Almost Hermitian Ricci flow

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Abstract
We introduce a new curvature flow which matches with the Ricci flow on metrics and preserves the almost Hermitian condition. This enables us to use Ricci flow to study almost Hermitian manifolds.

Keywords Ricci flow · Almost Hermitian geometry

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1 Introduction

The Ricci flow, introduced by Hamilton in [7], has proven to be a very useful tool in geometry and topology. Evidence of its utility can be seen through its application in Perelman’s solutions of the Poincaré conjecture and the Geometrization conjecture for 3-manifolds [10,11]. To exploit the power of this analytic tool, it is advantageous to identify settings where the flow respects and interacts well with the present geometric structures. For instance, the Ricci flow has been applied to manifolds with various positivity conditions on curvature (eg., manifolds with 1/4-pinched curvature, [1–3]). Another natural setting is on Kähler manifolds, where the Ricci flow has been applied to obtain numerous results. Given this, the Ricci flow proves a promising tool for the Analytic Minimal Model Program in the mission to classify compact Kähler manifolds birationally [14].

The use of the Ricci flow on Kähler manifolds so far indicates that, to some degree, the Ricci flow respects and ties in with the delicate interplay of a Riemannian metric...
and almost complex structure. How far does this utility go? Over the years it has been generally believed that the Ricci flow does not preserve non-Kähler Hermitian structures. Throughout the literature there are examples of flows on non-Kähler manifolds (e.g. almost Kähler [12] and Hermitian [6,13]) which apply techniques inspired by the Ricci flow to complex manifolds. In these settings, the metric flow must be modified to flow by some type of curvature quantity rather than the Ricci tensor. This modification helps ensure the flow preserves whatever desired structure is at play. As such, it was yet unclear to what extent the Ricci flow itself could be employed in such settings.

In contrast to the highly structured Kähler manifold, an almost Hermitian manifold is simply an almost complex manifold with compatibility imposed between the metric and complex structure. In this short paper, we demonstrate that given an almost complex manifold, by coupling with a flow on almost complex structures, the Ricci flow in fact does preserve this compatibility. This opens doors to applying the techniques of Ricci flow to studying properties of almost Hermitian structures with an eye towards identifying new classes of manifolds.

1.1 Outline of Paper and Statement of Main Results

Given an almost Hermitian manifold \((M, g, J)\) with Riemannian metric \(g\) and almost complex structure \(J\), we denote by \(\omega\) its Kähler form and by \(\nabla\) the associated Chern connection. We will define the Almost Hermitian Ricci flow to be a paired flow of the metric and almost complex structure given by

\[
\begin{align*}
\left(\frac{\partial g}{\partial t}\right)_{bc} &= -2R_{bc} \\
\left(\frac{\partial J}{\partial t}\right)^{c}_{a} &= -(P_{av} - 2J^{y}_{a} R_{cy})^{2,0+0,2} g^{vc} - \left((\mathcal{L}_{\partial} J)^{m}_{a} g^{mv}_{skew} g^{vc}ight) + (\kappa_{av} + 2\nabla^{e} (d\omega)^{3,0+0,3}) g^{vc}. 
\end{align*}
\] (1.1)

Here, \(P\) denotes the Chern Ricci tensor, \(R_{c}\) denotes the Ricci tensor, and \(\kappa\) is a lower order \(J\)-antiinvariant skew symmetric two cotensor. The first collection of terms in the evolution of the almost complex structure is in fact the main operator in the symplectic curvature flow setting (cf. [12, p. 182]). When viewed in the almost Hermitian setting (rather than almost Kähler), the first collection of terms may not assure that (1.1) is parabolic modulo diffeomorphisms except in dimension 4. This is why we need to modify by the addition of an extra term in order to create a flow in arbitrary dimension with desired behaviour. Note the resultant flow of the symplectic form is given by

\[
\left(\frac{\partial \omega}{\partial t}\right)_{ab} = -P_{ab} + \left(P^{1,1} - 2J^{x}_{a} R_{bx}^{1,1}\right) - \left((\mathcal{L}_{\partial} J)^{m}_{a} g^{mb}_{skew} + \kappa_{ab} + 2\nabla^{e} (d\omega)^{3,0+0,3}_{eab}\right). 
\]

Note in particular that if we define

\[
\bar{\omega}_{t} \triangleq \omega_{t} - \int_{0}^{t} \mathcal{C}(s) \, ds, 
\]

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it follows that $\frac{\partial \tilde{\omega}}{\partial t} = - P$, and thus $\tilde{\omega}$ stays closed along the flow. Here is our main result.

**Theorem A** For $(M^n, g, J)$, the almost Hermitian Ricci flow is a parabolic flow modulo diffeomorphisms which preserves the almost Hermitian condition.

### 2 Background on Almost Hermitian Manifolds

In this section we describe baseline concepts for almost Hermitian manifolds. In the first subsection we discuss key objects, while in the following subsection we use these objects to construct computational quantities and analyse their behaviours.

#### 2.1 Basics

We begin with a Riemannian mani $(M^n, g)$ equipped with an almost complex structure $J$ (that is, $J \in \text{End}(TM)$ such that $J^2 = - \text{Id}$). An almost Hermitian manifold $(M^n, g, J)$ satisfies the additional condition that $J$ is $g$-compatible, namely that

$$g(X, Y) = g(JX, JY)$$

(2.1)

From $g$ and $J$ arise a natural symplectic form $\omega$ given by

$$\omega(X, Y) = g(JX, Y)$$

(2.2)

In coordinates, resultant identities are as follows:

$$g_{ij} = J_a^i \omega_{js} \quad g^{ij} = J^i_s \omega^{js} \quad \omega_{ij} = J^i_s g^{js} = \omega_{ks} g^{js} \quad \omega^{ac} \omega_{cb} = \delta^a_b.$$  

To clarify, we denote by $\omega^{ij}$ those entries of the inverse $\omega^{-1}$ instead of the raising of $\omega$ by the metric $g$. We have the following decomposition of $TM_{\otimes 2}$,

$$A_{ij}^{1,1} = \frac{1}{2} \left( A_{ij} + A_{ab} J_i^a J_j^b \right),$$

$$A_{ij}^{2,0+0,2} = \frac{1}{2} \left( A_{ij} - A_{ab} J_i^a J_j^b \right).$$

Likewise for $TM_{\otimes 3}$, we have subsets of $TM_{\otimes 3}$ given by $TM_{\otimes 3}^{2,1+1,2}$ and $TM_{\otimes 3}^{3,0+0,3}$ with projections

$$B_{ijk}^{2,1+1,2} \equiv \frac{3}{4} B_{ijk} + \frac{1}{4} \left( J_j^b J_k^c B_{abc} + J_i^a J_k^c B_{ajc} + J_i^a J_j^b B_{abk} \right),$$

$$B_{ijk}^{3,0+0,3} \equiv \frac{1}{4} \left( B_{ijk} - J_i^a J_j^b B_{abk} - J_j^b J_k^c B_{abc} - J_i^a J_k^c B_{ajc} \right).$$  

**Remark 2.1** Note as stated in ([5, p. 263]) , in dim $M = 4$ we have $\Lambda_3^{3,0+0,3}$ is trivial.
Henceforth, we adhere to conventions of Gauduchon and Kobayashi–Nomizu ([9, Chapter IX], [5, p. 259] above (1.1.3)) regarding the corresponding relational identities. We define the $d^c$ operator acting on $\eta \in \Lambda_m(M)$ as

$$(d^c \eta)_{a_1 \cdots a_{(m+1)}} \triangleq -J_{a_1}^{b_1} \cdots J_{a_{(m+1)}}^{b_{(m+1)}} (d\eta)_{b_1 \cdots b_{(m+1)}}.$$  

The Nijenhuis tensor is given by the following,

$$N^i_{jk} \triangleq 2 \left( J^p_j \left( \partial_p J^i_k \right) - J^p_k \left( \partial_p J^i_j \right) - J^1_p \left( \partial_j J^i_k \right) + J^1_p \left( \partial_k J^i_j \right) \right).$$ (2.4)

The Nijenhuis tensor nearly satisfies a Bianchi identity up to a correction ((2.2.4) of [5]),

$$N_{ijk} + N_{jki} + N_{kij} = 8 (d^c \omega)_{ijk}^{3,0+0,3}.$$ (2.5)

Our Nijenhuis tensor conventions match those of [9], which differs with [5] by a factor of 8.

