Abstract We study the scalar electrodynamics ($SQED_4$) and the spinor electrodynamics ($QED_4$) in the null-plane formalism. We follow Dirac’s technique for constrained systems to analyze the constraint structure in both theories in detail. We impose the appropriate boundary conditions on the fields to fix the hidden subset first class constraints that generate improper gauge transformations and obtain a unique inverse of the second-class constraint matrix. Finally, choosing the null-plane gauge condition, we determine the generalized Dirac brackets of the independent dynamical variables, which via the correspondence principle give the (anti)-commutators for posterior quantization.

Keywords Null-plane coordinates · Constraint analysis · Null-plane gauge · Dirac brackets.

1 Introduction
Late in the 1940s, Dirac [1] proposed three different forms of relativistic dynamics depending on the types of surfaces where the independent degree of freedom was initiated. The first one, named the instant form, occurs when a space-like surface is chosen to establish the fundamental Poisson brackets or commutations relations. This procedure, so far the one most frequently used, is usually called equal-time quantization. The second form, the point form, amounts to taking a branch of the hyperbolic surface $x^\mu x^\mu = \kappa^2$ with $x^0 > 0$. And, the third form, front form or light front, occurs when we choose the surface of a single light wave to study the field dynamics; this is the null-plane formalism, which was first applied to physical phenomena almost three decades after Dirac’s idea came to light. An important advantage, pointed out by Dirac, is that seven of the ten Poincaré generators are kinematical on the null-plane, while only six have this property in the conventional theory constructed from the instant form. Another noteworthy feature of relativistic theories on the null-plane is that they give origin to singular Lagrangians, e.g., constrained dynamical systems. Dirac’s procedure [2] can therefore be employed to analyze the constraint structure of a given theory. In general, this approach reduces the number of independent field operators in the corresponding phase space.

At equal-time, any two different points are space-like separated. Therefore, the fields defined at these points are naturally independent quantities. On a null-plane surface, the situation is different, because the microcausality principle leads to a locality requirement for only the transversal components, and the longitudinal component becomes
non-local in the theory, although such situation would not be unexpected [3]. It is possible to verify that the transformation from the usual coordinates to the null-plane coordinates is not a Lorentz transformation and the structure of the phase space is different when we compare it with the conventional one. The description of a physical system in the null-plane formalism could therefore give additional information to that provided by the conventional formalism [3]. For example, the momentum four-vector is \( (k^+, k^-, k^T) \) where \( k^+ \) is the null-plane energy, while \( k^T \) and \( k^- \) indicate the transverse and longitudinal components of the momentum. Therefore, a massive particle on the mass shell, \( k^- = \frac{m^2 + (k^T)^2}{2k^+} \), has positive-definite values for \( k^\pm \) in contrast to \(-\infty \leq k_{1,2,3}^\pm \leq \) for the usual components. Consequently, the vacuum on the null-plane quantized theory may become simpler than the one in the conventional (equal-time) theory, and in many cases the interacting-theory vacuum on the null-plane may be the same as the perturbation-theory vacuum. For example, the conservation of the total longitudinal momentum would bar the excitations of particle-antiparticle pairs by the null-plane vacuum (having \( k^+ = 0 \)) [4].

Quantization on the null-plane is equivalent to quantization on the characteristic surfaces of the classical field equations. One, therefore, has to specify the Cauchy data on both characteristics, \( x^+ = \) const and \( x^- = \) const. and not only on one simple null-plane [5]. In this context, the light-cone quantization of free massless fermions in \( (1+1) \)-dimensions on both characteristics shows that the procedure leads to the correct physical descriptions [6].

On the other hand, Reference [7] identified an important problem associated to quantization on the null-plane: after establishing the gauge fixing condition for the first-class constraints, and after the second-class constraints have seen handled through Dirac’s procedure, no proper gauge transformations, correspondent to first class constraints, can be carried out; however, there still remains in the analysis a species of improper gauge transformations associated to the existence of hidden first-class constraints, which are related to the zero mode of the longitudinal derivative \( \partial_- \) and appear because appropriate boundary conditions on the fields are missing [8]. This problem makes it impossible to define a unique inverse for the second-class constraint matrix used to define consistent Dirac brackets (DB). Therefore, to fix the improper gauge transformations, one must impose appropriate boundary conditions.

Here, adopting Dirac’s formalism for constrained systems, we study the constraint structure of the scalar and spinor electrodynamics on the null-plane. The paper is organized as follow: In Section 2, we study the SQED4, analyze its constraint structure in detail and determine the Hamiltonian. We classify the set of constraints and find that one of the first-class constraints, a linear combination of scalar and electromagnetic constraints, is a null vector of the respective constraint matrix. We invert the first-class constraints with the corresponding gauge conditions and find a unique inverse of the matrix of second-class constraints by imposing appropriate boundary conditions on the fields that eliminate the hidden first-class constraints. Next, we calculate Dirac’s brackets among the fundamental dynamical variables. In Section 3, we study the QED4 and show that the projection of the fermionic fields allows to observe the existence of only second-class constraints and that the first-class constraints in the fermionic sector are associated with the hidden subset of first-class constraints that generate improper gauge transformations. We also show that the fermionic constraints determine that the electron field is fully described by only two of the four components. We use the null-plane gauge to transform the set of first-class constraints to second class, and we obtain a graded algebra by imposing boundary conditions on the independent components. In the last section, we present remarks and conclusions.

2 Scalar Electroodynamics (SQED4): Constraint Structure

The gauge theory we are considering is defined by the following Lagrangian density in 4-dimensional space-time

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g \lbrack \mu A_\mu \phi \rbrack^2 - m^2 \phi \phi^*.
\] (1)

Here, \( \phi \) is one-component complex scalar field, \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( D_\mu \equiv \partial_\mu + ig A_\mu \) is the covariant derivative. The model is invariant under the following local \( U(1) \) gauge symmetry

\[
\phi \rightarrow e^{i\alpha(x)} \phi \quad , \quad \phi^* \rightarrow e^{-i\alpha(x)} \phi^* \quad , \quad A_\mu \rightarrow A_\mu - \frac{1}{g} \partial_\mu \alpha.
\] (2)

The field equations are given by the expressions

\[
\partial_\mu F^{\mu\alpha} + j^\alpha = 0
\] (3)

and

\[
\left( D_\mu^* D^{\mu} + m^2 \right) \phi^* = 0 \quad , \quad \left( D_\mu D^{\mu} + m^2 \right) \phi = 0
\] (4)

where \( j^\alpha \) is the current defined by

\[
j^\mu \equiv ig \left[ \phi \left( \partial_\mu \phi^* - ig A_\mu \phi^* \right) - \phi^* \left( \partial_\mu \phi + ig A_\mu \phi \right) \right].
\] (5)

The canonical conjugate momenta of the fields \( A_\mu \), \( \phi \) and \( \phi^* \) are

\[
\pi^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_+ A_\mu)} = F^{\mu+},
\] (6)

\[
\pi^* \equiv \frac{\partial \mathcal{L}}{\partial (\partial_+ \phi)} = (D_\mu \phi)^* \quad , \quad \pi \equiv \frac{\partial \mathcal{L}}{\partial (\partial_+ \phi^*)} = D_\mu \phi
\] (7)
respectively.

