Good orientations of 2T-graphs

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Abstract

Graphs which contain \(k\) edge-disjoint spanning trees have been characterized by Tutte. Out-branchings and in-branchings are natural analogues of spanning trees for digraphs. Edmonds has shown that it can be decided in polynomial time whether a digraph contains \(k\) arc-disjoint out-branchings or \(k\) arc-disjoint in-branchings. Somewhat surprisingly, Thomassen proved that deciding whether a digraph contains a pair of arc-disjoint out-branching and in-branching is an NP-complete problem. This problem has since been studied for various classes of digraphs, giving rise to NP-completeness results as well as polynomial time solutions.

In this paper we study graphs which admit acyclic orientations that contain a pair of arc-disjoint out-branching and in-branching (such an orientation is called good) and we focus on edge-minimal such graphs. A 2T-graph is a graph whose edge set can be decomposed into two edge-disjoint spanning trees. Vertex-minimal 2T-graphs with at least two vertices which are known as generic circuits play an important role in rigidity theory for graphs. We prove that every generic circuit has a good orientation. Using this result we prove that if \(G\) is 2T-graph whose vertex set has a partition \(V_1, V_2, \ldots, V_k\) so that each \(V_i\) induces a generic circuit \(G_i\) of \(G\) and the set of edges between different \(G_i\)'s form a matching in \(G\), then \(G\) has a good orientation. We also obtain a characterization for the case when the set of edges between different \(G_i\)'s form a double tree, that is, if we contract each \(G_i\) to one vertex, and delete parallel edges we obtain a tree. All our proofs are constructive and imply polynomial algorithms for finding the desired good orderings and the pairs of arc-disjoint branchings which certify that the orderings are good.

We also identify a structure which can be used to certify a 2T-graph which does not have a good orientation.

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1 Introduction

We consider graphs and digraphs which may contain parallel edges and arcs respectively but no loops, and generally follow the terminology in [3]. Our point of departure is the following theorem of Tutte, which characterizes graphs that contain \( k \) edge-disjoint spanning trees.

**Theorem 1.1.** [23] A graph \( G = (V, E) \) has \( k \) edge-disjoint spanning trees if and only if, for every partition \( \mathcal{F} \) of \( V \), \( e_\mathcal{F} \geq k(|\mathcal{F}| - 1) \) where \( e_\mathcal{F} \) is the number of edges with end vertices in different sets of \( \mathcal{F} \).

Using matroid techniques, one can obtain a polynomial algorithm which either finds a collection of \( k \) edge-disjoint spanning trees of a given graph \( G \) or a partition \( \mathcal{F} \) for which \( e_\mathcal{F} < k(|\mathcal{F}| - 1) \) that shows no such collection exists in \( G \) (see e.g. [20]).

Let \( D = (V, A) \) be a digraph and \( r \) be a vertex of \( D \). An **out-branching** (respectively, **in-branching**) in \( D \) is a spanning subdigraph \( B^+_r \) (respectively, \( B^-_r \)) of \( D \) in which each vertex \( v \neq r \) has precisely one entering (respectively, leaving) arc and \( r \) has no entering (respectively, leaving) arc. The vertex \( r \) is called the **root** of \( B^+_r \) (respectively, \( B^-_r \)). It follows from definition that the arc set of an out-branching (respectively, in-branching) of \( D \) induces a spanning tree in the underlying graph of \( D \). It is also easy to see that \( D \) has an out-branching \( B^+_r \) (respectively, an in-branching \( B^-_r \)) if and only if there is a directed path from \( r \) to \( v \) (respectively, from \( v \) to \( r \)) for every vertex \( v \) of \( D \). A well-known result due to Edmonds [13] shows that it can be decided in polynomial time whether a digraph contains \( k \) arc-disjoint out-branchings or \( k \) arc-disjoint in-branchings.

Somewhat surprisingly, Thomassen proved that the problem of deciding whether a digraph contains a pair of arc-disjoint out-branching and in-branching is NP-complete (see [1]). This problem has since been studied for various classes of digraphs, giving rise to NP-completeness results as well as polynomial time solutions [1, 4, 7, 8, 9, 10]. In particular, it is proved in [8] that the problem is polynomial time solvable for acyclic digraphs, and in [4] that every 2-arc-strong locally semicomplete digraph contains a pair of arc-disjoint out-branching and in-branching.

It turns out that acyclic digraphs which contain a pair of arc-disjoint out-branching and in-branching admit a nice characterization. Suppose that \( D = (V, A) \) is an acyclic digraph and that \( B^+_r, B^-_r \) are a pair of arc-disjoint out-branching and in-branching rooted at \( s, t \) respectively in \( D \). Then \( s \) must be the unique vertex of in-degree zero and \( t \) the unique vertex of out-degree zero in \( D \). Let \( X \subseteq V \setminus \{s\} \) and let \( X^- \) denote the set of vertices with at least one out-neighbour in \( X \). Since each vertex of \( x \in X \) has an in-coming arc in \( B^+_s \) and each vertex \( x' \in X^- \) has an out-going arc in \( B^-_t \), we must have

\[
\sum_{x \in X^-} (d^+(x) - 1) \geq |X|.
\]
The following theorem shows that these necessary conditions are also sufficient for the digraph \( D \) to have such a pair \( B^+_s, B^-_t \).

**Theorem 1.2.** \([8]\) Let \( D = (V, A) \) be an acyclic digraph in which \( s \) is the unique vertex of in-degree zero and \( t \) is the unique vertex of out degree zero. Then \( D \) contains a pair of arc-disjoint out-branching and in-branching rooted at \( s \) and \( t \) respectively if and only if \(^7\) holds for every \( X \subseteq V \setminus \{s\} \). Furthermore, there exists a polynomial algorithm which either finds the desired pair of branchings or a set \( X \) which violates \(^7\).

Every graph has an acyclic orientation. A natural way of obtaining an acyclic orientation of a graph \( G \) is to orient the edges according to a vertex ordering \( \prec \) of \( G \), that is, each edge \( uv \) of \( G \) is oriented from \( u \) to \( v \) if and only if \( u \prec v \). In fact, every acyclic orientation of \( G \) can be obtained in this way. Given a vertex ordering \( \prec \) of \( G \), we use \( D_{\prec} \) to denote the acyclic orientation of \( G \) resulting from \( \prec \), and call \( \prec \) good if \( D_{\prec} \) contains a pair of arc-disjoint out-branching and in-branching. We also call an orientation \( D \) of \( G \) good if \( D = D_{\prec} \) for some good ordering \( \prec \) of \( G \). Thus a graph has a good ordering if and only if it has a good orientation. We call such graphs good graphs. By Theorem 1.2, one can check in polynomial time whether a given ordering \( \prec \) of \( G \) is good and return a pair of arc-disjoint branchings in \( D_{\prec} \) if \( \prec \) is good. However, no polynomial time recognition algorithm is known for graphs that have good orderings.

An obvious necessary condition for a graph \( G \) to have a good ordering is that \( G \) contains a pair of edge-disjoint spanning trees. This condition alone implies the existence of a pair of arc-disjoint out-branching and in-branching in an orientation of \( G \). But such an orientation may never be made acyclic for certain graphs, which means that \( G \) does not have a good ordering. On the other hand, to certify that a graph has a good ordering, it suffices to exhibit an acyclic orientation of \( G \), often in the form of \( D_{\prec} \), and show it contains a pair of arc-disjoint out-branching and in-branching.

In this paper we focus on the study of edge-minimal graphs which have good orderings (or equivalently, good orientations).

**Definition 1.3.** A graph \( G = (V, E) \) is a 2T-graph if \( E \) is the union of two edge-disjoint spanning trees.

Clearly, a graph has a good ordering if and only if it contains a spanning 2T-graph which has a good ordering. A 2T-graph on \( n \) vertices has exactly \( 2n - 2 \) edges. The following theorem, due to Nash-Williams, implies a characterization of when a graph on \( n \) vertices and \( 2n - 2 \) edges is a 2T-graph. For a graph \( G = (V, E) \) and \( X \subseteq V \), the subgraph of \( G \) induced by \( X \) is denoted by \( G[X] \).

**Theorem 1.4.** \([18]\) The edge set of a graph \( G \) is the union of two forests if and only if
\[
|E(G[X])| \leq 2|X| - 2
\]
for every non-empty subset \( X \) of \( V \).

**Corollary 1.5.** A graph \( G = (V, E) \) is a 2T-graph if and only if \( |V| \geq 2, |E| = 2|V| - 2, \) and \(^2\) holds.
Generic circuits (see definition below) are important in rigidity theory for graphs. A celebrated theorem of Laman [16] implies that, for any graph $G$, the generic circuits are exactly the circuits of the so-called generic rigidity matroid on the edges of $G$. Generic circuits have also been studied by Berg and Jordán [11], who proved that every generic circuit is 2-connected and gave a full characterization of 3-connected generic circuits (see Theorem 2.3).

**Definition 1.6.** A graph $G = (V, E)$ is a **generic circuit** if it satisfies the following conditions:

(i) $|E| = 2|V| - 2 > 0$, and

(ii) $|E(G[X])| \leq 2|X| - 3$, for every $X \subset V$ with $2 \leq |X| \leq |V| - 1$.

Generic circuits are building blocks for 2T-graphs. According to Corollary[13], each generic circuit is a 2T-graph on two or more vertices with the property that no proper induced subgraph with two or more vertices is a 2T-graph. The only two-vertex generic circuit is the one having two parallel edges. Since no proper subgraph of a generic circuit is a generic circuit, every generic circuit on more than two vertices is a simple graph (i.e., containing no parallel edges). There is no generic circuit on three vertices and the only four-vertex generic circuit is $K_4$. The wheels $W_k$, $k \geq 4$ are all (3-connected) generic circuits. Berg and Jordán [11] proved that every 3-connected generic circuit can be reduced to $K_4$ by a series of so called Henneberg moves (see definition below). We shall use this to prove that every generic circuit has a good ordering.

