SOME FAMILIES OF SUPERCONGRUENCES INVOLVING
ALTERNATING MULTIPLE HARMONIC SUMS

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Abstract. Let $p$ be a prime. In this short note we study some families of supercongruences involving the following alternating sums

$$\sum_{j_1+j_2+\cdots+j_n=2p^r \atop p \nmid j_1 \cdots j_n} \frac{(-1)^{j_1+\cdots+j_n}}{j_1 \cdots j_n} \left( \mod p^r \right),$$

which extend similar statements proved by Shen and Cai who treated the cases when $n = 4, 5$. Our method works for arbitrary $n$.

1. Introduction

Over the past quarter of a century, multiple zeta values (MZVs) and their various generalizations have been intensively studied by many mathematicians and physicists due to their important applications in quite a few different areas of mathematics and theoretical physics. These values are infinite series whose finite sums are commonly called the multiple harmonic sums, defined as follows. Let $\mathbb{N}$ and $\mathbb{N}_0$ be the set of positive integers and nonnegative integers, respectively. For any $n, d \in \mathbb{N}$ and $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$, we define the multiple harmonic sums (MHSs) by

$$\mathcal{H}_n(s) := \sum_{n > k_1 > \cdots > k_d > 0} \frac{1}{k_1^{s_1} \cdots k_d^{s_d}}.$$ 

For example, $\mathcal{H}_{n+1}(1)$ is often called the $n$th harmonic number.

Very recently, a finite version of MZVs has emerged which has been conjectured to be closely related to MZVs, see [15, Ch. 8]. These values are essentially the MHSs truncated at different primes and then taken residues modulo the corresponding primes. Such congruences were first studied independently by the last author in [12, 13] and Hoffman in [3]. In general, it is well-known that Bernoulli numbers play a very important role in these congruences, see [8] for some classical results. As an application, in [11] the

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last author proved, by using some special properties of the double harmonic sums, that for every odd prime $p$

$$
\sum_{i,j,k \geq 1, i+j+k=p} \frac{1}{i^j k^k} \equiv -2B_{p-3} \pmod{p},
$$

(1)

where $B_k$ are Bernoulli numbers defined by the generating series

$$
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.
$$

Later, Ji gave an alternative simpler proof of (1) in [4] using some combinatorial techniques. Congruence (1) has since been generalized by either increasing the number of indices, changing the bound from $p$ to multiples of $p$ or $p$-powers, and/or considering the corresponding supercongruences (see [1, 5, 9, 10, 14, 16]), or even allowing the alternating version of MHSs (see [6, 7]).

Our main results of this short note concern the following type of sums. Let $P_p$ be the set of positive integers not divisible by $p$. For $m, n, r, N \in \mathbb{N}$, we define

$$
Z_n(N, p) := \sum_{l_1 + l_2 + \ldots + l_n = N, l_1, \ldots, l_n \in P_p} \frac{1}{l_1 l_2 \ldots l_n}
$$

for $p | N$,

$$
R_n^{(m)}(p^r) := \sum_{l_1 + l_2 + \ldots + l_n = mp^r, l_1, \ldots, l_n \in P_p} \frac{1}{l_1 l_2 \ldots l_n} = Z_n(mp^r, p)
$$

for $p \nmid m$,

$$
S_n^{(m)}(p^r) := \sum_{l_1 + l_2 + \ldots + l_n = mp^r, p^r > l_1, \ldots, l_n \in P_p} \frac{1}{l_1 l_2 \ldots l_n}
$$

for $p \nmid m$.

The primary goal of our study is to find nice and simple supercongruences involving alternating sums defined as follows:

$$
\sigma_n^{(b)}(N, p) := \sum_{l_1 + l_2 + \ldots + l_n = N, l_1, \ldots, l_n \in P_p} \frac{(-1)^{l_1+\ldots+l_n}}{l_1 l_2 \ldots l_n}
$$

for $p | N$.

We will reduce these congruences to those of $Z_n(N, p)$ whose special cases $R_n^{(m)}(p^r)$ are closely related to $S_n^{(m)}(p^r)$ by Proposition 2.3. These results are motivated by the recent work of Shen and Cai [6] who studied the above sums for $n = 3, 4$. In Theorem 3.4 we generalize this to arbitrary $n$ by using $R_n^{(m)}(p^r)$ with $m = 1, 2$.

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2. Some useful lemmas

We start with a formula expressing the sums \( Z_n(N, p) \) in terms of a modified version of multiple harmonic sums.

