Chaotic mixing using micro-rotors in a confined domain

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We study chaotic mixing induced by point micro-rotors in a two dimensional Stokes flow. We numerically demonstrate that the dynamics of micro-rotors is significantly influenced by the boundaries. While certain configurations of four or more point micro-rotors have been known to exhibit complex dynamics in an un-bounded domain, we find that assemblies of micro-rotors that are asymmetrically influenced by the boundaries exhibit complex behavior even in the simple two rotor case. The chaotic mixing capability of micro-rotors is demonstrated by using the minimal case of two micro-rotors. Unlike the case of the classic blinking vortex dynamics, the velocity field of the field is smooth in time and satisfies the no slip boundary condition. The mixing of fluid tracers as a function of relative positions of micro-rotors is studied using first return maps and fluid mixing is assessed by visualizing the deformation of an initially square blob of tracer particles. The decay of variance of a scalar concentration field is observed. Stirring using micro-rotors is proposed as a simple means to effectively mix fluids in low Reynolds number flows. This study can have important applications in the biomedical and pharmaceutical industries.

I. INTRODUCTION

Mixing and transport of fluids in the low Reynolds number regime is an important challenge that has gained considerable attention over the years. As a consequence of weak fluid inertia, viscous diffusion is the dominant mode of mixing in low Reynolds number flows. Viscous diffusion unfolds over microscopic length scales and by itself is insufficient to achieve effective mixing in most microfluidic applications which require rapid mixing. This underscores the importance of inertial-advective effects for transporting fluid and for enhancing viscous diffusion by increasing the area of contact between fluids being mixed. Various means of achieving rapid microfluidic mixing have been proposed over the years. Active mixing mechanisms induce large scale motions of the fluid by directly influencing the fluid by means of mixing elements driven by force fields such as electric/magnetic or acoustic fields that compensate for the lack of fluid inertia and can achieve high mixing quality especially in the vicinity of the mixing elements. Passive methods of mixing involve altering the boundary conditions of the flow so as to maximize the area of contact between fluids being mixed. This is usually achieved by incorporating geometric features in the path of the fluid flow or by altering the path itself such that fluid stretches and folds into finer scale structures. Passive methods are high throughput methods and achieve a more uniform mixing than active methods. The quality of passive mixing however, is directly dependent on the duration of time the fluid spends within the mixing geometry necessitating the need for longer microfluidic channels. In this work, we study a particularly simple model for active micro-mixers which are not localized in space and thus achieve efficient and homogeneous mixing at fast time scales.

Naturally occurring micro-swimmers have been known to induce mixing of the surrounding fluid during locomotion. They accomplish this by continuously stirring the fluid around them as they swim, thus making up for the weak inertia of the fluid. Inspired by these studies, artificial micro-swimmers have gained attention as a means to mix and

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predictably manipulate fluids at small scales. While many different micro-swimmer morphologies have been proposed in these works and others, one of the simplest synthetic micro-swimmers that can be studied for this purpose is a self propelled spinning particle. Grzybowski demonstrated the self-assembly of spinning rotors in 2D and later Campbell and Grzybowski used these ideas to experimentally demonstrate the possibility of mixing using magnetically driven rotors. More recently, Ballard et al. demonstrated microfluidic mixing due magnetic micro-beads using experiments and lattice Boltzmann simulations. These and other experimental investigations of magnetic or otherwise driven-particles have shown their effectiveness for micro-mixing applications.

The dynamics of such spinning particles or 'micro-rotors' were recently studied numerically where the flow generated by the micro-rotors were modeled using a rotlet velocity field. The rotlet or a couplet is a solution to the stokes equation with a point torque inhomogeneity Meleshko and Aref showed that "blinking" rotlets under circular confinement could be an effective model to study mixing in viscous flows where they present the blinking rotlet as an analogue of the paradigmatic blinking point vortex in inviscid flows pioneered by Aref. In these studies the location of the blinking singularities are fixed in the domain and the singularities themselves do not interact with each other.

