Quantum dynamics of a two-atom-qubit system

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Abstract. A physical model of the quantum information exchange between two qubits is studied theoretically. The qubits are two identical two-level atoms, the physical mechanism of the quantum information exchange is the mutual dependence of the reduced density matrices of two qubits generated by their couplings with a multimode radiation field. The Lehmberg–Agarwal master equation is exactly solved. The explicit form of the mutual dependence of two reduced density matrices is established. The application to study the entanglement of two qubits is discussed.

Keywords: Qubit, quantum information, transmission, density matrix.

1. Introduction

Quantum information processing devices and systems usually contain two-level quantum systems called qubits (quantum bits). The quantum information (QI) encoded into a qubit is its wave function (for a pure state) or its density matrix (for a mixed state), and the QI transfer or transmission between two qubits is the state transfer or transmission between them [1–6]. The physical origin of the state transfer between two qubits may be their direct coupling or their interactions with some intermediate physical fields while the QI transmission between two distant qubits is performed through the intermediary of some physical fields.

The present work is devoted to the theoretical study of the QI exchange in a two-qubit system consisting of two identical two-level atoms coupled to a multimode radiation field whose modes are initially in the vacuum state [7–11]. The exact solution of rate equations is established in section 2. The interpretation of this solution as the explicit form of quantum information transmission between two qubits is presented in section 3. The application to study the entanglement sudden death and revival is discussed in section 4. Section 5 is the conclusion.

2. Rate equations and their solution

Denote ρ the density matrix of this two-qubit system, E – the difference of two energy levels in each atom, σ±(σ3) and τ±(τ3) – the dipole rising (lowering) operators in two-level atoms in the form of the Pauli matrices. Together with σ± and τ± introduce also corresponding Pauli matrices σ3 and τ3 as well as the
unit matrices $\sigma_0 = \tau_0 = 1$. As a $4 \times 4$ matrix $\rho$ is expressed in terms of 15 generators of the SU(4) group. There are different choices of these SU(4) generators [12,13]. We use following operators

$$\Gamma_{(\alpha\beta)} = (\sigma_\alpha \otimes \tau_\beta), \quad \alpha, \beta = 0, +, -, 3, \quad (\alpha\beta) \neq (00),$$

and express density matrix $\rho$ in terms of them

$$\rho = \frac{1}{4} \sum_{(\alpha\beta) \neq (00)} \Gamma_{(\alpha\beta)} \rho_{(\alpha\beta)}.$$  

The time evolution of density matrix $\rho$ is determined by the Lehmberg–Agarwal [7, 8, 11] master equation

$$\frac{d\rho}{dt} = -i[H_S, \rho] + L\rho,$$

with

$$H_S = \frac{E}{2} [(\sigma_3 \otimes 1) + (1 \otimes \tau_3)] + J[(\sigma_+ \otimes \tau_-) + (\sigma_- \otimes \tau_+)],$$

where $J$ is the dipole–dipole coupling constant of two atoms, and

$$L\rho = \gamma \left\{ (\sigma_- \otimes 1)\rho(\sigma_+ \otimes 1) - \frac{1}{2} (\sigma_- \otimes 1)\rho(\sigma_- \otimes 1) \right\} +$$

$$+ \left\{ (1 \otimes \tau_-)\rho(1 \otimes \tau_+) - \frac{1}{2} (1 \otimes \tau_-)\rho(1 \otimes \tau_+) \right\} +$$

$$+ \chi \left\{ (\sigma_- \otimes 1)\rho(1 \otimes \tau_-) - \frac{1}{2} (\sigma_- \otimes 1)\rho(\sigma_- \otimes 1) \right\} +$$

$$+ \left\{ (1 \otimes \tau_-)\rho(\sigma_- \otimes 1) - \frac{1}{2} (1 \otimes \tau_-)\rho(\sigma_- \otimes 1) \right\},$$

where $\gamma$ is spontaneous decay rate of each atom caused by its coupling with the radiation field, and $\chi$ is the collective damping constant of two atoms. The expression of $J$ and $\chi$ in terms of $\gamma$ was given in reference [11].

