Quantum version of Prisoners’ Dilemma under interacting environment

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Abstract
Quantum game theory is a rapidly evolving subject that extends beyond physics. In this research work, a schematic picture of quantum game theory has been provided with the help of the famous game Prisoners’ Dilemma, where both of the prisoners possess an entangled qubit. Increasing the payoffs of the prisoners, quantum theory introduces a new Nash equilibrium. It has been considered that the shared qubits may interact with the environment, which is a bath of simple harmonic oscillators. This interaction introduces decoherence. Calculating the decoherence factor, we have shown that as time increases, the new Nash equilibrium disappears and accordingly the corresponding payoffs reduce. The decoherence factor has been analyzed for different time regions, and a characteristic time scale of decoherence has also been provided. A critical time scale, below which Alice and Bob can use the quantum strategy to achieve a higher payoff, has been described with its connection to the off-diagonal elements of the reduced density matrix of the prisoners. The critical time scale has also been discussed based on whether the decoherence occurred once or twice. A comparative discussion establishes that the process of decoherence is faster when it occurs twice. It needs to be mentioned here that the total time of decoherence is the same in both cases.

Keywords Quantum game · Quantum decoherence · Nash equilibrium · Quantum strategies

1 Introduction

Game theory has a special aura that enlightens biology, mathematics, social science and various fields of physics. In the situations where it is difficult to make a decision among multipartite systems, game theory provides a logical approach towards an acceptable
solution. Using quantum properties in game theory, a new area of physics, quantum game theory has been created, with enormous applications in quantum information and quantum computation. In quantum game theory, the participants can share an entangled quantum state which provides some advantages over the classical game theory.

Considering two vital properties of quantumness, superposition and entanglement, Eisert et al.[1] have introduced the quantum extension of the classical game Prisoners’ Dilemma (PD)[7]. But in the quantum version, the situation changes when the shared qubits of the prisoners interact with the environment. In that case, the state of the environment gets entangled with the qubit and as a consequence quantum decoherence starts ruling the composite state. This decoherence process destroys the quantumness of the system. As it is impossible to construct a perfectly closed system, the interaction of the environment is inevitable. In various literature, the interaction of the system with its environment has been called monitoring of the environment. Decoherence reduces the advantages of the players that they gained by the quantum properties. In this present work starting from the quantum version of PD, we have analyzed the payoffs of the prisoners after quantum decoherence. The discussions on the time scale and the corresponding decoherence function, for which the payoffs associated with the quantum strategies have advantages, have been made. Then, these results have been compared with those when decoherence occurs twice.

### 2 Quantum game

In the classical PD, two prisoners, Alice and Bob are separately interrogated. They may decide to choose cooperation (strategy C) or defection (strategy D) independently. But their payoffs will depend according to Table 1. The Nash equilibrium[2][3] for PD can be found as (D,D).

|            | Bob (C) | Bob (D) |
|------------|---------|---------|
| Alice (C)  | (3,3)   | (0,5)   |
| Alice (D)  | (5,0)   | (1,1)   |

Table 1: Payoffs for the Prisoners’ Dilemma. The numerics in the parenthesis represent the payoffs of Alice and Bob, respectively.

In its quantum version[1], let Alice and Bob initially share a 2 qubits state $|00\rangle$, where $|0\rangle$ (which denotes C) and $|1\rangle$ (which denotes D) are 2 basis states of a two dimensional Hilbert space. There is an unitary operator $\hat{J}$ which entangles the two states and known to both of the prisoners. Under the conditions $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$, two quantum strategies $\hat{U}_A$ (for Alice) and $\hat{U}_B$ (for Bob) can be found from,

$$\hat{U} (\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi} \cos(\theta/2) \end{pmatrix}. \quad (1)$$

Associating the operators for the strategies C and D as $\hat{C} = \hat{U}(0, 0)$ and $\hat{D} = \hat{U}(\pi, 0)$, respectively, the entanglement operator $\hat{J}$ can be represented by $$\hat{J} = \frac{1}{\sqrt{2}} (\hat{1} \otimes \hat{1} +$$

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$i\hat{D} \otimes \hat{D}$) for maximum entanglement, where $\hat{I}$ is the identity operator. The general process of a quantum game can be described in the following steps:

1. Initial density operator: $\hat{\rho}_0 = |00\rangle\langle 00|$  
2. Entanglement: $\hat{\rho}_1 = \hat{J}\hat{\rho}_0\hat{J}^\dagger$  
3. Application of strategies: $\hat{\rho}_2 = \left(\hat{U}_A \otimes \hat{U}_B\right)\hat{\rho}_1\left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger$  
4. Final density operator: $\hat{\rho}_f = \hat{J}^\dagger\hat{\rho}_2\hat{J}$

After successful execution of the above steps, the prisoners may apply suitable measurements to achieve the payoffs. One of the prisoners (say Alice) may expect the payoff as

$$R_A = \sum_{ij} a_{ij} P_{ij}$$  

where $P_{ij}$ corresponds to the probability for getting the state $|ij\rangle$ as a result of the measurement and $a_{ij}$ is the associated classical payoff of Alice. Disappearing the classical Nash equilibrium $(D,D)$ the intervention of quantum physics shows a new Nash equilibrium $(\hat{Q},\hat{Q})$ with higher payoff where $\hat{Q} = \hat{U}(0, \pi/2)$.

### 3 Model of quantum decoherence

Along with many advantages, quantum physics also brings some obstacles, e.g., quantum decoherence, as a result of which the quality of those advantages decreases. Quantum game theory falls into the trap of quantum decoherence through the environmental monitoring and lost its advantages due to nonunitary evolution. To observe the consequences of quantum decoherence, let us apply a simple process of decoherence between step 2 and step 3 of the general process of the quantum game discussed in Sect. 2. Suppose Alice and Bob share an entangled state of two spin $-\frac{1}{2}$ particles and the combined state is interacting with the environment which is a bath of simple harmonic oscillators. The total Hamiltonian $\hat{H}$\cite{4}\cite{5} of the system and the environment takes the form as,

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{AE} + \hat{H}_{BE} + \hat{H}_E$$

where $\hat{H}_A, \hat{H}_B$ and $\hat{H}_E$ denote the Hamiltonian corresponding to Alice, Bob and the environment, respectively. $\hat{H}_{AE}$ and $\hat{H}_{BE}$ are symbolized as the interaction Hamiltonian of Alice and Bob with the environment, respectively. Explicit expressions of the above Hamiltonians are

$$\hat{H}_j = \frac{1}{2}\omega \hat{\sigma}_z, \quad \hat{H}_{jE} = \hat{\sigma}_z \otimes \sum_i C_i \hat{x}_i, \quad \hat{H}_E = \sum_i \left(\frac{\hat{p}_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2 \hat{x}_i^2\right)$$

where $j$ may be treated as $A$ and $B$ for Alice and Bob, respectively, $\hat{\sigma}_z$ is one of the well-known Pauli matrices, the Larmour frequency $\omega$ is related to the difference between the energy levels of the two basis states of the associated spin $-\frac{1}{2}$ system, $C_i$ is
Fig. 1 Graphical representation of $ln D(t)$ for three different time regions. All the three graphs show that the approximated values of $D(t)$ in the respective time regions nicely match with the exact value according to the Eq. (15). Here the values of the constant $J_0$ and the thermal energy $\beta^{-1}$ of the environment have been chosen to be 1 and $\omega_c/100$, respectively.

The coupling factor between the system and $i^{th}$ harmonic oscillator of the environment, and any other terms associated with $i$ like $\hat{p}_i$, $\hat{x}_i$, $m_i$ and $\omega_i$ denote the momentum, position, mass and natural frequency of the $i^{th}$ harmonic oscillator of the environment, respectively.

With the help of creation operator($\hat{a}^\dagger$) and annihilation operator($\hat{a}$), the Hamiltonian and the time evolution operator in the interaction picture may be written as

$$\hat{H}_I(t) = (\hat{\sigma}_z A + \hat{\sigma}_z B) \otimes \sum_i \left( G_i \hat{a}_i^\dagger e^{i\omega_i t} + G_i^* \hat{a}_i e^{-i\omega_i t} \right)$$

(9)

and

$$\hat{U}_I(t) = \mathcal{P} e^{\frac{i}{2}(\hat{\sigma}_z A + \hat{\sigma}_z B) \otimes \sum_i \left( \eta_i(t) \hat{a}_i^\dagger - \eta_i^*(t) \hat{a}_i \right)},$$

