Spectral sum rules and finite volume partition function in gauge theories with real and pseudoreal fermions.

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Abstract

Based on the chiral symmetry breaking pattern and the corresponding low-energy effective lagrangian, we determine the fermion mass dependence of the partition function and derive sum rules for eigenvalues of the QCD Dirac operator in finite Euclidean volume. Results are given for $N_c = 2$ and for Yang-Mills theory coupled to several light adjoint Majorana fermions. They coincide with those derived earlier in the framework of random matrix theory.

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1 Introduction.

More than 20 years have passed since the advent of QCD, but the crucial physical questions about the mechanism of confinement and the structure of the QCD vacuum state still remain basically unanswered. The QCD sum rule method \[1\] proved to be very useful in understanding the hadron spectrum and dynamics, but only rather crude characteristics of the vacuum, such as the gluon condensate \(< G^a_{\mu\nu}G^a_{\mu\nu}>_0\) and quark condensate \(< \bar{q}q >_0\), could be determined in this way.

There are two other directions of research which seem now most promising: lattice numerical simulations \[2\] and model simulations \[3\] which can be used to confront assumptions on the form of the QCD vacuum functional with experiment. Naturally, both lattice and model simulations are performed in a finite volume.

Recently, it has been observed that, besides experimental data, there are some exact theoretical results specific for finite volume systems which follow from first principles and the assumption that chiral symmetry breaking occurs \[4\]. In particular, the light quark mass dependence of the finite volume QCD partition function has been determined and, on basis of that, sum rules for the eigenvalues of Euclidean Dirac operator have been derived. The simplest such sum rule is

\[
\left\langle \sum_n \frac{1}{\lambda^n_n} \right\rangle_\nu = \frac{(\Sigma V)^2}{4(|\nu| + N_f)},
\]

(1.1)

where \(V\) is the 4-dimensional volume, \(\Sigma = | < \bar{\psi}\psi >_0 |\), and \(\nu = (1/32\pi^2) \int d^4x G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\) is the topological charge of the gauge field configuration. The averaging goes over all gauge fields with given \(\nu\) with the weight

\[
\propto \exp \left\{ -\frac{1}{4g^2} \int d^4x G^a_{\mu\nu}G^a_{\mu\nu} \right\} [\text{det}'iD]^{N_f},
\]

where \(D\) is the massless Dirac operator and \(\text{det}'\) is the product of all its nonzero eigenvalues (the index theorem enforces the appearance of \(\nu\) zero fermion modes for each flavor in the gluon background with topological charge \(\nu\). They are shifted from zero when a small fermion mass \(m\) is switched on). The sum in LHS of (1.1) runs only over positive eigenvalues \(\lambda_n\) which are much less than the characteristic hadron scale \(\propto \Lambda_{QCD}\) (The region \(\lambda_n \gg \Lambda_{QCD}\) brings about the ultraviolet divergent contribution \(\propto \Lambda_{ultr}^2 V\) which
bears no interesting dynamic information). The result (1.1) is valid when the length of the box \( L \) is much larger than \( \Lambda_{QCD}^{-1} \).

The sum rule (1.1) may be checked by lattice and model simulations, or more correctly, be used as a test for the correctness of these simulations. Indeed, if we approximate the sum over all gauge configurations by a liquid of instantons, the sum rule (1.1) is satisfied [5].

The sum rule (1.1) has been derived under the assumption of the standard pattern of chiral symmetry breaking

\[
SU_L(N_f) \otimes SU_R(N_f) \to SU_V(N_f).
\] (1.2)

However, the breaking according to this scheme only occurs for fermions belonging to the complex fundamental representation of the color group. This is the case when \( N_c \geq 3 \). For \( N_c = 2 \), the fundamental representation is pseudoreal: quarks and antiquarks transform in the same way under the action of the gauge group and the pattern of chiral symmetry breaking is different leading to different sum rules [3]. A third pattern of chiral symmetry breaking is for fermions in the adjoint (real) representation leading to yet another class of sum rules [4, 6].

As we shall discuss in more detail in the next section, the true chiral symmetry group of the lagrangian of \( SU(2) \)–color with fundamental fermions is \( SU(2N_f) \) rather than just \( SU_L(N_f) \otimes SU_R(N_f) \) (it involves also the transformations that mix quarks with antiquarks). The pattern of the spontaneous symmetry breaking due to formation of a quark condensate is

\[
SU(2N_f) \to Sp(2N_f).
\] (1.3)

One of the main results of this paper is the derivation of the analog of sum rule (1.1) for the case \( N_c = 2 \):

\[
\left\langle \sum_n \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{(\Sigma V)^2}{4(|\nu| + 2N_f - 1)}.
\] (1.4)

It is of significant practical importance as numerical simulations of \( QCD \) with \( N_c = 2 \) are much easier and the sum rule (1.4) may be checked sooner.
In [4], the case of Majorana fermions belonging to adjoint representation of the color group has been also considered. The pattern of symmetry breaking is

$$SU(N_f) \rightarrow SO(N_f).$$

(1.5)

The analog of the sum rule (1.1) was derived in [4] for $N_f = 1$ and $N_f = 2$ which are technically simpler. In this paper we fill up this gap and derive the sum rule for arbitrary $N_f$:

$$\left\langle \sum_n \frac{1}{\lambda_n^2} \right\rangle_{\bar{\nu}} = \frac{(\Sigma V)^2}{4(|\bar{\nu}| + (N_f + 1)/2)},$$

(1.6)

where $\bar{\nu} = \nu N_c$. The sum runs over the positive doubly degenerate [4] pairs of eigenvalues of the adjoint Dirac operator.