This paper is organized as follows. In Section 2 we begin with some preliminary results on generic circuits from [11] and then prove a technical lemma that shows how to lift a good orientation of a 2T-graph resulted from a Henneberg move (Lemma 2.4). The lemma will be used in Section 3 for the proof of a statement which implies that every generic circuit has a good ordering (Theorem 3.2). Section 4 is devoted to the study of the structure of 2T-graphs. We show that every 2T-graph is built from generic circuits and is reducible to a single vertex by a sequence of contractions of generic circuits (Theorem 4.5). We also describe a polynomial algorithm which identifies all generic circuits of a 2T-graph (Theorem 4.6). This implies that the problem of deciding whether a graph is a disjoint union of generic circuits is polynomial time solvable for 2T-graphs (Theorem 4.8). We also show that the problem is NP-complete in general (Theorem 4.9). In Section 5, we explore properties of 2T-graphs which have good orderings and identify a forbidden structure for these graphs (Theorem 5.7). In Section 6, we restrict our study on 2T-graphs which are disjoint unions of generic circuits. We prove that if the edges connecting the different generic circuits form a matching, then one can always produce a good ordering (Theorem 6.1) and we also characterize when such an ordering exists if the graph reduces to a double tree by contraction (Theorem 6.3). Finally, in Section 7, we list some open problems and show that the problem of finding a so called $(s, t)$-ordering of a digraph is NP-complete (Theorem 7.5).

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1The wheel $W_k$ is the graph that one obtains for a cycle of length $k$ by adding a new vertex and an edge from this to each vertex of the cycle.
2 Lifting good orientations of a 2T-graph

Definition 2.1. Let $G = (V, E)$ be a generic circuit, let $z$ be a vertex with three distinct neighbours $u, v, w$. A Henneberg move from $z$ is the operation that deletes $z$ and its three incident edges from $G$ and adds precisely one of the the edges $uv, uw, vw$. A Henneberg move is admissible if the resulting graph, which we denote by $G^{uv}_z$, where $uv$ is the edge we added to $G - z$, is a generic circuit and a Henneberg move is feasible if it is admissible and $G^{uv}_z$ is a 3-connected graph.

Theorem 2.2. \cite{11} Let $G = (V, E)$ be a 3-connected generic circuit on $n \geq 5$ vertices. Then either $G$ has four distinct degree 3 vertices from which we can perform an admissible Henneberg move, or $G$ has 3 pairwise non-adjacent vertices, each of degree 3, so that we can perform a feasible Henneberg move from each of these.

Theorem 2.3. \cite{11} A 3-connected graph $G = (V, E)$ is a generic circuit if and only if $G$ can be reduced to (build from) $K_4$ by applying a series of feasible Henneberg moves (a series of Henneberg extensions$^2$).

It is easy to see that if $z$ is a vertex of degree three in a 2T-graph $G$ then we can obtain a new 2T-graph $G'$ by performing a Henneberg move from $z$. The following lemma shows that when the three neighbours of the vertex $z$ that we remove in a Henneberg move are distinct, we can lift back a good orientation of $G'$ to a good orientation of $G$.

Figure 1: How to lift a good ordering to a Henneberg extension as in Lemma 2.4. In-branchings are displayed solid, out-branchings are dashed. The first line displays the three possible orders of the relevant vertices (increasing left to right) as they occur in the proof. The second line displays the ordering and the modification of the branchings in the extension.

Lemma 2.4. Let $G$ be a 2T-graph on $n$ vertices and let $z$ be a vertex of degree 3 with three distinct neighbours $u, v, w$ from which we can perform an admissible Henneberg move to get $G^{uv}_z$. If $G^{uv}_z$ is good, then also $G$ is good.

\footnote{This is the inverse operation of a Henneberg move.}
Proof: Let $\prec' = (v_1, v_2, \ldots, v_n)$ be a good ordering of $G^w_z$ and let $\tilde{B}^+_{v_1}, \tilde{B}^-_{v_{n+1}}$ be arc-disjoint branchings of $D^w$.

Assume without loss of generality that $u = v_i$ and $v = v_j$ where $i < j$ (if this is not the case then consider the reverse ordering $\prec'$ which is also good). Let $k \in [n - 1]$ be the index of $w$ ($w = v_k$) and recall that $k \neq i, j$. Now there are 6 possible cases depending on the position of $w$ and which of the two branchings the arc $uv$ belongs to. In all cases we explain how to insert $z$ in the ordering $\prec'$ and update the two branchings which certifies that the new ordering $\prec$ is good.

- $uv$ is in $\tilde{B}^-_{v_{n+1}}$ and $j < k$. In this case we obtain $\prec$ from $\prec'$ by inserting $z$ anywhere between $v = v_j$ and $w = v_k$, replacing the arc $v_i v_j$ by the arcs $v_i z, z v_k$ and adding the arc $v_j z$ to $\tilde{B}^-_{v_1}$.
- $uv$ is in $\tilde{B}^-_{v_{n+1}}$ and $i < k < j$. In this case we obtain $\prec$ from $\prec'$ by inserting $z$ anywhere between $w = v_k$ and $v = v_j$, replacing the arc $v_i v_j$ by the arcs $v_i z, z v_j$ and adding the arc $v_k z$ to $\tilde{B}^+_{v_1}$.
- $uv$ is in $\tilde{B}^-_{v_{n+1}}$ and $k < i$. In this case we obtain $\prec$ from $\prec'$ by inserting $z$ anywhere between $u = v_i$ and $v = v_j$, replacing the arc $v_i v_j$ of $\tilde{B}^-_{v_{n+1}}$ by the arcs $v_i z, z v_j$ and adding the arc $v_k z$ to $\tilde{B}^+_{v_1}$.

The argument in the remaining three cases is obtained by considering $\prec'$ and noting that this switches the roles of the in- and out-branchings. $\Box$

3 Generic circuits are all good

In this section we show that every generic circuit has a good ordering. In fact we prove a stronger statement on generic circuits which turns out to be very useful in the study of 2T-graphs that have good orderings.

Let $H = (V, E)$ be 2-connected and let $\{u, v\}$ be a pair of non-adjacent vertices such that $H - \{u, v\}$ is not connected. Then there exists $X, Y \subset V$ such that $X \cap Y = \{u, v\}$, $X \cup Y = V$ and there are no edges between $X - Y$ and $Y - X$. A 2-separation of $H$ along the cutset $\{u, v\}$ is the process which replaces $H$ by the two graphs $H[X] + e$ and $H[Y] + e$, where $e$ is a new edge connecting $u$ and $v$. It is easy to show the following.

Lemma 3.1. [11] Let $G = (V, E)$ be a generic circuit. Then $G$ is 2-connected. Moreover, if $G - \{a, b\}$ is not connected, with connected components $X', Y'$, then $ab \notin E$ and both of the graphs $G_1 = G[X' \cup \{a, b\}] + ab$ and $G_2 = G[Y' \cup \{a, b\}] + ab$ are generic circuits.

Theorem 3.2. Let $G = (V, E)$ be a generic circuit, let $s, t$ be distinct vertices of $G$ and let $e$ be an edge incident with at least one of $s, t$. Then the following holds:

(i) $G$ has a good ordering $\prec$ with corresponding branchings $B^+, B^-$ in which $s$ is the root of $B^+$, $t$ is the root of $B^-$ and $e$ belongs to $B^+$.  

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(ii) \( G \) has a good ordering \( \prec \) with corresponding branchings \( B^+, B^- \) in which \( s \) is the root of \( B^+ \), \( t \) is the root of \( B^- \) and \( e \) belongs to \( B^- \).

**Proof:** The statement is clearly true when \( G \) has two vertices. So assume that \( G \) has more than two vertices. The proof is by induction on \( n \), the number of vertices of \( G \). The smallest generic circuit on \( n > 2 \) vertices is \( K_4 \) and we prove that the statement holds for \( K_4 \). By symmetry (reversing all arcs) it suffices to consider the case when \( e \) is incident with \( s \) (see Figure 2).

It is possible to order the vertices of \( K_4 \) as \( s = v_1, v_2, v_3, v_4 = t \) such that \( e \neq sv_2 \), implying that \( e = sv_3 \) or \( e = st \). Let \( B^+_{s,1} \) and \( B^+_{s,2} \) be the out-branchings at \( s \) formed by the arcs \( sv_2, v_2v_3, st \) and of \( sv_2, v_2t, sv_3 \), respectively. The three remaining edges form in-branchings \( B^-_{t,1}, B^-_{t,2} \) at \( t \), respectively. Since \( st \in B^+_{s,1} \) and \( sv_3 \in B^+_{s,2} \), we find the desired branchings containing \( e \) as in (i), (ii), respectively.

\[ B^+_{s,1}, B^-_{t,1} \]

\[ B^+_{s,2}, B^-_{t,2} \]

Figure 2: Illustrating the base case of the inductive proof of Theorem 3.2. All arcs are oriented from left to right, the prescribed edge is either \( sv_3 \) or \( st \), and the picture shows that we may force \( e \) to be in the (dashed) out-branching as well as in the (solid) in-branching.

Assume below that \( n > 4 \) and that the statement holds for every generic circuit on at most \( n - 1 \) vertices.

Suppose that \( G \) is not 3-connected. Then it has a separating set \( \{u, v\} \) of size 2 (Recall that, by Lemma 3.1, \( G \) is 2-connected). Let \( G_1, G_2 \) be obtained from \( G, u, v \) by 2-separation. Then each of \( G_1, G_2 \) are smaller generic circuits so the theorem holds by induction for each of these. Note that \( e \) is not the edge \( uv \) as this edge does not belong to \( G \) by Lemma 3.1.

Suppose first that \( s, t \) are both vertices of the same \( G_i \), say w.l.o.g. \( G_1 \). Then \( e \) is also an edge of \( G_1 \) and there are two cases depending on whether we want \( e \) to belong to the out-branching or the in-branching. We give the proof for the first case, the proof of the later is analogous.

