Parafermionic zero modes (PZMs) [19–21] (also known as fractional Majorana modes) provide an example of such computationally rich (still not universal) anyons. They can be thought of as $Z_n$ generalizations of MZMs. Similar to MZMs, $Z_n$ PZMs are associated with $n$ fold degeneracy of the ground state that is robust to all local perturbations. Due to fundamental restrictions set on possible topological phases in strictly one dimensional systems [22, 23], PZMs cannot exist in isolation. However, recently it was realized that boundaries of two dimensional systems can circumvent these restrictions. It was explicitly shown that PZMs emerge at the one dimensional boundary of two counter-propagating fractional quantum Hall (FQH) edges coupled with superconducting contacts [24–26]. These setups greatly resemble the canonical proposal used to realize MZMs [27], with the role of topological insulator played by a pair of opposite-chirality FQH states. All of the existing proposals (involving superconductors) require two main ingredients, induced superconductivity via coupling FQH edge state to a superconductor and crossed Andreev tunneling between two edges. The first requirement has already been achieved in experiments [28, 29]. However, the second requirement is likely to be difficult to achieve experimentally due to the disruption of FQH edge states placed adjacent to a superconductor. This is because strong coupling to the superconductor is likely to change the density in the surrounding 2D electron gas, pushing the FQH edges away from the superconductor. The amplitude of quasiparticle tunneling between the edges would then be reduced by the increased distance and the Fermi wave-vector in the intervening superconducting region.

In this letter we propose a practical scheme to realize $Z_3$ PZMs from the spin unpolarized 2/3 fractional quantum Hall state using superconducting contacts without cross-Andreev tunneling, which can be realized in present experiments. Our system is comprised of a linear array of superconductors coupled to loops of FQH edge states, different loops are connected via quasiparticle hopping.
merical methods to study this model and show that for realistic values of parameters, at relatively small chain lengths (order ten loops) this system can be tuned to a topological phase hosting $\mathbb{Z}_3$ PZMs.

Model.—We begin by studying a single loop coupled to a superconductor. Assuming $SU(2)$ symmetry, the effective Hamiltonian describing the charge part of the $\nu = 2/3$ FQH edge state is given by the following chiral boson theory [30, 31],

$$H_{\text{edge}} = \int_0^L dx [\frac{u}{4\pi \nu} (\partial_x \varphi(x))^2 - \frac{um\mu}{2L} \partial_x \varphi(x)]$$

(1)

where $L$ is the length of the loop, $u$ is the mode velocity, $m\mu$ is the gate controlled dimensionless chemical potential (as opposed to actual chemical potential $\mu = \frac{um\mu}{2L}$) and $\varphi(x)$ is the chiral boson field that is defined in terms of the charge density operator as $\rho = \frac{1}{2\pi} \partial_x \varphi(x)$. The $\varphi(x)$ field obeys the commutation relation $[\varphi(x), \varphi(y)] = 2\pi i \nu \delta(x-y)$. Using this relation we can write the charge $2/3$ spinless quasiparticle creation operator as $e^{i\varphi(x)}$ and charge 2 Cooper pair creation operator as $e^{3i\varphi(x)}$. The neutral mode, which does not couple to the SC and is expected to be non-degenerate and gapped will be ignored in the rest of this work. The edge Hamiltonian $H_{\text{edge}}$ can be diagonalized [32] by mode expanding $\varphi(x)$ as

$$\varphi(x) = \frac{2\pi \hat{n} \nu x}{L} + \hat{\varphi}_0 + \sum_{k=0}^{K_{\text{max}}} \left[ -i \sqrt{\frac{\nu}{k}} a_k e^{2\pi ikx/L} + i \sqrt{\frac{\nu}{k}} a_k e^{-2\pi ikx/L} \right]$$

(2)

where $a_k, a_k^\dagger$ are the $k$th momentum boson creation and annihilation operator for $k \in \mathbb{N}$, $\hat{\varphi}_0, \hat{n}$ are the zero mode phase and number operators, respectively, and $K_{\text{max}}$ is the momentum cutoff.