From Eqs. (6) and (7), we get one dynamical relation:
\[ \pi^- = \partial_+ A_+ - \partial_- A_+ \] (8)

and five primary constraints, three for the electromagnetic sector
\[ C \equiv \pi^+ \approx 0 \]
\[ \chi^k \equiv \pi^k - \partial_+ A_k + \partial_\mu A_\mu \approx 0 \] (9)

and, two for the scalar sector
\[ \Gamma \equiv - D_- \phi \approx 0 \]
\[ \Gamma^* \equiv \pi^* - (D_- \phi)^* \approx 0 \] (10)

Following Dirac’s procedure [2], we define the canonical Hamiltonian density by the equality
\[ \mathcal{H}_C = \frac{1}{2} (\pi^-)^2 + (\pi^- \partial_- + \pi^k \partial_k - j^3) A_+ - D_k \phi (D^k \phi)^* + m^2 \phi \phi^* + \frac{1}{4} F_{kl} F_{kl} , \] (11)

so that the canonical Hamiltonian is \( H_c = \int d^3 y \mathcal{H}_C \), with
\[ \int d^3 y = \int dy^1 dy^2 dy^- . \]

To define the primary Hamiltonian \( H_P \), we add to the canonical Hamiltonian the primary constraints with their corresponding Lagrange multipliers
\[ H_P = H_C + \int d^3 y \left( w_1 C + u_k \chi^k + \nu^\ast \Gamma + \Gamma^* \nu \right) , \] (12)

where \( w_1 \) and \( u_k \) are the multipliers related to the electromagnetic constraints, and \( \nu \) and \( \nu^* \) are the multipliers for the scalar constraints. The fundamental Poisson brackets (PB) between the fields are
\[ \{ A_\mu (x) , \pi^\mu (y) \} = \delta_\mu^\nu \delta^3 (x - y) , \] (13)
\[ \{ \phi (x) , \pi^* (y) \} = \delta^3 (x - y) , \quad \{ \phi^* (x) , \pi (y) \} = \delta^3 (x - y) . \] (14)

Dirac’s procedure tells us that, to preserve the primary constraints in time (consistence condition), under the time evolution generated by the primary Hamiltonian, we have to require that they have a weakly vanishing PB with \( H_P \). This requirement on the scalar constraints yields the equality
\[ \Gamma = - i g [ \phi \pi^- + 2 D_- (A_\mu \phi)] - D^k D_k \phi - m^2 \phi - 2 D_- \nu \approx 0 , \] (15)
\[ \Gamma^* = i g [ \phi^* \pi^- + 2 D_-^* (A_\mu \phi^*)] - (D^k D_k \phi)^* - m^2 \phi^* - 2 (D_- \nu^*)^* \approx 0 , \]

which determine the multipliers \( \nu \) and \( \nu^* \), respectively. There are no other constraints associated with the scalar sector. In the electromagnetic sector, the consistence condition for \( \chi^k \) yields the relation
\[ \dot{\chi}^k = \partial_k \pi^- + j^k + \partial_\mu F_{jk} - 2 \partial_- u_k \approx 0 , \] (16)

i.e., an equation for its associated multiplier \( u_k \). Finally, the consistence condition for \( \pi^+ \) yields a secondary constraint \( G \), defined by the equation
\[ \dot{G} = \partial_- \pi^- + \partial_k \pi^k + j^+ \approx 0 , \] (17)

It is easy to verify that there are no other constraints generated from the consistence condition for Gauss’s law, because \( G \) is automatically conserved
\[ \dot{G} \equiv i g [ \phi^* \Gamma - \phi \Gamma^* ] \approx 0 . \] (18)

There are, therefore, no additional constraints in the theory, the full set of constraints being given by Eqs. (9), (10) and (17).

2.1 Constraint Classification

The constraint \( \pi^+ \) has vanishing PB with all the other constraints; it is, therefore, a first-class constraint. The remaining set \( \Phi^a = \{ \Gamma , \Gamma^* , G , \chi^k \} \) is apparently of second-class. However, the determinant of its constraint matrix \( \{ \Phi^a (x) , \Phi^b (y) \} \) is zero (see Appendix C), because the matrix has a zero mode whose eigenvector gives a linear combination of constraints and this is yet another first-class constraint.

To use an alternative argument, the constraint \( G \), Gauss’s law, is the second first-class constraint for zero coupling constant: free Maxwell’s field theory. On the other hand, if \( G \) belonged to a minimal set of second-class constraints, it would be impossible to take the zero coupling-constant limit, because the DB would become undefined. Although the DB is defined with respect to a non-singular matrix, it could become singular when we go back to the free theory. In conclusion, there is a linear combination that is independent of \( \pi^+ \) and is a first-class constraint. That linear combination is
\[ \Sigma \equiv G - i g ( \phi^* \Gamma - \phi \Gamma^* ) . \] (19)

We, therefore, have the following set of second-class constraints:
\[ \Gamma \equiv \pi - D_- \phi \approx 0 \]
\[ \Gamma^* \equiv \pi^* - (D_- \phi)^* \approx 0 \]
\[ \chi^k \equiv \pi^k - \partial_+ A_k + \partial_\mu A_\mu \approx 0 \] (20)

and the set of first-class constraints
\[ C \equiv \pi^+ \approx 0 \]
\[ \Sigma \equiv G - i g ( \phi^* \Gamma - \phi \Gamma^* ) \approx 0 . \] (21)

We can be sure that this is the maximal number of first-class constraints because the quest for the time independence of \( \{ \Gamma , \Gamma^* , \chi^k \} \) leads to equations that determine their respective Lagrange multipliers. The second first-class constraint in Eq. (19) must be contrasted with the instant form analysis [9], in which the second first-class constraint is not a linear combination of electromagnetic and scalar
constraints, because in this formalism the scalar sector is unconstrained.

