Non-leading Logarithms in Principal Value Resummation

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Abstract

We apply the method of principal value resummation of large momentum-dependent radiative corrections to the calculation of the Drell Yan cross section. We sum all next-to-leading logarithms and provide numerical results for the resummed exponent and the corresponding hard scattering function.
1 Introduction

All-order resummation techniques have been in existence for some time now \[1, 2\], mainly in an attempt to control perturbative QCD corrections to specific hadron-hadron scattering cross sections which are numerically large at a fixed order of perturbation theory \[3, 4\]. Such resummation formulas \[5\] typically organize the large perturbative corrections in an exponential containing integrals of the QCD running coupling $\alpha_s(\mu^2)$ over various momentum scales and suffer from divergences associated with the Landau pole of $\alpha_s(\mu^2)$.

In \[6\] it was shown, in the context of the inclusive dilepton cross section, how to define a consistent resummation formula by a principal value prescription for the infrared (IR) coupling constant singularities of the exponent. This defines unambiguously the resummed perturbative series, as the exact exponent may be approximated by a finite series in the fixed coupling constant, thus avoiding an explicit and numerically important presence of higher-twist contributions through an arbitrary IR cut-off \[7\]. The principal value prescription then defines a best perturbative resummation in the following sense: A perturbative exponent can be constructed as an asymptotic approximation to the principal value exponent and its degree (in $\alpha_s(Q^2)$) is precisely determinable numerically as a function of $Q^2$ as well. Thus, the often quoted “asymptotic nature of the QCD perturbative series” is given a precise meaning.

In \[6\] it was also shown that all large momentum-dependent perturbative corrections were contained in leading and next-to-leading exponents ($E_L$ and $E_{NL}$ respectively). $E_L$ was studied in that work, both analytically and numerically. $E_{NL}$ is, however, also important for a precise perturbative description of the process and a full study of this exponent is necessary before the non-perturbative parton flux is convoluted with the resummed hard part in order to yield the theoretical prediction for the cross section.

The purpose of this work is to address these issues and perform a thorough analytical and numerical study of the exponentiated hard part, including all large momentum-dependent perturbative corrections. In a subsequent paper, we will use this hard part to calculate the dilepton cross sections at both fixed-target and collider energies and compare these with experimental results.

This paper is organized as follows: in section 2 we briefly summarize the new resummation formula and the expressions for $E_L$, both as an exact function of $Q^2$ and as an asymptotic perturbative series, i.e., as a polynomial in $\alpha_s(Q^2)$. This makes contact with earlier work \[8\] and defines notation and $E_{NL}$ as well. In section 3 we calculate $E_{NL}$, in the spirit of \[6\], both exactly and as an asymptotic perturbative series. The results are presented in both moment and momentum space, using a closed formula for the inverse Mellin transform, which has been derived elsewhere \[9\]. In section 4 we study the higher-twist inherent in the exact exponent of the resummation formula. We establish, in moment space, the asymptotic properties of the exponent in the non-perturbative region. These properties make evident the smooth behavior of the cross section for these extremely large moments. In section 5 we present numerical results for $E_L$ and $E_{NL}$ in the perturbative and non-perturbative region, along with the corresponding perturbative approximations. We also present the hard part in momentum space, for a variety of values of $Q^2$, including all large perturbative corrections. Finally in section 6 we summarize our findings and anticipate future, phenomenologically oriented work. Some technical details are given in an appendix.
2 Principal-Value Resummation

In this section we briefly summarize the principal-value resummation formula as applied to the dilepton production hard scattering function. Let us define kinematic variables for this reaction, denoted by
\[ h_1(p_1) + h_2(p_2) \rightarrow l\bar{l}(Q^\mu) + X. \] (1)

We denote
\[ s = (p_1 + p_2)^2, \quad \tau = Q^2/s, \quad z = \tau/x_a x_b \]
where \( x_a, x_b \) are the momentum fractions of the partons participating in the hard scattering. We shall concentrate on the diagonal-flavor quark-antiquark hard parts, which alone are singular in the soft-gluon limit \( z \to 1 \), at a fixed order of perturbation theory.

For quark-antiquark scattering, we obtain the principal-value resummation formula in moment space \[5, 6, 8\]
\[ \tilde{\omega}_{q\bar{q}}(n, \alpha) = A(\alpha) \tilde{I}(n, \alpha), \] (2)

where the function \( A(\alpha) \) will be fully analyzed in a future paper, and where \( \alpha \equiv \alpha_s(Q^2)/\pi \). All \( n \)-dependence in \( \omega_{q\bar{q}} \) is contained in the exponential
\[ \tilde{I}(n, \alpha) = \exp[E(n, \alpha)_L + E(n, \alpha)_{NL}] . \] (3)

In eq. (3), \( E(n, \alpha)_L \) includes the one-loop running coupling effects, and \( E(n, \alpha)_{NL} \) the two-loop effects. In \[5, 6, 8\] it was shown that, if we define the range of validity of perturbation theory \( \alpha_s(Q^2) \ln n < 1 \), this formula includes all large perturbative corrections. In particular, \( E_L \) resums all corrections containing more powers of \( \ln n \) than of \( \alpha \), while \( E_{NL} \) completes the resummation of corrections having equal powers of these two quantities. As usual, conclusions in momentum space may be drawn using the correspondence \[5, 9\] \( n \leftrightarrow 1/(1-z) \). For \( E_L \) in the Deep-Inelastic Scattering (DIS) scheme, we can write the expressions \[6\]
\[ E(n, \alpha)_L = \alpha(g_1^{(1)} I_1 - g_2^{(1)} I_2), \] (4)

with
\[ I_1 \equiv I_1(n, t) = 2I(n, t/2) - I(n, t), \] (5)

where
\[ I(n, t) = t \int_P d\zeta \left( \frac{\zeta^{n-1} - 1}{1 - \zeta} \right) \ln(1 + (1/t) \ln(1 - \zeta)), \] (6)

and
\[ I_2 \equiv I_2(n, t) = \int_P d\zeta \left( \frac{\zeta^{n-1} - 1}{1 - \zeta} \right) \frac{1}{1 + (1/t) \ln(1 - \zeta)}, \] (7)

and where we define
\[ t \equiv \frac{1}{\alpha b_2} = \ln \left( \frac{Q^2}{\Lambda^2} \right). \] (8)

Here \( g_1^{(1)}, g_2^{(1)} \) are numerical coefficients \[3\], while \( b_2 \) is the first coefficient of the QCD beta-function. The explicit expressions for these coefficients are given in section 5. \( E_{NL} \) is given by a similar, albeit more complicated, expression, which we will also give below.

\[1\] A precise definition of this range will be discussed below, and in section 4.
The principal value prescription makes the integrals in eqs. (3)-(7) well-defined at 
\(-1/\tau \ln(1 - \zeta) = 1\). \(P\) stands for an averaged mirror-symmetric contour connecting 0 and 
1 above and below the real axis. It will be useful to consider two such special contours 
for analytical and/or numerical results. The definition of \(P\), along with these two specific 
choices, appear in fig. 1.

