Classification of multipartite entangled states by multidimensional determinants

Akimasa Miyake

Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan and
Quantum Computation and Information Project, ERATO, Japan Science and Technology, Hongo 5-28-3, Bunkyo-ku, Tokyo 113-0033, Japan

(Dated: )

We find that multidimensional determinants "hyperdeterminants", related to entanglement measures (the so-called concurrence or 3-tangle for the 2 or 3 qubits, respectively), are derived from a duality between entangled states and separable states. By means of the hyperdeterminant and its singularities, the single copy of multipartite pure entangled states is classified into an onion structure of every closed subset, similar to that by the local rank in the bipartite case. This reveals how inequivalent multipartite entangled classes are partially ordered under local actions. In particular, the generic entangled class of the maximal dimension, distinguished as the nonzero hyperdeterminant, does not include the maximally entangled states in Bell’s inequalities in general (e.g., in the $n \geq 4$ qubits), contrary to the widely known bipartite or 3-qubit cases. It suggests that not only are they never locally interconvertible with the majority of multipartite entangled states, but they would have no grounds for the canonical $n$-partite entangled states. Our classification is also useful for the mixed states.

I. INTRODUCTION

Entanglement is the quantum correlation exhibiting nonlocal (nonseparable) properties. It is supposed to be never strengthened, on average, by local operations and classical communication (LOCC). In particular, entanglement in multi-parties is of fundamental interest in quantum many-body theory [1], and makes quantum information processing (QIP), e.g., distillation protocol, more efficient than that relying on entanglement only in two-parties [2]. Here, we classify and characterize the multipartite entanglement which has yet to be understood, compared with the bipartite one.

For the single copy of bipartite pure states on $\mathcal{H}(\mathbb{C}^{k+1})\otimes\mathcal{H}(\mathbb{C}^{k+1})$, we are interested in whether a state $|\Psi\rangle$ can convert to another state $|\Phi\rangle$ by LOCC. It is convenient to consider the Schmidt decomposition,

$$|\Psi\rangle = \sum_{i_1,i_2=0}^{k} a_{i_1,i_2} |i_1\rangle \otimes |i_2\rangle = \sum_{j=0}^{k} \lambda_j |e_j\rangle \otimes |e'_j\rangle,$$

(1)

where the computational basis $|i_j\rangle$ is transformed to a local biorthonormal basis $|e_j\rangle, |e'_j\rangle$, and the Schmidt coefficients $\lambda_j$ can be taken as $\lambda_j \geq 0$. We call the number of nonzero $\lambda_j$ the (Schmidt) local rank $r$. Then the LOCC convertibility is given by a majorization rule between the coefficients $\lambda_j$ of $|\Psi\rangle$ and those of $|\Phi\rangle$. This suggests that the structure of entangled states consists of partially ordered, continuous classes labeled by a set of $\lambda_j$. In particular, $|\Psi\rangle$ and $|\Phi\rangle$ belong to the same class under the LOCC classification if and only if all continuous $\lambda_j$ coincide.

Suppose we are concerned with a coarse grained classification by the so-called stochastic LOCC (SLOCC) [4, 5], where we identify $|\Psi\rangle$ and $|\Phi\rangle$ that are interconvertible back and forth with (maybe different) nonvanishing probabilities. This is because $|\Psi\rangle$ and $|\Phi\rangle$ are supposed to perform the same tasks in QIP although their probabilities of success differ. Later, we find that this SLOCC classification is still fine grained to classify the multipartite entanglement. Mathematically, two states belong to the same class under SLOCC if and only if they are converted by an invertible local operation having a nonzero determinant [6]. Thus the SLOCC classification is equivalent to the classification of orbits of the natural action: direct product of general linear groups $GL_{k+1}(\mathbb{C}) \times GL_{k+1}(\mathbb{C})$. The local rank $r$ in Eq. (1) equivalently the rank of $a_{i_1,i_2}$ is found to be preserved under SLOCC. A set $S_j$ of states of the local rank $j$ is a closed subvariety under SLOCC and $S_{j-1}$ is the singular locus of $S_j$. This is how the local rank leads to an "onion" structure (mathematically the stratification):

$$S_{k+1} \supset S_k \supset \cdots \supset S_1 \supset S_0 = \emptyset,$$

(2)

and $S_j - S_{j-1}$ ($j = 1, \ldots, k+1$) give $k+1$ classes of entangled states. Since the local rank can decrease by non-invertible local operations, i.e., general LOCC [4, 5], these classes are totally ordered such that, in particular, the outermost generic set $S_{k+1} - S_k$ is the class of maximally entangled states and the innermost set $S_1 = S_1 - S_0$ is that of separable states.

For the single copy of multipartite pure states,

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n=0}^{k_1,\ldots,k_n} a_{i_1,\ldots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle,$$

(3)

there are difficulties in extending the Schmidt decomposition for a multiorthonormal basis [4]. Moreover, an
attempt to use the tensor rank of \(a_{i_1 \ldots i_n}\) falls down since \(S_j\), defined by it, is not always closed \(10\) \(11\) \(12\). In the 3 qubits, Dür et al. showed that SLOCC classifies the whole states \(M\) into finite classes and in particular there exist two inequivalent, Greenberger-Horne-Zeilinger (GHZ) and W, classes of the tripartite entanglement \(13\). They also pointed out that this case is exceptional since the action \(GL_{k+1}(C) \times \cdots \times GL_{k_n+1}(C)\) has infinitely many orbits in general (e.g., for \(n \geq 4\)).

In this paper, we classify multipartite entanglement in a unified manner based on the hyperdeterminant. The advantages are three-fold. (i) This classification is equivalent to the SLOCC classification when SLOCC has finitely many orbits. So it naturally includes the widely known bipartite and 3-qubit cases. (ii) In the multipartite case, we need further SLOCC invariants in addition to the local ranks. For example, in the 3-qubit case \(5\), the 3-tangle \(\tau\), just the absolute value of the hyperdeterminant (see Eq. \(11\)), is utilized to distinguish GHZ and W classes. This work clarifies why the 3-tangle \(\tau\) appears and how these SLOCC invariants are related to the hyperdeterminant in general. (iii) Our classification is also useful to multipartite mixed states. A mixed state \(\rho\) can be decomposed as a convex combination of projection operators onto pure states. Considering how \(\rho\) needs at least the outer class in the onion structure of pure states, we can also classify multipartite mixed states into the totally ordered classes (for details, see Appendix \(13\)). We concentrate on the pure states here.

The rest of the paper is organized as follows. In Sec. \(11\), a duality between separable states and entangled states is introduced. We find that the hyperdeterminant, associated to this duality, and its singularities lead to the SLOCC-invariant onion-like structure of multipartite entanglement. The characteristics of the hyperdeterminant and its singularities are explained in Sec. \(13\). Classifications of multipartite entangled states are exemplified in Sec. \(14\) so as to reveal how they are ordered under SLOCC. Finally, the conclusion is given in Sec. \(15\).

