The Squared Coefficient of Variation for MMPP is Greater than Unity

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Abstract

Folklore often treats the Markov Modulated Poisson Process as bursty because the variance divided by the expectation of counts is greater than unity. When viewed through the lens of the inter-event process, this ideally corresponds to a squared coefficient of variation greater than unity. As this has not been proved to date, we provide a proof together with an associated stochastic order relation.

1 Introduction

The Markov Modulated Poisson Process (MMPP) is a typical example of a Cox process, also known as a doubly stochastic Poisson process, [7] and [16]. It is a highly popular model from both a theoretical and applicative point of view. For a detailed outline of a variety of classic MMPP results, see [6] and references therein. In terms of applications, MMPPs are useful for modelling phenomena where bursty point processes are present such as in telecommunications, health-care, earthquakes modelling and finance. To date, MMPPs have been used in thousands of research papers with hundreds of new papers appearing yearly. In addition, theoretical properties of MMPP generalizations are of recent research interest, as in [15]. However, are all the elementary properties of the basic MMPP known? In this paper, we establish an elementary result for MMPPs that has often been taken for granted in modelling folklore.

A common way to represent and analyse MMPPs is using the framework of Matrix Analytic Methods and considering the MMPP as a special case of a Markovian Arrival Process (MAP). See for example [8], [13] and Chapter XI in [1]. The characterizing feature of MAP processes is that as finite state background continuous time Markov chain evolves, its state and transitions affect the counting measure, \( N(\cdot) \) of the associated point process on the line.

Let \( N(A) \) denote the number of events of the MMPP occurring on the set \( A \subset \mathbb{R} \). As with general point processes, the MMPP is time-stationary if the distribution of \( N(A) \) and \( N(A + t) \) is the same, where \( A + t := \{ u : u - t \in A \} \). Further, use \( \{ T_n \} \) to denote the sequence of inter-event times. As with general simple point processes on the line, the MMPP is event-stationary if the joint distribution of \( T_{k_1}, \ldots, T_{k_m} \) is the same as that of \( T_{k_1 + \ell}, \ldots, T_{k_m + \ell} \) for any integer sequence of indices \( k_1, \ldots, k_m \) and any integer shift \( \ell \). As MMPP is a special case of a MAP, time-stationarity and event-stationarity are easily characterized by the initial distribution of the background chain. Details follow.

In describing and measuring simple stationary point processes on the line, first and second order quantities are often very useful. The first order measure is the rate of the point process, \( \lambda^* \), which specifies the mean number of event occurrences during a unit time. For the time stationary version, \( \mathbb{E}[N(t)] = \lambda^* t \) and for the event-stationary version, \( \mathbb{E}[T_n]^{-1} = \lambda^* \). Then, typical second order measures of interest are respectively the limiting index of dispersion of counts and the squared coefficient of variation,

\[
I = \lim_{t \to \infty} \frac{\text{Var}(N(t))}{\mathbb{E}[N(t)]}, \quad \text{and} \quad c^2 = \frac{\text{Var}(T_1)}{\mathbb{E}^2[T_1]},
\]

where \( T_1 \) is taken from the event-stationary version.

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In point process modelling, the Poisson process, often used as a benchmark, exhibits both $I = 1$ and $c^2 = 1$. Values greater than unity indicate high variability (burstiness) and values less than unity are nearing more deterministic arrival patterns. Modellers often use MMPPs to represent bursty temporal patterns as in for example [1], [11] and [14], among hundreds of other important research works. Much of the popularity of MMPP stems from the intuition that it serves as a good model of bursty traffic. In fact, showing that $I \geq 1$ is straightforward (e.g. Chapter 6 of [12]). However, to the best of our knowledge, a proof that for MMPPs, $c^2 \geq 1$ has been lacking to date and is not simple.

Our contribution in this note is a proof of this fact, together with stochastic order relations between $T_i^s$ and $T_i^e$, corresponding to the first inter-event time in the time-stationary and event-stationary cases respectively. Namely, we show that $T_i^s \geq_{st} T_i^e$, i.e., $\mathbb{P}(T_i^s > t) \geq \mathbb{P}(T_i^e > t)$ for all $t$.

The remainder of this paper is structured as follows. In Section 2 we represent MMPP as a MAP and provide the needed background. Section 3 presents the main result and we conclude in Section 4.

2 Background: MMPP as a MAP

A Markovian Arrival Process (MAP) of order $p$ is generated by a two-dimensional Markov process $Z(t) = (N(t), X(t))$ for $t \geq 0$ on the state space $\{0, 1, \cdots \} \times \{1, \cdots , p\}$. Here, the component $N(t)$ is a counting process that counts the number of events (arrivals) during $[0, t]$ with $\mathbb{P}(N(0) = 0) = 1$. The background process, $X(t)$, when considered in isolation, is an irreducible CTMC with state space $\{1, \ldots , p\}$, initial distribution $\eta$ and transition rate matrix $Q$.

