Self-similar solutions for active scalar equations in Fourier-Besov-Morrey spaces

Lucas C. F. Ferreira
Universidade Estadual de Campinas, IMECC- Departamento de Matemática,
Rua Sérgio Buarque de Holanda, 651, CEP 13083-859, Campinas-SP, Brazil.
email: lcff@ime.unicamp.br

Lidiane S. M. Lima
Universidade Estadual de Campinas, IMECC- Departamento de Matemática,
Rua Sérgio Buarque de Holanda, 651, CEP 13083-859, Campinas-SP, Brazil.
email: lidynet@hotmail.com

Abstract
We are concerned with a family of dissipative active scalar equation with velocity fields coupled via multiplier operators that can be of high-order. We consider sub-critical values for the fractional diffusion and prove global well-posedness of solutions with small initial data belonging to a framework based on Fourier transform, namely Fourier-Besov-Morrey spaces. Since the smallness condition is with respect to the weak norm of this space, some initial data with large $L^2$-norm can be considered. Self-similar solutions are obtained depending on the homogeneity of the initial data and couplings. Also, we show that solutions are asymptotically self-similar at infinity. Our results can be applied in a unified way for a number of active scalar PDEs like 1D models on dislocation dynamics in crystals, Burguer's equations, 2D vorticity equation, 2D generalized SQG, 3D magneto-geostrophic equations, among others.

AMS MSC: 35Q35, 35A01, 35C06, 35B40, 35R11, 42B35

Keywords: Active scalar equations; global well-posedness; self-similar solutions; asymptotic behavior

1 Introduction

Frameworks based on the Fourier transform give an elegant way of handling PDEs and may reveal underlying mathematical and physical aspects connected with low and high frequencies. Our main intent is to obtain a theory of self-similar solutions and stability in such type of framework that can be applied in a unified way for a number of active scalar PDEs.

For that matter, we consider a family of dissipative active scalar equations with velocity fields coupled via certain multiplier operators, which includes PDEs with hamiltonian or gradient flow
structure. Examples of them are given below together with the corresponding literature. In fact, nice physical models arise in dimension \( n = 1, 2, 3 \). That family was introduced in [11] (see also [10]) and reads as

\[
\begin{aligned}
\frac{\partial \theta}{\partial t} + \kappa (-\Delta)^\gamma \theta + \nabla_x \cdot (u \theta) &= 0, \quad x \in \mathbb{R}^n, \ t > 0, \\
\theta(x, 0) &= \theta_0(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where \( n \geq 1 \), \( \gamma > 1/2 \), and the fractional dissipation is defined by \( \hat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma} \hat{f}(\xi) \). Since we are concerned with the dissipative case, we assume \( \kappa = 1 \) for the sake of simplicity.

The velocity field \( u \) is coupled to the scalar \( \theta \) via the multiplier vector operator

\[
u = P[\theta] = (u_1, u_2, ..., u_n)
\]

where

\[
u_k = \sum_{j=1}^{n} a_{jk} R_j \Lambda^{1-1} P_j[\theta], \quad \text{for } 1 \leq k \leq n,
\]

\( \Lambda = (-\Delta)^{1/2} \), \( R_j = -\partial_j (-\Delta)^{-1/2} \) is the \( j \)-th Riesz transform, \( a_{jk} \)'s are constant and

\[
\hat{P}_j[\theta](\xi) = P_j(\xi) \hat{\theta}(\xi).
\]

In the case \( n = 1 \), \( R_j \)'s should be understood as the Hilbert transform

\[
\mathcal{H}(u)(x) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy.
\]

Let \( \mathcal{A} \) be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that \( \hat{f}|_{U} \in \mathcal{D}'(U) \) is a complex Radon measure on \( U \), for all bounded open \( U \), and the total variation \( |\hat{f}| \) (extended to \( \mathbb{R}^n \)) is a tempered measure, that is, \( \int_{\mathbb{R}^n}(1 + |x|)^{-N} |\hat{f}| \ < \infty \), for some \( N \geq 0 \). We assume that \( P_j \)'s are measurable functions such that

\[
|P_j(\xi)| \leq C|\xi|^{\beta}, \quad \text{a.e. } \xi \in \mathbb{R}^n, \quad (1.5)
\]

for all \( 1 \leq j \leq n \), where the parameter \( \beta \in [0, n+1) \) with \( \beta < 2\gamma \); and so \( P_j[\cdot] \) makes sense from \( \mathcal{A} \) to \( \mathcal{S}'(\mathbb{R}^n) \). In view of (1.3) and (1.4), we can write the Fourier transform of the velocity field \( u \) as

\[
\hat{u}(\xi) = P(\xi) \hat{\theta}(\xi)
\]

where \( P(\xi) = [\hat{P}_1, ..., \hat{P}_n] \) with

\[
\hat{P}_k(\xi) = \sum_{j=1}^{n} a_{jk} \frac{i \xi_j}{|\xi|^2} P_j(\xi), \quad 1 \leq k \leq n,
\]

and so

\[
|\hat{u}(\xi)| \leq C|\xi|^{\beta-1}|\hat{\theta}(\xi)|.
\]
Assuming that $P_j$’s are homogeneous of degree $\beta$, (1.1) has the scaling map

$$\theta \rightarrow \theta_\lambda = \lambda^{2\gamma-\beta}\theta(\lambda x, \lambda^{2\theta} t), \text{ for all } \lambda > 0,$$

which induces the scaling for the initial condition

$$\theta_0 \rightarrow \lambda^{2\gamma-\beta}\theta_0(\lambda x).$$

Even when $P_j$’s are not homogeneous, the map (1.8) works well as an intrinsic scaling for (1.1)-(1.2) in the sense that it is useful to identify suitable functional spaces for global existence and provides a notion of criticality. Indeed, we have three basic cases: sub-critical $\beta < 2\gamma$, critical $\beta = 2\gamma$, and super-critical $\beta > 2\gamma$, which correspond to the sign of the exponent in (1.9).

From (1.7), one can see that (1.1)-(1.2) takes into account the effect of couplings with positive order when $\beta > 1$, which are named here as high-order ones. They behave morally like a positive derivative of $(\beta - 1)$-order, and produce more difficulties in comparison with SQG $(\beta = 1)$ and $\beta < 1$, at least in the context of $H^s$ and $L^p$-theory.

Here we consider a functional setting based on Fourier transform in which the existence theory works in a unified way for both low ($\beta < 1$) and high-order couplings. More precisely, we prove global well-posedness and stability results for (1.1)-(1.2) with small initial data belonging to the framework of homogeneous Fourier-Besov-Morrey spaces $\mathcal{F}N^{s,\mu}_{p,\infty}$ (see (2.3) for the definition), where $s = n - \frac{n-\mu}{p} - (2\gamma - \beta)$, $rac{n-p}{n+\beta+1-2\gamma} < p \leq \infty$, $0 \leq \mu < n$, $\gamma \neq 1/2$ and $0 \leq \beta < 2\gamma < \frac{n+\beta+1}{2}$ (see Theorem 3.1). In view of the norm (2.5), our results allow to take some large data in $H^s$ and $L^p$-spaces (see Remark 3.3), and homogeneous functions of degree $-(2\gamma - \beta)$. In fact, in the scale of $\mathcal{F}N^{s,\mu}_{p,\infty}$-spaces, the last property occurs only when $s = n - \frac{n-\mu}{p} - (2\gamma - \beta)$. So, we obtain that solutions are self-similar, i.e. invariant under (1.8), provided that the symbols $P_j$’s are homogeneous of degree $\beta$ and $\theta_0$ is homogeneous of degree $-(2\gamma - \beta)$. Motivated by that, we also analyse the asymptotic stability of solutions and obtain a basin of attraction around each self-similar solution.