**Lemma 2.1.** Let \( n, N \in \mathbb{N} \) and \( p \) be a prime. If \( p \mid N \) then we have

\[
Z_n(N, p) = \frac{n!}{N} \sum_{\substack{1 \leq u_1 < \cdots < u_{n-1} < N \\ u_1, u_2 - u_1, \ldots, u_{n-1} - u_{n-2}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{u_1 \cdots u_{n-1}}.
\]  

(2)

**Proof.** First, noting that \( l_1 + l_2 + \cdots + l_n = N \), we have

\[
Z_n(N, p) = \frac{1}{N} \sum_{\substack{l_1 + l_2 + \cdots + l_n = N \\ l_1, \ldots, l_n \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \cdots l_n}.
\]

Then one writes

\[
\frac{1}{l_1 \cdots l_{n-1}} = \frac{l_1 + \cdots + l_{n-1}}{l_1 \cdots l_{n-1}(l_1 + \cdots + l_{n-1})}
\]

to get

\[
Z_n(N, p) = \frac{n(n-1)}{N} \sum_{\substack{l_1 + \cdots + l_{n-1} < u_{n-1} < N \\ l_1, \ldots, l_{n-2}, u_{n-1} - l_1 - \cdots - l_{n-2}, u_{n-1} \in \mathcal{P}_p}} \frac{1}{l_1 l_2 \cdots l_{n-2} u_{n-1}},
\]

and continues in this way by using the substitutions \( u_j = l_1 + \cdots + l_j \) for \( 1 \leq j < n \) to prove equation (2). This completes the proof of the lemma. \( \Box \)

**Lemma 2.2.** Suppose \( m, n, r \in \mathbb{N} \) and \( p \) is a prime with \( p > n + 1 \). Then we have

\[
S_n^{(m)}(p^{r+1}) \equiv (-1)^{m-1} \left( \frac{n-2}{m-1} \right) S_n^{(1)}(p^2)p^{r-1} \pmod{p^{r+1}}.
\]

**Proof.** For all \( n, a \in \mathbb{N} \), set

\[
\gamma_n(a) := (-1)^{n+1} \frac{(a-1)!(n-1-a)!}{(n-1)!}.
\]

By [5, Lemma 2.3], we have

\[
S_n^{(m)}(p^{r+1}) \equiv p \sum_{a=1}^{n-1} (-1)^{m-1} \left( \frac{n-2}{m-1} \right) \gamma_n(a) S_n^{(a)}(p^r) \pmod{p^{r+1}}
\]

\[
\equiv (-1)^{m-1} \left( \frac{n-2}{m-1} \right) S_n^{(1)}(p^{r+1}) \pmod{p^{r+1}}.
\]
So the lemma follows from [3 (1.3)] which says
\[ S_n^{(1)}(p^{r+1}) \equiv pS_n^{(1)}(p^r) \pmod{p^{r+1}} \]
for all \( r \geq 2 \). \( \square \)

**Proposition 2.3.** Let \( m, n, r \in \mathbb{N} \) with \( r \geq 2 \). Then we have
\[
R_n^{(m)}(p) \equiv \sum_{a=1}^{n-1} \binom{m + n - a - 1}{n - 1} S_n^{(a)}(p) \pmod{p}, \tag{3}
\]
\[
R_n^{(m)}(p^r) \equiv m \cdot S_n^{(1)}(p^2)p^{r-2} \pmod{p^r}. \tag{4}
\]

