Enhanced Phase Estimation in Parity-Detection-Based Mach–Zehnder Interferometer using Non-Gaussian Two-Mode Squeezed Thermal Input State

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While the quantum metrological advantages of performing non-Gaussian operations on two-mode squeezed vacuum (TMSV) states have been extensively explored, similar studies in the context of two-mode squeezed thermal (TMST) states are severely lacking. This paper explores the potential advantages of performing non-Gaussian operations on TMST state for phase estimation using parity detection-based Mach–Zehnder interferometry and compares it with the TMSV case. To this end, a realistic photon subtraction, addition, and catalysis model is considered. A unified Wigner function of the photon subtracted, photon added, and photon catalyzed TMST state is derived, which is used to obtain the expression for the phase sensitivity. The results show that performing non-Gaussian operations on TMST states can enhance the phase sensitivity for significant squeezing and transmissivity parameter ranges. Because of the probabilistic nature of these operations, it is of utmost importance to consider their success probability. When the success probability is considered, the photon catalysis operation performed using a high transmissivity beam splitter is the optimal non-Gaussian operation. This contrasts with the TMSV case, where photon addition is observed as the most optimal. Further, the derived Wigner function of the non-Gaussian TMST states will be useful for state characterization and various quantum protocols.

1. Introduction

Optical interferometers have played an important role in the advancement of quantum sensing.[1,2] In particular, the Mach–Zehnder interferometer (MZI) has been extensively utilized for phase sensitivity studies. While the phase sensitivity of MZI using classical resources is limited by so-called shot-noise limit (SNL),[3] quantum resources, such as squeezed states and entangled states, can break the SNL and achieve the Heisenberg limit (HL).[4–7] Many quantum states of light have been employed in parity measurement-based optical interferometry to reach HL with the notable example of N00N states.[6,8–20] However, photon loss renders N00N states fragile to environmental interactions. Moreover, it has been shown that two-mode squeezed vacuum (TMSV) states can even break the HL limit,[6] but their implementation is limited by the maximum achievable squeezing.[21]

To overcome the hindrance caused by the upper limit on the maximum achievable squeezing, one can resort to the engineering of non-Gaussian states by performing photon subtraction (PS), photon addition (PA), or photon catalysis (PC) operations on the TMSV state. These non-Gaussian states have enhanced squeezing and entanglement, which has been utilized for performance improvement in quantum key distribution,[22–27] quantum teleportation,[28–31] as well as quantum metrology.[34–39]

Ideally, one would like to use states free from thermal noise. However, even in the most carefully designed experiments, losses are inevitable, and we are bound to end up with mixed states. Thermal states are an important class of mixed Gaussian states, which have played a key role in quantum optics from its inception.[40] These states have since then been used in various practical applications, such as thermal lasers,[41] ghost imaging,[42–44] and quantum illumination.[45] Two-mode squeezed thermal (TMST) state[46] have been proposed for use in quantum phase estimation.[47] TMST states have also been realized experimentally.[48–51] Further, non-Gaussian operations on squeezed thermal states are also being considered actively. For instance, nonclassicality and entanglement have been explored in photon subtracted and added squeezed thermal states.[52–55] Similarly, photon-subtracted TMST (PSTMST) and photon-added TMST (PATMST) states have also been proposed as resources for quantum teleportation.[56] There have been experimental efforts in the preparation, reconstruction, and
statistical parameter estimation of multiphoton subtracted thermal states.\textsuperscript{[57, 58]}

Motivated by these studies, we aim to determine whether non-Gaussian operations such as PS, PA, and PC can improve the phase sensitivity of the original TMST state. To this end, we consider the experimental model of PS, PA, and PC operations\textsuperscript{[59]} (see Figure 2). Next, we obtain the Wigner function describing the PSTMST, PATMST, and photon-catalyzed TMST (PCTMST) states, which is utilized to evaluate the phase sensitivity of the parity-detection-based MZI. In this paper, we will collectively refer to PSTMST, PATMST, and PCTMST states as NGTMST states, where “NG” stands for non-Gaussian. Also note that from here on, the term non-Gaussian operations will be used only to refer to PA, PS, and PC operations unless stated otherwise.

We emphasize that the presence of an additional parameter in the TMST state (average number of photons in the thermal state) significantly enhances the complexity of analytical calculations of the involved quantities compared to the TMSV state. We employ the phase space formalism rather than the more often used operator formalism because of the computational ease provided by the former.\textsuperscript{[60]}

Our results demonstrate that the non-Gaussian operations can significantly enhance a TMST state’s phase sensitivity. To better understand the above results and to find out the squeezing and transmissivity values rendering an enhancement overlap between the phase sensitivity of the TMST and NGTMST states in the squeezing-transmissivity plane. Further, with an intent to identify optimal non-Gaussian operations, we consider the probabilistic nature of non-Gaussian operations. A careful analysis shows that only for photon catalysis, the parameter values corresponding to a significant enhancement overlap with those corresponding to a region of high success probability, which happens in a large transmissivity regime. Hence, it can be concluded that between all the non-Gaussian operations, implementing PC using a high transmissivity beam splitter is the optimal choice. This is contrary to the case involving non-Gaussian TMSV states, where PA operation turns out to be most optimal.\textsuperscript{[69]}

Also, the expression for a unified Wigner function of NGTMST states derived here does not appear in the existing literature to the best of our knowledge. This expression will be a welcome addition to existing literature and will find important use while dealing with various CV QIP protocols involving NGTMST states. Additionally, we supply a single expression for the parity-detection-based phase sensitivity to deal with all three non-Gaussian operations performed on the TMST state.

