Some Properties of Local Modules

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Abstract. In this paper, we study an important concept namely local module. Some new result and properties have been studied in this note. A module M is local if it has a unique maximal submodule. We prove that when M is projective and finitely generated with largest maximal submodule, this mean M is local.

Keyword: Local module, Indecomposable module, Maximal submodule, Projective module, Local ring.

1. Introduction

From [1], "any module M is called local if it has a largest submodule. Also, the author in [2], "showed every indecomposable module over a generalized uniserial ring is uniserial and so is local". Mekei and Wisbauer [3]; "showed any modules whose finitely generated submodules are finite as a sets". Note that the author in [4], "explained the locally direct summands of module through completely indecomposable module over local ring". In [5], "We found some properties of regular categories and some results". By [6], "defined. Local ring means a ring with unique Maximal ideal". In [7], "a module called uniserial if submodules are totally ordered by inclusion (other terminologies: serial or chain module). Evidently, a valuation ring R is uniserial as a module over itself, and its ring of quotients is likewise a uniserial R-module".

2. Main Results

Here we study all the properties of local module. Also, some new results have been added in this section. But before that we need some basic concepts.

2.1. Definition

Let R be a ring. Then R is called local if it has exactly one maximal ideal such that maximal ideal means; M is called maximal module, if ∃ any ideal J of R ⊆ M ⊆ J ⊆ R, then J=R.

2.2. Example

The best example of local ring is
R = (Z₄, +₄, .₄), because the ideal which generated by element 2 is only maximal ideal of R.

2.3. Remark

There are two ideals which generated by 2 and 3 are maximal ideals of Z₄ So Z₄ cannot be local ring.

Now we can present another definition of local ring by using two concepts the radical of ring and division ring.

2.4. Definition

[7] “Let R be a ring. Then the radical of R is the intersection of all maximal ideals of R”.

\[ \text{Rad} (R) = \cap \{ M \in R : M \text{ maximal ideal} \} \]

2.5. Definition

[7] “Let R be a ring, Then R is called division if it has a unye, R is integral domain and every element in R has inverse in R. Hence R is called local ring if \( \text{Rad}(M) \) is division ring”.

2.6. Definition

[7]. ”An R- module M is called uniserial module if the lattice of submodules is linearly ordered under inclusion and if a module is a direct sum of uniserial modules is called serial module”.

2.7. Theorem

If \( M \text{ Rad}(M) = N_1 \bigoplus N_2 ; \ N_1 , \ N_2 \) minimal submodules, then \( M \) is injective .

Proof: suppose \( M/N_1 \) is injective, hence M can embeds in \( E(M^{N_1}) \text{ Rad}(M) \bigoplus E(M^{N_2}) \text{ Rad}(M) \) such that \( E(M^{N_1}) \) is a composition length or exceptional of \( (N_1) \) so \( E(M^{N_2}) \) is an exceptional of \( N_2 \). But this contradicts and \( M_1; \ M_2 \) any two uniserial R-modules. So \( \text{M}_1\text{Rad}(M)\bigoplus\text{M}_2\text{Rad}(M) \not\supseteq \text{a submodule} \) is local and not uniserial .Thus \( (N_1) \) is injective .

2.8. Lemma

“Let M be a f.g. projective R-module. If M has largest maximal submodule, then M is a local module” [2].

2.9. Definition

“Let M be an R-module. If M has a unique maximal submodule N, then M is called indecomposable” [2].

Moreover M is called in decomposable module if there are summands \( M_1 \) and \( M_2 \) s.t \( M = M_1 \bigoplus M_2 \), \( M_1 \) local and \( M_1 \) does not have any maximal submodule .

2.10. Theorem

Every projective indecomposable module then M is local.

Proof:
Suppose that $M$ is a projective, so $M$ has maximal submodule (i.e $M = MJ(R)$, then for each $k \leq M, N \leq K$, implies that $M = 0$ or $M = K$).

We have $M$ is indecomposable module. Thus $M$ is a local.

2.11. Theorem

If $M$ is a free simple module, then $M$ is local module.

Proof: If $M$ is a free on set $s$.

Let $A$ and $B$ any two modules

If $f: A \rightarrow B$ be any homo.

