EIGENSPACES OF SYMMETRIC GRAPHS ARE NOT TYPICALLY IRREDUCIBLE

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Abstract. We construct rich families of Schrödinger operators on symmetric graphs, both quantum and combinatorial, whose spectral degeneracies are persistently larger than the maximal dimension of an irreducible representation of the symmetry group.

1. Introduction

In many circumstances, it has been established that a “typical” Schrödinger-type operator $L$ on the Hilbert space $H$ has simple spectrum. Classical results of Uhlenbeck [18] concern the Laplace operator on compact manifolds: the spectrum can be made simple by a small perturbation of the manifold’s metric. For quantum graphs, similar results have been established by Friedlander [9] and extended by Berkolaiko and Liu [6]. The role of the metric here is played by the lengths of the graph’s edges. In the case of a combinatorial graph Laplacian, the degeneracies in the spectrum can be lifted, for example, by changing the edge weights or by addition of the small potential (i.e. a diagonal matrix).

If $L$ commutes with a group of unitary operators $S$, it is easy to see that the eigenspace $E_{\lambda}(L)$ corresponding to the eigenvalue $\lambda$ is a representation of the group $S$ [19]. It is therefore expected that some of the eigenvalues will be degenerate, with the size of the degeneracies dictated by the degrees of the group’s irreducible representations. In analogy to the above results, it is natural to assume here that the eigenvalues will not be more degenerate than necessary: a perturbation with the same symmetry $S$ can ensure that every $E_{\lambda}(L)$ is irreducible. We will call this assumption “generic irreducibility”.

The only rigorous result establishing generic irreducibility is due to Zelditch [20], who considered the Laplace operator on finite $C^{\infty}$ Riemannian covers and established the positive result under the assumption that the dimension of the manifold is greater than or equal to the maximal dimension of irreducible representations. To quote [20], “[it] leaves open many interesting cases of the generic irreducibility question [...] in particular, it does not touch the case of graphs”. The purpose of this letter is to construct a rich family of graph examples (both quantum and combinatorial) on which generic irreducibility fails.

We stress that our examples are families of Schrödinger operators where we allow for perturbations not only of the metric (edge lengths or weights) but also of the potential, as long as the prescribed symmetry is preserved. This contrasts the positive result of [20] where only perturbations of the metric were enough to resolve the degeneracy.

We will start with the quantum graphs, on which our example was originally constructed and on which it has particularly rich structure. We will then point out how to translate our example to the case of combinatorial graph. We remark that a one-parameter family of

\[\text{It may happen that the space } H \text{ is not rich enough to support some of the representations; we will see an example in this letter.}\]
Hubbard Hamiltonians (a model similar to a combinatorial graph) with persistently reducible eigenstates has been previously considered by Heilmann and Lieb \[13\] (see also \[7\]).

2. Definitions

Let $G$ be a graph with each edge $e$ being identified with an interval $[0, \ell_e]$ of the real line. This gives us a local variable $x_e$ on the edge $e$ which can be interpreted geometrically as the distance from the initial vertex. Which of the two end-vertices is to be considered initial is chosen arbitrarily; the analysis is independent of this choice.

We are interested in the eigenproblem of the Schrödinger operator $L := -\Delta + Q$, namely

$$
- \frac{\partial^2}{\partial x^2} u_e(x) + Q_e(x)u_e(x) = \lambda u_e(x),
$$

where the potential $Q_e(x)$ is sufficiently regular to keep the problem self-adjoint, for instance a piecewise continuous function. The functions $u$ are assumed to belong to the Sobolev space $H^2(e)$ on each edge $e$. We will impose the so-called Neumann-Kirchhoff (NK) conditions at the vertices of the graph: we require that $u$ is continuous on the vertices, i.e. $u_{e_1}(v) = u_{e_2}(v)$ for each vertex $v$ and any two edges, $e_1, e_2$ incident to $v$, and that the current is conserved,

$$
\sum_{e \sim v} \frac{\partial}{\partial x} u_e(v) = 0 \quad \text{for all vertices } v,
$$

where the summation is over all edges incident to the vertex $v$ and the derivative is covariant into the edge (i.e. if $v$ is the final vertex for the edge $e$, the corresponding term gets a minus sign). Further information can be found in the review \[11\], the textbook \[5\] or a recent elementary introduction \[3\], among other sources.

