FANO HYPERSURFACES IN POSITIVE CHARACTERISTIC

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Abstract. We prove that a general Fano hypersurface in a projective space over an algebraically closed field of arbitrary characteristic is separably rationally connected.

1. Introduction

In this paper, we work with varieties over an algebraically closed field $k$ of arbitrary characteristic.

Definition 1.1 (Kol96 IV.3). Let $X$ be a variety defined over $k$.

A variety $X$ is rationally connected if there is a family of irreducible proper rational curves $g : U \to Y$ and an evaluation morphism $u : U \to X$ such that the morphism $u^{(2)} : U \times_Y U \to X \times X$ is dominant.

A variety $X$ is separably rationally connected if there exists a proper rational curve $f : \mathbb{P}^1 \to X$ such that the image lies in the smooth locus of $X$ and the pullback of the tangent sheaf $f^*TX$ is ample. Such rational curves are called very free curves.

We refer to Kollár’s book [Kol96] or the work of Kollár-Miyaoka-Mori [KMM92] for the background. If $X$ is separably rationally connected, then $X$ is rationally connected. The converse is true when the ground field is of characteristic zero by using the generic smoothness for the dominant map $u^{(2)}$. In positive characteristics, the converse statement is open.

In characteristic zero, a very important class of rationally connected varieties are Fano varieties, i.e., smooth varieties with ample anticanonical bundles. In positive characteristic, we only know that they are rationally chain connected.

Question 1.2 (Kollár). In arbitrary characteristic, are Fano varieties separably rationally connected?

The question is open even for Fano hypersurfaces in projective spaces. In this paper, we prove the following theorem.

Theorem 1.3. In arbitrary characteristic, a general Fano hypersurface of degree $n$ in $\mathbb{P}^n_k$ contains a minimal very free rational curve of degree $n$, i.e., the pullback of the tangent bundle has the splitting type $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(n-2)}$.

Theorem 1.4. In arbitrary characteristic, a general Fano hypersurface in $\mathbb{P}^n_k$ is separably rationally connected.

de Jong-Starr [dJS03] proved that every family of separably rationally connected varieties over a curve admits a rational section. Thus using Theorem 1.4 we give another proof of Tsen’s theorem.
Corollary 1.5. Every family of Fano hypersurfaces in $\mathbb{P}^n$ over a curve admits a rational section. \hfill \Box

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2. Typical Curves and Deformation Theory

Let $n$ be an integer $\geq 3$. Let $X$ be a hypersurface of degree $n$ in $\mathbb{P}^n$. Let $C$ be a smoothly embedded rational curve of degree $e$ in $X$. We have the normal bundle short exact sequence.

$$0 \longrightarrow TC \longrightarrow TX|_{C} \longrightarrow \mathcal{N}_{C|X} \longrightarrow 0$$

By adjunction, the degree of $TX|_{C}$ is the degree of $\mathcal{O}_{\mathbb{P}^n}(1)|_{C}$. Thus the degree of the normal bundle $\mathcal{N}_{C|X}$ is $e - 2$ and the rank is $n - 2$.

Definition 2.1. Let $e$ be a positive integer $\leq n$. A smoothly embedded rational curve $C$ of degree $e$ in $X$ is typical, if the normal bundle is the following:

$$\mathcal{N}_{C|X} \cong \left\{ \begin{array}{ll}
\mathcal{O}(e(n-3)) \oplus \mathcal{O}(-1), & \text{if } e = 1, \\
\mathcal{O}(e(n-e)) \oplus \mathcal{O}(1)^{\oplus(e-2)}, & \text{if } e \geq 2.
\end{array} \right.$$ 

The curve $C$ is a typical line if the degree of $C$ is one.

Note that when $e = n$, typical rational curves of degree $n$ are very free.

Lemma 2.2. Let $L$ be a smoothly embedded line in a hypersurface $X$ of degree $n$. Then $L$ is typical if and only if both of the following conditions hold:

1. $h^1(C, \mathcal{N}_{L|X}) = 0$,
2. $h^1(C, \mathcal{N}_{L|X}(-1)) \leq 1$.

