SINGULARITIES OF CERTAIN FINITE ENERGY SOLUTIONS TO THE NAVIER-STOKES SYSTEM

Grzegorz Karch
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Gruwaldzki 2/4, Wrocław, Poland

Maria E. Schonbek
University of California, Department of Mathematics
Santa Cruz, CA 95064, USA

Tomas P. Schonbek*
Florida Atlantic University, Department of Mathematical Sciences
Boca Raton, FL 33431, USA

(Communicated by José A. Carrillo)

ABSTRACT. We continue and supplement studies from [G. Karch and X. Zheng, Discrete Contin. Dyn. Syst. 35 (2015), 3039–3057] on solutions to the three dimensional incompressible Navier-Stokes system which are regular outside a curve in \((\gamma(t), t) \in \mathbb{R}^3 \times [0, \infty)\) and singular on it. We revisit some of the existence results as well as some of the asymptotic estimates obtained in that work in order prove that those solutions belongs to the space \(C([0, \infty), L^2(\mathbb{R}^3)^3)\).

1. Introduction.

Statement of the main result. In this paper, we extend results from the work [9] on singular solutions of the Navier-Stokes system

\[
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{div } u = 0
\]

in the three dimensional space. Here, the vector \(u = (u_1(x,t), u_2(x,t), u_3(x,t))\) denotes the unknown velocity field and the scalar function \(p = p(x,t)\) stands for the unknown pressure. The main result of this paper is contained in the following theorem on the existence of singular solutions in the energy space \(L^2(\mathbb{R}^3)^3\) to the Navier-Stokes system.

**Theorem 1.1.** Consider a curve \(\Gamma = \{ (\gamma(t), t) \in \mathbb{R}^3 \times \mathbb{R}^+ : t \geq 0 \}\), where \(\gamma : \mathbb{R}^+ \to \mathbb{R}^3\) is an arbitrary Hölder continuous function with a Hölder exponent \(\alpha \in (\frac{1}{2}, 1]\). There exists a divergence-free vector field \(u \in C([0, \infty), L^2(\mathbb{R}^3)^3)\) and the corresponding pressure \(p = p(x,t)\) which are regular and solve system (1) pointwise for all \((x,t) \in (\mathbb{R}^3 \times \mathbb{R}^+) \setminus \Gamma\) and which are singular on the curve \(\Gamma\).

2010 Mathematics Subject Classification. Primary: 35Q30; Secondary: 76D05, 35B40.

Key words and phrases. Navier-Stokes system, incompressible fluid, time-dependent singularity, Słotkkin-Landau solutions.

* Corresponding author: Tomas P. Schonbek.
Solutions to system (1) with singularities on the curve $\Gamma$ were constructed in paper [9] and the main contribution of this work is to show that those solutions have a finite energy and belong to the space $C([0, \infty), L^2(\mathbb{R}^3)^3)$.

The problem considered in [9] is a time dependent generalized version of the Landau stationary problem [11] which models a fluid into which a thin pipe discharges a jet oriented along the positive part of the $x$-axis. The solution studied by Landau seems to have been found originally by Słezkin [19] and a translation from the Russian of Słezkin’s original notes on this solution can be found as appendices in [5]. The stationary solution in question, to be called the Słezkin-Landau solution $(V^c, Q^c) \equiv (V_1^c, V_2^c, V_3^c, Q^c)$ is defined for each real constant $|c| > 1$ by the following formulas

$$V_1^c(x) = 2|c|^2 - 2x_1|c| + cx_1^2 |x| (c|x| - x_1)^2,$$

$$V_2^c(x) = 2x_2(cx_1 - |x|) |x| |c|x| - x_1|^2,$$

$$V_3^c(x) = 2x_3(cx_1 - |x|) |x| |c|x| - x_1|^2,$$

$$Q^c(x) = 4cx_1 - |x| |x| |c|x| - x_1|^2,$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is immediate to check that these locally integrable functions are in $C^\infty(\mathbb{R}^3 \setminus \{0\})$ and $V_1^c, V_2^c, V_3^c$ are homogeneous of degree $-1$. One can prove (see e.g. [1, p. 209] and [2, Prop. 2.1]) that $(V^c, Q^c)$ satisfies in the sense of distributions the following stationary Navier-Stokes system with a singular force

$$-\nabla^2 V^c + (V^c \cdot \nabla)V^c + V^c \nabla Q^c = \kappa \delta_0 e_1, \quad \text{div } V^c = 0,$$

where $e_1 = (1, 0, 0)$, the symbol $\delta_0$ denotes as usual the Dirac measure at the origin, and

$$\kappa = \kappa(c) = \frac{8\pi c}{3(c^2 - 1)} \left(2 + 6c^2 - 3c(c^2 - 1) \log \left(\frac{c + 1}{c - 1}\right)\right).$$

We will be using the following main property of this relation: $\lim_{|c| \to \infty} \kappa(c) = 0$.

The authors of the paper [9] developed tools to study analogous time dependent solutions of the following singular Navier-Stokes equations

$$u_t - \nabla u + (u \cdot \nabla) u + \nabla p = \kappa \delta_0 e_1,$$

$$\text{div } u = 0,$$

$$u(0) = 0$$

with a Hölder continuous function $\gamma : [0, \infty) \to \mathbb{R}^3$, the shifted Dirac delta function $\delta_{\gamma(t)} = \delta(-\gamma(t))$, and a sufficiently small constant $\kappa$. The paper [9] recalls the result from [2] on the existence of a solution $(u, p)$ to problem (4), however, the main goal in [9] is to show that $u = u(x, t)$ and $p = p(x, t)$ are locally bounded away from the graph of the curve $\Gamma = \{(\gamma(t), t) : t \geq 0\}$ and are singular along this curve. Moreover, the couple $(u, p)$ satisfies problem (4) in the sense of distributions. In addition, $u \in L^\infty([0, \infty), L^3(\mathbb{R}^3)^3)$ (here $L^{\infty}(\mathbb{R}^3)$ is the usual weak Marcinkiewicz
$L^p$-space), thus $p = \sum R_i R_j (u_j u_j)$ belongs to $L^\infty([0, \infty), L^2(\mathbb{R}^3))$ with the Riesz transforms $R_1, R_2, R_3$. These results only require $\gamma$ to be continuous, but the authors of [9] do assume $\kappa = \kappa(c)$ to be sufficiently small. Then, assuming $\gamma$ to be Hölder continuous with exponent $\alpha \in (\frac{1}{2}, 1]$, the paper [9] compares the solution $(u, p)$ of (4) to the Slëzkin-Landau solution made time dependent by translating the origin to $\gamma(t)$ proving that for all $t > 0$,

$$u(t) - V^c(\cdot - \gamma(t)) \in L^q(\mathbb{R}^3) \quad \text{for} \quad 3 < q < \frac{3}{2(1 - \alpha)}$$

and

$$p(t) - Q^c(\cdot - \gamma(t)) \in L^q(\mathbb{R}^3) \quad \text{for} \quad \frac{3}{2} < q < \frac{3}{3 - 2\alpha}.$$ 

In this work, we revisit some of the existence results as well as some of the asymptotic estimates obtained in paper [9]. We repeat some of the proofs, with slight variations, because we need to have solutions to problem (4) and their estimates expressed in a slightly different form from how it appeared in [9]. Here, estimates of the $L^2(\mathbb{R}^3)$-norms are essentially new and this is done in Sections 2 and 3. Section 4 contains our main result: we prove that the solution of problem (4) satisfies $u \in C([0, \infty), L^2(\mathbb{R}^3))$ for $\kappa = \kappa(c)$ with $|c|$ large enough.

**Related results.** The Slëzkin-Landau solution has appeared in several recent works. It is proved in [2] that they are asymptotically stable in a suitable Banach space of tempered distributions. They are also asymptotically stable under arbitrary large initial perturbations of finite energy, see [7, 8]. They appear in asymptotic expansions of solutions to initial-boundary value problems for the Navier-Stokes system (1), cf. [4, 10, 6, 13, 3]. Here, we also mention the article by Sverak [20] in which it is proved that the Slëzkin-Landau solution is the only stationary solution of the three dimensional Navier-Stokes system that is invariant under the natural scaling of the system.

Concerning other equations, solutions singular along curves, have been constructed by Sato and Yanagida [14, 15, 16, 17, 18] and by Takahashi and Yanagida [21, 22] for the linear and non-linear heat equation.

**Notation.** We use the notation from [9]. Specifically, a major role in all these considerations is played by the Banach space $PM^a$ for $a > 0$ defined as the space of all $w \in S'((\mathbb{R}^3)$ such that

$$||w||_{PM^a} = (2\pi)^{3/2} \sup_{\xi \in \mathbb{R}^3} |\xi|^a |\hat{w}(\xi)|.$$ 

Then, for a time dependent function $u : [0, T] \to PM^a$ with fixed $T \in (0, \infty]$ we define the scaling invariant norm

$$||u||_{a,T} = \sup_{0 < t < T} t^{a/2 - 1} ||u(t)||_{PM^a}.$$ 

The factor of $(2\pi)^{3/2}$ is added in the definition of $|| \cdot ||_{PM^a}$ because we are using the Fourier transform in the form

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx.$$ 

Thus, the factor $(2\pi)^{-3/2}$ appears in the well known formula

$$\mathcal{F}^{-1}(\hat{g}) = (2\pi)^{-3/2} f \ast g.$$
2. The integral formulation of the problem.

