Axially Symmetric Type N Space-Time with a Naked Curvature Singularity and Closed Timelike Curves

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Abstract—A family of type N exact solutions of the Einstein field equations, regular everywhere except on the symmetry axis, where it possesses a naked curvature singularity, is presented. The stress-energy tensor is that of an anisotropic fluid coupled with a pure radiation field and satisfies different energy conditions, and the physical parameters diverge as \( r \to 0 \). The space-time admits a non-expanding, non-twisting, and shear-free geodesic null congruence and belongs to a special class of type N Kundt metrics. The space-time is geodesically complete along the radial direction in the constant \( z \) planes and exhibits geometrically different properties from the known \( pp \)-wave space-times. The present family of solutions admits closed timelike curves (CTC) which appear after a certain time instant, and the present metric is a four-dimensional generalization of the Misner space metric in curved space-time.

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1. INTRODUCTION

All vacuum or non-vacuum solutions of Einstein’s field equations with or without a cosmological constant, which are of type N with non-twisting and shear-free geodesic null congruence have been extensively studied (see, [1, 2] and references therein). A complete class of non-twisting type N vacuum solutions with a nonzero cosmological constant was found in [3]. These solutions were further analyzed in [4] and subsequently by various authors (see [5–10]). Type N Kundt metrics [11, 12] are the general class of vacuum or non-vacuum solutions to the Einstein field equations which admit a nonexpanding, nontwisting, and shear-free geodesic null congruence. If this shear-free geodesic null congruence is a covariantly constant vector field, we called this solution a \( pp \)-wave or plane-wave space-time, see, e.g., [13–17] and references therein). In [18], a special class of type N Kundt nonvacuum solutions to the field equations was obtained, with a nonzero cosmological constant, which exhibit geometrically different properties from the known \( pp \)-waves. Subsequently, a type N vacuum solution with a nonzero cosmological constant, a special subcase of the Siklos class of solution, was obtained in [19]. Recently, a family of type N aligned pure radiation field solution with a nonzero cosmological constant was constructed in [1]. This family of solutions admits closed timelike curves and exhibits geometrically different properties from the known \( pp \)-wave space-time. Note that this family of solutions [1] is different from the Siklos class [18] in the sense that the first family of solutions admits closed timelike curves which violate the causality condition. The type N vacuum solution with a nonzero cosmological constant, obtained in [20], and the pure radiation field type N solution with a nonzero cosmological constant in [2], are a special subcase of this family of solutions [1]. It is worth mentioning that all the above type N vacuum or nonvacuum solutions are free from curvature singularities. A few of them violate the causality condition, and some others do not.

However, some other algebraically special solutions, showing geometrically different properties from \( pp \)-waves or plane-waves, possess naked curvature singularities, and among these, some solutions violate the causality condition. This type of solutions with a naked curvature singularity includes a type D vacuum space-time [21], axially symmetric type II null dust fluid space-time [22], a cylindrically symmetric type D non-vacuum space-time [23], type N non-vacuum space-time [24], and axially symmetric type II space-time [25] which violate the causality condition. Other investigations of gravitational collapse solutions with naked singularities, satisfying the causality condition, include a counter-rotating dust shell cylinder [26], the evolution of a cylindrical dust shell [27], the collapse of nonrotating and infinite dust cylinder [28], the high-speed collapse of a cylindrically symmetric thick shell model composed of dust [29], a perfect fluid solution with nonvanish-
The ranges of the coordinates are $X$ here, we have assumed that counter-rotating dust shells [31] = rotating curved space-time with the metric generalization of the Misner space metric in curved $N$ solutions under study here is a four-dimensional $pp$ the flatness condition [54] which states that in the limit of the rotation axis $r \to 0$,

$$g^\mu\nu(\nabla_\mu X)(\nabla_\nu X) \to 1. \quad (4)$$

Therefore, the symmetry axis $r = 0$ is singular, which is not well-defined. The singular axis may contain some line-like sources and/or conical singularities [54]. The appearance of space-time singularities on the axis would be considered as representing the existence of some kind of sources [55–58].