Proof. We may assume that $\mathcal{N}_{L|X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-2})$, where $a_1 \geq \cdots \geq a_{n-2}$. Condition (1) is equivalent to that $a_1 \geq \cdots \geq a_{n-2} \geq -1$. Together with condition (2), $a_{n-2}$ is either 0 or $-1$. When $a_{n-2} = 0$, $\mathcal{N}_{L|X}$ is semipositive, contradicting with the fact that the degree of $\mathcal{N}_{C|X}$ is $-1$. When $a_{n-2} = -1$, $\mathcal{N}_{L|X}/\mathcal{O}(a_{n-2})$ is semipositive. Because of the degree of the normal bundle, $L$ is typical. \hfill \Box

Lemma 2.3. Let $C$ be a smoothly embedded rational curve of degree $e$ in a hypersurface $X$ of degree $n$, where $2 \leq e \leq n$. Then $C$ is typical if and only if both of the following conditions hold:

1. $h^1(C, \mathcal{N}_{C|X}(-1)) = 0$,
2. $h^1(C, \mathcal{N}_{C|X}(-2)) \leq n - e$.

Proof. Recall that the rank of the normal bundle $\mathcal{N}_{C|X}$ is $n - 2$ and the degree is $e - 2$. We may assume that $\mathcal{N}_{C|X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-2})$, where $a_1 \geq \cdots \geq a_{n-2}$. Condition (1) is equivalent to that $a_{n-2} \geq 0$. Condition (2) implies that at most $n - e$ of $a_i$’s are 0. By degree count, $C$ is a typical rational curve of degree $e$. \hfill \Box

Typical rational curves in the hypersurface $X$ are deformation open as very free curves in the following sense.

Let $H_n$ be the Hilbert scheme of hypersurfaces of degree $n$ in $\mathbb{P}^n$. It is isomorphic to some projective space. Let $\mathcal{X} \to H_n$ be the universal hypersurface. The morphism $\mathcal{X} \to H_n$ is flat projective and there exists a relative very ample invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ on $\mathcal{X}$. 

Let $R_{e,n}$ be the Hilbert scheme parameterizing flat projective families of one-dimensional subschemes in $\mathcal{X}$ with the Hilbert polynomial $P(d) = ed + 1$. By [Kol96] Theorem 1.4, $R_{e,n}$ is projective over $H_n$.

Let $\mathcal{C}$ be the universal families over $R_{e,n}$, denoted by $\pi: \mathcal{C} \to R_{e,n}$. We have the following diagram,

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & R_{e,n} \times H_n \mathcal{X} \\
\downarrow \pi & & \downarrow \\
R_{e,n} & \leftarrow & R_{e,n} \times H_n \mathcal{X}
\end{array}
$$

where $i$ is a closed immersion.

**Proposition 2.4.** Let $e$ be a positive integer $\leq n$. There exists an open subset in $R_{e,n}$ parameterizing typical curves of degree $e$ in hypersurfaces of degree $n$.

**Proof.** Every typical curve of degree $e$ in a hypersurface of degree $n$ gives a point in $R_{e,n}$. Any small deformation of a smoothly embedded rational curve is still a smoothly embedded rational curve. Thus the proposition follows by Lemma 2.2 and the upper semicontinuity theorem [Har77] III.12.8. $\square$

**Lemma 2.5.** There exists an open subset in $R_{e,n}$ such that for every closed point $(C, X)$ in the open subset, $C$ lies in the smooth locus of $X$.

**Proof.** Let $S \subset \mathcal{X}$ be the relative singular locus in the universal hypersurface. $S$ is a closed subset of $\mathcal{X}$. Since $\pi$ is proper, the locus $\pi(i^{-1}(R_{e,n} \times H_n S))$ is a closed subset of $R_{e,n}$ parametrizing the point $(C', X')$ such that $C'$ intersects the singular locus of $X'$. Thus the complement $U$ is open in $R_{e,n}$ and satisfies the desired property. $\square$

Let $L$ be a typical line in a hypersurface $X$ of degree $n$ in $\mathbb{P}^n$. By definition, $N_{L/X} \cong \mathcal{O}(n-3) \oplus \mathcal{O}(-1)$. We have a canonically defined trivial subbundle $\mathcal{O}^{\oplus(n-2)}$ of $N_{L/X}$.

