Abstract: It is a natural question to ask whether two links are equivalent by the following moves \( t_k \) (where \( k \) is a fixed number of positive half twists) and if they are, how many moves are needed to go from one link to the other. In particular if \( k = 2 \) and the second link is a trivial link it is the question about the unknotting number. The new polynomial invariants of links often allow us to answer the above questions. Also the first homology groups of cyclic branch covers over links provide some interesting information.

Introduction. In the first part of the paper we apply the Jones-Conway (Homfly) and Kauffman polynomials to find whether two links are not \( t_k \) equivalent and if they are, to gain some information how many moves are needed to go from one link to the other.

In the second part we describe the Fox congruence classes and their relations with \( t_k \) moves. We use the Fox method to analyse relations between \( t_k \) moves and the first homology groups of branched cyclic covers of links.

In the third part we consider the influence of \( t_k \) moves on the Goeritz and Seifert matrices and analyse Lickorish-Millett [L-M-2] and Murakami [Mur-1, Mur-2] formulas from the point of view of \( t_k \) moves and illustrate them by various examples. At the end of the paper we outline some relations with signatures of links and non-cyclic coverings of link spaces.

Now we will formulate the basic definitions and state the main results of the paper concerning connections between \( t_k \) moves and the Jones-Conway polynomial invariants of links.
Consider diagrams of oriented links $L_0$ and $L_k$ which are identical, except the parts of the diagrams shown on Fig. 0.1.

\begin{center}
\includegraphics[width=0.7\textwidth]{fig01.png}
\end{center}

\textbf{Fig. 0.1}

\textbf{Definition 0.1.} The $t_k$ move (or $k$ twist) is the elementary operation on an oriented diagram $L_0$ resulting in $L_k$ (Fig. 0.1). Two oriented links $L$ and $L'$ are said to be $t_k$ equivalent ($L \sim t_k L'$) if one can go from $L$ to $L'$ using $t_k^{\pm 1}$ moves (and isotopy).

The $t_k$ distance between $t_k$ equivalent links $L$ and $L'$ (denoted $|L, L'|_{t_k}$) is defined to be the minimal number of $t_k^{\pm 1}$ moves needed to go from $L$ to $L'$.

The $t_k$ level distance between $L$ and $L'$ (denoted $|L, L'|_{lev}$) is defined to be the number of $t_k$ moves minus the number of $t_k^{-1}$ moves needed when we go from $L$ to $L'$ (we will show later (Corollary 1.2) that for $k > 2$ it does not depend on the choice of a path joining $L$ and $L'$).

The classical unknotting number is the $t_2$ distance from a given link to an unlink.

\textbf{Corollary 0.2.} Let $P_L(a, z)$ be a Jones-Conway polynomial described by the properties

(i) $P_{T_1}(a, z) = 1$, 

(ii) $P_L(a, z)$ is invariant under $t_k$ moves.
(ii) \( aP_{\overrightarrow{\Delta}}(a, z) + a^{-1}P_{\overleftarrow{\Delta}}(a, z) = zP_{\overleftarrow{\Delta}}(a, z) \),
where \( T_1 \) is a trivial knot. Then for \( z_0 = 2\cos(\pi m/k) \) \((z_0 \neq 0, \mp 2)\)
\[ P_{t_k(L)}(a, z_0) = (-1)^m a^{-k} P_L(a, z_0) \]
and for \( t_k \) equivalent links \( L \) and \( L' \)
\[ P_{L'}(a, z_0) = ((-1)^m a^{-k})^{L,L'}_{t_k} P_L(a, z_0) \]
and neither side is identically zero.

We can introduce a \( \overrightarrow{t}_k \) move and \( \overleftarrow{t}_k \) equivalence of oriented links \((\sim \overrightarrow{t}_k)\) similarly to the \( t_k \) move and \((\sim t_k)\) (see Fig. 0.2).

\[ (\overrightarrow{t}_k(L) \text{ is naturally oriented if } k \text{ is even}) \]

Fig. 0.2

**Corollary 0.3.** (1.8) If \( a_0^{2k} = (-1)^k \), \( a_0 \neq \mp i \), then
\[ P_{t_{2k}(L)}(a_0, z) = P_L(a_0, z). \]

**Corollary 0.4.** Let \( V_L(t) \) be the Jones polynomial described by the properties
(i) \( V_{T_1}(t) = 1 \),
(ii) \( t^{-1}V_{\overrightarrow{\Delta}}(t) - tV_{\overleftarrow{\Delta}}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{\overleftarrow{\Delta}}(t) \),
then

(a) If \( t^k = (-1)^k \) (i.e. \( t^{1/2} = e^{i\pi m/k} \)), \( t \neq -1 \), then a \( t_k \) move changes \( V_L(t) \) by \((-1)^m j^k\), that is

\[ V_{t_k(L)}(t) = (-1)^m j^k V_L(t). \]

(b) If \( t^{2k} = 1 \) (i.e. \( t = e^{i\pi m/k} \)), \( t \neq -1 \), then

\[ V_{t_{2k}(L)}(t) = V_L(t). \]

(c) Assume \( k \) is odd and \( t^k = -1 \). Then

\[ V_{\bar{t}_k(L)}(t) = \omega_{4k} V_L(t), \]

where \( \omega_{4k} \) is a properly chosen \( 4k \)-root of unity (depending also on the choice of the orientation of \( \bar{t}_k(L) \); see Theorem 1.13).

1. \( t_k \)-moves and Conway formulas for the Jones-Conway and Kauffman polynomials.

When one considers the sequence of links \( L, t_1(L), t_2(L), \ldots, (\overline{\cdots}, \overline{\cdots}) \) then the Jones-Conway (and Kauffman) polynomials \( P_L(a, z), P_{t_1(L)}(a, z), P_{t_2(L)}(a, z), \ldots \) form a (generalized) Fibonacci sequence. So one can expect that there is a nice formula which expresses \( P_{t_k(L)}(a, z) \) in terms of \( P_{t_1(L)}(a, z) \) and \( P_L(a, z) \) and in fact we have the following result:

**Theorem 1.1.** \( a^k P_{t_k(L)}(a, z) = av_1^{(k)}(z)P_{t_1(L)}(a, z) - v_1^{(k-1)}(z)P_{t_2(L)}(a, z), \) where \( v_1^{(k+2)}(z) = zv_1^{(k+1)}(z) - v_1^{(k)}(z) \) and \( v_1^{(-1)}(z) = -1, v_1^{(0)}(z) = 0, v_1^{(1)}(z) = 1. \) In particular if one substitutes \( z = p + p^{-1} \) one gets \( v_1^{(k)}(z) = \frac{p^k - p^{-k}}{p - p^{-1}}. \) [Added for e-print: \( v_1^{(k)}(z) \) is a variant of the Chebyshev polynomial of the second kind.]

**Proof.** We proceed by induction on \( k \). For \( k = 1, 2 \) the formula from Theorem 1.1 holds:

\[ aP_{t_1(L)}(a, z) = aP_{t_1(L)}(a, z) - 0 \cdot P_L(a, z) \]
and

\[ a^2 P_{t_2(L)}(a, z) = az P_{t_1(L)}(a, z) - P_L(a, z). \]

Assume that Theorem 1.1 holds for 1, 2, \ldots, k − 1, \((k > 2)\). Now one gets:

\[ a^k P_{t_k(L)}(a, z) = a^{k-1} z P_{t_{k-1}(L)}(a, z) - a^{k-2} z P_{t_{k-2}(L)}(a, z) = \]

\[ = z (a v_1^{(k-1)}(z) P_{t_1(L)}(a, z) - v_1^{(k-2)}(z) P_L(a, z)) - \]

\[ (a v_1^{(k-2)}(z) P_{t_1(L)}(a, z) - v_1^{(k-3)}(z) P_L(a, z)) = \]

\[ a(z v_1^{(k-1)}(z) P_{t_1(L)}(a, z) - v_1^{(k-2)}(z) P_{t_1(L)}(a, z)) - \]

\[ (z v_1^{(k-2)}(z) P_L(a, z) - v_1^{(k-3)}(z) P_L(a, z)) = \]

\[ a v_1^{(k)}(z) P_{t_1(L)}(a, z) - v_1^{(k-1)}(z) P_L(a, z)). \]

To see, that for \( z = p + p^{-1}, v_1^{(k)}(z) = \frac{p^{k-1} - p^{-k}}{p - p^{-1}} \) it is enough to observe that

\[ \frac{p^{k+2} - p^{-(k+2)}}{p - p^{-1}} = (p + p^{-1}) \frac{p^{k+1} - p^{-(k+1)}}{p - p^{-1}} = \frac{p^k - p^{-k}}{p - p^{-1}}. \]

\[ \blacksquare \]

**Corollary 1.2.** If \( p_0^{2k} = 1 \) (i.e. \( p_0 = e^{\pi i m/k} \), \( p_0 \neq \mp 1, \mp i \)) or equivalently \( z_0 = 2 \cos(\pi m/k); z_0 \neq 0, \mp 2 \) then

\[ P_{t_k(L)}(a, z_0) = (-1)^m a^{-k} P_L(a, z_0) \]

and for \( t_k \) equivalent links \( L \) and \( L' \)

\[ P_{L'}(a, z_0) = ((-1)^m a^{-k})^{[L,L']_{k}^{wv}} P_L(a, z_0) \] and neither side is identically zero.
Proof. Assume $p_0 \neq \mp 1, \mp i$. Then $v_1^{(k)}(z_0) = 0$ reduces to $p_0^{2k} = 1$, so $p_0 = e^{\pi im/k}$ and $z_0 = p_0 + p_0^{-1} = 2 \cos(\pi m/k)$. Now the equation from Theorem 1.1 reduces to

$$a^k P_{k(L)}(a, z_0) = -v_1^{(k-1)}(z_0)P_L(a, z_0) = -\frac{p_0^{-k} - p_0^{1-k}}{p_0 - p_0^{-1}} P_L(a, z_0) =$$

$$= p_0^k P_L(a, z_0) = (-1)^m P_L(a, z_0).$$

So the first part of Corollary 1.2 is proven.