By induction there is a good ordering \( \prec_1 \) of \( V(G_1) \) and arc-disjoint branchings \( B^+_{s,1}, B^-_{t,1} \) so that \( e \) belongs to \( B^+_{s,1} \). By interchanging the names of \( u, v \) if necessary, we can assume that the edge \( uv \) is oriented from \( u \) to \( v \) in \( D(\prec_1) \). Suppose first that the arc \( u \rightarrow v \) is used in \( B^+_{s,1} \). By induction, by specifying the vertices \( u, v \) as roots and \( e^+ = uv \) as the edge,
has a good ordering $\prec_2^+$ such that $D(\prec_2^+)$ has arc-disjoint branchings $B^+_{u,2}, B^-_{v,2}$ where the arc $uv$ is in $B^-_{v,2}$. Now it is easy to check that $B^+_u, B^-_v$ form a solution in $G$ if we let $A(B^+_u) = A(B^+_{u,1} - uv) \cup A(B^+_{u,2})$ and $A(B^-_v) = A(B^-_{v,1}) \cup A(B^-_{v,2} - uv)$. Here we used that there is no edge between $u$ and $v$ in $G$, so $e$ is not the removed arc above. The corresponding good ordering $\prec$ is obtained from $\prec_1, \prec_2^+$ by inserting all vertices of $V(G_2) - \{u, v\}$ just after $u$ in $\prec_1$. Suppose now that the arc $u \to v$ is used in $B^-_{t,1}$. By induction, by specifying the vertices $u, v$ as roots and $e = uv$ as the edge, $G_2$ has a good ordering $\prec_2$ and arc-disjoint branchings $B^+_{u,2}, B^-_{v,2}$ such that the arc $uv$ is in $B^+_{u,2}$. Again we obtain the solution in $G$ by combining the two orderings and the branchings. By similar arguments we can show that there is also a good ordering such that the edge $e$ belongs to $B^-_t$.

Suppose now that only one of the vertices $s, t$, say wlog. $s$ is a vertex of $G_1$ and $t$ is in $G_2$. Note that this means that $s, t \notin \{u, v\}$. Consider the graph $G_1$ with specification $s, v, e$. By induction $G_1$ has a good orientation $D_1$ with arc-disjoint branchings $B^+_{s,1}, B^-_{e,1}$ so that $e$ belongs to $B^+_{s,1}$. Note that, as $v$ is the root of the in-branching, the edge $uv$ is oriented from $u$ to $v$ in $D_1$. If the arc $uv$ belongs to $B^-_{t,1}$, then we consider $G_2$ with specification $u, t, uv$ where $uv$ should belong to the in-branching. By induction there exists a good orientation $D_2$ of $G_2$ with arc-disjoint branchings $B^+_{u,2}, B^-_{v,2}$ such that the arc $uv$ is in $B^-_{v,2}$. Now we obtain the desired acyclic orientation and arc-disjoint branchings by setting $A(B^+_u) = A(B^+_{u,1} - uv) \cup A(B^+_{u,2})$ and $A(B^-_v) = A(B^-_{v,1}) \cup A(B^-_{v,2} - uv)$. To see that we do not create any directed cycles by combining the acyclic orientations $D_1$ and $D_2$ it suffices to observe that $u$ has no arc entering in $D_2$ and $v$ has no arc leaving in $D_1$. If the arc $uv$ belongs to $B^-_{v,1}$, then we consider $G_2$ with specification $u, t, uv$ where $uv$ should belong to the out-branching. Again, by induction, there exists an acyclic orientation $D_2$ of $G_2$ with good branchings and combining the two orientations and the branchings as above we obtain the desired acyclic orientation of $G$ and good in- and out-branchings. By similar arguments we can show that there is also a good ordering such that the edge $e$ belongs to $B^-_t$.

It remains to consider the case when $G$ is 3-connected. By Theorem 2.2 there is an admissible Henneberg move $G \to G^w_z$ from a vertex $z \notin \{s, t\}$ which is not incident with $e$. Consider $G^w_z$ with specification $s, t, e$, where $e$ belongs to the out-branching. By induction there is an acyclic orientation $D'$ of $G^w_z$ and arc-disjoint branchings $B^+_{s,2}, B^-_{t,2}$ so that $e$ is in $B^+_{s,2}$. Now apply Lemma 2.4 to obtain an acyclic orientation $D$ of $G$ in which $s$ is the root of an out-branching $B^+$ which contains $e$ and $t$ is the root of an in-branching $B^-$ which is arc-disjoint from $B^+$. The proof of the case when $e$ must belong to $B^-_t$ is analogous.

We will see in Section 6 that Theorem 3.2 is very useful when studying good orderings of 2T-graphs. The result below shows that it can also be applied to an infinite class of graphs which are not 2T-graphs.

**Theorem 3.3.** Let $G$ be a 4-regular 4-connected graph in which every edge is on a triangle. Then $G - \{e, f\}$ is a spanning generic circuit for any two disjoint edges $e, f$. In particular, $G$ admits a good ordering.

**Proof.** Observe that $G$ is simple, as it is 4-regular and 4-connected. By Tutte’s Theorem, $H := G - \{e, f\}$ is a 2T-graph. Suppose, to the contrary, that it contains a 2T-graph $C$ as a proper subgraph. Then elementary counting shows that $C$ is an induced subgraph of $G$ whose edge-neighborhood $N$ consists of exactly four edges. (In particular, neither $e$ nor $f$
connects two vertices from $V(C)$. The endpoints of the edges from $N$ in $V(C)$ are pairwise distinct since $|V(C)| \geq 4$ and $G$ is 4-connected. Since $G - \{h, g\}$ is a 2T-graph for $h \neq g$ from $N$ we see that $\overline{C} := G - V(C)$ is a 2T-graph or consists of a single vertex only. If it is a 2T-graph then the endpoints of the edges from $N$ in $V(\overline{C})$ are pairwise distinct, too, contradicting the assumption that every edge is on at least one triangle. If, otherwise, $\overline{C}$ consists of a single vertex only then it is incident with both $e$ and $f$, contradicting the assumption that $e, f$ are disjoint. □

Thomassen conjectured that every 4-connected line graph is Hamiltonian [22]; more generally, Matthews and Sumner conjectured that every 4-connected claw-free graph (that is, a graph without $K_{1,3}$ as an induced subgraph) is Hamiltonian [17]. These conjectures are, indeed, equivalent [24], and it suffices to consider 4-connected line graphs of cubic graphs [15].

Theorem 3.3 shows that such graphs have a spanning generic circuit (that is, a spanning cycle in the rigidity matroid).

4 Structure of generic circuits in 2T-graphs

Every 2T-graph $G$ on two or more vertices contains a generic circuit as an induced subgraph. Indeed, any minimal set $X$ with $|X| \geq 2$ and $|E(G[X])| = 2|X| - 2$ induces a generic circuit in $G$. We say that $H$ is a generic circuit of a graph $G$ if $H$ is a generic circuit and an induced subgraph of $G$.

Proposition 4.1. Let $G = (V, E)$ be a 2T-graph. Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are distinct generic circuits of $G$. Then $|V_1 \cap V_2| \leq 1$ and hence $|E_1 \cap E_2| = 0$. In the case when $|V_1 \cap V_2| = 0$ (i.e., $G_1$ and $G_2$ have no vertex in common), let $k$ denote the

Proof: Suppose to the contrary that $|V_1 \cap V_2| \geq 2$. Since $G_1$ and $G_2$ are generic circuits, $|E_1| = 2|V_1| - 2$ and $|E_2| = 2|V_2| - 2$. Since $V_1 \cap V_2 \subset V_1$, we must have $|E_1 \cap E_2| = |E(G[V_1 \cup V_2])| \leq 2|V_1 \cap V_2| - 3$. But then

$$|E(G[V_1 \cup V_2])| \geq |E_1| + |E_2| - |E_1 \cap E_2|$$

$$= (2|V_1| - 2) + (2|V_2| - 2) - |E_1 \cap E_2|$$

$$\geq 2(|V_1| + |V_2|) - 4 - (2|V_1 \cap V_2| - 3)$$

$$= 2(|V_1| + |V_2| - |V_1 \cap V_2|) - 1$$

$$= 2(|V_1 \cup V_2|) - 1,$$

contradicting that $G$ is a 2T-graph and hence satisfies (2) (see Corollary 1.5). Hence $|V_1 \cap V_2| \leq 1$.

Suppose that $|V_1 \cap V_2| = 0$ (i.e., $G_1$ and $G_2$ have no vertex in common). Let $k$ denote the
number of edges between \(G_1\) and \(G_2\). Then
\[
k = |E(G[V_1 \cup V_2])| - |E_1 \cup E_2|\leq (2|V_1 \cup V_2| - 2) - (|E_1| + |E_2|) = 2(|V_1| + |V_2|) - 2 - (2|V_1| - 2 + 2|V_2| - 2) = 2.
\]

Proposition 4.2. Let \(r \geq 2\) and \(G_i = (V_i, E_i)\) where \(1 \leq i \leq r\) be generic circuits of a \(2T\)-graph \(G = (V, E)\). Suppose that \(|V_i \cap V_j| = 1\) if and only if \(|i - j| = 1\). Then there is no edge with one end in \(V_1 \setminus V_r\) and the other end in \(V_r \setminus V_1\).

Proof: Let \(k\) be the number of edges each has one end in \(V_1 \setminus V_r\) and the other end in \(V_r \setminus V_1\). We prove \(k = 0\) by induction on \(r\). When \(r = 2\),
\[
k = |E(G[V_1 \cup V_2])| - |E_1 \cup E_2|\leq (2|V_1 \cup V_2| - 2) - (|E_1| + |E_2|) = 2(|V_1| + |V_2| - 1) - 2 - (2|V_1| - 2 + 2|V_2| - 2) = 0.
\]

Assume \(r > 2\) and there is no edges with one end in \(V_i \setminus V_j\) and the other end in \(V_j \setminus V_i\) for all \(1 \leq |i - j| \leq r - 2\). By assumption \(|V_i \cap V_j| = 1\) if and only if \(|i - j| = 1\) and in particular \(|V_i \cap V_j| = 0\) if \(|i - j| > 1\). Hence
\[
k = |E(G[V_1 \cup \cdots \cup V_r])| - |E_1 \cup \cdots \cup E_r|\leq (2|V_1 \cup \cdots \cup V_r| - 2) - (|E_1| + \cdots + |E_r|) = 2(|V_1| + \cdots + |V_r| - (r - 1)) - 2 - (2|V_1| - 2 + \cdots + 2|V_r| - 2) = 0.
\]

This completes the proof. \(\square\)

Proposition 4.3. Let \(G_i = (V_i, E_i)\) where \(1 \leq i \leq r\) be the collection of generic circuits of a \(2T\)-graph \(G = (V, E)\) and let \(G = (V, E)\) be the hypergraph where \(E = \{V_i : 1 \leq i \leq r\}\). Then \(G\) is a hyperforest.

