Model Structures on Ind Categories and the Accessibility Rank of Weak Equivalences

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Abstract
In [BaSc1] we introduced a much weaker and easy to verify structure then a model category, which we called a "weak fibration category". Our main result in [BaSc1] is that a small weak fibration category can be "completed" into a full model category structure on its pro-category, provided the pro category satisfies a certain two out of three property. In this paper we give sufficient intrinsic conditions on a weak fibration category for this two out of three property to hold. We apply these results to prove theorems giving sufficient conditions for the finite accessibility of the category of weak equivalences in combinatorial model categories. We apply these theorems to the standard model structure on the category of simplicial sets, and deduce that its class of weak equivalences is finitely accessible. The same result on simplicial sets was recently proved also in [RaRo], using different methods.

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1 Introduction

In certain respect, according to the algebraic approach to homotopy theory, the basic object of study in homotopy theory is a category $\mathcal{C}$ endowed with a class of morphism $\mathcal{W}$ called weak-equivalences that should be considered as “isomorphisms honoris causa”. If the class of weak-equivalences is well behaved we say that $(\mathcal{C}, \mathcal{W})$ is a relative category:

**Definition 1.1.** A relative category is a pair: $(\mathcal{C}, \mathcal{W})$, consisting of a category $\mathcal{C}$, and a subcategory $\mathcal{W} \subseteq \mathcal{C}$ that contains all the isomorphisms and satisfies the 2 out of 3 property. $\mathcal{W}$ is called the subcategory of weak equivalences.

The data of a relative category is enough to define most of the different constructions needed in homotopy theory (such that mapping spaces, homotopy limits, derived functors, etc...) by universal properties. Alas, in a relative category it is in practice very hard to insure existence of wanted objects or to carry out any computations. Thus working effectively in a relative category $(\mathcal{C}, \mathcal{W})$ is usually achieved by adding some extra structure. The most prevalent example is the structure of a model category defined by Quillen in [Qui]. Model categories, albeit very useful, have quite a “heavy” axiomatization. A model category consists of relative category $(\mathcal{C}, \mathcal{W})$ together with two subcategories $\mathcal{F}, \mathcal{Cof}$ of $\mathcal{C}$ called cofibrations and fibrations. The quadruple $(\mathcal{C}, \mathcal{W}, \mathcal{F}, \mathcal{Cof})$ should satisfy many axioms (we refer the reader to [Hov] for the modern definition of a model category). The axioms for a model category are often very hard to verify, and furthermore, there are situations in which there is a natural definition of weak equivalences and fibrations, however, the resulting structure is not a model category (Note that the structure of a model category is determined by the classes of weak equivalences and fibrations, since the class of cofibrations is then determined by a left lifting property).

In [BaSc1] we introduced a much weaker and easy to verify structure then a model category, which we called a ”weak fibration category”. A weak fibration category consists of a relative category $(\mathcal{C}, \mathcal{W})$ together with one subcategory $\mathcal{F}$ of $\mathcal{C}$ called fibrations, satisfying certain axioms (see Definition 2.12). Our main result in [BaSc1] is that a small weak fibration category can be ”completed” into a full model category structure on its pro-category, provided the pro category satisfies a certain two out of three property. Namely, let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a weak fibration category. We say the a morphism in $\text{Pro}(\mathcal{C})$ is in $Lw^\equiv(\mathcal{W})$ if it is isomorphic, as a morphism in $\text{Pro}(\mathcal{C})$, to a natural transformation which is levelwise in $\mathcal{W}$. We say that the weak fibration category $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is pro-admissible if $(\text{Pro}(\mathcal{C}), Lw^\equiv(\mathcal{W}))$ is a relative category.

The main result of [BaSc1] (see Theorem 4.4 there) is:

**Theorem 1.2.** Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a small pro admissible weak fibration category. Then there exists a model category structure on $\text{Pro}(\mathcal{C})$ s.t:
1. The weak equivalences are $W := Lw^\approx(W)$.

2. The cofibrations are $C := \perp(F \cap W)$.

3. The fibrations are maps satisfying the right lifting property with respect to all acyclic fibrations.

Moreover, this model category is $\omega$-cocombinatorial, with $F$ as the set of generating fibrations and $F \cap W$ as the set of generating acyclic cofibrations.

Remark 1.3.

1. Note that by abuse of notation we consider morphisms of $C$ as morphisms of $\text{Pro}(C)$ indexed by the trivial diagram.

2. A more explicit description of the fibrations in this model structure can be given (see [BaSc] Theorem 4.4), but this requires some more definitions and we will not need it in this paper.

3. A model category is said to be cocombinatorial if its opposite category is combinatorial. Combinatorial model categories were introduced by J. H. Smith as model categories which are locally presentable and cofibrantly generated (see for instance the appendix of [Lur]). If $\gamma$ is a regular cardinal we also follow J. H. Smith and call a model category $\gamma$-combinatorial if it is combinatorial and both cofibrations and trivial cofibrations are generated by sets of morphisms having $\gamma$-presentable domains and codomains.

The pro admissibility condition on a weak fibration category $C$, appearing in Theorem 1.2, is not intrinsic to $C$. It is useful to be able to deduce the admissibility of $C$ only from conditions on $C$ itself. One purpose of this paper is to give one possible solution to this problem. This is done in Section 3.

Everything we have discussed so far is completely dualizable. Thus we can define the notion of an ind-admissible weak cofibration category, and show:

**Theorem 1.4.** Let $(M, W, C)$ be a small ind admissible weak cofibration category. Then there exists a model category structure on $\text{Ind}(M)$ s.t:

1. The weak equivalences are $W := Lw^\approx(W)$.

2. The fibrations are $F = (C \cap W)^\perp$.

3. The cofibrations are maps satisfying the left lifting property with respect to all acyclic fibrations.

Moreover, this model category is $\omega$-combinatorial, with $C$ as the set of generating cofibrations and $C \cap W$ as the set of generating acyclic cofibrations.

Model categories constructed using Theorem 1.4 have some further convenient property, namely, their class of weak equivalences is finitely accessible, when viewed as a full subcategory of the morphism category (we follow the terminology of [AR] throughout this paper). This means that a filtered colimit of
weak equivalences is a weak equivalence and every weak equivalence is a filtered colimit of weak equivalences between finitely presentable objects. This assertion follows immediately from [IsaL] Theorem 5.1 which says that the closure of $\mathcal{W}$ under filtered colimits in the morphism category $\text{Ind}(\mathcal{M})^{\rightarrow} \cong \text{Ind}(\mathcal{M}^{\rightarrow})$ is precisely $Lw^\approx(W)$ (Note that any object of $\mathcal{M}$ is finitely presentable as an object in $\text{Ind}(\mathcal{M})$).

In [BaSc1] we have applied Theorem 1.2 to a specific weak fibration category (namely, the category of simplicial sheaves over a Grothendieck site, where the weak equivalences and the fibrations are local in the sense of Jardine) to obtain a novel model structure in its pro-category.