2.2 Equations of Motion and Gauge Fixing Conditions

At this point, we need to check that we have the correct (Euler-Lagrange) equations of motion. To this end, we calculate the time derivative of the fields by computing their PB’s with the so-called extended Hamiltonian (\( H_E \)), which now generates the time translations or temporal evolutions. To obtain \( H_E \), we add to the primary Hamiltonian \( H_P \), all the first-class constraints. It results that

\[
H_E = \int d^4y \left[ \frac{1}{2} (\pi - \dot{X})^2 + \left( \pi \partial_x + \pi^x a_0 - \mu_0 \right) \partial_\mu \left( \partial_\mu \phi \right)^* + m^2 \phi^* \right] + \int d^4y \left[ w_1 C + w_2 \chi + v^\mu \Gamma + \Gamma^* v + w_3 \Sigma \right].
\] (22)

The time evolution of the dynamical variables of the electromagnetic field is described by the equalities

\[
\dot{A}_+ = w_1, \quad \dot{\pi} = v + ig w_2 \phi
\]

\[
\dot{\phi}^* = v^* - ig w_2 \phi^*
\]

and the time evolution of the scalar fields, by the equalities

\[
\dot{\pi} = -(D_+ D_\pi)^* + ig A_+ (D_+ \phi)^* - \left( D_\pi D_\phi \right)^* - m^2 \phi^* - ig \phi^* D_+ w_2 + 2ig w_3 D_+ \phi
\]

\[
\dot{\phi} = -D_+ D_\phi - i g \lambda_1 D_+ \phi - D_\pi D_\phi - m^2 \phi + ig \phi D_+ w_2 + 2ig w_3 D_\phi.
\]

from which it is easy to obtain that

\[
D_\mu D^\mu \phi + m^2 \phi \approx i g \phi D_+ w_2 + 2ig w_3 D_\phi ,
\]

\[
(D_\mu D^\mu \phi)^* + m^2 \phi^* \approx -i g \phi^* D_+ w_2 - 2ig w_3 (D_- \phi)^* ,
\]

\[
\partial_\mu F^{\mu \nu} + j^\nu \approx 0 ,
\]

2.3 Dirac’s Brackets (DB)

We follow the iterative method to calculate Dirac’s brackets. Thus, we first consider the set of the first-class constraints Eq. (21) and their gauge-fixing conditions Eq. (24)

\[
\Phi_1 \equiv \pi^+ , \quad \Phi_3 \equiv A_-
\]

\[
\Phi_2 \equiv \Sigma = G - i g (\phi^* \Gamma - \phi \Gamma^*) , \quad \Phi_4 \equiv \pi^- + \partial_- A_+ ,
\]

such that the set of constraints Eq. (25) is second class and such that the constraint matrix \( C_{ij}(x, y) \equiv \{ \Phi_i(x), \Phi_j(y) \} \) with components

\[
C_{ij}(x, y) = \begin{pmatrix}
0 & 0 & 0 & \partial_x^2 \\
0 & 0 & -\partial_y^2 & 0 \\
0 & -\partial_y^2 & 0 & 1 \\
\partial_x^2 & 0 & -1 & 0
\end{pmatrix} \delta^3 (x - y) ,
\] (26)

be regular.

In order to find the inverse of the constraint matrix Eq. (26), we need a suitable inverse for the longitudinal derivative \( \partial_- \). In general, the operator \( \partial_- \) has the following inverse:

\[
(\partial_-)^{-1} f(x^-) = \frac{1}{2} \int dy^- \epsilon (x^- - y^-) f (y^-, x^+, x^-) + F (x^+, x^-) ,
\]

where the function \( \epsilon(x) \) is

\[
\epsilon (x) = \begin{cases}
1 , & x > 0 \\
0 , & x = 0 \\
-1 , & x < 0
\end{cases}
\] (28)

and \( F (x^+, x^-) \) is an arbitrary independent function.

The last term on the right-hand side shows that the inverse of the constraint matrix in Eq. (26) is non-unique. Nevertheless, Dirac proved that the matrix formed by a complete set of second-class constraints should be unique. Therefore, it is said that the set of second-class constraints in Eq (25) is not purely second class.

Steinhardt [7] proved that the inverse matrix of Eq. (26) is non-unique because a subset of first-class constraints is hidden among the second-class constraints [8]. To unveil this subset of constraints, we note that the most general solution of Eqs. (15) and (16) is

\[
v (x) = \hat{v} (x) + s (x^+, x^-) ,
\]

\[
v^* (x) = \hat{v}^* (x) + s^* (x^+, x^-) ,
\]

\[
u_k (x) = \hat{u}_k (x) + s_k (x^+, x^-) ,
\]

\[
u^k (x) = \hat{u}^k (x) + s^k (x^+, x^-) ,
\]
where \( s(x^+, x^-), s^*(x^+, x^-) \) and \( s_k (x^+, x^-) \) are arbitrary functions of \( (x^+, x^-) \), and \( \nu (x) \), \( \nu^* (x) \) and \( \bar{u}_k (x) \) represent the “unambiguous” solutions.

If we substitute the right-hand sides of Eq. (29) for \( v, v^* \) and \( u_k \) in the extended Hamiltonian Eq. (22), the following expression results:

\[
H'_k = \int d^3y \left[ \frac{1}{2} \left( \pi^- \partial_+ + \pi^k \partial_k - j^- \right) A_+ - D_k \phi \left( D^k \phi \right)^* + m^2 \phi \phi^* + \frac{1}{4} F_{ij} F_{ij} \right] + \int d^3y \left[ w_1 C + \bar{u}_k \nu^k + \nu \Gamma^* + \Gamma^* \nu + w_2 \Sigma \right] + \int d^3y \left[ s_k \chi^k + s^* \Gamma + \Gamma^* s \right].
\]

(30)

Therefore, although the set of constraints (25) seems to be second class, the multipliers \( v, v^* \) and \( u_k \) are not completely fixed, which implies that the Hamiltonian still contains the arbitrary functions \( s, s^* \) and \( s_k \).

