Thermodynamics of chiral symmetry at low densities

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Abstract

The phase diagram of two-color QCD as a function of temperature and baryon chemical potential is considered. Using a low-energy chiral Lagrangian based on the symmetries of the microscopic theory, we determine, at the one-loop level, the temperature dependence of the critical chemical potential for diquark condensation and the temperature dependence of the diquark condensate and baryon density. The prediction for the temperature dependence of the critical chemical potential is consistent with the one obtained for a dilute Bose gas. The associated phase transition is shown to be of second order for low temperatures and first order at higher temperatures. The tricritical point at which the second order phase transition ends is determined. The results are carried over to QCD with quarks in the adjoint representation and to ordinary QCD at a non-zero chemical potential for isospin.

\textit{PACS:} 11.30.Rd, 12.39.Fe, 12.38.Lg, 71.30.+h

\textit{Keywords:} QCD partition function; Non-zero baryon density; Non-zero isospin density; QCD with two Colors; Adjoint QCD; QCD Dirac operator; Lattice QCD; Chiral Lagrangian; Chiral Perturbation Theory; Dilute Bose Gases.
1 Introduction

The properties of strongly interacting matter at finite density and temperature are presently being studied in a variety of ways. Phenomenologically, our understanding of this subject is based on heavy ion collisions and the properties of neutron stars. On the theoretical side several approaches are followed. An asymptotic expansion in the inverse baryon density can be derived in the framework of perturbative QCD [1, 2]. At realistic densities, first principle calculations are not possible and one has to rely on schematic models such as the Nambu-Jona-Lasinio model [3, 4, 5], Random Matrix Models [6, 7], or semiclassical methods such as the instanton liquid model [8]. Based on these studies a plethora of phases and phenomena have been proposed for QCD at non-zero baryon density. Among others, we mention a superconducting phase [1, 8, 3], a color-flavor locked phase [9], a crystalline phase [5], a tricritical point [7, 10] and continuity between quark and hadronic matter [11].

Because of the complex phase of the fermion determinant these scenarios cannot be confirmed by means of first principle lattice simulations of QCD. This makes it imperative to analyze closely related theories for which lattice simulations at realistic densities are possible. We will consider QCD with two colors at non-zero baryon density, QCD with three colors at non-zero isospin density (or phase quenched QCD at non-zero baryon density) or, in general, at nonzero flavor density, and QCD with gauge fields in the adjoint representation at nonzero baryon density. Chiral symmetry is broken spontaneously in the ground state of both QCD with two colors and QCD with three colors, and the low-energy excitations of these theories are given by a theory of weakly interacting Goldstone bosons which can be analyzed order by order in Chiral Perturbation Theory [12, 13]. Within this framework, a phase transition to a Bose condensed phase has been found to be induced by a chemical potential related to a quantum number carried by at least one of the Goldstone bosons [14, 15, 16, 17, 18, 19].

For QCD with three colors this corresponds to the inclusion of a chemical potential for isospin or strangeness leading to pion or kaon condensation on a scale determined by the mass the Goldstone modes [17, 19]. Because of the Pauli-Gürsey symmetry, Goldstone bosons with non-zero baryon number appear in QCD with two colors, leading to diquark condensation for a baryon chemical potential beyond half the mass of the diquark boson [14, 15, 17, 18, 20]. By including an isospin chemical potential the combined phase diagram can be studied within the framework of chiral perturbation theory [17, 18]. We emphasize that in the chiral limit these results are exact and should coincide with the microscopic theory.

In the color-flavor locked phase which has been proposed as a possible state of QCD at high density chiral symmetry is broken spontaneously as well. The corresponding Goldstone bosons can be described in terms of a chiral Lagrangian [1, 4, 21, 22] very much like the one for QCD with two colors [14, 13].

In this article we study chiral Lagrangians at non-zero chemical potential and temper-
ature. We mainly focus on QCD with two-colors for which also lattice QCD simulations are available [23, 24, 25], but we will discuss the extension of our results to QCD with adjoint quarks at finite baryon density and to QCD with three or more colors and fundamental quarks at non-zero isospin density. In a previous publication [20] we have shown that the chiral Lagrangian for QCD with two colors is renormalizable to one loop, i.e., all infinities that are generated by the one-loop diagrams can be absorbed into the coupling constants of the next to leading order terms in the chiral Lagrangian such that the constants are independent of the temperature and the chemical potential. In that article we have also shown that, to one-loop order, the critical chemical potential is given by half the renormalized pion mass.

Recently Kogut and Sinclair [26] have investigated the critical exponents of the phase transition at zero temperature in quenched QCD with three colors at finite isospin chemical potential, \( \mu_I \), and in QCD with two colors at finite baryon chemical potential, \( \mu_B \). From chiral perturbation theory at one-loop level we expect that at zero temperature the phase transition to the charged phase is characterized by mean field critical exponents. The measurements in [26] are consistent with mean field critical exponents. Lattice simulations [27] at non-zero baryon chemical potential and temperature, \( T \), indicate that QCD with two colors displays a rich and interesting phase diagram including a tricritical point in the (\( \mu_B, T \))-plane. In this article we explore this phase diagram by means of chiral perturbation theory by extending the 1-loop calculation in [20] to non-zero temperature. Our results support the existence of a tricritical point in the (\( \mu_B, T \))-plane.

The phase diagram of three-color QCD in the (\( \mu_I, T \))-plane has been studied recently on the lattice [28]. The phase diagram again includes a tricritical point. Extending the analytic results for \( N_c = 2 \) to \( N_c = 3 \) we confirm the existence of this tricritical point in the (\( \mu_I, T \))-plane.

The pattern of chiral symmetry breaking is determined by the Dyson index \( \beta_D \) of the Dirac operator. Its value denotes the number of independent degrees of freedom per matrix element. If the Dirac operator does not have anti-unitary symmetries its matrix elements are complex and \( \beta_D = 2 \). This is the case for QCD with three or more colors and fundamental fermions. Anti-unitary symmetry operators can be written as \( AK \) with \( A \) a unitary operator and \( K \) the complex conjugation operator. There are only two nontrivial possibilities for \( (AK)^2 \),

\[
(AK)^2 = 1 \quad \text{with} \quad \beta_D = 1, \quad (1.1)
\]

\[
(AK)^2 = -1 \quad \text{with} \quad \beta_D = 4. \quad (1.2)
\]

In the first case it is always possible to find a basis for which the Dirac operator is real. This case is realized for QCD with two colors and quarks in the fundamental representation. The second case, in which the matrix elements of the Dirac operator can be organized into selfdual quaternions, is realized for QCD with two or more colors and quarks in the adjoint representation. The addition of a chemical potential does affect this triality [29]. Below the critical value of the chemical potential, the number of Goldstone bosons is
determined by the pattern of chiral symmetry breaking at zero chemical potential. The only thing that happens at nonzero chemical potential is that degeneracies in the mass spectrum of the Goldstone bosons are lifted. The three different cases are very similar. The only difference is in the degeneracies of the Goldstone modes which only depend on the symmetry breaking pattern. For definiteness we analyze $\beta_D = 1$ in this article, and in sections 8.1 and 8.2, we discuss the changes that occur for $\beta_D = 4$ and $\beta_D = 2$, respectively.

The paper is organized as follows. To make the paper self-contained we first discuss QCD with two colors and the transition into the diquark condensed phase at zero temperature. Then we calculate the temperature dependence of the free energy in the two phases of the theory separately. Using this we determine the critical chemical potential for diquark condensation at non-zero temperature and the tricritical point at which the second order phase transition turns first order. We also study the order parameter and the number density.

2 The Symmetries of QCD with Two Colors

The Lagrangian that describes the low-energy limit of QCD with two colors is uniquely determined by the symmetries of the microscopic theory and the assumption of spontaneous breaking of chiral symmetry. We start by discussing the symmetries.

