C}^n) \otimes T((\mathbb{C}^n)^*)$$

where $T$ denote the algebra of tensors on $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$. The subalgebra of invariant tensors for the two groups are described up to mutations (i.e. permutations of covariant factors and permutations of contravariant factors) as follows:

(i) FFT for $SL(n, \mathbb{C})$: The space $T^{SL(n, \mathbb{C})}$ of invariant tensors is the $\mathbb{C}$-span of all mutations of tensor products of $det$, $det^*$, $I$, where

$$det := \sum_{\sigma \in S_n} \text{sign}(\sigma) e_{\sigma(1)}^* \otimes \cdots \otimes e_{\sigma(n)}^*, \quad I := \sum_{i=1}^n e_i \otimes e_i^*.$$  

(ii) FFT for $Sp(2l, \mathbb{C})$: The space $T^{Sp(2l, \mathbb{C})}$ of invariant tensors is the $\mathbb{C}$-span of all mutations of tensor products of $p$, $p^*$, $I$, where

$$p := \sum_{i=1}^l (e_i \otimes e_{l+i} - e_{l+i} \otimes e_i), \quad I := \sum_{i=1}^n e_i \otimes e_i^*.$$  

**Theorem 5.3.** (Based on Theorem 5.5.11 in [GW09]). Denote by $\omega_1, \ldots, \omega_{n-1}$ the fundamental weights of $SL(n, \mathbb{C})$, and $\Lambda^r \mathbb{C}^n$ the $r$th exterior power of the standard representation. The representation $\Lambda^r \mathbb{C}^n$ is irreducible for all $r = 1, \ldots, n$, with highest weight $\omega_r$ for $r = 1, \ldots, n - 1$.

**Theorem 5.4.** (Based on Theorem 5.5.13 in [GW09]). For $n = 2l$ or $2l+1$, let $\omega_1, \ldots, \omega_l$ denote the fundamental weights of $SO(n, \mathbb{C})$, and $\Lambda^l \mathbb{C}^n$ the $r$th exterior power of the standard representation.

(i) For $n = 2l+1 \geq 3$: The representation $\Lambda^l \mathbb{C}^n$ is irreducible for all $r = 1, \ldots, n$, with highest weight $\omega_r$ for $r = 1, \ldots, l - 1$ and $2\omega_l$ for $r = l$.

(ii) For $n = 2l \geq 4$: The representation $\Lambda^l \mathbb{C}^n$ is irreducible for $r = 1, \ldots, l - 1$, with highest weight $\omega_r$ for $r = 1, \ldots, l - 2$ and $\omega_{l-1} + \omega_l$ for $r = l - 1$. The representation $\Lambda^l \mathbb{C}^n$ splits as the direct sum of two irreducible submodules with highest weights $2\omega_{l-1}$ and $2\omega_l$ respectively.

Now we prove a general result regarding invariant tensors for two of the classical cases:

**Proposition 5.5.** If $(M = G/H, g, \xi, \eta, \varphi)$ is a homogeneous $3$-Sasakian space with $G = Sp(n)$ or $SU(n+1)$, then the algebra $T_{inv}(m_C^H)$ of invariant horizontal complex tensors is generated, up to mutations, by its degree 0 and 2 elements.

**Proof.** Complexifying the isotropy representation gives $m_C \simeq 3 \mathbb{C} \oplus 2U$ for some $\mathbb{C}^2$-module $U$ (see Remark 4.5 in [DOP19]), and in particular $m_C^2 \simeq 2U$. We treat the two cases for $G$ one at a time using the detailed descriptions from [DOP19] of the isotropy algebra, reductive complement, and module $U$ for each case.
Case 1: $G = \text{Sp}(n)$. Here the isotropy group is $H = \text{Sp}(n - 1)$, and we have $U = \mathbb{C}^{2n-2}$ (the standard module for $\mathfrak{h}^C = \mathfrak{sp}(2n-2, \mathbb{C})$). The space of horizontal complex tensors is

$$T(m^C_H) = T(2\mathbb{C}^{2n-2}) = \bigoplus_{k \geq 0} (2\mathbb{C}^{2n-2})^{\otimes k} = \bigoplus_{k \geq 0} 2^k (\mathbb{C}^{2n-2})^{\otimes k}.$$  

By Theorem 5.2(ii), the tensor invariants for the standard representation of $\text{Sp}(2n-2, \mathbb{C})$ are spanned by mutations of tensor powers of the 2-form defining the symplectic group. We conclude that any invariant tensor is a mutation of a tensor power of this 2-form (the number of factors used from the first and second summands in $m^C_H = \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ determines in which summand on the right hand side of (16) the resulting invariant tensor lies).

