\( \mathcal{N} = 2 \) conformal field theories from \( M^2 \)-branes at conifold singularities.

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Abstract

We make some comments on the derivation of \( \mathcal{N} = 2 \) super–conformal field theories with smooth gauge group from \( M^2 \)-branes placed at conifold singularities, giving a detailed prescription for two specific examples: the singular cones over the \( Q^{111} \) and \( M^{110} \) manifolds.
1 Introduction

The most important advances in the knowledge of the non–perturbative aspects in string–
theory have been due to the discovery and the study of $D$–branes and $M$–theory. The
first have revealed to be essential in the construction of a consistent theory of strings and
also very important to test geometry at sub–stringy scale. The latter has opened the
possibility to have interesting informations on the strong coupling limit of such theories.

A striking example of the new aspects discovered by considering $D$–branes as probes
of stringy interactions is that space–time is a derived rather than a primary concept,
arising as the moduli space of $D$–brane world–volume gauge theories. Some specific
analyses [1, 2, 3], where branes are placed at orbifold singularities, show indeed such an
unexpected feature.

Another important development that has recently emerged is the possible relation
between certain gauge theories, describing world–volume theories of large numbers of $D$
or $M$–branes, and superstring or $M$–theory on backgrounds of the factorised form $AdS \times H$,
with $H$ a compact manifold [4, 5].

In flat space the conjectured equivalence relation specifies $H$ to be a sphere, as
$AdS_{p+2} \times S^{d−p−2}$ arises as the horizon manifold for a $Dp$ or $Mp$–brane in $d$–dimensions.
This is no longer true if we place the branes on singular spaces. Since there is no ob-
struction to placing the branes at the singular point, the horizon $H$ will now be different
from the one we had at smooth points (i.e. the sphere). This leads to the possibility of
deriving superconformal gauge theories with less than the maximal supersymmetry.

In string theory, the SCFT describing $D$–branes at orbifold singularities can be derived
by projecting out the invariant states and the potential from the flat space theory [1, 2, 3].
On other kind of singularities, such as conifolds, one must find others and indirect ways
of derivation. Among many attempts regarding this kind of singularities [3, 4], the most
satisfying way seems to be the one outlined in [5]. If one can find an orbifold, from
which a partial resolution leads to the desired singularity, the corresponding gauge theory
can be derived from the orbifold one by giving appropriate expectation values to some of
the moduli fields and studying the reduced theory. The choice is obviously related to the
way of resolving the orbifold singularity and it is controlled by the Fayet–Ilioupulos terms
(governed by the twisted string sectors).

For $D3$–branes in type $IIB$ string theory, the relevant singularities are the Gorenstein
canonical singularities in $\mathbb{C}^3$ and a wide class is given by those which can be described by
toric geometry [6], some of which have been carefully studied in [6].

A natural generalization would be to study the fundamental branes of $M$–theory at
$\mathbb{C}^4/\Gamma$ singularities. This could lead to three–dimensional theories with $\mathcal{N} \leq 8$.

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2 In [3] it was considered for the first time the orbifold limit of ALE manifolds (or Taub Nut, or higher
dimensional generalizations) as a device to define the corresponding brane conformal field theory.
3 All singularities of the form $\mathbb{C}^n/\Gamma$ for finite abelian groups $\Gamma \subset U(n)$ are toric singularities.
There are many problems connected with this programme, from the absence of a complete classification of such Gorenstein singularities, to the possibility of having singular horizons [11]. Good and interesting cases are the ones where the horizon is given by a coset space of dimension seven. Generic coset manifolds admitting an Einstein metric have been described in [12], while the seven–dimensional ones have been completely classified in [13]. These spaces can be retrieved as horizon manifolds for M2–branes placed at the singular point of a space of the form $M_3 \times C(H)$ where $C(H)$ is the eight–dimensional cone over the coset–space [16].

