On models of non-Euclidean spaces generated by associative algebras

Maria Trnková

email: M.D.Trnkova@gmail.com

1Department of Algebra and Geometry, Faculty of Science, Palacky University in Olomouc, Tomkova 10, Olomouc, Czech Republic
2Department of Geometry, NIIMM, Kazan State University, prof. Nuzhina 1/37, Kazan, Russia

Abstract We present the non-trivial example how to generate non-Euclidean geometries from associative unital algebras. We consider bundles of the sphere of the degenerate non-Euclidean space and its two models. The first (conformal) model is obtained by the mapping \( S \) onto a plane pass through the origin. It is analogous to the stereographic mapping. The second model (projective) is constructed by the Norden normalization method, where we project the sphere onto a plane of normalization defining the metric and Christoffel symbols which allow us to find geodesic curves.

1. Introduction

A lot of models of non-Euclidean spaces were studied in the past, especially spaces of a constant curvature, projective spaces and the conformal planes (e.g. [1], [2], [3], [4]). There exists a lot of studies on how these models can be generated by algebras. It is well known that algebras define some structures in bundle manifolds of different types (e.g. [5], [6], [7]). In the literature, we can find many applications of this approach on the cases of non-Euclidean spaces (e.g. [8], [9], [10], [11], [12]).

We would like to present non-standard models within this framework. In the beginning, we describe how an associative algebra generates a vector space and we also discuss some of its properties. In the next section we define a sphere and the map \( S \) in this vector space and we use it to construct a conformal model. In the third section we remind some facts about the Norden normalization method [13] and we use it for the construction of a projective model.

Let us denote by \( \mathfrak{A} \) a unital associative \( n \)-dimensional algebra with the multiplication \( xy \), and by \( G \subset \mathfrak{A} \) the set of invertible elements. Then \( G \) is a Lie
group with the same multiplication rule. Let \( \mathfrak{B} \subset \mathfrak{A} \) be a unital subalgebra of an algebra \( \mathfrak{A} \) and \( H \subset G \) be the set of its invertible elements. So, \( H \) is a Lie subgroup of group \( G \) and \( G/H \) is the factor-space of right cosets. A bundle \((G, \pi, M = G/H)\) is a principal bundle with the structure group \( H \), where \( \pi \) is a canonical projection (for example, [14], [15]).

Foundations of the theory of finite-dimensional associative algebras were made by E. Cartan (1898), Wedderburn (1908) and F. E. Molin (1983), who discovered the structure of any algebra over an arbitrary base field [16]. E. Study and E. Cartan in [17] classified all 3 and 4-dimensional unital associative irreducible algebras up to an isomorphism. This classification could also be found in [18]. In this paper we consider only one type of 3-dimensional algebra \( \mathfrak{A} \). We leave a more complicated 4-dimensional case for a future work.

Let \( \{1, e_1, e_2\} \) be a basis of our algebra \( \mathfrak{A} \) with the identity element 1. The multiplication rules are:

\[
(e_1)^2 = 1, \quad (e_2)^2 = 0, \quad e_1 e_2 = -e_2 e_1 = e_2.
\]  

The matrix representation of an algebra \( \mathfrak{A} \) is a space of upper triangular matrices \( T_u = \{ \begin{pmatrix} x_0 & x_2 \\ 0 & x_1 \end{pmatrix} | x = x_0 + x_1 e_1 + x_2 e_2 \in \mathfrak{A} \} \) with the basic elements [16]

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

We consider the trivial conjugation \( x = x_0 + x^i e_i \rightarrow \overline{x} = x_0 - x^i e_i \) with the property \( \overline{xy} = \overline{y} \overline{x} \) and the bilinear form

\[
(x, y) = \frac{1}{2} (x \overline{y} + y \overline{x}).
\]

This form takes the real values and it determines a degenerate scalar product:

\[
(x, y) = x_0 y_0 - x_1 y_1.
\]

It defines a structure of degenerated pseudo-Euclidean vector spaces with rank 2 in the algebra \( \mathfrak{A} \). (It is also possible to call this space as "semi-pseudo-Euclidean", but later we will call it just "pseudo-Euclidean"). The set of invertible elements \( G = \{ x \in \mathfrak{A} | (x_0)^2 - (x_1)^2 \neq 0 \} \) is a non-Abelian Lie group. Its underlying manifold is \( \mathbb{R}^3 \) without two transversal 2-planes, hence it consists of 4 connected components.

