Non-relativistic four dimensional p-brane supersymmetric theories and Lie algebra expansion

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Abstract
We apply the Lie algebra expansion method to the $\mathcal{N} = 1$ super-Poincaré algebra in four dimensions. We define a set of p-brane projectors that induce a decomposition of the super-Poincaré algebra preparatory for the expansion. We show that starting from the $\mathcal{N} = 1$ supergravity action in four dimensions it is possible to obtain non-relativistic supersymmetric theories, describing strings or membranes.

Keywords: supergravity, Lie algebra, Newton–Cartan, supersymmetry, non-relativistic theories, p-brane

1. Introduction
In recent years non-relativistic gravity theories have received a growing interest for their role in a wide range of physical contexts, as condensed matter physics, where gravity is used as background to build effective theories, and holography, where, as an example, Newton–Cartan geometry has been recognized as the boundary geometry of a certain class of $z = 2$ Lifshitz spacetime [1–5].

Among the wide class of non-relativistic gravity theories a prominent role is covered by Newtonian gravity and its geometrical re-formulation, Newton–Cartan gravity [6, 7]. This theory has required a specific geometrical setting, Newton–Cartan geometry, that plays, for the Newtonian gravity the same role played by Riemannian geometry for general relativity [8–17].

In [18] it has been show how starting from the Bargmann algebra, the central extension of the Galilei algebra, through a gauging procedure, it is possible to obtain Newton–Cartan formulation of Newtonian gravity. This procedure has been generalized to the stringy case and extension of the Galilei algebra [19–21].

Recently this gauging procedure has been combined with the Lie algebra expansion method [22–27] to reproduce in a systematic way the results of [19–21] and to define a method to
build non-relativistic actions [28]. In particular it is possible, starting from the Poincaré group, and expanding it opportunitely, to generate an extension of the Galilei algebra. The expansion induced on the gauge fields defines the key to obtain non-relativistic actions from an initial relativistic action.

Along the the same way as for purely bosonic theories a raising attention has been direct to supersymmetric extension of non-relativistic theories. In particular, promoting the Bargmann algebra to a superalgebra, and refining the gauge procedure with opportune curvature constraints, this approach has led to the description of three dimensional non-relativistic supersymmetric theories [29–31]. In these cases the Galilei group of the bosonic case is replaced by the super-Galilei group. This suggests a natural way to extend the systematic procedure of [28]. In particular it is possible to start from the super-Poincaré algebra and use the Lie algebra expansion method to obtain the super-Galilei algebra and its extended versions. Several results have been obtained in this direction, via the Lie algebra expansion [32], or the semigroup expansion method [33, 34]. In particular in [32] the Lie algebra expansion has been applied to the \( \mathcal{N} = 2 \) super-Poincaré algebra in three dimensions and the Chern–Simons action.

In the present paper we extend the results of [32] to the \( \mathcal{N} = 1 \) four dimensional super-Poincaré algebra discussing string and membrane cases. In particular opposite to that work, starting with the \( \mathcal{N} = 1 \) theory we will find in our final theories fermions propagating half degrees of freedom of a Majorana fermion. We will see that both for strings and membranes it is possible to define non-relativistic actions. Furthermore our discussion will define a general method that could be applied to obtain different \( D \)-dimensional non-relativistic or ultra-relativistic \( p \)-brane supergravity theories.

The paper is organized as follows. In section 1.1 we describe the general setting; starting from the super-Poincaré algebra we define an opportune index splitting and derive the fundamental ingredients for the gauging procedure. In section 1.2 we specify our analysis to the four dimensional \( \mathcal{N} = 1 \) super-Poincaré algebra and we present our main results. We study the \( D = 4, \mathcal{N} = 1 \) supergravity action in 1.5 order formalism. Through the application of the Lie algebra expansion method to the four dimensional super-Poincaré algebra we derive two non-relativistic actions, one describing strings, the other describing membranes. In appendix A we study the technical details that are fundamental to define the Lie algebra expansion. In particular we define a set of \( p \)-brane projectors that provide us with a super-Poincaré algebra decomposition opportunely tailored to obtain a non-relativistic \( p \)-brane algebras after the expansion. Our discussions try to be as general as possible and could be immediately applied to higher dimensional cases and \( \mathcal{N} > 1 \) as well.

1.1. Lie algebra expansion: the super-Poincaré algebra

In this section we discuss the \( \mathcal{N} = 1 \) super-Poincaré algebra and prepare the setting for the analysis next to come. We perform a gauging of the superalgebra by introducing the associated gauge fields and parameters and describe the corresponding transformations. This is preparatory for the application of the expansion. In particular by applying the Lie algebra expansion method to the \( \mathcal{N} = 1 \) super-Poincaré algebra we construct superalgebras describing the non relativistic regime. The construction is based on an expansion of the generators of the algebra, in a parameter that could be identified with \( 1/c, c \) being the speed of light, and this induces an analogous expansions for gauge fields and curvatures. Our strategy to generate new Lagrangians describing the non-relativistic regime is to use these expansion directly into the the \( \mathcal{N} = 1 \) supergravity Lagrangian [27, 35]. By this way, due to the underlying algebraic structure provided by the Lie algebra expansion method we have full control of the symmetries of the new actions.
1.1.1. General setting and gauging. We are going to define our setup for the analysis of the $\mathcal{N} = 1$ super-Poincaré algebra in four dimensions and its expansion. We discuss our setting with a general approach, assuming the superalgebra we are going to study is $D$-dimensional in a space with signature $(+,−,\ldots,−)$. Although some case by case modifications are required for $D \neq 4$, for our purpose it is convenient to proceed as described, since our analysis lends itself to immediate generalizations.

We consider the following commutation relations for the $D$-dimensional $\mathcal{N} = 1$ super-Poincaré algebra:

\begin{align}
[P_A, P_B] &= 0 \\
[J_{\hat{A} \hat{B}}, P_C] &= 2\eta_{\hat{C} \hat{B}} P_{\hat{A}} \\
[J_{\hat{A} \hat{B}}, J_{\hat{C} \hat{D}}] &= 4\eta_{\hat{A} \hat{C}} J_{\hat{B} \hat{D}} \\
\{Q^\alpha, Q^\beta\} &= i(\gamma^\hat{\alpha} C^{-1})^{\alpha\beta} P_{\hat{A}} \\
[J_{\hat{A} \hat{B}}, Q^\alpha] &= -\frac{1}{2}(\gamma_{\hat{A} \hat{B}})^{\alpha\beta} Q^\beta,
\end{align}

where the hatted indices run over $\hat{A} = 0, \ldots, D - 1$ and the $Q$ are Majorana spinors.

We gauge the algebra associating to each generator a gauge field and a parameter:

Gauge Fields

\begin{align}
J_{\hat{A} \hat{B}} &\rightarrow \Omega^{\hat{A} \hat{B}}_\mu \\
P_{\hat{A}} &\rightarrow E^\hat{A}_\mu \\
Q^\alpha &\rightarrow \psi^\alpha_{\mu \alpha},
\end{align}

Gauge Parameters

\begin{align}
J_{\hat{A} \hat{B}} &\rightarrow \lambda^{\hat{A} \hat{B}} \\
P_{\hat{A}} &\rightarrow \eta^\hat{A} \\
Q^\alpha &\rightarrow \epsilon^\alpha.
\end{align}

The gauge curvatures are

\begin{align}
R^{\hat{A} \hat{B}}(J) &= d\Omega^{\hat{A} \hat{B}} - \Omega^{\hat{A} \hat{C}} \wedge \Omega_{\hat{C} \hat{B}} \\
R^\hat{A}(P) &= dE^\hat{A} + \Omega^{\hat{A} \hat{B}} \wedge E_{\hat{B}} - \frac{i}{2} \psi \wedge \gamma^{\hat{A}} \psi \\
R^\alpha(Q) &= d\psi^\alpha + \frac{1}{4} \Omega^{\hat{A} \hat{B}} \wedge (\gamma_{\hat{A} \hat{B}} \psi)^\alpha.
\end{align}

The transformations laws for the gauge fields read

1. Note that with this choice the generators $J$ are anti-Hermitian and $\Omega^{\hat{A} \hat{B}}_\mu$ is real.
2. This assumption clearly limits the possible dimensions.
\[ \delta \hat{\Omega}^{\hat{\alpha} \hat{\beta}} = d\hat{\gamma}^{\hat{\alpha} \hat{\beta}} - 2\hat{\eta}^{[\hat{\alpha}}\hat{\chi}^{\hat{\beta}]} \]  (5a)

\[ \delta E^{\hat{\alpha}} = d\hat{\gamma}^{\hat{\alpha}} - \hat{\gamma}^{\hat{\alpha} \hat{\beta}} E_{\hat{\beta}} + \hat{\eta}_{\hat{\beta}} \Omega^{\hat{\alpha} \hat{\beta}} + \hat{\tau} \hat{\gamma}^{\hat{\alpha}} \]  (5b)

\[ \delta \psi^\alpha = \left[ \left( d + \frac{1}{4} \hat{\gamma}^{\hat{a} \hat{b}} {\gamma}_{{\hat{a} \hat{b}}} \right) \epsilon^\alpha \right] - \frac{1}{4} \lambda^{\hat{a} \hat{b}} (\gamma_{\hat{a} \hat{b}} \psi)^\alpha \]  (5c)

and for the associated curvatures

\[ \delta R^{\hat{a} \hat{b} \hat{c}}(J) = -\lambda^{\hat{a} \hat{b} \hat{c}} R^{\hat{a} \hat{b} \hat{c}}(J) \]  (6a)

\[ \delta R^{\hat{a} \hat{b} \hat{c}}(P) = -\lambda^{\hat{a} \hat{b} \hat{c}} R^{\hat{a} \hat{b} \hat{c}}(J) + \hat{\gamma}^{\hat{a} \hat{b} \hat{c}} \gamma^{\hat{a} \hat{b} \hat{c}} \]  (6b)

\[ \delta R^{\hat{a} \hat{b} \hat{c}}(Q) = -\frac{1}{4} \lambda^{\hat{a} \hat{b}} (\gamma_{\hat{a} \hat{b}} R)(Q)^\alpha + \frac{1}{4} R^{\hat{a} \hat{b}}(J) (\gamma_{\hat{a} \hat{b}} \epsilon)^\alpha. \]  (6c)

Now we consider a decomposition of the indices as \( \hat{A} = \{ A, a \} \) with \( A = 0, \ldots, p \) and \( a = p + 1, \ldots, D - 1 \). This induces a decomposition of the generators as

\[ J_{\hat{A} \hat{B}} \rightarrow \{ J_{AB}, G_{AB}, J_{ab} \} \]  (7a)

\[ P_A \rightarrow \{ H_A, P_a \}. \]  (7b)

