A NON-COMMUTATIVE $F_5$ ALGORITHM WITH AN APPLICATION TO THE COMPUTATION OF LOEWY LAYERS

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Abstract. We provide a non-commutative version of the $F_5$ algorithm, namely for right-modules over path algebra quotients. It terminates, if the path algebra quotient is a basic algebra. In addition, we use the $F_5$ algorithm in negative degree monomial orderings to compute Loewy layers.

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1. Introduction

In this paper, we present a non-commutative version of the $F_5$ algorithm, in the setting of finitely generated sub-modules of free right-modules of finite rank over path algebra quotients. It terminates, if the path algebra quotient is a basic algebra. The $F_5$ algorithm is usually thought of as an algorithm for the computation of Gröbner bases. It improves efficiency of the computation by discarding many “useless” critical pairs, namely critical pairs whose S-polynomials would reduce to zero, in Buchberger’s algorithm. In addition, we show that the $F_5$ algorithm immediately yields the Loewy layers of a right-module over a basic algebra, provided that a negative degree monomial ordering is used.

A basic algebra is a finite dimensional quotient of a path algebra, by relations that are at least quadratic. Basic algebras are useful in the study of modular group algebras of finite groups: If $p$ is a prime dividing the order of a finite group $G$, then there is some finite extension field $K$ over $\mathbb{F}_p$, so that $KG$ is Morita equivalent to a basic algebra $\mathcal{A}$ over some finite quiver, whose connected components correspond to the $p$-blocks of $G$; see [Erd90]. In particular, the Ext algebra of $G$ with coefficients

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in $K$ can be computed by constructing a minimal projective resolution of the simple module associated to each vertex of the quiver corresponding to $\mathcal{A}$.

If an initial segment of a minimal projective resolution is given, then one needs to compute minimal generating sets of kernels of homomorphisms of free right-$\mathcal{A}$ modules, in order to compute the next term of the resolution. Three main computational approaches have been considered: Gröbner bases with respect to well-orderings in path algebras, linear algebra, and standard bases with respect to negative degree orderings in path algebras.

It is well known [GP08] that kernels of module homomorphisms can be computed by means of Gröbner bases. D. Farkas, C. Feustel, and E. Green [FFG93] have provided a Gröbner basis theory for path algebra quotients. In [FGKK93], it was shown how to obtain minimal generating sets from the Gröbner bases, in the case of basic algebras. See also [GSZ01].

However, since $\mathcal{A}$ is of finite dimension as a $K$-vector space, the problem can also be solved by linear algebra. This approach was exploited by J. Carlson [CVEZ03] in the first complete computation of the modular cohomology rings of all groups of order 64.

D. Green [Gre03] suggested to make use of negative monomial orderings; in this setting, one would talk about standard bases rather than Gröbner bases. His standard basis theory for sub-modules $\mathcal{M}$ of a free right-$\mathcal{A}$ module also takes into account containment in the radical of $\mathcal{M}$. The standard bases thus obtained are called heady. Green shows that a minimal generating set of $\mathcal{M}$ can easily be read off of a heady standard basis of $\mathcal{M}$. In examples, standard bases for group algebras are often much smaller than Gröbner bases, as was pointed out in [Gre09].

D. Green implemented his algorithms in C and has shown in many practical computations that heady standard bases often perform better than linear algebra. The author of this paper used Green’s programs in a modular group cohomology package [KG12] for the open source computer algebra system Sage [Sage12], which resulted in the first complete computation of the modular cohomology rings of all groups of order 128 [GK11] and of several bigger groups, for different primes, including the mod-2 cohomology of the third Conway group [KGE11].

Since we use negative degree orderings, our approach is in the spirit of [Gre03]. But [Gre03] focuses on modular group rings of groups of prime power order, which corresponds to basic algebras whose quivers have a single vertex. The first purpose of this paper is to lift that restriction and consider general basic algebras.

The algorithm for the computation of heady standard bases in [Gre03] is of Buchberger type. In particular, performance suffers when too many useless critical pairs are considered, i.e., critical pairs whose S-polynomials reduce to zero. The second purpose of this paper is to show that one can use an $F_5$ algorithm [Fau02] for right-modules over basic algebras. With the $F_5$ algorithm, the number of useless critical pairs can be drastically reduced.

In fact, the $F_5$ algorithm computes a generalisation of heady standard bases, which we call signed standard bases. The third purpose of this paper is to show: When a signed standard basis of a sub-module $\mathcal{M}$ of a free right-$\mathcal{A}$ module is known with respect to a negative degree monomial ordering, then one can not only read off of it a minimal generating set of $\mathcal{M}$, but one can read off a $K$-vector space basis for each Loewy layer of $\mathcal{M}$. So, the $F_5$ algorithm is not only more efficient than the
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Buchberger style algorithm from \cite{Gre03}, but it yields more information. Loewy layers are defined as follows.

**Definition 1** \cite{Ben98}. The radical $\text{Rad}(M)$ of $M$ is the intersection of all the maximal submodules of $M$. Define $\text{Rad}^0(M) = M$ and

$$\text{Rad}^d(M) = \text{Rad}\left(\text{Rad}^{d-1}(M)\right)$$

for $d = 1, 2, \ldots$. The $d$-th Loewy layer is $\text{Rad}^{d-1}(M)/\text{Rad}^d(M)$.

In particular, the first Loewy layer $M/\text{Rad}(M)$ is the head of $M$, and a minimal generating set of $M$ as a right-$A$ module corresponds to a $K$-vector space basis of $M/\text{Rad}(M)$.

This paper is organised as follows. In Section 2, we recall some notions from standard basis theory, adapted to path algebra quotients (not necessarily finite dimensional).

In Section 3, we show how Buchberger’s algorithm looks like for right-modules over path algebra quotients. The main difference is that one replaces S-polynomials by so-called topplings. Since our focus is on the $F_5$ algorithm, we skip most proofs in that section and refer to \cite{Gre03}. We conclude the section by discussing the chain criterion in the context of topplings.

Section 4 is devoted to signed standard bases of right-modules over path algebra quotients. The main result in that section is Theorem 2, providing a criterion for detection of signed standard bases that combines Faugère’s $F_5$ and rewritten criteria. The criterion is used in Algorithm 4.

The $F_5$ criterion helps to discard many critical pairs whose S-polynomials would reduce to zero in the Buchberger algorithm. Nonetheless, in general, zero reductions can not be totally avoided. But by an idea from \cite{AP11}, any occurring zero reduction can be used to improve the criterion and thus helps to avoid some other zero reductions.

Algorithm 4 may not terminate for general path algebra quotients, but if it terminates, then it returns a signed standard basis. We did not try to find the most general conditions for termination (in particular we do not prove that termination is granted for all noetherian path algebra quotients), since our work is motivated by the study of basic algebras, for which termination of Algorithm 4 is clear.

The final Section 5 shows how signed standard bases for negative degree monomial orderings can be used to compute Loewy layers.

## 2. Monomial orderings for quotients of path algebras

Let $\mathcal{P}$ be a path algebra over a field $K$, given by a finite quiver, $Q$. Let $x_1, \ldots, x_n$ be generators of $\mathcal{P}$ corresponding to the arrows $\alpha_1, \ldots, \alpha_n$ of $Q$. To each vertex $v$ of $Q$ corresponds an idempotent $1_v \in \mathcal{P}$, such that

$$1_v \cdot x_i = \begin{cases} x_i & \text{if } \alpha_i \text{ starts at } v \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_j \cdot 1_v = \begin{cases} x_j & \text{if } \alpha_j \text{ ends at } v \\ 0 & \text{otherwise} \end{cases}$$
Multiplication in $\mathcal{P}$ corresponds to concatenation of paths in $Q$. In particular, the directed path in $Q$ are in one-to-one correspondence with the power products of the generators $x_1, \ldots, x_n$, to which we refer to as the monomials of $\mathcal{P}$, the length of the path defining the degree of the monomial. If a monomial $b$ of $\mathcal{P}$ corresponds to a path in $Q$ with endpoint $v$ and a monomial $c$ of $\mathcal{P}$ corresponds to a path in $Q$ with startpoint $w \neq v$, then we define $b \cdot c = 0$. The trivial path consisting of vertex $v$ corresponds to the monomial $1_v$. The set $\text{Mon}(\mathcal{P})$ of monomials of $\mathcal{P}$ forms a basis of $\mathcal{P}$ as a $K$-vector space.

Let $\mathcal{A}$ be a quotient algebra of $\mathcal{P}$. In this paper, we focus on finitely generated sub-modules $M$ of a free right-$\mathcal{A}$ module $\mathcal{A}^r$ of rank $r$. In our main application, $\mathcal{A}$ is a basic algebra, hence finite dimensional over $K$. However, for the moment we make no restrictions. For the following notions related with the theory of standard bases, we adapt the notions from [GP08].

**Definition 2.** A monomial ordering of $\mathcal{P}$ is a total ordering $>$ on $\text{Mon}(\mathcal{P})$, such that

$$b_1 > b_2 \implies b_1 \cdot b > b_2 \cdot b \quad \text{and} \quad b' \cdot b_1 > b' \cdot b_2$$

for all $b_1, b_2, b, b' \in \text{Mon}(\mathcal{P})$ such that $b_1 \cdot b, b_2 \cdot b, b' \cdot b_1$ and $b' \cdot b_2$ are all non-zero.

**Definition 3.**

- A monomial ordering of $\mathcal{P}$ is called positive, or global, if $b > 1_v$ for any vertex $v$ of $Q$ and any monomial $b \in \text{Mon}(\mathcal{P})$ with $\deg(b) > 0$.
- A monomial ordering of $\mathcal{P}$ is called negative, or local, if $b < 1_v$ for any vertex $v$ of $Q$ and any monomial $b \in \text{Mon}(\mathcal{P})$ with $\deg(b) > 0$.
- A monomial ordering of $\mathcal{P}$ is called a positive degree ordering (resp. a negative degree ordering) if $\deg(b_1) > \deg(b_2)$ implies $b_1 > b_2$ (resp. $b_2 > b_1$), for all $b_1, b_2 \in \text{Mon}(\mathcal{P})$.