Case 2: $G = \text{SU}(n + 1)$. Here the isotropy group is $H = \text{SU}(n+1) \times \text{U}(1)$. Treating first the case $n \geq 3$, there is a decomposition $U = \mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*$ which is valid as modules for both $[\mathfrak{h}, \mathfrak{h}]^C = \mathfrak{su}(n-1, \mathbb{C})$ and $Z(\mathfrak{h})^C = \mathfrak{u}(1)^C$. We then have

$$T(m^C_H) = T(2\mathbb{C}^{n-1} + 2(\mathbb{C}^{n-1})^*) = T(2\mathbb{C}^{n-1}) \otimes T(2(\mathbb{C}^{n-1})^*) = \left( \bigoplus_{k \geq 0} 2^k (\mathbb{C}^{n-1})^{\otimes k} \right) \otimes \left( \bigoplus_{k \geq 0} 2^k ((\mathbb{C}^{n-1})^*)^{\otimes k} \right)$$

as $H$-modules, and examining the action of $u(1)^C$ (see Section 4.5 in [DOP19]) shows that any invariant form must have an equal number of $\mathbb{C}^{n-1}$ and $(\mathbb{C}^{n-1})^*$ factors. Combining this with Theorem 5.2(i), one sees that the invariant tensors are spanned by mutations of tensor products of $B := \sum_{i=1}^{n-1} e_i \otimes e_i^*$ and $\det \otimes \det^*$. The result for $n \geq 3$ then follows from the observation on p. 1312 of [Sch08] that $\det \otimes \det^*$ is a linear combination of mutations of a tensor power of $B$. Now we consider the case $n = 2$. Here the horizontal component of the (complexified) isotropy representation is $m^C_H = 2\mathcal{V}_1 \oplus 2\mathcal{V}_{-1}$, where $\mathcal{V}_x$ is the $\mathfrak{h}^C = \mathbb{C}$ representation $\mathbb{C}$ under the action $x \cdot z = \frac{3}{2} xz$ for all $x, z \in \mathbb{C}$ (see p.27 in [DOP19]). The space of horizontal complex tensors is given by

$$T(m^C_H) = T(2\mathcal{V}_1 \oplus 2\mathcal{V}_{-1}) = T(2\mathcal{V}_1) \otimes T(2\mathcal{V}_{-1}) = \left( \bigoplus_{k \geq 0} 2^k \mathcal{V}_1^{\otimes k} \right) \otimes \left( \bigoplus_{k \geq 0} 2^k \mathcal{V}_{-1}^{\otimes k} \right),$$

and it is easy to see that a tensor is invariant if and only if each term has the same number of factors from $\mathcal{V}_1$ and $\mathcal{V}_{-1}$. Let $v_k$ be a basis vector for (the 1-dimensional space) $\mathcal{V}_x \cong \mathbb{C}$. We then obtain

$$T_{\text{inv}}(m^C_H) = \bigoplus_{k \geq 0} 2^{2k} (\mathcal{V}_1^{\otimes k} \otimes \mathcal{V}_{-1}^{\otimes k}) = \bigoplus_{k \geq 0} 2^{2k} \mathbb{C} v_1 \otimes \cdots \otimes v_1 \otimes v_{-1} \otimes \cdots \otimes v_{-1},$$

and hence the space of invariant horizontal tensors is spanned by mutations of powers of the 2-tensor $v_1 \otimes v_{-1}$, completing the proof for this case.

As a corollary, we deduce:

**Theorem 5.6.** Let $(M = G/H, g, \xi_1, \eta_1, \phi_1)$ be a simply-connected homogeneous 3-Sasakian space with $G = \text{Sp}(n)$ or $\text{SU}(n+1)$. For any $k \in \mathbb{Z}_{\geq 0}$, the space $\Lambda^k_{\text{inv}} m$ of invariant $k$-forms is spanned by wedge products of invariant 1- and 2- forms. Explicitly, the invariant algebra $\Lambda^k_{\text{inv}} m$ is spanned by elements of the form

$$\tau_{e_a} := \xi_1^{e_1} \wedge \xi_2^{e_2} \wedge \xi_3^{e_3} \wedge (\Phi_0)^{a_0} \wedge (\Phi_1|\mathcal{H})^{a_1} \wedge (\Phi_2|\mathcal{H})^{a_2} \wedge (\Phi_3|\mathcal{H})^{a_3},$$

where $e_1, e_2, e_3 \in \{0, 1\}$, $a_0, a_1, a_2, a_3 \in \mathbb{Z}_{\geq 0}$, and $a_0 = 0$ for $G \neq \text{SU}(n+1)$.

**Proof.** Since the vertical space $m_V$ is the direct sum of three copies of the trivial $H$-module, we have

$$\Lambda^k_{\text{inv}} m_V = \Lambda^k m_V = \text{span}_{\mathbb{R}} \{\xi_1, \xi_2, \xi_3, \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_2 \wedge \xi_3, \xi_1 \wedge \xi_2 \wedge \xi_3\}. $$
Noting that the skew-symmetrization of a tensor and the skew-symmetrization of any mutation of that tensor agree up to a multiple of ±1, the result then follows from Proposition 5.3 by observing that
\[
Λ_{inv}^m = (Λ^*(m_\nu \oplus m_H))^H = ((Λ^*m_\nu) \otimes (Λ^*m_H))^H = ((Λ^*m_\nu)^H \otimes (Λ^*m_H))^H = (Λ^*m_\nu)^H \otimes (Λ^*m_H)^H.
\]

In fact, we expect that Theorem 5.6 should hold for all homogeneous 3-Sasakian spaces, however the arguments for the remaining cases become much harder (due to the appearance of extra tensors in certain dimensions for the \( G = \text{SO}(n + 3) \) case, and the lack of known invariant theory results for the exceptional cases), so other methods would likely be needed to prove such a result. For this reason we now prove a somewhat weaker result, but one which can be shown for all the cases and which will nonetheless be sufficient for the purpose of finding the spaces of invariant spinors:

**Proposition 5.7.** If \((M = G/H, g, ξ_1, η_2, ϕ_1)\) is a simply-connected homogeneous 3-Sasakian space, then the algebras \(Λ_{inv}^0(m_H^C), Λ_{inv}^0(m_H^C)\) of invariant horizontal \(ϕ_1\)-(anti-)holomorphic forms are generated by their degree 0 and 2 elements.

