Explosive instability of dust settling in a protoplanetary disc

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ABSTRACT

It is shown that gas-dust perturbations in a disc with dust settling to the disc midplane exhibit the non-linear three-wave resonant interactions between streaming dust wave (SDW) and two inertial waves (IW). In the particular case considered in this paper, SDW at the wavenumber \( k^* \) = \( 2\kappa/(g_t t_s) \), where \( \kappa, g_t \) and \( t_s \) are, respectively, epicyclic frequency, vertical gravitational acceleration and particle’s stopping time, interacts with two IW at the lower wavenumbers \( k' \) and \( k'' \) such that \( k' < k_{DSI} < k'' < k^* \), where \( k_{DSI} = \kappa/(g_t t_s) \) is the wavenumber of the linear resonance between SDW and IW associated with the previously discovered linear dust settling instability. The problem is solved analytically in the limit of the small dust fraction. As soon as the dynamical dust back reaction on gas is taken into account, \( k^*, k' \) and \( k'' \) become slightly non-collinear and the emerging interaction of waves leads to simultaneous explosive growth of their amplitudes. This growth is explained by the conservative exchange energy between the waves. The amplitudes of all three waves grow because the negative energy SDW transfers its energy to the positive energy IW. The product of the dimensionless amplitude of initially dominant wave and the time of explosion can be less than Keplerian time in a disc. It is shown that, generally, the three-wave resonance of an explosive type exists in a wide range of wavenumbers \( 0 < k^* \leq 2\kappa/(g_t t_s) \). An explosive instability of gas-dust mixture may facilitate the dust clumping and the subsequent formation of planetesimals in young protoplanetary discs.

Key words: hydrodynamics — instabilities — protoplanetary discs — accretion, accretion discs — waves — turbulence

1 INTRODUCTION

The streaming instability of gas-dust mixture associated with the dust radial drift in the midplane of protoplanetary disc has been discovered by Youdin & Goodman (2005) and further extensively investigated using the numerical simulations (Youdin & Johansen 2007; Johansen & Youdin 2007; Johansen et al. 2009; Carrera et al. 2015; Yang et al. 2017). The ability of the streaming instability to concentrate solids into high densities in the non-linear regime is accepted to be necessary for the formation of planetesimals. However, it has thresholds for small solids and low metallicities, which quantitatively depend on the details of numerical setup (Yang & Johansen 2014; Yang et al. 2017; Li et al. 2018; Li & Youdin 2021).

The amount of dust in the disc midplane can be probably increased through the dust settling instability (DSI) discovered by Squire & Hopkins (2018). DSI is caused by vertical rather than radial drift of the grains. So far, the only attempt to study the dynamics of finite amplitude perturbations presumably associated with non-linear stage of DSI has been made by Krapp et al. (2020). The numerical simulations performed with the multi-fluid code have shown either a weak or slow particles clumping with a caveat about the convergence of the maximum dust density.

This work is another effort to study the non-linear dynamics of gas-dust perturbations with the account of the dust back reaction on gas, which takes place in a disc with the dust settling to its midplane. Unlike most of the previous work on the non-linear gas-dust dynamics in protoplanetary discs, it employs essentially the analytical approach. Of course, the latter becomes possible due to several simplifications of the considered problem. First, perturbations are considered in a patch of disc much smaller than the disc scaleheight. It is assumed that perturbations are axisymmetric, while the dust behaves like the second pressureless fluid. The dust is coupled with the gas through the drag force parametrised by the stopping time of the particles (Squire & Hopkins 2018). Further, the model is restricted by the case of the small dust fraction. Finally, the major point is that two-fluid dynamics is considered within the weakly non-linear theory with only the quadratic interactions between the modes retained (Craik 1988). As far as the amplitudes of perturbations are small, the quadratic interactions are most efficient for modes satisfying the three-wave resonant conditions (Kadomtsev & Karpman 1971).

The three-wave resonant interactions are responsible for redistribution of energy over the different scales in various geophysical and astrophysical flows. They serve as main process coupling...
waves in a weak turbulence regime. The general theory of such a dynamics was introduced in physics of fluids by Phillips (1960) who was the first to search for three-wave resonance among surface gravity waves. This work was followed by many studies of three-wave resonance in various flows. For example, the three-wave resonance of capillary-gravity waves was investigated by McGoldrick (1965), see also its recent experimental verification by Haudin et al. (2016). The existence of resonant triads among three internal gravity waves, or alternatively, among two surface gravity waves and one internal gravity wave in stratified medium has been shown by Thorpe (1966). For the corresponding experimental study see, e.g., Joubaud et al. (2012) who performed the first measurement of the parametric subharmonic instability growth rate in a tank filled with stratified salt water.

Like internal waves, inertial waves (IW hereafter) propagating in a rotating fluid obey a similar anisotropic dispersion relation, which makes the frequency depend on the direction of propagation of wave rather than on its wavelength. Accordingly, it has been verified experimentally by Bordes et al. (2012) that subharmonic secondary waves excited due to three-wave resonance among IW propagate closer to the plane of rotation as compared with propagation of the primary wave. This intrinsic feature of resonance between plane IW gives rise to the anisotropic turbulent transfer of energy mainly in the direction perpendicular to the rotation axis and generation of columnar vortices in weakly turbulent rotating flows, see e.g. Smith & Waleffe (1999) and Galtier (2003).

There has been also much work on the three-wave resonances in stellar interiors. Vandakurov (1965) found the possibility of converting the radial pulsation of the star into two non-radial modes with a sum of frequencies close to the frequency of the radial pulsation. Later on, Dziembowski (1982) constructed the general theory of resonant interactions of stellar perturbations. He argued that three-wave resonance may be a mechanism limiting the amplitudes of modes. The variant of the theory of resonant interactions of stellar perturbations applied to distribution of modes having random phases has been developed by Kumar & Goldreich (1989), for the most recent example of work on such a problem for the red giants see Weinberg et al. (2021).

Three-wave resonance has a striking manifestation in non-equilibrium media. Since the work on the interaction of waves in plasma penetrated by a beam of charged particles or electrostatic waves propagating in magnetised plasma, see Dikasov et al. (1965) and further Coppi et al. (1969), Fukai et al. (1970), it has been known that the resonant interaction of positive and negative energy waves may lead to amplitudes of all waves growing up to infinity at finite time. Such kind of solutions has been referred to as an explosive instability. Afterwards, it has been detected in the laboratory plasma, see Nakamura (1977) and Sugaya et al. (1978). However, explosive instability of beam-plasma system proved difficult to investigate in laboratory. An illustrative example of explosive instability can be found in Jones & Fukai (1979), who studied its evolution employing one-dimensional cold fluid model of an electron beam as well as the corresponding particle-in-cell simulations, see e.g. Figure 3 of their paper. Perfect agreement of the fluid code solution with the growth of perturbations predicted by the standard mode coupling equations for the corresponding resonant triad implied that the higher order wave couplings did not saturate the instability. Saturation was revealed in particle-in-cell simulations, which demonstrated that the high enough perturbations cause mixing of electrons belonging to the beam and the surrounding plasma. This causes the heating of medium and the subsequent frequency mismatch in the resonant triad.

For the non-linear dynamical system of quite a general form, see Dougherty (1970), Davidson (1972) and Rabinovich & Reutov (1973), it was shown that for explosive instability to occur two conditions must be satisfied. First, one of the resonant waves must have the energy sign different from the energy signs of the other two waves. Second, this wave must take the highest frequency in the absolute value comparing with the frequencies of the other two waves. Cairns (1979) pointed out that these conditions are met for three-wave resonance of waves propagating in a three-layer flow with step-wise profiles of density and velocity. Shortly after that, Craik & Adam (1979) confirmed this prediction of explosive instability by the direct calculation of the corresponding interaction coefficients.

Explosive instability has been suggested to be responsible for some transient effects in terrestrial and space environment. For example, the waves of vorticity observed in alongshore oceanic currents can be generated by an explosive interactions within the corresponding resonant triads below the low-frequency threshold predicted by the linear stability theory but not seen in observational data, see Shira et al. (1997). Explosive instability of kink waves existing in magnetic flux tubes, see Ryu-tova (1988), can manifest itself in quiescent prominences of solar atmosphere. High-resolution space observations of Sun reveal the growing ripples at the prominence/corona interface, which end up with rapid formation of mushroom-like disturbances. The sudden formation of such structures can be naturally explained by an explosive growth of negative energy kink waves when magnetised flow is stable with respect to the linear Kelvin-Helmholtz instability, see Ryu-tova & Tarbell (2000) and Ryu-tova et al. (2010).

This work considers the non-linear stability of dust settling through the gas being in vertical hydrostatic equilibrium at some height above the protoplanetary disc midplane. Previously, Zhuravlev (2019), hereafter Z19, revealed that the linear perturbations of the dust density advected by the settling dust can be considered as the negative energy wave. This wave was referred to as the streaming dust wave (SDW hereafter). Its linear resonance with the positive energy IW propagating in the gas gives rise to DS1. Whether an explosive three-wave resonant interaction among SDW and IW is possible on the same background is an issue addressed in this study. At the same time, it is important to note that such an instability must be absent in the case of radial drift of the dust settled to the disc midplane because then there is no negative energy SDW, see Z19.

In Section 2 the dynamical equations for perturbations of gas-dust mixture are derived retaining the terms that are quadratic over the dimensionless amplitudes of perturbations. The reasonable assumptions of the small dust fraction and the small stopping time of the particles make the linear problem analytically tractable with the only additional restriction that the solution is sought sufficiently far from the linear resonance between SDW and IW. This is exposed in Section 3. The particular case of three-wave resonance between one SDW and two IW satisfying the general conditions of explosive instability is proposed in Section 4. In Section 5 an explosive interaction between these waves is derived using the linear solution at the corresponding frequencies obtained previously. A conservative type of wave interactions is checked in Section 6. At last, Section 7 is assigned for various estimates of the time of explosion. It is shown that explosive instability of gas-dust mixture can emerge at the physically reasonable time, which is much shorter than the timescale of the dust settling.
2 NON-LINEAR EQUATIONS FOR DYNAMICS OF GAS-DUST MIXTURE IN A DISC

Starting point of the present analysis is the set of two-fluid equations describing the local axisymmetric dynamics of a partially coupled gas-dust mixture in a protoplanetary disc with the dust back reaction on gas taken into account, see Z19:

$$\partial_t U - 2\Omega_0 U_y e_x + (2 - q)\Omega_0 U_x e_y + (U \nabla)U = \nabla p_0 \frac{\nabla (p + p_0)}{\rho},$$  

(1)

$$\nabla (p + p_0) = \frac{\nabla U}{\tau_0}.$$  

(2)

$$\nabla \cdot (U - \frac{\rho_0}{\rho} \rho U) = 0,$$  

(3)

$$\partial_t \rho + \nabla (\rho U) = 0.$$  

(4)

The notations for important variables are summarised in the Appendix A. Equations (1-4) are written in terms of the centre-of-mass velocity,

$$U \equiv \frac{\rho_0 U_0 + \rho U_p}{\rho},$$

where $U_{0,p}$ and $\rho_{0,p}$ are velocity and density of gas and dust, respectively, while $\rho \equiv \rho_0 + \rho_p$ is the total density of mixture. It is assumed that gas and dust velocities are measured with respect to the reference shear velocity $U_0 = -q\Omega_0 xe_y$ as defined in Z19, where $\Omega_0$ is angular velocity of the reference frame comoving with the certain patch of disc. The shear rate, $q$, takes approximately the Keplerian value, $q = 3/2$, in protoplanetary discs. The gas pressure, $p$, is measured with respect to the reference pressure $p_0$, which defines $U_0$, through the following equations

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial x} = -\frac{\partial \Phi}{\partial x} + \Omega_0^2 (r_0 + x) + 2\Omega_0 U_0,$$  

(5)

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} = -\frac{\partial \Phi}{\partial z},$$  

(6)

where $\Phi$ is the gravitational potential of the host star.

