Classical mechanics is not $\hbar \to 0$ limit of quantum mechanics

O. V. Man'ko, and V. I. Man’ko
Lebedev Physical Institute, Moscow, Leninskii pr., 53

Abstract
Both the set of quantum states and the set of classical states described by symplectic tomographic probability distributions (tomograms) are studied. It is shown that the sets have common part but there exist tomograms of classical states which are not admissible in quantum mechanics and vica versa, there exist tomograms of quantum states which are not admissible in classical mechanics. Role of different transformations of reference frames in phase space of classical and quantum systems (scaling and rotation) determining the admissibility of the tomograms as well as the role of quantum uncertainty relations is elucidated. Union of all admissible tomograms of both quantum and classical states is discussed in context of interaction of quantum and classical systems. Negative probabilities in classical mechanics and in quantum mechanics corresponding to the tomograms of classical states and quantum states are compared with properties of nonpositive and nonnegative density operators, respectively.

1 Introduction
The classical mechanics in quantum-like description was considered in [1]-[3]. The possibility to combine description of quantum and classical degrees of freedom was discussed in [4]. In standard formulation of classical mechanics of systems with fluctuations (classical statistical mechanics) one uses the notion of the system state expressed in terms of probability distribution function $f(q, p)$ in the phase space of the system. Here $q$ is the position and $p$ is the momentum, which are supposed to be simultaneously measurable. In standard formulation of quantum mechanics the notion of the system state is expressed in terms of complex wave function $\psi(q)$ which is interpreted as a "probability wave" (pure state). In case of mixed state it is described by density matrix which is a complex function of two variables $\rho(q, q')$ considered as matrix element $<q|\rho|q'>$ of the nonnegative hermitian density operator $\hat{\rho}$ in position representation. Recently a new formulation of notion of state in classical mechanics and in quantum mechanics was suggested [5, 1]. The state is described by positive measurable probability distribution function $w(X, \mu, \nu)$ (called tomogram) of random variable $X$ which has the physical meaning of the system position. This position $X$ is determined in the reference frame in the system phase space and the axis of the reference frame are scaled and rotated. The real parameters $\mu$ and $\nu$ label the reference frame in the phase space and the scaling parameters $\lambda$ and rotation angle $\theta$ are connected with the parameters $\mu$ and $\nu$ by the relation:

$$\mu = e^\lambda \cos \theta \quad \nu = e^{-\lambda} \sin \theta.$$ 

The position and momentum in quantum mechanics obey to uncertainty relation which prohibits simultaneous measuring these physical observables. The wave function, density matrix and tomogram of a quantum state depend on Planck constant $\hbar$. The Planck constant determines the uncertainty of the position and momentum in quantum mechanics. The less is the value of the Planck constant considered as a parameter, the less is the uncertainty of the
position and momentum connected with the quantum fluctuations. It means that one can scale Planck constant $\hbar \to \epsilon \hbar$. For $\epsilon = -1$ this transform is connected with time reverse or with transposition of density matrix. The Planck constant is not involved into description of states in classical mechanics. There is a common vision of the connection of the quantum and classical mechanics. In this common vision the classical mechanics is the limit of the quantum mechanics for Planck constant equal to zero. It means that condition $\hbar = 0$ reconstructs the simultaneous measurability of the position and momentum. The aim of our work is to discuss this problem. We will show that the picture of mutual relation of the quantum mechanics and classical mechanics differs from this simple $\hbar \to 0$ limit picture. The reason is that the sets of states (sets of tomograms) in classical and quantum mechanics are different though they have the common part. In the $\hbar \to 0$ limit set of quantum states does not coincide with the set of classical states. This picture became possible to clarify namely in the tomographic probability representation [5]-[18] for which both classical and quantum states are described by the same object (tomographic probability distribution). In representations using different objects (wave function and density matrix in quantum domain and probability distribution in phase space in classical domain) the comparison of situation in the limit $\hbar \to 0$ in the density matrix with classical situation is difficult.

Another goal of our paper is to construct formalism of classical mechanics using the operators in Hilbert space and their tomographic symbols like in quantum domain. The difference with quantum mechanics is connected with difference of product of the operators (star-product) used in classical domain which is commutative in this case. We will show that the notion of negative probabilities which usually is associated to the Wigner function in quantum mechanics can naturally appear in classical domain too. This follows from the fact that in quantum-like formalism of classical mechanics the density operators describing the classical states are hermitian but can be nonpositive. This means that in their spectral decomposition the eigenvalues (playing the role of probabilities) can take negative values. Thus one has interesting duality. In classical mechanics the probability distributions describing the classical states are nonnegative functions. But corresponding density operators describing the classical states in framework of quantum-like formalism can be nonpositive. In quantum domain the situation is opposite. The density operators describing the quantum states are nonnegative (eigenvalues in the spectral decomposition of the density operators playing the role of probabilities are mandatory positive). But the Wigner functions which play the role of probability density can take negative values in some domains of phase space.