In this work we use the theoretical model of rotlets to further investigate the rich dynamics of micro-rotors on the 2D plane in a circular-confined domain. Each micro-rotor or rotlet do not have a fixed location, but move due to the interaction with the domain boundary as well as with the other rotlets. We demonstrate that even the simple two micro-rotor case can exhibit complex dynamics due to the influence of the boundary on the micro-rotors. The effect of the boundary is first demonstrated by contrasting with the case of regular dynamics of two micro-rotors in an unbounded domain. In the bounded circular domain the chaotic dynamics of passive tracers advected by micro-rotor generated flows are studied using first return maps and the mechanism of mixing is explained. Microfluidic mixing efficiency is quantified by measuring the decay of variance of an initial concentration field.

FIG. 1: Two (left image) and three rotlet (right image) trajectories in an unbounded domain. The red filled circle in each plot shows the starting positions of the rotlets, while the dashed line shows the trajectory for an arbitrary finite time.

II. POINT MICRO-ROTORS IN 2D STOKES FLOWS

The flow due to a spinning micro-particle can be modeled as a singular torque in a quiescent, viscous fluid. In the vanishing Reynolds number limit, the governing equations of motion for fluid are the linearized Navier-Stokes equations

\[ \mu \nabla^2 u = \nabla \times \gamma \delta(x - x_0) \]
\[ \nabla \cdot u = 0. \]

The solution of this set of equations is called a rotlet and is given by,

\[ u(x) = -\gamma \hat{k} \times \frac{x - x_0}{||x - x_0||^2}. \]

Analogous to the classical point vortex which can be thought of as the intersection of a 3D vortex filament and a plane perpendicular to it, a point rotlet can be considered to be the intersection of an infinitesimal spinning particle and a horizontal plane passing through its center.
In the absence of any boundaries or other singularities in its vicinity, the rotlet is fixed in position and the flow is associated with circular streamlines around the location of the singularity.

In an assembly of $N$ rotlets in a plane with locations $(x_i, y_i)$ for $i \in [1, N]$, each rotlet is advected by a velocity that is a linear superposition of the velocities induced by all the other rotlets. The motion of the $i$-th rotlet can then be expressed by,

$$\frac{dx_i}{dt} = \sum_{j=1, j \neq i}^{j=N} \gamma_j \frac{(y_i - y_j)}{R_{ij}^2}$$

$$\frac{dy_i}{dt} = \sum_{j=1, j \neq i}^{j=N} -\gamma_j \frac{(x_i - x_j)}{R_{ij}^2}$$

where, the square of distance between the $i$-th and $j$-th rotlet is given by $R_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$.

Similar to the case of interacting point vortices in an inviscid flow, any initial configuration of three or fewer rotlets produces non chaotic dynamics\textsuperscript{19}. For instance for the case of two rotlets in an unbounded plane, it can be shown through a simple calculation that the distance between the rotlets are invariant,

$$\frac{dR_{12}^2}{dt} = \frac{d}{dt}(x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= 2(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + 2(y_1 - y_2)(\dot{y}_1 - \dot{y}_2)$$

$$= \frac{2}{R_{12}^2}(x_1 - x_2)(\gamma_2(y_1 - y_2) - \gamma_1(y_2 - y_1))$$

$$+ \frac{2}{R_{12}^2}(y_1 - y_2)(a\gamma_2(x_1 - x_2) + \gamma_1(x_2 - x_1))$$

$$= 0.$$

As an example, the dynamics of two rotlets with an equal clockwise spin in an unbounded domain is shown in Fig. 1(a). The two rotlets move on a circle whose diameter is equal to the distance between the two rotlets. When the rotlets are such that $\gamma_1 = -\gamma_2$, the two rotlets move along two parallel lines.

