ON THE UNIFORMIZATION OF A VECTOR FIELD AND THE INDEX ALONG A JORDAN CURVE

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Abstract. Let \( X = P \partial/\partial x + Q \partial/\partial y \) be a \( C^r \) vector field defined on a compact and connected plane domain \( D \subseteq \mathbb{R}^2 \). We define the \( C^r \)-uniformization over the torus by means of a constant vector field over the torus and investigate sufficient and necessary conditions for such \( C^r \)-uniformization. These conditions are related to the index of the vector field along suitable Jordan curves defined by the boundary of its domain. As an application we show that this provides us a sufficient condition for the existence of a first integral for the differential equation defined by \( X \) and the possibility of its computation.

1. Introduction and preliminaries

The existence of a first integral for a vector field is a classical problem in the theory of dynamical systems. One of the main tools to this aim is the study of the rectification or uniformization of a vector field. Roughly speaking, we understand as uniformization of a vector field over a plane domain, as the existence of a diffeomorphic map that brings such a vector field to a constant vector field. Local uniformization is a well known property of any vector field with mild conditions (see e.g. [4, Theorem 1.2]). Regarding global plane domains, there are several results that deal with several techniques in order to find a first integral, see for instance [2, 7, 8, 10, 11, 12]. The technique we develop in the present paper is based in the ideas presented by Finn in [6], who studied the behavior of the index of a plane direction field along a Jordan curve that represent the boundary of the plane domain. Around the year 2010, J.A. Martínez-Alfaro suggested the author a generalization of the uniformization of a vector field in the sense of Finn, as a vector field which is conjugate with a constant one, but they are not necessarily plane vector fields. This new point of view will allows us to obtain necessary conditions for such a uniformization and the existence of a first integral. In addition, we obtain a sufficient condition for the uniformization in this new sense. In the present paper we recover these original ideas taking a plane vector field and a vector field over the torus. To illustrate our election observe the following trivial fact:

There exist a unique compact, differentiable, connected and orientable 2-manifold without boundary which admit a global regular vector field. This manifold is the torus.

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Sketch of the proof. Let $T^2$ be the torus. It is not hard to define a constant global vector field without singular points. Assume that there is another compact, differentiable, connected and orientable 2-manifold without boundary $M^2$, necessarily it has Euler characteristic $\chi(M^2) \neq 0$. Thus, Poincaré-Hopf’s theorem implies that every vector field defined over $M^2$ has, at least, one singular point.

Regarding the index of a direction field along a Jordan curve. This concept was applied by Finn ([6]) in order to determine conditions for the uniformization of such a direction field, that relates the indexes 0 and 2 by means of a suitable diffeomorphism. Concretely, it is shown that we can uniformize a direction field, provided the boundary of a simply connected plane domain has index 0 or 2. In our case, we use the same idea for the case of index 1. However, we cannot ensure that a vector (or direction) field can be uniformized in the sense of Finn. For this reason we have extended the definition of uniformizable field to a uniformization over the torus. Despite the example exposed in this paper is developed by means of basic theory, we believe that this idea is very interesting for the study of dynamical systems over a general manifold.

We have organized the paper as follows. In Section 2 we define the $C^r$-uniformization of a vector field defined over a plane domain with respect to a domain over the torus. Then, in Theorem 2.3 we show that $C^r$-uniformization over the torus implies that the index of the vector field involved around a suitable Jordan curve is always equal to 1. We also provide a test for the existence of a first integral of a plane vector field. In Section 3, we explore if the condition given in the previous theorem could be sufficient, and obtain (Theorem 3.4) that there are several cases of plane vector fields that the condition on the index is sufficient for the $C^r$-uniformization of the vector field.

Let us introduce some terminology for the paper. Let $D \subseteq \mathbb{R}^2$ be an closed plane domain, recall that a vector field $X : D \to T(D)$ is said to be uniformizable if it is conjugate with a constant vector field $U : E \to T(E)$ in the plane, for a certain plane domain $E$, i.e. if there exists a diffeomorphic map $\varphi : D \to E$ so that $d\varphi \circ X = U \circ \varphi$ (see [6] Definition 3 and [4] Section 1.3). We denote the torus by $T^2$. The definition of the index of a direction field along a curve can be found in [6] Section 1. Let us introduce it for vector fields. Let $D$ be a plane domain and let $\Gamma : [0, 2\pi] \to D$ be a Jordan curve on $D$. Let $X$ be a vector field defined on $D$, we define the index of the vector field $X$ along the curve $\Gamma$ as $\text{ind}_\Gamma(X) := \text{deg}(X|_\Gamma/|X|_\Gamma)$. A well known formula to compute this index is:

$$\text{ind}_\Gamma(X) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\det(X|_\Gamma, (X|_\Gamma)')}{{\|X|_\Gamma\|^2}} dt,$$