QCD with two colors and fermions in the fundamental representation is special since the gauge group is pseudo real. Its generators, $\tau_k$, satisfy $-\tau_2\tau_k^\dagger\tau_2 = \tau_k$. This implies that the Dirac operator, $D$, satisfies the anti-unitary symmetry,

$$D\tau_2C\gamma_5 = \tau_2C\gamma_5D^\star, \quad (2.3)$$

where $C$ is the charge conjugation matrix. This symmetry follows immediately from the explicit form of the Dirac operator,

$$D = \gamma_\nu D_\nu + m + \mu_B\gamma_0. \quad (2.4)$$

Here, $D_\nu \equiv \partial_\nu + A_\nu$ is the gauge covariant derivative and $\gamma_\nu$ are the hermitian Euclidean Dirac matrices. The microscopic Lagrangian at $m = \mu_B = 0$ has an $SU(2N_f)$ chiral invariance, often referred to as the Pauli-Gürsey symmetry [30]. This flavor symmetry is manifest if we choose a basis given by left-handed quarks, $q_L$, and conjugate right-handed anti-quarks, $\sigma_2\tau_2q_R^\star$, ($\sigma_2$ acts in spin space),

$$\Psi \equiv \begin{pmatrix} q_L \\ \sigma_2\tau_2q_R^\star \end{pmatrix}. \quad (2.5)$$
The fermionic part of the Lagrangian for QCD with two colors at \( m = \mu_B = 0 \) in this basis is

\[
L_{\text{QCD}} = i\Psi^\dagger \sigma_\nu D_\nu \Psi,
\]

(2.6)

where \( \sigma_\nu = (-i, \sigma_b) \). It is invariant under global transformations

\[
\Psi \rightarrow V \Psi, \ \text{with} \ V \in SU(2N_f).
\]

(2.7)

The global \( SU(2N_f) \) chiral invariance is broken explicitly at nonzero baryon chemical potential, \( \mu_B \), quark mass, \( m \), or the diquark source, \( j \), which can be seen from the QCD Lagrangian in terms of the \( \Psi \)-fields,

\[
L_{\text{QCD}} = i\Psi^\dagger \sigma_\nu (D_\nu - \mu_B B \delta_0 \nu) \Psi - \left( \frac{1}{2} \Psi^T \sigma_2 \tau_2 M \Psi + \text{h.c.} \right).
\]

(2.8)

The matrix that includes the mass and the diquark source is defined by

\[
M \equiv \sqrt{m^2 + j^2 (\hat{M} \cos \phi + \hat{J} \sin \phi)}.
\]

(2.9)

where \( \tan \phi = j/m \). The baryon charge matrix \( B \), the mass matrix \( \hat{M} \), and the diquark source matrix \( \hat{J} \) are given by,

\[
B \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{M} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \hat{J} \equiv \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}
\]

(2.10)

with

\[
I \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(2.11)

The explicit breaking of the global \( SU(2N_f) \) symmetry of the Lagrangian can be compensated by a rotation of the source matrices,

\[
M \rightarrow V^* \hat{M} V^\dagger,
\]

(2.12)

\[
B \rightarrow VBV^\dagger.
\]

(2.13)

The combined transformation of \( \Psi, M, \) and \( B \) leaves \( L_{\text{QCD}} \) invariant. This global \( SU(2N_f) \) invariance of the Lagrangian can be extended to a local \( SU(2N_f) \) flavor symmetry by introducing the vector field \( B_\nu \equiv (B, 0, 0, 0) \) with transformation properties

\[
B_\nu \rightarrow VB_\nu V^\dagger - \frac{1}{\mu_B} V \partial_\nu V^\dagger.
\]

(2.14)
This invariance is vital as we impose that the chiral Lagrangian must be invariant under this local flavor symmetry.

3 The chiral Lagrangian

The global $SU(2N_f)$ chiral symmetry of the microscopic theory is believed to be spontaneously broken down to $Sp(2N_f)$ by the vacuum $^{32,33}$. Associated with this spontaneous breakdown is a set of $N_f(2N_f - 1) - 1$ Goldstone modes described by the field $\Sigma \in SU(2N_f)/Sp(2N_f)$. For simplicity, we will only consider an even number of degenerate flavors. For a discussion of non-degenerate flavors see $^{19}$.

Under the local $SU(2N_f)$ rotation the Goldstone field $\Sigma$ transforms as

$$\Sigma \rightarrow V\Sigma V^T, \quad V \in SU(2N_f).$$

(3.15)

The underlying microscopic theory is invariant under the combined local transformation (2.7), (2.12), (2.14) of $\Psi$, $B$, and $M$. The same symmetries should be present in the low-energy chiral Lagrangian and is thus invariant under the combined transformations (3.13), (2.12), (2.14) of $\Sigma$, $B$, and $M$. Therefore only the covariant derivative $^{14}$

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - \mu_B (B_\nu \Sigma + \Sigma B_\nu^T),$$

$$\nabla_\nu \Sigma^\dagger = \partial_\nu \Sigma^\dagger + \mu_B (B_\nu^T \Sigma^\dagger + \Sigma^\dagger B_\nu),$$

(3.16)

and products of $M$ and $\Sigma$ can enter in the chiral Lagrangian. The terms that can enter in the chiral Lagrangian are uniquely determined by this symmetry requirement and Lorentz invariance. However, as the number of allowed terms is infinite one needs to introduce a counting scheme which defines the expansion parameters of the theory. In this paper we extend the usual power counting of chiral perturbation theory giving the same weight to the square root of the quark mass, the square root of the diquark source, the chemical potential, and the momenta:

$$\mathcal{O}(p) \sim \mathcal{O}(\mu_B) \sim \mathcal{O}(\sqrt{m}) \sim \mathcal{O}(\sqrt{j}) \sim \frac{1}{L}.$$  

(3.17)

Within this counting scheme we obtain a well defined expansion.

Before describing this expansion it is instructive to take a closer look at the nature of the Goldstone manifold $SU(2N_f)/Sp(2N_f)$. This manifold has $N_f(2N_f - 1) - 1$ degrees
of freedom and can be represented by anti-symmetric special unitary matrices of size $2N_f \times 2N_f$. A convenient parameterization is given by,

$$\Sigma = U\Sigma U^T ,$$  \hspace{1cm} (3.18)

where,

$$U = \exp(i\Pi/2F) \quad \text{and} \quad \Pi = \pi_a X_a/\sqrt{2N_f} .$$  \hspace{1cm} (3.19)

Here, $F$ is the pion decay constant, the fields $\pi_a$ are the Goldstone modes and the $X_a$ are the $2N_f^2 - N_f - 1$ generators of the coset $SU(2N_f)/Sp(2N_f)$. They obey the relation $X_a\Sigma = \Sigma X_a^T$ and are normalized according to $\text{Tr}X_a X_b = 2N_f \delta_{ab}$. The constant antisymmetric special unitary matrix $\Sigma$ is the ground state about which the $\Pi$-fields fluctuate.

### 3.1 Leading Order Chiral Lagrangian

The chiral Lagrangian to leading order, $p^2$, in the momentum expansion with the required invariance properties is the non-linear $\sigma$-model [15],

$$\mathcal{L}^{(2)} = \frac{F^2}{2} \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger - \chi^\dagger \Sigma - \chi \Sigma^\dagger \right] ,$$  \hspace{1cm} (3.20)

where we have introduced the source term

$$\chi = \frac{G}{F^2} M^\dagger .$$  \hspace{1cm} (3.21)

The constant $G$ is the chiral condensate in the chiral limit. It is tied to the leading order pion decay constant $F$ and pion mass $M$ through the Gell-Mann–Oakes–Renner relation, $M^2 = mG/F^2$. The action $S_2 = \int dx \mathcal{L}^{(2)}$ at order $p^2$, given by the tree diagrams of $\mathcal{L}^{(2)}$, was studied in [14, 15].

### 3.2 Next-to-Leading Order Chiral Lagrangian

At next to leading order the total action is given by four terms [20],

$$S = S_2 + S_{1\text{-loop}} + S_4 + S_A .$$  \hspace{1cm} (3.22)
The contribution to the free energy from the one-loop diagrams of $\mathcal{L}^{(2)}$ is denoted $S_{1-\text{loop}}$, $S_4 = \int dx \mathcal{L}^{(4)}$, and $S_A$ is the Wess-Zumino functional. The part of the chiral Lagrangian that contributes to the free energy at order $p^4$ is given by [20],

$$
\mathcal{L}^{(4)} = -L_0 \text{Tr} \left[ \nabla_{\nu} \Sigma \nabla_{\tau} \Sigma^\dagger \nabla_{\nu} \Sigma \nabla_{\tau} \Sigma^\dagger \right] - L_1 \left( \text{Tr} \left[ \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right] \right)^2 - L_2 \text{Tr} \left[ \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right] \text{Tr} \left[ \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right] - L_3 \text{Tr} \left[ \left( \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right)^2 \right] + L_4 \text{Tr} \left[ \chi \Sigma^\dagger + \Sigma \chi^\dagger \right] \text{Tr} \left[ \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right] + L_5 \text{Tr} \left[ \left( \chi \Sigma^\dagger + \Sigma \chi^\dagger \right) \left( \nabla_{\nu} \Sigma \nabla_{\nu} \Sigma^\dagger \right) \right] - L_6 \left( \text{Tr} \left[ \chi \Sigma^\dagger + \Sigma \chi^\dagger \right] \right)^2 - L_7 \left( \text{Tr} \left[ \chi^\dagger \Sigma - \chi \Sigma^\dagger \right] \right)^2 - L_8 \text{Tr} \left[ \chi \Sigma^\dagger \chi^\dagger \Sigma^\dagger + \Sigma \chi^\dagger \Sigma \chi^\dagger \right] - H_2 \text{Tr} \left[ \chi \chi^\dagger \right] .
$$

The dimensionless constants $L_k$ and $H_2$ include counter terms that cancel the divergences that appear at one-loop level (see [20] for a detailed calculation). The contribution from the Wess-Zumino functional is independent of temperature to the 1-loop order and will be ignored in this article. For a discussion of the Wess-Zumino term at $\mu_B \neq 0$ we refer to [34].

