Correlation measures of binary sequences derived from Euler quotients

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Abstract

Fermat-Euler quotients arose from the study of the first case of Fermat’s
Last Theorem, and have numerous applications in number theory. Recently
they were studied from the cryptographic aspects by constructing many pseudo-
random binary sequences, whose linear complexities and trace representations
were calculated. In this work, we further study their correlation measures by
using the approach based on Dirichlet characters, Ramanujan sums and Gauss
sums. Our results show that the 4-order correlation measures of these sequences
are very large. Therefore they may not be suggested for cryptography.

Keywords. Euler quotient, binary sequence, correlation measure, character sum

1 Introduction

Let \( S = (s_0, s_1, \ldots) \) be a binary sequence over \( \mathbb{F}_2 = \{0, 1\} \) and \( k \) a positive integer.
Mauduit and Sarkozy [16] introduced the \((N\text{-th})\) correlation measure of order \( k \) for
the first \( N \) terms of \( S \), which is defined as

\[
C_k(S, N) = \max_{U,D} \left| \sum_{n=0}^{U-1} (-1)^{s_{n+d_1}+s_{n+d_2}+\ldots+s_{n+d_k}} \right|
\]

where the maximum is taken over all \( U \leq N - k + 1 \) and \( D = (d_1, d_2, \ldots, d_k) \) with
integers \( 0 \leq d_1 < d_2 < \ldots < d_k \leq N - U. \)
From the viewpoint of cryptography, it is excepted that the $N$-th correlation measure of order $k$ of sequences is as “small” (in terms of $N$, in particular, is $o(N)$ as $N \to \infty$) as possible. It was shown in [4] that for a “truely” random sequence $S$, $C_k(S, N)$ (for some fixed $k$) is around $N^{1/2}$ with “near 1” probability.

Additionally, if $S$ is $T$-periodic (in this case, we will denote $S$ as $S = (s_0, s_1, \ldots, s_{T-1})$), we use the following definition for the periodic correlation measure of order $k$ of $S$,

$$\theta_k(S) = \max_{D} \left| \sum_{n=0}^{T-1} (-1)^{s_{n+d_1} + s_{n+d_2} + \ldots + s_{n+d_k}} \right|,$$

where $D = (d_1, \ldots, d_k)$, with $0 \leq d_1 < d_2 < \ldots < d_k < T$. It is clear that $\theta_2(S)$ is the (classic) auto-correlation of $S$.

In this work, we mainly consider the periodic correlation measure of order $k$ for some binary sequences derived from Euler quotients studied recently.

Let $p$ be a prime and let $n$ be an integer with $\gcd(n, p) = 1$. From the Fermat’s Little Theorem we know that $n^{p-1} \equiv 1 \pmod{p}$. Then the Fermat quotient $Q_p(n)$ is defined as

$$Q_p(n) = \frac{n^{p-1} - 1}{p} \pmod{p}, \quad 0 \leq Q_p(n) < p.$$ 

We also define $Q_p(n) = 0$ if $\gcd(n, p) > 1$. Fermat quotients arose from the study of the first case of Fermat’s Last Theorem, and have many applications in number theory (see [2, 5, 12, 14, 17, 19–21] for details). Define the $p^2$-periodic binary sequence $\mathbf{s} = (s_0, s_1, \ldots, s_{p^2-1})$ by

$$s_t = \begin{cases} 
0, & \text{if } 0 \leq \frac{Q_p(t)}{p} < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq \frac{Q_p(t)}{p} < 1.
\end{cases}$$

The second author (partially with other co-authors) studied the well-distribution measure and correlation measure of order 2 of $\mathbf{s}$ by using estimates for exponential sums of Fermat quotients in [11], the linear complexity of $\mathbf{s}$ in [7, 10], and the trace representation of $\mathbf{s}$ by determining the defining pairs of all binary characteristic sequences of cosets in [6]. In [15] the first author with other co-author showed that the 4-order correlation measure of $\mathbf{s}$ is very large.