**Proof.** Employing the basis for \(\mathfrak{sp}(1) = \text{span}_\mathbb{R}\{ξ_1, ξ_2, ξ_3\}\) given in Equation (17) of [DOP19], the almost complex structure \(ϕ_1|_H = \text{ad}(ξ_1)\) acts on \((m_H^C) = (g_1)^C\) via \(i\text{Id}\) on \((1, 0) \otimes U\) and \(-i\text{Id}\) on \((0, 1) \otimes U\), hence the holomorphic (resp. anti-holomorphic) horizontal tangent vectors are given by \(Λ^0,1(m_H^C) = (1, 0) \otimes U \cong U\) (resp. \(Λ^{0,1}(m_H^C) = (0, 1) \otimes U \cong U\)). Therefore we have \(Λ^kU \cong Λ^{k,0}(m_H^C) \cong Λ^{0,k}(m_H^C)\) for all \(k \geq 0\), so it suffices to consider the invariant exterior forms on \(U\). We use this approach to treat the cases for \(G\) individually.

**Case 1: \(G = \text{Sp}(n)\).** Here the isotropy group is \(\text{Sp}(n-1)^C \cong \text{Sp}(2n-2, \mathbb{C})\) and we have \(U = \mathbb{C}^{2n-2}\) (the standard representation). It then follows from Theorem 5.2(ii) that \(Λ_{inv}^kU \cong Λ_{inv}^k\mathbb{C}^{2n-2}\) is generated by alternating tensor powers (i.e. wedge powers) of the 2-form \(p \in Λ^2\mathbb{C}^{2n-2}\) stabilized by \(\text{Sp}(2n-2, \mathbb{C})\).

**Case 2: \(G = \text{SU}(n + 1)\).** Proceeding as in proof of Proposition 5.5 (Case 2), we have:

\[
Λ^kU \cong Λ^k(\mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*) \cong \bigoplus_{p+q=k} (Λ^p\mathbb{C}^{n-1}) \otimes (Λ^q(\mathbb{C}^{n-1})^*),
\]

and examining the action of \(\mathfrak{u}(1)^C\) (see p.26 in [DOP19]) shows that any invariant element of \(Λ^kU\) must have the same number of \(\mathbb{C}^{n-1}\) and \((\mathbb{C}^{n-1})^*\) factors. By Theorem 5.3 the modules \(Λ^p\mathbb{C}^{n-1}\) and \(Λ^q(\mathbb{C}^{n-1})^*\) are irreducible and non-isomorphic unless \(p = q\), and it then follows from (18) that

\[
\dim_\mathbb{C} Λ_{inv}^k(U) = \begin{cases} 1 & \text{if } k \text{ is even and } k \leq \dim_\mathbb{C} U, \\ 0 & \text{otherwise.} \end{cases}
\]

In particular one checks in a basis that \(ω_{1,0} := (Φ_2|_H - iΦ_3|_H)\) (resp. \(ω_{0,1} := (Φ_2|_H + iΦ_3|_H)\)) is an element of \(Λ_{inv}^{2,0}(\mathbb{C}^C)\) (resp. \(Λ_{inv}^{0,2}(\mathbb{C}^C)\)), and that the top power of \(ω_{1,0}\) (resp. \(ω_{0,1}\)) is a \(ϕ_1\)-holomorphic (resp. \(ϕ_1\)-anti-holomorphic) volume form. It follows that lower powers are non-zero, hence span the 1-dimensional space of invariant \(ϕ_1\)-(anti-)holomorphic forms in the relevant dimension.

**Case 3: \(G = \text{SO}(n + 3)\).** Here the isotropy group is \(H = \text{SO}(n-1) \times \text{Sp}(1)\), and it is shown in Section 4.3 of [DOP19] that \(U = \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}\) as \(\text{SO}(n-1, \mathbb{C})\)-modules (two copies of the standard representation). Note that this decomposition does not hold as modules for \(\text{Sp}(1)^C \cong \text{Sp}(2, \mathbb{C})\), which instead acts on the rows rather than the columns. As modules for \(\text{SO}(n-1, \mathbb{C})\), the decomposition (18) therefore also holds in this case. We now split into two subcases based on the parity of \(n - 1\):

(i) If \(n-1\) is odd then, by Theorem 5.3(i), the modules \(Λ^p\mathbb{C}^{n-1}\) and \(Λ^q\mathbb{C}^{n-1}\) in (18) are irreducible and non-isomorphic unless \(p = q\), hence (19) also holds in this subcase. The result then follows by the same argument as in Case 2, however we will also give a more direct argument which will be useful in subcase (ii). When \(k = 2p \leq \dim_\mathbb{C} U\) is even, the 1-dimensional space of invariants is \((Λ^p\mathbb{C}^{n-1} \otimes Λ^p\mathbb{C}^{n-1})_\text{SO}(n-1, \mathbb{C})\) and is therefore spanned by the extension to \(Λ^p\mathbb{C}^{n-1}\) of the bilinear
form $\beta: \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \to \mathbb{C}$ defining the group $\text{SO}(n-1, \mathbb{C})$; this extension is precisely the $p$-fold tensor power of $\beta \in \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} = (\Lambda^1 \mathbb{C}^{n-1}) \otimes (\Lambda^1 \mathbb{C}^{n-1}) \subset \Lambda^2 U$. Using the notation

$$e_i \leftrightarrow (e_i, 0), \quad \widehat{e}_i \leftrightarrow (0, e_i)$$

to denote the standard basis vectors for the two factors in $U = \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$, we have

$$\beta = \sum_{i=1}^{n-1} (e_i \otimes \widehat{e}_i - \widehat{e}_i \otimes e_i).$$

One then easily sees from Equation (24) in [DOP19] that $\beta$ is annihilated by $\mathfrak{h}^C$, giving an alternative proof of the result in this subcase.