II. DUALITY BETWEEN SEPARABLE STATES AND ENTANGLED STATES

In this section, we find that there is a duality between the set of separable states and that of entangled states. This duality derives the hyperdeterminant our classification is based on.

A. Preliminary: Segre variety

To introduce our idea, we first recall the geometry of pure states. In a complex (finite) \(k+1\)-dimensional Hilbert space \(H(C^{k+1})\), let \(|\Psi\rangle\) be a (not necessarily normalized) vector given by \(k+1\)-tuple of complex amplitudes \(x_j (j = 0, \ldots, k) \in C^{k+1} - \{0\}\) in a computational basis. The physical state in \(H(C^{k+1})\) is a ray, an equivalence class of vectors up to an overall nonzero complex number. Then the set of rays constitutes the complex projective space \(CP^k\) (the projectivization of \(H(C^{k+1})\)), and \(x := (x_0 : \ldots : x_k)\), considered up to a complex scalar multiple, gives homogeneous coordinates in \(CP^k\).

For a composite system which consists of \(H(C^{k_1+1})\) and \(H(C^{k_2+1})\), the whole Hilbert space is the tensor product \(H(C^{k_1+1}) \otimes H(C^{k_2+1})\) and the associated projective space is \(M = CP^{(k_1+1)(k_2+1)−1}\). A set \(X\) of the separable states is the mere Cartesian product \(CP^{k_1} \times CP^{k_2}\), whose dimension \(k_1 + k_2\) is much smaller than that of the whole space \(M\), \((k_1 + 1)(k_2 + 1)−1\). This \(X\) is a closed, smooth algebraic subvariety (Segre variety) defined by the Segre embedding into \(CP^{(k_1+1)(k_2+1)−1}\) \(12\) \(13\).

\[
CP^{k_1} \times CP^{k_2} \ni (x_0^{(1)}, x_0^{(2)}) \mapsto (x_0^{(1)} : \ldots : x_k^{(1)}), (x_0^{(2)} : \ldots : x_k^{(2)}).
\]

Denoting homogeneous coordinates in \(CP^{(k_1+1)(k_2+1)−1}\) by \(b_{i_1 \ldots i_2} = x_{i_1}^{(1)} x_{i_2}^{(2)} (0 \leq i_1 \leq k_1, 0 \leq i_2 \leq k_2)\), we find that the Segre variety \(X\) is given by the common zero locus of \(k_1(k_1 + 1)k_2(k_2 + 1)/4\) homogeneous polynomials of degree 2:

\[
b_{i_1 \ldots i_2} b_{i_1' \ldots i_2'} - b_{i_1 \ldots i_2} b_{i_1' \ldots i_2'},
\]

where \(0 \leq i_1 < i_1' \leq k_1, 0 \leq i_2 < i_2' \leq k_2\). Note that this condition implies that all \(2 \times 2\) minors of "matrix" \(b_{i_1 \ldots i_2}\) equal 0; i.e., the rank of \(b_{i_1 \ldots i_2}\) is 1. Thus we have \(X = S_1\), which agrees with the SLOCC classification by the local rank in the bipartite case.

Now consider the multipartite Cartesian product \(X = CP^{k_1} \times \cdots \times CP^{k_\alpha}\) in the Segre embedding into \(M = CP^{(k_1+1)\cdots(k_\alpha+1)−1}\). Because this Segre variety \(X\) is the projectivization of a variety of the matrices \(b_{i_1 \ldots i_\alpha} = x_{i_1}^{(1)} \cdots x_{i_\alpha}^{(\alpha)}\), it gives a set of the completely separable states in \(H(C^{k_1+1}) \otimes \cdots \otimes H(C^{k_\alpha+1})\). By another Segre embedding, say \(X' = CP^{(k_1+1)(k_2+1)−1} \times \cdots \times CP^{k_\alpha}\), we also distinguish a set of separable states where only 1st and 2nd parties can be entangled, i.e., when we regard 1st and 2nd parties as one party, an element of this set is completely separable for "\(n-1\)" parties. This is how, also in the multipartite case, we can classify all kinds of separable states, typically lower dimensional sets. Note that, in the multipartite case, this check for the separability is more strict than the check by local ranks \(14\).

B. Main idea: duality

We rather want to classify entangled states, typically higher dimensional complementary sets of separable states. Our strategy is based on the duality in algebraic geometry; a hyperplane in \(CP\) forms the point of a dual projective space \(CP^*\), and conversely every point \(p\) of \(CP\) is tied to a hyperplane \(p^*\) in \(CP^*\) as the set of all hyperplanes in \(CP\) passing through \(p\). Remarkably, the
projective duality between projective subspaces, like the above example, can be extended to an involutive correspondence between irreducible algebraic subvarieties in $\mathbb{CP}^n$ and $\mathbb{CP}^n$. So we define a projectively dual (irreducible) variety $X^\vee \subset \mathbb{CP}^n$ as the closure of the set of all hyperplanes tangent to the Segre variety $X$.

Let us observe (and see the reason later) that, in the bipartite case seen in Sec. the variety $S_k$ of the degenerate $(k+1) \times (k+1)$ matrices $A = a_{i_1,i_2}$ is projectively dual to the variety $S_1 = X$ of the matrices $B = b_{i_1,i_2} = x^{(1)}_{i_1} x^{(2)}_{i_2}$. That is, $S_k$ is the dual variety $X^\vee$. Following an analogy with a 2-dimensional (bipartite) case, an $n$-dimensional matrix $A = a_{i_1,...,i_n}$ is called degenerate if and only if it (precisely, its projectivization) lies in the projectively dual variety $X^\vee$ of the Segre variety $X$. In other words, identifying the space of $n$-dimensional matrices with its dual by means of the pairing,

$$F(A, B) = \sum_{i_1,...,i_n=0}^{k_1,...,k_n} a_{i_1,...,i_n} b_{i_1,...,i_n},$$

we see that $A$ is degenerate if and only if its orthogonal hyperplane $F(A, B) = 0$ is tangent to $X$ at some nonzero point $x = (x^{(1)},...,x^{(n)})$. Analytically, a set of equations,

$$F(A, x) = \sum_{i_1,...,i_n=0}^{k_1,...,k_n} a_{i_1,...,i_n} x^{(1)}_{i_1} \cdots x^{(n)}_{i_n} = 0,$$

$$\frac{\partial}{\partial x^{(j)}_{i_j}} F(A, x) = 0 \quad \text{for all } j, i_j$$

($j = 1, ..., n$ and $0 \leq i_j \leq k_j$), has at least a nontrivial solution $x = (x^{(1)},...,x^{(n)})$ of every $x^{(j)} \neq 0$, and then $x$ is called a critical point. The above condition is also equivalent to saying that the kernel $\ker F$ of $F(A, x)$ is not empty, where $\ker F$ is the set of points $x = (x^{(1)},...,x^{(n)}) \in X$ such that, in every $j_0 = 1, ..., n$,

$$F(A, (x^{(1)},...,x^{(j_0-1)},z^{(j_0)},x^{(j_0+1)},...,x^{(n)})) = 0,$$

for the arbitrary $z^{(j_0)}$.

