Global weak solutions for a periodic two-component $\mu$-Hunter–Saxton system

Jingjing Liu · Zhaoyang Yin

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Abstract This paper is concerned with global existence of weak solutions for a periodic two-component $\mu$-Hunter–Saxton system. We first derive global existence for strong solutions to the system with smooth approximate initial data. Then, we show that the limit of approximate solutions is a global weak solution of the two-component $\mu$-Hunter–Saxton system.

Keywords A periodic two-component $\mu$-Hunter–Saxton system · Weak solutions · Global existence · Approximate solutions

Mathematics Subject Classification (2000) 35G25 · 35L05
1 Introduction

Recently, a new 2-component system was introduced by Zuo in [36] as follows:

\[
\begin{align*}
\mu(u)_t - u_{txx} &= 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x - \gamma u_{xxx}, \\
\rho_t &= (\rho u)_x + \gamma \rho_x, \\
u(0, x) &= u_0(x), \\
\rho(0, x) &= \rho_0(x), \\
u(t, x + 1) &= u(t, x), \\
\rho(t, x + 1) &= \rho(t, x),
\end{align*}
\]

where \(\mu(u) = \int_{\mathbb{R}/\mathbb{Z}} u \, dx\) with \(\mathbb{S} = \mathbb{R}/\mathbb{Z}\) and \(\gamma \in \mathbb{R}\). By integrating both sides of the first equation in the system (1.1) over the circle \(\mathbb{S} = \mathbb{R}/\mathbb{Z}\) and using the periodicity of \(u\), one obtain

\[
\mu(u_t) = \mu(u)_t = 0.
\]

This yields the following periodic 2-component \(\mu\)-Hunter–Saxton system:

\[
\begin{align*}
-u_{txx} &= 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \rho\rho_x - \gamma u_{xxx}, \\
\rho_t &= (\rho u)_x + \gamma \rho_x, \\
u(0, x) &= u_0(x), \\
\rho(0, x) &= \rho_0(x), \\
u(t, x + 1) &= u(t, x), \\
\rho(t, x + 1) &= \rho(t, x),
\end{align*}
\]

with \(\gamma \in \mathbb{R}\). This system is a 2-component generalization of the generalized Hunter–Saxton equation obtained in [23]. The author [36] shows that this system is both a bihamiltonian Euler equation and a bivariational equation. Moreover, the geometric background of the system (1.1) has been comprehensively studied by Escher in [11] recently.

For \(\rho \equiv 0\) and \(\gamma = 0\), and replacing \(t\) by \(-t\), the system (1.2) reduces to the generalized Hunter–Saxton equation (named \(\mu\)-Hunter–Saxton equation or \(\mu\)-Camassa–Holm equation) as follows [23]:

\[
-u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx},
\]

where \(\mu(u) = \int_{\mathbb{S}} u \, dx\) denotes the mean of \(u\). Moreover, the periodic \(\mu\)-Hunter–Saxton equation and the periodic \(\mu\)-Degasperis–Procesi equation have also been studied in [12,15,24] recently. It is worthy to note that the \(\mu\)-Hunter–Saxton equation lies mid-way between the periodic Hunter–Saxton and Camassa–Holm equations. For \(\mu(u) = 0\), the Eq. (1.3) reduces to the Hunter–Saxton equation [17]

\[
u_{txx} + 2u_xu_{xx} + uu_{xxx} = 0,
\]
modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. In the Hunter–Saxton equation [17], \( x \) is the space variable in a reference frame moving with the linearized wave velocity, \( t \) is a slow-time variable and \( u(t, x) \) is a measure of the average orientation of the medium locally around \( x \) at time \( t \). More precisely, the orientation of the molecules is described by the field of unit vectors \((\cos u(t, x), \sin u(t, x))\) [32]. The single-component model also arises in a different physical context as the high-frequency limit [9,18] of the Camassa–Holm equation for shallow water waves [3,7,22] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [6] with a bi-Hamiltonian structure [14] which is completely integrable [8]. The Hunter–Saxton equation also has a bi-Hamiltonian structure [22,28] and is completely integrable [1,18]. The initial value problem for the Hunter–Saxton equation (1.4) on the line (nonperiodic case) and on the unit circle \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \) were studied by Hunter and Saxton in [17] using the method of characteristics and by Yin in [32] using Kato semigroup method, respectively. Moreover, global dissipative and conservative weak solutions for the initial boundary value problem of the Hunter–Saxton equation on the half line were investigated extensively, c.f. [2,19,20,33–35].

For \( \rho \neq 0, \gamma = 0, \mu(u) = 0 \) and replacing \( t \) by \(-t\), the system (1.2) reduces to the two-component periodic Hunter–Saxton system, which is a short wave limit of the two-component periodic Camassa–Holm system. In the two-component periodic Camassa–Holm system, the variable \( u(t, x) \) describes the horizontal velocity of the fluid and the variable \( \rho(t, x) \) is in connection with the horizontal deviation of the surface from equilibrium. More presentation and study of the two-component periodic Hunter–Saxton system can be found in [5,30,31]. Moreover, the Cauchy problem and global weak solutions of two-component periodic Hunter–Saxton system have been discussed in [26,30] and [16] respectively. Furthermore, for a hydrodynamical justification of the two component system one can refer to [21].

Furthermore, the Cauchy problem of (1.2) has been discussed in [27] recently. The authors established the local well-posedness, derived the precise blow-up scenario for the system (1.2) and proved that the system (1.2) has global strong solutions and also finite time blow-up solutions. However, the existence of global weak solutions to the system (1.2) has not been studied yet. The aim of this paper is to present a global existence result of weak solutions to the system (1.2).

The main result of this paper is to give the existence of a global-in-time weak solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to the problem (1.2) with the initial \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in W^{1,\infty}(\mathbb{S}) \times L^\infty(\mathbb{S}) \).

Before giving the precise statement of the main result, we first introduce the definition of weak solution to the problem (1.2).

