1. Introduction

Aperiodic order refers to mathematical structures that are not periodic, but are nevertheless highly ordered and close to periodic in some way. Aperiodically ordered patterns gained increased interest among physicists and mathematicians upon the discovery of the quasicrystal in 1982, since these structures were useful in understanding the properties of quasicrystals. For an overview of aperiodic order and in particular its connection to crystallography, please consult [Moo97] [BG13] and [BG17].

These aperiodic ordered structures have long been associated with remarkable images. Among them are the iconic diffraction pattern of the quasicrystal with 10-fold rotational symmetry (Figure 1), the Penrose tiling, and the Hofstadter butterfly, which is a graphical solution of the Harper's equation. (The Harper's equation can be interpreted as a Schrödinger equation with aperiodically ordered potential).

Figure 1. Electron diffraction pattern. Image by Frederic Mompiou

Interest in these aperiodically ordered patterns emerged in the arts long before they were studied by scientists. Aperiodically ordered structures appear in medieval Islamic architecture [LS07], [AA12], [ATR14]. German renaissance artist Albrecht
Dürer also experimented with aperiodic tilings [Lüc00]. Aperiodic order has also emerged in music composition [Pru86], [Ong20], [Tre22].

In this paper, we present a novel approach to creating art from aperiodically ordered patterns. Unlike many of the examples cited above, this art is not based on aperiodically ordered tilings. Rather, it emerges from the symmetries inherent in these patterns that form the diffraction patterns similar to the ones in Figure 1.

This art is constructed from an infinite binary sequence known as the Thue-Morse (or Prouhet-Thue-Morse) sequence. This sequence will be defined in the next section, but for now we will just say that the first few terms are

\[
\text{abcbaabbababa}\ldots
\]

This sequence is not periodic, but is nevertheless close to periodic in the following way. Given an infinite sequence \(S\), we define its complexity sequence \(\{C_n^S\}\) as follows. The entry \(C_n^S\) for \(n \in \mathbb{Z}_+\) is the number of distinct subsequences of \(S\) of length \(n\). If \(S\) is a periodic or eventually periodic sequence, it is not hard to see that \(\{C_n^S\}\) is bounded. If \(S\) is a random binary sequence, almost surely \(\{C_n^S\} = 2^n\). According to the Morse-Hedlund theorem (e.g. Proposition 4.1 of [BG13]), if \(S\) is not periodic then its corresponding \(\{C_n^S\}\) grows at least linearly. It can be shown that if \(S\) is the Thue-Morse sequence, then its corresponding \(\{C_n^S\}\) grows linearly (more precisely, by Proposition 4.5 of [Brl89] \(C_n^S\) is bounded above by \(10n/3\)). In this sense, we can say the Thue-Morse sequence is aperiodic, but as close to periodic as possible. We thus describe it as an “aperiodically ordered” sequence.

We will look at the autocorrelation function corresponding to this Thue-Morse sequence. The autocorrelation function \(\eta(n)\) in essence measures how similar a sequence is to a copy of itself shifted \(n\) steps. It is used for calculating the diffraction pattern of a quasicrystal structure to obtain images like in Figure 1 For a random, independent identically distributed sequence the autocorrelation function is almost always constant. For a periodic sequence, the autocorrelation function \(\eta(n)\) is also periodic. But for an aperiodically ordered sequence, \(\eta(n)\) has a very complicated and interesting structure.

For our art project, we look at a generalization, the fourth order autocorrelation function \(\eta(m, n, k)\). This measures how much the Thue-Morse sequence and three copies of itself shifted \(m, n\) and \(k\) steps respectively are similar to each other. We fix the \(k\), and create a matrix whose \((i, j)\) entry contains \(\eta(i, j, k)\). We then assign colours to each matrix entry, creating an image in the fashion of a heatmap. As a preview, we will show an example image in Figure 2. More elaborate examples will be shown later in the paper. For more on the mathematical properties of this higher order autocorrelation function, please consult [BC22].
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2. Background

2.1. The Thue-Morse sequence. The Thue-Morse sequence (sequence A010060 in the Online Encyclopedia of Integer Sequences [Fou22]) is a sequence of a’s and b’s that is defined in the following way.