(ii) If $n - 1 = 2l$ is even then Theorem 5.4(ii) implies that, for $p, q \leq n - 1$,

$$\dim_{\mathbb{C}} \text{Hom}_{\text{SO}(n-1, \mathbb{C})}(\Lambda^p \mathbb{C}^{n-1}, \Lambda^q \mathbb{C}^{n-1}) = \begin{cases} 0 & \text{if } p \neq q, \\ 1 & \text{if } p = q \neq l, \\ 2 & \text{if } p = q = l. \end{cases}$$

One then sees immediately from [18] that only the trivial invariants are possible when $k$ is odd, so it suffices to consider the case where $k = 2p$ is even. If $p \neq l$ then the 1-dimensional space of degree $k$ invariants is spanned by the $p$-fold tensor power of $\beta$, by the same argument as in subcase (i). For $p = l$, we pass to the level of Lie algebras and note that the 2-dimensional space of invariants in the representation $\Lambda^l (\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$ is spanned by the $l$-fold tensor power of $\beta$ along with the “split determinant” tensor

$$\det_{t,l} := \sum_{\sigma \in S_{n-1}} \text{sign}(\sigma) e_{\sigma(1)} \otimes \ldots e_{\sigma(l)} \otimes \widehat{e}_{\sigma(l+1)} \otimes \ldots \otimes \widehat{e}_{\sigma(n-1)},$$

where $\widehat{e}_i$ denotes the standard basis vectors coming from the second summand of $U = \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$, as defined in subcase (i). From Equation (24) in [DOP19], the isotropy operator

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sp}(1)^C \subset \mathfrak{h}^C$$

acts on $U$ via $((\alpha, \beta) \in \mathbb{C}^n \times \mathbb{C}^n) \mapsto (0, \alpha)$, i.e. it acts on each row $(a_i, b_i) \cong \mathbb{C}^2$ via left multiplication by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus, the corresponding Lie group element $\exp(E)$ acts on each row via left multiplication by the matrix

$$\exp\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

sending $e_i \mapsto e_i + \widehat{e}_i$ and $\widehat{e}_i \mapsto \widehat{e}_i$. The action of $\exp(E) \in H^C$ on $\det_{t,l}$ is therefore given by

$$\exp(E) \cdot \det_{t,l} = \sum_{\sigma \in S_{n-1}} \text{sign}(\sigma) (e_{\sigma(1)} + \widehat{e}_{\sigma(1)}) \otimes \ldots (e_{\sigma(l)} + \widehat{e}_{\sigma(l)}) \otimes \widehat{e}_{\sigma(l+1)} \otimes \ldots \widehat{e}_{\sigma(n-1)} \neq \det_{t,l}.$$ 

This shows that the space of invariants is generated by $\beta$, hence (19) also holds in this subcase, completing the proof.

Case 4: The Five Exceptional Spaces. These are the five spaces from Theorem 2.8 with $G$ an exceptional Lie group. Following [DOP19], denote the corresponding 3-Sasakian data by

$$\mathfrak{g}^s = \mathfrak{g}_0^s \oplus \mathfrak{g}_1^s, \quad (\mathfrak{g}^s_1)^C \cong \mathbb{C}^2 \otimes U^s, \quad s = 1, 2, 3, 4, 5,$$

and we recall that the $(\mathfrak{h}^s)^C$-modules $U^s$ have been described in terms of highest weight modules on p.24 of [DOP19]. We summarize this information in Table 1.

Using the LiE computer algebra package (http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/), one checks that (19) holds for each $s = 1, 2, 3, 4, 5$, and the result in this case then follows by the same argument as in Case 2.\]
As a corollary, we obtain:

**Theorem 5.8.** If \((M = G/H, g, \xi, \eta, \varphi_1)\) is a simply-connected homogeneous 3-Sasakian space, then the \(\varphi_1\)-(anti-)holomorphic forms are given by

\[
\Lambda_{inv}^0(m^C) = \text{span}_\mathbb{C}\{\omega^k_{1,0}, y_1, \omega^{k,1}_{1,0}\}_{k=0},
\]

where

\[
y_{1,0} := (\xi_2 - i\xi_3), \quad y_{1,1} := y_1, \quad \omega_{1,0} := (\Phi_2|\mathcal{H} - i\Phi_3|\mathcal{H}), \quad \omega_{1,1} := \omega_{1,0}.
\]

**Proof.** Since the isotropy group acts trivially in the vertical directions, we have

\[
\Lambda_{inv}^0(m^C) = \Lambda_{inv}^0(m^C) \otimes \Lambda_{inv}^0(m^C) \cong [1 \otimes \Lambda_{inv}^0(m^C)] \oplus [y_{1,0} \otimes \Lambda_{inv}^0(m^C)],
\]

and the result then follows from Case 2 in the proof of the preceding proposition, since we have seen that \(19\) holds in each case. \(\square\)

Using Theorem 5.8, we are now ready to prove the main result of the paper:

**Theorem 5.9.** For a homogeneous 3-Sasakian manifold \((M = G/H, g, \xi, \eta, \varphi_1)\) of dimension \(4n - 1\), the invariant spinors are given with respect to an adapted basis by

\[
\Sigma_{inv} = \text{span}_\mathbb{C}\{\omega^k, y_1, \omega^{k,1}\}_{k=0},
\]

where \(\omega := \sum_{p=1}^{n-1} y_{2p} \wedge y_{2p+1}\).

