HAMiLTONiAN FORMALiSM OF THE iNVERSE PROBlEM USING DiRAC GEOMETRY AND ITS APPLICATION ON LINEAR SYSTEMS

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ABoRACT. We present Hamiltonian formalism for the inverse problem having Dirac\big-isotropic structures as underlying geometry. We used the same idea at [1] to treat replicator equations. Here we state the procedure used there for general vector fields that can be written in a gradient form. For a linear system, we show that if representing matrix of the system has at least one pair of positive-negative non-zero eigenvalues or in the case of eigenvalue zero, at least one three dimensional Jordan block associated to it, then the linear system has a Hamiltonian description with respect to a non-trivial Dirac\big-isotropic structure. More interestingly, we prove that every Hamiltonian linear system is Hamiltonian integrable. As a byproduct, we found a class of linear systems with eigenvalue zero that are Hamiltonian only with respect to a proper big-isotropic structure. Our approach also provides a clear picture for the alternative Hamiltonian descriptions of linear systems.

1. INTRODUCTION

Given equations of motion in Newtonian formalism, say \ddot{u} = F(u, \dot{u}) , the inverse problem in the Lagrangian formalism aims to find a Lagrangian function \mathcal{L}(u, \dot{u}) such that the given equations of motion are the Euler-Lagrange equations of Lagrangian \mathcal{L}(u, \dot{u}) . In the Hamiltonian version of this problem, in addition to finding a Hamiltonian function, one has to determine the geometric structure as well. If the Lagrangian function is regular it gives rise, through Legendre transformation, to a Hamiltonian description of the given equations of motion on the cotangent bundle having the canonical symplectic structure of the cotangent bundle as underlying geometry. However, for many interesting problems the Lagrangian is not regular and the Hamiltonian description can not be obtained from Lagrangian inverse problem. For this reason the Hamiltonian inverse problem should be considered independently, see [7] Introduction, for more on this matter.

The Hamiltonian formalism of the inverse problem is been studied in the context of symplectic and poisson geometries, see [3,7]. Here, we consider Dirac geometry, i.e. Dirac and big-isotropic structures (see Section 2 for preliminaries on these...
structures), as underlying geometry for our Hamiltonian descriptions. It includes all symplectic, presymplectic and poisson cases as well. We only consider the problem locally or equivalently on $\mathbb{R}^n$. The global aspects of the problem are left for future works.

The first inkling of the procedure used here occurred when we were trying to figure out the possibility of becoming Hamiltonian for a given replicator equation (or its equivalent Lotka-Volterra equation) with pay-off matrix which is not skew-symmetrizable. The outcome, which is published recently at [1], was enlargement of the set of conservative replicator and Lotka-Volterra equations. Here, we state the same approach for a general vector filed that can be written in a gradient form.

The idea is quite simple. Let $X = B\eta$ be a vector field, where $B$ and $\eta$ are a matrix valued, respectively, a vector valued functions. We will be looking for a matrix valued function $D$ such that $DB$ is skew-symmetric and $D^T\eta$, considered as a 1-form, is closed. This by itself yields constants of motion for the vector field $X$. If an additional integrability condition is satisfied, the pair $(B, D^T)$ generates a Dirac structure if $\text{ker } B \cap \text{ker } D^T = 0$ and a big-isotropic structure otherwise, see Theorem 3.3. In the case of replicator equations and linear systems this simply boils down to finding a particular type of constant matrices $D$ such that $DB$ is skew-symmetric. In our opinion, supported by the results on replicator equations, [1], and linear systems, stated here, the approach can be applied to other problems as well.

A linear systems is a vector field of the form

$$X(u) = B(u),$$

where $B$ is a constant matrix. It goes without mentioning that linear systems play an important role in the study of dynamical systems. For a good account of examples of linear systems aligned with our work here, see [3].

Hamiltonian linear systems, first studied by Williamson at [10]. By Hamiltonian he meant Hamiltonian with respect to canonical symplectic form on $\mathbb{R}^{2n}$. One main difficulty with his approach was that to put linear system $X = Bu$ in a simpler form, say to diagonalize or put $B$ in Jordan normal form, one needed to do so by symplectic, also known as canonical, changes of variables. The factorization approach presented at [7] and discussed in more details in the textbook [3], overcomes this problem. Our approach enjoys the same advantage of factorization, making it possible to consider $B$ in the canonical normal form, see Remark 4.2.

Factorization means finding skew-symmetric matrix $\Lambda$ and a symmetric matrix $H$ such that $B = \Lambda H$. The matrix $\Lambda$ is the representing matrix of a poisson structure and $H$ yields a quadratic Hamiltonian. When $B$ is invertible factorization yields a symplectic structure. For this reason, it is necessary for non-zero positive-negative eigenvalues to be pairable, i.e. have similar Jordan blocks. This restricts the scope of applicability, simply because it is not able to detect if the linear system is Hamiltonian with respect to a presymplectic structure, see (4.22). In the
case of eigenvalue zero, factorization requires also for pairs of Jordan blocks. Our approach removes all these conditions and is able to detect possible symplectic, presymplectic, poisson and Dirac structures. Furthermore, it detects possible big-isotropic structures which are even more general than Dirac structures. In this regard, an interesting outcome is that a linear system that contains a single even dimensional Jordan block associated to eigenvalue zero has Hamiltonian description only with respect to a proper big-isotropic structure, see Lemma 4.6. This shows that big-isotropic structures have to be considered as well.

Alternative, Hamiltonian descriptions for linear systems is an issue which becomes very clear in our approach, see Remark 4.5. We also show that a Hamiltonian linear system in our setting is always Hamiltonian integrable. This completes a similar result obtained by factorisation approach at [7].

Organization of the paper: In Section 2, we provide a simple introduction to (pre-)symplectic, poisson, Dirac and big-isotropic structures on \( \mathbb{R}^m \). In Section 3, we state our results on Hamiltonian formalism of the inverse problem. In Section 4, we discuss Hamiltonian linear systems. In section 5, we first provide a simple introduction to integrable systems and then show that every Hamiltonian linear system is Hamiltonian integrable.

2. Dirac and big-isotropic structures

In this section, we first provide a simple introduction to (pre-)symplectic, poisson, Dirac and big-isotropic structures, for more details see [1] and references therein.

(pre-)Symplectic structure: Let \( \omega \) be a closed two form on a manifold \( M \). It defines a linear vector bundle map \( \omega^\#: T^* M \to T^* M \) by \( X \mapsto \omega(X, .) \). In local coordinates and when \( M = \mathbb{R}^m \), we use the notation \( \omega^\#(u) \) for the representing matrix of the linear map \( \omega^\#: T_u M \to T_u^* M \). If \( \omega^\#(u) \) is invertible for every \( u \in \mathbb{R}^m \) then \( \omega \) is a symplectic structure on, necessarily, even dimensional manifold \( M \). Relaxing the invertibility condition on \( \omega^\#(u) \), the closed two form \( \omega \) is called a presymplectic structure. In both cases a Hamiltonian vector field \( X_H \) is defined by \( \omega^\#.X_H = dH \).

Poisson structure: Let \( \pi \) be a bivector on \( M \) i.e. a bilinear, antisymmetric map \( \pi : T^* M \times T^* M \to \mathbb{R} \). Similar to the (pre)-symplectic case, it defines a linear vector bundle map \( \pi^\#: T^* M \to TM \) by \( \alpha \mapsto \pi(\alpha, .) \). We use the notation \( \pi^\#(u) \) in the same manner as symplectic case. If \( \pi \) satisfies \([\pi, \pi] = 0\) where \([. , .]\) is Schouten bracket then it defines a poisson structure. The local expression for \([\pi, \pi] = 0\) which is known as Jacobi condition is

\[
\sum_{i=1}^m \left( \pi^\#_{ij} \frac{\partial \pi^\#_{lk}}{\partial u_l} + \pi^\#_{ik} \frac{\partial \pi^\#_{lj}}{\partial u_l} + \pi^\#_{lj} \frac{\partial \pi^\#_{ik}}{\partial u_l} \right) = 0 \quad \forall i, j, k
\] (2.1)

A Hamiltonian vector field $X_H$ is defined by $X_H = \pi^\# dH$.

An alternative definition for poisson manifold $M$ is a manifold equipped with a poisson bracket i.e. a bilinear skew-symmetric bracket $\{ f, g \} := (dg)^t \pi^\# df$ on $C^\infty(M)$ which satisfies Leibniz’s rule and Jacobi identity.

The Jacobi identity (2.1) guaranties the integrability of the distribution defined at every point $u \in \mathbb{R}^m$ by the image of the linear map $\pi^\#(u)$. Each leaf of this foliation have a symplectic structure induced by $\pi$. The dimension of the symplectic leaf passing through a given point is called the rank of the poisson structure at that point. The flow of $X_H$ preserves this foliation and its restriction to each one of these leaves is Hamiltonian in the symplectic sense. So in principle what one gets is a smooth bunch of Hamiltonian vector fields defined on the leaves of a symplectic foliation. The Hamiltonian evolutionary games discussed in [2] are of this type.

**Dirac structure:** Dirac structure, introduced in [4][5], unites and generalizes the poisson and presymplectic structures (hence their “intersection” i.e. symplectic structure). Let $M$ be a manifold, then the vector bundle $T^\prime M = T M \oplus T^* M$ is called the big tangent bundle or, in some literature, Pontryagin bundle. By $P_1 : T^\prime M \to TM$ and $P_2 : T^\prime M \to T^* M$ we, respectively, denote the projections on the first and second components. Denoting the natural pairing between vector field $X \in \mathfrak{X}(M)$ and 1-form $\alpha \in \Omega^1(M)$ by $\alpha(X)$, a natural pairing on the sections of $T^\prime M$ is defined by

$$\ll (X, \alpha), (Y, \beta) \gg = \frac{1}{2} (\beta(X) + \alpha(Y)). \quad (2.2)$$

Let $L$ be a linear subbundle of $T^\prime M$, its annihilator with respect to the pairing $\ll .., \gg$ is defined as

$$L^\perp := \{(X, \alpha) \in T^\prime M \mid \ll (X, \alpha), (Y, \beta) \gg = 0 \quad \forall (Y, \beta) \in L\}.$$ 

The pairing $\ll .., \gg$ is neither positive definite nor negative definite. As a consequence for a given linear subbundle $L$ of $T^\prime M$ the intersection $L \cap L^\perp$ can be non-empty. Having this in mind, a linear subbundle $L \subset T^\prime M$ is called isotropic if $L \subset L^\perp$. If $L = L^\perp$ then $L$ is called maximal isotropic. Maximal isotropy implies that the dimension of the fibers of $L$ is equal to the dimension of $M$.