\forall x \in S, \text{choose } ax \in A \ni j(x) = ax \quad (1)

\forall x \in F, g(x) \in B \text{ and } f: A \rightarrow B \text{ is onto}

\exists x \in F \ni g(x) \in B \quad (2)

\exists a \text{ unique homo } h: F \rightarrow A \ni h = \text{ homo} \quad (3)

T.P $foh = g$

Let $x \in F \ni x = \sum_{k=1}^{n} r_{k} x_{k} \quad (\text{since } F = \langle S \rangle \text{ and } r_{k} \in R)

\forall K = 1, 2, \ldots, n$

$(foh)(x) = f(h(\sum_{k=0}^{n} r_{k} x_{k})) = f(\sum_{k=0}^{n} r_{k} h(x_{k}))$ because $h$ is homo

(by free)

$(fog) = f(\sum_{k=1}^{n} r_{k} f(ax_{k}))$ since $f$ is homo.

\begin{align*}
&= f(\sum_{k=0}^{n} r_{k} f(ax_{k})) \quad (\text{from (2)})
&= \sum_{k=0}^{n} r_{k} f(ax_{k}) \quad (\text{from (1)})
&= \sum_{k=1}^{n} g(ax_{k}) \quad (\text{from (2)})
\end{align*}
\[ g \sum_{k=1}^{n} r_k x_k \sum_{k=1}^{m} r_k x_k \quad \text{since } g \text{ is homo} \]

\[ \text{foh} = g \]

\[ F \text{ is a projective.} \]

Let \( M \) be a simple module. We need to prove that \( M \) is indecomposable. Suppose that \( M \) is decomposable. So \( M = N_1 \oplus N_2 \ni N_1 \) and \( N_2 \) are submodules of \( M \). (That is, \( M = N_1 \oplus N_2 \) and \( N_1 \cap N_2 = \{ 0 \} \)). Thus \( M \) has a non-trivial summands different from \( M \) say \( N_1 \) and \( N_2 \). But \( M \) is simple module. So \( M \) has only submodules \( M \) itself and \( \{0\} \). This contradiction. Hence \( M \) is indecomposable. So \( M \) is local.

### 2.12 Corollary

Every projective \( R \)-module with unique maximal submodule is indecomposable. (local module).

Now we move to the study of another characteristic of local module \( M \), which is done by studying the submodule \( N \) of \( M \) that carries the characteristic finite as a set.

### 2.13 Definition

[3] “If any submodule \( N \) of \( M \) is f. generated, then \( M \) is called locally finite “.

Note that any ring \( R \) has faithful and locally f. module is called locally finite ring.

The following a good example of faithful locally finite modules over \( R \):

### 2.14 Example

\( R = \mathbb{Z}, M = \mathbb{Z}_p \infty \ni p \) is a prime.

### 2.15 Remark

Any ring \( R \) (M module) is called residually finite if any module \( M \) of \( R \) is the sub direct product of finite rings (module).

### 2.16 Theorem

Let \( R \) be a ring. If \( R \) satisfy the Remark (2.15), there any module \( M \) of \( R \) is faithful and locally finite \( R \)-module.

Proof: Suppose that \( R \) satisfy the Remark 2.15. So \( R = \pi R_i \ni R_i \) are finite rings. Let

\[ f_i: R = \pi R_i \rightarrow R_i \text{ be an epimorphism} \quad \forall i \in I. \]

Hence \( I_i \) = kernel of \( f_i \). \( M_i = ( I, +) \ni i \in I \) is comm. group. Note that any \( M_i \) is a finite \( R \)-module and \( M = \oplus M_i \ni i \in I \) is a faithful and locally finite.

In the next theorem we present a relationship between multiplication module over local ring and local module.
2.17. Definition.

Let $R$ be a comm. ring with identity and $M$ a unital $R$-module. Then $M$ is called a multiplication module provided for each submodule $N$ of $M$ there exists an $I$ of $R$ such that $N=IM$. 

2.18. Theorem

Every multiplication module, is cyclic and local.

Proof: It is clear the module $M$ is a multiplication $\frac{R}{\text{Ann}(M)}$-module. Since $\frac{R}{\text{Ann}(M)}$ is a local ring, then $M$ is cyclic $\frac{R}{\text{Ann}(M)}$-module, and so is cyclic $R$-module [8]. Now $M\cong \frac{\text{Ann(M)}}{\text{Ann}(M)}$, $M$ is local.

Recall that $I=(N:M)$ is a prime ideal such that $N$ is a proper of $M$ and any module $M$ is a f. generated and $\dim(M)=0$. By above concepts we can present the following result which explain the relationship between semi-multiplication module and local module.

2.19. Corollary

If $M$ is a semi multiplication module, then $M$ is local cyclic module.

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