Equation (2) represents the marginal case of particles tightly coupled to the gas so that there is no time-lag of their dynamical response to the change of the gas acceleration. This limit is usually referred to as the terminal velocity approximation, see Youdin & Goodman (2005). It is known that for the terminal velocity approximation to be valid the stopping time of the particles, $\tau_0$, must be so small that

$$\tau_0 \equiv t_s \max \{t_{\text{es}}^{-1}, \Omega_0 \} \ll 1,$$  

(7)

and

$$\frac{g_s t_{\text{es}}^2}{l_{\text{es}}} \ll 1,$$  

(8)

where $t_{\text{es}}$ and $l_{\text{es}}$ are, respectively, the characteristic time- and length-scales of gas-dust mixture dynamics, see the discussion of the general equations in Z19. An important parameter to be used below is the dimensionless stopping time akin to $\tau_0$

$$\tau \equiv t_s \Omega_0.$$  

(9)

### 2.1 Stationary solution

The dust settling solution is described by the following solution of equations (1-4), see Z19:

$$U = 0,$$  

(10)

$$\nabla (p + p_0) = g,$$  

(11)

$$V = t_s g,$$  

(12)

where $g = -g_s e_s$ is the vertical gravitational acceleration. Hence, the radial drift of the particles is neglected in this study.

Finally, equation (3) combined with equation (12) implies that $\rho_p = \text{const}$ on the local scale considered here.

### 2.2 Weakly non-linear equations for axisymmetric gas-dust perturbations

The Eulerian perturbation of the centre-of-mass velocity, $u$, the Eulerian perturbation of enthalpy of gas-dust mixture, $W \equiv p'/\rho$, where $p'$ is the Eulerian perturbation of gas pressure, and the relative perturbation of the dust density, $\delta \equiv \rho_p/\rho_p$, where $\rho_p$ is the Eulerian perturbation of the dust density, are imposed on the background given by equations (10-13). These perturbations obey the following weakly non-linear equations

$$\partial_t u - 2u_y e_x + \frac{\kappa^2}{2} u_x e_y + (u \cdot \nabla)u = -\nabla W - \frac{f}{\tau} \delta e_z + f \delta \nabla W,$$  

(14)

$$\partial_t \delta = -2\tau \partial_x u_y + (1 - f) \partial_x \delta + \tau \sum_{i,k=1} \partial_i u_i \partial_k u_k - \nabla \cdot (\delta u) f \partial_x \delta^2.$$  

(15)

The sum in the right-hand side (RHS) of equation (15) is done over $x$- and $z$-projections of the velocity perturbation. In equations (14-15) and below velocity, time and distance are measured in units of $g_s t_{\text{es}}, \Omega_0^{-1}$ and $g_s t_{\text{es}}/\Omega_0$, respectively. The dimensionless epicyclic frequency squared is denoted by $\kappa^2 \equiv 2(2 - q)$.

Equations (14-15) are derived up to the second order in the small ratios of the amplitudes of perturbations to the corresponding background quantities1. Equations (14-15) are valid up to the linear order in the small dust fraction

$$f \equiv \frac{\rho_p}{\rho_0} \ll 1$$  

(16)

as the terms of higher order in $f$ has been omitted. By the same reason, $W$ entering the non-linear term in RHS of equation (14) can be excluded according to the relation derived from the divergence of equation (14),

$$W \approx 2\nabla \cdot \partial_x u_y,$$  

(17)

which is valid to the zeroth order in $f$.

1 Note that the term $\tau \nabla^2 W$ coming to the right-hand side of equation (15) from equations (2-3) written for perturbations was excluded using the divergence of equation (14) up to the leading order in $\tau$. 

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3 THE LINEAR PROBLEM

3.1 Dispersion equation
Let perturbations be described by the vector of the state variables
\[ \chi \equiv \{ u_x, u_y, u_z, \delta \}. \] (18)
The particular solution for infinitesimal perturbations should be sought in the form of the Fourier harmonics
\[ \chi^i = \hat{\chi}^i \exp(-i\omega t + ikx), \] (19)
where \( kx = k_s x + k_z z. \)

The linearised equations (14-15) yield that \( \hat{\chi}^i \) obey the following linear algebraic set of equations2:
\[ \omega k_z \hat{u}_x - \omega k_z \hat{u}_z = 2 ik k_s \hat{u}_y + i \frac{f}{\tau} k_x \delta, \] (20)
\[ \omega k_z \hat{u}_y = \frac{ik^2}{2} k_s \hat{u}_x, \] (21)
\[ \omega k_z \hat{u}_z = \frac{ik^2}{2} k_s \hat{u}_x, \] (22)
\[ -i\omega \hat{\delta} = -2i\tau k_s \hat{u}_y + i(1 - f)k_s \delta, \] (23)
which give the dispersion equation
\[ D^+_{\omega}(\omega, k) \cdot D^-_{\omega}(\omega, k) \cdot D_{\omega}(\omega, k) = \epsilon(k), \] (24)

where
\[ D^+_{\omega}(\omega, k) \equiv \omega \mp \omega_i, \] (25)
\[ D^-_{\omega}(\omega, k) \equiv \omega - \omega_p, \] (26)
\[ \epsilon(k) \equiv f k^2 \frac{k_z^2}{k^2}k_s, \] (27)
with \( k^2 \equiv k_x^2 + k_z^2, \omega_p \equiv -k_s (1 - f), \) and \( \omega_i = k_s z / k. \)

As \( \epsilon \to 0, \) equation (24) splits into three separate dispersion equations explicitly given by the definitions (25-26), which describe two oppositely propagating IW and one SDW, see the details in Z19.

Equations (20-23) reduce to equations (31-34) of Z19 in the vicinity of the linear resonance, or alternatively, the mode crossing between SDW and IW, where the non-resonant correction \( \propto f \delta \) to the dynamics of SDW in RHS of equation (23) becomes small compared to the coupling term in RHS of the dispersion equation (24).

In the opposite case, when considering the solution of equation (24) sufficiently far from the mode crossing, although staying within the limit of the small dust fraction, this non-resonant correction should be retained along with the (non-resonant) correction due to the coupling term, \( \epsilon. \) Below, such a linear solution will be referred to as the non-resonant one in contrast to the resonant solution considered in Section 3.5 of Z19. It turns out that the resonant solution describes the growing modes of DSI with growth rates \( \propto \epsilon^{1/2}, \) while the non-resonant solution describes the neutral modes with real corrections to the frequencies \( \propto \epsilon, \) see the next Section.

3.2 Approximate non-resonant solution in the limit of small \( f \)
The solution of equation (24) can be sought in the form
\[ \omega = \pm \omega_i + \Delta_\pm \] (28)
as well as
\[ \omega = \omega_p + \Delta_p, \] (29)
where it is assumed that the corrections are small,
\[ |\Delta_\pm| \ll \omega_i, \] (30)
\[ |\Delta_p| \ll |\omega_p|. \] (31)

In this case, equation (24) can be reduced to the corresponding quadratic equations with respect to \( \Delta_p \) and \( \Delta_\pm. \) They read
\[ (\omega_p^2 - \omega_i^2 + 2\omega_p \Delta_p) \Delta_p = \epsilon \] (32)
and
\[ \pm 2\omega_i \Delta_\pm (\pm \omega_i - \omega_p + \Delta_\pm) = \epsilon. \] (33)

In the limit \( \pm \omega_i \to \omega_p, \) equations (32-33) have the same resonant solution \( \propto \epsilon^{1/2}, \) which describes DSI. In the opposite limit, when the solution to equations (32-33) is considered far from the mode crossing between SDW and IW, \( \pm \omega_i = \omega_p, \) so that
\[ 2\epsilon \ll (\omega_i \mp \omega_p)^2 \] (34)
for \( \Delta_\pm \) and
\[ 8\epsilon |\omega_p| \ll (\omega_i^2 - \omega_p^2)^2 \] (35)
for \( \Delta_p, \) respectively, one obtains to leading order in \( f \)
\[ \Delta_\pm \approx \frac{\epsilon}{2(\omega_i \mp \omega_p)} \omega_i \] (36)
and
\[ \Delta_p \approx \frac{\epsilon}{\omega_p^2 - \omega_i^2}. \] (37)
where it can be assumed that \( \omega_p = -k_s. \)

The inequalities (30-31) and (34-35) will be used in Section 7 to evaluate the bounds of the analytical model.

In this way, one finds the approximate solutions of equation (24) taking into account the non-zero coupling between SDW and IW provided that it occurs sufficiently far from the mode crossing. In what follows, it is assumed that \( k_s > 0. \) For the given wavenumber, \( k, \) explicitly,
\[ \omega(\omega, k) = \omega_p + f \tilde{\kappa}^2 \frac{k_z^2}{k^2} \left( 1 - \frac{\tilde{\kappa}^2}{k^2} \right)^{-1} \] (38)
corresponding to the slightly modified counterpart of SDW, and
\[ \omega(\omega, k)[2, \omega] \approx \omega_i \mp \frac{f}{2} \tilde{\kappa}^2 \frac{k_z^2}{k^2} \left( 1 + \frac{\tilde{\kappa}^2}{k^2} \right)^{-1} \] (39)
corresponding to the slightly modified counterparts of IW− and IW+ propagating, respectively, in the same and the opposite sense as SDW. Hereafter, the modes with frequencies (38) and (39) will be referred to as SDW and IW+, respectively. Note that, here and below, the index 1, 2, 3 in square brackets in the left-hand side of equations (38-39) stands, respectively, for SDW, IW− and IW+.
4 THREE-WAVE RESONANCE

Let the resonant triad contain one negative energy SDW, which will be additionally marked with an index ′ below. In this case, in order for the resonant triad to satisfy the conditions of explosive instability, the two remaining waves should have positive energies and propagate in the same sense as SDW. Therefore, along with SDW the explosive resonant triad should contain two IW′. These waves will be denoted as IW′ 1 and IW′ 2 hereafter. Thus, the three-wave resonance considered in this work has the following form

\[ SDW_r \rightarrow IW'_1 + IW''_1. \]  

(40)

The waves entering (40) must satisfy the following condition of three-wave resonance

\[ \omega(k^*)[1] = \omega(k')[2] + \omega(k'')[2], \]  

(41)

where

\[ k^* = k' + k'' \]  

(42)

are the wavenumbers to be determined. This condition comes from the requirement that each of three modes matches both spatial and temporal periodicities of the driving force arising due to interaction between the other two modes, see e.g. Kadomtsev & Karpman (1971) and Craik (1988).

