ADJOINT METHODS FOR STATIC HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We use the adjoint methods to study the static Hamilton-Jacobi equations and to prove the speed of convergence for those equations. The main new ideas are to introduce adjoint equations corresponding to the formal linearizations of regularized equations of vanishing viscosity type, and from the solutions $\sigma^\varepsilon$ of those we can get the properties of the solutions $u$ of the Hamilton-Jacobi equations. We classify the static equations into two types and present two new ways to deal with each type. The methods can be applied to various static problems and point out the new ways to look at those PDE.

1. INTRODUCTION

The theory of viscosity solutions for Hamilton-Jacobi equations, introduced by Crandall and Lions in [2] provides a body of simple and effective techniques for discovering the existence, uniqueness, and stability of the solutions. To date, many results concerning the speed of convergence for Hamilton-Jacobi equations of various types have been studied. However, all the methods seem to be in the indirect ways by using viscosity solution techniques and maximum principles.

Recently, Evans in his forthcoming paper [7] introduces some new methods to study Hamilton-Jacobi equations for the time-dependent case, including the nonlinear adjoint method. This method turns out to be very useful to observe various time-dependent problems of vanishing viscosity type. However, it may not work well when applying directly to time-independent problems because of some difficulties such as the existence, uniqueness of the solution of the adjoint equation as well as the nonnegative property as we will discuss below. In this present paper, we will introduce some new ideas to apply this method to study some time-independent PDE such as stationary Hamilton-Jacobi equation, Eikonal-like equation and effective Hamiltonian. We classify the problems into two classes: the class containing $u^\varepsilon$ in the regularized equation, the class not containing $u^\varepsilon$ in the regularized equation, and propose two different new methods to deal with each class. The main goals are to build up the constructive ways to look at those PDE directly through the adjoint equations and hence can be used not only here but also for other observations in the future.
We will here introduce the adjoint equation for each class of equation and then prove the speed of convergence. In fact, we can go further by using the constructions here and the Compensated compactness to get more properties of the solutions as in [7] and get some further results. We will explore those properties elsewhere in the future.

Outline of this paper: This paper contains three sections about three types of static Hamilton-Jacobi equations, which are quite interesting and familiar to the readers. Our purpose in this paper is expository, hence we try to make all the equations as simple as possible to show the key ideas instead of coming into complicated proofs. All the ideas work for general cases as well.

In section 2, we study the stationary problem in the whole space $\mathbb{R}^n$

$$u(x) + H(x, Du(x)) = 0 \quad \text{in } \mathbb{R}^n$$

by looking at the regularized problem

$$u^\varepsilon(x) + H(x, Du^\varepsilon(x)) = \varepsilon \Delta u^\varepsilon \quad \text{in } \mathbb{R}^n.$$ 

This problem is of the first type because the regularized equation contains $u^\varepsilon$. We have the general theme to deal with such problem like this by introduce the so-called fake parabolic adjoint equation as following:

$$\begin{cases}
-2\sigma^\varepsilon_t - \text{div}(D_pH(x, Du^\varepsilon))\sigma^\varepsilon = \varepsilon \Delta \sigma^\varepsilon \\
\sigma^\varepsilon|_T = \delta_{x_0}.
\end{cases}$$

Using $\sigma^\varepsilon$ we can prove that $|\frac{\partial u^\varepsilon}{\partial \varepsilon}| \leq C\varepsilon^{-1/2}$, which implies that $||u^\varepsilon - u||_{L^\infty} \leq C\varepsilon^{1/2}$.

Section 3 deals with the Eikonal-like equation in the bounded domain

$$\begin{cases}
H(Du(x)) = 0 \\
u(x) = 0
\end{cases} \quad \text{in } U,$$

and we also look at the following regularized problem:

$$\begin{cases}
H(Du^\varepsilon(x)) = \varepsilon \Delta u^\varepsilon(x) \\
u^\varepsilon(x) = 0
\end{cases} \quad \text{in } U.$$ 

This problem is of the second type since the regularized equation does not contain $u^\varepsilon$. The idea of dealing with this type of problem is much more different with the previous one since we could not switch the problem into parabolic type. It turns out that in this case, the adjoint equation is of elliptic type and is an analog of the time-dependent Hamilton-Jacobi
equation in [7] as following:

\[
\begin{cases}
-\text{div}(DH(Du^\varepsilon)\sigma^\varepsilon) = \varepsilon \Delta \sigma^\varepsilon + \delta_{x_0} & \text{in } U, \\
\sigma^\varepsilon = 0 & \text{on } \partial U.
\end{cases}
\]

Besides the beauty of this adjoint equation, we furthermore can also relax the convexity condition of $H$. Up to now, all the papers dealing with the Eikonal-like equation require the convexity condition of $H$ for the bounded properties and comparison properties hence the uniqueness of the solutions. However, we can see one of the good signal in the paper [5] of Ishii for the proof of uniqueness of $u$ where he only require condition (H4)' instead of convexity condition. In this section, we only require that $H$ has some kind of homogeneous condition, which is much weaker, and quite natural. We will have to reprove such comparison properties and uniqueness of solutions in this section as well, but they are not really a big deal for us. Finally, we get the same speed of convergence as in the case above. One interesting point is that we could not find such result in all of the references, so it may be the new one.

Finally, in the last section, we will study the effective Hamiltonian of homogenization of Hamilton-Jacobi equations:

\[ H(P + Dv, y) = \tilde{H}(P). \]

Instead of considering the normal regularized problem, we will look at the slightly different regularized problem, which includes the vanishing viscosity term, as following:

\[ \theta \varepsilon^\theta + H(P + D\varepsilon^\theta, y) = \theta^2 \Delta \varepsilon^\theta. \]

This problem hence is also of the first type since the regularized problem contains the $\varepsilon^\theta$ term. Similar to section 2 above, we will also introduce the fake parabolic adjoint equation as following:

\[
\begin{cases}
-2\theta \sigma_\tau^\theta - \text{div}(D\sigma^\theta H) = \theta^2 \Delta \sigma^\theta & (x, t) \in \mathbb{R}^n \times (0, T) \\
\sigma^\theta |_{T} = \delta_{x_0},
\end{cases}
\]

and we can prove that \( \frac{\partial (\theta \varepsilon^\theta)}{\partial \theta} \leq C \), therefore get \( \| \theta \varepsilon^\theta + \tilde{H}(P) \|_{L^\infty} \leq C \theta \). Notice that from this, we can imply the result of [4] as well.