In the case of $n = 2$, the condition for Eqs. 4 coincides with the usual notion of degeneracy and means that $A$ does not have the full rank. It shows that $X^\vee$ is nothing but $S_k$. In particular, $X^\vee$, defined by this condition, is of codimension 1 and is given by the ordinary determinant $\det A = 0$, or and only if $A$ is a square ($k_1 = k_2 = k$) matrix. In the $n$-dimensional case, if $X^\vee$ is a hypersurface (of codimension 1), it is given by the zero locus of a unique (up to sign) irreducible homogeneous polynomial over $\mathbb{Z}$ of $a_{i_1,...,i_n}$. This polynomial is the hyperdeterminant introduced by Cayley and is denoted by $\det A$. As usual, if $X^\vee$ is not a hypersurface, we set $\det A$ to be 1.

Remark that, in the bipartite case, we classify the states $S_{k+1} - S_k = M - X^\vee$ as the generic entangled states, the states $S_k - S_{k-1} = X^\vee - X^\vee_{\text{sing}}$ as the next generic entangled states, and so on. Likewise, we aim to classify the multipartite entangled states into the onion structure by the dual variety $X^\vee (\det A = 0)$, its singular locus $X^\vee_{\text{sing}}$, and so on (i.e., by every closed subvariety), instead of the tensor rank.

III. HYPERDETERMINANT AND ITS SINGULARITIES

In order to classify multipartite entanglement into the SLOCC-invariant onion structure, we explore the dual variety $X^\vee$ (zero hyperdeterminant) and its singular locus in this section.

A. Hyperdeterminant

We utilize the hyperdeterminant, the generalized determinant for higher dimensional matrices by Gelfand et al. $17\ 16$. Its absolute value is also known as an entanglement measure, the concurrence $C$ $17$ or 3-tangle $\tau$ $18$ for the 2 or 3-qubit pure case, respectively.

$$C = 2|\det A_2| = 2|\det A| = 2|a_{00}a_{11} - a_{01}a_{10}|,$$

$$\tau = 4|\det A_3| =$$

$$= 4(a^2_{000}a^2_{111} + a^2_{001}a^2_{101} + a^2_{010}a^2_{101} + a^2_{011}a^2_{101} - 2(a_{000}a_{001}a_{111} + a_{000}a_{010}a_{101} + a_{001}a_{010}a_{101} + a_{011}a_{010}a_{101}) + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{001}a_{101}a_{110} + a_{010}a_{001}a_{101}a_{110} + a_{010}a_{100}a_{101}a_{110})).$$

The following useful facts are found in $16$. Without loss of generality, we assume that $k_1 \geq k_2 \geq \cdots \geq k_n \geq 1$. The $n$-dimensional hyperdeterminant $\det A$ of format $(k_1+1) \times \cdots \times (k_n+1)$ exists, i.e., $X^\vee$ is a hypersurface, if and only if a ”polygon inequality” $k_1 \leq k_2 + \cdots + k_n$ is satisfied. For $n = 2$, this condition is reduced to $k_1 = k_2$ as desired, and $\det A$ coincides with det $A$. The matrix format is called boundary if $k_1 = k_2 + \cdots + k_n$ and interior if $k_1 < k_2 + \cdots + k_n$. Note that (i) The boundary format includes the ”bipartite cut” between the 1st party and the others so that it is mathematically tractable. (ii) The interior format includes the $n \geq 3$-qubit case. We treat hereafter the format where the polygon inequality holds and $X^\vee$ is the largest closed subvariety, defined by the hypersurface $\det A = 0$.

$\det A$ is relatively invariant (invariant up to constant) under the action of $GL_{k_1+1}(\mathbb{C}) \times \cdots \times GL_{k_n+1}(\mathbb{C})$. In particular, interchanging two parallel slices (submatrices with some fixed directions) leaves $\det A$ invariant up to sign, and $\det A$ is a homogeneous polynomial in the entries of each slice. Since it is ensured that $X^\vee$, $X^\vee_{\text{sing}}$, and further singularities are invariant under SLOCC, our classification is equivalent to or coarser than the SLOCC classification. Later, we see that the former and the lat-
ter correspond to the case where SLOCC gives finitely and infinitely many classes, respectively.

B. Schlöfli’s construction

It would not be easy to calculate $\text{Det}A$ directly by its definition that Eqs. 7 have at least one solution. Still, the Schlöfli’s method enables us to construct $\text{Det}A_n$ of format $2^n$ (n qubits) by induction on $n$.

For $n = 2$, by definition $\text{Det}A_2 = \text{det} A = a_{00}a_{11} - a_{01}a_{10}$. Suppose $\text{Det}A_n$, whose degree of homogeneity is $l$, is given. Associating an $n+1$-dimensional matrix $a_{i_0,i_1,\ldots,i_n}$ ($i_j = 0, 1$) to a family of $n$-dimensional matrices $A(x) = \sum_i a_{i_0,i_1,\ldots,i_n}x_{i_0} \cdots x_{i_n}$ linearly depending on the auxiliary variable $x_{i_0}$, we have $\text{Det}A(x)_n$. Due to Theorem 4.1 and 4.2 of [10], the discriminant $\Delta$ of $\text{Det}A(x)_n$ gives $\text{Det}A_{n+1}$ with an extra factor $R_n$. The Sylvester formula of the discriminant $\Delta$ for binary forms enables us to write $\text{Det}A_{n+1}$ in terms of the determinant of order $2l-1$:

$$\text{Det}A_{n+1} = \frac{\Delta(\text{Det}A(x)_n)}{R_n}$$

\[ \frac{1}{R_n c_l} \left| \begin{array}{cccc} c_0 & c_1 & \cdots & c_l-2 & c_l-1 & c_l & \cdots & 0 \\ 0 & c_0 & \cdots & c_l-2 & c_l-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 & \cdots & c_l \\ 0 & 0 & \cdots & 0 & 1c_1 & 2c_2 & \cdots & lc_l \\ \end{array} \right|, \quad (11) \]

where each $c_j$ is the coefficient of $x_0^{l-j}x_1^j$ in $\text{Det}A(x)_n$, i.e., $c_j = \frac{1}{(l-j)!j!} \frac{\partial^l}{\partial x_0^j \partial x_1^l} \text{Det}A(x)_n$.