The layout of this paper is as follows. In Section 2, we present a brief overview of the formalism for CV systems. In Section 3, we derive the unified Wigner function for all the NGTMST states. Section 4 contains a short description of parity-detection-based lossless MZI. We start by comparing the phase sensitivities of the TMSV and TMST states. We then show the advantages of using NGTMST states over TMST states to determine the phase for specifically chosen values of involved parameters. Next, we move on to a more involved analysis by studying the difference between the phase sensitivity of the TMST and NGTMST states and the success probability of corresponding non-Gaussian operations for experimentally reasonable values of squeezing and transmissivity. To gain a better perspective, we also compare our results to the case of the non-Gaussian TMSV state (Ref. [39]). Finally, in Section 5, we conclude the article by summarizing the main points and discussing possible future directions.

2. Brief Description of CV Systems

We consider an n-mode quantum optical system, whose th mode can be represented by a pair of Hermitian quadrature operators \( \hat{q}_i, \hat{p}_i \). For convenience, we define the following 2n-dimensional column vector comprising of n pairs of Hermitian quadrature operators:

\[
\hat{\xi} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T, \quad i = 1, 2, \ldots, 2n.
\]

In terms of the column vector \( \hat{\xi} \), the canonical commutation relation can be formulated as

\[
[i\Omega, \hat{\xi}] = i\Omega_{ij}, \quad (i, j = 1, 2, \ldots, 2n),
\]

where we have taken \( \hbar=1 \) and \( \Omega \) is a 2n \( \times \) 2n matrix given by

\[
\Omega = \bigoplus_{i=1}^n \omega = \begin{pmatrix}
\omega \\
\vdots \\
\omega
\end{pmatrix}, \quad \omega = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

The photon annihilation and creation operators for the th mode are defined in terms of the corresponding quadrature operators as follows:

\[
\hat{a}_i = \frac{1}{\sqrt{2}}(\hat{q}_i + i\hat{p}_i), \quad \hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(\hat{q}_i - i\hat{p}_i).
\]

For a quantum system with density operator \( \hat{\rho} \), we can define the Wigner distribution function as below:

\[
W(\xi) = \int \frac{d\xi \cdot \xi}{(2\pi)^n} \exp(-\frac{1}{2} \xi \cdot [\hat{\rho} + \frac{1}{2} \xi^2] \cdot \xi),
\]

where \( \xi = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T \in \mathbb{R}^{2n}, \quad \xi^\dagger = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T, \quad \hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)^T, \quad \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)^T \). The Wigner function of a Fock state \( |n\rangle \) can be evaluated using Equation (5) as

\[
W_{(n)}(\xi, p) = \frac{(-1)^n}{\pi} \exp(-\frac{\xi^2 + p^2}{2}) L_n[2(\xi^2 + p^2)],
\]

where \( L_n(\bullet) \) is the Laguerre polynomial of nth order. We can also reformulate the Wigner function in terms of the average of displaced parity operators\textsuperscript{[61]}

\[
W(\xi) = \frac{1}{\pi^n} \text{Tr}[\hat{\rho} D(\xi) \hat{D}(\xi)^\dagger],
\]

where \( \hat{D}(\xi) = \prod_{i=1}^n \exp(i\xi_i \hat{a}_i^\dagger \hat{a}_i) \) is the parity operator and \( D(\xi) = \exp[i\xi \Omega] \) is the displacement operator.

Gaussian states are an important class of CV-system states whose Wigner distribution is a Gaussian function. The Wigner
function (5) for n-mode Gaussian states simplifies to the following form \(^{(62)}\):

\[
W(\xi) = \frac{\exp[-(1/2)(\xi - d)^T V^{-1}(\xi - d)]}{(2\pi)^{n/2} \sqrt{|\det V|}}, \tag{8}
\]

where \(d = \text{Tr}[\rho \hat{\xi}]\) is the displacement vector and \(V\) is a \(2n \times 2n\) covariance matrix, whose element can be calculated as

\[
V_{ij} = \frac{1}{2} \langle [\Delta \xi_i, \Delta \xi_j] \rangle, \tag{9}
\]

where \(\Delta \xi_i = \xi_i - \langle \xi_i \rangle\), and \(\{ , \} \) denotes anti-commutator. The action of an infinite-dimensional unitary operator \(U^*\) on a density operator can be mapped to a symplectic transformation \(S \in \text{Sp}(2n, R)\) acting on the quadrature operators. Further, the state evolution \(\rho \rightarrow U^* U \rho U^* U\) can be rephrased as a symplectic transformation in phase space as follows: \(^{(60)}\)

\[
d \rightarrow SD, \quad V \rightarrow SVS^T, \quad \text{and} \quad W(\xi) \rightarrow W(S^{-1} \xi). \tag{10}
\]

