1. Introduction

The genesis of natural but noncommuting coordinates can be traced back to Heisenberg’s uncertainty principle in quantum mechanics, which limits the accuracy of the simultaneous determination of the position and momentum \((q, p)\) of a subatomic particle. As Heisenberg argued \[\text{[Hei27]}\] (and Kennard rigorously derived \[\text{[Ken27]}\]), the inherent imprecision of such a measurement “is a straightforward mathematical consequence of the quantum mechanical commutation rule for the position and the corresponding momentum operators \(qp - pq = i\hbar\), where \(\hbar = \frac{h}{2\pi}\) is the reduced Planck constant. Such an identity cannot be satisfied by matrices (over \(\mathbb{C}\)), which is obvious, but not even by bounded operators in Hilbert space. Assuming \(q\) and \(p\) self-adjoint, this can be seen by passing to the Weyl integrated form \[\text{[Wey28] \S 45,}\]

\[
V_s U_t = e^{2\pi i \hbar t s} U_t V_s, \quad t, s \in \mathbb{R}. \tag{1}
\]

Moreover, the latter relation determines a unitary representation \(\pi_\hbar\) of the (implicitly defined) group \(H_3(\mathbb{R})\), called the Heisenberg group. By a celebrated theorem of Stone and von Neumann all such irreducible representations are unitarily equivalent. The restriction to the lattice \(H_3(\mathbb{Z}) \subset H_3(\mathbb{R})\) of an irreducible unitary representation \(\pi_\theta\), \(\theta \in \mathbb{R}\), generates the \(C^\ast\)-algebra \(A_\theta\) nowadays known as the noncommutative torus of slope \(\theta\). When \(\theta \in [0, 1] \setminus \mathbb{Q}\), as shall be assumed throughout this paper, the \(C^\ast\)-algebra \(A_\theta\) is also known as the irrational rotation algebra and is the (unique up to isomorphism) \(C^\ast\)-algebra generated by a pair of unitary operators \(U_1, U_2\) satisfying

\[
U_2 U_1 = e^{2\pi i \theta} U_1 U_2. \tag{2}
\]

Moreover \(A_\theta\) is a simple \(C^\ast\)-algebra, thus typifying the coordinates of a purely noncommutative space. For this reason on the one hand, and due to its accessibility on the other, \(A_\theta\) has received much attention during the last several decades and has been a favorite testing ground for quite a number of fruitful mathematical investigations.

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Although the habitual geometric intuition is rendered utterly inoperative in a ‘space without points’ such as the one represented by $A_\theta$, the curvature as a “measure of deviation from flatness” (in Riemann’s own words) could still make some sense. It is the goal of this brief survey to review the recent developments that led to the emergence of a quantized version of Gaussian curvature for the noncommutative torus. Many of the essential ideas presented below have their origin in Alain Connes’ 1980 C. R. Acad. Sc. Paris Note $[\text{Con80}]$, which effectively constitutes the birth certificate of noncommutative differential geometry. That foundational article not only established the most basic geometric concepts and constructions, such as the geometric realization of the finitely generated projective modules over $A_\theta$, the explicit construction of constant curvature connections for them and the definition and calculation of their Chern classes, but also provided the crucial computational tool, in the form of a pseudodifferential calculus adapted to $C^*$-dynamical systems.

The specific line of research whose highlights we are about to summarize was sparked by a paper by Connes and Paula Cohen ($[\text{Conformal geometry of the irrational rotation algebra}]$, Preprint MPI Bonn, 1992-93) which showed how the passage from the (unique) trace of $A_\theta$ to a non-tracial conformal weight associated to a Weyl factor (or ‘dilaton’) gives rise to a non-flat geometry on the noncommutative torus, which can be investigated with the help of the adapted pseudodifferential calculus of $[\text{Con80}]$. In a later elaboration $[\text{CoTr11}]$ of that paper, the passage from flatness to conformal flatness was placed in the setting of spectral triples (see § 2.1 and § 2.2 below), which in the intervening years has emerged as the proper framework for the metric aspect in noncommutative geometry (cf. $[\text{Con94}]$, Ch. 6], $[\text{Con13}]$, $[\text{CoMo08}]$). Completing the calculations begun in the 1992 preprint they proved in $[\text{CoTr11}]$ an analogue of the Gauss-Bonnet formula for the conformally twisted (called ‘modular’) spectral triples. The full calculation of the modular Gaussian curvature was first done by A. Connes in 2009, with the aid of Wolfram’s Mathematica, and is included in $[\text{CoMo14}]$. Fathizadeh and Khalkhali $[\text{FaKh13}]$ independently performed the same calculation with the help of a different computing software.

Apart from computing the expression of the modular curvature (see § 2.3 below), Connes and Moscovici showed in $[\text{CoMo14}]$ that one can make effective use of variational methods even in the abstract operator-theoretic context of the spectral triple encoding the geometry of the noncommutative torus. After giving a variational proof of the modular Gauss-Bonnet formula which requires no computations (see § 2.4), they related the modular Gaussian curvature to the gradient of the Ray-Singer log-determinant of the Laplacian viewed as a functional on the space of Weyl factors. As a consequence, they obtained an a priori proof of an internal consistency relation for the constituents of the modular curvature. In addition they showed by purely operator-theoretic arguments that, as in the case of Riemann surfaces (cf. $[\text{OPS88}]$), the normalized log-determinant functional attains its extreme value only at the trivial Weyl factor, in other words for the flat ‘metric’ (see § 2.5 below).

For reasons which will soon become transparent (see § 3.1), the natural equivalence relation between noncommutative spaces is that of Morita equivalence between their
respective algebras of coordinates. For noncommutative tori the Morita equivalence is implemented by the Heisenberg bimodules described by Connes \cite{Con80} and Rieffel \cite{Rie81}. Lesch and Moscovici extended in \cite{LeMo16} to spectral triples on noncommutative tori associated to Heisenberg equivalence bimodules (see \S 3.2 and \S 3.3). Moreover, in doing so they managed to dispose of any computer-aided calculations (see \S 5.1). Most notably they showed (see \S 3.4 below) that whenever $A_{\theta}$ is realized as the endomorphism algebra of a Heisenberg $A_{\theta}'$-module endowed with the $A_{\theta}'$-valued Hermitian structure obtained by twisting the canonical one by a positive invertible element in $A_{\theta}$, the curvature of $A_{\theta}$ with respect to the corresponding spectral triple over $A_{\theta}'$ is equal to the modular curvature associated to the same element of $A_{\theta}$ viewed as conformal factor. In a certain sense this is reminiscent of Gauss’s Theorema Egregium “If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged”.

The fundamentals of Connes’ pseudodifferential calculus as well as its extension to twisted $C^*$-dynamical systems, which provide the essential device for proving all the above results, are explained in \S 4. Finally, \S 5 clarifies how to use the affiliated symbol calculus in order to compute the resolvent trace expansion, or equivalently the heat trace expansion, for the relevant Laplace-type operators.

2. Curvature of modular spectral triples

2.1. Flat spectral triples. In noncommutative geometry a metric structure on a space with $C^*$-algebra of coordinates $A$ is represented by a triad of data $(A, \mathcal{H}, D)$ called spectral triple, modeled on the Dirac operator on a manifold: $A$ is realized as a norm-closed subalgebra of bounded operators on a Hilbert space $\mathcal{H}$, $D$ is an unbounded self-adjoint operator whose resolvent belongs to any $p$-Schatten ideal with $p > d$ where $d > 0$ signifies the dimension, and $D$ interacts with the coordinates by having bounded commutators (or more generally bounded twisted commutators) $[D, a]$ for any $a$ in a dense subalgebra of $A$. The Dirac operator was chosen as model since it represents the fundamental class in $K$-homology and at the same time plays the role of a quantized inverse line element (see \cite{Con13}). In the case of $A_{\theta}$ one can obtain such a triad by simply reproducing the construction of the $\bar{\partial} + \bar{\partial}^*$ operator on the ordinary torus $T^2 = (\mathbb{R}/2\pi \mathbb{Z})^2$.

To fix the notation we briefly review some basic properties of the $C^*$-algebra $A_{\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

First of all, the torus $T^2$ acts on $A_{\theta}$ via the representation by automorphisms defined on the basis elements by

$$\alpha_{\mathbf{r}}(U_1^n U_2^m) = e^{i(r_1 n + r_2 m)} U_1^n U_2^m, \quad \mathbf{r} = (r_1, r_2) \in \mathbb{R}^2. \quad (3)$$

By analogy with the action of $T^2$ on $A_0 \equiv C(T^2)$, we call these automorphisms translations.

The smooth vectors of this representation of $T^2$ are precisely the elements of the form $a = \sum_{m, n \in \mathbb{Z}} a_{m, n} U_1^m U_2^n$ with rapidly decaying coefficients, i.e. such that
The space $\Omega^1_C(A)$ determines the unique normalized trace $\varphi_0$ of the $C^*$-algebra $A_\theta$.

The image of the differential of the above representation on $A_\theta$ is the Lie algebra generated by the outer derivations $\delta_1$ and $\delta_2$, uniquely determined by the relations

$$\delta_1(U_j) = \delta_2^j U_j, \quad i, j \in \{1, 2\}.$$ (4)

these are the counterparts of the operators $\frac{1}{\tau} (\partial/\partial x + \bar{\tau} \partial/\partial y)$, and $\frac{1}{\tau} (\partial/\partial x + \tau \partial/\partial y)$ acting on $C^\infty(\mathbb{T}^2)$.