Proof: Suppose to the contrary that \(G\) is not a hyperforest. Then there exist \(V_{i_1}, V_{i_2}, \ldots, V_{i_\ell}\) for some \(\ell \geq 3\) such that \(|V_{i_j} \cap V_{i_k}| = 1\) if and only if \(|j - k| = 1\) or \(\ell - 1\) and moreover, the common vertices between the hyperedges on the hypercycle are pairwise distinct. Thus
\[
\sum_{j=1}^{\ell}|V_{i_j}| = |V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_\ell}| + \ell.
\]

By Proposition 4.2 there is no edge with one end in \(V_{i_j} \setminus V_{i_k}\) and the other end in \(V_{i_k} \setminus V_{i_j}\).
for all \( j \neq k \). Hence

\[
|E(G[V_i \cup V_i \cup \cdots \cup V_i])| = \sum_{j=1}^{f} (2|V_i| - 2) = 2|V_i \cup V_i \cup \cdots \cup V_i|, 
\]

contradicting that \( G \) is a 2T-graph and hence satisfies (2).

Let \( G = (V, E) \) be a 2T-graph and let \( \tilde{G} = (V, \mathcal{E}) \) be the hypergraph defined in Proposition 4.3. Two generic circuits of \( G \) are connected if their vertex sets are in the same hypertree of \( G \). Not every vertex of \( G \) needs to be in a generic circuit of \( G \). A generic component of \( G \) is either a set consisting of a single vertex which is not in any generic circuit of \( G \) or the union of a maximal set of connected generic circuits. A generic component is called trivial if it consists of a single vertex and non-trivial otherwise. An edge of \( G \) is external if it is not contained in any generic circuit. By Proposition 4.2 there is no external edge in a generic component. Thus each generic component is a 2T-graph. Two generic components do not have a vertex in common. A similar proof as for Proposition 4.1 shows that there can be at most two external edges between two generic components. We summarize these properties below.

**Proposition 4.4.** Let \( G = (V, E) \) be a 2T-graph. Then the following statements hold:

1. there is no external edge in a generic component;
2. each generic component is a 2T-graph;
3. two generic components are vertex-disjoint;
4. there are at most two external edges between two generic components.

Thus every 2T-graph \( G \) partitions uniquely into pairwise vertex-disjoint generic components. The quotient graph \( \tilde{G} \) of \( G \) is the graph obtained from \( G \) by contracting each generic component to a single vertex (and deleting loops resulted from the contractions). It follows from Proposition 4.4 that every 2T-graph can be reduced to \( K_1 \) by successively taking quotients.

**Theorem 4.5.** Let \( G \) be a 2T-graph. Then there is a sequence of 2T-graphs \( G_0, G_1, \ldots, G_k \) where \( G_0 = G \), \( G_k = K_1 \), and \( G_i = \tilde{G}_{i+1} \) for each \( i = 1, 2, \ldots, k \). In particular, \( \tilde{G} \) is a 2T-graph.

**Theorem 4.6.** There exists a polynomial algorithm \( A \) which given a 2T-graph \( G = (V, E) \) as input finds the collection \( G_1, G_2, \ldots, G_r \), \( r \geq 1 \) of generic circuits of \( G \).

**Proof:** This follows from the fact that the subset system \( M = (E, \mathcal{I}) \) is a matroid, where \( E' \subseteq E \) is in \( \mathcal{I} \) precisely when \( E' = \emptyset \) or \( |E'| \leq 2|V(E')| - 3 \) holds, where \( V(E') \) is the set of vertices spanned by the edges in \( E' \). See [12] for a description of a polynomial independence oracle. The circuits of \( M \) are precisely the generic circuits of \( G \). Recall from matroid theory that an element \( e \in E \) belongs to a circuit of \( M \) precisely when there exists a base of \( M \) in
Thus we can produce all the circuits by considering each edge \( e \in E \) one at a time. If there is a base \( B \subset E - e \), then \( B \cup \{e\} \) contains a unique circuit \( C_e \) which also contains \( e \) and we can find \( C_e \) in polynomial time by using independence tests in \( M \). Since the generic circuits are edge-disjoint, by Proposition 4.1 we will find all generic circuits by the process above.

Proof: We first use the algorithm \( A \) of Theorem 4.6 to find the set \( G_1, G_2, \ldots, G_r \) of generic circuits of \( G \). If \( r = 1 \) we are done as our decomposition consists of that generic circuit alone (\( G \) is a generic circuit). So assume now that \( r \geq 2 \) and form the hypergraph \( G \) from \( G_1, G_2, \ldots, G_r \). Initialize \( H_1 \) as the graph \( G \) and \( G_1 \) as the hypergraph \( G \).

By Proposition 4.3 \( G_1 \) is a hyperforest and hence, by Proposition 4.1 it has an edge which has at most one vertex in common with the rest of the edges of \( G_1 \). Let \( G_i \) be a generic circuit corresponding to such an edge. Note that, as \( |V(G_i)| \geq 2 \) the generic circuit \( G_i \) must be part of any decomposition of \( V \) into generic circuits. Now let \( V_2 = V - V(G_i) \) and consider the induced subgraph \( H_2 = G[V_2] \) of \( G \) and the hypergraph \( G_2 = (V_2, E_2) \) that we obtain from \( G_1 \) by deleting the vertices of \( V(G_i) \) as well as every hyperedge that contains a vertex from \( V(G_i) \). If \( G_2 \) has at least one edge, we can again find one which intersects the rest of the edges in at most one vertex. Let \( G_i \) denote the corresponding generic circuit and add this to our collection. Form \( H_3, G_3 \) as above. Continuing this way we will either find the desired decomposition or we reach a situation where the current hypergraph \( G_k \) has at least one vertex but no edges. In this case it follows from the fact that the generic circuits we have removed so far are the only ones who could cover the corresponding vertex sets that \( G \) has no decomposition into generic circuits. As the number, \( r \), of generic circuits in \( G \) is bounded by \( |E|/2 \) since generic circuits are edge-disjoint, the process above will terminate in a polynomial number of steps and each step also take polynomial time.

The proof above made heavy use of the structure of generic circuits in 2T-graphs. For general graphs the situation is much worse.

Theorem 4.9. It is NP-complete to decide if the vertex set of a graph admits a partition whose members induce generic circuits.

Proof: Recall the problem exact cover by 3-sets which is as follows: given a set \( X \) with \( |X| = 3q \) for some integer \( q \) and a collection \( \mathcal{C} = Y_1, \ldots, Y_k \) of 3-element subsets of \( X \); does \( \mathcal{C} \) contain a collection of \( q \) disjoint sets \( Y_{i_1}, \ldots, Y_{i_q} \) such that each element of \( X \) is in exactly one of these sets? exact cover by 3-sets is NP-complete [14 Page 221]. Let exact cover by 4-sets be the same problem as above, except that \( |X| = 4q \) and each set in \( \mathcal{C} \) has size 4. It is easy to see that exact cover by 3-sets polynomially reduces to exact cover by 4-sets: Given an instance \( X, \mathcal{C} \) of exact cover by 3-sets we extend \( X \) to \( X' \) by adding \( q \) new elements \( z_1, z_2, \ldots, z_q \) and construct \( \mathcal{C}' \) by including the \( q \) sets \( Y \cup \{z_i\}, i \in [q] \) for each set \( Y \in \mathcal{C} \). It is easy to check that \( X, \mathcal{C} \) is a yes-instance of
Theorem 3.2, each $G$ (i.e., $G$ easy to see that the concatenation of these is a hyperpath and the choice of $G_i$ from $G_i = 1$, let $G$ be a 2T-graph. For simplicity we shall call a generic circuit of $G$ a circuit of $G$. Recall from Section 4 that each generic component of $G$ consists of either a single vertex or a set of circuits that form a hypertree in $G = (V, E)$. We call a generic component of $G$ a hyperpath if its circuits $G_1, G_2, \ldots, G_k$ ($k \geq 1$) satisfy the property that for all distinct $i, j$, $G_i$ and $G_j$ have a common vertex if and only if $|i - j| = 1$. Note that the common vertices between circuits are pairwise distinct and in particular, a generic component consisting of one circuit is a hyperpath. We call $G$ linear if every non-trivial generic component of $G$ is a hyperpath.

Proposition 5.1. Let $G$ be a 2T-graph which has one non-trivial generic component and no trivial generic component. Then $G$ has a good ordering if and only if $G$ is linear (i.e., $G$ is a hyperpath).

Proof: Suppose that $G$ is a hyperpath formed by circuits $G_1, G_2, \ldots, G_k$. For each $i = 1, 2, \ldots, k - 1$, let $v_i$ be the common vertex of $G_i$ and $G_{i+1}$. Arbitrarily pick a vertex $v_0$ from $G_1$ distinct from $v_1$ and a vertex $v_k$ from $G_k$ distinct from $v_{k-1}$. The assumption that $G$ is a hyperpath and the choice of $v_0, v_k$ ensure that $v_0, v_1, \ldots, v_k$ are pairwise distinct. By Theorem 3.2 each $G_i$ has a good ordering $\prec_i$ that begins with $v_{i-1}$ and ends with $v_i$. It is easy to see that the concatenation of these $k$ orderings gives a good ordering of $G$.