The master equation (3) has following exact solution:

$$\rho_{(+0)}(t) = \frac{1}{2} e^{-iEt} \left[ e^{\gamma t} \left( J + i\frac{\gamma t}{2} \right) e^{-iHt} + \left( J - i\frac{\gamma t}{2} \right) e^{iHt} e^{-\gamma t} \right] +$$

$$+ \frac{e^{-\gamma t}}{2J - i\gamma} \left[ J + i\frac{\gamma t}{2} e^{-iHt} + \left( J - i\frac{\gamma t}{2} \right) e^{iHt} e^{-\gamma t} \right] \rho_{(+0)}(0) +$$

$$+ \frac{1}{2} e^{-iEt} \left[ e^{-\gamma t} \left( J + i\frac{\gamma t}{2} \right) e^{-iHt} + \left( J - i\frac{\gamma t}{2} \right) e^{iHt} e^{-\gamma t} \right] -$$

$$- \frac{e^{-\gamma t}}{2J - i\gamma} \left[ J + i\frac{\gamma t}{2} e^{-iHt} + \left( J - i\frac{\gamma t}{2} \right) e^{iHt} e^{-\gamma t} \right] \rho_{(0+)}(0) -$$

$$- \frac{1}{2} e^{-iEt} \left( J - i\frac{\gamma t}{2} \right) \left[ e^{\gamma t} - e^{iHt} e^{-\gamma t} + e^{\gamma t} \left( J - i\frac{\gamma t}{2} \right) e^{-iHt} e^{-\gamma t} \right] \rho_{(+3)}(0) -$$

$$- \frac{1}{2} e^{-iEt} \left( J - i\frac{\gamma t}{2} \right) \left[ e^{\gamma t} - e^{iHt} e^{-\gamma t} + e^{\gamma t} \left( J - i\frac{\gamma t}{2} \right) e^{-iHt} e^{-\gamma t} \right] \rho_{(+3)}(0) -$$

$$- \frac{1}{2} e^{-iEt} \left( J + i\frac{\gamma t}{2} \right) \left[ e^{\gamma t} - e^{iHt} e^{-\gamma t} - e^{\gamma t} \left( J + i\frac{\gamma t}{2} \right) e^{-iHt} e^{-\gamma t} \right] \rho_{(-3)}(0).$$
\begin{align}
-\frac{1}{2} e^{-i\epsilon t} \left( J - i \frac{X}{2} \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J - i \frac{X}{2} \right) e^{-i\epsilon t} - \left( J + i \frac{X}{2} + i\epsilon \right) e^{i\epsilon t} \right] e^{i\epsilon t} \left[ \left( J + i \frac{X}{2} - i\epsilon \right) e^{-i\epsilon t} - e^{-i\epsilon t} e^{-i\epsilon t} \right] \rho_{(3+)0},
\end{align}

\begin{align}
\rho_{(3+)}(t) = & \frac{1}{2} e^{-i\epsilon t} \left( J - i \frac{X}{2} \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J - i \frac{X}{2} \right) e^{-i\epsilon t} + \left( J + i \frac{X}{2} + i\epsilon \right) e^{i\epsilon t} \right] + \\
& + \frac{1}{2} e^{-i\epsilon t} \left( J - i \frac{X}{2} \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J - i \frac{X}{2} \right) e^{-i\epsilon t} + \left( J + i \frac{X}{2} - i\epsilon \right) e^{-i\epsilon t} e^{-i\epsilon t} \right] \rho_{(3+)0} + \\
& + \frac{1}{2} e^{-i\epsilon t} \left( J - i \frac{X}{2} \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J + i \frac{X}{2} + i\epsilon \right) e^{i\epsilon t} + \left( J + i \frac{X}{2} - i\epsilon \right) e^{-i\epsilon t} \right] \rho_{(3+)0} - \\
& - \frac{1}{2} e^{-i\epsilon t} \left( J + i \frac{X}{2} + i\epsilon \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J - i \frac{X}{2} \right) e^{-i\epsilon t} + \left( J + i \frac{X}{2} - i\epsilon \right) e^{-i\epsilon t} \right] \rho_{(3+)0} - \\
& - \frac{1}{2} e^{-i\epsilon t} \left( J + i \frac{X}{2} + i\epsilon \right) e^{-i\gamma t} e^{-i\epsilon^2 t} \left[ \left( J + i \frac{X}{2} - i\epsilon \right) e^{i\epsilon t} + \left( J + i \frac{X}{2} - i\epsilon \right) e^{-i\epsilon t} \right] \rho_{(3+)0}.
\end{align}

