THE EULER EVALUATION ON MV-ALGEBRAS

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Abstract. Every finitely presented MV-algebra A has a unique idempotent valuation E assigning value 1 to every basic element of A. For each a ∈ A, E(a) turns out to coincide with the Euler characteristic of the open set of maximal ideals m of A such that a/m is nonzero.

To Ján Jakubík, on his 90th birthday

1. Introduction

For all unexplained notions concerning MV-algebras we refer to [3] and [8]. We refer to [6] for background on algebraic topology. A rational polyhedron in [0, 1]d is a finite union of simplexes with rational vertices in [0, 1]d. As shown in [5] §4 and [8] §3.4, rational polyhedra are dually equivalent to finitely presented MV-algebras. In the light of [8, 6.3], any finitely presented MV-algebra A can be identified with the MV-algebra M(P) of all McNaughton functions over some rational polyhedron P ⊆ [0, 1]d, for some d = 1, 2, . . . . In [2] and [8, 6.3] it is proved that A is finitely presented iff it has a basis B. An element x ∈ A is said to be basic if it belongs to some basis of A. We let µ(A) denote the maximal spectral space of the MV-algebra A. By [8, 4.16], for every maximal ideal m ∈ µ(A) the quotient MV-algebra A/m is uniquely isomorphic to a subalgebra J of the standard MV-algebra [0, 1]. Identifying A/m and J, for any a ∈ A, the element a/m becomes a real number. We write support(a) = {m ∈ µ(A) | a/m > 0}. When A = M(P) is finitely presented, support(a) is homeomorphic to the complementary set in P of a rational polyhedron in [0, 1]d (see [8, §4.5, 6.2]). By definition, the Euler characteristic χ(support(a)) is the alternating sum of the Betti numbers of support(a), as given by singular homology theory. This is homotopy invariant. For any piecewise linear continuous function l : [0, 1]d → [0, 1] the set \{x ∈ [0, 1]d | l(x) > 0\} is homotopy equivalent to \{x ∈ [0, 1]d | l(x) ≥ ε\} for all small enough ε > 0: an exercise in (piecewise linear) Morse theory. [11] shows that the latter is a deformation retract of the former. As a consequence,

\[ \chi(\{x ∈ [0, 1]^d | l(x) > 0\}) = \chi(\{x ∈ [0, 1]^d | l(x) ≥ ε\}), \forall ε > 0 \text{ small enough.} \quad (1) \]

As an MV-algebra variant of the main result of [9], in this paper we prove:

Theorem 1.1. For any finitely presented MV-algebra A let the map E: A → Z be given by E(a) = χ(support(a)), for all a ∈ A. Then E has the following properties:

(i) E(0) = 0.
(ii) (Normalization) E(b) = 1 for each basic element b of A.
(iii) (Idempotency) For all p, q ∈ A, E(p ⊕ q) = E(p ∨ q).
(iv) (Additivity) E is a valuation: for all p, q ∈ A, E(p ∨ q) = E(p) + E(q) − E(p ∧ q).

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Conversely, properties (i)-(iv) uniquely characterize $E$ among all real-valued functions defined on $A$.

The main reason of interest for this result, and of distinction from $[9]$, is the combination of MV-algebraic base theory $[8]$ §§5,6 with piecewise linear Morse theory $[11,13]$ to the construction of a Turing computable Euler valuation on every finitely presented MV-algebra $A$, and its characterization as the Euler characteristic $\chi$ of the support of all $a \in A$. The proofs in this paper use sophisticated MV-algebraic techniques that are not available in the context of Riesz spaces of $[9]$.

2. MV-algebraic lemmas

Lemma 2.1. ($[8]$ 6.3 and $[8]$ 6.4) A is a finitely presented MV-algebra iff $A$ has a basis off $A$ has the form $M(P)$ for some rational polyhedron $P \subseteq [0,1]^d$ in some euclidean space $\mathbb{R}^d$. Let $B$ be a basis of $A$. Then $P$ can be chosen so that there is a regular triangulation $\Delta$ of $P$ and an isomorphism of $A$ onto $M(P)$ sending the elements of $B$ one-one onto the elements of the Schauder basis $\mathcal{H}_\Delta$.