**Proposition 2.6.** Let $X$ be a hypersurface of degree $n$ in $\mathbb{P}^n$. Let $L$ and $M$ be two typical lines in $X$ intersecting transversally at only one point $p$. Assume that the following conditions hold:

1. the direction $T_p L$ is not in the trivial subbundle of $N_{M/X}$;
2. the direction $T_p M$ is not in the trivial subbundle of $N_{L/X}$.

Then the pair $(L \cup M, X) \in R_{2,n}$ can be smoothed to a pair $(C, X')$ where $C$ is a typical conic in $X'$. Furthermore, there exists an open neighborhood of $(L \cup M, X)$ in which any smoothing of $(L \cup M, X)$ is a typical conic.

**Proof.** Let $D$ be the union of the lines $L$ and $M$. Since $D$ is a local complete intersection and lies in the smooth locus of $X$, the normal bundle $N_{D/X}$ is locally free. We have the following short exact sequence.

$$
0 \longrightarrow N_{L/X} \longrightarrow N_{D/X}|_L \longrightarrow T_p L \otimes T_p M \longrightarrow 0
$$

By [GHS03] Lemma 2.6, the locally free sheaf $N_{D/X}|_L$ is the sheaf of rational sections of $N_{L/X}$ which has at most one pole at the direction of $T_p M$. Since $N_{L/X} \cong \mathcal{O}(n-3) \oplus \mathcal{O}(-1)$, condition (2) implies that $N_{D/X}|_L$ is isomorphic to $\mathcal{O}^{\oplus(n-2)}$. 


By the same argument, condition (1) implies that the sheaf \( \mathcal{N}_{D_1|X}|_M \) is isomorphic to \( \mathcal{O}^\oplus(n-2) \). Now we have the following short exact sequence.

\[
0 \longrightarrow \mathcal{N}_{D_1|X}|_M(-p) \longrightarrow \mathcal{N}_{D_1|X} \longrightarrow \mathcal{N}_{D_1|X}|_L \longrightarrow 0
\]

First we claim that \( D \) can be smoothed. Since \( h^1(D, \mathcal{N}_{D_1|X}) = 0 \), the pair \((D, X)\) is unobstructed in \( R_{2,n} \), cf. [Kol96] I.2. By [Sta09] Lemma 3.17, it suffices to show that the map

\[
H^0(D, \mathcal{N}_{D_1|X}) \to H^0(L, \mathcal{N}_{D_1|X}|_L) \to T_pL \otimes T_pM
\]

is surjective. Since \( H^1(M, \mathcal{N}_{D_1|X}|_M(-p)) = 0 \), the first map is surjective. Since \( H^1(L, \mathcal{N}_{D_1|X}|_L) = 0 \), the second map is surjective.

Let \( q, r \) be two distinct points on \( L - \{p\} \). By the long exact sequence associated to the above short exact sequence at \( h^1 \), we get \( h^1(D, \mathcal{N}_{D_1|X}(-q)) = 0 \) and \( h^1(D, \mathcal{N}_{D_1|X}(-q - r)) = n - 2 \).

Now for any smoothing \((D_t, X_t)\) of \((D, X)\) over \( T \), we can specify two distinct points \( p_t \) and \( q_t \) on \( D_t \) which specialize to \( q \) and \( r \) on \( D \). By Lemma 2.5, after shrinking \( T \), the conic \( D_t \) lies in the smooth locus of \( X_t \). Thus \( D_t \) is smoothly embedded. By the upper semicontinuity theorem and Lemma 2.3, \( D_t \) is a typical conic in \( X_t \).

**Definition 2.7.** Let \( X \) be a hypersurface of degree \( n \) in \( \mathbb{P}^n \). A **typical comb** with \( m \) teeth in \( X \) is a reduced curve in \( X \) with \( m + 1 \) irreducible components \( C, L_1, \ldots, L_m \) satisfying the following conditions:

1. \( C \) is a typical conic in \( X \);
2. \( L_1, \ldots, L_m \) are disjoint typical lines in \( X \) and each \( L_i \) intersects \( C \) transversally at \( p_i \).

