On the Randomized Complexity of Minimizing a Convex Quadratic Function

Max Simchowitz
UC Berkeley
msimchow@berkeley.edu

September 25, 2018

Abstract

Minimizing a convex, quadratic objective is a fundamental problem in machine learning and optimization. In this work, we study prove information-theoretic gradient-query complexity lower bounds for minimizing convex quadratic functions, which, unlike prior works, apply even for randomized algorithms. Specifically, we construct a distribution over quadratic functions that witnesses lower bounds which match those known for deterministic algorithms, up to multiplicative constants. The distribution which witnesses our lower bound is in fact quite benign: it is both closed form, and derived from classical ensembles in random matrix theory. We believe that our construction constitutes a plausible “average case” setting, and thus provides compelling evidence that the worst case and average case complexity of convex-quadratic optimization are essentially identical.

1 Introduction

The problem of minimizing convex, quadratic functions of the form \( f_{A,b}(x) := \frac{1}{2} x^\top A x - \langle b, x \rangle \) for \( A \succ 0 \) is a fundamental algorithmic primitive in the machine learning and optimization. Many popular approaches for minimizing \( f_{A,b} \) can be characterized as “first order” methods, or algorithms which proceed by querying the gradients \( \nabla f_{A,b}(x^{(i)}) \) at a sequence of iterates \( x^{(i)} \), in order to arrive at a final approximate minimum \( \tilde{x} \). Standard gradient descent, the heavy-ball method, Nesterov’s accelerated descent, and conjugate-gradient can be all be expressed in this form.

The seminal work of Nemirovskii et al. [1983] established that for a class of deterministic, first order methods, the number of gradient queries required to achieve a solution \( \tilde{x} \) which approximates \( x^* := \arg \min_x \frac{1}{2} x^\top A x - \langle b, x \rangle = A^{-1} b \) has the following scaling:

- **Condition-Dependent Rate:** To attain \( \| \tilde{x} - x^* \|_2 \leq \epsilon \), one needs \( \Theta \left( \sqrt{\text{cond}(A) \log(1/\epsilon)} \right) \), where \( \text{cond}(A) = \lambda_{\max}(A)/\lambda_{\min}(A) \).

- **Condition-Free Rate:** For any \( \epsilon > 0 \), there exists an \( A, b \) such that to obtain \( f_{A,b}(\tilde{x}) - f_{A,b}(x^*) \leq \epsilon \cdot \lambda_1(A) \| x^* \|^2 \), one needs \( \Theta \left( \sqrt{1/\epsilon} \right) \) queries.\(^1\)

It has long been wondered whether the above, worst-case lower bounds are reflective of the “average case” difficulty of minimizing quadratic functions, or if they are mere artifacts of uniquely

\(^1\)Note that \( \lambda_1(A) \) is precisely the Lipschitz constant of \( \nabla f_{A,b} \), and \( \| x^* \|^2 \) corresponds to the Euclidean radius of the domain over which one is minimizing; see Remark 2.2.
adversarial constructions. For example, one may hope that randomness may allow a first order algorithm to avoid querying in worst-case, uninformative directions, at least for the initial few iterations. Furthermore, quadratic objectives have uniform curvature, and thus local gradient exploration can provide global information about the function.

In this work, we show that in fact randomness does not substantially improve the query complexity of first order algorithms. Specifically, we show that even for randomized algorithms, (a) to obtain a solution $\|\tilde{x} - x_*\|_2 \leq \epsilon_0$, for a small but universal constant $\epsilon_0$, one needs $\Omega\left(\sqrt{\text{cond}(A)}\right)$ gradient queries, and, as a consequence, (b) for any $\epsilon > 0$, the condition-free lower bound of $\Omega(\epsilon^{-1/2})$ queries for an $\epsilon$-approximate solution holds as well. These lower bounds are attained by explicit constructions of distributions over parameters $A$ and $b$, which are derived from classical models in random matrix theory. Hence, not only do our lower bounds resolve the question of the complexity of quadratic minimization with randomized first-order queries; they also provide compelling evidence that the worst-case and “average-case” complexity of quadratic minimization coincide up to constant factors.

1.1 Proof Ideas and Organization

Our argument draws heavily upon a lower bound due to Simchowitz et al. [2018] for approximating the top eigenvector of a deformed Wigner model, $M := W + \lambda uu^\top$, given a matrix-vector multiplication queries of the form $w(i) = Mv(i)$. Here, $W$ is drawn from a Gaussian Orthogonal Ensemble (see Section 3.1), $u \sim \mathcal{N}(0, I/d)$, and $\lambda > 1$ is a parameter controlling $\text{gap}(M) := 1 - \lambda_2(M)/\lambda_1(M)$. That work showed that eigenvector approximation implies estimation of the so-called “plant” $u$, and showed that one required $\Omega(\text{gap}(M)^{-1/2} \log d)$ queries to perform the estimation appropriately.

In this work, we show an analogous reduction: one can estimate $u$ if one can minimize the function $f_{A,b}$, where $A = \gamma I - M$ for an appropriate $\gamma$, and $b$ is a Gaussian vector that is slightly correlated with $u$. We also consider matrix vector multiply queries $w(i) = Mv(i)$; these are equivalent both to querying $Am(i)$, and to querying $\nabla f(y(i))$ (see Remark 2.1).

The intuition behind our reduction comes from the Shift-and-Invert meta-algorithm introduced by Garber et al. [2016]. For epochs $s \in [S - 1]$ and $\tilde{y}^{(0)}$ uniform on the sphere, Shift-and-Invert calls a black-box quadratic solver to produce iterates $\tilde{y}^{(s+1)} \approx A^{-1}\tilde{y}^{(s)} = \arg\min_y f_{A,\tilde{y}^{(s)}}$. If the errors $\|\tilde{y}^{(s+1)} - A^{-1}\tilde{y}^{(s)}\|$ are sufficiently small and if $\gamma$ is tuned appropriately one can show that (a) $\text{cond}(A) \approx 1/\text{gap}(M)$ and (b) letting $v_1(M)$ denote the top eigenvector of $M$, the iterate $\tilde{y}^{(S)}$ satisfies

$$\langle \tilde{y}^{(S)}, v_1(M) \rangle^2 \geq 1 - \epsilon,$$

where $S = \Theta(\log(d/\epsilon))$ is independent of $\text{gap}(M)$.

In other words, Shift-and-Invert reduces approximating the eigenvector of $M$ to minimizing a sequence of $O(1)$ convex quadratic functions $\{f_{A,\tilde{y}^{(s-1)}}\}_{s \in [S]}$ with condition number $O\left(\frac{1}{\text{gap}(M)}\right)$. Applying the lower bound for estimating $u$ from Simchowitz et al. [2018], one should expect $\Omega\left(\frac{1}{\sqrt{\text{gap}(M)}}\right) = \tilde{\Omega}(\sqrt{\text{cond}(A)})$ queries on average to minimize these functions.

Unfortunately, applying the reduction in a black-box fashion requires high accuracy approximations of $\arg\min_y f_{A,\tilde{y}^{(s)}}$, and does not yield a single constructive ‘hard instance’. Our analysis therefore departs from the black-box reduction in that (a) we warm start $\tilde{y}^{(0)} \leftarrow b$ near the plant $u$ as opposed to uniformly on the sphere, (b) we effectively consider only the first iteration of the Shift-and-Invert scheme, corresponding to finding $\tilde{x} \approx A^{-1}b$, and (c) we directly analyze the overlap

---

2 In Simchowitz et al. [2018], $u$ was taken to be uniform on the sphere. This work chooses $u$ to be Gaussian order to prove Proposition 3.1.
between \( \hat{x} \) and the plant \( \mathbf{u} \), \((\hat{x}, \mathbf{u})^2\); the reduction is sketched in Section 3.1. Moreover, we modify information-theoretic lower bounds for the estimation of \( \mathbf{u} \) from queries of \( \mathbf{M} \) to account for the additional information conveyed by the linear term \( \mathbf{b} \) (see Section 3.2). Altogether, our reduction affords us simpler proofs and an explicit construction of a “hard instance”. The reduction also tolerates greater error between the approximate minimizer \( \hat{x} \) and the optimum \( \mathbf{x}_* = \mathbf{A}^{-1}\mathbf{b} \), which directly translates into stronger lower bounds.

In particular, to obtain a lower bound which matches known upper bounds up to constants, we to show that the error \( \hat{x} - \mathbf{x}_* \) cannot align to closely with \( \mathbf{u} \). Otherwise, one could obtain good approximations of \( \mathbf{x}_* \), namely \( \hat{x} \), which conveyed little information about \( \mathbf{u} \). Since \( \hat{x} - \mathbf{x}_* \) is independent of \( \mathbf{u} \) given \( \mathbf{M} \) and \( \mathbf{b} \), we can bound their overlap in terms of the quantity

\[
\text{overlap} := \max_{\hat{\mathbf{u}} = \mathbf{u}(\mathbf{M}, \mathbf{b})} \mathbb{E}_{\mathbf{M}, \mathbf{b}, \mathbf{u}}[(\hat{\mathbf{u}}, \mathbf{u})^2].
\]

Using a result due to Lelarge and Miolane [2016] regarding the minimum mean-squared error of estimating the plant \( \mathbf{u} \) in a deformed Wigner model, we prove Proposition 3.1, which gives an order-optimal bound on overlap in terms of relevant problem parameters, provided that the ambient dimension \( d \) is sufficiently large. We remark that the result of Lelarge and Miolane [2016] had been proven under additional restrictions by Barbier et al. [2016]; see Section 2.1 for related work and additional discussion.

The paper is organized as follows. In Section 2, we formally introduce our formal query model and state our results; Section 2.1 discusses related work. In Section 3, we sketch the main components of the proof. Section 3.1 formally introduces the distribution over \( \mathbf{A}, \mathbf{b} \) which witnesses our lower bound; it also presents Proposition 3.1, which bounds the term overlap, and gives the reduction from estimating the plant \( \mathbf{u} \) to approximately minimizing \( f_{\mathbf{A}, \mathbf{b}} \). Section 3.2 provides an information-theoretic lower bound for estimating \( \mathbf{u} \) in our query model, and Section 3.3 concludes the proofs of our main results. Section 4 gives a more in-depth proof roadmap for the reduction from estimation to optimization, and Section 5 fleshes out the proof of the lower bound for estimating \( \mathbf{u} \), and Section 6 provides background information and a proof sketch for our bounds on overlap.

### 1.2 Notation

We shall use bold upper case letters (e.g. \( \mathbf{M}, \mathbf{A}, \mathbf{W} \)) to denote (typically random) matrices related to a given problem instance, bold lower case letters (e.g. \( \mathbf{b}, \mathbf{u}, \mathbf{z} \)) to denote (typically random) vectors related to a problem instance, and lower case serif-font (\( \mathbf{v}^{(i)}, \mathbf{w}^{(i)}, \text{Alg}[\mathbf{x}] \)) to denote quantities related to a given algorithm. We use the standard notation \( \| \cdot \|_2, \| \cdot \|_{\text{op}}, \| \cdot \|_F \) for the Euclidean 2-norm, matrix \( \ell_2 \to \ell_2 \) operator norm, and matrix Frobenius norm, respectively. We let \( e_1, \ldots, e_d \in \mathbb{R}^d \) denote the canonical basis vectors in \( \mathbb{R}^d \), let \( \mathcal{S}^{d-1} := \{ x \in \mathbb{R}^d : \| x \|_2 = 1 \} \) denote the unit sphere, \( \mathcal{S}^d := \{ M \in \mathbb{R}^{d \times d} : M = M^\top \} \) the set of symmetric matrices, and \( \mathcal{S}^d_{++} := \{ M \in \mathcal{S}^d : M > 0 \} \) the set of positive definite matrices. For a matrix \( \mathbf{A} \in \mathcal{S}^d \), let \( \lambda_{\max}(\mathbf{A}) := \lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \cdots \geq \lambda_d(\mathbf{A}) = \lambda_{\min}(\mathbf{A}) \) denote its eigenvalues. For \( \mathbf{A} \in \mathcal{S}^d_{++} \) and \( \mathbf{b} \in \mathbb{R}^d \), we let \( \text{cond}(\mathbf{A}) := \lambda_1(\mathbf{A})/\lambda_d(\mathbf{A}) \), and \( f_{\mathbf{A}, \mathbf{b}}(x) := \frac{1}{2} x^\top \mathbf{A} x - \langle \mathbf{b}, x \rangle \). Given vectors \( v_1, \ldots, v_k \in \mathbb{R}^d \), we let \( \text{Proj}_{v_1, \ldots, v_k} \) denote the orthogonal projection onto \( \text{span}\{v_1, \ldots, v_k\} \). Lastly, given \( x \in \mathbb{R}^n \), we let \( \text{unit}(x) = x/\| x \| \) if \( x \neq 0 \), and unit \( (0) = 0 \).

### 2 Main Results

We begin by presenting a formal definition of our query model.
Remark 2.1. We remark that the above query model is equivalent to a querying exact gradient

\[ T \]

Since one needs to have

\[ \text{seed query complexity} \]

Definition 2.1

\[ \text{The queries} \]

Moreover, \( d \)

Typically, convex optimization lower bounds are stated in terms of a strong convexity

Remark 2.2. \( \lambda \) that

\( \tilde{\lambda} \) that both these quantities are concentrate sharply in our particular distribution over \((\tilde{A}, \tilde{b})\).

\( \alpha \) and a global minimizer, \( \text{iterates to lie in the Krylov space spanned by past queries as in Agarwal and Bottou [2014]}, \) and by not requiring iterates to lie in the Krylov space spanned by past queries as in Nemirovskii et al. [1983].

We now state our main result, which shows that there exists a distribution over instances \((\tilde{A}, \tilde{b})\) which matches the lower bounds of Nemirovskii et al. [1983]:

Theorem 2.1 (Main Theorem: Minimax Rate with Conjectured Polynomial Dimension). There exists a functions \( d_0, d_1 : \mathbb{R} \to \mathbb{N} \) and universal constants \( c_1, \ldots, c_4 > 0 \) such that the following holds. For \( \kappa \geq 20 \), and \( d \geq d_0(\kappa) \), there exists a joint distribution over instances \((\tilde{A}, \tilde{b}) \in \mathbb{S}^d_{++} \times \mathbb{R}^d \)

\( \text{such that (a) } \text{cond}(\tilde{A}) \leq \kappa \) and \( (b) \) for any \( d \geq d_1(\kappa) \) and any \( \text{RQA Alg with query complexity } T \leq c_1 \sqrt{\kappa} \), we have that for \( \tilde{x}_* := \tilde{A}^{-1} \tilde{b} \),

\[ \mathbb{P}_{\tilde{A}, \tilde{b}, \text{Alg}} \left[ \left\{ \| \tilde{x} - \tilde{x}_*\|^2 \leq c_2 \| \tilde{x}_*\|^2 \right\} \lor \left\{ f_{\tilde{A}, \tilde{b}}(\tilde{x}) - f_{\tilde{A}, \tilde{b}}(\tilde{x}_*) \leq c_2 \cdot \frac{\lambda_1(\tilde{A}) \| \tilde{x}_*\|^2}{\kappa} \right\} \right] \leq e^{-c_4 d^3} , \]

Moreover, \( d_0 = O(\text{poly}(\kappa)) \), and under a plausible conjecture, Conjecture 6.1, \( d_1(\kappa) = O(\text{poly}(\kappa)) \) as well. Here, \( \mathbb{P}_{\tilde{A}, \tilde{b}, \text{Alg}} \) refers to probability taken with respect to the random instance \( \tilde{A}, \tilde{b} \), and the random seed \( \xi \).

Remark 2.2. Typically, convex optimization lower bounds are stated in terms of a strong convexity \( \alpha \), a smoothness parameter \( \beta \), and the radius of the domain, or distance between the first iterate and a global minimizer, \( R = \| \tilde{x} - x(0)\|_2 \) (see e.g. Bubeck et al. [2015]). For quadratics, the strong convexity parameter is \( \alpha = \lambda_{\text{min}}(\tilde{A}) \) and the smoothness parameter is \( \beta = \lambda_{\text{max}}(\tilde{A}) \); one can show that both these quantities are concentrate sharply in our particular distribution over \((\tilde{A}, \tilde{b})\), and that \( \lambda_{\text{max}}(\tilde{A}) \) is at most a universal constant. As we are considering unconstrained optimization, the radius of the domain corresponds to \( R = \| \tilde{x}_*\|_2 \). Indeed, the distribution of \((\tilde{A}, \tilde{b})\) is rotationally symmetric, so a priori, the best estimate of \( \tilde{x}_* \) (before observing \( \tilde{b} \) or querying \( \tilde{A} \)) is \( \tilde{x} = 0 \). Hence the event \( \left\{ f_{\tilde{A}, \tilde{b}}(\tilde{x}) - f_{\tilde{A}, \tilde{b}}(\tilde{x}_*) \leq c_2 \lambda_1(\tilde{A}) \| \tilde{x}_*\|^2 \right\} \) can be interpreted as \( \left\{ f_{\tilde{A}, \tilde{b}}(\tilde{x}) - f_{\tilde{A}, \tilde{b}}(\tilde{x}_*) \leq c_2 R^2 \right\} \).

Since one needs to have \( T \geq c_1 \sqrt{\kappa} \), we have that, with high probability,

\[ f_{\tilde{A}, \tilde{b}}(\tilde{x}) - f_{\tilde{A}, \tilde{b}}(\tilde{x}_*) \geq \frac{c_2}{c_1} \cdot \frac{\beta R^2}{T^2} . \]
which is which is the standard presentation of lower bounds for convex optimization. Similarly, the complement of the event \( \{ \| \hat{x} - \tilde{x}_* \|_2^2 \leq c_2 \| \tilde{x}_* \|_2^2 \} \) can be rendered as

\[
\| \hat{x} - \tilde{x}_* \|_2 \geq c_2 \| \tilde{x}_* \|_2^2 \left( 1 - \frac{1}{\kappa} \right)^T \text{ for } T = c_1 \sqrt{\kappa},
\]

where \( \kappa = \text{cond}(\tilde{A}) \geq \beta/\alpha \) is an upper bound on condition number.

**Remark 2.3** (Scalings of \(d_0, d_1\)). In Theorem 2.1, the dimension \( d_0(\kappa) \) corresponds to how large the ambient dimension \( d \) needs to be in order for \( \tilde{A} \) to have the appropriate condition number, and for approximations of \( \tilde{A}^{-1} \tilde{b} \) to have sufficient overlap with \( \mathbf{u} \), assuming a bound on overlap. For the sake of brevity, we show that \( d_0 \) is an unspecified polynomial in \( \kappa \); characterizing the explicit dependence is possible, but would require great care, lengthier proofs, and would distract from the major ideas of the work.

The dimension \( d_1(\kappa) \) captures how large \( d \) must be in order to obtain the necessary bound on overlap. Though \( d_1(\kappa) \) is finite, we are only able to guarantee that the dependence on \( \kappa \) is polynomial under a plausible conjecture, Conjecture 6.1, which requires that either (a) minimum-mean squared error of the estimate of the planted solution in a deformed Wigner model, or (b) the mutual information between the deformed Wigner matrix and the planted solution, converge to their asymptotic values at a polynomial rate.

If non-conjectural bounds are desired which still guarantee that the dimension need only be polynomial in the condition number, we instead have the following theorem:

**Theorem 2.2** (Main Theorem: Weaker Rate with Guaranteed Polynomial Dimension). Let \( c_1, \ldots, c_4 \) be as in Theorem 2.1, and let \( d_0(\kappa) = \mathcal{O}(\text{poly}(\kappa)) \). Then for every \( \kappa \geq 20 \), there exists a distribution \( (\mathbf{A}, \mathbf{b}) \) such that \( (\mathbf{A}, \mathbf{b}) \in \mathcal{S}^d_{++} \times \mathbb{R}^d \) such that \( \mathbb{P}[\text{cond}(\mathbf{A}) \leq \kappa] \geq 1 - e^{-c_4 d^{3/2}} \) and for any \( d \geq d_0(\kappa) \) and any RQA \( \text{Alg} \) with query complexity \( T < c_1 \sqrt{\kappa} \), we have that

\[
\mathbb{P}_{\mathbf{A}, \mathbf{b}, \text{Alg}} \left[ \left\{ \| \hat{x} - \tilde{x}_* \|_2^2 \leq \frac{c_2}{\sqrt{\kappa}} \right\} \lor \left\{ f_{\mathbf{A}, \mathbf{b}}(\hat{x}) - f_{\mathbf{A}, \mathbf{b}}(\tilde{x}_*) \leq \frac{c_2}{\kappa^{3/2}} \right\} \right] \leq e^{-c_4 d^{3/2}},
\]

Note that Theorem 2.2 does not imply the minimax lower bound (1); however, it does show that to get to a modest accuracy in either \( \| \hat{x} - \tilde{x}_* \|_2 \) or \( f_{\mathbf{A}, \mathbf{b}}(\hat{x}) - f_{\mathbf{A}, \mathbf{b}}(\tilde{x}_*) \), one needs \( \Omega(\sqrt{\text{cond}(\mathbf{A})}) \) queries.

