Abstract

We develop the general Theory of Cayley Hamilton algebras using norms and compare with the approach, valid only in characteristic 0, using traces and presented in a previous paper [20].
Norms and Cayley Hamilton algebras

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November 11, 2020

To the memory of Edoardo Vesentini

Contents

1 Invariants and representations 4
  1.1 $n$–dimensional representations ......................... 4
  1.4 Generic matrices and invariants .......................... 5
  1.7 Symmetry ............................................. 6

2 $n$–Cayley–Hamilton algebras 9
  2.1 Polynomial laws and determinants ........................ 9
    2.3.1 Multiplicative polynomial laws ........................ 10
  2.4 $n$–Cayley–Hamilton algebras ............................. 11
  2.8 Determinants .......................................... 12
  2.12 Symbolic approach ..................................... 14
  2.21 The free $n$–Cayley–Hamilton algebra .................... 18
  2.26 Azumaya algebras ..................................... 20
  2.34 $\Sigma$–algebras ...................................... 25
    2.42.1 The first and second fundamental Theorem for ma-
      trix invariants revisited ................................ 27
    2.47.1 The main Theorem of Cayley–Hamilton algebras 29

3 Prime and simple Cayley Hamilton algebras 30
  3.1 General facts .......................................... 30
    3.1.1 The kernel and the radical .......................... 30
    3.9.1 Semisimple algebras ................................ 33
    3.14.1 An abstract Theorem ................................ 35
  3.16 General semisimple algebras ............................. 36
Foreword

A basic fact for an \( n \times n \) matrix \( a \) (with entries in a commutative ring) is the construction of its characteristic polynomial \( \chi_a(t) := \det(t - a) \), \( t \) a variable, and the Cayley Hamilton theorem \( \chi_a(a) = 0 \).

The notion of Cayley Hamilton algebra (CH algebras for short), see Definition 2.45, was introduced in 1987 by Procesi [16] as an axiomatic treatment of the Cayley Hamilton theorem. This was done in order to clarify the Theory of \( n \)-dimensional representations, cf. Definition 1.2, of an associative and in general noncommutative algebra \( R \). With 1, unless otherwise specified, (from now on just called algebra).

The theory was developed only in characteristic 0, for two reasons; the first being that at that time it was not clear to the author if the characteristic free results of Donkin [9] and Zubkov [33] were sufficient to found the theory in general. The second reason was mostly because it looked not likely that the main theorem 2.48 could possibly hold in general.

The first concern can now be considered to have a positive solution, due to the contributions of several people, and we may take the book [8] as reference. As for the second, that is the main theorem in positive characteristic, the issue remains unsettled. The present author feels that it should not be true in general but has no counterexamples.

A partial theory in general characteristics replacing the trace with the determinant appears already in Procesi [13] and [17].

For the general definition, see 2.6 for details, we follow Chenevier [6]:

**Definition 0.1.** Given an algebra \( R \) over a commutative ring \( A \) and \( n \in \mathbb{N} \), an \( n \)-norm is a multiplicative polynomial law, as \( A \)-modules, \( N : R \to A \) homogeneous of degree \( n \) (see Definition 2.2).

An algebra \( R \) over a commutative ring \( A \) with an \( n \)-norm \( N : R \to A \) is a Cayley Hamilton algebra if, for every commutative \( A \) algebra \( B \), each \( a \in B \otimes_A R \) satisfies its characteristic polynomial \( \chi_a(t) := N(t - a) \), that is \( \chi_a(a) = 0 \).

The original definition over \( \mathbb{Q} \) is through the axiomatization of a trace, and closer to the Theory of pseudocharacters see [20]. The definition via
trace is also closer to the language of Universal algebra, while the one using norms is more categorical in nature.

This paper is a continuation of [20] where we have developed the theory in characteristic 0 using the notion of trace algebra. Here instead the approach is characteristic free and through the axiomatization of Norms.

The first section of this paper forms an exposition of known results with one or two new facts or proofs. We suggest the reader to start at §2: Cayley–Hamilton algebras. In §3 we treat the general structure theory of Cayley–Hamilton algebras and some structure of their $T$–ideals.

1 Invariants and representations

1.1 n–dimensional representations

Let us recall some basic facts which are treated in detail in the forthcoming book with Aljadeff, Giambruno and Regev [1].

For a given $n \in \mathbb{N}$ and a ring $A$ by $M_n(A)$ we denote the ring of $n \times n$ matrices with coefficients in $A$, by a symbol $(a_{i,j})$ we denote a matrix with entries $a_{i,j} \in A$, $i, j = 1, \ldots, n$.

In particular we will usually assume $A$ commutative so that the construction $A \mapsto M_n(A)$ is a functor from the category $C$ of commutative rings to that $R$ of associative rings. To a map $f : A \to B$ is associated a map $M_n(f) : M_n(A) \to M_n(B)$ in the obvious way $M_n(f)((a_{i,j})) := (f(a_{i,j}))$.

**Definition 1.2.** By an $n$–dimensional representation of a ring $R$ we mean a homomorphism $f : R \to M_n(A)$ with $A$ commutative.

The set valued functor $A \mapsto \text{hom}_R(R, M_n(A))$ is representable. That is, there is a commutative ring $T_n(R)$ and a natural isomorphism $j_A$

$$\text{hom}_R(R, M_n(A)) \xrightarrow{j_A} \text{hom}_C(T_n(R), A), \quad j_A : f \mapsto \bar{f}$$

given by the universal map $j_R : R \to M_n(T_n(R))$ and a commutative diagram $f = M_n(\bar{f}) \circ j_R$:

$$\begin{array}{ccc}
R & \xrightarrow{j_R} & M_n(T_n(R)) \\
\downarrow{f} & & \downarrow{M_n(\bar{f})} \\
& M_n(A) &
\end{array}$$  \hspace{1cm} (1)
The map $j_R : R \to M_n(T_n(R))$ is called the \textit{universal n–dimensional representation of R} or \textit{the universal map into $n \times n$ matrices.}

Of course it is possible that $R$ has no $n$–dimensional representations, in which case $T_n(R) = \{0\}$.

Of course the same discussion can be performed when $R$ is in the category $R_F$ of algebras over a commutative ring $F$. In this case the functor $A \mapsto \text{hom}_{R_F}(R, M_n(A))$ is on commutative $F$ algebras and $T_n(R)$ is an $F$ algebra.

The construction of $j_R$ is in two steps. First one easily sees that when $R = F\langle x_i \rangle_{i \in I}$ is a free algebra then:

**Proposition 1.3.** $T_n(R) = F[\xi_{h,k}^{(i)}]$ is the polynomial algebra over $F$ in the variables $\xi_{h,k}^{(i)}$, $i \in I$, $h, k = 1, \ldots, n$ and $j_R(x_i) = \xi_i := (\xi_{h,k}^{(i)})$ the generic matrix with entries $\xi_{h,k}^{(i)}$.

For a general algebra $R$ one may present it as a quotient $R = F\langle x_i \rangle/I$ of a free algebra. Then $j_F(x_i)(I)$ generates in $M_n(F[\xi_{h,k}^{(i)}])$ an ideal which is, as any ideal in a matrix algebra, of the form $M_n(J)$, with $J$ an ideal of $F[\xi_{h,k}^{(i)}]$. Then the universal map for $R$ is given by

$$
\begin{array}{ccc}
F\langle x_i \rangle & \longrightarrow & M_n(F[\xi_{h,k}^{(i)}]) \\
\downarrow & & \downarrow \\
R & \xrightarrow{j_R} & M_n(F[\xi_{h,k}^{(i)}]/J).
\end{array}
$$

By the universal property this is independent of the presentation of $R$.

### 1.4 Generic matrices and invariants

**Definition 1.5.** The subalgebra $F\langle \xi_i | i \in I \rangle$ of $M_n(F[\xi_{h,k}^{(i)}])$, $i \in I$, $h, k = 1, \ldots, n$ generated by the matrices $\xi_i$ is called the algebra of \textit{generic matrices}.

A classical Theorem of Amitsur states that, if $F$ is a domain then $F\langle \xi_i | i \in I \rangle$ is a domain. If $I$ has $\ell$ elements we also denote $F\langle \xi_i | i \in I \rangle = F_n(\ell)$. If $\ell \geq 2$ then $F_n(\ell)$ has a division ring of quotients $D_n(\ell)$ which is of dimension $n^2$ over its center $Z_n(\ell)$. These algebras have been extensively studied. One defines first the commutative subalgebra $T_n(\ell) \subset Z_n(\ell)$ generated by
the coefficients $\sigma_i(a)$ of the characteristic polynomial $\det(t - a) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(a)t^{n-i}$, $\forall a \in F_n(\ell)$. Next define $S_n(\ell) = F_n(\ell)T_n(\ell) \subset D_n(\ell)$, one can understand $S_n(\ell)$ and $T_n(\ell)$ by invariant theory, see Remark 2.23.

The invariant theory involved is presented in [14] when $F$ is a field of characteristic 0, and may be considered as the first and second fundamental theorem of matrix invariants. For a characteristic free treatment see the book [8]. In general, for simplicity of exposition assume that $F$ is an infinite field:

**Theorem 1.6.** The algebra $T_n(\ell)$ is the algebra of polynomial invariants under the simultaneous action of $GL(n, F)$ by conjugation on the space $M_n(F)^\ell$ of $\ell$–tuples of $n \times n$ matrices.

The algebra $S_n(\ell)$ is the algebra of $GL(n, F)$–equivariant polynomial maps from the space $M_n(F)^\ell$ of $\ell$–tuples of $n \times n$ matrices to $M_n F$.

As usual together with a first fundamental theorem one may ask for a second fundamental theorem which was proved independently by Procesi [14] and Razmyslov [21] when $F$ has characteristic 0 and by Zubkov [33] in general. In this paper it will appear as characterization of free Cayley–Hamilton algebras, Theorem 2.22.

For a general algebra, quotient of the free algebra again one may add to $R$ the algebra $T_n(R)$ generated by the coefficients of the characteristic polynomial $\sigma_i(a)$, $\forall a \in \mathcal{J}_R(R)$.

1.7 Symmetry

The functor $\text{hom}_R(R, M_n(A))$ has a group of symmetries: the projective linear group $\text{PGL}(n)$.

It is best to define this as a representable group valued functor on the category $C$ of commutative rings. The functor associates to a commutative ring $A$ the group $\mathfrak{S}_n(A) := \text{Aut}_A(M_n(A))$ of $A$–linear automorphisms of the matrix algebra $M_n(A)$. One has a natural homomorphism of the general linear group $GL(n, A)$ to $\mathfrak{S}_n(A)$ which associates to an invertible matrix $X$ the inner automorphism $a \mapsto XaX^{-1}$.

The functor general linear group $GL(n, A)$ is represented by the Hopf algebra $\mathbb{Z}[x_{i,j}][d^{-1}]$, $i, j = 1, \ldots, n$ with $d = \det(X)$, $X := (x_{i,j})$ with the usual structure given compactly by comultiplication $\delta$, antipode $S$ and counit $\epsilon$:

$$\delta(X) = X \otimes X, \quad S(X) := X^{-1}, \quad \epsilon : X \to 1_n.$$
The functor $\mathfrak{S}_n(\mathcal{A})$ is represented by the sub Hopf algebra, of $GL(n, \mathcal{A})$, $P_n \subset \mathbb{Z}[x_{i,j}][d^{-1}]$ formed of elements homogeneous of degree 0. It has a basis, over $\mathbb{Z}$, of elements $ad^{-h}$ where $a$ is a doubly standard tableaux with no rows of length $n$ and of degree $h \cdot n$. For a proof see [1] Theorem 3.4.21.

Remark 1.8. Of course if we work in the category of commutative $F$ algebras we replace $P_n$ with $P_n \otimes_\mathbb{Z} F \subset F[x_{i,j}][d^{-1}]$.

The identity map $1_{P_n} : P_n \to P_n$ induces a generic automorphism of $M_n(P_n)$ which can be given by conjugation by the generic matrix $X$:

$$J : M_n(P_n) \to M_n(P_n), \quad J : a \mapsto XaX^{-1} \in M_n(P_n).$$

Observe that $X$ is not a matrix in $P_n$ only the entries of $XaX^{-1}$ are in $P_n$ i.e. are homogeneous of degree 0 in the entries of $X$. If we denote by $y_{i,j}$ the $i,j$ entry of $X^{-1}$ (given by the usual formula as the cofactor divided by the determinant), we have

$$Xe_{i,j}X^{-1} = (z^{(i,j)}), \quad z^{(i,j)}_{h,k} = x_{h,i}y_{j,k}.$$

In fact the entries $z^{(i,j)}_{h,k} = x_{h,i}y_{j,k}$ of the matrices $Xe_{i,j}X^{-1}$ generate $P_n$ as algebra and the ideal of relations is generated by the relations expressing the fact that the map $e_{i,j} \mapsto (z^{(i,j)}_{h,k})$ is a homomorphism (and then automatically an isomorphism).

Then given an automorphism $g \in Aut_A(M_n(\mathcal{A}))$ its associated classifying map $\bar{g} : P_n \to A$ fits in the commutative diagram:

$$M_n(P_n) \xrightarrow{J} M_n(P_n) \quad (3)$$

$$\xymatrix{ M_n(P_n) \ar[d]_{M_n(\bar{g})} \ar[rr]^J & & M_n(P_n) \ar[d]_{M_n(\bar{g})} \\ M_n(A) \ar[rr]^g & & M_n(A) \quad }$$

Finally we have an action of $\mathfrak{S}_n(\mathcal{A})$ on $\text{hom}_\mathcal{A}(R, M_n(\mathcal{A}))$ by composing a map $f$ with an automorphism $g$. One has a commutative diagram

$$\xymatrix{ R \ar[r]^{J_R} & M_n(T_n(R)) \ar[d]^g \\ M_n(A) \ar[u]^f & M_n(g \circ f) \ar[d] \\ M_n(A) \ar[r]^g & M_n(A) \quad (4) }$$

Assume now that $R$ is an $F$ algebra so also $T_n(R)$ is an $F$ algebra and $M_n(T_n(R)) = T_n(R) \otimes_F M_n(F)$. An automorphism $g$ of $M_n(F)$ induces
an automorphism $1 \otimes g$ of $M_n(T_n(R))$. Take $A = T_n(R)$ and $f = j_R$ in Formula (4) and set $\hat{g} := 1 \otimes g \circ j_R$, an automorphism of $T_n(R)$, so that 

$$M_n(\hat{g}_1 \circ \hat{g}_2) \circ j_R = 1 \otimes (g_1 \circ g_2) \circ j_R = 1 \otimes g_1 \circ M_n(\hat{g}_2) \circ j_R$$

implies $\hat{g}_1 \circ \hat{g}_2 = \hat{g}_2 \circ \hat{g}_1$. Which implies that the map $g \mapsto \hat{g}$ is an anti-

homomorphism from $\text{Aut}_F(M_n(F))$ to $\text{Aut}(T_n(R))$. Finally

**Proposition 1.9.** The map $g \mapsto g \otimes \hat{g}^{-1}$ is a homomorphism from the
group $\text{Aut}_F M_n(F)$ to the group of all automorphisms of $M_n(T_n(R))$. The image of $R$ under $j_R$ is formed of invariant elements.

This is particularly simple when $F$ is an infinite field. In this case the group $\text{Aut}_F M_n(F) = \text{PGL}(n, F)$ is Zariski dense in $\text{PGL}(n, \bar{F})$ with $\bar{F}$ an algebraic closure of $F$. Otherwise this can be set in the language of polynomial laws.

