Factors of certain sums involving central $q$-binomial coefficients

Victor J. W. Guo · Su-Dan Wang

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Abstract
Recently, Ni and Pan proved a $q$-congruence on certain sums involving central $q$-binomial coefficients, which was conjectured by Guo. In this paper, we give a generalization of this $q$-congruence and confirm another $q$-congruence, also conjectured by Guo. Our proof uses Ni and Pan’s technique and a simple $q$-congruence observed by Guo and Schlosser.

Keywords Congruences · $q$-binomial coefficients · Cyclotomic polynomials

Mathematics Subject Classification 11B65 · 05A10 · 05A30

1 Introduction

In 1914, Ramanujan [27] obtained a number of representations for $1/\pi$. One such instance, though not listed in [27], is the identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}, \quad (1.1)$$

which was first proved by Bauer [1] in 1859. Such formulas for $1/\pi$ gained popularity in 1980’s for the reason that they can be utilized to provide fast algorithms for computing decimal digits of $\pi$. See, for example, the Borwein brothers’ monograph [2]. Recently, Guiller [4] gave a general method to prove Ramanujan-type series. For a $q$-analogue of (1.1), see [15].
In 1997, Van Hamme [30] conjectured that 13 Ramanujan-type series possess nice $p$-adic analogues, such as
\[
\sum_{k=0}^{(p-1)/2} \frac{4k + 1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.2}
\]
where $p$ is an odd prime. The congruence (1.2) was confirmed by Mortenson [23] using a $_6F_5$ transformation, and was reproved by Zudilin [32] employing the WZ (Wilf–Zeilberger) method. In 2013, also via the WZ method, Sun [29] gave the following generalization of (1.2): for any integer $n \geq 2$,
\[
\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^3 (-64)^{n-k-1} \equiv 0 \pmod{2n \binom{2n}{n}}. \tag{1.3}
\]
Recently, among other things, Ni and Pan [24] proved the following extension of (1.3): for each $n \geq 2$ and $r \geq 1$,
\[
\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^r (-4)^r(n-k-1) \equiv 0 \pmod{2^{r-2}n \binom{2n}{n}}. \tag{1.4}
\]

In fact, Ni and Pan also gave a $q$-analogue of (1.4): for each $n \geq 2$ and $r \geq 2$, modulo $(1 + q^{n-1})2^{r-2}[n]_{q}^{[2n-1]}$,
\[
\sum_{k=0}^{n-1} (-1)^k q^{k^2+(r-2)k}[4k + 1]^{2r-1} (-q^{k+1}; q)^{4r-2}_{n-k-1} \equiv 0, \tag{1.5}
\]
\[
\sum_{k=0}^{n-1} q^{(r-2)k}[4k + 1]^{2r} (-q^{k+1}; q)^{4r}_{n-k-1} \equiv 0. \tag{1.6}
\]

Here $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ stands for the $q$-shifted factorial,
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, \quad \text{if } 0 \leq k \leq n,
\]
\[
0, \quad \text{otherwise},
\]
denotes the $q$-binomial coefficient, and $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$ is the $q$-integer. The $q$-congruences (1.5) and (1.6) were originally conjectured by the first author [6, Conjecture 5.4]. They are obviously true for $r = 1$ (the left-hand sides have closed forms; see [6]). The $r = 2$ case of (1.5) is a $q$-analogue of (1.3) and was first proved by the first author himself. The $r = 2$ case of (1.6) was first obtained by the present authors [13].

The first aim of this paper is to prove the following generalizations of (1.5) and (1.6).

**Theorem 1.1** For each $n \geq 2$, $r \geq 1$ and $s \geq 0$, modulo $(1 + q^{2n-2})2^{r-2}[n]_{q}^{[2n-1]_{q}^{2}}$,
\[
\sum_{k=0}^{n-1} (-1)^k q^{2k^2+(2r-4s-4)k}[4k + 1]^{2s}[4k + 1]_{q}^{2} \binom{2k}{k}^{2r-1} (-q^{2k+2}; q^{2})^{4r-2}_{n-k-1} \equiv 0, \tag{1.7}
\]
\[
\sum_{k=0}^{n-1} q^{(2r-4s-4)k}[4k + 1]^{2s}[4k + 1]_{q}^{2} \binom{2k}{k}^{2r} (-q^{2k+2}; q^{2})^{4r}_{n-k-1} \equiv 0. \tag{1.8}
\]
In particular, letting \( q \to 1 \) in (1.7) and (1.8), we obtain a generalization of (1.4).

