INVERSE SCATTERING BY THE STABILITY INDEX METHOD

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ABSTRACT. A novel numerical method for solving inverse scattering problem with fixed-energy data is proposed. The method contains a new important concept: the stability index of the inversion problem. This is a number, computed from the data, which shows how stable the inversion is. If this index is small, then the inversion provides a set of potentials which differ so little, that practically one can represent this set by one potential. If this index is larger than some threshold, then practically one concludes that with the given data the inversion is unstable and the potential cannot be identified uniquely from the data. Inversion of the fixed-energy phase shifts for several model potentials is considered. The results show practical efficiency of the proposed method. The method is of general nature and is applicable to a very wide variety of the inverse problems.

1. INTRODUCTION

Let \( q(x), x \in \mathbb{R}^3 \), be a real-valued potential with compact support. Let \( R > 0 \) be a number such that \( q(x) = 0 \) for \( |x| > R \). We also assume that \( q \in L^2(B_R), \ B_R = \{ x : |x| \leq R, x \in \mathbb{R}^3 \}. \) Let \( S^2 \) be the unit sphere, and \( \alpha \in S^2 \). For a given energy \( k > 0 \) the scattering solution \( \psi(x, \alpha) \) is defined as the solution of

\[
\Delta \psi + k^2 \psi - q(x) \psi = 0, \quad x \in \mathbb{R}^3
\]

satisfying the following asymptotic condition at infinity:

\[
\psi = \psi_0 + v, \quad \psi_0 := e^{ik\alpha \cdot x}, \quad \alpha \in S^2,
\]

\[
\lim_{r \to \infty} \int_{|x| = r} \left| \frac{\partial v}{\partial r} - ikv \right|^2 ds = 0.
\]

It can be shown, that

\[
\psi(x, \alpha) = \psi_0 + A(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad \text{as} \quad r \to \infty, \quad \frac{x}{r} = \alpha' \quad r = |x|.
\]

The function \( A(\alpha', \alpha, k) \) is called the scattering amplitude, \( \alpha \) and \( \alpha' \) are the directions of the incident and scattered waves, and \( k^2 \) is the energy, see [10], [12].

For spherically symmetric scatterers \( q(x) = q(r) \) the scattering amplitude satisfies \( A(\alpha', \alpha, k) = A(\alpha', \alpha', k) \). The converse is established in [14]. Following [14], the scattering amplitude for \( q = q(r) \) can be written as

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\[ A(\alpha', \alpha, k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l(k) Y_{lm}(\alpha') Y_{lm}(\alpha), \]

where \( Y_{lm} \) are the spherical harmonics, normalized in \( L^2(S^2) \), and the bar denotes the complex conjugate.

The fixed-energy phase shifts \(-\pi < \delta_l \leq \pi \) (\( \delta_l = \delta(l, k), k > 0 \) is fixed) are related to \( A_l(k) \) (see e.g., [19]) by the formula:

\[ A_l(k) = \frac{4\pi}{k} e^{i\delta_l} \sin(\delta_l). \]

In section 2 we give, following [1], formulas for calculating fixed-energy phase shifts for piecewise-constant compactly supported potentials. Let us denote this class of potentials by PC. Since an arbitrary integrable potential can be approximated with the prescribed accuracy by a PC potential, the class PC is sufficiently large for practical purposes.

In sections 3 and 4 a novel minimization method, the stability index method, is described. Our inversion procedure is based on this method. An important novel feature of this method, which seems not to have been present in other methods, is the concept of the stability index, which is a number characterizing the stability of the numerical inversion.

Several parameter-fitting procedures were proposed for calculating the potentials from the fixed-energy phase shifts, (by Fiedeldey, Lipperheide, Hooshyar and Razavy, Ioannides and Mackintosh, Newton, Sabatier, May and Scheid, Ramm, and others). These works are referenced and their results are described in [5] and [10]. Recent works [6]-[9] and [18]-[20] present new numerical methods for solving this problem.

In section 5 numerical examples of the inversion of the fixed-energy shifts are given for three potentials. Physical motivation for the choice of these potentials is given and directions for future research are suggested.

Section 6 contains brief conclusions.

2. Phase Shifts for Piecewise-Constant Potentials

Phase shifts for a spherically symmetric potential can be computed by a variety of methods, e.g. by a variable phase method described in [3]. The computation involves solving a nonlinear ODE for each phase shift. However, if the potential is compactly supported and piecewise-constant, then a much simpler method described in [1] can be used. It is summarized below. Since the set of compactly supported and piecewise-constant potentials is dense in the set \( L^1(0, \infty) \) potentials, it is quite reasonable to look for an approximate solution to the inverse scattering problem in the class of piecewise-constant compactly supported potentials.