The norm is defined as usual, \( |x, y|^2 = (x - y, x - y) \). The geodesic curves \( x(t) \) are then

\[
x_0 = a_0 t + b_0 \quad x_1 = a_1 t + b_1 \quad x_2 = f(t)
\]

where \( f(t) \) is an arbitrary function of \( t \) and \( a_0, a_1, b_0, b_1 \) are the numerical coefficients.

\footnote{Irreducible means indecomposable into a direct sum of algebras.}
In the basis (2) we can find two subalgebras: $R(e_1)$ with basis $\{1, e_1\}$, it is an algebra of double numbers, and a subalgebra $R(e_2)$ with basis $\{1, e_2\}$, it is an algebra of dual numbers. The set of their invertible elements $H_1 = \{x_0 + x_1e_1 \in R(e_1) \mid x_0^2 - x_1^2 \neq 0\}$ and $H_2 = \{x_0 + x_2e_2 \in R(e_2) \mid x_0 \neq 0\}$ are Lie subgroups of the Lie group $G$.

The space of right cosets $H_1 x$ defines a trivial principal bundle $(G, \pi, M = G/H_1)$ over the real line $\mathbb{R}$. The fiber is a plane without two transversal lines and the structure group is $H_1$. The coordinate view of the canonical projection $\pi$ is:

$$\pi(x) = \frac{x_2}{x_0 - x_1}. \quad (5)$$

The equation of fibers is:

$$u(x_0 - x_1) - x_2 = 0, \quad u \in \mathbb{R}. \quad (6)$$

Let us investigate $G$, the group of transformations of Lie group $G$. We can easily find that it has no dilations and inversions while there is a vertical translation $x \to x + a, a \in G$. Furthermore, $G$ includes the rotations, resp. anti-rotations, $x' = ax$ or $x' = xa$

with $|a|^2 = 1$, resp. $|a|^2 = -1$. These elements can be represented as:

$$a = \cosh \varphi \pm \sinh \varphi e_1 + u \sinh \varphi e_2, \quad \text{resp.} \quad a = \sinh \varphi \pm \cosh \varphi e_1 + u \cosh \varphi e_2,$$

where $u \in \mathbb{R}$. The anti-rotations transform the elements with the positive norms into the elements with the negative norms and visa versa.

The bilinear form (3) in the algebra $A$ takes the real values, therefore it is possible to present it as: $(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(|x|y + |y|x)$. Consequently, in the case of rotations the hyperbolic cosine of an angle between $x$ and $x'$ are equal to

$$\cosh(x, x') = \frac{(x, ax)}{|x||ax|} = \frac{1/2(x\overline{a}x + ax\overline{x})}{|x|^2} = \frac{1/2(x\overline{a}x + ax\overline{x})}{|x|^2} = \frac{1}{2}(\overline{a} + a) = \cosh \varphi,$$

and the same for the right multiplication. Similarly we get $\sinh \varphi$ for anti-rotations. Note that the angle $\varphi$ does not depend on $x$.

Transformations

$$x' = axb, \quad (8)$$

where $|a|^2 = \pm 1, \ |b|^2 = \pm 1$, are compositions of rotations and/or anti-rotations $x' = ax$ and $x' = xb$. We see that (8) defines proper rotations and anti-rotations.