To obtain the analogue of the corresponding flat metric on $\mathbb{T}^2$, we let $H_0 \equiv L^2(A_\theta, \varphi_0)$ denote the Hilbert space completion of $A_\theta$ with respect to the scalar inner product

$$\langle a, b \rangle = \varphi_0(a^* b), \quad a, b \in A_\theta.$$ (5)

The space $\Omega^1 A_\theta$ of formal 1-forms $\sum a \, db$, $a, b \in A_\theta$, is also endowed with a semi-definite inner product defined by

$$\langle adb, a'b'd'b' \rangle = \varphi_0(a^* (a')^* \delta_r (b')^* \delta_r b'^*), \quad a, a', b, b' \in A_\theta.$$ (6)

On completing its quotient modulo the subspace of those elements $\omega \in \Omega^1 A_\theta$ such that $\langle \omega, \omega \rangle = 0$, one obtains a Hilbert space denoted $H^{(1,0)}$. $H^{(1,0)}$ is also an $A_\theta$-bimodule under the natural left and right action of $A_\theta$. Both these actions are unitary. Moreover, the linear map from $\Omega^1 A_\theta$ to $A_\theta$ defined by sending the class of $\sum adb$ in $H^{(1,0)}$ to $\sum a \delta_r (b) \in H_0$ induces an $A_\theta$-bimodule isomorphism between $H^{(1,0)}$ and $H_0$.

Denoting by $\partial_r$ the closure of the operator $\delta_r : A_\theta \to H_0$ viewed as unbounded operator from $H_0$ to $H^{(1,0)}$ one obtains a spectral triple $(A_\theta, \mathcal{H}, D_r)$ by taking $\mathcal{H} = H_0 \oplus H^{(1,0)}$ and as unbounded self-adjoint operator $D_r = \left( \begin{array}{cc} 0 & \partial_r^* \\ \partial_r & 0 \end{array} \right)$. Concurrently, the triad $(A_\theta^p, \mathcal{H}, D_r)$ is a spectral triple with respect to the right action of $A_\theta$. One can turn it into a spectral triple for the left action of $A_\theta$ by passing to its transposed $(A_\theta, \mathcal{H}, \bar{D}_r)$ (see [CoMo14] §1.2 for the general definition), where $\mathcal{H}$ is the complex conjugate of $\mathcal{H}$ and $\bar{D}_r = \left( \begin{array}{cc} 0 & \bar{\partial}_r^* \\ \bar{\partial}_r & 0 \end{array} \right)$.

2.2. Modular spectral triples. To implement the analogue of a conformal change of metric structure, we choose a self-adjoint element $h = h^* \in A_\theta$ and use it to
replace the trace $\varphi_0$ by the positive linear functional $\varphi \equiv \varphi_h$ defined by
\[ \varphi(a) \equiv \varphi_h(a) = \varphi_0(a e^{-h}), \quad a \in A_\theta. \] (5)

Then $\varphi$ determines an inner product $\langle \cdot, \cdot \rangle_\varphi$ on $A_\theta$,
\[ \langle a, b \rangle_\varphi = \varphi(a^* b), \quad a, b \in A_\theta, \]
which by completion gives rise to a Hilbert space $H_\varphi$. The latter is again an $A_\theta$-bimodule but, since $\varphi$ is no longer tracial, the right action is no longer unitary.

The non-unimodularity of $\varphi$ is expressed by Tomita’s modular operator $\Delta$, which in this case is
\[ \Delta(x) = e^{-h} x e^h, \quad x \in A_\theta, \]
and gives rise to the 1-parameter group of inner automorphisms
\[ \sigma_t(x) = \Delta^{-it} = e^{ith} x e^{-ith}, \quad x \in A_\theta. \] (6)

Instead of the tracial property $\varphi$ satisfies the KMS condition
\[ \varphi(ab) = \varphi(b \sigma_i(a)) = \varphi(be^{-h} a e^h), \quad a, b \in A_\theta. \]

To restore the unitarity of the right action one redefines it by setting
\[ a^{op} := J_\varphi a^* J_\varphi \in \mathcal{L}(\mathcal{H}_\varphi), \quad a \in A_\theta, \]
where $J_\varphi(a) = \Delta^{1/2}(a^*) = k^{-1} a^* k$, $a \in A_\theta$, and $k = e^{h/2}$.

While keeping $\mathcal{H}^{(1,0)}$ unchanged, we now view $\delta_\tau$ as a densely defined operator from $\mathcal{H}_\varphi$ to $\mathcal{H}^{(1,0)}$. Its closure $\partial_\varphi$ is then used to define $D_\varphi = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix}$ giving rise to the triad $(A_\theta^{op}, \mathcal{H}_\varphi, D_\varphi)$, where $\mathcal{H}_\varphi = \mathcal{H} \oplus \mathcal{H}^{(1,0)}$. This is a twisted spectral triple (see [CoMo08] for the general definition) over $A^{op}$, with the twisted commutators $D_\varphi a^{op} - (k^{-1} a k)^{op} D_\varphi, \quad a \in A_\theta$ bounded. Its transposed, formed as in the flat case, yields the modular spectral triple over $A_\theta$, with operator $\tilde{D}_\varphi = \begin{pmatrix} 0 & k \partial_\varphi \\ \partial_\varphi^* k & 0 \end{pmatrix}$, where the conformal factor $k$ acts by left multiplication, and with underlying Hilbert space $\overline{\mathcal{H}_\varphi}$.

By a series of identifications, it is shown in [CoMo14, §1.3] that the modular spectral triple associated to $\varphi$, or equivalently to the conformal factor $k = e^{h/2}$, is canonically isomorphic to the twisted spectral triple $(A_\theta, \mathcal{H}_0, D_k)$ with $\mathcal{H}_0 := \mathcal{H} \oplus \mathcal{H}_0$ and $D_k := \begin{pmatrix} 0 & k \delta_\tau \\ \delta_\tau^* k & 0 \end{pmatrix}$.

We finally note that $D_k^2 = \triangle_k \oplus \triangle_k^{(0,1)}$, where
\[ \triangle_k := k \triangle_\tau k \equiv k \delta_\tau \delta_\tau^* k \quad \text{and} \quad \triangle_k^{(0,1)} = \delta_\tau^* k^2 \delta_\tau, \] (7)
are the counterparts of the Laplacian on functions, respectively the Laplacian on $(0,1)$-forms.
2.3. Modular curvature. The meaning of locality in noncommutative geometry is guided by the analogy with the Fourier transform, which interrelates the local behavior of functions with the decay at infinity of their coefficients. In a similar way, in the noncommutative formalism the local invariants of a spectral triple $(A, \mathcal{H}, D)$ are encoded in the high frequency behavior of the spectrum of the ‘inverse line element’ $D$ coupled with the action of the algebra of coordinates. For example, the local index formula in noncommutative geometry \cite[Part II]{CoMo95} expresses the Connes-Chern character of a spectral triple with finite dimension spectrum in terms of multilinear functionals given by residues of zeta functions defined by

$$z \mapsto \text{Tr} \left( a_0 [D, a_1]^{(k_1)} \cdots [D, a_p]^{(k_p)} |D|^{-z} \right), \quad \Re(z) \gg 0,$$

where $a_0, \ldots, a_p \in A$ and $[D, a]^{(k)} = [D^2, \ldots, [D^2, [D, a]] \cdots]$ with $D^2$ repeated $k$-times; the existence of the meromorphic continuation of such zeta functions is built in the definition of finite dimension spectrum for a spectral triple. Clearly, perturbing $D$ by a trace class operator will not affect these residue functionals, whence the local nature of the index formula described in their terms.

In the specific case of the noncommutative torus the concept of locality can be pushed much closer to the customary one. Namely, if $(A_\theta, \tilde{\mathcal{H}}, D_k)$ is a modular spectral triple as in §2.2, for its Laplacian ‘on functions’ there is an asymptotic expansion

$$\text{Tr} \left( a e^{-t\triangle_k} \right) \sim_{t \to 0} \sum_{q=0}^\infty a_{2q}(a, \triangle_k) t^{q-1}, \quad a \in A_\theta,$$

whose functional coefficients $a_{2q}$ are not only local in the above sense, but they are also absolutely continuous with respect to the unique trace, i.e. of the form

$$a_{2q} \in A_\theta \ni a \mapsto a_{2q}(a, \triangle_k) = \varphi_0 (a K_k^{(q)}), \quad K_k^{(q)} \in A_\theta,$$

with ‘Radon-Nikodym derivatives’ $K_k^{(q)} \in A_\theta$ computable by means of symbolic calculus. The technical apparatus which justifies the heat expansion Eq. (8) as well as the explicit computation of $K_k^{(0)}$ will be discussed in §§4-5.

In particular, the Radon-Nikodym derivative of the term $a_2$, which classically delivers the scalar curvature, was fully computed in \cite{CoMo14, FaKh13} and represents the modular scalar curvature. Abbreviating its notation to $K_k$ instead of $K_k^{(0)}$, it has the following expression:

$$K_k = -\frac{\pi}{237} \left( K_0(\nabla)(\triangle(h)) + \frac{1}{2} H_0(\nabla^{(1)}, \nabla^{(2)})(\square_{\mathbb{R}}(h)) \right),$$

where $\nabla = \log \Delta$ is the inner derivation implemented by $-h$,

$$\triangle(h) = \delta_r \delta_r^* = \delta_r^2(h) + 2\Re \tau \delta_1 \delta_2(h) + |\tau|^2 \delta_2^2(h),$$

$\square_{\mathbb{R}}$ is the Dirichlet quadratic form

$$\square_{\mathbb{R}}(\ell) := (\delta_1(\ell))^2 + \Re \tau \left( \delta_1(\ell) \delta_2(\ell) + \delta_2(\ell) \delta_1(\ell) \right) + |\tau|^2 (\delta_2(\ell))^2,$$
and $\nabla^{(i)}$, $i = 1, 2$, signifies that $\nabla$ is acting on the $i$th factor. The functions $K_0(s)$ and $H_0(s, t)$, whose expressions resulted from the symbolic computations, are given by

$$K_0(s) = \frac{-2 + s \coth \left( \frac{s}{2} \right)}{s \sinh \left( \frac{s}{2} \right)} \quad \text{and} \quad H_0(s, t) = \frac{t(s + t) \cosh(s) - s(s + t) \cosh(t) + (s - t)(s + t \sinh(s) + \sinh(t) - \sinh(s + t))}{st(s + t) \sinh \left( \frac{s}{2} \right) \sinh \left( \frac{t}{2} \right) \sinh \left( \frac{s + t}{2} \right)^2}.$$ 