**Definition 1.1** \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \in C(\mathbb{R}_+; H^1(\mathbb{S}) \times L^2(\mathbb{S})) \) is said to be an admissible weak solution to the problem (1.2) if

\[
z(t, x) \in L^\infty(\mathbb{R}_+; H^1(\mathbb{S}) \times L^2(\mathbb{S}))
\]
satisfies the system (1.2) and \( z(t, \cdot) \rightarrow z_0 \) as \( t \rightarrow 0^+ \) in the sense of distributions on \( \mathbb{R}_+ \times \mathbb{R} \). Moreover,

\[
\|u_x(t, \cdot)\|_{L^2(S)} + \|\rho(t, \cdot)\|_{L^2(S)} \leq \|u_{0x}\|_{L^2(S)} + \|\rho_0\|_{L^2(S)}.
\]

The main result of this paper can be stated as follows:

**Theorem 1.1** Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in W^{1,\infty}(S) \times L^\infty(S) \). If there exists an \( \alpha > 0 \) such that \( \rho_0(x) \geq \alpha \) for a.e. \( x \in S \), then the system (1.2) has an admissible weak solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \in C(\mathbb{R}_+; H^1(S) \times L^2(S)) \cap L^\infty(\mathbb{R}_+; H^1(S) \times L^2(S)) \)

in the sense of Definition 1.1 satisfies \( \mu(u)_t = 0 \). Furthermore,

\( u \in L^\infty_{loc}(\mathbb{R}_+; W^{1,\infty}(S)) \) and \( \rho \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(S)) \).

**Remark 1.1** If there exists an \( \alpha < 0 \) such that \( \rho_0(x) \leq \alpha \) for a.e. \( x \in S \), then the conclusions in Theorem 1.1 also hold.

The paper is organized as follows. In Sect. 2, we recall some useful lemmas and derive some priori estimates on global strong solutions to (1.2). In Sect. 3, we obtain the global existence of approximate solutions to (1.2) with smooth approximate initial data. In Sect. 4, acquiring the precompactness of approximate solutions, we prove the existence of the global weak solutions to (1.2).

## 2 Preliminaries

Since (1.1) is equivalent to (1.2) under the condition \( \mu(u)_t = \mu(u)_x = 0 \), to obtain the existence of global weak solutions to (1.2) we study (1.1) henceforth. Moreover, for the sake of convenience, we let

\[
\mu_0 = \mu(u_0) = \mu(u) = \int_S u(t, x)dx.
\]

We now provide the framework in which we shall reformulate the system (1.1). We rewrite the system (1.1) as follows:

\[
\begin{aligned}
&u_t - (u + \gamma)u_x = \partial_x A^{-1}(2\mu_0u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), t > 0, x \in \mathbb{R}, \\
&\rho_t - (\rho u)_x = \gamma \rho_x, \\
&u(0, x) = u_0(x), \\
&\rho(0, x) = \rho_0(x), \\
&u(t, x + 1) = u(t, x), t \geq 0, x \in \mathbb{R}, \\
&\rho(t, x + 1) = \rho(t, x), t \geq 0, x \in \mathbb{R},
\end{aligned}
\]

(2.1)
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where $A = \mu - \partial_x^2$ is an isomorphism between $H^s(\mathbb{S})$ and $H^{s-2}(\mathbb{S})$ with the inverse $v = A^{-1}w$ given explicitly by

$$
v(x) = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\frac{1}{0}\int_0^y w(s)dsdy
- \int_0^x \int_0^y w(s)dsdy + \frac{1}{0}\int_0^y \int_0^s w(r)drdsdy,
$$

(2.2)

which can be found in [12]. Since $A^{-1}$ and $\partial_x$ commute, the following identities hold

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)dx - \int_0^x w(x)dy + \int_0^x w(y)dydx,
$$

(2.3)

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)dx.
$$

(2.4)

If we rewrite the inverse of the operator $A = \mu - \partial_x^2$ in terms of a Green’s function, we find $(A^{-1}m)(x) = \int_0^1 g(x-x')m(x')dx' = (g*m)(x)$. So, we get another equivalent form:

$$\begin{align*}
\begin{cases}
u_t - (u + \gamma)\nu_x = \partial_x g * (2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), & t > 0, \ x \in \mathbb{R}, \\
\rho_t - (\rho u)_x = \gamma\rho_x, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}, \\
\rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
u(t, x + 1) = u(t, x), & t \geq 0, \ x \in \mathbb{R}, \\
\rho(t, x + 1) = \rho(t, x), & t \geq 0, \ x \in \mathbb{R},
\end{cases}
\end{align*}
$$

(2.5)

where the Green’s function $g(x)$ is given [24] by

$$g(x) = \frac{1}{2}x(x - 1) + \frac{13}{12} \quad \text{for} \ x \in [0, 1)
$$

(2.6)

and is extended periodically to the real line. In other words,

$$g(x - x') = \frac{(x - x')^2}{2} - \frac{|x - x'|}{2} + \frac{13}{12}, \quad x, x' \in [0, 1).
$$

In particular, $\mu(g) = 1$. 

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Lemma 2.1 [4, 15] If \( f \in H^1(S) \) is such that \( \int_S f(x) \, dx = 0 \), then we have

\[
\max_{x \in S} f^2(x) \leq \frac{1}{12} \int_S f^2(x) \, dx.
\]

Assume that \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) is a smooth solution of (2.1). For convenience, we let

\[
\mu_1 = \left( \int_S (u_{0,x}^2 + \rho_0^2) \, dx \right)^{\frac{1}{2}}.
\]

Using the system (1.1), a simple calculation implies

\[
\frac{d}{dt} \int_S (u_{0,x}^2 + \rho_0^2) \, dx = 0.
\]

So we have

\[
\mu_1 = \left( \int_S (u_{0,x}^2 + \rho_0^2) \, dx \right)^{\frac{1}{2}} = \left( \int_S (u_x^2 + \rho^2) \, dx \right)^{\frac{1}{2}}.
\] (2.7)

Note that \( \int_S (u(t, x) - \mu_0) \, dx = \mu_0 - \mu_0 = 0 \). By Lemma 2.1, we find that

\[
\max_{x \in S} [u(t, x) - \mu_0]^2 \leq \frac{1}{12} \int_S u_x^2(t, x) \, dx \leq \frac{1}{12} \mu_1^2.
\]

So we have

\[
\|u(t, \cdot)\|_{L^\infty(S)} \leq |\mu_0| + \frac{\sqrt{3}}{6} \mu_1.
\] (2.8)