Consider a finite set of points \( 0 = r_0 < r_1 < r_2 < \cdots < r_N = R \) and a piecewise-constant potential

\[ q(r) = q_i, \text{ on } [r_{i-1}, r_i) \text{ for } i = 1, \ldots, N, \text{ and } q = 0 \text{ for } r \geq R. \]

Denote

\[ \kappa_i^2 := k^2 - q_i, \]
where \( i = 1, \ldots, N, \) and \( k \) is some fixed positive number. Consider the following problem for the radial Schrödinger equation:

\[
(2.3) \quad \frac{d^2 \varphi_i}{dr^2} + \left( k^2 - \frac{l(l+1)}{r^2} \right) \varphi_i = q \varphi_i, \quad \lim_{r \to 0} (2l+1)!! r^{-l-1} \varphi_i(r) = 1,
\]

which we rewrite as:

\[
(2.4) \quad \frac{d^2 \varphi_i}{dr^2} + \left( \kappa_i^2 - \frac{l(l+1)}{r^2} \right) \varphi_i = 0
\]

on the interval \( r_{i-1} \leq r < r_i \). On \([r_{i-1}, r_i)\) one has the following general solution of (2.4)

\[
(2.5) \quad \varphi_i(r) = A_i j_l(\kappa_i r) + B_i n_l(\kappa_i r),
\]

where

\[
(2.6) \quad j_l(kr) = \frac{\sqrt{\pi kr}}{2} J_{l+1/2}(kr), \quad n_l(kr) = \frac{\sqrt{\pi kr}}{2} N_{l+1/2}(kr)
\]

and \( J_l, \ N_l \) are the Bessel and Neumann functions.

If \( \kappa_i^2 = k^2 - q_i \leq 0 \) for some \( i \), then the solution \( \varphi_i(r) \) in (2.4) can be expressed through some powers of \( r \) (for \( \kappa_i = 0 \)) or the modified Bessel and Neumann functions, and our approach is still valid with the appropriate changes.

From the regularity of \( \varphi_i \) at zero one gets \( B_i = 0 \). Denote \( x_i = B_i/A_i \), then \( x_1 = 0 \). We are looking for a continuously differentiable solution \( \varphi_i \). Thus, the following interface conditions hold:

\[
(2.7) \quad \frac{\kappa_i}{\kappa_{i+1}} [A_i j_l(\kappa_i r_i) + B_i n_l(\kappa_i r_i)] = A_{i+1} j_l(\kappa_{i+1} r_i) + B_{i+1} n_l(\kappa_{i+1} r_i).
\]

Therefore

\[
(2.8) \quad \begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = \frac{1}{\kappa_{i+1}} \begin{pmatrix} \alpha^1_{11} & \alpha^1_{12} \\ \alpha^2_{21} & \alpha^2_{22} \end{pmatrix} \begin{pmatrix} A_i \\ B_i \end{pmatrix},
\]

where the entries of the matrix \( \alpha^i \) can be written explicitly (see [1] for details).

Thus

\[
(2.9) \quad x_{i+1} = \frac{\alpha^1_{21} + \alpha^2_{22} x_i}{\alpha^1_{11} + \alpha^1_{12} x_i}, \quad x_i := \frac{B_i}{A_i}
\]

The phase shift \( \delta(k, l) \) is defined by

\[
(2.10) \quad \varphi_l(r) \sim \frac{|F(k, l)|}{k^{l+1}} \sin(kr - \frac{\pi l}{2} + \delta(k, l)) \quad r \to \infty,
\]

where \( F(k, l) \) is the Jost function. For \( r > R \) one has:

\[
(2.11) \quad \varphi_l(r) = A_{N+1} j_l(kr) + B_{N+1} n_l(kr),
\]

and \( j_l(kr) \sim \sin(kr - l\pi/2), \ n_l(kr) \sim -\cos(kr - l\pi/2), \) \( r \to \infty \), one gets:

\[
(2.12) \quad \tan \delta(k, l) = -\frac{B_{N+1}}{A_{N+1}} = -x_{N+1}.
\]
Finally, the phase shifts of the potential \( q(r) \) are calculated by the formula:

\[
\delta(k, l) = - \arctan x_{N+1}.
\]  

(2.13)

Let \( q_0(r) \) be a spherically symmetric piecewise-constant potential. Let \( \{ \delta(k, l) \}_{l=1}^{N} \) be the set of its phase shifts for a fixed \( k > 0 \) and a sufficiently large \( N \). Let \( q(r) \) be another potential, and let \( \{ \delta(k, l) \}_{l=1}^{N} \) be the set of its phase shifts.

The best fit to data function \( \Phi(q, k) \) is defined by

\[
\Phi(q, k) = \frac{\sum_{l=1}^{N} |\delta(k, l) - \tilde{\delta}(k, l)|^2}{\sum_{l=1}^{N} |\tilde{\delta}(k, l)|^2}.
\]  

(2.14)

The phase shifts are known to decay rapidly with \( l \), see [17]. Thus, for sufficiently large \( N \), the function \( \Phi \) is practically the same as the one which would use all the shifts in (2.14). The inverse problem of the reconstruction of the potential from its fixed-energy phase shifts is reduced to the minimization of the objective function \( \Phi \) over an appropriate admissible set.

