Abstract

Let $\mathcal{F}$ be any collection of linearly separable sets of a set $P$ of $n$ points either in $\mathbb{R}^2$, or in $\mathbb{R}^3$. We show that for every natural number $k$ either one can find $k$ pairwise disjoint sets in $\mathcal{F}$, or there are $O(k)$ points in $P$ that together hit all sets in $\mathcal{F}$. The proof is based on showing a similar result for families $\mathcal{F}$ of sets separable by pseudo-discs in $\mathbb{R}^2$. We complement these statements by showing that analogous result fails to hold for collections of linearly separable sets in $\mathbb{R}^4$ and higher dimensional euclidean spaces.
1 Introduction

Let $\mathcal{H} = (V, E)$ be a a hyper-graph. A hitting set for $\mathcal{H}$ is a subset of vertices which intersects every edge in $E$. A matching in $\mathcal{H}$ is a subset of mutually disjoint edges. Let $\tau(\mathcal{H})$ denote the size of a minimum hitting set of $\mathcal{H}$ and let $\nu(\mathcal{H})$ denote the size of a maximum matching of $\mathcal{H}$. The parameters $\tau(\mathcal{H}), \nu(\mathcal{H})$ were studied extensively in combinatorics and in computer science. $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ relate to each other. Indeed, every hitting set must contain a distinct element from each edge in any matching and therefore $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$. Moreover, by strong duality for linear programming it follows that the size of a minimum fractional hitting set, denoted by $\nu^*(\mathcal{H})$, is equal to the size of a maximum fractional matching, denoted by $\tau^*(\mathcal{H})$. So every hyper-graph $\mathcal{H}$ satisfies:

$$\nu(\mathcal{H}) \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq \tau(\mathcal{H}).$$

Hyper-graphs $\mathcal{H}$ for which $\tau(\mathcal{H}) = \nu(\mathcal{H})$ or for which $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ are close to each other have also been studied. See for example [4, 2, 3] and references within.

We study the gap between $\nu(\mathcal{H})$ and $\tau(\mathcal{H})$ for hyper-graphs which can be realized by an arrangement of half-spaces in $\mathbb{R}^d$ when $d$ is small. This property is quantified by the affine sign-rank. The affine sign-rank of a hyper-graph $\mathcal{H}$ is the minimum number $d$ for which there is an identification of $V(\mathcal{H})$ as points in $\mathbb{R}^d$ and of $E(\mathcal{H})$ as half-spaces in $\mathbb{R}^d$ such that for all $v \in V(\mathcal{H}), e \in E(\mathcal{H})$, $v \in e$ if and only if the point corresponding to $v$ is in the half-space corresponding to $e$. The affine sign-rank is closely related to the sign-rank of $\mathcal{H}$ which was studied in many contexts such as geometry [3], machine learning [14, 7, 15], communication complexity [23, 12, 13, 26] and more.

Hyper-graphs with small affine sign-rank have small VC dimension (at most the affine sign-rank plus one) and therefore, by [8, 11], for such hyper-graphs:

$$\tau(\mathcal{H}) \leq O(\tau^*(\mathcal{H}) \log \tau^*(\mathcal{H})).$$

How about $\nu(\mathcal{H})$? Is it also close to $\nu^*(\mathcal{H})$? In general, low VC dimension does not imply that $\nu(\mathcal{H})$ is close to $\nu^*(\mathcal{H})$. A simple example is given by $\mathcal{H} = (P, L)$ where $P$ and $L$ are the sets of points and lines in a projective plane of order $n$. Recall that in a projective plane of order $n$ $|P| = |L| = n^2 + n + 1$, each two lines intersect in a unique point, each two points have a unique line containing both of them, each line contain exactly $n + 1$ points and each point has exactly $n + 1$ lines containing it. Thus, its VC dimension is 2, $\nu(\mathcal{H}) = 1$ (since every two lines intersect) and $\nu^*(\mathcal{H}) \geq \frac{|L|}{n+1} = \frac{n^2 + n + 1}{n+1} = \Omega(n)$ as we may choose a $\frac{1}{n+1}$ fraction of every line so that every point is covered exactly once and the total weight of the fractional matching is $\frac{|L|}{n+1}$. However, since the affine sign-rank of $\mathcal{H}$ is $\Omega(n^{1/2})$ [13, 5] this example does not rule out the possibility that $\tau$ and $\nu$ are close for hyper-graphs of constant affine sign-rank.

We show that if the affine sign-rank of $\mathcal{H}$ is less than 4 then $\tau(\mathcal{H}) = \Theta(\nu(\mathcal{H}))$. We complement this by showing that there are hyper-graphs $\mathcal{H}$ with affine sign-rank 4 such that $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H})$ is arbitrarily large.

We note that the fact that $\tau(\mathcal{H}) = \Theta(\nu(\mathcal{H}))$ when the affine sign-rank is 2 is already known [10]. For completeness we add our alternative proof for it and show how this proof is generalized to capture the case of affine sign-rank 3.
2 Our results

For a set \( P \) of points in \( \mathbb{R}^d \) and a family \( F \) of ranges in \( \mathbb{R}^d \) we denote by \( \mathcal{H}(P, F) \) the hyper-graph on the set of vertices \( P \) whose edges consist of the sets \( \{P \cap F \mid F \in F\} \), without multiplicities. So, the affine sign-rank of \( \mathcal{H} \) is \( d \) if and only if there is a set \( P \) of points in \( \mathbb{R}^d \) and a family \( F \) of half-spaces in \( \mathbb{R}^d \) such that \( \mathcal{H} \) is isomorphic to \( \mathcal{H}(P, F) \).

2.1 The case of affine sign-rank 2 and pseudo-discs

As mentioned above, we show that if \( \mathcal{H} \) is a hyper-graph with affine sign-rank 2 then \( \tau(\mathcal{H}) = \Theta(\nu(\mathcal{H})) \). In fact, we prove it for a more general class of hyper-graphs: A family \( C \) of simple closed curves in \( \mathbb{R}^2 \) is called a family of pseudo-circles if every two curves in \( C \) are either disjoint or cross at two points. A family of circles, no two of which touch, is a natural example for such a family. A family of pseudo-discs is a family of compact sets whose boundaries form a family of pseudo-circles. Natural examples for families of pseudo-discs are translates of a fixed convex set in the plane as well as homothetic copies of a fixed convex set in the plane.

Note that if the affine sign-rank of \( \mathcal{H} \) is 2 then there is a set of points \( P \) in the plane and a family of pseudo-discs \( F \) such that \( \mathcal{H} \) is isomorphic \( \mathcal{H}(P, F) \) (just replace each half-space by a large enough circular disc).

**Theorem 1** ([10]). Let \( P \) be a set of points in the plane and let \( F \) be a family of pseudo-discs. Let \( \mathcal{H} \) be the hyper-graph \( \mathcal{H} = \mathcal{H}(P, F) \). Then for every integer \( k \geq 1 \) either \( \mathcal{H} \) has \( k \) pairwise disjoint edges, or one can find \( O(k) \) points in \( P \) that hit all the edges in \( \mathcal{H} \).