Given a monomial ordering $>$, each element $p \in \mathcal{P}$ can be uniquely written as

$$p = \alpha_1 b_1 + \ldots + \alpha_k b_k$$

with $\alpha_1, \ldots, \alpha_k \in K \setminus \{0\}$ and monomials $b_1 > b_2 > \ldots > b_k$. If $p \neq 0$, we define

- $\text{lm}(p) = b_1$, the leading monomial of $p$,
- $\text{lc}(p) = \alpha_1$, the leading coefficient of $p$,
- $\text{lt}(p) = \alpha_1 b_1$, the leading term or head of $p$,
- $\text{tail}(p) = p - \text{lt}(p)$, the tail of $p$.

Since we are focusing on right modules, our notion of divisibility of monomials prefers one side as well:

**Definition 4.** Let $b, c \in \text{Mon}(\mathcal{P})$ be monomials. We say that $b$ divides $c$ (denoted $b \mid c$), if there is a monomial $b' \in \text{Mon}(\mathcal{P})$ such that $b \cdot b' = c$.

**Definition 5.** Let $\psi : \mathcal{P} \to \mathcal{A}$ be the quotient map. Then $\text{stdMon}_A(\mathcal{P}) \subset \text{Mon}(\mathcal{P})$ is formed by all monomials that are not leading monomials of elements of $\ker(\psi)$.

We assume in this paper that the standard monomials are known. Note that the standard monomials could alternatively be obtained by linear algebra or from a two-sided standard basis of $\ker(\psi)$, if $\mathcal{A}$ is finite dimensional. One easily sees that $\psi$ is injective on $\text{stdMon}_A(\mathcal{P})$, and that $B_\psi(\mathcal{A}) = \psi(\text{stdMon}_A(\mathcal{P}))$ is a basis of $\mathcal{A}$ as a $K$-vector space. We call it the preferred basis, following [Gre03], and we call its elements the monomials of $\mathcal{A}$.
Definition 6. We define the lift \( \lambda(b) \) of a monomial \( b \in B_>(\mathcal{A}) \) of \( \mathcal{A} \) as the unique element of \( \text{stdMon}_f(\mathcal{P}) \) with \( \psi(\lambda(b)) = b \).

We can now define divisibility of elements of the preferred basis and an ordering on the preferred basis as follows.

Definition 7. Let \( b, b' \in B_>(\mathcal{A}) \). We say that \( b \) strictly divides \( b' \) (or \( b \parallel b' \)) if and only if \( \lambda(b) | \lambda(b') \), and define \( b > b' \) if and only if \( \lambda(b) > \lambda(b') \). We define \( \deg(b) = \deg(\lambda(b)) \).

Note that the existence of some \( s \in \mathcal{A} \) or even \( s \in B_>(\mathcal{A}) \) with \( b \cdot s = b' \) does not necessarily imply that \( b \parallel b' \). However, strict divisibility is a transitive relation.

We call the induced ordering on the preferred basis a monomial ordering on \( \mathcal{A} \), inheriting the properties “positive”, “negative” or “degree ordering” from the monomial ordering on \( \mathcal{P} \).

Definition 8. A monomial ordering on \( \mathcal{A} \) is admissible, if there is no infinite strictly decreasing sequence of monomials of \( \mathcal{A} \).

Since any element of \( f \in \mathcal{A} \) can be uniquely written as
\[ f = \alpha_1 b_1 + ... + \alpha_k b_k \]
with \( \alpha_1, ..., \alpha_k \in K \setminus \{0\} \) and \( b_1, ..., b_k \in B_>(\mathcal{A}) \) with \( b_1 > b_2 > ... > b_k \), we can define the leading monomial \( \text{lm}(f) = b_1 \), the leading coefficient \( \text{lc}(f) = \alpha_1 \), the leading term \( \text{lt}(f) = \alpha_1 b_1 \) and the tail \( \text{tail}(f) = f - \text{lt}(f) \) of \( f \), provided \( f \neq 0 \).

With leading monomials being defined in \( \mathcal{A} \), we can define another notion of divisibility on \( B_>(\mathcal{A}) \), that is weaker than strict divisibility.

Definition 9. Let \( b, b' \in B_>(\mathcal{A}) \). We say that \( b \) divides \( b' \) (or \( b | b' \)) if and only if there is some \( c \in B_>(\mathcal{A}) \) such that \( \text{lm}(b \cdot c) = b' \).

Obviously \( b \parallel b' \) implies \( b | b' \). Note that \( b | b' \) does not necessarily mean that there is some \( f \in \mathcal{A} \) with \( b \cdot f = b' \).

Divisibility and strict divisibility are interrelated by the following notion, that we adapt from \cite{Gre03}.

Definition 10. Let \( b \in B_>(\mathcal{A}) \) and \( t \in B_>(\mathcal{A}) \cup \{0\} \).

1. If there is a \( c \in B_>(\mathcal{A}) \) such that either \( \text{lm}(b \cdot c) = t \neq 0 \) and \( b \parallel t \), or \( b \cdot c = 0 = t \), then \( t \) is called a toppling of \( b \) with cofactor \( c \).
2. If \( c \in B_>(\mathcal{A}) \) is not cofactor of a toppling of \( b \) then we call \( c \) a small cofactor of \( b \).
3. Let \( t \) be a toppling of \( b \) with a cofactor \( c \). Assume that all \( c' \in B_>(\mathcal{A}) \) with \( c' \parallel c \) and \( c' \neq c \) are small cofactors of \( b \). Then \( t \) is called a minimal toppling of \( b \).

We make two easy observations:

Remark 1. Let \( b, c, c' \in B_>(\mathcal{A}) \). Then \( c \) is a small cofactor of \( b \) and \( c' \) is a small cofactor of \( b \cdot c \) if and only if \( c \cdot c' \) is a small cofactor of \( b \).

Remark 2. For any \( b, b' \in B_>(\mathcal{A}) \) with \( b \parallel b' \), there is a finite sequence \( b = b_0, b_1, b_2, ..., b_m \in B_>(\mathcal{A}) \) such that \( b_i \) is a minimal toppling of \( b_{i-1} \), for all \( i = 1, ..., m \), and \( b_m \parallel b' \).
Lemma 1. Let $a, a', b \in B_>(A)$ with $a > a'$, so that $b$ is a small cofactor of $a$, and $a' \cdot b \neq 0$. Then $\text{lm}(a \cdot b) > \text{lm}(a' \cdot b)$. In particular, if $f \in A$ and $b \in B_>(A)$ is a small cofactor of $\text{lm}(f)$, then $\text{lm}(f \cdot b) = \text{lm}(f) \cdot b$.

Proof. Let $\tilde{a} = \lambda(a)$, $\tilde{a}' = \lambda(a')$ and $\tilde{b} = \lambda(b)$. By definition of the monomial ordering on $A$, we have $\tilde{a}' < \tilde{a}$ and thus $\tilde{a}' \cdot \tilde{b} < \tilde{a} \cdot \tilde{b}$. Since $b$ is a small cofactor of $a$, we have $\tilde{a} \cdot \tilde{b} \in \text{stdMon}_A(P)$.

If $\tilde{a}' \cdot \tilde{b} \in \text{stdMon}_A(P)$ then $\text{lm}(a' \cdot b) = \psi(\tilde{a}') \cdot b = \psi(\tilde{a}' \cdot \tilde{b}) < \psi(\tilde{a} \cdot \tilde{b})$. Otherwise, let $\psi(\tilde{a}' \cdot \tilde{b}) = \psi(\tilde{a}') \cdot b = \alpha_1 b_1 + \ldots + \alpha_k b_k$ with $\alpha_1, \ldots, \alpha_k \in K \setminus \{0\}$ and $b_1, \ldots, b_k \in B_>(A)$ with $b_1 = \text{lm}(\psi(\tilde{a}' \cdot \tilde{b}))$, and let $\tilde{b}_i = \lambda(b_i)$ for $i = 1, \ldots, k$. We have $k > 0$, since $a' \cdot b \neq 0$ by hypothesis.

Since $z = \tilde{a}' \cdot \tilde{b} - (\alpha_1 b_1 + \ldots + \alpha_k \tilde{b}_k) \in \ker(\psi)$ and since the standard monomials $\tilde{b}_1, \ldots, \tilde{b}_k$ are by definition not leading monomials of any element of $\ker(\psi)$, it follows $\tilde{b}_i < \text{lm}(z) = \tilde{a}' \cdot \tilde{b} < \tilde{a} \cdot \tilde{b}$ for $i = 1, \ldots, k$. Hence, $b_1 = \text{lm}(\psi(\tilde{a}' \cdot \tilde{b})) = \text{lm}(a' \cdot b) < \text{lm}(\psi(\tilde{a} \cdot \tilde{b})) = \text{lm}(a \cdot b)$.

For the last statement of the lemma, we already know that $\text{lm}(f \cdot b) = \text{lm}(\text{lm}(f) \cdot b)$. There remains to observe that $\text{lm}(f \cdot b) = \psi(\tilde{a} \cdot \tilde{b})$, since $b$ is not cofactor of a toppling of $\text{lm}(f)$. Hence, $\text{lm}(\text{lm}(f) \cdot b) = \text{lm}(f) \cdot b$. □

3. Monomial orderings for right modules quotients of path algebras

We use the same notations as in the previous section and are now studying the free right-$A$ module $F = A'$ of rank $r$. Let $v_1, \ldots, v_r$ be free generators of $F$.

Definition 11. Let a monomial ordering $>$ on $A$ be given, so that the notion of leading monomial is defined in the quotient $A$ of $P$.

(1) The set of monomials $\text{Mon}(F)$ of $F$ is the set of all $v_i \cdot b$ with $i = 1, \ldots, r$ and $b \in B_>(A)$.

(2) A total ordering $>$ on $\text{Mon}(F)$ is called a monomial ordering compatible with $>$ on $A$, if it satisfies

(a) $b_1 > b_2 \iff v_i \cdot b_1 > v_i \cdot b_2$

(b) $v_i \cdot b_1 > v_j \cdot b_2 \iff v_i \cdot b_1 > v_j \cdot \text{lm}(b_2) \cdot b$ or $b_2 \cdot b = 0$

for all $i, j = 1, \ldots, r$ and all $b_1, b_2, b \in \text{stdMon}(A)$ so that $b$ is a small cofactor of $b_1$.

Since any $f \in F$ can be uniquely written as a linear combination of monomials of $F$, we obtain the notions of leading monomial $\text{lm}(f)$, leading coefficient $\text{lc}(f)$, leading term $\text{lt}(f)$ and tail $\text{tail}(f)$ as we did for elements of $A$.