**Definition 2.1.** A **Dirac structure** on a manifold $M$ is a maximal isotropic linear subbundle $L \subset T M \oplus T^* M$ such that for every given section $(X, \alpha), (Y, \beta) \in \Gamma(L)$

$$\left( [X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) \right) \in \Gamma(L), \quad (2.3)$$

where $[.,.]$ denotes the Lie bracket between vector fields and $\mathcal{L}$ stands for Lie derivative. The left hand side of (2.3) is called **Courant bracket** of two sections $(X, \alpha), (Y, \beta)$ and is denoted by $\ll [X, \alpha], (Y, \beta) \gg$. 
Furthermore, a vector field $X \in \mathfrak{X}(M)$ is called Hamiltonian with respect to Dirac structure $L$ if there exist a Hamiltonian $H \in C^\infty(M)$ such that $(X, dH) \in \Gamma(L)$.

Following example shows that Dirac structure unifies symplectic and poisson structures.

**Example 2.2.** A (pre-)symplectic form $\omega$ and a poisson $\pi$ define the Dirac structures $L_\omega = \{(X, \omega^\sharp(X))| X \in \mathfrak{X}(M)\}$, respectively, $L_\pi = \{((\pi^\sharp(\alpha)), \alpha) | \alpha \in \Omega^1(M)\}$.

The skew-symmetricness of $\omega$ and $\pi$ yields the maximal isotropy condition and closeness of $\omega$, respectively, the Jacobi identity (2.1) yield (2.3).

**Lemma 2.3.** Let $f \in C^\infty(M)$ and $(X, \alpha), (Y, \beta) \in TM$. Then

$$[(X, \alpha), f(Y, \beta)] = f[(X, \alpha), (Y, \beta)] + X(f). (Y, \beta) - \ll (X, \alpha), (Y, \beta) \gg (0, df).$$

**Proof.**

$$[(X, \alpha), f(Y, \beta)] = ([X, fY], \mathcal{L}_X(f\beta) - \mathcal{L}_Y\alpha + \frac{1}{2}d(\alpha(fY) - f\beta(X))$$

$$= ([X, fY] + X(f)Y, X(f)\beta + f\mathcal{L}_X\beta - f\mathcal{L}_Y\alpha - \alpha(Y).df$$

$$+ \frac{1}{2}f.d(\alpha(Y) - f\beta(X)) + \frac{1}{2}df.(\alpha(Y) - \beta(X)))$$

$$= f[(X, \alpha), (Y, \beta)] + X(f). (Y, \beta) - \ll (X, \alpha), (Y, \beta) \gg (0, df).$$

**Remark 2.4.** Note that if $L$ is isotropic the last term on the right hand side of equation (2.4) is zero. It means that if for $(X, \alpha), (Y, \beta) \in L$ we have that $[(X, \alpha), (Y, \beta)] \in L$ then $[(X, \alpha), f(Y, \beta)] \in L$. In other word, even though Courant bracket is not bilinear, closeness of the sections of an isotropic subbundle with respect to Courant bracket is bilinear. Therefore, one may check the integrability condition only on a basis of an isotropic subbundle $L$.

We also consider big-isotropic structure which is a generalization of Dirac structure. Up to our knowledge, not much has been done regarding the Hamiltonian systems with big-isotropic structures as underlying structure. In [8] the author studies the geometry of these structures and in [9] he studies Hamiltonian systems in this context, providing some reduction theorems for this type of Hamiltonian systems.

**Definition 2.5.** A big-isotropic structure is an isotropic linear subbundle $L \subset TM \oplus T^*M$ which satisfies (2.3) and a vector field $X \in \mathfrak{X}(M)$ is called Hamiltonian with respect to big-isotropic structure $L$ if there exist a Hamiltonian $H \in C^\infty(M)$ such that $(X, dH) \in L$.

**Definition 2.6.** A function $F$ is called a Casimir of Dirac\big-isotropic structure $L$ if $(0, dF) \in L$ and a vector field $X$ isotropic with respect to $L$ if $(X, 0) \in L$. 

Note that Casimirs are constants of motion for every Hamiltonian vector fields and adding an isotropic vector field \( X \) to a Hamiltonian vector field \( X_H \), one gets an other Hamiltonian vector field with respect to the same Hamiltonian.

A consequence of the fact that sections of a Dirac\-big-isotropic structure \( L \) are closed with respect to Courant bracket \((2.3)\) is the integrability of the (possibly singular) distribution \( P_1(L) \). Every leaf \( S \) of the foliation generated by \( P_1(L) \) is equipped with the closed two form
\[
\omega_S((P_1(X, \alpha), P_1(Y, \beta))) = \frac{1}{2}(\alpha(Y) - \beta(X)) \quad \forall (P_1(X, \alpha), P_1(Y, \beta)) \in TS,
\]
i.e. \( P_1(L) \) integrates to a presymplectic foliation.

3. Hamiltonian inverse problem

In this section we use Dirac\-big-isotropic structures introduced in Section 2 in order to discuss Hamiltonian inverse problem for vector fields of type
\[
X(u) = (B\eta)(u),
\]
where \( B \) is a matrix valued function on \( \mathbb{R}^m \) and \( \eta \) is a 1-form defined by the map \( \eta: \mathbb{R}^m \to \mathbb{R}^m \). We start by introducing the type of Dirac\-big-isotropic structures we will be using.

**Lemma 3.1.** Let \( L_{(B,D^t)} \) be the linear subbundle of \( T\mathbb{R}^m \oplus T^*\mathbb{R}^m \) which is defined by \( L_{(B,D^t)}(u) := \{(B(u)z, D^t(u)z) \mid \forall z \in \mathbb{R}^m \} \) at every point \( u \in \mathbb{R}^m \) where \( B \) and \( D \) are two \( m \times m \)-matrix valued functions on \( \mathbb{R}^m \). Then subbundle \( L_{(B,D^t)} \) is a big-isotropic structure if and only if

(i) \( (D^tB + B^tD^t)(u) = 0 \) for every point \( u \in \mathbb{R}^m \).

(ii) For every \( i, j = 1, \ldots, m \) we have
\[
\llbracket (E_i, \xi_i), (E_j, \xi_j) \rrbracket \in L_{(B,D^t)} \quad \text{where} \quad (E_i, \xi_i) = (Be_i, D^t e_i) \quad \forall i
\]
Furthermore it is a Dirac structure if it also satisfies

(iii) \( (\ker B(u)) \cap (\ker D^t(u)) = 0 \) for every point \( u \in \mathbb{R}^m \).

**Proof.** Item (i) is equivalent to \( L_{(B,D^t)} \) being isotropic and Item (iii) guaranties the maximality. Remark 2.4 together with Item (ii) yield the integrability condition \((2.3)\).

Following Corollary is an immediate consequence of Remark 2.4 and the fact that for a given non-singular matrix \( W \) we have
\[
DB \quad \text{is skew-symmetric} \quad \iff \quad W^t DBW \quad \text{is skew-symmetric.}
\]

**Corollary 3.2.** Let \( L_{(B,D^t)} \) be the linear subbundle of \( T\mathbb{R}^m \oplus T^*\mathbb{R}^m \) as defined at Lemma 3.1 and \( W(u) \) be a \( m \times m \) invertible matrix valued function on \( \mathbb{R}^m \). If \( L_{(B,D^t)} \) is a Dirac\-big-isotropic structure then \( L_{(BW,D^tW)} \) is a Dirac\-big-isotropic structure.
Two interesting cases are
i) If $B$ is invertible which yields $L_{(B,D^t)} = L_{(I,D^t(B)^{-1})}$. This is Dirac structure
generated by presymplectic form $\omega^# = D^t(B)^{-1}$.
ii) If $D^t$ is invertible then $L_{(B,D^t)} = L_{(B(D^t)^{-1},I)}$. This is Dirac structure generated
by poisson structure $\pi^# = B(D^t)^{-1}$.

Considering Dirac\-big-isotropic structure $L_{(B,D^t)}$, a pair $(X,dH)$ is a Hamiltonian system if and only if there exist a function $\eta : \mathbb{R}^m \to \mathbb{R}^m$ such that $X = B\eta$ and $dH = D^t\eta$. Now, we are ready to state the main result of this section.

**Theorem 3.3.** Let $X = B\eta$ be a vector field on $\mathbb{R}^m$. If there exist a matrix valued function $D$ such that

1) The 1-form $D^t\eta$ is closed.
2) $DB(u)$ is skew-symmetric for any $u \in \mathbb{R}^m$.

Then function $H$ where $dH = D^t\eta$ is a constant of motion for the vector field $X$. Furthermore, if for any $i, j$ there exists a map $c_{ij}$ such that

$$[(E_i, \zeta_i), (E_j, \zeta_j)] = (Bc_{ij}, D^t c_{ij}),$$

where $E_i = Be_i$ and $\zeta_i = D^t e_i$, then $X$ is Hamiltonian having the function $H$ as Hamiltonian function and Dirac\-big-isotropic structure $L_{(B,D^t)}$ as underlying structure.

- If $B(u)$ is invertible for every $u$ the underlying structure is the presymplectic structure $\omega^# = D^t(B)^{-1}$. If $D(u)$ is invertible as well $\omega$ is a symplectic structure.
- If $D(u)$ is invertible for every $u$ then the underlying structure is the poisson structure $\pi^# = B(D^t)^{-1}$.
- If $\ker B(u) \cap \ker D^t(u) = 0$ for every $u$ then $L_{(B,D^t)}$ is a Dirac structure, otherwise a big isotropic one.

**Proof.** The proof of the fact that $H$ is a constant of motion is

$$<X,dH> = <B\eta, D^t\eta> = \eta^t DB\eta = 0.$$ 

where we used the fact that $DB$ is skew-symmetric. The rest of the Theorem is an immediate corollary of Lemma 3.1 and Corollary 3.2. □

**4. Hamiltonian Linear Systems**

In this section, we apply our method to vector fields of type

$$X = Bu,$$  \hspace{1cm} (4.1)

where $B$ is a constant $m \times m$-matrix. Since $B$ is a constant matrix, the matrix valued function $D$ required by Theorem 3.3 can be assumed to be constant as well.
As it is shown in [1, Lemma 4.4], given a matrix $D$ such that $DB$ is skew-symmetric the linear subbundle of $T\mathbb{R}^m \oplus T^*\mathbb{R}^m$ defined by

$$L_{(B,D^t)}(u) := \{(Bz, D^t z) | \forall z \in \mathbb{R}^m\},$$

is a big isotropic structure and it is a Dirac structure if the extra maximality condition $\ker B \cap \ker D^t = 0$ holds. Theorem 3.3 also requires that the 1-form $D^t u$ be closed. This requirement forces $D$ to be symmetric. Summing up what have said, we have following corollary of Theorem 3.3.

**Corollary 4.1.** Let $B$ be a constant matrix. If there is a symmetric matrix $D$ such that $DB$ is skew-symmetric then linear system $X = Bu$ is Hamiltonian with respect to Dirac big-isotropic structure $L_{(B,D^t)}(u) := \{(Bz, D^t z) | \forall z \in \mathbb{R}^m\}$ having $H(u) = u^t Du$ as Hamiltonian function. Furthermore, if $B$ is invertible then $X$ is Hamiltonian with respect to constant presymplectic (symplectic if $D$ is invertible as well) structure $\omega^\sharp = DB^{-1}$ and if $D$ is invertible then $X$ is Hamiltonian with respect to constant poisson structure $\pi^\sharp = BD^{-1}$. In both cases with the same Hamiltonian function $H(u) = u^t Du$.