Furthermore performing the manipulations described in appendix A, using the projectors \( \Pi^\alpha_a \) (we always assume \( \gamma_0 \) to be in \( U \), see appendix A), we find, directly from equation (1), the following commutation rules,

\[ [J_{AB}, J_{CD}] = 4\eta_{(b(C|J_{D)|B]} \]  (8a)

\[ [J_{ab}, J_{cd}] = 4\eta_{(b|J_{d)}|a]} \]  (8b)

\[ [J_{AB}, G_{CD}] = 2\eta_{(c|G_{AB}|D]} \]  (8c)

\[ [J_{ab}, G_{cd}] = 2\eta_{(b|G_{c,d}|a]} \]  (8d)

\[ [G_{AB}, G_{cd}] = -\eta_{AB} J_{ab} - \eta_{ab} J_{AB} \]  (8e)

\[ [J_{AB}, H_c] = 2\eta_{AB} H_c] \]  (8f)

\[ [J_{ab}, P_a] = 2\eta_{ab} P_a] \]  (8g)

\[ [G_{AB}, H_c] = -\eta_{AB} P_a] \]  (8h)

\[ [G_{AB}, P_a] = \eta_{ab} H_a] \]  (8i)

\[ [J_{AB}, Q^\alpha_{\beta}] = -\frac{1}{2} (\gamma_{ab})^\alpha_{\beta} Q^\alpha_{\beta} \]  (8j)

\[ [J_{ab}, Q'] = -\frac{1}{2} (\gamma_{ab})^\alpha_{\beta} Q^\alpha_{\beta} \]  (8k)

\[ [G_{AB}, Q^\alpha_{\beta}] = -\frac{1}{2} (\gamma_{ab})^\alpha_{\beta} Q^\alpha_{\beta} \]  (8l)

\[ \{ Q^\alpha_{\beta}, Q^\beta_{\gamma} \} = i \{ \Pi^\gamma_{\gamma(a} C^{-1}]^\alpha_{\beta} H^H \]  (8m)

\[ \{ Q^\alpha_{\beta}, Q^\beta_{\gamma} \} = i \{ \Pi^\gamma_{\gamma(a} C^{-1}]^\alpha_{\beta} P^p \} , \]  (8n)

where \( \xi^U = (-1)^{\eta_{(b|J_{d)}|a]} \) and the matrix \( C \) appearing in the algebra is the charge conjugation matrix. We introduce the gauge fields defined by the index splitting.
The corresponding curvatures read (omitting form indices and spinor indices)

\[
\begin{align*}
R^{\alpha \beta} & = d\Omega^{\alpha \beta} - \Omega^{\alpha \gamma} \wedge \Omega^{\gamma \beta} - \Omega^{\alpha \beta} \wedge \Omega^{\gamma \delta} \\
R^{\alpha \beta} & = d\Omega^{\alpha \beta} - \Omega^{\alpha \gamma} \wedge \Omega^{\gamma \beta} - \Omega^{\alpha \beta} \wedge \Omega^{\gamma \delta} \\
R^{\alpha \beta} & = d\Omega^{\alpha \beta} + \theta^{\alpha \beta} \wedge \Omega^{\gamma} + \Omega^{\alpha \beta} \wedge \Omega^{\gamma} \\
R^{\alpha \beta} & = d\Omega^{\alpha \beta} + \theta^{\alpha \beta} \wedge \Omega^{\gamma} + \Omega^{\alpha \beta} \wedge \Omega^{\gamma} \\
R^{\alpha \beta} & = dE^{\alpha} + \Omega^{\alpha \beta} \wedge E^{\beta} - \Omega^{\alpha \beta} \wedge \tau^{A} - i2\psi \gamma^{A} \wedge \psi - i2\psi \gamma^{A} \wedge \psi \\
R^{\alpha \beta} & = dE^{\alpha} + \Omega^{\alpha \beta} \wedge E^{\beta} - \Omega^{\alpha \beta} \wedge \tau^{A} - i2\psi \gamma^{A} \wedge \psi - i2\psi \gamma^{A} \wedge \psi \\
R^{\alpha \beta} & = d\psi^{\alpha} + \frac{1}{4} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} - \frac{1}{4} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} + \frac{1}{2} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} - \frac{1}{2} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} \\
R^{\alpha \beta} & = d\psi^{\alpha} + \frac{1}{4} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} - \frac{1}{4} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} + \frac{1}{2} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A} - \frac{1}{2} \Omega^{\alpha \beta} \gamma_{AB} \wedge \psi^{A}.
\end{align*}
\]

Taking the following gauge parameters

\[
\begin{align*}
J^{\alpha \beta} & \rightarrow \lambda^{\alpha \beta} \\
J^{\alpha \beta} & \rightarrow \lambda^{\alpha \beta} \\
G^{\alpha \beta} & \rightarrow \lambda^{\alpha \beta} \\
P^{\alpha} & \rightarrow \eta^{\alpha} \\
H^{\alpha} & \rightarrow \eta^{\alpha} \\
Q^{\alpha \beta} & \rightarrow 1_{\alpha \beta}^{(-1)^{p}} \\
Q^{\alpha \beta} & \rightarrow 1_{\alpha \beta}^{(-1)^{p}}
\end{align*}
\]

we can write the transformation rules as

\[
\begin{align*}
\delta \Omega^{\alpha \beta} & = d\lambda^{\alpha \beta} + 2\lambda^{C} \Omega_{C}^{\alpha \beta} + 2\lambda^{B} \Omega_{B}^{\alpha \beta} \\
\delta \Omega^{\alpha \beta} & = d\lambda^{\alpha \beta} + 2\lambda^{C} \Omega_{C}^{\alpha \beta} + 2\lambda^{B} \Omega_{B}^{\alpha \beta} \\
\delta \Omega^{\alpha \beta} & = d\lambda^{\alpha \beta} + 2\lambda^{C} \Omega_{C}^{\alpha \beta} + 2\lambda^{B} \Omega_{B}^{\alpha \beta} \\
\delta E^{\alpha} & = d\eta^{\alpha} + \eta_{\beta} \Omega^{\alpha \beta} + \eta_{\beta} \Omega^{\alpha \beta} - \lambda^{\alpha \beta} \tau^{B} - \lambda^{\alpha \beta} \tau^{B} + \eta^{\alpha} \gamma^{A} \wedge \psi^{A} - \eta^{\alpha} \gamma^{A} \wedge \psi^{A}.
\end{align*}
\]
\[\delta \tau^A = dy^A + \eta_b \gamma^{AB} + \eta_b \Omega^{AB} - \lambda^{AB} \tau_B - \lambda^{AB} E_b + \bar{\psi}^+ \gamma^A \wedge \psi^+ + \bar{\psi}^- \gamma^A \wedge \psi^- (12e)\]

\[\delta \psi^+ = de^+ - \frac{1}{4} \lambda^{AB} \gamma_{AB} \psi^+ - \frac{1}{4} \Omega^{AB} \gamma_{AB} \psi^+ - \frac{1}{2} \lambda^{ab} \gamma_{ab} \psi^+ - \frac{1}{4} \Omega^{ab} \gamma_{ab} \psi^+ - \frac{1}{2} \lambda^{AB} \gamma_{AB} \epsilon^+ - \frac{1}{2} \Omega^{AB} \gamma_{AB} \epsilon^+ (12f)\]

\[\delta \psi^- = de^- - \frac{1}{4} \lambda^{AB} \gamma_{AB} \psi^- - \frac{1}{4} \Omega^{AB} \gamma_{AB} \epsilon^- - \frac{1}{2} \lambda^{ab} \gamma_{ab} \psi^- - \frac{1}{4} \Omega^{ab} \gamma_{ab} \epsilon^- - \frac{1}{2} \lambda^{AB} \gamma_{AB} \epsilon^- - \frac{1}{2} \Omega^{AB} \gamma_{AB} \epsilon^- + \frac{1}{2} \lambda^{ab} \gamma_{ab} \epsilon^- - \frac{1}{4} \Omega^{ab} \gamma_{ab} \epsilon^- - \frac{1}{2} \lambda^{AB} \gamma_{AB} \psi^- - \frac{1}{2} \Omega^{AB} \gamma_{AB} \psi^- + \frac{1}{2} \lambda^{ab} \gamma_{ab} \psi^- - \frac{1}{4} \Omega^{ab} \gamma_{ab} \psi^- + \frac{1}{2} \lambda^{AB} \gamma_{AB} \epsilon^- - \frac{1}{2} \Omega^{AB} \gamma_{AB} \epsilon^- - \frac{1}{2} \lambda^{ab} \gamma_{ab} \epsilon^- - \frac{1}{4} \Omega^{ab} \gamma_{ab} \epsilon^- . (12g)\]

We recall that
\[\chi^\pm = (\chi^\pm U^T) C, (13)\]
for any spinor \(\chi\), see subsection A.1. Having performed a gauging of the algebra now we have a correspondence between the generators and gauge fields that is fundamental for our purpose of deriving new Lagrangians, describing non-relativistic regime, from the \(N = 1\) four dimensional supergravity Lagrangian. Indeed all the manipulations we are going to perform on the algebra, and in particular the expansion procedure, could be directly transferred, in a consistent way, on the gauge fields.

### 1.1.2. Expansion

The Lie algebra expansion is a method to generate a new algebra, or superalgebra, usually bigger, from a starting one. In this subsection we briefly review the method before applying it to the super-Poincaré algebra and the associated gauge fields.

We consider a Lie superalgebra \(g\) with generators \(\{ T_a \} \quad a = 1, \ldots, \dim g\) and supercommutation relations
\[[T_a, T_b] = f_{ab}^c T_c, (14)\]
where \([,\] denotes super-commutator and \(f_{ab}^c\) are structure constants. We want to obtain a new algebra starting from \(g\) by applying the Lie algebra expansion method. In order to do this the first step is to decompose \(g\) in subspaces. This fixes uniquely the expansion [27]. In the present work we limit our attention to a decomposition in two subspaces
\[g = V_0 \oplus V_1, (15)\]
with symmetric space structure, i.e. satisfying the following relations
\[[V_0, V_0] \subseteq V_0 \quad (16a)\]
\[[V_0, V_1] \subseteq V_1 \quad (16b)\]
\[[V_1, V_1] \subseteq V_0. (16c)\]

Denoting with \(T_{a0}\) and \(T_{a1}\) the generators of \(g\) belonging to \(V_0\) and \(V_1\) respectively, a consistent expansion associated with the decomposition above is the following

\[\text{We remark that, by definition of superalgebra, there is an underlying } \mathbb{Z}_2 \text{ grading, } g = Z_0 \oplus Z_1 \text{ not to be confused with the decomposition preparatory for the expansion. Based on this grading } [,] \text{ denotes supercommutator}
\]\n\[[X, Y] = XY - (-1)^{\deg X \deg Y} YX, (17)\]
where \(\deg X = 0 \text{ or } 1\) if \(X\) belongs respectively to \(Z_0\) or \(Z_1\). This results in a commutator or anticommutator depending on the nature of the arguments.