\( \rho_{(0)-} \) and \( \rho_{(3)-} \) have similar expressions with the change of the signs of \( E \) and \( J, E \to -E, J \to -J \), and the substitution \( \rho_{(3+)0} \to \rho_{(3-)0} \), \( \rho_{(3+)0} \to \rho_{(3-)0} \),

\begin{align}
\rho_{(30)}(t) = & \frac{1}{4} \left[ \frac{\gamma + X}{\gamma} e^{-(\gamma-X)\mu} + \frac{\gamma - X}{\gamma} e^{-(\gamma+X)\mu} - 4 \frac{X^2}{\gamma^2 - X^2} e^{-2\gamma t} - 2 e^{-\gamma t} e^{2J t} \right] \rho_{(30)}(0) + \\
& + \frac{1}{4} \left[ \frac{\gamma - X}{\gamma} e^{-(\gamma-X)\mu} + \frac{\gamma + X}{\gamma} e^{-(\gamma+X)\mu} - 4 \frac{X^2}{\gamma^2 - X^2} e^{-2\gamma t} - 2 e^{-\gamma t} e^{2J t} \right] \rho_{(30)}(0) - \\
& - \frac{1}{8} \left[ e^{-(\gamma-X)\mu} - e^{-(\gamma+X)\mu} - 2 i e^{-\gamma t} \sin 2J t \right] \rho_{(3-)}(0) + \\
& - \frac{1}{8} \left[ e^{-(\gamma-X)\mu} - e^{-(\gamma+X)\mu} + 2 i e^{-\gamma t} \sin 2J t \right] \rho_{(3-)}(0) - \\
& - \frac{1}{2} \left[ \frac{\gamma + X}{\gamma} e^{-(\gamma-X)\mu} - \frac{\gamma - X}{\gamma} e^{-(\gamma+X)\mu} + 4 \frac{X^2}{\gamma^2 - X^2} e^{-2\gamma t} \right] \rho_{(33)}(0) + \\
& + \frac{1}{8} \left[ \frac{\gamma - X}{\gamma} e^{-(\gamma-X)\mu} + \frac{\gamma + X}{\gamma} e^{-(\gamma+X)\mu} - 4 \frac{X^2}{\gamma^2 - X^2} e^{-2\gamma t} \right] - 2 \right],
\end{align}
Suppose that at the initial time $t = 0$ the quantum states of two spin-qubits are independent, and $\rho$.

Quantum information encoded into each qubit, called $\sigma$-qubit or $\tau$-qubit, is its reduced density matrix $\rho^\sigma = T_T \rho$ or $\rho^\tau = T_\sigma \rho$, where $T_T$ or $T_\sigma$ denotes the trace with respect to the spin indices of spin-qubit $\tau$ or $\sigma$. We have

$$\rho^\sigma = \frac{1}{2} + \sum_\alpha \sigma_\alpha \rho^\sigma_\alpha, \quad \rho^\sigma_\alpha(t) = 2 \rho_{(\alpha0)}(t), \quad \rho^\tau = \frac{1}{2} + \sum_\alpha \tau_\alpha \rho^\tau_\alpha, \quad \rho^\tau_\alpha(t) = 2 \rho_{(0\alpha)}(t).$$