Lemma 2.2. Let $B = \mathcal{H}_\Delta$ be a Schauder basis of an MV-algebra $A = M(P)$, for some regular triangulation $\Delta$ of $P$. Let $b_0,b_1,\ldots,b_u$ be distinct elements of $B$, with their respective vertices $v_0, v_1, \ldots, v_u$. Let us define $b^0_0 = b_0$, $B^0 = B$, $\Delta(0) = \Delta$, and inductively, $c_k = b_k \wedge b^0_k^{-1}$, $b^0_k = b^0_k - c_k$, $a_k = b_k \ominus c_k$, and

$$
B^k = (B^k \setminus \{b^0_k, b_k\}) \cup \{a_k, b^0_k\} \cup \{c_k, \text{ if } c_k \neq 0\}, \quad k = 1, \ldots, u.
$$

(i) Then $b_0 \wedge \bigoplus_{i=0}^u b_i$ coincides with $c_1 \oplus \cdots \oplus c_u$, where any two nonzero $c_i, c_j$ are distinct elements of $B^u$.

(ii) For each $k = 1, \ldots, u$, $B^k = \mathcal{H}_{\Delta(k)}$, where the regular triangulation $\Delta(k)$ is obtained by blowing up $\Delta(k-1)$ at the Farey mediant $w_k$ of the segment $\text{conv}(v_0, v_k)$, provided $\text{conv}(v_0, v_k) \in \Delta(k-1)$. Otherwise, $\Delta(k) = \Delta(k-1)$.

Proof. The proof proceeds by induction on $k = 1, \ldots, u$. If $k = 1$, by definition, $b_0 \wedge b_1 = b_0 \wedge b_1 = c_1$. If $c_1 = 0$, then upon letting $B^1 = B$, (i)-(ii) are settled. If $c_1 \neq 0$, then $B^1 = B \setminus \{b_0, b_1\} \cup \{c_1, b_0 \ominus c_1, b_1 \ominus c_1\}$ is the Schauder basis $\mathcal{H}_{\Delta(1)}$ of the regular triangulation $\Delta(1)$ of $P$ obtained from $\Delta$ via a Farey blow-up at the Farey mediant $w_1$ of the 1-simplex $\text{conv}(v_0, v_1)$ of $\Delta$. Trivially, (i) is satisfied. By $[9]$ 9.2.1, $B^1 = \mathcal{H}_{\Delta(1)}$ satisfies (ii).

Inductively, assume the claim holds for all $l < k$. In particular, $b_0 \wedge \bigoplus_{i=1}^{k-1} b_i = \bigoplus_{i=1}^{k-1} c_i$, with $c_1, \ldots, c_{k-1} \in B^{k-1} \cup \{0\}$. Further, $B^{k-1}$ is a basis of $A$.

We will prove (i) arguing pointwise for each $p \in P$:

Case 1: $b_0 \leq \bigoplus_{i=1}^{k-1} b_i \leq \bigoplus_{i=1}^{k} b_i$.

If there is $j \in \{1, \ldots, k-1\}$ such that $b^{0,j-1}_0 \leq b_j$ at $p$, then $c_j = b^{0,j}_0$, and $b^{0,j}_0 - c_j = 0$, for all $i > j$. In particular $c_k = 0$. Therefore, $b_0 \wedge \bigoplus_{i=1}^{k} b_i = b_0 \wedge \bigoplus_{i=1}^{k-1} b_i = \bigoplus_{i=1}^{k-1} c_i = \bigoplus_{i=1}^{k} c_i$, at $p$, whence the identity (i) trivially holds at $p$.