(1.1)

see for instance [3]. It is also well known the Bendixon’s formula for the computation of the index of a vector field in a point. That is $\text{ind}_x X = 1 + (E - H)/2$, where $E$ is the number of elliptic sectors and $H$ is the number of hyperbolic sectors around a sufficiently small neighborhood of $x$. We will adapt this formula
for our purposes. We refer to [1, 4] for notation, definitions and basic results that do not appear here.

2. Necessary condition for \(C^\infty\)-uniformization over the torus

Recall that the type of a Jordan curve over the torus identified with the \(S^1 \times S^1\) is the number of laps around each circumference \(S^1\). This type can also be seen as an element of the homotopy group \(\pi_1(T^2)\). Let \(Z: T^2 \to T(T^2)\) be a constant vector field and \(\Gamma\) be a Jordan curve of \((p,q)\) type over the torus, where \(p,q \in \mathbb{Z} \setminus \{0\}\). Obviously \(\text{ind}_\Gamma Z = 0\). Then we consider the set \(T^2 \setminus \Gamma\) in order to transform it in a corona by means of a suitable diffeomorphism. Our result states that the index of the vector field along the boundary of such corona has index 1.

**Definition 2.1.** Let \(r \geq 1\) and \(D \subseteq \mathbb{R}^2\) be a compact and connected domain. Let \(X: D \to T(D)\) be a \(C^r\) vector field. We say that \(X\) is \(C^r\)-uniformizable over \(M^2\) if there exists a domain \(\Delta \subseteq T^2\) so that \(X\) is \(C^r\)-conjugate to a constant vector field \(Z: \Delta \to T(\Delta)\), i.e. there exists a diffeomorphism \(\psi: D \to \Delta\) such that \(d\psi \circ X = Z \circ \psi\).

Despite we only deal with plane domains and domains over the torus, it is not hard to generalize this definition to general vector fields and general manifolds which could be of interest for a future research.

**Example 2.2.** The constant vector field \(X = a \partial/\partial x + b \partial/\partial y\), where \(a,b \in \mathbb{R}\) so that \(a/b \in \mathbb{Q} \setminus \{0\}\), defined in a domain of \(\mathbb{R}^2\), is a clear example of \(C^\infty\)-uniformizable vector field over the torus and over the cylinder.

Let the torus \(T^2\) and \(Z: T^2 \to T(T^2)\) be a regular vector field. Let \(\Gamma\) be a fixed Jordan curve over \(T^2\) and \(\tilde{\Gamma}\) be a Jordan curve homotopic to \(\Gamma\) such that \(\tilde{\Gamma} \cap \Gamma = \emptyset\). We prove that there exists a diffeomorphism into its image \(\varphi_\Gamma: T^2 \setminus \Gamma \to D \subseteq \mathbb{R}^2\) such that the index of the vector field \(Y := X|_{T^2 \setminus \Gamma} \circ \varphi^{-1}\) along the curve \(\Lambda := \tilde{\Gamma} \circ \varphi^{-1}\) is one, i.e. \(\text{ind}_\Lambda Y = 1\).

**Theorem 2.3.** Let \(\Gamma\) be a \((p,q)\) type and \(C^r\) Jordan curve over the torus \(T^2\), where \(p,q \in \mathbb{Z} \setminus \{0\}\). Let \(Z: T^2 \to T(T^2)\) be a constant vector field. Then, there exist a Jordan curve \(\Gamma_0\) such that \(\Gamma \cap \Gamma_0 = \emptyset\), a domain \(D \subseteq \mathbb{R}^2\) and a diffeomorphism \(\varphi: T^2 \setminus \Gamma_0 \to D\) such that

\[
\text{ind}_{\varphi \Gamma}(d\varphi \circ Z \circ \varphi^{-1}) = 1.
\]