Start with the length-1 sequence $\sigma_1 = a$. Then to define $\sigma_j$ for $j \geq 2$, we replace every a in the string $\sigma_{j-1}$ with $ab$, and every b in the string $\sigma_{j-1}$ with $ba$. For example, $\sigma_2 = ab$ since we replaced the a in $\sigma_1$ with $ab$. Then $\sigma_3 = abba$ since we replaced the a in $\sigma_2$ with $ab$, and we replaced the b in $\sigma_2$ with $ba$. The first five $\sigma_j$ can then be defined similarly as follows:
We observe that the first entries of all the $\sigma_j$ are the same. This is easily proved by induction. This observation ensures that the following definition is well-defined:

**Definition 2.1** (Thue-Morse sequence). The Thue-Morse sequence is an infinite binary sequence $\sigma$ such that for any $k \in \mathbb{Z}_+$, the $k$th entry of $\sigma$ is the same as the $k$th entry of $\sigma_j$, for all $\sigma_j$ that have length at least $k$.

Thus

$$\sigma = abbabaabbaababa\ldots$$

See [BG13], and [AS99] for more extensive discussion on this Thue-Morse sequence.

2.2. **Thue-Morse Autocorrelation functions.** In this subsection, we introduce the autocorrelation function $\eta : \mathbb{Z} \to \mathbb{Q}$. This function arises from the mathematical theory of diffraction. To give a brief description of the physics. We are interested in how the aperiodically ordered lattice structure of a quasicrystal affects the diffraction of light that is shone through it. the Thue-Morse sequence is used as a 1-dimensional analogue of a quasicrystal lattice. The diffraction pattern of the quasicrystal is given by the Fourier transform of the autocorrelation measure of the Thue-Morse sequence, and this autocorrelation measure is a measure whose weights are determined by the autocorrelation function of the Thue-Morse sequence. In other words, the autocorrelation measure is $\sum_{m \in \mathbb{Z}} \eta(m)\delta_m$, where $\delta_m$ refers to a Dirac Delta at the location $m$. Details of this construction can be found in Section 9 of [BG13].

In this paper, whenever we discuss the $n$th entry of a string of symbols, we start counting from 0, i.e. the first entry is the 0th entry. Now for $n \in \mathbb{Z}_{\geq 0}$ let us define

$$\sigma(n) = \begin{cases} +1, & \text{when the } n\text{th entry of the Thue-Morse sequence is } a \\ -1, & \text{when the } n\text{th entry of the Thue-Morse sequence is } b \end{cases}$$

Now we can define the Thue-Morse autocorrelation function.

**Definition 2.2.** [Thue-Morse autocorrelation function] For $m \in \mathbb{Z}_{\geq 0}$, the Thue-Morse autocorrelation function is defined to be

$$\eta(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \sigma(i)\sigma(i + m).$$

This limit exists for any $m \in \mathbb{Z}_+$ (see the discussion around (2.2) of [BC22]). If $m$ is in $\mathbb{Z}_{<0}$, then we define $\eta(m) = \eta(-m)$. 

It is not too hard to prove the following:

**Proposition 2.3 ([Kak72]).** For all $m \in \mathbb{Z}_{\geq 0},$

$$
\eta(2m) = \eta(m), \quad \eta(2m + 1) = -\frac{1}{2}(\eta(m) + \eta(m+1)).
$$

This proposition gives us an alternate way of calculating $\eta(m)$. We may set $\eta(0) = 1$ and $\eta(1) = -1/3$ and use the recursion relations to define the other $\eta(m)$.

2.3. **Higher order autocorrelation.** We can modify the definition of Definition 2.2 so we are considering products of three or more Thue-Morse terms instead. For example,

**Definition 2.4.** [Order 3 Thue-Morse autocorrelation function] For $(m, n) \in \mathbb{Z}^2_{\geq 0},$ the order 3 Thue-Morse autocorrelation function is defined to be

$$
\eta(m, n) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \sigma(i)\sigma(i+m)\sigma(i+n).
$$

However, this definition is not very interesting! Corollary 4.2 of [BC22] says that $\eta(m, n) = 0$ for all $m$ and $n$. So we instead proceed to order 4 correlations.