I would like to express my appreciation to my advisor, Lawrence C. Evans for giving me the problems and plenty of fruitful discussions. I thank Charlie Smart for his nice suggestions and discussions.
2. STATIONARY PROBLEM IN $\mathbb{R}^n$

We are going to observe the properties of stationary Hamilton-Jacobi equation in $\mathbb{R}^n$:

(2.1) $u(x) + H(x, Du(x)) = 0$ in $\mathbb{R}^n$.

As usual, we consider first the regularized equation

(2.2) $u^\varepsilon(x) + H(x, Du^\varepsilon(x)) = \varepsilon \Delta u^\varepsilon$ in $\mathbb{R}^n$.

Let us for simplicity assume that $H$ is smooth and $H$ satisfies some conditions as in [10], [12]; or more explicitly

$$\begin{cases}
\sup_{x \in \mathbb{R}^n} |H(x,0)| \leq C \leq \infty; & \sup_{x \in \mathbb{R}^n} |D_x H(x, p)| \leq C(1 + |p|), \\
H(x, p) \to \infty \text{ as } |p| \to \infty \text{ uniformly for } x \in \mathbb{R}^n.
\end{cases}$$

Other conditions can be applied as well, we just want to make everything simple.

We have some well-known standard estimates from [10] as following:

(2.3) $||u^\varepsilon||_{L^\infty}, ||Du^\varepsilon||_{L^\infty} \leq C$.

Our main theorem of this section is

**Theorem 2.1.** There exists a constant $C > 0$ s.t.

(2.4) $||u^\varepsilon - u||_{L^\infty} \leq C\varepsilon^{1/2}$.

In fact, this theorem was proved long time ago, for instance in [3], [8], [12]. However, we propose here the new way to prove it by using adjoint method and furthermore a new way of thinking about the first type of the static case, the type containing $u^\varepsilon$ in the regularized equation.

**Lemma 2.2.** Let $w^\varepsilon = \frac{|Du^\varepsilon|^2}{2}$ then $w^\varepsilon$ satisfies:

(2.5) $2w^\varepsilon + D_pH(x, Du^\varepsilon) \cdot Dw^\varepsilon + D_xH(x, Du^\varepsilon) \cdot Du^\varepsilon = \varepsilon \Delta w^\varepsilon - \varepsilon |D^2u^\varepsilon|^2$.

**Proof**

Differentiate the equation (2.2) with respect to $x_i$ we get:

(2.6) $u^\varepsilon_{x_i} + H_{x_i}(x, Du^\varepsilon) + H_{p_k}(x, Du^\varepsilon)u_{x_kx_i}^\varepsilon = \varepsilon \Delta u^\varepsilon_{x_i}$.

Taking the product of (2.6) with $u^\varepsilon_{x_i}$ and summing over $i$ we get

(2.7) $|Du^\varepsilon|^2 + D_xH(x, Du^\varepsilon) \cdot Du^\varepsilon + H_{p_k}(x, Du^\varepsilon)(\frac{|Du^\varepsilon|^2}{2})_{x_k} = \varepsilon \Delta u^\varepsilon_{x_i} u^\varepsilon_{x_i}$. 
Furthermore, we note that:

\[
\Delta \varepsilon u_{x_i} u_{x_i} = u_{x_k x_i} u_{x_k} x_k - \sum_{i,k} |u_{x_k x_i}|^2 = \Delta \frac{|Du\varepsilon|^2}{2} - |D^2 u\varepsilon|^2.
\]

Combining those two calculations we get the lemma.

Now we introduce the new function \(v\varepsilon\) to change (2.5) into the fake parabolic type. We will explain later the reason why we have to switch to parabolic type. Let \(T > 0\) be a constant and let

\[
v\varepsilon(x,t) = e^t w\varepsilon(x) \quad (x,t) \in \mathbb{R}^n \times [0,T].
\]

Then from (2.5), we therefore get that \(v\varepsilon\) satisfies:

\[
2v\varepsilon_t + D_p H(x, Du\varepsilon) . Dv\varepsilon + e^t D_x H(x, Du\varepsilon) . Du\varepsilon = \varepsilon \Delta v\varepsilon - \varepsilon e^t |D^2 u\varepsilon|^2.
\]

**Adjoint method:** We now introduce the adjoint problem to the problem (2.8). For \(x_0 \in \mathbb{R}^n\), let \(\sigma\varepsilon\) be the solution of the following PDE:

\[
\begin{aligned}
-2\sigma\varepsilon_t - \text{div}(D_p H(x, Du\varepsilon) \sigma\varepsilon) &= \varepsilon \Delta \sigma\varepsilon \\
\sigma\varepsilon|_T &= \delta_{x_0}.
\end{aligned}
\]

From the solution \(\sigma\varepsilon\) of the adjoint equation, we can somehow figure out the properties of \(u\varepsilon\) as well as \(u\), which are our very important goals especially in the case that \(H\) is not convex in \(p\). However, let us point out some properties of \(\sigma\varepsilon\) first:

**Lemma 2.3. Properties of \(\sigma\varepsilon\)**

\[(i) \quad \sigma\varepsilon(x,t) \geq 0 \quad (x,t) \in \mathbb{R}^n \times (0,T),
(ii) \quad \int_{\mathbb{R}^n} \sigma\varepsilon(x,t) dx = 1 \quad t \in (0,T).\]

**Proof**

The proof is easy based on maximum principle and integration over \(\mathbb{R}^n\). We will not go into the details of the proof.

**Remark 2.4.**

As we can see, when we change the equation into the parabolic type then we automatically have the existence of the solution \(\sigma\varepsilon\) as well as the maximum principle can be applied with the only requirement of the boundedness of coefficients. Note that we need the property (i) of the above Lemma to do further derivations as you can see below.

Besides, one can write down the adjoint equation of (2.5) in form of elliptic equation and can see that the adjoint equation may not have the solution, the uniqueness of the solution as well as the required condition to apply the maximum principle.
Now, we start to observe properties and connections between $\sigma^\varepsilon$ and $u^\varepsilon$

**Lemma 2.5.** There exists a constant $C > 0$ s.t.

$$
\int_0^T \int_{\mathbb{R}^n} \varepsilon e^\varepsilon |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \, dt \leq C.
$$

