On List Decoding of Insertion and Deletion Errors

Shu Liu,† Ivan Tjuawinata ‡ and Chaoping Xing ‡

Abstract

Insertion and deletion (Insdel for short) errors are synchronization errors in communication systems caused by the loss of positional information of the message. Since the work by Guruswami and Wang [12] that studied list decoding of binary codes with deletion errors only, there have been some further investigations on the list decoding of insertion codes, deletion codes and insdel codes. However, unlike classical Hamming metric or even rank-metric, there are still many unsolved problems on list decoding of insdel codes.

The purpose of the current paper is to move toward complete or partial solutions for some of these problems. Our contributions mainly consist of three parts. Firstly, we provide an upper bound on the list decoding radius of an insdel code in terms of its rate. This bound provides some improvements when degenerated to insertions only and deletions only compared to the previous results in [17]. Secondly, we analyse the list decodability of random insdel codes. It shows that although there is a gap between the list decoding radius of random insdel codes and our upper bound on list decoding radius, when the alphabet size is sufficiently large, this gap no longer exists. In addition, we show that list decoding of random insdel codes surpasses the Singleton bound when there are more insertion errors than deletion errors and the alphabet size is sufficiently large. We also find that our results improve some previous findings in [17] and [12]. Furthermore, our results reveal the existence of an insdel code that can be list decoded against insdel errors beyond its minimum insdel distance while still having polynomial list size. Lastly, we construct a family of explicit insdel codes with efficient list decoding algorithm. As a result, we obtain a Zyablov-type bound for insdel errors.

1 Introduction

Insertion and deletion (Insdel for short) errors are synchronization errors [16], [17] in communication systems caused by the loss of positional information of the message. They have recently attracted many attention due to their applicabilities in many interesting fields such as DNA storage and DNA analysis [23], [38], race-track memory error correction [3] and language processing [2], [32].

The study of codes with insertion and deletion errors was pioneered by Levenshtein, Varshamov and Tenengolts in the 1960s [36], [27], [26] and [35]. This study was then further developed by Brakensiek, Guruswami and Zbarsky [1]. There have also been different directions for the study of insdel codes such as the study of some special forms of the insdel errors [34], [4], [25] and [31] as well as their relations with Weyl groups [18].

*Shu Liu is with the National Key Laboratory of Science and Technology on Communications, University of Electronic Science and Technology of China, Chengdu 611731, China, and also with the Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371 (email: shuliu@uestc.edu.cn).

†Ivan Tjuawinata is with the Strategic Centre for Research on Privacy-Preserving Technologies and Systems, Nanyang Technological University, Singapore 637553 (ivan.tjuawinata@ntu.edu.sg).

‡Chaoping Xing is with the Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371 (email: xingcp@ntu.edu.sg).
Previous results

Guruswami and Wang [12] studied list decoding of binary codes with deletion errors only. They provided a decoding radius for binary codes with deletion only. In addition, they explicitly constructed binary codes with decoding radius close to $\frac{1}{2}$ for deletion errors only. Wachter-Zeh [37] firstly considered the list decoding of insdel codes and provided a Johnson-type upper bound on list size in terms of minimum insdel distance of a given code in 2017. Hayashi and Yasunaga [19] provided some amendments on the result in [37] and derived a Johnson-type upper bound which is only meaningful when insertion occurs.

Based on the indexing scheme and concatenated codes, they further provided efficient encoding and decoding algorithms by concatenating an inner code achieving this Johnson-type bound and an outer list-recoverable Reed-Solomon code achieving the classical Johnson bound. In 2018, Haeupler, Shahrasbi and Sudan [17] constructed a family of list-decodable insdel codes through the use of synchronization strings with larger list decoding radius (beyond Johnson-type upper bound) for sufficiently large alphabet size and designed its efficient list decoding algorithm. Furthermore, instead of insdel errors, they derived some upper bounds on list decodability for insertion or deletion errors only. Lastly, they considered the list decodability of random codes with insertion or deletion errors only. Their results reveal that there is a gap between the upper bound on list decodability of insertion (or deletion) codes and list decodability of a random insertion (or deletion) code. Haeupler and Rubinstein [15] introduced probabilistic fast-decodable indexing schemes for insdel distance which reduces the computing complexity of list decoding algorithm in [17].

Previous findings that we have discussed above leave several problems: (i) what is the list decodability of a random insdel code? (ii) are there some reasonable upper bounds on list decoding radius of insdel codes in terms of rate? (iii) is there a Zyablov-type bound for insdel codes for small alphabet size $q$?

Our results

In this paper, we focus on the list decoding of insdel codes. Our results are mainly divided into three parts. Firstly, we establish an upper bound on list decodability of insdel codes.

Secondly, we analyse the list decodability of random insdel codes. It shows that although there is a gap between the list decodability of random insdel codes and the upper bound we have derived, this gap no longer exists when the alphabet size is sufficiently large. Interestingly, the list decodability of random insdel codes surpasses the Singleton bound when there are more insertion errors than deletion errors with the alphabet size is sufficiently large. This characteristic is not found in codes of many other metrics such as Hamming metric, rank-metric, cover-metric and symbol-pair metric. Another phenomenon that can be observed from the list decodability of random insdel codes is the existence of insdel codes that can be list decoded against insdel errors beyond its minimum distance when the alphabet size is sufficiently large. This does not happen for other metrics.

Lastly, we construct a family of $q$-ary insdel codes which can be efficiently list decoded. Since the construction of explicit insdel codes for sufficiently large $q$ has been discussed in [17], our construction focuses on smaller $q$, even when $q = 2$. As a result, we derive a Zyablov-type bound.

Our techniques

To obtain an upper bound on list decodability of insdel codes and derive list decodability of random insdel codes, the key part is to estimate the size of an insdel ball. This is much more complicated than classical Hamming or rank-metrics. We develop some tricks to get tighter bounds on size of insdel balls.
Firstly, we only estimate the number of vectors in the insdel ball with the same length as the code length. This directly eliminates all the other elements of the insdel ball with inappropriate lengths. Secondly, due to the minimality requirement of insdel distance and the commutativity of insertion and deletion operations up to some repositionings, to enumerate these vectors, we separate to two phases; insertion phase and deletion phase where we use existing estimates on both phases when only one of the operations occurs. In contrast to the insertion sphere size which can be calculated exactly, we only have an upper bound and a lower bound for the deletion sphere size which are not asymptotically tight and depend on the number of runs of the centre. To tighten these bounds, we classify the possible centres to several cases based on the number of runs that they have. This leads to asymptotically tighter bounds in all cases. Having these bounds on the estimate of insdel ball size, they are then used in the calculation of the upper bound of limit of list decoding of insdel codes and the list decodability of random insdel codes.

More specifically, the upper bound on list decodability of insdel codes is calculated through the following analysis. Assuming that an insdel code $C$ is list-decodable with normalized list decoding radius $\tau$ and list size $L$, we analyse the maximum rate of $C$ with respect to $\tau$ while keeping $L$ to be polynomial. The list decoding radius of a random insdel codes of rate $R$ can be analysed in two steps. Firstly, we compute the probability that a random code of rate $R$ is list-decodable up to normalized list-decoding radius $\tau$. Having this probability, we derive a restriction of $R$ and $\tau$ to make this probability negligibly close to 0.

As for our explicit construction, to increase the rate of our concatenated code, we reduced the indexing scheme size. Due to this reduction, after the inner decoding, we can no longer directly identify the correct position of each element in the list with respect to the outer codeword. We took an additional step to optimize the classification of the possible position lists. Using this technique, fixing the list-decoding radius, we obtained a code with higher rate. This is true even compared to a concatenated code in [19] with outer code and inner code being the ones used in our construction. Furthermore, compared to the construction in [19], instead of having a separate requirements on the number of insertion errors and deletion errors, our code only bounds the combined number of insertion and deletion errors, allowing our code to list decode a wider range of insertion and deletion errors.

**Comparisons**

Although the authors of [37] and [19] studied list decodability of insdel codes, they mainly focused on relation of distance and list decoding radius. As a random insdel code achieves the Gilbert-Varshamov bound, we can derive list decodability of a random code by plugging the minimum distance to the Johnson bound. In this sense, they derived a list decodability of a random code. Our approach is different, we provide list decodability of a random code directly. It shows that our bound is better than the bound derived from Johnson bound of a random code given in [19]. As other investigations [12], [17] considered insertion or deletion only, we have to degenerate our bounds on insdel errors to the insertion only case and deletion only case when comparing them with the previous results. When degenerating our result on list decodability of random binary codes to deletion only, we obtain the same result given in [12]. When our upper bound on list decodability of insdel errors is degenerated to insertion or deletion only, our result is better than those in [17] for some parameter regimes. Again when list decodability of random insdel codes is degenerated to insertion errors only, our result is better than those in [17] for some parameter regimes. When list decodability of random insdel codes is degenerated to deletion errors only, we get the same result as in [17].

Lastly for explicit construction, our Zyablov-type bound is better than Johnson-type bound given in [19]. This is due to the fact that both inner code and outer code chosen in [19] are worse than ours. When degenerating our explicit insdel codes to binary code with deletion only and decoding radius close to $\frac{1}{2}$, we
get the same result as in [12].

Organization

This paper is organized as follows. In Section 2 we introduce definitions of insdel codes and some preliminaries on list decoding. Section 3 contains the bounds on the number of fixed length words in an insdel ball. In Section 4 we find the maximum list decoding radius of insdel codes. Section 5 is dedicated to the analysis of the list decodability of random insdel codes. Lastly, the construction and decoding algorithm of our list-decodable insdel codes are provided in Section 6.

2 Preliminaries

Let Σ_q be a finite alphabet of size q and Σ_q^n be the set of all vectors of length n over Σ_q. For any positive real number i, we denote by [i] the set of integers \{1, \ldots, [i]\}.

Definition 1. (Insdel distance) The insdel distance d(a, b) between two words a ∈ Σ_q^n_1 and b ∈ Σ_q^n_2 (not necessarily of the same length) is the minimum number of insertions and deletions which is needed to transform a into b.

Note that for two vectors a of length n_1 and b of length n_2, d(a, b) is at least |n_1 - n_2| and is at most n_1 + n_2. The minimum insdel distance of a code C ⊆ Σ_q^n is defined as d(C) = \min_{a, b ∈ C, a ≠ b} \{d(a, b)\}. A code over Σ_q of length n with size M and minimum insdel distance d is called an \( (n, M, d)_{\Sigma_q} \)-insdel codes. Similar to classical Hamming metric codes, we can define the rate and the relative insdel distance of an \( (n, M, d)_{\Sigma_q} \)-insdel code C by

\[ R(C) = \frac{\log_q |C|}{n} \quad \text{and} \quad \delta(C) = \frac{d}{2n}. \]

The relative insdel distance is normalized by 2n instead of n since the insdel distance between two words in Σ_q^n takes a nonnegative integer value up to 2n.

The minimum insdel distance is one of the important parameters for an insdel code. So, it is desirable to keep minimum insdel distance d as large as possible for an insdel code with fixed length n. It has been shown [16] that an \( (n, M, d)_{\Sigma_q} \)-insdel code C must obey the following version of the Singleton bound.

Proposition 1. (Singleton Bound [16]) Let C ⊆ Σ_q^n be an \( (n, M, d)_{\Sigma_q} \)-insdel code of length n and minimum insdel distance 0 ≤ d ≤ 2n, then

\[ M \leq q^{n-d/2+1}. \]

An asymptotic way to state the Singleton bound for an insdel code C in term of its rate and relative minimum insdel distance is \( R(C) + \delta(C) \leq 1 \).

An \( [n, k, d]_{\Sigma_q} \)-insdel code is a \( \Sigma_q \)-linear code over Σ_q of length n, dimension k and minimum insdel distance d.

