Bosonization rules in $1/2 + 1$ dimensions

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Abstract

We derive the bosonization rules for free fermions on a half-line with physically sensible boundary conditions for Luttinger fermions. We use path-integral methods to calculate the bosonized fermionic currents on the half-line and derive their commutation relations for a system with a boundary. We compute the fermion determinant of the fermionic fluctuations for a system with a boundary using Forman’s approach. We find that the degrees of freedom induced at the boundary do not to modify the commutation relations of the bulk. We give an explicit derivation of the bosonization rules for the fermion operators for a system with boundaries. We derive a set of bosonization rules for the Fermi operators which include the explicit effect of the boundaries and of boundary degrees of freedom. As a byproduct, we calculate the one-particle Green’s function and determine the effects of the boundaries on its analytic structure.

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I. INTRODUCTION

The Fermi-Bose equivalence, commonly known as *bosonization*, is one of the most powerful approaches to extract the non-perturbative behavior of $1 + 1$-dimensional Field theories and strongly interacting Condensed Matter systems. This approach is an outgrowth of the work on the solution of the Luttinger-Thirring model by a long list of authors [1–6]. The Fermi-Bose equivalence is deeply rooted in the properties of the algebras of the currents and densities of the physical observables. An alternative picture of the same physics is also found in the functional integral approach [7,8]. In its standard form, bosonization maps a system of interacting fermions into a system of bosons. The system of fermions (as well as the equivalent bosonic theory) is usually defined on manifolds without boundaries. This is appropriate for the study of the thermodynamic limit. In the case of the simple Luttinger-Thirring model, which has an abelian $U(1)$ symmetry, the bosonic theory is essentially free and the solution of the interacting fermion theory is thus found. For more complicated situations, in which the symmetry is non-abelian, the bosonized theory is a non-trivial Conformal Field Theory [9–12].

There are a number of situations of physical interest in which systems with boundaries are important. Of particular interest are the so-called quantum wires. Also, there are a number of situations in which a *localized* degree of freedom interacts with an extended system of fermions. Examples are one-dimensional systems (as for instance, quantum wires) interacting with magnetic and non-magnetic impurities. The interplay of Luttinger-like behavior and the Kondo effect is a subject of considerable interest [13,14]. Similar type of physics is also found in the context of the Callan-Rubakov effect [15]. A number of authors have used bosonization methods to the study of semi-infinite systems [14] and systems with quantum impurities [16] such as the Kondo effect [17] and spin chains with impurities [18]. However, in all cases, the bosonization rules of the *bulk* system were used and little attention was paid to the role that the boundary conditions of the fermions may play in the bosonized theory.
In this paper we discuss the effects of the boundary conditions of the fermions and in which way they affect the physics of the bosonized theory. Since bosonization is seen most clearly in the case of free fermions, we consider here just the simplest case, namely non-interacting, abelian, theory of fermions on a half-line ("$1 + \frac{1}{2}$-dimensions"). In a separate publication [19] we will discuss the case of interactions and the Kondo effect, which involves systems with non-abelian symmetries. We consider systems of fermions with two types of boundary conditions: (a) $R = -L$ at the boundary and (b) $R(0, x_2) = -e^{2i\theta(x_2)}L(0, x_2)$, again at the boundary. This choice of boundary conditions is motivated by the underlying microscopic physics of the systems of interest. In the case of quantum wires, the microscopic system consists of non-relativistic fermions at finite density. At a physical edge (i.e. the boundary) the fermion amplitude $\Psi(0)$ must vanish. Upon a decomposition into Left and Right movers, we find that a sharp edge implies the choice (a), i.e. $R(0) = -L(0)$. If a localized quantum impurity, parametrized by a degree of freedom $\theta$, is placed at the edge, the boundary condition changes to case (b), i.e. $R(0, x_2) = -e^{2i\theta(x_2)}L(0, x_2)$. Physically, this boundary condition is caused by a coupling between the fermion charge density at the edge and the localized quantum impurity. We also introduce physically sensible models of the quantum dynamics of the impurity.

In section II, we study a system of free electrons confined to the half-line and derive an expression for its partition function. In order to write this partition function only in terms of bosonic operators, in Section III we compute the determinant of the $1 + \frac{1}{2}$-dimensional Dirac operator coupled to a source which is defined as a delta function at the boundary. We show that this problem is equivalent to the computation of the determinant of the Dirac operator plus a suitable boundary condition. This last determinant is solved by using Forman’s approach [20].

In section IV, we derive an expression for the partition function only in terms of bosonic degrees of freedom. In other terms, we present an alternative form of abelian bosonization in terms of functional integrals. Although this is a subject that has been discussed extensiveley in the literature, we present here a derivation of this classic result in a form that is suitable for
treating systems with boundaries. We use these results to calculate the bosonized currents, the current correlation functions and their anti-commutation relations paying close attention to the role of the boundary conditions of the fermions.

In section V, we derive the one particle fermionic Green’s function from the bosonized theory. We discuss two cases, (a) when the boundary condition is $R = -L$, and (b) when there is an extra degree of freedom at the boundary, i.e., $R(0, x_2) = -e^{2i\theta(x_2)}L(0, x_2)$. We focus on the way the one particle Green’s function is affected by the presence of the boundary and of a dynamical degree of freedom at the boundary. In section VI we discuss a model of free fermions coupled to a dynamical boundary degree of freedom, and calculate the fermion one-particle Green’s function for this case of interest. We consider two asymptotic regimes: (a) one in which the boundary degree of freedom is strongly pinned, and (b) the case in which it fluctuates wildly. In the Conclusions we give a brief summary of our results. In the appendices we give details of the computation of the boundary contributions to the fermion Jacobian, of the use of Forman’s method and to Euclidean-Minkowski correspondences are given in the Appendices.

II. FREE FERMIONS ON A HALF-LINE

In this section we study a system of free massless Dirac fermions confined to the half line (i.e., $1 + \frac{1}{2}$ dimensions) satisfying dynamical boundary conditions. Our goal is to compute the partition function in order to be able to derive the bosonizations rules for this system. Throughout, we work in Euclidean space. In practice, this means that the space-time manifold is a semi-infinite long cylinder, with its axis along the space direction $x_1$ and periodic (antiperiodic for fermi fields) boundary conditions along the imaginary time direction $x_2$ of perimeter $T$, with $T \to \infty$.

The (Euclidean) Lagrangian for the system (coupled to sources) is

$$\mathcal{L}_F = \bar{\psi} i\partial \psi + A_\mu J_\mu$$

where $A_\mu$ is a source (a background gauge field) which couples to the fermion current $J_\mu =$
In a Fermi theory in Euclidean space, not just the time coordinate has to be continued analytically to the imaginary axis. The structure of the $\gamma$-matrices and the requirement that the Euclidean Dirac operator be hermitean forces an analytic continuation of the chiral angle. Hence, the chiral transformations generated by $\gamma_5$ become complexified in Euclidean space. However, we must keep in mind that these two analytic continuations are logically distinct. Moreover, these two analytic continuations need to be done in order to have a physically sensible interpretation of the results and to recover results back in Minkowski space. Hence, the fermion boundary conditions themselves also have to be continued analytically. We thus demand that the fermions satisfy the boundary condition $R(0, x_2) = -e^{2\theta(x_2)} L(0, x_2)$.

The dynamics of the fermions is described by the functional integral

$$Z[A] = \int D\bar{\psi} D\psi \exp \left( -\int d^2x \mathcal{L}_F \right).$$  \hspace{1cm} (2.2)

In what follows we will use the path integral bosonization method based on Seeley’s expansions for complex powers of elliptic operators \cite{21}.

The first step in the bosonization process is to decouple the fermions from the sources by means of a combination of (suitably chosen) gauge and chiral smooth, single-valued transformations of the form

$$\psi(x) = e^{i\eta(x)+\gamma_5\phi(x)}\chi(x)$$  
$$\bar{\psi}(x) = \bar{\chi}(x) e^{-i\eta(x)+\gamma_5\phi(x)},$$  \hspace{1cm} (2.3)

provided that the vector potentials $A_\mu(x)$ can be written in the form

$$A_\mu(x) = \partial_\mu \eta(x) - \epsilon_{\mu\nu} \partial_\nu \phi(x).$$  \hspace{1cm} (2.4)

This requirement is satisfied by all topologically trivial configurations of the fields $A_\mu$. The usefulness of the transformations of Eq. (2.3) and Eq. (2.4) is that they completely decouple the fermions from the vector potentials. However, this transformation changes the fermionic measure in a non-trivial way. Namely, the change in the fermionic measure under this transformation is:
\[ \mathcal{D}\bar{\psi}\mathcal{D}\psi = J_F \mathcal{D}\bar{\chi}\mathcal{D}\chi. \] (2.5)

The Jacobian, \( J_F \), can be written as
\[
J_F = \frac{\text{Det}(i\slashed{\partial} + \bar{A})}{\text{Det}(i\slashed{\partial})} = e^{-\int_0^1 w'(t)dt} \] (2.6)

with
\[
w' = 2\int dx_2 dx_1 tr K_2[i\slashed{D}_t; x, x] \gamma^5 \phi(x) \] (2.7)

where \( i\slashed{D}_t = i\slashed{D}_t[(1 - t)A] \). The kernel \( K_2[i\slashed{D}_t; x, x] \) is the constant term in the analytic expansion of the Heat Kernel \( \langle x| e^{-s \mathcal{D}_t^2} |x \rangle \). In the case of the fermions on the full line (which can be considered as a close manifold if we impose vanishing boundary conditions at the infinity) the Jacobian is well known and it has been calculated by several authors \[7\].

In the case of a manifold with boundary we have to evaluate this Heat Kernel, following Atiyah-Patodi-Singer (APS) \[22\], separating the contributions of the boundary from the ones of the bulk. Hence, the decoupling of the gauge field has to be done in two steps, first we decouple the bulk and then the boundary. For this purpose it is convenient to write the field \( A_\mu \) as \( A_\mu = B_\mu + s_\mu \) where \( s_\mu \) is defined as the value of \( A_\mu \) at the boundary, and \( B_\mu \) as its value in the bulk. That is, \( s_\mu = A_\mu \delta(x_1) \) and \( B_\mu \) is zero at the origin.

We define a chiral and a gauge transformation as in Eq. (2.3), such that it only decouples the field \( B_\mu \) from the fermions. Therefore at this point we are going to assume that the field \( B_\mu \) is the one that can be written in terms of \( \phi \) and \( \eta \) as in Eq. (2.4). The partition function becomes
\[
Z[B_\mu, s_\mu] = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi J_F \exp \left( - \int_\Omega d^2 x [\bar{\chi}(i\slashed{\partial} + \bar{\gamma})\chi] \right) \] (2.8)

where the integral in the action is restricted to the half-plane \( \Omega = \{ \vec{x} \mid x_1 \geq 0 \} \). The Jacobian reads,
\[
J_F = \frac{\text{Det}(i\slashed{\partial} + B + \bar{\gamma})}{\text{Det}(i\slashed{\partial} + \bar{\gamma})} = e^{-\int_0^1 w'(t)dt}. \] (2.9)
The APS procedure to compute the Heat Kernel consists in dividing the domain \((\Omega)\) in two intervals, \((0, \epsilon)\) and \((\epsilon, \infty)\). In the interval \((\epsilon, \infty)\) we can compute \(w'(t)\) in a usual way and obtain the well-known result for closed compact manifolds

\[
J_F = e^{-\frac{1}{8\pi} \int_{\Omega(\epsilon)} (\partial_\nu \phi)^2 + \frac{1}{\pi} \int_{\Omega(\epsilon)} \epsilon_{\mu\nu} s_\mu \partial_\nu \phi}
\]  

(2.10)

where \(\Omega(\epsilon) = \{ \vec{x} \mid x_1 > \epsilon \}\). The contribution of the boundary to the jacobian is given by the regular term of the following expression, (in the limit \(s \to 0\))

\[
\delta J_F = \exp \left( \int_0^1 dt \int dx_2 \int_0^\epsilon dx_1 <x|e^{-sD_1 D_2}|x> \phi(x_1, x_2) \right) .
\]

(2.11)

We show in Appendix A that this contribution is given by

\[
\delta J_F = \exp \left( -\frac{1}{4\epsilon} \int dx_2 \phi(0, x_2) \right)
\]

(2.12)

Therefore, if we choose \(\phi(0, x_2) = 0\) the contribution of the boundary term to the jacobian is one. It is important to remark that this cancellation happens only because we made a chiral transformation that is trivial at the boundary. The price for this choice is that we still have the fermions coupled to the field \(s_\mu\) with support only on the boundary. We will show how to deal with this problem in the next section.

Then after the chiral transformation is performed the partition function is

\[
Z[B_\mu, s_\mu] = \int \mathcal{D}\bar{\chi} \mathcal{D} \chi \exp \left( -\int_{\Omega} d^2 x [\bar{\chi}(i\partial + \gamma) \chi] \right) \exp \left( -\frac{1}{2\pi} \int_{\Omega} (\partial_\mu \phi)^2 + \frac{1}{\pi} \int_{\Omega} \epsilon_{\mu\nu} s_\mu \partial_\nu \phi \right)
\]

(2.13)

where we took the limit \(\epsilon \to 0\).

