SOLUTION GENERATING
IN TEN DIMENSIONAL SUPERSYMMETRIC
CLASSICAL YANG–MILLS THEORIES

Contribution to the Dubna Memorial Volume
In honour of Mikhail V. Saveliev

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Abstract

In a recent paper (hep-th/9811108), Saveliev and the author showed that there exits an on-shell light cone gauge where the non-linear part of the field equations reduces to a (super) version of Yang’s equations which may be solved by methods inspired by the ones previously developed for self-dual Yang-Mills equations in four dimensions. Later on (hep-th/9903218), the analogy between these latter theories and the present ones was pushed further by writing down a set of super partial linear differential equations which are the analogues of the Lax pair of Belavin and Zakharov. Using this Lax representation, it is shown in the present article that solution-generating techniques are at work, which are similar to the ones developed for four dimensional self-dual Yang-Mills theories in the late seventies.

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In Memoriam Mikhail V. Saveliev

On the morning of September 21st 1998, I arrive in Durham UK to participate in the 2nd Annual TMR Conference. I have left Paris rather early in order to deliver my talk at 3 o’clock concerning the recent progress with Misha. I know that he has not been well and I am much worried about him; but when Ed. Corrigan greets me and says “do you know that Misha died yesterday?”, I am thunderstruck. I suddenly realise that we will never meet again. How terribly sad! Our last encounter in person was in Cambridge UK at the beginning of March 1997. He had accompanied me to the bus station in his usual attentionate and friendly manner, and we were enthusiastically discussing about future research developments. The evening before he had treated me to a hearty dinner cooked by him-self, which was the occasion of a very friendly evening as he enjoyed so much. At that time, I felt so sure we would soon meet again! We made plans to do that, in Brazil in Paris, or in Tbilissi, but time was always flying too fast. One always assume, wrongly that the good things will remain for ever for us to reach! Life is unfortunately too short.

How I miss his frequent email messages! They were so nice and stimulating. In looking back at them I am impressed by what they brought me day after day: new ideas, important remarks, key references to the scientific literature, results of painstaking calculations, some personal news (too rarely) which gave me a glimpse at the difficulties of his every day life. In return, and when it came to it, I tried as much as possible to express my sympathy and support but of course, concerning every day life, Russia is so different from what I know personally! On top of that, we were fighting to make progress in our ambitious program solely by exchanging ideas over the net, a notoriously difficult task. I am afraid that he was under too much pressure, and that this was an important factor in the deterioration of his health.

Some parts of his emails bring back so vividely the memory of the time already past: On February 1998, Misha wrote from Moscow

Dear Jean–Loup,

Let me answer on yr section “More difficulties (unfortunately)”. Definitely you are right, ... Now I understand that you was pretty right when said that we should meet personally to put all the things in order; unfortunately now it is too late, because I should leave for Brazil already on 25.03.98, Luiz already sent me tickets and my grant opens just that time. Nethertheless I am quite optimistic that we are on the right track and will succeeded in solving the problem in the nearest future....

He wrote the following from Brazil during Spring 1998:

Dear Jean–Loup,

Thanks for yr 3 messages (mail, debut and a latex-file). ..... I’ll study yr file in detail and then be back to you. I am alone here, mainly because Svetlana Jr has
her lessons at the University until June, then has exams, and since she is, let say a
"home child", we would prefer not to leave her alone, especially since Moscow is not
a quiet small place like Protvino. So it would be lone for me these three months,
but better time for work. I would say that here it is very nice, Luiz and other people
here, at IFT, are very kind and careful to me; meals in the town are nicely in test;
I am sure you would like brazilian food. What do you think about Tbilisi, will you
be able to come? In any case I am writing to my colleagues in Tbilisi to send you
an invitation letter.

With best regards, Misha

When my trip to Brazil was cancelled, he wrote

Dear Jean–Loup,

Thank your for your today message. That’s pity that you are not able to come
in Sao Paulo; hope that you’ll manage to arrange yr plans to visit Tbilisi, otherwise
I’ll try to come in Paris in the Fall at least for a short while since we have a lot
of things to discuss in person, I believe that see now some remarkable directions
in geometry of supermanifolds, relevant Pluckers, etc. And, as a joke, looking at
my directory GS (Gervais-Saveliev) containing a lot of messages we have exchanged
during this project, I think that we can publish a book of letters with a title ”How
many wrong & right ways might be in science”...

In retrospect, I am rather struck by the following premonatory part of a message
he sent me during May 1998:

P.S. Last evening I was very busy writing condolences to Russia concerning our
very eminent physicists David Kirzhnitz from Theory Div. of the Lebedev Inst.,
Polubarinov from Dubna (he was a coauthor of Victor Ogievetsky), whom both I
knew very good for some 30 years, and Boris Dzelepov, who have passed away two
days ago. That’s a great pity to loose so many colleagues & friends this year!