In [6], a series expression for the integrals in eqs. (4)-(7) was found, using the principal 
value contour proper, \(P\):

\[
E(n, \alpha)_L = -\alpha g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^2} 2E(mt/2) \\
+ \alpha g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^2} E(mt) \\
- \alpha g_2^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!} E(mt),
\]

(9)

where \((a)_m \equiv \Gamma(a + m)/\Gamma(a)\), \(\Gamma\) being the Euler Gamma function, with

\[
E(x) = xe^{-x} Ei(x),
\]

(10)

and where the exponential integral is defined by the principal value

\[
Ei(x) \equiv P \int_x^{-\infty} d\nu \frac{e^\nu}{\nu}.
\]

(11)

A perturbative version of eq. (9) may be obtained by performing an asymptotic expansion 
of the special functions \(E(mt), E(mt/2)\) for \(mt, mt/2 \gg 1\) in terms of \(\ln^j n/\nu^\rho\) and in a 
range where the \(\ln n\) behavior is suppressed for all terms in the sums. For example, the 
second sum in (9) can be written as

\[
\alpha g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^2} E(mt) = \alpha g_1^{(1)} \sum_{\rho=0}^{N} \frac{\rho!}{\nu^\rho} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^{\rho+2}},
\]

(12)

where use had been made of eq. (3.2) of [6]:

\[
E(mt) \simeq \sum_{\rho=0}^{N} \frac{\rho!}{(mt)^\rho}.
\]

(13)

Applying eq. (3.8) of [6],

\[
\sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^{\rho+1}} = (-1)^\rho \sum_{j=0}^{\rho+1} c_{\rho+1-j} \frac{(-1)^j}{j!} \ln^j n,
\]

(14)

to eq. (12), one then obtains

\[
\alpha g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m^2} E(mt) = \alpha g_1^{(1)} \sum_{\rho=0}^{N} \frac{\rho!}{\nu^\rho} (-1)^{\rho+1} \sum_{j=0}^{\rho+2} c_{\rho+2-j} \frac{(-1)^j}{j!} \ln^j n.
\]

(15)
From eq. (15), we see that for \( n \gg 1 \), the suppression of \( \ln n \) by \( 1/t^\rho \) can be quantified for a given power \( 1/t^\rho \), by constraining the corresponding monomial \( \ln^j n, \ j = \rho \). This leads to \( \alpha b_2 \ln n < 1 \) or, equivalently, \( ne^{-t} < 1 \), as the range of validity of perturbation theory. We will further discuss this argument from a different point of view, when we investigate the non-perturbative regime in section 4. Of course, in eq. (9) the sums have two different scales, the smaller one being \( t/2 \). Thus, if we want to define the perturbative regime for all three sums, we should restrict ourselves to \( n \) for which
\[
\frac{1}{2} \ln n < t \equiv \frac{1}{2\alpha b_2} \Leftrightarrow \frac{1}{2} \ll n < \frac{Q}{\Lambda}. 
\] (16)

For \( n \) in this range, a large-\( t \) asymptotic approximation leads to the perturbative formula
\[
E(n, \alpha)_L \simeq E(n, \alpha, N)_L = \sum_{i=1}^{2N_L+1} \sum_{\rho=1}^{\rho+1} \alpha^\rho \sum_{j=0}^{r_L}(i) \ln^j n + \sum_{\rho=1}^{\rho+1} \alpha^\rho \sum_{j=0}^{r_L}(3) \ln^j n, 
\] (17)
where the \( N_L \) determine the best asymptotic approximation, and where the coefficients are given by
\[
s_{j,\rho}(1) = -g_1^{(1)} b_2^{-\rho-1} (-1)^{\rho+j} (\rho-1)! j! 2^\rho c_{\rho+1-j}, 
\]
\[
s_{j,\rho}(2) = g_1^{(1)} b_2^{-\rho-1} (-1)^{\rho+j} (\rho-1)! j! c_{\rho+1-j}, 
\]
\[
s_{j,\rho}(3) = g_2^{(1)} b_2^{-\rho-1} (-1)^{\rho+j} (\rho-1)! j! c_{\rho-j}. 
\] (18)

The coefficients \( c_k \) are given by \( \Gamma(1+z) = \sum_{k=0}^{\infty} c_k z^k \). Notice that \( E_L \) in eq. (17) becomes a polynomial in \( \ln n \) whose precise powers are determined by the above large-\( t \) asymptotic approximation (perturbative approximation). Hence, the optimum number of resummed perturbative corrections, \( N \equiv \{N_L\} \), is also determined in principal-value resummation. Numerical results for those numbers were given in [6] and will be incorporated and enlarged upon in sec. 5 below.

### 3 Evaluation of the Next-to-Leading exponent in the perturbative regime

Following [3], we may write the non-leading exponent \( E_{NL} \) from eq. (3) as an integral over the function
\[
g_i^{NL}(\alpha[Q^2]) = -g_1^{(1)} \alpha^2 (b_1/b_2) \ln(1 + \alpha b_2 \ln \lambda) (1 + \alpha b_2 \ln \lambda)^2 + g_2^{(1)} \alpha^2 \frac{1}{(1 + \alpha b_2 \ln \lambda)^2} , \ i = 1, 2. 
\] (19)

In these terms, \( E_{NL} \) is
\[
E(n, \alpha)_{NL} = - \int d\zeta \left\{ \int_0^\zeta \frac{dy}{1-y} g_1^{NL}(\alpha[(1-\zeta)(1-y)Q^2]) + g_2^{NL}(\alpha[(1-\zeta)Q^2]) \right\}. 
\] (20)
From eqs. (19), (20) we may write the next-to-leading exponent as

$$E(n, \alpha)_{NL} = \alpha(g_1^{(1)} J_1 - g_2^{(1)} J_2) + \alpha^2(g_1^{(2)} K_1 - g_2^{(2)} K_2),$$

with

$$J_1 \equiv (\alpha b_3/b_2) \int_P d\zeta \left( \frac{\zeta^{-1} - 1}{1 - \zeta} \right) \int_0^\zeta \frac{dy}{1 - y} \ln[(1 + (1/t) \ln((1 - \zeta)(1 - y)))]^{2},$$

$$J_2 \equiv - (\alpha b_3/b_2) \int_P d\zeta \left( \frac{\zeta^{-1} - 1}{1 - \zeta} \right) \ln(1 + (1/t) \ln(1 - \zeta))^{2},$$

$$K_1 \equiv - \int_P d\zeta \left( \frac{\zeta^{-1} - 1}{1 - \zeta} \right) \int_0^\zeta \frac{dy}{1 - y} \frac{1}{(1 + (1/t) \ln(1 - \zeta))^{2}},$$

$$K_2 \equiv \int_P d\zeta \left( \frac{\zeta^{-1} - 1}{1 - \zeta} \right) \frac{1}{(1 + (1/t) \ln(1 - \zeta))^{2}}.$$

These integrals may be readily computed by separating the moment variable dependence from the coupling constant dependence, as in fig. 1, and using the contour $P$ of fig. 1. Care must be taken, however, regarding the singularity structure of the integrals over the various pieces of the contour. The semicircle contributions will now contain singular pieces of the form $\ln' \delta/\delta^j$, $i, j = 0, 1, ..$ which will cancel with similar singularities coming from the straight-line contributions. Let us consider, for example, the integral $J_2$. Using the binomial expansion, we write

$$J_2 = \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!} J_2^m,$$

with

$$J_2^m = -(\alpha b_3/b_2) \int_P d\zeta \zeta^{m-1} \frac{\ln(1 + (1/t) \ln \zeta)}{(1 + (1/t) \ln \zeta)^2}.$$

The semicircle contributions each contain an imaginary part, which cancels upon addition, and a singular real part:

$$\frac{1}{2} J_2^m (\cap) + \frac{1}{2} J_2^m (\cup) = -(b_3/b_2^2) t \zeta_t^m \left( - \frac{2 \ln(\delta/(t \zeta_t))}{\delta/\zeta_t} + \frac{2}{\delta/\zeta_t} + \frac{m \pi^2}{2} \right),$$

with $\zeta_t \equiv e^{-t}$. The straight-line integrations, on the other hand, give a combined contribution:

$$\frac{1}{2} J_2^m (+i\epsilon) + \frac{1}{2} J_2^m (-i\epsilon) = -(b_3/b_2^2) t \zeta_t^m \left( \frac{2 \ln(\delta/(t \zeta_t))}{\delta/\zeta_t} + \frac{2}{\delta/\zeta_t} \right) - \frac{1}{t \zeta_t^m} - \frac{m}{2} \int_{-\infty}^{mt} dx e^{x} \ln^2(|x|/(mt)) + m \mathcal{P} \int_{-\infty}^{mt} dx e^{x} \ln^2(|x|/(mt)) + m \mathcal{P} \int_{-\infty}^{mt} dx e^{x} \ln^2(|x|/(mt)),$$

Upon addition we obtain the finite result

$$J_2^m = \frac{1}{2} \left( J_2^m (+i\epsilon) + J_2^m (\cap) \right) + \frac{1}{2} \left( J_2^m (-i\epsilon) + J_2^m (\cup) \right)$$

$$= \left( b_3/b_2^2 \right) (1 - \mathcal{E}(mt)) - \left( b_3/(2b_2^2) \right) \Lambda(mt),$$

(30)
with
\[ \Lambda(x) \equiv xe^{-x}(\pi^2 - \int_{-\infty}^{x} dy e^y \ln^2(|y|/x)). \] (31)

The function \( \Lambda(x) \) is expressible in terms of Incomplete Gamma functions, as we will show later in this section.