From the above family of solutions (1), one can construct a family of type $N$ solutions to the field equations. In this work, we attempt to construct a family of type $N$ solutions with or without a cosmological constant. A family of type $N$ aligned pure radiation field solution with a nonzero cosmological constant was constructed in [1]. In this work, we attempt to construct a family of type $N$ solutions, especially non-vacuum ones, of the field equations with zero cosmological constant, violating the causality condition and possessing a naked singularity.

We choose the following metric functions for a zero cosmological constant nonvacuum solution for the metric (1):

$$g_{rr}(r) = A(r),$$
$$A(r) = [B'(r)]^2 B(r),$$
$$g_{\psi\psi}(t,r) = -c_0 \dot{a}(t) B(r),$$
$$g_{t\psi}(t,r) = -c_0 \dot{a}(t) C(r),$$
$$g_{z\psi}(r) = c_1 B(r),$$
$$g_{zz}(r) = B(r). \quad (5)$$

Here the prime denotes an ordinary derivative with respect to $r$, $c_0 > 0$, $c_1 > 0$, $c_0 > 0$ are constants, $B(r)$, $C(r)$ are free functions, and the dot stands for an ordinary derivative in $t$. The space-time metric (1) with (5) can be express as

$$ds^2 = B(r) [B'(r) dr^2 + dz^2 - 2c_0 \dot{a}(t)dtdz \psi' + 2c_0 \dot{a}(t) dzd\psi], \quad (6)$$

Replacing

$$r \to \tilde{r} := B(r), \quad t \to T := a(t) \quad (7)$$

in the above metric (6) and after a redefinition of $C(r)$ as $H(r)$, we get a family of $A = 0$ type $N$ solutions (finally dropping the bar)

$$ds^2 = r [dr^2 + dz^2 - 2c_0 T dz \psi - T dz^2 + 2c_1 zd\psi + 2c_2 H(r)drdz]. \quad (8)$$

The solution (8) is our main aim that we shall below discuss in detail.
where

\[ T^\mu \nu = \mu k^\mu k^\nu + (\rho + p_t)U^\mu U^\nu + p_t \, g^{\mu \nu} \]
\[ T_3^3 = p_T. \]  

From the field equations (13), using Eqs. (10) and (21), after simplification we have the following non-zero physical parameters:

\[ \rho = p_r = -p_t = \frac{3}{4r^3}, \quad \mu = \frac{1}{2c_0} \left[ c_1 + c_2 \left( \frac{H(r)}{r} - H'(r) \right) \right]. \]  

The stress-energy tensor satisfies the following energy conditions [59]:

- WEC: \( \rho > 0, \quad \mu > 0, \)
- WEC_r: \( \rho + p_t = 0, \)
- WEC_\phi: \( \rho + p_r > 0, \)
- SEC: \( \rho + p_r + 2p_t = 0, \)
- DEC: \( \rho \geq |p_t|, \quad i=t,r,z. \)

The physical parameters (\( \rho, p_r, p_t \)) are singular on the symmetry axis \( r = 0. \) Therefore, the stress-energy tensor of the anisotropic fluid diverges on the symmetry axis, and a naked singularity is formed.

**A few examples of this family of type N solutions are:**

**Example 1:** Let us choose the function \( H(r) = c_3/r \) in the metric (8), where \( c_3 > 0. \) In that case, the solution (8) represents a Petrov type N space-time with an anisotropic fluid coupled with a constant-energy-density radiation field. The physical parameters are

\[ \rho = p_r = -p_t = \frac{3}{4r^3}, \quad \mu = \frac{c_1}{2c_0}. \]  

Recently, the author constructed a type N solution with an anisotropic fluid coupled with a radiation field [24], a special subcase of the present family of type N solutions (8).

**Example 2:** Let us choose the function \( H(r) = -c_1 r/2c_2 \) in the metric (8). In that case the solution (8) represents a Petrov type N space-time with only an anisotropic fluid. The physical parameters

\[ \rho = p_r = -p_t = \frac{3}{4r^3} \]  

diverge on the symmetry axis \( r = 0. \)

**Example 3:** If one chooses the function \( H(r) = c_3/r - c_1 r/(2c_2) \) in the metric (8), where \( c_3 > 0, \) the solution (8) also represents a Petrov type N space-time with only an anisotropic fluid with the physical parameters (25).