The conic \( C \) is called the **handle** of the comb and \( L_i \)'s are called the **teeth**.

**Proposition 2.8.** Let \( X \) be a hypersurface of degree \( n \) in \( \mathbb{P}^n \). Let \( D = C \cup L_1 \cup \cdots \cup L_{n-2} \) be a typical comb with \( n - 2 \) teeth in \( X \). Let \( p_i \) be the intersection point \( L_i \cap C \). Assume that the following conditions hold:

1. the direction \( T_pC \) is not in the trivial subbundle of \( \mathcal{N}_{L_1|X} \);
2. the directions \( T_pL_i \) are general in \( \mathcal{N}_{C|X} \) such that the sheaf \( \mathcal{N}_{D_1|X}|_C \) is isomorphic to \( \mathcal{O}(1)^\oplus(n-2) \).

Then the pair \((D, X)\) \( \in R_{n,n} \) can be smoothed to a pair \((C', X')\) where \( C' \) is a very free curve in \( X' \).

**Proof:** The proof is very similar to the proof of Proposition 2.6. Here we only sketch the proof. Condition (1) implies that the sheaf \( \mathcal{N}_{D_1|X}|_{L_i} \) is isomorphic to \( \mathcal{O}^\oplus(n-2) \) for each \( i \). We have the following short exact sequence.

\[
0 \longrightarrow \cup_i \mathcal{N}_{D_1|X}|_{L_i}(-p) \longrightarrow \mathcal{N}_{D_1|X} \longrightarrow \mathcal{N}_{D_1|X}|_C \longrightarrow 0
\]

\[
\mathcal{O}^\oplus(n-2) \longrightarrow \mathcal{O}^\oplus(n-2)
\]
Since $H^1(D, N_{D|X}) = 0$, $D$ is unobstructed. By diagram chasing, the map $H^0(D, N_{D|X}) \to \bigcup T_p, C \otimes T_p, L_i$ is surjective. Thus we can smooth the typical comb $D$.

Now we may choose a smoothing $(D_t, X_t)$ and specify two distinct points $(q_t, r_t)$ which specialize to two distinct points $(q, r)$ on $C - \{p_1, \cdots, p_{n-2}\}$. By the long exact sequence, we know that $h^1(D, N_{D|X}(-q - r)) = 0$. By Lemma 2.5 and the upper semicontinuity theorem, a general smoothing of the pair $(D, X)$ gives a very free curve in a general hypersurface. □

3. An Example

In this section, we construct a hypersurface of degree $n$ in $\mathbb{P}^n$, which contains a special configuration of lines. Later we will use this example to produce a very free curve in a general hypersurface.

Let $n$ be an integer $\geq 4$. Let $[x_0 : \cdots : x_n]$ be the homogeneous coordinates for $\mathbb{P}^n$. Let $X$ be a hypersurface of degree $n$ in the projective space $\mathbb{P}^n$ defined by the following equation.

\[
x_0^{n-1}x_n + x_1^{n-3}x_n^2x_0 + (x_1^{n-1} + x_0x_1^{n-2} + \cdots + x_0^{n-3}x_1^2)x_2 + (x_2^{n-1} + x_0x_2^{n-2} + \cdots + x_0^{n-3}x_2^2)x_3 + \cdots
\]

\[
+ x_1x_n^{n-2}x_{n-2} + (x_0^{n-4}x_1^3 + x_0^{n-3}x_1^2)x_{n-2} + (x_0^{n-4}x_2^3 + x_0^{n-3}x_2^2)x_{n-1} + \cdots
\]

\[
+ x_n^{n-1}x_{n-1} + x_0^{n-3}x_n^2x_{n-1} + x_0^{n-3}x_2^3x_1 + \cdots
\]

**Notation 3.1.** Let $p$ be the point $[1 : 0 : \cdots : 0]$ and $q$ be the point $[0 : 1 : 0 : \cdots : 0]$. Let $L_i$ be the line spanned by $\{e_0, e_i\}$ for $i = 1, \ldots, n-1$ and $L_n$ be the line spanned by $\{e_1, e_n\}$. It is easy to check that they all lie in the hypersurface $X$. Let $C$ be the union of $L_1, \cdots, L_n$. The following picture shows the configuration of the points and the lines in the projective space.