**Proof.** We will use the representation of the torus by means of the usual equivalence relation in \([0,2\pi]^2\), that is \(T^2 \equiv [0,2\pi]^2/\sim\). Since the type of a curve is invariant by diffeomorphism, without loss of generality, we can assume that \(\Gamma(t) = (pt,qt)\) for \(t \in [0,2\pi]\). Again, without loss of generality we take the constant vector field \(Z\) defined as \(Z(x,y) := \partial/\partial x\). Thus, \(\Gamma\) has type \((p,q)\) and the representation of \(\Gamma\) in \([0,2\pi]^2/\sim\) is the segment between \((0,0)\) and \((2p\pi,2q\pi)\) in \(\mathbb{R}^2\). The angle of this segment with respect to the positive abscissa is \(\arctan\left(\frac{2}{p}\right)\).
Let the Jordan curve $\Gamma_0(t) := (pt + 1/2, qt + 1/2)$, clearly $\Gamma \cap \Gamma_0 = \emptyset$. We define the diffeomorphism $\varphi : T^2 \setminus \Gamma_0 \to \mathbb{R}^2$ given by

$$\varphi := \pi \circ \tau \circ \rho_{p,q},$$

where $\rho_{p,q} := \frac{1}{\sqrt{p^2 + q^2}} \left( \begin{array}{c} p \\ -q \end{array} \right)$ is the clockwise rotation centered at the origin and angle $\arctan \left( \frac{q}{p} \right)$, $\tau$ is the translation $\tau(x, y) = (x, y + 2\pi)$, which translate the segment in the plane $[0, 2\pi(p^2 + q^2)^{1/2}] \times \{0\}$ to the segment $[0, 2\pi(p^2 + q^2)^{1/2}] \times \{2\pi\}$, and $\pi$ is the change to polar variables, that is $\pi(x, y) = (y \cos(x), y \sin(x))$. Therefore

$$\varphi(x, y) = \frac{1}{p^2 + q^2} \left((py - qx + 2\pi) \cos(px + qy), (py - qx + 2\pi) \sin(px + qy)\right).$$

Let us obtain the index of the vector field $d\varphi \circ Z \circ \varphi^{-1}$ along $\varphi \circ \Gamma$, by a direct computation, using the formula (1.1). On the one hand we have that

$$(py - qx)|_{\Gamma(t)} = p(qt) - q(pt) = 0 \quad \text{and} \quad (px + qy)|_{\Gamma(t)} = (p^2 + q^2)t.$$  \hfill (2.1)

For the aim of brevity we define $H := p^2 + q^2$, $c := \cos(Ht)$ and $s := \sin(Ht)$. Hence

$$\varphi \circ \Gamma = \frac{2\pi}{H}(c, s),$$

defines a curve on $D \subset \mathbb{R}^2$. On the other hand we get the following commutative diagram

\[\begin{array}{ccc}
[0, 2\pi] & \xrightarrow{\Gamma} & T^2 \\
\downarrow{\varphi \circ \Gamma} & & \downarrow{\varphi} \\
D \subset \mathbb{R}^2 & \xrightarrow{\varphi \circ \Gamma} & T(D)
\end{array}\]

Therefore,

$$(d\varphi \circ Z \circ \varphi^{-1})|_{\varphi \circ \Gamma} = d\varphi \circ Z|_{\Gamma} = d\varphi(\partial/\partial x)|_{\Gamma} = \frac{\partial \varphi}{\partial x}|_{\Gamma}.$$

The partial derivative of $\varphi$ with respect to $x$ is

$$\frac{\partial \varphi}{\partial x}(x, y) = \frac{1}{H} \left( -q \cos(px + qy) - p(py - qx + 2\pi) \sin(px + qy), \right.$$

$$\left. -q \sin(px + qy) - p(py - qx + 2\pi) \cos(px + qy) \right).$$

Thus, taking into account (2.1) and the given notation, we obtain that

$$\left. \frac{\partial \varphi}{\partial x} \right|_{\Gamma} = \frac{1}{H}(-qc - 2\pi ps, -qs - 2\pi pc).$$

Now, having in mind that $c' = -Hs$ and $s' = Hc$, we can compute the derivative

$$\left( \left. \frac{\partial \varphi}{\partial x} \right|_{\Gamma} \right)' = (qs + 2\pi pc, -qc - 2\pi ps),$$

On the other hand we have the square norm is

$$\left\| \left. \frac{\partial \varphi}{\partial x} \right|_{\Gamma} \right\|_2^2 = \frac{1}{H} ((qc + 2\pi ps)^2 + (qs + 2\pi pc)^2),$$
in consequence
\[ \det\left( \frac{\partial^2}{\partial x \partial t}, \frac{\partial^2}{\partial x \partial \tau} \right) = \frac{1}{H} \left( (qc + 2p\pi s)^2 + (qs + 2p\pi c)^2 \right) = \left\| \frac{\partial^2}{\partial x \partial t} \right\|_2^2. \]
This equality together with (2.2), implies that
\[ \text{ind}_{\varphi_1}(d\varphi \circ Z \circ \varphi^{-1}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\det\left( \frac{\partial^2}{\partial x \partial t}, \frac{\partial^2}{\partial x \partial \tau} \right)'}{\left\| \frac{\partial^2}{\partial x \partial t} \right\|_2^2} \, dt = 1, \]
which is the claim.

