A NOTE ON EDGE ORIENTED REINFORCED RANDOM WALKS AND RWRE.

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Abstract. This work introduces the notion of edge oriented reinforced random walk which proposes in a general framework an alternative understanding of the annealed law of random walks in random environment.

1. Introduction

The aim of this text is to present in a general framework a precise and explicit correspondance between a class of edge oriented reinforced random walks and random walks in random environment (RWRE). In other words the study of an edge oriented reinforced random walks, which is strongly non-Markovian, is equivalent by introducing an external randomness, to a Markovian problem. This relation already appeared, as a tool, in a work of Pemantle (cf [3]), for the study of reinforced random walks on binary trees, for very special laws of reinforcement.

The essence of the result is that the law of an edge oriented reinforced random walk coincides with the annealed laws of a RWRE if the law of reinforcement satisfies the following condition: at one point the probability of a sequence of successive moves depends only on the number of each type of move. This condition, we call admissibility, is expressed as the closedness of a certain discrete form. The proof of our result is based on a theorem on the moment problem for random variables in $[0, 1]^d$. As a corollary, the uniqueness of this moment problem implies that the annealed law determines the quenched law for RWRE.

2. Definitions and statement of the result

Definition 1. We call law of reinforcement with $d$ neighbours a function $V$:

$$V : \mathbb{Z}_+^d \mapsto T_d := \{(x_1, \ldots, x_d) \in [0, 1]^d, \text{ s.t.}, \sum_{i=1}^{d} x_i = 1\}$$

$$\vec{p} = (p_1, \ldots, p_d) \rightarrow (V_1(\vec{p}), \ldots, V_d(\vec{p}))$$

We now define for a law of reinforcement the notion of admissibility, which will play a key role in the following. Let $(e_i)_{1 \leq i \leq d}$ denote the canonical basis of $\mathbb{Z}_+^d$. 
Definition 2. Let us consider the graph $\mathbb{Z}_+^d$ whose edges $(\vec{p}, \vec{p} + e_i)$ are oriented from $\vec{p}$ to $\vec{p} + e_i$. Let $V$ be a law of reinforcement on $\mathbb{Z}_+^d$ and $w$ be the 1-form defined by $w((\vec{p}, \vec{p} + e_i)) = \ln V_i(\vec{p})$.

We shall say that $V$ is admissible when $w$ is a closed form.

We consider now a countable graph $G$ having at any point a finite number of neighbours. For any vertex $x$ of $G$ we denote by $d(x)$ the cardinal of the neighbours of $x$ and $\{e(x, i), i = 1, ..., d(x)\}$ the neighbours of $x$. At any vertex we suppose given a law of reinforcement $V(x)$ with $d(x)$ neighbours: $V(x) : \vec{p} \rightarrow (V_i(x, \vec{p}))_{1 \leq i \leq d(x)}$.

Definition 3. We call reinforced random walk with law of reinforcement $V(x)$, the random walk defined by the family of laws on the trajectories starting at $x_0$, $(\hat{P}_{x_0})_{x_0 \in G}$ given by

$$\hat{P}_{x_0}(X_{n+1} = e(x, i)|\sigma(X_n = x) \land \sigma(X_k, k \leq n - 1)) = V_i(x, \vec{N}(x))$$

where $\vec{N}(x) = (N_i(x))_{1 \leq i \leq d(x)}$ and $N_i(x) = \sum_{l=0}^{d(x)-1} 1\{X_l = x, X_{l+1} = e(x, i)\}$.

Definition 4. A reinforced random walk is called admissible when $V(x)$ is admissible for all vertex $x$ of $G$.

Let us now introduce random walks in random environment on the graph $G$.

We define an environment as an element $\omega = (\omega(x))_{x \in G}$ where at any vertex $x$, $\omega(x)$ is in $T_{d(x)}$. At any vertex $x$ of $G$, we consider a probability measure $\mu_x$ on $T_{d(x)}$ and we set $\mu := \otimes_{x \in G} \mu_x$, so that $\mu$ is a probability measure on the environments such that $(\omega(x))_{x \in G}$ are independent random variables of law $\mu_x$.

We denote by $P_{x_0, \omega}$ the law of the Markov chain in the environment $\omega$ starting at $x_0$ defined by:

$$\forall x_0 \in G, \forall k \in \mathbb{N}, \ P_{x_0, \omega}(X_{k+1} = e(x, i)|X_k = x) = \omega(x, i).$$

Finally we denote by $P_x$ the annealed measure i.e. $P_x = \mu \otimes P_{x, \omega}$. We are now able to state our main result:

Theorem 1. For any countable graph $G$

i) For all law of environment $\mu = \otimes_{x \in G} \mu_x$ the law of the reinforced random walk $(\hat{P}_{x_0})_{x_0 \in G}$ associated with the law of reinforcement $V$ given by:

$$V_i(x, p_1, ..., p_{d(x)}) = \frac{E_{\mu_x}[\omega(x, i) \prod_{j=1}^{d(x)} \omega(x, j)^{p_j}]}{E_{\mu_x}[\prod_{j=1}^{d(x)} \omega(x, j)^{p_j}]}$$

where $E_{\mu_x}$ denotes the expectation under the law $\mu_x$ coincides with the annealed law of the RWRE $(P_{x_0})_{x_0 \in G}$. Moreover the law $(V(x))$ is admissible.

ii) Conversely, if $V$ is an admissible law of reinforcement on $G$, then there exists a unique law of environment $\mu = \otimes_{x \in G} \mu_x$ for which equality (1) is
satisfied, thus for which the law of the reinforcement random walk \((\hat{P}_{x_0})_{x_0 \in G}\) coincides with the annealed law \((P_{x_0})_{x_0 \in G}\).