### 3. Stability Index Minimization Method

Let the minimization problem be

\[
\min \{ \Phi(q) : q \in A_{adm} \}
\]  

(3.1)

Let \( \tilde{q}_0 \) be its global minimizer. Typically, the structure of the objective function \( \Phi \) is quite complicated: this function may have many local minima. Moreover, the objective function in a neighborhood of minima can be nearly flat resulting in large minimizing sets defined by

\[
S_\epsilon = \{ q \in A_{adm} : \Phi(q) < \Phi(\tilde{q}_0) + \epsilon \}
\]  

(3.2)

for an \( \epsilon > 0 \).

Given an \( \epsilon > 0 \), let \( D_\epsilon \) be the diameter of the minimizing set \( S_\epsilon \), which we call the **Stability Index** \( D \) of the minimization problem (3.1). The usage of the letter \( D \) for this index is explained in formula (3.6) below.

One would expect to obtain stable identification for minimization problems with small stability indices. However, the minimization problems with large stability indices have distinct minimizers with practically the same values of the objective function. If no additional information is known, one has an uncertainty of the minimizer’s choice. The stability index provides a quantitative measure of this uncertainty or instability of the minimization.

The basic idea of the Stability Index minimization method is to iteratively estimate normalized stability indices of a minimization problem, and, based on this information, to conclude if the method has achieved a stable minimum.

A particular implementation of the Stability Index method used here employs a Hybrid Stochastic-Deterministic (HSD) approach. The stochastic part explores the entire admissible set, while the deterministic local minimization finds the best fit in a neighborhood of the chosen in the stochastic part of the search initial guesses. The HSD approach has proved to be successful for a variety of problems in inverse quantum scattering (see [7, 18]) as well as in other applications (see [8, 6]). A somewhat different implementation of the Stability Index Method is described in [9].

We seek the potentials \( q(r) \) in the class of piecewise-constant, spherically symmetric real-valued functions. Let the admissible set be

\[
A_{adm} \subset \{(r_1, r_2, \ldots, r_M, q_1, q_2, \ldots, q_M) : 0 \leq r_1 \leq R, \ q_{\text{low}} \leq q_m \leq q_{\text{high}} \}.
\]  

(3.3)
where the bounds $q_{low}$ and $q_{high}$ for the potentials, as well as the bound $M$ on the expected number of layers are assumed to be known.

A configuration $(r_1, r_2, \ldots, r_M, q_1, q_2, \ldots, q_M)$ corresponds to the potential

\[ q(r) = q_m, \quad \text{for} \quad r_{m-1} \leq r < r_m, \quad 1 \leq m \leq M, \tag{3.4} \]

where $r_0 = 0$ and $q(r) = 0$ for $r \geq r_M = R$.

Note, that the admissible configurations must also satisfy

\[ r_1 \leq r_2 \leq r_3 \leq \cdots \leq r_M. \tag{3.5} \]

First we describe the global (stochastic) part of the algorithm, which can be called the Iterative Reduced Random Search (IRRS) method. This description is followed by its iterative version, and a Local Minimization Method (LMM) incorporating a Reduction procedure.

Let a batch $H$ of $L$ trial points be generated in $A_{adm}$ using a uniformly distributed random variable. In our case $A_{adm}$ is a box in $\mathbb{R}^{2M}$. The uniform random variable is called $2M$ times to produce a point (configuration representing a potential) in this box (after the appropriate rescaling in each dimension). Finally, the obtained values of $r_i$ are rearranged in the ascending order to satisfy (3.5).

A certain fixed fraction $\gamma$ of the original batch of $L$ points is used to proceed with the local searches. Typically, $L = 5000$ and $\gamma = 0.01$. This reduced sample $H_{red}$ of $\gamma L$ points is chosen to contain the points with the smallest $\gamma L$ values of $\Phi$ among the original batch $H$. The local searches (the LMM procedure) are started from every point in this reduced sample $H_{red}$. This way only the points that seem to be in a neighborhood of the global minimum are used for an expensive local minimization, and the computational time is not wasted on less promising candidates.

Let $H_{min}$ be the $\gamma L$ points obtained as the result of the local minimizations ($\gamma L = 50$ in our computations). Let $S_{min}$ be the subset of $H_{min}$ containing points $\{p_i\}$ with the smallest $\nu \gamma L$ (0 $< \nu < 1$, we used $\nu = 0.16$) values in $H_{min}$. We call $S_{min}$ the minimizing set. The choice of $\nu$ determines a representative sample of global minimizers. If all these minimizers are close to each other, then the objective function $\Phi$ is not flat near the global minimum. That is, the method identifies the minimum consistently. To define this consistency in quantitative terms, let $\|\cdot\|$ be a norm in the admissible set.

Let

\[ D = diam(S_{min}) = \max\{\|p_i - p_j\|/d_{av} : p_i, p_j \in S_{min}\}, \tag{3.6} \]

where $d_{av}$ is the average norm of the elements in $H_{min}$. The normalization by $d_{av}$ is introduced to provide comparable results for different potentials. Thus $D$ is an estimate for the (normalized) Stability Index of the minimization problem. The identification is considered to be stable if the Stability Index $D < \epsilon$. Otherwise, another batch of trial points is generated, and the process is repeated as follows.