This theorem establish a necessary condition for the $C^r$-uniformization over the torus of any vector space defined over a connected domain in $\mathbb{R}^2$.

**Remark 2.4.** Let $D \subseteq \mathbb{R}^2$ be a connected domain and $X$ be a vector field over $D$. If $X$ is $C^r$-uniformizable over the torus by means of $\psi$, then there exist a vector field $\tilde{X} : T^2 \setminus \Gamma \to T(T^2)$ (extension of $d\psi \circ X$) where $\Gamma$ is a Jordan curve and also exists a diffeomorphism $\varphi : T^2 \setminus \Gamma \to D_0 \subseteq \mathbb{R}^2$, such that $D_0$ is defined as the region between suitable Jordan curves $\Gamma_0$ and $\Gamma_1$, and also
\[ \text{ind}_{\Gamma_i} d\varphi \circ \tilde{X} = 1, \quad i = 0, 1. \]
Observe that we have assumed that $d\varphi \circ \tilde{X}$ can be extended to its closure. Conversely, in case that any diffeomorphism $\psi : D \to \Delta \subseteq T^2$ and any extension of $\tilde{X}$ and diffeomorphism $\varphi$ as above satisfies that
\[ \text{ind}_{\Gamma_i} d\varphi \circ \tilde{X} \neq 1, \]
for $i = 0$ or $i = 1$. Then $X$ cannot be $C^r$-uniformizable over the torus.

We now provide an application to the existence of a first integral.

**Proposition 2.5.** Let $D \subseteq \mathbb{R}^2$ be a compact connected domain and $X$ be a vector field over $D$ with Hausdorff flow. Assume that $X$ is $C^r$-uniformizable over the torus by means of $\psi = (f, g)$, so that there is a pair $(a, b) \in \mathbb{R}^2 \setminus 0$ such that the P.D.E.
\[
\begin{cases}
  h_x = bf_x - af_y \\
  h_y = bg_x - ag_y
\end{cases}
\] (2.3)
has solution $h$. Then $X$ has a first integral $h$.

**Proof.** Let us denote by $(u, v)$ the coordinates over $T^2$ and by $(x, y)$ the coordinates over $D$. Since $X$ is $C^r$-uniformizable by means of the diffeomorphism $\psi : D \subseteq \mathbb{R}^2 \to \Delta \subseteq T^2$, there exists $a, b \in \mathbb{R}^r \setminus \{0\}$ and a constant vector field $Z := a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$ over torus, such that $d\psi \circ X = Z \circ \psi$. Let us denote $X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ and consider that $x$ and $y$ so that $\dot{x} = P(x, y)$ and $\dot{y} = Q(x, y)$. On the one hand, since $Z$ is constant, there exists $H$ a first integral of $Z$. We can choose, for instance, $H(u, v) = bu - av$, since $H_u = b$ and $H_v = -a$ and $Z(H) = aH_u + bH_v = 0$. On the other hand we can choose the variables $u$ and $v$ so that $\psi^{-1}(u, v) = (x, y)$, hence
\[ d\psi \circ X \circ \psi^{-1} = (f_x \dot{x} + g_x \dot{y})\frac{\partial}{\partial u} + (f_y \dot{x} + g_y \dot{y})\frac{\partial}{\partial v}. \]
Therefore
\[
d\psi \circ X \circ \psi^{-1}(H) = (f_x \dot{x} + g_x \dot{y})H_u + (f_y \dot{x} + g_y \dot{y})H_v
= \dot{x}(H_u f_x + H_v f_y) + \dot{y}(H_u g_x + H_v g_y)
= \dot{x}(bf_x - af_y) + \dot{y}(bg_x - ag_y) = 0.
\]
(2.4)

By hypothesis, \(h\) is solution of (2.3), hence by (2.4)
\[
X(h) = \dot{x}h_x + \dot{y}h_y = \dot{x}(bf_x - a f_y) + \dot{y}(bg_x - a g_y) = 0.
\]
That means that \(h\) is a first integral for \(X\). \(\square\)

