FAMILIES OF EMBEDDINGS OF THE ALTERNATING GROUP OF RANK 5 INTO THE CREMONA GROUP

I. KRYLOV

Abstract. I study embeddings of alternating group of rank five into the Cremona group of rank three. I find all embeddings induced by $A_5Q$-del Pezzo fibrations and I study their conjugacy. As an application, I show that there is a series of continuous families of pairwise non-conjugate embeddings of alternating group of rank five into $Cr_3(C)$.

1. Introduction

The Cremona group $Cr_n$ of rank $n$ is a group of birational transformations of $P^n$. It is natural to study the group by studying its subgroups and finite subgroups in particular. The classification of finite subgroups in $Cr_2$ up to conjugacy is almost complete [7]. It is not feasible to achieve a classification for $Cr_3$, nevertheless we can say something about its finite subgroups, for example we know that $Cr_n$ is Jordan [14].

If we limit the group types, for example to $p$-subgroups ([15]) or simple non-abelian, then the problem becomes more manageable. The motivating problem for me is the following.

Problem 1.1. Classify the embeddings of finite simple non-abelian subgroups of $Cr_3$ up to conjugacy.

The isomorphism types of simple non-abelian groups were classified in [11, Theorem 1.3], the possibilities are: $A_5$, $A_6$, $A_7$, $PSL_2(7)$, $SL_2(8)$, and $PSp_4(3)$. In this paper I study the families of subgroups of $Cr_3$.

Question 1.2. Let $G$ be a group. Is there a continuous family of embeddings $G \hookrightarrow Cr_n$ which are not pairwise conjugate to each other?

In general, one expects that the answer is positive for small groups and negative for big groups. For example there are huge families of pairwise non-conjugate embeddings of $Z/2Z$ induced by Bertini and Geiser involutions [13]. Also it was recently shown in [1] that there is a continuous family of pairwise non-conjugate embeddings of $S_4$ into $Cr_3$.

On the other hand, finite non-abelian groups are big thus one expects that there should be no continuous families of pairwise non-conjugate embeddings of these groups. Indeed, there are only six simple non-abelian subgroups of $Cr_2$ up to conjugacy: three subgroups isomorphic to $A_5$, two subgroups isomorphic to $PSL_2(7)$, and one subgroup isomorphic to $A_6$. In dimension three we know by [11, Theorem 1.5] that there are only finitely many embeddings for the three largest finite simple non-abelian subgroups. I study the embeddings of the smallest finite simple non-abelian subgroup: $A_5$.

Theorem 1.3. For each $k \geq 2$ there is a $2k - 3$-dimensional family of embeddings $A_5 \hookrightarrow Cr_3$ which are not pairwise conjugate to each other.

The remaining cases are $A_6$ and $PSL_2(7)$. Some progress toward classification the embeddings of these groups has been made in [3] and [10], so far the results agree with the following expectation.

Conjecture 1.4. The embeddings into $Cr_3$ of $A_6$ and $PSL_2(7)$ up to conjugation form discrete families.

1.1. $GQ$-Mori fiber spaces. The study of rational group action on $P^3$ may be replaced with the study of regular group actions on suitable rational varieties.

Definition 1.5. I say that $\pi: Y \to Z$ is a $GQ$-Mori fiber space if

- $dim Z < dim Y$ and $\pi$ has connected fibers,
• $G$-action on $Y$ is faithful and $Z$ admits a $G$-action such that the map $\pi$ is $G$-equivariant,
• $Y$ is terminal and $GQ$-factorial, that is $G$-invariant divisors on $Y$ are $Q$-factorial,
• $-K_Y$ is $\pi$-ample and the relative $G$-invariant Picard rank is $\rho^G(Y/Z) = 1$.

When $Y$ is Gorenstein I will omit $Q$ and say that $\pi: Y \to Z$ is a $G$-Mori fiber space. If $\dim Y - \dim Z = 2$, then I say that $\pi: Y \to Z$ is a $GQ$-del Pezzo fibration.

By [12, Proposition 1.2] for a rational action of $G$ on a rationally connected variety $X$ there is a $GQ$-Mori fiber space $\pi: X \to Y$ such that $X$ is $G$-equivariantly birational to $Y$, hence the $G$-action on $Y$ induces a conjugate embedding of $G$ into $\text{Bir}(X)$. The classification of subgroups of $C_r$ up to conjugation is equivalent to the classification of the rational $GQ$-Mori fiber space up to $G$-equivariant birational equivalence. To show that two subgroups of $C_r$ isomorphic to $G$ are not conjugate, one needs to show that there are no $G$-equivariant birational maps between the $GQ$-Mori fiber spaces inducing them.

Now I give examples of $A_5$-del Pezzo fibrations.

**Example 1.6.** The projective plane with $A_5$-action induced by a three-dimensional representation $W_3$ is an $A_5$-del Pezzo surface. A del Pezzo surface $S_5$ of degree 5 satisfies $A_5 \subset \text{Aut} S_5$ and is $A_5$-minimal, hence it is an $A_5$-del Pezzo surface.

Let $S$ be a $G$-del Pezzo surface, then $S \times \mathbb{P}^1/\mathbb{P}^1$ is a $G$-del Pezzo fibration. It follows that $\mathbb{P}^2 \times \mathbb{P}^1/\mathbb{P}^1$ and $S_5 \times \mathbb{P}^1/\mathbb{P}^1$ are $A_5$-del Pezzo fibrations.

**Example 1.7.** Let $W_3$ and $W'_3$ be the 3-dimensional representations of $A_5$. I consider two $A_5$-linearized vector bundles defined as

$$\mathcal{E}_n = (W_3 \times \mathbb{P}^1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \quad \text{and} \quad \mathcal{E}'_n = (W'_3 \times \mathbb{P}^1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

Let $T_n = \mathbb{P}(\mathcal{E}_n)$ and $T'_n = \mathbb{P}(\mathcal{E}'_n)$ and denote by $\pi_T$ and $\pi'_T$ the corresponding projections onto $\mathbb{P}^1$.

I introduce coordinates on $T_n$ (resp. $T'_n$) as follows. Let $u,v$ be the coordinates on the base $\mathbb{P}^1$, $x, y, z$ be the coordinates on $W_3$ (resp. $W'_3$), and let $w$ be the coordinate on the fiber of $\mathcal{O}_{\mathbb{P}^1}(-n) \to \mathbb{P}^1$.

The degrees of the coordinates are given by

$$\begin{pmatrix}
    u & v & x & y & z & w \\
    0 & 0 & 1 & 1 & 1 & 1 \\
    1 & 1 & 0 & 0 & 0 & -n
\end{pmatrix}$$

There is a unique $A_5$-invariant conic $\Delta$ on $\mathbb{P}(W_3)$ (resp. $\mathbb{P}(W'_3)$). Up to a change of coordinates on $W_3$ (resp. $W'_3$) we may assume that its equation is $xz - y^2 = 0$. Let $X_n$ (resp. $X'_n$) be the hypersurface in $T_n$ (resp. $T'_n$) given by the equation

$$a_{2n}(u,v)w^2 = xz - y^2,$$

where $\deg a_{2n} = 2n$. Observe that this equation is $A_5$-invariant, thus $X_n$ and $X'_n$ admit a faithful $A_5$-action. The $A_5$-varieties $X_n$ and $X'_n$ are smooth if and only if $a_{2n}$ has no multiple factors. If $l(u,v)$ is a multiple factor of $a_{2n}$, then $X_n$ and $X'_n$ have a $cA_1$-singularity at $x = y = z = l(u,v) = 0$. We have $\rho(X_n) = \rho(X'_n) = 2$ if $a_{2n}$ is not a square (Lemma 3.4). The restriction $\pi$ (resp. $\pi'$) of $\pi_T$ (resp. $\pi'_T$) to $X_n$ (resp. $X'_n$) induces the structure of $A_5$-del Pezzo fibration on $X_n$ (resp. $X'_n$).