Lemma 2.2 [27] Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2(S) \times H^1(S) \). If \( \rho_0(x) \neq 0 \) for all \( x \in S \), then the corresponding strong solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (1.1) exists globally in time, i.e. \( z \in C(\mathbb{R}^+; H^2(S) \times H^1(S)) \cap C^1(\mathbb{R}^+; H^1(S) \times L^2(S)) \). Moreover, there exists \( \beta > 0 \) such that for all \( t \in \mathbb{R}^+ \),

\[
\|u_x(t, \cdot)\|_{L^\infty(S)} \leq \frac{1}{2\beta} (1 + \|\rho_0\|_{L^\infty(S)}^2 + \|u_{0,x}\|_{L^\infty(S)}^2) \cdot e^{(4\mu_0^2 + \frac{1}{2} \mu_1^2 + \frac{\sqrt{3}}{2} |\mu_0| \mu_1 + \frac{1}{2})t} := C_1(t)
\]

and

\[
\|\rho(t, \cdot)\|_{L^\infty(S)} \leq e^{C_1(t)t} \|\rho_0\|_{L^\infty(S)} := C_2(t),
\]

where \( \beta = \inf_{x \in S} |\rho_0(x)| \).
Lemma 2.3 [29] Let \( X, B \) and \( Y \) are Banach spaces, \( X \subset B \subset Y \) with compact imbedding \( X \to B \), \( 1 \leq p \leq \infty \). Assume that

1. \( F \) is bounded in \( L^p(0, T; X) \),

2. \( \| f(\cdot + h) - f(\cdot) \|_{L^p(0, T-h; Y)} \to 0 \) as \( h \to 0 \) uniformly for \( f \in F \).

Then \( F \) is relatively compact in \( L^p(0, T; B) \) (and in \( C([0, T]; B) \) if \( p = \infty \)).

Lemma 2.4 (Appendix C of [25]) Let \( X \) be a separable reflexive Banach space, \( Y \) is a Banach space such that \( X \hookrightarrow Y \), \( Y' \) is separable and dense in \( X' \). Assume that for some \( T \in (0, \infty) \)

1. \( f^n \in L^\infty(0, T; X) \cap C([0, T]; Y) \), \( \| f^n \|_{L^\infty(0, T; X)} \leq C, \forall n \geq 1 \),

2. \( \forall \phi \in Y', (\phi, f^n(t))_{Y' \times Y} \) is uniformly continuous in \( t \in [0, T] \) and uniformly in \( n \geq 1 \).

Then \( f^n \) is relatively compact in \( C^w([0, T]; X) \), the space of continuous functions from \([0, T]\) with values in \( X \) when the latter space is equipped with its weak topology.

Remark 2.1 If the conditions which \( f^n \) satisfies in Lemma 2.4 are replaced by the following conditions:

\[ f^n \in L^\infty(0, T; X), \quad \partial_t f^n \in L^p(0, T; Y) \text{ for some } p \in (1, \infty], \]

and

\[ \| f^n \|_{L^\infty(0, T; X)}, \quad \| \partial_t f^n \|_{L^p(0, T; Y)} \leq C, \quad \forall n \geq 1, \]

then the conclusion of Lemma 2.4 holds true.

3 Global approximate solutions

In the section, we first prove the existence of approximate solutions. Then, with the basic estimates given in Sect. 2, we will give some useful estimates to the approximate solutions.

Let \( z_0 = \begin{pmatrix} u_0^n \cr \rho_0^n \end{pmatrix} \in W^{1,\infty}(\mathbb{S}) \times L^\infty(\mathbb{S}) \) and there exists an \( \alpha > 0 \) such that \( \rho_0(x) \geq \alpha \) for a.e. \( x \in \mathbb{S} \). Define \( z^n_0 := \begin{pmatrix} \phi_n * u_0^n \cr \phi_n * \rho_0^n \end{pmatrix} = \begin{pmatrix} u_0^n \cr \rho_0^n \end{pmatrix} \in H^2(\mathbb{S}) \times H^1(\mathbb{S}), \) for \( n \geq 1, \) here \( \{\phi_n\}_{n \geq 1} \) are the mollifiers.

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\[
\phi_n(x) := \left( \int_{\mathbb{R}} \phi(\xi)d\xi \right)^{-1} n\phi(nx), \quad x \in \mathbb{R}, \ n \geq 1,
\]

where \( \phi \in C_c^\infty(\mathbb{R}) \) is defined by
\[
\phi(x) = \begin{cases} 
e^{-1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}
\]

In view of \( \rho_0(x) \geq \alpha > 0 \), for a.e. \( x \in \bar{S} \) and \( \phi_n(x) \geq 0 \), we have
\[
\rho_n^0(x) = \phi_n * \rho_0(x) \geq \alpha \int \phi_n(y)dy = \alpha > 0, \ \forall x \in \bar{S}.
\]
Clearly, we also have
\[
u_0^n \to u_0 \text{ in } H^1(\bar{S}), \quad \rho_0^n \to \rho_0 \text{ in } L^2(\bar{S}), \quad \text{as } n \to \infty \quad (3.1)
\]
and
\[
\|u_0^n\|_{L^2(\bar{S})} \leq \|u_0\|_{L^2(\bar{S})}, \quad \|u_{0,x}^n\|_{L^2(\bar{S})} \leq \|u_{0,x}\|_{L^2(\bar{S})}, \quad \|\rho_0^n\|_{L^2(\bar{S})} \leq \|\rho_0\|_{L^2(\bar{S})}. \quad (3.2)
\]
Thus, we obtain the corresponding solution \( z^n \in C(\mathbb{R}_+; H^2(\bar{S}) \times H^1(\bar{S})) \cap C(\mathbb{R}_+; H^1(\bar{S}) \times L^2(\bar{S})) \) to the system (1.1) with initial data \( z_0^n \) by Lemma 2.2 under the condition \( \mu(u^n)_t = \mu(u^n) = 0. \)

For given \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in W^{1,\infty}(\bar{S}) \times L^{\infty}(\bar{S}) \), we set
\[
\mu_0^n := \mu(u_0^n) = \mu(u^n) = \int_{\bar{S}} u^n(t,x)dx,
\]
\[
\mu_1^n := \left( \int_{\bar{S}} (u_{0,x}^n)^2 + (\rho_0^n)^2 dx \right)^{\frac{1}{2}} = \left( \int_{\bar{S}} (u_0^n)^2 + (\rho_0^n)^2 dx \right)^{\frac{1}{2}}
\]
Then we have the following remark.