Since

\[
B_\mu = -(\epsilon_{\mu\nu} \partial_\nu \phi - \partial_\mu \eta).
\]

(2.14)

we can write

\[
\partial_\nu \partial_\nu \phi(x) = \epsilon_{\alpha\mu} \partial_\alpha B_\mu(x)
\]

(2.15)
with the boundary condition

$$\phi(0, x_2) = 0$$  \hspace{1cm} (2.16)

Therefore

$$\phi(x) = \int_{\Omega} d^2 y \ G_0(x, y) \epsilon_{\alpha \mu} \partial_\alpha B_\mu(y)$$  \hspace{1cm} (2.17)

where $G_0(x, y)$ is the Green’s function defined as

$$\begin{cases} 
\partial_x^2 G_0(x, y) = \delta(x - y) \\
G_0(0, x_2; y) = 0.
\end{cases}$$  \hspace{1cm} (2.18)

Since $B_\mu$ is only defined for $x_1 > 0$, we have to use the Green’s function for the half line (Eq. (2.18)) which is given by

$$G_0(x, y) = g(x_2 - y_2, x_1 - y_1) - g(x_2 - y_2, x_1 + y_1)$$  \hspace{1cm} (2.19)

where

$$g(u, v) = \frac{1}{4\pi} \ln \left( \frac{u^2 + v^2 + a^2}{a^2} \right)$$  \hspace{1cm} (2.20)

with $a$ acting as a regulator.

Using these expressions to write $\phi(x)$ in terms of $B_\mu$, the jacobian $J_F$ can be written as

$$\ln J_F = -\frac{1}{2\pi} \int_{\Omega} (\partial_\mu \phi)^2 + \frac{1}{\pi} \int_{\Omega} \epsilon_{\mu \nu} s_\mu \partial_\nu \phi$$

$$= \frac{1}{2\pi} \int_{\Omega} dx \int_{\Omega} dy (B_\mu(x) \Gamma_{\mu \nu}(x, y) B_\nu(y) + 2B_\mu(x) \Gamma_{\mu \nu}(x, y) s_\nu(y))$$

$$= \frac{1}{2\pi} \int_{\Omega} dx \int_{\Omega} dy (B_\mu(x) + s_\mu(x)) \Gamma_{\mu \nu}(x, y) (B_\nu(x) + s_\nu(x))$$

$$- \frac{1}{2\pi} \int_{\Omega} dx \int_{\Omega} dy s_\mu(x) \Gamma_{\mu \nu}(x, y) s_\nu(y)$$  \hspace{1cm} (2.21)

where $\Gamma_{\mu \nu}(x, y)$ is defined as

$$\Gamma_{\mu \nu}(x, y) = \left( \partial^{(x)}_\alpha \partial^{(y)}_\beta \delta_{\mu \nu} - \partial^{(y)}_\mu \partial^{(x)}_\nu \right) G_0(x, y)$$  \hspace{1cm} (2.22)

We can linearize the quadratic term in $B_\mu(x) + s_\mu(x)$ by introducing a bosonic field $\omega$ defined in $\Omega$ such that $\omega(0, x_2) = 0$. It is easy to see that Eq. (2.21) can be expressed as
\[ \ln J_F = \mathcal{K} \int \mathcal{D}\omega \exp \left( -\frac{1}{2\pi} \int_\Omega (\partial_\mu \omega)^2 + \frac{i}{\pi} \int_\Omega \epsilon_{\mu\nu} \partial_\nu \omega (s_\mu + B_\mu) \right) \]
\[ \exp \left( -\frac{1}{2\pi} \int_\Omega dx \int_\Omega dy \left( \bar{s}_\mu(x) \Gamma_{\mu\nu}(x, y) s_\nu(y) \right) \right) \]

(2.23)

Since by definition, \( s_\mu(x) = s_\mu(x^2) \delta(x_1) \), we need the value of \( \Gamma(x, y) \) at \( x_1 = y_1 = 0 \). It is easy to check that \( \Gamma_{12}|_{x_1, y_1=0} = \Gamma_{21}|_{x_1, y_1=0} = 0 \). Similarly, by using the expression for the Green’s function, Eqs. (2.14-2.18), and for the kernels \( \Gamma_{\mu\nu}(x, y) \), Eq. (2.22), we also find that, in the limit \( a \to 0 \), \( \Gamma_{11}|_{x_1, y_1=0} = 0 \). In the same way, we find that the value of \( \Gamma_{22}(x, y)|_{x_1, y_1=0} \) is given by

\[ \Gamma_{22}(x, y)|_{x_1, y_1=0} = -\frac{1}{\pi} \mathcal{P} \frac{1}{(x_2 - y_2)^2 + a^2} + \frac{1}{a} \delta(x_2 - y_2) \]

(2.24)

where \( \mathcal{P} \) denotes the principal value. From now on we will assume a principal value in this type of expressions. Then the partition function becomes

\[ Z[b_\mu, s_\mu] = \mathcal{K} \int \mathcal{D}\bar{\chi} \mathcal{D}\chi \mathcal{D}\omega \exp \left( -\int_\Omega d^2x \left[ \bar{\chi}(i\partial_\mu + s_\mu)\chi \right] \right) \]
\[ \exp \left( -\frac{1}{2\pi} \int_\Omega (\partial_\mu \omega)^2 + \frac{i}{\pi} \int_\Omega \epsilon_{\mu\nu} \partial_\nu \omega (s_\mu + B_\mu) \right) \]
\[ \exp \left( -\frac{1}{4\pi^2} \int_\Omega \frac{(s_2(y_2) - s_2(x_2))^2}{(x_2 - y_2)^2} \right) \].

(2.25)

Now we have solved completely the decoupling of the gauge field \( B_\mu \). But we still have a fermionic integration left to perform. This integration will be done in the next section.

### III. FREE FERMIONS IN A \( \delta(X_1) \)-POTENTIAL

In this section we calculate the determinant of the \( (1 + \frac{1}{2}) \)-dimensional Dirac operator coupled to a \( \delta(x_1) \) potential. Hence we have to consider the Dirac operator, \( [\gamma^\mu (i\partial_\mu + s_\mu)] \), where \( s_\mu = s_\mu(x^2) \delta(x_1) \). The formal way to compute the determinant of this operator is by computing its eigenvalues. First, it is useful to regularize the \( \delta \) function as the limit \( \epsilon \to 0 \) of the function \( v_\epsilon(x) \) where \( \int_{-\frac{\epsilon}{2}}^{\epsilon} v_\epsilon(x) = 1 \). It is clear that when \( \epsilon, \delta \to 0 \), the solutions of

\[ [\gamma^\mu (i\partial_\mu + s_\mu)]\psi = \lambda \psi \]

(3.1)
in $x_1 \geq 0$ will be the same as the solutions of the equation

$$\gamma^\mu (i \partial_\mu) \psi = \lambda \psi$$  \hspace{1cm} (3.2)

plus a suitable chosen boundary condition. To find such a boundary condition we observe that integrating Eq. (3.1) over the interval $(\epsilon - \frac{\delta}{2}, \epsilon + \frac{\delta}{2})$ and assuming that the field $\psi$ is finite in such interval, we obtain for small $\delta$

$$-\gamma_1 (\psi(\epsilon^+) - \psi(\epsilon^-)) + i (\gamma_1 s_1(x_2) + \gamma_2 s_2(x_2)) \int_{\epsilon - \frac{\delta}{2}}^{\epsilon + \frac{\delta}{2}} dx_1 \psi(x_1) v_\epsilon(x_1) = 0.$$  \hspace{1cm} (3.3)

Such a field can be found if we look at solutions of

$$\partial_1 \psi = (\gamma_5 s_2(x_2) + i s_1(x_2)) v_\epsilon(x_1) \psi(x_1).$$  \hspace{1cm} (3.4)

Explicitly,

$$\psi(x) = \hat{P}_{x_1} \exp[(\gamma_5 s_2(x_2) + is_1(x_2)) \int_{\epsilon^-}^{x_1} dy v_\epsilon(y)] \psi(\epsilon^-)$$  \hspace{1cm} (3.5)

where $\hat{P}_{x_1}$ is the spatial ordering operator. Since the operator in the exponent of Eq. (3.5) commutes at spatially separated points because of the shape of $v_\epsilon(x)$, we may set $\hat{P}_x = 1$, and taking the limit $\delta \to 0$ we get

$$\psi(\epsilon^+) = e^{\gamma_5 s_2(x_2) + is_1(x_2)} \psi(\epsilon^-).$$  \hspace{1cm} (3.6)

It is clear now that, in the limit $\epsilon \to 0$, the boundary condition we are looking for becomes

$$\psi(\epsilon^+) = e^{-\gamma_5 s_2(x_2) + is_1(x_2)} \psi(\epsilon^-).$$  \hspace{1cm} (3.7)

Recalling that at $\epsilon^-$ the fields also satisfy $B e^{-\gamma_5 \theta(x_2)} \psi(\epsilon^-) = 0$ with $B$ the matrix given by

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$  \hspace{1cm} (3.8)

then, in the limit $\epsilon \to 0$, the boundary condition for the field $\psi$ is

$$Bu^{-1}(x_2) \psi|_{x_1=0} = 0$$  \hspace{1cm} (3.9)
where $\mathcal{U}(x_2) = e^{\gamma_5(\theta(x_2)+s_2(x_2))}$. This equation tells us that the source $s_2(x_2)$ can be thought as a dynamical degree of freedom at the boundary that interacts with the fermions through the above boundary condition. In other words, the determinant we have to compute is $\text{det}(\gamma^\mu(i\partial_\mu))_{\mathcal{B}\mathcal{U}^{-1}}$ where

$$\gamma^\mu(i\partial_\mu)_{\mathcal{B}\mathcal{U}^{-1}} = \begin{cases} 
\gamma^\mu(i\partial_\mu) \text{ with } \psi \text{ such that } \mathcal{B}\mathcal{U}^{-1}(x_2)\psi|_{x_1=0} = 0. 
\end{cases}$$

The solution for this kind of determinants was first given by Forman [20]. His theorem relates the determinant of the above differential operator with the determinant of an (infinity dimensional) matrix,

$$\mathcal{D}\text{et}(\gamma^\mu(i\partial_\mu))_{\mathcal{B}\mathcal{U}^{-1}(x=0)} \sim \int ds \text{Tr}\left(\frac{d}{ds}H(s)H^{-1}(s)\right)$$

The matrix $H$ is a functional of the function $\theta(x_2) + s_2(x_2)$.

In order to apply Forman’s methods we introduce an auxiliary parameter $\tau$ such that

$$\mathcal{U}(\tau) = e^{\tau\gamma_5\alpha}$$

where we have defined $\alpha(x_2) = \theta(x_2) + s_2(x_2)$. This method relies on the knowledge of the space of solutions of the homogeneous equation $(i\gamma^\mu \partial_\mu)\psi = 0$. We must consider a complete, but otherwise arbitrary, system of functions in the kernel of this differential operator, which do not need to satisfy any particular boundary condition. In order to simplify the calculations, we choose the basis that expands the kernel of $(i\gamma^\mu \partial_\mu)\psi = 0$, satisfying

$$\psi_n(x_1,-\frac{T}{2}) = -\psi_n(x_1,\frac{T}{2}),$$

$$B\psi_n(x_2,0) = B\psi_n(x_2,L) = 0.$$  

Later on we will send $L,T \to \infty$. The basis is

$$\psi_n(x_2,x_1) = e^{iw_n[x_2-i\gamma_5(x_1-L)]} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, n \in \mathbb{Z}$$

where $w_n = \frac{(2n+1)\pi}{L}$. Forman’s Theorem [20] relates the problem of evaluating the functional determinant of a differential operator with a parameter-dependent boundary condition (such
as the one defined by Eq. (3.10)), to that of obtaining the determinant of an operator $\Phi_\mu(\tau)$, acting on functions defined on the boundary:

$$\frac{d}{d\mu} \ln \left( \frac{\text{Det}(i\partial)_{BU^{-1}(\mu+\tau)}}{\text{Det}(i\partial)_{BU^{-1}(\mu)}} \right) = \frac{d}{d\mu} \ln \det \Phi_\mu(\tau). \quad (3.15)$$

Here the determinants on the left-hand side are defined through the $\zeta$-function regularization, while the right-hand side is a well-defined quantity that can be evaluated using any basis at the boundary. The operator $\Phi_\mu(\tau)$ is defined as follows. We first define the Poisson map $P_B$. If the boundary conditions are defined with the operator $B$, then $P_B$ is such that the unique solution of

$$\begin{cases}
O f(x) = 0 & x \in \Omega \\
B f(x) = h(x) & x \in \partial \Omega
\end{cases} \quad (3.16)$$

where $O$ is an arbitrary differential operator, is given by $f(x) = P_B h(x)$. If $A$ is another boundary condition, then $\Phi_{AB} = AP_B$. Even though $P_B$ is difficult to evaluate, $\Phi_{AB}$ is not because of its main property

$$A\psi = \Phi_{AB}B\psi \quad (3.17)$$

for any solution of $O\psi = 0$. In our case $\Phi_\mu(\tau)$ satisfies

$$BU^{-1}(\mu + \tau)\psi = \Phi_\mu(\tau)BU^{-1}(\mu)\psi \quad (3.18)$$

for any $\psi$ that obeys the free Dirac equation. Let us define $h_\mu^n(x_2)$ as

$$h_\mu^n(x_2) = BU^{-1}(\mu)\psi_n(x_2) \quad (3.19)$$

where $\psi_n(x_2) = \psi_n(0, x_2)$. They satisfy $h_\mu^n(x_2) = \Phi_\mu(\tau)h_\mu^n(x_2)$. If we were able to write $h_\mu^n(x_2)$ and $h_{\mu+n}(x_2)$ in terms of a complete set of functions in $[0, L]$ we would have the (infinite) matrix $\Phi_\mu(\tau)$ expressed in this basis, and by using conventional methods we could compute its determinant. The basis will be $\psi_n(L, x_2) = e^{iw_n x_2}$ with $w_n = \frac{(2n+1)\pi}{L}$ ($n \in \mathbb{Z}$). It is easy to compute $h_\mu^n(x_2)$ and $h_{\mu+n}(x_2)$ in such basis. Following the definition,
\[ h^n_\mu(x_2) = \cosh(w_n L + \mu \alpha)e^{iw_n x_2}, \]
\[ h^{n+\tau}_\mu(x_2) = \cosh(w_n L + (\mu + \tau) \alpha)e^{iw_n x_2}. \] (3.20)