On the other hand suppose that $G$ is not a hyperpath but has an acyclic orientation with arc-disjoint branchings $B^+_s, B^-_s$. Since $G$ is not a hyperpath, either there are three circuits intersecting at the same vertex or there are three pairwise non-intersecting circuits each

5 Properties of good 2T-graphs

Let $G$ be a 2T-graph. For simplicity we shall call a generic circuit of $G$ a circuit of $G$. Recall from Section 4 that each generic component of $G$ consists of either a single vertex or a set of circuits that form a hypertree in $G = (V, E)$. We call a generic component of $G$ a hyperpath if its circuits $G_1, G_2, \ldots, G_k$ ($k \geq 1$) satisfy the property that for all distinct $i, j$, $G_i$ and $G_j$ have a common vertex if and only if $|i - j| = 1$. Note that the common vertices between circuits are pairwise distinct and in particular, a generic component consisting of one circuit is a hyperpath. We call $G$ linear if every non-trivial generic component of $G$ is a hyperpath.

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On the other hand suppose that $G$ is not a hyperpath but has an acyclic orientation with arc-disjoint branchings $B^+_s, B^-_s$. Since $G$ is not a hyperpath, either there are three circuits intersecting at the same vertex or there are three pairwise non-intersecting circuits each
intersecting with a fourth circuit. In either case, one of the three circuits contains neither $s$ nor $t$. This would imply that the arc sets of $B^+_H, B^-_H$ restricted to this circuit contains a directed cycle, a contradiction to the fact that the orientation of $G$ is acyclic.

**Proposition 5.2.** If a 2T-graph $G$ has a good ordering, then $G$ is linear.

**Proof:** Suppose that $D$ is a good orientation of $G$ with arc-disjoint branchings $B^+_s, B^-_t$. Consider a non-trivial generic component $H$ of $G$ and its orientation $D'$ induced by $D$ which is clearly acyclic. Since $H$ has $2|V(H)| - 2$ edges, $A(D') \cap A(B^+_H)$ and $A(D') \cap A(B^-_H)$ induce arc-disjoint branchings in $D'$, certifying that $D'$ is a good orientation of $H$. By Proposition 5.1, $H$ is a hyperpath. Hence every non-trivial generic component of $G$ is a hyperpath and therefore $G$ is linear.

In view of Proposition 5.2, we only need to consider linear 2T-graphs for possible good orderings or good orientations. Suppose that $D$ is a good orientation of a 2T-graph $G$ with arc-disjoint branchings $B^+_s, B^-_t$. Let $H$ be a generic component of $G$. Then the proof of Proposition 5.2 shows that $D' = D[V(H)]$ is a good orientation of $H$ with arc-disjoint branchings $B^+_s, B^-_t$ which are the restrictions of $B^+_H, B^-_H$ to $V(H)$. We refer $s, t$ to as global roots and $s', t'$ as to local roots (of the corresponding branchings in $H$). The external degree of a vertex $x$ in $G$ is the number of external edges incident with $x$ in $G$ and the external degree of $H$ is the sum of external degrees of the vertices of $H$.

**Lemma 5.3.** Let $G$ be a 2T-graph which has a good orientation with arc-disjoint branchings $B^+, B^-$. Then every non-trivial generic component has distinct local roots. Suppose that $H, H'$ are generic components of $G$ and $xy$ is an external edge where $x, y$ are vertices in $H, H'$ respectively. Then one of the following holds:

(a) $xy \in A(B^-)$ and $x$ is the local root of the in-branching $B^-_H$ in $H$ which is the restriction of $B^-$ to $V(H)$;

(b) $xy \in A(B^+)$, $y$ is the local root of the out-branching $B^+_H$ in $H'$ which is the restriction of $B^+$ to $V(H')$ and if the external degree of $x$ is one, then $x$ is either the root of $B^+$ (and hence the local root of $B^+_H$ which is the restriction of $B^+$ to $V(H)$) or not a local root.

(c) $yx \in A(B^+)$ and $x$ is the local root of the out-branching $B^+_H$ in $H$ which is the restriction of $B^+$ to $V(H)$

(d) $yx \in A(B^-)$, $y$ is the local root of the in-branching $B^-_H$ in $H'$ which is the restriction of $B^-$ and if the external degree of $x$ is one, then $x$ is either the root of $B^-$ (and hence the local root of $B^-_H$ which is the restriction of $B^-$ to $V(H)$) or not a local root.

In particular, if the external degrees of $x, y$ are both one, and neither $H$ nor $H'$ contains a global root, then either $x$ is a local root in $H$ or $y$ is a local root in $H'$ but not both.

**Proof:** Suppose that $D$ is a good orientation of $G$ with arc-disjoint branchings $B^+, B^-$. Then, as we mentioned above, for every non-trivial generic component $H$ the restrictions of $B^+, B^-$ to $V(H)$ form a pair of arc-disjoint branchings in $D[V(H)]$ and since $D$ is acyclic, the roots of these branchings must be distinct. Thus the first part of the lemma holds. This
implies that the digraph \( \tilde{D} \) that we obtain by contracting each non-trivial generic component to one vertex is a good orientation of the quotient \( \tilde{G} \) of \( G \) and the digraphs \( \tilde{B}^+, \tilde{B}^- \) that we obtain from \( B^+, B^- \) via this contraction are arc-disjoint in- and out-branchings of \( \tilde{D} \). As every vertex which is not the root of an in-branching (out-branching) has exactly one arc leaving it (entering it) this implies that if some arc \( uv \) of \( B^+ \) (\( B^- \)) enters (leaves) a non-trivial generic component, then \( v (u) \) is the local out-root (in-root) of that component. Now it is easy to see that (a)-(d) hold. The last claim is a direct consequence of these and the fact that \( B^+ \) and \( B^- \) are arc-disjoint.

We say that a subset \( X \subset V \) with \( 2 \leq |X| \leq |V| - 2 \) is pendant at \( x \) in \( G \) if all edges between \( X \) and \( V(G) - X \) are incident with \( x \). Note that \( X \) is pendant at \( x \) in \( G \) if and only if \( V - X \) is pendant at \( x \) in \( G \).

**Lemma 5.4.** If \( X \) is pendant at \( x \) in a good 2T-graph \( G \), then every good orientation \( D \) of \( G \) will have \( |X \cap \{s, t\}| = 1 \), where \( s \) and \( t \) are the roots of arc-disjoint branchings \( B^+_s, B^-_t \) that certify that \( D \) is good. That is, \( X \) contains precisely one global root.

**Proof:** Let \( B^+_s, B^-_t \) be a pair of arc-disjoint branchings that certify that \( D \) is good and suppose that none of \( s, t \) are in \( X \). Let \( z \in X - x \) (such a vertex exists as \( |X| > 1 \)). As \( X \) is pendant in \( x \) the \((s, z)\)-path in \( B^+_s \) passes through \( x \) and the \((z, t)\)-path in \( B^-_t \) also passes through \( x \), but then \( D \) contains a directed cycle, contradicting that it is acyclic. Since \( V - X \) is also pendant at \( x \), we see that \( |X \cap \{s, t\}| = 1 \) must hold.

Let \( G \) be a 2T-graph. Suppose that \( H \) is a generic component of \( G \) which is a hyperpath formed by circuits \( G_1, G_2, \ldots, G_k \). Then \( H \) is called pendant if one of following conditions holds:

- \( V(H) \) is a pendant set in \( G \);
- all vertices of \( G_1 \) have external degree zero or all vertices of \( G_k \) have external degree zero.

**Corollary 5.5.** If \( H \) is a pendant generic component of a 2T-graph \( G \), then \( H \) must contain a global root.

**Proof:** If \( H \) is the only generic component in \( G \), then clearly it contains a global root. So assume that \( G \) has at least two generic components. We show that \( V(H) \) contains a pendant set. If all vertices of \( G_1 \) have external degree zero, then \( H \) has at least two circuits and \( V(G_1) \) is a pendant set in \( G \). Similarly, if all vertices of \( G_k \) have external degree zero, then \( V(G_k) \) is a pendant set in \( G \). In any case \( V(H) \) contains a pendant set and hence a global root by Lemma 5.4.

**Corollary 5.6.** If \( G \) contains three or more pairwise disjoint pendant subsets \( X_1, X_2, X_3 \), then \( G \) has no good orientation. In particular, a 2T-graph has a good ordering then it contains at most two pendant generic components.

**Proof:** This follows immediately from Lemma 5.4 and Corollary 5.5.

**Theorem 5.7.** Suppose that there are vertex-disjoint circuits \( G_{i_0}, G_{i_1}, \ldots, G_{i_p}, p \geq 1 \) of a 2T-graph \( G \) such that
Figure 3: Example of a 2T-graph $G$ whose vertex set is partitioned in circuits but which has no good ordering. By Corollary 5.5, in any good orientation of $G$, the global roots $s, t$ are necessarily contained in $G_1$ and $G_4$. Now Lemma 5.3 implies that the two vertices of attachment of $G_2, G_3$ must be local roots (of $G_2, G_3$, respectively) but not global roots in any good orientation. However, if two such local roots from distinct circuits are incident with only one external edge, then, by Lemma 5.3, these edges cannot be the same, implying that $G$ has no good ordering.

- Each $G_{ij}$ has external degree 3
- Some vertex $x_0 \in V(G_{i0})$ has external degree 2 and the third external edge goes between a vertex $y_0 \in V(G_{i0}) - x_0$ and a vertex $z_1 \in V(G_{i1})$
- Some vertex $x_p \in V(G_{ip})$ has external degree 2 and the third external edge is adjacent to a vertex $y_p \in V(G_{ip}) - x_p$ and a vertex $z_{p-1} \in V(G_{ip-1})$, where $z_{p-1} \neq x_0$ if $p = 1$.
- For each $j \in [p-1]$ there is exactly one external edge between $V(G_{ij})$ and $V(G_{ij+1})$: $y_j z_{j+1}$ with $y_j \in V(G_j)$ and $z_{j+1} \in V(G_{j+1})$.

If $G$ has a good ordering $\prec$, then some vertex of $V(G_{i0}) \cup V(G_{i1}) \cup \ldots V(G_{ip})$ is the first or the last vertex according to $\prec$ (that is, at least one of the global roots $s, t$ belongs to that vertex set).