**Definition 2.5.** [Order 4 Thue-Morse autocorrelation function] For $(m, n, k) \in \mathbb{Z}^3_{\geq 0},$ the order 4 Thue-Morse autocorrelation function is defined to be

$$
\eta(m, n, k) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \sigma(i)\sigma(i+m)\sigma(i+n)\sigma(i+k).
$$

Let us first verify that Definition 2.5 is not as trivial as Definition 2.4. It is easy to check, for instance that

$$
\eta(m, m, k) = \eta(m, k, m) = \eta(k, m, m) = \eta(k).
$$

We can also develop a recursion algorithm to calculate $\eta(m, n, k)$.

**Proposition 2.6.**

\[
\eta(m, n, k) = \frac{(-1)^{m+n+k}}{2} \left( \eta \left( \left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{k}{2} \right\rfloor \right) + \eta \left( \left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{k}{2} \right\rceil \right) \right)
\]

**Remark 2.7.** A generalized version of the above proposition can be found in [BC22], where they discuss order $n$ autocorrelations.

**Proof.** We will use (4.14) of [BG13] which states

$$
\sigma(2j) = \sigma(j) \text{ and } \sigma(2j + 1) = -\sigma(j).
$$

We can rewrite this as

$$
\sigma(2i + j) = (-1)^j\sigma(i + \left\lfloor \frac{j}{2} \right\rfloor)
$$
for all $j \in \mathbb{Z}_{\geq 0}$. By the Birkhoff Ergodic Theorem and the fact that all the infinite sums are absolute continuous (and therefore rearrangement is allowed) we can calculate

$$\eta(m,n,k)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \sigma(i)\sigma(i+m)\sigma(i+n)\sigma(i+k)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{i=0,i \ even}^{N-1} \sigma(i)\sigma(i+m)\sigma(i+n)\sigma(i+k)$$

$$+ \lim_{N \to \infty} \frac{1}{N} \sum_{i=0,i \ odd}^{N-1} \sigma(i)\sigma(i+m)\sigma(i+n)\sigma(i+k)$$

$$= \lim_{N \to \infty} \left( -1 \right)^{m+n+k} \frac{1}{N} \sum_{i=0,i \ even}^{N-1} \sigma(i/2)\sigma(i/2+\lfloor m/2 \rfloor)\sigma(i/2+\lfloor n/2 \rfloor)\sigma(i/2+\lfloor k/2 \rfloor)$$

$$+ \lim_{N \to \infty} \left( -1 \right)^{m+n+k} \frac{1}{N} \sum_{i=0,i \ odd}^{N-1} \sigma(i/2)\sigma(i/2+\lceil m/2 \rceil)\sigma(i/2+\lceil n/2 \rceil)\sigma(i/2+\lceil k/2 \rceil)$$

$$\times \sum_{i=0,i \ odd}^{N-1} \sigma(i/2)\sigma(i/2+\lfloor m/2 \rfloor)\sigma(i/2+\lfloor n/2 \rfloor)\sigma(i/2+\lfloor k/2 \rfloor)$$

$$= \lim_{N \to \infty} \left( -1 \right)^{m+n+k} \frac{1}{N} \sum_{j=0}^{\lfloor (N-1)/2 \rfloor} \sigma(j)\sigma(j+\lfloor m/2 \rfloor)\sigma(j+\lfloor n/2 \rfloor)\sigma(j+\lfloor k/2 \rfloor)$$

$$+ \lim_{N \to \infty} \left( -1 \right)^{m+n+k} \frac{1}{N} \sum_{j=0}^{\lfloor (N-2)/2 \rfloor} \sigma(j)\sigma(j+\lfloor m/2 \rfloor)\sigma(j+\lfloor n/2 \rfloor)\sigma(j+\lfloor k/2 \rfloor)$$