For the second part it is enough to show that for each link $L$ and any complex number $z_0$ ($z_0 \neq 0$) $P_L(a, z_0)$ is never identically zero. It follows from the fact that $P_L(a, a + a^{-1}) \equiv 1$ (see [LM-1] or [P-1], or apply the standard induction: it holds for trivial links and whenever it holds for $L_-(\rightarrow)$ and $L_0(\rightarrow)$ it holds for $L_+(\rightarrow)$ and if it holds for $L_+$ and $L_0$ it holds for $L_-$). □

If $a = i$, $t^{1/2} = -ip$ in the Jones-Conway polynomial $P_L(a, z)$, ($z = p + p^{-1}$), we get the (normalized) Alexander polynomial $\Delta_L(t)$ which satisfies:

(i) $\Delta_{T_1}(t) = 1,$
(ii) $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}(t).$

**Corollary 1.3.** [Fe-1, Ki] If $t^k = (-1)^k$ (i.e. $t^{1/2} = -ie^{\pi im/k}$), $t \neq -1$ then $\Delta_{k(L)}(t) = (-1)^m(-i)^k \Delta_L(t)$.

Proof. It follows immediately from Corollary 1.2. One have only additionally notice that the formula from Corollary 1.2 remains true for $a = \mp i$, $p = \pm i$. □

When we substitute $a = it^{-1}$, $p = it^{1/2}$ in $P(a, z)$ ($z = p + p^{-1}$) we get the Jones polynomial $V_L(t)$ which satisfies:

(i) $V_{T_1}(t) = 1,$
(ii) $\frac{1}{t} V_{L_+}(t) - t V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}}) V_{L_0}(t).$

There has been some confusion as to the conventions. We use that of [Jo-2].
Corollary 1.4. If \( t^k = (-1)^k \) (i.e. \( t^{1/2} = -ie^{\pi i m/k} \), \( t \neq -1 \), then a \( t_k \) move changes \( V_L(t) \) by \( (-1)^m i^k \) that is

\[
V_{t_k}(t) = (-1)^m i^k V_L(t).
\]

Proof. It is true for \( t = 1 \) (then \( k \) is even). For \( t \neq \mp 1 \) it follows immediately from Corollary 1.2. \( \square \)

Corollary 1.5. If \( p_0 = \varepsilon = \mp 1 \) (so \( z_0 = 2\varepsilon = \pm 2 \) then

(a) \( a^k P_{t_k(L)}(a, z_0) = \varepsilon a^k P_{t_k(L)}(a, z_0) - \varepsilon (k-1) P_L(a, z_0) \) and

(b) \( P_{t_k(L)}(a, z_0) \equiv \varepsilon a^{-k} P_L(a, z_0) \pmod {k/2} \) i.e. the equality holds if \( P_L(a, z_0) \)
is understood to be a Laurent polynomial in \( a \) with coefficients in the ring \( \mathbb{Z}[1/2]/k\mathbb{Z}[1/2] \).

Corollary 1.6. (Generalized Conway formula). The following formula holds for the Jones-Conway polynomial:

\[
a^k P_{L+k}(a, z) + a^{-k} P_{L-k}(a, z) = w_1^{(k)}(z) P_L(a, z),
\]

where \( w_1^{(0)} = 2, w_1^{(1)} = z, w_1^{(k)} = zw_1^{(k-1)} - w_1^{(k-2)} \). After substituting \( z = p + p^{-1} \)
one gets \( w_1^{(k)} = p^k + p^{-k} \).

Proof. From Theorem 1.1 one gets:

\[
a^k P_{L+k} = a v_1^{(k)} P_{L+1} - v_1^{(k-1)} P_L.\]

and

\[
a^{-k} P_{L-k} = a^{-1} v_1^{(k)} P_{L-1} - v_1^{(k-1)} P_L.\]

Adding these equations by sides one gets:

\[
a^k P_{L+k} + a^{-k} P_{L-k} = v_1^{(k)} (a P_{L+1} + a^{-1} P_{L-1}) - 2v_1^{(k-1)} P_L =
\]
\[ J.H.\text{Przytycki} = (zv_1^{(k)} - 2v_1^{(k-1)})P_{L_0}. \]

Now substituting \( w_1^{(k)} = zv_1^{(k)} - 2v_1^{(k-1)} \) one gets the equation from Corollary 1.6 (notice that \( v_1^{(-1)} = -1 \)). \[ \square \]

Now we will get formulas for \( \bar{t}_k \) moves analogous to those for \( t_k \) moves.

**Theorem 1.7.** \( P_{\bar{t}_{2k}(L)}(a, z) = (-1)^k a^{2k} P_{L_0}(a, z) + zu_1^{(2k)}(a) P_L(a, z) \) where \( \bar{t}_{2k}(L), L \) are oriented diagrams which are identical, except the parts of the diagrams shown on Fig. 1.1, and \( u_1^{(0)} = 0, u_1^{(2)} = a, u_1^{(2k)} = -a^2 u_1^{(2(k-1))} + a \) or equivalently \( u_1^{(2k)} = (-1)^{k+1} a^{k+1} \frac{a^{k+(-1)^{k+1}a^{-k}}}{a+a^{-1}} \).

![Fig. 1.1](image)

Proof. We proceed by induction on \( k \). For \( k = 0, 1 \) the formula from Theorem 1.7 holds:

\[ P_{L_0}(a, z) = P_L(a, z) + z \cdot 0 \cdot P_L(a, z) \]

and

\[ P_{\bar{t}_{2k}(L)}(a, z) = -a^2 P_{L_0}(a, z) + zaP_L(a, z). \]

Assume that Theorem 1.7 holds for 0, 1, \ldots, \( k - 1 \) \((k \geq 2) \). Now one gets:
\[
P_{t_{2k}(L)} = -a^2 P_{t_{2(k-1)}(L)} + za P_L = -a^2 ((-1)^{k-1} a^{2(k-1)} P_L)
\]
\[
- a^2 (zu_1^{(2(k-1))} P_L) + za P_L = (-1)^{k} a^{2k} P_L
\]
\[
z(-a^2 u_1^{(2(k-1))} + a) P_L = (-1)^k a^{2k} P_L + zu_1^{(2k)} P_L.
\]

\[\square\]

**Corollary 1.8.** If \(a_0^{2k} = (-1)^k (a_0 \neq \mp i)\), then

\[P_{t_{2k}(L)}(a_0, z) = P_L(a_0, z).\]

**Corollary 1.9.** If \(a_0 = \varepsilon i = \mp i\) then

(a) \(P_{t_{2k}(L)}(a_0, z) = P_L(a_0, z) + z\varepsilon ik P_L\) \((a_0, z)\) and

(b) \(P_{t_{2k}(L)}(a_0, z) \equiv P_L(a_0, z) \pmod k\) i.e. equality holds if \(P_L(a_0, z)\) is understood to be the Laurent polynomial in \(z\) with coefficients in the ring \(\mathbb{Z} + i\mathbb{Z}/k(\mathbb{Z} + i\mathbb{Z})\).

(c) \(\text{[Fo-1]}\) If \(a = i\), \(t^{1/2} = -ip (z = p + p^{-1})\) one gets the (normalized) Alexander polynomial and (b) reduces to \(\Delta_{t_{2k}(L)}(t) \equiv \Delta_L(t) \pmod k\) i.e. the equality holds if \(\Delta_L(t)\) is reduced to a polynomial in \(\mathbb{Z}_k[\sqrt{t^{1/2}}]\).

For the Jones polynomial \((a = it^{-1}, p = it^{1/2})\), Corollary 1.8 reduces to:

**Corollary 1.10.** If \(t^{2k} = 1\), \(t \neq -1\) then

\[V_{t_{2k}(L)}(t) = V_L(t).\]

**Proof.** It is true for \(t = 1\). For \(t \neq \mp 1\) it follows from Corollary 1.8. \(\square\)
Corollary 1.11. (Generalized Conway formula). The following formulas hold for the Jones-Conway polynomial:

(i) \( a^{-2k}P_{t_{2k}}(a, z) + a^{2k}P_{t_{2k}^{-1}}(a, z) = (-1)^k 2P_L(a, z) + \left((\frac{a^{k+1}a^{-k}}{a+a^{-1}})^2P_L(a, z), \right) \)

(ii) \( a^{-k}P_{t_{2k}}(a, z) + (-1)^k a^k P_{t_{2k}^{-1}}(a, z) = ((-1)^k a^k + a^{-k})P_L(a, z), \)

(iii) \( a^{-k}P_{t_k}(a, z) + (-1)^{k+1}a^k P_{t_k^{-1}}(a, z) = z(\frac{a^{-k+\varepsilon(k)a^k}}{a+a^{-1}})P_L(a, z), \)

where

\[ \varepsilon(k) = \begin{cases} 
-1 & \text{if } k + 2 \text{ is a multiple of } 4, \\
1 & \text{otherwise}
\end{cases} \]

Proof. (i) follows immediately from Theorem 1.7; one has to add equations for \( a^{-2k}P_{t_{2k}}(L) \) and for \( a^{2k}P_{t_{2k}^{-1}}(L), \)

(ii) follows from Theorem 1.7, by adding equations for \( a^{-k}P_{t_{2k}}(L) \) and for \( (-1)^k a^k P_{t_{2k}^{-1}}(L). \)

(iii) \( (k \text{ even}) \) follows from Theorem 1.7 by adding equations for \( a^{-2k}P_{t_k}(L) \) and \( (-1)^{k+1}a^k P_{t_k^{-1}}(L). \) If \( k \text{ is odd} \) then from Theorem 1.7 one gets

\[ a^{-(2k+1)}P_{t_{2k+1}}(a, z) = (-1)^ka^{-1}P_{t_1}(L) + za^{-(2k+1)}u_1^{(2k)}(a)P_L(a, z) \]

and

\[ a^{2k+1}P_{t_{2k+1}^{-1}}(a, z) = (-1)^kaP_{t_1^{-1}}(a, z) + za^{2k+1}((-1)^{k+1}a^{-2k}u_1^{(2k)}(a))P_L(a, z) \]

(in the last equality we use the fact that \( u_1^{(2k)}(1/a) = (-1)^{k+1}a^{-2k}u_1^{(2k)}(a) \))

Adding the above equalities one gets:

\[ a^{-2k+1}P_{t_{2k+1}}(L) + a^{2k+1}P_{t_{2k+1}^{-1}}(L) = (-1)^k(a^{-1}P_{t_1}(L) + aP_{t_1^{-1}}(L)) + \]

\[ + (z(1)^{-k-1}a^{-k} + (-1)^{k+1}a^{-k}) \frac{a^k + (-1)^{k+1}a^{-k}}{a + a^{-1}} + za^{k+1} \frac{a^k + (-1)^{k+1}a^{-k}}{a + a^{-1}})P_L(a, z) \]
\[
\begin{align*}
    z P_L \cdot \left( (-1)^k + \frac{(-1)^{k+1}a^{-1} + a^{-(2k+1)} + a^{2k+1} + (-1)^{k+1}a}{a + a^{-1}} \right) &= \\
    = z\left( \frac{a^{2k+1} + a^{-(2k+1)}}{a + a^{-1}} \right) P_L.
\end{align*}
\]

We worked, till now, with \( \bar{t}_k \) moves for \( k \) even, and the reason for this was that if \( L \) is oriented then \( \bar{t}_k(L) \) has no any natural orientation for \( k \) odd. For the Jones polynomial, however, one has Jones reversing result (see [1] or [2]) so one can still find how \( V_L(t) \) is changed under a \( \bar{t}_k \) move.