Similarly,

$$x' = a\overline{xb} \quad (9)$$

are compositions of the reflection $x' = \overline{x}$ and transformations (8). These are improper rotations and anti-rotations.

**Lemma** Any proper or improper rotation/anti-rotation of the pseudo-Euclidean space $G$ can be represented by (8) or (9).
Proof Rotations and anti-rotations (8), (9) are compositions of odd and even numbers of reflections of planes passing through the origin. To each plane corresponds its orthonormal vector \( n \). If vectors \( x_1 \) and \( n \) are collinear, then \( \overline{x_1}n = -nx_1 \) and \( x_1' = -n\overline{x_1}n = -nx_1 = -x_1 \). If vectors \( x_2 \) and \( n \) are orthogonal, then \( \overline{x_2}n = n\overline{x_2} = 0 \) and \( x_2' = -n\overline{x_2}n = -nx_2 = x_2 \). On the other hand, any vector \( x \) can be represented as a sum of vectors \( x_1 \) and \( x_2 \). It means, that a reflection of the plane is: \( x' = -nxn \). Therefore, the composition of even, resp. odd number of reflections of planes are transformation (8), resp. (9). □

Translations and rotations/anti-rotations are then isometries. All transformations can be written in a known form (for further discussion see e.g. [4])

\[
\begin{align*}
    x_0' &= x_0 \cosh \varphi + x_1 \sinh \varphi + a_0 \\
    x_1' &= x_1 \cosh \varphi + x_0 \sinh \varphi + a_1 \\
    x_2' &= u_0x_0 + u_1x_1 + u_2x_2 + a_2
\end{align*}
\]

(10)

where \( a = a_ie_i \in G \) and \( u_i \in \mathbb{R} \).

Let us introduce adapted coordinates \((u, \lambda, \varphi)\) of the bundle in semi-Euclidean space, here \( u \) is a basic coordinate, \( \lambda, \varphi \) are fiber coordinates. If \(|x|^2 > 0\), we denote \( \lambda = \pm \sqrt{x_0^2 - x_1^2} \neq 0 \), the sign of \( \lambda \) is equal to the sign of \( x_0 \). The adapted coordinates of the bundle in this case are:

\[
    \begin{align*}
        x_0 &= \lambda \cosh \varphi, \\
        x_1 &= \lambda \sinh \varphi, \\
        x_2 &= u\lambda \exp \varphi,
    \end{align*}
\]

(11)

where \( \lambda \in \mathbb{R}_0 \), \( u, \varphi \in \mathbb{R} \).

If \(|x|^2 < 0\), then we write \( \lambda = \pm \sqrt{x_1^2 - x_0^2} \), the sign of \( \lambda \) is equal to the sign of \( x_1 \):

\[
    \begin{align*}
        x_0 &= \lambda \sinh \varphi, \\
        x_1 &= \lambda \cosh \varphi, \\
        x_2 &= u\lambda \exp \varphi.
    \end{align*}
\]

(12)

The structure group acts as follows:

\[
    u' = u, \quad \lambda' = \lambda \rho, \quad \varphi' = \varphi + \psi,
\]

(13)

where the element \( a(0, \rho, \psi) \) of the structure group acts on the element \( x(u, \lambda, \varphi) \in G \). This group consists of 4 connected components.

2. Conformal model of a sphere

We call semi-Euclidean sphere with an unit radius the set of all elements of algebra \( \mathfrak{A} \) whose square is equal to one,

\[
S^2(1) = \{ x \in \mathfrak{A} \mid x_0^2 - x_1^2 = 1 \}.
\]

Analogously, the set of elements with an imaginary unit module \(|x|^2 = -1\) we call semi-Euclidean sphere with an imaginary unit radius \( S^2(-1) \). One of these spheres can be obtained from another one by the rotation.

The transformations (10) are now constrained by additional relation \( x_0^2 - y_0^2 = 1 \), therefore, only rotations and vertical translations remain, \( a_0 = a_1 = 0 \).

