Integrable nonlocal asymptotic reductions of physically significant nonlinear equations

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Received 13 January 2019, revised 28 February 2019
Accepted for publication 11 March 2019
Published 21 March 2019

Abstract
Quasi-monochromatic complex reductions of a number of physically important equations are obtained. Starting from the cubic nonlinear Klein–Gordon (NLKG), the Korteweg–de Vries (KdV) and water wave equations, it is shown that the leading order asymptotic approximation can be transformed to the well-known integrable AKNS system (Ablowitz \textit{et al} 1974 \textit{Stud. Appl. Math.} \textbf{53} 249) associated with second order (in space) nonlinear wave equations. This in turn establishes, for the first time, an important physical connection between the recently discovered nonlocal integrable reductions of the AKNS system and physically interesting equations. Reductions include the parity-time, reverse space-time and reverse time nonlocal nonlinear Schrödinger equations.

Keywords: integrable systems, nonlinear waves, solitons and inverse scattering transform, nonlinear Schrödinger equation

1. Introduction
Ever since the seminal work establishing that the KdV, nonlinear Schrödinger (NLS), sine-Gordon equations and many others are integrable [1–6], there has been an enormous effort directed at finding solutions and understanding the mathematical and physical properties of these equations see [7–12]. Methods include the inverse scattering transform (IST) and direct methods such as Hirota, Darboux and Bäcklund transformations [13–15] amongst many others. From a physics point of view, the classical NLS equation (given in normalized units)
\[
iq_t = q_{xx} - 2\sigma q^* q, \quad \sigma = \mp 1,
\]
(1)

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$(q^*$ is the complex conjugate of $q$) plays a crucial role in modeling physical systems ranging from photonics to Bose–Einstein condensation to deep water fluid dynamics to mention a few [16–19]. This NLS equation is universal, namely, it arises generically as the slowly varying wave envelope approximation of a uniform wave train solution of the underlying governing equation. This is frequently referred to as the quasi-monochromatic approximation see [19]. Importantly, this equation was deduced in [6] as a special symmetry reduction of a coupled system of evolution equations given by

$$iqt = q_{xx} - 2q^2 r,$$  \hspace{1cm} (2)

$$-ir_t = r_{xx} - 2r^2 q,$$  \hspace{1cm} (3)

where $q(x,t)$ and $r(x,t)$ are potentials of the well-known AKNS $2 \times 2$ linear scattering problem [6]. Indeed, when $r = \sigma q^*$, one recovers equation (1).

For a few decades, it was thought that $r = \sigma q^*$ is the only interesting reduction of the AKNS scattering problem. Surprisingly, in 2013 Ablowitz and Musslimani [20] showed that there was another interesting reduction given by $r(x,t) = \sigma q^*(-x,-t)$ giving rise to the integrable nonlocal NLS equation

$$iqt(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(-x,-t).$$  \hspace{1cm} (4)

Motivated by studies of parity-time (PT) symmetry in quantum physics and optics see [21–28], they termed it the PT symmetric NLS equation, (or PTNLS for short). The IST theory associated with PTNLS equation was studied in detail in [29]. Soon afterwards, Ablowitz and Musslimani found that there were yet two more reductions of the AKNS scattering problem leading to interesting nonlocal NLS type equations: these are $r(x,t) = \sigma q(-x,-t)$ and $r(x,t) = \sigma q(x,-t)$ giving rise to the so-called reverse space-time NLS (RSTNLS) and reverse time NLS (RTNLS) equations, respectively given by [30].

$$iqt(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(-x,-t),$$  \hspace{1cm} (5)

$$iqt(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(x,-t).$$  \hspace{1cm} (6)

These findings have ignited great interest in the emerging field of nonlocal PT symmetric integrable systems focusing on its mathematical structure and physical properties [31–51]. A natural question is whether and how such reductions can come out of physically interesting nonlinear wave equations, thus establishing connections between nonlocal integrable reductions and physically relevant models. In this paper, we address this very important issue and show that all of the above equations, i.e. the PTNLS, RSTNLS and RTNLS can be obtained as asymptotic quasi-monochromatic reductions from the nonlinear Klein–Gordon equation with a cubic nonlinear term, the KdV equation which has a quadratically nonlinear term and the more complicated nonlinear water wave equations. To do this, we modify the standard assumption of the form of the leading order solution, allowing it to be complex. This results in the remarkable observation that one gets a system of equations that is transformable to the standard AKNS $q,r$ system for second order spatial systems. Furthermore this observation connects the integrable KdV equation with all the above NLS type integrable flows.