**Iterative Reduced Random Search (IRRS).** (at the $j$–th iteration).

1. Fix $0 < \gamma, \nu < 1$, $\beta > 1$, $\epsilon > 0$ and $j_{max}$.
2. Generate another batch $H^j$ of $L$ trial points (configurations) in $A_{adm}$ using a uniform random distribution.
3. Reduce $H^j$ to the reduced sample $H^j_{red}$ of $\gamma L$ points by selecting the points in $H^j$ with the smallest $\gamma L$ values of $\Phi$.
4. Apply the Local Minimization Method (LMM) starting it at each of the $\gamma L$ points in $H^j_{red}$, and obtain the set $H^j_{min}$ consisting of the $\gamma L$ minimizers.
4. Combine $H^j_{\text{min}}$ with $H^{j-1}_{\text{min}}$ obtained at the previous iteration. Let $S^j_{\text{min}}$ be the set of $\nu \gamma L$ points from $H^j_{\text{min}} \cup H^{j-1}_{\text{min}}$ with the smallest values of $\Phi$. (Use $H^1_{\text{min}}$ for $j = 1$).

5. Compute the Stability Index (diameter) $D^j$ of $S^j_{\text{min}}$ by $D^j = \max \{ \| p_i - p_k \| / d_{av} : p_i, p_k \in S^j_{\text{min}} \}$, where $d_{av} = \sum \| p_k \| / \gamma L$, $p_k \in H^1_{\text{min}}$ is the average norm of the elements of $H^1_{\text{min}}$. (Thus, $d_{av}$ is computed only once at $j = 1$ and this value is used for all subsequent iterations).

6. Stopping criterion.
   
   Let $p \in S^j_{\text{min}}$ be the point with the smallest value of $\Phi$ in $S^j_{\text{min}}$ (the global minimizer).
   
   If $D^j \leq \epsilon$, then stop. The global minimum is $p$. The minimization is stable.
   
   If $D^j > \epsilon$ and $\Phi(q) \leq \beta \Phi(p) : q \in S^j_{\text{min}}$, then stop. The minimization is unstable. The Stability Index $D^j$ is the measure of the instability of the minimization.
   
   Otherwise, return to step 1 and do another iteration, unless the maximum number of iterations $j_{\text{max}}$ is exceeded.

We used $\beta = 1.1$, $\epsilon = 0.02$ and $j_{\text{max}} = 30$. The choice of these and other parameters ($L = 5000$, $\gamma = 0.01$, $\nu = 0.16$ $\epsilon_r = 0.1$ (used in LMM)) is dictated by their meaning in the algorithm and the comparative performance of the program at their different values. As usual, some adjustment of the parameters, stopping criteria, etc., is needed to achieve the optimal performance of the algorithm.

4. Local Minimization Method

The Hybrid Stochastic Deterministic Method couples the Stochastic part described in the previous section with a deterministic Local Minimization Method. Numerical experiments show that the objective function $\Phi$ is relatively well behaved in this problem: while it contains many local minima and, at some points, $\Phi$ is not differentiable, standard minimization methods work well here. A Newton-type method for the minimization of $\Phi$ is described in [1]. We have chosen to use a variation of Powell’s minimization method which does not require the computation of the derivatives of the objective function. Such method needs a minimization routine for a one-dimensional minimization of $\Phi$, which we do using a Bisection or a Golden Rule method. See [3] or [7] for a complete description of our method.

Now we can describe our Basic Local Minimization Method in $\mathbb{R}^{2M}$, which is a modification of Powell’s minimization method [3]. It is assumed here that the starting position (configuration) $Q_0 \in A_{\text{adm}}$ is supplied by the procedure LMM (see below), and the entry to LMM is provided by the global minimization part (IRRS).

**Basic Local Minimization Method.**

1. Choose the set of directions $u_i$, $i = 1, 2, \ldots, 2M$, to be the standard basis in $\mathbb{R}^{2M}$

   $$u_i = (0, 0, \ldots, 1, \ldots, 0),$$

   where 1 is in the $i$-th place.

2. Save your starting configuration supplied by LMM as $Q_0$.

3. For each $i = 1, \ldots, 2M$ move from $Q_0$ along the line defined by $u_i$ and find the point of minimum $Q^i_t$. This defines $2M$ temporary points of minima.

4. Re-index the directions $u_i$, so that (for the new indices) $\Phi(Q^1_t) \leq \Phi(Q^2_t) \leq \ldots, \Phi(Q^t_{2M}) \leq \Phi(Q_0)$.

5. For $i = 1, \ldots, 2M$ move from $Q_{i-1}$ along the direction $u_i$ and find the point of minimum $Q_i$.

6. Set $v = Q_{2M} - Q_0$.

7. Move from $Q_0$ along the direction $v$ and find the minimum. Call it $Q_0$ again. It replaces $Q_0$ from step 2.

8. Repeat the above steps until a stopping criterion is satisfied.
Note, that we use the temporary points of minima $Q_i$ only to rearrange the initial directions $u_i$ in a different order. The stopping criterion is the same as the one in [11, Subroutine Powell].