Now we can rewrite Eq. (1) as,

$$H_{\text{edge}} = \frac{um\nu}{L} (\hat{n}^2 - m\mu \hat{n} + \frac{2}{\nu} \tilde{P}) + \text{const}$$

(3)

where $\tilde{P} = \sum_{k=0}^{K_{\text{max}}} ka_k^\dagger a_k$ is the total momentum operator. When the dimensionless chemical potential $m\mu$ is tuned by a gate voltage to integer values the spectrum is invariant under changing $n = m \rightarrow n = -m + m\mu$. For odd $m\mu$ this translates into a two fold ground state degeneracy. This degeneracy survives the addition of superconductivity and will play a crucial role later on.

The effect of the superconductor on a single loop can be modeled by the Hamiltonian

$$H_{sc} = \int_0^L dx \frac{\Delta(x)}{L} e^{-\frac{2\pi i m\mu}{L} \varphi(x)} + h.c$$

(4)

describing the tunneling of Cooper pairs to and from the superconductor. Here $\Delta(x)$ corresponds to the position-dependent Cooper pair tunneling amplitude and $e^{-\frac{2\pi i m\mu x}{L}}$ is the phase factor taking into account the chemical potential mismatch between the FQH edge and the (grounded) superconductor. Fourier transforming $\Delta(x) = \sum_k \Delta(k) e^{\frac{2\pi ikx}{L}}$ and mode expanding $\varphi(x)$ (as in Eq. (2)) allows us to write the only nonzero matrix elements of $H_{sc}$:

$$\langle n_0 \pm 3, \{m_k\}| H_{sc} | n_0, \{n_k\} \rangle = \sum_k \Delta(k) \delta(\Delta E \pm 3k)$$

(5)

$$\times \langle n_0 \pm 3, \{m_k\}| e^{\pm 3i\varphi(0)} | n_0, \{n_k\} \rangle,$$

where $\Delta E = E(n_0 \pm 3, \{m_k\}) - E(n_0, \{n_k\})$ is the energy difference between the initial and the final state, and $E = \frac{um\nu}{L} (n^2 - m\mu n + 3P)$ is the bare edge energy of each state in accordance with Eq. (3). Equation (5) implies that the special case of uniform superconductivity $\Delta(x) = \Delta_0$ leads to the additional conserved quantity $H_{\text{edge}}$, as $[H_{sc}, H_{\text{edge}}] = 0$. Though convenient, this symmetry is not generic and is therefore not used in this letter.

The inclusion of $H_{sc}$ reduces the conservation of fractional charge $n$ to conservation of $q = \text{mod}(n, 3)$, which only takes three values $q = 0, 1, 2$. Using this restriction we can divide the system into three independent charge sectors with $q = 0, 1, 2$. For integer $m\mu$, the $n = 3m$ to $n = -3m + m\mu$ symmetry of the non-superconducting edge now translates into degeneracy of two of the charge sectors, for example at $m\mu = 1$ the two sectors $q = 0, 1$ will become degenerate.

Using Eq. (5) we can numerically calculate the spectrum of a single loop $H = H_{\text{edge}} + H_{sc}$. We set $m\mu = 1$ and assume a “pseudo-point-like” superconductivity such as a function of $\Delta_0 / L$. Here $m\mu = 1$, $K_{\text{max}} = 4$ and $q$ is the fractional charge modulo three $q = \text{mod}(n, 3)$. All red circles are two fold degenerate, blue circles are non-degenerate.
that $\Delta(k) = \Delta$ for $|k| \leq K_{\text{max}}$, where $K_{\text{max}}$ is the momentum cutoff defined in Eq.(2). The low-lying part of the spectrum is plotted in Fig. 2. This plot shows that the ground state is separated from the rest of the spectrum by a gap for a finite range of $\Delta$. However, the ground state degeneracy (between $q = 0, 1$) remains twofold with $q = 2$ split.

**Effective Hamiltonian.**—In the absence of superconductivity and for odd values of the dimensionless chemical potential ($m_\mu = 2n - 1$), the ground-state energy of the system is twofold degenerate and is separated from the excited states by a gap for a range of $\Delta$. The two ground states can be labeled by fractional charge $q = 0, 1$. Therefore as long as we choose $\Delta$ in this range and restrict the ratio of hopping amplitude to the energy gap $t/\Delta E$ to be small, we can apply a Schrieffer-Wolff transformation to obtain an effective Hamiltonian defined in the Hilbert space spanned by ground states of single loops. This emergent Hilbert space has only two states per site ($q = 0, 1$) and therefore can be thought of as a chain of spin 1/2 particles, where the states with spin up/down correspond to the single loop ground states with $q = 1, 0$.