Steinhardt has shown that this hidden subset of first-class constraints is associated with improper gauge transformations [7]. Unlike the first-class constraints Eq. (21), an improper gauge transformation cannot be identified with generators of gauge transformations [8]. Since constraints of this kind of constraints are tied to boundary conditions, they can map a given physical solution onto another with different boundary conditions, which is not equivalent to the former [8, 9]. It is, therefore, impossible to eliminate the improper gauge transformations by means of gauge conditions since such procedure would exclude configurations physically allowed to the system. These hidden constraints can be eliminated by fixing appropriated boundary conditions on the fields in order to turn the total Hamiltonian into a true generator of the time evolution of the physical system.

Thus, in order to explicitly evaluate the inverse of the matrix of second-class constraints Eq. (26) and ensure its uniqueness, we must determine \( F (x^+, x^-) \). This function can be evaluated if we impose the appropriate boundary conditions on the fields \( \phi, \phi^* \) and \( A_k \) in Reference [5]. Under such boundary conditions, the inverse of the operator \( \partial_- \) is defined for all integrable functions \( f(x^-) \) that are less singular than \( \frac{1}{x^-} \) and vanish faster than \( \frac{1}{x^-} \) for large \( x^- \), namely

\[
(\partial_-)^{-1} f(x^-) = \frac{1}{2} \int dy^- \epsilon (x^- - y^-) f(y^-).
\]

(31)

With this, we find a unique inverse of Eq (26) which is given by the equation

\[
C_{ij} = \frac{1}{2} \begin{pmatrix}
0 & -|x^- - y^-| & 0 & \epsilon (x^- - y^-) \\
|x^- - y^-| & 0 & -\epsilon (x^- - y^-) & 0 \\
0 & \epsilon (x^- - y^-) & 0 & 0 \\
\epsilon (x^- - y^-) & 0 & 0 & 0
\end{pmatrix} \delta^2 (x^+ - y^+).
\]

Alternatively, we can determine the inverse by insisting that the DB’s satisfy Jacobi identities [11], which leads to the same result.

Using the inverse defined by Eq. (32), the first set of DB’s \( \{ \cdot, \cdot \}_{D1} \), for two given dynamical variables \( A (x) \) and \( B (y) \) is calculated by the equality

\[
\{ A (x), B (y) \}_{D1} = \{ A (x), B (y) \} - \int d^3u d^3v \{ A (x), \Phi_i (u) \} C_{ij}^{-1} (u, v) \{ \Phi_j (v), B (y) \}.
\]

(33)

Thus, the nonzero DB’s are

\[
\{ \phi (x), A_+ (y) \}_{D1} = \frac{i g}{2} \phi (x) [x^+ - y^-] \delta^2 (x^+ - y^+) ,
\]

\[
\{ \phi^* (x), A_+ (y) \}_{D1} = -\frac{i g}{2} \phi^* (x) [x^+ - y^-] \delta^2 (x^+ - y^+) ,
\]

\[
\{ A_k (x), A_+ (y) \}_{D1} = -\frac{1}{2} |x^+ - y^-| \delta^2 (x^+ - y^+) .
\]

(34)

Now, following with the iterative procedure to calculate DB [9], we consider the subset of the remaining second-class constraints that under the brackets DB1 are given by the expressions

\[
\Psi_1 \equiv \Gamma = \pi - \partial_- \phi , \quad \Psi_2 \equiv \Gamma^* = \pi^* - \partial_- \phi^* ,
\]

\[
\Psi_3 \equiv \chi^1 = \chi^2 \quad , \quad \Psi_4 \equiv \chi^3
\]

where \( X^k = \pi^k - \partial_- A_k \).

The constraint matrix from this set is defined as

\[
D_{ij} (x, y) \equiv \{ \Psi_i (x), \Psi_j (y) \}_{D1} .
\]

(36)

Using the boundary conditions on the fields accepted to calculate Eq. (32), we compute the inverse \( D^{-1} \) and thus we define the second set of DB’s \( \{ \cdot, \cdot \}_{D2} \),

\[
\{ A (x), B (y) \}_{D2} = \{ A (x), B (y) \}_{D1} - \int d^3u d^3v \{ A (x), \Psi_i (u) \}_{D1}
\]

\[
D_{ij}^{-1} (u, v) \{ \Psi_j (v), B (y) \}_{D1} .
\]

(37)

We then obtain the final DB among the fundamental dynamical variables of the theory

\[
\{ A_k (x), A_l (y) \}_{D2} = -\frac{1}{4} \delta_k^l \epsilon (x^+ - y^-) \delta^2 (x^+ - y^+) ,
\]

\[
\{ \phi (x), \phi^* (y) \}_{D2} = -\frac{1}{4} \epsilon (x^+ - y^-) \delta^2 (x^+ - y^+) .
\]

(38)
\{ \phi( x ), A_+( y ) \}_{D_2} \\
= \frac{i}{2} g \phi( x ) | x^- - y^- | \delta^2 \left( x^+ - y^+ \right) - \frac{i}{8} g \delta^2 \left( x^+ - y^+ \right) \\
\int dv^- \epsilon ( x^- - v^- ) \phi \left( x^+, v^- \right) \epsilon ( v^- - y^- ) ,

(39)

\{ \phi^*( x ), A_+( y ) \}_{D_2} \\
= - \frac{i}{2} g \phi^*( x ) | x^- - y^- | \delta^2 \left( x^+ - y^+ \right) + \frac{i}{8} g \delta^2 \left( x^+ - y^+ \right) \\
\int dv^- \epsilon ( x^- - v^- ) \phi^* \left( x^+, v^- \right) \epsilon ( v^- - y^- ) .

(40)

From the correspondence principle, we obtain the following commutators among the fields:

\[ [ A_k ( x ), A_l ( y ) ] = - \frac{i}{4} \delta^k_l \epsilon ( x^- - y^- ) \delta^2 \left( x^+ - y^+ \right) , \]

(41)

\[ [ \phi ( x ), \phi^*( y ) ] = - \frac{i}{4} \epsilon ( x^- - y^- ) \delta^2 \left( x^+ - y^+ \right) , \]

(42)

\[ [ \phi ( x ), A_+( y ) ] \\
= - \frac{i}{2} g \phi( x ) | x^- - y^- | \delta^2 \left( x^+ - y^+ \right) + \frac{1}{8} g \delta^2 \left( x^+ - y^+ \right) \\
\left( x^+ - y^+ \right) \int dv^- \epsilon ( x^- - v^- ) \phi \left( x^+, v^- \right) \epsilon ( v^- - y^- ) ,

(43)

\[ [ \phi^*( x ), A_+( y ) ] = \frac{1}{2} g \phi^*( x ) | x^- - y^- | \delta^2 \left( x^+ - y^+ \right) - \frac{1}{8} g \delta^2 \left( x^+ - y^+ \right) \\
\int dv^- \epsilon ( x^- - v^- ) \phi^* \left( x^+, v^- \right) \epsilon ( v^- - y^- ) .