It is not difficult to consider the existence of resonant triads (40) on the plane of wavenumbers in the limit of negligible \( f \rightarrow 0 \). Let \( \theta^*, \theta' \) and \( \theta'' \) be the angles of \( k^*, k' \) and \( k'' \), respectively, with respect to the radial direction. Equations (41) and (42) yield in this case

\[ \frac{k^*}{\sin \theta^*} = \frac{k'}{\sin \theta'} + \frac{k''}{\sin \theta''}, \]  

(43)

\[ k' \cos \theta' + k'' \cos \theta'' = k \cos \theta^*. \]  

(44)

The set of equations (43) has the following solution provided that the wave angles are known

\[ k^* = 2k \cos \left( \frac{\beta + \beta''}{2} \right), \]  

\[ k' = k^* \frac{\sin(\beta')}{\sin(\beta + \beta'' / 2)}, \]  

\[ k'' = k^* \frac{\sin(\beta'')}{\sin(\beta + \beta' / 2)}, \]  

(44)

where \( k^*, k' \) and \( k'' \) stand for the wavenumbers of SDW, IW′ 1 and IW′ 2, respectively, while \( \beta' = \theta' - \theta^* \) and \( \beta'' = \theta'' - \theta^* \) are the angles between the resonant waves. In order to satisfy the general conditions of explosive instability, the resonant triad (44) is obtained assuming that \( 0 < \theta''' < \theta^* < \theta' < \pi \). It can be seen that the existence of this triad is quite general. It follows that the wavenumber of SDW, covers the range \( (0, 2k) \), while the wavenumbers of the two other resonant waves take values from 0 up to the wavenumber of SDW, depending on the ratio between the angles \( \beta' \) and \( \beta'' \). At the same time, as soon as the angles \( \beta' \) and \( \beta'' \) are specified, the resonant triad exists regardless of the direction of wave propagation.

The solution (44) should be used to find the small corrections to \( k^*, k' \) and \( k'' \) caused by the non-zero dust fraction. These are determined by an equation (41) combined with the approximate frequencies (38-39). This exercise is straightforward, however, it seems to entail formidable algebraic calculations of interaction between the resonant waves, see Section 5. As the present study aims to treat the interactions within the resonant triad by the analytical means, it is confined below to a simple particular case of collinear waves, \( \beta', \beta'' \rightarrow 0 \). For the special choice \( a/\beta' = (1 - a)\beta'' \), the solution (44) yields

\[ k' = a k^*, \quad k'' = (1 - a) k^* \quad \text{and} \quad |k^*| = 2 \tilde{k}, \]  

(45)

where the dimensionless free parameter \( a \) is assumed to be enclosed in the range \( 0 < a < 1/2 \). Thus, IW′ 1 and IW′ 2 are both collinear to SDW, but \( k > k' > k'' \), see the left panel in Figure 1. Note that as \( a \rightarrow 1/2 \), \( k' \rightarrow k'' \rightarrow k_{DSI} \), where \( k_{DSI} = \tilde{k} \), which means that IW′ 1 and IW′ 2 approach each other at the mode crossing with another SDW defined by the equality \( -\omega = \omega_p \). In this study \( a \) cannot take value too close to 1/2 for the sake of possibility of the analytical treatment, see Section 3.2. Note that for waves from the solution (44) as well as (45) the requirement \( a \ll 1 \) is sufficient to satisfy the conditions (7-8) of the terminal velocity approximation.

It is not difficult to see that interaction between the waves of the resonant triad vanishes for \( f \rightarrow 0 \), since in this limit the operator u ∇ acting on any field collinear to u results with the zero value for the divergence-free u, what implies that non-linear terms in equations (14-15) vanish.

As soon as the dust fraction is not negligible, \( f > 0 \), SDW, should interact with slightly non-collinear IW′ 1 and IW′ 2, having wavenumbers \( k' + \Delta k \) and \( k'' - \Delta k \), respectively, see the right panel in Figure 1. To keep the notations simple, everywhere below it is assumed that \( k' \) and \( k'' \) contain the small correction \( \Delta k \), which is proportional to \( f \). This correction is derived from equation (41) combined with equations (38), (39) and (45) to leading order in the small dust fraction. The corresponding small differences to the resonant frequencies

\[ \Delta \omega_{SDW} = \Delta \omega_{IW1} + \Delta \omega_{IW''}. \]  

(46)

are explicitly

\[ \Delta \omega_{SDW} = f k^* \left( 1 + \frac{k''^2}{3k^2} \right), \]  

(47)

\[ \Delta \omega_{IW1} = \frac{a}{1 - 2a} \frac{k^2}{2k^2} - \frac{k^2}{8a^2k^2} (k^* \Delta k_z - k'' \Delta k_x) \]  

(48)

and \( \Delta \omega_{IW''} \) is given by equation (48) with the replacement \( a \rightarrow 1 - a \). Equation (46) leads to the following condition on \( \Delta k \) specified by its own cross product with \( k^* \):

\[ k^* \Delta k_z - k'' \Delta k_x = f \tilde{k}^2 \frac{8a(1-a)}{2a - 1} \left( \frac{k^*}{k^2} + \frac{5k^4}{6k^6} \right). \]  

(49)

In order to obtain the frequencies of the resonant triad for \( f > 0 \), one should use equations (38) and (39) in combination with the conditions (45) and (49), see Appendix B for the resulting...
expressions. With the triad frequencies at hand, the linear equations (20-23) provide the eigen-vectors, \( \chi \), corresponding to SDW, IW', and IW''\(_s\), see Appendix C for the resulting expressions.

Complementary eigen-frequencies and eigen-vectors provided, respectively, by equations (38 - 39) and (20-23) at each of the triad wavenumbers are obtained in a similar way, see Appendices B and C. For the triad wavenumbers, the corresponding sets of eigen-vectors construct the three different bases necessary to obtain the coupling coefficients of wave-wave interaction, see the next Section.

5 INTERACTION OF WAVES

The finite-amplitude perturbations can be considered in the form of the spatial Fourier harmonics

\[ \chi^i = \psi^i_k \exp(ikx) \]

at the arbitrary wavenumber \( k \).

It follows from equations (14-15) that \( \psi_k \), satisfies an equation

\[ \partial_t \psi^i_k = \sum_{j=1}^{3} L^i_{k, j} \psi^j_k + \frac{1}{2\pi} \sum_{j,k=1}^{3} \int N^i_{k, j, k} \psi^j_{\bar{k}} \psi^0_{\bar{l}} \ dl, \]

where \( L^i_{k, j} \) specifies the linear dynamics of perturbations, while \( N^i_{k, j, k} \) describes their quadratic interaction. Explicitly,

\[ L^i_{k, j} = \begin{pmatrix} 0 & 2k_x^2 & 0 & \frac{k_x k_y}{k^2} \\ -\frac{k_y}{k^2} & 0 & 0 & \frac{k_y k_z}{k^2} \\ -\frac{k_z}{k^2} & 0 & 0 & \frac{k_z k_x}{k^2} \\ 0 & \frac{k_z}{k^2} & \frac{k_x}{k^2} & 0 \end{pmatrix}, \]

\[ N^1_{k, j, k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -i k_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ N^2_{k, j, k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ N^3_{k, j, k} = \begin{pmatrix} -\tau(k_x - l_x) i k_x & 0 & 0 & 0 \\ 0 & -\tau(k_x - l_x) i k_x & 0 & 0 \\ 0 & 0 & -\tau(k_x - l_x) i k_x & 0 \\ 0 & 0 & 0 & i f k_x \end{pmatrix}. \]

The general solution of equation (50) can be sought as the combination of linear modes

\[ \psi^i_k = \sum_{s=1}^{3} A^i_{k}[s] (t) \phi^i_{k}[s] \exp(-i \omega_k[s] t), \]

with amplitudes \( A^i_{k}[s] \) evolving due to interaction of modes at different wavenumbers. Note that, in accordance with the notations in equations (38-39), the values of the new index in the square brackets, \( s = 1, 2, 3 \), correspond to SDW, IW', and IW''\(_s\), respectively.

Written in this way, vectors \( \phi^i_{k}[s] \) are equivalent to \( \chi^i \) being the solution of the linear system (20-23). Consequently, they are the eigen-vectors of \( L^i_{k, j} \),

\[ -i \omega_k[s] \phi^i_{k}[s] = \sum_{j=1}^{4} L^i_{k, j} \phi^j_{k}[s], \]

(57)

These eigen-vectors construct basis in the linear space of vectors \( \psi \) corresponding to the particular wavenumber. Note that equation (57) is satisfied separately for each \( s = 1, 2, 3 \).

Equation (50) comes to

\[ \sum_{s=1}^{3} \frac{\partial_t A_{k}[s] \phi^i_{k}[s] \exp(-i \omega_k[s] t)}{2\pi} = \frac{1}{2\pi} \sum_{p,q=1}^{3} \int N^i_{k, l, k} A_{k-[p]} A_{l-[q]} \phi^i_{k-[p]} \phi^j_{l-[q]} \exp(-i \omega_{k-[p]} t - i \omega_{l-[q]} t) dl, \]

(58)

The linear space of vectors represented by equation (56) can be normalised according to the following inner product

\[ \langle \psi_1, \psi_2 \rangle = \sum_{i=1}^{4} \psi_1^i(k) \psi_2^*_{2k}, \]

(59)

where the asterisk denotes complex conjugation. The rule (59) allows one to introduce the dual basis

\[ (\phi_{[p]}, \tilde{\phi}_{[q]}) = \delta_{pq}, \]

(60)

where \( \delta_{pq} \) is the Kronecker delta.

