Baryon mass extrapolation

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Consideration of the analytical properties of pion-induced baryon self-energies leads to new functional forms for the extrapolation of light baryon masses. These functional forms reproduce the leading non-analytic behavior of chiral perturbation theory, the correct heavy-quark limit and have the advantage of containing information on the extended structure of hadrons. The forms involve only three unknown parameters which may be optimized by fitting to present lattice data. Recent dynamical fermion results from CP-PACS and UK-QCD are extrapolated using these new functional forms. We also use these functions to probe the limit of the chiral perturbative regime and shed light on the applicability of chiral perturbation theory to the extrapolation of present lattice QCD results.

1. FORMALISM

In recent years there has been tremendous progress in the computation of baryon masses within lattice QCD. Still, it remains necessary to extrapolate the calculated results to the physical pion mass ($\mu = 140$ MeV) in order to make a comparison with experimental data. In doing so one necessarily encounters some non-linearity in the quark mass (or $m_\pi^2$), including the non-analytic behavior associated with dynamical chiral symmetry breaking. We recently investigated this problem for the case of the nucleon magnetic moments [1]. It is vital to develop a sound understanding of how to extrapolate to the physical pion mass.

1.1. Self-Energy Contributions

Chiral symmetry is dynamically broken in QCD and the pion alone is a near Goldstone boson. It is strongly coupled to baryons and plays a significant role in $N$ and $\Delta$ self-energies. The one-loop pion induced self-energies of the $N$ and $\Delta$ are given by the processes shown in Fig. 1.

In the standard heavy baryon limit, the analytic expression for the pion cloud correction to the masses of the $N$ and $\Delta$ are of the form

$$\delta M_N = \sigma_{NN} + \sigma_{N\Delta}; \quad \delta M_\Delta = \sigma_{NN} + \sigma_{N\Delta}, \quad (1)$$

where

$$\sigma_{NN} = \frac{3g_A^2}{16\pi^2f^2_\pi} \int_0^\infty dk \frac{k^4u_{NN}^2(k)}{w^2(k)}, \quad (2)$$

$$\sigma_{N\Delta} = \frac{6g_A^2}{25\pi^2f^2_\pi} \int_0^\infty dk \frac{k^4u_{N\Delta}^2(k)}{w(k)(\Delta M + w(k))}, \quad (3)$$

$$\sigma_{\Delta N} = \frac{3g_A^2}{50\pi^2f^2_\pi} \int_0^\infty dk \frac{k^4u_{\Delta N}^2(k)}{w(k)(\Delta M + w(k))}. \quad (4)$$

Here $\Delta M = M_\Delta - M_N$, $g_A = 1.26$ is the axial charge of the nucleon, $w(k) = \sqrt{k^2 + m_\pi^2}$ is the pion energy and $u_{NN}(k), u_{N\Delta}(k), \ldots$ are the $NN\pi, N\Delta\pi, \ldots$ form factors associated with the emission of a pion of three-momentum $k$. The

Figure 1. One-loop pion induced self-energy of the nucleon and the delta.
form factors reflect the finite size of the baryonic source of the pion field and suppress the emission probability at high virtual pion momentum. As a result, the self-energy integrals are not divergent.

The leading non-analytic (LNA) contribution of these self-energy diagrams is associated with the infrared behavior of the corresponding integrals; i.e. the behavior as \( k \to 0 \). As a consequence, the leading non-analytic behavior does not depend on the details of the form factors. Indeed, the well known results of chiral perturbation theory \([2,4]\) are reproduced even when the form factors are approximated by \( u(k) = \theta(\Lambda - k) \).

Of course, our concern with respect to lattice QCD is not so much the behavior as \( m_\pi \to 0 \), but the extrapolation from high pion masses to the physical pion mass. In this context the branch point at \( m_\pi^2 = \Delta M^2 \), associated with transitions of \( N \to \Delta \) or \( \Delta \to N \), is at least as important as the LNA behavior near \( m_\pi = 0 \).

Heavy quark effective theory suggests that as \( m_\pi \to \infty \) the quarks become static and hadron masses become proportional to the quark mass. In this spirit, corrections are expected to be of order \( 1/m_q \) where \( m_q \) is the heavy quark mass. The presence of a cut-off associated with the form factor acts to suppress the pion induced self energy for increasing pion masses, as evidenced by the \( m_\pi^2 \) in the denominators of Eqs. (3) and (4). While some \( m_\pi^2 \) dependence in the form factor is expected, this is a second-order effect and does not alter the qualitative feature of the self-energy corrections tending to zero as \( 1/m_\pi^2 \) in the heavy quark limit.

Rather than simplifying our expressions to just the LNA terms, we retain the complete expressions \([2,4]\) as they contain important physics that would be lost by making a simplification. We note that keeping the entire form is not in contradiction with \( \chi PT \). However, as one proceeds to larger quark masses, differences between the full forms and the expressions in the chiral limit will become apparent, highlighting the importance of the branch point and the form factor reflecting the finite size of baryons.