Let us first give the results for the integrals \( J_1 \), \( K_1 \), \( K_2 \) appearing in eqs. (22)-(25).

Performing the \( y \)-integration for the integral \( J_1 \) and expanding in terms of the “Pochhammer symbol” \((1 - n)_m\) as before, we have
\[
J_1 = \frac{b_3}{b_2} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!} \left\{ J_{1;1}^m(t/2) - J_{1;1}^m(t) + J_{1;2}^m(t/2) - J_{1;2}^m(t) \right\},
\] (32)

where we define
\[
J_{1;1}^m(t) \equiv \int_P d\zeta \zeta^{m-1} \ln(1 + (1/t) \ln \zeta) = \frac{1}{2m} \Lambda(mt),
\] (33)

and
\[
J_{1;2}^m(t) \equiv \int_P d\zeta \zeta^{m-1} \frac{1}{1 + (1/t) \ln \zeta} = \frac{1}{m} \mathcal{E}(mt).
\] (34)

Hence we obtain
\[
J_1 = \frac{b_3}{b_2} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left\{ \frac{1}{2} \left( \Lambda(mt/2) - \Lambda(mt) \right) + \mathcal{E}(mt/2) - \mathcal{E}(mt) \right\}.
\] (35)

Working similarly for the integrals \( K_1 \), \( K_2 \), eqs. (5.4), (5.5), we find
\[
K_1 = -\frac{1}{ab_2} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left\{ \mathcal{E}(mt/2) - \mathcal{E}(mt) \right\},
\] (36)

and
\[
K_2 = \frac{1}{ab_2} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left\{ \mathcal{E}(mt) - 1 \right\}.
\] (37)

Putting everything together we find for the next-to-leading exponent the expression
\[
E(n, \alpha)_{NL} = (ab_3/b_2^2) \left\{ g_1^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left[ \mathcal{E}(mt/2) - \mathcal{E}(mt) + \frac{1}{2} \left( \Lambda(mt/2) - \Lambda(mt) \right) \right] \right. \\
+ g_2^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left[ \mathcal{E}(mt) - 1 + \frac{1}{2} \Lambda(mt) \right] \right\} \\
+ (\alpha/b_2) \left\{ -g_1^{(2)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left[ \mathcal{E}(mt/2) - \mathcal{E}(mt) \right] \\
- g_2^{(2)} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m!m} \left[ \mathcal{E}(mt) - 1 \right] \right\}.
\] (38)

To recover the resummed perturbative series, we have to perform asymptotic expansions on the functions \( \mathcal{E}(x) \), and \( \Lambda(x) \), eq. (31).
As with the function $E(x) \equiv xe^{-x}Ei(x)$ (see eq. (31)), we will now derive an asymptotic formula for the function $\Lambda(x)$, eq. (31). The definition of this function is

$$\Lambda(x) \equiv xe^{-x}(\pi^2 - \int_{-\infty}^{x} dy y^2 \ln^2(|y|x)) \simeq -xe^{-x}\left\{\gamma_2(x) - 2\ln x\gamma_1(x) + \ln^2 x\gamma_0(x)\right\},$$

where

$$\gamma_n(x) \equiv \int_{0}^{x} dy y^n \ln^n y,$$

the difference between the two expressions in eq. (39) being of order $(\ln Q)_{\Lambda Q}$. The functions $\gamma_n(x)$ may be expressed in terms of Incomplete Gamma functions [10]

$$\gamma_n(x) = \lim_{\epsilon \to 0^+} \frac{\partial}{\partial \epsilon} \int_{0}^{x} dy y^n \epsilon^{\epsilon} \Gamma(\epsilon + 1)\gamma^*(\epsilon + 1, -x).$$

Using the asymptotic expansion for the Incomplete Gamma function, or simply integrating eq. (40) by parts, we obtain

$$\gamma_n(x) \simeq \lim_{\epsilon \to 0^+} \frac{\partial}{\partial \epsilon} x^{\epsilon} e^x \left\{1 + \sum_{\rho=1}^{N'-1} \frac{(-\epsilon)^\rho}{x^\rho} \right\},$$

where $N'$ will be defined shortly. Realizing that $(-\epsilon)^\rho$ is a polynomial of $\rho$ degree in $\epsilon$, with roots $\epsilon_j = j, j = 0, ... \rho - 1$, we finally obtain

$$\gamma_0(x) \simeq e^x,$$

$$\gamma_1(x) \simeq e^x \left[\ln x - \frac{1}{x} \sum_{\rho=0}^{N'-1} \frac{\rho!}{x^\rho} \right],$$

$$\gamma_2(x) \simeq e^x \left[\ln^2 x - \frac{2\ln x}{x} \sum_{\rho=0}^{N'-1} \frac{\rho!}{x^\rho} + \frac{2}{x} \sum_{\rho=1}^{N'-1} \frac{\Psi(\rho + 1) + \gamma_0}{x^\rho} \right].$$

Putting everything together we obtain the asymptotic expression

$$\Lambda(x) \simeq -2 \sum_{\rho=1}^{N'} \frac{[\Psi(\rho + 1) + \gamma_0] \rho!}{x^\rho},$$

where $\Psi$ is the logarithmic derivative of the $\Gamma$-function. As with $N \equiv \{N^L\}$ in eq. (31), $N'$ depends on $x$ and different $N'$s will be determined for the different sums in eq. (38). A similar method was introduced in [6] for $E_L$. Let us separate, therefore, the sums in eq. (38) and derive optimum numbers of perturbative terms by direct optimization of each sum separately. Using eq. (44) along with the corresponding asymptotic formula for $E(x)$, [6], we obtain the perturbative asymptotic approximation:

$$E(n, \alpha)_{NL} \simeq E(n, \alpha, N')_{NL} = (\alpha b_3/b_2^2)g_1^{(1)} \sum_{\rho=0}^{N^L} \frac{\rho!}{t^\rho} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m! m^{\rho+1}}$$

$$- (\alpha b_3/b_2^2)g_1^{(1)} \sum_{\rho=0}^{N^L} \frac{\rho!}{t^\rho} \sum_{m=1}^{\infty} \frac{(1 - n)_m}{m! m^{\rho+1}}.$$
where the perturbative coefficients are

\[-(a/b_3/b_2^2)g_1^{(1)} \sum_{\rho=1}^{N_1^{NL}} \frac{\rho!2^\rho}{t_0^\rho} \Psi(\rho + 1) + \gamma \sum_{m=1}^{\infty} \frac{(1 - n)^m}{m!m^{\rho+1}}\]

\[+(a/b_3/b_2^2)g_1^{(1)} \sum_{\rho=0}^{N_1^{NL}} \frac{\rho!}{t_0^\rho} \Psi(\rho + 1) + \gamma \sum_{m=1}^{\infty} \frac{(1 - n)^m}{m!m^{\rho+1}}\]

\[+(a/b_3/b_2^2)g_2^{(1)} \sum_{\rho=0}^{N_2^{NL}} \frac{\rho!}{t_0^\rho} \sum_{m=1}^{\infty} \frac{(1 - n)^m}{m!m^{\rho}}\]

\[-(a/b_3/b_2^2)g_2^{(1)} \sum_{m=1}^{\infty} \frac{(1 - n)^m}{m!}\]