**Proposition 1.10.** For every commutative $F$ algebra $B$, an automorphism $g \in \text{Aut}_B M_n(B)$ induces an automorphism of the $B$ algebra $T_n(R) \otimes_F B$. The map $g \mapsto g \otimes \hat{g}^{-1}$ is a homomorphism from the group $\text{Aut}_B M_n(B)$ to the group of all automorphisms of $M_n(B) \otimes_B (T_n(R) \otimes_F B) = M_n(T_n(R) \otimes_F B)$. The image of $gR$ under $j_R \otimes 1_B$ is formed of invariant elements.

The functor $\text{G}_n(A) \times \text{hom}_R(R_n, M_n(A))$ is represented by $P_n \otimes T_n(R)$. Given $f \in \text{hom}_R(R_n, M_n(A))$ associated to $\tilde{f} : T_n(R) \rightarrow A$ and an automorphism $g$ of $M_n(A)$ associated to $\hat{g} : P_n \rightarrow A$ the pair is associated to

$$\rightarrow A \otimes A \xrightarrow{m} A, \quad m(a \otimes b) := ab.$$ 

The natural transformation $\text{G}_n(A) \times \text{hom}_R(R_n, M_n(A)) \rightarrow \text{hom}_R(R_n, M_n(A))$ induces the coaction on the classifying rings $\eta : T_n(R) \rightarrow P_n \otimes T_n(R)$.

Thus the composition $g \circ f \in \text{hom}_R(R_n, M_n(A))$ is associated to

$$\eta \circ \tilde{f} : T_n(R) \xrightarrow{\eta} P_n \otimes T_n(R) \xrightarrow{\hat{g} \otimes \tilde{f}} A \otimes A \xrightarrow{m} A, \quad m(a \otimes b) := ab.$$ 

In particular for $R = \mathbb{Z}(x_i)_{i \in I}$ we have $T_n(\mathbb{Z}(x_i)_{i \in I})$ is the polynomial

ring in the entries $\xi^{(k)}_{i,j}$ of the generic matrices $\xi_k$ and the action is via the formula

$$\xi_k \mapsto X \xi_k X^{-1}, \quad \xi^{(k)}_{i,j} \mapsto \sum_{a,b} x_{i,a} \xi^{(k)}_{a,b} y_{b,j}, \quad X^{-1} = (y_{i,j}).$$
We have $x_{i,a}y_{b,j} \in P_n$. The induced map $T_n(Z(x_i)_{i \in I}) \to P_n \otimes T_n(F(x_i)_{i \in I})$ can be identified to the coaction.

\[ \eta : T_n(R) \to P_n \otimes T_n(R), \quad \eta(\xi_{i,j}) = \sum_{a,b=1}^n x_{i,a} \xi_{a,b} y_{b,j}. \] (5)

Remark 1.11. In general the invariants are the elements invariant under the generic automorphism. In our case:

\[ T_n(R)^{\text{inv}} := \{ a \in T_n(R) \mid \eta(a) = 1 \otimes a \}. \] (6)

2 \hspace{1em} n–Cayley–Hamilton algebras

2.1 Polynomial laws and determinants

Given a commutative ring $F$, an $F$ module $M$, and a commutative $F$ algebra $B$ one has the base change functor from $F$–modules to $B$–modules:

\[ _BM := B \otimes_F M. \] (7)

Recall that, in [25] and [26], Roby defines:

**Definition 2.2.** A polynomial law between two $F$ modules $M, N$ is a natural transformation of the two set valued functors on the category $\mathcal{C}_F$ of commutative $F$ algebras:

\[ f_B : _BM \to _BN, \quad B \in \mathcal{C}_F. \] (8)

Such a law is homogeneous of degree $n$ if:

\[ f_B(ba) = b^n f_B(a), \quad \forall b \in B, \quad \forall a \in _BM, \quad \forall B \in \mathcal{C}_F. \]

Given $M$ let us denote, for $N$ any module, by $\mathcal{P}_n(M,N)$ the polynomial laws homogeneous of degree $n$ from $M$ to $N$. This, by Roby's Theory, is a set valued functor on modules $N$ and it is representable.

This is done by constructing the divided powers $\Gamma_n(M) = \Gamma_{n,F}(M)$ (over the base $F$) together with the map $i_M : m \mapsto m^{[n]}$ of $M$ to $\Gamma_n(M)$.

The divided powers $\Gamma_n(M)$ are constructed by generators and relations. The construction is compatible with base change, that is given a commutative $F$ algebra $B$ we have:

\[ \Gamma_{n,B}(_BM) = _B\Gamma_{n,F}(M). \]
In fact in most applications there is a more concrete description of \( \Gamma_n(M) \). For instance if \( M \) is a free (or just projective) \( F \) module one describes the divided power as symmetric tensors:

\[
\Gamma_n(M) \simeq (M^\otimes n)^{S_n}, \quad M^\otimes n = M \otimes_F M \otimes_F \cdots \otimes_F M.
\] (9)

From now on we assume to be in this case. Notice that by change of coefficient rings if \( F \to A \) is a map of commutative rings we have \( A\, M := A \otimes_F M \) and

\[
A\, M^\otimes n = A \otimes_F M^\otimes n, \quad A\, \Gamma_n(M) = \Gamma_n(A\, M).
\]

**Remark 2.3.** In general a polynomial law is not determined by the map \( f : M \to N \). Under further hypothesis this is the case, the simplest being that the base commutative ring \( F \) contains an infinite field.

### 2.3.1 Multiplicative polynomial laws

If we have two \( F \)-algebras \( R, S \) we have the notion of multiplicative polynomial law \( d : R \to S \) that is

\[
d(ab) = d(a)d(b), \quad \forall a, b \in BR, \quad \forall B.
\]

One proves that, if \( R \) is an algebra, then \( \Gamma_n(R) \) is also an algebra, which we call the \( n \)th Schur algebra of \( R \), see [8], and \( i_R \) is a universal multiplicative polynomial map, homogeneous of degree \( n \). That is any multiplicative polynomial law, homogeneous of degree \( n \) from \( R \) to an algebra \( S \) factors through \( i_R \) and a homomorphism of \( \Gamma_n(R) \) to \( S \).

Denoting by \( \mathcal{M}_n(R, S) \) the set of multiplicative polynomial laws homogeneous of degree \( n \) from \( R \) to \( S \) we have the isomorphism:

\[
\varphi : d \mapsto d \circ i_R, \quad \hom_R(\Gamma_n(R), S) \xrightarrow{\sim} \mathcal{M}_n(R, S), \quad S \in R.
\] (10)

When we have \( \Gamma_n(R) = (R^\otimes n)^{S_n} \), Formula (9), the algebra structure on \( \Gamma_n(R) \) is induced by the tensor product of algebras.

If \( S \) is any algebra we denote by \( S_{ab} \) its abelianization, that is \( S \) modulo the ideal \( C_S \) generated by all commutators \([x, y] = x \cdot y - y \cdot x\).

When we restrict, in Formula (10), to \( S \) commutative, this functor, on commutative algebras, is represented by \( \pi : \Gamma_n(R) \to \Gamma_n(R)_{ab} \) that is the abelianization of \( \Gamma_n(R) \).

\[
\mathcal{M}_n(R, A) \simeq \hom_C(\Gamma_n(R)_{ab}, A), \quad A \in C.
\] (11)

This is an object studied by Roby in [26], and by Ziplies in [32] and discussed in some detail in my book with De Concini [8].
2.4 $n$–Cayley–Hamilton algebras

Consider an algebra $R$, over a commutative ring $Z$.

**Definition 2.5.** A multiplicative polynomial law, of $Z$ algebras, homogeneous of degree $n$, $N : R \to Z$ will be called a norm.

One may apply Roby’s theory. First $N(r) = \tilde{N}(r^\otimes n)$ then, by Formula (11), the map $\tilde{N}$ factors through the abelianization $\Gamma_n(R)_{ab}$ of $\Gamma_n(R)$ (we still denote it by $\tilde{N}$).

$R \xrightarrow{r \mapsto r^\otimes n} \Gamma_n(R) \xrightarrow{\pi} \Gamma_n(R)_{ab} \quad \Gamma_n(R)$ the $n^{th}$ divided power.

For $a \in B \otimes_Z R$ and a commutative variable $t$ we have $t - a \in B[t] \otimes_Z R$ and can define the characteristic polynomial of $a$ as $\chi_a(t) := N(t - a) \in B[t]$.

**Definition 2.6.** We say that $R$ is an $n$–Cayley–Hamilton algebra if one has the analogue of the Cayley–Hamilton Theorem $\chi_a(a) = 0$, $\forall a \in B R$, $\forall B$.

Sometimes we will use the short notation CH–algebra for Cayley–Hamilton algebra.

**Remark 2.7.** In [19] we treated the Theory in characteristic 0. In this case it is better to use instead of the norm the trace. This at the same time simplifies the treatment but also yields stronger results due to the linear reductivity of the linear group.

We start with an important example.

If $F$ is an infinite field and $V$ a vector space of some finite dimension $m$ over $F$ one has for $R = \text{End}_F(V)$ that

$$(\text{End}_F(V)^\otimes n)_{S_n} = \text{End}_F(V^\otimes n)_{S_n} = \text{End}_{F[S_n]}(V^\otimes n).$$

The algebra $\text{End}_{F[S_n]}(V^\otimes n)$ is spanned by the elements $g^\otimes n$ where $g \in \text{GL}(V)$.

If the characteristic of $F$ is 0 one has the decomposition

$$V^\otimes n = \bigoplus_{\lambda - n|\text{ht}(\lambda) \leq m} M_\lambda \otimes S_\lambda(V)$$
where $M_\lambda$ is the irreducible representation of $S_n$ corresponding to $\lambda$ and $S_\lambda(V)$, a Schur functor, the corresponding irreducible representation of $GL(V)$. One proves that

$$\text{End}_F(V^{\otimes n})^{S_n} = \oplus_{\lambda \vdash n, \text{ht} \lambda \leq m} \text{End}(S_\lambda(V)).$$

Each summand of this decomposition is a simple algebra and abelian only when $S_\lambda(V)$ is 1–dimensional. This happens only if $n = im$ and $S_\lambda(V) = \wedge^m(V)^\otimes i$. In this case

$$\text{End}_F(V^{\otimes n})^{S_n}_{ab} = \text{End}((\wedge^m(V)^\otimes i), N(a) = \det(a)^i, a \in \text{End}_F(V).$$

By the remarkable theory of good filtrations or combinatorially of standard double tableaux, in positive characteristic or over the integers the previous formula can be replaced by a canonical filtration of which Formula (13) is the associated graded representation. From this one could deduce a general form of Formula (14).

### 2.8 Determinants

Given an algebra $R$ the composition of a homomorphism $j : R \to M_n(A)$ with the determinant $\det \circ j : R \to M_n(A) \overset{\text{det}}{\to} A$ is a multiplicative polynomial law, homogeneous of degree $n$. One then obtains a natural transformation of functors:

$$\text{hom}_{R,F}(R, M_n(A)) \to M_n(R, A),$$

and a commutative diagram.

$$R \xrightarrow{i_R} \Gamma_n(R) \xrightarrow{\pi} \Gamma_n(R)_{ab} \xrightarrow{D} T_n(R), \ D \circ \pi \circ i_R = \det \circ j_R.$$
Remark 2.9. [Problem] One of the main problems of the theory is to understand when is that $D : \Gamma_n(R)_{ab} \to T_n(R)_{\Phi_n}$ is an isomorphism.

This happens in several cases. In particular we have a fundamental result, Theorem 20.24 of [8].

Theorem 2.10. Consider the free algebra $A(X)$ in some set of variables $X = \{x_i\}_{i \in I}$ with $A$ a field or the integers.

Then $D : \Gamma_n(A(X))_{ab} \to T_n(A(X))_{\Phi_n}$ is an isomorphism.

Notice that $T_n(A(X))_{\Phi_n}$ is the ring of invariants of $X$-tuples of $n \times n$ matrices.

Theorem 2.10 is based on a Theorem of Procesi [14] and Razmyslov [23] characterizing trace identities of matrices (cf. Theorem 2.43), and proved by Zieplies [31] and Vaccarino [28], when $A = \mathbb{Q}$.

The general case is fully treated in [8]. It is based on the characteristic free results of Donkin [9] and Zubkov [33] on the invariants of matrices and a careful combinatorial study of $\Gamma_n(A(X))$ inspired by the work of Zieplies.

In fact since $A(X)$ is a free $A$ module, its divided power is more conveniently described as the symmetric tensors:

$$\Gamma_n(A(X)) \simeq (A(X)^{\otimes n})^{S_n}.$$ 

Since, using the basis of monomials, the space $A(X)^{\otimes n}$ is a permutation representation of $S_n$, one has a combinatorial description of $(A(X)^{\otimes n})^{S_n}$. The abelian quotient, isomorphic to the ring of invariants of matrices, does not have a combinatorial description and it is a rather hard object to study.

Example 2.11. If $X = \{x\}$ is a single variable we have that $A(X) = A[x]$ is the commutative ring of polynomials, $A[x]^{\otimes n} = A[x_1, \ldots, x_n]$ so $\Gamma_n(A[x])$ is the algebra of symmetric polynomials in $n$-variables, commutative.

Consider the elementary symmetric function $e_i$ defined by:

$$(t - x)^{\otimes n} = \prod_{i=1}^{n} (t - x_i) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \ldots + (-1)^n e_n. \quad (17)$$

The map $D$ maps $(t - x)^{\otimes n}$ to $\det(t - \xi) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(\xi) t^{n-i}$, the characteristic polynomial of a generic matrix $\xi = (\xi_{i,j})$.

So the elementary symmetric function $e_i$ maps to $\sigma_i(\xi)$, the generators of invariants of a single matrix.

This is a very special case of Theorem 2.10.
Although usually one deduces the functions $\sigma_i(x)$ for an $n \times n$ matrix $x$ from the determinant of $t - x$ one sees immediately that embedding the $n \times n$ matrices into $n + 1 \times n + 1$ matrices as upper left corner

$$i_n(x) := \begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix}, \quad \det(t - i_n(x)) = \det \begin{vmatrix} t - x & 0 \\ 0 & t \end{vmatrix} = \det(t - x)t$$

implies $\sigma_i(x) = \sigma_i(i_n(x))$ so the functions $\sigma_i(x)$ are in fact also defined for infinite matrices with only finitely many non 0 entries $\bigcup_n M_n(F)$. It is thus important to understand the symbolic calculus on these functions and this is the theme of the next section.

### 2.12 Symbolic approach

The construction $R \mapsto \Gamma_n(R)$ is functorial in $R$ so given $r \in R$ the map $A[x] \to R, \ x \to r$ induces a map $\Gamma_n(A[x]) \to \Gamma_n(R)$ under which $e_i \to \tau_i(r), \ (1 + r)^\otimes n = 1 + \tau_1(r) + \tau_2(r) + \ldots + \tau_n(r)$.

**Proposition 2.13.** It is proved, in [8] Lemma 20.12, that, given a basis $a_i$ of $R$, the elements $\tau_i(a_j)$ generate $(R^\otimes n)^{S_n}$. Once we pass to the abelianization and set $\sigma_i(a)$ to be the class of $\tau_i(a)$ we have $\sigma_i(ab) = \sigma_i(ba), \ \forall a, b$, see [8] Proposition 20.20.