(10)

respectively, where $G_i$ is considered as complex for generic case and related to $C_i$ as $C_i \hat{x}_i = (G_i \hat{a}_i^\dagger + G_i^* \hat{a}_i)$. $\eta_i(t)$ can be found as $\eta_i(t) = 2 \frac{G_i}{\omega_i} (1 - e^{i\omega_i t})$ and $\mathcal{P}$ is a global phase term which has no importance for our case and may be neglected.
Fig. 2 Dependence of Alice’s payoff on $\hat{U}_A$ and $\hat{U}_B$ for different times, where $\hat{D} = \hat{U}(\pi, 0)$, $\hat{C} = \hat{U}(0, 0)$ and $\hat{Q} = \hat{U}(0, \pi/2)$. A single parameter, $p \in [-1, 1]$, has been used to calculate $\hat{U}(\theta, \phi)$ such that $\hat{D} = \hat{U}(0, -p\pi/2)$ for $p \in [-1, 0]$ and $\hat{C} = \hat{U}(p\pi, 0)$ for $p \in [0, 1]$. The values of $J_0$ and $\beta^{-1}$ have been kept the same as in Fig. 1. It is clear that there is a new Nash equilibrium $(\hat{Q}, \hat{Q})$ at $t = 0.0[\omega_c^{-1}]$ but it decreases with time and disappears at large time scale, e.g., $t = 1000[\omega_c^{-1}]$. 

(a) at $t = 0.0[\omega_c^{-1}]$ 
(b) at $t = 0.1[\omega_c^{-1}]$ 
(c) at $t = 3.0[\omega_c^{-1}]$ 
(d) at $t = 6.9[\omega_c^{-1}]$ 
(e) at $t = 10.0[\omega_c^{-1}]$ 
(f) at $t = 1000[\omega_c^{-1}]$
Now let us suppose that after application of the entanglement operator $\hat{J}$ the combined state of Alice and Bob becomes $|\psi(0)\rangle = \frac{1}{\sqrt{2}} [|00\rangle + i|11\rangle]$ which is maximally entangled. If the environment is in thermal equilibrium at temperature $T$, then the combined density operator for Alice, Bob and the environment will be,

$$\hat{\rho}_{ABE}(0) = \frac{1}{2} [|00\rangle\langle 00| - i|00\rangle\langle 11| + i|11\rangle\langle 00| + |11\rangle\langle 11|] \otimes \frac{e^{-\beta \omega_i \hat{a}_i^{\dagger} \hat{a}_i}}{Tr(e^{-\beta \omega_i \hat{a}_i^{\dagger} \hat{a}_i})}$$

where $\beta = 1/k_B T$ and $k_B$ is the Boltzmann constant. After certain time $t$, the total density operator will be evolved to $\hat{\rho}_{ABE}(t) = \hat{U}_I(t) \hat{\rho}_{ABE}(0) \hat{U}_I^{-1}(t)$ and the reduced density operator for Alice and Bob will be $\hat{\rho}_{AB}(t) = Tr_E [\hat{\rho}_{ABE}(t)]$. Calculations show that the diagonal elements of $\hat{\rho}_{AB}(t)$ remain constant. But in case of the off-diagonal elements the situation is different. The off-diagonal elements of $\hat{\rho}_{AB}(t)$ are

$$\left[ \hat{\rho}_{AB}(t) \right]_{kl} = \left[ \hat{\rho}_{AB}(0) \right]_{kl} \prod_i e^{\pm \frac{1}{2} [\eta_i(t) \hat{a}_i^{\dagger} - \eta_i^*(t) \hat{a}_i]} E_i$$ (11)

where $k$ and $l$ can have values 0 or 1 but $k \neq l$. In various literature [8][9] it can be found that,

$$\prod_i e^{-\frac{1}{2} [\eta_i(t) \hat{a}_i^{\dagger} - \eta_i^*(t) \hat{a}_i]} E_i = e^{-|\eta_i(t)|^2 \coth\left(\frac{\beta \omega_i}{2}\right)}$$ (12)

and the term,

$$\prod_i e^{-|\eta_i(t)|^2 \coth\left(\frac{\beta \omega_i}{2}\right)} = D(t)$$ (13)

is responsible for the time dependence of the off-diagonal elements of the reduced density operator and is called decoherence factor or decoherence function.