**Proof**

(2.11) \[
\frac{d}{dt} \int_{\mathbb{R}^n} 2\sigma^\varepsilon v^\varepsilon = \int_{\mathbb{R}^n} 2\sigma_i^\varepsilon v_i^\varepsilon + 2\sigma^\varepsilon v^\varepsilon
\]

\[
= \int_{\mathbb{R}^n} 2\sigma_i^\varepsilon v_i^\varepsilon + \int_{\mathbb{R}^n} (-D_p H.Dv^\varepsilon - \varepsilon' D_x H.Du^\varepsilon + \varepsilon \Delta v^\varepsilon - \varepsilon e^\varepsilon |D^2 u^\varepsilon|^2) \sigma^\varepsilon
\]

\[
= \int_{\mathbb{R}^n} (2\sigma_i^\varepsilon + \text{div}(D_p H \sigma^\varepsilon) + \varepsilon \Delta \sigma^\varepsilon) v^\varepsilon - \int_{\mathbb{R}^n} (\varepsilon' D_x H.Du^\varepsilon + \varepsilon \varepsilon' |D^2 u^\varepsilon|^2) \sigma^\varepsilon
\]

\[
= - \int_{\mathbb{R}^n} (\varepsilon' D_x H.Du^\varepsilon + \varepsilon \varepsilon' |D^2 u^\varepsilon|^2) \sigma^\varepsilon.
\]

Now we integrate (2.11) from 0 to $T$:

(2.12) \[
\int_{\mathbb{R}^n} 2\sigma^\varepsilon(x,T) v^\varepsilon(x,T) \, dx - \int_{\mathbb{R}^n} 2\sigma^\varepsilon(x,0) v^\varepsilon(x,0) \, dx
\]

\[
= - \int_0^T \int_{\mathbb{R}^n} \varepsilon' D_x H.Du^\varepsilon \sigma^\varepsilon - \int_0^T \int_{\mathbb{R}^n} \varepsilon \varepsilon' |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \, dt.
\]

Hence we get:

(2.13) \[
\int_0^T \int_{\mathbb{R}^n} \varepsilon e^\varepsilon |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \, dt
\]

\[
\leq |2v^\varepsilon(x_0,T)| + \int_{\mathbb{R}^n} 2\sigma^\varepsilon(x,0) v^\varepsilon(x,0) \, dx + \int_0^T \int_{\mathbb{R}^n} \varepsilon' D_x H.Du^\varepsilon \sigma^\varepsilon |x_0 |
\]

\[
\leq 2e^T C + 2C + C(e^T - 1) \leq C.
\]

We get the lemma.

Notice that all of the estimates here are independent of the choice of $x_0$. More precisely, for any $x_0 \in \mathbb{R}^n$ and the corresponding $\sigma^\varepsilon$, the estimates stay the same with the same constants. More generally, we also have all such estimates if we assume $\sigma^\varepsilon|_T = \nu$ for $\nu$ is a probability measure, but we do not really use the general probability measure here.

**Definition 2.6.** Define $u^\varepsilon(x) = \frac{\partial u^\varepsilon}{\partial \varepsilon}(x)$.

We have the following theorem:
**Theorem 2.7.** There exists a constant $C > 0$ s.t.

\[(2.14) \quad |u^e(x)| \leq Ce^{-1/2}.\]

**Proof**

According to standard elliptic estimates, the function $u^e$ is smooth in the parameter $\varepsilon$ away from $\varepsilon = 0$. Differentiate (2.2) with respect to $\varepsilon$ we get

\[(2.15) \quad u^e_{\varepsilon} + D_pH(x,Du^e).Du^e_{\varepsilon} = \varepsilon \Delta u^e_{\varepsilon} + \Delta u^e.\]

Define $z^e : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ s.t. $z^e(x,\cdot) = e^t u^e(x)$. Then $z^e$ satisfies the following PDE:

\[(2.16) \quad z^e_t + D_pH(x,Du^e).Dz^e = \varepsilon \Delta z^e + e^t \Delta u^e.\]

Notice that the coefficients of (2.16) is slightly different with those of the adjoint equation. Playing the same tricks as in Lemma (2.5) we have:

\[(2.17) \quad \frac{d}{dt} \int_{\mathbb{R}^n} 2\sigma^e \xi \, dx = \int_{\mathbb{R}^n} 2\sigma^e u^e + \sigma^e \xi^e + \int_{\mathbb{R}^n} e^t u^e \sigma^e
\]

\[\quad = \int_{\mathbb{R}^n} 2\sigma^e u^e + \int_{\mathbb{R}^n} (-D_pH.D \xi^e + \varepsilon \Delta \xi^e + e^t \Delta u^e) \sigma^e + \int_{\mathbb{R}^n} e^t u^e \sigma^e
\]

\[\quad = \int_{\mathbb{R}^n} (2\sigma^e \xi^e + \text{div}(D_pH \sigma^e) + \varepsilon \Delta \sigma^e) \xi^e + \int_{\mathbb{R}^n} (e^t \Delta u^e \sigma^e + e^t u^e \sigma^e)
\]

\[\quad = \int_{\mathbb{R}^n} (e^t \Delta u^e \sigma^e + e^t u^e \sigma^e).\]

Now, we choose $x_0$ such that $|u^e(x_0)| = \max_{\mathbb{R}^n} |u^e(x)|$.

(In fact, if the maximum does not attain, we can choose $x_0$ such that $|u^e(x_0)| \geq \sup_{\mathbb{R}^n} |u^e(x)| - \theta$, and then let $\theta$ goes to 0. Let’s make everything simple and clear.)

Integrate (2.17) from 0 to $T$ we have:

\[(2.18) \quad \int_{\mathbb{R}^n} 2\sigma^e(x,T)z^e(x,T) \, dx - \int_{\mathbb{R}^n} 2\sigma^e(x,0)z^e(x,0) \, dx
\]

\[\quad = \int_0^T \int_{\mathbb{R}^n} e^t \Delta u^e \sigma^e + \int_0^T \int_{\mathbb{R}^n} e^t u^e \sigma^e \, dxdt.
\]

Substitute the condition $\sigma^e|_T = \delta_{x_0}$ into the equation above, we get:

\[(2.19) \quad |2e^T u^e(x_0) - \int_{\mathbb{R}^n} 2\sigma^e(x,0)z^e(x,0) - \int_0^T \int_{\mathbb{R}^n} e^t u^e \sigma^e| = \int_0^T \int_{\mathbb{R}^n} e^t \Delta u^e \sigma^e|.\]
Furthermore, we can control the LHS of (2.19)

\[
LHS = |2e^T u^e(x_0) - \int_{\mathbb{R}^n} 2\sigma^e(x,0)z^e(x,0) - \int_0^T \int_{\mathbb{R}^n} e^T u^e \sigma^e| \\
\geq 2e^T |u^e(x_0)| - \int_{\mathbb{R}^n} 2\sigma^e(x,0)|u^e(x_0)| - \int_0^T \int_{\mathbb{R}^n} e^T |u^e(x_0)|\sigma^e \\
= |u^e(x_0)|(2e^T - 2 - (e^T - 1)) = |u^e(x_0)|(e^T - 1).
\]

Besides, by using Lemma 2.5 and Holder’s inequality, we can estimate the RHS of (2.19):

\[
RHS = |\int_0^T \int_{\mathbb{R}^n} e^T \Delta u^e | \leq C \int_0^T \int_{\mathbb{R}^n} |D^2 u^e| \sigma^e \\
\leq C \left\{ \int_0^T \int_{\mathbb{R}^n} |D^2 u^e|^2 \sigma^e \right\}^{1/2} \left\{ \int_0^T \int_{\mathbb{R}^n} \sigma^e \right\}^{1/2} \leq Ce^{-1/2}.
\]

So we get the theorem.