**Proof.** It is well-known that the 3-Sasakian structure on \(M\) gives a reduction of the structure group of the tangent bundle to \(\text{Sp}(n-1) \subset \text{SO}(4n-1)\) (Theorem 5 in [Kuo70]). Furthermore, since the 3-Sasakian structure on \(M = G/H\) is assumed to be homogeneous, we have that the image of the isotropy representation \(\iota\) is contained in this reduction:

\(\iota(H) \subset \text{Sp}(n-1) \subset \text{SO}(4n-1)\).

Noting that operators in the symplectic group are traceless, the action of the spin lift of \(\text{Sp}(n-1)\) on \(\Sigma\) is given explicitly in Proposition 4.4 of [AHL22]—it is simply the standard action on differential forms (via exterior powers of the dual of the standard representation). Thus the spin lift of \(\iota(H)\) acts on the spinor module \(\Sigma = \Lambda^* L^*\) via the standard action on \(\varphi_1\)-anti-holomorphic differential forms, i.e. \(\Sigma \cong \Lambda_{C}^0\otimes (M)\) as homogeneous graded vector bundles. The result then follows from Theorem 5.8 by noting that \(y_1 = \frac{1}{\sqrt{2}} y_{0,1}\) and \(\omega = -\frac{1}{2} \omega_{0,1}\). \(\square\)

**Remark 5.10.** The motivation for Theorem 5.9 can be found in Equation (36) and Theorem 4.11 in [AHL22], which proved the result for the case \(S^{4n-1} = Sp(n)/Sp(n-1)\). Indeed, these spheres are in many ways the prototypical examples of homogeneous 3-Sasakian spaces, so it is not surprising that the result holds in general.
In the proof of Theorem 5.9 we relied on the identification $\Sigma \cong \Lambda^{\bullet}_{\operatorname{inv}}(M)$ between invariant $\varphi_1$-anti-holomorphic forms and invariant spinors. A natural question then arises as to the relationship between invariant real differential forms and invariant spinors. The next few results are devoted to the exploration of this topic, and from this point forward we take the Clifford algebra representation associated to an adapted basis, so that the invariant spinors take the form (24).

**Lemma 5.11.** Let $(M = G/H, g)$ be a homogeneous 3-Sasakian manifold of dimension $4n - 1$. Then for any integer $k \geq 0$ we have

\begin{align}
\Phi_0 \cdot \omega^k &= 0, \\
(\Phi_1|_H) \cdot \omega^k &= 2i(2k - n + 1)\omega^k, \\
(\Phi_2|_H) \cdot \omega^k &= 2(\omega^{k+1} - k(n - 1)\omega^{k-1}), \\
(\Phi_3|_H) \cdot \omega^k &= -2i(\omega^{k+1} + k(n - 1)\omega^{k-1}),
\end{align}

where $\omega := \sum_{p=1}^{n-1} y_{2p} \wedge y_{2p+1}$ and we use the convention $\omega^0 = 1$.

**Proof.** These identities follow from a straightforward calculation in the spin representation. As an example, we present here the argument for $\Phi_0 = \sum_{p=1}^{n-1} (e_{4p} \wedge e_{4p+1} - e_{4p+2} \wedge e_{4p+3})$. For any $k \in \mathbb{Z}_{\geq 0}$, one has

\begin{align*}
\Phi_0 \cdot \omega^k &= \sum_{p=1}^{n-1} (e_{4p} \wedge e_{4p+1} - e_{4p+2} \wedge e_{4p+3}) \cdot \omega^k \\
&= i \sum_{p=1}^{n-1} [(x_{2p,1} + y_{2p}) \wedge y_{2p+1} - x_{2p+1,1} \wedge y_{2p+1} + y_{2p+1} \wedge -x_{2p,1}] \omega^k \\
&= i \sum_{p=1}^{n-1} [x_{2p,1} \omega^k - y_{2p} \wedge (x_{2p,1} \wedge \omega^k) - x_{2p+1,1} \omega^k + y_{2p+1} \wedge (x_{2p+1,1} \wedge \omega^k)] \\
&= i \sum_{p=1}^{n-1} [\omega^k - 2ky_{2p} \wedge y_{2p+1} + \omega^{k-1} - \omega^k + 2ky_{2p+1} \wedge (-y_{2p})] \\
&= 0.
\end{align*}

The calculations for $(\Phi_i|_H) \cdot \omega^k$, $i = 1, 2, 3$ are analogous. \hfill \Box

We immediately deduce:

**Corollary 5.12.** For $i \in \{0, 1, 2, 3\}$, let $S_i$ denote the complex span of the spinors $(\Phi_i|_H)^k \cdot 1$ with $k = 1, \ldots, 2n - 1$. We have:

$S_0 = \{0\}$, \hspace{1em} $S_1 = \operatorname{span}_C\{1\}$, \hspace{1em} $S_2 = S_3 = \operatorname{span}_C(\omega^k)_{k=0}^{n-1}$.