Integral version of the Navier-Stokes initial value problem. For two vector fields \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \), we denote by \( u \otimes v \) the \( 3 \times 3 \) matrix whose \((\ell, k)\) entry is \( u_\ell v_k \). We denote the heat kernel by \( F(x, t) = (4\pi t)^{-3/2} \exp(-|x|^2/4t) \). With \( R_j \) denoting the \( j\)-th Riesz transform, (that is \( R_j f(\xi) = -i(\xi_j/|\xi|) \hat{f}(\xi) \)) we define \( \mathcal{R} \) to be the \( 3 \times 3 \) matrix of operators \( (R_j R_k) \). Thus,

\[
(\mathcal{R}(u \otimes v))_{jk} = \sum_{\ell=1}^{3} R_j R_{\ell}(u_\ell v_k).
\]

For \( t \geq 0 \), with a slight abuse of notation, we define the “convolution” \( K(t) \ast (u \otimes v) \) to be the vector whose \( j\)-th component is given by

\[
[K(t) \ast (u \otimes v)]_j = \sum_{k=1}^{3} \frac{\partial F(t)}{\partial x_k} \ast \{|I + \mathcal{R}|(u \otimes v)\}_{jk},
\]

where we have a sum of actual convolutions on the right hand side. The basic bilinear form is now defined by the formula

\[
B(u, v)(t) \equiv \int_0^t K(t - s) \ast (u(s) \otimes v(s)) \, ds.
\]

Introducing for \( k \in \{1, 2, 3\} \) the \( 3 \times 3 \) matrix \( K_k = (K_{k\ell}) \), where

\[
K_{k\ell}(\xi, t) = i(2\pi)^{-3/2} \xi_k e^{-t|\xi|^2} \left( \delta_{j\ell} - \frac{\xi_j \xi_\ell}{|\xi|^2} \right),
\]

we obtain

\[
\mathcal{F}\{e^{(t-s)\Delta}\left(u \cdot \nabla v_j(s) + \sum_{k=1}^3 R_j R_k(u \cdot \nabla v_k)\right)(\xi)\}
\]

\[
= (2\pi)^{3/2} \sum_{k=1}^3 \hat{K}_k(t-s)(u \otimes v)(s)_{jk}.
\]

Now, the integral version of the Navier-Stokes initial value problem

\[
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f,
\]

\[
\text{div } u = 0,
\]

\[
u(\cdot, 0) = u_0
\]

can then be written in the form

\[
u(t) = e^{t\Delta} u_0 - B(u, u)(t) + \int_0^t e^{(t-s)\Delta} (f(s) + \mathcal{R}f(s)) \, ds.
\]

The matrix \( K = K(t) = K(x, t) \) with the coefficients defined by formula (6) is sometimes referred to as the Oseen tensor, see e.g. [12]. It is not too hard to prove (see e.g. [12, Sec. 4.5]) that there is a constant \( C > 0 \) such that

\[
|K(x, t)| \leq \frac{C}{(|x|^2 + t)^2}
\]

for all \( x \in \mathbb{R}^3, \ t > 0 \), so that we have the estimate

\[
\|K(t)\|_{L^p(\mathbb{R}^3)} \leq C t^{-\frac{3}{2} - \frac{3}{p'}}
\]

for all \( t > 0 \), where, as usual, \( 1/p + 1/p' = 1 \).
Heat equation with singular force. Let $\gamma \in C([0, \infty), \mathbb{R}^3)$. For $(x, t) \neq (\gamma(t), t)$ we define
\[
\varphi(x, t) \equiv \int_0^t e^{(t-s)\Delta} \delta_{\gamma(s)} \, ds = \int_0^t \frac{1}{(4\pi(t-s))^{3/2}} e^{-\frac{|x-\gamma(s)|^2}{4(t-s)}} \, ds.
\] (9)

By direct calculation, the function $\varphi \in C^\infty((\mathbb{R}^3 \times [0, \infty)) \setminus \{ (\gamma(t), t) \mid t \geq 0 \})$ is locally integrable in $\mathbb{R}^3 \times [0, \infty)$ and satisfies the following inhomogeneous heat equation
\[
\frac{\partial \varphi}{\partial t} - \Delta \varphi = \delta_{\gamma(t)}
\]
in the sense of distributions (see [21] for other properties of this function). Thus, for every test function $\psi \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))$ we have the equalities
\[
\left< \left( \frac{\partial}{\partial t} - \Delta \right) \varphi, \psi \right> = \left< \varphi, \frac{\partial \psi}{\partial t} - \Delta \psi \right>
\]
\[
= \int_{\mathbb{R}^3 \times (0, \infty)} \varphi(x, t) \left( \frac{\partial \psi}{\partial t}(x, t) - \Delta \psi(x, t) \right) \, dx \, dt
\]
\[
= \int_0^\infty \psi(\gamma(t), t) \, dt.
\]
The Fourier transform of $\varphi$ is given by the formula
\[
\hat{\varphi}(\xi, t) = (2\pi)^{-3/2} e^{-t|\xi|^2} \int_0^t e^{s|\xi|^2} e^{-i\gamma(s) \cdot \xi} \, ds
\]
so that
\[
|\hat{\varphi}(\xi, t)| \leq (2\pi)^{-3/2} e^{-t|\xi|^2} \int_0^t e^{s|\xi|^2} \, ds = (2\pi)^{-3/2} \frac{1 - e^{-t|\xi|^2}}{|\xi|^2},
\]
which implies the inequality
\[
\|\varphi(t)\|_{PM^2} = (2\pi)^{3/2} \sup_{x \in \mathbb{R}^3} |\xi|^2 |\hat{\varphi}(\xi, t)| \leq 1 \quad \text{for all} \quad t \geq 0.
\] (10)

We also remark that $\varphi(x, 0) = 0$ if $x \neq \gamma(0)$.

Navier-Stokes system with singular force. We consider now the integral equation (8) with $u_0 = 0$ and $f(t) = \kappa \delta_{\gamma(t)} e_1$, where $e_1 = (1, 0, 0)$, $\kappa \in \mathbb{R}$, and $\gamma \in C([0, \infty), \mathbb{R}^3)$. Here, we have
\[
\int_0^t e^{(t-s)\Delta} f(s) \, ds = \kappa \Phi(t), \quad \int_0^t e^{(t-s)\Delta} \mathcal{R} f(s) \, ds = -\kappa \Psi(t),
\] (11)
where
\[
\Phi(x, t) = \begin{pmatrix} \varphi(x, t) \\ 0 \\ 0 \end{pmatrix},
\]
with the function $\varphi(x, t)$ defined by (9) and with $\Psi = (\psi_1, \psi_2, \psi_3)$, where for $j = 1, 2, 3$ we define
\[
\psi_j(t) = -R_1 R_j \varphi(t), \quad \text{hence} \quad \hat{\psi}_j(\xi, t) = (2\pi)^{-n/2} \frac{\xi_j}{|\xi|^2} \int_0^t e^{-(t-s)\xi_j^2} e^{-i\gamma(s) \cdot \xi} \, ds.
\]

Notice that by inequality (10), we have $\|\Phi(t)\|_{PM^2} \leq 1$ and, since obviously $\|R_j f\|_{PM^2} \leq \|f\|_{PM^2}$, we also obtain $\|\Psi(t)\|_{PM^2} \leq 1$. Thus,
\[
\|\kappa(\Phi(t) - \Psi(t))\|_{PM^2} \leq 2|\kappa| \quad \text{for all} \quad t \geq 0.
\] (12)
In this notation, the integral formulation of problem (4) becomes
\[ u(t) = -B(u,u)(t) + \kappa (\Phi(t) - \Psi(t)). \] (13)

**Estimates of the bilinear form.** Our next goal is to prove suitable estimates of the bilinear form \( B(\cdot, \cdot) \) defined in (5). First, however, we recall an inequality from [9].

**Lemma 2.1** ([9, Lemma 3.5]). Let \( a, b \in (0, 3) \) and \( a + b > 3 \). There exists a constant \( C > 0 \) such that for all \( u \in \mathcal{PM}^a \) and \( v \in \mathcal{PM}^b \)
\[ \|uv\|_{\mathcal{PM}^{a+b}} \leq C\|u\|_{\mathcal{PM}^a}\|v\|_{\mathcal{PM}^b}. \]

In the following lemma, we gather those estimates of \( B(\cdot, \cdot) \) which will be required in the proof of the main result.

**Lemma 2.2.** Let \( u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in C([0, \infty), \mathcal{S}(\mathbb{R}^3)) \) be arbitrary.