The set of numbers $N' \equiv \{N_i^{NL}\}, i = 1,...6$, is determined by direct optimization of eqs. (38), (45), as in the leading case. Then, performing a Stirling approximation on the $n$-dependent summations we observe that the fifth and sixth sums contain at most one less power of $\ln n$ than of coupling constant and hence we drop these for consistency. Therefore we end up with the perturbative formula

\[E(n, \alpha, N')_{NL} = \sum_{i=1}^{4} \sum_{\rho=2}^{N^{NL}+1} \alpha^\rho \sum_{j=0}^{\rho} s_{j,\rho}^{NL}(i) \ln^j n,\]

where the perturbative coefficients are

\[s_{j,\rho}^{NL}(1) = -g_1^{(1)}(b_3/b_2^2) - g_1^{(2)}(b_2)b_2^{\rho-1}(-1)^{\rho+j} \frac{(\rho - 1)!}{j!} 2^{\rho-1} c_{\rho-j},\]

\[s_{j,\rho}^{NL}(2) = g_1^{(1)}(b_3/b_2^2) - g_1^{(2)}(b_2)b_2^{\rho-1}(-1)^{\rho+j} \frac{(\rho - 1)!}{j!} c_{\rho-j},\]

\[s_{j,\rho}^{NL}(3) = g_1^{(1)}(b_3/b_2^2)b_2^{\rho-1}(-1)^{\rho+j} \frac{(\rho - 1)!}{j!} 2^{\rho-1} \Psi(\rho) + \gamma) c_{\rho-j},\]

\[s_{j,\rho}^{NL}(4) = -g_1^{(1)}(b_3/b_2^2)b_2^{\rho-1}(-1)^{\rho+j} \frac{(\rho - 1)!}{j!} \Psi(\rho) + \gamma) c_{\rho-j}.\]
The perturbative exponent in moment space, resumming all large perturbative corrections, becomes, after eqs. (17), (18), (46), (47):

\[ E(n, \alpha, \mathcal{N}) = E(n, \alpha, N)_L + E(n, \alpha, N')_{NL}, \]  

(48)

with \( \mathcal{N} = \mathcal{N}(t) \equiv \{N; N'\} = \{N_1^L, N_2^L, N_3^L; N_1^{NL}, N_2^{NL}, N_3^{NL}, N_4^{NL}\} \), all numbers being functions of \( t \) obeying the relations

\[ N_1^L(t) = N_2^L(t/2), \quad N_1^{NL}(t) = N_2^{NL}(t/2) = N_3^L(t), \quad N_3^{NL}(t) = N_4^{NL}(t/2), \]  

(49)

which give a total of three independent functions. The resummed hard parts in momentum space can now be obtained, given the exponent eq. (48). In the perturbative regime, a closed formula for the hard parts giving all large perturbative corrections is [8]:

\[
I(z, \alpha) = \delta(1-z) - \left[ \frac{e^{\frac{1-z}{1-z^\alpha}}}{\pi(1-z)} \Gamma \left( \frac{1}{1-z}, \alpha, \mathcal{N} \right) \right] 
\]

(50)

with

\[ P_1(n, \alpha, \mathcal{N}) = \frac{\partial}{\partial \ln n} E(n, \alpha, \mathcal{N}). \]

(51)

The range of validity of eq. (50) is \( 2\alpha b_2 \ln(1/(1-z)) < 1 \) and in fact, numerical calculations in [8] suggest that this upper limit is in good agreement with the peak position of the exact leading exponent, \( \alpha b_2 \ln(1/(1-z_L)) \simeq 0.6 \). For larger \( 1/(1-z) \), the higher-twist inherent in the resummation formula, takes over, causing the perturbative approximation to break down. In the next section we will determine the behavior of the higher twist analytically, as a large-\( n \) asymptotic approximation in a regime complementary to the perturbative one, namely for \( \alpha b_2 \ln(1/(1-z)) > 1 \). In section 5 we will present numerical results for \( E(n, \alpha)_L/N_L \) and \( E(n, \alpha, N/N')_{L/N_L} \) along with the sets of numbers \( N \) and \( N' \), for various values of \( \alpha \).

4 Behavior of the exponent in the non-perturbative regime

Before proceeding to numerical results, it is important to understand analytically the behavior of the exponent for asymptotically large values of \( n \)-beyond the perturbative regime. This will reveal both the behavior of the higher twist inherent in our resummation formalism, as well as that of the cross section itself at the very edge of phase space \( z \to 1 \), or equivalently, \( n \to \infty \). It will also confirm and extend the corresponding numerical calculations, which are quite sensitive at this range. In [8] a heuristic argument for the behavior of \( E(n, \alpha)_L \) was given, which was verified by numerical results showing that

\[ \lim_{n \to \infty} E(n, \alpha)_L = -\infty. \]

Here we will elaborate on this argument, developing asymptotic formulas for both \( E(n, \alpha)_L \) and \( E(n, \alpha)_{NL} \) in the non-perturbative range \( 1 < \alpha b_2 \ln n < \infty \). These formulas will be checked against numerical results in section 5.
Let us begin with the leading exponent, eq. (4). We will concentrate on the integral $I(n, t)$, eq. (6). After a change of variables $\zeta \to 1 - \zeta$, it reads

$$I(n, t) = t \int_{\mathcal{P}} \frac{d\zeta}{\zeta} W(n, \zeta) \ln(1 + (1/t) \ln \zeta), \quad (52)$$

with the $n$-dependent weight given by

$$W(n, \zeta) \equiv [(1 - \zeta)^{n-1} - 1] = e^{-(n-1)\ln(1/\zeta)} - 1. \quad (53)$$

The branch-point singularity of the above integrand is at $\zeta = e^{-t}$. On the other hand, the weight approaches the value $-1$ when $2\zeta > 1 - e^{-(n-1)} \equiv n_0(n)$. (54)

As $n$ varies, we may have either $\zeta_t < n_0(n)$ or $\zeta_t > n_0(n)$. The former inequality is satisfied whenever

$$n < 1 + \frac{1}{\ln \left( \frac{1}{1-e^{-t}} \right)} \equiv \zeta_1(t). \quad (55)$$

In fact, for reasonably small values of $\alpha$, $\zeta_1(t) \simeq \zeta_t^{-1} = e^t = Q^2/\Lambda^2$ and hence eq. (55) is equivalent to $\alpha b_2 \ln n < 1$, which we recognize as our perturbative regime for this particular integral. As before, some of the integrals in $E_L$ and $E_{NL}$ contain the scale $t/2$, but again we take the intersection of the regions defined by $\zeta_1(t)$, $\zeta_1(t/2)$ to define the narrower non-perturbative regime $n > \zeta_1(t) \simeq e^t$. This leaves a rather uninteresting intermediate region $Q/\Lambda < n < Q^2/\Lambda^2$, where various scales impose different asymptotic behaviors to different pieces of the exponent.