The symmetries we consider are induced by the graph’s global isometries, with the standard definition of the graph metric (the length of the shortest path). Namely, given an isometry $s : G \to G$, the corresponding operator $S$ on the Hilbert space of $L^2$ function on $G$ acts as $(Su)(x) = u(s^{-1}(x))$. It is easy to see that an isometry maps vertices to vertices, preserving the degree and edges to edges, preserving the length. Therefore the group of all isometries of a metric graph coincides with the group of symmetries of the corresponding edge-weighted discrete graph (the edge lengths become weights; symmetry transformations must preserve weights).

As has been observed before (see, e.g. \[13\]), one can easily enrich the group of Hilbert space symmetries \textit{a posteriori}, by considering all unitary operators leaving the eigenspaces invariant. The isometries, however, are the natural choice of \textit{a priori} symmetries which one would expect to explain all degeneracies in the spectrum of a simple operator such as Schrödinger.

3. Quantum Graph Example

The starting point of our considerations was an observation that the regular tetrahedron graph (the complete graph on four vertices $K_4$ with all edge lengths equal to $a$) and no potential, $Q_e(x) \equiv 0$, has eigenvalue $\lambda_a = (2\pi/a)^2$ with multiplicity 4. This is more than the maximal degree of an irreducible representation (irrep): the symmetry group of a tetrahedron is the symmetric group $S_4$ acting on the graph by permuting its vertices which has irreps of
dimensions 1, 1, 2, 3 and 3. Therefore, the eigenspace of \( \lambda_a \) cannot be irreducible.\(^2\) However, this example is less than satisfactory, due to the paucity of the space of possible perturbations. The only free parameter is the length \( a \) and the entire spectrum changes trivially when all lengths are scaled by the same factor.

It is instructive to look at the eigenspace of \( \lambda_a \). Its basis can be chosen as follows: one eigenfunction is equal to \( \cos(2\pi x/a) \) on every edge of the graph (and is 1 at every vertex); three more eigenfunctions are equal to \( \sin(2\pi x/a) \) on the edges bounding one of the faces of the tetrahedron and are identically zero on the other edges (on every vertex they are equal to 0). The tetrahedron has four faces and one can construct four corresponding functions, but one of them can be obtained as the sum of the other three. Eigenfunctions of this type give rise to many interesting phenomena in quantum graphs, for instance to emergence of “topological resonances” \(^{10, 8}\), to special terms in the zeta function of equilateral quantum graphs \(^{12}\), and to masking of the poles of the Titchmarsh–Weyl function \(^{14}\). And it is these eigenfunctions which will lead us to a better example.

Let us inscribe a tetrahedron into a cube, see Fig. 1. We denote this graph by \( G \) and stress that the embedding of \( G \) into \( \mathbb{R}^3 \) is done for visualisation reasons only; we do not assume any relation between the length \( a \) of the tetrahedron’s edge and the length \( b \) of the cube edge.\(^3\) The resulting graph has the symmetry of the tetrahedron. In fact, we will allow any potential \( Q_a(x) \) and \( Q_b(x) \) on the edges of length \( a \) and \( b \), as long as the symmetry of the graph is preserved. This condition only restricts \( Q_a(x) \) to be even and forces the orientation of the cube edges to be chosen consistently: all edges oriented from the odd-numbered vertex (where the tetrahedron edges are incident) to the even-numbered vertex. The orientation of an edge serves to prescribe how the potential is placed on the edge.

This graph turns out to have eigenvalues of multiplicity at least 5, which we will demonstrate by constructing the eigenfunctions. We summarize this discussion as a theorem.

\(^{2}\)It can be shown to be the sum of the standard and the identity representations of \( S_4 \).

\(^{3}\)i.e. we reserve the right to change the metric on the edges; equivalently, we can allow the edges to be curved.
Theorem 3.1. Consider the graph $\Gamma$ depicted in Fig. 1. Let all tetrahedron edges (i.e. those which connect odd-numbered vertices) have length $a$ and support potential $Q_a(x)$ which is even, $Q_a(a-x) = Q_a(x)$. Let all cube edges (i.e. those that connect an odd-numbered vertex to an even-numbered one) have length $b$ and support potential $Q_b(x)$ of arbitrary form which is placed so that $x = 0$ corresponds to the odd-numbered vertex and $x = b$ corresponds to the even-numbered vertex.