Definition 12. Let $v_i \cdot b_1, v_j \cdot b_2 \in \text{Mon}(F)$. We say that $v_i \cdot b_1$ divides (resp. strictly divides) $v_j \cdot b_2$, denoted $v_i \cdot b_1 | v_j \cdot b_2$ (resp. $v_i \cdot b_1 \parallel v_j \cdot b_2$), if and only if $i = j$ and $b_1 | b_2$ (resp. $i = j$ and $b_1 \parallel b_2$).

Lemma 2. Let $f \in F$ with $\text{lm}(f) = v_i \cdot b_f$, and let $b \in B_>(A)$ be a small cofactor of $b_f$. Then $\text{lm}(f \cdot b) = \text{lm}(f) \cdot b$.

Proof. Lemma 1 can be applied component-wise. □
Definition 13. Let \( G \) be a finite subset of \( F \setminus \{0\} \). An element \( f \in F \) has a standard representation with respect to \( G \) if and only if \( f \) can be written as

\[
f = \sum_{i=1}^{m} \alpha_i g_i \cdot c_i
\]
such that \( \alpha_i \in K \setminus \{0\}, g_i \in G, c_i \in B_>(A) \) is a small cofactor of \( \text{lm}(g_i) \), and \( \text{lm}(g_i) \cdot c_i = \text{lm}(g_i \cdot c_i) \leq \text{lm}(f) \), for all \( i = 1, ..., m \).

Definition 14. A normal form \( NF \) on \( F \) assigns to any \( f \in F \) and any finite subset \( G \subset F \setminus \{0\} \) an element \( NF(f, G) \in F \), such that the following holds.

1. \( NF(0, G) = 0 \).
2. If \( NF(f, G) \neq 0 \) then \( \text{lm}(g) \parallel \text{lm}(NF(f, G)) \), for all \( g \in G \).
3. \( f - NF(f, G) \) has a standard representation with respect to \( G \).

Algorithm 1: A normal form algorithm

\[
\text{Data: } f \in F \text{ and a finite subset } G \subset F \setminus \{0\} \\
\text{Result: } NF(f, G)
\]

begin

\[
f_r \leftarrow f
\]

while \( f_r \neq 0 \) and there is some \( g \in G \) with \( \text{lm}(g) \parallel \text{lm}(f_r) \) do

Let \( c \in B_>(A) \) be the small cofactor of \( \text{lm}(g) \) with \( \text{lm}(f_r) = \text{lm}(g) \cdot c \)

\[
f_r \leftarrow f_r - \frac{\text{lc}(f_r)}{\text{lc}(g)} g \cdot c
\]

end

return \( f_r \)

Lemma 3. If the monomial ordering on \( A \) is admissible, then Algorithm 1 computes a normal form on \( F \).

Proof. In the while-loop of Algorithm 1 \( \text{lm}(f_r) \) strictly decreases, since \( \text{lm}(g \cdot c) = \text{lm}(g) \cdot c = \text{lm}(f_r) \), by Lemma 2. Since \( F \) is of finite rank and since the monomial ordering on \( A \) is admissible, Algorithm 1 terminates in finite time.

By construction, the leading monomial of the returned element \( f_r \) is not strictly divisible by the leading monomial of any \( g \in G \). Moreover, \( f - f_r \) has a standard representation with respect to \( G \), since all cofactors in the algorithm are small, and since in the while loop holds \( \text{lm}(g \cdot c) = \text{lm}(g) \cdot c \) by Lemma 2 and \( \text{lm}(g) \cdot c = \text{lm}(f_r) \leq \text{lm}(f) \) by construction.

3.1. Standard bases for right modules over quotients of path algebras.

Let \( P, A \) and \( F \) be as in the previous section. We are now studying submodules \( M \subset F \) and assume that we have an admissible monomial ordering on \( A \).

Definition 15. Let \( G \subset M \setminus \{0\} \) be a finite subset. An element \( f \in F \setminus \{0\} \) is called reducible with respect to \( G \), if there is some \( g \in G \) such that \( \text{lm}(g) \parallel \text{lm}(f) \). Otherwise, it is called irreducible with respect to \( G \).

A finite subset \( G \subset M \setminus \{0\} \) is called interreduced, if every \( g \in G \) is irreducible with respect to \( G \setminus \{g\} \).

A finite subset \( G \subset M \setminus \{0\} \) is called a standard basis of \( M \), if every \( f \in M \setminus \{0\} \) is reducible with respect to \( G \).

Lemma 4. If \( G \) is a standard basis of \( M \), then every element of \( M \) has a standard representation with respect to \( G \). In particular, \( G \) generates \( M \) as a right-\( A \) module, and \( f \in M \) if and only if \( NF(f, G) = 0 \).
Proof. Let \( f \in F \). Since \( G \subset M \), \( \text{NF}(f, G) = 0 \) implies that \( f \in M \).

Now, assume that \( f \in M \). Since \( f \in M \), the element \( f_r \) in the while-loop of Algorithm 1 is in \( M \) as well. Hence, by definition of a standard basis, there is \( g \in G \) whose leading monomial is a strict divisor of the leading monomial of \( f_r \). Hence, the algorithm will continue until \( f_r = 0 \). \( \square \)

Lemma 5. Let \( G \subset M \setminus \{0\} \) be a finite subset. If \( f \in M \setminus \{0\} \) is reducible with respect to \( G \) then it is reducible with respect to

\[
\{\text{NF}(g, G \setminus \{g\})\} \cup G \setminus \{g\}
\]

for all \( g \in G \).

Proof. If \( \text{lm}(f) \) is strictly divisible by the leading monomial of some element of \( G \setminus \{g\} \) or if \( \text{lm}(\text{NF}(g, G \setminus \{g\})) = \text{lm}(g) \), then there is nothing to show.

Otherwise, \( \text{lm}(g) \parallel \text{lm}(f) \) and there is some \( g' \in G \setminus \{g\} \) with \( \text{lm}(g') \parallel \text{lm}(g) \). Hence, \( \text{lm}(g') \parallel \text{lm}(f) \), since strict divisibility is transitive. \( \square \)

Corollary 1. For any finite subset \( G \subset M \setminus \{0\} \), there is an interreduced finite subset \( \text{interred}(G) \subset M \setminus \{0\} \) such that if \( f \) is reducible with respect to \( G \) then \( f \) is reducible with respect to \( \text{interred}(G) \), for all \( f \in F \).

Proof. If \( G \) is not interreduced, there is some \( g \in G \) such that \( g' = \text{NF}(g, G \setminus \{g\}) \neq g \) and \( \text{lm}(g') < \text{lm}(g) \). We replace \( G \) by \( \{g'\} \cup G \setminus \{g\} \). By the preceding Lemma, the change of \( G \) does not decrease the set of elements of \( F \) that are reducible with respect to \( G \).

Since the leading monomial strictly decreases and we only have finitely many monomials, we obtain an \( \text{interred}(G) \) after finitely many steps. \( \square \)

Definition 16. We say that a finite subset \( G \subset M \setminus \{0\} \) satisfies property \((T)\), if it is interreduced, and \( g \cdot c \) has a standard representation with respect to \( G \), for every \( g \in G \) and the cofactor \( c \) of every minimal toppling of \( \text{lm}(g) \).

Theorem 1 (\cite{Gre03}). If a finite subset \( G \subset M \setminus \{0\} \) generates \( M \) as a right-\( A \) module and has property \((T)\), then it is a standard basis of \( M \).

Our main result, Theorem 2, is a generalisation of Theorem 1. Therefore, for the sake of brevity, we do not include a proof of Theorem 1. Note that \cite{Gre03} obtains a similar result for so-called heady standard bases, which in turn is a special case of signed standard bases.

3.2. A short account on the chain criterion. We now focus on the case that \( A \) is finite dimensional over \( K \). Then, there is a standard basis of \( M \), simply since \( \text{Mon}(F) \) is finite. But finding a standard basis by an enumeration of leading monomials would certainly not be very efficient. Theorem 1 provides a more efficient
algorithm for the computation of a standard basis of $M$, similar to Buchberger’s algorithm.

Algorithm 2: A Buchberger style computation of a standard basis

**Data:** $G = \{g_1, ..., g_k\}$, generating $M$ as a right-$A$ module.

**Result:** An interreduced standard basis of $M$.

**begin**

while There is some $g \in G$ and a cofactor $c$ of a minimal toppling of $\text{lm}(g)$ such that $g \cdot c$ has no standard representation with respect to $G$ do

$G \leftarrow \text{interred}(G \cup \{\text{NF}(g \cdot c, G)\})$

**return** $G$

**Lemma 6.** Algorithm 2 computes an interreduced standard basis of $M$, in the case that $A$ is finite dimensional over $K$.

**Proof.** In the while-loop of the algorithm, the number of monomials of $M$ that are strictly divisible by the leading monomial of an element of $G$ strictly increases. Since there are only finitely many monomials, the computation terminates in finite time. The result is a standard basis, by Corollary 1 and Theorem 1. $\square$

Algorithm 2 is certainly more efficient than an attempt to directly enumerate all leading monomials of a sub-module $M \subset F$. However, there is a common problem in the computation of standard bases: In the head of the while-loop in Algorithm 2 one needs to test whether $g \cdot c$ has a standard representation with respect to $G$, i.e., whether $\text{NF}(g \cdot c, G) = 0$. If $\text{NF}(g \cdot c, G) = 0$, then the pair $(g, c)$ does not contribute to the standard basis. Since the computation of $\text{NF}(g \cdot c, G)$ is a non-trivial task, it would be nice to have a criterion that disregards the pair $(g, c)$ without computation of a normal form.

There are several criteria known from the computation of Gröbner bases in the commutative case, such as Buchberger’s product or chain criteria [Buc79], or Faugère’s $F_5$ and rewritten criteria [Fau02].

We discuss how these criteria apply in our non-commutative context. First of all, since we consider modules of rank greater than one, the product criterion would not hold, even in the commutative case [GP08, Remark 2.5.11]. Faugère’s criteria are the subject of the next section. Here, we argue that in fact we are already using some kind of chain criterion.

Our rings being highly non-commutative is not the only problem with using existing criteria: The computation of standard bases usually is based on $S$-polynomials, which are not even mentioned in Algorithm 2. However, a different point of view shows that $S$-polynomials are hidden in the notion of a toppling. This point of view appears in more detail in [Gre03], by describing the computation of normal forms as “two-speed reduction”.