Note that the varieties $X_1$ and $X'_1$ are unique while for $n \geq 2$ the families of varieties $X_n$ and $X'_n$ are of dimension $2n - 3$.

I show that these are essentially the only $A_5Q$-del Pezzo fibrations with the trivial action on the base.

**Theorem 1.8.** Let $\pi: V \to \mathbb{P}^1$ be an $A_5Q$-del Pezzo fibration and suppose $A_5$ acts trivially on the base. Then one of the following holds:

1. The general fiber of $\pi$ is $\mathbb{P}^2$ and $V$ is $A_5$-equivariantly birational to $\mathbb{P}^2 \times \mathbb{P}^1$ with $A_5$ acting only on the first factor;
2. $V \cong A_5 \times \mathbb{P}^1$, where $S_5$ is the del Pezzo surface of degree 5 and $A_5$ acts only on the first factor;
3. The general fiber of $\pi$ is a quadric and $V$ is $A_5$-equivariantly birational to a smooth $X_n$. 


The varieties from case (3) are inducing families embeddings of $A_5$ into $Cr_5$. In order to show that these embeddings are not pairwise conjugate I use the notion of $G$-equivariant birational superrigidity.

1.2. $G$-equivariant birational superrigidity.

**Definition 1.9.** Let $\pi_X : X \to S$ and $\pi_Y : Y \to Z$ be $GQ$-Mori fiber spaces. A $G$-equivariant birational map $\chi : X \dasharrow Y$ is called *square* if it fits into a commutative diagram

$$
\begin{array}{c}
X \xrightarrow{\chi} Y \\
\pi_X \downarrow \quad \quad \quad \quad \quad \quad \downarrow \pi_Y \\
S \xrightarrow{g} Z,
\end{array}
$$

where $g$ is birational and, in addition, the induced map on the generic fibers $\chi_\eta : X_\eta \to Y_\eta$ is isomorphism of $G$-varieties. In this case I say that $X/S$ and $Y/Z$ are $G$-equivariantly square birational.

**Definition 1.10.** I say that $\pi_X : X \to S$ is $G$-equivariantly superrigid if for any $GQ$-Mori fiber space $\pi_Y : Y \to Z$ the variety $Y$ is $G$-equivariantly birational to $X$ if and only if $X/S$ and $Y/Z$ are $G$-equivariantly square birational.

We know all the varieties in the $A_5$-birational equivalence class of smooth $X_n$ (resp. $X_n'$) for $n \geq 2$.

**Theorem 1.11.** Suppose $X_n$ (resp. $X_n'$) is smooth and $n \geq 2$, then $X_n$ (resp. $X_n'$) is $A_5$-equivariantly birationally superrigid. Moreover, any $A_5\mathbb{Q}$-Mori fiber space in the $A_5$-equivariant birational equivalence class is isomorphic to $X_m$ for some $a_{2m} = a_{2n}b_{m-n}$, where $\deg b = m - n$.

Theorem 1.11 implies the following result.

**Corollary 1.12.** Suppose $X_n$, $X_k$, $X_m'(\prime)$, and $X_1'$ are smooth and $n, m, k, l \geq 2$, then:

1. The varieties $X_n$ and $X_m'$ are not $A_5$-equivariantly birational to $\mathbb{P}^2 \times \mathbb{P}^1$ and $S_5 \times \mathbb{P}^1$;
2. The variety $X_n$ is not $A_5$-equivariantly birational to $X_m'$;
3. The variety $X_n$ is $A_5$-equivariantly birational to $X_k$ if and only if $X_n$ is $A_5$-equivariantly isomorphic to $X_k$ and the same holds for $X_m'$ and $X_1'$;
4. For $n \geq 2$ the family of $A_5$-del Pezzo fibrations $X_n$ (or $X_m'$) induces a $2n - 3$-dimensional family (resp. $2m - 3$-dimensional) of pairwise non-conjugate embeddings $A_5 \hookrightarrow Cr_3$.

The assertion (4) is Theorem 1.3.

**Acknowledgments.** The author would like to thank Andrey Trepalin and Constantin Shramov for pointing out how to work with del Pezzo surfaces of degree 5 over non-closed fields. The author was supported by KIAS individual grant MG069801 at Korea Institute for Advanced Study.

**Notations and conventions.** I work over $\mathbb{C}$ unless stated otherwise. All varieties are considered to be projective and normal. I denote the symmetric and alternating groups of rank $n$ by $S_n$ and $A_n$ respectively. I denote the del Pezzo surface of degree 5 by $S_5$ and Clebsch cubic by $S_3$. I denote the linear equivalence of divisors by $\sim$, the numerical equivalence of cycles by $\equiv$, and $A_5$-equivariant biregular equivalence by $\cong_{A_5}$. Given a curve $\Delta$ I denote its orbit by $\Delta$. By a slight abuse of notations I denote the curve $\sum_{C \in \Delta} C$ by $\Delta$ as well.

2. **The representatives of $A_5$-equivariant birational classes of $A_5\mathbb{Q}$-del Pezzo fibrations with trivial action on the base**

2.1. **Representations of $A_5$ and its central extension.** Recall that $A_5$ has the following irreducible representations: $I = I_1, W_3, W_4', W_4$, and $W_5$ (see [4, Sections 5.2 and 5.6], for details). The lower index is the dimension of the representation. The action of $A_5$ on $\mathbb{P}^1$ is induced by irreducible representations of the central extension $2.A_5$: $U_2$ and $U_2'$.

I recall some facts about the action of $A_5$ on smooth curves and surfaces

**Lemma 2.1** ([4, Lemmas 5.1.3, 5.1.4 and Section 5.2]). Let $S$ be a smooth surface and $C$ be a smooth curve with a non-trivial action of $A_5$. 

---

3

(1) Let $\Sigma$ be an $A_5$-orbit on $C$, then $|\Sigma| = 12, 20, \text{ or } 60$.
(2) Let $\Sigma$ be an $A_5$-orbit on $S$, then $|\Sigma| \geq 5$.

Proposition 2.2 ([4, Chapter 6]). Let $\pi : X \to \mathbb{P}^1$ be an $A_5Q$-del Pezzo fibration and let $F$ be a general fiber of $\pi$. Then one of the following holds

1. $F \cong A_5 P(W_3)$ or $F \cong A_5 P(W''_3)$,
2. $F \cong A_5 S_5$,
3. $F \cong A_5 S_3$,
4. $F \cong A_5 P(U_2) \times P(U_2)$ or $F \cong A_5 P(U''_2) \times P(U''_2)$, in this case I say $F$ is a quadric with a diagonal $A_5$-action,
5. $F \cong A_5 P(U_2) \times P(U''_2)$, this case I say that $F$ is a quadric with a twisted diagonal $A_5$-action,
6. $F \cong A_5 P(U_2) \times P(I \oplus I)$ or $F \cong A_5 P(U''_2) \times P(I \oplus I)$, in this case I say that $F$ is a quadric with a one-factor $A_5$-action.

Knowing the induced $A_5$-action on $\mathbb{P}^3$ is very useful for studying cases (4a), (4b), and (4c).