We consider the subbundle of the bundle \((G, \pi, M = G/H_1)\) of semi-Euclidean sphere \( S^2(1) \), i.e. the bundle \( \pi : S^2(1) \to M \). The fibers of the new bundle
are intersections of $S^2(1)$ and planes $\{1\}$. The restriction of the group of double numbers $H_1$ to $S^2(1)$ is a Lie subgroup $S_1$ of double numbers with an unit module

$$S_1 = \{a_0 + a_1e_1 \in H_1 \mid a_0^2 - a_1^2 = 1\}.$$ 

This group consists of two connected components. The bundle $(S^2(1), \pi, M)$ is a principal bundle of the group $S^2(1)$ by the Lie subgroup $S_1$ to right cosets.

We define coordinates adapted to the bundle on semi-Euclidean sphere $S^2(1)$. If $x \in S^2(1)$ then from (11) we get $\lambda = \varepsilon$, $\varepsilon = \pm 1$. The parametric equation of semi-Euclidean sphere in the adapted coordinates $(u, \varphi)$ is:

$$r(u, \varphi) = \varepsilon \cosh \varphi, \sinh \varphi, u \exp \varphi,$$

where $u$ is a basis coordinate, $\varphi$ is a fiber coordinate. Different values of $\varepsilon$ correspond to different connected components of semi-Euclidean sphere $S^2(1)$.

Let us define the action of the structure group $S_1$ on semi-Euclidean sphere. From (13) and using the adapted coordinates of elements $a(0, \varepsilon_1, \psi)$, $x(u, \varepsilon, \varphi) \in S^2(1)$ we get:

$$u' = u, \quad \varepsilon' = \varepsilon \varepsilon_1, \quad \varphi' = \varphi + \psi.$$

This group also consists of two connected components.

The metric tensor for semi-Euclidean sphere has the matrix representation:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The linear element of the metric is:

$$ds_1^2 = -d\varphi^2.$$ 

Now, we want to define the conformal model of the bundle $(S^2(1), \pi, \mathbb{R})$. For that we need to introduce the conformal map of the sphere to a disconnected plane $f : S^2(1) \to Q \in \mathbb{R}^2$. $Q$ is located at $x_0 = 0$. We know that the sphere consists of two disconnected components, one with $x_0 > 0$, and other with $x_0 < 0$. We choose a pole at the first one, $N(1, 0, 0)$. All points of $S^2(1)$ except the line through the pole $N$ are stereographically projected to $Q$ such that the first component of the sphere with $x_0 > 0$ is mapped on $x_1 = (-\infty, -1) \cup (1, \infty)$ while the second component with $x_0 < 0$ is mapped on the strip $x_1 = (-1, 1)$. We denote $x, y$ coordinates on $Q$ such that the $x$ axis lies along $x_1$ while the $y$ axis along $x_2$. Then

$$x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0},$$

An inverse map $f^{-1} : \mathbb{R}^2 \to S^2(1)$ where $x \neq \pm 1$ is:

$$x_0 = -\frac{1 + x^2}{1 - x^2}, \quad x_1 = \frac{2x}{1 - x^2}, \quad x_2 = \frac{2y}{1 - x^2}.$$
If we substitute formulas (16) into (14) then we obtain the relations between coordinates $x, y$ and adapted coordinates $u, \varphi$ which are on semi-Euclidean sphere:

$$
\begin{align*}
  f: & \quad x = \frac{\sinh \varphi}{\varepsilon - \cosh \varphi}, \\
  & \quad y = \frac{u \exp \varphi}{\varepsilon - \cosh \varphi}.
\end{align*}
$$

Then the inverse map is:

$$
\begin{align*}
  \varphi &= \ln \left( \frac{\varepsilon x - 1}{x + 1} \right), \\
  u &= -\frac{2y}{(1 - x)^2}.
\end{align*}
$$

(18)

Note that the lines $x = \pm 1$ are not included in the mapping and $Q$ consists of three disconnected components. Also, the line $x_0 = 1, x_1 = 0$ has no image in this mapping. We add it by hand, identifying the image of this line with the points $x = \pm \infty$ on $Q$. Then two disconnected parts $x = (-\infty, -1)$ and $x = (1, \infty)$ are connected and we call this plane $C^2$.