**Proof.** The cases $S_0$ and $S_1$ are clear from the preceding lemma. From (22) we note that the Clifford product of the form $\frac{1}{2}\Phi_2|_H$ with the spinor $\omega^k$ is a monic degree $(k + 1)$ polynomial in $\omega$. Together with the base case $(\Phi_2|_H) \cdot 1 = 2\omega$, applying the preceding Lemma inductively shows that $\frac{1}{2^k}(\Phi_2|_H)^k \cdot 1$ is a monic degree $k$ polynomial in $\omega$, and the result for $S_2$ follows. The proof for $S_3$ is analogous. \hfill \Box

This also gives a nice description of the invariant spinors in terms of the invariant real differential forms:

**Theorem 5.13.** The space $\Sigma_{\operatorname{inv}}$ of invariant spinors on a homogeneous 3-Sasakian manifold is spanned by Clifford products of invariant differential forms with the invariant spinor $1 \in \Sigma_{\operatorname{inv}}$. 


Proof. In light of Theorem [5.9] it suffices to show that spinors of the form $\omega^k$ and $y_1 \wedge \omega^k$ can be obtained as linear combinations of Clifford products of invariant differential forms with $1 \in \Sigma_{inv}$. This follows from Corollary [5.12] by noting that $\omega^k \in S_2$ and $y_1 \wedge \omega^k \in \xi_2 \cdot S_2$. \hfill \Box

Remark 5.14. We would like to point out that the results of this section so far also hold in the more general setting of homogeneous $3-(\alpha, \delta)$-Sasaki spaces, which were introduced in [AD20]. The reason for this is that the generalized $3$-$\alpha, \delta$-Sasaki structures coincides, in the case of a compact space, with the notion of $3$-$\alpha, \delta$-Sasaki data. In particular, the isotropy representation of a family of compact homogeneous $3$-$\alpha, \delta$-Sasaki spaces parameterized by $\alpha, \delta > 0$ is isomorphic to the isotropy representation of the corresponding homogeneous $3$-Sasakian space obtained by setting $\alpha = \delta = 1$.

6. The Space of Riemannian Killing Spinors

We conclude the paper with an explicit description of the Riemannian Killing spinors on a homogeneous $3$-Sasakian space:

**Theorem 6.1.** Any Killing spinor on a homogeneous $3$-Sasakian manifold $(M^{4n-1} = G/H, g, \xi_i, \eta_i, \varphi_i)$ is $G$-invariant. If $n \geq 2$ then the space of Killing spinors has a basis given by

$$\psi_k := \omega^{k+1} - i(k + 1)y_1 \wedge \omega^k, \quad -1 \leq k \leq n - 1,$$

where we use the conventions $\omega^{-1} = 0$ and $\omega^0 = 1$. If $n = 1$ then the space of Killing spinors has a basis given by $1, y_1$.

**Proof.** Let $\Lambda, \Lambda^g : m \times m \to m$ denote the Nomizu maps of the canonical and Levi-Civita connections respectively. First we consider the horizontal directions $X \in \mathcal{H}$. From Equation [5] we have $\Lambda(X) = 0$ for all $X \in \mathcal{H}$, and thus any Riemannian Killing spinor $\psi$ satisfies the algebraic equation

$$0 = \tilde{\Lambda}(X) \cdot \psi = \tilde{\Lambda}^g(X) \cdot \psi + \frac{1}{4}(X,T) \cdot \psi = \frac{1}{2}X \cdot \psi + \frac{1}{2} \sum_{i=1}^{3} \xi_i \cdot \varphi_i(X) \cdot \psi \quad \text{for all } X \in \mathcal{H}. \quad (25)$$

Assume first that $n \geq 2$. Calculating in an adapted basis $\{e_{4p}, e_{4p+1}, e_{4p+2}, e_{4p+3}\}$, we find:

$$\frac{1}{2}e_{4p} \cdot \omega^k + \frac{1}{2} \sum_{i=1}^{3} \xi_i \cdot \varphi_i(e_{4p}) \cdot \omega^k = \frac{1}{2} \left( e_{4p} \cdot \omega^k + \xi_1 \cdot e_{4p+1} \cdot \omega^k + \xi_2 \cdot e_{4p+2} \cdot \omega^k + \xi_3 \cdot e_{4p+3} \cdot \omega^k \right) = \frac{1}{2} \left( 2ix_{2p} \cdot \omega^k + \xi_1 \cdot e_{4p+1} \cdot \omega^k + \xi_2 \cdot e_{4p+2} \cdot \omega^k + \xi_3 \cdot e_{4p+3} \cdot \omega^k \right) = ix_{2p} \cdot \omega^k - \frac{1}{2} \left( y_1 \wedge (y_{2p+1} \wedge x_{2p+1}) \omega^k + y_1 \wedge (y_{2p+1} \wedge -x_{2p+1}) \omega^k \right) = ix_{2p} \cdot \omega^k - y_1 \wedge (x_{2p+1} \omega^k)$$

and similarly,

$$\frac{1}{2}e_{4p} \cdot (y_1 \wedge \omega^k) + \frac{1}{2} \sum_{i=1}^{3} \xi_i \cdot \varphi_i(e_{4p}) \cdot (y_1 \wedge \omega^k) = iy_{2p} \wedge (y_1 \wedge \omega^k) + y_{2p+1} \wedge \omega^k.$$