Namely, one could model $F = A^r$ as $P^r / \ker(\psi)^r$, using a two-sided standard basis $S$ of $\ker(\psi)$. Computing a standard basis of a sub-module $M \subset F$ could be done by lifting the generators of $M$ to elements of $\text{stdMon}_A(P)^r$, and adding to the lifted generators a copy of $S$ in each component of the free module.

Let $\tilde{g} \in P^r$ be the lift of a generator $g$ of $M$ with $\text{lm}(\tilde{g})$ in the $i$-th component, and $s \in S$. If there are monomials $a, b, c$ of $P$ such that $v_i \cdot a \cdot c = \text{lm}(\tilde{g})$ and
vanishes, because now apply \( \psi \).

Note that this only occurs if \( \text{lm}(\hat{g}) \cdot c \) does not belong to \( \text{stdMon}_{\mathcal{A}}(\mathcal{B})^r \). When we now apply \( \psi^r \), then the \( S \)-polynomial is mapped to \( g \cdot \psi(c) \). The second summand vanishes, because \( s \) belongs to a two-sided standard basis of \( \ker(\psi) \). Hence, the \( S \)-polynomial in \( \mathcal{P}^r \) corresponds to multiplying an element of \( F \) with a toppling \( \psi(c) \) of its leading monomial.

Let \( g, \hat{g}, s, a, b, c \) be as above. Assume that there is \( s \neq s' \in S \) and monomials \( a', b', c', d \) of \( \mathcal{P} \) such that \( u_i \cdot a' \cdot b' = \text{lm}(\hat{g}) \) and \( b' \cdot c' = \text{lm}(s') \) and \( a' \cdot \text{lm}(s') \cdot d = a \cdot \text{lm}(s) \). Then the chain criterion says that the \( S \)-polynomial of the pair \((\hat{g}, u_i \cdot s)\) does not need to be considered.

In our context, if we find \( s' \) as above, then \( c = c' \cdot d \) is not cofactor of a minimal toppling of \( \text{lm}(g) \), since \( c' \) is a cofactor of a toppling as well. Hence, the chain criterion corresponds to the fact that we only consider \emph{minimal} topplings in Algorithm 2.

4. Signatures

As before, let \( M \) be a finitely generated sub-module of a free right-\( \mathcal{A} \) module \( F \) of rank \( r \). We fix a finite ordered generating set \( \{g_1, ..., g_m\} \) of \( M \). Let \( E = \mathcal{P}^m \) be a free right-\( \mathcal{P} \) module with free generators \( \epsilon_1, ..., \epsilon_m \). We fix a monomial ordering on \( \mathcal{P} \) that induces an admissible monomial ordering on \( \mathcal{A} \), and fix a monomial ordering on \( E \) compatible with the monomial ordering on \( \mathcal{P} \).

By applying \( \psi \) and evaluating at the generating set of \( M \), we obtain a map \( ev: E \to M \) with \( ev(\epsilon_i \cdot c) = \hat{g}_i \cdot \psi(c) \), for all monomials \( \epsilon_i \cdot c \) of \( E \).

**Definition 17.** A signed element of \( M \) is a pair \((f_u, s)\) with \( f_u \in M \) and \( s \in \text{Mon}(E) \), such that there is some \( f \in E \) with \( \text{lm}(f) = \sigma \) and \( ev(f) = f_u \).

For a signed element \( f = (f_u, s) \) of \( M \), we define the unsigned element \( \text{poly}(f) = f_u \) and the signature \( \sigma(\overline{f}) = s \).

A signed subset of \( M \) is a set formed by signed elements of \( M \).

If \( b \) is a monomial of \( \mathcal{A} \) and \( f \) is a signed element of \( M \), then for all \( \tilde{b} \in \mathcal{P} \) with \( \psi(\tilde{b}) = b \), \( (\text{poly}(f) \cdot b, \text{lm}(\sigma(f) \cdot \tilde{b})) \) is a signed element of \( M \) as well. For simplicity, we write \( \text{lm}(f) = \text{lm}(\text{poly}(f)) \) and \( \text{lc}(f) = \text{lc}(\text{poly}(f)) \).

**Definition 18.** Let \( f \in M \) and \( s \in \text{Mon}(E) \). If there is a signed element \( g \) of \( M \) with \( f = \text{poly}(g) \) and \( \sigma(g) < s \), then \( f \) is dominated by \( s \). A signed element \( f \) of \( M \) is suboptimal, if \( \text{poly}(f) \) is dominated by \( \sigma(f) \).

**Definition 19.** Let \( G \) be a finite signed subset of \( M \), let \( f \in F \), and let \( s \in \text{Mon}(E) \). An \emph{s-standard representation with respect to \( G \) of \( f \)} is a list of triples \((\alpha_1, g_1, c_1), ..., (\alpha_k, g_k, c_k)\), where \( \alpha_i \in K \), \( g_i \in G \), and \( c_i \in \mathcal{B}_{>}(\mathcal{A}) \) is a small cofactor of \( \text{lm}(g_i) \), for \( i = 1, ..., k \), such that

\[
\text{poly}(f) = \sum_{i=1}^{k} \alpha_i \text{poly}(g_i) \cdot c_i
\]

and \( s > \text{lm}(\sigma(g_i) \cdot \lambda(c_i)) \neq 0 \) for \( i = 1, ..., k \).
Note that we do not bound \( \text{lm}(\text{poly}(g_i) \cdot c_i) \) in terms of \( f \). For simplicity, if \( f \) is a signed element of \( M \), then a standard representation of \( f \) shall denote a \( \sigma(f) \)-standard representation of \( \text{poly}(f) \). Clearly, if a signed element \( f \) of \( F \) has a standard representation with respect to a signed subset of \( M \), then it is a suboptimal signed element of \( M \), and \( \sigma(f) \) is the leading monomial of an element of \( \text{ker}(ev) \).

**Definition 20.** Let \( G \) be a signed subset of \( M \setminus \{0\} \), let \( f \in F \), and let \( s \in \text{Mon}(E) \).

- We say that \( f \) is \( s \)-reducible (resp. weakly \( s \)-reducible) with respect to \( G \), if there is some \( g \in G \) and a small cofactor \( c \) of \( \text{lm}(g) \) such that \( \text{lm}(g) \cdot c = \text{lm}(f) \) and \( \sigma(g) \cdot \lambda(c) < s \) (resp. \( \sigma(g) \cdot \lambda(c) \leq s \)).
- We say that \( f \) is (weakly) \( s \)-reducible with respect to \( M \), if there is some signed element \( g \) of \( M \setminus \{0\} \) such that \( f \) is (weakly) \( s \)-reducible with respect to \( \{g\} \).
- We say that \( f \) is \( s \)-irreducible with respect to \( G \), if \( \text{poly}(f) = 0 \) or \( f \) is not \( s \)-reducible.

**Lemma 7.** Let \( f \in F \) and let \( s \in \text{Mon}(E) \). If \( f \) is (weakly) \( s \)-reducible with respect to \( M \) then there is some signed element \( g \) of \( M \setminus \{0\} \) such that \( \text{poly}(g) \) is \( \sigma(g) \)-irreducible with respect to \( M \) and \( f \) is (weakly) \( s \)-reducible with respect to \( \{g\} \).

**Proof.** By definition, there is some signed element \( g \) of \( M \setminus \{0\} \) such that there is a small cofactor \( c \) of \( \text{lm}(g) \) with \( \text{lm}(f) = \text{lm}(g) \cdot c \) and \( \sigma(g) \cdot \lambda(c) < s \) (resp. \( \sigma(g) \cdot \lambda(c) \leq s \)). We will show that one can choose \( g \) so that \( \text{poly}(g) \) is \( \sigma(g) \)-irreducible with respect to \( M \).

Assume that \( g \) is \( \sigma(g) \)-reducible with respect to \( M \). Then there is some signed element \( g' \) of \( M \setminus \{0\} \) and a small cofactor \( c' \) of \( \text{lm}(g') \) such that \( \text{lm}(g') \cdot c' = \text{lm}(g) \) and \( \sigma(g') \cdot \lambda(c') < \sigma(g) \). Since \( c' \cdot c \) is a small cofactor of \( \text{lm}(g') \) and \( \sigma(g') \cdot \lambda(c' \cdot c) < \sigma(g) \cdot \lambda(c) \), we can replace \( g \) by \( g' \).

Since \( \lambda(c' \cdot c) < \sigma(g) \cdot \lambda(c) < s \) and \( < \) is supposed to be a well-ordering on \( \text{Mon}(E) \), a replacement of \( g \) by \( g' \) can only occur finitely many times. Hence, eventually we find \( g \) so that \( \text{poly}(g) \) is \( \sigma(g) \)-irreducible with respect to \( M \). \( \square \)

**Definition 21.** A signed normal form on \( F \) assigns to any \( f \in F \), any finite signed subset \( G \) of \( F \setminus \{0\} \) and any \( s \in \text{Mon}(E) \) an element \( \text{NF}_s(f,G) \in F \), such that the following holds.

1. If \( f = 0 \) then \( \text{NF}_s(f,G) = 0 \).
2. \( \text{NF}_s(f,G) \) is \( s \)-irreducible with respect to \( G \).
3. \( f - \text{NF}_s(f,G) \) has an \( s \)-standard representation with respect to \( G \).

If \( f \) is a signed element of \( M \), then we implicitly assume \( s = \sigma(f) \) in the two preceding definitions, unless stated otherwise. Note that a signed element \( f \) of \( M \) is
irreducible with respect to \{f\}. We denote \(\text{NF}(f, G) = (\text{NF}_{\sigma(f)}(\text{poly}(f), G), \sigma(f))\); it is easy to see that this is a signed element.

**Algorithm 3**: A signed normal form algorithm

**Data**: \(f \in F\), a finite signed subset \(G \subseteq F \setminus \{0\}\) and \(s \in \text{Mon}(s)\)

**Result**: \(\text{NF}_s(f, G)\)

```
begin
  \(f_r \leftarrow f\)
  while \(\text{poly}(f_r) \neq 0\) and there is some \(g \in G\) and a small cofactor \(c\) of \(\text{lm}(g)\) with \(\text{lm}(g) \cdot c = \text{lm}(f_r)\) and \(\sigma(g) \cdot \lambda(c) < s\) do
    \(f_r \leftarrow f_r - \frac{\text{lc}(f_r)}{\text{lc}(g)} \text{poly}(g) \cdot c\)
  return \(f_r\)
```

**Lemma 8.** Algorithm 3 computes a signed normal form on \(F\).

**Proof.** The proof of the lemma is essentially as the proof of Lemma 3. \(\square\)

### 4.1. Signed standard bases.

The following definitions were adapted from [AP11].