In this paper we deal with such singular cones when the corresponding supergravity theory preserves $\mathcal{N} = 2$ supersymmetry. The language we will use is that of toric geometry. This is indeed an effective way of describing the classical $D$–brane moduli space as pointed out in [1, 17] and also a very simple tool to deal with ADE singularities. For an introduction to toric geometry for physicists see [18]. A more rigorous exposition can be found in the book by Fulton [19] and recent developments are exposed in [20]. Some useful considerations about equations defining toric varieties and combinatorial aspects can be found in [21, 22].

In the next section we are going to describe the $\mathcal{N} = 2$ $C(H)$ cones in this language, specifying the charge matrices for their definition as a symplectic quotient. In section three we show which are the classes of Gorenstein $\mathbb{C}^4/\Gamma$ singularities allowed by the physical consistency requirements and show which resolutions can lead to the conifold singularities we want to describe. The last section contains some comments on the related $\mathcal{N} = 2$ field theories.

2 The cones over coset manifolds

Eleven–dimensional supergravity can be spontaneously compactified through the Freund–Rubin mechanism to a space of the form $AdS_4 \times H^7$, where $H^7$ is one of the coset spaces classified in [13]. The cases we want to analyse are the $\mathcal{N} = 2$ ones, namely the $Q^{ppr}$ [14], the $M^{ppr}$ [15] and the $V_{5,2}$, which are defined as the following group quotients

$$\frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}, \quad \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)}, \quad \text{and} \quad \frac{SO(5)}{SO(3)}$$

For the sake of simplicity, for the first two classes of manifolds, we are going to choose a single representative given by $Q^{111}$ and $M^{110}$. It is worth pointing out that all the other $M^{ppr}$ manifolds can be derived from $M^{110}$ by quotienting with a suitable cyclic group [13].

Since we want to use the language of toric geometry we have to give a description of these manifolds in such a language. The starting point to make this construction is a theorem reported in [23] which states that if $S$ is a compact quasi-regular homogeneous Sasakian–Einstein manifold, then $S$ is a circle-bundle over a generalized flag manifold. Since our seven–dimensional manifolds are Sasaki–Einstein we can apply such a theorem.
The $Q^{111}$ manifold could be recovered as a circle bundle over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $M^{110}$ fibering over $\mathbb{P}^2 \times \mathbb{P}^1$ and the Stiefel manifold $V_{5,2}$ fibering over the real Grassmanian of projective lines in the real $\mathbb{P}^4$.

The essential requirement is that these manifolds admit a toric description. We will see that $Q^{111}$, $M^{110}$ and their cones can be easily described by toric geometry, but this is not the case for the Stiefel manifold. This manifold can be described as the following surface embedded in $\mathbb{P}^4$:

$$\sum_{i=1}^{5} z_i^2 = 0. \quad (2.1)$$

We also know that a toric manifold can always be described by embedding equations given by one monomial equals another monomial. Since the quadric (2.1) cannot have such a description in $\mathbb{P}^4$ without a change in its degree, we conclude that this is not a toric variety.

We thus limit ourselves to the $Q$ and $M$ cases.

### 2.1 Toric description of $Q^{111}$

The first manifold we describe is the $Q^{111}$, which can be found as the unit circle bundle inside the $L_Q = O_{\mathbb{P}^1}(-1) \otimes O_{\mathbb{P}^1}(-1) \otimes O_{\mathbb{P}^1}(-1)$ line bundle.

$(\mathbb{P}^1)^3$ as a toric variety can be described by a fan generated by $\{\pm e_1, \pm e_2, \pm e_3\}$. From the fan, one can simply deduce the matrix describing the combinatorics for this variety

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}$$

and one can also associate six homogeneous coordinates to this space as described in [24]. In this case they are just the couples of homogeneous coordinates describing the three projective lines.

