MARKOV LOOPS AND RENORMALIZATION

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Abstract

We study the Poissonnian ensembles of Markov loops and the associated renormalized self intersection local times.

1 Introduction

The purpose of this paper is to explore some simple relations between Markovian path and loop measures, spanning trees, determinants, and Markov fields such as the free field. The main emphasis is put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [9] for planar Brownian motion in relation with SLE processes and in [10] for simple random walks. They appeared informally already in [24]. For half integral values $\frac{k}{2}$ of the intensity parameter $\alpha$, these occupation fields can be identified with the sum of squares of $k$ copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin’s isomorphism (cf [2], [17], [13]). We first present the results in the elementary framework of symmetric Markov chains on a finite space, proving also en passant several interesting results such as the relation between loop ensembles and spanning trees. Then we show some results can be extended to more general Markov processes. There are no
essential difficulties when points are not polar but other cases are more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We can show that the renormalised powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the two dimensional case and that they can be identified with higher even Wick powers of the free field when \( \alpha \) is a half integer.

\section{Symmetric Markov processes on finite spaces}

Notations: Functions and measures on finite (or countable) spaces are often denoted as vectors and covectors.

The multiplication operator defined by a function \( f \) acting on functions or on measures is in general simply denoted by \( f \), but sometimes it will be denoted \( M_f \). The function obtained as the density of a measure \( \mu \) with respect to some other measure \( \nu \) is simply denoted \( \frac{\mu}{\nu} \).

Our basic object will be a finite space \( X \) and a set of non negative conductances \( C_{x,y} = C_{y,x} \), indexed by pairs of distinct points of \( X \).

We say \( \{x, y\} \) is a link or an edge iff \( C_{x,y} > 0 \) and an oriented edge \( (x, y) \) is defined by the choice of an ordering in an edge. We set \(- (x, y) = (y, x)\) and if \( e = (x, y) \), we denote it also \((e^-, e^+)\).

The points of \( X \) together with the set of non oriented edges \( E \) define a graph \( (X, E) \). We \emph{assume it is connected}. The set of oriented edges is denoted \( E^o \).

An important example is the case in which conductances are equal to zero or one. Then the conductance matrix is the adjacency matrix of the graph: \( C_{x,y} = 1_{\{x,y\} \in E} \).

\subsection{Energy}

Let us consider a nonnegative function \( \kappa \) on \( X \). Set \( \lambda_x = \kappa_x + \sum_y C_{x,y} P^x_y = \frac{C_{x,x}}{\lambda_x} \). \( P \) is a \( \lambda \)-symmetric (sub) stochastic transition matrix: \( \lambda_x P^x_y = \lambda_y P^y_x \) with \( P^x_x = 0 \) for all \( x \) in \( X \) and it defines a symmetric irreducible Markov chain \( \xi_n \).

We can define above it a \( \lambda \)-symmetric irreducible Markov chain in continuous time \( x_t \), with exponential holding times, of parameter 1. We have \( x_t = \xi_{N_t} \), where \( N_t \) denotes a
Poisson process of intensity 1. The infinitesimal generator writes \( L^y_y = P^x_y - \delta^x_y \).

We denote by \( P_t \) its (sub) Markovian semigroup \( \exp(Lt) = \sum \frac{t^k}{k!} L^k \). \( L \) and \( P_t \) are \( \lambda \)-symmetric.

We will consider the Markov chain associated with \( C, \kappa \), sometimes in discrete time, sometimes in continuous time (with exponential holding times).

Recall that for any complex function \( z^x, x \in X \), the "energy"

\[
e(z) = \langle -Lz, z \rangle_\lambda = \sum_{x \in X} -(Lz)^x \lambda^x
\]

is nonnegative as it can be written

\[
e(z) = \frac{1}{2} \sum_{x,y} C_{x,y}(z^x - z^y)(\overline{z}^x - \overline{z}^y) + \sum_x \kappa^x z^x \overline{z}^x = \sum_x \lambda^x z^x \overline{z}^x - \sum_{x,y} C_{x,y} z^x \overline{z}^y
\]

The Dirichlet space ([4]) is the space of real functions equipped with the energy scalar product defined by polarization of \( e \).

Note that the non-negative symmetric "conductance matrix" \( C \) and the non-negative equilibrium or "killing" (or "equilibrium") measure \( \kappa \) are the free parameters of the model.

We have a dichotomy between:

- the recurrent case where 0 is the lowest eigenvalue of \(-L\), and the corresponding eigenspace is formed by constants. Equivalently, \( P1 = 1 \) and \( \kappa \) vanishes.

- the transient case where the lowest eigenvalue is positive which means there is a "Poincaré inequality": For some positive \( \varepsilon \), the energy \( e(f, f) \) dominates \( \varepsilon \langle f, f \rangle_\lambda \) for all \( f \). Equivalently, \( \kappa \) does not vanish.

We will now work in the transient case. We denote by \( V \) the associated potential operator \( (\lambda) = \int_0^\infty P_t dt \). It can be expressed in terms of the spectral resolution of \( L \).

We denote by \( G \) the Green function defined on \( X^2 \) as \( G^{x,y} = \frac{V^x_y}{\lambda^y} = \frac{1}{\lambda^y}[(I - P)^{-1}]^x_y \)

i.e. \( G = (M - C)^{-1} \). It induces a linear bijection from measures into functions. We set

\[
(G\mu)^x = \sum_y G^{x,y} \mu_y
\]

Note that \( e(f, G\mu) = \langle f, \mu \rangle \) (i.e. \( \sum_x f^x \mu_x \)) for all function \( f \) and measure \( \mu \). In particular \( G\kappa = 1 \) as \( e(1, f) = \sum f^x \kappa_x = \langle f, 1 \rangle_\kappa \).

See ([4]) for a development of this theory in a more general setting.
In the recurrent case, the potential operator $V$ operates on the space $\lambda^\perp$ of functions $f$ such that $\langle f, 1 \rangle_\lambda = 0$ as the inverse of the restriction of $I - P$ to $\lambda^\perp$. The Green operator $G$ maps the space of measures of total charge zero onto $\lambda^\perp$. Setting for any signed measure $\nu$ of total charge zero $G\nu = V\nu_{\lambda^\perp}$, we have for any function $f$, $\langle \nu, f \rangle = e(G\nu, f)$ (as $e(G\nu, 1) = 0$) and in particular $f^x - f^y = e(G(\delta_x - \delta_y), f)$.

2.2 Feynman-Kac formula

For the continuous time Markov chain $x_t$ (with exponential holding times) and $k(x)$ any non negative function, we have the Feynman Kac formula:

$$E_x(e^{-\int_0^t k(x_s)ds}1_{(x_t=y)}) = [\exp(t(L - M_k))]_{xy}.$$

For any nonnegative measure $\chi$, set $V\chi = (-L + M_\chi)^{-1}$ and $G\chi = V\chi M_\chi = (M_\chi + M_\chi - C)^{-1}$. It is a symmetric nonnegative function on $X \times X$. $G_0$ is the Green function $G$, and $G\chi$ can be viewed as the Green function of the energy form $e_\chi = e + \| \|_{L^2(\chi)}^2$.

Note that $e_\chi$ has the same conductances $C$ as $e$, but $\chi$ is added to the killing measure. Note also that $V\chi$ is not the potential of the Markov chain associated with $e_\chi$ when one takes exponential holding times of parameter 1 but the Green function is intrinsic i.e. invariant under a change of time scale. Still, we have by Feynman Kac formula

$$\int_0^\infty E_x(e^{-\int_0^t \chi(x_s)ds}1_{(x_t=y)})dt = [V\chi]_{xy}.$$

We have also the "resolvent" equation $V - V\chi = VM_\chi V = V_\chi M_\chi V$. Then,

$$G - G\chi = GM_\chi G\chi = G_\chi M_\chi G.$$

2.3 Countable spaces

The assumption of finiteness of $X$ can be relaxed. On countable spaces, the previous results extend easily when under spectral gap conditions. In the transient case, the Dirichlet space $\mathbb{H}$ is the space of all functions $f$ with finite energy $e(f)$ which are limits in energy norm of functions with finite support. The energy of a measure is defined as $\sup_{f \in \mathbb{H}} \frac{\mu(f)^2}{e(f)}$. It includes Dirac measures. The potential $G\mu$ is well defined in $\mathbb{H}$ for all finite energy measures $\mu$, by the identity $e(f, G\mu) = \langle f, \mu \rangle$, valid for all $f$ in the Dirichlet space.

Most important cases are the non ramified covering of finite graphs.
3  Loop measures

3.1  A measure on based loops

We denote $\mathbb{P}^x$ the family of probability laws on piecewise constant paths defined by $P_t$.

$$\mathbb{P}^x(\gamma(t_1) = x_1, \ldots, \gamma(t_h) = x_h) = P_t(x, x_1) P_{t_2-t_1}(x_1, x_2) \ldots P_{t_h-t_{h-1}}(x_{h-1}, x_h)$$

Denoting by $p(\gamma)$ the number of jumps and $T_i$ the jump times, we have:

$$\mathbb{P}_x(p(\gamma) = k, \gamma_{T_1} = x_1, \ldots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \ldots, T_k \in dt_k) = \frac{C_{x_1x_2} \ldots C_{x_{k-1}x_k} \kappa_{x_k}}{\lambda_{x_2} \lambda_{x_3} \ldots \lambda_{x_k}} 1_{\{0 < t_1 < \ldots < t_h\}} e^{-\int \sum dt_1 \ldots dt_k}$$

For any integer $p > 2$, let us define a based loop with $p$ points in $X$ as a couple $l = (\xi, \tau) = (\xi_1, x \ldots \xi_{p+1})$ in $X^p \times \mathbb{R}^{p+1}$, and set $\overline{\xi} = \xi_{p+1}$ (equivalently, we can parametrize the the discrete based loop by $\mathbb{Z}/p\mathbb{Z}$). The integer $p$ represents the number of points in the discrete based loop $\xi = (\xi_1, \ldots, \xi_{p+1})$ and will be denoted $p(\xi)$. Note two time parameters are attached to the base point since the based loops do not in general end or start with a jump.

Based loops with one point ($p = 1$) are simply given by a pair $(\xi, \tau)$ in $X \times \mathbb{R}_+$. Based loops have a natural time parametrization $l(t)$ and a time period $T(\xi) = \sum_{i=1}^{p(\xi)+1} \tau_i$. If we denote $\sum_{i=1}^m \tau_i$ by $T_m$: $l(t) = \xi_m$ on $[T_{m-1}, T_m)$ (with by convention $T_0 = 0$ and $\xi_0 = \xi_p$).

A $\sigma$-finite measure $\mu$ is defined on based loops by

$$\mu = \sum_{x \in X} \int_0^\infty \frac{1}{t} \mathbb{P}^x \lambda_x dt$$

where $\mathbb{P}^x_t$ denotes the (non normalized) ”law” of a path from $x$ to $y$ of duration $t$: If $t_1 < t_2 \ldots < t_h < t$,

$$\mathbb{P}^x_t(l(t_1) = x_1, \ldots, l(t_h) = x_h) = [P_t]_{x_1} \ldots [P_{t-h}]_{x_h} \frac{1}{\lambda_y}$$

Its mass is $p^x_t = \frac{[P_t]_{\overline{\xi}}}{\lambda_y}$. And for any measurable set $A$ of piecewise constant paths indexed by $[0, t]$, we can also write

$$\mathbb{P}^x_t(A) = \mathbb{P}_x(A \cap \{x_t = y\}) \frac{1}{\lambda_y}$$
From the first expression, we see that by definition of \( \mu \), if \( t_1 < t_2 < \ldots < t_h < t \),
\[
\mu(l(t_1)) = x_1, \ldots, l(t_h) = x_h, T \in dt = \left[ P_{t_1+t-t_h} x_1 \right] x_1 \ldots \left[ P_{t_h-t_{h-1}} x_h \right] x_h - \frac{1}{t} dt \tag{1}
\]

Note also that for \( k > 1 \), using the second expression of \( P_t \) and the fact that conditionally to \( N_t = k \), the jump times are distributed like an increasingly reordered \( k \)-uniform sample of \([0, t]\)
\[
\lambda_t \mathbb{P}_t (p = k, \xi_2 = x_2, \ldots, \xi_k = x_k, T_1 \in dt_1, \ldots, T_k \in dt_k) = P_x \mathbb{P}_x (p = k, \xi_2 = x_2, \ldots, \xi_k = x_k, T_1 \in dt_1, \ldots, T_k \in dt_k) e^{-t dt_1 \ldots dt_k}
\]

Therefore
\[
\mu(p = k, \xi_1 = x_1, \ldots, \xi_k = x_k, T_1 \in dt_1, \ldots, T_k \in dt_k, T \in dt) \tag{2}
\]
\[
= P_{x_2} \mathbb{P}_{x_2} (p = k, \xi_2 = x_2, \ldots, \xi_k = x_k, T_1 \in dt_1, \ldots, T_k \in dt_k) e^{-t dt_1 \ldots dt_k} \tag{3}
\]

for \( k > 1 \).
Moreover, for one point-loops, \( \mu\{p(\xi) = 1, \xi_1 = x_1, \tau_1 \in dt\} = \frac{e^{-t}}{t} dt \)

### 3.2 First properties

Note that the loop measure is invariant under time reversal.

If \( D \) is a subset of \( X \), the restriction of \( \mu \) to loops contained in \( D \), denoted \( \mu^D \) is clearly the loop measure induced by the Markov chain killed at the exit of \( D \). This can be called the restriction property.

Let us recall that this killed Markov chain is defined by the restriction of \( \lambda \) to \( D \) and the restriction \( P^D \) of \( P \) to \( D^2 \) (or equivalently by the restriction \( e_D \) of the Dirichlet norm \( e \) to functions vanishing outside \( D \)).

As \( \int \frac{t^{k-1}}{k!} e^{-t} dt = \frac{1}{k} \), it follows from (2) that for \( k > 1 \), on based loops,
\[
\mu(p(\xi) = k, \xi_1 = x_1, \ldots, \xi_k = x_k) = \frac{1}{k} P_{x_2}^{x_1} \ldots P_{x_k}^{x_k} \tag{4}
\]

In particular, we obtain that, for \( k \geq 2 \)
\[
\mu(p = k) = \frac{1}{k} \text{Tr}(P^k)
\]
and therefore, as $Tr(P) = 0$,

$$
\mu(p > 1) = \sum_{2}^{\infty} \frac{1}{k} Tr(P^k) = -\log(\det(I - P)) = \log(\det(G) \prod_{x} \lambda_{x})
$$

since (denoting $M_{\lambda}$ the diagonal matrix with entries $\lambda_{x}$), we have

$$
\det(I - P) = \frac{\det(M_{\lambda} - C)}{\det(M_{\lambda})}
$$

Moreover

$$
\int p(l)1_{\{p > 1\}} \mu(dl) = \sum_{2}^{\infty} Tr(P^k) = Tr((I - P)^{-1}P) = Tr(GC)
$$

### 3.3 Loops and pointed loops

It is clear on formula 1 that $\mu$ is invariant under the time shift that acts naturally on based loops.

A loop is defined as an equivalence class of based loops for this shift. Therefore, $\mu$ induces a measure on loops also denoted by $\mu$.

