Strata of vector spaces of forms in $R = k[x, y]$, and of rational curves in $\mathbb{P}^k$.

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Abstract

Consider the polynomial ring $R = k[x, y]$ over an infinite field $k$ and the subspace $R_j$ of degree-$j$ homogeneous polynomials. The Grassmanian $G = \text{Grass}(R_j, d)$ parametrizes the vector spaces $V \subset R_j$ having dimension $d$. The strata $\text{Grass}_H(R_j, d) \subset G$ determined by the Hilbert function $H = H(R/(V))$ or, equivalently, by the Betti numbers of the algebra $R/(V)$, are locally closed and irreducible of known dimension. They satisfy a frontier property that the closure of a stratum is its union with lower strata $[I1, I2]$. The strata are determined also by the decomposition of the restricted tangent bundle $T_V = \phi^*_V(T)$ where $T$ is the tangent bundle on $\mathbb{P}^{d-1}$ and $\phi_V$ is the rational curve determined by $V$. Each stratum corresponds to a partition, and the poset of strata under closure is isomorphic to the poset of corresponding partitions in the Bruhat order. They are coarser than the strata defined by D. Cox, A. Kustin, C. Polini, and B. Ulrich [CKPU] and studied further in [KPU], that are determined in part by singularities of $\phi_V$. We explain these results and give examples to make them more accessible. We also generalize a result of D. Cox, T. Sederberg, and F. Chen and another of C. D’Andrea concerning the dimension and closure of $\mu$ families of parametrized rational curves from planar [CSC, D] to higher dimensional embeddings (Theorem 1.11).

Consider the polynomial ring $R = k[x, y]$ with maximum ideal $m = (x,y)$ over an infinite field $k$, and write $R = \bigoplus_{j=0}^{\infty} R_j$. Let $V \subset R_j$ be a $d$-dimensional vector space of degree-$j$ forms; then $V$ determines a rational curve $\phi_V : \mathbb{P}^1 \to \mathbb{P}^{d-1}$. We denote by $\text{Grass}(R_j, d)$ the Grassmann variety parametrizing all such $d$-dimensional subspaces of $R_j$. The Hilbert function $H(A), A = R/(V)$ is the sequence

$$H(A) = (1, 2, \ldots, j, j + 1 - d, \ldots, h_i, \ldots), h_i = \dim_k(A_i).$$

The sequences $H$ possible for such a Hilbert function are well-known (Lemma 0.2). We denote by $\text{Grass}_H(R_j, d) \subset \text{Grass}(R_j, d)$ the corresponding parameter space.

C. Polini described in her talk [Pol] her study with collaborators D. Cox, A. Kustin, and B. Ulrich [CKPU] [KPU] of certain locally closed subfamilies of $\text{Grass}(R_j, d)$ determined by the singularity types of the associated rational planar and space curves $\phi_V$. They relate these
singularity types to strata of normalized relation matrices for the ideal \((V)\) by relation degrees, then by zeroes of entries and equalities between entries. The purpose of this note is to recall and illustrate some of the properties of the coarser stratification by Hilbert function \(H(R/(V))\) – or relation degrees of \((V)\) – studied in [I1] GIS [I2]. We hope that others might wish to extend these combinatorial descriptions and deformation properties to the finer stratification described by C. Polini in her talk. In section [I1] we generalize a result of D. Cox, T. Sederberg, and F. Chen [CSC], and of C. D’Andrea [D] on \(\mu\)-bases to higher embedding dimensions. In Section 1 we briefly mention analogous results for rational scrolls.

We denote by \(T\) the tangent bundle on \(\mathbb{P}^{d-1}\). The restricted tangent bundle \(T_{V} = \phi^{*}_{V}(T)\) on \(\mathbb{P}^{1}\) has a decomposition \(T_{V} = \oplus \mathcal{O}(k_{i})\) into a direct sum of line bundles: knowing the degrees \(K = \{k_{i}\}\) of the summands is equivalent to knowing the Betti numbers in a minimal resolution of a basis of \(V\), or equivalently, to knowing the column degrees \(D\) of the \(d \times (d-1)\) Hilbert-Burch relation matrix \(M\) among the generators. This decomposition is also equivalent to simply knowing the Hilbert function of the algebra \(R/(V)\) [Asc1 Asc2 GIS CHI Ber2].