A space naturally related to $\mathcal{F}N^{s,\mu}_{p,\infty}$ is the homogeneous Fourier-Besov space $\mathcal{F}B^{s}_{p,\infty}$ which was introduced by Konieczny-Yoneda [35] in order to study the Navier-Stokes-Coriolis system in $\mathbb{R}^3$ with $s = 2 - \frac{3}{p}$ and $p > 3$. Inspired by [35], we introduce $\mathcal{F}N^{s,\mu}_{p,\infty}$ that is larger than the Fourier-Besov space with the same scaling and seems to be new in analysis of PDEs, at least to the best of our knowledge. Precisely, we have the continuous inclusion $\mathcal{F}B^{s_2}_{p_2,\infty} \subset \mathcal{F}N^{s_1}_{p_1,\mu,\infty}$ for $1 \leq p_1 \leq p_2 < \infty$, $0 \leq \mu < n$ and $s_2 + \frac{n}{p_2} = s_1 + \frac{n-\mu}{p_1}$. Taking $\mu = 0$ in (2.3), the Morrey space $M_{p,\mu}$ coincides with $L^p$, and so $\mathcal{F}N^{0,0}_{p,\infty} = \mathcal{F}B^s_{p,\infty}$. In [3] the Navier-Stokes system was studied in $PM^{n-1}$ which is a subspace of $\mathcal{F}B^{n-1-n/p}_{p,\infty}$ with norm defined via Fourier transform too. The space $\mathcal{F}N^{s,\mu}_{p,\infty}$ with $s = n - 1 - \frac{n-\mu}{p}$ could be employed to provide a larger initial data class for uniform solvability (with respect to angular velocity) of the Navier-Stokes-Coriolis system. They also can be seen as a counterpart in Fourier variables of the homogeneous Besov-Morrey spaces $N^{s,\mu}_{p,\infty}$ introduced by [30] in order to study the Navier-Stokes equations (see Remark 2.1 for the definition). We also refer
Dissipative active scalar equations

the reader to [38] for useful properties, bilinear and pseudo-differential estimates in $\mathcal{N}^s_{p,\mu,\infty}$ spaces, and an extension of the analysis to compact Riemannian manifolds. Due to lack of Hausdorff-Young inequality on Morrey spaces, it seems that there are no inclusion relations between $\mathcal{F}\mathcal{N}^s_{p_1,\mu_1,\infty}$ and $\mathcal{N}^s_{p_2,\mu_2,\infty}$ for $1 \leq p_1, p_2 < \infty$, $0 < \mu_1, \mu_2 < n$, and $s_2 - \frac{n - \mu_2}{p_2} = s_1 + \frac{n - \mu_1}{p_1} - n$ (same scaling).

Active scalar equations arise in a number of important mathematical toy or physical models. For instance, in the case $n = 1$ we have Burguer’s equation ($\beta = 1$ and $u = \theta$) and the transport equation

$$\theta_t + (\mathcal{H}(\theta)\theta)_x + \kappa(-\Delta)\gamma \theta = 0 \tag{1.10}$$

where $\beta = 1$ and $u = \mathcal{H}(\theta)$ is the Hilbert Transform, a zero-order multiplier operator. For the former example, we refer the reader to [33] and [21] for results on blow-up, global existence and regularity of solutions belonging to Lebesgue or smooth spaces. Results on global existence, self-similarity, finite-time singularity and asymptotic behavior of solutions for (1.10) have also been obtained by several authors, see e.g. [6], [7], [8], [42], [9], and their references. Replacing in (1.10) $\mathcal{H}(\theta)\theta$ by $\theta \Lambda^\alpha \mathcal{H}(\theta)$, it leads to

$$\theta_t + (\theta \Lambda^\alpha \mathcal{H}(\theta))_x + \kappa(-\Delta)\gamma \theta = 0 \tag{1.11}$$

which has the form (1.1)-(1.2) with $\beta = \alpha + 1$ and $u = \Lambda^\alpha \mathcal{H}(\theta)$. Equations (1.10) and (1.11) are related to models on dislocation dynamics in crystals with $\theta$ representing the number density of fractures per unit length in the material (see e.g. [29], [19]). Self-similar asymptotic behavior of solutions for (1.11) with $u_0 \in BUC(\mathbb{R})$ and $\kappa = 0$ was proved in [4] showing an important role of self-similar solutions in the description of asymptotics for this and related models (see also [29]).

In the 2D case, we have the vorticity equation $u = \nabla^\perp(-\Delta)^{-\beta} \theta$ and SQG equation $u = \nabla^\perp((\Delta)\gamma \theta).$ The former is a very famous fluid dynamical model while the latter has been the object of a lot of papers concerning existence, uniqueness, regularity and asymptotic behavior of solutions in the inviscid case $\kappa = 0$ or in the subcritical ($1/2 < \gamma < 1$), critical ($\gamma = 1/2$) and supercritical ($\gamma \in (0, 1/2)$) ranges (see e.g. [2], [3], [15], [16], [17], [20], [30], [31], [32], [33], [43], [45], and their references).

In the 3D case, an example of active scalar PDE comes from magneto-geostrophic dynamics in a physical situation of fast rotating electrically conducting fluids, reading as

$$\frac{\partial \theta}{\partial t} + u \nabla_x \theta + (-\Delta)^\beta \theta = 0,$$ with $div(u) = 0$ and $u_j = \sum_{k=1}^3 \partial_k T_{kj} \theta,$ \tag{1.12}

where $(T_{kj})_{3 \times 3}$ is a matrix of Calderón-Zygmund singular integral operator (see [41], [27]). Notice that (1.11)-(1.2) recovers (1.12) with $n = 3$ and $\beta = 2$. In fact, (1.12) is a generalization of the physical case $\gamma = 1$ (which is critical) to include fractional dissipation. An important feature is that the presence of the underlying magnetic field generates a non-isotropic structure for the symbol of $T_{kj}$, and then the associated symbols $P_j$’s in (1.3) are non-radially symmetric (see [26]) for an explicit expression of $T_{kj}$). Global existence of smooth solutions with $\gamma = 1$ and $L^2$ initial data can be found
Dissipative active scalar equations

in [26] by using De Giorgi techniques in order to reach Hölder continuity of weak solutions. See also [25, 28] and their references for further results on well-posedness, regularity and stability in a framework of \( L^2 \) or smooth functions.

The dynamical of solutions naturally depend of the coupling between the velocity and active scalar. So the PDE (1.1) allows to analyse many kinds of dynamics by varying the symbol \( P_j \).

The results of [11] considered \( P[j] \) in (1.2) such that \( P_j \in C^\infty(\mathbb{R}^n\setminus\{0\}) \), \( P_j \) is radially symmetric, nondecreasing in \(|\xi|\), and satisfying a Hörmander-Mikhlin type condition. There the authors showed existence of global solutions in \( L^\infty((0, \infty); Y) \), where \( Y = L^1 \cap L^\infty \cap B_{s, M}^{q, \infty} \) with \( s > 1 \) and \( 2 \leq q \leq \infty \), by means of a priori estimates in Besov spaces and a successive approximation scheme. The space \( B_{s, M}^{q, \infty} \) is an extension of the classical Besov space \( B_{s, q}^{\infty} \) whose the norm increases depending on the growth of \( M \). A technical growth condition depending on \( M \) is also assumed for \( P_j \). Their results can be applied for several active scalars. For instance, for the generalized SQG

\[
\begin{align*}
\theta & = \Lambda^\beta - 1(\nabla \perp - 1 \nabla \theta), \quad (1.13)
\end{align*}
\]