A loop is defined by the discrete loop $\xi^\circ$ formed by the $\xi_i$ in circular order, (i.e. up to translation) and the associated scaled holding times. We clearly have:

$$
\mu(\xi^\circ = (x_1, x_2, ..., x_k)^\circ) = P_{x_2}^{x_1}...P_{x_k}^{x_1}
$$

However, loops are not easy to parametrize, that is why we will work mostly with based loops or pointed loops. These are defined as based loops ending with a jump, or as loops with a starting point. They can be parametrized by a based discrete loop and by the holding times at each point. Calculations are easier if we work with based or pointed loops, even though we will deal only with functions independent of the base point.

The parameters of the pointed loop naturally associated with a based loop are $\xi_1, ..., \xi_p$ and

$$
\tau_1 + \tau_{p+1} = \tau_1^*, \tau_i = \tau_i^*, \ 2 \leq i \leq p
$$

An elementary change of variables, shows the expression of $\mu$ on pointed loops writes:
\[
\mu(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = P_{x_1}^{x_1} P_{x_2}^{x_k} \sum_{t_i} e^{-\sum t_i dt_1...dt_k} dt_i
\]

(5)

Trivial \( (p = 1) \) pointed loops and trivial based loops coincide.

Note that loop functionals can be written
\[
\Phi(l) = \sum_1^1 \{p = k\} \Phi_k((\xi_i, \tau_i^*), i = 1, ...k)
\]
with \( \Phi_k \) invariant under circular permutation of the variables \((\xi_i, \tau_i^*)\).

Then, for non negative \( \Phi_k \)
\[
\int \Phi_k(l^o) \mu(dl) = \int \sum_{x_i} \Phi_k(x_i, t_i) P_{x_1}^{x_1} P_{x_2}^{x_k} e^{-\sum t_i} \sum_{t_i} dt_1...dt_k
\]
and by invariance under circular permutation, the term \( t_1 \) can be replaced by any \( t_i \).

Therefore, adding up and dividing by \( k \), we get that
\[
\int \Phi_k(l^o) \mu(dl) = \int \frac{1}{k} \sum_{x_i} \Phi_k(x_i, t_i) P_{x_1}^{x_1} P_{x_2}^{x_k} e^{-\sum t_i} dt_1...dt_k
\]

The expression on the right side, applied to any pointed loop functional defines a different measure on pointed loops, we will denote by \( \mu^* \). It induces the same measure as \( \mu \) on loops.

We see on this expression that conditionally to the discrete loop, the holding times of the loop are independent exponential variables.

\[
\mu^*(p = k, \xi_i = x_i, \tau_i^* \in dt_i) = \frac{1}{k} \prod_{i\in\mathbb{Z}/p\mathbb{Z}} C_{\xi_i, \xi_{i+1}} e^{-t_i} dt_i
\]

(6)

Conditionally to \( p(\xi) = k \), \( T \) is a gamma variable of density \( \frac{k-1}{(k-1)!} e^{-t} \) on \( \mathbb{R}_+ \) and \((\tilde{T}, 1 \leq i \leq k) \) an independent ordered \( k \)-sample of the uniform distribution on \((0, T)\) (whence the factor \( \frac{1}{k} \)). Both are independent, conditionally to \( p \) of the discrete loop. We see that \( \mu \), on based loops, is obtained from \( \mu \) on the loops by choosing the based point uniformly. On the other hand, it induces a choice of \( \xi_1 \) biased by the size of the \( \tau_i^* \)'s, different of \( \mu^* \) (whence the factor \( \frac{1}{k} \)). But we will consider only loop functionals.

It will be convenient to rescale the holding time at each \( \xi_i \) by \( \lambda_{\xi_i} \) and set \( \tilde{\tau}_i = \frac{\tau_i^*}{\lambda_{\xi_i}} \).
The discrete part of the loop is the most important, though we will see that to establish a connection with Gaussian fields it is necessary to consider occupation times. The simplest variables are the number of jumps from \( x \) to \( y \), defined for every oriented edge \((x, y)\)

\[
N_{x,y} = \# \{ i : \xi_i = x, \xi_{i+1} = y \}
\]

(recall the convention \( \xi_{p+1} = \xi_1 \)) and

\[
N_x = \sum_y N_{x,y}
\]

Note that \( N_x = \# \{ i \geq 1 : \xi_i = x \} \) except for trivial one point loops for which it vanishes.

Then, the measure on pointed loops (5) can be rewritten as:

\[
\mu^*(p = 1, \xi = x, \hat{\tau} \in dt) = e^{-\lambda_x dt} \quad \text{and}
\]

\[
\mu^*(p = k, \xi_i = x_i, \hat{\tau}_i \in dt_i) = \frac{1}{k} \prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{-N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t_i dt_i}
\]

Another bridge measure \( \mu^{x,y} \) can be defined on paths \( \gamma \) from \( x \) to \( y \): \( \mu^{x,y}(d\gamma) = \int_0^\infty P^{x,y}(d\gamma) dt \).

Note that the mass of \( \mu^{x,y} \) is \( G^{x,y} \). We also have, with similar notations as the one defined for loops, \( p \) denoting the number of jumps

\[
\mu^{x,y}(p(\gamma) = k, \gamma_{T_1} = x_1, \ldots, \gamma_{T_{k-1}} = x_{k-1}, T_1 \in dt_1, \ldots, T_{k-1} \in dt_{k-1}, T \in dt)
\]

\[
= \frac{C_{x_1,x_2} C_{x_2,x_3} \cdots C_{x_{k-1},y}}{\lambda_x \lambda_{x_2} \cdots \lambda_y} 1_{\{0 < t_1 < \ldots < t_k < t\}} e^{-t} dt_1 \ldots dt_k dt
\]

### 3.4 Occupation field

To each loop \( l \) we associate local times, i.e. an occupation field \( \{ \hat{l}_x, x \in X \} \) defined by

\[
\hat{l}_x = \int_0^{T(l)} 1_{\{\xi(s) = x\}} \frac{1}{\lambda_{\xi(s)}} ds = \sum_{i=1}^{p(l)} 1_{\{\xi_i = x\}} \hat{\tau}_i
\]

for any representative \( l = (\xi, \tau) \) of \( l^o \).

For a path \( \gamma, \hat{\gamma} \) is defined in the same way.

Note that

\[
\mu((1 - e^{-\alpha \hat{l}}) 1_{p=1}) = \int_0^\infty e^{-t} (1 - e^{-\alpha \hat{l}}) \frac{dt}{t} = \log(1 + \frac{\alpha}{\lambda_x})
\]
(by expanding $1 - e^{-\frac{\alpha}{\lambda}t}$ before the integration, assuming first $\alpha$ small and then by analyticity of both members, or more elegantly, noticing that $\int_a^b (e^{-cx} - e^{-dx}) \frac{dx}{x}$ is symmetric in $(a, b)$ and $(c, d)$).

In particular, $\mu(\hat{l}^x 1_{\{p=1\}}) = \frac{1}{\lambda x}$.

From formula 5, we get easily that the joint conditional distribution of $(\hat{l}^x, x \in X)$ given $(N_x, x \in X)$ is a product of gamma distributions. In particular, from the expression of the moments of a gamma distribution, we get that for any function $\Phi$ of the discrete loop and $k \geq 1$,

$$\mu(\hat{l}^x)^k 1_{\{p>1\}} \Phi(x) = \lambda_x^{-k} \mu((N_x + k - 1)(N_x + 1)N_x \Phi(x))$$

In particular, $\mu(\hat{l}^x) = \frac{1}{\lambda x} [\mu(N_x) + 1] = G^x$. 

Note that functions of $\hat{l}$ are not the only functions naturally defined on the loops. Other such variables of interest are, for $n \geq 2$, the multiple local times, defined as follows:

$$\hat{l}^{x_1, \ldots, x_n} = \sum_{j=0}^{n-1} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left\{ \begin{array}{l}
\xi(t_1) = x_1 + j, \ldots, \xi(t_{n-1}) = x_{n-1}, \ldots, \xi(t_n) = x_n \end{array} \right\} \prod_{i=1}^{n} \frac{1}{\lambda x_i} dt_i$$

It is easy to check that, when the points $x_i$ are distinct,

$$\hat{l}^{x_1, \ldots, x_n} = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \ldots < i_n \leq p(l)} \prod_{i=1}^{n} \frac{1}{\lambda x} \hat{l}^{x_{i_1}, \ldots, x_{i_n}}.$$  

Note that in general $\hat{l}^{x_1, \ldots, x_k}$ cannot be expressed in terms of $\hat{l}$.

If $x_1 = x_2 = \ldots = x_n$, $\hat{l}^{x_1, \ldots, x_n} = \frac{1}{(n-1)!} [\hat{l}^x]^n$. It can be viewed as a $n$-th self intersection local time.

One can deduce from the definitions of $\mu$ the following:

**Proposition 1** $\mu(\hat{l}^{x_1, \ldots, x_n}) = G^{x_1, x_2} G^{x_2, x_3} \ldots G^{x_n, x_1}$

**Proof.** Let us denote $\frac{1}{\lambda y} [P_t^x]_y$ by $p_t^{x,y}$ or $p_t(x, y)$. From the definition of $\hat{l}^{x_1, \ldots, x_n}$ and $\mu$, $\mu(\hat{l}^{x_1, \ldots, x_n})$ equals:

$$\sum_x \lambda_x \sum_{j=0}^{n-1} \int \int_{\{0<t_1<\ldots<t_{n-1}<t\}} \frac{1}{\lambda t} p_t(x, x_{1+j}) \cdots p_{t-t_n}(x_{n+j}, x) \prod dt_idt$$
where sums of indices $k + j$ are computed mod($n$). By the semigroup property, it equals

$$\sum_{j=0}^{n-1} \int \int_{\{0 < t_1 < \ldots < t_n < t\}} \frac{1}{t} p_{t_2-t_1}(x_{1+j}, x_{2+j}) \cdots p_{t_1+t-t_n}(x_{n+j}, x_{1+j}) \prod dt_1 dt.$$ 

Performing the change of variables $v_2 = t_2 - t_1, \ldots, v_n = t_n - t_{n-1}, v_1 = t_1 + t - t_n,$ and $v = t_1$, we obtain:

$$\sum_{j=0}^{n-1} \int \int_{\{0 < v < v_1, 0 < v_i\}} \frac{1}{v_1 + \ldots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \cdots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i dv$$

$$= \sum_{j=0}^{n-1} \int \int_{\{0 < v_i\}} \frac{v_1}{v_1 + \ldots + v_n} p_{v_2}(x_{1+j}, x_{2+j}) \cdots p_{v_1}(x_{n+j}, x_{1+j}) \prod dv_i$$

$$= \sum_{j=1}^{n} \int \int_{\{0 < v_i\}} \frac{v_j}{v_1 + \ldots + v_n} p_{v_2}(x_1, x_2) \cdots p_{v_1}(x_n, x_1) \prod dv_i$$

$$= \int \int_{\{0 < v_i\}} p_{v_2}(x_1, x_2) \cdots p_{v_1}(x_n, x_1) \prod dv_i$$

$$= G^{x_1,x_2} G^{x_2,x_3} \ldots G^{x_n,x_1}.$$ 

Note that another proof can be derived from formula (10) $\blacksquare$

Let us come back to the occupation field to compute its Laplace transform. From the Feynman-Kac formula, it comes easily that, denoting $M_\chi$ the diagonal matrix with coefficients $\chi_{x\lambda}$

$$P_t^x(x^{\langle i, \chi \rangle} - 1) = \frac{1}{\lambda_x} (\exp(t(P - I - M_\chi))^{x}_x - \exp(t(P - I))^{x}_x).$$

Integrating in $t$ after expanding, we get from the definition of $\mu$ (first for $\chi$ small enough):

$$\int (e^{-\langle i, \chi \rangle} - 1) d\mu(l) = \sum_{k=1}^{\infty} \int_0^\infty [Tr((P - M_\chi)^k) - Tr((P)^k)] t^{k-1} k! e^{-t} dt$$

$$= -Tr(\log(I - P + M_\chi)) + Tr(\log(I - P))$$

$$= 11.$$
Hence, as $Tr(\log) = \log(\det)$

$$\int (e^{-<\tilde{L},\chi>} - 1)d\mu(l) = \log[\det(-L(-L + M_{\chi/\lambda})^{-1})] = -\log \det(I + VM_{\chi/\lambda})$$

which now holds for all non negative $\chi$ as both members are analytic in $\chi$. Besides, by the ”resolvent” equation:

$$\det(I + GM_{\chi})^{-1} = \det(I - G_{\chi}M_{\chi}) = \frac{\det(G_{\chi})}{\det(G)}$$

(11)

Note that $\det(I + GM_{\chi}) = \det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$ and $\det(I - G_{\chi}M_{\chi}) = \det(I - M_{\sqrt{\chi}}G_{\chi}M_{\sqrt{\chi}})$, so we can deal with symmetric matrices. Finally we have the

**Proposition 2** $\mu(e^{-<\tilde{L},\chi>} - 1) = -\log(\det(I + M_{\sqrt{\chi}}GM_{\sqrt{\chi}})) = \log(\frac{\det(G_{\chi})}{\det(G)})$

Note that in particular $\mu(e^{-\tilde{L}x} - 1) = -\log(1 + tG^{x,x})$.

Note finally that if $\chi$ has support in $D$, by the restriction property

$$\mu(1_{\{\tilde{L}(X) \in D\}})(e^{-<\tilde{L},\chi>} - 1)) = -\log(\det(I + M_{\sqrt{\chi}}G_{\sqrt{\chi}}M_{\sqrt{\chi}})) = \log(\frac{\det(G_{\chi}^{D})}{\det(G^{D})})$$

Here the determinants are taken on matrices indexed by $D$ and $G^{D}$ the Green function of the process killed on leaving $D$.

For paths we have $\mathbb{P}^{x,y}_{t}(e^{-<\tilde{L},\chi>}) = \frac{1}{\lambda_{y}} \exp(t(L - M_{\chi}^{x,y}))$. Hence

$$\mu^{x,y}(e^{-<\tilde{L},\chi>}) = \frac{1}{\lambda_{y}}((I - P + M_{\chi/m})^{-1})_{x,y} = [G_{\chi}]^{x,y}.$$

Also $\mathbb{E}^{x}(e^{-<\tilde{L},\chi>}) = \sum_{y}[G_{\chi}]^{x,y}K_{\chi}$ i.e.$[G_{\chi}K]^{x}$.

4 Poisson process of loops

4.1 Definition

Still following the idea of [9], which was already implicitly in germ in [24], define, for all positive $\alpha$, the Poissonian ensemble of loops $\mathcal{L}_{\alpha}$ with intensity $\alpha \mu$. We denote by $\mathbb{P}$ or $\mathbb{P}_{\mu}$ its distribution.
Recall it means that for any functional $\Phi$ on the loop space, vanishing on loops of arbitrary small length,

$$E(e^{i\sum_{l \in \mathcal{L}_\alpha} \Phi(l)}) = \exp(\alpha \int (e^{i\Phi(l)} - 1)\mu(dl))$$

Note that by the restriction property, $\mathcal{L}_\alpha^D = \{l \in \mathcal{L}_\alpha, l \subseteq D\}$ is a Poisson process of loops with intensity $\mu^D$, and that $\mathcal{L}_\alpha^D$ is independent of $\mathcal{L}_\alpha \setminus \mathcal{L}_\alpha^D$.