Note that every element $\eta \in \ker B$ which does not belong to the ker $D$ yields a non-trivial linear Casimir $H_\eta$ defined by $dH_\eta = D\eta$ and every element $\xi \in \ker D$ which is not in the ker $B$ yields a non-trivial isotropic vector field $X_\xi = B\xi$.

**Remark 4.2.** Under a linear change of coordinates $u = Tv$ matrices $B$ and $D$ are transformed as following

$$X(v) = T^{-1}BTv, \quad \text{and} \quad H(v) = v^t T^t DTv.$$

Clearly,

- $D$ is symmetric if and only if $T^t DT$ is so.
- $DB$ is skew-symmetric if and only if $T^t DT.T^{-1}BT$ is so.

Then, instead of $B$ we could consider any other member of its conjugacy class.

Let $I_n, 0_{m \times n}$ denote $n \times n$ identity matrix, respectively, $m \times n$ matrix with all its components equal zero. We will omit the dimension subscripts when there is no ambiguity. For real number $\lambda$ and $s_j \neq 1$ we define

$$J_{s_j}(\lambda) := \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{s_j \times s_j},$$

(4.2)
and for the pair of complex numbers $a \pm bi$

$$J_{2s_j}(a \pm bi) = \begin{pmatrix}
B_{(a \pm bi)} & I_2 & \cdots & 0 \\
0 & B_{(a \pm bi)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{(a \pm bi)} \\
0 & 0 & \cdots & 0 & B_{(a \pm bi)}
\end{pmatrix}_{2s_j \times 2s_j} \tag{4.3}
$$

where $B_{(a \pm bi)} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

We use the notation $\text{diag}(A_1, A_2, ..., A_k)$ for a matrix with diagonal blocks $A_1, A_2, ..., A_k$. As it is known:

a) For any real eigenvalue $\lambda$ of a given matrix its Jordan normal form contains a block of the form

$$J_s(\lambda) = \text{diag}(\lambda I_r, J_{s_1}(\lambda), \ldots, J_{s_k}(\lambda))$$

where $J_{s_j}(\lambda)$ is defined at (4.2).

b) For any pair of complex eigenvalues $a \pm bi$ of a given matrix its Jordan normal form contains a block of the form

$$J_{2s}(a \pm bi) = \text{diag}(I_{2r}(a \pm bi), J_{2s_1}(a \pm bi), \ldots, J_{2s_k}(a \pm bi))$$

where $J_{2s_j}(a \pm bi)$ is defined at (4.3) and

$$I_{2r}(a \pm bi) = \text{diag}(B_{(a \pm bi)}, \ldots, B_{(a \pm bi)})$$.

**Remark 4.3.** Even though we put the non-degenerate eigenvalues all together in one block, we will treat each one of them as one dimensional Jordan blocks. Furthermore, by an abuse of notation we, sometimes, use the same symbol for possibly different numbers in the indexes of $J_s$.

Solving the algebraic system of equations to find symmetric matrix $D$ which makes $DB$ skew-symmetric is very cumbersome for general matrix $B$. However, by virtue of Remark 4.2 one could consider matrix $B$ to be of the form

$$\text{diag}(J_s(0), J_{s^+}(\pm \lambda), \ldots, J_{s_k}(\pm bi), \ldots, J_{2s_+}(\pm (a \pm bi)), \ldots)$$

real nonzero eigenvalues  

pure imaginary eigenvalues  

complex eigenvalues with $a \neq 0$
where

\[
\mathbf{J}_s(0) := \text{diag}(\mathbf{0}_{r \times r}, J_{s_1}(0), \ldots, J_{s_{k-1}}(0), J_{s_k}(0)),
\]

\[
\mathbf{J}_{s^r,s^l}(\pm \lambda) := \text{diag}(\lambda \mathbf{I}_r, J_{s_1}(\lambda), \ldots, J_{s_l}(\lambda), -\lambda \mathbf{I}_r, J_{s_1}(-\lambda), \ldots, J_{s_l}(-\lambda))
\]

\[
\mathbf{J}_{s}(\pm bi) := \text{diag}(\mathbb{I}_r(\pm bi), J_{s_1}(\pm bi), \ldots, J_{s_k}(\pm bi), J_{s_k}(\pm bi)),
\]

\[
\mathbf{J}_{s^r,s^l}(\pm (a \pm bi)) := \text{diag}(\mathbb{I}_r(\pm bi), J_{s_1}(a \pm bi), \ldots, J_{s_l}(a \pm bi), \mathbb{I}_r(-\lambda(a \pm bi)), J_{s_1}(-\lambda(a \pm bi)), \ldots, J_{s_l}(-\lambda(a \pm bi)))
\]

In this case, the system of algebraic equations gets simplified significantly. Following Lemma describes the symmetric matrix $D$ that makes $DB$ skew-symmetric.

**Theorem 4.4.** Let $B$ the matrix defined at (4.3) then the symmetric matrix $D$ which makes $DB$ skew-symmetric is of the form

\[
D = \text{diag}(D_s(0), D_{s^r,s^l}(\pm \lambda), \ldots, D_{2s}(\pm bi), \ldots, D_{2s^r,2s^l}(\pm (a \pm bi)), \ldots),
\]

where

i) The matrix $D_s(0)$ is of the form

\[
\begin{pmatrix}
D_r & D_{r,s_1} & D_{r,s_2} & \ldots & D_{r,s_k} \\
D_{r,s_1}^t & D_{s_1} & D_{s_1,s_2} & \ldots & D_{s_1,s_k} \\
D_{r,s_2}^t & D_{s_1,s_2}^t & D_{s_2} & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
D_{r,s_k}^t & D_{s_1,s_k}^t & \ldots & \ldots & D_{s_k}
\end{pmatrix}
\]

(4.10)

where

1) $D_r$ is an arbitrary symmetric matrix.
2) $D_{r,s_i} = \begin{pmatrix} \mathbf{0} \\ d_{r,s_i} \end{pmatrix}$, $r \times s_j$ where $d_{r,s_i}$ is a $r \times 1$ arbitrary vector.
3) $D_{s_j} = [d_{kl}]$ where

\[
d_{rk} = 0 \quad \text{for} \quad k = 1, \ldots, s_j, \quad l = 1, \ldots, (s_j - k)
\]

\[
d_{(s_j - v)s_j} = 0 \quad \text{for} \quad v = 1, 3, 5, \ldots
\]

\[
d_{(s_j - v)(s_j - u)} = -d_{(s_j - v - 1)(s_j - u + 1)} \quad \text{for} \quad v = 0, 1, \ldots, (s_j - 2), \quad u = 1, 2, \ldots, (s_j - v - 1).
\]

For example

\[
D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{42} \\
0 & 0 & -d_{42} & 0 \\
0 & d_{42} & 0 & d_{44} \end{pmatrix}, \quad D_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} \\
0 & 0 & 0 & -d_{15} & 0 \\
0 & 0 & d_{15} & 0 & d_{45} \\
0 & -d_{15} & 0 & -d_{35} & 0 \\
d_{15} & 0 & d_{35} & 0 & d_{55} \end{pmatrix}.
\]
4) For \( s_i \leq s_j \) \( D_{s_i,s_j} = [d_{ki}] \) where

\[
d_{vu} = 0 \quad \text{for } v = 1, \ldots, s_i, \ u = 1, \ldots, (s_j - v)
\]

\[
d_{(s_i-v)(s_j-u)} = -d_{(s_i-v-1)(s_j-u+1)} \quad \text{for } v = 0, 1, \ldots, (s_i - 2), \ u = 1, 2, \ldots, (s_i - v - 1).
\]

For example

\[
D_{4,5} = \begin{pmatrix}
0 & 0 & 0 & 0 & d_{15} \\
0 & 0 & 0 & -d_{15} & d_{25} \\
0 & 0 & d_{15} & -d_{25} & d_{35} \\
0 & -d_{15} & d_{25} & -d_{35} & d_{45}
\end{pmatrix}
\]

ii) The matrix \( D_{s_i,s_j}(\pm \lambda) \) is of the form

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} \\
0 & 0 & \cdots & 0 & D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} \\
D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} & 0 & 0 & \cdots & 0 \\
D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D_{r^+,-} & D_{r^+,-} & \cdots & D_{r^+,-} & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

where

1) \( D_{r^+,-} \) is an arbitrary matrix.

2) \( D_{s_i,s_j}(\pm \lambda) = (0 \ d_{r^+,-} \cdots \ d_{r^+,-}) \) where \( \ast = +,- \) and \( d_{r^+,-} \) is a \( r^+ \times 1 \) arbitrary vector.

3) For \( s_i^+ \leq s_j^- \), \( D_{s_i,s_j} = [d_{ki}] \) where

\[
d_{vu} = 0 \quad \text{for } v = 1, \ldots, s_i^+, \ u = 1, \ldots, (s_j^- - v)
\]

\[
d_{(s_i^+-v)(s_j^-u)} = -d_{(s_i^+-v-1)(s_j^-u+1)} \quad \text{for } v = 0, 1, \ldots, (s_i^+ - 2), \ u = 1, 2, \ldots, (s_i^+ - v - 1).
\]

This is the same matrix as in Item i-4. For \( s_i^+ \geq s_j^- \), one takes the first zero columns and put them at end of matrix as lines.

iii) The matrix \( D_{2s}(\pm bi) \) is of the form

\[
\begin{pmatrix}
D_{2r} & D_{2r,2s_1} & D_{2r,2s_2} & \cdots & D_{2r,2s_k} \\
D_{2r,2s_1} & D_{2s_1} & D_{2s_1,2s_2} & \cdots & D_{2s_1,2s_k} \\
D_{2r,2s_2} & D_{2s_1,2s_2} & \ddots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
D_{2r,2s_k} & D_{2s_1,2s_k} & \cdots & D_{2s_k}
\end{pmatrix}
\]

where

\[1\text{Clearly, we could } s_1 \leq s_2 \leq \ldots \leq s_k.\]
1) \[ D_{2r} = \begin{pmatrix} d_{11}I_2 & B_{(\alpha_{12}+\beta_{12}t)} & \cdots & B_{(\alpha_{1r}+\beta_{1r}t)} \\ B_{(\alpha_{12}+\beta_{12}t)}^t & d_{22}I_2 & \cdots & B_{(\alpha_{2r}+\beta_{2r}t)} \\ \vdots & \vdots & \ddots & \vdots \\ B_{(\alpha_{1r}+\beta_{1r}t)}^t & B_{(\alpha_{2r}+\beta_{2r}t)} & \cdots & d_{rr}I_2 \end{pmatrix} \]
wherein \( B_{(\alpha_{ij}+\beta_{ij}t)} = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{pmatrix} \).