with \( 0 \leq \beta < 2\gamma < 1 \) and \( n = 2 \). By varying the parameter \( \beta \) from 0 to 1, this equation interpolates 2D vorticity and SQG equations, and it is called modified SQG in the critical case \( \beta = 2\gamma \). The equation (1.1) with (1.13) has been studied for instance in [10], [14], [33], [37], [39], [40] where one can find existence and regularity results with data in Sobolev spaces \( H^m \) for \( m \geq 0 \). The conditions \( \kappa > 0 \), \( \beta \in [0, 1] \) and \( \beta = 2\gamma \) were assumed in [14], [33], [37], [39], and \( \kappa > 0 \) and \( 1 \leq \beta < 2\gamma < 2 \) in [40]. Also, local well-posedness of \( H^m \)-solutions for (1.1)-(1.13) was proved in the inviscid case \( \kappa = 0 \) for \( \beta \in [1, 2] \) and \( m \geq 4 \). Another application is for log-type couplings such as

\[
\begin{align*}
P_j(\xi) &= |\xi|^\alpha (\log(1 + |\xi|^2))^\chi, \quad \chi > 0, \\
P_j(\xi) &= |\xi|^\alpha (\log(1 + \log(1 + |\xi|^2)))^\chi, \quad \chi > 0,
\end{align*}
\]

which are interesting in view of the numerical evidences presented in [44] that, even for \( \kappa = 0 \), (1.1) with velocity

\[
\begin{align*}
\theta & = \nabla \perp (\log(I - \Delta))^{\chi} \theta, \quad \chi > 0, \\
\end{align*}
\]

may be globally well-posed. As a first step for this conjecture, the paper [10] proved local well-posedness of \( H^4 \)-solutions for (1.1)-(1.16) with \( \gamma > 0 \). In the case \( \alpha = 0 \) and \( n = 2 \), (1.14) and (1.15) correspond to log and log-log Navier-Stokes, respectively, which are intermediate models between 2D vorticity and SQG equations. Considering the inviscid case \( \kappa = 0 \), that is, log and log-log Euler equations, the authors of [12] proved global existence results for \( 0 \leq \chi \leq 1 \) and initial data \( \theta_0 \in L^1 \cap L^\infty \cap B_{s, \infty}^q \), where \( s > 1 \), \( q > 2 \) and \( B_{s, \infty}^q \) is the inhomogeneous Besov space.

Some symbols with log-type growth also provide examples of slightly supercritical active scalar equations that behave nicely from the viewpoint of global existence of smooth solutions. Relying on
the method of [31] based on modulus of continuity, the work [18] showed existence and uniqueness of global smooth solutions for (1.1) with $n = 2$, $\gamma = 1/2$, and a multiplier coupling operator

$$u = \nabla^\perp \Lambda^{-1} m(\Lambda) \theta,$$

(1.17)

where $m$ is a smooth, radial, nondecreasing function such that $m(\xi) \geq 1$, for all $\xi \in \mathbb{R}^2$. The authors also assumed a Hörmander-Mikhlin type condition and the growth hypothesis

$$\frac{m(\xi)}{\log \log |\xi|} \to 0 \text{ as } |\xi| \to \infty.$$

(1.18)

In a certain sense, the coupling (1.17) is critical at origin and supercritical at infinity at most by a $\log\log$ factor.

In [24], the authors tackled (1.1) with high-order couplings and showed global existence and decay of solutions in the critical Lebesgue space $L^{\frac{2n}{2n-\beta}}(\mathbb{R}^n)$. There they worked within a sub-range of $1 \leq \beta < 2\gamma < n + 1$ and considered $P_j \in C^{\left[\frac{n}{2}\right]+1}(\mathbb{R}^n \setminus \{0\})$ such that

$$\left| \frac{\partial^\alpha P_j}{\partial \xi^\alpha}(\xi) \right| \leq C|\xi|^{\beta-|\alpha|},$$

(1.19)

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq \left[\frac{n}{2}\right] + 1$ and $\xi \neq 0$. The condition (1.19) is also of Hörmander-Mikhlin type and allows non-radial symbols. The solutions in [24] are $C^\infty$-smooth for $t > 0$, but they do not necessarily belong to $L^2$ because $2 < \frac{n}{2\gamma-\beta}$ for either $n = 2, 3$ or $1/2 < \gamma < 1$. The approach in [24] relies on a combination of time-weighted Kato type norms, scaling arguments, $L^q$-maximum principles, and arguments of the type parabolic De Giorgi-Nash-Moser.

Due to the large number of works cited above, in what follows, we highlight the novelties of the present paper for the reader convenience. We provide self-similar solutions and asymptotic behavior results in a new framework outside $L^2$ by means of an approach based on Fourier transform. These results can be applied in a unified way for the above mentioned couplings in the subcritical range $\beta < 2\gamma$. Our conditions cover symbols $P_j \notin C(\mathbb{R}^n \setminus \{0\})$ and non-radial ones like that of the magneto-geostrophic equation (1.12). As already pointed out above, some initial data with large $L^2$-norm can be considered because the smallness condition is with respect to the weak norm of $\mathcal{F} \mathcal{N}^{s}_{p,\mu,\infty}$.

The outline of this paper is as follows. In section 2, we introduce the functional setting and recall an useful abstract point fixed lemma. Our results are stated in section 3. Estimates for the nonlinear term associated to the mild formulation of (1.1)-(1.2) are obtained in section 4. Finally, the results are proved in section 5.

2 Preliminares

In this section we introduce Fourier-Besov-Morrey spaces and recall an abstract fixed point lemma that will be useful for our purposes.
2.1 Fourier-Besov-Morrey spaces

We start by recalling Morrey spaces $M_{p,\lambda} = M_{p,\lambda}(\mathbb{R}^n)$ (see [46], [34] for more details). For $1 \leq p \leq \infty$ and $0 \leq \mu < n$, the space $M_{p,\mu}$ is defined as

$$M_{p,\mu} = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{M_{p,\mu}} < \infty \right\},$$

(2.1)

where

$$\|f\|_{M_{p,\mu}} = \sup_{x_0 \in \mathbb{R}^n, 0 < R < \infty} \left( R^{-\mu/p} \|f\|_{L^p(B_R(x_0))} \right)$$

(2.2)

and $B_R(x_0) \subset \mathbb{R}^n$ is the open ball with center $x_0$ and radius $R$. The space $M_{p,\mu}$ endowed with $\|\cdot\|_{M_{p,\mu}}$ is a Banach space. For $p = \infty$, $M_{p,\mu}$ becomes $L^\infty$. In the case $p = 1$, $M_{p,\mu}$ should be understood as a space of Radon measures and the $L^1$-norm in (2.2) as the total variation of the measure $f$ on $B_R(x_0)$.

If $1 \leq p_i \leq \infty$ and $0 \leq \mu_i < n$ with $1/p_3 = 1/p_1 + 1/p_2$ and $\mu_3 = \mu_1/p_1 + \mu_2/p_2$, then we have the Hölder type inequality

$$\|fg\|_{M_{p_3,\mu_3}} \leq \|f\|_{M_{p_1,\mu_1}} \|g\|_{M_{p_2,\mu_2}}.$$  

(2.3)

Also, for $1 \leq p \leq \infty$ and $0 \leq \mu < n$,

$$\|\varphi * g\|_{M_{p,\mu}} \leq \|\varphi\|_1 \|g\|_{M_{p,\mu}},$$

(2.4)

for all $\varphi \in L^1$ and $g \in M_{p,\mu}$.