**Proof of Theorem 2.1**

By using Theorem 2.7, we immediately get the result.

### 3. EIKONAL-LIKE EQUATION IN BOUNDED DOMAIN

We are going to observe the properties of Eikonal-like Hamilton-Jacobi equation in the smooth bounded domain \(U\):

\[
\begin{aligned}
H(Du(x)) &= 0 \quad \text{in } U, \\
u(x) &= 0 \quad \text{on } \partial U.
\end{aligned}
\]

(3.1)

Our approach, as usual, is to consider regularized problem:

\[
\begin{aligned}
H(Du^e(x)) &= \varepsilon \Delta u^e(x) \quad \text{in } U, \\
u^e(x) &= 0 \quad \text{on } \partial U.
\end{aligned}
\]

(3.2)

Crandall and Lions study this equation in sense of viscosity solution first in [2] and Lions in [10]. After that Fleming and Souganidis study this in more details and also give some asymptotic series of the solutions of the regularized problem in [9]. Then Ishii gives a simple and direct proof of the uniqueness of the solution in [5]. We here base on the conditions given in [9], [10] and we refer the readers to [5], [9] and [10] for more details.

Our goal here is not only to prove the speed of convergence but also to relax the convexity conditions of \(H\). Obviously we cannot relax the convexity condition without require some sufficient conditions as we will see in the counter-example below. But the condition we need is much more weaker and quite natural like the homogenous condition. We assume the following conditions:
(H1) $H$ smooth and $H(0) < 0$,

(H2) $H$ is coercive, i.e. $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty$,

(H3) There exist $\gamma, \delta > 0$ s.t. $DH(p).p - \gamma H(p) \geq \delta > 0 \quad \forall \ p \in \mathbb{R}^n$.

The condition (H3) is used to replace the convexity condition. We will discuss about this condition later. We just make an obvious observation that if $H$ is convex then we have (H3) with $\gamma = 1$ and $\delta = -H(0)$.

**Theorem 3.1.** We have some well-known standard estimates as following:

(3.3) $\|u^\varepsilon\|_{L^\infty}, \|Du^\varepsilon\|_{L^\infty} \leq C$.

**Proof**
In the case where $H$ is convex then this theorem is proved in [9] by Lemma 1.1 and 1.2 or in [10]. Here, we follow almost all of the proofs and just need to slightly change some estimates that use the convexity condition.

By Lemma 1.1 and the first part of Lemma 1.2 in [9], there exists a const $C > 0$ s.t. $0 \leq u^\varepsilon \leq C$ in $\bar{U}$ and $|Du^\varepsilon| \leq C$ on $\partial U$.

To complete the proof we only need to bound $|Du^\varepsilon|$ in $U$.

Using the same idea like [9], [10], let $w = |Du^\varepsilon| - \mu u^\varepsilon$, where $\mu$ is to be a suitably chosen constant. Suppose that $w$ has a positive maximum at an interior point $x_0 \in U$. At $x_0$ we have:

$$0 = w_{x_i} = \sum_k u^\varepsilon_{x_k} u^\varepsilon_{x_kx_i} - \mu u^\varepsilon_{x_i},$$

Hence we get:

$$\sum_i (\sum_k u^\varepsilon_{x_k} u^\varepsilon_{x_kx_i})^2 = \mu^2 |Du^\varepsilon|^4.$$

Furthermore, we also have:

$$0 \leq -\varepsilon \Delta w = \varepsilon \sum_{i\neq k} (\frac{u^\varepsilon_{x_k} u^\varepsilon_{x_kx_i}}{|Du^\varepsilon|^3})^2 - \frac{\varepsilon \sum_{i\neq k} (u^\varepsilon_{x_kx_i})^2}{|Du^\varepsilon|} + \sum_k \frac{u^\varepsilon_{x_k} (-\varepsilon \Delta u^\varepsilon)_{x_k}}{|Du^\varepsilon|} + \mu (\varepsilon \Delta u^\varepsilon),$$

By using the inequality $\frac{(\Delta u^\varepsilon)^2}{n} \leq \sum_{i,k} (u^\varepsilon_{x_i} u^\varepsilon_{x_k})^2$ and (3.2) we get

$$0 \leq \varepsilon \mu^2 |Du^\varepsilon| - \frac{H^2}{n\varepsilon |Du^\varepsilon|} - \mu DH.Du^\varepsilon + \mu H.$$

Besides, by (H3) we get

$$\mu DH.Du^\varepsilon - \mu \gamma H > \delta \mu > 0,$$
Thus,
\[ \frac{H^2}{|Du^\varepsilon|^2} \leq n\mu^2\varepsilon^2 + n\varepsilon(\mu - \mu\gamma) \frac{H}{|Du^\varepsilon|} \leq n\mu^2\varepsilon^2 + n\varepsilon\mu(1 + \gamma) \frac{|H|}{|Du^\varepsilon|}. \]
Choose \( \mu = \frac{1}{2n(1 + \gamma)} \) then for \( \varepsilon < 1 \), we get the estimate:
\[ \frac{H^2}{|Du^\varepsilon|^2} \leq 1 + \frac{|H|}{|Du^\varepsilon|}. \]
By the coercivity condition (H2) we finally get \( |Du^\varepsilon| \) is bounded independently of \( \varepsilon \). We get the theorem.

**Remark 3.2.**

The existence of the solution of (3.2) follows directly from [9] with some changes of conditions as in the proof of Theorem 3.1 above.

Now we discuss about the uniqueness of the viscosity solution \( u \) of (3.1).

For \( p \in \mathbb{R}^n \) we consider the function \( \phi \) from \((0, \infty) \) to \( \mathbb{R} \)
\[ \phi(t) = t^{-\gamma}H(tp) \quad \forall \ t > 0, \]
then we have:
\[ \phi'(t) = t^{-\gamma-1}(DH(tp).tp - \gamma H(tp)) > t^{-\gamma-1}\delta > 0. \]
Hence \( \phi \) is strictly increasing and for \( t < 1 \) we have furthermore:
\[ \phi(1) - \phi(t) = \int_t^1 \phi'(s)ds > \int_t^1 s^{-\gamma-1}\delta ds = \frac{\delta}{\gamma+1}(t^{-\gamma} - 1) > 0, \]
Thus,
\[ H(tp) \leq t^\gamma H(p) - \frac{\delta}{\gamma+1}(1 - t^\gamma) = t^\gamma H(p) + \frac{-\delta}{(\gamma+1)H(0)}(1 - t^\gamma)H(0). \]
By (H1) we have that \( H(0) < 0 \). So we have all the conditions satisfy the conditions (H1)-(H3) and (H4)' of [5] with \( \phi = 0 \). Hence we get the uniqueness of viscosity solution of (3.1).