2. Nonlinear Klein–Gordon reductions

We begin by considering the NLKG equation (sometimes called the $\phi^4$ model in physics)
\[ \phi_{tt} - \phi_{xx} + \phi - \sigma \phi^3 = 0, \]  
(7)

which, for \( \sigma = -1 \), can be obtained as an approximation to the sine-Gordon equation \( \phi_{tt} - \phi_{xx} + \sin \phi = 0 \) by keeping the first two terms in the expansion of \( \sin \phi \) for small \( \phi \). Here and below we consider \( x, t \) to be real. We look for a ‘small’ quasi-monochromatic solution to equation (7) in the form

\[ \phi = \phi(\theta, X, T; \epsilon), \quad \theta = kx - \omega t, X = \epsilon x, T = \epsilon t, \]
(8)

with the linear dispersion relation and group velocity given by \( \omega^2 - k^2 = 1, \quad \omega'(k) = \frac{k}{2} \).

Substituting equation (8) in (7), expanding \( \phi \) in a series in \( \epsilon \), i.e. \( \phi = \epsilon \phi_0 + \epsilon^2 \phi_1 + \epsilon^3 \phi_2 + \cdots \), and setting all coefficients to zero yields a sequence of equations at \( O(\epsilon^j) \), \( j = 0, 1, 2, \ldots \)

\[ L \phi_0 \equiv (\partial_\theta^2 + 1)\phi_0 = 0, \]
(9)

\[ L \phi_1 = (2k \partial_\theta \partial_X + 2\omega \partial_\theta \partial_T)\phi_0, \]
(10)

\[ L \phi_2 = (2k \partial_\theta \partial_X + 2\omega \partial_\theta \partial_T)\phi_1 - (\partial_\theta^2 - \partial_X^2)\phi_0 + \sigma \phi_0^3. \]
(11)

The general solution to the leading order equation (9) is

\[ \phi_0 = A(X, T)e^{i\theta} + B(X, T)e^{-i\theta}. \]
(12)

Note that here we do not necessarily require \( B = A^* \). Hence these solutions are generally complex by considering complex initial data. We point out that complexifying physically interesting equations either through its independent variables (e.g. Painlevé type equations [52]) or by allowing the dependent variables (wave functions) to become complex valued [53] has been the subject of great interest in recent years [54–58]. Doing so leads to a range of novel phenomena [59]. Substituting the solution given by (12) into (10) leads to

\[ L \phi_1 = 2i(k \partial_X A + \omega \partial_T A)e^{i\theta} - 2i(k \partial_X B + \omega \partial_T B)e^{-i\theta}. \]

To remove secular terms we take

\[ 2i(k \partial_X A + \omega \partial_T A) = \epsilon g_1 + \epsilon^2 g_2 + \cdots, \]
(13)

\[ -2i(k \partial_X B + \omega \partial_T B) = \epsilon h_1 + \epsilon^2 h_2 + \cdots, \]
(14)

and require \( \phi_1 = 0 \) (all homogeneous solutions are incorporated into the leading order). The terms \( g_1, g_2, \ldots, h_1, h_2, \ldots \) are determined by removal of secular terms at higher order. In this way the appropriate higher order nonlinear equations are obtained. The equation at \( O(\epsilon^3) \) is now given by

\[ L \phi_2 = -\left(\partial_\theta^2 - \partial_X^2\right)\phi_0 + \sigma \phi_0^3 + g_1 e^{i\theta} + h_1 e^{-i\theta}. \]
(15)

Substituting the solution for \( \phi_0 \) from equation (12) determines \( g_1, h_1 \) to be

\[ g_1 = \left(\partial_\theta^2 - \partial_X^2\right)A - 3\sigma A^2 B, \]
(16)

\[ h_1 = \left(\partial_\theta^2 - \partial_X^2\right)B + 3\sigma A^2 B. \]
(17)