There have been many further studies of the geometry of this decomposition, as [BR Ber1 Ber2 Cl EV1 EV2 G GS GHI He HeK Ka PS Ra1 Ra2 Ra3 Ran1 Ran2].

In [I1, Section 4B] and [I2] we studied the stratification of \(G = \text{Grass}(R_{j}, d)\) by the Hilbert function \(H(H(R/(V)))\). We determined which sequences \(H\) can occur, and gave a 1-1 correspondence \(H \rightarrow P_{H}\) between the Hilbert function strata Grass\(_{H}(R_{j}, d)\) and certain partitions of \(j + 1 - d\) [I2, Lemma 2.23] (see Theorem 1.6 below). We also determined the codimension of the \(H\) stratum in \(G\) [I1, Proposition 4.7], [I2 Thm. 2.17] (see (1.12) below).

We denote by \(G_{H}\) the projective variety parametrizing the graded ideals \(I \subset R\) such that \(H(R/I) = H\). We set \(c_{H} = \lim_{i \rightarrow \infty} h_{i};\) it is the degree of the common factor of each ideal \(I \in G_{H}\). Given a sequence \(H = (\cdots h_{i}, \cdots)\) let \(g(H) = \min\{i \mid h_{i} \neq i + 1\}\), and set \(E_{H} = (e_{0}, \ldots, e_{i}, \ldots), e_{i} = \Delta H = h_{i} - h_{i+1}\). Then \(H\) occurs as a Hilbert function \(H(R/I)\) for a graded ideal \(I\) of \(R\) if \(e_{i} \geq 0\) for \(i \geq g(H)\) [Mac]. We showed

**Theorem 0.1.** [I1, Theorem 2.12], [I2, Theorem 1.10]. The variety \(G_{H}\) when nonempty is an irreducible nonsingular projective variety of dimension

\[
\dim G_{H} = c_{H} + \sum_{i \geq g(H)} (e_{i} + 1) \cdot (e_{i+1}). \tag{0.2}
\]

Now we set \(\varrho = \), so \(E_{H} = (e_{j}, e_{j+1}, \ldots), e_{i} = h_{i} - h_{i+1}\), and let \(\tau_{H} = e_{j+1} + 1\). We denote by Grass\(_{H}(R_{j}, d)\) the subvariety of \(G = \text{Grass}(R_{j}, d)\) parametrizing the vector spaces \(V \subset R_{j}\) of Hilbert function \(H(R/(V)) = H\). It is locally closed in \(G\).

**Lemma 0.2.** [I1, Proposition 4.6] The stratum Grass\(_{H}(R_{j}) \subset G\) is nonempty if and only if \(1 \leq \tau_{H} \leq \min(d, j + 2 - d)\), and \(E_{H}\) is non-increasing. Then it is irreducible and open dense in \(G_{H}\).

We denote by \(\mathcal{H}(j, d)\) the set of sequences \(H = (h_{j}, h_{j+1}, \ldots)\) satisfying the conditions of Lemma 0.2. For \(H', H \in \mathcal{H}(j, d)\) we define the termwise partial order

\[
H' \geq H \iff h'_{i} \geq h_{i} \text{ for } j \leq i < \infty. \tag{0.3}
\]

These strata satisfy a frontier property that the closure of a stratum is a union of strata.

\(^{1}\)Warning: the \(H\) used here is written \(T\) in [I2]; it is the tail (higher degree) part of the Hilbert function of the full ancestor ideal, for which there are similar results.
Theorem 0.3. [I1, Theorem 4.10], [I2, Theorem 2.32] The closure $\text{Grass}_H(R_j, d)$ of each stratum of $G$ satisfies
\[
\text{Grass}_H(R_j, d) = \bigcup_{H' \in \mathcal{H}(j, d) \mid H' \geq H} \text{Grass}_{H'}(R_j, d)
\]
the union of the strata whose Hilbert function is greater or equal termwise to $H$. The projection $p : G_H \to \text{Grass}_H(R_j, d)$ where $p(I) = I_j$ is a desingularization of the Zariski closure $\text{Grass}_H(R_j, d) \subset G$.

Showing that $p$ is surjective involves constructing for each given ideal $I' = (V')$ of Hilbert function $H' = H(R/I')$ a new ideal $I \supset I'$ such that $H(R/I) = H$; this is accomplished step by step using properties of the invariant $\tau(V)$ of (1.3) below [I1, Proposition 4.9], [I2, Lemma 2.30]. The frontier property (0.4) then follows from the known non-singularity and irreducibility of $G_H$ (Theorem 0.1).