A MAP is characterised by parameters $(\eta, C, D)$, where the matrix $C$ has negative diagonal elements and non-negative off-diagonal elements, recording the background phase transitions with no arrival. The matrix $D$ has non-negative elements and describes the changes of the background process with an arrival (an increase of $N(t)$ by 1) or (as in the case of MMPP) arrivals without a background phase change. Moreover, we have $Q = C + D$. More details are in [1] (Chapter XI) and [9] (Chapter 2). The generator of $Z(t)$ is a block, bi-diagonal matrix with diagonal block elements $C$ and upper blocks $D$.

Since $Q$ is assumed irreducible and finite, it has a unique stationary distribution $\pi$ satisfying $\pi Q = 0^t$, $\pi 1 = 1$ (note that we take probability vectors as rows and other vectors as columns). The irreducibility of $Q$ also implies that the matrix $-C$ is non-singular with $(-C)^{-1}$ having non-negative elements. This follows since $-C$ is an M-matrix, see for example [3]. Of further interest is the embedded discrete-time Markov chain with irreducible stochastic matrix $P = (-C)^{-1}D$, and stationary distribution $\alpha$, where $\alpha P = \alpha$ and $\alpha 1 = 1$.

Observe the relationship between the (continuous time and discrete time) stationary distributions $\pi$ and $\alpha$:

$$\alpha = \frac{\pi D}{\pi D 1} \quad \text{and} \quad \pi = \frac{\alpha (-C)^{-1}}{\alpha (-C)^{-1} 1} = \lambda^* \alpha (-C)^{-1},$$

where $\lambda^* = \pi D 1 = \pi (-C) 1$.

The following known proposition, as distilled from the literature (see for example [1], Chapter XI) provides the key results of MAPs that we use in this paper.

Proposition 2.1. Consider a MAP with parameters $(\eta, C, D)$, then

$$\mathbb{P}(T_1 > t) = \eta e^{Ct} 1.$$  \hspace{1cm} (3)

Further, if $\eta = \pi$ then the MAP is time-stationary and if $\eta = \alpha$ it is event stationary, where $\pi$ and $\alpha$ are associated stationary distributions.

It is well known that MMPPs are a special case of a MAP where in the standard parametrization, $D$ is a diagonal matrix with entries $\lambda = (\lambda_1, \cdots , \lambda_p)$. If $\lambda_i = \lambda$ for all $i$ then $N(\cdot)$ is trivially a Poisson process with rate $\lambda$. In this case, all our results below obviously hold. We thus focus on the case where there exists $i \neq j$ with $\lambda_i \neq \lambda_j$.

For general MAPs, observing the phase-type structure [3] makes computation of moments of $T_1$ straightforward, see for example [8]. In our context, we obtain a matrix formula for the squared coefficient of variation,

$$c^2 = 2\pi (-C) 1 \pi (-C)^{-1} 1 - 1.$$  \hspace{1cm} (4)
Illustrating this for an MMPP with $p=2$ (taken from [10]) and

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -\sigma_1 - \lambda_1 & \sigma_1 \\ \sigma_2 & -\sigma_2 - \lambda_2 \end{pmatrix},$$

results in

$$c^2 = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_2\sigma_1 + \lambda_1(\lambda_2 + \sigma_2))}.$$ 

Thus it is evident that $c^2 > 1$ as long as $\lambda_1 \neq \lambda_2$ and otherwise $c^2 = 1$. However, for MMPPs of arbitrary $p$, the fact that $c^2 \geq 1$ is not immediacy evident from (1), hence the main result of our paper.

3 Main Result

Our main result dealing with the stochastic order implies $c^2 \geq 1$ as presented below.

**Theorem 3.1.** For MMPPs, $T_1^a \succeq_{st} T_1^b$.

**Proof.** Using (3), the claim of the theorem is,

$$\pi e^{Ct} 1 \geq \alpha e^{Ct} 1, \quad \forall t \geq 0,$$

we assume by contradiction that $\pi e^{Ct} 1 < \alpha e^{Ct} 1$ for all $t \geq 0$. Hence,

$$\pi \int_0^\infty e^{Ct} dt 1 < \alpha \int_0^\infty e^{Ct} dt 1.$$ 

Now since $C$ is non-singular we use $(-C)^{-1} = \int_0^\infty e^{Ct} dt$ and \(\{\} \) reads as $\pi(-C)^{-1} 1 < \alpha(-C)^{-1} 1$.

But as we show below,

$$\pi(-C)^{-1} 1 \geq \alpha(-C)^{-1} 1.$$ 

Hence a contradiction and \(\{\} \) holds.

For the sake of simplicity, denote the row vector $(\pi - \alpha)(-C)^{-1}$ by $\omega$, then, the claim of (7) is equivalent to show that $\omega 1 \geq 0$. By applying the definition of the probability transition matrix $P = (-C)^{-1} D$, using the fact that $P$ is a stochastic matrix, and applying $\alpha 1 = \pi 1 = 1$, we have

$$0 = (\pi - \alpha) 1 = (\pi - \alpha) P 1 = (\pi - \alpha)(-C)^{-1} D 1 = \omega D 1.$$ 

Or equivalently:

$$0 = \sum_{i=1}^p \omega_i \lambda_i,$$

where $\{\lambda_i\}$ are elements of the matrix $D = \text{diag}(\lambda)$ of the MMPP.