In this paper we also consider an application of Theorem 1.2 (or rather of its dual version, Theorem 1.4), but in a reverse direction. Namely, we begin with an $\omega$-combinatorial model category $\mathcal{M}$ and ask whether the model structure on $\mathcal{M}$ is induced, via Theorem 1.4 from a weak cofibration structure on its full subcategory of finitely presentable objects. The main conclusion we wish to deduce from this is the finite accessibility of the class of weak equivalences in $\mathcal{M}$, as explained above.

While we were writing the first draft of this paper, Raptis and Rosicky published a paper with some related results [RaRo]. In their paper, Raptis and Rosicky mention that while the class of weak equivalences in any combinatorial model category is known to be accessible, the known estimates for the accessibility rank are generally not the best possible. In their paper they prove theorems giving estimates for the accessibility rank of weak equivalences in various cases. Their main application is to the standard model structure on simplicial sets. They show that the class of weak equivalences in this model structure is finitely accessible.

The purpose of this paper is the same, as well as the main example. Namely, we prove theorems giving estimates for the accessibility rank of weak equivalences in various cases, and our main example is the category of simplicial sets on which we achieve a similar estimate as [RaRo]. However, our theorems, as well as the methods of proof, are completely different. Since our basic tool is based on applying Theorem 1.4 as explained above, our estimates only concern finite accessibility. We do believe, however, that Theorem 1.4 and thus also our results here, can be generalized to an arbitrary cardinal instead of $\omega$. On the other hand, our theorems apply also in cases where the theorems in [RaRo] do not.

We will now state our main results. For this, we first need a definition:

**Definition 1.5.** Let $(\mathcal{C}, \mathcal{W})$ be a relative category. A map $f : A \to B$ in $\mathcal{C}$ will be called right proper, if for every pull back square of the form:

$$
\begin{array}{ccc}
C & \rightarrow & D \\
\downarrow j & & \downarrow i \\
A & \rightarrow & B
\end{array}
$$

s.t. $i$ is a weak equivalence, the map $j$ is also a weak equivalence.
We can now state our first criterion for the finite accessibility of the category of weak equivalences (see Theorem 1.6):

**Theorem 1.6.** Let \((\mathcal{M}, W, F, C)\) be an \(\omega\)-combinatorial left proper model category. Let \(\mathcal{M}\) denote the full subcategory of \(\mathcal{M}\) spanned by the finitely presentable objects.

Suppose we are given a cylinder object in \(\mathcal{M}\), that is, for every object \(B\) of \(\mathcal{M}\) we are given a factorization in \(\mathcal{M}\) of the fold map \(B \sqcup B \to B\) into a cofibration followed by a weak equivalence:

\[
B \sqcup B \xrightarrow{\left( i_0, i_1 \right)} I \otimes B \xrightarrow{p} B.
\]

(Note that we are not assuming any simplicial structure, \(I \otimes B\) is just a suggestive notation).

We make the following further assumptions:

1. The category \(\mathcal{M}\) has finite limits.
2. Every object in \(\mathcal{M}\) is cofibrant.
3. For every morphism \(f : A \to B\) in \(\mathcal{M}\) the map \(B \sqcup A (I \otimes A) \to B\), induced by the commutative square:

\[
\begin{array}{ccc}
A & \xrightarrow{i_0} & I \otimes A \\
\downarrow f & & \downarrow f \circ p \\
B & = & B,
\end{array}
\]

is a right proper map in \((\mathcal{M}, W)\).

Then the full subcategory of the morphism category of \(\mathcal{M}\), spanned by the class of weak equivalences, is finitely accessible.

Our second criterion can be shown using the first one (see Theorem 1.7):

**Theorem 1.7.** Let \((\mathcal{M}, W, F, C)\) be an \(\omega\)-combinatorial left proper model category. Let \(\mathcal{M}\) denote the full subcategory of \(\mathcal{M}\) spanned by the finitely presentable objects. Assume that the category \(\mathcal{M}\) has finite limits and let \(*\) denote the terminal object in \(\mathcal{M}\).

Suppose we are given a factorization in \(\mathcal{M}\) of the fold map \(* \sqcup * \to *\) into a cofibration followed by a weak equivalence:

\[
* \sqcup * \to I \to *.
\]

We make the following further assumptions:

1. For every morphism \(Y \to B\) in \(\mathcal{M}\) the functor:

\[
Y \times_B (-) : \mathcal{M}/B \to \mathcal{M}
\]

commutes with finite colimits.
2. Every object in $M$ is cofibrant.

3. For every object $B$ in $M$ the functor:

$$B \times (-) : M \to M$$

preserves cofibrations and weak equivalences.

Then the full subcategory of the morphism category of $M$, spanned by the class of weak equivalences, is finitely accessible.

It is not hard to verify that the standard model structure on the category of simplicial sets satisfies the hypothesis of the previous theorem (see Theorem 4.6). Thus we obtain:

**Theorem 1.8.** The full subcategory of the morphism category of $S$, spanned by the class of weak equivalences, is finitely accessible.

### 1.1 Organization of the paper

In Section 2 we bring a short review of the necessary background on pro-categories and model structures on them. Everything in this section dualizes easily to ind-categories. In Section 3 we prove a theorem giving sufficient intrinsic conditions for the admissibility of a weak cofibration category. We also define an auxiliary notion that generalizes the notion of a model category. The results and definitions of Section 3 will be used in Section 4 where we prove our main results of this paper, namely, a series of criterions for the finite accessibility of the category of weak equivalences.

### 1.2 Acknowledgments

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### 2 Preliminaries: model structures on pro-categories

In this section we bring a review of the necessary background on model structures on pro-categories. We state the results without proof, for later reference. For proofs and more information the reader is referred to [AM], [Isa], [BaSc] and [BaSc1].

#### 2.1 Pro categories

**Definition 2.1.** A category $I$ is called directed if the following conditions are satisfied:

1. $I$ is non-empty.
2. for every pair of objects $s, t \in I$, there exists an object $u \in I$, together with morphisms $u \to s$ and $u \to t$.

3. for every pair of morphisms $f, g : s \to t$ in $I$, there exists a morphism $h : u \to s$ in $I$, s.t. $f \circ h = g \circ h$.

A category is called small if it has only a set of objects and a set of morphisms.

**Definition 2.2.** Let $C$ be a category. The category $\text{Pro}(C)$ has as objects all diagrams in $C$ of the form $I \to C$ s.t. $I$ is small and directed (see Definition 2.1). The morphisms are defined by the formula:

$$\text{Hom}_{\text{Pro}(C)}(X, Y) := \lim_s \colim_t \text{Hom}_C(X_t, Y_s).$$

Composition of morphisms is defined in the obvious way.

