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Realizing quantum advantage without entanglement in single-photon states

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Abstract

We propose an optical circuit for attaining quantum measurement advantage in a system that has no quantum entanglement. Our device produces symmetric two-qubit X-states with controllable antidiagonal elements, and does not require entangled states as input. We discuss the use of this device in a two-qubit quantum game. When entanglement is absent, the maximum quantum advantage in this game is 1/3 bit. A slightly diminished quantum advantage, 0.311 bit, can be realized with a simplified transaction protocol.

1. Introduction

The concept of quantum advantage can be illustrated by an example from dense coding [1]. Alice and Bob share a pair of qubits in a maximally entangled state, such as a Bell state. By performing one-qubit unitary operations, Alice can transmit two bits of information to Bob while sending him only one physical qubit. The information advantage of having shared access to a maximally entangled state is thus one bit in this case. That advantage is due to the presence of entanglement [1].

Quantum discord [3, 4] measures the difference between the total quantum mutual information and the classical mutual information of a system [5]. Although entanglement has been widely used as a measure of a given state’s potential quantum advantage, recent studies have suggested that quantum discord can be a richer resource [6–14]. Gu et al. [15] studied the relationship between quantum advantage and quantum discord in an encoding/decoding protocol. In particular, they explain it through an example using Gaussian discord for continuous variables, showing that the maximum quantum advantage that can be harvested in the optimal encoding is equal to the initial amount of quantum discord. Questions remain about the relationship between quantum correlations and quantum advantage in the general case.

Almeida et al. [16] showed how to use quantum advantage to prove that an untrusted party can perform entangling gates. They consider a particular X-state that corresponds to an equal mixture of three Bell states. Such a realization relies on a source of entangled photons to study the ideal case where the encoding is optimal. This result poses questions about the generalization of the problem to correlations among degrees of freedom of a single particle, and to non-ideal scenarios where it is not possible perform an optimal encoding.

Here we discuss the role of quantum correlations [17], in particular quantum discord, in the realization of quantum advantage in various encoding/decoding scenarios. We introduce a realistic experimental setup that can generate a broad class of two-qubit states, and harvest significant quantum advantage even when there is no entanglement.

Many two-qubit states belong to the class of `X-states’ discussed by Yu and Eberly [18] and Rau [19]. This class includes the Bell and Werner states [1, 20]. Both quantum discord and the optimal projective measurements can be determined analytically for arbitrary rank 2 X-states. For higher ranks it is necessary to solve a set of transcendental equations [21–24] or assume an error of up to 0.0021 bits [25]. It is challenging to...
design a feasible experimental system that can generate arbitrary $X$-states, because that requires a programmable decoherence mechanism.

Here, we introduce an optical device that generates symmetric two-qubit $X$-states. It can serve as the basis for making arbitrary $X$-states. By restricting detailed discussion to symmetric $X$-states, we are able to illustrate clearly the physical and operational meaning of quantum advantage. Symmetric $X$-states are incoherent superpositions of maximally entangled Bell states. They can also represent a Bell state subject to decoherence in quantum information processing protocols [18, 26].

Our device consists of a single-photon source and a Mach–Zehnder-like interferometer, using only passive optical components. The two qubits of this system are the polarization and the optical path of the photon. Our device can generate mixed two-qubit states with a wide range of entanglement and quantum discord. It can also be applied to encoding/decoding protocols of a classical random variable $K$ [27]. We describe in detail a series of transactions between Alice and Bob in which Alice encodes $K$ in an $X$-state and Bob attempts to decode $K$ from that state. When quantum discord is present, Bob can better estimate $K$ than he could by using only local measurements and one-qubit operations. This quantum advantage exists even when the two qubits are unentangled.