$$= \frac{(-1)^{m+n+k}}{2} \left( \eta(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor, \lfloor k/2 \rfloor) + \eta(\lfloor m/2 \rfloor, \lfloor n/2 \rfloor, \lfloor k/2 \rfloor) \right)$$
Remark 2.8. An example where these higher order correlation functions appear in the mathematical physics literature is in [Luc89]. One key object in that paper is the complex Lyapunov exponent $\Omega(E)$. Luck considers a discrete Schrödinger operator acting on $l^2(\mathbb{Z}_{\geq 0})$,

$$
(H\psi)_n := -\psi_{j+1} - \psi_{j-1} + V_j \psi_j, j \geq 0
$$

treating $\psi_{-1}$ as 0. Here $V_j = V \sigma(j)$, where $V$ is a positive number and $\sigma(j)$ is defined in (3).

When we choose $\psi$ to be a formal eigenvector of $H$ corresponding to an eigenvalue $E \in \mathbb{R}$, we can define

$$
\Omega(E) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{\psi_{j+1}}{\psi_j}.
$$

Using the Schrödinger equation $H\psi = E\psi$ and (17), we expand $\Omega(E)$ in the variable $V$:

$\Omega(E) = \Omega^{(0)} + \Omega^{(1)} + \omega^{(2)} + \ldots + \omega^{(n)} + \ldots$ where for every $n$, $\Omega^{(n)}$ is a multiple of $V^n$ but not $V^{n+1}$. Luck calculates $\Omega^{(0)}$, $\Omega^{(1)}$, $\Omega^{(2)}$ and $\Omega^{(3)}$. The calculations (2.23) and (2.26) in [Luc89] demonstrate respectively that $\Omega^{(2)}$ can be expressed in terms of the form $\eta(n)$ and $\Omega^{(3)}$ can be expressed in terms of the form $\eta(m, n)$. It can be analogously calculate that $\Omega^{(4)}$ can be expressed in terms of the form $\eta(m, n, k)$ and so on.

This construction is useful, because Luck uses perturbative behaviour of $\Omega(E)$ to understand the appearance of spectral gaps at $E$ as the variable $V$ varies.

In fact, the motivation for this project emerged as I was attempting to understand the behaviour of $\Omega(E)$ in finer detail than presented in [Luc89]. I was using Microsoft Excel to record values of $\eta(x, y, z)$ in order to find patterns. I would fix a $z$, and let $x$ and $y$ each vary from 0 to $z$. The columns of the excel sheet would represent $x$, and the rows of the excel sheet would represent $y$, with $z$ fixed. I would then fill in the spreadsheet cell with the corresponding value of $\eta(x, y, z)$. In order to make it easier to process the data, I would colour-code the values in the excel sheet, for example making every cell with a 0 in it light blue, every cell with a $-1/4$ in it red, and so on. While doing this I realised that the Excel spreadsheet was producing rather striking images, and this prompted my curiosity as to what pictures could be created if I used values of $z$ that were in the hundreds and thousands. This motivated the art in the following section.
3. Art generated from 4th Order Thue-Morse correlation functions

The process for creating our Thue-Morse autocorrelation art is simply to extend the process in Figure to larger values of $z$. That is, fix $z$ to be a large integer. Then generate a $(z + 1) \times (z + 1)$ matrix where the entry in the $x$th row and $y$th column is $\eta(x, y, z)$ (here we start counting from the $x = 0$th row and the $y = 0$th column). Then create a color assignment function $f$ whose domain is $\mathbb{R}$ and whose codomain is a set of colors. We can then generate an image $z + 1$ pixels wide and $z + 1$ pixels high, where the pixel in $(x, y)$ has colour given by $f(\eta(x, y, z))$. Even for very simple functions $f$, this process can generate some striking images! We will demonstrate a few examples.

The following images are joint work with one of my undergraduate students, Loh Jia Jun. They were created using this colour assignment function. The output of the function is described using the format XXYYZZ in terms of RGB values, where XX represents the strength of the red component, YY represents the strength of the green component, and ZZ represents the strength of the blue component. The numbers XX, YY and ZZ are written in two-digit hexadecimal notation. Thus for example, FFFFFF represents black, 000000 represents white, and 00FF00 represents green.

The following images were created with the colour assignment function described above, with differing values for $z$. 