We denote by $\mathcal{D}\mathcal{L}_\alpha$ the set of non trivial discrete loops in $\mathcal{L}_\alpha$. Then,

$$\mathbb{P}(\mathcal{D}\mathcal{L}_\alpha = \{l_1, l_2, \ldots l_k\}) = e^{-\alpha \mu(p>0)} \alpha^k \mu(l_1) \ldots \mu(l_k) = \alpha^k \left[ \prod_x \lambda_x \right]^\alpha \prod_{x,y} C_{x,y}^{N_{x,y}^{(\alpha)}} \prod_x \lambda_x^{-N_x^{(\alpha)}}$$

with $N_x^{(\alpha)} = \sum_{l \in \mathcal{L}_\alpha} N_x(l)$ and $N_{x,y}^{(\alpha)} = \sum_{l \in \mathcal{L}_\alpha} N_{x,y}(l)$, when these loops are distinct.

We can associate to $\mathcal{L}_\alpha$ a $\sigma$-finite measure (in fact as we will see, finite when $X$ is finite, and more generally if $G$ is trace class) called local time or occupation field

$$\widehat{\mathcal{L}}_\alpha = \sum_{l \in \mathcal{L}_\alpha} \xi_l$$

Then, for any non-negative measure $\chi$ on $X$

$$\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle}) = \exp(\alpha \int (e^{-\langle \xi, \chi \rangle} - 1)d\mu(l))$$

and therefore by proposition 2 we have

**Corollary 3** $\mathbb{E}(e^{-\langle \widehat{\mathcal{L}}_\alpha, \chi \rangle}) = \det(I + \sqrt{\chi} GM\sqrt{\chi})^{-\alpha} = \left( \frac{\det(G\chi)}{\det(G)} \right)^\alpha$

Many calculations follow from this result.

It follows that $\mathbb{E}(e^{-t\widehat{\mathcal{L}}_\alpha^x}) = (1 + tG^{x,x})^{-\alpha}$. We see that $\widehat{\mathcal{L}}_\alpha^x$ follows a gamma distribution $\Gamma(\alpha, G^{x,x})$, with density $1_{\{x>0\}} \frac{e^{-\xi/\lambda_x}}{\Gamma(\alpha)} G^{x,x}^{-1}$ (in particular, an exponential distribution of mean $G^{x,x}$ for $\alpha = 1$). When we let $\alpha$ vary as a time parameter, we get a family of gamma subordinators, which can be called a "multivariate gamma subordinator".

We check in particular that $\mathbb{E}(\widehat{\mathcal{L}}_\alpha^x) = \alpha G^{x,x}$ which follows directly from $\mu(\xi_x) = G^{x,x}$.
Note also that for $\alpha > 1$,

$$\mathbb{E}((1 - \exp(-\hat{\mathcal{L}}^x_\alpha / G_{x,x}))^{-1}) = \zeta(\alpha).$$

More generally, for two points:

$$\mathbb{E}(e^{-t\hat{\mathcal{L}}^x_\alpha} e^{-s\hat{\mathcal{L}}^y_\alpha}) = ((1 + tG^{x,x})(1 + sG^{y,y}) - st(G^{x,y})^2)^{-\alpha}$$

This allows to compute the joint density of $\hat{\mathcal{L}}^x_\alpha$ and $\hat{\mathcal{L}}^y_\alpha$ in terms of Bessel and Struve functions.

We can condition the loops by the set of associated non trivial discrete loop by using the restricted $\sigma$-field $\sigma(\mathcal{DL}_{\alpha})$ which contains the variables $N_{x,y}$. We see from 9 and 7 that

$$\mathbb{E}(e^{-\langle \hat{\mathcal{L}}^x_\alpha, \chi \rangle}_{\mathcal{DL}_{\alpha}}) = \prod_x (\frac{\lambda_x}{\lambda_x + \chi_x})^{N_{x}\alpha}_x + 1$$

The distribution of $\{N^{(\alpha)}_x, x \in X\}$ follows easily, from corollary 3 in terms of generating functions:

$$\mathbb{E}(\prod_x (s^{N^{(\alpha)}_x} + 1) = \det(\delta_{x,y} + \sqrt{\frac{\lambda_x \lambda_y (1 - s_x)(1 - s_y)}{s_x s_y} G_{x,y}})^{-\alpha} \tag{12}$$

so that the vector of components $N^{(\alpha)}_x$ follows a multivariate negative binomial distribution (see for example [26]).

It follows in particular that $N^{(\alpha)}_x$ follows a negative binomial distribution of parameters $-\alpha$ and $\frac{1}{\lambda_x G_{x,x}}$. Note that for $\alpha = 1$, $N^{(1)}_x + 1$ follows a geometric distribution of parameter $\frac{1}{\lambda_x G_{x,x}}$.

### 4.2 Moments and polynomials of the occupation field

It is easy to check (and well known from the properties of the gamma distributions) that the moments of $\hat{\mathcal{L}}^x_\alpha$ are related to the factorial moments of $N^{(\alpha)}_x$:

$$\mathbb{E}((\hat{\mathcal{L}}^x_\alpha)^k | \mathcal{DL}_{\alpha}) = \frac{(N^{(\alpha)}_x + k)(N^{(\alpha)}_x + k - 1)\ldots(N^{(\alpha)}_x + 1)}{k! \lambda_x^k}$$
It is well known that Laguerre polynomials \( L_k^{(\alpha-1)} \) with generating function

\[
\sum_{k=0}^{\infty} t^k L_k^{(\alpha-1)}(u) = \frac{e^{-\frac{ut}{1-t}}}{(1-t)^{\alpha}}
\]

are orthogonal for the \( \Gamma(\alpha, 1) \) distribution with density \( \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}1_{\{u>0\}} \). They have mean zero and variance \( \Gamma(\alpha+k) k! \). Hence if we set \( \sigma_x = G^{x,x} \) and \( P_k^{\alpha,\sigma}(x) = (-\sigma)^k L_k^{(\alpha-1)}(\frac{x}{\sigma}) \), the random variables \( P_k^{\alpha,\sigma}(\hat{L}_{\alpha}^x) \) are orthogonal with mean 0 and variance \( \sigma^2 k \Gamma(\alpha+k) k! \), for \( k > 0 \).

Note that \( P_k^{\alpha,\sigma}(\hat{L}_{\alpha}^x) = \hat{L}_{\alpha}^x - \alpha \sigma x - \mathbb{E}(\hat{L}_{\alpha}^x) \). It will be denoted \( \tilde{\hat{L}}_{\alpha}^x \).

Moreover, we have

\[
\sum_{k=0}^{\infty} t^k P_k^{\alpha,\sigma}(u) = \sum (-\sigma t)^k L_k^{(\alpha-1)}(\frac{u}{\sigma}) = \frac{e^{-\frac{ut}{1+\sigma t}}}{(1+\sigma t)^{\alpha}}
\]

Therefore, we get, by developing in entire series in \((s,t)\) and identifying the coefficients:

\[
\mathbb{E}(P_k^{\alpha,\sigma}(\tilde{\hat{L}}_{\alpha}^x), P_I^{\alpha,\sigma}(\tilde{\hat{L}}_{\alpha}^y)) = \delta_{k,l} (G^{x,y})^{2k} \frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!}
\]

(13)

Let us stress the fact that \( G^{x,x} \) and \( G^{y,y} \) do not appear on the right side of this formula. This is quite important from the renormalisation point of view, as we will consider in the last section the two dimensional Brownian motion for which the Green function diverges on the diagonal.

More generally one can prove similar formulas for products of higher order.

Note that since \( G_x M_x \) is a contraction, from determinant expansions given in [25] and [26], we have

\[
\det(I + M_{\sqrt{\pi}}G M_{\sqrt{\pi}})^{-\alpha} = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{\chi_1...\chi_k} \text{Per}_{\alpha}(G_{i_1,i_m}, 1 \leq l, m \leq k)
\]

and then, from corollary 3, it comes that:

\[
\mathbb{E}((\tilde{\hat{L}}_{\alpha}, \chi)^k) = \sum_{\chi_{i_1...\chi_k}} \text{Per}_{\alpha}(G_{i_l,i_m}, 1 \leq l, m \leq k)
\]

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Here the $\alpha$-permanent $Per_\alpha$ is defined as \[
\sum_{\sigma \in S_k} \alpha^{m(\sigma)} G_{i_1,i_\sigma(1)} \cdots G_{i_k,i_\sigma(k)}
\] with $m(\sigma)$ denoting the number of cycles in $\sigma$.

Note that from this determinant expansion follows directly (see [26]) an explicit form for the multivariate negative binomial distribution, and therefore, a series expansion for the density of the multivariate gamma distribution.

It is actually not difficult to give a direct proof of this result. Thus, the Poisson process of loops provides a natural probabilistic proof and interpretation of this combinatorial identity (see [26] for an historical view of the subject).

We can show in fact that:

**Proposition 4** For any $(i_1, \ldots i_k)$ in $X^k$, $\mathbb{E}(\widehat{L}_{\alpha}^{-i_1} \cdots \widehat{L}_{\alpha}^{-i_k}) = Per_\alpha(G_{i_1,i_1}, 1 \leq l, m \leq k)$

**Proof.** The cycles of the permutations in the expression of $Per_\alpha$ are associated with point configurations on loops. We obtain the result by summing the contributions of all possible partitions of the points $i_1 \ldots i_k$ into a finite set of distinct loops. We can then decompose again the expression according to ordering of points on each loop. We can conclude by using the formula $\mu(\widehat{L}_{x_1} \cdots \widehat{L}_{x_m}) = G_{x_1,x_2}G_{x_2,x_3} \cdots G_{x_m,x_1}$ and the following property of Poisson measures (Cf formula 3-13 in [6]): For any system of non negative loop functionals $F_i$

\[
\mathbb{E}\left(\sum_{l_1 \neq l_2 \cdots \neq l_k \in L_{\alpha}} \prod F_i(l_i)\right) = \prod \alpha \mu(F_i)
\]

■

**Remark 5** We can actually check this formula in the special case $i_1 = i_2 = \ldots = i_k = x$. From the moments of the Gamma distribution, we have that $\mathbb{E}(\widehat{L}_{\alpha}^x)^n = (G^x)^n \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ and the $\alpha$-permanent writes $\sum d(n,k)\alpha^k$ where the coefficients $d(n,k)$ are the numbers of $n-$permutations with $k$ cycles (Stirling numbers of the first kind). One checks that $d(n+1,k) = nd(n,k) + d(n,k-1)$.

Let $S_k^0$ be the set of permutations of $k$ elements without fixed point. They correspond to configurations without isolated point.

Set $Per_{\alpha}^0(G^{i_1,i_m}, 1 \leq l, m \leq k) = \sum_{\sigma \in S_k^0} \alpha^{m(\sigma)} G_{i_1,i_{\sigma(1)}} \cdots G_{i_k,i_{\sigma(k)}}$. Then an easy calculation shows that:

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Corollary 6 \[ E(\tilde{\mathcal{L}}_{\alpha}^{-i_1}...\tilde{\mathcal{L}}_{\alpha}^{-i_k}) = \text{Per}_\alpha^0(G^{i_1;i_m}, 1 \leq l, m \leq k) \]

**Proof.** Indeed, the expectation writes
\[ \sum_{p \leq k} \sum_{l \subseteq \{1,...,k\}, |l|=p} (-1)^{k-p} \prod_{i \in I} G^{i_1;i_i} \text{Per}_\alpha(G^{i_1;i_i}, a, b \in I) \]
and
\[ \text{Per}_\alpha(G^{i_1;i_1}, a, b \in I) = \sum_{J \subseteq I} \prod_{j \in J} G^{j,j} \text{Per}_\alpha^0(G^{i_1;i_i}, a, b \in J). \]

Then, expressing \( E(\tilde{\mathcal{L}}_{\alpha}^{-i_1}...\tilde{\mathcal{L}}_{\alpha}^{-i_k}) \) in terms of \( \text{Per}_\alpha^0 \)'s, we see that if \( J \subseteq \{1,...,k\} \), \( |J| < k \), the coefficient of \( \text{Per}_\alpha^0(G^{i_1;i_1}, a, b \in J) \) is
\[ \sum_{I \subseteq J} (-1)^{k-|J|} \prod_{j \in J} G^{i_1;i_j} \]
which vanishes as \( (-1)^{|J|} = (-1)^{|J|}(-1)^{|J|-|I|} = (1-1)^{|J|-|I|} = 0. \]

Set \( Q_k^{\alpha,\sigma}(u) = \text{Per}_\alpha^0(u + \alpha \sigma) \) so that \( \text{Per}_\alpha^0(G^{x;i}) = Q_k^{\alpha,\sigma}(\tilde{\mathcal{L}}_{\alpha}^{x}) \). This quantity will be called the \( n \)-th renormalized self intersection local time or the \( n \)-th renormalized power of the occupation field and denoted \( \tilde{\mathcal{L}}_{\alpha}^{x,n} \).

From the recurrence relation of Laguerre polynomials
\[ nL_n^{(\alpha-1)}(u) = (-u + 2n + \alpha - 2)L_{n-1}^{(\alpha-1)} - (n + \alpha - 2)L_n^{(\alpha-1)}, \]
we get that
\[ nQ_n^{\alpha,\sigma}(u) = (u - 2\sigma(n-1))Q_{n-1}^{\alpha,\sigma}(u) - \sigma^2(\alpha + n - 2)Q_{n-2}^{\alpha,\sigma}(u) \]
In particular \( Q_2^{u,\sigma}(u) = \frac{1}{2}(u^2 - 2\sigma u + \alpha \sigma^2) \).

We have also, from (13)
\[ E(Q_k^{\alpha,\sigma_x}(\tilde{\mathcal{L}}_{\alpha}^{x}), Q_l^{\alpha,\sigma_y}(\tilde{\mathcal{L}}_{\alpha}^{y})) = \delta_{k,l}(G^{x,y})^{2k} \alpha(\alpha+1)...(\alpha+k-1) \]
(14)

The comparison of the identity (14) and corollary 6 yields a combinatorial result which will be fundamental in the renormalizing procedure presented in the last section.

The identity (14) can be considered as a polynomial identity in the variables \( \sigma_x, \sigma_y \) and \( G^{x,y} \).

If \( Q_k^{\alpha,\sigma_x}(u) = \sum_{m=0}^{k} q_{m}^{\alpha,k} u^{m} \sigma_x^{k-m} \), if we denote \( N_{n,m,r,p} \) the number of ordered configurations of \( n \) black points and \( m \) red points on \( r \) non trivial oriented cycles, such that only \( 2p \) links are between red and black points, we have
\[
\mathbb{E}(L_{\alpha}^x)^m (L_{\alpha}^y)^n = \sum_r \sum_{p \leq \inf(m,n)} \alpha^r N_{n,m,r,p} \langle G_{x,y} \rangle^{2p} \langle \sigma_x \rangle^{n-p} \langle \sigma_y \rangle^{m-p}
\]

and therefore

\[
\sum_r \sum_{p \leq k \leq l} \sum_{p \leq m \leq l} \alpha^r q_{m,n}^\alpha q_{n,m}^\alpha N_{n,m,r,p} = 0 \text{ unless } p = l = k. \tag{15}
\]

\[
\sum_r \alpha^r q_{k,n}^\alpha q_{n,k}^\alpha N_{k,k,r,k} = \frac{\alpha(\alpha + 1) \ldots (\alpha + k - 1)}{k!} \tag{16}
\]

Note that one can check directly that \( q_{k,n}^\alpha \) = \( \frac{1}{k!} \), and \( N_{k,k,1,k} = k!(k-1)! \), \( N_{k,k,k,k} = k! \) which confirms the identity (16) above.