Note that because for $n = 2$ or $3$, the extra factor $R_n$ is just a nonzero constant, $\text{Det}A_4$ for the 3 or 4 qubits is readily calculated, respectively. It would be instructive to check that $\text{Det}A_3$ in Eq. (10) is obtained in this way. On the other hand, for $n \geq 4$, $R_n$ is the Chow form (related resultant) of irreducible components of the singular locus $X_{\text{sing}}$. These are due to the fact that $X_{\text{sing}}$ has codimension 2 in $M$ for any format of the dimension $n \geq 3$ except for the format $2^3$ (3-qubit case), which was conjectured in [15] and proved in [24]. So we have to explore $X_{\text{sing}}$ not only to classify entangled states in the n qubits, but to calculate $\text{Det}A_{n+1}$ inductively. Although $\text{Det}A_{n\geq 5}$ has yet to be written explicitly, only its degree $l$ of homogeneity is known (in Corollary 2.10 of [14]) to grow very fast as $2, 4, 24, 128, 880, 6816, 60032, 589312, 6384384$ for $n = 2, 3, \ldots, 10$.

C. Singularities of the hyperdeterminant

We describe the singular locus of the dual variety $X^\vee$. The technical details are given in [20]. It is known that, for the boundary format, the next largest closed subvariety $X_{\text{sing}}^\vee$ is always an irreducible hypersurface in $X^\vee$; in contrast, for the interior one, $X_{\text{sing}}^\vee$ has generally two closed irreducible components of codimension 1 in $X^\vee$, node-type $X_{\text{node}}^\vee$ and cusp-type $X_{\text{cusp}}^\vee$ singularities. The rest of this subsection can be skipped for the first reading. It is also illustrated for the 3-qubit case in Appendix A.

First, $X_{\text{node}}^\vee$ is the closure of the set of hyperplanes tangent to the Segre variety $X$ at more than one point (cf. Fig. 1). $X_{\text{node}}^\vee$ can be composed of closed irreducible subvarieties $X_{\text{node}}^\vee(J)$ labeled by the subset $J \subset \{1, \ldots, n\}$, including $\emptyset$. Indicating that two solutions $x = (x^{(1)}, \ldots, x^{(j)}, \ldots, x^{(n)})$ of Eq. (8) coincide for $j \in J$, the label $J$ distinguishes the pattern in these solutions. In order to rewrite $X_{\text{node}}^\vee(J)$, let us pick up a point $x^o(J)$ such that its homogeneous coordinates $x^{(j)}_i = \delta_{i,j,0}$ for $j \in J$ and $\delta_{i,j,k}$ for $j \notin J$. It is convenient to label the positions of 1 in each $x^{(j)}$ by a multi-index $[i_1, \ldots, i_n]$. For example, $x^o(1)$ is labeled by $[0, k_2, \ldots, k_n]$ and $x^o(1, \ldots, n)$ is just written by $x^o$. According to Eqs. 7, $X_{\text{node}}^\vee(J)$, tangent to $X$ at $x^o(J)$, consists of the matrices $A$ of all $a_{i_1,\ldots,i_n} = 0$ such that $[i_1, \ldots, i_n]$ differs from $[i_1, \ldots, i_n]$ of $x^o(J)$ in at most one index. Then we can define $X_{\text{node}}^\vee(J)$ as

$$X_{\text{node}}^\vee(J) = (X^\vee|_{x^o} \cap X^\vee|x^o(J)) \cdot G, \quad (12)$$

where $G = GL_{k_1+1} \times \cdots \times GL_{k_n+1}$ acts on $M$ from the right and the bar stands for the closure.

Second, $X_{\text{cusp}}^\vee$ is the set of hyperplanes having a critical point which is not a simple quadratic singularity (cf. Fig. 1). Precisely, the quadric part of $F(A, x)$ at $x^o$ is a matrix $y_{(j,i), (j', i', j')} = (\partial^2/\partial x_j^{(j')} \partial x_i^{(i')}) F(A, x^o)$, where the pairs $(j, i), (j', i', j') \ (1 \leq i \leq k_j, 1 \leq i' \leq k_{j'})$ are the row and the column index, respectively. Denoting by $X_{\text{cusp}}^\vee|x^o$ the variety of the Hessian $\text{det} y = 0$ in $X^\vee|x^o$, we

![FIG. 1: Two types of singularities of $X^\vee$. $X_{\text{node}}^\vee$ corresponds to the bitangent of $X$, where both tangencies are of the first order. $X_{\text{cusp}}^\vee$ corresponds to the tangent at an inflection point of $X$, where its tangency is of the second order.](image-url)
can define $X^\vee_{\text{cusp}}$ as
\[
X^\vee_{\text{cusp}} = X^\vee_{\text{cusp}}|_{x^\circ \cdot G}.
\] (13)
This $X^\vee_{\text{cusp}}$ is already closed without taking the closure.

IV. CLASSIFICATION OF MULTIPARTITE ENTANGLEMENT

According to Sec. III and III we illustrate the classification of multipartite pure entangled states for typical cases.

A. 3-qubit (format 3\(^2\)) case

The classification of the 3 qubits under SLOCC has been already done in [2] [4]. Surprisingly, Gelfand et al. considered the same mathematical problem by DetA3 in Example 4.5 of [16]. Our idea is inspired by this example. We complement the Gelfand et al.’s result, analyzing additionally the singularities of $X^\vee$ in detail. The dimensions, representatives, names, and varieties of the orbits are summarized as follows. The basis vector $|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle$ is abbreviated to $|i_1i_2i_3\rangle$.

dim 7: $|000\rangle + |111\rangle$, GHZ $\in M(=\mathbb{CP}^7) - X^\vee$.

dim 6: $|001\rangle + |010\rangle + |100\rangle$, W $\in X^\vee - X^\vee_{\text{sing}} = X^\vee - X^\vee_{\text{cusp}}$.

dim 4: $|001\rangle + |010\rangle$, $|001\rangle + |100\rangle$, $|010\rangle + |100\rangle$, biseparable $B_j \in X^\vee_{\text{node}}(j) - X$ for $j = 1, 2, 3$.

$X^\vee_{\text{node}}(j) = \mathbb{CP}^1 \cdot \mathbb{CP}^1 \cdot \mathbb{CP}^1$ are three closed irreducible components of $X^\vee_{\text{sing}} = X^\vee_{\text{cusp}}$.

dim 3: $|000\rangle$, completely separable $S$.