Actually, the results (1.4) and (1.6) are not new. They have been obtained earlier using universality arguments and random matrix theory [6]. In that approach it was argued that the spectrum of the Dirac operator near zero virtuality is universal, i.e. it is completely determined by the symmetries of the system and can therefore be obtained by a random matrix theory with only these symmetries as input. The aim of this paper is to illustrate once more that both approaches (stochastic matrices and chiral lagrangian) are physically equivalent. The results (1.1), (1.4) and (1.6) follow from only the symmetry properties of the theory which are properly accounted for in both ways of reasoning.

The structure of the paper is the following. In the next section, we discuss symmetry breaking patterns for $QCD$ with $N_c = 2$ and for the theory with adjoint fermions. In section 3, we discuss the technically simpler case $N_c = N_f = 2$, where a nice closed expression for the finite volume partition function can be derived. In section 4, we derive sum rules (1.4) and (1.6) for any $N_f$ and any topological charge sector. In section 5, we derive expressions for partition function in the sector with a given topological charge for the case of equal fermion masses both for $N_c = 2$ with fundamental fermions and for adjoint theories. The results, which generalize the corresponding expression for $QCD$ with $N_c \geq 3$ derived in [4], can be expressed in terms of a determinant of antisymmetric matrices made of Bessel functions (fundamental fermions with $N_c = 2$) or certain integrals related to series of Bessel functions (adjoint fermions).
2 Chiral symmetry and its breaking.

In this section we discuss the chiral symmetry breaking pattern for $SU(2)$ color with fundamental fermions and for adjoint fermions with $SU(N_c)$, $N_c \geq 2$. The discussion up to (2.9) will be in Minkowsky space. All other sections of this paper and the remainder of this section starting from (2.10) will be in Euclidean space time.

The fermion part of the QCD lagrangian involving $N_f$ massless quark Dirac fields belonging to the fundamental representation of the color group is habitually written as

$$L_{ferm} = i \sum_{f=1}^{N_f} \left[ \bar{q}_f^L D q_f^L + \bar{q}_f^R D q_f^R \right], \quad (2.1)$$

where $q_{L,R} = \frac{1}{2}(1 \pm \gamma^5)q$ and $D$ is the Dirac operator. In this basis the $U(N_f) \otimes U(N_f)$ symmetry of the lagrangian is immediately obvious. Of course, a $U_A(1)$ subgroup is broken explicitly by the anomaly.

However, for $N_c = 2$, the symmetry is higher. To see that, one should note that the combination

$$\tilde{q}^f_{iL} = \epsilon_{ij} C \bar{q}^f_{jR} \quad (2.2)$$

is a left-handed spinor that transforms according to the same representation of the $SU(2)$ color group as $q^f_{iL}$ ($i$ is the color index and $C$ is the charge conjugation matrix). It is convenient to write (2.1) in terms of $N_f + N_f$ two-component Weyl spinors $w^f_{\alpha}$

$$L_{ferm} = i \sum_{f=1}^{2N_f} \bar{w}^f D w^f. \quad (2.3)$$

Obviously, this lagrangian enjoys the $U(2N_f)$ symmetry. The $U(1)$ part of it is anomalous and only the $SU(2N_f)$ symmetry is left in the full quantum theory. A subgroup is broken spontaneously, however, by the formation of the fermion condensate. The condensates with maximal flavor symmetry [7] are given by

$$\langle \epsilon^{ij} \epsilon^{\alpha\beta} w^f_{i\alpha} w^f_{j\beta} \rangle_0 = \frac{\Sigma}{2} I^{ff'} \quad (2.4)$$

The matrix $I$ is the $2N_f \times 2N_f$ antisymmetric matrix

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.5)$$
where $1$ is the $N_f \times N_f$ unit matrix. The fact that $I^{ff'}$ is antisymmetric follows just from the Grassmann nature of the fields $w^f_{i\alpha}$. The specific form (2.3) (It is fixed up to an arbitrary transformation $I \rightarrow UIU^T$ with $U \in SU(2N_f)$, $U^T$ is the transpose of $U$. Multiplying of the condensate matrix by an overall $U(1)$ phase factor is also possible but it corresponds to going into a sector with a different vacuum angle $\theta$.) follows from the requirement that the formation of the condensate does not lead to the spontaneous breaking of the vector symmetry $SU_V(N_f)$. This is a consequence of the Vafa-Witten theorem [8]: the spontaneous breaking of a vector flavor symmetry would result in the appearance of scalar Goldstone particles. However, the vector-like nature of QCD enforces the lightest particle in the spectrum to be a pseudoscalar [9,10,11]. The same conclusion follows from an analysis of the effective potential for $N_c \rightarrow \infty$ [12].

Naturally, the mere fact that the condensate is formed and that the symmetry is broken cannot be proven rigorously and is an assumption. For standard QCD with $N_c = 3$, the breaking of chiral symmetry is an experimental fact. It is natural to think that a similar (not yet quite disclosed) mechanism also leads to the formation of a chiral condensate in the other theories discussed above. The transformations of $SU(2N_f)$ which leave the form $I^{ff'}w^f_iw^{f'}_j$ invariant constitute the symplectic group $Sp(2N_f)$ (of which $SU_V(N_f)$ is a subgroup), and we arrive at the chiral breaking pattern (1.3).

This result is known in the literature. People were interested with nonstandard patterns of chiral breaking mainly in association with technicolor models [7,14]. In a recent work [15], this question has also been addressed, but the authors assumed a flavor asymmetric condensate formation with a different symmetry breaking pattern.

The breaking (1.3) leads to the appearance of

$$(2N_f)^2 - 1 - (2N_f^2 + N_f) = 2N_f^2 - N_f - 1$$

Goldstone bosons which are parameterized by the coset $SU(2N_f)/Sp(2N_f)$. For $N_f = 2$, we have 5 instead of the usual 3 Goldstone bosons.