The proof of the uniqueness of \( u^\varepsilon \) is quite complicated and follows the key idea of this section. Therefore we put it in the appendix at the end of this paper.

Our main theorem of this section is

**Theorem 3.3.** There exists a constant \( C > 0 \) s.t.
\[ ||u^\varepsilon - u||_{L^\infty} \leq C\varepsilon^{1/2}. \]
Some of the lemmas below will be quite familiar with the lemmas in section 2. Therefore, we will only state those lemmas (without the proofs).

**Lemma 3.4.** Let $w^\varepsilon = \frac{|Du^\varepsilon|^2}{2}$ then $w^\varepsilon$ satisfies:

\begin{equation}
DH(Du^\varepsilon)Dw^\varepsilon = \varepsilon \Delta w^\varepsilon - \varepsilon |D^2u^\varepsilon|^2.
\end{equation}

Note that in (3.5) we do not have the term $w^\varepsilon$, hence we cannot convert this equation to the parabolic type as in section 2. We need some other new idea to deal with this type of equation.

**Adjoint method:** We now introduce the adjoint problem to the problem (3.5). For each $x_0 \in U$, we consider the equation:

\begin{equation}
\begin{cases}
-\text{div}(DH(Du^\varepsilon)\sigma^\varepsilon) = \varepsilon \Delta \sigma^\varepsilon + \delta_{x_0} & \text{in } U, \\
\sigma^\varepsilon = 0 & \text{on } \partial U.
\end{cases}
\end{equation}

The adjoint equation here is very nice and somehow similar to the one that Evans introduce in [7] of the time-dependent case. From the adjoint function $\sigma^\varepsilon$, we can somehow figure out the properties of $u^\varepsilon$ as well as $u$, which are our very important goals especially in the case that $H$ is not convex in $p$. However, the problem is, like what we have mentioned in the Remark 2.4 above, we do not know about the existence, uniqueness of (3.6) as well as the nonnegative property of $\sigma^\varepsilon$, which we really need. It’s quite interesting that to observe $\sigma^\varepsilon$, we once again need the adjoint equation of (3.6):

For each $f \in L^2(U)$ and $f \geq 0$, we consider the following equation

\begin{equation}
\begin{cases}
DH(Du^\varepsilon)Dv^\varepsilon = \varepsilon \Delta v^\varepsilon + f & \text{in } U, \\
v^\varepsilon = 0 & \text{on } \partial U.
\end{cases}
\end{equation}

For $f = 0$ then it’s obvious by the Maximum principle that $v^\varepsilon = 0$. Hence by Fredholm alternative we get both (3.7) and then (3.6) have unique solutions.

Furthermore, by Maximum principle again, we get that $v^\varepsilon \geq 0$.

**Lemma 3.5.** We have:

\begin{equation}
\int_U f \sigma^\varepsilon dx = v^\varepsilon(x_0) \geq 0.
\end{equation}

Hence in particular, $\sigma^\varepsilon \geq 0$. 

Proof
We have:

\[
\int_U f \sigma^\varepsilon dx = \int_U DH(Du^\varepsilon) \cdot Dv^\varepsilon \sigma^\varepsilon - \varepsilon \Delta v^\varepsilon \sigma^\varepsilon \\
= \int_U (\varepsilon \Delta v^\varepsilon) v^\varepsilon = v^\varepsilon(x_0) \geq 0.
\]

We therefore get the lemma.

From the above lemma, we can easily derive some properties for \( \sigma^\varepsilon \):

Lemma 3.6. Properties of \( \sigma^\varepsilon \)

(i) \( \sigma^\varepsilon \geq 0 \). In particular, \( \frac{\partial \sigma^\varepsilon}{\partial n} \leq 0 \) on \( \partial U \).

(ii) \( \int_{\partial U} \varepsilon \frac{\partial \sigma^\varepsilon}{\partial n} dS = -1 \).

Lemma 3.7. There exists a constant \( C > 0 \) s.t.

\[
\int_U \varepsilon|D^2 u^\varepsilon|^2 \sigma^\varepsilon dx \leq C.
\]

Proof
By (3.5), we have:

\[
\int_U (DH(Du^\varepsilon) \cdot Dw^\varepsilon - \varepsilon \Delta w^\varepsilon) \sigma^\varepsilon dx = -\int_U \varepsilon|D^2 u^\varepsilon|^2 \sigma^\varepsilon dx.
\]

Furthermore, we integrate by parts the LHS:

\[
LHS = \int_U -\operatorname{div}(DH(Du^\varepsilon)\sigma^\varepsilon)w^\varepsilon - \varepsilon \Delta \sigma^\varepsilon w^\varepsilon + \int_{\partial U} \varepsilon \frac{\partial \sigma^\varepsilon}{\partial n} w^\varepsilon \\
= \int_U (\varepsilon \Delta \sigma^\varepsilon)w^\varepsilon + \int_{\partial U} \varepsilon \frac{\partial \sigma^\varepsilon}{\partial n} w^\varepsilon \\
= w(x_0) + \int_{\partial U} \varepsilon \frac{\partial \sigma^\varepsilon}{\partial n} w^\varepsilon.
\]

So, by using Lemma 3.6 we get the lemma.

As normal, if we can bound \( \int_U \sigma^\varepsilon dx \) independently of \( \varepsilon \) then everything is fine, we can get the result. However, it’s really a big deal here. We will show the reasons why in a moment.

Choose \( f = 1 \) then (3.7) reads

\[
\left\{
\begin{array}{ll}
DH(Du^\varepsilon) \cdot Dv^\varepsilon = \varepsilon \Delta v^\varepsilon + 1 & \text{in } U, \\
\varepsilon \Delta v^\varepsilon = 0 & \text{on } \partial U.
\end{array}
\right.
\]
And also Lemma 3.5 reads

\[ \int_U \sigma^\varepsilon \, dx = v^\varepsilon(x_0) \geq 0. \]

Hence, in order to bound \( \int_U \sigma^\varepsilon \, dx \), we need to bound \( v^\varepsilon(x_0) \). So, our hope is that \( \max_U v^\varepsilon \) is bounded uniformly independently of \( \varepsilon \).

