Abstract—A perfect secret-sharing scheme is a method of distributing a secret among a set of participants in such a way that only qualified subsets of participants can recover the secret and the participants in any unqualified subset cannot obtain any information about the secret. The collection of all qualified subsets is called the access structure of the scheme. In a graph-based access structure, each vertex of a graph G represents a participant and each edge of G represents a minimal qualified subset. The average information ratio of a perfect secret-sharing scheme realizing a given access structure is the ratio of the average length of the shares given to the participants to the length of the secret. The infimum of the average information ratio of all possible perfect secret-sharing schemes realizing an access structure is called the optimal average information ratio of that access structure. In this paper, we study the optimal average information ratio of access structures based on coalescence graphs. We investigate how the optimal average information ratio changes under graph coalescence.

Index Terms—secret-sharing scheme, average information ratio, star covering

I. INTRODUCTION

In a secret-sharing scheme, there is a dealer who has a secret, a finite set P of participants and a collection Γ of subsets of P called the access structure. Each subset in Γ is a qualified subset. An access structure must be monotone, that is, any subset of P containing a qualified subset must also be qualified. A secret-sharing scheme is a method by which the dealer distributes a secret among the participants in P such that only the participants in a qualified subset can recover the secret from the shares they received. If, in addition, the participants in any unqualified subset cannot get any information about the secret, then the secret-sharing scheme is called perfect. Since all secret-sharing schemes considered in this paper are perfect, we will simply use "secret-sharing scheme" for "perfect secret-sharing scheme". Therefore, an access structure Γ is completely determined by the family of all its minimal subsets which is called the basis of Γ.

In 1979, the first kind of secret-sharing schemes called the (t,n)-threshold schemes was introduced independently by Shamir [1] and Blakley [2]. In such a scheme, the basis of the access structure consists of all t-subsets of the participant set of size n. Their work has raised a great deal of interest in the research of many aspects of secret-sharing problems. The reader is referred to [3] for a survey on recent progress and applications on the topic. The information ratio and the average information ratio of secret-sharing schemes have been the main subjects of discussion. The information ratio of a secret-sharing scheme is the ratio of the maximum length (in bits) of the share given to a participant to the length of the secret, while the average information ratio of a secret-sharing scheme stands for the ratio of the average length of the shares given to the participants to the length of the secret. Since these ratios respectively represent the maximum and the average number of bits a participant has to remember for each bit of the secret, they are expected to be as low as possible. Constructing secret-sharing schemes with the lowest ratios becomes an important task to achieve. Given an access structure Γ, the infimum of the (average) information ratio of all possible secret-sharing schemes realizing this access structure Γ is referred to as the optimal (average) information ratio of Γ.

In this paper, we consider graph-based access structures. Given a simple graph G, let the vertex set V(G) of G be the set of participants. The access structure based on G has the edge set E(G) of G as its basis. A secret-sharing scheme Σ for the access structure based on G is a collection of random variables ζv for v ∈ V(G) with a joint distribution such that

(i) ζv is the secret and ζv is the share of v;
(ii) If uv ∈ E(G), then ζu and ζv together determine the value of ζu;
(iii) If A ⊆ V(G) is an independent set in G, then ζA and the collection {ζv | v ∈ A} are statistically independent.

Given a discrete random variable X with possible values {x1, x2, ..., xn} and a probability distribution (p(x))n, the Shannon entropy of X is defined as

H(X) = −Σi=1,...,n p(xi) log p(xi)

which is a measure of the average uncertainty associated with X. This value reflects the average number of bits needed to represent the element in X faithfully [4]. Using this well-known Shannon entropy, the information ratio of the scheme Σ can be defined as RΣ = maxv∈V(G) {H(ζv)/H(ζv)} and