Therefore the non-perturbative asymptotics are described by the conditions

$$n > \zeta_1(t) \simeq e^t = \frac{Q^2}{\Lambda^2}, \quad W(n, \zeta) \simeq -1, \quad \zeta > n_0(n) \simeq \frac{1}{n}, \quad (56)$$

where the asymptotic behavior of $I(n, t)$ is given by

$$I(n, t) \simeq -t \int_{\mathcal{P}_0} \frac{d\zeta}{\zeta} \ln(1 + (1/t) \ln \zeta), \quad (57)$$

with $\mathcal{P}_0$ the principal-value contour proper, fig. 1b, but starting from $n_0(n)$, rather than 0. We have computed the integral on this contour in the appendix. The result is

$$I(n, t) \simeq t \left\{ -tL(n, t) \ln L(n, t) + \ln(1/n_0(n)) \right\} \equiv tG(n, t), \quad (58)$$

with

$$n_0(n) = 1 - e^{-\frac{1}{n-1}} \simeq \frac{1}{n}, \quad L(n, t) \equiv \frac{1}{t} \ln \left( \frac{1}{n_0(n)} \right) - 1 \simeq \frac{\ln n}{t} - 1, \quad n > \zeta_1(t) = 1 + \frac{1}{\ln \left( \frac{1}{1-e^{-t}} \right)} \simeq e^t. \quad (59)$$

\footnote{This value is the source of divergent behavior as $n \to \infty$.}
Let us now digress for a while, to complete the discussion in section 2 regarding the definition of the perturbative regime. We will confine our discussion to the integral \( I(n, t) \) as given by eq. (6). In section 2, we showed that a general expression for this integral is

\[
I(n, t) = -\sum_{m=1}^{\infty} \frac{(1-n)m}{m!m^2} \mathcal{E}(mt) .
\]

We also showed that for \( t > 1, \ 1 \ll n < e^t \),

\[
I(n, t) \simeq -\sum_{\rho=0}^{N} \rho! (-1)^{\rho+1} \sum_{j=0}^{\rho+2} c_{\rho+2-j} \frac{(-1)^j}{j!} \ln^j n \equiv I(n, t, N) .
\]

Even though the second condition in eq. (61) is the one which suppresses the logarithms and hence makes an asymptotic expansion possible in \( I(n, t, N) \), only the first condition in eq. (61) was used to derive this approximation. Hence, the question might arise whether the perturbative approximation (62) is valid independently of the second condition in (61), which defines the perturbative regime in \( n \). We now see from eqs. (56), (58), (59), however, that for \( n > \zeta_1(t) \approx e^t \), the integral \( I(n, t) \) is approximated by

\[
I(n, t) > \simeq t \ln L(n, t) \equiv H(n, t) ,
\]

with \( L(n, t) \) given by eq. (59). Our leading exponent is then given, in this region, by

\[
E(n, \alpha)_{L>} = \alpha g_1^{(1)} [2I(n, t/2) > - I(n, t)>] - \alpha g_2^{(1)} I_2(n, t> ) .
\]

From eqs. (58,62) we find that, when \( n > e^t \),

\[
G(n, t) \simeq -\ln n \ln \left( \frac{\ln n}{t} \right) ,
\]

and hence

\[
E(n, \alpha)_{L>} \simeq - \frac{g_1^{(1)}}{b_2} \ln 2 \ln n - \frac{g_2^{(1)}}{b_2} \ln \left( \frac{\ln n}{t} \right) .
\]

The first term above dominates the behavior as a logarithmic divergence in \( n \), with a negative coefficient. This point was also made in [6], but the precise asymptotic behavior, eqs. (58)- (63), valid in almost all of the non-perturbative range \( n > \zeta_1(t) \), was not given.

\[
^3I(n, t)> , \ on \ the \ other \ hand, \ is \ not \ even \ defined \ for \ n < e^t .
\]
We can similarly find the large-$n$ asymptotics of the next-to-leading exponent,

\[ E(n, \alpha)_{NL} = \alpha[g_1(1)J_1(n, t) - g_2(1)J_2(n, t)] + \alpha^2[g_1(2)K_1(n, t) - g_2(2)K_2(n, t)]. \]  

(66)

Concentrating on the integrals $J_1$ and $K_1$, eq. (21), and using the results of the appendix, we can write

\[ J_1(n, t) \sim b_3 \frac{t^2}{b_2^2} \{ F(n, t/2) - F(n, t) + H(n, t/2) - H(n, t) \}, \]  

(67)

with $H(n, t)$ defined in eq. (63), where

\[ F(n, t) = \frac{t}{2} \left\{ \ln^2(L(n, t)) - \pi^2 \right\}. \]  

(68)

and

\[ K_1(n, t) \sim -t \left\{ H(n, t/2) - H(n, t) \right\}. \]  

(69)

Similarly, for $J_2$ and $K_2$ we find

\[ J_2(n, t) \sim -b_3 \frac{t^2}{b_2^2} P(n, t), \]  

(70)

where

\[ P(n, t) = \frac{1}{L(n, t)} \ln L(n, t) + \frac{1}{L(n, t)} + 1, \]  

(71)

and

\[ K_2(n, t) \sim t \left\{ \frac{1}{L(n, t)} + 1 \right\} \equiv Q(n, t). \]  

(72)

Notice from the above expressions that the quantities $J_2, K_2$, which were subleading in the perturbative regime, remain subleading in the non-perturbative regime as well. In fact, the dominant large-$n$ behavior in $E(n, \alpha)_{NL}$ comes from the function $F(n, t)$, eq. (68). For values

\[ n > 1 + \frac{1}{\ln \left( \frac{1}{1 - e^{-[1+\alpha]} t} \right)} \sim e^{[1+\alpha] t}, \]  

(73)

the next-to-leading exponent is dominated by

\[ E(n, \alpha)_{NL} > \sim \alpha g_1(1) b_3 \frac{t^2}{b_2^2} \left\{ \ln \frac{2 \ln n}{t} - \frac{t}{2} \ln^2 \left( \frac{\ln n}{t} \right) \right\} \sim -g_1(1) b_3 \frac{t^2}{4b_2^2} \ln^2 \left( \frac{\ln n}{t} \right). \]  

(74)

Therefore, $E(n, \alpha)_{NL}$ also tends to $-\infty$, though more slowly than $E(n, \alpha)_{L>}$. This only happens, however, for huge values of $n$, eq. (73), quite inaccessible even to numerical calculations. As we shall see in section 5, within the numerically accessible range of $n$, the singularity manifests itself through the presence of $\pi^2$, eq. (68), which dominates, making $E(n, \alpha)_{NL}$ look like it is tending to $+\infty$ within that range. This is a demonstration of the importance of the above asymptotic formulas where numerical calculations break down.
5 Numerical results for the Exponent and the Hard Part

In this section we present numerical results for the exponent of the resummation formula as well as the $q\bar{q}$ hard part, at various values of the kinematic variables, and including the large perturbative corrections. In our opinion, these results exhaust our knowledge of the perturbative phase of the theory. The various constants appearing throughout this work, are

\begin{align}
  g_1^{(1)} &= 2C_F, \quad g_2^{(1)} = -\frac{3}{2}C_F, \quad g_1^{(2)} = C_F \left[ C_A \left( \frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5n_f}{9} \right], \\
  b_2 &= (11C_A - 2n_f)/12, \quad b_3 = (34C_A^2 - (10C_A + 6C_F)n_f)/48.
\end{align}

(75)

For the rest of this paper, we will use the values $\Lambda = 0.2\text{GeV}$, $n_f = 4$ and $Q = 5, 10, 90\text{GeV}$, which correspond to $\alpha = 0.075, 0.061, 0.039$ respectively.