![Diagram showing the configuration of points and lines](image)

**Lemma 3.2.**

1. Both $p$ and $q$ lie in the smooth locus of $X$.
2. The tangent space $T_pX$ is the hyperplane $\{x_n = 0\}$, which is spanned by the lines $L_1, \ldots, L_{n-1}$.
3. The tangent space of $T_qX$ is the hyperplane $\{x_2 = 0\}$. 
Proof. By taking the partial derivatives of $F$, we have $\frac{\partial F}{\partial x_i}(p) = 0$ for $i = 0, \cdots, n-1$ and $\frac{\partial F}{\partial x_n}(p) = 1$. Similarly, we have $\frac{\partial F}{\partial x_i}(q) = 0$ for $i \neq 2$ and $\frac{\partial F}{\partial x_2}(q) = 1$. □

Lemma 3.3. The lines $L_1, \cdots, L_{n-1}$ are in the smooth locus of $X$.

Proof. We will prove the case for line $L_1$. The remaining cases can be computed directly by the same method. Denote $L_1 = \{[x_0 : x_1 : 0 : \cdots : 0] \in \mathbb{P}^n\}$. By restricting the partial derivatives of the defining equation of the hypersurface $X$ on $L_1$, we get the following.

\begin{align}
\frac{\partial F}{\partial x_1}|_{L_1} &= x_1^{n-2} + x_0 x_1^{n-2} + \cdots + x_0^{n-3} x_1^3 \\
\frac{\partial F}{\partial x_2}|_{L_1} &= x_0 x_1^{n-2} + \cdots + x_0^{n-3} x_1^2 \\
&\vdots \\
\frac{\partial F}{\partial x_{n-2}}|_{L_1} &= x_0^{n-4} x_1^3 + x_0^{n-3} x_1^2 \\
\frac{\partial F}{\partial x_{n-1}}|_{L_1} &= x_0^{n-3} x_1^2
\end{align}

(3.1)

For points on $L_1$ with $x_0 \neq 0$, we have $\frac{\partial F}{\partial x_i}|_{L_1} \neq 0$. At the point $q$, $\frac{\partial F}{\partial x_2}|_{L_1} \neq 0$. Hence every point on the line $L_1$ is a smooth point of $X$. □

Lemma 3.4. The line $L_n$ is in the smooth locus of $X$.

Proof. By restricting the partial derivatives of the defining equation of $X$ on $L_n$, we get the following.

\begin{align}
\frac{\partial F}{\partial x_1}|_{L_n} &= x_1^{n-2} x_n^2 \\
\frac{\partial F}{\partial x_2}|_{L_n} &= x_1^{n-3} x_n^2 \\
&\vdots \\
\frac{\partial F}{\partial x_{n-1}}|_{L_n} &= x_1 x_n^{n-2} \\
\frac{\partial F}{\partial x_n}|_{L_n} &= x_n^{n-1}
\end{align}

(3.2)

For points on $L_n$ with $x_1 \neq 0$, we have $\frac{\partial F}{\partial x_i}|_{L_n} \neq 0$. For points on $L_n$ with $x_n \neq 0$, we have $\frac{\partial F}{\partial x_n}|_{L_n} \neq 0$. Hence every point on the line $L_n$ is a smooth point of $X$. □

Proposition 3.5. With the notations as above, $X$ satisfies the following properties.

1. The lines $L_1, \cdots, L_n$ are typical in $X$.

2. For $i = 1, \cdots, n-1$, the trivial subbundle of the normal bundle $N_{L_i|X}$ at $p$ is generated by $\partial_{i+1} - \partial_{i+2}, \cdots, \partial_{i+n-2} - \partial_{i+n-1}$, where $j$ takes value in $1, \cdots, n-1$ mod $n-1$.