**Example 2.6.** Let \(D \subseteq \mathbb{R}^2\) be a plane domain and \(X: D \to T(D)\) be vector field. Assume that there is a diffeomorphism \(\psi: D \to T^2\) as was defined above \(\psi := (f, g)\) and assume that \(X := P \partial/\partial x + Q \partial/\partial y\) is such that

\[
P := \frac{ag_y - bf_y}{f_x g_y - f_y g_x}
\]

and

\[
Q := \frac{bf_x - ag_x}{f_x g_y - f_y g_x},
\]

where the numerators does not vanishes at the same time in \(D\) (that ensures that the vector field is regular) and also the denominator \(f_x g_y - f_y g_x\) does not vanishes in \(D\). This vector field is \(C^r\)-uniformizable by means of the constant vector field \(a \partial/\partial u + b \partial/\partial v\), where \(u, v\) denotes the coordinate variables over the torus.

Nevertheless, a priori at least, the \(C^r\)-uniformization over the torus seems not be easy to establish for general vector fields. In the next section we prove that there are cases in what we can obtain the converse of Theorem 2.3, giving us a useful tool in the study of several vector fields by means of the index of a vector field along a Jordan curve.

### 3. Sufficient condition for \(C^r\)-uniformization over the torus

In the previous section we have seen that a vector field defined on a suitable domain such that is \(C^r\)-uniformizable over the torus, must have index 1 along the boundary of such domain. It would be interesting if this condition is also sufficient. We now see that there are some cases in which we can obtain a rectification of the vector field over a certain plane domain. This rectification brings us to the wanted \(C^r\)-uniformization over the torus. Nevertheless, there are cases of regular vector fields that we cannot include, for example those that have a periodic orbit or those that have infinite tangent points with the boundary of its domain. In these cases we probably could make a perturbation of the vector field, but we do not have studied this case in this paper. In order to apply our technique we need some preliminary definitions.

Let us first adapt Bendixon’s formula in order to compute the index of a vector field around a Jordan curve as follows. Let \(D\) be a compact simply connected plane domain such such that the boundary \(\Gamma := \partial D\) is a Jordan curve. Let \(X\) be a regular vector field with Hausdorff flow over \(D\). With an abuse of notation we
identify set $\Gamma$ with a regular parametrization defined on the interval $I := [0, 1]$. We say that a point in $\Gamma$ is tangent or transverse to $X$ in the following sense. Given a $p \in \Gamma$, it is said that $p$ is tangent to $X$ if $\langle X(p) \rangle = \langle \Gamma'(p) \rangle$, and it is said that $p$ is transverse to $X$ in the other case. The number of tangent points to $X$ of $\Gamma$ is called the contact of $X$ with respect to (w.r.t.) $\Gamma$. We say that a tangent point (to $X$), $p \in \Gamma$ is elliptic (for $X$) if there is a neighborhood $N$ of $p$ in $D$ such that

$$(\beta_p \setminus \{p\}) \cap N \text{ is disconnected},$$

where $\beta_p$ is an integral curve of $X$ through the point $p$ and we identify the set curve with its parametrization $\beta_p$. A tangent point $q \in \Gamma$ (to $X$) is hyperbolic (for $X$) if for every neighborhood $N$ of $p$ in $D$ we have that

$$\beta_p \cap N = \{p\}.$$ 

Taking into account these definitions we establish that the index of $\Gamma$ with respect to $X$, that we assume has finite contact with respect to $\Gamma$. Assume that $X$ has finite contact with respect to $\Gamma$. We say that the index of $X$ with respect to $\Gamma$ is

$$\text{ind}_\Gamma X = 1 + (e - h)/2, \quad (3.1)$$

In view of the Bendixson’s formula, it is clear that the index of a vector field along a curve only depends on the behavior of the integral curves that are tangent to the boundary $\Gamma$, as these curves determine the number of elliptic and hyperbolic points of the vector field. From the definition of hyperbolic point of $\Gamma = \partial D$ we easily deduce the following result the characterizes such points.

Observe that a $C^r$ and regular vector field with Hausdorff flow, every integral curve must be injective. Nevertheless, for an hyperbolic point $p$, an integrable curve through $p$ satisfies that $\gamma_p = \{p\}$, so it is constant, hence its derivative vanishes and the vector field is regular. This implies that such vector field cannot have integral curve through $p$. The converse is also true. With this we establish the following characterization of hyperbolic points.

**Lemma 3.1.** Following the notation above. Let $X$ be a regular vector field over $D$ with Hausdorff flow. Then, $p$ is an hyperbolic point of $\partial D$ if and only if it does not exist integral curve for $p$.