Then the symmetry group of $\Gamma$ is the symmetric group $S_4$ which has irreducible representations of degrees 1, 1, 2, 3 and 3, yet for any choice of $a$, $b$, $Q_a$ and $Q_b$, the graph $\Gamma$ has infinitely many eigenvalues of multiplicity at least 5. The corresponding eigenspaces must therefore be reducible.

Proof. Let $\lambda$ be an eigenvalue of the Dirichlet problem on the interval of length $b$, namely
\[
-\psi''(x) + Q_b(x)\psi(x) = \lambda\psi(x), \quad \psi(0) = \psi(b) = 0,
\]
with $\psi(x)$ the corresponding eigenfunction. Take a cycle on the graph $G$ consisting of the cube edges only, for example, the cycle on the vertices 1, 4, 5, 8. Place the function $\psi$ on the edges of the cycle so that the derivative is continuous along the cycle. Namely, we place $\psi(x)$ on the edge (1,4), $-\psi(b-x)$ on the edge (4,5), then again $\psi(x)$ on (5,8) and so on, see Fig. 2. We extend the function by 0 to the rest of the graph. It is easy to see that the resulting function is continuous on the entire graph and satisfies condition (2) on the vertices. By virtue of equation (3) it also satisfies the Schrödinger equation on the entire graph with the eigenvalue $\lambda$.

Obviously, we can repeat this process for every cycle consisting of edges of length $b$. The number of the linearly independent functions that can be produced in this way is equal to the number of the linearly independent cycles on the cube subgraph of $G$. This, in turn, is given by the first Betti number of the subgraph, $\beta = E - V + 1$, which for the cube is equal to 5. Informally, the boundaries of 5 faces of the cube are independent, while the sixth one is given by their sum.

To summarize, each eigenvalue $\lambda$ of (3) is an eigenvalue of multiplicity at least 5 of the graph $G$ and its eigenspace is a reducible representation of the group of symmetries of $G$. 

4. VARIATIONS ON THE EXAMPLE AND ITS ANALYSIS

We note that the role played by the tetrahedron subgraph of the graph $G$ is a very limited one: it serves to restrict the symmetry group of the resulting graph and adds the freedom of choosing the metric on its edges (equivalently, their length) and the potential. We can dispense with this subgraph altogether and consider the cube graph $G_c$ with the odd-numbered
vertices distinguished from even-numbered. This can be done by choosing the potential $Q_b(x)$ which is not even, i.e. $Q_b(b-x) \neq Q_b(x)$, or by changing the vertex conditions at even-numbered vertices to $\delta$-type with non-zero parameter. We will assume the former method is used and not dwell on the latter.

It is also interesting to note that it is not necessary to restrict the symmetry of the cube: the group of cube’s symmetries, the full octahedral group, has representations of degrees up to 3 and therefore the persistent eigenspace of dimension 5 still provides a valid counter-example to the generic irreducibility conjecture. However, this further restricts the space of available perturbations and makes the forthcoming analysis unwieldy due to the large symmetry group.

We now consider the graph $G_c$ with the symmetry group $S_4$ and identify the decomposition of the domain of the Schrödinger operator into the subspaces corresponding to the irreducible representations of $S_4$. More precisely, denote by $\mathcal{H}(G_c)$ the functions on the edges of $G_c$ that belong to the Sobolev space $H^2$ on every edge and satisfy continuity condition and condition (2) on the vertices of the graph. For a representation $\rho$, let $M^\rho_g$ denote the matrix corresponding to the group element $g \in S_4$. We are looking for tuples $(\psi_1, \ldots, \psi_d)^T$ of functions from $\mathcal{H}(G_c)$ which satisfy the intertwining condition

$$\begin{pmatrix}
\psi_1(gx) \\
\vdots \\
\psi_d(gx)
\end{pmatrix} = M^\rho_g \begin{pmatrix}
\psi_1(x) \\
\vdots \\
\psi_d(x)
\end{pmatrix}.$$  \(\tag{4}\)