To calculate the low-energy effective Hamiltonian, we start with the Hamiltonian describing quasiparticle hopping between different loops,

$$H_{\text{tunnel}} = t \sum_i e^{i\varphi_i(L/2)} e^{-i\varphi_{i+1}(0)} + \text{H.c.} \quad (6)$$

Note that $e^{i\varphi_i(x)}$ is an anyonic operator and has nontrivial commutation relation with other anyonic operators. For different sites we can write,

$$e^{i\varphi_i(x)} e^{i\varphi_j(x')} = e^{i\varphi_j(x')} e^{i\varphi_i(x)} e^{i\pi \text{sgn}(j-i)}. \quad (7)$$

Using a generalized Jordan-Wigner string we can define the new field variables $\tilde{\varphi}(x)$,

$$e^{i\tilde{\varphi}_i(x)} = e^{i\pi \nu \sum_{j<i} \delta_{ji}} e^{i\varphi_i(x)} \quad (8)$$

The advantage of these new field variables is that they commute trivially between different sites [$e^{i\tilde{\varphi}(n), e^{i\tilde{\varphi}(n')}] \propto \delta_{nn'}$, and therefore act strictly on the local loop Hilbert space. Now we can rewrite $H_{\text{tunnel}}$ as,

$$H_{\text{tunnel}} = t \sum_i e^{i\tilde{\varphi}_i(L/2)} e^{-i\pi \varphi_{n_0,i}} e^{-i\varphi_{i+1}(0)} + \text{H.c.} \quad (9)$$

To second order in perturbation theory we can write the low energy effective Hamiltonian as,

$$H_{\text{eff}} = PH_{\text{tunnel}} P - PH_{\text{tunnel}} \frac{(1-P)}{H_0} H_{\text{tunnel}} P + O(t^3/\Delta E^2) \quad (10)$$

where $P$ is the single loop ground state projection operator defined as $P = \sum_i |(q = i)(q = i)\rangle$, and $H_0$ is the shifted single loop Hamiltonian (shifting to set ground state energy to zero).

Putting everything together we get (Details of this calculation can be found in the supplementary material, here we just quote the final results),

$$H_{\text{eff}} = \sum_i \left[ (t\alpha_0 e^{i\pi/3} - t^2 \alpha_1 e^{-2i\pi/3}) \sigma_i^+ \sigma_{i+1}^- + t^2 \lambda \sigma_i^+ \sigma_{i+1}^- \right] + H.c. + O(t^3/\Delta E^2) \quad (11)$$

where $\sigma$’s are the usual Pauli matrices. Note that all terms in the Hamiltonian conserve fractional charge (spin) modulo three and are also $Z_2$ symmetric under $\sigma_z \rightarrow -\sigma_z$. This $Z_2$ symmetry can be associated with the $\varphi \rightarrow -\varphi + \frac{2\pi m_{\mu} \nu_2}{L}$ symmetry of the original Hamiltonian (that we have already discussed). Note that presence of the term $t^2 \lambda \sigma_i^+ \sigma_{i+1}^- \sigma_i^+ \sigma_{i+1}^+$ requires non-zero superconductivity, since without $\Delta$ fractional charge (spin) has to be conserved. As seen in Fig. 3, this term, which arises at second order in tunneling, requires $H_{\text{sc}}$ so that $q=3$ may be converted to $q=0$ by the removal of a Cooper pair.

**Analysis.**—In the absence of the superconductivity ($\sigma_z$ non-conserving terms), the conservation of $\sigma_z$ ensures a gapless state with low-energy Luttinger liquid Hamiltonian where $\sigma_z \sim \nabla \phi$. In this Luttinger liquid description, the superconducting term $t^2 \lambda \sigma_i^+ \sigma_i^+ \sigma_i^+ \sigma_{i+1}^-$ is represented as $g \cos(3\theta)$, which converts the Luttinger liquid to a Sine-Gordon model. For the correct choice of parameters the superconductivity induced term $g \cos(3\theta)$ becomes perturbatively relevant [33] and gaps out the system into a topological phase with a $Z_3$ parafermionic degeneracy, where each ground state corresponds to the phase $\theta$ being locked at one of the three minima of the

| Chain length | Ground state energy | 1st excited state energy | 2nd excited state energy | 3rd excited state energy |
|-------------|---------------------|--------------------------|--------------------------|--------------------------|
| 10          | -6.818              | -6.696                   | -6.696                   | -6.149                   |
| 40          | -29.681             | -29.677                  | -29.677                  | -29.384                  |
| 100         | -75.524             | -75.524                  | -75.524                  | -75.267                  |