$x = \bigcap_{j=1,2,3} X^\vee_{\text{node}}(j) = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

$G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2$ has the onisphere structure of six orbits on $M$ (see Fig. 2), by excluding the orbit $\emptyset (= X^\vee_{\text{node}}(\emptyset))$. The dual variety $X^\vee$ is given by DetA3 = 0 (cf. Eq. (11)). Its dimension is $7 - 1 = 6$. The outside of $X^\vee$ is generic tripartite entangled class of the maximal dimension, whose representative is GHZ. This suggests that almost any state in the 3 qubits can be locally transformed into GHZ with a finite probability, and vice versa. Next, we can identify $X^\vee_{\text{sing}}$ as $X^\vee_{\text{cusp}}$, which is the union of three closed irreducible subvarieties $X^\vee_{\text{node}}(j)$ for $j = 1, 2, 3$ (also see Appendix A). For example, $X^\vee_{\text{node}}(1)$ means by definition that, in addition to the condition of $X^\vee$ in Sec. III there exists some nonzero $x^{(1)}$ such that $F(A, x) = 0$ for any $x^{(2)}, x^{(3)}$; i.e., a set of linear equations $y_{i_2i_3}(x^{(1)}) = (\partial^2/\partial x_{i_2} \partial x_{i_3}) F(A, x) = 0$ for $i_3 = 0, 1$ has a nontrivial solution $x^{(1)}$. This indicates that the ”bipartite” matrix
\[
\begin{pmatrix}
 a_{000} & a_{001} & a_{010} & a_{011} \\
 a_{100} & a_{101} & a_{110} & a_{111}
\end{pmatrix}
\] (14)
ever has the full rank (i.e., six 2x2 minors in Eq. (14) are zero). We can identify $X^\vee_{\text{node}}(1)$ as the set $\mathbb{CP}^1 \times \mathbb{CP}^1$.

seen in Sec. II A of biseparable states between the 1st party and the rest of the parties. Its dimension is $1 + 3 = 4$. Likewise, $X^\vee_{\text{node}}(j)$ for $j = 2, 3$ gives the biseparable class for the 2nd or 3rd party, respectively. So, the class of $X^\vee - X^\vee_{\text{sing}}$ is found to be tripartite entangled states, whose representative is W. We can intuitively see that, among genuine tripartite entangled states, W is rare, compared to GHZ [2]. Finally, the intersection of $X^\vee_{\text{node}}(j)$ is the completely separable class $S$, given by the Segre variety $X$ of dimension 3. Another intuitive explanation about this procedure is seen in Appendix A.

Now we clarify the relationship of six classes by non-invertible local operations. Because noninvertible local operations cause the decrease in local ranks [21], the partially ordered structure of entangled states in the 3 qubits, included in Fig. 4, appears. Two inequivalent tripartite entangled classes, GHZ and W, have the same local ranks (2, 2, 2) for each party so that they are not interconvertible by the noninvertible local operations (i.e., general LOCC). Two classes hold different physical properties [5]: the GHZ representative state has the maximal amount of generic tripartite entanglement measured by the 3-tangle $\tau\alpha[\text{DetA3}], while the W representative state has the maximal amount of (average) 2-partite entanglement distributed over 3 parties (also [22]). Under LOCC, a state in these two classes can be transformed into any state in one of the three biseparable classes $B_j$ ($j = 1, 2, 3$), where the $j$-th local rank is 1 and the others are 2. Three classes $B_j$ never convert into each other. Likewise, a state in $B_j$ can be locally transformed into any state in the completely separable class $S$ of local ranks (1, 1, 1).

This is how the onion-like classification of SLOCC orbits reveals that multipartite entangled classes constitute the partially ordered structure. It indicates significant
differences from the totally ordered one in the bipartite case. (i) In the 3-qubit case, all SLOCC invariants we need to classify is the hyperdeterminant \(\text{Det}A_3\) in addition to local ranks. (ii) Although noninvertible local operations generally mean the transformation further inside the onion structure, an outer class can not necessarily be transformed into the *neighboring* inner class. A good example is given by GHZ and W, as we have just seen.

**B. Format 3 × 2 × 2 case**

Before proceeding to the \(n \geq 4\)-qubit case, we drop in the format 3 × 2 × 2, which would give an insight into the structure of multipartite entangled states when each party has a system consisting of more than two levels. This case is interesting since on the one hand (contrary to the 3-qubit case), it is typical that GHZ and W are included in \(X_{\text{sing}}\); on the other hand (similarly to the bipartite or 3-qubit cases), SLOCC has still finite classes so that it becomes another good test for the equivalence to the SLOCC classification. Besides, it is a boundary format so that several subvarieties can be explicitly calculated, and enables us to analyze entanglement in the qubits system using an auxiliary level, like ion traps.

![Diagram](image)

**FIG. 3:** The onion-like classification of SLOCC orbits in the 3×2×2 format. Although this resembles Fig. 2 in the order of SLOCC orbits (two orbits are added outside), it is worth while to note that singularities of \(X^v\), which classify the SLOCC orbits, have a different order.

\[
\begin{align*}
\text{GHZ} & : |001> + |100> + |110> + |211> \quad (3,2,2) \\
\text{W} & : |010> + |101> + |111> \quad (2,2,2) \\
\text{GHZW} & : |000> + |101> + |110> + |211> \quad (3,2,2) \\
\end{align*}
\]

**FIG. 4:** The partially ordered structure of multipartite pure entangled states in the 3 × 2 × 2 format, including the 3-qubit case. Each class, corresponding to the SLOCC orbit, is labeled by the representative, local ranks, and the name. Non-invertible local operations, indicated by dashed arrows, degrade "higher" entangled classes into "lower" entangled ones. i.e., all four 3 × 3 minors \(m_j\) in Eq. (16) are zero. The SLOCC orbits which appear inside \(X_{\text{sing}}\) are essentially the same as the 3-qubit case.

Thus we obtain the partially ordered structure of multipartite entangled states as Fig. 3. The tripartite entanglement consists of four classes. Because the class of \(M - X^v\), whose representative is \(|000> + |101> + |110> + |211>\), and that of \(X^v - X_{\text{sing}}\), whose representative is \(|000> + |101> + |211>\), have the same local ranks (3, 2, 2), they do not convert each other in the same reason as GHZ and W do not. However, the former two classes of the local ranks (3, 2, 2) can convert to the latter two classes of (2, 2, 2) by noninvertible local operations (i.e., LOCC). And we

\[
\text{det}A = m_1m_4 - m_2m_3
\]

of degree 6, where \(m_j\) \((j = 1, 2, 3, 4)\) is the \(3 \times 3\) minor of

\[
\begin{pmatrix}
 a_{000} & a_{001} & a_{010} & a_{011} \\
 a_{100} & a_{101} & a_{110} & a_{111} \\
 a_{200} & a_{201} & a_{210} & a_{211}
\end{pmatrix}
\]

without the \(j\)-th column, respectively. Next, it is characteristic that \(X^v_{\text{sing}}\) is \(X_{\text{node}}(1)\) \[21\]. Similarly to the 3-qubit case in Sec. IV A, \(X_{\text{node}}(1)\) means that the "bipartite" matrix in Eq. (16) does not have the full rank,
can "degrade" these tripartite entangled classes into the
biseparable or completely separable classes by LOCC in
a similar fashion to the 3 qubits.

