Black hole-naked singularity dualism and the repulsion of two Kerr black holes due to spin-spin interaction

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We report about the possibility for interacting Kerr sources to exist in two different states – black holes or naked singularities – both states characterized by the same masses and angular momenta. However, as will be shown in the present letter, a liding constituents, as was surmised by Hawking [8] long ago. It appears that an important physical information pertinent to the case of the head-on collisions of black holes was also supplied by the well-known double-Kerr action force) is that the spin-spin repulsion in a binary black hole system is still possible and is based on the gravitational attraction, which prevents the two black holes from merging into a single black hole. Moreover, we have discovered that the nonuniqueness also includes a surprising phenomenon when the same mass and angular momentum determine simultaneously a black-hole and a hyper-extreme configuration of two Kerr sources, which allows to speak about the black hole-naked singularity (BH/NS) dualism as a new highly nonlinear effect occurring in the binary systems subject to a very strong gravity. We now turn to the details.

The configurations of two equal Kerr sources.—The family of two equal Kerr sources kept apart by a massless strut is described by the metric

\[ ds^2 = f^{-1}[\epsilon^2(r^2 + d\tau^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \]

\[ f = \frac{A\bar{A} - \bar{B}B}{(A + \bar{B})(A + B)}, \quad \epsilon^{2\gamma} = \frac{A\bar{A} - \bar{B}B}{16\lambda_0 \lambda_0 R_1 R_2 R_3 R_4}, \]

\[ \omega = \omega_0 - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{AA - BB}, \]

\[ A = (R_1 - R_2)(R_3 - R_4) - 4\sigma(1 - \alpha_1), \]

\[ B = 2\sigma[(1 - 2\sigma)(R_1 - R_4) - (1 + 2\sigma)(R_2 - R_3)], \]

\[ G = -sB + s\alpha[2R_1 R_3 - 2R_2 R_4 - 4\sigma(R_1 R_2 - R_3 R_4)], \]

where the functions \( R_i \) are defined by the expressions

\[ R_i = X_1 \sqrt{\rho^2 + (z - \alpha_i)^2}, \]

\[ X_1 = -1/X_4 = \phi(\mu + \sqrt{\mu^2 - 1}), \]

\[ X_2 = -1/X_3 = \phi(\mu - \sqrt{\mu^2 - 1}), \]

\[ \alpha_1 = -\alpha_4 = s\left(1/2 + \sigma\right), \quad \alpha_2 = -\alpha_3 = s\left(1/2 - \sigma\right), \]

\[ \sigma = -\frac{i\sqrt{\mu^2 - 1}}{\nu}, \]

\[ \mu = \frac{(\phi^2 + 1)[\phi(\nu^2 - 4) + i\nu(\phi^2 - 1)]}{2(\nu^2 + 1)^2 - 4\nu(\phi^2 - 1)} \]

and the constants \( \lambda_0 \) and \( \omega_0 \) have the form

\[ \lambda_0 = \frac{1}{\nu^2}(1 - \phi^2)[(1 + 4\phi^2)^2 - \nu^2\phi^2], \]

\[ \omega_0 = \frac{2is(1 - \phi^2 - 2i\mu\nu\phi^2)}{(1 + \phi^2)^2 - \nu^2\phi^2}. \]

The above formulas (1)-(5) involve three arbitrary parameters, which are the coordinate distance \( s \) between the centers of the Kerr constituents (see Fig. 1), the real
positive constant $\nu$, and the unitary complex parameter $\phi$, so that $\phi \bar{\phi} = 1$; the black-hole configurations correspond to real positive $\sigma$, while the pure imaginary $\sigma$ describe the binary systems of hyperextreme Kerr objects (naked singularities). Instead of $\nu$ and $\phi$, however, one may use the dimensionless individual mass $m$ and angular momentum per unit mass $a$ of each constituent (these are related to the total dimensional quantities by the relations $M_T = 2ms$, $J_T = 2ma^2$), and finding $\phi$ in terms of $m$ and $a$ then reduces to solving the cubic equation

$$Z^3 - pZ^2 + pZ - 1 = 0, \ Z \equiv \phi^2,$$

$$p \equiv p_x + ip_y = -\frac{m + 2}{m} - i\frac{(2m + 1)^2}{ma}, \ (4)$$

while the dependence $\nu(m, a)$ is determined by the equation

$$\nu = -\frac{i[m(\phi^2 + 1)^2 + 2\phi^2]}{m\phi(\phi^2 - 1)}, \ (5)$$

into which the explicit form of $\phi$ should be inserted after the resolution of equation (4), taking into account that the correct choice of sign in $\phi = \pm \sqrt{Z}$ must ensure $\nu > 0$. Obviously, for $m$ and $a$ the following relations hold:

$$m = -\frac{2}{p_x + 1}, \ a = -\frac{(p_x - 3)^2}{2p_y(p_x + 1)} \ (6)$$

As has been established in [9], equation (4) may have, for given values of $m$ and $a$, up to three unitary roots, and the nonuniqueness zone is defined by the “tanga curve” depicted in Fig. 2, inside of which (and on the curve itself) the values of $m$ and $a$ do not lead to unique configurations of the Kerr constituents. The only example of nonuniqueness considered in [9] involved the hyperextreme Kerr sources because at that time we did not yet know how to extend our analysis to the black-hole case. The difficulty that exhibits the latter case is the need, in addition to finding configurations with real positive $m$ and $\sigma$, to satisfy also the condition $\sigma^2 < \frac{1}{4}$ ensuring the separation of the black holes on the symmetry axis. The strategy that eventually allowed us to get the desired two different binary black-hole configurations defined by the same values of $m$ and $a$ consisted in first identifying one physically meaningful black-hole configuration on the tanga curve (where only two unitary $Z$ are possible) and then searching the roots of the cubic equation (4) inside the curved triangle which would be slightly different from the boundary values. Below we give the numerical value of $p$ found in this way which determines two different black-hole configurations and one hyperextreme configuration of two Kerr constituents, all three configurations sharing the same mass and angular momentum:

$$p = -1.146560133416505 + 2.152383532475482i. \ (7)$$

The corresponding values of $m$ and $a$ obtainable from (6) are the following:

$$m = 13.64628, \ a = -27.25276, \ (8)$$

and these are given up to five decimal places (like all the remaining quantities hereafter). Then the first binary configuration of black holes is described by the set of the parameters

$$\begin{align*}
\phi &= -0.67856 - 0.73455i, \ \nu = 1.35343, \\
\mu &= -0.83296, \ \sigma = 0.40884, \\
X_1 &= 0.97166 + 0.23637i, \\
X_2 &= 0.15875 + 0.98732i,
\end{align*} \ (9)$$

while the second binary black-hole configuration is defined by the values

$$\begin{align*}
\phi &= -0.67819 - 0.73489i, \ \nu = 1.35145, \\
\mu &= -0.86653, \ \sigma = 0.36933, \\
X_1 &= 0.954475 + 0.29829i, \\
X_2 &= 0.22086 + 0.97531i.
\end{align*} \ (10)$$

Remarkably, the above two subextreme configurations are also accompanied by a binary system of hyperextreme Kerr sources, namely,

$$\begin{align*}
\phi &= -0.07962 - 0.99683i, \ \nu = 0.08623, \\
\mu &= -2.07558, \ \sigma = -21.09186i, \\
X_1 &= 0.02044 + 0.25597i, \\
X_2 &= 0.31007 + 3.88202i, \ (11)
\end{align*}$$

so that, on the one hand, we have demonstrated the nonuniqueness of binary black-hole configurations at certain values of masses and angular momenta, and on the other hand have established the subextreme/hyperextreme duality of the binary configurations which means that in a highly nonlinear regime a pair of interacting Kerr sources can exist either as a black-hole binary or a system of two naked singularities, both states characterized by the same masses and angular momenta of the constituents.

From (8) it follows that $J_T^2/M_T^4 = 0.99709$, and hence all the configurations (9)-(11) would resemble a nearly extreme single Kerr black hole to a distant observer.

The interaction force.—An outstanding feature of the particular configurations presented above is that the spin-spin repulsion in them exceeds the gravitational attraction, so that the constituents repel each other. This can be seen by analyzing the formula of the interaction force which has the form

$$F = \frac{1}{4} (e^{-2\gamma_0} - 1), \ (12)$$

with $\gamma_0$ denoting the value of the metric function $\gamma$ on the part of the symmetry axis separating the two constituents. Then, taking into account that

$$e^{2\gamma_0} = g_+^2/g_-, \ g_\pm = \phi^4 \pm (\nu^2 - 2)\phi^2 + 1, \ (13)$$

we get for the three configurations (9)-(11), labeled respectively with subindices I, II and III, the following values of $F$:

$$F_I = -0.24235, \ F_{II} = -0.24, \ F_{III} = -0.24887, \ (14)$$
and one can see that these values differ only insignificantly from each other. This means that the specific type of a Kerr source – black hole or naked singularity – does not really have any serious effect on the interaction force in a binary configuration if the subextreme and hyperextreme constituents have the same masses and angular momenta and are separated by the same coordinate distance.

**Physical implications.**—The existence of the repulsion between two subextreme Kerr black holes gives a clear indication that the spin-spin interaction should not be neglected in the analysis of the head-on collision of corotating black holes because this interaction is able to prevent the colliding black-hole binary from forming a single black hole. Indeed, our findings suggest that the following dynamical scenario for such a collision should not be excluded in principle: The two initially separated black holes approach each other under the action of the gravitational attraction which is inversely proportional to $r^2$; the contribution of the spin-spin force at large separation is negligible, being as is well known \[12\] inversely proportional to $s^4$. However, at a small separation, the spin-spin repulsion becomes dominant, and the black holes will bounce off each other, moving away from each other to a distance where the gravitational attraction prevails again and provokes a next cycle of approaching and bouncing of the black holes. Since such a nonstationary process is axially symmetric, no angular momentum is radiated away, while the loss of mass through gravitational radiation in the case of separated colliding black holes is insignificant \[13\], and hence we end up with a binary system of oscillating black holes, that loses very slowly its mass, as a possible final state of the head-on collision.