We can extend these definitions to a compact non-simply connected domain taking into a count some mild conditions on the domain. Assume that $D$ is an $n$-connected domain with connected interior. We assume that $\partial D = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_n$, where $\Gamma_i$ are Jordan curves and the compact region delimited by $\Gamma_0$ contains $\Gamma_i$ for $i = 1, \ldots, n$. It is not hard to show that for every $k > 0$ there exists $\Gamma_i'(k) \subset D$ such that $0 < d(\Gamma_i, \Gamma_i'(k)) < 1/k$ for $i = 1, \ldots, n$. This allows us to define elliptic and hyperbolic points in the simply connected regions defined by $\Gamma_i'(k)$. By this way we provide a sequence of hyperbolic or elliptic points. If $p \in \Gamma_i$ is limit of a sequence of hyperbolic/elliptic points obtained by this procedure, we say the $p$ is hyperbolic/elliptic in $\Gamma_i$ for $i = 1, \ldots, n$. 


The index along a Jordan curve is invariant by a diffeomorphic map, so without loss of generality we will use a corona as standard domain. Let $\delta > 0$ and $x \in \mathbb{R}^2$, let us denote by $B_\delta(x)$ the open ball centered at $x$ with radius $\delta$. We define the following sets: $B_1 := B_1(0)$, $B_2 := B_2(0)$ and the corona $C := \overline{B_2} \setminus B_1$, which is a closed domain defined by the curves $\partial B_1$ and $\partial B_2$. This domain is compact non-simply connected. Let $\Gamma$ be an injective curve, we define $d$ as the restriction of the usual distance to the set $\Gamma$. Let $p$ that $\Gamma$ is a closed domain defined by the curves $\partial B$.

We also write $(\Gamma)$ for the section of curve in $\partial B$ contact with respect to $\partial B$. Let $\delta > 0$ be the section of curve in $\partial B$ contact with respect to $\partial B$. Let $O. GALDAMES-BRAVO$

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$$[p_1, p_2]_\Gamma := \{ p \in \Gamma : d_\Gamma(p_1, p_2) = d_\Gamma(p_1, p) + d_\Gamma(p, p_2) \}.$$ 

We also write $(p_1, p_2)_\Gamma$, $[p_1, p_2)_\Gamma$ and $(p_1, p_2)_\Gamma$, for the cases in what the points $p_1$ and $p_2$ are included or not into such a set. We say that any $p \in (p_1, p_2)_\Gamma$ is between $p_1$ and $p_2$ along $\Gamma$ and we say that a point between $p_1$ and $p_2$ is the middle point of $[p_1, p_2]_\Gamma$ if $d_\Gamma(p_1, p) = d_\Gamma(p, p_2)$.

Let $X$ be a regular vector field over $C$ such that its flow is Hausdorff and has positive contact with respect to $\partial C$. Let $\gamma : I \to C$ be an integral curve of $X$ such that $\gamma(0), \gamma(1) \in \partial B_2$ ($\partial B_1$) and $\gamma(t) \notin \partial B_2$ ($\partial B_1$) for every $0 < t < 1$. This integral curve divides $C$ in two regions and one of them is simply connected. Let us denote this region by $C^\gamma_\$, $\gamma$. Observe that in case that an integral curve has contact greater that 2 we can take the piece of that that satisfies such conditions. The proof of the following lemma is based on a proof of the intermediate value theorem.

**Lemma 3.2.** Let $X$ be a regular vector field over $C$ such that its flow is Hausdorff and has positive contact with respect to $\partial C$. Let $\gamma : I \to C$ be an integral curve of $X$ such that $\gamma(0), \gamma(1) \in \partial B_2$ ($\partial B_1$), $\gamma(0) \neq \gamma(1)$ and $\gamma(t) \notin \partial B_2$ ($\partial B_1$) for every $0 < t < 1$. Assume that $X$ has not an injective integral curve with positive contact with respect to $\partial B_2$ ($\partial B_1$) inside $C^\gamma_\$. Then, there is a unique hyperbolic (elliptic) point between them $\gamma(0)$ and $\gamma(1)$ along $\partial B_2$ ($\partial B_1$).