**Example 4:** If one chooses the function \( H(r) = c_1 r c_2 \) in the metric (8), the solution (8) represents a conformally flat type O space-time with an anisotropic fluid coupled with a constant-energy-density radiation field. The physical parameters are

\[ \rho = p_r = -p_t = \frac{3}{4r^3}, \quad \mu = \frac{c_1}{2c_0}. \]  

**2.2. Geodesic Analysis and Strength of the Naked Singularities**

In this subsection, we focus on the radial geodesics, which necessarily hit the singularity \( r = 0 \) [60], and finally the strength of the naked singularities.

The Lagrangian for geodesics in the metric (8) is given by

\[ L = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} r \dot{\psi}^2 + z^2 - c_0^2 \dot{T}^2 \]
\[ + 2c_1 z \dot{\psi} + 2c_2 H(r) \dot{r}, \]  

where the dot stands for a derivative in an affine parameter \( \lambda. \) From (18), it is clear that \( \psi \) is a cyclic coordinate. There exist constants of motion corresponding to this cyclic coordinate, i.e., the azimuthal angular momentum \( p_\phi \) which is a constant given by

\[ p_\phi = -T \dot{\psi} - c_0 \dot{T} + c_1 z \dot{z} + c_2 H(r) \dot{r}. \]
\[ = \text{const.} \]  

For the metric (8), the geodesic equation for the coordinates \( T \) and \( r \) in an explicit form are

\[ \ddot{T} = \frac{1}{2c_0^2 r} \left[ -rT \dot{\psi}^2 - c_2^2 r \dot{z}^2 \dot{\psi}^2 \right. \]
\[ + c_2 H^2(r) \dot{\psi}(2c_0 r - r \dot{\psi}) + 2c_0 c_1 r \dot{z}^2 \]
\[ \left. \quad - 2c_0^2 r \dot{T} - 2c_0 r \dot{\psi} \dot{T} + c_0 c_2 H(r) \right] \]
\[ \times \left\{ \dot{r}^2 - T \dot{\psi}^2 + 2c_1 z \dot{\psi} \dot{z} + \dot{z}^2 - 2c_0 \dot{T}^2 \right\} \]
\[ + 2c_0 c_2 r \dot{T}^2 H'(r), \]  

\[ \ddot{r} = \frac{1}{2c_0^2 r} \left[ c_2 H(r) \dot{\psi}(2c_0 r - r \dot{\psi}) \right. \]
\[ - c_0 (\dot{r}^2 + T \dot{\psi}^2 - 2c_1 z \dot{\psi} \dot{z} \]
\[ \left. \quad - \dot{z}^2 + 2c_0 \dot{T} \dot{r} \right]. \]  

For a radial geodesic \( \dot{z} = 0 = \dot{\phi}. \) We have chosen \( z = \text{const} \) planes defined by \( z = z_0 = 0, \) then from Eqs. (29) and (30) we have

\[ \ddot{T} = \frac{1}{2c_0^2 r} \left[ -2c_0^2 r \ddot{T} + c_0 c_2 H(r) \ddot{r} \right] \]
\[ + 2c_0 c_2 r \ddot{r}^2 H'(r), \]
\[ \ddot{r} = -\frac{\dot{r}^2}{2r}. \]
The solution for \( r \) gives
\[
\dot{r}(s) = \frac{c_4}{\sqrt{r}} \Rightarrow r(s) = \left[ \frac{3}{2}(c_4 s + c_5) \right]^{2/3},
\]
where \( c_4, c_5 \) are constants. From Eq. (31) we have
\[
\ddot{T} + \frac{\dot{r}}{r} \dot{T} = \frac{c_4^2 c_5^2}{2c_0 r} \left( 2H'(r) + \frac{H(r)}{r} \right).
\]
From Eq. (33) one can easily check for the chosen function \( H(r) \) that the radial geodesics path \( T \) is bounded for a finite value of the affine parameter \( s \) including \( s = 0 \) (since \( r(s = 0) = \text{const} \neq 0 \) provided that \( c_5 \neq 0 \)), and thus the solution under study is radially geodesically complete.