**Remark 3.1** By (2.7), (2.8) and (3.2), we have
\[
\mu_0^n \to \mu_0 \text{ as } n \to \infty, \quad (\mu_1^n)^2 \to \mu_1^2 \text{ as } n \to \infty \quad (3.3)
\]
and
\[
\|u_0^n(t,\cdot)\|_{L^2(\bar{S})}^2 + \|\rho_0^n(t,\cdot)\|_{L^2(\bar{S})}^2 = (\mu_1^n)^2 \leq \mu_1^2, \quad \forall t \in \mathbb{R}_+.
\]
Moreover, we get

\[ |\mu^n_0| \leq \int_S |u^n_0| dx \leq \|u^n_0\|_{L^2(S)} \leq \|u_0\|_{L^2(S)}, \tag{3.5} \]

\[ \|u^n(t, \cdot)\|_{L^\infty(S)} \leq |\mu^n_0| + \frac{\sqrt{3}}{6} \mu^n_1 \leq \|u_0\|_{L^2(S)} + \frac{\sqrt{3}}{6} \mu_1. \tag{3.6} \]

Furthermore, by Lemma 2.2, we obtain

\[ \|u^n_x(t, \cdot)\|_{L^\infty(S)} \leq \frac{1}{2\beta} (1 + \|\rho_0\|_{L^\infty(S)} + \|u_{0,x}\|_{L^\infty(S)}) \cdot e^{(4\|u_0\|^2_{L^2(S)} + \frac{1}{3} \mu_1 + \frac{\sqrt{3}}{2} \|u_0\|_{L^2(S)} \mu_1 + \frac{1}{3} \mu_1) t := \tilde{C}_1(t)} \tag{3.7} \]

and

\[ \|\rho^n(t, \cdot)\|_{L^\infty(S)} \leq e^{\tilde{C}_1(t) T} \|\rho_0\|_{L^\infty(S)} := \tilde{C}_2(t), \tag{3.8} \]

where \( \beta = \inf_{x \in S} |\rho_0(x)|. \)

### 4 The existence of global weak solution

In this section, with the basic energy estimate in Sect. 3, we are ready to obtain the necessary compactness of the approximate solutions \( z^n(t, x) \). Acquiring the precompactness of approximate solutions, we prove the existence of the global weak solutions to the system (1.1).

**Lemma 4.1** For any fixed \( T > 0 \), there exist a subsequence \( \{z^{nk}(t, x)\} \) of the sequence \( \{z^n(t, x)\} \) and some function \( z(t, x) \in L^\infty([0, \infty); H^1(S) \times L^2(S)) \cap (H^1([0, T] \times S) \times L^2([0, T] \times S)) \) such that

\[ z^{nk} \rightharpoonup z \text{ in } H^1([0, T] \times S) \times L^2([0, T] \times S) \text{ as } n_k \to \infty, \forall T > 0, \tag{4.1} \]

and

\[ u^{nk} \to u \text{ in } L^\infty([0, T] \times S) \text{ as } n_k \to \infty. \tag{4.2} \]

Moreover, \( u(t, x) \in C([0, T]; L^\infty(S)) \).

**Proof** It follows from Remark 3.1 that \( \{z^n(t, x)\} \) is uniformly bounded in \( L^\infty([0, \infty); H^1(S) \times L^2(S)) \).

We will prove that the sequence \( \{z^n(t, x)\} \) is uniformly bounded in the space \( H^1([0, T] \times S) \times L^2([0, T] \times S) \). Firstly, we claim that \( \{u^n(t, x)\} \) is uniformly bounded in the space \( H^1([0, T] \times S) \). Using Remark 3.1, we have
\[\| (u^n + \gamma) u^n_x \|_{L^2([0, T] \times \mathbb{S})} \leq \| u^n u^n_x \|_{L^2([0, T] \times \mathbb{S})} + |\gamma| \| u^n_x \|_{L^2([0, T] \times \mathbb{S})}\]
\[\leq (\| u_0 \|_{L^2(\mathbb{S})} + \sqrt{\frac{3}{6}} \mu_1 + |\gamma|) \mu_1 \sqrt{T},\]
and
\[\| g_x \ast (2 \mu_0^2 u^n + \frac{1}{2} (u^n_x)^2 + \frac{1}{2} (\rho^n)^2) \|_{L^2([0, T] \times \mathbb{S})}\]
\[\leq \| g_x \|_{L^2([0, T] \times \mathbb{S})} \| 2 \mu_0^2 u^n + \frac{1}{2} (u^n_x)^2 + \frac{1}{2} (\rho^n)^2 \|_{L^2([0, T] \times \mathbb{S})}\]
\[\leq \frac{T}{12} \left( \int_0^T (\mu_0^n)^2 dx dt + \int_0^T (u^n)^2 dx dt + \frac{1}{2} \int_0^T (\rho^n)^2 dx dt \right)\]
\[\leq \frac{T}{12} \left[ \| u_0 \|_{L^2(\mathbb{S})}^2 T + \left( \| u_0 \|_{L^2(\mathbb{S})}^2 + \frac{\sqrt{3}}{6} \mu_1 \right)^2 + \frac{1}{2} \mu_1^2 T \right].\]

Then, by the first equation in (2.2), we know that \( \{u^n(t, x)\} \) is uniformly bounded in \( L^2([0, T] \times \mathbb{S}) \). Combining this conclusion with Remark 3.1, we obtain \( \{u^n(t, x)\} \) is uniformly bounded in the space \( H^1([0, T] \times \mathbb{S}) \). Furthermore, from Remark 3.1, one can easily obtain that \( \rho^n \) is uniformly bounded in the space \( L^2([0, T] \times \mathbb{S}) \), and thus (4.1) follows.

Observe that, for each \( 0 \leq s, t \leq T \),
\[\| u^n(t, \cdot) - u^n(s, \cdot) \|_{L^2(\mathbb{S})}^2 = \int_\mathbb{S} \left( \int_0^t \frac{\partial u^n}{\partial \tau} (\tau, x) \, d\tau \right)^2 dx \leq |t - s| \int_0^T (u^n_s)^2 dx dt.\]

Moreover, \( \{u^n(t, x)\} \) is uniformly bounded in \( L^\infty(0, T; H^1(\mathbb{S})) \) and \( H^1(\mathbb{S}) \subset L^2(\mathbb{S}) \), then (4.2) and \( u(t, x) \in C([0, T]; L^\infty(\mathbb{S})) \) is consequence of Lemma 2.3.