---

**Figure 3.** Colour-coded excel screenshot corresponding to calculating $\eta(x, y, z)$ with $z = 20$
Table 1. The color assignment function

| $x$         | $f(x)$          |
|-------------|-----------------|
| $(-\infty, -0.1)$ | E8BA00         |
| $[-0.1, -0.05)$ | 6C89EE         |
| $(-0.05, -0.025)$ | FF750E         |
| $(-0.025, -0.01)$ | 0000FF         |
| $(-0.01, -0.008)$ | FF0000         |
| $[-0.008, 0)$ | 56BEE9         |
| $(0, 0.008]$ | 83FE93         |
| $(0.008, 0.01]$ | FF0000         |
| $(0.01, 0.025]$ | 00FF00         |
| $(0.025, 0.05]$ | 00FFFF         |
| $(0.05, 0.1]$ | 84FF00         |
| $(0.1, \infty)$ | 84FF00         |
| $\{-0.05, 0\}$ | white          |

Figure 4. Image with $z = 1023$
Figure 5. Image with $z = 1200$
Figure 6. Image with $z = 1475$
Appendix A. Code

The code to generate these images is written in Java 8. There are two separate programs. The first generates a .csv file, which is an array in which for a fixed $z$ the $m$th row and $n$th column is filled with the number $\eta(m,n,z)$ defined in Definition 2.5. The latest version of the file is found in the github repository here: https://doi.org/10.5281/zenodo.7060457. To change the $z$-value, simply adjust the integer in line 19 of the program below.

```java
import java.util.Scanner;
import java.io.FileNotFoundException;
import java.io.FileOutputStream;
import java.io.OutputStream;
import java.math.BigInteger;

public class TMcor {

    public static void main(String[] args) throws FileNotFoundException {
        final PrintStream oldStdout = System.out;
        System.setOut(new PrintStream(new FileOutputStream("ArtOutput.csv"))); // Above gives the filename of the output file

        int[] input = new int[4];
        input[0]=1; // don't change this
        input[1]=0; // starting column, default is 0
        input[2]=0; // starting row, default is 0
        input[3]=120; // value of z

        for(int j=input[1];j<1+input[3];j++)
            { String outstring="";

                for(int k=input[2];k<1+input[3];k++)
                    {
                        int[] loopinput= new int[4];
                        loopinput[0]=input[0];
                        loopinput[1]=j;
                        loopinput[2]=k;
                        loopinput[3]=input[3];

                        int[][] output=tree(loopinput);

                        int lastlevel=1+(int)Math.ceil(Math.log((double)input[3])/Math.log(2));
                        int oneplus=0;
                        int oneminus=0;
```
int zeroplus = 0;
int zerominus = 0;

for (int i = 0; i < Math.pow(2, lastlevel - 1); i++) {
    int sum = output[1][i][lastlevel - 1] + output[2][i][lastlevel - 1] + output[3][i][lastlevel - 1];
    if (sum % 2 == 0 & output[0][i][lastlevel - 1] == 1) zeroplus += 1;
    if (sum % 2 == 0 & output[0][i][lastlevel - 1] == -1) zerominus += 1;
    if (sum % 2 == 1 & output[0][i][lastlevel - 1] == 1) oneplus += 1;
    if (sum % 2 == 1 & output[0][i][lastlevel - 1] == -1) oneminus += 1;
}
int zerocalc = zeroplus - zerominus;
int onecalc = oneplus - oneminus;

double answer = (zerocalc * 1.0 - onecalc * (1.0 / 3)) / Math.pow(2, lastlevel - 1);
outstring = outstring + "" + answer + ",";