Lemma 2.3. Let $F$ be a quadric with an action of $A_5$, let $V$ be the 4-dimensional representation dual to $H^0(F, -\frac{1}{2}K_F)$, and let $g : F \to \mathbb{P}(V)$ be the $A_5$-equivariant embedding induced by $|\frac{1}{2}K_F|$. Then

1. the $A_5$-action on $F$ is diagonal if and only if $V \cong A_5 W_3 \oplus I$ or $V \cong A_5 W''_3 \oplus I$;
2. the $A_5$-action on $F$ is twisted diagonal if and only if $V \cong A_5 W_4$;
3. the $A_5$-action on $F$ is a one-factor action if and only if $V \cong A_5 U_2 \oplus U_2$, or $V \cong A_5 U''_2 \oplus U''_2$.

Proof. Assertion (1) is equivalent to [4, Lemma 6.3.3, (i)].

Clearly, $\mathbb{P}(V)$ has no invariant lines and planes if and only if $F$ has no invariant divisors of degree $(1,1)$ or $(0,1)$. It follows that the action on $F$ is twisted diagonal if and only if $V \cong A_5 W_4$.

Suppose $F \cong A_5 P(U_2) \times P(I \oplus I)$. Let $L_1$ and $L_2$ be the curves of bi-degree $(0,1)$ on $F$. Then $L_1 \cong A_5 P(U_2)$ and $L_2 \cong A_5 P(U_2)$. Since $g(L_1)$ and $g(L_2)$ are $A_5$-invariant skew lines we see that $V \cong U_2 \oplus U_2$. To prove the converse, it is enough to show that there is the unique $A_5$-invariant quadric on $P(U_2 \oplus U_2)$. There are $A_5$-invariant skew lines $Z_1$ and $Z_2$ and any $A_5$-invariant quadric must contain both $Z_1$ and $Z_2$ since there are no orbits of length $\leq 2$ on $\mathbb{P}^1$. On the other hand, the space of quadrics containing a pair of $A_5$-invariant skew lines is isomorphic to $\mathbb{P}(W)$ for some 4-dimensional representation of $A_5$ or $2A_5$. The image on $\mathbb{P}(V)$ of the quadric with a one-factor action corresponds to an $A_5$-fixed point on $\mathbb{P}(W)$. It follows that $W \cong A_5 W_3 \oplus I$ or $W \cong A_5 W''_3 \oplus I$ Thus the unique $A_5$-invariant quadric on $P(U_2 \oplus U_2)$ is unique.

2.2. The general fiber is $\mathbb{P}^2$.

Lemma 2.4. Let $\pi : X \to \mathbb{P}^1$ be an $A_5Q$-del Pezzo fibration of degree 9 and suppose $A_5$ acts trivially on the base. Then $X$ is $A_5$-equivariantly birational to $\mathbb{P}^2 \times \mathbb{P}^1$ with a trivial action of $A_5$ on $\mathbb{P}^1$.

Proof. Let $X_\eta$ be the generic fiber of $\pi$. It is a form of $\mathbb{P}^2$ and has a point, thus it is $\mathbb{P}^2$. The isomorphism $X_\eta \cong \mathbb{P}^2$ induces the $A_5$-equivariant birational map to $\mathbb{P}^2 \times \mathbb{P}^1$. Note that $\mathbb{P}^2 \times \mathbb{P}^1$ is not the only $A_5Q$-del Pezzo fibration of degree 9 with a trivial action on the base.

Example 2.5. Let $\Delta$ be a conic in a fiber $F \subset \mathbb{P}^2 \times \mathbb{P}^1$ over the point $P \in \mathbb{P}^1$. Then we may blow up $\mathbb{P}^2 \times \mathbb{P}^1$ at $\Delta$ and then contract the proper transform of $F$ to acquire a variety $V_1$ with a $\frac{1}{2}(1,1,1)$-singularity. The composition $\mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow V_1$ of these maps is an elementary $A_5$-equivariant Sarkisov link of $A_5Q$-del Pezzo fibrations. The new fiber over $P$ is isomorphic to $\mathbb{P}(1,1,4)$.

Example 2.6. Consider a toric variety $R_\eta$ with Cox $R_\eta = \mathbb{C}[u,v,x,y,z,w]$, the irrelevant ideal $I = \langle u,v \rangle \cap \langle x,y,z,w \rangle$, and the grading given by

$$
\begin{pmatrix}
  u & v & x & y & z & w \\
  0 & 0 & 1 & 1 & 1 & 2 \\
  1 & 1 & 0 & 0 & 0 & -n
\end{pmatrix}
$$
I assume that $A_5$-action on $R_n$ comes from the identification of $\mathbb{C}^3_{x,y,z} \cong W_3$. Suppose that the equation of the $A_5$-invariant quadric on $W_3$ is $xz - y^2 = 0$. Then consider an $A_5$-invariant hypersurface $V_n$ in $R_n$ given by the equation

$$a_n(u,v)w = xz - y^2.$$ 

There is a natural $A_5$-equivariant projection $\pi_R: R_n \to \mathbb{P}^{1}_{u,v}$ which is a $\mathbb{P}(1,1,1,2)$-bundle. The restriction $\pi = \pi_R|_{V_n}: V_n \to \mathbb{P}^{1}_{u,v}$ is an $A_5$-del Pezzo fibration of degree 9 with the trivial action on the base. The variety $V_n$ has 2-Goreinstein singularities at $x = y = z = a_n(u,v) = 0$. For every simple root of $a_n(u,v)$ we get $\frac{1}{2}(1,1,1)$-singularity and for a root of multiplicity $k$ we get $cA_1/\mu_2$-singularity, where $\mu_2$ is the cyclic group of order 2.

**Question 2.7.** Let $\pi: X \to \mathbb{P}^1$ be an $A_5\mathbb{Q}$-del Pezzo fibration and suppose that the general fiber of $\pi$ is $\mathbb{P}^2$. Suppose $A_5$ acts trivially on the base. Is it true that $X \cong V_n$ for some $a_n(u,v)$?

2.3. The general fiber is $S_5$.

**Proposition 2.8.** Let $\pi: X \to \mathbb{P}^1$ be an $A_5\mathbb{Q}$-del Pezzo fibration of degree 5. Then it is $A_5$-equivariantly birational to $S_5 \times \mathbb{P}^1$.

**Proof.** Consider the generic fiber $X_\eta$ of $\pi$. It is a del Pezzo surface of degree 5 over $\mathbb{C}(t)$ which admits an action of $A_5$. Recall, that a del Pezzo surface of degree 5 over an algebraically closed field is unique and it admits a faithful $A_5$-action. Hence the $A_5$-action on $Pic(X_\eta)$ is faithful and induces an embedding $A_5 \hookrightarrow W(A_4) \cong S_5$, where $A_4$ is the root lattice.

Let $\Gamma = \text{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$, then the action of $\Gamma$ and $A_5$ on $X_\eta$ commutes. It follows that $\Gamma$ commutes with $W(A_4) = S_5$, therefore all $(-1)$-curves of $X_\eta$ are defined over $\mathbb{C}(t)$. Thus $X_\eta$ is acquired by a blow up of $\mathbb{P}^2$ at 4 points defined over $\mathbb{C}(t)$. In particular a del Pezzo surface of degree 5 with an $A_5$-action is unique.

On the other hand the generic fiber of the projection $S_5 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a del Pezzo surface of degree 5 with an action of $A_5$, hence isomorphic to $X_\eta$. The isomorphism induces the $A_5$-equivariant birational map $X \dashrightarrow S_5 \times \mathbb{P}^1$.

Unlike the the degree 9 case, one can show that there are no other $A_5\mathbb{Q}$-del Pezzo fibrations of degree 5.

**Lemma 2.9.** The threefold $S_5 \times \mathbb{P}^1$ is the unique $A_5\mathbb{Q}$-del Pezzo fibration of degree 5.

**Proof.** It is well known that $\text{det} A_5 S_5 \geq 1$, thus the statement follows from [2, Theorem 1.5] (see [10, Theorem 3.6] for $G$-invariant version). 