6
\[ T_{a_0} = \sum_{k \in \mathbb{Z}_{+}}^{\infty} k \cdot T_{a_0}^{(k)} = T_{a_0}^{(0)} + \lambda T_{a_0}^{(2)} + \cdots \]  

(18a)

\[ T_{a_1} = \sum_{k \in \mathbb{Z}_{+} + 1}^{\infty} k \cdot T_{a_1}^{(k)} = \lambda T_{a_1}^{(1)} + \lambda^3 T_{a_1}^{(3)} + \cdots . \]  

(18b)

Now, in the spirit of the method, we can consider each \( T_{a}^{(k)} \) appearing at different order in the expansion as an independent generator in a new infinite dimensional superalgebra with super-commutation relations:

\[ [T_{a}^{(m)}, T_{b}^{(n)}] = f_{ab}^{(m+n)} T_{c}. \]  

(19)

This infinite dimensional superalgebra could be reduced to a finite dimensional one by truncating the expansion as

\[ T_{a_0} = \sum_{k=0}^{N_0} k \cdot T_{a_0}^{(k)}, \]  

(20a)

\[ T_{a_1} = \sum_{k=1}^{N_1} k \cdot T_{a_1}^{(k)}, \]  

(20b)

where \( N_0 \) and \( N_1 \) define the order of the truncation and are even and odd non-negative integers respectively. The truncation is consistent, i.e. the set of generators \( T_{a}^{(k)} \) appearing on the right hand sides of equation (20) defines a finite dimensional Lie superalgebra if one of the following relations between \( N_0 \) and \( N_1 \) holds

\[ N_1 = N_0 \pm 1. \]  

(21)

The superalgebra obtained by this way, after the truncation is denoted with \( g(N_0, N_1) \) and it has generators \{\( T_{a_0}^{(0)}, \ldots, T_{a_0}^{(N_0)}, T_{a_1}^{(1)}, \ldots, T_{a_1}^{(N_1)} \)\} and non-zero super-commutation relations

\[ [T_{a}^{(m)}, T_{b}^{(n)}] = f_{ab}^{(m+n)} T_{c}. \]  

(22)

Note in particular that if the order \( n + m \) exceeds the truncation order the super-commutator vanishes, since there is no \( T_{c} \) in the right hand side of the expression above. For further details we refer to [27]. We remark that the expansion is induced by the initial decomposition of the algebra. A fundamental ingredient guiding us in defining the opportune decomposition to obtain superalgebras describing a certain physical regime is that the lowest order truncation, \( g(0, 1) \) is always the Inonu-Wigner contraction of \( g \) with respect to \( V_0 \). Since we are interested in a non-relativistic algebra we would choose \( V_0 \) in such a way that \( g(0, 1) \) corresponds to the super-Galilei algebra. As discussed in appendix A this request could only be satisfied if

\[ \xi^U = (-1)^{\frac{n+m+1}{2}} + 1, \]  

(23)
that corresponds to the cases $p = 1$ and $p = 2$, for $p < 4$. Using these information and the setting defined in the previous subsection we can choose the decomposition as

\begin{align}
V_0 &= \{ J_{AB}, J_{ab}, H_A, Q^{++} \} \\
V_1 &= \{ G_{AB}, P_a, Q^{+-} \}
\end{align}

(24a) (24b)

We recognize that it satisfies the symmetric space structure. It is then straightforward to expand the corresponding generators as

\begin{align}
J^{AB} &= \sum_{k=0}^{N_0} \lambda^k J^{AB} = (0)^{AB} + \lambda^2 J^{AB} + \ldots \tag{25a} \\
\Omega^{ab}_{\mu} &= \sum_{k=0}^{N_0} \lambda^k J^{ab} = (0)^{ab} + \lambda^2 J^{ab} + \ldots \tag{25b} \\
H^A &= \sum_{k=0}^{N_0} \lambda^k H^A = (0)^{A} + \lambda^2 H^A + \ldots \tag{25c} \\
Q^+ &= \sum_{k=0}^{N_0} \lambda^k Q^+ = (0)^+ + \lambda^2 Q^+ + \ldots \tag{25d} \\
P^a &= \sum_{k=1}^{N_1} \lambda^k P^a = (0)^a + \lambda^3 P^a + \ldots \tag{25e} \\
G^{ab} &= \sum_{k=1}^{N_1} \lambda^k G^{ab} = (0)^{ab} + \lambda^3 G^{ab} + \ldots \tag{25f} \\
Q^- &= \sum_{k=1}^{N_1} \lambda^k Q^- = (0)^- + \lambda^3 Q^- + \ldots \tag{25g}
\end{align}

This induces an analogous expansion of the associated gauge fields

\begin{align}
\Omega^{AB}_{\mu} &= \sum_{k=0}^{N_0} \lambda^k \Omega^{AB}_{\mu} = (0)^{AB}_{\mu} + \lambda^2 \Omega^{AB}_{\mu} + \ldots \tag{26a} \\
\Omega^{ab}_{\mu} &= \sum_{k=0}^{N_0} \lambda^k \Omega^{ab}_{\mu} = (0)^{ab}_{\mu} + \lambda^2 \Omega^{ab}_{\mu} + \ldots \tag{26b} \\
\tau^A_{\mu} &= \sum_{k=0}^{N_0} \lambda^k \tau^A_{\mu} = (0)^A_{\mu} + \lambda^2 \tau^A_{\mu} + \ldots \tag{26c}
\end{align}
\[ \psi_{\alpha}^+ = \sum_{k=0}^{N_0} \lambda^k \psi_{\alpha}^{(k)} + \psi_{\alpha}^{(2)} + \cdots \] (26d)

\[ E_{\mu}^a = \sum_{k=1}^{N_1} \lambda^k E_{\mu}^{(k)} = \lambda E_{\mu}^{(1)} + \lambda^3 E_{\mu}^{(3)} + \cdots \] (26e)

\[ \Omega_{\mu}^{ab} = \sum_{k=1}^{N_1} \lambda^k \Omega_{\mu}^{(k)} = \lambda \Omega_{\mu}^{(1)} + \lambda^3 \Omega_{\mu}^{(3)} + \cdots \] (26f)

\[ \psi_{\alpha}^- = \sum_{k=1}^{N_1} \lambda^k \psi_{\alpha}^{-(k)} + \psi_{\alpha}^{-(3)} + \cdots \] (26g)

The expansion parameter \( \lambda \) in this context could be considered as \( 1/c \), with \( c \) the speed of light.

The explicit commutation rules of the expanded superalgebra and the explicit form of these for the contraction \( g(0, 1) \) can be found in subsection A.3.

1.2. \( \mathcal{N} = 1 \) \( D = 4 \) super-Poincaré algebra

In this section we focus on the \( \mathcal{N} = 1 \) super-Poincaré algebra in four dimensions and apply the Lie algebra expansion method. We will define a non-relativistic supersymmetric action for strings and membranes. By studying these particular actions and their invariance we will be able to define a general method to construct action term for strings and membranes associated to an expanded algebra \( g(N_0, N_1) \). We remark that the analysis of this section is specific of four dimensions and we are going to use gamma matrices and spinor properties that holds only in four dimensions. Generalizations to different dimensions, although straightforward, require a careful analysis of these aspects.

1.2.1. The algebra. We report here, for convenience, the commutation relations of the \( \mathcal{N} = 1 \) four dimensional super-Poincaré algebra as derived in appendix A for \( p = 1, 2 \) with the use of the projectors \( \Pi^U_\pm \) defined in equation (76),

\[ \{ Q^{\alpha \pm}, Q^{\beta \pm} \} = i [i \Pi^U_\pm \gamma_A C^{-1}]^{\alpha \beta} H^A \] (27a)

\[ \{ Q^{\alpha \pm}, Q^{\beta \mp} \} = i [i \Pi^U_\pm \gamma_a C^{-1}]^{\alpha \beta} P^a \] (27b)

\[ [J_{AB}, Q^{\alpha \pm}] = \frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^{\beta \pm} \] (27c)

\[ [J_{ab}, Q^{\alpha \pm}] = -\frac{1}{2} (\gamma_{ab})^\alpha_\beta Q^{\beta \mp} \] (27d)

\[ [G_{AB}, Q^{\alpha \pm}] = \frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^{\beta \mp} \] (27e)

where \( Q \) is a Majorana spinor,

\[ \bar{Q} = Q^C \] (28)
We recall that the charge conjugation matrix in four dimensions satisfies
\[ C\gamma^A C^{-1} = -\gamma^{A'} \]  
(29a)
\[ C^2 = -1 \]  
(29b)
\[ C' = -C. \]  
(29c)

We can choose the gamma matrices to be purely imaginary, an explicit choice is
\[ \gamma_0 = \sigma_1 \otimes \sigma_2 \]  
(30a)
\[ \gamma_1 = i\sigma_1 \otimes \sigma_3 \]  
(30b)
\[ \gamma_2 = -i\sigma_2 \otimes \sigma_2 \]  
(30c)
\[ \gamma_3 = -i\sigma_1 \otimes \sigma_1 \]  
(30d)
\[ \gamma_5 = \sigma_3 \otimes \sigma_2 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 \]  
(30e)
\[ C = C_{(-)} = -i\sigma_1 \otimes \sigma_2 = -i\gamma_0 \]  
(30f)
\[ C_{(+)} = i\sigma_2 \otimes \sigma_1 = \gamma_2 \gamma_1. \]  
(30g)

where \(\sigma_i\) are the Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
(31a)
(31b)
(31c)

We assume \(\gamma_0\) to be always in \(U\). In four dimensions the projectors \(\Pi_U^\mu\) satisfy the following relations
\[ \Pi_U^{\mu, \gamma_A} = \gamma_A \Pi_U^{\mu, (-1)p} \]  
(32a)
\[ \Pi_U^{\mu, \gamma_5} = \gamma_5 \Pi_U^{\mu, (-1)p+1} \]  
(32b)
\[ \Pi_U^{\mu, \gamma_5} = \gamma_5 \Pi_U^{\mu, (-1)p+1}. \]  
(32c)

Explicitly for, \(p = 1\) and \(p = 2\) we have respectively \(U = -\gamma_0 \gamma_1\) and \(U = i\gamma_0 \gamma_1 \gamma_2\). Having defined the setting for the Lie algebra expansion of the four dimensional super-Poincaré algebra now we turn our attention to the construction of non-relativistic supersymmetric actions.

1.2.2. The action. Our starting point to build a non-relativistic supersymmetric action in four dimension is the \(\mathcal{N} = 1\) \(D = 4\) supergravity action in 1.5 order formalism [37, 38], i.e. we treat spin connection and vielbein as independent fields, but we require the spin connection to be on-shell to guarantee the invariance of the action under the supersymmetry transformations. Explicitly the Lagrangian reads
\[ \mathcal{L} = EE^A_F E^\mu_B \rho^{\lambda \mu} (J) + 2\epsilon^{\mu \nu \lambda \sigma} \psi^\lambda \gamma_5 \gamma_\sigma D_\nu \psi^\nu, \]  
(33)
that could be rewritten, up to a multiplicative constant factor, as

\[ \mathcal{L} = \epsilon_{\hat{A}B\hat{C}}R^{\hat{A}\hat{B}}(J) \wedge E^{\hat{C}} \wedge E^{\hat{B}} + 4\overline{\psi} \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge D\psi, \]  
(34)

where we have omitted the form indices. The Rarita–Schwinger term could be put in the form

\[ \overline{\psi}\gamma_5 \gamma_\lambda \wedge E^\lambda \wedge D\psi = \overline{\psi} \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R(Q). \]  
(35)

Now we, guided by the results of the previous section, being interested in the nonrelativistic regime, we consider the string case, \( p = 1 \), and the membrane case, \( p = 2 \). We will see how the expansion of the gauge fields will lead us to two non-relativistic supersymmetric actions, preserving after an opportune truncation, in a sense that will be discussed in the next, the invariance properties of the initial action. We will examine how this procedure could be generalized and define a correspondence between the truncated algebras and the actions.