Up to now, we have considered only massless fermions. The mass term

$$\mathcal{L}_m = \epsilon^{ij} \epsilon^{\alpha\beta} M_{ff'} w^f_{i\alpha} w^{f'}_{j\beta} + c.c.$$  

(2.6)

\footnote{Earlier, only the case $N_c \geq 3$ when the fermions belong to the essentially complex representation has been considered, but the theorem can be easily generalized to the case when the representation is real or pseudoreal [13].}
(where $\mathcal{M}_{f'f}$ is an arbitrary antisymmetric complex matrix $2N_f \times 2N_f$) breaks the $SU(2N_f)$ symmetry explicitly and gives masses to the Goldstone bosons which are, however, small if the matrix elements $\mathcal{M}_{f'f}$ are much smaller than $\Lambda_{QCD}$ which we will always assume.

Let us discuss now the symmetry breaking pattern in the case when the fermions belong to the adjoint representation of the color group which is real [7, 14, 4]. The fermion part of the lagrangian [1] is

$$L_{\text{ferm}}^a = i \sum_{f=1}^{N_f} \bar{w}_f D w_f,$$

where $N_f$ is now the number of massless Weyl adjoint fermions $w^{fa}$. The lagrangian (2.7) possesses a $U(N_f)$ symmetry of which the $U(1)$ part is anomalous. In this case the Vafa-Witten theorem dictates [13] the condensate to be flavor-symmetric (see also [4]),

$$\langle \epsilon^{\alpha\beta} w^{af}_\alpha w^{af'}_\beta \rangle_0 = \frac{\sum}{2} \delta^{ff'},$$

(This form is fixed up to an arbitrary unitary transformation from $SU(N_f) : \delta^{ff'} \rightarrow (UU^T)^{ff'}$.). The formation of the condensate breaks down the symmetry to $SO(N_f)$ (Only orthogonal transformations leave the form $\epsilon^{\alpha\beta} w^{af}_\alpha w^{af'}_\beta$ invariant). This leads to the appearance of

$$N_f^2 - 1 = \frac{N_f(N_f - 1)}{2} = \frac{N_f(N_f + 1)}{2} - 1$$

Goldstone bosons. They acquire small masses after switching on the quark mass term

$$L_m^a = \epsilon^{\alpha\beta} \mathcal{M}_{f'f} w^{af}_\alpha w^{af'}_\beta + \text{c.c.},$$

(2.9)

(where $\mathcal{M}_{f'f}$ is now a symmetric complex $N_f \times N_f$) matrix.

Before proceeding further, let us discuss the simplest case $N_f = 1$ and remind simultaneously some relevant arguments from [4] where the reader can find the details. It is well known that, in standard QCD with only one light quark, no spontaneous chiral symmetry breaking occurs at all. After explicit breaking of $U_A(1)$ due to the axial anomaly,

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3 The lagrangian (2.7) can also be expressed in terms of the Majorana 4-component fields

$$\lambda_M = \left( \begin{array}{c} w \\ -\sigma_2 \bar{w} \end{array} \right),$$

(see e.g. [16]).
the symmetry of the quantum lagrangian is just $U_V(1)$ which is not broken spontaneously by the formation of a condensate.

Also in the other theories discussed above there is no spontaneous symmetry breaking for one flavor. For the adjoint $N_f = 1$ case (for zero fermion mass, this theory is just $N = 1$ supersymmetric Yang-Mills), no symmetry is left right from the beginning. For $N_c = 2, N_f = 1$, the chiral symmetry is $SU(2)$ but it is not broken by the formation of the condensate $<\epsilon^{ij}\epsilon^{\alpha\beta}w_{i\alpha}w_{j\beta}>$ which is the group invariant.

That means that the spectrum of all $N_f = 1$ theories involves a gap and, when the size of the system $L$ is much larger than the characteristic scale $\Lambda_{QCD}^{-1}$, the extensive property for the partition function holds:

$$Z(m, \theta) = \exp(-\epsilon_{\text{vac}}(m, \theta)V) \sim \exp(\Sigma m \cos \theta V), \quad (2.10)$$

where $m \ll \Lambda_{QCD}$ is the quark mass and $\theta$ is the vacuum angle. The normalization $\epsilon_{\text{vac}}(0, \theta) = 0$ is chosen. The particular combination $m \cos \theta$ appears in the first term of the Taylor expansion of $\epsilon_{\text{vac}}$ in $m$ due to Ward identities which enforce all physical quantities to depend on the parameters $m$ and $\theta$ in the combination $me^{i\theta}$ and the requirement of reality of $\epsilon_{\text{vac}}$. The parameter $\Sigma$ has the meaning of the quark condensate. For definiteness, we have written (2.10) for theories with fundamental fermions. For adjoint fermions, the combination $me^{i\theta}/N_c$ enters and one should substitute $\theta \rightarrow \theta/N_c$ in (2.10).

The partition function in the sector with a given topological charge $\nu$ is given by the Fourier integral

$$Z_\nu(m) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iv\theta} d\theta Z(m, \theta) = I_\nu(m\Sigma V), \quad (2.11)$$

where $I_\nu(x) = I_{-\nu}(x)$ is the exponentially rising modified Bessel function. In the adjoint case, the integral over $\theta$ extends up to $2\pi N_c$ and the same result holds but with rescaling $\nu \rightarrow \tilde{\nu} = \nu N_c$. The admissible topological numbers $\nu$ are then integer multiples of $1/N_c$ \[17\]. The sum rules (1.1), (1.4), and (1.6) are obtained from the expansion of $Z_\nu(m)$ in $m$ and comparing it with the expansion of (the Euclidean gamma matrices are anti-hermitean)

$$\left< \frac{\det'(i\mathcal{D} - m)}{\det'(i\mathcal{D})} \right>_\nu = \left< \prod_{\lambda_n > 0} \left(1 + \frac{m^2}{\lambda_n^2}\right) \right>_{\nu} \quad (2.12)$$

They all coincide for $N_f = 1$. 