Still another refinement of the local phase is necessary to produce a successful minimization. The admissible set $A_{adm}$, see (3.3)-(3.5), belongs to a $2M$ dimensional minimization space $\mathbb{R}^{2M}$. The dimension $2M$ of this space is chosen a priori to be larger than $2N$, where $N$ is the number of layers in the original potential. We have chosen $M = 2$ in our numerical experiments. However, since the sought potential may have fewer than $M$ layers, we found that conducting searches in lower-dimensional subspaces of $\mathbb{R}^{2M}$ is essential for the local minimization phase. A variation of the following "reduction" procedure has also been found to be necessary in [8] for the search of small subsurface objects, and in [9] for the identification of multilayered scatterers.

If two adjacent layers in a potential have values $v_{i-1}$ and $v_i$ and the objective function $\Phi$ is not changed much when both layers are assigned the same value $v_i$ (or $v_{i-1}$), then these two layers can be replaced with just one layer occupying their place. The change in $\Phi$ is controlled by the parameter $\epsilon_r$. We used $\epsilon_r = 0.1$. This value, found from numerical experiments, seems to provide the most consistent identification. The minimization problem becomes constrained to a lower dimensional subspace of $\mathbb{R}^{2M}$ and the local minimization is done in this subspace.

**Reduction Procedure.** Let $\epsilon_r$ be a positive number.

1. Save your starting configuration $Q_0 = (r_1, r_2, \ldots, r_M, v_1, v_2, \ldots, v_M) \in A_{adm}$ and the value $\Phi(Q_0)$. Let the $(M+1)$-st layer be $L_{M+1} = \{r_M \leq |x| \leq R\}$ and $v_{M+1} = 0$.

2. Let $2 \leq i \leq M + 1$. Replace $v_{i-1}$ in the layer $L_{i-1}$ by $v_i$. This defines a new configuration $Q_i^d$, where the layers $L_{i-1}$ and $L_i$ are replaced with one new layer. Here $d$ stands for the downward adjustment. Compute $\Phi(Q_i^d)$ and the difference $c_i^d = |\Phi(Q_0) - \Phi(Q_i^d)|$. Repeat for each layer in the original configuration $Q_0$.

3. Let $1 \leq i \leq M$. Replace $v_{i+1}$ in the layer $L_{i+1}$ by $v_i$. This defines a new configuration $Q_i^u$, where the layers $L_i$ and $L_{i+1}$ are replaced with one new layer. Here $u$ stands for the upward adjustment. Compute $\Phi(Q_i^u)$ and the difference $c_i^u = |\Phi(Q_0) - \Phi(Q_i^u)|$. Repeat for each layer in the original configuration $Q_0$.

4. Find the smallest among the numbers $c_i^d$ and $c_i^u$. If this number is less than $\epsilon_r \Phi(Q_0)$, then implement the adjustment that produced this number. The resulting new configuration has one less layer than the original configuration $Q_0$.

5. Repeat the above steps until no further reduction in the number of layers is occurring.

Note, that an application of the Reduction Procedure may or may not result in the actual reduction of the number of layers.

Finally, the entire Local Minimization Method (LMM) consists of the following:

**Local Minimization Method (LMM).**

1. Let your starting configuration supplied by IRRS be $Q_0 = (r_1, r_2, \ldots, r_M, v_1, v_2, \ldots, v_M) \in A_{adm}$.

2. Apply the Reduction Procedure to $Q_0$, and obtain a reduced configuration $Q_0'$ containing $M'$ layers.

3. Apply the Basic Minimization Method in $A_{adm} \cap \mathbb{R}^{2M'}$ with the starting configuration $Q_0'$, and obtain a configuration $Q_1$.

4. Apply the Reduction Procedure to $Q_1$, and obtain a final reduced configuration $Q_i$.

As we have already mentioned, LMM is used as the local phase of the global minimization.