If there is no $j \in \{1, \ldots, k-1\}$ such that $b^{0,j-1}_0 \leq b_j$ at $p$ then $b_i < b^{0,i}_0 \leq b_0$ for all $i = 1, \ldots, k-1$. As a consequence, $c_i = b_i$ and $b^{0}_0 = b_0 - \bigoplus_{i=1}^{k} b_i$, for all $i = 1, \ldots, k-1$. It follows that $b^{0,k-1}_0 = b_0 - \bigoplus_{i=1}^{k-1} b_i = 0$, and again $c_k = 0$. Then $b_0 \wedge \bigoplus_{i=1}^{k} b_i = b_0 = b_0 \wedge \bigoplus_{i=1}^{k-1} b_i = \bigoplus_{i=1}^{k-1} c_i = \bigoplus_{i=1}^{k} c_i$, and (i) is satisfied at $p$.

Case 2: $\bigoplus_{i=1}^{k-1} b_i \leq \bigoplus_{i=1}^{k} b_i < b_0$.

By way of contradiction, suppose there is a (smallest) index $j$ such that $b^{0,j-1}_0 < b_j$.

Then $j > 1$, $b_0 \leq b^{0,j-1}_0$, and $c_i = b_i$, for all $i < j$. As a consequence, $b^{0}_0 = b_0 - \bigoplus_{i=1}^{j-1} b_i$, whence $b^{0,j}_0 - b_j = b_0 - \bigoplus_{i=1}^{j-1} b_i > 0$ and $b^{0,j}_0 > b_j$, which is a contradiction.
We have just shown that \( b_i \leq b_0^{k-1} \) for all \( i = 1, \ldots, k \). In particular \( c_k = b_k \), whence 
\[
\bigoplus_{i=1}^k b_i = \bigoplus_{i=1}^k b_i = \bigoplus_{i=1}^{k-1} b_i \oplus b_k = (b_0 \bigoplus \bigoplus_{i=1}^{k-1} b_i) \oplus b_k = \bigoplus_{i=1}^{k-1} c_i \oplus c_k = \bigoplus_{i=1}^k c_i,
\]
thus showing that identity (i) holds at \( p \).

Case 3: \( \bigoplus_{i=1}^{k-1} b_i < b_0 \leq \bigoplus_{i=1}^k b_i \).

As in the previous case, for all \( i = 1, \ldots, k-1 \) we have \( b_i \leq b_0^{k-1} \), whence \( c_i = b_i \) and \( b_0^k = b_0 - \bigoplus_{i=1}^{k-1} b_i \). Moreover, \( b_0^{k-1} = b_0 - \bigoplus_{i=1}^{k-1} b_i \) and \( c_k = b_k \bigcap (b_0 - \bigoplus_{i=1}^{k-1} b_i) \).

Therefore, \( b_0 \bigoplus \bigoplus_{i=1}^{k-1} b_i = (b_0 - \bigoplus_{i=1}^{k-1} b_i) \bigcap (b_k \bigoplus \bigoplus_{i=1}^{k-1} b_i) = ((b_0 - \bigoplus_{i=1}^{k-1} b_i) \bigcap b_k) \bigoplus \bigoplus_{i=1}^{k-1} c_i = \bigoplus_{i=1}^k c_i \), which shows that (i) holds at \( p \).

We have shown that identity (i) holds on \( P \).

We now prove that \( B^k \) satisfies (ii). By induction hypothesis, \( B^k = \mathcal{H}_{\Delta(k-1)} \),
where \( \Delta(k-1) \) is obtained from \( \Delta(0) = \Delta \) via Farey blow-ups.

If \( c_k = 0 \), then \( B^k = B^{k-1} = \mathcal{H}_{\Delta(k-1)} \). We show that \( \Delta(k-1) = \Delta(k) \).

By way of contradiction, let \( \text{conv}(v_0, v_k) \) be a simplex of \( \Delta(k-1) \).

Since both \( b_0^{k-1} \) and \( b_k \) are linear on \( \text{conv}(v_0, v_k) \), then \( v_0 \) is the unique point of \( \text{conv}(v_0, v_k) \) where \( b_k \) vanishes, and \( v_k \) is the unique point of \( \text{conv}(v_0, v_k) \) where \( b_0 \) vanishes.

