Domain Walls and Flux Tubes

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Abstract

We present a new vortex solution made of a domain wall compactified into a cylinder and stabilized by the magnetic flux within. When the thickness of the wall is much less than the radius of the vortex some precise results can be obtained, such as the tension spectrum and profile functions. This vortex can naturally end on the wall that has created it, making the simplest junction between a wall and a vortex. We then classify every kind of junction between a flux tube and domain wall. The criteria for classification are as follows: a flux can or can not end on the wall, and when it ends, the flux must go somewhere. Various examples are discussed, including abelian and nonabelian theories, as well as supersymmetric and non-supersymmetric theories.

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1 Introduction

We present a new kind of vortex which is formed by a wall. It occurs whenever a theory has two degenerate vacua, one in the Coulomb phase, and one in the Higgs phase. The ordinary vortex in the Higgs vacuum \cite{1, 2} can be thought of as the domain wall interpolating between the two vacua compactified into a cylinder. The rolled wall is stabilized by the magnetic field inside.\footnote{This is reminiscent of the relation between $Q$-kinks and $Q$-lumps discussed in \cite{3}.}

We then study the extent to which the thickness of the wall $\Delta_W$ is much less than the radius of the vortex $R_V$. This limit is reached for a large number of quanta of the magnetic flux. Within these limits the spectrum of vortices is simple to compute by means of classical arguments. They are particular kind of type I superconducting vortices with a tension that scales like $n^{2/3}$. In addition, the profile functions of the vortex are exactly computable within this limit.

This kind of flux tube naturally ends on the wall that has created it. When a wall vortex ends on its wall, the magnetic flux it carries is spread radially through the Coulomb vacuum on the other side of the wall. The point on the wall at which the vortex ends is seen from the Coulomb vacuum as a monopole of double charge. The junction between the wall and the wall vortex can be thought of as the final stage of a process in which a monopole in the Coulomb phase is pushed against the wall.

This kind of wall vortex arises in the nonabelian gauge theories every time a domain wall interpolates between two vacua with different confinement indices.

We then classify the junctions between domain walls and flux tubes. The previously discussed junction is only one particular variety, and perhaps the most natural. The two basic criteria for distinguish junctions are:

- The tube can or can not end on the wall.
- Where the flux goes.

When the vortex ends on the wall and spread its flux across the 2-dimensional surface of the wall is where we find a D-brane junction. This example has been widely studied in recent years \cite{3, 4, 5}. The driving force behind these studies was, in fact, to find some field theoretical analogues of the D-brane physics in string theory.
This variety of wall, in order to completely resemble a D-brane, must also supports massless gauge fields in the effective low-energy action.

Another type of junction is found when the string ends on the wall and the flux is captured by a particle confined within the wall. In this case, there is no localization of the gauge field inside the wall. The first example has been found in [7] where it was shown that the \( \mathbb{Z}_N \) strings of pure \( SU(N) \) \( \mathcal{N} = 1 \) super Yang-Mills can end on the domain wall that interpolates between two chirally adjacent vacua.

The last type is a cross-junction, which is characterized by the tube, rather than ending on the wall, crossing it instead and becoming another flux tube in the opposite vacuum. We can find examples of it in pure \( SU(N) \) \( \mathcal{N} = 1 \) SYM when the domain wall interpolates between two vacua whose phase shift has a common divisor with \( N \).

Finally we put our classification to the test in a particular theory. In \( \mathcal{N} = 2 \) pure SYM broken to \( \mathcal{N} = 1 \) by a generic superpotential for the adjoint field, there are a lot of vacua with different confinement index [8]. We provide examples in which all three types of junctions are presented simultaneously.

The paper is divided into two parts. Section 2 is devoted to the study of the wall vortex. In 2.1 we consider the abelian gauge theory and its supersymmetric extension. In 2.2 we study the wall vortex in nonabelian gauge theories. In Section 3 we classify the domain wall/flux tube junctions. In 3.1 we consider a flux tube which ends on the wall, and in 3.2 a flux tube which crosses the wall. Finally, in 3.3 we give an example in which every junction studied in the paper is presented simultaneously. In Appendix A we provide a mechanism to obtain a flux tube/flux tube junction, and we use it to build a Y junction for baryons in \( SU(3) \).

2 The Vortex Formed by a Wall

We consider a flux tube that can be thought as made of wall. The simplest example in which it can arise is a \( U(1) \) gauge theory that has two degenerate vacua: one in the Coulomb phase and the other in the Higgs phase. Consider the domain wall of tension \( T_W \) that interpolates between these two vacua. We can build a flux tube rolling the wall in a cylinder of radius \( R \), keeping the Coulomb phase inside the tube and turning on a magnetic flux inside (see Figure 1). The tension of the wall gives a contribution \( T_W 2\pi R \) to the energy density. The magnetic flux is \( \Phi_B = B\pi R^2 \) where
Figure 1: The wall vortex. A wall of thickness $\Delta W$ is compactified on a circle of radius $R_V$ and stabilized by the magnetic field inside.

$B$ is the magnetic field. Varying the radius $R$ the flux $\Phi_B$ must remain constant, so the contribution of the flux to the energy density is $\Phi_B/2\pi R^2$. The magnetic flux is quantized in integer values: $\Phi_B = 2\pi n/e$, where $e$ is the coupling constant. The tension of the tube is the sum of two pieces, one that comes from the flux and the other that comes from the wall:

$$T(R) = \frac{\Phi_B^2}{2\pi R^2} + T_W 2\pi R \ .$$

The stable configuration is the one that minimizes the tension:

$$R_V = 3\sqrt{2} \sqrt[3]{\frac{n^2}{e^2 T_W}} \ , \quad T_V = 3\sqrt{2}\pi \left(\frac{nT_W}{e}\right)^{2/3} .$$

This result can be trusted only when the thickness of the wall $\Delta W$ is much less than the radius of the vortex. In general the profile of the vortex will be a mixture of the magnetic field and the wall. In any case this kind of flux tube can be though as made of the same stuff of the wall.
2.1 Abelian Higgs-Coulomb model

Now we are going to consider an explicit example. The simplest that we can imagine is a $U(1)$ gauge theory coupled to a charged scalar $q$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} |(\partial_{\mu} - ieA_{\mu})q|^2 - V(|q|) ,$$  \hspace{1cm} (2.3)

and a suitable potential (see Figure 2) that has two degenerate minima, one in the Coulomb phase $q = 0$ and the other in the Higgs phase $|q| = q_0$. This theory admits a kink that interpolates between the two vacua. The profile of the kink, $q(z)$, is constant in $x, y$ and has boundary conditions $q(-\infty) = 0$ and $q(+\infty) = q_0$. The profile $q(z)$ has constant phase since the potential is a real function that depends only on $|q|$. In the Higgs phase there is a vortex obtained by choosing an element $n$ of the homotopy group $[n] \in \pi_1(S^1)$:

$$q = e^{in\theta} q(r) ,$$  \hspace{1cm} (2.4)

$$A_{\theta} = \frac{n}{e} r A(r) ,$$

where we have used cylindrical coordinates $r, \theta, z$ instead of $x, y, z$. Clearly the wall and the tube are continuously related. Consider a kink in the radial direction at a radius much greater than its thickness. We can roll it around the axe $z$ giving it a gauge rotation $e^{in\theta}$. This automatically turns on a pure gauge field at large distance.