Namely let \( L = \{ L_1, \ldots, L_i, \ldots, L_n \} \) be an oriented link of \( n \) components and \( L' = \{ L_1, \ldots, -L_i, \ldots, L_n \} \), i.e. the orientation of \( L_i \) is reversed, and let \( \lambda = \text{lk}(L_i, L - L_i) \). Then

\begin{enumerate}
    \item \( V_{L'}(t) = t^{-3\lambda}V_L(t) \).
\end{enumerate}

**Theorem 1.13.** Consider a \( \bar{t}_k \) move on an oriented link \( L \), and assume \( k \) is odd. We have two cases:

(i) \( c(\bar{t}_k(L)) < c(L) \), where \( c(L) \) denotes the number of components. That is two components of \( L \), say \( L_i \) and \( L_j \), are involved in the \( \bar{t}_k \) move (see Fig. 1.2). Let \( \lambda = \text{lk}(L_i, L - L_i) \). Then for \( t^k = (-1)^k \) (i.e. \( t^{1/2} = -ie^{\pi i m/k} \)), \( i \neq -1 \):

\[
    V_{\bar{t}_k(L)}(t) = (-1)^m t^{-3\lambda}V_L(t) = (-1)^{m+\lambda} t^k e^{-6\pi i m \lambda / k} V_L(t),
\]

where the orientation of \( \bar{t}_k(L) \) is chosen so that it does not agree with the orientation of \( L_i \).

(ii) \( c(\bar{t}_k(L)) = c(L) \). That is one component of \( L \) is involved in the \( \bar{t}_k \) move. Let \( L' \) denote the smoothing of \( L \) (Fig. 1.3). \( L' \) has more components than \( L \) and let \( L_i, L_j \) be the new components of \( L' \) (Fig. 1.3). Let \( \lambda = \text{lk}(L_i, L - L_i) \) and assume that \( \bar{t}_k(L) \) is oriented in such a way that its orientation agrees with that of \( L_j \) (with exception of \( L_i \). Then for \( t^k = (-1)^k \) (i.e. \( t^{1/2} = -ie^{\pi i m/k} \)), \( i \neq -1 \):
\[ V_{t_k(L)}(t) = t^{-3\lambda}V_L(t) = (-1)^{\lambda}e^{-6\pi im\lambda/k}V_L(t). \]

**Proof.**

(i) We use the Jones reversing result and Corollary 1.4 and we get (see Fig. 1.2)

\[ V_{t_k(L)}(t) = (-1)^{m_i}V_{L'}(t) = (-1)^{m_i}t^{-3\lambda}V_L(t) = (-1)^{m_i}e^{6\pi im\lambda/k}V_L(t). \]

![Fig. 1.2](image)

(ii) We use Corollary 1.4 and the part (i) and we get (see Fig. 1.4):

\[ V_{t_k(L)}(t) = (-1)^{m_i}t^{-3\lambda+k}V_{L'}(t) = (-1)^{m_i}t^{-3\lambda+k}(-1)^{m_i}V_L(t) = t^{-3\lambda}V_L(t) =
\]

\[ (-1)^{\lambda}e^{-6\pi im\lambda/k}V_L(t). \]

![Fig. 1.3](image)
It is possible to get Theorem 1.13 by considering the variant of the Jones polynomial which is an invariant of regular isotopy and does not depend on orientation (\cite{Ka-1}).

We will use this idea considering how the Kauffman polynomial changes under \(t_k\) and \(\overline{t}_k\) moves.
Two diagrams of links are regularly isotopic iff one can be obtained from the other by a sequence of Reidemeister moves of type $\Omega_2^\pm$ and isotopy of the projection plane (see Fig. 1.5).

\begin{align*}
\begin{array}{c}
\text{Fig. 1.5} \\
\end{array}
\end{align*}

The Kauffman polynomial of regular isotopy of unoriented diagrams is defined by (see [Ka-3]; also [P-1]):

\begin{enumerate}
\item $\Lambda_{T_1}(a, x) = a^{\text{tw}(T_1)}$, where $T_1$ is a diagram representing the trivial knot (up to isotopy) and $\text{tw}(T_1) = \sum \text{sgn } p$ where the sum is taken over all crossings of $T_1$.
\item $\Lambda_{\bigotimes}(a, x) + \Lambda_{\bigotimes}(a, x) = x\Lambda_{\bigotimes}(a, x) + x\Lambda_{\bigotimes}(a, x)$.
\end{enumerate}

The Kauffman polynomial of oriented links is defined by

$$F_L(a, x) = a^{-\text{tw}(L)}\Lambda_L(a, x).$$

**Theorem 1.14.** $\Lambda_{\underbrace{\bigotimes\cdots\bigotimes}_k}(a, x) = v_1^{(k)}(a, x)\Lambda_{\bigotimes}(a, x) - v_1^{(k-1)}(x)\Lambda_{\bigotimes}(a, x) + xv_2^{(k)}(a, x)\Lambda_{\bigotimes}(a, x)$, where $v_1^{(k)}(a, x)$ is the same as in Theorem 1.1 and $v_2^{(0)}(a, x) = 0$, $v_2^{(1)}(a, x) = 0$, $v_2^{(2)}(a, x) = a^{-1}$, $v_2^{(k)}(a, x) = xv_2^{(k-1)}(a, x) - v_2^{(k-2)}(a, x) + a^{1-k}$. In particular for $x = p+p^{-1}$ one gets $v_1^{(k)}(a, x) = \frac{p^k-p^{-k}}{p-p^{-1}}$, $v_2^{(k)} = ((p-p^{-1})(a+a^{-1}-(p+p^{-1}))^{-1}(-a^{-1}(p^k-p^{-k})+p(a^{-k}-p^{-k})-p^{-1}(a^{-k}-p^k))$. 


**Proof.** We proceed by induction on $k$. For $k = 0, 1, 2$ the formula from Theorem 1.14 holds. Assume that it holds for $0, 1, \ldots, k - 1$ ($k > 2$). Now one gets:

$$
\begin{align*}
\Lambda_{\infty-\infty} = & \ x\Lambda_{\infty-\infty} - \Lambda_{\infty-\infty} + xa^{1-k} \Lambda 
= & \ x(v^{(k-1)}_1 \Lambda \bigotimes - v^{(k-1)}_1 \Lambda \bigotimes + xv^{(k-1)}_2 \Lambda \bigotimes ) - (v^{(k-2)}_1 \Lambda \bigotimes - v^{(k-3)}_1 \Lambda \bigotimes + xv^{(k-2)}_2 \Lambda \bigotimes ) + xa^{1-k} \Lambda \\
= & \ (xv^{(k-1)}_1 - v^{(k-2)}_1) \Lambda \bigotimes - (xv^{(k-2)}_1 - v^{(k-3)}_1) \Lambda \bigotimes + x(xv^{(k-2)}_1 - v^{(k-2)}_2 + a^{1-k}) \Lambda \\
= & \ v^{(k)}_1 \Lambda \bigotimes - v^{(k-1)}_1 \Lambda \bigotimes + xv^{(k)}_2 \Lambda \bigotimes .
\end{align*}
$$

The formula for $v^{(k)}_2(a, p + p^{-1})$ may be verified directly but we omit this tedious task by considering the trivial links of Fig. 1.6. From this figure we get immediately that

$$
a^{-k} = \frac{p^k - p^{-k}}{p - p^{-1}} a^{-1} - \frac{p^{k-1} - p^{-(k-1)}}{p - p^{-1}} + (p + p^{-1})v^{(k)}_2(a, x) \frac{a + a^{-1} - (p + p^{-1})}{p + p^{-1}},
$$

and it finishes the proof of Theorem 1.14. \qed

Fig. 1.6
Corollary 1.15. (a) If \( p_0^2 = 1 \) (i.e. \( p_0 = e^{\pi i m/k} \), \( p_0 \neq \mp 1, \mp i \) or equivalently \( x_0 = 2 \cos(\pi m/k) \), \( x_0 \neq 0, \mp 2 \), then

\[
\Lambda_{\infty-x} (a, x_0) = (-1)^m \Lambda_{\infty-x} (a, x_0) + \frac{a^{-k} - p_0^{-k}}{(a - p_0)(1 - a^{-1}p_0^{-1})} \Lambda (a, x_0).
\]

(b) If \( p_0^2 = 1, p_0 \neq \mp 1, \mp i, a^k = p_0^k, a_0 \neq p_0^\mp 1 \) then

\[
\Lambda_{\infty-x} (a_0, x_0) = (-1)^m \Lambda_{\infty-x} (a_0, x_0).
\]

(c) If \( p_0 = \varepsilon = \mp 1 \) (so \( x_0 = 2 \varepsilon = \mp 2 \)) and \( a^k = \varepsilon^k, a_0 \neq \varepsilon \) then

\[
\Lambda_{\infty-x} (a_0, x_0) \equiv \varepsilon \Lambda_{\infty-x} (a_0, x_0) (\text{mod } k/2^i),
\]

i.e. equality holds in the ring \( \mathbb{Z}[a_0/2]/k\mathbb{Z}[a_0/2] \).

Proof. It follows from Theorem 1.14 similarly as Corollaries 1.2 and 1.5 followed from Theorem 1.1.