While the dynamics of three rotlets in a plane are integrable, the calculation of their trajectories is not trivial except for special cases. One such special case of three rotlet dynamics is shown in Fig. 1(b). The three rotlets are of equal clockwise spin with initial positions that coincide with the vertices of an equilateral triangle. In this case the rotlets move on a circle with a radius equal to the circumradius of the equilateral triangle. Other special cases of the three rotlet problem are discussed in\textsuperscript{19}. On the other hand, four or more rotlets may not exhibit periodic trajectories\textsuperscript{19}.

In the context of chaotic mixing of a fluid using the motion of rotlets as stirrers, this suggests that in general multiple rotlets are necessary to mix fluid in an unbounded domain. However, mixing studies performed in a bounded domain are a more realistic model of practical scenarios where such mixing might be performed. This suggests that the dynamics of micro-rotors as microfluidic mixers are best studied inside a closed contour such as a circular boundary with no-penetration and viscous no-slip boundary conditions. We show that when the fluid domain is bounded, as it is in practical applications, fewer rotlets can show chaotic dynamics that leads to chaotic mixing. Specifically we consider the case of two rotlets confined to the interior of a unit circle.
III. THE N-ROTLET MODEL UNDER CIRCULAR CONFINEMENT

Here, we use the model for a rotlet under circular confinement which was independently derived by Ranger and Meleshko, to study the dynamics of a single and multiple-interacting micro-rotors.

This model incorporates the effects of a circular boundary of radius “a” around a point rotlet of strength $\gamma$ located at a position $r = b$ and $\theta = 0$ inside the circular boundary. This is done by placing an image system of singularities of strength $-\gamma$ at a location $r = a^2 b$ and $\theta = 0$ outside the boundary such that the normal velocity due to the original rotlet is identically canceled at every point on its perimeter. The no-slip boundary condition is satisfied by analytically constructing a flow field that cancels out the tangential velocity at every point on the circular boundary while not violating the normal velocity boundary condition. The stream function for such a flow is given as

$$\psi = \frac{\gamma}{2} \left[ \log \frac{A}{B} + \frac{C}{B} \right]$$

with,

$$A = r^2 - 2bx + b^2$$
$$B = a^2 - 2bx + \left( \frac{b^2 r^2}{a^2} \right)$$
$$C = (1 - \frac{r^2}{a^2}) \left( a^2 - \frac{b^2 r^2}{a^2} \right).$$

When multiple rotlets are present inside a circle, the velocity of the $i$-th rotlet is a linear superposition of the velocity due to the other $j = 1..(N-1)$ rotlets in the domain and due to the image singularities of all the $N$ rotlets. This velocity is given by:

$$\frac{dx_i}{dt} = \sum_{j=1, j\neq i}^{j=N} \frac{\gamma_j}{2} \left[ \frac{B_j}{B^2_j} \right] \left( \frac{\partial}{\partial x} \right) \left( B_j A_{jy} - A_j B_{jy} \right) + \frac{B_j C_{jy} - C_j B_{jy}}{B^2_j}$$

$$+ \frac{\gamma_i}{2} \left( -\frac{B_{iy}}{B_i} \right) + \frac{B_i C_{iy} - C_i B_{iy}}{B^2_i}$$

$$\frac{dy_i}{dt} = \sum_{j=1, j\neq i}^{j=N} -\frac{\gamma_j}{2} \left[ \frac{B_j}{B^2_j} \right] \left( \frac{\partial}{\partial y} \right) \left( B_j A_{jx} - A_j B_{jx} \right) + \frac{B_j C_{jx} - C_j B_{jx}}{B^2_j}$$

$$- \frac{\gamma_i}{2} \left( -\frac{B_{ix}}{B_i} \right) + \frac{B_i C_{ix} - C_i B_{ix}}{B^2_i}$$