For a large and dense environment, the environmental frequency modes can be considered as continuous. If we use Ohmic spectral density with a high cutoff frequency $\omega_c$, then Eq. (13) takes the form as

$$D(t) = \exp\left[-J_0 \int_0^\infty \frac{d\omega}{\omega} \coth\left(\frac{\beta \omega}{2}\right)(1 - \cos \omega t) e^{\omega t}(\omega / \omega_c)\right]$$ (14)

where $J_0$ is a constant and it contains the coupling factor between the system and the oscillators of the environment. If we consider that, $\beta^{-1} \ll \omega_c$, i.e., the thermal energy is quite lower than the cutoff frequency associated with the spectral density of the environment then $D(t)$ can be written as

$$D(t) = \exp\left[-\frac{J_0}{2} \ln(1 + \omega_c^2 t^2) - J_0 \ln\left\{\frac{\sinh(\pi t / \beta)}{\pi t / \beta}\right\}\right].$$ (15)

Calculations show that for three different time regions, (i) $t \ll \omega_c^{-1}$ and $t \ll \beta$, (ii) $t \gg \omega_c^{-1}$ but $t \ll \beta$, and (iii) $t \gg \beta$, the decoherence function can be approximated.
Fig. 3 Representation of $S_A$ according to inequality (18) with increasing time $t$. The values of $\beta^{-1}$ and $J_0$ corresponding to the four figures have been written in the concerned sub-captions. All these figures show that after certain time scale (i.e., critical time scale $\tau_c$) associated with the figures the value of $S_A$ becomes negative. This means that the quantum Nash equilibrium $(\hat{Q}, \hat{Q})$ disappears after $\tau_c$. It is also clear that $\tau_c$ increases with decreasing $J_0$.

This exponential decay of $D(t)$ causes a decrease in the entanglement between the qubits of Alice and Bob. For very long time region, i.e., $t \gg \beta$, the typical time scale of decoherence or the characteristic time scale of decoherence

$$\tau_D = \frac{\beta}{\pi J_0}$$

which can be found from the approximated equation of $D(t)$. 

(a) for $\beta^{-1} = \omega_c/100$ and $J_0 = 1$

(b) for $\beta^{-1} = \omega_c/1000$ and $J_0 = 1$

(c) for $\beta^{-1} = \omega_c/100$ and $J_0 = 0.5$

(d) for $\beta^{-1} = \omega_c/100$ and $J_0 = 0.25$
Figure 1 shows the graphical representations of $\ln D(t)$ for different time regions. To ensure the condition $\beta^{-1} \ll \omega_c$, we have chosen $\beta^{-1} = \omega_c/100$. We have set the value of the constant $J_0$ to 1. It is clear from the figure that all the three approximated expressions of $D(t)$ are compatible with Eq. (15) for the respective time regions. Figure 1c shows that for very large time region, $\ln D(t)$ can be approximated by a straight line. The value of the tangent to the straight portion of the curve associated with the exact calculation is $0.0314[\omega_c]$, and inverse of this value, i.e., $31.84[\omega_c^{-1}]$, is the exact characteristic time scale of decoherence. This value of characteristic time scale is approximately equal to $31.83[\omega_c^{-1}]$, which is obtained from Eq. (16). Thus, the expression of the decoherence time scale is verified.

So, if the entangled qubits of Alice and Bob interact with the environment according to the model discussed above, the entanglement tends to decrease through decoherence and for a long time region ($t \gg \beta$), the characteristic time scale of decoherence is $\beta/(\pi J_0)$.

4 Payoffs of the prisoners under decoherence

After decoherence, if the prisoners apply the quantum strategies $\hat{U}_A$ and $\hat{U}_B$ to their respective qubits, then one of the prisoners (say Alice) can have the payoff as

$$R_A = [2 + D(t)\cos(2(\phi_A + \phi_B))]\cos^2\left(\frac{\theta_A}{2}\right)\cos^2\left(\frac{\theta_B}{2}\right)$$

$$+ [2 - D(t)]\sin^2\left(\frac{\theta_A}{2}\right)\sin^2\left(\frac{\theta_B}{2}\right)$$

$$- [5\sin(\phi_A - \phi_B) + \sin(\phi_A + \phi_B)[D(t) + 2]]$$

$$\sin\left(\frac{\theta_A}{2}\right)\sin\left(\frac{\theta_B}{2}\right)\cos\left(\frac{\theta_A}{2}\right)\cos\left(\frac{\theta_B}{2}\right)$$
Fig. 5 Time dependence of Alice’s payoff when both of the prisoners apply $\hat{Q}$ to their respective qubits. It shows that the payoff reduces faster when the decoherence occurs twice, i.e., before the application of the strategies and after the application of the strategies though the total time of decoherence is the same for all three cases

$$+rac{5}{2} \left[ \{1 - D(t)\cos(2\phi_A)\} \cos^2 \left( \frac{\theta_A}{2} \right) \sin^2 \left( \frac{\theta_B}{2} \right) - \{1 + D(t)\cos(2\phi_B)\} \sin^2 \left( \frac{\theta_A}{2} \right) \cos^2 \left( \frac{\theta_B}{2} \right) \right].$$  \hspace{1cm} (17)