Let us now recall the Littlewood-Paley decomposition (see e.g. [4], [36], [22], [38]). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radially symmetric function with support in $\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \text{ for all } \xi \neq 0.$$  

Consider the family of functions $\{\varphi_k\}_{k \in \mathbb{Z}}$ defined by $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, for every $k \in \mathbb{Z}$. Let $\mathcal{P}$ be the set of all polynomials with $n$ variables. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, the homogeneous Fourier-Besov-Morrey space $\mathcal{F}N^s_{p,\mu,q}(\mathbb{R}^n)$ is the Banach space of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ such that the norm $\|f\|_{\mathcal{F}N^s_{p,\mu,q}}$ is finite, where

$$\|f\|_{\mathcal{F}N^s_{p,\mu,q}} = \begin{cases} \left( \sum_{k \in \mathbb{Z}} 2^{ksq} \|\varphi_k \hat{f}\|_{M_{p,\mu}}^q \right)^{1/q}, & q < \infty, \\ \sup_{k \in \mathbb{Z}} 2^{ks} \|\varphi_k \hat{f}\|_{M_{p,\mu}}, & q = \infty. \end{cases}$$

(2.5)

Remark 2.1 Let us observe that Besov-Morrey spaces $N^s_{p,\mu,q}$ studied in [30], [38] are obtained by replacing in (2.5) $\varphi_k \hat{f}$ by $(\varphi_k \hat{f})^\vee = [(\varphi_k)^\vee * f]$.
In order to study how the product acts on Fourier-Besov-Morrey space, we need to recall the Bony’s paraproduct formula (see [13],[22],[38]). Firstly define the localization operators

\[ \Delta_j f = \varphi_j(D)f \text{ and } S_j f = \sum_{k \leq j-1} \Delta_k f, \text{ for every } j \in \mathbb{Z}. \]  

A direct computation gives the equalities

\[ \Delta_j \Delta_k f = 0, \text{ if } |j - k| \geq 2, \]  

\[ \Delta_j (S_{k-1} f \Delta_k g) = 0, \text{ if } |j - k| \geq 5. \]  

Given \( f \in \mathcal{S}'(\mathbb{R}^n) \), the paraproduct operator \( T_f \) is defined as

\[ T_f g = \sum_{k \in \mathbb{Z}} S_{k-1} f \Delta_k g, \text{ for all } g \in \mathcal{S}'(\mathbb{R}^n). \]  

Using (2.9) we can write

\[ fg = T_f g + T_g f + R(f, g) \]  

where

\[ R(f, g) = \sum_{k \in \mathbb{Z}} \Delta_k f \tilde{\Delta}_k g \text{ and } \tilde{\Delta}_k g = \sum_{|k' - k| \leq 1} \Delta_{k'} g. \]  

It follows from (2.7)-(2.8) that

\[ \Delta_j (fg) = \sum_{|k - j| \leq 4} \Delta_j (S_{k-1} f \Delta_k g) + \sum_{|k - j| \leq 4} \Delta_j (S_{k-1} g \Delta_k f) + \sum_{k \geq j-2} \Delta_j (\Delta_k f \tilde{\Delta}_k g). \]  

We finish this subsection with a Bernstein type lemma in Fourier variables in Morrey spaces.

**Lemma 2.2** Let \( 1 \leq q \leq p \leq \infty, 0 \leq \mu_1, \mu_2 < n, \frac{n-\mu_2}{p} \leq \frac{n-\mu_1}{q} \), and let \( \alpha \) be a multiindex. If \( \text{supp}(\hat{f}) \subset \{ |\xi| \leq A2^j \} \) then there is a constant \( C > 0 \) independent of \( f \) and \( j \) such that

\[ \| (i\xi)^\alpha \hat{f} \|_{M_{q,\mu_2}} \leq C 2^j |\alpha| + j \left( \frac{n-\mu_2}{q} - \frac{n-\mu_1}{p} \right) \| \hat{f} \|_{M_{p,\mu_1}}. \]  

**Proof.** From the condition on support of \( \hat{f} \) and (2.8), we obtain

\[ \| (i\xi)^\alpha \hat{f} \|_{M_{q,\mu_2}} \leq C 2^j |\alpha| \| 1 \cdot \hat{f} \|_{M_{q,\mu_2}} \leq C 2^j |\alpha| 2^j \left( \frac{n-\mu_2}{q} - \frac{n-\mu_1}{p} \right) \| \hat{f} \|_{M_{p,\mu_1}}, \]  

which gives (2.13).
2.2 A fixed point lemma

In order to avoid extensive point fixed arguments we are going to use the following abstract lemma.

**Lemma 2.3** (see [22]) Let $X$ be a Banach space with norm $\| \cdot \|_X$, and $B : X \times X \to X$ be a continuous bilinear map, i.e., there exists $K > 0$ such that $\| B(x_1, x_2) \|_X \leq K \| x_1 \|_X \| x_2 \|_X$, for all $x_1, x_2 \in X$. Given $0 < \varepsilon < \frac{1}{4K}$ and $y \in X$ such that $\| y \|_X \leq \varepsilon$, there exists a unique solution $x \in X$ for the equation $x = y + B(x, x)$ in the closed ball $\{ x \in X : \| x \|_X \leq 2\varepsilon \}$. Moreover, the solution $x$ depends continuously on $y$ in the following sense: If $\| \tilde{y} \|_X \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\| \tilde{x} \|_X \leq 2\varepsilon$, then

$$\| x - \tilde{x} \|_X \leq \frac{1}{1 - 4K\varepsilon} \| y - \tilde{y} \|_X.$$  

\hspace{1cm} (2.14)

3 Results

In this work we consider solutions for (1.1)-(1.2) which satisfy an integral equation come from Duhamel’s principle. This equation reads as

$$\theta(t) = G_\gamma(t)\theta_0 + B(\theta, \theta)(t),$$  

where $G_\gamma(t)\theta_0 = g_\gamma(\cdot, t) * \theta_0$ with $g_\gamma(\xi, t) = e^{-t|\xi|^{2\gamma}}$ and

$$B(\theta, \varphi)(t) = -\int_0^t G_\gamma(t-\tau) (\nabla_x \cdot (P[\theta] \varphi)) (\tau) d\tau.$$  

\hspace{1cm} (3.2)

In what follows, we state our well-posedness and self-similarity results.

**Theorem 3.1** Let $n \geq 1$, $\gamma > 1/2$, $0 \leq \beta < 2\gamma < \frac{n+\beta+1}{2}$, $\gamma < \frac{p-n}{n+1-4\gamma} < p \leq \infty$, and $0 \leq \mu < n$. Suppose $\theta_0 \in \mathcal{F}\mathcal{N}_{p,\mu,\infty}^s$ with $s = n - \frac{n-\mu}{p} - (2\gamma - \beta)$. Let $K$ be as in Lemma 4.1 and $0 < \varepsilon < \frac{1}{4K}$.

(i) (Well-posedness) If $\| \theta_0 \|_{\mathcal{F}\mathcal{N}_{p,\mu,\infty}^s} \leq \varepsilon$ then (3.1) has a unique global solution

$$\theta \in BC((0, \infty); \mathcal{F}\mathcal{N}_{p,\mu,\infty}^s(\mathbb{R}^n))$$

such that $\sup_{t > 0} \| \theta(t) \|_{\mathcal{F}\mathcal{N}_{p,\mu,\infty}^s} \leq 2\varepsilon$. The data-solution map $\theta_0 \to \theta(x, t)$ from

$$\left\{ f \in \mathcal{F}\mathcal{N}_{p,\mu,\infty}^s : \| f \|_{\mathcal{F}\mathcal{N}_{p,\mu,\infty}^s} \leq \varepsilon \right\}$$

\hspace{1cm} to $BC((0, \infty); \mathcal{F}\mathcal{N}_{p,\mu,\infty}^s(\mathbb{R}^n))$ is Lipschitz continuous. Moreover $\theta(x, t) \to \theta_0$ in $S'(\mathbb{R}^n)$ as $t \to 0^+$. 