To determine the strength of the naked singularities, we consider the criterion developed in [61, 62]. A sufficient condition for a singularity to be strong [61, 63] is that
\[
\lim_{s \to a} s^2 R_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 (> 0),
\]
where \( dx^\mu/ds \) is the tangent vector to the radial geodesics, and \( R_{\mu\nu} \) is the Ricci tensor. The weaker condition, which we call the limiting focusing condition [62], is defined by
\[
\lim_{s \to 0} s R_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.
\]

For the solution (8) under study we have
\[
\lim_{s \to a} s^2 \left[ R_{rr} \left( \frac{dr}{ds} \right)^2 + R_{\psi\psi} \left( \frac{d\psi}{ds} \right)^2 \right] = \lim_{s \to 0} \frac{3 s^2}{2r^2} \dot{r}^2, \quad (\text{since } \dot{\psi} = 0)
\]
\[
= \frac{2 c_4^2}{3} \lim_{s \to 0} \frac{s^2}{(c_5 + c_4 s)^2}
\]
\[
= 0 \quad \text{if} \quad c_5 \neq 0,
\]
\[
= \text{const.} \quad \text{if} \quad c_5 = 0.
\]
Similarly, the solution under study does not satisfy the limiting focusing condition. Thus the naked singularity (NS) that is formed due to the divergence of the scalar curvature does not satisfy either the strong curvature condition or the limiting focusing condition provided that we have assumed the constant \( c_5 > 0 \), otherwise only the strong curvature condition holds true for \( c_5 = 0 \). In that case \( c_5 = 0 \), the solution under study is geodesically incomplete.

2.3. Generalization of the Misner Space Metric and the Petrov Classification

We show below that the solution (8) is a four-dimensional generalization of the Misner space metric to curved space-time. Before that, we discuss the Misner space metric in two dimensions, given by [64]
\[
ds^2_{\text{Misner}} = -2dT d\psi - T d\psi^2,
\]
where \(-\infty < T < \infty\), and the \( \psi \) coordinate is periodic. The Misner space metric in two dimensions is locally flat and regular everywhere. The curves defined by \( T = T_0 > 0 \), where \( T_0 \) is a constant, being timelike and closed on account of the periodicity of \( \psi \), are formed as closed timelike curves (CTC). The null curve \( T = T_0 = 0 \) serves as a chronology horizon (since \( g^{TT} = 0 \) at \( T = T_0 = 0 \)), which divides the space-time into a chronal region without CTC and a nonchronal region with CTC. There is a Cauchy horizon at \( T = \text{const} = T_0 = 0 \) for any such spacelike \( T = \text{const} = T_0 < 0 \) hypersurface (since \( g^{TT} = T < 0 \) at \( T = T_0 < 0 \)). Hence the space-time evolves from an initial spacelike \( T = \text{const} = T_0 < 0 \) hypersurface in a causally well-behaved manner, up to a moment, i.e., a null hypersurface \( T = T_0 = 0 \), and the formation of CTC takes place from causally well-behaved initial conditions. The Misner space metric in two dimensions is a prime example of space-time where closed timelike curves develop at some particular moment. Levanyov et al. [65] generalized this flat Misner space to three and four-dimensional flat space.

For constant \( r, z \), from the metric (8) we get (finally dropping the bar)
\[
ds^2 = r(-2c_0dT d\psi - T d\psi^2) = \Omega ds^2_{\text{Misner}},
\]
a conformal Misner space metric, where \( \Omega = r \) is the conformal factor, and we have performed the transformation \( T \to c_0^2 T \) and \( \psi \to c_0 \psi \).

To study the Petrov classification of the solution (8), one can construct a set of tetrad vectors \( (k, l, m, \bar{m}) \) [54]. This set is given by
\[
k_\mu = (0, 0, 1, 0),
\]
\[
l_\mu = r c_0, -c_2 H(r), \frac{T}{2}, -c_1 z \),
\]
\[
m_\mu = \sqrt{\frac{r}{2}} (0, 1, 0, i),
\]
\[
\bar{m}_\mu = \sqrt{\frac{r}{2}} (0, 1, 0, -i),
\]
where \( i = \sqrt{-1} \). The above set of tetrad vectors is such that the metric tensor for the line element (8) can be expressed as
\[
g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu,
\]
where \( -k_\mu l_\mu = m_\mu \bar{m}_\mu = 1 \), others are all vanishing.