### 4 Zero Temperature

We now review the predictions obtained from next-to-leading order chiral perturbation theory at zero temperature [20]. These results are the basis for the present work. As usual in 1-loop calculations, a non-zero temperature does not introduce additional divergences in the action. Once we have renormalized the theory to 1-loop order at $T = 0$, it remains finite at nonzero temperature and chemical potential. In the present paper we study the $T \neq 0$ effects at order $p^4$. We first review the order $p^2$ results and the corrections at zero temperature at order $p^4$.

At $T = 0$ the orientation of the vacuum to order $p^2$ is given by,

$$
\Sigma(\alpha) = \Sigma_c \cos \alpha + \Sigma_d \sin \alpha ,
$$

where,

$$
\alpha = 0 \quad \text{if} \quad \mu_B < M/2 ,
$$

$$
\cos \alpha = \frac{\mu^2}{4\eta^2} \quad \text{if} \quad \mu_B > M/2 ,
$$

(4.24)
\[ \Sigma_c \equiv I \quad \text{and} \quad \Sigma_d \equiv \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix} \quad (4.26) \]

A non-zero value of \( \alpha \) corresponds to diquark condensation \cite{15}. The diquarks are color neutral and the condensed phase is a superfluid. We refer to the phases as the normal phase (\( \alpha = 0 \)) and the diquark- or superfluid phase (\( \alpha \neq 0 \)). At non-zero \( j \), the diquark condensate is always non-vanishing. In this case, the orientation of the vacuum is still given by \( \Sigma = \Sigma_c \cos \alpha + \Sigma_d \sin \alpha \), but \( \alpha \) is determined by the saddle point equation,

\[ 4\mu_B^2 \cos \alpha \sin \alpha = \tilde{M}^2 \sin(\alpha - \phi) \quad (4.27) \]

where \( \tilde{M} \) is the leading order pion mass at diquark source \( j \neq 0 \),

\[ \tilde{M}^2 = \frac{G \sqrt{m^2 + j^2}}{F^2} \quad (4.28) \]

To order \( p^4 \) calculations we need the propagators of \( \mathcal{L}^{(2)} \). In the normal phase we have \( N_f^2 - 1 \) modes with zero baryon charge, which we will call \( P \)-modes and \( N_f(N_f - 1) \) diquark modes, which we will call \( Q \) modes. Their inverse propagators are given by \cite{15, 20},

\[ D^P = p^2 + M^2 \quad (4.29) \]

and,

\[ D^Q = \begin{pmatrix} p^2 + M^2 - 4\mu_B^2 & 4i\mu_B p_0 \\ 4i\mu_B p_0 & p^2 + M^2 - 4\mu_B^2 \end{pmatrix} \quad (4.30) \]

In the phase of condensed diquarks four different types of modes can be distinguished: \( N_f(N_f + 1)/2 \) \( P_S \)-modes, \( (N_f^2 - N_f - 2)/2 \) \( P_A \)-modes and \( N_f(N_f - 1) \) complex \( Q \)-modes. The inverse propagators read,

\[ D^{P_S} = p^2 + M_1^2 + \frac{1}{4}M_3^2 \quad , \]

\[ D^{P_A} = p^2 + M_2^2 + \frac{1}{4}M_3^2 \quad , \]

\[ D^Q = \begin{pmatrix} p^2 + M_1^2 & iM_3 p_0 \\ iM_3 p_0 & p^2 + M_2^2 \end{pmatrix} \quad (4.31) \]
where we have introduced,

\[
\begin{align*}
M_1^2 &\equiv \bar{M}^2 \cos(\alpha - \phi) - 4\mu_B^2 \cos 2\alpha, \\
M_2^2 &\equiv \bar{M}^2 \cos(\alpha - \phi) - 4\mu_B^2 \cos^2 \alpha, \\
M_3^2 &\equiv 16\mu_B^2 \cos^2 \alpha.
\end{align*}
\] (4.32)

The \(Q\)-modes are mixed, but the mixing matrix can be diagonalized analytically with eigenmodes denoted by \(\tilde{Q}\) and \(\tilde{Q}^\dagger\). The \(\tilde{Q}\)-modes are of special interest since they are the Goldstone modes of the superfluid phase at \(j = 0\) and \(\mu_B > M/2\). In particular, these diquark modes are massless even though \(m \neq 0\).

In \cite{20}, it was found that at \(T = 0\) the \(p^4\) action is renormalizable at the 1-loop level. The renormalized \(p^4\) action predicts that for \(\mu_B \geq \mu_c = m/2\) the diquark condensate is nonzero. The quantity \(m_\pi\) is the physical pion mass at 1-loop order expressed in terms of the renormalized couplings of \(L^{(4)}\), the ultraviolet cutoff \(\Lambda\), and \(M^2 = Gm/F^2\) \cite{20},

\[
m_\pi^2 = M^2 \left(1 + \frac{M^2}{F^2} \left[4(-2N_f L_4 - L_5^r + 4N_f L_6^r + 2L_8^r) + \frac{N_f + 1}{64\pi^2 N_f} \ln \frac{M^2}{\Lambda^2}\right]\right). \] (4.33)

The difference between \(M^2\) and \(m_\pi^2\) is of order \(p^4\). Replacing \(M^2\) with \(m_\pi^2\) in the calculation of \(S_{1\text{-loop}}\) and \(S_4\) therefore only leads to a higher order correction. We shall make this replacement throughout the calculation.

## 5 Non-Zero Temperature

The temperature dependence in the chiral expansion enters through \(S_{1\text{-loop}}\) since this is the only part which involves an explicit momentum integration. In the following sections we calculate the temperature dependence of \(S_{1\text{-loop}}\) in the normal phase and in the diquark phase, in a similar way as for usual Chiral Perturbation Theory at nonzero temperature and zero chemical potential \cite{35}. This subsequently allows us determine the critical chemical potential, \(\mu_c(T)\), which separates the two phases.

### 5.1 The Normal Phase at \(T \neq 0\)

The free energy at 1-loop order in the normal phase is given by

\[
\Omega = -2N_f F^2 M^2 - \frac{1}{2}(N_f^2 - 1)G_0^P(m_\pi, T) - \frac{1}{4}N_f(N_f - 1)G_0^O(m_\pi, \mu_B, T) \] (5.34)
\[-2N_f m^4_\pi \left( 8N_f L_6 + 2L_8 + H_2 \right) + \mathcal{O}(p^6) , \]

with

\[ G^P_0 (m_\pi, T) = - \frac{d^dp}{(2\pi)^d} \ln(p^2 + m^2_\pi), \quad (5.35) \]
\[ G^Q_0 (m_\pi, \mu_B, T) = - \frac{d^dp}{(2\pi)^d} \ln \left( (p^2 + m^2_\pi - 4\mu^2_B) + 16\mu^2_B p^2_0 \right), \quad (5.36) \]

In the expressions above we have introduced the notation

\[ \frac{d^dp}{(2\pi)^d} f(p_0, \vec{p}) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} f(2\pi n T, \vec{p}). \quad (5.37) \]

As shown below, the 1-loop contribution can be decomposed in the following way:

\[ G_0(T) = \Delta_0 + g_0(T), \quad (5.38) \]

where \( \Delta_0 \) is the zero temperature propagator at the origin, and \( g_0(T) \) contains the full temperature dependence. The same decomposition holds in the diquark condensation phase and we shall employ it throughout this paper. The propagator of the \( P \)-mode at finite temperature can be rewritten as,

\[ G^P_0 (m_\pi, T) = \int_0^\infty dt \frac{e^{-tm^2_\pi}}{(4\pi)^{d/2}d^{d/2+1}} \sum_{n=-\infty}^{\infty} e^{-n^2/4dT^2}. \quad (5.39) \]

The \( T \)-independent part is given by the \( n = 0 \) term and can be written as,

\[ \Delta^P_0 = - \int \frac{d^dp}{(2\pi)^d} \ln(p^2 + m^2_\pi) = \frac{1}{(4\pi)^{d/2}} \Gamma(-\frac{d}{2}) m^d_\pi. \quad (5.40) \]

This divergence is canceled by the counter terms in \( \mathcal{L}^{(4)} \) as was explicitly shown in [20]. After performing the \( t \)-integration over the \( n \neq 0 \) terms we find for the temperature dependent part,

\[ g^P_0 (T) = 16 \frac{m^2_\pi T^2}{(4\pi)^2} \sum_{n=1}^{\infty} \frac{K_2(m_\pi n)}{n^2}, \quad (5.41) \]

where \( K_2 \) is the modified Bessel function. To evaluate \( G^Q_0 \) it is helpful to use the identity,

\[ (p^2 + m^2_\pi - 4\mu^2_B)^2 + 16\mu^2_B p^2_0 \]
\[ = \left( (p_0 - 2i\mu_B)^2 + \vec{p}^2 + m^2_\pi \right) \left( (p_0 + 2i\mu_B)^2 + \vec{p}^2 + m^2_\pi \right). \quad (5.42) \]
This results in,
\[
G_0^Q = \frac{2}{(4\pi)^{d/2}} \int_0^\infty dt \frac{e^{-tm^2}}{t^{1+d/2}} \sum_{n=-\infty}^{\infty} \cosh\left(\frac{2\mu_B n}{T}\right) e^{-n^2/4T^2}. \tag{5.43}
\]

The \(T\)-independent part of this function diverges in 4 dimensions,
\[
\Delta_0^Q = -\int \frac{d^d p}{(2\pi)^d} \ln \left((p^2 + m_\pi^2 - 4\mu_B^2)^2 + 16\mu_B^2 \rho_0^2\right) = \frac{2}{(4\pi)^{d/2}} \Gamma\left(-\frac{d}{2}\right) m^d, \tag{5.44}
\]
and is renormalized by counter terms in \(\mathcal{L}^{(4)}\). The temperature dependent part reads,
\[
g_0^Q(T) = 32 \frac{m^2 T^2}{(4\pi)^2} \sum_{n=1}^\infty K_2\left(\frac{m_\pi T}{n^2}\right) \cosh\left(\frac{2\mu_B n}{T}\right). \tag{5.45}
\]

The temperature dependent parts \(g_0^P(T)\) and \(g_0^Q(T)\) are finite but are not known analytically for arbitrary mass and chemical potential. However, two interesting limits can be analyzed analytically: \(T \ll \mu_B, m_\pi\) and \(T \gg \mu_B, m_\pi\).