### 4.3 Hitting probabilities

Let \([H^F]_y^x = \mathbb{P}_x(x_T = y)\) be the hitting distribution of \( F \) by the Markov chain starting at \( x \). Set \( D = F^c \) and denote \( e^D, P^D = P|_{D \times D}, V^D = [(I - P^D)]^{-1} \) and \( G^D = [(M^C - C)|_{D \times D}]^{-1} \) the energy, the transition matrix, the potential and the Green function of the process killed at the hitting of \( F \). Recall that

\[
[H^F]_y^x = 1_{\{x = y\}} + \sum_1^\infty \sum_{z \in D} \langle (P^D)^k \rangle^x_z \langle P^D \rangle^y_z = 1_{\{x = y\}} + \sum_1^\infty \sum_{z \in D} \langle V^D \rangle^x_z \langle P^D \rangle^y_z.
\]

Moreover we have by the strong Markov property, \( V = V^D + H^F V \) and therefore \( G = G^D + H^F G \).

(Here we extend \( V^D \) and \( G^D \) to \( X \times X \) by adding zero entries outside \( D \times D \)).

As \( G \) and \( G^D \) are symmetric, we have \( [H^F]_y^x = [H^F]_x^y \) so that for any measure \( \nu \),

\[
H^F(G\nu) = G(\nu H^F).
\]

Therefore we see that for any function \( f \) and measure \( \nu \),

\[
e(H^F f, G^D \nu) = e(H^F f, G\nu) - e(H^F f, H^F G\nu) = \langle H^F f, \nu \rangle - e(H^F f, G(\nu H^F)) = 0 \text{ as } (H^F)^2 = H^F.
\]

Equivalently, we have the following:

**Proposition 7** For any \( g \) vanishing on \( F \), \( e(H^F f, g) = 0 \) so that \( I - H^F \) is the \( \nu \)-orthogonal projection on the space of functions supported in \( D \).

For further developments see for example ( [12]) and its references.

The restriction property holds for \( \mathcal{L}_\alpha \) as it holds for \( \mu \). The set \( \mathcal{L}^D_\alpha \) of loops inside \( D \) is associated with \( \mu^D \) and independent of \( \mathcal{L}_\alpha - \mathcal{L}^D_\alpha \). Therefore, we see from corollary 3 that

\[
\mathbb{E}(e^{-\langle \tilde{\mathcal{L}}_\alpha - \tilde{\mathcal{L}}^D_\alpha \rangle}) = \left( \frac{\det(G) \det(G^D)}{\det(G^\alpha) \det(G^D)} \right)^\alpha.
\]
From the support of the Gamma distribution, we see that \( \mu(\hat{l}(F) > 0) = \infty \). But this is clearly due to trivial loops as it can be seen directly from the definition of \( \mu \) that in this simple framework they cover the whole space \( X \).

Note however that

\[
\mu(\hat{l}(F) > 0, p > 1) = \mu(p > 1) - \mu(\hat{l}(F) = 0, p > 1) = \mu(p > 1) - \mu^D(p > 1)
\]

\[
= -\log\left(\frac{\det(I - P)}{\det_D(I - P)}\right) = -\log\left(\frac{\det(G^D)}{\prod_{x \in F} \lambda_x \det(G)}\right).
\]

It follows that the probability no non trivial loop (i.e. a loop which is not reduced to a point) in \( \mathcal{L}_\alpha \) intersects \( F \) equals

\[
\exp(-\alpha \mu(\{l, p(l) > 1, \hat{l}(F) > 0\})) = \left(\frac{\det(G^D)}{\prod_{x \in F} \lambda_x \det(G)}\right)^\alpha.
\]

Recall that by Jacobi’s identity, for any \( (n+p, n+p) \) invertible matrix \( A \),

\[
det(A^{-1}) det(A_{ij}, 1 \leq i, j \leq n) = det((A^{-1})_{k,l}, n \leq k, l \leq n+p).
\]

In particular, \( \det(G^D) = \frac{\det(G)}{\det(G|_{F \times F})} \), so we have the

**Proposition 8** The probability that no non trivial loop in \( \mathcal{L}_\alpha \) intersects \( F \) equals

\[
\left[\prod_{x \in F} \lambda_x \det(G|_{F \times F})\right]^{-\alpha}.
\]

Moreover \( \mathbb{E}(e^{-\langle \mathcal{L}_\alpha - \mathcal{L}^D_\alpha, x \rangle}) = \left(\frac{\det(G)}{\det(F_{x \times F}(G\chi))}\right)^\alpha \)

In particular, it follows that the probability no non trivial loop in \( \mathcal{L}_\alpha \) visits \( x \) equals \( \left(\frac{1}{\lambda_x G^{x,x}}\right)^\alpha \) which is also a consequence of the fact that \( N_x \) follows a negative binomial distribution of parameters \(-\alpha\) and \( \frac{1}{\lambda_x G^{x,x}} \).

Also, if \( F_1 \) and \( F_2 \) are disjoint,

\[
\mu(\hat{l}(F_1) \hat{l}(F_2) > 0) = \mu(\hat{l}(F_1) > 0, p > 1) + \mu(\hat{l}(F_2) > 0, p > 1) - \mu(\hat{l}(F_1 \cup F_2) > 0, p > 1)
\]

\[
= \log\left(\frac{\det(G) \det(G^{D_1 \cap D_2})}{\det(G^{D_1}) \det(G^{D_2})}\right).
\]
Therefore the probability no non trivial loop in $L_\alpha$ intersects $F_1$ and $F_2$ equals

$$\exp(-\alpha \mu(\{l,p(l) > 1, \prod \hat{l}(F_i) > 0\})) = \left( \frac{\det(G) \det(G^{D_1 \cap D_2})}{\det(G^{D_1}) \det(G^{D_2})} \right)^{-\alpha}$$

It follows that the probability no non trivial loop in $L_\alpha$ visits two distinct points $x$ and $y$ equals $(\frac{G^{x,y} - (G^{x,y})^2}{G^{x,x} G^{y,y}})^{\alpha}$ and in particular $1 - (\frac{G^{x,y}}{G^{x,x} G^{y,y}})^2$ if $\alpha = 1$. This formula can be easily generalized to $n$ disjoint sets.

5 The Gaussian free field

5.1 Dynkin’s Isomorphism

By a well known calculation, if $X$ is finite, for any $\chi \in \mathbb{R}^X_+$,

$$\sqrt{\frac{\det(M_\lambda - C)}{(2\pi)^{|X|/2}}} \int (e^{-\frac{1}{2} < z, \chi >} e^{-\frac{1}{2} \phi(z)} \prod_{u \in X} dz^u) = \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

and

$$\sqrt{\frac{\det(M_\lambda - C)}{(2\pi)^{|X|/2}}} \int (z^x z^y (e^{-\frac{1}{2} < z^2, \chi >} e^{-\frac{1}{2} \phi(z)} \prod_{u \in X} dz^u) = (G_\chi)^{x,y} \sqrt{\frac{\det(G_\chi)}{\det(G)}}$$

This can be easily reformulated by introducing the Gaussian field $\phi$ defined by the covariance $\mathbb{E}(\phi^x \phi^y) = G^{x,y}$ (this reformulation cannot be dispensed with when $X$ becomes infinite).

So we have $\mathbb{E}(e^{-\frac{1}{2} < \phi^2, \chi >}) = \det(I + GM_\chi)^{-\frac{1}{2}} = \sqrt{\det(G_\chi G^{-1})}$ and

$$\mathbb{E}(e^{\frac{1}{2} \phi^2} e^{-\frac{1}{2} < \phi^2, \chi >}) = (G_\chi)^{x,y} \sqrt{\det(G_\chi G^{-1})}$$

Then as sums of exponentials of the form $e^{-\frac{1}{2} < \cdot, \chi >}$ are dense in continuous functions on $\mathbb{R}^X_+$ the following holds:

Theorem 9  a) The fields $\mathcal{L}_{\phi}$ and $\frac{1}{2} \phi^2$ have the same distribution.

b) $\mathbb{E}(\phi^x \phi^y F(\frac{1}{2} \phi^2)) = \int \mathbb{E}(F(\mathcal{L}_{\phi} + \gamma)) \mu^{x,y}(d\gamma)$ for any bounded functional $F$ of a non negative field.

Remarks:
a) This is a version of Dynkin’s isomorphism (Cf [2]). It can be extended to non symmetric generators (Cf [14]).

b) An analogous result can be given when \( \alpha \) is any positive half integer, by using real vector valued Gaussian field, or equivalently complex fields for integral values of \( \alpha \) (in particular \( \alpha = 1 \)).

c) Note it implies immediately that the process \( \phi^2 \) is infinitely divisible. See [3] and its references for a converse and earlier proofs of this last fact.

### 5.2 Fock spaces and Wick product

The Gaussian space \( \mathcal{H} \) spanned by \( \{ \phi^x, x \in X \} \) is isomorphic to the Dirichlet space \( \mathbb{H} \) by the linear map mapping \( \phi^x \) on \( G^x \) which extends into an isomorphism between the space of square integrable functionals of the Gaussian fields and the symmetric Fock space obtained as the closure of the sum of all symmetric tensor powers of \( \mathbb{H} \) (Bose second quantization: See [22], [18]). We have seen in theorem 9 that \( L^2 \) functionals of \( \hat{L}_1 \) can be represented in this symmetric Fock space.

In order to prepare the extension of these isomorphisms to the more difficult framework of continuous spaces (which can often be viewed as scaling limits of discrete spaces), including especially the planar Brownian motion considered in [9], we shall introduce the renormalized (or Wick) powers of \( \phi \). We set: \( (\phi^x)^n := (G^x)^{\frac{n}{2}} H_n(\phi^x/\sqrt{G^{xx}}) \) where \( H_n \) is the \( n \)-th Hermite polynomial (characterized by \( \sum_{t=0}^{n!} H_n(u) = e^{tu-\frac{t^2}{2}} \)). It is the inverse image of the \( n \)-th tensor power of \( G^{xx} \) in the Fock space.

Setting as before \( \sigma_x = G^{xx} \), from the relation between Hermite polynomials \( H_{2n} \) and Laguerre polynomials \( L_{\frac{n}{2}}^{\frac{1}{2}} \),

\[
H_{2n}(x) = (-2)^n n! L_n^{\frac{1}{2}}(\frac{x^2}{2})
\]

it comes that:

\[
: (\phi^x)^{2n} := 2^n n! P_n^{\frac{1}{2}, \frac{1}{2}} ((\frac{\phi^x}{2})^2)
\]

More generally, if \( \phi_1, \phi_2, \ldots, \phi_k \) are \( k \) independent copies of the free field, we can define

\[
: \prod_{j=1}^{k} \phi_j^{n_j} := \prod_{j=1}^{k} : \phi_j^{n_j} :. \]

Then it comes that:

\[
: \left( \sum_{j=1}^{k} \phi_j^2 \right)^n := \sum_{n_1+\ldots+n_k=n} \frac{n!}{n_1! \ldots n_k!} \prod_{j=1}^{k} \phi_j^{2n_j} :.
\]
From the generating function of the polynomials $P_n^k\sigma$,

$$P_n^k\sigma(\sum_1^k u_j) = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1!\cdots n_k!} \prod_{j=1}^k P_{n_j}^\sigma(u_j).$$

Therefore

$$P_n^{k\sigma}(\frac{1}{2}(\sum_1^k \phi_j)^2) = \frac{1}{2^n n!} : (\sum_1^k \phi_j^2)^n : . \quad (17)$$

Note that : $\sum_1^k \phi_j^2 := \sum_1^k \phi_j^2 - \sigma$ These variables are orthogonal in $L^2$. Let $\tilde{L}^x = \tilde{L}^x - \sigma$ be the centered occupation field. Note that an equivalent formulation of theorem 9 is that the fields $\frac{1}{2} : \sum_1^k \phi_j^2 :$ and $\tilde{L}_k^\sigma$ have the same law.

Let us now consider the relation of higher Wick powers with self intersection local times.

Recall that the renormalized $n$-th self intersections field $\tilde{L}_k^\sigma = P_n^{\sigma\sigma}(\tilde{L}^x_\alpha) = Q_n^{\sigma\sigma}(\tilde{L}^x_\alpha)$ have been defined by orthonormalization in $L^2$ of the powers of the occupation time.

Then comes the

**Proposition 10** The fields $\tilde{L}_k^\sigma$ and $\frac{1}{2^n n!} : (\sum_1^k \phi_j^2)^n :$ have the same law.

This follows directly from (17).

**Remark 11** As a consequence, it can be shown that:

$$E(\prod_{j=1}^r q_{k_j}^{\alpha\sigma x_j}(\tilde{L}^x_{\sigma})) = \sum_{\sigma \in S_{k_1,k_2,\ldots,k_j}} (2\alpha)^{m(\sigma)} G_{i_1,i_{\sigma(1)}} \cdots G_{i_k,i_{\sigma(k)}}$$

where $S_{k_1,k_2,\ldots,k_j}$ is the set of permutations $\sigma$ of $k = \sum k_j$ such that

$\sigma(\{\sum_{l=1}^{j-1} k_l + 1, \ldots, \sum_{l=1}^{j-1} k_l + k_j\} \cap \{\sum_{l=1}^{j-1} k_l + 1, \ldots, \sum_{l=1}^{j-1} k_l + k_j\})$ is empty for all $j$.

The identity follows from Wick’s theorem when $\alpha$ is a half integer, then extends to all $\alpha$ since both members are polynomials in $\alpha$. The condition on $\sigma$ indicates that no pairing is allowed inside the same Wick power.
6 Energy variation and currents

The loop measure $\mu$ depends on the energy $e$ which is defined by the free parameters $C, \kappa$. It will sometimes be denoted $\mu_e$. We shall denote $Z_e$ the determinant $\det(G) = \det(M_\lambda - C)^{-1}$. Then $\mu(p > 0) = \log(Z_e) + \sum \log(\lambda_x)$.

$Z^\alpha_e$ is called the partition function of $L^\alpha$.

The following result is suggested by an analogy with quantum field theory (Cf [5]).

**Proposition 12**

i) $\frac{\partial \mu}{\partial \kappa} = \hat{\mu}(\hat{t}^x)$

ii) If $C_{x,y} > 0$, $\frac{\partial \mu}{\partial C_{x,y}} = -T_{x,y} \mu$ with $T_{x,y}(l) = (\hat{t}^x + \hat{t}^y) - \frac{N_{x,y}}{C_{x,y}}(l) - \frac{N_{y,x}}{C_{x,y}}(l)$.

Note that the formula i) would be a direct consequence of the Dynkin isomorphism if we considered only sets defined by the occupation field.

**Proof.** Recall that by formula (7): $\mu^*(p = 1, \xi = x, \hat{\tau} \in dt) = e^{-\lambda_x t dt}$ and $\mu^*(p = k, \xi_i = x, \hat{\tau}_i \in dt_i) = \frac{1}{k} \prod_{x,y} C_{x,y}^{N_{x,y}} \prod_x \lambda_x^{N_x} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} \lambda_{\xi_i} e^{-\lambda_{\xi_i} t dt_i}$.

Moreover we have $C_{x,y} = C_{y,x} = \lambda_x P_y$ and $\lambda_x = \kappa_x + \sum_y C_{x,y}$

The two formulas follow by elementary calculation. ■

Recall that $\mu(\hat{t}^x) = G^{x,x}$ and $\mu(N_{x,y}) = G^{x,y}C_{x,y}$.