**Corollary 1.2** For each \( n \geq 2, r \geq 1 \) and \( s \geq 0 \),

\[
\sum_{k=0}^{n-1} (4k + 1)^{2s+1} \binom{2k}{k} (-4)^r(n-k-1) \equiv 0 \pmod{2^{r-2} n \binom{2n}{n}}.
\]

The first author [7, Theorem 1.1] proved that, for any positive odd integer \( n \),

\[
\sum_{k=0}^{n-1} [3k + 1] (q; q^2)^3 q^{-(\frac{k+1}{2})} \equiv [n] q^{(1-n)/2} \pmod{[n] \Phi_n(q)^2},
\]

where \( \Phi_n(q) \) is the \( n \)-th cyclotomic polynomial in \( q \), i.e.,

\[
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(n,k)=1} (q - \zeta^k)
\]

with \( \zeta \) being an \( n \)-th primitive root of unity.

The second aim of this paper is to prove the following \( q \)-congruence, which was originally conjectured by the first author [7, Conjecture 1.7].

**Theorem 1.3** Let \( n \geq 2 \) be an integer. Then

\[
\sum_{k=0}^{n-1} [3k + 1] \binom{2k}{k}^3 (-q^{k+1}; q^2)^4 q^{-(\frac{k+1}{2})} \equiv 0 \pmod{1 + q^{n-1} [n] \binom{2n-1}{n-1}}.
\]

Letting \( q \to 1 \) in (1.10), we get the following conclusion, which was conjectured by Sun [28, Conjecture 5.1(i)] and was recently proved by Mao and Zhang [22].

**Corollary 1.4** For each \( n \geq 2 \),

\[
\sum_{k=0}^{n-1} (3k + 1) \binom{2k}{k}^3 16^{n-k-1} \equiv 0 \pmod{2 n \binom{2n}{n}}.
\]

We point out that some other interesting congruences and \( q \)-congruences can be found in [5,8–12,14,16–21,25,31,33].

**2 Proof of Theorem 1.1**

We need the following lemma, which is a special case of [24, Lemma 3.2] and can also be deduced from the \( q \)-Lucas theorem (see [26]).

**Lemma 2.1** Let \( s \) and \( t \) be non-negative integers with \( 0 \leq t \leq d - 1 \). Then

\[
\frac{(q; q^2)_{sd+t}}{(q^2; q^2)_{sd+t}} = \frac{1}{4^s} \binom{2s}{s} \frac{(q; q^2)^t}{(q^2; q^2)^t} (\mod \Phi_d(q)).
\]
We adopt some notation similar to that used by Ni and Pan [24]. For any positive integer \( n \), let

\[
S(n) = \left\{ d \geq 3 : d \text{ is odd and } \left\lfloor \frac{n - \frac{d+1}{2}}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},
\]

where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). It is easy to see that, for any integer \( d > 2n - 1 \), the number \( (d + 1)/2 \) is greater than \( n \), and so \( d \notin S(n) \). This means that \( S(n) \) only contains finite elements. Let

\[
A_n(q) = \prod_{d \in S(n)} \Phi_d(q),
\]

\[
C_n(q) = \prod_{d \mid n, d > 1 \text{ odd}} \Phi_d(q).
\]

It is clear that, if \( d \mid n \), then \( d \notin S(n) \). Therefore, the polynomials \( A_n(q) \) and \( C_n(q) \) are relatively prime.

We now give the key lemma, which is similar to [24, Theorem 2.1].

**Lemma 2.2** Let \( v_0(q), v_1(q), \ldots \) be a sequence of rational functions in \( q \) such that, for any positive odd integer \( d > 1 \),

(i) \( v_k(q) \) is \( \Phi_d(q^2) \)-integral for each \( k \geq 0 \), i.e., the denominator of \( v_k(q) \) is relatively prime to \( \Phi_d(q^2) \);

(ii) for any non-negative integers \( s \) and \( t \) with \( 0 \leq t \leq d - 1 \),

\[
v_{s d + t}(q) \equiv \mu_s(q)v_t(q) \pmod{\Phi_d(q^2)},
\]

where \( \mu_s(q) \) is a \( \Phi_d(q^2) \)-integral rational function only dependent on \( s \);