Looking first at the \(T \ll \mu_B, m_\pi\) limit, we make use of the asymptotic series of \(K_2\) and find,
\[
g_0^Q(T) = \sqrt{2} T^4 \left(\frac{m_\pi}{\pi T}\right)^{3/2} \sum_{n=1}^\infty \frac{1}{n^{5/2}} e^{(2\mu_B - m_\pi) n/T} \left[1 + \frac{15T}{8nm_\pi} + \ldots\right] \tag{5.46}
\]
\[
= \sqrt{2} T^4 \left(\frac{m_\pi}{\pi T}\right)^{3/2} \left[\text{Li}_{5/2}\left(e^{(2\mu_B - m_\pi)/T}\right) + \frac{15T}{8m_\pi} \text{Li}_{7/2}\left(e^{(2\mu_B - m_\pi)/T}\right) + \ldots\right].
\]

The polylogarithm in the second line is defined by,
\[
\text{Li}_n(z) \equiv \sum_{k=1}^\infty \frac{z^k}{k^n}. \tag{5.47}
\]

For \(T \ll m_\pi - 2\mu_B\), the sum is dominated by the \(n = 1\) term. When \(2\mu_B \to m_\pi\), the difference \(m_\pi - 2\mu_B\) can become comparable to or even smaller than \(T\). If so, more terms will become important in the sum over \(n\). This is not the case for the \(P\) modes as their masses are independent of \(\mu_B\) when \(\alpha = 0\). Hence, for all \(T \ll m_\pi\) we find,
\[
g_0^P = \frac{T^4}{\sqrt{2}} \left(\frac{m_\pi}{\pi T}\right)^{3/2} e^{-m_\pi/T} \left[1 + \frac{15T}{8m_\pi} + \ldots\right]. \tag{5.48}
\]

11
In the limit $T \gg \mu_B$, the integrated propagators are given by,

$$
\begin{align*}
    g_0^P &= -\frac{T}{\pi^2} \int dp \, p^2 \ln\left[1 - \exp(-\sqrt{p^2 + m_\pi^2}/T)\right] = \frac{\pi^2 T^4}{45} - \frac{m_\pi^2 T^2}{12} + \ldots, \\
    g_0^Q &= -\frac{T}{\pi^2} \int dp \, p^2 \left( \ln\left[1 - \exp((-2\mu_B - \sqrt{p^2 + m_\pi^2})/T)\right] + \ln\left[1 - \exp((2\mu_B - \sqrt{p^2 + m_\pi^2})/T)\right] \right) + \ldots \\
    &= \frac{2\pi^2 T^4}{45} - \frac{m_\pi^2 T^2}{6} + \frac{4\mu_B^2 T^2}{3} + \ldots.
\end{align*}
$$

(5.49)

(5.50)

The two terms in $g_0^Q$ give the respective contributions from the diquarks and the antidiquarks.

Having calculated the non-zero temperature contributions to the free energy in the normal phase we now turn to the condensates. The baryon density is defined as,

$$
n_B \equiv -\frac{d\Omega}{d\mu_B}.
$$

(5.51)

In the normal phase only the $Q$-modes depend on $\mu_B$. In the low temperature limit, $T \ll m_\pi$, we thus find,

$$
n_B = \frac{1}{4} N_f (N_f - 1) \left(\frac{2m_\pi T}{\pi}\right)^{3/2} \text{Li}_{3/2}(e^{(2\mu_B - m_\pi)/T}).
$$

(5.52)

This result agrees with the standard form of the density of a relativistic Bose gas (see eg. [36]).

The calculation of the temperature dependence of the chiral condensate to 1-loop order is straightforward as well. Using the definition,

$$
\langle \bar{\psi}\psi \rangle \equiv -\frac{\partial \Omega}{\partial m},
$$

(5.53)

we find,

$$
\langle \bar{\psi}\psi \rangle(T) = \langle \bar{\psi}\psi \rangle_0 + \frac{1}{2} (N_f^2 - 1) e^{-m_\pi/T} \frac{G}{2F^2 \sqrt{m_\pi}} \left(\frac{T}{2\pi}\right)^{3/2} (3T - 2m_\pi) \\
    + \frac{1}{4} N_f (N_f - 1) \frac{G}{F^2 \sqrt{m_\pi}} \left(\frac{T}{2\pi}\right)^{3/2} \left(3TLi_{5/2}(e^{(2\mu_B - m_\pi)/T}) - 2m_\pi Li_{3/2}(e^{(2\mu_B - m_\pi)/T})\right).
$$

(5.54)
The zero temperature part, \( \langle \bar{\psi} \psi \rangle_0 \), was evaluated in [20],

\[
\langle \bar{\psi} \psi \rangle_0 = 2N_f G \left( 1 + 2 \frac{M^2}{F^2} \left[ 8N_f L_6 + 2L_8^r + H_2^r - \frac{2N_f^2 - N_f - 1}{128 \pi^2 N_f} \ln \frac{M^2}{\Lambda^2} \right] \right). \tag{5.55}
\]

Since we are studying the limit \( T \ll m_\pi \), eq. (5.55) can be simplified to,

\[
\langle \bar{\psi} \psi \rangle(T) = \langle \bar{\psi} \psi \rangle_0 - \frac{1}{2} N_f (N_f - 1) e^{-m_\pi/T} \frac{G \sqrt{m_\pi} G}{F^2} \left( \frac{T}{2\pi} \right)^{3/2} - \frac{1}{2} N_f (N_f - 1) \frac{G \sqrt{m_\pi} G}{F^2} \left( \frac{T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{(2\mu_B - m_\pi)/T} \right). \tag{5.56}
\]

The \( T \neq 0 \) correction is negative. The \( \mu_B \) dependence of the \( T \neq 0 \) correction is through the \( \text{Li}_{3/2} \left( e^{(2\mu_B - m_\pi)/T} \right) \) factor of the \( Q \) modes. The corrections are finite and for \( (m_\pi - 2\mu_B)/T \sim 1 \) the \( \mu_B \)-dependent term dominates the exponentially suppressed contribution of the \( P \)-modes.

### 5.2 The Phase with Condensed Diquarks at \( T \neq 0 \)

In the phase of condensed diquarks, the free energy is found to be given by [20],

\[
\Omega = -2N_f F^2 \left( \frac{1}{2} (M_1^2 + M_2^2) + \frac{1}{4} M_3^2 \right) - \frac{1}{4} N_f (N_f + 1) G_0^{P_S} - \frac{1}{4} N_f (N_f - 1) - 2G_0^{P_A} - \frac{1}{4} N_f (N_f - 1) G_0^Q - 2N_f (M_1^2 - M_2^2)^2 \left( L_0 + 2N_f L_4 + 2N_f L_2 + L_3 \right) - 4N_f (M_1^2 - M_2^2)(M_2^2 + \frac{1}{4} M_3^2) \left( 2N_f L_4 + L_5 \right) - 8N_f (M_2^2 + \frac{1}{4} M_3^2) \left( 2N_f L_6 + L_8 \right) - 2N_f \tilde{M}^4 \left( -2L_8 + H_2 \right) + O(p^6). \tag{5.57}
\]

The one-loop integrals of the inverse propagators (4.31) are given by,

\[
G_0^{P_S} = - \oint \frac{d^d p}{(2\pi)^d} \ln \left( p^2 + M_1^2 + \frac{1}{4} M_3^2 \right), \tag{5.58}
\]

\[
G_0^{P_A} = - \oint \frac{d^d p}{(2\pi)^d} \ln \left( p^2 + M_2^2 + \frac{1}{4} M_3^2 \right), \tag{5.59}
\]

13
and,

\[
G_0^Q = -\oint \frac{d^4p}{(2\pi)^d} \ln \left( (p^2 + M_1^2)(p^2 + M_2^2) + p_0^2 M_3^2 \right).
\]  

(5.60)

The decomposition of \(G_0^P\) and \(G_0^P A\) into a temperature independent part \(\Delta_0\) and a temperature dependent part \(g_0(T)\) is completely analogous to the evaluation of the \(P\)-modes in the normal phase, see (5.40), (5.41). The \(Q\)-modes, however, do not have the standard \(p^2 + m^2\) form, and the separation \(G_0^Q(T) = \Delta_0^Q + g_0^Q(T)\) is non-trivial. Fortunately this technical problem can be dealt with in the physically interesting region for which \(\mu B \simeq m_\pi / 2 = \mu_c(T = 0)\). In this region the rotation angle \(\alpha\) introduced in (4.24) is necessarily small and may be used as an expansion parameter. In the following we focus on this regime.