Thus, if $X : I \to C$ and $Y : J \to C$ are objects in $\text{Pro}(C)$, giving a morphism $X \to Y$ means specifying for every $s$ in $J$ an object $t$ in $I$ and a morphism $X_t \to Y_s$ in $C$. These morphisms should of course satisfy some compatibility condition. In particular, if the indexing categories are equal: $I = J$, any natural transformation: $X \to Y$ gives rise to a morphism $X \to Y$ in $\text{Pro}(C)$. More generally, if $p : J \to I$ is a functor, and $\phi : p^*X := X \circ p \to Y$ is a natural transformation, then the pair $(p, \phi)$ determines a morphism $\nu_{p, \phi} : X \to Y$ in $\text{Pro}(C)$ (for every $s$ in $J$ we take the morphism $\phi_s : X_{p(s)} \to Y_s$). In particular, taking $Y = p^*X$ and $\phi$ to be the identity natural transformation, we see that $p$ determines a morphism $\nu_{p, X} : X \to p^*X$ in $\text{Pro}(C)$.

The word pro-object refers to objects of pro-categories. A **simple** pro-object is one indexed by the category with one object and one (identity) map. Note that for any category $C$, $\text{Pro}(C)$ contains $C$ as the full subcategory spanned by the simple objects.

**Definition 2.3.** Let $p : J \to I$ be a functor between small categories. The functor $p$ is said to be (left) cofinal if for every $i$ in $I$ the over category $p/\downarrow i$ is nonempty and connected.

Cofinal functors play an important role in the theory of pro-categories mainly because of the following well known lemma (see for example [AM]):

**Lemma 2.4.** Let $p : J \to I$ be a cofinal functor between small directed categories, and let $X : I \to C$ be an object in $\text{Pro}(C)$. Then the morphism in $\text{Pro}(C)$ that $p$ induces: $\nu_{p, X} : X \to p^*X$, is an isomorphism.

The following lemma can be found in [AM] Appendix 3.2. See also [BaSc] Corollary 3.26 for a stronger result.

**Lemma 2.5.** Every morphism in $\text{Pro}(C)$ is isomorphic, in the category of morphisms in $\text{Pro}(C)$, to a morphism that comes from a natural transformation (that is, to a morphism of the form $\nu_{id, \phi}$, where $\phi$ is a natural transformation).
**Definition 2.6.** Let \( C \) be a category, \( M \subseteq Mor(C) \) a class of morphisms in \( C \), \( I \) a small category, and \( F : X \rightarrow Y \) a morphism in \( C^I \). Then \( F \) will be called a levelwise \( M \)-map, if for every \( i \in I \): the morphism \( X_i \rightarrow Y_i \) is in \( M \). We will denote this by \( F \in Lw(M) \).

**Definition 2.7.** Let \( C \) be a category, and \( M \subseteq Mor(C) \) a class of morphisms in \( C \). Denote by:

1. \( \perp M \) the class of morphisms in \( C \) having the left lifting property w.r.t. any morphism in \( M \).
2. \( M\perp \) the class of morphisms in \( C \) having the right lifting property w.r.t. any morphism in \( M \).
3. \( Lw^\cong(M) \) the class of morphisms in \( Pro(C) \) that are isomorphic to a morphism that comes from a natural transformation which is a level-wise \( M \)-map.

Everything we did so far (and throughout this paper) is completely dualizable. Thus we can define:

**Definition 2.8.** A category \( I \) is called codirected if the following conditions are satisfied:

1. \( I \) is non-empty.
2. for every pair of objects \( s, t \in I \), there exists an object \( u \in I \), together with morphisms \( s \rightarrow u \) and \( t \rightarrow u \).
3. for every pair of morphisms \( f, g : s \rightarrow t \) in \( I \), there exists a morphism \( h : t \rightarrow u \) in \( I \), s.t. \( h \circ f = h \circ g \).

The dual to the notion of a pro category is the notion of an ind category:

**Definition 2.9.** Let \( C \) be a category. The category \( Ind(C) \) has as objects all diagrams in \( C \) of the form \( I \rightarrow C \) s.t. \( I \) is small and codirected (see Definition 2.8). The morphisms are defined by the formula:

\[
\text{Hom}_{Pro(C)}(X, Y) := \lim_{s} \text{colim}_{t} \text{Hom}_{C}(X_{s}, Y_{t}).
\]

Composition of morphisms is defined in the obvious way.

Clearly for every category \( C \) we have a natural isomorphism of categories: \( Ind(C)^{op} \cong Pro(C^{op}) \).

We are not going to write the dual to every definition or theorem explicitly, only in certain cases.
2.2 From a weak fibration category to a model category

We now bring the definition of a weak fibration category, after two preliminary definitions:

Definition 2.10. Let $\mathcal{C}$ ba a category, and let $M, N$ be classes of morphisms in $\mathcal{C}$. We will denote by $\text{Mor}(\mathcal{C}) = M \circ N$ the assertion that every map $A \to B$ in $\mathcal{C}$ can be factored as $A \xrightarrow{f} C \xrightarrow{g} B$, where $f$ is in $N$ and $g$ is in $M$.

Definition 2.11. Let $\mathcal{C}$ be category with finite limits, and let $M \subseteq \mathcal{C}$ be a subcategory. We say that $M$ is closed under base change if whenever we have a pullback square:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^g & & \downarrow^f \\
C & \rightarrow & D
\end{array}
\]

such that $f$ is in $M$, then $g$ is in $M$.

Definition 2.12. A weak fibration category is a category $\mathcal{C}$ with an additional structure of two subcategories:

$$\mathcal{F}, \mathcal{W} \subseteq \mathcal{C}$$

that contain all the isomorphisms, such that the following conditions are satisfied:

1. $\mathcal{C}$ has all finite limits.
2. $\mathcal{W}$ has the 2 out of 3 property.
3. The subcategories $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under base change.
4. $\text{Mor}(\mathcal{C}) = \mathcal{F} \circ \mathcal{W}$.

The maps in $\mathcal{F}$ are called fibrations, and the maps in $\mathcal{W}$ are called weak equivalences.

Definition 2.13. A relative category is a pair: $(\mathcal{C}, \mathcal{W})$, consisting of a category $\mathcal{C}$, and a subcategory $\mathcal{W} \subseteq \mathcal{C}$ that contains all the isomorphisms and satisfies the 2 out of 3 property. $\mathcal{W}$ is called the subcategory of weak equivalences.

Remark 2.14. Any weak fibration category, is naturally a relative category, when ignoring the fibrations.

Definition 2.15. We will denote by $\rightarrow$ the category consisting of two objects and one non identity morphism between them. Thus, if $\mathcal{C}$ is any category, the functor category $\mathcal{C}^{\rightarrow}$ is just the category of morphisms in $\mathcal{C}$.