**Definition 22.** Let \(G\) be a finite signed subset of \(M \setminus \{0\}\).

1. If every \(g \in G\) is irreducible with respect to \(G\), and there is no \(g' \in G\) with a small cofactor \(c\) of \(\text{lm}(g')\) such that \(\text{lm}(g') \cdot c = \text{lm}(g)\) and \(\sigma(g') \cdot \lambda(c) = \sigma(g)\), then \(G\) is called *interreduced*.

2. Assume that each signed element \(f\) of \(M \setminus \{0\}\) that is \(\sigma(f)\)-irreducible with respect to \(M \setminus \{0\}\) is weakly \(\sigma(f)\)-reducible with respect to \(G\). Then \(G\) is a *signed standard basis* of \(M\).

Rephrasing the definition, if \(G\) is a signed standard basis of \(M\) and \(f\) is a \(\sigma(f)\)-irreducible signed element of \(M \setminus \{0\}\), then there is some \(g \in G\) and a small cofactor \(c\) of \(\text{lm}(g)\) such that \(\text{lm}(g) \cdot c = \text{lm}(f)\) and \(\sigma(g) \cdot \lambda(c) = \sigma(f)\).

**Lemma 9.** Let \(G\) be a signed standard basis of \(M\), let \(f \in F \setminus \{0\}\) and let \(s \in \text{Mon}(E)\).

1. If \(f\) is \(s\)-reducible with respect to \(M\) then it is \(s\)-reducible with respect to \(G\).
2. \(f\) is an element of \(M\) that is dominated by \(s\) if and only if \(\text{NF}_s(f, G) = 0\).
3. \(G' = \{\text{poly}(g) : g \in G\}\) is a standard basis of \(M\).

**Remark 3.** The unsigned normal form \(\text{NF}(\text{poly}(g), G')\) can be zero, even if \(g \in G\) is irreducible with respect to \(G\). Hence, \(G'\) is usually not a minimal standard basis, and not even interreduced.

**Proof.**

1. By Lemma 8, we can assume that there is some signed element \(h\) of \(M \setminus \{0\}\) such that \(\text{poly}(h)\) is \(\sigma(h)\)-irreducible with respect to \(M\), and there is a small cofactor \(c\) of \(\text{lm}(h)\) such that \(\text{lm}(h) \cdot c = \text{lm}(f)\) and \(\sigma(h) \cdot \lambda(c) < s\).

   By Definition 22, there is some \(g\) in \(G\) and a small cofactor \(b\) of \(\text{lm}(g)\) such that \(\text{lm}(g) \cdot b = \text{lm}(h)\) and \(\sigma(g) \cdot \lambda(b) = \sigma(h)\). Since \(b \cdot c\) is a small cofactor of \(\text{lm}(g)\), \(\text{lm}(g) \cdot b \cdot c = \text{lm}(h) \cdot c = \text{lm}(f)\) and \(\sigma(g) \cdot \lambda(b \cdot c) = \sigma(h) \cdot c < s\), we find that \(f\) is \(s\)-reducible with respect to \(G\).

2. If \(\text{NF}_s(f, G) = 0\) then \(f\) has an \(s\)-standard representation with respect to \(G\). Since \(G\) is formed by signed elements of \(M\), we find \(f \in M\), and by the definition of an \(s\)-standard representation it is dominated by \(s\).
If $f \in M$ is dominated by $s$, then the first statement of the lemma implies that Algorithm 5 will end only when $f_r = 0$. Hence, $\text{NF}_s(f, G) = 0$.

(3) Let $f \in M \setminus \{0\}$. We have to show that $f$ is reducible with respect to $G'$, i.e., there is some $g \in G$ and a small cofactor $c$ of $\text{lm}(g)$ such that $\text{lm}(g) \cdot c = \text{lm}(f)$. Since the map $\text{ev}: E \to F$ is surjective, there is some $s \in \text{Mon}(E)$ such that $(f, s)$ is a signed element.

If $f$ is $s$-reducible with respect to $M$, then it is $s$-reducible with respect to $G$ by the first part of the lemma, and thus $f$ is reducible with respect to $G'$. Otherwise, Definition 22 ensures that $f$ is reducible with respect to $G'$. Hence, $G'$ is a standard basis of $M$.

Lemma 10. Let $G$ be a finite signed subset of $M \setminus \{0\}$, $f \in F$ and $s \in \text{Mon}(E)$.

(1) If $f$ is (weakly) $s$-reducible with respect to $G$ then it is (weakly) $s$-reducible with respect to $\tilde{G} = \{\text{NF}(g, G \setminus \{g\})\} \cup G \setminus \{g\}$ for all $g \in G$.

(2) Let $g \in G$ such that there is some $g' \in G$ and a small cofactor $c$ of $\text{lm}(g')$ such that $\text{lt}(g) = \text{lt}(g') \cdot c$ and $\sigma(g) = (\sigma(g') \cdot c)\cdot \lambda(c)$. $f$ is (weakly) $s$-reducible with respect to $G$ if and only if it is (weakly) $s$-reducible with respect to $G \setminus \{g\}$.

Proof.

(1) If $f$ is $s$-reducible (resp. weakly $s$-reducible) with respect to $G \setminus \{g\}$ or if $\text{NF}(g, G \setminus \{g\}) = g$, then there is nothing to show.

Otherwise, there is a small cofactor $c$ of $\text{lm}(g)$, such that $\text{lm}(g) \cdot c = \text{lm}(f)$ and $\sigma(g) \cdot \lambda(c) < s$ (resp. $\sigma(g) \cdot \lambda(c) \leq s$), and there is some $g' \in G \setminus \{g\}$ and a small cofactor $c'$ of $\text{lm}(g')$ such that $\text{lm}(g') \cdot c' = \text{lm}(g)$ and $\sigma(g') \cdot \lambda(c') < \sigma(g)$.

The monomial $c' \cdot c$ is a small cofactor of $\text{lm}(g')$. Since $\sigma(g') \cdot \lambda(c' \cdot c) = (\sigma(g') \cdot \lambda(c') \cdot \lambda(c) < \sigma(g) \cdot \lambda(c) < s$ (resp. $\ldots \leq s$), we find that $f$ is (weakly) $s$-reducible with respect to $\{g'\}$ and thus with respect to $G$.

(2) If $f$ is (weakly) $s$-reducible with respect to $G \setminus \{g\}$ then it is (weakly) $s$-reducible with respect to $G$.

If $f$ is (weakly) $s$-reducible with respect to $\{h\}$ for some $h \in G \setminus \{g\}$ then $f$ is (weakly) $s$-reducible with respect to $G \setminus \{g\}$. Otherwise, if $f$ is (weakly) $s$-reducible with respect to $G$ then there is a small cofactor $d$ of $\text{lm}(g)$ such that $\text{lm}(g) \cdot d = \text{lm}(f)$ and $\sigma(g) \cdot \lambda(d) < s$ (or $\ldots \leq s$). It follows that $c \cdot d$ is a small cofactor of $\text{lm}(g')$, and $\text{lm}(g') \cdot (c \cdot d) = \text{lm}(f)$ and $\sigma(g') \cdot \lambda(c \cdot d) < \sigma(g) \cdot \lambda(d) < s$ (or $\ldots \leq s$). Hence, $f$ is (weakly) $s$-reducible with respect to $\{g'\}$ and thus with respect to $G \setminus \{g\}$.

Corollary 2. For any finite signed subset $G$ of $M \setminus \{0\}$, there is an interreduced finite signed subset $\text{interred}(G)$ of $M \setminus \{0\}$ such that if $f$ is (weakly) $s$-reducible with respect to $G$ then $f$ is (weakly) $s$-reducible with respect to $\text{interred}(G)$, for all $f \in F$ and $s \in \text{Mon}(E)$.

Proof. If there is some $g \in G$ such that $g' = \text{NF}(g, G \setminus \{g\}) \neq g$ and $\text{lm}(g') < \text{lm}(g)$, then we replace $G$ by $\{g'\} \cup G \setminus \{g\}$. By the preceding Lemma, the change of $G$ does not decrease the set of elements of $F$ that are (weakly) $s$-reducible with respect to $G$, for all $s \in \text{Mon}(E)$. 

If there is some \( g, g' \in G \) and a small cofactor \( c \) of \( \text{lm}(g') \) such that \( \text{lm}(g') \cdot c = \text{lm}(g) \) and \( \sigma(g') \cdot \lambda(c) = \sigma(g) \), then we replace \( G \) by \( G \setminus \{g\} \). By the preceding lemma, the set of elements of \( F \) that are (weakly) \( s \)-reducible with respect to \( G \) does not change, for all \( s \in \text{Mon}(E) \).

Since the leading monomials strictly decrease and we only have finitely many monomials occurring in \( G \), we obtain \( \text{interred}(G) \) after finitely many steps.

\[ \square \]

**Lemma 11.** Let \( G \) be an interreduced finite signed subset of \( M \). If \( g_1, g_2 \in G \), \( g_1 \neq g_2 \), and \( c_1, c_2 \) are small cofactors of \( \text{lm}(g_1), \text{lm}(g_2) \) such that \( g_i \cdot c_i \) is \( \sigma(g_i) \cdot \lambda(c_i) \)-irreducible with respect to \( G \), for \( i = 1, 2 \), then \( \text{lm}(g_1) \cdot c_1 \neq \text{lm}(g_2) \cdot c_2 \).

**Proof.** To obtain a contradiction, we assume that \( g_1, g_2, c_1, c_2 \) satisfy the hypothesis of the lemma, but \( \text{lm}(g_1) \cdot c_1 = \text{lm}(g_2) \cdot c_2 \).