8
When \( N_f \geq 2 \), spontaneous chiral symmetry breaking occurs and Goldstone modes appear. The presence of quasi-massless particles in the spectrum brings about long-distance correlations which do not allow one to write down the finite volume partition function in the extensive form (2.10). However, the properties of Goldstone bosons are known, and one can take them into account explicitly in the path integral for the partition function. When \( L \) is much less than the inverse Goldstone mass \( \sim (m_{\Lambda_{QCD}})^{-1/2} \), it suffices to take into account only the zero spatial Fourier harmonics of the Goldstone fields and the path integral is reduced to a finite dimensional integral over the coset space where the Goldstone modes live.

The case \( N_f = 2 \) is particularly simple because of the simple structure of these cosets. The integrals can be done explicitly. For \( N_c = 3 \) with fundamental fermions, the Goldstone modes live on \( SU(2) \equiv S^3 \). In the adjoint case, the coset is \( SU(2)/SO(2) = S^2 \). These two cases have been treated in [4]. The analysis of the \( N_c = N_f = 2 \) theory is performed along the same lines.

Again, the coset is simple:

\[
\mathcal{K} = SU(4)/Sp(4) \equiv SO(6)/SO(5) = S^5.
\]

The corresponding low-energy chiral effective lagrangian belongs to the class considered in [18]. It can be formulated in terms of the 6-dimensional vector \( \mathbf{n} \) with the constraint \( n^2 = 1 \). When \( L \ll (m_{\Lambda_{QCD}})^{-1/2} \), the kinetic term \( \sim (\partial_\mu \mathbf{n})^2 \) in the effective lagrangian can be neglected, and only the mass term is relevant. If all quark masses are equal and real, the mass term is just \( 2m \cos \frac{\theta}{2} \sum n_6 \). As earlier, the particular dependence on the vacuum angle \( \theta \) follows from the Ward identities. The partition function can be written as

\[
Z(m, \theta) \propto \int d\Omega_5 e^{2m \cos \frac{\theta}{2} \sum V \cos \Phi},
\]

where \( \Phi \) is the polar angle on \( S^5 \) (so that \( d\Omega_5 \propto \sin^4 \Phi \)). The integral is elementary and we get

\[
Z(m, \theta) = \frac{2I_2(2m \cos \frac{\theta}{2} \sum V)}{(m \cos \frac{\theta}{2} \sum V)^3},
\]
(we choose the normalization $Z(0, \theta) = 1$). The partition function in a given topological sector $\nu$ is the coefficient of $e^{i\nu \theta}$ in the Fourier expansion of $Z(m, \theta)$:

$$Z_{\nu} = 2 \sum_{k=0}^{\infty} \frac{(x/2)^{2(\nu|+k)}}{k!(\nu|+k)!(\nu|+k+2)!(2\nu|+k)!} \frac{[2(\nu|+k)]!}{k!((\nu|+k)!}$$

(3.4)

with $x = m \Sigma V$. Comparing the coefficients of $x^{\nu|}$ and $x^{\nu|+2}$, we arrive at the sum rule (1.4) with $N_f = 2$.

A more explicit expression for the partition function $Z_{\nu}$ is obtained is we integrate over $\theta$ first. As a result we find

$$Z_{\nu} = \frac{8}{3\pi} \int_0^{\pi} d\Phi \sin^4 \Phi I_{2\nu}(2x \cos \Phi).$$

(3.5)

By writing $\sin^4 \Phi = \frac{1}{8}(\cos 4\Phi - 4 \cos 2\Phi + 3)$, this integral can be expressed into known integrals [24]. The result is

$$Z_{\nu} = \frac{1}{3} \left[ 3I_{\nu}^2(x) - 4I_{\nu-1}(x)I_{\nu+1}(x) + I_{\nu-2}(x)I_{\nu+2}(x) \right].$$

(3.6)

In section 5 we will show that this formula can be generalized to arbitrary $N_f$ in the form of the Pfaffian of an anti-symmetric matrix to be defined later.

4 Sum rules for any $N_f$.

Let us consider first the case of adjoint fermions with arbitrary number of flavors $N_f$. The coset $SU(N_f)/SO(N_f)$ is spanned by the symmetric unimodular matrices $S$ which can be presented in the form $S = UU^T$ with $U \in SU(N_f)$ . The partition function $Z(\mathcal{M}, \theta)$ represents the invariant integral over the coset which, for convenience, will be traded off for the integral over the full group $SU(N_f)$:

$$Z(\mathcal{M}, \theta) = \int_{SU(N_f)} dU \exp \{ V \Sigma Re [ Tr(\mathcal{M} e^{i\theta/N_c N_f} UU^T) ] \}.$$

(4.1)

Here, $dU$ is the Haar measure on the group, and due to Ward identities, the mass matrix $\mathcal{M}$ and the vacuum angle $\theta$ enter in a particular combination $\mathcal{M} \exp(i\theta/N_c N_f)$ only [4]. The integral (4.1) involves also dummy variables spanning the stability subgroup of orthogonal $SO(N_f)$ matrices on which the integrand does not depend.
The partition function $Z_\nu(\mathcal{M})$ in the sector with a given topological charge $\nu$ is given by the Fourier integral

$$ Z_\nu(\mathcal{M}) = \frac{1}{2\pi N_c} \int_0^{2\pi N_c} e^{-i\nu \theta} d\theta Z(\mathcal{M}, \theta). \quad \text{(4.2)} $$

It can be written as the integral over the full $U(N_f)$ group as ($\bar{\nu} = \nu N_c$)

$$ Z_{\bar{\nu}}(X) = \int_{U(N_f)} dU (\det U)^{-2\bar{\nu}} \exp \left( \text{Re}(\text{Tr} X U U^T) \right), \quad \text{(4.3)} $$

where $X \equiv V\Sigma \mathcal{M}$. The mass matrix $\mathcal{M}$ is symmetric and complex.