On the other hand, I was far from realising that his health had so much deterio-
rated. He was always blaming his old car accident. For instance, he wrote en August
28th:

Dear Jean-Loup,

Sorry for a long silence. I was in a rather bad shape; just after my last message
to you I had several vascular spasms and swoons, once even with a loss of memory,
true for a short time. Presumably, it is caused by my neck small bell problem after
the car crash in 1988. Now I am slightly better, and am beginning to work, but feel
necessity to make some medical treatments in Moscow. .... ”

Going further back in time, it is a pleasure to remember that I met Misha for
the first time in 1992 when his famous work with Lesnov had already proven to be
so important for two dimensional conformal/integrable systems. We immediately
started to collaborate and have done so, at least on a part time basis, ever since.
His contribution to our research program has been invaluable. After a series of pa-
pers concerned with two dimensional Toda theories —where we successively discussed
black hole solutions, W geometries, and higher grading generalisations— we more
recently turned to the explicit classical integration of supersymmetric theories with
local symmetries in more than two dimensions, a very ambitious program which is
still far from completion at the present time. Working with Misha has been a won-
derful experience which terminated so abruptly! I will always remember our excited
and friendly discussions, his kindness and enthousiam, his fantastic knowledge of
the scientific literature! We, at École Normale, were lucky enough to invite him
for several extended visits which were extremely fruitful. Misha and I met in other places, but altogether much too rarely.

I will always remember the fun we had in discussing physics; but I now regret that, although we were very good friends, we seldom took time to socialise outside research. These few very warm and friendly encounters are dear to my memory, especially when his Svetlana’s (as he used to say) were present. This happened in particular for a day in my country house during one of his stay in Paris and on one evening in his apartment when he was visiting Cambridge UK. It is good to remember how happy he was on these occasions, how affectionate and (rightly) proud he was with his wife and daughter, how friendly and warmly he behaved!

M. Saveliev was great both as a scientist and as a human being. He was obviously such a good father, husband, friend!

Jean-Loup Gervais

1 Introduction

In recent times we turned\cite{1} to the classical integration of theories in more than two dimensions with local extended supersymmetries. Our motivation was twofold. On the one hand this problem is very important for the recent developments in duality and M theory. On the other hand, the recent advances initiated by Seiberg and Witten indicate that these theories are in many ways higher dimensional analogues of two dimensional conformal/integrable systems, so that progress may be expected. Since fall 1997, we have studied super Yang-Mills theories in ten dimensions. There, it was shown by Witten\cite{3} that the field equations are equivalent to flatness conditions. This is a priori similar to well known basic ones of Toda theories, albeit no real progress could be made at that time, since the corresponding Lax type equations involve an arbitrary light like vector which plays the role of a spectral parameter. At first, we reformulated the field equations in a way which is similar to a super version of the higher dimensional generalisations of Toda theories developed by Razumov and Saveliev\cite{2}, where the Yang-Mills gauge algebra is extended to a super one. This has not yet been published since, contrary to our initial hope, the two types of theories do not seem to be equivalent. I hope to return to this problem in a near future. In the mean time, we found the existence of an on-shell gauge, in super Yang-Mills where the field equations simplify tremendously and where the first similarity with self-dual Yang-Mills in four dimensions came out\cite{1}. More recently\cite{4}, I was able to write down a set of super partial linear differential equations whose consistency conditions may be derived from the SUSY Y-M equations in ten dimensions, and which are the analogues of the Lax pair of Belavin and Zakharov\cite{6}. 

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As is well known, super Yang-Mills theories in ten dimensions just describes a standard non-abelian gauge field coupled with a charged Majorana-Weyl spinor field in the adjoint representation of the gauge group. The dynamics is thus specified by the standard action

\[ S = \int d^{10}x \text{Tr} \left\{ -\frac{1}{4} Y_{mn} Y^{mn} + \frac{1}{2} \bar{\phi} \left( \Gamma^m \partial_m \phi + [X_m, \phi]_\perp \right) \right\}, \]  

(1.1)

\[ Y_{mn} = \partial_m X_n - \partial_n X_m + [X_m, X_n]_\perp. \]  

(1.2)

The notations are as follows\(^2\): \(X_m(x)\) is the vector potential, \(\phi(x)\) is the Majorana-Weyl spinor. Both are matrices in the adjoint representation of the gauge group \(G\). Latin indices \(m = 0, \ldots, 9\) describe Minkowski components. Greek indices \(\alpha = 1, \ldots, 16\) denote chiral spinor components. We will use the superspace formulation with odd coordinates \(\theta^\alpha\). The super vector potentials, which are valued in the gauge group, are noted \(A_m(x, \theta)\), \(A_\alpha(x, \theta)\). As shown in refs. \(3\), \(5\), we may remove all the additional fields and uniquely reconstruct the physical fields \(X_m, \phi\) from \(A_m\) and \(A_\alpha\) if we impose the condition \(\theta^\alpha A_\alpha = 0\) on the latter.