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the average information ratio of the scheme \( \Sigma \) is 
\[
AR_{\Sigma} = \sum_{v \in V(G)} \left( |H(\zeta_v)| / |V(G)| \right) \left( |H(\zeta_v)| / |V(G)| \right).
\]
For simplicity, with the same symbol \( G \), we will denote both the graph as well 
as the access structure based on it. For instance, “a 
secret-sharing scheme on \( G \)” refers to “a secret-sharing 
scheme for the access structure based on \( G \).” 
As mentioned above, the optimal information ratio \( R(G) \) of \( G \) 
and the optimal average information ratio \( AR(G) \) of \( G \) are 
the infimum of the information ratio \( R_{\Sigma} \) and 
the average information ratio \( AR_{\Sigma} \) over all possible secret-sharing 
schemes \( \Sigma \) on \( G \) respectively. It is well known that 
\( R(G) \geq AR(G) \geq 1 \) [5] and that \( R(G)=1 \) if and only if 
\( AR(G)=1 \). A secret-sharing scheme \( \Sigma \) with the optimal 
ratio \( R_{\Sigma}=1 \) or \( AR_{\Sigma}=1 \) is called ideal. An access 
structure \( G \) is ideal if there exists an ideal secret-sharing 
scheme on it.

In this paper we investigate the average information 
ratio of coalescence of graphs. Let \( G \) and \( H \) be two 
graphs. The coalescence of \( G \) and \( H \), denoted as \( G \times_{vw} H \) 
where \( u \in V(G) \) and \( v \in V(H) \), is the graph obtained by 
making coalescence of these two graphs through 
identifying the vertices \( u \) and \( v \) as a new vertex \( w_{(u,v)} \) in 
the resulting graph. Therefore, the coalescence graph 
\( G \times_{vw} H \) of \( G \) and \( H \) is a graph with vertex set 
\( V(G \times_{vw} H) = (V(G) \setminus \{u\}) \cup (V(H) \setminus \{v\}) \cup \{w_{(u,v)}\} \), 
and the edge set \( E(G \times_{vw} H) = (E(G) \setminus \{e \in E_G(u)\}) \cup 
(E(H) \setminus \{e \in E_H(v)\}) \cup \{w_{(u,v)} e' \mid z \in N_G(u) \cup N_H(v) \} \} \).

Determining the exact values of \( R(G) \) and \( AR(G) \) is 
quite challenging. Most known results give bounds on 
them [6-11, 12-18]. The exact values of the optimal 
average information ratio of most graphs of order no 
more than five and the optimal information ratio of most 
graphs of order no more than six have been determined 
[9,14,15]. Before 2007, apart from a specially defined 
class of graphs [6], the paths and cycles are the only 
infinite classes of graphs which have known exact values 
of the optimal information ratio and the optimal average 
information ratio. Csiszar and Tardos's [19] determined 
the exact values of the optimal information ratio of all 
trees in 2007. In 2009, Csiszar and Ligeti [20] made an 
even greater achievement by showing that \( R(G)=2-k/k \), 
where \( k \) is the maximum degree of \( G \), for any graph \( G \) 
satisfying the following properties: (i) every vertex has at 
most one neighbor of degree one; (ii) vertices of degree at 
least three are not connected by an edge, and (iii) the 
girth of \( G \) is at least six. In 2012, Lu and Fu [21] went on 
settling the exact values of the optimal average 
information ratio of all trees. In this paper, we 
consider the optimal average information ratio of coalescence 
graphs.

This paper is organized as follows. In Section II, we 
recall basic definitions and restate some known results to 
be used in our discussion. In Section III, we investigate 
how the optimal average information ratio changes under 
graph coalescence. Subsequently, the exact values or 
bounds on the optimal average information ratio of some 
classes of graphs are obtained. A final remark is given in 
the conclusion.
subgraph in $G$; (ii) each vertex $v \in V_0$ has a designated outside neighbor $v^*$ which is defined as a neighbor of $v$ which is outside $V_0$ and is not adjacent to any other vertex in $V_0$, and (iii) $|v^*| \in V_0$ is an independent set in $G$. A collection $\Omega = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_l\}$ of cores of $G$ such that $\epsilon_1, \epsilon_2, \ldots, \epsilon_l$ is a partition of $(V(G))^2$ is called a core cluster of $G$ of size $c_\Omega = l$ [21]. We also denote the minimum size of a core cluster of $G$ as $c^*(G)$ for $G \neq K_1,1$, called the core number of $G$. The core number of $K_1,1$ is naturally defined as $c^*(K_1,1) = 0$. A core cluster of size $c^*(G)$ is called an optimal core cluster of $G$.