I expect that $S_5 \times \mathbb{P}^1$ is the unique $A_5\mathbb{Q}$-del Pezzo fibration in its birational class but it is $A_5$-equivariantly birational to other $A_5$-Mori fiber spaces. We can easily construct many $\mathbb{P}^1$-bundles $A_5$-equivariantly birational to $S_5 \times \mathbb{P}^1$ by blowing up orbits of fibers of the projection onto $S_5$.

**Question 2.10.** Does there exist an $A_5\mathbb{Q}$-Mori fiber space which is not $A_5$-equivariantly square birational to $S_5 \times \mathbb{P}^1/\mathbb{P}^1$ or $S_5 \times \mathbb{P}^1/S_5$ but is $A_5$-equivariantly birational to $S_5 \times \mathbb{P}^1$?

2.4. The general fiber is $S_3$.

**Lemma 2.11.** Let $X_\eta$ be a cubic over $\mathbb{C}(t)$ admitting a faithful $A_5$-action, then $\rho^{A_5} X_\eta = 2$. In particular, the $A_5\mathbb{Q}$-del Pezzo fibrations of degree 3 do not exist.

**Proof.** There is the unique $A_5$-invariant cubic $S_3$ in $\mathbb{P}^3$: Clebsh cubic. It follows that there is a unique $A_5$-invariant cubic $X_\eta$ in $\mathbb{P}^2_{C(t)}$. We may realize $X_\eta$ as the generic fiber of the projection $S_3 \times \mathbb{P}^1 \to \mathbb{P}^1$. Thus $\rho^{A_5} (X_\eta) = 2$. 

\[ \square \]
2.5. The general fiber is a quadric. Recall that there are three types of $A_5$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ as defined in Proposition 2.2: diagonal, twisted diagonal, and one-factor action. The $A_5$-quadric fibration exist only for the former type.

Lemma 2.12. Let $\pi : X \to \mathbb{P}^1$ be an $A_5$-equivariant map such that the $A_5$-action on a general fiber $F$ of $\pi$ is either twisted diagonal or a one-factor action. Let $X_\eta$ be the generic fiber of $\pi$, then $\rho(X_\eta) = 2$.

Proof. Suppose the $A_5$-action on $F$ is twisted diagonal, then $F \subset \mathbb{P}(W_4)$ by Lemma 2.3. Since $\text{Sym}^2(W_4) \cong W_5 \oplus W_4 \oplus I$, there is the unique $A_5$-invariant quadric $S_\eta$ in $\mathbb{P}(W_4 \oplus \mathbb{C}(t))$. On the other hand, $S_\eta$ is also a general fiber of the projection
\[ \mathbb{P}(U_2) \times \mathbb{P}(U_2) \times \mathbb{P}(I \oplus I) \to \mathbb{P}(I \oplus I), \]
thus we see that $\rho^A(S_\eta) = 2$.

Suppose the $A_5$-action on $F$ is a one-factor action corresponding to the representation $U_2$ of $A_5$. Then $F \subset \mathbb{P}(U_2 \oplus U_2)$ is the unique $A_5$-invariant quadric by Lemma 2.3. Similarly to the twisted diagonal case I conclude that $\rho^A(S_\eta) = 2$. \qed

Recall the construction of the $A_5\mathbb{Q}$-quadric fibrations $X_n$ and $X'_n$ from Example 1.7.

Proposition 2.13. Let $\pi_V : V \to \mathbb{P}^1$ be an $A_5\mathbb{Q}$-quadric fibration. Suppose the action on a general fiber $F$ of $\pi_V$ is diagonal. Then $V$ is $A_5$-equivariantly birational to a smooth $X_n$ or $X'_n$ for some $a_{2n}(u,v)$.

Proof. The fiber $F$ embeds into $\mathbb{P}(W_3 \oplus I)$ or $\mathbb{P}(W'_3 \oplus I)$ by Lemma 2.3. Suppose the former, the latter case is analogous, then
\[ X_\eta \hookrightarrow \mathbb{P}((W_3 \oplus I) \otimes \mathbb{C}(t)). \]
Let $x, y, z$ be the coordinates on $W_3$ and let $w$ be the coordinate on $I$. Then, up to a change of coordinates on $W_3$, every $A_5$-invariant quadric in $\mathbb{P}(W_3 \oplus I)$ has the equation
\[ \lambda(t)w^2 = \mu(t)(xz - y^2). \]
It follows that up to a change of coordinates in $\mathbb{P}((W_3 \oplus I) \otimes \mathbb{C}(t))$ the equation of $X_\eta$ is
\[ b(t)w^2 = xz - y^2, \]
where $b(t)$ has no multiple roots.

Set $a(u,v)$ to be the homogeneous polynomial without multiple factors satisfying $a(t,1) = b(t)$. Then the general fiber of $X_n \to \mathbb{P}^1$ is $A_5$-equivariantly isomorphic to $X_\eta$. This isomorphism of the general fibers induces the $A_5$-equivariant birational map. \qed

This completes the proof of Theorem 1.8.

3. Rigidity results

This section is devoted to proving Theorem 1.11. First, I present some elementary results on geometry of varieties $X_n$. Next, I prove that $X_n$ and $X'_n$ are $A_5$-birationally superrigid for $n \geq 2$.

3.1. The geometry of $X_n$. Note that $T_n \cong T'_n$ and for a fixed $a_{2n}$ we have $X_n \cong X'_n$. These isomorphisms are not $A_5$-equivariant. From now on I work with $T_n$ and $X_n$, the proofs for $X'_n$ are identical. Note that in this section I do not assume that $a_{2n}(u,v)$ has no multiple factors. In that case $X_n$ is no longer smooth. Indeed, for each multiple linear factor $l(u,v)$ of $a_{2n}(u,v)$ the point $(x = y = z = l(u,v) = 0)$ is a $cA_1$-singularity.

Denote by $H_T$ the divisor class of $(x = 0)$ and by $F_T$ the divisor class of $(u = 0)$ on $T_n$. Denote $H = H_T|_{X_n}$ and $F = F_T|_{X_n}$. Let $s \in A^3(T_n)$ and $s \in A^2(X_n)$ be the classes of the curve $x = y = w = 0$. Let $f_T \in A^3(T_n)$ and $f \in A^2(X_n)$ be the classes of the curve $x = y = u = 0$. Using the fact that
\[ (x = y = z = w = 0) = \emptyset \quad \text{and} \quad (x = y = z = u = 0) = pt \]
we compute the intersections on $T_n$.

Lemma 3.1. The following holds for $T_n$:

1. The classes $s_T$ and $f_T$ generate the cone of effective curves on $T_n$;
(2) $H^2_T \cdot F_T \equiv f_T$;
(3) $H^3_T \equiv s_T + nf_T$;
(4) $H^4_T \cdot F_T = 1$;
(5) $H^4 = n$.

**Corollary 3.2.** The following holds for $X_n$:

1. $H \cdot F \equiv 2f$;
2. $H^2 \equiv 2s + 2nf$;
3. $H^2 \cdot F = 2$;
4. $H^3 = 2n$;
5. $K_{X_n} \sim -2H + (n - 2)F$;
6. $K^2_{X_n} \equiv 8s + 24f - 8nf$.

In order to better understand a singular $X_n$, it is useful to know how it is related to a smooth $X_m$.

**Lemma 3.3.** Let variety $X_n$ correspond to $a_{2n}(u,v)$ and $X_{n+1}$ to $u^2a_{2n}(u,v)$. Denote the fiber $u = 0$ by $F$. Let $\sigma: \tilde{X} \to X_n$ be the blow up of $X_n$ along the curve $u = w = 0$. Then there is a map $\psi: \tilde{X} \to X_{n+1}$ contracting $\sigma^{-1} F$ to a $cA_1$ singularity $x = y = z = u = 0$. The composition $\varphi_u = \psi \circ \sigma^{-1}$ is an $A_5$-equivariant elementary Sarkisov link.