**Remark 2.4** (The distributions \( (\mathbf{A}, \mathbf{b}) \) and \( (\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \)). The distributions over \( (\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \) from Theorem 2.1 and \( (\mathbf{A}, \mathbf{b}) \) from Theorem 2.2 differ subtly. The form of the distribution over \( (\mathbf{A}, \mathbf{b}) \) is given explicitly at the beginning of Section 3.1, and is specialized for Theorem 2.2 by appropriately tuning parameters \( \lambda = 1 + \sqrt{\frac{20}{\kappa}} \) and \( \tau_0 = (\lambda - 1)^2 \). The distribution over \( (\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \) is obtained by conditioning \( (\mathbf{A}, \mathbf{b}) \) on a constant-probability, \( (\mathbf{A}, \mathbf{b}) \)-measurable event \( \mathcal{E} \) (see remarks following Proposition 3.2). If one prefers, one can express Theorem 2.1 as saying that, for the distribution \( (\mathbf{A}, \mathbf{b}) \) as in Section 3.1 and Theorem 2.2, any algorithm with \( T \leq c_1 \sqrt{\kappa} \) has a large error with constant probability. However, by distinguishing between \( (\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \) and \( (\mathbf{A}, \mathbf{b}) \), we ensure that any algorithm incurs error with overwhelming, rather than just constant, probability.

### 2.1 Related Work

It is hard to do justice to the vast body of work on quadratic minimization and first order methods for optimization. We shall restrict the present survey to the lower bounds literature.
Lower Bounds for Convex Optimization: The seminal work of Nemirovskii et al. [1983] established tight lower bounds on the number of gradient queries required to minimize quadratic objectives, in a model where the algorithm was (a) required to be deterministic (and was analyzed for a worst-case initialization), and (b) the gradient queries were restricted to lie in the linear span of the previous queries, known as the Krylov space. Agarwal and Bottou [2014] showed that deterministic algorithms can be assumed to query in the Krylov space without loss of generality, but did not extend their analysis to randomized methods. Woodworth and Srebro [2016] proved truly lower bounds against randomized first-order algorithms for finite-sum optimization of convex functions, but their constructions require non-quadratic objectives. Subsequent works generalized these constructions to query models which allow for high-order derivatives [Agarwal and Hazan, 2017, Arjevani et al., 2017]; these lower bounds are only relevant for non-quadratic functions, since a second order method can, by definition, minimize a quadratic function in one iteration.

All aforementioned lower bounds, as well as those presented in this paper, require the ambient problem dimension to be sufficiently large as a function of relevant problem parameters; another line of work due to Arjevani and Shamir [2016] attains dimension-free lower bounds, but at the expense of restricting the query model.

Lower Bounds for Stochastic Optimization: Lower bounds have also been established in the stochastic convex optimization [Agarwal et al., 2009, Jamieson et al., 2012], where each gradient-or function-value oracle query is corrupted with i.i.d. noise, and Allen-Zhu and Li [2016] prove analogues of these bounds for streaming PCA. Other works have considered lower bounds which hold when the optimization algorithm is subject to memory constraints [Steinhardt et al., 2015, Steinhardt and Duchi, 2015, Shamir, 2014]. While these stochastic lower bounds are information-theoretic, and thus unconditional, they are incomparable to the setting considered in this work, where we are allowed to make exact, noiseless queries.

Query Complexity: Our proof casts eigenvector computation as a sequential estimation problem. These have been studied at length in the context of sparse recovery and active adaptive compressed sensing [Arias-Castro et al., 2013, Price and Woodruff, 2013, Castro and Tánczos, 2017, Castro et al., 2014]. Due to the noiseless oracle model, our setting is most similar to that of Price and Woodruff [Price and Woodruff, 2013], whereas other works [Arias-Castro et al., 2013, Castro and Tánczos, 2017, Castro et al., 2014] study measurements contaminated with noise. More broadly, query complexity has received much recent attention in the context of communication-complexity [Anshu et al., 2017, Nelson et al., 2017], in which lower bounds on query complexity imply corresponding bounds against communication via lifting theorems.

Estimation in the Deformed Wigner Model: As mentioned in As mentioned in Section 1.1, we require a result due to Lelarge and Miolane [2016] regarding the minimum mean squared error of estimation in a deformed Wigner model; this is achieved by establishing that the replica-symmetric formula for mutual information in the deformed Wigner model holds in broad generality. The replica-symmetric formula had been conjectured by the statistical physics community (see Lesieur et al. [2015]), and Barbier et al. [2016] and Krzakala et al. [2016] had rigorously proven this formula under the restriction that the entries of the plant \( \mathbf{u} \) have finite support. In our application, \( \mathbf{u} \) has Gaussian entries, which is why we need the slightly more general result of Lelarge and Miolane [2016]. Later, Alaoui and Krzakala [2018] give a concise proof of the replica-symmetric formula, again under the assumption that \( \mathbf{u} \) has finite support.
3 Proof Roadmap

3.1 Reduction to Estimation in the Deformed Wigner Model

Our random instances will be parameterized by the quantities $\lambda > 0$, $\tau_0 > 0$, and $d \in \mathbb{N}$; we shall tune the parameters $\lambda, \tau_0$ for the proofs of Theorem 2.1 and 2.2 in Section 3.3. We say $c$ is a universal constant if it does not depend on $\lambda, \tau_0, d$, and write $f(\lambda, \tau_0, d) \lesssim g(f, \lambda, \tau_0, d)$ if there exists a universal constant $c > 0$ such that $f(\lambda, \tau_0, d) \leq c \cdot g(f, \lambda, \tau_0, d)$. For each $\lambda \in (1, 2]$ and $d \in \mathbb{N}$, consider the deformed Wigner model

$$M := \lambda uu^\top + W,$$

where $u \sim \mathcal{N}(0, I/d)$ is called the plant, and $W$ is a GOE matrix, with $W_{ii} \sim \mathcal{N}(0, 2)$ for $i \in [d]$, $W_{ij} \sim \mathcal{N}(0, 1)$ and $W_{ji} := W_{ij}$ for $i \neq j \leq d$. With $u$ and $M$ defined above, we define our random instance $(A, b)$ as

$$A := (2(\lambda + \lambda^{-1}) - 2)I - M \quad \text{and} \quad b | W, u \sim \mathcal{N}(\sqrt{\tau_0}u, I/d),$$

and let $x_\ast := A^{-1}b \in \mathbb{R}^d$ denote the vector which exists almost surely, and when $A \in \mathbb{S}^d_{++}$, is the unique minimizer of the quadratic objective $f_{A,b}(x) := \frac{1}{2}x^\top Ax - \langle b, x \rangle$. The goal of this Section is to present Propositions 3.2 and 3.3, which show that if the output $\tilde{x}$ of RQA is close to $x_\ast$ in $\|\cdot\|_2$ then $\tilde{x}$ has a large inner product with $u$. Thus, we show a reduction to approximate quadratic minimization from estimation of $u$, for which we provide lower bounds in Section 3.2.

The parameter $\lambda \in (1, 2]$ gives us a knob to control the condition number of $A$, and $\tau_0 \leq (\lambda - 1)^{-2}$ gives us control over to what extent we “warm-start” the algorithm near the true planted solution $u$. Specially, Proposition 4.2 implies that $\text{cond}(A)$ will concentrate below

$$\text{cond}(\lambda) := \frac{2(\lambda^2 + 1)}{(\lambda - 1)^2} \leq \mathcal{O}\left((\lambda - 1)^{-2}\right),$$

and standard concentration implies that $\langle u, b \rangle^2$ concentrates around $\tau_0$. In Proposition 4.3, we show that if $\tau_0$ is is in some desired range, then then $x_\ast$ satisfies

$$\langle \text{unit}(x_\ast), u \rangle^2 \geq \frac{\tau_0}{\lambda - 1} \quad \text{with high probability.}$$

In other words, the solution $x_\ast$ is about $1/(\lambda - 1)$-times more correlated with the plant $u$ than is $b$.

Whereas (5) controls the overlap between $x_\ast$ and $u$, we are more precisely interested in the overlap between $\tilde{x}$ and $u$. If the error $\tilde{x} - x_\ast$ could align arbitrarily well with $u$, then we would only be able to tolerate small errors $\tilde{x} - x_\ast$ to ensure large correlations $\langle \text{unit}(x_\ast), u \rangle^2$. However, we observe that both $x_\ast$ and $\tilde{x}$ are conditionally independent of $u$, given $A, b$. Hence, we can bound the alignment between $\tilde{x} - x_\ast$ and $u$ in terms of

$$\text{ovlap}_{d, \lambda}(\tau_0) := \mathbb{E}_{A,b} \max_{u \in \mathbb{S}^{d-1}} \mathbb{E}_u \left[ (\tilde{u}, u)^2 | M, b \right],$$

where $\text{ovlap}_{d, \lambda}(\tau_0)$ controls the largest possible alignment between $u$ and any vector $\tilde{u}$ depending on a total observation of $M, b$. Leveraging a recent result regarding the asymptotic error of plant estimation in a deformed Wigner model Lelarge and Miolane [2016], we can bound $\text{ovlap}_{d, \lambda}(\tau_0) \lesssim \lambda - 1$ when $\tau_0 \leq (\lambda - 1)^2$ and $d$ is sufficiently large:
Proposition 3.1. Suppose that \( \tau_0 \leq (\lambda - 1)^2 \). Then, there exists a \( d_1 = d_1(\lambda, \tau_0) \) such for all \( d \geq d_1 \), \( \text{ovlap}_{d, \lambda}(\tau_0) \leq 5(\lambda - 1) \). Moreover, under Conjecture 6.1, \( d_1 \leq O\left(\log(\frac{1}{\lambda - 1}, \frac{1}{\tau_0})\right) \).

The proof and intuition for the above proposition are deferred to Section 6. Lastly, when the approximation \( \text{ovlap}_{d, \lambda} \approx (\lambda - 1) \) holds, we have the following reduction from plant estimation to quadratic optimization:

Proposition 3.2. There exists universal constants \( c_1, \ldots, c_5 > 0 \) such that the following is true. Let \( \lambda \in (1, 2) \), \( \tau_0 = (\lambda - 1)^2 \) and suppose \( \text{ovlap}_{d, \lambda}(\tau_0) \leq K(\lambda - 1) \) for some \( K > 0 \). Then, there exists a distribution \( D \) of instances \((\mathbf{A}, \mathbf{b})\) with \( \mathbb{P}[\mathbf{A} > 0 \cap \text{cond}(\mathbf{A}) \leq 2\text{cond}(\lambda)] = 1 \) such that, for \( \tilde{x}_* = \mathbf{A}^{-1}\mathbf{b} \)

\[
\mathbb{P}_{\mathbf{A}, \mathbf{b}, \mathbf{u}, \text{Alg}} \left[ \frac{\tilde{x}}{\|\mathbf{x}\|} \cdot \mathbf{u} \geq c_1 (\lambda - 1) \right] \geq \frac{1}{4} \mathbb{P}_{\mathbf{A}, \mathbf{b} \sim \mathcal{D}} \mathbb{P}_{\text{Alg}} \left[ \frac{\|\mathbf{x} - \tilde{x}_*\|_2}{\|\mathbf{x}_*\|_2} \leq \frac{c_2}{K} \right] - e^{c_5 d^{-\epsilon_3}(\lambda - 1)^{c_4}}.
\]

A couple remarks are in order. First, the ‘hard distribution’ \( \mathcal{D} \) is obtained by taking the distribution \( \mathbf{M}, \mathbf{b} \) and conditioning on the events where (a) \( \mathbf{M} \) is well conditioned and (b) where the posterior on \( \mathbf{u} \), given \( \mathbf{M}, \mathbf{b} \) is such that \( \max_{\mathbf{u} \in S^{d-1}} E_{\mathbf{u}}[\langle \mathbf{u}, \mathbf{u} \rangle^2 | \mathbf{M}, \mathbf{b}] \leq \text{ovlap}_{d, \lambda}(\tau_0) \). The first event has high probability by Proposition 4.2, and the second event occurs with constant probability by Markov’s inequality (Section 4.3); thus, the conditional distribution is well-defined.

Secondly, Theorem 6.1 shows that we can take \( K = 5 \) in Proposition 3.2, as long as \( d \) is sufficiently large as a function of \( \lambda \) and \( \tau_0 \). If explicit and non-conjectural bounds on the dimension \( d \) are desired, we can instead opt to use the trivial estimate \( \text{ovlap}_{d, \lambda}(\tau_0) \leq \mathbb{E}\|\mathbf{u}\|_2^2 = 1 \), at the expense of requiring higher accuracy solutions. In this setting, we have the following proposition:

Proposition 3.3. There exists universal constants \( c_0, c_1, \ldots, c_5 > 0 \) such that the following is true. For all \( \lambda \in (1, 2) \) and \( \tau_0 \in [d^{-\epsilon_1}, (\lambda - 1)^2] \), then \( \mathbf{A}, \mathbf{b} \) as defined above satisfy

\[
\mathbb{P}_{\mathbf{A}, \mathbf{b}, \mathbf{u}, \text{Alg}} \left[ \frac{\tilde{x}}{\|\mathbf{x}\|} \cdot \mathbf{u} \geq c_1 \frac{\tau_0}{\lambda - 1} \right] \geq \mathbb{P}_{\mathbf{A}, \mathbf{b}, \text{Alg}} \left[ \frac{\|\tilde{x} - \tilde{x}_*\|_2}{\|\tilde{x}_*\|_2} \leq \frac{c_2}{\lambda - 1} \right] - e^{c_5 d^{-\epsilon_3}(\lambda - 1)^{c_4}},
\]

and \( \mathbb{P}[\text{cond}(\mathbf{A}) \leq 2\text{cond}(\lambda)] \leq e^{c_5 d^{-\epsilon_3}(\lambda - 1)^{c_4}}. \)

Remark 3.1. Proposition 3.3 requires that \( \|\tilde{x} - \mathbf{x}_*\| \) to be small than that of Proposition 3.2. However, unlike Proposition 3.2, Proposition 3.3 allows the \( \tau_0 \), the parameter controlling the correlation between \( \mathbf{b} \) and \( \mathbf{u} \), to be vanishingly small in the dimension. In fact, the condition \( \tau_0 \geq d^{-\epsilon} \) can be replaced by \( \tau_0 \geq d^{1-\epsilon} \) for any \( \epsilon > 0 \), provided that the constants \( c_1, \ldots, c_4 \) are ammended accordingly. Thus, our \( \Omega(\sqrt{\text{cond}(\mathbf{A})}) \) lower bounds hold even when the linear term \( \mathbf{b} \) and the plant \( \mathbf{u} \) have little correlation, providing the solution accuracy is sufficiently high.

3.2 Lower Bound for Estimation of \( \mathbf{u} \)

Having reduced the problem to estimating \( \mathbf{u} \), we conclude by bounding on the number of queries required to ensure that \( \frac{\tilde{x}}{\|\mathbf{x}\|} \in S^{d-1} \) has a sufficiently large inner product with the planted solution \( \mathbf{u} \). To do so, we observe that an RQA interacting with an instance \((\mathbf{A}, \mathbf{b})\) is equivalent to interacting with an instance \((\mathbf{M}, \mathbf{b})\), since \( \mathbf{A} = (2(\lambda + \lambda^{-1}) - 2)I - \mathbf{M} \). Moreover, without loss of generality we can bound

\[
\frac{\langle \tilde{x}, \mathbf{u} \rangle}{\|\tilde{x}\|} \leq \|\text{Proj}_{v^{(1)}, \ldots, v^{(T-1)}} \mathbf{u}\|_2^2,
\]

since we can assume without loss of generality that the \( T + 1 \)-st query made by our algorithm \text{Alg} to be \( v^{(T+1)} = \tilde{x} \). With this reduction in place, we have the following theorem:
Hence, setting \( \lambda \in (1, 2] \), \( \tau_0 \leq (\lambda - 1)^2 \), and \( d \geq \frac{16\lambda^4}{\tau_0(\lambda - 1)^2} \), and let \( \mathbf{u}, \mathbf{M} \) and \( \mathbf{b} \) be as in Section 3.1. Then for any RQA Alg interacting with the instances \((\mathbf{M}, \mathbf{b})\),

\[
\mathbb{P}_{\mathbf{u, M, b, Alg}} \left[ \| \text{Proj}_{\mathbf{v}(1), \ldots, \mathbf{v}(T+1)} \mathbf{u} \|_2^2 > 2\lambda^{4T+2} \cdot \tau_0 \sum_{j=1}^T \lambda^{4j} \right] \leq T^2 e^{-d\lambda^2 \tau_0 (\lambda - 1)/16} + e^{-d^{1/4}/8}.
\]

where the probability is taken over the randomness of the algorithm, and over \( \mathbf{u, b, W} \).

For intuition, we recall that \( \tau_0 \) controls the initial information about \( \mathbf{u} \) conveyed by \( \mathbf{b} \), via \( \mathbb{E}(\mathbf{b, u})^2 = \tau_0 + o(1) \). Moreover, \( \| \text{Proj}_{\mathbf{v}(1), \ldots, \mathbf{v}(T+1)} \mathbf{u} \|_2^2 \) controls the amount of information about \( \mathbf{u} \) acquired by the queries \( \mathbf{v}(1), \ldots, \mathbf{v}(T+1) \). Hence, Theorem 3.4 says that the rate of information acquired grows as the first \( \mathcal{O}(T) \) terms of a geometric series with base \( \lambda^{\mathcal{O}(1)} \), multiplied by the initial information \( \tau_0 \). In particular, if \( c_1 \) is the constant from Proposition 3.2 (or Proposition 3.3), and if \( T \leq \frac{1\wedge c_1}{16(\lambda - 1)} \), we have that

\[
2\lambda^{4T+2} \cdot \tau_0 \sum_{j=1}^T \lambda^{4j} \leq 2\tau_0 T \lambda^{8T+2} \leq \frac{\tau_0 \lambda^{5/2}}{8c_1(\lambda - 1)} \leq \frac{\tau_0}{c_1(\lambda - 1)}.
\]

Hence, by Theorem 3.4 and absorbing constants in the probability, we have that for \( T \leq \frac{1\wedge c_1}{16(\lambda - 1)} \),

\[
\mathbb{P}_{\mathbf{u, M, b, Alg}} \left[ \| \text{Proj}_{\mathbf{v}(1), \ldots, \mathbf{v}(T+1)} \mathbf{u} \|_2^2 > \frac{\tau_0}{c_1(\lambda - 1)} \right] \leq e^{-d\lambda^2 \tau_0 (\lambda - 1)/16} + e^{-d^{1/4}/8}.
\]

(7)

### 3.3 Proof of Main Results

**Proof of Theorem 2.1.** Fix \( \kappa \geq 20 \), and let \( \tau_0 = (\lambda - 1)^2 \), and let \( \lambda = 1 + \sqrt{\frac{20}{\kappa}} \); note that \( \lambda \in (1, 2]^3 \). There exists a \( d_1 = d_1(\lambda) = d_1(\kappa) \) such that for all \( d \geq d_1(\kappa) \), the conclusion of Proposition 3.1 holds. For \( d \geq d_1(\kappa) \), we combine (7) with Proposition 3.2, taking \( K = 5 \), letting \( c_1, \ldots, c_5 \) and \( \mathbf{A}, \mathbf{b}, \mathbf{x}, \mathbf{b}, \mathbf{x} \) be as in Proposition 3.2. These imply that for \( T \leq \frac{1\wedge c_1}{16(\lambda - 1)} \),

\[
\frac{1}{4} \mathbb{P}_{\mathbf{A, b} \sim \mathcal{D}} \mathbb{P}_{\text{Alg}} \left[ \| \mathbf{x} - \mathbf{x}_* \|_2^2 \leq \frac{c_2}{5} \right] \leq e^{-d\lambda^2 (\lambda - 1)^2/16} + e^{-d^{1/4}/8} + e^{c_5 d^{-c_3}(\lambda - 1)^{c_4}}.
\]

By Proposition 3.2, we have that with probability 1,

\[
\text{cond}(\mathbf{A}) \leq 2\text{cond}(\lambda) := 2 \cdot \frac{2(\lambda^2 + 1)}{(\lambda - 1)^2} \leq 20/(\lambda - 1)^2 \text{ for } \lambda \in (1, 2].
\]

Hence, setting \( \lambda = 1 + \frac{1}{20\kappa} \), \( c_1' = \frac{1\wedge c_1}{16\sqrt{20}} \), \( c_2' = c_2/5 \), we find that for \( T \leq c_1' \sqrt{\kappa} \), and \( c_3, \ldots, c_5 \) as in Proposition 3.2,

\[
\mathbb{P}_{\mathbf{A, b} \sim \mathcal{D}} \mathbb{P}_{\text{Alg}} \left[ \| \mathbf{x} - \mathbf{x}_* \|_2^2 \leq \frac{c_2'}{5} \right] \leq 4 \left\{ e^{-d\lambda^2 (\lambda - 1)^2/16} + e^{-d^{1/4}/8} + e^{c_5 d^{\kappa c_3}(\lambda - 1)^{c_4}} \right\}
\]

\[
\leq c_3' e^{-d_3' c_5 d^{\kappa c_3}}.
\]

---

3The choice of \( \kappa \geq 20 \) is arbitrary, and can be replaced by any constant bounded away from 1.
where $c'_1, \ldots, c'_6$ are universal constants. In particular, there is a $d_0(\kappa) = \mathcal{O}(\text{poly}(\kappa))$ such that for $d \geq d_0(\kappa)$,

$$
P_\mathbf{A, b} \sim \mathcal{D}^{\mathcal{P}_{\text{Alg}}}[\|\bar{x} - \bar{x}_*\|^2 \geq c'_2\|\bar{x}_*\|^2] \leq e^{-c'_d d^{c'_d}}.$$

However, we have that

$$
f_{\mathbf{A, b}}(\bar{x}) - f_{\mathbf{A, b}}(\bar{x}_*) \geq \lambda_d(\mathbf{A})\|\bar{x} - \bar{x}_*\|^2 = \frac{\lambda_1(\mathbf{A})}{\text{cond}(\mathbf{A})}\|\bar{x} - \bar{x}_*\|^2 \geq \frac{\lambda_1(\mathbf{A})}{\kappa}\|\bar{x} - \bar{x}_*\|^2,$$

which implies that

$$
P_\mathbf{A, b} \sim \mathcal{D}^{\mathcal{P}_{\text{Alg}}}[\|\bar{x} - \bar{x}_*\|^2 \leq c'_2\|\bar{x}_*\|^2] \lor \left\{ f_{\mathbf{A, b}}(\bar{x}) - f_{\mathbf{A, b}}(\bar{x}_*) \leq c'_2\frac{\lambda_1(\mathbf{A})}{\kappa}\|\bar{x}_*\|^2 \right\}
= P_\mathbf{A, b} \sim \mathcal{D}^{\mathcal{P}_{\text{Alg}}}[\|\bar{x} - \bar{x}_*\|^2 \geq c'_2\|\bar{x}_*\|^2] \leq e^{-c'_d d^{c'_d}}.$$

Theorem 2.2 now follows by relabeling universal constants appropriately. Note that under Conjecture 6.1, Proposition 3.1 implies that $d_1(\lambda) = \mathcal{O}(\text{poly}(\lambda - 1))$, which by our choice of $\kappa$ implies that we can write $d_1 = d_1(\kappa) = \mathcal{O}(\text{poly}(\kappa))$.