With this condition, it was shown in refs. \(3\), \(5\), that the field equations derived from the Lagrangian (1.1) are equivalent to the flatness conditions

\[ F_{\alpha\beta} = 0, \]  

(1.3)

where \(F\) is the supercovariant curvature

\[ F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha + [A_\alpha, A_\beta] + 2 (\sigma^m)_{\alpha\beta} A_m. \]  

(1.4)

\(D_\alpha\) denote the superderivatives

\[ D_\alpha = \partial_\alpha - (\sigma^m)_{\alpha\beta} \theta^\beta \partial_m, \]  

(1.5)

and we use the Dirac matrices

\[ \Gamma^m = \begin{pmatrix} 0_{16 \times 16} & (\sigma^m)_{\alpha\beta} \\ ((\sigma^m)_{\alpha\beta})^T & 0_{16 \times 16} \end{pmatrix}, \quad \Gamma^{11} = \begin{pmatrix} 0_{16 \times 16} & 0 \\ 0 & -1_{16 \times 16} \end{pmatrix}. \]  

(1.6)

Throughout the paper, it will be convenient to use the following particular realisation:

\[ ((\sigma^9)_{\alpha\beta}) = \begin{pmatrix} 0_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix}, \quad (\sigma^9)_{\alpha\beta} = \begin{pmatrix} 0_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix} \]  

(1.7)

\[ ((\sigma^0)_{\alpha\beta}) = -((\sigma^0)_{\alpha\beta}) = \begin{pmatrix} 0_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix}, \quad (\sigma^0)_{\alpha\beta} = \begin{pmatrix} 0_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix} \]  

(1.8)

\[ ((\sigma^i)_{\alpha\beta}) = ((\sigma^i)_{\alpha\beta}) = \begin{pmatrix} 0 & \gamma^i_\mu \gamma^\mu_\nu \\ (\gamma^i_\nu)^T \gamma^\mu_\mu \end{pmatrix}, \quad i = 1, \ldots, 8. \]  

(1.9)

The convention for greek letters is as follows: Letters from the beginning of the alphabet run from 1 to 16. Letters from the middle of alphabet run from 1 to 8. In this way, we shall separate the two spinor representations of \(O(8)\) by rewriting \(\alpha_1, \ldots, \alpha_{16}\) as \(\mu_1, \ldots, \mu_8, \bar{\mu}_1, \ldots, \bar{\mu}_8\)\(^2\)They are essentially the same as in ref.\(1\).
Using the above explicit realisations on sees that the equations to solve take the form

\[ F_{\mu \nu} \equiv D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu] = 2 \delta_{\mu \nu} (A_0 + A_9) \] (1.10)
\[ F_{\mu \rho} \equiv D_\mu A_\rho + D_\rho A_\mu + [A_\mu, A_\rho] = 2 \delta_{\mu \rho} (A_0 - A_9) \] (1.11)
\[ F_{\mu \nu} \equiv D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu] = -2 \sum_{i=1}^{8} A_i \gamma_i^\mu \gamma_i^\nu \] (1.12)

In my last paper with M. Saveliev [1], these flatness conditions in superspace were used to go to an on-shell light-cone gauge where half of the superfields vanish. After reduction to (1 + 1) dimensions, the non-linear part of the equations was transformed into equations for a scalar superfield which are (super) analogues of the so called Yang equations which were much studied in connection with solutions of self-dual Yang-Mills equations in four dimensions. The main differences between the two type of relations is that derivatives are now replaced by superderivatives, that there are sixteen equations instead of four, and that the indices are paired differently. Nevertheless, it was found that these novel features are precisely such that the equations may be solved by methods very similar to the ones developed in connection with self-dual Yang-Mills in four dimensions. The aim of the present paper is to push this analogy much further, by deriving the analogues of the Lax pair of Belavin Zakharov[6] which was instrumental for deriving multi-instanton solutions at the end of the seventies.

2 The Lax representation

For completeness, let us repeat the essential points of ref.[1]. The original theory is $O(9, 1)$ invariant, but the choice of Dirac matrices just summarized is covariant only under a particular $O(8)$ subgroup. The Lax representation comes out after picking up a particular $O(7)$ subgroup of the latter. This done simply by remarking that we may choose one $\gamma^i$ to be the unit matrix, in which case the others are antisymmetric and obey the $O(7)$ Dirac algebra. This is so, for instance in the following explicit representation of the $O(8)$ gamma matrices, where $\gamma^8$ is equal to one, which we will use throughout:

\[ \gamma^1 = \tau_1 \otimes \tau_3 \tau_1 \otimes 1 \]
\[ \gamma^2 = 1 \otimes \tau_1 \otimes \tau_3 \tau_1 \]
\[ \gamma^3 = \tau_3 \tau_1 \otimes 1 \otimes \tau_1 \]
\[ \gamma^4 = \tau_3 \tau_1 \otimes \tau_3 \tau_1 \otimes \tau_3 \tau_1 \]
\[ \gamma^5 = \tau_3 \otimes \tau_3 \tau_1 \otimes 1 \]
\[ \gamma^6 = 1 \otimes \tau_3 \otimes \tau_3 \tau_1 \]
\[ \gamma^7 = \tau_3 \tau_1 \otimes 1 \otimes \tau_3 \]
\[ \gamma^8 = 1 \otimes 1 \otimes 1. \] (2.1)

With this choice, it follows from equations (1.10) and (1.11) that

\[ F_{\mu \nu} = 2 \delta_{\mu \nu} (A_0 + A_9), \quad F_{\mu \rho} = 2 \delta_{\mu \rho} (A_0 - A_9), \quad F_{\mu \nu} + F_{\nu \mu} = -4 \delta_{\mu \nu} A_8. \] (2.2)