In the limit \( L \to \infty \), \( \tanh(w_n L) \) goes to \( \text{sign}(w_n) \), where
\[
\text{sign}(w_n) = \text{sign}(2n + 1) = \begin{cases} 
1, & n \geq 0 \\
-1, & n < 0 
\end{cases} \] (3.21)

Then, up to a normalization constant, we can write
\[ h^n_\mu(x_2) = e^{\text{sign}(w_n)\mu \alpha(x_2)}e^{iw_n x_2} \] (3.22)

Note that the \( m \)-th component of \( h^n_\mu(x_2) \) in the basis under consideration is
\[ h^{n,m}_\mu(x_2) = \frac{1}{2T} \int_{-T}^{T} dx_2 \ e^{[i(w_n-w_m)x_2]} e^{[\text{sign}(w_m)\mu \alpha(x_2)]} \] (3.23)

We can think of \( h^{n,m}_\mu(x_2) \) as matrix elements, in the space spanned by the basis \( \psi_n(0, x_2) \), of a matrix \( h(\tau) \). In this way,
\[ < \psi_n | h(\tau) | \psi_m > = h^{n,m}_\mu \] (3.24)

and the operator \( \Phi_\mu(\tau) \) can be written as
\[ h(\mu + \tau)h^{-1}(\tau) = \Phi_\mu(\tau) \] (3.25)

Then
\[ \frac{d}{d\mu} \ln \det \Phi_\mu(\tau) = Tr \frac{d}{d\mu} [h(\mu + \tau)h^{-1}(\mu + \tau)] - \frac{d}{d\mu} [h(\mu)h^{-1}(\mu)]. \] (3.26)

We have reduced the computation of \( \frac{d}{d\mu} \ln \det \Phi_\mu(\tau) \) to the computation of a quantity of the form
\[ Tr \left[ \frac{d}{d\mu} h(\mu)h^{-1}(\mu) \right] \] (3.27)

at \( \mu = 1 \), where \( \mu \) is an arbitrary parameter, and \( h(\mu) \) is a matrix of the form
\[ h(\mu) = \begin{pmatrix} [e^{-\alpha}]_{--} & [e^\alpha]_{-+} \\
[e^{-\alpha}]_{+-} & [e^\alpha]_{++} \end{pmatrix} \] (3.28)
(see Appendix B for notation). At this point it is important to make a connection between
the matrix \( h(\mu) \) and the so-called Toeplitz matrices. A Toeplitz matrix of order \( N \) is the
name for a matrix \( T^{(N)} \) of the form

\[
T^{(N)}_{m,n} = c_{m-n} \quad m, n = 0, \ldots, N - 1
\]

(3.29)

where \( c_p \) are arbitrary complex numbers \( (p = 0, \pm 1, \ldots, \pm (N - 1)) \). Let us denote \( D_N \) as
its determinant. Giving a set of complex numbers \( c_p, p \geq 0 \), and under suitable conditions
Szego’s Theorem \([23]\) gives an expression for

\[
\lim_{N \rightarrow \infty} \frac{D_N}{\mu^N}
\]

(3.30)

Clearly \( \mu^N \) acts as a regulator. If

\[
c_p = \frac{1}{2\pi} \int_0^{2\pi} dt \ e^{-ipt} \ C(e^{it})
\]

(3.31)

and

\[
\mu = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} dt \ C(e^{it}) \right\}
\]

(3.32)

and

\[
g_p = \frac{1}{2\pi} \int_0^{2\pi} dt \ e^{-ipt} \ \ln C(e^{it})
\]

(3.33)

then, Szego’s Theorem states that

\[
\lim_{N \rightarrow \infty} \frac{D_N}{\mu^N} = \exp\left\{ \sum_{p=1}^{\infty} p g_{-p} g_p \right\}.
\]

(3.34)

It is obvious that \([e^{-\alpha}]_-\) and \([e^{\alpha}]_+\) are both Toeplitz blocks in the sense that they satisfy
the definition of a Toeplitz matrix. Notice that if \( h(\mu) \) were made up by only one of these
blocks, the determinant (6.19) would be exactly the one given by Szego’s Theorem. We are
going to show that, even though the off-diagonal blocks are non-vanishing, and the indices
that label the matrix entries are integers numbers, Szego’s Theorem is still valid up to an
overall factor of 2. We can also see that Forman’s Theorem provides a natural cancelation
for the regularization factors in the Szego’s Theorem.
In Appendix B we show that

\[ Tr\left[ \frac{d}{d\mu} h(\mu) h^{-1}(\mu) \right] = tr \{ \alpha \}_{++} - \{ \alpha \}_{--} \]

\[ + \{ \alpha \}_{+-} (- [e^{\alpha}]_{-+} [e^{-\alpha}]_{+-} + [e^{\alpha}]_{--} [e^{-\alpha}]_{-+} [e^{-\alpha}]_{+-} [e^{\alpha}]_{++}) \]

\[ + \{ \alpha \}_{-+} ([e^{-\alpha}]_{++} [e^{\alpha}]_{-+} - [e^{-\alpha}]_{++} [e^{\alpha}]_{-+} [e^{-\alpha}]_{-+} [e^{\alpha}]_{--}) \]  \hspace{1cm} (3.35)

After anti-transform Fourier the above expression, we see that the first two terms cancell with each other. The third term becomes

\[ Tr\{ [\alpha]_{+-}[e^{\alpha}]_{-+}[e^{-\alpha}]_{++} \} = \sum_{n \geq 0, p < 0} \sum_{p \geq 0, q = 0} \sum_{q = 0}^{p-1} [\alpha]_{n-p} [e^{\alpha}]_{p-q} [e^{-\alpha}]_{q-n} \]

\[ = \sum_{n \geq 0} \sum_{p \geq 0} \sum_{q = 0}^{p-1} [\alpha]_{n+p} [e^{\alpha}]_{-p+q} [e^{-\alpha}]_{-q-n} \]

\[ = \oint [dz_1] \frac{dz_1}{\Pi i} \sum_{n \geq 0} \sum_{p \geq 0} \sum_{q = 0}^{p-1} \alpha_+ (z_1) e^{\alpha-(z_2)} e^{-\alpha-(z_1)} z_1^{-(n+p)} z_2^{(p-q)} z_3^{q+n} \]

\[ = - \oint [dz_1] [\alpha (z_1)] \oint [dz_2] \frac{e^{\alpha-(z_2)}}{(z_2 - z_1)} \oint [dz_3] \frac{e^{-\alpha-(z_3)}}{(z_3 - z_1)^2} \]

\[ = - \oint [dz_1] [\alpha_+ (z_1)] e^{\alpha-(z_2)} (- \partial z_1 \alpha_+(z_1)) e^{-\alpha-(z_2)} \]

\[ = \frac{1}{2\pi i} \int dx_2 \alpha_+(x_2) \partial_2 \alpha_-(x_2). \] \hspace{1cm} (3.36)

Following the same procedure it is easy to see that the fourth term gives exactly the same contribution. The two terms with \([\alpha]_{-+}\) contribute with the same kind of expression but with “+” interchanged with “-” and with opposite sign. Therefore

\[ Tr\left[ \frac{d}{d\mu} h(\mu) h^{-1}(s\mu) \right] = -\frac{1}{\pi i} \int dx_2 [\alpha_+(x_2) \partial_2 \alpha_-(x_2) - \alpha_-(x_2) \partial_2 \alpha_+(x_2)] \]

\[ = \frac{1}{2\pi^2} \int dx_2 dy_2 \frac{(\alpha(y_2) - \alpha(x_2))^2}{(x_2 - y_2)^2}. \] \hspace{1cm} (3.37)

Eq. (3.37) coincides up to a factor of 2 with the expression given by the Szego’s Theorem for each one of the diagonal blocks. The regularization factors for the determinant are \([\alpha]_{++}\) and \([\alpha]_{--}\) and we see that they cancel with each other in a natural way. This expression coincides with the result obtained by Falomir et al. [24], using a perturbative approach. Note that our result does not involve any approximation. By inserting this result in Eq. (3.15) and recalling that \(\alpha = \theta + s_2\) we obtain
\[
\frac{d}{d\mu} \ln \left( \frac{\text{Det}(i\partial)_{\mu} B^{\mu} - 1}{\text{Det}(i\partial)_{\mu}} \right) \bigg|_{\mu=0, \tau=1} = \frac{1}{2\pi^2} \int dx_2 dy_2 \frac{[(\theta + s_2)(y_2) - (\theta + s_2)(x_2)]^2}{(x_2 - y_2)^2}.
\]

(3.38)

Following Falomir et al. [24] we obtain the value of the ratio of the determinants which is given by

\[
\ln \left( \frac{\text{Det}(i\partial)_{\mu} B^{\mu} - 1}{\text{Det}(i\partial)_{\mu}} \right) = \frac{1}{4\pi^2} \int dx_2 dy_2 \frac{[(\theta + s_2)(y_2) - (\theta + s_2)(x_2)]^2}{(x_2 - y_2)^2}.
\]

(3.39)

This kind of non-local expressions have been used by Anderson [25] and Nozieres and Dominicis in the X-ray absorption and emission in metals [26] and in the context of the Kondo problem [27]. We have then computed the last ingredient we needed in Eq (2.25) to complete the bosonization scheme. In the next section we describe the bosonized theory starting from that expression for the partition function.

**IV. BOSONIZED THEORY**

Once the fermionic generating functional is known, any correlation function can be obtained by functional differentiation. In particular, vacuum expectation values of arbitrary products of fermion currents

\[
< \prod_{i=1}^{N} J_\mu(x_i) > = < \prod_{i=1}^{N} \bar{\psi}(x_i) \gamma_\mu \psi(x_i) >
\]

(4.1)

can be obtained from the fermionic partition function

\[
\mathcal{Z}[A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( - \int d^2x \bar{\psi} (i\partial + A) \psi \right)
\]

(4.2)

by using the regularization prescription for defining the fermion determinant, as follows

\[
< J_\mu(x) > = \frac{1}{\mathcal{Z}[A]} \frac{\delta \mathcal{Z}[A]}{\delta A_\mu(x)} \bigg|_{A=0}.
\]

(4.3)

Since, as we have shown in section II, we have to split the external field \( A_\mu \) into two fields, \( B_\mu \) and \( s_\mu \), we define two currents, one in the bulk

\[
< J_\mu(x) > = \frac{1}{\mathcal{Z}[b_\mu, s_\mu]} \frac{\delta \mathcal{Z}[b_\mu, s_\mu]}{\delta B_\mu(x)} \bigg|_{B, s=0}
\]

(4.4)
for $x_1 > 0$ and the other at the boundary

$$
<J_\mu(0, x_2) > = \frac{1}{\mathcal{Z}[b_\mu, s_\mu]} \delta \mathcal{Z}[b_\mu, s_\mu] \bigg|_{B, s=0}. 
$$

(4.5)

By using Eq. (2.25) and Eq. (3.39), the partition function is

$$
\mathcal{Z}[B_\mu, s_\mu] = \mathcal{K} \int D\omega \exp \left( -\frac{1}{2\pi} \int (\partial_\mu \omega)^2 + \frac{i}{\pi} \int \epsilon_{\mu\nu} \partial_\nu \omega (s_\mu + B_\mu) \right)
\exp \left( \frac{1}{4\pi^2} \int \left[ (s_2(y_2) - s_2(x_2))^2 - \frac{[(\theta + s_2)(y_2) - (\theta + s_2)(x_2)]^2}{(x_2 - y_2)^2} \right] \right) 
$$

(4.6)

which can be written as

$$
\mathcal{Z}[B_\mu, s_\mu] = \mathcal{K} \int D\omega \exp \left( -\frac{1}{2\pi} \int (\partial_\mu \omega)^2 + \frac{i}{\pi} \int \epsilon_{\mu\nu} \partial_\nu \omega (s_\mu + B_\mu) \right)
\exp \left( \frac{1}{2\pi} \int dx_2 dy_2 \theta(x_2) K(x_2, y_2) \theta(y_2) \right)
\exp \left( \frac{1}{\pi} \int dx_2 dy_2 \theta(x_2) K(x_2, y_2) s_2(y_2) \right) 
$$

(4.7)

The kernel $K(x_2, y_2)$ is defined as

$$
K(x_2, y_2) = \left( -\frac{1}{\pi} \mathcal{P} \frac{1}{(x_2 - y_2)^2} + \frac{1}{a} \delta(x_2 - y_2) \right). 
$$

(4.8)

Note that this action is already linear in the external fields. Then the currents can be easily read from the partition function,

$$
J_\mu(x) = \frac{i}{\pi} \epsilon_{\mu\nu} \partial_\nu \omega , \text{ for } x_1 > 0, 
$$

(4.9)

which is the usual bosonization formula for the fermionic current in the bulk (in Euclidean space). The value of the current at the boundary is

$$
J_\mu(0, x_2) = \frac{i}{\pi} \epsilon_{\mu\nu} \partial_\nu \omega|_{x_1=0} + \frac{\delta \mu_2}{\pi} \int dy_2 K(x_2, y_2) \theta(y_2). 
$$

(4.10)

Let us now compute the current-current correlation functions. They are given by

$$
<J_\mu(x)J_\nu(y) >|_{x_1, y_1 > 0} = \frac{1}{\mathcal{Z}[b_\mu, s_\mu]} \delta^2 \ln \mathcal{Z}[b_\mu, s_\mu] \bigg|_{B=0}
$$