Definition 2.16. A relative category $(\mathcal{C}, \mathcal{W})$ will be called:

1. pro admissible, if $Lw^{\oplus}(\mathcal{W}) \subseteq Pro(\mathcal{C})^{\rightarrow}$ satisfies the 2-out-of-3 property.
2. ind admissible, if $Lw^\cong(W) \subseteq Ind(C)^\rightarrow$ satisfies the 2-out-of-3 property.
3. admissible, if it both pro and ind admissible.

We now state the main Theorem in [BaSc1] giving a model structure on $Pro(C)$:

**Theorem 2.17.** Let $(C, W, F)$ be a small pro admissible weak fibration category. Then there exists a model category structure on $Pro(C)$ s.t:

1. The weak equivalences are $W := Lw^\cong(W)$.
2. The cofibrations are $C := \perp(F \cap W)$.
3. The fibrations are maps satisfying the right lifting property with respect to all acyclic fibrations.

Moreover, this model category is $\omega$-categorial, with $F$ as the set of generating fibrations and $F \cap W$ as the set of generating acyclic fibrations.

The dual to the notion of a weak fibration category is a weak cofibration category. Namely, a weak cofibration category is a category $M$ together with two subcategories: $C, W$ such that $(M^{op}, C^{op}, W^{op})$ is a weak fibration category. The following is a dual formulation of Theorem 2.17:

**Theorem 2.18.** Let $(M, W, C)$ be a small ind admissible weak cofibration category. Then there exists a model category structure on $Ind(M)$ s.t:

1. The weak equivalences are $W := Lw^\cong(W)$.
2. The fibrations are $F = (C \cap W)^\perp$.
3. The cofibrations are maps satisfying the left lifting property with respect to all acyclic fibrations.

Moreover, this model category is $\omega$-combinatorial, with $C$ as the set of generating cofibrations and $C \cap W$ as the set of generating acyclic cofibrations.

### 3 Proper morphisms

#### 3.1 A criterion for the 2 out of 3 property

The pro admissibility condition on a relative category $C$, appearing in Theorem 2.17, is not intrinsic to $C$ (see Definition 2.16). It is useful to be able to deduce the admissibility of $C$ only from conditions on $C$ itself. In this subsection we give one possible solution to this problem. The idea is a very straightforward generalization of an idea of Isaksen ([Isa], section 3).

**Definition 3.1.** Let $(C, W)$ be a relative category. A map $f : A \to B$ in $C$ will be called:
1. Left proper, if for every push out square of the form:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^i & & \downarrow^j \\
C & \xrightarrow{\ } & D
\end{array}
\]

s.t. \(i\) is a weak equivalence, the map \(j\) is also a weak equivalence.

2. Right proper, if for every pull back square of the form:

\[
\begin{array}{ccc}
C & \xrightarrow{\ } & D \\
\downarrow^j & & \downarrow^i \\
A & \xrightarrow{f} & B
\end{array}
\]

s.t. \(i\) is a weak equivalence, the map \(j\) is also a weak equivalence.

We denote by \(LP\) the class of left proper maps in \(C\) and by \(RP\) the class of right proper maps in \(C\).

Remark 3.2. The notion of a right proper map is related to the notion of a sharp map defined by Rezk in [Rez]. A sharp map is a map such that all its base changes are right proper. In other words the class of sharp maps is the largest class of maps that is contained in the right proper maps and is closed under base change (see Definition 2.11). A sharp map is called a weak fibration by Cisinski and a fibrillation by Barwick and Kan.

Example 1. Let \(M\) be a model category. Then:

1. Every acyclic cofibration in \(M\) is a left proper map in \((M,W)\).

2. Every acyclic fibration in \(M\) is a right proper map in \((M,W)\).

3. The model category \(M\) is left proper iff every cofibration in \(M\) is a left proper map in \((M,W)\).

4. The model category \(M\) is right proper iff every fibration in \(M\) is a right proper map in \((M,W)\).

Definition 3.3. Let \((C,W)\) be a relative category. Then \((C,W)\) will be said to have proper factorizations, if the following hold:

1. \(\text{Mor}(C) = RP \circ LP\).

2. \(\text{Mor}(C) = RP \circ W\).

3. \(\text{Mor}(C) = W \circ LP\).

Lemma 3.4. Let \(M\) be a proper model category. Then the relative category \((M,W)\) has proper factorizations.
Proof. .

1. \( \text{Mor}(\mathcal{M}) = RP \circ LP \) is shown by factoring every map into a cofibration followed by an acyclic fibration (see Example 1).

2. \( \text{Mor}(\mathcal{C}) = RP \circ W \) is shown by factoring every map into an acyclic cofibration followed by a fibration (see Example 1).

3. \( \text{Mor}(\mathcal{C}) = W \circ LP \) is shown by factoring every map into a cofibration followed by an acyclic fibration (see Example 1).

The following is shown in \( \text{Isa} \) Lemma 3.2 (see Remark 3.3):

**Lemma 3.5.** Let \( \mathcal{C} \) be a category, and let \( N \) and \( M \) be classes of morphisms in \( \mathcal{C} \), such that \( \text{Mor}(\mathcal{C}) = M \circ N \). Let \( T \) be a directed category and let \( f : \{ X_t \}_{t \in T} \rightarrow \{ Y_t \}_{t \in T} \) be a natural transformation, that is, a map in the functor category \( \mathcal{C}^T \). Suppose that \( f \) is an isomorphism as a map in \( \text{Pro}(\mathcal{C}) \) (or \( \text{Ind}(\mathcal{C}) \)).

Then there exist a directed category \( J \), a cofinal functor \( p : J \rightarrow T \) and a factorization \( p^* X \xrightarrow{g} H \xrightarrow{h} p^* Y \) of \( p^* f : p^* X \rightarrow p^* Y \) in the category \( \mathcal{C}^J \) such that \( h \) is a level-wise \( M \) map, \( g \) is a levelwise \( N \) map, and \( g, h \) are isomorphisms as maps in \( \text{Pro}(\mathcal{C}) \) (or \( \text{Ind}(\mathcal{C}) \)).

The following proposition is our main motivation for introducing the concepts of left and right proper morphisms:

**Proposition 3.6.** Let \( (\mathcal{C}, W) \) be a relative category, and let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be a pair of composable morphisms in \( \text{Pro}(\mathcal{C}) \) (or \( \text{Ind}(\mathcal{C}) \)). Then:

1. If \( \mathcal{C} \) has finite limits and colimits, and \( \text{Mor}(\mathcal{C}) = RP \circ LP \), then \( f, g \in Lw_\sim(W) \) implies that \( g \circ f \in Lw_\sim(W) \).

2. If \( \mathcal{C} \) has finite limits, and \( \text{Mor}(\mathcal{C}) = RP \circ W \), then \( g, g \circ f \in Lw_\sim(W) \) implies that \( f \in Lw_\sim(W) \).