**Corollary 1.** The annealed law determines the quenched law for RWRE, i.e. there can not exist two different laws of environment \(\mu\) and \(\mu'\) with the same annealed law \((P_{x_0})_{x_0 \in G}\).

Example 1: At any point \(x\), choose a vector \((\alpha(x,1), \ldots, \alpha(x,d(x)))\) in \((\mathbb{R}_+^*)^{d(x)}\). The law of reinforcement \(V_t(x, p_1, \ldots, p_{d(x)}) = \frac{\alpha(x,i) + p_i}{\sum_{j=1}^{d(x)} \alpha(x,j) + p_j}\) is an admissible law of reinforcement associated with the environment \((\mu_x)_{x \in G}\) where \(\mu_x\) is a Dirichlet law with parameters \((\alpha(x,1), \ldots, \alpha(x,d(x)))\), i.e. \(\mu_x\) is the law on \(T_{d(x)}\) with density

\[
\mu_x(t_1, \ldots, t_{d(x)}) = \frac{\Gamma(\alpha(x,1) + \ldots + \alpha(x,d(x)))}{\prod_{i=1}^{d(x)} \Gamma(\alpha(x,i))} \prod_{i=1}^{d(x)} t_i^{\alpha(x,i)-1}.
\]

This is the meaning of the classical Polya’s urn scheme (cf [1], section VI.12), if we put at all sites an independent urn which will define the move at that site.

Example 2: We can generalize the previous example as follows: at all point \(x\) in \(G\), choose not only a vector \((\alpha(x,1), \ldots, \alpha(x,d(x)))\) in \((\mathbb{R}_+^*)^{d(x)}\), but also an integer \(n(x)\) and a homogeneous polynomial \(P(x, t_1, \ldots, t_{d(x)})\) of degree \(n(x)\) of the form:

\[
P(x, t_1, \ldots, t_{d(x)}) = \sum_{k_1, \ldots, k_{d(x)} = n(x)} a_{k_1, \ldots, k_{d(x)}}(x) t_1^{k_1} \ldots t_{d(x)}^{k_{d(x)}},
\]

where the \(a_{k_1, \ldots, k_{d(x)}}(x)\)’s are non-negative reals (not all null). Then we define \(\mu_x\) as the law on \(T_{d(x)}\) with density

\[
\mu_x(t_1, \ldots, t_{d(x)}) = \frac{\left(\prod_{i=1}^{d(x)} t_i^{\alpha(x,i)-1}\right) P(x, t_1, \ldots, t_{d(x)})}{\int_{T_{d(x)}} \left(\prod_{i=1}^{d(x)} t_i^{\alpha(x,i)-1}\right) P(x, t_1, \ldots, t_{d(x)})}.
\]

On the other side, consider the polynomials

\[
Q(x, y_1, \ldots, y_{d(x)}) = \sum_{k_1, \ldots, k_{d(x)} = n(x)} a_{k_1, \ldots, k_{d(x)}}(x) \prod_{i=1}^{d(x)} (y_i, k_i),
\]

where we write \((y, k)\) for the product \(y \cdot \ldots \cdot (y + k - 1)\). Then the law \((\mu_x)\) is associated with the law of reinforcement \(V(x)\), where \(V_t(x, p_1, \ldots, p_{d(x)})\) is given by (to simplify, we forget the \(x\) dependance in the next formula, and simply write \(\alpha_i\) for \(\alpha(x,i)\) and \(n\) for \(n(x)\))

\[
\left(\frac{\alpha_i + p_i}{\sum_{j=1}^{d} \alpha_j + p_j + n}\right) \frac{Q(\alpha_1 + p_1, \ldots, \alpha_i + p_i + 1, \ldots, \alpha_d + p_d)}{Q(\alpha_1 + p_1, \ldots, \alpha_d + p_d)}.
\]
where $N$ the variables $\omega$ the moments of $\mu$ implies the solvability of the moment problem for $y$ with $V$.

We can see that the condition is expressed in terms of the moments of $\mu$, therefore the aim is to prove that the assumption of admissibility on $V$ implies the solvability of the moment problem for $\mu$.

For that purpose we introduce the quantities which are intended to be the moments of $\mu$.