Using the set of null tetrad vectors (39), we have calculated the five Weyl scalars. These are given by
\[
\Psi_0 = \Psi_1 = 0 = \Psi_2 = \Psi_3,
\]
\[ \Psi_4 = -\frac{1}{4c_0} [c_1 - c_2 H'(r)]. \]  

Physically, the nonzero Weyl scalars \( \Psi_4 \) corresponds to transverse wave components propagating along the principal null direction \( k \) of multiplicity 4. Therefore the null vector (39) is aligned with the principal null directions (PND) along which the radiation propagates. In addition, the Weyl tensor \( C_{\mu \nu \rho \sigma} \) satisfies the following Bel criterion

\[ C_{\mu \nu \rho \sigma} k^\sigma = 0. \]  

Thus the metric (8) is of type N in the Petrov classification scheme. The complex scalar quantities \( \Phi_{AB} = \Phi^A_B, A, B = 0, 1, 2 \) associated with the Ricci tensor \( R_{\mu \nu} \) are

\[
\begin{align*}
\Phi_{02} &= \frac{1}{4} R_{\mu \nu} m^\mu m^\nu - \frac{1}{2} \rho, \\
\Phi_{11} &= \frac{1}{4} R_{\mu \nu} (k^\mu l^\nu + m^\mu m^\nu) - \frac{1}{4} \rho, \\
\Phi_{22} &= \frac{1}{2} R_{\mu \nu} l^\mu l^\nu \\
&= \frac{1}{4c_0} \left[ c_1 + c_2 \left\{ \frac{H(r)}{r} + H'(r) \right\} \right],
\end{align*}
\]  

while the others are all vanishing. The null vector \( k_\mu \) satisfies the geodesics condition \( k_{\mu;\nu} k^{\nu} = 0 \) with

\[
\Theta = \frac{1}{2} k^\mu;\mu = 0,
\]

\[
\omega^2 = \frac{1}{2} k^\mu;\nu k_{\mu;\nu} = 0,
\]

\[
|\sigma|^2 = \frac{1}{2} k^\mu;\nu k_{\mu;\nu} - \Theta^2 = 0.
\]

But the null vector field \( k \) is not a covariantly constant vector field (CCNV), i.e.,

\[ k_{\mu;\nu} \neq 0. \]

That means the space-time (8) admits a non-expanding, non-twisting, shear-free null vector field which is geodesic. This geodesic null congruence is not formed by a covariantly constant null vector field, and therefore, the space-time under study exhibits geometrically different properties than the known \( pp \)-waves or plane waves.

3. CONCLUSIONS

In this paper, a family of type N exact nonvacuum solution to the Einstein's field equations with zero cosmological constant, satisfying different energy conditions, has been studied. This family of solutions is regular everywhere except on the symmetry axis, where it possesses a naked curvature singularity. The stress-energy tensor of this solution is that of an anisotropic fluid coupled with a pure radiation field, and it satisfies different energy conditions. The physical parameters such as the fluid energy-density \( \rho \), the radial pressure \( p_r \) and the tangential pressures \( p_t \) diverge on the symmetry axis \( r = 0 \). The radiation energy-density may be constant or a function of the spatial coordinate \( r \) for the chosen function \( H(r) \) and satisfied the null energy condition.

A few type N and type O solutions are presented in Examples 1–4, which are special subcases of the solution under study. In Subsection 2.2, we have studied the geodesic completeness of the type N solution. We have seen that this solution is radially geodetically complete in the \( z = \text{const} \) plane, provided that \( c_5 > 0 \). Also, we have studied the strength of the naked singularity which is present due to the divergence of the scalar curvature invariants constructed from the Riemann tensor \( R_{\mu \nu \rho \sigma} \) and/or divergence of the fluid energy-density \( (\rho) \). We have shown that the naked singularity satisfies neither the strong curvature condition according to Tipler [61] nor the limiting focusing condition according to Krolak [62], provided we assume the arbitrary parameter \( c_5 > 0 \), otherwise only the strong curvature condition holds.