Theorem 1 implies that every \( \mathcal{H} \) with affine sign-rank 2 has \( \tau(\mathcal{H}) = \Theta(\nu(\mathcal{H})) \). Theorem 1 was proved by Chan and Har-Peled in [10], however the proof that we present here is based on a different approach. Our methods are useful also in the case when the affine sign-rank is 3. The proof of Theorem 1 is based on the following Theorem:

**Theorem 2.** Let \( F \) be a family of pseudo-discs in the plane. Let \( P \) be a finite set of points in the plane and consider the hyper-graph \( \mathcal{H} = \mathcal{H}(P, F) \). There exists an edge \( e \) in \( \mathcal{H} \) such that the maximum cardinality of a matching among the edges in \( \mathcal{H} \) that intersect with \( e \) is at most 156.

Theorem 2 implies Theorem 1 as follows. Apply Theorem 2 to find an edge \( e \) in \( \mathcal{H} \) such that among those edges intersecting it there are at most 156 pairwise disjoint ones. Delete \( e \) and those edges intersecting it from \( \mathcal{H} \). Repeat this until the graph is empty. If this continues \( k \) steps, then we find \( k \) pairwise disjoint edges. Otherwise, we decompose \( \mathcal{H} \) into less than \( k \) families, \( \mathcal{H}_1, \ldots, \mathcal{H}_\ell \), of edges such that in each family \( \mathcal{H}_i \) there are at most 156 pairwise disjoint edges.

We then show that for every \( 1 \leq i \leq \ell \) the edges in \( \mathcal{H}_i \) can be pierced by \( O(1) \) points. This will conclude the proof of Theorem 1. In order to show that each \( \mathcal{H}_i \) is indeed pierced by \( O(1) \) points, we rely on the techniques of Alon and Kleitman in [1] by proving a \((p, q)\) Theorem for each of the \( \mathcal{H}_i \) (see the proof of Theorem 1).

Theorem 2 is a discrete version (and therefore also generalization) of Theorem 1 in [24], in which the set \( P \) is the entire plane. The proof of Theorem 2 follows the proof of Theorem 1 in [24] with some suitable adjustments.

The result in Theorem 2 (and also Theorem 1 in [24]) can be interpreted as saying that in every family of pseudo-discs there is a so called “small” pseudo-disc. Indeed, notice that in every
family of circular discs, the disc of smallest area, $D$, has the property that the maximum number of mutually disjoint discs from the family that intersect with it is at most $O(1)$ (see the introduction in [24] and the references therein for more details). Theorem 2 implies that the same phenomenon happens in every family of pseudo-discs.

The authors of [10], in which Theorem 1 was first proved, explicitly note that one of the challenges they overcome is the absence of a “smallest pseudo-disc”. In this paper and in [24] the existence of such pseudo-disc is proved. We prove Theorems 1 and 2 in Section 3.

2.2 The case of affine sign-rank

**Theorem 3.** Let $P$ be a set of points in $\mathbb{R}^3$ and let $\mathcal{F}$ be a family of half-spaces. Let $\mathcal{H}$ be the hyper-graph $\mathcal{H} = \mathcal{H}(P, \mathcal{F})$. Then for every integer $k \geq 1$ either $\mathcal{H}$ has $k$ pairwise disjoint edges, or one can find $O(k)$ points in $P$ that hit all the edges in $\mathcal{H}$.

Like in the case of affine sign-rank 2, the proof of Theorem 3 is based on the following theorem that is an analogue of Theorem 2:

**Theorem 4.** Let $P$ be a set of points in $\mathbb{R}^3$ and let $\mathcal{F}$ be a family of half-spaces. Let $\mathcal{H}$ be the hyper-graph $\mathcal{H} = \mathcal{H}(P, \mathcal{F})$. Then there exists an edge in $\mathcal{H}$ such that the cardinality of the maximum matching among the edges in $\mathcal{H}$ intersecting it is at most 156.

We prove Theorems 3 and 4 in Section 4.

2.3 The case of affine sign-rank

We show that the analogous result to Theorems 1 and 3 fails for affine sign-rank greater than 3.

**Theorem 5.** For every $n \in \mathbb{N}$ there exists a set $P$ of $N = \binom{n}{2}$ points and a set $\mathcal{F}$ of $n$ half-spaces in $\mathbb{R}^4$ such that:

1. Every two edges in $\mathcal{H}(P, \mathcal{F})$ have a non-empty intersection (which implies that $\nu(\mathcal{H}) = 1$).
2. Any subset of $P$ which pierce all edges in $\mathcal{H}(P, \mathcal{F})$ has at least $\frac{n-1}{2}$ points in it (i.e. $\tau(\mathcal{H}) \geq \frac{n-1}{2}$).

We prove Theorem 5 in Section 5.

2.4 Connection to $\epsilon$-nets

Theorems 1 and 3 immediately imply a result from [21] about the existence of an $\epsilon$-net of size linear in $\frac{1}{\epsilon}$ for hyper-graphs $\mathcal{H}(P, \mathcal{F})$, where $\mathcal{F}$ is a family of pseudo-discs in $\mathbb{R}^2$ (hence also the special case where $\mathcal{F}$ is a family of half-planes) or half-spaces in $\mathbb{R}^3$. Indeed, given such a hyper-graph $\mathcal{H}$ and $\epsilon > 0$, we delete from $\mathcal{H}$ all the edges of cardinality smaller than $\epsilon|P|$. Set $k = \frac{1}{\epsilon}$. Notice that now $\mathcal{H}$ does not contain $k$ pairwise disjoint edges simply because every edge is of cardinality greater than $\epsilon|P|$. It follows that one can find $O(k) = O(\frac{1}{\epsilon})$ points in $P$ that meet all the edges in $\mathcal{H}$.
Pach and Tardos \cite{pach-tardos} have recently shown that for every \( \epsilon > 0 \) and large enough \( n \), there is a collection of \( n \) points, \( P \), in \( \mathbb{R}^4 \) and a collection of half spaces, \( F \), such that every \( \epsilon \)-net for \( \mathcal{H}(P,F) \) has size \( \Omega(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \). This corresponds to Theorem 5, and in fact implies some variant of it.

### 2.5 An algorithmic application

An immediate algorithmic application of Theorems 1 and 3 is a polynomial constant factor approximation algorithm for finding maximum matching in hyper-graphs of the form \( \mathcal{H}(P,F) \) where \( F \) is a set of pseudo-discs (or half-planes) and \( P \subseteq \mathbb{R}^2 \) or \( F \) is a set of half-spaces in \( \mathbb{R}^3 \) and \( P \subseteq \mathbb{R}^3 \). Indeed, given such a hyper-graph \( \mathcal{H} \), we can repeatedly find a “small” edge \( e \in E(\mathcal{H}) \) in the sense of Theorems 1 and 3, add it to the matching and then delete \( e \) and those edges intersecting it from \( \mathcal{H} \) and continue until all the edges of \( \mathcal{H} \) are consumed. The final maximal (with respect to set containment) matching \( M \) has size which is at least \( \frac{1}{156} \) of the size of a maximum matching. We note that Chan and Har-Peled \cite{chan-har-peled} give a PTAS for maximum matching among pseudo-discs, with a different constant, also for the weighted case.