3. If \( \mathcal{C} \) has finite colimits, and \( \text{Mor}(\mathcal{C}) = W \circ LP \), then \( f, g \circ f \in Lw_\sim(W) \) implies that \( g \in Lw_\sim(W) \).

**Proof.** For simplicity of writing we only write the \( \text{Pro}(\mathcal{C}) \) case.

We show 1. The proof is a straightforward generalization of the proof of \( \text{Isa} \) Lemma 3.5.

Since \( f, g \in Lw_\sim(W) \) there exists a diagram in \( \text{Pro}(\mathcal{C}) \):

\[
\begin{array}{ccc}
X'' & \xrightarrow{f} & Y'' \\
& \equiv & \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
& \equiv & \\
Y' & \xrightarrow{g} & Z'
\end{array}
\]
such that the vertical maps are isomorphisms in $\text{Pro}(\mathcal{C})$ and such that $Y' \to Z'$ is a natural transformation indexed by $I$ that is level-wise in $\mathcal{W}$ and $X'' \to Y''$ is a natural transformation indexed by $J$ that is level-wise in $\mathcal{W}$.

Let $Y' \xrightarrow{\cong} Y''$ denote the composition $Y' \xrightarrow{\cong} Y \xrightarrow{\cong} Y''$. It is an isomorphism in $\text{Pro}(\mathcal{C})$ (but not necessarily a level-wise isomorphism). It follows from [AM] Appendix 3.2 that there exists a directed category $K$, cofinal functors $p : K \to I$ and $q : K \to J$, and a map in $\mathcal{C}^K$:

$$q^*Y' \to p^*Y''$$

such that there is a commutative diagram in $\text{Pro}(\mathcal{C})$:

$$\begin{array}{ccc}
Y' & \xrightarrow{\cong} & Y'' \\
\downarrow \cong & & \downarrow \cong \\
Y' & \xrightarrow{\cong} & Y''
\end{array}$$

with all maps isomorphisms. Thus we have a diagram in $\mathcal{C}^K$:

$$p^*X'' \to p^*Y'' \xleftarrow{\cong} q^*Y' \to q^*Z'$$

such that the first and last maps are level-wise in $\mathcal{W}$ and the middle map is an isomorphism as a map in $\text{Pro}(\mathcal{C})$ (but not necessarily a level-wise isomorphism).

Since $\text{Mor}(\mathcal{C}) = RP \circ LP$ we get by Lemma [AM] applied for $M = RP$ and $N = LP$, that after pulling back by a cofinal functor $T \to K$ we get a diagram in $\mathcal{C}^T$:

$$A \to B \xleftarrow{\cong} E \xleftarrow{\cong} C \to D$$

such that the first and last maps are level-wise in $\mathcal{W}$, the second map is level-wise right proper and an isomorphism in $\text{Pro}(\mathcal{C})$ and the third map is level-wise left proper and an isomorphism in $\text{Pro}(\mathcal{C})$.

By Corollary 3.19 of [BaSc], since $\mathcal{C}$ has finite limits and colimits, the pull back and push out in $\text{Pro}(\mathcal{C})$ of a diagram in $\mathcal{C}^T$ can be computed level-wise. We thus get the following diagram in $\mathcal{C}^T$:

$$\begin{array}{ccc}
A & \xrightarrow{\text{Lw}(\mathcal{W})} & B \\
\cong \downarrow & & \cong \downarrow \\
A \times_B E & \xrightarrow{\text{Lw}(\mathcal{W})} & E \coprod_C D \\
\cong \downarrow & & \cong \downarrow \\
C & \xrightarrow{\text{Lw}(\mathcal{W})} & D
\end{array}$$

where $\cong$ indicates isomorphism in $\text{Pro}(\mathcal{C})$.
We thus get that the composition

\[ A \times_B E \xrightarrow{Lw(W)} E \xrightarrow{Lw(W)} E \coprod C D \]

is a level-wise \( W \) map that is isomorphic, as a map in \( \text{Pro}(C) \), to the composition \( g \circ f \). It follows that \( g \circ f \in Lw^\approx(W) \).

It is not hard to show (2) and (3) using the same type of generalization to the proof of [Isa] Lemma 3.6. \( \square \)

**Corollary 3.7.** Let \((C, W)\) be a relative category that has finite limits and colimits and proper factorizations. Then \((C, W)\) is admissible (see Definition 3.10). In particular, if \( C \) is a proper model category then \((C, W)\) is admissible.

### 3.2 Almost model categories

Corollary 3.7 gives sufficient conditions for the admissibility of a relative category and in particular of a weak cofibration category. However in some interesting examples these conditions are too restrictive. Namely, in some situations there is a natural mapping cylinder factorization (see the proof of Theorem 4.4) which can be shown to give factorizations of the forms \( \text{Mor}(M) = RP \circ LP \) and \( \text{Mor}(M) = W \circ LP \) but not \( \text{Mor}(M) = RP \circ W \). We will therefore need to use an auxiliary notion that is more general than a model category, which we call an almost model category.

**Definition 3.8.** An almost model category is a quadruple \((M, W, F, C)\) satisfying all the axioms of a model category, except (maybe) the two out of three property for \( W \). More precisely, an almost model category satisfies:

1. \( M \) is complete and cocomplete.
2. \( W, F, C \) are subcategories of \( M \) that are closed under retracts.
3. \( C \cap W \subseteq \perp F \) and \( C \subseteq \perp (F \cap W) \).
4. There exist functorial factorizations in \( M \) into a map in \( C \cap W \) followed by a map in \( F \) and into a map in \( C \) followed by a map in \( F \cap W \).

The following lemma can be proven just as in the case of model categories (see for example [Hov] Lemma 1.1.10):

**Lemma 3.9.** In an almost model category \((M, W, F, C)\) we have:

1. \( C \cap W = \perp F \).
2. \( C = \perp (F \cap W) \).
3. \( F \cap W = C \perp \).
4. \( F = (C \cap W) \perp \).
Definition 3.10. A relative category \((\mathcal{C}, \mathcal{W})\) will be called almost pro admissible, if \(Lw^\approx(\mathcal{W}) \subseteq Pro(\mathcal{C})^{-}\) satisfies the following portion of the 2-out-of-3 property:

For every pair of composable morphisms in \(Pro(\mathcal{C})\): \(X \xrightarrow{f} Z \xrightarrow{g} Y\) we have:
1. If \(f, g\) belong to \(Lw^\approx(\mathcal{W})\) then \(g \circ f \in Lw^\approx(\mathcal{W})\).
2. If \(g, g \circ f\) belong to \(Lw^\approx(\mathcal{W})\) then \(f \in Lw^\approx(\mathcal{W})\).

Theorem 3.11. Let \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) be a small almost pro admissible weak fibration category. Then there exist an almost model category structure on \(Pro(\mathcal{C})\) s.t:

1. The weak equivalences are \(\mathcal{W} := Lw^\approx(\mathcal{W})\).
2. The fibrations are \(\mathcal{C} := \perp(\mathcal{F} \cap \mathcal{W})\).
3. The cofibrations are maps satisfying the right lifting property with respect to all acyclic fibrations.