We have symmetrized the mixed (last) equations so that the right-hand sides only involve Kronecker delta’s in the spinor indices. By taking $\gamma^8$ to be the unit matrix, we have introduced a mapping between overlined and non overlined indices. Accordingly, in the previous equation and hereafter, whenever we write an overlined index and non overlined one with the same letter (such as $\mu$ and $\overline{\mu}$) we mean that
they are numerically equal, so that $\gamma^8_{\mu\bar{\nu}} = 1$. Next, in parallel with what was done for self-dual Yang-Mills in four dimensions, it is convenient to go to complex (super) coordinates. Thus we introduce, with $i$ the square root of minus one,

\[ G_{\mu\nu} = F_{\mu\nu} - iF_{\mu\bar{\nu}} + iF_{\bar{\mu}\bar{\nu}}, \]
\[ G_{\bar{\mu}\bar{\nu}} = F_{\bar{\mu}\bar{\nu}} - iF_{\mu\bar{\nu}} - iF_{\bar{\mu}\bar{\nu}}, \]
\[ G_{\mu\bar{\nu}} = F_{\mu\bar{\nu}} + iF_{\mu\bar{\nu}} - iF_{\bar{\mu}\bar{\nu}}, \]
\[ G_{\bar{\mu}\nu} = F_{\bar{\mu}\nu} + iF_{\bar{\mu}\nu} + iF_{\mu\nu}. \]  

(2.3)

\[ \Delta_{\mu} = D_{\mu} + iD_{\bar{\mu}}, \]
\[ \Delta_{\bar{\mu}} = D_{\bar{\mu}} - iD_{\mu}, \]
\[ B_{\mu} = A_{\mu} + iA_{\bar{\mu}}, \]
\[ B_{\bar{\mu}} = A_{\bar{\mu}} - iA_{\mu}. \]  

(2.4)

A straightforward computation shows that

\[ [\Delta_{\mu}, \Delta_{\nu}]_+ = 4\delta_{\mu\nu} (\partial_9 - i\partial_8), \]
\[ [\Delta_{\bar{\mu}}, \Delta_{\bar{\nu}}]_+ = 4\delta_{\mu\nu} (\partial_9 + i\partial_8), \]
\[ [\Delta_{\mu}, \Delta_{\bar{\nu}}]_+ + [\Delta_{\bar{\mu}}, \Delta_{\bar{\nu}}]_+ = 8\delta_{\mu\nu}\partial_0. \]  

(2.6)

Consider, now the system of differential equations

\[ D_{\mu} \Psi (\lambda) \equiv (\Delta_{\mu} + \lambda\Delta_{\bar{\mu}} + B_{\mu} + \lambda B_{\bar{\mu}}) \Psi(\lambda) = 0, \mu = 1, \ldots, 8. \]  

(2.7)

Of course, although we do not write it for simplicity of notations, $\Psi(\lambda)$ is a superfield function of $x$ and $\theta$. The parameter $\lambda$ is an arbitrary complex number. The consistency condition of these equations is

\[ [D_{\mu}, D_{\nu}]_+ \Psi(\lambda) = 0. \]  

(2.8)

This gives

\[ \{4\delta_{\mu\nu} (\partial_9 - i\partial_8) + G_{\mu\nu}\} \Psi + \lambda \{8\delta_{\mu\nu}\partial_0 + G_{\bar{\nu}\bar{\mu}} + G_{\bar{\mu}\nu}\} \Psi 
+ \lambda^2 \{4\delta_{\mu\nu} (\partial_9 + i\partial_8) + G_{\bar{\mu}\bar{\nu}}\} \Psi = 0. \]

Thus we correctly get that, for $\mu \neq \nu$

\[ G_{\mu\nu} = G_{\bar{\mu}\bar{\nu}} = G_{\mu\bar{\nu}} + G_{\bar{\mu}\nu} = 0, \]

and that $G_{\mu\mu}, G_{\bar{\mu}\bar{\mu}}, G_{\mu\bar{\mu}}$ do not depend upon $\mu$. Thus these consistency conditions are equivalent to the symmetrized dynamical equations 2.2.

### 3 Solution generating mechanism

In this section we discuss a solution generating mechanism analogous to the one developed for self-dual Yang-Mills in four dimensions in ref[7]. Although this is not absolutely necessary, we will assume in order to simplify the discussion that there is no dependence upon $x^i$, for $i = 1, \ldots, 8$. Thus, besides the odd fermionic variables, the superfields only depends upon $x^0 = x^0 \pm x^9$. In ref[4], the hermiticity conditions for superfields were established with $SU(N)$ as the gauge group, assuming

\[ \text{For the new symbols, the group theoretical meaning of the fermionic indices } \mu, \bar{\nu} \text{ is lost. We adopt this convention to avoid clusy notations.} \]
to avoid complications that only look at solutions such that $\phi^\alpha = 0$. For these purely bosonic solutions $A_\alpha$ and $A_m$ only involve odd and even powers of $\theta$ respectively. The hermiticity condition on superfields is that

$$A_\alpha^\dagger = -KA_\alpha K, \quad A_m^\dagger = -KA_m K,$$

(3.1)

with

$$K = (-1)^{R(R-1)/2}.$$

(3.2)