We here relate these results to the partitions determining the minimal resolution of $A = R/(V)$ and give examples that we hope will make these results more accessible.

1 Restricted tangent bundle strata of $\text{Grass}(R_j, d)$.

We first state some relevant results of [I2] in Theorem 1.6. We then illustrate these results by two examples: $(j, d) = (6, 3)$ (Example 1.8) and $(j, d) = (8, 3)$ in Example 1.9. The latter is the example C. Polini gave in her talk, that of a three-dimensional vector space $V = \langle f_1, f_2, f_3 \rangle$ with no common factor (base point). In her discussion the singularity type corresponds to several invariants of the $3 \times 2$ “Hilbert-Burch” matrix $M$ of relations among the generators of $I = (V) = (f_1, f_2, f_3)$: these include the set of zero entries and also the equalities between entries in a normal form for $M$, and as well more subtle invariants having to do with factors of the entries.

For $(j, d) = (8, 3)$, when there are no base points, the possibilities for the column degrees of $M$ are $(3, 3), (4, 2)$ and $(5, 1)$. C. Polini in her talk discussed the behavior under deformation of singularity types within the first, generic family of $(3, 3)$ relation degrees. Our approach applies to all pairs of positive integers $(j, d), d \leq j + 1$ and to all sequences $D$ but deals with the coarser invariant $H$. We begin by describing properties of the $\tau$ invariant: as we shall see, $d - \tau(V)$ is just the number of relations of $(V)$ in degree $j + 1$, so is equal to the number of 1’s in the relation degrees $D$ (equation (1.7)).

Definition 1.1. Let $V \subset R_j$ be a vector subspace. We denote by $\overline{V}$ the ancestor ideal of $V$:
\[
\overline{V} = V : R_j + \cdots + V : R_1 + (V),
\]
where $V : R_k = \{f \in R_{j-k} \mid R_k \cdot f \subset V\}$. It satisfies
\[
\overline{V} \cap m^2 = (V)
\]
and is the largest ideal satisfying (1.2). The $\tau$ invariant of $V$ is
\[
\tau(V) = \dim_k R_1 \cdot V - \dim_k V = 1 + \text{cod}_k V - \text{cod}_k R_1 V = 1 + e_{j+1}
\]

The definition of $\overline{V}$ is independent of the number of variables; but the results concerning $\tau(V)$ require the ring $R = k[x, y]$. 
where $\text{cod}_k V = j + 1 - d$ and $\text{cod}_k R_1 V = \dim_k R_{j+1}/R_1 V = H(R/(V))_{j+1}$.

**Lemma 1.2.** [I2, Lemma 2.2] The integer $\tau(V)$ is the number of generators of the ancestor ideal $\overline{V}$. We have

$$1 \leq \tau(V) \leq \min\{\dim_k V, 1 + \text{cod}_k V\}.$$  \hfill (1.4)

**Example 1.3.** For $(j, d) = (j, 3)$ we have $1 \leq \tau(V) \leq 3$:

- $\tau(V) = 1 \Leftrightarrow V = \langle x^2, xy, y^2 \rangle \cdot f$ where $f \in R_{j-2}$;
- $\tau(V) \leq 2 \Leftrightarrow V = \langle x \cdot f, y \cdot f, f_3 \rangle$ where $f \in R_{j-1}$;
- $\tau(V) = 3$ otherwise.

We denote by $\text{Grass}_\tau(R_j, d)$ the subfamily of $\text{Grass}(R_j, d)$ parametrizing vector spaces with $\tau(V) = \tau$. It is the subfamily of those $V$ with $d - \tau$ linear relations (see (1.9)).

**Lemma 1.4.** [I2, Theorem 2.17] The codimension of $\text{Grass}_\tau(R_j, d)$ in $\text{Grass}(R_j, d)$ satisfies

$$\text{cod} \text{Grass}_\tau(R_j, d) = (\dim_k V - \tau)(\dim_k V - (\tau - 1)) = (d - \tau)(j + 2 - d - \tau).$$  \hfill (1.5)

For an integer $n \in \mathbb{Z}$ we denote by $|n|^+$ the integer $n$ if $n \geq 0$ and 0 otherwise. We write $P \vdash n$ for “$P$ partitions $n$”. Recall the Bruhat or orbit closure partial order on partitions of $n$

$$P \geq P' \iff \forall k \in \mathbb{N}, \quad \sum_{i=0}^{k} P_i \geq \sum_{i=0}^{k} P'_i.$$  \hfill (1.6)

We denote by $\mathcal{P}(a)$ the poset of partitions of $a$ and by $\mathcal{P}(a, b)$ the poset of partitions of $a$ into exactly $b$ non-zero parts, under the Bruhat partial order; set $\mathcal{P}(a, \leq b) = \mathcal{P}(a, 1) \cup \cdots \cup \mathcal{P}(a, b)$. Recall that $c_H, H = H(R/(V))$ is the degree of $\gcd(V)$ – the number of base points.