The second function is related to the first by the functional identity

$$-\frac{1}{2} \tilde{H}_0(s_1, s_2) = \frac{\tilde{K}_0(s_2) - \tilde{K}_0(s_1)}{s_1 + s_2} + \frac{\tilde{K}_0(s_1 + s_2) - \tilde{K}_0(s_2) - \tilde{K}_0(s_1 + s_2) - \tilde{K}_0(s_1)}{s_2}.$$ 

where

$$\tilde{K}_0(s) = 4 \frac{\sinh(s/2)}{s} K_0(s) \quad \text{and} \quad \tilde{H}_0(s, t) = 4 \frac{\sinh((s + t)/2)}{s + t} H_0(s, t).$$ 

A noteworthy feature of the main curvature-defining function is that, up to a constant factor, $\tilde{K}_0$ is a generating function for the Bernoulli numbers; precisely,

$$\tilde{K}_0(t) = 8 \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n-2}.$$ 

2.4. Modular Gauss-Bonnet formula. Since the $K$-groups of the noncommutative torus are the same as of the ordinary torus, its Euler characteristic vanishes. Thus, the analogue of the Gauss-Bonnet theorem for the modular spectral triple is the identity

$$\varphi_0 \left( K_k^{(0)} \right) = 0.$$ 

This can be directly checked by making use of the fact that the group of modular automorphisms $\sigma_t$ (cf. Eq. (6)) preserves the trace $\varphi_0$ and fixes the dilaton $h$, in conjunction with the ‘integration by parts’ rule

$$\varphi_0(a \delta_j(b)) = -\varphi_0(\delta_j(a)b), \quad a, b \in A_\theta.$$ 

(See [CoMo14] Lemma 4.2 for the precise identity to be used).

An alternative variational argument, given in [CoMo14] §1.4], runs as follows. Consider the family of Laplacians

$$\triangle_s := k^s \triangle k^s = e^{\frac{4i}{\tau}} \triangle e^{\frac{4i}{\tau}}, \quad s \in \mathbb{R}.$$ 

(13)
One has $\frac{d}{ds}\triangle_s = \frac{1}{2}(h \triangle_s + \triangle_s h)$. By Duhamel’s formula one can interchange the derivative with the trace. Hence

$$\frac{d}{ds} \operatorname{Tr}(e^{-t \triangle_s}) = -t \operatorname{Tr}(h \triangle_s e^{-t \triangle_s}) = t \frac{d}{dt} \operatorname{Tr}(h e^{-t \triangle_s}).$$

Differentiating term-by-term the asymptotic expansion Eq. (8) (with $a = 1$ omitted in notation) yields

$$\frac{d}{ds} a_j(\triangle_s) = \frac{1}{2}(j - 2) a_j(h, \triangle_s), \quad j \in \mathbb{Z}^+. $$

In particular, $a_2(\triangle_s) = a_2(\triangle_\tau)$. The latter vanishes because $\triangle_\tau$ is isospectral to the Laplacian of the ordinary torus with the same complex structure and, as is well-known, if $\triangle_M$ is the Laplacian on a Riemann surface then $a_2(\triangle_M) = \frac{\chi(M)}{6}$, where $\chi(M)$ is the Euler characteristic of $M$.

2.5. Variation of determinant and modular Gaussian curvature. The zeta function $\zeta_{\triangle_k}(a, z) = \operatorname{Tr}(a \triangle_k^{-2}(1 - P_k))$, $\Re z > 2$ where $P_k$ stands for the orthogonal projection onto $\ker \triangle_k$, is related to the corresponding theta function by the Mellin transform

$$\zeta_{\triangle_k}(a, z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \operatorname{Tr}(a(e^{-t \triangle_k} - P_k)) \, dt. $$

The asymptotic expansion Eq. (8) ensures that it has meromorphic continuation and its value at 0 is

$$\zeta_{\triangle_k}(a, 0) = a_2(a, \triangle_\varphi) - \operatorname{Tr}(P_k a P_k) = a_2(a, \triangle_\varphi) - \frac{\varphi_0(ak^{-2})}{\varphi_0(k^{-2})}. \quad (14)$$

In particular for $a = 1$ (suppressed in notation), one has

$$\zeta_{\triangle_k}(0) = -1, \quad (15)$$

and also the Ray-Singer log-determinant is well-defined:

$$\log \operatorname{Det} \triangle_k := -\zeta'_{\triangle_k}(0).$$

Differentiating the 1-parameter family of zeta functions corresponding to (13) one obtains the identity

$$\frac{d}{ds} \zeta_{sh}(z) = -z \zeta_{sh}(h, z), \quad \forall z \in \mathbb{C},$$

which in turn yields the variation formula

$$-\frac{d}{ds} \zeta_{sh}(0) = \zeta_{sh}(h, 0).$$

From Eq. (14) and Eq. (9) applied to the weights $\varphi_s$ with dilaton $sh$ one obtains

$$\log \operatorname{Det} \triangle_k = \log \operatorname{Det} \triangle + \log \varphi(1) - \frac{\pi}{3s} \int_0^1 \varphi_0(h(sK_0(s\nabla)(\triangle(\log k))
+ s^2 H_0(s\nabla^{(1)}, s\nabla^{(2)})(\square_R(\log k))) \, ds$$
The first term is the same as for the corresponding elliptic curve and by the Kronecker limit formula has the expression (cf. [RASY73, Theorem 4.1])

\[
\log \operatorname{Det} \Delta = - \frac{d}{ds} \bigg|_{s=0} \sum_{(n,m) \neq (0,0)} |n + m \tau|^{-2s} = \log \left( 4\pi^2 |\eta(\tau)|^4 \right),
\]

where \( \eta \) is the Dedekind eta function \( \eta(\tau) = e^{\pi i \tau} \prod_{n \geq 0} (1 - e^{2\pi in\tau}) \). After a series of technical manipulations of the last term (see [CoMo14, §4.1]), one obtains the modular analogue of Polyakov’s anomaly formula:

\[
\log \operatorname{Det} \Delta_k = \log \left( 4\pi^2 |\eta(\tau)|^4 \right) + \log \varphi(1) - \frac{\pi}{43\tau} \varphi_0 \left( K_+(\nabla^{(1)})(\Box^R(h)) \right), \quad (16)
\]

where \( K_+(s) := 4\frac{\cosh(\frac{x}{v})}{v^2} \geq 0, s \in \mathbb{R} \). Furthermore, it is shown in [CoMo14, Proof of Theorem 4.6] that the positivity of the function \( K_+ \) can be upgraded to operator positivity, implying the inequality

\[
\varphi_0 \left( K_+(\nabla^{(1)})(\Box^R(\log k)) \right) \geq 0, \quad (17)
\]

with equality only for \( k = 1 \).

The (negative of) log-determinant can be turned into a scale invariant functional by adding the area term:

\[
F(\log k) := \zeta'_{\Delta_k}(0) + \log \varphi(1) = - \log \operatorname{Det}(\Delta_k) + \log \varphi(1). \quad (18)
\]

Due to the equality Eq. (14), the corrected functional \( F \) remains unchanged when the Weyl factor \( k \) is multiplied by a scalar. In the new notation the identity Eq. (16) reads as follows:

\[
F(h) = - \log \left( 4\pi^2 |\eta(\tau)|^4 \right) + \frac{\pi}{43\tau} \varphi_0 \left( K_+(\nabla^{(1)})(\Box^R(h)) \right). \quad (19)
\]

In view of the inequality Eq. (17) one concludes that, as in the case of the ordinary torus (cf. [OPS88]), the scale invariant functional \( F \) attains its extremal value only for the trivial Weyl factor, in other words at the flat metric.

The gradient of \( F \) is defined by means of the inner product of \( L^2(\mathcal{A}_\theta, \varphi_0) \) via the pairing

\[
\langle \text{grad}_h F, a \rangle \equiv \varphi_0(a \text{ grad}_h F) = \frac{d}{d\varepsilon} |_{\varepsilon=0} F(h + \varepsilon a), \quad a = a^* \in \mathcal{A}_\theta.
\]

A direct computation of the gradient, using the definition Eq. (18) combined with the identities Eq. (17) and Eq. (9), yields the following explicit expression (cf. [CoMo14, Theorem 4.8]):

\[
\text{grad}_h F = \frac{\pi}{43\tau} \left( \tilde{K}(\nabla)(\Delta(h)) + \tilde{H}(\nabla^{(1)}, \nabla^{(2)})(\Box^R(h)) \right). \quad (20)
\]

In the case of the ordinary torus the gradient of the corresponding functional (cf. [OPS88, (3.8)]) gives precisely the Gaussian curvature. This makes it compelling to take the above formula as definition of the modular Gaussian curvature.
Finally, computing the gradient of $F$ out of its explicit formula Eq. (19), and then comparing with the expression Eq. (20), produces the functional identity Eq. (10) relating $H$ and $K$.

3. Morita invariance of the modular curvature

3.1. Foliation algebras and Heisenberg bimodules. The most suggestive description of the noncommutative torus was given by Connes in [CON82], where he described it as the “space of leaves” for the Kronecker foliation $\mathcal{F}_\theta$ of the ordinary torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by the differential equation $dy - \theta dx = 0$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The holonomy groupoid $\mathcal{G}_\theta$ of this foliation identifies with the smooth groupoid determined by the flow of the above equation. Its convolution $C^*$-algebra $C^*(\mathcal{G}_\theta)$, which represents the (coordinates of the) space of leaves, coincides with the crossed product $C(T^2) \rtimes_\theta \mathbb{R}$, where the action of $\mathbb{R}$ on $T^2$ is given by the flow Eq. (3). $C^*(\mathcal{G}_\theta)$ is isomorphic to $A_\theta \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the $C^*$-algebra of compact operators, and thus strongly Morita equivalent to $A_\theta$.