(iii)

\[
\sum_{k=0}^{(d-1)/2} \frac{(q^2; q^4)_k}{(q^4; q^4)_k} v_k(q) \equiv 0 \pmod{\Phi_d(q^2)}.
\]

Then, for all positive integers \( n \),

\[
\sum_{k=0}^{n-1} \frac{(q^2; q^4)_k}{(q^4; q^4)_k} v_k(q) \equiv 0 \pmod{A_n(q^2)C_n(q^2)}.
\]

**(Proof** Our proof is very similar to that of [24, Theorem 2.1]. In order to make the paper more readable, we provide it here. For \( d \in S(n) \), we can write \( n = ud + v \) with \( (d + 1)/2 \leq v \leq d - 1 \). Thus, for any \( v \leq t \leq d - 1 \), the expression \( (q^2; q^4)_t/(q^4; q^4)_t \) is congruent to 0 modulo \( \Phi_d(q^2) \). In view of Lemma 2.1, we obtain

\[
\frac{(q^2; q^4)_{ud+t}}{(q^4; q^4)_{ud+t}} \equiv 0 \pmod{\Phi_d(q^2)}.
\]
Applying Lemma 2.1 again, we conclude that
\[
\sum_{k=0}^{n-1} \frac{(q^2; q^4)_k}{(q^4; q^4)_k} \nu_k(q) \equiv \sum_{s=0}^{u} \sum_{t=0}^{d-1} \frac{(q^2; q^4)_{s+1} t}{(q^4; q^4)_{s+1} t} \nu_{s+1} t(q) \\
= \sum_{s=0}^{u} \frac{1}{4^s} \left(\frac{2s}{s}\right) \mu_s(q) \sum_{t=0}^{d-1} \frac{(q^2; q^4)_t}{(q^4; q^4)_t} \nu_t(q) \\
= 0 \pmod{\Phi_d(q^2)},
\]
where we have used the condition (iii) and the fact that \((q^2; q^4)_t \nu_t(q)/(q^4; q^4)_t\) is congruent to 0 modulo \(\Phi_d(q^2)\) for \((d + 1)/2 \leq t \leq d - 1\). This proves that (2.1) is true modulo \(A_n(q^2)\).

On the other hand, for \(d \mid n\), we assume that \(u = n/d\). Still by Lemma 2.1, we have
\[
\sum_{k=0}^{n-1} \frac{(q^2; q^4)_k}{(q^4; q^4)_k} \nu_k(q) = \sum_{s=0}^{u-1} \sum_{t=0}^{d-1} \frac{(q^2; q^4)_{s+1} t}{(q^4; q^4)_{s+1} t} \nu_{s+1} t(q) \\
= \sum_{s=0}^{u-1} \frac{1}{4^s} \left(\frac{2s}{s}\right) \mu_s(q) \sum_{t=0}^{d-1} \frac{(q^2; q^4)_t}{(q^4; q^4)_t} \nu_t(q) \\
= 0 \pmod{\Phi_d(q^2)}.\]

This proves that (2.1) is also true modulo \(C_n(q^2)\).\qed

We also require the following easily proved result, which is due to Guo and Schlosser [11, Lemma 3.1].

**Lemma 2.3** Let \(d\) be a positive odd integer. Then, for \(0 \leq k \leq (d - 1)/2\), we have
\[
\frac{(q; q^2)^{d-1/2-k}}{(q^2; q^4)^{d-1/2-k}} \equiv (-1)^{d-1/2-2k} \frac{(q; q^2)^{d-1-k/4+k}}{(q^2; q^4)^{d-1-k/4+k}} \pmod{\Phi_d(q^2)}.
\]

In order to simplify the proof of Theorem 1.1, we need to present the following result.

**Theorem 2.4** Let \(n \geq 2\), \(r \geq 1\) and \(s \geq 0\) be integers. Then
\[
\sum_{k=0}^{n-1} (-1)^k q^{2k^2 + 2(r - 4s - 4)k} [4k + 1]^{2s}[4k + 1] q^2 \frac{(q^2; q^4)_k^{2r - 1}}{(q^4; q^4)_k^{2r - 1}} \equiv 0 \pmod{A_n(q^2)C_n(q^2)},
\]
(2.2)
\[
\sum_{k=0}^{n-1} q^{2(r - 4s - 4)k} [4k + 1]^{2s}[4k + 1] q^2 \frac{(q^2; q^4)_k^{2r}}{(q^4; q^4)_k^{2r}} \equiv 0 \pmod{A_n(q^2)C_n(q^2)}.
\]
(2.3)