2) \[ D_{2r,2s_i} = \begin{pmatrix} 0 & \cdots & 0 & B_{(\alpha_{1j}+\beta_{1j}t)} \\ 0 & \cdots & 0 & B_{(\alpha_{2j}+\beta_{2j}t)} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{(\alpha_{sj}+\beta_{sj}t)} \end{pmatrix}_{r \times s_j} \]

3) \[ D_{2s_j} = [G_{kl}]_{k,l=1,...,s_j} \text{ such that} \]

\[ G_{ik} = 0_{2 \times 2} \quad \text{for} \quad k = 1, ..., s_j, \quad l = 1, ..., (s_j - k) \]

\[ G_{(s_j-w)s_j} = \alpha_w I_2 \quad \text{for} \quad w = 0, 2, 4, \ldots \]

\[ G_{(s_j-w)s_j} = B_{(\beta_w t)} \quad \text{for} \quad w = 1, 3, 5, \ldots \]

\[ G_{(s_j-v)(s_j-u)} = -G_{(s_j-v-1)(s_j-u+1)} \quad \text{for} \quad v = 0, 1, ..., (s_j - 2), \quad u = 1, 2, ..., (s_j - v - 1). \]

For example

\[ D_{s} = \begin{pmatrix} 0 & 0 & B_{(\beta_j t)} \\ 0 & 0 & -B_{(\beta_j t)} \\ -B_{(\beta_j t)} & \alpha_{j2}I_2 & B_{(\beta_j t)} \end{pmatrix}, \quad D_{10} = \begin{pmatrix} 0 & 0 & 0 & -\alpha_{41}I_2 & B_{(\beta_j t)} \\ 0 & 0 & \alpha_{41}I_2 & -B_{(\beta_j t)} & \alpha_{42}I_2 \\ -\alpha_{41}I_2 & \alpha_{42}I_2 & -B_{(\beta_j t)} & B_{(\beta_j t)} \\ 0 & 0 & \alpha_{41}I_2 & -B_{(\beta_j t)} & \alpha_{42}I_2 \\ \alpha_{41}I_2 & -B_{(\beta_j t)} & \alpha_{42}I_2 & -B_{(\beta_j t)} & \alpha_{42}I_2 \end{pmatrix} \]

4) For \( s_i \leq s_j \)

\[ D_{2s_i,2s_j} = [G_{kl}] \]

\[ G_{ik} = 0_{2 \times 2} \quad \text{for} \quad k = 1, ..., s_i, \quad l = 1, ..., (s_j - k) \]

\[ G_{(s_i-w)s_j} = B_{(\alpha_{w+1}+\beta_{w+1} t)} \quad \text{for} \quad w = 0, 1, 2, 3, \ldots, 4, 5, \ldots \]

\[ G_{(s_i-v)(s_j-u)} = -G_{(s_i-v-1)(s_j-u+1)} \quad \text{for} \quad v = 0, 1, ..., (s_i - 2), \quad u = 1, 2, ..., (s_i - v - 1). \]

For example

\[ D_{s,10} = \begin{pmatrix} 0 & 0 & B_{(\alpha_{i+1}+\beta_{i+1} t)} \\ 0 & 0 & -B_{(\alpha_{i+1}+\beta_{i+1} t)} \\ -B_{(\alpha_{i+1}+\beta_{i+1} t)} & B_{(\alpha_{i+1}+\beta_{i+1} t)} \\ 0 & B_{(\alpha_{i+1}+\beta_{i+1} t)} & -B_{(\alpha_{i+1}+\beta_{i+1} t)} \\ 0 & -B_{(\alpha_{i+1}+\beta_{i+1} t)} & B_{(\alpha_{i+1}+\beta_{i+1} t)} \end{pmatrix} \]

\(^2\)Clearly, we could \( s_1 \leq s_2 \leq \ldots \leq s_k.\)
iv) The matrix $D_{2s_+^*,2s_-^*}(\pm(a \pm bi))$ is of the form

\[
\begin{pmatrix}
0_{2r^+ \times 2r^+} & 0 & \ldots & 0 & D_{2r^+,-} & D_{2r^+,-} & \ldots & D_{2r^+,-} \\
0 & 0_{2s^+_i \times 2s^+_i} & \ldots & 0 & D_{2s^+_i,-} & D_{2s^+_i,-} & \ldots & D_{2s^+_i,-} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0_{2s^+_k \times 2s^+_k} & D_{2s^+_k,-} & D_{2s^+_k,-} & \ldots & D_{2s^+_k,-} \\
D_{2s^+_k,-} & D_{2s^+_k,-} & \ldots & D_{2s^+_k,-} & 0_{2s^+_k \times 2s^+_k} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D_{2s^+_k,-} & D_{2s^+_k,-} & \ldots & D_{2s^+_k,-} & 0 & 0 & \ldots & 0_{2s^+_k \times 2s^+_k} \\
\end{pmatrix}
\]

where

1) $D_{2r^+,-} = \begin{pmatrix} B_{(a_{11}+\beta_{11}i)} & B_{(a_{12}+\beta_{12}i)} & \ldots & B_{(a_{1r^-}+\beta_{1r^-}i)} \\ B_{(a_{21}+\beta_{21}i)} & B_{(a_{22}+\beta_{22}i)} & \ldots & B_{(a_{2r^-}+\beta_{2r^-}i)} \\ \vdots & \vdots & \ddots & \vdots \\ B_{(a_{r^+1}+\beta_{r^+1}i)} & B_{(a_{r^+2}+\beta_{r^+2}i)} & \ldots & B_{(a_{r^+r^-}+\beta_{r^+r^-}i)} \end{pmatrix}$

2) $D_{2s^+_i,-} = \begin{pmatrix} 0 & \ldots & 0 & B_{(a_{11}\pm\beta_{11}i)} \\ 0 & \ldots & 0 & B_{(a_{12}\pm\beta_{12}i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & B_{(a_{r^+2}\pm\beta_{r^+2}i)} \end{pmatrix}_{2r^+ \times 2s^+_i}$

3) For $s^+_i \leq s^-_j$, $D_{2s^+_i,2s^-_j} = [G_{sl}]$ where

$G_{1k} = 0_{2 \times 2}$ for $k = 1, \ldots, s^+_i$, $l = 1, \ldots, (s^-_j - k)$

$G_{(s^+_i - w)s^-_j} = B_{(a_w\pm\beta_{w1})}$ for $w = 0, 1, 2, 3, \ldots, 4, 5, \ldots$

$G_{(s^+_i - v)(s^-_j - u)} = -G_{(s^+_i - v-1)(s^-_j - u+1)}$ for $v = 0, 1, \ldots, (s^+_i - 2)$, $u = 1, 2, \ldots, (s^+_i - v - 1)$.

This is the same matrix as in Item iii-4. For $s^+_i \geq s^-_j$, one takes the first zero columns and put them at end of matrix as lines.

Proof. We first prove that $D$ has the diagonal form [4.19]. In order to do so, we need to discuss following three cases.

1) $B = \text{diag}(J_{s_1}(\lambda_1), J_{s_2}(\lambda_2))$ where $\lambda_1 \neq -\lambda_2$. We write Matrix $D$ in the block form

\[
\begin{pmatrix}
Z_{s_1} & Z \\
Z^t & Z_{s_2} \\
\end{pmatrix}
\]

Now, we need to show that $ZJ_{s_2}(\lambda_2) = -J_{s_1}^t(\lambda_1)Z$. Rewriting this equation, we have

\[
(\lambda_2 + \lambda_1)Z = -(H_{s_1}^tZ + ZH_{s_2}),
\]

(4.14)
where

\[ H_{s_i} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \]  

(4.15)

Multiplying both sides of (4.14) by \((\lambda_2 + \lambda_1)\) and replacing all the term \((\lambda_2 + \lambda_1)Z\) in the right side of obtained equation by \(-\left(H_{s_i}^t Z + Z H_{s_2}\right)\) we get:

\[ (\lambda_2 + \lambda_1)^2 Z = \left(H_{s_1}^t\right)^2 + 2H_{s_1}^t Z H_{s_2} + Z(H_{s_2})^2. \]  

(4.16)

Repeating this process, we obtain for every \(p = 1, 2, 3, \ldots\)

\[ (\lambda_2 + \lambda_1)^p Z = \sum_{q=1}^{p} \binom{p}{q} \left(H_{s_1}^t\right)^q Z H_{s_2}^{p-q}. \]

Choose \(p\) large enough that either \(\left(H_{s_1}^t\right)^q = 0\) or \(H_{s_2}^{p-q} = 0\) for every \(q = 1, \ldots, p\). Then the right hand side of Equation (4.16) is zero, and since \(\lambda_2 + \lambda_1 \neq 0\), we find that \(Z = 0\).

2) \(B = \text{diag}(J_{2s_1}(a_1 + b_1 i), J_{2s_2}(a_2 + b_2 i))\) where \((a_1 + a_2) \neq 0\) or \((b_1^2 - b_2^2) \neq 0\). As in Item (1) we need to show that

\[ Z J 2s_2(a_2 + b_2 i) = - J 2s_1(a_1 + b_1 i) Z. \]  

(4.17)

We divide \(Z\) into \(2 \times 2\) blocks \(Z_{ij}\) where \(i = 1, \ldots, s_1\) and \(j = 1, \ldots, s_2\). An straightforward calculation shows that the equation

\[ Z_{ij} B_{(a_2 + b_2 i)} = - B_{(a_2 + b_2 i)} Z_{ij} \]

has nontrivial solution for \(Z_{ij}\) if and only if \((a_1 + a_2) = 0\) and \((b_1^2 - b_2^2) = 0\) which is not the case here. Now, Equation (4.17) implies that:

\[ Z_{11} B_{(a_2 + b_2 i)} = - B_{(a_2 + b_2 i)} Z_{11}; \]

\[ Z_{11} + Z_{12} B_{(a_2 + b_2 i)} = - B_{(a_2 + b_2 i)} Z_{12}; \]

\[ \vdots \]

\[ Z_{1(s_j-1)} + Z_{1s_j} B_{(a_2 + b_2 i)} = - B_{(a_2 + b_2 i)} Z_{1s_j}. \]

Clearly, \(Z_{11} = Z_{12} = \ldots = Z_{1s_j} = 0\). The first "line" of \(Z\) being zero implies that the second "line" is zero and so on.

3) \(B = \text{diag}(J_{s_1}(\lambda_1), J_{2s_2}(a_2 + b_2))\) where \(b_2 \neq 0\). Similarly, we have to show that

\[ Z J 2s_2(a_2 + b_2 i) = - J 2s_1(\lambda_1) Z. \]  

(4.18)

We divide \(Z\) into \(s_1 \times 2\) blocks \(Z^j, j = 1, \ldots, s_2\). It is very easy to see that equation

\[ Z^j B_{(a_2 + b_2 i)} = - J 2s_1(\lambda_1) Z^j, \]
has nontrivial solution if and only if \((a_1 - \lambda_1)^2 + b_1^2 = 0\) which is not the case here. Equation (4.18) implies
\[
Z^1 B_{(a_2 + b_2)} = -J_{s_1}^i (\lambda_1) Z^1,
\]
\[
Z^1 + Z^2 B_{(a_2 + b_2)} = -J_{s_1}^i (\lambda_1) Z^2,
\]
\[
\vdots
\]
\[
Z^{s_j-1} + Z^{s_j} B_{(a_2 + b_2)} = -J_{s_1}^i (\lambda_1) Z^{s_j}.
\]

Again, it is clear that \(Z = 0\).