where, $A_x, A_y, B_x, B_y, C_x, C_y$ etc represent the derivatives of terms A,B and C with respect to spatial variables x and y respectively. Here, the velocity due to the $j$-th rotlet is computed in a rotated frame of reference such that $\theta_j = 0$ in the rotated coordinates and the computed velocity is then rotated back to the original frame of reference.
The first observation that can be made is that while a single rotlet in the unbounded plane remains stationary, a single rotlet inside a circular boundary has a net translation as a result of boundary interactions, Fig. 2. The influence of boundaries on producing quasiperiodic and chaotic rotlet dynamics is easily seen by taking the case of two interacting rotlets. If the initial position of one of the rotlets is closer to the no-slip boundary than the other, their otherwise regular-periodic trajectory is perturbed, thus making their dynamics aperiodic. Figure 2 shows where the rotlet closer to the center does not return to its initial position when the rotlet closer to the boundary returns to its initial position. Similarly, the relative equilibrium configuration of the 3 rotlets in the unbounded plane shown in Fig. 1 is no longer a relative equilibrium configuration in a circular domain as shown by the quasiperiodic trajectories of the three rotlets in Fig. 2. In the rest of the paper we will restrict our investigation to the two rotlet system in a circular domain, where the rotlets’ motion can exhibit regimes of quasiperiodic dynamics. As discussed in the next few sections such quasiperiodic dynamics of the rotlets are key to induce mixing of the fluid.

IV. CHAOTIC MIXING USING SAME SPIN MICRO-ROTORS

To investigate mixing of the fluid in a circular domain, the equations of motion of the rotlets, (7), have to be augmented by the advection equation for a fluid tracer that is not colocated with any of the N rotlets. A fluid tracer is advected with a velocity that is a superposition of the velocities induced by each rotlet and each of the corresponding image singularities. If the location of a fluid tracer is denoted by \((x, y)\), then the advection equation for the tracer is

\[
\begin{align*}
\frac{dx}{dt} &= u_x = \sum_{j=1}^{j=N} \gamma_j \left( \frac{B_j B_j A_{1y} - A_{1j} B_{1jy}}{A_j B_j^2} + \frac{B_j C_j y - C_j B_{1jy}}{B_j^2} \right), \\
\frac{dy}{dt} &= u_y = \sum_{j=1}^{j=N} \gamma_j \left( -\frac{B_j B_j A_{1x} - A_{1j} B_{1jx}}{A_j B_j^2} + \frac{B_j C_j x - C_j B_{1jx}}{B_j^2} \right). \tag{8}
\end{align*}
\]

Two interpretations are possible for the equations of advection of a fluid particle in the domain. The advection equations (8) can model a two dimensional time dependent dynamical system, where the explicit time dependence is through the positions of the N rotlets, which can be obtained from (7). Following this the trajectories of fluid particles are allowed to intersect in the domain. A second possible interpretation is that equations (7), (8) define a time independent \(2N + 2\) dimensional dynamical system. In this case too trajectories of fluid particles projected onto two dimensional circular domain can intersect. Following either interpretation chaotic dynamics of fluid particles in the circular domain cannot be ruled out. We will adopt the first view and treat the advection of fluid particles as a two dimensional time dependent dynamical system. The velocity of the fluid in the circular domain is then defined by the vector field \((u_x, u_y)\). Let the time dependent flow map for this dynamical system be denoted by \(\phi_{t_0}^t(x(t_0), y(t_0)) \rightarrow (x(t), y(t))\). The flow map \(\phi_{t_0}^t\) takes as input an initial condition \((x_{t_0}, y_{t_0})\) and maps it to the solution of the dynamical system at time \(t\).