1. Assume \( 0 \leq \delta < 1, 0 < T \leq \infty \). There exists a constant \( \eta_0 > 0 \) such that
\[ \|B(u,v)\|_{L^{2+\delta,T}} \leq \eta_0 \|u\|_{L^{2+\delta,T}}\|v\|_{L^{2+\delta,T}}. \] (14)

2. Assume \( 0 < \delta < 1 \). There exists a constant \( C_\delta > 0 \) such that
\[ \|B(u,v)(t)\|_{\mathcal{PM}^{2+\delta}} \leq C_\delta \int_0^t (t-s)^{\frac{\delta}{2}-1}\|u(s)\|_{\mathcal{PM}^{2+\delta}}\|v(s)\|_{\mathcal{PM}^{2+\delta}} \ ds \] (15)
for all \( t > 0 \).

3. Assume \( 0 < \delta < 1 \). There exists a constant \( C_\delta \) such that
\[ \|B(u,v)(t)\|_{L^2(\mathbb{R}^3)} \leq C_\delta \int_0^t (t-s)^{\frac{\delta}{2}-1}\|u(s)\|_{\mathcal{PM}^{2+\delta}}\|v(s)\|_{L^2(\mathbb{R}^3)} \ ds \] (16)
for all \( t > 0 \).

**Proof.** Part 1 and essentially Part 2 are proved in [9], however, in the case of a slightly different bilinear form obtained after applying the Leray projector. We recall that reasoning for the completeness of exposition.

Assume \( 0 \leq \delta \leq 1 \). To avoid repetitions in the proof, we denote either \( \alpha = 2, \beta = 2 + \delta \) for Part 1 or \( \alpha = 2 - \delta, \beta = 1 + 2\delta \) for Part 2. Notice that in both cases \( \alpha + \beta = 3 + \delta \). In addition, let \( a = 2, b = 2 + \delta \) for Part 1 and \( a = b = 2 + \delta \) for Part 2. In either case, \( a + b - 3 = \beta \) so that, by Lemma 2.1, we have
\[ \|uv_{k}(s)\|_{\mathcal{PM}^a} \leq C_\delta \|u(s)\|_{\mathcal{PM}^a}\|v(s)\|_{\mathcal{PM}^b}, \]
with \( C_\delta \) denoting a constant that may depend only on \( \delta \in [0, 1) \), not necessarily the same one in each inequality. Thus, using formula (7), we obtain
\[ \left| \xi \right|^{2+\delta} \mathcal{F} \left\{ \int_0^t K_{k\ell}(t-s) \ast (u_{\ell}v_k) \right\}(\xi) \ ds \]
\[ = \left| \xi \right|^{2+\delta} |\xi_k| \left| \int_0^t e^{-(t-s)}|\xi|^2 \left( \delta_{\ell \ell} - \frac{\xi_j \xi_{\ell}}{|\xi|^2} \right) \hat{u}_{\ell} \hat{v}_k(\xi) \ ds \right| \]
\[ \leq 2\left| \xi \right|^{3+\delta} \int_0^t e^{-(t-s)}|\xi|^2 |\hat{u}_{\ell} \hat{v}_k(s, \xi)| \ ds \]
\[ \leq 2\left| \xi \right|^{\alpha} \int_0^t e^{-(t-s)}|\xi|^2 \|u_{\ell}v_k(s)\|_{\mathcal{PM}^a} \ ds \]
\[ \leq C_\delta \left| \xi \right|^{\alpha} \int_0^t e^{-(t-s)}|\xi|^2 \|u_{\ell}(s)\|_{\mathcal{PM}^a}\|v_k(s)\|_{\mathcal{PM}^b} \ ds. \] (17)
Choosing \( \alpha = 2, b = 2 + \delta \) we get

\[
\|B(u, v)(t)\|_{P^{2_\alpha+\delta}} \leq C_\delta \sup_{\xi \in \mathbb{R}^3} \int_0^t |\xi|^2 e^{-(t-s)}|\xi|^2 \|u(s)\|_{P_{\xi}M^{2_\alpha}} \|v(s)\|_{P_{\xi}M^{2_\alpha+\delta}} \, ds,
\]

thus, for \( 0 \leq t < T \), we obtain

\[
\|B(u, v)(t)\|_{P^{2_\alpha+\delta}} \leq C_\delta \|u\|_{2, T} \|v\|_{2_\alpha+\delta, T} \sup_{\xi \in \mathbb{R}^3} |\xi|^2 \int_0^t e^{-(t-s)}|\xi|^2 s^{-\delta/2} \, ds
\]

\[
= C_\delta t^{-\delta/2} \|u\|_{2, \infty} \|v\|_{2_\alpha+\delta, \infty} \sup_{\xi \in \mathbb{R}^3} (|t| \xi|^2) \int_0^1 \sigma^{-\delta/2} e^{-(1-\sigma)t|\xi|^2} \, d\sigma.
\]

Since the function \( r \mapsto r \int_0^1 \sigma^{-\delta/2} e^{-(1-\sigma)r} \, d\sigma : [0, \infty) \to \mathbb{R} \) is bounded and a continuous function converging to zero when either \( r \to 0 \) or \( r \to \infty \), we deduce that

\[
\sup_{\xi \in \mathbb{R}^3, t > 0} \left( |t| \xi|^2 \int_0^1 \sigma^{-\delta/2} e^{-(1-\sigma)t|\xi|^2} \, d\sigma \right) < \infty.
\]

In this way, we have completed the proof of estimate (14) with \( \eta_\delta = C_\delta \).

Assume now \( 0 < \delta < 1 \). Returning to estimate (17) with \( \alpha = 2 - \delta \) and \( b = 1 + 2\delta \) (so \( a = b = 2 + \delta \)), and using the estimate \( |\xi|^{2-\delta} e^{-(t-s)|\xi|^2} \leq C_\delta (t-s)^{\frac{\delta}{2}-1} \) we get the estimate

\[
\left|\xi|^{2+\delta} \mathcal{F} \left\{ \int_0^t K_{jk\ell}(t-s) \ast (u_k v_k) \right\} (\xi) \, ds \right|
\]

\[
\leq C_\delta \int_0^t (t-s)^{\frac{\delta}{2}-1} \|u_k(s)\|_{P^{2_\alpha+\delta}} \|v_k(s)\|_{P^{2_\alpha+\delta}} \, ds,
\]

proving immediately inequality (15).

For the proof of Part 3 we recall that for \( 0 < \alpha < 3 \), the Riesz potential of a function \( f \) is defined by

\[
I_\alpha f(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-\alpha}} \, dy
\]

and satisfies

\[
\|I_\alpha f\|_{L^r(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)} \quad \text{if} \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{3},
\]

for some constant \( C \) independent of \( f \).

Since \( |\delta_{j\ell} - \xi_j \xi_\ell/|\xi|^2| \leq 2 \) we obtain

\[
\|B(u, v)(t)\|_{L^2(\mathbb{R}^3)} \leq C \int_0^t \max_{k, \ell} \left\| |\xi| e^{-(t-s)}|\xi|^2 \hat{u}_k(s) * \hat{v}_\ell(s)(\xi) \right\|_{L^2(\mathbb{R}^3)} \, ds.
\]

Moreover, we have

\[
\left\| |\xi| e^{-(t-s)}|\xi|^2 \hat{u}_k(s) * \hat{v}_\ell(s)(\xi) \right\|_{L^2(\mathbb{R}^3)}^2
\]

\[
\leq \int_{\mathbb{R}^3} |\xi|^2 e^{-2(t-s)|\xi|^2} \left( \int_{\mathbb{R}^3} |\hat{u}(\xi - \eta, s)| \, d\eta \right)^2 d\xi
\]

\[
\leq \int_{\mathbb{R}^3} |\xi|^2 e^{-2(t-s)|\xi|^2} \left( \int_{\mathbb{R}^3} |\xi - \eta|^{2+\delta} |\hat{u}(\xi - \eta, s)| \left| \frac{\hat{v}(\eta, s)}{|\xi - \eta|^{2+\delta}} \right| \, d\eta \right)^2 d\xi
\]

\[
\leq \|u(s)\|_{P^{2_\alpha+\delta}}^2 \int_{\mathbb{R}^3} |\xi|^2 e^{-2(t-s)|\xi|^2} I_{1,-\delta}(\hat{v})(\xi)^2 \, d\xi.
\]
Now, we apply the Hölder inequality with \( q = 3/(1 + 2\delta) \), \( q' = 3/[2(1 - \delta)] \) and estimate for the Riesz potential (19) with \( \alpha = 1 - \delta \) (so \( 3 - \alpha = 2 + \delta \)), \( p = 2 \), and \( r = 6/1 + 2\delta \) to get
\[
\left\| \xi e^{-(t-s)|\xi|^2}\hat{u}_k(s) \ast \hat{u}_l(s)(\xi) \right\|_{L^2(\mathbb{R}^3)}^2
\leq \left\| u(s) \right\|_{\mathcal{PM}^{2+s}}^2 \left( \int_{\mathbb{R}^3} \left\| e^{\frac{3}{1-\delta}|\xi|^2} d\xi \right\|^{\frac{2(1-\delta)}{3}} \right) \left\| I_{1-\delta} \hat{\vartheta} \right\|_{L^\infty(\mathbb{R}^3)}^2
\leq C \left\| u(s) \right\|_{\mathcal{PM}^{2+s}}^2 \left\| v(s) \right\|_{L^2(\mathbb{R}^3)}^2 (t-s)^{\delta-2}.
\] (21)

Combining inequality (20) with the one in (21) we complete the proof of estimate (16).

\[\square\]

**Remark 1.** Note the following immediate consequence of estimate (16):
\[
\|B(u,v)(t)\|_{L^2(\mathbb{R}^3)} \leq C t^{5/2} \sup_{0 \leq s \leq t} \left\| u(s) \right\|_{\mathcal{PM}^{2+s}} \left\| v(s) \right\|_{L^2(\mathbb{R}^3)}.
\] (22)

We conclude this section by recalling a theorem on the existence of solutions to the singular Navier-Stokes system (4).