Then, we provide the definitions of an insertion (or deletion) sphere and an insdel ball.
Definition 2. (Sphere) For a word \( u \in \Sigma_q^n \) and a nonnegative real number \( z \), the deletion sphere centered at \( u \) with radius \( z \) is defined by

\[
S_D(u, z) = \{ v \in \Sigma_q^{n-z} : v \text{ can be obtained from } u \text{ by } z \text{ deletions} \}.
\]

Insertion sphere, denoted by \( S_I(u, z) \), can be defined similarly.

The insdel ball, as an analogue to the Hamming metric ball, is used to count the number of words within a given insdel distance.

Definition 3. (Insdel Ball) For a word \( u \in \Sigma_q^n \) and a nonnegative real number \( z \), the insdel ball centered at \( u \) with radius \( z \) is defined by

\[
B(u, z) = \bigcup_{i=\max\{n-z,0\}}^{n+z} \Sigma_q^i : d(u, v) \leq z \bigg\}.
\]

We now proceed to the definition of list decodability of insdel codes.

Definition 4. For a real \( \tau \geq 0 \), an insdel code \( C \subseteq \Sigma_q^n \) is said to be \((\tau n, L)\)-list-decodable, if for every nonnegative integer \( m \in [n - \tau n, n + \tau n] \) and every \( r \in \Sigma_q^m \),

\[
|B(r, \tau n) \cap C| \leq L.
\]

Then, we introduce the entropy function.

Definition 5. (q-ary Entropy Function) Let \( q \) be an integer and \( x \) be a real number such that \( q \geq 2 \) and \( 0 < x < 1 \). The q-ary entropy function, \( H_q(x) \) is defined as follows

\[
H_q(x) = x \log_q(q - 1) - x \log_q(x) - (1 - x) \log_q(1 - x).
\]

By convention, we define \( H_q(0) = H_q(1) = 0 \).

For the analysis of the results presented in comparison to the results provided in [17], an approximation of the entropy function is sometimes done when \( q \) is sufficiently large. This approximation is based on the following result.

Proposition 2. (see in [33 Proposition 3.3.2]) For small enough \( \epsilon \), for any \( 0 \leq x \leq 1 - \frac{1}{q} \), we have \( H_q(x) \leq x + \epsilon \) if and only if \( q = 2^{\Omega(\frac{1}{\epsilon})} \).

In the same range of \( q \), it can also be readily verified that \( H_q(x) \geq x \). Combined with the upper bound provided in Proposition 2, \( H_q(x) \) can then be approximated simply by \( x \) with arbitrarily small error \( \epsilon \) given that \( q = 2^{\Omega(\frac{1}{\epsilon})} \).

Finally, we provide the definition of a list-recoverable code. List-recoverable codes are used in the explicit construction discussed in Section 6. The study of list-recoverable codes was inspired by Guruswami-Sudan’s list decoding algorithm for Reed-Solomon codes [9]. Many list-recoverable codes have been constructed such as [8], [11], [13], [14], [20], [21] and [24]. In this paper, we use an alternative definition of list-recoverable code.

Definition 6. Let \( 0 < \alpha < 1 \) be a real number, \( \ell \) and \( L \) be two positive integers. A code \( C \subseteq \Sigma_q^n \) is said to be \((\alpha, \ell, L)\)-list-recoverable if for any given of \( n \) sets \( S_1, \cdots, S_n \subseteq \Sigma_q \) such that \( \sum_{i=1}^n |S_i| \leq \ell \), we have

\[
|\{ x = (x_1, \cdots, x_n) \in C : |\{ i \in [n] : x_i \in S_i \}| \geq \alpha n \}| \leq L.
\]
3 Analysis of insdel ball

Note that our interest is in the number of codewords in an insdel ball and our insdel code $C$ is over $\Sigma_q^n$. So to have an estimate that is independent of the actual $C$, this paper focuses on the set of vectors of length $n$ in the insdel ball, $B(r, \tau n) \cap \Sigma_q^n$.

The insertion and deletion operations are commutative up to some adjustments. So, the order of the operations from one word to the other does not matter as long as the number of deletions and insertions are the same. Let $C \subseteq \Sigma_q^n$ be an insdel code and $r \in \Sigma_q^n$ be the received word. We assume that an insdel error of size at most $\tau n$ occurs during transmission for some $\tau \geq 0$. We further assume that $\gamma$ fraction of insertions and $\kappa$ fraction of deletions occurred to obtain $r$, where $\gamma \geq 0$ and $0 \leq \kappa \leq 1$. So, we have

$$B(r, \tau n) \cap \Sigma_q^n = \bigcup_{\gamma + \kappa \leq \tau} \bigcup_{c' \in S_{D}(c', \gamma n)} S_{D}(c', \gamma n) = \bigcup_{\gamma + \kappa \leq \tau} \bigcup_{c' \in S_{D}(r', \gamma n)} S_{D}(c', \kappa n),$$

where $\gamma \leq \frac{n-m}{2n}$ and $\kappa \leq \frac{n-n-m}{2n}$. Define $\gamma^* = \frac{n-m}{2n}$ and $\kappa^* = \frac{n-n-m}{2n}$.

**Definition 7.** For any sequence $s$ and a non-negative integer $n \geq 0$, we define $s^n$ as follows:

$$s^n := \begin{cases} \xi & \text{if } n = 0, \\ (s, s, \ldots, s) & \text{otherwise.} \end{cases}$$

Here $\xi$ represents the empty sequence of length 0. For $s \in \Sigma_q$, to avoid confusion, we will define the repetition sequence by $(s)^n$ instead of $s^n$.

**Definition 8.** For a positive integer $n$, we define the repetition set $R_q(n) \subseteq \Sigma_q^n$ as

$$R_q(n) = \{ (\alpha)^n \in \Sigma_q^n : \alpha \in \Sigma_q \}.$$

Note that for any $c = (c_1, \ldots, c_n) \in C$, the vector $v \triangleq (v^*)^n$ can be obtained from $c$ by performing $(q-1)n$ insertion operations where $v^* \triangleq (0, 1, \ldots, q-1) \in \Sigma_q^q$. Similarly, for any $c \in C$, there must exist an element $x \in \Sigma_q$ which appears for at least $n/q$ times in $c$. So by using $\frac{q-1}{q} n$ deletion operations, any $c \in C$ can always be transformed to a repetition form $(x)^n/q$. So if $|B(r, \tau n) \cap \Sigma_q^n| = poly(n)$ for all $r$, the fraction of insertions $\gamma$ cannot be beyond $q-1$ and the fraction of deletions $\kappa$ cannot be beyond $\frac{q-1}{q}$.

We consider two cases to discuss the bounds on the size of $B(r, \tau n) \cap \Sigma_q^n$ depending on the form of the received word $r \in \Sigma_q^m$.

Firstly, we consider $|B(r, \tau n) \cap \Sigma_q^n|$ when the received word $r \in R_q(m)$.

**Lemma 3.** Given a received word $r \in R_q(m) \subseteq \Sigma_q^m$ with $m \in [n-\tau n, n+\tau n]$ and $\kappa^* = \frac{n-m}{2} \leq \frac{q-1}{q}$, we have

$$|B(r, \tau n) \cap \Sigma_q^n| = q^n H_q(\kappa^*) + O(1).$$

**Proof.** Without loss of generality, assume $r = (0)^m$. Note that for any $x \in B(r, \tau n) \cap \Sigma_q^n$ with Hamming weight $w$, all of the non-zero elements must appear in the insertion phase from $r$. Since we can insert at most $m$ symbols, we have $w_{H}(x) = w \leq \kappa n \leq \frac{n-n-m}{2}$. The last inequality comes from the fact that $\gamma + \kappa \leq \tau$ and $\kappa n - \gamma n = \gamma n = n - m$. Hence, the insertion and deletion processes can be regrouped to two
main steps: adjusting the number of zeros with the appropriate number of insertion or deletion and then inserting the non-zero symbols. So, when we enumerate the number of elements of $\mathcal{B}(r, \tau n) \cap \Sigma_q^n$ with weight $w$, we first transform $(0)^{m}$ to $(0)^{n-w}$ before inserting all the $w$ non-zeros. Enumerating all possible $x \in \mathcal{B}(r, \tau n) \cap \Sigma_q^n$, we have

$$|\mathcal{B}(r, \tau n) \cap \Sigma_q^n| = \sum_{w=0}^{\frac{\tau n + \eta - m}{2}} |\{ e \in \mathcal{S}_q((0)^{n-w}, w), \mathrm{wt}_H(e) = w \}| = \sum_{w=0}^{\frac{\tau n + \eta - m}{2}} \binom{n}{w}(q-1)^w.$$

Since the maximum term is when $w = \frac{\tau n + \eta - m}{2} = \kappa n$, the maximum summand in the last term is $(\binom{n}{\kappa n})(q-1)^{\kappa n} = q^{nH_q(\kappa^n) + O(1)}$. Hence $|\mathcal{B}(r, \tau n) \cap \Sigma_q^n|$ can be bounded by

$$q^{nH_q(\kappa^n) + O(1)} \leq |\mathcal{B}(r, \tau n) \cap \Sigma_q^n| \leq q^{n\left(H_q(\kappa^n) + O\left(\frac{\log_q(n)}{n}\right)\right) + O(1)}.$$

Note that when $n$ is sufficiently large, the two bounds are the same, which is $q^{nH_q(\kappa^n) + O(1)}$. Hence asymptotically, the bounds become equality $|\mathcal{B}(r, \tau n) \cap \Sigma_q^n| = q^{nH_q(\kappa^n) + O(1)}$.

Then, we consider when the received word $r \in \Sigma_q^n \setminus R_q(m)$. In the remainder of this section, we denote $w = \mathrm{wt}_H(r)$. In general, the received word $r$ has the following form

$$r = ((0)^{a_1}, x_1, (0)^{a_2}, x_2, \ldots, (0)^{a_w}, x_w, (0)^{a_{w+1}}),$$

where $a_1, \ldots, a_{w+1} \geq 0$, $a_1 + \cdots + a_{w+1} = m - w$ and $x_1, \ldots, x_w \in \Sigma_q \setminus \{0\}$.

Define $\varphi(r)$ be the number of runs in $r$ where a run in $r$ is a maximum consecutive identical symbol in $r$. For example, the number of runs in $r = (0, 1, 1, 1, 0)$ is 3 while the number of runs in $r = (0, 1, 0, 1)$ is 4. Furthermore, define $t = |\{i \in \{1, \ldots, w+1\} : a_i = 0\}|$.

**Lemma 4.** Assuming $q \geq 3$, $\varphi(r)$ can be tightly bounded by

$$2(w-t) + 1 \leq \varphi(r) \leq 2w - t + 1.$$

*Proof.* Note that when $t = 0$, $\varphi(r) = 2w + 1$. Having one of these $a_i$ to be 0 will decrease $\varphi(r)$ by at least 1, since the run $(0)^{a_i}$ itself is removed. On the other hand, the most reduction to the number of runs that $a_i = 0$ can cause is 2. It happens when $2 \leq i \leq w$ and $x_{i-1} = x_i$. Thus, $2w + 1 - 2t \leq \varphi(r) \leq 2w + 1 - t$. Noting that all non-zero elements must be the same when $q = 2$, the argument above provides us with $\varphi(r) \leq 2(w-t) + 1$. It is easy to see that the upper bound is tight when $t \geq 2$.

To prove the bounds are tight, consider $q \geq 3$, it is sufficient to construct two $r$ with number of runs achieving the two bounds. Consider $a_2 = a_3 = \cdots = a_{t+1} = 1$ and $x_1 = \cdots = x_{t+1} = 1$. Then, the received word $r = ((0)^{a_1}, (1)^{t+1}, (0)^{a_{t+2}}, x_{t+2}, \ldots, (0)^{a_w}, x_w, (0)^{a_{w+1}})$, where $\varphi(r) = 2(w-t) + 1$ proving the tightness of the lower bound. Note that this also proves the tightness of the lower bound for $q = 2$. Consider $a_2 = \cdots = a_{t-1} = 0$, $x_1 = x_2 = \cdots = x_{2i+1} = \cdots = 1$ and $x_2 = x_4 = \cdots = x_{2i} = \cdots = \alpha$ for some non-zero $\alpha \in \Sigma_q$ with $\alpha \neq 1$. So, we have the received word $r = (1, \alpha, 1, \alpha, \cdots, (0)^{a_1}, x_1, \cdots, (0)^{a_w}, x_w)$, where $\varphi(r) = 2w - t + 1$ proving the tightness of the upper bound when $q \geq 3$. \hfill \square
For our analysis in the remainder of this section, we will be using the following two results regarding the size of insertion and deletion spheres.