**Proof.** Let \( p \) be a prime number such that \( p > n + 1 \). For any \( n \)-tuples \((l_1, \ldots, l_n)\) of integers in \( \mathcal{P}_p \) satisfying \( l_1 + \cdots + l_n = mp^r \), we rewrite
\[ l_i = x_ip^r + y_i, \quad x_i \geq 0, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq n. \]
Since
\[ \left( \sum_{i=1}^{n} x_i \right)p^r + \sum_{i=1}^{n} y_i = mp^r, \]
there exists \( 1 \leq a < n \) such that
\[
\begin{cases} 
  x_1 + \cdots + x_n = m - a, \\
  y_1 + \cdots + y_n = ap^r.
\end{cases}
\]
For \( 1 \leq a < n \), the equation \( x_1 + \cdots + x_n = m - a \) has \( \binom{m + n - a - 1}{n - 1} \) nonnegative integer solutions. Hence, for all \( r \geq 1 \),
\[
R_n^{(m)}(p^r) = \sum_{l_1 + \cdots + l_n = mp^r} \frac{1}{l_1l_2 \cdots l_n},
\]
\[
= \sum_{a=1}^{n-1} \sum_{\substack{x_1 + \cdots + x_n = m - a \\
y_1 + \cdots + y_n = ap^r}} \frac{1}{(x_1p^r + y_1) \cdots (x_n p^r + y_n)}
\]
\[
\equiv \sum_{a=1}^{n-1} \binom{m + n - a - 1}{n - 1} S_n^{(a)}(p^r) \pmod{p^r}
\]
\[
\equiv \sum_{a=1}^{n-1} \binom{m + n - a - 1}{n - 1} (-1)^a \binom{n - a - 2}{a - 1} S_n^{(1)}(p^2)p^{r-2} \pmod{p^r},
\]
by Lemma 2.2. Note that the penultimate step holds for \( r = 1 \) which implies (3). However, the last step is valid only when \( r \geq 2 \). So (4) follows.
immediately from
\[
\sum_{a=1}^{n-1} (-1)^{a-1} \binom{n-2}{a-1} \binom{m+n-a-1}{n-1} = \sum_{a=1}^{n-1} \binom{a+1-n}{a-1} \binom{m+n-a-1}{m-a} = m
\]
by the famous Chu–Vandermonde identity. □

3. ALTERNATING SUMS

We now define the alternating version of the multiple harmonic sums. For convenience, we denote by \( \bar{s} \) a signed integer for every \( s \in \mathbb{N} \) and set \( |\bar{s}| = s \) and \( \text{sgn}(\bar{s}) = -1 \). Let \( s_j \) be either a positive integer or a signed integer for all \( j = 1, \ldots, d \). For any \( n \in \mathbb{N} \), the alternating MHS is defined by

\[
\mathcal{H}_n(s_1, \ldots, s_d) := \sum_{n > k_1 > \ldots > k_d > 0} \text{sgn}(s_1)^{k_1} \cdots \text{sgn}(s_d)^{k_d}.
\]

For example, \( \lim_{n \to \infty} \mathcal{H}_n(\bar{1}) \) is just the well-known alternating harmonic series.

As variations of alternating MHSs, we have defined that

\[
\sigma_n^{(b)}(N, p) = \sum_{l_1 + l_2 + \cdots + l_n = N} \frac{(-1)^{l_1+\cdots+l_n}}{l_1l_2\ldots l_n} \quad \text{for } p|N.
\]

In this section, for each fixed \( n \geq 4 \), we will study some suitable linear combinations of \( \sigma_n^{(b)}(N, p) \) for \( b = 1, \ldots, n-1 \). To this end, for any \( a \geq b \geq 0 \), \( d \geq 0 \) and \( s = (s_1, \ldots, s_d) \in \{1, \bar{1}\}^d \), we define

\[
F_a^{(b)}(s, N, p) := \sum_{N > i_1 > \cdots > i_d > 1} \frac{\text{sgn}(s_1)^{i_1} \cdots \text{sgn}(s_d)^{i_d} (-1)^{i_1+\cdots+i_d}}{i_1 \cdots i_d l_1 \ldots l_a}.
\]

Then it is easy to see that if \( N \) is even then

\[
N \sigma_n^{(b)}(N, p) = (n-b)F_{n-1}^{(b)}(\emptyset, N, p) + bF_{n-1}^{(n-b)}(\emptyset, N, p), \quad (5)
\]
\[
F_a^{(b)}(s, N, p) = (a-b)F_{a-1}^{(b)}((s, 1), N, p) + bF_{a-1}^{(a-b)}((s, \bar{1}), N, p), \quad a \geq 1. \quad (6)
\]

Here, we have abused the notation by writing \( (s, 1) = (s_1, \ldots, s_d, 1) \) and \( (s, \bar{1}) = (s_1, \ldots, s_d, \bar{1}) \). For \( m \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \), put

\[
X_m := (1_m), \quad Z_n := ((\bar{1}), 1_{n-1}).
\]