3. The trivial subbundle of the normal bundle $N_{L_i|X}$ at $q$ is generated by $\partial_3, \cdots, \partial_{n-1}$.

4. The trivial subbundle of the normal bundle $N_{L_n|X}$ at $q$ is generated by $\partial_3, \cdots, \partial_{n-1}$.

Proof. Let $L$ be a line in $X$. We have the following short exact sequences.

\[
\begin{array}{cccccc}
0 & \longrightarrow & N_{L|X}(-1) & \longrightarrow & N_{L|P^n}(-1) & \longrightarrow & N_{X|P^n}|L(-1) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & N_{L|X}(-1) & \longrightarrow & \mathcal{O}_L^{\oplus(n-1)} & \longrightarrow & \mathcal{O}_L(n-1) & \longrightarrow & 0
\end{array}
\]
The associated long exact sequence is the following.

\[ H^0(L, N_{L|X}(-1)) \to k^n \overset{\alpha}{\longrightarrow} H^0(L, \mathcal{O}(n-1)) \to H^1(L, N_{L|X}(-1)) \to 0 \]

where the map \( \alpha \) sends the natural basis of \( k^n \) to the derivatives of \( F \) with respect to the normal directions of \( L \) in \( P^n \). By Lemma 2.2, \( L \) is typical if and only if the image of \( \alpha \) is of codimension one in \( H^0(L, \mathcal{O}(n-1)) \).

When \( L = L_1 \), by Lemma 3.8, \( \frac{\partial F}{\partial x_1} |_{L_1}, \ldots, \frac{\partial F}{\partial x_n} |_{L_1} \) form a codimensional-one subspace of \( H^0(L_1, \mathcal{O}_{L_1}(n-1)) \). Thus we get \( H^1(L_1, N_{L_1|X}(-1)) \) is one dimensional, i.e., \( L_1 \) is typical in \( X \).

By the short exact sequence above, \( N_{L_1|X}(-1) \) is a subbundle of the trivial bundle \( \mathcal{O}_{L_1}^{(n-1)} \) which maps to 0 in \( \mathcal{O}_{L_1}(n-1) \). Let \( \partial_2, \ldots, \partial_n \) be the generators of \( \mathcal{O}_{L_1}^{(n-1)} \). We get \( N_{L_1|X}(-1) \) is generated by \( x_0(\partial_2 - \partial_3) - x_1(\partial_3 - \partial_4), \ldots, x_0(\partial_{n-2} - \partial_{n-1}) - x_1(\partial_{n-1}) - x_2^2 \partial_n - x_1^2 \partial_n \) as an \( \mathcal{O}_{L_1} \)-module. If we restrict the bundle at \( p \) and \( q \), we get (2) and (3) for \( L_1 \).

When \( L = L_2, \ldots, L_{n-1} \), we can prove in a similar way. When \( L = L_n \), (4) follows from the same computation as above by applying (3.2).

With the description of the trivial subbundles of the normal bundles of lines in \( X \) as above, we get the following corollaries.

**Corollary 3.6.** We have the following statements.

1. The lines \( L_1 \) and \( L_n \) are typical in \( X \).
2. The direction \( T_p L_1 \) is not in the trivial subbundle of \( N_{L_1|X} \).
3. The direction \( T_p L_n \) is not in the trivial subbundle of \( N_{L_n|X} \).

**Corollary 3.7.** We have the following statements.

1. The lines \( L_2, \ldots, L_{n-2} \) are typical in \( X \).
2. The direction \( T_p L_1 \) is not in the trivial subbundle of \( N_{L_1|X} \) for \( 2 \leq i \leq n-1 \).
3. The directions \( T_p L_2, \ldots, T_p L_{n-1} \) span the normal bundle \( N_{L_i|X} \) at \( p \).

**4. Proof of the Main Theorem**

**Lemma 4.1.** (Har92 Ex 13.8). Let \( C \) be the union of \( n \) lines \( L_1, \ldots, L_n \) in \( P^n \) as in Notation 3.1. Then the Hilbert polynomial of \( C \) is \( P(d) = nd + 1 \). In particular, the arithmetic genus of \( C \) is 0.