**Lemma 1.5.** [I2, Lemma 2.23,(2.45)] The minimal resolutions that occur for algebras $R/(V)$ for which $\tau(V) = \tau$ and $c_H = 0$ can be written

$$0 \to \sum_{i=1}^{\tau-1} R(-j - 1 - \lambda_i) \oplus R(-j - 1)^{d-\tau} \to R(-j)^d \to R \to R/(V) \to 0,$$  \hfill (1.7)

and correspond 1-1 to the partitions

$$\lambda = (\lambda_1, \ldots, \lambda_{\tau-1}), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\tau-1} > 0 \text{ of } j + 1 - d \text{ into } \tau - 1 \text{ parts}.$$  \hfill (1.8)

We define the partition

$$D = D(\lambda) = (\lambda + 1, 1^{d-\tau}) = (\lambda_1 + 1, \ldots, \lambda_{\tau-1} + 1, 1, \ldots, 1)$$  \hfill (1.9)

of $j$ into $d - 1$ parts, which gives the complete set of relation degrees in (1.7), relative to $j$. Thus, $D$ is the column degrees of the Hilbert-Burch matrix $M_I$ used in [Pol] and determines a Hilbert function $H(D)$. We denote by $\text{MR}_D(R_j, d) = \text{Grass}_{H(D)}(R_j, d)$ the stratum of $\text{Grass}(d, R_j)$ parametrizing $V$ satisfying (1.7), with $D$ from (1.8), (1.9). When $c_H \neq 0$ there is a projection $V \to V : f$ where $f = \gcd(V)$. Then, $\lambda$ partitions $j + 1 - c_H - d$ into $\tau - 1$ parts and $D$ partitions $j - c_H$ into $d - 1$ parts. Note that $\tau(H) = 1 \Leftrightarrow c_H = j + 1 - d \neq 0$. We denote by $\lambda^\vee$ the conjugate partition to $\lambda$ – switch rows and columns in the Ferrers diagram of $\lambda$. 


Theorem 1.6. [Section 2.2, Lemma 2.23, Theorem 2.24, (2.48),(2.50), (2.54)]. The relation between the partition $\lambda$ and the corresponding Hilbert function $H_{\lambda} = H(R/(V))$ is

$$H_{\lambda} : \quad H_{\lambda,j+i} = c_{H} + \sum_{u} |\lambda_{u} - i|^{+} \text{ for } i \geq 0; \text{ also } \Delta H_{\geq j} = \lambda^{\vee}. \quad (1.10)$$

We have for $c_{H_{\lambda}} = c_{H'} = 0$,

$$H_{\lambda} \leq H_{\lambda'} \iff \lambda \leq \lambda' \iff \lambda^{\vee} \geq (\lambda')^{\vee} \iff D(\lambda) \leq D(\lambda') \quad (1.11)$$

in the Bruhat order of partitions.

The stratum $MR_{D}(R_{j},d)$ is irreducible and has codimension in $G = \text{Grass}(R_{j},d)$,

$$\ell(D) = c_{H(D)} \cdot (d - 1) + \sum_{u \leq v} (D_{u} - D_{v} - 1)^{+}. \quad (1.12)$$

Its codimension in $\text{Grass}_{\tau}(R_{j},d)$ is $\ell(\lambda) = c_{H(D)} \cdot (d - 1) + \sum_{u \leq v} (\lambda_{u} - \lambda - 1)^{+}$.

Theorem 0.3 and (1.11) of Theorem 1.6 imply

**Corollary 1.7.** Fix $(j,d)$. The poset of closures $\{\text{Grass}_{H_{\lambda}}(R_{j},d) \mid \lambda$ from (1.8) under inclusion is the poset of those Hilbert functions of Lemma 0.2 with $c_{H} = 0$ under the termwise partial order. It is isomorphic under the map $H_{\lambda} \to D(\lambda)$ to the poset $P(j,d - 1)$ and is isomorphic under the map $H_{\lambda} \to \lambda$ to the poset $P(j + 1 - d, (d - 1))$.