**Lemma 5.** (see in [23]) For any non-negative integer \( n_2 \) and a vector \( s \in \Sigma_q^{n_1} \), the size of \( S_1(s, n_2) \) can be exactly calculated by

\[
|S_1(s, n_2)| = \sum_{i=0}^{n_2} \binom{n_1 + n_2}{i} (q - 1)^i.
\]

**Lemma 6.** (see in [23]) For any non-negative integer \( n_2 \leq n_1 \) and a vector \( s \in \Sigma_q^{n_1} \), the size of \( S_D(s, n_2) \) can be tightly bounded by

\[
\sum_{i=0}^{n_2} \binom{\varphi(s) - n_2}{i} \leq |S_D(s, n_2)| \leq \binom{\varphi(s) + n_2 - 1}{n_2}.
\]

**Remark 1.** Having the tight bounds of \( \varphi(r) \) in Lemma 4 and \( S_D(r, \gamma n) \) in Lemma 5 these bounds result in the following tight bounds

\[
q^{(2w - 2t + 1 - \gamma n)H_q\left(\frac{\gamma n}{2w - t + 1}\right) + O(1)}(q - 1)^{-\gamma n} \leq |S_D(r, \gamma n)| \leq q^{(2w - t + \gamma n)H_q\left(\frac{\gamma n}{2w - t + 1}\right) + O(1)}(q - 1)^{-\gamma n}
\]

for \( q \geq 3 \) and

\[
2^{(2w - 2t + 1 - \gamma n)H_2\left(\frac{\gamma n}{2w - t + 1}\right) + O(1)} \leq |S_D(r, \gamma n)| \leq 2^{(2w - t + 3 + 2\gamma n)H_2\left(\frac{\gamma n}{2w - t + 1}\right) + O(1)}
\]

when \( q = 2 \). The tightness here is in the sense that they are the maximum and minimum values of \( |S_D(r, \gamma n)| \) for \( r \in \Sigma_q^m \setminus R_q(m) \).

**Lemma 7.** Let \( C \subseteq \Sigma_q^n \) be an insdel code and \( r \in \Sigma_q^n \setminus R_q(m) \) be a received word with the form in [2], such that \( m \in \left[ n - \kappa n, n + \gamma n \right] \), \( 0 \leq \kappa < \frac{q - 1}{q}, \gamma < q - 1 \) and \( \kappa + \gamma = \tau \geq 0 \). Let \( \gamma^* = \frac{\tau n - m + \kappa}{2 n} \) and \( \kappa^* = \frac{n + \gamma n - m}{2 n} \). Then, for \( q \geq 3 \), the size of \( B(r, \tau n) \cap \Sigma_q^n \) is upper bounded by

\[
|B(r, \tau n) \cap \Sigma_q^n| \leq q^{(2w - t + \gamma^* n)H_q\left(\frac{\gamma^* n}{2w - t + 1}\right) - \gamma^* n \log_q(q - 1) + nH_q(\kappa^*) + O(1)}.
\]

When \( q = 2 \), the size of \( B(r, \tau n) \cap \Sigma_q^n \) is upper bounded by

\[
|B(r, \tau n) \cap \Sigma_q^n| \leq 2^{(2w - t + 2 + \gamma^* n)H_2\left(\frac{\gamma^* n}{2w - t + 1}\right) + nH_2(\kappa^*)} + O(1).
\]

**Proof.** The following proof works for \( q \geq 3 \). The proof can also be applied for \( q = 2 \) by using Inequality 4. Let \( r \) be a received word of length \( m \). During the transmission, suppose that \( \gamma n \) insertions and \( \kappa n = n + \gamma n - m \) deletions occur. Thus, \( n + \gamma n - \kappa n = m \). To enumerate the elements in the ball, first we enumerate the elements in \( S_D(r, \gamma n) \). Then the size of \( S_1(c', \kappa n) \) is calculated for \( c' \in S_D(r, \gamma n) \).
Since \((2w - t + \gamma n)H_q\left(\frac{\gamma n}{2w - t + \gamma n}\right) - \gamma n \log_q (q - 1)\) and \(nH_q(\kappa)\) are increasing functions on \(\gamma \leq \gamma^*\) and \(\kappa \leq \kappa^*\) respectively, we have

\[
|\mathcal{B}(r, \tau n) \cap \Sigma^n_q| \leq q^{(2w - t + \gamma^* n)H_q\left(\frac{\gamma^* n}{2w - t + \gamma^* n}\right) - \gamma^* n \log_q (q - 1) + nH_q(\kappa^*) + O(1)}
\]

Now, we will give some definitions and lemmas before providing the lower bound for the size of \(\mathcal{B}(r, \tau n) \cap \Sigma^n_q\).

**Definition 9.** Let \(m\) be a positive integer and \(v = (v_1, \ldots, v_m) \in \Sigma^n_q\). For any \(\alpha \in \Sigma_q\), we define \(n_\alpha(v)\) to be the number of entries of \(v\) that has value \(\alpha\). That is,

\[
n_\alpha(v) = |\{i \in \{1, \ldots, m\} : v_i = \alpha\}|
\]

If there is no confusion on the value of \(v\), we omit it from the notation and just write \(n_\alpha\).

**Lemma 8.** Without loss of generality, we can assume that the Hamming weight of the received word \(r\), \(w = \text{wt}_H(r) \leq m - \frac{m}{q}\).

**Proof.** Let the received word \(r = (r_1, \ldots, r_m)\) and \(\alpha^* \in \Sigma_q\) be the element of \(\Sigma_q\) that occurs most frequently in \(r\). Since \(r \not\in R_q(m)\), so \(n_{\alpha^*} \leq m - 1\). We relabel the elements in \(r\) with value \(\alpha^*\) to 0 and the ones with value \(0\) to \(\alpha^*\). After relabelling, we have \(n_0 = \max\{n_\alpha : \alpha \in \Sigma_q\}\). By Pigeonhole Principle, \(n_0 \geq \frac{m}{q}\) and hence \(w = \text{wt}_H(r) \leq m - \frac{m}{q}\).

**Remark 2.** Note that given the values of \(w\) and \(t\), we have two cases to consider: \(m - w \geq w + 1\) and \(m - w \leq w + 1\). When \(m - w \geq w + 1\), we have \(w \leq \frac{m - 1}{2}\) and \(0 \leq t \leq w\). On the other hand, if \(m - w \leq w + 1\), based on Lemma 8 we have \(\frac{m - 1}{2} \leq w \leq m \left(1 - \frac{1}{q}\right)\) and \(2w + 1 - m \leq t \leq w\). In both cases, we have that \(2w - t \leq m - 1\), which will be used in analysing the performance of list decodability of random insdel codes.

**Lemma 9.** Let \(C \subseteq \Sigma^n_q\) be an insdel code and \(r \in \Sigma^n_q \setminus R_q(m)\) be a received word with the form (2), such that \(m \in [n - \kappa n, n + \gamma n]\), \(0 \leq \kappa < \frac{q - 1}{q}, \gamma < q - 1\) and \(\kappa + \gamma = \tau \geq 0\). Then, the size of \(\mathcal{B}(r, \tau n) \cap \Sigma^n_q\) is lower bounded by

\[
|\mathcal{B}(r, \tau n) \cap \Sigma^n_q| \geq q^{(m + \kappa^* n)H_q\left(\frac{\kappa^* n}{m + \kappa^* n}\right) + (2w - t + 1 - \gamma^* n)H_q\left(\frac{\gamma^* n}{2(w - t) + 1 - \gamma^* n}\right) - \gamma^* n \log_q (q - 1) + O(1)},
\]

where \(\gamma^* = \frac{\tau n - n + m}{2n}\) and \(\kappa^* = \frac{\tau n + n - m}{2n}\).

**Proof.** During the transmission, suppose that \(\gamma n\) insertions and \(\kappa n = n + \gamma n - m\) deletions occur. Thus, \(n + \gamma n - \kappa n = m\). Since \(\varphi(r) \geq 2w - 2t + 1\) from the Lemma 4 based on the Lemma 6 we have

\[
|\mathcal{S}_D(r, \gamma n)| \geq \sum_{j=0}^{\gamma n} \binom{2(w - t) + 1 - \gamma n}{j}.
\]
Proposition 10. Let \( \binom{m}{i} \) be the size of \( B \) currently not possible to derive the actual Gilbert-Varshamov bound. However, utilizing the upper bound of \( q^{(m+\kappa n)H_q\left(\frac{\kappa n}{m+n}\right)} + (2w-2t+1-\gamma n)H_q\left(\frac{\gamma n}{2w-2t+1-\gamma n}\right), (q-1)-\gamma n \)

Due to the lack of exact estimate of the size of insdel ball when there are non-zero deletions, it is currently not possible to derive the actual Gilbert-Varshamov bound. However, utilizing the upper bound of \( q^{(m+\kappa n)H_q\left(\frac{\kappa n}{m+n}\right)} + (2w-2t+1-\gamma n)H_q\left(\frac{\gamma n}{2w-2t+1-\gamma n}\right), (q-1)-\gamma n \)

\[ |B(r, \tau n) \cap \Sigma_q^n| = \sum_{\gamma = \max \left\{ \frac{m-n}{0} \right\}}^{\gamma_n} \sum_{j=0}^{\gamma_n} \binom{m+\kappa n}{j} (q-1)^i \sum_{j=0}^{\gamma_n} \left( \frac{2w-t+1-\gamma n}{j} \right) \]

Remark 3. Based on Remark 4, the bounds of Lemmas 7 and 9 are tight. When \( q \) is sufficiently large, the bounds of Lemmas 7 and 9 are asymptotically the same,

\[ \lim_{n \to \infty} \frac{\log_q |B(r, \tau n) \cap \Sigma_q^n|}{n} = \frac{\kappa n}{n} = \kappa^* \]

which is the same as the bound in Lemma 3.

4 Limit to list decoding of insdel codes

In this section, we find the maximum list decoding radius of insdel codes, namely the limit of list decodability of insdel codes. The idea of our proof is based on counting argument on the number of words of fixed length in an insdel ball.

It is interesting to look at the Gilbert-Varshamov bound for the insdel codes. The Gilbert-Varshamov bound is known as an upper bound on the list decoding radius under Hamming metric codes [10], rank-metric codes [5], cover-metric codes [29] and symbol-pair metric codes [30]. Thus, any codes under Hamming metric, rank-metric, cover-metric or symbol-pair metric that are list decoded beyond this bound will output an exponential list size. A natural question is whether the Gilbert-Varshamov bound is also the limit to the list decoding of insdel codes.