Line bundles over this space can be described in toric geometry by including a seventh affine coordinate. The $L$ line–bundle is indeed described by the matrix

$$A = \begin{pmatrix}
A_1 & A_2 & B_1 & B_2 & C_1 & C_2 & p \\
1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & -1
\end{pmatrix}, \quad (2.2)$$

which, with a change of basis, can be presented as

$$A' = \begin{pmatrix}
1 & 1 & 1 & 1 & -2 & -2 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1
\end{pmatrix}. \quad (2.3)$$

Since $Q^{111}$ is the circle bundle inside this $\mathbb{C}^*$–bundle, it can be retrieved fixing the absolute value of $p$, e.g. $|p|^2 = 1$. 

Thus, $Q^{111}$ is the submanifold of the $A$-generated manifold described by the $D$–term equations
\[|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = |C_1|^2 + |C_2|^2 = 1, \tag{2.4}\]
and quotiented by the two $U(1)$ actions given by the first two rows of (2.3)
\[
(A, B, C) \rightarrow (e^{i\alpha} A, e^{-2i\alpha} B, e^{-i\alpha} C), \tag{2.5a}
\]
\[
(A, B) \rightarrow (e^{i\alpha} A, e^{-i\alpha} B). \tag{2.5b}
\]
It has therefore a description in terms of $(S^3 \times S^3 \times S^3)/U(1 \times U(1))$.

At this point we can also build the invariant coordinates to obtain an explicit embedding of $C(Q^{111})$ inside $\mathbb{C}^8$. These coordinates, invariant under the $C^*$ action described by (2.3), are
\[
z_0 = A_1 B_1 C_1; \quad z_4 = A_2 B_1 C_1; \quad z_1 = A_1 B_1 C_2; \quad z_5 = A_2 B_1 C_2; \quad z_2 = A_1 B_2 C_1; \quad z_6 = A_2 B_2 C_1; \quad z_3 = A_1 B_2 C_2; \quad z_7 = A_2 B_2 C_2; \tag{2.6}
\]
and a set of independent embedding equations is thus given by
\[
\begin{align*}
z_0 z_5 &= z_1 z_4, \\
z_2 z_7 &= z_3 z_6, \\
z_0 z_3 &= z_1 z_2, \\
z_4 z_7 &= z_5 z_6. \tag{2.7}
\end{align*}
\]

We would like to point out here that these can be viewed as the equations defining $Q^{111}$ in $\mathbb{P}^7$ and that they are simply the equations for the Segre embedding $(\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7$. $C(Q^{111})$ is then the affine cone over the projectively embedded variety.

In this description, $R$–symmetry corresponds to $p$–coordinate rotations. Since we can use the last row $U(1)$ action to gauge–fix $p = 1$, choosing an appropriate gauge for the remaining $U(1)$’s, we can deduce the $R$–symmetry charges of the various coordinate fields. This shows that not all the coordinates have the same charges and therefore some of them must be composite states of the fundamental fields.

### 2.2 Toric description of $M^{110}$

As we have just done for the $Q^{111}$ manifold, we can give an analogous description of the $M^{110}$ manifold as the circle bundle inside the canonical line bundle $L_M = \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$.

The fact that this is the correct line bundle can be easily derived by the construction presented in [3], where a generic $M^{pq0}$ is seen as a $S^5 \times S^3/U(1)$ manifold. Indeed $S^5$ is a $U(1)$–bundle over $\mathbb{P}^2$ and $S^3$ is a $U(1)$–bundle over $\mathbb{P}^1$. The $U(1)$ at the denominator identifies the two bundles with an action of the type $(e^{2i\alpha} U, e^{-3i\alpha} V)$, where $U$ are the $\mathbb{P}^2$ homogeneous coordinates and $V$ the $\mathbb{P}^1$ ones. It is straightforward then to derive the toric description of such a bundle for $p = q = 1$. 
Repeating the construction of the last section, \( \mathbb{P}^2 \times \mathbb{P}^1 \) as a toric variety can be described by a fan generated by \( \{ e_1, e_2, -(e_1 + e_2), \pm e_3 \} \). From this, one can deduce the matrix encoding the combinatorics for this variety and describe the line–bundle \( \mathcal{L} \) as

\[
\mathcal{A} = \begin{pmatrix}
U_1 & U_2 & U_3 & V_1 & V_2 & p \\
1 & 1 & 1 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 1 & -2
\end{pmatrix}, \quad (2.8)
\]