We notice that 3 grades in the 3-qubit case changed to
4 grades in the 3×2×2 (1-qutrit and 2-qubit) case. In
general, the partially ordered structure becomes "higher", as
the system of each party becomes the higher dimensional
one. We also see how the tensor rank 11 is inadequate for
the onion-like classification of SLOCC orbits.

C. $n \geq 4$-qubit (format 2) case

Further in the $n \geq 4$-qubit case, our classification
works. The outermost class $M(=CP^{n-1})-X^v$ of generic
$n$-partite entangled states is given by $DetA_n \neq 0$. In
$n=4$, $DetA_4$ of degree 24 is explicitly calculated by the
Schläfli’s construction in Sec. IIIB. It would be sugges-
tive local operations, the "representative" of the outermost class by invert-
to the "representative" of the outermost class by invert-
ary to the "representative" of the outermost class by invert-
ary local operations,

$$
\alpha(|0000\rangle + |1111\rangle) + \beta(|0011\rangle + |1100\rangle) \\
+ \gamma(|0101\rangle + |1010\rangle) + \delta(|0110\rangle + |1001\rangle),
$$

(17)

where the continuous complex coefficients $\alpha, \beta, \gamma, \delta$
should satisfy

$$
DetA_4 = \alpha^2 \beta^2 \gamma^2 \delta^2 (\alpha + \beta + \gamma + \delta)^2(\alpha + \beta + \gamma - \delta)^2 \\
(\alpha + \beta - \gamma + \delta)^2(\alpha - \beta + \gamma + \delta)^2(-\alpha + \beta + \gamma + \delta)^2 \\
(\alpha + \beta - \gamma - \delta)^2(\alpha - \beta + \gamma - \delta)^2(\alpha - \beta - \gamma + \delta)^2 \neq 0.
$$

(18)

Thus three complex parameters remain in the outer-
most class (since we consider rays rather than normalized
state vectors). This means that there are infinitely many
same dimensional SLOCC orbits in the 4 qubits, and the
SLOCC orbits never locally convert to each other when
their sets of the parameters are distinct. It is also the
case for the $n>4$ qubits. Note that, in $n=4$, this outer-
most class $M-X^v$ corresponds to the family of generic
states in Verstraete et al.’s classification of the 4 qubits
by a different approach (generalizing the singular value
decomposition in matrix analysis to complex orthogonal
equivalence classes), and $X^v$ contains their other special
families.

The next outermost class is $X^v-X^v_{sing}$. In the 4 qubits,$X^v_{sing}$ is shown to consist of eight closed irreducible
components of codimension 1 in $X^v$, $X^v_{cusp}$, $X^v_{node(\emptyset)}$, and six
$X^v_{node(j_1,j_2)}$ for $1 \leq j_1 < j_2 \leq 4$. They neither contain
nor are contained by each other. Their intersections also
give (finitely) many lower dimensional genuine 4-partite
tangled classes. Since the 4-partite entangled classes
necessarily have the same local ranks (2, 2, 2, 2), these
classes are not interconvertible by noninvertible local
operations (i.e., any LOCC). As typical examples, GHZ
(i.e., $a_{0000} = a_{1111} \neq 0$ and the others are 0) is included
in the intersection of $X^v_{node(\emptyset)}$ and six $X^v_{node(j_1,j_2)}$, but
is excluded from $X^v_{cusp}$. In contrast, $W$,

$$
|W\rangle = |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle,
$$

(20)

(i.e., $a_{0001} = a_{0010} = a_{0100} = a_{1000} \neq 0$ and the others
are 0) is included in the intersection of $X^v_{cusp}$ and six
$X^v_{node(j_1,j_2)}$ but is excluded from $X^v_{node(\emptyset)}$.

In the $n>4$ qubits, $X^v_{sing}$ is shown to consist of just
two closed irreducible components $X^v_{cusp}$ and $X^v_{node(\emptyset)}$
$[20]$. We find that GHZ and $W$ are contained not only in
$X^v$ ($DetA_n = 0$) but in $X^v_{sing}$; i.e., they have nontrivial
solutions in Eqs. (17), satisfying the singular conditions.
They correspond to different intersections of further sing-
ularities, similarly to the 4 qubits. In other words, they
are peculiar, living in the border dimensions between en-
tangled states and separable one.

In brief, the dual variety $X^v$ and its singularities lead to
the coarse onion-like classification of SLOCC orbits,
when SLOCC gives infinitely many orbits. The partially
ordered structure of multipartite pure entangled states
becomes "wider", as the number $n$ of parties increases.
Although many inequivalent $n$-partite entangled classes
appear in the $n$ qubits, they never locally convert to each
other, as observed in [3]. In particular, the majority of
the $n$-partite entangled states never convert to GHZ (or
$W$) by LOCC, and the opposite conversion is also not possible.
This is a significant difference from the bipartite
or 3-qubit case, where almost any entangled state and the
maximally entangled state (GHZ) can convert to each
other by LOCC with nonvanishing probabilities.

V. CONCLUSION

We have presented the onion-like classification of mul-
tipartite entanglement (SLOCC orbits) by the dual vari-
ety $X^v$, i.e., the hyperdeterminant $DetA$. It leads to the
partially ordered structure, such as Fig. 4 of inequal-
ent multipartite entangled classes of pure states, which
is significantly different from the totally ordered one in the
bipartite case. Local ranks are not enough to distinguish
these classes, and we need to calculate SLOCC invariants
associated with $DetA$. In other words, the generic entan-
gled class of the maximal dimension (the outermost class)
is given by the outside of $X^v$ ($DetA \neq 0$), and other mul-
tipartite entangled classes appear as $X^v$ or its different
singularities. Analytically, the classification of multipar-
tite entanglement corresponds to that of the number and
pattern of the solutions in Eqs. (17).

This work reveals that the situation of the widely
known bipartite or 3-qubit cases, where the maximally
entangled states in Bell’s inequalities belong to the
generic class, is exceptional. Lying far inside the onion
structure, the maximally entangled states (GHZ) are in-
cluded in the lower dimensional peculiar class in general,
e.g., for the $n \geq 4$ qubits. It suggests two points. The
The onion-like classification seems to be reasonable in the sense that it coincides with the SLOCC classification when SLOCC gives finitely many orbits, such as the bipartite or 3-qubit cases. So two states belonging to the same class can convert each other by invertible local operations with nonzero probabilities. On the other hand, when SLOCC gives infinitely many orbits, this classification is still SLOCC-invariant, but may contain in one class infinitely many same dimensional SLOCC orbits which can not locally convert to each other even probabilistically. For example, in the 4-qubit case, the generic entangled class in Eq. (14) has three nonlinear continuous parameters. Note that it can be possible to make the onion-like classification finer, by characterizing the nonlinear continuous parameters in each class.