**Proof:** Assume that $V(G_{i0}) \cup V(G_{i1}) \cup \ldots V(G_{ip})$ does not contain any global root. The two local roots of $G_{i0}$ are $x_0$ and $y_0$. So $z_1$ can not be a local root of $G_{i1}$. Then $y_1$ is a local root of $G_{i1}$, and $z_2$ is not a local root of $G_{i2}$. Following the argument, $z_p$ is not a local root of $G_{ip}$, but then it has only one local root, a contradiction. 

We call $G_{i0}, G_{i1}, \ldots, G_{ip}$ as above a **conflict** of $G$.

We say that two conflicts are **disjoint** if no circuit is involved in both of them.

The following is immediate from Corollary 5.6 and Theorem 5.7. For an example, see Figure 3.
Figure 4: The figure above shows part of a graph $G$ whose vertex set is partitioned in circuits together with all the external edges connecting them to other circuits. Assume that we have a good ordering and that the seven circuits displayed in the configuration do not contain any of the two global roots. Consider the external edge $xy$ between $C$ and $C'$. By Lemma 5.3, exactly one of $x$ and $y$ must be a local root. Say, w.l.o.g. that $x$ is a local root so $y$ cannot be a local root of $C'$. We encode this fact by a white arrow from $C$ to $C'$ in the quotient graph (lower left figure). Now the other two vertices displayed in $C'$ must be its local roots, so that, following our drawing convention, we need to orient the remaining two edges incident with $C'$ in the quotient away from it. Processing this way all the six circuits in the upper row we get the lower right figure and deduce that, finally, there is no way to place two local roots in the circuit $C''$. Thus the conclusion is that if $G$ has a good ordering, then at least one of the global roots must be a vertex of one of the circuits in the upper part of the figure.
**Corollary 5.8.** Let $G$ be a 2T-graph. If $G$ has 3 disjoint conflicts, then $G$ has no good ordering.

Even if the graph has no conflict, then it is possible that it has no good orientation. Indeed, using the example in Figure 4 we can now construct a complex example in Figure 5 below of a 2T-graph whose vertex set partitions into vertex sets of disjoint circuits so that $G$ has no good ordering. Note that it is necessary for the conclusion that there must be a global root in each of the three locally identical pieces of the graph that all of the circuits at the rim have exactly three vertices that are incident with external edges.

![Figure 5: Example of a 3-connected 2T-graph $G$ such that the set of external edges almost form a matching and $G$ has no good ordering. The solid and dashed edges illustrate two spanning trees along the external edges which can be extended arbitrarily into the circuits. Note that there are 22 circuits and 42 external edges connecting these so all of these are needed by Theorem 1.1. It also follows from Theorem 1.1 applied to the partition consisting of the seven circuits appearing from (roughly) 2 o’clock to 6 o’clock in the figure and the union of the remaining 15 circuits that the 4 external edges between these two collections are all needed and since they are incident with only 3 vertices of the seven circuits, there will be two external edges incident with the same vertex. One gets further examples by enlarging the three paths on the rim of the figure.](image)

6 2T-graphs which are disjoint unions of circuits

In this section we consider 2T-graphs whose generic components are circuits. When we speak of a good orientation $D_\prec$ of a 2T-graph $G$, we use $s$ to denote the root of the out-branching $B^+$ and $t$ to denote the root of the in-branching $B^-$, where $B^+, B^-$ certify that $D$ is good (so $s$ is the first and $t$ is the last vertex in the ordering $\prec$).
A circuit $H$ of $G$ is called a leaf if there are exactly two external edges between $H$ and some other circuit, that is, $H$ corresponds to a vertex in $G$ incident with two parallel edges, otherwise $H$ is called internal.

**Theorem 6.1.** Let $G = (V, E)$ be a $2T$-graph whose generic components are circuits. If the external edges in $G$ form a matching, then $G$ has a good ordering.

**Proof:** Let $G_1, G_2, \ldots, G_k$ be the circuits of $G$. We prove the theorem by by induction on $k$. When $k = 1$, $G$ is itself a circuit and the result follows from Theorem 3.2. So assume $k \geq 2$. Suppose first that some circuit $G_i$ is a leaf. By relabelling the circuits we may assume that $i = k$ and that $G_k$ is connected to $G_{k-1}$ by a matching of 2 edges $uv, zw$, where $u, z \in V(G_{k-1})$ and $v, w \in V(G_k)$. By induction $G - V(G_k)$ has a good ordering $\prec'$. By renaming if necessary we can assume $u \prec' z$. By Theorem 3.2 $G_k$ has a good ordering $\prec''$ such that $v$ is the first vertex and $w$ the last vertex of $\prec''$. Now we obtain a good ordering by inserting all the vertices of $\prec''$ just after $u$ in $\prec'$. Note that this corresponds to taking the union of the branchings $B^+_s, B^-_s$ that correspond to $\prec'$ and the branchings $B^+_w, B^-_w$ that correspond to $\prec''$ by letting $A(B^+_s) = A(B^+_s) \cup A(\hat{B}^+_s) \cup \{uv\}$ and $A(B^-_s) = A(B^-_s) \cup A(\hat{B}^-_s) \cup \{wz\}$.

Suppose now that every $G_i$, $i \in [k]$ is internal. As $G$ and hence its quotient $\hat{G}$ is a $2T$-graph, there is a circuit $G_j$ such that there are exactly 3 edges $u_1v_1, u_2v_2, u_3v_3$, with $v_i \in V(G_j)$ connecting $V(G_j)$ to $V(V(G_j))$. Again we may assume that $j = k$. We may also assume w.l.o.g. that for some pair of spanning trees $T_1, T_2$ of $G$, the edges $u_1v_1, u_2v_2$ belong to $T_1$ and $u_3v_3$ belongs to $T_2$ (so the vertex in $\hat{G}$ corresponding to $G_k$ is a leaf in $T_2$). Note that this means that $u_1, u_2$ belong to different circuits $G_a, G_b$. Now let $H$ be obtained from $G$ by deleting the vertices of $V(G_k)$ and adding the edge $u_1u_2$. Then $V(H)$ decomposes into a disjoint union of vertex sets of circuits and set of edges connecting these form a matching. By induction there is a good ordering $\prec$ of $H$. Let $B^+_{s,0}, B^-_{s,0}$ be a pair of arc-disjoint branchings that certify that $\prec$ is a good ordering of $H$. We are going to show how to insert the vertices of $V(G_k)$ so that we obtain a good ordering of $G$. By renaming $u_1, u_2, v_1, v_2$ and possibly considering the reverse ordering $\prec$ if necessary we can assume that $u_1 \prec u_2$ and that the arc $u_1u_2$ belongs to $B^+_s$. We now consider the three possible positions of $u_3$ in the ordering $\prec$ (see Figure 6).

- $u_3 \prec u_1 \prec u_2$. By Theorem 3.2 $G_k$ has a good ordering $\prec_1$ of $G_k$ such that $v_3$ is the initial vertex and $v_5$ is the terminal vertex of $\prec_1$. Let $B^+_{v_3,1}, B^-_{v_2,1}$ be arc-disjoint branchings (on $V(G_k)$) certifying that $\prec_1$ is good. Then we obtain a good ordering of $G$ by inserting all the vertices of $\prec_1$ just after $u_1$ in $\prec$ and we obtain the desired branchings $B^+_s, B^-_s$ by letting $A(B^+_s) = A(B^+_s) \cup A(B^+_{v_3,1}) \cup \{u_3v_3\}$ and $A(B^-_s) = A(B^-_s) \cup A(\hat{B}^-_s) \cup \{u_1v_1, v_2u_2\}$.

- $u_1 \prec u_2 \prec u_3$. By Theorem 3.2 $G_k$ has a good ordering $\prec_2$ of $G_k$ such that $v_2$ is the initial vertex and $v_3$ is the terminal vertex of $\prec_2$. Let $B^+_{v_2,2}, B^-_{v_2,2}$ be arc-disjoint branchings (on $V(G_k)$) certifying that $\prec_2$ is good. Then we obtain a good ordering of $G$ by inserting all the vertices of $\prec_2$ just after $u_2$ in $\prec$ and we obtain the desired branchings $B^+_s, B^-_s$ by letting $A(B^+_s) = A(B^+_s) \cup A(B^+_{v_2,2}) \cup \{u_2v_2\}$ and $A(B^-_s) = A(B^-_s) \cup A(\hat{B}^-_s) \cup \{u_1v_1, v_2u_2\}$.
• $u_1 \prec u_3 \prec u_2$. Consider again the good ordering $\prec_1$ above and the branchings $B_{v_{3,1}}^+, B_{v_{2,1}}^-$. Then we obtain a good ordering of $G$ by inserting all the vertices of $\prec_1$ just after $u_3$ in $\prec$ and we obtain the desired branchings $B_{v_{3,1}}^+$ by letting $A(B_{v_3}^+) = A(B_{v_3}^+) \cup A(B_{v_{3,1}}^+) \cup \{v_3u_3\}$ and $A(B_{v_2}^-) = A(B_{v_2}^-) \cup A(B_{v_{2,1}}^-) \cup \{u_1u_1, v_2u_2\}$.

As we saw, in all the possible cases we obtain a good ordering of $G$ together with a pair of arc-disjoint branchings which certify that the ordering is good so the proof is complete.

Figure 5 shows an example of a $2T$-graph $G$ whose vertex set partitions into vertex sets of generic circuits such that the set of edges between different circuits almost forms a matching and the graph $G$ has no good ordering.

Figure 6: How to lift a good ordering to a new circuit as in the proof of Theorem 6.1. In-branchings are displayed solid, out-branchings are dashed. The first line displays the three possible orders of the relevant vertices (increasing left to right) as they occur in the proof. The second line displays the ordering of the augmented graph and how the branchings lead into and out of the new circuit; its local out- and in-root is the leftmost and rightmost $v_i$, respectively.

A **double tree** is any graph that one can obtain from a tree $T$ by adding one parallel edge for each edge of $T$. A **double path** is a double tree whose underlying simple graph is a path.