It turns out that this fact is not true for general \( H \). For example, when \( DH(p) = 0 \) for all \( p \), then we can see that there is a problem. More precisely, let’s consider the following ODE:

\[ \begin{cases} 
\varepsilon \Delta v^\varepsilon + 1 = 0 & \text{in } (0,1), \\
v^\varepsilon(0) = v^\varepsilon(1) = 0. 
\end{cases} \]

Then we get \( v^\varepsilon(x) = \frac{1}{2\varepsilon} (x - x^2) \), hence \( \max_{[0,1]} v^\varepsilon = \frac{1}{8\varepsilon} \), which is really dangerous.

Heuristically, this counter-example shows that we need to have some conditions that allow us to control the \( DH(p) \).

We introduce the second example, which is a good signal, as following:

\[ \begin{cases} 
(\varepsilon v^\varepsilon)' = \varepsilon \Delta v^\varepsilon + 1 & \text{in } (0,1), \\
v^\varepsilon(0) = v^\varepsilon(1) = 0. 
\end{cases} \]

Then we get

\[ v^\varepsilon(x) = x - \frac{e^{\varepsilon x} - 1}{e^{1/\varepsilon} - 1}. \]

Then \( \max_{[0,1]} v^\varepsilon \leq 1 \), which is great and a good signal telling us that we can have the uniformly boundedness of \( \max_U v^\varepsilon \) independent of \( \varepsilon \) with some appropriate conditions.

If we have the convexity-like condition

\( DH(p).p - H(p) \geq -H(0) > 0 \quad \forall \, p \in \mathbb{R}^n, \)

then in fact we can prove the statement above. However, we don’t like this type of condition and we want to relax the convexity condition. So, our idea here is to introduce condition (H3) as we state above:

(H3) There exist \( \gamma, \delta > 0 \) s.t. \( DH(p).p - \gamma H(p) \geq \delta > 0 \quad \forall \, p \in \mathbb{R}^n. \)

In fact, this required condition is similar to the homogenous condition. It’s natural and it works well for a lot of cases where \( H \) is not convex. For example, for \( n = 1 \), let’s consider the following function:

\[ H(p) = (p^2 - 1)^2 - 2 = p^4 - 2p^2 - 1, \]

then \( H \) is not convex and we have

\[ DH(p).p - 2H(p) = (4p^4 - 4p^2) - 2(p^4 - 2p^2 - 1) = 2p^4 + 2 \geq 2 > 0. \]
It’s easy to see that $H$ therefore satisfies all the required conditions of our problem even though $H$ is not convex. We can also see that this condition is suitable and fit well for every required step of our problem. The following lemma is the key lemma of this section, it shows the way to bound $\max_U v^\varepsilon$: 

**Lemma 3.8.** Let $\alpha, \beta \in \mathbb{R}$ and $z(x) = \alpha x.Du^\varepsilon(x) + \beta u^\varepsilon(x)$ then

\[
DH(Du^\varepsilon).Dz - \varepsilon \Delta z = (\alpha + \beta)DH(Du^\varepsilon).Du^\varepsilon - (2\alpha + \beta)\varepsilon \Delta u^\varepsilon.
\]

**Proof**

It’s enough to work with $z(x) = x.Du^\varepsilon(x) = x_iu^\varepsilon_{x_i}$. Thus,

\[
\begin{align*}
z_{x_i} &= u^\varepsilon_{x_i} + x_iu^\varepsilon_{x_i x_i}, \\
z_{x_i x_i} &= u^\varepsilon_{x_i x_i} + u^\varepsilon_{x_i x_i} + x_iu^\varepsilon_{x_i x_i x_i}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
Dz &= Du^\varepsilon + x_iDu^\varepsilon_{x_i}, \\
\Delta z &= 2Du^\varepsilon + x_i\Delta u^\varepsilon.
\end{align*}
\]

Besides, differentiate (3.2) with respect to $x_i$ we get

\[
Dh(Du^\varepsilon).Du^\varepsilon_{x_i} = \varepsilon \Delta u^\varepsilon_{x_i}.
\]

Hence:

\[
\begin{align*}
DH(Du^\varepsilon).Dz - \varepsilon \Delta z &= DH(Du^\varepsilon).Du^\varepsilon - 2\varepsilon \Delta u^\varepsilon + x_i(DH(Du^\varepsilon).Du^\varepsilon_{x_i} - \varepsilon \Delta u^\varepsilon_{x_i}) \\
&= DH(Du^\varepsilon).Du^\varepsilon - 2\varepsilon \Delta u^\varepsilon.
\end{align*}
\]

We get the lemma.

This lemma gives us the key idea to find the supersolution of (3.13) of the type $z$, hence we can get the result by using Maximum principle.

We can choose appropriate $\alpha, \beta$ s.t. $\alpha + \beta > 0$ and $\frac{2\alpha + \beta}{\alpha + \beta} = \gamma$. Then we get:

\[
DH(Du^\varepsilon).Dz - \varepsilon \Delta z = (\alpha + \beta)(DH(Du^\varepsilon).Du^\varepsilon - \gamma \varepsilon \Delta u^\varepsilon)
\]

\[
\geq (\alpha + \beta)(\gamma H(Du^\varepsilon) + \delta - \gamma \varepsilon \Delta u^\varepsilon) = (\alpha + \beta)\delta > 0.
\]

Hence for $k = \frac{1}{(\alpha + \beta)\delta}$ and $y(x) = k(z(x) + M)$ for $M > 0$ large enough so that $y|_{\partial U} \geq 0$ and we also get $y$ is the supersolution of (3.13), i.e.

\[
DH(Du^\varepsilon).Dy - \varepsilon \Delta y \geq 1.
\]

Thus, by the Maximum principle, we easily get:

\[
0 \leq v^\varepsilon \leq y.
\]
So, finally we get, there exists $C > 0$ s.t. $0 \leq v^\varepsilon \leq C$.

**Proof of Theorem 3.3**

Differentiate (3.2) with respect to $\varepsilon$ we get

$$
\begin{aligned}
DH(Du^\varepsilon).Du^\varepsilon &= \varepsilon A_{\varepsilon} + A_{\varepsilon} \quad \text{in } U, \\
\varepsilon \Delta u^\varepsilon &= 0 \quad \text{on } \partial U.
\end{aligned}
$$

Doing the same steps like in the previous chapter we get the result.

4. **Homogenization: The New Proposed Model and the Speed of Convergence to the Effective Hamiltonian**

In this section, we point out the new way of approximation to calculate the effective Hamiltonian.