Section 2 discusses the theory involved in computing quantum discord. In particular, we provide discord and concurrence for symmetric two-qubit $X$-states. We also define the concept of quantum advantage in the context of the encoding protocol. In section 3 we describe the optical device. We discuss the generation of symmetric two-qubit $X$-states and the encoding/decoding protocol. In section 4 we analyze the properties of the state of the system before and after the encoding. We study the behavior of quantum correlations for the optimal and quasi-optimal encoding. We focus on the cases where there is significant quantum advantage without entanglement.

2. Theory

2.1. Quantum discord and two-qubit $X$-states

Quantum discord is the difference between the total mutual information shared by two qubits, $I$, and their locally accessible (classical) mutual information $J[1, 5]$. In light of the optical application that we propose in section 3, we designate the two qubits by the labels $s$ and $p$, which we use below to distinguish photon spin (polarization) from interferometric path, but which can be taken to describe any two-qubit system. For a system described by the density matrix $\rho_{sp}$ in a composed Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_p$, the total mutual information between the two qubits is

$$ I(\rho_{sp}) = S(\rho_s) + S(\rho_p) - S(\rho_{sp}), $$

where $S$ denotes the von Neumann entropy and $\rho_i = \text{Tr}_j(\rho_{sp})$ is the partial density matrix of the subsystem $i$, with $i \neq j = s, p$ [1]. After a projective measurement [28] on the first qubit $\Pi_{\pm} = |\pm\rangle \langle \pm|$, the conditional state of $p$ is given by $\rho_{sp|\pm} = \text{Tr}_s(\Pi_{\pm} \otimes I_p) \rho_{sp} / p_{p|\pm}$, where $p_{p|\pm} = \text{Tr}(\Pi_{\pm} \otimes I_p)$, where $I_p$ is the identity for qubit $p$. The classical mutual information, $I(\rho_{sp|\pm})$, is the amount by which the uncertainty of system $s$ is reduced after measuring $s$ [5], and is

$$ I(\rho_{sp|\pm}) = S(\rho_p) - \inf_{\Pi_{\pm}} \sum_s S(\rho_{sp|\pm}), $$

where the average conditional entropy, $\sum_s S(\rho_{sp|\pm}) = p_{p|+} S(\rho_{sp|+}) + p_{p|-} S(\rho_{sp|-})$, is minimized over all possible projective measurements. Finally, quantum discord [5], $D(\rho_{sp|\pm})$, corresponds to the shared information between $s$ and $p$, that cannot be obtained by measuring $s$,

$$ D(\rho_{sp|\pm}) = S(\rho_p) - S(\rho_{sp}) + \inf_{\Pi_{\pm}} \sum_s S(\rho_{sp|\pm}). $$

The analogous expressions for $I(\rho_{sp})$ and $D(\rho_{sp|\pm})$ can be obtained by interchanging the roles of the qubits when computing the average conditional entropy.

In this work we restrict ourselves to a particular class of states that are generated by an incoherent superposition of maximal entangled states [1]. However, the same analysis can be performed for any symmetric two-qubit $X$-state. In the computational basis these states take the form

$$ \rho_{sp} = \begin{pmatrix} a & 0 & 0 & w^* \\ 0 & b & z^* & 0 \\ 0 & z & b & 0 \\ w & 0 & 0 & a \end{pmatrix}, $$

with $b = 1/2 - a$. We show below that without loss of generality, the coherences $w \in [0, a]$ and $z \in [0, b]$ can be taken to be real and positive. The concurrence $C(\rho_{sp})$ and entanglement of formation $E(\rho_{sp})$ are given by the
Wooters formulae [2]:

\[ G(\rho_p) = 2 \max\{0, w - b, z - a\}, \]

\[ E(C(\rho_p)) = h(1 + \sqrt{1 - C^2})/2, \]

where \( h(x) = -x \log_x 2 - (1 - x) \log_2 (1 - x) \). Since entanglement is a monotonic function of concurrence they have the same minima and maxima, which are 0 and 1 respectively.