### 5. Numerical Results

We studied the performance of the algorithm for 3 different potentials $q_i(r), \ i = 1, 2, 3$ chosen from the physical considerations.
In nuclear physics one measures the length in units of \( \text{fm} = 10^{-15}\text{m} \) and the quantity \( q_3 \) in units of \( 1/\text{fm}^2 \), and the wave number in units of \( 1/\text{fm} \). The physical potential and incident energy are given by \( V(r) = \frac{\hbar^2}{2\mu} q_3(r) \) and \( E = \frac{k^2\hbar^2}{2\mu} \), respectively. Here \( \hbar = \frac{\hbar}{2\pi} \), \( \hbar = 6.62510^{-27} \text{ erg}\cdot\text{s} \) is the Planck constant, \( \hbar c = 197.32 \text{ MeV}\cdot\text{fm} \), \( c = 3 \cdot 10^8 \text{ m/sec} \) is the velocity of light, and \( \mu \) is the mass of a neutron. By choosing the mass \( \mu \) to be equal to the mass of a neutron \( \mu = 939.6 \text{ MeV}/c^2 \), the potential and energy have the values of \( V(r) = -207.2 \text{ MeV} \) for \( 0 \leq r < 8 \text{ fm} \) and \( E(k = 1/\text{fm}) = 20.72 \text{ MeV} \). In atomic physics one uses atomic units with the Bohr radius \( a_0 = 0.529 \cdot 10^{-10} \text{m} \) as the unit of length. Here, \( r, k \) and \( q_3 \) are measured in units of \( a_0 \), \( 1/a_0 \) and \( 1/a_0^2 \), respectively. By assuming a scattering of an electron with mass \( m \) and energy as follows: \( V(r) = -136 \text{ eV} \) for \( 0 \leq r < 8a_0 = 4.23 \cdot 10^{-10} \text{m} \) and \( E(k = 1/a_0) = 13.6 \text{ eV} \). These numbers give motivation for the choice of examples applicable in nuclear and atomic physics.

The method used in this paper deals with finite-range (compactly supported) potentials. One can use this method for potentials with the Coulomb tail or other potentials of interest in physics, which are not of finite range. This is done by using the phase shifts transformation method which allows one to transform the phase shifts corresponding to a potential, not of finite range, whose behavior is known for \( r > a \), where \( a \) is some radius, into the phase shifts corresponding to a potential of finite range \( a \) (see [2], p.156).

In practice differential cross section is measured at various angles, and from it the fixed-energy phase shifts are calculated by a parameter-fitting procedure. Therefore, we plan in the future work to generalize the stability index method to the case when the original data are the values of the differential cross section, rather than the phase shifts.

By the physical reasons discussed above, we choose the following three potentials:

\[
q_1(r) = \begin{cases} 
-2/3 & 0 \leq r < 8.0 \\
0 & r \geq 8.0
\end{cases}
\]

\[
q_2(r) = \begin{cases} 
-4.0 & 0 \leq r < 8.0 \\
0 & r \geq 8.0
\end{cases}
\]

\[
q_3(r) = \begin{cases} 
-10.0 & 0 \leq r < 8.0 \\
0 & r \geq 8.0
\end{cases}
\]

In each case the following values of the parameters have been used. The radius \( R \) of the support of each \( q_i \) was chosen to be \( R = 10.0 \). The admissible set \( A_{adm} \) was defined with \( M = 2 \). The Reduced Random Search parameters: \( L = 5000, \gamma = 0.01, \nu = 0.16, \epsilon = 0.02, \beta = 1.10, J_{max} = 30 \). The value \( \epsilon_r = 0.1 \) was used in the Reduction Procedure during the local minimization phase. The initial configurations were generated using a random number generator with seeds determined by the system time. A typical run time was about 10 minutes on a 333 MHz PC, depending on the number of iterations in IRRS. The number \( N \) of the shifts used in (2.14) for the formation of the objective function \( \Phi(q) \) was 31 for all the wave numbers. As it can be seen from Table 1 the shifts for the potential \( q_1 \), decay rapidly for \( k = 1 \), but they remain large for \( k = 4 \). The upper and lower bounds for the potentials \( q_{low} = -20.0 \) and \( q_{high} = 0.0 \) used in the definition of the admissible set \( A_{adm} \) were chosen to reflect a priori information about the potentials.
Table 1. Phase shifts $\delta(k, l)$ of $q_3(r)$.

| $l$ | $k = 1.0$ | $k = 4.0$ |
|-----|-----------|-----------|
| 0   | -0.66496  | -0.62217  |
| 1   | -0.31009  | -0.64598  |
| 2   | -0.72324  | -0.65824  |
| 3   | -0.88586  | -0.64604  |
| 4   | -0.74713  | -0.74239  |
| 5   | 1.15839   | -0.65260  |
| 6   | 1.54292   | -0.86826  |
| 7   | -0.56945  | -0.69851  |
| 8   | -0.38745  | -1.00663  |
| 9   | 0.16888   | -0.83757  |
| 10  | -0.02690  | -1.08641  |
| 11  | -0.01261  | -1.09074  |
| 12  | 0.00017   | -1.08645  |
| 13  | -0.00010  | -1.39603  |
| 14  | -0.00004  | -1.24536  |
| 15  | 0.00000   | -1.55399  |
| 16  | 0.00000   | 1.49036   |
| 17  | 0.00000   | 1.56437   |
| 18  | 0.00000   | 1.11836   |
| 19  | 0.00000   | 1.12265   |
| 20  | 0.00000   | 1.11829   |
| 21  | 0.00000   | 0.60125   |
| 22  | 0.00000   | 0.58327   |
| 23  | 0.00000   | 0.59973   |
| 24  | 0.00000   | 0.00875   |
| 25  | 0.00000   | -0.18826  |
| 26  | 0.00000   | -0.11221  |
| 27  | 0.00000   | -0.47503  |
| 28  | 0.00000   | -1.22725  |
| 29  | 0.00000   | -1.29222  |
| 30  | 0.00000   | -1.25626  |

The identification was attempted with 3 different noise levels $h$. The levels are $h = 0.00$ (no noise), $h = 0.01$ and $h = 0.1$. More precisely, the noisy phase shifts $\delta_h(k, l)$ were obtained from the exact phase shifts $\delta(k, l)$ by the formula

$$\delta_h(k, l) = \delta(k, l)(1 + (0.5 - z) \cdot h),$$

where $z$ is the uniformly distributed on $[0, 1]$ random variable.