(ii) (Self-Similarity) If the symbol \( P_j(\xi) \) is homogeneous of degree \( \beta \), for \( j = 1, 2, ..., n \), and if \( \theta_0 \) is a homogeneous distribution of degree \(- (2\gamma - \beta)\) then the solution \( \theta \) is self-similar, that is, 
\[
\theta = \theta_\lambda := \lambda^{2\gamma - \beta} \theta(\lambda x, \lambda^{2\gamma} t), \quad \text{for all } \lambda > 0.
\]

Remark 3.2 (Local-in-time solutions) Assume the same conditions on \( n, \beta, \gamma, p, \mu \) given in Theorem \ref{thm:local_in_time} and let \( s > n - \frac{n-\mu}{p} - (2\gamma - \beta) \) and \( \theta_0 \in \mathcal{F}\mathcal{N}_s^{p,\mu,\infty} \). A local-in-time well-posedness result in \( BC((0,T);\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}(\mathbb{R}^n)) \) could be proved by assuming a small condition on \( T > 0 \) and regardless the size of the initial data. Again, the solution \( \theta \) would satisfy the initial condition in the sense of distributions, that is, \( \theta(x,t) \to \theta_0 \) in \( \mathcal{S}'(\mathbb{R}^n) \) as \( t \to 0^+ \).

Remark 3.3 (Data with large \( L^2 \)-norm) Let \( \theta_0, \psi_0 \in \mathcal{S}'(\mathbb{R}^n) \) be such that
\[
\hat{\theta}_0(\xi) = \delta |\xi|^{-(n-(2\gamma-\beta))} 1_{|\xi|<R_1}, \quad \text{and} \quad \hat{\psi}_0(\xi) = \delta |\xi|^{-(n-(2\gamma-\beta))} 1_{|\xi|>R_2}, \tag{3.3}
\]
where \( \delta > 0 \) and \( 1_A \) stands for the characteristic function of set \( A \). Recalling that \((|\xi|^{-(n-(2\gamma-\beta))})^\gamma = C|\xi|^{-(2\gamma-\beta)}\), it follows from (2.5) that
\[
\|\theta_0\|_{\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}}, \|\psi_0\|_{\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}} \leq \|\delta C|x|^{-(2\gamma-\beta)}\|_{\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}} = C\delta,
\]
where \( C > 0 \) is independent of \( R_1, R_2 \). We can apply Theorem \ref{thm:local_in_time} with \( \theta_0 \) and \( \psi_0 \) for all \( R_1, R_2 > 0 \) by choosing \( \delta > 0 \) in such a way that \( \delta = \frac{C}{R_1^\gamma} \). On the other hand, if \( n + 2\beta < 4\gamma \), then \( \theta_0 \in L^2 \) with \( \|\theta_0\|_{L^2}^2 = C(\delta) R_1^{4\gamma-(n+2\beta)} \) for all \( R_1 > 0 \), and if \( n + 2\beta > 4\gamma \) then \( \psi_0 \in L^2 \) with \( \|\psi_0\|_{L^2}^2 = C(\delta) R_2^{4\gamma-(n+2\beta)} \), for all \( R_2 > 0 \). Then \( \theta_0 \) and \( \psi_0 \) can have arbitrarily large \( L^2 \)-norm by making \( R_1 \to \infty \) and \( R_2 \to 0 \), respectively.

Now we present a result of asymptotic stability of solutions in the framework of Fourier-Besov-Morrey spaces. Here we follow ideas from \cite{3} where the authors studied Navier-Stokes equations in the framework of PM\(^a\)-spaces.

**Theorem 3.4 (Asymptotic stability)** Under the hypotheses of Theorem \ref{thm:asymptotic_stability} Assume that \( \theta \) and \( \phi \) are solutions for (3.1) given by Theorem \ref{thm:asymptotic_stability} with small initial data \( \theta_0 \) and \( \phi_0 \in \mathcal{F}\mathcal{N}_s^{p,\mu,\infty}(\mathbb{R}^n) \), respectively. We have that
\[
\lim_{t \to \infty} \|\theta(\cdot,t) - \phi(\cdot,t)\|_{\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}} = 0 \tag{3.4}
\]
if and only if
\[
\lim_{t \to \infty} \|G_\gamma(t)(\theta_0 - \phi_0)\|_{\mathcal{F}\mathcal{N}_s^{p,\mu,\infty}} = 0. \tag{3.5}
\]

Remark 3.5 Theorem \ref{thm:asymptotic_stability} provides a class of solutions asymptotically self-similar at infinity. In fact, when \( \phi_0 = \theta_0 + \varphi \) with \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) and \( \theta_0 \) is homogeneous of degree \(-(2\gamma - \beta)\), the corresponding solution \( \phi(x,t) \) is attracted to the self-similar solution \( \theta(x,t) \) in the sense of (3.4).
4 Bilinear estimate

In order to perform a fixed point argument in $BC((0,T); \mathcal{F}_p^s)$, we need to obtain estimates for the bilinear part $B(\cdot, \cdot)$ of (3.1) on this space.

**Lemma 4.1** Under the hypothesis of Theorem 3.1, there exists a constant $K > 0$ such that

$$\sup_{t>0} \|B(\theta, \phi)\|_{\mathcal{F}_p^s} \leq K \sup_{t>0} \|\theta\|_{\mathcal{F}_p^s} \sup_{t>0} \|\phi\|_{\mathcal{F}_p^s},$$

for all $\theta, \phi \in L^\infty((0,\infty); \mathcal{F}_p^s(\mathbb{R}^n))$.

**Proof.** Let us first prove that there exists $K_1 > 0$ such that

$$\sup_{t>0} \|B(\theta, \phi)\|_{\mathcal{F}_p^s} \leq K_1 \sup_{t>0} \|P[\theta]\phi\|_{\mathcal{F}_p^{s-2\gamma+1}}$$

For this let $k \in \mathbb{Z}$ be fixed. Since $\text{supp}(\varphi_k) \subset \{\xi \in \mathbb{R}^n : 2^k - 1 \leq |\xi| \leq 2^{k+1}\}$ we have

$$2^{ks} \|\varphi_k B(\theta, \phi)\|_{M_{p,\mu}} \leq 2^{ks} \int_0^t \|\xi e^{-(t-\tau)|\xi|^2} (P[\theta]\phi)(\xi, \tau)\varphi_k(\xi)\|_{M_{p,\mu}} d\tau$$

$$\leq \int_0^t 2^{k+1} e^{-(t-\tau)2^{2\gamma(k-1)}} 2^{2\gamma(k-1)} 2^{(s-2\gamma+1)k} \|(P[\theta]\phi)(\cdot, \tau)\varphi_k(\cdot)\|_{M_{p,\mu}} d\tau$$

$$\leq 2^{2\gamma+1} \int_0^t 2^{2\gamma(k-1)} e^{-(t-\tau)2^{2\gamma(k-1)}} 2^{(s-2\gamma+1)k} \|(P[\theta]\phi)(\cdot, \tau)\varphi_k(\cdot)\|_{M_{p,\mu}} d\tau$$

$$\leq C \int_0^t 2^{2\gamma(k-1)} e^{-(t-\tau)2^{2\gamma(k-1)}} \|(P[\theta]\phi)(\cdot, \tau)\|_{\mathcal{F}_p^{s-2\gamma+1}} d\tau$$

$$\leq K_1 \sup_{t>0} \|(P[\theta]\phi)(t)\|_{\mathcal{F}_p^{s-2\gamma+1}}.$$ (4.3)

Taking the supremum over $k \in \mathbb{Z}$ and afterwards over $t > 0$, we obtain (4.2).