### 3 The case of affine sign-rank 2 and pseudo-discs

In this section we prove Theorem 1 and Theorem 2.

We start with the proof of Theorem 2 and then use this result to prove Theorem 1.

An important special case of Theorem 2 in which the set \( P \) is the set of all point in \( \mathbb{R}^2 \) is shown in \cite{lachner}. The proof of Theorem 2 will follow the same lines of the proof in \cite{lachner} with some suitable adjustments.

The idea of the proof is to show that if \( B \) is a maximum matching in \( \mathcal{H} \) then on average over all edges \( e \in B \) the cardinality of a maximum matching among the edges in \( \mathcal{H} \) that intersects with \( e \) is less than 157. This means that there exists an edge in \( B \) with the desired property.

We will make use of the following lemma that is in fact Corollary 1 in \cite{lachner}:

**Lemma 1.** Let \( B \) be a family of pairwise disjoint sets in the plane and let \( F \) be a family of pseudo-discs. Let \( D \) be a member of \( F \) and suppose that \( D \) intersects exactly \( k \) members of \( B \) one of which is the set \( e \in B \). Then for every \( 2 \leq \ell \leq k \) there exists a set \( D' \subseteq D \) such that \( D' \) intersects \( e \) and exactly \( \ell - 1 \) other sets from \( B \), and \( F \cup \{D'\} \) is again a family of pseudo-discs.

We will also need the next lemma that is parallel to (and will take the place of) Lemma 2 in \cite{lachner}.

**Lemma 2.** Let \( F \) be a family of pseudo-discs in the plane. Let \( P \) be a finite set of points in the plane and consider the hyper-graph \( \mathcal{H} = \mathcal{H}(P,F) \). Assume \( B \) is a subgraph of \( \mathcal{H} \) consisting of pairwise disjoint hyper-edges. Consider the graph \( G \) whose vertices correspond to the edges in \( B \) and connect two vertices \( e, e' \in B \) by an edge if there is an edge in \( \mathcal{H} \) that has a nonempty intersection with \( e \) and with \( e' \) and has an empty intersection with all other edges in \( B \). Then \( G \) is planar.

**Proof.** We draw \( G \) as a topological graph in the plane as follows. From every edge \( e \in B \) we pick one vertex, that we denote by \( v(e) \), and the collection of all these vertices is the set \( V \) of vertices
of $G$. Denote by $\mathcal{H}_2$ the set of all edges in $H$ that have a non-empty intersection with precisely two of the edges in $B$. For every pair of edges $e$ and $e'$ in $B$ that are intersected by some edge $f$ (possibly such an edge $f$ is not unique) in $\mathcal{H}_2$ we draw an edge between $v(e)$ and $v(e')$ as follows. Pick a vertex $x \in e \cap f$ and a vertex $x' \in e' \cap f$. Recall that $f$ is the intersection of $P$ with some pseudo-disc $D$ in $\mathcal{F}$. Similarly, let $C$ and $C'$ be two pseudo-discs in $\mathcal{F}$ whose intersection with $P$ is equal to $e$ and $e'$, respectively. Let $W_{xx'}$ be an arc, connecting $x$ and $x'$, that lies entirely in $D$. Let $W_{v(e)x}$ be an arc connecting $v(e)$ to $x$ that lies entirely in $C$. Let $W_{v(e')x'}$ be an arc connecting $v(e')$ to $x'$ that lies entirely in $C'$. Finally, we draw the edge in $G$ connecting $v(e)$ and $v(e')$ as the union (or concatenation) of $W_{v(e)x}, W_{xx'},$ and $W_{x'v(e')}$. We will show that any two edges in $G$ that do not share a common vertex are drawn so that they cross an even number of times. The Hanani-Tutte Theorem ([16] [23]) then implies the planarity of $G$.

We will use the following elementary lemma from [9]:

**Lemma 3** (Lemma 1 in [9]). Let $D_1$ and $D_2$ be two pseudo-discs in the plane. Let $x$ and $y$ be two points in $D_1 \setminus D_2$. Let $a$ and $b$ be two points in $D_2 \setminus D_1$. Let $\gamma_{xy}$ be any Jordan arc connecting $x$ and $y$ that is fully contained in $D_1$. Let $\gamma_{ab}$ be any Jordan arc connecting $a$ and $b$ that is fully contained in $D_2$. Then $\gamma_{xy}$ and $\gamma_{ab}$ cross an even number of times.

Let $v(e), v(e'), v(k), v(k')$ be four distinct vertices of $G$. This means in particular that $e, e', k,$ and $k'$ are four pairwise disjoint hyper-edges in $B$. Suppose that $v(e)$ and $v(e')$ are connected by an edge in $G$. This means that there are $x \in e$ and $x' \in e'$ and $f \in \mathcal{H}_2$ such that $x \in e \cap f$ and $x' \in e' \cap f$. Let $E, E'$, and $F$ in $\mathcal{F}$ be the pseudo-discs such that $e = E \cap P,$ $e' = E' \cap P,$ and $f = F \cap P$. Suppose also that $v(k)$ and $v(k')$ are connected by an edge in $G$. This means that there are $y \in k$ and $y' \in k'$ and $q \in \mathcal{H}_2$ such that $y \in k \cap q$ and $y' \in k' \cap q$. Let $K, K'$, and $Q$ in $\mathcal{F}$ be the pseudo-discs such that $k = K \cap P,$ $k' = K' \cap P,$ and $q = Q \cap P$.

By Lemma 3 $W_{v(e)x}$ and $W_{v(y)y}$ cross an even number of times. Indeed, $E$ contains $v(e)$ and $x$ and does not contain $v(k)$ and $y$. $K$ contains $v(k)$ and $y$ and does not contain $v(e)$ and $x$. Similarly, each of $W_{v(e)x}, W_{xx'},$ and $W_{v(e')x'}$ crosses each of $W_{v(k)y}, W_{yy'},$ and $W_{v(k')y'}$ an even number of times. We conclude that the edge in $G$ connecting $v(e)$ and $v(e')$ crosses the edge in $G$ connecting $v(k)$ and $v(k')$ an even number of times, as desired. 

**Proof of Theorem 2** The proof goes almost verbatim as the proof of Theorem 1 in [24]. Lemma 2 in [24] is replaced by the above Lemma 2.

Let $B$ be a collection of pairwise disjoint edges in $\mathcal{H}$ of maximum cardinality and let $n = |B|$. For every $e \in B$ denote by $\alpha_1(e)$ the size of a maximum matching among those edges in $\mathcal{H}$ that intersect with $e$ but with no other edge in $B$. Denote by $\alpha_2(e)$ the size of a maximum matching among those edges in $\mathcal{H}$ that intersect with $e$ and with precisely one more edge in $B$. Denote by $\alpha_3(e)$ the size of a maximum matching among those edges in $\mathcal{H}$ that intersect with $e$ and with at least two more edges in $B$. Observe that it is enough to show that $\sum_{e \in B} \alpha_1(e) + \alpha_2(e) + \alpha_3(e) < 157n$.