In [11], Lions, Papanicolaou and Varadhan show the way to find the effective Hamiltonian by consider the following equation

$$(4.1) \quad \varepsilon v^\varepsilon + H(P + Dv^\varepsilon, y) = 0,$$

and they prove that $\varepsilon v^\varepsilon$ converges to $-\check{H}(P)$ as $\varepsilon$ goes to 0 and we also have:

$$(4.2) \quad H(P + Dv, y) = \check{H}(P),$$

where $H$ here is $\mathbb{T}^n$-periodic in the second variable and satisfies some properties as in Section 2.

Recently, Capuzzo-Dolcetta and Ishii prove the speed of convergence of this problem is $O(\varepsilon)$ in [4]. More precisely, there exists a const $C > 0$ s.t.

$$(4.3) \quad |\varepsilon v^\varepsilon + \check{H}(P)| \leq C(1 + |P|)\varepsilon.$$

The proof of this estimate is really simple and only based on some comparison principles. However, there are still some big issues remain. The most difficult one is that even though $\check{H}(P)$ is unique, $v$ in general is not. Also in practice, it’s hard to calculate the solution of (4.1). Our way of approaching this problem is nothing fancy, just again look back at the vanishing viscosity method:

$$(4.4) \quad \varepsilon v^{\varepsilon, \delta} + H(P + Dv^{\varepsilon, \delta}, y) = \delta \Delta v^{\varepsilon, \delta}.$$

As we have already proved, there exists a const $C > 0$ s.t.

$$(4.5) \quad |\varepsilon v^{\varepsilon, \delta} - v^\varepsilon| \leq C \delta^{1/2}.$$

Hence in general we have:

$$(4.6) \quad |\varepsilon v^{\varepsilon, \delta} + \check{H}(P)| \leq C \delta^{1/2} + C\varepsilon,$$
In particular, if we choose $\delta = \varepsilon^2$ then
\begin{equation}
|\varepsilon v^{\varepsilon, \varepsilon^2} + \bar{H}(P)| \leq C\varepsilon.
\end{equation}
This is the motivation for us to consider a slightly different class of approximate equations and show the speed of convergence in this case. Let $z^\theta = v^{\theta, \theta^2}$ then:
\begin{equation}
\theta z^\theta + H(P + Dz^\theta, y) = \theta^2 \Delta z^\theta.
\end{equation}
Firstly, we have some standard observations: $z^\theta$ is unique hence $T_n$-periodic and from [10], there exists a constant $C > 0$ such that
\[ ||z^\theta||_{L^\infty}, ||Dz^\theta||_{L^\infty} \leq C. \]
Also from the $T^n$-periodic property and the boundedness of $||Dz^\theta||_{L^\infty}$, we have one more nice observation
\[ |z^\theta(x) - z^\theta(y)| \leq C ||Dz^\theta||_{L^\infty} \leq C \quad \forall x, y \in \mathbb{R}^n. \]
Like in the the previous sections, our goal here is to prove:
\[ \left| \frac{\partial (z^\theta)}{\partial \theta} \right| \leq C. \]
Although our method is slightly complicated than in [4], it creates a constructive way to study the effective Hamiltonian and we will use this to study the weak KAM theory elsewhere in the future.

Lemma 4.1. Let $w^\theta = \frac{|Dz^\theta|^2}{2}$ then
\begin{equation}
2\theta w^\theta + D_pH.Dw^\theta + D_xH.Dz^\theta = \theta^2 \Delta w^\theta - 2\theta^2|D^2z^\theta|^2.
\end{equation}

The equation here is of first type since (4.9) contains $w^\theta$. Using the same method as in Section 2, we introduce the fake time-dependent function $v^\theta$ s.t. $v^\theta(x, t) = e^{t} w^\theta(x)$ for $t \in [0, T]$ for some $T > 0$ fixed, then $v^\theta$ satisfies
\begin{equation}
2\theta v^\theta_t + D_pH.Dv^\theta + e^t D_xH.Dz^\theta = \theta^2 \Delta v^\theta - 2\theta^2 e^t|D^2z^\theta|^2.
\end{equation}

Adjoint method: We now introduce the adjoint equation of (4.10):
\begin{equation}
\begin{cases}
-2\theta \sigma^\theta_t - \text{div}(D_pH\sigma^\theta) = \theta^2 \Delta \sigma^\theta & (x, t) \in \mathbb{R}^n \times (0, T), \\
\sigma^\theta|_T = \delta_{x_0}.
\end{cases}
\end{equation}
Similarly, we have some properties of $\sigma^\theta$ like above:
Lemma 4.2. Properties of $\sigma^\theta$

(i) $\sigma^\theta(x,t) \geq 0 \quad (x,t) \in \mathbb{R}^n \times (0,T)$,

(ii) $\int_{\mathbb{R}^n} \sigma^\theta(x,t) dx = 1 \quad t \in (0,T)$.

Lemma 4.3. There exists a const $C > 0$ s.t.

$$\theta^2 \int_0^T \int_{\mathbb{R}^n} |D^2 z^\theta|^2 \sigma^\theta \leq C. \quad (4.12)$$

Again, all the estimates here don’t depend on the choice of $x_0$ as stated carefully in section 2.

Theorem 4.4. There exists a const $C > 0$ s.t.

$$| (\theta z^\theta)_{\theta}(x) | \leq C. \quad (4.13)$$

Proof

Firstly, differentiate (4.8) with respect to $\theta$ we get

$$z^\theta + \theta z^\theta_0 + D_p H Dz^\theta_0 = \theta^2 \Delta z^\theta_0 + 2\theta \Delta z^\theta. \quad (4.14)$$

Doing the same steps as in Theorem 2.7, we get the following

$$2\theta (e^T z^\theta_0(x_0) - \int_{\mathbb{R}^n} z^\theta_0(x) \sigma^\theta(x,0) dx) - \theta \int_0^T \int_{\mathbb{R}^n} z^\theta_0 \sigma^\theta dx dt +$$

$$+ \int_0^T \int_{\mathbb{R}^n} e^T z^\theta_0 \sigma^\theta dx dt = 2\theta \int_0^T \int_{\mathbb{R}^n} e^T \Delta z^\theta_0 \sigma^\theta dx dt. \quad (4.15)$$

Let

$$A = 2\theta (e^T z^\theta_0(x_0) - \int_{\mathbb{R}^n} z^\theta_0(x) \sigma^\theta(x,0) dx) - \theta \int_0^T \int_{\mathbb{R}^n} z^\theta_0 \sigma^\theta dx dt,$$

$$B = \int_0^T \int_{\mathbb{R}^n} e^T z^\theta_0 \sigma^\theta dx dt.$$

Notice that we have $|A + B| \leq C$ for some positive constant $C$ independently of the choice of $x_0$ by Lemma 4.3.