(44)

The first two relations were specified by Neville and Rohrlich [12]. To derive the pair of results, Reference [12] started from the free-field operators commutation relations for unequal times \( x^+, y^+ \) and then calculated the commutators on the null-plane at equal time, i.e., \( x^+ = y^+ \). The commutation relations involving the field operators \( \hat{A}_+( x ) \) were not obtained in [12], and it was stated that they must be derived by solving a quantum constraint. However, as we have shown, one can obtain these last commutation relations at the classical level by following carefully Diracs analysis of the constraint structure of the model.

3 Spinor Electrodynamics (\( Q E D_4 \)): Constraint Structure

The Lagrangian density of the spinor Electrodynamics written in terms of the light cone projections\(^1\) of the fermionic fields is of the fermionic fields is

\[
\mathcal{L} = \bar{\psi}_+ \left( \frac{i}{2} \gamma^+ \rightarrow A_+ + g A_+ \gamma^+ \right) \psi_+ + \bar{\psi}_- \left( \frac{i}{2} \gamma^- \rightarrow A_- - g A_- \gamma^- \right) \psi_-
\]

\[
+ \bar{\psi}_+ \left( \frac{i}{2} \gamma_+ - g \hat{A} \gamma - m \right) \psi_+ + \bar{\psi}_- \left( \frac{i}{2} \gamma_- - g \hat{A} \gamma - m \right) \psi_-
\]

(45)

where we have defined \( \gamma^k A_k \equiv A \) for \( k = 1, 2, \ldots \). The corresponding field equations are the following:

\[
\partial_\mu F^{\mu\nu} - g \bar{\psi} \gamma^\nu \psi = 0 \\
( i \partial_\mu - g A_\mu ) \gamma^\nu \psi_+ + [ i \sigma - g A - m ] \psi_- = 0
\]

(46)

and

\[
( i \partial_\mu + g A_\mu ) \bar{\psi}_+ \gamma^\nu + \bar{\psi}_- \left( i \sigma + g A + m \right) = 0 \\
( i \partial_\mu + g A_\mu ) \bar{\psi}_- \gamma^\nu - \bar{\psi}_+ \left( i \sigma + g A + m \right) = 0 ,
\]

and the canonical momenta for the fields are given by the expression

\[
\pi^\mu = F^{\mu\nu} = \partial_\nu A^\mu - \partial^\nu A^\mu
\]

(47)

and

\[
\tilde{\phi}_{+a} = - \frac{i}{2} \bar{\psi}_+ ( y^+ )_{b} \gamma_a , \quad \phi_{+a} = - \frac{i}{2} ( \gamma^+ )_{ab} \psi_{+b} ,
\]

\[
\tilde{\phi}_{-a} = 0 , \quad \phi_{-a} = 0 ,
\]

(48)

where \( a, b = 1, 2, 3, 4 \).

From the canonical-momenta equations, we observe that only one of the equalities in Eq. (47) is dynamical

\[
\pi^\nu = F^{\nu\nu} = \partial_\nu A^\nu - \partial^\nu A^\nu
\]

(49)

while all the other equations coming from Eqs. (47) and (48) give a set composed of three primary bosonic constraints:

\[
C \equiv \pi^+ \approx 0 , \quad C^k \equiv \pi^k + \partial_\nu A^\nu - \partial^\nu A^\nu \approx 0 \quad k = 1, 2
\]

(50)

and four fermionic constraints:

\[
\Gamma_{+a} \equiv \phi_{+a} + \frac{1}{4} ( \gamma^+ )_{ab} \psi_{+b} \approx 0 , \quad \Gamma_{+a} \equiv \phi_{+a} + \frac{1}{4} \psi_{+b} \gamma^b ( \gamma^+ )_{ab} \approx 0
\]

\[
\Gamma_{-a} \equiv \psi_{-a} \approx 0 , \quad \Gamma_{-a} \equiv \phi_{-a} \approx 0 .
\]

(51)

\(^1\)See the definitions in Appendix A.
The canonical Hamiltonian density is given by the equality [13]

\[ \mathcal{H}_c = \frac{1}{2} \left( \pi^2 + \left[ \pi \partial \pi - e_3 \dot{\theta} + g \psi^+ \psi + \bar{\psi} \right] \right) A_4 + \frac{1}{2} (F_{12})^2 \]

and the primary Hamiltonian takes the form

\[ H_P = H_c + \int dy^3 \left[ u C + u_k C_k + \bar{\Gamma}_{a_1} v_{1a} + \bar{\Gamma}_{a_2} v_{2a} - \bar{v}_{1a} \Gamma_{a_1} - \bar{v}_{2a} \Gamma_{a_2} \right]. \]  

Where \( u \) and \( u_k \) are bosonic Lagrange multipliers and, \( v_1, v_2, \bar{v}_1 \) and \( \bar{v}_2 \) are fermionic multipliers.

The fundamental Poisson brackets are

\[
\{ A_\mu (x), \pi^\nu (y) \} = \delta_\mu^\nu \delta^3 (x - y), \\
\{ \psi^a (x), \phi_{ab} (y) \} = -\delta_{ab} \delta^3 (x - y), \\
\{ \bar{\psi}^a (x), \phi_{ab} (y) \} = -\delta_{ab} \delta^3 (x - y), \\
\{ C^k (x), C^j (y) \} = -2 \delta^k_j \delta^\gamma_\delta \delta^3 (x - y). 
\]

Next, we list the non null PB’s between the primary constraints:

\[
\{ \Gamma_{+a} (x), \Gamma_{+b} (y) \} = -i \left( \gamma^a_{ab} + \delta^a_3 \right) (x - y), \\
\{ \bar{\Gamma}_{+a} (x), \Gamma_{+b} (y) \} = -i \left( \gamma^a_{ab} + \delta^a_3 \right) (x - y), \\
\{ C^a (x), \Gamma_{+b} (y) \} = -2 \delta^a_j \delta^3_\delta (x - y). 
\]

3.1 The Fermionic Sector

To preserve the primary constraints in time, we compute the existence condition for the fermionic constraints Eq. (51). For \( \Gamma_+ \), we obtain the relation

\[ \hat{\Gamma}_+ = (i \bar{\theta} - g A - m) \psi_+ - g A + \gamma^+ \psi_+ + i \gamma^+ v_1 \approx 0. \]