**Definition 2.2** The Chern connection is the unique connection $\nabla = \partial + \Upsilon$ such that

$$\nabla g \equiv 0, \quad \nabla J \equiv 0, \quad \tau^{1,1} \equiv 0,$$ (2.6)

where $\tau \in \Lambda_2 \otimes TM$ denotes the torsion of $\nabla$.

**Remark 2.3** The Levi-Civita connection will be denoted by $D = \partial + \Gamma$.

We begin by analysing key quantities of almost Hermitian manifolds with respect to the conventions of [5]. Set $\psi \equiv d\omega$.

**Definition 2.4** The Lee form $\vartheta$ is given by

$$\vartheta_k \triangleq \frac{1}{2} \omega^{ij} \psi_{ijk}.$$  

### 2.2 Computational Relations

Given the objects established in the prior section, we now formulate key computational quantities and related identities which will be essential.
Lemma 2.5 For \((M^n, \omega, J)\) almost Hermitian, the negative contorsion tensor of the Chern connection is

\[
\Theta_{ijk} \triangleq (D - \nabla)_{ijk} = \frac{1}{8} N_{jki} + \frac{1}{2} J_i^p \Psi_{pjk},
\]

where

\[
\Psi_{ijk} \triangleq \frac{1}{2} \left( \psi_{ijk} + J_i^q J_k^r \psi_{iqr} + J_j^q J_k^r \psi_{ajr} + J_j^p J_k^r \psi_{pjk} \right) \in \Lambda_3.
\]

(2.7)

Proof Starting from formula (2.5.3) of [5] which presents a family of Hermitian connections parametrized by the variable \(t\), taking \(t = 1\) for the Chern connection we obtain

\[
(\nabla - D)_{ijk} = -\frac{1}{8} N_{jki} + \frac{1}{2} \left( (d^c \omega)^{3,0+0,3}_{ijk} + (d^c \omega)^{2,1+1,2}_{abc} J_j^b J_k^c \right).
\]

(2.8)

Thus rearranging yields

\[
\Theta_{ijk} = \frac{1}{8} N_{jki} - \frac{1}{2} \left( (d^c \omega)^{3,0+0,3}_{ijk} + (d^c \omega)^{2,1+1,2}_{abc} J_j^b J_k^c \right).
\]

We expand out the latter terms on the right hand side. First,

\[
(d^c \omega)^{2,1+1,2}_{ijk} = \frac{3}{4} \left( (d^c \omega)_{ijk} + \frac{1}{4} \left( J_j^b J_k^c (d^c \omega)_{abc} + J_j^q J_k^r (d^c \omega)_{ajc} + J_j^a J_b^i (d^c \omega)_{abk} \right) \right)
\]

\[
= -\frac{3}{4} J_i^p J_j^q J_k^r \psi_{pqr} + \frac{1}{4} \left( J_j^p \psi_{pjk} + J_j^q \psi_{iqk} + J_j^r \psi_{ijr} \right).
\]

(2.9)

Next we compute

\[
(d^c \omega)^{3,0+0,3}_{ijk} = \frac{1}{4} \left( (d^c \omega)_{ijk} - J_i^a J_j^b (d^c \omega)_{abk} - J_j^b J_k^c (d^c \omega)_{abc} - J_k^a J_j^c (d^c \omega)_{ajc} \right)
\]

\[
= \frac{1}{4} \left( -J_i^p J_j^q J_k^r \psi_{pqr} + J_j^q \psi_{ijr} + J_i^p \psi_{pjk} + J_j^q \psi_{ijr} \right) + \frac{1}{4} \left( J_i^m J_j^n J_k^p \psi_{pmn} + J_j^m J_i^n \psi_{ijm} + J_i^m J_j^n \psi_{imk} \right).
\]

(2.10)

Modifying (2.9) with action by the almost complex structure yields:

\[
J_i^m J_j^n \left( d^c \omega \right)_{i_mn}^{2,1+1,2}
\]

\[
= -\frac{3}{4} J_i^m J_j^n J_k^p \psi_{pqr} - \frac{1}{4} J_j^m J_k^n \left( J_i^p \psi_{pmn} + J_i^n \psi_{imn} + J_j^m \psi_{imn} \right)
\]

\[
= -\frac{3}{4} J_i^p \psi_{pjk} - \frac{1}{4} \left( J_j^m J_k^n J_i^p \psi_{pmn} + J_j^p \psi_{ijm} \right).
\]

(2.11)

Combining (2.11) and (2.10) yields

\[
J_i^m J_j^n \left( d^c \omega \right)_{i_mn}^{2,1+1,2} + \left( d^c \omega \right)_{ij}^{3,0+0,3}
\]
\[ \begin{align*}
&= -\frac{1}{4} J_i^p J_j^q J_k^r \psi_{pqr} + \frac{1}{4} J_k^r \psi_{ijr} + \frac{1}{4} J_i^p \psi_{pjk} + \frac{1}{4} J_j^q \psi_{iqk} \\
&\quad - \frac{3}{4} J_i^p \psi_{pjk} - \frac{1}{4} J_m^m J_k^m J_i^p \psi_{pmn} + \frac{1}{4} J_k^m \psi_{ijn} + \frac{1}{4} J_j^m \psi_{imk} \\
&= -\frac{1}{2} J_i^p \left( J_j^q J_k^r \psi_{pqr} + J_p^y J_k^r \psi_{sjr} + J_p^y J_j^q \psi_{yqk} + \psi_{pjk} \right).
\end{align*} \]

Inserting this into (2.8) yields the result. \(\square\)

**Corollary 2.6** Updating (2.5) with (2.10) yields that

\[ N_{ijk} + N_{jki} + N_{kij} = 2 \left( J_k^r \psi_{ijr} + J_i^p \psi_{pjk} + J_j^q \psi_{iqk} - J_i^p J_j^q J_k^r \psi_{pqr} \right). \]

With the tensors \(\Theta\) and \(\Psi\) classified above, we record some basic properties of each which will be crucial to upcoming computations. Note that \(\Psi\) was identified as a natural generalisation of \(\psi\); the below properties for \(\Psi\) hold for \(\psi\), and for \(\dim M = 4\), we have \(\Psi \equiv \psi\).

**Lemma 2.7** For \((M^n, \omega, J)\) almost Hermitian,

1. \(\Theta_{ijk}\) is skew in \((jk)\).
2. \(\omega^{ij} \Psi_{ijk} = -2\partial_k\) and \(g^{ij} \Psi_{ijk} = 0\).
3. \(\omega^{ij} \Theta_{ijk} = 0\), \(\omega^{ik} \Theta_{ijk} = 0\), and \(\omega^{jk} \Theta_{ijk} = -J_i^p \partial_p\).
4. \(g^{ij} \Theta_{ijk} = -\partial_k\), \(g^{ik} \Theta_{ijk} = \partial_j\), and \(g^{jk} \Theta_{ijk} = 0\).