In Subsection 2.3, we have shown that the solution under study is a four-dimensional generalization of the Misner space metric in curved space-time, admitting closed timelike curves which appear after a certain instant of time, thus making a time-machine space-time. Furthermore, by constructing a set of null tetrad vectors, we have classified the solution under study. We have seen that the nonzero Weyl scalar is only \( \Psi_4 \neq 0 \), while all the rest vanish. Also, the Weyl tensor satisfies the Bel criterion, namely, \( C_{\mu \nu \rho \sigma} k^\sigma = 0 \), where \( k^\mu \) is the null vector field aligned with the principal null directions (PNDs) along which the radiation propagates. Thus we have confirmed that the solution is clearly of type N in the Petrov classification scheme. The studied family of type N solutions admits a non-expanding, non-twisting, and shear-free geodesic null congruence. But this shear-free null geodesic vector field is not covariantly constant, and thus the rays of the gravitational waves are not parallel, i.e., our type N solution exhibits geometrically different properties as compared to the known \( pp \)-waves or plane waves.

The known special class of type N Kundt metrics with or without a cosmological constant consists of either \( pp \)-waves or plane-waves and/or different from both of these (e.g., the Siklos solution and its subcases). In this study, we have seen that our family of type N nonvacuum space-times exhibits geometrically different properties from the known \( pp \)-waves and belongs to a special class of type N Kundt metrics. Besides, this family of type N solutions is a
four-dimensional generalization of the Misner space metric in curved space-time and behaves as a time machine in the sense that this space-time admits closed timelike curves which appear after a certain instant of time. Therefore the type N non-vacuum solution under study also is different from the known class of Siklos solutions. The family of non-twisting type N non-vacuum solutions under study, with zero cosmological constant, violates the causality condition and forms a special class of type N Kundt metrics. Furthermore, the time-dependent metric studied here, being a four-dimensional generalization of the Misner space in curved space-time, also represents a cosmic time machine.