Two of the standard tools in the studies of laminar mixing of fluids are Poincaré maps and tracking the evolution of a generic blob of fluid. The blob is assumed to contain tracers and if the blob filaments through stretching and folding and eventually spreads across the domain, the fluid flow is mixing. It should be noted that in volume conserving fluid flows, the initial area or volume of a blob cannot change, and it cannot occupy the entire domain. Instead one is interested in a measure of the spread of the blob, such as the variance. In the present problem we first define a blob of a fluid at initial time \(t_0\), as an open set, \(B(t_0)\), in the circular domain not containing any rotlet. The initial blob is mapped to the set \(B(t) = \phi_{t_0}^t(B(t_0))\), such that every point \((x(t_0), y(t_0)) \in B(t_0)\) is mapped to \((x(t), y(t)) = \phi_{t_0}^t(x(t_0), y(t_0))\). Analytical solution of the flow map is nontrivial and we use numerical simulations of (8) to investigate Poincaré maps and the evolution of arbitrary blobs of fluid.

We first discuss the mixing dynamics in the circular domain due to a single rotlet. A single rotlet in the domain moves only due to the influence of the boundary, i.e., its advection is due to the image singularity system alone. The velocity of the rotlet \((\dot{x}_1, \dot{y}_1)\) given by (7) is always tangential to the radial line joining the center of the circle. It can be easily verified that the projection of the velocity of the rotlet along a unit radial vector is zero,
\[
(\dot{x}_1, \dot{y}_1) \cdot \left( \frac{x}{r}, \frac{y}{r} \right) = \frac{\gamma}{2} \left( B \frac{x}{B} + \frac{BC y - CB y}{B^2} \right) - \frac{\gamma}{2} \left( B \frac{y}{B} + \frac{BC x - CB x}{B^2} \right) = 0.
\]

(9)

A stirring mechanism that produces good mixing is characterized by stretching and folding of large scale coherent structures of the flow into finer structures that are intertwined such that the surface area of contact between them is increased manifold, enabling faster diffusion. However, it is possible that such effective mixing is not uniform, resulting in islands of poor mixing embedded within a chaotic region. To identify such regions of poor mixing, it is very useful to construct time-T Poincaré-surface of section of the time dependent dynamical system (8). A single rotlet in the circular domain moves with a constant speed along a circular path with time period \(T\). The time period \(T\) depends on the initial distance of the rotlet from the center of the circle. Such a motion of the single rotlet does not produce mixing of the fluid in the entire domain. This can be demonstrated through a computation of \(T\)-period Poincaré maps for several initial conditions of a tracer particle. The images in fig. 3 are \(T\)-period Poincaré maps for different distances of the rotlet from the center of the circle. In almost all cases, very large open sets of the domain exist which are not visited by the tracer. The Poincaré maps provide numerical evidence that a single rotlet in a circular domain cannot mix the fluid in the entire domain.

![Fig. 3: Time T-Poincare maps for the evolution of a tracer particle.](image)

FIG. 3: Time \(T\)-Poincare maps for the evolution of a tracer particle. The distance, \(d\), of the rotlet from the center varies in the four cases shown. In all cases large subsets of the fluid domain are not visited by a fluid tracer with arbitrarily chosen initial location.

A. Chaotic mixing by two same spin rotlets

When two rotlets with the same spin are present in the circular domain, more complex dynamics of the rotlets as well as the fluid particles ensue. These dynamics depend upon the initial relative positions of the rotlets. The trajectories of the two rotlets themselves are shown for various initial configurations in fig. 4 with one of the trajectories shown in blue and the other in black. The leftmost column in this figure shows the initial positions of the two rotlets, represented by two small filled circles. The next three columns show the trajectories of the rotlets at time periods \(T\), \(2T\) and a \(nT\) where \(n\) is a number ranging from 40 to 50 in different cases. The evolution of the rotor positions in each case are not periodic, instead, they display an almost-periodic behavior.
FIG. 4: The first column shows the initial rotlet locations (filled red circles) along with the instantaneous streamlines for the initial configuration. The second column shows the rotlet trajectories for some time period $T$ while the third column shows the almost periodic behavior at time $2T$ and the fourth column shows the dense trajectories after several cycles of time $T$.
This time period $T$ is different in each case and is chosen to represent the time period of the nearly periodic trajectory of one or both of the rotlets in the respective cases.