From Eq. (56), we can derive two relations with help of the null-plane \( \gamma \)-algebra. First, we compute \( \frac{i}{2} \gamma^+ \hat{\Gamma}_+ \) to obtain one component of the multiplier \( v_1 \)

\[ \Delta^+ v_1 = - \frac{i}{2} \bar{\gamma}^+ \left( i \bar{\theta} - g A - m \right) \psi_+ + i g A_+ \psi_+ \approx 0. \]

The second relation is obtained from the projection \( \Delta^+ \hat{\Gamma}_+ \), which yields a secondary constraint

\[ \Delta^+ \hat{\Gamma}_+ = (i \bar{\theta} - g A - m) \Delta^+ \psi_+ \approx 0, \]

and since \( (i \bar{\theta} - g A - m) \) is an invertible operator, we can set the secondary constraint in the form

\[ \Phi = \Delta^+ \psi_+ \approx 0. \]

For \( \hat{\Gamma}_+ \), we get the expression

\[ \hat{\Gamma}_+ = \bar{\psi}_+ \left( i \bar{\theta} + g A + m \right) + g A_+ \psi_+ - i \bar{\psi}_1 \gamma^+ \approx 0. \]
\[ \dot{\Omega}_- = -(i \partial_+ + g A_-) \tilde{v}_2 \gamma^- - \tilde{v}_1 \left( i \overleftrightarrow{\partial} + g A + m \right) = 0, \]  

(68)

respectively.

We can now show that the set of Eq. (57), (61), (66), (67) and (68) allows us to define completely all the fermionic Lagrange multipliers. To this end, we compute \( \Delta^- \hat{\Omega}_- \) to get the other component for \( v_1 \):

\[ \Delta^- v_1 = 0 , \]  

(69)

Then, using that \( \Delta^+ v_1 + \Delta^- v_1 = v_1 \), we determine the multiplier \( v_1 \):

\[ v_1 = -\frac{i}{2} \gamma^- (i \overleftrightarrow{\partial} - g A - m) \psi_- + i g A_+ \psi_+ . \]  

(70)

From Eq. (67), we compute \( \frac{1}{2} \gamma^+ \dot{\Omega}_- \) to determine the component \( \Delta^- v_2 \) with the equation

\[ (i \partial_+ - g A_-) (\Delta^- v_2) = \frac{i}{2} \left[ -(i \overleftrightarrow{\partial} - g A)^2 + m^2 \right] \psi_- - \frac{i}{2} g A_+ \gamma^+ (i \overleftrightarrow{\partial} - g A - m) \psi_+ , \]

which combined with the component \( \Delta^+ v_2 \) in Eq. (66) gives the result

\[ (i \partial_+ - g A_-) v_2 = \frac{i}{2} \left[ -(i \overleftrightarrow{\partial} - g A)^2 + m^2 \right] \psi_- - \frac{i}{2} g A_+ \gamma^+ (i \overleftrightarrow{\partial} - g A - m) \psi_+ . \]  

(71)

A similar procedure determines the multipliers \( \tilde{v}_1 \) and \( \tilde{v}_2 \):

\[ \tilde{v}_1 = -\frac{i}{2} \tilde{\psi}_- \left( i \overleftrightarrow{\partial} + g A + m \right) \gamma^- - i g A_+ \tilde{\psi}_+ , \]  

(72)

and

\[ (i \partial_+ + g A_-) \tilde{v}_2 = \frac{i}{2} \tilde{\psi}_- \left[ -(i \overleftrightarrow{\partial} + g A)^2 + m^2 \right] + \frac{i}{2} g A_+ \tilde{\psi}_+ \left( i \overleftrightarrow{\partial} + g A + m \right) \gamma^+ , \]  

(73)

We have, therefore, determined all the fermionic Lagrange multipliers. The set of primary and secondary fermionic constraints are second class according to Dirac’s procedure [9]. The use of the projection of the fermionic fields has allowed us to observe clearly the existence of fermionic secondary constraints, which shows that the field is fully described by only two of their four components.

3.2 The Electromagnetic Sector

The consistent condition for \( C^k \) yields the equality

\[ C^k = \left( \delta^k_1 \partial_1 - \delta^k_1 \partial_2 \right) F_{12} - g \left( \tilde{\psi}_+ \gamma^k \psi_- + g \tilde{\psi}_- \gamma^k \psi_+ \right) + \partial_i \pi^- - 2 \partial_- v_k \approx 0 , \]  

(74)

which is a differential equation for the \( u_k \) Lagrange multipliers. The consistence condition for \( \pi^+ \) produces the following secondary constraint:

\[ G \equiv \pi^+ = \partial_- \pi^- + \partial_k \pi^k - g \tilde{\psi} \]  

(75)

which is Gauss’s law, the consistence condition for which shows that

\[ \dot{G} = ig \left( \hat{\Gamma}_+ \psi_+ + \hat{\psi}_+ \Gamma_+ + \hat{\psi}_- \Gamma_- + \hat{\Gamma}_- \psi_- \right) \approx 0 , \]  

(76)

3.3 Constraint Classification and Gauge Fixing Conditions

The full set of primary and secondary constraints is given by Eqs. (50), (51), (59), (63), (64), (65) and (75):

\[ C_-, C^k, \Gamma_+, \Gamma_- , \Gamma_+, \Gamma_-, \Omega_+ , \Omega_- , \Phi , \hat{\Omega}_+ , \hat{\Phi} , G . \]  

(77)

The first, \( C = \pi^+ \), has a vanishing PB with each of the constraints and is therefore a first-class constraint. Apparently, the remaining subset of constraints is second class. Nonetheless, they form a singular constraint matrix with a zero mode, and the associated eigenvector yields a linear combination that is another first-class constraint (see Appendix C). Alternatively, we must observe that as in the fermionic case, the electromagnetic sector must maintain its free constraint structure, because the interaction term cannot change the first-class structure of the free theory into second-class ones, because the DB would be undefined in the limit of zero coupling constant. Thus, the combination, a first-class constraint that is independent of \( \pi^+ \), is given by the expression

\[ \Sigma \equiv G - ig \left[ \tilde{\psi}_+ \Gamma_+ + \tilde{\Gamma}_+ \psi_+ + \tilde{\psi}_- \Gamma_- + \tilde{\Gamma}_- \psi_- \right] . \]  

(78)

Thus, we have the following set of second-class constraints

\[ \Gamma_+ , \Gamma_- , \Omega_+ , \Omega_- , \Phi , \hat{\Gamma}_+ , \hat{\Gamma}_- , \hat{\Omega}_+ , \hat{\Phi} , C^k \]  

(79)

and the set of first-class constraints

\[ C_-, \Sigma . \]  

(80)

This is the maximal number of first-class constraints, and the consistence condition on the second-class constraints leads to expressions for the Lagrange multiplier correspondent to each.