For \( s = (X_{w_1}, Z_{w_2}, \ldots, Z_{w_l}) \), we set \( W_s := (w_1, w_2, \ldots, w_l) \), \( \text{len}(W_s) := l \) and

\[
A_\emptyset := 0, \quad B_\emptyset := 0, \quad W_\emptyset := (0), \quad P_\emptyset := b. \quad (7)
\]
Otherwise, for $s \neq \emptyset$, we define

$$A_s := \sum_{2^i} w_i, \quad B_s := \sum_{2^i} w_i, \quad P_s := \begin{cases} a - b - A_s & \text{if } 2 | \text{len}(W_s); \\ b - B_s & \text{if } 2 \nmid \text{len}(W_s). \end{cases}$$

Finally, for all fixed $a \geq b \geq 0$ and $A, B \geq 0$, we put

$$C_{a,b}(A, B) = C(A, B) := \begin{cases} 1 & \text{if } A, B = 0; \\ (b - A)^{(a - B)} & \text{if } A = 0, B > 0; \\ (a - b)_A & \text{if } A > 0, B = 0; \\ (a - b)_A(b - B) & \text{if } A > 0, B > 0, \end{cases}$$

where $(x)_\alpha = x(x - 1) \cdots (x - \alpha + 1)$ is the Pochhammer symbol for the falling factorial.

**Lemma 3.1.** Let $a, b \in \mathbb{N}_0$. Then for any fixed nonnegative integer $d \leq a$,

$$F^{(b)}_a(\emptyset, N, p) = \sum_{s \in \{1, 1\}^d} C_{a,b}(A_s, B_s) F^{(P_s)}_{a-A_s-B_s}(s, N, p).$$

**Proof.** We will prove this by induction on $d$. If $d = 0$ then there is only one term in the sum corresponding to $s = \emptyset$. Then the lemma holds by (11). Now let $d \geq 1$ and suppose the lemma is true when $d$ is replaced by $d - 1$. Observe that any composition in $\{1, 1\}^d$ is produced by either $(s, 1)$ or $(s, 1)$ for a unique $s \in \{1, 1\}^{d-1}$. Further, it is easy to see that

$$(A_{(s,1)}, B_{(s,1)}) = \begin{cases} (A_s + 1, B_s) & \text{if } 2 \nmid \text{len}(W_s); \\ (A_s, B_s + 1) & \text{if } 2 | \text{len}(W_s), \end{cases}$$

$$(A_{(s,1)}, B_{(s,1)}) = \begin{cases} (A_s, B_s + 1) & \text{if } 2 \nmid \text{len}(W_s); \\ (A_s + 1, B_s) & \text{if } 2 | \text{len}(W_s). \end{cases}$$

If $d < a$ and $2 \nmid \text{len}(W_s)$, then by (6)

$$C(A_s, B_s) F^{(b - B_s)}_{a-A_s-B_s}(s, N, p) = C(A_s, B_s) \left[ (a - b - A_s) F^{(b - B_s)}_{a-A_s-B_s-1}((s, 1), N, p) \\ + (b - B_s) F^{(a - b - A_s)}_{a-A_s-B_s-1}((s, 1), N, p) \right]$$

$$= C(A_s + 1, B_s) F^{(P_{(s,1)})}_{a-(A_s+1)-B_s}((s, 1), N, P)$$
$$+ C(A_s, B_s + 1) F^{(P_{(s,1)})}_{a-A_s-(B_s+1)}((s, 1), N, p)$$
$$= C(A_{(s,1)}, B_{(s,1)}) F^{(P_{(s,1)})}_{a-A_{(s,1)}-B_{(s,1)}}((s, 1), N, P)$$
$$+ C(A_{(s,1)}, B_{(s,1)}) F^{(P_{(s,1)})}_{a-A_{(s,1)}-B_{(s,1)}}((s, 1), N, p).$$
If $d < a$ and $2 | \text{len}(W_s)$, then by (3) again
\[
C(A_s, B_s) F_{a-A_s-B_s}^{(a-b-A_s)}(s, N, p)
\]
\[
= C(A_s, B_s) \left[ (b - B_s) F_{a-A_s-B_s-1}^{(a-b-A_s)}((s, 1), N, p) + (a - b - A_s) F_{a-A_s-B_s-1}^{(b-B_s)}((s, 1), N, p) \right]
\]
\[
= C(A_s, B_s + 1) F_{a-A_s-B_s+1}^{(P_{a,b})}((s, 1), N, P)
\]
\[
+ C(A_s + 1, B_s) F_{a-A_s-B_s+1}^{(P_{a,b})}((s, 1), N, P)
\]
\[
= C(A_{(s,1)}, B_{(s,1)}) F_{a-A_{(s,1)}-B_{(s,1)}}^{(P_{a,b})}((s, 1), N, P)
\]
\[
+ C(A_{(s,1)}, B_{(s,1)}) F_{a-A_{(s,1)}-B_{(s,1)}}^{(P_{a,b})}((s, 1), N, p).
\]
This finishes the induction proof of the lemma. \qed