In particular, after enlarging $Q$ into $C^2$ by the infinitely distant point and ideal line crossing this point, then the stereographic map $f$ becomes diffeomorphism $S$. Note that the infinitely distant point is the image of point $N$. The ideal line is the image of the straight line belonging to $S^2(1)$ and crossing the pole: $x_0 = 1, x_1 = 0$.

Let us now consider the commutative diagram:

$$
\begin{array}{c}
  S^2(1) \xrightarrow{\pi} \mathbb{R} \\
  \downarrow S \quad \nearrow p \\
  C^2 \end{array}
$$

The map $p = \pi \circ S^{-1} : C^2 \to \mathbb{R}$ is defined by this diagram. We find the coordinate form of this map:

$$
u = -\frac{2y}{(1 - x)^2}.
$$

The map $p : C^2 \to \mathbb{R}$ defines the trivial principal bundle with the base $\mathbb{R}$ and the structure group $S_1$.

**Theorem** Let $S$ is the map $S^2(1) \to C^2$ as described before. Then $S$ is a conformal map.

**Proof** The metric on $G$ induces the metric on $C^2$. In the coordinates $x, y$ it has the form:

$$
ds^2 = -dx^2.
$$

(19)

Let us find the metric of semi-Euclidean sphere from the metric on $C^2$. From (18) we get $d\varphi = \frac{2}{x^2 - 1}dx$ and using (15) and (19) we find:

$$
ds_1^2 = \frac{4}{(x^2 - 1)^2}ds^2.
$$

Hence, the linear element of semi-Euclidean sphere differs from the linear element of $C^2$ by a conformal factor and therefore, the map $S$ is conformal. \(\Box\)
We find the equation of fibers on $C^2$. The 1-parametric fibers family of the bundle $(S^2(1), \pi, \mathbb{R})$ in the adaptive coordinates (14) is: $u = c$, $c \in \mathbb{R}$. From (18) we get the image of this family under the map $S$:

$$y = -c/2 \cdot (x - 1)^2.$$  \hspace{1cm} (20)

The $C^2$ plane is also fibred by this 1-parametric family of parabolas.

3. THE PROJECTIVE CONFORMAL MODEL

Now we construct the projective semi-conformal model of the sphere $S^2(1)$ and the principal bundle on it. We use a normalization method of A.P.Norden [13], [19]. A. P. Shirokov in his work [20] constructed conformal models of Non-Euclidean spaces with this method.

In a projective space $P_n$ a hypersurface $X_{n-1}$ as an absolute is called normalized if with every point $Q \in X_{n-1}$ there is associated:
1) a line $P_I$ which has the point $Q$ as the only intersection with the tangent space $T_{n-1}$,
2) a linear space $P_{n-2}$ that belongs to $T_{n-1}$, but it does not contain the point $Q$.
We call them normals of first and second types, $P_I$ and $P_{II}$.

In order to have a polar normalization, $P_I$ and $P_{II}$ must be polar with respect to the absolute $X_{n-1}$.