**Proof** Identity (1) is apparent by the definition of \(\Theta\). For (2), we compute the corresponding contractions of \(\Psi\), beginning with contractions by \(\omega\),

\[ \begin{align*}
\omega^{ij} \Psi_{ijk} &= \frac{1}{2} \left( J_q^i \omega^{ij} J_k^r \psi_{iqr} + J_i^p \omega^{ij} J_k^r \psi_{pjr} + J_j^q \omega^{ij} J_k^r \psi_{pkq} + \omega^{ij} \psi_{ijk} \right) \\
&= \frac{1}{2} \left( g^{qj} J_k^r \psi_{iqr} - g^{pj} J_k^r \psi_{pjr} + \omega^{pq} \psi_{pqk} + \omega^{ij} \psi_{ijk} \right) \\
&= -2\partial_k.
\end{align*} \]

The \(g\) contraction follows easily from the expression of \(\Psi\) in (2.7). For (3) we investigate \(\omega\) contractions of \(\Theta\). First,

\[ \begin{align*}
\omega^{ij} \Theta_{ijk} &= 0 + \frac{1}{2} J_i^p \omega^{ij} \Psi_{pjk} \\
&= 0.
\end{align*} \]

The next identity, \(\omega^{ik} \Theta_{ijk} = 0\) follows by (1). Lastly we have

\[ \begin{align*}
\omega^{ik} \Theta_{ijk} &= 0 + \frac{1}{2} J_i^p \omega^{ik} \Psi_{pjk} \\
&= -J_i^p \partial_p.
\end{align*} \]
Finally, we have that for (4),
\[ g^{ij} \Theta_{ijk} = 0 + \frac{1}{2} J^p_i g^{ij} \Psi_{pjk} = -\vartheta_k. \]

The last two identities follow by (1) again. We conclude the result. \[\square\]

\section*{2.3 Chern Curvature Identities}

Here we record some main identities concerning the Chern connection’s associated curvature $\Omega$, torsion $\tau$ and contorsion tensor $-\Theta$.

\begin{lemma}
Let $(M^n, g, J)$ be almost Hermitian. Then
\[ \Omega_{ijkl} = -\Omega_{jikl} = -\Omega_{ijlk} = \Omega_{iab} J^a_k J^b_l. \]
\end{lemma}

Aspects of the Riemann curvature tensor translate with some residual torsion terms. Define Chern–Ricci curvature $(P)$ and a computational intermediary $(V)$ by
\[ P_{ab} \triangleq \omega^{cd} \Omega_{abcd}, \quad V_{ab} \triangleq \Omega^r_{r ab}, \quad (2.12) \]
and finally, the Chern scalar curvature is
\[ \varrho \triangleq \omega^{ba} P_{ab}. \]

\begin{proposition}
For $(M^n, \omega, J)$ almost Hermitian,
\[ Rm_{ijkl} = \Omega_{ijkl} + \nabla_i \Theta_{jkl} - \nabla_j \Theta_{ikl} + \Theta_{isl} \Theta^s_{jk} - \Theta_{jsl} \Theta^s_{ik} + \tau_{ij} \Theta_{skl}. \quad (2.13) \]
\end{proposition}

\begin{proof}
This follows by expanding out the formulas for $Rm$ and $\Omega$ in terms of connection coefficients. \[\square\]

\begin{corollary}
For $(M^n, \omega, J)$ almost Hermitian,
\[ Rc_{jk} = V_{jk} + g^{il} \nabla_i \Theta_{jkl} - \nabla_j \vartheta_k + \vartheta_s \Theta^s_{jk} - \Theta^s_{ij} \Theta^i_{sk}. \]
\end{corollary}

\begin{proof}
Take the trace of Proposition 2.9 and simplify using Lemma 2.7. \[\square\]

\begin{lemma}
For $(M^n, J, \omega)$ almost Hermitian,
\[ \Omega_{ijkl} + \Omega_{jikl} + \Omega_{kijl} = (\nabla_i \tau_{jkl} + \nabla_j \tau_{ikl} + \nabla_k \tau_{ijl}) + \left( \tau^s_{jk} \tau_{s il} + \tau^s_{ki} \tau_{sjl} + \tau^s_{ij} \tau_{s kl} \right). \]
\end{lemma}

\begin{proof}
Expand the Bianchi identity for Riemannian curvature using Proposition 2.9. \[\square\]
Lemma 2.12 For \((\mathcal{M}^n, J, \omega)\) almost Hermitian we have that

\[
\Omega_{klij} - \Omega_{ijkl} \triangleq T_{klij} = \nabla_i \Theta_{jkl} - \nabla_j \Theta_{ikl} - \nabla_k \Theta_{lij} + \nabla_l \Theta_{kij} + \Theta_{ils} \Theta_{sij} - \Theta_{ksj} \Theta_{si} + \Theta_{lsj} \Theta_{sk} + \tau_{ij} \Theta_{s}^{skl} - \tau_{kl} \Theta_{s}^{si}.
\]

Where it follows that

\[
T_{ijkl} = -T_{klij}, \quad T_{ijkl} = -T_{jikl} = -T_{ijlk}.
\] (2.14)

Proof This follows by computing \(R_{mijkl} - R_{mkljj}\) using (2.13). \(\square\)

3 Relation of Ricci Tensors

To show that (1.1) preserves the almost Hermitian structure, we need to obtain the explicit form of the difference between the Ricci and Chern Ricci tensors.

Lemma 3.1 For \((\mathcal{M}^n, J, \omega)\) almost Hermitian,

\[
V_{jk} = \frac{1}{2} J_{k}^{a} P_{ja} + J_{k}^{a} \omega^{br} \nabla_{b} \Theta_{jar} + \frac{1}{2} \nabla_{j} \partial_{k} + \frac{1}{2} J_{k}^{a} J_{i}^{u} \nabla_{a} \partial_{u}
\]

\[
+ J_{k}^{a} \omega^{br} \left( \Theta_{ar}^{s} - \Theta_{ra}^{s} \right) \Theta_{jbs} + J_{k}^{a} \omega^{br} \left( \Theta_{jb}^{s} - \Theta_{bj}^{s} \right) \Theta_{s}^{ar}
\]

\[
+ \frac{1}{2} J_{k}^{a} J_{p}^{s} \partial_{p} \left( \Theta_{ja}^{s} - \Theta_{aj}^{s} \right).
\]

Proof We model our argument on Lemma 2.16 of [8]. Using Lemma 2.11 and manipulating (with an important step highlighted),

\[
V_{jk} = g_{re} \Omega_{rjke}
\]

\[
= J_{k}^{a} J_{e}^{b} g^{re} \Omega_{rjab}
\]

\[
= J_{k}^{a} \omega^{br} \Omega_{rjab}
\]

\[
= -J_{k}^{a} \omega^{br} \left( \Omega_{jarb} + \Omega_{arjb} \right) + J_{k}^{a} \omega^{br} \left( \nabla_{j} \tau_{ar} + \nabla_{a} \tau_{rjb} + \nabla_{r} \tau_{jab} \right)
\]

\[
- J_{k}^{a} \omega^{br} g_{bs} \left( \tau_{re}^{s} \tau_{ja}^{e} + \tau_{je}^{s} \tau_{ar}^{e} + \tau_{ae}^{s} \tau_{rj}^{e} \right).
\]

We leave the latter lines alone and manipulate the first,

\[
-J_{k}^{a} \omega^{br} \left( \Omega_{jarb} + \Omega_{arjb} \right) = J_{k}^{a} P_{ja} - J_{k}^{a} \omega^{br} \Omega_{arjb}
\]

\[
= J_{k}^{a} P_{ja} - J_{k}^{a} \omega^{br} T_{arjb} - \Omega_{jbar}
\]

\[
= J_{k}^{a} P_{ja} - J_{k}^{a} \omega^{br} T_{arjb} - \Omega_{jbar}.
\]
Given the reappearance of the indicated term, we rearrange and obtain

\[ V_{jk} = J_k^a \omega^{br} \Omega_{rjab} \]

\[ = \frac{1}{2} J_k^a P_{ja} - \frac{1}{2} J_k^a \omega^{br} T_{arjb} + \frac{1}{2} J_k^a \omega^{br} (\nabla_j \tau_{arb} + \nabla_a \tau_{rjb} + \nabla_r \tau_{jab}) \]

\[ - \frac{1}{2} J_k^a \omega^{br} g_{bs} \left( \tau_{re} \tau_{ja} + \tau_{je} \tau_{ar} + \tau_{ae} \tau_{rj} \right). \]