**Theorem 2.4** [21] If $\Omega$ is a core cluster of a graph $G$, then

$$AR(G) = \frac{|V(G)| + |(V(G))^2| - c^*(G)}{|V(G)|}$$

**Theorem 2.5** [21] The inequality $c^*(G) \geq d_n$ holds for any star covering $\Pi$ and core cluster $g$ of a graph $G$. In particular, $c^*(G) \geq d^*(G)$.

**Theorem 2.6** [21] If there exists a star covering $\Pi$ and a core cluster $g$ of a graph $G$ such that $c^*_\Pi = d^*_n$, then $c^*(G) = d^*(G)$ and

$$AR(G) = \frac{|V(G)| + |(V(G))^2| - d^*(G)}{|V(G)|}$$

As indicated above, the equality $c^*(G) = d^*(G)$ makes a criterion for examining whether the lower bound and the upper bound on $AR(G)$ will match. We call $G$ realizable if $c^*(G) = d^*(G)$ holds.

In what follows, $N_k(v)$ denotes the set of all neighbors of $v$ in $G$. A vertex $v$ is called a $k$-vertex of $G$ if $deg_k(v) = k$. For any $W \subseteq V(G)$, the subgraph $G-W$ is obtained by removing all vertices in $W$ and their incident edges from $G$.

### III. Main Result

Now, we discuss the average information ratio of the coalescence graph of two graphs.

**Theorem 3.1** Suppose that $G$ and $H$ are graphs with optimal star coverings $\Pi_G$ and $\Pi_H$ and optimal core clusters $\Omega_G$ and $\Omega_H$ respectively.

1. If neither $u$ nor $v$ is a 1-vertex, then there exists a star covering $\Pi$ and a core cluster $\Omega$ of $G+_{w_H}H$ such that $d_\Pi = d^*(G) + d^*(H) - 2$ and $c^*_\Omega = c^*(G) + c^*(H) - 1$.

In addition, if both $u$ and $v$ are centers of some stars in $\Pi_G$ and $\Pi_H$ respectively, then there is a star covering $\Pi'$ of $G+_{w_H}H$ with deduction $d_{\Pi'} = d^*(G) + d^*(H) - 1$.

(2) If $v$ is a 1-vertex and $u$ is not, then there exists a star covering $\Pi$ and a core cluster $\Omega$ of $G+_{w_H}H$ such that $d_\Pi = d^*(G) + d^*(H) - 1$ and $c^*_\Omega = c^*(G) + c^*(H) - 1$.

In addition, if both $u$ and $v$ are centers of some stars in $\Pi_G$ and $\Pi_H$ respectively, then there is a star covering $\Pi'$ of $G+_{w_H}H$ with deduction $d_{\Pi'} = d^*(G) + d^*(H) - 1$.

(3) If both $u$ and $v$ are 1-vertices, then there exists a star covering $\Pi$ and a core cluster $\Omega$ of $G+_{w_H}H$ such that $d_\Pi = d^*(G) + d^*(H) - 1$ and $c^*_\Omega = c^*(G) + c^*(H) - 1$.

In addition, if both $u$ and $v$ are centers of some stars in $\Pi_G$ and $\Pi_H$ respectively, then there is a star covering $\Pi'$ of $G+_{w_H}H$ with deduction $d_{\Pi'} = d^*(G) + d^*(H) - 1$.

If $\Omega_H$ is also a core cluster of $H-v$, then there is a core cluster $\Omega'$ of $G+_{w_H}H$ with size $c_{\Omega'} = c^*(G) + c^*(H) - 1$.

**Proof.** In this proof, we denote the graph $G+_{w_H}H$ as $K$. Let us assume that $u$ is a vertex of a star $S_u$ in the star covering $\Pi_G$ and $v$ is a vertex of the star $S_v$ in the star covering $\Pi_H$.

1. Consider the star covering of $K$ defined as $\Pi = \Pi_G \cup \Pi_H$, then the vertex-number sum $m_\Pi$ of $\Pi$ is $m_\Pi = m_{\Pi_G} + m_{\Pi_H}$.