**Remark 4.1** By Remark 3.1 and the above argument, there exists a subsequences of \( \{ (u^n_s)^2 \, , \, (\rho^n_s)^2 \} \), denoted again by \( \{ (u^n_{s_k})^2 \, , \, (\rho^n_{s_k})^2 \} \), converging weakly in \( L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}) \), where \( 1 < p < \infty \), i.e. there exist \( u^2_{x_k} \in L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) and \( \rho^2 \in L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) such that
\[ (u^n_{s_k})^2 \rightharpoonup u^2_{x_k} \text{ and } (\rho^n_{s_k})^2 \rightharpoonup \rho^2 \text{ in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}). \]

Moreover, we have
\[ u^n_{x_k} \rightharpoonup u_x \text{ in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}) \text{ and } u^n_{x_k} \rightharpoonup^* u_x \text{ in } L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{S})), \]
\[ \rho^n_{s_k} \rightharpoonup \rho \text{ in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}) \text{ and } \rho^n_{s_k} \rightharpoonup^* \rho \text{ in } L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{S})), \]
Furthermore, we have
\[ u_x^2(t, x) \leq u_x^2(t, x), \quad \rho^2(t, x) \leq \overline{\rho}^2(t, x) \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}. \] (4.3)

In view of (3.7) and (3.8), we have
\[ \|u_x(t, \cdot)\|_{L^\infty(\mathbb{S})} \leq \overline{C}_1(t), \quad \|\rho(t, \cdot)\|_{L^\infty(\mathbb{S})} \leq \overline{C}_2(t), \quad \forall t \in \mathbb{R}_+, \] (4.4)

where \( \overline{C}_1(t) \) and \( \overline{C}_2(t) \) are given in Remark 3.1.

**Lemma 4.2** There hold
\[ \lim_{t \to 0^+} \int_\mathbb{S} u_x^2(t, x) dx = \lim_{t \to 0^+} \int_\mathbb{S} u_x^2(t, x) = \int_\mathbb{S} u_{0,x}^2(x) dx \] (4.5)

and
\[ \lim_{t \to 0^+} \int_\mathbb{S} \rho^2(t, x) dx = \lim_{t \to 0^+} \int_\mathbb{S} \rho^2(t, x) = \int_\mathbb{S} \rho_0^2(x) dx. \] (4.6)

**Proof** By Lemma 4.1, for any \( T > 0 \), we have \( u^n \in L^\infty(0, T; H^1(\mathbb{S})) \), \( u^n_t \) are uniformly bounded in \( L^\infty(0, T; L^2(\mathbb{S})) \). By Lemma 2.2, we have \( u^n \in C([0, T]; H^1(\mathbb{S})) \).

Then in view of Lemma 2.4, Remark 2.1 and the proof of Lemma 4.1, we have \( \{u^n\} \) contains a subsequence, denoted again by \( \{u^{nk}\} \), converging weakly in \( H^1(\mathbb{S}) \) uniformly in \( t \in [0, T] \). The limit function is \( u \). This implies that \( u \) is weakly continuous from \( [0, T] \) into \( H^1(\mathbb{S}) \), i.e.,
\[ u \in C^w([0, T]; H^1(\mathbb{S})). \] (4.7)

Similarly, as \( \rho^n \in L^\infty(0, T; L^2(\mathbb{S})) \), in view of (3.4) and (3.6), we get that for all \( t \in [0, T] \),
\[ \|\rho^n_t(t, \cdot)\|_{H^{-1}(\mathbb{S})} = \sup_{\|\phi\|_{H^1(\mathbb{S})}=1} \int_\mathbb{S} ((u^n_x \rho^n) \phi + \gamma \rho^n \phi) dx \]
\[ \leq \|u^n \rho^n\|_{L^2(\mathbb{S})} + |\gamma| \|\rho^n\|_{L^2(\mathbb{S})} \]
\[ \leq (\|u_0\|_{L^2(\mathbb{S})} + \sqrt{3}/6 \mu_1 + |\gamma|) \mu_1. \]

This shows that \( \rho^n_t \) is uniformly bounded in \( L^\infty(0, T; H^{-1}(\mathbb{S})) \). Then by Lemma 2.4 and Remark 2.1, we have \( \{\rho^n\} \) contains a subsequence, denoted again by \( \{\rho^{nk}\} \), converging weakly in \( L^2(\mathbb{S}) \) uniformly in \( t \). The limit function is \( \rho \). This implies that \( \rho \) is weakly continuous from \( [0, T] \) into \( L^2(\mathbb{S}) \), i.e.,
\[ \rho \in C^w([0, T]; L^2(\mathbb{S})). \] (4.8)
Then by (4.7) and (4.8), we get
\[
\rho(t, \cdot) \rightharpoonup \rho_0 \quad \text{and} \quad u_x(t, \cdot) \rightharpoonup u_{0,x} \quad \text{in} \quad L^2(S) \quad \text{as} \quad t \to 0^+.
\]
Thus, we have
\[
\liminf_{t \to 0^+} \int_S \rho^2(t, x) \, dx \geq \int_S \rho_0^2(x) \, dx
\]
and
\[
\liminf_{t \to 0^+} \int_S u_{x}^2(t, x) \, dx \geq \int_S u_{0,x}^2(x) \, dx.
\]
Therefore, we deduce
\[
\liminf_{t \to 0^+} \int_S (u_{x}^2(t, x) + \rho^2(t, x)) \, dx
\]
\[
\geq \liminf_{t \to 0^+} \int_S u_{x}^2(t, x) \, dx + \liminf_{t \to 0^+} \int_S \rho^2(t, x) \, dx
\]
\[
\geq \int_S (u_{0,x}^2(x) + \rho_0^2(x)) \, dx. \tag{4.9}
\]
Moreover, by Remark 4.1 and (3.2), we infer
\[
\int_S \left( \overline{(u_{x}^n(t, x) + \rho^n(t, x))} \right) \, dx
\]
\[
\leq \liminf_{n_k \to \infty} \int_S \left( (u_{x}^n)^2(t, x) + (\rho^n)^2(t, x) \right) \, dx
\]
\[
= \liminf_{n_k \to \infty} \int_S \left( (u_{0,x}^n)^2(x) + (\rho_0^n)^2(x) \right) \, dx
\]
\[
\leq \int_S (u_{0,x}^2(x) + \rho_0^2(x)) \, dx.
\]
Thus, we obtain
\[
\limsup_{t \to 0^+} \int_S \left( \overline{(u_{x}^2(t, x) + \rho^2(t, x))} \right) \, dx \leq \int_S (u_{0,x}^2(x) + \rho_0^2(x)) \, dx. \tag{4.10}
\]
In view of (4.3), (4.9) and (4.10), we get
\[
\liminf_{t \to 0^+} \int_S (u^2_x(t, x) + \rho^2(t, x))dx \geq \int_S (u^2_{0,x}(x) + \rho^2_0(x))dx \\
\geq \limsup_{t \to 0^+} \int_S (\overline{u^2_x}(t, x) + \overline{\rho^2}(t, x))dx \\
\geq \limsup_{t \to 0^+} \int_S (u^2_x(t, x) + \rho^2(t, x))dx.
\]