If we define,

\[
\bar{\mu} \equiv (\mu_B - m_\pi / 2) / m_\pi,
\]  

(5.61)

we know from the zero-temperature results [13, 20] that \(\bar{\mu}_B \sim \alpha^2\) and that \(\phi \sim \alpha^3\). We now compute the temperature-dependent part of (5.60) up to sixth order in \(\alpha\). This allows us to make the decomposition \(G_0^Q(T) = \Delta_0^Q + g_0^Q(T)\) in the region \(\mu_B \simeq \mu_c(T = 0)\).

The expansion of \(G_0^Q\) is given by,

\[
G_0^Q = -\oint \frac{d^4p}{(2\pi)^d} \ln \left[ (p_0 + im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right] \left[ (p_0 - im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right] \left( (p_0 + im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right) - \frac{1}{4}(M_1^2 - M_2^2)^2 + O(\alpha^8)
\]  

(5.62)

\[
= -\oint \frac{d^4p}{(2\pi)^d} \ln \left[ (p_0 + im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right] \left( (p_0 - im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right) + \frac{1}{4}(M_1^2 - M_2^2)^2 \oint \frac{d^4p}{(2\pi)^d} \left( (p_0 + im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right) \left( (p_0 - im_\pi y)^2 + \bar{p}^2 + m_\pi^2 z^2 \right) + O(\alpha^8),
\]  

(5.63)

where we have defined,

\[
y \equiv \frac{1}{2} \frac{M_3}{m_\pi},
\]  

(5.64)

\[
z \equiv \frac{1}{m_\pi} \sqrt{\frac{1}{2} (M_1^2 + M_2^2) + \frac{1}{4} M_3^2}.
\]
The first term looks exactly like the \(Q\)-propagator at the origin in the normal phase (cf. (5.43)). We find that the \(T\)-dependent part of the first integral in (5.63) is given by

\[
\mathcal{I}(T) \equiv -\int \frac{d^dp}{(2\pi)^d} \ln \left[ \left( (p_0 + im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \left( (p_0 - im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \right] = 2 \pi \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{m_\pi z T}{n} \right)^{2} \cosh \left( \frac{m_\pi y n T}{T} \right) K_2 \left( \frac{m_\pi z n}{T} \right). \tag{5.65}
\]

The \(n \neq 0\) terms in the second integral in (5.63) are finite and can be rewritten as,

\[
\mathcal{J}(T) \equiv \int \frac{d^dp}{(2\pi)^d} \left( (p_0 + im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \left( (p_0 - im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \frac{1}{(p_0^2)} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p_0^2} \left( \frac{1}{z} \frac{\partial}{\partial z} + \frac{1}{y} \frac{\partial}{\partial y} \right) \ln \left[ \left( (p_0 + im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \left( (p_0 - im_\pi y)^2 + \vec{p}^2 + m_\pi^2 z^2 \right) \right] = -\frac{1}{4\pi^2 m_\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2\pi n T)^2} \left( \frac{1}{z} \frac{\partial}{\partial z} + \frac{1}{y} \frac{\partial}{\partial y} \right) \left( \frac{m_\pi z T}{n} \right)^{2} \cosh \left( \frac{m_\pi y n T}{T} \right) K_2 \left( \frac{m_\pi z n}{T} \right) \tag{5.66}
\]

To go from the second to the third line, we have used the same steps as in (5.45) and (5.65).

To order \(\alpha^4\) we find that the temperature-dependent contribution of the \(Q\)-modes for \(\mu_B \sim m_\pi/2\) is given by

\[
g_0^Q(T) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{m_\pi z T}{n} \right)^{2} \cosh \left( \frac{m_\pi y n T}{T} \right) K_2 \left( \frac{m_\pi z n}{T} \right) + (M_1^2 - M_2^2)^2 \frac{m_\pi z}{64 \pi^4 y T} \sum_{n=1}^{\infty} n^{-3} \left[ y \cosh \left( \frac{m_\pi y n T}{T} \right) K_1 \left( \frac{m_\pi z n}{T} \right) - z \sinh \left( \frac{m_\pi y n T}{T} \right) K_2 \left( \frac{m_\pi z n}{T} \right) \right]. \tag{5.67}
\]

Inserting this expression into \(\Omega\) of (5.57) gives the free energy for \(\mu_B \sim \mu_c(T = 0)\). The free energy gives an effective potential for \(\alpha\) as a function of \(T\) and \(\mu_B\). Using this effective potential we now study the diquark phase transition in the \((\mu_B, T)\)-plane.
6 Second Order Phase Transition and Tricritical Point

Recent lattice simulations \[27\] of two-color QCD at non-zero temperature and baryon chemical potential have shown that the phase transition between the normal phase and the superfluid phase is of second order for small temperatures and chemical potentials close to \(\frac{\pi}{2}\) and that this second order phase transition line in the \((\mu_B, T)\)-plane ends in a tricritical point. In this section we show that Chiral Perturbation Theory confirms the existence of this tricritical point. We shall use the effective potential for the orientation angle \(\alpha\). For the shortcomings and problems associated with the use of the effective potential approach to determine the properties of the phase diagram, we refer to \[37\].

The sum in (5.67) is not in general analytic but can be evaluated in the high and low temperature limits. Only the low temperature limit is of relevance in this case since the critical temperature, \(T_c\), is much smaller than \(\frac{\pi}{2}\) for the values of \(\mu_B\) under consideration, that is \(T_c(\mu_B \sim \frac{\pi}{2}) \ll \frac{\pi}{2}\). (This is a well known property of Bose condensation: \(T_c \ll M\) if \(n_B \ll M^3\), see eg. \[38\].) In setting up the low temperature limit there are two choices, \(T(\mu_B \sim \frac{\pi}{2}) \ll \frac{\pi}{2}(z - y) \ll \frac{\pi}{2}\) or \(\frac{\pi}{2}(z - y) \ll T \ll \frac{\pi}{2}z\). In order to separate them it is useful to know the leading order in the expansion of \(y\) and \(z\) and \(z - y\) in powers of \(\alpha\) and \(\bar{\mu}\). To order \(\alpha^4\) they are given by,

\[
\begin{align*}
z &= 1 + \bar{\mu}\alpha^2 - \frac{1}{16}\alpha^4 + \frac{1}{2}\alpha \phi + \ldots, \\
y &= 1 + 2\bar{\mu} - \frac{1}{2}\alpha^2 - \bar{\mu}\alpha^2 + \frac{1}{24}\alpha^4 + \ldots, \\
z - y &= -2\bar{\mu} + \frac{1}{2}\alpha^2 + \frac{1}{2}\phi \alpha + 2\bar{\mu}\alpha^2 - \frac{5}{48}\alpha^4 + \ldots, \\
(M_1^2 - M_2^2)^2 &= m_\pi^4 \alpha^4 + \ldots.
\end{align*}
\]

The critical chemical potential at \(T \neq 0\) is the value of \(\mu\) for which the vacuum angle \(\alpha\) becomes non-zero. The angle \(\alpha\) is determined by minimizing the free energy with respect to \(\alpha\). It is thus essential that we have not fixed \(\alpha\) at the \(T = 0\) saddle point (4.27). The dimensionless quantities we expand in here are \(\alpha, \bar{\mu} \sim \alpha^2, \phi \sim \alpha^3\). The resulting free energy corresponds to that of a Landau-Ginzburg approach with \(\alpha\) being the relevant order parameter.

The critical temperature, \(T_c\), is the temperature for which the free energy is minimized by \(\alpha = 0\). From (6.68) we have that \(z - y < 0\) for \(\alpha = 0\). The critical temperature thus violates the bound \(T_c \ll m_\pi(z - y)\). Consequently we can not consistently evaluate \(T_c\) in this limit. Instead we focus on the physically relevant limit \(m_\pi(z - y) \ll T \ll m_\pi z\).