For such states, which are described by density matrices of the form shown in equation (4), the conditional states after a measurement of one qubit (i.e. a local measurement) are independent of the qubit that was measured, \( \rho_{p1} \neq \rho_{p2} \). Because of this, the average conditional entropy is also symmetric and therefore \( J(\rho_{p1}) = J(\rho_{p2}) = J(\rho_p) \) and \( D(\rho_{p1}) = D(\rho_{p2}) = D(\rho_p) \). Following the criteria introduced by Maldonado-Trapp et al [21], the discord for this family of states is given by

\[ D(\rho_p) = \begin{cases} 1 - S(\rho_p) + S_{\sigma_2}(\rho_{p1}) & \text{if } 0 \leq u \leq v, \\ 1 - S(\rho_p) + S_{\sigma_2}(\rho_{p2}) & \text{if } v \leq u \leq 1, \end{cases} \]

where \( u = 2(w + z) \) and \( v = |4a - 1| \). The quantities \( S_{\sigma_2}(\rho_{p1}) \) and \( S_{\sigma_2}(\rho_{p2}) \) correspond to the minimal average conditional entropies, when the optimal measurements are along \( \sigma_2 \) and \( \sigma_2 \) respectively, where \( \sigma_i \) are the Pauli matrices [1]. Explicitly, these entropies are given by

\[ S_{\sigma_2}(\rho_{p1}) = - \frac{1 - v}{2} \log_2 \frac{1 - v}{2} - \frac{1 + v}{2} \log_2 \frac{1 + v}{2}, \]

\[ S_{\sigma_2}(\rho_{p2}) = - \frac{1 - u}{2} \log_2 \frac{1 - u}{2} - \frac{1 + u}{2} \log_2 \frac{1 + u}{2}. \]

When \( u = v \) the average conditional entropy does not depend on the measurement. Note that if \( wz < 0 \) the optimal measurement \( \sigma_2 \) must be replaced by \( \sigma_2 \) [21]. Although the optimal measurement changes, the value of the average conditional entropy remains the same. Since correlations do not change under local operations, the case \( wz < 0 \) can be avoided by considering a rotation over the two qubits of the form \( \exp(i\phi_2 \sigma_2 / 2) \otimes \exp(i\phi_3 \sigma_2 / 2) \). Thus, as noted above, we can always choose \( \phi_2 \) and \( \phi_3 \) so that \( w \) and \( z \) are real and positive.

2.2. Encoding process and quantum advantage

We now consider a scenario in which two independent parties, Alice and Bob, have access to a source of discordant \( D(\rho_p) > 0 \) two-qubit symmetric X-states that are fully known to both of them. Alice and Bob engage in a sequence of encoding/decoding transactions, each of which starts with the same X-state. Alice generates a random variable \( K \) that takes values \( k = (b_1, b_2) \) with a probability distribution \( \rho_{k,b} \), where \( b_1 \) and \( b_2 \) are random classical bits [1, 27]. She encodes \( K \) in the qubit \( s \) and then challenges Bob to estimate \( K \) by measuring the encoded state.

In figure 1 we show a quantum circuit that illustrates the protocol described above. For each transaction, Alice applies a local unitary operation \( U_k = \sigma_{k,b} \sigma_{k,b} \) to qubit \( s \) and sends the state to Bob. The four unitary operations that she can apply are \( U_{0} = 1, U_{1} = \sigma_z, U_{2} = \sigma_x \) and \( U_{3} = \sigma_x \sigma_z \), where \( 1 \) is the identity operator for the qubit \( s \). The ensemble received by Bob is thus described by the density matrix

![Quantum circuit for the encoding/decoding protocol](image)
\[ \tilde{\rho}_p = \sum_{k=1}^{4} p_k \rho_k = \sum_{k=1}^{4} p_k U_k \otimes \mathbb{1}_p \rho_{AB} U_k^\dagger \otimes \mathbb{1}_p. \] 

(10)

By performing a decoding protocol after each transaction, Bob constructs a new random variable \( K^* \). Bob then estimates the value of \( k \) that was sent by Alice, and records his estimate as \( k^* = (b_1^*, b_2^*) \in K^* \). We will see that the accuracy of Bob’s estimation is determined by his access to two-qubit operations.