**Theorem 2.3** (see [2, Thm. 4.1] and [9, Thm. 5.1]). Assume \( |\kappa| < 1/(8\eta_0) \), where \( \eta_0 \) is the constant on the right hand side of (14) corresponding to \( \delta = 0 \). Equation (13) has a unique solution \( u \in C([0, \infty), L^3_{\infty}) \). This solution satisfies
\[
\left\| u \right\|_{2,\infty} \leq 4 |\kappa| \leq \frac{1}{2\eta_0}.
\]

This theorem is an immediate consequence of the Banach fixed point theorem applied to the equation \( u = B(u,u) + z \) using estimates from (12) and (14) with \( \delta = 0 \).

From now on, we denote by \( u = u(x,t) \) the solution provided by Theorem 2.3 and our goal is to show that \( u \in C([0, \infty), L^2(\mathbb{R}^3)) \).

3. **Asymptotics by the Slęzkin-Landau solution.** In this section, we compare our solution \( u \) given by Theorem 2.3 with the Slęzkin-Landau solution made time dependent by translating the origin to \( \gamma(t) \) for \( t \geq 0 \). Thus we introduce the function \( h(x) = 1/(c|x| - x_1) \) which allows us to write the following equalities
\[
V_1^c = 2h - 2x_1 \frac{\partial h}{\partial x_1}, \quad V_2^c = -2x_2 \frac{\partial h}{\partial x_1}, \\
V_3^c = -2x_3 \frac{\partial h}{\partial x_1}, \quad Q^c = -4 \frac{\partial h}{\partial x_1}.
\]

Using these formulas and writing \( h \) in the form
\[
h(x) = \frac{1}{c|x|} \left( \frac{1}{1 - \frac{x_1}{c|x|}} \right),
\]
it is easy to prove that for each multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) there exists a constant \( C_\alpha \), depending on \( \alpha \) but not on \( c \) (recall that \( |c| > 1 \)), such that
\[
|D^\alpha V^c(x)| \leq \frac{C_\alpha}{|c|} |x|^{-1-|\alpha|} \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}
\] (23)
and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) with \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq 3 \). These are pointwise estimates of the Slęzkin-Landau solutions which extends those in [7, Lemma 3.1] and the
following lemma is a consequence of them. It appears in [9, Lemma 3.10], however, for the sake of completeness, we provide a proof that is somewhat simpler than the one in [9].

**Lemma 3.1.** There exists a constant $K$ independent of $c$ such that

$$\|V^c\|_{PAM^2} \leq \frac{K}{|c|}. \tag{24}$$

**Proof.** It suffices to prove that if a function $W \in C^3(\mathbb{R}^3 \setminus \{0\})$ satisfies the estimate $|D^\alpha W(x)| \leq (C_\alpha/c)|x|^{-1-|\alpha|}$ for $|\alpha| \leq 3$ with $C_\alpha$ independent of $c$, then $\|W\|_{PAM^2} \leq K/c$ for some constant $K$ independent of $c$.

Let $\xi \in \mathbb{R}^3$, $\xi \neq 0$, it will remain fixed for most of the proof. Let $\chi \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1$, $\chi(x) = 0$ if $|x| > 2$. Let $\chi_{\xi}(x) = \chi(|\xi|x)$. We write

$$W = \chi_{\xi} W + (1 - \chi_{\xi}) W = W_1 + W_2.$$

Now

$$|\hat{W}_1(\xi)| \leq \int_{\mathbb{R}^3} |W_1(x)| \, dx = \int_{|x| \leq 2/|\xi|} |W(x)| \, dx \leq \frac{C_0}{|c|} \int_{|x| \leq 2/|\xi|} |x|^{-1} \, dx = \frac{8\pi c_0}{|c||\xi|^2}.$$

Next for $j = 1, 2, 3,$

$$|\xi^j W_2(\xi)| = \left| \mathcal{F}(\frac{\partial^3 W_2}{\partial x_j^3})(\xi) \right|$$

$$\leq \int_{\mathbb{R}^3} \left( \sum_{k=0}^2 \binom{3}{k} \left| \frac{\partial^{3-k} \chi_{\xi}}{\partial x_j^{3-k}} \right| \left| \frac{\partial^k W}{\partial x_j^k} \right| + (1 - \chi_{\xi}) \left| \frac{\partial^3 W}{\partial x_j^3} \right| \right) \, dx.$$

Let $C = \max_{|\alpha| \leq 3} C_\alpha$; then for $k = 0, 1, 2,$ since $D^\alpha \chi = 0$ for $|x| \leq 1$ and for $|x| \geq 2$ if $\alpha \neq 0$,

$$\int_{\mathbb{R}^3} \left| \frac{\partial^{3-k} \chi_{\xi}}{\partial x_j^{3-k}} \right| \left| \frac{\partial^k W}{\partial x_j^k} \right| \, dx = \int_{\frac{1}{|c|} \leq |x| \leq \frac{2}{|c|}} \left| \frac{\partial^{3-k} \chi_{\xi}}{\partial x_j^{3-k}} \right| \left| \frac{\partial^k W}{\partial x_j^k} \right| \, dx$$

$$= |\xi|^{3-k} \int_{|x| \leq \frac{2}{|c|}} \left| \frac{\partial^{3-k} \chi_{\xi}(|\xi|x)}{\partial x_j^{3-k}} \right| \left| \frac{\partial^k W}{\partial x_j^k}(|\xi|x) \right| \, dx$$

$$\leq \frac{C}{|c|} |\xi|^{3-k} \int_{|x| \leq \frac{2}{|c|}} \left| \frac{\partial^{3-k} \chi_{\xi}(|\xi|x)}{\partial x_j^{3-k}} \right| \left| \frac{\partial^k W}{\partial x_j^k}(|\xi|x) \right| \, dx$$

$$\leq \frac{C}{|c|} |\xi|^4 \int_{|x| \leq \frac{2}{|c|}} \left| \frac{\partial^{3-k} \chi_{\xi}(|\xi|x)}{\partial x_j^{3-k}} \right| \, dx$$

$$= \frac{C}{|c|} |\xi| \int_{|x| \leq 2} \left| \frac{\partial^{3-k} \chi_{\xi}(|\xi|x)}{\partial x_j^{3-k}} \right| \, dx$$

$$= \frac{K}{|c|} |\xi|,$$

where $K$ is a constant that does not depend on $c$ or $\xi$, not necessarily the same from expression to expression. Finally

$$\int_{\mathbb{R}^3} (1 - \chi_{\xi}) \left| \frac{\partial^3 W}{\partial x_j^3} \right| \, dx \leq \frac{K}{|c|} \int_{|x| \geq 1/|\xi|} |x|^{-4} \, dx = \frac{K}{|c|} |\xi|.$$

Putting it all together, we proved for $j = 1, 2, 3$ the inequality

$$|\xi_j|^3 |\hat{W}_2(\xi)| \leq \frac{K}{|c|} |\xi| \quad \text{or equivalently} \quad |\xi|^2 |\hat{W}_2(\xi)| \leq \frac{K}{|c|}.$$
Combining it with the estimate on \( W_1 \), we complete the proof. \( \square \)

Let \( \gamma : [0, \infty) \rightarrow \mathbb{R}^3 \). For each \((x, t) \in \mathbb{R}^3 \times [0, \infty)\), we define the translated Slézkin-Landau solution

\[
V_c^\gamma(x, t) = V^c(x - \gamma(t)) \quad \text{and} \quad Q_c^\gamma(t) = Q^c(x - \gamma(t)).
\]

which satisfies, for each fixed \( t > 0 \), the system

\[
- \Delta V_c^\gamma(t) + V_c^\gamma(t) \cdot \nabla V_c^\gamma(t) + \nabla Q_c^\gamma(t) = \kappa \delta_{\gamma(t)} \delta_1.
\]

Since \( \hat{V}_c^\gamma(\xi, t) = e^{-\gamma(t) \cdot \xi} \hat{V}_c^\gamma(\xi) \), it is clear that

\[
\|V_c^\gamma(t)\|_{PM^2} = \|V_c^\gamma\|_{PM^2} \quad \text{for all} \quad t \geq 0.
\]

**Lemma 3.2.** With the vector field \( V_c^\gamma \) from (25) and the bilinear form \( B \) defined in (5), we have the equality

\[
B(V_c^\gamma, V_c^\gamma)(t) = W^c(t) + \kappa(\Phi(t) - \Psi(t)),
\]

where

\[
W^c(t) = \int_0^t e^{(t-s)\Delta} \Delta V_c^\gamma(s) \, ds
\]

or equivalently

\[
\hat{W}_c^\gamma(\xi, t) = -|\xi|^2 \hat{V}_c^\gamma(\xi) \int_0^t e^{-(t-s)|\xi|^2} \hat{V}_c^\gamma(s) \, ds
\]

and the functions \( \Phi(t) \) and \( \Psi(t) \) are defined in (11).