Finer geometric representations of the space of leaves are obtained by passing to reduced $C^*$-algebras associated to complete transversals. Any pair of relatively prime integers $(d, c) \in \mathbb{Z}^2$ determines a family of lines of slope $\frac{d}{c}$, which project onto simple closed geodesics in the same free homotopy class, and the free homotopy classes of closed geodesics on $T^2$ are parametrized by the rational projective line $P^1(\mathbb{Q}) \equiv \mathbb{Q} \cup \{\frac{1}{0}\}$. Letting $N_{c,d}$ denote the primitive closed geodesic of slope $\frac{d}{c}$ passing through the base point of $T^2$, one obtains a complete transversal for $\mathcal{F}_\theta$.

The convolution algebra of the corresponding étale holonomy groupoid identifies with the crossed product algebra $C(\mathbb{R}/\mathbb{Z}) \rtimes_{\theta'} \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts by the rotation of angle $\theta' = \frac{ad + b}{ac + d}$ with $a, b \in \mathbb{Z}$ chosen such that $ad - bc = 1$. This $C^*$-algebra is none other than $A_{\theta'}$. In particular $A_\theta = C(\mathbb{R}/\mathbb{Z}) \rtimes_\theta \mathbb{Z}$ is the reduced $C^*$-algebra associated to $N_{0,1}$. By construction all algebras $A_{\theta'}$ with $\theta' = g \cdot \theta, g \in \text{SL}(2, \mathbb{Z})$, are Morita equivalent, and they actually exhaust (cf. [RIES]) all the noncommutative tori Morita equivalent to $A_\theta$.

In the same framework Connes [CON82 §13] gave a geometric description of the $(A_{\theta'}, A_\theta)$-bimodules $E(g, \theta)$ implementing the Morita equivalence of $A_\theta$ with $A_{\theta'}$. $E(g, \theta)$ is a completion of the $(A_{\theta'}, A_\theta)$-bimodule $\mathcal{E}(g, \theta) := S(\mathbb{R})^{[c]} \equiv S(\mathbb{R} \times \mathbb{Z}_c)$, $\mathbb{Z}_c := \mathbb{Z}/c\mathbb{Z}$, with the actions defined as follows:

$$(fu_1)(t, \alpha) := e^{2\pi i(t - \frac{ad}{c})} f(t, \alpha), \quad (fu_2)(t, \alpha) := f(t - \frac{c\theta + d}{c}, \alpha - 1);$$

$$(v_1f)(t, \alpha) := e^{2\pi i(\frac{t}{c} - \frac{d}{c})} f(t, \alpha), \quad (v_2f)(t, \alpha) := f(t - \frac{1}{c}, \alpha - a).$$

If $c = 0$ then $E(g, \theta) = A_{\theta'}^\text{op}$ is the trivial $(A_{\theta'}^\text{op}, A_\theta)$-bimodule. By analogy with the vector bundles over elliptic curves, one defines the rank, degree and slope of $\mathcal{E}(g, \theta)$ by $\text{rk}(g, \theta) = c\theta + d$, $\deg(g, \theta) = c$, resp. $\mu(g, \theta) := \frac{\deg(g, \theta)}{\text{rk}(g, \theta)}$. 

The $L^2$-scalar product on $E(g, \theta)$

$$< f_1, f_2 > := \int_{\mathbb{R} \times \mathbb{Z}_c} f_1(t, \alpha) f_2(t, \alpha) dt d\alpha$$

where the integration is with respect to the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\mathbb{Z}_c$, determines uniquely $A_\theta$-valued and $A_{\theta'}$-valued inner products satisfying the double equality

$$|\text{rk}(g, \theta)| \varphi_0(A_{\theta'} < f_2, f_1 >) = < f_1, f_2 > = \varphi_0(< f_1, f_2 >_{A_{\theta}}), \quad (21)$$

where $\varphi_0$ stands for the trace of $A_{\theta'}$. The completion $E(g, \theta)$ with respect to $| \cdot |_{A_\theta}^{1/2}$ is a full right $C^*$-module over $A_\theta$, and $\text{End}_{A_\theta} = A_{\theta'}$. In addition, $\mathcal{H}_0(g, \theta) := E(g, \theta) \otimes_{A_\theta} L^2(A_\theta, \varphi_0)$ is the Hilbert space $L^2(\mathbb{R} \times \mathbb{Z}_c)$.

Instead of the $\mathbb{R}^2$-action Eq. (3), the non-trivial bimodules $\mathcal{E}(g, \theta)$ are acted upon by the Heisenberg group $H_3(\mathbb{R})$. Equivalently, $\mathbb{R}^2$ acts projectively on $\mathcal{E}(g, \theta)$, and this action is compatible with the natural $\mathbb{R}^2$-actions on $A_\theta$ and $A_{\theta'}$. At the Lie algebra level, this action gives rise to the standard connection $\nabla^E$ on $\mathcal{E}(g, \theta)$, given by the derivatives $(\nabla_1 f)(t, \alpha) = \partial_{\alpha} f(t, \alpha)$, $(\nabla_2 f)(t, \alpha) = 2\pi i t \mu(g, \theta) f(t, \alpha)$ with constant curvature: $[\nabla_1, \nabla_2] = 2\pi i \mu(g, \theta) \text{Id}$. Furthermore, this connection is bi-Hermitian, in the sense that it preserves both the $A_\theta$-valued and the $A_{\theta'}$-valued inner product.

### 3.2. Modular Heisenberg spectral triples.

Each bimodule $\mathcal{E} = \mathcal{E}(g, \theta)$ gives rise to a double spectral triple, by coupling it with the flat Dirac $D_c$ by means of its standard connection. Specifically, $\nabla^E$ splits into holomorphic and anti-holomorphic components, $\nabla^E = \partial_{\xi}^c \oplus \tilde{\partial}_{\xi}^c$, where $\partial_{\xi} := \nabla_1 + \tau \nabla_2$. One then forms the operator $D_\xi = \begin{pmatrix} 0 & \partial_{\xi}^c \\ \partial_{\xi} & 0 \end{pmatrix}$ acting on the Hilbert space $\tilde{\mathcal{H}}(g, \theta) = \mathcal{H}_0(g, \theta) \oplus \mathcal{H}^{(1,0)}(g, \theta)$, where $\mathcal{H}^{(1,0)}(g, \theta) := E(g, \theta) \otimes_{A_\theta} \mathcal{H}^{(1,0)}(A_\theta)$. Together with the natural right action of $A_\theta$, these data define a spectral triple of constant curvature $\left( A_{\theta}^{\text{op}}, \tilde{\mathcal{H}}(g, \theta), D_\xi \right)$.

We note that from the spectral point of view the operator $D_\xi$ resembles the Hodge-de Rham operator of an elliptic curve with coefficients in a line bundle. In particular its Laplacian $\Delta_\xi = \partial_{\xi}^c \partial_{\xi}$ is a direct sum of $|\text{deg}(\xi)|$ copies of the harmonic oscillator

$$H := -\frac{d^2}{dt^2} + 4\pi^2 \mu(\xi)^2 |\tau|^2 t^2 - 4\pi i \mu(\xi) \Re(\tau) t \frac{d}{dt} - 2\pi i \mu(\xi) \overline{\tau} \text{Id}.$$

Now turning on the conformal change Eq. (5) from $\varphi_0$ to $\varphi_k$, one replaces $D_\xi$ by $D_{\xi, \varphi}$ in the same way as in §2.2. The resulting spectral triple over the algebra $A_{\theta}^{\text{op}}$ is again a twisted one. After correcting for the lack of unitarity of the action of $A_{\theta}^{\text{op}}$ again as in §2.2 the operator $D_{\xi, \varphi}$ is being canonically identified with $D_{\xi, k} := \begin{pmatrix} 0 & R_k \partial_{\xi} \\ \partial_{\xi} R_k & 0 \end{pmatrix}$ acting on $\tilde{\mathcal{H}}_0(g, \theta) = \mathcal{H}_0(g, \theta) \oplus \mathcal{H}_0(g, \theta)$. 
The appropriate transposed in this setting is constructed using the canonical anti-isomorphism from \( E = \mathcal{E}(g, \theta) \) to \( E':= \mathcal{E}(g^{-1}, \theta') \),
\[
J_{g,\theta}(f)(x, \alpha) = \frac{f((c\theta + d)x, -d^{-1} \alpha)}{f(\mathcal{E}(g, \theta)), \quad f \in \mathcal{E}(g, \theta),
\]
which switches the \((A_\theta', A_\theta')\)-action on the first into the \((A_\theta, A_\theta')\)-action on the second. We thus arrive at the modular Heisenberg spectral triple \( \left( A_\theta, \mathcal{H}_0(g^{-1}, \theta'), \mathcal{D}_{E',k} \right) \) with \( \mathcal{D}_{E',k} = -\operatorname{rk}(E') \begin{pmatrix} 0 & k\partial_{E'} \\ \partial_{E'},k & 0 \end{pmatrix} \). Its Laplacian on sections is \( \triangle_{E',k} = \operatorname{rk}(E')^2 k\partial_{E'}\partial_{E',k} \).