The rest of the proof requires cumbersome but strait forward calculations. For any Item we provide an outline of the proof without going into detailed calculations.

i) The block \(D_r\) could be any symmetric matrix since it is associated to zero matrix. Regarding the symmetric diagonal blocks \(D_{s_i}, i = 1, ..., k\) the required equation is \(H_{s_i}^i D_{s_i} = -D_{s_i} H_{s_i}\). Note that the product \(H_{s_i}^i D_{s_i}\) is obtained from \(D_{s_i}\) by shifting all the rows one place downward and filling the first row with zeros. Similarly, \(D_{s_i} H_{s_i}\) is obtained from \(D_{s_i}\) by shifting all the columns one place to the right and filling the first column with zeros. Considering these two facts it is quite strait forward to verify that the matrix \(D_{s_i}\) has the form prescribed in the Lemma. The off-diagonal blocks \(D_{s_i, s_j}\) should satisfy \(H_{s_i}^i D_{s_i} = -D_{s_j} H_{s_j}\). The matrix \(D_{s_i, s_j}\) is not square for \(s_i \neq s_j\) and when \(s_i = s_j\) it is not required to be symmetric. Again, applying the downward and rightward shifts one can verify that \(D_{s_i, s_j}\) has the form stated in the Lemma. The same reasoning applies to \(D_{r, s_i}\).

ii) The required equation here is
\[
(\lambda_2 + \lambda_1) Z = -(H_{s_1}^i Z + Z H_{s_2}),
\]
where \(\lambda_1, \lambda_2 = \pm \lambda\). When \(\lambda_1 = \lambda_2\), by the same reasoning as Theorem 4.4, the block \(Z\) should be zero. When \(\lambda_2 = -\lambda_1\), an other use of the shifting properties of \(H_{s_i}\), through some strait forward calculations, show that \(D_{r, s_i}^*\) and \(D_{s_i, s_j}^*\) have the form given in the Lemma. For \(D_{r, r}\) the dimension of Jordan blocks (associated to non-degenerate eigenvalues) involved are one i.e. \(H_{s_i} = 0\) so it could be arbitrary.

iii) We divide the matrix \(D_{2s}\) into \(2 \times 2\) blocks \(Z_{ij}\). with \(2 \times 2\) blocks. The \(2 \times 2\) blocks of \(D_{2r}\) should satisfy \(Z_{ij} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} Z_{ij}\). Solving this equation, one gets the symmetric blocks \(Z_{ii} = d_{ii} I\) and blocks \(Z_{ij} = B_{(\alpha_{ij} + \beta_{ij})}, i < j\) as claimed.
The first "line" (in the two by two block form) of $D_{2s_i}$ and $D_{2s_{i,2s_j}}$ should satisfy

$$Z_{11} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = - \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} Z_{11},$$

$$Z_{11} + Z_{12} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = - \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} Z_{12},$$

$$\vdots$$

$$Z_{1(s_j-1)} + Z_{1s_j} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = - \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} Z_{1s_j}.$$  

Again solving these equations for symmetric matrix $D_{2s_i}$ and continuing on with other lines give the form of $D_{2s_i}$. The same applies for arbitrary matrix $D_{2s_{i,2s_j}}$ and similar process works for $D_{2r,2s_i}$.

iv) The proof of this Item is similar to Item (3).

The proof of Theorem 4.4 is similar to the proof of [6, Theorem (9.1.1)]. We will state and use a modified version of [6, Theorem (9.1.1)], in Section 5 (Theorem 5.4). Following comments are some outcomes of our analysis.

**Remark 4.5.** For a given vector field $X = Bu$, defined at (4.4), we have:

1) If matrix $B$ has at least one pair of positive-negative eigenvalues $\pm \lambda \neq 0$, real or complex, then there is a Hamiltonian description for $X = Bu$ with quadratic Hamiltonian $H(u) = u^tDu$ and underlying non-trivial Dirac structure where non-trivial means that the skew-symmetric matrix $DB$ is not null. Furthermore, if the representing matrix $B$ is invertible, i.e. the eigenvalue zero is not around, then the underlying structure is a presymplectic one.

2) In the case of eigenvalue zero, presence of a three dimensional Jordan block guarantees a Hamiltonian description with respect to a non-trivial Dirac\big-isotropic structure. In fact, we will show in Lemma 4.6 that a vector field which has only eigenvalue zero becomes Hamiltonian only with respect to either a poisson structure or a proper big-isotropic structure. For the case when both non-zero and zero eigenvalues are present the underlying structure is Dirac or big-isotropic.

3) Let $d_{ij}$ (or $\alpha_{ij}$ or $\beta_{ij}$) be one of the free variables of $D$. We define $D_{ij}$ to be the matrix obtained from matrix $D$ putting $d_{ij} = 1$ and all other free variables equal to zero. Clearly, the function $F_{ij} = u^tD_{ij}u$ is a constant of motion for the vector field $X$ and the set of $\{F_{ij}\}_{ij}$, where $i, j$ run over free variables of $D$, generates all constants of motions obtained by our algorithm. We will use this fact to discuss integrability of the vector field $X$.

4) It is clear that play around with free variables of $D$ yields alternative Hamiltonian description for $X$. 

□
We continue analysing the outcomes a bit more.

**Eigenvalue zero.** Following Lemma shows that in the case of unique eigenvalue zero the underlying structure of the Hamiltonian description is either a poisson structure or a proper big-isotropic one.

**Lemma 4.6.** Matrix $D_s(0)$ in Item (i) of Theorem 4.4 can either be chosen to be invertible or $\ker D_s(0) \cap \ker J_s(0) \neq 0$. It can be chosen invertible if and only if it contains only odd dimensional Jordan blocks $J_s(0)$ or pairs $(J_{s_1}, J_{s_2})$ where $s_j$ is even.

**Proof.** The first block of $D_s(0)$ i.e. $D_r$ is an arbitrary symmetric matrix, so it can be chosen to be invertible. In the rest of the proof we always assume that $D_r$ is invertible. As we mentioned before without losing generality, we may assume that $s_1 \leq s_2 \leq \ldots \leq s_k$. Our proof goes by induction on the number of degenerate Jordan blocks. For $k = 1$ if $J_{s_1}(0)$ is even dimensional then the counter diagonal and all the upper sub-counter diagonals of $D_{s_1}$ are zero. For example for $s_1 = 4$ we have

$$D_s(0) = \begin{pmatrix}
D_r & 0 & 0 & 0 & d_{r,4} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{24} \\
0 & 0 & 0 & -d_{24} & 0 \\
d_{r,4} & 0 & d_{24} & 0 & d_{44}
\end{pmatrix}$$

This means that $D_s(0)$ can not be chosen to be invertible since the column $r + 1$ is always zero. Furthermore,

$$\epsilon_{r+1} \in \ker D_s(0) \cap \ker J_s(0) \neq 0.$$ 

If $s_1$ is odd dimensional, the matrix $D_{s_1}$ can be chosen to be invertible. Simply, set the counter diagonal different than zero and the rest of its elements equal to zero. Setting $d_{r,s_1} = 0$, we get $D_s(0) = \text{diag}(D_r, D_{s_1}(0))$, which means that $D_s(0)$ can be chosen to be invertible. Now assume that Lemma holds when $J_{s_j}(0)$ has $k - 1$ degenerate Jordan blocks. We consider two cases.

1) If $s_k$ is odd dimensional, we chose $D_{s_k}$ to be invertible. Then we write $D_s(0)$ in the following block form

$$\begin{pmatrix}
D_1 & D_2 \\
D_2^t & D_{sk}
\end{pmatrix}.$$ 

For invertible matrix $T = \begin{pmatrix} I_{(s-k)} & 0 \\ -D_{sk}^{-1}D_2^t & I_{sk} \end{pmatrix}$ we have

$$T^t D_s(0) T = \begin{pmatrix}
D_1 - D_2 D_{sk}^{-1} D_2^t & 0 \\
0 & D_{sk}
\end{pmatrix}.$$ 

Note $D_{sk}$ is symmetric so is its inverse. Furthermore, $T^{-1} = \begin{pmatrix} I_{(s-k)} & 0 \\ D_{sk}^{-1}D_2^t & I_{sk} \end{pmatrix}$. 

$$\text{(4.19)}$$
We show that
\[ T^{-1}J_s(0)T = J_s(0). \] (4.20)

Considering \( J_s(0) \) in the block form \( \begin{pmatrix} J_1 & 0 \\ 0 & J_{s_k} \end{pmatrix} \), we only need to show that
\[
D_{s_k}^{-1}D_2^{-1}J_1 - J_{s_k}D_{s_k}^{-1}D_2^t = 0. \tag{4.21}
\]

Two consequences of \( D_s(0)J_s(0) \) being skew-symmetric are \( D_2^{-1}J_1 = -(D_2J_{s_k})^t \) and \( J_{s_k}^tD_{s_k} + D_{s_k}J_{s_k} = 0 \). By the first one, the right hand side of (4.21) is equal to
\[
-(D_{s_k}^{-1}J_{s_k}^t + J_{s_k}D_{s_k}^{-1})D_2^t,
\]
and the second one implies that (4.21) holds. Equations (4.19) and (4.20) together with Remark 4.2 show that doing a change of variable by \( T \) we can restrict ourself to the matrix \( J_1 \) which has \((k-1)\) degenerate Jordan blocks. This in turn proves Lemma by induction.

2) If \( s_k \) is even then \( D_{s_k} \) is singular for sure. Now if \( s_{k-1} \neq s_k \) then the column \( s - s_k + 1 \) of \( D_s(0) \) is zero and
\[
eq 0.
\]

For the case where \( s_k = s_{k-1} \), we write \( D_s(0) \) in the following block form
\[
\begin{pmatrix}
D_1 & D_2 \\
D_1^t & D_4
\end{pmatrix},
\]
where \( D_4 = \begin{pmatrix} D_{s_k-1} & D_{s_k-1,s_k} \\ D_{s_k-1,s_k}^t & D_{s_k} \end{pmatrix} \). The matrix \( D_4 \) can be chosen to be invertible. Simply, set \( D_{s_k-1} = D_{s_k} = 0 \) and the counter diagonal of \( D_{s_k-1,s_k} \) different than zero and the rest of its elements equal to zero. Now, we can repeat a similar change of variable as in Item (1) and restrict ourself to a matrix with \((k-2)\) Jordan blocks which proves Lemma by induction.

\[ \square \]

Remark 4.7. Since any Dirac structure restricted to one point is of the form we discussed here, Lemma 4.6 yields examples of vector fields (linear ones) that have Hamiltonian description only with respect to a proper big-isotropic structures.