**Example 1.8.** (As in C. Polini’s talk). Let $(j,d) = (6,3)$, so $\text{cod}_{k}(V) = 6 + 1 - 3 = 4$, and $\dim\text{Grass}(R_{6},3) = 3 \cdot 4 = 12$. By equation (1.4) we have $1 \leq \tau(V) \leq 3$. Assume first that there are no base points: $c_{V} = 0$. When $\tau = 3$ the Hilbert functions $H_{\lambda}$ are determined by the partitions of $\text{cod}_{k}V = 4$ into $\tau - 1 = 2$ parts. The generic case has partition $\lambda = (2,2)$, and corresponds to relation degrees $D = (3,3)$ and to Hilbert function $(H_{\lambda})_{\geq 6} = (4,2,0)$. The special case for $\tau = 3$ has partition $\lambda' = (3,1)$, and corresponds to relation degrees $D' = (4,2)$ and to $(H_{\lambda'})_{\geq 6} = (4,2,1,0)$. By (1.12) $\text{Grass}_{H_{\lambda'}}(R_{j},d)$ has codimension 1 in $\text{Grass}(R_{6},3)$. When $\tau = 2$ the unique partition is $\lambda = (4)$: the relation degrees $D'' = (5,1)$, the Hilbert function $(H_{\lambda''})_{\geq 6} = (4,3,2,1,0)$; and by (1.5) $\text{Grass}_{H_{\lambda''}}(R_{6},3) = \text{Grass}_{2}(R_{6},3)$ has codimension 3 in $\text{Grass}(R_{6},3)$. Here $D''$ is the most special case with $c_{H} = 0$.

To summarize, an open dense $U \subset \text{Grass}(R_{6},3)$ parametrizes $V$ having no base points, and is decomposed into three $H$-strata corresponding to the three partitions $\lambda, \lambda', \lambda''$ of four or, equivalently by Corollary 1.7, the partitions $D, D'$ and $D''$ of six into 2 parts.

When $c_{H} = 1$ there is a single base point: then $V = \ell \cdot V'$ where $\ell \in R_{1}$ and $V' \in \text{Grass}(R_{5},3)$ has codimension 3 in $R_{5}$. The two strata correspond to the two partitions of $3 = \text{cod}_{k}V'$ into at most $\text{cod}_{k} - 1 = 2$ parts: so $\tau = 3 : \lambda''' = (2,1)$ and $(H_{\lambda''''})_{\geq 6} = (4,2,1)$; and $\tau = 2 : \lambda^{(4)} = (3)$ and $(H_{\lambda^{(4)}})_{\geq 6} = (4,3,2,1)$.

When $c_{H} = 2$ the two Hilbert functions are $H_{\geq 6} = (4,2)$ for $\tau = 3$ and partition $\lambda = (1,1)$ and $H_{\geq 6} = (4,3,2)$ for $\tau = 2$ and $\lambda = (2)$. There are two more (see Table 1.1). Theorem 0.3 and a comparison of the Hilbert functions show that there are two maximal chains in $H(6,3)$, between $H_{\geq 6} = (4,3,2)$ and $(4,2,1,0)$, one corresponding to the chain $\lambda^{\vee} = (1,1) \leq (1,1,1) \leq (1,1,1,1) \leq (2,1,1)$ and the other to the chain $\lambda^{\vee} = (1,1) \leq (2) \leq (2,1) \leq (2,1,1)$. 

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we homogenize with respect to a new variable relatively prime triples $(P_{\lambda}$, then $c_H = 0$ the partitions $\lambda$ are $(3,3) \leq (4,2) \leq (5,1) \leq (6)$ with the $(3,3)$ stratum being open-dense. The Hilbert functions $H(R/(V))_{\geq 8}$ are specified in Table 1.2. Here the conjugate partition $\lambda'$ satisfies

$$\lambda' = \Delta H_{\geq 8} = (e_8(H), e_9(H), \ldots)$$

(1.13)

the first differences of $H_{\geq 8}$. Since $e_8 = 2$, we have for the $\lambda = (5,1)$ stratum $H_{\geq 8} = (6,4,3,2,1,0)$ that $\lambda' = (2,1,1,1,1)$ and by (0.2),

$$\dim \text{Grass}_H(R_8,3) = (3 \cdot 2 + 3 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1) = 15.$$

The closure of the $\lambda = (5,1)$ stratum is its union with the eleven strata whose Hilbert functions $H_{\geq 8}$ lie above $(6,4,3,2,1,0)$: we have indicated these strata with $\ast$ in the rightmost column of Table 1.2. By Theorem 1.6 this closure has desingularization the family $G_H, H = (1,2,3,4,5,6,7,8,6,4,3,2,1,0) -$ the family of all graded ideals in $R$ satisfying $H(R/I) = H$.