We first note that for every $e \in B$ we must have $\alpha_1(e) \leq 1$. Indeed, otherwise one can find two disjoint edges $e'$ and $e''$ in $H$ that do not intersect with any edge in $B$ but $e$. The set $B \cup \{e', e''\} \setminus \{e\}$ contradicts that maximality of $B$.

Next, we show that $\sum_{e \in B} \alpha_2(e) \leq 12n$. Consider the graph $G$ whose vertices correspond to the edges in $B$ and connect two vertices $e, e' \in B$ by an edge if there is an edge in $\mathcal{H}$ that has a nonempty intersection with $e$ and with $e'$ and has an empty intersection with all other edges in $B$. By Lemma 2 $G$ is planar. Therefore, $G$ has at most $3n$ edges. For every $e \in B$ denote by $d(e)$ the
degree of \( e \) in \( G \). Therefore,

\[
\sum_{e \in B} d(e) \leq 6n. \tag{1}
\]

We claim that for every \( e \in B \) we have \( \alpha_2(e) \leq 2d(e) \). Indeed, otherwise by the pigeonhole principle one can find three pairwise disjoint edges \( g, g', \) and \( g'' \) in \( \mathcal{H} \) and an edge \( e' \) in \( B \) such that each of \( g, g', \) and \( g'' \) intersects \( e \) and \( e' \) but no other edge in \( B \). In this case \( B \cup \{g, g', g''\} \setminus \{e, e'\} \) contradicts the maximality of \( B \).

Inequality (1) implies now \( \sum_{e \in B} \alpha_2(e) \leq 12n \). It remains to show that \( \sum_{e \in B} \alpha_3(e) < 144n \). The derivation of this inequality is more involved than the derivation of the inequalities regarding \( \alpha_1, \alpha_2 \). We will show that if it is not the case that \( \sum_{e \in B} \alpha_3(e) < 144n \), then we can derive an (impossible) embedding of \( K_{3,3} \) in the plane.

Denote by \( \mathcal{F}_3 \) the subfamily of \( \mathcal{F} \) that consists of pseudo-discs in \( \mathcal{F} \) that intersect with three or more edges in \( B \). Using repeatedly Lemma 1 with \( \mathcal{F} = \mathcal{H}_3 \) and with \( \ell = 3 \), we can find, for every \( D \in \mathcal{H}_3 \) and every \( e \in B \) that is intersected by \( D \), a (new) pseudo-disc \( D^e \subset D \) that intersects with \( e \) and with exactly two more sets from \( B \). Moreover, the collection of all the new sets \( D^e \) obtained in this way is a family of pseudo-discs. We denote this family of pseudo-discs by \( \mathcal{D} \). Let \( T \) denote the set of all triples of edges in \( B \) that are intersected by a pseudo-disc in \( \mathcal{D} \).

We denote by \( Z \) the collection of all pairs of sets from \( B \) that appear together in some triple in \( T \). We claim that \( |Z| < 12n \): Pick every set in \( B \) with probability \( \frac{1}{2} \). Call a pair \( \{e, e'\} \) in \( Z \) good if both \( e \) and \( e' \) were picked and an edge \( f \in B \) such that \( e, e' \), and \( f \) is a triple in \( T \) was not picked. The expected number of good pairs in \( Z \) is at least \( 1/8 \) of the pairs in \( Z \). On the other hand, by Lemma 2 the set of good pairs in \( Z \) is the set of edges of a planar graph (on an expected number of \( n/2 \) vertices) and therefore the expected number of good pairs is less than \( 3 \cdot \frac{n}{2} \).

Now consider the graph \( K \) whose set of vertices is the edges in \( B \) and whose set of edges is \( Z \). For every \( e \in B \) denote by \( d(e) \) the degree of \( e \) in this graph. Notice that, in view of the above, \( \sum_{e \in B} d(e) = 2|Z| < 24n \).

Fix \( e \in B \). Define a graph \( K^e \) on the set of neighbors of \( e \) in \( K \) where we connect two neighbors \( e_1, e_2 \) of \( e \) in \( K \) by an edge in \( K^e \) if and only if \( \{e, e_1, e_2\} \) is a triple in \( T \). This is equivalent to that there is \( D \in \mathcal{D} \) that intersects with \( e, e_1 \), and with \( e_2 \). Denote by \( t(e) \) the number of edges in \( K^e \). By ignoring the set \( e \) and applying Lemma 2 we see that \( K^e \) is planar. \( K^e \) has \( d(e) \) vertices and is planar and therefore \( t(e) < 3d(e) \).

We claim that for every \( e \in B \) we must have \( \alpha_3(e) \leq 2t(e) \). Indeed, assume to the contrary that \( \alpha_3(e) > 2t(e) \). Then there is a collection \( Q \) of at least \( 2t(e) + 1 \) pairwise disjoint edges of \( \mathcal{H} \), each of which has a non-empty intersection with \( e \) and with at least two more edges in \( B \). Because of Lemma 1 every edge in \( Q \) must have a non-empty intersection with \( e \) and with at least two edges \( e' \) and \( e'' \) that form a pair in \( Z \). The hyper-edges \( e' \) and \( e'' \) are therefore connected by an edge in \( K^e \). By the pigeonhole principle, because there are only \( t(e) \) edges in \( K^e \) while \( |Q| \geq 2t(e) + 1 \), there exist \( e' \) and \( e'' \) that are connected by an edge in \( K^e \) such that \( e, e' \), and \( e'' \) are all intersected by three (pairwise disjoint) edges \( g_1, g_2, g_3 \in \mathcal{D} \). This is impossible as it gives an embedding of the graph \( K_{3,3} \) in the plane. To see this, recall that also the sets \( e, e_1, e_2 \) are pairwise disjoint. For every \( 1 \leq i, j \leq 3 \) add a small pseudo-disc surrounding one point in the intersection of \( e_i \) and \( g_j \). Lemma 2 implies now an (impossible) embedding of \( K_{3,3} \) in the plane.
We conclude that
\[ \sum_{e \in B} \alpha_3(e) < \sum_{e \in B} 2t(e) \leq \sum_{e \in B} 6d(e) \leq 6 \cdot 24n = 144n. \]

The proof is now complete as we have
\[ \sum_{e \in B} \alpha_1(e) + \alpha_2(e) + \alpha_3(e) < n + 12n + 144n = 157n \]

This implies the existence of \( e \in B \) such that \( \alpha_1(e) + \alpha_2(e) + \alpha_3(e) \leq 156 \).

Having proved Theorem 2, we are now ready to prove Theorem 1.

**Proof of Theorem 1.** Repeatedly apply Theorem 2 and find an edge \( e \) in \( H \) such that among those edges intersecting it there are at most 156 pairwise disjoint ones. Then delete \( e \) and those edges intersecting it from \( H \) and continue. If we can continue \( k \) steps, then we find \( k \) pairwise disjoint edges. Otherwise, we decompose \( H \) into less than \( k \) families, \( H_1, \ldots, H_\ell \), of edges such that in each family \( H_i \) there are at most 156 pairwise disjoint edges.

We will now show that for every \( 1 \leq i \leq \ell \) the edges in \( H_i \) can be pierced by \( O(1) \) points. This will conclude the proof of Theorem 1.