Quantum optical mode in thermal equilibrium with a bath at a given temperature is said to be in a thermal state. This mode can be thought of as a classical mixture of different photon number states with weight factors given by the Bose distribution. As stated earlier in this paper, we consider the TMST state, which can be generated by subjecting two uncorrelated thermal modes to a two-mode squeezing transformation. The following covariance matrix describes the TMST state represented by the density operator \(\rho_{A_1 A_2}\):

\[
V_{A_1 A_2} = S_{A_1 A_2}(r) V_{\text{th}} S_{A_1 A_2}(r)^T, \tag{11}
\]

where \(V_{\text{th}} = (n_{\text{th}} + 1/2) I_4\) is the covariance matrix of the two uncorrelated thermal modes with \(n_{\text{th}}\) being the average number of photons in the thermal state. Further, \(S_{A_1 A_2}(r)\) is the two-mode squeezing transformation given by

\[
S_{A_1 A_2}(r) = \begin{pmatrix}
\cosh r & 0 & \sinh r & Z \\
0 & 1 & 0 & -1
\end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}. \tag{12}
\]

Putting \(n_{\text{th}} = 0\) in Equation (12), we obtain the covariance matrix of the TMSV state. Since TMST state is a Gaussian state with zero mean and covariance matrix given by Equation (12), the Wigner function of the TMST state can be readily evaluated using Equation (8):

\[
W(\xi) = \frac{1}{(2\pi \kappa)^2} \exp \left[ -\frac{(q_1^2 + p_1^2 + q_2^2 + p_2^2) \cosh(2\kappa)}{2\kappa} \right] + (q_1, q_2 - p_1, p_2) \sinh(2\kappa) / \kappa, \tag{13}
\]

where \(\kappa = (n_{\text{th}} + 1/2)\).

Consideration of mixed states, for instance, TMST states are not a matter of choice but a necessity at a more practical level. Ideally one would ideally like to use states free from any thermal noise such as TMSV states. However even in the most carefully designed experiments, losses are inevitable, and we are bound to end up with a mixed state. This makes studies like this not only very useful but necessary. In the process of generating two-mode squeezed vacuum state, losses due to different imperfections such as misalignment, absorption by optical elements, and mode mismatch transform the initial TMSV state to a TMST state. This provides a further significant motivation to study squeezed thermal states. Below, we present a discussion showing that the initial TMSV state under linear losses can be mapped to a TMST state.

We aim to relate the squeezing \(r_o\) of the TMSV state interacting with a lossy channel (modeled via a beam splitter of transmissivity \(\eta\)) to the squeezing \(r\) and thermal coefficient \(\kappa\) of the resulting TMST state, \(^{(64-66)}\) that is,

\[
\text{TMSV state with lossy channels} \Rightarrow \text{TMST state}. \tag{14}
\]

To this end, we make use of the following two facts: (i) Both modes of the TMSV state independently interacting with lossy channels (Figure 1a) is equivalent to the case of two single-mode squeezed vacuum states interacting with independent lossy channels followed by a 50 : 50 beam splitter (Figure 1b)\(^{(67)}\) and (ii) The TMST state (11) can be generated by mixing two single-mode squeezed thermal state using a 50 : 50 beam splitter. The covariance matrix for a single-mode squeezed vacuum state interacting with a lossy channel is

\[
V_{r, \eta} = \frac{1}{2} \begin{pmatrix}
\eta e^{-2r} + 1 - \eta & 0 \\
0 & \eta e^{2r} + 1 - \eta
\end{pmatrix}. \tag{15}
\]

To obtain the relation between the parameters \((r_0, \eta)\) and \((r, \kappa)\), we equate the covariance matrix for a single-mode squeezed vacuum state interacting with a lossy channel (15) and a single-mode squeezed thermal state with covariance matrix \(\text{diag}(e^{-2r} \kappa, e^{2r} \kappa)\). This leads to the following two equations: \(^{(64)}\)

\[
\eta e^{-2r_0} + 1 - \eta = 2e^{-2r} \kappa, \quad \text{and} \quad \eta e^{2r_0} + 1 - \eta = 2e^{2r} \kappa. \tag{16}
\]
For given $k$ and $r$, we can solve the above equations to obtain the corresponding $r_j$ and $n_j$. For instance, solving the above equations for $k = 1$ and $r = 1$ gives $r_1 = 1.47$ and $n_j = 0.77$. However, it should be noted that solutions to the above Equation (16) do not exist for any arbitrary value of $r$ and $k$, that is, there exist TMST states that cannot be viewed as TMSV states interacting with linear lossy channels.

3. Wigner Function of NGTMS states

The schematic for the generation of NGTMS states is shown in Figure 2. The mode $A_2$ of the TMST state is mixed with an auxiliary mode $F$, which is in Fock state $|m\rangle$, using a beamsplitter of transmissivity $\tau$. Photon number measurement is performed on the output auxiliary mode to obtain the NGTMS states.

A photon number resolving detector, represented by the positive-operator-valued measure (POVM) $\{\Pi_\lambda = |n\rangle \langle n|, \lambda \neq \Pi_\lambda\}$ is employed to measure the mode $F$. When the POVM element $\Pi_\lambda$ clicks, the non-Gaussian operation is performed successfully on the mode $A_2$. The unnormalized state obtained after the detection of $n$ photons is given by

$$\rho_{A_2}^\lambda = \text{Tr}_F(|n\rangle \langle n|) \otimes \rho_{A_2}^\lambda \rho_{A_2}^\lambda,$$

which corresponds to NGTMS state. The partial tracing of mode $F$ in Wigner formalism is equivalent to integrating the product of the Wigner distributions with respect to the phase space variables $F^i$ [73-75]

$$\tilde{W}_{\Delta A_2}^\lambda(\xi_1, \xi_2) = 2\pi \int d^2 \xi_3 \left< \xi_3 | \rho_{A_2}^\lambda | \xi_3 \right> \times \left< \xi_3 | \rho_{A_2}^\lambda | \xi_3 \right>$$

(21)

By carefully selecting the values of the parameters $(m, n)$, we can perform the required non-Gaussian operations. For PS operation, $m < n$, whereas for PA operation, $m > n$. Lastly, for PC operation, $m = n$. PS operation on TMST states yields PSTMS states. Similarly, PA and PC operations on TMST states yield PATMS and PCTMS states, respectively.