We have the deduction

$$d_\Pi = |V(K)| + |(V(K))^2| - m_\Pi = |V(G)| + |(V(G))^2| - m_\Pi$$

$$= |V(G)| + |(V(G))^2| - m_{\Pi_G} - m_{\Pi_H} + 2$$

$$= d_{\Pi_G} + d_{\Pi_H} - 2$$

On the other hand, let us consider the core cluster of $K$. Suppose that $u$ belongs to the core $\gamma^G_1$ in the core cluster $\Omega_G$ and $v$ belongs to the core $\gamma^H_1$ in the core cluster $\Omega_H$, then the core cluster $\Omega$ of $K$ can be defined as $\Omega = \Omega_G \setminus \{\gamma^G_1\} \cup \Omega_H \setminus \{\gamma^H_1\} \cup \{\gamma^G_1 \setminus \{u\} \cup \{\gamma^H_1 \setminus \{v\} \cup \{w_{\Pi_H}\}\})$ because the designated outside neighbor of $u$ in $G$ can serve as the one for $w_{\Pi_H}$ in $K$.

The size of $\Omega$ is $c_\Omega = c^*(G) + c^*(H) - 1$.

In addition, if $u$ and $v$ are centers of the stars $S_u$ and $S_v$ in $\Pi_G$ and $\Pi_H$ respectively, then $\Pi' = (\Pi_G \setminus \{S_u\}) \cup \Pi_H$.
\(\{(\Omega \setminus \{g\}) \cup (\{g\} \setminus \{u\})\} \cup \Omega_{H}\). Then \(\Omega\) has size \(c_{\Omega} = c^{*}(G) + c^{*}(H)\) as desired.

In addition, if \(u\) and \(v\) are centers of the stars \(S_{u}\) and \(S_{v}\) in \(\Pi_{G}\) and \(\Pi_{H}\) respectively, then the star covering \(\Pi' = (\Omega_{G} \setminus \{S_{u}\}) \cup (\Pi_{H} \setminus \{S_{v}\}) \cup \{(S_{u} + w_{vw})\}\) of \(K\) has deduction

\[
d_{\Pi'} = |V(K)| + |(V(K))^{2} | - m_{t}\n\]

\[
= |(V(G)) + |V(H)| - 1| + |(V(G))^{2} | + |(V(H))^{2} | + 1)(m_{tg} + m_{th})
\]

\[
= d_{tg} + d_{th} + 1.
\]

In the case where \(\Omega_{H}\) is a core cluster of \(H-v\), let \(\gamma^{H}_{1}\) be the core in \(\Omega_{H}\) which contains the unique neighbor \(v'\) of \(v\). The set \(\gamma^{H}_{1} \cup \{w_{uv}\}\) is a core of \(K\) because the unique neighbor \(u'\) of \(u\) in \(G\) serves as the one for \(w_{uv}\) in \(K\) and the designated outside neighbor of \(v'\) in \(H-v\) serves as the one for \(v'\) in \(K\). Therefore, the collection \(\Omega_{G} = \Omega_{G} \cup (\Omega_{H} \setminus \{\gamma^{H}_{1}\}) \cup (\{\gamma^{G}_{1} \cup \{w_{uv}\}\})\) is a core cluster of \(K\) of size \(c_{\Omega} = c^{*}(G) + c^{*}(H)\).

This result naturally leads to a bound on the optimal average information ratio of coalescence graphs as follows.

**Corollary 3.2** If \(G\) and \(H\) are two graphs with \(u \in V(G)\) and \(v \in V(H)\), then

\[
\frac{|V(G)| + |V(H)| + |(V(G))^{2} | + |(V(H))^{2} | - c^{*}(G) - c^{*}(H) - 1}{|V(G)| + |V(H)| - 1} \leq AR(G +_{u} H)
\]

\[
\leq \frac{|V(G)| + |V(H)| + |(V(G))^{2} | + |(V(H))^{2} | - d^{*}(G) - d^{*}(H) + 2}{|V(G)| + |V(H)| - 1}
\]

Theorem 3.1 also enables us to obtain realizable coalescence graphs from realizable graphs.