} System.out.println(outstring);
}

System.setOut(oldStdout);
System.out.println("The program ran successfully");

public static int[][] recursion (int[] start) {
    int[][] placeholder = new int[4][2];
    placeholder[1][0] = (int) Math.floor(((float) start[1]) / 2);
    placeholder[2][0] = (int) Math.floor(((float) start[2]) / 2);
    placeholder[3][0] = (int) Math.floor(((float) start[3]) / 2);
    placeholder[1][1] = (int) Math.ceil(((float) start[1]) / 2);
    placeholder[2][1] = (int) Math.ceil(((float) start[2]) / 2);
    placeholder[3][1] = (int) Math.ceil(((float) start[3]) / 2);

    placeholder[0][0] = start[0];
    if (placeholder[1][0] != placeholder[1][1]) placeholder[0][0] *= -1;
    if (placeholder[2][0] != placeholder[2][1]) placeholder[0][0] *= -1;
    if (placeholder[3][0] != placeholder[3][1]) placeholder[0][0] *= -1;

    placeholder[0][1] = placeholder[0][0];
    return placeholder;
The output of tree is \[\text{sign}, x, y, z][\text{horizontal level}][\text{vertical level}]

```java
public static int[][][] tree (int[] root){
    int levels=1+(int)Math.ceil(Math.log ((double) root[3])/Math.log(2));
    int[][][] x=new int[4][(int)Math.pow(2,levels-1)][levels];
    for(int i=0;i<4;i++) x[i][0][0]=root[i];
    for(int ell=1; ell<levels;ell++) {
        int horz=(int)Math.pow(2,ell);
        for(int j=0;j<horz/2;j++) {
            int[] temp1= new int[4];
            int[][] temp2= new int[4][2];
            for(int i=0;i<4;i++) temp1[i]=x[i][j][ell-1];
            temp2 = recursion (temp1);
            for (int i =0; i <4; i++) {
                x[i][2*j][ell]=temp2[i][0];
                x[i][2*j+1][ell]=temp2[i][1];
            }
        }
    }
    return x;
}
```