We assume now without loss of generality that there is an order $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ (with $\lambda_i \neq \lambda_j$ for some $i, j$). In case that for $1 < p' < p$, $0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{p'-1}$, and $0 < \lambda_{p'}$, then (8) reduces to,

$$0 = \sum_{i=p'}^p \omega_i \lambda_i.$$

In the rest of the proof, we assume that $p' = 1$, meaning that all $\lambda_i$ are strictly positive. Adapting to the case of $p' > 1$ is straightforward. Now, \{\lambda' - \lambda_i\}_{i=1,\ldots,p} is a non-increasing sequence and therefore in the sequence \{\pi_i - \alpha_i\} = \{\frac{\pi}{\alpha}(\lambda' - \lambda_i)\} when an element $\pi_k - \alpha_k$ is negative, all the elements $\pi_i - \alpha_i$ for $i \geq k$ are negative. Moreover, both $\pi$ and $\alpha$ are probability vectors, so $(\pi - \alpha) 1 = \sum_i (\pi_i - \alpha_i) = 0$. Therefore, at least the first element in the sequence \{\pi_i - \alpha_i\} = \{\frac{\pi}{\alpha}(\lambda' - \lambda_i)\} is positive. Hence, there exists an index $1 < k \leq p$ such that $\pi_i - \alpha_i$ for $i = 1, \cdots, k-1$ is non-negative and for $i = k, \cdots, p$ is negative. Therefore, since elements of the matrix $(-C)^{-1}$ are all non-negative, we have:

$$(\pi - \alpha)(-C)^{-1} 1 = \omega 1 = \sum_{i=1}^{k-1} \omega_i \pi_i + \sum_{i=k}^p \omega_i$$

where $\omega_i$ is non-negative and $\omega_i$ is negative.
Now if we were to assume that $\omega_1 < 0$, then from the above equation:

$$\sum_{i=1}^{k-1} \omega_i < \sum_{i=k}^{p} \omega_i.$$ 

Then, since $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$, the above equation results in (note that we multiply the left hand side with values of $\lambda_i$ that are all less than the values of $\lambda_i$ multiplied in the right hand side):

$$\sum_{i=1}^{k-1} \omega_i \lambda_i < \sum_{i=k}^{p} \omega_i \lambda_i.$$ 

But, from (8), we have:

$$\sum_{i=1}^{k-1} \omega_i \lambda_i = \sum_{i=k}^{p} \omega_i \lambda_i.$$ (9) 

Consequently, the assumption $\omega_1 < 0$ is not true and hence (7) holds.

The following lemma is a well-known consequence of point process theory on the line.

**Lemma 3.2.** Consider a simple non-transient point process on the line, and let $T^a_1, T^b_1$ represent the first inter-event time in the time-stationary case and event-stationary case respectively. Then $c^2 \geq 1$ if and only if $E[T^a_1] \geq E[T^b_1]$.

**Proof.** From point process theory (see for example, Eq. (3.4.17) of [5]), it holds

$$E[T^a_1] = \frac{1}{2} \lambda^* E[(T^b_1)^2],$$

where,

$$\lambda^* = \lim_{t \to \infty} \frac{E[N[0,t]]}{t} = \frac{1}{E[T^b_1]}.$$ 

Now,

$$c^2 = \frac{E[(T^b_1)^2] - (E[T^b_1])^2}{(E[T^b_1])^2} = 2 \frac{E[T^a_1]}{E[T^b_1]} - 1,$$

and we obtain the result.

This now gives the key finding of this paper.

**Corollary 3.3.** For MMPPs, $c^2 \geq 1$.

**Proof.** Using Theorem 3.3 we get $E[T^a_1] \geq E[T^b_1]$. Now applying Lemma 3.2 we get the result.

## 4 Conclusion

Having analysed MMPPs in depth and establishing $c^2 \geq 1$, we encountered a few related open questions dealing with MMPPs as well as the more general class of doubly stochastic Poisson processes (Cox processes).

First and foremost, for a general Cox process on the line (see [7]), we conjecture that whenever $E[(T^b_1)^2]$ is finite, it holds that $c^2 \geq 1$. Establishing such a result would clearly use different methods than the matrix analytic methods used in this note.

The second branch of questions deals with characterizing the Poisson process via $c^2 = 1$ and considering when an MMPP is Poisson. For example, for the general class of MAPs, the authors of [2]
provide a condition for determining if a given MAP is Poisson. It is not hard to construct a MAP with \( c^2 = 1 \) that is not Poisson. But, we believe that all MMPPs with \( c^2 = 1 \) are Poisson. Yet, we don’t have a proof. Further, we believe that for an MMPP, if \( c^2 = 1 \) then all \( \lambda_i \) are equal (the converse is trivially true). We don’t have a proof of this either. Related questions also hold for the more general Cox Processes.

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