Due to the lack of exact estimate of the size of insdel ball when there are non-zero deletions, it is currently not possible to derive the actual Gilbert-Varshamov bound. However, utilizing the upper bound of the size of \( B(r, \tau n) \cap \Sigma_q^n \) we have derived in Lemma 7, we can provide an estimate of Gilbert-Varshamov bounds of Lemmas 7 and 9 are asymptotically the same, \( \delta \) is sufficiently large, the

\[ |B(r, \tau n) \cap \Sigma_q^n| \geq \sum_{\gamma = \max \left\{ \frac{m-n}{0} \right\}}^{\gamma_n} \sum_{j=0}^{\gamma_n} \binom{m+\kappa n}{j} (q-1)^i \sum_{j=0}^{\gamma_n} \left( \frac{2w-t+1-\gamma n}{j} \right) \]

Proposition 10. Let \( 0 < \delta \leq \frac{q-1}{q} \), then

\[ \lim_{n \to \infty} \frac{\log_q A_q(n, 2\delta n)}{n} \geq 1 - (1 + \delta)H_q\left(\frac{\delta}{1+\delta}\right) + \delta \log_q(q-1) - H_q(\delta). \]

Proof. Consider \( |B(r, 2\delta n - 1) \cap \Sigma_q^n| \) with \( r \in \Sigma_q^n \), we must have \( \gamma = \kappa \leq \delta - \frac{1}{2} < \frac{q-1}{q} \). For any \( r \in \Sigma_q^n \setminus R_q(n) \), we can simplify the upper bound of Lemma 7 to

\[ |B(r, 2\delta n - 1) \cap \Sigma_q^n| \leq q^{n+\delta n-\frac{1}{2}}H_q\left(\frac{\delta n-\frac{1}{2}}{n+\delta n-\frac{1}{2}}\right) - (\delta n-\frac{1}{2}) \log_q(q-1) + nH_q(\delta) + O(1) \].

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Note that this bound also applies to \( q = 2 \) although it is not tight. For any \( r_1, r_2 \in R_q(n), r_1 \neq r_2 \), we have \( d(r_1, r_2) = 2n > 2\delta n \). So, we can have an insdel code \( C \) such that \( R_q(n) \subseteq C \). Note that this eliminates at most \( q \cdot q^{nH_q(\delta) + O(1)} \) elements of \( \Sigma_q^n \) from being in \( C \). Thus, we have

\[
A_q(n, 2\delta n) \geq q + \frac{q^n - q^{nH_q(\delta) + 1 + O(1)}}{q^{(n-1+\delta n - \frac{3}{2})H_q \left( \frac{\delta n - \frac{1}{2}}{n-1+\delta n} \right) + O(1) - (\delta n - \frac{1}{2}) \log_q(q-1) + nH_q(\delta)}} - \frac{q^n}{q^{(n-1+\delta n - \frac{3}{2})H_q \left( \frac{\delta n - \frac{1}{2}}{n-1+\delta n} \right) + O(1) - (\delta n - \frac{1}{2}) \log_q(q-1) + nH_q(\delta)}} \\
\geq q \left(1 - (1 - \frac{3}{2n} + \delta)H_q \left( \frac{\delta n - \frac{1}{2}}{1 - \frac{3}{2n} + \delta} \right) - O(1) \right) + \frac{q^n}{q^{nH_q(\delta)}} \\
= q \left(1 - (1 - \frac{3}{2n} + \delta)H_q \left( \frac{\delta n - \frac{1}{2}}{1 - \frac{3}{2n} + \delta} \right) - O(1) \right) + \delta \log_q(q-1) - H_q(\delta),
\]

Then, taking limit as \( n \) tends to \( \infty \),

\[
\lim_{n \to \infty} \frac{\log_q A_q(n, 2\delta n)}{n} \geq 1 - (1 + \delta)H_q \left( \frac{\delta}{1 + \delta} \right) + \delta \log_q(q-1) - H_q(\delta),
\]

we obtain the desired result. \( \square \)

We also consider the value of \( A_q(n, 2\delta n) \) when \( \frac{q-1}{q} < \delta < 1 \).

**Proposition 11.** Let \( \frac{q-1}{q} < \delta < 1 \), then \( A_q(n, 2\delta n) = q \) which directly implies \( \lim_{n \to \infty} \frac{\log_q A_q(n, 2\delta n)}{n} = 0 \).

**Proof.** For \( v \in \Sigma_q^n \), denote by \( n_v \triangleq \max_{\alpha \in \Sigma_q} \{n_\alpha(v)\} \) the largest occurrence of any element of \( \Sigma_q \) in \( v \). Furthermore, for any \( \alpha \in \Sigma_q \), denote by \( V_\alpha(q, n) \triangleq \{v \in \Sigma_q^n : n_\alpha(v) \geq \frac{n}{q}\} \) the set of all vectors over \( \Sigma_q^n \) of length \( n \) which has at least \( \frac{n}{q} \) entries having value \( \alpha \). By Pigeonhole Principle, we have that for any \( v \in \Sigma_q^n \), there must exist \( \alpha \in \Sigma_q \) such that \( n_\alpha(v) \geq \frac{n}{q} \). This implies \( \Sigma_q^n = \bigcup_{\alpha \in \Sigma_q} V_\alpha(q, n) \).

Now note that for any \( \alpha \in \Sigma_q \) and \( u, v \in V_\alpha(q, n) \) two distinct elements of \( V_\alpha(q, n) \), from \( u \), we can simply delete all entries except \( \frac{n}{q} \) occurrences of \( \alpha \) (which is guaranteed by the fact that \( u, v \in V_\alpha(q, n) \)) and then insert the appropriate entries to obtain \( v \). So \( d(u, v) \leq \frac{2n(q-1)}{q} \). This shows that if \( C \subseteq \Sigma_q^n \) is an insdel code of minimum insdel distance \( 2\delta n \), we have \( \|C\| \leq 2n(q-1) \). Hence \( A_q(n, 2\delta n) \leq q \).

To show equality, we construct an insdel code \( C \subseteq \Sigma_q^n \) of minimum insdel distance \( 2\delta n \). Consider the following code

\[ C = \{(0)^{n-\delta n}(\alpha)^{\delta n}, \alpha \in \Sigma_q\}. \]

It is easy to see that \( |C| = q \) and \( d(C) = 2\delta n \). This shows the existence of an insdel code of minimum insdel distance \( 2\delta n \) over \( \Sigma_q^n \) proving that \( A_q(n, 2\delta n) = q \), concluding the proof. \( \square \)

**Remark 4.** Proposition 11 shows that for relative minimum insdel distance larger than \( \frac{q-1}{q} \), there does not exist any asymptotically good code. Hence the asymptotic behaviour of codes of relative minimum insdel
distance beyond $\frac{\theta - 1}{q}$ is not of interest and all our analysis will be done with the estimate of the bounds when
the relative minimum indel distance is at most $\frac{\theta - 1}{q}$, which is derived in Proposition 10.

**Remark 5.** Proposition 10 is obtained by combining the upper bound in Lemma 7 with the generic method in deriving the classical Gilbert-Varshamov bound. For any $\epsilon \in (0, 1)$, when $q = 2^{\Omega(1/\epsilon)}$, the lower bound in Proposition 10 can be rewritten as $R \geq 1 - \delta - \epsilon$.

Now we investigate the list decodability of indel codes.

**Theorem 1.** Let $C \subseteq \Sigma_q^n$ be a $(\tau n, L)$-list-decodable indel code of rate $R$ with polynomial list size $L = \text{poly}(n)$. Then, for any $\gamma$ fraction of insertions and $\kappa$ fraction of deletions with $\kappa + \gamma = \tau$, we must have

$$ R \leq 1 - (1 + \gamma - \kappa)H_2 \left( \frac{2\theta}{1 + \gamma - \kappa} \right) \log_q 2 - (2\theta - \gamma)H_q \left( \frac{\gamma}{2\theta - \gamma} \right) + (1 + \gamma - \kappa) $$

$$ - (1 + \gamma)H_q \left( \frac{\kappa}{1 + \gamma} \right) + \gamma \log_q (q - 1) - \theta \log_q (q - 1) $$

with $\theta := \theta(q, \gamma, \kappa) \triangleq \frac{2\gamma + \sqrt{q - 1}(1+2\gamma - \kappa) + \sqrt{(2\gamma + \sqrt{q - 1}(1+2\gamma - \kappa))^2 - \sqrt{q - 1}(4+4\gamma - 1)\gamma(1+\gamma - \kappa)}}{4(1+\sqrt{q - 1})}$.

**Proof.** Let $C$ be an indel code of rate $R$ in $\Sigma_q^n$. Suppose that $C$ is $(\tau n, L)$-list-decodable with $R \geq 1 - (1 + \gamma - \kappa)H_2 \left( \frac{2\theta}{1 + \gamma - \kappa} \right) \log_q 2 - (2\theta - \gamma)H_q \left( \frac{\gamma}{2\theta - \gamma} \right) + (1 + \gamma - \kappa)$. Then, for any $\gamma$ fraction of insertions and $\kappa$ fraction of deletions with $\kappa + \gamma = \tau$ and $\theta = \theta(q, \gamma, \kappa)$. Thus, we have

$$ \sum_{r \in \Sigma_q^n} |B(r, \tau n) \cap C| = \frac{q^{Rn}}{q^n} \left( \sum_{r \in R_q(m)} |B(r, \tau n) \cap \Sigma_q^n| + \sum_{r \in \Sigma_q^n \setminus R_q(m)} |B(r, \tau n) \cap \Sigma_q^n| \right). $$

By Lemmas 3 and 7, we can obtain the lower bound of Equation (6). Note that the lower bound of Lemma 7 depends on $w$ and $t$. Let $\ell = w - t$. For all possible $\ell$, define $A_\ell = |\{ r \in \Sigma_q^m \setminus R_q(m) : r \text{ has exactly } \ell \text{ runs of zero symbols}\}|$. Thus,

$$ A_\ell = \sum_{w \geq \ell} \binom{w + 1}{\ell} \cdot \binom{m - w - 1}{\ell} (q - 1)^w $$

$$ \geq (q - 1)^\ell \sum_{w \geq \ell} \binom{w}{\ell} \cdot \binom{m - w - 1}{\ell} $$

$$ = (q - 1)^\ell \binom{m}{2\ell}. $$

We have

$$ \sum_{r \in \Sigma_q^n \setminus R_q(m)} |B(r, \tau n) \cap \Sigma_q^n| $$

$$ \geq \sum_{r \in \Sigma_q^n \setminus R_q(m)} q^{(m+n)H_q \left( \frac{n}{m+n} + (2(w-t)+1-\gamma n)H_q \left( \frac{n}{2(w-t)+1-\gamma n} \right) - \gamma n \log_q (q-1) + O(1) \right)} $$

$$ \geq q^{(m+n)H_q \left( \frac{n}{m+n} \right) - n \log_q (q-1) + O(1)} \sum_\ell (q - 1)^\ell \binom{m}{2\ell} q^{(2\ell+1-\gamma n)H_q \left( \frac{n}{2\ell+1-\gamma n} \right)} $$

$$ \geq q^{n \left( (1+\gamma)H_q \left( \frac{\kappa}{1+\gamma} \right) - \gamma \log_q (q-1) + \theta \log_q (q-1) + (1+\gamma - \kappa) \right) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) + (2\theta - \gamma)H_q \left( \frac{\gamma}{2\theta - \gamma} \right) $$

$$ \boxed{\geq q^{n \left( (1+\gamma)H_q \left( \frac{\kappa}{1+\gamma} \right) - \gamma \log_q (q-1) + \theta \log_q (q-1) + (1+\gamma - \kappa) \right) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) + (2\theta - \gamma)H_q \left( \frac{\gamma}{2\theta - \gamma} \right)}}.
where \( \ell = \theta n \). By finding the critical point of \( \sum_{l}(q-1)^{\ell} \binom{m}{2\ell} q^{-\gamma \ell} \), the summand reaches its maximum when \( \theta = \theta(q, \gamma, \kappa) \). Hence, we have

\[
\sum_{r \in \Sigma_q^n} |B(r, \tau n) \cap C| \\
\geq q^n \left( R - 1 + (1 + \gamma) H_q \left( \frac{\kappa}{1+\gamma} \right) - \gamma \log_q (q-1) + (1 + \gamma - \kappa) H_q \left( \frac{2\theta}{1+\gamma} \right) \log_q 2 + (2\theta - \gamma) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) \right).
\]

By the Pigeonhole Principle, there exists \( r \in \Sigma_q^n \) such that

\[
|B(r, \tau n) \cap C| \\
\geq q^n \left( R - 1 + (1 + \gamma) H_q \left( \frac{\kappa}{1+\gamma} \right) - \gamma \log_q (q-1) + (1 + \gamma - \kappa) H_q \left( \frac{2\theta}{1+\gamma} \right) \log_q 2 + (2\theta - \gamma) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) \right) - m \\
\geq q^n \left( R - 1 + (1 + \gamma) H_q \left( \frac{\kappa}{1+\gamma} \right) - \gamma \log_q (q-1) + (1 + \gamma - \kappa) H_q \left( \frac{2\theta}{1+\gamma} \right) \log_q 2 + (2\theta - \gamma) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) - (1 + \gamma - \kappa) \right) \\
\geq q^n \epsilon.
\]

\( \square \)

**Corollary 12.** Let \( C \subseteq \Sigma_q^n \) be a \((\tau n, L)\)-list-decodable insdel code of rate \( R \) with polynomial list size \( L = \text{poly}(n) \). Then, for any \( \gamma \) fraction of insertions and \( \kappa \) fraction of deletions with \( \kappa + \gamma = \tau \), we must have \( R \leq 1 - \kappa \), if \( n \) and \( q \) are sufficiently large.