1.2.2.1 String case \( p = 1 \). In this case, for the Rarita–Schwinger term, we obtain (see appendix A) (note that the \( \psi^\pm \) in this case are the fields associated with \( Q^\pm \) respectively)

\[ \overline{\psi} \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R(Q) = (\psi^+) \gamma_5 \gamma_\lambda \wedge \tau^\lambda \wedge R^\lambda(Q) + (\psi^-) \gamma_5 \gamma_\lambda \wedge \tau^\lambda \wedge R^\lambda(Q) \]

\[ + (\psi^+) \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R^\lambda(Q) + (\psi^-) \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R^\lambda(Q). \]  
(36)

Then for the full Lagrangian we can write

\[ \mathcal{L} = \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} [R^{\hat{A}\hat{B}}(J) \wedge \tau^\hat{C} \wedge \tau^\hat{D} + 4R^{\hat{A}\hat{B}}(G) \wedge E^\hat{C} \wedge \tau^\hat{D} + R^{\hat{A}\hat{B}}(J) \wedge E^\hat{C} \wedge E^\hat{D}] \]

\[ + 4 \left( \psi^+ \gamma_5 \gamma_\lambda \wedge \tau^\lambda \wedge R^\lambda(Q) + \psi^- \gamma_5 \gamma_\lambda \wedge \tau^\lambda \wedge R^\lambda(Q) \right) \]

\[ + (\psi^+) \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R^\lambda(Q) + (\psi^-) \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R^\lambda(Q). \]  
(37)

We can expand this Lagrangian by taking the algebra \( g(2,3) \), namely imposing the truncation \( N_0 = 2 \) and \( N_1 = 3 \). We choose this truncation because it does not coincide with the contraction and furthermore, as we will discuss later in subsubsection 1.2.2.3, the form of the Lagrangian is the same that would have appeared considering the order \( g(2,1) \) but this does not affects the invariance properties. As we are going to discuss the procedure is completely general and could be straightforwardly extended to higher order. At the lowest orders we have

\[ (0) = \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} [R^{\hat{A}\hat{B}}(J) \wedge \tau^\hat{C} \wedge \tau^\hat{D} - 4\tau^\hat{A} \wedge (\psi^-) \gamma_5 \gamma_\lambda \wedge R^\lambda(Q)] \]  
(38a)

\[ (0) = \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} [R^{\hat{A}\hat{B}}(J) \wedge \tau^\hat{C} \wedge \tau^\hat{D} + 4R^{\hat{A}\hat{B}}(G) \wedge E^\hat{C} \wedge \tau^\hat{D} + (\psi^-) \gamma_5 \gamma_\lambda \wedge E^\lambda \wedge R^\lambda(Q)] \]  
(38b)
Among the terms discussed above our candidate action is given by taking the Lagrangian density $\mathcal{L}$. Our choice is dictated by the criteria described in [28]. In particular in $\mathcal{L}$ only fields of the algebra $g(2, 3)$ appear and, furthermore, the truncation to $g(2, 3)$ does not affect their form if compared with the untruncated algebra. These two properties guarantee that the expanded action remains invariant under the expanded transformations corresponding to the invariance of the initial action. We recognize that in the action $\mathcal{L}$ the fields $\Omega^{AB}$, $\Omega^A$, $\psi^-$ and $E^a$ do not appear; as explained in [28] this does not affect the invariance under the associated gauge transformation. Indeed it could be immediately checked that under the gauge transformations corresponding to the generators associated with the fields above,

\[
\begin{align*}
\delta (2) \Omega^{AB} &= d \lambda^{AB} + 2 \lambda_c C^{(0)} \Omega^c, \\
\delta (3) \Omega^A &= d \lambda^A + \lambda^{A(0)} \Omega^B - \lambda^{B(0)} \Omega^A, \\
\delta (3) \psi^+ &= d \epsilon^A - \frac{1}{2} \lambda^{AB} \gamma_{AB} (0)^+ - \frac{1}{2} \lambda_{AB} \gamma^{AB} (0)^+ - \frac{1}{2} \lambda_{AB} \gamma_{AB} (0)^-, \\
\delta (2) \psi^- &= \frac{\lambda^{AB} \gamma_{AB} (0)}{4},
\end{align*}
\]

the Einstein–Hilbert term and the Rarita–Schwinger are separately invariant (note that only the last two fields above appear in the action). Thus we have obtained the following non-relativistic supersymmetric action for string super-Galilei algebra in four dimensions

\[
\mathcal{L}^{p=1} = \epsilon_{ABd} \left[ R^{AB}(J) \wedge (0) + E^d \wedge (1) + 4 R^{AB}(G) \wedge (1) + (0) \wedge E^d \wedge (0) B \right]
\]

\[
+ \left[ R^{AB}(J) \wedge (0) + (0) B \wedge (0) + 4 R^{AB}(J) \wedge (2) + (0) B \wedge (2) \right]
\]

\[
- \frac{1}{4} \left[ \tau^A \wedge R^+(Q) + \tau^A \wedge (0) \right] \left[ \psi^- \gamma_5 \gamma_a \wedge R^-(Q) \right]
\]

\[
+ \left[ (2)^A \wedge (0) \right] \gamma_5 \gamma_a \wedge R^- + \left[ (0) \right] \gamma_5 \gamma_a \wedge R^+(Q)
\]

\[
+ \left[ (1) \right] \gamma_5 \gamma_a \wedge R^+(Q) + \left[ (1) \gamma^+ \right] \gamma_5 \gamma_a \wedge R^-(Q). \tag{40}
\]

It is interesting to compare this result with those of [28]. In particular the non-supersymmetric truncation obtained by modding out the fermions fulfills the invariance conditions described in [28],

\[
n \leq N_0, \tag{41}
\]

where $n$ is the order of the term, since in this case $n = 2$ and $N_0 = 2$. We study the invariance conditions for the full Lagrangian in subsection 1.2.3.3. This holds true also for the other models we discuss in the present work.
1.2.2.2. Membrane case $p = 2$. In this case for the Rarita–Schwinger term we obtain

$$\bar{\psi} \gamma_5 \gamma_A \wedge E^3 \wedge R(Q) = (\psi^+) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^-(Q) + (\psi^-) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^+(Q)$$

$$+ (\psi^+) C_{\gamma_5 \gamma_3} \wedge E \wedge R^+(Q) + (\psi^-) C_{\gamma_5 \gamma_3} \wedge E \wedge R^-(Q),$$

(42)

where $E = E^3$. Then for the full Lagrangian we can write

$$\mathcal{L} = 2 \epsilon_{ABC} \left[ R^{AB}(J) \wedge \tau^C \wedge E + R^A(G) \wedge \tau^B \wedge \tau^C \right]$$

$$+ 4 \left[ (\psi^+) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^-(Q) + (\psi^-) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^+(Q) + (\psi^+) C_{\gamma_5 \gamma_3} \wedge E \wedge R^+(Q) + (\psi^-) C_{\gamma_5 \gamma_3} \wedge E \wedge R^-(Q) \right].$$

(43)

Expanding at the lowest orders we have

$$\mathcal{L}^{(1)} = 2 \epsilon_{ABC} \left[ R^{AB}(J) \wedge \tau^A \wedge E + R^A(G) \wedge \tau^B \wedge \tau^C \right]$$

$$- 4 \left[ (\psi^+) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^-(Q) + (\psi^-) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^+(Q) + (\psi^+) C_{\gamma_5 \gamma_3} \wedge E \wedge R^+(Q) + (\psi^-) C_{\gamma_5 \gamma_3} \wedge E \wedge R^-(Q) \right].$$

(44a)

$$\mathcal{L}^{(3)} = 2 \epsilon_{ABC} \left[ R^{AB}(J) \wedge \tau^A \wedge E + R^A(G) \wedge \tau^B \wedge \tau^C \right]$$

$$+ 4 \left[ (\psi^+) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^-(Q) + (\psi^-) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^+(Q) + (\psi^+) C_{\gamma_5 \gamma_3} \wedge E \wedge R^+(Q) + (\psi^-) C_{\gamma_5 \gamma_3} \wedge E \wedge R^-(Q) \right].$$

(44b)

For the same reasons considered in the previous subsection, if we consider a truncation at order $p(2, 3)$, our action is given by the order three term,

$$\mathcal{L}^{p=2} = 2 \epsilon_{ABC} \left[ R^{AB}(J) \wedge \tau^A \wedge E + R^A(G) \wedge \tau^B \wedge \tau^C \right]$$

$$- 4 \left[ (\psi^+) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^-(Q) + (\psi^-) C_{\gamma_5 \gamma_A} \wedge \tau^A \wedge R^+(Q) + (\psi^+) C_{\gamma_5 \gamma_3} \wedge E \wedge R^+(Q) + (\psi^-) C_{\gamma_5 \gamma_3} \wedge E \wedge R^-(Q) \right].$$
for an analysis of a Lagrangian term with the lowest order truncation under which it reproduces the invariance properties of the initial action. We also note that the non-supersymmetric truncation of the Lorentz generators leaves the action invariant (see also [28] for further details).

We remark that in principle, as will be discussed in full generality in subsubsection 1.2.3.3 also the first order term \( \mathcal{L} \) share the same invariance properties of \( \mathcal{L} \) under \( \mathfrak{g}(2, 3) \), but this is trivially verified since in \( \mathcal{L} \) there are only fields already appearing in \( \mathfrak{g}(0, 1) \). Thus we associate a Lagrangian term with the lowest order truncation under which it reproduces the invariance properties of the initial action. We also note that the non-supersymmetric truncation of the action above fulfills the invariance condition,

\[
n \leq N_1
\]

described in [28], since \( n = 3 \) and \( N_1 = 3 \). We refer to subsubsection 1.2.3.3 for an analysis of the invariance for the full Lagrangian.

1.2.3. Invariance and transformation rules. In this subsection we discuss the invariance of the action under the algebra \( \mathfrak{g}(2, 3) \) that we have obtained. We focus on P-type symmetries, i.e. the transformations descending from space and time translations after the expansion and under supersymmetry since it is immediate to recognize that the transformations descending from Lorentz generators leave the action invariant (see also [28] for further details).

1.2.3.1. P-type symmetries. The action is not invariant under P-type symmetries, i.e. the transformations generated by \( \hat{P}_A \) (space and time translations), however we could apply an argument similar to that described in [28] to see that this is not an independent symmetry and to derive the modified P-transformations leaving the action invariant up to a term proportional to the equation of motion of the spin connection. The action is invariant under Lorentz, general coordinate transformations and a trivial symmetry, or equation of motion symmetry; these last transforms the fields in terms proportional to the equation of motion. In particular if we write the variation as

\[
\delta \mathcal{L} = A^\mu_A E^a_{\mu} + B^\mu_{AB} \delta \gamma^A_{\mu} + \bar{C}^a_{\mu} \delta \psi_{\mu},
\]

where explicitly

\[
A^\mu_A = E \left[ R(J)E^\mu_{\c} - 2 R^\mu_{\c}(J) \right] + \epsilon^{\mu\nu\lambda\chi} \gamma_3 \gamma_5 \gamma_\chi R_{\mu\nu}(Q)
\]

(48a)

\[
B^\mu_{AB} = E \left[ 2 E^\mu_{\c} R^\mu_{\c}(P) + E^\mu_{\c} R^\mu_{\c}(P) \right]
\]

(48b)

\[
C^a_{\mu} = \epsilon^{\mu\nu\lambda\sigma} [2 \gamma_5 \gamma_\lambda R_{\mu\nu}(Q) + R^a_{\mu\nu}(Q) \gamma_5 \gamma_\lambda \psi_{\nu}] .
\]

(48c)

We recognize, among others, the following trivial symmetry

\[
\delta E^a_{\mu} = 2 ER^a_{\mu\nu}(P) \sigma^\nu
\]

(49a)
\[ \delta \bar{\Omega}^{\hat{h}} = -2A^{\hat{h}}_{\mu} \rho_{\hat{h}} - E^{\hat{h}}_{\mu} \lambda^{\hat{h}} + E^{\hat{h}}_{\mu} A^{\hat{c}}_{\hat{c}} + E^{\hat{h}}_{\sigma} A^{\hat{c}}_{\hat{c}} \]  
\[ \delta \bar{\psi}^{\sigma} = 0. \]  

In appendix C we describe in details the trivial symmetries. The vielbein is the only field transforming under P-type symmetries. The full transformations of the vielbein, apart from P-type transformation are given by the trivial symmetry. Lorentz, supersymmetry and general coordinate transformations

\[ \delta E^{\hat{A}}_{\mu} = 2E^A_{\mu}(P) \sigma^\nu - \lambda^{\hat{h}} E^{\hat{h}}_{\mu} + \xi^{\nu} \partial_\nu E^{\hat{A}}_{\mu} + \partial_\nu \xi^{\nu} E^{\hat{A}}_{\mu} + \bar{\sigma} \gamma^\nu \psi_{\nu}. \]  

By choosing the parameters as

\[ \xi^{\nu} = 2E \sigma^\nu \]
\[ \lambda^{\hat{h}} = -\xi^{\hat{h}} \xi^{\nu} \]
\[ \epsilon = \xi^{\nu} \psi_{\nu}, \]

the transformation above reduces to

\[ \delta E^{\hat{A}}_{\mu} = \partial_\mu \xi^{\hat{A}} + \Omega^{\hat{h}} \xi^{\hat{h}}, \]

where \( \xi^{\hat{A}} = E^A_{\mu} \xi^{\nu} \). We recognize this as the P-type transformation with parameter \( \eta^{\hat{A}} = \xi^{\hat{A}} \).

