A Mathematical Model for Inventory and Price-Dependent Demand with All-Units Discount

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Abstract. Inventory becomes one of the factors a company especially a retailer must consider. A good inventory management will guarantee a retailer in running the business. However, inventory management is a difficult task since it is affected by many factors such as the demand fluctuation, deterioration, discount, sales price, uncertain delivery from the supplier and any other factors. Some factors that affect the consumers demand behavior is the quantity of displayed (available) goods and the sales price. When the customers see that there are still a lot of goods available, there is a perception that the goods available is new and fresh, therefore this will induce demand compared to the least available goods on display. On the other aspect, discount offered by the supplier can be exercised to minimize the total inventory cost by ordering more goods. This paper deals with inventory model where demand depends on the inventory level and sales price and also considering all-units discount from the supplier. From the model, we will find the optimum order quantity and sales price per unit that minimize the total inventory cost.

Keywords: inventory model, inventory-dependent demand, sales price, all-units discount

1. Introduction

Economic Order Quantity (EOQ) is the simplest inventory model since it is assumed that demand rate is constant from time to time. In reality demands are not constant and they are depended on some factors such as the quantity ordered, and the sales price. For example, in the supermarket when there are still many goods available (on display), consumer demands will be higher compared to a condition where there are only a few available goods on display. Customers have perception that a lot of goods available on display related to the freshness of the goods and they are willing to buy them. Less available goods on display will give an impression that the goods are not fresh and unsold. Levin, et al. (1972) have analysed that a lot of goods displayed on the shelves will influence the customer to buy more. This fact is also found by Silver and Peterson (1985) that states the amount of sales corresponds to the amount of goods displayed or owned by the company. There are a lot of research in this area analysed from different angles. Nagare and Dutta (2012) have developed an inventory model for deteriorated item with inventory-dependent demand feature, while Clarabella (2016) modified the model by adding discount factors such as all-units discount and incremental discount. Ricardo, et.al. (2017) also extended the Nagare and Dutta (2012) model by assuming a quadratic function for holding cost and adding time dependent deterioration rate. Chang, et. al. (2010) have proposed an inventory model with price and inventory-dependent demand for deteriorated items with shelf limitation.
Setiawan, et.al (2017) have discussed an inventory-dependent demand model for deteriorated items and considering return for deteriorated items to the supplier. Meanwhile, Neilshan, et. al. (2018) have considered a model with deterioration, all-units discount and return. For probabilistic demand, model development has been conducted by Limansyah and Lesmono (2012) using Gamma lead-time demand and Lesmono and Limansyah (2014) for continuous discount function.

Instead of the available inventory, sales price per unit is also affecting the customers demand. As in the economic theory, it states that when the price per unit is cheaper then demand will increase. With the available goods on display and cheaper sales price, demand will increase making the inventory depletes faster. Also, discount offered by the supplier will induce demand from the retailer in order to minimize the retailer’s total inventory cost, especially in purchasing cost. Considering all the above factors, this paper will develop a mathematical model for price and inventory-dependent demand with all-units discount. From this model we can obtain the optimum order quantity and sales price per unit that minimizes the total inventory cost.

2. Assumptions and Notations
The assumptions used in developing our model are
1. There is no shortage.
2. There is no lead time.
3. The discount offered by the supplier is all-units discount.
4. The warehouse capacity is unlimited.
5. The deterministic demand rate follows the following equation
   \[ D(I(t), p) = f(p) + \beta I(t) \]
   where \( f(p) \) is the sales price function and \( f'(p) \) is a monotonically decreasing function.
6. The decision variable of sales price per unit \( (p) \) must be higher than the purchasing price per unit \( (P) \).

The notations used in this model are as follows.
\[
\begin{align*}
D(t) & = \text{Demand at time } t. \\
\beta & = \text{Increasing demand factor } (\beta > 0). \\
p & = \text{Sales price per unit.} \\
S & = \text{Ordering cost per replenishment.} \\
h & = \text{Holding cost fraction per unit per planning period.} \\
P_i & = \text{Purchasing price per unit.} \\
Q & = \text{Optimum order quantity.} \\
T & = \text{Time between replenishment..} \\
TC & = \text{Total inventory cost.} \\
R & = \text{Reorder point.}
\end{align*}
\]

3. The Model Formulation
An inventory and price-dependent demand model is a model that can be found in big stores such as supermarket and department stores.
At the beginning, there are \( Q \) units of goods in the warehouse. As time goes by, the inventory level decreases in accordance with demand at time \( t \). The higher the inventory level the faster it will deplete due to the higher demand. At \( t = t_1 \), the inventory level becomes zero and at the same time there are replenishment of \( Q \) units of goods. The company will reorder when the level of its inventory reaches the reorder point of \( R \) units. This cycle is repeated from time to time.