Proof of Theorem 2.2. The proof of Theorem 2.2 is almost identical to that of Theorem 2.2. The only differences are that (a) we use Proposition 3.3, which translates into an upper bound bound on the event $\{\|\bar{x} - \bar{x}_*\|^2 \leq \frac{1}{\lambda - 1}\} = \{\|\bar{x} - \bar{x}_*\|^2 \leq \frac{1}{\sqrt{\kappa}}\}$, (b) we use the original distribution $(\mathbf{A, b})$ rather than the conditioned-distribution $(\bar{\mathbf{A}}, \bar{\mathbf{b}})$, (c) because of not conditioning, we have the guarantee $\mathbb{P}[	ext{cond}(\mathbf{A}) \leq \kappa] \geq 1 - e^{-c_3 d^{c_3}}$ (but not with probability 1), and (d) because we don’t need Proposition 3.1, there is no need to ensure $d \geq d_1(\kappa)$.

4 Reduction from Estimation to Minimization: Proof of Propositions 3.2 and 3.3

In what follows, we let $\nu > 1$ denote a parameter representing a multiplicative error in our deviation bounds; one can take $\nu = 2$ without affecting the scaling of the results. Moreover, we let $\delta_{\nu, \lambda}(d)$ denote a term which is bounded above by $c_1 \exp(-c_1 d^{c_2} \cdot (\nu - 1)^{c_3} \cdot (\lambda - 1)^{c_4})$ for some universal constants $c_1, c_2, c_3, c_4 > 0$. We shall prove the following theorem, from which Propositions 3.2 and 3.2 as special cases:

Theorem 4.1. Let $c_0$ be a universal constant, $\lambda \in (1, 2]$, and let $d^{-c_0} \leq \tau_0 \leq (\lambda - 1)^2$, $\nu > 1$, and let $\bar{x} \perp u | \sigma(\mathbf{A}, \mathbf{b})$. Then,

1. $\mathbb{P}[\text{cond}(\mathbf{A}) \leq \nu \cdot \text{cond}(\lambda)] \geq 1 - \delta_{\nu, \lambda}(d)]$

2. Define $\mathcal{E}_{\text{err}}(\nu) := \left\{ \|\bar{x} - \bar{x}_*\|^2 \leq \frac{\tau_0}{4\nu(\lambda - 1)} \right\}$. Then

$$
\mathbb{P}\left[\left\langle \frac{\bar{x}}{\|\bar{x}\|}, u \right\rangle \geq \frac{1}{2} \sqrt{\frac{\tau_0}{3\nu(\lambda - 1)}} \right] \geq \mathbb{P}[\mathcal{E}_{\text{err}}(\nu)] - \delta_{\nu, \lambda}(d).$$
3. Define the event $\mathcal{E}_{\text{err}}(\nu, t) := \left\{ \frac{\|x - x^{\text{true}}\|^2}{\|x\|^2} \leq \sqrt{\frac{\tau_0}{3\nu(t-\mathbb{L})\text{ovlp}_{d,\lambda}}} \right\}$. There exists a $\sigma(A, b)$-measurable event with probability $\mathbb{P}[\{\text{ovlp}(t)\} \geq 1 - t^{-1/2}]$ such that

$$
\mathbb{P}\left[ \frac{\|\hat{x}\|}{\|x\|} \geq \frac{1}{2} \sqrt{\frac{\tau_0}{3\nu(\lambda - 1)}} \right] \geq \left(1 - \frac{1}{\sqrt{t}}\right)^2 \mathbb{P}\left[ \mathcal{E}_{\text{err}}(\nu, t) \right] \left\{ \text{cond}(A) \leq \nu \cdot \text{cond}(\lambda) \right\} \cap \mathcal{E}_{\text{ovlp}}(t) - \delta_{\nu, \lambda}(d). \tag{8}
$$

Proposition 3.3 follows by directly applying Parts 1 and 2 of the above theorem with $\nu = 2$ and absorbing universal constants. Proposition 3.2 follows by applying Parts 1 and 3 of the above theorem with $\nu = 2$ and $t = 1/4$. In this case $\mathcal{E}_{\text{err}}(2, 1/4)$ can be rendered as $\left\{ \frac{\|x - x^{\text{true}}\|^2}{\|x\|^2} \leq \left(\frac{\tau_0}{c(\lambda - 1)\text{ovlp}_{d,\lambda}}\right) \right\}$ for some absolute constant $c$. Substituting in $\tau_0 = (\lambda - 1)^2$ and $\text{ovlp}_{d,\lambda} = K(\lambda - 1)$, $\mathcal{E}_{\text{err}}(2, 1/4)$ reduces to the $\left\{ \frac{\|x - x^{\text{true}}\|^2}{\|x\|^2} \leq c_2 \right\}$, which appears on the right hand side of the display in Proposition 3.2. The distribution over $(\hat{A}, \tilde{b})$ in Proposition 3.2 is just that of $(A, b) | \mathcal{E}_{\text{ovlp}}(t) \cap \{\text{cond}(A) \leq 2\text{cond}(\lambda)\}).^4$

4.1 The Condition Number of $A$

In this section, we sketch the proof of the following proposition, which controls $\text{cond}(A)$:

**Proposition 4.2.** Let $\lambda \in (1, 2]$. Then the event

$$
\mathcal{E}_A(\nu, \lambda) := \left\{ \frac{(\lambda - 1)^2}{\lambda \nu^{1/2}} \leq \lambda_d(A) \leq \lambda_1(A) \leq \nu^{1/2} \cdot 2(\lambda + \lambda^{-1}) \right\}, \tag{9}
$$

occurs with probability at least $1 - \delta_{\nu, \lambda}(d)$.

To understand the proof of Proposition 4.2, we remark that the spectrum of $M$ is well studied in random matrix theory Pêché [2006], Féral and Pêché [2007], Anderson et al. [2010], Benaych-Georges and Nadakuditi [2011]. In particular, as $d \to \infty$, we have

$$
\lambda_1(M) \xrightarrow{\text{prob.}} \lambda + \lambda^{-1} \quad \text{and} \quad \lambda_d(M) \xrightarrow{\text{prob.}} -2. \tag{10}
$$

Setting $A = (2(\lambda + \lambda^{-1}) - 2)I - M$ we have that

$$
\lambda_1(A) \xrightarrow{\text{prob.}} 2(\lambda + \lambda^{-1}) \quad \text{and} \quad \lambda_d(A) \xrightarrow{\text{prob.}} \lambda + \lambda^{-1} - 2 = \lambda^{-1}(\lambda - 1)^2 \tag{11}
$$

To prove Proposition 4.2, we invoke non-asymptotic analogues of the above asymptotic convergence results, derived in Simchowitz et al. [2018]. The details are carried out in Appendix B.1.

4.2 Overlap of $x_*$ and $u$

In this section, we prove Proposition 4.3, which proves that the true minimizer $x_*$ overlaps with $u$. Throughout, it will be convenient for us to render $b = \sqrt{\tau_0}u + z$, where $z \sim \mathcal{N}(0, I/d)$ is independent of $W, u$. Our main result is as follows:

---

^4Note that this is well defined, since $\mathbb{P}[\{\text{ovlp}(t) \cap \{\text{cond}(A) \leq 2\text{cond}(\lambda)\}\}]$ occurs with non-zero probability as long as $1 - \sqrt{t} - \delta_{\nu, \lambda}(d) > 0$, which holds whenever of the hand side of the display in Proposition 3.2 is nonzero.
Proposition 4.3. There exists universal constants $c_1,c_2,c_3 > 0$ such that, for all $\delta > 0$, $\lambda > 1$, and all $d^{-9} \leq \tau_0 \leq (\lambda - 1)^2$, then the event

$$
E_{x_+}(\nu) := \left\{ \frac{\langle x_+, u \rangle}{\|x_+\|_2^2} \geq \frac{\tau_0 \nu}{3\lambda - 1} \right\}
$$

occurs with probability at least $1 - \delta_{\nu,\lambda}(d) - \delta$

Remark 4.1. In the limit of $\lambda \to 1$, the constant of $\frac{1}{3}$ can be improved to $\frac{2\sqrt{2}}{(\lambda - 1) + \tau_0/(\lambda - 1)^2} + O(\lambda - 1)$.

The proof of proposition 4.3 is quite technical, but we outline the main ideas. We will introduce the notation $\overline{\sigma}_d(1)$ to denote a term which satisfies $P[\overline{\sigma}_d(1) \leq \nu - 1] \leq \delta_{\nu,\lambda}(d)$, and let $\gamma := 2(\lambda + \lambda^{-1}) - 2$ denote the factor such that $A = \gamma I - M$. In the appendix, we show that

$$
\frac{\langle x_+, u \rangle}{\|x_+\|_2^2} = \frac{\tau_0 (u^T A^{-1} u)^2 - \overline{\sigma}_d(1)}{\tau_0 u^T A^{-2} u + z^T A^{-2} z + \overline{\sigma}_d(1)}
$$

We then unpack $A^{-1}$ and $A^{-2}$ using the Sherman-Morrison-identity, and relate the above expression to terms depending on $z^T(\gamma I - W)^{-1}z$, $z^T(\gamma I - W)^{-2}z$, and analogous terms with $z$ replaced by $u$. Since $W$ is independent of $z$ and $u$, Hanson-Wright implies

$$
z^T(\gamma I - W)^{-1}z = \text{tr}(\gamma I - W)^{-1} + \overline{\sigma}_d(1) \text{ and } z^T(\gamma I - W)^{-2}z = \text{tr}(\gamma I - W)^{-2} + \overline{\sigma}_d(1),$$

and similarly for terms involving $u$. Asymptotic expressions for $\text{tr}(\gamma I - W)^{-1}$ and $\text{tr}(\gamma I - W)^{-2}$ are well-studied in the literature Anderson et al. [2010], Peché [2006], Féral and Peché [2007], Benaych-Georges and Nadakuditi [2011]. In Appendix B.2 prove the following, quantitative convergence result:

Proposition 4.4. The following bounds hold:

$$
\text{tr}(\gamma I - W)^{-1} = s(\gamma) + \overline{\sigma}_d(1), \text{ where } s(\gamma) := \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}
$$

$$
\text{tr}(\gamma I - W)^{-2} = q(\gamma) + \overline{\sigma}_d(1), \text{ where } q(\gamma) := \frac{-d}{d\gamma} s(\gamma)
$$

The function $s(\gamma)$ is known as the Stieljes transform of the Wigner Semicircle law Anderson et al. [2010], and is a central object in the study of random matrices. The estimate $\text{tr}(\gamma I - W)^{-1} = s(\gamma) + \overline{\sigma}_d(1)$ is a direct consequence of a non-asymptotic convergence result from Simchowitz et al. [2018]; the estimate for $\text{tr}(\gamma I - W)^{-2}$ follows from a quantitative version (Lemma B.4) of a classical lemma regarding the convergence of derivatives of concave functions. Putting things together, we show in Appendix A that

$$
\frac{\langle x_+, u \rangle}{\|x_+\|_2^2} = \tau_0 \cdot \frac{1 + \overline{\sigma}_d(1)}{s(\gamma)^{-2} \cdot q(\gamma) (\tau_0 + (1 - \lambda s(\gamma))^2) + \overline{\sigma}_d(1)},
$$

Lastly, we bound $s(\gamma)^{-2}q(\gamma) \leq 3/2(\lambda - 1)$ and $1 - \lambda s(\gamma) \leq \lambda - 1$ (Lemma A.6) which implies Proposition 4.3, after some elementary computations completed in Appendix A.
4.3 Proof of Theorem 4.1, Parts 2 and 3

We start off by proving the more involved version of the Theorem 4.1, Part 3. In Section 4.3.1, we then modify the proof in the simpler setting of Part 2. Let \( \mathbf{x} := x/\|x\| \). Let \( \Delta := \text{unit}(\mathbf{x}) - \text{unit}(\mathbf{x}_*) \) denote the unit vector pointing in the direction of \( \text{unit}(\mathbf{x}) - \text{unit}(\mathbf{x}_*) \). We can lower bound the overlap between \( \widehat{\mathbf{x}} \) and \( \mathbf{u} \) via

\[
|\langle \text{unit}(\widehat{\mathbf{x}}), \mathbf{u} \rangle| \geq |\langle \text{unit}(\mathbf{x}_*), \mathbf{u} \rangle| - \|\text{unit}(\mathbf{x}) - \text{unit}(\mathbf{x}_*)\|_2 |\langle \Delta, \mathbf{u} \rangle| \\
\geq |\langle \text{unit}(\mathbf{x}_*), \mathbf{u} \rangle| - \frac{2\|\mathbf{x}_* - \widehat{\mathbf{x}}\|_2}{\|\mathbf{x}_*\|_2} |\langle \Delta, \mathbf{u} \rangle| , \tag{16}
\]

where we verify \((i)\) in Section 4.3.2. In order to control \( |\langle \Delta, \mathbf{u} \rangle| \), we introduce the \((M, b)\)-measurable event

\[
\mathcal{E}_{\text{ovlp}}(t) := \left\{ \sup_{\mathbf{u} = \text{unit}(M, b)} \mathbb{P}_u [\langle \mathbf{u}, \mathbf{u} \rangle^2 > t \cdot \text{ovlap}_{d, \lambda}(\tau_0) | M, b ] \leq t^{-1/2} \right\}
\]

For ease of notation, we shall drop the dependence on the parameter \( \nu \) in the definitions of the events \( \mathcal{E}_A \), \( \mathcal{E}_{\text{err}} \), and \( \mathcal{E}_{\text{X}^c} \), and use the shorthand \( \text{ovlap}_{d, \lambda} := \text{ovlap}_{d, \lambda}(\tau_0) \). Starting from (16), we have the following probabilistic lower bound, for any \( t > 1 \):

\[
P \left[ |\langle \text{unit}(\widehat{\mathbf{x}}), \mathbf{u} \rangle| \geq \frac{1}{2} \sqrt{\frac{\tau_0}{3(\lambda - 1)\nu}} \right] \\
\geq P \left[ 2 \cdot \frac{\|\mathbf{x}_* - \widehat{\mathbf{x}}\|_2}{\|\mathbf{x}_*\|_2} \cdot |\langle \Delta, \mathbf{u} \rangle| \leq \frac{1}{2} \sqrt{\frac{\tau_0}{3(\lambda - 1)\nu}} \right] - P \left[ |\langle \text{unit}(\mathbf{x}_*), \mathbf{u} \rangle| \geq \frac{1}{2} \sqrt{\frac{\tau_0}{3(\lambda - 1)\nu}} \right] \\
\geq P \left[ 2 \cdot \frac{\|\mathbf{x}_* - \widehat{\mathbf{x}}\|_2}{\|\mathbf{x}_*\|_2} \cdot |\langle \Delta, \mathbf{u} \rangle| \leq \frac{1}{2} \sqrt{\frac{\tau_0}{3(\lambda - 1)\nu}} \right] - P[\mathcal{E}_{\text{X}_*}] \\
= P \left[ \mathcal{E}_{\text{err}}(t, \nu) \cap \left\{ \langle \Delta, \mathbf{u} \rangle^2 \leq t \cdot \text{ovlap}_{d, \lambda} \right\} \right] - P[\mathcal{E}_{\text{X}_*}] \\
\geq P \left[ \mathcal{E}_{\text{err}}(t, \nu) \cap \left\{ \langle \Delta, \mathbf{u} \rangle^2 \leq t \cdot \text{ovlap}_{d, \lambda} \right\} \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t) \right] - P[\mathcal{E}_{\text{X}_*}] .
\]

Now, we have

\[
P \left[ \mathcal{E}_{\text{err}}(t) \cap \left\{ |\langle \mathbf{u}, \mathbf{u} \rangle| \leq t \cdot \text{ovlap}_{d, \lambda} \right\} \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t) \right] \\
= \mathbb{E}_{M, b} \mathbb{E}_u \left[ \mathbb{I} (\mathcal{E}_{\text{err}}(t) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)) \cdot \mathbb{I} (|\langle \mathbf{u}, \mathbf{u} \rangle| \leq t \cdot \text{ovlap}_{d, \lambda}) \right] |M, b] \\
\overset{(i)}{=} \mathbb{E}_{M, b} \mathbb{I} (\mathcal{E}_{\text{err}}(t) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)) \cdot \mathbb{E}_u \mathbb{I} (|\langle \mathbf{u}, \mathbf{u} \rangle| \leq t \cdot \text{ovlap}_{d, \lambda}] \\
\overset{(ii)}{\geq} \mathbb{E}_{M, b} \mathbb{I} (\mathcal{E}_{\text{err}}(t) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)) \cdot (1 - t^{-1/2}) \\
= (1 - t^{-1/2}) \mathbb{P}_{M, b} \mathbb{I} (\mathcal{E}_{\text{err}}(t) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)) \\
= (1 - t^{-1/2}) \mathbb{P}_{M, b} \mathbb{I} (\mathcal{E}_{\text{err}}(t) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)) \cdot \mathbb{P}_{M, b} \mathbb{I} (\mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}(t)] ,
\]

where \((i)\) uses the fact that \( \mathcal{E}_A \), \( \mathcal{E}_{\text{ovlp}}(t) \), and \( \mathcal{E}_{\text{err}}(t) \) are all \( \sigma(M, b) \)-measurable events, and \((ii)\) uses the definition of \( \mathcal{E}_{\text{ovlp}} \). To conclude, we lower bound \( \mathbb{E}_{\text{ovlp}} \) by Markov’s inequality:

**Lemma 4.5.** \( \mathbb{P}[\mathcal{E}_{\text{ovlp}}(t)] \geq 1 - t^{-1/2} \).
Proof. With two applications of Markov’s inequality,

\[
\mathbb{P}_{M,b}\left\{ \max_{\hat{u} = \hat{u}(M,b)} \mathbb{P}_u[\langle \hat{u}, u \rangle^2 \geq \text{ovlap}_{d,\lambda}(\tau)|M, b] \geq t^{-1/2} \right\}
\leq \frac{1}{t^{1/2}} \mathbb{E}_{M,b}\left\{ \max_{\hat{u} = \hat{u}(M,b)} \mathbb{E}_u[\langle \hat{u}, u \rangle^2|M, b] \geq t^{-1/2} \right\}
\leq \frac{1}{t^{1/2}} \mathbb{E}_{M,b}\left\{ \max_{\hat{u} = \hat{u}(M,b)} \mathbb{E}_u[\langle \hat{u}, u \rangle^2|M, b] \right\} = \frac{1}{t^{1/2}}.
\]

\[
4.3.1 \text{ Replacing ovlap}_{d,\lambda}(\tau) \text{ by } 1
\]