We apply this construction to the free algebra. The grading of the algebra $A\langle X \rangle$ induces a grading of $\Gamma_n(A(X))$. Recall that a monomial $M$ of positive length, is called primitive if it is not a power $N^k, k > 1$.

From Proposition 2.13 and Formula (22) one sees that the elements $\tau_i(M), \ i = 1, \ldots, n$ as $M$ runs over the primitive monomials generate $\Gamma_n(A(X))$. The elements $\tau_i(M)$ satisfy complicated relations which are not fully understood.

**Definition 2.14.** Denote by $S_{n,A}$ the abelian quotient of $\Gamma_n(A(X))$ and by $\sigma_i(M)$ the class of $\tau_i(M)$ in $S_{n,A}$.

One can prove, Proposition 20.20 of [8], that if $M = AB$ one has $\sigma_i(AB) = \sigma_i(BA)$, one says that $AB$ and $BA$ are cyclically equivalent.

A Lyndon word, is a primitive monomial minimal, in the lexicographic order, in its class of cyclic equivalence.

As for the theory of symmetric functions one can pass to the limit, as $n \to \infty$, of the algebras $\Gamma_n(A(X))$ and their abelian quotients.
If \( \epsilon : A\langle X \rangle \to A \) is the evaluation of \( X \) in 0, we have the map
\[
\pi_n : A\langle X \rangle \otimes^{n+1} A = A\langle X \rangle \otimes^n, \quad \pi_n(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = a_1 \otimes \ldots \otimes a_n \otimes \epsilon(a_{n+1}).
\]
This induces a map, still called \( \pi_n : \Gamma_{n+1}(A\langle X \rangle) \to \Gamma_n(A\langle X \rangle) \). We have
\[
\pi_n(\tau_i(M)) = \begin{cases} 
\tau_i(M) & \text{if } i \leq n \\
0 & \text{if } i = n+1
\end{cases}.
\]

**Definition 2.15.** One can then define a limit algebra \( \Gamma_\infty(A\langle X \rangle) \) generated by the elements \( \tau_i(M) \), \( i = 1, \ldots, \infty \) as \( M \) runs over the primitive monomials and its abelian quotient \( S_A\langle X \rangle \) denoted often just \( S_A \), generated by the classes \( \sigma_i(M) \) of \( \tau_i(M) \).

The maps \( \pi_n \) give rise to limit maps:
\[
\tau_n : \Gamma_\infty(A\langle X \rangle) \to \Gamma_n(A\langle X \rangle), \quad \tau_n : S_A\langle X \rangle \to S_{n,A}.
\]

In [8] we have proved:

**Corollary 20.15** The algebra \( \Gamma_\infty(Q\langle X \rangle) \) is the universal enveloping algebra of \( Q^+_\langle X \rangle \) considered as a Lie algebra.

As for the structure of \( S_{n,A} \), Theorem 20.22 of [8] (due to Zieplies) states that, in the commutative algebra \( S_A \) one has \( \sigma_i(AB) = \sigma_i(BA) \) for all monomials \( A, B \) and finally that \( S_A = A[\sigma_i(M)] \), \( M \) a Lyndon word, is the free polynomial ring in the variables \( \sigma_i(M) \) as \( M \) varies among the Lyndon words.

Let \( T_A(X) \) denote the monoid of endomorphisms of \( A\langle X \rangle \) given by mapping each variable \( x_i \in X \) to some element \( f_i \in A\langle X \rangle_+ \), the ideal kernel of \( \epsilon \) of elements with no constant term, this condition means that these endomorphisms commute with the map \( \epsilon \). Each such endomorphism induces an endomorphism of each \( \Gamma_n(A\langle X \rangle) \) compatible with the maps \( \pi_n \) and hence an endomorphism on \( S_{n,A} := \Gamma_n(A\langle X \rangle)_{ab} \) and on \( \Gamma_\infty(A\langle X \rangle) \) and \( S_A \).

**Definition 2.16.** A \( T \)-ideal of \( A\langle X \rangle \) or of \( \Gamma_n(A\langle X \rangle) \), or \( S_{n,A} \), \( n = 1, \ldots, \infty \) is a multigraded ideal \( I \) closed under all endomorphisms induced by \( T_A(X) \).

**Remark 2.17.** The condition of \( I \) to be multigraded can be replaced by the condition that, for every commutative \( A \) algebra \( B \), the ideal \( _BI := B \otimes_A I \) is closed under all endomorphisms induced by \( T_B(X) \). This is in the spirit of polynomial laws.
For each $i = 1, 2, \ldots$ we have the maps
\[ f \mapsto \tau_i(f), \ A\langle X \rangle_+ \to \Gamma_\infty(A\langle X \rangle); \quad f \mapsto \sigma_i(f), \ A\langle X \rangle_+ \to S_A. \] (19)
They are both polynomial laws homogeneous of degree $i$ which commute with the action of the endomorphisms $T(X)$.

There is an explicit Formula, see [8] Theorem 4.15 p. 37, which allows us to compute these laws for $S_A$. It is due to Amitsur, [2], (who stated it for matrix invariants), and later independently by Reutenauer and Schützenberger [24].

**Theorem 2.18.** Given an $n \in \mathbb{N}$, non commutative variables $x_i$ and commutative parameters $t_i$:

\[
\sigma_n(\sum_i t_i x_i) = \sum_{(p_1 < \ldots < p_k) \in W_0, \ j_1, \ldots, j_k \in \mathbb{N}, \ \sum j_i \ell(p_i) = n} (-1)^{n-\sum j_i \sum_{i=1}^k j_i \nu(p_i)} \sigma_{j_1}(p_1) \cdots \sigma_{j_k}(p_k)
\] (20)

Here $W_0$ denotes the set of Lyndon words ordered by the degree lexicographic order. For a word $p$, $\nu(p)$ is the vector $(a_1, \ldots, a_n)$ with $a_i$ counting how many times the variable $x_i$ appears in $p$. Finally $t^{(a_1, \ldots, a_n)} := \prod_{i=1}^n t_i^{a_i}$.

In particular one can collect, in Formula (20) the terms of the same degree in the variables $x_i$ and have an explicit expression of the **polarized** forms of $\sigma_n(x)$:

\[
\sigma_n(\sum_i t_i x_i) = \sum_{(a_1, \ldots, a_n) \mid \sum a_i = n} \prod_{i=1}^n t_i^{a_i} \sigma_{n;a_1, \ldots, a_n}(x_1, \ldots, x_n).
\] (21)

Substituting for a variable $x$ a linear combination $\sum_j t_j M_j$ of monomials and applying Formula (20) to $\sigma_n(\sum_j t_j M_j)$ one obtains an element of $S_A$ provided one has a further law. In fact a primitive word computed in monomials need no more be primitive so we also need the expression of the elements $\sigma_i(x^j)$ in terms of the $\sigma_k(x), k \leq i \cdot j$. These Formulas arise from Example 2.11 as the (stable) universal polynomial formulas in the algebra of symmetric functions expressing

\[ e_i(x_1^j, \ldots, x_n^j) = P_{i,j}(e_1, \ldots, e_{i,j}) \] (22)
in terms of the elementary symmetric functions $e_k(x_1, \ldots, x_n)$ for $n > i \cdot j$.

One has
Theorem 2.19. The Kernels of the maps \( \Gamma_\infty(A(X)) \to \Gamma_n(A(X)) \), respectively \( S_A \to S_{n,A} \) are the \( T \)-ideals generated by all the elements \( \tau_i(f) \), \( i > n \), respectively the \( T \)-ideal generated by all the elements \( \sigma_i(f) \), \( i > n \), \( f \in A(X) \).

In other words, in the case \( S_{n,A} \), the Kernel of \( \pi_n \) is the ideal generated by all the polarized forms \( \sigma_{m,a_1,\ldots,a_n}(p_1,\ldots,p_m) \), \( m > n \) with \( p_1,\ldots,p_m \) monomials.

The Theorem of Zubkov then states that the ring of invariants of matrices has the same generators and relations as \( S_{n,A} \), when \( A = \mathbb{Z} \) or a field, hence the isomorphism of Theorem 2.10.

The following example shows for \( n = 2 \) an explicit deduction of the multiplicative nature of the determinant from these relations.

\[
\sigma_{3;1,1,1}(a,b,ba) = \\
+ \sigma_1(a)\sigma_1(b)\sigma_1(ab) - \sigma_1(a)\sigma_1(ab^2) - \sigma_1(b)\sigma_1(a^2b) + \sigma_1(a^2b^2) - 2\sigma_2(ab) \\
\sigma_{4;2,2}(a,b) = \\
- \sigma_1(a)\sigma_1(b)\sigma_1(ab) + \sigma_1(a)\sigma_1(ab^2) + \sigma_1(b)\sigma_1(a^2b) - \sigma_1(a^2b^2) \\
+ \sigma_2(ab) + \sigma_2(a)\sigma_2(b) \\
\sigma_{3;1,1,1}(a,b,ba) + \sigma_{4;2,2}(a,b) = \sigma_2(ab) - \sigma_2(a)\sigma_2(b).
\]

One can in fact take the basic identity for invariants of \( n \times n \) matrices.

\[
\det(ab) = \det(a)\det(b) \iff \sigma_n(ab) - \sigma_n(a)\sigma_n(b) = 0. \tag{23}
\]

Consider the polynomial ring \( A[\sigma_i(p)] \), \( p \in W_0 \), \( i \leq n \). Using Formula (20) define, for each \( f = \sum_i t_i M_i \in A(X) \), the element \( \sigma_k(f) \), \( k \leq n \) as follows.

If one substitutes each \( x_i \) with \( M_i \) in Formula (20) one has a formal expression containing symbols \( \sigma_i(M) \), \( i \leq k \) where \( M \) may be an arbitrary monomial (including 1). Then \( M \) is cyclically equivalent to some power \( N^j \) with \( N \in W_0 \) a Lyndon word. One then considers in the algebra of symmetric functions in exactly \( n \) variables the Formula (22) with \( e_i = 0 \), \( \forall i > n \) i.e. \( e_i(x_1^i,\ldots,x_n^i) = P_{i,j}(e_1,\ldots,e_n,0,\ldots,0) \) in terms of the elementary symmetric functions \( e_k(x_1,\ldots,x_n) \), \( k \leq n \). One then sets

\[
\sigma_i(N^j) = P_{i,j}(\sigma_1(N),\ldots,\sigma_n(N)), \quad \sigma_i(1) := \binom{n}{i}.
\]

Given \( f = \sum_i u_i M_i \), \( g = \sum_i v_i M_i \in A(X) \) one may consider

\[
\sigma_n(fg) - \sigma_n(f)\sigma_n(g) = \sum_{\hat{h}\hat{k}} \varphi_{\hat{h}\hat{k}} \in \varphi_{\hat{h}\hat{k}} \in A[\sigma_i(p)]. \tag{25}
\]
Evaluating the variables $\xi_i \in X$ in the generic $n \times n$ matrices one has a homomorphism $\rho : A\langle X \rangle \to A[\xi_i]$ to the algebra of generic matrices which extends to a homomorphism of the symbolic algebra $\rho : A[\sigma_i(p)] \to A[\xi_{i,j}]^{PGL(n)}$ to the ring of invariants of matrices. By the Theorem of Donkin this is surjective. Moreover clearly the identity given by (24) holds for the corresponding matrix invariants. As for (25) we have that $\sigma_n(\rho(fg)) = \sigma_n(\rho(f))\sigma_n(\rho(g)) = \det(\rho(f)\rho(g)) = \det(\rho(f))\det(\rho(g)) = 0$ so all the elements $\varphi_{\bar{h},\bar{k}}$ map to 0.

**Theorem 2.20.** The Kernel of $\rho$ is the ideal $K$ of $A[\sigma_i(p)]$ generated by the elements $\varphi_{\bar{h},\bar{k}}$ of Formula (25), when computed using Formula (24) and (20).

**Proof.** The previous relations express the identity $\sigma_n(fg) = \sigma_n(f)\sigma_n(g)$. Consider the algebra $A[\sigma_i(p)]/K$, and the map $A\langle X \rangle \to A[\sigma_i(p)]/K$ mapping $f \in A\langle X \rangle$ to the class $\bar{\sigma}_n(f)$ of $\sigma_n(f)$ modulo $K$.

By construction this is a multiplicative map homogeneous of degree $n$ so it factors through a map $A\langle X \rangle \to \Gamma_n(A\langle X \rangle)_{ab} \xrightarrow{\bar{\rho}} A[\sigma_i(p)]/K$. On the other hand $\Gamma_n(A\langle X \rangle)_{ab}$ is generated by the elements $\sigma_i(p)$ and the generators of $K$ are 0 in $\Gamma_n(A\langle X \rangle)_{ab}$ hence $\bar{\rho}$ is an isomorphism and so the claim follows from Theorem 2.10.

\[ \square \]

### 2.21 The free $n$–Cayley–Hamilton algebra

$n$–Cayley–Hamilton algebras over a commutative ring $F$ form a category, where a map is assumed to commute with the norm.

If the base ring $F$ is either $\mathbb{Z}$ or a field, one has a particularly useful description of the free $n$–Cayley–Hamilton algebra in any set of variables $X = \{x_i\}_{i \in I}$. It will be given in Corollary 2.24.

Recall the definition 1.5 of $F(\xi_i) \subset M_n(F[\xi_{h,k}])$, $i \in I$, $h, k = 1, \ldots, n$ the algebra of generic matrices, and the action of $PGL$ on $M_n(F[\xi_{h,k}])$ and on $F[\xi_{h,k}]$. The algebra $F[\xi_{h,k}]^{PGL}$ is by definition the algebra of invariants of $X$–tuples of $n \times n$ matrices. The algebra $M_n(F[\xi_{h,k}])^{PGL}$ is by definition the algebra of equivariant maps (polynomial laws) from $X$–tuples of $n \times n$ matrices to $n \times n$ matrices.

One starts from the free algebra $F\langle x_i \rangle_{i \in I}$ and the map to the generic matrices $j := j_{F\langle x_i \rangle} : F\langle x_i \rangle \to F\langle \xi_i \rangle \subset M_n(F[\xi_{h,k}])$, and then compose this
with the determinant

$$F\langle x_i \rangle \xrightarrow{\det} M_n(F[\xi_{h,k}]) \xrightarrow{\det} F[\xi_{h,k}]^{PGL}$$

**Theorem 2.22.** 1. In the commutative diagram

$$\begin{array}{c}
\Gamma_n(F\langle x_i \rangle_{i \in I}) \xrightarrow{\pi} N \xrightarrow{1} F[\xi_{h,k}]^{PGL}
\end{array}$$

the last map $\Gamma_n(F\langle x_i \rangle_{i \in I})_{ab} \cong F[\xi_{h,k}]^{PGL}$ is an isomorphism.

2. Extend the map $\det \circ j_{F(x_i)}$ to a norm $F\langle x_i \rangle_{i \in I} \otimes_F F[\xi_{h,k}]^{PGL} \to F[\xi_{h,k}]^{PGL}$. Then this induces a norm compatible homomorphism

$$\tilde{j}_{F(x_i)}: F\langle x_i \rangle \otimes_F F[\xi_{h,k}]^{PGL} \xrightarrow{\otimes 1} F\langle \xi_i \rangle \otimes_F F[\xi_{h,k}]^{PGL} \xrightarrow{m} M_n(F[\xi_{h,k}])^{PGL},$$

with $m$ the multiplication.

3. $\tilde{j}_{F(x_i)}$ is surjective and its kernel $K$ is generated by the evaluation of the $n$ characteristic polynomial of all its elements.