The maximal accessible information [1] that Bob can obtain about \( K \) corresponds to the Holevo information which for an ensemble \( \epsilon = \{ p_k, \rho_k \} \) is given by

\[ I_q = S(\tilde{\rho}_p) - S(\rho_p). \]

(11)

Its minimum and maximum values are given by zero and two bits respectively. When \( I_q = 2 \) the bits \( b_1 \) and \( b_2 \) can be determined deterministically, i.e. Bob can recover \( K \) with certainty. An example of this case is superdense coding [1, 29].

We note that when the protocol is applied to a symmetric X-state, the average state after the encoding \( \tilde{\rho}_p \) is also symmetric. Considering this, if Bob makes a projective measurement \( \Pi_s \) on qubit \( s \), the accessible information that he can obtain from the qubit \( p \) is equivalent to the information that he can obtain from \( s \) given a measurement on \( p \), which is

\[ I_s = \sup_{p} \left( \sum_{\pm} S(\rho_{p|\pm}) - \sum_k p_k S(\rho_{p|\pm}^k) \right). \]

(12)

where \( \rho_{p|\pm} = \text{Tr}(\Pi_{\pm} \otimes \mathbb{1}_p \rho_p / P_{\pm}) \) and \( \rho_{p|\pm}^k = \text{Tr}(U_k^\dagger \Pi_{\pm} U_k \otimes \mathbb{1}_p \rho_p / P_{\pm}) \) are the conditional states for qubit \( p \). As for quantum discord, this quantity is non-trivial to compute since it requires the optimization of conditional entropies over all possible projective measurements.

Quantum advantage, \( \Delta I \), is defined as the difference between \( I_s \) and \( I_q \), \( \Delta I = I_q - I_s \) [15, 16]. It corresponds to the extra information that Bob can gain by performing two-qubit operations prior to making local measurements of each qubit. When the random variable \( K \) is encoded in \( \rho_p \), decoherence is induced in the system and therefore the correlations between \( s \) and \( p \) are modified. The discord consumption [15] is defined as the difference between quantum discord before and after the encoding, \( \Delta D(\rho_p) = D(\rho_p) - D(\tilde{\rho}_p) \). Gu et al [15] showed that the quantum advantage of an encoding protocol and the discord consumption are related by the following inequality:

\[ \Delta D(\rho_p) - J(\tilde{\rho}_p) \leq \Delta I \leq \Delta D(\rho_p), \]

(13)

where \( J(\tilde{\rho}_p) \) is the classical mutual information between \( s \) and \( p \) after the protocol. Optimal encoding is the encoding that maximizes the quantum advantage [15]. It corresponds to the one in which the total mutual information is consumed, i.e. \( I(\tilde{\rho}_p) = D(\tilde{\rho}_p) = 0 \). In this case the quantum advantage is equal to the initial amount of quantum discord, \( \Delta I = D(\rho_p) \).

3. Experimental proposal

In this section we propose an optical implementation for generating symmetric two-qubit X-states, as in equation (4), and use them in an encoding protocol. The detailed setup, shown in figure 2, consists of three parts: an X-state source, an encoding and decoding mechanism, and a measurement process.

We use a linear polarization basis with horizontal and vertical components designated by \( |h⟩ \) and \( |v⟩ \) respectively, and a path basis designed by \( |0⟩ \) and \( |1⟩ \), which in the computational basis corresponds to

\[ |h⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |v⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |0⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

(14)

In this basis, an X-state can be thought as an incoherent superposition of two Bell-like states, \( |Φ⟩ = c_{00} |h⟩ |0⟩ + c_{11} |v⟩ |1⟩ \) and \( |Ψ⟩ = c_{01} |h⟩ |1⟩ + c_{10} |v⟩ |0⟩ \), where the coefficients \( c_{ij} \) are the probability amplitudes of the states \( |i⟩ |j⟩ \). It is well known in optics that a polarizer beam splitter (PBS) can create such states for a single incoming photon by properly choosing the PBS input port. Then, by combining two incoherent paths into each port of a PBS the output state will be X-states of the form (4).