This completes the proof of the lemma. \(\square\)

**Lemma 4.3** There holds
\[
\frac{\partial}{\partial t} (u^2_x + \rho^2) - \frac{\partial}{\partial x} \left[ (u + \gamma)(u^2_x + \rho^2) \right] = -4\mu_0 uu_x + 4\mu_0^2 u_x + u_x \mu_1^2
\]
in the sense of distributions on \([0, T] \times \mathbb{R}\).

**Proof** Note that \(z^{nk}\) is the solution of the system (2.1) with the initial \(z^{nk}_0\). Differentiating the first equation in (2.1) with respect \(x\) and using \(\partial^2_x A^{-1} w = -w + \mu(w)\), we have
\[
u^n - \mu^n + \gamma \mu^n x = -2\mu_0^n u^n + \frac{1}{2}(u^n_x)^2 - \frac{1}{2}(\rho^n)^2 + 2(\mu_0^n)^2 + \frac{1}{2}(\mu_1^n)^2.
\]
Then
\[
\frac{\partial}{\partial t} (u^n_x)^2 - \frac{\partial}{\partial x} ((u^n + \gamma)(u^n_x)^2) = -4\mu_0^n u^n_x u^n_k - u^n_k (\rho^n_k)^2 \\
+4(\mu_0^n)^2 u^n_x + u^n_k (\mu_1^n)^2
\]
and
\[
\frac{\partial}{\partial t} (\rho^n_k)^2 - \frac{\partial}{\partial x} ((u^n + \gamma)(\rho^n_k)^2) = u^n_k (\rho^n_k)^2.
\]
Adding the above two equalities and letting \(n_k \to \infty\), in view of Lemma 4.1 and Remark 4.1, we get (4.11). \(\square\)

**Lemma 4.4** There holds
\[
\frac{\partial}{\partial t} (u^2_x + \rho^2) - \frac{\partial}{\partial x} \left[ (u + \gamma)(u^2_x + \rho^2) \right] = u^3_x - 4\mu_0 uu_x - u_x \rho^2 - u_x \rho^2 + 4\mu_0^2 u_x + \mu_1^2 u_x + u_x \rho^2
\]
in the sense of distributions on \([0, T] \times \mathbb{R}\).
Proof Since \( z^n_k \) satisfy
\[
 u^n_t - ((u^n + \gamma) u^n_x)_x = -2\mu^0 u^n \frac{u^n}{u^n_x} - \frac{1}{2} (u^n_x)^2 - \frac{1}{2} (\rho^n)^2 + 2(\mu^0)^2 + \frac{1}{2} (\mu^1)^2
\]
and
\[
 \rho^n_t - (\rho^n u^n)_x = \gamma \rho^n_x.
\]
In view of Lemma 4.1 and Remark 4.1, letting \( n_k \rightarrow \infty \), we obtain
\[
 u^n_t - ((u + \gamma) u^n)_x = -2\mu^0 u - \frac{1}{2} u^n_x^2 - \frac{1}{2} \rho^2 + 2(\mu^0)^2 + \frac{1}{2} (\mu^1)^2
\]
and
\[
 \rho^n_t - (\rho u)_x = \gamma \rho_x.
\]
in the sense of distributions on \((0, T] \times \mathbb{R})\).

Denote \( u^n_{n, x}(t, x) := ((u_{x}(t, \cdot) * \phi_n)(x) \) and \( \rho^n_{n, x}(t, x) := (\rho(t, \cdot) * \phi_n)(x) \). According to Lemma II.1 of [10], it follows from (4.13) and (4.14) that \( u^n_{n, x} \) and \( \rho^n_{n, x} \) solve
\[
 \frac{\partial u^n_{n, x}}{\partial t} - (u + \gamma) \frac{\partial u^n_{n, x}}{\partial x} = (u^n_x^2 - 2\mu^0 u - \frac{1}{2} u^n_x^2 - \frac{1}{2} \rho^2 + 2\mu^2 + \frac{1}{2} \mu^2_1) * \phi_n + \tau_n
\]
and
\[
 \frac{\partial \rho^n_{n, x}}{\partial t} - (u + \gamma) \frac{\partial \rho^n_{n, x}}{\partial x} = (u\rho) * \phi_n + \sigma_n,
\]
where the errors \( \tau_n = (u \frac{\partial u_{n, x}}{\partial x}) * \phi_n - u \cdot \frac{\partial u_{n, x}}{\partial x} \) and \( \sigma_n = (u \frac{\partial \rho}{\partial x}) * \phi_n - u \cdot \frac{\partial \rho_{n, x}}{\partial x} \) tend to zero in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \). Using (4.15) and (4.16), we get
\[
 \frac{\partial (u^n_{n, x})^2}{\partial t} - \frac{\partial}{\partial x} ((u + \gamma) u^n_{n, x})^2 - 2u_{n, x} \left( (u^n_x^2 - 2\mu^0 u - \frac{1}{2} u^n_x^2 - \frac{1}{2} \rho^2 + 2\mu^2 + \frac{1}{2} \mu^2_1) * \phi_n + \tau_n \right) - u_{x} u^n_{n, x}^2.
\]
and
\[
 \frac{\partial (\rho^n_{n, x})^2}{\partial t} - \frac{\partial}{\partial x} ((u + \gamma) \rho^n_{n, x})^2 = 2\rho_n ((u \rho) * \phi_n + \sigma_n) - u_{x} \rho^n_{n, x}^2.
\]
Sending \( n \rightarrow \infty \) in (4.17) and (4.18) and adding the results yield (4.12).