The generating function for the Laguerre polynomials

$$L_n[2(q^2 + p^2)] = \hat{D} \exp \left[ \frac{s}{2} + s(q + ip) - t(q - ip) \right].$$

(22)

with

$$\hat{D} = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial q^n} \frac{\partial^n}{\partial p^n} [\cdot]_{s=t=0}$$

(23)

can be used to convert Equation (21) into a Gaussian integral. On integration of Equation (21), we get

$$\tilde{W}_{\Delta A_2}^\lambda(\xi_1, \xi_2) = \frac{\exp \left( \xi_i M_i \xi \right)}{a_0} \hat{D}_1 \exp \left[ -a_1 u_1 v_1 + a_2 u_2 v_1 + a_3 u_2 v_2 + a_4 u_1 v_2 + v_1 v_2 \right],$$

(24)

where the column vector $\xi$ is defined as $(q_1, p_1, q_2, p_2)^T$, and the coefficients $a_1$ and the matrix $M_i$ are defined in Equations (A1) and (A3) of the Appendix A, respectively. Further, the differential operator $\hat{D}_1$ is given as

$$\hat{D}_1 = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial u_1^n} \frac{\partial^n}{\partial v_1^n} \frac{\partial^n}{\partial u_2^n} \frac{\partial^n}{\partial v_2^n} [\cdot]_{s=t=0}$$

(25)

We can express Equation (24) in terms of two-variable Hermite polynomials $H_{m,n}(x,y)$ [76]

$$\tilde{W}_{\Delta A_2}^\lambda(\xi_1, \xi_2) = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial u_1^n} \frac{\partial^n}{\partial v_1^n} \frac{\partial^n}{\partial u_2^n} \frac{\partial^n}{\partial v_2^n} [\cdot]_{s=t=0}$$

(26)

$$\times H_{m,n}(x,y)$$

$$\times \frac{a_1}{\sqrt{a_1}} \frac{a_2}{\sqrt{a_2}} \frac{a_3}{\sqrt{a_3}} \frac{a_4}{\sqrt{a_4}}$$

$$\times H_{m,n}(x,y)$$

$$\times \frac{a_1}{\sqrt{a_1}} \frac{a_2}{\sqrt{a_2}} \frac{a_3}{\sqrt{a_3}} \frac{a_4}{\sqrt{a_4}}$$

(26)
The probability of a successful non-Gaussian operation can be evaluated by taking the trace of the density operator $\rho_{A'_1A'_2}$ (20):

$$P_{NG} = \text{Tr}(\rho_{A'_1A'_2}) = \int d^2\xi_1 d^2\xi_2 \overline{W}_{A'_1A'_2}^{NG}.$$  

(27)

where the column vector $u$ is defined as $(u_1, v_1, u_2, v_2)^T$ and the matrix $M_i$ and coefficient $\alpha_i$ are defined in Equations (A5) and (A6) of the Appendix A. The normalized Wigner function $\overline{W}_{A'_1A'_2}^{NG}$ of the non-Gaussian NGTMST state is obtained as

$$\overline{W}_{A'_1A'_2}^{NG}(\xi_1, \xi_2) = \langle P_{NG} \rangle^{-1} \overline{W}_{A'_1A'_2}^{NG}(\xi_1, \xi_2).$$  

(28)

The Wigner function of several special cases can be readily derived from Equation (28). For example, in the unit transmissivity limit $\tau \to 1$ with $m = 0$, we obtain the Wigner function of the ideal $n$-PSTMST state $A'_1A'_2$ (TMST) with $A'_n$ being the normalization factor. Similarly, in the unit transmissivity limit $\tau \to 1$ with $m = 0$, we obtain the Wigner function of the ideal $m$-PATMST state $A'_nA'_m$ (TMST) with $A'_n$ being the normalization factor. We note that the Wigner function of ideal phonon subtracted and added TMST states have been derived in Ref. [56]. Further, setting $\kappa = 1/2$ (equivalently $n_{th} = 0$) in Equation (28) yields the Wigner function of the non-Gaussian TMSV state.

4. Parity Detection Based Phase Sensitivity in MZI

We consider a lossless MZI comprised of two 50:50 beam splitters and two phase shifters, as shown in Figure 3. The NGTMST states are introduced as input in the interferometer. The annihilation operators $\hat{a}_1$ and $\hat{a}_2$ represent the two input modes. This setup is used for estimating the unknown phase $\phi$ introduced via the two phase shifters by measuring the parity operator on the output mode $\hat{a}_2$.