The partition function (4.3) is an invariant function of $X$:

$$ Z_{\bar{\nu}}(V^T XV) = (\det V)^{2\bar{\nu}} Z_{\bar{\nu}}(X), \quad \text{(4.4)} $$

where $V$ is a unitary matrix. This enables us to expand the partition function for $\bar{\nu} \geq 0$ as

$$ Z_{\bar{\nu}}(X) = N_{\bar{\nu}} (\det(X))^{\bar{\nu}} (1 + a_{\bar{\nu}} \text{tr} X^\dagger X + \mathcal{O}(X^4)). \quad \text{(4.5)} $$

The normalization factor is denoted by $N_{\bar{\nu}}$. (For $\bar{\nu} < 0$ the first factor in the expansion should be replaced by $(\det(X^\dagger))^{-\bar{\nu}}$.) Such integrals have been considered before in [4] for the $SU_L(N_f) \otimes SU_R(N_f) \to SU_V(N_f)$ chiral symmetry breaking scheme. A similar analysis will be applied to the present partition function. The matrix $X$ can be decomposed as

$$ X = \sum_{a=1}^{M^s} x_a t^a, \quad \text{(4.6)} $$

where $t^a$ is an orthonormal set of symmetric real matrices which are the generators for the symmetric unitary matrices. They satisfy the identities

$$ \text{Tr}\{t^a t^b\} = \frac{1}{2} \delta^{ab}, \quad a = 1, \cdots, M^s, $$

$$ \sum_a t^a t^a = \frac{M^s}{2N_f}. \quad \text{(4.7)} $$

The total number of generators $M^s = N_f(N_f + 1)/2$. The coefficients $x_a$ in (4.6) are complex numbers. For any symmetric matrices $A$ and $B$, we have the identity

$$ \sum_{a=1}^{M^s} \text{Tr}(t^a A) \text{Tr}(t^a B) = \frac{1}{2} \text{Tr}(AB). \quad \text{(4.8)} $$
This enables us to show that the partition function (4.3) satisfies the differential equation

$$
\sum_a \partial_{x_a} \partial_{\bar{x}_a} Z_\nu = \frac{N_f}{8} Z_\nu.
$$

(4.9)

Substituting here the expansion (4.5), the coefficients $a_\nu$ can be easily found:

$$
a_\nu = \frac{1}{4 \left( |\nu| + (N_f + 1)/2 \right)}.
$$

(4.10)

Comparing (4.5) with the mass expansion of the averaged Dirac determinant [4], one immediately gets to the spectral sum rule (1.6) which agrees with the result obtained by using random matrix theory [6].

The calculation of the sum rule for fermions in the fundamental $SU(2)$−color representation is analogous. In this case the partition function corresponds the chiral symmetry breaking scheme $SU(2N_f) \rightarrow Sp(2N_f)$. The coset is spanned by the antisymmetric unimodular matrices $A$ which can be presented in the form $A = UIU^T$ where $U \in SU(N_f)$ and $I$ is the symplectic matrix defined in (2.5). In this case the mass enters only in the combination $M \exp(i\theta/N_f)$. The partition function in the sector with a given topological charge $\nu$ is given by the integral over the coset and vacuum angle $\theta$ which can be expressed in terms of the integral over the full $U(2N_f)$ group :

$$
Z_\nu(X) = \int_{U(2N_f)} dU (\det U)^{-\nu} \exp \left( \frac{1}{2} \text{Re}(\text{Tr} XUIU^T) \right).
$$

(4.11)

The matrix $X = V\Sigma M$ is now an anti-symmetric complex matrix. This partition function also satisfies an invariance relation

$$
Z_\nu(V^TXV) = (\det V)^\nu Z_\nu(X),
$$

(4.12)

(cf. (4.4)), and therefore can be expanded in group invariants. The expansion is a little bit modified as compared to (4.5) and for $\nu \geq 0$ it has the form

$$
Z_\nu(X) = N_\nu [\text{Pf}(X)]^\nu (1 + \frac{1}{2} a_\nu \text{tr} X^\dagger X + \mathcal{O}(X^4)).
$$

(4.13)

where $\text{Pf}(X) = \sqrt{\det(X)}$ is the Pfaffian of the antisymmetric matrix $X$. (For $\nu < 0$ the first factor in this expansion has to be replaced by $[\text{Pf}(X^\dagger)]^{-\nu}$.)

The matrices $UIU^T$ are anti-symmetric complex unitary matrices. Their generators are anti-symmetric real $2N_f \times 2N_f$ matrices which we use as a basis for the expansion of
\[ X = \sum_{a=1}^{M_{as}} x_a t^a. \]  

(4.14)

In this case the total number of generators is \( M_{as} = N_f (2N_f - 1) \). Instead of (4.7) we have the relations

\[ \text{Tr} \{ t^a t^b \} = \frac{1}{2} \delta^{ab}, \quad a = 1, \ldots, M_{as}, \]

\[ \sum_a t^a t^a = \frac{M_{as}}{4N_f}, \]  

(4.15)

and (4.8) is also valid for anti-symmetric matrices \( A \) and \( B \). In this case the sum over \( a \) runs over the anti-symmetric generators. The partition function (4.11) satisfies the differential equation

\[ \sum_a \partial_x a \partial_{\bar{x}_a} Z_\nu = \frac{N_f}{16} Z_\nu. \]  

(4.16)

(cf. (4.9)). Substitution of the expansion of \( Z_\nu \) in powers of \( X \) yields

\[ a_\nu = \frac{1}{4(|\nu| + 2N_f - 1)}. \]  

(4.17)

The corresponding sum-rule has the form (1.4) which agrees with the result obtained from random matrix theory.