Now we convert all \( \tau \) terms to \( \Theta \), noting that \( \tau_{ijk} \equiv \Theta_{ijk} - \Theta_{ijk} \),

\[ V_{jk} = \frac{1}{2} J_k^a P_{ja} - \frac{1}{2} J_k^a \omega^{br} T_{arjb} \]

\[ + \frac{1}{2} J_k^a \omega^{br} (\nabla_j \Theta_{arb} + \nabla_a \Theta_{rjb} + \nabla_r (\Theta_{arb} - \Theta_{rjb})) \]

\[ - \frac{1}{2} J_k^a \omega^{br} \left( (\Theta_{erb} - \Theta_{reb}) (\Theta^c_{ja} - \Theta^c_{ja}) + (\Theta_{ejb} - \Theta_{jeb}) (\Theta^c_{ra} - \Theta^c_{ar}) \right) \]

\[ - \frac{1}{2} J_k^a \omega^{br} \left( (\Theta_{eab} - \Theta_{aeb}) (\Theta^c_j - \Theta^c_{rj}) \right). \]

We now simplify each labelled row beginning with Row 1. Using Lemmata 2.7 and 2.12,

\[ \omega^{br} T_{arjb} = -\omega^{br} T_{arjb} \]

\[ = -\omega^{br} \nabla_b \Theta_{jar} + \omega^{br} \nabla_j \Theta_{bar} + \omega^{br} \nabla_a \Theta_{rjb} - \omega^{br} \nabla_r \Theta_{abj} \]

\[ - \omega^{br} \Theta_{bsr} \Theta^s_{ja} + \omega^{br} \Theta_{asj} \Theta^s_{rb} + \omega^{br} \Theta_{jsr} \Theta^s_{ba} - \omega^{br} \Theta_{rsj} \Theta^s_{ab} \]

\[ - \omega^{br} \tau_{bj} \Theta_{sar} + \omega^{br} \tau_{s \phi} \Theta^s_{sbj} \]

\[ = -\omega^{br} \nabla_b \Theta_{jar} + \omega^{br} \nabla_b \Theta_{arj} - \omega^{br} \Theta_{jsb} \Theta^s_{ra} + \omega^{br} \Theta_{bsj} \Theta^s_{ar} \]

\[ - \omega^{br} \Theta^s_{jb} \Theta_{sar} + \omega^{br} \Theta^s_{sjb} \Theta_{sar} + \omega^{br} \Theta^s_{sbj} \Theta_{sar} - \omega^{br} \Theta^s_{sbj} \Theta_{sar}. \]

Therefore it follows that

\[ (R1)_{jk} - \frac{1}{2} J_k^a P_{ja} = -\frac{1}{2} J_k^a \omega^{br} T_{arjb} \]

\[ = \frac{1}{2} J_k^a \omega^{br} \nabla_b \Theta_{jar} - \frac{1}{2} J_k^a \omega^{br} \nabla_j \Theta_{bar} + \frac{1}{2} J_k^a \omega^{br} \Theta_{jsb} \Theta^s_{ra} \]

\[ - \frac{1}{2} J_k^a \omega^{br} \Theta_{bsj} \Theta^s_{ar} + \frac{1}{2} J_k^a \omega^{br} \Theta^s_{sbj} \Theta_{sar} \]

\[ = \frac{1}{2} J_k^a \omega^{br} \nabla_b \Theta_{jar} - \frac{1}{2} J_k^a \omega^{br} \nabla_b \Theta_{arj} + \frac{1}{2} J_k^a \omega^{br} \Theta_{jsb} \Theta^s_{ra} \]

\[ - \frac{1}{2} J_k^a \omega^{br} \Theta_{bsj} \Theta^s_{ar} + \frac{1}{2} J_k^a \omega^{br} \left( \Theta^s_{jb} - \Theta^s_{bj} \right) \Theta_{sar} \]
We now address Row 2. For this, applying Lemma 2.7.

\[(R2)_{jk} = \frac{1}{2} J^a_k \omega^{br} \left( -\nabla_j \Theta_{arb} + \nabla_a \Theta_{jrb} + \nabla_r (\Theta_{ajb} - \Theta_{jab}) \right) \]
\[= \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J^a_k J^b_j \nabla_a \vartheta_u - \frac{1}{2} J^a_k \omega^{br} \nabla_b \Theta_{ajr} + \frac{1}{2} J^a_k \omega^{br} \nabla_b \Theta_{jar}.\]

Lastly, for Row 3, we expand out and search for simplifications

\[(R3)_{jk} = -\frac{1}{2} J^a_k \omega^{br} \left( (\Theta_{srb} - \Theta_{rsb}) \left( \Theta^s_{aj} - \Theta^s_{ja} \right) + (\Theta_{sjb} - \Theta_{sjb}) \left( \Theta^s_{ra} - \Theta^s_{ar} \right) \right) \]
\[+ \frac{1}{2} J^a_k \omega^{br} \left( (\Theta_{sar} - \Theta_{asr}) \left( \Theta^s_{jb} - \Theta^s_{bja} \right) \right) \]
\[= -\frac{1}{2} J^a_k J^b_s \nabla_p \left( \Theta^s_{aj} - \Theta^s_{ja} \right) - \frac{1}{2} J^a_k \omega^{br} \Theta_{srb} \left( \Theta^s_{ra} - \Theta^s_{ar} \right) \]
\[+ \frac{1}{2} J^a_k \omega^{br} \Theta_{srb} \left( \Theta^s_{ra} - \Theta^s_{ar} \right) + \frac{1}{2} J^a_k \omega^{br} \Theta_{sar} \left( \Theta^s_{rb} - \Theta^s_{bja} \right) \]
\[+ \frac{1}{2} J^a_k J^b_s \nabla_p \left( \Theta^s_{ja} - \Theta^s_{aj} \right).\]

Combining everything we have that

\[V_{jk} = \frac{1}{2} J^a_k P_{ja} + J^a_k \omega^{br} \nabla_b \Theta_{jar} + \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J^a_k J^b_j \nabla_a \vartheta_u \]
\[+ \frac{1}{2} J^a_k \omega^{br} (\Theta^s_{ar} - \Theta^s_{ra}) \Theta_{jbs} + \frac{1}{2} J^a_k \omega^{br} (\Theta^s_{bj} - \Theta^s_{bja}) \Theta_{sar} \]
\[+ \frac{1}{2} J^a_k J^b_s \nabla_p \left( \Theta^s_{ja} - \Theta^s_{aj} \right),\]
as desired. \hfill \Box

**Corollary 3.2** For \((M^n, \omega, J)\) almost Hermitian,

\[\text{Re}_{jk} = \frac{1}{2} J^a_k P_{ja} - \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J^a_k J^b_j \nabla_a \vartheta_u + g^{il} \nabla_l \Theta_{jkl} + J^a_k \omega^{br} \nabla_b \Theta_{jar} \]
\[+ J^a_k \omega^{br} (\Theta^s_{ar} - \Theta^s_{ra}) \Theta_{jbs} + J^a_k \omega^{br} (\Theta^s_{bj} - \Theta^s_{bja}) \Theta_{sar} \]
\[+ \frac{1}{2} J^a_k J^b_s \nabla_p \left( \Theta^s_{ja} - \Theta^s_{aj} \right) + \vartheta_s \Theta^s_{jk} - \Theta^s_{ij} \Theta^i_{sk}.\]

**Proof** Combining Lemma 3.1 with Corollary 2.10 yields the result. \hfill \Box

Provided the explicit comparison formula of the Ricci tensors for both connections, we now analyse the highest order terms in preparation for future symbol computations. We let \([\cdot]_2\) denotes the projection to second derivatives of the primitives \((g, J)\).
Proposition 3.3 For \((M^n, \omega, J)\) almost Hermitian,
\[
\left[ \text{Rc}_{jk} - \frac{1}{2} J_k^a P_{ja} \right]_2 = \left[ \frac{1}{4} \nabla^l N_{klij} - \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J_k^a J_j^b \nabla_a \vartheta_y - \frac{1}{2} \nabla^e \left( J_e^r \psi_{jkr} + J_k^a \psi_{jae} \right) \right]_2.
\] (3.1)

**Proof** Isolating the highest order terms of Corollary 3.2 yields
\[
\left[ \text{Rc}_{jk} \right]_2 = \left[ \frac{1}{2} J_k^a P_{ja} - \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J_k^a J_j^u \nabla_a \vartheta_u + g^{il} \nabla_i \Theta_{jkl} + J_k^a \omega^{br} \nabla_b \Theta_{jar} \right]_2.
\] (3.2)