So we have $\mu(T_{x,y}) = G^{x,x} + G^{y,y} - 2G^{x,y}$.

Then, the above proposition allows to compute all moments of $T$ and $\hat{t}$ relative to $\mu_e$ (they could be called Schwinger functions). The above proposition gives the infinitesimal form of the following formula.

**Proposition 13** Consider another energy form $e'$ defined on the same graph. Then we have the following identity:

$$\frac{\partial \mu_{e'}}{\partial \mu_e} = e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \hat{t}^x}$$

Consequently

$$\mu_e \left( e^{\sum N_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}}) - \sum (\lambda'_x - \lambda_x) \hat{t}^x} - 1 \right) = \log \left( \frac{Z_{e'}}{Z_e} \right) \quad (18)$$

**Proof.** The first formula is a straightforward consequence of (7). The proof of (18) goes by evaluating separately the contribution of trivial loops, which equals $\sum_x \log(\frac{\lambda'_x}{\lambda^x})$. 23
Indeed,

$$\mu_e\left((e^{\sum_{x,y} \log(\frac{C'_{x,y}}{C_{x,y}})} - \sum (\lambda'_x - \lambda_x))^2 - 1\right) = \mu_e(p > 1) - \mu_e(p > 1) + \mu_e(1_{\{p=1\}}(e^{\sum (\lambda'_x - \lambda_x)^2} - 1)).$$

The difference of the first two terms equals $\log(Z_{e'}) + \sum \log(\lambda'_x) - (\log(Z_e) - \sum \log(\lambda_x))$.

The last term equals $\sum_x \int_0^\infty (e^{-\frac{\lambda'_x - \lambda_x}{\lambda_x} t} - 1) \frac{e^{-t}}{t} dt$ which can be computed as before:

$$\mu_e(1_{\{p=1\}}(e^{\sum (\lambda'_x - \lambda_x)^2} - 1)) = -\sum \log(\frac{\lambda'_x}{\lambda_x})$$

\[19\]

**Remark 14** (h-transforms) Note that if $C'_{x,y} = h^x h^y C_{x,y}$ and $\kappa'_x = -h L h \lambda$ for some positive function $h$ on $E$ such that $L h \leq 0$, as $\lambda' = h^2 \lambda$ and $[P'']_{x,y} = \frac{1}{h^2} P''_{x,y} h^y$, we have $[G'']^{x,y} = \frac{G^{x,y}}{h^x h^y}$ and $Z'_{e'} = \frac{1}{[h(x)]^2}$.

**Remark 15** Note also that $[\frac{Z_{e'}}{Z_e}]^{\frac{1}{2}} = \mathbb{E}(e^{-\frac{1}{2}(|e' - e)(\phi)|})$, if $\phi$ is the Gaussian free field associated with $e$.

Integrating out the holding times, formula (18) can be written equivalently:

$$\mu_e\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} \prod_x \left[\frac{\lambda_x}{\lambda'_x}\right]^{N_x+1} - 1\right) = \log(\frac{Z_{e'}}{Z_e})$$

and therefore

$$\mathbb{E}_{\mathcal{L}_e}\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} \prod_x \left[\frac{\lambda_x}{\lambda'_x}\right]^{N_x+1} - 1\right) = \mathbb{E}_{\mathcal{L}_e}\left(\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} e^{-\langle \lambda' - \lambda, \mathcal{L}_e \rangle} \right) = \left(\frac{Z_{e'}}{Z_e}\right)^\alpha$$

Note also that $\prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}} = \prod_{(x,y)} \left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y} + N_{y,x}}$.  

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Remark 16 These \( \frac{Z_{e'}}{Z_e} \) determine, when \( e' \) varies with \( \frac{e'}{C} \leq 1 \) and \( \frac{\lambda'}{\lambda} = 1 \), the Laplace transform of the distribution of the traversal numbers of non oriented links \( N_{x,y} + N_{y,x} \).

Other variables of interest on the loop space are associated with elements of the space \( \Lambda^- \) of odd functions \( \omega \) on oriented links: \( \omega_{x,y} = -\omega_{y,x} \). Let us mention a few elementary results.

The operator \( [P(\omega)]_{y} = P_{y}^{x} \exp(i\omega_{x,y}) \) is also self adjoint in \( L^{2}(\lambda) \). The associated loop variable writes \( \sum_{x,y}^{\omega_{x,y}} N_{x,y}(l) \). We will denote it \( \int_{l} \omega \). Note it is invariant if \( \omega_{x,y} \) is replaced by \( \omega_{x,y} + g_{y} - g_{x} \) for some \( g \). Set \( [G(\omega)]_{x,y} = \frac{[I-P(\omega)]^{-1}}{\lambda_{y}} \). By an argument similar to the one given above for the occupation field, we have:

\[
P^{t}_{x,x}(e^{i\int_{l}^{x} \omega} - 1) = \exp(t(P(\omega) - I))_{x,x} - \exp(t(P - I))_{x,x}.
\]

Integrating in \( t \) after expanding, we get from the definition of \( \mu \):

\[
\int (e^{i\int_{l}^{x} \omega} - 1) d\mu(l) = \sum_{k=1}^{\infty} \frac{1}{k} \left[ \text{Tr}((P(\omega))^{k}) - \text{Tr}((P)^{k}) \right]
\]

Hence

\[
\int (e^{i\int_{l}^{x} \omega} - 1) d\mu(l) = \log|\det(-L(I - P(\omega))^{-1})|
\]

Hence \( \int (e^{i\int_{l}^{x} \omega} - 1) d\mu(l) = \log|\det(-L(I - P(\omega))^{-1})| \) and

\[
\int (\exp(i\int_{l}^{x} \omega) - 1) d\mu(dl) = \log|\det(G(\omega)G^{-1})|
\]

We can now extend the previous results (18) and (20) to obtain, setting \( \det(G(\omega)) = Z_{e,\omega} \)

\[
\mu_e(e^{-\sum_{x,y} N_{x,y} \log(\frac{e^{i\omega_{x,y}}}{C_{x,y}}) - \sum_{x,y} (\lambda'_{y} - \lambda_{y}) \int_{x} \omega - 1}) = \log\left(\frac{Z_{e',\omega}}{Z_e}\right)
\]

and

\[
E\left( \prod_{x,y} \left( \frac{C'_{x,y}}{C_{x,y}} e^{i\alpha_{x,y}} \right)^{N_{x,y}^{(\alpha)}} \right) = \left( \frac{Z_{e',\omega}}{Z_e}\right)^{\alpha}
\]

Let us now introduce a new

Definition 17 We say that sets \( \Lambda_{i} \) of non trivial loops are equivalent when the associated occupation fields are equal and when the total traversal numbers \( \sum_{l \in \Lambda_{i}} N_{x,y}(l) \) are equal
for all oriented edges \((x, y)\). Equivalence classes will be called loop networks on the graph. We denote \(\overline{\Lambda}\) the loop network defined by \(\Lambda\).

Similarly, a set \(L\) of non-trivial discrete loops defines a discrete network characterized by the total traversal numbers.

Note that these expectations determine the distribution of the network \(L_\alpha\) defined by the loop ensemble \(L_\alpha\). We will denote \(B^{e,e',\omega}\) the variables

\[
\prod_{x,y} \left[ \frac{C'_{x,y}}{C_{x,y}} e^{i\omega_{x,y}} \right]^{N_{x,y}^{(o)}} e^{-\sum(x'_ax_ay)}L_{\alpha}^{x,y}. 
\]

**Remark 18** This last formula applies to the calculation of loop indices: If we have for example a simple random walk on an oriented planar graph, and if \(\omega'\) is a point of the dual graph \(X'\), \(\omega'\) can be chosen such that \(\int \omega'\) is the winding number of the loop around a given point \(z'\) of the dual graph \(X'\). Then \(e^{i\pi \sum \int_{l} \omega'_{z}}\) is a spin system of interest. We then get for example that

\[
\mu(\int \omega' \neq 0) = -\frac{1}{2\pi} \int_{0}^{2\pi} \log(\det(G(2\pi u \omega'_{z})G^{-1}))du 
\]

and hence

\[
P(\sum_{l \in L_\alpha} |\int l \omega'_{z}| = 0) = e^{\frac{\alpha}{2\pi} \int_{0}^{2\pi} \log(\det(G(2\pi u \omega'_{z})G^{-1}))du} 
\]

Conditional distributions of the occupation field with respect to values of the winding number can also be obtained.

## 7 Loop erasure and spanning trees.

Recall that an oriented link \(g\) is a pair of points \((g^-, g^+)\) such that \(C_g = C_{g^-, g^+} \neq 0\). Define \(-g = (g^+, g^-)\).

Let \(\mu_{x,y}^{x,y}\) be the measure induced by \(C\) on discrete self-avoiding paths between \(x\) and \(y\):

\[
\mu_{x,y}^{x,y}(x, x_2, ..., x_{n-1}, y) = C_{x,x_2}C_{x_2,x_3}...C_{x_{n-1},y}. 
\]

Another way to define a measure on discrete self-avoiding paths from \(x\) to \(y\) is loop erasure (see [7], [19] and [8]). In this context, the loops can be trivial as they correspond to a single holding times, and loop erasure produces a discrete path without holding times.

We have the following:
Proposition 19  The image of $\mu^{x,y}$ by the loop erasure map $\gamma \rightarrow \gamma^{BE}$ is $\mu^{x,y}_{BE}$ defined on self avoiding paths by $\mu^{x,y}_{BE}(\eta) = \mu^{x,y}_{\neq}(\eta) \frac{\det(G)}{\det(G[\eta])}$ (Here $\{\eta\}$ denotes the set of points in the path $\eta$)

Proof. If $\eta = (x_1 = x, x_2, \ldots, x_n = y)$ and $\eta_m = (x, \ldots, x_m)$,

$$\mu^{x,y}(\gamma^{BE} = \eta) = \frac{\delta_y^x}{\lambda_y} + \sum_{k=2}^{\infty} [P^k |x| x_2 P^2 |x_2| x (\gamma^{BE} = \theta \eta)]$$

where $\mu^{x,y}_{\{x\}c}$ denotes the bridge measure for the Markov chain killed as it hits $x$ and $\theta$ the natural shift on discrete paths. By recurrence, this clearly equals

$$V^x P^x V^2 |x_2| \ldots |V^{\eta_{m-1}}| \frac{1}{y} \lambda_y^{-1} = \mu^{x,y}_{\neq}(\eta) \frac{\det(G)}{\det(G[\eta])}$$

as

$$[V^{\eta_{m-1}}]^{c|\{x\}c} = \frac{\det([(I - P)|\{\eta_m\}c \times \{\eta_m\}c])}{\det([(I - P)|\{\eta_{m-1}\}c \times \{\eta_{m-1}\}c])} = \frac{\det(V^{\eta_{m-1}})}{\det(V^{\eta_{m-1}})} = \frac{\det(G^{\eta_{m-1}})}{\det(G^{\eta_{m-1}})} \lambda_{x_m}.$$ 

for all $m \leq n - 1$. 

Also, by Feynman-Kac formula, for any self-avoiding path $\eta$:

$$\int e^{-<\hat{\gamma}, \lambda>} 1_{\{\gamma^{BE} = \eta\}} \mu^{x,y}(d\gamma) = \frac{\det(G_x)}{\det(G^{\eta})} \mu^{x,y}_{\neq}(\eta) = \det(G_x |\{\eta\} \times \{\eta\}) \mu^{x,y}_{\neq}(\eta)$$

$$= \frac{\det(G_x |\{\eta\} \times \{\eta\})}{\det(G[\eta])} \mu^{x,y}_{BE}(\eta).$$

Therefore, recalling that by the results of section 4.3 conditionally to $\eta$, $\hat{L}_1$ and $\hat{L}_{1}^{(n)c}$ are independent, we see that under $\mu^{x,y}$, the conditional distribution of $\gamma$ given $\gamma^{BE} = \eta$ is the distribution of $\hat{L}_1 - \hat{L}_{1}^{(n)c}$ i.e. the occupation field of the loops of $L_1$ which intersect $\eta$.

More generally, it can be shown that

Proposition 20  The conditional distribution of the network $\overline{L_1}$ defined by the loops of $\gamma$, given that $\gamma^{BE} = \eta$, is identical to the distribution of the network defined by $L_1 / L_{1}^{(n)c}$ i.e. the loops of $L_1$ which intersect $\eta$. 

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Proof. Recall the notation \( \mathcal{Z} = \det(G) \). First an elementary calculation using (7) shows that \( \mu^{x,y}(e^{i \int \gamma \omega 1 \{ \gamma_{BE} = \eta \}}) \) equals

\[
\mu^{x,y}(1 \{ \gamma_{BE} = \eta \}) \prod \left[ C'u_{\xi, \xi+1} e^{i \omega_{\xi, \xi+1} \lambda'_{\xi+1} \lambda_{\xi}} \right] \frac{C'_{x,x_2} C'_{x_1,x_3} \cdots C'_{x_{n-1},y} e^{i \int \omega \mu^{x,y} \left( \prod \left[ C'u_{u,v} e^{i \omega_{u,v}} \right] N_{u,v}(L'_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \} \right)}}{C_{x,x_2} C_{x_1,x_3} \cdots C_{x_{n-1},y} e^{i \int \omega \mu^{x,y} \left( \prod \left[ C'u_{u,v} e^{i \omega_{u,v}} \right] N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \} \right)}}.
\]

(Note the term \( e^{-\langle \lambda' - \lambda, \gamma \rangle} \) can be replaced by \( \prod \frac{e^{i \omega_{u,v}}}{e^{i \omega_{u,v}}} \).)

Moreover, by the proof of the previous proposition, applied to the Markov chain defined by \( e' \) perturbed by \( \omega \), we have also

\[
\mu^{x',y}(e^{i \int \gamma \omega 1 \{ \gamma_{BE} = \eta \}}) = C'_{x,x_2} C'_{x_1,x_3} \cdots C'_{x_{n-1},y} e^{i \int \omega \mu^{x,y} \left( \prod \left[ C'u_{u,v} e^{i \omega_{u,v}} \right] N_{u,v}(L'_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \} \right)}/Z_{e', \omega}.
\]

Therefore

\[
\mu^{x,y}(\prod \frac{C'u_{u,v} e^{i \omega_{u,v}}}{C_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}) = \frac{Z_{e'} \prod C'u_{u,v} e^{i \omega_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}}{Z_{e'} \prod C'u_{u,v} e^{i \omega_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}}.
\]

Moreover, by (21) and the properties of the Poisson processes,

\[
\mathbb{E}(\prod \frac{C'u_{u,v} e^{i \omega_{u,v}}}{C_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}) = \frac{Z_{e'} \prod C'u_{u,v} e^{i \omega_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}}{Z_{e'} \prod C'u_{u,v} e^{i \omega_{u,v}} N_{u,v}(L_{\gamma}) e^{-\langle \lambda' - \lambda, \gamma \rangle} 1 \{ \gamma_{BE} = \eta \}}.
\]

It follows that the joint distribution of the traversal numbers and the occupation field are identical for the set of erased loops and \( L_{\gamma} \). ■

Similarly one can define the image of \( \mathbb{P}^e \) by \( BE \) which is given by

\[
\mathbb{P}^e_{BE}(\eta) = C_{x_1,x_2} \cdots C_{x_{n-1},x_n} \kappa_{x_n} \det(G_{\{\eta\}}) \times \det(G_{\{\eta\}}) \eta_{\{\eta\}},
\]

for \( \eta = (x_1, \ldots, x_n) \), and get the same results.