Thus the P-type symmetries are not independent and could be written as a combination of supersymmetry, Lorentz, general coordinate transformations and the trivial symmetry. Rewriting the P-type transformation as combination of the other symmetries require the a modification of the P-type transformation of the gravitino and of the spin connection as follows

\[ \delta E^{\hat{A}}_{\mu} = \partial_\mu \xi^{\hat{A}} + \Omega^{\hat{h}} \xi^{\hat{h}} \]
\[ \delta \psi_{\mu} = \left( \partial_\mu + \frac{1}{4} \Omega^{\hat{h}} \gamma^{\hat{h}} \right) (\eta^{\hat{c}} \psi^{\hat{c}}) + \frac{1}{4} \eta^{\nu} \Omega^{\hat{h}} \gamma^{\hat{h}} \psi_{\nu} \]
\[ \delta \bar{\Omega}^{\hat{h}} = \frac{1}{2E} \left[ -2A^{\hat{h}}_{\mu} \eta^{\hat{h}} - E^{\hat{h}}_{\mu} \eta^{\hat{h}} A^{\hat{c}}_{\hat{c}} + E^{\hat{h}}_{\sigma} A^{\hat{c}}_{\hat{c}} \right] - \partial_\mu (\Omega^{\hat{h}} \eta^{\hat{c}}) - 2 \Omega^{\hat{h}} \gamma^{\hat{h}} \eta^{\hat{c}} \eta^{\hat{h}}. \]

Since of the transformations we have written the P-type transformations only super-symmetry leave the action not invariant, this means that the invariance of the action is directly related to the behavior under supersymmetry that we are going to discuss in the next section. In particular putting the spin connection on-shell guarantees the invariance under supersymmetry that in turn, for the argument just exposed, implies the invariance under equation (53).

1.2.3.2. Supersymmetry. The Lagrangian we have considered [37, 38],

\[ S = \epsilon_{\hat{A}B\hat{C}\hat{D}} R^{\hat{A}B}(J) \wedge E^{\hat{C}} \wedge E^{\hat{D}} + 4 \bar{\psi} \gamma_5 \gamma^{\hat{A}} \wedge E^{\hat{A}} \wedge D \psi, \]

is invariant under the supersymmetry transformations listed in subsection 1.1.1 provided that the following constraint is imposed

\[ R^{\hat{A}}(P) = 0, \]

i.e. the spin connection is put on-shell. This means that the actions that we have found are invariant under the following supersymmetry transformations, for the \( p = 1 \) and \( p = 2 \) cases, when
the corresponding constraints are satisfied, i.e. when the fields coming from the spin connection through the expansion, are put on-shell. For the case $p = 1$ and $p = 2$ the supersymmetry transformations rules and constraints are

**Strings, $p = 1$.** The supersymmetry transformation rules are

\[
\delta (1) E^a = i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{56a}
\]

\[
\delta (1) \psi^- = i (\epsilon^-)^- \gamma_5 \gamma^a \psi^- - (\epsilon^-)^- \gamma_5 \gamma^a \psi^+ + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{56b}
\]

\[
\delta (0) \tau = (\epsilon^-)^- \gamma_5 \gamma^a \psi^- \tag{56c}
\]

\[
\delta (2) \psi^- = i (\epsilon^-)^- \gamma_5 \gamma^a \psi^- + i (\epsilon^-)^- \gamma_5 \gamma^a \psi^+ + i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{56d}
\]

\[
\delta (0) \psi^+ = d (\epsilon^-) - \frac{i}{4} \Omega^{ab} \gamma_{ab} \psi^+ \tag{56e}
\]

\[
\delta (2) \psi^+ = d (\epsilon^-) - i (\epsilon^-) \gamma_5 \gamma^a \psi^- - \frac{1}{4} \Omega^{ab} \gamma_{ab} \psi^+ - \frac{1}{2} \gamma_5 \gamma_{ab} \psi^+ \tag{56f}
\]

\[
\delta (1) \psi^+ = d (\epsilon^+)_+ - \frac{i}{4} \Omega^{ab} \gamma_{ab} \psi^+ + \frac{1}{2} \gamma_5 \gamma_{ab} \psi^+ \tag{56g}
\]

\[
\delta (3) \psi^+ = d (\epsilon^+)_+ - \frac{1}{4} \Omega^{ab} \gamma_{ab} \psi^+ - \frac{1}{2} \gamma_5 \gamma_{ab} \psi^+ - \frac{1}{2} \gamma_5 \gamma_{ab} \psi^+ \tag{56h}
\]

while the curvature constraints read

\[
R^{(0)}(P) = d (1) E^a + \Omega^{ab} \wedge E_b - \Omega^{0a} \wedge \tau - i (\psi^-) \gamma^a \wedge \psi^+ = 0 \tag{57a}
\]

\[
R^{(1)}(P) = d (3) E^a + \Omega^{ab} \wedge E_b + \Omega^{0a} \wedge \tau - i (\psi^-) \gamma^a \wedge \psi^- \tag{57b}
\]

\[
R^{(0)}(H) = d (0) \tau - \frac{i}{2} (\psi^-) \gamma_5 \gamma^0 \wedge \psi^- = 0 \tag{57c}
\]

\[
R^{(2)}(H) = d (2) + \Omega^{0a} \wedge E_a - i (\psi^-) \gamma_5 \gamma^0 \wedge \psi^+ - \frac{1}{2} (\psi^-) \gamma_5 \gamma^0 \wedge \psi^- = 0 \tag{57d}
\]

**Membranes, $p = 2$.** In this case the supersymmetry transformation rules are

\[
\delta (1) E^a = i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{58a}
\]

\[
\delta (3) E^a = (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + i (\epsilon^-)^- \gamma_5 \gamma^a \psi^+ + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^- + (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{58b}
\]

\[
\delta (2) A = i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ \tag{58c}
\]

\[
\delta (1) \psi^+ = i (\epsilon^-)^- \gamma_5 \gamma^a \psi^+ + i (\epsilon^-)^+ \gamma_5 \gamma^a \psi^+ + i (\epsilon^-)^- \gamma_5 \gamma^a \psi^+ \tag{58d}
\]
particular truncating the algebra it is possible that, for a given order properties of the initial Lagrangian are reproduced order by order if one considers an infinite expansion, thus the only issues could arise truncating the algebra and thus the fields. In

\[ R^{a}(P) = d^{(0)}{\bar{\psi}}^{+} + \frac{(0)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(0)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(P) = d^{(2)}{\bar{\psi}}^{+} + \frac{(2)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(2)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(P) = d^{(3)}{\bar{\psi}}^{+} + \frac{(3)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(3)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(H) = d^{(0)}{\bar{\psi}}^{+} + \frac{(0)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(0)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(H) = d^{(2)}{\bar{\psi}}^{+} + \frac{(2)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(2)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

while the curvature constraints read

\[ R^{a}(P) = d^{(1)}{\bar{\psi}}^{+} + \frac{(1)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(1)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(P) = d^{(3)}{\bar{\psi}}^{+} + \frac{(3)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(3)}{4} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(H) = d^{(1)}{\bar{\psi}}^{+} + \frac{(1)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(1)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

\[ R^{a}(H) = d^{(3)}{\bar{\psi}}^{+} + \frac{(3)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} - \frac{(3)}{2} \Omega^{ab} \gamma^{ab} \phi^{+} \]  

What we have obtained is a set of two actions, one describing a non-relativistic supersymmetric string theory, the other non-relativistic supersymmetric theory of membranes. These action are invariant under the expanded supersymmetry transformations up to terms proportional to the expanded curvature \( R(P) \). We remark that we have been able to obtain these actions, starting from the \( \mathcal{N} = 1 \) theory, considering projected Majorana fermions propagating half degrees of freedom of four dimensional Majorana fermions. It is also interesting to note that the expanded action are now on-shell invariant under a four fermionic parameters set of transformations.

In the next subsection we analyze the invariance conditions pointing out that the procedures described above could be easily carried out at higher order giving non-relativistic Lagrangians transforming, in the way just described, under the corresponding expanded algebra.

1.2.3.3. Invariance conditions. Now we study the invariance conditions for the expanded action at order \( n \) under the algebra \( g(N_0, N_i) \) and define how these induce a relations between the order \( n \) and the order of truncation. The starting point is to note that the trasformation properties of the initial Lagrangian are reproduced order by order if one considers an infinite expansion, thus the only issues could arise truncating the algebra and thus the fields. In particular truncating the algebra it is possible that, for a given order \( n \) term in the expansion
of the Lagrangian, \( \mathcal{L} \), there are some terms missing with respect to the untruncated expansion [28]. In order to understand how this could happen it is useful to consider a toy model that fully captures the picture. Let us take a Lagrangian term \( \mathcal{B} \) given by the product of two fields \( a, b \),

\[
\mathcal{B} = ab. \tag{60}
\]

We can expand the two fields as

\[
a = \sum_{k=0}^{\infty} \lambda^k (^{(a)}\! a), \tag{61a}
\]

\[
b = \sum_{k=0}^{\infty} \lambda^k (^{(b)}\! b), \tag{61b}
\]

Then if we consider the order \( n \) in the corresponding expansion of \( \mathcal{B} \), just obtained by substituting the expansion above in the initial Lagrangian, it will be given by

\[
(^{(n)}\! \mathcal{B}) = \sum_{k=0}^{n} \lambda^k (^{(a)}\! (a)^{(n-k)}\! b + \cdots + (^{(a)}\! (b)^{(n-1)}\! a + (^{(b)}\! (a)^{(n)}\! b. \tag{62}
\]

We can truncate the expansion of the fields at certain orders \( N_a \) and \( N_b \) as

\[
a = \sum_{k=0}^{N_a} \lambda^k (^{(a)}\! a), \tag{63a}
\]

\[
b = \sum_{k=0}^{N_b} \lambda^k (^{(b)}\! b). \tag{63b}
\]