We have several observations below:

(i) Take any $x' \in \mathbb{R}^n$, we can control $B$ in term of $z^\theta(x')$ by using the property $|z^\theta(x) - z^\theta(x')| \leq C$ for all $x \in \mathbb{R}^n$. More explicitly,

$$|B - (e^T - 1)z^\theta(x')| \leq \int_0^T \int_{\mathbb{R}^n} e^T |z^\theta(x) - z^\theta(x')| \sigma^\theta dx dt \leq C(e^T - 1) = C. \quad (4.16)$$
(ii) Now we need to control $A$. There exist $x_1, x_2 \in \mathbb{R}^n$ s.t.
\[ z_0^\theta(x_1) = m = \min_{\mathbb{R}^n} z_0^\theta(x) \leq z_0^\theta(x) \leq \max_{\mathbb{R}^n} z_0^\theta(x) = M = z_0^\theta(x_2). \]

(Again, if the maximum or minimum does not attain, we can deal with the problem like what we do in Section 2. Let’s make everything simple and clear.)

Let $\sigma_0^\theta, x_1$ be the solution of the adjoint equation corresponding to $x_1$ (here we need to change from $x_0$ to $x_1$), then we have:
\[ A = 2\theta(e^T z_0^\theta(x_1) - \int_{\mathbb{R}^n} z_0^\theta(x) \sigma_0^\theta, x_1(x,0)dx) - \theta \int_0^T \int_{\mathbb{R}^n} z_0^\theta \sigma_0^\theta, x_1 dxdt \leq (e^T - 1) \theta z_0^\theta(x_1). \]

Therefore, by both of our observations, we have
\[ -C \leq A + B \leq (e^T - 1) \theta z_0^\theta(x_1) + B \leq (e^T - 1) m \theta + (e^T - 1) z_0^\theta(x') + C, \]

which implies that
\begin{equation}
(4.17) \quad -C \leq (e^T - 1)(m \theta + z_0^\theta(x')) \quad \forall x' \in \mathbb{R}^n.
\end{equation}

Similarly, for $\sigma_0^\theta, x_2$ be the solution of the adjoint equation corresponding to $x_2$ (here we need to change from $x_0$ to $x_2$), then we also have:
\[ A = 2\theta(e^T z_0^\theta(x_2) - \int_{\mathbb{R}^n} z_0^\theta(x) \sigma_0^\theta, x_2(x,0)dx) - \theta \int_0^T \int_{\mathbb{R}^n} z_0^\theta \sigma_0^\theta, x_2 dxdt \geq (e^T - 1) \theta z_0^\theta(x_2). \]

Therefore, we will also have
\begin{equation}
(4.18) \quad (e^T - 1)(M \theta + z_0^\theta(x')) \leq C \quad \forall x' \in \mathbb{R}^n.
\end{equation}

From (4.17) and (4.18) we get
\begin{equation}
(4.19) \quad (e^T - 1)|\theta z_0^\theta(x) + z_0^\theta(x)| \leq C \quad \forall x \in \mathbb{R}^n.
\end{equation}

We get the theorem.

5. APPENDIX

We will prove the uniqueness of $u^\epsilon$ in (3.2).

**Theorem 5.1.** If $u$ and $v$ are the solutions of (3.2) then we get $u = v$.

**Proof**

It’s enough to prove that $u \leq v$.

If we have $H(Du) - \epsilon \Delta u < H(Dv) - \epsilon \Delta v$ in $U$ and $u \leq v$ on $\partial U$ then we easily get $u \leq v$ in $U$.

In general, we will go exactly the same way as the normal way to prove the weak Maximum principle. The strategy is to find a sequence of functions $\{z_0^\theta\}$ such that $H(Du) - \epsilon \Delta u < H(Dz_0^\theta) - \epsilon \Delta z_0^\theta$ in $U$ and $u \leq z_0^\theta$ on $\partial U$ and $z_0^\theta$ converges to $v$. Hence we get $u \leq z_0^\theta$.
for all $\theta$, which implies $u \leq v$.
Recall from Remark 3.2 the properties of $\phi$, we get for $t > 1$ then
\[
H(tp) \geq t^\gamma H(p) + \frac{\delta}{\gamma+1} (t^\gamma - 1).
\]

Let $z = sv + t(x.Dv + M)$ where $M > 0$ is a constant will be chosen later. Our goal here is
to find $z$ such that $H(Dz) - \varepsilon \Delta z > 0$ corresponding to $s \to 1$ and $t \to 0$, we then derive the
result. We can see that the function $z$ here is really similar to the one we need in pointing
out the super-solution in Lemma 3.8. We have:
\[
Dz = (s+t)Dv + tx_iDv_{x_i},
\]
\[
\Delta z = (s+2t)\Delta v + tx_i\Delta v_{x_i}.
\]
For $s$ very close to 1, for $t > 0$ very close to 0 and $s+t > 1$ we have:
\[
H(Dz) - \varepsilon \Delta z = H((s+t)Dv + tx_iDv_{x_i}) - \varepsilon (s+2t)\Delta v + tx_i\Delta v_{x_i}
\]
\[
= H((s+t)Dv) + tDH((s+t)Dv).(x_iDv_{x_i}) + t^2 O(1) - \varepsilon (s+2t)\Delta v + tx_i\Delta v_{x_i}
\]
\[
= H((s+t)Dv) + tDH(Dv).(x_iDv_{x_i}) + t((s+t) - 1)O(1) + t^2 O(1) -
\]
\[
- \varepsilon (s+2t)\Delta v + tx_i\Delta v_{x_i}
\]
\[
\geq (s+t)^\gamma H(Dv) + \frac{\delta}{\gamma+1} ((s+t)^\gamma - 1) + t((s+t) - 1)O(1) + t^2 O(1) -
\]
\[
- \varepsilon (s+2t)\Delta v.
\]
Hence if we choose $s$ and $t$ such that $(s+t)^\gamma = s + 2t$ then everything seems to be nice.
For $\theta > 0$, we let: $t = (1+\theta)^\gamma - (1+\theta)$ and $s = 2(1+\theta) - (1+\theta)^\gamma$.
Then we get $(s+t)^\gamma = s + 2t = (1+\theta)^\gamma$ and for $\theta$ small enough we also have:
\[
\frac{\delta}{\gamma+1} ((s+t)^\gamma - 1) + t((s+t) - 1)O(1) + t^2 O(1) > 0.
\]
Hence we get $H(Dz) - \varepsilon \Delta z > 0$.
Finally, choose $M$ large enough to guarantee $z \geq u$ on $\partial U$, we get the theorem.

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