Having the vectors of dual basis at hand, one finds its inner product with equation (58) in the following form

\[ \partial_t A_{k}[s] \phi^i_{k}[s] \exp(-i \omega_k[s] t) = \frac{1}{2\pi} \sum_{p,q=1}^{3} \int Q_{k, k-[p]} [\phi^i_{k-[p]} \phi^j_{l-[q]} \exp(-i \omega_{k-[p]} t - i \omega_{l-[q]} t)] dl, \]

(61)

where

\[ Q_{k, k-[p]} [\phi^i_{k-[p]} \phi^j_{l-[q]}] \equiv \sum_{i,j,k=1}^{3} N^i_{k, l, k} \tilde{\phi}^i_{k}[s] \phi^j_{l-[q]} \phi^s_{l-[p]} \phi^0_{l-[q]}, \]

(62)

From now on, it is assumed that the modes are excited only at the wavenumbers of the three-wave resonance proposed in Section 4. The eigen-vectors \( \phi^i_{k}[s][p], \phi^i_{k}[s][q] \) and \( \phi^j_{k}[s][p] \) obtained at these wavenumbers can be found in Appendix C. Accordingly, the necessary vectors of dual basis are given in Appendix D. Equation (61) yields the following set of evolutionary equations for amplitudes of the resonant triad. First, for SDW,

\[ \partial_t A_{k}[s][1] = Q_1 A_{k}[s][2] A_{k}[s][2], \]

(63)

where

\[ Q_1 = \frac{1}{2\pi} \sum_{i,j,k=1}^{4} \left[ N^i_{k, k'[s]} j k' \phi^i_{k}[s][j] \phi^j_{k}[s][j] + N^i_{k, k'[s]} j k' \phi^i_{k}[s][j] \phi^j_{k}[s][j] \right] \tilde{\phi}^i_{k}[s][1]. \]

(64)

Next, for IW',

\[ \partial_t A_{k}[s][2] = Q_2 A_{k}[s][1] A_{k}[s][2], \]

(65)
Explosive instability of gas-dust mixture

where

\[ Q_2 = \frac{1}{2\pi} \sum_{i,j,k=1}^{4} \left[ N_i^{(k^*)} N_k^{(j^*)} \phi_{k^* i}^{(j^*)} \phi_{i k^*}^{(j^*)} + N_k^{(j^*)} N_j^{(k^*)} \phi_{j^* k}^{(k^*)} \phi_{k j^*}^{(k^*)} \right] \] (66)

Last, for \( IW'' \)

\[ \partial_t A_{k^* [j]} = Q_3 A_{k^* [j]} A_{k^* [j]}, \] (67)

where

\[ Q_3 = \frac{1}{2\pi} \sum_{i,j,k=1}^{4} \left[ N_i^{(k^*)} N_k^{(j^*)} \phi_{k^* i}^{(j^*)} \phi_{i k^*}^{(j^*)} + N_k^{(j^*)} N_j^{(k^*)} \phi_{j^* k}^{(k^*)} \phi_{k j^*}^{(k^*)} \right] \] (68)

In order to shorten the notations further on, the following changes are made

\[ A_{k^* [j]} \rightarrow A_1, \]
\[ A_{k^* [j]} \rightarrow A_2, \]
\[ A_{k^* [j]} \rightarrow A_3, \]

which end up in equations for three-wave resonance

\[ \partial_t A_1 = Q_1 A_2 A_3, \]
\[ \partial_t A_2 = Q_2 A_1 A_3, \]
\[ \partial_t A_3 = Q_3 A_2 A_1. \] (69)

Note that \( A_1 \) is the relative dust density perturbation, \( \delta \), in SDW, while \( A_2 \) and \( A_3 \) are the radial projections of the velocity perturbation, \( \hat{u} \), in \( IW' \) and \( IW'' \), respectively.

Derivation of the coupling coefficients is straightforward and its details can be found in Appendix E. Explicitly,

\[ Q_1 = -\frac{10}{3\pi} f^2 \kappa^3 a(1 - a) \left( \frac{20 \kappa^2 + k_2^2}{2a - 1} \right), \] (70)

\[ Q_2 = Q_3 = \frac{1}{3\pi} f^2 \kappa^3 a(1 - a) \left( \frac{20 \kappa^2 + k_2^2}{2a - 1} \right). \] (71)

Note that the superscript \( \bullet \) after \( k_2 \) and \( k_3 \) is omitted in equations (70), (71) and everywhere below.

Examination of equations (64), (66), (68) along with the structure of eigen-vectors, see Appendixes C and D, reveals the general physics of the interaction between the resonant waves, which is illustrated by the scheme in Figure 2. Namely, the first equation of (69) drives the perturbation of the dust density in SDW, caused mainly by the second non-linear term in equation (15) being the product of the velocity perturbation of one \( IW' \), and the dust density perturbation of the other \( IW' \), induced aerodynamically by the gradient of its own perturbation of pressure. The second and the third equations of (69) drive perturbations of velocity in either \( IW' \), caused mainly by the advection term in the left-hand side of equation (14) being the product of the velocity perturbation of the other \( IW' \) and the velocity perturbation of SDW, induced by its own perturbation of the dust density via the dust back reaction on the gas.

There is a caution concerning the dependence of the coupling coefficients on the dust fraction. Indeed, an extra order of \( f \) in all \( Q_{1, 2, 3} \) comes from the geometry of the triad, i.e. its collinearity in the limit \( f \rightarrow 0 \). The coupling coefficients driving \( IW'' \) of some other triad (40) consisting of the non-collinear waves in the zeroth order in \( f \) should be as small as \( Q_{2, 3} \sim f \), while SDW, should be driven already in the limit of negligible \( f \rightarrow 0 \).

In this work, perturbation of velocity and therefore the amplitudes of \( IW' \) are measured in units of the dust settling velocity. In general, this is a physically justified choice, since the restriction \( u \lesssim 1 \) implies that the leading non-linear terms, \( (u \cdot \nabla) u \) in equation (14) and \( \nabla \cdot (u \delta u) \) in equation (15), are weaker than the linear terms in these equations at \( t_{sw} \sim \Omega_0^{-1} \) and \( t_{sw} \sim g_{d} t_{sw}, \Omega_0 \) corresponding, respectively, to time- and length-scales of the resonant triad. Therefore, \( u \lesssim 1 \) corresponds to a weakly non-linear regime of perturbation dynamics. However, this restriction seems to be excessive in the particular case of perturbations, which construct SDW, \( IW' \) and \( IW'' \). That is it by the following reasons. First, the leading non-linear terms vanish for a single incompressible linear mode due to \( \nabla \cdot u = 0 \), while the non-vanishing terms are smaller at least by factor \( f \ll 1 \), see the last terms in RHS of equations (14) and (15). Consequently, \( IW' \) or \( IW'' \) alone can safely propagate in a mixture with amplitudes \( u > 1 \). This is even more true for SDW, as its perturbation velocity induced by the dust back reaction on gas \( \sim f \). Second, interaction between the resonant modes is weakened by the factors either \( \sim \tau \) or \( \sim f \) because it proceeds, respectively, either via aerodynamical concentration of dust or via the dust back reaction on gas, see Figure 2. Moreover, an additional factor of \( f \) weakening their interaction comes from the collinearity of the particular resonant triad considered in this work, see the caution made here above. In this situation, it is plausible to measure velocity perturbations in the usual units of sound speed, i.e. \( c_s \equiv g_{d}/\Omega_0 \), which is larger than the settling rate by factor \( \tau^{-1} \).

The change to the units of \( c_s \) leads to the replacement \( Q_{1} \rightarrow Q_{1}/\tau^{2} \) in the first equation of (69). That is, all coupling coefficients diverge as \( \tau \rightarrow 0 \) implying that the resonant interaction of modes becomes infinitely strong for small particles in spite of the vanishing settling as well as aerodynamic clumping. However, as \( \tau \rightarrow 0 \), the dimensional wavelengths of resonant waves, \( \sim g_{d} t_{sw}/\Omega_0 \), vanish. Thus, the corresponding unbounded amplification of resonant interaction is associated with the gradient standing in the leading non-linear terms of equations (14)- (15) and working at the vanishing scale of the three-wave resonance. Dissipative forces existing in a real disc should suppress the interaction of waves at scales smaller than some threshold scale. The corresponding threshold value of \( \tau \equiv \tau_* \) is estimated below in Section 7.3. Note that the units of \( c_s \) for the amplitudes of velocity perturbations will be used in Section 7.2 and there below.

Equations (70) and (71) show that \( Q_{1, 2, 3} \) are negative-definite for the accepted values of \( a \). In this case, the following replacement

\[ A_1 \rightarrow \frac{-\tilde{A}_1}{\sqrt{-Q_2 \sqrt{-Q_3}}} \]
\[ A_2 \rightarrow \frac{-\tilde{A}_2}{\sqrt{-Q_1 \sqrt{-Q_3}}} \]
\[ A_3 \rightarrow \frac{-\tilde{A}_3}{\sqrt{-Q_1 \sqrt{-Q_2}}} \] (72)

leads to the set of equations of the standard type

\[ \partial_t \tilde{A}_1 = \tilde{A}_2 \tilde{A}_3, \]
\[ \partial_t \tilde{A}_2 = \tilde{A}_1 \tilde{A}_3, \]
\[ \partial_t \tilde{A}_3 = \tilde{A}_1 \tilde{A}_2. \] (73)

The latter, of course, does not mean that the dynamical loop shown in Figure 2 persists for non-collinear triad in the limit \( f \rightarrow 0 \).
and it is assumed that each of $r_{1,2,3}$ equals to $\tilde{A}_{2,3,4}^2(0)$ in such an order that $r_3 > r_2 > r_1 > 0$. Since the solution (76) describes the case of real and negative $A_{1,2,3}$ only, the absolute values of these amplitudes are used hereafter $|A_{1,2,3}| = -A_{1,2,3}$. For brevity, the corresponding replacement $A_{1,2,3} \rightarrow -A_{1,2,3}$ is assumed everywhere below.

The solution (76) shows that as $t \rightarrow t_e$, the amplitudes of SDW, IW, and IW $\overset{\circ}{\tau}$ blow up to infinity irrespectively of their initial values, which is a manifestation of an explosive instability. The particular curves of $A_{1,2,3}(t)$ produced for feasible $\tau = 0.001$ and $f = 0.01$ can be found in Figures 3 and 4. As the initial amplitudes take the equally large values, the time of explosion is rather short being much less than the characteristic settling time $\sim \tau^{-1}$. It can be seen that the time of explosion becomes moderately larger while the only one amplitude remains dominant at $t = 0$. The dominant IW makes the amplitudes to blow up far longer than that for dominant SDW, though, it occurs still within the settling time. This difference is expected from the dependence of $Q_1$ and $Q_{2,3}$ on $\tau$ and $f$, see equations (70) and (71), which show that SDW interacts with either of IWs stronger than IW $\overset{\circ}{\tau}$ interacts with IW $\overset{\circ}{\tau}$. As discussed below the equations (70) and (71), this is because the amplitudes of IW $\overset{\circ}{\tau}$ and IW $\overset{\circ}{\tau}$ are bounded by the value of settling velocity $\propto \tau$ according to the units chosen in this work, and in particular, in Figures 3 and 4. Also, the curves in Figures 3 and 4 show that the growth rate of $A_{1,2,3}$ increases as amplitudes approach $t \rightarrow t_e$. This suggests that the final stage of explosive instability becomes inconsistent with the underlying terminal velocity approximation. See the next Section, which elucidates this issue.

5.1 Compliance with terminal velocity approximation

The characteristic time of gas-dust dynamics due to interaction of resonant waves becomes smaller as $t \rightarrow t_e$, what can be seen in Figures 3 and 4. Consequently, $\tau$, can be substantially larger than $\tau$, and the requirement $\tau \ll 1$ may become insufficient to satisfy the terminal velocity approximation, which is the case for non-interacting (linear) resonant waves, see Section 4. Let $t_e$ be defined as

$$t_e = \min \left\{ \frac{A_1}{A_1}, \frac{A_2}{A_2}, \frac{A_3}{A_3} \right\} > 2 \frac{A}{A}$$

(79)

The solution (76) has a simple asymptotics close to the time of explosion,

$$\tilde{A} \approx \frac{1}{(t_e - t)^2}$$

(80)

as soon as

$$\frac{1}{(t_e - t)^2} \gg r_3 - r_1$$

(81)

and additionally

$$r_3 - r_1 \sim O(r_3).$$

(82)

At the same time, the inverse Jacobi elliptic function in equation (78) takes value of order of unity under the condition (82), what leads to an order-of-magnitude estimate

$$t_e \sim (r_3 - r_1)^{-1/2}.$$

(83)
Equation (83) implies that the restriction (81) is equivalent to
\[ t_e \gg t_c - t. \tag{84} \]

Thus, it follows from equation (80) that \( t_{ev} \gg t_e - t \) under the restriction (84). An explosive growth of resonant waves under the terminal velocity approximation requires that \( t_{ev} \gg t_e \), which, in turn, implies the corresponding necessary condition
\[ \frac{t_e}{t_c} \ll 1. \tag{85} \]

This condition guarantees that the significant stage of an explosive growth of resonant triad proceeds under the terminal velocity approximation. On the other hand, the consideration above shows that the terminal velocity approximation is always violated sufficiently close to \( t_e \), what occurs when \( t_e - t \sim t_e \) and the corresponding growth factor of resonant waves attains the order of \( t_e/t_s \).