It remains to prove the product estimate

$$\sup_{t>0} \|P[\theta]\phi\|_{\mathcal{F}_p^{s-2\gamma+1}} \leq K_2 \sup_{t>0} \|\theta\|_{\mathcal{F}_p^s} \sup_{t>0} \|\phi\|_{\mathcal{F}_p^s}.$$ (4.5)

From estimate (1.7), we get

$$\|\varphi_k(\xi)\hat{u}(\xi, t)\|_{M_{p,\mu}} \leq C 2^{k(\beta-1)} \|\varphi_k(\xi)\hat{\theta}(\xi, t)\|_{M_{p,\mu}},$$

because $\text{supp}(\varphi_k) \subset \{\xi \in \mathbb{R}^n : 2^k - 1 \leq |\xi| \leq 2^{k+1}\}$. From definition (2.5) and (2.6) we have that

$$\|P[\theta]\phi\|_{\mathcal{F}_p^{s-2\gamma+1}} = \sup_{j \in \mathbb{Z}} 2^{j(s-2\gamma+1)} \|\Delta_j(P[\theta]\phi)\|_{M_{p,\mu}}.$$
Let $j \in \mathbb{Z}$ be fixed. Using the decomposition $(2.10)$ with $f$ and $g$ replaced by $P[\theta]$ and $\phi$, respectively, and taking the norm $\| \cdot \|_{M_{p,\mu}}$, we get

$$
\| \varphi_j P[\theta] \|_{M_{p,\mu}} \leq \sum_{|k-j| \leq 4} \| \varphi_j (S_{k-1} P[\theta]) * (\varphi_k \hat{\phi}) \|_{M_{p,\mu}} + \sum_{|k-j| \leq 4} \| \varphi_j (S_{k-1} \phi) * (\varphi_k P[\theta]) \|_{M_{p,\mu}} 
+ \sum_{k \geq j-2} \| \varphi_j (\varphi_k P[\theta]) * (\varphi_k \hat{\phi}) \|_{M_{p,\mu}} = I_1 + I_2 + I_3.
$$

(4.7)

In order to estimate $I_1$ we use Young’s inequality in Morrey spaces $(2.4)$ to obtain

$$
2^{j(s-2\gamma+1)} I_1 \leq 2^{j(s-2\gamma+1)} \sum_{|k-j| \leq 4} \| S_{k-1} P[\theta] \|_{L^1} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}}.
$$

(4.8)

Recalling the definition of $S_j$, Lemma $2.7$ with $|\alpha| = 0$ implies that

$$
\| S_{k-1} P[\theta] \|_{L^1} \leq \sum_{k' < k} 2^{k'(n-\frac{\alpha}{p})} \| \varphi_{k'} P[\theta] \|_{M_{p,\mu}}.
$$

(4.9)

Since $n - \frac{n-\mu}{p} = s + 2\gamma - \beta$ and $2^k \sim 2^j$ when $|k-j| \leq 4$, it follows from $(4.9)$ that

$$
2^{j(s-2\gamma+1)} \sum_{|k-j| \leq 4} \| S_{k-1} P[\theta] \|_{L^1} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}} 
\leq 2^{j(s-2\gamma+1)} \sum_{|k-j| \leq 4} \sum_{k' < k} 2^{k'(s+2\gamma-\beta)} \| \varphi_{k'} P[\theta] \|_{M_{p,\mu}} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}}
\leq 2^{j(s-2\gamma+1)} C \sum_{|k-j| \leq 4} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}} \sum_{k' < k} 2^{k'(2\gamma-1)} 2^{k'(s-\beta+1)} 2^{k'(\beta-1)} \| \varphi_{k'} P[\theta] \|_{M_{p,\mu}}
\leq 2^{j(s-2\gamma+1)} C \left( \sum_{|k-j| \leq 4} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}} \sum_{k' < k} 2^{k'(2\gamma-1)} \right) \left( \sup_{k' < k} 2^{k's} \| \varphi_{k'} P[\theta] \|_{M_{p,\mu}} \right)
\leq C \| \theta \|_{F^s_{N_{p,\mu},\infty}} \sum_{|k-j| \leq 4} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}} 2^{k(2\gamma-1)}
\leq C \| \theta \|_{F^s_{N_{p,\mu},\infty}} \sum_{|k-j| \leq 4} 2^{ks} \| \varphi_k \hat{\phi} \|_{M_{p,\mu}}
\leq C \sup_{t > 0} \| \theta \|_{F^s_{N_{p,\mu},\infty}} \sup_{t > 0} \| \phi \|_{F^s_{N_{p,\mu},\infty}}.
$$

(4.11)

(4.12)

where we have used $(4.6)$ in $(4.10)$, and $\gamma > 1/2$ in $(4.11)$. Inserting $(4.12)$ into $(4.8)$ we obtain

$$
2^{j(s-2\gamma+1)} I_1 \leq C \sup_{t > 0} \| \theta \|_{F^s_{N_{p,\mu},\infty}} \sup_{t > 0} \| \phi \|_{F^s_{N_{p,\mu},\infty}}.
$$

(4.13)

By using that $\beta < 2\gamma$, similarly one can show that

$$
2^{j(s-2\gamma+1)} I_2 \leq C \sup_{t > 0} \| \theta \|_{F^s_{N_{p,\mu},\infty}} \sup_{t > 0} \| \phi \|_{F^s_{N_{p,\mu},\infty}}.
$$

(4.14)
Now we deal with the third term in (4.7). Note that \( s - 2\gamma + 1 > 0 \) due to the condition \( \frac{n-\mu}{n+\beta+1-4\gamma} < p \). Again employing Young's inequality and Lemma 2.2, we get

\[
2^{j(s-2\gamma+1)} I_3 \leq 2^{j(s-2\gamma+1)} \sum_{k \geq j-2} \|\varphi_k \hat{\theta}\|_{L^1} \|\varphi_k \hat{\phi}\|_{M_{p,\mu}}
\]

\[
\leq \sum_{k \geq j-2} 2^{j(s-2\gamma+1)} 2^{k(\beta-1)+k(n-\frac{\mu}{p})} \|\varphi_k \hat{\theta}\|_{M_{p,\mu}} \|\varphi_k \hat{\phi}\|_{M_{p,\mu}}
\]

\[
\leq \sum_{k \geq j-2} 2^{j(s-2\gamma+1)} 2^{(s+2\gamma-1)k} \|\varphi_k \hat{\theta}\|_{M_{p,\mu}} \|\varphi_k \hat{\phi}\|_{M_{p,\mu}}
\]

\[
\leq C \sum_{k \geq j-2} 2^{(j-k)(s-2\gamma+1)} \left( 2^{ks} \|\varphi_k \hat{\theta}\|_{M_{p,\mu}} \right) \left( 2^{ks} \|\varphi_k \hat{\phi}\|_{M_{p,\mu}} \right)
\]

\[
\leq C \sup_{t>0} \|\theta\|_{\mathcal{F}^s_{p,\mu,\infty}} \sup_{t>0} \|\phi\|_{\mathcal{F}^s_{p,\mu,\infty}}, \quad (4.15)
\]

because \( \sum_{k \geq j-2} 2^{(j-k)(s-2\gamma+1)} < \infty \). The estimates (4.13), (4.14) and (4.15) yield (4.5), as required.

\section{5 Proof of Theorems}

In this section we prove Theorems \textbf{3.1} and \textbf{3.4}. The proof of the former is the subject of the next subsection.