**Corollary 3.2.** Let $a, b \in \mathbb{N}$ with $a \geq b$. For all $s \in \{1, \overline{1}\}^a$, we have
\[
C_{a,b}(A_s, B_s) = \begin{cases} 
(a - b)!b & \text{if } A_s = a - b, B_s = b; \\
0 & \text{if } A_s \neq a - b, B_s \neq b.
\end{cases}
\]

**Proof.** It is easy to see that $A_s + B_s = |W_s| = |s| = a$. If $C_{a,b}(A_s, B_s) \neq 0$, then by its definition
\[
a - b - A_s + 1 > 0, b - B_s + 1 = -(a - b - A_s) + 1 > 0,
\]
which imply that $A_s = a - b, B_s = b$ and $C_{a,b}(A_s, B_s) = (a - b)!b$. \qed

**Corollary 3.3.** For all fixed $a \in \mathbb{N}$, we have
\[
\sum_{b=0}^{a} \binom{a}{b} F_a^{(b)}(0, 2N, p) = \frac{N}{a + 1} Z_{a+1}(N, p).
\]

**Proof.** By Corollary 3.2, $C(A_s, B_s) \neq 0$ for one and only one $b$ for every $s \in \{1, \overline{1}\}^a$. Thus,
\[
\sum_{b=0}^{a} \binom{a}{b} F_a^{(b)}(0, 2N, p) = \sum_{s \in \{1, \overline{1}\}^a} a! F_0^{(0)}(s, 2N, p)
\]
\[
= a! \sum_{\substack{2N > i_1 > \cdots > i_a > 0 \\
i_1, i_1-1, \ldots, i_a-1 \in \mathbb{P}_p}} \frac{(1 + (-1)^{i_1}) \cdots (1 + (-1)^{i_a})}{i_1 \cdots i_a}
\]
As the term is nonzero only when all indices are even, we get
\[
\sum_{b=0}^{a} \binom{a}{b} F_a^{(b)}(0, 2N, p) = a! \sum_{\substack{N > i_1 > \cdots > i_a > 0 \\
i_1, i_1-1, \ldots, i_a-1 \in \mathbb{P}_p}} \frac{1}{i_1 \cdots i_a}.
\]
We can now finish the proof of the corollary by applying Lemma 2.1.

**Theorem 3.4.** Let $n, N$ be two positive integers and $p$ a prime. If $p|N$ then we have

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N,p) = \frac{1}{2} Z_n(N,p) - Z_n(2N,p),$$

where $\alpha_{n,b} = 1$ except for $\alpha_{n,n/2} = 1/2$ when $n$ is even. In particular, for every $r \in \mathbb{N}$ and prime $p$ we have

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2p^r, p) = \frac{1}{2} \left( R_n^{(1)}(p^r) - R_n^{(2)}(p^r) \right) - \frac{3}{2} \sigma_n^{(1)}(p^2) p^{r-1} \pmod{p^r}.$$

**Proof.** For even $N$, we have $\sigma_n^{(b)}(N,p) = \sigma_{n-b}^{(n-b)}(N,p)$ and therefore we get

$$2N \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N,p) = \frac{1}{2} \sum_{b=0}^{n} \binom{n}{b} 2N \sigma_n^{(b)}(2N,p)$$

$$= \frac{1}{2} \sum_{b=0}^{n} \binom{n}{b} (n-b) F_{n-1}^{(b)}(\emptyset, 2N,p) + \frac{1}{2} \sum_{b=0}^{n} \binom{n}{b} b F_{n-1}^{(n-b)}(\emptyset, 2N,p)$$

by (5). Using substitution $b \rightarrow n - b$ in the second sum, we get

$$2N \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N,p) = \sum_{b=0}^{n} (n-b) \binom{n}{b} F_{n-1}^{(b)}(\emptyset, 2N,p)$$

$$= n \sum_{b=0}^{n-1} \binom{n-1}{b} F_{n-1}^{(b)}(\emptyset, 2N,p) = NZ_n(N,p),$$