**TABLE I:** DMRG calculation results for “pseudo point-like” superconductivity (defined earlier) at $t = 1$, $\Delta/L = 0.046$ and momentum cutoff $K_{\text{max}} = 4$.
cos(3θ) term. To check whether this degeneracy occurs in a our system (i.e. the Hamiltonian in Eq. (11)) with realistic values of the parameters, we numerically study the Hamiltonian in Eq. (11) using the DMRG method. DMRG calculations were performed using the ITensor C++ library[34]. Sample results of this calculation are shown in Table. I. These results confirm existence of a three-fold degeneracy for reasonable parameters. The degeneracy is weakly split for small chain lengths N ≈ 10 and is more pronounced at longer lengths, as expected from a true topological degeneracy. We further emphasize that despite our calculation being done for a relatively small momentum cutoff $K_{\text{max}} = 4$ (chosen for calculational simplicity), we expect our results to hold quite generically, independent of the value of $K_{\text{max}}$.

An alternative interesting limit is that of “true” point-like superconducting contacts (as opposed to pseudo point like defined before), that is $\Delta(k) = \Delta \forall k$. This limit is particularly appealing, as in this case analytical results may be obtained for large values of $\Delta$. Following the formalism developed in Ref.[35], we show (with details in the Supplementary Material) that in the large $\Delta$ limit the system is described by set of decoupled harmonic oscillators, and that in this limit all three fractional charge sectors become degenerate. Analogous to the previous case (small $\Delta$) as long as $t/\Delta E$ is small, we can use Wolf transformation to find an effective Hamiltonian. In this case the effective Hilbert space of each site is three dimensional (corresponding to three fractional charge sectors) and can be thought of as a three state clock model. In this limit, it’s useful to define,

$$\alpha_{2j-1} = e^{-i\varphi_j(0)} A(0) \alpha_{2j} = e^{-i\varphi_j(L/2)} A(L/2)$$

where $A(x) = (\prod_q |q = i| e^{-i\varphi(q)} |q = i + 1\rangle)$ is a normalization factor. Within the effective Hilbert space these operators are the usual parafermionic operators, that is $\alpha_j^\dagger = 1$ and $\alpha_j \alpha_{j'} = \alpha_j \alpha_{j'} e^{i\frac{\pi}{3} \left(\text{sgn}(j'-j)\right)}$. Using these variables and the Hamiltonians in Eqs.(6) and (10) we arrive at,

$$H_{\text{eff}} = t \sum_i (A^\dagger(L/2) A(0) \alpha_{2j}^\dagger \alpha_{2j+1} + H.c.) + O(t^2/\Delta E)$$

(13)

in this form presence of parafermionic edge zero modes $(\alpha_1, \alpha_{2N+1})$ is already manifest[20]. Note that in contrast to Eq.(11), here the calculation has been done to first order in $t$. Using the usual clock model variables[36] we can write the Hamiltonian in a more familiar form,

$$H_{\text{eff}} = \sum_i (-J \sigma_j^\dagger \sigma_{j+1} + H.c.) + O(t^2/\Delta E)$$

(14)

where $J = t A^\dagger(L/2) A(0) e^{i\pi/3}$ and

$$\sigma = \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix}, \omega = e^{2\pi i/3}$$

(15)

is the “clock” operator. In the large but finite $\Delta$ regime, three fold ground state degeneracy of single loops is not exact. We can take this energy difference into account by adding a term $h(\tau_{j} + \tau_{j}^*)$ to Eq.(14) where $h$ is the energy difference of charge sectors $q = 0, 1$ with the charge sector $q = 2$. Estimates for value of $h$ can be found in the supplementary material. Putting everything together we have,

$$H_{\text{eff}} = \sum_i (-J \sigma_j^\dagger \sigma_{j+1} + H.c.) + h(\tau_i + \tau_{i}^*) + O(t^2/\Delta E)$$

(16)

Note that $\sigma_\tau$ is a non-local operator in the physical system of interest. This is an important point as locality prevents the introduction of a Hamiltonian term proportional to $\sigma_\tau$. With this constraint and for small values of $h$ ($h$ can be made arbitrarily small by choosing large enough $\Delta$) the Hamiltonian in Eq.(16) is well known to be in a topological phase with three-fold degeneracy[19].