Again, with a change of basis we get

\[
\mathcal{A}' = \begin{pmatrix}
2 & 2 & 2 & -3 & -3 & 0 \\
0 & 0 & 0 & 1 & 1 & -1
\end{pmatrix}, \quad (2.9)
\]

The \( M^{110} \) manifold is then the horizon described by

\[
|U_1|^2 + |U_2|^2 + |U_3|^2 = 3, \quad |V_1|^2 + |V_2|^2 = 2, \quad (2.10)
\]

and quotiented by the \( U(1) \)–action

\[
(U, V) \rightarrow (e^{2i\alpha}U, e^{-3i\alpha}V), \quad (2.11)
\]

exactly the desired \( (S^5 \times S^3)/U(1) \) description, with the \( (2.11) \) \( U(1) \)–action.

With the same procedure of the last section, we can again build the invariant coordinates

\[
\begin{align*}
    z_0 &= U_3^2 V_1^2; \\
    z_1 &= U_1^2 U_2^2 V_1^2; \\
    z_2 &= U_1 U_2 U_3^2 V_1^2; \\
    z_3 &= \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
    z_{27} &= U_1 U_2 U_3 V_2^2; \\
    z_{28} &= \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
    z_{29} &= U_3^2 V_2^2; \\
\end{align*}
\]

and the embedding equations in \( \mathbb{C}^{30} \), which we will not specify here\(^4\).

## 3 \( \mathbb{C}^4/\Gamma \) orbifold singularities

The essential idea behind the construction of effective field theories for \( D \)–branes (at singularities) is that the fields describing their degrees of freedom are related to fundamental strings stretched between the branes. This is indeed how the right gauge group and superpotential is chosen.

This cannot surely happen for the fundamental objects in \( M \)–theory: the \( M2 \)–branes. The picture just described is no longer valid since there are no strings in the eleven–dimensional theory.

The way to overcome this obstacle is to think of \( M \)–theory as the strong–coupling limit of type \( IIA \) string theory. If this is allowed, then one can try to find the corresponding ten–dimensional configuration and see how to describe the low energy effective field theory

\(^4\)The embedding equations for the \( \mathcal{L}_M \) line bundle have been independently derived by [25] who also claim that they describe the cone over \( M^{11r} \) for any \( r \).
for this latter. It has been shown \cite{26} that $M2$–branes at orbifold singularities arise as a particular phase in the diagram describing a more complex situation where one has to deal with $D2$–branes of type $IIA$ theory localized onto $D6$–branes, localized or smeared $M2$–branes and various other field–theory phases.

In particular, if one studies $N_2$ $D2$–branes over $N_6$ $D6$–branes, when $N_6 \ll N_2$, the effective description is that of $M2$–branes at an $A_{N_6-1}$ singularity $\mathbb{C}^2/\mathbb{Z}_{N_6}$.

The fact that one obtains such a singularity can be understood by the supergravity solution representing these $D2$–branes localized within $D6$–brane in the decoupling limit \cite{27}. This solution is given by a Minkowski space $M(10,1)$ (with one direction compactified) with $\mathbb{Z}_{N_6}$ identifications over four dimensions. If we call $\alpha$ the $N_6$–th root of unity and we complexify the four real dimensions on which we make the identification, the identification we have to perform is

$$(z_1, z_2) \sim (\alpha z_1, \alpha z_2). \quad (3.1)$$

This kind of action means that we have a Gorenstein canonical singularity only if $N_6 = 2$ and thus we have to restrict ourselves to orbifold singularities of the form

$$\mathbb{C}^4/ \mathbb{Z}_2 \times \Gamma'.$$

According to what we have just said, the $\mathbb{Z}_2$ action must be chosen to be $(-, -, +, +)$ while the $\Gamma'$ action must be of the form $(\omega^{a_1}, +, \omega^{a_2}, \omega^{a_3})$ because this is the only action leading to a Gorenstein canonical singularity, and compatible with a consistent compactification down to ten dimensions.