It is then possible that the term \( (^{(a)}\! b \) for the untruncated expansion, in equation (62), is modified by this truncation. Indeed if \( N_b < n \) the first term in equation (62) does not exist anymore. The same is true for the last term if \( N_a < n \) and in general depending on how \( N_a \) and \( N_b \) are fixed there could be more or less terms missing in \( \mathcal{B} \) with respect to the same Lagrangian term with the fully untruncated expansion for the \( a, b \) fields, equation (62). The form of \( \mathcal{B} \) after the truncation is just

\[
(\mathcal{B}) = \sum_{k= \max(0, n-N_b)}^{\min(N_a, n)} \lambda^k (^{(a)}\! (a)^{(n-k)}\! b. \tag{64}
\]

This is exactly what could happen in the expansion of the \( \mathcal{N} = 1 \) four dimensional supergravity Lagrangian and it could potentially spoil the initial invariance or transformation properties. Thus, given the truncated algebra \( g(N_0, N_1) \), we want to define up to which order \( n \) there are no missing term in the Lagrangian \( \mathcal{L} \) and we want this expressed in the form of relations between
\( n \) and the truncation orders \( N_0, N_1 \). Looking back at our toy model a sufficient condition to be sure that there are no missing terms after the truncation in the order \( n \) Lagrangian, \( b \) is given by
\[
 n \leq \min\{N_a, N_b\}. \tag{65}
\]
There are two main differences between the toy model and the expansion that we have described for the supergravity action; the first is that in general we deal with terms with more than two fields, the second is that in the expansion of some fields the starting order is not zero. While for the first difference we could just in turn group more terms in two sets and repeat the analysis discussed for the toy model, the second difference could be easily taken into account just shifting \( N_a \) or \( N_b \) in the condition above. Then we can study the conditions separately for the Einstein–Hilbert and Rarita–Schwinger terms. For the Einstein–Hilbert term we get (compare with [28])
\[
 p = 1 \quad n \leq N_0 \tag{66a}
\]
\[
 p = 2 \quad n \leq N_1 \tag{66b}
\]
For the Rarita–Schwinger term we find the following conditions
\[
 p = 1 \quad n \leq \min\{N_0, N_1 + 1\} \tag{67a}
\]
\[
 p = 2 \quad n \leq \min\{N_0 + 1, N_1\} \tag{67b}
\]
Since the truncation condition reads \( N_1 = N_0 + 1 \) then it is immediate to recognize that these reduce to equation (66). Thus we deduce that the action term at order \( n \) shares the same invariance properties of the initial Lagrangian under the expanded algebra \( g(N_0, N_1) \) if the following conditions are satisfied
\[
 p = 1 \quad n \leq N_0 \tag{68a}
\]
\[
 p = 2 \quad n \leq N_1 \tag{68b}
\]
This implies, as an example, that in the string case the term \( L^{(2)} \) is invariant under \( g(2, 1) \) or higher order algebras. It is worth nothing that in the string case the condition is only on \( N_0 \), while, in the membrane case, it is only on \( N_1 \) (note that in the string case the expansion of the action contains only even order terms while in the membrane case only odd order term). This implies that there is a degeneracy, i.e. the highest order \( n \) for the action term for which invariance are not destroyed by truncation at \( g(N_0, N_1) \) is the same for \( g(n, n \pm 1) \) and \( g(n \pm 1, n) \) for the string and membrane cases respectively.

2. Discussion and conclusions

In this work we have applied the procedure of Lie algebra expansion to the \( \mathcal{N} = 1 \) super-Poincaré algebra in four dimensions, generalizing the results of [32] for the \( \mathcal{N} = 2 \) \( p = 0 \) three dimensional super-Poincaré algebra.

In section 1.1 we have defined the general setting for the application of the Lie algebra expansion method. In particular we have considered the \( \mathcal{N} = 1 \) \( p \)-brane super-Poincaré algebra in \( D \)-dimensions, \( g \). We have then defined a decomposition in subspaces \( g = V_0 \oplus V_1 \). In order to obtain a non-relativistic algebra after the expansion, and to be able to apply the Lie algebra expansion method, we have required this decomposition to satisfy two requirements,
the first is that it induces a symmetric space structure, the second is that the Inonu-Wigner contraction of \( g \) with respect to the subalgebra \( V_0 \) should be the \( p \)-brane super-Galilei algebra. The decomposition preparatory to the expansion method has required a splitting of the spinors using two sets of \( p \)-brane projectors. These projectors have been defined in appendix A, where we describe in detail their application to the superalgebra and their properties. The projectors provide a completely general tool that could be immediately applied or straightforwardly generalized to different cases, including different signatures and \( N > 1 \). We have realized that, using this approach, it is possible to define a decomposition of the \( N = 1 \) super-Poincaré algebra such that we obtain two non-relativistic algebras for \( p = 1, 2 \) while for particle the algebra obtained is ultra-relativistic.

In section 1.2 we have applied the expansion procedure to the \( N = 1 \) four dimensional super-Poincaré algebra focusing on the action. Expanding the gauge fields associated to the expanded \( p \)-brane super-Poincaré algebra we have obtained two actions, for strings and membranes, describing two non-relativistic supersymmetric theories. The actions that we find contain Fermion fields that have half the number of degrees of freedom of four dimensional Majorana spinors. We have shown, subsection 1.2.3 that these actions satisfy the invariance properties of the initial action, in particular these are on-shell invariant under the expanded supersymmetry transformations. In subsubsection 1.2.3.3 we have described the general invariance conditions for an \( n \) order action term under the expanded algebra \( g(N_0, N_1) \). This defines a completely general method to find non-relativistic supersymmetric theories of strings and membranes.

The actions that we have explicitly studied and the systematic that we have defined are particularly relevant from the physical point of view, since they describe four dimensional gravity models with a non-relativistic symmetry, an extension of the stringy or \( p = 2 \) super-Galilei algebra, with fermion fields. Furthermore the method itself that has been used, assures a full control on the symmetries of the final theories, even considering higher truncation, i.e. theories with a great number of fields. All these features make our models particularly interesting not only for possible applications in several different context but also as prototypes of a procedure to investigate different regimes in gravity theories with fermions.

The most natural extension of the present results is the application of the Lie algebra expansion method, trough the formalism we provide, to the ultra-relativistic case. It is also interesting to use our setting to study \( N > 1 \) supersymmetric theories.

We believe also that our models could find a natural application in the context of holography and condensed matter physics [39]. It has been shown that asymptotically locally Lifshitz space–times are holographically dual to field theories with exhibit Schrödinger invariance [4, 40, 41]. In this setting the geometry induced by the bulk on the boundary is torsional Newton–Cartan. Our symmetries being supersymmetric extension of the expanded \( p = 1, 2 \) Galilei algebra we believe that can have an interesting application in this context. In particular it would be extremely interesting to understand of which kind of space our models could be thought as boundary geometry of, and the presence of supersymmetry-like symmetry could provide a strong help in this direction.

Analogously we believe that our work could have relevant applications in hydrodynamics [39]. In particular the systematic nature of the method that we propose and the possibility to have a full control of the symmetries after the expansion could provide a very powerful tool to investigate all these applications.
Acknowledgments

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Appendix A. Supersymmetry algebra and $p$-Brane projectors

In this section we discuss how to act on the super-Poincaré algebra \( g \) in order to decompose it in a way that is useful for the Lie algebra expansion. In particular what we want to obtain is a decomposition in two subspaces \( g = V_0 \oplus V_1 \) such that they respect the following symmetric space structure

\[
[V_0, V_0] \subseteq V_0 \quad (69a) \\
[V_0, V_1] \subseteq V_1 \quad (69b) \\
[V_1, V_1] \subseteq V_0, \quad (69c)
\]

where \([\cdot, \cdot] \) denotes the supercommutator. Furthermore since \( g(0, 1) \) will be the Inonu–Wigner contraction of \( g \) with respect to \( V_0 \) and we are interested in obtaining, through the expansion, a non-relativistic algebra we would like to choose \( V_0 \) such that \( g(0, 1) \) is the superGalilei algebra. By this way the higher order expanded algebras \( g(N_0, N_1) \) will be an extension of the \( p \)-brane super-Galilei algebras.

We consider a \( D \) dimensional space with signature \(+, -, \ldots, -\) and a flat index decomposition \( \hat{A} = \{A, a\} \) in general not related to the signature, with the index \( A \) taking \( p + 1 \) values, say \( A = 0, 1, \ldots, p \), and the index \( a \) the remaining \( D - p - 1 \) values. In this case we can always choose the gamma matrices such that \( \gamma_0 = \gamma_0 \) and \( \gamma_i = -\gamma_i \) for \( i = 1, \ldots, D - 1 \). We define the following matrices

\[
U = \alpha \gamma_0 \gamma_1 \ldots \gamma_p \\
V = \beta \gamma_{p+1} \ldots \gamma_{D-1}, \quad (70, 71)
\]

where \( \alpha \) and \( \beta \) are two phase factors, and see how they act on the gamma matrices. We note that

\[
V = \pm \gamma_i U. \quad (72)
\]

where \( \gamma_i = \eta \gamma_0 \ldots \gamma_{D-1} \) and \( \eta \) is a phase factor. These matrices satisfy

\[
U^\dagger = (\alpha^*)^2 (-1)^{\frac{D(p+1)}{2}} U \quad (73a) \\
U^\dagger U = U U^\dagger = 1 \quad (73b) \\
V^\dagger = (\beta^*)^2 (-1)^{\frac{(D-p)(D-1)}{2}} V \quad (73c) \\
V^\dagger V = V V^\dagger = 1 \quad (73d)
\]

and
\[ U_{\gamma A} U^\dagger = ( -1)^p \gamma_A \] (74a)
\[ U_{\gamma a} U^\dagger = ( -1)^{p+1} \gamma_a \] (74b)
\[ V_{\gamma A} V^\dagger = ( -1)^{D-p-1} \gamma_A \] (74c)
\[ V_{\gamma a} V^\dagger = ( -1)^{D-p} \gamma_a \] (74d)

We have thus built two matrices that act discriminating the two subsets of our index splitting \( \hat{A} = \{ A, a \} \). We can then define the following projectors

\[ \Pi_U^\pm = \frac{1}{2} (1 \pm U) \] (75)
\[ \Pi_V^\pm = \frac{1}{2} (1 \pm V) \] (76)

but in order for this to be projectors we should require \( UU = 1 \) and \( VV = 1 \) that will fix the phase factors to

\[ \alpha = ( -1)^{p(\frac{p+1}{4})} \] (77a)
\[ \beta = ( -1)^{(D-p)(\frac{D-p-1}{4})} \] (77b)

With these choices we immediately recognize that

\[ U^\dagger = U \] (78a)
\[ V^\dagger = V \] (78b)

The definition of the projectors induces the following definitions

\[ Q^\pm = \Pi_U^\pm Q \] (79a)

or

\[ Q^\pm = \Pi_V^\pm Q \] (79b)

The Dirac adjoint is defined by

\[ \overline{\psi} = \psi^\dagger D \] (80)

where we consider \( D \) as in [36]

\[ D = \delta \gamma_0 \] (81)

and \( \delta \) is a phase factor. This matrix satisfies

\[ UD = ( -1)^p DU \] (82a)
\[ VD = ( -1)^{D-p-1} DV \] (82b)

implying, in the two cases

\[ \overline{Q}^\pm = (\Pi_U^\pm Q) = \overline{D} \Pi_U^{\dagger( -1)^p} \] (83a)
\[ \overline{Q}^\pm = (\Pi_\pm^U Q) = \overline{\Pi}_{(\pm 1)\rho - 1}^U. \] (83b)