There is a result about eigenvalue zero in [7] and repeated in [3, Theorem 4.2]. An example which is discussed in [7] is the vector field \( X = Bu \) where \( B = \text{diag}(G, G, ..., G) \) with \( G = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \). It is shown there that \( X \) is Hamiltonian.
with respect to symplectic structure

\[
\Omega = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & I_2 \\
\cdots & \cdots & \cdots & -I_2 & 0 \\
\cdots & I_2 & 0 & \cdots \\
\cdots & -I_2 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

and Hamiltonian given by symmetric matrix \(\tilde{\Omega}\tilde{B}\). The matrix \(\tilde{B}\) is not in the Jordan canonical form. To do a comparison, we consider \(\tilde{B} = G\) i.e. with only one block. The Hamiltonian given by \(\Omega \tilde{B}\) is then \(H_1(u) = -\frac{1}{2}(u_3^2 + u_4^2)\). To apply our approach, we do the change of variable \(u = Tv\) with \(T = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}\). Then

\[
X(v) = (T^{-1}.G.T) v = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} v,
\]

where \(B = \text{diag}(J_2(0), J_2(0))\). By Item (i) of Theorem 4.4 we have:

\[
D = \begin{pmatrix}
0 & 0 & 0 & d_{14} \\
0 & d_{22} & -d_{14} & d_{24} \\
0 & -d_{14} & 0 & 0 \\
d_{14} & d_{24} & 0 & d_{44}
\end{pmatrix}
\]

If \(d_{14} \neq 0\) then \(D\) is invertible and \(X(v)\) is Hamiltonian with respect to Poisson structure \(B.D^{-1}\) and Hamiltonian function

\[
H(v) = \frac{1}{2}(v^t Dv) = \frac{1}{2}(d_{22}v_2^2 + d_{44}v_4^2 + 2d_{14}(v_1v_4 - v_2v_3)).
\]

Linear Casimirs given by the element of the ker \(B\) are \(v_2(= u_4)\) and \(v_4(= u_3)\). Interestingly enough that \(H_1(u)\) is a Casimir in our setting, i.e. in [7], the vector field \(\tilde{B}u\) is paired with a Casimir trough a symplectic structure. As mentioned there, this is only possible when zero eigenvalues have even multiplicity i.e. Jordan blocks of the eigenvalue zero come in pairs. Our approach works for any dimension and any type of Jordan blocks, detecting Casimirs. For the example above, one may cut off Casimirs and consider the Hamiltonian \(H_{\text{red}}(v) = d_{14}(v_1v_4 - v_2v_3)\).
Non-zero Eigenvalues. If $B$ is invertible i.e zero is not an eigenvalue then the underlying structure for Hamiltonian description is the presymplectic structure $\omega^\# = D^T B^{-1}$. If the dimension of $B$ is odd then it is not possible to have symplectic structure. In even dimension it is clear that if all eigenvalues of $B$ come in pairs $\pm \lambda_j$ or quadruples $\pm(a_j \pm b_j i)$ with the same dimensional Jordan blocks i.e. $s_j^+ = s_j^-$, $\forall j$ then the underlying structure could be chosen to be symplectic structure. It is very cumbersome to check directly that the structure can not be chosen symplectic otherwise. However, it is a conclusion of the known fact that for a Hamiltonian vector field $X$, in the context of symplectic geometry, the Jordan block belonging to a real or complex eigenvalue $\lambda$ has the same structure as the Jordan block belonging to $-\lambda$, see [7, page 453] and reference therein. One example in even dimension that can not have symplectic structure as underlying structure is the vector field

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2} (w + z) \frac{\partial}{\partial z} + \frac{1}{2} (z - 3w) \frac{\partial}{\partial w}. \tag{4.22}$$

This example is taken from [7] where it is been used as an example which is not Hamiltonian with respect to any constant symplectic structure. Diagonalizing the representing matrix we have

$$X(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} v.$$

By Item $(ii)$ of Theorem 4.4 we have

$$D = \begin{pmatrix} 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & 0 \\ d_{14} & d_{24} & 0 & 0 \end{pmatrix}.$$

The matrix $D$ is clearly singular and the presymplectic structure is

$$DB^{-1} = \begin{pmatrix} 0 & 0 & 0 & -d_{14} \\ 0 & 0 & 0 & -d_{24} \\ 0 & 0 & 0 & 0 \\ d_{14} & d_{24} & 0 & 0 \end{pmatrix}.$$

Kernel of $D$ is generated by $\xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} d_{24} \\ d_{14} \\ 1 \\ 0 \end{pmatrix}$, then

$$X_{\xi_1} = (0, 0, -1, 0) \quad \text{and} \quad X_{\xi_2} = (-d_{24}/d_{14}, 1),$$

are the isotropic vector fields.
5. Integrability of Linear Systems

In this Section, we discuss integrability of linear Hamiltonian vector fields. We provide a short introduction to integrable systems, in general, following [11].

Definition 5.1. Let $M$ be an $m$-dimensional manifold and $p \geq 1, q \geq 0$ such that $p + q = m$. A $m$ tuple $(X_1, \ldots, X_p, F_1, \ldots, F_q)$, where $X_i \in \mathfrak{X}(M)$ and $F_j \in C^\infty(M)$, is called an integrable system of type $(p, q)$ on $M$ if it satisfies the following conditions:

i) $[X_i, X_j] = 0$ \quad $\forall i, j = 1, \ldots, p$,

ii) $X_i(F_j) = 0$ \quad $\forall i \leq p, j \leq q$,

iii) $X_1 \wedge \ldots \wedge X_p \neq 0$ and $dF_1 \wedge \ldots \wedge dF_q \neq 0$ almost everywhere on $M$.

A vector field $X$ on a manifold $M$ is called integrable if there is an integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ of some type $(p, q)$ on $M$ with $X_1 = X$.

The system defined above is called regular on a level set $N$ of first integrals, i.e. a level set of the map $(F_1, \ldots, F_q) : M \to \mathbb{R}^q$, if conditions (i) and (ii) hold everywhere on $N$. An integrable system in the sense of Definition 5.1 has action-angle variables (also known as Liouville system of coordinates) around any compact regular level set $N$, see [11, Theorem 2.1].

Definition 5.1 ignores the geometric structure underlying Hamiltonian system and only considers commuting flows and first integrals, for that reason it is also called non-Hamiltonian integrability. The linear vector fields we discussed in Section 4 are Hamiltonian with respect to presymplectic, poisson, Dirac or proper big-isotropic structures. We provide integrability definitions taking these structures in account as well. The additional requirement is that the underlying structure should be preserved by the commuting flows.

Definition 5.2. An integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ is called Hamiltonian integrable on a manifold $M$ equipped with one of presymplectic, poisson, Dirac or proper big-isotropic structures if there exist $H_1, \ldots, H_q \in C^\infty$ such that for any $i = 1, \ldots, q$ the vector field $X_i$ is Hamiltonian with respect to the geometric structure that $M$ is equipped with, having $H_i$ as Hamiltonian, i.e. for any $i = 1, \ldots, q$:

- On a presymplectic manifold $(M, \omega)$: $dH_i = \omega^\#(X_i)$.
- On a poisson manifold $(M, \pi)$: $X_i = \pi^\#(dH_i)$.
- On Dirac\big-isotropic manifold $(M, L)$: $(X_i, dH_i) \in L$.

A vector field $X$ on a manifold $M$ is called Hamiltonian integrable if there is a Hamiltonian integrable system $(X_1, \ldots, X_p, F_1, \ldots, F_q)$ of some type $(p, q)$ on $M$ with $X_1 = X$.

Interested readers are referred to [11] for more details on action-angle variables for presymplectic, poisson and Dirac structures. For big-isotropic structures, we are not aware of any work on action-angle variable or even on integrable systems on them. Regarding Hamiltonian integrable systems on big-isotropic manifolds we
use the same definition as Dirac one. We also believe that the results of [11] can be proved easily for big-isotropic structure as well but it is beyond the scope of this work.

**Definition 5.3.** We will refer to

1) A matrix $T_{m \times n}$ as upper triangular Toeplitz matrix associated to real eigenvalues if for $m = n$, it is upper triangular with the same values along bands i.e.

$$T_m = \begin{pmatrix} c_0 & c_1 & \ldots & c_{m-1} \\ 0 & c_0 & \ldots & c_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_0 \end{pmatrix},$$

for $m < n$: $T_{m \times n} = (0 \ T_m)$ and for $m > n$: $T_{m \times n} = \begin{pmatrix} T_m \\ 0 \end{pmatrix}$.

For Toeplitz matrix $T_m$, we also will use the notation $\sum_{i=0}^{m-1} c_i H_m(i)$ where $H_m(i)$ is the $m \times m$ matrix with ones on the $i^{th}$ diagonal and zeros elsewhere i.e. $H_m(0) = I_m$, $H_m(1) = H_m$ as in (4.15) and so on.

2) A matrix $T_{2m \times 2n}$ as upper triangular Toeplitz matrix associated to complex eigenvalues if for $m = n$ it is of the form

$$T_{2m} = \begin{pmatrix} C_1 & C_2 & \ldots & C_m \\ 0 & C_1 & \ldots & C_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_1 \end{pmatrix},$$

where $C_l = \begin{pmatrix} \alpha_l & \beta_i \\ -\alpha_l & \beta_i \end{pmatrix}$, $l = 1, \ldots, m$. For $m < n$: $T_{2m \times 2n} = (0 \ T_{2m})$. and for $m > n$: $T_{2m \times 2n} = \begin{pmatrix} T_{2m} \\ 0 \end{pmatrix}$.

Following Theorem is a modified version of Theorem (9.1.1) of [6]. We did a slight modification to that Theorem to be able to use for the Jordan blocks, defined at (4.7) and (4.8), associated to complex eigenvalues. The proof of the Theorem is essentially same as the one given at [6]. It only requires a slight modification.

**Theorem 5.4.** Let $B$ the matrix defined at (4.4) and $C = (C_{\alpha \beta})_{\alpha, \beta}$ be an $n \times n$ matrix where $\alpha$, $\beta$ run over the set of Jordan blocks $J_s(\xi)$ where

$$\xi \in \{0, \lambda_1, -\lambda_1, \ldots, \pm b_1 i, \pm b_2 i, \ldots, (a_1 \pm b_1 i), -(a_1 \pm b_1 i), \ldots\},$$

and every non-degenerate eigenvalue is considered as a one dimensional Jordan block. Then $C$ commutes with $B$ if and only if $C_{J_{\alpha}(\xi_1) J_{\beta}(\xi_2)} = 0$ when $\xi_1 \neq \xi_2$ and $C_{J_{\alpha}(\xi) J_{\beta}(\xi)}$ is

- a Toeplitz matrix as defined in the item (1) of Definition 5.3 when $\xi$ is a real eigenvalue,
- a Toeplitz matrix as defined in the item (2) of Definition 5.3 when \( \xi \) is a conjugate pair of complex eigenvalues.