The remaining terms in equation (15) satisfy

\[ L \phi_2 = \sigma \left(A^3 e^{3i\theta} + B^3 e^{-3i\theta}\right), \]
(18)

whose solution is given by

\[ \phi_2 = -\frac{\sigma}{8} \left(A^3 e^{3i\theta} + B^3 e^{-3i\theta}\right). \]
(19)
Using $g_1, h_1$ from equations (13) and (14) we find the nonlinear equations
\begin{equation}
2i(\omega \partial_x A + k \partial_y A) - \epsilon((\partial_x^2 - \partial_y^2)A + 3\sigma A^2B) = 0,
\end{equation}
\begin{equation}
-2i(\omega \partial_x B + k \partial_y B) - \epsilon((\partial_x^2 - \partial_y^2)B + 3\sigma B^2A) = 0.
\end{equation}
The above equations are coupled NLS equations which we will transform to a more convenient form. Using the dispersion relation and the transformations $\xi = x - \omega'(k) T, \tau = \epsilon T$, the nonlinear equations (20) and (21) now read
\begin{equation}
i\partial_x A + \frac{\omega''(k)}{2} \partial_x^2 A + \frac{3}{2\omega} \sigma A^2 B = 0,
\end{equation}
\begin{equation}
i\partial_x B + \frac{\omega''(k)}{2} \partial_x^2 B + \frac{3}{2\omega} \sigma B^2 A = 0,
\end{equation}
where we used $\omega''(k) = (1 - \omega^2)/\omega = 1/\omega^2$. By rescaling $\xi = x, \tau = \gamma t, A = aq, B = br$ with $\gamma = 2\omega^3, 3\sigma \omega^2 ab = -2$, these equations transform to the $q, r$ system (2) and (3). These equations are the integrable AKNS pair that leads to NLS and its reductions see [11, 30]. In particular, as stated in the introduction, equations (2) and (3) have exact reductions to the classical NLS (1) when $r = \sigma q^*$, the PTNLS (4) when $r(x,t) = \sigma q^*(-x, -t)$, the RSTNLS (5) when $r(x,t) = \sigma q(-x, -t)$ and the RTNLS equation (6) when $r(x,t) = \sigma q(x, t)$ see [11]. Solutions to the nonlocal equations (4)–(6) were analyzed for decaying data in [11, 30] and for non decaying initial conditions in [41, 42].

3. Korteweg–de Vries reductions

In this section we will consider the following normalized KdV equation
\begin{equation}
u_t + 6\nu u_x + u_{xxx} = 0,
\end{equation}
to derive the $q, r$ system. Similar to the nonlinear KG case, we will look for a ‘small’ quasimonochromatic solution to the KdV in the form given by equation (8), with the linear dispersion relation and group velocity given by $\omega = -k^3, \omega'(k) = -3k^2$. Expanding $u$ in a series in $\epsilon : u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ yields a sequence of equations, of which the first three at $O(\epsilon^j), j = 1, 2, 3$, are given by
\begin{equation}
\hat{L} u_0 \equiv \partial_x(\partial_x^2 + 1) u_0 = 0,
\end{equation}
\begin{equation}
\hat{L} u_1 = -\partial_x u_0 - 6ku_0 \partial_x u_0 - 3k^2 \partial_x^2 \partial_x u_0,
\end{equation}
\begin{equation}
\hat{L} u_2 = -\partial_x u_1 - 6ku_0 \partial_x u_1 - 6u_0 \partial_x u_0 - 6u_0 \partial_x u_0 - 3k^2 \partial_x^2 \partial_x u_1 - 3k \partial_x \partial_x^2 u_0.
\end{equation}
The solution to the leading order equation is given by
\begin{equation}
u_0 = A(X, T)e^{i\theta} + B(X, T)e^{-i\theta} + M(X, T).
\end{equation}
Here, we note that the solution to the leading order KdV asymptotic series is different from that of the nonlinear KG by adding a required mean term $M(X, T)$. Substituting this solution into (26) yields
\[
\hat{L}u_1 = -(A_T + \omega'(k)A_X + 6ikMA)e^{i\theta} - (B_T + \omega'(k)B_X - 6ikMB)e^{-i\theta} - M_T - 6ikA^2e^{2i\theta} + 6ikB^2e^{-2i\theta}.
\]