Then, we may ask, what is the physical interpretation of the onion-like classification in the case of infinitely many SLOCC orbits? Although a simple answer has yet to be found, we discuss two points. (i) Let us consider global unitary operations which create the multipartite entanglement. On the one hand, states in distinct classes would have the different complexity of the global operations, since they have the distinct number and pattern of nonlocal parameters. On the other hand, states in one class are supposed to have the equivalent complexity, since they just correspond to different "angles" of the global unitary operations. (ii) We can consider the case where more than one state are shared, including the asymptotic case. Even in two shared states, there can exist a local conversion which is impossible if they are operated separately, such as the catalysis effect [26]. So we can expect that we do it more efficiently in this situation, and the coarse classification may have some physical significance. This problem remains unsettled even in the bipartite case.

Finally, two related topics are discussed. (i) The absolute value $|\text{Det}A_n|$ of the hyperdeterminant, representing the amount of generic entanglement, is an entanglement monotone by Vidal [27]. This never conflicts with the property that the maximally entangled states in Bell’s inequalities (GHZ) generally have a zero Det$A_n$. A single entanglement monotone is insufficient to judge the LOCC convertibility, and generic entangled states of the nonzero Det$A_n$ can not convert to GHZ in spite of decreasing $|\text{Det}A_n|$. (ii) The 3-tangle $\tau = 4|\text{Det}A_3|$ first appeared in the context of so-called entanglement sharing [18]; i.e., in the 3 qubits, there is a constraint (trade-off) between the amount of 2-partite entanglement and that of 3-partite entanglement. By using the entanglement measure (concurrence $C$) for the 2-qubit mixed entangled states, this is written as $C^2_{123} \geq C^2_{12} + C^2_{13}$, and $\tau$ is defined by $\tau = C^2_{123} - C^2_{12} - C^2_{13}$ for the 3-qubit pure entangled states. We expect that, in turn, the hyperdeterminant Det$A_n$ gives a clue to find the entanglement measure of more than 2-qubit mixed states.

Acknowledgments

The author would like to thank M. Wadati, M. Murao, G. Kato, the members in the ERATO Project of Quantum Computation and Information, and the participants in the Sixth Quantum Information Technology Symposium in Japan on May (2002) for the most helpful discussions. The work is partially supported by the ERATO Project.

APPENDIX A: REPRESENTATIVES OF THE 3-QUBIT ENTANGLED CLASSES

In Sec. 16 we have classified entangled classes (SLOCC orbits), utilizing SLOCC-invariant closed sub-varieties such as the dual variety $X^\vee$ and its singularities. In this appendix, we give an intuitive explanation about our technique. We obtain entangled classes by their representatives. The 3-qubit case is exemplified, and the notation and terminology of Sec. 11C is followed.

As the representative of the outermost generic entangled class, almost any state (indeed, satisfying Det$A_3 \neq 0$) in the whole space $M = CP^7$ is qualified. The GHZ state $|000\rangle + |111\rangle$ is chosen among them, since it can be seen as the multidimensional analog of the identity matrix.

We look for the representative of the dual variety $X^\vee$, which is qualified as that of the next outermost entangled class. When $X^\vee$ is the hyperplane tangent to the Segre variety $X$ at $x^o$ such that $x^o_{ij} = \delta_{ij,0}$ $(j = 1, 2, 3)$, the "$x^o$-section" of $X^\vee$ is given as

$$X^\vee|_{x^o} = \{ a_{000} = a_{001} = a_{010} = a_{100} = 0 \}, \quad (A1)$$

in order that Eqs. (7) have the nontrivial solution $x^o$. This suggests that the representative of $X^\vee$ is the W state $|001\rangle + |010\rangle + |100\rangle$, since the states given by Eq. (A1) and W convert to each other under some invertible local
operations \( G \in GL_2 \times GL_2 \times GL_2 \) as
\[
a_{011} |011\rangle + a_{101} |101\rangle + a_{110} |110\rangle + a_{111} |111\rangle
\]
\[
\succeq |011\rangle + |101\rangle + |110\rangle + |111\rangle
\]
\[
\succeq |001\rangle + |010\rangle + |100\rangle.
\]
(A2)

The candidates for the next outer entangled class are two, node-type \( X^\vee_{\text{node}} \) and cusp-type \( X^\vee_{\text{cusp}} \), singularity sets of \( X^\vee \). We first consider the representative of \( X^\vee_{\text{node}}(1) \). According to Eq. (12), the \( x^o \)-section of \( X^\vee_{\text{node}}(1) \) is given as
\[
X^\vee_{\text{node}}(1)|_{x^o} = X^\vee|_{x^o} \cap X^\vee|_{x^o(1)} = \{ a_{000} = a_{001} = a_{010} = a_{100} = a_{111} = 0 \}.
\]
(A3)

We find that the representative of \( X^\vee_{\text{node}}(1) \) is the biseparable state \( |001\rangle + |010\rangle \) in \( B_1 \), checking that
\[
a_{101} |101\rangle + a_{110} |110\rangle \succeq |001\rangle + |010\rangle.
\]
(A4)

In the same manner, \( X^\vee_{\text{node}}(2) \) or \( X^\vee_{\text{node}}(3) \) represents the biseparable class \( B_2 \) or \( B_3 \), respectively.

Let us second analyze the representative of \( X^\vee_{\text{cusp}} \). In terms of the quadratic part \( y \) of \( F(A, x) \) at \( x^o \):
\[
y = \begin{pmatrix}
0 & a_{110} & a_{101} \\
 a_{110} & 0 & a_{011} \\
 a_{101} & a_{011} & 0
\end{pmatrix},
\]
(A5)

the \( x^o \)-section of \( X^\vee_{\text{cusp}} \) is given as
\[
X^\vee_{\text{cusp}}|_{x^o} = \{ a_{000} = a_{001} = a_{010} = a_{100} = 0, \}
\]
\[
det y = 2a_{011}a_{101}a_{110} = 0.
\]
(A6)

We have three possibilities for \( \det y = 0 \). In the case of \( a_{011} = 0 \), this component of \( X^\vee_{\text{cusp}} \) represents the biseparable class \( B_1 \), since
\[
a_{101} |101\rangle + a_{110} |110\rangle + a_{111} |111\rangle \succeq |001\rangle + |010\rangle.
\]
(A7)

Likewise, in the case of \( a_{010} = 0 \) or \( a_{110} = 0 \), each component of \( X^\vee_{\text{cusp}} \) corresponds to the biseparable class \( B_2 \) or \( B_3 \), respectively. Remembering that each \( B_j \) is characterized by \( X^\vee_{\text{node}}(j) \) for \( j = 1, 2, 3 \), we have shown that \( X^\vee_{\text{cusp}} \) has three irreducible components \( X^\vee_{\text{node}}(j) \). Thus, the next outer entangled classes are three biseparable classes \( B_j \), which never contain nor are contained by each other.