The sum rules (1.1), (1.4), (1.6) can be written universally as

\[ \left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{(\Sigma V)^2}{4 |\nu| + (\text{dim(coset)} + 1)/N_f}, \]  

with the rescaling \( \nu \rightarrow \bar{\nu} \) and counting in the sum only one eigenvalue of each degenerate pair in the adjoint case.

### 5 Partition function for equal masses

In this section, we evaluate the partition function for the special case when all masses are equal to \( m \). We use the notation \( x = m V \Sigma \). First, we consider the case of adjoint fermions. The matrix \( UUT^T \) in (4.1) is a symmetric unitary matrix which can be diagonalized by an orthogonal matrix. To evaluate the integral we use the eigenvalues \( \exp(i\theta_k) \) and the
eigenvectors of $UU^T$ as new integration variables. The Jacobian of this transformation has been used extensively in random matrix theory. It has the form

$$J_s(\theta_1, \ldots, \theta_N) \propto \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|.$$  

(5.1)

The partition function can be rewritten as

$$Z_\nu(x) = \left. \frac{\pi^{N_f/2} N_f!}{2^{N_f} \Gamma(1 + N_f/2)} \right| \frac{\prod_{k=1}^{N_f} d\theta_k \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}| \exp \left( x \sum_{k} \cos(\theta_k) + i\overline{\nu} \sum_{k} \theta_k \right)}{\exp \left( x \sum_{k} \cos(\theta_k) + i\overline{\nu} \sum_{k} \theta_k \right)} \right| \left(1 + x \sum_{k} \cos(\theta_k) + i\overline{\nu} \sum_{k} \theta_k \right).$$  

(5.2)

The normalization is such that $Z(0) = Z_0(0) = 1$. Integrals of these type have been studied extensively by Mehta [20, 21]. The result can be expressed as a Pfaffian (see appendix A)

$$Z_\nu(x) = \frac{\pi^{N_f/2} N_f!}{2^{N_f} \Gamma(1 + N_f/2)} \text{Pf}(A).$$  

(5.3)

For even $N_f$, the matrix elements of the anti-symmetric matrix $A$ are given by

$$A_{pq} = -i \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \epsilon(\theta - \phi) \exp(i(p\phi + q\theta)) \exp(x \cos \phi + x \cos \theta + i\overline{\nu}(\phi + \theta)).$$  

(5.4)

The indices $p$ and $q$ run between $-\frac{N_f}{2} + \frac{1}{2}$ and $\frac{N_f}{2} - \frac{1}{2}$.

For $\nu = 0$ and $x = 0$, the Pfaffian in (5.3) can be calculated readily. The result is

$$\text{Pf}(A)|_{\nu=0, x=0} = \frac{2^{N_f/2}}{\pi^{N_f/2}(N_f - 1)!!},$$  

(5.5)

which gives the correct normalization for the partition function.

The simplest non-trivial case is $N_f = 2$ which describes two Majorana adjoint flavors (or one Dirac flavor). In this case, one can derive

$$Z_\nu(x) = \frac{\pi}{2} A_{\frac{1}{2}} = \frac{x^{|\nu|}}{(2|\overline{\nu}| + 1)!} \left(1 + \frac{x^2}{2|\overline{\nu}| + 3} + O(x^4)\right),$$  

(5.6)

$^4$Expanding $\epsilon(\theta - \phi)$ in the Fourier series on the interval $-2\pi \leq \theta - \phi \leq 2\pi$, one can present the integral (5.4) in the form

$$A_{pq} = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k + \frac{1}{2}} I_{\overline{\nu} + p + k + \frac{1}{2}}(x) I_{\overline{\nu} + q - k - \frac{1}{2}}(x).$$

14
which agrees with results obtained in \cite{4}. For odd values of $N_f$ the partition function is equal to the Pfaffian of the matrix $A$ supplemented by an additional row and column (see appendix A)

$$Z_{\bar{\nu}} = \frac{\pi^{N_f/2}N_f!}{2^N_f \Gamma(1+N_f/2)} \text{Pf}\left( \begin{array}{cc} A & c \\ -c^T & 0 \end{array} \right), \quad (5.7)$$

where $c$ is a vector of length $N_f$ defined by

$$c_p = (-1)^{(N_f-1)/2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \exp(ip\theta) \exp(x \cos \theta + i\bar{\nu} \theta). \quad (5.8)$$

Again, the Pfaffian can be calculated easily for $\nu = 0$ and $x = 0$:

$$\text{Pf}(A)|_{\nu=0, x=0} = \frac{2^{(N_f-1)/2}}{\pi^{(N_f-1)/2}(N_f-1)!}, \quad (5.9)$$

which results in a partition function that is normalized to one.

The small $x$ expansion of the partition functions (5.3), (5.7) starts from the term $\propto x^{N_f|\bar{\nu}|}$ which corresponds to the presence of $N_f|\bar{\nu}|$ pairs of fermion zero modes in the gauge field background with topological charge $\nu$. For $x \gg 1$,

$$Z_{\nu} \propto \exp(N_fx) \quad (5.10)$$

for all $\bar{\nu}$ which means that, in the infinite volume limit, the contributions of all topological charges are of the same order \cite{4}.