**Proof.** We only prove the case for $\partial B_2$, the case for $\partial B_1$ is analogous. For the sake of simplicity, we parametrize $\gamma$ so that $\gamma(0), \gamma(1) \in \partial B_2$ are the two consecutive points along $\gamma$, it suffices to find a non-injective integral curve inside $C^\gamma_\$. Let $\Gamma$ be the section of curve in $\partial B_2$ from $\gamma(0)$ until $\gamma(1)$. Let $p_0$ be the middle point between $\gamma(0)$ and $\gamma(1)$ along $\Gamma$, and let $\gamma_0$ be the integral curve of $X$ through $p_0$. If $\gamma_0(0) = \gamma_0(1)$, $\gamma_0$ must be constant, since the flow is Hausdorff and the vector field is regular, thus there is not integral curve for $p_0$, thanks to Lemma 3.1 we have that $p_0$ is hyperbolic. If $\gamma_0(0) \neq \gamma_0(1)$, we take $p_1$ be the middle point of $[\gamma_0(0), \gamma_0(1)]_\Gamma$. Let $\gamma_1$ be the integral curve through $p_1$. Since there is not any injective integral curve with positive contact inside $C^\gamma_\$ and $X$ is regular, necessarily $\gamma_1$ can be parametrized so that $\gamma_1(0) = p_1 \neq \gamma_1(1) \in \Gamma$. Assume that we cannot establish that $\gamma_1(0) = p_1$, thus $\gamma_1(0) \neq p_1$, then $p_1 = \gamma_1(t_1)$ for some $t_1 \in (0,1)$. Since the flow of $X$ is Hausdorff, $\gamma_1(0) \in \Gamma$ and also $\gamma_1(t_1)$ must be parallel to $\Gamma$, which is in contradiction with the hypothesis “there is not integral curves of positive contact inside $C^\gamma_\$.”
We repeat this process with $\gamma_1$. By this way we obtain two cases: either we find an hyperbolic point or we construct a sequence of points $(p_i)_i$ such that $p_i = \gamma_i(0) \neq \gamma_i(1)$, where $\gamma_i$ is the integral curve of $X$ through $p_i$. Assume the second case and let $h := d_\Gamma(\gamma_i(0), \gamma_i(1))$. Thus, by construction we have that $d_\Gamma(\gamma_i(0), \gamma_i(1)) < h/2^i \downarrow 0$. By continuity of $\Gamma$ we have that $d_\Gamma(\gamma_i(0), \gamma_i(1)) \downarrow 0$ implies that $\lim_i \gamma_i(0) = \lim_i \gamma_i(1) =: p \in \Gamma$. On the one hand we have that (formally speaking) $p_i \in \{\gamma_i(0), \gamma_i(1)\} \to \{p\}$, thus $\lim_i \gamma_i(0) = \lim_i p_i := p = \gamma_p(0)$ for a suitable parametrization of $\gamma_p$, the integral curve of $X$ through $p \in D$. Observe that $\gamma_i = \gamma_{p_i}$, hence

$$\gamma_p(0) = \gamma_{\lim_{i} p_i}(0) = \lim_i \gamma_i(0) = \lim_i \gamma_i(1) = \gamma_{\lim_{i} p_i}(1) = \gamma_p(1).$$

Again thanks to Lemma 3.1 we obtain our goal. Let us prove the uniqueness. Suppose that there are two hyperbolic points between $\gamma(0)$ and $\gamma(1)$ along $\Gamma$. Without loss of generality we can assume that there is not another hyperbolic point between them along $\Gamma$. Assume that $q_0, q_1 \in \Gamma$ are such hyperbolic points. Since there is not integral curves with positive contact in $C^*_\gamma$, every $q \in [q_0, q_1]_\Gamma$ cannot be elliptic, so they are all transverse to $\Gamma$. Therefore every integral curve $\gamma_{q_2}$ for $q_2 \in [q_0, q_1]_\Gamma$ satisfies either $q_0$ or $q_1$ belongs to the connected component $C^*_{\gamma_{q_2}}$. Assume that $q_1 \in C^*_{\gamma_{q_2}}$, thus there exists $q_3 \in [q_0, q_2]_\Gamma$ such that $q_0 \in C^*_{\gamma_{q_3}}$. Now we choose the middle point $q_4 \in [q_2, q_3]_\Gamma$ and repeat the procedure. Following this method we construct a sequence $(q_i)_i$ such that $q_i \in [q_{i-2}, q_{i-1}]_\Gamma$, where $q_1 \in C^*_{\gamma_{q_{i-2}}}$ and $q_0 \in C^*_{\gamma_{q_{i-1}}}$. Moreover, $d_\Gamma(q_{i-2}, q_{i-1}) \downarrow 0$, hence there exists $\lim_i q_i = q \in \Gamma$. Assume that $q_1 \in C^*_{\gamma_{q_{i}}}$, thus for every $r \in [q_0, q]_\Gamma$, $q_0 \in C^*_{\gamma_{q_{i}}}$, which is a contradiction with the hypothesis that $X$ is regular and has Hausdorff flow.