Next, we give the main lemma of this section.

\[ \square \]
Lemma 4.5 There hold
\[ \overline{u}_x^2(t, x) = u_x^2(t, x) \quad \text{and} \quad \overline{\rho}^2(t, x) = \rho^2(t, x) \quad \text{a.e. on} \quad [0, T] \times \mathbb{R}. \] (4.19)

Proof Subtracting (4.12) from (4.11) and integrating the obtained equality over \((\varepsilon, t) \times S\) give
\[
\int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right)(t, x) dx
= \int_{\varepsilon}^t \int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right) u_x(t, x) dx ds
+ \int_S \left( (\overline{u}_x^2(\varepsilon, x) - u_x^2(\varepsilon, x) + \overline{\rho}^2(\varepsilon, x) - \rho^2(\varepsilon, x)) \right) dx,
\]
for almost all \(t \in [0, T]\). Letting \(\varepsilon \to 0\) and using Lemma 4.2 and (4.4), we get
\[
\int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right)(t, x) dx \leq \tilde{C}_1(T) \int_{0}^{t} \int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right) dx ds.
\]
Using Gronwall’s inequality, we obtain
\[
\int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right)(t, x) dx \leq 0.
\]
By (4.3), we deduce
\[
0 \leq \int_S \left( (\overline{u}_x^2 - u_x^2 + \overline{\rho}^2 - \rho^2) \right)(t, x) dx \leq 0,
\]
which yields
\[
\int_S (\overline{u}_x^2 - u_x^2)(t, x) dx = \int_S (\overline{\rho}^2 - \rho^2)(t, x) dx = 0.
\]
In view of \(u^n_x, u_x\) and \(\rho^n, \rho\) being periodic with respect to \(x\), we deduce that (4.19) holds.

To prove our main theorem, we need the following lemma.

Lemma 4.6 [13] Let \(U\) be the bounded domain in \(\mathbb{R}^n\). Assume that the sequence \(\{f_k\}_{k=1}^\infty\) is bounded in \(L^\infty(U; \mathbb{R}^m)\). Then there exist a subsequence \(\{f_{k_j}\}_{j=1}^\infty \subset \mathbb{R}^m\).
\( \{ f_k \}_{k=1}^\infty \) and for a.e. \( x \in U \), a Borel probability measure \( \nu_x \) on \( \mathbb{R}^m \) such that for \( F \in C(\mathbb{R}^m) \) we have

\[
F(f_k) \rightharpoonup^* F \quad \text{in} \quad L^\infty(U),
\]

where \( F(x) = \int_{\mathbb{R}^m} F(y) d\nu_x \) for a.e. \( x \in U \). Moreover, if \( \nu_x \) is a unit point mass for a.e. \( x \in U \), then \( f_k \to f \) in \( L^2(U; \mathbb{R}^m) \), here \( f(x) = \int_{\mathbb{R}^m} y d\nu_x \).

By (3.7) and (3.8), for any \( T > 0 \) and \( a < b \), \( \{ u^n_k \} \) and \( \{ \rho^n_k \} \) are uniformly bounded on \( (0, T) \times (a, b) \). Using Lemma 4.6, there exist subsequences \( \{ u^n_{k_1} \} \subset \{ u^n_k \} \) and two Borel probability measures \( \mu(t, x), \nu(t, x) \) on \( \mathbb{R} \) such that for each \( F \in C(\mathbb{R}) \) we have

\[
F(u^n_{k_1}) \rightharpoonup^* F_1, \quad F(\rho^n_{k_1}) \rightharpoonup^* F_2 \quad \text{in} \quad L^\infty((0, T) \times (a, b)),
\]

where \( F_1(t, x) = \int_{\mathbb{R}} F(y) d\mu(t, x) \) and \( F_2(t, x) = \int_{\mathbb{R}} F(y) d\nu(t, x) \) for a.e. \( (t, x) \in (0, T) \times (a, b) \). By Lemma 4.5, we have that \( \mu(t, x) = \delta_{u^n(t, x)} \) and \( \nu(t, x) = \delta_{\rho^n(t, x)} \) for a.e. \( (t, x) \in (0, T) \times (a, b) \). Then, in view of Lemma 4.6 and from the arbitrariness of \( T, a, b \), we obtain

\[
\begin{align*}
\{ u^n_{k_1}, \rho^n_{k_1} \}_{k_1=1}^\infty &\subset \{ u^n_k, \rho^n_k \}_{n=1}^\infty \quad \text{for a.e.} \quad (t, x) \in (0, T) \times (a, b) \\
\end{align*}
\]

and two Borel probability measures \( \mu(t, x), \nu(t, x) \) on \( \mathbb{R} \) such that for each \( F \in C(\mathbb{R}) \) we have

\[
F(u^n_k) \rightharpoonup^* F_1, \quad F(\rho^n_k) \rightharpoonup^* F_2 \quad \text{in} \quad L^\infty((0, T) \times (a, b)),
\]

where \( F_1(t, x) = \int_{\mathbb{R}} F(y) d\mu(t, x) \) and \( F_2(t, x) = \int_{\mathbb{R}} F(y) d\nu(t, x) \) for a.e. \( (t, x) \in (0, T) \times (a, b) \). By Lemma 4.5, we have that \( \mu(t, x) = \delta_{u^n(t, x)} \) and \( \nu(t, x) = \delta_{\rho^n(t, x)} \) for a.e. \( (t, x) \in (0, T) \times (a, b) \). Then, in view of Lemma 4.6 and from the arbitrariness of \( T, a, b \), we obtain

\[
u^n \to \nu \quad \text{in} \quad L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}). \quad (4.20)
\]