2 Classical states and probability distribution in phase space

First we discuss notion of classical states. For simplicity we consider a particle with one degree of freedom with mass $m = 1$ which has position $-\infty < q < \infty$ and the velocity $-\infty < \dot{q} < \infty$. The state of the particle is completely determined by these two quantities $q$ and $\dot{q}$. We introduce the momentum $p = m\dot{q}$ which in our units ($m = 1$) gives $p = \dot{q}$. Thus the state of the particle is identified with the point in phase space (plane) with coordinates $q, p$. Evolution of the particle state is described by a trajectory in the phase space $q(t), p(t)$. If one has the classical particle
inside of some environment (interacting with other particles, e.g. in a gas with temperature $T$) the position $q$ and momentum $p$ fluctuate. In view of these fluctuations the state of the particle is described by a probability distribution function $f(q, p)$ which is nonnegative

$$f(q, p) \geq 0$$

and normalized

$$\int f(q, p) \, dq \, dp = 1.$$ (2)

All the states of classical particles are described by the probability distribution functions belonging to a set. There are some other properties of the probability distribution functions from this set. These properties seem to be obvious according to our classical intuition. For example, we assume that to a state of the particle with velocity $\dot{q}$ (momentum $p$) and position $q$ corresponds another state with opposite velocity $-\dot{q}$ (momentum $-p$) and the same position $q$. In the case of states with fluctuations it means that if the function $f(q, p)$ belongs to the set of admissible probability distribution functions the function $f_l(q, p) = f(q, -p)$ also belongs to the set. It is the property of time reversibility which we assume for the classical states. Analogously, the reflection in mirror operation for the position $q \rightarrow -q$ combined with unchanged momentum provides the following property. If a function $f(q, p)$ belongs to the set of admissible classical probability distributions the function $f_-(q, p) = f(-q, p)$ also belongs to this set. Combination of these two properties provides the result that if the function $f(q, p)$ belongs to the set of the probability distributions the function $f(-q, -p)$ also belongs to the same set. The possibility to shift the origin of the reference frame in the phase space of the particle which we accept intuitively as obvious one means that the probability distribution function $f(q + q_o, p + p_o)$ with arbitrary real shift parameters $q_o$ and $p_o$ also belongs to the set of admissible classical probability distributions. The discussed transforms belong to real symplectic group acting on classical phase space with adding the reflections.

3 Scaling transform

Now we will discuss the behaviour of probability distribution function of a classical state with respect to independent scaling of position and momentum

$$q \rightarrow \lambda_q q,$$
$$p \rightarrow \lambda_p p,$$ (3)

where $\lambda_q$ and $\lambda_p$ are arbitrary real parameters. In fact, we consider reference frames in phase space of classical particle where the position and momentum axes are scaled. The transform (3) provides the change of probability distribution function

$$f(q, p) \rightarrow f_s(q, p) = N(\lambda_q, \lambda_p) f(\lambda_q q, \lambda_p p).$$ (4)

Normalisation constant $N(\lambda_q, \lambda_p)$ gives the equality

$$\int f_s(q, p) dq \, dp = 1,$$ (5)
and it reads
\[ N(\lambda_q, \lambda_p) = |\lambda_q \lambda_p| \] (6)

In integral form one has the relation
\[ f_s(q, p) = \int |\lambda_q \lambda_p| f(q', p') \delta(q' - \lambda_q q) \delta(p' - \lambda_p p) dq' dp', \] (7)

where the kernel of the integral scaling transform reads
\[ K_s(q, p, q', p') = |\lambda_q \lambda_p| \delta(q' - \lambda_q q) \delta(p' - \lambda_p p). \] (8)

The change of signs of \( q \) and \( p \) is the partial case of the transform (7) for \( \lambda_q = -1, \lambda_p = -1 \). In classical case if the distribution function \( f(q, p) \) in phase space is admissible the functions \( f_s(q, p) \) (7) are also admissible for all real values of parameters \( \lambda_q \neq 0 \) and \( \lambda_p \neq 0 \). The real numbers form two-parameter commutative Lie group with product of group elements defined by standard multiplication rule of the real numbers. Geometrically the real numbers are determined by points on the plane \( \lambda_q, \lambda_p \) with excluded points on the axes.

If the variances and covariance of two fluctuating observables \( q \) and \( p \) in the state with the probability distribution density \( f(q, p) \) are equal \( \sigma_{qq}, \sigma_{pp} \) and \( \sigma_{qp} \), respectively, the variances and covariance in the state with the probability distribution density \( f_s(q, p) \) (7) read
\[ \sigma_{qq}^{(s)} = \lambda_q^{-2} \sigma_{qq}, \sigma_{pp}^{(s)} = \lambda_p^{-2} \sigma_{pp}, \sigma_{qp}^{(s)} = (\lambda_q \lambda_p)^{-1} \sigma_{qp}. \] (9)

For symmetric dispersion matrix of the classical state with probability distribution \( f(q, p) \), i.e.,
\[ \sigma = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{pp} \end{pmatrix} \] (10)

one can calculate the determinant
\[ d = \sigma_{qq} \sigma_{pp} - \sigma_{qp}^2 \] (11)

and the trace
\[ T = \sigma_{qq} + \sigma_{pp} \] (12)

The dispersion matrix is nonnegative, i.e.,
\[ \sigma_{qq} \geq 0, \sigma_{pp} \geq 0, d \geq 0, T \geq 0. \] (13)