We remove secular terms by taking
\[
\begin{align*}
A_T + \omega'(k)A_X + 6ikMA &= \epsilon \hat{g}_1 + \epsilon^2 \hat{g}_2 + \cdots , \\
B_T + \omega'(k)B_X - 6ikMB &= \epsilon \hat{h}_1 + \epsilon^2 \hat{h}_2 + \cdots , \\
M_T &= \epsilon \hat{f}_1 + \epsilon^2 \hat{f}_2 + \cdots ,
\end{align*}
\]

and the solution \(u_1\) is given by
\[
\begin{align*}
u_1 &= \alpha_1 e^{2i\theta} + \beta_1 e^{-2i\theta} , \\
\alpha_1 &= \frac{A^2}{k^2} , \beta_1 = \frac{B^2}{k^2} .
\end{align*}
\]

The equation at \(O(\epsilon^3)\) is therefore given by
\[
\hat{L}u_2 = \hat{R},
\]
where \(\hat{R}\) (not given here due to size) depends on the amplitudes and their derivatives. To remove secular terms, consequently we find
\[
\begin{align*}
\hat{g}_1 &= -6i\frac{A^2B}{k} - 3ikA_{XX} - 6(AM)_X , \\
\hat{h}_1 &= 6i\frac{AB^2}{k} + 3ikB_{XX} - 6(BM)_X , \\
\hat{f}_1 &= 6(AB)_X - 6MM_X ,
\end{align*}
\]
hence the equations (30)–(32) yield
\[
\begin{align*}
A_T + \omega'(k)A_X + 6ikMA &= \epsilon \left(-6i\frac{A^2B}{k} - 3ikA_{XX} - 6(AM)_X \right) \\
B_T + \omega'(k)B_X - 6ikMB &= \epsilon \left(6i\frac{AB^2}{k} + 3ikB_{XX} - 6(BM)_X \right) \\
M_T &= \epsilon \left(-6(AB)_X - 6MM_X \right).
\end{align*}
\]

Employing traveling coordinates, these equations can be put in the form given by equations (22) and (23) with the nonlinear coefficient \(3\sigma/(2\omega)\) being replaced by \(6/k(k \neq 0)\). Thus the integrable KdV equation is directly connected to this integrable \(q, r\) system (2), (3) and its reductions: classical NLS, PTNLS, RSTNLS, RTNLS.

### 4. Water wave reductions

In this section, we show how one can obtain the AKNS reduction starting from the classical water wave equations governing ideal incompressible surface gravity waves. To do so, we start from the one dimensional, deep water wave equations given by
\[
\begin{align*}
\phi_{xx} + \phi_{zz} &= 0 , \quad -\infty < z < \eta ; \quad \lim_{z \to -\infty} \phi_x = 0 ,
\end{align*}
\]
\[ \eta_t + \eta_x \phi_x = \phi_z, \text{ on } z = \eta, \]  
\[ \phi_t + \phi_x^2 + \phi_z^2 + g \eta = 0, \text{ on } z = \eta, \]  
(42)  
(43)

where \( \eta \) and \( \phi \) are the fluid free surface elevation and velocity potential respectively. For simplicity, deep water is considered. We assume that the amplitude is small. To analyze this system one expands \( \phi \) around the small amplitude free surface \( z = \eta \) in equations (42) and (43) as follows

\[ \phi(x, \eta) = \phi(x, 0) + \phi_x(x, 0) \eta + \frac{1}{2} \phi_{xx}(x, 0) \eta^2 + \cdots. \]  
(44)

Then we expand \( \phi, \eta \) in the following asymptotic series

\[ \phi = \epsilon(A e^{i\theta + |k| z} + B e^{-i\theta + |k| z}) + \epsilon^2(A_2 e^{2i\theta + 2|k| z} + B_2 e^{-2i\theta + 2|k| z} + \tilde{\phi}) + \cdots \]  
(45)

\[ \eta = \epsilon(C e^{i\theta} + D e^{-i\theta}) + \epsilon^2(C_2 e^{2i\theta} + D_2 e^{-2i\theta} + \tilde{\eta}) + \cdots \]  
(46)