With the above preparations, we now conclude the proof of the Theorem 1.1. Let \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) be the limit of the approximate solutions \( z^n_k \) as \( n_k \to \infty \). It then follows from Remark 3.1 and Lemma 4.1 that \( z \in L^\infty(\mathbb{R}^+, H^1(S) \times L^2(S)) \) holds. From (4.20), we have that

\[
\partial_x g * \left( 2\mu_0 u^n + \frac{1}{2} (u^n)^2 + \frac{1}{2} (\rho^n)^2 \right) \to \partial_x g * \left( 2\mu_0 u + \frac{1}{2} u^2 + \frac{1}{2} \rho^2 \right)
\]

in the sense of distributions on \( \mathbb{R}^+ \times \mathbb{R} \). This shows that \( z \) solves (2.5) in the sense of distributions on \( \mathbb{R}^+ \times \mathbb{R} \). Moreover, combining

\[
\int_S u^n(x) dx = \int_S u^n_0(x) dx \to \int_S u_0(x) dx = \mu_0
\]

and

\[
\int_S u^n(x) dx \to \int_S u(x) dx
\]

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as \( n \to \infty \), we know that \( u \) satisfies \( \mu(u)_t = 0 \). Furthermore, (4.4) implies

\[
u \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{1,\infty}(\mathbb{S})) \text{ and } \rho \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^\infty(\mathbb{S})).
\]

From Lemma 4.2, we have \( z \in C^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{S}) \times L^2(\mathbb{S})) \). Consequently, we will prove that \( z \in C([0, \infty); L^\infty(\mathbb{S})) \). Note that \( u \in C((0, \infty); L^\infty(\mathbb{S})) \), it is enough to show that \( \int_\mathbb{S} (u_x^2 + \rho^2)dx \) is conserved in time. Indeed, if this holds, then

\[
\|z(t) - z(s)\|_{H^1(\mathbb{S}) \times L^2(\mathbb{S})}^2
= \|u(t) - u(s)\|_{L^2(\mathbb{S})}^2 + \|u_x(t) - u_x(s)\|_{L^2(\mathbb{S})}^2 + \|\rho(t) - \rho(s)\|_{L^2(\mathbb{S})}^2
\]

\[
= \|u(t) - u(s)\|_{L^2(\mathbb{S})}^2 + \|u_x(t)\|_{L^2(\mathbb{S})}^2 + \|u_x(s)\|_{L^2(\mathbb{S})}^2 - 2(u_x(s), u_x(t))_{L^2(\mathbb{S})} \\
+ \|\rho(t)\|_{L^2(\mathbb{S})}^2 + \|\rho(s)\|_{L^2(\mathbb{S})}^2 - 2(\rho(s), \rho(t))_{L^2(\mathbb{S})}
\]

\[
= \|u(t) - u(s)\|_{L^2(\mathbb{S})}^2 + 2(\|u_0\|_{L^2(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2) \\
- 2((u_x(s), u_x(t))_{L^2(\mathbb{S})} + (\rho(s), \rho(t))_{L^2(\mathbb{S})}),
\]

\( \forall t, s \in \mathbb{R}^+ \). Since \( \|u(t) - u(s)\|_{L^2(\mathbb{S})}^2 \to 0 \) and

\[
(u_x(s), u_x(t))_{L^2(\mathbb{S})} + (\rho(s), \rho(t))_{L^2(\mathbb{S})} \to \|u_x(t)\|_{L^2(\mathbb{S})}^2 + \|\rho(t)\|_{L^2(\mathbb{S})}^2
\]

\[
= \|u_0\|_{L^2(\mathbb{S})}^2 + \|\rho_0\|_{L^2(\mathbb{S})}^2,
\]

as \( s \to t \), we have \( z \in C(\mathbb{R}^+; H^1(\mathbb{S}) \times L^2(\mathbb{S})) \).

The conservation of \( \int_\mathbb{S} (u_x^2 + \rho^2)dx \) in time is proved by a regularization technique. Denote \( f_n = f \star \phi_n \). By Lemma 4.5 and (4.15), (4.16), we have

\[
\frac{\partial u_{n,x}}{\partial t} - (u + \gamma) \frac{\partial u_{n,x}}{\partial x} = \left(\frac{1}{2} u_x^2 - 2\mu_0 u - \frac{1}{2} \rho^2 + 2\mu_0^2 + \frac{1}{2} \mu_1^2\right) \star \phi_n + \tau_n \quad (4.21)
\]

and

\[
\frac{\partial \rho_n}{\partial t} - (u + \gamma) \frac{\partial \rho_n}{\partial x} = (u_x \rho) \star \phi_n + \sigma_n, \quad (4.22)
\]

Multiplying (4.21) with \( u_{n,x} \), we obtain by integration

\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{S} u_{n,x}^2 dx - \frac{1}{2} \int_\mathbb{S} (u + \gamma)(u_{n,x})_x dx
\]

\[
= \int_\mathbb{S} u_{n,x} \left(\frac{1}{2} u_x^2 - 2\mu_0 u - \frac{1}{2} \rho^2 + 2\mu_0^2 + \frac{1}{2} \mu_1^2\right) \star \phi_n dx + \int_\mathbb{S} u_{n,x} \cdot \tau_n dx.
\]

(4.23)
Multiplying (4.22) with $\rho_n$, we get by integration

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} \rho_n^2 dx - \frac{1}{2} \int_{\mathbb{S}} (u + \gamma)(\rho_n^2)_x dx = \int_{\mathbb{S}} \rho_n(u_x \rho) * \phi_n dx + \int_{\mathbb{S}} \rho_n \sigma_n dx.
$$

(4.24)

Adding (4.23) and (4.24), we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} (u_{n,x}^2 + \rho_n^2) dx = \frac{1}{2} \int_{\mathbb{S}} (u + \gamma)(u_{n,x}^2)_x dx
$$

$$
+ \int_{\mathbb{S}} u_{n,x} \left( \frac{1}{2} u_{x}^2 - 2\mu_0 u - \frac{1}{2} \rho^2 + 2\mu_0^2 + \frac{1}{2} \mu_1^2 \right) * \phi_n dx
$$

$$
+ \int_{\mathbb{S}} u_{n,x} \cdot \tau_n dx + \frac{1}{2} \int_{\mathbb{S}} (u + \gamma)(\rho_n^2)_x dx + \int_{\mathbb{S}} \rho_n(u_x \rho) * \phi_n dx + \int_{\mathbb{S}} \rho_n \sigma_n dx.
$$

As for fixed $T > 0$, $u$, $u_x$ and $\rho$ are bounded in $[0, T] \times \mathbb{S}$. Let $n \to \infty$, on account of Lebesgue’s dominated convergence theorem, we get

$$
\frac{d}{dt} \int_{\mathbb{S}} (u_x^2 + \rho^2) dx = 0.
$$

This completes the proof of Theorem 1.1.

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