Fix \( \delta > 0 \) to be chosen later, and the events \( \mathcal{E}_{\text{ovlp}}'(\nu, \delta) := \{\mathbb{P}[\max_{\hat{u}} \langle \hat{u}, u \rangle^2 \geq \nu|M, b] \leq \delta\} \), and \( \mathcal{E}_{\text{err}}'(\nu) := \left\{\frac{\|x_{\ast} - \hat{x}\|_2}{\|x_{\ast}\|_2} \leq 1 - \delta, \frac{7\nu}{3(\lambda - 1)\text{ovlap}_{d,\lambda}} \right\} \). Then, the same line of arguments show

\[
\mathbb{P}\left[\|\text{unit}(\hat{x}), u\| \geq \frac{1}{2} \sqrt{\frac{7\nu}{3(\lambda - 1)}}\right] \geq (1 - \delta)\mathbb{P}\left[\mathcal{E}_{\text{err}}'(\nu) \cap \mathcal{E}_A \cap \mathcal{E}_{\text{ovlp}}'(\nu, \delta) - \mathbb{P}[\mathcal{E}_{\text{ovlp}}'(\nu, \delta) \cap \mathcal{E}_A]ight] - \mathbb{P}[\mathcal{E}_{\text{err}}'(\nu)] - \mathbb{P}[\mathcal{E}_{\text{ovlp}}'(\nu, \delta)] - \mathbb{P}[\mathcal{E}_{\text{err}}'(\nu)] - \mathbb{P}[\mathcal{E}_{\text{err}}'(\nu) - \mathcal{E}_A] - \mathbb{P}[\mathcal{E}_{\text{err}}'(\nu) - \mathcal{E}_A]
\]

Again, we bound \( \mathcal{E}_{\text{ovlp}}'(\delta) \) by Markov’s inequality

\[
\mathbb{P}_{M,\perp}\left[\{\mathbb{P}[\max_{\hat{u} \in S^{d-1}} \langle \hat{u}, u \rangle^2 \geq \nu|M, b] \geq \delta\}\right] \leq \mathbb{P}_{M,\perp}\left[\{\mathbb{P}[\|u\|_2^2 \geq \nu|M, b] \geq \delta\}\right] \leq \frac{1}{\delta} \mathbb{E}_{M,\perp}[\mathbb{P}[\|u\|_2^2 \geq \nu|M, b]]]
\]

Choosing \( \delta = \sqrt{\mathbb{P}[\|u\|_2^2 \geq \nu]} \), we have

\[
\mathbb{P}\left[\|\text{unit}(\hat{x}), u\| \geq \frac{1}{2} \sqrt{\frac{7\nu}{3(\lambda - 1)}}\right] \geq \mathbb{P}[\mathcal{E}_{\text{err}}'(\nu)] - \mathbb{P}[\mathcal{E}_A] - 2\sqrt{\mathbb{P}[\|u\|_2^2 \geq \nu]}.
\]

To conclude, we observe that \( \mathbb{P}[\mathcal{E}_A] \leq \delta_{\nu,\lambda}(d) \) by Proposition 4.2, and that \( \mathbb{P}[\|u\|_2^2 \geq \nu] \leq \delta_{\nu,\lambda}(d) \) by standard \( \chi^2 \)-concentrated (e.g. Lemma A.4).

\[
4.3.2 \text{ Proof of } (16)
\]

Note that with probability 1, \( x_{\ast} \neq 0 \). Moreover, if \( \hat{x} = 0 \), then (16) follows immediately from the triangle inequality. Otherwise,

\[
\left\|\frac{x_{\ast} - \hat{x}}{\|x_{\ast}\|_2} - \frac{\hat{x}}{\|\hat{x}\|_2}\right\|_2 \leq \left\|\frac{x_{\ast} - \hat{x}}{\|x_{\ast}\|_2}\right\|_2 + \left\|\frac{\hat{x}}{\|\hat{x}\|_2} - \frac{\hat{x}}{\|\hat{x}\|_2}\right\|_2 \leq 2 \left\|\frac{x_{\ast} - \hat{x}}{\|x_{\ast}\|_2}\right\|_2.
\]
5 Lower Bound for Plant Estimation

For simplicity, it will be easier to consider lower bounding the setting where the plant \( u \) is uniform on the sphere, rather than Gaussian:

**Proposition 5.1.** Let \( M := \lambda uu^T + W \), where \( W \) is a GOE matrix, \( u \sim S^{d-1} \), and \( b|W, u \sim \mathcal{N}(\sqrt{\tau_0}u, I/d) \). Then, for any \( \tau_0 \geq \frac{\lambda^2}{d(\lambda-1)^2} \) and any randomized query algorithm \( \text{Alg} \),

\[
\mathbb{P}_{u,M,b} \left[ \| \text{Proj}_{v(1),\ldots,v(T+1)}u \|^2_2 > 2\tau_0 \sum_{j=1}^T \lambda^4 j \right] \leq T^2 e^{-d\lambda^2 \tau_0 (\lambda-1)},
\]

where the probability is taken over the randomness of the \( \text{Alg} \), and over \( u, b, W \).

**Recovering Theorem 3.4 from Proposition 5.1.** To recover the case where \( u \sim \mathcal{N}(0, I/d) \), we invoke a data-processing argument. Suppose that, as in the setting of Theorem 3.4, \( u \sim \mathcal{N}(0, I/d) \), but consider the query model where \( \text{Alg} \) is given both \( b \) and \( \| u \|_2 \) before it makes queries of \( M \). This is strictly more information that in the query model of Definition 2.1, and any lower bound which holds in this setting will hold a fortiori in the setting of Definition 2.1.

We now observe that, conditioned on \( \| u \|_2 \), the setting described above is equivalent to the original query model of Definition 2.1, but with an instance \( M \) and \( b \), distributed as:

\[
\tilde{M} := W + \tilde{\lambda} uu^T, \quad \tilde{b} \sim \mathcal{N}(\sqrt{\tau_0}u, I/d), \quad \tilde{u} \sim S^{d-1},
\]

where \( \tilde{\lambda} := \| u \|^2_2 \lambda, \quad \tau_0 \sim \| u \|^2_2 \tau_0 \). In particular, if \( \mathcal{I}_d := \{ (\tilde{\lambda}, \tilde{\tau}_0) : \tilde{\lambda} = \nu \lambda, \tilde{\tau}_0 = \nu \tau_0, \nu \geq 1 - d^{-1/4} \} \), then in Appendix D.1, we show that Proposition 5.1 implies

\[
\mathbb{P}_{u,M,b} \left[ \| \text{Proj}_{v(1),\ldots,v(T+1)}u \|^2_2 > 2(1 + d^{-1/4})^4T^2 \cdot \tau_0 \sum_{j=1}^T \lambda^4 j \right] \leq \max_{(\tilde{\lambda}, \tilde{\tau}_0) \in \mathcal{I}_d} T^2 e^{-d\lambda^2 \tilde{\tau}_0 (\lambda-1)} + \mathbb{P}[\| u \|^2_2 \in \mathcal{I}_d].
\]

In Appendix D.1, we complete the proof of Theorem 3.4 by verifying that \( \tau_0 \geq \frac{\lambda^2}{d(\lambda-1)^2} \) holds under the given conditions on \( d, \tau_0, \lambda \), and appropriately bounding (19), and bounding the RHS of (19). Finally, under out condition on \( d \geq \frac{16\lambda^2}{\tau_0(\lambda-1)^2} \), we have \( (1 + d^{-1/4}) \leq \lambda \), which implies \( (1 + d^{-1/4})^4k^2 \leq \lambda^4k^2 \).

The proof of Proposition 5.1 draws heavily from the lower bounds in Simchowitz et al. [2018]; the key difference is that, in our setting, the algorithm \( \text{Alg} \) has access to the side information \( b \). As in Simchowitz et al. [2018], we without loss of generality that the queries \( v(1), \ldots, v(T+1) \) form an orthonormal basis; we let \( V_k \in \mathbb{R}^{d \times k} \) denote the whose columns \( v(1), \ldots, v(k) \). We then define the potential function

\[
\Phi(V_k; u) := u^TV_kV_k^Tu = \| \text{Proj}_{v(1),\ldots,v(k)}u \|^2_2,
\]

Next, since the distribution over \( M \) and \( u \) is fixed, it suffices to prove Proposition 5.1 for deterministic algorithms\(^5\)

Our central technical result is a recursion which bounds the probability that \( \Phi(V_{k+1}; u) \) exceeds a threshold \( \tau_k \), under the event that \( \Phi(V_k; u) \) is beneath a threshold \( \tau_k \).

\(^5\)Indeed, for a randomized algorithm, one can always construct a deterministic algorithm by randomized algorithm with the seed which yields the greatest value of \( \mathbb{P}_{u \sim S^{d-1}} [ \| \text{Proj}_{v(1),\ldots,v(T+1)}u \|^2_2 > (1 + c)\tau_0 \sum_{j=1}^T \lambda^2 j ] \)
Proposition 5.2. Under the randomness of \( u, W \) and \( \text{Alg} \), one has the bound

\[
P[\{ \Phi(V_k; \bar{u}) \leq \tau_k \} \cap \{ \Phi(V_{k+1}; \bar{u}) > \tau_{k+1} \}]
\leq \exp \left\{ \frac{\lambda - 1}{2\lambda} \left( d\lambda^3 (\tau_k + \tau_0) - \left( \sqrt{d\tau_{k+1}} - \sqrt{2k + 2} \right)^2 \right) \right\}
\] (21)

Remark 5.1. Proposition 5.2 coincides with Proposition 3.1 in Simchowitz et al. [2018], with the choice \( \eta = \lambda \), and the additional factor \( \tau_0 \) in the exponential.

Let’s now prove Proposition 5.1. We fix \( \delta := T^2e^{-c\lambda^2\tau_0(\lambda - 1)/2} \). It suffices to contract sequence of \( \tau_1, \tau_2, \ldots \) such that, for each \( k \geq 1 \), the right hand side of (21) is at most \( \delta/k^2 \). Indeed, when \( k = 0 \), \( \Phi(V_0; \bar{u}) = 0 \), so we can choose \( \tau_0 = 0 \) (since \( P[\Phi(V_0; \bar{u}) \leq 0] = 1 \)). Therefore, summing up, this will prove

\[
P[\Phi(V_{T+1}; \bar{u}) > \tau_{T+1}]
= \sum_{k=0}^{T} \sum_{k=1}^{\infty} \delta/k^2 \leq 2\delta.
\]

To choose \( \{ \tau_k \} \), suppose for the moment that that we can ensure that, for all \( k \geq 0 \),

\[
\left( \sqrt{d\tau_{k+1}} - \sqrt{2k + 2} \right)^2 \geq d\tau_{k+1}/\lambda
\] (22)

Then, it suffices to choose \( \tau_{k+1} \) such that \( \exp \left\{ \frac{\lambda - 1}{2\lambda} \left( d\lambda^4 (\tau_k + \tau_0) - d\tau_{k+1} \right) \right\} = \delta/k^2 \). Solving for \( \tau_{k+1} \) in terms of \( \tau_k \) and \( \tau_0 \), we find

\[
\tau_{k+1} := \frac{2\lambda^2}{d(\lambda - 1)} \log(k^2/\delta) + \lambda^4(\tau_k + \tau_0).
\]

We can give a closed form upper bound for \( \tau_{k+1} \) via

\[
\tau_{k+1} = \sum_{j=1}^{k} \lambda^{4(k-j)} \left( \lambda^4 \tau_0 + \frac{2\lambda^2}{d(\lambda - 1)} \log(j^2/\delta) \right)
\leq \left( \lambda^4 \tau_0 + \frac{2\lambda^2 \log(k^2/\delta)}{d(\lambda - 1)} \right) \sum_{j=1}^{k} \lambda^{4(k-j)} = \left( \tau_0 + \frac{2\log(k^2/\delta)}{d(\lambda - 1)} \right) \sum_{j=1}^{k} \lambda^{4j}.
\]

In particular, taking \( k = T \) if \( \delta := T^2e^{-\lambda^2\tau_0(\lambda - 1)} \), we have that

\[
P \left[ \Phi(V_{T+1}; \bar{u}) > 2\tau_0 \sum_{j=1}^{k} \lambda^{4j} \right] \leq T^2e^{-\lambda^2\tau_0(\lambda - 1)/2}.
\]

To see that our chosen sequence of \( \tau_k \) actually satisfies (22), we note that \( \tau_k \) satisfies the lower bound \( \tau_k \geq \lambda^{4k} \tau_0 \). As we show in Section D.2, this lower bound implies that Equation (22) is satisfied for all \( k \geq 0 \).
6 Upper Bound on $\text{ovlap}_{d,\lambda}$

In this section, we present an asymptotic bound on

$$\text{ovlap}_{d,\lambda}(\tau_0) := \mathbb{E}_{M,b,\hat{u} \in S^{d-1}} \mathbb{E}_u [\langle \hat{u}, u \rangle^2 | M, b],$$

stated as follows:

**Theorem 6.1** (Asymptotic Bound on $\text{ovlap}_{d,\lambda}(\tau_0)$). For $u, b, M$ and $\text{ovlap}_{d,\lambda}(\tau_0)$ defined in Section 3.1, we have for $\lambda \in (1, 2]$ that

$$\lim_{d \to \infty} \text{ovlap}_{d,\lambda}(\tau_0) \leq 1 - \frac{1}{\lambda^2} + \tau_0 + \frac{\sqrt{\tau_0}}{\lambda}.$$ 

In particular, if $\tau_0 = (\lambda - 1)^2$, then the above reduces to

$$\lim_{d \to \infty} \text{ovlap}_{d,\lambda}(\tau_0) \leq (\lambda - 1) \left( \frac{\lambda + 1}{\lambda^2} + (\lambda - 1) + \frac{1}{\lambda} \right) \leq \frac{9}{2}(\lambda - 1).$$

This implies the following corollary, which proves the first part of Proposition 3.1:

**Corollary 6.2.** There exists a $d_0 = d_0(\lambda, \tau_0)$ such that for all $d \geq d_0$, $\text{ovlap}_{d,\lambda}(\tau_0) \leq 5(\lambda - 1)$.

In other words, for $d$ sufficiently large, we can take $K$ in Proposition 3.2 to be a universal constant. For intuition about Theorem 6.1, consider the setting where we do not have access to side information $b$, that is, $\tau_0 = 0$. Perhaps the most natural estimator of $u \sim \mathcal{N}(0, I/d)$ is the top eigenvector of $v_1(M)$, and it is known (see, e.g. Péché [2006]) that, for any $\lambda \geq 0$,

$$\lim_{d \to \infty} \langle v_1(M), u \rangle^2 = \max \{ 1 - \lambda^{-2}, 0 \}.$$

Nevertheless, one may still wonder if there exists a more sophisticated (maybe computationally infeasible!) estimator $\hat{u}$ has a larger expected overlap with $u$ than does $v_1(M)$.

Beautiful recent results due to Barbier et al. [2016] and Lelarge and Miolane [2016] show in fact that this is not the case. These works show an explicit and very general formula for the mutual information between $M$ and $u$. Barbier et al. [2016] applies when the entries of $u$ have a finite (discrete) support, and Lelarge and Miolane [2016] when $u$ is drawn according to any distribution with i.i.d. coordinates whose second moments are bounded. Due to a correspondence between mutual information and MMSE in a Gaussian channel [Guo et al., 2005], these works use this formula to derive the following asymptotic expression for the minimum mean square error (MMSE) for estimating $uu^\top$ given $M := W + uu^\top$, defined as:

$$\text{MMSE}_{d,\lambda}(uu^\top | M) := \mathbb{E}_u \left[ ||uu^\top - \mathbb{E}[uu^\top | M]||_F^2 | M \right].$$

By relating the optimal overlap to the MMSE$_{d,\lambda}$, Lelarge and Miolane [2016] conclude that, in the special case that $P_0 = \mathcal{N}(0, 1)$, $v_1(M)$ indeed attains the optimal asymptotic overlap of $1 - \lambda^{-2}$.

Unlike the setting of Lelarge and Miolane [2016], we need to account for the additional side information given in $b$. This is achieved by noticing that, conditioning on $b$ amounts to changing the conditional distribution of $u$; by conjugacy, $u|b$ is still Gaussian, and its covariance is isotropic (Lemma C.1). Lastly, by a symmetry argument, we show without loss of generality $\mathbb{E}[u|b]$ is aligned with the all-ones vector. Thus, the coordinates of $u$ given $b$ can be assumed to be i.i.d, returning us to the setting of Lelarge and Miolane [2016]. The proof of Theorem 6.1 is formally given in Section 6.2 below.
6.1 Conjectures for Non-Asymptotic Bound on $\text{ovlap}_{d,\lambda}$

We now introduce a conjecture under which we can bound $d$ by being polynomially large in relevant problem parameters.

**Conjecture 6.1** (Non-Asymptotic Convergence). There exists universal constants $c_0, \ldots, c_3$ such that, for all $\lambda \in (1, 2]$, all $\mu \in (0, 1)$, $d \geq d_0$, and $\hat{u} \sim \mathcal{N}(\mu/\sqrt{d}, 1/d)$, either (a)

$$\text{MMSE}_{d,\lambda}(\hat{u}^\top | M) \geq \lim_{d \to \infty} \text{MMSE}_{d,\lambda}(\tilde{u}^\top | M) - c_0 d^{-c_1} \cdot (\lambda - 1)^{-c_2} \cdot \mu^{-c_3},$$

or (b), the mutual information $i(W + \lambda \hat{u}^\top, \tilde{u}^\top)$ between $W + \lambda \hat{u}^\top$ and $\tilde{u}^\top$ satisfies

$$|i(W + \lambda \hat{u}^\top; \tilde{u}^\top) - \lim_{d \to \infty} i(W + \lambda \tilde{u}^\top; \tilde{u}^\top)| \leq c_0 d^{-c_1} \cdot (\lambda - 1)^{-c_2} \cdot \mu^{-c_3}.$$

The above conjecture simply says that the relevant information-theoretic quantities converge to their asymptotic values at polynomial rates in relevant problem conjectures. The author believes that the dependence on $\mu \in (0, 1)$ is not needed, but we accomodate this dependence in the conjecture because it does not affect what follows. In Section C.5, we that the above conjecture implies the desired bound non-asymptotic on $\text{ovlap}_{d,\lambda}$:

**Proposition 6.3.** Conjecture 6.1 part (b) implies Conjecture 6.1 part (a), and Conjecture 6.1 part (a) implies that there exists constants $c_1, c_2, c_3, c_0 > 0$ for which

$$\text{ovlap}_{d,\lambda}(\tau_0) \leq 1 - \frac{1}{\lambda^2} + \tau_0 + \frac{\sqrt{\tau_0}}{\lambda} + c_0 d^{-c_1} (\lambda - 1)^{-c_2} \tau_0^{-c_3}. \quad (24)$$

In particular, if $\tau_0 = (\lambda - 1)^2$, we get the following analogue of Corollary 6.2, which proves the second part of Proposition 3.1.

**Corollary 6.4.** If either Part (a) or (b) of Conjecture 6.1 hold, then there exists universal constants $c_0, c_1 > 0$, $d \geq c_0 (\lambda - 1)^{-c_1}$, $\text{ovlap}_{d,\lambda}(\tau_0) \leq 5(\lambda - 1)$.

6.2 Proof of Theorem 6.1

Fix $\lambda \in (1, 2]$ and $\tau_0 \leq (\lambda - 1)^2$. To prove Theorem 6.1, we relate $\text{ovlap}_{d,\lambda}(\tau)$ to the Minimum Mean Squared Error of estimating $u^\top$ given $M$ and $b$. Define the conditional MMSE

$$\text{MMSE}_{d,\lambda}(u^\top; M, b) := \mathbb{E}_u \left[ \|u^\top - \mathbb{E}[u^\top | M, b]\|_2^2 | M, b \right], \quad (25)$$

which is the minimum mean squared error attainable by any estimate of $u^\top$ given access to $M$ and $b$. $\text{ovlap}_{d,\lambda}(\tau_0)$ is controlled by $\text{MMSE}_{d,\lambda}(u^\top; M, b)$ via the following estimate (proved in Section C.1)

**Lemma 6.5.** There exists universal constant $c_1, c_2$ such that for any estimator $\hat{u} = \hat{u}(M, b) \in S^{d-1}$,

$$\mathbb{E}_{M,b}[\|\hat{u} - u\|^2 | M, b] \leq \sqrt{\mathbb{E}[[u]^2] - \mathbb{E}_{M,b}[\text{MMSE}_{d,\lambda}(u^\top | M, b)]} + c_1 d^{-c_2}.$$

By Jensen’s inequality, we upper bound the above display by the minimum mean-squared error, conditioned on $b$

$$\mathbb{E}_{b}[\|u\|^2] - \mathbb{E}_{M,b}[\text{MMSE}_{d,\lambda}(u^\top | M, b)] \leq \mathbb{E}_{b}[\mathbb{E}_{M,u}[\|u\|^2] - \mathbb{E}_M[\text{MMSE}_{d,\lambda}(u^\top | b, M)]]$$

In Lemma C.1, we compute the conditional distribution $u|b \sim \mathcal{N} \left( \frac{\sqrt{\tau_0} b}{1 + \tau_0} \cdot \frac{1}{d} \right)$. By rotation invariance, we argue that we may assume that $b$ is alinged with the all ones vector. This, combined with some truncation, lets us bound $\text{ovlap}_{d,\lambda}(\tau_0)$ in terms of a MMSE parameterized by the conditioned mean of $u$. Specifically, we have the following bound, proved in Section C.2
Proposition 6.6. Define the mean parametrized minimum mean squared error:

$$\text{MMSE}_{d,\lambda}(\mu) := \mathbb{E}_{\tilde{u},\tilde{M}} \left[ \|\tilde{u}\tilde{u}^\top - \mathbb{E}[\tilde{u}\tilde{u}^\top]\tilde{M}\|_F^2 \right]$$

where $\tilde{M} := W + \lambda \tilde{u}\tilde{u}^\top \sim \mathcal{N}(\mu/\sqrt{d}, 1/d)$.