Example 1.9. For $(j,d) = (8,3)$ we have $\tau \leq 3$ so for $c_H = 0$ the partitions $\lambda$ are $(3,3) \leq (4,2) \leq (5,1) \leq (6)$ with the $(3,3)$ stratum being open-dense. The Hilbert functions $H(R/(V))_{\geq 8}$ are specified in Table 1.2. Here the conjugate partition $\lambda'$ satisfies

$$\lambda' = \Delta H_{\geq 8} = (e_8(H), e_9(H), \ldots)$$

(1.13)

the first differences of $H_{\geq 8}$. Since $e_8 = 2$, we have for the $\lambda = (5,1)$ stratum $H_{\geq 8} = (6,4,3,2,1,0)$ that $\lambda' = (2,1,1,1,1)$ and by (0.2),

$$\dim \text{Grass}_H(R_8,3) = (3 \cdot 2 + 3 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1) = 15.$$

Example 1.10. For $(j,d) = (9,4)$ the poset of $H$-strata with $c_H = 0$ under closure is by Corollary 1.7 isomorphic to the poset $P(6, \leq 3)$ of partitions $\lambda$ of 6 into at most 3 parts, or, equivalently, to the poset $P(9,3)$ of partitions $D$ of 9 into exactly 3 parts. The strata for $\lambda = (4,1,1)$ and for $\lambda' = (3,3)$ are incomparable: $H_{\geq 9}(\lambda) = (6,3,2,1,0)$ and $H_{\geq 9}(\lambda') = (6,4,2,0)$. By Theorems 0.3 and 1.6 the incomparability of partitions $\lambda, \lambda'$ in the Bruhat order is equivalent to incomparability of the Hilbert functions $H_\lambda, H_{\lambda'}$ in the termwise partial order, and implies that no closure of a subfamily of one of the strata can intersect the other stratum.

1.1 Parametrization of rational curves

David Cox after seeing an earlier version of this note remarked on the connections to his work with T. Sederberg and F. Chen. He writes that “the paper [CSC] defines $P_n$ as the set of all relatively prime triples $(a,b,c)$ with $a,b,c \in k[t]$ and $n = \max(\deg(a), \deg(b), \deg(c))$. Thus, if we homogenize with respect to a new variable $u$, then $P_n$ is canonically isomorphic to the set

$$\{(A,B,C) \in k[t,u]^3 \mid A, B, C \text{ homogeneous of degree } n \text{ and } gcd(A, B, C) = 1\}$$

(1.14)
Table 1.2: The Grass$H(R_8, 3)$ strata.

Then the subset $P^\mu_n$ is naturally isomorphic to the subset of $P_n$ consisting of triples $(A, B, C)$ for which the ideal $I = (A, B, C)$ has a free resolution, taking here $R = k[t, u]$.

$$0 \to R(-n - \mu) + R(-2n + \mu) \to R(-n)^3 \to 0.$$ (1.15)

with $\mu \leq n - \mu.$” Their main result is the irreducibility and dimension of the closure $\overline{P}^\mu_n$.

**Theorem.** [CSC, Theorem 1.1] For each $\mu, 0 \leq \mu \leq \lfloor n/2 \rfloor$ the closure $\overline{P}^\mu_n$ is irreducible of dimension

$$\dim \overline{P}^\mu_n = \begin{cases} 3n + 3 & \text{if } \mu = \lfloor n/2 \rfloor \\ 2n + 2\mu + 4 & \text{if } \mu < \lfloor n/2 \rfloor \end{cases}$$ (1.16)

Carlos D’Andrea then gave an explicit approximation of $P^\nu_n$ by a sequence in $P^{\nu+1}_n$, showing