3. Quantum Information Exchange

Quantum information encoded into each qubit, called $\sigma$-qubit or $\tau$-qubit, is its reduced density matrix $\rho^\sigma = T_T \rho$ or $\rho^\tau = T_\sigma \rho$, where $T_T$ or $T_\sigma$ denotes the trace with respect to the spin indices of spin-qubit $\tau$ or $\sigma$. We have

$$\rho^\sigma = \frac{1}{2} + \sum_\alpha \sigma_\alpha \rho^\sigma_\alpha, \quad \rho^\sigma_\alpha(t) = 2 \rho_{(\alpha0)}(t), \quad \rho^\tau = \frac{1}{2} + \sum_\alpha \tau_\alpha \rho^\tau_\alpha, \quad \rho^\tau_\alpha(t) = 2 \rho_{(0\alpha)}(t).$$

Suppose that at the initial time $t = 0$ the quantum states of two spin-qubits are independent, and therefore
\[ \rho_{(a0)}(0) = \frac{1}{2} \rho^\sigma_a(0), \quad \rho_{(0a)}(0) = \frac{1}{2} \rho^\tau_a(0). \]
\[ \rho_{(a\beta)}(0) = \rho^\sigma_a(0) \rho^\tau_\beta(0), \quad \alpha \neq 0, \beta \neq 0. \]

Then, from the expressions of \( \rho_{(20)}(t) \) and \( \rho_{(30)}(t) \) presented in preceding Section we derive following formulae

\[
\rho^\sigma_a(t) = \frac{1}{2} e^{\gamma iEt} \left\{ \frac{e^{-(\gamma - \chi)t}}{2J + i\gamma} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \right. \\
\left. + \frac{e^{-(\gamma - \chi)t}}{2J + i\gamma} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \rho^\sigma_a(0) + \\
\right. \\
\left. + \frac{1}{2} e^{\gamma iEt} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \rho^\tau_\beta(0) \right\} \\
(13)
\]

\[
\rho^\tau_\beta(t) = \frac{1}{2} e^{\gamma iEt} \left\{ \frac{e^{-(\gamma - \chi)t}}{2J + i\gamma} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \right. \\
\left. + \frac{e^{-(\gamma - \chi)t}}{2J + i\gamma} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \rho^\sigma_a(0) \rho^\tau_\beta(0) - \\
\right. \\
\left. - \frac{1}{2} e^{\gamma iEt} \left[ (J + i\frac{\chi}{2} - i\gamma) e^{2i\gamma t} + (J + i\frac{\chi}{2} + i\gamma) e^{-2i\gamma t} \right] \rho^\tau_\beta(0) \rho^\sigma_a(0) \right\} \\
(14)
\]

\[
\rho^\sigma_a(t) = \frac{1}{4} \left[ \frac{\gamma - \chi}{\gamma + \chi} e^{-(\gamma - \chi)t} + \frac{\gamma + \chi}{\gamma - \chi} e^{-(\gamma + \chi)t} - 4 \frac{\chi^2}{\gamma^2 - \chi^2} e^{-2\gamma t} + 2 e^{-\gamma t} \cos 2Jt \right] \rho^\sigma_a(0) + \\
\left. + \frac{1}{4} \left[ \frac{\gamma - \chi}{\gamma + \chi} e^{-(\gamma - \chi)t} + \frac{\gamma + \chi}{\gamma - \chi} e^{-(\gamma + \chi)t} - 4 \frac{\chi^2}{\gamma^2 - \chi^2} e^{-2\gamma t} - 2 e^{-\gamma t} \cos 2Jt \right] \rho^\tau_\beta(0) - \\
\right. \\
\left. - \frac{1}{4} e^{-(\gamma - \chi)t} \rho^\sigma_a(0) \rho^\tau_\beta(0) - \\
\right. \\
\left. - \frac{1}{4} e^{-(\gamma + \chi)t} \rho^\sigma_a(0) \rho^\tau_\beta(0) \right\} \\
(15)
\]