Without loss of generality, let \( \deg(\text{lm}(g_1)) \leq \deg(\text{lm}(g_2)) \). Since the monomial \( \text{lm}(g_1) \cdot c_1 = \text{lm}(g_2) \cdot c_2 \) corresponds to a unique path in the quiver, we can write \( c_1 = c_1' \cdot c_2 \) with a small cofactor \( c_1' \) of \( \text{lm}(g_1) \) such that \( \text{lm}(g_2) = \text{lm}(g_1) \cdot c_1' \). Since \( G \) is interreduced, we have \( \sigma(g_2) \neq \sigma(g_1) \cdot \lambda(c_1) \). Hence, \( \sigma(g_2) \cdot \lambda(c_2) \neq \sigma(g_1) \cdot \lambda(c_1) \). Since \( \text{lm}(g_i \cdot c_1) = \text{lm}(g_i) \cdot c_1 = \text{lm}(g_2) \cdot c_2 = \text{lm}(g_2 \cdot c_2) \), it follows that either \( g_1 \cdot c_1 \) is \( \sigma(g_1) \cdot \lambda(c_1) \)-reducible or \( g_2 \cdot c_2 \) is \( \sigma(g_2) \cdot \lambda(c_2) \)-reducible with respect to \( G \), which is impossible by construction.

\[ \square \]

**Definition 23.** Let \( G \) be a finite signed subset of \( M \setminus \{0\} \) and \( g \in G \).

1. If \( c \) is the cofactor of a minimal toppling of \( \text{lm}(g) \), we call \( (g, c) \) a critical pair of type \( T \) with cofactor \( c \) of \( G \).
2. If \( p = (g, c) \) is a critical pair of type \( T \) of \( G \), we define the \( S \)-polynomial of \( p \) as \( S(p) = (\text{poly}(g) \cdot c, \sigma(g) \cdot \lambda(c)) \).
3. If \( c \) is a small cofactor of \( \text{lm}(g) \) and there is some \( g' \in G \) such that \( \text{lm}(g') = \text{lm}(g) \cdot c \) and \( \sigma(g') < \sigma(g) \cdot \lambda(c) \), then we call \( (g, g') \) a critical pair of type \( S \) with cofactor \( c \) of \( G \).
4. If \( p = (g, g') \) is a critical pair of type \( S \) with cofactor \( c \) of \( G \), we define the \( S \)-polynomial of \( p \) as

\[
S(p) = \left( \text{poly}(g) \cdot c - \frac{\text{lc}(g')}{\text{lc}(g)} g', \sigma(g) \cdot \lambda(c) \right)
\]

When we do not name the type of a critical pair, it can be either type. By construction, if \( p \) is a critical pair of \( G \), then the \( S \)-polynomial \( S(p) \) is a signed element. Note that for type \( S \), the unsigned element \( \text{poly}(g') \) is reducible with respect to \( \{\text{poly}(g)\} \), but the signed element \( g' \) is not reducible with respect to \( \{g\} \), because its signature is too small. This is why we need to take into account critical pairs of type \( S \). Indeed, in the unsigned case, the critical pairs of type \( T \) would actually be enough to construct standard bases; see Theorem 14 and [Gre03].

**Definition 24.** Let \( G \) be a finite signed subset of \( M \setminus \{0\} \).

1. A critical pair \( (g, c) \) of \( G \) of type \( T \) is normal, if \( g \) is irreducible with respect to \( G \), and \( \sigma(g) \cdot \lambda(c) \) is not leading monomial of an element of \( \ker(ev) \).
2. A critical pair \( (g, g') \) of type \( S \) with cofactor \( c \) of \( G \) is normal, if both \( g \) and \( g' \) are irreducible with respect to \( G \), and \( \sigma(g) \cdot \lambda(c) \) is not leading monomial of an element of \( \ker(ev) \).

**Definition 25.** A finite signed subset \( G \) of \( M \setminus \{0\} \) satisfies the \( F_5 \) criterion, if for each normal critical pair \( p \) of \( G \), there is some \( g \in G \) and a small cofactor \( c \) of
$\text{l}(g)$ such that $\sigma(g) \cdot \lambda(c) = \sigma(S(p))$ and $\text{poly}(g) \cdot c$ is $\sigma(S(p))$-irreducible with respect to $G$.

**Lemma 12.** Let $G$ be a finite signed subset of $M \setminus \{0\}$ that satisfies the $F_5$ criterion and is irreduced. Let $\tau$ be a monomial of $E$ that is not leading monomial of an element of $\ker(ev)$.

Assume that there is some $g \in G$ and some monomial $b \in B_>(A)$ such that $\sigma(g) \cdot \lambda(b) = \tau$. One can choose $g$ and $b$ such that $b$ is a small cofactor of $\text{l}(g)$ and $\text{poly}(g) \cdot b$ is $\tau$-irreducible with respect to $G$.

**Proof.** The proof is by induction on $\text{deg}(b)$.

If there are $g$ and $b$ such that $\text{deg}(b) = 0$, then $\text{poly}(g) \cdot b = \text{poly}(g)$ and $\sigma(g) \cdot \lambda(b) = \sigma(g) = \tau$. Since $G$ is irreduced, $\text{poly}(g)$ is $\tau$-irreducible with respect to $G$.

By now, let $\text{deg}(b) > 0$, so that we can write $b = b' \cdot x$ for some $x \in B_>(A)$ with $\text{deg}(x) = 1$. Let $\tau' = \sigma \cdot \lambda(b')$. Since $\tau = \tau' \cdot \lambda(x)$ is not leading monomial of an element of $\ker(ev)$, its divisor $\tau'$ isn't either. Thus, by induction, there is some $g_0 \in G$ and a small cofactor $c_0'$ of $\text{l}(g_0)$ such that $\sigma(g_0) \cdot \lambda(c_0') = \tau'$ and $\text{poly}(g_0) \cdot c_0'$ is $\tau'$-irreducible with respect to $G$. Either $x$ is a toppling of $\text{l}(g_0) \cdot c_0'$, or it is a small cofactor of $\text{l}(g_0) \cdot c_0'$.

Assume that $x$ is the cofactor of a toppling of $\text{l}(g_0) \cdot c_0'$. Since $x$ is of degree one, the toppling is minimal. Since $c_0'$ is a small cofactor of $\text{l}(g_0)$, it follows that $c_0 = c_0' \cdot x$ is a minimal toppling of $\text{l}(g_0)$. Since $G$ is irreduced, $\text{poly}(g_0)$ is $\sigma(g_0)$-irreducible with respect to $G$. Moreover, $\sigma(g_0) \cdot \lambda(c_0) = \tau$ is not the leading monomial of an element of $\ker(ev)$. Hence, $(g_0, c_0)$ is a normal critical pair of $G$ of type $T$.

By the $F_5$ criterion, there is some $g_1 \in G$ and a small cofactor $c_1$ of $\text{l}(g_1)$ such that $\sigma(g_1) \cdot \lambda(c_1) = \sigma(S(g_0, c_0))$ and $\text{poly}(g_1) \cdot c_1$ is $\sigma(S(g_0, c_0))$-irreducible with respect to $G$. Since $\sigma(S(g_0, c_0)) = \sigma(g_0) \cdot \lambda(c_0' \cdot x) = \tau$, the statement of the lemma holds in this case.

There remains to study the case that $x$ is a small cofactor of $\text{l}(g_0) \cdot c_0'$, which implies that $c_0 = c_0' \cdot x$ is a small cofactor of $\text{l}(g_0)$. We have $\sigma(g_0) \cdot \lambda(c_0) = \tau$.

Assume that $\text{poly}(g_0) \cdot c_0$ is $\tau$-reducible with respect to $G$. That means, there is some $h \in G$ and a small cofactor $d$ of $\text{l}(h)$ such that $\text{l}(h) \cdot d = \text{l}(g_0) \cdot c_0$ and $\sigma(h) \cdot \lambda(d) < \tau$. The monomial $\text{l}(h) \cdot d$ is not leading monomial of an element of $\ker(ev)$. Hence, $(g_0, c_0)$ is a normal critical pair of $G$ of type $S$ with cofactor $c_0$. Both $g_0$ and $h$ are irreducible with respect to $G$, since $G$ is irreduced. Moreover, $\sigma(g_0) \cdot \lambda(c_0) = \tau$ is not leading monomial of an element of $\ker(ev)$. Hence, $(g_0, h)$ is a normal critical pair. We have $\sigma(S(g_0, h)) = \tau$. Hence, by the $F_5$ criterion, there is some $\bar{g} \in G$ and a small cofactor $\bar{c}$ of $\text{l}(\bar{g})$ such that $\sigma(\bar{g}) \cdot \lambda(\bar{c}) = \tau$ and $\text{poly}(\bar{g}) \cdot \bar{c}$ is $\tau$-irreducible with respect to $G$. $\square$
Recall that we have provided our free module $E$ with generators $e_1, \ldots, e_m$ that are mapped by $ev$ to the originally given generators $g_1, \ldots, g_m$ of $M$. We now come to the main theorem of this section.

**Theorem 2.** Let $G$ be an interreduced finite signed subset of $M \setminus \{0\}$. Assume that for each $i = 1, \ldots, m$ so that $e_i$ is not leading monomial of an element of $\ker(ev)$, there is some $g \in G$ with $\sigma(g) = e_i$. Then $G$ is a signed standard basis of $M$ if and only if $G$ satisfies the $F_5$ criterion.

**Proof.** If $G$ is a signed standard basis, then any signed element of $M \setminus \{0\}$ is weakly reducible with respect to $G$. In particular, this holds for $S$-polynomials of critical pairs. Hence, $G$ satisfies the $F_5$ criterion.

Now suppose that $G$ satisfies the $F_5$ criterion. We prove by contradiction that $G$ is a signed standard basis. Assume that there is a signed element $f$ of $M \setminus \{0\}$ so that $\operatorname{pol}(f)$ is not even reducible with respect to $M \setminus \{0\}$ and is not weakly $\sigma(f)$-reducible with respect to $G$.

Since $f$ is a signed element, there is some $\tilde{f} \in E$ with $\lambda(\tilde{f}) = \sigma(f)$ and $ev(\tilde{f}) = \operatorname{pol}(f)$. Assume that $\sigma(f)$ is the leading monomial of an element $z \in \ker(ev)$. Then let $f' = f - \frac{\lambda(\tilde{f})}{\lambda(z)} \cdot z$. Apparently $\lambda(\tilde{f}) < \lambda(\tilde{f})$, but $ev(f') = ev(\tilde{f}) = ev(f)$. Hence, $\operatorname{pol}(f)$ is $\sigma(f)$-reducible with respect to $M \setminus \{0\}$. This contradiction implies that $\sigma(f)$ is not the leading monomial of an element of $\ker(ev)$.