**Theorem.** [D, Thm. 1.2, (2)] For $0 \leq \mu \leq \lfloor n/2 \rfloor$ the closure $\overline{P}^\mu_n$ in $P_n$ satisfies

$$\overline{P}^\mu_n = \bigcup_{\nu \leq \mu} P^\nu_n.$$ (1.17)

We now compare to our notation: given $(n, \mu)$ the relation degree partition is $D = (d_1, d_2) = (n - \mu, \mu)$, so by [1.12] the codimension of the stratum MR$_D(R_j, \overline{d})$ is $(n - 2\mu - 1)^+$: this is the codimension $3n + 3 - (2n + 2\mu + 4)$ given in (1.16). Next, we generalize.
The affine space $A^N, N = d(n+1)$ parametrizes $d$-tuples of degree-$n$ polynomials in $k[x]$ and an open dense $A^N_U$ parametrizes those that form linearly independent sets. Given a partition $D$ of an integer $k, d-1 \leq k \leq n$ into $d-1$ parts, we define a subscheme of $A^N_U$,

$$
\mathcal{CP}^D_n = \{(A = (a_1, a_2, \ldots, a_d), a_i \in k[x]) \subset A^N_U, N = d(n+1),
$$

(1.18)

comprised of those ordered sequences of $d$ linearly independent polynomials in $k[x]$, each of degree less or equal $n$, that parametrize rational curves in $\mathbb{P}^{d-1}$ whose relation degrees are $D$. Here $k = n - c$ where $c$ is the degree of $\gcd(A)$. When $D$ partitions $n$ (we write $D \vdash n$) the open dense subscheme $\mathcal{P}^D_n \subset \mathcal{CP}^D_n$ parametrizes those $d$-tuples that are relatively prime and have highest degree $n$; we set $\mathcal{P}^D_{n,d} = \bigcup_{D \vdash n} \mathcal{P}^D_n$. The general linear group $\text{Gl}_d(k)$ acts freely on $\mathcal{CP}^D_n$ and we have a surjective projection with fibre $\text{Gl}_d(k)$

$$
\pi : \mathcal{CP}^D_n \to \text{MR}_D(R_j, d) : \pi(A) = \langle A \rangle.
$$

(1.19)

**Theorem 1.11.** Let $D$ partition $n-c$ into $d-1$ non-zero parts. Then $\mathcal{CP}^D_n$ is an irreducible locally closed subvariety of $A^N$ having dimension (where $\ell(D)$ is from (1.12))

$$
N - \ell(D).
$$

(1.20)

The closure $\overline{\mathcal{CP}^D_n}$ in $A^{(n+1)d}_U$ satisfies

$$
\overline{\mathcal{CP}^D_n} = \bigcup_{H(D') \geq H(D)} \overline{\mathcal{CP}^{D'}_n}.
$$

(1.21)

For $D \vdash n$ the closure of $\mathcal{P}^D_n$ in $\mathcal{P}^D_{n,d}$ is $\overline{\mathcal{P}^D_n} = \bigcup_{D' \geq D, D' \vdash n} \overline{\mathcal{P}^{D'}_n}$.

**Proof.** This is a consequence of Theorems 0.3 and 1.6 and the property of the fibration $\pi$, that preserves codimensions and closures. \qed

There are now many articles on $\mu$ bases, as [SC, WJG] for space curves. We hope this extension to all embedding dimensions might be useful.

## 2 Rational scroll strata for vector spaces of forms.

Recall that the ancestor ideal $V$ defined in (1.1) is the largest ideal satisfying $V \cap m^j = (V)$. We define the “nose” or “scroll” Hilbert function $N(V)$:

$$
N(V) = H_{\leq j}(R/V),
$$

(2.1)

which satisfies $e_{i-1}(N) \leq e_i(N)$ for $i < j$. Let $N(j, d)$ denote the set of possible nose sequences. Here $N$ is determined by a partition $A_N$ of $d$ into $\tau$ parts. Setting $I = V$, we have

$$
\dim_k I_{j-i} = \sum_u | a_u - i |^+.
$$

(2.2)

The partition invariant $A_V = A_{N(V)}$ determines the minimum rational scroll on which the rational curve determined by $V$ lies. Here $N'$ more special is equivalent to $N' \leq N$ and
\[\begin{array}{cccccc}
A & \tau & c_H & H_{6,7,8,9} & \dim_k V_{6,7,8,9} & A^\vee & \dim \\
(1,1,1,1) & 4 & 0 & (7,8,9,6) & (0,0,0,4) & (4) & 24 \\
(2,1,1) & 3 & 0 & (7,8,8,6) & (0,0,1,4) & (3,1) & 20 \\
(2,2) & 2 & 0 & (7,8,7,6) & (0,0,2,4) & (2,2) & 14 \\
(3,1) & 2 & 0 & (7,7,7,6) & (0,1,2,4) & (2,1,1) & 13 \\
(4) & 1 & 6 & (6,6,6,6) & (1,2,3,4) & (1,1,1,1) & 6 \\
\end{array}\]

Table 2.1: Nose Hilbert function Grass\(_{N_A}(R_9, 4)\) strata.