### 6 CONSERVATION OF ENERGY

The displacement for perturbations of gas-dust mixture can be introduced through the common kinematic relation with its centre-of-mass velocity
\[ \frac{d\xi}{dt} = u + (\xi \cdot \nabla) U. \tag{86} \]

Equation (86) yields the projections
\[ \partial_t \xi_x = u_x, \tag{87} \]
\[ \partial_t \xi_y = u_y - q\xi_z, \tag{88} \]
\[ \partial_t \xi_z = u_z, \tag{89} \]
which allow one to rewrite dynamical equations for linear gas-dust perturbations in terms of the displacement. Explicitly,
\[ \partial_t \xi_x + \tilde{\kappa}^2 \xi_x = c_s^2 \partial_x (\nabla \cdot \xi), \tag{90} \]
\[ \partial_t \xi_z = c_s^2 \partial_z (\nabla \cdot \xi) - \frac{\delta}{\tau} \partial_z D, \tag{91} \]
\[ \partial_t \xi_x = \tilde{\kappa} \partial_x (\nabla \cdot \xi) - \frac{\delta}{\tau} \partial_x D, \tag{92} \]
where \( \nabla \cdot \xi = \partial_x \xi_x + \partial_z \xi_z \) and it was additionally assumed that \( p' = c_s^2 \rho' \) with \( \rho' \) being the perturbation of the gas density. Note that equations (90-92) are considered in the limit of incompressible dynamics, i.e. it is assumed that \( c_s \to \infty \), whereas \( \rho' \to 0 \) and \( \nabla \cdot \xi = 0 \) leaving the pressure term in RHS of equations (90) and (91) finite. Equations (90-92) contain a new variable, by definition,
\[ \partial_D \equiv \delta. \tag{93} \]

Equations (90-92) follow from the requirement that the action
\[ S = \int \mathcal{L}(\chi', \partial_t \chi') \, d^3 x \, dt \tag{94} \]
with \( d^3 x = dx dy dz \) and the Lagrangian density \( \mathcal{L} \) be stationary with respect to arbitrary variations of \( \chi' \), which is defined as \( \chi' \equiv \{ \xi_x, \xi_z, D \} \) in this Section. If so, equations (90-92) are identical to the corresponding Euler-Lagrange equations produced by the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \left[ (\partial_t \xi_x)^2 + (\partial_t \xi_z)^2 - \kappa^2 \xi_x^2 \right] - \frac{c_s^2}{2} \left[ (\partial_x \xi_x)^2 + 2 \delta \xi_x \xi_z + (\partial_z \xi_z)^2 \right] + \frac{f}{\tau} \left[ (1 - f) \frac{\delta^2}{2\tau^2 \kappa^2} - \frac{\delta \partial_t D}{2\tau^2} + D \partial_t \xi_z \right]. \tag{95} \]

Note that the thermal terms in the second square brackets in equation (95) vanish in the considered incompressible limit.

The symmetry of \( \mathcal{L} \) with respect to translations in time leads to conservation of energy, \( E \equiv \int \mathcal{E} \, d^3 x \), where the energy density of perturbations, \( \mathcal{E} \), is introduced as
\[ \mathcal{E} = -L + \frac{\delta \mathcal{L}}{\delta (\partial_t \chi_i)} \partial_t \chi_i. \tag{96} \]

Equation (95) yields
\[ \mathcal{E} = \frac{u_x^2}{2} + \frac{u_z^2}{2} + \frac{f}{\tau^2 \kappa^2} (1 - f) \frac{\delta^2}{2} - \frac{f}{\tau^2} D \partial_t \xi_z, \tag{96} \]
which is averaged over the mode phase in order to obtain the energy density of a plane wave,
\[ \mathcal{E} = \frac{u_x^2}{4} + \frac{u_z^2}{2} + \frac{f}{\tau^2 \kappa^2} (1 - f) \frac{\delta^2}{4} - \frac{f}{\tau \omega} \frac{\delta u_x}{2}. \tag{97} \]

To leading order in \( f \), equation (97) gives the following expressions
\[ \mathcal{E}_1 \approx -\frac{f}{\tau^2 \kappa^2} A_1^2 + O(f^2), \tag{98} \]
\[ \mathcal{E}_{2,3} \approx \frac{8 \kappa^2 A_{1,3}^2}{k_0^2} + O(f) \tag{98} \]
for SDW, IW', and IW", respectively.

Equations (98) combined with equations (69) show that the total energy of the resonant triad, \( \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \), is conserved during the explosive growth of waves. Therefore, explosive instability of the triad (40) is driven by the conservative transfer of energy from SDW, to both IW' and IW".

### 7 TIME OF EXPLOSION

The relevance of an explosive solution (76) in protoplanetary discs can be assessed by comparing the time of explosion, \( t_e \), with characteristic timescales in a disc.

Exact values of \( t_e \) in units of \( \Omega_*^{-1} \) according to equation (78) are shown in Figure 5 for the particular case of the amplitudes of resonant waves equal to each other. As expected, the time of explosion decreases for smaller particles, see discussion below the equations (70) and (71). Also, there is a sharp increase dependence of \( t_e \) on the dust fraction. As \( f \) increases up to 0.1, it becomes shorter than the Keplerian time. Additionally, \( t_e \) decreases for IW', approaching the linear resonance between IW and SDW, \( a \to 1/2 \). Note that the analytical derivation of interaction between the resonant waves breaks as \( a \to 1/2 \), see text below in this Section for the corresponding estimates.

\[ \chi' \equiv \{ \xi_x, \xi_z, D \} \]
Figure 3. Behaviour of the interacting resonant modes according to the solution (76). The amplitudes $A_1$, $A_2$ and $A_3$ are shown vs. time for their various initial values. Solid, dashed and dot-dashed curves stand for SDW$_r$, IW$_r'$ and IW$_r''$, respectively. Upon the increase of the time of explosion: \{ $A_1(0) = 0.3$, $A_2(0) = 0.3$, $A_3(0) = 0.3$ \}, \{ $A_1(0) = 0.3$, $A_2(0) = 0.3$, $A_3(0) = 0.03$ \}, \{ $A_1(0) = 0.3$, $A_2(0) = 0.03$, $A_3(0) = 0.003$ \}. The other parameters are $\kappa = 1$, $k_x = k_z = 0.01$, $\theta = 0.4$, $\tau = 0.001$, $f = 0.01$. The left (right) cross represents estimate of $t_e$ according to equation (99) taken for $A_1(0) = 0.3$ and $A_2(0) = 0.3(0.03)$.

Figure 4. Behaviour of the interacting resonant modes according to the solution (76). The amplitudes $A_1$, $A_2$ and $A_3$ are shown vs. time for their various initial values. Solid, dashed and dot-dashed curves stand for SDW$_r$, IW$_r'$ and IW$_r''$, respectively. Upon the increase of the time of explosion: \{ $A_1(0) = 0.01$, $A_2(0) = 0.3$, $A_3(0) = 0.3$ \}, \{ $A_1(0) = 0.003$, $A_2(0) = 0.3$, $A_3(0) = 0.3$ \}, \{ $A_1(0) = 0.01$, $A_2(0) = 0.3$, $A_3(0) = 0.01$ \}, \{ $A_1(0) = 0.003$, $A_2(0) = 0.3$, $A_3(0) = 0.003$ \}. The other parameters are $\kappa = 1$, $k_x = k_z = 0.01$, $\theta = 0.4$, $\tau = 0.001$, $f = 0.01$. The left (right) star represents estimate of $t_e$ according to equation (101) taken for $A_1(0) = 0.01(0.003)$ and $A_2(0) = 0.3$.

7.1 Analytical approximations of $t_e$

Simple estimates of $t_e$ can be made in the limiting case when the amplitudes of resonant waves strongly differ from each other at $t = 0$. That is, there is one dominant mode, while one of the rest minor modes prevails the other one. In this case, $r_3 \gg r_2 \gg r_1$ and the Jacobi function standing in equation (78) exhibits approximately logarithmic growth giving $t_e \approx r_3^{-1/2} \ln \left( \frac{4 \sqrt{r_3} / r_2}{4 \sqrt{r_3} / r_2} \right)$. Accordingly, the following general approximations of $t_e$ are obtained.

7.1.1 Dominant SDW$_r$ and sub-dominant IW$_r'$

$t_e \approx t_e^{SDW} \equiv \frac{3\pi \tau}{A_1(0) f^2} \frac{(2a - 1)^2}{a(1 - a)} \frac{k_z^2}{20k_x^2 + k_z^2} L^{SDW}$, \hspace{1cm} (99)

where

$L^{SDW} = \ln (\frac{A_1(0)}{A_2(0)} f^{1/2} k_z) \frac{\tau}{k_z^2}$

provided that

$A_2(0) \gg A_3(0)$

and

$\frac{A_1(0)}{A_2(0)} \gg \frac{4\pi}{f^{1/2} k_z}$

Note that as far as $\tau$ is sufficiently small, this regime can be valid also for comparable $A_1(0) \sim A_2(0)$, see the accordance of equation (99) with an exact analytical solution in Figure 3. The case of sub-dominant IW$_r''$ is considered similarly with the replacements $A_{2,3}(0) \rightarrow A_{3,2}(0)$.

7.1.2 Dominant IW$_r'$ and subdominant SDW$_r$

$t_e \approx t_e^{IW} \equiv \frac{3\pi}{4A_2(0)} \frac{1}{f^{1/2} k_z^2} \frac{(2a - 1)^2}{a(1 - a)} \frac{k_z^2}{20k_x^2 + k_z^2} L^{IW}_1$, \hspace{1cm} (101)

where

$L^{IW}_1 = \ln \left( \frac{16A_2(0)}{A_1(0)} \frac{\tau}{k_z} \right) f^{1/2}$

provided that

$\frac{A_1(0)}{A_2(0)} \gg \frac{4\pi}{f^{1/2} k_z}$

and

$\frac{A_2(0)}{A_1(0)} \gg \frac{f^{1/2} k_z}{4\pi}$

Note that as far as $\tau$ is sufficiently small, this regime can be valid also for comparable $A_1(0) \sim A_2(0)$, see the accordance of equation (99) with an exact analytical solution in Figure 3. The case of sub-dominant IW$_r''$ is considered similarly with the replacements $A_{2,3}(0) \rightarrow A_{3,2}(0)$.
Note that as far as $\tau$ is sufficiently small, this regime can be valid also for comparable $A_1(0) \sim A_3(0)$, while $A_2(0)$ should significantly exceed $A_1(0)$, see the accordance of equation (101) with an exact analytical solution in Figure 4.