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REFERENCES

1. F. Ahmed, Prog. Theor. Exp. Phys. 2019, 013E03 (2019).
2. F. Ahmed, Eur. Phys. J. C 78, 385 (2018).
3. A. Garcia Diaz and J. F. Plebanski, J. Math. Phys. 22, 2655 (1981).
4. I. Oszváth, I. Robinson, and K. Rozga, J. Math. Phys. 26, 1775 (1985).
5. J. Bičák and J. Podolský, J. Math. Phys. 40, 4495 (1999).
6. J. Bičák and J. Podolský, J. Math. Phys. 40, 4506 (1999).
7. J. Podolský, Ph.D. thesis, Department of Theoretical Physics, Charles University, Prague (1993).
8. N. Van den Bergh, E. Gunzig, and P. Nardone, Class. Quantum Grav. 7, L175 (1990).
9. H. Salazar, A. Garcia Diaz, and J. F. Plebanski, J. Math. Phys. 24, 2191 (1983).
10. J. Podolský and M. Ortaggio, Class. Quantum Grav. 20, 1685 (2003).
11. W. Kundt, Z. Phys. 163, 77 (1961).
12. W. Kundt, Proc. R. Soc. A 270, 328 (1962).
13. D. Sarma, M. Patgiri, and F. Ahmed, Gen. Rel. Grav. 46, 1633 (2014).
14. F. Ahmed, Ann. Phys. (N. Y.) 386, 25 (2017).
15. F. Ahmed, Prog. Theor. Exp. Phys. 2017, 043E02 (2017).
16. F. Ahmed, Theor. Math. Phys. 195, 916 (2018).
17. D. Sarma, M. Patgiri, and F. Ahmed, Ann. Phys. 329, 179 (2013).
18. S. T. C. Siklos, in Galaxies, Axisymmetric Systems and Relativity, (Ed. M. A. H. MacCallum Cambridge University Press, Cambridge, 1985).
19. V. R. Kaigorodov, Sov. Phys. Doklady 7, 893 (1963).
20. F. Ahmed, Ann. Phys. (N.Y.) 382, 127 (2017).
21. D. Sarma, F. Ahmed, and M. Patgiri, Adv. HEP 2016, 2546186 (2016).
22. F. Ahmed, Adv. High Energy Phys. 2017, 3587018 (2017).
23. F. Ahmed, Adv. High Energy Phys. 2017, 7943649 (2017).
24. F. Ahmed, Prog. Theor. Exp. Phys. 2017, 083E03 (2017).
25. F. Ahmed, Int. J. Geom. Meth. Mod. Phys. 15, 1850153 (2018).
26. T. A. Apostolatos and K. S. Thorne, Phys. Rev. D 46, 2435 (1992).
27. F. Echeverria, Phys. Rev. D 47, 2271 (1993).
28. S. Guttia, T. P. Singh, P. A. Sundararaj, and C. Vaz, gr-qc/0212089.
29. K. Nakao and Y. Morisawa, Class. Quantum Grav. 21, 2101 (2004).
30. K. Nakao and Y. Morisawa, Prog. Theor. Phys. 113, 73 (2005).
31. S. Gonsalves and S. Jhingan, Int. J. Mod. Phys. D 11, 1469 (2002).
32. B. C. Nolan, Phys. Rev. D 65, 104006 (2002).
33. P. R. C. T. Periera and A. Wang, Phys. Rev. D 62, 124001 (2000).
34. A. Wang, Phys. Rev. D 68, 064006 (2003).
35. A. Wang, Phys. Rev. D 72, 108501 (2005).
36. F. Ahmed and F. Rahaman, Adv. HEP. 2018, 7839619 (2018).
37. F. Ahmed and F. Rahaman, Eur. Phys. J. A 54, 52 (2018).
38. F. Ahmed, F. Rahaman, and S. Sarkar, Eur. Phys. J. A 54, 224 (2018).
39. F. Ahmed, Eur. Phys. J. C 79, 493 (2019).
40. T. Chiba, Prog. Theor. Phys. 95, 321 (1996).
41. K. Nakao, Phys. Rev. D 71, 124007 (2005).
42. T. Piran, Phys. Rev. Lett. 41, 1085 (1978).
43. J. M. M. Senovilla and R. Vera, Class. Quantum Grav. 17, 2843 (2000).
44. H. Bondi, Proc. R. Soc. Lond. A 427, 259 (1990).
45. K. S. Thorne, Phys. Rev. 138, B251 (1965).
46. T. A. Morgan, Gen. Rel. Grav. 4, 273 (1973).
47. P. S. Letelier and A. Wang, Phys. Rev. D 49, 5105 (1994).
48. M. A. Melvin, Phys. Lett. 8, 65 (1964).
49. M. A. Melvin, Phys. Rev. 139, B225 (1965).
50. R. Penrose, Rivista del Nuovo Cimento 1, 252 (1969).
51. R. Penrose, Singularities and time-asymetry, in General Relativity: An Einstein Centenary Survey (Ed. S. W. Hawking et al., Cambridge University Press, Cambridge, 1979), pp. 581–638.
52. R. Penrose, The Question of Cosmic Censorship, in Black Holes and Relativistic Stars (Ed. R. M. Wald, Chicago University Press, Chicago, 1994).
53. A. Ori, Phys. Rev. Lett. 95, 021101 (2005).
54. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions to Einstein's Field Equations* (Cambridge University Press, Cambridge, 2005).

55. W. B. Bonnor, Gen. Rel. Grav. 24, 551 (1992).

56. W. B. Bonnor, J. B. Griffiths, and M. A. H. MacCallum, Gen. Rel. Grav. 26, 687 (1994).

57. M. F. A. da Silva, L. Herrera, F. M. Paiva and N. O. Santos, J. Math. Phys. 36, 3625 (1995).

58. S. Haggag and F. Dsokey, Class. Quantum Grav. 13, 3221 (1996).

59. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

60. Y. Kurita and K. Nakao, Phys. Rev. D 73, 064022 (2006).

61. F. J. Tipler, Phys. Lett. A 64, 8 (1977).

62. A. Krolak, J. Math. Phys. 28, 138 (1987).

63. C. J. S. Clarke and A. Krolak, J. Geom. Phys. 2, 127 (1986).

64. C. W. Misner, in *Relativity Theory and Astrophysics I: Relativity and Cosmology* (Ed. J. Ehlers, Lectures in Applied Mathematics, vol. 8, American Mathematical Society, Providence, 1967).

65. D. Levanony and A. Ori, Phys. Rev. D 83, 044043 (2011).