As a result of these considerations, we propose to use the analytic expressions for the self-energy integrals corresponding to a sharp cut-off in order to incorporate the correct LNA structure in a simple three-parameter description of the \( m_\pi \) dependence of the \( N \) and \( \Delta \) masses. In the heavy quark limit hadron masses become proportional to the quark mass. Hence we can simulate a linear dependence of the baryon masses on the quark mass, \( m_q \), in this region, by adding a term involving \( m_\pi^2 \). The functional form for the mass of the nucleon suggested by this analysis is then:

\[
M_N = \alpha_N + \beta_N m_\pi^2 + \sigma_{NN}(\Lambda_N) + \sigma_{N\Delta}(\Lambda_N),
\]

while that for the \( \Delta \) is:

\[
M_\Delta = \alpha_\Delta + \beta_\Delta m_\pi^2 + \sigma_{\Delta\Delta}(\Lambda_\Delta) + \sigma_{\Delta N}(\Lambda_\Delta).
\]

### 1.2. Model Dependence

The use of a sharp cut-off, \( u(k) = \theta(\Lambda - k) \), as a form factor may seem somewhat unfortunate given that phenomenology suggests a dipole form factor better approximates the axial-vector form factor. However, the sensitivity to such model-dependent issues is shown to be negligible in Fig. 3. There, the self-energy contribution \( \sigma_{NN}(= \sigma_{\Delta\Delta}) \) for a 1 GeV dipole form factor (solid curve) is compared with a sharp cut-off form factor combined with the standard \( \alpha + \beta m_\pi^2 \) terms of \([5,6]\) or \([3,4]\). Optimizing \( \Lambda, \alpha, \) and \( \beta \) provides the fine-dash curve of Fig. 3. Differences are at the few MeV level indicating negligible sensitivity to the actual analytic structure of the form factor.

Here we have focused on the pion self-energy contribution to the \( N \) and \( \Delta \) form factors. Only the pion displays a rapid mass dependence as the chiral limit is approached. Other mesons participating in similar diagrams do not give rise to such rapidly changing behavior and can be accommodated in the \( \alpha + \beta m_\pi^2 \) terms of \([5,6]\) or \([3,4]\). Moreover, the form factor suppresses the contributions from more massive intermediate states including multiple pion dressings. Other multi-loop pion contributions renormalize the vertex and hence we use the renormalized coupling \( g_A \) as a measure of the pion-nucleon coupling.

### 2. ANALYSIS

We consider two independent dynamical-fermion lattice simulations of the \( N \) and \( \Delta \)
The self-energy contribution $\sigma_{NN}$ for a 1 GeV dipole form factor (solid curve) is compared with a sharp cut-off form factor $\theta(\Lambda_N - k)$ (fine-dash curve). Self-energy contributions $\sigma_{N\Delta}$ (dot-dash) and $\sigma_{\Delta N}$ (long-dash) for a 1 GeV dipole are also illustrated.

masses. We select results from CP-PACS’s $12^3 \times 32$ and $16^3 \times 32$ simulations at $\beta = 1.9$, and UKQCD’s $12^3 \times 24$ simulations at $\beta = 5.2$.

Figure 3 displays fits of (5) to the lattice data. In order to perform fits in which $\Lambda$ is unconstrained, it is essential to have lattice simulations at light quark masses approaching $m_\pi^2 \sim 0.1$ GeV$^2$.

It is common to see the use of the following $\chi$PT-motivated expression for the mass dependence of hadron masses,

$$M_N = \alpha + \beta m_\pi^2 + \gamma m_\pi^3.$$  

(7)

The result of such a fit for the $N$ is shown as the dashed curve in Fig. 3. The coefficient of the $m_\pi^3$ term in a three parameter fit is $-0.761$. This disagrees with the coefficient of $-5.60$ known from $\chi$PT (which is correctly incorporated in (6)) and illustrated as the solid and dash-dot curves of Fig. 3) by almost an order of magnitude. This clearly indicates the failings of (6).

The dotted curve of Fig. 3 indicates the leading non-analytic term of the chiral expansion dominates from the chiral limit up to the branch point at $m_\pi = \Delta M \simeq 300$ MeV, beyond which $\chi$PT breaks down. The curvature around $m_\pi = \Delta M$, neglected in previous extrapolations of the lattice data, leads to shifts in the extrapolated masses of the same order as the departure of lattice estimates from experimental measurements.

REFERENCES
1. D. B. Leinweber, D. H. Lu, and A. W. Thomas, Phys. Rev. D60 (1999) 034014, hep-lat/9810005.
2. D.B. Leinweber, A.W. Thomas, K. Tsushima and S.V. Wright, “Baryon masses from lattice QCD: Beyond the perturbative chiral regime,” hep-lat/9906027.
3. E. Jenkins, Nucl. Phys. B368, 190 (1992).
4. R.F. Lebed, Nucl. Phys. B430, 295 (1994).
5. CP-PACS-Collaboration (S. Aoki et al.), UTCCP-P-61, hep-lat/9902018.
6. UKQCD Collaboration (C. R. Alton et al.), Phys. Rev. D60 (1999) 034507, hep-lat/9808016.