Recall that a subset $X \subset V$ with $2 \leq |X| \leq |V| - 2$ is pendant at $x$ in $G$ if all edges between $X$ and $V(G) - X$ are incident with $x$.

**Definition 6.2.** Let $G$ be a $2T$-graph whose quotient graph is a double tree $T$. An **obstacle** in $G$ is a subgraph $H$ consisting of a subset of the circuits of $G$ and the edges between these such that the quotient graph of $G[V(H)]$ is a double path $T_H$ of $T$ so that

• $H$ contains circuits $C, C'$, possibly equal, and vertices $x \in C, y \in C'$, such that $x = y$ if $C = C'$ and there is an $(x, y)$-path $P$ in $H$ which uses only external edges of $H$ (so $P$ is also a path in $T_H$ between the two vertices corresponding to the circuits $C, C'$).
• $T - V(T_H)$ has at least two connected components $A, B$ and $V_A$ is pendant at $x$ and $V_B$ is pendant at $y$ in $G$, where $V_A$ (resp. $V_B$) is the union of those circuits of $G$ that correspond to the vertex set $A$ (resp. $B$) in $T$.

**Theorem 6.3.** Let $G$ be a $2T$-graph whose quotient is a double tree $T$, then $G$ has a good ordering if and only if

1. $G$ has at most two pendant circuits and
2. $G$ contains no obstacle.

**Proof:** By Corollary 5.6 we see that (i) must hold if $G$ has a good ordering.

Suppose that $G$ contains an obstacle $H$ but there exists a good ordering $\prec$ with associated branchings $B^+_s, B^-_s$ in $D = D_\prec$. Let $x, y$ be the special vertices according to the definition. Suppose first that $x = y$ and let $C$ be the circuit that contains $x$, let $v_C$ be the vertex of $T$ that corresponds to $C$ and let $V_A, V_B$ be the union of the vertex sets of circuits of $G$ so that these correspond to distinct connected components $A, B$ of $T - v_C$ and both $V_A$ and $V_B$ are pendant at $x$ in $G$. By Lemma 5.4 we may assume w.l.o.g that $s \in V_A$ and $t \in V_B$. Then it is easy to see that $D$ contains two arcs $a_1x, a_2x$ from $V_A$ to $x$ and the two arcs $xb_1, xb_2$ from $x$ to $V_B$ and precisely one of the arcs $a_1x, a_2x$ is in $B^+_s$ and the other is in $B^-_s$ and the same holds for the arcs $xb_1, xb_2$. Now consider a vertex $z \in C - x$. The $(s, z)$-path in $B^+_s$ contains $x$ and the $(z, t)$-path in $B^-_t$ also contains $x$ so $D$ is not acyclic, contradiction.

Hence we must have $x \neq y$ and $x, y$ are in different circuits (so $C \neq C'$). Again we let $V_A$ be the union of vertices of circuits of $G$ so that $V_A$ is pendant at $x$ and similarly let $V_B$ be the union of vertices of circuits of $G$ so that $V_B$ is pendant at $y$. Again by Lemma 5.4 we may assume w.l.o.g that $s \in V_A$ and $t \in V_B$. As above $D$ must contain two arcs $a_1x, a_2x$ from $V_A$ to $x$ and the two arcs $yb_1, yb_2$ from $y$ to $V_B$ and precisely one of the arcs $a_1x, a_2x$ is in $B^+_s$ and the other is in $B^-_s$ and the same holds for the arcs $yb_1, yb_2$. Let $C_1, C_2, \ldots, C_r$, $r \geq 2$ be circuits of $G$ so that $C = C_1, C' = C_r$ and $v_{C_1}, v_{C_2}, \ldots, v_{C_r}$ is a path in $T$ which corresponds to the $(x, y)$-path $P = x_1x_2 \ldots x_r$, where $x = x_1, y = x_r$, that uses only edges between different circuits in $G$ (by the definition of an obstacle). As $s \in V_A$ and $t \in V_B$ the path $P$ must be a directed $(x_1, x_r)$-path in $D$ and using that $D[V(C_1)]$ are $D[V(C_i)]$ are acyclic we can conclude as above that the arc $x_1x_2$ is an arc of $B^+_s$ and the arc $x_{r-1}x_r$ is an arc of $B^-_s$ (if $x_1x_2$ would not be an arc of $B^+_s$, then $D[V(C_1)]$ would contain a directed path from $x_1$ to the end vertex $z$ of the other arc leaving $V(C_1)$ and also a directed path from $z$ to $x_1$, implying that $D[V(C_1)]$ would not be acyclic). Thus it follows that for some index $1 < j < r$ the arc $x_{j-1}x_j$ is an arc of $B^+_s$ and the arc $x_jx_{j+1}$ is an arc of $B^-_s$. This implies that for every $z \in C_j$ the $(s, z)$-path of $B^+_s$ and the $(z, t)$-path of $B^-_s$ contains $x_j$, contradicting that $D$ is acyclic.

Suppose now that $G$ satisfies (i) and (ii). We shall prove by induction on the number, $k$, of circuits in $G$ that $G$ has a good orientation. The base case $k = 1$ follows from Theorem 3.2 so we may proceed to the induction step.

Suppose first that $G$ has a leaf circuit $G_h$ that is not pendant. Let $v_h$ be the neighbour of $v_h$ in $G$ and let $G_{h'}$ be the circuit of $G$ corresponding to $v_{h'}$. As $G_h$ is not pendant the two edges $xx', zz'$ between $G_h$ and $G_{h'}$ have distinct end vertices $x, z$ in $V(G_h)$ and
distinct end vertices \( x', z' \) in \( V(G_h') \). By induction \( G - G_h \) has a good orientation \( D' \) and we may assume, by reversing all arcs, if necessary, that \( x' \) occurs before \( z' \) in the ordering \( \prec' \) that induces \( D' \). By Theorem 3.2, \( G_h \) has a good orientation \( D'' \) where \( x \) is the out-root and \( z \) is the in-root. Now we obtain a good orientation \( D \) by adding the two arcs \( x'x \) and \( zz' \) and using the first in the out-branching rooted at \( x \) and the later in the in-branching.

Thus we can assume from now on that every leaf component of \( G \) is pendant and now it follows from Corollary 5.6 that \( G \) is a double path whose circuits we can assume are ordered as \( G_1, G_2, \ldots, G_k \) in the ordering that the corresponding vertices \( v_1, v_2, \ldots, v_k \) appear in the quotient \( G \).

We prove the following stronger statement which will imply that \( G \) has a good orientation.

**Claim 1.** Let \( G \) be a double path having no obstacle and whose circuits are ordered as \( G_1, G_2, \ldots, G_k \). Let \( s \in V(G_1) \) be any vertex, except \( a \) if \( G_1 \) is pendant at \( a \) \( \in V(G_1) \) and let \( t \in V(G_k) \) be any vertex except \( b \) if \( G_k \) is pendant at \( b \in V(G_k) \) (such vertices are called **candidates for roots**). Then has a good orientation \( D \prec \) so that \( s \) is the first vertex (root of the out-branching) and \( t \) is the last vertex in \( \prec \) if and only if none of the following hold.

(a) There is an \((s,t)\)-path \( P \) in \( G \) which uses only external edges.

(b) \( t \) is an end vertex of one of the edges from \( G_{k-1} \), there is an index \( i \in [k - 1] \) so that the two edges from \( G_i \) to \( G_{i+1} \) are incident with the same vertex \( x \) of \( G_{i+1} \) and there is an \((x,t)\)-path in \( G \) which uses only external edges.

(c) \( s \) is an end vertex of one of the edges from \( G_1 \) to \( G_2 \), there is an index \( j \in [k] \setminus \{1\} \) so that the two edges from \( G_{j-1} \) to \( G_j \) are incident with the same vertex \( y \) of \( G_{j-1} \) and there is an \((s,y)\)-path in \( G \) which uses only external edges.

**Proof of claim:** Note that if \( \prec : v_1, \ldots, v_n \) is a good ordering with \( s = v_1 \) and \( t = v_n \), then, in the corresponding acyclic orientation \( D \prec \), the two edges between \( G_i \) and \( G_{i+1} \) are both oriented towards \( G_{i+1} \) and for every pair of arc-disjoint branchings \( B^+_{s}, B^-_{t} \) in \( D \), exactly one of these arcs belong to \( B^+_{s} \) and the other to \( B^-_{t} \).

We first show that if \( G, s, t \) satisfy any of (a)-(c), then there is no good ordering \( v_1, \ldots, v_n \) with \( s = v_1 \) and \( t = v_n \).

Suppose that \( G \) has a good ordering \( v_1, \ldots, v_n \) with \( s = v_1 \) and \( t = v_n \) and let \( B^+_{s}, B^-_{t} \) be a pair of arc-disjoint branchings in the acyclic digraph \( D = D \prec \).

If (a) holds, then let \( P = x_1x_2 \ldots x_k \) be a path from \( s = x_1 \) to \( t = x_k \) so that each edge \( x_i x_{i+1}, i \in [k - 1] \) has one end vertex in \( G_i \) and the other in \( G_{i+1} \). As \( B^+_{s} \) induces and out-branching from \( s \) in the acyclic digraph \( D[V(G_1)] \), we must have that the arc \( s x_2 \) belongs to \( B^+_{s} \). By a similar argument, the arc \( x_{k-1} t \) belongs to \( B^-_{t} \). Hence there is an index \( 1 < i < k \) such that the arc \( x_{i-1} x_i \) is in \( B^+_{s} \) and the arc \( x_i x_{i+1} \) is in \( B^-_{t} \). However this implies that \( x_1 \) is both an out-root and an in-root in \( D[V(G_i)] \), contradicting that \( D \) is acyclic. So (a) cannot hold if there is a good ordering.
Suppose now that none of (a)-(c) hold. We prove the existence of a good orientation by induction on \( k \). For \( k = 1 \) the claim follows from Theorem 3.2.