Given these restrictions and limiting the analysis to cyclic groups, we have found only two classes of Gorenstein canonical singularities of the form\footnote{A recent paper \cite{30} studies a singularity of the form $\mathbb{C}^4/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ with action given by $g_1 = (-, -, +, +)$ and $g_2 = (-, +, \omega^{m-1}, \omega); \ g_3 = (-, +, \omega, +, +)$. But this means that one has to quotient $\mathbb{C}^4$ also by the composite generator $g_1 g_2 g_3 = (-, -, +, +)$ which has exactly the form of the quotient $\mathbb{C}^4/\mathbb{Z}_2$. This is known to be a terminal singularity and thus admits no Calabi–Yau resolutions.} $\mathbb{C}^4/(\mathbb{Z}_2 \times \Gamma')$ satisfying the above requirements. These are given by

$$\mathbb{C}^4/ \mathbb{Z}_2 \times \mathbb{Z}_{2m} \quad \text{and} \quad \mathbb{C}^4/ \mathbb{Z}_2 \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2m}, \quad (3.2)$$

with $m \geq 2$. The first is chosen with an action given by $g_1 = (-, -, +, +)$ and $g_2 = (-, +, \omega^{m-1}, \omega); \ g_3 = (-, - +, +, +, +)$, the second with action $g_1 = (-, -, +, +), \ g_2 = (\omega^{2m-1}, +, \omega, +)$ and $g_3 = (\omega^{2m-1}, +, +, +, \omega)$, with $\omega$ the $2m$–th root of the unity.

- The orbifold $\mathbb{C}^4/ \mathbb{Z}_2 \times \mathbb{Z}_{2m}, \ m \geq 2$

The toric data for the $\mathbb{C}^4/(\mathbb{Z}_2 \times \mathbb{Z}_{2m})$ orbifold can be deduced by one of the recipes
described in [3] or (chapter 2 of) [19] and are contained in the following matrix:

\[
A = \begin{pmatrix}
1 & 1 & 0 & -m & 1 & 0 & 0 \\
0 & -2 & 0 & 2m & -1 & 1 & 0 \\
0 & 0 & 1 & 1 - m & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}.
\] (3.3)

From \(A\) one can derive the charge matrix \(Q\) [10], which is

\[
Q = \begin{pmatrix}
m & 0 & m - 1 & 1 & 0 & -2m & 0 \\
1 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 & 1 & -1
\end{pmatrix} \begin{pmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{pmatrix},
\] (3.4)

with an extra column including the \(D\)-terms.

This simple matrix allows us to see that there are no interesting resolutions of this singularity (for any \(m\)) leading to the \(C(Q^{111})\) or \(C(M^{110})\) cones. We can find partial resolutions giving some of the lower dimensional singularities described in [10], like the conifold \((\zeta_2 = 0 \text{ or } \zeta_3 = 0)\), the suspended pinch point \((\zeta_2 = 0 \text{ and } \zeta_3 = 0)\) and the \(\mathbb{Z}_2\) orbifold singularities \((\zeta_2 + \zeta_3 = 0)\). All these imply that now \(\mathbb{C}^4/\Gamma\) is at least reduced to \(\mathbb{C} \times \mathbb{C}^3/\Gamma\).