Let $m \geq 2$ be an odd number and let $n$ be an integer coprime to $m$. The Euler’s theorem says that $n^{\phi(m)} \equiv 1 \pmod{m}$, where $\phi$ is the Euler’s totient function. Then the Euler quotient $Q_m(n)$ is defined as

$$Q_m(n) = \frac{n^{\phi(m)} - 1}{m} \pmod{m}, \quad 0 \leq Q_m(n) < m.$$ 

We also define $Q_m(n) = 0$ if $\gcd(n, m) > 1$. Agoh, Dilcher and Skula [1] studied the detailed properties of Euler quotients. For example, from Proposition 2.1 of [1] we
have

\[ Q_m(n_1n_2) \equiv Q_m(n_1) + Q_m(n_2) \pmod{m} \quad \text{for} \quad \gcd(n_1n_2, m) = 1, \quad (1) \]

and

\[ Q_m(n + cm) \equiv Q_m(n) + cn^{-1}\phi(m) \pmod{m} \quad \text{for} \quad \gcd(n, m) = 1. \quad (2) \]

Recently many binary sequences were constructed from Euler quotients. For example, let \( m = p^\tau \) for a fixed number \( \tau \geq 1 \), the \( p^{\tau + 1} \)-periodic sequence \( \tilde{s} = (\tilde{s}_0, \tilde{s}_1, \cdots, \tilde{s}_{p^{\tau+1} - 1}) \) is defined by

\[
\tilde{s}_t = \begin{cases} 
0, & \text{if } 0 \leq \frac{Q_{p^\tau}(t)}{p^{\tau}} < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq \frac{Q_{p^\tau}(t)}{p^{\tau}} < 1.
\end{cases}
\quad (3)
\]

The linear complexity of \( \tilde{s} \) had been investigated in [13] and the trace representation of \( \tilde{s} \) was given in [8].

Moreover, let \( m = pq \) be a product of two distinct odd primes \( p \) and \( q \) with \( p \mid (q - 1) \), the \( pq^2 \)-periodic sequence \( \hat{s} = (\hat{s}_0, \hat{s}_1, \cdots, \hat{s}_{pq^2 - 1}) \) is defined by

\[
\hat{s}_t = \begin{cases} 
0, & \text{if } 0 \leq \frac{Q_{pq}(t)}{pq} < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq \frac{Q_{pq}(t)}{pq} < 1.
\end{cases}
\quad (4)
\]

Very recently the minimal polynomials and linear complexities were determined in [22] for \( \hat{s} \), and the trace representation of \( \hat{s} \) has been given in [23] provided that \( 2^{q - 1} \not\equiv 1 \pmod{q^2} \).

In this work, we shall further study the (periodic) correlation measures of \( \tilde{s} \) and \( \hat{s} \) by using the approach based on Dirichlet characters, Ramanujan sums and Gauss sums. We state below the main result.

**Theorem 1.** Let \( m \geq 2 \) be an odd number. Suppose that \( Q_m(n) \) is \( km \)-periodic with \( k > 3 \) and \( k \mid m \). Define the \( km \)-periodic sequence \( s = (s_0, s_1, \cdots, s_{km - 1}) \in \{0, 1\}^{km} \) by

\[
s_t = \begin{cases} 
0, & \text{if } 0 \leq \frac{Q_m(t)}{m} < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq \frac{Q_m(t)}{m} < 1.
\end{cases}
\quad (5)
\]

Then we have

\[
\sum_{t=0}^{km-1} (-1)^{s_t + s_t + m + s_t + 2m + s_t + 3m} = km - \frac{2}{3}k\phi(m) + O\left(\phi(m)\phi(k)^{-1}k^{\frac{5}{2}}\log(m)^4\right).
\]

Taking special values of \( m \) and \( k \) in Theorem 1, we immediately get the correlation measures of \( \tilde{s} \) and \( \hat{s} \).