### 7.1.3 Dominant IW’ and sub-dominant IW”

$$t_e \approx t_{e,2}' \equiv \frac{3\pi}{4A_2(0)} \left[ \frac{1}{f^{3/2}k^2} \left( \frac{2\alpha - 1}{2} \right)^{2} \frac{k_2}{a(1 - a)} \right] \frac{k_2}{\kappa} L_2^{\text{IW}}, \quad (103)$$

where

$$L_2^{\text{IW}} = \ln \left( \frac{4A_2(0)}{A_3(0)} \right) \quad (104)$$

provided that

$$\frac{A_3(0)}{A_1(0)} \gg \frac{f^{1/2}k_2}{4\pi k^2},$$

and

$$\frac{A_2(0)}{A_3(0)} \gg 1.$$  

Consideration of dominant IW’ is identical to Sections 7.1.2 and 7.1.3 with the replacements $A_{2,3}(0) \rightarrow A_{1,2}(0)$.

### 7.2 Lower estimates of $t_e$

As expected, $t_e$ becomes shorter as the inverse initial amplitude of the dominant mode. At the same time, it increases in gas-dust mixture with smaller dust fraction, however, becoming shorter for smaller particles$^6$, see the discussion below the equations (70-71).

The time of explosion can be additionally decreased for the triads containing inertial waves located closer to the band of DSI, so for $a \rightarrow 1/2$. Alternatively, time of explosion decreases for almost radially propagating resonant modes, $k_z \ll 1$. The corresponding lower estimate of $t_e$ can be obtained using the marginal condition of the validity of the analytical approximation employed in this work. Namely, the analytical form of waves involved in the resonant triad is valid under the restrictions (30-31) and (34-35). It can be checked that taken at the resonant wavenumbers they are satisfied together provided that the overall condition

$$\frac{4f\kappa^2}{(1 - 2\alpha)^2 k_2^2} \ll 1 \quad (105)$$

is true. An additional restriction comes from the condition that $\Delta k/k \ll 1$. Equation (49) yields

$$\frac{8f\kappa^2}{1 - 2\alpha} \left( \frac{1}{k_2^2} + \frac{5}{6} \frac{1}{k_2^2} \right) \ll 1. \quad (106)$$

The restrictions (105) and (106) put the lower limit on the value of $|2\alpha - 1|k_z$, or alternatively, on the values of $[2a - 1]$ and $k_z$ separately, which enter the numerators of equations (99), (101) and (103). They provide the lower estimates of $t_{e,SDW}^{SW}$ and $t_{e,SDW}^{\text{IW}}$. It is convenient to formulate these estimates separately for different cases defined by the ratio between $k_z$ and $k_e$.

Note that logarithmic factors entering equations (99), (101) and (103) are omitted in the following estimates, thus, $t_{e,1}^{\text{IW}} \approx t_{e,2}' \rightarrow t_{e,2}$.

### 7.3 Keplerian disc

The lower bound of the time of explosion over all limiting cases given above can be estimated in a Keplerian disc, where $k = 1$, as

Hereafter it is assumed that $A_{2,3}(0)$ is measured in units of $g_z/S_0$, see discussion below the equations (70-71) and the footnote 6.

### 7.2.1 Almost radially propagating modes $k_z \ll k_e$

$$t_{e,SDW}^{SW} \gtrsim \frac{48\pi}{5A_1(0)} \frac{k^2}{f^2}, \quad t_{e,SDW}^{\text{IW}} \gtrsim \frac{24\pi}{A_2(0)} \frac{k^2}{f}. \quad (107)$$

In equation (107) it is assumed that combinations of $a$ entering denominators of (99) and (101) take approximately their largest values at $0 < a < 1/2$.

There should be a caution about the lower estimate of $t_{e}^{\text{IW}}$ from equation (107), which formally provides the existence of an explosive instability for $f \rightarrow 0$. The inspection of the coupling coefficients shows that this issue originates from the divergence of $Q_1$ as soon as $f \rightarrow 0$ and $k_z \sim O(f^{1/2})$, which is marginally allowed by the non-resonant linear solution for IW, in the case of almost radially propagating waves. As was discussed in Section 5, the interaction between IW’ and IW” produced mainly by the velocity perturbation of one IW, and the aerodynamically induced perturbation of the dust density of the other IW’s. However, equation (23) indicates that the latter diverges in the limit $k_z \rightarrow 0$. Such a singularity should be removed with the account of the dust diffusion. The further study of an explosive instability should check that explosion time of the resonant triad consisting of the radially propagating waves tends to infinity as $f \rightarrow 0$ in the system with non-zero dust diffusion.

### 7.2.2 Almost vertically propagating modes $k_z \approx k_e$

$$t_{e,SDW}^{SW} \gtrsim \frac{8\pi}{A_1(0)} \frac{k^4}{f^2}, \quad t_{e,SDW}^{\text{IW}} \gtrsim \frac{4\pi}{A_2(0)} \frac{k^4}{f}. \quad (108)$$

From the derivation of equation (108) it follows that $a$ can be approximately set to $1/2$ in this case.

### 7.3 Modes with $k_z \approx k_e$

$$t_{e,SDW}^{SW} \gtrsim \frac{48\pi}{11A_1(0)} \frac{k^2}{f}, \quad t_{e,SDW}^{\text{IW}} \gtrsim \frac{48\sqrt{2\pi}}{44A_2(0)} \frac{k^2}{f^{3/2}}. \quad (109)$$

From the derivation of equation (109) it follows that $a$ can be approximately set to $1/2$ in this case.

It can be seen that for reasonable values of $f$ the estimates (108) give the least lower limits on the time of explosion in gas-dust mixture with small dust fraction. Equation (108) is used below in the next Section.
the following
\[ t_e^{SDW} > \tau_e^{SDW} \equiv 20 \frac{\tau}{A_1(0)f^{2/3}}, \]
\[ t_e^{IW} > \tau_e^{IW} \equiv 10 \frac{\tau}{A_2(0)f^{1/6}}, \]  
(110)

which is measured in units of the Keplerian time for initially dominant SDW $t_e$ and IW $t_e$, respectively. This choice corresponds to waves propagating almost vertically.

7.3.1 On excitation of dominant IW $t_e$

Equations (110) indicate that the case of dominant IW $t_e$ looks preferable to the case of dominant SDW, with respect to transition to an explosive instability, because of quite a weak dependence of the time of explosion on the dust fraction. Therefore, the question arises about an excitation of IW $t_e$, with the amplitude sufficient to trigger explosive instability.

One possibility is a preliminary linear growth of IW $t_e$ due to DSI. However, the resonant triad considered analytically in this work contains IW, located far from the linear resonance between IW and SDW, which gives rise to the leading order DSI, see the corresponding condition (34). Therefore, the leading order DSI cannot be responsible for the production of such a finite-amplitude IW $t_e$. Nevertheless, exact solution of the general dispersion equation in Squire & Hopkins (2018) shows that the particular curves of DSI growth rate have broad wings of the growth rate outside of the main band of DSI associated with the linear resonant coupling between SDW and IW. The following remains to be checked, however, such a 'residual' linear instability may occur due to some additional mechanism responsible for slow growth of the coupled IW and SDW in the non-resonant range of wavenumbers satisfying the condition (34). If so, IW at these wavenumbers may become subject for further explosive growth due to the non-linear resonant interaction with the corresponding seeded SDW and other IW having much smaller amplitudes. On the other hand, the condition (34) can be relaxed in the sequel studies of an explosive instability, in which case the coupling coefficients for three-wave resonance can be obtained numerically. Provided that explosive instability keeps its strength for IW also from the band of the leading order DSI considered by Z19, it may be exactly the non-linear stage of DSI.

Besides, IW are known to be excited in turbulent rotating fluids. This process has long been observed and simulated in the laboratory tanks, see e.g. Hopfinger et al. (1982), Godeferd & Lollini (1999), Bewley et al. (2007), Lamriben et al. (2011). Some kind of turbulence pre-existing in the gas component including those generated by the shear of Keplerian motion may also be responsible for generation of finite amplitude IW subject to explosive instability. The energy spectrum of such waves in a disc must be a special issue. However, Kolmogorov cascade seems to be unsuitable to trigger an explosive instability. Indeed, the usual assumption that turbulence is characterised by the largest velocity fluctuations $V_1 \sim \alpha^{1/2} c_s$, at the outer scale $L_1 \sim \alpha^{1/2} h$, where $\alpha$ and $h$ are the disc scaleheight and the Shakura-Sunyaev parameter, leads to the turbulent velocity fluctuations $v_1 \sim \alpha^{1/3} \Omega h^{1/3} c_s$, and its correlation time $t_{corr} \sim \alpha^{-1/3} \Omega^{2/3} \Omega_0^{-1}$ evaluated at the scale of DSI, which is $k_{DSI} \sim 1/\langle \tau \rangle$7 in the dimensional form. If the amplitude entering $t_e^{IW}$ from equation (110) is supposed to be identified with $v_1$, the time of explosion of the corresponding explosive instability is found to be larger than $t_{corr}$. The latter implies that IW should disappear due to interaction with other modes of turbulent cascade before it could be amplified by some resonant SDW. Note that such a turbulent excitation of explosive instability would be possible at the sufficiently small scales corresponding to $\tau \lesssim \alpha^{1/2}$ as seen by comparing $L_1^{-1}$ with $k_{DSI}$. At large scales, $\tau \gtrsim \alpha^{1/2}$, turbulence has a damping effect. In this regime, the corresponding time of linear damping, $t_\nu$, must be larger than $t_{corr}$. The next Section is devoted to evaluation of the lower $\tau = \tau_{ei}$ corresponding to damping of explosive instability as far as $t_\nu \gtrsim t_{corr}$.