- **The orbifold** \(\frac{\mathbb{C}^4}{\mathbb{Z}_2 \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2m}}\), \(m \geq 2\)

The toric data for the \(\mathbb{C}^4/(\mathbb{Z}_2 \times \mathbb{Z}_{2m} \times \mathbb{Z}_{2m})\) orbifold are contained in the following matrix:

\[
A = \begin{pmatrix}
1 & -1 & 1 - 2m & 1 - 2m & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\
1 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 2m & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 2m & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix},
\] (3.5)

and we can derive again the charge matrix with the \(D\)-terms:

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & \zeta_1 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & \zeta_2 \\
1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & \zeta_3 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & \zeta_4 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & \zeta_5 \\
2m - 1 & 0 & 1 & 0 & 0 & -2m & 0 & 0 & 0 & 0 & 0 & \zeta_6 \\
2m - 1 & 0 & 0 & 1 & 0 & 0 & -2m & 0 & 0 & 0 & 0 & \zeta_7
\end{pmatrix}.
\] (3.6)

This reveals to be the right choice to obtain the desired conifold resolution. The cone over the \(Q^{111}\) manifold [23] can indeed be obtained by partially resolving this singularity\(^6\) keeping \(\zeta_2 - \zeta_3 = 0\) and \(\zeta_4 = 0\). We recognize in these two rows

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & -2 & -2 \\
1 & 1 & -1 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\zeta_2 - \zeta_3 = 0 \\
\zeta_4 = 0
\end{pmatrix},
\]

\(^6\)In [28] \(C(Q^{111})\) was obtained from the resolution of the factorised \(\mathbb{C}^2 / \mathbb{Z}_2 \times \mathbb{C}^2 / \mathbb{Z}_2\), claiming that \(Q^{111}\) is topologically a trivial \(S^3\) bundle over \(S^2 \times S^2\).
the $[2,3]$ data.

Again, we can find many $\mathbb{C} \times \mathbb{C}^3/\Gamma$ singularities such as the conifolds, the $\mathbb{Z}_2$ orbifolds and the suspended pinch points. The still missing resolution is the one leading to $C(M^{110})$. It seems that the only way to find such a partial resolution is to look at configurations of $M2$–branes at singularities which do not admit a ten–dimensional description.

4 Some comments on the gauge theories

Now that we have outlined the $D$–terms and resolutions necessary to obtain the $C(Q^{111})$ conifold from an orbifold, we would like to derive the gauge theory by "resolving" the orbifold gauge theory. As already said, for $D$–branes in ten–dimensional string theory the orbifold theory is derived from the theory on the smooth covering space by a suitable projection, and the charge matrices we have used to find the resolutions tell us which fields have to acquire a vev.

Unfortunately, in our case, the only information we can have is about which is the correct orbifold to partially solve, but the derivation of the field theory has to be found in a more indirect way.

We have to perform a double projection, the first to quotient the smooth $\mathbb{C}^4$ to $\mathbb{C}^4/\mathbb{Z}_2$ and the second to derive the $\mathbb{C}^4/(\mathbb{Z}_2 \times \Gamma')$ we are to study. The first step is accomplished by passing from the eleven–dimensional system of $M2$–branes at a $\mathbb{C}^2/\mathbb{Z}_2$ singularity to the ten–dimensional one, involving $D2$–branes on two $D6$–branes. This yields automatically the projected theory, with no need to perform the projection by hand.

Depending on the $\Gamma'$ chosen, we will have to study the field theory describing the low energy limit of $Nk$ (where $k = |\Gamma'|$) $D2$–branes over two $D6$–branes. We will choose the $D2$–branes to lie in the $x_1$, $x_2$ directions, while the $D6$–branes will be stretched in the $x_1, \ldots, x_6$ directions. Once reduced to such a configuration in ten dimensions, $\Gamma'$ will act on the $x_3, \ldots, x_8$ directions with the proper action.

The coordinates transverse to the $D6$–branes are non–dynamical degrees of freedom as seen from the $D2$–branes point of view. If we call indeed $g_3$ the effective coupling constant for the fields on the $D2$ and $g_7$ that for the ones on the $D6$–branes, these have to be related by

$$g_3^2 = \frac{g_7^2}{V_{3456}},$$

where $V_{3456}$ is the $D6$ volume transverse to the $D2$–brane. Now, sending $V \to \infty$, the kinetic energy of these fields explodes to infinity and thus they have to be treated as classical degrees of freedom frozen at a specific value.