To describe the mode transformations induced by the optical elements in the MZI, it is convenient to use the Schwinger representation of SU(2) algebra. The generators of the SU(2) algebra, also known as angular momentum operators, can be written in terms of the annihilation and creation operators of the input modes as follows:

$$\hat{J}_1 = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger),$$  

$$\hat{J}_2 = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger),$$  

$$\hat{J}_3 = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2).$$  

(29)

These angular momentum operators follow the commutation relations $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$. The infinite-dimensional unitary transformations of the first and the second beam splitters are given by $e^{-i(\phi/2)\hat{J}_3}$ and $e^{i(\phi/2)\hat{J}_3}$, respectively. Further, the cumulative action of both the phase shifters is given by $e^{i\phi\hat{J}_3}$. Hence, the overall action of the lossless MZI is given by

$$U(S_{MZI}) = e^{-i(\phi/2)\hat{J}_3} e^{i\phi\hat{J}_3} e^{i(\phi/2)\hat{J}_3} = e^{i\phi\hat{J}_3}.$$  

(30)

The resultant symplectic transformation $S_{MZI}$ of the MZI acting on the phase space variables $(\xi_1, \xi_2)^T$ is given by

$$S_{MZI} = \begin{pmatrix} \cos(\phi/2) & 1 - \sin(\phi/2) \frac{1}{2} \\ \sin(\phi/2) & \cos(\phi/2) \frac{1}{2} \end{pmatrix}. $$  

(31)

The transformation of the Wigner function of the input state due to the MZI can be written using Equation (10) as

$$W_{in}(\xi) \rightarrow W_{in}(S_{MZI}^{-1}\xi) = W_{out}(\xi).$$  

(32)

As shown in the schematic of MZI (Figure 3), we perform parity measurement on the output mode $\hat{a}_2$, which basically differentiates between even and odd photons numbers Fock state. The corresponding photon number parity operator is written as

$$\hat{P}_n = \exp\left(i\pi \hat{a}_2^\dagger \hat{a}_2\right) = (-1)^n \hat{a}_2^\dagger \hat{a}_2.$$  

(33)

Therefore, the average value of the parity measurement operator can be evaluated using Equation (7): \cite{78}

$$\langle \hat{P}_n \rangle = f(\phi) = \pi \int d^2\xi_1 W_{out}(\xi_1, 0).$$  

(34)
Using the Wigner function of the NGTMST state (28), the average of the parity measurement operator comes out to be

\[
f(\phi) = \frac{e_\phi \hat{D}_\phi \exp(\sqrt{\lambda^2} M_\phi u)}{d_\phi \hat{D}_\phi \exp(\sqrt{\lambda^2} M_\phi u)},
\]

(35)

where the matrix \(M_\phi\) and coefficient \(e_\phi\) are defined in Equations (A7) and (A8) of the Appendix A. Taking the unit transmissivity limit under the condition \(m = n\) in Equation (35), yields the average of the parity operator for the case of input TMST state:

\[
f(\phi)_{\text{TMST}} = \frac{1 - \lambda^2}{2\lambda^2[1 + \lambda^4 - 2\lambda^2 \cos(2\phi)]^{1/2}},
\]

(36)

with \(\lambda = \tanh r\). Further, setting \(\kappa = 1/2\) in Equation (36) renders the average of the parity operator for an input TMSV state.

The error propagation formula allows us to write the phase uncertainty or sensitivity as

\[
\Delta \phi = \sqrt{1 - f(\phi + \pi/2)^2} \left|\frac{\partial f}{\partial \phi}\right|.
\]

(37)

The phase uncertainty for the input TMST state can be written using Equation (36) as

\[
\Delta \phi_{\text{TMST}} = \sqrt{\frac{\lambda^2 - (1 - \lambda^2)^2}{1 + \lambda^4 - 2\lambda^2 \cos(2\phi)^2} \left|\frac{\lambda^2 - (1 - \lambda^2) \sin(2\phi)}{1 + \lambda^4 - 2\lambda^2 \cos(2\phi)^2}\right|^{1/2}}.
\]

(38)

We note that the phase uncertainty depends on the following parameters:

(i) Squeezing of the TMST state, referred to as squeezing from here on.

(ii) Transmissivity of the beam splitter used in the implementation of the non-Gaussian operations, referred to as transmissivity from now onward.

(iii) Magnitude of unknown phase introduced, hereby referred to as phase.

(iv) Average number of photons in the thermal state.

We now numerically study the dependence of the phase uncertainty of the input TMSV and TMST states on squeezing, phase, and thermal parameter. The results are shown in Figure 4. As is evident from Equation (38), the phase uncertainty for the input TMST state will be larger than the input TMSV state, which can also be seen in Figure 4a,b. As is shown in Figure 4a, the phase uncertainty blows up at \(r = 0\) for both the TMSV and TMST states and attains minima at \(r = 2.65\) and \(r = 2.80\), respectively. However, such large values of squeezing cannot be achieved with current technology.\(^{[21]}\) We note here that these specific numerical values of squeezing are for \(\phi = 0.01\).

As is evident from Equation (38), the phase uncertainty varies periodically with period \(\pi\) as a function of phase, which can also be seen in Figure 4b. The phase uncertainty for the TMSV state is minimized at \(\phi = 0, \pm \pi, \pm 2\pi, \ldots\), whereas for the TMST state, it blows up at these values with a pair of minima appearing in the surrounding regions.

To explore the effect of the variation of average photon number \(\langle n_{av}\rangle\) on the phase uncertainty for the TMST state, we plot phase uncertainty as a function of \(\kappa = n_{av} + \frac{1}{2}\) in Figure 4c. As expected, the phase uncertainty increases monotonically with \(\kappa\), and equivalently, with average photon number.