**Conclusion.**— In this work we have considered a linear array of superconducting “quantum dot”-like holes on a spin singlet 2/3 fractional quantum Hall sample and showed that for both large and small values of induced superconductivity $\Delta$, this system can be tuned to a topological phase hosting $\mathbb{Z}_3$ PZMs. Unlike earlier proposed ideas to realize PZMs, our approach does not rely on Andreev back-scattering between two fractional quantum Hall edges. We believe this feature makes our proposal suitable for realization in experiments using ingredients that have already been demonstrated.

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Effective Spin Model Parameters

We start by calculating the first term in Eq. (10), we can graphically represent this term as,

Algebraically we can write,

$$\alpha_0 = \langle 1 | e^{i\tilde{\phi}(L/2)} | 0 \rangle \langle 0 | e^{-i\tilde{\phi}(0)} | 1 \rangle e^{-i\pi/3}. \quad (S1)$$

Second term in Eq. (10) can be represented with four different diagrams corresponding to parameters $\alpha_1, \beta, \gamma, \lambda$. For $\alpha_1$ we have the following Figure,

note that this term also breaks conservation of fractional charge and therefore is only allowed at nonzero $\Delta$. In algebraic form we have,

$$\alpha_1 = e^{2i\pi/3} \sum_{j, j'} \langle 1 | e^{-i\tilde{\phi}(L/2)} | 2(j) \rangle \langle 2(j) | e^{-i\tilde{\phi}(L/2)} | 0 \rangle \frac{E(2j') + E(2j) - E(2j') - E(2j)}{E(2j')} \langle 0 | e^{i\tilde{\phi}(0)} | 2(j') \rangle \langle 2(j') | e^{i\tilde{\phi}(0)} | 1 \rangle \quad (S2)$$

where $|2(j)\rangle$ corresponds to $j$'th state (arbitrary ordering) with $q = 2$, and the energy $E$ is the bare $\Delta = 0$ energy of the state with respect to the ground state energy. For $\beta$ we have the diagram,
We study the Hamiltonian $H = H_{edge} + H_{sc}$ in the limit of strong and point-like superconductivity. Note that the Hamiltonian commutes with the operator $T = e^{-i\pi n\hat{\nu}}$, $[H, T] = 0$. Eigenvalue of $T$ can be shifted by a unitary transformation $e^{i\tilde{\varphi}_0}$ as can be seen from the commutation $T e^{i\tilde{\varphi}_0} = e^{iq\tilde{\varphi}_0} T e^{-i\pi \nu}$, i.e. if we define $H_{q=0}$ as the Hamiltonian $H$ in the charge sector $q = 0$, other charge eigenvalues $q$ can be generated from the transformed Hamiltonian,

$$H_q(m_\mu) = e^{-iq\tilde{\varphi}_0} H_{q=0}(m_\mu + q) e^{iq\tilde{\varphi}_0}. \quad (S6)$$

We emphasize that $H_{q=0}$ is defined in the Hilbert space where the wavefunctions are eigenstates of $T$ with eigenvalue 1 which in turn implies that the wavefunctions are periodic under the translation $\tilde{\varphi}_0 \to \tilde{\varphi}_0 + \pi \nu$ (same as saying allowed charges are multiples of three).
In the range of periodicity of the wave-functions, the potential $H_{sc}$ has a single minimum. In the limit of large $\Delta$, $H_{sc}$ we can expand around this minimum to obtain the harmonic approximation of $H_{sc}$,

$$H_{sc} \sim \Delta(1 - \frac{9}{2} \varphi^2(0)).$$  \hspace{1cm} (S7)

Note that the continuous nature of the harmonic approximation means that wave-functions would in general violate the periodicity condition as we change $\varphi_0 \rightarrow \varphi_0 + \pi \nu$. However, this boundary condition is expected to be irrelevant, for calculating ground state energies, in large delta regime where $\varphi_0$ becomes strongly localized around zero. In the next section, we’ll use an instanton approximation to see how enforcing periodic boundary conditions modifies our results.