(4.11)
\[ <J_\mu(x)J_\nu(y)>|_{x_1,y_1=0} = \frac{1}{Z[b_\mu, s_\mu]} \delta^2 \ln Z[b_\mu, s_\mu](x_2)\delta s_\nu(y_2) \left|_{s_\nu=0} \right. \]  

\[ <J_\mu(x)J_\nu(y)>|_{x_1>0,y_1=0} = \frac{1}{Z[b_\mu, s_\mu]} \delta^2 \ln Z[b_\mu, s_\mu] \delta B_\mu(x_2)\delta s_\nu(y_2) \left|_{B, s=0} \right. \]  

Before computing the functional differentiation, we have to integrate the bosonic fields. Instead of using Eq. (4.7) for the partition function, we go a step backwards and recall that the integral over \( w \) comes from Eq. (2.21) as an alternative way of writing the Jacobian \( J_\nu \) (Eq. (2.10)). By using Eqs. (2.13) and (3.39) we can also express the partition function as

\[ Z[B_\mu, s_\mu] = \exp \left( \frac{1}{2\pi} \int (B_\mu(x) + s_\mu(x))\Gamma_{\mu\nu}(x,y)(B_\nu(y) + s_\mu(x)) \right) \]

\[ \exp \left( \frac{1}{2\pi} \int dx_2 dy_2 \theta(x_2)K(x_2, y_2)\theta(y_2) \right) \]

\[ \exp \left( \frac{1}{\pi} \int dx_2 dy_2 \theta(x_2)K(x_2, y_2)s_\mu(y_2) \right) \]  

Then, the current-current correlation functions are

\[ <J_\mu(x)J_\nu(y)>|_{x_1,y_1>0} = \frac{1}{\pi} \Gamma_{\mu\nu}(x,y) \]  

\[ <J_\mu(x)J_\nu(y)>|_{x_1,y_1=0} = \frac{1}{\pi} \delta_{\mu,\nu} \delta_{\nu,2} K(x_2, y_2) \]  

\[ <J_\mu(x)J_\nu(y)>|_{x_1>0,y_1=0} = \frac{1}{\pi} \Gamma_{\mu\nu}(x,y)|_{x_1=0}. \]  

We follow now the method suggested in reference [7] to obtain the commutation relations from the current-current correlation functions at equal time.

For \( x, y \) in the bulk, we find

\[ [J_2(x), J_1(y)] = \frac{1}{2\pi}(2\Gamma_{21}(x,y)) \left|_{x_2=y_2} \right. \]

\[ = \frac{1}{\pi} \partial_1 \delta(x_1 - y_1) \]  

which is the usual commutation relation (up to a factor of \( i \) which is absent in Euclidean space). Close to the boundary, the only non-vanishing commutator is

18
\[ [J_2(x), J_1(0, y_2)] = \frac{1}{\pi} \frac{1}{2} \Gamma_{21} (x, y) \bigg|_{x_2 = y_2, y_1 = 0} \]
\[ = \frac{1}{\pi} \partial_1 \delta(x_1) \] (4.19)

since at equal time \( \Gamma_{21}(x, y)|_{y_1=0} = -\frac{1}{2} \partial_1 \delta(x_1) \). Hence, we see that the presence of the boundary changes the currents but it does not change the commutation relations.

In next section we will see how the boundary and its degree of freedom modify the one-particle Green’s function.

V. BOSONIZATION OF THE FERMI OPERATORS

In this section we derive a set of bosonization rules for the fermion operators for systems with two types of boundary conditions: a) fixed \( R(0, x_2) = -L(0, x_2) \) and b) dynamical boundary conditions \( R(0, x_2) = -\exp(2\theta(x_2))L(0, x_2) \), where \( \theta(x_2) \) is a boundary degree of freedom. Our strategy is to first calculate the one-particle Green’s function in \( 1 + \frac{1}{2} \) dimensions with both types of boundary conditions and to use these results to determine the bosonization rules. We will see that we can obtain a generalization of the Mandelstam bosonization rules for a semi-infinite system with specific boundary conditions. In other words, we are going to find bosonic variables, \( \omega(x) \) defined in the bulk, and \( \theta(x_2) \) defined at the boundary, with an action \( S_B(\omega, \theta) \), such that in the framework of this bosonic theory there exist operators with the same expectation values that those of the fermionic operators, \( R \) and \( L \) in the framework of the fermionic theory.

A. Boundary Condition \( R=-L \)

The Fermionic one particle Green’s function is

\[ S_{\alpha\beta}(x, y) = \langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle \] (5.1)

and it satisfies the Dirac equation.
\[
\begin{aligned}
&\left\{ \begin{array}{l}
    i\partial_x S_0(x, y) = \delta(x - y) \\
    S_0^{11}(0, x_2; y) = -S_0^{21}(0, x_2; y)
\end{array} \right. \\
&\text{(5.2)}
\end{aligned}
\]

This boundary condition implies precisely \( R(0, x_2) = -L(0, x_2) \). The solution for this differential equation is

\[
S_0(x, y) = S_F(x, y) - \gamma_2 S_F(x^*, y)
\]

where \( S_F \) is the free fermionic Green’s function, and \( x^* = (-x_1, x_2) \). Explicitly,

\[
S_0(x, y) = \frac{1}{2\pi} \begin{pmatrix}
-\frac{1}{w} & -\frac{1}{z} \\
\frac{1}{z} & \frac{1}{w}
\end{pmatrix}
\]

with \( z = (x_1 - y_1) + i(x_2 - y_2) \) and \( w = (-x_1 - y_1) + i(x_2 - y_2) \).

We will now proceed to find a bosonized expression for the fermion operator consistent with the boundary condition \( R = -L \). The method that we follow is a generalization of the procedure used by Fradkin and Kadanoff (FK) in two-dimensional critical phenomena \cite{FK1, FK2}. It consists of finding a set of operators in a theory of a free boson field \( \omega \) whose correlation function are given by Eq. (5.4). Following FK, we define a bosonic action coupled to a background gauge field as

\[
S_B(\omega, A_\mu) = -\frac{1}{2\pi} \int_\Omega d^2 x \left( \partial_\mu \omega + A_\mu \right)^2
\]

where \( \omega \) is a bosonic variable defined in \( \Omega \) which satisfies vanishing boundary conditions. Here \( A_\mu \) is a vector field such that

\[
B(z) \equiv \epsilon_{\mu\nu} \partial_\mu A_\nu(z) = \pi(-\delta(z-x) + \delta(z-y)).
\]

The gauge field \( A_\mu \) represents two flux tubes, each of flux \( \pi \) and \(-\pi\) respectively, at points \( \vec{x} \) and \( \vec{y} \). In other terms, we have two disorder operators. Hereafter, we will denote the insertion at \( x \) of a disorder operator of flux (or vorticity) \( \pm \pi \) by the symbol \( \mathcal{K}_{\pm\pi}(x) \). For infinite systems, FK found that the fermion operators are represented by suitable chosen linear combinations of products of disorder operators \( \mathcal{K}_{\pm\pi}(x) \) and order (or charge) operators...
exp((\pm i\omega(x')) (for \ x \rightarrow x'). This construction is the euclidean analog of the Mandelstam operators [1] of abelian bosonization. We will now investigate how does the presence of a boundary (as well as specific boundary conditions) alter these rules.

We are interested in computing the expectation values of operators of the form $e^{(\pm i\omega(x)+i\omega(y))}$ in the presence of disorder operators (i.e. gauge fields of the form of Eq. (5.8)). Therefore we need to compute bosonic partition functions of the form

$$Z[A_\mu, J] = \int \mathcal{D}\omega \ \exp \left( S_B + i \int_{\Omega} d^2z J(z)\omega(z) \right) \quad (5.7)$$

where the source $J$ has the form

$$J(z) = a\delta(z-x) + b\delta(z-y) \quad (5.8)$$

and $a,b$ take the values $\pm 1$. We will show that the expectation values of these operators in the bosonic theory give the components of the one particle fermionic Green’s function (5.4).

Since the action above is quadratic in $\omega$ it can be integrated explicitly

$$Z[A_\mu, J] = K \exp \left( -\frac{1}{2\pi} \int_{\Omega} d^2x A_\mu^2 \right) \exp\left( -\frac{\pi}{2} \int_{\Omega} d^2x d^2y \left( \frac{1}{\pi} \partial_\mu A_\mu + iJ(x)G_0(x,y)\frac{1}{\pi} \partial_\mu A_\mu + iJ(y) \right) \right) \quad (5.9)$$

where the Green’s function $G_0(x,y)$ was defined in Eq. (2.18). Note that, as it has been discussed before, this Green’s function is consistent with the boundary condition imposed on $\omega$. After some algebra (notice that the integration by parts can be safely done because of the boundary conditions chosen) we get for $Z[A_\mu, J]$

$$Z[A_\mu, J] = K \exp \left( -\frac{1}{2\pi} \int_{\Omega} d^2x d^2y A_\mu(x)\Gamma_{\mu\nu}(x,y)A_\nu(y) \right) \exp\left( \frac{i}{2} \int_{\Omega} d^2x d^2y \left[ A_\mu(x)\partial_\mu G_0(x,y)J(y) + J(x)\partial_\nu G_0(x,y)A_\nu(y) \right] \right) \quad (5.10)$$

where the function $\Gamma_{\mu\nu}(x,y)$ is defined as

$$\Gamma_{\lambda\alpha}(x,y) = \left( \partial_\mu^{(x)} \partial_\mu^{(y)} \delta_{\alpha\lambda} + \delta_{\lambda}^{(x)} \partial_\alpha^{(y)} \right) G_0(x,y) \quad (5.11)$$
The Green’s function of the half plane $G_0(x, y)$ is defined in terms of $g_1$ and $g_2$, the real parts of the analytic functions $\ln[(x_1 - y_1) + i(x_2 - y_2)]$ and $\ln[(-x_1 - y_1) + i(x_2 - y_2)]$ respectively. The imaginary parts of these functions are $\Phi_1(x, y) = \frac{1}{2\pi} \arctan\left(\frac{x_2 - y_2}{x_1 - y_1}\right)$ and $\Phi_2(x, y) = \frac{1}{2\pi} \arctan\left(\frac{x_2 - y_2}{x_1 - y_1}\right)$ respectively. Therefore we can use the Cauchy-Riemann equations to get

$$\partial^{(x)}_\mu g_1(x, y) = \epsilon_{\mu\nu} \partial^{(x)}_\nu \Phi_1(x, y)$$

and

$$\partial^{(x)}_\mu g_2(x, y) = -\epsilon_{\mu\nu} \partial^{(x)}_\nu \Phi_2(x, y).$$

With these relations we write the partition function as

$$Z[A_\mu, J] = K \exp \left( -\frac{1}{2\pi} \int_{\Omega} d^2x \, d^2y \, A_\mu(x) \hat{\Gamma}_{\mu\nu}(x, y) A_\nu(y) \right)$$

$$\exp \left( \frac{i}{2} \int_{\Omega} d^2x \, d^2y \left[ B(x) \Phi_1(x, y) J(y) + J(x) \Phi_1(x, y) B(y) \right] \right)$$

$$\exp \left( \frac{i}{2} \int_{\Omega} d^2x \, d^2y \left[ B(x) \Phi_2(x, y) J(y) - J(x) \Phi_2(x, y) B(y) \right] \right)$$

$$\exp \left( \frac{\pi}{2} \int_{\Omega} d^2x \, d^2y \, J(x) G_0(x, y) J(y) \right)$$

(5.14)

We can use again the fact that $B(x) = \epsilon_{\mu\nu} \partial_\mu A_\nu(x)$ to obtain

$$Z[A_\mu, J] = K \exp \left( \frac{1}{2} \int_{\Omega} d^2x \, d^2y \left[ \frac{1}{\pi} B(x) g_1(x, y) B(y) + \pi J(x) g_1(x, y) J(y) \right] \right)$$

$$\exp \left( \frac{1}{2} \int_{\Omega} d^2x \, d^2y \left[ i B(x) \Phi_1(x, y) J(y) + i J(x) \Phi_1(x, y) B(y) \right] \right)$$

$$\exp \left( \frac{1}{2} \int_{\Omega} d^2x \, d^2y \left[ \frac{1}{\pi} B(x) g_2(x, y) B(y) - \pi J(x) g_2(x, y) J(y) \right] \right)$$

$$\exp \left( \frac{1}{2} \int_{\Omega} d^2x \, d^2y \left[ i B(x) \Phi_2(x, y) J(y) - i J(x) \Phi_2(x, y) B(y) \right] \right).$$

(5.15)

Replacing $J(z) = a\delta(z - x) + b\delta(z - y)$ and $B(z) = \pi(-\delta(z - x) + \delta(z - y))$ in the integrands, the partition function

$$Z[A_\mu, J] = K \exp \left( \pi(-1 + ab) g_1(x, y) + i\pi(-b + a) \Phi_1(x, y) - i\pi(b + a) \Phi_2(x, y) \right)$$

$$\exp \left( \frac{\pi}{4} (1 - a^2) g_2(x, y) + \frac{\pi}{4} (1 - b^2) g_2(y, y) - \pi(1 + ab) g_2(x, y) \right).$$

(5.16)
\begin{equation}
\langle \exp(ia\omega(x) + ib\omega(y)) \rangle_{\text{bos}} = \frac{Z[A_\mu, J]}{Z[0]}
= \exp(\pi(-1 + ab)g_1(x, y) + i\pi(-b + a)\Phi_1(x, y)) \exp(-\pi(1 + ab)g_2(x, y) - i\pi(b + a)\Phi_2(x, y))
\exp\left(\frac{\pi}{4}(1 - a^2)g_2(x, x) + \frac{\pi}{4}(1 - b^2)g_2(y, y)\right)
\end{equation}

There are four possible combinations of $a$ and $b$, which correspond to the four matrix elements of $S_0(x, y)$. If $a = 1, b = -1$ we get

\begin{equation}
\langle \exp(i\omega(x) - i\omega(y)) \rangle = \exp\left[-\ln\left((x_2 - y_2)^2 + (x_1 - y_1)^2\right)\right] + i\arctan\left(\frac{x_2 - y_2}{x_1 - y_1} + i\pi\right) = -\frac{1}{z}.
\end{equation}

Comparing with Eq. (5.14), we see that this term corresponds, in the fermionic theory, to $S_0^{12}(x, y) = \langle R(x)R^\dagger(y) \rangle$. In the same way, with $a = b = -1$ we get

\begin{equation}
\langle \exp(-i\omega(x) - i\omega(y)) \rangle = \exp\left[-\ln\left((x_2 - y_2)^2 + (-x_1 - y_1)^2\right)\right] + i\arctan\left(\frac{x_2 - y_2}{-x_1 - y_1}\right) = \frac{1}{w}
\end{equation}

which corresponds to $S_0^{22}(x, y) = \langle L(x)R^\dagger(y) \rangle$.