**Proof.** Suppose \( C \) is \((\tau n, L)\)-list-decodable with \( R > 1 - \kappa \). Then, there exists a real \( \epsilon \in (0, 1) \) such that \( R \geq 1 - \kappa + \epsilon \). Note that when \( q = 2^{\Omega(1/\epsilon)} \), we have \( H_q(x) - \epsilon \leq x \leq H_q(x) \). This implies that the lower bound of \( |B(r, \tau n) \cap C| \) in Lemma 9 can be simplified to \( |B(r, \tau n) \cap C| \geq q^n \epsilon \). We can proceed as in Theorem 1 to obtain the result. \( \square \)

**Lemma 13.** Let \( \epsilon \in (0, 1) \) be small and the alphabet \( q = 2^{\Omega(1/\epsilon)} \). If there are \( \gamma \) fraction of insertions and \( \kappa \) fraction of deletions with \( \gamma + \kappa = \tau \) and \( \gamma > \kappa \), the limit of list decodability of insdel codes surpasses the Singleton bound.

**Proof.** Assume \( \tau n = 2\delta n \). Since \( \gamma > \kappa \), so \( \delta = \frac{\gamma + \kappa}{2} > \kappa \). We can obtain \( 1 - \kappa > 1 - \delta \). \( \square \)

**Remark 6.** Lemma 13 shows that the limit of list decodability of insdel codes surpasses the Singleton bound. This can not be found in other metrics such as Hamming metric, rank-metric, cover-metric and symbol-pair metric. We would like to note that this limit can be achieved by the list decoding radius of random insdel codes. This observation is further discussed in Remark 10.

Next, we discuss two degenerations of Theorem 1 when there are insertions only or deletions only.

**Corollary 14.** (Insertions only) Let \( C \subseteq \Sigma_q^n \) be a code of rate \( R \) that is list-decodable against any \( \gamma \) fractions of insertions with polynomial list list \( L = \text{poly}(n) \). Then, for any \( \gamma < q - 1 \), we must have

\[
R \leq 1 - \theta \log_q (q-1) - (1 + \gamma) H_q \left( \frac{2\theta}{1+\gamma} \right) \log_q 2 - (2\theta - \gamma) H_q \left( \frac{\gamma}{2\theta - \gamma} \right) + 1 + \gamma + \gamma \log_q (q-1)
\]

with \( \theta = \theta(q, \gamma, 0) \) where \( \theta(q, \gamma, \kappa) \) is defined in Theorem 1.
Remark 7. Corollary 14 provides an upper bound for the rate depending on the values of $q$ and $\gamma$. Denote this upper bound by $R_1(q, \gamma)$. Similarly, denote by $R_2(q, \gamma)$ the upper bound for the rate given in [17, Theorem 1.2]. Observing the plot of the two curves $R_1(q, \gamma)$ and $R_2(q, \gamma)$ for various $q$, we observe that for $q \geq 2$, $R_1(q, \gamma) \geq R_2(q, \gamma)$ for smaller value of $\gamma$ while $R_1(q, \gamma) \leq R_2(q, \gamma)$ for larger value of $\gamma$. This transition happens approximately when $2 \leq \gamma \leq 3$. In particular, this implies that when $q = 2$ and $3$, $R_1(q, \gamma)$ is always worse than $R_2(q, \gamma)$. This phenomenon can be observed in Figures 1a and 1b.

![Graphs showing the comparison between Corollary 14 and Theorem 1.2 for different values of $q$.](image)

(a) $q = 2$  
(b) $q = 10$

Figure 1: Comparison Figures of Remark 7 for insertions only

Corollary 15. (Deletions only) Let $C \subseteq \Sigma_q^n$ be a code of rate $R$ that is list-decodable against any $\kappa$ fractions of deletions with polynomial list size $L = \text{poly}(n)$. Then, for any $0 \leq \kappa < \frac{q-1}{q}$, we must have

$$R \leq 1 - (1 - \kappa)H_2 \left( \frac{2\theta}{1 - \kappa} \right) \log_q 2 + (1 - \kappa) - H_q(\kappa) - \theta \log_q (q - 1)$$

with $\theta = \theta(q, 0, \kappa)$ where $\theta(q, \gamma, \kappa)$ is defined in Theorem 1. When $q$ is sufficiently large, by Corollary 12 we have $R \leq 1 - \kappa$.

Remark 8. The result in Corollary 15 is then compared with [17, Theorem 1.3] for various $q$. Observing the graphs of the two upper bounds of $R$ for different values of $\kappa$, the following comparison can be observed: When $q = 2$, the upper bound in Corollary 15 is better than the upper bound provided in [17, Theorem 1.3]. This can be observed in Figure 2a. When $q = 3, 4$ and $5$ the upper bound in Corollary 15 is better than the upper bound in [17, Theorem 1.3] if and only if $\kappa \leq \kappa_1(q)$ or $\kappa \geq \kappa_2(q)$ where $\kappa_1(q)$ and $\kappa_2(q)$ are functions of $q$. These values can be found in Table 1. Figures 2b, 2c and 2d provide illustrations on this observation. When $q \geq 6$, the upper bound in Corollary 15 is always worse than the upper bound in [17, Theorem 1.3]. The case when $q = 6$ can be observed in Figure 2e.
Table 1: Table of values of $\kappa_1$ and $\kappa_2$ for $q = 3, 4$ and 5

| $q$ | $\kappa_1(q)$ | $\kappa_2(q)$ |
|-----|---------------|---------------|
| 3   | 0.09          | 0.59          |
| 4   | 0.24          | 0.624         |
| 5   | 0.42          | 0.55          |
5 List decoding of random insdel codes

In this section, we investigate the list decodability of random insdel codes.

**Theorem 2.** Let \( q \geq 3 \). For any \( 0 \leq \gamma < q - 1 \) fraction of insertions, \( 0 \leq \kappa < \frac{q-1}{q} \) fraction of deletions with \( \gamma + \kappa = \tau \) and for every small \( \epsilon \in (0, 1) \), with probability at least \( 1 - q^{-n} \), a random insdel code \( C \subseteq \Sigma_n^q \) of rate

\[
R = 1 - \left( 2\gamma - \kappa + 1 \right) H_q \left( \frac{\gamma}{2\gamma - \kappa + 1} \right) + \gamma \log_q (q - 1) - H_q(\kappa) - \epsilon
\]

is \((\tau n, O(1/\epsilon))\)-list-decodable for all sufficiently large \( n \).

**Proof.** Let \( \mathcal{L} = \left\lfloor \frac{2\gamma - \kappa + 1}{\epsilon} \right\rfloor - 1 \) and \( n \) be a sufficiently large positive integer. Pick an insdel code \( C \) with size \( q^{Rn} \) uniformly at random. We calculate the probability that \( C \) is not \((\tau n, \mathcal{L})\)-list-decodable.

If \( C \) is not \((\tau n, \mathcal{L})\)-list-decodable, there exists a word \( r \in \Sigma^m_q \) for a positive integer \( m \in [n - \tau n, n + \tau n] \) and a subset \( S \subseteq C \) with \( |S| = \mathcal{L} + 1 \) such that \( S \subseteq B(r, \tau n) \).

If \( r \in \Sigma^m_q \), by Lemma 7, the probability that one codeword \( c \in C \) is contained in \( B(r, \tau n) \) is at most

\[
q^{(m-1+\gamma^* n)H_q \left( \frac{\gamma^* n}{m-1+\gamma^* n} \right) - \gamma^* n \log_q (q-1) + n H_q(\kappa^*) + O(1)} - n.
\]

Together with Lemma 3, for a uniformly sampled
r, we have

\[
\Pr[c \in B(r, \tau n)] = \frac{|B(r, \tau n) \cap \Sigma_q^n|}{q^n} \leq q \cdot \frac{|B(r \in R_q(m), \tau n) \cap \Sigma_q^n|}{q^m + q \cdot \frac{|B(r \in \Sigma_q^n \setminus R_q(m), \tau n) \cap \Sigma_q^n|}{q^m}} \leq q^{nH_q(\gamma^n) + 1 - m - n + O(1)} \left(q + (q^m - q)q^{(m-1+\gamma^n)nH_q\left(\frac{n^{\gamma n}}{m-1+\gamma^n}\right)}\right) \leq q^{(m-1+\gamma^n)nH_q\left(\frac{n^{\gamma n}}{m-1+\gamma^n}\right)} - \gamma^n n \log_q(q-1) + nH_q(\kappa^*) - n + O(1)
\]

(8)

where \(\gamma^n = \frac{r-n+m}{2}\) and \(\kappa^* = \frac{r+n-m}{2}\).

Let \(E_{r,S}\) be the event that all codewords in \(S\) are contained in \(B(r, \tau n)\). By Equation (8), we have

\[
\Pr[E_{r,S}] \leq \left(\frac{|B(r, \tau n) \cap \Sigma_q^n|}{q^n}\right)^{\mathcal{L} + 1} \leq \left(q^{(m-1+\gamma^n)nH_q\left(\frac{n^{\gamma n}}{m-1+\gamma^n}\right)} - \gamma^n n \log_q(q-1) + nH_q(\kappa^*) - n + O(1)\right)^{\mathcal{L} + 1}.
\]

Note that since \(\gamma^* \leq \gamma \) and \(\kappa^* \leq \kappa\), this probability achieves its maximum when \(\gamma^* = \gamma, \kappa^* = \kappa\) and hence \(m = (\gamma - \kappa + 1)n\). Taking the union bound over all choices of \(m, q^m\) choices for \(r\) and \(S\) over any \((\mathcal{L} + 1)\)-subsets of \(\mathcal{C}\), we have

\[
\sum_{r,S} \Pr[E_{r,S}] \leq \sum_{m=n-\tau n}^{n+\tau n} q^m \left(\frac{|C|}{\mathcal{L} + 1}\right)^{\mathcal{L} + 1} q^{(m-1+\gamma^n)nH_q\left(\frac{n^{\gamma n}}{m-1+\gamma^n}\right)} - \gamma^n n \log_q(q-1) + nH_q(\kappa^*) - n + O(1)n^{\mathcal{L} + 1} \leq q^{(\gamma-\kappa+1)n}\left|C\right|^{\mathcal{L} + 1} q^{(2\gamma n - \kappa n + n - 1)H_q\left(\frac{n}{2\gamma - \kappa + 1}\right)} - \gamma n \log_q(q-1) + nH_q(\kappa) - n + O(1))^{(\mathcal{L} + 1)} \leq q^{n(\mathcal{L} + 1)} q^{\frac{n}{2\gamma - \kappa + 1} + R'(\gamma-\kappa)H_q\left(\frac{n}{\gamma - \kappa + 1}\right)} - \gamma \log_q(q-1) + H_q(\kappa) - 1 + O(\frac{1}{n}) \leq q^{-n}.
\]

The techniques used in Theorems 1 and 2 can also be used to find a similar result for \(q = 2\).