Figure 2: A potential with two degenerate vacua. $q = 0$ is in the Coulomb phase while $|q| = q_0$ is in the Higgs phase.
that, for continuity, must have a flux inside. This configuration in general is not a minimum of the energy density, so it will start to loose energy until it reach the stable vortex.

### 2.1.1 The wall limit

The differential equations for the profile functions of the vortex (2.4) are

\[
\frac{d^2 q}{dr^2} + \frac{1}{r} \frac{dq}{dr} - n^2 \frac{(1 - A)^2}{r^2} q - \frac{\delta V}{\delta q} = 0 , \\
\frac{d^2 A}{dr^2} - \frac{1}{r} \frac{dA}{dr} + e^2 (1 - A) q^2 = 0 ,
\]

where \( n \) is the winding number. We are looking for some limit of the parameters so that the vortex really looks like a rolled wall. In this limit the profile functions should be:

\[
q(r) = q_0 \theta_H (r - R_V) ,
\]

\[
A(r) = \begin{cases} 
\frac{r^2}{R_V^2} & 0 \leq r \leq R_V , \\
1 & r \geq R_V ,
\end{cases}
\]

where \( \theta_H \) is the step function. First of all we manipulate a bit the differential equations (2.5) to simplify them. The potential can be written as a dimensionless function

\[
V(q) = v_0 V \left( \frac{q}{q_0} \right) ,
\]

where \( v_0 \) is the scale of the potential and \( q_0 \) the vev. In the following we will not use the shape of the dimensionless potential \( V \). The only important thing is that \( V(0) = V(1) = 0 \) and its height is of order 1 (the simplest example is \( V(\chi) = \chi^2 (\chi^2 - 1)^2 \)). We also scale the scalar field \( q = q_0 \chi \). After these scalings the equations (2.5) for the profiles becomes:

\[
\frac{d^2 \chi}{dr^2} + \frac{1}{r} \frac{d\chi}{dr} - n^2 \frac{(1 - A)^2}{r^2} \chi - \alpha \frac{\delta V}{\delta \chi} = 0 , \\
\frac{d^2 A}{dr^2} - \frac{1}{r} \frac{dA}{dr} + \beta (1 - A) \chi^2 = 0 .
\]

So there are three parameters that enter the game:

\[
n , \quad \alpha = \frac{v_0}{q_0^2} , \quad \beta = e^2 q_0^2 .
\]
The domain wall tension and thickness are respectively:$$\begin{align*}
T_W &\sim q_0\sqrt{v_0} , \\
\Delta W &\sim \frac{q_0}{\sqrt{v_0}} .
\end{align*}$$ (2.11)

Now comes the first non trivial hint. If the wall limit exists, then formula (2.2) can be trusted in this limit. But the radius of the vortex $R_V$ comes out from equations (2.5) and so must depend only on the three relevant parameters $n, \alpha, \beta$. In general a function of $n, e, v_0, q_0$ cannot be expressed as a function of $n, \alpha, \beta$, but for (2.2) this is possible:

$$R_V \sim \frac{n^{2/3}}{\alpha^{1/6} \beta^{1/3}} .$$ (2.12)

If we wouldn’t have found such expression, we would have concluded that the wall limit doesn’t exist. This result encourages us to go on.

Now we discuss the wall limit. In this paragraph we look for a limit in which the radius of the vortex remain constant and the solution approaches the wall vortex (2.6) (our goal is described in Figure 3). The $\chi$ profile must become a step function:

![Figure 3: Increasing the winding number $n$ and keeping fixed the radius of the vortex $R_V$, the profile functions approach the wall vortex.](image)

it is zero inside $R_V$, it goes from zero to one in a distance $\Delta W$, and then remains constant at one. Thus $\chi''$ in equation (2.8) develops a $\delta'(r - R_V)$ singularity or, thinking in terms of $\Delta W$, $\chi'' \sim 1/\Delta W^2$. Thus to counterbalance this divergence in (2.8) we must have $\alpha$ that goes to infinity like $1/\Delta W^2$. The encouraging fact is that $\alpha \frac{\delta V}{\delta \chi}$ really resembles a $\delta'(r - R_V)$ in this limit (note that this is true only if the two

\[\text{For this qualitative result it is sufficient to take the wall lagrangian } L = -\partial q \partial q - V(q) \text{ and then write the tension as a function of the thickness } T(\Delta) \sim \frac{q_0}{\sqrt{v_0}} + v_0 \Delta. \text{ Minimizing } T(\Delta) \text{ with respect to } \Delta \text{ we find (2.11).}\]
vacua are exactly degenerate). Now consider the second equation \( (2.9) \) where \( A''(R_V) \) has a \((2/R_V)\delta(r-R_v)\) singularity, or thinking in terms of \( \Delta_W \), \( A''(R_V) \sim 1/(R_V\Delta_W) \). Since \((1-A)\chi\) is of order \( \Delta_W/R_v \) around \( R_v \), we must also send \( \beta \) to infinity like \( 1/\Delta_W^2 \). Now we can make the conjecture of the wall limit.