This is proved in [8], Theorem 18.17 and Remark 18.18 based on the theorems of Donkin [9] and Zubkov [33].

**Remark 2.23.** 1. If the set of variables has $\ell$ elements, then the algebra $F[\xi_{h,k}]^{PGL}$ equals the algebra $T_n(\ell)$ and the algebra $M_n(F[\xi_{h,k}])^{PGL}$ equals $S_n(\ell)$ of page 6.

2. Using Theorem 2.19 part 3. can be equivalently stated as follows:

Consider the map $\rho_n: \mathcal{S}_n \xrightarrow{\tau} S_n \xrightarrow{\rho_n} F[\xi_{h,k}]^{PGL}$ and:

$$F\langle x_i \rangle_{i \in I} \otimes \mathcal{S}_n \xrightarrow{\rho_n \otimes \rho_n} F\langle \xi_i \rangle_{i \in I} \otimes_F F[\xi_{h,k}]^{PGL} \xrightarrow{m} M_n(F[\xi_{h,k}])^{PGL}.$$

This map is surjective and its kernel is the ideal generated by the evaluation of the $m$ characteristic polynomials of all its elements for all $m > n$ and all the corresponding evaluations of the $\sigma_i$ in this ideal.

As a Corollary one has
Corollary 2.24. The algebra $M_n(F[\xi_{h,k}])^{PGL}$ is a free algebra on the generators $\xi_i$ in the category of $n$ Cayley–Hamilton $F$–algebras.

Proof. Let $R$ be an $n$ Cayley–Hamilton $F$–algebra, with norm algebra $A$ and consider a set $r_i, i \in I$ of elements of $R$.

Then one deduces a homomorphism $f : F\langle x_i \rangle_{i \in I} \to R, x_i \mapsto r_i$. Next one has a commutative diagram

$$
\begin{array}{ccc}
F\langle x_i \rangle & \xrightarrow{f} & R \\
\downarrow & & \downarrow \\
\Gamma_n(F\langle x_i \rangle) & \xrightarrow{\bar{f}} & \Gamma_n(R) \\
& \xrightarrow{N} & A
\end{array}
$$

From this one has a homomorphism $F[\xi_{h,k}]^{PGL} \to A$ and one of norm algebras $\bar{f} : F\langle x_i \rangle_{i \in I} \otimes_F F[\xi_{h,k}]^{PGL} \to R$.

Clearly (by Theorem 2.22 3)) $\bar{f}$ vanishes on the kernel $K$ of the quotient map $\bar{\pi}_n : F\langle x_i \rangle_{i \in I} \otimes_F F[\xi_{h,k}]^{PGL} \to M_n(F[\xi_{h,k}])^{PGL}$ so $\bar{f}$ factors through a norm compatible map $M_n(F[\xi_{h,k}])^{PGL} \to R$.

Corollary 2.25. Every $n$ Cayley–Hamilton $F$–algebra $R$ is the quotient, in the category of $n$ Cayley–Hamilton $F$–algebras, of an algebra $M_n(F[\xi_{h,k}])^{PGL}$ in some generators $\xi_i, i \in I$.

In particular every $n$ Cayley–Hamilton $F$–algebra $R$ satisfies all polynomial identities with coefficients in $F$ of $n \times n$ matrices.

2.26 Azumaya algebras

An important class of CH–algebras are Azumaya algebras, [5], [4]. For our purpose we may take as definition:

Definition 2.27. An algebra $R$ with center $Z$ is Azumaya of rank $n^2$ over $Z$ if there is a faithfully flat extension $Z \to W$ so that $W \otimes Z R \simeq M_n(W)$.

Then it is easily seen that the determinant, of the matrix algebra $M_n(W)$ restricted to $R$ maps to $Z$ giving rise to a multiplicative map called the reduced norm $N : R \to Z$. Then $R$, with this norm, is an $n$–CH algebra.

We want to see that this is essentially the only norm on $R$.

The next Theorem is due in part to Ziplies (but his proof is quite complicated and very long) [30].
We claim that $\sigma$.

Apply now Amitsur’s Formula (20) to the $\sigma_j(C)$, $C$ the permutation matrix $C := e_{1,2} + e_{2,3} + \ldots + e_{n-1,n} + e_{n,1}$, of the full cycle $(1, 2, \ldots, n) \in S_n$.

The only non zero monomials in the $x_i = e_{i,i+1}$ have either value some $e_{i,j}$, $j \neq i$ or some $e_{h,h}$, but of these the only Lyndon word is $x_1 x_2 \ldots x_n = e_{1,1}$ of degree $n$.

Since $\sigma_j(a) = 0, \forall j > n$ we deduce

$$\sigma_i(e_{1,1}) = \sigma_i(x_1 x_2 \ldots x_n) = (-1)^{(n-i)} \sigma_{i-n}(e_{1,2} + \ldots + e_{n-1,n} + e_{n,1}) = 0, \forall i > 1.$$

We claim that $\sigma_1(e_{1,1}) = (-1)^{n-1} \sigma_n(C) = 1$, which completes the computation.

Now $\sigma_n$ is a multiplicative map and so it is 1 on the alternating group $A_n \subset S_n$. If $n$ is odd then $C \in A_n$ so $\sigma_1(e_{h,h}) = 1$. If $n = 2k$ then $C^2 \in A_n$.

Set $a := \sigma_n(C)$, so $\sigma_1(e_{1,1}) = -a$, we have $a^2 = \sigma_n(C^2) = 1$.

Let $A = -e_{1,1} + \sum_{i=2}^{n} e_{i,i}$

Apply to $A$ Formula (20), a primitive monomial in the terms of $A$ vanishes unless it is one $e_{j,j}$. So

$$\sigma_n(A) = -\prod_{h=1}^{2k} \sigma_1(e_{h,h}) = -a^{2k} = -1.$$

Now $\det(AC) = 1$ so if $n \geq 4$ it is a product of commutators so

$$1 = \sigma_n(AC) = -a \implies \sigma_1(e_{h,h}) = 1.$$

For $n = 2$ one may check directly that

$$A = (1 + e_{1,2})(1 + e_{2,1})(1 - e_{1,2})(1 - e_{2,1}) = 3e_{1,1} - e_{1,2} + e_{2,1}$$
is a commutator and deduce from Amitsur’s Formula for $1 = \sigma_2(A)$:

$$1 = -\sigma_1(-e_{1,2}e_{2,1}) = \sigma_1(e_{1,1}).$$

Next consider $(M_n(\mathbb{Z})^{\otimes k})_{ab}^1$ for $k$ not a multiple of $n$.

The same argument shows that in this case $\sigma_k(C) = 0$. But $C^n = 1$ and $\sigma_k$ is multiplicative so $1 = \sigma_k(C^n) = 0$.

The case $k = in$ seems to be difficult to attack with this method so we use a different approach. \qed

**Lemma 2.29.** Let $F$ be an algebraically closed field, $A$ a commutative algebra over $F$ and $f : M_n(F) \to A$ be a multiplicative polynomial map of degree $m$. Then $m = in$ and $f(a) = \det(a)^i 1_A$.

**Proof.** We have already shown that multiplicative maps can exist only for degrees multiples of $n$ so assume that $m = in$. For $\lambda$ a scalar matrix, since $f(1) = 1_A$ one must have $f(\lambda) = \lambda^m 1_A$. If $a \in SL(n, F)$ then $a$ is a product of commutators so that $f(a) = 1_A$.

If $a \in GL(n, F)$ we can write $a = b\lambda$, $b \in SL(n, F)$ and $\lambda$ a scalar so that $f(a) = \lambda^m 1_A = \det(a)^i 1_A$. Since $GL(n, F)$ is Zariski dense in $M_n(F)$ and $f$ is a polynomial it follows that for all matrices $a$ we have $f(a) = \det(a)^i 1_A$. \qed

We claim that $A := (M_n(F)^{\otimes in})_{ab}^1 = F$. Let $\pi$ be the projection of $(M_n(F)^{\otimes in})_{ab}^1$ to $A$. If $a \in M_n(F)$ the map $a \mapsto \pi (a^{\otimes in})$ is a multiplicative polynomial map so it is $\det(a)^i 1_A$ and maps $M_n(F)$ to $F 1_A$. Now the elements $a^{\otimes in}$ span $(M_n(F)^{\otimes in})_{ab}^1$ and by construction $\pi$ is surjective so the claim follows.

Now let us pass to the general case $A := (M_n(\mathbb{Z})^{\otimes in})_{ab}^1$. We have $A = (M_n(\mathbb{Z})^{\otimes in})_{ab}^1 / J$ where $J$ is the ideal generated by commutators. The algebra $A$ as abelian group if finitely generated. For any field $F$ we have the exact sequence

$$0 \longrightarrow J \longrightarrow (M_n(\mathbb{Z})^{\otimes in})_{ab}^1 \longrightarrow A \longrightarrow 0$$

$$F \otimes J \overset{i}{\longrightarrow} F \otimes (M_n(\mathbb{Z})^{\otimes in})_{ab}^1 \overset{\pi}{\longrightarrow} F \otimes A \longrightarrow 0$$

Now $F \otimes (M_n(\mathbb{Z})^{\otimes in})_{ab}^1 = (M_n(F)^{\otimes in})_{ab}^1$ and $i(F \otimes J)$ is the ideal of $(M_n(F)^{\otimes in})_{ab}^1$ generated by commutators and $\pi$ is surjective so by the previous Lemma $F \otimes A = F$.

Since this is true for all $F$ of all characteristics one must have $A = \mathbb{Z}$.

22
In particular the first exact sequence splits and so for all commutative rings $B$ one has

$$0 \longrightarrow B \otimes J \overset{i}{\longrightarrow} (M_n(B) \otimes_{\text{in}})S_{\text{in}} \overset{\pi}{\longrightarrow} B \longrightarrow 0 \quad (29)$$

and

$$B = (M_n(B) \otimes_{\text{in}})_{\text{ab}}.$$

### Azumaya algebras and invariants

For Azumaya algebras the Problem of Remark 2.9 has a positive answer:

**Theorem 2.30.** If $R$ is a rank $n^2$ Azumaya algebra, over its center $Z$ the map $D : Z = \Gamma_n(R)_{\text{ab}} \rightarrow T_n(R)_{\text{ab}}$ is an isomorphism.

Assume first that $R = M_n(A)$. A map $M_n(A) \rightarrow M_n(B)$ consists of a morphism $f : A \rightarrow B$ and then an automorphism $g : B \otimes Z M_n(A) \rightarrow M_n(B)$, this functor is thus classified by $T_n(M_n(A)) = P_n \otimes A$ and we have the universal map $j : M_n(A) \rightarrow M_n(P_n \otimes A) \simeq P_n \otimes M_n(A)$. The map $j$ is also given by Formula (2):

$$a \mapsto XaX^{-1} \in M_n(P_n \otimes A), \ a \in M_n(A). \quad (30)$$

The condition for an element $u \in T_n(M_n(A)) = P_n \otimes A$ to be invariant is from Formula (6):

$$\Delta(u) = 1 \otimes u, \ \Delta : P_n \otimes A \rightarrow P_n \otimes P_n \otimes A, \ \text{comultiplication}$$

By the Hopf algebra properties if $S$ is the antipode we have $m \circ S \otimes 1 \circ \Delta = \epsilon$ which in group terms just means $f(x^{-1}x) = f(1)$

$$\Rightarrow m \circ S \otimes 1 \circ \Delta(u) = m \circ S \otimes 1(1 \otimes u) = m(1 \otimes u) = u = \epsilon(u) \in A.$$

In order to analyze the universal map $R \rightarrow M_n(T_n(R))$ for an Azumaya algebra $R$ we use a fact from the Theory of polynomial identities.

One knows [1] Theorem 10.2.19, that there is a multilinear non commutative polynomial $\varphi(x_1, \ldots, x_k)$ with coefficients in $\mathbb{Z}$ which, when evaluated in matrices $M_n(A)$ over any commutative ring $A$ does not vanish and takes values in the center $A$. From this it follows that:
Lemma 2.31. When we evaluate $\varphi$ in any Azumaya algebras $R$ of rank $n^2$ over its center $Z$ its values lie in $Z$ and every element of $Z$ is obtained as a sum of evaluations of $\varphi$.

Proof. In fact from the faithfully flat splitting $W \otimes_Z R \simeq M_n(W)$ it follows that $\varphi$ takes values in $Z$ and it does not vanish on $R$ otherwise it would vanish on $M_n(W)$. Since $\varphi$ is multilinear the sums of evaluations of $\varphi$ form an ideal $J$ of $Z$ and, if $J \neq Z$, then $\varphi$ vanishes on the Azumaya algebras $R/JR$ of rank $n^2$ over its center $Z/J$ a contradiction. \hfill \Box

Lemma 2.32. If $f: R_1 \to R_2$ is a ring homomorphism of two rank $n^2$ Azumaya algebras with centers $Z_1, Z_2$ then $f(Z_1) \subset Z_2, R_2 \simeq Z_2 \otimes_{Z_1} R_1$.

Proof. Given $a \in Z_1$ we have $a$ is a sum of evaluations $\varphi(a_1, \ldots, a_k)$ for some elements $a_i \in R_1$ so $f(a)$ is a sum of evaluations $\varphi(f(a_1), \ldots, f(a_k)) \in Z_2$. \hfill \Box

We have that $T_n(R)$ classifies the functor $\text{hom}_R(R, M_n(B))$ and under such a map $f: R \to M_n(B)$ we have $f(Z) \subset B$ and $B \otimes_Z R \simeq M_n(B)$ which is an isomorphism. We may, to begin with, assume that instead of working in the category of rings we work in that of $Z$ algebras $Z$. Then $\text{hom}_Z(R, M_n(B))$ is the set of isomorphisms $g: B \otimes_Z R \simeq M_n(B)$.

This set, for a given $B$, may be empty but when it is non empty we say that $B$ splits $R$ and on it the group $PGL(n, -)$ acts in a simply transitive way that is, by definition, we have that $T_n(R)$ is a torsor for $PGL(n, -)$.

Definition 2.33. Let $\mathcal{C}$ be a category and $G(-), F(-)$ be respectively a covariant group valued and a set valued functor, together with a group action $\mu: G(-) \times F(-) \to F(-)$.

We say that $F$ is a torsor over $G$ if for all $X \in \mathcal{C}$ given $x, y \in F(X)$ there is a unique element $g \in G(X)$ with $gy = x$.

If $G, F$ are represented by two objects $\mathcal{G}, \mathcal{F}$ and $\mathcal{C}$ has products we say that $F$ is a torsor over the group like object $\mathcal{G}$ (cf. [11]).

Notice again that when $F(X) = \emptyset$ the condition is void.

The previous condition can be conveniently reformulated as:

the natural transformation of functors:

$$G(X) \times F(X) \xrightarrow{1 \times \delta} G(X) \times F(X) \times F(X) \xrightarrow{\mu \times 1} F(X) \times F(X) \quad (31)$$

is an isomorphism (with $\delta$ the diagonal).
One knows that there is a faithfully flat extension \( j : Z \to W \) so that \( W \otimes_Z R \cong M_n(W) \). One deduces for the universal object \( T_n(R) \) that
\[
W \otimes_Z T_n(R) \cong W \otimes P_n
\]
and so that \( T_n(R) \) is faithfully flat over \( Z \).