Then, letting $\alpha$ have the distribution of $\|x\|$ for $x \sim \mathcal{N}(0, I/d)$, we have

$$\text{overlap}_{d,\lambda}(\tau_0) \leq \sqrt{\mathbb{E}_x I(\alpha - 1) \leq d^{-1}} \cdot \left( (1 + \tau_0 \alpha^2)^2 - \text{MMSE}_{d,\lambda}/(1 + \tau_0) \left( \sqrt{\tau_0} \alpha \right) \right) + c_1 d^{-c_2}.$$

for universal constants $c_1, c_2$.

The above proposition truncates to $I(|\alpha - 1| \leq d^{-1})$ for minor technical reasons. The upshot of using the mean-parametrized MMSE is that it is defined in terms of the random vector $\tilde{u} \sim \mathcal{N}(0, I/d)$, which has independent and similarly distributed coordinates. This allows us to use Theorem 1 in Lelarge and Miolane [2016], which gives an exact expression for the asymptotic value of the $\mathbb{E}_{\tilde{u},\tilde{M}} \left[ \|\tilde{u}\tilde{u}^\top - \mathbb{E}[\tilde{u}\tilde{u}^\top]\tilde{M}\|_F^2 \right]$ when $\tilde{u}$ has i.i.d. coordinates with finite second moments. When specialized to our setting, the bound yields the following estimate (see Sec C.3):

Corollary 6.7. For any fixed $\lambda \geq 0$ and $\mu \in \mathbb{R}$,

$$f(1 + \mu^2)^2 - \lim_{d \to \infty} \text{MMSE}_{d,\lambda}(\mu) \leq \left( 1 - \frac{1}{\lambda^2} + \mu^2 + \frac{|\mu|}{\lambda} \right)^2.$$

We may now conclude the proof of Theorem 6.1:

Proof of Theorem 6.1. For $d \in \mathbb{N}$ and $\lambda \in (1, 2]$, we define the functions

$$F_{d,\lambda}(\alpha) := I(|\alpha - 1| \leq d^{-1}) \cdot \left( (1 + \tau_0 \alpha^2)^2 - \text{MMSE}_{d,\lambda}/(1 + \tau_0) \left( \sqrt{\tau_0} \alpha \right) \right)$$

Note then that by Proposition 6.6, one has

$$\text{overlap}_{d,\lambda}(\tau) \leq \frac{\sqrt{\mathbb{E}_x F_{d,\lambda}(\alpha)}}{1 + \tau_0} + c_1 d^{-c_2}.$$

For $|\alpha - 1| \leq +d^{-1}/4$, we have

$$F_{d,\lambda}(\alpha) = \left( 1 - \frac{1}{\lambda^2} + \alpha^2 \tau_0 + \frac{\alpha \sqrt{\tau_0}}{\lambda} \right)^2 + \text{Err}_1(d; \alpha, 0).$$

We define $F_{d,\lambda}(\alpha) = \left( 1 - \frac{1}{\lambda^2} + \alpha^2 \tau_0 + \frac{\alpha \sqrt{\tau_0}}{\lambda} \right)^2 + \text{Err}_1(d; \alpha, 0) + \text{Err}_2(d),$

where (a) $\lim_{d \to \infty} \text{Err}_1(d; \alpha, 0) = 0$ for any choice of $\alpha, \lambda$, by Corollary 6.7 and (b) $\lim_{d \to \infty} \text{Err}_2(d) = 0$ (uniformly in $\alpha, \lambda$) by the assumption $|\alpha - 1| \leq d^{-1}/4$. Moreover, by writing out the explicit conditional expectation, it is straightforward to verify that $F_{d,\lambda}(\alpha)$ is continuous on $[1 - d^{-1/4}, 1 + d^{1/4}]$. Hence, given that $[1 - d^{-1/4}, 1 + d^{-1/4}]$ is compact, the error term $\alpha \lambda(d)$ can be chosen to be uniform in $\alpha$. Moreover, $F_{d,\lambda}(\alpha) = 0$ for $|\alpha - 1| > d^{-1/4}$. We therefore conclude that

$$\lim_{d \to \infty} F_{\alpha \geq 0} Y_{d,\lambda}(\alpha) \leq \left( 1 - \frac{1}{\lambda^2} + \tau_0 + \frac{\sqrt{\tau_0}}{\lambda} \right)^2.$$

(27)
Hence,

\[
\lim_{d \to \infty} \text{ovlap}_{d, \lambda}(\tau) = \lim_{d \to \infty} \left( \frac{\sqrt{E_{\alpha} Y_{d, \lambda}(\alpha)}}{1 + \tau_0} + c_1 d^{-c_2} \right) = \frac{\sqrt{\lim_{d \to \infty} E_{\alpha} Y_{d, \lambda}(\alpha)}}{1 + \tau_0}
\]

\[
\leq \left( 1 - \frac{1}{\lambda^2} + \tau_0 + \frac{\sqrt{\tau_0}}{\lambda} \right) \leq 1 - \frac{1}{\lambda^2} + \tau_0 + \frac{\sqrt{\tau_0}}{\lambda}.
\]
References

Alekh Agarwal and Leon Bottou. A lower bound for the optimization of finite sums. arXiv preprint arXiv:1410.0723, 2014.

Alekh Agarwal, Martin J Wainwright, Peter L Bartlett, and Pradeep K Ravikumar. Information-theoretic lower bounds on the oracle complexity of convex optimization. In Advances in Neural Information Processing Systems, pages 1–9, 2009.

Naman Agarwal and Elad Hazan. Lower bounds for higher-order convex optimization. arXiv preprint arXiv:1710.10329, 2017.

Ahmed El Alaoui and Florent Krzakala. Estimation in the spiked wigner model: A short proof of the replica formula. arXiv preprint arXiv:1801.01593, 2018.

Zeyuan Allen-Zhu and Yuanzhi Li. First efficient convergence for streaming k-pca: a global, gap-free, and near-optimal rate. arXiv preprint arXiv:1607.07837, 2016.

Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. An introduction to random matrices, volume 118. Cambridge university press, 2010.

Amurag Anshu, Naresh B Goud, Rahul Jain, Srijita Kundu, and Priyanka Mukhopadhyay. Lifting randomized query complexity to randomized communication complexity. arXiv preprint arXiv:1703.07521, 2017.

Ery Arias-Castro, Emmanuel J Candes, and Mark A Davenport. On the fundamental limits of adaptive sensing. IEEE Transactions on Information Theory, 59(1):472–481, 2013.

Yossi Arjevani and Ohad Shamir. On the iteration complexity of oblivious first-order optimization algorithms. In International Conference on Machine Learning, pages 908–916, 2016.

Yossi Arjevani, Ohad Shamir, and Ron Shiff. Oracle complexity of second-order methods for smooth convex optimization. Mathematical Programming, pages 1–34, 2017.

Jean Barbier, Mohamad Dia, Nicolas Macris, Florent Krzakala, Thibault Lesieur, and Lenka Zdeborová. Mutual information for symmetric rank-one matrix estimation: A proof of the replica formula. In Advances in Neural Information Processing Systems, pages 424–432, 2016.

Florent Benaych-Georges and Raj Rao Nadakuditi. The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices. Advances in Mathematics, 227(1):494–521, 2011.

Sébastien Bubeck et al. Convex optimization: Algorithms and complexity. Foundations and Trends® in Machine Learning, 8(3-4):231–357, 2015.

Rui M Castro and Ervin Tánczos. Adaptive compressed sensing for support recovery of structured sparse sets. IEEE Transactions on Information Theory, 63(3):1535–1554, 2017.

Rui M Castro et al. Adaptive sensing performance lower bounds for sparse signal detection and support estimation. Bernoulli, 20(4):2217–2246, 2014.

Delphine Féral and Sandrine Péché. The largest eigenvalue of rank one deformation of large Wigner matrices. Communications in Mathematical Physics, 272(1):185–228, 2007.
Dan Garber, Elad Hazan, Chi Jin, Sham M Kakade, Cameron Musco, Praneeth Netrapalli, and Aaron Sidford. Faster eigenvector computation via shift-and-invert preconditioning. 2016.

Dongning Guo, Shlomo Shamai, and Sergio Verdú. Mutual information and minimum mean-square error in gaussian channels. *IEEE Transactions on Information Theory*, 51(4):1261–1282, 2005.

Kevin G Jamieson, Robert Nowak, and Ben Recht. Query complexity of derivative-free optimization. In *Advances in Neural Information Processing Systems*, pages 2672–2680, 2012.

Florent Krzakala, Jiaming Xu, and Lenka Zdeborová. Mutual information in rank-one matrix estimation. *arXiv preprint arXiv:1603.08447*, 2016.

Beatrice Laurent and Pascal Massart. Adaptive estimation of a quadratic functional by model selection. *Annals of Statistics*, pages 1302–1338, 2000.

Marc Lelarge and Léo Miolane. Fundamental limits of symmetric low-rank matrix estimation. *arXiv preprint arXiv:1611.03888*, 2016.

Thibault Lesieur, Florent Krzakala, and Lenka Zdeborová. Mnse of probabilistic low-rank matrix estimation: Universality with respect to the output channel. *arXiv preprint arXiv:1507.03857*, 2015.

Jelani Nelson, Jakub Pachocki, and Zhengyu Wang. Optimal lower bounds for universal relation, samplers, and finding duplicates. *arXiv preprint arXiv:1703.08139*, 2017.

Arkadii Nemirovskii, David Borisovich Yudin, and Edgar Ronald Dawson. Problem complexity and method efficiency in optimization. 1983.

Sandrine Péché. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probability Theory and Related Fields*, 134(1):127–173, 2006.

Eric Price and David P Woodruff. Lower bounds for adaptive sparse recovery. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 652–663. Society for Industrial and Applied Mathematics, 2013.

Mark Rudelson, Roman Vershynin, et al. Hanson-wright inequality and sub-gaussian concentration. *Electronic Communications in Probability*, 18, 2013.

Ohad Shamir. Fundamental limits of online and distributed algorithms for statistical learning and estimation. In *Advances in Neural Information Processing Systems*, pages 163–171, 2014.

Max Simchowitz, Ahmed El Alaoui, and Benjamin Recht. Tight query complexity lower bounds for pca via finite sample deformed wigner law. *arXiv preprint arXiv:1804.01221*, 2018.

Jacob Steinhardt and John C Duchi. Minimax rates for memory-bounded sparse linear regression. In *COLT*, pages 1564–1587, 2015.

Jacob Steinhardt, Greg Valiant, and Stefan Wager. Memory, communication, and statistical queries. In *Electronic Colloquium on Computational Complexity (ECCC)*, volume 22, pages 1–2, 2015.

Blake E Woodworth and Nati Srebro. Tight complexity bounds for optimizing composite objectives. In *Advances in neural information processing systems*, pages 3639–3647, 2016.
A Proof of Proposition 4.3

Notation: Let $\gamma := 2(\lambda + \lambda^{-1}) - 2$. Recall the notation that $Z = \mathfrak{d}_d(1)$ if $\mathbb{P}[\mathfrak{d}_d(1) \geq \nu - 1] \leq \delta_{\nu, \lambda}(d)$, or equivalently, for any $\epsilon > 0$,

$$\mathbb{P}[|Z| \geq \epsilon] \leq \exp(-d^3 \epsilon c^2 (\lambda - 1)^{c_1})$$  \hfill (28)

for constants $c_0, c_1, c_2, c_3 > 0$. We will also use the notation $\delta_{\lambda}(d)$ to denote a term which is at most $\exp(-c_0 d^3 (\lambda - 1)^{c_2})$. Finally, we say $W = \overline{\mathfrak{d}}_d(1)$ if there is are constants $c_0, c_1, c_2, c_3$ such that $\mathbb{P}[|W| \geq (\lambda - 1)^{c_1}] \leq \exp(-c_0 d^3 (\lambda - 1)^{c_2})$. We shall use the following observation throughout:

Fact A.1. If $W = \overline{\mathfrak{d}}_d(1)$ and $Z = \mathfrak{d}_d(1)$, then $WZ = \mathfrak{d}_d(1)$, and $W + Z = \overline{\mathfrak{d}}_d(1)$. Moreover, $|Z|^p = \mathfrak{d}_d(1)$ for any fixed constant $p > 0$, and if $Z' = \mathfrak{d}_d(1)$, $ZZ' = \mathfrak{d}_d(1)$.

Proof of Proposition 4.3. We begin by writing out

$$\langle x_*, u \rangle = \langle A^{-1} b, u \rangle = \sqrt{\gamma_0} u^\top A^{-1} u + z^\top A^{-1} u$$  \hfill (29)

and

$$\|x\|^2_2 = b^\top A^{-2} b = \gamma_0 u^\top A^{-2} u + z^\top A^{-2} z + 2\sqrt{\gamma_0} u^\top A^{-2} z$$  \hfill (30)

The following lemma (proof in Section A.1.1) shows that $z^\top A^{-1} u$ and negligible $z^\top A^{-2} u$:

Lemma A.2. $z^\top A^{-1} u = \mathfrak{d}_d(1)$ and $u^\top A^{-2} z = \mathfrak{d}_d(1)$. More precisely, there exists a constant $c_1, c_2 > 0$ such that, the event $\mathcal{E}_{\text{cross}}(\delta) := \{z^\top A^{-1} u \leq \overline{\mathfrak{d}}_d(1) \cdot (d \log(1/\delta))^{-1/2}\}$ occurs with probability at least $1 - \delta$.

Next, we unpack our terms via the Sherman-Morrison identity, which states that any invertible $A \in \mathbb{R}^{d \times d}$, and $x, y \in \mathbb{R}^d$, one has

$$(A + xy^\top) = A^{-1} - \frac{A^{-1} xy^\top A^{-1}}{1 + y^\top A^{-1} x}$$

In particular, define the denominator term $\text{denom} := 1 - \lambda u^\top (\gamma I - W) u$, we have

$$A^{-1} = (\gamma I - W - \lambda uu^\top) = (\gamma I - W)^{-1} + \frac{\lambda (\gamma I - W) uu^\top (\gamma I - W)}{\text{denom}}$$  \hfill (31)

and thus, with probability at least $1 - \delta$

$$\langle x_*, u \rangle \overset{\text{Lem. A.2}}{=} \sqrt{\gamma_0} u^\top A^{-1} u + \overline{\mathfrak{d}}_d(1) \cdot (d \log(1/\delta))^{-1/2}$$

$$= \sqrt{\gamma_0} \left\{ u^\top (\gamma I - W)^{-1} u + \frac{\lambda (u^\top (\gamma I - W) u)^2}{\text{denom}} \right\} + \overline{\mathfrak{d}}_d(1) \cdot (d \log(1/\delta))^{-1/2}$$

$$\overset{(i)}{=} \sqrt{\gamma_0} u^\top (\gamma I - W)^{-1} u \cdot \left\{ 1 + \frac{\lambda u^\top (\gamma I - W) u}{\text{denom}} \right\} + \overline{\mathfrak{d}}_d(1) \cdot (d \log(1/\delta))^{-1/2}$$

$$= \sqrt{\gamma_0} u^\top (\gamma I - W)^{-1} u \frac{\text{denom}}{\text{denom}} + \overline{\mathfrak{d}}_d(1) \cdot (d \log(1/\delta))^{-1/2},$$  \hfill (32)

where $(i)$ uses $\lambda u^\top (\gamma I - W) u = 1 - \text{denom}$. To bound (30), we need to control $u^\top A^{-2} u$ and $z^\top A^{-2} z$. This is achieved by the following lemma, proved in Section A.2.
Lemma A.3. The following estimates hold:

\[ u^\top A^{-2} u = u^\top (\gamma I - W)^{-2} u \cdot \frac{1}{\text{denom}^2} \]

\[ z^\top A^{-2} z = z^\top (\gamma I - W)^{-2} z + \bar{\sigma}_d(1) \left\{ \frac{|\text{denom}| + u^\top (\gamma I - W)^{-2} u}{\text{denom}^2} \right\} \]

Inspecting Lemma A.3 and (A.2), we see that the terms we must control are \( u^\top (\gamma I - W)^{-1} u \), \( z^\top (\gamma I - W)^{-1} z \), and \( z^\top (\gamma I - W)^{-1} z \). Our first step is to invoke the Hanson-Wright inequality (see Section A.1.2 for proof):

Lemma A.4. \( z^\top (\gamma I - W)^{-1} z = \frac{1}{2} \text{tr}(\gamma I - W)^{-1} + \bar{\sigma}_d(1) \), \( u^\top (\gamma I - W)^{-1} u = \frac{1}{d} \text{tr}(\gamma I - W)^{-1} + \bar{\sigma}_d(1) \), and \( u^\top (\gamma I - W)^{-2} u = \frac{1}{d} \text{tr}(\gamma I - W)^{-1} + \bar{\sigma}_d(1) \)

Using the bounds \( \text{tr}(\gamma I - W)^{-1} = s(\gamma) + \bar{\sigma}_d(1) \) and \( \text{tr}(\gamma I - W)^{-2} = q(\gamma) + \bar{\sigma}_d(1) \) from Proposition 4.4, we have the following estimates:

\[ \langle x_\star, u \rangle = \sqrt{\tau_0} \frac{s(\gamma) + \bar{\sigma}_d(1)}{\text{denom}} + \bar{\sigma}_d(1) \cdot (d \log(1/\delta))^{-1/2} . \]

\[ u^\top A^{-1} u = (q(\gamma) + \bar{\sigma}_d(1)) \cdot \frac{1}{\text{denom}^2} . \]

\[ z^\top A^{-1} z = q(\gamma) + \bar{\sigma}_d(1) + \bar{\sigma}_d(1) \cdot \frac{|\text{denom}| + (q(\gamma) + \bar{\sigma}_d(1))}{\text{denom}^2} . \]

We can see that \( \text{denom} = 1 - \lambda s(\gamma) + \bar{\sigma}_d(1) \), and using the fact that \( s(\gamma), 1/s(\gamma), 1 - \lambda s(\gamma) \) and \( q(\gamma) \) are all \( \bar{\sigma}_d(1) \) (deterministically!):

Lemma A.5.