**Mathematical Conjecture:** Consider the succession of parameters \( \alpha_n = n^{4/3}\alpha_1 \) and \( \beta_n = n^{4/3}\beta_1 \) and call the solution of \((2.8)\) and \((2.9)\) with the vortex boundary conditions, \( \chi_{n,\alpha_n,\beta_n}(r) \) and \( A_{n,\alpha_n,\beta_n}(r) \). In the limit \( n \to \infty \)

\[
\lim_{n \to \infty} \chi_{n,\alpha_n,\beta_n}(r) \to \theta_H(r-R_v), \tag{2.13}
\]

\[
\lim_{n \to \infty} A_{n,\alpha_n,\beta_n}(r) \to \begin{cases} r^2/R_V^2 & 0 \leq r \leq R_V, \\ 1 & r > R_V. \end{cases}
\]

This limit has been chosen so that the radius of the vortex remain constant \( R_V \sim n^{2/3}\alpha_n^{-1/3}\beta_n^{-1/6} = \alpha_1^{-1/3}\beta_1^{-1/6} \) and also the ratio \( \alpha_n/\beta_n \) remains constant. The ratio \( \alpha/\beta \) disappears in the wall limit and is related only to the shape of the limiting functions \( \chi_{n,\alpha_n,\beta_n}(r) \) and \( A_{n,\alpha_n,\beta_n}(r) \). Probably a stronger version of the conjecture is true: the ratio \( \alpha_n/\beta_n \) is kept limited from above and from below during the limit, so that it doesn’t go neither to infinity nor to zero.

The limit discussed above has been called mathematical since it is easy to express in a mathematical language. In this paragraph we discuss a more physical situation in which we keep fixed the parameters of the theory and we only change the winding number \( n \). When \( n \) is increased the radius of the vortex becomes large while the thickness of the wall remains constant since it doesn’t depend on \( n \). We are going to use \((2.2)\) in an self consistent way. Suppose that \( n \) is enough large so that \((2.2)\) is true, then we use the expression of \( R_V \) to obtain the condition for \( n \) so that \( R_V \gg \Delta_W \).

**Physical Conjecture:** We keep \( \alpha \) and \( \beta \) fixed and increase \( n \). The condition \( R_V \gg \Delta_W \) is \( n \gg (\beta/\alpha)^{1/4} \). When this is satisfied the vortex resembles a wall vortex and its tension is given by

\[
T_V = 3\sqrt{2\pi} \left( \frac{n T_W}{e} \right)^{2/3}. \tag{2.14}
\]

### 2.1.2 Pushing a monopole against the wall

There is another intuitively way in which we can see the wall continuously transformed into the vortex. Consider a monopole in the Coulomb phase. The lines of its magnetic
flux will be tangent to the wall surface. Now we move the monopole in the wall direction as in Figure 4. The wall is repelled and change its shape. Far from the $x, y$

Figure 4: A monopole in the Coulomb phase is pushed against that domain wall. At the end of the process we are left with a monopole in the Higgs phase confined on a vortex that ends on the wall.

coordinate of the monopole the shape of the wall is logarithmic $z \propto \log (x^2 + y^2)$. If we continue to push the monopole in the negative $z$ the configuration will resemble more and more a monopole in Higgs phase, attached to a flux tube that ends on the wall. This physical picture well explains a wall that is continuously transformed into a vortex. It’s also clear that this kind of vortices can naturally end on the wall that has created them since they are made of the same stuff.

A more detailed and quantitative analysis of this junction will appear in [9].

2.1.3 $\mathcal{N} = 2$ super QED broken to $\mathcal{N} = 1$

Now we consider $\mathcal{N} = 2$ SQED broken to $\mathcal{N} = 1$ by a superpotential for the adjoint field. The $U(1)$ gauge multiplied is composed by the superfields $W_\alpha$ and $\Phi$, while the matter superfields are $Q$ of charge +1 and $\tilde{Q}$ of charge −1. The Lagrangian is the
\[ L = \int d^2\theta \frac{1}{4e^2} W^\alpha W_\alpha + h.c. \]  
\[ + \int d^2\theta d^2\bar{\theta} \left( \frac{1}{e^2} \Phi^\dagger \Phi + Q^\dagger e^Y Q + \bar{Q}^\dagger e^{-Y} \bar{Q} \right) \]  
\[ + \int d^2\theta \sqrt{2} (\bar{Q} \Phi Q - m \bar{Q} Q + W(\Phi)) + h.c. . \]  

The potential for the scalar fields is

\[ V = 2|\phi - m| q|^2 + 2|\phi - m| \bar{q}|^2 + 2e^2|\bar{q}q + W'(\phi)|^2 + \frac{e^2}{2} (|q|^2 - |\bar{q}|^2)^2 . \]  

This potential has two kind of minima: one is the Higgs vacuum

\[ \phi = m , \quad |q| = |\bar{q}| , \quad \bar{q}q = -W'(m) , \]  

and the others are a set of Coulomb vacua, each for every stationary point of the superpotential

\[ \phi = a_i , \quad q = \bar{q} = 0 . \]

In the Higgs vacuum the gauge group \( U(1) \) is completely broken by the quark condensate, so the theory admits vortex solutions that belongs to the homotopy group \( \pi_1(U(1)) = \mathbb{Z} \). We don’t lose any information choosing

\[ \bar{q} = -q^\dagger \frac{W'(\phi)}{|W'(\phi)|} . \]  

We are left with the following theory:

\[ L = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{e^2} \partial_\mu \phi^\dagger \partial^\mu \phi - (D_\mu q)^\dagger (D^\mu q) - V(\phi, |q|) , \]  

where the potential is

\[ V(\phi, |q|) = 2|\phi - m| q|^2 + \frac{e^2}{2} (|q|^2 - 2|W'(\phi)|^2)^2 . \]  

The vortex solution in terms of the profile functions is:

\[ q = e^{in\theta} q(r) , \]  
\[ \phi = \phi(r) , \]  
\[ A_\theta = \frac{n}{er} A(r) . \]
The boundary conditions for $q(r)$ and $A(r)$ are the usual ones. The boundary conditions for $\phi(r)$ are $\phi(\infty) = m$ and $\phi(0) = a_j$ where the root $a_j$ will be the one that makes the vortex lighter. When all the roots $a_i$ are reals, it is clear that $a_j$ is the one that minimize $|m - a_i|$. This vortex can be considered as made off the wall connecting the Higgs vacuum and the $j$-Coulomb vacuum. It is thus straightforward to generalize the physical conjecture made in 2.1.1. For $n$ sufficiently large so that $R_V \gg \Delta_W$, the spectrum of these vortices is (2.14).