Then the commutation relations of the super-Poincaré algebra involving the Fermionic generator

\[
\{ Q^{\alpha}, \overline{Q}_\beta \} = k (\gamma_A)^\alpha_\beta P_A
\]

(84a)

\[
[ J_{AB}, Q^\alpha ] = -\frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^\beta
\]

(84b)

\[
[ J_{AB}, \overline{Q}_\alpha ] = +\frac{1}{2} (\overline{\gamma}_{AB})_\alpha
\]

(84c)

applying the projector \( \Pi_\pm^U \) and \( \Pi_\pm^V \), become

\[ \Pi_\pm^U \]

\[
\{ Q^{\alpha \pm}, \overline{Q}_\beta \} = k [\Pi_\pm^U \gamma_A]_\beta^\alpha H^A
\]

(85a)

\[
\{ Q^{\alpha \pm}, \overline{Q}^\alpha \} = k [\Pi_\pm^U \gamma_A]_\beta^\alpha P^\alpha
\]

(85b)

\[
[ J_{AB}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^{\beta \pm}
\]

(85c)

\[
[ J_{ab}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{ab})^\alpha_\beta Q^{\beta \pm}
\]

(85d)

\[
[ G_{Ab}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{Ab})^\alpha_\beta Q^{\beta \mp}
\]

(85e)

\[ \Pi_\pm^V \]

\[
\{ Q^{\alpha \pm}, \overline{Q}^\alpha \} = k [\Pi_\pm^V \gamma_A]_\beta^\alpha H^A
\]

(86a)

\[
[ J_{AB}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^{\beta \mp}
\]

(86c)

\[
[ J_{ab}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{ab})^\alpha_\beta Q^{\beta \mp}
\]

(86d)

\[
[ G_{Ab}, Q^{\alpha \pm} ] = -\frac{1}{2} (\gamma_{Ab})^\alpha_\beta Q^{\beta \pm}.
\]

(86e)

Now we collect some useful relations before we discuss the case of Majorana spinors,

\[
\Pi_\pm^U \gamma_5 = \gamma_5 \Pi_\pm^U (\pm 1)^{p+1}(D-1)
\]

(87a)

\[
\Pi_\pm^U \gamma_4 = \gamma_4 \Pi_\pm^U (\pm 1)^p
\]

(87b)

\[
\Pi_\pm^U \gamma_\Lambda = \gamma_\Lambda \Pi_\pm^U (\pm 1)^{p+1}
\]

(87c)

\[
(\Pi_\pm^U) = \Pi_\pm^U
\]

(87d)

\[
\Pi_\pm^V \gamma_5 = \gamma_5 \Pi_\pm^V (\pm 1)^{D+1}(D-1)
\]

(87e)
For a complex spinor
\[
\psi^\pm = \Pi_\pm^U \psi
\]  
\[
(\psi^\pm)^\dagger = \psi^\dagger \Pi_\pm^V
\]  
\[
\bar{\psi}^\pm = \Pi_\pm^U \psi
\]  
\[
(\bar{\psi}^\pm)^\dagger = \psi^\dagger \Pi_\pm^V
\]  
This implies, considering the following bilinears, with the two different projectors, that we can write
\[
\Pi_\pm^V \gamma_a = \gamma_a \Pi_\pm^V (\delta_{a,-1})^{\delta_{\rho,-1}}
\]  
\[
(\Pi_\pm^V)^\dagger = \Pi_\pm^V
\]  
\[
\text{A.1. Majorana spinor}
\]
Now we consider the case in which \( Q \) is a Majorana spinor. This means
\[
\overline{Q} = Q^C = Q C_{(i)}
\]  
where \( C_{(i)} \) is the similarity matrix (charge conjugation matrix) defined by
\[
C_{(i)} \gamma_a C_{(i)}^{-1} = x \gamma_a
\]  
and \( x = \pm 1 \). The Majorana conjugate spinor \( Q^C \) is in components
\[
Q_a^C = Q^i (C_{(i)})_{\alpha a}
\]
In this case we can find a basis such that all the gamma matrices are purely imaginary and from now on we adopt this as our basis. We stress that not both $C_{(±)}$ exist for any dimension and furthermore that, when they both exist the consistency of the Majorana condition, and thus the supersymmetry algebra, require a precise choice. The argument that we present in this section is completely general but, due to the remarks just done should be carefully applied.

We can write

$$C_{(x)} \gamma_0 C_{(x)}^{-1} = -x \gamma_0 \quad (94a)$$
$$C_{(x)} \gamma_0 C_{(x)}^{-1} = +x \gamma_0 \quad (94b)$$

for $i = 1, \ldots, D - 1$. Thus we have

$$C_{(x)} U C_{(x)}^{-1} = x^p U \quad (95a)$$
$$C_{(x)} V C_{(x)}^{-1} = x^{D-p-1} V, \quad (95b)$$

that is

$$C_{(x)} U = x^p U C_{(x)} \quad (96a)$$
$$C_{(x)} V = x^{D-p-1} V C_{(x)} \quad (96b)$$

The projectors thus satisfy

$$C_{(x)} \Pi^U_{±} = \Pi^U_{± x} C_{(x)} \quad (97a)$$
$$C_{(x)} \Pi^V_{±} = \Pi^V_{± x} C_{(x)} \quad (97b)$$

then in the two cases we have

$$\overline{U}^\pm = \overline{\Pi}^U_{± x} C_{(x)} = Q \Pi^U_{± x} C_{(x)} \quad (98a)$$
$$\overline{V}^\pm = \overline{\Pi}^V_{± x} C_{(x)} = Q \Pi^V_{± x} C_{(x)} \quad (98b)$$

Since

$$U^t = +(-1)^\frac{D(D-1)}{2} U \quad (99a)$$
$$V^t = +(-1)^\frac{(D-p-1)(D-2)}{2} V. \quad (99b)$$

we get

$$\Pi^U_{±} \quad (100a)$$
$$\Pi^V_{±} \quad (100b)$$

In particular, using $C = C_{(-)}$,
\[\Pi_U^\pm = \left(Q^{\pm (-1)^{\frac{p(p+1)}{2}+1}}\right)^t C \]

\[\Pi_V^\pm = \left(Q^{\pm (-1)^{\frac{D(D-1)(p-p+3)}{2}}/2}\right)^t C.\]  

(102a)

(103a)

We can thus define

\[\xi_U = (-1)^{\frac{p(p+1)}{2}+1} \]

\[\xi_V = (-1)^{\frac{(D-p)(D-p+2)}{4}}.\]  

(104a)

(104b)

Finally the anticommutators of the algebra, using the two different projectors, could be put in the form

\[\{Q^\alpha, Q^\beta \pm \xi_U\} = k\left[\Pi_U^\pm \gamma_\alpha C^{-1}\right]^{\alpha\beta} H^A \]

\[\{Q^\alpha, Q^\beta \pm \xi_V\} = k\left[\Pi_V^\pm \gamma_\alpha C^{-1}\right]^{\alpha\beta} P^\sigma.\]  

(105a)

(105b)

(106a)

(106b)

By this way we can recognize that the interesting case for our purpose is that with \(\xi_U = +1\), i.e. \(p = 1, 2\). In these cases the decomposition of the superalgebra satisfying our requests, with the use of the projectors \(\Pi_U^\pm\) is \(V_0 = \{J_{AB}, J_{ab}, H^A, Q^+\}\). We note that the case \(\xi_U = -1\) corresponds to the ultra-relativistic case; we will not treat this case in the present work. We summarize our results in the following table (table A1).

We specialize now the spinor bilinear discussed in the previous section to the case in which they are Majorana. We have using \(\Pi_U^\pm\) or \(\Pi_V^\pm\)

\[\Psi \gamma_{A_1} \ldots \gamma_{A_p} \gamma_{a_1} \ldots \gamma_{a_j} \chi = \left(\psi^{+(1)} \right)^t C\gamma_{A_1} \ldots \gamma_{A_p} \gamma_{a_1} \ldots \gamma_{a_j} \chi^{+(1)^p+j+1+j} + \left(\psi^{-(1)} \right)^t C\gamma_{A_1} \ldots \gamma_{A_p} \gamma_{a_1} \ldots \gamma_{a_j} \chi^{-(1)^p+j+1+j}\]  

(107a)
respectively for the two projectors $\psi^U$ and $\psi^V$.

Table A1. Values of $\xi^U$ and $\xi^V$ in four dimensions, for different $p$. $\pm$ imply respectively that the corresponding algebra is non-relativistic/ultra-relativistic. Note that $V$ does not exist for $p = 3$.

| $D=4$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ |
|-------|-------|-------|-------|-------|
| $\xi^U$ | -     | +     | +     | -     |
| $\xi^V$ | -     | -     | +     | +     |

\[
\psi^U \gamma^i A_1 \cdots \gamma^j a_1 \cdots \gamma^l x_j
= \left( \psi^U (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j + (-1)^p (-1)^{D-1} \gamma^i A_1 \cdots \gamma^j a_1 \cdots \gamma^l x_j
+ \left( \psi^U (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j
= \left( \psi^U (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j + (-1)^p (-1)^{D-1} \gamma^i A_1 \cdots \gamma^j a_1 \cdots \gamma^l x_j
+ \left( \psi^U (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j (107b)
\]

In particular in this last case we specify to $D = 4$, $i = 1$, $j = 0$ and $i = 0$, $j = 1$ we get respectively for the two projectors

\[
\Pi^U_{\pm}
\]

\[
\Pi^V_{\pm}
\]

\[
\psi^V \gamma^i A_1 \cdots \gamma^j a_1 \cdots \gamma^l x_j
= \left( \psi^V (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j
= \left( \psi^V (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j + (-1)^p (-1)^{D-1} \gamma^i A_1 \cdots \gamma^j a_1 \cdots \gamma^l x_j
+ \left( \psi^V (-1)^{\frac{D(D-1)}{2}} \right)^t C_{\gamma^i A_1} \cdots \gamma^j a_1 \cdots \gamma^l x_j (108b)
\]
\[ \psi^\dagger C \gamma^a \chi = (\psi^\dagger \psi - (1)_{(3-p)(6-p)}^2) t C \gamma^a \chi , \tag{117a} \]

\[ [J_{AB}, J_{CD}] = 4 \eta_{[A[C} (J_{D][B]}^{(i+j)} - (i+j)^{k-l}) \tag{117b} \]

\[ [J_{ab}, J_{cd}] = 4 \eta_{[a[c} (J_{d][b]}^{(i+j)} - (i+j)^{k-l}) \tag{117b} \]

\[ [J_{AB}, G_{CD}] = 2 \eta_{[A[C} (G_{D][B]}^{(i+j)} - (i+j)^{k-l}) \tag{117c} \]

\[ [J_{ab}, G_{cd}] = 2 \eta_{[a[c} (G_{d][b]}^{(i+j)} - (i+j)^{k-l}) \tag{117d} \]
\begin{align}
\{Q^{\alpha+}, Q^{\beta+}\} &= i \left[ \eta^{\alpha\beta} \right] \eta_{\alpha\beta}^{(i+j)} H^4 \\
\{Q^{\alpha+}, Q^{\beta-}\} &= i \left[ \eta^{\alpha\beta} \right] \eta_{\alpha\beta}^{(i+j)} H^4 \\
\{Q^{\alpha-}, Q^{\beta+}\} &= i \left[ \eta^{\alpha\beta} \right] \eta_{\alpha\beta}^{(i+j)} H^4 \\
\{Q^{\alpha-}, Q^{\beta-}\} &= i \left[ \eta^{\alpha\beta} \right] \eta_{\alpha\beta}^{(i+j)} H^4
\end{align}

where $i, j \in 2\mathbb{Z}_+$ and $k, p \in 2\mathbb{Z}_+ + 1$ and it is understood that if the order of a generator exceeds the truncation order it is to be set to zero in the relations above.