**Proof.** Computing the divergence of both sides of the equation

\[
- \Delta V^c + V^c \cdot \nabla V^c + \nabla Q = \kappa \delta_0 \delta_1
\]

and using \( \text{div} \, V^c = 0 \) we get the following equation understood in the sense of distributions

\[
\Delta Q = -\text{div} \, (V^c \cdot \nabla V^c) + \kappa \frac{\partial \delta}{\partial x_1} = -\sum_{k, \ell} \frac{\partial^2 (V_k^c V_\ell^c)}{\partial x_k \partial x_\ell} + \kappa \frac{\partial \delta_0}{\partial x_1}
\]

Thus

\[
Q = (-\Delta)^{-1} \left( \sum_{k, \ell} \frac{\partial^2 (V_k^c V_\ell^c)}{\partial x_k \partial x_\ell} - \kappa \frac{\partial \delta}{\partial x_1} \right) = \sum_{k, \ell} R_k R_\ell (V_k^c V_\ell^c) - \kappa (-\Delta)^{-1} \frac{\partial \delta_0}{\partial x_1}
\]

where \( R_j \) are the Riesz transforms. Moreover, it follows that

\[
\frac{\partial Q}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \sum_{k, \ell} R_k R_\ell (V_k^c V_\ell^c) \right) - \kappa R_j R_1 \delta = \left( \sum_{k, \ell} \frac{\partial}{\partial x_k} R_j R_\ell (V_k^c V_\ell^c) \right) - \kappa R_j R_1 \delta_0
\]

since \( \frac{\partial}{\partial x_j} R_k = \frac{\partial}{\partial x_k} R_j \). Now, we write equation (26) in the following form

\[
- \Delta V_{j, \gamma}^c + V_{\gamma}^c \cdot \nabla V_{j, \gamma}^c + \frac{\partial Q^\gamma}{\partial x_j} = \kappa \delta_\gamma \delta_{j_1} \quad \text{for} \quad j = 1, 2, 3,
\]
with the Kronecker delta δ_{j1}. Applying the heat semigroup $e^{(t-s)\Delta}$ to both sides of equation for $\frac{\partial Q}{\partial x_j}$ (translated by $\gamma(s)$) and integrating with respect to $s$ from 0 to $t$ we get (after a slight rearrangement) the following equation

$$\int_0^t e^{(t-s)\Delta} \left(V_{\gamma}^c(s) \cdot \nabla V_{\gamma}^c(s) + \sum_{k,l} \frac{\partial}{\partial x_k} R_j R_l (V_{k,\gamma}^c(s) V_{l,\gamma}^c(s)) \right) ds$$

$$= \int_0^t e^{(t-s)\Delta} V_{\gamma}^c(s) ds + \kappa \int_0^t e^{(t-s)\Delta} R_j R_k \delta_\gamma(s) ds$$

$$+ \kappa \int_0^t e^{(t-s)\Delta} \delta_\gamma(s) \delta_{j1} ds.$$ 

In the equality above, the left hand side is the $j$-th component of $B(V_{\gamma}^c, V_{\gamma}^c)(t)$ and the terms on the right hand side correspond to $W^c(t)$, $-\kappa \Psi(t)$, and $\kappa \Phi(t)$, respectively.

**Remark 2.** Returning to the expression for $W^c$, we see that

$$||W^c(t)||_{PM^2} \leq |\xi|^2 |\hat{\nu}^c(\xi)| \int_0^t e^{-(t-s)|\xi|^2} ds \leq ||V^c||_{PM^2}.$$

Comparing equation (27) with the one in (13), we can write

$$u = -W^c + B(V_{\gamma}^c, V_{\gamma}^c) - B(u, u).$$

(29)

Now, we improve estimates from [9, Theorem 5.3 and Corollary 2].

**Theorem 3.3.** Let $w = u - V_{\gamma}^c$. Then there exists $c_0 > 0$ such that if $|c| \geq c_0$, then

$$||w(t)||_{PM^2} \leq \frac{4||V^c||_{PM^2}}{1 - 2\eta_0||V^c||_{PM^2} + \sqrt{1 + 4\eta_0^2||V^c||_{PM^2}^2 - 12\eta_0||V^c||_{PM^2}}}$$

(30)

for all $t \geq 0$.

**Proof.** By equation (29),

$$w = u - V_{\gamma}^c = B(V_{\gamma}^c, V_{\gamma}^c) - B(u, u) - (W^c + V_{\gamma}^c)$$

$$= B(V_{\gamma}^c, V_{\gamma}^c - u) + B(V_{\gamma}^c - u, V_{\gamma}^c) - B(V_{\gamma}^c - u, V_{\gamma}^c - u) - (W^c + V_{\gamma}^c)$$

$$= -B(V_{\gamma}^c, w) - B(w, V_{\gamma}^c) - B(w, w) - (W^c + V_{\gamma}^c).$$

Estimating by (14) with $\delta = 0$, considering that $||V_{\gamma}||_{2,\infty} = ||V||_{PM^2}, ||W^c + V_{\gamma}^c||_{PM^2} \leq 2||V^c||_{PM^2}$, we get, after some rearranging that $||w||_{2,\infty}$ satisfies the following quadratic inequality

$$\eta_0 ||w||_{2,\infty}^2 \geq (1 - 2\eta_0||V^c||_{PM^2}) ||w||_{2,\infty}^2 + 2||V^c||_{PM^2} \geq 0.$$

It follows that $||w||_{PM^2}$ cannot be between the roots of the corresponding quadratic equation; that is, we either have

$$||w||_{2,\infty} \leq \frac{1 - 2\eta_0||V^c||_{PM^2} - \sqrt{1 + 4\eta_0^2||V^c||_{PM^2}^2 - 12\eta_0||V^c||_{PM^2}}}{2\eta_0}$$

or

$$||w||_{2,\infty} \geq \frac{1 - 2\eta_0||V^c||_{PM^2} + \sqrt{1 + 4\eta_0^2||V^c||_{PM^2}^2 - 12\eta_0||V^c||_{PM^2}}}{2\eta_0},$$

assuming, as we will, that $|c|$ is large enough so that $1 - 2\eta_0||V^c||_{PM^2} > 0$. 
However, if $|\alpha|$ is sufficiently large we will have
\[
\|W^c\|_{PM^2} < \frac{1}{4\eta_0} \sqrt{1 + 4\eta_0^2 \|V^c\|_{PM^2}^2 - 12\eta_0 \|V^c\|_{PM^2}}
\]
and then
\[
\|w\|_{2,\infty} \leq \|w\|_{PM^2} + \|V^c\|_{PM^2} \leq 4\kappa + \|V^c\|_{PM^2} \leq \frac{1}{2\eta_0} \|V^c\|_{PM^2} < 2\eta_0 \|V^c\|_{PM^2}
\]
so the first possibility prevails. To simplify the notation, let us set
\[
\nu = \frac{1 - 2\eta_0 \|V^c\|_{PM^2} - \sqrt{1 + 4\eta_0^2 \|V^c\|_{PM^2}^2 - 12\eta_0 \|V^c\|_{PM^2}}}{2\eta_0}
\]
so that $\|w\|_{2,\infty} \leq \nu$. Returning to the first chain of equalities in this proof, we can now estimate
\[
\|w\|_{2,\infty} \leq 2\eta_0 \|V^c\|_{PM^2} + 2\eta_0 \|w\|_{2,\infty} + 2\|V^c\|_{PM^2},
\]
from which we can solve to get for all $t \geq 0$ the required inequality (30).

Let us again recall results from [9]. Assuming (as we will from now on) that $\gamma$ is H"older continuous with exponent $\alpha \in (\frac{1}{2}, 1)$, i.e. it satisfies
\[
|\gamma(t) - \gamma(s)| \leq A|t - s|^\alpha
\]
for some $\alpha \in (\frac{1}{2}, 1)$ and $A \geq 0$, it is proved in [9, Theorem 5.4] that $w(t) = u(t) - V^c_\gamma(t) \in PM^{2+\delta}$ for $\delta = 2\alpha - 1$ (so $0 < \delta \leq 1$) and $t > 0$, as long as $|\alpha|$ large enough. Notice that $V^c_\gamma(t) \notin PM^{2+\delta}$ if $\delta > 0$. Moreover, it is also proved in [9] that $\|w(t)\|_{PM^{2+\delta}} = O(t^{-\frac{3}{2}})$ for $t \to 0$. We need these results in a somewhat more precise form so we will prove them again in these pages.