Our strategy is to show that the edges in \( H_i \) have the so called \((p, q)\) property for some \( p \) and \( q \). That is, out of every \( p \) sets in \( H_i \) there are \( q \) that have a non-empty intersection. In fact, by the definition of \( H_i \), it has the \((157, 2)\) property because there are at most 156 sets in \( H_i \) that are pairwise disjoint. This is the first step. The next step is to show a \((p, q)\) (for the same \( q \) above, that is \( q = 2 \)) theorem for hyper-graphs \( H(\mathcal{P}, \mathcal{F}) \) where \( \mathcal{F} \) is a family of pseudo-discs. This means that we will need to show that for a family of pseudo-discs \( \mathcal{F} \) if \( H(\mathcal{P}, \mathcal{F}) \) has the \((p, q)\) property, then one can find a constant number of points in \( \mathcal{P} \) that together pierce all edges in \( H(\mathcal{P}, \mathcal{F}) \).

In order to complete the second step we will rely on the techniques of Alon and Kleitman in [4]. Rather than repeating their proof and adjusting it to our case, we observe, following Alon et. al in [3] and Matoušek in [20] that it is enough to show that the edges of \( H(\mathcal{P}, \mathcal{F}) \) have fractional Helly number 2 (see below) and have a finite VC-dimension, which implies the existence of an \( \epsilon \)-net of size that depends only on \( \epsilon \). These two ingredients are enough to show that \( H(\mathcal{P}, \mathcal{F}) \) has a \((p, 2)\) theorem for every \( p > 2 \).

We recall that a hyper-graph \( H \) is said to have a fractional Helly number \( k \) if for every \( \alpha > 0 \) there is \( \beta > 0 \) such that for any \( n \) and any collection of \( n \) sets in \( \mathcal{F} \) in which there are at least \( \alpha \binom{n}{k} \) \( k \)-tuples that have nonempty intersection one can find a point incident to at least \( \beta n \) of the sets. Here \( \beta \) may depend only on \( \alpha \) (and the hyper-graph \( H \)) but not on \( n \). In our setting the hyper-graph \( H \) is of the form \( H(\mathcal{P}, \mathcal{F}) \) where \( \mathcal{F} \) is a set of pseudo discs and \( \mathcal{P} \) is a set of points. We will see that every such \( H \) has fractional Helly number 2 and that the corresponding \( \beta \) does not depend on \( \mathcal{P} \) nor on \( \mathcal{F} \) (it will only depend on certain combinatorial properties that are possessed by every family of pseudo-discs).

We recall also the notion of union complexity of a family of sets. We denote by \( U(\mathcal{F}, m) \) the maximum complexity (that is, number of faces of all dimensions) of the boundary of the union of any \( m \) members of \( \mathcal{F} \). We will need the following well known result from [18] saying that for a family \( \mathcal{F} \) of pseudo-discs we have \( U(\mathcal{F}, m) \leq 12m \).
We will use the following theorem from [25] (see Theorem 1 there) relating the notion of fractional Helly number with that of union complexity.

**Theorem 6.** Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to \infty} g(x) = 0$. Suppose that $\mathcal{F}$ is a family of geometric objects in $\mathbb{R}^d$ in general position, (that is, no point belongs to the intersection of more than $d$ boundaries of sets in $\mathcal{F}$) such that $U_\mathcal{F}(m) \leq g(m)m^k$ for every $m \in \mathbb{N}$. Then for every set of points $P$ the family $\mathcal{F}_P$ has fractional Helly number at most $k$ and this is in a way that depends only on the function $g$ and not on $\mathcal{F}$ or $P$.

To be more precise, for every $\alpha > 0$ there is a $\beta > 0$ such that for any family $\mathcal{F}$ satisfying the conditions in the theorem and a set of points $P$ in $\mathbb{R}^d$ the following is true: For any collection of $n$ sets in $\mathcal{H}(\mathcal{F}, P)$ in which there are at least $\alpha \binom{n}{k}$ $k$-tuples that have nonempty intersection one can find a point in $P$ incident to at least $\beta n$ of the sets.

Theorem 3 (with $d = 2$ and $k = 2$) and the linear bound on the union complexity of pseudo-discs [18] imply that $\mathcal{H}(P, \mathcal{F})$ has fractional Helly number at most 2. (Notice that we may assume without loss of generality that the sets in $\mathcal{F}$ are indeed in general position and therefore Theorem 6 applies here.)

It is well known and not hard to show (see for example Theorem 9 in [9]) that for a family $\mathcal{F}$ of pseudo-discs and a set $P$ of points the hyper-graph $\mathcal{H}(P, \mathcal{F})$ has a bounded VC-dimension (in fact at most 3). Therefore, each $\mathcal{H}_i$ has an $\epsilon$-net of size that depends only on $\epsilon$ (see [17]). The method of Alon and Kleitman [4] implies that each $\mathcal{H}(P, \mathcal{F})$ satisfies a $(p, 2)$ theorem. That is, if any subset $S$ of edges in $\mathcal{H}(P, \mathcal{F})$ satisfies the $(p, 2)$ property (from every $p$ sets in $S$ there are 2 sets that intersect), then there are $c(p)$ (a constant that depends only on $p$) vertices that together pierce all the sets in $S$ (see Theorem 4 and the discussion around it in [20]).

By our assumption each, $\mathcal{H}_i$ has the $(p, 2)$-property for $p = 157$. It follows that one can find a set of points of cardinality at most $c(157)k$ that together intersect all the edges in $\mathcal{H}$. [1]

4 The case of half-spaces in $\mathbb{R}^3$.

In this section we prove Theorem 3. The proof follows the same trajectory as the proof of Theorem 1 with analogous lemmata. Technically, the challenge in this case is to derive the analogous lemmata for half-spaces in $\mathbb{R}^3$.

For the proof of Theorem 3 we will need a corresponding three dimensional version of Lemma 2.

**Lemma 4.** Let $\mathcal{F}$ be a family of half-spaces in $\mathbb{R}^3$. Let $P$ be a finite set of points in $\mathbb{R}^3$ and consider the hyper-graph $\mathcal{H} = \mathcal{H}(P, \mathcal{F})$. Assume $B$ is a subgraph of $\mathcal{H}$ consisting of pairwise disjoint hyper-edges. Consider the graph $G$ whose vertices correspond to the edges in $B$ and connect two vertices $e, e' \in B$ by an edge if there is an edge in $\mathcal{H}$ that has a nonempty intersection with $e$ and with $e'$ and has an empty intersection with all other edges in $B$. Then $G$ is planar.

**Proof.** We notice that if the points of $P$ are in (strictly) convex position, then Lemma 2 follows almost right away from Lemma 2. To see this let $S$ denote the convex hull of $P$ and for every half-space $F$ in $\mathcal{F}$ let $F^S$ denote the intersection of $F$ with the boundary of $S$. Then the collection $\{F^S \mid F \in \mathcal{F}\}$ is a family of pseudo-discs lying on the boundary of $S$. Now Lemma 4 follows from Lemma 2 that, although stated in the plane, applies also to the boundary of $S$ (homeomorphic to the two dimensional sphere).
When the points of $P$ are not in convex position such a simple reduction is not possible anymore. Nevertheless, we will be able to make use of Lemma 2 after some suitable modifications.