On the $D2$–branes there lives an $\mathcal{N} = 8$ gauge theory, whose content is given by the gauge field $A^\mu_{ij}$ ($\mu = 0, 1, 2$) and the transverse coordinates fluctuations $\Phi^I_{ij}$ ($I = 3, \ldots, 9$) sitting in the adjoint of $U(Nk)$. This is exactly the same superconformal field theory which lives on an $M2$–brane placed on flat space $\mathbb{B}^4$. There we had a theory with eight scalars,
but it is easily seen that in three dimensions one can dualize one of these scalars to obtain a vector field. Since these D2–branes are placed onto two D6–branes, supersymmetry is broken down to $\mathcal{N} = 4$ and the $\Phi^I_{ij}$ have to be split in those of the new vector multiplets $(A^\mu_{ij}, \Phi^I_{ij}, \Phi^8_{ij}$ and $\Phi^9_{ij})$ and those of the hypermultiplets $(X^3_{ij}, X^5_{ij})$.

These are not the only fields living on the D2–branes. We also have to consider the strings stretched among the D2 and D6–branes. These carry a $U(Nk)$ color and a $U(2)$ flavour index and, from the three–dimensional point of view, they can be seen as quark fields $Q_i^{\mathcal{A}}$ and $ar{Q}_i^{\dot{\mathcal{A}}}$ ($\mathcal{A} \in \mathcal{2}$).

We could then write the superpotential for such a theory as it is derived from the tree–level stringy interactions. This theory will then be the superconformal field theory describing the low–energy limit of a system of $Nk$ M2–branes placed at the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity.

We chose $Nk$ and not simply $N$ branes, because now we want to perform the projection needed to obtain the theory on $\mathbb{C}^4/(\mathbb{Z}_2 \times \Gamma')$.

At this point indeed, if we choose $g \in \Gamma'$ and denote its actions on $\mathbb{C}^4$ and on the Chan–Paton factors by $R(g)$ and $S(g)$ respectively, we can determine which are the surviving fields for the projected theory. The surviving components $\Phi^I_{ij}$ of the gauge field $A$ will be the ones for which

$$A^\mu_{ij} = (S(g)A^\mu S^{-1}(g))_{ij} \quad (4.1)$$

and the surviving scalars must satisfy

$$\Phi^I_{ij} = R^I_{J}(g)(S(g)\Phi^J S^{-1}(g))_{ij},$$

$$X^I_{ij} = R^I_{J}(g)(S(g)X^J S^{-1}(g))_{ij} \quad (4.2)$$

We also have to consider the quark fields $Q$ and $\bar{Q}$ which are again to be projected under the $\Gamma'$ action. This means that we will keep only the components invariant under

$$Q_{i\mathcal{A}} = (S(g)Q)_{i\mathcal{A}},$$

$$\bar{Q}_{i\dot{\mathcal{A}}} = (\bar{Q}S^{-1}(g))_{i\dot{\mathcal{A}}} \quad (4.3)$$

One then has to substitute the surviving fields into the $\mathcal{N} = 4$ super Yang–Mills theory describing the D2–D6 system. We already discussed which would be the $D$–flatness conditions for the unprojected theory, we thus have to add the $F$–flatness conditions which now arise from solving the equations deriving from the minimization of the superpotential under its variation over the moduli fields ($\Phi, X$).

All these conditions together allow to derive the complete theory, while the geometric informations of section two are not sufficient to completely determine the gauge theory.

Here we limit ourselves to the above considerations leaving to a future work the exact computation of the SCFT Lagrangian and possibly the extension of what just said to $\mathcal{N} = 1$ or $\mathcal{N} = 3$ cases.

Our hope is that such constructions could bring us to the ”experimental” test of the $AdS/CFT$ correspondence through the $(AdS$ masses$)/(CFT$ weights$)$ relations following
the lines of [32], since the Kaluza Klein spectrum on such manifolds can be completely
determined and it is actually under computation [33].

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