**Theorem 3.** For any \(0 \leq \gamma < 1\) fraction of insertions, \(0 \leq \kappa < \frac{1}{2}\) fraction of deletions with \(\gamma + \kappa = \tau\) and for every small \(\epsilon \in (0, 1)\), with probability at least \(1 - 2^{-n}\), a random binary insdel code \(C \subseteq \Sigma_q^n\) of rate

\[
R = 1 - (2\theta + \gamma)H_2\left(\frac{\gamma}{2\theta + \gamma}\right) - H_2(\kappa) + (1 + \gamma - \kappa) - (1 + \gamma - \kappa)H_2\left(\frac{2\theta}{1 + \gamma - \kappa}\right) - \epsilon
\]

(9)

is \((\tau n, O(1/\epsilon))\)-list-decodable for all sufficiently large \(n\) with \(\tau := \theta^*(\gamma, \kappa)\), where

\[
\theta^*(\gamma, \kappa) = 1 + 2\gamma - \kappa + \sqrt{(1 + \gamma - \kappa)^2 + 10\gamma(1 + \gamma - \kappa) + \gamma^2}
\]

\[
8
\]

**Remark 9.** Theorem 2 improves the list decoding radius of random insdel codes in [19] for any \(q\). This can be observed in Figure 3.
Corollary 16. For any $0 \leq \gamma < q - 1$ fraction of insertions, $0 \leq \kappa < \frac{q-1}{q}$ fraction of deletions with $\gamma + \kappa = \tau$ and for every small $\epsilon \in (0, 1)$, with probability at least $1 - q^{-n}$, a random insdel code $C \subseteq \Sigma_q^n$ of rate $R = 1 - \kappa - \epsilon$ is $(\tau n, O(1/\epsilon))$-list-decodable for $q = 2^{\Omega(1/\epsilon)}$ and all sufficiently large $n$.

Remark 10. For any $\epsilon \in (0, 1)$, when $q = 2^{\Omega(1/\epsilon)}$, the list decoding radius of random insdel codes can achieve the limit of list decoding radius, which is $R = 1 - \kappa - \epsilon$.

Lemma 17. Let $\epsilon \in (0, 1)$ be small and the alphabet $q = 2^{\Omega(1/\epsilon)}$. If there are $\gamma$ fraction of insertions and $\kappa$ fraction of deletions with $\gamma + \kappa = \tau$ and $\gamma > \kappa + 2\epsilon$, there exists a $(\tau n, O(1/\epsilon))$-list-decodable insdel code with list decoding radius $\tau n$ beyond the minimum insdel distance $d$.

Proof. When $q = 2^{\Omega(1/\epsilon)}$, there exists a $(\tau n, O(1/\epsilon))$-list-decodable insdel code with rate $R = 1 - \kappa - \epsilon$ from Corollary 16. Since $\gamma > \kappa + 2\epsilon \Leftrightarrow \epsilon < \frac{\gamma - \kappa}{2}$, so $R > 1 - \frac{\gamma + \kappa}{2}$. By the Singleton bound, we can obtain the list decoding radius of insdel codes $\tau n$ is larger than $d$. \qed

Remark 11. Generally, the list decoding radius of codes cannot break the minimum distance barrier. This is true for codes in Hamming metric, rank-metric, symbol-pair and cover-metric. Interestingly, under insdel distance, some insdel codes can be list decoded beyond the minimum insdel distance with polynomial list size.

Based on Theorems 2 and 3, we have the following corollaries when only insertions (or deletions) occur.

Corollary 18. (Insertions only) For every small $\epsilon \in (0, 1)$ and $0 \leq \gamma < q - 1$ fraction of insertions, with probability at least $1 - q^{-n}$, a random code $C \subseteq \Sigma_q^n$ of rate

$$R = \begin{cases} 
1 - (1 + 2\gamma) H_q \left( \frac{\gamma}{1 + 2\gamma} \right) + \gamma \log_q(q - 1) - \epsilon & \text{if } q \geq 3 \\
1 - (2\theta + \gamma) H_2 \left( \frac{\gamma}{2\theta + \gamma} \right) + (1 + \gamma) - (1 + \gamma) H_2 \left( \frac{2\theta}{1 + \gamma} \right) - \epsilon & \text{if } q = 2
\end{cases}$$

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is list-decodable against any $\gamma n$ insertions for all sufficiently large $n$ with list size $L = O(1/\epsilon)$ and $\theta = \theta^*(\gamma, 0)$, where $\theta^*(\gamma, \kappa)$ is defined in Theorem 3.

**Remark 12.** We compare Corollary 18 with the result in [17, Theorem 1.7] in three cases; $q = 2, q \geq 3$ and $q = 2^{\Omega(1/\epsilon)}$. Firstly, let $q = 2$. Fixing the value of $\epsilon > 0$ and the list size guaranteed by the two bounds, plotting the two curves provides that the rate in Corollary 18 is better than the rate in [17, Theorem 1.7]. This can be observed from Figure 4. When $q \geq 3$, it can be shown that with fixed $\epsilon$ and the list size $L = \frac{\gamma + 1}{\epsilon} - 1$, the rate provided in Corollary 18 is worse than the rate provided in [17, Theorem 1.7]. Lastly, when $q = 2^{\Omega(1/\epsilon)}$, fixing the values of $\epsilon$ and $R = 1 - \epsilon$, the list size required in Corollary 18 is $\left\lceil \frac{\gamma + 1}{\epsilon} \right\rceil - 1$ while [17, Theorem 1.7] requires $L > \frac{\gamma + 1}{\epsilon} - 1$. So the two list size requirements differ by at most 1, which happens when $\gamma + 1$ is an integer multiple of $\epsilon$.

![Figure 4: Comparison Figure of Remark 12 for insertions only with $q = 2$ and $\epsilon = 0.0001$.](image)

**Remark 13.** When the alphabet size $q$ is sufficiently large, the limit of list decodability of insertion codes from Corollary 14 and [17, Theorem 1.2] are the same and can be achieved by the list decoding radius of random insertion codes.

**Corollary 19.** (Deletions only) For every small $\epsilon \in (0, 1)$ and $0 \leq \kappa < 1$ with probability at least $1 - q^{-\kappa n}$, a random code $C \subseteq \Sigma_q^n$ of rate $R = 1 - H_q(\kappa) - \epsilon$ is list-decodable against any $\kappa$ fraction of deletions for all sufficiently large $n$ with list size $L = O(1/\epsilon)$.

**Remark 14.** The comparison between Corollary 19 and [17, Theorem 1.6] reveals the following result. Fix $\epsilon > 0$ and list size $L = \left\lceil \frac{1+\gamma}{\epsilon} \right\rceil - 1$, the rates of the random list-decodable code $C$ reaches the same value $R = 1 - H_q(\kappa) - \epsilon$. Considering the list decodability of random binary deletion codes, the same analysis reveals that the list decoding radius in Corollary 19 is same as that in [12, Theorem 26]. These observations are illustrated in Figure 5.
Remark 15. When the alphabet size $q$ is sufficiently large, the limit of list decodability of deletion codes from Corollary 15 and [17, Theorem 1.3] are the same and can be achieved by the list decoding radius of random deletion codes. Note that for every small $\epsilon \in (0, 1)$, our case $q = \Omega(1/\epsilon)$ does not depend on $\kappa$ rather than $q = \frac{1}{1-\kappa} \frac{1}{\epsilon}$ in [17, Theorem 1.3], where $\kappa$ is the fraction of deletions.

Reducing the sample space from arbitrary insdel codes to arbitrary $\Sigma_q$-linear insdel code in the above theorems, we have the following results.

Theorem 4. Let $q \geq 3$. For any $0 \leq \gamma < q - 1$ fraction of insertions, $0 \leq \kappa < \frac{q-1}{q}$ fraction of deletions with $\gamma + \kappa = \tau$ and for every small $\epsilon \in (0, 1)$, with probability at least $1 - q^{-n}$, a random $\Sigma_q$-linear insdel code $C \subseteq \Sigma_q^n$ of rate

$$R = 1 - (2\gamma - \kappa + 1) H_q \left( \frac{\gamma}{2\gamma - \kappa + 1} \right) + \gamma \log_q (q - 1) - H_q(\kappa) - \epsilon$$

is $(\tau n, \exp(O(1/\epsilon)))$-list-decodable for all sufficiently large $n$.

Proof. Let $L = \lfloor \frac{2\gamma - \kappa + 1}{\epsilon} \rfloor - 1$ and $n$ be a sufficiently large integer. Then $\log_q (L + 1) = \left\lfloor \frac{2\gamma - \kappa + 1}{\epsilon} \right\rfloor$ and $L = \exp(O(\frac{1}{\epsilon}))$. Pick $Rn$ $\Sigma_q$-linearly independent words uniformly at random from $\Sigma_q^n$. The $\Sigma_q$-linear insdel code $\mathcal{C}$ spanned by these words has rate $R$. If $\mathcal{C}$ is not $(\tau n, L)$-list-decodable, then there exists a word $r \in \mathbb{F}_q^m$ for a positive integer $m \in [n - \tau n, n + \tau n]$ and a subset $\mathcal{S} \subseteq C$ with $|\mathcal{S}| = L + 1$ such that $\mathcal{S} \subseteq B(r, \tau n)$. There are at least $L' = \log_q (L + 1) = \left\lfloor \frac{2\gamma - \kappa + 1}{\epsilon} \right\rfloor$ codewords in $\mathcal{S}$ which are $\Sigma_q$-linearly independent. Let $\mathcal{S}'$ be the $\Sigma_q$-linear span of these $L'$ codewords, thus $\mathcal{S}' \subseteq \mathcal{S}$. Then, $\Pr[E_{r, \mathcal{S}}] \leq \Pr[E_{r, \mathcal{S}'}]$
and

\[
\Pr[E_{r,S}] \leq \left( \frac{|B(r, \tau n)|}{q^n} \right)^{L'} \\
= \left( \frac{q}{q^m} \cdot \frac{|B(r \in R_q(m), \tau n) \cap \Sigma_q^n|}{q^n} + \frac{q^m - q}{q^m} \cdot \frac{|B(r \in \Sigma_q^m \setminus R_q(m), \tau n) \cap \Sigma_q^n|}{q^n} \right)^{L'} \\
\leq \left( q^{(m-1+\gamma^*)n}H_q\left( \frac{\gamma^*}{m-1+\gamma^*} \right) - \gamma^* n \log q(q-1)+nH_q(\kappa^*) - n + O(1) \right)^{L'}.
\]

Note that since \( \gamma^* \leq \gamma \) and \( \kappa^* \leq \kappa \), this probability achieves its maximum when \( \gamma^* = \gamma \), \( \kappa^* = \kappa \) and hence \( m = (\gamma - \kappa + 1)n \). Taking the union bound over all choices of \( m \), \( q^m \) choices for \( r \) and any \( L', \sum_q \)-linearly independent words from \( C \), we can derive the following probability.

\[
\sum_{r,S} \Pr[E_{r,S}] \leq \sum_{m = n-\tau n}^{n+\tau n} q^m \left( \frac{|C|}{L'} \right) \left( q^{(m-1+\gamma^*)n}H_q\left( \frac{\gamma^*}{m-1+\gamma^*} \right) - \gamma^* n \log q(q-1)+nH_q(\kappa^*) - n + O(1) \right)^{L'} \\
\leq q^{(\gamma - \kappa + 1)n}|C|L' q \left( 2\gamma n - \kappa n + n - 1 \right) H_q \left( \frac{\gamma}{2\gamma - \kappa + 1} \right) - \gamma n \log q(q-1)+nH_q(\kappa) - n + O(1) \right)^{L'} \\
\leq q^{nL'} \left( \frac{2\gamma - \kappa + 1}{L'} + R + (2\gamma - \kappa + 1)H_q \left( \frac{\gamma}{2\gamma - \kappa + 1} \right) - \gamma \log q(q-1)+H_q(\kappa) - 1 + O(\frac{1}{n}) \right) \\
\leq q^{-n}. 
\]

\[\square\]

**Theorem 5.** For any \( 0 \leq \gamma < 1 \) fraction of insertions, \( 0 \leq \kappa < \frac{1}{2} \) fraction of deletions with \( \gamma + \kappa = \tau \) and for every small \( \epsilon \in (0,1) \), with probability at least \( 1 - 2^{-n} \), a random binary \( \Sigma_2 \)-linear insdel code \( C \subseteq \Sigma_2^n \) of rate

\[
R = 1 - (2\theta - \gamma)H_2 \left( \frac{\gamma}{2\theta + \gamma} \right) - H_2(\kappa) + (1 + \gamma - \kappa) - (1 + \gamma - \kappa)H_2 \left( \frac{2\theta}{1 + \gamma - \kappa} \right) - \epsilon
\]

is \((\tau n, \exp(O(1/\epsilon)))\)-list-decodable for all sufficiently large \( n \) where \( \theta = \theta^*(\gamma, \kappa) \) as defined in Theorem 3.