**Proof** We only give a proof of (2.2), since the proof of (2.3) is exactly the same as that of (2.2). For any non-negative integer \(k\), let
\[
\nu_k(q) = (-1)^k q^{2k^2 + 2(r - 4s - 4)k} [4k + 1]^{2s}[4k + 1] q \frac{(q^2; q^4)_k^{2r - 2}}{(q^4; q^4)_k^{2r - 2}}.
\]

Then, for any odd \(d\), the rational function \(\nu_k(q)\) is \(\Phi_d(q^2)\)-integral, since
\[
\frac{(q^2; q^4)_k}{(q^4; q^4)_k} = \left[\frac{2k}{k}\right] \frac{1}{q^{2}(-q^2; q^2)_k^2},
\]
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and \((-q^2; q^2)_k\) is relatively prime to \(\Phi_d(q^2)\). Using \(q^{2d} \equiv 1 \pmod{\Phi_d(q^2)}\) and applying Lemma 2.1 with \(q \mapsto q^2\), we have
\[
v_{sd+t}(q) = (-1)^{d+t} q^{2(sd+t)^2+(2r-4s-4)(sd+t)}[4(sd + t) + 1]^{2t}[4(sd + t) + 1] q^{2d+t} \equiv (-1)^s \frac{1}{q^{2r-2}} v_t(q) \pmod{\Phi_d(q^2)}
\]
for non-negative integers \(s\) and \(t\) with \(0 \leq t \leq d - 1\). Thus, the sequence \(v_0(q), v_1(q), \ldots\) satisfies the requirements (i) and (ii) of Lemma 2.2.

We now verify the requirement (iii) of Lemma 2.2, i.e.,
\[
\sum_{k=0}^{(d-1)/2} (-1)^k q^{2k^2+(2r-4s-4)k}[4k + 1]^{2t}[4k + 1] q^{2k} \equiv 0 \pmod{\Phi_d(q^2)} \tag{2.4}
\]
In fact, by Lemma 2.3 with \(q \mapsto q^2\), it is easy to verify that, for \(0 \leq k \leq (d - 1)/2\), the \(k\)-th and \(((d - 1)/2 - k)\)-th terms on the left-hand side of (2.4) cancel each other modulo \(\Phi_d(q^2)\), i.e.,
\[
(-1)^{(d-1)/2-k} q^{2k^2+(2r-4s-4)k}[4k + 1]^{2t}[4k + 1] q^{2k} \equiv (-1)^k q^{2k^2+(2r-4s-4)k}[4k + 1]^{2t}[4k + 1] q^{2k} \pmod{\Phi_d(q^2)}.
\]
This proves (2.4). By Lemma 2.2, we conclude that (2.2) is true modulo \(A_n(q^2)C_n(q^2)\). \(\square\)

We collected adequate ingredients and are able to prove Theorem 1.1.

**Proof of Theorem 1.1** For \(0 \leq k \leq n - 2\), the \(q\)-factorial \((-q^{2k+2}; q^2)_{n-k-1}\) contains the factor \(1 + q^{2n-2}\), and for \(k = n - 1\), we have
\[
\begin{aligned}
\binom{2k}{k} q^2 &= \binom{2n-2}{n-1} q^2 = (1 + q^{2n-2})\binom{2n-3}{n-2} q^2.
\end{aligned}
\]
Therefore, the left-hand sides of (1.7) and (1.8) are both divisible by \((1 + q^{2n-2})^{2r-1}\).

In what follows, we shall prove that the left-hand sides of (1.7) and (1.8) are both divisible by \([n]q^2[n-1]q^2\). It is easy to see that the \(q\)-binomial coefficient \(\binom{n}{k}_q\) has the following factorization (see [3, Lemma 1]):
\[
\binom{n}{k}_q = \prod_{d \in D_{n,k}} \Phi_d(q^2),
\]
where
\[
D_{n,k} := \left\{ d \geq 2 : \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor < \left\lfloor \frac{n}{d} \right\rfloor \right\}.
\]
It is easily seen that \(1 < d \in D_{2n-1,n-1}\) is odd if and only if \(d \in S(n)\), and so
\[
[n]q^2 \binom{2n-1}{n-1} q^2 = A_n(q^2)C_n(q^2) \prod_{d \mid n \text{ } d \geq 2 \text{ is even}} \Phi_d(q^2) \prod_{\substack{d \in D_{2n-1,n-1} \\ d \text{ is even}}} \Phi_d(q^2). \tag{2.5}
\]
Note that
\[ \left[ \frac{2k}{k} \right] \frac{(-q^{2k+2}; q^2)_n}{(q^4; q^4)_k} = \frac{(q^2; q^4)_k}{(q^4; q^4)_k} \frac{(-q^2; q^2)^2}{(q^2; q^2)_n}. \]