The next step is to impose gauge conditions, one for each first-class constraint, so that the set of gauge-fixing conditions and first-class constraints defines a second-class set. Care should be taken to define gauge conditions that are compatible with the Euler-Lagrange equations of motion. We, therefore, choose a set of conditions known as
the null-plane gauge, which is defined by the following relations:

\[ B = A_\perp \approx 0 \ , \ K \equiv \pi_\perp + \partial_\perp A_\perp \approx 0. \]  \hfill (81)

With this, the set of first-class constraints and gauge-fixing conditions is now of second class. If the photon field were coupled to the fermion field, we would consider \( A_\perp \) as a possible gauge condition, but that would make it impossible to find a second gauge condition compatible with the equations of motion. This is similar to what happens with the radiation gauge in the instant-form formalism, i.e., \( x^0 = \text{const.} \) plane \[9].

### 3.4 Dirac’s Brackets

The explicit evaluation of the inverse of the full matrix of second-class constraints involves an arbitrary function, to determine which we impose appropriate boundary conditions on the fields \[5]. The inverse of the operator \( \partial_- \) is then defined as in Eq. (31). After tedious manipulations, we obtain the graded Lie algebra for the dynamical variables of the spinor electrodynamics, in the form

\[
\{ A_k (x) , \ A_j (y) \}_D = - \frac{1}{4} \delta^k_j \epsilon (x^-, y^-) \delta^2 \left( x^+ - y^+ \right)
\]  \hfill (82)

\[
\{ \psi_a (x) , \ \bar{\psi}_b (y) \}_D = - \frac{i}{2} \ (\gamma^-)_{ab} \delta^3 (x - y) - \frac{1}{4} \epsilon (x^- - y^-) \ (i \partial^+_a + m)_{ab} \delta^2 \left( x^+ - y^+ \right) + \frac{i}{8} \ |x^- - y^-| \delta^2 \left( x^+ - y^+ \right)
\]  \hfill (83)

\[
\{ A_\perp (x) , \ \psi (y) \}_D = - \frac{i}{2} g \ |x^- - y^-| \delta^2 \left( x^+ - y^+ \right) \psi (y)
\]  \hfill (84)

\[
\{ A_\perp (x) , \ \bar{\psi} (y) \}_D = \frac{i}{2} g |x^- - y^-| \delta^2 \left( x^+ - y^+ \right) \bar{\psi} (y)
\]  \hfill (85)

Our two first expressions, Eq. (82) and (83), are in accord with the result obtained by Rohrlich \[5\], and Kogut and Soper \[14\] when the correspondence principle is applied. Reference \[5\] has determined these relations from the (anti)-commutator between the field operators for unequal times, while Reference \[14\] postulated the (anti)-commutators for the annihilation and creation operators for the quantum fields and transformed those relations back to coordinate space to obtain the equal-time (anti)-commutators.

Equations (84) and (85), associated with the field \( A_\perp \), were derived from careful application of Dirac’s procedure to the null-plane. The correspondence principle shows that...
these relations are equivalent to the expressions derived by Kogut and Soper [14].

4 Remarks and Conclusions

We have carried out the constraint analysis of the scalar and spinor electrodynamics on the null-plane. Several characteristics or features are in contrast with the customary space-like hyper-surface formulation.

We have shown that the $SQED_4$ has a first-class constraint, Gauss’s law, which comes from the linear combination of electromagnetic and scalar constraints given by the zero-mode eigenvector of the constraint matrix. This is a consequence of the constraints associated with the scalar sector. By contrast, in the instant form-analysis, the second first-class constraint receives no contributions from scalar constraints because the scalar sector is free of constraints in this formalism.

After selecting the null-plane gauge conditions to transform the first-class constraints into second-class ones, we had to impose appropriate boundary conditions on the fields to fix a hidden subset of first-class constraints allowing us to find a unique inverse of the second-class constraint matrix. Next, we have obtained the DB’s of the theory and quantized it via the correspondence principle, Eqs. (41), are consistent with the results in the literature [12]. Although it did not derive the relations Eq. (44), which involve the field operator $A_+$, Reference [12] stated that they could be obtained from the solution of a quantum constraint. Instead, to calculate the commutators, we have quantized the DB’s derived at the classical level from a careful analysis of the $SQED_4$ constraint structure.

In the case of the $QED_4$, our careful analysis of the fermionic-sector constraint structure showed that it has only second-class constraints. A hidden subset of first-class constraints nonetheless exists [7], which generates improper gauge transformations [8]. This first class subset is associated with the impossibility to define a unique inverse for the operator $\partial_-$, in turn related to the insufficiency of boundary conditions to solve the Cauchy data problem. Appropriate boundary conditions have to be imposed on the fields to insure the uniqueness of the inverse.

To fix the first-class set of the $QED_4$, we have chosen the null-plane gauge conditions and followed Dirac’s procedure to we obtain the graded algebra Eqs. (82)–(85) for the canonical variables. Via the correspondence principle, Eqs. (82) and (83) reproduce the canonical (anti-)commutation relations for the quantum fields derived in [5, 14]. In addition, Eqs. (84) and (85), associated with the field $A_+$, derived following application of Dirac’s procedure, reproduce the expressions derived by Kogut and Soper [14] at the quantum level.

Recently [18], the null-plane Hamiltonian structure of (1+1)-dimensional electrodynamics, the Schwinger model, has been studied. Dirac’s procedure [2] was followed, and the hidden first-class constraints were dealt with [7, 8]. The study showed that the fermionic sector has only second-class constraints, as in the instant formalism. And as it can be shown, the $QED_4$ fermionic sector in the null-plane also presents only a second-class constraint structure. We conclude that fermionic fields satisfying a first-order Dirac equation in the null-plane or instant-form formalisms have only second-class constraints structure.

We are currently studying the constraint structure of the pure Yang-Mills fields and also the Hamiltonian structure of the theory resulting from the interaction between the Yang-Mils fields and complex scalar fields. Reports on this research will be communicated elsewhere.