The images in Fig. 4 show that there exist two distinct modes of motion. The first mode of motion seen in Fig. 4(a), (b) and (f) is when each rotlet exhibits a trajectory which is similar to that of the other. This results in a dense set of overlapping trajectories such that the trajectory of one rotlet is indistinguishable from that of the other after a sufficient time has elapsed, as in the images in Fig. 4(a4), (b4) and (f4). For these cases, numerics suggest that for long integration times, any open subset of the domain that is visited by one rotlet is also visited by the other rotlet at a different time. The second mode of motion, is when the two rotlets have trajectories that do not cross each other as in Fig. 4(c), (d) and (e). In these cases the fluid can be divided into disjoint subsets each of which has the trajectory of only rotlet in its interior. This is most clearly seen in Fig. 4(c) where one rotlet is confined to a small region around the center of the circle and the other is confined to a thin annulus far from the center.

The mixing of the fluid in the domain cannot however be concluded from the motion of the rotors themselves. Figure 5 shows the time $T$-Poincare maps for the rotlet configuration corresponding to these in 4. The first three maps 5(a)-(c) corresponding to 4(a)-(c) respectively show large distinct spatial bands and islands shown in (black, blue and red). Fluid from these disjoint sets does not mix well with the rest of the domain. The latter three cases 5(d)-(f) corresponding to 4(d)-(f) exhibit a Poincare map which covers the a very large part of the domain.

The mixing (or lack thereof) observed in the Poincare maps can be quantified by measuring how uniformly the iterates of the Poincare map are distributed in the domain. The spatial variation in the distribution of a large number of iterates of a Poincare map is a measure of mixing. Smaller variations indicate better mixing of the fluid in the domain. A more standard method to measure mixing is via the variations in the spatial distribution of an initially coherent blob of fluid rather than time $T$-Poincare map of a single initial condition. In the numerical simulations the variance is computed in a discrete manner, by first dividing the domain into $N$ subsets, $b_1, \ldots, b_N$. These subsets are squares except close to the boundary. At the boundary of the circle the discretization produces subsets of varying sizes. The initial coherent blob is represented by a uniform distribution.
of a large number, \( n_p \), of fluid particles that lie in the interior of the blob. The blob is thus the set of \( n_p \) discrete fluid particles, \( B(0) = (x_k(0), y_k(0) : k = 1, \ldots, n_p) \). The blob at a later time is the merely the set \( B(t) = (x_k(t), y_k(t) : k = 1, \ldots, n_p) \). The number of the fluid particles that lie in subset \( b_k \) is denoted by \( N_k(t) \) and the area of the subset is denoted by \( A_k \). Then an ideal uniform distribution of the iterates would lead to \( A_k \) \( n_p \) iterates in the \( k \)th box, where \( A = \sum A_k \) is the area of the unit circle. The variance in the distribution of the blob is then

\[
\sigma^2(t) = \sum_{k=1}^{k=N} \left( \frac{N_k(t)}{A_k} - \frac{n_p}{A} \right)^2.
\]  

(10)

A plot of the evolution of this variance for the six cases is shown in fig. 6. The computation is done with \( n_p = 40000 \) particles initial distributed uniformly in a square region of size \( 0.1 \times 0.1 \) centered at \((x, y) = (0.5, 0.5)\). The domain is discretized into \( N = 1681 \) boxes. The initial variance is very large, \( \sigma^2(t = 0) = 3.82 \times 10^5 \) and decreases quickly in all the cases. The variance of the iterates of the Poincare maps decay the fastest in cases (d)-(f), with the fastest decay occurring in case (f), while the variance decays very slowly in cases (a) and (c). The rapid mixing produced by the rotors is illustrated in fig. 7 by the stretching and folding of different two blobs. This results in alternating layers of thin-folded strips of the blobs called “striations” as seen in Figs. 7. A movie of this mixing process is available as supplementary material.