where the coefficients of \( \phi : A, B, A_2, B_2, \tilde{\phi} \) depend on slow variables \( X = \epsilon x, Z = \epsilon z, T = \epsilon t \) while the coefficients of \( \eta : C, D, A_2, D_2, \tilde{\eta} \) depend only on \( X = x, T = t \); the rapid phase is, as usual, given by \( \theta = kx - \omega t \). We substitute (45) and (46) into the water wave equations (41) and (42), see [19]. The leading order problem shows that the dispersion relation satisfies \( \omega^2 = g|k| \), and \( A = gC/(i\omega), B = -gD/(i\omega) \). At higher order we calculate \( A_2, B_2, C_2, D_2 \) in terms of the first harmonics \( C, D \) and it is found that the mean term \( \tilde{\eta} \) (like the KdV case) is small; we find: \( C_2 = \frac{d}{dx}CA, A_2 = O(\epsilon), D_2 = -\frac{d}{dx}BD, B_2 = O(\epsilon), \tilde{\eta} = O(\epsilon), \partial_x \tilde{\eta} - \frac{2|k|}{\omega}(AB)_x = O(\epsilon) \).

Amplitude equations are obtained for the first harmonics. In traveling wave coordinates the leading order slowly varying first harmonic amplitudes \( C, D \) associated with the wave elevation \( \eta \) satisfy the same equations as (22) and (23) only with the nonlinear coefficient \( 3\sigma/(2\omega) \) being replaced by \( n_2 = -2k^2\omega \). Rescaling these equations yield the general \( q, r \) system (2) and (3).

Some additional remarks are in order. The only differences between deep water and finite depth \( h \) are that the dispersion relation and nonlinear coefficient \( n_2 \) are different see [7]. We also note that after inspecting the above reduction the quasi-monochromatic limit of two-dimensional shallow water waves with surface tension in shallow water \( (kh \ll 1) \) will follow along the same lines (see also [7, 8, 60, 61]). One expects to find a system closely related to AKNS systems associated with second order nonlinear wave equations in two space dimensions which has reductions to the integrable classical Davey–Stewartson equation and the PT symmetric Davey–Stewartson equation see [30]. However details of this calculation are outside the scope of this paper.

5. Conclusion

Quasi-monochromatic complex reductions of a cubic nonlinear Klein–Gordon, the KdV and water waves equations are considered. It is found that the asymptotic reductions satisfy the well-known AKNS ‘\( q, r \)’ system (2) and (3) for second order in space integrable nonlinear wave equations. As such, all integrable nonlocal reductions, recently reported in the literature, are contained. This includes the FTNLS, reverse space-time and reverse time NLS equations.
Acknowledgment

MJA was partially supported by NSF under Grant No. DMS-1712793.