4.1. Advantages of Non-Gaussian Operations on TMST State in Phase Estimation

In this section, we explore the advantages of performing non-Gaussian operations on the TMST state in the context of phase estimation. As we shall see, various NGTMST states outperform the original TMST state. We first analyze the dependency of the phase uncertainty on initial squeezing \(r\) while transmissivity \(\tau\), magnitude of phase \(\phi\), and thermal parameter \(\kappa\) are fixed at suitably chosen values (see Figure 5). In Figure 5a, we observe that 1-PSTMST and 2-PSTMST states improve the phase sensitivity over the original TMST states. It is also interesting to note that the value of \(\Delta \phi_{\text{PS}}\) blows up at \(r = 0.3\). Similar features can also be observed for other NGTMST states if parameter values are appropriately chosen. As seen in Figure 5b, both the 1-PATMST and 2-PATMST states outperform the original TMST state. We also notice that for up to \(r = 1\), 1-PATMST state outperforms 2-PATMST state, which is in contrast with Figure 5a, where 2-PSTMST state outperforms 1-PSTMST state. Figure 5c shows that 1-PCTMST and 2-PCTMST states outperform the TMST state up to \(r = 1.3\). While 1-PCTMST state performs better for \(r \leq 1.0\), it is outperformed by the 2-PCTMST state beyond this squeezing.

In Figure 6, we show the phase uncertainty with respect to transmissivity at a given value of squeezing, thermal parameter, and phase. It can be clearly seen that for given values of parameters, all NGTMST states outperform the TMST state for all transmissivity values. We also notice that the phase sensitivity improves with increasing \(\tau\) for PSTMST and PATMST states. Moreover, as we can see in Figure 6c, either the 1-PCTMST or 2-PCTMST states can provide better phase sensitivity depending on the value of transmissivity. Also, the phase uncertainty of the PCTMST states approaches that of the TMST state in the unit transmissivity limit.

In Figure 7, we plot the phase uncertainty with respect to phase magnitude at a given squeezing and transmissivity values. While PSTMST and PATMST states improve the phase sensitivity for small phase values, PCTMST states can enhance the phase sensitivity even for larger phase values.

Figure 8 displays the phase uncertainty as a function of the thermal parameter \(\kappa\) at a specified squeezing, transmissivity, and phase value. While 1-PSTMST and 1-PATMST states outperform the TMST state for all values of \(\kappa\), 2-PSTMST and 2-PATMST states underperform as compared to the TMST state near \(\kappa = 1/2\). Further, we observe that photon catalysis on TMST state also outperforms the TMST state except for \(\kappa \approx 1/2\). These results at \(\kappa = 1/2\) do corroborate with those obtained in Ref. [39] for non-Gaussian operations on TMSV state. Moreover, we observe that the relative improvement in phase sensitivity by non-Gaussian operations increases with an increase in the average number of thermal photons.
Figure 4. Phase uncertainty $\Delta \phi$ as a function of a) squeezing parameter $r$, b) phase $\phi$, and c) thermal parameter $\kappa$. While $\Delta \phi$ and $\phi$ axes are in rad, $r$ and $\kappa$ axis are dimensionless.
Figure 5. Phase uncertainty $\Delta \phi$ as a function of the squeezing parameter $r$ for TMST and various NGTMST states. We have set $\tau = 0.9$, $\phi = 0.01\text{ rad}$, and $\kappa = 1$ for all the panels. While $\Delta \phi$ axis is in rad, $r$ axis is dimensionless.
Figure 6. Phase uncertainty $\Delta \phi$ as a function of the transmissivity $\tau$ for TMST and various NGTMST states. We have set the squeezing as $r = 1$ for all the panels. Further, we have taken $\phi = 0.01$ rad, and $\kappa = 1$ in all the panels. While $\Delta \phi$ axis is in rad, $\tau$ axis is dimensionless.
Figure 7. Phase uncertainty $\Delta \varphi$ as a function of the phase $\varphi$ for TMST and various NGTMST states. We have set $\tau = 0.9$, $r = 1$, and $\kappa = 1$ for all the panels. Both $\Delta \varphi$ and $\varphi$ axes are in rad.
Figure 8. Phase uncertainty $\Delta \phi$ as a function of the thermal parameter $\kappa$ for TMST and various NGTMST states. We have set $\tau = 0.9$, $\phi = 0.01$ rad, and $r = 1$ for all the panels. While $\Delta \phi$ axis is in rad, $\kappa$ axis is dimensionless.
4.2. Success Probability and Relative Performances of NGTMSV States in Phase Estimation

In the previous section, we explored the advantages of using NGTMSV states over the original TMST state for certain specifically chosen values of state parameters (r and r) and phase φ. In order to gain a comprehensive insight into the comparative performances of NGTMSV and TMST states in the context of phase estimation, we now explore the advantages of using non-Gaussian resources for a reasonable range of transmissivity and squeezing parameters for a given value of phase. For this purpose, we start by defining a figure of merit, $D^{NG}$, as the difference of Δφ between TMST and NGTMSV states:

$$D^{NG} = \Delta \phi_{TMST} - \Delta \phi_{NGTMSV}$$

(39)

where the subscript T stands for thermal in the TMST state. The region of positive $D^{NG}$ corresponds to those values of transmissivity and squeezing for which the NGTMSV states perform better than the TMST state.