Hence, we can identify

\begin{align*}
R(x) &\sim \exp(+i\omega(x)) K_{-\pi}(x) \\
R^\dagger(x) &\sim \exp(-i\omega(x)) K_{+\pi}(x) \\
L(x) &\sim \exp(-i\omega(x)) K_{+\pi}(x) \\
L^\dagger(x) &\sim \exp(+i\omega(x)) K_{-\pi}(x)
\end{align*}

where $\sim$ means that we identify products of operators (in the sense of the operator product expansion) inside arbitrary matrix elements of the fermionic and bosonic theories respectively. In the same way we can identify the remaining matrix elements. We conclude that, for the case of the boundary condition $R = -L$, the fermion operator on the half-plane is constructed in the same way as in FK [23]. The boundary condition appears only through the presence of diagonal operators which mix $L$ and $R$ components of the Fermi field.
It is worthwhile to note that, under global chiral (euclidean) transformations, $R \rightarrow e^{\theta_0} R$ and $L \rightarrow e^{-\theta_0} L$, the off-diagonal matrix elements of the fermion propagator remain invariant but the diagonal matrix elements do not. In the bosonized theory, global chiral transformations are represented by $\omega \rightarrow \omega + \theta_0$. This representation of the matrix elements clearly satisfies this symmetry. Notice, however, that a global chiral transformation requires that the boundary condition be modified since the boundary condition itself breaks the chiral symmetry explicitly. Below we will come back on this issue.

**B. Dynamical Boundary Conditions**

Up to this point, we have shown how, following FK, we can calculate the one particle Green’s function in its bosonized form for the boundary conditions $R(x_2) = -L(x_2)$. The next step is the calculation of the Green’s function with dynamical boundary conditions. In other words, we are looking for the solutions of

\[
\begin{cases}
  i\partial_x S_F(x, y|\theta) = \delta(x - y) \\
  S_{11}^F(0, x_2; y|\theta) = -e^{2\theta(x_2)} S_{21}^F(0, x_2; y|\theta).
\end{cases}
\]  

(5.21)

Here, $\theta(x_2)$ represents a quantum mechanical degree of freedom localized at the boundary, i.e., a quantum impurity. For the rest of this section we will take $\theta(x_2)$ to be arbitrary but fixed. In the next section we will consider a model with a fully dynamical, quantum mechanical boundary degree of freedom.

The solution of Eq. (5.21) is

\[
S_F(x, y|\theta) = \frac{1}{2\pi} \left( \frac{-b(x)a^{-1}(y)}{w} - \frac{b(x)b^{-1}(y)}{z} \right)
\]

(5.22)

The functions $a(x)$ and $b(x)$ have to be chosen in such a way that the differential equation and the boundary condition of Eq. (5.21) are satisfied. In particular, it has to be $(\partial_{x_2} + i\partial_{x_1}) b(x) = (\partial_{x_2} - i\partial_{x_1}) a(x) = 0$.

To find the explicit form of $a(x)$ and $b(x)$ we will use the FK approach. In the method developed in the previous subsection, the bosonic field satisfies vanishing bound-
ary conditions. This reflects the fact that the boundary conditions for the fermions were $R(0, x_2) = -L(0, x_2)$. But we have yet to determine how dynamical boundary conditions may affect the boundary conditions for the bosons. Our strategy will be to relate $S_F(x, y|\theta)$ with a Green’s function that satisfies non-twisted, $R = -L$, boundary conditions. For that purpose we define $S(x, y)$ as a solution of the following equation

\[
\begin{cases}
(i\partial_x + \bar{A})S(x, y) = \delta(x - y) \\
S^{11}(0, x_2; y) = -S^{21}(0, x_2; y)
\end{cases}
\]

(5.23)

Now, it is obvious that we can find $S(x, y)$ by using the method described in the previous subsection. To establish a relation between $S(x, y)$ and $S_F(x, y|\theta)$ we write $\bar{A}_\mu$ as

\[
\bar{A}_\mu = \epsilon_{\mu\nu} \partial_\nu \rho
\]

(5.24)

where $\rho(x)$ is a function such that $\rho(0, x_2) = \theta(x_2)$. It can be seen that if we take,

\[
S_F(x, y|\theta) = e^{\gamma_5 \rho(x)} S(x, y) e^{\gamma_5 \rho(y)}
\]

(5.25)

then $S_F(x, y|\theta)$ is a solution of Eq.(5.21). Following the same steps that lead to the calculation of $S_0$, we define

\[
Z[A_\mu, J, \bar{A}_\mu] = \int D\omega \exp S_B(\omega, A_\mu) \\
\exp \left( i \int_{\Omega} J(z) \omega(z) + \int_{\Omega} J_\mu(z) \bar{A}_\mu(z) \right) \\
\exp \left( \frac{1}{2\pi} \int dx_2 dy_2 \theta(x_2) K(x_2, y_2) \theta(y_2) + \frac{1}{\pi} \int dx_2 dy_2 \bar{A}_2(x_2) K(x_2, y_2) \theta(y_2) \right)
\]

(5.26)

Here, $A_\mu$ and $J$ are the external sources defined in Eqs. (5.6) and (5.8) respectively, and $J_\mu(z)$ is the current defined in (4.9). If we define $S_B(\theta, \bar{A}_2)$ as

\[
S_B(\theta, \bar{A}_2) = \frac{1}{2\pi} \int dx_2 dy_2 \theta(x_2) K(x_2, y_2) \theta(y_2) + \frac{1}{\pi} \int dx_2 dy_2 \bar{A}_2(x_2) K(x_2, y_2) \theta(y_2)
\]

(5.27)

the partition function can be written as
\[ Z[A_\mu, J, \bar{A}_\mu] = e^{S_B(\theta, \bar{A}_z)} \int D\omega \exp \left( S_B(\omega, A_\mu) + i \int_{\Omega} [J(z) - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \bar{A}_\mu(z)] \omega(z) \right) \]
\[ = e^{S_B(\theta, \bar{A}_z)} \int D\omega \exp \left( S_B + i \int_{\Omega} \tilde{J}(z) \omega(z) \right) \]  

(5.28)

with \( \tilde{J}(z) = J(z) - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \bar{A}_\mu(z) \). This expression is formally equal to Eq. (5.7). Hence, we can follow the same steps that took this equation into Eq. (5.14) to obtain

\[ Z[A_\mu, J, \bar{A}_\mu] = e^{S_B(\theta, \bar{A}_z)} Z[A_\mu, J] \]

\[ \exp \left( \frac{i}{2\pi} \int_{\Omega} [\partial_\mu A_\mu(x)G_0(x, y) \nabla^2 \rho(y) + \nabla^2 \rho(x)G_0(x, y) \partial_\mu A_\mu(y)] \right) \]
\[ \exp \left( \frac{-1}{2} \int_{\Omega} [J(x)G_0(x, y) \nabla^2 \rho(y) + \nabla^2 \rho(x)G_0(x, y)J(y)] \right) \]
\[ \exp \left( \frac{1}{2\pi} \int \bar{A}_\mu(x)\Gamma_{\mu\nu}(x, y)A_\nu(y) \right) \]  

(5.29)

where \( Z[A_\mu, J] \) is exactly equal to Eq. (5.13) and the other factors come from the extra term in \( \tilde{J}(x) \). These two factors can be computed with the same techniques used before giving

\[ Z[A_\mu, J, \bar{A}_\mu] = e^{S_B(\theta, \bar{A}_z) + \frac{i}{2\pi} \int \bar{A}_\mu(x)\Gamma_{\mu\nu}(x, y)A_\nu(y) Z[A_\mu, J] \]
\[ \quad \exp (-a\rho(x) - b\rho(y)) \exp \left( \frac{1}{\pi} \int dz_2 \rho(0, z_2) \frac{ax_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \]
\[ \exp \left( \frac{1}{\pi} \int dz_2 \rho(0, z_2) \frac{by_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right). \]  

(5.30)

Then,

\[ \langle \exp (i\omega(x) + ib\omega(y)) \rangle \equiv \frac{Z[A_\mu, J, \bar{A}_\mu]}{Z[A_\mu]} \]
\[ = \exp (\pi(-1 + ab)g_1(x, y) + i\pi(-b + a)\Phi_1(x, y)) \]
\[ \exp (-\pi(1 + ab)g_2(x, y) - i\pi(b + a)\Phi_2(x, y)) \]
\[ \exp \left( \frac{1}{\pi} \int dz_2 \rho(0, z_2) \frac{ax_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \]
\[ \exp \left( \frac{1}{\pi} \int dz_2 \rho(0, z_2) \frac{by_1 - i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right) \]
\[ \exp (-a\rho(x) - b\rho(y)) \]  

(5.31)

In this case also, there are four combinations of \( a \) and \( b \) which give the four matrix elements of \( S(x, y) \). Up to a normalization constant the one particle Green’s function is
\[ S(x, y) = \frac{1}{2\pi} \begin{pmatrix} -e^{-\rho(x)b(x)a^{-1}(y)e^{-\rho(y)}} & -e^{-\rho(x)b(x)b^{-1}(y)e^{\rho(y)}} \\ e^{\rho(x)a(x)a^{-1}(y)e^{-\rho(y)}} & e^{\rho(x)a(x)b^{-1}(y)e^{\rho(y)}} \end{pmatrix} \]  

(5.32)

where

\[ a(x) = \exp \left( \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{-x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \]  

(5.33)

\[ b(y) = \exp \left( \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{y_1 + i(y_2 - z_2)}{y_1^2 + (y_2 - z_2)^2} \right) \]  

(5.34)

and

\[ \rho(x) = \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{x_1}{x_1^2 + (x_2 - z_2)^2}. \]  

(5.35)

Note that \( \rho(x_1, x_2) \) satisfies

\[ \begin{cases} 
\nabla^2 \rho(x) = 0 \text{ in } x_1 > 0 \\
\rho(0, x_2) = \theta(x_2)
\end{cases} \]  

(5.36)

It can be seen that

\[ \ln a(x) = -2 \int_{\omega < 0} \frac{d\omega}{2\pi} \rho(\omega) e^{i\omega(x_2 - ix_1)} \equiv -2\rho_-(x_2 - ix_1) \]  

(5.37)

Hence, \( \ln a(x) \) is the anti-analytic extension of the negative frequency part of \( \rho(0, x_2) = \theta(x_2) \). In the same way,

\[ \ln b(x) = 2 \int_{\omega > 0} \frac{d\omega}{2\pi} \rho(\omega) e^{i\omega(x_2 + ix_1)} \equiv 2\rho_+(x_2 + ix_1) \]  

(5.38)

is the analytic extension of the positive frequency part of \( \rho(0, x_2) = \theta(x_2) \). Note that \( \rho(x_1, x_2) \) is the real part of both, \( \rho_+(x_1 + ix_2) \) and \( \rho_-(x_1 - ix_2) \) and that

\[ 2\rho_+(x_2 + ix_1) + 2\rho_-(x_2 - ix_1) = 2\rho(x_1, x_2) \]  

(5.39)
Using the relation between $S_F(x,y|\theta)$ and $S(x,y)$ (Eq. (5.25)), we get an expression in the bosonized theory for the fermion Green’s function which satisfies dynamical boundary conditions (Eq. (5.22)).