For the classical state with the probability distribution \( f_s(q, p) \) one has
\[ d^{(s)} = |\lambda_q \lambda_p|^{-2} d, \quad T^{(s)} = \lambda_q^{-2} \sigma_{qq} + \lambda_p^{-2} \sigma_{pp}. \] (14)

For all classical states of the particle the parameters \( d^{(s)} \) and \( T^{(s)} \) satisfy the inequalities (13). This property corresponds to the obvious statement that in classical case one can have states without fluctuations and correlations of position and momentum as well as the states with fluctuations and correlations obeying only to constraints (13).
4 Tomographic representation

Let us discuss the property of classical states considered in previous sections using tomographic probability representation introduced in [1]. Following this work we introduce the tomogram (called also tomographic probability distribution or marginal distribution)

\[ \omega(X, \mu, \nu) = \int f(q, p)\delta(X - \mu q - \nu p)dq dp, \]  

(15)

where \( \mu \) and \( \nu \) are real parameters \(-\infty < \mu, \nu < \infty\). One has normalisation condition

\[ \int \omega(X, \mu, \nu) dX = 1 \]  

(16)

for arbitrary parameters \( \mu \) and \( \nu \). Also one has the inverse formula

\[ f(q, p) = \frac{1}{2\pi} \int \omega(X, \mu, \nu)e^{i(X-\mu q - \nu p)}dX d\mu d\nu. \]  

(17)

The tomogram has the homogeneity property

\[ \omega(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|}\omega(X, \mu, \nu). \]  

(18)

It is equal to probability density of the position \( X \) measured in reference frame in phase space with rotated and scaled axes. The rotation angle \( \theta \) and scaling parameter \( e^\lambda \) are connected with parameters \( \mu \) and \( \nu \) as it is discussed in the Introduction. The tomogram completely determines the classical state of the particle. The dispersions of the position and momentum can be calculated using the dispersion of the variable \( X \) given by formula

\[ \sigma_{XX}(\mu, \nu) = \int X^2\omega(X, \mu, \nu)dX - (\int X\omega(X, \mu, \nu)dX)^2. \]  

(19)

Thus, one has

\[ \sigma_{qq} = \sigma_{XX}(1, 0), \sigma_{pp} = \sigma_{XX}(0, 1). \]  

(20)

The covariance of position and momentum has the form

\[ \sigma_{qp} = 2\pi \int dX \frac{\partial^2 \omega(X, \mu, \nu)}{\partial \mu \partial \nu} \bigg|_{\mu = 0 \atop \nu = 0} \]  

(21)

Let us discuss now the properties of the tomogram. The change \( p \to -p \) in the probability distribution provides the change \( \nu \to -\nu \) in the tomogram. The change \( q \to -q \) provides the change \( \mu \to -\mu \) in the tomogram. The scaling transform \( q \to \lambda_q q, p \to \lambda_p p \) and \( f(q, p) \to f_s(q, p) \) provides the transform of the tomogram

\[ \omega(X, \mu, \nu) \to \omega_s(X, \mu, \nu) = \omega(X, \frac{\mu}{\lambda_q}, \frac{\nu}{\lambda_p}). \]  

(22)

The kernel of the above scaling transform in integral form reads

\[ \omega(X, \mu, \nu) \to \omega_s(X, \mu, \nu) = \int \omega(X', \mu', \nu')K_s(X, \mu, \nu, X', \mu', \nu')dX', d\mu', d\nu', \]  

(23)

\[ K_s(X, \mu, \nu, X', \mu', \nu') = \delta(X - X')\delta(\frac{\mu}{\lambda_q} - \mu')\delta(\frac{\nu}{\lambda_p} - \nu'). \]
5 Uncertainty relations

In quantum mechanics there exist some inequalities called uncertainty relations. The most important are uncertainty relations for position and momentum introduced by Heisenberg (see, for example [19]) and by Schrodinger and Robertson [20, 21]. Review of the uncertainty relations is given in [22]-[24]. Let us derive the uncertainty relations. We use the obvious inequality that for an arbitrary operator $\hat{F}$ one has

$$<\hat{F}^+\hat{F}> \geq 0. \quad (24)$$

Here we consider the mean value either for pure state $|\psi>$, i.e.,

$$<\psi|\hat{F}^+\hat{F}|\psi> \geq 0, \quad (25)$$

or for mixed state with nonnegative density operator $\hat{\rho}$, i.e.

$$\text{Tr}(\hat{\rho}\hat{F}^+\hat{F}) \geq 0. \quad (26)$$

Let us construct a special operator $\hat{F}$ considering $N$ operators $\hat{Q}_\alpha$, where $\alpha = 1, 2, ..., N$, in the form

$$\hat{F} = \sum_{\alpha=1}^{N} C_\alpha \hat{Q}_\alpha. \quad (27)$$

Here $C_\alpha$ are arbitrary complex numbers. Inequality (24) yields

$$\sum_{\alpha,\beta=1}^{N} C_\alpha^* <\hat{Q}_\alpha^+\hat{Q}_\beta> C_\beta \geq 0. \quad (28)$$