For simplicity, our X-state source is fed by an input photon with a polarization state \( |ψ_{in}⟩ = (|h⟩ + |v⟩) / \sqrt{2} \). The state is then split in two incoherent paths. This can be done by a random switch choosing which path the photon goes through, or by a beam splitter (BS) and a subsequent path delay longer than the coherence length of the photon, or by adding a random noise in one of the paths. In particular, we will describe this process by assuming that the input photon first encounters a BS with transmission and reflection coefficients \( T \) and \( R \) respectively, where \( T + R = 1 \), and introducing a random source of phase noise in one path, see figure 2. It provides a phase shift \( \exp(i\beta/\beta) \) where \( \beta \) is a random variable with a Gaussian probability
distribution $p(\beta) = \frac{\exp(-\beta^2/(2\sigma^2))}{\sqrt{2\pi\sigma^2}}$ and standard deviation $\sigma$. We consider $\sigma$ to be sufficiently large such that the average of different phases $\langle e^{\pm i\beta} \rangle = e^{-\sigma^2}$ can be neglected, producing two incoherent beams. Paths $|0\rangle$ and $|1\rangle$ are recombined with a PBS; all PBSs in this apparatus transmit polarization $|h\rangle$ and reflect polarization $|v\rangle$. The PBS acts as a control (CNOT) gate with polarization and path as control and target qubits [1]. In path $|0\rangle$ we place a PBS that introduces one auxiliary path (black path in figure 2). In those paths we add a controllable time delay, $\tau_h$ or $\tau_v$, which allows us to control the coherences of the $|h\rangle$ and $|v\rangle$ components of $|0\rangle$. With this, the anti-diagonal term associated with $\{ |h\rangle |0\rangle \}$ and $\{ |v\rangle |0\rangle \}$ decreases by a factor of $\kappa_h = e^{-\tau_h}$ and $\kappa_v = e^{-\tau_v}$ respectively, where $\tau$ is the coherence time of the photons. The resulting density matrix can be written in the $\{ |h\rangle |0\rangle, |h\rangle |1\rangle, |v\rangle |0\rangle, |v\rangle |1\rangle \}$ basis as

$$
\rho = \frac{1}{2} \begin{pmatrix}
R & 0 & 0 & -i\kappa_h \\
0 & T - i\kappa_v & 0 & 0 \\
i\kappa_h & 0 & 0 & R \\
0 & 0 & 0 & T
\end{pmatrix}.
$$

(15)

To encode the random variable $K$, both paths $|0\rangle$ and $|1\rangle$ go through a PC. The PC allows Alice to arbitrarily rotate the polarization state of the photon. We assume that Alice has a random number generator that tells her which set of bits $(b_1, b_2)$ to send in each transaction. While to send the bits $(0, 0)$ she does nothing to the polarization state, to send $(0, 1), (1, 0)$ or $(1, 1)$, she applies a rotation at $\pi$ around the axes $X, Y, and Z$ in the polarization Bloch sphere [1]. The decoding process is performed by a PBS and half wave plate (HWP) which acts as a CNOT and a Hadamard gate in polarization respectively [1]. The HWP is set at an angle $\pi/8$ with respect to the horizontal. Measurement: two PBS and four photon detectors are placed to measure in the $\{ |h\rangle |0\rangle, |h\rangle |1\rangle, |v\rangle |0\rangle, |v\rangle |1\rangle \}$ basis.