![Diagram]

**Proof.** Elementary calculations.

Thus we see that any singular $X_n$ is $A_5$-equivariantly square birational to a smooth $X_m$ with $m = n - 2k$ for some $k > 0$.

**Lemma 3.4.** Suppose $a_{2n}(u,v)$ is not a square, then

1. $\text{Pic} X_n \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot F$,
2. The classes $s$ and $f$ generate the cone of effective curves $\text{NE}(X_n)$.

**Proof.** The assertion (1) holds since $\text{Pic} X_n \cong \text{Pic} X_\eta \oplus \text{Pic} \mathbb{P}^1$ and $\text{Pic} X_\eta$ is generated by the hyperplane section if and only if $a_{2n}(u,v)$ is not a complete square.

The assertion (2) follows from (1) and Lemma 3.1 (1) for smooth $X_n$ by Poincaré duality. Any $X_n$ is related to some smooth $X_m$ by a sequence of elementary Sarkisov links described in Lemma 3.3. The elementary links preserve the dimension of $\text{NE}(X_n)$, hence the assertion holds by Lemma 3.1 (1).

**Lemma 3.5.** Suppose $n \geq 2$, then $X_n$ satisfies the $K$-condition, that is $-K_{X_n}$ is not in the interior of the cone of movable divisors.

**Proof.** The linear system $|H|$ defines a divisorial contraction $\sigma: X_n \to Y_n$, where $Y_n$ is a hypersurface in $\mathbb{P}(1,1,n,n,n)$ given by the equation $a_{2n}(u,v) = xz - y^2$.

Thus the cone of movable divisors of $X_n$ is generated by $H$ and $F$ which implies the statement of the lemma.

### 3.2. Towards birational superrigidity.

For definitions of canonical singularities of pairs we refer the reader to [9, pages 16-17] and [5, Definition 2.1].

Suppose we are given a birational map $\chi: X \dasharrow Y$ to a Mori fiber space $\pi_Y: Y \to Z$. Let $\mathcal{M}_Y$ be a very ample complete linear system on $Y$. I say $\mathcal{M} = \chi^{-1}\mathcal{M}_Y$ is a mobile linear system associated to $f$. There are numbers $\lambda \in \mathbb{Q}_+$ and $l \in \mathbb{Q}$ such that $\lambda \mathcal{M} \sim -K_X + lF$. The Noether-Fano inequality is the essential result used to prove birational rigidity-type results.
Theorem 3.6 (Noether-Fano inequality, [6, Theorem 4.2]). Suppose \( l > 0 \) and \((X, \lambda M)\) has canonical singularities, then \( \chi \) is isomorphism.

Set \( X = X_n \) for some \( a_{2n}(u, v) \) that is not a square and let \( \chi, M, \lambda, l \) be as above. I now examine at which subvarieties the pair \((X, \lambda M)\) could be non-canonical.

Let \( C \) be an irreducible curve on \( X \), I say \( C \) is horizontal if \( \pi(C) = \mathbb{P}^1 \) and vertical if \( \pi(C) \) is a point. I say that a curve \( C \) is horizontal (resp. vertical) if every irreducible component of \( C \) is horizontal (resp. vertical).

Let \( C \subset X \) be a horizontal or a vertical curve. I define its degree as follows

\[
\text{deg}(C) = \begin{cases} 
-K_X \cdot C/2, & \text{if } C \text{ is vertical} \\
C \cdot F, & \text{if } C \text{ is horizontal}
\end{cases}
\]

Lemma 3.7. Suppose \((X, \lambda M)\) is not canonical at a curve \( \Delta \), then \( \Delta \) is an \( A_5 \)-invariant vertical curve of degree 2.

Proof. Let \( \overline{\Delta} \) be the \( A_5 \)-orbit of \( \Delta \). The pair \((X, \lambda M)\) is not canonical at a curve \( \Delta \) if and only if \( \text{mult}_{\Delta} \lambda M > 1 \) and hence if and only if \( \text{mult}_{\overline{\Delta}} \lambda M > 1 \).

Suppose \( \Delta \) is horizontal, let \( F \) be a general fiber of \( \pi \) and let \( D_1, D_2 \in M \) be general divisors. Then the set-theoretic intersection

\[
\Sigma = F \cap \overline{\Delta} \subset F \cap D_1 \cap D_2
\]

is a union of orbits on \( F \). Hence

\[
8 = F \cdot \lambda D_1 \cdot \lambda D_2 > \lambda^2 F \cdot \overline{\Delta} \geq |\Sigma| \geq 12,
\]

a contradiction.

Suppose \( \overline{\Delta} \subset F \) and let \( D \) be general in \( M \). Let \( H_F = H|_F \), then \( D|_F \sim 2H_F \) and \( \text{ord}_F D|_F > 1 \).

It follows that \( \text{deg} \overline{\Delta} \leq 4 \) and [4, Lemma 6.4.4] implies that \( \overline{\Delta} = \Delta \) is the \( A_5 \)-invariant curve of degree 2.

Suppose \( F \) is singular, then it is a cone over \( \mathbb{P}^1 \) with \( A_5 \)-action inherited from \( \mathbb{P}^1 \). Let \( C \) be the unique \( A_5 \)-invariant hyperplane section of \( F \). Suppose \( \Delta \neq C \) and denote \( \Delta \sim kH_F \). Set \( \Sigma = \Delta \cap C \), it is a union of \( A_5 \)-orbits on \( C \) hence \( |\Sigma| \geq 12 \) by Lemma 2.1. It follows that \( \Delta \cdot C \geq 12 \) and \( k \geq 6 \). On the other hand, \( D|_F \sim 2H_F \), a contradiction. \( \square \)

Lemma 3.8. Let \( P \) be an \( A_5 \)-invariant point and suppose \( X \) is smooth at \( P \). Then pair \((X, \lambda M)\) is canonical at \( P \).

Proof. Suppose \((X, \lambda M)\) is not canonical at \( P \) and let \( E_0 \) be the divisorial valuation over \( X \) such that \( a(E_0, X, \lambda M) < 0 \).

First, observe that a fiber \( F \) of \( \pi \) containing \( P \) is a quadratic cone and \( P \) is its vertex. Let \( \sigma: \tilde{X} \to X \) be the blow up at \( P \) and let \( E \) be the exceptional divisor of \( \sigma \). Let \( L \) be a general line through \( P \), then for general \( D \in M \)

\[
\text{mult}_P \lambda D \leq L \cdot \lambda D = 2.
\]

It follows that \( a(E, X, \lambda M) \geq 0 \), hence the center \( B \) of \( E_0 \) on \( \tilde{X} \) is a point or a curve on \( E \).

Note that the action of \( A_5 \) on \( E \) is non-trivial. Indeed, the point \( P \) up to a change of coordinates on \( \mathbb{P}^1_{u,v} \) has the equations \( u = x = y = z = 0 \) and the local equation of \( X \) near \( P \) is \( u = 0 \), thus \( E \cong A_5 \mathbb{P}(W_3) \). Denote \( \tilde{M}_E = (\sigma^{-1} M)|_E \), then \( \text{mult}_B \tilde{M}_E > 1 \) and \( \deg \tilde{M}_E = \text{mult}_P \lambda M \leq 2 \). On the other hand, if \( B \) is a curve, then \( \deg B \geq 2 \), a contradiction.