We will expand out the last two terms involving \(\Theta\). Using Lemma 2.5 we have
\[
g^{il} \nabla_i \Theta_{jkl} = \frac{1}{8} \nabla^l N_{klij} + \frac{1}{2} J_j^a \nabla^l \Psi_{uakl}.
\]

Next we compute
\[
J_k^a \omega^{br} \nabla_b \Theta_{jar} = J_k^a \omega^{br} \nabla_b \left( \frac{1}{8} N_{arj} + \frac{1}{2} J_j^u \Psi_{uar} \right)
= J_k^a \omega^{br} \nabla_b \left( \frac{-1}{8} J_a^m J_j^n N_{mnj} + \frac{1}{2} J_j^u \Psi_{uar} \right)
= \frac{1}{8} \nabla^l N_{klij} + \frac{1}{2} J_k^a \omega^{br} J_j^u \nabla_b \Psi_{uar}.
\]

Thus, updating (3.2) we conclude that
\[
\left[ \text{Rc}_{jk} - \frac{1}{2} J_k^a P_{ja} \right]_2 = \left[ \frac{1}{4} \nabla^l N_{klij} - \frac{1}{2} \nabla_j \vartheta_k + \frac{1}{2} J_k^a J_j^r \nabla_a \vartheta_y + \frac{1}{2} J_j^u \left( \nabla^l \Psi_{uakl} - \frac{1}{2} J_k^a J_e^r \nabla^e \Psi_{uar} \right) \right]_2.
\]

We will address the last quantity. For this, first consider
\[
\nabla^e \Psi_{uakl} = \frac{1}{2} \nabla^e \left( J_k^a J_e^r \psi_{uakr} + J_u^a J_e^r \psi_{ykr} + J_a^b J_k^a \psi_{yae} + \psi_{uakl} \right).
\]

Next we compute out,
\[
J_k^a J_e^r \nabla^e \Psi_{uakl}
= \frac{1}{2} \nabla^e \left( J_k^a J_e^r J_a^b \psi_{uqlh} + J_k^a J_e^r J_j^b \psi_{yqh} + J_a^b J_k^a J_e^r \psi_{yqh} + J_k^a J_e^r \psi_{uqlh} \right)
= \frac{1}{2} \nabla^e \left( \psi_{uakl} - J_k^a J_u^a \psi_{yae} - J_e^r J_u^a \psi_{ykr} + J_k^a J_e^r \psi_{uakr} \right).
\]
Combining we obtain that

$$\nabla^e \Psi_{uke} - J^c_k J^r_e \nabla^e \Psi_{uar} = \nabla^e \left( J^y_u J^r_e \Psi_{ykr} + J^y_u J^a_k \Psi_{yae} \right).$$

Combining yields the result. \qed

### 4 Construction of the Flow

We aim to derive the structure of a flow for almost complex structures which matches with Ricci flow on the metrics (as displayed in (1.1)). To do so, we begin investigating the operator introduced in the work of the symplectic curvature flow [12, p. 182] but rather in the almost Hermitian setting and modify accordingly. Let us write the flow we will construct as follows:

\[
\begin{aligned}
&\left\{ \frac{\partial g}{\partial t} \right\}_{bc} = -2 \text{Rc}_{bc} \\
&\left\{ \frac{\partial J}{\partial t} \right\}_a = - \left( P_{av} - 2 J^e_a \text{Rc}_{ev} \right) g^{vc} + \partial\text{a}v g^{vc},
\end{aligned}
\]  

(4.1)

where $\partial \in TM \otimes 2$ will be determined. The first term in the flow on $J$ corresponds to the operator

$$J^c_a \mapsto J^k_a \left( 2 \text{Rc}_{jk} - J^a_k P_{ja} \right) g^{je}. \quad (4.2)$$

In order to preserve the properties of $J_t$ as an almost complex structure, the right side of the flow on $J$ in (4.1) must be $J$-antiinvariant after we lower the superscript to subscript by the metric $g$. On other hand, if we require (4.1) to preserve the Hermitian structure, we have

\[
0 = \frac{\partial}{\partial t} \left[ g_{ab} - g_{pq} J^p_a J^q_b \right]
= -4 \left( \text{Rc}_{ab} \right)^{2,0+0,2} - g_{pq} g^{vp} \left( - P_{av}^{2,0+0,2} + 2 J^e_a \text{Rc}_{ev}^{2,0+0,2} + \partial\text{a}v \right) J^q_b
- g_{pq} g^{qv} J^p_a \left( - P_{bv}^{2,0+0,2} + 2 J^e_b \text{Rc}_{ev}^{2,0+0,2} + \partial\text{b}v \right) J^q_b
= -4 \text{Rc}_{ab}^{2,0+0,2} - g_{pq} g^{vp} \left( - P_{av}^{2,0+0,2} + 2 J^e_a \text{Rc}_{ev}^{2,0+0,2} + \partial\text{a}v \right) J^q_b
- g_{pq} g^{qv} J^p_a \left( - P_{bv}^{2,0+0,2} + 2 J^e_b \text{Rc}_{ev}^{2,0+0,2} + \partial\text{b}v \right)
= -4 \text{Rc}_{ab}^{2,0+0,2} \left( - P_{aq}^{2,0+0,2} + 2 J^e_a \text{Rc}_{eq}^{2,0+0,2} + \partial\text{a}q \right) J^q_b
- J^a_p \left( - P_{bp}^{2,0+0,2} + 2 J^e_b \text{Rc}_{ep}^{2,0+0,2} + \partial\text{b}p \right)
= - J^q_b \left( \partial\text{a}q + \partial\text{qa} \right).
\]

Therefore

$$\left( \partial\text{a}v \right)_{\text{sym}} = 0.$$
This means that $\mathcal{U}$ must be skew-symmetric, but we have freedom in choosing $\mathcal{U}$ so long it is skew-symmetric and $J$-antiinvariant. As we tailor $\mathcal{U}$ to produce the proper (pseudo)parabolic flow, we need to compute the symbol of the operator in (4.2). Using Proposition 3.3 above we reexpress this, up to highest order, as

$$
\begin{split}
2J^k_a\left(R_{jk} - \frac{1}{2} J^e_k P_{ju} \right)^{2,0+0.2} g^{ic} \\
= \left[ -\frac{1}{2} J^k_a \nabla N^c_{lk} + \frac{1}{2} (\mathcal{A}(J))_a^c + \frac{1}{2} (\mathcal{B}(J))_a^c \right]^{2,0+0.2},
\end{split}
$$

(4.3)

where we have set

$$
\begin{align}
\frac{1}{2} (\mathcal{A}(J))_a^c & \triangleq 2\omega^{cy} \left( \nabla_y \vartheta_a \right)^{2,0+0.2}, \\
\frac{1}{2} (\mathcal{B}(J))_a^c & \triangleq g^{ic} \left( \nabla^e \psi_{jae} - J^e_a J^f_c \nabla^e \psi_{jkr} \right)^{2,0+0.2}.
\end{align}
$$

(4.4)

Note that both $\mathcal{A}$ and $\mathcal{B}$ vanish in the almost Kähler setting. Since we know the behaviour of the first term via Proposition 5.4 of [12], we will investigate these latter terms. We begin with an identification of $\mathcal{A}$ which will be relevant to us later.