5.1 The resummed exponent

Let us first present numerical results for the exponent in moment-space. Approximate conclusions in momentum space may be reached through the correspondence $n \leftrightarrow 1/(1-z)$. As in [6], let us first give the optimum numbers of asymptotic terms for eqs. (45), (46), for the above values of $Q$, computed at the intermediate value $n = 30$. As we have noted before, the set $\mathcal{N}$ depends on $t = 1/(\alpha b_2) = \ln(Q^2/\Lambda^2)$.

| Optimum numbers of asymptotic terms for $E(n, \alpha)_L, E(n, \alpha)_{NL}$ | \(N_1^L\) | \(N_2^L\) | \(N_3^L\) | \(N_1^{NL}\) | \(N_2^{NL}\) | \(N_3^{NL}\) | \(N_4^{NL}\) |
|---|---|---|---|---|---|---|
| \(0.075(6.4)\) | 1 | 5 | 5 | 1 | 5 | 1 | 5 |
| \(0.061(7.8)\) | 2 | 6 | 6 | 2 | 6 | 1 | 6 |
| \(0.039(12.2)\) | 4 | 12 | 11 | 4 | 11 | 4 | 11 |

Given these values we may compute the perturbative exponent as given in eqs. (48), (17), (18), (46), (47) and compare it with the exact exponent $E(n, \alpha) = E(n, \alpha)_L + E(n, \alpha)_{NL}$. The latter is computed numerically on the box contour $\mathcal{P}$, fig. 1c. In fig. 2 we have plotted the result of these comparisons for the above values of $\alpha$. For completeness we have also included comparisons of the leading exponents alone. Notice that we expect the peak of the exact exponents to grow as $\alpha$ gets smaller. This is due to the fact that the higher-twist tends to suppress the exponent, hence, the smaller the $\alpha$, the less the suppression. Notice also that the exact and perturbative curves are in excellent agreement near the peak of the exact curves. We will comment on this in some detail later. Once past the peak of $E(n, \alpha)$, the higher-twist completely dominates the exponent.

Using the results of sec. 4 we may compare the exact exponents with their large-$n$ asymptotic expressions, eqs. (65), (74) in the non-perturbative regime. This is done in fig. 3 for various values of $\alpha$, within the maximum range of $n$ attainable before we run into

\footnote{As with the leading exponent, there is very little dependence of these numbers on $n$, when the latter is well within the perturbative regime. Strictly speaking, for $Q = 5\text{GeV}$, $n = 30$ reaches the limit $Q/\Lambda$ for $N_1^L, N_1^{NL}, N_3^{NL}$. However, these numbers are unchanged for lower values of $n$, as well.}
round-off errors for the exact exponents. Note that the agreement between the exact curves and the analytical approximations improves as \( n \) increases.

In fig. 4a we have plotted separately \( E(n, \alpha)_{NL} \) and the corresponding perturbative approximation for various values of \( Q \). We see that both curves have the same shape for all values of \( Q \) used, but that at low \( Q \)-values numerical agreement is only fair. We can trace this to the determination of \( N' \) and \( N_{NL}^3 \) in particular, see eqs. (38), (45), and table 1. Using the exact expression for \( \Lambda(x) \) as given by the first equality of eq. (39), we see that the \( \pi^2 \)-factor is obviously of higher-twist origin and cannot be expressed as a polynomial in \( \alpha \).

However, its effect on the third sum of eq. (38) may be isolated. The equation determining \( N_{NL}^3 \) may be written

\[
\frac{1}{2} \sum_{m=1}^{n-1} \frac{(1-n)_m}{m!m} \Lambda(mt/2) \simeq - \sum_{\rho=1}^{N_{NL}^3} \frac{\rho!2^\rho}{\rho^\rho} \left[ \Psi(\rho+1) + \gamma \right] \sum_{m=1}^{n-1} \frac{(1-n)_m}{m!m^{1+\rho}}.
\]

(76)

The LHS gives, for \( Q = 10\text{GeV} \), the values \( \{1.798, 0.363\} \) for \( n = \{10, 30\} \) respectively. At the same time the best approximation on the RHS is given for \( N_{NL}^3 = 1 \) and equals the minimum possible values \( \{1.219, 2.212\} \) respectively, which is admittedly not a very good perturbative approximation, with an error relative to the perturbative value of \( \{47\%, 83\%\} \) respectively. If, on the other hand, we subtract the effect of the \( \pi^2 \)-term in the LHS of eq. (76) that enters the definition of \( \Lambda(x) \) in eq. (31), which may be computed analytically to be

\[
\frac{\pi^2 t}{4} \left[ (1 - e^{-t/2})^{n-1} - 1 \right],
\]

we obtain the values \( \{5.008, 8.923\} \). These can be approximated by the perturbative expression for \( N_{NL}^3 = 3 - 4 \) yielding the values \( \{5.568, 8.395\} \), in very good agreement with the previous ones, with an error of only \( \{10\%, 6\%\} \). For values of \( Q \) near the \( Z \)-mass, on the other hand, numerical agreement between the two curves is excellent in the perturbative regime. To summarize, the accuracy of the perturbative approximation for \( E(n, \alpha)_{NL} \) is fair for low \( Q \) because the higher-twist is much more important, and is suppressed more slowly with increasing \( Q \), than in the integrals giving the leading exponent, \( E(n, \alpha)_{L} \).

On the other hand, in fig. 4b we have plotted separately \( E(n, \alpha)_{NL} \) in the nonperturbative regime and have compared it with the corresponding asymptotic approximation \( E(n, \alpha)_{NL>^2} \), up to the maximum values of \( n \) where a reliable numerical calculation of the former is possible. Notice the excellent agreement as well as the fact that these quantities have a positive increasing behavior in this range, something exhibited by the dominance of the \( \pi^2 \)-term in eq. (68), as remarked in sec. 4. Of course, for even larger values of \( n \), the next-to-leading exponent tends to \(-\infty\) as suggested by eqs. (73), (74).

Let us now return to fig. 2 and discuss in more detail the peak positions of the exact exponents. In [6] it was found that the peak of \( E(n, \alpha)_{L} \) is pretty close to the perturbative boundary \( n \simeq Q/\Lambda \). Adding \( E(n, \alpha)_{NL} \) decreases both the magnitude and the position of the peak, as seen in the above figures. Let us therefore define some characteristic points in these curves which will describe, within a controlled arbitrariness, the interface between

\footnote{\textsuperscript{5}}These values are inaccessible to an exact numerical calculation but are very easily handled numerically through the approximate expressions, eqs. (66-72).
the perturbative and non-perturbative regime and a corresponding theoretical uncertainty in the resummed perturbative cross section. One such point that we have already mentioned several times, \( e^{t/2} = Q/\Lambda \), separates the perturbative from the exact analytical expression for the exponent. This, from an analytical point of view, is clearly the boundary between the perturbative and non-perturbative regimes. A second point, already mentioned in [6], is the peak position of the exponent, \( \zeta_p : E(\zeta_p, \alpha) = \text{max} \). In fact we may define separately the peak positions for the leading exponent, \( \zeta_p^L \), and the full one, \( \zeta_p \). The table that follows gives these numbers as a function of \( Q \).

| \( Q (\text{GeV}) \) | \( Q/\Lambda \) | \( \zeta_p^L \) | \( \zeta_p \) |
|-----------------|-------------|-------------|-------------|
| 5               | 25          | 37          | 14          |
| 10              | 50          | 100         | 37          |
| 90              | 450         | 1875        | 570         |

Notice in the above table that inclusion of \( E_{NL} \) creates good agreement between the order of \( Q/\Lambda \) and \( \zeta_p(Q) \). That is, while \( 2\alpha b_2 \ln \zeta_p^L \) is close to unity with good accuracy, ranging from 1.12 (\( Q = 5\text{GeV} \)) to 1.24 (\( Q = 90\text{GeV} \)), inclusion of the next-to-leading exponent moves the peak so that \( 2\alpha b_2 \ln \zeta_p(5; 10; 90\text{GeV}) = 0.82 ; 0.92 ; 1.03 \), is close to 1. In fact these results strongly support the view that the peak of the exponent occurs at the limit of validity of perturbation theory and \emph{approximately coincides} with the analytical result, namely \( Q/\Lambda \). The fact that we have two different scales in the problem, \( t \) and \( t/2 \), is reflected in the minor differences between \( Q/\Lambda \) and \( \zeta_p \) in table 2. This difference may be used to provide a small theoretical uncertainty in the resummed perturbative hard part but, for the purpose of this work we will consider \( e^{t/2} = Q/\Lambda \) to be the boundary of the perturbative regime in moment space. In momentum space, this converts into a perturbative regime given by

\[
0 < z < 1 - \frac{\Lambda}{Q},
\]

and this is what we will use for defining all resummed perturbative quantities below.