A moment of reflection shows that replacing \( \varphi_0 \) by \( \varphi \) is equivalent to changing the Hermitian structure on \( E' \) by \( k^{-2} \in A_\theta \equiv \operatorname{End}_{A_\theta'}(E') \). Indeed, in view of Eq. (21) applied to \( E' \), the passage to the \( A_\theta' \)-valued Hermitian inner product
\[
(f_1', f_2')_{A_\theta',k} = |\operatorname{rk}(E')|^{-1} < k^{-2} f_1', f_2' >_{A_\theta'}, \quad f_1', f_2' \in E',
\]
has the same effect on the \( L^2 \)-inner product, since
\[
\varphi_0' \left( (f_1', f_2')_{A_\theta',k} \right) = |\operatorname{rk}(E')|^{-1} \varphi_0'(< k^{-2} f_1', f_2' >_{A_\theta'}) = \varphi_0(\operatorname{rk}(E') k^{-2} f_1', f_2') = \varphi(\operatorname{rk}(E') k^{-1} f_1', f_2').
\]

In conclusion, the passage from the ‘constant curvature metric’ on \( A_\theta \) represented by the Heisenberg spectral triple \( \left( A_\theta'^{\text{op}}, \mathcal{H}(g, \theta), \mathcal{D}_{E} \right) \) to the ‘curved metric’ represented by the modular Heisenberg spectral triple \( \left( A_\theta, \mathcal{H}_0(g^{-1}, \theta'), \mathcal{D}_{E',k} \right) \) can be interpreted as being effect ed by changing the Hermitian structure of \( E' \) according to Eq. (22). Note that this interpretation remains valid even when \( c = 0 \), i.e. for \( E = A_\theta \).

The extended version of Connes’ pseudodifferential calculus (see 33) allows to establish the heat asymptotic expansion
\[
\operatorname{Tr} \left( a e^{-t\triangle_{E',k}} \right) \sim_{t \downarrow 0} \sum_{q=0}^{\infty} a_{2q}(a, \triangle_{E',k}) t^{q-1}, \quad a \in A_\theta,
\]
and express its functional coefficients in local form. In particular, the curvature functional is of the form
\[
a_{2}(a, \triangle_{E',k}) = \frac{1}{4\pi^3 \Gamma} \varphi_{E'}(a \mathcal{K}_{E',k}) = \frac{1}{4\pi^3 \Gamma} \operatorname{rk}(E') \varphi_{0}(a \mathcal{K}_{E',k}), \quad a \in A_\theta,
\]
where \( \varphi_{E'} := \operatorname{rk}(E') \varphi_{0} \) is the natural trace on \( A_\theta = \operatorname{End}_{A_\theta'}(E') \), and the curvature density has the expression (cf. [LemO16] Theorem 2.12)
\[
\mathcal{K}_{E',k} = K(\nabla)(\triangle(h)) + H(\nabla^1, \nabla^2) \left( \Box^R(h) \right) + \mu(E') 1.
\]
3.3. Ray-Singer determinant vs. Yang-Mills functional. To obtain the variation formula of the Ray-Singer log-determinant functional

\[ A^\Theta \ni h^* = h \mapsto \log \text{Det}(\Delta_{E',k}) := -\zeta'_{\Delta_{E',k}}(0), \]

one proceeds as in §2.5, starting with the insertion of the curvature expression \((24)\) in the derivative

\[ -\frac{d}{ds}\zeta'_{\Delta_{E',k}}(s) = \zeta_{\Delta_{E',k}}(h,0). \]

After integrating the resulting expression one arrives at the following exact formula for the Ray-Singer determinant (cf. [LeMo16, Theorem 2.15])

\[
\log \text{Det}(\Delta_{E',k}) = \frac{1}{2} |\text{deg}(E')| \log (2|\mu(E')|3(\tau)) - \frac{1}{2} |\text{deg}(E')| \varphi_0(h) - \frac{1}{16\pi^3} \left( \frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left( K_2(\nabla^1_h)(\square^\mathbb{R}(h)) \right) \right). 
\]

The scale invariant form of the functional is

\[ F_{E'}(h) = -\log \text{Det}(\Delta_{E',k}) - \frac{1}{2} |\text{deg}(E')| \varphi_0(h). \]

Using the preceding formula, it’s exact expression is seen to be

\[
F_{E'}(h) = -\log \text{Det}(\Delta_{E'}) + \frac{1}{16\pi^3} \left( \frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left( K_2(\nabla^1_h)(\square^\mathbb{R}(h)) \right) \right). \tag{25}
\]

When viewed as a functional on the (positive cone of) metrics on the Heisenberg left \(A^\Theta\)-module \(E', F_{E'}\) attains its minimum only at the metric whose corresponding connection compatible with the holomorphic structure has constant curvature (cf. [LeMo16, Theorem 2.16]).

Thus the Ray-Singer functional behaves in the same manner as the Yang-Mills functional of Connes and Rieffel (cf. [CoRi87]), which however is defined on the space of connections on the noncommutative torus.

3.4. Invariance of the Gaussian curvature. The gradient of the functional \(F_{E'}\), now defined via the equation

\[ \langle \text{grad}_h F_{E'}, a \rangle_{E'} \equiv \frac{1}{4\pi^3} \varphi_{E'}(a \cdot \text{grad}_h F_{E'}) := \frac{d}{de} |_{e=0} F(h + ea), \]

Its explicit expression can be computed as in [CoMo14, §4.2] and the answer turns out to be exactly the same as in the case of trivial coefficients, cf. Eq. (20):

\[ \text{grad}_h F_{E'} = \frac{\pi}{43(\tau)} \left( \bar{K}(\nabla)(\Delta(h)) + \bar{H}(\nabla^{(1)}, \nabla^{(2)})(\square^\mathbb{R}(h)) \right) = \text{grad}_h F. \]

This result can be interpreted as expressing the invariance of the modular Gaussian curvature under Morita equivalence in two different ways. First it shows that the Gaussian curvature associated to a change of Hermitian metric on a Heisenberg equivalence bimodule \(E'\) by a fixed positive invertible \(k \in A^\Theta\), viewed as an element of \(\text{End}_{A^\Theta}(E')\), is independent of \(E'\). Secondly, regarding the Heisenberg spectral triples with inverse line-element \(D_{E',k}\) as right spectral triples, conferring metrics to \(A^\Theta\), it
proves that the entire collection of Morita equivalent algebras \( \{ A_{g, \theta}; \theta \in \text{SL}(2, \mathbb{Z}) \} \) inherits the same modular curvature as the intrinsic one of \( A_g \).

4. Pseudodifferential multipliers and symbol calculus

The main technical device which was used for proving the above results is a pseudodifferential calculus adapted to twisted \( C^* \)-dynamical systems, extending the well-known calculi due to Connes [Con80].

Originally, pseudodifferential operators (ΨDO) were invented (see Kohn and Nirenberg [Kon65] or for a textbook Shubin [Shu01]) to study elliptic partial differential operators. ΨDO form an algebra which contains differential operators and the parametrices to elliptic differential operators. They come with a symbolic calculus: while the (complete) symbols of differential operators are polynomials in the covariables, ΨDO are obtained by allowing more general types of symbol functions, e.g. of Hörmander type.

4.1. Ordinary ΨDO in \( \mathbb{R}^n \) from the point of view of \( C^* \)-dynamical systems.

4.1.1. Standard representation on the \( L^2 \)-space (GNS space). Connes’ pseudodifferential calculus on a \( C^* \)-dynamical system \((A, \mathbb{R}^n, \alpha)\) should be viewed as a pseudodifferential calculus on \( \mathbb{R}^n \). To motivate the defining formulas and to connect to the standard pseudodifferential calculus, we briefly recast the latter in the language of \( C^* \)-dynamical systems such that the link becomes apparent. Recall that for a suitably nice symbol function \( \sigma(\xi, s), s, \xi \in \mathbb{R}^n \) one defines the pseudodifferential operator with complete symbol \( \sigma \) as

\[
\left( \text{Op}(\sigma)u \right)(s) := \int_{\mathbb{R}^n} e^{i(s, \xi)} \sigma(\xi, s) \hat{u}(\xi) d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(s-y, \xi)} \sigma(\xi, s) u(y) dy d\xi. \tag{26}
\]

Now let us abuse this formula a little. Let \( \mathcal{A}^\infty := \mathcal{S}(\mathbb{R}^n) \subset C_0(\mathbb{R}^n) \) be the Schwartz space viewed as a *-subalgebra of \( C_0(\mathbb{R}^n) \). It acts on itself by left multiplication. Furthermore, there is a one parameter group of *-automorphisms \( \alpha_x(f) := f(\cdot - x) \) and a one parameter family of operators \( \pi_x(f) := f(\cdot - x) \) satisfying \( \pi_ya\pi_{-y} = \alpha_{-y}(a), a \in \mathcal{A}^\infty \). This gives rise to a covariant representation of the dynamical system \((\mathcal{A}^\infty, \mathbb{R}^n, \alpha)\) on the Hilbert space \( L^2(\mathbb{R}^n) \) which is the GNS space of the \( \alpha \)-invariant tracial weight \( \varphi_0(f) = \int_{\mathbb{R}^n} f \), i.e. the completion of \( \mathcal{A}^\infty \) with respect to the inner product \( \langle f, g \rangle_{\varphi_0} = \varphi_0(f^*g) = \int_{\mathbb{R}^n} \overline{f} g \).

\footnote{For consistency with the later exposition we deliberately use a somewhat unusual order and naming convention for the variables \( \xi, s, \xi \). \( s \) plays the role of the covariable and the spacial variable \( s \) is normally called \( x \) in ΨDO textbooks. We want to view the function \( \sigma(\xi, \cdot) \) as an algebra valued function on \( \mathbb{R}^n_\xi \).}
Now given \( u \in \mathcal{A}^\infty \) and a symbol \( \sigma \in \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty) = \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) we continue from Eq. (26) and compute

\[
(\text{Op}(\sigma)u)(s) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{i(y,\xi)} \sigma(\xi, s) d\xi \right) u(s - y) dy \\
= \int_{\mathbb{R}^n} (\mathcal{T}^{-1}_{\xi \to y} \sigma(y))(s) u(s - y) dy \\
= \int_{\mathbb{R}^n} \sigma^\vee_{\xi \to y}(y) \pi_y u dy(s),
\]

with \( \sigma^\vee_{\xi \to y} := \mathcal{T}^{-1}_{\xi \to y} \sigma \).

Thus symbols in the Schwartz space \( \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty) \) act, after a Fourier transform in the first variable, covariantly with respect to the natural representation of the covariance algebra \( \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty) \rtimes_\alpha \mathbb{R} \) on the GNS space of the weight \( \varphi_0 \).