\(A' \geq A\), the former from the inequality \(H' \geq H\) in Lemma \[0.3\]. We have \[12\] Lemma 2.30, Theorem 2.32] the analogue for \(N\) of Theorem \[0.3\],

\[
\text{Grass}_N(R_j, d) = \bigcup_{N' \in N(j,d), N' \leq N} \text{Grass}_{N'}(R_j, d). \quad (2.3)
\]

The codimension of Grass\(_{N_A}(R_j, d)\) in Grass\(_{\tau}(R_j, d)\) satisfies (ibid. Theorem 2.24)

\[
\text{codim} \text{Grass}_{N_A}(R_j, d) = \ell(A) = \sum_{u \leq v} (a_u - a_v - 1)^+. \quad (2.4)
\]

We showed analogues in \[12\] for \(N_A\) and also for the pair \((N_A, H_A)\) of each statement in Theorem \[1.6\]. We state the codimension formulas in Grass\(_{\tau}(R_j, d)\) since the Grass\(_{N_A}(R_j, d)\) and Grass\(_{H_V}(R_j, d)\) strata intersect properly in their corresponding Grass\(_{\tau}(R_j, d)\).

**Example 2.1.** Let \(j = 9, d = 4\), and fix \(\tau = 3\). Consider the partition \(A = (2,1,1)\), where \(\dim_k V_{6,7,8,9} = (0,0,1,4)\) and \(N_V = (1,2,3,4,5,6,7,8,8,6)\). The closure of the stratum \(N_A\) is all strata \(N_A'\) that are termwise smaller or equal to \(N_A\): such that \(A' \geq A\) in the orbit closure order of partitions of 4. A generic element \(V_{f,W}\) of Grass\(_{N_A}(R_9, 4)\) is determined by a pair \((f,W)\) \mid \(f \in R_8\) is generic, and \(W\) is a generic 2-dimensional subspace of \(R_9/(R_1 \cdot f))\): so Grass\(_{N_A}(R_9, 4)\) is fibred over \(\mathbb{P}_8 = \mathbb{P}(R_8)\) by an open in the Grassmanian Grass\(_{8}(2,2)\). Its dimension is \(8 + 12 = 20\), which agrees with the dimension of Grass\(_{3}(R_9, 4)\) from \[1.5\]. Similar arguments give the structure of other nose strata of Grass\(_{\tau}(R_9, 4)\).

### 2.1 Problems

We pose some open questions that warrant further exploration.

**Question 1.** How do the frontier and desingularization properties of the stratification of Grass\(_{\tau}(R_j, d)\) by \{Grass\(_H(R_j, d)\)\} extend to the finer stratification of rational curves by singularity types?

**Question 2.** The behavior of the closures Grass\(_H(R_j, d)\) as the limit vector spaces pick up base points is described in Theorems \[0.3\] and \[1.6\]. How does this extend to the finer stratification?
**Question 3.** The variety structure of the closures $\text{Grass}_H(R_j, d)$ is not known; these are not Schubert varieties [I1] (the count is wrong) but are they related to Schubert varieties? What are their cohomology classes in $G$? We know $\text{Grass}_X(R_j, d)$ (Section 2) and $\text{Grass}_H(R_j, d)$, intersect properly on $\text{Grass}_r(R_j, d)$. Do they intersect transversely? Apply [HN]?

**Question 4.** Given an Artin algebra $A = R/I$ and a linear element $\ell \in R_1$ there is a partition $\pi_{\ell, A}$ giving the Jordan type of multiplication by $\ell$ in $A$ [HW, BIK]. The set $\pi(A) = \{\pi_{\ell, A} | \ell \in R_1\}$. How does $\pi(A), A = R/(V)$ behave as $V$ deforms in $\text{Grass}(R_j, d)$?

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