Using the identity

$$\hat{Q}_\alpha^+\hat{Q}_\beta = \frac{1}{2}[\hat{Q}_\alpha^+\hat{Q}_\beta] + \frac{1}{2}\{\hat{Q}_\alpha^+\hat{Q}_\beta\} \quad (29)$$

and introducing the notation

$$\frac{1}{2} <\{\hat{Q}_\alpha^+\hat{Q}_\beta\}> = \sigma_{\alpha\beta}, \quad (30)$$

$$\frac{1}{2} <[\hat{Q}_\alpha^+\hat{Q}_\beta]> = \Sigma_{\alpha\beta}, \quad (31)$$

one has inequality (28) in the form

$$\sum_{\alpha,\beta=1}^{N} C_\alpha^*(\sigma_{\alpha\beta} + \Sigma_{\alpha\beta})C_\beta \geq 0. \quad (32)$$

The nonnegativity condition for this quadratic form implies the hermitian matrix of this form satisfies the positivity condition, i.e.,

$$\sigma_{\alpha\beta} + \Sigma_{\alpha\beta} \geq 0. \quad (33)$$

The derived inequality is general one. Let us consider important partial cases. For one degree of freedom ($\alpha = 1, 2$) let us take hermitian operators

$$\hat{Q}_1 = \hat{q} - <\hat{q}>, \quad \hat{Q}_2 = \hat{p} - <\hat{p}>. \quad (34)$$
In this case the matrix $\sigma_{\alpha\beta}$ is the dispersion matrix for position and momentum. The matrix $\Sigma_{\alpha\beta}$ has the form proportional to Pauli matrix $\sigma_y$, i.e.

$$\Sigma_{\alpha\beta} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}. \quad (35)$$

(We take below $\hbar = 1$).

The inequality (33) means that the matrix

$$\begin{pmatrix} \sigma_{qq} & \sigma_{qp} + \frac{i}{2} \\ \sigma_{qp} - \frac{i}{2} & \sigma_{pp} \end{pmatrix} \geq 0. \quad (36)$$

The criterion of positivity of the matrix means

$$\sigma_{qq} \geq 0, \quad \sigma_{pp} \geq 0 \quad (37)$$

and

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 - \frac{1}{4} \geq 0. \quad (38)$$

Inequality (38) is the Schrödinger–Robertson uncertainty relation [22]

$$\sigma_{qq}\sigma_{pp} \geq \frac{1}{4(1 - r^2)}. \quad (39)$$

Here $r$ defines the correlation of position and momentum

$$r = \frac{1}{\sqrt{\sigma_{qq}\sigma_{pp}}} \sigma_{qp}. \quad (40)$$

For $n$ degrees of freedom let us take $N = 2n$ operators $\hat{\mathbf{Q}}_{\alpha}$ in the form

$$\hat{\mathbf{Q}}_1 = \hat{q}_1 - \langle \hat{q}_1 \rangle, \quad \hat{\mathbf{Q}}_2 = \hat{q}_2 - \langle \hat{q}_2 \rangle, \quad \ldots, \quad \hat{\mathbf{Q}}_n = \hat{q}_n - \langle \hat{q}_n \rangle,$$

$$\hat{\mathbf{Q}}_{n+1} = \hat{p}_1 - \langle \hat{p}_1 \rangle, \quad \hat{\mathbf{Q}}_{n+2} = \hat{p}_2 - \langle \hat{p}_2 \rangle, \quad \ldots, \quad \hat{\mathbf{Q}}_N = \hat{p}_n - \langle \hat{p}_n \rangle. \quad (41)$$

In this case the Robertson uncertainty relation [19] expressed as the condition of positivity of $2n \times 2n$-matrix (33), where $\sigma_{\alpha\beta}$ is multimode dispersion matrix of positions and momenta and the $2n \times 2n$-matrix $\Sigma_{\alpha\beta}$ has the form (35) with $n \times n$-blocks proportional to identity matrix in $n$ dimensions.

The positivity condition (33) means that all the major minors of the $2n \times 2n$-matrix are nonnegative. The determinant of the matrix (33) can be calculated and one has the uncertainty relation in the form of the inequality

$$\det(\sigma_{\alpha\beta}) \geq \frac{1}{4^n}. \quad (42)$$

One can take the operator $\hat{\mathbf{Q}}_{\alpha}$ as angular momentum operators (spins). In this case one has inequalities which give some constraints for spin states.
6 Admissibility of quantum states

In the case of classical statistical system with \( n \) degrees of freedom one can determine the dispersion matrix of position and momentum

\[
\sigma_{\alpha\beta} = \int f(\vec{q}, \vec{p}) Q_\alpha Q_\beta d\vec{q} d\vec{p}.
\] (43)

Here \( f(\vec{q}, \vec{p}) \) is probability density in the phase space of the system,

\[
\int f(\vec{q}, \vec{p}) d\vec{q} d\vec{p} = 1.
\]