By Theorem 2.4, the left-hand sides of (1.7) and (1.8) are both divisible by \( A_n(q^2)C_n(q^2) \). It remains to show that (1.7) and (1.8) also hold modulo
\[ \prod_{d \mid n \text{ is even}} \Phi_d(q^2) \cdot \prod_{d \in D_{2n-1}, n-1 \text{ is even}} \Phi_d(q^2). \]

Firstly, let \( d \mid n \) be an even integer. Then
\[ 1 + q^d = \frac{1 - q^{2d}}{1 - q^d} \equiv 0 \pmod{\Phi_d(q^2)}. \]

Thus, for \( 0 \leq k < n - d/2 \), the \( q \)-shifted factorial \((-q^{2k+2}; q^2)_n\) contains the factor \( 1 + q^{2n-d} \) and is congruent to 0 modulo \( \Phi_d(q^2) \). On the other hand, for \( n - d/2 \leq k \leq n - 1 \), we have \( d \in D_{2k}, k \), i.e.,
\[ \left[ \frac{2k}{k} \right] \frac{(-q^{2k+2}; q^2)_n}{(q^4; q^4)_k} \equiv 0 \pmod{\Phi_d(q^2)}. \quad (2.6) \]

Hence, for \( 0 \leq k \leq n - 1 \), we always have
\[ \left[ \frac{2k}{k} \right] \frac{(-q^{2k+2}; q^2)_n}{(q^4; q^4)_k} \equiv 0 \pmod{\Phi_d(q^2)}. \quad (2.7) \]

This proves that (1.7) and (1.8) are true modulo \( \prod_{d \mid n \text{ is even}} \Phi_d(q^2) \). Secondly, we consider the case where \( d \in D_{2n-1}, n-1 \) is even. Write \( n = ud + v \) with \( 0 \leq v \leq d - 1 \). Then \( v > d/2 \). Thus, for \( 0 \leq k < ud + d/2 \), the polynomial \((-q^{2k+2}; q^2)_n\) has the factor \( 1 + q^{2ud+d} \) and is divisible by \( \Phi_d(q^2) \). Moreover, for \( ud + d/2 \leq k \leq n - 1 \), we have \( d \in D_{2k}, k \), and so (2.6) holds. Therefore, for \( 0 \leq k \leq n - 1 \), the \( q \)-congruence (2.7) also holds in this case. This proves that (1.7) and (1.8) are true modulo \( \prod_{d \in D_{2n-1}, n-1 \text{ is even}} \Phi_d(q^2) \).

Noticing that
\[ [n]_q [2n - 1] \frac{2n - 1}{n - 1} q^2 = (1 + q^{2n-2})[2n - 1] \frac{2n - 3}{n - 2} q^2, \]
and \((1 + q^{2n-2})\) is relatively prime to \([2n - 1]q^2\), the least common multiple of \((1 + q^{2n-2})^{2r-1}\) and \([n]_q \frac{2n-1}{n-1} q^2\) is just \((1 + q^{2n-2})^{2r-2}[n]_q \frac{2n-1}{n-1} q^2\). This completes the proof of the theorem.

3 Proof of Theorem 1.3

We first give the following result.

**Lemma 3.1** Let \( d \) be a positive odd integer. Let \( s \) and \( t \) be non-negative integers with \( 0 \leq t \leq d - 1 \). Then
\[ (-q; q)_{sd+t} \equiv 2^t (-q; q)_t \pmod{\Phi_d(q)}. \]
Proof It is easy to see that \((-q; q)_{d-1} \equiv 1 \pmod{\Phi_d(q)}\) (see, for example, [13, Lemma 3.2]). Hence, for \(0 \leq t \leq d - 1\), we have
\[
(-q; q)_{sd+t} = (-q^{sd+1}; q) \prod_{j=0}^{s-1} (-q^{jd+1}; q)_d
\]
\[
\equiv (-q; q)_t (-q; q)_d^s
\]
\[
\equiv 2^s (-q; q)_t \pmod{\Phi_d(q)},
\]
where we have used \(q^d \equiv 1 \pmod{\Phi_d(q)}\). 