*Proof.* This can be computed directly. For any \( d > 0 \), when \( i = 1, \ldots, n-1 \), the homogeneous polynomials of degree \( d \) that do not vanish on \( L_i \) are generated by \( \{ x_0^d, x_0^{d-1} x_i, \ldots, x_i^d \} \). The homogeneous polynomials of degree \( d \) that do not vanish on \( L_n \) are generated by \( \{ x_1^d, x_1^{d-1} x_i, \ldots, x_n^d \} \). Thus when \( d \gg 0 \), \( P(d) = H^0(C, \mathcal{O}_C(d)) = nd + 1 \).

The curve \( C \) is an example of curves with rational \( n \)-fold point, cf. CCC11 3.7.

The following lemma is an analogue of CCC11 Lemma 3.8.

**Lemma 4.2.** With the same notations as in Lemma 4.1, the following properties hold for \( C \) for every positive integer \( d \):

1. \( h^0(C, \mathcal{O}_C(d)) = nd + 1 \) and \( h^1(C, \mathcal{O}_C(d)) = 0 \).
2. \( h^1(C, L_C(d)) = 0 \).
3. \( h^0(C, L_C(d)) = h^0(P^n, \mathcal{O}(d)) - nd - 1 \).
Lemma 4.4. By Riemann-Roch and Lemma 4.1, we have

\[ h^0(C, \mathcal{O}_C(d)) \geq \chi(\mathcal{O}_C(d)) = nd + 1 - p_g(C) = nd + 1. \]

On the other hand, every global section of \( \mathcal{O}_C(d) \) is obtained by gluing global sections on each component, which imposes at least \( n - 1 \) linear conditions. Since we have \( h^0(L_i, \mathcal{O}(d)) = d + 1 \) for every \( i \),

\[ h^0(\mathcal{O}_C(d)) \leq n(d + 1) - (n - 1) = nd + 1. \]

Therefore, \( h^0(C, \mathcal{O}_C(d)) = nd + 1 \) for every positive integer \( d \). Since the image of \( r : H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(C, \mathcal{O}_C(d)) \) has dimension \( nd + 1 \) as in Lemma 4.1, the map \( r \) is surjective. The lemma follows by considering the long exact sequence associated to the ideal sheaf of \( C \).

**Proof.**

Construction 4.3. Let \( C \) be the union of \( n \) lines \( L_1, \ldots, L_n \) in \( \mathbb{P}^n \) as in Notation 3.1. If we consider \( L_1 \cup L_n \) as a conic in \( \mathbb{P}^n \), there exists a smooth affine pointed curve \( (T, 0) \) and a smoothing \( D' : (T, 0) \to (T, 0) \) satisfying the following conditions:

1. The special fiber \( D'_0 \) is \( L_1 \cup L_n \);
2. For any \( t \in T - \{0\} \), \( D'_t \) is a smooth conic contained in the plane spanned by \( L_1 \) and \( L_n \).

We can choose \( n - 2 \) sections \( s_i : (T, 0) \to D' \) for \( i = 1, \ldots, n - 2 \) such that \( s_i(0) = p \) for all \( i \)'s and for \( t \in T - \{0\} \), \( s_i(t) \)'s are all distinct on \( D'_t \).

For any \( s_i(t) \), there exists a unique line \( L_{t,i+1}(t) \) through \( s_i(t) \) parallel to \( L_{t,i+1} \). After gluing the families of lines \( L_{t,i+1}(t) \) on \( D'_t \) at \( s_i(t) \) for all \( i \)'s, we get a family of reducible curves \( \pi : D \to (T, 0) \) satisfying the following conditions:

1. The special fiber \( D_0 \) is \( C \) as constructed in 4.1.
2. For any \( t \in T - \{0\} \), \( D_t \) is a comb with the handle \( D'_t \) and with the teeth lines.

The family \( \pi : D \to (T, 0) \) is flat by Lemma 4.1. We have the following diagram.

\[
\begin{array}{ccc}
D_0 &=& C \\
\downarrow & & \downarrow \\
0 &=& (T, 0) \\
\end{array}
\]

\[
\begin{array}{ccc}
D &=& \mathbb{P}^n \\
\pi & & \pi \\
\downarrow & & \downarrow \\
(T, 0) &=& (T, 0) \\
\end{array}
\]

**Lemma 4.4.** Let \( \mathcal{I}_D \) be the ideal sheaves of \( D \) in \( \mathbb{P}^n \). The sheaf \( \pi_* \mathcal{I}_D(d) \) is locally free on \( T \) for any \( d > 0 \).