From Formula (31) the property of being a torsor can be stated in terms of the coaction of \( P_n \otimes Z \) over \( T_n(R) \), as in Formula (5): \[
\eta : T_n(R) \to (P_n \otimes Z) \otimes_Z T_n(R) = P_n \otimes T_n(R).
\]
Since \( F \) and \( G \) are controvariant functors Formula (31) gives a dual formula for their representing objects. This is the fact that the map \( 1 \otimes m \circ \eta \otimes 1 \) (where \( m \) is the multiplication \( m(a \otimes b) = ab \)) is an isomorphism:
\[
1 \otimes m \circ \eta \otimes 1 : T_n(R) \otimes T_n(R) \xrightarrow{\eta \otimes 1} P_n \otimes T_n(R) \otimes T_n(R) \xrightarrow{1 \otimes m} P_n \otimes T_n(R).
\]
This formula at the level of points, in any commutative \( Z \)-algebra \( A \), means exactly that given two points \((x, y)\) in \( \text{hom}(T_n(R), A) \) there is a unique \( g \in \mathcal{G}(A) \) with \( x = gy \), or \((x, y) = (gy, y)\).

If \( u \in T_n(R) \) we have
\[
1 \otimes m \circ \eta \otimes 1(u \otimes 1) = 1 \otimes m \circ \eta(u) \otimes 1 = \eta(u)
\]
\[
1 \otimes m \circ \eta \otimes 1(1 \otimes u) = 1 \otimes m \circ 1 \otimes u = 1 \otimes u.
\]
Since the map \( 1 \otimes m \circ \eta \otimes 1 \) is an isomorphism, the condition of invariance \( \eta(u) = 1 \otimes u \) is equivalent to \( u \otimes 1 = 1 \otimes u \) which by faithfully flat descent is equivalent to \( u \in Z \). This proves Theorem 2.30.

### 2.34 Σ–algebras

We have seen in Formula (19) the maps \( \sigma_i(f) \), \( A\langle X \rangle_+ \to S_A \), which are polynomial laws homogeneous of degree \( i \) which commute with the action of the endomorphisms \( T(X) \). We can view the operators \( \sigma_i \) as homogeneous polynomial maps, with respect to \( S_A \), of degree \( i \) of \( A\langle X \rangle_+ \otimes S_A \) to the center \( S_A \) which satisfy the Amitsur identity (20). Thus we may set the following definition:

**Definition 2.35.** 1) A \( \Sigma \) algebra \( R \) is an algebra over a commutative ring \( A \) equipped with polynomial laws \( \sigma_i : R \to R \) which satisfy:
\[
[s_i(a), b] = 0, \quad \sigma_i(\sigma_j(a)b) = \sigma_j(\sigma_i(a))b, \quad \forall a, b \in R, \quad \forall B \in C_A.
\]
\[ \sigma_i(ab) = \sigma_i(ba), \quad \sigma_i(a^j) = P_{i,j}(\sigma_1(a), \ldots, \sigma_{i-1}(a)) \] (33)

and that also satisfy Amitsur’s Formula (20).

2) The \( \sigma \)-algebra \( \sigma(R) \) of \( R \) is the algebra generated over \( A \) by the elements \( \sigma_j(a) \), \( j \in \mathbb{N}, a \in R \).

It is a subalgebra of the center of \( R \) closed under the operations \( \sigma_i \).

**Definition 2.36.** An ideal \( I \subset R \) in a \( \Sigma \)-algebra is a \( \sigma \)-ideal if it is closed under the maps \( \sigma_i \).

**Proposition 2.37.** If \( I \) is a \( \sigma \)-ideal of \( R \) the maps \( \sigma_i \) pass to the quotient \( R/I \) which is thus also a \( \Sigma \)-algebra.

If \( I \) is a \( \sigma \)-ideal of \( R \) and \( B \) is a commutative \( A \) algebra, then \( B/I \) is a \( \sigma \)-ideal of \( B \).

**Proof.** Given \( a \in R \) and \( b \in I \) we need to see that \( \sigma_i(a + b) - \sigma_i(a) \in I \). This follows from Amitsur’s Formula. The second part also follows from Amitsur’s Formula. \( \square \)

**Remark 2.38.** Since the \( \sigma_i \) are polynomial laws the previous statements extend to all \( B R \).

Then \( \Sigma \)-algebras also form a category, where \( \text{hom}_\Sigma(R, S) \) denotes the set of homomorphisms \( f : R \to S \) commuting with the operations \( \sigma_i \), i.e. \( f(\sigma_j(r)) = \sigma_j(f(r)), \forall j \in \mathbb{N}, \forall r \in R \).

The kernel of a \( \sigma \)-homomorphism is a \( \sigma \)-ideal and the usual homomorphism theorem holds.

This category has free algebras namely \( A(\langle X \rangle) \otimes S_A \) if we do not consider algebras with 1 or \( A(\langle X \rangle) \otimes S_A[\sigma_i(1)] \) by declaring the elements \( \sigma_i(1) \) to be independent variables.

**Definition 2.39.** A \( \Sigma \)-identity for a \( \Sigma \)-algebra \( R \) is an element of the free algebra \( f \in A(\langle X \rangle) \otimes S_A \) which vanishes under all evaluations of \( X \) in \( R \).

By abuse of notations we denote by \( S_A \) the algebra \( S_A[\sigma_i(1)] \), \( i = 1, \ldots \). As in the Theory of polynomial identities one has then the notions of \( T \)-ideal of \( A(\langle X \rangle) \otimes S_A \), of variety of \( \Sigma \)-algebras and of \( \Sigma \) or PI equivalence of \( \Sigma \)-algebras.

**Remark 2.40.** 1) In this language one defines a formal Cayley Hamilton polynomial in the free algebra by \( CH_n(x) := x^n + \sum_{i=1}^{n} (-1)^i \sigma_i(x)x^{n-i} \).

Then an \( n \)-Cayley Hamilton \( A \)-algebra \( R \) can be also defined as a \( \Sigma \) algebra satisfying the following conditions, which we will refer to as
Definition 2.41. three \( CH_n \) conditions:
1) \( \sigma_i(x) = 0, \forall i > n \), 2) \( CH_n(x) = 0, \forall x \in BR \), and \( B \) any commutative \( A \)-algebra, and 3) \( \sigma_i(1) = \binom{n}{i}, \forall i \leq n \).

Remark 2.42. It then follows from the Theorem of Zubkov and Zieplies that \( \sigma_n \) is a norm and that \( CH_n(x) \) is the evaluation for \( t = x \) of \( \sigma_n(t - x) \), see [7] and also [19]. In particular all the maps \( \sigma_i(x) \) are deduced from \( \sigma_n(x) \) via the formulas \( \sigma_n(t - x) = t^n + \sum_{i=1}^{n} (-1)^i \sigma_i(x) t^{n-i} \) and \( \sigma_i(x) = 0, \forall i > n \).

2.42.1 The first and second fundamental Theorem for matrix invariants revisited

The first and second fundamental Theorem for matrix invariants for algebras may be viewed as the starting point of the Theory of Cayley—Hamilton algebras, in all characteristics. It is Theorem 2.22 which can be interpreted best in the language of \( \Sigma \)-algebras. Let \( F = \mathbb{Z} \), or a field:

**Theorem 2.43.** The algebra \( F_{\Sigma,n}(X) \) of equivariant polynomial maps from \( X \)-tuples of \( n \times n \) matrices, \( M_n(F)^X \) to \( n \times n \) matrices \( M_n(F) \), is the free \( \Sigma \)-algebra \( F(X) \otimes S_A \) modulo the \( T \)-ideal generated by the three \( CH_n \) conditions of Definition 2.41

\[
F_{\Sigma,n}(X) := F(X) \otimes S_A / \langle CH_n(x), \sigma_i(x) = 0, \forall i > n, \sigma_i(1) = \binom{n}{i} \rangle.
\]

(34)

To be concrete if \( X \) has \( \ell \) elements, let \( A_{\ell,n} \) denote the polynomial functions on the space \( M_n(F)^\ell \) (that is the algebra of polynomials over \( F \) in \( mn^2 \) variables \( \xi_{i,(j,h)} \); \( i = 1, \ldots, m; j, h = 1, \ldots, n \)).

On this space, and hence on \( A_{\ell,n} \), acts the group \( PGL(n,-) \) by conjugation.

The space of polynomial maps from \( M_n(F)^\ell \) to \( M_n(F) \) is

\[
M_n(A_{\ell,n}) = M_n(F) \otimes A_{\ell,n}.
\]

This is a \( \Sigma \)-algebra in an obvious way, and on this space acts diagonally \( PGL(n,-) \) commuting with the \( \Sigma \)-operators and the invariants

\[
F_{\Sigma,n}(x_1, \ldots, x_\ell) = M_n(A_{\ell,n})^{PGL(n,-)} = (M_n(F) \otimes A_{\ell,n})^{PGL(n,-)}
\]

27
are a $\Sigma$–subalgebra which is the relatively free algebra in $\ell$ variables in the variety of $\Sigma$–algebras satisfying the three $CH_n$ conditions.

For the $\sigma$–algebra of $F_{\Sigma,n}(X)$ we have $T_n(\ell) = A_{\ell,n}^{PGL(n,-)}$. Of course we may let $\ell$ be also infinity (of any type) and have

$$F_{\Sigma,n}(X) = M_n(A_{x,n})^{PGL(n,-)} = (M_n(F) \otimes A_{x,n})^{PGL(n,-)}$$

where $A_{x,n}$ is the polynomial ring on $M_n(F)^X$.

**Remark 2.44.** If $F$ is an infinite field one may take for $PGL(n,-)$ the actual group $PGL(n,F)$, otherwise one has two options. The first option is to take for $PGL(n,-)$ the group $PGL(n,G)$ for $G$ any infinite field containing $F$ or take the categorical notion, valid also over $\mathbb{Z}$, taking the coaction $\eta : A_{x,n} \otimes M_n(F) \to P_n \otimes A_{x,n} \otimes M_n(F)$ (with $P_n$ the coordinate ring of $PGL(n,-)$) set

$$(A_{x,n} \otimes M_n(F))^{PGL(n,-)} := \{ r \in (A_{x,n} \otimes M_n(F)) \mid \eta(r) = 1 \otimes r \}.$$ 

By the universal properties one sees that $\eta(r) = 1 \otimes r$ is equivalent to $g(r) = 1 \otimes r$, $\forall g \in PGL(n,B)$, $\forall B$.

The reader will understand at this point that the approach with trace and that with norm, in characteristic 0, are equivalent.

For a proof of these Theorems in all characteristics or even $\mathbb{Z}$–algebras, the Theorem of Zubkov, the reader may consult [8].

One can then reformulate the definition of $n$–Cayley–Hamilton algebra in this language:

**Definition 2.45.** A $\Sigma$–algebra satisfying the three $CH_n$ conditions

$$\langle CH_n(x), \sigma_i(x) = 0, \forall i > n, \sigma_i(1) = \binom{n}{i} \rangle$$

will be called an $n$–Cayley–Hamilton algebra or $n$–CH algebra.

In other words an $n$–Cayley–Hamilton algebra $R$ is a quotient, as $\Sigma$–algebra, of one free algebra $F_{\Sigma,n}(X)$.

For $n = 1$ a 1–CH algebra is just a commutative algebra in which the norm is the identity map or $\sigma_i(a) = 0$, $\forall i > 1$ and $\sigma_1(a) = a$. Therefore the Theory of $n$–Cayley–Hamilton algebras may be viewed as a generalization of Commutative algebra.
Remark 2.46. For an \( n \)-Cayley–Hamilton algebra \( R \) the map \( \sigma_n \) is a norm and, from Remark 2.42 it follows that the \( \sigma \) algebra \( \sigma(R) \) coincides with its norm algebra.

Remark 2.47. An \( n \)-Cayley–Hamilton algebra \( R \) satisfies all the polynomial identities of \( M_n(\mathbb{Z}) \).

2.47.1 The main Theorem of Cayley–Hamilton algebras

Let \( R \) be any \( n \)-Cayley–Hamilton algebra over a commutative ring \( F \).

Choosing a set of generators \( X \) for \( R \) we may present \( R \) as a quotient of a free algebra \( F_{\Sigma,n}(X) = (M_n(F) \otimes A_{X,n})^{PGL(n,\mathbb{C})} \) modulo a \( \Sigma \)-ideal \( I \). Now \( (M_n(F) \otimes A_{X,n})I(M_n(F) \otimes A_{X,n}) \) is an ideal of \( M_n(F) \otimes A_{X,n} = M_n(A_{X,n}) \) so there is an ideal \( J \subset A_{X,n} \), which is \( PGL(n,\mathbb{C}) \) stable with \( (M_n(F) \otimes A_{X,n})I(M_n(F) \otimes A_{X,n}) = M_n(J) \)

from which one has a commutative diagram:

\[
\begin{array}{ccc}
F_{\Sigma,n}(X) & \xrightarrow{i} & M_n[A_{X,n}]^{PGL(n,\mathbb{C})} \\
\downarrow & & \downarrow \\
R & \xrightarrow{i_R} & M_n[A_{X,n}/J]^{PGL(n,\mathbb{C})}
\end{array}
\]

If \( F \supset \mathbb{Q} \) then \( i_R \) is an isomorphism, [Strong embedding Theorem].

The fact that in characteristic 0, \( i_R \) is an isomorphism depends upon the fact that \( GL(n) \), in characteristic 0, is linearly reductive, and then the proof, see [16] or [1] Theorem 14.2.1, of this Theorem is based on the so-called Reynold’s identities.

In general the nature of \( i_R \) is not known, we do not know if parts of this statement are true. This is in my opinion the main open problem of the Theory.
3 Prime and simple Cayley Hamilton algebras

3.1 General facts

3.1.1 The kernel and the radical

Definition 3.2. 1. a simple $\Sigma$ algebra is one with no proper $\sigma$ ideals,

2. a prime $\Sigma$ algebra is one in which if $I, J$ are two $\sigma$ ideals with $IJ = 0$ then either $I = 0$ or $J = 0$.

3. Finally a semiprime $\Sigma$ algebra is one in which if $I$ is an ideal with $I^2 = 0$ then $I = 0$.

Notice that prime implies semiprime.

Definition 3.3. 1) Given a $\Sigma$–algebra $R$ the set

$$K_R := \{x \in R \mid \sigma_i(xy) = 0, \forall y \in R, \forall i\}$$

will be called the kernel of the $\Sigma$–algebra.

$R$ is called nondegenerate if $K_R = 0$.

2) The set

$$\tilde{K}_R := \{x \in R \mid \sigma_i(x) \text{ is nilpotent, } \forall y \in R, \forall i\}$$

will be called the radical of the $\Sigma$–algebra.

$R$ is called regular if $\tilde{K}_R = 0$.

By Amitsur’s Formula (20) both $K_R$ and $\tilde{K}_R$ are $\sigma$–ideals of $R$.

Moreover if $B$ is a commutative A algebra $B \tilde{K}_R \subseteq K_{BR}$, $B \tilde{K}_R \subseteq \tilde{K}_{BR}$.

If $I$ is a $\sigma$–ideal in a $\Sigma$–algebra $R$ we set $K(I) \supset I$ (resp. $\tilde{K}(I)$) to be the ideal such that $R/K(I) = K_{R/I}$ (resp. $R/\tilde{K}(I) = \tilde{K}_{R/I}$).

We call $K(I)$ the radical kernel of $I$ and $\tilde{K}(I)$ the nil kernel of $I$, both $\sigma$–ideals.