Acknowledgments BMP thanks CNPq for partial support. GERZ thanks CNPq (grant 142695/2005-0) for full support.

Appendix A: Notation

The null-plane time $x^+$ and longitudinal coordinate $x^-$ are defined as

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}} \text{ and } x^- = \frac{x^0 - x^3}{\sqrt{2}},$$

(86)

respectively, with the transverse coordinates $x^\perp \equiv (x^1, x^2)$ kept unchanged.

Hence, in the four-vector space $x = (x^+, x^1, x^2, x^-)$, the metric is

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$  

(87)

Explicitly,

$$x^+ = x_+ , \quad x^- = x_- , \quad x \cdot y = x^+ y^- + x^- y^+ - x^\perp \cdot y^\perp ,$$

(88)

where the derivatives with respect to $x^+ ex^-$ are defined as

$$\partial_+ \equiv \frac{\partial}{\partial x^+} , \quad \partial_- \equiv \frac{\partial}{\partial x^-}$$

(89)

with $\partial^+ = \partial_-$. Here, we have used the following relations

$$\delta^4(x - y) = \delta(x^+ - y^+)\delta^2(x^\perp - y^\perp)\delta(x^-- y^-).$$

$$\frac{1}{2} \frac{d}{dx^-} \epsilon(x^- - y^-) = \delta(x^- - y^-),$$

$$\frac{1}{2} \int dy^- \epsilon(x^- - y^-)\epsilon(y^- - z^-) = |x^- - y^-|$$

(90)
The same orthogonal transformation is applied to the Dirac matrices that still obey
\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}. \] (91)
This makes \( \gamma^+ \) and \( \gamma^- \) singular matrices.
Since
\[ (\gamma^+)^k = \gamma^- , \quad (\gamma^-)^k = \gamma^+ \quad (\gamma^k)^k = -\gamma^k \quad k = 1, 2, \ldots \] (92)
we define the Hermitian matrices
\[ \Delta^\pm = \frac{1}{2} \gamma^\mp \gamma^\pm \] (93)
which are projection operators:
\[ (\Delta^\pm)^2 = \Delta^\pm , \quad \Delta^\pm \Delta^\mp = 0 , \quad \Delta^- + \Delta^+ = 1 . \] (94)

The eigenvector, therefore, is
\[ U = (1, ig\bar{\psi}_+, ig\bar{\psi}_-, -ig\psi_+, -ig\psi_-, 0, 0, 0, 0, 0)^T , \] (101)
gives the second first class
\[ \Sigma = G - ig \left[ \bar{\psi}_+ \Gamma_+ + \bar{\psi}_- \Gamma_- + \bar{\psi}_- \Gamma_+ + \bar{\psi}_+ \Gamma_- \right] . \] (102)

Appendix C: Grassmann Algebras

A Grassmann algebra contains bosonic (self-commuting) and fermionic (self-anticommuting) variables [15]:
\[ FB = (-1)^{n_a n_b} B F , \] (103)
where \( n = 0 \) for a bosonic and \( n = 1 \) for a fermionic variable. Note that the product of two fermionic variables is bosonic, and the product of a fermionic and a bosonic variables is fermionic. The left derivative of a fermionic variable is defined as
\[ \frac{\partial}{\partial \psi_\alpha} \left\{ \psi_{\alpha_1} \psi_{\alpha_2} \cdots \psi_{\alpha_n} \right\} = -\delta_{\alpha_1\alpha_2} \psi_{\alpha_3} \cdots \psi_{\alpha_n} \cdots + \delta_{\alpha_2\alpha_3} \psi_{\alpha_1} \psi_{\alpha_4} \cdots \psi_{\alpha_n} + \cdots + (-1)^n \delta_{\alpha_n\alpha_1} \psi_{\alpha_1} \psi_{\alpha_2} \cdots \psi_{\alpha_{n-1}} . \] (104)

The Poisson Brackets can be defined in analogy to ordinary mechanics [16, 17]. The phase space is spanned by
,q_i, p^j \text{ which are bosons and } \psi_\alpha \text{ and } \pi^\alpha, \text{ fermions. Denote by } B(F) \text{ a bosonic (Fermionic) element of the Grassmann algebra, then}

\[ \{B_1, B_2\} = -\{B_2, B_1\} \]

\[ = \left\{ \frac{\partial B_1}{\partial q_i} \frac{\partial B_2}{\partial p^j} - \frac{\partial B_2}{\partial q_i} \frac{\partial B_1}{\partial p^j} \right\} + \left\{ \frac{\partial B_1}{\partial \phi_\alpha} \frac{\partial B_2}{\partial \pi^\alpha} - \frac{\partial B_2}{\partial \phi_\alpha} \frac{\partial B_1}{\partial \pi^\alpha} \right\} \]

\[ \{F, B\} = -\{B, F\} \]

\[ = \left\{ \frac{\partial F}{\partial q_i} \frac{\partial B}{\partial p^j} - \frac{\partial B}{\partial q_i} \frac{\partial F}{\partial p^j} \right\} - \left\{ \frac{\partial F}{\partial \phi_\alpha} \frac{\partial B}{\partial \pi^\alpha} + \frac{\partial B}{\partial \phi_\alpha} \frac{\partial F}{\partial \pi^\alpha} \right\} \]

\[ \{F_1, F_2\} = \{F_2, F_1\} \]

\[ = \left\{ \frac{\partial F_1}{\partial q_i} \frac{\partial F_2}{\partial p^j} + \frac{\partial F_2}{\partial q_i} \frac{\partial F_1}{\partial p^j} \right\} - \left\{ \frac{\partial F_1}{\partial \phi_\alpha} \frac{\partial F_2}{\partial \pi^\alpha} + \frac{\partial F_2}{\partial \phi_\alpha} \frac{\partial F_1}{\partial \pi^\alpha} \right\} . \]

(105)

It follows from its definition that the Poisson brackets have the properties

\[ \{A, B\} = -(-1)^{a_1 a_2} \{B, A\} \]

\[ \{A, B + C\} = \{A, B\} + \{A, C\} \]

\[ \{A, BC\} = \{A, B\} C + A \{B, C\} \]

\[ \{AB, C\} = (-1)^{a_1 a_2} \{A, C\} B + A \{B, C\} \]

\[ (-1)^{a_1 a_3} \{A, \{B, C\}\} + (-1)^{a_2 a_3} \{B, \{C, A\}\} \]

\[ + (-1)^{a_3 a_2} \{C, \{A, B\}\} = 0 . \]

(106)

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