The functions \( Q_\alpha, \alpha = 1, 2, \ldots, N = 2n \) are defined as

\[
Q_1 = q_1 - <q_1>, \quad Q_2 = q_2 - <q_2>, \ldots, \quad Q_n = q_n - <q_n>,
\]

\[
Q_{n+1} = p_1 - <p_1>, \quad Q_{n+2} = p_2 - <p_2>, \ldots, \quad Q_N = p_n - <p_n>.
\] (44)

and

\[
<q_s> = \int f(\vec{q}, \vec{p}) q_s d\vec{q} d\vec{p},
\]

\[
<p_s> = \int f(\vec{q}, \vec{p}) p_s d\vec{q} d\vec{p},
\]

\[
s = 1, 2, \ldots, n.
\] (45)

Intuitively it is clear that one can make scaling transform of the probability density

\[
f(\vec{q}, \vec{p}) \rightarrow f_s(\vec{q}, \vec{p}) = \int K(\vec{q}, \vec{p}, \vec{q}', \vec{p}') f(\vec{q}', \vec{p}') d\vec{q}' d\vec{p}',
\] (46)

where the kernel has the form

\[
K(\vec{q}, \vec{p}, \vec{q}', \vec{p}') = \prod_{s=1}^{n} |\lambda_{qs} \lambda_{ps}| \delta(q'_s - \lambda_{qs} q_s) \delta(p'_s - \lambda_{ps} p_s).
\] (47)

The \( 2n \) real parameters \( \lambda_{qs}, \lambda_{ps} \) are arbitrary nonzero numbers. These parameters can be considered as abelian group which is direct product on \( n \) abelian groups discussed for one degree of freedom. The distribution (46) is admissible to describe a classical state. Quantum states with density operator \( \hat{\rho} \) can be described by the Wigner function \( [25] W(\vec{q}, \vec{p}) \). This function is connected with the density matrix in position representation \( \rho(\vec{x}, \vec{x}') \) by the formulae

\[
W(\vec{q}, \vec{p}) = \int \rho(\vec{q} + \frac{\vec{u}}{2}, \vec{q} - \frac{\vec{u}}{2}) e^{-i\vec{p}\vec{u}} d\vec{u},
\]

\[
\rho(\vec{x}, \vec{x}') = \frac{1}{(2\pi)^n} \int W\left(\frac{\vec{x} + \vec{x}'}{2}, \vec{p}\right) e^{i\vec{p}(\vec{x} - \vec{x}')/2} d\vec{p}.
\] (48)

The mean values and dispersion matrix for positions and momenta of quantum system can be calculated using formulas (45) and (43) with replacements

\[
f(\vec{q}, \vec{p}) \rightarrow \frac{W(\vec{q}, \vec{p})}{(2\pi)^n}.
\] (49)
The Wigner function describing the quantum state must satisfy the uncertainty relations. Due to this if one has scaling transform

\[ W(\vec{q}, \vec{p}) \rightarrow W_s(\vec{q}, \vec{p}) = \int K(\vec{q}, \vec{p}, \vec{q}', \vec{p}') W(\vec{q}', \vec{p}') \, dq' \, dp' \]  

(50)

the kernel of scaling transform is given by (47) but the parameters \( \lambda_p \) and \( \lambda_q \) must satisfy constraints providing the uncertainty relations

\[ |\lambda_p \lambda_q| \geq 1. \]

It means that in comparison with classical states where the parameters form the commutative group in quantum state the parameters form the semigroup of real numbers which is direct product of semigroups corresponding to one degrees of freedom. The numbers \( \lambda_p, \lambda_q \) are situated on the plane of these parameters. For classical states the forbidden numbers \( \lambda_p, \lambda_q \) are situated on the ”classical cross” which is composed by two axes of reference frame in the plane. The thickness of the cross components is zero. In quantum case one has ”quantum cross” where the values of parameters \( \lambda_p, \lambda_q \) are forbidden by the uncertainty relations. The boundary of the cross is formed by four hyperbolas. Outside of the ”quantum cross” one has admissible values of parameters \( \lambda_p, \lambda_q \) and all points of the plane in the admissible domain form the semigroup of numbers. Thus transition from classical domain to quantum domain can be associated with operation of reducing the group of real numbers with natural product formula to semigroup of these numbers (outside of quantum cross) with the same natural product formula. The thickness of the quantum cross is nonzero and it is proportional to Planck constant. The classical states in classical statistical mechanics have the property that scaling of any subsystem is permitted. In quantum domain one has some states which behave differently with respect to scaling of some subsystem due to specific quantum correlations.

If one uses the tomographic representation of quantum states the states are associated with symplectic tomogram

\[ W(\vec{X}, \vec{\mu}, \vec{\nu}) = \int W(\vec{q}, \vec{p}) \left( \prod_{s=1}^{n} \delta(X_s - \mu_s q_s - \nu_s p_s) \right) \frac{dq \, dp}{(2\pi)^n}. \]  

(51)

The tomogram is probability distribution function of position \( \vec{X} \) depending on extra real parameters \( \vec{\mu} \) and \( \vec{\nu} \). The scaling transform of the tomogram reads

\[ \omega(\vec{X}, \vec{\mu}, \vec{\nu}) \rightarrow \omega_s(\vec{X}, \vec{\mu}, \vec{\nu}) = \int W(\vec{X}', \vec{\mu}', \vec{\nu}') \left( \prod_{s=1}^{n} \delta(X_s - X_s') \right) \times \delta(\mu_s' - \frac{\mu_s}{\lambda_p}) \delta(\nu_s' - \frac{\nu_s}{\lambda_q}) \, d\vec{\mu}' \, d\vec{\nu}' \, d\vec{X}'. \]  