**Proof.** By the cohomology and base change theorem [Har77] III.12.9, it suffices to show that \( h^0(\mathbb{P}^n, \mathcal{I}_D(d)) \) is constant. By upper semicontinuity and Lemma 4.2, we have \( h^0(\mathbb{P}^n, \mathcal{I}_D(d)) \leq h^0(\mathbb{P}^n, \mathcal{O}(d)) - nd - 1 \). On the other hand, for any \( t \in T - \{0\} \), the curve \( D_t \) is a local complete intersection and \( \mathcal{O}_{D_t}(d) \) is a positive bundle on \( D_t \). Thus we have \( h^1(D_t, \mathcal{O}_{D_t}(d)) = 0 \) and \( h^0(D_t, \mathcal{O}_{D_t}(d)) = nd + 1 \). Consider the following exact sequence.

\[
0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{I}_{D_t}(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(D_t, \mathcal{O}_{D_t}(d)) \longrightarrow 0.
\]

We get \( h^0(\mathbb{P}^n, \mathcal{I}_{D_t}(d)) \geq h^0(\mathbb{P}^n, \mathcal{O}(d)) - nd - 1 \).

**Proof of Theorem 1.3.** The theorem is trivial for \( n = 2, 3 \). We can assume that \( n \geq 4 \). By [Kol96] IV.3.11 and Lemma 2.5, it suffices to produce one very free curve in a hypersurface of degree \( n \). By Lemma 4.4, after shrinking \( T \), hypersurfaces of
degree $n$ containing $D_t$ in $\mathbb{P}^n_t$ form a trivial projective bundle over $(T, 0)$. Thus the family $\pi : D \to (T, 0)$ admits a lifting to a flat family of pairs $\pi : (D, \mathcal{X}_t) \to (T, 0)$ in $R_{n,n}$ such that the special fiber $(D_0, \mathcal{X}_0)$ is $(C, X)$ which is constructed in Section 3.

\[
\begin{array}{ccc}
D & \xrightarrow{i} & \mathcal{X}_T \\
\downarrow \pi & & \downarrow \pi \\
(T, 0) & \xrightarrow{\pi} & \mathbb{P}^n_T
\end{array}
\]

All the following steps of the proof requires to shrink $T$ if necessary. By Proposition 2.6 and Corollary 3.6 we may assume that the handle $D'_t$ is a typical conic in $\mathcal{X}_t$ for every $t \in T - \{0\}$. By Proposition 2.4 and Corollary 3.7 (1), all the teeth of the comb $D_t$ are typical. Thus for every $t \in T - \{0\}$, we get a typical comb $D_t$ as in Definition 2.7. Now the theorem follows if we verify the two conditions in Proposition 2.3. Since they are open conditions, it suffices to check on the special fiber $(C, X)$, which is proved in Corollary 3.7. □

Proof of Theorem 1.4. By [Kol96] IV.3.11 and Lemma 2.5, it suffices to produce one very free curve in a hypersurface of degree $d$. Let $Y$ be a general smooth Fano hypersurface of degree $d$ in $\mathbb{P}^n$. When $d = n$, this is proved in Theorem 1.3. When $d < n$, we may choose a general linear subspace $L$ of dimension $d$ such that $Y \cap L$ is smooth and contains a very free curve $f : \mathbb{P}^1 \to Y \cap L$ by Theorem 1.3. By the normal bundle exact sequence,

\[
0 \longrightarrow T(Y \cap L) \longrightarrow TY \longrightarrow N_{Y\cap L|Y} \longrightarrow 0
\]

the sheaf $f^*T(Y \cap L)$ is positive and the sheaf $N_{Y\cap L|Y}$ is isomorphic to $N_{L|\mathbb{P}^n}$, which is $\mathcal{O}(1)^{\oplus (n-d)}$. Therefore the pullback bundle $f^*TY$ is positive. Thus $f : \mathbb{P}^1 \to Y \cap L \to Y$ is a very free curve in $Y$. □

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