**Proof.** The proof of this Theorem is very similar to the one of Theorem 4.4 with minor difference that \( C \) is not required to be symmetric. Considering appropriate block form for \( C \), every \( Z_{rl} \) should satisfy an equations of the form

\[
Z_{rl} J_{s_i}(\xi_2) = J_{s_i}(\xi_1) Z_{rl}.
\]

For the rest of the proof one uses nilpotent and row-column shifting properties of \( H_{s_i} \) and the fact that equation

\[
Z_{rl} B_{(a_2 + b_2)} = B_{(a_2 + b_2)} Z_{rl},
\]

has trivial solution if \( (a_2 + b_2) \neq (a_1 + b_1) \) and solutions of the form \( B_{(a_1 + \beta, r_1 \ell)} \) otherwise.

We now proceed to discuss integrability of linear Hamiltonian systems. As in Section 4 we consider the linear system \( X = Bu \) where \( B \) is in Jordan canonical form \( 4.4 \). We will take the constant of motions obtained from \( D \), see Remark 4.5 as first integrals and will look for commuting flows that preserve these quantities. We start with the presymplectic case i.e. when \( B \) is invertible.

**Lemma 5.5.** For matrix \( B \), invertible, let \( F_{i_1 r_1} = u^t D_{i_1 r_1} u, \ldots, F_{i_q r_q} = u^t D_{i_q r_q} u \), be \( q \geq 0 \) almost everywhere linearly independent constants of motion obtained from \( D \) as mentioned in Item (3) of Remark 4.5. Setting \( C_1 := B \), if there exists matrices \( C_1, \ldots, C_p \) such that \( p + q = m \) and

1) For the vector fields \( X_j = C_j u, j = 1, \ldots, p \), one has \( X_1 \wedge \ldots \wedge X_p \neq 0 \) almost everywhere on \( \mathbb{R}^m \).

2) \([C_i, C_j] = C_j C_i - C_i C_j = 0, \forall i, j = 1, \ldots, p.\]

3) \( D_{i r} C_i \) is skew-symmetric for every \( i = 1, \ldots, q \) and \( j = 1, \ldots, p \).

Then \( X \) is Hamiltonian integrable on \( \mathbb{R}^m \) equipped with presymplectic structure \( \omega^\#_0 = D_0 B^{-1} \) where \( D_0 = \sum_{j=1}^q D_{i_j r_j} \). Furthermore, for each \( i = 2, \ldots, p \) the vector field \( X_i = C_i u \) is Hamiltonian with respect to presymplectic form \( D_0 B^{-1} \) having function \( H_i \) defined by \( dH_i = (D_0 B^{-1}) C_i u \) as its Hamiltonian function. Equivalently, the pairs \( (D^{-1} B C_i u, D_0^{-1} B^{-1} C_i u) \) are Hamiltonian with respect to Dirac structure \( L_{(B, D_0)} \).

**Proof.** Note that \( [X_i, X_j] = 0 \) is equivalent to \([C_i, C_j] = 0\). Furthermore, assumption (3) of the theorem yields \( X_i(F_j) = 0 \) for every \( i, j \). So \( (X_1, \ldots, X_p, F_1, \ldots, F_q) \) is an integrable system in the sense of Definition 5.1. In order to prove that it is Hamiltonian integrable, according to Definition 5.2 we only need to show that the matrix \((D_0 B^{-1}) C_i \) is symmetric for \( H_i \) to be well-defined. Following equation finishes the proof.

\[
(D_0 B^{-1} C_i)^t = C_i^t (D_0 B^{-1})^t = -C_i^t D_0 B^{-1} = (-1)^2 D_0 C_i B^{-1} = D_0 B^{-1} C_i,
\]
where we used the facts that $D_0 B^{-1}$ and $D_j C_i$ are skew-symmetric, $D_0 = \sum_{j=1}^q D_j$ and 

$$[B, C_i] = 0 \iff [B^{-1}, C_i] = 0.$$ 

\[\square\]

**Remark 5.6.** Note that

$$\omega_0(X_i, X_j) = u^t C_i D_0 B^{-1} C_j u = -u^t D_0 C_j B^{-1} C_i u = -u^t D_0 B^{-1} C_j C_i u$$

$$= -\mathcal{L}_{X_i}(H_j) = -\frac{1}{2} u^t (D_0 B^{-1} C_j C_i + (D_0 B^{-1} C_j C_i)^\dagger) u$$

$$= \frac{1}{2} u^t (D_0 B^{-1} C_j C_i + C_i^t C_j B^{-1} D_0) u = \frac{1}{2} u^t (D_0 B^{-1} C_j C_i + C_i^t D_0 B^{-1} C_j) u$$

$$= \frac{1}{2} u^t (D_0 B^{-1} C_j C_i - C_i^t B^{-1} D_0 C_j) u = \frac{1}{2} u^t (D_0 B^{-1} (C_j C_i - C_i C_j)) u = 0.$$ 

This means that the vector fields $X_i, i = 1, ..., p$ generate an isotropic subspace with respect to $\omega_0$, so $p \leq \text{Rank}(\omega_0) + \text{dim(\ker \omega_0)}$. It also shows that any function $H_i, i = 1, ..., p$ is also a constant along every $X_j, j = 1, ..., p$.

**Theorem 5.7.** Every Hamiltonian linear system $X = Bu$ where $B$ is invertible, is Hamiltonian integrable with respect to a Dirac structure $L_{(B,D_0)}$ and, equivalently, with respect to presymplectic structure $\omega = D_0 B^{-1}$, (in the sense of Definition 5.2).

**Proof.** We decompose $B$ into blocks of types $J_{s^+, s^-}(\lambda \pm a bi)$, $J_{2a^+, 2b^-}(\pm (a \pm bi))$, see [4.6], and similar notation for $J_{s^+, 2b^-}(a bi)$ where $-z$ is not an eigenvalue. Then prove the result for each block. Our system is the direct product of these systems and clearly integrable if all of them are so. We start with block 

$$J_{s^+, s^-}(\pm \lambda) = \text{diag}(\lambda I_{r^+}, J_{s^+}^{r^+}(-\lambda), ..., J_{s^+}^{r^+}(-\lambda), ..., J_{s^+}^{r^+}(-\lambda)).$$

Without any loss of generality we assume that $r^+ + k \leq r^- + l$. For every $i = 1, ..., r^+$, we consider the $i^{th}$ non-degenerate eigenvalue $\lambda$ as a one dimensional Jordan block and denote it by $J_{n_i^+}^{s^+}(\lambda)$ and similar notation $J_{n_i^-}^{s^-}(-\lambda)$ for the $j^{th}$ non-degenerate eigenvalue $-\lambda$. Then we pair each block of the set 

$$J^{\lambda} := \{J_{n_i^+}^{r^+}(\lambda), ..., J_{n_i^-}^{r^-}(\lambda), J_{s^+}^{s^+}(\lambda), ..., J_{s^+}^{s^+}(\lambda)\}$$

with an element of the set 

$$J^{-\lambda} := \{J_{n_i^-}^{s^-}(-\lambda), ..., J_{n_i^-}^{s^-}(-\lambda), J_{s^+}^{s^-}(-\lambda), ..., J_{s^+}^{s^-}(-\lambda)\}$$

in a way that minimizes the sum of the differences between dimensions of the pairs. We leave the extra blocks of eigenvalue $-\lambda$ alone. In order to use Lemma 5.3 for every pair $(J_{\xi}(\lambda), J_{\xi}(-\lambda))$ we pick constants of motion $F_1, ..., F_{r(\xi, \lambda)}$ associated to
free of
\[
D_{\xi,\chi} := \begin{cases}
\text{element } d_{ij} \text{ of } D_{r_i r_j}^{-} & \text{if } \xi = n^+_i, \chi = n^-_j, \\
\text{element } d_{is_j} \text{ of } D_{r_i s_j}^{-} & \text{if } \xi = n^+_i, \chi = s^-_j, \\
\text{element } d_{js_i^*} \text{ of } D_{r_i s_i^*}^{-} & \text{if } \xi = s^+_i, \chi = n^-_j, \\
\text{the block } D_{s_i^* s_j^*}^{-} & \text{if } \chi = s^+_i, \chi = s^-_j,
\end{cases}
\]

By the form of the matrix \( D_\lambda(\lambda) \) it is clear that \( q(\xi,\chi) = \min\{\xi, \chi\} \), and these functions are independent. Furthermore, putting all these functions together yields \( q = \sum_{\nu=1}^{r-1} q(\xi,\chi) \)-independent constants of motion. Our choice of submatrices of \( D_{\xi,\chi} \), makes it possible to decompose \( J \cdot (-\lambda) \) into blocks associated to pairs \( (J_{\xi}(\lambda), J_{\chi}(\lambda)) \) and the ones that are left alone (with associated submatrix zero to the ones left alone). We show that these blocks are integrable which consequently implies that vector field \( J \cdot (-\lambda)u \) is so.

For a pair \( (J_{\xi}(\lambda), J_{\chi}(\lambda)) \), let \( p(\xi,\chi) = \max\{\xi, \chi\} \). For a moment, we assume that \( p(\xi,\chi) = \chi \), the other case is similar. Now, let \( D_{\nu r_i}, i = 1, \ldots, q(\xi,\chi) \) be the sub-matrices of \( D_0 = \begin{pmatrix} 0 & D_{\xi,\chi}^{-} \\ D_{\xi,\chi}^{t} & 0 \end{pmatrix} \). Then the matrix \( C = \text{diag}(T_{\xi}, T_{\chi}) \) where

\[
T_{\xi} = \sum_{i=0}^{\xi-1} c_{i+1} H_{\xi}(i),
\]

and \( c_0, \ldots, c_{\chi-1} \) are arbitrary numbers, satisfies the condition (3) of Lemma 5.5 i.e. \( D_{\nu r_i} C \) is skew-symmetric for every \( i = 1, \ldots, q(\xi,\chi) \). By Theorem 5.4 it also commutes with \( B = \text{diag}(J_{\xi}(\lambda), J_{\chi}(\lambda)) \). Now, we define \( E_j, j = 1, \ldots, p(\xi,\chi) - 1 \) to be the matrix obtained from \( C \) by setting \( e_j = 1 \) and the rest of free variables equal to zero. Note that \( \text{diag}(J_{\xi}(\lambda), J_{\chi}(\lambda)) = \lambda E_0 + E_1 \). The matrices

\[
C_1 = \lambda E_0 + E_1, C_2 = E_1, \ldots, C_{p(\xi,\chi)} = E_{p(\xi,\chi) - 1}
\]

satisfy assumptions of Lemma 5.5 i.e. every pair \( (J_{\xi}(\lambda), J_{\chi}(\lambda)) \) is integrable. For a left alone Jordan block \( J_{\chi}(\lambda) \) since the associated matrix \( D_0 \) is zero, the Toeplitz matrix \( T_{\chi} \) yields, in the same manner as above, matrices \( C_1, \ldots, C_{\chi} \) that satisfy assumptions of Lemma 5.5.