### 5.2 The resummed hard part

A formula for the hard part of the \( q\bar{q} \) cross section, which certainly contains all the perturbative corrections that eq. (50) does, but where the higher-twist is not separated, may be obtained by “extending” eq. (50) throughout the whole range of \( z \), by numerically calculating both the full exponent \( E(1/(1-z), \alpha) \) and its derivative

\[
P_1 \left( \frac{1}{1-z}, \alpha \right) = \frac{\partial}{\partial \ln(1/(1-z))} E \left( \frac{1}{1-z}, \alpha \right),
\]

\[\text{This point was denoted by} \ q_1 \ \text{in} \ [8].\]

\[\text{Strictly speaking, the inversion of the Mellin transform} \ [8], \ \text{was performed only for the perturbative exponent. The corresponding formula for the hard parts, involving power-counting arguments for all large perturbative corrections, simplifies into eq. (50) only for the polynomial approximation of the exponent as powers of} \ \alpha \ \text{and} \ \ln n.\]
on the “box” contour \( \bar{P} \), fig. 1:

\[
I(z) = \delta(1 - z) - \left[ \frac{e^{E(1 - z, \alpha)}}{\pi(1 - z)} \Gamma \left(1 + P_1 \left(\frac{1}{1 - z}, \alpha\right)\right) \sin \left(\pi P_1 \left(\frac{1}{1 - z}, \alpha\right)\right) \right]_+. \tag{78}
\]

As is discussed in [8], in the physical cross section we may perform an integration by parts and explicitly get rid of the plus-distributions as well as the \( \delta \)-function in the above formula. Then, the relevant momentum-dependent quantity contributing directly to the cross section as the perturbative hard part is

\[
H(z, \alpha) = \int_0^z dz' \frac{e^{E(1 - z', \alpha)}}{\pi(1 - z')} \Gamma \left(1 + P_1 \left(\frac{1}{1 - z'}, \alpha\right)\right) \sin \left(\pi P_1 \left(\frac{1}{1 - z'}, \alpha\right)\right) . \tag{79}
\]

On the other hand, from the previous section, and using the results in tables 1 and 2, we can now find the resummed perturbative hard part by separating the higher twist. Since the latter has the effect of reducing the exponent till, as \( n \to \infty \), it goes to \(-\infty\), we may remove its effect by setting the perturbative hard part equal to zero in the non-perturbative regime \( 1 - \Lambda/Q \leq z \leq 1 \). We thus arrive at the following perturbative formula:

\[
H(z, \alpha, \mathcal{N}) = \Theta \left(1 - \frac{\Lambda}{Q} - z\right) \int_0^z dz' \frac{e^{E(1 - z', \alpha, \mathcal{N})}}{\pi(1 - z')} \Gamma \left(1 + P_1 \left(\frac{1}{1 - z'}, \alpha, \mathcal{N}\right)\right) \times \sin \left(\pi P_1 \left(\frac{1}{1 - z'}, \alpha, \mathcal{N}\right)\right) . \tag{80}
\]

To avoid repetition, let us define the notation that, for any resummed quantity of interest \( R, R(\mathcal{N})_L \) stands for this quantity computed using eq. (80) with \( E(\mathcal{N})_L \) only, \( R(\mathcal{N}) \) using the same equation but with the full \( E(\mathcal{N}) \) and similarly for \( R_L \) and \( R \), but using eq. (79) instead. We will now present numerically the hard part computed according to these various definitions.

In fig. 5 we have plotted the functions \( H(z, \alpha)_L, H(z, \alpha) \) as well as their perturbative counterparts. Some comments on these curves are in order. As a general observation, the hard part calculated with the total exponent agrees very well with its perturbative approximation in almost all of the \( z \)-range and the agreement improves with increasing \( Q \). In particular, near the \( Z \)-mass the agreement is excellent. In the neighborhood of the non-perturbative regime, \( z > 1 - \Lambda/Q \), we observe some interesting behavior, naturally more pronounced at low \( Q \)-values. Let us first concentrate on \( H(z, \alpha, \mathcal{N})_L \), computed from eq. (80). Going for simplicity to \( n \)-space, we observe that for any value of \( Q \), the polynomial \( P_1(n, \alpha, \mathcal{N}) \) is a positive increasing function of \( n \) in the perturbative regime but may be of order unity for sufficiently large values of \( Q \). This was actually anticipated in [8], and does not contradict the power-counting of \( \alpha \) and \( \ln n \) developed in that reference, which established that \( P_1 \) contains at most equal powers of these quantities and hence is anticipated to be \( \leq 1 \) in the range \( 2a_b q \ln n < 1 \). In fact this expectation is numerically accurate for the normalized series \( P_1/g_1^{(1)} \), for all values of \( Q \). The somewhat large (and arbitrary from a resummation point of view) value of the color factor \( g_1^{(1)} = 2C_F = 8/3 \), which is automatically included in the resummation as an overall multiplicative factor of \( E(n, \alpha, \mathcal{N})_L \), as it should, can make \( P_1 \) somewhat larger. To be concrete, for \( Q = 5, 10, 90 \text{GeV}, P_1(n =
For the values \( P_1(n, \alpha, N)_L > 1 \) at corresponding values of \( Q \), the sinusoidal dependence of the hard part will produce a decreasing behavior which may even become oscillatory for sufficiently large \( Q \). The decreasing behavior is shown in fig. 5b, whereas the oscillatory behavior occurs for \( Q = 90 \text{GeV} \) but is not shown. These effects are obviously concentrated in the neighborhood of \( z \approx 1 - \Lambda/Q \) in momentum space, but are still part of the perturbative regime due primarily to the overall color factor.

Similar remarks apply to the "exact" curves, produced through the formula (79), where the sign of the integrand will become negative whenever \( P_1 > 1 \) or \( P_1 < 0 \). We note that, at \( Q = 5 \text{GeV} \) both \( P_1^L \) and \( P_1 \) are less than 1 in the entire perturbative regime, at \( Q = 10 \text{GeV} \) \( P_1^L > 1 \) for \( 15 < n < 38 \) while \( P_1 < 1 \) in the entire perturbative regime and at \( Q = 90 \text{GeV} \) \( P_1^L; P_1 > 1 \) for \( 39 < n < 935 \) and \( 27 < n < 319 \) respectively. These ranges will create oscillations in the corresponding curves for the hard part, as is shown in fig. 5b for \( H(z, \alpha)_L \).

As we tend towards the peak position of the corresponding exponent, table 1, \( P_1^L; P_1 \) will decrease again, change sign as they cross the peak position, and become negative in the non-perturbative regime. There is no danger from the poles of the \( \Gamma \)-function when \( P_1 \) becomes a negative integer, because \( \sin(\pi P_1) \), that multiplies it, cancels these poles. The corresponding curves will decrease thereafter, reaching a finite value at \( z = 1 \) since, as we have seen in sec. 4, the exponent of the resummation formula (79) tends to \(-\infty\) as \( z \to 1 \). We indeed find numerically that, for \( Q = 5, 10, 90 \text{GeV} \), \( H(z \approx 1, \alpha)_L \approx -2.17; -10.85; -644.0 \) respectively, while \( H(z \approx 1, \alpha) \approx -0.72; -2.23; -151.4 \).

The corresponding curves for the cross section will be bounded for values of \( \tau \) near the edge of phase space \( \tau \approx 1 \).