By using Eqs.(5.33-5.34) we can rewrite Eq.(5.22) as

$$S_F(x,y|\theta) = \frac{1}{2\pi} \left( -\frac{1}{x - i(x_2 - z_2)} + \frac{1}{y + i(y_2 - z_2)} \right)$$

where

$$h_{11}(z_2) = \frac{1}{x - i(x_2 - z_2)} + \frac{1}{y + i(y_2 - z_2)}$$

$$h_{12}(z_2) = \frac{1}{x - i(x_2 - z_2)} - \frac{1}{y - i(y_2 - z_2)}$$

$$h_{21}(z_2) = -\frac{1}{x + i(x_2 - z_2)} + \frac{1}{y + i(y_2 - z_2)}$$

$$h_{22}(z_2) = -\frac{1}{x + i(x_2 - z_2)} - \frac{1}{y - i(y_2 - z_2)}$$

(5.41)

We can summarize these results in the form of a set of bosonization rules for the fermion operators, which now include the effects of the presence of a boundary degree of freedom. As in the previous case, i. e. for $R(x_2) = -L(x_2)$, we can identify the fermionic operators, $R(x_2)$ and $L(x_2)$, with the corresponding bosonic operators. From Eq. (5.22) we see that

$$R(x_2) \sim \exp \left( i\omega(x) + \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \mathcal{K}_{-\pi}(x)$$

$$R(x_2)^\dagger \sim \exp \left( -i\omega(x) - \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \mathcal{K}_{+\pi}(x)$$

$$L(x_2) \sim \exp \left( -i\omega(x) + \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{-x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \mathcal{K}_{+\pi}(x)$$

$$L(x_2)^\dagger \sim \exp \left( +i\omega(x) - \frac{1}{\pi} \int dz_2 \theta(z_2) \frac{-x_1 + i(x_2 - z_2)}{x_1^2 + (x_2 - z_2)^2} \right) \mathcal{K}_{-\pi}(x)$$

(5.42)

where, as before, $\sim$ means that the bosonic operators yield the same correlation functions (with the same boundary conditions) as the operators $R$ and $L$ in the Fermi theory. This result completes the bosonization construction. Notice that the main modification is the presence of a boundary operator in the definition of the fermion operator of this generalization.
VI. FREE FERMIONS COUPLED TO A DYNAMICAL BOUNDARY DEGREE OF FREEDOM

Now we will compute the one particle Green’s function in the case that \( \theta \) is a quantum mechanical degree of freedom. That is, instead of regarding \( \theta(x_2) \) as a prescribed classical parameter, it has to be regarded of as an extra degree of freedom of the full system. In order to account for the effects of the dynamics of \( \theta \), we have to specify its dynamics through an extra term in the lagrangian and we must also integrate over all possible configurations of \( \theta \). Hence, the full one particle Green’s function becomes

\[
S_F(x, y) = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}\theta \, \psi(x)\bar{\psi}(y) \, e^{iS(\bar{\psi}, \psi, \theta) + iS_\theta(\theta)}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{iS(\bar{\psi}, \psi, \theta)}} = \frac{\int \mathcal{D}\theta \, S_F(x, y|\theta) \, \text{Det}(\theta) \, e^{iS_\theta(\theta)}}{\int \mathcal{D}\theta \, \text{Det}(\theta) \, e^{iS_\theta(\theta)}} \quad (6.1)
\]

where \( S(\bar{\psi}, \psi, \theta) \) is the action for the massless fermions in the half line coupled to a dynamical degree of freedom at the boundary. \( \text{Det}(\theta) \) is the determinant calculated in Section III (Eq. (3.39)). Upon exponentiation, it yields an induced term in the action \( S_{\text{ind}}(\theta) \) of the form

\[
S_{\text{ind}}(\theta) \equiv \ln \left( \frac{\text{Det}(i\partial \theta)_{\text{B}}^{-1}}{\text{Det}(i\partial \theta)_B} \right) = \frac{1}{4\pi^2} \int dx_2 dy_2 \frac{[\theta(y_2) - \theta(x_2)]^2}{(x_2 - y_2)^2 + a^2}. \quad (6.2)
\]

which represents the dynamics of \( \theta \) induced by the fermionic degree of freedom. The intrinsic dynamics of the boundary degree of freedom is represented by the term \( S_\theta(\theta) \) in the action.

At this point, it is convenient (and helpful to understand the physics) to make an analytic continuation of the boundary degree of freedom \( \theta \rightarrow i\theta \). In this way, we recover the usual periodicity property \( \theta \rightarrow \theta + 2\pi \). In Euclidean space this symmetry is not manifest since
the group of chiral transformations is non-compact in Euclidean space. This change has
for effect to (a) change the sign of the induced action \( S_{ind}(\theta) \) and (b) the Green function
amplitudes need to be analytically continued (Eq. (5.40)).

The intrinsic dynamics of the boundary degree of freedom \( \theta \) is determined by the un-
derlying microscopic system from which the Luttinger-type model is derived. A simple and
still realistic model of this dynamics consists in regarding \( \theta \) as a coordinate for a single
degree of freedom moving on a ring. This is the situation in the Caldeira-Leggett model [30]
which describes the effects of a local degree of freedom coupled to a “fermion bath”. From
the point of view of a Hubbard model or a quantum wire, \( \theta \) represents a boundary effect
of electron-electron correlations. A set of “interesting” terms that can be incorporated in
\( S_\theta(\theta) \) are (in imaginary time)

\[
S_\theta(\theta) = \int dx_2 \left( \frac{M}{2} \left( \frac{\partial \theta}{\partial x_2} \right)^2 - V(\theta) \right)
\]

(6.3)

where \( M \) is the mass of the boundary degree of freedom. In quantum wire problems, the
potential \( V(\theta) \) is the backscattering amplitude at the boundary. This term is of the form

\[
V(\theta) = G \cos(\beta\theta)
\]

(6.4)

where \( \beta \) is a parameter which, if interactions are present, depends on the electron-electron
coupling constant.

By simple inspection we can see that all terms of the action, except for the potential
term, are invariant under a global chiral rotation of the boundary conditions, \( \theta(x_2) \to \theta(x_2) + \theta_0 \). The potential term reduces this continuous symmetry to a discrete subgroup
\( \theta(x_2) \to \theta(x_2) + 2\pi n/\beta \), where \( n \) is an arbitrary integer. Similarly, the off-diagonal terms
of the Green function are \textit{invariant} under global chiral rotations whereas the diagonal terms
\textit{are not}. These symmetries have a very simple interpretation in the bosonized theory. Since
the chiral symmetry of the fermions translates into the invariance of the bosonized theory
by constant shifts of the field \( \omega \), the boundary condition \( \omega = 0 \) at \( x_1 = 0 \) reflects the fermion
boundary condition \( R = -L \). The boundary degree of freedom \( \theta \) can be viewed as the
value of \( \omega \) at the boundary. Hence, a path integral in which one integrates over all values of \( \theta \) without restrictions should correspond to free boundary conditions for \( \omega \). We will see below that this is indeed the case and that, in the absence of potential terms, the boundary conditions are free and the system behaves as in the case of an infinite line. Conversely, potential terms act to “pin” the boundary value of the bosonic field \( \omega \) to a finite set of possible values. This behavior is closely related to the picture of Affleck and Ludwig of the Kondo problem as a crossover in boundary conditions \[17\].

In the framework of macroscopic quantum tunneling and coherence, Fisher and Zwerger \[31\] have studied systems with actions of this same type. By means of renormalization group arguments, they concluded that the kinetic energy term is \textit{irrelevant} at low frequencies (or long times), that the induced term in the action is a strictly marginal operator and that the cosine terms of the potential are relevant. Fisher and Zwerger refer to this strong-coupling fixed point as to the \textit{localization transition}. The induced term in the action plays the same role in this problem as the free massless scalar theory (or Gaussian model) does in two-dimensional critical systems and in conformal field theory.

We are interested in the computation of the matrix elements of the fermion one-particle Green’s function for a system with dynamical boundary conditions. To simplify the discussion we will include the effects of the potential term only within a linearized approximation in which \( V(\theta) \approx -\frac{G}{2} \theta^2 \equiv -\bar{G} \theta^2 \). This approximation misses the important periodicity property and overemphasizes the physics of localization. Recent work by Ludwig and collaborators \[32\] and by Tsvelik \[33\] indicates that there is interesting physics which this fixed point does not describe correctly.

Hence, we conclude that the effective dynamics of the boundary degree of freedom \( \theta \) is described by the action

\[
S_{\text{eff}}(\theta) = \frac{1}{2\pi} \int dx_2 dy_2 \theta(x_2) K(x_2, y_2) \theta(y_2) + \int dx_2 \left( \frac{M}{2} \left( \frac{\partial \theta}{\partial x_2} \right)^2 + \frac{\bar{G}}{2} \theta^2 \right)
\]

where \( K(x_2, y_2) \) is given by Eq.\[4.8\]. In order to compute the one-particle fermion Green’s function we need to evaluate functional integrals over \( \theta \) of the form
\[ W_{\alpha\beta} \equiv \langle \exp \left( \frac{i}{\pi} \int dz_2 \theta(z_2) h_{\alpha\beta}(z_2) \right) \rangle_{\theta} = \frac{\int D\theta \exp \left( -S_{\text{eff}}(\theta) + \frac{i}{\pi} \int dz_2 \theta(x_2) h_{\alpha\beta}(z_2) \right)}{\int D\theta \exp \left( -S_{\text{eff}}(\theta) \right)} \] (6.6)

Since the exponent is quadratic in \( \theta \), these integrals can be done explicitly. In Fourier space, the action \( S_{\text{eff}}(\theta) \) is

\[ S_{\text{eff}}(\theta) = \int \frac{d\omega}{2\pi} |\theta(\omega)|^2 \frac{K_{\text{eff}}(\omega)}{2\pi} \] (6.7)

where the kernel \( K_{\text{eff}}(\omega) \) is given by

\[ K_{\text{eff}}(\omega) = \left. \frac{1}{a} - e^{-a|\omega|} + \pi M\omega^2 + \pi \bar{G} \right. \] (6.8)

The form of the kernel \( K_{\text{eff}}(\omega) \) shows that the inertial term acts like a high frequency cutoff. At low frequencies we can use the approximation \( K_{\text{eff}}(\omega) \approx |\omega| + \pi \bar{G} \). The expectation values of the form of Eq. (6.4) become

\[ W_{\alpha\beta} = \exp \left( -\frac{1}{2\pi} \int \frac{d\omega}{2\pi} h_{\alpha\beta}(-\omega) \frac{1}{K_{\text{eff}}(\omega)} h_{\alpha\beta}(\omega) \right) \] (6.9)

where \( h_{\alpha\beta}(\omega) \) is the Fourier transform of \( h_{\alpha\beta}(z_2) \). By making use of the expressions for the amplitudes \( h_{\alpha\beta}(z_2) \) given in Eq. (5.41), we find that their Fourier transforms can be written in terms of the functions

\[ F_{\pm}(\omega; u, v) = \int_{-\infty}^{+\infty} d\tau \frac{e^{i\omega \tau}}{u \pm i(\tau - v)} = 2\pi e^{-|\omega|(u \mp iv)} \Theta(\pm \omega) \] (6.10)

where \( u > 0 \) and \( \Theta(x) \) is the step function. The Fourier transforms become

\[
\begin{align*}
    h_{11}(\omega) &= F_+(\omega; x_1, x_2) + F_-(\omega; y_1, y_2) = 2\pi e^{-|\omega|(x_1-i-x_2)} \Theta(\omega) + 2\pi e^{-|\omega|(y_1+i+y_2)} \Theta(-\omega) \\
    h_{12}(\omega) &= F_+(\omega; x_1, x_2) - F_-(\omega; y_1, y_2) = 2\pi \left( e^{-|\omega|(x_1-i-x_2)} - e^{-|\omega|(y_1-i-y_2)} \right) \Theta(\omega) \\
    h_{21}(\omega) &= -F_-(\omega; x_1, x_2) + F_-(\omega; y_1, y_2) = 2\pi \left( -e^{-|\omega|(x_1+i+x_2)} + e^{-|\omega|(y_1+i+y_2)} \right) \Theta(-\omega) \\
    h_{22}(\omega) &= -F_-(\omega; x_1, x_2) - F_+(\omega; y_1, y_2) = -2\pi e^{-|\omega|(x_1+i+x_2)} \Theta(-\omega) - 2\pi e^{-|\omega|(y_1-i-y_2)} \Theta(\omega)
\end{align*}
\] (6.11)

We can now write down an explicit formula for the logarithm of the amplitudes \( W_{\alpha\beta} \). We find that the \textit{off-diagonal} matrix elements vanish identically, \( \ln W_{12} = \ln W_{21} \equiv 0 \) and that the \textit{diagonal} matrix elements are given by

\[ \ln W_{11} = \ln W_{22} = \int D\theta \ln \left( \frac{\int D\theta \exp \left( -S_{\text{eff}}(\theta) + \frac{i}{\pi} \int dz_2 \theta(x_2) h_{\alpha\beta}(z_2) \right)}{\int D\theta \exp \left( -S_{\text{eff}}(\theta) \right)} \right) \]

\[ \ln W_{12} = \ln W_{21} \equiv 0 \]

\[ \ln W_{11} \] and \( \ln W_{22} \) are given by the logarithms of the respective kernels.
\[ \ln \bar{W} = \ln W_{11} = \ln W_{22} = -2 \int_0^\infty d\omega \frac{e^{-\omega((x_1+y_1)-i(x_2-y_2))}}{K_{\text{eff}}(\omega)} \tag{6.12} \]

The explicit computation of the integral in eq. 6.12 yields the result

\[ \ln \bar{W} = 2e^{\pi \bar{G} \bar{w}} Ei(-\pi \bar{G} \bar{w}) \tag{6.13} \]

where \( Ei(-x) \) is the exponential-integral function. This result yields the asymptotic behaviors of \( \ln \bar{W} \) as

\[ \ln \bar{W} \approx 2 \left( C + \ln(- \bar{\omega} \pi \bar{G}) \right), \text{ for } |\bar{w}|\pi \bar{G} \ll 1 \]
\[ \ln \bar{W} \approx -\frac{2}{\pi G \bar{w}}, \text{ for } |\bar{w}|\pi \bar{G} \gg 1 \tag{6.14} \]

where \( C \) is Euler’s constant. In the limit \( x_1 + y_1 \to 0 \), these results hold provided that \( x_1 + y_1 \) is replaced by the smallest of the cutoff \( a \) and \( \sqrt{\pi M} \).

With these results at hand we find that, for dynamical boundary conditions, the matrix elements of the Fermion Green’s function are given by

\[ S_F(x, y) = \frac{1}{2\pi} \begin{pmatrix} -\frac{W}{\bar{w}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{W}{\bar{w}} \end{pmatrix} \tag{6.15} \]

where \( \bar{W} \) is given in Eq.( 6.12).