LISTING 1. Program to Generate .csv array

The second program assigns colors to each entry in the array according to the function in Table 1. To change the color assignments, adjust the code from lines 65 to 130.

```java
import java.awt.image.BufferedImage;
import java.util.*;
import javax.imageio.ImageIO;
import java.lang.*;
import java.io.*;
public class TMart{
    public static void main (String[] args) throws java.lang.Exception{
        ...
    }
}
String csvFile = "ArtOutput.csv";
BufferedReader br = null;
String line = "";
String cvsSplitBy = ",";

BufferedReader brcount = new BufferedReader(new FileReader(csvFile));
int count = 0;
while (brcount.readLine() != null)
{
    count++;
}
brcount.close();

double[][] DoubleMatrix = new double[count][count];
try {
    int row= 0;
    br = new BufferedReader(new FileReader(csvFile));
    while ((line = br.readLine()) != null) {

        // use comma as separator
        String[] country = line.split(cvsSplitBy);

        for(int i=0;i<count;i++) DoubleMatrix[row][i]= Double.parseDouble(country[i]);
        row=row+1;
    }
}

} catch (FileNotFoundException e) {
    e.printStackTrace();
} catch (IOException e) {
    e.printStackTrace();
} finally {
    if (br != null) {
        try {
            br.close();
        } catch (IOException e) {
            e.printStackTrace();
        }
    }
}

int[][] r = new int[count][count];
int[][] g = new int[count][count];
int[][] b = new int[count][count];
for(int j=0; j<count; j++)
{
    for (int k=0; k<count; k++) {
        if(DoubleMatrix[j][k]>0.008 && DoubleMatrix[j][k] <=0.01) {
            r[j][k]=0xFF;
            g[j][k]=0x00;
            b[j][k]=0x00;
        }
        else if(DoubleMatrix[j][k]<=-0.008&& DoubleMatrix[j][k] >-0.01) {
            r[j][k]=0xFF;
            g[j][k]=0xFF;
            b[j][k]=0x00;
        }
        else if(DoubleMatrix[j][k] >0.01 && DoubleMatrix[j][k] <=0.025) {
            r[j][k]=0x00;
            g[j][k]=0xFF;
            b[j][k]=0x00;
        }
        else if(DoubleMatrix[j][k] < -0.01 && DoubleMatrix[j][k] >= -0.025) {
            r[j][k]=0x00;
            g[j][k]=0x00;
            b[j][k]=0xFF;
        }
        else if(DoubleMatrix[j][k] >0.025 && DoubleMatrix[j][k] <=0.05) {
            r[j][k]=0x00;
            g[j][k]=0xFF;
            b[j][k]=0xFF;
        }
        else if(DoubleMatrix[j][k] < -0.025 && DoubleMatrix[j][k] > -0.05) {
            r[j][k]=0xFF;
            g[j][k]=0xC7;
            b[j][k]=0x50;
        }
        else if(DoubleMatrix[j][k] >0 && DoubleMatrix[j][k] <=0.008) {
            r[j][k]=0x83;
            g[j][k]=0xFE;
            b[j][k]=0x93;
        }
        else if(DoubleMatrix[j][k] <0 && DoubleMatrix[j][k] >= -0.008) {
            r[j][k]=0x00;
            g[j][k]=0xFF;
            b[j][k]=0x00;
        }
    }
}
else if(DoubleMatrix[j][k]>0.008 && DoubleMatrix[j][k] <=0.01) {
    r[j][k]=0xFF;
    g[j][k]=0x00;
    b[j][k]=0x00;
}
else if(DoubleMatrix[j][k] >0.01 && DoubleMatrix[j][k] <=0.025) {
    r[j][k]=0x00;
    g[j][k]=0xFF;
    b[j][k]=0x00;
}
else if(DoubleMatrix[j][k] >0.025 && DoubleMatrix[j][k] <=0.05) {
    r[j][k]=0x00;
    g[j][k]=0xFF;
    b[j][k]=0xFF;
}
else if(DoubleMatrix[j][k] < -0.025 && DoubleMatrix[j][k] > -0.05) {
    r[j][k]=0xFF;
    g[j][k]=0xC7;
    b[j][k]=0x50;
}
else if(DoubleMatrix[j][k] >0 && DoubleMatrix[j][k] <=0.008) {
    r[j][k]=0x83;
    g[j][k]=0xFE;
    b[j][k]=0x93;
}
else if(DoubleMatrix[j][k] <0 && DoubleMatrix[j][k] >= -0.008) {
    r[j][k]=0x00;
    g[j][k]=0xFF;
    b[j][k]=0x00;
}
ABSTRACT ART GENERATED BY THUE-MORSE CORRELATION FUNCTIONS

101 r[j][k]=0x56;
102 g[j][k]=0xBE;
103 b[j][k]=0xE9;
104 }
105 else if(DoubleMatrix[j][k]<-0.05&& DoubleMatrix[j][k]>-0.1) {
106  r[j][k]=0x6C;
107  g[j][k]=0x89;
108  b[j][k]=0xEE;
109 }
110 else if(DoubleMatrix[j][k]>0.05 && DoubleMatrix[j][k]<=0.1) {
111  r[j][k]=0x84;
112  g[j][k]=0xFF;
113  b[j][k]=0x00;
114 }
115 else if(DoubleMatrix[j][k]<-0.1) {
116  r[j][k]=0xE8;
117  g[j][k]=0xBA;
118  b[j][k]=0x00;
119 }
120 else if(DoubleMatrix[j][k]>0.1) {
121  r[j][k]=0xFF;
122  g[j][k]=0x76;
123  b[j][k]=0x5D;
124 }
125 else {
126  r[j][k]=0x00;
127  g[j][k]=0x00;
128  b[j][k]=0x00;
129 }
130 }
131 }
132 }
133 }
134 }
135 }
136 int width = count;
137 int height = count;
138 }
139 BufferedImage image = new BufferedImage(width, height, BufferedImage.TYPE_INT_RGB);
140 }
141 for (int y = 0; y < height; y++) {
142  for (int x = 0; x < width; x++) {
143   int rgb = r[y][x];
144   rgb = (rgb << 8) + g[y][x];
Listing 2. Program to generate colors from the .csv array

Running these two programs in succession without changes will generate the image in Figure 5.

A.1. Simple Example. Let us go through a simple example. We modify line 19 of Listing 1 to set \texttt{input[3]= 3}. When we run the program, it will generate the following .csv file:

![ArtOutput.csv file generated by Listing 1 with input[3]= 3](image)

Then, using this .csv file as the input for the program in Listing 2, we obtain the following image:
Figure 8. .bmp image file generated by Listing 2 using the .csv file in Figure A.1 as input.

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