Lemma 3.4. Let $R$ be a n–CH algebra. An element $r \in R$ is nilpotent if and only if all the $\sigma_i(r)$ are nilpotent.
Proof. In one direction every element \( r \in R \) satisfies its characteristic polynomial, if \( \sigma_i(r) \) is nilpotent for all \( i \) we have \( r^n \) is a linear combination of the commuting nilpotent elements \( \sigma_i(r)r^{n-1} \) hence the claim. Assume now \( r \) nilpotent. The elements \( \sigma_i(r^k), i \cdot k \leq N \) satisfy the relations of the corresponding symmetric functions \( e_i(X^k) := e_i(x_1^k, x_2^k, \ldots, x_N^k) \).

Now, for each \( k \) the polynomial ring \( \mathbb{Z}[x_1, x_2, \ldots, x_N] \) is integral over the subring \( \mathbb{Z}[e_1(X^k), e_2(X^k), \ldots, e_N(X^k)] \) so \( \sigma_i(r) \) satisfies a monic polynomial of some degree \( \ell \) whose coefficients are polynomials in the elements \( \sigma_j(r^k) \) (and with 0 constant coefficient). If \( r^k = 0 \) these coefficients are all 0 so \( \sigma_i(r)\ell = 0 \).

Proposition 3.5. 1. \( K_R \) is the maximal \( \sigma \)-ideal \( J \) where \( \sigma(J) = 0 \).

2. If \( R \) is an \( n \)-CH algebra we have \( \tilde{K}_R \) is the maximal ideal \( I \) with the property that \( B \) is nil for all \( B \).

3. If \( R \) is an \( n \)-CH algebra \( R/\tilde{K}_R \) is regular, i.e \( \tilde{K}_{R/\tilde{K}_R} = 0 \).

4. If \( \sigma_i(a) \) is nilpotent, then \( \sigma_i(a) \in \tilde{K}_R \).

5. If \( R \) is an \( n \)-CH algebra, over some commutative ring \( A \), and \( I \) is a nil ideal of \( R \) then \( B/I \) is a nil ideal of \( BR \) for every commutative \( B \) algebra.

Proof. 1) The first part is clear. 2) Follows from the previous Lemma 3.4.

3) If the class of \( r \in R \) is in the radical \( \tilde{S} \) of \( S := R/\tilde{K}_R \) we have that for each \( y \in B R, \sigma_i(ry) \) is nilpotent, in \( B S \) hence \( \sigma_i(ry) \) is nilpotent also in \( B R \) and so \( r \in \tilde{K}_R \).

4) As for the last statement, \( \sigma_j(\sigma_i(a)y) = \sigma_i(a)^j \sigma_j(y) \) is nilpotent.

5) This follows from the previous Lemma 3.4 and Amitsur’s Formula.

Corollary 3.6. Let \( R \) be a \( n \)-CH algebra with \( \sigma \)-algebra reduced (no nonzero nilpotent elements) then if \( r \in R \) is nilpotent we have \( r^n = 0 \).

In particular we have

Corollary 3.7. 1) An \( n \)-CH algebra \( R \) is semiprime if and only if its \( \sigma \)-algebra is reduced and the radical \( \tilde{K}_R = 0 \).

2) An \( n \)-CH algebra \( R \) is prime if and only if its \( \sigma \)-algebra is a domain and \( \tilde{K}_R = 0 \).
3) An n–CH algebra $R$ is simple if and only if its $\sigma$–algebra is a field and $\overline{K}_R = 0$.

Proof. 1) Assume $R$ semiprime. If the $\sigma$–algebra contains a non zero nilpotent element $a$ then $Ra$ is a nilpotent ideal a contradiction. Since $R$ is a PI algebra, it is semiprime if and only if it does not contain a nonzero nil ideal. So since $\overline{K}_R$ is nil and $R$ is semiprime $\overline{K}_R = 0$.

Conversely if $R$ has an ideal $I \neq 0$ with $I^2 = 0$ then for each $a \in B\overline{I}$ we have $\sigma_i(a)$ is nilpotent for all $i$, by Lemma 3.4 then $I \subset \overline{K}_R$.

2) As for the second statement let us show that the given conditions imply $R$ prime. In fact given two $\sigma$–ideals $I, J$ with $IJ = 0$ since $\sigma(I) \subset I$, $\sigma(J) \subset J$ we have $\sigma(I)\sigma(J) = 0$. Since these are ideals and $\sigma(R)$ is a domain one of them must be 0. If $\sigma(I) = 0$ then $I \subset \overline{K}_R = \{0\}$ since $R$ is semiprime and by the previous statement.

Conversely if $R$ is prime in particular it is semiprime so we must have $\overline{K}_R = 0$. If we had two non zero elements $a, b \in \sigma(R)$ with $ab = 0$ we would have $Ra \cdot Rb = 0$ and $Ra$, $Rb$ are $\sigma$ ideals, a contradiction.

3) If $R$ is simple it is prime so $\sigma(R)$ is a domain. We need to show that if $a \in \sigma(R)$ then $a$ is invertible, and the element $b$ with $ab = 1$ is in $\sigma(R)$.

First the ideal $aR$ is $\sigma$ stable so it must be $R$ and there is an element $b$ with $ab = ba = 1$. Thus the field of fractions $K$ of $\sigma(R)$ is contained in $R$, but since $K$ is integral over $\sigma(R)$, by the going up Theorem it coincides with $\sigma(R)$.

Conversely if $\sigma(R)$ is a field and $I$ is a nonzero proper $\sigma$ ideal, for every $a \in I$ we must have $\sigma_i(a) \in I \implies \sigma_i(a) = 0$, $\forall i$. Then $I \subset \overline{K}_R = \{0\}$.

\[ \square \]

**Proposition 3.8.** 1) If $R$ is a semiprime $n$ Cayley-Hamilton algebra with $a \in \sigma(R)$ not a zero divisor in $\sigma(R)$ then $a$ is not a zero divisor in $R$.

2) If $R$ is a prime $\Sigma$–algebra, $\sigma(R)$ is a domain and $R$ is torsion free relative to $\sigma(R)$.

Proof. 1) Let $J := \{ r \in R \mid ar = 0 \}$, then $J$ is an ideal and we claim it is nil hence by hypothesis 0. In fact taking one of the functions $\sigma_i$ we have $0 = \sigma_i(ar) = a^r\sigma_i(r)$ implies $\sigma_i(r) = 0$ for all $r \in J$, since $r$ satisfies its CH it must be $r^n = 0$.

2) If $a \in \sigma(R)$ and $J := \{ r \in R \mid ar = 0 \}$ then both $J$ and $Ra$ are ideals closed under the $\Sigma$ operations and $JRa = 0$. Since $R$ is prime and $Ra \neq 0$ it follows that $J = 0$. \[ \square \]
Finally the local finiteness property:

**Proposition 3.9.** An $n$–CH algebra $R$ finitely generated over its $\sigma$–algebra $\sigma(R)$ is a finite $\sigma(R)$ module.

**Proof.** The Cayley Hamilton identity implies that each element of $R$ is integral over $\sigma(R)$ of degree $\leq n$ then this is a standard result in PI rings consequence of Shirshov’s Lemma, [1] Theorem 8.2.1.

### 3.9.1 Semisimple algebras

In this section $F$ denotes an infinite field, this hypothesis could be removed but it simplifies the treatment. We want to study general CH algebras over $F$ such that the values of the norm and hence of all the $\sigma_i$ are in $F$.

First a simple fact. Let $R = R_1 \oplus R_2$ be an $F$ algebra and $N : R \to F$ a multiplicative polynomial map. Then setting $e_1, e_2$ the two unit elements of $R_1, R_2$ we have $N(a, b) = N((a, e_2)(e_1, b)) = N(a, e_2)N(e_1, b)$.

Set $N_1(a) := N(a, e_2), N_2(b) := N(e_1, b)$. Clearly $N_i : R_i \to F, i = 1, 2$ are both multiplicative polynomial maps and $N(a, b) = N_1(a)N_2(b)$.

**Proposition 3.10.** If $N$ is homogeneous of degree $n$ there are two positive integers $h_1, h_2 > 0$ with $h_1 + h_2 = n$ and $N_1(a)$ is homogeneous of degree $h_1$ while $N_2(b)$ is homogeneous of degree $h_2$.

**Proof.** Restrict $N$ to $F \oplus F$ then $N$ factors through a homomorphism of $[(F \oplus F)^{\otimes n}]^{S_n} \to F$.

Now $[(F \oplus F)^{\otimes n}]^{S_n}$ is the direct sum of $n + 1$ copies of $F$, each with unit element the symmetrization $e_1^h e_2^{n-h}$ of $e_1^\otimes h \otimes e_2^{\otimes n-h}$.

A homomorphism $[(F \oplus F)^{\otimes n}]^{S_n} \to F$ thus factors though the projection to one of this summands. The claim then follows from the remark that $(ae_1 + \beta e_2)^{\otimes n} = \sum_{h=0}^n \alpha^h \beta^{n-h} e_1^h e_2^{n-h}$. \qed

**Corollary 3.11.** Under the previous hypotheses, $N : R \to F$ a multiplicative polynomial map of degree $n$. If $R = \oplus_{i=1}^k R_i$ then $k \leq n$.

**Proof.** $N$ is a product of the norms $N_i$ each with some degree $h_i > 0$ and $n = \sum_i h_i$. \qed

**Corollary 3.12.** Under the previous hypotheses, $R$ is an $n$–CH algebra if and only if $R_i$ is an $h_i$ CH–algebra for $i = 1, 2$. 

33
Proof.

\[ \chi_{(a,b)}(t) = N((te_1 - a, te_2 - b) = N_1(te_1 - a)N_2(te_2 - b) = \chi_a^1(t)\chi_b^2(t), \]

so if \( R_i \) is an \( h_i \) CH–algebra for \( i = 1, 2 \) we have

\[ \chi_{(a,b)}((a, b)) = (\chi_a^1(a), \chi_a^1(b))(\chi_b^2(a), \chi_b^2(b)) = 0. \quad (39) \]

Conversely assume \( \chi_{(a,b)}((a, b)) = 0 \). Given \( a \in R_1 \) take \( \alpha \in F \) so

\[ \chi_a^2(t) = (t - \alpha)^{h_2} \]

and

\[ \chi_{(a,\alpha)}((a, \alpha)) = \chi_a^1(a)(a - \alpha)^{h_2} = 0, \forall \alpha \in F. \]

The minimal polynomial \( f(t) \) of \( a \) thus divides \( \chi_a^1(t)(t - \alpha)^{h_2}, \forall \alpha \in F \) hence it divides \( \chi_a^1(t) \) so \( \chi_a^1(a) = 0 \) and similarly for \( b \).

\[ \square \]

A semisimple algebra \( S \) finite dimensional over a field \( F \) is isomorphic to the direct sum \( S = \oplus_i M_{k_i}(D_i) \) of matrix algebras over division rings which are finite dimensional over \( F \).

We treated the theory in characteristic 0 in [20], so we assume that \( F \) has some positive characteristic \( p > 0 \). Let \( G_i \) denote the center of \( D_i \).

If \( F \) is separably closed then all the \( D_i = G_i \) are fields, purely inseparable over \( F \) (this follows from the fact that a division ring \( D \) of degree \( n \) has a separable element of degree \( n \) over the center, see for instance Saltmen [27]). We ask in general which norms exist on \( S \) with values in \( F \) which make \( S \) an \( n \)–Cayley–Hamilton algebra.

We start with a special case.

Given two lists \( m := m_1, \ldots, m_k \) and \( a := a_1, \ldots, a_k \) of positive integers with \( \sum_j m_j a_j = n \) consider the algebra with norm \( N \)

\[ F(m; a) := \oplus_{i=1}^k M_{m_i}(F), \subset M_n(F), \quad N(r_1, \ldots, r_k) = \prod_{i=1}^k \det(r_i)^{a_i} \quad (40) \]

where the \( i^{th} \) block is repeated \( a_i \).

\( F(m; a) \) is a subalgebra (of block diagonal matrices) of \( M_n(F) \) and then the norm \( N \) equals the determinant, hence it is an \( n \) Cayley–Hamilton algebra, and, as \( \sigma \)– algebra, it is simple.

Conversely we have the standard:
Proposition 3.13. If $F$ is algebraically closed and $S \subset M_n(F)$ is a semisimple algebra then it is one of the algebras $F(m_1, \ldots , m_k; a_1, \ldots , a_k)$.

Proof. A semisimple algebra $S$ over $F$ is of the form $S = \oplus_{i=1}^{k} M_{m_i}(F)$.

An embedding of $S$ in $M_n(F)$ is a faithful $n$–dimensional representation of $S$. Now the representations of $S$ are direct sums of the irreducible representations $F^{m_i}$ of the blocks $M_{m_i}(F)$, and a faithful $n$–dimensional representation of $S$ is thus of the form

$$\oplus_i (F^{m_i})^{\oplus a_i}, \quad a_i \in \mathbb{N}, \quad a_i > 0, \quad \sum_i a_i m_i = n.$$ 

For this representation the algebra $S$ appears as block diagonal matrices, with an $m_i \times m_i$ block repeated $a_i$ times. The norm is the determinant described by Formula (40).

Formula (40) can be made axiomatic. Assume $F$ is an infinite field.

Theorem 3.14. Suppose that $F(m) := \oplus_{i=1}^{q} M_{m_i}(F)$ is equipped with a Norm $N$ of degree $n$ with values in $F$. Then there are positive integers $a_i$ with $\sum_i a_i m_i = n$ so that

$$N(r_1, \ldots , r_q) = \prod_{i=1}^{q} \det(r_i)^{a_i}. \quad (41)$$

Hence it is the $n$ Cayley–Hamilton algebra $F(m; a)$.

Proof. From Corollary 3.12 we are reduced to the case $R = M_h(F)$. In this case the statement is a special case of Theorem 2.28.

\square

3.14.1 An abstract Theorem

We have an even more abstract Theorem:

Theorem 3.15. Let $F$ be an algebraically closed field and $S$ an $n$ Cayley–Hamilton algebra with norm in $F$ and radical $K_S$.

Then $S/K_S$ is finite dimensional, simple, and isomorphic to one of the algebras $F(m_1, \ldots , m_k; a_1, \ldots , a_k)$ as algebra with norm.
Proof. First remark that, since the values of $\sigma_i$ are all in $F$ we have that the kernel equals the radical $K_S$.

Passing to $S/K_S$ we may thus assume that $K_S = 0$. Let us first assume that $S$ is finite dimensional, then by Proposition 3.5 we have that $S$ is a semisimple algebra so it is of the form $S = \bigoplus_{i=1}^k M_{m_i}(F)$. Since it is an $n$ Cayley–Hamilton algebra the statement follows from Theorem 3.14.

Now let us show that it is finite dimensional. For any choice of a finite set of elements $A = \{a_1, \ldots, a_k\} \subset S$ let $S_A$ be the subalgebra generated by these elements, since each $\sigma$ takes values in $F$ this is also a $\Sigma$–subalgebra.

By a standard theorem of PI theory since $S$ is algebraic of bounded degree each $S_A$ is finite dimensional. Then if $J_A$ is the radical of $S_A$ we have by the previous part that $\dim S_A/J_A \leq n^2$. Let us choose $A$ so that $\dim S_A/J_A$ is maximal. We claim that $S = S_A$ and $J_A = 0$.