Furthermore we have: \(\mathcal{C} \cap \mathcal{W} = \perp \mathcal{F}\).

Proof. The main theorem of [BaSc1] (namely Theorem 4.4) says that if \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) is a small pro admissible weak fibration category, then \(Pro(\mathcal{C})\) is a model category, if we define the weak equivalences fibrations and cofibrations as in Theorem 3.11.

Going over the proof of that theorem we find that we can show all the axioms of a model category for \(Pro(\mathcal{C})\), accept the two out of three property for \(Lw^\approx(\mathcal{W})\), using only the fact that \(\mathcal{C}\) is almost pro admissible (In fact, the only place where we use the fact that \(Lw^\approx(\mathcal{W})\) stisfies the two out of three property is in Lemma 4.10, where we only use the portion of the 2-out-of-3 property given in Definition 3.11). \(\square\)

We can dualize the above:

Definition 3.12. A relative category \((\mathcal{C}, \mathcal{W})\) will be called almost ind admissible, if \(Lw^\approx(\mathcal{W}) \subseteq Ind(\mathcal{C})^{-}\) satisfies the following portion of the 2-out-of-3 property:

For every pair of composable morphisms in \(Ind(\mathcal{C})\): \(X \xrightarrow{f} Z \xrightarrow{g} Y\) we have:
1. If \(f, g\) belong to \(Lw^\approx(\mathcal{W})\) then \(g \circ f \in Lw^\approx(\mathcal{W})\).
2. If \(f, g \circ f\) belong to \(Lw^\approx(\mathcal{W})\) then \(g \in Lw^\approx(\mathcal{W})\).

Theorem 3.13. Let \((\mathcal{M}, \mathcal{W}, \mathcal{C})\) be a small almost ind admissible weak cofibration category. Then there exist an almost model category structure on \(Ind(\mathcal{M})\) s.t:

1. The weak equivalences are \(\mathcal{W} := Lw^\approx(\mathcal{W})\).
2. The fibrations are \(\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^\perp\).
3. The cofibrations are maps satisfying the left lifting property with respect to all acyclic fibrations.

Furthermore we have: \(\mathcal{F} \cap \mathcal{W} = \mathcal{C}^\perp\).
4 Criterions for finite accessibility

In this last section we will state our main results of this paper, namely, a series of criterions for the finite accessibility of the category of weak equivalences. The criterions are stated in a decreasing level of generality (each criterion being an application or a special case of the previous one) but in an increasing level of convenience of verification and applicability. Our only example in this paper is the category of simplicial sets, which is an example of applying the third and last criteron. However, the authors are aware of an example where the second criteron applies but not the third. This is a non standard model structure on the category of chain complexes of modules over a ring, and will be treated in a future paper.

Definition 4.1. A category is called finitely accessible if it has directed colimits end there is a small set of finitely presentable object that generate it under directed colimits.

The following lemma explains the relevance of Theorem 2.18 to the finite accessibility of the category of weak equivalences.

Lemma 4.2. Let \((\mathcal{M}, W, C)\) be a small ind admissible weak cofibration category. Consider the model structure induced on \(\text{Ind}(\mathcal{M})\) by Theorem 2.18. Then the full subcategory of \(\text{Ind}(\mathcal{M})\) spanned by the class of weak equivalences, is finitely accessible (see Definition 4.1).

Proof. It is enough to show that a filtered colimit in \(\text{Ind}(\mathcal{M})\) of weak equivalences is a weak equivalence and every weak equivalence is a filtered colimit of weak equivalences between finitely presentable objects. This assertion follows immediately from [IsaL] Theorem 5.1 which says that the closure of \(W\) under filtered colimits in the morphism category \(\text{Ind}(\mathcal{M})\) is precisely \(Lw^\Sigma(W)\) (Note that any object of \(\mathcal{M}\) is finitely presentable as an object in \(\text{Ind}(\mathcal{M})\)). \qed

We now come to our first criteron:

Proposition 4.3. Let \((\mathcal{M}, W, C)\) be an \(\omega\)-combinatorial model category. Let \(\mathcal{M}\) denote the full subcategory of \(\mathcal{M}\) spanned by the finitely presentable objects. Let \(W, C\) denote the classes of weak equivalences and cofibrations between object in \(\mathcal{M}\), respectively. We denote by \(LP\) the class of left proper maps in \((\mathcal{M}, W)\) and by \(RP\) the class of right proper maps in \((\mathcal{M}, W)\).

We make the following further assumptions:

1. The category \(\mathcal{M}\) has finite limits.
2. \(\text{Mor}(\mathcal{M}) = W \circ C\).
3. \(\text{Mor}(\mathcal{M}) = W \circ LP\).
4. \(\text{Mor}(\mathcal{M}) = RP \circ LP\).
Then \((\mathcal{M}, \mathcal{W}, \mathcal{C})\) is an ind admissible weak cofibration category and the induced model structure on \(\text{Ind}(\mathcal{M})\), given by Theorem 2.18, coincides with \((\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})\), under the natural equivalence \(\mathcal{M} \cong \text{Ind}(\mathcal{M})\).

In particular, it follows from Lemma 4.2 that the full subcategory of \(\mathcal{M}^{-}\), spanned by the class of weak equivalences, is finitely accessible.

**Proof.** Since \(\mathcal{M}\) is locally finitely presentable (being \(\omega\)-combinatorial) it follows that its full subcategory \(\mathcal{M}\) is essentially small, closed under finite colimits, and we have a natural equivalence of categories \(\text{Ind}(\mathcal{M}) \cong \mathcal{M}\) given by taking colimits (see [AR]).

It is now trivial to verify, using assumption 2 above, that \((\mathcal{M}, \mathcal{W}, \mathcal{C})\) is a weak cofibration category. Using assumptions 1,3 and 4 we get by Proposition 3.6 that \((\mathcal{M}, \mathcal{W}, \mathcal{C})\) is almost ind admissible (see Definition 3.12). Thus, by Theorem 3.13 there exists an almost model category structure on \(\mathcal{M} \cong \text{Ind}(\mathcal{M})\) s.t:

1. The weak equivalences are \(\mathcal{W} := Lw^{\cong}(\mathcal{W})\).
2. The fibrations are \(\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^\perp\).

Furthermore we have: \(\mathcal{F} \cap \mathcal{W} = \mathcal{C}^\perp\).