Let us define the amount of demand at time \( t \) as
\[
D(t) = f(p) + \beta I(t).
\]

Therefore,
\[
\frac{dI(t)}{dt} = -f(p) - \beta I(t), \quad 0 \leq t \leq t_1
\]
\[
I(t) = -\frac{f(p)}{\beta} + ce^{-\beta t}
\]

Since at \( t = t_1 \) it is known that \( I(t_1) = 0 \), then
\[
I(t) = f(p) \left( e^{\beta t_1} - 1 \right)
\]  
(1)

Notice that at \( t = 0 \), the beginning inventory level will equal the amount of replenishment of \( Q \) units, therefore
\[
Q = I(0) = f(p) \left( e^{\beta t_1} - 1 \right)
\]  
(2)

Note that \( 0 \leq t \leq t_1 = T \), then as a consequence, equation (2) can be written as
\[
Q = I(0) = f(p) \left( e^{\beta T} - 1 \right)
\]  
(3)

1. Purchasing cost is a cost incurred to buy goods. In this model, considering the discount offered by the supplier, then the amount of purchasing cost per unit can be defined as follows.
\[
P_i = \begin{cases} 
  a_0 & \text{for } U_0 \leq Q < U_1 \\
  a_1 & \text{for } U_1 \leq Q < U_2 \\
  \vdots & \\
  a_j & \text{for } U_j \leq Q < U_{j+1}
\end{cases}
\]
where \( a_k > a_{k+1}, k = 0,1,2,3, ..., j - 1 \) for each unit of goods. The amount of purchasing cost in a period is \( P_i \cdot \frac{f(p)}{\beta} (e^{\beta T} - 1) \)

2. Ordering cost occurs every time replenishment is made. If the amount of cost occurred in each replenishment is \( S \), then the ordering cost in one period is \( \frac{S}{T} \)

3. Holding cost is usually used for storing and maintaining goods. The amount of holding cost in one period is given by \( hP_i \cdot \left( \frac{f(p)}{\beta} - \frac{f(p)(1+e^{\beta T})}{\beta^2} \right) \)

Therefore, the total inventory cost in one period in this model is

\[
TC(p,T) = P_i \cdot \frac{f(p)}{\beta} (e^{\beta T} - 1) + \frac{S}{T} + hP_i \cdot \left( \frac{f(p)}{\beta} - \frac{f(p)(1+e^{\beta T})}{\beta^2} \right)
\] (4)

The next step in to find the minimum total inventory cost. Two necessary conditions that must be satisfied are

\[
\frac{\partial TC}{\partial p} = 0 \quad \text{and} \quad \frac{\partial TC}{\partial T} = 0.
\] (5)

It is quite difficult to find the solution of these two equations of \( \frac{\partial TC}{\partial p} = 0 \) and \( \frac{\partial TC}{\partial T} = 0 \) manually, therefore we use MAPLE to solve them.

We also develop an algorithm to find the solution of the model which includes the following steps. In each of price breaks (price offered by the supplier), then

1. Using MAPLE, find the sales price per unit \( (p) \) and time between replenishment \( (T) \) according to equation (5)

2. Calculate the value of \( Q \) using \( Q = \frac{f(p)}{\beta} (e^{\beta T} - 1) \).

3. Compare \( Q \) with \( U \). If \( Q \) is within the interval \( U_j \leq Q < U_{j+1} \), then \( Q \) is valid. Otherwise it is not valid and we can ignore it.