Denote by $M$ the union of all edges in $B$. We say that a point of $M$ is extreme if it lies on the boundary of the convex hull of $M$.

**Lemma 5.** Let $e_1$ and $e_2$ be two edges in $B$. Suppose that there exists an edge $f \in \mathcal{H}$ such that $f$ has a nonempty intersection with $e_1$ and with $e_2$ and $f$ does not intersect any other edge in $B$. Then there exists a half-space $F'$, not necessarily in $F$, such that both intersections of $F'$ with $e_1$ and with $e_2$ contain extreme points of $M$ and still $F'$ does not intersect any other edge in $B$ but $e_1$ and $e_2$.

**Proof.** We shall use the following basic fact several times: Any half-space that has a non-empty intersection with $M$ contains an extreme point of $M$. Let $F$ denote the half-space that contains at least one extreme vertex of $M$. Because $F \cap M \subseteq e_1 \cup e_2$ we conclude that there is an extreme vertex of $M$ either in $F \cap e_1$, or in $F \cap e_2$ (if there is an extreme vertex of $M$ in both, then we are done with $F' = F$). Without loss of generality assume that $F \cap e_2$ contains an extreme vertex of $M$. Let $E_1 \in F$ be the half-space such that $e_1 = E_1 \cap P$. $E_1$ contains an extreme vertex of $M$ that belongs to $e_1$. Let $\ell$ denote the line of intersection of the boundaries of $E$ and $E_1$. Notice that $(F \cup E_1) \cap M \subseteq e_1 \cup e_2$. Take $F' = F$ and start rotating $F'$ about the line $\ell$ such that at each moment $F' \subset F \cup E_1$. At each moment of the rotation until $F'$ coincides with $E_1$, the half-space $F'$ contains the intersection $F \cap E_1$ and therefore $F'$ has a nonempty intersection with $e_1$. We stop at the last moment where $F'$ still contains an extreme vertex of $M$ that belongs to $e_2$. At this moment $F'$ must also contain a vertex of $e_1$ that is extreme in $M$. This is because at each moment $F'$ must contain an extreme vertex of $M$. This completes the proof of the lemma. ☐

Going back to the proof of Lemma 4 let $S$ denote the convex hull of $M$. For every edge $e$ in $B$ let $F(e) \in F$ be the half-space in $F$ such that $e = F(e) \cap P$. Denote by $\tilde{e}$ the set of extreme vertices of $M$ in $e$. Notice that for every $e \in B$ we have $\tilde{e} \neq \emptyset$ because every edge in $B$ is the intersection of $P$ with some half-space (in $F$). Let $\tilde{M}$ denote the set of extreme points in $M$. Because $M$ is just the union of all edges in $B$, we have $\tilde{M} = \bigcup_{e \in B} \tilde{e}$. Observe that $\{ \tilde{e} \mid e \in B \}$ is the set of edges of the hyper-graph $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(\tilde{M}, \{ F(e) \mid e \in B \})$. For every pair of hyper edges $e, e' \in B$ that are neighbors in the graph $G$ (defined in the statement of Lemma 4) let $F(e, e') \in F$ denote some half-space in $F$ that has a nonempty intersection only with the edges $e$ and $e'$ from $B$. By Lemma 5 there exists a half-space that, with a slight abuse of notation, we denote by $F(\tilde{e}, e')$, not necessarily in $F$, such that $F(\tilde{e}, e')$ has a non-empty intersection only with $\tilde{e}$ and with $e'$ from the collection $\{ \tilde{f} \mid f \in B \}$.

Let

$$F' = \{ F_e \mid e \in B \} \cup \{ F(\tilde{e}, e') \mid (e, e') \text{ is an edge in } G \}.$$  

We define now a graph $G'$ whose set of vertices is $B' = \{ \tilde{e} \mid e \in B \}$. We connect $\tilde{e}$ and $\tilde{e}'$ in $B'$ by an edge in $G'$ if there is an edge $f$ in the hyper-graph $\tilde{\mathcal{H}}(\tilde{M}, F')$ such that $f$ has a nonempty intersection with $\tilde{e}$ and with $\tilde{e}'$ and $f$ has an empty intersection with all other sets in $B'$. It follows from the discussion above that if $e$ and $e'$ are two sets in $B$ that are connected by an edge in $G$, then $\tilde{e}$ and $\tilde{e}'$ in $B'$ are connected by an edge in $G'$.

Because $\tilde{M}$ is in convex position, the hyper-graph $\tilde{\mathcal{H}}(\tilde{M}, F')$ can be presented as a hyper-graph on the set of vertices $\tilde{M}$ whose set of edges correspond to pseudo-discs on $S$, where $S$ is the boundary of the convex hull of $M$. We then apply Lemma 2 (where $B$ is replaced by $\{ \tilde{e} \mid e \in B \}$ and $F$ is
replaced by $F'$) and conclude that $G'$ is planar. The planarity of $G$ follows because $G$ is a subgraph of $G'$. Q.E.D.

We are now ready to prove Theorem 4. The proof will follow the lines and will have a similar structure as of the proof of the corresponding theorem for pseudo-discs in the plane, namely Theorem 2.

**Proof of Theorem 4.** As in the proof of Theorem 2 let $B$ be a maximum (in cardinality) collection of pairwise disjoint edges in $\mathcal{H}$ and let $n = |B|$. For every $e \in B$ denote by $\alpha_1(e)$ the maximum cardinality of a matching among those edges in $\mathcal{H}$ that intersect with $e$ but with no other edge in $B$. Denote by $\alpha_2(e)$ the maximum cardinality of a matching among those edges in $\mathcal{H}$ that intersect with $e$ and with precisely one more edge in $B$. Denote by $\alpha_3(e)$ the maximum cardinality of a matching among those edges in $\mathcal{H}$ that intersect with $e$ and with at least two more edges in $B$. It is enough to show that $\sum_{e \in B} \alpha_1(e) + \alpha_2(e) + \alpha_3(e) < 157n$.

For every $e \in B$ we must have $\alpha_1(e) \leq 1$, or else we get a contradiction to the maximality of $B$ (as in the proof of Theorem 2).

Next we show that $\sum_{e \in B} \alpha_2(e) \leq 12n$. Consider the graph $G$ whose vertices correspond to the edges in $B$ and connect two vertices $e, e' \in B$ by an edge if there is an edge in $\mathcal{H}$ that has a nonempty intersection with $e$ and with $e'$ and has an empty intersection with all other edges in $B$. By Lemma 1, $G$ is planar. Therefore, $G$ has at most $3n$ edges. For every $e \in B$ denote by $d(e)$ the degree of $e$ in $G$. Therefore,

$$\sum_{e \in B} d(e) \leq 6n. \quad (2)$$

We claim that for every $e$ in $B$ we have $\alpha_2(e) \leq 2d(e)$. Indeed, otherwise, by the pigeonhole principle, one can find three pairwise disjoint edges $g, g', g''$ in $\mathcal{H}$ and an edge $e'$ in $B$ such that each of $g, g'$, and $g''$ intersects $e$ and $e'$ but no other edge in $B$. In this case $B \cup \{g, g', g''\} \setminus \{e, e'\}$ contradicts the maximality of $B$. Inequality (2) implies now $\sum_{e \in B} \alpha_2(e) \leq 12n$.