Since the model category \((\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})\) is \(\omega\)-combinatorial we have that:
\[
\mathcal{F} \cap \mathcal{W} = \mathcal{C}^\perp = \mathcal{F} \cap \mathcal{W}.
\]
\[
\mathcal{F} : = (\mathcal{C} \cap \mathcal{W})^\perp = \mathcal{F}.
\]
Thus, using Lemma 3.9 we also obtain:
\[
\overline{\mathcal{C}} \cap \mathcal{W} = \mathcal{C}^\perp = \mathcal{F} \cap \mathcal{W}.
\]
\[
\overline{\mathcal{C}} := (\mathcal{F} \cap \mathcal{W})^\perp = (\mathcal{F} \cap \mathcal{W}) = \mathcal{C}.
\]

It is now easy to show that \(\mathcal{W} = \mathcal{W}\). We will show that \(\mathcal{W} \subseteq \mathcal{W}\), and the other direction can be shown similarly.

Let \(f : X \to Y\) be an element in \(\mathcal{W}\). We decompose \(f\), in the almost model category \((\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})\), into an acyclic cofibration followed by a fibration:
\[
X \xrightarrow{h \in \overline{\mathcal{C}} \cap \mathcal{W}} Z \xrightarrow{g \in \mathcal{F}} Y.
\]

Since the weak cofibration category \((\mathcal{M}, \mathcal{W}, \mathcal{C})\) is almost ind admissible we have that \(g\) also belongs to \(\mathcal{W}\). Thus we have:
\[
h \in \overline{\mathcal{C}} \cap \mathcal{W} = \mathcal{C} \cap \mathcal{W},
\]
\[
g \in \mathcal{F} \cap \mathcal{W} = \mathcal{F} \cap \mathcal{W}.
\]
It follows that \(f \in \mathcal{W}\), because \(\mathcal{W}\) is closed under composition. \(\square\)

We now come to our second criterion for the finite accessibility of the category of weak equivalences.
Theorem 4.4. Let $(\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ be an $\omega$-combinatorial left proper model category. Let $\mathcal{M}$ denote the full subcategory of $\mathcal{M}$ spanned by the finitely presentable objects. Let $\mathcal{W}, \mathcal{C}$ denote the classes of weak equivalences and cofibrations between object in $\mathcal{M}$, respectively.

Suppose we are given a cylinder object in $\mathcal{M}$, that is, for every object $B$ of $\mathcal{M}$ we are given a factorization in $\mathcal{M}$ of the fold map $B \sqcup B \to B$ into a cofibration followed by a weak equivalence:

$$B \sqcup B \xrightarrow{(i_0, i_1)} I \otimes B \xrightarrow{p} B.$$ (Note that we are not assuming any simplicial structure, $I \otimes B$ is just a suggestive notation).

We make the following further assumptions:

1. The category $\mathcal{M}$ has finite limits.
2. Every object in $\mathcal{M}$ is cofibrant.
3. For every morphism $f : A \to B$ in $\mathcal{M}$ the map $B \sqcup_A (I \otimes A) \to B$, induced by the commutative square:

$$\begin{array}{ccc}
A & \xrightarrow{i_0} & I \otimes A \\
\downarrow f & & \downarrow f \circ p \\
B & = & B,
\end{array}$$

is a right proper map in $(\mathcal{M}, \mathcal{W})$.

Then $(\mathcal{M}, \mathcal{W}, \mathcal{C})$ is an ind admissible weak cofibration category and the induced model structure on $\text{Ind}(\mathcal{M})$, given by Theorem 2.18, coincides with $(\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$, under the natural equivalence $\mathcal{M} \simeq \text{Ind}(\mathcal{M})$.

In particular, it follows from Lemma 4.3 that the full subcategory of $\mathcal{M}^{op}$, spanned by the class of weak equivalences, is finitely accessible.

Proof. We will verify that all the conditions of Proposition 4.3 are satisfied. We only need to check the existence of factorizations of the form:

1. $\text{Mor}(\mathcal{M}) = \mathcal{W} \circ \mathcal{C}$.
2. $\text{Mor}(\mathcal{M}) = \mathcal{W} \circ \text{LP}$.
3. $\text{Mor}(\mathcal{M}) = \text{RP} \circ \text{LP}$.

All the factorizations above will be given by the same factorization which we now describe. This is just the mapping cylinder factorization relative to our given cylinder object for $\mathcal{M}$.

It is not hard to show that for any $B \in \mathcal{M}$ the maps $i_0, i_1 : B \to I \otimes B$ are acyclic cofibrations (for more details see [EnSc1] in the discussion following Definition 7.2).
Let \( f : A \to B \) be a morphism in \( \mathcal{M} \). We define the mapping cylinder of \( f \) to be the push out:

\[
\begin{array}{c}
A \xrightarrow{i_0} I \otimes A \\
\downarrow f \\
B \xrightarrow{\sim} C(f).
\end{array}
\]

We define a morphism \( q : C(f) = B \bigsqcup_A (I \otimes A) \to B \) to be the one induced by the commutative square:

\[
\begin{array}{c}
A \xrightarrow{i_0} I \otimes A \\
\downarrow f \\
B \xrightarrow{f \circ p} \sim B.
\end{array}
\]

We define a morphism \( i : A \to C(f) = B \bigsqcup_A (I \otimes A) \) to be the composition:

\[
A \xrightarrow{i_1} I \otimes A \xrightarrow{f} C(f).
\]

Clearly \( f = qi \), and we call this the mapping cylinder factorization.

The map \( q \) is a left inverse to \( j \) defined by the mapping cylinder push out back square:

\[
\begin{array}{c}
A \xrightarrow{i_0} I \otimes A \\
\downarrow f \\
B \xrightarrow{j} C(f).
\end{array}
\]

Since \( i_0 \) is an acyclic cofibration we get that \( j \) is also an acyclic cofibration, and in particular \( q \) is a weak equivalence.

The map \( i \) is a cofibration, being a composite of two cofibrations:

\[
A \xrightarrow{1} I \otimes A \xrightarrow{f \circ p} B \bigsqcup_A (I \otimes A).
\]

These maps are cofibrations because of the following push out squares:

\[
\begin{array}{c}
\phi \\
\downarrow \\
A \bigsqcup A \xrightarrow{(i_0, i_1)} I \otimes A.
\end{array}
\]

Since the map \( i \) is a cofibration and \( \mathcal{M} \) is left proper, we get that the map \( i \) is also left proper. By Assumption 3, \( q \) is right proper.

We now come to our third and last criterion.
Theorem 4.5. Let \((M, W, F, C)\) be an \(\omega\)-combinatorial left proper model category. Let \(M\) denote the full subcategory of \(M\) spanned by the finitely presentable objects. Assume that the category \(M\) has finite limits and let \(*\) denote the terminal object in \(M\). Let \(W, C\) denote the classes of weak equivalences and cofibrations between object in \(M\), respectively.

Suppose we are given a factorization in \(M\) of the fold map \(* \sqcup * \to *\) into a cofibration followed by a weak equivalence:

\[* \sqcup * \to I \to *.*

We make the following further assumptions:

1. For every morphism \(Y \to B\) in \(M\) the functor:
   
   \[Y \times_B (-) : M_B \to M\]

   commutes with finite colimits.