We will write in general the above hermiticity conditions under the form $A_\alpha^\dagger = -\bar{A}_\alpha$, $A_m^\dagger = -\bar{A}_m$ and so on. Thus

$$\bar{B}_\mu = -B_{\bar{\mu}}.$$

Assume that we only consider solutions of equation 2.7 which are regular at $\lambda = 0$. Then it follows that we may write

$$B_\mu = - (\Delta_\mu \Lambda) \Lambda^{-1}, \quad \Lambda = \Psi(0).$$

(3.3)

Then

$$B_{\bar{\mu}} = - K \Lambda^\dagger \Lambda^{-1} \left( \Delta_{\bar{\mu}} \Lambda^\dagger \right) K = \left( \Delta_{\bar{\mu}} \bar{\Lambda}^{-1} \right) \bar{\Lambda}.$$

An easy computation then shows that we may change gauge by replacing $\Psi \to \bar{\Lambda} \Psi$, in such a way that the Lax equations become

$$(\Delta_\mu + \lambda \Delta_{\bar{\mu}} - a_\mu) \Psi(\lambda) = 0$$

(3.4)

where $a_\mu = (\Delta_\mu g) g^{-1}$, and

$$g = \bar{\Lambda} \Lambda.$$

(3.5)

In this gauge, at $\lambda = 0$, we get

$$\Delta_\mu \Psi(0) = (\Delta_\mu g) g^{-1} \Psi_0,$$

so that

$$\Psi(0) = g.$$

Following the discussion of ref.[7] closely, the input is a solution $\Psi_0$ of equations 3.4, such that $a_\mu^0 = (\Delta_\mu \Psi_0(0)) \Psi(0)^{-1}$. We look for a solution of equations 3.4 in the following form

$$\Psi(\lambda) = \chi(\lambda) \Psi_0(\lambda),$$

(3.6)

where

$$\chi(\lambda) = 1 + \sum_{k=1}^{n} \frac{R_k}{\lambda - \lambda_k}.$$}

(3.7)

The quantities $R_k$ are superfields which are $N \times N$ matrices, and $\lambda_k$ are superfields independent from $\lambda$. Hence $\chi$ is meromorphic in the $\lambda$ plane. Substitute the ansatz 3.6 into equations 3.4. One gets

$$(\Delta_\mu + \lambda \Delta_{\bar{\mu}}) \chi \chi^{-1} + \chi a_\mu^0 \chi^{-1} = a_\mu$$

(3.8)

Since $\chi(\lambda) \chi^{-1}(\lambda) = 1$, the ansatz 3.7 immediately implies that

$$R_k \chi^{-1}(\lambda_k) = 0.$$
Thus we may write
\[
(R_k)_{ab} = \sum_{j=1}^{s_k} n_{\alpha}^{(k,j)} m_{b}^{(k,j)}, \quad (\chi^{-1}(\lambda_k))_{ab} = \sum_{\ell=s_k+1}^{N} q_{\alpha}^{(k,\ell)} p_{b}^{(k,\ell)}, \quad (3.10)
\]
\[
m^{(k,j)} q^{(k,\ell)} \equiv \sum_{a=1}^{N} m_{a}^{(k,j)} q_{a}^{(k,\ell)} = 0, \quad (3.11)
\]
where \( s_k \) are the dimensions of the subspaces upon which \( R_k \) projects. We write the above as
\[
R_k = \sum_{j=1}^{s_k} n^{(k,j)} \otimes m^{(k,j)}, \quad \chi^{-1}(\lambda_k) = \sum_{\ell=s_k+1}^{N} q^{(k,\ell)} \otimes p^{(k,\ell)}.
\]
The left hand side of equation 3.8 would contain second and first order poles at \( \lambda = \lambda_k \), whereas the right-hand side is analytic.

**Absence of double pole**  This leads to the conditions
\[
\Delta_\mu \lambda_k + \lambda_k \Delta_\overline{\mu} \lambda_k = 0. \quad (3.12)
\]

Let us proceed by analogy with the bosonic case. Consider an arbitrary superfield \( h(\lambda, x_+, x_-, \theta^1, \ldots, \theta^8, \bar{\theta}^1, \ldots, \bar{\theta}^8) \), noted \( h(\lambda) \) for brevity satisfying
\[
(\Delta_\mu + \lambda \Delta_\overline{\mu}) h(\lambda) = 0. \quad (3.13)
\]

Note that, in terms of \( D_\mu \) and \( D_{\overline{\mu}} \) this means that
\[
(D_\mu + i D_{\overline{\mu}}) h + \lambda (D_\mu - i D_{\overline{\mu}}) h, \]
so that
\[
(1 + \lambda) D_\mu h + i (1 - \lambda) D_{\overline{\mu}} h = 0,
\]
\[
D_\mu h + i \frac{1 - \lambda}{1 + \lambda} D_{\overline{\mu}} h = 0.
\]