(52)

The tomogram of the quantum states are admissible if the values of parameters yield the uncertainty relation. It means that the linear map of the density operator \( \hat{\rho} \) of the quantum state with the tomogram \( \omega(\vec{X}, \vec{\mu}, \vec{\nu}) \) [26] is positive map. The states can be devided into two classes. Some states are admissible if coordinates of any subsystem \( \mu_s, \nu_s \) are scaled. In other cases scaling of subsystem coordinates \( \mu_s, \nu_s \) provides the nonpositive density operator.
7 Kernel of commutative star-product of density matrices of classical states

In quantum mechanics observables and states are described by hermitian operators. The states are described by nonnegative density operators $\hat{\rho}$. The Weyl symbol $W(q, p)$ of arbitrary operator $\hat{A}$ is given in terms of the matrix of the operator $A(x, x') = \langle x | \hat{A} | x' \rangle$ in position representation as

$$W_A(q, p) = \int A(q + \frac{u}{2}, q - \frac{u}{2})e^{-ipu} du.$$  \hfill (53)

The inverse of this formula reads

$$A(x, x') = \frac{1}{2\pi} \int W_A \left( \frac{x + x'}{2}, p \right) e^{ip(x-x')} dp.$$  \hfill (54)

The density operator $\hat{\rho}$ is nonnegative hermitian operator which means that

$$\text{Tr}(\hat{\rho}|\psi><\psi|) \geq 0$$ \hfill (55)

for arbitrary projector $|\psi><\psi|$. The Wigner function $W(q, p)$ of the quantum state must satisfy the condition

$$\int \psi^*(x)W \left( \frac{x + x'}{2}, p \right) e^{ip(x-x')} \psi(x') dp \, dx \, dx' \geq 0$$ \hfill (56)

for arbitrary wave function $\psi(x)$. The positivity condition means that the quantum uncertainty relation calculated by means of the given Wigner function is necessary condition for admissible quantum state.

For classical state the positive probability distributions $f(q, p)$ (states) and the functions in phase space $A(q, p)$ (observables) can be considered as Weyl symbols of some hermitian operators $\hat{\rho}_{cl}$ and $\hat{A}_{cl}$, respectively. Since the uncertainty relation can be violated in classical states it means that the corresponding density operators $\hat{\rho}_{cl}$ of classical state can be considered as nonpositive operator. In the spectral decomposition of the classical density operator

$$\hat{\rho}_{cl} = \int W_{cl}(j)|\psi_j><\psi_j| dj,$$  \hfill (57)

the weight function $W_{cl}(j)$ playing the role of probability density can take negative values which is mandatory for the classical state with density $f(q, p)$ violating quantum uncertainty relation. Thus we have the property. Positive probability density of the classical state $f(q, p)$ being expressed in terms of density operator $\hat{\rho}_{cl}$ has real quantum-like probability density $W_{cl}(j)$ which can take negative values. This property is dual to the property of Wigner function. For quantum state $\hat{\rho}$ (nonnegative density operator) the Wigner function $W(q, p)$ playing the role of classical-like probability density in phase space can take negative values (negative probability). Thus, the “negative probabilities” can be associated both to classical and to quantum states. Classical states with positive densities $f(q, p)$ in quantum-like description by means of nonpositive density operators are associated with “negative probabilities” $W_{cl}(j)$ in spectral decomposition of the density operators violating the uncertainty relation. Quantum state have
only positive probabilities associated with spectral decomposition of nonnegative density operators but they demonstrate "negative probabilities" which are Wigner functions associated with the quantum states. The Weyl symbols of two quantum observables $W_A(q, p)$ and $W_B(q, p)$ are multiplied using the kernel

$$ W_A \star W_B(q, p) = \int W_A(q_1, p_1)W_B(q_2, p_2)K(q_1, p_1, q_2, p_2, q, p) dq_1 dq_2 dp_1 dp_2. \quad (58) $$

The product of Weyl symbols of quantum observables is noncommutative since, see, e.g. [27]

$$ K(q_1, p_1, q_2, p_2, q, p) = \frac{1}{\pi^2} \exp\{2i[q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2)]\}. \quad (59) $$

If one considers the product of matrices of operators $\hat{A}$ and $\hat{B}$ in position representation $A(x, x')$ and $B(x, x')$ corresponding to classical observable $A(q, p)$ and $B(q, p)$ with commutative pointwise product rule the kernel of the star-product

$$ A \star B(x, x') = \int A(x_1, x_2)B(x'_1, x'_2)K(x_1, x_2, x'_1, x'_2, x, x')dx_1dx_2dx'_1dx'_2 \quad (60) $$

has the form

$$ K(x_1, x_2, x'_1, x'_2, x, x') = \int du_1 du_2 \delta(x - x' - u_1 - u_2)\delta(x_1 - \frac{x + x'}{2} - \frac{u_1}{2}) $$

$$ \delta(x_2 - \frac{x + x'}{2} + \frac{u_1}{2})\delta(x'_1 - \frac{x + x'}{2} - \frac{u_2}{2})\delta(x'_2 - \frac{x + x'}{2} + \frac{u_2}{2})(61) $$

Thus we obtain the kernel of commutative star-product of matrices of operators $\hat{A}$ and $\hat{B}$ in position representation for classical observables $A(q, p)$ and $B(q, p)$ with pointwise product.