For $N_c = 2$ with fundamental fermions, we proceed in exactly the same way. In this case the anti-symmetric unitary matrix $UIU^T$ can also be brought into the canonical form by an orthogonal transformation. The result is an anti-symmetric tri-diagonal matrix with off-diagonal matrix elements $\pm \exp(i\theta_k)$. Again we use the angles $\theta_k$ as new integration variables. The Jacobian of this transformation is

$$J_{as}(\theta_1, \cdots, \theta_{N_f}) \propto \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^4, \quad (5.11)$$

leading to the partition function

$$Z_{\nu} = \frac{1}{(2N_f - 1)!N_f!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{k=1}^{N_f} d\theta_k 2\pi \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^4 \exp \left( x \sum_k \cos(\theta_k) + i\nu \sum_k \theta_k \right). \quad (5.12)$$
The normalization factors have been chosen such that $Z_0(0) = 1$. Also this partition function can be rewritten as a Pfaffian (see appendix B):

$$Z_\nu = \frac{1}{(2N_f - 1)!!} \text{Pf}(A), \quad (5.13)$$

where the matrix elements of $A$ are given by

$$A_{pq} = (q - p) \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \exp(i(q + p)\theta) \exp(x\cos(\theta) + iv\theta)$$

$$= (q - p)I_{p+q+\nu}, \quad (5.14)$$

where the indices $p$ and $q$ run between $-(N_f - \frac{1}{2})$ and $N_f - \frac{1}{2}$. Note that $A$ is a $2N_f \times 2N_f$ dimensional matrix.

For $x \ll 1$, $Z_\nu(x) \propto x^{N_f|\nu|}$ which agrees with the zero modes counting. The large $x$ asymptotics of $Z_\nu(x)$ is given by (5.10) and is universal for all $\nu$.

As an example, let us consider $N_f = 2$. In this case the matrix $A$ is given by

$$A = \begin{pmatrix}
0 & I_{\nu-2}(x) & 2I_{\nu-1}(x) & 3I_{\nu}(x) \\
-I_{\nu-2}(x) & 0 & I_{\nu}(x) & 2I_{\nu+1}(x) \\
-2I_{\nu-1}(x) & -I_{\nu}(x) & 0 & I_{\nu+2}(x) \\
-3I_{\nu}(x) & -2I_{\nu+1}(x) & -I_{\nu+2}(x) & 0
\end{pmatrix}. \quad (5.15)$$

The Pfaffian of this matrix is given by

$$\text{Pf}(A) = 3I_{\nu}^2(x) - 4I_{\nu-1}(x)I_{\nu+1}(x) + I_{\nu-2}(x)I_{\nu+2}(x), \quad (5.16)$$

and $Z_\nu(x)$ coincides with our previous result (3.6) derived by performing the integral over the angles directly in (3.5). Expansion of $Z_\nu(x)$ in powers of $x$ reproduces the sum-rule derived in previous sections.

6 Discussion

Depending on whether the representation of the color group is real ($SU(N_c)$, $N_c \geq 2$ with adjoint fermions), complex ($SU(N_c)$, $N_c \geq 3$) or pseudoreal ($SU(2)$ with fundamental fermions), chiral symmetry breaking is realized in a different way leading to different Goldstone sectors. They are parameterized by $SU(N_f)/SO(2N_f)$, $SU_R(N_f) \times$
$SU_L(N_f)/SU(N_f)$ and $SU(2N_f)/Sp(2N_f)$, respectively. For each of the three cases we have determined explicitly the dependence of the partition function in finite space-time volume on the fermion mass $m$. As was shown in 4, the comparison of the chiral expansion of these partition functions with the chiral expansion of the Dirac determinant in the original path integral leads to sum rules for the inverse powers of the eigenvalues of the massless Dirac operator in QCD. In other words, we have very specific correlations between the eigenvalues of the Dirac operator.

The same triality is known from random matrix theory. The matrix elements of the three classical random matrix ensembles are real, complex or quaternion real. These ensembles can be generalized to include the chiral structure of QCD, and it is possible to derive the spectral density and its correlation functions. Previously it had been verified that for complex representations of the color group, this leads to exactly the same sum rules as obtained from the finite volume partition function 23. In the framework of random matrix theory it is straightforward to derive sum rules for the two other ensembles as well 6. (The theories with real, complex, and pseudoreal fermions correspond to setting $\beta = 4$, $\beta = 2$, and $\beta = 1$ in the universal formula (14) of Ref.6).

This leads to the question whether the sum rules for the two remaining cases are indeed the ones that follow from the effective partition function that is based on general assumptions only. This question has been answered affirmatively in this paper. Since both approaches are based on symmetry this result did not come as a surprise. Apart from this, several other new results were derived. In particular, we found explicit expressions for the partition function for equal quark masses which generalize results already known for the complex case. This does not yet answer whether all spectral correlations obtained from random matrix theory are also properties of the QCD partition function. However, from the study of spectra of classically chaotic systems, we know that correlations on the order of one or several level spacings are universal 19. It are precisely such correlations that are responsible for the spectral sum rules. This leads to the conjecture that the microscopic spectral density is universal.

In this paper, we always assumed that the parameters of the theory were in the range

\[
m \ll \Lambda_{QCD},
\]

\[
\Lambda_{QCD}^{-1} \ll L \ll m_\pi^{-1} \sim (m\Lambda_{QCD})^{-1/2}.
\]  

(6.1)
The upper bound for the length of the box $L$ was essential because, if $L$ would be larger than the Compton wavelength of the Goldstone modes, $\sim (m\Lambda_{QCD})^{-1/2}$, nonstatic modes of the Goldstone fields would be excited and the path integral could not be reduced to a finite dimensional group integral. The calculation is also possible for larger $L$ by exploiting the fact that the characteristic fluctuations of Goldstone fields are small in that case [26]. But, as we are mainly interested in the derivation of sum rules (1.4) and (1.6) which are formulated for theories with massless fermions, results in the range (6.1) are quite sufficient for our purposes.

From the practical point of view, the results for the theory with $N_c = N_f = 2$ analyzed in section 3 are perhaps the most important. This theory bears all features of standard $QCD$ but involves fewer degrees of freedom so that it will be easier to confront the sum rules with lattice simulations.

In this paper, the effective partition functions were postulated on the basis of chiral symmetry and other general arguments. Our results suggest that the static limit of the finite volume partition function can also be derived directly from random matrix theory. Work in this direction is under way.