To better understand the impact of the probabilistic nature of non-Gaussian operations, we plot the success probability alongside the $D^{NG}$ for the same state parameter ranges. Success probability is also a good measure of resource utilization and can be defined as the fraction of successful non-Gaussian operations.

For comparative purposes, we reproduce the results of Ref. [39] in Figure 9, which demonstrates the advantages of performing non-Gaussian operations on the TMSV state in the context of phase estimation. The equivalent figure of merit (39) for NGTMSV states can be written as

$$D^{NG} = \Delta \phi_{TMSV} - \Delta \phi_{NGTMSV}$$

(40)

where the subscript V stands for vacuum in the TMSV state. In the left panels of Figure 9, success probability for various non-Gaussian operations are plotted; whereas the right panels show the plot for different fixed values of corresponding $D^{NG} (= 0.0, 0.1, 0.5, 1, 2, 3)$ as a function of the transmissivity τ and squeezing parameter r. Positive $D^{NG}$ corresponds to that region in the (τ, r) space where the NGTMSV states outperform the TMSV state. A careful comparison of different non-Gaussian operations shows that only for photon addition, the region of positive $D^{NG}$ overlaps with the corresponding region of high success probability. Therefore, it can be concluded that out of all the non-Gaussian operations considered, only photon addition offers an advantage while taking the probabilistic nature of these operations into account. Reference [39] provides a detailed discussion of these results.

We now proceed to analyze the advantages rendered by the NGTMSV states compared to the TMST state. As shown in Figure 9, we plot the success probability for various non-Gaussian operations in the left panel of Figure 10. In contrast, the right panel shows the plot for different fixed values of corresponding $D^{NG} (= 0, 1, 5, 20, 50, 100)$ as a function of the transmissivity τ and squeezing parameter r. There is a considerable enhancement in the magnitude of $D^{NG}$ in comparison to $D^{NG}$. This signifies the fact that the incremental advantage of performing non-Gaussian operations on the TMST state is much more when compared to performing these operations on the TMSV state.

On careful comparison of the left panels of Figures 9 and 10, we observe that the maximum achievable success probability for PS and PC operations on the TMST state is approximately the same as on the TMSV state, whereas it decreases considerably for PA operation. Similarly, a careful comparison of the right panels reveals that the region of positive $D^{NG}$ in r and r space increases for all three non-Gaussian operations. While for PC operation, this change is due to an increase in the allowed values of both parameters r and r; for PA and PS operations, this is not the case as there is no scope for increment in r and only the allowed range of r is increased.

Taking the success probability considerations into account, we observe that only for photon catalysis, the region of positive $D^{NG}$ overlaps with the corresponding region of high success probability (large r regime). Therefore, photon catalysis offers maximum advantage whilst taking the probabilistic nature of non-Gaussian operations into account.

5. Conclusion

In this article, we showed that performing non-Gaussian operations on TMST states enhances the phase sensitivity in parity detection-based MZI. To this end, we derived a unified Wigner distribution function describing PSTMST, PACTMST, and PCTMST states altogether. We utilize this function to obtain a single phase sensitivity expression for all three cases mentioned above. By appropriately changing the number of input and detected photons, we can perform PS, PA, and PC operations. The Wigner function and the phase sensitivity depend on the initial squeezing of the TMST state, the average number of photons in the thermal state, the transmissivity of the beam splitter used in the implementation of the non-Gaussian operations, and the entanglement parameter. A meticulous analysis involving the probabilistic nature of non-Gaussian operations reveals the photon catalysis operation as the most optimal non-Gaussian operation. These optimal conditions are achieved while working in a high transmissivity regime.

It is clear that the results of this study will be of significant relevance for any future phase estimation experiments involving TMST states. Recent experiments implementing PS operations on thermal states signal that our proposal could be implemented in lab. Further, the derived Wigner function will be of great use in the state characterization via quantifying nonclassicality, entanglement, non-Gaussianity, and nonlocality. As mentioned earlier, we overcame the calculational challenges involved in dealing with non-Gaussian operations on the TMST state by following phase space formalism instead of the traditional operator method. The phase space method utilized in this work can be valuable in circumventing calculations challenges in various other problems as well, for instance, quantum teleportation via PCTMST states, a problem posed in Ref. [56]. While the scope of this paper is limited to exploring the advantages of performing non-Gaussian operations on a single mode of the TMST state, the effects of performing these non-Gaussian operations on both modes remain to be investigated.
Figure 9. Left panels show the success probability as a function of the transmissivity $\tau$ and squeezing parameter $\lambda$. Right panels show the curves of fixed $D^{NG}_V$, the difference of $\Delta \phi$ between TMSV and NGTMSV states, as a function of $\tau$ and $\lambda$. We have considered NGTMSV states generated by performing non-Gaussian operations on one of the modes of the TMSV state, and the corresponding values of the parameters $(m, n)$ have been shown. The legend on the right panel shows the plotted values of $D^{NG}_V$. The phase, $\phi$, is taken to be 0.01. Both the axes are dimensionless.
Figure 10. Left panels show the success probability as a function of the transmissivity $\tau$ and squeezing parameter $\lambda$. Right panels show the curves of fixed $D_{\chi}^{NC}$, the difference of $\Delta \phi$ between TMST and NGTMST states, as a function of $\tau$ and $r$. We have considered NGTMST states generated by performing non-Gaussian operations on one of the modes of the TMST state, and the corresponding values of the parameters $(m, n)$ have been shown. The phase, $\phi$, is taken to be 0.01. Solid black, large dashed red, dashed green, dotted orange, dot dashed cyan, and double dot dashed purple curves represent $D_{\chi}^{NC} (= 0, 1, 5, 20, 50, 100)$, respectively. Both the axes are dimensionless.
Appendix A: Coefficients Appearing in the Wigner Function, Probability, and Average of Parity Operator