Therefore, the solution of this equation was already derived in ref.\[1\]. Equation 3.12 is satisfied if \( \lambda_k \) are superfields, that is,
\[
\lambda_k \left( x_+, x_-, \theta^1, \ldots, \theta^8, \bar{\theta}^1, \ldots, \bar{\theta}^8 \right),
\]
which are such that
\[
h \left( x_+, x_-, \theta^1, \ldots, \theta^8, \bar{\theta}^1, \ldots, \bar{\theta}^8, \lambda_k(x_+, x_-, \theta^1, \ldots, \theta^8, \bar{\theta}^1, \ldots, \bar{\theta}^8) \right) \equiv 0. \quad (3.14)
\]

**Proof**  Indeed, since we get identically zero, we have
\[
\Delta_\mu h(\lambda_k) = 0 = \Delta_\mu h|_{\lambda_{\text{fixed}}=\lambda_k} + \frac{\partial h}{\partial \lambda}(\lambda_k) \Delta_\mu \lambda_k,
\]
\[
\Delta_{\overline{\mu}} h(\lambda_k) = 0 = \Delta_{\overline{\mu}} h|_{\lambda_{\text{fixed}}=\lambda_k} + \frac{\partial h}{\partial \lambda}(\lambda_k) \Delta_{\overline{\mu}} \lambda_k.
\]
The result follows from the equation \( \Delta_\mu h + \lambda \Delta_{\overline{\mu}} h = 0 \).
Absence of first order poles  This leads to the condition
\[
(\Delta \mu R_k + \lambda \Delta \sigma R_k) \chi^{-1}(\lambda_k) + R_k a_0^0 \chi^{-1}(\lambda_k) = 0.  \tag{3.15}
\]
Substitute equation 3.10. One has
\[
(\Delta \mu R_k) \chi^{-1}(\lambda_k) = \Delta \mu \left( \sum_{j=1}^{s_k} n^{(k,j)} \otimes m^{(k,j)} \right) \sum_{\ell=s_k+1}^{N} q^{(k,\ell)} \otimes p^{(k,\ell)}
\]
\[
(\Delta \sigma R_k) \chi^{-1}(\lambda_k) = \sum_{j=1}^{s_k} n^{(k,j)} \otimes \left( \Delta \sigma m^{(k,j)} \right) \sum_{\ell=s_k+1}^{N} q^{(k,\ell)} \otimes p^{(k,\ell)}.
\]
Thus we get
\[
\sum_{j=1}^{s_k} n^{(k,j)} \otimes \left\{ (\Delta \mu + \lambda_k \Delta \sigma) m^{(k,j)} \right\} \sum_{\ell=s_k+1}^{N} q^{(k,\ell)} \otimes p^{(k,\ell)}
\]
\[
+ \left( \sum_{j=1}^{s_k} n^{(k,j)} \otimes m^{(k,j)} \right) a_0^0 \left( \sum_{\ell=s_k+1}^{N} q^{(k,\ell)} \otimes p^{(k,\ell)} \right) = 0
\]
Thus we conclude that we have to solve the equations
\[
\left\{ (\Delta \mu + \lambda_k \Delta \sigma) m^{(k,j)} \right\} q^{(k,\ell)} + m^{(k,j)} a_0^0 q^{(k,\ell)} = 0.  \tag{3.16}
\]
Solution of these equations  We observe that \( \Psi_0(\lambda_k) \) satisfies
\[
\left( \Delta \mu + \lambda \Delta \sigma - a_0^0 \right) \Psi_0(\lambda) = 0.
\]
Thus we get
\[
- (\Delta \mu + \lambda \Delta \sigma) \Psi_0^{-1}(\lambda) - \Psi_0^{-1}(\lambda) a_0^0 = 0,
\]
and, setting \( \lambda = \lambda_k \),
\[
- (\Delta \mu + \lambda_k \Delta \sigma) \Psi_0^{-1}(\lambda_k) - \Psi_0^{-1}(\lambda_k) a_0^0 = 0.
\]
Thus equations 3.16 are solved by
\[
m^{(k,j)}_\mu = M^{(k,j)}_b \left( \Psi_0^{-1} \right)_{ba}(\lambda_k),  \tag{3.17}
\]
where \( M^{(k,j)}_b \) is a solution of the equation
\[
(\Delta \mu + \lambda_k \Delta \sigma) M^{(k,j)}_b = 0.  \tag{3.18}
\]
Hermiticity  According to equation (3.5)
\[ g = \tilde{g}. \]  (3.19)

A straightforward computation gives
\[ (\Delta_\mu + \lambda \Delta_\pi - \Delta_\mu g \cdot g^{-1}) g \tilde{\Psi}^{-1}(1/\lambda^*) = 0. \]

We see that \( g \tilde{\Psi}^{-1}(1/\lambda^*) \) and \( \Psi(\lambda) \) satisfy the same equation. Thus we may assume that \( g \tilde{\Psi}^{-1}(1/\lambda^*) = \Psi(\lambda) \). Let us assume that the simple solution satisfies the same condition: \( g_0 \tilde{\Psi}_0^{-1}(1/\lambda^*) = \Psi_0(\lambda) \). Using the ansatz (3.7), we see that
\[ g \tilde{\chi}^{-1}(1/\lambda^*) \tilde{\Psi}^{-1}(1/\lambda) = \chi(\lambda) \Psi_0(\lambda). \]