We note furthermore, that for \( \sigma \in \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) the operator \( \text{Op}(\sigma) \) is trace class and from the calculation Eq. (27) we see that the Schwartz kernel of \( \text{Op}(\sigma) \) on the diagonal is given by \( \int_{\mathbb{R}^n} \sigma(\xi, \cdot) d\xi \), hence

\[
\text{Tr}(\text{Op}(\sigma)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi, s) ds d\xi = \varphi_0((\sigma^\vee_{\xi \to y}(0)) = \int_{\mathbb{R}^n} \varphi_0(\sigma(\xi, \cdot) d\xi. \quad (28)
\]

We now take the Schwartz functions \( \sigma^\vee_{\xi \to y} \) as basic objects. Identifying \( f \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) with \( \pi(f) = \int_{\mathbb{R}^n} f(x) \pi_x dx \) the space \( \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) becomes a \(*\)-algebra with \(*\)-representation \( f \mapsto \pi(f) \) on \( L^2(\mathbb{R}^n) \). Explicitly, \( \pi(f) \circ \pi(g) = \pi(f * g) \) and \( \pi(f^*) = \pi(f^*) \), where

\[
f^*(x) = \alpha_x(f(-x)^*), \quad (f * g)(x) = \int_{\mathbb{R}^n} f(y) \alpha_y(g(x - y)) dy, \quad (29)
\]

resp. with the second variables spelled out, \( \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) becomes a \(*\)-algebra with involution and product given by

\[
f^*(x, s) = f(-x, s - x), \quad (f * g)(x, s) = \int_{\mathbb{R}^n} f(y, s) g(x - y, s - y) dy. \quad (30)
\]

4.1.2. Pseudodifferential multipliers. We now lift the previous \(*\)-representation to a “universal” multiplier representation as follows:

\( \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) is a pre-C*-module with inner product \( \langle f, g \rangle = \int_{\mathbb{R}^n} f(x)^* g(x) dx \).

Put

\[
\langle a f \rangle(x) = \alpha_{-x}(a) f(x), \langle U_y f \rangle(x) = f(x - y), \quad a \in \mathcal{S}(\mathbb{R}^n). \quad (31)
\]

Since \( U_x a U_{-x} = \alpha_x(a) \) this gives rise to a covariant representation of the \(*\)-algebra \( \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) by associating to \( f \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) the multiplier \( M_f = \int_{\mathbb{R}^n} f(x) U_x dx \).

If \( \varphi \) is a \( \alpha \)-invariant trace on \( \mathcal{S}(\mathbb{R}^n) \) then the dual trace \( \widehat{\varphi} \) on \( \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^n)) \) is given by

\[
\widehat{\varphi}(f) = \varphi(f(0)) = \int_{\mathbb{R}^n} \varphi(f(\xi)) d\xi. \quad (32)
\]
Note that \( d\xi \) is the Plancherel measure of the dual group \((\mathbb{R}^n)^\wedge\) w.r.t. the duality pairing \((x, \xi) \mapsto e^{i(x, \xi)}\).

In case of the trace \( \varphi_0 = \int_{\mathbb{R}^n} \) from the previous section the dual trace equals the trace Eq. (28) on the Hilbert space representation \( L^2(\mathbb{R}^n)\). This equality should be viewed as a coincidence. In general the dual trace does not coincide with the Hilbert space trace on a representation, resp. this depends on the representation, see [5.2].

By associating to \( f \in S(\mathbb{R}^n, \mathcal{A}^\infty)\) the multiplier \( M_f = \int_{\mathbb{R}^n} f(x) U_x dx \) the space \( S(\mathbb{R}^n, \mathcal{A}^\infty)\) becomes a \(*\)-algebra. Putting \( P_f := M_f\) and allowing \( f \) to be a symbol of Hörmander class \( S^\alpha(\mathbb{R}^n, \mathcal{A}^\infty)\) we obtain an algebra of multipliers which, via the representation \( \pi \) from above, is isomorphic to an algebra of pseudodifferential operators in \( \mathbb{R}^n\). We deliberately say “an” and not “the” here as in \( \mathbb{R}^n\) there are various versions of such algebras which differ only by the behavior of symbols as the spacial variable \( s \to \infty\), cf. [Shu01 Chap. IV].

4.2. Pseudodifferential multipliers on twisted crossed products. The action of the Heisenberg group on \( \mathcal{E}(g, \theta)\) induces a \( C^*\)-dynamical system \((\mathcal{A}, \mathbb{R}^{n=2}, \alpha)\) \((\mathcal{A} = \mathcal{A}_B \text{ or } \mathcal{A} = \mathcal{A}_G)\). Equivalently, \( \mathbb{R}^n\) acts by a projective representation with cocycle \( e(x, y) := e^{i(Bx, y)}\), with a skew-symmetric matrix \( B = (b_{kl})_{k,l=1}^n\). In order to construct the resolvent of elliptic differential operators (i.e. Laplacians) on Heisenberg modules one therefore extends the previous considerations to twisted \( C^*\)-dynamical systems. In the previous two sections we have formulated the standard pseudodifferential operator conventions in such a way that they carry over almost ad verbatim to the twisted case.

Consider a \( C^*\)-dynamical system \((\mathcal{A}, \mathbb{R}^n, \alpha)\) with, now for simplicity, unital \( \mathcal{A}\). Furthermore, let

\[
e(x, y) := e^{i\sigma(x, y)} = e^{i(Bx, y)}, \quad \sigma(x, y) := \langle Bx, y \rangle
\]

with a skew-symmetric real \( n \times n\)-matrix \( B = (b_{kl})_{k,l=1}^n\). By \( \mathcal{A}^\infty\) we denote the smooth subalgebra, i.e. those \( a \in \mathcal{A}\) for which \( t \mapsto \alpha_t(a)\) is smooth.

As before the Schwartz space \( S(\mathbb{R}^n, \mathcal{A}^\infty)\) is a pre-\( C^*\)-module with inner product \( \langle f, g \rangle = \int_{\mathbb{R}^n} f(x)^* g(x) dx\). Putting \( (U_y f)(x) = e(x, -y) f(x - y)\) we obtain a projective family of unitaries \( U_x^* = U_{-x}^* = U_{y}^* U_x = e(x, y)U_{x+y}, \quad x, y \in \mathbb{R}^n\), \( U_x a U_{-x} = \alpha_x(a), \quad a \in \mathcal{A}^\infty\). Together with \( (a f)(x) = \alpha_{-x}(a) f(x), \quad a \in \mathcal{A}^\infty\) and associating to \( f \in S(\mathbb{R}^n, \mathcal{A}^\infty)\) the multiplier \( M_f = \int_{\mathbb{R}^n} f(x) U_x dx\) the space \( S(\mathbb{R}^n, \mathcal{A}^\infty)\) becomes a \(*\)-algebra. Explicitly, cf. Eq. (29)

\[
f^*(x) = \alpha_x(f(-x)^*), \quad (f * g)(x) = \int_{\mathbb{R}^n} f(y) \alpha_y(g(x-y)) e(y, x) dy.
\]

Note that the formula for \( f^*\) is the same as in the untwisted case.

As in the untwisted case, a \( \alpha\)-invariant trace \( \psi \) on \( \mathcal{A}\) induces a dual trace \( \hat{\psi} \) on \( S(\mathbb{R}^n, \mathcal{A}^\infty)\) which is given by the same formula as Eq. (32).

To define the pseudodifferential operator convention we now read Eq. (27) backwards. Namely, given Schwartz functions \( f, u \in S(\mathbb{R}^n, \mathcal{A}^\infty)\), and abbreviating
\( f^\vee := \mathcal{F}^{-1}_{\xi \to y} f \) the inverse Fourier transform of \( f \), we find

\[
(M_{f^\vee} u)(x) := \left( \int_{\mathbb{R}^n} f^\vee(y) U_y u \, dy \right)(x)
\]

\[
= \int_{\mathbb{R}^n} \alpha_{-x}(f^\vee(y)) u(x-y) e(x,-y) \, dy = \int_{\mathbb{R}^n} \alpha_{-x}(f^\vee(x-y)) u(y) e(x,y) \, dy
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - Bx)} \alpha_{-x}(f(\xi)) u(y) \, dy \, d\xi
\]

\[
= \int_{\mathbb{R}^n} e^{i(x \cdot \xi)} \alpha_{-x}(f(\xi)) u(\xi - Bx) \, d\xi = \int_{\mathbb{R}^n} e^{i(x \cdot \xi)} \alpha_{-x}(f(\xi + Bx)) u(\xi) \, d\xi
\]

\[
=: (P_f u)(x)
\]

and call the so defined multiplier \( P_f \) a \textit{(twisted) pseudodifferential multiplier with symbol} \( f \). This should be compared to Eq. [26].

Strictly speaking, so far we have only dealt with smoothing operators as all symbols were Schwartz functions. One now has to extend \( P_f \) to a larger class of functions \( f \). The purpose of the somewhat lengthy exposition so far was to show that, at least in \( \mathbb{R}^n \) but there in a rather broad sense, smoothing operators are nothing but convolution operators and their symbols are obtained by applying a partial Fourier transform. General \( \Psi DO \) are therefore nothing but \textit{singular} convolution operators. This is not surprising as \( \Psi DO \) may, via the Schwartz Kernel Theorem, also be viewed as singular integral operators.

The extension to general symbol functions now follows the standard route. Putting \( P_f := M_{f^\vee} \) and allowing \( f \) to be a symbol of Hörmander class \( \mathcal{S}^m(\mathbb{R}^n, \mathcal{A}^\infty) \) we obtain a class of multipliers extending the pseudodifferential multipliers à la Connes [CON80] and Baaj’s [BA88A] [BA88B]. Later we will also need the so called classical (1-step polyhomogeneous) symbols \( f \in \mathcal{CS}^m(\mathbb{R}^n, \mathcal{A}^\infty) \) which have an asymptotic expansion

\[
f \sim \sum_{j=0}^\infty f_{m-j}
\]

with \( f_{m-j}(\lambda \xi) = \lambda^{m-j} \cdot f_{m-j}(\xi), |\xi| \geq 1, \lambda \geq 1 \).