The distance $d(p_1(r), p_2(r))$ for potentials in step 5 of the IRRS algorithm was computed as

$$d(p_1(r), p_2(r)) = \|p_1(r) - p_2(r)\|$$

where the norm is the $L_2$-norm in $\mathbb{R}^3$.

The results of the identification algorithm (the Stability Indices) for different iterations of the IRRS algorithm are shown in Tables 2-4.
Table 2. Stability Indices for \( q_1(r) \) identification at different noise levels \( h \).

| Iteration | \( h = 0.00 \) | \( h = 0.01 \) | \( h = 0.10 \) |
|-----------|----------------|----------------|----------------|
| \( k = 1.00 \) | 1.256985 0.592597 1.953778 | 0.538440 0.133685 0.799142 | 0.014616 0.123247 |
| \( k = 2.00 \) | 0.000020 0.002094 0.000960 | 0.000000 0.014553 0.046275 | 0.000000 0.000501 0.096444 |
| \( k = 2.50 \) | 0.000000 0.000511 0.004635 | 0.000000 0.014553 0.046275 | 0.000000 0.000501 0.096444 |
| \( k = 3.00 \) | 0.000000 0.022935 0.027214 | 0.000000 0.014553 0.046275 | 0.000000 0.000501 0.096444 |

For example, Table 4 shows that for \( k = 2.5, \ h = 0.00 \) the Stability Index has reached the value 0.013621 after 2 iterations. According to the Stopping criterion for IRRS, the program has been terminated with the conclusion that the identification was stable. In this case the potential identified by the program was

\[
p(r) = \begin{cases} 
-10.000024 & 0 \leq r < 7.999994 \\
0.0 & r \geq 7.999994
\end{cases}
\]

which is very close to the original potential

\[
q_3(r) = \begin{cases} 
-10.0 & 0 \leq r < 8.0 \\
0.0 & r \geq 8.0
\end{cases}
\]

On the other hand, when the phase shifts of \( q_3(r) \) were corrupted by a 10\% noise \( k = 2.5, \ h = 0.10 \), the program was terminated (according to the Stopping criterion) after 4 iterations with the Stability Index at 0.079241. Since the Stability Index is greater than the a priori chosen threshold of \( \epsilon = 0.02 \) the conclusion is that the identification is unstable. A closer look into this situation reveals that the values of the objective function \( \Phi(p_i) \), \( p_i \in S_{\text{min}} \) (there are 8 elements in \( S_{\text{min}} \)) are between 0.0992806 and 0.100320. Since we chose \( \beta = 1.1 \) the values are within the required 10\% of each other. The actual potentials for which the normalized distance is equal to the Stability Index 0.079241 are

\[
p_1(r) = \begin{cases} 
-9.997164 & 0 \leq r < 7.932678 \\
-7.487082 & 7.932678 \leq r < 8.025500 \\
0.0 & r \geq 8.025500
\end{cases}
\]

and

\[
p_2(r) = \begin{cases} 
-9.999565 & 0 \leq r < 7.987208 \\
-1.236253 & 7.987208 \leq r < 8.102628 \\
0.0 & r \geq 8.102628
\end{cases}
\]

with \( \Phi(p_1) = 0.0992806 \) and \( \Phi(p_2) = 0.0997561 \). One may conclude from this example that the threshold \( \epsilon = 0.02 \) is too tight and can be relaxed, if the above uncertainty is acceptable.
Table 3. Stability Indices for $q_2(r)$ identification at different noise levels $h$.