The inverse propagators of the \(P_S\) and the \(P_A\) modes have the standard \(p^2 + m_\pi^2\) form.
(cf. \(\text{(4.31)}\)), and the functions \(g^P_0\) and \(g^A_0\) in the \(T \ll m_\pi z\) limit are given by

\[
g^P_0 = T^4 \frac{1}{\sqrt{2}} \left( \frac{\sqrt{M_1^2 + M_2^2/4}}{\pi T} \right)^{3/2} e^{-\sqrt{M_1^2 + M_2^2/4}/T} \tag{6.69}
\]

\[
g^A_0 = T^4 \frac{1}{\sqrt{2}} \left( \frac{\sqrt{M_1^2 + M_2^2/4}}{\pi T} \right)^{3/2} e^{-\sqrt{M_1^2 + M_2^2/4}/T}. \tag{6.70}
\]

As compared to \(g^Q_0\) these contributions are exponentially suppressed and can be neglected.

Consider the function \(g^Q_0\) given in \(\text{(5.67)}\) in the limit \(T \ll m_\pi\) but \(T \gg m_\pi (z - y)\). For \(T \ll m_\pi z\), the asymptotic expansion of \(K_2\) can be used, which results in,

\[
g^Q_0 = \sqrt{\frac{2m_\pi^2 z^3 T^5}{\pi^3}} \sum_{n=1}^{\infty} n^{-5/2} \cosh \left( \frac{m_\pi y n}{T} \right) \exp \left[ - \frac{m_\pi n z}{T} \right] \left[ 1 + \frac{15T}{8m_\pi zn} + \ldots \right] + \frac{(M_1^2 - M_2^2)^2}{512y \sqrt{2\pi^7} z m_\pi T} \sum_{n=1}^{\infty} n^{-9/2} \exp \left[ - \frac{m_\pi n z}{T} \right] \left[ y(3T + 8nm_\pi) \cosh \left( \frac{m_\pi y n}{T} \right) \right]
\]

\[
- z(5T + 8nm_\pi) \sinh \left( \frac{m_\pi y n}{T} \right). \tag{6.71}
\]

Since \(z - y \ll z + y\) and using the definition of the polylogarithm function \(\text{(5.47)}\) we find that,

\[
g^Q_0 = \sqrt{\frac{m_\pi^2 z^3 T^5}{2\pi^3}} \left[ \text{Li}_{5/2}(e^{-m_\pi(z-y)/T}) + \frac{15T}{8m_\pi z} \text{Li}_{7/2}(e^{-m_\pi(z-y)/T}) + \ldots \right] - \frac{(M_1^2 - M_2^2)^2}{1024y \sqrt{2\pi^7} z m_\pi T} \left[ 8m_\pi z (z - y) \text{Li}_{7/2}(e^{-m_\pi(z-y)/T}) \right.
\]

\[
\left. + 3T(5z - y) \text{Li}_{9/2}(e^{-m_\pi(z-y)/T}) \right] + \ldots. \tag{6.72}
\]

Now since \(m_\pi(z - y) \ll T\) we can expand \(\text{Li}_k(e^{-x})\) about the origin, see \(\text{(39)}\). For the leading order term we get that,

\[
g^Q_0 = \sqrt{\frac{m_\pi^2 z^3 T^5}{2\pi^3}} \left[ \zeta(5/2) - \zeta(3/2) \left( \frac{m_\pi (z - y)}{T} \right) - \Gamma(-3/2) \left( \frac{m_\pi (z - y)}{T} \right)^{3/2} \right] + \frac{1}{2} \zeta(1/2) \left( \frac{m_\pi (z - y)}{T} \right)^2 - \frac{3(M_1^2 - M_2^2)^2}{256} \zeta(9/2) \sqrt{\frac{T}{2\pi^7 m_\pi z}} + O(\alpha^6). \tag{6.73}
\]

From now on, and for simplicity, we set \(\phi = 0\). We are now ready to evaluate the free
energy,

\[
\Omega = -N_f F^2 \left( \text{const} + (4\mu_B^2 - m_B^2)\alpha^2 - \frac{1}{12}(16\mu_B^2 - M^2)\alpha^4 + 2a_4 \frac{m_\pi^4}{F^2} \alpha^4 \right) + \frac{1}{4F^2} (N_f - 1) g_0^Q ,
\]

where the \( T = 0 \) part, evaluated in [20], is given by,

\[
a_4 = -\frac{5}{3}(2N_f L_4' + L_6') + \frac{4}{3}(2N_f L_6' + L_8') + (L_6 + 2N_f L_1 + 2N_f L_2' + L_5') \quad (6.75)
\]

\[
- \frac{N_f - 1}{512\pi^2 N_f} + \frac{N_f + 1}{384\pi^2 N_f} \ln \frac{M^2}{\Lambda^2}.
\]

The minimum of the free energy determines the ground state of the theory. With the free energy (6.74), the roots of the equation \( \partial \Omega / \partial \alpha = 0 \) are given by \( \alpha = 0 \) and the roots of the following quadratic equation in \( \alpha^2 \)

\[
c_2 + c_4 \alpha^2 + c_6 \alpha^4 = 0 \quad (6.76)
\]

where, to leading order, we find that

\[
c_2 = -\left(32\sqrt{2}\pi^3 F^2 \bar{\mu} - (N_f - 1) \zeta(\frac{3}{2}) \sqrt{m_\pi T^3}\right)^2 + \cdots, \quad (6.77)
\]

\[
c_4 = \frac{1}{4}(N_f - 1)^2 m_\pi T^2 \left(8m_\pi - 2m_\pi \zeta(\frac{1}{2}) \zeta(\frac{3}{2}) - 3T \zeta^2(\frac{3}{2})\right)
\]

\[
+ 8\sqrt{2F^2\pi^{3/2}} \left(32\sqrt{2}\pi^3 F^2 \bar{\mu} - (N_f - 1) \zeta(\frac{3}{2}) \sqrt{m_\pi T^3}\right)
\]

\[
\left(1 - \frac{3(N_f - 1) \sqrt{m_\pi T}}{256\sqrt{2}\pi F^2} \left(16\pi^2 \zeta(\frac{1}{2}) - \zeta(\frac{9}{2}) - \frac{a_4 m_\pi^2}{2\pi^2 F^2}\right)\right)
\]

\[
+ \frac{19}{12} \left(32\sqrt{2}\pi^3 F^2 \bar{\mu} - (N_f - 1) \zeta(\frac{3}{2}) \sqrt{m_\pi T^3}\right)^2 + \cdots
\]

\[
c_6 = -32\pi^3 F^4 + \cdots.
\]

An effective potential of the form \( \Omega = c_0 + \frac{1}{2} c_2 \alpha^2 + \frac{1}{4} c_4 \alpha^4 + \frac{1}{6} c_6 \alpha^6 \) would lead to the same equation as (6.76). As it is well known from the literature, the signs of the coefficients \( c_2 \) and \( c_4 \) determine the order of the phase transition (see e.g. [19]). If \( c_2 = 0 \) and \( c_4 > 0 \), the phase transition is second order, and if \( c_2 = 0 \) and \( c_4 = 0 \), there is a tricritical point.
From $c_2 = 0$, we find that the phase transition occurs at,

$$\bar{\mu}_{\sec} = \frac{N_f - 1}{32 F^2_\pi} \frac{3}{2} \sqrt{\frac{m_\pi T^3}{2\pi^3}}.$$  \hfill (6.79)

From $c_2 = 0$ and $c_4 = 0$, we find a tricritical point at,

$$\bar{\mu}_{\tri} = \frac{(N_f - 1)m^2_{\pi}}{6\sqrt{3\zeta^2(\frac{3}{2})} F^2_\pi} \left( 1 - \frac{\zeta(1/2)\zeta(3/2)}{4\pi} \right)^{3/2},$$  \hfill (6.80)

$$T_{\tri} = 2m_{\pi} \frac{4\pi - \zeta(1/2)\zeta(3/2)}{3\zeta^2(3/2)}.$$  \hfill (6.81)

For $T < T_{\tri}$, we therefore find a second order phase transition at,

$$\mu_{\sec}(T) = m_{\pi} \left( \frac{1}{2} + \bar{\mu}_{\sec}(T) \right)$$  \hfill (6.82)

$$= \frac{m_{\pi}}{2} + \frac{1}{32 F^2_\pi} (N_f - 1) \sqrt{\frac{m^2_{\pi} T^3}{2\pi^3} \zeta(\frac{3}{2})}.$$  \hfill (6.83)

Alternatively this equation gives $T_{\sec}$ as a function of $\mu_B$.

It is instructive to compare the results obtained with the predictions of a dilute Bose gas in the canonical ensemble. In that approach the baryon density, $n_B$, is fixed and the interaction is determined by the quartic terms in the chiral Lagrangian. In order to make the comparison recall that near the phase transition point at $T = 0$ we have to leading order in $\mu_B - M/2$ \cite{15},

$$n_B = 32N_f F^2(\mu_B - \frac{M}{2}) = 32N_f F^2 M \bar{\mu}.$$  \hfill (6.84)

Inserting this relation in (6.83) we find that the critical temperature and the number density are related by,

$$n_B = 2N_f (N_f - 1) \zeta(\frac{3}{2}) \left( \frac{m_{\pi} T_{\sec}(n_B)}{2\pi} \right)^{3/2},$$  \hfill (6.85)

which exactly corresponds to the relation found in the semi-classical approach \cite{11}.