and similar ones with the interchange \( \sigma \leftrightarrow \tau \). They show that reduced density matrix \( \rho^\sigma(t) \) of \( \sigma \)-qubit at \( t > 0 \) depends not only on its initial state \( \rho^\sigma(0) \), but also on the initial state \( \rho^\tau(0) \) of \( \tau \)-qubit, and vice-versa. This mutual dependence of reduced density matrices of two qubits exhibits a physical mechanism of the QI exchange between them: read-out QI of \( \sigma \)-qubit at \( t > 0 \) reflects in-put QI encoded onto \( \tau \)-qubit.
at $t = 0$, and vice-versa. Note that even in the special case with $\rho^\tau(0) = 0$, $\alpha = \pm 3$, the time evolution of reduced density matrix $\rho^\tau(t)$,

$$\rho^\tau_\pm(t) = \frac{1}{2} e^{\mp iE_\pm t} \left[ e^{-\gamma \chi} e^{\pm i \gamma / 2} \left( J \pm i \frac{\chi}{2} \pm i \gamma \right) e^{\mp i \gamma / 2} e^{\pm i \gamma / 2} \right] +$$

$$+ e^{-(\gamma - \chi)^2 / 2} \left[ \frac{J \pm i \frac{\chi}{2} \mp i \gamma}{2} e^{\mp i \gamma / 2} + \frac{J \mp i \frac{\chi}{2} \mp i \gamma}{2} e^{i \gamma / 2} \right] \rho^\tau_\pm(0),$$

$$\rho^\tau_\pm(t) = \frac{1}{4} \left[ \frac{\gamma - \chi}{\gamma + \chi} e^{-\gamma \chi} + \frac{\gamma + \chi}{\gamma - \chi} e^{-(\gamma + \chi)^2 / 2} e^{-2 \gamma^2 / 2} + 2 e^{-\gamma^2 / 2} \cos 2\chi t \right] \rho^\tau_\pm(0) +$$

is different from that of a single qubit

$$\rho^\tau_\pm(t) = e^{\gamma E_{\pm} t} e^{-\gamma} \rho^\tau_\pm(0),$$

$$\rho^\tau_\pm(t) = e^{-\gamma} \rho^\tau_\pm(0) + \frac{1}{2} (e^{-\gamma} - 1).$$

In general, by comparing the time evolution of reduced density matrix $\rho^\tau(t)$ with that of a single qubit $\rho^\tau(t)$, we can discover the presence of $\tau$-qubit and also determine its initial state $\rho^\tau(0)$.

4. Entanglement

For studying the entanglement between two qubits we can apply reasonings of reference [11] and use the collective Dick states [14]

$$|e\rangle = |11\rangle, \quad |g\rangle = |22\rangle,$$

$$|s\rangle = \frac{1}{\sqrt{2}} \left[ |21\rangle + |12\rangle \right], \quad |a\rangle = \frac{1}{\sqrt{2}} \left[ |21\rangle - |12\rangle \right]$$

as the basis vectors. In this basis $\rho$ has following matrix elements

$$\rho_{ee}(t) = \rho_{ee}(0) e^{-2\gamma t},$$

$$\rho_{eg}(t) = \rho_{eg}(0) e^{-\gamma E t},$$

$$\rho_{ge}(t) = \rho_{ge}(0) e^{-\gamma E t},$$

$$\rho_{gg}(t) = 1 + \frac{1}{2} \left[ e^{-\gamma E t} + e^{-(\gamma + \chi)^2 / 2} \right] \rho_{gg}(0) +$$

$$+ \frac{1}{2} \left[ \frac{\gamma^2 + 3 \chi^2 - 3 e^{-2 \gamma t} \chi + 3 \chi - \gamma e^{-(\gamma - \chi)^2 / 2} \gamma - \chi}{\gamma^2 - \chi^2} e^{-2 \gamma t} \gamma + \chi \right] \rho_{ee}(0) +$$

$$+ \frac{1}{2} \left[ e^{-(\gamma - \chi)^2 / 2} \left[ \rho_{ss}(0) - \rho_{ss}(0) \right] - \frac{1}{2} \left[ e^{-(\gamma - \chi)^2 / 2} + e^{-(\gamma + \chi)^2 / 2} \right] \right].$$
\[\rho_{ss}(t) = \frac{1}{2} \left[ \frac{Y-3X}{Y-X} e^{-(y+x)t} + \frac{Y+X}{Y-X} e^{-2y} \right] \rho_{ss}(0) + \frac{1}{2} \left[ \frac{Y+X}{Y-X} e^{-2y} - e^{-(y+x)t} \right] \rho_{aa}(0) - \rho_{ee}(0) + \rho_{gg}(0) - 1, \]