Wilson’s algorithm (see [16]) iterates this construction, starting with \( x' \)'s in arbitrary order. Each step of the algorithm reproduces the first step except it stops when it hits the already constructed tree of self avoiding paths. It provides a construction of a random spanning tree. Its law is a probability measure \( \mathbb{P}^e_{ST} \) on the set \( ST_{X,\Delta} \) of spanning trees of
$X$ rooted at the cemetery point $\Delta$ defined by the energy $e$. The weight attached to each oriented link $g = (x, y)$ of $X \times X$ is the conductance and the weight attached to the link $(x, \Delta)$ is $\kappa_x$ we can also denote by $C_{x,\Delta}$. As the determinants simplify, the probability of a tree $\Upsilon$ is given by a simple formula:

$$P^e_{ST}(\Upsilon) = Z_e \prod_{\xi \in \Upsilon} C_{\xi}$$

(22)

It is clearly independent of the ordering chosen initially. Now note that, since we get a probability

$$Z_e \sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y) \in \Upsilon} C_{x,y} \prod_{x,(x,\Delta) \in \Upsilon} \kappa_x = 1$$

or equivalently

$$\sum_{\Upsilon \in ST_{X,\Delta}} \prod_{(x,y) \in \Upsilon} P^x \prod_{x,(x,\Delta) \in \Upsilon} P^x = \frac{1}{\prod_{x \in X} \lambda_x Z_e}$$

Then, it comes that, for any $e'$ for which conductances (including $\kappa'$) are positive only on links of $e$,

$$E^e_{ST}\left( \prod_{(x,y) \in \Upsilon} \frac{P^x_{y}}{P^x} \prod_{x,(x,\Delta) \in \Upsilon} \frac{P^x_{\Delta}}{P^x} \right) = \frac{\prod_{x \in X} \lambda_x Z_e}{\prod_{x \in X} \lambda'_{x} Z'_e}$$

and

$$E^e_{ST}\left( \prod_{(x,y) \in \Upsilon} \frac{C'_{x,y}}{C_{x,y}} \prod_{x,(x,\Delta) \in \Upsilon} \frac{\kappa'_x}{\kappa_x} \right) = \frac{Z_e}{Z'_e}$$

(23)

Note also that in the case of a graph (i.e. when all conductances are equal to 1), all spanning trees have the same probability. The expression of their cardinal as the determinant $Z_e$ is Cayley’s theorem (see for exemple [16]).

**Corollary 21** The network defined by the random set of loops $L_W$ constructed in this algorithm is independent of the random spanning tree, and independent of the ordering. It has the same distribution as the network defined by the loops of $L_1$.

This result follows easily from proposition 20.
8 Decompositions

Note first that with the energy $e$, we can associate a rescaled Markov chain $\tilde{x}_t$ in which holding times at any point $x$ are exponential times of parameters $\lambda_x$: $\tilde{x}_t = x_{\tau_t}$ with $\tau_t = \inf(s, \int_0^s \frac{1}{\lambda_{x_u}} du = t)$. For the rescaled Markov chain, local times coincide with the time spent in a point and the duality measure is simply the counting measure. The Markov loops can be rescaled as well and we did it in fact already when we introduced pointed loops. More generally we may introduce different holding times parameters but it would be essentially useless as the random variables we are interested into are intrinsic, i.e. depend only on $e$.

If $D \subset X$ and we set $F = D^c$, the orthogonal decomposition of the energy $e(f, f) = e(f) + e(f - H^F f) + e(H^F f)$ leads to the decomposition of the Gaussian field mentioned above and also to a decomposition of the rescaled Markov chain into the rescaled Markov chain killed at the exit of $D$ and the trace of the rescaled Markov chain on $F$, i.e. $\tilde{x}^{(F)}_t = \tilde{x}_{S^F_t}$, with $S^F_t = \inf(s, \int_0^s 1_F(\tilde{x}_u) du = t)$.

Proposition 22 The trace of the rescaled Markov chain on $F$ is the rescaled Markov chain defined by the energy functional $e^{(F)}(f) = e(H^F f)$, for which

$$C^{(F)}_{x,y} = C_{x,y} + \sum_{a,b \in D} C_{x,a}C_{b,y}[G^D]^{a,b}$$

$$\lambda^{(F)}_x = \lambda_x - \sum_{a,b \in D} C_{x,a}C_{b,x}[G^D]^{a,b}$$

and

$$Z_e = Z_{e,D}Z_{e,(F)}$$

Proof. For the second assertion, note first that for any $y \in F$,

$$[H^F]_y^x = 1_{x=y} + 1_D(x) \sum_{b \in D} [G^D]^{x,b}C_{b,y}.$$ 

Moreover, $e(H^F f) = e(f, H^F f)$ and therefore

$$\lambda^{(F)}_x = e^{(F)}(1 \{x\}) = e(1 \{x\}, H^F 1 \{x\}) = \lambda_x - \sum_{a \in D} C_{x,a}[H^F]^a_{1 \{x\}} = \lambda_x(1 - p^{(F)}_x)$$

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where \( p_z^F = \sum_{a,b \in D} P^x_a [G^D]^{a,b} C_{b,x} = \sum_{a \in D} P^x_a [H^F]_x^a \) is the probability that the Markov chain starting at \( x \) will return to \( x \) after an excursion in \( D \).

Then for distinct \( x \) and \( y \) in \( F \),

\[
C^{(F)}_{x,y} = -e^{(F)}(1_{\{x\}}, 1_{\{y\}}) = -e(1_{\{x\}}, H^F 1_{\{y\}}) = C_{x,y} + \sum_a C_{x,a}[H^F]_y^a = C_{x,y} + \sum_{a,b \in D} C_{x,a} C_{b,y}[G^D]^{a,b}.
\]

Note that the graph defined on \( F \) by the non vanishing conductances \( C^{(F)}_{x,y} \) has in general more edges than the restriction to \( F \) of the original graph.

For the third assertion, note also that \( G^{(F)} \) is the restriction of \( G \) to \( F \) as for all \( x,y \in F \), \( e^{(F)}(G\delta_y|F, 1_{\{x\}}) = e(G\delta_y, [H^F 1_{\{x\}}]) = 1_{\{x=y\}} \). Hence the determinant decomposition already used in section 4.3 yields the final formula. The cases where \( F \) has one point was already treated in section 4.3.

Finally, for the first assertion note the transition matrix \( [P^{(F)}]_{xy} \) can be computed directly and equals

\[
P^x_y + \sum_{a,b \in D} P^x_a P^b_y V^{D \cup \{x\}}_b = P^x_y + \sum_{a,b \in D} P^x_a C_{b,y}[G^{D \cup \{x\}}]^{a,b}.
\]

It can be decomposed according whether the jump to \( y \) occurs from \( x \) or from \( D \) and the number of excursions from \( x \) to \( x \):

\[
[P^{(F)}]_{xy} = \sum_{k=0}^\infty (\sum_{a,b \in D} P^x_a [V^D]_b^a P^b_y)^k (P^x_y + \sum_{a,b \in D} P^x_a [V^D]_b^a P^b_y)
\]

\[
= \sum_{k=0}^\infty (\sum_{a,b \in D} P^x_a [G^D]^{a,b} C_{b,x})^k (P^x_y + \sum_{a,b \in D} P^x_a [G^D]^{a,b} C_{b,y}).
\]

The expansion of \( C^{(F)}_{x,y} \) in geometric series yields the exactly the same result.

Finally, remark that the holding times of \( \hat{x}_t^{(F)} \) at any point \( x \in F \) are sums of a random number of independent holding times of \( \hat{x}_t \). This random integer counts the excursions from \( x \) to \( x \) performed by the chain \( \hat{x}_t \) during the holding time of \( \hat{x}_t^{(F)} \). It follows a geometric distribution of parameter \( 1 - p_z^{(F)} \). Therefore, \( \frac{1}{\lambda_z^{(F)}} = \frac{1}{\lambda_z (1-p_z)} \) is the expectation of the holding times of \( \hat{x}_t^{(F)} \) at \( x \).

If \( \chi \) is carried by \( D \) and if we set \( e^z_\chi = e + ||\chi||_{L^2(\chi)} \) and denote \( [e^z_\chi]^{(F)} \) by \( e^{(F,\chi)} \) we have

\[
C^{(F,\chi)}_{x,y} = C_{x,y} + \sum_{a,b} C_{x,a} C_{b,y}[G^D]^{a,b}, \quad p_z^{(F,\chi)} = \sum_{a,b \in D} P^x_a [G^D]^{a,b} C_{b,x}.
\]

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and $\lambda_2^{(F_x)} = \lambda_x(1 - p_2^{(F_x)})$.

More generally, if $e^\#$ is such that $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on $F$ we have:

$$C_{x,y}^\#(F) = C_{x,y} + \sum_{a,b} C_{x,a}^\# \lambda^\#_{a,b}[G^\#_{a,b}], \quad p_x^\#(F) = \sum_{a,b \in D} p_{a,x}^\# \lambda^\#_{a,b} \mu_{b,x}$$

and $\lambda^\#_{x}(F) = \lambda_x(1 - p_x^\#(F))$.

A loop in $X$ which hits $F$ can be decomposed into a loop $l(F)$ in $F$ and its excursions in $D$ which may come back to their starting point. Let $\mu_{D}^{a,b}$ denote the bridge measure (with mass $[G^\#_{a,b}]$) associated with $e^\#$.

Set

$$\nu_{x,y}^D = \frac{1}{C_{x,y}} [C_{x,y} \delta_0 + \sum_{a,b \in D} C_{x,a} C_{b,y} \mu_{a,b}^D], \quad p_x^D = \sum_{n=1}^{\infty} \frac{1}{\lambda_x p_x^D} \left( \sum_{a,b \in D} C_{x,a} C_{b,x} \mu_{a,b}^D \right)$$

and $\nu_x^D = \frac{1}{1 - \rho_x^D} \delta_0 + \sum_{n=1}^{\infty} [p_x^D]^{n} \rho_x^D = \nu_x^D(1) = \nu_x^D(1) = 1$.

Note that $\rho_x^D(1) = \nu_{x,y}^D(1) = \nu_x^D(1) = 1$.

A loop $l$ can be decomposed into its restriction $l(F) = (\xi, \tau)$ in $F$ (possibly a one point loop), a family of excursions $\gamma_{\xi,\xi+1}$ attached to the jumps of $l(F)$ and systems of i.i.d. excursions $(\gamma_{h, \xi}, h \leq n_{\xi})$ attached to the points of $l(F)$. Note the set of excursions can be empty.

We get a decomposition of $\mu$ into its restriction $\mu^D$ to loops in $D$ (associated to the process killed at the exit of $D$), the loop measure $\mu(F)$ defined on loops of $F$ by the trace of the Markov chain on $F$, probability measures $\nu_{x,y}^D$ on excursions in $D$ indexed by pairs of points in $F$ and measures $\rho_x^D$ on excursions in $D$ indexed by points of $F$. Moreover, the integers $n_{\xi}$ follow a Poisson distribution of parameter $\lambda_{\xi}^{\#} \tau_1$ and the conditional distribution of the rescaled holding times in $\xi$ before each excursion $\gamma_{\xi}$ is the distribution $\beta_{n_{\xi}, \tau}$ of the increments of a uniform sample of $n_{\xi}$ points in $[0, \tau]$ put in increasing order. We denote these holding times by $\tau_{\xi,h}$ and set $l = \Lambda(l(F), (\gamma_{\xi,\xi+1}), (n_{\xi}, \gamma_{h, \xi}, \tau_{h}))$.

Then $\mu - \mu^D$ is the image measure by $\Lambda$ of

$$\mu(F)(dl(F)) \prod (\nu_{\xi,\xi+1}^D)(d\gamma_{\xi,\xi+1}) e^{-\lambda_{\xi}^{\#} \tau_1} \sum_{k=0}^{\infty} \frac{\lambda_{\xi}^{\#} \tau_1^k}{k!} \delta_{l_{\xi} = k} \beta_{\gamma_{h, \xi}, \tau_{h}}(dl_{\xi,\xi+1})$$

The Poisson process $L_{\alpha}^D = \{l(F), l \in L_{\alpha}\}$ has intensity $\mu(F)$ and is independent of $L_{\alpha}^D$. 

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Note that $\mathcal{L}_{\alpha}^{[F]}$ is the restriction of $\widehat{\mathcal{L}}_{\alpha}$ to $F$. In particular, if $\chi$ is a measure carried by $D$, we have:

$$
\mathbb{E}(e^{-\langle \mathcal{L}_{\alpha,\chi} \rangle}|\mathcal{L}_{\alpha}^{[F]}) = \mathbb{E}(e^{-\langle \mathcal{L}_{\alpha}^{D} \rangle}|\mathcal{L}_{\alpha}^{[F]}) \left( \prod_{x,y \in F} e^{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x,y)} \right) \prod_{x \in F} e^{\lambda_{\alpha}^{[F]}_{x,x}(\mathcal{L}_{\alpha}^{[F]})(x)}
$$

(recall that $\mathcal{L}_{\alpha}^{[F]}$ is the restriction of $\widehat{\mathcal{L}}_{\alpha}$ to $F$). Also, if we condition on the set of discrete loops $\mathcal{D}\mathcal{L}_{\alpha}^{[F]}$

$$
\mathbb{E}(e^{-\langle \mathcal{L}_{\alpha,\chi} \rangle}|\mathcal{D}\mathcal{L}_{\alpha}^{[F]}) = \left[ \frac{Z_{e,D}}{Z_{e,F}} \right]^{\alpha} \left( \prod_{x,y \in F} \left[ \frac{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x,y)}{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x)} \right] \prod_{x \in F} e^{\lambda_{\alpha}^{[F]}_{x,x}(\mathcal{L}_{\alpha}^{[F]})(x)} \right) + 1
$$

where the last exponent $N_x + 1$ is obtained by taking into account the loops which have a trivial trace on $F$ (see formula (19)).

More generally we can show in the same way the following

**Proposition 23** If $C^\# = C$ on $F \times F$, and $\lambda = \lambda^\#$ on $F$, we denote $B^{e,e^\#}$ the multiplicative functional $\prod_{x,y} (\frac{C_{x,y}^\#}{C_{x,y}})^{N_{x,y}} e^{\sum_{x \in D} \hat{c}_x (\lambda_{x}^\# - \lambda_y)}$.