| $k$ | Iteration | $h = 0.00$ | $h = 0.01$ | $h = 0.10$ |
|-----|-----------|-----------|-----------|-----------|
| 1.00 | 1         | 0.774376  | 0.598471  | 0.108902  |
|      | 2         | 0.773718  | 1.027345  | 0.023206  |
|      | 3         | 0.026492  | 0.025593  | 0.023206  |
|      | 4         | 0.020522  | 0.029533  | 0.024081  |
|      | 5         | 0.020524  | 0.029533  | 0.024081  |
|      | 6         | 0.000745  | 0.029533  |           |
|      |           | 7         | 0.029533  |           |
|      |           | 8         | 0.029533  |           |
|      |           | 9         | 0.029533  |           |
|      |           | 10        | 0.029533  |           |
|      |           | 11        | 0.029619  |           |
|      |           | 12        | 0.025816  |           |
|      |           | 13        | 0.025816  |           |
|      |           | 14        | 0.008901  |           |
| 2.00 | 1         | 0.863796  | 0.799356  | 0.981239  |
|      | 2         | 0.861842  | 0.799356  | 0.029445  |
|      | 3         | 0.008653  | 0.000993  | 0.029445  |
|      | 4         |           | 0.029445  |           |
|      | 5         |           | 0.026513  |           |
|      | 6         |           | 0.026513  |           |
|      | 7         |           | 0.024881  |           |
| 2.50 | 1         | 1.848910  | 1.632298  | 0.894087  |
|      | 2         | 1.197131  | 1.632298  | 0.507953  |
|      | 3         | 0.580361  | 1.183455  | 0.025454  |
|      | 4         | 0.030516  | 0.528979  |           |
|      | 5         | 0.016195  | 0.032661  |           |
| 3.00 | 1         | 1.844702  | 1.849016  | 1.708201  |
|      | 2         | 1.649700  | 1.782775  | 1.512821  |
|      | 3         | 1.456026  | 1.782775  | 1.412345  |
|      | 4         | 1.410253  | 1.457020  | 1.156964  |
|      | 5         | 0.624358  | 0.961263  | 1.156964  |
|      | 6         | 0.692080  | 0.961263  | 0.902681  |
|      | 7         | 0.692080  | 0.961263  | 0.902681  |
|      | 8         | 0.345804  | 0.291611  | 0.902474  |
|      | 9         | 0.345804  | 0.286390  | 0.159221  |
|      | 10        | 0.345804  | 0.260693  | 0.154829  |
|      | 11        | 0.043845  | 0.260693  | 0.154829  |
|      | 12        | 0.043845  | 0.260693  | 0.135537  |
|      | 13        | 0.043845  | 0.260693  | 0.135537  |
|      | 14        | 0.043845  | 0.260693  | 0.135537  |
|      | 15        | 0.042080  | 0.157024  | 0.107548  |
|      | 16        | 0.042080  | 0.157024  |           |
|      | 17        | 0.042080  | 0.157024  |           |
|      | 18        | 0.000429  | 0.157024  |           |
|      | 19        |           | 0.022988  |           |
| 4.00 | 1         | 0.000000  | 0.000674  | 0.050705  |
Table 4. Stability Indices for $q_3(r)$ identification at different noise levels $h$.

| $k$  | Iteration | $h = 0.00$  | $h = 0.01$  | $h = 0.10$  |
|------|------------|-------------|-------------|-------------|
| 1.00 | 1          | 0.564168    | 0.594314    | 0.764340    |
|      | 2          | 0.024441    | 0.028558    | 0.081888    |
|      | 3          | 0.024441    | 0.014468    | 0.050755    |
|      | 4          | 0.024684    |             |             |
|      | 5          | 0.024684    |             |             |
|      | 6          | 0.005800    |             |             |
| 2.00 | 1          | 0.684053    | 1.450148    | 0.485783    |
|      | 2          | 0.423283    | 0.792431    | 0.078716    |
|      | 3          | 0.006291    | 0.457650    | 0.078716    |
|      | 4          | 0.023157    | 0.078716    |             |
|      | 5          |             | 0.078716    |             |
|      | 6          |             | 0.078716    |             |
|      | 7          |             | 0.078716    |             |
|      | 8          |             | 0.078716    |             |
|      | 9          |             | 0.078716    |             |
|      | 10         |             | 0.078716    |             |
|      | 11         |             | 0.078716    |             |
| 2.50 | 1          | 0.126528    | 0.993192    | 0.996519    |
|      | 2          | 0.013621    | 0.105537    | 0.855049    |
|      | 3          | 0.033694    | 0.849123    |             |
|      | 4          | 0.026811    | 0.079241    |             |
| 3.00 | 1          | 0.962483    | 1.541714    | 0.731315    |
|      | 2          | 0.222880    | 0.164744    | 0.731315    |
|      | 3          | 0.158809    | 0.021775    | 0.072009    |
|      | 4          | 0.021366    |             |             |
|      | 5          | 0.021366    |             |             |
|      | 6          | 0.001416    |             |             |
| 4.00 | 1          | 1.714951    | 1.413549    | 0.788434    |
|      | 2          | 0.033024    | 0.075503    | 0.024482    |
|      | 3          | 0.018250    | 0.029385    |             |
|      | 4          | 0.029421    |             |             |
|      | 5          | 0.029421    |             |             |
|      | 6          | 0.015946    |             |             |

6. Conclusions

A novel numerical method for solving inverse scattering problem with fixed-energy data is proposed. The method contains a new important concept: the stability index of the inversion problem. This index is a number, computed from the data, which shows how stable the inversion is. If this index is small, then the inversion provides a set of potentials which differ so little, that practically one can represent this set by one potential. If this index is larger than some threshold, then one concludes that practically, with the given data, the inversion is unstable and the potential cannot be identified uniquely from these data. Inversion of the fixed-energy phase shifts for several model potentials is considered. The results show practical efficiency of the proposed method. The method is of general nature and is applicable to a very wide variety of inverse problems.
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