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Corollary 1. Let $p > 2$ be a prime and let $\tau \geq 1$ be a fixed integer. Let the $p^{\tau+1}$-periodic sequence $\tilde{s} = (\tilde{s}_0, \tilde{s}_1, \cdots, \tilde{s}_{p^{\tau+1} - 1})$ be defined as in (3). Then we have

$$
\sum_{t=0}^{p^{\tau+1} - 1} (-1)^{\tilde{s}_t + \tilde{s}_{t+p^\tau} + \tilde{s}_{t+2p^\tau} + \tilde{s}_{t+3p^\tau}} = \frac{1}{3}p^{\tau+1} + O\left(p^{\tau+\frac{1}{2}}(\log p)^4\right).
$$

Corollary 2. Let $p$ and $q$ be two distinct odd primes with $p \mid (q - 1)$, and let the $pq^2$-periodic sequence $\hat{s} = (\hat{s}_0, \hat{s}_1, \cdots, \hat{s}_{pq^2 - 1})$ be defined as in (4). Then we have

$$
\sum_{t=0}^{pq^2 - 1} (-1)^{\hat{s}_t + \hat{s}_{t+pq} + \hat{s}_{t+2pq} + \hat{s}_{t+3pq}} = \frac{1}{3}pq^2 + \frac{2}{3}q^2 + O\left(pq^{\frac{3}{2}}(\log pq)^4\right).
$$

Our results indicate that the correlation measures of order 4 of $\tilde{s}$ and $\hat{s}$ are very large. Therefore these sequences may not be suggested for cryptography.

To prove Theorem 1, we introduce basic properties of Dirichlet characters, Ramanujan sums and Gauss sums, and then prove two lemmas on the mean values of character sums in Sect. 2. We express $(-1)^{s_t}$ in terms of character sums in Sect. 3 to finish the proof of Theorem 1 by using the results showed in Sect. 2.

## 2 Dirichlet characters and Gauss sums

Let $N > 1$ be an integer. The Ramanujan sum is denoted by

$$
c_N(n) = \sum_{\substack{t=0 \atop \text{gcd}(t,N)=1}}^{N-1} e_N(tn),
$$

where $e_N(x) = e^{2\pi \sqrt{-1}x/N}$. We have

$$
c_N(n) = \mu\left(\frac{N}{\text{gcd}(n,N)}\right) \phi(N)^{-1} \left(\frac{N}{\text{gcd}(n,N)}\right),
$$

where $\mu$ is the Möbius function.

If the function $\chi$ satisfies the following conditions:

(i). $\chi(n_1n_2) = \chi(n_1)\chi(n_2)$ for all integers $n_1, n_2$,

(ii). $\chi(n + N) = \chi(n)$ for all integers $n$,

(iii). $\chi(n) = 0$ for $\text{gcd}(n,N) > 1$,

then $\chi$ is one of the Dirichlet characters modulo $N$. When $\chi(n) = 1$ for all $n$ with $\text{gcd}(n,N) = 1$ we say $\chi$ is the trivial character modulo $N$. The integer $d$ is called an induced modulus for $\chi$ if $\chi(a) = 1$ whenever $\text{gcd}(a,N) = 1$ and $a \equiv 1 \pmod d$. 


A Dirichlet character \( \chi \mod N \) is said to be primitive mod \( N \) if it has no induced modulus \( d < N \). The smallest induced modulus \( d \) for \( \chi \) is called the conductor of \( \chi \). Every non-trivial character \( \chi \) modulo \( N \) can be uniquely written as \( \chi = \chi_0 \chi^* \), where \( \chi_0 \) is the trivial character modulo \( N \) and \( \chi^* \) is the primitive character modulo the conductor of \( \chi \).