Suppose \( B \) is a point and let \( \overline{B} \) be the \( A_5 \)-orbit of \( B \). Then \( |\overline{B}| \geq 6 \) and there are 4 points \( P_1, P_2, P_3, P_4 \in \overline{B} \subset E \cong \mathbb{P}^2 \) in general position. I claim that

\[
\sum_{i=1}^{4} \text{mult}_{P_i} C \leq 2 \text{deg} C
\]
for any curve $C \subset E$. Indeed, denote by $L_{ij}$ the line on $E$ passing through $P_i$ and $P_j$. Decomposing $C = C' + \sum \alpha_{ij} L_{ij}$ and counting multiplicities I conclude the inequality (1). Thus
\[
4 < 4 \text{mult}_{B} \lambda \mathcal{M}_{E} = \sum_{i=1}^{4} \text{mult}_{P_i} \lambda \mathcal{M}_{E} \leq 4,
\]
a contradiction. \hfill \square

Recall that in Lemma 3.3 I have constructed an elementary Sarkisov link between $X_n$ and $X_{n+1}$. I now show that a composition of these links “untwists” vertical curves of degree 2.

Let $b(u, v)$ be a polynomial of degree $k$. Then there is the associated map $\varphi_{b}: X \to X_{b}$, where $X_{b}$ is a hypersurface in $T_{n+k}$ given by the equation $a(u, v)(b(u, v))^{2}w^{2} = xz - y^{2}$. The map $\varphi_{b}$ is the composition of elementary links described in Lemma 3.3. Denote $\mathcal{M}_{b} = \varphi_{b}^{-1}\mathcal{M}$.

**Proposition 3.9.** Suppose $X$ is smooth, then there is $b(u, v)$ such that the pair $(X_{b}, \lambda \mathcal{M}_{b})$ is canonical at curves and $A_{5}$-invariant points.

**Proof.** I prove the proposition by playing the two-ray game. Suppose $(X, \lambda \mathcal{M})$ is not canonical at a curve $\Delta_{1}$. By Lemma 3.7 the curve $\Delta_{1}$ is of degree 2 and is $A_{5}$-invariant, hence its equations are $l_{1}(u, v) = w = 0$ for some linear $l_{1}(u, v)$. The elementary Sarkisov link starting at $\Delta_{1}$ is the map $\varphi_{l_{1}}$. Let $\mathcal{M}_{l_{1}} = \varphi_{l_{1}}^{-1}\mathcal{M}$. If the pair $(X_{l_{1}}, \mathcal{M}_{l_{1}})$ is canonical at curves, then I am done. Otherwise there is a curve $\Delta_{2}$ and an elementary Sarkisov link $\varphi_{l_{2}}: X_{l_{1}} \to X_{l_{2}}$, and I repeat the process as many times as required. The process terminates by [6, Theorem 6.1] and I set $b = l_{1}l_{2} \ldots l_{k}$. The pair $(X_{b}, \lambda \mathcal{M}_{b})$ is canonical at curves by construction and Lemma 3.7. The pair $(X_{b}, \lambda \mathcal{M}_{b})$ is canonical at smooth $A_{5}$-invariant points by Lemma 3.8. I will now show that the pair $(X_{b}, \lambda \mathcal{M}_{b})$ is canonical at singular $A_{5}$-invariant points as well.

Recall that by [8, Theorem 1.1] if the pair $(X_{b}, \lambda \mathcal{M}_{b})$ is not canonical at a $cA_{1}$-point $P$ with the local equation
\[
xx + y^{2} + uy^{N} = 0,
\]
then $a(E_{0}, X_{b}, \lambda \mathcal{M}_{b}) < 0$, where $E_{0}$ is the exceptional divisor of a $(s, t, 2t - s, 1)$-weighted blow up at $P$ (or $(1, 3, 5, 2)$-weighted blow up if $N = 3$) for some coprime $s \leq t \leq N/2$. Note that $a(E_{0}, X_{b}, \lambda \mathcal{M}_{b}) > 0$ for $s = t = 1$ by construction. Thus, if $N = 2$, we are done. We proceed by induction. If $N = 3$, then $a(E_{0}, X_{b}, \lambda \mathcal{M}_{b}) < 0$ for $(1, 3, 5, 2)$-weighted blow up at $P$. Then the pair $(X_{b}, \lambda \mathcal{M}_{b})$ is not canonical at the invariant $A_{5}$-invariant point in the fiber $u = 0$, which contradicts Lemma 3.8.

Similarly if $N \geq 4$, then $a(E_{0}, X_{b}, \lambda \mathcal{M}_{b}) \geq 0$ for $s \geq 2$, otherwise the pair $(X_{b}, \lambda \mathcal{M}_{b})$ is not canonical at the invariant $A_{5}$-invariant point in the fiber $u = 0$. Thus we may assume that $s = 1$ and $t > 1$. Let $\sigma: \tilde{X}_{b} \to X$ be the blow up of at $P$ and let $\tilde{\mathcal{M}}_{b} = \sigma^{-1}\mathcal{M}_{b}$. Then the pair $(\tilde{X}_{b}, \lambda \tilde{\mathcal{M}}_{b})$ is not canonical at a line $L$ on the exceptional divisor $E_{0}$ of $\sigma$. Hence, it is not canonical at each line in the orbit of $L$. It follows that $\text{deg} \tilde{\mathcal{M}}_{b}|_{E_{0}} > 6$ since the length of the orbit of $L$ is at least 12, which contradicts $a(E_{0}, X_{b}, \mathcal{M}_{b}) > 0$. \hfill \square

It remains to exclude the points that are not fixed by the $A_{5}$-action.

3.2.1. **Excluding orbits of points.** I am going to choose a surface $S$ containing some points in the orbit of $P$. Then I am going to use multiplicities of $\mathcal{M}|_{S}$ at points infinitely close to $P_1$ to compute bounds on multiplicities of $\mathcal{M}$ at subvarieties infinitely close to $P_1$. These bounds contradict the Noether-Fano inequality.

First, I construct the suitable surface $S$. Let $P$ be a point, let $F$ be the fiber of $\pi$ containing $P$, and let $\overline{F}$ be the orbit of $P$. Suppose that $|\overline{F}| > 1$, then no two points in $\overline{F}$ lie on the same line in $F$.

**Lemma 3.10.** Let $P_{1}, \ldots, P_{9}$ be the points on $F$ such that no two points lie on a line. Then there is a curve $\Gamma$ of degree $\gamma$ such that
\begin{enumerate}
\item $\gamma = 2$ or $\gamma = 4$,
\item $\Gamma$ passes through $2\gamma$ points among $P_{1}, \ldots, P_{9}$,
\item $\Gamma$ is smooth at these points.
\end{enumerate}
Proof. For any 8 points on $F$ there is a unique quadric section passing through them. Let $\Gamma$ be the quadric section through $P_1, \ldots, P_8$, clearly $\deg \Gamma = \gamma = 4$. Suppose it is not smooth at one of the points, for example $P_1$.

First, suppose $\Gamma$ is reducible, then clearly $\Gamma$ has at most 3 components. If $\Gamma$ has 3 components, then there is a conic $\Gamma'$ containing at least 6 points among $P_1, \ldots, P_8$. Since $\Gamma'$ is irreducible and hence smooth we are done.

Suppose $\Gamma = \Gamma_1 + \Gamma_2$ where $\Gamma_1, \Gamma_2$ are irreducible. In that case $\Gamma_1 \cong \Gamma_2 \cong \mathbb{P}^1$. If $\deg \Gamma_1 = \Gamma_2 = 2$, then there is a curve of degree 2 containing at least 4 points among $P_1, \ldots, P_8$. If $\deg \Gamma_1 = 1$ and $\deg \Gamma_2 = 3$, then $P_1 \in \Gamma_1$ and $P_1, \ldots, P_8 \in \Gamma_2$. Then $\Gamma_2 + L$, where $L$ is a general line is the curve we are looking for.