Within the harmonic approximation $H_{q=0}$ can be turned into a set of decoupled harmonic oscillators with trivial spectrum. A particularly nice feature of this transformation is that the spectrum of $H_{q=0}$ will not depend on $q$ (spectrum of $H_{q=0}$ will not depend on $m_\mu$). Degeneracy of the three charge sectors in the large $\Delta$ regime follows from this. We’ll now show this by explicitly diagonalizing the Hamiltonian.

To diagonalize the Hamiltonian, it’s useful to define the following generalized “position” and “momentum” operators,

$$\hat{\beta}_k \equiv i \sqrt{\frac{L}{2u\pi k}} (\hat{a}_k - \hat{a}^\dagger_k); \quad \hat{\alpha}_k \equiv \sqrt{\frac{u\pi k}{2L}} (\hat{a}_k + \hat{a}^\dagger_k).$$  \hspace{1cm} (S8)

These operators satisfy,

$$[\hat{\beta}_k, \hat{\alpha}_{k'}] = i \delta_{kk'}; \quad [\hat{\beta}_k, \hat{\beta}_{k'}] = [\hat{\alpha}_k, \hat{\alpha}_{k'}] = 0. \hspace{1cm} (S9)$$

In these variables and within harmonic approximation $H_{q=0}$ becomes (ignoring constants),

$$H_{q=0,h} = \hat{X}^T H_X \hat{X} + \hat{P}^T \hat{P} \quad \text{with},$$

$$\hat{X} \equiv \left( \sqrt{\frac{L}{u\pi \nu}} \hat{\varphi}_0, \hat{\beta} \right); \quad \hat{P} \equiv \left( \sqrt{\frac{u\pi \nu}{L}} (\hat{n} - m_\mu), \hat{\alpha} \right). \hspace{1cm} (S10)$$

Where(setting $m_\mu = 1$),

$$H_X = \frac{9u\Delta \pi \nu}{2L}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 + \frac{1}{2u\pi k L \Delta \nu^2} \delta_{kk'} \end{pmatrix}.$$  \hspace{1cm} (S11)

Note that since the commutator $[\hat{\varphi}_0, \hat{n} - m_\mu] = i$ is independent of $m_\mu$, the spectrum of $H_{q=0,h}$ does not depend on $m_\mu$. Which in turn implies that the spectrum of $H_{q=0,h}$ does not depend on $q$.

In its diagonal form this Hamiltonian describes a set of decoupled harmonic oscillators with different frequencies. In Fig. S6 we have plotted the energy gap $\delta_0$ as a function of $\Delta$.

**FIG. S6:** Plot of the energy gap $\delta_0$ as a function of $\Delta$ for various total number of modes $K_{max}$ shown in caption. Here numerical values of $m_\mu, u$ and $L$ are set to one.

We find $\varphi(0)$ is localized for large values of $\Delta$. Therefore, for large enough value of $\Delta$, harmonic approximation is justified. Here numerical values of $m_\mu, u$ and $L$ are set to one.

This completes our discussion of the spectrum within harmonic approximation.

As a measure of validity for our approximation, we calculate the expectation value of $\langle \varphi^2(0) \rangle_0$. As shown in Fig. S7 we find that the fluctuations are monotonically decreasing function of the coupling strength $\Delta$. In other words, one can always choose the value of $\Delta$ to be large enough to get phase fluctuations in $\varphi(0)$ to be much smaller than $2\pi$ which justifies the use of harmonic approximation.

**FIG. S7:** $\langle \varphi^2(0) \rangle$ as a function of $\Delta$ for various total number of modes $K_{max}$ shown in caption. We find $\varphi(0)$ is localized for large values of $\Delta$. Therefore, for large enough value of $\Delta$, harmonic approximation is justified. Here numerical values of $m_\mu, u$ and $L$ are set to one.
FIG. S8: Logarithm of ground-state energy splitting between charge sectors $q = 2$ and $q = 0, 1$ (they’re degenerate) as a function of coupling strength $\Delta$, for different total number of modes $K_{\text{max}}$ shown in the inset. Notice that the energy splitting goes to zero at large $\Delta$. Here numerical values of $m_q$, $u$ and $L$ are set to one.