**Corollary 20.** For any \( 0 \leq \gamma < q - 1 \) fraction of insertions, \( 0 \leq \kappa < \frac{q-1}{q} \) fraction of deletions with \( \gamma + \kappa = \tau \) and for every small \( \epsilon \in (0,1) \), with probability at least \( 1 - q^{-n} \), an \( \Sigma_q \)-linear random insdel code \( C \subseteq \Sigma_q^n \) of rate \( R = 1 - \kappa - \epsilon \) is \((\tau n, \exp(O(1/\epsilon)))\)-list-decodable for \( q = 2^{\Omega(1/\epsilon)} \) and all sufficiently large \( n \).

### 6 Explicit insdel codes with list decoding algorithm

In this section, we provide an explicit construction of a family of insdel codes that has an efficient decoding algorithm. Similar to [7], [12] and [19], the construction is done by concatenation method and indexing scheme.

In [19], they constructed a family of insdel codes with list decoding radius up to the Johnson-type bound and designed its efficient algorithm. The construction done by Haeupler, Shahrasbi and Sudan in [17] provided a family of list-decodable insdel codes when the alphabet size is sufficiently large. The construction considered there has list decoding radius achieving the limit of list decoding radius that we have derived in
Corollary 12. This paper focuses on the construction of a family of explicit list-decodable insdel codes for smaller alphabet size, even when \( q = 2 \).

Denote our concatenated code by \( C_{\text{conc}} \), with inner code \( C_{\text{in}} \) and outer code by \( C_{\text{out}} \). The outer code \( C_{\text{out}} \) is chosen to be a \( p \)-ary code of length \( N \) and rate \( R_{\text{out}} \) that is \((\alpha_{\text{out}}, \ell_{\text{out}}, \mathcal{L}_{\text{out}})\)-list-recoverable. The inner code \( C_{\text{in}} \) is chosen to be a random \( q \)-ary code of length \( n \) and rate \( R_{\text{in}} \). By Theorems 2 and 3, \( C_{\text{in}} \) is \((\tau_{\text{in}} n, O(1/\epsilon_{\text{in}}))\)-list-decodable with rate \( R_{\text{in}} \). To obtain the codewords in \( c \in C_{\text{conc}} \) from the outer codeword \( c_{\text{out}} = (c_1, \cdots, c_N) \in C_{\text{out}} \), index each \( c_i \) by \( i \mod \epsilon_{\text{cont}} N + 1 \) for some values \( \epsilon_{\text{cont}} \) that will be determined later and encode \( (i \mod \epsilon_{\text{cont}} N + 1, c_i) \) with the encoding function \( \varphi_{\text{in}} : [\epsilon_{\text{cont}} N] \times \Sigma_p \rightarrow \Sigma_q^n \) of \( C_{\text{in}} \), \( c = (\varphi_{\text{in}}(1, c_1), \cdots, \varphi_{\text{in}}((N \mod \epsilon_{\text{cont}} N) + 1, c_N)) \).

Let \( c \in C_{\text{conc}} \subseteq \Sigma_q^n \) be the sent codeword and \( M \in [\max\{0, nN - \tau nN\}, nN + \tau nN] \) be the length of the received word \( r = (r_1, \cdots, r_M) \) such that \( d(c, r) \leq \tau nN \). Denote \( c = (v_1, \cdots, v_N) \) where \( v_i \in C_{\text{in}} \) is the \( i \)-th block of \( c \) and \( r = (w_1, \cdots, w_N) \) such that \( w_i \) is obtained from \( v_i \). Denote by \( \tau_i n \) the length of the received word \( r \).

Then \( \sum_{i=1}^N \tau_i n \leq \tau n N \).

\[
\begin{array}{cccccc}
\text{c} & \vdots & \vdots & \vdots & \vdots & \vdots \\
| & \hat{v}_1 + \\tau_n n & + & \cdots & + & \hat{v}_i + \cdots + \tau_n n \leq \tau n N \\
\downarrow & \hat{w}_1 & + & \\cdots & + & \hat{w}_i & \cdots & + & \cdots & + & \hat{w}_N \\
\end{array}
\]

Figure 6: Partition of the sent codeword \( c \) and the received word \( r \)

### 6.1 Construction of subsequences of the received word

Let \( 0 < \tau^* < \tau_{\text{in}} \) and \( \hat{\tau} \triangleq \tau_{\text{in}} - \tau^* \). Define the following set of subsequences of \( r \) :

\[
S \triangleq \left\{ \left( r_{1+\Phi}, \cdots, r_{\Phi+\Lambda} \right) : \Phi = \lambda \hat{\tau} n, \Lambda = \mu \hat{\tau} n, \lambda, \mu \in \mathbb{Z}, 0 \leq \lambda \leq 1 + \frac{M}{\max(0, 1 - \tau^*)}, \max(0, 1 - \tau^*) \leq \mu \leq 1 + \frac{1 + \tau^*}{\hat{\tau}} \right\}. \tag{10}
\]

**Lemma 21.** Take a subsequence \( w \) of \( r \) with \( w = (r_{1+sp}, \cdots, r_{sp+len}) \). Then there exists a subsequence \( s = (r_{1+\Phi}, \cdots, r_{\Phi+\Lambda}) \in S \) such that both \( \Phi \) and \( \Lambda \) are integer multiples of \( \hat{\tau} n \) and \( d(s, w) \leq \hat{\tau} n \).

**Proof.** Let \( \varphi_{sp} \) and \( \varphi_{len} \) be non-negative integers and \( sp \hat{\tau} n \) and \( len \hat{\tau} n \) be non-negative real numbers such that \( sp = \varphi_{sp} \hat{\tau} n + sp \hat{\tau} n \) and \( len = \varphi_{len} \hat{\tau} n + len \hat{\tau} n \) where \( 0 \leq sp \hat{\tau} n, len \hat{\tau} n < \hat{\tau} n \). Utilizing these notations, \( w \) can be rewritten as \( w = (r_{1+sp \varphi_{sp} \hat{\tau} n + sp \hat{\tau} n}, \cdots, r_{(sp + \varphi_{len} + 1) \hat{\tau} n - (sp + len \hat{\tau} n)}) \). By assumption, we obtain \( 0 \leq sp \hat{\tau} n + len \hat{\tau} n \leq 2 \hat{\tau} n \) and divide into two cases to discuss

1. When \( 0 \leq sp \hat{\tau} n + len \hat{\tau} n < \hat{\tau} n \), then \( (\varphi_{sp} + \varphi_{len}) \hat{\tau} n + (sp \hat{\tau} n + len \hat{\tau} n) < (\varphi_{sp} + \varphi_{len} + 1) \hat{\tau} n \). Set \( \Phi = \varphi_{sp} \hat{\tau} n \) and \( \Lambda = (\varphi_{len} + 1) \hat{\tau} n \). This implies \( s = (r_{\varphi_{sp} \hat{\tau} n}, \cdots, r_{(\varphi_{sp} + \varphi_{len} + 1) \hat{\tau} n - 1}) \). In this case \( w \) is a subsequence of \( s \). Hence \( d(w, s) \) is at most the difference in the lengths, \( d(w, s) = (\varphi_{len} + 1) \hat{\tau} n - (\varphi_{len} \hat{\tau} n + len \hat{\tau} n) = \hat{\tau} n - len \hat{\tau} n \leq \hat{\tau} n \).

2. When \( \hat{\tau} n \leq sp \hat{\tau} n + len \hat{\tau} n < 2 \hat{\tau} n \), then \( (\varphi_{sp} + \varphi_{len}) \hat{\tau} n + (sp \hat{\tau} n + len \hat{\tau} n) \geq (\varphi_{sp} + \varphi_{len} + 1) \hat{\tau} n \). Set \( \Phi = (\varphi_{sp} + 1) \hat{\tau} n \) and \( \Lambda = \varphi_{len} \hat{\tau} n \). This implies \( s = (r_{(\varphi_{sp} + 1) \hat{\tau} n}, \cdots, r_{(\varphi_{sp} + \varphi_{len} + 1) \hat{\tau} n - 1}) \). In this case \( s \) is a subsequence of \( w \). Hence \( d(w, s) \) is at most the difference in the lengths, \( d(w, s) = (\varphi_{len} \hat{\tau} n + len \hat{\tau} n) - (\varphi_{len} \hat{\tau} n) = len \hat{\tau} n \leq \hat{\tau} n \).

\[\square\]
Note that \(|S| \leq \left( \frac{(1+\tau)N - \max(0,1-\tau^*)}{\ell_{in} - \tau_n^*} \right)^{\min(2\tau^*,1+\tau^*)} = O(N)\). Using the list decodability of \(C_{in}\), each \(s = (r_{1+sp}, \ldots, r_{sp+\ell}) \in S\) can be list decoded to a list of size \(O\left(\frac{1}{\epsilon_{in}}\right)\). Recall that the domain of the inner encoding has an indexing scheme with indices from 1 to \(\epsilon_{cont}N\). So any element of the resulting list is in the form \((i, c) \in [\epsilon_{cont}N] \times \Sigma_p^m\). Based on the index introduced, \(c\) is then a possible value of the entries of the outer codeword in the indices that is \(i \mod \epsilon_{cont}N + 1\). Now we consider whether \(c\) is a possible value of all entries of the outer codeword in the indices that is \(i \mod \epsilon_{cont}N + 1\).

Let \(j \in [N]\) such that \(j-1 \equiv i \mod \epsilon_{cont}N\). Then there exists a non-negative integer \(j_N\) such that \(j = 1 + i + j_N\epsilon_{cont}N\). Fix the notations \(v_j^{(L)} = (v_1, \ldots, v_{j-1}), s_j^{(L)} = (r_1, \ldots, r_{sp}), v_j^{(R)} = (v_{j+1}, \ldots, v_N)\) and \(s_j^{(R)} = (r_{1+sp+\ell}, \ldots, r_M)\). Lastly, denote by \(\tau_j n = d(v_j, s), \tau_i n = d(v_i, s_i)\) and \(\tau_j n = d(v_j, s_j)\). Then \(\tau_j n + \tau_j n + \tau_j n = d(c, r) \leq \tau n N\). This leads to the following requirements that \((sp, len)\) needs to satisfy

1. \(\tau_{in} n \geq |n - len|,\)
2. \(\tau n N - |n - len| \geq \tau_j n \geq |sp - (j-1)n|,\)
3. \(\tau n N - |n - len| - |sp + (j-1)n| \geq \tau_j n \geq |(N - j)n - sp - \ell|\) and
4. \(0 \leq sp \leq M - len\) since we have \(1 \leq 1 + sp \leq sp + len \leq M\).