Therefore, \( c_k > 0 \) on the (nonempty) set \( \text{conv}(v_0, v_k) \setminus \{v_0, v_k\} \), a contradiction.

Having thus shown that \( \text{conv}(v_0, v_k) \notin \Delta(k-1) \), we obtain \( \Delta(k-1) = \Delta(k) \) and \( B^k = \mathcal{H}_{\Delta(k)} \).

If \( c_k \neq 0 \) then by [3] 9.2.1 \( B^k = \mathcal{H}_{\Delta(k)} \).

The proof is complete. \( \square \)

**Remark 2.3.** In Lemma \([2,2]\) each \( \oplus \) operation is only applied to Schauder hats belonging to the same basis. Therefore, each \( \oplus \) symbol in the statement and in the proof of the lemma can be replaced by the addition \( + \) symbol. Similarly, in all expressions \( b_0^{k-1} \oplus c_k \) and \( b_k \oplus c_k \), each \( \oplus \) operation can be replaced by the subtraction \( - \) operation, because it is always the case that \( c_k \leq b_0^{k-1} \) and \( c_k \leq b_k \).

### 3. The open support of a finitely presented MV-algebra

Let \( A \) be a finitely presented MV-algebra. By Lemma \([2,3]\) we can identify \( A \) with \( \mathcal{M}(P) \) for some rational polyhedron \( P \subseteq [0,1]^d \). For each element \( a \in A \) the rational polyhedron \( \text{oneset}(a) \subseteq [0,1]^d \) is defined by

\[
\text{oneset}(a) = \{ x \in P \mid a(x) = 1 \} = \{ m \in \mu(A) \mid a/m = 1 \}.
\]

For each \( \lambda \in [0,1] \), we also let

\[
\text{uplevelset}_\lambda(a) = \{ x \in P \mid a(x) \geq \lambda \} = \{ m \in \mu(A) \mid a/m \geq \lambda \}.
\]

Following [8], we let \( 2 \cdot a = a \oplus a \) and inductively, \( (n+1) \cdot a = a \oplus n \cdot a \).

**Lemma 3.1.** With the above notation we have

\[
\chi(\mu(A)) = \chi(\text{oneset}(1)) = \chi(\text{support}(1)) = \chi(P). \tag{2}
\]

Further, for any \( a \in A \), the open set support \( \text{support}(a) \) is homotopy equivalent to the rational polyhedron \( \text{uplevelset}_{1/n}(a) \), for all large integers \( n \); as a matter of fact, the latter is a deformation retract of the former. Thus, in particular,

\[
\chi(\text{support}(a)) = \lim_{n \to \infty} \chi(\text{oneset}(n \cdot a)).
\]

**Proof.** The verification of \([2]\) is trivial. To prove the remaining statements, for all \( 0 < m \leq n \) we have \( \text{oneset}(m \cdot a) \subseteq \text{oneset}(n \cdot a) \). For all large \( n \), \( \text{oneset}((n + 1) \cdot a) \) collapses into \( \text{oneset}(n \cdot a) \). Thus all these rational polyhedra are homotopic, and \( \lim_{n \to \infty} \chi(\text{oneset}(n \cdot a)) = \lim_{n \to \infty} \chi(\text{uplevelset}_{1/n}(a)) \). (Intuitively, a retraction is given by the map sending each point at the boundary of \( \text{oneset}((n + 1) \cdot b) \) to the point at the boundary of \( \text{oneset}(n \cdot b) \) given by the line originating from the vertex of the simplex of the free face of this point.)
Lemma 3.2. For each element $a$ of a finitely presented MV-algebra $A = M(P)$, the integer $\chi(\text{support}(a))$ coincides with the Euler characteristic of the supplement in $P = \mu(A)$ of the set

$$a^{-1}(0) = \{x \in P \mid a(x) = 0\} = \{m \in \mu(A) \mid a/m = 0\} = P \setminus \text{support}(a).$$

Proof. From the first part of the proof of [6, 5.3.9].