Thus, for an invariant spinor

$$\psi := \sum_{k=0}^{n-1} \lambda_k \omega^k + \sum_{k=0}^{n-1} \lambda'_k (y_1 \wedge \omega^k),$$

we have

$$\frac{1}{2}e_{4p} \cdot \psi + \frac{1}{2} \sum_{i=1}^{3} \xi_i \cdot \varphi_i(e_{4p}) \cdot \psi = \sum_{k=0}^{n-1} \lambda_k [ix_{2p} \cdot \omega^k - y_1 \wedge (x_{2p+1} \omega^k)] + \sum_{k=0}^{n-1} \lambda'_k [y_{2p+1} \wedge \omega^k + iy_{2p} \wedge (y_1 \wedge \omega^k)]. \quad (26)$$
For \( k = l \), the degrees of the four terms are \( 2l - 1, 2l, 2l + 1 \) and \( 2l + 2 \) respectively. Considering separately the even and odd degree parts of (26), we are seeking solutions of
\[
\sum_{k=0}^{n-1} [i \lambda_k x_{2p} \omega^k + \lambda_k y_{2p+1} \wedge \omega^k] = 0 = \sum_{k=0}^{n-1} [-\lambda_k y_1 \wedge (x_{2p+1} \omega^k) + i \lambda_k y_{2p} \wedge (y_1 \wedge \omega^k)],
\]
or equivalently, solutions of the linear system of equations
\[
\lambda'_k = -i(k+1) \lambda_{k+1}, \quad -1 \leq k \leq n-1.
\]
This gives \( n + 1 \) linearly independent spinors
\[
\psi_k := \omega^{k+1} - i(k+1)y_1 \wedge \omega^k, \quad -1 \leq k \leq n-1,
\]
and a straightforward calculation of the other horizontal derivatives (by substituting \( X = e_{4p+1}, e_{4p+2}, \) and \( e_{4p+3} \) into (23)) shows that these spinors satisfy the Killing equation in the horizontal directions.

In the vertical directions, one sees from Proposition 4.2.2 in [ADS21] that the Nomizu map for the Levi-Civita connection is given by
\[
\Lambda^g(\xi_i^j) = \xi_j^k \wedge \xi_k
\]
for any even permutation \((i, j, k)\) of \((1, 2, 3)\). Considering the spin lifts \( \tilde{\Lambda}^g(\xi_i^j) = \frac{1}{2} \xi_j^k \wedge \xi_k \), another straightforward calculation in the spin representation shows that any spinor of the form \( \omega^k \) or \( y_1 \wedge \omega^k \) satisfies the Killing equation in the vertical directions, and we conclude that the \( \psi_k \) are Killing spinors in the case \( n \geq 2 \).

Assuming now that \( n = 1 \), the horizontal distribution is trivial, and consequently Equation (25) does not apply. In this dimension the spinor bundle has complex dimension equal to \( n = 1 + 1 = 2 \), hence it is spanned by \( 1, y_1 \), and the above argument for the vertical directions shows that these are Killing spinors. The dimension of the space of Killing spinors on a 3-Sasakian manifold \((M^{4n-1}, g)\) is equal to \( n + 1 \) (see [Bar93]), and the result follows.

Finally, we explore which of the Killing spinors from the preceding theorem induce the invariant 3-Sasakian structure on \((M = G/H, g)\) via the construction in Section 3. By Theorem 3.9 and the proof of Theorem 3.7 we see that in order to recover the Sasakian structure \((\xi_i, \eta_i, \varphi_i)\) it suffices to find a Killing spinor \( \Psi \) such that \( \Psi' := -\xi^i \cdot \Psi \) is also a Killing spinor. Thus we consider the subbundles \( \mathcal{E}_i \) of the spinor bundle consisting of spinors with this property:
\[
\mathcal{E}_i := \{ \Psi \in \kappa(M, g) : \Psi' := -\xi^i \cdot \Psi \in \kappa(M, g) \}, \quad i = 1, 2, 3,
\]
where \( \kappa(M, g) := \text{span}_\mathbb{C}\{\psi_k\}_{k=-1}^{n-1} \) denotes the space of Killing spinors. In fact, it turns out that these subbundles coincide with the \( \mathcal{E}_i^- \) in the homogeneous setting, as the following proposition shows:

**Proposition 6.2.** If \((M = G/H, g)\) is a homogeneous 3-Sasakian space, then \( \mathcal{E}_i = \mathcal{E}_i^- \), \( i = 1, 2, 3 \). Furthermore, bases of these spaces are given by
\[
\mathcal{E}_i = \text{span}_\mathbb{C}\{\Psi_{\mathcal{E}_i,0}, \Psi_{\mathcal{E}_i,1}\},
\]
where
\[
\Psi_{\mathcal{E}_i,0} := 1, \quad \Psi_{\mathcal{E}_i,1} := y_1 \wedge \omega^{n-1}, \quad \Psi_{\mathcal{E}_2,0} := \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} \psi_{2k},
\]
\[
\Psi_{\mathcal{E}_2,1} := \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^k}{(2k)!} \psi_{2k-1}, \quad \Psi_{\mathcal{E}_3,0} := \sum_{k=0}^{n-1} \frac{1}{(2k+1)!} \psi_{2k}, \quad \Psi_{\mathcal{E}_3,1} := \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{(2k)!} \psi_{2k-1}.
\]