First let us show that $J_A \subset K_S$ the Kernel of $S$. Let $a \in J_A$ and $r \in S$; we need to show that $\sigma_i(ra) = 0$, $\forall i$. If $r \in S_A$ this is the previous statement, if $r \notin S_A$ then $S_{A,r} \supseteq S_A$ and we claim that $J_{A,r} \supseteq J_A$, in fact otherwise $\dim S_{A,r}/J_{A,r} > \dim S_A/J_A$ a contradiction.

Then by the previous argument $\sigma_i(ar) = 0$ so $a \in K_S$ but since $S$ is simple $K_S = 0$ and $J_A = 0$. Next if $S_A \neq S$ we have again some $S_{A,r} \supseteq S_A$ and now $J_{A,r} \neq 0$ a contradiction. \qed

3.16 General semisimple algebras

If $R$ is a simple PI algebra, over a field $F$, we have $R = M_k(D)$ with $D$ a division ring finite dimensional over its center $G \supseteq F$ (Theorem 11.2.1 of [1]), let $\dim_G D = h^2$. If furthermore $R$ is finite dimensional over $F$ let $\dim_F G = \ell$.

In this last case the algebra $R$ is endowed with a canonical Norm homogeneous of degree $kh\ell$ which is a composition of two norms

$$N_{R/F} = N_{R/G} \circ N_{G/F}$$

The norm $N_{R/G}$ can be defined as follows. We take a maximal subfield $M \subset D$ separable over $G$ then:

$$M_k(D) \otimes_G M = M_{k \cdot h}(M).$$

If $a \in M_k(D)$ define as Norm $N_{R/G}(a) := \det(a \otimes 1)$ as matrix.

It is a standard fact that $N_{R/G}(a) \in G$. As for $N_{G/F}(g)$, $g \in G$ one takes the determinant of the multiplication by $g$ a $\ell \times \ell$ matrix over $F$. 36
Assume $G$ is separable over $F$ and $\bar{F} \supset G$ is the separable closure of $G$ and $F$. We have the $\ell$, $F$–embeddings of $G$ in $\bar{F}$, $\gamma_1, \ldots, \gamma_\ell$ given by Galois Theory:

$$G \otimes_F \bar{F} = \bar{F}^\ell, \ g \otimes 1 = (\gamma_1(g), \ldots, \gamma_\ell(g)) \implies N(g) = \prod_i \gamma_i(g) \in F. \ (42)$$

Notice that, in this case we also have a trace $tr(a) \in F$, $\forall a \in R$, the separability condition is given by the fact that the trace form $tr(ab)$ is non degenerate.

In general let $L$ be the separable closure of $F$ in $G$, and $a = [L : F], \ p^k = [G : L]$. Let $\bar{F}$ be a separable closure of $F$ we still have that

$$L \otimes_F \bar{F} = F^{\otimes a}, \ D \otimes_F \bar{F} = D \otimes_L (L \otimes_F \bar{F}) = \oplus_i D \otimes_L \bar{F}$$

where in the summands $L$ embeds in $\bar{F}$ by the $a$ different embeddings given by Galois Theory as in Formula (42).

Moreover $G \otimes_L \bar{F}$ is a field by Theorem 9 page 163 of [10]. Finally

**Lemma 3.17.** $D \otimes_L \bar{F} = D \otimes_G (G \otimes_L \bar{F}) \simeq M_h(G \otimes_L \bar{F})$ and

$$M_k(D) \otimes_F \bar{F} = M_k(D) \otimes_G (G \otimes_F \bar{F}) = M_k(D) \otimes_G (G \otimes_L (L \otimes_F \bar{F}))$$

$$= \oplus_i M_{hk}(G \otimes_G \bar{F}). \ (43)$$

**Proof.** $D$ contains a maximal subfield $M$ separable over $G$ so generated by a single element $a$ satisfying an irreducible separable polynomial $f(x) = x^h + \sum_{i=1}^h \alpha_i x^{h-i}$, $f(a) = 0$ with coefficients $\alpha_i$ in $G$. We have for some power $a^{p^k}$ that $a^{p^k}$ satisfies $x^h + \sum_{i=1}^h \alpha_i^{p^k} x^{h-i}$ with coefficients $\alpha_i^{p^k}$ in $\bar{F}$. Since $M$ is purely inseparable over $G[a^{p^k}]$ we have that $a^{p^k}$ is also a generator of $M$ but being separable over $\bar{F}$ it is in $\bar{F}$. Therefore $G \otimes_L \bar{F}$ is a splitting field for $D$. \qed

As for the norm $N_{M_k(D)/F}$ one has to embed $G \otimes_L \bar{F} \subset M_\ell(\bar{F})$ and then in the embedding

$$M_k(D) \otimes_F \bar{F} \subset \oplus_i M_{hk \ell}(\bar{F}) \subset M_{hk \ell}(\bar{F}), \ a \mapsto (a_1, \ldots, a_\ell)$$

we have

$$N_{M_k(D)/F}(a) = \prod_i \det(a_i).$$

37
Proposition 3.18. Under this norm $M_k(D)$ is a $k \cdot h \cdot p^h \cdot \mathfrak{e} \cdot \text{CH algebra}$.

Proof. The norm is induced by the determinant of the previous Formula.

We ask now what is the general form of a norm and we will see that, contrary to what happens in characteristic 0 in general there are norms which have degree strictly less than that of the canonical norm.

We now do not even assume that $D$ is finite dimensional over $F$.

Let again $L$ be the separable closure of $F$ in $G$, and consider a norm $N : R = M_k(D) \to F$ that is a multiplicative polynomial map homogeneous of some degree $n$.

Theorem 3.19. $[L : F] = \mathfrak{e} < \infty$. There is a minimum integer $k$ such that $g^k \in L$, $\forall g \in G$. There is a $b \in \mathbb{N}$ such that $n = khbp^h\mathfrak{e}$. The norm $N$ depends only upon $b$ and will be denoted by $N_b$ and maps as

$$N_b : M_k(D) \xrightarrow{N_{R/G}} G \xrightarrow{a \mapsto abp^h} L \xrightarrow{N_{L/F}} F.$$ (44)

That is $N_b = N_{1b}$. The proof is in several steps. Denote again by $\bar{F}$ a separable closure of $F$. Restricting the norm to $L \otimes_F \bar{F}$ we have, by Corollary 3.11, that $[L : F] = \mathfrak{e}$ is finite. The $\mathfrak{e}$ divides $n$ by a combination of Formula (42) and Theorem 3.14. Lemma 3.17 still holds.

The norm $N$ induces a norm $N : M_k(D) \otimes_F \bar{F} \to \bar{F}$ which, by Proposition 3.10 is the product of the norms $N_i$ in the $\mathfrak{e}$ summands of Formula (43). If $\gamma$ is an automorphism of $\bar{F}$ over $F$ we have $N \circ 1 \otimes \gamma = \gamma \circ N$. This allows us to say that the norms $N_i$ induced according to Proposition 3.10 in the $a$ summand of Formula (43) have all the same degree $m$ and $n = ma$.

Set $m = hk$ we have to analyze the norms $N : M_m(G \otimes_L \bar{F}) \to \bar{F}$ which by abuse of notation we still think of degree $n$.

The first case is when $m = 1$.

Changing notations let $G \supset F$ be purely inseparable over $F$.

Lemma 3.20. The map $\pi_n : G^{\otimes n} \to G$, $a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto \prod_i a_i$ is a surjective homomorphism with kernel the nil radical of $G^{\otimes n}$.

Proof. By induction on $n$. The map $\pi : G \otimes G \to G$, $\pi(a \otimes b) = ab$ is a homomorphism with kernel the ideal generated by $a \otimes 1 - 1 \otimes a$ which is nilpotent since, for some $h$ we have $a^h \in F$ so $(a \otimes 1 - 1 \otimes a)^h = a^h \otimes 1 - 1 \otimes a^h = 0$. Then by induction $\pi_n$ must factor through $J \otimes G$ with $J$ the radical of $G^{\otimes n-1}$.
But $G^\otimes n / J \otimes G = G \otimes G$ and we are at the beginning of the induction.

\[\square\]

**Lemma 3.21.** If $N : G \to F$ is a multiplicative polynomial map of degree $n$ then there is a minimal $k$ so that $a^{p^k} \in F$, $\forall a \in G$, $n = bp^k$, $b \in \mathbb{N}$ and $N(a) = a^n$.

**Proof.** Let $a \in G$ be such that $a^{p^h} \in F$ with $h$ minimal, restrict $N$ first to $G = F[a]$. A multiplicative polynomial map then factors through $N$:

\[N : G \xrightarrow{\text{w-\otimes}_{G}^n} (G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n} \xrightarrow{\bar{N}} F\]  

(45)

with $\bar{N}$ a homomorphism which thus vanishes on the radical.

Now the map $\pi : G^\otimes n \to G$, $a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto \prod_i a_i$ is a surjective homomorphism with kernel the radical of $G^\otimes n$. It follows that $\pi$ restricted to $(G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n}$ induces an isomorphism of $(G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n}$ modulo its radical with a field $L$ with $F \subset L \subset G$. Thus if $\bar{N}$ exists, since it factors through $(G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n}$ we must have $L = F$.

Next $(G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n}$ is generated by the elementary symmetric functions in the elements $1 \otimes a \otimes 1 \otimes \cdots$ so the image of $\pi$ is generated by the elements $\binom{n}{j} a^j$. In particular if they have to lie in $F$ we must have that $a^n \in F$ so $n$ is a multiple of the minimal $p^h$ for which $a^{p^h} \in F$.

For general $G$ the statement follows by restricting to all subfields $F[a]$ of previous type, assuming the existence of $N$ we must in particular have that for each $a \in G$ we have $[F[a], F] = p^d$ with $p^d$ dividing $n$. Therefore there is a minimum $k$ so that $a^{p^k} \in F$, $\forall a \in G$ and $n$ is a multiple of $p^k$.

Finally $N$ is the canonical norm $G \to (G \otimes_{F} G \otimes \cdots \otimes_{F} G)^{S_n} = F$ which is $a \mapsto a^n$.

\[\square\]

**Remark 3.22.** If $\bar{F}$ is an algebraic closure of $F$, setting $y = x - a$

\[\bar{F} \otimes_{F} F[a] = \bar{F}[x]/(x^{p^k} - a^{p^k}) \simeq \bar{F}[y]/y^{p^k}.\]

The norm $N$ extends to

\[N : \bar{F}[y]/y^{p^k} \xrightarrow{\text{w-\otimes}_{\bar{F}}^{S_n}} [(\bar{F}[y]/y^{p^k})^{\otimes_{\bar{F}}}^{S_n}] \xrightarrow{\bar{N}} \bar{F}.\]

(46)

There is a unique $\bar{N}$ in Formula (46) which vanishes on the radical.
We pass to general \( m \). Let \( N : M_{m}(G) \to F \) be a multiplicative polynomial map, over \( F \), of degree \( n \). Consider the subspace of diagonal matrices \( G^m \subset M_{m}(G) \), the norm \( N \) restricted to \( G^m \) is a product of norms \( N_i \) for the various summands, but the symmetric group \( S_m \subset M_{m}(G) \) permutes the summands so the norms are all the same and by Lemma 3.21 there is \( b \) so that \( N(a_1, \ldots, a_m) = \prod_i a_i^{b^k} = \det((a_1, \ldots, a_m))^{b^k} \).

Next consider elementary matrices \( e_{i,j}(\alpha) := 1 + \alpha e_{i,j}, \; i \neq j, \; \alpha \in G \).

We have \( e_{i,j}(\alpha)e_{i,j}(\beta) = e_{i,j}(\alpha + \beta) \) so \( N \) on this subgroup is a multiplicative polynomial map from \( G \), as additive \( F \) vector space, to the multiplicative \( F \). In a basis of \( G \) over \( F \) it is thus a polynomial of some degree \( k \) with

\[
 f(x_1 + y_1, \ldots, x_h + y_h) = f(x_1, \ldots, x_h)f(y_1, \ldots, y_h)
\]

this by degree implies that \( f = 1 \). Then by the usual Gaussian elimination any matrix \( A \) is a product of a diagonal matrix times elementary matrices so that hence \( N(A) = \det(A)^{b^k} \) for all \( A \).

Finally the norm \( N_b \) of Formula 44 becomes under tensor product with \( F \) the one of Theorem 3.19 so the Theorem follows.

Let us summarize what we proved. Consider a general semisimple algebra over \( F \) \( R = \otimes_{i=1}^{p} M_{k_i}(D_i) \), \( \dim_{G_i} D_i = h_i^2 \) where \( G_i \) is the center of the division algebra \( D_i \). Let \( L_i \subset G_i \) be the separable closure of \( F \), \( \dim_F L = \ell_i \) and \( k_i \) the minimum integer such that \( a^{p^{k_i}} \in L_i, \; \forall a \in G_i \). Given positive integers \( b_i, \; i = 1, \ldots, p \) we may define the norm

\[
 N(a_1, \ldots, a_p) := \prod_{i=1}^{p} N_{b_i}(a_i), \; a_i \in M_{k_i}(D_i), \; N_{b_i}(a_i) \quad \text{Formula (44).} \quad (47)
\]

**Theorem 3.23.** The algebra \( R = \otimes_{i=1}^{p} M_{k_i}(D_i) \) with the previous norm is an \( n \) CH algebra with \( n = \sum_i k_i h_i p^{k_i} \ell_i \).

Conversely any norm on \( R \) is of the previous form.

**Proof.** From Corollary 3.12 we may reduce to \( R = M_k(D) \) and then, by splitting to the case in which \( G \) is purely inseparable over \( F \), \( R = M_k(G) \) and \( N : M_k(G) \to F, \; N(a) = \det(a)^{b^k} \). The characteristic polynomial is thus \( \chi(t) = \det(t - a)^{b^k} \). Then clearly this vanishes for \( t = a \) by the usual CH Theorem for \( M_k(G) \).

**Corollary 3.24.** Let \( R = \otimes_{i=1}^{p} M_{k_i}(D_i) \) be an \( n \) CH algebra over a field \( F \) as in Theorem 3.23. If for one \( i \) we have that the dimension of \( M_{k_i}(D_i) \) over its center \( G_i \) equals \( n^2 \) then \( R = M_{k_i}(D_i), \; F = G_i \) and the norm is the reduced norm.
For the next Theorem we need:

**Lemma 3.25.** Let $S$ be a semiprime algebra with center an infinite field $C$ and each element satisfies an algebraic equation of degree $n$ over $C$. Then $S$ is a finite dimensional central simple algebra.

**Proof.** $S$ satisfies a polynomial identity so it is enough to prove that $S$ is a prime algebra (cf. [1] Theorem 11.2.6).

Let $P$ be a prime ideal of $S$, the prime algebra $S/P$ is algebraic over $C$ so its center is a field and being a PI ring it is a simple algebra, thus isomorphic to $M_k(D)$ with $D$ a division algebra finite dimensional over its center. In particular $P$ is a maximal ideal. By [1] Theorem 1.1.41 we have $\{0\} = \bigcap P$ is the intersection of all prime (maximal ideals). If there are only finitely many maximal ideals $P_1 \cap P_2 \cap \ldots \cap P_k = \{0\}$ then $S = \oplus_i S/P_i$ and its center is a field only if $m = 1$. But if we have $m > n$ maximal ideals we still have $S/ \cap_{i=1}^m P_i = \oplus_{i=1}^m S/P_i$ contains $C^m$ which contains elements which are not algebraic of degree $n$ over $C$. $\square$

**Theorem 3.26.** If $S$ is a $\sigma$–simple $n$ Cayley–Hamilton algebra then $S$ is isomorphic to one of the algebras of Theorem 3.23.