## 8 Quantum and classical subsystems

In this section we discuss a possibility to describe interaction of classical and quantum systems. For this we start from the state of two independent systems. First system is considered as quantum one and the second system is considered as classical one. The tomographic description of the combined system can be given in terms of tomogram $\omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)$. If there is no correlations of quantum ($X_1$) and classical ($X_2$) degrees of freedom the tomogram has the factorized form

$$ \omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \omega_q(X_1, \mu_1, \nu_1)\omega_{cl}(X_2, \mu_2, \nu_2). \quad (62) $$

The tomogram $\omega_q$ satisfies the uncertainty relation. The classical tomogram $\omega_{cl}$ has no such constraints. The form (62) corresponds to factorized form of the density operator of combined system

$$ \hat{\rho} = \hat{\rho}_q\hat{\rho}_{cl}. \quad (63) $$

One can construct some evolution equation for the tomogram containing the classical and the quantum parts. This problem needs further investigation. Another problem is $h \to 0$ limit of quantum mechanics. Since all the density operators of quantum states are nonnegative and the density operators of classical states (in quantum-like description) can be nonpositive there
is no possibility in the limit $\hbar \to 0$ to get from nonnegative operator the nonpositive ones. In framework of tomograms the quantum tomograms in classical limits give the result which can be not admissible. On the other hand there are classical tomograms which are not admissible in quantum mechanics. Due to this these tomograms can not be obtained by the limit procedure $\hbar \to 0$ from the quantum tomogram. This discussion can be elucidated also in terms of transition group $\to$ semigroup. As we clarified the uncertainty relation in quantum mechanics implied the change of scaling group by scaling semigroup. This operation shows that we change the mathematical nature of object (scaling transform) considering transition from classical to quantum domain. To conclude one can consider union of admissible tomograms as probability description of combined classical-quantum system. This kind of speculation gives a possibility in natural way to describe interaction of quantum and classical degrees of freedom.

9 Conclusion

We summarize the main points of our consideration.

1. In classical statistical mechanics the states can be associated in usual representation with probability distribution function $f(q,p)$ in phase space of the system. In quantum-like representation the states in classical statistical mechanics can be associated with density operators $\hat{\rho}_{cl}$ acting in a Hilbert space. In tomographic representation the states in classical statistical mechanics can be associated with probability distribution function $\omega(X,\mu,\nu)$ of position $X$ measured in ensemble of reference frames labelled by real parameters $\mu$ and $\nu$. The density operators $\hat{\rho}_{cl}$ can be nonpositive. The distribution functions $f(q,p)$ and tomograms $\omega(X,\mu,\nu)$ are nonnegative functions.

The observables in classical statistical mechanics are associated with functions in phase space $A(q,p)$ in usual representation. In quantum-like representation the observables in classical statistical mechanics are described by hermitian operators $\hat{A}_{cl}$. The density operators $\hat{\rho}_{cl}$ and the observables in classical statistical mechanics can be given in position representation by the matrices $\hat{\rho}_{cl}(x,x') = <x|\hat{\rho}_{cl}|x'>$, $A_{cl}(x,x') = <x|\hat{A}_{cl}|x'>$. For example, in classical statistical mechanics the position and momentum operators have the form $\hat{q}_{cl} = x$ and $\hat{p}_{cl} = -i\frac{\partial}{\partial x}$ or in matrix representation $q(x,x') = <x|\hat{q}_{cl}|x'> = x\delta(x-x')$, $p_{cl}(x,x') = <x|\hat{p}_{cl}|x'> = -i\delta'(x-x')$. The commutative product of the operator matrices is determined by means of the kernel given by (61). The classical evolution equation for classical density operators does not preserve the positivity of the initial density operator. This preservation of positivity takes place only for linear systems. In tomographic representation the observables in classical statistical mechanics are described by their tomograms $A(X,\mu,\nu)$. The commutative product of the tomograms corresponds to pointwise product of the observables $A(q,p)$ in usual representation and to commutative product of the operators $\hat{A}_{cl}$ in quantum-like representation. The tomograms and density operators of states in classical statistical mechanics must satisfy the condition of positivity of the probability distributions $f(q,p)$ describing the states in usual representation. But uncertainty relations of quantum mechanics can be violated.