Thus we get
\[ g \tilde{\chi}^{-1}(1/\lambda^*) g_0^{-1} \Psi_0(\lambda) = \chi(\lambda) \Psi_0(\lambda). \]

Therefore, we arrive at the condition
\[ g \tilde{\chi}^{-1}(1/\lambda^*) g_0^{-1} = \chi(\lambda). \]  (3.20)

Writing equivalently
\[ g = \chi(\lambda) g_0 \tilde{\chi}(1/\lambda^*), \]  (3.21)

we see that, at the poles \( \chi(1/\lambda_k^*) = 0 \). Thus \( \chi^{-1}(\lambda) \) has poles at \( \lambda = 1/\lambda_k^* \). Take the residue of equation (3.21) at \( \lambda = \lambda_k^{*-1} \). According to equation (3.7),
\[ \chi^\dagger(1/\lambda_k^*) = 1 + \sum_{\ell=1}^{n} \frac{R^{\dagger}_{\ell}}{\lambda_k^{*-1} - \lambda_\ell}, \]
which gives, according to equation (3.10),
\[ (\chi(1/\lambda_k^*))_{ab} = \delta_{ab} + \sum_{\ell=1}^{n} \sum_{j=1}^{s_\ell} \frac{n_a^{(\ell,j)} m_b^{(\ell,j)}}{\lambda_k^{*-1} - \lambda_\ell}. \]

The residue must vanish. This gives
\[ 0 = \chi(1/\lambda_k^*) g_0 R_k^{\dagger} = \left( \delta_{ad} + \sum_{\ell=1}^{n} \sum_{j=1}^{s_\ell} \frac{n_a^{(\ell,j)} m_d^{(\ell,j)}}{\lambda_k^{*-1} - \lambda_\ell} \right) (g_0)_{dc} (R_k)^*_{bc} . \]

Thus we find that
\[ \left( \delta_{ad} + \sum_{\ell=1}^{n} \sum_{j=1}^{s_\ell} \frac{n_a^{(\ell,j)} m_d^{(\ell,j)}}{\lambda_k^{*-1} - \lambda_\ell} \right) (g_0)_{dc} m_c^{*(k,\ell,\ell)} = 0, \]
that is,
\[ \frac{1}{\lambda_k} (g_0)_{ac} m_c^{*(k,\ell,\ell)} + \sum_{\ell=1}^{n} \sum_{j=1}^{s_\ell} \frac{n_a^{(\ell,j)} m_d^{(\ell,j)}}{1 - \lambda_k^* \lambda_\ell} (g_0)_{dc} m_c^{*(k,\ell,\ell)} = 0. \]

Define
\[ \Gamma^{(\ell,\ell,\ell,\ell)} \equiv \frac{m_d^{(\ell,j)}}{1 - \lambda_k^* \lambda_\ell}. \]  (3.22)
We have now
\[
\frac{1}{\lambda_k} (g_0)_{ac} m_c^{* (k,j_k)} + \sum_{\ell=1}^{n} \sum_{j_{\ell}=1}^{n} n_a^{(\ell,j_{\ell})} \Gamma^{(\ell,j_{\ell},k,j_k)} = 0.
\]
Thus,
\[
n_a^{(k,j_k)} = - \sum_{\ell=1}^{n} \sum_{j_{\ell}=1}^{n} \frac{1}{\lambda_{\ell}^2} (g_0)_{ac} m_c^{* (\ell,j_{\ell})} \Gamma^{-1 (\ell,j_{\ell},k,j_k)}.
\]
Substitute finally into equation 3.7, which gives
\[
g = \Psi(0) = \chi(0) g_0 = \left(1 - \sum_k \frac{R_k}{\lambda_k}\right) g_0;
\]
that is, according to equation 3.10,
\[
(g)_{ab} = (g_0)_{ab} - \sum_k \sum_{j_k=1}^{n} \frac{n_a^{(k,j_k)} m_c^{(k,j_k)}}{\lambda_k} (g_0)_{cb}.
\]
Substitute equation 3.23. One gets
\[
(g)_{ab} = (g_0)_{ab} + \sum_{k=1}^{n} \sum_{j_k=1}^{n} \sum_{\ell=1}^{n} \sum_{j_{\ell}=1}^{n} \frac{1}{\lambda_{k}\lambda_{\ell}} (g_0)_{ad} m_d^{* (\ell,j_{\ell})} \Gamma^{-1 (\ell,j_{\ell},k,j_k)} m_c^{(k,j_k)} (g_0)_{cb}.
\]
Define
\[
N_b^{(k,j_k)} = m_c^{(k,j_k)} (g_0)_{cb}.
\]
Since \( g_0 = g_0 \), we have \( (g_0)_{ab} = (g_0)_{ba} \). Thus
\[
\tilde{N}_b^{(k,j_k)} = \tilde{m}_c^{(k,j_k)} (g_0)_{bc}.
\]
Thus we finally arrive at the following expression for \( g \)
\[
(g)_{ab} = (g_0)_{ab} + \sum_{k=1}^{n} \sum_{j_k=1}^{n} \sum_{\ell=1}^{n} \sum_{j_{\ell}=1}^{n} \frac{1}{\lambda_{k}\lambda_{\ell}} \tilde{N}_a^{(\ell,j_{\ell})} \Gamma^{-1 (\ell,j_{\ell},k,j_k)} N_b^{(k,j_k)}.
\]
Concerning \( \Psi \), we obtain
\[
\Psi(\lambda) = \chi(\lambda) \psi_0(\lambda) = \left(1 + \sum_k \frac{R_k}{\lambda - \lambda_k}\right) \psi_0(\lambda),
\]
that is, according to equation 3.10,
\[
\Psi_{ab}(\lambda) = \chi(\lambda)_{ac} \psi_0_{cb}(\lambda) = \left(\delta_{ac} + \sum_k \sum_{j_k=1}^{n} n_a^{(k,j_k)} m_c^{(k,j_k)} \frac{1}{\lambda - \lambda_k}\right) \psi_{0,cb}(\lambda).
\]
Substitute equation 3.23. One gets
\[
\Psi_{ab}(\lambda) = \\
\left(\delta_{ac} - \sum_{k=1}^{n} \sum_{j_k=1}^{n} \sum_{\ell=1}^{n} \sum_{j_{\ell}=1}^{n} \frac{1}{\lambda_{1}(\lambda - \lambda_k)} (g_0^{-1})_{ad} \tilde{m}_d^{(\ell,j_{\ell})} \Gamma^{-1 (\ell,j_{\ell},k,j_k)} m_c^{(k,j_k)}\right) \psi_{0,cb}(\lambda).
\]
**Question of determinant** Since the gauge group is taken to be $SU(N)$ we have to impose that $\det g = 1$. Let us try to impose this condition, since is not satisfied yet. For this purpose we want to compute the determinant of equation (3.25). Consider the case of only one pole first. One has, in that case,