Hence, and invoking Fact A.1 to simplify terms in the denominator, we have

\[ \frac{\langle x_\star, u \rangle^2}{\|x_\star\|^2} = \frac{\text{denom}^2 \langle x_\star, u \rangle^2}{\text{denom}^2 \|x_\star\|^2} = \frac{\tau_0 \cdot (s(\gamma) + \bar{\sigma}_d(1)) \pm \bar{\sigma}_d(1) \cdot (d \log(1/\delta))^{-1/2}^2}{q(\gamma)(\tau_0 + (1 - \lambda s(\gamma))^2) + \bar{\sigma}_d(1)} . \]

\[ = \frac{\tau_0 \cdot (1 + \bar{\sigma}_d(1) \pm \bar{\sigma}_d(1) \cdot (\tau_0 d \log(1/\delta))^{-1/2})^2}{s(\gamma)^2 \cdot q(\gamma)(\tau_0 + (1 - \lambda s(\gamma))^2) + \bar{\sigma}_d(1)} , \]

where in the last line, we divided the numerator and denominator both by \( s(\gamma) \), using the fact that \( 1/s(\gamma) = \bar{\sigma}_d(1) \) (see (33)), and simplifying with Fact A.1. Let’s simplify the numerator a bit. As long as \( \tau_0 \geq d^{-9} \) and taking \( \delta = e^{-0.05d} \), we can see that, with probability

\[ 1 + \sigma_d(1) \pm \bar{\sigma}_d(1) \cdot (\tau_0 d \log(1/\delta))^{-1/2} = 1 + \bar{\sigma}_d(1) . \]

We now introduce a lemma which allows us to

Lemma A.6. \( q(\gamma)s(\gamma)^{-2} \leq \frac{3}{2(\lambda - 1)} \) and \( 1 - \lambda s(\gamma) \leq (\lambda - 1) \).
Moreover, since \( \tau_0 \leq (\lambda - 1) \), we conclude that 
\[
\frac{\langle x_*, u \rangle^2}{\|x_*\|^2} = \tau_0 \cdot \frac{1 + \tilde{O}_d(1)}{3(\lambda - 1) + \tilde{O}_d(1)},
\]
which implies the Proposition. \( \square \)

## A.1 Supporting Proofs

We shall introduce the notation here \( \delta_\lambda(d) \), which denotes a term which is at most \( \exp(-d^{c_1}(\lambda - 1)^{c_2}) \) for universal constants \( c_1, c_2 > 0 \). Note that \( W = \tilde{O}_d(1) \) if \( \mathbb{P}[|W| \geq (\lambda - 1)^{-c_1}] \leq \delta_\lambda(d) \). We shall also find useful the following explicit expression for \( q(a) \):

**Lemma A.7.** \( q(a) := -\frac{d}{da} s(a) = \frac{s(a)}{\sqrt{a^4 - 1}} \).

**Proof.** Recalling \( s(a) = \frac{a - \sqrt{a^2 - 1}}{2(\sqrt{a^2 - 1})} \), we have \( q(a) = -\frac{d}{da} s(a) = \frac{1}{2} \left( 1 - \frac{a}{\sqrt{a^2 - 1}} \right) \). Rearranging, we find \( \frac{1}{\sqrt{a^2 - 1}} (\frac{\sqrt{a^2 - 1}}{2} - 1) \), and we recognize \( \sqrt{a^2 - 1} := s(a) \). \( \square \)

### A.1.1 Proof of Lemma A.2

For \( \ell \in \{1, 2\} \), we can write \( z^T A^{-\ell} u = \|A^{-\ell} u\|_2 \cdot \langle \text{unit}(A^{-\ell} u), z \rangle \). By standard Gaussian concentration, and the fact that \( A^\ell u \perp z \), \( \langle \text{unit}(A^{-\ell} u), z \rangle = \tilde{O}_d(1) \). Hence, by Fact A.1, it suffices to show that \( \|A^{-\ell} u\|_2 \leq \tilde{O}_d(1) \). To this end

\[
\mathbb{P}[\|A^{-\ell} u\|_2 \geq 2(\frac{\sqrt{2} \lambda}{(\lambda - 1)^2})] \leq \mathbb{P}[\|u\|_2 \geq 2] + \mathbb{P}[\|u\|_2 \geq (\frac{\sqrt{2} \lambda}{(\lambda - 1)^2})^{-1}] \leq \delta_\lambda(d),
\]

where the last inequality is standard gaussian concentration for \( \|u\|_2 \), and Proposition 4.2 for bounding \( \mathbb{P}[\|A\|_2 \geq (\frac{\sqrt{2} \lambda}{(\lambda - 1)^2})^{-1}] \).

### A.1.2 Proof of Lemma A.4

By Theorem B.1 (which bounds \( \|W\| \leq 2 + d^{-\Omega(1)} \) with high probability), we see that \( \gamma I - W \gtrsim (\lambda - 1)^2 \) with probability \( 1 - \delta_\lambda(d) \). The bounds now follow from a routine application of the Hanson-Wright inequality (see, e.g. Rudelson et al. [2013]) on the event \( \{\gamma I - W \gtrsim (\lambda - 1)^2\} \), and noting that \( u \) and \( z \) are both independent of \( W \).

### A.2 Proof of Lemma A.3

In light of (31), we have that

\[
A^{-1} = \left( (\gamma I - W)^{-1} + \frac{(\gamma I - W)^{-1}(uu^T)(\gamma I - W)^{-1}}{\text{denom}} \right)^2
\]

\[
= (\gamma I - W)^{-2} + 2\lambda \text{Symm} \left( \frac{(\gamma I - W)^{-2}(uu^T)(\gamma I - W)^{-1}}{\text{denom}} \right)
\]

\[
+ \lambda^2 u^T (\gamma I - W)^{-2} u \frac{(\gamma I - W)^{-1}uu^T(\gamma I - W)^{-1}}{\text{denom}^2}
\]
A. Computing $u^\top B^{-2}u$. Using the above, we have that $u^\top B^{-2}u$

\[
u^\top A^{-2}u = u^\top (\gamma I - W)^{-2}u + 2\lambda \frac{u^\top (\gamma I - W)^{-2}(uu^\top)(\gamma I - W)^{-1}u}{\text{denom}} + \lambda^2 \frac{u^\top (\gamma I - W)^{-2}uu^\top(\gamma I - W)^{-1}u}{\text{denom}^2} = u^\top (\gamma I - W)^{-2}u \cdot \left\{ 1 + 2\lambda \frac{u^\top (\gamma I - W)^{-1}u}{\text{denom}} + \lambda^2 \left( \frac{u^\top (\gamma I - W)^{-1}u}{\text{denom}} \right)^2 \right\}
\]

\[= u^\top (\gamma I - W)^{-2}u \cdot \left\{ 1 + \frac{\lambda u^\top (\gamma I - W)^{-1}u}{\text{denom}} \right\}^2 \]

\[= u^\top (\gamma I - W)^{-2}u \cdot \left( \frac{1}{\text{denom}} \right)^2 \left( \lambda u^\top (\gamma I - W)u = 1 - \text{denom}. \right) \]

B. Computing $z^\top B^{-2}z$. We now compute

\[z^\top (\gamma I - W - \lambda uu^\top)z \]

\[= z^\top (\gamma I - W)^{-2}z + 2\lambda \frac{z^\top (\gamma I - W)^{-2}(uu^\top)(\gamma I - W)^{-1}z}{\text{denom}} + \lambda^2 \frac{z^\top (\gamma I - W)^{-2}zu^\top(\gamma I - W)^{-1}z}{\text{denom}^2} \]

By Lemma A.2, $z^\top (\gamma I - W)^{-2}u = \bar{\sigma}_d(1)$ and $u^\top (\gamma I - W)^{-1}z = \bar{\sigma}_d(1)$. Thus,

\[z^\top (\gamma I - W - \lambda uu^\top)z = z^\top (\gamma I - W)^{-2}z + \frac{2\lambda \bar{\sigma}_d(1)\bar{\sigma}_d(1)}{\text{denom}} + \frac{\lambda u^\top (\gamma I - W)^{-1}u}{\text{denom}^2} + \frac{\bar{\sigma}_d(1)\cdot u^\top (\gamma I - W)^{-2}u}{\text{denom}^2} \]

where the last step uses $\bar{\sigma}_d(1)\cdot \bar{\sigma}_d(1) = \bar{\sigma}_d(1)$ by Fact A.2, and the fact that $\lambda \leq 2$. Factoring out the $\bar{\sigma}_d(1)$ term yields

\[z^\top (\gamma I - W - \lambda uu^\top)z = z^\top (\gamma I - W)^{-2}z + \bar{\sigma}_d(1)\cdot \left\{ \frac{\text{denom} + u^\top (\gamma I - W)^{-2}u}{\text{denom}^2} \right\} \]

A.3 Proof of Lemma A.6

Upper bound on $q(\gamma)$: We begin to upper bound $q(\gamma)$ by recalling the formula $q(\gamma) = \frac{s(\gamma)}{\sqrt{\gamma^2 - 4}}$. Hence,

\[q(\gamma)s(\gamma)^{-2} = \frac{1}{s(\gamma)\sqrt{\gamma^2 - 4}} \]

Moreover, noting that $\gamma = 2(\lambda + \lambda^{-1}) - 2 \in [2, 3]$ for $\lambda \in (1, 2]$,

\[s(\gamma) = \frac{\gamma - \sqrt{\gamma^2 - 4}}{2} = \frac{\gamma^2 - (\gamma^2 - 4)}{2(\gamma + \sqrt{\gamma^2 - 4})} = \frac{4}{2(\gamma + \sqrt{\gamma^2 - 4})} \geq \frac{1}{\gamma} \geq \frac{1}{3}. \quad (33) \]
Letting $\text{gap} = \lambda + \lambda^{-1} - 2$, we have

\[
\sqrt{\gamma^2 - 4} = \sqrt{(\lambda + \lambda^{-1} + \text{gap})^2 - 4} = \sqrt{(\lambda + \lambda^{-1})^2 - 4 + 2\text{gap}(\lambda + \lambda^{-1}) + \text{gap}^2} = \sqrt{(\lambda + \lambda^{-1} - 2)(\lambda + \lambda^{-1} + 2) + 2\text{gap}(\lambda + \lambda^{-1}) + \text{gap}^2} = \sqrt{\text{gap}(\lambda + \lambda^{-1} + 2) + 2\text{gap}(\lambda + \lambda^{-1}) + \text{gap}^2} = \sqrt{\text{gap} \cdot (3\lambda + \lambda^{-1}) + 2 + \text{gap}} = \sqrt{\text{gap} \cdot 4(\lambda + \lambda^{-1})}
\]

\[= 2(\lambda - 1)\sqrt{1 + \lambda^{-2}}
\]

(34)

Hence, we conclude

\[
q(\gamma)s(\gamma)^{-2} = \frac{1}{s(\gamma)\sqrt{\gamma^2 - 4}} \leq \frac{3}{2(\lambda - 1)}
\]

**Upper Bound for $1 - \lambda s(\gamma)$.** We begin by upper bound $s(\gamma)$ via

\[
1 - \lambda s(\gamma) = 1 - \lambda \frac{\gamma - \sqrt{\gamma^2 - 4}}{2} = 1 - \frac{\lambda(\lambda + \lambda^{-1} + \text{gap}) - \lambda\sqrt{\gamma^2 - 4}}{2} = 1 - \frac{\lambda^2 + 1 + \lambda(\text{gap}) - \sqrt{\gamma^2 - 4}}{2} = 1 - \frac{\lambda^2 + 1 + \lambda(\text{gap} - \sqrt{\gamma^2 - 4})}{2}
\]

\[
= \lambda\sqrt{\gamma^2 - 4} - \lambda\text{gap} - (\lambda^2 - 1) = 1 - \frac{\lambda^2 + 1 + \lambda(\text{gap} - \sqrt{\gamma^2 - 4})}{2} = 1 - \frac{\lambda^2 + 1 + \lambda(\text{gap} - \sqrt{\gamma^2 - 4})}{2} = 1 - \frac{\lambda^2 - 1 - (\lambda^2 - 1)}{2} = 1 - \frac{\lambda^2 - 1 - (\lambda^2 - 1)}{2} = \frac{\lambda^2 - 1 - (\lambda^2 - 1)}{2} = \frac{\lambda^2 - 1 - (\lambda^2 - 1)}{2} = (\lambda - 1)(\sqrt{\lambda^2 + 1} - \lambda) \leq (\lambda - 1).
\]

where (i) uses (34).

**B Random Matrix Theory**

**B.1 Proof of Proposition 4.2**

Recall the $\overline{\sigma}_d(1)$-notation from (28), that $Z = \overline{\sigma}_d(1)$ if $P[|Z| \geq \epsilon] \leq \exp(-d^{c_3}\epsilon^{c_2}(\lambda - 1)^{c_3})$. Moreover, observe the equivalence that if $W$ is a random quantity, and $W_0$ is deterministic, and if, $W_0 \geq (\lambda - 1)^c$ for some constant $c$, then $W - W_0 = \overline{\sigma}_d(1)$ implies $P[\nu^{-1}W \leq W_0 \leq \nu W] = \delta_{\nu,\lambda}(d)$ for any $\nu > 1$. Thus, to prove Proposition 4.2, it suffices to show

\[
\lambda_1(A) \leq 2(\lambda + \lambda^{-1}) + \overline{\sigma}_d(1) \quad \text{and} \quad \lambda_d(A) \geq (\lambda - 1)^2/\lambda + \overline{\sigma}_d(1)
\]

Further, we observe that

\[
\lambda_1(A) = 2(\lambda + \lambda^{-1}) - 2 - \lambda_d(W + \lambda uu^\top) \leq 2(\lambda + \lambda^{-1}) - 2 - \lambda_d(W) \leq 2(\lambda + \lambda^{-1}) + (\|W\|_{\text{op}} - 2),
\]

(\text{op})
where (i) is by eigenvalue interlacing. Moreover, we have that
\[ \lambda_d(A) = (\lambda + \lambda^{-1} - 2) + \lambda + \lambda^{-1} - \lambda_1(M). \]

Hence, to conclude, it suffices to verify that \( \|W\|_{op} - 2 = \bar{\sigma}_d(1) \) and \( \lambda + \lambda^{-1} - \lambda_1(M) = \bar{\sigma}_d(1) \). This is a direct consequence of the following finite sample convergence bound from Simchowitz et al. [2018]:

**Theorem B.1** (Rank-1 Specialization of Theorem 6.1 in Simchowitz et al. [2018]). There exists a universal constant \( C \geq 0 \) such that the following holds. Let \( M = W + \lambda uu^T \), and let \( \text{gap} := \frac{(\lambda - 1)^2}{\lambda^2 + 1} \).

Let \( \kappa \leq 1/2, \epsilon \leq \text{gap} \cdot \min\{\frac{1}{2}, \frac{1}{\sqrt{\lambda - 1}}\} \), and \( \delta > 0 \). Then for
\[
d \geq C \left( \frac{(q + \log(1/\delta))}{\text{gap}^2} + (\kappa \text{gap})^{-3} \log(1/\kappa \text{gap}) \right),
\]
the event the event \( \mathcal{E}_M \) defined below holds with probability at least \( 1 - 9\delta \):
\[
\mathcal{E}_M := \{ ||W||_{op} \leq 2 + \kappa(\lambda + \lambda^{-1} - 2) \} \cap \{ \lambda_1(M) \in (\lambda + \lambda^{-1})[1 - \epsilon, 1 + \epsilon] \}.
\]

**B.2 Proof of Proposition 4.4**

Before showing proving Proposition 4.4, we will reducing bounding \( |q(a) - \text{tr}(aI - W)^{-2}| \) to bound \( |q(a) - \text{tr}(aI - W)^{-1}| \). Throughout, we shall take \( \lambda \in (1, 2], \gamma = 2(\lambda + \lambda^{-1}) - 2 \), The reduction if facilitated by the following proposition:

**Proposition B.2.** Let \( C \geq 8 \) denote a universal constant, and fix \( \epsilon \leq (\lambda - 1) \). Then then, there exists a (deterministic) \( t = t(\lambda, \epsilon) \) such that (a) \( t \leq \frac{\gamma - 2}{2} \) and (b) on the event
\[
\{ ||W||_{op} \geq \gamma - t \} \cap \left\{ \max_{a \in [\gamma - t, \gamma + t]} |\text{tr}(aI - W)^{-1} - s(a)| \leq \epsilon \right\}
\]
it holds that \( |\text{tr}(aI - W)^{-2} - q(a)| \leq 2\sqrt{2C(\lambda - 1)^{-3}}\epsilon \).

**Proof of Proposition B.2.** Let \( C \geq 2 \) be a constant defined in Lemma B.3 below, let \( L := C(\lambda - 1)^{-3} \), and let \( t := \sqrt{2\epsilon/L} = \sqrt{2\epsilon(\lambda - 1)^{-3}}/C \). Observe that, since \( \epsilon \leq \lambda - 1 \) and \( C \geq 2 \), we have that
\[
t \leq \lambda + \lambda^{-1} - 2 = \frac{\gamma - 2}{2}.
\]

We now assume that the following event holds:
\[
\{ ||W||_{op} \geq \gamma - t \} \cap \left\{ \max_{a \in [\gamma - t, \gamma + t]} |\text{tr}(aI - W)^{-1} - s(a)| \leq \epsilon \right\}
\]
If we define the maps
\[
f(a) := -s(a) \quad \text{and} \quad g(a) := -\text{tr}(aI - W)^{-1} = -\sum_{i=1}^{d} \frac{1}{a - \lambda_i(W)},
\]
we observe that on the event \( \{ ||W||_{op} < \gamma - t \} \), \( g(a) \) is concave and differentiable on \( [\gamma - t, \infty) \), with \( g'(a) = \text{tr}(aI - W)^{-2} \), and \( f(a) \) is differentiable on \( (2, \infty) \), with \( f'(a) = q(a) \). The following lemma shows in addition that \( f'(a) \) is \( L \) Lipschitz for \( a \in [\gamma - t, \gamma + t] \):

\[
28
\]
Lemma B.3. Let $\lambda \leq 2$, $\gamma = 2(\lambda + \lambda^{-1}) - 2$, and $t \leq (\gamma - 2)/2$. Then there is a universal constant $C \geq 8$ for which

$$\max_{a \in \{\gamma - t, \gamma + t\}} |q'(a)| \leq C(\lambda - 1)^{-3}.$$

To conclude, we invoke the following approximation bound for concave functions:

Lemma B.4. Let $L > 0$ and $\epsilon > 0$, and set $t = \sqrt{2\epsilon / L}$. Then if $g, f : [x - t, x + t] \to \mathbb{R}$ are such that (a) $g$ be a concave, differentiable function on $[x - t, x + t]$, (b) $f'(x)$ exists and is $L$-Lipschitz $[x - t, x + t]$, and (c) for all $a \in \{x - t, x + t\}$, $|f(a) - g(a)| \leq \epsilon$, then $|f'(x) - g'(x)| \leq 2\sqrt{2L\epsilon}$.

Proof of Proposition 4.4. The estimate $\text{tr}(\gamma I - W)^{-1} = s(\gamma) + \overline{\sigma}_d(1)$ follows immediately from the following finite sample bound:

Theorem B.5 (Specialization of Proposition 6.5 in Simchowitz et al. [2018]). Fix $\delta \in (0, 1)$, let $p = e^{-d^{1/3}}$, and let $z^* := 23d^{-1/3}(\log 2)^{2/3}(d)$. Fix an $a \in (2 + \frac{1}{32}(z^* - 2), d)$, and assume that $\tau := (d(a - z^*)^2)^{-1/2}$ satisfies $\overline{\tau}^2 < \min\{\sqrt{\frac{1}{16\sqrt{2}}, \frac{a - 2}{32}}, \}$, and $p^{1/3} \leq \overline{\tau}/8$. Then with probability at least $1 - \delta - p$,

$$|\text{tr}(aI - W)^{-1} - s(a)| \leq c_6\overline{\tau}^2 + 8d^{3/2}p^{1/6}, \text{ where } c_6 := 4\sqrt{2} + 2\sqrt{\log(2/\delta)}.$$

For the estimate $\text{tr}(\gamma I - W)^{-2} = q(\gamma) + \overline{\sigma}_d(1)$, note that for $\lambda \in (1, 2]$ and $t \leq \frac{\gamma - 2}{2}$ as in Proposition B.2, we have that

$$[\gamma - t, \gamma + t] \subset \left[2 + \frac{(\lambda - 1)^2}{2}, 5\right] \quad (37)$$

Hence, we have that for any $a \in \{\gamma - t, \gamma + t\}$, $\text{tr}(\gamma I - W)^{-1} = s(a) + \overline{\sigma}_d(1)$. By Proposition B.2 and some algebraic manipulations, we see that the equality $(i)$ in

$$\text{tr}(\gamma I - W)^{-2} = q(\gamma) + 2\sqrt{2\overline{\sigma}_d(1)}(C(\lambda - 1)^{-3} \overset{\text{Fact A.1}}{=} q(\gamma) + \overline{\sigma}_d(1))$$

will follow as soon as we can bound $\mathbb{P}[\|W\|_{\text{op}} \geq \gamma - t] \leq \exp(-c_0d^{c_1}(\lambda - 1)^{c_2})$. Since $\gamma - t \geq 2 + (\lambda - 1)^2/2$, it suffices only to show that, for universal constants $c_0, c_1, c_2 > 0$,

$$\mathbb{P}\left[\|W\|_{\text{op}} < 2 + \frac{(\lambda - 1)^2}{\lambda}\right] \geq 1 - \exp(-c_0d^{c_1}(\lambda - 1)^{c_2})$$

The above display is direct consequence of the following proposition:

Proposition B.6 (Specialization of Proposition 6.3 in Simchowitz et al. [2018]). Let $d \geq 250$, and fix a $p \in (0, 1)$. Then, $\mathbb{P}[\|W\|_{\text{op}} > z^*] \leq e^{-d^{1/3}}$, where $z^* = 23d^{-1/3}(\log 2)^{2/3}(d)$.