Explicitly if we consider the truncation $g(0, 1)$ we find the following commutations relations:

\begin{align}
\{J_{AB}, J_{CD}\} &= 4 \eta_{AB[CD]} J_{[CD]} \\
\{J_{ab}, J_{cd}\} &= 4 \eta_{ad[bc]} J_{[bc]} \\
\{J_{AB}, G_{CD}\} &= 2 \eta_{[CD]BA} \\
\{J_{ab}, G_{cd}\} &= 2 \eta_{[cd]ba} \\
\{G_{Aa}, G_{Bb}\} &= 0 \\
\{J_{AB}, H_C\} &= 2 \eta_{[AB]CH} \\
\end{align}
We recognize that this is the Inonu–Wigner contraction of the super-Poincaré algebra with respect to $V_0$, i.e. the $p$-brane super-Galilei algebra. In particular we could identify $g(N_0, N_1)$ as an extension of the $p$-brane super-Galilei algebra.

Appendix B. Useful relations

In this section we list some relations useful in studying the invariance of the action and the derivation of the equation of motion.

\begin{align}
\gamma^A \gamma^B \gamma^C &= \eta_{AB} \gamma^C + \eta_{BC} \gamma^A - \eta_{AC} \gamma^B - i \epsilon_{ABC} \gamma^D \gamma^5 \\
\gamma^A \gamma^B \gamma^C &= 2 \eta_{AB} \gamma^C - 2 \eta_{AC} \gamma^B + 2 \eta_{BC} \gamma^A - 2 \eta_{AB} \gamma^B \gamma^C - \eta_{ABC} \gamma^D \\
\gamma^a \gamma^b \gamma^c &= 2 \eta_{ab} \gamma^c - 2 \eta_{ac} \gamma^b + 2 \eta_{bc} \gamma^a - 2 \eta_{ab} \gamma^c + \gamma^a \gamma^b \gamma^c \\
\gamma^a \gamma^b \gamma^c &= \gamma_{aB} + 2 E_{a[B} \gamma_{B]} = i \epsilon_{a[B} \gamma_{BCD} \gamma^7 \gamma^5 + 2 E_{a[B} \gamma_{B]} \\
\overline{\psi} \gamma^A \chi &= \nabla_{\gamma^A} \psi \\
\overline{\psi} \gamma^A \chi &= \nabla_{\gamma^A} \chi \\
\overline{\psi} \gamma^A \gamma^B \gamma^C \chi &= - \chi \gamma^A \gamma^B \gamma^C \psi.
\end{align}

(119a) (119b) (119c) (119d) (119e) (119f) (119g)
The variation under susy in differential form language

\[ \delta S = 2R^\lambda(P) \wedge \overline{\psi}_\lambda \gamma_\sigma \wedge R(Q) \]

**Appendix C. Trivial symmetry**

In this section we study the trivial symmetries of the $\mathcal{N} = 1$ supergravity action in four dimensions. The trivial symmetries or equation of motion symmetries are define by the transformations of the fields to be proportional to the equations of motion. As a starting point we take the action in the following form

\[ \mathcal{L} = EE^\mu E^\nu R^\lambda_{\mu \nu}(J) + 2\epsilon^{\mu \nu \lambda \sigma} \overline{\psi}_\lambda \gamma_\sigma \gamma_\gamma D_\mu \psi_\nu \]

and we compute the variations with respect to the three fields, obtaining

\[ \delta L = E \left[ R(J)E^\mu - 2R^\mu_{\nu}(J) \right] \delta E^\nu + \epsilon^{\mu \nu \lambda \sigma} \overline{\psi}_\lambda \gamma_\sigma \gamma_\gamma R_{\mu \nu}(Q) \delta \psi_\nu \]

\[ \delta \Omega = 2\epsilon^{\mu \nu \lambda \sigma} \overline{\psi}_\lambda \gamma_\sigma \gamma_\gamma \overline{\psi_\mu} \gamma_\sigma \gamma_\gamma \Omega \]

\[ \delta \psi = 2\epsilon^{\mu \nu \lambda \sigma} \overline{\psi}_\lambda \gamma_\sigma \gamma_\gamma \overline{\psi_\mu} \gamma_\sigma \gamma_\gamma \]

We denote the full variation of the Lagrangian as

\[ \delta \mathcal{L} = A^\mu_{\lambda} \delta E^\nu_{\mu} + B^\mu_{\lambda \beta} \delta \Omega^\lambda_{\beta \mu} + \overline{C}^\mu_{\sigma} \delta \psi_{\sigma} \]

where $A^\mu_{\lambda}$, $B^\mu_{\lambda \beta}$ and $\overline{C}^\mu_{\sigma}$ could be immediately read from equation (121). We will define three trivial symmetries to which we will refer as type $AB$, $BC$ and $AC$ from the equations of motion involved in the transformations.

**C.1. Type AB**

We consider the following transformation

\[ \delta E^\lambda_{\mu} = aB^\lambda_{\mu \beta} \sigma^\beta + bE^\lambda_{\mu \beta}B^\beta_{\nu \sigma} \sigma^\sigma + cB^\beta_{\mu \beta} \sigma^\lambda \]

\[ \delta \Omega^\lambda_{\mu \beta} = -aA^\mu_{\lambda \beta} \sigma^\beta - bE^\mu_{\lambda \beta}A^\lambda_{\nu \sigma} \sigma^\sigma - cB^\beta_{\mu \beta} \sigma^\lambda \]

\[ \delta \psi_{\mu} = 0, \]

where $a, b, c$ are arbitrary constants. We note that taking $a = 2, b = 1, c = -1$ we have

\[ \delta E^\lambda_{\mu} = 2B^\lambda_{\mu \beta} \sigma^\beta + E^\lambda_{\mu \beta}B^\beta_{\nu \sigma} \sigma^\sigma - B^\beta_{\mu \beta} \sigma^\lambda = 2ER^\lambda_{\mu \beta}(P) \sigma^\sigma. \]
C.2. Type BC

For the type BC trivial symmetry we have the following transformations

\[ \delta E^{\hat{A}}_\mu = 0 \]  

\[ \delta \Omega^{\hat{A}\hat{B}}_\mu = -aE^{\hat{B}}_\mu \lambda^\mu_{\hat{A}} C^\nu - bC^{\hat{A}}_\mu \lambda^\mu_{\hat{B}} C - cC^{\hat{A}}_\mu \lambda^\mu_{\hat{B}} \]  

\[ \delta \psi^{\alpha}_\mu = a\lambda^{\alpha}_{\mu} \hat{B}^{\hat{A}} + b\lambda^{\alpha}_{\mu} \hat{B}^{\hat{B}} + c\lambda^{\alpha}_{\mu} \hat{B}^{\hat{C}} \]  

With \( c = 2, a = 1, b = -1 \) we have

\[ \delta \psi^{\alpha}_\mu = 2ER^{\hat{B}}_\mu (P) \chi^{\nu}_{\hat{B}}. \]  

C.3. Type AC

Finally for the type AC we have

\[ \delta E^{\hat{A}}_\mu = aC^{\hat{A}}_\mu \chi^\mu_{\hat{A}} + bE^{\hat{B}}_\mu C^{\hat{B}}_\mu \]  

\[ \delta \Omega^{\hat{A}\hat{B}}_\mu = 0 \]  

\[ \delta \psi^{\alpha}_\mu = -a\chi^{\alpha}_{\mu} \hat{A}^{\hat{A}} + b\chi^{\alpha}_{\mu} \hat{A}^{\hat{A}} \]  

If we set \( a = 2, b = -1 \) we get

\[ \delta \psi^{\alpha}_\mu = 4ER^{\hat{A}}_\mu (P) \chi^{\alpha}_{\hat{B}}. \]  

C.4. Lorentz + general coordinate transformations

Under Lorentz and general coordinate transformations we have

\[ \delta E^{\hat{A}}_\mu = -\lambda^{\hat{B}}_\mu E^{\hat{B}} + \xi^\nu \partial_\nu E^{\hat{A}}_\mu + \partial_\mu \xi^\nu E^{\hat{A}}_\nu \]  

\[ \delta \Omega^{\hat{A}\hat{B}}_\mu = \partial_\mu \lambda^{\hat{B}}_\nu - 2\Omega^{\hat{A}}_\mu \lambda^{\hat{B}}_\nu + \xi^\nu \partial_\mu \Omega^{\hat{A}\hat{B}}_\nu + \partial_\mu \xi^\nu \Omega^{\hat{A}\hat{B}}_\nu \]  

\[ \delta \psi^{\alpha}_\mu = -\frac{1}{4} \lambda^{\hat{B}}_\mu \gamma^{\alpha \hat{B}} \psi^\mu + \xi^\nu \partial_\nu \psi_\mu + \partial_\mu \xi^\nu \psi_\nu. \]

Appendix D. Equations of motion of the spin connection and expansion

In this section we list the equations of motion of the fields coming from the spin connection after the expansion for the actions equations (40) and (45). The equation of motion for the spin connection before the expansion and splitting of the indices is given by

\[ \delta \Omega L = 2\epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} (P) \wedge E^{\hat{B}} \wedge \delta \Omega^{\hat{C}\hat{D}} \]  

The equations of motion for the fields coming from the spin connection in the two cases, \( p = 1, 2 \), are
Strings, $p = 1$

\[
\epsilon_{ab} \delta \Omega^{(1)}_{AB} \wedge R^a(P) \wedge E^b = 0 \quad (131a)
\]

\[
\epsilon_{ab} \delta \Omega^{(0)}_{AB} \wedge (0)^a H \wedge (0)^b = 0 \quad (131b)
\]

\[
\epsilon_{ab} \delta \Omega^{(0)}_{ab} \wedge \left[ (2)^a H \wedge (0)^b + (0)^a H \wedge (2)^b \right] = 0 \quad (131c)
\]

\[
\epsilon_{ab} \delta \Omega^{(1)}_{ab} \wedge \left[ (0)^a H \wedge (1)^b - (1)^a H \wedge (0)^b \right] = 0. \quad (131d)
\]

Note that in the $p = 1$ case the fields $\Omega^{(1)}_{AB}$ and $\Omega^{(2)}_{AB}$ do not appear in the Lagrangian.

Membranes, $p = 2$

\[
\epsilon_{ABC} \delta \Omega^{(2)}_{AB} \wedge \left[ (0)^C H \wedge (1)^E - (1)^E H \wedge (0)^C \right] = 0 \quad (132a)
\]

\[
\epsilon_{ABC} \delta \Omega^{(0)}_{AB} \wedge \left[ (2)^C H \wedge (1)^E + (0)^C H \wedge (3)^E - (3)^E H \wedge (0)^C \right] = 0 \quad (132b)
\]

\[
\epsilon_{ABC} \delta \Omega^{(1)}_{A} \wedge \left[ (2)^B H \wedge (0)^C + (0)^B H \wedge (2)^C \right] = 0 \quad (132c)
\]

\[
\epsilon_{ABC} \delta \Omega^{(3)}_{A} \wedge (0)^B H \wedge (0)^C = 0. \quad (132d)
\]

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