4. Theoretical results

4.1. Quantum correlations before encoding

Since the von Neumann entropy is invariant under unitary local operations, we apply to the density matrix of the experimental X-states the following rotational operation, $e^{i\pi/4} \otimes I_p e^{-i\pi/4} \otimes I_p$, such that the density matrix elements of the rotated state are real and positive. The correlations present in the original, $\rho$, and rotated state are the same and they can be computed by using the formulas discussed in section 2.1. In particular, concurrence is given by $C(\rho) = \max\{0, \kappa_h R - T, \kappa_v T - R\}$ and it vanishes when $\kappa_h R \leq T$ and $\kappa_v T \leq R$, or when $R = T$. The analytical expression for quantum discord is given by equation (7), with $u = \kappa_h R + \kappa_v T$ and $v = |2R - 1|$. By using the second derivative test [27] we found that, with respect to variations of $\kappa_h$ and $\kappa_v$, quantum discord is maximized in two cases: $\kappa_v = 1$ and $\kappa_h = 0$; and $\kappa_v = 0$ and $\kappa_h = 1$.

Figures 3(a) and (b) show the behavior of the pre-encoding concurrence and discord, $C(\rho)$ and $D(\rho)$, as a function of $R$ and $\kappa_h$ when $\kappa_v = 0$. When $\kappa_h = 0$, the corresponding dependence on $R$ and $\kappa_v$ can be found by
reflecting these figures about the line R = 1/2. From figure 3(a) we note concurrence increases monotonically with \( \kappa_h \) and decreases monotonically with \( R \). Concurrence is zero at the left region of the dashed line, \( (1 + \kappa_h)R = 1 \). It reaches its maximum value, \( C = 1 \), when the photon is completely reflected onto path \( |0\rangle \), \( R = 1 \), and there is no time delay in the auxiliary path, \( \kappa_h = 1 \). This maximum corresponds to the Bell states given by the red line, \( |\psi_+\rangle = (|h0\rangle + |v1\rangle)/\sqrt{2} \) [1]. From figure 3(b) we note that quantum discord also increases monotonically with \( \kappa_h \). Discord only vanishes when the photon is completely transmitted, \( R = 0 \), or when the coherence time of the photon is much less than the time delay, \( \kappa_h = 0 \). It also reaches its maximum value for Bell states, \( D = 1 \), at \( R = 1 \) and \( \kappa_h = 1 \). Besides the global maximum, discord also has local maxima along the black boundaries. The highest value that discord reaches without entanglement is represented by the red dot and its value is \( D = 1/3 \) at \( \kappa_h = 1 \) and \( R = 1/3 \). In figure 3(c) the gray and white regions show where the optimal measurements are \( \sigma_Z \) and \( \sigma_X \) respectively. These are the measurements of polarization that maximize the average conditional entropy and therefore that least disturb the system but allow us to obtain more information about the system. The conditional entropy is independent of the measurement for the states along the black lines. The states along the black solid line are the Werner states [1, 20] which can be written as \( \rho_W = (1 - \kappa_h)1_d/4 + \kappa_h|\psi_+\rangle\langle \psi_+| \). We defined the Werner-like states to be the states along the black dashed line, which can be written as \( (1 + \kappa_h)1_d/4 - \kappa_h|\psi_+\rangle\langle \psi_+| \).

4.2. Quantum correlations after encoding

In order to maximize quantum correlations we consider the pre-encoding state as the one that satisfies \( \kappa_h = 1 \) and \( \kappa_v = 0 \), this is

\[
\rho_p = \frac{1}{2} \begin{pmatrix}
R & 0 & 0 & R \\
0 & T & 0 & 0 \\
0 & 0 & T & 0 \\
R & 0 & 0 & R
\end{pmatrix}.
\]

After each transaction of the bit pair \((b_1, b_2)\) according to the encoding protocol described in section (2.2), the density matrix of the post-encoding state becomes

\[
\rho_k = \frac{1}{4} \begin{pmatrix}
1 - (-1)^{b_1}(T - R) & 0 & 0 & (-1)^{b_2}(1 + (-1)^{b_1})R \\
0 & 1 + (-1)^{b_1}(T - R) & (-1)^{b_2}(1 - (-1)^{b_1})R & 0 \\
0 & 0 & 1 + (-1)^{b_2}(T - R) & 0 \\
(-1)^{b_2}(1 + (-1)^{b_1})R & 0 & 0 & 1 - (-1)^{b_1}(T - R)
\end{pmatrix}
\]