\subsection{5.1 Proof of Theorem \textbf{3.1}}

\textbf{Part (i):} Let us define the Banach space

\[\mathcal{X} = BC((0, \infty); \mathcal{F}^s_{p,\mu,\infty}(\mathbb{R}^n))\]

with norm given by

\[\|\theta\|_{\mathcal{X}} = \sup_{t>0} \|\theta(\cdot, t)\|_{\mathcal{F}^s_{p,\mu,\infty}}.\]

Rewriting Lemma 4.1 with the norm \( \|\cdot\|_{\mathcal{X}} \), we obtain

\[\|\mathcal{B}(\theta, \phi)\|_{\mathcal{X}} \leq K \|\theta\|_{\mathcal{X}} \|\phi\|_{\mathcal{X}}. \quad (5.1)\]

The linear part in (3.4) can be handle as

\[\|G_\gamma(t) \theta_0\|_{\mathcal{X}} = \sup_{t>0} \left( \sup_{k \in \mathbb{Z}} 2^{ks} \|\varphi_k e^{-t(\cdot)^\gamma} \theta_0\|_{M_{p,\mu}} \right) \]

\[
\leq \sup_{t>0} \left( \sup_{k \in \mathbb{Z}} 2^{ks} \|\varphi_k \theta_0\|_{M_{p,\mu}} \right) \]

\[= \|\theta_0\|_{\mathcal{F}^s_{p,\mu,\infty}}. \quad (5.2)\]
So, taking $0 < \varepsilon < \frac{1}{10K}$ and $\|\theta_0\|_{\mathcal{F}N^s_{p,\mu,\infty}} \leq \varepsilon$, Lemma 2.3 together with the estimates (5.1) and (5.2) assures that there is a unique solution $\theta \in \mathcal{X}$ for (3.1) such that $\|\theta\|_{\mathcal{X}} \leq 2\varepsilon$. The Lipschitz continuity is obtained from (2.14). The weak convergence as $t \to 0^+$ follows by standard arguments and the reader is referred to [47].

\section*{Part (ii):}
Since the symbol $P_j$ is homogeneous of degree $\beta$, for $j = 1, 2, \ldots, n$, and $\theta_0$ homogeneous of degree $-(2\gamma - \beta)$, a simple computation shows that $\theta_\lambda = \lambda^{2\gamma - \beta} \theta(\lambda x, \lambda^{2\gamma} t)$ verifies (3.1), for all $\lambda > 0$, provided that $\theta$ does so. Moreover, due to the scaling invariance of the $\mathcal{X}$-norm, it follows that $\|\theta_\lambda\|_{\mathcal{X}} = \|\theta\|_{\mathcal{X}} \leq 2\varepsilon$. The uniqueness result contained in item (i) gives us that $\theta_\lambda = \theta$, for all $\lambda > 0$. 

\section{5.2 Proofs of Theorem 3.4}
We prove that (5.5) implies (5.4). The converse follows similarly and is left to the reader. Subtracting the corresponding integral equations verified by $\theta$ and $\phi$ and afterwards computing the $\mathcal{F}N^s_{p,\mu,\infty}$-norm, we obtain

\begin{align*}
\|\theta(t) - \phi(t)\|_{\mathcal{F}N^s_{p,\mu,\infty}} &\leq \|G_\tau(t)(\theta_0 - \phi_0)\|_{\mathcal{F}N^s_{p,\mu,\infty}} + \|B(\theta - \phi, \theta) + B(\phi, \theta - \phi)\|_{\mathcal{F}N^s_{p,\mu,\infty}} \\
&\leq \|G_\tau(t)(\theta_0 - \phi_0)\|_{\mathcal{F}N^s_{p,\mu,\infty}} + \left\| \int_0^t G_\tau(t) \nabla_x \cdot \left( P[\theta - \phi] \theta(\tau) + P[\phi] (\theta - \phi)(\tau) \right) d\tau \right\|_{\mathcal{F}N^s_{p,\mu,\infty}} \\
&\quad + \left\| \int_0^t G_\tau(t - \tau) \nabla_x \cdot \left( P[\theta - \phi] \theta(\tau) + P[\phi] (\theta - \phi)(\tau) \right) d\tau \right\|_{\mathcal{F}N^s_{p,\mu,\infty}} \\
&\quad := I_0 + I_1 + I_2, \tag{5.3}
\end{align*}

where $\delta > 0$ will be chosen later. Proceeding as in (4.3) and using (4.5), we have that

\begin{align*}
I_1 &\leq C \sup_{k \in \mathbb{Z}} \int_0^{\delta t} 2^{2\gamma(k-1)} e^{-(t - \tau)2^{2\gamma(k-1)}} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}N^s_{p,\mu,\infty}} \left( \|\theta(\tau)\|_{\mathcal{F}N^s_{p,\mu,\infty}} + \|\phi(\tau)\|_{\mathcal{F}N^s_{p,\mu,\infty}} \right) d\tau \\
&\leq 4\varepsilon K \sup_{k \in \mathbb{Z}} \int_0^{\delta t} 2^{2\gamma(k-1)} e^{-(t - \tau)2^{2\gamma(k-1)}} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}N^s_{p,\mu,\infty}} d\tau, \tag{5.4}
\end{align*}

where $K > 0$ is as in Lemma 4.1 and above we have used that

\begin{align*}
\sup_{t > 0} \|\theta(t)\|_{\mathcal{F}N^s_{p,\mu,\infty}} &\leq 2\varepsilon \quad \text{and} \quad \sup_{t > 0} \|\phi(t)\|_{\mathcal{F}N^s_{p,\mu,\infty}} \leq 2\varepsilon. \tag{5.5}
\end{align*}

Making the change of variables $\tau = tz$ in (5.4) and using the equality

\begin{align*}
\sup_{k \in \mathbb{Z}} 2^{2\gamma(k-1)} e^{-(t - \tau)2^{2\gamma(k-1)}} = C \frac{C}{t(1 - \tau)},
\end{align*}
it follows that
\[
I_1 \leq 4\varepsilon K \sup_{k \in \mathbb{Z}} \int_0^\delta 2^{2\gamma(k-1)} e^{-(t-\tau)2^{2\gamma(k-1)}} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}_{p,p,\infty}^s} d\tau
\]
\[
= \sup_{k \in \mathbb{Z}} \int_0^\delta t^{2^{2\gamma(k-1)}} e^{-(1-\tau)2^{2\gamma(k-1)}} \|\theta(t\tau) - \phi(t\tau)\|_{\mathcal{F}_{p,p,\infty}^s} d\tau
\]
\[
\leq 4\varepsilon KC \int_0^\delta (1-\tau)^{-1} \|\theta(t\tau) - \phi(t\tau)\|_{\mathcal{F}_{p,p,\infty}^s} d\tau. \tag{5.6}
\]

For \( I_2 \), we have that
\[
I_2 \leq C \sup_{k \in \mathbb{Z}} \int_0^t 2^{2\gamma(k-1)} e^{-(t-\tau)2^{2\gamma(k-1)}} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}_{p,p,\infty}^s} \left( \|\theta(\tau)\|_{\mathcal{F}_{p,p,\infty}^s} + \|\phi(\tau)\|_{\mathcal{F}_{p,p,\infty}^s} \right) d\tau
\]
\[
\leq 4\varepsilon K \left( \sup_{k \in \mathbb{Z}} \int_0^t 2^{2\gamma(k-1)} e^{-(t-\tau)2^{2\gamma(k-1)}} d\tau \right) \left( \sup_{\delta t < \tau < t} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}_{p,p,\infty}^s} \right)
\]
\[
\leq 4\varepsilon K \sup_{\delta t < \tau < t} \|\theta(\tau) - \phi(\tau)\|_{\mathcal{F}_{p,p,\infty}^s}. \tag{5.7}
\]

Finally, taking
\[
L = \limsup_{t \to \infty} \|\theta(\cdot, t) - \phi(\cdot, t)\|_{\mathcal{F}_{p,p,\infty}^s},
\]
notice that \( 0 \leq L \leq 4\varepsilon \) in view of (5.5). Computing the \( \limsup \) in (5.3), the inequalities (5.6) and (5.7) give us
\[
L \leq \left( C4\varepsilon K \log\left( \frac{1}{1-\delta} \right) + 4\varepsilon K \right) L
\]
which implies \( L = 0 \), because we can choose \( \delta > 0 \) sufficiently small so that
\[
C4\varepsilon K \log\left( \frac{1}{1-\delta} \right) + 4\varepsilon K < 1.
\]

\section*{References}

[1] P. Biler, G. Karch, R. Monneau, Nonlinear diffusion of dislocation density and self-similar solutions, Comm. Math. Phys. 294 (2010), 145–168.