Hence, we find that the off diagonal matrix elements of the fermion Green’s function, \( i.e. \) the propagators \( \langle R(x)R^\dagger(y) \rangle \) and \( \langle L(x)L^\dagger(y) \rangle \) respectively, are \textit{equal} to their value in the infinite plane and are unaffected by dynamical boundary conditions. Notice, however, that for a \textit{specified} configuration of the boundary chiral angle \( \theta(x_2) \), these matrix elements are modified. In contrast, we find that the diagonal matrix elements, \( i.e. \) the propagators \( \langle R(x)L^\dagger(y) \rangle \) and \( \langle L(x)R^\dagger(y) \rangle \) respectively, are modified by dynamical boundary conditions. However, these changes are only significant either close to the boundary and at short times. At long times and far away from the boundary, the off diagonal matrix elements behave exactly as in the case of \( R = -L \) boundary conditions. Close to the boundary, we find power law corrections of the form
\[ \langle R(x)L^\dagger(y) \rangle \sim -\text{const.} \frac{(\pi \bar{G} \bar{w})^2}{2\pi w} \]  

(6.16)

If the potential term is absent ($\bar{G} \to 0$), the boundary degree of freedom $\theta$ is unpinned. In this regime, the diagonal matrix elements of the fermion one-particle Green’s function vanish identically, even away from the boundary,

\[ S_F(x,y) = \frac{1}{2\pi} \begin{pmatrix} 0 & -\frac{1}{z} \\ \frac{1}{z} & 0 \end{pmatrix} \]  

(6.17)

In this limit the boundary degree of freedom fluctuates wildly and it effectively removes the boundary condition $\omega = 0$ for the bosonized theory. In other terms, the bosonized theory has free boundary conditions instead of Dirichlet. In this regime, the fermions are not scattered by the boundary. In a sense, they are “eaten” by the boundary degree of freedom. The fermion Green’s function in this limit is equal to the Green’s function in the infinite plane (i.e. the full line). Once again, this result agrees with the picture of Affleck and Ludwig.

Finally, let us note that the amplitude $\bar{W}$ has a clear physical meaning. If $\bar{W}$ was a constant, it would have the straightforward physical interpretation as a backscattering amplitude. However, it is both position and time-dependent. Hence, the backscattering amplitude determined from $\bar{W}$ is both momentum and energy (frequency) dependent. The actual form of this dependence is rather involved, as can be seen by inspecting the form of $\bar{W}$ in real time. The frequency dependence of $\bar{W}$ is simply a manifestation of the crossover between the two asymptotic behaviors shown in eq. 6.14. Indeed, at frequencies high compared with $1/(\pi \bar{G})$, $\bar{W}$ scales to zero at low frequencies, i.e. it behaves asymptotically as in the case in which the boundary degree of freedom is unpinned ($\bar{G} \to 0$). In this regime the effective value of the diagonal matrix elements of the fermion Green’s function scale to zero and the effective boundary condition is free. Conversely, at low frequencies and for $\bar{G}$ fixed, $\bar{W}$ scales to a constant. This is the behavior of the pinned boundary degree of freedom and it amounts to a boundary condition $R = -L$ for the fermions.
VII. CONCLUSIONS

In this paper we have reexamined the bosonization of a theory of free fermions in the
presence of boundaries and with specific boundary conditions. This is a very simple system in
which the role of boundary conditions and of boundary degrees of freedom can be investigated
quite explicitly.

We have exhibited the role of the boundary conditions of the fermions in the bosonized
theory. We used the Forman’s method for the computation of fermion determinants to
derive the action of the bosonized theory which exhibits the effects of the boundary degrees
of freedom. This action allowed us to determine the form of the currents and found that
the currents acquire additive corrections due to the boundary degree of freedom. These
corrections are needed in order to to satisfy the conservation laws. In contrast, we find that
the current algebra is not affected by the presence of boundaries and of boundary degrees of
freedom. We also constructed the bosonized form of the Fermi operators . We showed that
the boundary degrees of freedom enter explicitly in the definition of the Fermi operators.
Finally, we used these results to calculate the fermion one-particle Green’s function for a
theory with a dynamical boundary degree of freedom.

The methods described in this paper can be generalized so as to include the effects
of interactions and of non-abelian symmetries. It is also possible to use these methods
to investigate the interesting non-perturbative structure recently found , even in abelian
systems, by means of the thermodynamic Bethe Ansatz \cite{32,33}. In a separate publication \cite{19}
we apply our methods to the case of interacting fermions, in particular to the Luttinger model
and study the interplay between the Kondo effect and interactions in 1 + 1-dimensional
systems.
VIII. ACKNOWLEDGEMENTS

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APPENDIX A: BOUNDARY JACOBIAN

The aim of this appendix is the computation of

$$\ln \delta J_F = \lim_{s, \epsilon \to 0} \left( \int_0^1 dt \int_{x_1}^x dx_2 \int_0^\epsilon dx_1 < x | e^{-s \mathcal{D}_t} \mathcal{D}_t | x > \phi(x_1, x_2) \right)$$

$$= -2 \lim_{s, \epsilon \to 0} \left( \int_0^1 dt \int_{x_1}^x dx_2 \int_0^\epsilon dx_1 \text{tr}[K_2(i \mathcal{D}_t; x, x) \gamma_5 \phi(x_1, x_2)] \right). \quad (A1)$$

In order to do this computation we have to find the heat kernel $K_2$ in $(0, \epsilon)$. The operator $i \mathcal{D}$ can be written as

$$i \mathcal{D} = \begin{pmatrix} 0 & -\partial_1 + B \\ \partial_1 + B & 0 \end{pmatrix} \quad (A2)$$

with $B = i \partial_2 + A_2$. Following APS [22] we make the approximation $A_2(x_1, x_2) \approx A(0, x_2)$. In order to compute the trace, we need to find a basis of eingestates of $i \mathcal{D}_t$. We construct such a basis as follows. Let us call $e_\lambda(x_2)$ the eigenvector of eigenvalue $\lambda$ of the operator $B$. That is,

$$B e_\lambda(x_2) = \lambda e_\lambda(x_2). \quad (A3)$$

It can be shown that there exist $w_{k, \lambda}$ and a set of functions $f_{k, \lambda}(x_1)$ and $g_{k, \lambda}(x_1)$ such that

$$\begin{cases} 
  i \mathcal{D} \psi_{k, \lambda}(x_1, x_2) = w_{k, \lambda} \psi_{k, \lambda}(x_1, x_2) \\
  f_{k, \lambda}(0) = -g_{k, \lambda}(0)
\end{cases} \quad (A4)$$

where by definition $\psi_{k, \lambda}(x_1, x_2)$ is

$$\psi_{k, \lambda}(x_1, x_2) = \begin{pmatrix} f_{k, \lambda}(x_1) e_\lambda(x_2) \\
  g_{k, \lambda}(x_1) e_\lambda(x_2) 
\end{pmatrix} = \begin{pmatrix} R_{k, \lambda}(x_1, x_2) \\
  L_{k, \lambda}(x_1, x_2) \end{pmatrix}. \quad (A5)$$

Specifically,

$$\psi_{k, \lambda}(x_1, x_2) = \frac{e_\lambda(x_2)}{(2(\lambda + w_{k, \lambda}) w_{k, \lambda})^{1/2}} \begin{pmatrix} (\lambda + w_{k, \lambda}) \sin k x_1 - k \cos k x_1 \\
  (\lambda + w_{k, \lambda}) \sin k x_1 + k \cos k x_1 \end{pmatrix} \quad (A6)$$
form the complete set that satisfies Eq. (A4) if we take \( w_{k,\lambda} = (k^2 + \lambda^2)^{\frac{1}{2}} \) \((k \geq 0)\), and 
\[ \lambda = \frac{2\pi}{T}(n + \frac{1}{2} + \Lambda) \] where \( \Lambda \) is the flux of the gauge field through the boundary.

Then,
\[
\ln \delta J_F = \lim_{s, \epsilon \to 0} \left( -2 \int dx_2 \phi(0, x_2) \int_0^\epsilon dx_1 \left[ \sum_{k, \lambda} R_{k,\lambda}^\dagger(x_1) \phi e^{-s(\partial^2_x B^2)} R_{k,\lambda}(x_1) - \sum_{k, \lambda} L_{k,\lambda}^\dagger(x_1) \phi e^{-s(\partial^2_x B^2)} L_{k,\lambda}(x_1) \right] \right) 
= \lim_{s, \epsilon \to 0} \int dx_2 \phi(0, x_2) \int_0^\epsilon \frac{dk}{\pi} \sum_{\lambda} \left( \frac{e^{-s(k^2 + \lambda^2)}}{(k^2 + \lambda^2)^{\frac{3}{2}}} \right) \cos(2k\epsilon - 1). \tag{A7}
\]

It is important to note that in the above limit, \( s \ll \epsilon \). In the limit \( T \to \infty \) the sum over \( \lambda \) can be replaced by an integral. By using the Euler’s formula it can be shown that
\[
\ln \delta J_F = \frac{-2T}{\pi^2} \lim_{s, \epsilon \to 0} \int dx_2 \phi(0, x_2) \int_0^\infty \frac{dk}{\pi} \int_0^\infty du \left( \frac{e^{-s(k^2 + u^2)}}{(k^2 + u^2)^{\frac{3}{2}}} \right) \cos(2k\epsilon - 1) \tag{A8}
\]
The result of this last integral is
\[
\ln \delta J_F = \frac{-T}{4\pi} \lim_{s, \epsilon \to 0} \int dx_2 \phi(0, x_2) \Gamma(\frac{1}{2}) \sqrt{s} \left\{ e^{(\frac{\delta^2}{4} + \epsilon^2)} \left( 1 - \Phi \left( \sqrt{\frac{\delta^2}{4} + \epsilon^2} \right) \frac{1}{\sqrt{s}} \right) - e^{(\frac{\epsilon^2}{4})} \left( 1 - \Phi \left( \frac{\delta^2}{2\sqrt{s}} \right) \right) \right\} \tag{A9}
\]
where \( \delta \) is a Schwinger parameter that goes to zero and \( \Phi(x) \) is the error function. The order of the limits is \( \delta \ll s \ll \epsilon \). Then in the first term of Eq. (A9) we use the asymptotic expansion of \( \Phi(\sqrt{x}) \). But in the second term we use the limit for \( x \ll 0 \) of \( \Phi(x) \). Hence Eq. (A9) becomes
\[
\ln \delta J_F = \frac{-T}{4\pi} \lim_{s, \epsilon \to 0} \int dx_2 \phi(0, x_2) \frac{\Gamma(\frac{1}{2}) \sqrt{s}}{\sqrt{s}} \left( \frac{\Gamma(\frac{1}{2}) \sqrt{s}}{\epsilon} - 1 \right). \tag{A10}
\]
The second term in Eq. (A10) tells us that there is a singularity in \( s \). According to the definition of the heat kernel regularization \[7\] we have to keep only the finite part of (A10). Therefore, the properly regularized jacobian at the boundary is
\[
\delta J_F = \exp \left( -\frac{T}{4\epsilon} \int dx_2 \phi(0, x_2) \right). \tag{A11}
\]
Since \( \phi(0, x_2) = 0 \), the contribution of this jacobian to the partition function is one.
APPENDIX B: FORMAN’S DETERMINANT

In this appendix we are going to set up the notation and and show some of the calculations that lead to eq. (3.33) First note that we are working in the space spanned by \( e^{i\omega_n x_0} \) with \( n \) running over the integers. This space splits naturally into the one expanded by \( e^{i\omega_n x_0} \) with \( n \geq 0 \) (the space of functions with positive frequencies) and the space spanned by \( e^{i\omega_n x_0} \) with \( n < 0 \). Indeed, each \( \alpha \) can be decomposed into \( \alpha_+ \) and \( \alpha_- \) such that \( \alpha = \alpha_+ + \alpha_- \) and

\[
\alpha_+(x_2) = \sum_{n \geq 0} \alpha_n e^{i\omega_n x_2} \\
\alpha_-(x_2) = \sum_{n < 0} \alpha_n e^{i\omega_n x_2}.
\]

(B1)

In the same spirit, we can decompose operators \( \mathcal{A} \) acting in such space into blocks:

\[
\mathcal{A} = \begin{pmatrix}
[\mathcal{A}]_{--} & [\mathcal{A}]_{+-} \\
[\mathcal{A}]_{-+} & [\mathcal{A}]_{++}
\end{pmatrix} \tag{B2}
\]

where \([\mathcal{A}]_{--}\) means \([\mathcal{A}]_{m-n}\) with \( m, n < 0 \) and so on.