**Almost-BPS solution**

The theory under consideration has been widely studied since it arise as an effective description of more complicated nonabelian theories [10, 11, 12, 13]. This SQED is sometimes magnetic dual with respect to the original degrees of freedom and thus our vortices describes confinement of quarks. Usually these vortices have been studied in an almost BPS regime, that is by neglecting the second derivative of the superpotential. In this limit we can make a stronger ansatz for the vortex solution:

$$\phi = m \, .$$

Thus the Lagrangian becomes the usual one in the BPS limit [14, 15]:

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - (D_\mu q)^\dagger (D^\mu q) - \frac{e^2}{2} (|q|^2 - 2|W'|)^2 \, .$$

The spectrum of vortices is the well known BPS proportionality between tension and charge:

$$T = 4\pi |n W'(m)| \, .$$

Clearly (2.23) is only an approximation. It is enough to look at the equation of motion for $\phi$ and see that it is not satisfied inside the vortex:

$$\frac{\Box \phi}{e^2} = 2(\phi - m)|q|^2 + e^2 W''(\phi)^\dagger (|q|^2 - 2|W'|)^2 \frac{W'(\phi)}{W''(\phi)} \, .$$

An important point is to find the condition under which (2.23) is a good approximation to the spectrum. In [16] we have done this for the winding number $n = 1$, and the condition has been found to be:

$$\frac{e^2 W''}{W'} \ll 1 \, .$$

It is clear also from [16] that as the winding $n$ becomes bigger, so do the non-BPS corrections. For consistency with the wall vortex limit, we should have a spectrum
Figure 5: Spectrum of vortices in $\mathcal{N} = 2$ SQED broken by a superpotential. If \( \frac{e^2 W''}{W} \ll 1 \) an almost BPS region is present for small winding numbers. Increasing \( n \) non-BPS corrections become larger and not negligible. When the winding \( n \) is sufficiently big we are in the wall vortex regime.

like Figure 5. For sufficiently small \( n \) the tension is given by the BPS formula (2.28) and is proportional to \( n \). For sufficiently big \( n \) the tension is given by (2.14) and is proportional to \( n^{2/3} \). There must not be superposition between the two regions.

$\mathcal{N} = 2$ SQED broken by adjoint mass

Now we make the consistency check the the two regions do not overlapped. For simplicity we limit ourselves to the quadratic superpotential \( W(\Phi) = \frac{1}{4} \mu \Phi^2 \). The energy density for the wall is

\[
T = \int dx \left( \frac{1}{e^2} \partial_x \phi^\dagger \partial_x \phi + \partial_x q^\dagger \partial_x q + 2|\phi - m|q|^2 + \frac{e^2}{2} (|q|^2 - \mu \phi)^2 \right) .
\]

(2.28)

We rewrite it in terms of the dimensionless variables \( \phi = m \varphi \) and \( q = \sqrt{\mu m} \chi \):

\[
T = \int dx \left( \frac{m^2}{e^2} \partial_x \varphi^\dagger \partial_x \varphi + \partial_x \chi^\dagger \partial_x \chi + 2m^2 \mu |(\varphi - 1)\chi|^2 + \frac{e^2 m^2 \mu^2}{2} (|\chi|^2 - \varphi^2) \right) .
\]

(2.29)

The condition for the existence of an almost BPS region is

\[
\frac{e^2 \mu}{m} \ll 1.
\]

(2.30)

The \( n = 1 \) vortex would also belong to the wall vortex region is the condition \( R_V \gg \Delta_W \) would be satisfied. In terms of our variables this condition would be

\[
\frac{1}{e^{2/3} T_W^{1/3}} \gg \Delta_W .
\]

(2.31)
We are going to prove that, if (2.30) is satisfied, then (2.31) cannot be satisfied.

Under the assumption that (2.30) is satisfied, the tension (2.29) can be approximated by

$$T \sim \frac{m^2}{e^2 \Delta} + m^3 \mu \Delta,$$  \hspace{1cm} (2.32)

where $\Delta$ is the length scale of variation of the fields. Minimizing (2.32) with respect to $\Delta$ we obtain:

$$\Delta_W \sim \frac{1}{e \sqrt{\mu m}}, \quad T_W \sim \frac{m^{5/2} \mu^{1/2}}{e}.$$ \hspace{1cm} (2.33)

Thus the condition for the wall vortex is

$$\frac{1}{e^{2/3} T_W^{1/3} \Delta_W} \sim \left( \frac{e^2 \mu}{m} \right)^{1/3} \gg 1,$$ \hspace{1cm} (2.34)

and it is clearly not satisfied if (2.30) is true.

### 2.1.4 Quantum corrections

The non-supersymmetric theory (2.3) with the potential of Figure 2 has a problem: it is not stable under quantum corrections since there is no spontaneously broken discrete symmetry that can relate an Higgs vacuum with a Coulomb vacuum. Only with a fine tuning of the parameters we can have this degeneracy.

Degeneracy of vacua, not related by any spontaneously broken discrete symmetry, is a common feature of supersymmetric gauge theories. In fact in the supersymmetric theory described in 2.1.3 the degeneracy is protected by non renormalization theorems. We can thus ask what happens to the relations (2.2) when we consider quantum corrections. Note that to obtain (2.2) we have used only general principles and so they should be valid also in a full quantum theory, even if the various parameters that enter in the game $T_V, R_V, T_W, \Delta_W$ are subject to quantum corrections. Special attention should be given to the coupling constant $e(\mu)$ since in the full quantum theory depends on the energy scale $\mu$. But since we want to study vortices with large $n$ and so large radius $R_V$, we should keep the low-energy coupling constant $e(m_{\text{lightest}})$ where $m_{\text{lightest}}$ is the mass of the lightest charged particle.

### 2.1.5 Zero modes

The low-energy dynamics of a $p$-soliton is described by an effective $p + 1$ dimensional field theory on the world volume of the soliton. By low-energy we mean length greater
that the thickness of the soliton. The target space of the effective action is the space of zero modes. This space can be divided in displacement zero modes, the ones that describe fluctuation of the brane in the space, and internal zero modes. Up to now it is the standard technique for describe soliton dynamics. In the case of the wall vortex it is not hard to imagine that there should be a relation between the effective $2 + 1$ theory of the domain wall and the effective $1 + 1$ theory of the vortex. In fact the vortex dynamics is described by the $2 + 1$ theory of the wall with a spatial direction compactified on a circle of radius $R_V$. For wave lengths $\lambda$ so that $\Delta W \ll \lambda \ll R_V$, is a vibrating membrane and for wave lengths $\lambda \gg R_V$ it is a vibrating string. In Figure 6 we have the displacement zero modes in the two different regimes. It is also clear that the wall vortex share the same internal zero modes target space with the wall that makes it.

![Figure 6](image-url)

**Figure 6**: Fluctuation of the displacement zero modes. For wave length $\lambda \gg R_V$ we have a vibrating string. For wave length $\Delta W \ll \lambda \ll R_V$ we have a vibrating wall compactified on a circle.