4. Proof of Theorem: Uniqueness

Lemma 3.1 yields an integer $d > 0$ together with a rational polyhedron $P \subseteq [0, 1]^d$ such that $A$ can be identified with $M(P)$ without loss of generality. Let $E_1$ and $E_2$ be real-valued functions on $A$ satisfying (i)-(iv). By (i), they agree at 0. Let $a$ be a nonzero element of $A$. By [5, §6.2,6.3], there is a regular triangulation $\Delta$ of $P$ such that $a$ (linear over every simplex of $\Delta$, and) can be written as

$$a = \bigoplus_{i=0}^u m_i \cdot b_i = \sum_{i=0}^u m_i b_i,$$

for distinct Schauder hats $b_0, b_1, \ldots, b_u$ of $\Delta$ and integers $m_0, m_1, \ldots, m_u > 0$. Since $E_2$ satisfies the idempotency condition (iii),

$$E_2(a) = E_2(\sum_{i=0}^u m_i b_i) = E_2(\sum_{i=0}^u b_i) = E_2(b_0 + \sum_{i=1}^u b_i) = E_2(b_0 \lor \sum_{i=1}^u b_i).$$

Since $E_2$ satisfies the additivity property (iv),

$$E_2(b_0 \lor \sum_{i=1}^u b_i) = E_2(b_0) + E_2(\sum_{i=1}^u b_i) = E_2(b_0 \land \sum_{i=1}^u b_i).$$

By Lemma 2.2(i) there is a Schauder basis $B'$ together with elements $b'_1, \ldots, b'_u \in B' \cup \{0\}$ such that $b_0 \land \sum_{i=1}^u b_i = \sum_{j=1}^u b'_j$. Consequently,

$$E_2(a) = E_2(b_0) + E_2(\sum_{i=1}^u b_i) - E_2(\sum_{j=1}^u b'_j).$$

Arguing by induction on $u$ we conclude that the value $E_2(a)$ is computed by a linear polynomial function $\rho$ (uniquely determined by $\Delta$) of the values of $E_2$ on finitely many basic elements of $A$. The same holds for $E_1$, with the same $\Delta$ and $\rho$. By the normalization condition (ii), $E_1$ and $E_2$ agree on basic elements, whence they agree on $a$. Therefore $E_1 = E_2.$

5. End of Proof of Theorem: $E$ has properties (i)-(iv)

(i) Trivially, $\chi(\text{support}(0)) = \chi(\emptyset) = 0$.

(ii) Since $A$ is finitely presented, by [5, 6.3] $A$ has a basis $B$. Writing as above $A = M(P)$, each basic element of $M(P)$ becomes a Schauder hat $b$ in the Schauder basis $B$. For all large $n$ the rational polyhedron oneset$(n \cdot b)$ is contractible to the vertex of the Schauder hat $b$. Therefore, $\lim_{n \to \infty} \chi(\text{oneset}(n \cdot b)) = 1$, and by Lemma 3.1 $\chi(\text{oneset}(b)) = 1 = E(b)$, as desired.

(iii) Trivially, $\text{support}(p \lor q) = \text{support}(p \oplus q)$.

(iv) In view of Lemma 3.1 let the integer $n$ be so large that $E(p) = \chi(\text{oneset}(n \cdot p)), E(q) = \chi(\text{oneset}(n \cdot q)), E(p \lor q) = \chi(\text{oneset}(n \cdot (p \lor q)))$. 

\[E(p) = \chi(\text{oneset}(n \cdot p)), E(q) = \chi(\text{oneset}(n \cdot q)), E(p \lor q) = \chi(\text{oneset}(n \cdot (p \lor q))) \]
and $E(p \land q) = \chi(\text{oneset}(n \cdot (p \land q)))$. Then we can write

\[
E(p) + E(q) = \chi(\text{oneset}(n \cdot p)) + \chi(\text{oneset}(n \cdot q))
\]

\[
= \chi(\text{oneset}(n \cdot p) \cup \text{oneset}(n \cdot q)) + \chi(\text{oneset}(n \cdot p) \cap \text{oneset}(n \cdot q))
\]

\[
= \chi(\text{oneset}(n \cdot p \lor n \cdot q)) + \chi(\text{oneset}(n \cdot p \land n \cdot q))
\]

\[
= \chi(\text{oneset}(n \cdot (p \lor q))) + \chi(\text{oneset}(n \cdot (p \land q)))
\]

\[
= E(p \lor q) + E(p \land q).
\]