Observe that the result above ensure us that there is at least one hyperbolic/elliptic point “between” two elliptic/hyperbolic points of the same integral curve. Further, this hyperbolic/elliptic point is unique if there is not another integral curve with positive contact in the simply connected region delimited by such an integral curve and the boundary of $C$ (between these two elliptic/hyperbolic points). We will call these hyperbolic/elliptic points as generated by the integral curve, that depends on the connected component of $\partial C$ from such points are viewed: $B_2/B_1$. This allows us to obtain the following

**Corollary 3.3.** Let $X$ be a regular vector field over $C$ that has Hausdorff flow and positive contact w.r.t. $\partial C$. Let $\gamma: I \to C$ be an integral curve with positive contact w.r.t. $\partial C$. Let $e (h)$ be the number of elliptic (hyperbolic) points of $\gamma$ w.r.t. $\partial B_2 (\partial B_1)$. Let $h (e)$ be the number of the hyperbolic points generated by $\gamma$. Then the following statements hold:

1. $\gamma$ starts and ends in $\partial B_2$ if and only if $h = e + 1$.
2. $\gamma$ starts and ends in $\partial B_1$ if and only if $e = h + 1$.
3. $\gamma$ starts or ends (one of them) in $\partial B_2$ if and only if $h = e$.
4. $\gamma$ starts or ends (one of them) in $\partial B_1$ if and only if $e = h$.
5. $\gamma$ does neither starts nor ends in $\partial B_2$ if and only if $h = e - 1$.
6. $\gamma$ does neither starts nor ends in $\partial B_1$ if and only if $e = h - 1$. 
Under the hypothesis of this corollary we can easily deduce that the index of a vector field defined on $\mathcal{C}$, is the same w.r.t. $\partial B_1$ and $\partial B_2$.

To finish the paper we provide a theorem that allows us to verify if a vector field over $\mathcal{C}$ can be uniformized over the torus. Recall that this may be useful for the existence and computation of a first integral for such a vector field. We apply a technique of cut and paste similar to the one given in [5]. We just give a sketch of the proof.

**Theorem 3.4.** Let $X: \mathcal{C} \to T(\mathcal{C})$ be a regular $C^r$ vector field ($r \geq 1$) with finite contact with respect to $\partial \mathcal{C} = \partial B_1 \cup \partial B_2$ such that its flow is Hausdorff. Assume that $\text{ind}_{\partial B_2} X = 1$. Then $X$ is $C^r$-uniformizable over the torus and $\text{ind}_{\partial B_1} X = 1$.

**Sketch of the proof.** It suffices to deform the vector field over the plane. Then we use the diffeomorphism given in Theorem 2.3 to obtain the $C^r$-uniformization. We proceed by induction of the number or contact of the vector field with respect to $\partial B_2$. Assume that the contact is 1, hence we have two cases described by the diagrams:

!(a)

Diagram (a) corresponds to the case where the contact is 1, hence we have two cases described by the diagrams:

![Diagram](image)

**Figure 1.** Integral curve with contact 1.

Taking into account Corollary 3.3 and the Bendixson’s formula (3.1), only the case (a) satisfies that $\text{ind}_{\partial B_2} X = 1$. In such a case we proceed as in the following schema.

![Diagram](image)

**Figure 2.** Rectification procedure.

We rectify the integral curves of the domain $\mathcal{C}$. To this aim we have deformed the original domain. We choose an arbitrary integral curve $\alpha$ that starts/ends in...
$B_i$ and ends/starts in $B_j$, where $i \neq j$ and $i, j \in \{1, 2\}$. Cut by such an integral curve making up the corresponding identification of points. Then we rectify the vector field following the unique integral curve $\gamma$ that has contact with $B_2$. Once we have this new domain, taking into account the correspondence of points of the curve $\alpha$, we apply the diffeomorphism given in Theorem 2.3 to bring this new domain to the torus.

We now assume that the contact of $X$ w.r.t. $B_2$ is $n + 1$. Thanks to the induction hypothesis, there is a sector where we have $n$ points of contact, and it rest only one point of contact around which we have to rectify the vector field. We explain this situation taking two consecutive points of contact. In such a case, we can reduce the possibilities to the following six cases that we present below together with a description of the deformation that we have to do in each case.
To prove that the index of $X$ along $\partial B_1$ is also equal to 1 is a simple consequence of Corollary 3.3.

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