Consider $s \in \{1, ..., d\}^\mathbb{N}$ and the path $U(s, \cdot)$ on $\mathbb{Z}_+^d$ defined by:

$$U(s, n) := \sum_{i=0}^{n-1} e_{s(i)}$$

and let $M(s, n) := \prod_{i=0}^{n-1} V(s(i))(U(s, i))$.

The fact that $V$ is admissible implies that $M(s, n)$ depends only on $U(s, n)$, indeed $M(s, n) = \exp(\int_0^{U(s, n)} w)$ (with the notations of definition 2). So let us introduce the sequence with $d$ indices:

$$v_{k_1, ..., k_d} := M(s, n) \text{ for } U(s, n) = k_1 e_1 + ... + k_d e_d$$

What remains to prove is that $v$ is the sequence of moments of a $T_{d(x)}^\mu$-valued variable. For that purpose we use the generalization of the Hausdorff criterium, concerning the existence of a (unique) solution to the moment problem for random variables on $[0, 1]^d$. For any $\hat{h} = h_1, \ldots, h_d$ we define the operator $\Delta^\hat{h}$ on real sequences indexed by $\mathbb{Z}_+^d$, i.e. $\Delta^\hat{h} : \mathbb{R}^{\mathbb{Z}_+^d} \rightarrow \mathbb{R}^{\mathbb{Z}_+^d}$ and defined recursively by

$$\Delta^{e_i}(u) = (u_{k+e_i} - u_k)_{k \in \mathbb{Z}_+^d},$$

and

$$\Delta^{k+e_i} = \Delta^{e_i} \circ \Delta^k.$$

(Remark that this definition is valid since the $\Delta^{e_i}$'s commute). We recall here the result of Hildebrandt and Schoenber (cf [2]): a sequence $(u_k) \in [0, 1]^{\mathbb{Z}_+^d}$ is the moment sequence of a probability measure on $[0, 1]^d$, i.e. $u_k = \int t_1^{k_1} \cdots t_d^{k_d} d\mu(t_1, \ldots, t_d)$ if and only if for all $\hat{h}$ and $\tilde{k}$ in $\mathbb{Z}_+^d$, $(-1)^{\sum h_i} \Delta^\hat{h}(u)(\tilde{k})$ is positive.

Proof:

i) For any vertices $x_0, x$ of $G$, $\forall 1 \leq i \leq d(x), \forall n \in \mathbb{N}$,

$$P_{x_0}(X_{n+1} = e(x,i)|(X_n = x) \wedge \sigma(X_k, k \leq n-1))$$

$$= \frac{E[\omega(x,i) \prod_{y \in G} N_j^d(y) \omega(y,j)N_j^d(y)]}{E[\prod_{y \in G} \prod_{j=1}^d \omega(y,j)N_j^d(y)]},$$

where $N_j^d(y)$ is as defined in definition 3. Now using the independance of the variables $\omega(y,i)$ for different vertices $y$, the terms depending on $\omega(y,j)$ for $y \neq x$ cancel in the previous ratio and we get i).

ii) The only point is to prove that for any admissible law of reinforcement $V$ with $d$ neighbourhoods there exists a probability measure $\mu$ on $T_d$, such that a $T_d^\mu$-valued random variable $\bar{X} := (X_1, ..., X_d)$ of law $\mu$ satisfies $V(p_1, ..., p_d) = E[M(s, n)] = \exp(\int_0^{U(s, n)} w)$ (with the notations of definition 2).
Let us verify this for the sequence $v_k$ introduced previously. Since for all $k$, $\sum_{i=1}^{d} V_i(k) = 1$ we have:

$$-\Delta^e_i(v)(k) = \sum_{j=1}^{d} v_{k_1, \ldots, k_{j-1}+1, \ldots, k_d}.$$ 

Hence, by composition $(-1)^{h_1 + \cdots + h_d} \Delta^{h_1, \ldots, h_d}(v_{k_1, \ldots, k_d})$ will always be a linear combination with positive coefficients of the terms of the sequence $v$. So the condition of the criterium is satisfied. Hence $\mu$ exists and is unique as a solution of the moment problem whose support is compact. The last thing to check is that it is supported by $T_d$. Let $X$ be a random variable with law $\mu$. The only thing to check is that $\sum_{i=1}^{d} X_i = 1$ $\mu$-almost surely. Since the law of $\sum_{i=1}^{d} X_i$ has compact support this is equivalent to show that all the moments of $\sum_{i=1}^{d} X_i$ are equal to 1. Using the fact that $\sum_{i=1}^{d} V_i(k) = 1$ at all point $k$ we know that for all integer $n$

$$1 = \sum_{s \in \{1, \ldots, d\}^n} M(s, n)$$

$$= \sum_{k_1 + \cdots + k_d = n} \# \{ s \in \{1, \ldots, d\}^n, U(s, n) = (k_1, \ldots, k_d) \} v_{k_1, \ldots, k_d}$$

$$= \sum_{k_1 + \cdots + k_d = n} \frac{(k_1 + \cdots + k_d)!}{k_1! \cdots k_d!} v_{k_1, \ldots, k_d},$$

and this last expression is the $n$-th moment of $(X_1 + \ldots + X_d)$.

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