We will call the functions satisfying this condition the *equivariant functions* for the representation $\rho$; the subspace of $\mathcal{H}(G_c)$ spanned by them is called the *isotypic component* of the representation $\rho$ and denoted by $\mathcal{H}_\rho(G_c)$. Once the space $\mathcal{H}_\rho(G_c)$ identified, one can restrict the operator $L$ to this space and unitarily reduce it to a simpler problem. This procedure, pioneered on quantum graphs by Band, Parzanchevski and Ben-Shach \[2, 16\] for their study of isospectrality is called “quotient graph construction”. We refer the interested reader to these papers as well as to the forthcoming work \[1\] where it is formalized in terms of the scattering matrices.
As mentioned above, $S_4$ has 5 irreps of degrees 1, 1, 2, 3 and 3. The identity representation maps every $g \in S_4$ to multiplication by one,

$$R_i = \{(1 3) \mapsto (1), \quad (1 5) \mapsto (1), \quad (1 7) \mapsto (1)\}.$$  

In this notation, for each $g$ from a set of generators of $S_4$ (here we took $(1 3), (1 5)$ and $(1 7)$; we remind that the symmetry transformations act as permutations on the odd-numbered vertices) we specify a $1 \times 1$ matrix (in this case, multiplication by 1). The intertwining condition becomes $\psi(gx) = \psi(x)$ for every $g \in S_4$ which is satisfied by $\psi(x)$ which are equal to the same function, which we denote by $f$, on every edge oriented from odd to even-numbered vertex, see Fig. 3. Not every $f$ is admissible: condition (2) at vertex 1, for example, becomes the condition $3f'(v_1) = 0$; the same at vertex 8. The continuity at each vertex is, of course, automatic. The space of functions of this form on $G_c$ is the isotypic component of $R_i$ denoted by $\mathcal{H}_i(G_c)$. For the function from $\mathcal{H}_i(G_c)$ to be an eigenfunction of our Schrödinger operator $L$ on the graph $G_c$, $f$ must be an eigenfunction of the Neumann problem

$$-\psi''(x) + Q_b(x)\psi(x) = \lambda\psi(x), \quad \psi'(0) = \psi'(b) = 0$$

on the interval $[0, b]$. The spectrum of (6) coincides with the spectrum of $L$ restricted to the space $\mathcal{H}_i(G_c)$ (actually, the corresponding operators are unitarily equivalent).

The sign representation maps every $g \in S_4$ into multiplication by the sign of the permutation $g$,

$$R_s = \{(1 3) \mapsto (-1), \quad (1 5) \mapsto (-1), \quad (1 7) \mapsto (-1)\}.$$  

It is easy to see that the space of functions satisfying (4) with representation $R_s$ is the trivial space, $\mathcal{H}_s(G_c) = \{0\}$. Indeed, taking for example the edge $(1, 4)$, we observe that it is fixed by the reflection $(5 7)$, therefore the component of $\psi$ on this edge must satisfy $\psi_{(1,4)}(x) = -\psi_{(1,4)}(x)$, and therefore $\psi_{(1,4)} \equiv 0$. It is easy to verify that each edge is similarly fixed by some transposition, so $\psi$ must be 0 on every edge. Therefore, the operator $L$ has no eigenvalues corresponding to the sign representation!
Figure 5. The structure of the functions on the graph $G_c$ that transform according to the representation $R_{3d,1}$, equation (9).

The next representation is the **irreducible representation of degree 2**, given in the matrix form by

$$R_{2d} = \begin{cases} (13) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & (15) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, & (17) \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \end{cases}.$$  

Note that the matrices in this presentation are not unitary, but can be made so using a change of basis. However, with matrices in this form, the pairs of functions from $\mathcal{H}(G_c)$ transforming according to this representation have especially simple form, depicted in Fig. 4. It is immediate from the figure that for $f$ to be admissible, it must satisfy Dirichlet problem (3). Each admissible $f$ gives rise to a two-dimensional eigenspace of $L$.