We enlarge the semi-Euclidean space $E^3$ to a projective space $P^3$. Here $kE^3$ denotes a $n$-dimensional semi-Euclidean space with the metric tensor of rank $k$, and $l$ is the number of negative inertia index in a quadric form. We consider homogeneous coordinates $(y_0 : y_1 : y_2 : y_3)$ in $P^3$, where $x_i = \frac{y_i}{y_3}, i = 0, 1, 2$. Thus $S^2(1) : x_0^2 - x_1^2 = 1$ describes the hyperquadric in $P^3$:

$$y_0^2 - y_1^2 - y_2^2 = 0.$$  \hspace{1cm} (21)

Here the projective basis $(E_0, E_1, E_2, E_3)$ is chosen in the following way. The vertex $E_0$ of basis is inside the hyperquadric. The other vertices $E_1, E_2, E_3$ are on its polar plane, $y_0 = 0$. The line $E_0E_3$ crosses the hyperquadric at poles $N(1 : 0 : 0 : 1), N'(1 : 0 : 0 : -1)$. Vertices $E_1, E_2$ lie on the polar of the line $E_0E_3$. The vertex of the hyperquadric coincides with the vertex $E_2$.

The stereographic map of the projective plane $P^2 : y_0 = 0$ to the hyperquadric (21) from the pole $N(1 : 0 : 0 : 1)$ is shown on the picture. Let $U(0 : y_1 : y_2 : y_3) \in P^2$. If $y_3 = 0$, then the line $UN$ belongs to the tangent plane $T_N : y_0 - y_3 = 0$ of the hyperquadric (21) at the point $N$ and in this case the intersection point of the line $UN$ with the hyperquadric is not uniquely determined. If $y_3 \neq 0$, then the intersection point of the line $UN$ with the hyperquadric is unique. So, we choose the line $E_1E_2 : y_3 = 0$ as the line at infinity. In the area $y_3 \neq 0$ we consider the Cartesian coordinates $x_1 = \frac{y_2}{y_3}, x_2 = \frac{y_2}{y_3}$. Then the plane $\alpha : y_0 = 0, y_3 \neq 0$ becomes a plane with an affine structure $A^2$. It is possible to introduce the
structure of semi-Euclidean plane $\mathbb{E}^2$ with the linear element

$$ds_0^2 = dx_1^2. \quad (22)$$

The hyperquadric and the plane $\alpha$ do not intersect or intersect in two imaginary parallel lines

$$x_1^2 = -1. \quad (23)$$

The restriction of the stereographic projection to the plane $\alpha$ maps the point $U(0 : x_1 : x_2 : 1)$ into the point $X_1$

$$X_1(-1 - x_1^2 : 2x_1 : 2x_2 : 1 - x_1^2). \quad (24)$$

So, the Cartesian coordinates $x_i$ can be used as the local coordinates at the hyperquadric except the point of its intersection with the tangent plane $T_N$.

We construct an autopolar normalization of the hyperquadric. As a normal of the first type we take lines with the fixed center $E_0$ and as a normal of the second type we take their polar lines which belong to the plane $\alpha$ and cross the vertex $E_2$ of the hyperquadric. The line $E_0X_1$ intersects the plane $\alpha$ at the point

$$X(0 : 2x_1 : 2x_2 : 1 - x_1^2).$$

In this normalization the polar of the point $X$ intersects the plane $\alpha$ on the normal $P_{11}$. Thus for any point $X$ in the plane $\alpha$ there corresponds a line which does not cross this point. It means that the plane $\alpha$ is also normalized. The normalization of $\alpha$ is defined by an absolute quadric \[[23]\].

We consider the derivative equations of this normalization. If we take normals of the first type with fixed center $E_0$, then the derivative equations \[[13]\], p.204)
have the form:

\[ \partial_i X = Y_i + l_i X, \]
\[ \nabla_j Y_i = l_j Y_i + p_{ji} X. \]  \hspace{1cm} (25)

The points \( X, Y_i, E_0 \) define a family of projective frames. Here \( Y_i \) are generating points of the normal \( P_{II} \).

We can calculate the values \((X, X), (X, Y_i)\) on the plane \( \alpha \) using the quadric form, which is in the left part of equation (21). So, \((X, X) = -(1 + x_1^2)^2 \).