II. Finite $\Delta$ corrections to ground state energies

As mentioned earlier wave functions found within harmonic approximation violate the periodicity condition as we change $\varphi_0 \rightarrow \varphi_0 + \pi \nu$. In this section we use an instanton approximation to see how this periodicity changes our previous results.

We’ll enforce periodicity by externally projecting the states into the physical Hilbert space,

$$|q = 0\rangle = \mathcal{P}|q = 0\rangle_h,$$  \hspace{1cm} (S12)

where $|q = 0\rangle_h$ is the $q = 0$ eigenstate within the harmonic approximation and $\mathcal{P}$ is an operator that projects into the sector that obeys the periodicity condition as $\varphi_0 \rightarrow \varphi_0 + \pi \nu$. We then calculate the $q$ and $\Delta$ dependence of ground-state energy using,

$$E_q = \frac{\langle q|H|q\rangle/\langle q|q\rangle}{\sum_{m \in \mathbb{Z}} e^{-i \pi \nu q m} N(m)} \sum_{m \in \mathbb{Z}} e^{-i \pi \nu q m} D(m),$$  \hspace{1cm} (S13)

where $N(m) = N_1(m) + N_2(m) + N_3(m) + N_4(m)$ with the definitions,

$$N_1(m) = \left( \sum_{\nu} (\lambda_{\nu}^0)^2 - \left( \pi \nu m \sum_{\nu} (\lambda_{\nu}^0)^2 \right)^2 \right) \prod_n e^{-\left( \frac{i}{2} h_{nm} \right)^2}$$

$$N_2(m) = -\prod_n e^{-\left( \frac{i}{2} h_{nm} \right)^2} \left( \sum_{q} \left( \frac{2 \pi u k}{L} \left( \sum_n (\lambda_{q}^0)^2 \right) + (\pi \nu m)^2 \left( \sum_{q} \lambda_{q}^0 \lambda_{q}^0 \right) \right) \right)$$

$$N_3(m) = \left( \sum_{q} \frac{u \pi k}{L} \right) \prod_n e^{-\left( \frac{i}{2} h_{nm} \right)^2}$$

$$N_4(m) = \frac{9 \Delta}{2} \left( \prod_n e^{-\left( \frac{i}{2} (\lambda_{nm}^0 - f_1)^2 \right) - h_{nm} f_n} \right.$$

$$\left. + \prod_n e^{-\left( \frac{i}{2} (\lambda_{nm}^0 - f_1)^2 \right) + h_{nm} f_n} \right),$$

$$D(m) = \prod_n e^{-\left( \frac{i}{2} h_{nm} \right)^2},$$  \hspace{1cm} (S14)

with,

$$f_n = \frac{2}{\nu} \left( \sum_{k} \sqrt{\frac{\nu}{k}} (\lambda_{kn} + \lambda_{kn}^0) \right)$$

$$h_{nm} = \pi \nu m \lambda_{nm}^0$$

$$\lambda_{0n}^0 = \frac{1}{2} \sqrt{\frac{L}{u \pi \nu}} U_{0n} \sqrt{\omega_n}$$

$$\lambda_{-n}^- = i \sqrt{\frac{u \pi \nu}{L}} U_{0n} \frac{1}{\sqrt{\omega_n}}$$

$$\lambda_{+n}^+ = \frac{i}{2} \sqrt{\frac{L}{2 \pi u k}} U_{kn} \sqrt{\omega_n} + \frac{i}{2} \sqrt{\frac{2 \pi u k}{L}} U_{kn} \frac{1}{\sqrt{\omega_n}}$$

$$\lambda_{-n}^- = -\frac{i}{2} \sqrt{\frac{L}{2 \pi u k}} U_{kn} \sqrt{\omega_n} + \frac{i}{2} \sqrt{\frac{2 \pi u k}{L}} U_{kn} \frac{1}{\sqrt{\omega_n}}$$  \hspace{1cm} (S15)

where, $U_{mn}$ and $\omega_n$ are the matrix element of unitary matrix $U$, diagonalizing $H_X$ and the $n$th eigenvalue of $H$, respectively (see Eq. (S10)).