## 2.2 Nonabelian theories and the existence at strong coupling

It is a natural question if these wall vortices can arise in nonabelian theory and at strong coupling. The answer is yes and, as we will see in a moment, it is enough to
consider confining vacua with different confinement index.

Take a generic $SU(N)$ gauge theory. The confinement index $t$ is defined as the minimal integer so that $Q^t$ in unconfined. By $Q^t$ we mean the representation of $t$ antisymmetrized quarks $Q$ in the fundamental of the gauge group. When $t = N$ the theory is completely confined, like ordinary QCD. When $t = 1$ the theory is completely unconfined. $t$ must be a divisor of $N$. The confinement is caused by strings with topological number $\mathbb{Z}_t$. A $k$-string, with $k \in \mathbb{Z}_t$, confines the $Q^k$ representation. The $t$-string is trivial and in fact $Q^t$ is unconfined.

Now consider a case in which this theory has one vacuum $A$ with confinement index $t_A$ and another vacuum $B$ with confinement index $t_B > t_A$. Suppose that exists a domain wall interpolating between these two vacua. There will be at least one representation $Q^k$ that is confined in one vacuum and not confined in the other. For example $Q^{t_A}$ is not confined in $A$ but confined in $B$. We can make a continuous transformation, such as the one previously described in Figure 4. We take a $Q^{t_A}$ particle in vacuum $A$ and we push it against the wall. We will end with a $Q^{t_A}$ in vacuum $B$ connected to a $t_A$-string that ends on the wall. This continuous interpolation between the wall and the string shows that the $t_A$-string is indeed a wall vortex.

In general every $k$-string in vacuum $B$ is a wall vortex with respect to the $A/B$ wall if $k$ is a multiple of $t_A$ but not of $t_B$. The same can be said on the other side of the wall. Every $k$-string in vacuum $A$ is a wall vortex with respect to the $B/A$ wall if $k$ is a multiple of $t_B$ but not of $t_A$. To be more precise we should think of the strings as living in an extended group $\mathbb{Z}_{\text{lcm}(t_A, t_B)}$ that contains both $\mathbb{Z}_{t_A}$ and $\mathbb{Z}_{t_B}$. Strings in vacuum $A$ that belong to the subgroup $\mathbb{Z}_{\text{lcm}(t_A, t_B)}/t_B$ are wall vortices with respect to the $A/B$ wall. Strings in vacuum $B$ that belong to the subgroup $\mathbb{Z}_{\text{lcm}(t_A, t_B)}/t_A$ are wall vortices with respect to the $B/A$ wall. We will came back to this point in 3.3.

In general we can prove that a domain wall interpolating between vacua with different confinement index exists. Consider the set of vacua of the theory and group them in sets with the same confinement index like in Figure 7. It is not true that for any couple of vacua $A$ and $B$ the domain wall interpolating between them exist. But, for sure, it will exist at least a path connecting the two vacua so that in every segment of the path the domain wall exist and it’s stable. In this path there will be some domain wall interpolating between vacua with different confinement index.

The study of the wall limit for nonabelian strings, such as the one done in 2.1.1, will appear in [17].
Figure 7: In general it not true that given any couple of vacua $A$ and $B$ a domain wall that interpolates between them exist. But if it doesn’t exist then it decays in a path of stable domain walls that connect $A$ and $B$ passing through other vacua. If the $t_A \neq t_B$, surely there will be a stable domain wall at some point of the path that connects vacua with different confinement index.

3 Classification of the Domain Wall/Flux Tube Junctions

The wall vortex studied in Section 2 is one particular case of wall/tube junction. The purpose of this Section is to classify the various kind of junctions between walls and tubes. The junctions can be distinguished by simple properties:

- If the tube can end or not on the wall.
- Where the flux goes into.

In this spirit, the Coulomb junction of Section 2 can be schematically represented as in Figure 8. The tube ends on the wall and the flux goes in the opposite half space. To analyze other examples we make the first distinction. In 3.1 we consider vortices that can end on the wall. In 3.2 we consider vortices that cross the wall and continue on the other side.
3.1 The Flux Tube Ends on the Wall

In string theory, a string can have Dirichlet boundary conditions if it ends on a dynamical object called D-brane. By analogy, in some QFT, similar phenomenon can happen when a flux tube ends on the wall. We will briefly describe two examples in which a flux tube can terminate on a wall. In the first case there is a localization of the flux on the wall. Thus the tube spread out its flux into the wall. In the second example there is a bound state confined on the wall that captures the flux. The bound state is an object composed by two pieces that belong to the two different vacua.

3.1.1 Localization of the flux

This case is probably the more studied, particularly in the last few years. We have a domain wall interpolating between two vacua both in the Higgs phase. The magnetic strings on both side of the wall can end on it and spread the flux inside it (see Figure 9). Inside the wall there must be a localized gauge theory. Note that if we are able to prove that the $2 + 1$ dimensional theory contains the massless gauge field, the confining string must end on the wall (the other possible configurations would be not favorite energetically).

Recently there has been research in the the so called non-abelian Higgs model.
Figure 9: The D-brane junction. A vortex ends on a wall and spread its flux into it. The effective theory on the wall must contain massless gauge field. The point where the tube ends is an electric charge with respect to the $2+1$ effective gauge theory.

It is a $U(N_c)$ $\mathcal{N} = 2$ gauge theory with $N_f$ hypermultiplets of mass $m_i$. The theory is then broken to $\mathcal{N} = 1$ adding a Fayet-Iliopoulos term for the $U(1)$ subgroup. When $N_f \geq N_c$ the theory has $\binom{N_f}{N_c}$ vacua since every color must be locked to some flavor. This theory is a nice lab because it possesses all the three kinds of solitons: monopoles, vortices and domain walls, that can appear in various junctions that preserves supersymmetry. Consider in particular a fundamental wall that separates two vacua $\ldots, \hat{i}, i + 1, \ldots$ and $\ldots, \hat{i}, i + 1, \ldots$ (with the hat we indicate the flavors that are not locked). In the first vacua a $U(1)$ is locked to the flavor $i$ while the flavor $i + 1$ is unlocked, the opposite in the other vacua. In both vacua the $U(1)$ is Higgsed, in the first by the condensation of the flavor $Q_i$ and in the second by the flavor $Q_{i+1}$. They admit a magnetic vortex that breaks $1/2$ of supersymmetries and saturates the BPS bound. The central charge for the vortex is given essentially by the Fayet-Iliopoulos term. It has been shown that the walls are also $1/2$-BPS and in particular the one under consideration (from $i, \hat{i} + 1$ to $\hat{i}, i + 1$), has a Coulomb phase inside with respect to this $U(1)$. The junction vortex-wall has been shown to be $1/4$-BPS. For generalizations to nonabelian strings and nonabelian walls see [18].