Then,

$$
\mathbb{E}(B^{e,e^\#}|\mathcal{L}_{\alpha}^{[F]}) = \left[ \frac{Z_{e,D}}{Z_{e,F}} \right]^{\alpha} \left( \prod_{x,y \in F} \left[ \frac{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x,y)}{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x)} \right] \prod_{x \in F} e^{\lambda_{\alpha}^{[F]}_{x,x}(\mathcal{L}_{\alpha}^{[F]})(x)} \right) + 1
$$

and

$$
\mathbb{E}(B^{e,e^\#}|\mathcal{D}\mathcal{L}_{\alpha}^{[F]}) = \left[ \frac{Z_{e,D}}{Z_{e,F}} \right]^{\alpha} \left( \prod_{x,y \in F} \left[ \frac{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x,y)}{\lambda_{\alpha}^{[F]}_{x,y}(\mathcal{L}_{\alpha}^{[F]})(x)} \right] \prod_{x \in F} e^{\lambda_{\alpha}^{[F]}_{x,x}(\mathcal{L}_{\alpha}^{[F]})(x)} \right) + 1
$$

These decomposition and conditional expectation formulas extend to include a current $\omega$. Note that $e^{(F)}$ will depend on $\omega$ unless it is closed (i.e. vanish on every loop) in $D$. In particular, it allows to define $\omega^F$ such that:

$$
Z_{e,\omega} = Z_{e,D} Z_{e,F} |_{\omega^F}
$$

The previous proposition implies the following **Markov property**:
Remark 24 If \( D = D_1 \cup D_2 \) with \( D_1 \) and \( D_2 \) strongly disconnected, (i.e. such that for any \((x, y, z) \in D_1 \times D_2 \times F\), \( C_{x,y} \) and \( C_{x,z}C_{y,z} \) vanish), the restrictions of the network \( L_\alpha \) to \( D_1 \cup F \) and \( D_2 \cup F \) are independent conditionally to the restriction of \( L_\alpha \) to \( F \).

Proof. It follows from the fact that as \( D_1 \) and \( D_2 \) are disconnected, any excursion measure \( \nu^D_{x,y} \) or \( \rho^D_x \) from \( F \) into \( D = D_1 \cup D_2 \) is an excursion measure either in \( D_1 \) or in \( D_2 \).

Branching processes with immigration An interesting example can be given after extending slightly the scope of the theory to countable transient symmetric Markov chains: We can take \( X = \mathbb{N} - \{0\} \), \( C_{n,n+1} = 1 \) for all \( n \geq 1 \) and \( \kappa_1 = 1 \) and \( P \) to be the transfer matrix of the simple symmetric random walk killed at 0.

Then we can apply the previous considerations to check that \( \hat{L}_\alpha^n \) is a branching process with immigration.

The immigration at level \( n \) comes from the loops whose infimum is \( n \) and the branching from the excursions of the loops existing at level \( n \) to level \( n + 1 \). Set \( F_n = \{1, 2, \ldots n\} \) and \( D_n = F_n^{c} \).

The immigration law (on \( \mathbb{R}^+ \)) is a Gamma distribution \( \Gamma(\alpha, G^{1,1}) \). It is the law of \( \hat{L}_\alpha^1 \) and also of \([\hat{L}_\alpha^{D_{n-1}}]^n\) for all \( n > 1 \). From the above calculations of conditional expectations, we get that for any positive parameter \( \gamma \),

\[
\mathbb{E}(e^{-[\gamma L_\alpha^1 \hat{L}_\alpha^n]|L_\alpha^{\{F_n-1\}}}) = \mathbb{E}(e^{-[\gamma L_\alpha^{D_{n-1}}]^n})e^{\lambda_{n-1}^{\{F_n-1,\gamma\delta_n\}} - \lambda_{n-1}^{\{F_n-1\}}\hat{L}_\alpha^{n-1}}
\]

From this formula, it is clear that \( \hat{L}_\alpha^n \) is a Markov process. To be more precise, note that for any \( n, m > 0 \), \( V_m^n = 2(n \wedge m) \) and \( \lambda_n = 2 \), that \( G_{\gamma \delta_n} = G^{1,n} - G^{1,1}G^{1,n}_{\gamma \delta_n} \) so that \( G_{\gamma \delta_1}^{1,n} = \frac{1}{1+\gamma} \) and that for any \( n > 0 \), the restriction of the Markov chain to \( D_n \) is isomorphic to the original Markov chain. Then it comes that for all \( n \), \( p_n^{\{F_n\}} = \frac{1}{2} \), \( \lambda_n^{\{F_n\}} = 1 \), \( p_n^{\{F_{n,\gamma \delta_{n+1}}\}} = \frac{1}{2(1+\gamma)} \) and \( \lambda_n^{\{F_{n,\gamma \delta_{n+1}}\}} = \frac{2^\gamma+1}{1+\gamma} \) so that the Laplace exponent of the convolution semigroup \( \nu_t \) defining the branching mechanism equals \( \frac{\gamma}{1+\gamma} = \int (1 - e^{-\gamma s})e^{-s}ds \). It is the semigroup of a compound Poisson process whose Levy measure is exponential. The conditional law of \( \hat{L}_\alpha^{n+1} \) given \( \hat{L}_\alpha^n \) is the convolution of the immigration law \( \Gamma(\alpha,1) \) with \( \nu^{\hat{L}_\alpha^n} \).
Alternatively, we can consider the integer valued process $N_n(L_n^{(F_0)}) + 1$ which is a Galton Watson process with immigration. In our example, we find the reproduction law $\pi(n) = 2^{-n-1}$ for all $n \geq 0$ (critical binary branching).

If we consider the occupation field defined by the loops going through 1, we get a branching process without immigration: it is the classical relation between random walks local times and branching processes.

9 The case of general Markov processes

We now explain briefly how some of the above results will be extended to a symmetric Markov process on an infinite space $X$. The construction of the loop measure as well as a lot of computations can be performed quite generally, using Markov processes or Dirichlet space theory (Cf for example [4]). It works as soon as the bridge or excursion measures $P^{x,y}_t$ can be properly defined. The semigroup should have a locally integrable kernel $p_t(x,y)$.

Let us consider more closely the occupation field $\hat{I}$. The extension is rather straightforward when points are not polar. We can start with a Dirichlet space of continuous functions and a measure $m$ such that there is a mass gap. Let $P_t$ the associated Feller semigroup. Then the Green function is well defined as the mutual energy of the Dirac measures $\delta_x$ and $\delta_y$ which have finite energy. It is the covariance function of a Gaussian free field $\phi(x)$, which will be associated to the field $\hat{L}_x^1$ of local times of the Poisson process of random loops whose intensity is given by the loop measure defined by the semigroup $P_t$. This will apply to examples related to one dimensional Brownian motion or to Markov chains on countable spaces.

When we consider Brownian motion on the half line, we get a continuous branching process with immigration, as in the discrete case.

When points are polar, one needs to be more careful. We will consider only the case of the two and three dimensional Brownian motion in a bounded domain $D$ killed at the boundary, i.e. associated with the classical energy with Dirichlet boundary condition. The Green function does not induce a trace class operator but it is still Hilbert-Schmidt which allows to define renormalized determinants $\det_2$ (Cf [21]).

If $A$ is a symmetric Hilbert Schmidt operator, $\det_2(I + A)$ is defined as $\prod(1 + \lambda_i)e^{-\lambda_i}$ where $\lambda_i$ are the eigenvalues of $A$. 

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The Gaussian field (called free field) whose covariance function is the Green function is now a generalized field: Generalized fields are not defined pointwise but have to be smeared by a test function $f$. Still $\phi(f)$ is often denoted $\int \phi(x)f(x)dx$.

Wick powers : $\phi^n$ of the free field can be defined as generalized field by approximation as soon as the $2^n$-th power of the Green function, $G(x, y)^{2n}$ is locally integrable (Cf [22]). This is the case for all $n$ for Brownian motion in dimension two, as the Green function has only a logarithmic singularity on the diagonal, and for $n = 2$ in dimension three as the singularity is of the order of $1/\|x - y\|$. More precisely, taking for example $\pi_x^\varepsilon(dy)$ to be the normalized area measure on the sphere of radius $\varepsilon$ around $x$, $\phi(\pi_x^\varepsilon)$ is a Gaussian field with covariance $\sigma_x^\varepsilon \pi_x^\varepsilon(dx)$.

In these cases, we can extend the statement of theorem 9 to the renormalized occupation field $\hat{\mathcal{L}}_{x}$ and the Wick square $\phi^2$ of the free field.
Lemma 25 For any non negative function $f$,
\[
\mu(\langle \hat{t}, f \rangle^n) = (n-1)! \int G(x_1, x_2)f(x_2)G(x_2, x_3)f(x_3)\ldots G(x_n, x_1)f(x_1) \prod_{i=1}^{n} dx_i
\]

One can define in a similar way the analogous of multiple local times, and get for their integrals with respect to $\mu$ a formula analogous to the one obtained in the discrete case.

Let $G$ denote the operator on $L^2(D, dx)$ defined by $G$. Let $f$ be a non negative continuous function with compact support in $D$.

Note that $\langle \hat{t}, f \rangle$ is $\mu$-integrable only in dimension one as then, $G$ is locally trace class.

In that case, using for all $x$ an approximation of the Dirac measure at $x$, local times $\hat{t}^x$ can be defined in such a way that $\langle \hat{t}, f \rangle = \int \hat{t}^x f(x)dx$.

$\langle \hat{t}, f \rangle$ is $\mu$-square integrable in dimensions one, two and three, as $G$ is Hilbert-Schmidt if $D$ is bounded, since $\int \int_{D \times D} G(x, y)^2 dxdy < \infty$, and otherwise locally Hilbert-Schmidt.

N.B.: Considering distributions $\chi$ such that $\int \int (G(x, y)^2 \chi(dx) \chi(dy) < \infty$, we could see that $\langle \hat{t}, \chi \rangle$ can be defined by approximation as a square integrable variable and $\mu(\langle \hat{t}, \chi \rangle^2) = \int (G(x, y)^2 \chi(dx) \chi(dy)$.

Let $z$ be a complex number such that $\text{Re}(z) > 0$.

Note also that $e^{-z \langle \hat{t}, f \rangle} + z \langle \hat{t}, f \rangle - 1$ is bounded by $\frac{|z|^2}{2} \langle \hat{t}, f \rangle^2$ and expands as an alternating series $\sum_{n=2}^{\infty} \frac{z^n}{n!} (-\langle \hat{t}, f \rangle)^n$, with $|e^{-z \langle \hat{t}, f \rangle} - 1 - \sum_{n=1}^{N} \frac{z^n}{n!} (-\langle \hat{t}, f \rangle)^n| \leq \frac{|z \langle \hat{t}, f \rangle|^{N+1}}{(N+1)!}$.

Then, for $|z|$ small enough, it follows from the above lemma that
\[
\mu(e^{-z \langle \hat{t}, f \rangle} + z \langle \hat{t}, f \rangle - 1) = \sum_{n=2}^{\infty} \frac{z^n}{n!} Tr((M_{\sqrt{\mathcal{J}}}G_{\sqrt{\mathcal{J}}})^n)
\]

As $M_{\sqrt{\mathcal{J}}}G_{\sqrt{\mathcal{J}}}$ is Hilbert-Schmidt $\det_2(I + z M_{\sqrt{\mathcal{J}}}G_{\sqrt{\mathcal{J}}})$ is well defined and the second member writes $-\log(\det_2(I + z M_{\sqrt{\mathcal{J}}}G_{\sqrt{\mathcal{J}}}))$.

Then the identity
\[
\mu(e^{-z \langle \hat{t}, f \rangle} + z \langle \hat{t}, f \rangle - 1) = -\log(\det_2(I + z M_{\sqrt{\mathcal{J}}}G_{\sqrt{\mathcal{J}}}))
\]
extends, as both sides are analytic as locally uniform limits of analytic functions, to all complex values with positive real part.

The renormalized occupation field \( \widetilde{L}_\alpha \) is defined as the compensated sum of all \( \widetilde{l} \) in \( \mathcal{L}_\alpha \) (formally, \( \widetilde{L}_\alpha = \mathcal{L}_\alpha - \int f^{T(i)}_0 \delta_{l_i} ds \mu(dl) \)) By a standard argument used for the construction of Levy processes,

\[
\langle \widetilde{L}_\alpha, f \rangle = \lim_{\varepsilon \to 0} \left( \sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds - \alpha \mu(1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds) ) \right)
\]

We can denote \( \lim_{\varepsilon \to 0} \langle \widetilde{L}_{\alpha, \varepsilon}, f \rangle \) which converges a.s. and in \( L^2 \), as

\[
E( (\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds - \alpha \mu(1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds))^2 ) = \alpha \int (1_{\{T > \varepsilon\}} \int_0^T f(\gamma_s) ds)^2 \mu(dl)
\]

and \( \mathbb{E}(\langle \widetilde{L}_\alpha, f \rangle^2) = Tr((\sqrt{\mathcal{L}}G\sqrt{\mathcal{L}})^2) \). Note that if we fix \( f \), \( \alpha \) can be considered as a time parameter and \( \langle \widetilde{L}_{\alpha, \varepsilon}, f \rangle \) as Levy processes with discrete positive jumps approximating a Levy process with positive jumps \( \langle \widetilde{L}_\alpha, f \rangle \). The Levy exponent \( \mu(1_{\{T > \varepsilon\}}(e^{-\langle \widetilde{l}, f \rangle} + \langle \widetilde{l}, f \rangle - 1)) \) of \( \langle \widetilde{L}_{\alpha, \varepsilon}, f \rangle \) converges towards the Levy exponent of \( \langle \widetilde{L}_\alpha, f \rangle \) which is \( \mu((e^{-\langle \widetilde{l}, f \rangle} + \langle \widetilde{l}, f \rangle - 1)) \).

And, from the identity \( E(e^{-\langle \widetilde{L}_\alpha, f \rangle}) = e^{-\alpha \mu(\langle \widetilde{l}, f \rangle + \langle \tilde{I}, f \rangle - 1)} \), we get the

**Theorem 26** Assume \( d \leq 3 \). Denoting \( \widetilde{L}_\alpha \) the compensated sum of all \( \widetilde{l} \) in \( \mathcal{L}_\alpha \), we have

\[
E(e^{-\langle \widetilde{L}_\alpha, f \rangle}) = \det_\alpha(I + M\sqrt{\mathcal{L}}GM\sqrt{\mathcal{L}})^{-\alpha}
\]

Moreover \( e^{-\langle \widetilde{L}_{\alpha, \varepsilon}, f \rangle} \) converges a.s. and in \( L^1 \) towards \( e^{-\langle \widetilde{L}_\alpha, f \rangle} \).

Considering distributions of finite energy \( \chi \) (i.e. such that \( \int (G(x, y))^2 \chi(dx) \chi(dy) < \infty \)), we can see that \( \langle \widetilde{L}_\alpha, \chi \rangle \) can be defined by approximation as \( \lim_{\lambda \to \infty} \langle \widetilde{L}_\alpha, \lambda G \chi \rangle \) and

\[
\mathbb{E}(\langle \widetilde{L}_\alpha, \chi \rangle^2) = \alpha \int (G(x, y))^2 \chi(dx) \chi(dy).
\]

Specializing to \( \alpha = \frac{k}{2} \), \( k \) being any positive integer we have:
Corollary 27  The renormalized occupation field $\tilde{L}_k$ and the Wick square $\frac{1}{2} : \sum_1^k \phi_i^2$ have the same distribution.

If $\Theta$ is a conformal map from $D$ onto $\Theta(D)$, it follows from the conformal invariance of the Brownian trajectories that a similar property holds for the Brownian loop soup (Cf [9]). More precisely, if $c(x) = \text{Jacobian}_{\Theta}(x)$ and, given a loop $l$, if $T^c(l)$ denotes the reparametrized loop $l_{\tau_s}$ with $\int_0^T c(l_u) du = s$, $\Theta T^c(\mathcal{L}_\alpha)$ is the Brownian loop soup of intensity parameter $\alpha$ on $\Theta(D)$. Then we have the following:

Proposition 28  $\Theta(c\tilde{L}_\alpha)$ is the renormalized occupation field on $\Theta(D)$.