For all possible values of \(M\), these requirements are met if and only if \(\max(n - \tau_{in} n, n - |M - N n|) \leq len \leq \min(n + \tau_{in} n, n + |M - N n|)\) and \(\max(0, (j-1)n - \frac{(1+\tau)nN - M}{2} + \max(n - len, 0)) \leq sp \leq \min\left(M - len, (j-1)n + \frac{M - (1-\tau)nN}{2} + \min(n - len, 0)\right)\). Here \(W_i\) can be constructed depending on the values of \(M\) and \(\tau^*\). In general, this requirements are equivalent to \(\max(0, n - \tau_{in} n) \leq len \leq n + \tau_{in} n\) and \((j-1)n - \frac{(1+\tau)nN - M}{2} + \max(n - len, 0) \leq sp \leq (j-1)n + \frac{M - (1-\tau)nN}{2} + \min(n - len, 0)\). Setting \(sp = \lambda \tau n, len = \mu \tau n, j = 1 + i + j_N\epsilon_{cont}N\), for some non-negative integers \(\lambda\) and \(\mu\) the requirements become

\[
\frac{\max(0, 1 - \tau_{in})}{\tau} \leq \mu \leq \frac{1 + \tau_{in}}{\tau} \tag{11}
\]

and

\[
\frac{i + j_N\epsilon_{cont}N - \frac{(1+\tau)nN - M}{2}}{\tau n} + \max\left(\frac{1}{\tau} - \mu, 0\right) \leq \lambda \leq \frac{i + j_N\epsilon_{cont}N + \frac{M - (1-\tau)nN}{2}}{\tau n} + \min\left(\frac{1}{\tau} - \mu, 0\right) \tag{12}
\]

Fixing \(i, \lambda, \mu\). This gives us the following requirement on the value of \(j_N\).

\[
\max\left(0, \frac{(1-\tau)nN - M + \lambda \tau n - \min(n - \mu \tau n, 0) - i}{\epsilon_{cont}N}\right) \leq j_N \leq \frac{(1+\tau)nN - M + \lambda \tau n - \max(n - \mu \tau n, 0) - i}{\epsilon_{cont}N} \tag{13}
\]

So fixing \(i, \lambda, \mu\), the number of possible \(j_N\) is at most \(\frac{\tau}{\epsilon_{cont}}\). In total, the sum of the sizes of the positional lists \(\ell_{out}\) is at most \(O\left(\frac{\tau N}{\epsilon_{in} \epsilon_{cont}}\right)\).
6.2 Construction and list decoding algorithm

**Theorem 6.** Let $\epsilon_{conc}, \epsilon_{cont}, \epsilon_{out}, R_{out}, \epsilon_{in}, R_{in} \in (0, 1)$ be positive real numbers. Furthermore, take $m = \frac{\epsilon_{out}}{N \epsilon_{cont}}$. Let $C_{out} \subseteq \mathbb{F}_q^{N_{2m}}$ be a code of rate $R_{out}$ and $(\alpha_{out} := R_{out} + \epsilon_{out}, \ell_{out}, C_{out} := N \frac{2^{\ell_{out}}}{N \epsilon_{out}})$-list-recoverable for some $\ell_{out} = O(N)$ and $\zeta$ satisfying $(\alpha_{out} - \frac{\epsilon_{out}}{1 - \frac{\epsilon_{out}}{R_{out}}}) < \alpha_{out} - \epsilon_{out}$ with list-recovering complexity $T(N)$. Set $C_{in} \subseteq \Sigma_q^m$ to be an insdel code by Theorems 2 and 3 depending on the value of $q$ that has rate $R_{in}$ and is $(\tau_{in} n, O\left(\frac{1}{\epsilon_{in}}\right))$ where the relation between $R_{in}$ and $\tau_{in}$ is determined by Equation (9) if $q = 2$ and Equation (7) otherwise. Lastly, choose $0 < \tau^* < \tau_{in}$ such that $\tau_{in} - \frac{\epsilon_{conc}}{1 - \alpha_{out}} \leq \tau^*$.

Using the concatenation method described above, $C_{conc}$ is a list-decodable insdel code of rate $R_{conc} = R_{out} R_{in} - \epsilon$ and it is $\left((1 - \alpha_{out}) \tau_{in} - \epsilon_{conc} \right)nN, N \Omega\left(\frac{1}{\epsilon_{in} \epsilon_{out}}\right)$ -list-decodable for some small $\epsilon$. Furthermore, the list decoding algorithm has complexity $\text{poly}(N) + T(N)$.

**Proof.** First, we discuss the rate of the concatenated code. Based on the concatenation method described above, $R_{in} = \frac{\log_q \epsilon_{cont} + (1 + 2m) \log_q N}{n}$ and $R_{conc} = \frac{R_{out}}{1 + \frac{2m}{n}} \left( R_{in} - \frac{\log_q \epsilon_{cont}}{n} \right)$. For any $\epsilon > 0$, $\epsilon_{cont}$ and $\epsilon_{out}$ can be chosen such that $R_{conc} \geq R_{out} R_{in} - \epsilon$.

Then we discuss the decoding algorithm. The idea is to list decode sufficiently many “windows” from $S$ we described in Subsection 6.1. Apply the list-recovering algorithm for the outer code to the resulting lists for each entries $A_1, \cdots, A_N$. The full algorithm can be found in Algorithm 1.

**Algorithm 1 List Decoding Algorithm for $C_{conc}$**

Require: Received word $r \in \Sigma_q^M$, max$\{0, (1 - \tau)nN\} \leq M \leq (1 + \tau)nN$.
1: Set $A_1, \cdots, A_N \leftarrow \emptyset$;
2: Construct $S$ as discussed in Subsection 6.1;
3: for $s \in S$ do
4: for $(i, \alpha) \in [\epsilon_{cont} N] \times \Sigma N_{2m}$ do
5: Calculate $c_\alpha = \varphi_{in}(i, \alpha)$;
6: if $d(c_\alpha, s) \leq \tau_{in} n$ then
7: for $j = 0, \cdots, \frac{1}{\epsilon_{cont}} - 1$ do
8: if $j$ satisfies the requirement in [13] then
9: $A_{j \epsilon_{cont} N + i + 1} \leftarrow A_{j \epsilon_{cont} N + i + 1} \cup \{\alpha\}$;
10: end if
11: end for
12: end if
13: end for
14: end for
15: Apply list-recovering algorithm for $C_{out}$ with positional lists $A_1, \cdots, A_N$ to get $L_{out} \subseteq C_{out}$ and apply $\varphi_{in}$ for each codewords to get $L \subseteq C_{conc}$ of the same size;
16: return $L$;

Next we discuss the correctness of the decoding algorithm. An index $i$ is said to be “good” if $d(v_i, w_i) = \tau_i n \leq \tau^* n$ and “bad” otherwise. Since $\sum_{i=1}^N \tau_i \leq \tau N$, if there are $h$ “bad” indices, $\tau N > h \tau^*$. This implies that there are at most $\frac{\tau}{\tau^*} N$ “bad” indices. By the property of $S_i, v_i \in A_i$ for at least $\left(1 - \frac{\tau}{\tau^*}\right) N \geq \alpha_{out} N$
indices. As discussed in Subsection 6.1 \(\sum_{i=1}^{N} |A_i| \leq |S| = O\left(\frac{N}{\epsilon_{in}}\right) = \ell_{out}.\) Together with the list-recoverability of \(C_{out}\), inputting \(A_1, \ldots, A_N\) to the list-recovering algorithm of \(C_{out}\) yields a list \(L_{out}\) of size at most \(N^{2\ell_{out}}\) codewords of \(C_{out}\). Applying \(\varphi_{in}\) to each codeword in \(L_{out}\) gives a list \(L\) of codewords of \(C_{conc}\) of the same size. As discussed above, \(v_i \in A_i\) for at least \(\alpha_{out}N\) indices. So the original sent codeword \(e \in L\), proving the correctness of the decoding algorithm.

We consider the decoding complexity. Construction of \(S\) has complexity \(O(N)\). Next, for each \(s \in S\) and \((i, \alpha) \in \epsilon_{cont, N} \times \Sigma_N^{2m}\), calculation of \(d(e_{\alpha}, s)\) requires finding the longest common subsequence of two strings of length \(n\). This takes \(O(n^2)\). So if \(T_{in} = O(\epsilon_{cont} N^{1+2m})\), is the encoding time of the inner code, the complexity of the comparison is \(O(n^2|S|\epsilon_{cont} N^{1+2m}) = O\left(n^2 N^{2+\frac{2}{\epsilon_{cont}}\epsilon_{in}^2}\right)\). Next, by the list decodability of \(C_{in}\), we have at most \(|S|\) elements of these comparisons resulting in distance of at most \(\tau_{in}n\). Hence the assignment step of \(\alpha\) to the \(O\left(\frac{\tau}{\epsilon_{in} \epsilon_{cont}}\right)\) positional lists has complexity \(O\left(\frac{\tau N}{\epsilon_{in} \epsilon_{cont}}\right)\). Lastly, encoding each element of \(L_{out}\) takes \(O\left(N^{1+\frac{1}{\epsilon_{out} \epsilon_{in} \epsilon_{cont}}\epsilon_{in}^2}\right)\). Since every step has complexity \(\text{poly}(N)\), the total complexity is \(\text{poly}(N) + T(N)\).

For our final construction, \(C_{out}\) is chosen from the family of list-recoverable \(p\)-ary codes of length \(N\) and rate \(R_{out}\) that can be derived from [6, Theorem 10]. In this construction, instead of making the codes to be over any \(\Sigma_p\), it is required that \(\Sigma_p\) is a finite field of \(p\) elements, which is denoted by \(\mathbb{F}_p\). This result can be transformed to a construction of list-recoverable code as can be observed in Theorem 22.

**Lemma 22.** (Adapted from [6, Theorem 10]) For \(R_{out}, \epsilon_{out} > 0\), a sufficiently large \(N\), \(\ell_{out} = O(N)\), there exists \(\zeta > 0, m = \frac{\ell_{out}}{N\zeta^2}\) and a prime power \(p = O(N^2)\) such that a folded Reed-Solomon code \(C_{out} \subseteq \mathbb{F}_p^N\) of rate \(R_{out}\) is \(\left(\alpha_{out} \epsilon_{out}, \ell_{out}, \epsilon_{out}, \epsilon_{out} p^{\frac{N}{2\zeta}}\right)\)-list-recoverable. Here \(\zeta\) is chosen to be small enough such that \(\frac{(\alpha_{out} - \zeta)(1 - \zeta)}{1 - \epsilon_{out}} < \alpha_{out} - \epsilon_{out}\). Furthermore, the basis of the list \(L_{out}\) can be recovered in complexity \(O((mN \log(p))^2)\) and \(L_{out}\) can be recovered with time complexity \(O\left(p^{\frac{\epsilon_{out}}{N\zeta}}\right)\).

The construction in Theorem 6 provides the following family of insdel codes over small alphabet size that are list-decodable up to a Zyablov-type bound.

**Theorem 7.** (Zyablov-type bound) For every prime power \(q\), real numbers \(0 < R, \epsilon < 1\) and sufficiently large \(N\), there exists a family of list-decodable insdel codes of rate \(R\), length \(N\) and is \(\left(\tau N, N^{O(\frac{1}{\epsilon})}\right)\)-list-decodable where

\[
\tau = \max_{0 < R_{in}, R_{in} < 1, (1 - R_{out}) f^{-1}(R_{in}) - \epsilon.}
\]

The function \(R_{in} = f(\tau_{in})\) is defined as Equation 9 if \(q = 2\) and Equation 7 otherwise. Lastly, this family of insdel codes can be list-decoded in \(\text{poly}(N)\) time.

Finally, we provide some comparisons between our construction and some existing constructions of insdel codes for small values of \(q\).
Remark 16. The list decoding radius of our construction in Theorem 7 is beyond the Johnson-type bound, which improves the explicit construction of insdel codes designed by Hayashi and Yasunaga [19] for small alphabet size even in the binary case. The improvement when $q = 2$ can be observed in Figure 7.

We showed the comparison with the binary deletion codes constructed in [12].

Remark 17. Guruswami and Wang [12] provided an explicit construction of binary deletion codes with list decoding radius \((\frac{1}{2} - \epsilon)nN\) and polynomial list size. For deletions only, our construction has a larger range of list decoding radius \(\tau \in (0, \frac{1}{2})\) with polynomial list size.

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