Apparently, the nonlinear effects are strongest at the smallest separation distances between the components of the binary system, when the BH/NS dualism is most likely to show itself up. Actually, it can be shown by passing to the dimensional quantities that the large $s$ approximation withdraws the binary system from the nonuniqueness zone, which explains in particular why the dualism phenomenon has not been discovered earlier. It is also clear that at short distances the individual stationary limit surfaces of the interacting Kerr constituents may form a combined ergoregion, as it happens for instance in the configurations \[9-11\]. Interestingly, the binaries with attracting constituents ($\mathcal{F} > 0$) are then distinguishable from the configurations with repelling constituents ($\mathcal{F} < 0$) by the presence of a massless ring singularity off the symmetry axis in the latter configurations which has a ‘benign character’ using Wald’s terminology \[14\] and just signals that the common ergosurface is about to be divided into two disconnected parts or has already started such a division. We emphasize that in the fully dynamical binary configurations, which are mimicked by our solutions and could be simulated in the framework of numerical relativity \[15\], neither of the two aforementioned singularities would appear.

Let us also mention the relation of our analytical results to the discussion held during the last decade in the framework of a test particle approximation about the possibility to convert a spinning black hole into a naked singularity \[16-21\]. The similarity between the paper \[16\] and our work is that the black holes in both approaches must be almost extreme for being able to be turned into the naked singularities, the transition itself involving a very fine tuning of the parameters. At the same time, our research based on the exact solution of Einstein’s equations specifies and clarifies various important aspects connecting the black holes and naked singularities that cannot be taken into account by the approximation scheme of Refs. \[16-21\]. First of all, the binary systems in our analysis involve interacting black holes, with full account of the spin-spin repulsion contribution, when the values of the individual angular momenta defining the extremality condition can exceed considerably the angular momentum of a single Kerr extreme black hole \[22\] (the analogous change of the extremality condition for binaries in electrostatics was discussed in \[23\]). As a consequence, for the configurations \[9-11\] the ratio $|a|/m$ is equal to 1.99712, being almost twice greater than that of a single extreme black hole, which illustrates well that in the binary systems the inequality $|a|/m > 1$ may determine both the naked singularities and the rapidly rotating black holes. Moreover, it is really remarkable that the same values of $m$ and $a$ can define a pair of subextreme or hyperextreme constituents characterized by the presence or absence of the event horizons, and it seems that the transition from the black-hole to the naked-singularity state (and vice versa) in such binaries, to which in particular belong the configurations \[9-11\], presumably occurs in a spontaneous way. In this respect we would like to remark that the conversion of a Kerr naked singularity into a black hole by means of test bodies would seem to us a simpler process than destroying the black hole horizon through overspinning discussed in \[16\], since the former requires exclusively an increase in mass, and hence it would be plausible to suppose that some astrophysical black holes might have been initially formed as naked singularities and only later became black holes after consuming a necessary amount of surrounding matter. Let us also note for completeness that in the extreme limit ($\sigma = 0$) the metric \[1\] reduces to a special subfamily of the well-known Kinnersley-Chitre solution \[24\] determining two identical corotating extreme black holes which was identified and analyzed in the paper \[25\]. Apparently, the degenerated solution is unsuitable for treating any nonextreme configurations, and in particular, if used in a gedanken experiment to destroy the horizons of extreme black holes with test bodies, it would give the same negative result as earlier obtained by Wald \[26\] with the aid of a single Kerr extreme solution. At the same time, the extreme binary configurations from \[24\] can be shown to provide analytical examples of two repelling extreme black holes, thus supporting our results concerning the more general non-extreme case.

Therefore, we have shown that the spin-spin interac-
tination in binary configurations of Kerr sources can play an important role at the last stages of the merging or colliding processes, giving rise to various interesting nonlinear effects. This interaction turns out to be a major factor able to prevent two colliding corotating black holes from forming a single black hole; alternatively, it may lead to the formation of the two-component oscillators as the end state of the collision, and in principle the latter binary oscillating systems might be detectable through the astrophysical observations. Our results also suggest that taking into account of the spin-spin interaction could cause an elongation in time of the gravitational wave signals which are being received from the generic

merging black holes or neutron stars. Lastly, the present research clearly demonstrates that the physics of binary black holes is richer than that of single black hole space-times, and we expect that a clever amalgamation of the numerical and analytical approaches to the study of the binary configurations will be able to shed more light on the properties of the interacting black holes in the future.

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FIG. 1: Two possible configurations of a pair of equal nonextreme Kerr sources: (a) a black-hole binary, (b) a system of two naked singularities.

FIG. 2: The curved triangle separating the unique configurations with particular $m$ and $a$ from those for which the same particular values of mass and angular momentum are shared by three different configurations (the latter lie inside the tanga curve).