Proof.  We have to show that the compensated sum is the same if we perform it after or before the time change. For this it is enough to check that

$$E\left(\left[\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \eta\}} - 1_{\{T \leq \varepsilon\}}) \int_0^T f(\gamma_s) ds - \alpha \int (1_{\{T > \eta\}} - 1_{\{T \leq \varepsilon\}}) \int_0^T f(\gamma_s) ds \mu(d\gamma)\right]^2\right)$$

$$= \alpha \int (1_{\{T > \eta\}} - 1_{\{T \leq \varepsilon\}}) \int_0^T f(\gamma_s) ds^2 \mu(d\gamma)$$

and

$$E\left(\left[\sum_{\gamma \in \mathcal{L}_\alpha} (1_{\{T > \varepsilon\}} - 1_{\{T \leq \eta\}}) \int_0^T f(\gamma_s) ds - \alpha \int (1_{\{T > \varepsilon\}} - 1_{\{T \leq \eta\}}) \int_0^T f(\gamma_s) ds \mu(d\gamma)\right]^2\right)$$

$$\alpha \int (1_{\{T > \varepsilon\}} - 1_{\{T \leq \eta\}}) \int_0^T f(\gamma_s) ds^2 \mu(d\gamma)$$

converge to zero as $\varepsilon$ and $\eta$ go to zero. It follows from the fact that:

$$\int (1_{\{T \leq \varepsilon\}}) \int_0^T f(\gamma_s) ds^2 \mu(d\gamma)$$

and

$$\int (1_{\{T \leq \eta\}}) \int_0^T f(\gamma_s) ds^2 \mu(d\gamma)$$

converge to 0. The second follows easily from the first if $c$ is bounded away from zero. We can always consider the "loop soups" in an increasing sequence of relatively compact open subsets of $D$ to reduce the general case to that situation. ■
As in the discrete case (see corollary 6), we can compute product expectations. In dimensions one and two, for \( f_j \) continuous functions with compact support in \( D \):

\[
\mathbb{E}(\langle \tilde{L}_\alpha, f_1 \rangle ... \langle \tilde{L}_\alpha, f_k \rangle) = \int \text{Per}_\alpha^0(G(x_l, x_m), 1 \leq l, m \leq k) \prod f_j(x_j)dx_j \tag{24}
\]

10 Renormalized powers

In dimension one, powers of the occupation field can be viewed as integrated self intersection local times. In dimension two, renormalized powers of the occupation field, also called renormalized self intersections local times can be defined as follows:

**Theorem 29** Assume \( d = 2 \). Let \( \pi^x_\varepsilon(dy) \) be the normalized arclength on the circle of radius \( \varepsilon \) around \( x \), and set \( \sigma^x_\varepsilon = \int G(y, z)\pi^z_\varepsilon(dy)\pi^y_\varepsilon(dz) \). Then, \( \int f(x)Q^\alpha,\sigma^x_\varepsilon(\langle \tilde{L}_\alpha, \pi^x_\varepsilon \rangle)dx \) converges in \( L^2 \) for any bounded continuous function \( f \) with compact support towards a limit denoted \( \langle \tilde{L}_\alpha, f \rangle \) and

\[
\mathbb{E}(\langle \tilde{L}_\alpha, f \rangle \langle \tilde{L}_\alpha, h \rangle) = \delta_{l,k} \frac{\alpha^{(\alpha+1)} - \alpha^{(\alpha+k-1)}}{k!} \int G^{2k}(x, y)f(x)h(y)dxdy.
\]

**Proof.** The idea of the proof can be understood by trying to prove that \( \mathbb{E}(\int f(x)Q^\alpha,\sigma^x_\varepsilon(\langle \tilde{L}_\alpha, \pi^x_\varepsilon \rangle)dx)^2 \) remains bounded as \( \varepsilon \) decreases to zero. The idea is to expand this expression in terms of sums of integrals of product of Green functions and check that the combinatorial identities (15) imply the cancelation of the logarithmic divergences.

This is done by showing (as done below in the proof of the theorem) one can modify slightly the products of Green functions appearing in \( \mathbb{E}(Q^\alpha,\sigma^x_\varepsilon(\langle \tilde{L}_\alpha, \pi^x_\varepsilon \rangle)Q^\alpha,\sigma^y_\varepsilon(\langle \tilde{L}_\alpha, \pi^y_\varepsilon \rangle)) \) to replace them by products of the form \( G(x, y)\Sigma^i (\sigma^x_\varepsilon)^i(\sigma^y_\varepsilon)^h \). The cancelation of terms containing \( \sigma^x_\varepsilon \) and/or \( \sigma^y_\varepsilon \) then follows directly from the combinatorial identitites.

Let us now prove the theorem. Consider first, for any \( x_1, x_2, ... x_n \), \( \varepsilon \) small enough and \( \varepsilon \leq \varepsilon_1, ... \varepsilon_n \leq 2\varepsilon \), with \( \varepsilon_i = \varepsilon_j \) if \( x_i = x_j \), an expression of the form:

\[
\Delta = \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma^x_{\varepsilon_i})^{m_i} - \int G(y_1, y_2) ... G(y_n, y_1)\pi^x_{\varepsilon_1}(dy_1) ... \pi^x_{\varepsilon_n}(dy_n)
\]
in which we define $m_i$ as $\sup(h,x_{i+h}=x_i)$.

In the integral term, we first replace progressively $G(y_{i-1}, y_i)$ by $G(x_{i-1}, x_i)$ whenever $x_{i-1} \neq x_i$, using triangle, then Schwartz inequality, to get an upper bound of the absolute value of the difference made by this substitution in terms of a sum $\Delta'$ of expressions of the form

$$\prod_l G(x_l, x_{l+1}) \sqrt{\int (G(y_1, y_2) - G(x_1, x_2))^2 \pi^{x_1}_{\varepsilon_1}(dy_1) \pi^{x_2}_{\varepsilon_2}(dy_2) \int \prod \pi^{x_k}_{\varepsilon_k}(dy_k)}.$$ 

The expression obtained after these substitutions can be written

$$W = \prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i) \int G(y_1, y_2) \ldots G(y_{m_i-1}, y_{m_i}) \pi^{x_1}_{\varepsilon_1}(dy_1) \ldots \pi^{x_i}_{\varepsilon_i}(dy_{m_i})$$

and we see the integral terms could be replaced by $(\sigma^{x_i})^{m_i}$ if $G$ was translation invariant. But as the distance between $x$ and $y$ tends to 0, $G(x, y)$ is equivalent to $G_0(x, y) = \frac{1}{2\pi} \log(||x - y||)$ and moreover, $G(x, y) = G_0(x, y) - H(x, dz)G_0(z, y)$, $H$ denoting the Poisson kernel on the boundary of $D$. As our points lie in a compact inside $D$, it follows that for some constant $C$, for $||y_1 - x|| \leq \varepsilon$, $\int (G(y_1, y_2) \pi^{x_2}_{\varepsilon}(dy_2) - \sigma^{x_2}) < C \varepsilon$.

Hence, the difference $\Delta''$ between $W$ and $\prod_{i, x_{i-1} \neq x_i} G(x_{i-1}, x_i)(\sigma^{x_i})^{m_i}$ can be bounded by $\varepsilon W'$, where $W'$ is an expression similar to $W$.

To get a good upper bound on $\Delta$, using the previous observations, by repeated applications of Hölder inequality, it is enough to show that for $\varepsilon$ small enough, $C$ and $C'$ denoting various constants:

1) $\int (G(y_1, y_2) - G(x_1, x_2))^2 \pi^{x_1}_{\varepsilon_1}(dy_1) \pi^{x_2}_{\varepsilon_2}(dy_2) < C(\varepsilon 1_{||x_1 - x_2|| \geq \sqrt{\varepsilon}} + (G(x_1, x_2)^2 + \log(\varepsilon)^2)1_{||x_1 - x_2|| < \sqrt{\varepsilon}})$

2) $\int G(y_1, y_2)^k \pi^{x_1}_{\varepsilon_1}(dy_1) \pi^{x_2}_{\varepsilon_2}(dy_2) < C \log(\varepsilon)^k$

3) $\int G(y_1, y_2)^k \pi^{x_1}_{\varepsilon_1}(dy_1) \pi^{x_2}_{\varepsilon_2}(dy_2) < C \log(\varepsilon)^k$

As the main contributions come from the singularities of $G$, they follow from the following simple inequalities:

1') $\int |\log(\varepsilon^2 + 2R\varepsilon \cos(\theta) + R^2) - \log(R)|^2 d\theta$

$= \int |\log((\varepsilon/R)^2 + 2(\varepsilon/R) \cos(\theta) + 1)|^2 d\theta < C((\varepsilon 1_{R \geq \sqrt{\varepsilon}}) + \log^2(R/\varepsilon)1_{R < \sqrt{\varepsilon}}))$
2') \( \int |\log(2 + 2 \cos(\theta))|^k d\theta \leq C |\log(\varepsilon)|^k \)

3') \( \int |\log(\varepsilon_1 \cos(\theta_1) + \varepsilon_2 \cos(\theta_2) + r)^2 + (\varepsilon_1 \sin(\theta_1) + \varepsilon_2 \sin(\theta_2))^2|^k d\theta_1 d\theta_2 \leq C(|\log(\varepsilon)|^k). \)

Indeed, developing the polynomials and using formula (24) we can express this expectation as a linear combination of integrals under identity (15) to get the value \( \delta_{t,k} G(x, y) \frac{2k\alpha(\alpha + 1)\cdots(\alpha + k - 1)}{k!} \)

\[ C \log(\varepsilon)^{N_{i,k}}(\sqrt{\varepsilon} + G(x, y)^{2k}1_{||x,y||<\sqrt{\varepsilon}}) \] (25)

Let us now show that for \( \varepsilon \leq \varepsilon_1, \varepsilon_2 \leq 2\varepsilon \), we have, for some integer \( N_{n,k} \)

\[
\left| \mathbb{E}(Q_k^{\alpha_1,\sigma_1}(\langle \tilde{L}_\alpha, \pi_{x_1}^x \rangle)Q_{l,k}^{\alpha_2,\sigma_2}(\langle \tilde{L}_\alpha, \pi_{y_2}^y \rangle)) - \delta_{t,k} G(x, y) \frac{2k\alpha(\alpha + 1)\cdots(\alpha + k - 1)}{k!} \right| 
\leq C \log(\varepsilon)^{N_{i,k}}(\sqrt{\varepsilon} + G(x, y)^{2k}1_{||x,y||<\sqrt{\varepsilon}})
\]

Indeed, developing the polynomials and using formula (24) we can express this expectation as a linear combination of integrals under products of \( G(x_i, y_i), G(x_j, y_j) \) and \( G(y_j, y_{j'}) \) as we did in the discrete case. If we replace each \( G(x_i, y_j) \) by \( G(x, y) \), each \( G(x_i, x_i') \) by \( \sigma_{x_1}^x \) and each \( G(y_j, y_j') \) by \( \sigma_{y_2}^y \), we can use the combinatorial identity (15) to get the value \( \delta_{t,k} G(x, y) \frac{2k\alpha(\alpha + 1)\cdots(\alpha + k - 1)}{k!} \). Then, the above results allow to bound the error made by this replacement.

The bound (25) is uniform in \( (x, y) \) only away from the diagonal as \( G(x, y) \) can be arbitrarily large but we conclude from it that for any bounded integrable \( f \) and \( h \),

\[
\left| \int \mathbb{E}(Q_k^{\alpha_1,\sigma_1}(\langle \tilde{L}_\alpha, \pi_{x_1}^x \rangle)Q_{l,k}^{\alpha_2,\sigma_2}(\langle \tilde{L}_\alpha, \pi_{y_2}^y \rangle)) - \delta_{t,k} G(x, y) \frac{2k\alpha(\alpha + 1)\cdots(\alpha + k - 1)}{k!} f(x)h(y)dx\,dy \right| 
\leq C' \sqrt{\varepsilon} \log(\varepsilon)^{N_{i,k}}(\sqrt{\varepsilon} + G(x, y)^{2k}1_{||x,y||<\sqrt{\varepsilon}})
\]

(as \( \int G(x, y)^{2k}1_{||x,y||<\sqrt{\varepsilon}}dx\,dy \) can be bounded by \( C\varepsilon^{\frac{2}{3}} \), for example).

Taking \( \varepsilon_n = 2^{-n} \), it is then straightforward to check that \( \int f(x)Q_k^{\alpha_1,\sigma_1}(\langle \tilde{L}_\alpha, \pi_{x_n}^x \rangle)dx \)

is a Cauchy sequence in \( L^2 \). The theorem follows.

Specializing to \( \alpha = \frac{k}{2} \), \( k \) being any positive integer as before, Wick powers of \( \sum_{j=1}^{k} \sigma_j^2 \)

are associated with self intersection local times of the loops. More precisely, we have:
Proposition 30  The renormalized self intersection local times $\tilde{L}^{n}_{k}$ and the Wick powers $\frac{1}{2^n n!} : (\sum_{l=1}^{k} \phi_{l}^{2})^{n} :$ have the same joint distribution.

The proof is similar to the one given in [13] and also to the proof of the above theorem, but simpler. It is just a calculation of the $L^{2}$-norm of

$$\int \left[ (\phi^{2})^{n} : (x) - Q_{n}^{\frac{1}{2}} \sigma_{x}^{2} (\phi_{2}^{2} : (\pi_{x}^{2}) ) \right] f(x) dx$$

which converges to zero with $\varepsilon$.

Final remarks:

a) These generalized fields have two fundamental properties:

Firstly they are local fields (or more precisely local functionals of the field $\tilde{L}^{k}_{\alpha}$ in the sense that their values on functions supported in an open set $D$ depend only on the trace of the loops on $D$.

Secondly, noting we could use different regularizations to define $\tilde{L}^{k}_{\alpha}$, the action of a conformal transformation $\Theta$ on these fields is given by the $k$-th power of the conformal factor $c = Jacobian(\Theta)$. More precisely, $\Theta(c^{k} \tilde{L}^{k}_{\alpha})$ is the renormalized $k$-th power of the occupation field in $\Theta(D)$.

b) It should be possible to derive from the above remark the existence of exponential moments and introduce non trivial local interactions as in the constructive field theory derived from the free field (Cf [22]).

c) Let us also briefly consider currents. We will restrict our attention to the one and two dimensional Brownian case, $X$ being an open subset of the line or plane. Currents can be defined by vector fields, with compact support.

Then, if now we denote by $\phi$ the complex valued free field (its real and imaginary parts being two independent copies of the free field), $\int_{1}^{\omega}$ and $\int_{X} (\phi \partial_{\omega} \phi - \phi \partial_{\omega} \phi) dx$ are well defined square integrable variables in dimension 1 (it can be checked easily by Fourier series). The distribution of the centered occupation field of the loop process "twisted" by the complex exponential $\exp(\sum_{l=1}^{k} \int_{1}^{1} i \omega + \frac{1}{2} (\| \omega \|^{2})$) appears to be the same as the distribution of the field $\phi \phi$ : ”twisted” by the complex exponential $\exp(\int_{X} (\phi \partial_{\omega} \phi - \phi \partial_{\omega} \phi) dx )$ (Cf[14]).

In dimension 2, logarithmic divergences occur.
d) There is a lot of related investigations. The extension of the properties proved here in the finite framework has still to be completed, though the relation with spanning trees should follow from the remarkable results obtained on SLE processes, especially [11]. Note finally that other essential relations between SLE processes, loops and free fields appear in [27], [20] and [1].

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