By our assumption on the monomial ordering, any descending sequence of monomials of $E$ ends among the leading monomials of $\ker(ev)$ after finitely many steps. Hence, since $\sigma(f)$ is not leading monomial of an element of $\ker(ev)$, we can choose $f$ such that $\sigma(f)$ is minimal.

We show: If $f'$ is a (not necessarily irreducible) signed element of $M \setminus \{0\}$ with $\sigma(f') < \sigma(f)$, then $f'$ is weakly reducible with respect to $G$. Namely, choose a signed element $f''$ of $M \setminus \{0\}$ with $\lambda(f'') = \lambda(f')$, so that $\sigma(f'')$ is minimal. Then, $f''$ is reducible with respect to $M \setminus \{0\}$, and $\sigma(f'') \leq \sigma(f') < \sigma(f)$. Hence, by the choice of $f$, $f''$ is weakly reducible with respect to $G$, and since $\lambda(f') = \lambda(f'')$ and $\sigma(f') \geq \sigma(f'')$, $f'$ is weakly reducible with respect to $G$ as well.

Since $\sigma(f)$ is not the leading monomial of an element of $\ker(ev)$, we can write it as $\sigma(f) = \epsilon \cdot \lambda(c)$ for some monomial $c$ of $A$. By the hypothesis of this theorem, there is some $g \in G$ such that $\sigma(g) \cdot \lambda(c) = \sigma(f)$. By Lemma 12, we can choose $g$ and $c$ such that $c$ is a small cofactor of $\lambda(g)$ and $\operatorname{pol}(g) \cdot c$ is $\sigma(f)$-irreducible with respect to $G$.

Our aim is to show that $\lambda(g) \cdot c = \lambda(f)$, which means that $\operatorname{pol}(f)$ is weakly $\sigma(f)$-reducible with respect to $G$ and hence finishes the proof.

Assume that $\lambda(g) \cdot c < \lambda(f)$. Since $g$ is a signed element, there is some $\tilde{g} \in E$ such that $\lambda(\tilde{g}) = \sigma(g)$ and $ev(\tilde{g}) = \operatorname{pol}(g)$. Then, let $\tilde{f} = \tilde{f} - \frac{\lambda(\tilde{g})}{\lambda(f)} \cdot \lambda(c)$. By assumption, we have $\lambda(\tilde{f}) < \lambda(\tilde{f})$ and $\lambda(\operatorname{ev}(\tilde{f})) = \lambda(\operatorname{ev}(\tilde{f})) = \lambda(f)$. But that is a contradiction to $\operatorname{pol}(f)$ being $\sigma(f)$-irreducible with respect to $M \setminus \{0\}$.

Hence, $\lambda(g) \cdot c \geq \lambda(f)$. Assume that $\lambda(g) \cdot c > \lambda(f)$. Then, let $\tilde{g}' = \tilde{g} \cdot \lambda(c) - \frac{\lambda(\tilde{g})}{\lambda(f)} \cdot f$. We have $\lambda(\tilde{g}') < \lambda(\tilde{g}' \cdot \lambda(c)) = \sigma(f)$ and $\lambda(\operatorname{ev}(\tilde{g}')) = \lambda(\operatorname{ev}(\tilde{g} \cdot \lambda(c))) = \lambda(g) \cdot c$.

By the same argument as above, since $\sigma(f)$ is minimal and since $\tilde{g}'$ yields a signed element of signature $\lambda(\tilde{g}') < \sigma(f)$, we obtain that $\lambda(\tilde{g}')$ is weakly $\lambda(\tilde{g}')$-reducible with respect to $G$, and is thus $\sigma(f)$-reducible with respect to $G$. But
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lm$(ev(\tilde{g}')) = lm(g \cdot c)$, and thus $g \cdot c$ is $\sigma(f)$-reducible with respect to $G$. This is a contradiction to the choice of $g$ and $c$.

To summarise, both $lm(g) \cdot c < lm(f)$ and $lm(g) \cdot c > lm(f)$ yield a contradiction. Hence $lm(g) \cdot c = lm(f)$, and thus $f$ is weakly reducible with respect to $G$. □

4.2. Computing signed standard bases. Theorem 2 provides a way to compute signed standard bases — with the complication that one needed to know ker$(ev)$ in advance, in order to decide whether a critical pair is normal. This is often unfeasible or impossible. Therefore, we use a weakened version of the $F_5$ criterion. The basic idea is to use partial knowledge of the leading monomials of ker$(ev)$, and increase the partial knowledge on the fly. We have learnt this idea from [AP11].

In the rest of this section, let $L$ be a finite set of monomials of $E$ such that each element of $L$ is leading monomial of some element of ker$(ev)$.

**Definition 26.** A monomial $\epsilon_i \cdot c$ of $E$ is called standard relative to $L$, if $c \in \text{stdMon}_A(P)$ and there are no monomials $c', b, d$ of $P$ with $\epsilon_i \cdot c' \in L$ and $c = b \cdot c' \cdot d$.

**Remark 4.** If $\epsilon_i \cdot m$ is not the leading monomial of an element of ker$(ev)$ then it is standard relative to $L$, for any finite set $L$ of leading monomials of elements of ker$(ev)$.

**Definition 27.** A critical pair $(g, c)$ of type T (resp. a critical pair $(g, g')$ of type S) with cofactor $c$ of $G$ is called normal relative to $L$, if $\sigma(g) \cdot \lambda(c)$ is standard relative to $L$, and $g$ is irreducible (resp. both $g$ and $g'$ are irreducible) with respect to $G$.

**Remark 5.** If a critical pair of $G$ is normal, then it is normal relative to $L$, for any choice of $L$.

**Algorithm 4:** The $F_5$ algorithm, computing signed standard bases

**Data:** A finite ordered subset $\{\hat{g}_1, ..., \hat{g}_d\}$ of $M$, generating $M$ as a right-$A$ module.

**Result:** An interreduced signed standard basis of $M$.

```
begin
G ← interred ($\{(\hat{g}_i, \epsilon_i): i = 1, ..., d\}$)
L = ∅
while there is a critical pair $p$ of $G$ that is normal relative to $L$ so that it is impossible to find $g \in G$ and a small cofactor $c$ of lm$(g)$ such that
$\sigma(g) \cdot \lambda(c) = \sigma(S(p))$ and poly$(g) \cdot c$ is $\sigma(S(p))$-irreducible with respect to $G$
do
s ← NF$(S(p), G)$
if poly$(s) = 0$ then $L ← L \cup \sigma(s)$
else $G ←$ interred $(G \cup \{s\})$, enlarging $L$ if a zero reduction occurs
return $G$
```

**Theorem 3.** If Algorithm 4 terminates, then it returns an interreduced signed standard basis of $M$. It terminates in finite time, if and only if the while loop is executed only finitely many times.
Remark 6. The number of normal critical pairs of type T usually is much smaller than the number of topplings considered in Algorithm 2. Hence, the $F_5$ criterion discards many topplings whose S-polynomials would reduce to zero.

One should note, however, that some additional critical pairs need to be considered, namely those of type S. However, in practical computations, we found that Algorithm 2 is a lot more efficient than Algorithm 3.

Proof. Let $p$ be as in the while-loop of Algorithm 4. If $\text{NF}(\sigma(p), G) = 0$, then $\sigma(\sigma(p))$ is the leading monomial of an element of $\text{ker}(ev)$. Hence, in all steps of the algorithm, the set $L$ is formed by some leading monomials of elements of $\text{ker}(ev)$.

By the preceding remark, if $p$ is normal, then it is normal relative to $L$.

By the initial definition of $G$, there is some $g \in G$ with $\sigma(g) = \epsilon_i$, for any $\epsilon_i$. Of course, this still holds when adding an element to $G$ in the while-loop. We show: If there is some $g \in G$ with $\sigma(g) = \epsilon_i$ then either $\epsilon_i$ is the leading monomial of an element of $\text{ker}(ev)$, or $\text{interred}(G)$ contains an element of signature $\epsilon_i$. Namely, when interreducing $G$, either $\text{poly}(\text{NF}(g, G)) = 0$ and $g$ is removed from $G$, or $g$ is replaced by $\text{NF}(g, G)$. In the former case, $\sigma(g) = \epsilon_i$ turns out to be the leading monomial of an element of $\text{ker}(ev)$. In the latter case, it suffices to note that $\sigma(g) = \sigma(\text{NF}(g, G)) = \epsilon_i$. Hence, $G$ satisfies the hypothesis of Theorem 2 which means that $G$ is a signed standard basis if and only if it satisfies the $F_5$ criterion.

If $G$ is not a signed standard basis, then the $F_5$ criterion does not hold for $G$, and we will find a critical pair $p$ satisfying the hypothesis of the while-loop of Algorithm 4 so that $G$ will be enlarged by $\text{NF}(\sigma(p)) \neq 0$. In particular, the algorithm does not terminate yet. If $G$ is a signed standard basis, then the $F_5$ criterion holds for $G$, and Algorithm 4 terminates, potentially after verifying that the S-polynomials of all remaining normal critical pairs relative to $L$ (which are finite in number) reduce to zero.

Therefore, if Algorithm 4 terminates, then it returns an interreduced signed standard basis. For the second statement, we note that each computation in the while-loop of Algorithm 4 is finite, since we use an admissible monomial ordering.

To test the condition of the while-loop of Algorithm 4 is certainly computationally complex. However, if $p$ is a critical pair, it helps that $\sigma(\sigma(p))$ can be read off of $p$ without computing $\sigma(p)$. In practical implementations, one computes a list of normal critical pairs and updates that list whenever $L$ or $G$ change. We do not go into detail, but remark that the following lemma helps to keep the list short.

Lemma 13. If $p$ and $p'$ are two critical pairs of $G$ such that $\sigma(\sigma(p)) = \sigma(\sigma(p'))$ and $p$ is considered in the while-loop of Algorithm 4, then $p'$ will not be considered in the while-loop. In particular, each critical pair will be considered at most once during the algorithm.

Remark 7. The fact expressed in this lemma is usually referred to as the rewritten criterion [Pou02].

Proof. When $p$ is considered in the while-loop, two possibilities occur, depending on whether $\text{poly}(\text{NF}(\sigma(p), G)) = 0$ or not.

If $\text{poly}(\text{NF}(\sigma(p), G)) = 0$, then $\sigma(\sigma(p))$ will be added to $L$, and thus $p'$ will be discarded since it is not normal relative to the enlarged set $L$.