Finally, we use the notation \( \alpha \equiv \alpha(\mu, x_2) \equiv \mu \alpha(x_2) \) The first step is the calculation of the inverse of

\[
h(\mu) = \begin{pmatrix}
[e^{-\alpha}]_{--} & [e^{\alpha}]_{+-} \\
[e^{-\alpha}]_{-+} & [e^{\alpha}]_{++}
\end{pmatrix} \tag{B3}
\]

Formally it has the form

\[
h^{-1}(\mu) = \begin{pmatrix}
[U]_{--} & [U]_{+-} \\
[U]_{-+} & [U]_{++}
\end{pmatrix} \tag{B4}
\]

where

\[
[U]_{--} = \{1 - ([e^{-\alpha}]_{--})^{-1}[e^{\alpha}]_{+-}([e^{\alpha}]_{++})^{-1}[e^{-\alpha}]_{++}\}^{-1}([e^{-\alpha}]_{--})^{-1}, \\
[U]_{+-} = \{1 - ([e^{\alpha}]_{++})^{-1}[e^{-\alpha}]_{-+}([e^{-\alpha}]_{--})^{-1}[e^{\alpha}]_{--}\}^{-1}([e^{\alpha}]_{++})^{-1}, \\
[U]_{-+} = ([e^{-\alpha}]_{--})^{-1}[e^{\alpha}]_{-+}(([e^{\alpha}]_{++})^{-1}[e^{-\alpha}]_{+-}([e^{-\alpha}]_{--})^{-1}[e^{\alpha}]_{--} - 1)^{-1}([e^{\alpha}]_{++})^{-1}, \\
[U]_{++} = ([e^{\alpha}]_{++})^{-1}[e^{-\alpha}]_{++}(([e^{-\alpha}]_{--})^{-1}[e^{\alpha}]_{-+}([e^{\alpha}]_{+-})^{-1}[e^{-\alpha}]_{+-} - 1)^{-1}([e^{\alpha}]_{++})^{-1}. \tag{B5}
\]
We need to compute
\[
Tr\left[\frac{d}{d\mu} h(\mu) h^{-1}(\mu)\right] = Tr \left( \frac{d}{d\mu} ([e^{-\alpha}]_-_- [U]_-_- + \frac{d}{d\mu} ([e^\alpha]_+_- [U]_+_- \\
+ \frac{d}{d\mu} ([e^{-\alpha}]_+-_+ [U]_+-_+ + \frac{d}{d\mu} ([e^\alpha]_+-_+ [U]_+-_+) \right) =
\]
\[
\left\{ \left( \frac{d}{d\mu} ([e^{-\alpha}]_-_-) - \frac{d}{d\mu} ([e^\alpha]_+_-) \right) \right\} =
\]
\[
\{ 1 - ([e^{-\alpha}]_-_-)^{-1} ([e^\alpha]_+_-)^{-1} ([e^{-\alpha}]_-_-)^{-1} ([e^\alpha]_+_-)^{-1} \}
\]
\[
+ Tr \left\{ \left( \frac{d}{d\mu} ([e^\alpha]_+_-) - \frac{d}{d\mu} ([e^{-\alpha}]_+_-) \right) \right\} =
\]
\[
\{ 1 - ([e^\alpha]_+_-)^{-1} ([e^{-\alpha}]_+_-)^{-1} ([e^\alpha]_+_-)^{-1} ([e^{-\alpha}]_+_-)^{-1} \}. \tag{B6}
\]

In order to compute this last expression it is usefull to derive some idenities relating the
matrix elements used above. Note that for deriving this expressions we are going to use the
fact that $\alpha$ is an abelian variable. Apart from that $\alpha$ could be any antiperiodic function in
$[0, \beta]$. The main propertie that we are using here is the fact that $[e^{\alpha_-}]_p = 0$ for $p \geq 0$ and
$[e^{\alpha_+}]_n = 0$ for $n < 0$. Having this in mind the following identities are straightforward
\[
[e^{\alpha_-}]_+_- = 0 \tag{B7}
\]
\[
[e^{\alpha_+}]_-_- = 0 \tag{B8}
\]
\[
[e^{\alpha_-}]_+_- [e^{\alpha_+}]_+_- = [e^{-\alpha} - e^{\alpha_+}]_+_- \tag{B9}
\]
\[
[e^{\alpha_-}]_+_- [e^{-\alpha_-}]_+_- = 1_+_- \tag{B10}
\]
\[
[e^{\alpha_+}]_+_- [e^{-\alpha_+}]_+_- = 1_+_- \tag{B11}
\]
\[
[e^{\alpha_-}]_-_- [e^{\alpha_-}]_-_- = [e^\alpha]_-_- \tag{B12}
\]
\[
[e^{\alpha_+}]_-_- [e^{-\alpha_+}]_-_- = 1_-_- \tag{B13}
\]
\[
[e^{\alpha_-}]_-_- [e^{-\alpha_-}]_-_- = 1_-_- \tag{B14}
\]
\[
[e^{\alpha_-}]_-_- [e^{\alpha_+}]_+_- = [e^{-\alpha} - e^{\alpha_+}]_-_- \tag{B15}
\]
\[
[e^{\alpha_+}]_-_- [e^{-\alpha_-}]_-_- = [e^\alpha]_-_- \tag{B16}
\]
\[
[e^{\alpha_-}]_+_- [e^{\alpha_+}]_-_- = [e^{-\alpha} - e^{\alpha_+}]_+_- \tag{B17}
\]
\[
[e^{\alpha_+}]_+_- [e^{-\alpha_-}]_+_- = [e^\alpha]_+_- \tag{B18}
\]
With a change $\alpha \to -\alpha$ more identities can be derived. It is immediate to realize that

\[
([e^{-\alpha}]_{-})^{-1} = [e^{\alpha}]_{-}[e^{\alpha}]_{-}
\]

\[
([e^{\alpha}]_{++})^{-1} = [e^{-\alpha}]_{++}[e^{-\alpha}]_{++}. \tag{B21}
\]

As an example of how to use the identities (B7)-(B20) we compute the following expression

\[
1 - ([e^{-\alpha}]_{-})^{-1}[e^{\alpha}]_{-}([e^{\alpha}]_{++})^{-1}([e^{-\alpha}]_{-})^{-1}
\]

We define $[M]_{--}$ as

\[
[M]_{--} \equiv ([e^{-\alpha}]_{-})^{-1}[e^{\alpha}]_{-}([e^{\alpha}]_{++})^{-1}[e^{-\alpha}]_{--}. \tag{B22}
\]

By using Eq.(B21) we obtain

\[
[M]_{--} = [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--} \tag{B23}
\]

With Eqs.(B15) and (B18) we can write the factors of the form $[e^{\alpha}]$ in terms of $[e^{\alpha}]$ and $[e^{-\alpha}]$

\[
[M]_{--} = [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--} = [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}.
\]

After rearranging some factors using Eqs.(B16) and (B17) $[M]_{--}$ can be written as follows

\[
[M]_{--} = [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}.
\]

Hence

\[
1 - [M]_{--} = [e^{\alpha}]_{--}(1 - [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--})
\]

\[
= [e^{\alpha}]_{--}[e^{\alpha}]_{--}[e^{-\alpha}]_{--}[e^{-\alpha}]_{--}. \tag{B24}
\]

By using eqs.(B14) the inverse of the above expression reads
\[(1 - [M]_{--})^{-1} = [e^{-\alpha}]_{--}([e^{-\alpha}]_{--})^{-1}([e^\alpha]_{--})^{-1}[e^\alpha]_{--}. \quad \text{(B25)}\]

The factors \([e^{-\alpha}]_{--}\) and \([e^\alpha]_{--}\) can be computed with the help of eqs. (B12)-(B14). The result is

\[(e^{-\alpha}]_{--}\) and \([e^\alpha]_{--}\) can be computed with the help of eqs. (B12)-(B14). The result is

\[(e^{-\alpha}]_{--}(e^{-\alpha}]_{--}^{-1} = [e^{-\alpha}]_{--}[e^\alpha]_{--}^{-1}[e^{-\alpha}]_{--}^{-1}[e^\alpha]_{--}. \quad \text{(B26)}\]

Therefore, the expression we were calculating turns out to be

\[1 - ([e^{-\alpha}]_{--}^{-1}([e^\alpha]_{--})^{-1}([e^{-\alpha}]_{--})^{-1}([e^\alpha]_{--}]) = [e^{-\alpha}]_{--}[e^\alpha]_{--}^{-1}[e^{-\alpha}]_{--}^{-1}[e^\alpha]_{--}. \quad \text{(B27)}\]

Following the same steps we can see that

\[1 - ([e^\alpha]_{++})^{-1}([e^{-\alpha}]_{--}^{-1}([e^{-\alpha}]_{--})^{-1}([e^\alpha]_{--}]) = [e^{-\alpha}]_{++}[e^{-\alpha}]_{++}^{-1}[e^{-\alpha}]_{++}^{-1}[e^\alpha]_{++}. \quad \text{(B28)}\]

The next step is the calculation of the derivatives.

\[
\left( \frac{d}{d\mu} [e^\alpha]_{n-m} \right) = -\int e^{-2\pi i (n-m)x^2} \alpha(x_2) e^{-\mu \alpha(x_2)}
\]

\[
= -\sum_p [e^{-\mu \alpha(x_0)}]_{n-m-p}[\alpha]_p
\]

\[
= -\sum_q [e^{-\mu \alpha(x_0)}]_{n-q}[\alpha]_{q-m}. \quad \text{(B29)}
\]

And in our notation,

\[
\left( \frac{d}{d\mu} [e^\alpha]_{--} \right) = -([e^{-\alpha}]_{--}[\alpha]_{--} + [e^{-\alpha}]_{++}[\alpha]_{++}). \quad \text{(B30)}
\]

In the same way,

\[
\frac{d}{d\mu} [e^{-\alpha}]_{++} = ([e^\alpha]_{--}[\alpha]_{++} + [e^\alpha]_{++}[\alpha]_{++})
\]

\[
\frac{d}{d\mu} [e^\alpha]_{++} = -([e^{-\alpha}]_{--}[\alpha]_{--} + [e^{-\alpha}]_{++}[\alpha]_{++})
\]

\[
\frac{d}{d\mu} [e^{-\alpha}]_{++} = ([e^\alpha]_{--}[\alpha]_{--} + [e^\alpha]_{++}[\alpha]_{++}). \quad \text{(B31)}
\]

By inserting (B27)-(B31) into (B6) we obtain
\[
Tr\left[\frac{d}{d\mu} h(\mu) h^{-1}(\mu)\right] = \\
Tr \left[\alpha\right]_{-+} \left\{ [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^\alpha]_{+-} - \\
[e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^\alpha]_{+-} \right\} + \\
Tr \left[\alpha\right]_{++} \left\{ [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^{-\alpha}]_{++} [e^\alpha]_{+-} \right\} + \\
Tr \left[\alpha\right]_{-+} \left\{ [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^\alpha]_{+-} \right\} + \\
Tr \left[\alpha\right]_{--} \left\{ [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^\alpha]_{+-} \right\}.
\]

\text{(B32)}

By using the identities (B7)-(B20) we can see that the factor of \([\alpha]_{--}\) contributes with \(-1\) and the factor of \([\alpha]_{++}\) contributes with \(1\). The other two terms are more complicated to calculate but it can be done. The final result is

\[
Tr\left[\frac{d}{d\mu} h(\mu) h^{-1}(\mu)\right] = tr \left\{ [\alpha]_{++} - [\alpha]_{--} \right. \\
\left. + [\alpha]_{+-} (-[\alpha]_{--})_{++} + [\alpha]_{--} ([e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^\alpha]_{+-}) \right. \\
\left. + [\alpha]_{--} ([e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^{-\alpha}]_{--} [e^\alpha]_{+-}) \right\}. \quad \text{(B33)}
\]

**APPENDIX C: EUCLIDEAN-MINKOWSKI CORRESPONDENCES FOR DIRAC FERMIONS**

Upon the analytic continuation from real time \(x_0\) to imaginary time \(x_2 = ix_0\), a number of quantities are transformed. Firstly, the Minkowski space metric \(g_{\mu\nu}\)

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(C1)}
\]

becomes, in euclidean space
\[ g_{\mu\nu} = -\delta_{\mu\nu}. \]  

(C2)

The Minkowski space \( \gamma \)-matrices are (in the chiral basis) are

\[ \gamma_0 = \sigma_1 \quad \gamma_1 = -i\sigma_2 \quad \gamma_5 = \sigma_3 \]  

(C3)

and have the property

\[ \text{tr}(\gamma_\mu \gamma_\nu) = 2 \ g_{\mu\nu}. \]  

(C4)

The euclidean \( \gamma \)-matrices are chosen to be hermitean, in which case they are

\[ \gamma_1 = \sigma_2 \quad \gamma_2 = \sigma_1 \quad \gamma_5 = \sigma_3 \]  

(C5)

and they satisfy the relations

\[ \text{tr}(\gamma_\mu \gamma_\nu) = 2 \ \delta_{\mu\nu}, \quad \gamma_5 \gamma_\mu = -i \epsilon_{\mu\nu} \gamma_\nu. \]  

(C6)

We also give the correspondence for

\[ \bar{\psi} = \psi^\dagger \gamma_0 \rightarrow i\bar{\psi} \]  

(C7)

as well as for vector fields, which transform like

\[ A_0 \rightarrow iA_2 \quad A_1 \rightarrow A_1. \]  

(C8)

With this notation the covariant derivatives become

\[ i\dot{\psi} + \mathcal{A} = \begin{pmatrix} 0 & \partial_z - iA_z \\ -\partial_z + iA_z & 0 \end{pmatrix} \]  

(C9)

where we have set

\[ \partial_z = \partial_1 - i\partial_2 \quad , \quad \partial_\bar{z} = \partial_1 + i\partial_2 \]  

\[ A_z = A_1 - iA_2 \quad , \quad A_\bar{z} = A_1 + iA_2. \]  

(C10)

Using these identifications the Minkowski space actions \( S_M \) and \( S_E \) are related in the usual manner
For the specific case of a massive Dirac fermion coupled to a background gauge field, the Lagrangian \( \mathcal{L}_M \),

\[
\mathcal{L}_M = \bar{\psi} \left( i \partial + A \right) \psi - M \bar{\psi} \psi - iM_5 \bar{\psi} \gamma_5 \psi
\]  

becomes, in euclidean space,

\[
\mathcal{L}_E = \bar{\psi} \left( i \partial + A \right) \psi + iM \bar{\psi} \psi - M_5 \bar{\psi} \gamma_5 \psi.
\]

Notice that the factor of \( i \) in the mass term results from our choice of hermitean euclidean \( \gamma \)-matrices. Had we chosen antihermitean \( \gamma \)-matrices the factor of \( i \) would have been absent from the fermion mass term.

Finally we note that the Minkowski space fermion propagator

\[
S_{\alpha\beta}^M = -i \langle \hat{T} \psi_\alpha(x) \bar{\psi}_\beta(x') \rangle
\]  

(where \( \langle A \rangle \) is the ground state expectation value of \( A \) and \( \hat{T} \) is the time ordering operator) becomes, in euclidean space,

\[
S_{\alpha\beta}^E = \langle \psi_\alpha(x) \bar{\psi}_\beta(x') \rangle.
\]
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