Suppose next that \( k = 2 \). Let \( u_1u_2 \) and \( v_1, v_2 \) with \( u_1, v_1 \in V(G_1) \) be the two edges between \( G_1 \) and \( G_2 \). Suppose first that \( t \not\in \{u_2, v_2\} \). Since \( G_1 \) is not pendant at \( s \), we can assume w.l.o.g. that \( s \neq v_1 \). By Theorem 3.2 there is a good orientation of \( G_1 \) in which \( s \) is the out-root and \( v_1 \) is the in-root and a good orientation of \( G_2 \) in which \( u_2 \) is the out-root and \( t \) is the in-root. Thus we obtain the desired orientation by adding the arc \( u_1v_2 \) to the union of the two out-branchings and the arc \( v_1v_2 \) to the union of the two in-branchings. Suppose now that \( t \in \{u_2, v_2\} \). Without loss of generality \( t = u_2 \). Since (a) does not hold, we know that \( s \neq u_1 \). By Theorem 3.2 \( G_1 \) has a good orientation with \( s \) and out-root and \( u_1 \) as in-root and \( G_2 \) has a good orientation with \( v_2 \) as out-root and \( t \) as in-root. Now we obtain the desired branchings by adding the arc \( u_1u_2 \) to the union of the two in-branchings and the arc \( v_1v_2 \) to the union of the two out-branchings.

Assume that \( k \geq 3 \) and that the claim holds for all double paths which satisfy none of (a)-(c) and have fewer than \( k \) circuits. Let \( s \in V(G_1), t \in V(G_k) \) be candidates for roots and let \( xx', zz' \) be the two edges between \( G_1 \) and \( G_2 \). Without loss of generality we have \( s \neq z \). Note that (b) cannot hold for \( s', t \) in \( G' = G - G_1 \) when \( s' \in \{x', z'\} \) because \( G' \) is an induced subgraph of \( G \). Suppose that (a) holds for \( (G', x', t) \). Then \( z' \neq x' \) as (b) does not hold for \( G \). Now (a) cannot hold for \( (G', z', t) \) as this would imply that (b) holds in \( G \). For the same reason (c) cannot hold for \( (G', z', t) \). Thus if (a) holds for \( (G', x', t) \), then none of (a)-(c) hold for \( (G', z', t) \). If (c) holds for \( (G', x', t) \) we conclude that none of (a),(c) hold for \( (G', z', t) \), because both would imply that \( G \) contains an obstacle. Let \( s' = x' \) unless one of (a)-(c) holds for \( x' \) and in that case \( s \neq x \) must hold and we let \( s' = z' \). By the arguments above, none of (a)-(c) hold for \( (G', s', t) \).

By induction \( G' \) has a good orientation where \( s' \) is the out-root and \( t \) is the in-root and by Theorem 3.2 \( G_1 \) has a good orientation in which \( s \) is the out-root and \( z \) is the in-root. Let \( a \) be the arc \( xx' \) if \( s' = x' \) and otherwise let \( a \) be the arc \( zz' \). Now adding \( a \) to the union of the two out-branchings and the other arc from \( G_1 \) to \( G_2 \) to the union of the two in-branchings, we obtain the desired good orientation. This completes the proof of Claim 1.

Now we are ready to conclude the proof of Theorem 6.3. As \( G_1, G_k \) are circuits they both have at least 2 vertices. If we can choose \( s \in V(G_1) \) and \( t \in V(G_k) \) so that none of these two vertices are incident with edges to the other circuits, then we are done by the Claim 1 so either \( |V(G_1)| = 2 \) or \( |V(G_k)| = 2 \) or both. Suppose w.l.o.g. that \( |V(G_1)| = 2 \) and that the two edges from \( G_1 \) to \( V - G_1 \) are incident with different vertices \( u, v \in V(G_1) \). As \( V(G_1) \) is pendant, these two edges end in the same vertex \( x \). If we can choose \( t \in V(G_k) \) so
that it is not incident with any of the edges between $G_{k-1}$ and $G_k$, then we are done, so we may assume that we also have $V(G_k) = \{z, w\}$ and that the edges between $G_{k-1}$ and $G_k$ are $yz, yw$ for some $y \in V(G_{k-1})$ ($G_k$ is pendant). Now it follows from the fact that (ii) holds that every $(x, y)$-path in $G$ uses an edge which is inside some $G_i$, we can take $s$ and $t$ freely among $u, v$, respectively $z, w$ and conclude by the claim (none of (a)-(c) can hold). □

7 Remarks and open problems

Let us start by recalling that the following is an immediate consequence of Theorem 3.2 as we first find a good ordering of the circuit and then orient the remaining edges according to that ordering.

**Corollary 7.1.** Every graph which contains a circuit as a spanning subgraph has a good ordering.

**Conjecture 7.2.** There exists a polynomial algorithm for deciding whether a 2T-graph has a good ordering.

**Problem 7.3.** What is the complexity of deciding whether a given graph has a good ordering?

Two of the authors of the current paper proved the following generalization of Theorem 3.3. Note that its proof is more complicated than that of Theorem 3.3.

**Theorem 7.4.** Every 4-regular 4-connected graph has a good orientation.

Let $D = (V, A)$ be a digraph and let $s, t$ be distinct vertices of $V$. An $(s, t)$-ordering of $D$ is an ordering $v_1, v_2, \ldots, v_n$ with $v_1 = s, v_n = t$ such that every vertex $v_i$ with $i < n$ has an arc to some $v_j$ with $i < j$ and every vertex $v_p$ with $p > r$ has an arc from some $v_p$ with $p < r$. It is easy to see that $D$ has such an ordering if and only if it has a spanning acyclic digraph in with branchings $B^+_s, B^-_t$. These branchings are not necessarily arc-disjoint but it is clear that if $D$ has a good ordering with $s$ as the initial and $t$ as the terminal vertex then this ordering is also an $(s, t)$-ordering. Hence having an $(s, t)$-ordering is a necessary condition for having a good ordering with $s$ as the initial and $t$ as the terminal vertex.

**Theorem 7.5.** It is NP-complete to decide whether a digraph $D = (V, A)$ with prescribed vertices $s, t \in V$ has an $(s, t)$-ordering.

**Proof:** The so called BETWEENNESS problem is as follows: given a set $S$ and a collection of triples $(x_i, y_i, z_i), i \in [m]$, consisting of three distinct elements of $S$: is there a total order on $S$ (called a betweenness order on $S$) so that for each of the triples we have either $x_i < y_i < z_i$ or $z_i < y_i < x_i$? BETWEENNESS is NP-complete [19]. Given an instance $[S, (x_i, y_i, z_i), i \in [m]]$ of BETWEENNESS we construct the following digraph $D$. The vertex set $V$ of $D$ is constructed as follows: first take $5m$ vertices

$$a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m, d_1, \ldots, d_m, e_1, \ldots, e_m$$

where $\{a_i, b_i, c_i\}$ corresponds to the triple $(x_i, y_i, z_i)$ and then identify those vertices in the set $\{a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m\}$ that correspond to the same element of $S$. Then,
add two more vertices: s and t. The arc set of D consists of an arc from s to each vertex of \{a_1, \ldots, a_m, c_1, \ldots, c_m\}, an arc from each such vertex to t and the following 6m arcs which model the betweenness conditions: for each triple \((x_i, y_i, z_i)\) D contains the arcs \((a_id_i, c_id_i), (d_ib_i, b_ie_i), (e_ia_i, e_ic_i)\). Clearly D can be constructed in polynomial time. We claim that D has an (s, t)-ordering if and only if there is a betweenness total ordering of S. Suppose first that D has an (s, t)-ordering. The vertices \(d_i, b_i, c_i\) must occur in that order as \(b_i\) is the unique out-neighbour (in-neighbour) of \(d_i (e_i)\). As each \(a_i, c_i\) are the only in-neighbours (out-neighbours) of \(d_i (e_i)\) in D the vertices \(a_i, c_i\) cannot both occur after (before) \(d_i (e_i)\) so the vertices in \(\{a_i, b_i, c_i\}\) will occur either in the order \(a_i, b_i, c_i\) or in the order \(c_i, b_i, a_i\). Thus taking the same order for the elements in S as for the corresponding vertices of D, we obtain a betweenness total order. Conversely, if we are given a betweenness total order for S we just place the vertices in \(\{a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m\}\) in the order that the corresponding elements of S occur and then insert each vertex \(d_i (e_i)\) anywhere between \(a_e\) and \(b_i\) (or \(c_i\)) if the triple \((x_i, y_i, z_i)\) is ordered as \(x_i < y_i < z_i\) and otherwise we insert \(d_i (e_i)\) anywhere between \(c_i\) and \(b_i\) (or \(a_i\)). Finally insert s as the first element and t as the last element. Now every vertex different from s, t has an earlier in-neighbour and a later out-neighbour, so it is an (s, t)-ordering.

If D is semicomplete digraph, that is, a digraph with no pair of non-adjacent vertices, then D has an (s, t)-ordering for a given pair of distinct vertices s, t if and only if D has a Hamiltonian path from s to t \([2, 21]\). It was shown in \([6]\) that there exists a polynomial algorithm for deciding the existence of such a path in a given semicomplete digraph so for semicomplete digraphs the (s, t)-ordering problem is polynomially solvable.

**Corollary 7.6.** It is NP-complete to decide if a strong digraph D = (V, A) has an (p, q)-ordering for some choice of distinct vertices p, q ∈ V

**Proof:** Let D′ be the digraph that we obtain from the digraph D in the proof above by adding the arcs ts. Then D′ is strong and it is easy to see that the only possible pair for which there could exists a (p, q)-ordering is the pair p = s, q = t: for each triple \((x_i, y_i, z_i)\) the corresponding vertices in D must occur either in the order \(a_i, d_i, b_i, c_i\) or in the order \(c_i, d_i, b_i, c_i, a_i\) and in both cases s must be before all these vertices and t must be after all these vertices.

**Problem 7.7.** What is the complexity of deciding whether a digraph which has a pair of arc-disjoint branchings \(B^+_1, B^-_1\) has such a pair whose union (of the arcs) is an acyclic digraph?

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