The proof of integrability for block \( J_{2a^*,2a^-}(\pm(a \pm bi)) \) is exactly the same as block \( J_{a^*,a^-}(\pm\lambda) \) with only difference that we treat the \((2 \times 2)\) blocks \( B_{(a+bi)} \) like a number. Note that \((2 \times 2)\) matrices of this type commute with each other. We are only left to proof integrability for the block

\[
J_a(\pm bi) := \text{diag}(I_2, J_{2a}(\pm bi), \ldots, J_{2a}(\pm bi), J_{si}(\pm bi)).
\]

For this case we choose the constants of motion to be generated by the free components of \( D_0 = \text{diag}(D_{2r}^0, D_{2a_1}^0, \ldots, D_{2a_k}^0) \) where \( D_{2r}^0 = \text{diag}(d_{11} I_2, \ldots, d_{rr} I_2) \) and...
$D^0_{2s_j} = [G_{kl}]_{k,l=1,...,s_j}$ such that

$G_{ik} = 0_{2 \times 2}$ for $k = 1, ..., s_j$, $l = 1, ..., (s_j - k)$

$G_{(s_j-w)s_j} = 0_{2 \times 2}$ for $w = 0, 2, 4, ...$

$G_{(s_j-w)s_j} = B(\beta_w)$ for $w = 1, 3, 5, ...$

$G_{(s_j-v)(s_j-u)} = -G_{(s_j-v-1)(s_j-u+1)}$ for $v = 0, 1, ..., (s_j - 2)$, $u = 1, 2, ..., (s_j - v - 1)$.

For example

$D^0_s = \begin{pmatrix} 0 & 0 & 0 & B(\beta_0) \\ 0 & B(\beta_0) & -B(\beta_0) & 0 \\ -B(\beta_0) & 0 & B(\beta_0) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $D^0_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B(\beta_0) & 0 \\ 0 & 0 & 0 & -B(\beta_0) & 0 \\ 0 & -B(\beta_0) & 0 & B(\beta_0) & 0 \\ -B(\beta_0) & 0 & -B(\beta_0) & 0 & 0 \end{pmatrix}$.

Let $C = \text{diag}(C_{2r}, C_{2s_1}, ..., C_{2s_2})$ where $C_{2r} = \text{diag}(B(\beta_{2i}), ..., B(\beta_{2i}))$ and $C_{2s_j} = B(\beta_{2i})H_{2s_j}(0) + B(\beta_{2i})H_{2s_j}(2) + ...$, for example

$D^0_s = \begin{pmatrix} B(\beta_{2i}) & 0 & 0 & B(\beta_{2i}) \\ 0 & B(\beta_{2i}) & 0 & B(\beta_{2i}) \\ 0 & 0 & B(\beta_{2i}) & B(\beta_{2i}) \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $D^0_{10} = \begin{pmatrix} B(\beta_{2i}) & 0 & 0 & 0 \\ 0 & B(\beta_{2i}) & 0 & B(\beta_{2i}) \\ 0 & 0 & B(\beta_{2i}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It is very easy to see that $D_0C$ is anti-symmetric. $D_0$ yields required constants of motion and $C$ yields required vector fields for $J_s(2bi)u$ to be integrable.

Finally, for the blocks $J_s(z)$ where $-z$ is not an eigenvalue, we may use Theorem 5.4 to get $s$ commuting independent vector fields $C_1, ..., C_s$ with $C_1 = J_s(z)$. This finishes the proof of the Theorem. \[\square\]

**Remark 5.8.** For matrix $B = \text{diag}(\lambda I_r, -\lambda I_r)$, using the $(r^+ + r^- - 1)$ almost linearly independent constants of motion

$F := \{(F_{1,r^++1} = u^tD_{1,r^++1}u)_{l=1,...,r^-}, (F_{j,r^++1} = u^tD_{j,r^++1}u)_{j=2,...,r^+}\}$,

one gets the Hamiltonian integrable system $(Bu, F)$. This shows that for a given linear system $X = Bu$, one may get more than one Hamiltonian integrable system.

**Theorem 5.9.** Every linear system of the form $X = Bu$ where

$B = (0_{r \times r}, J_{2k_1+1}(0), ..., J_{2k_1+1}(0), (J_{2l_1}(0), J_{2l_1}(0)), ..., (J_{2l_f}(0), J_{2l_f}(0)))$.

is Hamiltonian integrable with respect to a Dirac structure $L(B, D_0)$ where $D_0$ is invertible. In other word, it is Hamiltonian integrable with respect to the Poisson structure $\pi^# = BD_0^{-1}$.

**Proof.** In our choice of $D_0$, we set all components related to interaction among these (pairs of) blocks to be zero i.e. consider it of the form

$\text{diag}(D^0_1, D^0_{2k_1+1}, ..., D^0_{2k_1+1}, D^0_{p1}, ..., D^0_{pf}).$
where
\[
D^0_{pi} = \begin{pmatrix}
D^0_{2i} & D^0_{2i,2t_i} \\
(D^0_{2i,2t_i})^t & D^0_{2t_i}
\end{pmatrix}.
\]

This way, we will be able to discuss them separately. The first block 0_{0x0} generates no dynamics. Setting \(D^r_r = I_r\), we get \(r\) independent constants of motion. This constants of motions are actually Casimirs.

In the rest of the proof we use the fact that \(J_s(0) = H_s(1)\) is nilpotent, see Definition 5.3. For the odd dimensional block \(J_{2k_i+1}\) we set
\[
D^0_{2k_i+1} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & -1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

Now, Let
\[
X_1 = J_{2k_i+1}u, \quad X_2 = (J_{2k_i})^3u, \ldots, X_{k_i} = (J_{2k_i+1})^{(2k_i-1)}u,
\]
and
\[
F_1 = \frac{1}{2} u^t D^0_{2k_i+1}u, \quad F_2 = \frac{1}{2} u^t D^0_{2k_i+1}(J_{2k_i+1})^2u, \ldots,
\]
\[
F_{k_i} = \frac{1}{2} u^t D^0_{2k_i+1}(J_{2k_i+1})^{(2k_i-2)}u, \quad F_{k_i+1} = \frac{1}{2} u^t D^0_{2k_i+1}(J_{2k_i+1})^{2k_i}u.
\]

It is very easy to check that \((X_1, \ldots, X_{k_i}, F_1, \ldots, F_{k_i+1})\) is an integrable system in the sense of Definition 5.1. Furthermore, \((X_j, dF_j) \in L(J_{2k_i+1}, D^0_{2k_i+1})\) for \(j = 1, \ldots, k_i\) and \((0, dF_{k_i+1}) \in L(J_{2k_i+1}, D^0_{2k_i+1})\) i.e. this system is Hamiltonian integrable as well. In this setting, the function \(F_{k_i+1} = u^2_{2k_i+1}\) is a Casimir of Dirac structure \(L(J_{2k_i+1}, D^0_{2k_i+1})\) and the equivalent poisson structure.

For pair of even dimensional block \((J_{2t_i}, J_{2t_i})\) we set
\[
D^0_{pi} = \begin{pmatrix}
0 & D^0_{2i,2t_i} \\
(D^0_{2i,2t_i})^t & 0
\end{pmatrix},
\]
where
\[
D^0_{2i,2t_i} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & -1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

Again, by a strait forward calculation one can see that the vector fields
\[
X_1 = (J_{2t_i}, J_{2t_i})u, \quad X_2 = ((J_{2t_i})^3, J_{2t_i})u, \ldots, X_{t_i} = ((J_{2t_i+1})^{(2t_i-1)}, J_{2t_i})u,
\]
\[
X_{t_i+1} = (J_{2t_i}, (J_{2t_i})^3)u, \quad X_{(t_i+2)} = (J_{2t_i}, (J_{2t_i})^5)u, \ldots, X_{(2t_i-1)} = (J_{2t_i}, (J_{2t_i+1})^{(2t_i-1)})u,
\]
and functions
\[ F_1 = \frac{1}{2} u^t D_{pt}^0 u, \quad F_2 = u^t D_{pt}^0 ((J_{2t})^2, 0) u, \quad \ldots, \quad F_t = \frac{1}{2} u^t D_{pt}^0 ((J_{2t})^{2(t-2)}, 0) u, \]
\[ F_{t+1} = \frac{1}{2} u^t D_{pt}^0 (0, (J_{2k+1})^2) u, \quad \ldots, \quad F_{2t-1} = \frac{1}{2} u^t D_{pt}^0 (0, (J_{2k})^{2k-2}) u \]
constitute an integrable system. For the last two functions, note that \( dF_{2t_i} (J_{2t_i}, J_{2t_i}) = dF_{2t_i+1} (J_{2t_i}, J_{2t_i}) = 0. \)

For every \( i = 1, \ldots, 2t_i - 1 \) we have
\[ (X_i, dF_i) \in L_{((J_{2t_i}, J_{2t_i}), D_{pt}^0)}. \]
Furthermore, since \( dF_{2t_i} = D_{pt}^0 (-e_{(2i+1)}) \), \( dF_{2t_i+1} = D_{pt}^0 e_1 \) and both \( e_{(2i+1)}, e_1 \) are in the kernel of \( (J_{2t_i}, J_{2t_i}) \), we have \( (0, dF_{2t_i}), (0, dF_{2t_i+1}) \in L_{((J_{2t_i}, J_{2t_i}), D_{pt}^0)}. \) This means that the integrable system we provided for \( (J_{2t_i}, J_{2t_i}) u \) is Hamiltonian integrable as well. The functions \( F_{2t_i}, F_{2t_i+1} \) are Casimirs.

**Theorem 5.10.** Every linear system of the form \( X = J_{2t}(0) u \) is Hamiltonian integrable with respect to a proper big-isotropic structure \( L_{(B, D_0)}. \)

**Proof.** We set
\[ D_{2t}^0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{pmatrix}. \]  
(5.1)

Then
\[ X_1 = J_{2t} u, \quad X_2 = (J_{2t})^3 u, \ldots, \quad X_t = (J_{2t})^{(2t-1)} u, \\]
and
\[ F_1 = \frac{1}{2} u^t D_{2t}^0 u, \quad F_2 = \frac{1}{2} u^t D_{2t}^0 (J_{2t})^2 u, \ldots, \quad F_{k_i} = \frac{1}{2} u^t D_{2t}^0 (J_{2t})^{(2(t-1)-2)} u, \]
constitute a Hamiltonian integrable system with respect to proper big-isotropic structure \( L_{(J_{2t}(0), D_{2t}^0)}. \) □

Putting Theorems 5.7, 5.9 and 5.10 all together, we have proved:

**Theorem 5.11.** Every Hamiltonian linear system is Hamiltonian integrable with respect to at least one Dirac\( \setminus \) big-isotropic structure.

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