If $\text{poly}(\text{NF}(\sigma(p), G)) \neq 0$, then we add to $G$ the signed element $\text{NF}(\sigma(p), G)$, that is irreducible with respect to $G$ and has the same signature as $\sigma(p)$ and thus
the same signature as $S(p')$. Hence, $p'$ will be discarded in this case as well, by the $F_5$ criterion. □

5. Loewy Layers of Right Modules over Basic Algebras

In this section, let $P$ be a path algebra over a field $K$, and let $A$ be a basic algebra. Hence, $A$ is finite-dimensional over $K$, and if $\psi : P \to A$ is the quotient map, then $\ker(\psi) \subset P^2$. In particular, any choice of a monomial ordering on $P$ induces an admissible monomial ordering on $A$.

Recall that a generating set of $P$ as a $K$-algebra is given by idempotents $1_v$ corresponding to the vertices of a quiver $Q$, and elements $x_i$ of degree one corresponding to the arrows of $Q$. Since the defining relations for $A$ are at least quadratic,

$$\text{Rad}(A) = \{\psi(x) : x \in P, \deg(x) = 1\} \cdot A$$

and $\ker(\psi)$ does not contain elements of degree zero or one.

Apart from the additional assumption on the relations, we use the same notations as in the preceding sections. Hence, we have a sub-module $M$ of a free right-$A$ module $F$ of rank $r$, generated by $\{g_1, \ldots, g_m\}$, and a free right-$A$ module $E$ of rank $m$ whose generators $e_1, \ldots, e_m$ are mapped to $g_1, \ldots, g_m$ by a homomorphism $ev : E \to M$. We have $\text{Rad}^k(M) = M \cdot (\text{Rad}(A))^k$ for all $k = 1, 2, \ldots$.

**Lemma 14.** Under the hypotheses of this section, Algorithm 3 terminates.

**Proof.** In each repetition of the while-loop in Algorithm 3 there is only a finite number of critical pairs to be considered, simply since $G$ is finite and the number of minimal topplings of each monomial is finite.

If $p$ is the critical pair considered in the while-loop, and poly$(\text{NF}(p, G)) = 0$, then $G$ does not change. In particular, no additional critical pair emerges. On the contrary, some normal critical pairs may become non-normal relative to the now enlarged $L$.

Otherwise, the set

$$\{\sigma(g) \cdot \lambda(m) : g \in G, m \text{ a small cofactor of } \text{lm}(g)\}$$

strictly increases. Since $E$ only has finitely many monomials that are not leading monomials of elements of $\ker(ev)$, this can only happen finitely many times. □

**Definition 28.** Let $\tau$ be a monomial of $E$. We define the $\tau$-layer of $M$ as

$$\mathcal{L}_\tau(M) = \{\text{ev} (\hat{f}) : \hat{f} \in E, \text{lm}(\hat{f}) \leq \tau\}.$$

**Lemma 15.** Assume that the monomial ordering on $E$ is a negative degree ordering. Let $\tau$ be the greatest monomial of $E$ such that $\deg(\tau) = d$, for some non-negative integer $d$. Then, $\text{Rad}^d(M) = \mathcal{L}_\tau(M)$.

**Proof.** Since $\text{Rad}^d(M) = M \cdot (\text{Rad}(A))^d$ and $\text{Rad}(A)$ is generated by the monomials of $A$ of degree one, it follows that $f \in M$ belongs to $\text{Rad}^d(M)$ if and only if there is some $\hat{f} \in E$ whose monomials are all of degree at least $d$, and $\text{ev}(\hat{f}) = f$.

Since we assume that the monomial ordering on $E$ is a negative degree ordering, the monomials of $\hat{f}$ are all of degree at least $d$ if and only if $\text{lm}(\hat{f}) \leq \tau$. □

**Lemma 16.** Let $\tau$ be a monomial of $E$. Let $G$ be an interreduced signed standard basis of $M$. Let $B_\tau(M, G)$ be the set of all $\text{poly}(g) \cdot c$ with $g \in G$ and a small cofactor $c$ of $\text{lm}(g)$ such that $\sigma(g) \cdot \lambda(c) \leq \tau$ and $\text{poly}(g) \cdot c$ is $\sigma(g) \cdot \lambda(c)$-irreducible with respect to $G$. Then, $B_\tau(M, G)$ is a $K$-vector space basis of $\mathcal{L}_\tau(M)$. 

Proof. If \( f \in \mathcal{L}_\tau(M) \), then there is some monomial \( \sigma \leq \tau \) of \( E \) such that \((f, \sigma)\) is a signed element of \( M \). Since \( G \) is a signed standard basis of \( M \), \( f \) is weakly \( \sigma \)-reducible with respect to \( G \). Hence, we find \( g \in G \) and a small cofactor \( c \) of \( \text{lm}(g) \) such that \( \sigma(g) \cdot \lambda(c) \leq \sigma \leq \tau \) and \( \text{lm}(g) \cdot c = \text{lm}(f) \). When we choose \( \sigma(g) \cdot \lambda(c) \) minimal, then \( \text{poly}(g) \cdot c \) is \( \sigma(g) \cdot \lambda(c) \)-irreducible with respect to \( G \). Hence, \( B_{d}(M,G) \) generates \( L_\tau(M) \) as a \( K \)-vector space.

The leading monomials of the elements of \( B_\tau(M,G) \) are pairwise distinct, by Lemma 11, since \( G \) is interreduced. Hence, \( B_\tau(M,G) \) is \( K \)-linearly independent. \( \square \)

Remark 8. If \( \tau, \tau' \) are monomials of \( E \) and \( \tau' \leq \tau \), then \( B_\tau(M,G) \subseteq B_{\tau'}(M,G) \), for any interreduced signed standard basis \( G \) of \( M \).

Recall that the \( d \)-th Loewy layer of \( M \) is \( \text{Rad}^{d-1}(M)/\text{Rad}^{d}(M) \), for \( d = 1, 2, \ldots \). For \( f \in \text{Rad}^{d-1}(M) \), we denote the equivalence class of \( f \) in \( \text{Rad}^{d-1}(M)/\text{Rad}^{d}(M) \) by \([f]\).

Theorem 4. Suppose that the monomial ordering on \( E \) is a negative degree ordering. Let \( d \) be some positive integer, let \( \tau \) be the greatest monomial of \( E \) such that \( \deg(\tau) = d-1 \), and let \( \tau' \) be the greatest monomial of \( E \) such that \( \deg(\tau') = d \). Let \( G \) be an interreduced signed standard basis of \( M \). Then
\[
\{ [f] : f \in B_\tau(M,G) \setminus B_{\tau'}(M,G) \}
\]
is a \( K \)-vector space basis of the \( d \)-th Loewy layer of \( M \).

Proof. By the choice of \( \tau \) and \( \tau' \) and by Lemmas 13 and 16, \( B_\tau(M,G) \) is a basis of \( \text{Rad}^{d-1}(M) \), and its subset \( B_{\tau'}(M,G) \) is a basis of \( \text{Rad}^{d}(M) \). The claim directly follows. \( \square \)

Corollary 3. If \( G \) is an interreduced signed standard basis of \( M \). The elements of \( G \) whose signatures are of degree zero form a minimal generating set of \( M \).

Proof. By Theorem 4, the elements of \( G \) with signatures of degree zero yield a basis of the first Loewy layer of \( M \), i.e., of the head of \( M \). Hence, they form a minimal generating set of \( M \). \( \square \)

References

[AP11] Alberto Arri and John Perry. The F5 criterion revised. *J. Symbolic Comput.*, 46(9):1017–1029, 2011.
[Ben98] D. J. Benson. *Representations and cohomology. I.* Cambridge Studies in Advanced Math., vol. 30. Cambridge University Press, Cambridge, second edition, 1998.
[Buc79] B. Buchberger. A criterion for detecting unnecessary reductions in the construction of Gröbner-bases. In *Symbolic and algebraic computation (EUROSAM '79, Internat. Sympos., Marseille, 1979)*, volume 72 of *Lecture Notes in Comput. Sci.*, pages 3–21. Springer, Berlin, 1979.
[CVEZ03] Jon F. Carlson, Luis Valero-Elizondo, and Mucheng Zhang. Calculations of the cohomology rings of groups of order dividing 64. In *Cohomology Rings of Finite Groups*, volume 3 of *Algebras and Applications*, pages 337–760. Kluwer Academic Publishers, Dordrecht, 2003.
[Erd90] Karin Erdmann. *Blocks of tame representation type and related algebras*, volume 1428 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990.

[Fau02] Jean-Charles Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero ($F^5$). In *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation*, pages 75–83 (electronic), New York, 2002. ACM.

[FFG93] Daniel R. Farkas, C. D. Feustel, and Edward L. Green. Synergy in the theories of Gröbner bases and path algebras. *Canad. J. Math.*, 45(4):727–739, 1993.

[FGKK93] Charles D. Feustel, Edward L. Green, Ellen Kirkman, and James Kuzmanovich. Constructing projective resolutions. *Comm. Algebra*, 21(6):1869–1887, 1993.

[GK11] David J. Green and Simon A. King. The computation of the cohomology rings of all groups of order 128. *J. Algebra*, 325:352–363, 2011.

[GP08] Gert-Martin Greuel and Gerhard Pfister. *A Singular introduction to commutative algebra*. Springer, Berlin, extended edition, 2008. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX).

[Gre03] David J. Green. Gröbner bases and the computation of group cohomology, volume 1828 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[Gre09] David J. Green. Gröbner bases for $p$-group algebras. arXiv:0910.1699 [math.GR], 2009.

[GSZ01] E. L. Green, O. Solberg, and D. Zacharia. Minimal projective resolutions. *Trans. Amer. Math. Soc.*, 353(7):2915–2939 (electronic), 2001.

[KG12] Simon A. King and David J. Green. *p-Group Cohomology Package (Version 2.1.3)*, 2012. Peer-reviewed optional package for Sage [S+12]. http://sage.math.washington.edu/home/SimonKing/Cohomology/.

[KGE11] Simon A. King, David J. Green, and Graham Ellis. The mod-2 cohomology ring of the third Conway group is Cohen-Macaulay. *Algebr. Geom. Topol.*, 11(2):719–734, 2011.

[S+12] W. A. Stein et al. *Sage Mathematics Software (Version 5.4)*. The Sage Development Team, 2012. http://www.sagemath.org.