Corollary 1.16. (Generalized Conway formula)

\[
\Lambda_{\infty-x}^{(k)} (a, x) + \Lambda_{\infty-x}^{(k)} (a, x) = w_1^{(k)} (a, x) + xw_2^{(k)} (a, x) \Lambda (a, x),
\]

where \( w_1^{(k)} (x) = w_1^k (z) \) from Corollary 1.6, and

\[
w_2^{(0)} (a, x) = 0, \; w_2^{(1)} (a, x) = 1, \; w_2^{(k)} (a, x) = xw_2^{(k-1)} (a, x) - w_2^{(k-2)} (a, x) + a^{k-1} + a^{1-k};
\]

when one substitutes \( x = p + p^{-1} \) then

\[
w_1^{(k)} (x) = p^k + p^{-k}
\]

and

\[
w_2^{(k)} (a, x) = a^{-(k-1)} p^{-(k-1)} \frac{a^k - p^k}{a - p} \left( 1 - a^k p^k \right). \]
proof. From Theorem 1.14 one gets:

\[ \Lambda_{\infty - \infty} = v_1^{(k)} \Lambda_\infty - v_1^{(k-1)} \Lambda_{\infty \cdot \infty} + x v_2^{(k)}(a, x) \Lambda \setminus \{ \}
\]

and

\[ \Lambda_{\infty - \infty} = v_1^{(k)} \Lambda_\infty - v_1^{(k-1)} \Lambda_{\infty \cdot \infty} + x v_2^{(k)}(a^{-1}, x) \Lambda \setminus \{ \}
\]

Adding the above equations by sides one gets:

\[ \Lambda_{\infty - \infty} + \Lambda_{\infty - \infty} = v_1^{(k)} (\Lambda_\infty + \Lambda_{\infty \cdot \infty} ) - 2v_1^{(k-1)} \Lambda_{\infty \cdot \infty} +
\]

\[ + x (v_2^{(k)}(a, x) + v_2^{(k)}(a^{-1}, x)) \Lambda \setminus \{ \}
\]

\[ = (x v_1^k - 2v_1^{(k-1)}) \Lambda_{\infty \cdot \infty} +
\]

\[ + x (v_1^{(k)} + v_2^{(k)}(a, x) + v_2^{(k)}(a^{-1}, x)) \Lambda \setminus \{ \}
\]

Now substituting \( w_1^{(k)} = x v_1^k - 2v_1^{(k-1)} \) and \( w_2^{(k)} = v_1^k(a) + v_2^{(k)}(a, x) + v_2^{(k)}(a^{-1}, x) \)

one gets the equation from Corollary 1.16.

We end this part of the paper by translating Corollary 1.15(b) into the Kauffman polynomial of oriented links.

\[ \text{Corollary 1.17. If } p_0^{2k} = 1, p_0 \neq \pm 1, \; \mp i \text{ and } a_0^k = p_0^k, a_0 \neq p_0^{\mp 1} \text{ then}
\]

(a) \( F_{\infty - \infty}(a_0, p_0) = a_0^k a_0^{tw(\infty \cdot \infty)} F_{\infty \cdot \infty}(a_0, p_0); \)

In particular

(b) \( F_k(L)(a_0, p_0) = F_L(a_0, p_0), \)

(c) \( F_{\bar{t}_k(L)}(a_0, p_0) = F_L(a_0, p_0), \)

(d) \( F_{\bar{t}_k(L)}(a_0, p_0) = a^{4k} F_L(a_0, p_0), \) where \( k \) is odd and \( \lambda \) defined as follows (compare Theorem 1.13):

(i) If \( L \) has more components than \( \bar{t}_k(L) \) and \( L_i \) is the only component of \( L \) such that the chosen orientation on \( \bar{t}_k(L) \) does not agree with that of \( L_i \) then \( \lambda = lk(L_i, L - L_i) \) (compare Fig. 1.2).
(ii) If $L$ has the same number of components as $\tilde{t}_k(L)$, consider the smoothing $L$ of $L$ (Fig. 1.3). Let $L_i$ be the only component of $L$ such that the chosen orientation on $\tilde{t}_k(L)$ does not agree with that of $L_i$ then $\lambda = lk(L_i, L - L_i)$ (compare Fig. 1.3).

Proof. (a) follows immediately from Corollary 1.15(b) and the definition of $F_L(a, x)$. (b) and (c) hold because in these cases $tw(\overline{\infty}) - tw(\infty^{-x}) = \mp k$;

(d) $tw(\overline{\infty}) - tw(\infty^{-x}) = \begin{cases} 4\lambda - k & \text{in the case (i)} \\ 4\lambda + k & \text{in the case (ii)} \end{cases}$

so the equality (d) holds.

When one substitutes $a = t^{-3/4}$, $x = -(t^{1/4} + t^{-1/4})$ in the Kauffman polynomial $F(a, x)$ one gets the Jones polynomial $V(t)$ ([Li], see also [P-1]). Corollary 1.15 gives, therefore, some information about the behaviour of $V(t)$ under $t_k$ and $\tilde{t}_k$ moves. It happens, however, that one gets no new information comparing with Corollaries 1.4, 1.10, and Theorem 1.13.

Theorem 1.1 and 1.14 can be stated as one theorem if one uses the three variable polynomial $J_L(a, x, z)$ which generalizes the Jones-Conway and Kauffman polynomials (see [P-1]), however, one cannot gain any new information from this approach.
2. Historical Background (Fox Congruence Classes).

The unknotting number of a knot was considered probably before knot theory became a science. It was a natural question to ask how many times one has to "cheat" to get from a knot an unknot. K. Reidemeister wrote in 1932 in his book [Re]: "It is very easy to define a number of knot invariants so long as one is not concerned with giving algorithms for their computation ... One can change each knot projection into projection of circle by reversing the overcrossings and undercrossings at, say, $k$ double points of the projection. The minimal number $u(K)$ of these operations, that is, the minimal number of self-piercings, by which a knot is transformed into a circle, is a natural measure of knottedness".

The first interesting results about unknotting number were found by H. Wendt [We] in 1937. Namely Wendt proved that if $u(K)$ is the unknotting number of $K$ and $e_s$ is the minimal number of generators of the group $H_1(M_s^K, Z)$, where $M_s^K$ is the cyclic, $s$-fold branched cover of $(S^3, K)$ then

$$e_s \leq u(K)(s - 1).$$

$t_k$ and $\bar{t}_k$ moves ($\rightsquigarrow \overline{\rightsquigarrow} \overline{\overline{\rightsquigarrow}} \ldots \overline{\rightsquigarrow}$) appear to have been first explicitly considered by S. Kinoshita in 1957 [Kin-1], who observed that the Wendt inequality is also valid if we allow all $t_{2k}$ and $\bar{t}_{2k}$ moves, not only $t_2$ moves (see Corollary 2.6(b)). The following year, 1958, R. Fox [Fo-1] considered twists of knots and congruence of knots modulo $(n, q)$; the notion which is closely related, and in some sense more general, than $t_{2k}$ and $\bar{t}_{2k}$ moves. Congruence modulo $(n, q)$ was chosen so that the Alexander polynomial (or more generally Alexander module) is a good tool to study this.

The same year (1958), S. Kinoshita [Kin-2] used the Fox twists to generalize once more the Wendt inequality (see Corollary 2.6). The Fox approach is related to ours so we will present it here with some details. We follow the Fox paper [Fo-1] taking into account the corrections made by Kinoshita [Kin-3] and Nakanishi and Suzuki [N-S]. I am grateful to K. Murasugi and H. Murakami for informing me about the Fox paper and about the Kawauchi and Nakanishi conjectures.
Consider the following homeomorphism $\tau$ of a 3-disk $D^3 = \langle 0, 1 \rangle \times D^2$: $\tau(t, z) = (t, e^{2\pi i t} z)$. It is the natural extension to $\langle 0, 1 \rangle \times D^2$ of the Dehn twist on the annulus $\langle 0, 1 \rangle \times \partial D^2$ (see Fig. 2.1).

We call $\tau$ a simple twist or a Dehn twist. Now whenever we have a properly embedded 2-disk in a 3-manifold $M$ (and either $M$ or a tubular neighbourhood of the disk is oriented), we have uniquely (up to isotopy) associated with the disk the Dehn twist (the twist is carried by a tubular neighbourhood of the disk). In particular for an oriented solid torus there is only one nontrivial Dehn twist, because there is only one, up to isotopy, nontrivial proper disk in it.

Now let $L$ be a link in $\Sigma^3$ (we will assume $\Sigma^3 = S^3$, but in fact $\Sigma^3$ can be any homology 3-sphere), and $D^2$ a disk which cuts $L$ transversely. Let $V_2$ be the solid torus - a small tubular neighbourhood of $\partial D^2$ in $\Sigma^3$, and $V_1$ the closure of its complement ($V_1 = \Sigma^3 - V_2$). If $\Sigma^3 = S^3$, $V_1$ is a solid torus too. Now perform the Dehn twist on $V_1$ using the disk $D^2$. The twist restricted to the link $L$ is denoted by $t_{2,q}$ where $q \geq 0$ is the absolute value of the crossing number of $D^2$ and $L$. By $t_{2n,q}$ we denote $t_{2,q}^n$. Notice that our $t_{2n}^\alpha$ move is special case of $t_{2n,2}$ move, and $t_{2n}$ move is a special case of $t_{2n,0}$ move. Two oriented links $L_1$ and $L_2$ are called, by Fox, \textit{congruent modulo $n,q$} ($L_1 \equiv L_2 \pmod{n,q}$) if one can go from $L_1$ to $L_2$ using $t_{2n,q}^{\pm 1}$ moves (and isotopy), where $q'$ can vary but is always a multiple of $q$. If we allow only
$t_{2n,q}^\pm$ moves then we say, after Nakanishi and Suzuki, that $L_1$ and $L_2$ are $q$-congruent modulo $n$ ($L_1 \equiv_q L_2 \pmod{n}$) or that they are $t_{2n,q}$ equivalent ($L_1 \sim_{t_{2n,q}} L_2$). The Alexander polynomial (and module) is a nice tool for distinguishing nonequivalent links because $L$ and $t_{2n,q}(L)$ are the same outside the ball in which the move occurs.

\textbf{Theorem 2.1.} (a) $t_{2n,q}$ equivalent links have the same Alexander module modulo $\frac{(t-1)(t^{nq}-1)}{t^q-1}$, in particular

(b) for $t_0^{nq} = 1$ ($t_0^q \neq 1$ or $t_0 = 1$) $\Delta_L(t) \equiv \Delta_{t_{2n,q}(L)}(t)$.