The proof of Theorem \ref{thm:equivalent-def} is now complete. \hfill \Box

By \cite{[8]} 3.4, every finitely presented MV-algebra $A$ is the Lindenbaum algebra of some formula $\theta(X_1, \ldots, X_n)$ in Lukasiewicz logic $\mathcal{L}_\infty$. The map $E$ of Theorem \ref{thm:equivalent-def} determines the integer-valued map $E'$ from all formulas $\phi(X_1, \ldots, X_n)$, by the stipulation $E'(\phi) = E(\phi/\theta)$, where $\phi/\theta$ is the equivalence class of $\phi$ modulo $\theta$.

**Corollary 5.1.** $E'$ is Turing computable.

**Proof.** With reference to the proofs of Lemmas \ref{lem:equivalent} and \ref{lem:equivalent-def} (or alternatively, Lemma \ref{lem:equivalent-def}), let $a \in A$ and let $B$ a basis in $A$ such that $a = \bigoplus b_i m_i$, $b_i = \sum b_i$, for distinct Schauder hats $b_0, b_1, \ldots, b_u \in B$ and integers $m_0, m_1, \ldots, m_u > 0$. Let $1/d_i$ be the maximum value of $b_i$, for $i = 0, \ldots, u$. As in \cite{[8]} Definition 5.7, each $d_i$ is a nonzero integer. Since the McNaughton function $a$ is linear on every simplex of $B$, each nonzero local minimum and each nonzero local maximum of $a$ is of the form $m_i/d_i$, for some $i = 0, \ldots, u$. Let $n_i$ be the smallest integer such that $n_i \cdot m_i/d_i \geq 1$, and $n = \max(n_0, \ldots, n_u)$. For all integers $k \geq n$ each nonzero local minimum and each nonzero local maximum of $k \cdot a$ is equal to 1. As a consequence, $a$ has no local minima nor local maxima in $(0, 1/n)$. For each real number $0 < \delta < 1/(2n)$ the function $a$ can be slightly perturbed to obtain a smooth function $a_\delta$ with no local minima nor local maxima in $(\delta, 1/n - \delta)$ and such that $\text{uplevelset}_{\delta}(a_\delta)$ is homotopy equivalent to $\text{uplevelset}_{\lambda}(a)$, for any $\lambda \in (\delta, 1/n - \delta)$. We can now apply Morse Theory (e.g. \cite{[7]} Theorem 3.1) to the functions $a_\delta$, defined by $a_\delta(p) = -a(p)$ at each point $p$ where $a_\delta$ is defined, to obtain the homotopy equivalence of $\text{uplevelset}_{\lambda_1}(a_\delta)$ and $\text{uplevelset}_{\lambda_2}(a_\delta)$ for all $\lambda_1, \lambda_2 \in (\delta, 1/n - \delta)$. As a consequence, we get the homotopy equivalence of $\text{uplevelset}_{\delta}(a)$ and $\text{uplevelset}_{1/n}(a)$ for all $k_1, k_2 > n$. This ensures that $\text{oneset}(k_1 \cdot a)$ and $\text{oneset}(k_2 \cdot a)$ are homotopy equivalent for all $k_1, k_2 > n$. In conclusion, $\chi(\text{support}(a)) = \chi(\text{oneset}(n + 1 \cdot a))$. Perusal of the proofs of Lemmas \ref{lem:equivalent} and \ref{lem:equivalent-def} in combination with \cite{[8]} 18.1, shows that given (a formula for) $a$, some Turing machine will output (formulas for) the basis $B$, along with the integer $n$, and the integer $E(a) = \chi(\text{oneset}(n + 1 \cdot a))$. \hfill \Box

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