**Lemma 3.4.** Let $\gamma : [0, \infty) \to \mathbb{R}^3$ be H"older continuous with exponent $\alpha \in (\frac{1}{2}, 1]$. Consider $W^c$ defined in (37) and $V^c_\gamma - (25)$. There exists a constant $\mu_0 = \mu_0(c) > 0$ satisfying $\lim_{|\alpha| \to \infty} \mu_0 = 0$ such that
\[
\|W^c(t) + V^c_\gamma(t)\|_{PM^{2+\delta}} \leq \begin{cases} \mu_0 t^{-\delta/2}, & \text{if } 0 < t \leq 1, \\ \mu_0, & \text{if } t > 1. \end{cases}
\]

**Proof.** We have
\[
\|\hat{W}^c(t) + \hat{V}^c_\gamma(t)\| = \|\hat{V}^c(\xi)\| \left| e^{-it\gamma(t) \cdot \xi} - \xi \right| \int_0^t e^{-(t-s)|\xi|^2 - i\gamma(s) \cdot \xi} ds
\]
\[
= |\hat{V}^c(\xi)| \left| e^{-t|\xi|^2 + i\gamma(t) \cdot \xi} + \xi \int_0^t e^{-(t-s)|\xi|^2} \left( e^{-i\gamma(s) \cdot \xi} - e^{-i\gamma(t) \cdot \xi} \right) ds \right|
\]
\[
\leq |\hat{V}^c(\xi)| \left( e^{-t|\xi|^2} + A|\xi|^3 \int_0^t e^{-(t-s)|\xi|^2} (t-s)^\alpha ds \right)
\]
\[
\leq |\hat{V}^c(\xi)| \left( e^{-t|\xi|^2} + C|\xi|^{1-2\alpha} \right)
\]
Taking $\delta = 2\alpha - 1$, we get
\[
|\xi|^{2+\delta} \|\hat{W}^c(t) + \hat{V}^c_\gamma(t)\| \leq |\xi|^2 |\hat{V}^c(\xi)| \left( |\xi|^\delta e^{-t|\xi|^2} + C \right) \leq C \|V^c\|_{PM^2} (1 + t^{-\delta/2}).
\]
The lemma follows with $\mu_0 = C \|V^c\|_{PM^2}$. \qed
We can now state the estimate we had in mind.

**Theorem 3.5.** Let $\gamma : [0, \infty) \to \mathbb{R}^2$ be Hölder continuous with exponent $\alpha \in (\frac{1}{2}, 1]$. Set $\delta = 2\alpha - 1$. There exists $c_0 > 0$ such that if $|c| \geq c_0$ then the function $w = u - V_{\gamma}^c$ satisfies

$$
\|w(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}} \leq \begin{cases} 
2\mu_0 t^{-\delta/2}, & \text{if } 0 < t \leq 1, \\
2\mu_0, & \text{if } t > 1.
\end{cases}
$$

(32)

**Proof.** We begin with a simple estimate that will be needed in our reasoning. Assume $0 < \delta < 1$. There exists a constant $C$ (depending on $\delta$) such that

$$
|\xi|^2 \int_0^t e^{-(t-s)}|\xi|^{2-\delta/2} ds \leq C t^{-\delta/2}
$$

(33)

for all $\xi \in \mathbb{R}^3$, $t > 0$. In fact, using the estimate $|\xi|^2 e^{-t}|\xi|^{2-\delta/2} \leq C t^{-\delta/2}$ for every $t > 0$, we get

$$
|\xi|^2 \int_0^{t/2} e^{-(t-s)}|\xi|^{2-\delta/2} ds \leq |\xi|^2 e^{-t/2} \int_0^{t/2} s^{-\delta/2} ds \leq C t^{-1}(t/2)^{1-\delta/2} = Ct^{-\delta/2},
$$

while

$$
|\xi|^2 \int_{t/2}^t e^{-(t-s)}|\xi|^{2-\delta/2} ds \leq |\xi|^2 (t/2)^{-\delta/2} \int_0^{t/2} e^{-t/2} ds \leq t^{-\delta/2}.
$$

Thus, inequality (33) follows.

We recall that, by Theorem 2.3 and remarks following it, the function $u = u(x,t)$, as a solution of the fixed point problem (13), is the limit in $\mathcal{P}\mathcal{M}^2$-norm of the sequence defined by

$$
u_1 = \kappa(\Phi - \Psi), \quad u_{n+1} = \kappa(\Phi - \Psi) - B(u_n, u_n) \quad \text{for } n \geq 1.
$$

Setting $w_n = u_n - V_{\gamma}^c$ and using the same type of decomposition used at the beginning of the proof of Theorem 3.3, we have that

$$
w_{n+1} = -(V_c^c + V_{\gamma}^c) - B(V_{\gamma}^c, w_n) - B(w_n, V_{\gamma}^c) - B(w_n, w_n).
$$

We proceed by induction. Consider $0 < t \leq 1$, first, and assume the inequality $\|w_n(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}} \leq \mu t^{-\delta/2}$ for some $n \geq 1$, $\mu > 0$. In view of inequalities (18) and (33),

$$
\|B(V_{\gamma}^c, w_n)(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}}
\leq C_\delta \mu \|V_c^c\|_{\mathcal{P}\mathcal{M}^2} \sup_{\xi \in \mathbb{R}^3} \int_0^{t} |\xi|^2 e^{-(t-s)}|\xi|^{\gamma} s^{-\delta/2} ds \leq C_\delta \mu \|V_c^c\|_{\mathcal{P}\mathcal{M}^2} t^{-\delta/2}.
$$

We use similar estimates for $B(w_n, V_{\gamma}^c)$ and $B(w_n, w_n)$ obtaining

$$
\|w_{n+1}(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}} \leq \mu_0 t^{-\delta/2} + C_\delta \mu \left(\|V_c^c\|_{\mathcal{P}\mathcal{M}^2} + \|w_n\|_{L^2,\infty} t^{-\delta/2}.
$$

The expression $\|V_c^c\|_{\mathcal{P}\mathcal{M}^2} + \|w_n\|_{L^2,\infty}$ can be made arbitrarily small by selecting $|c|$ large enough to obtain

$$
2\mu_0 \left(\|V_c^c\|_{\mathcal{P}\mathcal{M}^2} + \|w_n\|_{L^2,\infty}\right) \leq \mu_0.
$$

With this choice of $c$, we see that the assumption $\|w_n(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}} \leq 2\mu_0 t^{-\delta/2}$ implies the inequality $\|w_{n+1}(t)\|_{\mathcal{P}\mathcal{M}^{2+\delta}} \leq 2\mu_0 t^{-\delta/2}$. 

For \( t > 1 \), we proceed similarly. Using the inequality
\[
|\xi|^2 \int_0^t e^{-(t-s)|\xi|^2} \max(1, s^{-\delta/2}) \, ds \leq C \quad \text{for all } t > 0, \, \xi \in \mathbb{R}^3
\]
with some constant \( C > 0 \), we get that if \(|\xi|\) is large enough, then \( \| w_{n+1}(t) \|_{\mathcal{P}M^{2+s}} \leq 2\mu_0 \), completing the proof of the theorem.

**Remark 3.** If we restrict our attention to the solution in an interval \( 0 \leq t \leq T \) with \( 0 < T < \infty \), we can define \( C(T) = \max \{ \mu_0, \mu_0 T^{3/2} \} \) and then we can state the result of Theorem 3.5 in the following form. There exists a constant \( C(T) \), depending on \( T \) such that
\[
\| w(t) \|_{\mathcal{P}M^{2+s}} \leq C(T) t^{-\delta/2}, \quad 0 < t \leq T.
\]
Assuming \( q \geq 2 \), recalling that \( w(t) \in \mathcal{P}M^2 \) for all \( t \geq 0 \), we have the following inequalities with \( a_q = (2\pi)^{-\frac{3}{4} + \frac{q'}{q}} \)
\[
\| w(t) \|_{L^q(\mathbb{R}^3)} \leq a_q \| \hat{w}(t) \|_{L^{q'}(\mathbb{R}^3)} = a_q \left( \int_{|\xi| \leq 1} |\hat{w}(\xi, t)|^{q'} \, d\xi + \int_{|\xi| \geq 1} |\hat{w}(\xi, t)|^{q'} \, d\xi \right)^{\frac{1}{q'}} \\
\leq a_q \left( \| w(t) \|_{\mathcal{P}M^2}^{q'} \int_{|\xi| \leq 1} |\xi|^{-2q'} \, d\xi + \| w(t) \|_{\mathcal{P}M^{2+s}}^{q'} \int_{|\xi| \geq 1} |\xi|^{-2(q+\delta)q'} \, d\xi \right)^{\frac{1}{q'}} \\
\leq C(T) \left( \| w(t) \|_{\mathcal{P}M^2} + \| w(t) \|_{\mathcal{P}M^{2+s}} \right)
\]
where \( C(T) < \infty \) as long as \( 2q' < 3 < (2 + \delta)q' = (2\alpha + 1)q' \); in other words, as long as \( 3 < q < 3/(2(1 - \alpha)) \).

4. Estimates of the \( L^2(\mathbb{R}^3) \)-norm. Now, we are in a position to show the main result of this work asserting \( L^2 \)-estimates of solutions to the singular Navier-Stokes initial value problem (4). We begin with a simple estimate that will be needed below.