Now we may assume that some quadric section through any of the 8 points among $P_1, \ldots, P_8$ is reducible and singular at one of these points. If these quadric sections coincide, that is there is a quadric section through all 9 points, then it is singular at most at one point, hence we are done. We now assume that all of these quadric sections are different curves.

Let $\Gamma'$ be the quadric section through $P_1, \ldots, P_7, P_9$. Since $\Gamma \cdot \Gamma' = 8$, the $\Gamma'$ must be singular at $P_9$. Similarly the quadric section $\Gamma''$ through $P_1, \ldots, P_6, P_8, P_9$ is singular at $P_9$. It follows that

$$8 = \Gamma'' \cdot \Gamma' \geq \sum \mult_{P_i} \Gamma'' \mult_{P_i} \Gamma' \geq 10,$$

a contradiction. \qed

It follows that for any orbit $\overline{P}$ on $F$ such that $|\overline{P}| > 1$ there is a curve $\Gamma$ of degree $\gamma$ containing 2$\gamma$ points from $\overline{P}$. Let $u = f(x, y, z, w) = 0$ be the equations of $\Gamma$.

Consider the linear system $\mathcal{S} \subset |\gamma H + F|$ of divisors containing $\Gamma$. Let $S$ be general in $\mathcal{S}$ and let $D$ be general in $\mathcal{M}$. Denote $D_S = D|_S$ and $\mult_D D = \alpha$, then $\ord_D D_S = \alpha$. Indeed, any $S \in \mathcal{S}$ has the equation $a(u, v) f(x, y, z, w) + a p(x, y, z, w)$ for some $p$ of degree $\gamma$ and linear $a$. Thus for general $S_1, S_2 \in \mathcal{S}$ we have $\ord_D S_1 \cap S_2 = 1$. It follows that we may decompose

$$D_S = \alpha \Gamma + D'_S.$$

I now recall the result of Kawakita on extremal contractions.

**Theorem 3.11 (Kawakita).** Let $\sigma : Y \to X$ be an extremal divisorial contraction in Mori category to a smooth point $P$. Then $\sigma$ is a weighted blow up of $X$ at $P$ with weights $(1, K, N)$ for some coprime $K, N$.

We use this as follows. By Noether-Fano inequality there is a divisorial valuation $E$ such that $a(E, X, \lambda \mathcal{M}) < 0$. It is well known that there is $E$ such that it is an exceptional divisor of an extremal divisorial contraction. Consider the tower of blow ups

$$X_N \to \cdots \to X_K \to \cdots \to X_1 \to X_0 = X,$$

where $\sigma_i$ is the blow up of $X_{i-1}$ at the center $B_{i-1}$ of $E$, $E_i$ is the exceptional divisor of $\sigma_i$. By Theorem 3.11 and elementary toric calculations we have

1. $E_N = E$ as divisorial valuations of $\mathbb{C}(X)$,
2. $B_i$ is a point for $i < K - 1$ and is a curve for $i \geq K$,
3. $B_j \cap E_j^{(i-1)} = \varnothing$.

For an object $A$ on $X_i$ I denote its proper transform on $X_j$ by $A^{(j)}$. Denote $\nu_i = \lambda \mult_{B_i} D^{(i-1)}$, then $a(E, X, \lambda \mathcal{M}) < 0$ if and only if

$$\nu_1 + \cdots + \nu_N > N + K.$$

Denote $\tilde{B}_j = B_j \cap \Gamma^{(j)}$ and set $L = \min\{j \mid \tilde{B}_{j-1} \neq \varnothing\}$, then $\tilde{B}_j$ is a point for $j \leq L$. Denote $\tilde{\nu}_i = \lambda \mult_{\tilde{B}_i} D_S^{(j)}$, then we may bound $\nu_i$ using $\tilde{\nu}_i$. But first, an auxiliary lemma.

**Lemma 3.12.** Let $B$ be a point or a smooth curve on a smooth threefold $X$. Let $S$ be a surface on $X$, suppose $\tilde{B} = B \cap S$ is a point and suppose $S$ is smooth at $\tilde{B}$. Let $\sigma : Y \to X$ be the blow up at $B$, let $E$
be its exceptional divisor, let \( S_Y = \sigma^{-1}S \) and let \( e = E \cap S_Y \). Let \( D \) be an effective divisor on \( X \) and let \( D_Y = \sigma^{-1}D \), then

\[
\text{mult}_B D|_S = \text{mult}_B D + \text{ord}_e D_Y|_{S_Y}.
\]

**Proof.** Elementary calculations in local coordinates. \( \square \)

**Lemma 3.13.** For any \( k \leq L \) we have

\[
\nu_1 + \cdots + \nu_k \leq \tilde{\nu}_1 + \cdots + \tilde{\nu}_k.
\]

**Proof.** Let \( e_i = S^{(i)} \cap E_i \) and denote \( m_i = \text{ord}_{e_j} \lambda D^{(j)}|_{S^{(i)}} \). Then

\[
D^{(j)}|_{S^{(i)}} = D_{S^{(i)}}^{(j)} + m_1 e_1^{(j)} + \cdots + m_j e_j.
\]

By the structure of the tower of blow ups we get

\[
\text{mult}_{B_j} \lambda D^{(j)}|_{S^{(i)}} = \tilde{\nu}_{j+1} + m_j \quad \text{for} \quad j \geq 1.
\]

On the other hand by Lemma 3.12

\[
\text{mult}_{B_j} \lambda D^{(j)}|_{S^{(i)}} = \nu_{j+1} + m_{j+1}.
\]

Thus we get the equalities

\[
\begin{align*}
\tilde{\nu}_1 &= \nu_1 + m_1 \\
\tilde{\nu}_2 + m_1 &= \nu_2 + m_2 \\
&\quad \cdots \\
\tilde{\nu}_L + m_{L-1} &= \nu_L + m_L,
\end{align*}
\]

which together imply the statement of the lemma. \( \square \)

We have a tower of blow ups of the form (2) for each of the points \( P_i \). In principle the numbers \( \nu_i, \tilde{\nu}_i, N, \) and \( K \) are different for the tower over \( P_i \) and over \( P_j \) for \( i \neq j \). But since \( P_i \) lie in the same orbit, the multiplicities and the resolutions are identical, hence I do not introduce new notations for them.

Instead assume that in (2) the map \( \sigma_i \) is the blow up at the orbit of \( B_i \). I denote the centers by \( B_{i,j} \) and their intersections with \( L^{(i)} \) by \( \tilde{B}_{i,j} \); the indices signify that \( B_{i,j} \in X_j \) and that \( B_{i,j} \) is in the preimage of \( P_i \).

**Proposition 3.14.** There is a bound

\[
\tilde{\nu}_1 + \cdots + \tilde{\nu}_N \leq L + 1 + \frac{N - L}{L}
\]

**Proof.** The curve \( \Gamma \) is smooth at \( P_i \), thus

\[
\tilde{\nu}_j = \lambda \alpha + \lambda \text{mult}_{\tilde{B}_{i,j}} D_{S^{(i)}}^{(j)} \quad \text{for} \quad j \leq L \quad \text{and}
\]

\[
\tilde{\nu}_j = \lambda \text{mult}_{B_{i,j}} D_{S^{(i)}}^{(j)} \quad \text{for} \quad j > L.
\]

On the other hand \( S \cap F = \Gamma \), therefore

\[
\lambda \sum_{i=1}^{2\gamma} \sum_{j=1}^{L} \text{mult}_{\tilde{B}_{i,j}} D_{S^{(i)}}^{(j)} \leq \lambda D_S \cdot F = \lambda D \cdot F \cdot S = 2\gamma.
\]

Observe that this implies \( \lambda \text{mult}_{\tilde{B}_{i,j}} D_{S^{(i)}}^{(j)} \leq \frac{1}{L} \) for \( j > L \).