4. Calculate \( TC \) for each valid \( Q \) and choose the minimum \( TC \).

4. Numerical Example

We consider the following scenario : \( f(p) = \frac{200}{p^2}, \beta = 0.4, S = $25, h = 0.05 \), and discount offered by the supplier for purchasing cost \( P = \begin{cases} $3, & Q < 15 \\ $2, & Q \geq 15 \end{cases} \)

Using MAPLE, we find the solution, depicted in Table 1 below.

| \( P \) | \( T \) (year) | \( p \) ($/unit) | \( Q \) (unit) | Information | \( TC \) ($) |
|---|---|---|---|---|---|
| 3 | 0.7054676067 | 4.18698462 | 9.299933290 | Q valid | 35.43748816 |
| 2 | 0.7054676067 | 3.418424980 | 13.94989993 | Q is not valid | - |

From Table 1, it is found that the optimal time between replenishment is 0.7054 year with the optimum order quantity of 9.2999 units. The selling price $4 per unit, making the total inventory cost of $35.437
In Table 2, we give another illustration for \( f(p) = \frac{200}{p^{2.5}} \) with the same parameters as Table 1.

| \( p \) | \( f(p) \) | \( \beta \) | \( T \) (year) | \( P \) (S/unit) | \( Q \) (unit) | Information | TC (S) |
|-------|--------|---------|-------------|--------------|-------------|------------|--------|
| 3     | \( \frac{200}{p^{2.5}} \) | 0.4     | 0.7054676067 | 3.144107209 | 9.29933290 | Q valid    | 35.43748816 |
| 2     | \( \frac{200}{p^{2.5}} \) | 0.4     | 0.7054676067 | 2.673380911 | 13.94989994 | Q is not valid | -     |

From Table 2, we can see that as the rate of the price function decreases faster, then the sales price per unit becomes cheaper, that is to the amount of $3.1441.

We also give results for the above examples for \( \beta = 0.2 \) and \( \beta = 0.3 \) with the price function \( f(p) = \frac{200}{p^{2}} \) and \( f(p) = \frac{200}{p^{2.5}} \). We can see from Table 3 and 4 that larger value of \( \beta \) will cause shorter time between replenishment and consequently larger total inventory cost.

| \( p \) | \( f(p) \) | \( \beta \) | \( T \) (year) | \( P \) (S/unit) | \( Q \) (unit) | Information | TC (S) |
|-------|--------|---------|-------------|--------------|-------------|------------|--------|
| 3     | \( \frac{200}{p^{2}} \) | 0.2     | 3.146942701 | 9.349359489 | 10.02697179 | Q valid    | 7.944218368 |
| 2     | \( \frac{200}{p^{2}} \) | 0.2     | 3.146942701 | 7.633720057 | 15.04045768 | Q valid    | 7.944218370 |
| 3     | \( \frac{200}{p^{2.5}} \) | 0.2     | 3.146942701 | 5.978957787 | 10.02697179 | Q valid    | 7.944218368 |
| 2     | \( \frac{200}{p^{2.5}} \) | 0.2     | 3.146942701 | 5.083806167 | 15.04045768 | Q valid    | 7.944218370 |

| \( p \) | \( f(p) \) | \( \beta \) | \( T \) (year) | \( P \) (S/unit) | \( Q \) (unit) | Information | TC (S) |
|-------|--------|---------|-------------|--------------|-------------|------------|--------|
| 3     | \( \frac{200}{p^{2}} \) | 0.3     | 1.302659231 | 5.770626207 | 9.572727386 | Q valid    | 19.19151179 |
| 2     | \( \frac{200}{p^{2}} \) | 0.3     | 1.302659231 | 4.711696568 | 14.35909108 | Q not valid | -     |
| 3     | \( \frac{200}{p^{2.5}} \) | 0.3     | 1.302659231 | 4.064230727 | 9.572727389 | Q valid    | 19.19151179 |
| 2     | \( \frac{200}{p^{2.5}} \) | 0.3     | 1.302659231 | 3.455746297 | 14.35909109 | Q not valid | -     |

5. Conclusions and Further Research

In this paper we have developed an inventory model with inventory and price-dependent demand and all-units discount. Numerical examples for sales price per unit \( f(p) \) dan demand increasing factor \( (\beta) \) of 0.2 and 0.3 has been conducted. From that examples it can be concluded that as the decreasing rate of the price function becomes faster the sales price per unit is also decreasing (the price becomes cheaper). Also, as the value \( \beta \) becomes larger, the time between replenishment becomes shorter since the larger value of \( \beta \) cause larger demand and as a consequence the inventory will deplete faster. Considering probabilistic demand will become other avenue for further research due to the realistic situation in practice.
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