7.3.2 Viscous threshold for explosive instability

The threshold $\tau \equiv \tau_{ei}$ corresponding to damping of an explosive instability by dissipative processes in a disc can be estimated using the corresponding an order-of-magnitude condition known from the theory of resonant interaction between waves with linear damping, see e.g. Wilhelmsson et al. (1970) and Wilhelmsson (1970). As soon as $t_\nu \gtrsim t_\nu$, where $t_\nu$ is the characteristic time of linear damping, an explosive growth of waves does not exist anymore. The time of linear damping on the scale of resonant triad reads
\[ t_\nu \sim \frac{\tau^2}{\alpha}, \]  
(111)
where it is assumed that $\alpha$ characterises the disc effective viscosity via the common relation for kinematic viscosity $\nu = \alpha l_0 h^2$. Note that equation (111) is obtained according to the assumption that $g_z \simeq \Omega_0^2 h$, so takes its maximum value in a disc, which provides the largest $t_\nu$ for the given $\tau$. Accordingly, the smallest particles and the corresponding lower bound of the resonant length-scales subject to explosive instability in a viscous disc are introduced by
\[ \tau_{ei} \simeq \frac{10\alpha}{A_2(0)f^{1/6}}, \]  
(112)
which is obtained equating $\tilde{t}_e^{IW}$ from equation (110) with $t_\nu$. The corresponding lower bound for $\tilde{t}_e^{IW}$ reads
\[ \tilde{t}_e^{IW} \simeq \frac{100\alpha}{A_2(0)f^{1/6}}. \]  
(113)

7.3.3 Setting threshold for explosive instability

Conversely, the biggest particles and the corresponding upper bound of the resonant length-scales subject to explosive instability are specified by the restriction that the time of explosion cannot be larger than settling time of the particles, $t_{set} \simeq \tau^{-1}$. Equating $t_{set}$ with $\tilde{t}_e^{IW}$ from equation (110) one obtains the corresponding largest $\tau$ as the following
\[ \tau_{set} \simeq \frac{A_2(0)^{1/2} f^{1/12}}{10^{1/2}}. \]  
(114)

The corresponding upper bound for $\tilde{t}_e^{IW}$ reads
\[ \tilde{t}_e^{IW} \simeq \frac{10^{1/2}}{A_2(0)^{1/2} f^{1/12}}. \]  
(115)

7.3.4 Threshold for an amplitude exciting explosive instability

Generally, $t_e^{IW}$ and the corresponding $\tau$ cover the range
\[ \tau_{set} \lesssim \tau \lesssim \tau_{ei}, \]
\[ t_e^{IW} \simeq \tilde{t}_e^{IW} \simeq \tilde{t}_e^{IW}. \]  
(116)
In a weakly viscous disc with small $\alpha$ as well as for $A_{2,3}(0)$ close to unity, $\tau$ covers a range from the value much smaller than one up to the value slightly less than one. As the initial amplitude decreases, the allowed area for $\tau$ subject to an explosive instability reduces. The lower threshold value of $A_{2,3}(0)$ subject to an explosive instability is estimated from the condition $\tau_{\text{crit}} \gtrsim \tau_{\nu}$:

$$A_{2,3}(0) \gtrsim \frac{10\alpha^{2/3}}{f^{1/6}}. \quad (117)$$

Below this value the particular case of explosive instability considered in this work cannot exist in a viscous gas-dust medium. The corresponding marginal values of $\tau$ and $l_{\nu}^{\text{IW}}$ are, respectively, the following

$$\tau \simeq \alpha^{1/3}, \quad l_{\nu}^{\text{IW}} \simeq \alpha^{-1/3}. \quad (118)$$

Note that the formal restriction of the weakly non-linear theory, $A_{2,3}(0) \lesssim 1$, provides the upper limit of the viscosity parameter,

$$\alpha \lesssim \frac{f^{1/4}}{10^{0.7}} \simeq 0.01, \quad (119)$$

obtained from equation (117) for $f \simeq 0.01$. This restriction also follows from the condition that $\tau_{\text{crit}} \gtrsim \tau_{\nu}$ or $l_{\text{crit}} \gtrsim l_{\nu}^{\text{IW}}$ taken with $A_{2,3}(0) \simeq 1$. Viscous discs where the restriction (119) is violated are stable with respect to an explosive instability.

### 8 CONCLUSIONS

This work is focused on possibility of weakly non-linear instability of gas-dust mixture with the dust settling through the horizontally rotating gas under the vertical hydrostatic equilibrium. It is revealed that such a flow is subject to explosive instability provided that the dust back reaction on gas is taken into account. The physics of instability is considered through the particular example of three-wave resonance among axisymmetric gas-dust waves, which are the counterparts of one SDW and two IW modified by the small amount of dust in a mixture. The fundamental reason that causes an explosive instability is the energy of SDW, which becomes negative provided that the dust settling is sufficiently fast, see Z19. This enables the unbounded growth of resonant waves, while the energy is transferred from SDW to two IW. At the same time, it is shown that interaction between the waves conserves the total energy of the resonant triad.

The main application of the considered model is a small patch of protoplanetary disc above the disc midplane. However, it can also be applied to other situations with the rotational profile different from the Keplerian one. For example, the considered model may be applied to local environments of the dust-laden envelopes of the rotating giant planets forming through the pebble accretion, see e.g., Johansen & Lambrechts (2017). Explosive instability of gas-dust mixture can be one more physical effect that accompanies complicated process of accretion of small solids inside the envelope, for the recent account see e.g. Popovas et al. (2018), Johansen & Nordlund (2020) and references therein. Indeed, the only parameter here, which describes the deviation from the rigid rotation is the dimensionless epicyclic frequency changing from two to one while replacing the rigid rotation by the Keplerian rotation. The coupling coefficients obtained for the resonant triad keep their negative signs, or equivalently, an explosive type irrespective of the rotational profile, whereas the analytical estimates exposed in Section 7 show that the time of explosion weakly depends on epicyclic frequency. Along with DSI, this makes explosive instability a generic process in rotating dusty astrophysical flows.

This work deals with particularly simple variant of resonant triad, which consists of collinear waves as $f \to 0$ and allows for fully analytical treatment. However, Section 4 also introduces the general resonant triad spanning a wide range of wavenumbers, $(0, 2k)$. Derivation of the non-linear coupling between the corresponding resonant waves is relegated to the future work. However, it should produce explosive instability widespread in phase space covering the linearly stable wavenumbers, where DSI is absent.

It was found that the conservative three-wave resonant interaction tends to infinity as $\tau \to 0$. This is explained by the shift of the resonant scales to infinitely small lengths, which makes the characteristic ‘non-linear frequency’ $\sim A/l_{\nu}^{\text{IW}}$ entering the non-linear terms of equations (14-15) diverge. In this situation, the lower spatial scale of three-wave resonance should be defined by viscous damping. The corresponding estimates lead to the overall conclusion that explosive instability can operate in discs with the usual dimensionless viscosity less than 0.01. The plausible scenario for transition of gas-dust mixture to explosive instability is briefly discussed in Section 7.3.1.

Subsequent studies of explosive instability in protoplanetary discs should be expanded to other resonances of an explosive type, which exist in gas-dust mixture with dust settling to the disc midplane. An important issue is to understand the possible connection between DSI and explosive instability, in particular, whether explosive instability can serve directly as the non-linear stage of DSI. The latter would imply that the dust overdensities can reach values at least comparable to the background value of the dust density. On that way, one of the necessary steps would be semi-analytical study of the particular three-wave resonance proposed in this work with IW, located close to the linear resonance with SDW, therefore, inside the band of the leading order DSI. Also, saturation of explosive instability should be examined employing the corresponding numerical simulations. The more challenging task is to study the saturation of explosive instability analytically employing the weakly non-linear theory in the third order over the amplitudes of SDW and IW. The corresponding resonant tetrads may define the fate of dust clumps as they attain sufficiently large amplitudes.

On the other side, the simple background solution used in this work should be generalised onto the settling of particles combined with their radial drift. The dispersion equation for SDW accounting for the dust radial drift may substantially affect the resonant triad as well as the interactions between the resonant waves. The settling of particles combined with their radial drift is the case considered by Squire & Hopkins (2018) as they found DSI. Later on, Z19 suggested that the corresponding small scale asymptotics of DSI exhibiting an unbound growth rate is produced by the triple linear coupling of one negative energy SDW with two positive energy IW. Krapp et al. (2020) studied dynamics of the non-linear gas-dust perturbations trying to determine the saturation level of the dust clumping after the linear growth of dust density perturbations caused by this branch of DSI. At the same time, the authors report that the large scale asymptotics of DSI determined solely by the settling of dust requires substantially higher numerical resolution, see their Figure C1 and also the left panel in their Figure 5. The latter suggests that simulations do not reproduce DSI for $k_{\parallel} \lesssim k_{\perp} = k_{\text{DSI}}$ even though the DSI growth rate must

\footnote{Here $A$ and $l_{\nu}^{\text{IW}}$ are, respectively, the velocity amplitude of resonant wave and resonant lengthscale.}
be of order of $\Omega$. Similarly, at least a longer wavelength mode of the resonant triad considered here may be numerically inhibited in the results of Krapp et al. (2020). Additionally, the start of simulations from white noise of velocity should involve IW, in interactions with multiple high-amplitude small-scale IW, which may suppress explosive instability similarly to the action of turbulent damping. Detailed analysis of the power spectrum of evolving perturbations is required in order to resolve these issues. Nevertheless, note that Krapp et al. (2020) obtained the dust overdensities much larger than the background value of the dust density reaching fully non-linear regime of the dust clumping in the majority of runs. It is possible that explosive instability operates at early stage of simulations. Whether its contribution to dust clumping or/and the transition to turbulence revealed by Krapp et al. (2020) is substantial should be addressed in the future work.

At last, the particular analytical solution obtained in this work can serve a good test for the numerical schemes employed to simulate the non-linear dynamics of gas-dust mixtures.

**DATA AVAILABILITY**

No new data were generated or analysed in support of this research.

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### APPENDIX A: DESCRIPTION OF MAIN VARIABLES

| Symbol | Meaning |
|--------|---------|
| $\rho_g$ | gas volume density |
| $\rho_p$ | dust volume density |
| $\rho$ | total density of gas-dust mixture |
| $f$ | dust fraction |
| $p$ | gas pressure |
| $U_g$ | velocity of gas |
| $U_p$ | velocity of dust |
| $U$ | centre-of-mass velocity of gas-dust mixture |
| $V$ | relative velocity of gas-dust mixture |
| $u$ | the Eulerian perturbation of $U$ |
| $\rho_p'$ | the Eulerian perturbation of $\rho_p$ |
| $p'$ | the Eulerian perturbation of $p$ |
| $\delta$ | the relative perturbation of $\rho_p$ |
| $p$ | gas pressure |
| $U_g$ | velocity of gas |
| $U_p$ | velocity of dust |
| $U$ | centre-of-mass velocity of gas-dust mixture |
| $V$ | relative velocity of gas-dust mixture |
| $\hat{u}_{x,y,z}$ | Fourier harmonics of $u_{x,y,z}$ |
| $\hat{\delta}$ | Fourier harmonic of $\delta$ |
| $\Omega_0$ | local angular velocity of disc |
| $g_z$ | vertical component of the stellar gravity |
| $q$ | local disc shear rate |
| $\kappa$ | epicyclic frequency in units of $\Omega$ |
| $t_s$ | particle stopping time |
| $\tau$ | the Stokes number |
| $k_x$ | radial wavenumber of mode |
| $k_z$ | vertical wavenumber of mode |
| $k$ | absolute value of wavenumber of mode |
| $\omega$ | the mode frequency |
| $\omega_i$ | frequency of inertial wave in the limit $f \to 0$ |
| $\omega_p$ | frequency of the streaming dust wave in the limit $f \to 0$ |
| $\epsilon$ | coupling term of the linear dispersion equation |
| $\Delta_p$ | non-resonant correction to $\omega_p$ due to the dust back reaction on gas |
| $\Delta_i^\pm$ | non-resonant correction to $\omega_i$ due to the dust back reaction on gas |
| $\hat{\delta}_{x,y,z}$ | Fourier harmonics of $\delta$ |
| $\hat{\delta}$ | Fourier harmonic of $\delta$ |
| $\Omega_{SDW_r}$ | resonant streaming dust wave |
| $\Omega_{IW_r'}$ | the first resonant inertial wave |
| $\Omega_{IW_r''}$ | the second resonant inertial wave |
| $k_{DSI}$ | wavenumber of the mode crossing, which gives rise to DSI |
| $\omega^*$ | wavenumber of $\Omega_{SDW_r}$ |
| $\omega_r'$ | wavenumber of $\Omega_{IW_r'}$ |
| $\omega_r''$ | wavenumber of $\Omega_{IW_r''}$ |
| $a$ | free parameter setting the resonant triad |
| $A_1$ | amplitude of the relative dust density perturbation in $\Omega_{SDW_r}$ |
| $A_2$ | amplitude of the radial projection of $u$ in $\Omega_{IW_r'}$ |
| $A_3$ | amplitude of the radial projection of $u$ in $\Omega_{IW_r''}$ |
| $Q_1$ | coefficient for non-linear coupling between $\Omega_{IW_r'}$ and $\Omega_{IW_r''}$ |
| $Q_2$ | coefficient for non-linear coupling between $\Omega_{SDW_r}$ and $\Omega_{IW_r'}$ |
| $Q_3$ | coefficient for non-linear coupling between $\Omega_{SDW_r}$ and $\Omega_{IW_r''}$ |
| $t_e^{SDW_r}$ | explosion time needed by $A_{1,2,3}$ to blow up to infinity |
| $t_e^{SDW_r}$ | the value of $t_e$ for the dominant $\Omega_{SDW_r}$ |
| $t_e^{SDW_r}$ | the lower estimate of $t_e^{SDW_r}$ |
| $t_e^{SDW_r}$ | the lower estimate of $t_e^{SDW_r}$ |
| $c_s$ | local speed of sound |
| $h$ | the disc scaleheight |
| $\alpha$ | the Shakura-Sunyaev viscosity parameter |
| $c_{st}$ | the lower estimate of $c_s$ |
| $\tau_v$ | the lower value of $\tau$ subject to explosive instability in a viscous disc |
| $\tau_{stl}$ | the upper value of $\tau$ comparable to the settling time |
| $\tau_{stl}$ | the upper value of $\tau$ subject to explosive instability quenched by fast dust settling |