The coefficients $a_i$ appearing in Equation (24) are given as

$$a_0 = e^2(k(2k^2 T + \Lambda r^2) \mu^2),$$
$$a_1 = kr^2(2k^2 \Lambda + \Lambda^2) b_0^{-1},$$
$$a_2 = (2b_1(q_1 - i \rho) + b_2(q_2 + i \sigma_2)) b_0^{-1},$$
$$a_3 = - (2b_1(q_1 + i \rho) - b_2(p_2 + i \sigma_2)) b_0^{-1},$$
$$a_4 = - kr^2(2k^2 \mu^2 - \Lambda) b_0^{-1},$$
$$a_5 = (2b_1(q_1 - i \rho_1) + b_4(q_2 + i \sigma_2)) b_0^{-1},$$
$$a_6 = - (2b_1(q_1 + i \rho_1) - b_4(p_2 + i \sigma_2)) b_0^{-1},$$
$$a_7 = 4k^2 \mu^2 t b_0^{-1},$$

where $\mu = \sqrt{1 - \lambda^2}$, $t = \sqrt{1 - r}$, $r = \sqrt{1 - t^2}$, $\Lambda = (1 + \lambda^2)$, and

$$b_0 = -2k(2k^2 T + \Lambda r^2),$$
$$b_1 = 2k \lambda r t,$$
$$b_2 = - kr(2k^2 \mu^2 + \Lambda),$$
$$b_3 = 2k \lambda r,$$
$$b_4 = k r s(2k^2 \mu^2 - \Lambda).$$

Further the matrix $M_1$ appearing in Equation (24) is given as

$$M_1 = \frac{1}{b_0} \begin{pmatrix}
  c_1 & 0 & c_2 & 0 \\
  0 & c_1 & 0 & -c_2 \\
  c_2 & 0 & c_1 & 0 \\
  0 & -c_2 & 0 & c_3
\end{pmatrix},$$

where

$$c_1 = k \lambda T + \mu^2 r^2,$$
$$c_2 = -4k \lambda t,$$
$$c_3 = k \lambda T + 2k^2 \mu^2 r^2.$$

The explicit form of matrix $M_2$ appearing in Equation (27) is given as

$$M_2 = \frac{1}{d_4} \begin{pmatrix}
  0 & d_1 & 0 & d_2 \\
  d_1 & 0 & d_2 & 0 \\
  0 & d_2 & 0 & d_1 \\
  d_2 & 0 & d_3 & 0
\end{pmatrix},$$

where

$$d_0 = \frac{2 \mu^2}{2k \lambda T + \mu^2 T},$$
$$d_1 = - kr^2 (\mu^2 - 2k \lambda),$$
$$d_2 = -2 \mu^2 t,$$
$$d_3 = - r^2 (\mu^2 - 2k \lambda),$$
$$d_4 = 8k \lambda T - 2 \mu^2 T.$$

The explicit form of matrix $M_3$ appearing in Equation (35) is given as

$$M_3 = \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 \\
  e_2 & e_1 & e_4 & e_3 \\
  e_3 & e_4 & e_1 & e_2 \\
  e_4 & e_3 & e_2 & e_1
\end{pmatrix},$$

where

$$e_0 = 8 \mu^2 e_1^{1/2},$$
$$e_1 = 8k \lambda r^2 (s_2 f_2 - 2s_1 f_1 \mu^2),$$
$$e_2 = 4k \lambda r^2 (f_1 \mu^2 - c_1 f_1),$$
$$e_3 = 8k \lambda r^2 (4k \lambda r^2 - f_1 \mu^2 T),$$
$$e_4 = -8k [\lambda \mu^2 r^2 (c_1 f_1 + f_2) + 4k T (\lambda^2 - 4 \lambda^4) c_1],$$
$$e_5 = 8k \lambda r^2 (s_2 f_2 - 2s_1 f_1 \mu^2),$$
$$e_6 = 4k \lambda r^2 (f_1 \mu^2 - c_1 f_1),$$
$$e_7 = 4 \left(4k \lambda T - f_1 \mu^2 r^2 \right)^2 - (16k \lambda s_1 t^2).$$

where $c_1 = \cos \phi$, $s_1 = \sin \phi$, $c_2 = \cos (2 \phi)$, and $s_2 = \sin (2 \phi)$ and

$$f_1 = 4k^2 - 1, \quad f_4 = 2 \mu^2 + 4k \lambda,$$
$$f_2 = 4k^2 + 1, \quad f_5 = 2 \mu^2 - 4k \lambda,$$
$$f_3 = c_1 f_1 - f_2, \quad f_6 = 4 \mu^2 r^2 - 4k \lambda T.$$

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.
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Mach–Zehnder interferometers, non-Gaussian operations, parity detection, quantum metrology, two mode squeezed thermal states

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