For a Dirichlet character \( \chi \mod N \), the Gauss sum associated with \( \chi \) is defined by

\[
G(n, \chi) = \sum_{t=0}^{N-1} \chi(t)e_N(tn).
\]

Let \( N^* \) be the conductor for \( \chi \) and let \( \chi^* \) be the induced primitive character. Let \( N_1 \) be the maximal divisor of \( N \) such that \( N_1 \) and \( N^* \) have the same prime divisors. Then we have

\[
G(n, \chi) = \begin{cases} 
(\chi^*)^{-1} \left( \frac{n}{\gcd(n,N)} \right) \chi^* \left( \frac{N}{N^* \gcd(n,N)} \right) \mu \left( \frac{N}{N^* \gcd(n,N)} \right) \\
\times \phi(N) \phi^{-1} \left( \frac{N}{\gcd(n,N)} \right) G(1, \chi^*), & \text{if } N^* = \frac{N_1}{\gcd(n,N_1)} \\
0, & \text{if } N^* \neq \frac{N_1}{\gcd(n,N_1)}.
\end{cases}
\]  

(7)

See Chapter 8 of [3] or Chapter 1 of [18] for more details of Dirichlet characters, Ramanujan sums and Gauss sums.

Now we prove two lemmas on the mean values of characters sums.

**Lemma 1.** Let \( m \geq 2 \) be an odd number and let \( k > 3 \) be an integer with \( k \mid m \). Let \( \chi \) be a Dirichlet character modulo \( km \) with \( \chi^m \) being trivial. For integers \( a_1, a_2, a_3 \) and \( a_4 \) we have

\[
\sum_{t=0}^{km-1} \chi(t) = \begin{cases} 
k \phi(m), & \text{if } m \mid (a_1 + a_2 + a_3 + a_4) \text{ and } k \mid (a_2 + 2a_3 + 3a_4), \\
O \left( \phi(m) \phi(k)^{-1}k^2 \right), & \text{otherwise.}
\end{cases}
\]
Proof. By the properties of residue systems we get
\[
\sum_{t=0}^{km-1} \chi(t^{a_1}(t + m)^{a_2}(t + 2m)^{a_3}(t + 3m)^{a_4})
\]
\[
= \sum_{y=0}^{m-1} \sum_{z=0}^{k-1} \chi((y + zm)^{a_1}(y + zm + m)^{a_2}(y + zm + 2m)^{a_3}(y + zm + 3m)^{a_4})
\]
\[
= \sum_{y=0}^{m-1} \sum_{z=0}^{k-1} \chi((y^{a_1} + a_1 y^{a_1-1}zm)(y^{a_2} + a_2 y^{a_2-1}(z + 1)m))
\]
\[
\times \chi((y^{a_3} + a_3 y^{a_3-1}(z + 2)m)(y^{a_4} + a_4 y^{a_4-1}(z + 3)m))
\]
\[
= \sum_{y=0}^{m-1} \chi(y^{a_1+a_2+a_3+a_4})
\]
\[
\times \sum_{z=0}^{k-1} \chi((1 + a_1 y^{-1}zm)(1 + a_2 y^{-1}(z + 1)m)(1 + a_3 y^{-1}(z + 2)m)(1 + a_4 y^{-1}(z + 3)m)).
\]
Clearly \(\chi(1 + nm)\) is a primitive additive character modulo \(k\), so there is uniquely an integer \(\beta\) such that \(1 \leq \beta \leq k\), \(\gcd(\beta, k) = 1\) and \(\chi(1 + nm) = e_k(\beta n)\). Hence,
\[
\sum_{t=0}^{km-1} \chi(t^{a_1}(t + m)^{a_2}(t + 2m)^{a_3}(t + 3m)^{a_4})
\]
\[
= \sum_{y=0}^{m-1} \chi(y^{a_1+a_2+a_3+a_4})
\]
\[
\times \sum_{z=0}^{k-1} e_k(\beta(a_1 y^{-1}z + a_2 y^{-1}(z + 1) + a_3 y^{-1}(z + 2) + a_4 y^{-1}(z + 3)))
\]
\[
= \begin{cases} 
  k \sum_{y=0}^{m-1} \chi^{-(a_1+a_2+a_3+a_4)}(y) e_k(\beta(a_2 + 2a_3 + 3a_4)y), & \text{if } k \mid (a_1 + a_2 + a_3 + a_4), \\
  0, & \text{if } k \nmid (a_1 + a_2 + a_3 + a_4).
\end{cases}
\]