Thus for \( f \in \mathcal{S}^m(\mathbb{R}^n, \mathcal{A}^\infty) \) we obtain a well-defined multiplier \( P_f \) acting on the pre-\( C^* \)-module \( \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty) \) with \textit{complete symbol} \( f \). The usual stationary phase arguments (e.g. [SHU01] § I.3) then allow to prove that the space \( \mathcal{L}_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty) = \bigcup_{\delta \in \mathbb{Z}} \mathcal{L}_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty) \) of twisted pseudodifferential multipliers (as well as its classical counterpart where the symbols \( f \) are 1-step polyhomogeneous) is a \( \ast \)-algebra.

For symbols \( f \in \mathcal{S}^m(\mathbb{R}^n, \mathcal{A}^\infty), g \in \mathcal{S}^m(\mathbb{R}^n, \mathcal{A}^\infty) \) the composition \( P_f \circ P_g \) is a pseudodifferential multiplier with symbol \( h \in \mathcal{S}^{m+m'}(\mathbb{R}^n, \mathcal{A}^\infty) \) and \( h \) has the asymptotic expansion

\[
h(t) \sim \sum_{\gamma} \frac{t^{-|\gamma|}}{\gamma!} (\partial^n f)(t) \partial_y^\gamma \bigg|_{y=0} \left( \alpha_{-y}(g(t + B y)) \right).
\]
Furthermore, \( P_f^* \) is a pseudodifferential multiplier with symbol
\[
\sigma(P_f^*) \sim \sum_{\gamma} \frac{1}{\gamma!} \partial_y^\gamma \delta^y f(t)^*.
\] (40)

Here \( \delta^\gamma \) denotes the basic derivative on \( A \) induced by the flow \( \alpha \): For \( a \in A^\infty \) and a multiindex \( \gamma \in \mathbb{Z}^n_+ \) it is defined by
\[
\delta^\gamma a := i^{\gamma|} \partial_x^\gamma |_{x=0} a_x(a) = i^{-\gamma|} \partial_x^\gamma |_{x=0} a_{-x}(a).
\] (41)
\( \delta^\gamma \) plays the role of the partial derivative \( i^{-\gamma|} \partial_x^\gamma \).

4.3. Differential multipliers. In the standard calculus differential operators are characterized as those pseudodifferential operators whose complete symbols are polynomial in the covariables \( \xi \). Adopting this as a definition for differential multipliers it turns out that the (multiplier counterparts) of the natural first and second order differential operators discussed in Section 2 are differential multipliers in this sense.

Somewhat more formally we call \( P f \in L^*_\sigma(\mathbb{R}^n, A^\infty) \) a differential multiplier of order \( \text{m} \) if \( f(\xi) = \sum_{\gamma \leq \text{m}} a_\gamma \xi^\gamma \); \( a_\gamma \in A^\infty \),
\[
f \in A^\infty[\xi_1, \ldots, \xi_n] \text{ is a polynomial of degree at most } \text{m}. \text{ Here the sum runs over all multiindices } \gamma \in \mathbb{Z}^n_+ \text{ with } |\gamma| \leq \text{m}. \] Clearly, polynomials in \( \xi \) are 1-step polyhomogeneous and hence differential operators are classical pseudodifferential multipliers.

Recall that in the ordinary pseudodifferential calculus the symbol of the basic derivatives \( i^{-\gamma|} \partial_x^\gamma \) is given by \( \xi^\gamma \). Therefore, for a multiindex \( \gamma \) we put \( \partial^\gamma := P_{\xi^\gamma} \). Explicitly, we find from Eq. (37) for \( u \in S(\mathbb{R}^n, A^\infty) \)
\[
(\partial^\gamma u)(x) = (P_{\xi^\gamma} u)(x) = \int_{\mathbb{R}^n} e^{i(x,y)\xi} (\xi + Bx)^\gamma \hat{u}(\xi) \, d\xi
= i^{-\gamma|} \partial_y^\gamma |_{y=0} \int_{\mathbb{R}^n} e^{i(x+y,\xi+Bx)} \hat{u}(\xi) \, d\xi
= i^{-\gamma|} \partial_y^\gamma |_{y=0} (e(x,y)u(x+y)) = i^{\gamma|} \partial_y^\gamma |_{y=0} U_y u(x).
\] (43)

It is important to note that due to the twisting in general \( \partial^\gamma \partial^\gamma' \neq \partial^{\gamma+\gamma'} \), as can be seen either directly or by the just proved product formula.

As in the ordinary pseudodifferential calculus it is in general not true that \( P_f^* = P_f \). However, \( \sigma(P_f)^* = \sigma(P_f^*) \mod S^{m-1}(\mathbb{R}^n, A^\infty) \).

Furthermore, \( \partial^\gamma \) is formally self-adjoint and thus for any differential multiplier we have indeed \( P_f^* = P_f \).
4.4. **Differential multipliers of order 1 and 2.** We look more closely at the most relevant case of differential multipliers of order 1 and 2. Let \( e_j, j = 1, \ldots, n \) be the canonical basis vectors of \( \mathbb{R}^n \). We abbreviate \( \partial_j := \partial^e_j \) and recall that \( b_{jk} \) denotes the entries of the skew-symmetric structure matrix of the twisting Eq. (33). Then by Eq. (43)

\[
\partial_j u(x) = i^{-1} \partial_{yj} \big|_{y=0} e^{i(Bx,y)} u(x+y) = \left( \frac{1}{i} \partial_{x_j} + b_{ji} x_i \right) u(x),
\]

(44)

were summing over repeated indices is understood. Thus

\[
\partial_j \partial_k = -\partial_{x_j} \partial_{x_k} - ib_{js} x_s \partial_{x_k} - ib_{ks} x_s \partial_{x_j} - ib_{kj} + b_{js} b_{kr} x_r x_s.
\]

(45)

In particular we have the “curvature identity”

\[
[\partial_j, \partial_k] = 2ib_{jk}.
\]

(46)

The twisting and the non-commutativity has an interesting effect on the symbol calculus. The symbol of \( \partial_j \cdot \partial_k \) is not \( \xi_j \cdot \xi_k \) but rather it is a consequence of the formula Eq. (39) that

\[
\sigma(\partial_j \cdot \partial_k) = \xi_j \cdot \xi_k + ib_{jk} = \sigma(\partial^{e_j+e_k}) + ib_{jk},
\]

(47)

hence \( \partial^{e_j+e_k} = \partial_j \cdot \partial_k - ib_{jk} \). From this the curvature identity Eq. (46) also follows.

4.4.1. **Differential multipliers in dimension \( n = 2 \).** Specializing further to dimension \( n = 2 \) it is most convenient to make use of the complex Wirtinger derivatives. Furthermore, the structure matrix \( b_{jk} \) has only one interesting entry \( b_{12} \). Fixing \( \tau \in \mathbb{C} \) with \( \Im \tau > 0 \) (a complex structure!) we have the following basic differential multipliers:

\[
\partial_\tau := \partial_1 + \tau \partial_2, \quad \partial_\tau^* = \partial_1 + \tau \partial_2, \quad \partial_1 := \partial^{1,0}, \partial_2 := \partial^{0,1}
\]

\[
[\partial_\tau, \partial_\tau^*] = -4 \Im \tau \cdot b_{12} =: c_\tau,
\]

\[
\Delta_\tau := \frac{1}{2} (\partial_\tau^* \partial_\tau + \partial_\tau \partial_\tau^*) = \partial_1^2 + |\tau|^2 \partial_2^2 + \Re \tau (\partial_1 \partial_2 + \partial_2 \partial_1).
\]

We will first analyze these operators acting as multipliers on the Hilbert module completion of \( S(\mathbb{R}^n, A^\infty) \). Lateron we will have to pass to their concrete counterparts acting on the Heisenberg modules.

5. **The resolvent expansion and trace formula**

The resolvent trace, or equivalently the heat trace, expansion for second order Laplace type operators goes back at least to Minakshisundaram and Pleijel [MiPl49]. Via Karamata’s tauberian theorem there is a connection to the eigenvalue counting function whose asymptotic analysis is quite subtle. The best remainder term for the counting function of general elliptic operators led Hörmander to develop his beautiful theory of Fourier integral operators [Hör71]. Later the resolvent trace (aka heat equation) method led to the development of local index theory [ABP73] with an enormous flow of publications.
In our opinion, by now the most streamlined approach to the resolvent expansion of elliptic differential operators is the calculus of parameter dependent pseudodifferential operators which essentially goes back to Seeley’s seminal complex powers paper \[\text{see} \text{See67}\] and which is presented very nicely in Shubin’s book \[\text{see} \text{Shu01}\] § II.9.  