Sample results of this calculation are shown in Fig. S8. This results can be used as an estimate for the value of the parameter $h$ in the clock model Hamiltonian (16).

III. Quasiparticle matrix elements

Let us consider quasiparticle tunneling elements between the ground states,

$$M^q(x) = \langle q + 1|e^{i \varphi(x)}|q\rangle.$$  \hspace{1cm} (S16)

In terms of states in the harmonic approximation,

$$M^q(x) = \langle q + 1|\hat{P}^\dagger e^{-i(q+1)\varphi_0} e^{i \varphi(x)} e^{i \varphi_0} \mathcal{P}|q\rangle_h.$$  \hspace{1cm} (S17)
Setting $m_\mu = 1$ and substituting $\mathcal{P} = N \sum_m e^{i \pi \nu m \hat{n}}$ ($N$ being the normalization constant) in the harmonic approximation for $|q\rangle = e^{-i q \hat{\phi}_0} |0\rangle$, we get,

$$M^q = \mathcal{A}^2 \sum_{p,m \in \mathbb{Z}} e^{i \pi \nu (p-m)(q+1)} \langle 0 | e^{i \pi \nu \hat{n}(m-p)} e^{i \nu(x)} | 0 \rangle$$

(S18)

with,

$$M^q_m = e^{i \pi \nu m (q+1)} \langle 0 | e^{-i \pi \nu m \hat{n}} e^{i \nu(x)} | 0 \rangle = e^{i \pi \nu m (q+1/2)}$$

$$\times \langle 0 | e^{i \hat{\psi}_0 + \pi \nu (2x/L-m) \hat{n} + i \sum_k \sqrt{\theta} (a_k e^{2i\pi x \xi k} - a_k^\dagger e^{-2i\pi x \xi k}) | 0 \rangle$$

(S19)

This explicit form makes clear the following identity,

$$M^q_{m+2}(x + L) = e^{i \pi \nu m (2q+1)} M^q_m(x)$$

$$\Rightarrow M^q(x + L) = e^{i \pi \nu m (2q+1)} M^q(x)$$

(S20)

Using the expressions,

$$\hat{a}_k^\dagger = \sum_n (\lambda_n^k A_n^\dagger + \lambda_n^{-k} A_n)$$

$$\hat{a}_k = - \sum_n (\lambda_n^k A_n^\dagger + \lambda_n^{-k} A_n)$$

$$\hat{n}_{q=0} = \sqrt{\frac{L}{\nu \pi u}} \frac{-i}{2} \sum_n U_{0n}^\dagger \sqrt{\omega_n} (A_n^\dagger - A_n)$$

$$\hat{\varphi}_0 = \frac{\pi \nu u}{L} \sum_n U_{0n} \frac{1}{\sqrt{\omega_n}} (A_n^\dagger + A_n)$$

(S21)

where $A_n$ is the $n^{th}$ component of the operator valued vector $A \equiv \frac{1}{\sqrt{2}} (\hat{X} + i \hat{P})$. $M^q_m$ can be expressed as,

$$M^q_m = e^{i \pi \nu m q} \prod_n e^{-\frac{1}{2} |\xi_n|^2} \langle 0 | e^{-\nu \xi_n^* A_n^\dagger e^{\nu \xi_n} A_n} | 0 \rangle$$

(S22)

with,

$$\eta_n^m(x) = \left( \frac{2 \pi \nu x}{L} - \pi \nu m \right) \frac{1}{2} \sqrt{\frac{L}{\nu \pi u}} U_{0n}^\dagger \sqrt{\omega_n}$$

$$+ i \sqrt{\frac{\pi \nu u}{L}} U_{0n} \sqrt{\frac{1}{\omega_n}} + \sum_k \sqrt{\frac{\nu}{k}} (\lambda_k e^{-2i\pi x \xi k/L} + \lambda_k^* e^{2i\pi x \xi k/L})$$

(S23)

We use this expression to calculate $|M^q|$ as a function of number of modes and coupling strength, $\Delta$. Since $|M^q=1| = |M^q=2|$, we plot the results for $|M^q=0|$ and $|M^q=1|$. The results are shown in Figs. S9 and S10.

These results can be used to calculate the parameter $A(x)$ in Eq.(13).