$$\Psi_{ab}(\lambda) = \left( \delta_{ac} - \sum_{j_1} \sum_{j_1'} \frac{1}{\lambda_1 (\lambda - \lambda_1)} (g_0)_{ad} m_d^{s(1,j_1')} \Gamma^{-1}(1,j_1',1,j_1) m_c^{(1,j_1)} \right) \Psi_{0cb}(\lambda).$$

Consider

$$P_{ac} \equiv \frac{1}{1 - \lambda_1 \lambda_1^*} \sum_{j_1} \sum_{j_1'} (g_0)_{ad} m_d^{s(1,j_1')} \Gamma^{-1}(1,j_1',1,j_1) m_c^{(1,j_1)}.$$

It is easily verified that it is a projector such that

$$\text{Tr } P \equiv \sum_a P_{aa} = \text{Tr } (\Gamma^{-1} \Gamma) = s_1.$$

We may write

$$\Psi(\lambda) = \left( 1 - \frac{1 - \lambda_1 \lambda_1^*}{\lambda_1^* (\lambda - \lambda_1)} P \right) \psi_0(\lambda).$$

After diagonalising, it is trivial that, if there is only one pole,

$$\det \Psi(\lambda) = \left( \frac{\lambda - \lambda_1^{-1}}{\lambda - \lambda_1} \right)^{s_1} \det \Psi_0. \quad (3.27)$$

If there are several poles, we may include them in succession by writing

$$1 + \sum_k \frac{R_k}{\lambda - \lambda_k} = \prod_k \left( 1 + \frac{R_k}{\lambda - \lambda_k} \right).$$

Then we get

$$\det \Psi(\lambda) = \prod_k \left( \frac{\lambda - \lambda_k^{-1}}{\lambda - \lambda_k} \right)^{s_k} \det \Psi_0, \quad (3.28)$$

$$\det g = \prod_k |\lambda_k|^{-2s_k} \det \Psi_0. \quad (3.29)$$

Thus the determinant is not one. Fortunately, if we have $(\Delta_{\mu} + \lambda \Delta_{\mu} - a_\mu) \Psi = 0$, we have

$$(\Delta_{\mu} + \lambda \Delta_{\mu}) \ln \det \Psi = \sum_{ab} (\Psi^{-1})_{ab} (\Delta_{\mu} + \lambda \Delta_{\mu}) (\Psi)_{ba}$$

$$= \sum_{ab} (\Psi^{-1})_{ab} (a_\mu \Psi)_{ba} = \sum_a (a_\mu)_{aa} = - \sum_{ab} (g)_{ab} \Delta_{\mu} (g^{-1})_{ba} = -\Delta_{\mu} \ln \det g.$$

Thus $\tilde{\Psi} = (\det \Psi)^\nu \Psi$ also solves equations (3.4), but now with $g (\det g)^{-\nu}$. Accordingly, we define the physical $g$ as

$$g_p = g \prod_{k=1}^n |\lambda_k|^{2s_k}. \quad (3.30)$$

This terminates the derivation of the multipole solution.
4 Outlook

We have verified that the solution generating method developed for the self-dual Yang-Mills in four dimensions may be straightforwardly applied to our case. This is yet another indication that the present system of symmetrised equations 2.2 is indeed completely and explicitly integrable.

Concerning the full Yang-Mills equations or equivalently the unsymmetrised equations 1.10–1.12, any solution is also a solution of the symmetrised equations 2.2. Thus we should be able to derive solutions of the latter which are general enough so that we may impose that they be solutions of the former. This problem is currently under investigation.

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