We know that \(\chi^{a_1+a_2+a_3+a_4}\) is a multiplicative character modulo \(m\) if \(k \mid a_1 + a_2 + a_3 + a_4\). Then from (6), (7) and the properties of Ramanujan sums and Gauss sums
we get
\[ \sum_{t=0}^{km-1} \chi(t^{a_1}(t+m)^{a_2}(t+2m)^{a_3}(t+3m)^{a_4}) \]
\[ = \begin{cases} k\phi(m), & \text{if } m \mid (a_1 + a_2 + a_3 + a_4) \text{ and } k \mid (a_2 + 2a_3 + 3a_4), \\ O\left(\phi(m)\phi(k)^{-1}k^{2}\right), & \text{otherwise}. \end{cases} \]

\[ \square \]

**Lemma 2.** Let \( m \geq 2 \) be an odd number and let \( k > 3 \) be an integer with \( k \mid m \).
Define
\[ \Xi_{m,k} := \sum_{1 \leq |a_1|,|a_2|,|a_3|,|a_4| \leq \frac{m-1}{2}} \sum_{l_1=m+1}^{m-1} e_m(-a_1 l_1) \sum_{l_2=m+1}^{m-1} e_m(-a_2 l_2) \times \sum_{l_3=m+1}^{m-1} e_m(-a_3 l_3) \sum_{l_4=m+1}^{m-1} e_m(-a_4 l_4). \]

Then we have
\[ \Xi_{m,k} = \frac{1}{48} m^4 + O\left(\frac{m^4(\log m)^3}{k}\right). \]

**Proof.** For absolute constant \( c > 0 \) we get
\[ \sum_{ck \leq |a_1| \leq \frac{m-1}{2}} \sum_{1 \leq |a_2|,|a_3|,|a_4| \leq \frac{m-1}{2}} \sum_{l_1=m+1}^{m-1} e_m(-a_1 l_1) \sum_{l_2=m+1}^{m-1} e_m(-a_2 l_2) \times \sum_{l_3=m+1}^{m-1} e_m(-a_3 l_3) \sum_{l_4=m+1}^{m-1} e_m(-a_4 l_4) \]
\[ \ll \sum_{1 \leq |a_2| \leq \frac{m-1}{2}} \sum_{1 \leq |a_3| \leq \frac{m-1}{2}} \sum_{|a_4| \leq \frac{m-1}{2}} \sum_{ck \leq |a_1| \leq \frac{m-1}{2}} \frac{m}{k} \]
\[ \ll \frac{m^4(\log m)^3}{k}. \]
Hence,

\[
\Xi_{m,k} := \sum_{1 \leq |a_1|, |a_2| \leq \frac{k}{32}} \sum_{1 \leq |a_3|, |a_4| \leq \frac{k}{32}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} e_m (-a_1 l_1) \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m (-a_2 l_2) \times \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m (-a_3 l_3) \sum_{l_4 = \frac{m+1}{2}}^{m-1} e_m (-a_4 l_4) + O \left( \frac{m^4 (\log m)^3}{k} \right)
\]

\[
= \sum_{1 \leq |a_1|, |a_2| \leq \frac{k}{32}} \sum_{1 \leq |a_3|, |a_4| \leq \frac{k}{32}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} e_m (-a_1 l_1) \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m ((2a_3 + 3a_4) l_2) \times \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m (-a_3 l_3) \sum_{l_4 = \frac{m+1}{2}}^{m-1} e_m (-a_4 l_4) + O \left( \frac{m^4 (\log m)^3}{k} \right).
\]