Finally, notice that both $\bar{\mu}_{\tri}$ and $T_{\tri}$ vanish in the chiral limit. This is in complete agreement with the results obtained within Chiral Perturbation Theory at zero temperature in the chiral limit: the phase transition is first order and $\mu_c = \bar{\mu}_c = 0$ \cite{14, 15}.
Notice that the tricritical temperature is of the same order as $m_\pi$. The approximation that we used in the last stage of the derivation of (6.76) is not fully justified for such high temperatures. The tricritical temperature given above is therefore approximate. In order to get a more precise answer, one needs to numerically compute the different integrals involved in the free energy. The most important point here is that Chiral Perturbation Theory confirms the existence of a tricritical point.

### 7 Order Parameter and Number Density

The order parameter of the diquark-condensation phase is the diquark condensate $\langle \psi \bar{\psi} \rangle = \langle \bar{\psi} \psi \rangle_0 \sin \alpha$, where $\langle \bar{\psi} \psi \rangle_0$ is the quark-antiquark condensate at zero temperature and chemical potential. The value of the order parameter close to the second order phase transition in the $(\mu_B, T)$-plane is therefore given by the root of (6.76). To leading order, we obtain,

$$\alpha^2 = 8(\bar{\mu} - \bar{\mu}_{\text{sec}}) + \cdots$$

$$= \frac{8}{m_\pi} \left( \mu_B - \frac{m_\pi}{2} - \frac{N_f - 1}{32 F_\pi^2} \left( \frac{3}{2} \right) \frac{m_\pi^3 T^3}{2\pi^3} \right).$$

(7.86)

In the diquark condensation phase, the number density is given by,

$$n_B = -\frac{d\Omega}{d\mu_B} = -\frac{1}{M} \frac{d\Omega}{d\bar{\mu}} = 4 N_f M F_\pi^2 (\alpha^2 + 4 \bar{\mu}_{\text{sec}}).$$

(7.87)

Both $\alpha$ and $n_B$ decrease with increasing $T$ at fixed $\bar{\mu} > \bar{\mu}_c$. In the zero temperature limit, we recover the results found in [21] for both the order parameter and the number density.

### 8 Other ensembles

So far we have studied two-color QCD with quarks transforming under the fundamental representation of the gauge group. The low-energy chiral Lagrangian based on the assumption of the spontaneous breaking of the $SU(2N_f)$ classical chiral symmetry leaving $Sp(2N_f)$ invariant. In the classification given in the introduction this is the universality class labeled by $\beta_D = 1$. There exist two other classes of chiral Lagrangians based on spontaneous breaking of chiral symmetry. The symmetry breaking patterns are
SU(2N_f) \rightarrow SO(2N_f) \text{ and } SU(N_f) \times SU(N_f) \rightarrow SU(N_f). \text{ As discussed in the introduction, the first case (} \beta_D = 4 \text{) applies to QCD with quarks in the adjoint representation of the gauge group, whereas the second case (} \beta_D = 2 \text{) corresponds to ordinary QCD with } N_c \geq 3. \text{ It is straightforward to extend the results obtained above to these two classes of theories and we do so in the following two subsections.}

8.1 Quarks in the adjoint representation of the gauge group

The chiral Lagrangian based on the coset SU(2N_f)/SO(2N_f) is similar in almost all respects to the chiral Lagrangian for two-color QCD. In both theories some Goldstone modes correspond to diquark states. The diquarks have baryon charge 2 and consequently there is a phase transition at \( \mu_B = m_\pi / 2 \). From the study to leading order in \(^1\) we know that the propagators in the two theories are identical, and only the degeneracy of the modes are different. Extending the results obtained above to \( \beta_D = 4 \) thus simply amounts to changing appropriate combinatorial factors. For the convenience of the reader we explicitly report the results.

In the \( \alpha = 0 \) phase the baryon density is (this is the analogue of (5.52)),

\[
n_B \equiv \frac{1}{4} N_f(N_f + 1) \left( \frac{2m_\pi T}{\pi} \right)^{3/2} \text{Li}_{3/2}(e^{(2\mu_B - m_\pi)/T}). \tag{8.88}
\]

The chiral condensate is (\( \alpha = 0 \)),

\[
\langle \bar{\psi}\psi \rangle(T) = \langle \bar{\psi}\psi \rangle_0 + \frac{1}{2}(N_f^2 - 1)e^{-m_\pi/T} \frac{G}{2F^2 \sqrt{m_\pi}} \left( \frac{T}{2\pi} \right)^{3/2} \left( 3T - 2m_\pi \right) \\
+ \frac{1}{4} N_f(N_f + 1) \frac{G}{F^2 \sqrt{m_\pi}} \left( \frac{T}{2\pi} \right)^{3/2} \left( 3T \text{Li}_{5/2}(e^{(2\mu_B - m_\pi)/T}) \right) \\
- 2m_\pi \text{Li}_{3/2}(e^{(2\mu_B - m_\pi)/T}). \tag{8.89}
\]

The free energy of the diquark condensation phase is given by,

\[
\Omega = -N_f F^2 \left( \text{const} + (4\mu_B^2 - m_\pi^2)\alpha_2 - \frac{1}{12}(16\mu_B^2 - M^2)\alpha_4 + \frac{1}{4F^2}(N_f + 1)g_0^Q \right) \tag{8.90}
\]

where \( g_0^Q \) is the same as in the two-color case and is given in (6.73). We ignored the zero-temperature contribution that corresponds to (6.75) since, as in the two-color case studied in the previous sections, it does not contribute to the critical points to leading
order. This free energy is extremely similar to (6.74). The analysis of the phase diagram proceeds in the same way as in Section 6.

We find that the critical chemical potential for the second order phase transition is given by,

$$\mu_{\text{sec}}(T) = \frac{m_\pi}{2} + \frac{N_f + 1}{32 F_\pi^2} \sqrt{\frac{m_\pi^3 T^3}{2\pi^3}} \zeta(\frac{3}{2}),$$  \hfill (8.91)$$

and that the phase transition turns first order at the tricritical point given by,

$$\mu_{\text{tri}} = \frac{m_\pi}{2} + \frac{(N_f + 1)m_\pi^2}{6\sqrt{3}\zeta^2(\frac{3}{2})F_\pi^2} \left(1 - \frac{\zeta(1/2)\zeta(3/2)}{4\pi}\right)^{3/2},$$  \hfill (8.92)$$

$$T_{\text{tri}} = \frac{4\pi - \zeta(1/2)\zeta(3/2)}{3\zeta^2(3/2)}.$$  \hfill (8.93)$$

The order parameter close to the second order phase transition is readily obtained in the same way as in the two-color case. To leading order, we obtain,

$$\alpha^2 = 8(\bar{\mu} - \bar{\mu}_{\text{sec}}),$$  \hfill (8.94)$$

where $\bar{\mu}_{\text{sec}}$ is given by (8.91), (5.61). In the diquark condensation phase, the number density is given by,

$$n_B = 4N_f MF_\pi^2(\alpha^2 + 4\bar{\mu}_{\text{sec}}).$$  \hfill (8.95)$$

### 8.2 Ordinary QCD with $\mu_I \neq 0$

In QCD with $N_c \geq 3$ and quarks in the fundamental representation of the gauge group the Goldstone manifold is $SU(N_f)$. The diquarks are no longer Goldstone modes and the chiral Lagrangian does not couple to $\mu_B$. The pions do however have non-trivial isospin and consequently the chiral Lagrangian couples to a chemical potential, $\mu_I = (\mu_u - \mu_d)/2$, for the third component of isospin. This theory is equivalent to QCD at nonzero baryon chemical potential but with a fermion determinant replaced by its absolute value. It was shown in [16, 17] that pion condensation occurs for $\mu_I \geq m_\pi$ at $T = 0$. An isospin chemical potential also has non-trivial effects for $\beta_D = 1$ and $\beta_D = 4$. As was shown in [18] the competition between $\mu_I$ and $\mu_B$ can lead to novel first order phase transitions. Similar phenomena occur in QCD with $N_c = 3$ and $N_f = 3$ [19]. In this system the competition
is established between $\mu_I$ and the strangeness chemical potential $\mu_S$. Here we shall give the leading temperature effects for QCD with $N_c \geq 3$ and quarks in the fundamental representation with a generalized isospin chemical potential. For an even number of flavors $N_f = 2n$ we take $n$ flavors with chemical potential $\mu_I$ and $n$ flavors with chemical potential $-\mu_I$.

The particular interest in this system comes from that fact that the fermion determinant in the path integral of the Euclidean partition function is positive, see [43]. The first lattice study of QCD at non-zero $\mu_I$ and $T$ appeared recently [28]. The similarity with two color QCD at $\mu_B \neq 0$ is striking. Again the chiral Lagrangian is completely analogous to QCD with two colors and describes the same physics. At $T = 0$ and $\mu_I = m_\pi$ there is a second order phase transition to a pion condensed phase and with increasing temperature this phase transition turns first order.