Recall that \( M \) is canonical at \( \Gamma \) by Proposition 3.9, that is \( \lambda \alpha \leq 1 \). Thus putting the bounds together we get the statement of the lemma. \( \square \)

**Corollary 3.15.** Let \( P \) be a point which is not \( \mathcal{A}_5 \)-fixed. Then the pair \( (X, \lambda M) \) is canonical at \( P \).

**Proof.** The bound (4) contradicts Noether-Fano inequality (3). \( \square \)

At last I am ready to prove Theorem 1.11.
Proof of Theorem 1.11. Let $\chi: X_n \dashrightarrow Y$ be an $A_5$-equivariant birational map to a $A_5Q$-Mori fiber space $Y/Z$ and let $M$ be the mobile $A_5$-invariant linear system associated $\chi$. Define the numbers $\lambda$ and $l$ by the equivalence $\lambda M + K_{X_n} \sim lF$. By Lemma 3.9 there is another birational model $\pi_b: X_b \to \mathbb{P}^1$ and an $A_5$-equivariant square birational map $\varphi_b: X_n \to X_b$ such that the corresponding pair $(X_b, \lambda M_b)$ is canonical at curves and $A_5$-fixed points. By Corollary 3.15 the pair $(X_b, \lambda M_b)$ is also canonical at points not fixed by $A_5$-action. Thus by Noether-Fano inequality the map $\chi \circ \varphi_b^{-1}: X_b \to Y$ is an isomorphism. I have shown that $X_n$ is $A_5$-equivariantly birational only to $X_m$ with $a_{2m} = a_{2n}b_m^{2m-n}$, in particular I have shown that $\chi$ is $A_5$-equivariantly square birational, hence $X_n$ is $A_5$-equivariantly birationally superrigid. □

4. The $A_5$-equivariant birational geometry of $X_1$

In this section I describe some $A_5$-Mori fiber spaces $A_5$-equivariantly birational to the variety $X_1$. Let $Q \subset \mathbb{P}(W_3 \oplus I \oplus I)$ be a smooth $A_5$-invariant quadric. Let $x, y, z$ be the coordinates on $W_3$ and $u, v$ be the coordinates on $I \oplus I$. Let $\Gamma$ be the $A_5$-invariant point curve of degree 2, then it has equations $(xz - y^2 = u = v = 0)$ after a change of coordinates on $W_3$. Thus the blow up of $Q$ at $\Gamma$ is $\sigma:\Gamma \to Q$. On the other hand $Q$ is a quadric with an $A_5$-invariant point, hence it is $A_5$-equivariantly birational to $\mathbb{P}(W_3 \oplus I)$.

Consider the blow up at the $A_5$-invariant point $\sigma_Y: Y_1 \to \mathbb{P}(W_3 \oplus I)$, where

$$Y_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}(W_3)} \oplus \mathcal{O}_{\mathbb{P}(W_3)}(-1)).$$

The $\mathbb{P}^1$-bundle $\tau: Y_1 \to \mathbb{P}(W_3)$ has many $A_5$-equivariant birational models. For example, an $A_5$-invariant conic on the exceptional divisor of $\sigma_Y$ induces the $A_5$-equivariant elementary Sarkisov link $Y_1 \dashrightarrow Y_3$, where

$$Y_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}(W_3)} \oplus \mathcal{O}_{\mathbb{P}(W_3)}(-3)).$$

Similarly, $Y_1$ is $A_5$-equivariantly birational to $Y_{2k+1}$ for any $k$. Alternatively, we can take a fiber $f$ of $\tau$ and the blow up $\sigma: \tilde{Y} \to Y_1$ at the orbit of $f$, it is easy to see that $\tilde{Y}$ admits an $A_5$-equivariant $\mathbb{P}^1$-bundle.

As we can see, $X_1$ has a rich $A_5$-equivariant birational geometry with the following $A_5$-Mori fiber spaces structures:

1. $A_5$-Fano variety $Q \subset \mathbb{P}(W_3 \oplus I \oplus I)$,
2. $A_5$-Fano variety $\mathbb{P}(W_3 \oplus I)$,
3. $A_5$-del Pezzo fibration $\pi: X_1 \to \mathbb{P}^1$,
4. $A_5$-conic bundle $\tau: Y_1 \to \mathbb{P}(W_3)$.

There is some evidence to suggest that these are the only $A_5$-Mori fiber space structures. Suppose $\chi: Q \to Y$ is a birational $A_5$-equivariant map to a $A_5Q$-Mori fiber space $\pi_Y: Y \to Z$ Let $M_Q$ be the associated mobile linear system. If $(Q, \lambda Q, M_Q)$ is not canonical at the $A_5$-invariant conic, then the map $\chi$ factors through $X_1$. Let $M$ be the corresponding mobile linear system on $X_1$. Elementary calculations show that $M \sim -K_{X_1} + lF$, where $l > 0$. It follows that the results of Section 3 are applicable and $Y/Z$ is $A_5$-equivariantly square birational to $X_1/\mathbb{P}^1$.

**Question 4.1.** Does there exist an $A_5Q$-Mori fiber space $A_5$-equivariantly birational to $Q$ which is not $A_5$-equivariantly square birational to $X_1/\mathbb{P}^1$, $Y_1/\mathbb{P}(W_3)$, $Q$, or $\mathbb{P}(W_3 \oplus I)$?

**References**

1. Hamid Ahmadinezhad, Ivan Cheltsov, Jihun Park, and Constantin Shramov, *Double veronese cones with 28 nodes*, 2019.
2. I. Cheltsov, *On singular cubic surfaces*, Asian J. Math. 13 (2009), no. 2, 191–214. MR 2559108
3. Ivan Cheltsov and Constantin Shramov, *Fives embeddings of one simple group*, Trans. Amer. Math. Soc. 366 (2014), no. 3, 1289–1331. MR 3145732
4. ______, *Cremona groups and the icosahedron*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016. MR 3444095
5. A. Corti, *Singularities of linear systems and 3-fold birational geometry*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 259–312.
6. Alessio Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254. MR 1311348
7. Igor V. Dolgachev and Vasily A. Iskovskikh, *Finite subgroups of the plane Cremona group*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 443–548. MR 2641179
8. Masayuki Kawakita, *Divisorial contractions in dimension three which contract divisors to compound A1 points*, Compositio Math. **133** (2002), no. 1, 95–116. MR 1918291
9. János Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR 1492525
10. Igor Krylov, *Birational geometry of del Pezzo fibrations with terminal quotient singularities*, J. Lond. Math. Soc. (2) **97** (2018), no. 2, 222–246. MR 3789845
11. Yuri Prokhorov, *Simple finite subgroups of the Cremona group of rank 3*, J. Algebraic Geom. **21** (2012), no. 3, 563–600. MR 2914804
12. , *G-Fano threefolds, I*, Adv. Geom. **13** (2013), no. 3, 389–418. MR 3100917
13. , *On stable conjugacy of finite subgroups of the plane cremona group, ii*, The Michigan Mathematical Journal **64** (2015), no. 2, 293–318.
14. Yuri Prokhorov and Constantin Shramov, *Jordan property for Cremona groups*, Amer. J. Math. **138** (2016), no. 2, 403–418. MR 3483470
15. , *p-subgroups in the space Cremona group*, Math. Nachr. **291** (2018), no. 8-9, 1374–1389. MR 3817323

Igor Krylov
Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea
IKrylov@kias.re.kr