The rapid mixing of the two blobs of fluid seen in 7 can be attributed to two mechanisms. The inner rotlet (the one closer to the center) moves around a large area of the domain, see fig. 4(f4)), tugged by the outer rotlet. The instantaneous streamlines form a large circulating cell around this inner rotlet (seen in fig. 4(f1)) and the two tracer blobs are stretched and wound around this rotlet that moves in a large part of the domain. This mechanism by itself however does not mix the fluid in the domain. This inadequacy is explained in fig. 8. Figure 8(a)-(b) shows the motion of a blob that stretches and winds around the inner rotlet but after an initial stretching the blob does not significantly deform. The original blob of fluid in fig. 8(a) is captured by the inner rotlet and moves as a satellite belt around this rotlet. To produce further mixing the role of the outer rotlet becomes important. When the inner and outer rotlet pass close to each other as in fig. 8(c)-(d) the blob of fluid around the inner rotlet is stretched out further and brought close to the boundary. The shear stretching induced by boundary produces a slowly moving anchoring point for a filament of this blob. The inner rotlet moves away from the boundary being tugged by the outer rotlet, in the process the blob experiences enhanced stretching and folding as in fig. 8(d)-(e). This process repeats each time the inner and outer rotlet pass each other as in fig. 8(f)-(g). The blob is spread almost through out the domain in only ten passes of the two rotlets fig. 8(h).

The outer rotlet also plays a role in mixing the fluid close to the boundary. The velocity of the fluid in a small neighborhood of this rotlet is very high, while at the same time decays to zero at the boundary. This very large velocity gradient around the moving rotlet stretches the fluid material lines into a long comet like tail and smears them along the boundary. This motion of the outer rotlet also enhances mixing within a thin strip adjacent to the boundary.
FIG. 7: Mixing of two distinct blobs of fluid by the rotors. The rapid stretching and folding of the blobs results in long filaments of the blobs that are eventually spread over the domain.

FIG. 8: Mixing by the rotlet pair. The outer rotlet moves around a large part of the domain. It captures nearby fluid (a) into a set that stretches and winds around it (b). (c)-(d) When the inner rotlet passes close to the outer rotlet the blob is forced to stretch and spread into a larger region. (e) The pull by the outer rotlet creates a slowly moving anchor point for the blob which enhances stretching. (f)-(g) A second close pass of the rotlets again enhances the stretching and spread of the blob. (h) After ten passes of the rotlets the blob has spread almost throughout the domain.

V. CONCLUSION

The dynamics of multiple interacting rotlets in an un-bounded domain has been recently studied as a model for micro-rotors in low Reynolds number flows. Micro-rotors can be considered a particularly simple micro-swimmer morphology to be used in microfluidic mixing. This paper studies the dynamics of two micro-rotors in the presence of a circular no-slip and no-penetration boundary and identifies qualitatively different dynamics from the un-bounded case. The motion of microrotors in a bounded domain is important from the point of view of microfluidic applications. The use of simple assemblies of a pair of micro-rotors to microfluidic mixing has been explored. It is shown that even in the minimal case of only two micro-rotors in the limit of vanishing Reynolds
number chaotic mixing is possible in a bounded domain. Larger number of micro-rotors and their mutual interactions can be expected to produce faster and more efficient mixing but those numerics are not explored here. The case of micro-rotors with opposing directions of rotation which is a very interesting theoretical possibility, is also deferred to future investigations. The mixing dynamics explored in this paper due to hydrodynamically interacting micro-rotors can combine the principles of passive chaotic advection with active mixing techniques. This can be achieved by short bursts of controlling the motion of one or more of the microrotors for achieving configurations that enhance mixing or to preferentially mix some regions of the fluid. Motion planning for such micro rotors will be explored in future investigations.

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