□
B.2.1 Proof of Lemma B.3

We see that for all \( a \geq 2 \)

\[
|q'(a)| = \left| \frac{d}{dx} \frac{g(x)}{\sqrt{x^2 - 4}} \right| \\
= \left| \frac{-q(a)\sqrt{a^2 - 4} - as(a)}{(x^2 - 4)^{3/2}} \right| = \left| \frac{-s(a) + 2xs(x)}{a^2 - 4} \right| \\
\leq s(a) \cdot \{1 + a\} \cdot (\min\{a^2 - 4, 1\})^{-3/2} \\
\leq s(a) \cdot (a + 2)^2 \cdot (\min\{a - 2, 1\})^{-3/2} \\
\leq (a + 2)^2 \cdot (\min\{a - 2, 1\})^{-3/2},
\]

where the last line uses that \( s(a) \) is decreasing (as \(-\frac{d}{da} q(a) > 0\)) for \( a \in (0, 2] \), so \( s(a) \leq s(2) = 1 \). In particular, suppose \( \lambda \leq 2 \), so that \( \gamma := 2(\lambda + \lambda^{-1}) - 2 \leq 3 \), and choose \( t \leq (\gamma - 2)/2 \) and \( \gamma \leq 3 \). Then,

\[
\max_{a \in [\gamma - t, \gamma + t]} \left| \frac{d}{da} q(a) \right| \leq (\gamma + 2 + t)^2 (\min\{\gamma - t - 2, 1\})^{-3/2} \leq C \min\left\{1, \frac{\gamma - 2}{2} \right\}^{-3/2} \\
= C \min\{1, (\lambda - 1)^2/\lambda\}^{-3/2} \leq C'(\lambda - 1)^{-3},
\]

where \( C, C' \) are universal constants

B.2.2 Proof of Lemma B.4

Let \( t = \sqrt{2\epsilon/L} \). Since \( g \) is concave and differentiable on \([x - 2\sqrt{2\epsilon/L}, \infty)\), we have that

\[
g(x) - g(x - t) \over t \geq g'(x) \geq \frac{g(x + t) - g(x)}{t}
\]

Moreover, if \( f' \) is \( L \)-Lipschitz on \([x - t, x + t]\), then

\[
f'(x) + tL \geq \frac{f(x) - f(x - t)}{t} \quad \text{and} \quad f'(x) - tL \leq \frac{f(x + t) - f(x)}{t}
\]

Hence,

\[
g'(x) \leq f'(x) + tL + \frac{(f - g)(x) - (f - g)(x - t)}{t}
\]

\[
g'(x) \geq f'(x) - tL + \frac{(f - g)(x + t) - (f - g)(x)}{t}.
\]

Thus, as \( \|g(u) - f(u)\| \leq \epsilon \) for all \( u \in \{x - t, x, x + t\} \), then by the choice of \( t = \sqrt{2\epsilon/L} \), we have

\[
f'(x) - 2\sqrt{2L\epsilon} = f'(x) - tL - \frac{2\epsilon}{t} \leq g'(x) \leq f'(x) + tL + \frac{2\epsilon}{t} = f'(x) + 2\sqrt{2L\epsilon}, \tag{38}
\]

whence \( |g'(x) - f'(x)| \leq 2\sqrt{2L\epsilon} \).
C Appendix for Proof of Theorem 6.1

C.1 Proof of Lemma 6.5

To turn an upper bound on MMSE\(_{d,\lambda}\) into a lower bound into inner product upper bounds, observe that for (M, b)-measurable \(\tilde{x}\) of the form \(\tilde{x} = \|\tilde{x}\| \tilde{u}\) and \(\tilde{u} \in \mathcal{S}^{d-1}\), one has (conditioning on M and b)

\[
\text{MMSE}_{d,\lambda}(uu^T|M, b) \leq E_u \left[ \|uu^T - \tilde{x}\|_F^2 | M, b \right] = E_u \left[ \|u\|_2^2 | M, b \right] - 2E_u[\langle \tilde{x}, u \rangle^2 | M, b] + E[\|\tilde{x}\|_2^4]
\]

In particular, setting

\[
\|\tilde{x}\|^2 := \sqrt{E[\|\tilde{u}\|_2^4 | M, b] - \text{MMSE}_{d,\lambda}(uu^T|M, b)} ,
\]

we have

\[
E_{\tilde{u}}[|\tilde{u}, \tilde{u}|^2 | M, b] \leq \sqrt{E[\|\tilde{u}\|_2^4 | M, b] - \text{MMSE}_{d,\lambda}(uu^T|M, b)} . 
\]  

Hence, we can bound

\[
E_{M,b}E\left(|\hat{u}, \hat{u}|^2 | M, b \right) \leq E_{M,b}\sqrt{E[\|\hat{u}\|_2^4 | M, b] - \text{MMSE}_{d,\lambda}(uu^T|M, b)}
\]

\[
\leq \sqrt{E_{M,b} \left[ E[\|\hat{u}\|_2^4 | M, b] - \text{MMSE}_{d,\lambda}(uu^T|M, b) \right]}
\]

\[
\leq \sqrt{E_{M,b} \left[ E[\|\hat{u}\|_2^4 | M, b] - \text{MMSE}_{d,\lambda}(uu^T|M, b) \right]}
\]

\[
\leq \sqrt{\left[ E[\|u\|_2^2]^2 + \text{Var}[\|u\|_2^2] - E_{M,b}[\text{MMSE}_{d,\lambda}(uu^T|M, b)] \right]}
\]

\[
\leq \sqrt{\left[ E[\|u\|_2^2]^2 - E_{M,b}[\text{MMSE}_{d,\lambda}(uu^T|M, b)] \right] + \text{Var}[\|u\|_2^2]}
\]

\[
\leq \sqrt{\left[ E[\|u\|_2^2]^2 - E_{M,b}[\text{MMSE}_{d,\lambda}(uu^T|M, b)] \right]} + c_1d^{-c_2} ,
\]

where (i) and (ii) are Cauchy Schwartz, (iii) is the inequality \(\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}\) for \(a, b \geq 0\), and (iv) uses standard Guassian moment bounds to bound \(\text{Var}[\|u\|_2^2]\). It remains to bound \(E[\|u\|_2^2]^2 - E[\text{MMSE}_{d,\lambda}(uu^T|M, b)]\).

C.2 Proof of Proposition 6.6

By Lemma 6.5, it suffices to bound \(E[\|u\|_2^2]^2 - E[\text{MMSE}_{d,\lambda}(uu^T|M, b)]\). Define

\[
\alpha := \frac{\|b\|}{\sqrt{1 + \sigma_0}} \quad \text{and} \quad \mathcal{E}_\alpha := \{1 - d^{-1/4} \leq \alpha \leq 1 + d^{-1/4}\} ,
\]

If \(\bar{u}\) denote the distribution of \(u|b\), then

\[
E[\|u\|_2^2]^2 - E[\text{MMSE}_{d,\lambda}(uu^T|M, b)] \leq E_E\left[\|\bar{u}\|_2^2|^2 \mathbb{I}(\mathcal{E}_\alpha)\right] + E_E\left[\|\bar{u}\|_2^2|^2 \mathbb{I}(\mathcal{E}_\alpha)\right] + E_{\mathcal{E}_\alpha}\left[\|\bar{u}\|_2^2|^2 \mathbb{I}(\mathcal{E}_\alpha)\right] + E_{\mathcal{E}_\alpha}\left[\|\bar{u}\|_2^2|^2 \mathbb{I}(\mathcal{E}_\alpha)\right]
\]

31
The following lemma characterizes the distribution of \( \tilde{u} \)

**Lemma C.1.** Conditioned on \( b \), \( u \) has the distribution \( \tilde{u} \sim \mathcal{N} \left( \frac{\sqrt{\tau_0}}{1+\tau_0}, 1_{d} \cdot \frac{1}{d} \right) \).

In particular, \( \mathbb{E}[\|u\|_2^2 | b] = \frac{1+\alpha_0}{1+\tau_0} \). Since \( \alpha \) has distribution of \( \|x\| \) for \( x \sim \mathcal{N}(0, I/d) \), standard concentration implies the bound \( \mathbb{E}_b[\mathbb{E}[\|u\|_2^2 | b] \cdot I(\mathcal{E}_\alpha)] \leq O \left( d^{-\Omega(1)} \right) \). Hence,

\[
\mathbb{E}_{\tilde{u}}(\tilde{u}, \tilde{u}^2 | M, b) \leq \mathbb{E}_{M,b}[\Delta] + O \left( d^{-\Omega(1)} \right)
\]

\[
\leq \sqrt{\max_{b: \mathcal{E}_\alpha \text{ holds}} \left( \mathbb{E}[\|u\|_2^2 | b]^2 - \mathbb{E}_M \text{MMSE}_{d,\lambda}(\tilde{u} \tilde{u}^T | M, b) \right)} + O \left( d^{-\Omega(1)} \right)
\]

\[
\leq \sqrt{\mathbb{E}_b \left[ \mathbb{E} \left[ \|\tilde{u}\|_2^2 | b \right]^2 - \mathbb{E}_M \text{MMSE}_{d,\lambda}(\tilde{u} \tilde{u}^T | M, b) \cdot |b| \cdot I(\mathcal{E}_\alpha) \right]} + O \left( d^{-\Omega(1)} \right)
\]

To conclude, observe that conditioned on any \( b \), the term \( \mathbb{E}[\|u\|_2^2 | b] - \mathbb{E}_M \text{MMSE}_{d,\lambda}(\tilde{u} \tilde{u}^T | M, b) \) and the noise \( W \) is invariant to orthogonal change of basis; hence, we may assume without loss of generality that \( b \) is aligned with the ones unit vector \( 1/\sqrt{d} \). Moreover, precisely, we may assume without loss of generality that \( b/\sqrt{1+\tau_0} = \alpha 1/\sqrt{d} \), in which case

\[
\tilde{u} = \frac{d}{\sqrt{1+\tau_0}} \tilde{u}(\alpha), \text{ where } \tilde{u}(\alpha) \sim \mathcal{N} \left( \alpha \sqrt{\tau_0/d}, 1/d \right)
\]  

(41)

Letting \( \text{MMSE}_{d,\lambda}(X; Y) \) denote the Frobenius MMSE of a random matrix \( X \) given observations \( Y \), we

\[
\mathbb{E}[\|u\|_2^2 | b] - \mathbb{E}_M \text{MMSE}_{d,\lambda}(\tilde{u} \tilde{u}^T | M, b)
\]

\[
= \mathbb{E}[\|\tilde{u}\|_2^2 | b] - \text{MMSE} \left( \tilde{u} \tilde{u}^T; M = W + \lambda \tilde{u} \tilde{u}^T \right) | b
\]

\[
= \mathbb{E}[\| \frac{1}{\sqrt{1+\tau_0}} \tilde{u}(\alpha) \|_2^2 | b] - \text{MMSE} \left( \tilde{u}(\alpha) \tilde{u}(\alpha)^T; M = W + \lambda \tilde{u}(\alpha) \tilde{u}(\alpha)^T \right)
\]

\[
= \frac{1}{(1+\tau_0)^2} \mathbb{E}[\|\tilde{u}(\alpha)\|_2^2 | b] - \text{MMSE} \left( \tilde{u}(\alpha) \tilde{u}(\alpha)^T; M = W + \lambda \tilde{u}(\alpha) \tilde{u}(\alpha)^T \right)
\]

\[
= \frac{(1+\tau_0\alpha^2)^2 - \text{MMSE}_{d,\lambda}(1+\tau_0) \left( \tilde{u}(\alpha) \tilde{u}(\alpha)^T \right)}{(1+\tau_0)^2}
\]

The bound now follows from Lemma 6.5.

**C.3 Proof of Corollary 6.7**

We begin by stating a general result due to Lelarge and Miotane [2016].

**Theorem C.2** (Theorem 1 in Lelarge and Miotane [2016]). Let \( D \) be a distribution on \( \mathbb{R} \) with finite second moment. For each dimension \( d \) and parameter \( \lambda \), suppose that \( \tilde{u} \) is a random variable with \( \sqrt{d}\tilde{u}_i \overset{i.i.d.}{\sim} D \), and \( \tilde{M} = \lambda \tilde{u} \tilde{u}^T + W \), where \( W \) is a GOE matrix. \( F \) Then, provided that \( \arg \max_{q\geq 0} F(q; \lambda) \) is unique for \( F(\cdot, \cdot) \) defined below, one has the asymptotic equality

\[
\lim_{d \to \infty} \mathbb{E}_{\tilde{u}, \tilde{M}} \left[ \|\tilde{u} \tilde{u}^T - \mathbb{E}[\tilde{u} \tilde{u}^T | \tilde{M}] \|_F^2 \right] = \mathbb{E}[X_0^2]^2 - \left( \arg \max_{q\geq 0} F(q; \lambda) \right)^2
\]

where \( F(q; \lambda) := \frac{\lambda^2 q^2}{2} \left( \mathbb{E}[X_0^2] - \frac{q}{2} \right) - \mathbb{I}(X_0, \lambda \sqrt{q} X_0 + Z_0) \),

where \( X_0 \sim D \), \( Z_0 \sim \mathcal{N}(0, 1) \), \( X_0 \perp Z_0 \) and \( \mathbb{I}(\cdot, \cdot) \) denote the mutual information between the first and second argument.
In particular, if $X_0 \sim \mathcal{N}(\mu, 1)$, we have

$$
\lim_{d \to \infty} \text{MMSE}_{d, \lambda}(\mu) = \mathbb{E}[X_0^2] - \left( \arg \max_{q \geq 0} F(q; \lambda) \right)^2
$$

where $F(q; \lambda) := \frac{\lambda^2 q}{2} \left( \mathbb{E}[X_0^2] - \frac{q}{2} \right) - i(X_0, \lambda \sqrt{q} X_0 + Z_0)$.

Hence, as $\mathbb{E}[X_0^2] = 1 + \mu^2$, we have

$$(1 + \mu^2)^2 - \lim_{d \to \infty} \text{MMSE}_{d, \lambda}(\mu) = \left( \arg \max_{q \geq 0} F(q; \lambda) \right)^2 \quad (42)$$

To compute $\arg \max_{q \geq 0} F(q; \lambda)$, observe that for any $\gamma$, we have

$$
i(X_0, \sqrt{\gamma} X_0 + Z_0) = i(X_0 - \mathbb{E}[X_0], \sqrt{\gamma}(X_0 - \mathbb{E}[X_0]) + Z_0)
= i(X_0', \sqrt{\gamma} X_0' + Z_0) \text{ where } X_0' \sim \mathcal{N}(0, 1)
= \frac{1}{2} \log(1 + q\lambda^2),$$

where the last line is a standard identity (see e.g. Equation 11 in Guo et al. [2005]). We may then compute

$$F(q; \lambda) := \frac{\lambda^2 q}{2} \left( \mathbb{E}[X_0^2] - \frac{q}{2} \right) - \frac{1}{2} \log(1 + q\lambda^2)$$

$$F'(q; \lambda) = \frac{\lambda^2}{2} (\mathbb{E}[X_0^2] - q) - \frac{\lambda^2}{2(1 + q\lambda^2)}$$

$$= \frac{\lambda^2}{2} \left( (\mathbb{E}[X_0^2] - q) - \frac{1}{1 + q\lambda^2} \right).$$

Setting $F'(q; \sqrt{\lambda}) = 0$, we see that

$$0 = (1 + q\lambda^2) \left( (\mathbb{E}[X_0^2] - q) - 1 \right)$$

$$= \left( \frac{1}{\lambda^2} + q \right) (\mathbb{E}[X_0^2] - q) - \frac{1}{\lambda^2}$$

$$= - \left\{ q^2 - q \left( \mathbb{E}[X_0^2] - \frac{1}{\lambda^2} \right) + \frac{1}{\lambda^2} \left( \mathbb{E}[X_0^2] - 1 \right) \right\}.$$"
We therefore conclude:

\[
\text{arg max } F(q; \lambda) = \frac{\mathbb{E}[X_0^2] - \frac{1}{X^2} + \sqrt{\left(\mathbb{E}[X_0^2] - \frac{1}{X^2}\right)^2 + \frac{4}{X^2} (\mathbb{E}[X_0^2] - 1)}}{2} \]

\[
= 1 + \mu^2 - \frac{1}{\lambda^2} + \sqrt{(1 + \mu^2 - \frac{1}{\lambda^2})^2 + \frac{4\mu^2}{\lambda^2}} \]  

\leq 1 + \mu^2 - \frac{1}{\lambda^2} + \frac{|\mu|}{\lambda}.

(43)

And hence,

\[
(1 + \mu^2)^2 - \lim_{d \to \infty} \text{MMSE}_{d, \lambda}(\mu) = (\text{arg max } F(q; \lambda))^2 \leq (1 - \frac{1}{\lambda^2} + \mu^2 + \frac{|\mu|}{\lambda})^2.

C.4 Proof of Lemma C.1

We observe that the posterior distribution of \(u|b\) is equivalent to the posterior distribution of \(u|b/\sqrt{\tau_0}\), which is

\[
u|b \sim \mathcal{N}\left(\left(\frac{I}{d}\right)^{-1} + \left(\frac{I}{d\tau_0}\right)^{-1}\right)^{-1} \left(\frac{I}{d\tau_0}\right)^{-1} \frac{b}{\sqrt{\tau_0}}, \left(\frac{I}{d\tau_0}\right)^{-1}\right)
\]

\[
= \mathcal{N}\left(\frac{\sqrt{\tau_0}b}{1 + \tau_0}, \frac{1}{d(1 + \tau_0)}\right)
\]

C.5 Proof of Proposition 6.3

The proof that Part (a) of Conjecture 6.1 follows along the lines of the proof of Theorem 6.1 while keeping track of the error terms to ensure that they remain polynomial. Let’s prove that Part (b) implies Part (a). Fix \(\mu > 0\), and for each dimension \(d\), define the function

\[
G_d(\lambda) := 4i(\bar{u}\bar{u}^\top, W + \sqrt{\lambda}\bar{u}\bar{u}^\top)
\]  

(44)

where \(\bar{u} \sim \mathcal{N}(\mu/\sqrt{d}, 1/d)\). By rescaling the diagonals of \(\bar{u}\bar{u}^\top\) factor of \(1/\sqrt{2}\) and using the fact that the lower diagonal entries, we see that estimating \(\bar{u}\bar{u}^\top\) from \(W + \sqrt{\lambda}\bar{u}\bar{u}^\top\) is equivalent to estimating the vector \(x\) from \(z + \sqrt{\lambda}x\), where \(x, z \in \mathbb{R}^{(n+1)/2}\) are defined as

\[
z_{ij} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1/d), \quad x_{ij} = \begin{cases} 
\bar{u}_i\bar{u}_j & i < j \\
\bar{u}_i^2/\sqrt{2} & i < j
\end{cases}
\]

and as above, \(\bar{u} \sim \mathcal{N}(\mu/\sqrt{d}, 1/d)\). Hence,

\[
G_d(\lambda) = 4i(x, \sqrt{\lambda}x + z),
\]

Having represented \(G_d(\lambda)\) as a mutual information in a standard Gaussian channel, Corollary 1 in Guo et al. [2005] implies that \(G_d(\lambda)\) is concave, and Theorem 1 in Guo et al. [2005] implies the
first equality in the following display:

\[
G'_d(\lambda) = 2 \mathbb{E}\left[ \sum_{i \leq j} \left( x_{ij} - \mathbb{E}[x_{ij} | \sqrt{\lambda}x + z] \right)^2 \right] \\
= 2 \mathbb{E}\left[ \sum_{1 \leq i < j \leq d} \left( \tilde{u}_i \tilde{u}_j - \mathbb{E}[\tilde{u}_i \tilde{u}_j | \sqrt{\lambda} \tilde{u} \tilde{u}^\top + W] \right)^2 + \frac{1}{2} \sum_{i=1}^{d} \left( \tilde{u}_i^2 - \mathbb{E}[\tilde{u}_i | \sqrt{\lambda} \tilde{u} \tilde{u}^\top + W] \right)^2 \right] \\
= \text{MMSE}_{d, \sqrt{\lambda}}(\tilde{u} \tilde{u}^\top; W + \sqrt{\lambda} \tilde{u} \tilde{u}^\top) \\
:= \text{MMSE}_{d, \sqrt{\lambda}}(\mu) \quad (45)
\]

Define the function \( G_\infty(\lambda) = \lim_{d \to \infty} G_d(\lambda) \). By Theorem 1 and the discussion in Section 3.2 in Lelarge and Miolane [2016], \( G_\infty(\lambda) \) is differentiable for all \( \lambda > 0 \) for which arg max \( q \geq 0 F(q, \lambda) \) as defined in Theorem C.2 is unique. Examining the the proof of Corollary 6.7 (Section C.3), we verified that this holds for all \( \lambda > 1 \); hence \( G_d(\lambda) \) is differentiable for \( \lambda > 1 \). Combining with the fact that \( G_\infty(\lambda) = \lim_{d \to \infty} G_d(\lambda) \) and \( G_d(\lambda) \) is differentiable, a standard analysis fact (Lemma 18 in Lelarge and Miolane [2016]) implies that

\[
G'_\infty(\lambda) = \lim_{d \to \infty} G'_d(\lambda) 
\]

\[
(45) \equiv \text{MMSE}_{d, \sqrt{\lambda}}(\mu) 
\]

\[
(43) \equiv \frac{1 + \mu^2 - \frac{1}{\lambda} + \sqrt{(1 + \mu^2 - \frac{1}{\lambda})^2 + \frac{4\mu^2}{\lambda}}}{2} 
\]