It remains to show that $\sum_{e \in B} \alpha_3(e) < 144n$. Denote by $F_3$ the subfamily of $F$ that consists of half-spaces in $F$ that intersect with three or more edges in $B$. Like in the proof of Theorem 2 this part is more involved. Similarly, we will show that if it is not the case that $\sum_{e \in B} \alpha_3(e) < 144n$, then we derive an (impossible) embedding of $K_{3,3}$ in an arrangement of hyper-planes in $\mathbb{R}^3$ (see Claim 1).

For every $F \in F_3$ and every $e \in B$ that is intersected by $F$, we find a (new) half-space $F^e$ that intersects with $e$ and with exactly two more edges in $B$. To do this, let $v \in e$ be an extreme vertex of $P$ and let $h$ be a hyper-plane supporting the convex hull of $P$ at $v$. Let $\ell$ denote the line of intersection of $h$ and the boundary of $F$. Rotate $F$ about the line $\ell$ until $F$ intersects only three edges in $B$ one of which must be $e$ because at all times of rotation we have $v \in F$.

We denote the family of all new half-spaces obtained this way by $\mathcal{D}$. Let $T$ denote the set of all triples of edges in $B$ that are intersected by half-spaces in $\mathcal{D}$.

We denote by $Z$ the collection of all pairs of sets from $B$ that appear together in some triple in $T$. One can show that $|Z| < 12n$: Pick every set in $B$ with probability $\frac{1}{2}$. Call a pair $\{e, e'\}$ in $Z$ *good* if both $e$ and $e'$ were picked and an edge $f \in B$ such that $e, e'$, and $f$ is a triple in $T$ was not picked. The expected number of good pairs in $Z$ is at least $1/8$ of the pairs in $Z$. On the other hand, by Lemma 4 the set of good pairs in $Z$ is the set of edges of a planar graph (on an expected
number of \( n/2 \) vertices). (We refer the reader to the proof of Theorem 2 to see this argument a bit more detailed.)

Now consider the graph \( K \) whose set of vertices is the edges in \( B \) and whose edges are those pairs in \( Z \). For every \( e \in B \) denote by \( \delta(e) \) the degree of \( e \) in this graph. Notice that, in view of the above, \( \sum_{e \in B} \delta(e) = 2|Z| < 24n \).

Fix \( e \in B \). Define a graph \( K^e \) on the set of neighbors of \( e \) in \( K \) where we connect two neighbors \( e_1, e_2 \) of \( e \) in \( K \) by an edge in \( K^e \) if and only if \( \{e, e_1, e_2\} \) is a triple in \( T \). This is equivalent to that there is \( D \in \mathcal{D} \) that intersects with \( e, e_1, \) and \( e_2 \). Denote by \( t(e) \) the number of edges in \( K^e \).

By ignoring the set \( e \) and applying Lemma 4, we see that \( K^e \) is planar. \( K^e \) has \( \delta(e) \) vertices and is planar and therefore \( t(e) < 3\delta(e) \).

We claim that for every \( e \in B \) we must have \( \alpha_3(e) \leq 2t(e) \).

Indeed, assume to the contrary that \( \alpha_3(e) > 2t(e) \). Then there is a collection \( Q \) of at least \( 2t(e) + 1 \) pairwise disjoint edges of \( H \), each of which has a non-empty intersection with \( e \) and with at least two more edges in \( B \). Every edge in \( Q \) has a non-empty intersection with \( e \) and with at least two edges \( e' \) and \( e'' \) that form a pair in \( Z \). The hyper-edges \( e' \) and \( e'' \) are therefore connected by an edge in \( K^e \). By the pigeonhole principle, because there are only \( t(e) \) edges in \( K^e \) while \( |Q| \geq 2t(e) + 1 \), there exist \( e' \) and \( e'' \) that are connected by an edge in \( K^e \) such that \( e, e', \) and \( e'' \) are all intersected by three (pairwise disjoint) edges \( g_1, g_2, g_3 \in Q \subset \mathcal{H} \). We claim that this situation is impossible. This follows directly from the following claim

**Claim 1.** It is impossible to find three half-spaces \( u_1, u_2, u_3 \) in \( \mathbb{R}^3 \) and another three half-spaces \( w_1, w_2, w_3 \) such that there are nine points \( g_{ij} \) for \( 1 \leq i, j \leq 3 \) satisfying \( g_{ij} \) lies only in \( u_i \) and \( w_j \) from the half-spaces \( u_1, u_2, u_3, w_1, w_2, w_3 \).

**Proof.** Considering the dual problem, it is enough to show that one cannot find three points \( u_1, u_2, u_3 \) in \( \mathbb{R}^3 \) and another three points \( w_1, w_2, w_3 \) in \( \mathbb{R}^3 \) such that there for every \( 1 \leq i, j \leq 3 \) there is a half-space containing only \( u_i \) and \( w_j \) from the points \( u_1, u_2, u_3, w_1, w_2, w_3 \).

Without loss of generality we assume that all the points are in general position. We may also assume that one of the triangles \( \Delta u_1 u_2 u_3 \) or \( \Delta w_1 w_2 w_3 \) is not a face of the convex hull of \( \{u_1, u_2, u_3, w_1, w_2, w_3\} \). Otherwise, the points \( u_1, u_2, u_3, w_1, w_2, w_3 \) are in convex position and each of the segments \( [u_i, w_j] \) is an edge of this convex polytope (because by assumption each pair of vertices \( w_i, u_j \) is separable from the rest of the vertices by a hyper-plane). The skeleton graph of a three dimensional convex polytope is planar and therefore cannot contain \( K_{3,3} \) as a subgraph. Therefore, without loss of generality we assume that the hyper-plane through \( u_1, u_2, \) and \( u_3 \) separates two of the points \( w_1, w_2, \) and \( w_3 \). Let \( h \) denote this hyper-plane and assume without loss of generality that \( w_1 \) and \( w_2 \) lie above \( h \) while \( w_3 \) lies below \( h \). We observe that the line through \( w_1 \) and \( w_2 \) must cross triangle \( \Delta u_1 u_2 u_3 \) for otherwise \( u_1, u_2, u_3, w_1, w_2 \) are in convex position and the edge-graph of their convex hull is the non-planar \( K_5 \). Without loss of generality assume that \( w_1 \) lies closer than \( w_2 \) to triangle \( \Delta u_1 u_2 u_3 \). Denote by \( O \) the point of intersection of the line through \( w_1 \) and \( w_2 \) with \( h \). For \( i = 1, 2, 3 \) let \( Q_i \) be a half-space containing only \( w_i \) and \( u_i \) from \( u_1, u_2, u_3, w_1, w_2, w_3 \). Observe that all three half-spaces \( Q_1, Q_2, \) and \( Q_3 \) must contain the point \( O \) (as they separate \( w_1 \) and \( w_2 \)) and, assuming \( h \) is horizontal, their supporting hyper-planes must all lie above \( O \). This implies that \( Q_1, Q_2, \) and \( Q_3 \) cover the whole half-space below \( h \) which is impossible as none of \( Q_1, Q_2, \) and \( Q_3 \) may contain \( w_3 \). ■

**Remark.** Although it is tempting to believe that the collection of all 2-sets (that is, sets of two points separable by a half-space) of a set of points in \( \mathbb{R}^3 \) is the set of edges of a planar graph, this
is not the case. One can check that $K_5$ can be realized in this way. Claim 1 shows that $K_{3,3}$ cannot be realized in this way.