3.1.2 Bound state localized on the wall

Another example of strings ending on walls is provided by $\mathcal{N} = 1$ $SU(N)$ super Yang-Mills. This theory has $N$ vacua obtained by the spontaneous breaking of the anomaly free residual $R$-symmetry. We label these vacua with the index $0 \leq r < N$. 

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(see Figure 10). Every vacuum has confinement index \( t = N \) and so there are \( \mathbb{Z}_N \)

\[
\begin{align*}
\mathcal{N} = 1 \text{ pure Super Yang Mills has } N \text{ different vacua labelled with } r. \\
\text{The confinement in each vacuum is explained by the condensation of a particle that carries magnetic} \\
\text{charge 1 and electric charge } Q^r. \text{ In every vacuum strings belong to the topological group } \mathbb{Z}_N. \text{ A} \\
k\text{-string connects a } Q^k \text{ with a } Q^k \text{ and creates a meson.}
\end{align*}
\]

strings responsible for the confinement of sources in any non trivial representation of
the center of the gauge group. The string \( k \in \mathbb{Z}_N \) confines the representation \( Q^k \).
The confinement in the \( r \) vacua can be understood as caused by the condensation a
monopole bounded together with a \( Q^r \). We indicate this object with \( (Q^r, 1) \) where
the 1 refers to the magnetic charge.

It has been shown in \cite{7} that the domain wall\(^3\) interpolating between two adjacent
vacua \( r \) and \( r + 1 \) is a D-brane for the \( k \)-strings. This phenomenon happens also
for every wall interpolating between a vacuum \( r_1 \) and a vacuum \( r_2 \) so that \( r_2 - r_1 \) is
prime with \( N \). The easiest way to understand it is the Rey’s argument reported in
\cite{7}. The confinement in the \( r \)-vacuum is due to the condensation of a \( (Q^r, 1) \) particle.
To prove that the \( k \)-string can end on the wall we need a bound state confined on
the wall that carries the same charge of \( Q^{-k} \). As showed in Figure 11 this object can
be obtained putting together \( k \) \( (Q^r, 1) \) and \(-k \) \( (Q^{r+1}, 1) \) on the opposite sides of the
wall.\(^4\)

\(^3\)Domain walls in \( \mathcal{N} = 1 \) super Yang-Mills have been studied in \cite{19, 20, 21}. The effective action
on these walls has been found in \cite{22}.

\(^4\)Actually, to have a rigorous prove, we should show the composite object has a binding energy.
3.2 The Flux Tube Crosses the Wall

When a flux tube ends on a wall, the flux must go somewhere. There are only two possibility: it goes on the other half space (see Section 2), or it is confined inside the wall (see 3.1). Now we consider the case in which the vortex cannot end on the wall and so must continue into another vortex in the other vacua (see Figure 12). They could also bring different fluxes since there could be a particle localized on the wall, in the junction between the two vortices, that carries the difference of the fluxes.

The first example is the trivial one. Take the usual $U(1)$ gauge theory coupled to a charged scalar in the Higgs phase. Then add a real scalar field responsible for the creation of a domain wall. We don’t put interactions between the sector of the theory responsible for the vortex and the sector responsible for the wall. The domain wall is transparent to the vortex that can cross it without any modification. This trivial junction also arise in the non-abelian Higgs model considered in 3.1.1. For example, we saw that the wall interpolating between the vacua $\ldots, i, i+1, \ldots$ and $\ldots, \hat{i}, i, i+1, \ldots$, forms a D.brane junction with respect to the vortex that carry the flux of the $U(1)$ that changes locking. The same wall makes a trivial junction with
Figure 12: The Cross junction. The flux tube crosses the wall and becomes a vortex in the other vacuum.

respect to the other $U(1)$’s that are left unchanged.

3.2.1 A vortex with a wall around it

Now we move to a non trivial case. Consider the $U(1)$ gauge theory $[2,3]$, where the potential has two different minima at $q_A < q_B$ (see Figure 13). There is a wall

![Figure 13: A potential with two degenerate Higgs vacua.](image)

$V(|q|)$

$q_A$ $q_B$ $|q|$
that interpolates between the two vacua, that has a profile \( q = q(z) \) with boundary conditions \( q(-\infty) = q_A \) and \( q(+\infty) = B \). The wall has not have a Coulomb phase inside it, since \( q(z) \) is always different from zero. Both the vacua \( A \) and \( B \), admit magnetic flux tubes and the tension \( T_A \) will be smaller than the tension \( T_B \). The Coulomb junction is excluded since \( A \) and \( B \) are both in the Higgs phase. The D-brane junction is also excluded since there is not a Coulomb phase inside the wall. The only possibility is that the vortex in \( A \) crosses the wall and becomes the vortex in \( B \) (see Figure 14). Since they carry the same flux there non need of any particle localized on the wall. There is a nice interpretation of the fact that \( T_A \) is smaller than \( T_B \). We can imagine that the vortex in \( B \) is composed by a vortex in \( A \) and the wall rolled around it. Thus the vortex in \( B \) is more heavy because it has a wall around.

### 3.2.2 Crossing in nonabelian theories

The last example comes from \( \mathcal{N} = 1 \) \( SU(N) \) super Yang-Mills. We choose \( r_A \), for the vacuum \( A \), and \( r_B \), for the vacuum \( B \), so that \( r_B - r_A \) is a divisor of \( N \). We use again the Rey’s argument to determine which string ends on the wall and which one crosses the wall. Is possible to build a bound state on which a \( k \)-string could end, only if \( k \) is a multiple of \( \gcd(r_B - r_A, N) \). Thus we have a subgroup \( \mathbb{Z}_{\gcd(r_B - r_A, N)} \subset \mathbb{Z}_{N} \) of
strings that can end on the wall with an End junction. Strings in the quotient group \( \mathbb{Z}_{N/\gcd(r_B-r_A,N)} \) instead, cross the wall and go into another string (Cross junction). Like in Figure 15 there can be a bound state multiple of \( \gcd(r_B-r_A,N) \) that change the flux of the string but cannot absorb completely its flux.

Figure 15: The vortex crosses the wall. A bound state confined on the wall can change the flux of the vortex. This happens whenever \( k_A - k_B \) is not a multiple of \( \gcd(r_A - r_B, N) \).