It can be understood as follows: $\Delta_L(t)$ and $\Delta_{t_{2n,q}(L)}(t)$ are equal as elements of the ring $R = \mathbb{Z}[t^{\pm 1}]/(t-1)(1+t^q+\ldots+t^{(n-1)q})$, up to multiplication by invertible elements of $R$ (in fact up to multiplication by classes of invertible elements in $\mathbb{Z}[t^{\pm 1}]$ i.e. $\mp t^p$).

If we substitute $a = i$ and $p = it^{1/2}$ in the Jones-Conway polynomial then we get the (normalized) Alexander polynomial $\Delta(t) \in \mathbb{Z}[t^{\pm 1}] \cup \sqrt{t}\mathbb{Z}[t^{\pm 1}]$. From our Corollary 1.3 follows that if $t^{2k} = 1$, $t \neq -1$, then $t_{2k}$ move changes $\Delta_L(t)$ by the factor $\varepsilon = t^k = \mp 1$ (i.e. $\Delta_{t_{2k}(L)}(t) = \varepsilon \Delta_L(t)$). Therefore $t_{2k}$ moves have, more less, the same influence on $\Delta_L(t)$ as more general $t_{2k,2}$ moves; however it is not true that every $t_{2k,2}$ move is a combination of $t_{2k}$ moves (see Example 3.8(b)).
Proof. Consider a small ball $B^3$ in which $t_{2n,q}$ move takes place (Fig. 2.2). $B^3 \cap L$ consists of $m$ parallel strings.

$m = 3$

Fig. 2.2

$\Sigma^3 - L - \text{int}B^3$ is homeomorphic to $\Sigma^3 - t_{2n,q}(L) - \text{int}B^3$ and the fundamental groups of these spaces have the following presentation:

$$\{a_1, \ldots, a_{m-1}, a'_1, \ldots, a'_m, x_1, \ldots : r_1, \ldots\},$$

where $a_1, \ldots, a_{m-1}, a'_1, \ldots, a'_m$ form a basis of the free group $\pi_1(\partial B^3 - L)$; see Fig. 2.2(a).

$\Sigma^3 - L$ and $\Sigma^3 - t_{2n,q}(L)$ can be obtained from $\Sigma^3 - L - B^3$ by adding $m - 1$ two-disks in the appropriate way (Fig. 2.2(b)). Therefore

$$\pi_1(\Sigma^3 - L) = \{a_1, \ldots, a_{m-1}, a'_1, \ldots, a'_m, x_1, \ldots : u_1, u_2, \ldots, u_{m-1}, r_1, \ldots\},$$

where $u_i = a'_i a_i^{-1}$, $i = 1, \ldots, m - 1$ (see Fig. 2.2(b)), and

$$\pi_1(\Sigma^3 - t_{2n,q}(L)) = \{a_1, \ldots, a_{m-1}, a'_1, \ldots, a'_m, x_1, \ldots : \tau(u_1), \tau(u_2), \ldots, \tau(u_{m-1}), r_1, \ldots\},$$

where $\tau(u_i) = (a'_m, \ldots, a'_1)^n a'_i (a'_m, \ldots, a'_1)^{-n} a'_i^{-1}$. 
Consider the natural projections \( p = p_2p_1 : \pi_1(\Sigma^3 - L) \xrightarrow{p_2} H_1(\Sigma^3 - L) \xrightarrow{p_1} \mathbb{Z} \), where \( p \) sends meridians of \( L \) onto \( t \) - a generator of integers, and \( p' : \pi_1(\Sigma^3 - t_{2n,q}(L)) \to \mathbb{Z} \). Then

\[
p(a_i) = p(a'_i) = t^{i+1} \quad \text{and} \quad p(a'_m a_{m-1} \ldots a'_1) = t^q = p(a_m a_{m-1} \ldots a_1) \quad \text{(without the loss of generality one can assume that the crossing number of } D^2 \text{ and } L \text{ is nonnegative so equal to } q).
\]

In particular if \( i \) and \( i' \) are embeddings of \( \Sigma^3 - L - B^3 \), in \( \Sigma^3 - L \) and \( \Sigma^3 - t_{2n,q}(L) \) respectively then \( p_i = p_i' \) (lack of this condition was the source of the mistake in the Fox paper [Fo-1]).

Now one can use Fox calculus to find Alexander-Fox modules of group representations \( p : \pi_1(\Sigma^3 - L) \to \mathbb{Z} \) and \( p' : \pi_1(\Sigma^3 - t_{2n,q}(L)) \to \mathbb{Z} \), and because

\[
p_*\left( \frac{\partial u}{\partial a'_j} \right) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}
\]

\[
p'_*\left( \frac{\partial \tau(u_i)}{\partial a'_{j+n}} \right) = \begin{cases} \frac{(1-t)(1-t^{nq})t^b_j}{1-t^q}, & \text{where } t^b_j = p(a'_m \ldots a'_{j+n}) \quad \text{if } i \neq j \\ \frac{(1-t)(1-t^{nq})t^b_j}{1-t^q} + t^{nq} & \text{if } i = j \end{cases}
\]

therefore

\[
p_*\left( \frac{\partial u}{\partial a'_j} \right) = p'_*\left( \frac{\partial \tau(u_i)}{\partial a'_{j+n}} \right) \mod \frac{(1-t)(1-t^{nq})}{1-t^q} \text{ and one gets:}
\]

**Lemma 2.2.** The Alexander-Fox modules of \( p : \pi_1(\Sigma^3 - L) \to \mathbb{Z} \) and \( p' : \pi_1(\Sigma^3 - t_{2n,q}(L)) \to \mathbb{Z} \) can be represented by the matrices which are the same modulo \( \frac{(1-t)(1-t^{nq})}{1-t^q} \).

Theorem 2.1 follows immediately from the lemma.
Corollary 2.3.  (a) Let \( qn \) be a multiple of \( s \) and \( k = \frac{n \cdot \gcd(q,s)}{s} \); where \( \gcd() \) is the greatest common divisor, then a \( t_{2n,q} \) move does not change \( H_1(M_s^L, \mathbb{Z}_k) \).

In particular

(b) If \( q \) is a multiple of \( s \) (e.g. \( q = 0 \)) then a \( t_{2n,q} \) move does not change \( H_1(M_s^L, \mathbb{Z}_n) \).

(c) Let \( n \) be a multiple of \( s \), and \( s \) and \( q \) are coprime then a \( t_{2n,q} \) move does not change \( H_1(M_s^L, \mathbb{Z}) \).

Proof. Alexander matrices can be used to describe \( H_1(M_s^L, \mathbb{Z}) \) as \( \mathbb{Z}[\mathbb{Z}_s] \) module \((t^s = 1)\). Then we use Lemma 2.2. \( \square \)

Corollary 2.4.  (a) Let \( \overline{t}_m(L) \) denote the minimal number of \( t_{2n,q} \) moves (we allow different \( n \) or \( q \)) but the number of strings involved in a \( t_{2n,q} \) must be less or equal \( m \) which are needed to change a given link \( L \) into unlink then

\[
|e_s - s(c(L) - 1)| \leq (s - 1)(m - 1)\overline{t}_m(L),
\]

where \( c(L) \) is the number of components of \( L \) and \( e_s \) is the minimal number of generators of \( H_1(M_s^L, \mathbb{Z}) \). In particular for \( \overline{u}(L) = \overline{t}_2(L) \) one gets:

(b) \([\text{Kin-1}]\) The minimal number of \( t_{2n} \) or \( \overline{t}_{2n} \) moves which are needed to change a given link \( L \) into unlink (\( \overline{u}(L) \)), satisfies:

\[
|e_s - s(c(L) - 1)| \leq (s - 1)\overline{u}(L).
\]

Proof. If \( T_n \) is a trivial link of \( n \) components then \( e_s(T_n) = s(n - 1) \). By the proof of Lemma 2.2, \( H_1(M_s^{t_{2n,q}L}, \mathbb{Z}) \) has a presentation which differs from a presentation of \( H_1(M_s^{L}, \mathbb{Z}) \) at most in \((s - 1)(m - 1)\) rows (we use additionally the fact that \( \frac{t^s - 1}{t - 1} = 1 + \ldots t^{s-1} \) is an annihilator of \( H_1(M_s^L, \mathbb{Z}) \) (see \([\text{B-Z}]\)), so

\[
|e_s(t_{2n,q}(L)) - e_s(L)| \leq (s - 1)(m - 1).
\]

\( \square \)
The Fox method (and Lemma 2.2) can be modified so that one can get the result about $t_k$ moves, for $k$ odd, analogous to Corollary 2.3 (compare [Ki]).

Consider a small ball $B^3$ in which a $t_k$ move takes place (Fig. 2.3).

\[ \Sigma^3 - L - \text{int}B^3 \text{ is homeomorphic to } \Sigma^3 - t_k(L) - \text{int}B^3 \text{ and the fundamental groups of them have the following presentation:} \]

\[ \{a, b, c, x_1, x_2, \ldots : r_1, r_2\}, \]

where $a$, $b$ and $c$ are classes of curves (generators of $\pi_1(\partial B^3 - L)$) shown on Fig. 2.4.
If we add a 2-handle along \( u = ac^{-1} \) we get \( \Sigma^3 - L \) and if we add a 2-handle along 
\[(ba)^{(k-1)/2} b (ba)^{-(k-1)/2} c^{-1} \]
we get \( \Sigma^3 - t_k(L) \); therefore
\[
\pi_1(\Sigma^3 - L) = \{a, b, c, x_1, x_2, \ldots : ac^{-1} = 1, r_1, r_2, \ldots\},
\]
\[
\pi_1(\Sigma^3 - t_k(L)) = \{a, b, c, x_1, x_2, \ldots : (ba)^{(k-1)/2} b (ba)^{-(k-1)/2} c^{-1}, r_1, r_2, \ldots\}.
\]
Consider the natural projections \( p : \pi_1(\Sigma^3 - L) \to \mathbb{Z} \) and \( p' : \pi_1(\Sigma^3 - t_k(L)) \to \mathbb{Z} \).
Then we have:
\[
p(a) = p(b) = p(c) = t, \quad p'(a) = p'(b) = p'(c) = t.
\]
Now we calculate that \( (r_0 = (ba)^{(k-1)/2} b (ba)^{-(k-1)/2} c^{-1}) \):
\[
\begin{align*}
p\left(\frac{\partial ac^{-1}}{\partial a}\right) &= 1, \quad p'(\frac{\partial r_0}{\partial a}) = 1 - \frac{t^k + 1}{t + 1}, \\
p\left(\frac{\partial ac^{-1}}{\partial b}\right) &= 0, \quad p'(\frac{\partial r_0}{\partial b}) = \frac{t^k + 1}{t + 1}, \\
p\left(\frac{\partial ac^{-1}}{\partial c}\right) &= -1, \quad p'(\frac{\partial r_0}{\partial c}) = -1,
\end{align*}
\]
and we get:

**Lemma 2.5.** The Alexander-Fox modules of \( L \) and \( t_k(L) \) can be presented by the following matrices.

| \( L \) | \( a \) | \( b \) | \( c \) | \( x_1 \) | \( x_2 \) | \( \ldots \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( ac^{-1} = 1 \) | 1 | 0 | -1 | 0 | 0 | \( \ldots \) |
| \( r_1 \) | * | * | * | * | | |
| \( \ldots \) | * | * | * | * | * | |

| \( t_k(L) \) | \( a \) | \( b \) | \( c \) | \( x_1 \) | \( x_2 \) | \( \ldots \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( r_0 \) | 1 - \( \frac{t^k + 1}{t + 1} \) | \( \frac{t^k + 1}{t + 1} \) | -1 | 0 | 0 | \( \ldots \) |
| \( r_1 \) | * | * | * | * | * | |
| \( \ldots \) | | * | * | | * | |

**Corollary 2.6.** (a) For \( \frac{t^k + 1}{t + 1} = 0 \) (k-odd) \( t_k \)-equivalent links have the same Alexander module, in particular
(b) $\Delta_t(L)(t) \equiv \mp i \Delta_L(t) (\mod \frac{t^k + 1}{t+1})$.

In fact from Corollary 1.3 follows that for a normalized Alexander polynomial $\Delta_L(t) \equiv \mp i \Delta_t(L)(t) (\mod \frac{t^k + 1}{t+1})$ or precisely

$$\Delta_L(t) \equiv t^{k/2} \Delta_{t_k}(L)(t) (\mod \frac{t^k + 1}{t+1}).$$

We can slightly generalize the results of Wendt and Kinoshita using Lemma 2.5.

**Corollary 2.7.** Let $\bar{u}_n(L)$ denote the minimal number of $t_{2k}$ or $t_k$ moves which are needed to change a given oriented link $L$ into unlink of $n$ components, then

$$|e_s - s(n - 1)| \leq (s - 1)\bar{u}_n(L),$$

where $e_s$ is the minimal number of generators of $H_1(M_L^{(s)}, \mathbb{Z})$.

### 3. Applications and Speculations

We start this part by proving two ”folklore” results which link Goeritz and Seifert matrices with $t_k$ or $\bar{t}_k$ moves.

**Theorem 3.1.** (a) There exist Goeritz’s matrices for $L$ and $t_k(L)$ (or $\bar{t}_k(L)$) which are the same modulo $k$.
(b) $t_k$ and $\bar{t}_k$-moves preserves $H_1(M_L^{(2)}, \mathbb{Z})$.

**Proof.** For the convenience we start from the definition of Goeritz’s matrix ([Goe, Gor]). Colour the regions of the diagram of an unoriented link alternately black and white, the unbounded region $X_0$ being coloured white, and number the other white
regions $X_1, \ldots, X_n$. Assign an incidence number $\eta(p) = \mp 1$ to each crossing point $p$ as shown in Fig. 3.1. Then define $n \times n$ Goeritz’s matrix $G = (g_{ij})$ by

$$
\sum \eta(c) \text{ summed over crossings points } p \text{ adjacent to } X_i \text{ and } X_j \\
\sum \eta(c) \text{ summed over crossings points } p \text{ adjacent to } X_i \text{ and to some } X_j \text{ (} i \neq j \text{) if } i = j \text{ (} i \geq 1 \text{).}
$$

Now consider the Fig. 3.2 with white regions $X_i$ and $X_j$.
t_k moves links

There are two possible cases:

(i) $X_i = X_j$, then $G_L = G_L'$; in fact $L$ is isotopic to $L'$.
(ii) $X_i \neq X_j$, we can assume that $i = 0$ and $j = 1$ then

$$G_{L'} = \begin{bmatrix} g_{11} + k, & g_{12} & \cdots & g_{1n} \\ \cdots \\ g_{n1}, & g_{n2} & \cdots & g_{nn} \end{bmatrix} \text{ where } G_L = (g_{ij})$$

The part (b) of the Theorem 3.1 follows from the fact that $G_L$ is a presentation matrix for $H_1(M_L^{(2)}, \mathbb{Z})$. □

An alternative proof of (b) can be given by considering Dehn surgery on $M_L^{(2)}$ corresponding to $t_k$ or $\bar{t}_k$ move on $L$.

**Theorem 3.2.** (a) Consider a $t_{2k,0}$ move of Fox (e.g. $\bar{t}_{2k}$ move), then there exist Seifert matrices for $L$ and $t_{2k,0}(L)$ which are the same modulo $k$.
(b) $t_{2k,0}$ move preserves $H_1(M_L^{(s)}, \mathbb{Z}_k)$ for any $s$. 
Proof. One can find a Seifert surface $S$ for $L$ which cuts the disk $D^2$ which supports the $t_{2k,0}$ move, as shown in Fig. 3.3. Then the Seifert matrix for $L$ defined by $S$ and for $t_{2k,0}(L)$ defined by $t_{2k,0}(S)$ satisfy the condition (a).

(b) follows from (a) because a presentation matrix for $H_1(M_L^{(s)}, \mathbb{Z})$ can be built of blocks of the shape $\mp V, \mp V^T, \mp (V + V^T)$, where $V$ is a Seifert matrix of $L$ and $V^T$ its transpose. On the other hand, (b) is a special case of Theorem 2.3(b).

Example 3.3.  

(a) The trivial knot ($T_1$) and the (right handed) trefoil knot ($3_1$) are $t_4$ equivalent. The figure eight knot ($4_1$) and the $5_2$ knot are $t_4$ equivalent however they are not $t_4$ equivalent to $T_1$ or $3_1$.

(b) $T_1$ and $5_2$ are $\bar{t}_4$ equivalent. $3_1$ and $4_1$ are $\bar{t}_4$ equivalent but they are not $\bar{t}_4$ equivalent to $T_1$ or $5_2$. 
First parts of (a) and (b) are illustrated in Fig. 3.4.

The second parts follow from Corollaries 1.2 and 1.8 ($t_4$ move changes $P_L(a, \sqrt{2})$ by the factor $-a^{-4}$ and $t_4$ preserves $P_L(1, z)$) and the following computation:

$$P_{T_1}(a, z) = 1$$

$$P_{3_1}(a, z) = -a^{-4} - 2a^{-2} + z^2a^{-2}; \quad P_{3_1}(a, \sqrt{2}) = -a^{-4}, \quad P_{3_1}(1, z) = z^2 - 3$$

$$P_{4_1}(a, z) = -a^{-2} - 1 - a^2 + z^2; \quad P_{4_1}(a, \sqrt{2}) = -a^{-2} + 1 - a^2; \quad P_{4_1}(1, z) = z^2 - 3$$

$$P_{5_2}(a, z) = -a^{-2} + a^4 + a^6 + z^2(a^2 - a^4); \quad P_{5_2}(a, \sqrt{2}) = -a^4(-a^{-2} + 1 - a^2); \quad P_{5_2}(1, z) = 1.$$  

**Example 3.4.** Every closed 3-braid knot is $t_4$ equivalent to the trivial knot or the figure eight knot. It is not an unexpected result because the quotient group $B_3/(\delta_1^4)$ is finite [Cox]. In fact a calculation shows that $B_3/(\delta_1^4)$ has only two classes of
knots (represented by $T_1$ and $A_1$). Because all presentations of $A_1$ as a 3-braid (e.g. $\delta_1\delta_2^{-1}\delta_1\delta_2^{-1}$) have the same exponent sum (equal to 0) therefore for every knot $K$ which is $t_4$ equivalent to $A_1$, each of its presentation as a 3-braid has the same exponent sum (equal to $4|A_1, K^{lev}$; compare Corollary 1.2). More in this direction can be got using other $t_k$ moves, compare Example 3.11, however it has been generally proved by H.Morton [Mo] and J.Birman that if $L$ is not a $(2,k)$ torus link then the exponent sum of $L$ does not depend on the presentation of $L$ as a closed 3-braid.

Example 3.5. Consider the following theorem of H.Murakami [Mur-1] (see also [L-M-2]):

$$P_L(1, \sqrt{2}) = V_L(i) = \begin{cases} (\sqrt{2})^{c(L)-1}(-1)^{\text{Arf}(L)} & \text{if Arf}(L) \text{ exists} \\ 0 & \text{otherwise} \end{cases},$$

where $c(L)$ denotes the number of components of $L$, $\text{Arf}(L)$ is the Arf (or Robertello) invariant (see [Rob] or [Ka-2]), and $t = i$ in $V_L(t)$ should be understood as $t^{1/2} = -e^{\pi i/4}$. Notice that our convention differs slightly from that of [L-M-1] or [L-M-2] namely $P_L(a, z) = P_L(\ell, -m) = (-1)^{c(L)-1}P_L(\ell, m)$.

It follows from Corollaries 1.2 and 1.8 that $t_4$ move changes $P_L(1, \sqrt{2})$ by factor $-1$ and $\bar{t}_4$ move preserves $P_L(1, \sqrt{2})$. Furthermore for $T_n$ - the trivial link of $n$ components $P_{T_n}(1, \sqrt{2}) = (\sqrt{2})^{n-1}$. On the other hand the Arf invariant of a trivial link is equal to zero, $t_4$ move changes the Arf invariant (if defined) and $\bar{t}_4$ move preserves it (see [Ka-2]). Therefore the Murakami theorem follows immediately from the above observations for a link which is $t_4$, $\bar{t}_4$ equivalent to a trivial link (i.e. a link which can be obtained from a trivial one using $t_4$ and $\bar{t}_4$ moves). This should be confronted with the following conjecture:
Conjecture 3.6. *(Kawauchi - Nakanishi)*

(a) If two links \(L_1\) and \(L_2\) are homotopic then they are \(t_4, \bar{t}_4\) equivalent\(^1\). In particular:
(b) Every knot is \(t_4, \bar{t}_4\) equivalent to the unknot.