It is not hard to show that

\[
\sum_{\frac{k}{32} < |a_3| \leq \frac{m-1}{2}} \sum_{1 \leq |a_4| \leq \frac{k}{32}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} e_m (-a_3 + 2a_4) l_1 \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m ((2a_3 + 3a_4) l_2) \times \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m (-a_3 l_3) \sum_{l_4 = \frac{m+1}{2}}^{m-1} e_m (-a_4 l_4)
\]

\[
\ll \sum_{\frac{k}{32} < |a_3| \leq \frac{m-1}{2}} \sum_{1 \leq |a_4| \leq \frac{k}{32}} m \cdot \left| \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m ((2a_3 + 3a_4) l_2) \right| \cdot \frac{m}{k} \cdot \frac{m}{|a_4|}
\]

\[
\ll \frac{m^3}{k} \sum_{1 \leq |a_4| \leq \frac{k}{32}} \sum_{\frac{k}{32} < |a_3| \leq \frac{m-1}{2}} \left| \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m ((2a_3 + 3a_4) l_2) \right|
\]

\[
\ll \frac{m^4 (\log m)^2}{k}.
\]
Therefore

\[
\Xi_{m,k} = \sum_{1 \leq |a_3|, |a_4| \leq \frac{m-1}{2}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} e_m \left( -(a_3 + 2a_4)l_1 \right) \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m \left( (2a_3 + 3a_4)l_2 \right)
\times \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m \left( -a_3l_3 \right) \sum_{l_4 = \frac{m+1}{2}}^{m-1} e_m \left( -a_4l_4 \right) + O \left( \frac{m^4(\log m)^3}{k} \right)
\]

\[
= \sum_{\frac{m+1}{2} \leq l_1, l_2, l_3, l_4 \leq m-1} \sum_{|a_3| \leq \frac{m-1}{2}} e_m \left( -(l_1 + 2l_2 - l_3)a_3 \right)
\times \sum_{1 \leq |a_4| \leq \frac{m-1}{2}} e_m \left( -(2l_1 + 3l_2 - l_4)a_4 \right)
\]

\[
-\sum_{\frac{m+1}{2} \leq l_1, l_2, l_3, l_4 \leq m-1} \sum_{|a_3| \leq \frac{m-1}{2}} e_m \left( -(l_1 + 2l_2 - l_3)a_3 \right)
\]

\[
-\sum_{\frac{m+1}{2} \leq l_1, l_2, l_3, l_4 \leq m-1} \sum_{|a_4| \leq \frac{m-1}{2}} e_m \left( -(2l_1 + 3l_2 - l_4)a_4 \right)
\]

\[
+\sum_{\frac{m+1}{2} \leq l_1, l_2, l_3, l_4 \leq m-1} 1 + O \left( \frac{m^4(\log m)^3}{k} \right)
\]

\[
= m^2 \sum_{\frac{m+1}{2} \leq l_1, l_2, l_3, l_4 \leq m-1} 1 - \frac{m(m-1)}{2} \sum_{\frac{m+1}{2} \leq l_1, l_2, l_3 \leq m-1} 1
\times \sum_{2l_2 \equiv l_1 + l_3 \, (\text{mod} \, m)} 1 + \frac{(m-1)^4}{16}
\]

\[
+O \left( \frac{m^4(\log m)^3}{k} \right).
\]

By using the idea of Lemma 2.2 of [15] we have

\[
\sum_{\frac{m+1}{2} \leq l_1, l_2, l_4 \leq m-1} 1 = \frac{m^2}{8} + O(m),
\]

(9)
\[ \sum_{m+1 \leq l_1, l_2, l_4 \leq m-1 \atop 3l_2 \equiv 2l_1 + l_4 \pmod{m}} 1 = \frac{m^2}{8} + O(m) \tag{10} \]

and

\[ \sum_{m+1 \leq l_1, l_2, l_3, l_4 \leq m-1 \atop 2l_2 \equiv l_1 + l_3 \pmod{m} \atop 3l_2 \equiv 2l_1 + l_4 \pmod{m}} 1 = \frac{m^2}{12} + O(m). \tag{11} \]

Combining (8)-(11) we immediately get

\[
\Xi_{m,k} = m^2 \left( \frac{m^2}{12} + O(m) \right) - 2 \cdot \frac{m(m-1)}{2} \left( \frac{m^2}{8} + O(m) \right) + \frac{(m-1)^4}{16} + O \left( \frac{m^4 (\log m)^3}{k} \right)
\]