At $\mu_I = 0$ and $m = 0$ this system has an $SU(N_f) \times SU(N_f) \times U(1)$ classical symmetry which is assumed to be spontaneously broken to $SU(N_f) \times U(1)$ by the vacuum. There are $N_f^2 - 1 \equiv 4n^2 - 1$ Goldstone modes which organize themselves into $n^2$ isotriplets and $n^2 - 1$ isosinglets. At $\mu_I \neq 0$, the $SU(N_f) \times U(1)$ is explicitly broken down to $SU(n) \times SU(n) \times U(1)$ leaving only internal rotations in the $u$ and the $d$ sector invariant. At $\mu_I = m_\pi$ the $SU(n) \times SU(n) \times U(1)$ invariance is spontaneously broken down to $SU(n)$ leaving $n^2$ massless modes. The leading order chiral Lagrangian is [16, 17],

$$L^{(2)} = \frac{F^2}{4} \text{Tr} \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger - \frac{F^2 M^2}{2} \text{ReTr} \Sigma,$$

(8.96)

where,

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - \delta_{\mu\nu} \frac{\mu_I}{2} (\tau_3 \Sigma - \Sigma \tau_3).$$

(8.97)

The isospin charge matrix is denoted $\tau_3$. Since the isospin charge of the pions is equal to one we redefine $\bar{\mu}$ as,

$$\bar{\mu} \equiv (\mu_I - m_\pi)/m_\pi.$$

(8.98)

We obtain the free energy for $T << M$,

$$\Omega = -N_f F^2 \left( \frac{1}{2} (M_1^2 + M_2^2) + \frac{1}{4} M_3^2 \right) - \frac{1}{2} n G_0^Q,$$

(8.99)

where, as in [18],

$$M_1^2 \equiv \bar{M}^2 \cos(\alpha - \phi) - \mu_I^2 \cos 2\alpha ,$$

$$M_2^2 \equiv \bar{M}^2 \cos(\alpha - \phi) + \mu_I^2 \cos 2\alpha ,$$

$$M_3^2 \equiv \bar{M}^2 \sin(\alpha - \phi).$$

(8.101)
\[ M_2^2 \equiv \tilde{M}^2 \cos(\alpha - \phi) - \mu_I^2 \cos^2 \alpha, \]  
\[ M_3^2 \equiv 4\mu_I^2 \cos^2 \alpha. \]  

In the \( \alpha = 0 \) phase the isospin density is,

\[ n_I \equiv \frac{1}{8} n^2 \left( \frac{2m_\pi T}{\pi} \right)^{3/2} \text{Li}_{3/2}(e^{(\mu_I - m_\pi)/T}). \]  

\[ n_I \equiv \frac{1}{8} n^2 \left( \frac{2m_\pi T}{\pi} \right)^{3/2} \text{Li}_{3/2}(e^{(\mu_I - m_\pi)/T}). \]  

The free energy of the diquark condensation phase is given by,

\[ \Omega = -\frac{n}{2} F^2 \left( \text{const} + (\mu_I^2 - m_\pi^2) \alpha^2 - \frac{1}{12} (4\mu_I^2 - M^2) \alpha^4 + \frac{n}{F_\pi^2} g_0^Q \right), \]  

where \( g_0^Q \) is the same as in the two-color case and is given in (6.73), but with,

\[ z = 1 + \frac{1}{2} \tilde{\mu} \alpha^2 - \frac{1}{16} \alpha^4 + \ldots, \]  
\[ y = 1 + \tilde{\mu} - \frac{1}{2} \alpha^2 - \frac{1}{2} \tilde{\mu} \alpha^2 + \frac{1}{24} \alpha^4 + \ldots, \]  

\[ M_1^2 - M_2^2 = m_\pi^2 \alpha^2 + \ldots, \]  

instead of (6.68) and \( \tilde{\mu} \) as defined in (8.98). We have ignored the zero-temperature contribution that corresponds to (6.75) since it does not contribute to the critical points to leading order as found in Section 6. This free energy is extremely similar to (6.74). The analysis of the phase diagrams proceeds in the same way as in Section 6.

We find that the critical isospin chemical potential for the line of second order phase transitions is given by,

\[ \mu_{\sec}(T) = m_\pi + \frac{n}{4 F_\pi^2} \left( \frac{m_\pi^2 T^3}{2 \pi^3} \zeta \left( \frac{3}{2} \right) \right), \]  

\[ \mu_{\sec}(T) = m_\pi + \frac{n}{4 F_\pi^2} \left( \frac{m_\pi^2 T^3}{2 \pi^3} \zeta \left( \frac{3}{2} \right) \right), \]  

\[ \mu_{\sec}(T) = m_\pi + \frac{n}{4 F_\pi^2} \left( \frac{m_\pi^2 T^3}{2 \pi^3} \zeta \left( \frac{3}{2} \right) \right) \]
and that this second order phase transition line ends in a tricritical point at

\[
\mu_{\text{tri}} = m_\pi + 4 \frac{nm_\pi^3}{3 \sqrt{3} \zeta^2\left(\frac{3}{2}\right) F_\pi^2} \left(1 - \frac{\zeta(1/2)\zeta(3/2)}{4\pi}\right)^{3/2},
\]

(8.106)

\[
T_{\text{tri}} = 2m_\pi \frac{4\pi - \zeta(1/2)\zeta(3/2)}{3\zeta^2(3/2)}.
\]

(8.107)

The order parameter close to the second order phase transition is easily obtained in the same way as in the two-color case. To leading order, we obtain,

\[
\alpha^2 = 4(\bar{\mu} - \bar{\mu}_{\text{sec}}),
\]

(8.108)

where \(\bar{\mu}_{\text{sec}}\) is given by (8.105), (8.98). In the superfluid phase the number density is given by,

\[
n_f = nMF_\pi^2(\alpha^2 + 2\bar{\mu}_{\text{sec}}).
\]

(8.109)

9 Summary and Conclusions

In this article we have studied the phase diagram of various QCD-like theories at non-zero chemical potential and non-zero temperature. These studies have been performed by means of chiral Lagrangians with the symmetries of the microscopic theories. The chiral Lagrangians describe the low-energy dynamics of the Goldstone modes and are limited to phases where the symmetry of the microscopic Lagrangian is spontaneously broken by the ground state. Although such theories cannot be used to study the high temperature restoration of the broken symmetry, they are very powerful to study the different low-energy phases.

Using Chiral Perturbation Theory, we have demonstrated that the phase diagram of QCD with two colors and quarks in the fundamental representation at non-zero temperature and baryon chemical potential has a very rich structure. At low temperature, there are two different phases: a “normal” phase with a non-zero quark-antiquark condensate, and a superfluid diquark condensation phase with a non-zero diquark condensate. For small enough temperatures, we have found that a second order phase transition separates these two phases. Our result for this second order phase transition line in the \((\mu_B, T)\)-plane are in complete agreement with the semi-classical dilute gas approximation. However, we have shown analytically that this second order phase transition line ends in a tricritical point when the chemical potential is increased. Above this point, the two phases are separated by a first order phase transition. All our results are solely based on symmetry and
are therefore very robust. They are in complete agreement with recent lattice simulations [14, 44] of this theory at non-zero temperature and chemical potential and with recent Random Matrix [45] studies.

We have also analyzed the phase diagram of theories that share many properties of QCD with two colors and quarks in the fundamental representation at non-zero temperature and baryon chemical potential. We have shown that the phase diagram of QCD with any number of colors and quarks in the adjoint representation at non-zero temperature and baryon chemical potential, as well as the phase diagram of QCD with three colors and quarks in the fundamental representation at non-zero temperature and isospin chemical potential are very similar to the phase diagram of QCD with two colors and quarks in the fundamental representation at non-zero temperature and baryon chemical potential that was described above. We again found a second order phase transition separating the normal phase from a superfluid phase at small temperature that ends in a tricritical point when the temperature is increased.

These different QCD-like theories all contain Goldstone modes in their spectrum at zero temperature and zero chemical potential. They all react similarly upon the introduction of a chemical potential associated with a quantum number carried by at least one of these Goldstone modes: At low enough temperature and high enough chemical potential, the ground state of the theory is a superfluid. The normal and superfluid phases are separated by a second order phase transition at small temperatures. This second order phase transition lines ends in a tricritical point and the phase transition is first order when the temperature increases. This remains true for any physical system with spontaneous breaking of a global symmetry, and their phase diagram at non-zero temperature and chemical potential will be similar to the one described above. In all these theories, the order parameter of the superfluid phase and the number density also behave similarly. These results are again in complete agreement with available lattice simulations. The results of this article could be used as non-trivial tests for new simulation schemes developed to solve the sign problem [46].

Acknowledgments: G. Baym, B. Klein, J. Kogut, J.T. Lenaghan, R. Pisarski, B. Vanderheyden, are acknowledged for useful discussions. D.T. is supported in part by “Holderbank”-Stiftung. This work was partially supported by the US DOE grant DE-FG-88ER40388 and by the NSF grant NSF-PHY-0102409.

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