Let us find coordinates of the metric tensor on the plane \( \alpha \). Hence, we take the Weierstrass standardization

\[ (\tilde{X}, \tilde{X}) = -1, \quad \tilde{X} = \frac{X}{1 + x_1^2}. \]

Then the coordinates of the metric tensor are the scalar products of partial derivatives \( g_{ij} = -(\partial_i \tilde{X}, \partial_j \tilde{X}) \):

\[ (g_{ij}) = \begin{pmatrix} \frac{4}{(1 + x_1^2)^2} & 0 \\ 0 & 0 \end{pmatrix}. \]

We got the conformal model of the polar normalized plane \( \alpha : y_0 = 0, y_3 \neq 0 \) with a linear element

\[ ds^2 = \frac{dx_1^2}{(1 + x_1^2)^2}. \]  \hspace{1cm} (26)

It means that this non-Euclidean plane is conformally equivalent to semi-Euclidean plane \( 1E^2 \).

The points \( X \) and \( Y_i \) are conjugated with respect to the polar (21) and \((X, Y_i) = 0\). From this equation and the derivative equations (25) we can get the non-zero connection coefficients:

\[ \Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{2x_1}{1 + x_1^2}, \quad \Gamma_{11}^2 = \frac{2x_2}{1 + x_1^2}. \]

The sums \( \Gamma_{ks}^a = \partial_k \ln \frac{c}{(1 + x_1^2)^2} \) (\( c = \text{const} \)) are gradients, so the connection is equiaffine. Curvature tensor has the following non-zero elements:

\[ R_{121}^2 = -R_{211}^2 = -\frac{4}{(1 + x_1^2)^2}. \]

Ricci curvature tensor \( R_{sk} = R_{isk}^i \) is symmetric: \( R_{11} = \frac{4}{(1 + x_1^2)^2} \). Metric \( g_{ij} \) and curvature \( R_{rsk}^i \) tensors are covariantly constant in this connection: \( \nabla_k g_{ij} = 0, \nabla_l R_{rsk}^i = 0 \). The infinitesimal linear operators for the quadric are

\[ \begin{cases} 
L_1 = y_0 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_0} \\
L_2 = y_0 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_0} \\
L_3 = y_1 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_1} 
\end{cases} \]  \hspace{1cm} (27)
Solving geodesic equations we find parametric solutions

\[
\begin{align*}
    x_1 &= \tan(\omega t + \phi) \\
    x_2 &= (c_1 e^{2i\omega t} + c_2 e^{-2i\omega t}) \sec^2(\omega t + \phi)\
\end{align*}
\]  

(28)

where \(c_1, c_2, \omega, \phi\) are integration constants. Eliminating the parameter \(t\) we can rewrite these equations in a simple form

\[
x_2 = A(x_1^2 - 1) + Bx_1
\]

where \(A\) and \(B\) are arbitrary constants. We see that the solution represents parabolas and lines in \(x_2x_1\) plane.

Let us consider the bundle of this plane by the double numbers subalgebra. We write the equations of fibers of semi-Euclidean sphere \(S^2(1)\) in homogeneous coordinates:

\[
\begin{align*}
    (y_0 - y_1)v - y_2 &= 0, \\
    y_0^2 - y_1^2 - y_3^2 &= 0,
\end{align*}
\]

(29)

This 1-parametric family of curves fibers the hyperquadric and it defines a bundle on it. The image of these fibers under the stereographic projection from the pole \(N\) to the plane \(\alpha\) is:

\[
x_2 = -v/2 \cdot (x_1 + 1)^2.
\]

It is 1-parametric family of parabolas.

**Remark**

We would obtain the similar results for the space of right cosets by the Lie subgroup \(H_2\) (it is the subgroup of invertible dual numbers) and the bundle of the group \(G\) by \(H_2\). However, \(H_2\) is a normal divisor of the group \(G\). Therefore, the spaces of right and left cosets coincide.