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Appendix A

In this appendix we calculate integrals of the type

$$
\rho(u) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{k=1}^{N} \frac{d\theta_k}{2\pi} \prod_{k<l}^{N} |e^{i\theta_k} - e^{i\theta_l}|,
$$

(A.1)

where $u(\theta)$ is an arbitrary function of $\theta$. We only give the main steps and refer to Dyson's
original paper [22] or the book of Mehta [20] for further details. Because the integrand is a symmetric function of the integration variables, the integration region can be restricted to the domain $D$ defined by $-\pi \leq \theta_1 \leq \cdots \leq \theta_N \leq \pi$ at the expense of a factor $N!$. For angles inside this domain we have the identity

$$
\prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}| = i \frac{N(N-1)}{2} \det \{\phi_p(\theta_j)\},
$$

(A.2)

where the determinant is over an $N \times N$ matrix with matrix elements given by

$$
\phi_p(\theta_j) = \exp(ip\theta_j), \quad p = -\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2}.
$$

(A.3)

To proceed we use the method of integration over alternate variables originally due to [25]. The result for even $N$ is

$$
\rho(u) = i \frac{N(N-1)}{2} N! \int_D \prod_{k \text{ even}} \frac{d\theta_k}{2\pi} u(\theta_k) \left| \begin{array}{ccccc}
S_{\frac{1-N}{2}}(\theta_2) & \phi_{\frac{1-N}{2}}(\theta_2) & \cdots & S_{\frac{1-N}{2}}(\theta_N) & \phi_{\frac{1-N}{2}}(\theta_N) \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
S_{\frac{N-1}{2}}(\theta_2) & \phi_{\frac{N-1}{2}}(\theta_2) & \cdots & S_{\frac{N-1}{2}}(\theta_N) & \phi_{\frac{N-1}{2}}(\theta_N)
\end{array} \right|.
$$

(A.4)

The quantities $S_p(\theta_k)$ are defined as

$$
S_p(\theta_k) = \int_{-\pi}^{\theta_k} u(\phi)\phi_p(\phi) \frac{d\phi}{2\pi}.
$$

(A.5)

For odd $N$, the last column is $S_{\frac{1-N}{2}}(\pi), \cdots, S_{\frac{N-1}{2}}(\pi)$. The integrand in (A.4) is a symmetric function of the integration variables, which permits us to extend the integration range of each variable to $[-\pi, \pi]$ and use the integration theorem of appendix (A.7) of ref. [20]. As a result we obtain

$$
\rho(u) = N! \text{Pf}(A),
$$

(A.6)

where the matrix elements of the anti-symmetric matrix $A$ are given by

$$
A_{pq} = -i \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [S_p(\theta)\phi_q(\theta) - S_q(\theta)\phi_p(\theta)]
= -i \int_{-\pi}^{\pi} \frac{d\phi \, d\theta}{2\pi \sqrt{2}} (\theta - \phi)u(\theta)u(\phi)[\phi_p(\phi)\phi_q(\theta) - \phi_q(\phi)\phi_p(\theta)].
$$

(A.7)
(the identity $i^{-\frac{N(N-1)}{2}} = (-i)^{\frac{N}{2}}$, even $N$, has been used). The Pfaffian of an $N \times N$ ($N$ even) anti-symmetric matrix is defined as

$$\text{Pf}(A) = \frac{1}{(N/2)!} \sum_{\sigma} \text{sign}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(N-1)\sigma(N)},$$

(A.8)

where the sum is over all permutations $\sigma$ of $\{1, 2, \cdots, N\}$. For Pfaffians we also have a Laplace expansion (see p. 393 of [27])

$$\text{Pf}(A) = (-1)^{r+1} [A_{r1} \text{Pf}(A_{1r,1r}) - A_{r2} \text{Pf}(A_{2r,2r}) - \cdots + (-1)^{N+1} A_{rN} \text{Pf}(A_{Nr,Nr})],$$

(A.9)

where the matrix $A_{kr,kr}$ is obtained from matrix $A$ by deleting the both the $k$'th and the $r$'th rows and columns.

For odd $N$, we expand the determinant with respect to the last column. Then the theorem from appendix (A.7) of [20] can be applied to each of the minors after extending the integration domain of each of the integration variables to $[-\pi, \pi]$. The extra factor $1/[(N-1)/2]!$ cancels against the same factor from the integration theorem in [20]. Using the inverse of the Laplace expansion (A.9) we find (with the use of the identity $i^{-\frac{N(N-1)}{2}} = (i)^{\frac{N-1}{2}}$, odd $N$)

$$\rho(u) = N! \text{Pf} \left( \begin{array}{cc} A & c \\ -c^T & 0 \end{array} \right),$$

(A.10)

where the matrix elements of $A$ are as defined above and

$$c_p = (-1)^{\frac{N+1}{2}} S_p(\pi).$$

(A.11)

Appendix B

In this appendix we calculate the integral

$$\rho(u) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_k \frac{d\theta_k}{2\pi} \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^4.$$  

(B.1)

According to Mehta, the Jacobian can be rewritten as a determinant of a $2N \times 2N$ matrix:

$$\prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^4 = \det\{\chi_p(\theta_j), p\chi_p(\theta_j)\},$$

(B.2)
where
\[ \chi_p(\theta_j) = e^{ip\theta_j}, \quad p = -N + \frac{1}{2}, \ldots, N - \frac{1}{2}. \] (B.3)

In (B.2), the rows of the determinant are indexed by \( p \), whereas the columns are indexed by \( j \). According to the theorem in appendix (A.7) of [20], the integral (B.1) can now be expressed as a Pfaffian
\[ \rho(u) = N! \text{Pf}(A), \] (B.4)

where the matrix elements of \( A \) are given by
\[ A_{pq} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} u(\theta) (\chi_p(\theta)\chi_q(\theta) - \chi_q(\theta)\chi_p(\theta)). \] (B.5)
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