We analyze the simplest example. Take \( N \) even and a wall between \( r_A \) and \( r_B = r_A + 2 \). Even strings can end on the wall. Odd strings must cross the wall and they becomes odd strings on the other side. For example a 1-string can end on a bound state that carries charge \( Q^{-2} \) and then emerges on the other side like a \(-1\)-string.

3.3 A complete example

An interesting theory is \( U(N) \mathcal{N} = 2 \) super Yang-Mills broken to \( \mathcal{N} = 1 \) by a generic superpotential. This theory is enough rich to present all the junctions just described.

The simplest example is a cubic superpotential so that \( W' = (\Phi - a_1)(\Phi - a_2) \) has two roots. For simplification we will consider the regime \( \Lambda \gg a_1, a_2, a_1 - a_2 \). In this case the theory must distribute the eigenvalues of \( \Phi \) at a scale where the gauge group
is still weakly coupled, and at this scale, $U(N)$ is broken down to $U(N_1) \times U(N_2)$. At the scale $\Lambda$ the gauge group confines and we have in global $N_1 \cdot N_2$ vacua labelled by two integers $r_{N_1}$ and $r_{N_2}$. A complete characterization of a vacua is thus given specifying three integers $N_1$, $r_1$ and $r_2$. The confinement index $t$ is the greatest common divisor between $N_1$, $N$, and $r_1 - r_2$. A domain wall is given choosing a couple of vacua, call the first vacuum $A$ and the second vacuum $B$.

### 3.3.1 All the junctions simultaneously

The simplest case in which all the three kind of junctions appear simultaneously is the one schematically represented if Figure 16. It is a $U(16)$ theory that breaks down to $U(8) \times U(8)$. For vacuum $A$ we chose $r_{A1} = r_{A2} = 0$ and the confinement index is $t_A = 8$. For vacuum $A$ we chose $r_{B1} = 2$, $r_{B2} = 6$ and the confinement index is $t_B = 4$. Vacuum $A$ has $\mathbb{Z}_8$ strings while vacuum $B$ has $\mathbb{Z}_4$ strings. The state $Q^4$ is confined in $A$ and non confined in $B$, thus we have a $\mathbb{Z}_2$ subgroup of strings in $A$ that make a Coulomb junction with the wall. The quotient $\mathbb{Z}_8/\mathbb{Z}_2 = \mathbb{Z}_4$ is now equal to the strings in vacuum $B$. The subgroup $\mathbb{Z}_2$ that confines even powers of $Q$ can end on the wall and thus they form an End junction. The quotient $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$ are the
strings that confine odd powers of $Q$ and they form a Cross junction.

In general all the three kind of junctions appear when we have vacua with different confinement indices. What we are going to say is presented schematically in Figure 17. The confining strings $\mathbb{Z}_{t_A}$ of vacuum $A$ and $\mathbb{Z}_{t_B}$ of vacuum $B$, must be though embedded in the smallest group that contains both of them $\mathbb{Z}_{\text{lcm}(t_A, t_B)}$. Strings that are multiple of $t_B$ but not of $t_A$ are wall vortices in vacuum $B$ and they belongs to the group $\mathbb{Z}_{\text{lcm}(t_A, t_B)}/t_B$. In the same way strings that are multiple of $t_A$ but not of $t_B$ are wall vortices in vacuum $A$ and they belongs to the group $\mathbb{Z}_{\text{lcm}(t_A, t_B)}/t_A$. For the remaining strings we must decide if they form an End junction or a Cross junction. The Cross-junctions has a nice interpretation. They can be thought as the stable strings in the domain wall background. So in the same way $\mathbb{Z}_{t_A}$ is the group of string in vacuum $A$ and $\mathbb{Z}_{t_B}$ is the group of strings in vacuum $B$, so $\mathbb{Z}_{\text{cross}}$ is the group of

![Figure 17: The web of groups that describe wall vortex junctions between vacua of a $SU(N)$ gauge theory with confinement index $t_A$ and $t_B$. The smallest group that contains both the $A$ and the $B$ strings is $\mathbb{Z}_{\text{lcm}(t_A, t_B)}$. The subgroup $\mathbb{Z}_{\text{lcm}(t_A, t_B)}/t_B$ identifies strings in vacuum $A$ that end on the wall and forms a Coulomb junction. The opposite is true for strings in vacuum $B$. $\mathbb{Z}_{\text{cross}}$ is the group of stable strings in the domain wall background, that is the group of strings that cross the wall. Strings in vacuum $A$ that belongs to the subgroup $\mathbb{Z}_{t_A/\text{cross}} \in \mathbb{Z}_{t_A}$ end on the wall and form an end junction. The group of crossing strings $\mathbb{Z}_{\text{cross}}$ is the quotient $\mathbb{Z}_{t_A}/\mathbb{Z}_{t_A/\text{cross}}$. The opposite is true for strings in vacuum $B$.](image-url)
stable strings in the domain wall background where \textit{cross} must be a divisor of both $t_A$ and $t_B$. The remaining strings $\mathbb{Z}_{t_A/\text{cross}}$ are End junction in vacuum $A$ and $\mathbb{Z}_{t_B/\text{cross}}$ are End junctions in vacuum $B$. In Figure 18 we give two examples of the groups relation of the scheme in Figure 17

$U(N)$ \quad $U(N) \rightarrow U(N_1) \times U(N_2)$

\[
\begin{array}{c}
\mathbb{Z}_N \quad \mathbb{Z}_N \\
\mathbb{Z}_{\gcd (r_A-r_B, N)} \quad \mathbb{Z}_{\gcd (r_{B1}-r_{B2}, N_1, N_2)} \\
\mathbb{Z}_{\gcd (r_{A1}, r_{A2}, N_1, N_2)} \quad \mathbb{Z}_{\gcd (r_{B1}-r_{B2}, N_1, N_2)} \\
\end{array}
\]

Figure 18: Two examples of web of groups in Figure 17. The first refers to $U(N)$ super Yang-Mills where vacuum $A$ has index $r_A$ and vacuum $B$ has index $r_B$. The second example is $U(N) \rightarrow U(N_1) \times U(N_2)$ with indices $r_{A1}, r_{A2}$ for vacuum $A$ and $r_{B1}, r_{B2}$ for vacuum $B$.