A routine computation shows that for \( \mu \leq 1 \), there exists constants \( c_1, c_2 > 0 \) such that \( |G''_\infty(\lambda)| \leq c_1(\lambda - 1)^{\frac{3}{2}} \). Hence, Lemma B.4 can be used to show that there are constants \( c_3, c_4 > 0 \)

\[
|G'_d(\lambda) - G'_\infty(\lambda)| \leq (c_1(\lambda - 1)^{\frac{3}{2}})^c_3 \cdot |G(\lambda) - G_\infty(\lambda)|^{c_4}. 
\]

This readily implies that Part (b) of Conjecture 6.1 imples Part (a).
D Supplementary Proofs for Theorem 3.4

D.1 Formal Proof of Theorem 3.4 from Proposition 5.1

Recall the definition $I_d := \{(\lambda, \tilde{\tau}_0) : \tilde{\lambda} = q\lambda, \tilde{\tau}_0 = q\tilde{\tau}_0, q \geq 1 - d^{-1/4}\}$. We begin by proving (19)

$$P_{u,M,b} \left[ \left\| \text{Proj}_{v(1),\ldots,v(T+1)}(u) \right\|_2^2 > 2(1 + d^{-1/4})^{4T+2}\tilde{\tau}_0 \sum_{j=1}^{T} \lambda^{4j} \right]$$

$$E_{u,M,b} \left[ \left\| \text{Proj}_{v(1),\ldots,v(T+1)}(u) \right\|_2^2 > 2(1 + d^{-1/4})^{4T+2}\tilde{\tau}_0 \sum_{j=1}^{T} \lambda^{4j} | \left\| u \right\|_2 \right]$$

$$= E_{u,M,b} \left[ \left\| u \right\|_2^2 \cdot \left\| \text{Proj}_{v(1),\ldots,v(T+1)}(\tilde{u}) \right\|_2^2 > 2(1 + d^{-1/4})^{4T+2}\tilde{\tau}_0 \sum_{j=1}^{T} \lambda^{4j} | \left\| u \right\|_2 \right]$$

$$\leq \max_{(\lambda, \tilde{\tau}_0) \in I_d} \left[ \frac{T^2 e^{-d\lambda^2/2\tilde{\tau}_0(\lambda-1)}}{d(\lambda-1)^3} + \mathbb{P}[\left\| u \right\|_2^2 \notin I_d] \right]$$

where the first line involves a probability taken over $u, M, b$ in the original query model and distribution, all subsequent lines involve probabilities taken over the instance defined in (18), and the last line holds Proposition 5.1 with $c = 1$, as long as $\tilde{\tau}_0 \geq \frac{\tilde{\lambda}^2}{d(\lambda-1)^3}$ for all appropriate $\tilde{\lambda}, \tilde{\tau}_0 \in I_d$. In light of Equation (19), it suffices to show that the following hold, under the assumptions of Theorem 3.4. Then

1. For all $(\lambda, \tilde{\tau}_0) \in I_d$, $\tilde{\tau}_0 \geq \frac{\tilde{\lambda}^2}{d(\lambda-1)^3}$. Indeed, by Equation (19), this implies that the RHS of the display in Theorem 3.4 is bounded by

$$\max_{(\lambda, \tilde{\tau}_0) \in I_d} \left[ \frac{T^2 e^{-d\lambda^2/2\tilde{\tau}_0(\lambda-1)/2}}{d(\lambda-1)^3} + \mathbb{P}[\left\| u \right\|_2^2 \notin I_d] \right] .$$

2. $\max_{(\lambda, \tilde{\tau}_0) \in I_d} e^{-d\lambda^2/2\tilde{\tau}_0(\lambda-1)/2} \leq e^{-d\lambda^2(\lambda-1)/16}$.

3. To verify that $\mathbb{P}[\left\| u \right\|_2^2 - 1] \leq d^{-1/4}] \geq 1 - e^{-d^{1/4}/8}$.

Proof of Point 1: Observe that, for $\tau_0 \leq (\lambda - 1)$ and $\lambda \geq 1$,

$$d \geq \frac{16\lambda^4}{\tau_0(\lambda-1)^3} = \max \left\{ 2^4, \left( \frac{2\lambda}{(\lambda-1)} \right)^4, \frac{16\lambda^2}{\tau_0(\lambda-1)^3} \right\} .$$

Hence, $1 - d^{-1/4} \geq 1/4$ and $\lambda - 1 - d^{-1/2} \lambda \geq \frac{1}{4}(\lambda - 1)$. Now, for $\tilde{\lambda} = q\lambda$ and $\tau_0 = q\tau_0$, the condition $\tilde{\tau}_0 \geq \frac{\tilde{\lambda}^2}{d(\lambda-1)^3}$ is equivalent to $\tau_0 \geq \frac{\lambda^2}{d(q\lambda-1)^3q}$. Hence, if $q \geq 1 - d^{-1/4}$,

$$\frac{\lambda^2}{d(q\lambda-1)^3q} \leq \frac{\lambda^2}{d(\lambda - 1 - d^{-1/4}\lambda)^3(1 - d^{-1/4})} \leq 2^4 \cdot \frac{\lambda^2}{d(\lambda-1)^3}$$
which is at most \( \tau_0 \) by assumption on \( d \).

**Proof of Point 2:** For \((\tilde{\lambda}, \tilde{\tau}_0) \in I_d\), ensures

\[
\tilde{\tau}_0 \tilde{\lambda}^2 (\tilde{\lambda} - 1) = \tau_0 \lambda^2 (1 - \sqrt{d})^3 (1 - \lambda - \lambda \sqrt{d}) \geq 2^4 \tau_0 \lambda^2 (\lambda - 1)
\]

Hence, we have 

\[
-c \tilde{\lambda}^2 \tilde{\tau}_0 (\tilde{\lambda} - 1)/2 \geq -cd \lambda^2 \tau_0 (\lambda - 1)/16.
\]

**Proof of Point 3:** We begin by recalling a standard result about concentration of a \( \chi^2 \) random variable:

**Lemma D.1** (\( \chi^2 \)-Squared Concentration, Lemma 1 in Laurent and Massart [2000]). Let \( \mathbf{u} \sim \mathcal{N}(0, I/d) \).

\[
\mathbb{P}[d \| \mathbf{u} \|_2^2 \geq d + 2t + 2 \sqrt{dt}] \leq \exp(-t).
\]

The bound now follows from choosing \( t = d^{1/4}/8 \).

**D.2 Verifying (22)**

Using the lower bound \( \tau_{k+1} \geq \lambda^{4k} \tau_0 \), we have that

\[
\left( \sqrt{d \tau_{k+1} - 2k + 2} \right)^2 \geq d \tau_{k+1}/\lambda \quad \text{iff} \quad (1 - 1/\lambda)d \tau_{k+1} - \sqrt{2(k + 1)d \tau_{k+1}} + 2k + 2 \geq 0
\]

\[
\iff (1 - 1/\lambda)d \tau_{k+1} - \sqrt{2(k + 1)/d \tau_{k+1}} \geq 0
\]

\[
\iff \sqrt{d \tau_{k+1}} \geq \sqrt{2(k + 1)/(1 - 1/\lambda)} \geq 0
\]

\[
\iff d \lambda^{4k} \tau_0 \geq \lambda^2 (2k + 1)/(\lambda - 1)^2 \geq 0
\]

\[
\iff d \tau_0 \geq \frac{2 \lambda^2}{(\lambda - 1)^2} \cdot \max_{k \geq 0} \lambda^{-4k} (k + 1) \geq 0.
\]

Now, we can compute that \( \max_{k \geq 0} \lambda^{-4k} (k + 1) = \exp(-4k \log \lambda + \log(k + 1)) \). The function \( x \mapsto -4x \log \lambda + \log(1 + x) \) is concave, and maximized when \( 1/(1 + x) = 4 \log \lambda \), that is, \( 1 + x = 1/4 \log \lambda \). Since \( \lambda^{-4k} \leq 1 \), we \( \max_{k \geq 0} \lambda^{-4k} (k + 1) \leq 1/4 \log \lambda \). Thus, it suffices that

\[
\tau_0 \geq \frac{\lambda^2}{2d(\lambda - 1)^2 \log \lambda}.
\]

Moreover, we have that bound that, for \( \lambda \in (1, 2] \), \( \log \lambda \geq \frac{\lambda - 1}{2} \), so in fact, its enough to take \( \tau_0 \geq \frac{\lambda^2}{d(\lambda - 1)^2} \).

**D.3 Proof of Proposition 21**

To begin, we can assume without loss of generality that \(\text{Alg}\) is deterministic. We let \( Z_k := \{b, v^{(1)}, w^{(1)}, \ldots, v^{(k)}, w^{(k)}\} \) denote the information collected by \(\text{Alg}\) up to round \( k \). Moreover, we let \( \mathbf{P}_u \) denote the distribution of \( Z_k \) given \( u = u \).

We recall the following data-processing inequality from Simchowitz et al. [2018]:
Proposition D.2. Let $\mathcal{D}$ be any distribution supported on $S^{d-1}$. Then for any $\tau_k \leq \tau_{k+1}$ and $\eta > 0$,

$$\mathbb{E}_{u \sim \mathcal{D}} P_u \left[ \{ \Phi(V_k; u) \leq \tau_k \} \cap \{ \Phi(V_{k+1}; u) > \tau_{k+1} \} \right] \leq \left( \mathbb{E}_{u \sim \mathcal{D}} \mathbb{E}_{Z_k \sim P_0} \left[ \left( \frac{dP_u(Z_k)}{dP_0(Z_k)} \right)^{1+\eta} \mathbb{I}(\{ \Phi(V_k; u) \leq \tau_k \}) \right] \cdot \sup_{V \in \mathcal{O}(d,k+1)} P_{u \sim \mathcal{D}}[\Phi(V; u) > \tau_{k+1}]^\eta \right)^{1/(1+\eta)}.$$

Proposition D.2 recursively controls the probability that the $\Phi(V_{k+1}, u)$ is above the threshold $\tau_{k+1}$, on the “good event” that $\Phi(V_k, u) \leq \tau_k$, in terms of two quantities: (a) an information-theoretic term that depends on the likelihood ratios and (b) a “best-guess” probability which upper bounds the largest value of $\Phi(V_{k+1}, u)$ if $V_{k+1}$ were selected only according to the prior on $u$, without any posterior knowledge of $Z_k$.

The best-guess probability with the following lemma, which we recall from Simchowitz et al. [2018]:

Lemma D.3. For any $V \in \mathcal{O}(d, k+1)$ and $d\tau_{k+1} \geq \sqrt{2(k+1)}$, we have

$$\mathbb{P}_{u \sim S^{d-1}}[u^\top V^\top V u \geq \tau_{k+1}] \leq \exp \left\{ -\frac{1}{2} \left( \sqrt{d\tau_{k+1}} - \sqrt{2(k+1)} \right)^2 \right\} \quad (47)$$

The likelihood term is a bit more effort to control. The following bound mirrors Proposition 3.4 in Simchowitz et al. [2018], but with the additional subtlety of taking the dependence on $\tau_0$ into account; the proof is in Section D.4.

Proposition D.4. For any $\tau_k \geq 0$ and any $u \in S^{d-1}$, we have

$$\mathbb{E}_{P_0} \left[ \left( \frac{dP_u(Z_k)}{dP_0(Z_k)} \mathbb{I}(\{ \Phi(V_k; u) \leq \tau_k \}) \right)^{1+\eta} \right] \leq \exp \left( \frac{\eta(1+\eta)}{2} \lambda^2(\tau_k + \tau_0) \right) \quad (48)$$

Putting the pieces together, we have

$$\mathbb{E}_{u \sim \mathcal{D}} P_u \left[ \{ \Phi(V_k; u) \leq \tau_k \} \cap \{ \Phi(V_{k+1}; u) > \tau_{k+1} \} \right] \leq \left( \exp \left( \frac{\eta(1+\eta)}{2} \lambda^2(\tau_k + \tau_0) \right) \cdot \exp \left\{ -\frac{\eta}{2} \left( \sqrt{d\tau_{k+1}} - \sqrt{2(k+1)} \right)^2 \right\} \right)^{1/(1+\eta)} = \exp \left( \frac{\eta}{2(1+\eta)} \left( \lambda^2(1+\eta)(\tau_k + \tau_0) - \left( \sqrt{d\tau_{k+1}} - \sqrt{2(k+1)} \right)^2 \right) \right).$$

Choosing $\eta = \lambda - 1$ concludes the proof of Proposition 21.

D.4 Proof of Proposition D.4

The proof of Proposition D.4 mirrors the proof of Proposition 3.4 in Simchowitz et al. [2018], with minor modifications to take into account the additional side information $b$. The next subsection first collects necessarily preliminary results, and the second concludes the proof.

D.4.1 Preliminary Results for Proposition D.4

We need to start by describing the likelihood ratios associated with the algorithm history $Z_k$:
Lemma D.5 (Conditional Likelihoods). Let \( P_i := I - V_i V_i^\top \) denote the orthogonal projection onto the orthogonal complement of \( \text{span}(v^{(1)}, \ldots, v^{(i)}) \). Under \( P_u \) (the joint law of \( M, b \) and \( Z_T \) on \( \{ u = u \} \)), we have

\[
(P_{i-1}) M v^{(i)} | Z_{i-1}, u = u \sim \mathcal{N} \left( \lambda (u^\top v^{(i)}) P_{i-1} u, \frac{1}{d} \Sigma_i \right) 
\]

where \( \Sigma_i := P_{i-1} \left( I_d + v^{(i)} v^{(i)\top} \right) P_{i-1} \). \hspace{1cm} (49)

In particular, \( w^{(i)} \) is conditionally independent of \( w, v^{(1)}, \ldots, v^{(i-1)} \) given \( v^{(1)}, \ldots, v^{(i)} \) and \( u = u \).

Proof. The lemma was proven in Lemma 2.4 Simchowitz et al. [2018] in the case where there was no initial side-information. When there is side information, we just need to argue that \( Z_i | Z_{i-1} \) is independent of \( w \), conditioned on \( u \). Since \( \text{EigSlv} v \) is deterministic by assumption, \( v^{(i)} \) is a measurable function of \( Z \), and thus \( Z_i | Z_{i-1} \) is a measurable function of \( w^{(i)} = (\lambda u u^\top + W) v^{(i)} \). Hence, conditioned on \( Z_{i-1} \) and \( u \), \( w^{(i)} \) is measurable function of \( W \), which is independent of \( w \). \hspace{1cm} \( \square \)

The next proposition is copied verbatim from Lemma 3.5 in Simchowitz et al. [2018], with the exceptions that the indices \( i \) are allowed to range from 0 to \( k \) (rather than 1 to \( k \)) to account for an initial round of side information. It’s proof is identical:

Proposition D.6 (Generic Upper Bound on Likelihood Ratios). Fix an \( s, r_0 \geq 0 \). For \( i \geq 0 \) and \( \tilde{V}_i \in \mathcal{O}(d,i) \), define the likelihood.

\[
g_i(\tilde{V}_i) := \mathbb{E}_{P_0} \left[ \frac{dP_u(Z_i | Z_{i-1})^{r_u} dP_s(Z_i | Z_{i-1})^{r_s}}{dP_0(Z_i | Z_{i-1})^{r_0}} | \tilde{V}_i = \tilde{V}_i \right]. \hspace{1cm} (50)
\]

Then for any \( V_k \subset \mathcal{O}(d,k) \), we have

\[
\mathbb{E}_{P_0} \left[ \frac{dP_u(Z_k)^{r_u} dP_s(Z_k)^{r_s}}{dP_0(Z_k)^{r_0}} \mathbb{I}(V_k \subset V_k) \right] \leq \sup_{\tilde{V}_k \in \mathcal{V}_{1:i}} \prod_{i=0}^k g_i(\tilde{V}_{1:i}) , \hspace{1cm} (51)
\]

where \( \tilde{V}_{1:i} \) denotes the first \( i \) columns of \( \tilde{V}_k \).

Lastly, we recall the following elementary computation, stated as Lemma 3.6 in Simchowitz et al. [2018]:

Lemma D.7. Let \( P \) denote the distribution \( \mathcal{N}(\mu_1, \Sigma) \) and \( Q \) denote \( \mathcal{N}(\mu_2, \Sigma) \), where \( \mu_1, \mu_2 \in \ker \Sigma \). Then

\[
\mathbb{E}_Q \left[ \left( \frac{dP}{dQ} \right)^{1+\eta} \right] = \exp \left( \frac{\eta(1 + \eta)}{2} (\mu_1 - \mu_2)^\top \Sigma \lambda (\mu_1 - \mu_2) \right) \hspace{1cm} (52)
\]

D.4.2 Concluding the proof of Proposition D.4

Fix a \( u \in \mathcal{S}^{d-1} \), and we shall and apply Proposition D.6 with \( r_u = r_0 = 1 + \eta \) and \( r_s = 0 \). In the language of Proposition D.6, we have

\[
g_i(V_i) = \mathbb{E}_{P_0} \left[ \left( \frac{dP_u(Z_i | Z_{i-1})}{dP_0(Z_i | Z_{i-1})} \right)^{1+\eta} | V_i \right] = \mathbb{E}_{P_0} \left[ \left( \frac{dP_u(w^{(i)} | Z_{i-1})}{dP_0(w^{(i)} | Z_{i-1})} \right)^{1+\eta} | V_i \right]
\]
Now, observe that, \( d\mathbf{P}_u(w^{(i)}|Z_{i-1}) \) is the density of \( \mathcal{N}(\lambda \langle u, v^{(i)} \rangle \cdot P_{i-1}^{-1} u, \frac{1}{d} \Sigma_i) \) and \( d\mathbf{P}_0(w^{(i)}|Z_{i-1}) \) is the density of \( \mathcal{N}(0, \frac{1}{d} \Sigma_i) \). Since \( \Sigma_i = P_{i-1}^{-1} (I_d + v^{(i)}v^{(i)\top}) P_{i-1}^{-1} \), we have \( P_{i-1}^{-1} \Sigma_i^{-1} P_{i-1} = P_{i-1} \preceq I \).

Thus,

\[
  u^\top P_{i-1}(\Sigma_i/d) \Sigma_i^{-1} P_{i-1} u \leq d\|u\|^2 = d \quad \forall u \in \mathbb{S}^{d-1}.
\]  

(53)

Hence, by Lemma D.7, we have for all \( i \in [k] \) that

\[
  g_i(V_i) \overset{\text{Lemma D.7}}{=} \exp \left( \frac{\eta(1 + \eta) \lambda^2 \langle u, v^{(i)} \rangle^2}{2} u^\top P_{i-1}(\Sigma_i/d) \Sigma_i^{-1} P_{i-1} u \right)
\]

Eq. (53) \[\leq\]

\[
  \exp \left( \frac{\eta(1 + \eta) \lambda^2 \cdot d\langle u, v^{(i)} \rangle^2}{2} \right)
\]

For \( i = 0 \), we have that \( w \sim \mathcal{N}(\sqrt{\tau_0} u, I/d) \). Thus,

\[
  g_0(\{\}) \overset{\text{Lemma D.7}}{=} \exp \left( \frac{\eta(1 + \eta)}{2} (\sqrt{\tau_0} u)^\top (I/d)^{-1} (\sqrt{\tau_0} u) \right) = \exp \left( \frac{d\eta(1 + \eta) \tau_0}{2} \right)
\]  

(54)

Hence, if \( \mathcal{V}_k := \{ \tilde{\mathcal{V}}_k \in \mathcal{O}(d; k) : \Phi(\tilde{\mathcal{V}}_k; u) \leq \tau_k \} \), then Proposition D.6 implies

\[
  \mathbb{E}_{\mathbf{P}_0} \left[ \left( \frac{d\mathbf{P}_u(Z_k)}{d\mathbf{P}_0(Z_k)} \right)^{1+\eta} I(\mathcal{V}_k \in \mathcal{V}_k) \right] \leq \exp \left( \frac{d\eta(1 + \eta) \tau_0}{2} \right) \cdot \sup_{\tilde{\mathcal{V}}_k \in \mathcal{V}_k} \prod_{i=1}^k \exp \left( \frac{\eta(1 + \eta) \lambda^2 \cdot d\langle u, \tilde{\mathcal{V}}_k[i] \rangle^2}{2} \right)
\]

\[
  = \exp \left( \frac{d\eta(1 + \eta) \tau_0}{2} \right) \sup_{\tilde{\mathcal{V}}_k \in \mathcal{V}_k} \exp \left( \frac{d\eta(1 + \eta) \lambda^2 \Phi(\tilde{\mathcal{V}}_k; u)}{2} \right)
\]

\[
  \leq \exp \left( \frac{d\eta(1 + \eta) \tau_0}{2} \right) \exp \left( \frac{d\eta(1 + \eta) \lambda^2 \tau_k}{2} \right)
\]

\[
  \leq \exp \left( \frac{d\eta(1 + \eta) \lambda^2 (\tau_k + \tau_0)}{2} \right) \quad \text{since } \lambda \geq 1.
\]