Note that the superscript $\bullet$ is omitted after $k_x$ and $k_z$ throughout all Sections of the Appendix.
APPENDIX B: SET OF EIGEN-FREQUENCIES AT RESONANT WAVENUMBERS

I) At the wavenumber of SDW_r.

i) SDW

\[ \omega = -k_x \left( 1 - f - \frac{k_x^2}{3k_z^2} \right). \]  

(B1)

ii) IW^-

\[ \omega = -\frac{k_z}{2} \left( 1 + \frac{k_x^2}{k_z^2} \right). \]  

(B2)

iii) IW^+

\[ \omega = \frac{k_z}{2} \left( 1 + \frac{f k_x^2}{3 k_z^2} \right). \]  

(B3)

II) At the wavenumber of IW'_r.

i) SDW

\[ \omega = -(a k_z + \Delta k_z) + f a k_x \left( 1 + \frac{k_x^2}{k_z^2 (4a^2 - 1)} \right). \]  

(B4)

ii) IW^-

\[ \omega = \frac{k_z}{2} - f \frac{k_z}{2a - 1} \left( 1 - a \right) \left( \frac{k_x}{k_z} + \frac{5 k_k}{6 k_z} + \frac{k_x a}{k_z} \right). \]  

(B5)

iii) IW^+

\[ \omega = \frac{k_z}{2} + f k_x \left[ \frac{1 - a}{2a - 1} \left( \frac{k_x}{k_z} + \frac{5 k_k}{6 k_z} + \frac{k_x a}{k_z} \right) + \frac{k_x a}{2(2a + 1)} \right]. \]  

(B6)

At the wavenumber of IW''_r the frequencies are obtained by the replacement \( a \rightarrow 1 - a \) and \( \Delta k_z \rightarrow -\Delta k_z \) in equations (B4-B6).

APPENDIX C: SET OF EIGEN-VECTORS AT RESONANT WAVENUMBERS

I) At the wavenumber of SDW_r.

i) SDW

\[ \begin{pmatrix} -\frac{f}{3} \frac{k_x}{k_z} \\ \frac{f}{6} \frac{k_x}{k_z} \\ -\frac{1}{1} \end{pmatrix} \]  

(C1)

ii) IW^-

\[ \begin{pmatrix} \frac{1}{k_x} \\ \frac{i k_x^2}{k_z} \left( 1 - f \frac{k_x^2}{k_z^2} \right) \\ -\frac{k_x}{k_z} \\ 4i \tau k_x (1 + 2f) \end{pmatrix} \]  

(C2)
iii) IW⁺

\[ \left\{ \begin{array}{l}
- \frac{i \kappa^2}{k_x} \left( 1 - \frac{f k_x^2}{3f k_x^2} \right), \\
- \frac{k_x}{k_x}, \\
- \frac{4i \pi \kappa^2 k_x}{3k_x^2} \left( 1 - \frac{4 f k_x^2}{3f k_x^2} \right) \\
\end{array} \right. \]  \hspace{1cm} (C3)

II) At the wavenumber of IW⁺.

i) SDW

\[ \left\{ \begin{array}{l}
- \frac{f k_x}{\tau \kappa^2 4a^2 - 1}, \\
\frac{f}{\tau 2(4a^2 - 1) k_x}, \\
\frac{i f k_x^2}{\tau k_x \kappa^2 4a^2 - 1}, \\
1. \\
\end{array} \right. \]  \hspace{1cm} (C4)

ii) IW⁻

\[ \left\{ \begin{array}{l}
\frac{i \kappa^2}{k_x} \left( 1 - \frac{2f k_x}{(2a - 1) k_x} \right) \left( 1 - a \right) \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) + a \frac{k_x}{2 k_x} \right), \\
- \frac{k_x}{k_x} \left[ 1 - f \frac{8 \kappa^2 (1 - a)}{(2a - 1) k_x k_x} \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) \right], \\
\frac{4i \pi \kappa^2 a k_x}{(2a - 1) k_x^2} \left( 1 + \frac{4 f (1 - a) k_x}{(2a - 1)^2 k_x} \left( 1 - a \right) \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) + a \frac{k_x}{2 k_x} \right) + \frac{2 f a}{2a - 1} - \frac{2 \Delta k_x}{k_x} + \frac{\Delta k_x}{a k_x} \right). \\
\end{array} \right. \]  \hspace{1cm} (C5)

iii) IW⁺

\[ \left\{ \begin{array}{l}
- \frac{i \kappa^2}{k_x} \left( 1 - \frac{2f k_x}{(2a - 1) k_x} \right) \left( 1 - a \right) \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) + a \frac{k_x}{2(2a + 1) k_x} \right), \\
- \frac{k_x}{k_x} \left[ 1 - f \frac{8 \kappa^2 (1 - a)}{(2a - 1) k_x k_x} \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) \right], \\
\frac{4i \pi \kappa^2 a k_x}{(2a + 1) k_x^2} \left( 1 - \frac{4 f (a + 1) k_x}{(2a + 1) k_x} \left( 1 - a \right) \left( \frac{k_x}{k_x} + \frac{5 k_x}{6 k_x} \right) + a \frac{k_x}{2(2a + 1) k_x} \right) + \frac{2 f a}{2a + 1} - \frac{2 \Delta k_x}{k_x} + \frac{\Delta k_x}{a k_x} \right). \\
\end{array} \right. \]  \hspace{1cm} (C6)

At the wavenumber of IW⁺, the eigen-vectors of SDW, IW⁻ and IW⁺ are obtained, respectively, from equations (C4), (C5) and (C6) by the replacements \( a \to 1 - a \) and \( \Delta k_{\kappa, \chi} \to -\Delta k_{\kappa, \chi} \).

**APPENDIX D: NECESSARY DUAL EIGEN-VECTORS**

I) At the wavenumber of SDW⁺.
Let the eigen-vectors of the resonant triad, 
\[ \phi_{(k_{(i+2)}/2]} \text{ and } \phi_{(k_{(i+2)}/2]} \], consist of the following components

\[
\begin{align*}
\phi_{(k_{(i+2)}/2] =} {} & \{ \hat{u}_y, \hat{u}_y', \hat{u}_y'', \hat{\delta} \}^T, \\
\phi_{(k_{(i+2)}/2]} =} & \{ \hat{u}_x', \hat{u}_x'' \}^T, \\
\phi_{(k_{(i+2)}/2]} =} & \{ \hat{u}_x', \hat{u}_x'' \}^T,
\end{align*}
\]

(E1)

explicitly given by the respective equations (C1), (C5) and (C5) with the replacements \( a \rightarrow 1 - a \) and \( \Delta k_{x,z} \rightarrow -\Delta k_{x,z} \).

Similarly, let the corresponding dual eigen-vectors, \( \phi_{(k_{(i+2)}/2] \text{ and } \phi_{(k_{(i+2)}/2]} \}, consist of the following components

\[
\begin{align*}
\hat{\phi}_{(k_{(i+2)}/2] =} {} & \{ \hat{u}_x, \hat{u}_y, \hat{u}_y', \hat{\delta} \}^T, \\
\hat{\phi}_{(k_{(i+2)}/2]} =} & \{ \hat{u}_x', \hat{u}_x'' \}^T, \\
\hat{\phi}_{(k_{(i+2)}/2]} =} & \{ \hat{u}_x', \hat{u}_x'' \}^T,
\end{align*}
\]

(E2)

explicitly given by the respective equations (D1), (D2) and (D2) with the replacement \( a \rightarrow 1 - a \).

The notations in RHS of equations (E1) and (E2) are used in this Section only, in order to outline the derivation of equations (70) and (71).

Inspection of equations (64), (66) and (68) shows that, to leading order in \( \tau \) and \( f \), the relevant terms can be arranged into combinations containing either

\[
D'' \equiv k_x u''_x + k_y u''_y + k_z u''_z = f k^2 \frac{8a}{2a - 1} k_z \frac{20k_x^2 + k_z^2}{6k_z k_z^2}
\]

(E3)

or

\[
D' \equiv k_x' u'_x + k_y' u'_y + k_z' u'_z = f k^2 \frac{8(1 - a)}{2a - 1} k_z \frac{20k_x^2 + k_z^2}{6k_z k_z^2}
\]

(E4)

As expected, equation (E4) can be obtained from equation (E3) by the replacement \( a \rightarrow 1 - a \). Note that \( D', D'' \sim O(f) \).

In this way,

\[
Q_{1} = -i(u''_x u_x' + u''_y u_y' + u''_z u_z') D'' - i(u'_x u'_x + u'_y u'_y + u'_z u'_z) D',
\]

(E5)

where combinations in front of \( D' \) and \( D'' \) are taken to the zeroth order in \( f \). Equation (E5) shortly leads to equation (70).
Further on,

\[ Q_2 = -i(\hat{u}_x \hat{u}_x^* + \hat{u}_y \hat{u}_y^* + \hat{u}_z \hat{u}_z^* + \hat{\delta} \hat{\delta}^*) D'' + i(\hat{u}_x'' \hat{u}_x''^* + \hat{u}_y'' \hat{u}_y''^* + \hat{u}_z'' \hat{u}_z''^*) (k_x'' \hat{u}_x + k_z'' \hat{u}_z), \]  

(E6)

where the combination in front of \( D'' \) is taken to the first order in \( f \), whereas the part of this equation standing after \( D'' \) yields zero to order of \( f^2 \). Equation (E6) shortly leads to equation (71). Derivation of \( Q_3 \) is identical to \( Q_2 \) with the replacement of superscripts \( t \leftrightarrow n \) in equation (E6).