From equation (17) we immediately note that \( b_2 \) only appears in the non-diagonal terms. Without joint measurements Bob cannot determine the value of \( b_2 \), thus he will not be able to estimate \( K \) with certainty.
After averaging out a series of transactions, on average Bob receives the state
\[
\tilde{\rho}_p = \frac{1}{2} \left( (p_1 + p_3)T + (p_1 + p_2)R \right)
\]
\[
\times \begin{pmatrix}
(p_1 + p_3)R & 0 & 0 & (p_1 - p_2)R \\
0 & (p_1 + p_3)R + (p_1 + p_2)T & (p_3 - p_2)R & 0 \\
0 & (p_3 - p_2)R & (p_3 + p_2)R + (p_3 + p_2)T & 0 \\
(p_1 - p_2)R & 0 & 0 & (p_3 + p_2)T + (p_3 + p_2)R \\
\end{pmatrix},
\]
where \(p_1, p_2, p_3, \) and \(p_4\) are the probabilities described in section 2.2. Without loss of generality we consider only positive coherences \(p_1 \geq p_2\) and \(p_3 \geq p_4\). For this ensemble, concurrence is given by
\[
C(\tilde{\rho}_p) = \max\{0, (2p_1 - 1)R - (p_1 + p_2)T, (2p_3 - 1)R - (p_3 + p_2)T\}.
\]
Quantum discord is given by equation (7), with \(u = 2(p_1 + p_3 - 1)R\) and \(v = |(p_3 + p_2)(1 - 2R) + R|\).

4.3. Quantum advantage and quasi-optimal encoding

As noted in section 2.2, an optimal encoding is the one that consumes all correlations, \(I(\tilde{\rho}_p) = 0\) [15]. This is satisfied when the final average state is \(\frac{1}{2} I_k \otimes I_k\); in other words, it is the encoding that introduces the greatest amount of decoherence to the system. From equation (18) we note that an optimal encoding must satisfy one of the following conditions: \(p_1 = p_2 = p_3 = p_4 = 1/4\), or \(R = T\). Thus, optimal encoding is that which yields a maximally mixed encoded state.

Aside from the noted relationship [15] between initial quantum discord and the quantum advantage in an optimal encoding, \(\Delta I = D(\tilde{\rho}_p)\), we found that in this case there is a direct relationship between the initial amount of information and the quantities \(I_q, I_c\) and \(I_T\). The total accessible information, \(I_{total}\) can be conceived as the amount of mutual information that has been removed from the initial state, \(I_q = I(\tilde{\rho}_p) - I(\rho_0)\). Since for optimal encodings the total mutual information becomes zero we can state that these are the encodings that extract all the information from the system and therefore \(I_q = I(\tilde{\rho}_p)\). In addition to this, from equation (11) we note that \(I_c\) is also related to the randomness that has been introduced to the system. For example, when only one of the events \(k\) has preference, say \(p_2 = 1\), no randomness is introduced and \(I_q\) is zero, while in an optimal encoding all the events have the same probability, \(p_2 = 1/4\), and no more randomness can be introduced. Since for optimal encodings \(\Delta I = D(\tilde{\rho}_p)\) and \(I_q = I(\rho_0)\), it is clear that the locally accessible information coincides with the initial classical information, \(I_c = I(\rho_0)\).

Although the optimal encoding is the one that maximizes the quantum advantage it is also interesting to study more general encodings. In particular, we study a group of encodings that we call ‘quasi-optimal’. All quantum discord is consumed in quasi-optimal encodings, \(D(\tilde{\rho}_p) = 0\), but classical mutual information is not necessarily consumed, \(I(\tilde{\rho}_p) = I(\rho_0)\). Both quantum discord and classical mutual information are consumed only for a truly optimal encoding. Of the cases discussed above, all encodings with \(p_1 = p_2\) and \(p_3 = p_4\) are quasi-optimal, including the special case \(p_1 = p_2 = p_3 = p_4\). The behavior of quantum correlations and advantage can be described by one variable. We choose the variable to be \(p_1\) where \(p_1 \in [0, 1/2]\).