In Figure 2 the payoff according to Eq. (17) is represented pictorially. Figure 2a shows the dependence of Alice’s payoff on $\hat{U}_A$ and $\hat{U}_B$ at $t = 0.0[\omega_c^{-1}]$, which is similar to Eisert et al[1], and that after $t = 1000[\omega_c^{-1}]$ is shown in Fig. 2f, which is consistent with Huang and Qiu[6]. The Nash equilibrium $(\hat{Q}, \hat{Q})$ disappears through decoherence as the time goes from 0 to some large time scale ($t \gg \beta$) with respect to the decoherence time scale ($\tau_D$).

Figure 2b–e shows that Alice’s payoff for $(\hat{Q}, \hat{Q})$ decreases with an increase in time and for the time scale of $t > 6.9[\omega_c^{-1}]$, $(\hat{Q}, \hat{Q})$ will no longer be the Nash equilibrium. This time scale has also been verified from the definition of Nash equilibrium[10], which states that if $(\hat{Q}, \hat{Q})$ is a Nash equilibrium, then the following conditions must be satisfied,

$$S_A = R_A(\hat{Q}, \hat{Q}) - R_A(\hat{U}_A, \hat{Q}) \geq 0$$  \hspace{1cm} (18)

and

$$S_B = R_B(\hat{Q}, \hat{Q}) - R_B(\hat{Q}, \hat{U}_B) \geq 0.$$  \hspace{1cm} (19)

Figure 3 shows some schematic representations of $S_A$ to study the critical time scale ($\tau_c$) below which $(\hat{Q}, \hat{Q})$ remains the Nash equilibrium. In Figure 3a we can observe that $S_A \geq 0$ for $t < 7.0[\omega_c^{-1}]$. Thus, below that time scale, there is no way for Alice to increase her payoff by changing the strategy $\hat{Q}$, while Bob has chosen the strategy $\hat{Q}$. We can also obtain a similar conclusion from a similar representation of $S_B$ for Bob. Thus, only below $\tau_c \approx 7.0[\omega_c^{-1}]$, we can find $(\hat{Q}, \hat{Q})$ as the Nash equilibrium.
equilibrium and the prisoners can still obtain higher payoffs. It should be noted that
\( \tau_c (\approx 7.0[\omega_c^{-1}]) < \tau_D (\approx 31.8[\omega_c^{-1}]) \).

Figure 3b with lower thermal energy of the environment, i.e., \( \beta^{-1} = \omega_c/1000 \),
shows almost similar \( \tau_c \) but the critical time scale increases with a decrease in \( J_0 \) as can be seen in Fig. 3c, d. Since \( J_0 \) contains the coupling factor between the system and the environment, the lower the value of \( J_0 \), the lower the interaction with the environment, which in turn increases \( \tau_c \). If we put the values of the constants \( J_0 \) and \( \beta^{-1} \) in the Eq. (15) for each representation of Fig. 3, we obtain that in each case

\[
D(t)|_{t=\tau_c} \approx 0.14. \tag{20}
\]

Therefore, in other words, if the off-diagonal elements of the reduced density matrix
of Alice and Bob, \( \hat{\rho}_{AB}(t) \), fall below 14% of its initial value, Alice and Bob will no longer have a higher payoff using quantum entanglement.