**Acknowledgement**

I would like to thank Professor Jiri Vanzura for fruitful discussions and support with writing this paper. This work was supported in part by grant No. 201/05/2707 of The Czech Science Foundation and by the Council of the Czech Government MSM 6198959214.

**References**

[1] Redei, L., Foundations of Euclidean and non-Euclidean geometries according to F. Klein, Oxford, New York, Pergamon Press (1968)

[2] Rosenfeld, B. A., A history of non-euclidean geometry: evolution of the concept of a geometric space; translated by Abe Shenitzer with the editorial assistance of Hardy Grant, New York: Springer-Verlag, (1988).

[3] Rosenfeld, B. A., Geometry of Lie Groups, Boston: Kluwer (1991).

[4] Yaglom, I. M., A simple non-Euclidean geometry and its physical basis: an elementary account of Galilean geometry and the Galilean principle of relativity; translated by Abe Shenitzer with the editorial assistance of Basil Gordon, New York: Springer (1979).
[5] Shirokov, A. P., Spaces over algebras and their applications. Geometry, 7. J. Math. Sci. (New York) 108 (2002), no. 2, 232–248.

[6] Pavlov, E. V., Hopteriev, H. T., Manifolds with an algebraic structure and CH-mapping, Bulg. Acad. of Science press, 1982, T.35, N2, 141-144; MR0666267 (83i:53041)

[7] Malakhal’tsev, M. A. A class of manifolds over the algebra of dual numbers. (Russian) Trudy Geom. Sem. Kazan. Univ. No. 21 (1991), 70–79; MR1195520 (94f:53053)

[8] Vishnevskii, V. V. Integrable affinor structures and their plural interpretations. Geometry, 7. J. Math. Sci. (New York) 108 (2002), no. 2, 151–187.

[9] Kirichenko, V. F., Arseneva, O. E., Differential Geometry of generalized almost quaternionic structures, eprint arXiv: [dg-ga/9702013]

[10] Morimoto, A., Prolongation of connections to bundles of infinitely near points, J. Diff. Geometry, 1976, 11, N4, 479-498

[11] Shurygin, V. V. Manifolds over local algebras that are equivalent to jet bundles. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1992, , no. 10, 68–79 (1993); translation in Russian Math. (Iz. VUZ) 36 (1992), no. 10, 66–77.

[12] Yano, K., Differential Geometry on complex and almost complex spaces, N. J., 1965

[13] Norden, A. P., Prostranstva affinnoi sviaznosti [in Russian] (Spaces with affine connection), Moscow: Nauka (1976). MR0467565 (57 #7421)

[14] Bourbaki, N., Elements of mathematics, Algebra I, Paris, Hermann; Reading, Mass., Addison-Wesley (1989)

[15] Husemoller, N., Fibre bundles, New York: McGraw-Hill (1966).

[16] Encyclopaedia of mathematical sciences, vol. 18, Algebra II. Noncommutative rings, identities/ A.I. Kostrikin, I.R. Shafarevich, eds. Berlin; New York: Springer-Verlag (1991)

[17] Study, E., Cartan, E., Nombres complexes, Encyclopedie des sciences mathematiques pures et appliquees, t.1. Vol.1. (1908) 329-468.

[18] Vishnevskii, V. V.; Shirokov, A. P.; Shurygin, V. V. Prostranstva nad algebrami. (Russian) [Spaces over algebras] Kazanskii Gosudarstvennyi Universitet, Kazan’, 1985. 264 pp.; MR0928390 (89m:53002)

[19] Norden, A. P., A generalization of the fundamental theorem of the theory of normalization. (Russian) Izv. Vys. Ucebn. Zaved. Matematika 1966 1966 no. 2 (51), 78–82; MR0196663 (33 #4850)

[20] Shirokov, A. P., Neevkliidovy prostranstva [in Russian] (Non-Euclidean spaces), Izdat. Kazan. univ. (1997). Zbl 0933.51011