**Lemma 4.1.** There exists \( \lambda > 0 \) such that
\[
\int_0^1 (t-s)^{\frac{3}{2} - 1}s^{-\frac{3}{2} + \frac{1}{4}} \, ds \leq \lambda t^{-1 + \frac{1}{4}} \quad \text{for all } t \geq 1.
\]

**Proof.** This estimate follows immediately from the fact that the functions
\[
h(t) = \int_0^1 (t-s)^{\frac{3}{2} - 1}s^{-\frac{3}{2} + \frac{1}{4}} \, ds = t^{\frac{5}{2}} \int_0^{1/t} (1-s)^{\frac{3}{2} - 1}s^{-\frac{3}{2} + \frac{1}{4}} \, ds,
\]
by L’Hôpital’s rule, satisfies
\[
\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{\int_0^{1/t} (1-s)^{\frac{3}{2} - 1}s^{-\frac{3}{2} + \frac{1}{4}} \, ds}{t^{-\frac{5}{2} + \frac{1}{4}}} = \lim_{\sigma \to 0^+} \frac{\int_0^\sigma (1-s)^{\frac{3}{2} - 1}s^{-\frac{3}{2} + \frac{1}{4}} \, ds}{\sigma^{-\frac{5}{2} + \frac{1}{4}}} \\
= \lim_{\sigma \to 0^+} \frac{(1-\sigma)^{\frac{3}{2} - 1}\sigma^{-\frac{3}{2} + \frac{1}{4}}}{(\frac{3}{4} - \frac{3}{2})\sigma^{-\frac{3}{2} + \frac{1}{4}}} = \left( \frac{5}{4} - \frac{\delta}{2} \right)^{-1}.
\]
We are in a position to prove the main result of this work.
Theorem 4.2. There exists $c_0 > 0$ such that if $|c| > c_0$ then the solution of the singular problem (4) exists and satisfies $u \in C([0, \infty), L^2(\mathbb{R}^3))$. Moreover, with $\mu_0$ as defined in Lemma 3.4, we have

$$
\|u(t)\|_{L^2(\mathbb{R}^3)} \leq (\mu_0 + 1) t^{1/2}, \quad \text{for } 0 \leq t \leq 1.
$$

(36)

Proof. Let $w = u - V_0$ as usual. Define a sequence $\{v_n\} \subset C([0, \infty), L^2(\mathbb{R}^3) \cap \mathcal{P}M^2)$ inductively by

$$
v_0 = -W^c \quad \text{and} \quad v_{n+1} = -W^c - B(w, v_n) - B(v_n, w) \quad \text{for } n \geq 0.
$$

Using estimates which lead to the proof of Theorem 2.3 it is easy to see that this sequence converges in $\mathcal{Y}_2^1 = C([0, \infty), \mathcal{P}M^2)$. By the uniqueness of the fixed point of the equation

$$
u = -W^c - B(w, u) - B(u, w)
$$

(if $|c|$ is large enough), it follows that the limit is $u = u(x, t)$. We are going to prove that the sequence $\{v_n\}$ also stays and converges in $L^2(\mathbb{R}^3)$.

Indeed, since

$$
|\hat{W}^c(\xi, t)| \leq |\xi|^2 |\hat{V}^c(\xi)| \int_0^t e^{-(t-s)|\xi|^2 - i\gamma(s)\cdot \xi} \, ds \leq \|V^c\|_{\mathcal{P}M^2} \left( \frac{1 - e^{-t|\xi|^2}}{|\xi|^2} \right),
$$

we get

$$
\|W^c(t)\|_{L^2(\mathbb{R}^3)} = \|W^c(t)\|_{L^2(\mathbb{R}^3)} \leq \|V^c\|_{\mathcal{P}M^2} \left( \int \left( \frac{1 - e^{-t|\xi|^2}}{|\xi|^2} \right)^2 \, d\xi \right)^{1/2} = C t^{1/4} \|V^c\|_{\mathcal{P}M^2}.
$$

Recalling the definition of $\mu_0$ in Lemma 3.4, we can write this, with a possibly slightly larger $\mu_0$, in the form

$$
\|W^c(t)\|_{L^2(\mathbb{R}^3)} \leq \mu_0 t^{1/4}.
$$

(37)

This shows that $v_0(t) = -W^c(t)$ is in $L^2(\mathbb{R}^3)$ for all $t \geq 0$ and it is clearly continuous as a function of $t$ with values in $L^2(\mathbb{R}^3)$. Setting $v_{-1} \equiv 0$ we get, in view of Theorem 3.5 and (16),

$$
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq \|B(w, v_n - v_{n-1})(t)\|_{L^2(\mathbb{R}^3)} + \|B(v_n - v_{n-1}(t), w)(t)\|_{L^2(\mathbb{R}^3)}
\leq 2\mu_0 C_\delta \int_0^t (t-s)^{\delta - 1} \max(1, s^{-\delta}) \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} \, ds
$$

(38)

for $n = 0, 1, 2, \ldots$.

Let

$$
\rho = 2\mu_0 C_\delta \max \left( B \left( \frac{\delta}{2}, \frac{5}{4} - \frac{\delta}{2} \right), \lambda \right).
$$

(39)

where $B$ is the Beta function, the number $C_\delta$ is as in (16), and $\lambda$ as as in (35). We select $|c|$ large enough (hence $\mu_0$ small enough) so that $\rho < 1$, and $\rho/(1 - \rho) \leq 1$.

Assume first that $0 \leq t \leq 1$. We prove by induction on $n \geq 0$ that

$$
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq \rho^{n+1} t^{1/2}.
$$

(40)

By (38) with $n = 0$ and by the estimate on $W^c = v_0$, we get

$$
\|v_1(t) - v_0(t)\|_{L^2(\mathbb{R}^3)} \leq 2\mu_0 C_\delta \int_0^t (t-s)^{\delta - 1} s^{-\delta} s^{1/2} \, ds = 2\mu_0 C_\delta B \left( \frac{\delta}{2}, \frac{5}{4} - \frac{\delta}{2} \right) t^{1/4}
$$

(41)
proving the case \( n = 0 \) in inequality (40). Assuming (40) to be proved up to \( n - 1 \) for some \( n \geq 1 \), we have by (38) and the induction hypothesis,

\[
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq 2\mu_0 C_\delta \int_0^t (t-s)^{\frac{5}{2} - \frac{1}{2}} s^{-\frac{5}{2}} \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq 2\mu_0 C_\delta \rho^n \int_0^t (t-s)^{\frac{5}{2} - \frac{1}{2}} s^{-\frac{5}{2}} s^{\frac{1}{2}} ds = \rho^n t^\frac{1}{2}
\]

completing the proof of inequality (40). Because \( \rho < 1 \), the sequence \( \{v_n\} \) converges uniformly in \( C([0,1], L^2(\mathbb{R}^3)) \) to a function \( v \in C([0,1], L^2(\mathbb{R}^3)) \). As mentioned above, \( v = u \) and this also proves

\[
\|u(t)\|_{L^2(\mathbb{R}^3)} \leq \|v_0(t)\|_{L^2(\mathbb{R}^3)} + \sum_{n=0}^\infty \|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)}
\]

\[
\leq \left( \mu_0 + \sum_{n=1}^\infty \rho^n \right) t^\frac{1}{2} = \left( \mu_0 + \frac{\rho}{1-\rho} \right) t^\frac{1}{2},
\]

and estimate (36) follows since \( \rho/(1-\rho) \leq 1 \).

Consider next \( t \geq 1 \). By estimates (38) and (40), we have

\[
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq 2\mu_0 C_\delta \int_0^1 (t-s)^{\frac{5}{2} - \frac{1}{2}} s^{-\frac{5}{2}} \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} ds
\]

\[
+ 2\mu_0 C_\delta \int_1^t (t-s)^{\frac{5}{2} - \frac{1}{2}} \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq 2\mu_0 C_\delta \rho^n \int_0^1 (t-s)^{\frac{5}{2} - \frac{1}{2}} s^{-\frac{5}{2}} + \frac{1}{2} ds
\]

\[
+ 2\mu_0 C_\delta \int_1^t (t-s)^{\frac{5}{2} - \frac{1}{2}} \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} ds.
\]

Then, by inequality (35) and the definition of \( \rho \),

\[
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq \rho^{n+1} t^{-1 + \frac{5}{2}} + 2\mu_0 C_\delta \int_1^t (t-s)^{\frac{5}{2} - 1} \|v_n(s) - v_{n-1}(s)\|_{L^2(\mathbb{R}^3)} ds.
\]

(41)

Define the following sequences for \( n = 1, 2, 3 \ldots \) and \( k = 1, 2, 3, \ldots \)

\[
a_n = \mu_0 \frac{(2C_\delta \mu_0 \Gamma \left( \frac{5}{4}\right))^n \Gamma \left( \frac{5}{4} + \frac{k}{2}\right)}{\Gamma \left( \frac{5}{4} + \frac{k}{2}\right)}, \quad b_k = \frac{\left( \Gamma \left( \frac{5}{4}\right)\right)^k}{\Gamma \left( \frac{5}{4} + \frac{k}{2}\right)}.
\]