Now suppose the entangled quantum states of Alice and Bob interact with the same
environment for the second time between step 3 and step 4 of Sect. 2. Then, again we need to perform the tedious job of Sect. 3, and finally, the payoff of Alice will be obtained as,
\[ R_A = \cos^2 \left( \frac{\theta_A}{2} \right) \cos^2 \left( \frac{\theta_B}{2} \right) \{2 + D_1(t_1)D_2(t_2)\cos(2\phi_A + 2\phi_B)\} \\
+ \sin^2 \left( \frac{\theta_A}{2} \right) \sin^2 \left( \frac{\theta_B}{2} \right) \{2 - D_1(t_1)D_2(t_2)\} -\sin \left( \frac{\theta_A}{2} \right) \sin \left( \frac{\theta_B}{2} \right) \\
\cos \left( \frac{\theta_A}{2} \right) \cos \left( \frac{\theta_B}{2} \right) \{\sin(\phi_A + \phi_B) (D_1(t_1) + 2D_2(t_2))\} \\
+ 5\sin(\phi_A - \phi_B)D_2(t_2) \\
+ \frac{5}{2} \left[ \cos^2 \left( \frac{\theta_A}{2} \right) \sin^2 \left( \frac{\theta_B}{2} \right) \{1 - D_1(t_1)D_2(t_2)\cos(2\phi_A)\} \right] \\
+ \sin^2 \left( \frac{\theta_A}{2} \right) \cos^2 \left( \frac{\theta_B}{2} \right) \{1 + D_1(t_1)D_2(t_2)\cos(2\phi_B)\} \right] \] (21)

where \(D_1(t_1)\) and \(D_2(t_2)\) are the decoherence factors for the decoherence processes that occurred before the application of the strategies (i.e., step 3 of Sect. 2) and after the application of the strategies, respectively, and \(t_1\) and \(t_2\) are the respective time elapsed during the decoherence processes.

Figure 4 depicts the situation of Eq. (21) after the total decoherence time \(t = 1000[\omega^{-1}]\), where \(t = t_1 + t_2\) and this situation is similar to Fig. 2f. That is, let a system goes through the decoherence process once (as in Fig. 2) or twice (as in Fig. 4); if in both cases the process runs for a long time, then the final result will be the same in both cases. But things change when the time scale is very small. These situations are represented in Fig. 5. In this figure, the time dependence of Alice’s payoff has been shown when both of the prisoners apply \(\hat{Q}\) to their respective qubits. It reflects the two different situations depending upon the occurrence of decoherence:

**Case-I** When decoherence occurs only before the application of the strategies or only after the application of the strategies.

**Case-II** When it takes place both before the application of the strategies and after the application of the strategies.

It can be concluded from Fig. 5 that, for Case-I the payoffs follow the same curve but the total effect of decoherence is faster for Case-II than Case-I though the total time of decoherence is equal for all the cases.

For Case-II, \(S_A\) is represented in Fig. 6 to study the comparison with Fig. 3 where decoherence occurred only before the application of the strategies. The \(\tau_c\)’s for all the sub-figures of Fig. 6 are less than the corresponding sub-figures of Fig. 3. This verifies the conclusion made from Fig. 5. If we evaluate the total decoherence function or the effective decoherence function at \(t = \tau_c\) for all the sub-figures of Fig. 6, we again obtain

\[ D_{\text{eff}}(t)|_{t=\tau_c} = D_1(t_1)D_2(t_2) \approx 0.14 \] (22)

where \(t = t_1 + t_2\), for all the four sub-figures. We have presented all the results for Alice. The same results can be achieved also for Bob.

Thus, according to our model of decoherence, Alice and Bob can have a higher payoff if the off-diagonal elements of their reduced density matrix, \(\hat{\rho}_{AB}(t)\), are greater than 14% of its initial value, no matter whether decoherence occurs once or twice.
5 Conclusions

Since no ideally closed system is possible, monitoring of the environment always affects the quantumness of a quantum system through decoherence. In this work, presenting a model for quantum decoherence and applying the model to the quantum version of PD it has been shown that the advantages of quantum game theory disappear with time. For this model of phase damping, we have studied the decoherence time scale in different time regions. Introducing a critical time scale $\tau_c$, we have analyzed different cases to show that below $\tau_c$ the prisoners can have a higher payoff relative to the classical version of PD. $\tau_c$ increases with decrease in the coupling strength $J_0$ between the system and the environment. The evaluation of the decoherence function at the critical time scale implies that the prisoners can have higher payoff until the off-diagonal elements of the reduced density matrix of Alice and Bob drop below 14% of its initial value. We have shown that the payoffs for the quantum Nash equilibrium decrease faster when decoherence is applied twice than once, keeping the total time of decoherence fixed. Finally, it has been concluded that, for the model discussed in this paper, Alice and Bob can have the higher payoff related to the Nash equilibrium $(\hat{Q}, \hat{Q})$ until the off-diagonal elements of their reduced density matrix drop below 14% of its initial value and it does not matter whether decoherence occurs once or twice.

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