by Corollary 3.3 with $a = n - 1$. Therefore,

$$\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N,p) = \sum_{b=0}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(b)}(2N,p) - \sigma_n^{(0)}(2N,p)$$

$$= \frac{1}{2} Z_n(N,p) - Z_n(2N,p)$$

since $\sigma_n^{(0)}(2N,p) = Z_n(2N,p)$. The final congruence of the theorem follows easily from Proposition 2.3. This completes the proof of the theorem. \(\square\)
Corollary 3.5. Let \( n \in \mathbb{N} \) and \( p \) be a prime such that \( p > n + 1 \). Then we have

\[
\sum_{b=1}^{\lfloor n/2 \rfloor} \alpha_{n,b} \binom{n}{b} \sigma_n^{(0)}(2p, p) \equiv \begin{cases} 
\frac{n!}{2} B_{p-n} & \text{mod } p, 
\text{if } 2 \nmid n; \\
-\frac{n!}{2} \sum_{a+b=n, \, a,b \geq 3} \frac{B_{p-a}B_{p-b}}{ab} & \text{mod } p, 
\text{if } 2|n.
\end{cases}
\]

Proof. This follows easily from Theorem 3.4, \[16\] Main Theorem, \[5\] Lemma 3.5 and Corollary 3.6] (for \( n \) odd) and \[10\] Theorem 1 and Corollary 1] (for \( n \) even). \( \square \)

Corollary 3.6. Let \( r \in \mathbb{N} \) and \( p > 4 \) be a prime. We have

\[
\sigma_4^{(1)}(2p^r, p) + 3\sigma_4^{(2)}(2p^r, p) \equiv 0 \pmod{p^r}, \tag{8}
\]

\[
\sigma_5^{(1)}(2p^r, p) + 2\sigma_5^{(2)}(2p^r, p) \equiv 6B_{p-5}p^{r-1} \pmod{p^r}. \tag{9}
\]

Proof. It follows from \[10\] Theorem 1], \[16\] Theorem 1.1], \[16\] Main Theorem], and \[9\] Theorem 2] that

\[
S_4^{(1)}(p^2) \equiv 0, \quad S_5^{(1)}(p^2) \equiv -20B_{p-5}p \pmod{p^2}.
\]

So Theorem 3.4 yields the corollary immediately. \( \square \)

In fact, this note was motivated by Shen and Cai’s proof of (8) and a finer version of (9) in \[7\]. Now it follows from \[10\] Theorem 4] and \[5\] Theorem 1.1] that

\[
S_6^{(1)}(p^2) \equiv -\frac{20}{3}B_{p-3}^2, \quad S_7^{(1)}(p^2) \equiv -504B_{p-7}p \pmod{p^2},
\]

and, by similar computation (see \[2\] for details)

\[
S_8^{(1)}(p^2) \equiv -\frac{1792}{5}B_{p-3}B_{p-5}p, \quad S_9^{(1)}(p^2) \equiv -\frac{32}{3}(2283B_{p-9} + 7B_{p-3}^3)p \pmod{p^2}.
\]

Therefore, by Theorem 3.4] modulo \( p^r \) (\( r \geq 2 \)), we have

\[
6\sigma_6^{(1)}(2p^r, p) + 15\sigma_6^{(2)}(2p^r, p) + 10\sigma_6^{(3)}(2p^r, p) \equiv 10B_{p-3}^2p^{r-1},
\]

\[
7\sigma_7^{(1)}(2p^r, p) + 21\sigma_7^{(2)}(2p^r, p) + 35\sigma_7^{(3)}(2p^r, p) \equiv 756B_{p-7}p^{r-1},
\]

\[
8\sigma_8^{(1)}(2p^r, p) + 28\sigma_8^{(2)}(2p^r, p) + 56\sigma_8^{(3)}(2p^r, p) + 35\sigma_8^{(4)}(2p^r, p) \equiv 2688\frac{1}{5}B_{p-3}B_{p-5}p^{r-1},
\]

\[
9\sigma_9^{(1)}(2p^r, p) + 36\sigma_9^{(2)}(2p^r, p) + 84\sigma_9^{(3)}(2p^r, p) + 126\sigma_9^{(4)}(2p^r, p) \equiv 16(2283B_{p-9} + 7B_{p-3}^3)p^{r-1}.
\]
By combining Theorem 3.4 and the numerical results of \( S_n^{(1)}(p^2) \) obtained in [2], one can derive easily similar explicit formulas for all \( n \leq 12 \).

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