We prove by induction on \( n \geq 1 \) that

\[
\|v_n(t) - v_{n-1}(t)\|_{L^2(\mathbb{R}^3)} \leq a_n t^{\frac{1}{2} + \frac{\alpha}{2}} + \rho^n \sum_{k=1}^n b_k t^{-1 + \frac{\alpha}{2}}.
\]

(42)

Note that for \( \alpha, \beta > 0 \) and \( t \geq 1 \), we have

\[
\int_1^t (t-s)^{\alpha-1} s^{\beta-1} \leq \int_0^t (t-s)^{\alpha-1} s^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} t^{\alpha + \beta - 1}.
\]
Using inequalities (41) with \( n = 0 \) and (37) (and recalling that \( v_0 = -W^c \)), we obtain the estimates

\[
\|v(t) - v_0(t)\|_{L^2(\mathbb{R}^3)} \leq \rho t^{-1 + \frac{n}{4}} + 2\mu_0 C_\delta a_0 \int_1^t (t-s)^{\frac{3}{4}-1} s^\frac{n}{4} ds \\
\leq \rho t^{-1 + \frac{n}{4}} + 2\mu_0^2 C_d \frac{\Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} \right) t^{\frac{1}{4} + \frac{n}{4}}}{\Gamma \left( \frac{3}{4} + \frac{(n+1)\delta}{2} \right)} \\
= b_1 \rho t^{-1 + \frac{n}{4}} + a_1 t^{\frac{1}{4} + \frac{n}{4}},
\]
proving the case \( n = 1 \) of (42). Now, assume (42) to be proved up to some \( n \geq 1 \). Then inequality (41) implies

\[
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq \rho^{n+1} t^{-1 + \frac{n}{4}} + 2\mu_0 C_\delta a_n \int_1^t (t-s)^{\frac{3}{4}-1} s^\frac{n}{4} ds \\
+ 2\mu_0 C_\delta \rho^n \sum_{k=1}^n b_k \int_1^t (t-s)^{\frac{3}{4}-1} s^\frac{n}{4} ds \\
\leq \rho^{n+1} t^{-1 + \frac{n}{4}} + 2\mu_0 C_\delta a_n \frac{\Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right) t^{1 + \frac{(n+1)\delta}{2}}}{\Gamma \left( \frac{3}{4} + \frac{(n+1)\delta}{2} \right)} \\
+ 2\mu_0 C_\delta \rho^n \sum_{k=1}^n b_k \frac{\Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right) t^{1 + \frac{(n+1)\delta}{2}}}{\Gamma \left( \frac{3}{4} + \frac{(n+1)\delta}{2} \right)}.
\]

Using the relations

\[
b_1 = 1, \quad 2\mu_0 C_\delta a_n \frac{\Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right)}{\Gamma \left( \frac{3}{4} + \frac{(n+1)\delta}{2} \right)} = a_{n+1},
\]
and

\[
2\mu_0 C_\delta \rho^n b_k \frac{\Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right)}{\Gamma \left( \frac{3}{4} + \frac{(n+1)\delta}{2} \right)} = 2\mu_0 C_\delta \rho^n b_{k+1} \leq \rho^{n+1} b_{k+1},
\]
we proved that

\[
\|v_{n+1}(t) - v_n(t)\|_{L^2(\mathbb{R}^3)} \leq a_{n+1} t^{1 + \frac{(n+1)\delta}{2}} + \rho^{n+1} \sum_{k=1}^n b_k t^{-1 + \frac{n\delta}{2}},
\]
which is the \( n+1 \) case of (42). The proof by induction is complete.

It is easy to see that the following formulas hold true

\[
\sum_{n=1}^\infty a_n t^{1 + \frac{n\delta}{2}} = \mu_0 \frac{\Gamma \left( \frac{5}{4} \right) t^{\frac{1}{4}}}{\Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right)} \sum_{n=1}^\infty \left( \frac{2C_\delta \mu_0 \Gamma \left( \frac{5}{4} \right) t^{\frac{1}{4}}} {\Gamma \left( \frac{3}{4} + \frac{n\delta}{2} \right)} \right)^n < t
\]
and

\[
\sum_{n=1}^\infty \rho^n \sum_{k=1}^n b_k t^{-1 + \frac{k\delta}{2}} = \frac{1}{t(1-\rho)} \sum_{k=1}^\infty b_k \left( \rho t^{\frac{1}{2}} \right)^k = \frac{\lambda}{t(1-\rho)} \sum_{k=1}^\infty \left( \frac{\Gamma \left( \frac{5}{4} \right) \rho t^{\frac{1}{2}}} {\Gamma \left( \frac{3}{4} + \frac{k\delta}{2} \right)} \right)^k < \infty
\]
for all \( t > 0 \). This proves that the series

\[
\sum_{n=1}^\infty \|v_n(t) - v_{n-1}(t)\|_{L^2(\mathbb{R}^3)}
\]
converges uniformly in finite subintervals of \([1, \infty)\) proving that the limit \(u\) of the sequence \(\{v_n\}\) is also a continuous \(L^2(\mathbb{R}^3)\) valued function in \([1, \infty)\).

REFERENCES

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, paperback ed., 1999.

[2] M. Cannone and G. Karch, Smooth or singular solutions to the Navier-Stokes system?, *J. Differential Equations*, 197 (2004), 247–274.

[3] A. Decaster and D. Ifimie, On the asymptotic behaviour of solutions of the stationary Navier-Stokes equations in dimension 3, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34 (2017), 277–291.

[4] R. Farwig, G. P. Galdi and M. Kyed, Asymptotic structure of a Leray solution to the Navier-Stokes flow around a rotating body, *Pacific J. Math.*, 253 (2011), 367–382.

[5] V. A. Galaktionov, On blow-up “twistors” for the Navier–Stokes equations in \(\mathbb{R}^3\): A view from reaction-diffusion theory, preprint, arXiv:0901.4286.

[6] K. Kang, H. Miura and T.-P. Tsai, Asymptotics of small exterior Navier-Stokes flows with non-decaying boundary data, *Comm. Partial Differential Equations*, 37 (2012), 1717–1753.

[7] G. Karch and D. Pilarczyk, Asymptotic stability of Landau solutions to Navier-Stokes system, *Arch. Ration. Mech. Anal.*, 202 (2011), 115–131.

[8] G. Karch, D. Pilarczyk and M. E. Schonbek, \(L^2\)-asymptotic stability of singular solutions to the Navier-Stokes system of equations in \(\mathbb{R}^3\), *J. Math. Pures Appl.*, 108 (2017), 14–40.

[9] G. Karch and X. Zheng, Time-dependent singularities in the Navier-Stokes system, *Discrete Contin. Dyn. Syst.*, 35 (2015), 3039–3057.

[10] A. Korolev and V. Šverák, On the large-distance asymptotics of steady state solutions of the Navier-Stokes equations in 3D exterior domains, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 303–313.

[11] L. Landau, A new exact solution of Navier-Stokes equations, *C.R. (Doklady) Acad. Sci. URSS (N.S.)*, 43 (1944), 286–288.

[12] P. G. Lemarié-Rieusset, *The Navier-Stokes Problem in the 21st Century*, CRC Press, Boca Raton, FL, 2016.

[13] H. Miura and T.-P. Tsai, Point singularities of 3D stationary Navier-Stokes flows, *J. Math. Fluid Mech.*, 14 (2012), 33–41.

[14] S. Sato and E. Yanagida, Solutions with moving singularities for a semilinear parabolic equation, *J. Differential Equations*, 246 (2009), 724–748.

[15] S. Sato and E. Yanagida, Forward self-similar solution with a moving singularity for a semilinear parabolic equation, *Discrete Contin. Dyn. Syst.*, 26 (2010), 313–331.

[16] S. Sato and E. Yanagida, Singular backward self-similar solutions of a semilinear parabolic equation, *Discrete Contin. Dyn. Syst. Ser. S*, 4 (2011), 897–906.

[17] S. Sato and E. Yanagida, Appearance of anomalous singularities in a semilinear parabolic equation, *Commun. Pure Appl. Anal.*, 11 (2012), 387–405.

[18] S. Sato and E. Yanagida, Asymptotic behavior of singular solutions for a semilinear parabolic equation, *Discrete Contin. Dyn. Syst.*, 32 (2012), 4027–4043.

[19] N. A. Slëzkin, On an integrability case of full differential equations of the motion of a viscous fluid, *Moskov. Gos. Univ. Uč. Zap.*, 2 (1934), 89–90.

[20] V. Šverák, On Landau’s solutions of the Navier-Stokes equations, *J. Math. Sci. (N.Y.)*, 179 (2011), 208–228.

[21] J. Takahashi and E. Yanagida, Time-dependent singularities in the heat equation, *Commun. Pure Appl. Anal.*, 14 (2015), 969–979.

[22] ———, Time-dependent singularities in a semilinear parabolic equation with absorption, *Commun. Contemp. Math.*, 18 (2016), 1550077, 27pp.

Received December 2018; revised June 2019.

E-mail address: grzegorz.karch@uwr.edu.pl
E-mail address: maria.schonbek@gmail.com
E-mail address: schonbek@fau.edu