**Proof.** By Corollary 3.7 the algebra $\sigma(R) = F$ is a field. Let $C$ be the center of $R$ it is also an $n$ Cayley–Hamilton algebra, commutative and with no nilpotent elements, we claim it is a finite direct sum of fields.

In fact we claim that in $C$ there are at most $n$ orthogonal idempotents. In fact if $e_1, \ldots, e_m$ are orthogonal idempotents we have that $S = \oplus_i Se_i$, by Corollary 3.10 we have $n = \sum h_i, h_i > 0$ and this claim follows.

Then if $1 = e_1 + \cdots + e_m$ is a decomposition into primitive orthogonal idempotents we have $C = \oplus_i C_i$, $C_i := Ce_i$. Each $C_i$ has no nilpotent elements, no non trivial idempotents and it is algebraic over $F$ then it is a field.

We claim that $S_i := Se_i$ is an $h_i$ simple CH algebra.

In fact clearly an ideal of $S_i$ is an ideal of $S$ so there are no nil ideals, the $\sigma$ algebra is contained in $F$ which is a field so, by the argument of Corollary 3.7 is also a field and the conclusion of Corollary 3.7 applies.

In positive characteristic the $\sigma$ algebra of $S_i$ need not be $F$ as the following example shows. Let $F$ be of positive characteristic $p$, consider on $R = F \oplus F$ the norm $N(a, b) = a^p b$, the $\sigma$ algebra of $R$ is $F$ but that of the first summand is $F^p$ which may be different from $F$.

Now, changing notations we may assume that the center of $S$ is a field $C \supset F = \sigma(S)$ and conclude by Lemma 3.25.
Recall that, by Proposition 3.8 a \(\sigma\)–prime \(n\) Cayley–Hamilton algebra \(S\) is torsion free over \(\sigma(R)\) which is a domain. Thus we can embed \(S\) in \(S \otimes_{\sigma(S)} K\).

**Corollary 3.27.** If \(S\) is a \(\sigma\)–prime \(n\) Cayley–Hamilton algebra and \(K\) is the field of fractions of \(\sigma(S)\) then \(S \otimes_{\sigma(S)} K \simeq \bigoplus_{i=1}^{p} M_{k_{i}}(D_{i})\) is a \(\sigma\)–simple \(n\) Cayley–Hamilton algebra isomorphic to one of the algebras of Theorem 3.23 and containing \(S\).

There are finitely many minimal prime ideals \(P_{j} = S \cap \bigoplus_{i=1}^{p} M_{k_{i}}(D_{i})\) in \(S\) with intersection \(\{0\}\).

### 3.28 The Spectrum

In any associative algebra \(R\) one can define the spectrum of \(R\) as the set of all its prime ideals, it is equipped with the Zariski topology.

For commutative algebras the spectrum is a contravariant functor with \(f : A \to B\) giving \(P \mapsto f^{-1}(P)\). But in general a subalgebra of a prime algebra need not be prime and the functoriality fails.

For \(\Sigma\)–algebras \(R\) we may define:

\[
Spec_{\sigma}(R) := \{P \mid P \text{ is a prime } \sigma\text{–ideal}\}.
\]

For an \(n\) Cayley-Hamilton algebra \(R\) we have, by Corollary 3.7 the map \(j : Spec_{\sigma}(R) \to Spec(\sigma(R)), P \mapsto P \cap \sigma(R)\) and the remarkable fact:

**Proposition 3.29.** The map \(j : Spec_{\sigma}(R) \to Spec(\sigma(R)), P \mapsto P \cap \sigma(R)\) is a homeomorphism, its inverse is \(p \mapsto \tilde{K}(pR)\).

**Proof.** First we have, for any \(\Sigma\)–algebra \(R\), any \(I \subset \sigma(R)\) an ideal of \(\sigma(R)\) that \(IR\) is a \(\sigma\)–ideal and \(IR \cap \sigma(R) = I\). In fact if \(r \in I \cap \sigma(R)\) we have \(r = \sum a_{i}s_{i}, a_{i} \in I, s_{i} \in R\) and, by Amitsur’s formula \(\sigma_{j}(r) = \sigma_{j}(\sum a_{i}s_{i})\) is a polynomial in elements \(\sigma_{k}(u)\) with \(u\) a monomial in the elements \(a_{i}s_{i}\). So \(u = a_{s}, a \in I\) and \(\sigma_{k}(u) = a^{b}\sigma_{k}(s) \in I\).

Let \(p \subset \sigma(R)\) be a prime \(\sigma\)–ideal. Since \(\sigma(R/pR) = \sigma(R)/p\) is a domain we have also \(\tilde{K}(pR)/pR \cap \sigma(R)/p = \{0\}\) hence \(\sigma(R/\tilde{K}(pR)) = \sigma(R)/p\).

From Corollary 3.7 2) we have that the ideal \(\tilde{K}(pR)\), is prime. In fact \(\sigma(R/\tilde{K}(pR)) = \sigma(R)/p\) a domain and also the radical of \(R/\tilde{K}(pR)\) is \(\{0\}\), by 3) of Proposition 3.5.

\[\square\]
So the composition in one direction is the identity $p = \tilde{K}(pR) \cap \sigma(R)$.

If $P$ is a prime $\sigma$–ideal we need to show that $P = \tilde{K}((P \cap \sigma(R))R)$.

We certainly have $P \supset \tilde{K}((P \cap \sigma(R))R)$ so it is enough to show that, if $P \supset Q$ are two prime $\sigma$–ideals and $P \cap \sigma(R) = Q \cap \sigma(R)$ then $P = Q$. In fact in $R/Q$, a prime $\sigma$–algebra, we have $\sigma(P/Q) = 0$ which implies $P/Q \subset K_{R/Q} = 0$.

So for $n$ Cayley-Hamilton algebras the spectrum is also a contravariant functor setting

$$f : A \to B, \ P \mapsto \tilde{K}(f^{-1}(P)).$$

Some consequences Assume that $R$ is an $n$ Cayley-Hamilton algebra. Let $p$ be a prime $\sigma$–ideal of $\sigma(R)$ and consider the local algebra $\sigma(R)_p$ and

$$R_p := R \otimes_{\sigma(R)} \sigma(R)_p$$

we have that

**Proposition 3.30.** $R_p$ is local, as $\sigma$–ring, with maximal $\sigma$–ideal $\tilde{K}(R_p)$. Theorem 3.23 gives the possible residue $\sigma$–simple algebras $R(p) := R_p/\tilde{K}(R_p)$.

A special case is when $R(p)$ is simple as algebra and of rank $n^2$ over its center. In this case one can apply Artin’s characterization of Azumaya algebras (cf. [1] Theorem 10.3.2) and deduce that

**Proposition 3.31.** If $R(p)$ is simple as algebra and of rank $n^2$ over its center than $R_p$ is a rank $n^2$ Azumaya algebra over its center $\sigma(R)_p$.

Let us analyze general prime ideals, not necessarily $\sigma$–ideals.

**Proposition 3.32.** Let $S$ be an $n$ Cayley–Hamilton algebra with $\sigma$–algebra $A$, $P$ an algebra ideal of $S$ which is prime and $p = P \cap A$. Let $F$ be the field of fractions of $A/p$.

Then $S/P \otimes_A F = S/P \otimes_{A/p} F$ is a simple algebra and $P$ is one of the minimal primes of the prime $\sigma$–algebra $S/\tilde{K}(pS)$ (Corollary 3.27).

**Proof.** We have $P \supset \tilde{K}(pS)$ since $\tilde{K}(pS)$ is nil modulo $pS$.

Thus we have a surjective map $S/\tilde{K}(pS) \to S/P$ which induces a surjective map $S/\tilde{K}(pS) \otimes_A F \to S/P \otimes_A F$ with $F$ the field of fractions of $A/p$.

Since $S/P \otimes_A F = S/P \otimes_{A/p} F$ is a prime algebra and, Corollary 3.27, $S/\tilde{K}(pS) \otimes_A F = \oplus_i S_i$ is a direct sum of simple algebras we must have $S/P \otimes_A F = S_i$ for one of the summands and the claim follows.
This is a strong form of going up and lying over of commutative algebra in this general setting.

**Corollary 3.33.** Let $S$ be an $n$ Cayley–Hamilton algebra with $\sigma$–algebra $A$, $M$ an algebra ideal of $S$ which is maximal and $m = M \cap A$. Then $m$ is a maximal ideal of $A$.

*Proof.* We have that $S/M$ is a simple algebra integral over $A/m$ hence its center, a field, is integral over $A/m$. Thus $A/m$ is a field from the going up theorem. \qed

**Lemma 3.34.** Let $S$ be an $n$ Cayley–Hamilton algebra with $\sigma$–algebra $A$, $M$ an algebra ideal of $S$ so that $S/M$ is simple of dimension $n^2$ over its center and $m = M \cap A$. Then $M = mS$.

*Proof.* The algebra $\tilde{S} := S/mS$ is an $n$ Cayley–Hamilton algebra with $\sigma$–algebra the field $F = A/mA$ by the previous Lemma.

We have that $\tilde{S}/\tilde{K}(\tilde{S})$ is a simple $\sigma$–algebra to which we can apply Theorem 3.23. From Corollary 3.24 the fact that one of the simple summands of $R := \tilde{S}/\tilde{K}(\tilde{S})$ is $S/M = M_k(D)$ simple of dimension $n^2$ over its center we have that $R = M_k(D)$ with center $F$. Since $S$ satisfies the PI’s of $n \times n$ matrices, by Artin’s characterization of Azumaya algebras we have that $S$ is Azumaya over its center $A$. By Theorem 2.28 the norm takes as values the center $A$ which thus must equal $F$ and so $S = M_k(D)$. \qed

**Theorem 3.35.** Let $S$ be an $n$ Cayley–Hamilton algebra with $\sigma$–algebra $A$ a local ring with maximal ideal $m$. Let $M$ be an algebra ideal of $S$ so that $S/M$ is simple of dimension $n^2$ over its center, then $M = mS$ and $S$ is a rank $n^2$ Azumaya algebra over $A$.

### 3.36 $T$–ideals and relatively free algebras

One purpose of this paper is to classify and then study prime and semiprime $T$–ideals in a free $CH_n$–algebra. In order to explain the meaning and the results of this program we recall first the classical theory of polynomial identities of which the present paper is a natural development.

Recall that, given an associative algebra $R$ over an infinite field $F$, a *polynomial identity* of $R$ is an element $f(x_1, \ldots, x_\ell) \in F(x_1, \ldots, x_\ell, \ldots)$ of the free algebra in countably many variables which vanishes under all evaluations in $R$, $x_i \mapsto r_i \in R$ of the variables $x_i$. 44
The set \( I \) of polynomial identities of \( R \) is an ideal of \( F \langle x_1, \ldots, x_\ell, \ldots \rangle \) with the special property of being closed under all substitutions of the variables \( x_i \mapsto g_i \in F \langle x_1, \ldots, x_\ell, \ldots \rangle \). Such an ideal is called a \( T \)--ideal. Conversely any \( T \)--ideal \( I \) is the ideal of polynomial identities of some algebra \( R \), and we can take as \( R = F \langle x_1, \ldots, x_\ell, \ldots \rangle / I \).

We want to investigate now the structure of prime \( T \)--ideals (in the sense of \( \Sigma \)--algebras) in the free \( n \) Cayley Hamilton algebra \( F_{\Sigma,n}(X) \). For simplicity let us assume that all algebras are over an infinite field, though this is not strictly necessary as the reader may show.

If \( P \) is such an ideal then it is the \( T \)--ideal of \( \Sigma \)--identities of the prime \( \Sigma \)--algebra \( R := F_{\Sigma,n}(X) / P \). These identities are also satisfied by the algebra of fractions of \( R \) so that finally a prime \( T \)--ideal of \( \Sigma \)--identities is the \( T \)--ideal of \( \Sigma \)--identities of a simple \( \Sigma \)--algebra \( S \) with \( \sigma(S) = L \supset F \) a field. Replace \( S \), with \( \tilde{S} := S \otimes_L \tilde{L} \) where \( \tilde{L} \) is an algebraic closure of \( L \). Although this algebra is not necessarily simple the \( \Sigma \)--identities with coefficients in \( F \) of \( S \) and \( \tilde{S} \), coincide.

If \( \tilde{L} \subset \tilde{L} \) is the separable closure of \( L \) we have that \( \tilde{S} := S \otimes_L \tilde{L} = \oplus_i M_m(G_i) \) with \( G_i \) purely inseparable over \( \tilde{L} \) and furthermore there is a minimum \( k_i \) with \( g p^{k_i} \in \tilde{L} \), \( \forall g \in G_i \).

The structure of \( \tilde{S} \otimes_{\tilde{L}} \tilde{L} = \oplus_i M_m(G_i \otimes_{\tilde{L}} \tilde{L}) \) can be deduced from Remark 3.22. The commutative algebras \( G_i \otimes_{\tilde{L}} \tilde{L} \) are obtained from the algebraically closed field \( \tilde{L} \) by adding nilpotent elements \( a_i, i \in I, \) of degrees \( p^{k_i}, h_i \leq k_i, \) i.e. \( G_i \otimes_{\tilde{L}} \tilde{L} = \tilde{L}[a_i], i \in I, \) \( a_i^{p^{k_i}} = 0. \) The norm \( N \) is a product of the norms \( N_i : M_m(\tilde{L}[a_i]) \rightarrow \tilde{L}[a_i] \) is given by \( A \mapsto \det(A)^{p^{k_i}}. \) Here \( A \in M_m(\tilde{L}[a_i]) \) is a matrix with entries \( a_{i,j} + u_{i,j}, a_{i,j} \in \tilde{L} \) and \( u_{i,j}^{p^{k_i}} = 0. \) So that \( \det(A) = \det((a_{i,j}))+u, \) \( u^{p^{k_i}} = 0 \) and finally \( \det(A)^{p^{k_i}} = \det((a_{i,j}))^{p^{k_i}}. \)

This means that the norm \( N_i \) is obtained from the norm \( \det(A)^{p^{k_i}} \) of \( M_m(\tilde{L}) \) by extending the coefficients, thus as far as \( \Sigma \)--identities, the algebra \( \tilde{S} := \oplus_i M_m(\tilde{L}[a_i]) \) is equivalent to \( \oplus_i M_m(\tilde{L}). \) Finally, since \( F \) is an infinite field, this is PI equivalent to \( \oplus_i M_m(F) \).

The possible norms on \( \oplus_i M_m(F) \) are given by Theorem 3.14 and are of the form given by Formula (41), thus depend only on \( q \) integers \( m_i, a_i \) with \( \sum_{i=1}^q m_ia_i = n. \)

To complete the classification of prime \( T \)--ideals we have only to show that two algebras associated to two different sequences of \( q \) integers \( m_i, a_i \) with \( \sum_{i=1}^q m_ia_i = n \) are not PI equivalent. For this it is enough to see that the corresponding relatively free algebras contain the information on these
numbers. This part is identical to that developed in the previous paper [19] to which we refer.

**Theorem 3.37.** Prime $T$–ideals (in the sense of $\Sigma$–algebras) in the free $n$ Cayley Hamilton algebra $F_{\Sigma,n}(X)$ are classified by sequences of $q$ integers $m_i, a_i$ with $\sum_{i=1}^{q} m_i a_i = n$. To such a sequence one associates the $T$–ideal of identities of the algebra of Formula (40) $F(m; a)$.

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47
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48