This completes the proof of Lemma 2. \qed

3 Correlation measures of order 4

Now we prove Theorem 1. By the orthogonality relations of additive character sums we get

\[
s_t = \frac{1}{m} \sum_{|a| \leq \frac{m-1}{2}} \sum_{l=\frac{m+1}{2}}^{m-1} e_m (a(Q_m(t) - l)).
\]

Hence,

\[
(-1)^{s_t} = 1 - 2s_t = -\frac{2}{m} \sum_{1 \leq |a| \leq \frac{m-1}{2}} \sum_{l=\frac{m+1}{2}}^{m-1} e_m (-al) e_m (aQ_m(t)) + \frac{1}{m}.
\]

Define

\[
\chi_{km}(n) = \begin{cases} e_m (Q_m(n)), & \text{if } \gcd(n, m) = 1, \\ 0, & \text{if } \gcd(n, m) > 1. \end{cases}
\]

Clearly \( \chi_{km}(n + km) = \chi_{km}(n) \), and by (1) we have \( \chi_{km}(n_1n_2) = \chi_{km}(n_1)\chi_{km}(n_2) \). Then \( \chi_{km}(n) \) is a Dirichlet character modulo \( km \) with \( \chi_{km}^m \) being trivial. Therefore

\[
(-1)^{s_t} = -\frac{2}{m} \sum_{1 \leq |a| \leq \frac{m-1}{2}} \sum_{l=\frac{m+1}{2}}^{m-1} e_m (-al) \chi_{km} (n^t) + \frac{1}{m}. \tag{12}
\]
By (12), Lemmas 1 and 2 we get
\[
\sum_{t=0}^{km-1} (-1)^{s_1 + s_1 + m + s_1 + 2m + s_1 + 3m} = \sum_{t=0}^{km-1} (-1)^{s_1 + s_1 + m + s_1 + 2m + s_1 + 3m} + \sum_{t=0}^{km-1} 1
\]
\[
= \frac{2^4}{m^4} \sum_{1 \leq |a_1| \leq m-1 \overline{1}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} \sum_{1 \leq |a_2| \leq m-1 \overline{1}} \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m(-a_1 l_1) e_m(-a_2 l_2) \times \sum_{1 \leq |a_3| \leq m-1 \overline{1}} \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m(-a_3 l_3) e_m(-a_4 l_4) \times \sum_{t=0}^{km-1} \chi_{km}(t^{a_1}(t + m)^{a_2}(t + 2m)^{a_3}(t + 3m)^{a_4}) + \sum_{t=0}^{km-1} 1 + O(k \log m)^3
\]
\[
= \frac{2^4k_\phi(m)}{m^4} \sum_{1 \leq |a_1|, |a_2|, |a_3|, |a_4| \leq \frac{m-1}{2}} \sum_{l_1 = \frac{m+1}{2}}^{m-1} \sum_{l_2 = \frac{m+1}{2}}^{m-1} e_m(-a_1 l_1) e_m(-a_2 l_2) \times \sum_{l_3 = \frac{m+1}{2}}^{m-1} e_m(-a_3 l_3) e_m(-a_4 l_4) \times k(m - \phi(m)) + O(\phi(m)\phi(k)^{-1}k^{\frac{3}{2}}(\log m)^4) + O(k(\log m)^3)
\]
\[
= km - \frac{2}{3} k_\phi(m) + O(\phi(m)\phi(k)^{-1}k^{\frac{3}{2}}(\log m)^4) .
\] (13)

This proves Theorem 1.

4 Final remarks

In this work, we have claimed that two families of binary sequences (see (3) and (4)) studied in the past several years have ‘large’ values on the correlation measures of order 4. They would be very vulnerable if used in cryptography.

It seems interesting to consider the case when the full peaks on the periodic correlation measure of these sequences appear, i.e., their periodic correlation measure
of order \( k \) equals to the period, see [9]. Such problem may be related to their linear complexity.

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