Conjecture 3.6 has been verified for the 2-bridge links, closed 3-string braids and pretzel links.

Example 3.7. Consider the following \(t_{\Delta^2}\)-move (\(\Delta^2\)-twist) on oriented diagrams of links (Fig. 3.5).

---

\(^1\)Added for e-print: The conjecture has been disproved in [D-P-2](#) for links of three or more components. For two component links it is still an open problem whether any such link is \(t_4, \bar{t}_4\) equivalent to \(T_2\) or the Hopf link.
A $t_4$ move can be obtained from a $t_{\Delta^2}$-move (and isotopy) as it is illustrated in Fig. 3.6.

J.Birman and B.Wajnryb [B-W] have proven that two links are $t_{\Delta^2}$ equivalent iff they have the same number of components and the same number of components with odd linking number with the rest of the link. Because a $t_4$ move preserves the number of components and all linking numbers modulo 2, therefore it can be obtained as a combination of $t_{\Delta^2}^{\pm 1}$ moves. In fact it follows from [B-W] that in order to get $t_4$ move we can always use an even number of $t_{\Delta^2}^{\pm 1}$ moves. Furthermore a $t_{\Delta^2}$ move changes the Arf invariant (if it exists) and therefore $V_L(i) = -V_{t_{\Delta^2}(L)}(i)$. The last equality can be also proven elementary without using [B-W]. Finally observe that not every $t_{\Delta^2}$ move is a combination of $t_4$, $\bar{t}_4$ moves. The reason is that $t_4$ and $\bar{t}_4$ moves...
preserve all linking numbers mod 2 but it is not always the case for a $t_{\Delta^2}$ move (see Fig. 3.7 for an example of links which are $t_{\Delta^2}$ equivalent but not $t_4$, $t_4$ equivalent).

**Fig. 3.7**

**Example 3.8.** (a) A $t_{\Delta^2}$ move is a special case of $t_{2,3}$ moves of Fox but it follows from [B-W] that any $t_{2,3}$ move is a combination of $t_{\Delta^2}$ moves. In fact, every $t_{2,3}$ move preserves the number of components and the number of components with odd linking number with the rest of the link. Similarly any $t_{2,2q+1}$ is a combination of $t_{\Delta^2}$ moves.
(b) a $t_4$ move is a special case of $t_{4,2}$ moves of Fox. There are $t_{4,2}$ equivalent links which are not $t_4$ equivalent. (Fig. 3.8).

![Fig. 3.8](image)

Two links of Fig. 3.8 are not $t_4$ equivalent because their sublinks of Fig. 3.9 are not $t_4$ equivalent.

![Fig. 3.9](image)

Namely $P_{T_2}(a, \sqrt{2}) = \frac{a + a^{-1}}{\sqrt{2}}$ and $P_K(a, \sqrt{2}) = \frac{a^{-5} - a^{-3} + 2a^{-1}}{\sqrt{2}}$, therefore by Corollary 1.2 $T_2$ and $K$ are not $t_4$ equivalent.
Example 3.9.  
(a) The square knot \((3_1\#\overline{3}_1)\), the (right-handed) granny knot \((3_1\#3_1)\), and \(T_3\) (the trivial \(3\)-component link) are \(t_3\) equivalent.
(b) The trefoil knot \((3_1)\) and \(T_2\) (the trivial link of \(2\)-components) are \(t_3\) equivalent.
(c) The knots \(5_2, 6_3\), the Hopf link \((2_1^2)\), the Borromean rings \((6_2^3)\) and the unknot \((T_1)\) are \(t_3\) equivalent.
(d) The figure eight knot \((4_1)\) and the knot \(9_{42}\) (in the Rolfsen notation \([\text{Rol}]\)) are \(t_3\) equivalent.
(e) No links from different classes ((a), (b), (c), (d)) are \(t_3\) equivalent however links of (c) and (d) are \(t_3, \overline{t}_3\) equivalent (i.e. there is a sequence of \(t_3^{\pm 1}\) or \(\overline{t}_3^{\pm 1}\) moves which lead from one link to another) and there is no more \(t_3, \overline{t}_3\) equivalences among the above links.

The \(t_3\) and \(t_3, \overline{t}_3\) equivalences are illustrated in Fig. 3.10, 3.11, 3.12 and 3.13.

---

**Fig. 3.10**
Fig. 3.11

Borromean rings ($6^3_2$)  

Hopf link ($2^2_1$)  

Fig. 3.12
Fig. 3.13

The first part of (e) follows Corollary 1.2 (a $t_3$ move changes $P_L(a, 1)$ by the factor $-a^{-3}$) and the following computation:

\[
P_{T_1}(a, 1) = 1, \quad P_{T_2}(a, 1) = a + a^{-1}, \quad P_{T_3}(a, 1) = a^{-2} + 2 + a^2, \quad P_{T_4}(a, 1) = -a^{-2} - a^2.
\]

The last statement of (e) follows from the fact that different trivial links are not $t_3$, $\bar{t}_3$ equivalent (see Lemma 3.10(c) below).

**Lemma 3.10.** Consider the Jones polynomial $V_L(t)$ for $t = e^{\pi i/3}$ $(t^{1/2} = -e^{\pi i/6})$, then

(a) $V_{t_3(L)}(e^{\pi i/3}) = iV_L(t)$

(b) $V_{\bar{t}_3(L)}(e^{\pi i/3}) = \begin{cases} (-1)^\lambda iV_L(t) & \text{if two components of } L \text{ are involved in } \bar{t}_3 \text{ move} \\ (-1)^\lambda V_L(t) & \text{if one component of } L \text{ is involved in } \bar{t}_3 \text{ move} \end{cases}$

$\lambda$ depends on the linking numbers of components of $L$ and $\bar{t}_3(L)$ and on an orientation of $\bar{t}_3(L)$ (see Theorem 1.13).
(c) The trivial links $T_k$ and $T_j$ ($k \neq j$) are not $t_3, \bar{t}_3$ equivalent and $V_{T_k}(e^{\pi i/3}) = (\sqrt{3})^{k-1}$.

Proof. (a) follows from Corollary 1.4, and (b) from Theorem 1.13. (c) follows from (a) and (b). \hfill \square

Example 3.11. Every closed 3-braid link is $t_3$ equivalent to $T_1$, $T_2$, $T_3$ or the figure eight knot (in fact one can go from any closed 3-braid link to one of these links using $t_3$ moves and regular isotopy). Because all presentations of $4_1$ and $T_3$ as closed 3-braids have the same exponent sum (equal to 0) therefore for any link $L$ which is $t_3$ equivalent to $4_1$ or $T_3$, each of its presentation as a 3-braid has the same exponent sum (equal to $3|4_1, L|_{t_3}^{\text{lev}}$ or $3|T_3, L|_{t_3}^{\text{lev}}$). Consider, for example, the closed 3-braid knot $\delta_1^4 \delta_2^{-1} \delta_1 \delta_2^{-4}$ (Fig. 3.14). It is $t_3$ equivalent to the figure eight knot so now we know that all presentations of this knot as a 3-braid have the exponent sum equal to zero; on the other hand, the knot is $t_4$ equivalent to the unknot so the method of Example 3.4 would not suffice to get the unique exponent sum.

Fig. 3.14
Example 3.12. Consider the following theorem of W.B.R.Lickorish and K.Millett [L-M-2] (conjectured by J.Birman and partially proved by V.Jones).

\[ P_L(e^{\pi i/6}, 1) = V_L\left((e^{\pi i/3}) = \mp i^{c(L)}-1(i\sqrt{3})\dim H_1(M^{(2)}_L, \mathbb{Z}_3), \right. \]

where \(c(L)\) denotes the number of components of \(L\) and \(t^{1/2} = e^{-\pi i/6}\) in \(V_L(t)\).

It follows from Lemma 3.10 that \(t_3\) and \(\bar{t}_3\) moves change \(V_L((e^{\pi i/3})\) by factors \(\mp 1\) or \(\mp i\) and the second case happens if the move changes the number of components. On the other hand \(t_3\) and \(\bar{t}_3\) moves preserve \(H_1(M^{(2)}_L, \mathbb{Z}_3)\) (Theorem 3.1(b)) and for the trivial link \(T_n\), \(\dim H_1(M^{(2)}_L, \mathbb{Z}_3) = n - 1\). Therefore the formula of Lickorish-Millett holds immediately from the above observations for a link which is \(t_3, \bar{t}_3\) equivalent to a trivial link (the sign in formula can be found using Lemma 3.10; it was identified generally by A.Lipson [Lip]). This should be confronted with the following conjecture.

Conjecture 3.13. (Montesinos-Nakanishi). Every link is \(t_3, \bar{t}_3\) equivalent to a trivial link.²

It is an easy (but tedious) task to check the conjecture for closed \(n\)-braids (\(n \leq 5\)) and \(n\)-bridge links (\(n \leq 3\)) because for the braid group \(B_n\) (\(n \leq 5\)) the group \(B_n/\langle \delta_1^3 \rangle\) is finite (Cox), however the author did it only for closed 3-braids and 2-bridge links.³

Example 3.14. Consider the following theorem of Lickorish and Millett [L-M-2] and H. Murakami [Mur-2]:

²Added for e-print: The conjecture has been disproved in [D-P-1]. The smallest known counter-example has 20 crossings.

³Added for e-print: The conjecture holds for 4-bridge links [P-TS] [Ts]. Furthermore every closed 5-braid is \(t_3, \bar{t}_3\) equivalent to a trivial link or to the closure of the 5-string braid \((\delta_1 \delta_2 \delta_3 \delta_4)^{10}\) [Chen]. The last link is a counter-example to Montesinos-Nakanishi conjecture [D-P-1].
\[ P_L(1, 1) = (2)^{(1/2) \dim H_1(M^{(3)}_L, Z_2)}. \]

It follows from Corollaries 1.2 and 1.8 that \( t_3 \) and \( \bar{t}_4 \) moves preserve \( P_L(1, 1) \). Furthermore, \( P_{T_n}(1, 1) = 2^{n-1} \). On the other hand \( \dim H_1(M^{(3)}_{T_n}, Z_2) = 2(n - 1) \) and \( \bar{t}_4 \) moves preserve \( H_1(M^{(3)}_L, Z_2) \). It can be shown, using the Fox approach that \( t_3 \)-moves preserve \( H_1(M^{(3)}_L, Z_2) \) \(([P-3])\). Therefore the formula of Lickorish-Millett-Murakami follows immediately from the above observations for a link which is \( t_3, \bar{t}_4 \) equivalent to a trivial link (i.e. a link which can be got from a trivial one using \( t_3 \) and \( \bar{t}_4 \) moves and isotopy). This leads to the following conjecture.