**Lemma 4.1** We have that

$$
\frac{1}{2} (\mathcal{A}(J))_a^c = \omega^{cy} \left( \mathcal{L}_{\vartheta} g \right)^{2,0+0.2}_{\gamma a} - \frac{1}{8} \omega^{cy} \vartheta^d \left( N_{dy} + N_{day} \right).
$$

**Proof** We first write the Lie derivative using the Levi-Civita connection and then convert to the Chern connection using Lemma 2.5.

$$
\begin{align}
(\mathcal{L}_{\vartheta} g)_{ij} &= (D_i \vartheta_j) + (D_j \vartheta_i) \\
&= (\nabla_i \vartheta_j) + (\nabla_j \vartheta_i) - \Theta_{ij} \vartheta^s - \Theta_{js} \vartheta^s \\
&= (\nabla_i \vartheta_j) + (\nabla_j \vartheta_i) - \left( \frac{1}{8} N_{dij} + \frac{1}{2} J^p_i \Psi_{pid} \right) \vartheta^d \\
&\quad - \left( \frac{1}{8} N_{idj} + \frac{1}{2} J^p_j \Psi_{pid} \right) \vartheta^d \\
&= \left( (\nabla_i \vartheta_j) + (\nabla_j \vartheta_i) \right) + \frac{1}{8} \vartheta^d \left( N_{dij} + N_{dij} \right) + \vartheta^d \left( J^p_i \Psi_{pid} + J^p_j \Psi_{pdi} \right).
\end{align}
$$

(4.5)

Then, projecting onto the $(2, 0) + (0, 2)$ component we have that

$$
\begin{align}
(\mathcal{L}_{\vartheta} g)^{2,0+0.2}_{ij} &= \left( (\nabla_i \vartheta_j) + (\nabla_j \vartheta_i) \right)^{2,0+0.2} + \frac{1}{8} \vartheta^d \left( N_{dij} + N_{dij} \right) + \vartheta^d \left( J^p_i \Psi_{pid} + J^p_j \Psi_{pdi} \right)^{2,0+0.2}.
\end{align}
$$
Now, for the final term we note that in fact
\[
2 \left( J_i^p \Psi_{pdj} + J_j^p \Psi_{pdi} \right)^{2,0+0,2} = J_i^p \Psi_{pdj} + J_j^p \Psi_{pdi} - J_i^m J_j^m \Psi_{pdm} - J_i^m J_j^m \Psi_{pdm} = J_i^p \Psi_{pdj} + J_j^p \Psi_{pdi} + J_j^p \Psi_{jdp} + J_i^p \Psi_{jdp} = J_i^p \Psi_{pdj} + J_j^p \Psi_{pdi} - J_j^p \Psi_{pdi} - J_i^p \Psi_{pdj} = 0.
\]
So updating (4.5) further yields
\[
(L_\vartheta g)^{2,0+0,2}_{ij} = 2 \left( (\nabla_i \vartheta_j) \right)_{sym}^{2,0+0,2} + \frac{1}{8} \vartheta^d \left( N_{dji} + N_{dij} \right). 
\]
In particular, it follows that
\[
\frac{1}{2} (\mathcal{A}(J))^c_a = 2 \omega^{cb} (\nabla_y \vartheta_a)^{2,0+0,2}_{sym} = \omega^{cb} (L_\vartheta g)^{2,0+0,2}_{ya} - \frac{1}{8} \omega^{cy} \vartheta^d \left( N_{dya} + N_{day} \right),
\]
which yields the result.

For the following computations we now focus on analysing the symbol of the operator \( \mathcal{A}(J) \). Let \([\cdot]_{2/g} \) denote the projection onto second order operators in \( J \), excluding second order operators on \( g \).

**Proposition 4.2** The symbol of the operator \( J \mapsto \mathcal{A}(J) \) is trivial.

**Proof** For this we expand out
\[
(\mathcal{A}(J))^c_a = 4 \omega^{cb} (\nabla_b \vartheta_a)^{2,0+0,2}_{sym} = 2 \omega^{cb} (\nabla_a \vartheta_b + \nabla_b \vartheta_a)^{2,0+0,2} = (\mathcal{A}_1(J))^c_a + (\mathcal{A}_2(J))^c_a. 
\]

For all terms present, we will need to understand derivatives of \( \vartheta \) up to highest order.

With this in mind, observe that
\[
\nabla_a \vartheta_b = \frac{1}{2} \omega^{ij} \nabla_a \Psi_{ijb} = \frac{1}{2} \omega^{ij} \nabla_a \left( \partial_i \omega_{jb} + \partial_j \omega_{bi} + \partial_b \omega_{ij} \right) = \omega^{ij} \nabla_a \left( \partial_i \omega_{jb} \right) + \frac{1}{2} \omega^{ij} \nabla_a \left( \partial_b \omega_{ij} \right).
\]
Then, we have that

\[
\left[\nabla_a \vartheta_b\right]_{2/g} = \left[-\omega^{ji} g_{ju} \nabla_a \left(\partial_i J^u_b + \frac{1}{2} \omega^{ji} \nabla_a \left(\partial_b \omega_{ij}\right)\right)\right]_{2/g}
\]

\[
= \left[J^u_a \nabla_a \left(\partial_i J^u_b\right) - \frac{1}{2} J^i_u \nabla_a \left(\partial_b J^u_i\right)\right]_{2/g}
\]

\[
= \left[-J^u_b \nabla_a \left(\partial_i J^u_i\right) - \frac{1}{2} J^i_u \nabla_a \left(\partial_b J^u_i\right)\right]_{2/g}.
\]

(4.7)

Now with reference to (4.6), for \(\mathcal{A}_1(J)\), we expand out

\[
\mathcal{A}_1(J)^c_a = 2\omega^{cb} (\nabla_a \vartheta_b)^{2,0+0,2} = \omega^{cb} (\nabla_a \vartheta_b - J^w_a J^v_b \nabla_w \vartheta_v)
\]

\[
= \underbrace{-\omega^{cb} (\nabla_a \vartheta_b)}_{\mathcal{A}_{11}(J)} + \underbrace{\omega^{cb} \left(J^w_a J^v_b \nabla_w \vartheta_v\right)}_{\mathcal{A}_{12}(J)}.
\]

We address each subterm. First we have that

\[
\mathcal{A}_{11}(J)_{2/g} = \left[-J^b_v g^{cv} (\nabla_a \vartheta_b)\right]_{2/g}
\]

\[
= \left[-J^b_v g^{cv} \left(-J^w_a \nabla_a \left(\partial_i J^w_b\right) - \frac{1}{2} J^i_u \nabla_a \left(\partial_b J^u_i\right)\right)\right]_{2/g}
\]

\[
= \left[-g^{cv} \nabla_a \left(\partial_i J^w_b\right) + \frac{1}{2} J^b_v g^{cv} J^i_u \nabla_a \left(\partial_b J^u_i\right)\right]_{2/g}.
\]

Likewise we compute

\[
\mathcal{A}_{12}(J)_{2/g} = \left[J^b_r g^{cr} \left(J^w_a J^v_b \nabla_w \vartheta_v\right)\right]_{2/g}
\]

\[
= \left[-J^w_a g^{cv} (\nabla_a \vartheta_v)\right]_{2/g}
\]

\[
= \left[-J^w_a g^{cv} \left(-J^v_w \nabla_w \left(\partial_i J^w_a\right) - \frac{1}{2} J^i_u \nabla_w \left(\partial_v J^u_i\right)\right)\right]_{2/g}
\]

\[
= \left[J^w_a J^v_a g^{cv} \nabla_w \left(\partial_i J^w_a\right) + \frac{1}{2} J^w_a g^{cv} J^i_u \nabla_w \left(\partial_v J^u_i\right)\right]_{2/g}.
\]

We now the linearisation of each and compute the off diagonal inner product with another \(J\)-variation, \(H\). Keep in mind that for all \(J\)-variations, we have that \(J^a_e H^b_e = -H^a_e J^b_e\).

\[
\left(\mathcal{L}^k_{\mathcal{A}_{11}}, H\right) = -g^{cv} \left(\xi^a \xi^i K^i_v\right) g^{ar} H^r_g c e
\]

\[
= -\left(\xi^r \xi^i K^i_v\right) H^r_v.
\]

\[
\left(\mathcal{L}^k_{\mathcal{A}_{12}}, H\right) = J^w_a J^v_a g^{cv} \left(\xi^u \xi^i K^i_u\right) g^{ar} H^r_g c e
\]

\[
\xi^r \xi^i K^i_v\right) H^r_v.
\]
\[ -H^u_v \left( \xi^v_{\xi^i} K_i^j \right). \]