Figure 4(a) shows consumed discord \(\Delta D\) (red) and quantum advantage \(\Delta I\) (blue) for the quasi-optimal encoding as a function of \(R\) and \(p_1\). Overall, quantum advantage is always less than discord consumption, in agreement with inequality (13). In (a) and (b) the gray point denotes the Bell state \(|\psi_+\rangle\), while the white and black points denote the maximum value of quantum advantage \(\Delta I\) when there is no entanglement for \(p_1 = 0.25\) and \(p_1 = 0.5\) respectively. Since quasi-optimal encoding consumes all discord, the consumed discord \(\Delta D\) equals the quantum discord of the pre-encoding state and is independent of \(p_1\). From figure 4(a) we note that quantum advantage (blue) is maximal for optimal encodings, \(p_1 = 0.25\), and it decreases symmetrically around \(p_1 = 0.25\). Away from \(p_1 = 0.25\) (optimal encoding), quantum advantage’s behavior is drastically different. Instead of increasing monotonically with \(R\), quantum advantage peaks at a certain \(R\) and then decreases as \(R\) increases. In the most extreme cases, where \(p_1 = 0\) (encoding by only applying \(\sigma_z\)), quantum advantage becomes zero for \(R = 1\), even though \(R = 1\) corresponds to the Bell state \(|\psi_+\rangle\) where both discord and entanglement is maximal. Such a difference is also evident in figure 4(b).

Figure 4(b) shows the behavior of concurrence of the pre-encoding state, \(C(\rho_{AB})\), with quantum advantage, \(\Delta I\) as a function of \(R\), for the optimal encoding, \(p_1 = 0.25\) (black line), and for a particular non-optimal encoding, \(p_1 = 0.5\) (black dashed line), which corresponds to encoding only in \(\sigma_z\). For \(R \leq 0.5\), the pre-encoding state \(\rho_{AB}\) is not entangled (zero concurrence). In this regime, the maximum advantage for using the optimal encoding is \(1/3\) bits at \(R = 1/3\) (white dot), while for the non-optimal encoding it is \(\Delta I = 3(2 - \log_3) / 4 \approx 0.311\) bit at \(R = 0.5\) (black dot). The difference between optimal and non-optimal encoding in this region is small. The maximal quantum advantage between optimal and non-optimal is as low as
approximately 0.022 bits. For $R \geq 0.5$, the pre-encoding state is entangled and is maximally entangled for $R = 1$ (the Bell state $|\psi_+\rangle$). In this regime, quantum advantage differs significantly between $p_1 = 0.25$ (optimal) and $p_1 = 0.5$ (non-optimal). For the non-optimal encoding $\Delta I$ decreases until it reaches zero bits at $R = 1$, and for the optimal one $\Delta I$ increases until it reaches 1 bit at $R = 1$ (gray dot).

5. Conclusions

In this work, we propose a versatile optical realization of symmetric two-qubit $X$-states. Studying the properties of these states allows us to generalize the applications of Bell states. We prove that $1/3$ bit is the maximum value of quantum advantage that can be attained without entanglement. We find that significant quantum advantage can be realized with simplified encoding protocols. Specifically, the quasi-optimal encoding with $p_1 = 0.5$ allows us to achieve a quantum advantage of 0.311 bit. The convenience of this quasi-optimal encoding over the optimal, comes from the easiness of its implementation, since it implies encoding only in $\sigma_z$. In addition we also show that the total accessible mutual information between the random variables is equal to the initial mutual information between qubits. This result can be used as a simple and powerful tool to test the performance of a sender/receiver in a communication protocol avoiding an optimization process involved in computing quantum discord or mutual classical information.

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