6 Conclusions

In this paper we have completed the principal-value resummation of the Drell Yan cross section in the DIS scheme, by including the next-to-leading exponent to previously existing results [6]. The resulting exponent \( E = E_L + E_{NL} \) includes the large perturbative corrections appearing as plus-distributions in the perturbative calculation of the cross section and produces a resummed hard part [8] that is finite throughout the entire kinematic range and independent of IR cutoffs. This principal-value exponent \( E \) may be approximated by a perturbative expression, \( E(N) \), obtained in moment space as an asymptotic series in the perturbative regime \( 1 << n < Q/\Lambda \), and by a non-perturbative expression, \( E(n)_> \) when \( n > Q^2/\Lambda^2 \). The former contains a finite number \( N(t) \) of perturbative terms, which depends on three independent functions of \( t \equiv \ln(Q^2/\Lambda^2) \), while the latter tends to \(-\infty\) as \( n \to \infty \). This makes the hard part a finite function of \( z \) and the resummed cross section a bounded function of \( \tau \).

The accuracy of the perturbative approximation of the exponent improves with \( Q \), as expected, being good at fixed-target values and excellent near the Z-mass. \( E_{NL}(N) \) approximates fairly its exact version at low \( Q \)-values, due to large higher-twist contributions, and

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*The values quoted for \( H \) are taken at the highest value of \( z \), equal to 0.999999, before machine round-off errors are encountered by the program.*
very well near the Z-mass. In short, $E_{NL}$ is numerically significant, as expected, in all its versions and all regions.

The hard part for the Drell-Yan cross section, $\mathcal{H}(z, \alpha)$, whose general form was given in $\S$, is reproduced accurately in the perturbative regime by its perturbative approximation $\mathcal{H}(z, \alpha, N)$ except near the edge $z = 1 - \Lambda/Q$, where both expressions peak and then oscillate (reaching negative values) due to higher-twist and/or large color factors. As before, agreement improves substantially with increasing $Q$, making the theoretical uncertainty in the resummation, if defined roughly as the difference between the above two quantities, a decreasing function of $Q$.

We are now in a position to input this resummed hard part, with the properties described above, directly to a convolution with the non-perturbative parton flux for various experimental situations, to obtain the corresponding resummed cross section. Due to the multitude of experimental cases to be covered, the associated numerical complication of the corresponding calculations, the need for an accurate definition of the uncertainty in the resummation and the essentially phenomenological character of these matters, we reserve them for a subsequent publication.

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APPENDIX A Example of a large-\(n\) asymptotic calculation

In this appendix, we will show in some detail how to obtain the asymptotic expressions of the various integrals in the exponent, in the non-perturbative regime, \(n > Q^2 / \Lambda^2\). As an example we will work out the integral \(J_1\), eq. (22). Performing the \(y\)-integration, we have

\[
J_1(n, t) = \frac{b_3}{b_2} \left\{ J_{1;1}(n, t/2) - J_{1;1}(n, t) + J_{1;2}(n, t/2) - J_{1;2}(n, t) \right\},
\]

where, after a change of variables \(\zeta \to 1 - \zeta\),

\[
J_{1;1}(n, t) = \int_{\mathcal{P}_0} d\zeta W(n, \zeta) \ln \left( 1 + \frac{1}{t} \ln \zeta \right)
\]

and

\[
J_{1;2}(n, t) = \int_{\mathcal{P}_0} d\zeta W(n, \zeta) \frac{1}{1 + (1/t) \ln \zeta}.
\]

\(W(n, \zeta)\) is given by eq. (53).

Focusing on the first of these functions we can write, after the discussion in sec. 4, its large-\(n\) approximation as

\[
J_{1;1}(n, t) \equiv -\int_{\mathcal{P}_0} d\zeta \ln \left( 1 + \frac{1}{t} \ln \zeta \right)
\]

where the contour \(\mathcal{P}_0\) extends from \(n_0(n)\) to 1 above and below the real axis and comprises a straight section and a semicircle around the singularity, with radius \(\delta\), see fig. 1a.

The semicircle contributions to the integral are

\[
\frac{1}{2} J_{1;1}(i\epsilon) + \frac{1}{2} J_{1;1}(-i\epsilon) = -\text{Re} \left\{ \int_{\mathcal{C}_0} d\zeta \ln \left( 1 + \frac{1}{t} \ln \zeta \right) \right\} = -\frac{\pi^2}{2}.
\]

On the other hand, the straight-line contributions are

\[
\frac{1}{2} J_{1;1}(+i\epsilon) + \frac{1}{2} J_{1;1}(-i\epsilon) = -\text{Re} \left\{ \int_{n_0(n)}^{1} \frac{d\zeta}{\zeta} \ln \left( 1 + \frac{1}{t} \ln \zeta \right) \right\} = -\frac{\pi^2}{2} + \frac{\pi^2}{2} \ln \left( \frac{n_0(n)}{n_0(n)} \right) - 1.
\]

From eqs. (85), (86), we obtain the desired asymptotic formula for the integral in the non-perturbative regime:

\[
J_{1;1}(n, \alpha) \equiv -\frac{t}{2} \left\{ \ln^2 \left( \frac{1}{t} \ln \left( \frac{n_0(n)}{n_0(n)} \right) \right) - 1 \right\} \equiv F(n, t).
\]

Notice that the functional form of the divergent \(n\)-behavior in eq. (87), in terms of the quantity

\[
L(n, t) \equiv \frac{1}{t} \ln \left( \frac{1}{n_0(n)} \right) - 1,
\]

which contains the perturbative/nonperturbative interface, is dictated by the structure of the IR singularities in eq. (86) (which, of course, cancel). The rest of the integrals in sec. 4 can be worked out in exactly the same way.
FIGURE CAPTIONS

Figure 1. Contours for evaluating the Principal-Value exponent.
(a) General definition.
(b) Principal-Value contour proper, $\mathcal{P}$.
(c) “Box” contour $\bar{\mathcal{P}}$, for numerical evaluation.

Figure 2. Exact and perturbative exponents.
(a) $Q = 5\text{GeV}$: Dot=$E(n, \alpha)_L$; Dash=$E(n, \alpha, \mathcal{N})_L$; Solid=$E(n, \alpha)$; Dot-dash=$E(n, \alpha, \mathcal{N})$.
(b) Same as (a) but for $Q = 10\text{GeV}$.
(c) Same as (a) but for $Q = 90\text{GeV}$.

Figure 3. Exact and nonperturbative (large-$n$) exponents.
(a) $Q = 5\text{GeV}$: Dot=$E(n, \alpha)_L$; Dash=$E(n, \alpha)_L>$; Solid=$E(n, \alpha)$; Dot-dash=$E(n, \alpha)>$.
(b) Same as (a) but for $Q = 10\text{GeV}$.
(c) Same as (a) but for $Q = 90\text{GeV}$.

Figure 4a. Exact and perturbative next-to-leading exponents.
Solid=$E(n, 0.075)_NL$; Dot=$E(n, 0.075, \mathcal{N})_NL$; Short dash=$E(n, 0.061)_NL$; Long dash=$E(n, 0.061, \mathcal{N})_NL$; Dot-short dash=$E(n, 0.039)_NL$; Dot-long dash=$E(n, 0.039, \mathcal{N})_NL$.

Figure 4b. Exact and non-perturbative (large-$n$) next-to-leading exponents.
Notation as in 4a, but with the substitution $E(n, \alpha, \mathcal{N})_NL \rightarrow E(n, \alpha)_NL>$.

Figure 5. The resummed hard function $\mathcal{H}(z, \alpha)$.
(a) $Q = 5\text{GeV}$: Dot=$\mathcal{H}(z, \alpha)_L$; Dash=$\mathcal{H}(z, \alpha, \mathcal{N})_L$; Solid=$\mathcal{H}(z, \alpha)$; Dot-dash=$\mathcal{H}(z, \alpha, \mathcal{N})$.
(b) Same as (a) but for $Q = 10\text{GeV}$.
(c) Same as (a) but for $Q = 90\text{GeV}$.

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