II. In quantum mechanics the states are described by nonegative density operators $\hat{\rho}$ in usual representation. In phase-space representation the states are associated with Wigner function $W(q,p)$ which can take negative values. In tomographic representation the quantum states are described by the tomogram which is nonegative probability distribution function of the
position $X$ measured in ensemble of reference frames in phase space labelled by real parameters $\mu$ and $\nu$. The density operator must be nonegative. The observables can be described by their tomograms which correspond to standard operators describing the observable in usual representation of quantum mechanics. In phase space representation the observables are described by the functions $A(q,p)$ which are Weyl symbols of the operators $\hat{A}$. The product of these functions is noncommutative and it is described by known kernel. The uncertainty relation provides constraints on admissible tomograms and Wigner functions of the quantum states. These constraints are associated with commutative semigroup of scaling transforms of tomograms (and Wigner functions) in quantum mechanics. In classical statistical mechanics the admissible tomograms (and Wigner functions) of the states are related by commutative group of scaling transforms. The quantum evolution equations preserve the nonnegativity of the quantum density operators. This corresponds to respecting the uncertainty relations in the process of quantum evolution. There exists a possibility to consider a union of all tomograms admissible both in quantum mechanics and in classical statistical mechanics to describe the interaction of classical and quantum systems. In the standard limit procedure $\hbar \to 0$ the set of quantum states (quantum tomograms) does not coincide with the set of classical tomograms. After the work was finished authors became aware of the work [28] where analogous aspects of connection of Heisenberg uncertainty relation with nonnegativity of classical density operator were discussed.

10 Acknowledgements

O.V.M. is grateful to the Russian Foundation for Basic Research for partial support of the work under Project No. 03-02-16408.

References

[1] Olga Man’ko, and V. I. Man’ko, J. Russ. Laser Res., 18, 407 (1997); 21, N 5, 411 (2000)
[2] J. G. Muga, and R. F. Snider, Europhys. Lett., 19, 569 (1992)
[3] V. I. Man’ko, Theory of Nonclassical States of Light, ed. V. V. Dodonov, and V. I. Man’ko, Taylor and Francis, London and New York, 2003, pp.219-240.
[4] Y. M. Shirokov, Teor. Mat. Fiz., 28, 308 (1976)[Theor. Math. Phys., 28, 806 (1976)
[5] S. Mancini, V. I. Man’ko, P. Tombesi, Phys. Lett., 213 A, 1 (1996)
[6] S. Mancini, V. I. Man’ko, P. Tombesi, Quantum Semiclass. Opt., 7, 615 (1995)
[7] V. V. Dodonov, and V. I. Man’ko, Phys. Lett., 239A, 335 (1997)
[8] V. I. Man’ko, and O. V. Man’ko, JETP, 85, 430 (1997)
[9] O. V. Man’ko, V. I. Man’ko, and G. Marmo, Phys. Scripta, 62, 446 (2000)
[10] O. V. Man’ko, V. I. Man’ko, and G. Marmo, J. Phys., 35 A, 699 (2002)
[11] S. Mancini, O. V. Man’ko, V. I. Man’ko, P. Tombesi, J. of Physics A, 34, 3461 (2001)
[12] A. B. Klimov, O. V. Man’ko, V. I. Man’ko, Yu. F. Smirnov, and V. N. Tolstoy, J. Phys., 35 A, 6101 (2002)
[13] V. A. Andreev, O. V. Man’ko, V. I. Man’ko, and S. S. Safonov, J. Russ. Laser Res., 19, 340 (1998)
[14] O. V. Man’ko, V. I. Man’ko, and S. S. Safonov, Theor. Math. Phys., 115, 185 (1998)
[15] V. I. Man’ko, S. S. Safonov, Yad. Fiz., 4, 658 (1998)
[16] V. A. Andreev, V. I. Man’ko, JETP, 87, 239 (1998)
[17] O. Castanos, R. Lopes-Pena, M. A. Man’ko, and V. I. Man’ko, J. Phys., 36 A, 4677 (2003)
[18] M. A. Man’ko, J. Russ. Laser Res., 22, 168 (2001); S. De Nicola, R. Fedele, M. A. Man’ko, V. I. Man’ko, Eur. Phys. J. B, 36, 385 (2003)
[19] V. V. Dodonov, and V. I. Man’ko, Generalization of uncertainty relation in quantum mechanics. In: Invariants and the Evolution of Nonstationary Quantum Systems. Proc. Lebedev Phys. Inst., vol. 183 (M. A. Markov, ed.), pp.5-70. Nauka, Moscow, 1987 [translated by Nova Science, Commack, 1989, pp.3-101]
[20] E. Schrödinger, Zum Heisenbergschen Unschärfeprinzip, Ber. Kgl. Acad. Wiss. Berlin, 24, 296 (1930)
[21] H. P. Robertson, Phys. Rev., 35, 667 (1930)
[22] V. V. Dodonov, E. V. Kurmushev, and V. I. Man’ko, Phys. Lett A, 79, 150 (1980)
[23] V. V. Dodonov, and V. I. Man’ko, Theory of Nonclassical States of Light, ed. V. V. Dodonov, and V. I. Man’ko, Taylor and Francis, London and New York, 2003, pp.1-94
[24] V. V. Dodonov, J. Opt. B, 4, S98 (2002)
[25] E. Wigner, Phys. Rev., 40, 749 (1932)
[26] V. I. Man’ko, G. Marmo, E. C. G. Sudarshan, F. Zaccaria, J. Russ. Laser Res., 24, 507 (2003); Phys. Lett A 327, 353 (2004)
[27] O. V. Man’ko, V. I. Man’ko, G. Marmo, Phys. Scr., 62, 446 (2000); J. Phys. A: Math. Gen., 35, 699 (2002)
[28] A. J. Bracken, J. G. Wood, quant-ph/0407052 (2004)