**Corollary 3.3** Suppose that \(G\) and \(H\) are realizable graphs with \(u \in V(G)\) and \(v \in V(H)\). Let \(\Omega_{H}\) be an optimal core clusters of \(H\). (1) if \(u\) and \(v\) are not 1-vertices and both \(u\) and \(v\) are centers of some stars in some optimal star coverings of \(G\) and \(H\) respectively, or (2) if \(v\) is a 1-vertex and \(u\) is not, and \(\Omega_{H}\) is also a core cluster of \(H-v\), then \(G +_{uv} H\) is realizable and

\[
d^{*}(G +_{uv} H) = c^{*} + c^{*}(H) - 1
\]

\[
d^{*}(G +_{uv} H) = c^{*} + c^{*}(H) - 1
\]

\[
d^{*}(G +_{uv} H) = c^{*} + c^{*}(H) - 1
\]

\[
d^{*}(G +_{uv} H) = c^{*} + c^{*}(H) - 1
\]
In the following corollary, \( C_k \) represents the cycle of length \( k \). Let \( u_i \in V(C_{2k}) \), \( i=1, 2, \ldots, m \). We define a "flower", denoted \( F(2k_1, 2k_2, \ldots, 2k_m) \), as the graph obtained by making coalescence of the even cycles \( C_{2k_1}, C_{2k_2}, \ldots, C_{2k_m} \) through identifying all the vertices \( u_i, u_{i+1}, \ldots, u_m \).

**Corollary 3.4** If \( u_i \in V(C_{2k_i}) \) and \( k_i \geq 3 \) for \( i = 1, 2, \ldots, m \), then the graph \( F(2k_1, 2k_2, \ldots, 2k_m) \) is realizable and

\[
AR(F(2k_1, 2k_2, \ldots, 2k_m)) = \frac{3(k_1 + k_2 + \ldots + k_m) - (m-1)}{2(k_1 + k_2 + \ldots + k_m) - (m-1)}.
\]

**Proof.** Since even cycles are realizable and \( d^*(C_{2k}) = c^*(C_{2k}) = k \), applying Corollary 3.3 successively, we have

\[
d^*(F(2k_1, 2k_2, \ldots, 2k_m)) = (k_1 + k_2 + \ldots + k_m) - (m-1).
\]

Therefore,

\[
AR(F(2k_1, 2k_2, \ldots, 2k_m)) = \frac{2V(F(2k_1, 2k_2, \ldots, 2k_m)) - |d^*(F(2k_1, 2k_2, \ldots, 2k_m))|}{V(F(2k_1, 2k_2, \ldots, 2k_m))} = \frac{3(k_1 + k_2 + \ldots + k_m) - (m-1)}{2(k_1 + k_2 + \ldots + k_m) - (m-1)}.
\]

Trees have also been shown to be realizable [21], we therefore have the following results.

**Corollary 3.5** Suppose that \( u \) is a vertex of \( C_{2k} \) and \( T \) is a tree. If there exists a vertex \( v \) of \( T \) which has two neighbors \( v \) and \( v' \), then the coalescence \( C_{2k} + w_T \) is realizable, \( c^*(C_{2k} + w_T) = d^*(C_{2k} + w_T) = c^*(T) + k - 1 \) and

\[
AR(C_{2k} + w_T) = \frac{|V(T)| + |V(T)|^2 + 3k - c^*(T) + 1}{|V(T)| + 2k - 1}.
\]

**Proof.** Let \( S_i \) be the star centered at \( v \) and having all \( v_i \) as its vertices. We know from [21] that there exists an optimal core container containing the star \( S_i \). By Corollary 3.3, \( C_{2k} + w_T \) is realizable and \( d^*(C_{2k} + w_T) = c^*(C_{2k} + w_T) = c^*(T) + k - 1 \). Therefore, we have

\[
AR(C_{2k} + w_T) = \frac{2|V(C_{2k})| + |V(T)| + |V(T)|^2 - c^*(C_{2k} + w_T)}{|V(C_{2k})| + |V(T)| - 1} = \frac{2(2k) + |V(T)| + |V(T)|^2 - c^*(T) + k - 1}{2k + |V(T)| - 1} = \frac{|V(T)| + |V(T)|^2 + 3k - c^*(T) + 1}{|V(T)| + 2k - 1}.
\]

**Corollary 3.6** Suppose that \( G \) is a realizable graph with \( v \in V(G) \) and \( T \) is a tree. If there is a vertex of \( T \) which has two leaf neighbors \( v \) and \( v' \), then \( G + w_T \) is realizable.

**Proof.** Let \( v_0 \) be the vertex which has two leaf neighbors \( v \) and \( v' \). Since \( v_0 \) has two leaf neighbors, the vertex \( v' \) can be the designated outside neighbor for \( v_0 \) in any core cluster. Any optimal core cluster of \( T \) is also a core cluster of \( T - v \). We then have the result by Corollary 3.3.