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References

[1] Korteweg D J and de Vries G 1895 Phil. Mag. Ser. 39 422
[2] Benney D J and Newell A C 1967 J. Math. Phys. 39 133
[3] Bour E 1862 J. Ecole Imperiale Polytech. 19 1
[4] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Phys. Rev. Lett. 19 1095
[5] Zakharov V E and Shabat A B 1972 Sov. Phys.—JETP 34 62
[6] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 Stud. Appl. Math. 53 249
[7] Ablowitz M J and Segur H 1981 Solitons and Inverse Scattering Transform (SIAM Studies in Applied Mathematics vol 4) (Philadelphia, PA: SIAM)
[8] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge: Cambridge University Press)
[9] Calogero F and Degasperis A 1982 Spectral Transform and Solitons I (Amsterdam: North-Holland)
[10] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 Theory of Solitons. The inverse Scattering Method (New York: Plenum)
[11] Ablowitz M J, Prinari B and Trubatch A D 2004 Discrete and Continuous Nonlinear Schrödinger Systems (Cambridge: Cambridge University Press)
[12] Yang J 2010 Nonlinear Waves in Integrable and Nonintegrable Systems (Philadelphia, PA: SIAM)
[13] Hirota R 2004 The Direct Method in Soliton Theory (Cambridge Tracts in Mathematics) (Cambridge: Cambridge University Press)
[14] Matveev V B and LaSalle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
[15] Rogers C and Schief W K 2002 Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory (Cambridge Texts in Applied Mathematics) (Cambridge: Cambridge University Press)
[16] Agrawal G P 1989 Nonlinear Fiber Optics (San Diego, CA: Academic)
[17] Kivshar Y S and Agrawal G P 2003 Optical Solitons: from Fibers to Photonic Crystals (New York: Academic)
[18] Pethick C J and Smith H 2008 Bose–Einstein Condensates in Dilute Gases (Cambridge: Cambridge University Press)
[19] Ablowitz M J 2011 Nonlinear Dispersive Waves Asymptotic Analysis and Solitons (Cambridge Texts in Applied Mathematics) (Cambridge: Cambridge University Press)
[20] Ablowitz M J and Musslimani Z H 2013 Phys. Rev. Lett. 110 064105
[21] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[22] Makris K G, El Ganainy R, Christodoulides D N and Musslimani Z H 2008 Phys. Rev. Lett. 100 103904
[23] Musslimani Z H, Makris K G, El Ganainy R and Christodoulides D N 2008 Phys. Rev. Lett. 100 030402
[24] El Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Opt. Lett. 32 2632
[25] Konotop V V, Yang J and Zeyzulin D A 2016 Rev. Mod. Phys. 88 035002
[26] Christodoulides D N and Yang J (ed) 2018 Parity-time Symmetry and its Applications (Berlin: Springer)
[27] Gbur G and Makris K G 2018 Introduction to non-Hermitian photonics in complex media: PT-symmetry and beyond Photon. Res. 6 PTS1–PTS3
[28] El Ganainy R, Makris K G, Khajavikhan M, Musslimani Z H, Rotter S and Christodoulides D N 2018 Nat. Phys. 14 11
[29] Ablowitz M J and Musslimani Z H 2016 Nonlinearity 29 915
[30] Ablowitz M J and Musslimani Z H 2017 Stud. Appl. Math. 139 7
[31] Lou S Y and Huang F 2017 Sci. Rep. 7 869
[32] Yang J 2018 Phys. Rev. E 98 042202
[33] Yang J 2018 Phys. Lett. A 383 328
[34] Yang B and Yang J 2018 Stud. Appl. Math. 140 178
[35] Yang B and Yang J 2019 Lett. Math. Phys. 109 945–73
[36] Xu Z and Chow K 2016 Appl. Math. Lett. 56 72
[37] Wen X, Yan Z and Yang Y 2016 Chaos 26 063123
[38] Fokas A S 2016 Nonlinearity 29 319
[39] Ma L, Shen S and Zhu Z 2017 J. Math. Phys. 58 103501
[40] Gadzhimuradov T A and Agalarov A M 2016 Phys. Rev. A 93 062124
[41] Ablowitz M J, Luo X-D and Musslimani Z H 2018 J. Math. Phys. 59 0011501
[42] Ablowitz M J, Feng B-F, Luo X-D and Musslimani Z H 2018 Theor. Math. Phys. 59 1241
[43] Feng B F, Luo X D, Ablowitz M J and Musslimani Z H 2018 Nonlinearity 31 5385
[44] Ablowitz M J, Feng B-F, Luo X-D and Musslimani Z H 2018 Stud. Appl. Math. 141 267
[45] Chen K, Deng X, Lou S Y and Zhang D J 2018 Stud. Appl. Math. 141 113
[46] Li M and Xu T 2015 Phys. Rev. E 91 033202
[47] Gurses M and Pekcan A 2018 Symp. Differ. Equ. Appl. 266 27
[48] Gurses M and Pekcan A 2018 J. Math. Phys. 59 051501
[49] Ji J L and Zhu Z N 2017 Commun. Nonlinear Sci. 42 699
[50] Yang B and Chen Y 2018 Nonlinear Dyn. 94 489–502
[51] Rao J G, Cheng Y and He J S 2017 Stud. Appl. Math. 139 568
[52] Ablowitz M J and Fokas A S 2003 Complex Variables: Introduction and Applications 2nd edn (Cambridge: Cambridge University Press)
[53] Mason L J and Woodhouse N M J 1996 Integrability, Self-Duality and Twistor Theory (Oxford: Oxford University Press)
[54] Fokas A S 2006 Phys. Rev. Lett. 96 190201
[55] Fokas A S and van der Weele M C 2018 J. Math. Phys. 59 091413
[56] Bender C M, Hook D W and Kooner K 2010 J. Phys. A: Math. Theor. 43 165201
[57] Bender C M and Feinberg J 2008 J. Phys. A: Math. Theor. 41 244004
[58] Bender C M, Holm D D and Hook D W 2007 J. Phys. A: Math. Theor. 40 F793
[59] Kevrekidis P G, Siettos C I and Kevrekidis Y G 2017 Nat. Commun. 8 1562
[60] Benney D J and Roskes G J 1969 Stud. Appl. Math. 48 377
[61] Davey A and Stewartson K 1974 On three dimensional packets of surface waves Proc. R. Soc. A 338 101