(25)

\[\rho_{sa}(t) = \rho_{sa}(0)e^{-yi} e^{-2jt}, \]

(26)

\[\rho_{as}(t) = \rho_{as}(0)e^{-yi} e^{2jt}, \]

(27)

\[\rho_{aa}(t) = \left( \frac{Y+X}{Y-X} \right) e^{-(y+x)t} + \frac{Y-X}{Y-X} \right] \rho_{aa}(0) + \frac{1}{2} \left[ \frac{Y-Y-X}{Y-X} e^{2yi} - e^{-(y+x)t} \right] \right| \rho_{ss}(0) - \rho_{ee}(0) + \rho_{gg}(0) - 1, \]

(28)

\[\rho_{ss}(t) = \rho_{ss}(0)e^{-(3y+x)t/2} e^{-i(E-J)t}, \]

(29)

\[\rho_{se}(t) = \rho_{se}(0)e^{-(3y+x)t/2} e^{i(E-J)t}, \]

(30)

\[\rho_{ea}(t) = \rho_{ea}(0)e^{-(3y-x)t/2} e^{-i(E+J)t}, \]

(31)

\[\rho_{ce}(t) = \rho_{ce}(0)e^{-(3y-x)t/2} e^{i(E+J)t}, \]

(32)

\[\rho_{gs}(t) = e^{-(y+x)t/2} e^{-i(E+J)t}, \]

(33)

\[\rho_{sg}(t) = e^{-(y-x)t/2} e^{i(E-J)t}, \]

(34)

\[\rho_{ga}(t) = e^{-(y-x)t/2} e^{-i(E-J)t}, \]

(35)

\[\rho_{ag}(t) = e^{-(y+x)t/2} e^{i(E+J)t}, \]

(36)

Consider a special class of quantum states satisfying conditions

\[\rho_{ss} = \rho_{ee} = \rho_{aa} = \rho_{gg} = \rho_{gg} = \rho_{gg} = \rho_{aa} = \rho_{aa} = 0 \]

(37)

and

\[\rho_{ge} = \rho_{eg}^* \]

(38)

at \( t = 0 \). From the explicit expressions of the elements of density matrix \( \rho \), equations (21)–(36), it follows that the same conditions still hold at any \( t > 0 \). For this class the formulae of the concurrence [15, 16] were derived by Ficek and Tanas [11]. Similarly, if the elements of \( \rho \) satisfy conditions

\[\rho_{ge} = \rho_{ge} = \rho_{ge} = \rho_{ge} = \rho_{ge} = \rho_{ge} = \rho_{ge} = \rho_{ge} = 0 \]

(39)

and

\[\rho_{eg} = \rho_{ge}^* \]

(40)

at \( t = 0 \), then these conditions still hold at any \( t > 0 \). For this class, the formulae of the concurrence were derived by Yu and Eberly [17]. Using formulae given in references [11, 17] it can be shown that for suitable initial values of elements of \( \rho \) there may take place the entanglement sudden death and revival, as this phenomenon was discovered in references [11, 17] in the cases of some specific states.
5. Conclusion
A model two-atom-qubit system was investigated. It consists of two identical two-level atoms coupled to a multimode radiation field whose modes are initially in the vacuum state. Its quantum dynamics is described by a density matrix satisfying the Lehmberg-Agarwal equation, which was exactly solved. Its solution demonstrates a physical mechanism of the QI exchange between two qubits. It would be also useful for the study of the entanglement sudden death and revival in two wide classes of two-qubit systems satisfying definite initial conditions.

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