Taking the sum we conclude that for \( \mathcal{A}_1 \),

\[ \left\{ \mathcal{L}_{\mathcal{A}_1}^k (K), H \right\} = -2H^u_v \left( \xi^v_{\xi^i} K_i^j \right). \]

Next we address term \( \mathcal{A}_2 \) of (4.6),

\[ \mathcal{A}_2(J) = 2\omega^{cb} (\nabla_b \partial_a)^{2.0+0.2} = \omega^{cb} \left( \nabla_b \partial_a - J^w_b J_a^u \nabla_w \partial_v \right) \]

\[ = \omega^{cb} \left( \nabla_b \partial_a \right)_{\mathcal{A}_21(J)} - \omega^{cb} \left( J^w_b J_a^u \nabla_w \partial_v \right)_{\mathcal{A}_22(J)}. \]

For the first term we compute

\[ [\mathcal{A}_21(J)]_{2/g} = \left[ \omega^{cb} \left( \nabla_b \partial_a \right) \right]_{2/g} \]

\[ = [-J^b_y g^{cy} \left( -J^u_a \nabla_b \left( \partial_i J_i^j - \frac{1}{2} J^i_u \nabla_b \left( \partial_a J_i^u \right) \right) \right)]_{2/g} \]

\[ = \left[ J^b_y g^{cy} J^u_a \left( \nabla_b \partial_i J_i^j \right) + \frac{1}{2} J^b_y g^{cy} J^u_i \nabla_b \left( \partial_a J_i^u \right) \right]_{2/g}. \]

Next we have that

\[ [\mathcal{A}_22(J)]_{2/g} = \left[ -\omega^{cb} \left( J^w_b J_a^u \nabla_w \partial_v \right) \right]_{2/g} \]

\[ = [-g^{uc} J^u_a \left( \nabla_w \partial_v \right)]_{2/g} \]

\[ = \left[ -g^{uc} J^u_a \left( -J^u_v \nabla_w \left( \partial_i J_i^j \right) - \frac{1}{2} J^i_u \nabla_w \partial_v J_i^u \right) \right]_{2/g} \]

\[ = \left[ -g^{uc} \left( \nabla_w \partial_i J_i^j \right) + \frac{1}{2} g^{uc} J^u_a J^i_w \nabla_v J_i^u \right]_{2/g}. \]

Therefore it follows that

\[ \left\{ \mathcal{L}_{\mathcal{A}_21}^k (K), H \right\} = J^b_y g^{cy} J^u_a \left( \xi^b_{\xi^i} K_i^j \right) g_{cm} g^{an} H^m_n \]

\[ = -H^b_y \left( \xi^b_{\xi^i} K_i^j \right) g^{yu}. \]

\[ \left\{ \mathcal{L}_{\mathcal{A}_22}^k (K), H \right\} = -g^{uc} \left( \xi^w_{\xi^i} K_i^j \right) g_{cm} g^{an} H^m_n \]

\[ = -\left( \xi^m_{\xi^i} K_i^j \right) g^{an} H^m_n. \]

Taking the sum of the two, we have

\[ \left\{ \mathcal{L}_{\mathcal{A}_2}^k (K), H \right\} = -2H^b_y \left( \xi^b_{\xi^i} K_i^j \right) g^{yu}. \]
Thus we conclude that, noting that \( Hg \) is skew-symmetric for variations where \( g \) is fixed,

\[
\left( \mathcal{L}_{\mathcal{A}}^k (K), H \right) = -2 H^b_y \left( \xi_b \xi^i K_u^i \right) g^{yu} - 2 H_u^a \left( \xi^u \xi^i K_a^i \right) = \mathcal{L}_{\mathcal{A}}^k (K, H) = 0.
\]

The result follows. \( \Box \)

Since \( \mathcal{A} \) is addressed, we next manipulate \( \mathcal{B} \) of (4.4) and observe that

\[
(\mathcal{B}(J))_a^c = 2 g^{jc} \left( \nabla^e \psi_{jae} - J^k_a J^r_e \nabla^e \psi_{jkr} \right)^{2,0+0,2} = g^{jc} \left( \psi_{jae} - J^k_a J^r_e \psi_{jkr} - J^m_j J^k_a \psi_{mke} + J^m_j J^k_a \psi_{mhr} \right) = 4 g^{jc} \left( \psi^3_{jae} \right).
\]

Since \( \Lambda^3_{3,0+0,3} \) is trivial in dimension 4, symbol of \( \mathcal{B} \) is as well. However, when \( \dim M > 4 \), \( \mathcal{B}(J) \) has nontrivial symbol and will destroy the parabolicity without the help of \( \mathcal{O} \). Observing that \( \nabla^e \psi^3_{jae} \) is skew-symmetric and \( J \)-antiinvariant, we will include this in our choice of \( \mathcal{O} \). Additionally, we will insert one final term whose presence will be justified in § 4.1.

\[
\mathcal{O}_{av} \triangleq \kappa_{av} - 2 \nabla^e \psi^3_{va e} - \left( (\mathcal{L}_\theta J)^m_g m v \right)_{skew}.
\]

equivalently, we have

\[
\frac{\partial J}{\partial t}^c_a = - (P_{av} - 2 J^y_a \mathcal{R}_y) g^{vc} + \left( \kappa_{av} - 2 \nabla^e \psi^3_{va e} \right) g^{vc} - \left( (\mathcal{L}_\theta J)^m_g m v \right)_{skew} g^{vc}.
\]

where \( \kappa \in TM_{\otimes 2} \) is a skew-symmetric and \( J \)-antiinvariant function of \( N \) and \( \psi \), in particular, it is of lower order in both \( g \) and \( J \). Summarizing the above, we see that the symbol of the above flow is given by that of

\[
- \frac{1}{2} J^k_a \nabla^l N_{lk} - \left( (\mathcal{L}_\theta J)^m_g m v \right)_{skew} g^{vc},
\]

which was computed in the almost Hermitian case in Proposition 5.4 of [12].
4.1 Proof of Main Theorem

We now look at the gauge modified version of the flow. First, letting $\nabla$ be some background torsion-free connection, set

$$Z^p \equiv (Z(\omega, J, \nabla))^p \triangleq \omega^{kl} \nabla_k J^p_l = g^{kl} \left( \Gamma^p_{kl} - \Gamma^p_{kl} \right) + \vartheta^p,$$  \hspace{1cm} (4.12)

where the final equality follows by Lemma 4.4 in the appendix, where we improve the statement of (5.13) of [12]. In particular, up to addition by the Lie form, this vector field is the same used in the short time existence proof for Ricci flow, which we will set as

$$X^p \equiv g^{kl} \left( \Gamma^p_{kl} - \Gamma^p_{kl} \right).$$

Note in particular that $Z = -X + \vartheta$. We now consider the following gauge modification of (1.1).

$$\left\{ \begin{aligned}
\left( \frac{\partial J}{\partial t} \right)^c_a & = - (P_{av} - 2J^l_a \nabla_y) ^{2.0+0.2} g^v c + (\kappa_{av} - 2\nabla^e \psi_v^e) ^{3.0+0.3} g^{ev} \\
- (L_{\partial} J)^m_a g^{mv} & \equiv (L_{\partial} J)^c_a \\
\end{aligned} \right\} \triangleq \mathcal{D}(g, J),$$  \hspace{1cm} (4.13)

Via (4.1) and (4.3), up to highest order,

$$\left[ (\mathcal{D} g)^{2.0+0.2} \right]_i^c = J^b_j (L_{\partial} J)^c_i \text{ sym}.$$  \hspace{1cm} (4.14)

Referring to Lemma 3.3 of [4], translated in coordinates we have that

$$\left( \mathcal{L}_\partial g \right)^{2.0+0.2}_{ij} = J^b_j (L_{\partial} J)^c_i g_{ab} \text{ sym}.$$
Note that since \((\mathcal{L}_\vartheta J) g\) is of type \((2, 0) + (0, 2)\), this means that the right side term is indeed of the same type (when lowered by the metric). It is also worth noting that \(((\mathcal{L}_\vartheta J) g)_{\text{skew}}\) corresponds with \((\mathcal{L}_\vartheta \omega)^{2,0+0,2}\). This justifies the choice of \(\tilde{\Omega}\) in (4.9).