**Conjecture 3.15.** Every link is \( t_3, \bar{t}_4 \) equivalent to a trivial link.

The author has checked the conjecture for closed 3-braid links (see the remark after Conjecture 3.13).

For \( t_5 \) and \( \bar{t}_4 \) moves the analogy of Conjecture 3.15 does not hold. For example, trivial links, the trefoil knot, 8₅ knot and 8₁₈ knot (\([Rc]\) see Fig. 3.15) are not pairwise \( t_5, \bar{t}_4 \) equivalent. The reason is that by Corollary 1.8 a \( \bar{t}_4 \) move does not change \( P_L(1, z) \) and by Corollary 1.2 \( t_5 \) move changes \( P_L(1, \frac{1 + \sqrt{5}}{2}) \) by the factor \(-1\) (notice that \( 2 \cos(\pi/5) = \frac{1 + \sqrt{5}}{2} \)); on the other hand all mentioned above links have pairwise different absolute values of \( P_L(1, \frac{1 + \sqrt{5}}{2}) \):

\[
P_{T_n}(1, \frac{1 + \sqrt{5}}{2}) = (\sqrt{5} - 1)^{n-1}
\]

\[
P_{3_1}(1, \frac{1 + \sqrt{5}}{2}) = \frac{-3 + \sqrt{5}}{2}, \quad P_{8_5}(1, \frac{1 + \sqrt{5}}{2}) = -4 + \sqrt{5},
\]

\[
P_{8_{18}}(1, \frac{1 + \sqrt{5}}{2}) = \frac{1 - 2\sqrt{5}}{2}.
\]

\(^4\)Added for e-print: It has been checked for closed 4-braid links \([Chen]\).
There is no chance for anything analogous to Conjectures 3.6 or 3.13 for \( t_k, \bar{t}_k \) moves, \( k \geq 5 \) (i.e. that all links are \( t_k, \bar{t}_k \) equivalent to the trivial links). In particular V.Jones ([Jo-3; Corollary 14.7] proved that the set \( \{ |V_L(e^{\pi i/5})| : \text{L is a link} \} \) is dense in \( (0, \infty) \). On the other hand \( t_5 \) and \( \bar{t}_5 \) moves do not change the absolute value of \( V_L(e^{\pi i/5}) \) (see Corollary 1.4 and Theorem 1.13), and for trivial links, the values \( |V_{T_n}(e^{\pi i/5})| = (2 \cos \pi/10)^{n-1} \) are discrete in \( (0, \infty) \).

There are natural relations between \( t_k \) moves and signatures of links; we will list here some examples of such relations. For convenience, we start from the definition of the Tristram-Levine signature (see [Gor, P-1 or P-2]). Let \( A_L \) be a Seifert matrix of a link \( L \). For each complex number \( \xi (\xi \neq 1) \) consider Hermitian matrix \( A_L(\xi) = (1 - \bar{\xi})A_L + (1 - \xi)A_L^T. \) The signature of this matrix, \( \sigma_L(\xi) \) is called the Tristram-Levine signature of the link \( L \). The classical signature \( \sigma \) satisfies \( \sigma_L = \sigma_L(0) \).

**Theorem 3.16.** (a) For any \( t_k \) move on an oriented link \( L \)

\[
k - 2 \leq \sigma_L - \sigma_{t_k(L)} \leq k,
\]

(b) \( 0 \leq \sigma_{t_{2k}(L)} - \sigma_L(\xi) \leq 2 \) if \( \text{Re}(1 - \xi) \geq 0 \),

(c) \( \sigma_L(\xi_0) - \sigma_{t_4}(\xi_0) = 2 \) if \( P(i, \sqrt{2}) \neq 0 \) and \( \xi_0 = 1 - e^{\pi i/4} = \frac{2 - \sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \),
(d) If two links $L_1$ and $L_2$ are $t_4$ equivalent then $|L_1, L_2|^{\text{lev}} = (1/2)(\sigma_{L_1}(\xi_0) - \sigma_{L_2}(\xi_0))$, provided $P_{L_1}(i, \sqrt{2}) \neq 0$.

Proof. (a) We use the formula of C.McA.Gordon, R.A.Litherland and A.Marin, which links the signature of Goeritz matrix of a link with a classical signature. We use the same notation as in the proof of Theorem 3.1. Divide the crossings of a given oriented link $L$ into two types as shown in Fig. 3.16.

![Fig. 3.16](image)

Define $\mu = \sum \eta(p)$, summed over all crossing points of type II then $\sigma_L = \sigma(G_L) - \mu(L)$. Furthermore we have $\mu(t_k(L)) - \mu(L) = k$ (see Fig. 3.2), and from the form of the matrices $G_{L'} = G_{t_k(L)}$ and $G_L$ (see proof of Theorem 3.1) follows that $-2 \leq \sigma(G_L) - \sigma(G_{t_k(L)}) \leq 0$ and therefore $-2 \leq \sigma_L + \mu(L) - \sigma_{t_k(L)} - \mu(t_k(L)) \leq 0$ and Theorem 3.16(a) follows.

To prove (b), we have to choose a proper Seifert surface from which we will find the adequate Seifert matrix so one could easily compare the Levine-Tristram signature
for \( L \) and \( \bar{t}_{2k}(L) \). We can assume that Seifert surfaces for \( L \) and \( \bar{t}_{2k}(L) \) looks locally as on Fig. 3.17 (or \( \bar{t}_{2k}(L) \) s isotopic to \( L \)).

![Fig. 3.17](image)

Then the Seifert matrices (in appropriate basis) are of the form:

\[
A_{\bar{t}_{2k}(L)} = \begin{bmatrix}
A_L \{ \alpha \\
\beta & q + k
\end{bmatrix},
\]

\[
A_L = \begin{bmatrix}
A_L \{ \alpha \\
\beta & q
\end{bmatrix},
\]

where \( A_L \) is the Seifert matrix of \( L \), \( \alpha \) is a column, \( \beta \) is a row and \( q \) is a number (compare \([Ka-1, P-T-2]\) or \([P-1]\)). Therefore

\[
A_{\bar{t}_{2k}(L)}(\xi) = \begin{bmatrix}
A_L \{ \xi \\
a \alpha^{-T} & m + k(2 - \xi - \bar{\xi})
\end{bmatrix},
\]

\[
A_L(\xi) = \begin{bmatrix}
A_L \{ \xi \\
a \alpha^{-T} & m
\end{bmatrix},
\]

where \( a = (1 - \xi)\alpha + (1 - \xi)\beta^T \) and \( m = ((1 - \xi) + (1 - \xi))q \). Because \( 2 - \xi - \bar{\xi} \geq 0 \), so \( 0 \leq \sigma(A_{\bar{t}_{2k}(L)}(\xi)) - \sigma(A_L(\xi)) \leq 2 \) and the proof of (b) is finished.

To prove (c) we need further characterization of the Tristram-Levine signature, given in \([P-T-2]\) (see also \([P-1]\)); Assume \(|1 - \xi| = 1\), we have:

(i) \( \det iA_L(\xi) = P_L(i, 2 - \xi - \bar{\xi}) = \Delta_L(t') \) (for \( \sqrt{t'} = -i(1 - \xi) \),
(ii) \( i^{\sigma_L(\xi)} = \frac{\Delta_L(t')}{\Delta_{t_{4(L)}}(t')} \) if \( \Delta_L(t') \neq 0 \),
(iii) \( 0 \leq \sigma_L(\xi) - \sigma_{t_{4(L)}}(\xi) \leq 4 \) if \( \text{Re}(1 - \xi) \geq 0 \).

((iii) can be got using (b) two times with \( k = 1 \); \( \bar{t}_2 \) moves are equivalent to \( t_2 \) moves).

Now consider the case when \( 2 - \xi_0 - \bar{\xi}_0 = \sqrt{2} (1 - \xi_0 = e^{\pi i/4}) \). Then by Corollary 1.3, \( \Delta_{t_{4(L)}}(t') = -\Delta_L(t') \). Therefore by (ii) and (iii) \( \delta_L(\xi_0) - \delta_{t_{4(L)}}(\xi_0) = 2 \). (d) follows immediately from (c).

One can expect interesting relations between \( t_k \) moves and non-cyclic coverings of links. We limit ourself to two examples, first of which was suggested by R.Campbell.

**Example 3.17.**  
(a) A link diagram is 3-coloured if every overpass is coloured, say, red, yellow or blue, at least two colours are used and at any given crossing either all three colours appear or only one colour appears \([Fo-2]\). Then if a link \( L_1 \) is \( t_3, \bar{t}_3 \) equivalent to \( L_2 \) then either both links are 3-coloured or none of them are 3-coloured. In particular a link which is \( t_3, \bar{t}_3 \) equivalent to a trivial link of more than one component is 3-coloured. The proof is illustrated in Fig. 3.18. The link \( 6^2_3 \) \([Rol]\) is 3-coloured in Fig. 3.19.

![Fig. 3.18](image-url)
(b) 3-colouring corresponds to an epimorphism $\pi_1(S^3 - L) \to S_3$; more generally we have: If a knot $K_1$ is $t_{2p}$, $\bar{t}_{2p}$ equivalent to $K_2$ ($p$ - prime) then either both knots or none of them have dihedral representations i.e. epimorphism

$$\pi_1(S^3 - K) \to D_{2p} = \{a, b : a^2 = 1, b^p = 1, aba = b^{-1}\}.$$ 

It follows from the fact that $t_{2p}$, $\bar{t}_{2p}$ moves preserve $H_1(M_K^{(2)}, \mathbb{Z}_p)$ (Theorem 3.1(b)) and from the result of Fox that the epimorphism exists iff $H_1(M_K^{(2)}, \mathbb{Z}_p)$, is nontrivial [Fo-2] (see also [B-Z]; 14.8).⁵

Fig. 3.19

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