**Proof.** We have already seen in the proof of Theorem 3.7 that \( \mathcal{E}_i^- \subseteq \mathcal{E}_i \). Thus in order to show that \( \mathcal{E}_i^- = \mathcal{E}_i \) it suffices to show that \( \text{rank}(\mathcal{E}_i) \leq 2 \). To find elements \( \Psi \in \mathcal{E}_i \), we write \( \Psi = \sum_{k=-1}^{n-1} \lambda_k \psi_k \).
as a linear combination of our basis of Killing spinors and seek to determine for which values of \(\lambda_1, \ldots, \lambda_{n-1}\) there exist \(\Theta_{n-1}, \ldots, \Theta_{n-1}\) \(\in\mathbb{C}\) satisfying

\[\xi_i \cdot \Psi = \sum_{k=1}^{n-1} \Theta_k \psi_k.\]  

We treat the cases \(i = 1, 2, 3\) one at a time:

(i) \(i = 1\): First, we note that

\[\xi_1 \cdot \psi_k = i\omega^{k+1} - (k+1)y_1 \wedge \omega^k.\]

Substituting this into (27), we are looking for solutions of

\[\sum_{k=1}^{n-1} \lambda_k \left[i\omega^{k+1} - (k+1)y_1 \wedge \omega^k\right] = \sum_{k=1}^{n-1} \Theta_k \left[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k\right],\]

or equivalently, solutions of the linear equations

\[i\lambda_{n-1} = \Theta_{n-1}, \quad \lambda_{n-1} = i\Theta_{n-1}, \quad i\lambda_k = \Theta_k, \quad \lambda_k = i\Theta_k, \quad \text{for } k = 0, \ldots, n-2.\]

Solutions of this system of equations necessarily have \(\lambda_k = \Theta_k = 0\) for all \(k \neq -1, n-1\). Therefore \(\mathcal{E}_1\) has rank \(\leq 2\) and is contained in the span of the spinors \(1, y_1 \wedge \omega^{n-1}\), completing the proof in this case.

(ii) \(i = 2\): Proceeding similarly as in the previous case, we first note that

\[\xi_2 \cdot \psi_k = iy_1 \wedge \omega^{k+1} + (k+1)\omega^k.\]

Substituting this into (27), we have

\[\sum_{k=1}^{n-1} \lambda_k \left[iy_1 \wedge \omega^{k+1} + (k+1)\omega^k\right] = \sum_{k=1}^{n-1} \Theta_k \left[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k\right],\]

or equivalently, the linear system

\[(k+1)\lambda_k = \Theta_{k-1}, \quad \lambda_{k-1} = -(k+1)\Theta_k, \quad \text{for } k = 0, \ldots, n-1.\]

These equations give rise to the recursive relation

\[\lambda_k = \frac{1}{k+1} \Theta_{k-1} = \frac{-1}{k(k+1)} \lambda_{k-2},\]

whose space of solutions lies within the span of the spinors \(\Psi_{\mathcal{E}_{2,0}}, \Psi_{\mathcal{E}_{2,1}}\). In particular \(\text{rank}(\mathcal{E}_2) \leq 2\), completing the proof in this case.

(iii) \(i = 3\): Similarly to the previous two cases, we first note that

\[\xi_3 \cdot \psi_k = y_1 \wedge \omega^{k+1} + i(k+1)\omega^k.\]

Substituting this into (27) gives

\[\sum_{k=1}^{n-1} \lambda_k \left[y_1 \wedge \omega^{k+1} + i(k+1)\omega^k\right] = \sum_{k=1}^{n-1} \Theta_k \left[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k\right],\]

which is equivalent to the linear system

\[i(k+1)\lambda_k = \Theta_{k-1}, \quad \lambda_{k-1} = -(k+1)\Theta_k, \quad \text{for } k = 0, \ldots, n-1.\]

These give the recursive relation

\[\lambda_k = \frac{-i}{k+1} \Theta_{k-1} = \frac{1}{k(k+1)} \lambda_{k-2},\]

whose space of solutions lies within the span of \(\Psi_{\mathcal{E}_{3,0}}, \Psi_{\mathcal{E}_{3,1}}\). In particular this shows that \(\text{rank}(\mathcal{E}_3) \leq 2\), completing the proof.
As noted above, it immediately follows that these spinors recover the homogeneous 3-Sasakian structure via the construction in Section 3, giving a full picture of this construction in the homogeneous 3-Sasakian setting:

**Theorem 6.3.** If \((M = G/H, g, \xi_i, \eta_i, \varphi_i)\) is a homogeneous 3-Sasakian space then, for each \(i \in \{1, 2, 3\}\), the Sasakian structure \((\xi_i, \eta_i, \varphi_i)\) arises from the spinors \(\Psi_i := \Psi_{E_i,0}\) and \(\Psi_i' := -\xi_i \cdot \Psi_{E_i,0}\) (or \(\Psi_i := \Psi_{E_i,1}\) and \(\Psi_i' := -\xi_i \cdot \Psi_{E_i,1}\)) via the construction in Section 3.

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