**IV. CONCLUSION**

In this paper, we discuss the optimal average information ratio of coalescence graphs and subsequently obtain the exact value or bounds on the optimal average information ratio of some classes of graphs. Our result also suggests that while considering the optimal average information ratio of large graphs, it is possible to view the large graph as the coalescence of some small graphs. Therefore, the exact value or bounds on the optimal average information ratio of the large graph can be obtained through examining small graphs.

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**REFERENCES**

[1] A. Shamir, “How to share a secret,” Commun. ACM, vol. 22, pp. 612-613, 1979.

[2] G. R. Blakley, “Safeguarding cryptographic keys,” Amer. Fed. Inf. Process. Soc., Proc., vol. 48, pp. 313-317, 1979.

[3] A. Beimel, “Secret-sharing schemes: A survey,” in Proc. 3rd Int. Workshop Coding and Cryptography, Lecture Note in Computer Science, vol. 6639, 2011, pp. 11-46.

[4] I. Csiszár and J. Körner, Information Theory. Coding Theorems for Discrete Memoryless Systems, New York: Academic Press, 1981.

[5] L. Csirmaz, “The size of a share must be large,” J. Cryptol., vol. 10, pp. 223-231, 1997.

[6] C. Blundo, A. De Santis, R. De Simone, and U. Vaccaro, “ Tight bounds on the information rate of secret sharing schemes,” Des. Codes Cryptogr., vol. 11, pp. 107-122, 1997.

[7] C. Blundo, A. De Santis, L. Gargano, and U. Vaccaro, “On the information rate of secret sharing schemes,” Theor. Comp. Soc., vol. 154, pp. 283-306, 1996.

[8] C. Blundo, A. De Santis, A. Giorgio Gaggian, and U. Vaccaro, “New bounds on the information rate of secret sharing schemes,” IEEE Trans. Inf. Theory, vol. 41, pp. 549-554, 1995.

[9] C. Blundo, A. De Santis, D. R. Stinson, and U. Vaccaro, “Graph decompositions and secret sharing schemes,” J. Cryptol., vol. 8, pp. 39-64, 1995.

[10] E. F. Brickell and D. M. Davenport, “On the classification of ideal secret sharing schemes,” J. Cryptol., vol. 4, pp. 123-134, 1991.

[11] E. F. Brickell and D. R. Stinson, “Some improved bounds on the information rate of perfect secret sharing schemes,” J. Cryptol., vol. 5, pp. 153-166, 1992.

[12] L. Csirmaz, “An impossibility result on graph secret sharing,” Des. Codes Cryptogr., vol. 53, pp. 195-209, 2009.

[13] L. Csirmaz, “Secret sharing schemes on graphs,” Studia Mathematica Hungarica, vol. 10, pp. 297-306, 1997.

[14] M. V. Dijk, “On the information rate of perfect secret sharing schemes,” Des. Codes Cryptogr., vol. 6, pp. 143-169, 1995.

[15] W. A. Jackson and K. M. Martin, “Perfect secret sharing schemes on five participants,” Des. Codes Cryptogr., vol. 9, pp. 267-286, 1996.

[16] D. R. Stinson, “An explication of secret sharing schemes,” Des. Codes Cryptogr., vol. 2, pp. 357-390, 1992.

[17] D. R. Stinson, “New general lower bounds on the information rate of perfect secret sharing schemes,” in Proc. Advances in Cryptology – CRYPTO ’92, Lecture Notes in Computer Science, vol. 740, 1993, pp. 168-182.

[18] D. R. Stinson, “Decomposition constructions for secret sharing schemes,” IEEE Trans. Inf. Theory, vol. 40, pp. 118-125, 1994.
[19] L. Csirmaz and G. Tardos, “Optimal information rate of secret sharing schemes on trees,” IEEE Trans. Inf. Theory, vol. 59, no. 4, pp. 2527-2530, 2013.
[20] L. Csirmaz and P. Ligeti, “On an infinite families of graphs with information ratio 2-1/k,” Computing, vol. 85, pp. 127-136, 2009.
[21] H. C. Lu and H. L. Fu, “The exact values of the average information ratio of perfect secret-sharing schemes for tree-based access structures,” Des. Codes Cryptogr., vol. 73, no. 1, pp. 37-46, 2014.

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