Going back to the proof of Theorem 4, we have:

$$\sum_{e \in B} \alpha_3(e) < \sum_{e \in B} 2t(e) \leq \sum_{e \in B} 6d(e) \leq 6 \cdot 24n = 144n.$$  

The proof is now complete as we have

$$\sum_{e \in B} \alpha_1(e) + \alpha_2(e) + \alpha_3(e) < n + 12n + 144n = 157n,$$

and this implies the existence of $e \in B$ such that $\alpha_1(e) + \alpha_2(e) + \alpha_3(e) \leq 156$.  

In the same way that Theorem 1 is a corollary of Theorem 2, we conclude Theorem 3 from Theorem 4.

**Proof of Theorem 3.** Repeatedly apply Theorem 4 and find an edge $e$ in $\mathcal{H}$ such that among those edges intersecting it there are at most 156 pairwise disjoint ones. Then delete $e$ and those edges intersecting it from $\mathcal{H}$ and continue. If we can continue $k$ steps, then we find $k$ pairwise disjoint edges. Otherwise, we decompose $\mathcal{H}$ into less than $k$ families, $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$, of edges such that in each family $\mathcal{H}_i$ there are at most 156 pairwise disjoint edges.

The boundary of the union of $m$ half-spaces in $\mathbb{R}^3$ is the boundary of a polyhedron with at most $m$ facets, which in turn has complexity linear in $m$. It now follows from Theorem 6 that each of the families $\mathcal{H}$ has fractional Helly number 2 in a way that is independent of $P$, as described in the statement of Theorem 6. It is well known that families of half-spaces (in any fixed dimension) have bounded VC-dimension. Hence each $\mathcal{H}_i$ has a bounded VC-dimension (in fact bounded by 4). Therefore, each $\mathcal{H}_i$ has an $\epsilon$-net of size that depends only on $\epsilon$ (see [17]). The method of Alon and Kleitman [4] implies that each $\mathcal{H}_i$ satisfies a $(p, 2)$ theorem. That is, if a subset $S$ of edges in $\mathcal{H}$ satisfies the $(p, 2)$ property (that is, from every $p$ sets in $S$ there are 2 sets that intersect), then there are $c(p)$ (a constant that depends only on $p$) vertices that together pierce all the sets in $S$.

By our assumption, each $\mathcal{H}_i$ has the $(p, 2)$-property for $p = 157$. It follows that one can find a set of points of cardinality at most $c(157)k$ that together pierce all the edges in $\mathcal{H}$.  

## 5 The case of half-spaces in $\mathbb{R}^d$ where $d \geq 4$

In this section we prove Theorem 5.

For every $n \in \mathbb{N}$ we need to construct a set $P$ of $N = \binom{n}{2}$ points and a set $\mathcal{F}$ of $n$ half-spaces in $\mathbb{R}^4$ such that:

1. Every two edges in $\mathcal{H}(P, \mathcal{F})$ have a non-empty intersection
2. Any subset of $P$ which pierce all edges in $\mathcal{H}(P, \mathcal{F})$ must consist of at least $\frac{n-1}{2}$ points.

The next lemma will be our main tool in constructing $\mathcal{H}(P, \mathcal{F})$. This lemma is a slight variation of an argument which was used by [1] to upper bound the sign-rank of a hyper-graph.
Lemma 6. Let $\mathcal{H}$ be a hypergraph such that every $v \in V(\mathcal{H})$ belongs to at most $d$ hyper-edges. Then $\mathcal{H}$ can be realized by points and half-spaces in $\mathbb{R}^{2d}$. That is, $\mathcal{H}$ is isomorphic to $\mathcal{H}(P, F)$ for some set $P$ of points in $\mathbb{R}^{2d}$ and a family $F$ of half spaces in $\mathbb{R}^{2d}$.

Proof. Pick some enumeration of $E(\mathcal{H})$, $e_1, e_2, \ldots, e_m$ where $m = |E(\mathcal{H})|$. For every $v \in V$ pick some real univariate polynomial $P_v(x)$ such that

- $P_v(0) = -1$,
- $P_v(i) > 0$ if $v \in e_i$ and $P_v(i) < 0$ if $v \notin e_i$, and
- $\deg(P_v) \leq 2d$.

It is not hard to see that such a polynomial always exists: For example, the polynomial $P_v(x) = -\frac{Q_v(x)}{Q_v(0)}$, where $Q_v(x) = \prod_{i : v \in e_i} \left( x - (i + \frac{1}{4}) \right) \left( x - (i - \frac{1}{4}) \right)$ satisfies the above requirements. For every $v \in V$ let $p_{v,i}, i = 0, \ldots, 2d$ denote the coefficients of $P_v(x)$. Notice that $p_{v,0} = -1$ for all $v$.

Every $v \in V$ will correspond to the point $x_v = (p_{v,1}, \ldots, p_{v,2d})$ and every $e_i$ correspond to the half-space $H_i = \{ x : \langle x, n_i \rangle \geq 1 \}$, where $n_i = (i, i^2, \ldots, i^{2d})$. Observe that $\langle x_v, n_i \rangle = P_v(i) + 1$ and therefore $v \in e_i$ if and only if $x_v \in H_i$ as required.

We now construct an hyper-graph $\mathcal{H}$ with $\binom{n}{2}$ vertices such that every vertex belongs to precisely two edges, every two edges have a non-empty intersection (that is, any matching in $\mathcal{H}$ is of size at most 1), and finally, any set of vertices that pierces all edges must consist of at least $\frac{n-1}{2}$ vertices. Once we introduce such a hyper-graph, it follows from Lemma 6 that it can be realized in $\mathbb{R}^4$ by points and half-spaces.

We take the vertices of $\mathcal{H}$ to be the edges of a complete simple graph on $n$ vertices $K_n$. Let us denote the vertices of $K_n$ by $x_1, \ldots, x_n$. Then $\mathcal{H}$ has $\binom{n}{2}$ vertices. The hyper-graph $\mathcal{H}$ will consist of $n$ edges $e_1, \ldots, e_n$ defined as follows. For every $1 \leq i \leq n$ the edge $e_i$ is the collection of all edges in $K_n$ incident to $x_i$.

It is easy to check that indeed every two sets in $\mathcal{H}(P, F)$ have a non-empty intersection and that any set of vertices of $\mathcal{H}$ that pierces all the edges of $\mathcal{H}$ must have size of at least $\frac{n-1}{2}$, as desired.

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