The **standard representation** of $S_4$ is a representation of degree 3, given by

$$R_{3d,1} = \begin{cases} (13) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & (15) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & (17) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{cases}.$$  

The triple of functions transforming according to this representation is schematically represented in Fig. 5. For $f$ to be admissible, it must again satisfy Dirichlet problem (3). We
remark that the first two equivariant functions are similar in structure to the equivariant functions we found for the representation $R_{2d}$ but differ from them in sign distribution.

Finally, the last irreducible representation is the product of the standard and sign representations. It has degree 3 and is given by

$$R_{3d,2} = \left\{ (13) \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (15) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad (17) \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\}.$$  

The triple of functions transforming according to this representation is schematically represented in Fig. A. Assume $x = 0$ at an odd-numbered vertex and $x = b$ at an even-numbered vertex. Then, for $f$ and $g$ to be admissible, they must satisfy the following problem

$$-f''(x) + Q_b(x)f(x) = \lambda f(x), \quad f(0) = g(0), \quad 2f'(0) + g'(0) = 0,$$

$$-g''(x) + Q_b(x)g(x) = \lambda g(x), \quad f(b) = -g(b), \quad 2f'(b) - g'(b) = 0.$$  

This problem is self-adjoint in an appropriately weighted $L^2 \times L^2$ space.
Eigenvalues of symmetric graphs are not typically irreducible.

Figure 7. An example of a combinatorial graph whose Schrödinger operator has a reducible eigenspace for all values of the vertex potential $a$, $b$, and $c$ and for all coupling weights $\alpha$ and $\beta$.

To summarize, the eigenvalues $\lambda$ of the Dirichlet problem (3) on a single edge are also present in the spectrum of the operator $L$ on the graph $G_c$ with multiplicity at least 5. Their subspaces reduce into the direct sum of the degree-two and the standard representations of the symmetry group of the underlying graph.

5. An example of a combinatorial graph with reducible eigenspaces

It is easy to construct an example of a combinatorial graph with reducible eigenspaces by analogy with the quantum graph $G_c$. The simplest such example is shown in Fig. 7. The corresponding Schrödinger operator is a $20 \times 20$ self-adjoint matrix with 5 real parameters. Namely, we can choose the potential at 3 types of vertices ($a$, $b$, and $c$ in the picture) and the coupling weights corresponding to two types of edges ($\alpha$ and $\beta$ in the picture). One can construct eigenvectors by choosing a face and placing alternating $\pm 1$ on the $c$-type vertices around that face; all other entries of the vector are zero. It is easy to check that it is indeed an eigenvector with the eigenvalue equal to the potential $c$. Similarly to above, there are 5 such independent eigenvectors.

It is straightforward to deduce the quotient eigenvalue problems as was done for the quantum graph $G_c$ above. The result is analogous and we leave the details to reader; the theory of constructing quotient combinatorial graphs will be formalized in [1].

6. Concluding remarks

Since the operators we considered have real coefficients, they are also symmetric with respect to complex conjugation. The choice of the field ($\mathbb{R}$ or $\mathbb{C}$) over which the representation is irreducible can play an important role (see the remarks in the end of [20], Sec. 1b; see also [4] for a different example). However, for the symmetry group in our example, the irreducible representations of the symmetry group over real numbers and over complex numbers coincide.

It is unclear at the moment if it is possible to predict (without direct computation) that the quotient graphs by $R_{2d}$ and by $R_{3d,1}$ will have coinciding spectra.

One may speculate that the large multiplicities in the examples we constructed may be viewed as traces of larger symmetry groups of 2-dimensional graph-like manifolds that were
shrunk to the graph limit (see [17] and references therein). As a starting point for this process one may take the celebrated Klein’s quartic, a compact Riemann surface in the shape of a tetrahedron with the highest possible order (namely, 168) automorphism group for its genus [15].

However, we feel that the central role in this example is played not by 1-dimensionality of the edges, but by vertices: quantum graphs are not 1-dimensional manifolds as they singular at the vertices. As a consequence, the unique continuation principle fails on graphs. In fact, one can create a host of similar examples by modifying a graph with a large symmetry group with a choice of few rank one perturbations at the vertices (for example, changing NK conditions to Dirichlet). Few well-placed modifications can completely break the symmetry yet each rank-one perturbation will split off only a single eigenvalue from each degenerate group.

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