### 3.3.2 Decay of junctions

In the theory at hand, strings can appear in different regimes distinguished by the magnitude of the superpotential. $\sqrt{W'}$ is the energy scale where $N = 2$ is broken and $W'$ is the tension scale of the strings. Thus we have three cases. $\sqrt{W'} \ll \Lambda \ll a$ is the strong coupling regime, where the superpotential is a small perturbation that can be added to the low energy dynamics of $N = 2$. In this case flux tubes are nothing but ANO vortices of the low-energy $U(1)$’s. In the intermediate regime $\Lambda \ll \sqrt{W'} \ll a$, the dynamics is approximatively that of various independent $U(N_i)$, note that the number of strings is the same as before but now they are governed by the group $\oplus_i \mathbb{Z}_{N_i}$. The transition between the strong coupling regime and the pure $N = 1$ regime has been studied in [13] and there is nothing new to say about it. The last regime is $\Lambda \ll a \ll \sqrt{W'}$, where some strings that were metastable before now can decay. Their dynamics is governed by the group $\mathbb{Z}_{\gcd (N_i, b_i)}$. 

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We consider in detail the case $N = 4$. We choose both the vacua $A$ and $B$ so that classically $U(4) \rightarrow U(2) \times U(2)$. In vacuum $A$ both the $U(2)$ factors are in the monopole vacuum $r_{A1} = r_{A2} = 0$. In vacuum $B$ the first $U(2)$ is in the monopole vacuum $r_{B1} = 0$, and the second in the dyon vacuum $r_{B2} = 1$ (see Figure 19).

![Diagram](image)

Figure 19: $U(4) \rightarrow U(2) \times U(2)$.

$\Lambda \ll \sqrt{W'} \ll a$. This is the most easiest region to analyze. Both vacua have two confining strings that are the non trivial elements of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The first group refers to the first $U(2)$ and, since $A$ and $B$ are both in the monopole vacua, these strings cannot end on the wall. So we are forced to conclude that they form a Cross junction. The second $\mathbb{Z}_2$ refers to the second $U(2)$. Since vacuum $A$ is in the monopole vacuum and $B$ in the dyon vacuum, the strings can end on the wall and so they forms an End junction.

$\Lambda \ll a \ll \sqrt{W'}$. In this region of parameters the two theories are governed by the confinement index. Vacuum $A$ has $\mathbb{Z}_2$ strings and so only one non trivial element. The interpretation is that the two elements founded in the previous regime belongs to the same topological element and can decay one into the other. Vacuum $B$ has $t_B = 1$ so is completely unconfined. The two strings founded previously are only metastable. The conclusion is obvious, we are in
the presence of a Coulomb junction. Thus we have seen an example of the decay schematically presented in Figure 20.

![Figure 20: Decay of junctions.](image)

A The Flux Tube/Flux Tube Junction

In this Appendix we describe, using the things studied in this paper, a way to obtain a junction between a flux tube and another flux tube. This junction can be used to build a baryon vertex in some particular confining gauge theory. In the string model of hadrons, mesons are identified with a string with a quark and an antiquark attached at the endpoints. For the baryons things are not so natural and various configurations are possible. One of the hypothesis is the so called “Y” junction: the baryon is composed by three strings with one end in common and three quarks at the other ends [24]. In general the construction of the baryon vertex is not at all trivial (see for example [13] for MQCD and [25] for the AdS/CFT correspondence). In what follows we describe another possible mechanism for the formation of the Y junction using the flux tube/flux tube junction.
Suppose to have a theory with two gauge groups $U(1)_1$ and $U(1)_2$, and two discrete vacua $A$ and $B$. For the first $U(1)_1$ both vacua are in the Higgs phase and, on the domain wall interpolating between the two, there is a localization of the gauge field. Thus we have a wall/vortex junction like a D-brane. For the second group $U(1)_2$ the vacuum $A$ is in the Higgs phase, while vacuum $B$ is in the Coulomb phase and we can have a vortex made of wall like the one studied in Section 2. The vortex in vacuum $A$ that carries the magnetic flux of $U(1)_1$, can end on the domain wall and, as a consequence, can also end on the magnetic vortex that carries flux under $U(1)_2$. In fact the last one is nothing but the wall rolled in a cylinder. In this way we we have obtained a “Y” junction.

Now we are going to show take a particular confining $SU(3)$ gauge theory that admits the Y junction previously constructed as its baryon vertex.

The theory that we consider is $\mathcal{N} = 2$ $SU(3)$ SQCD with two hypermultiplets in the foundamental representation. the hypermultiplets have mass $m_A$ and $m_B$. In a generic point of the moduli space (2 complex dimensions) the theory is in a $U(1)_1 \times U(1)_2$ Coulomb phase. At some critical lines (1 complex dimension), there are masses hypermultiplets for one of the $U(1)$’s. At some critical points (0 complex dimensions), both the $U(1)$’s have charged massless hypermultiplets. We take vacuum $A$ to be the critical point where $U(1)_1$ is locked to the flavor $m_A$, and some massless monopole is charged under $U(1)_2$. We choose vacuum $B$ to be on the critical line where $U(1)_1$ is locked to $m_2$, while $U(1)_2$ has no massless charged particles. We know that is possible to choose some superpotential $W(\Phi)$ so that vacua $A$ and $B$ are among the ones that survive the perturbation. After the perturbation, charged massless particles condense and creates flux tubes. If these particles have magnetic charge, the flux tubes carry electric flux and are responsible for confinement.

It is possible to send the mass $m_1$ into the strong coupling region ($\sim \Lambda$), so that the charge of the locked massless flavor becomes magnetic, and vacuum $A$ is in the confining phase.

With respect to $U(1)_1$ both vacua are both in the Higgs phase but the breaking of the gauge group is due to two different charged particles. Thanks to the same mechanism of [5], the $U(1)_1$ gauge field is localized on the wall that interpolates between $A$ and $B$. With respect to $U(1)_2$, vacuum $A$ is in the Higgs phase while vacuum $B$ in the Coulomb phase. The two flux tubes in vacuum $A$ can thus form a Y junction like the one previously described. So we can have a baryon vertex like the
one of Figure 21

Figure 21: A Y junction for the baryon vertex. The three quarks are chosen so that the fundamental of $SU(3)$ is $(Q_1, Q_2, Q_3)$. $Q_1$ and $Q_2$ are connected by a wall vortex that carries the flux of the $t_3$ generator. The quark $Q_3$ is connected to another tube that carries the $t_8$ charge makes a junction like $[92x796]3.1.1$ with the wall vortex. The charge of $Q_3$ is divided in two equal parts when the $t_8$ tube join the $t_3$ wall vortex, and this is consistent with the fact that $t_8 \propto \text{diag}(1, 1, -2)$.

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