[2] L.A. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 171 (3) (2010), 1903–1930.

[3] M. Cannone, G. Karch, Smooth or singular solutions to the Navier-Stokes system, J. Differential Equations 197 (2004), 247–274.

[4] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations, Handbook of mathematical fluid dynamics, Vol. III, 161–244, North-Holland, Amsterdam, 2004.
Dissipative active scalar equations

[5] J. A. Carrillo, L.C.F. Ferreira, The asymptotic behaviour of subcritical dissipative quasi-geostrophic equations, Nonlinearity 21 (5) (2008), 1001–1018.

[6] J. A. Carrillo, L.C.F. Ferreira, J.C. Precioso, A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity, Adv. Math. 231 (1) (2012), 306–327.

[7] A. Castro, D. Córdoba, Global existence, singularities and ill-posedness for a nonlocal flux, Adv. Math. 219 (6) (2008), 1916–1936.

[8] A. Castro, D. Córdoba, Infinite energy solutions of the surface quasi-geostrophic equation, Adv. Math. 225 (4) (2010), 1820–1829.

[9] D. Chae, A. Córdoba, D. Córdoba, M.A. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, Adv. Math. 194 (1) (2005), 203–223.

[10] D. Chae, P. Constantin, D. Cordoba, F. Gancedo, J. Wu, Generalized surface quasi-geostrophic equations with singular velocities, Communications on Pure and Applied Mathematics 65 (8) (2012), 1037-1066.

[11] D. Chae, P. Constantin, J. Wu, Dissipative models generalizing the 2D Navier-Stokes and the surface quasi-geostrophic equations, to appear in Indiana University Mathematics Journal.

[12] D. Chae, P. Constantin, J. Wu, Inviscid models generalizing the 2D Euler and the surface quasi-geostrophic equations, Archive for Rational Mechanics and Analysis, 202 (1) (2011), 35-62.

[13] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. de l’Ecole Norm. Sup., 14 (1981), 209-246.

[14] P. Constantin, G. Iyer, J. Wu, Global regularity for a modified critical dissipative quasi-geostrophic equation, Indiana University Mathematics Journal, 57 (6) (2008), 2681-2692.

[15] P. Constantin, A. Majda, E. Tabak, Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar, Nonlinearity 7 (1994), 1459–1533.

[16] P. Constantin, D. Córdoba, J. Wu, On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. 50 (2001), 97–107.

[17] P. Constantin, J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal. 30 (1999), 937–948.

[18] Dabkowski, Michael; Kiselev, Alexander; Vicol, Vlad Global well-posedness for a slightly supercritical surface quasi-geostrophic equation, Nonlinearity 25 (5) (2012), 1525–1535.
[19] J. Deslippe, R. Tedstrom, M. Daw, D. Chrzan, T. Neeraj, M. Mills, Dynamics scaling in a simple one-dimensional model of dislocation activity, Phil. Mag. 84, (2004) 2445–2454.

[20] H. Dong, D. Du, Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space, Discrete Contin. Dyn. Syst. 21 (4) (2008), 1095–1101.

[21] H. Dong, D. Du, D. Li, Finite time singularities and global well-posedness for fractal Burgers equations, Indiana Univ. Math. J. 58 (2) (2009), 807–821.

[22] P.G. Lemarie-Rieusset, Recent developments in the Navier-Stokes equations, Chapman and Hall, Research Notes in Maths. (431), 2002.

[23] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004), 511–528.

[24] L.C.F. Ferreira, L.S.M. Lima, Global well-posedness and symmetries for dissipative active scalar equations with higher-order couplings, arXiv:1305.2987.

[25] S. Friedlander, V. Vicol, Higher regularity of Hölder continuous solutions of parabolic equations with singular drift velocities, J. Math. Fluid Mech. 14 (2) (2012), 255–266.

[26] S. Friedlander, V. Vicol, Global well-posedness for an advection-diffusion equation arising in magnetogeostrophic dynamics, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2) (2011), 283–301.

[27] S. Friedlander, W. Rusin, V. Vicol, On the supercritically diffusive magnetogeostrophic equations, Nonlinearity 25 (11) (2012), 3071–3097.

[28] S. Friedlander, V. Vicol, On the ill/well-posedness and nonlinear instability of the magnetogeostrophic equations, Nonlinearity 24 (11) (2011), 3019–3042.

[29] A.K. Head, Dislocation group dynamics III. Similarity solutions of the continuum approximation, Phil. Mag. 26, (1972) 65–72.

[30] N. Ju, The Maximum Principle and the Global Attractor for 2D Dissipative Quasi-Geostrophic Equations, Comm. Math. Phys. 255 (2005), 161-181.

[31] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (3) (2007), 445–453.

[32] A. Kiselev, Regularity and blow up for active scalars, Math. Model. Nat. Phenom. 5 (4) (2010), 225–255.

[33] A. Kiselev, Nonlocal maximum principles for active scalars, Adv. Math. 227 (5) (2011), 1806-1826.

[34] T. Kato, Strong solutions of the Navier-Stokes equations in Morrey spaces, Bol. Soc. Brasil Mat. 22 (2) (1992) 127-55.
[35] P. Konieczny, T. Yoneda, On dispersive effect of the Coriolis force for the stationary Navier-Stokes equations, J. Differential Equations 250 (10) (2011), 3859–3873.

[36] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations 19 (5-6) (1994), 959–1014.

[37] R. May, Global well-posedness for a modified dissipative surface quasi-geostrophic equation in the critical Sobolev space $H^1$, J. Differential Equations 250 (1) (2011), 320–339. Stationary Navier-Stokes equations. J. Differential Equations 250 (2011), no. 10, 3859–3873.

[38] A.L. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE, Trans. Amer. Math. Soc. 355 (4) (2003), 1297–1364.

[39] C. Miao, L. Xue, Global well-posedness for a modified critical dissipative quasi-geostrophic equation, J. Differential Equations 252 (1) (2012), 792–818.

[40] C. Miao, L. Xue, On the regularity of a class of generalized quasi-geostrophic equations, J. Differential Equations 251 (10) (2011), 2789–2821.

[41] H.K. Moffatt. Magnetostrophic turbulence and the geodynamo. In IUTAM Symposium on Computational Physics and New Perspectives in Turbulence, volume 4 of IUTAM Bookser., p. 339-346. Springer, Dordrecht, 2008.

[42] A.C. Morlet, Further properties of a continuum of model equations with globally defined flux, J. Math. Anal. Appl. 221 (1) (1998), 132–160

[43] C.J. Niche, M.E. Schonbek, Decay of weak solutions to the 2D dissipative quasi-geostrophic equation, Comm. Math. Phys. 276 (2007), 93–115.

[44] K. Ohkitani, Dissipative and ideal surface quasi-geostrophic equations, Lecture presented at ICMS, Edinburgh, 2010.

[45] M.E. Schonbek, T.P. Schonbek, Asymptotic behavior to dissipative quasi-geostrophic flows, SIAM J. Math. Anal. 35 (2003), 357–375

[46] M.E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Comm. Partial Differential Equations 17 (1992), 1407–56.

[47] M. Yamazaki, The Navier-Stokes equations in the weak-$L^n$ space with time-dependent external force, Math. Ann. 317 (4) (2000), 635–675.