$\text{We will come back to this soon. Our goal here is to show that this calculus carries over to twisted pseudodifferential multipliers and that the second coefficient in the expansion can be calculated quite easily without any computer aid.}$

$\text{We consider the differential multiplier } P = P_{\xi_1, \xi_2} := k^2 \Delta + \varepsilon_1 (\partial_\tau k^2) \partial_\tau + \varepsilon_2 (\partial_\xi^2 k^2) \partial_\xi + a_0, \text{ where } a_0 \in A_\infty \text{ and } \varepsilon_1, \varepsilon_2 \text{ are real parameters. This multiplier contains all conformal Laplace type multipliers, which occur on Heisenberg modules over noncommutative tori, as special cases.}$

$\text{The symbol of } P \text{ takes the form } \sigma_P(\xi) := a_2(\xi) + a_1(\xi) + a_0, \text{ where } a_0 \in A_\infty \text{ is the same as above and}$

\begin{align*}
a_2(\xi) &= k^2 |\xi_1 + \tau \xi_2|^2 = k^2 |\eta|^2, \\
a_1(\xi) &= \varepsilon_1 (\partial_\tau k^2) \eta + \varepsilon_2 (\partial_\xi^2 k^2) \eta, \quad \eta := \xi_1 + \tau \xi_2, \\
&= : g_1 \eta + g_2 \eta, \quad g_1 := \varepsilon_1 \partial_\tau k^2, \quad g_2 := \varepsilon_2 \partial_\xi^2 k^2.
\end{align*}

$\text{The resolvent } (P - \lambda)^{-1} \text{ belongs to the parameter dependent pseudodifferential calculus and therefore its symbol has a polyhomogeneous expansion } \sigma_{P(\lambda)}^{-1} \sim b_{-2} + b_{-3} + b_{-4} + \ldots, \text{ where } b_{-k}(\xi, \lambda) \in A_\infty \text{ depends smoothly on } (\xi, \lambda) \text{ and is homogenous of degree } -k: b_{-k}(r \xi, r^2 \lambda) = r^{-k} b(\xi, \lambda). \text{ As a consequence we obtain for the } a \in A_\infty \text{ with respect the dual trace Eq. (52) } \hat{\varphi}_0 (\varphi_0 \text{ is the invariant trace on } A_\infty) \text{ an asymptotic expansion}$

$$
\varphi_0 \left( e^{-tP} \right) \sim t^{-\lambda}_0 \sum_{j=0}^\infty a_{2j}(P, a) t^{j-1},
$$

where it follows from the homogeneity $^3$

$$
a_{2j}(P, a) = \int_{\mathbb{R}^2} \int_C e^{-t\lambda} \varphi_0 \left( (b_{-2j}(\xi, gl)) \right) d\lambda d\xi = \int_{\mathbb{R}^2} \varphi_0 \left( b_{-2j-2}(\xi, -1) \right) d\xi \int_C e^{-t\lambda} (-\lambda)^{-j} d\lambda.
$$

Here $C$ is a contour in the complex plane encircling the positive semiaxis clockwise such that $\int_C e^{-t\lambda} (r - \lambda)^{-1} d\lambda = e^{-tr}$. The second line is a consequence of the homogeneity of the $b_{-k}$ (see \text{[COMO95]} §6). For the second nontrivial heat coefficient

$^2$In Seeley’s paper a subtle oversight caused a certain confusion which, at least among non-experts, seems to exist to this day. The resolvents of elliptic pseudodifferential operators in general only belong to a “weakly parametric” calculus. This difference between the resolvent calculi for differential resp. true pseudodifferential operators was clarified almost 30 years after Seeley’s original paper \[\text{see} \text{GrSe95}\].

$^3$Note that heat/resolvent invariants are enumerated from 0. We are after $a_2$ which is the second nontrivial heat invariant, as $a_1$ is always 0 for differential operators, but in the counting of the recursion system it is the third term.
one therefore obtains up to a sign (see loc. cit.)
\[ a_2(P,a) = \int_{\mathbb{R}^2} \varphi_0(b_{-4}(\xi, -1))d\xi. \]

Due to this formula, it will be convenient to compute \( b_{-4}(\xi, -1) \) modulo functions of total \( \xi \)-integral 0. Up to a function of total \( \xi \)-integral 0 we have the following closed formulas for the first three terms in the symbol expansion of \( (P - \lambda)^{-1} \):

\[
\begin{align*}
  b_{-2} &= b = (k^2|\eta|^2 - \lambda)^{-1}, \quad b_{-3} = -bk^2(\eta\partial_\tau^2 + \eta\partial_\tau) - ba_1b, \\
  b_{-4} &= (2bk^2|\eta|^2 - 1 - \varepsilon_1 - \varepsilon_2)bk^2\Delta_\tau b + \lambda bk^2((\partial_\tau^*b)(\partial_\tau b) + (\partial_\tau b)(\partial_\tau^*b)) \\
  &\quad \quad + \varepsilon_1 \cdot \lambda b(\partial_\tau^2k^2)b\partial_\tau^2b + \varepsilon_2 \cdot \lambda b(\partial_\tau^*k^2)b\partial_\tau^2b \\
  &\quad \quad + \varepsilon_1\varepsilon_2 \cdot |\eta|^2b \cdot ((\partial_\tau^2k^2)(\partial_\tau^*k^2) + (\partial_\tau^*k^2)(\partial_\tau k^2)) \cdot b - ba_0b.
\end{align*}
\]

The proof is straightforward, completely computer free, and fits on two pages, cf. [LEMO16, §3.3].

5.1 Second heat coefficient. Integrating \( b_{-4} \) over \( \xi \) is still a little involved and it requires the Rearrangement Lemma [CoMo95, §6.2]. This was recast and generalized in [Les17]. The calculus of divided differences allows to compute the many explicit integrals in a systematic way. As a result there exist entire functions \( K(s), H^R(s, t), H^S(s, t) \), such that with \( h := \log k^2 \) the second heat coefficient of \( P \) (w.r.t. the natural dual trace on the twisted crossed product) takes the form

\[
a_2(P,a) = \frac{1}{4\pi|3\pi|} \varphi_0 \left[ a\left(K(\nabla)(\triangle, h) - k^{-2}a_0 \right. \right.
\]
\[
\left. \left. + H^R(\nabla^{(1)}, \nabla^{(2)})(\square^R(h)) + H^S(\nabla^{(1)}, \nabla^{(2)})(\square^S(h)) \right) \right].
\]

Here, \( \square^{R/3}(h) := \frac{1}{2}(\partial_{\tau}h \cdot \partial_{\tau}^* \pm \partial_{\tau}^*h \cdot \partial_{\tau}h) \), \( \nabla = -\text{ad}(h) \), and \( \nabla^{(i)} \) signifies that it acts on the \( i \)-th factor (cf. [COMO14, Les17]).

The functions \( K, H^R, H^S \) depend only on \( P \) but not on \( \tau \). They can naturally be expressed in terms of simple divided divided differences of \( \log \).

5.2 Effective pseudodifferential operators and trace formulas. We consider the noncommutative torus \( A_\theta \) with generators \( U_1, U_2 \) and normalized trace \( \varphi_0 \). Let \( f : \mathbb{R}^2 \to \mathcal{A}_\infty \) be a symbol function (or Schwartz function) of sufficiently low order. Recall the trace Eq. (32) of the multiplier \( P_f : \hat{\nu}(P_f) = \int_{\mathbb{R}^2} \varphi_0(f(x))dx. \) However, the multiplier \( P_f \) is canonically represented as an operator on the GNS space \( L^2(A_\theta, \varphi_0) \) by \( \text{Op}(f) = \int_{\mathbb{R}^2} f^\nu(x)\pi_xdx, \) where \( \pi_x(U_1^{n_1}U_2^{n_2}) = e^{i(x,n)U_1^{n_1}U_2^{n_2}} \) is the unitary which implements the natural \( \mathbb{R}^2 \)-action on \( A_\theta \), cf. Eq. (3). \( \text{Op}(f) \) acts as a trace class operator on \( L^2(A_\theta, \varphi_0) \). More concretely, one computes \( \text{Op}(f)U_1^{n_1}U_2^{n_2} = f(-n_1, -n_2)U_1^{n_1}U_2^{n_2} \). Since \( (U_1^{n_1}U_2^{n_2})_{n\in\mathbb{Z}^2} \) is an orthonormal basis
of $L^2(A^\infty, \varphi_0)$ we obtain the trace formula

$$\text{Tr}(\text{Op}(f)) = \sum_{n \in \mathbb{Z}^2} \langle U_1^{n_1} U_2^{n_2}, f(-n_1, -n_2) U_1^{n_1} U_2^{n_2} \rangle$$

$$= \sum_{n \in \mathbb{Z}^2} \varphi_0 \left( (U_1^{n_1} U_2^{n_2})^* f(-n_1, -n_2) U_1^{n_1} U_2^{n_2} \right)$$

$$= \sum_{n \in \mathbb{Z}^2} \varphi_0(f(n)).$$

This looks a little different from the formula for the dual trace. However, for a parameter dependent symbol $f(x, \lambda)$ we can take advantage of the Poisson summation formula. Then we find

$$\sum_{n \in \mathbb{Z}^2} f(k, \lambda) = \sum_{k \in \mathbb{Z}} \hat{f}(2\pi k, \lambda)$$

$$= \hat{f}(0, \lambda) + \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{f}(2\pi k, \lambda)$$

$$= \int_{\mathbb{R}^2} f(\xi, \lambda) d\xi + O(\lambda^{-N})$$

for any $N$. The latter follows from integration by parts in the Fourier transform and the symbol estimates.

Thus the upshot is that for trace class symbols in the parameter dependent calculus the multiplier trace and the trace in the Hilbert space representation coincide only asymptotically. However, for computing heat and resolvent trace asymptotics this is good enough.

Furthermore, this observation has a far reaching generalization. Namely, the effective implementation of the pseudodifferential calculus amounts to passing from its realization on multipliers to a direct action on projective representation spaces (Heisenberg modules) or on $L^2(A, \varphi_0)$ itself. More concretely, let $\pi : G \to L(\mathcal{H})$ be a projective unitary representation of $G = \mathbb{R}^n \times (\mathbb{R}^n)^\wedge$. For a symbol $f \in S^m(\mathbb{R}^n, A^\infty)$ the assignment $S^m(\mathbb{R}^n, A^\infty) \ni f \mapsto \text{Op}(f) := \int_G f^\vee(y) \pi(y) dy$ represents pseudodifferential multipliers as concrete operators in $\mathcal{H}$.

By exploiting the representation theory of the Heisenberg group we are able to relate the Hilbert space trace of parameter dependent pseudodifferential operators to the trace of the corresponding multiplier acting on $S(\mathbb{R}^n, A^\infty)$. For details see [LeMo16 § 5 and Appendix A].

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