We also need the following result, which is similar to Lemma 2.2 and is a special case of [24, Theorem 2.1].

Lemma 3.2 Let \(v_0(q), v_1(q), \ldots\) be a sequence of rational functions in \(q\) such that, for any positive odd integer \(d > 1\),

(i) \(v_k(q)\) is \(\Phi_d(q)\)-integral for each \(k \geq 0\), i.e., the denominator of \(v_k(q)\) is relatively prime to \(\Phi_d(q)\);

(ii) for any non-negative integers \(s\) and \(t\) with \(0 \leq t \leq d - 1\),
\[
v_{sd+t}(q) \equiv \mu_s(q)v_t(q) \pmod{\Phi_d(q)},
\]
where \(\mu_s(q)\) is a \(\Phi_d(q)\)-integral rational function only dependent on \(s\);

(iii)
\[
\sum_{k=0}^{d-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} v_k(q) \equiv 0 \pmod{\Phi_d(q)}.
\]

Then, for all positive integers \(n\),
\[
\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} v_k(q) \equiv 0 \pmod{A_n(q)C_n(q)}.
\]

We can now prove Theorem 1.3.

Proof of Theorem 1.3 Similarly as before, for \(0 \leq k \leq n - 2\), we have
\[
(-q^{k+1}; q)_{n-k-1} \equiv 0 \pmod{1 + q^{n-1}},
\]
and for \(k = n - 1\), there holds \(\left[\frac{2k}{n}\right]q \equiv 0 \pmod{1 + q^{n-1}}\). This means that the left-hand side of (1.10) is divisible by \((1 + q^{n-1})^3\).

In what follows, we shall prove that the left-hand side of (1.10) is divisible by \([n]\left[\frac{2n-1}{n-1}\right]\).

Letting \(q^2 \mapsto q\) in (2.5), we have
\[
[n]\left[\frac{2n-1}{n-1}\right] = A_n(q)C_n(q) \prod_{d \mid n, d \geq 2 \text{ is even}} \Phi_d(q) \cdot \prod_{d \in \mathbb{D}_{2n-1,n-1}, d \text{ is even}} \Phi_d(q).
\]

For any non-negative integer \(k\), let
\[
v_k(q) = q^{-\left(k^2 + 1\right) / 2} [3k + 1] \frac{(q; q^2)_k^2(-q; q)_k^2}{(q^2; q^2)_k^2}.
\]
Then, applying Lemmas 2.1 and 3.1, for any positive odd integer \( d \) and non-negative integers \( s \) and \( t \) with \( 0 \leq t \leq d - 1 \), we have

\[
v_{sd+t}(q) = q^{-(d+t+1)/2} \left[ 3(sd + t) + 1 \right] \frac{(q^2)^2_{sd+t} \left(-q^2\right)^2_{sd+t}}{(q^2)^2_{2sd+t}} \equiv \frac{1}{4^s} \left(\frac{2s}{s}\right)^2 v_t(q) \pmod{\Phi_d(q)}.
\]

Moreover, by (1.9), we obtain

\[
d - 1 \sum_{k=0}^{d-1} q^{-(k+1)/2} [3k + 1] \frac{(q^2)^3_k \left(-q^2\right)^3_k}{(q^2)^2_{2k}} \equiv 0 \pmod{\Phi_d(q)}.
\]

Thus, we may apply Lemma 3.2 to deduce that

\[
\sum_{k=0}^{n-1} q^{-(k+1)/2} [3k + 1] \frac{(q^2)^3_k \left(-q^2\right)^3_k}{(q^2)^2_{2k}} \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{3.1}
\]

Multiplying the left-hand side of (3.1) by \((-q^2;q^2)^4_{n-1}\), we conclude that (1.10) is true modulo \(A_n(q)C_n(q)\).

It remains to show that (1.10) is also true modulo

\[
\prod_{d \mid n \atop d \geq 2 \text{ is even}} \Phi_d(q) \cdot \prod_{d \in D_{2n-1,n-1} \atop d \text{ is even}} \Phi_d(q).
\]

This is exactly the same as the proof of Theorem 1.1 and is omitted here. \(\square\)

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