\[
\left( [\mathcal{D}_1(g, J)]_a^c \right)_2 = \left[ -\frac{1}{2} f_a^k \nabla^l N_{ik}^c - (\mathcal{L}_{Z(g, J)} J)_a^c + (\mathcal{A}(J))_a^c \right]_2. \tag{4.15}
\]

By Proposition 4.2 and (4.11), combined with Proposition 5.4 of [12], it follows that

\[
\sigma \left[ \mathcal{L}_J \mathcal{D}_1 \right] (K)_i^j = |\xi|^2 K_i^j.
\]

Now we consider the next operator, \(\mathcal{D}_2\).

\[
(\mathcal{D}_2(g, J))_{bc} = -2R_{bc} + (\mathcal{L}_X g)_{bc}.
\]

Note that the Ricci tensor is completely independent of \(J\) and there is only lower order dependence in the remnant Lie derivative \(\mathcal{L}_X g\), and obviously has the correct symbol. In particular

\[
\sigma \left[ \mathcal{L}_J \mathcal{D}_2 \right] (K)_i^j = 0, \quad \sigma \left[ \mathcal{L}_g \mathcal{D}_2 \right] (h)_{ij} = |\xi|^2 h_{ij}.
\]

In particular, then we have that

\[
\sigma \left[ \mathcal{L} \mathcal{D} \right] (K, h) = \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \begin{pmatrix} K \\ h \end{pmatrix}.
\]

It follows that (4.13) is a strictly parabolic system of equations. Hence, the short time existence follows from the standard theory for parabolic flows. We can also follow arguments in the proof of Theorem 1.1 in [12] to complete the proof.

**Remark 4.3** We observe that by freezing the metric this flow is reduced to one on purely almost complex structures, of the form

\[
\left( \frac{\partial J}{\partial t} \right)_a^c = -p_{av}^{20+02} g^{vc} - (\mathcal{L}_\vartheta J)_a^m g_{mv} \right)_{\text{skew}} g^{vc} + \left( \kappa_{av} + 2\nabla^e (d\omega)_e^{30+03} \right) g^{vc}.
\]

This in fact is a *parabolic* flow on almost complex structures, which we will investigate in future work.

**Appendix**

**Lemma 4.4** For \((M^n, J, \omega)\) almost Hermitian, the vector field defined in (4.12) is

\[
Z^p = g^{uk} \left( \overline{\Gamma}_k^p - \Gamma_k^p \right) + \vartheta^p.
\]
Proof Starting with (4.12) we compute
\[
Z^p = \omega^{kl} \left( \nabla_k J_l^p \right)
= \omega^{kl} \left( \partial_k J_l^p \right) - \omega^{kl} \Gamma_{km}^m J_m^p + \omega^{kl} \Gamma_{km}^p J_m^m
= \omega^{kl} \left( \nabla_k J_l^p + \gamma_{kl}^u J_u^p - \gamma_{kl}^u J_u^m \Gamma_{km}^p \omega^{kl} + \Gamma_{km}^p g^{mk} \right).
\]
(4.16)

Let’s investigate these terms. For the first, by Lemma 2.7,
\[
\omega^{kl} \gamma_{kl}^u J_u^p = \omega^{kl} \left( \Gamma_{kl}^u - \Theta_{kl}^u \right) J_u^p
= 0.
\]

For the next term
\[
\gamma_{kl}^u g^{uk} = \left( \Gamma_{ku}^p - \Theta_{ku}^p \right) g^{uk}
= g^{uk} \Gamma_{ku}^p + \vartheta^p.
\]

Inserting these into (4.16) yields the result. \(\Box\)

We now justify (2.3) above regarding the type decompositions of \(TM \otimes \mathbb{C}^3\) by determining the corresponding projections into \(TM^{1,2+2,1}_{\mathbb{C}}\) and \(TM^{3,0+0,3}_{\mathbb{C}}\).

Lemma 4.5 For \((M^n, \omega, J)\) almost Hermitian,
\[
F_{i,j,k}^{3,0+0,3} = \frac{1}{4} \left( F_{ijk} - J_i J^b_{j} F_{ibc} - J_i J^c_{k} F_{ajc} - J_i J^a_{j} F_{abk} \right),
\]
\[
F_{i,j,k}^{2,1+1,2} = \frac{3}{4} F_{ijk} + \frac{1}{4} \left( J_i J^b_{j} F_{ibc} + J_i J^c_{k} F_{ajc} + J_i J^a_{j} F_{abk} \right).
\]

Proof For a vector field \(X \in TM\), define the projections
\[
\Pi_{1,0} X \triangleq \frac{1}{2} \left( X + i J X \right), \quad \Pi_{0,1} X \triangleq \frac{1}{2} \left( X - i J X \right).
\]
With this we have that
\[
\left( \Pi_{3,0} F \right) (X, Y, Z)
= F \left( \Pi_{1,0} X, \Pi_{1,0} Y, \Pi_{1,0} Z \right)
= \frac{1}{8} F (X, Y, Z) + i \frac{1}{8} F (X, Y, J Z) + i \frac{1}{8} F (X, J Y, Z) + (i)^2 \frac{1}{8} F (X, J Y, J Z)
+ i \frac{1}{8} F (J X, Y, Z) + (i)^2 \frac{1}{8} F (J X, Y, J Z)
+ (i)^3 \frac{1}{8} F (J X, J Y, Z) + (i)^3 \frac{1}{8} F (J X, J Y, J Z).
\]
\[ \begin{align*}
\frac{1}{8} F (X, Y, Z) + i \frac{1}{8} F (X, Y, JZ) + i \frac{1}{8} F (X, JY, Z) - \frac{1}{8} F (X, JY, JZ) \\
+ i \frac{1}{8} F (JX, Y, Z) - \frac{1}{8} F (JX, Y, JZ) - \frac{1}{8} F (JX, JY, Z) \\
- i \frac{1}{8} F (JX, JY, JZ).
\end{align*} \]

Similarly, we have that
\[ \begin{align*}
(\Pi_{0,3} F) (X, Y, Z) \\
= F \left( \Pi_{0,1} X, \Pi_{0,1} Y, \Pi_{0,1} Z \right) \\
= \frac{1}{8} F (X, Y, Z) - i \frac{1}{8} F (X, Y, JZ) - i \frac{1}{8} F (X, JY, Z) + (i)^2 \frac{1}{8} F (X, JY, JZ) \\
- i \frac{1}{8} F (JX, Y, Z) + (i)^2 \frac{1}{8} F (JX, Y, JZ) \\
+ (i)^2 \frac{1}{8} F (JX, JY, Z) - (i)^3 \frac{1}{8} F (JX, JY, JZ) \\
= \frac{1}{8} F (X, Y, Z) - i \frac{1}{8} F (X, Y, JZ) - i \frac{1}{8} F (X, JY, Z) - \frac{1}{8} F (X, JY, JZ) \\
- i \frac{1}{8} F (JX, Y, Z) - \frac{1}{8} F (JX, Y, JZ) - \frac{1}{8} F (JX, JY, Z) \\
+ i \frac{1}{8} F (JX, JY, JZ).
\end{align*} \]

Therefore we have that
\[ \begin{align*}
(\Pi_{3,0} F + \Pi_{0,3} F) (X, Y, Z) \\
= \frac{1}{4} \left( F (X, Y, Z) - F (X, JY, JZ) - F (JX, Y, JZ) - F (JX, JY, Z) \right).
\end{align*} \]

Now we compute
\[ \begin{align*}
(\Pi_{2,1} F + \Pi_{1,2} F) (X, Y, Z) \\
= \left( F (X, Y, Z) - (\Pi_{3,0} F + \Pi_{0,3} F) \right) (X, Y, Z) \\
= \frac{3}{4} F (X, Y, Z) + \frac{1}{4} \left( F (X, JY, JZ) + F (JX, Y, JZ) + F (JX, JY, Z) \right).
\end{align*} \]

Converting to coordinates yields the result. \(\Box\)

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