In general, remaining entangled classes are given by further singularities of \( X^\vee \) such as combinations of the above \( X^\vee_{\text{node}} \) and \( X^\vee_{\text{cusp}} \), or genuine higher singularities. In the 3-qubit case, since we see that \( X^\vee_{\text{node}}(j) \), representing the biseparable class \( B_j \), is just characterized as \( CP^1_{j=1} \times CP^3 \), there remains just one smaller closed irreducible subvariety \( CP^1 \times CP^1 \times CP^1 = X \) as their intersection \( \bigcap_{j=1,2,3} X^\vee_{\text{node}}(j) \). This Segre variety \( X \) represents the completely separable class \( S \), whose representative is \( |000\rangle \).

In the text, we have carried out the above procedure in the “\( x^o\)-free” manner (\( x^o \) should be taken as any state on \( X \)), and have obtained entangled classes as (difference) subsets. It enables us to decide readily which entangled class a given state \( |\Psi\rangle \) belongs to. After the classification of entangled classes, we can clarify their partially ordered structure under noninvertible local operations in the same manner as in the text.

APPENDIX B: CLASSIFICATION OF MULTIPARTITE MIXED STATES

The onion structure is also useful for the SLOCC-invariant classification of mixed entangled states. A mixed state \( \rho \) can be written as a convex combination of projectors onto pure states (extremal points),
\[
\rho = \sum_\mu p_\mu |\Psi_\mu(O_\lambda)\rangle\langle\Psi_\mu(O_\lambda)|, \quad p_\mu > 0,
\]
(B1)

where \( |\Psi_\mu(O_\lambda)\rangle \) is the pure state belonging to the SLOCC orbit \( O_\lambda \) of an index \( \lambda \). \( \lambda \) is labeled by the closed subvariety \( \overline{O_\lambda} \) (i.e., the closure of \( O_\lambda \) such as \( X^\vee, X^\vee_{\text{node}}(J), X^\vee_{\text{cusp}}, \) and \( X \)). Note that, in the multipartite case, there can be many closed subvarieties \( \overline{O_\lambda} \) which never contain nor are contained by each other; for example, \( X^\vee_{\text{node}}(1), X^\vee_{\text{node}}(2), \) and \( X^\vee_{\text{node}}(3) \) in Fig. 2. So, by taking the union of these “competitive” closed subvarieties \( \overline{O_\lambda} \) (it will form their convex hull in the space of \( \rho \)), we pick up only totally ordered ones, e.g., \( M, X^\vee, X^\vee_{\text{cusp}} = \bigcup_{j=1,2,3} X^\vee_{\text{node}}(j), \) and \( X \) in Fig. 2 for convenience later. Now, we are concerned with at most how various classes of pure entangled states the mixed state \( \rho \) consists of. We take the maximal closure of \( \overline{O_\lambda} \) appeared in Eq. (11) and denote it by \( \overline{O_{\max}} \). However, since \( \rho \) can be decomposed into the form of Eq. (11) in infinitely many ways, we should take the minimal closure of \( \overline{O_{\max}} \) over all possible decompositions, and write it as \( \min \overline{O_{\max}} \). Every convex subset \( S_\lambda \) of \( \lambda = \min \overline{O_{\max}} \) is closed such that \( S_\lambda \) of the smaller \( \lambda \) is contained by that of the larger one. In other words, \( S_\lambda \) of the larger \( \lambda \) consists of more classes (SLOCC orbits) of pure entangled states. That is how the mixed state \( \rho \) is classified into the closed convex subsets \( S_\lambda \) under SLOCC.

In the bipartite case, \( \min \overline{O_{\max}} \) is called the Schmidt number \( 2^k \) (since \( \overline{O_\lambda} \) is just labeled by the Schmidt local rank, as seen in Sec. 10). Also in the 3-qubit case, this kind of the classification has been done in [11], and four classes appear, following the above recipe:

(i) GHZ class \( S_M - S_X^\vee \) (consisting of all pure states);
(ii) W class \( S_X^\vee - S_{X_{\text{cusp}}} \) (consisting of the pure W, biseparable, or separable states);
(iii) biseparable class \( S_{X_{\text{cusp}}} - S_X \) (consisting of the pure biseparable or separable states);
(iv) separable class \( S_X \) (consisting of only the pure separable states).
I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Adv. in Math. 96, 226 (1992).

[1] A. Peres, Quantum Theory: Concepts and Methods (Kluwer Academic Publishers, 1995).
[2] C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996); M. Murao et al., *ibid.* 57, R4075 (1998).
[3] M.A. Nielsen, Phys. Rev. Lett. 83, 436 (1999); G. Vidal, *ibid.* 83, 1046 (1999).
[4] C.H. Bennett et al., Phys. Rev. A 63, 012307 (2000).
[5] W. Dürr, G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000).
[6] SLOCC was tied to $SL \times SL$ in $\mathbb{R}$. Since we treat $|\Psi\rangle$ as a ray, i.e., an unnormalized vector, we rather relate SLOCC to $GL \times GL$.
[7] The local rank can be also defined as the rank of the reduced density matrix traced out for all except one party. Note that this definition is applicable to the multipartite case.
[8] H.K. Lo and S. Popescu, Phys. Rev. A 63, 022301 (2001).
[9] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 363, 012307 (2000).
[10] The tensor rank means the number of terms in the minimal decomposition of $|\Psi\rangle$ by separable states. It coincides with the local rank in the bipartite case.
[11] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001).
[12] J.-L. Brylinski, quant-ph/0008031.
[13] A. Miyake and M. Wadati, Phys. Rev. A 64, 042317 (2001); D.C. Brody and L.P. Hughston, J. Geom. Phys. 38, 19 (2001); I. Bengtsson, J. Bräunlund, and K. Życzkowski, quant-ph/0108064.
[14] For example, let us consider two Einstein-Podolsky-Rosen (EPR) pairs $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{34}$ in the 4 qubits. Since their local ranks are (2,2,2,2), we can not distinguish this state from genuine 4-partite entangled states (cf. Sec. [V.C]). In contrast, we readily find that the state is included in $CP^3 \times CP^3$, so that it is separable (not genuine 4-partite entangled).
[15] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Adv. in Math. 96, 226 (1992).