2. Every object in \(M\) is cofibrant.

3. For every object \(B\) in \(M\) the functor:
   
   \[B \times (-) : M \to M\]

   preserves cofibrations and weak equivalences.

Then \((M, W, C)\) is an ind admissible weak cofibration category and the induced model structure on \(\text{Ind}(M)\), given by Theorem 2.18, coincides with \((M, W, F, C)\), under the natural equivalence \(M \simeq \text{Ind}(M)\).

In particular, it follows from Lemma 4.2 that the full subcategory of \(M^{\to}\), spanned by the class of weak equivalences, is finitely accessible.

Proof. We will verify that all the conditions of Theorem 4.4 are satisfied. For every object \(B\) of \(M\) we have that the induced diagram:

\[B \sqcup B \cong (* \times B) \sqcup (* \times B) \cong (* \sqcup *) \times B \to I \times B \to * \times B \cong B\]

is a factorization in \(M\) of the fold map \(B \sqcup B \to B\) into a cofibration followed by a weak equivalence (Note that here \(\times\) denotes the actual categorical product and is not just a suggestive notation).

Thus, we only need to check that for every morphism \(f : A \to B\) in \(M\) the map \(q : B \coprod_A (I \times A) \to B\), induced by the commutative square:

\[\begin{array}{ccc}
A & \xrightarrow{\iota_0} & I \times A \\
\downarrow^f & & \downarrow^{f_{\text{op}}} \\
B & \xrightarrow{=} & B,
\end{array}\]

is a right proper map in \((M, W)\).
We will use the same notation as in the proof of Theorem 4.4 regarding the mapping cylinder factorization.

Let:

\[
\begin{array}{ccc}
C(f) \times_B X & \longrightarrow & X \\
\downarrow j & & \downarrow i \\
C(f) & \longrightarrow & B
\end{array}
\]

be a pull back square in \( \mathcal{M} \), s.t. \( i \) is a weak equivalence. We need to show that \( j \) is a weak equivalence. Using condition 1 we get natural isomorphisms:

\[
C(f) \times_B X = (B \coprod_A (I \times A)) \times_B X \cong (B \times_B X) \coprod_{A \times_B X} ((I \times A) \times_B X) \cong
\]

\[
\cong (X \coprod_{A \times_B X} (I \times (A \times_B X))) = C(k),
\]

where: \( k : A \times_B X \to X \) is the natural map. By condition 3 and the proof of Theorem 4.4 we get that the natural map \( C(k) \cong C(f) \times_B X \to X \) is a weak equivalence. By the 2 out of 3 property we get that \( j \) is also a weak equivalence.

We now turn to our main example:

**Theorem 4.6.** Let \( S \) denote the category of simplicial sets with its standard model structure. Let \( S_f \) denote the full subcategory of \( S \) spanned by the finitely presentable objects. Let \( \mathcal{W}, \mathcal{C} \) denote the classes of weak equivalences and cofibrations between object in \( S_f \), respectively.

Then \((S_f, \mathcal{W}, \mathcal{C})\) is an ind admissible weak cofibration category and the induced model structure on \( \text{Ind}(S_f) \), given by Theorem 2.18, coincides with the standard model structure on \( S \), under the natural equivalence \( S \cong \text{Ind}(S_f) \).

In particular, it follows from Lemma 4.2 that the full subcategory of \( S^{-\to} \), spanned by the class of weak equivalences, is finitely accessible.

**Proof.** We will verify that all the conditions of Theorem 4.5 are satisfied. The model category \( S \) is \( \omega \)-combinatorial and left proper.

We first sketch a proof showing that the subcategory \( S_f \) of \( S \) is closed under finite limits.

Let \( X \) be a finite simplicial set. It is not hard to verify that there exist a finite diagram \( F : D \to \{ \Delta^0, \Delta^1, \Delta^2, \ldots \} \) such that:

\[
X \cong \text{colim}_D F.
\]

We now note the following facts:

1. In the category \( S \), pull backs commute with colimits.
2. For all \( n, m \geq 0 \): \( \Delta^n \times \Delta^m \) belongs to \( S_f \) (by direct computation).
3. A sub simplicial set of a finite simplicial set is also finite.
4. The colimit in \( \mathcal{S} \), of a finite diagram in \( \mathcal{S}_f \), belongs to \( \mathcal{S}_f \). Using these facts it is not hard to check that the pull back (in \( \mathcal{S} \)) of objects in \( \mathcal{S}_f \), belongs to \( \mathcal{S}_f \). Since the terminal object in \( \mathcal{S} \) also belongs to \( \mathcal{S}_f \), it follows that the subcategory \( \mathcal{S}_f \) of \( \mathcal{S} \) is closed under finite limits.

In particular this shows that \( \mathcal{S}_f \) admits finite limits and they can be calculated in \( \mathcal{S} \). This also gives condition 1 of Theorem 4.4 (as this condition is known to hold in \( \mathcal{S} \)).

Clearly every object in \( \mathcal{S}_f \) is cofibrant so condition 2 is satisfied.

Let \( B \) be an object in \( \mathcal{S}_f \). Since \( B \) is cofibrant and \( \mathcal{S} \) is a simplicial model category we get that the functor:

\[
B \times (-) : \mathcal{S} \to \mathcal{S}
\]

is a left Quillen functor and thus preserves cofibrations and weak equivalences between cofibrant objects. Since every object in \( \mathcal{S}_f \) is cofibrant we get that

\[
B \times (-) : \mathcal{S}_f \to \mathcal{S}_f
\]

preserves cofibrations and weak equivalences. This gives condition 3.

Finally, we may take the factorization of the fold map:

\[
* \sqcup * \to I \to *
\]

to be

\[
\Delta^{(0)} \sqcup \Delta^{(1)} \to \Delta^1 \to \Delta^0.
\]

\[\square\]

**Remark 4.7.** Let \( f : X \to Y \) be a morphism in \( \mathcal{S}_f \). In the proof of Theorem 4.4 we considered the mapping cylinder factorization of \( f : X \xrightarrow{h} C(f) \xrightarrow{g} Y \). We showed that \( g \) is right proper. Note that \( g \) is not, in general, a fibration in \( \mathcal{S} \). Consider the map \( f : \Delta^n \to \Delta^0 \) (\( n \geq 0 \)). Then the mapping cylinder factorization of \( f \) is just:

\[
\Delta^{[1, \ldots, n+1]} \to \Delta^{n+1} \to \Delta^0.
\]

But \( \Delta^{n+1} \to \Delta^0 \) is not a Kan fibration, since \( \Delta^{n+1} \) is not a Kan complex. Thus we see that we are using the extra generalization provided by Proposition 3.6 over Isaksen’s results (Lemmas 3.5 and 3.6 in [Isa]).

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