Abstract—This paper aims to construct optimal Z-complementary code set (ZCCS) with non-power-of-two (NPT) lengths to enable interference-free multicarrier code-division multiple access (MC-CDMA) systems. The existing ZCCSs with NPT lengths, which are constructed from generalized Boolean functions (GBFs), are sub-optimal only with respect to the set size upper bound. For the first time in the literature, we advocate the use of pseudo-Boolean functions (PBFs) (each of which transforms a number of binary variables to a real number as a natural generalization of GBF) for direct constructions of optimal ZCCSs with NPT lengths.

Index Terms—Multicarrier code-division multiple access (MC-CDMA), generalized Boolean function (GBF), pseudo-Boolean function (PBF), Z-complementary code set (ZCCS), zero correlation zone (ZCZ)

I. INTRODUCTION

MULTICARRIER code-division multiple access (MC-CDMA) has been one of the most widely adopted wireless techniques in many communication systems/standards owing to its efficient fast Fourier transform (FFT) based implementation, resilience against intersymbol interference, and high spectral efficiency [1]. That being said, MC-CDMA may suffer from multiple-access interference (MAI) [2] and multipath interference (MPI) [3]. A promising way to address both MAI and MPI is to adopt proper spreading codes, such as complete complementary codes (CCC) [4] and Z-complementary code sets (ZCCSs) [5]. This paper focuses on efficient construction of ZCCSs with a new tool, called pseudo-Boolean functions (PBFs), to enable interference-free quasi-synchronous MC-CDMA systems.

In 2007, Z-complementary pairs (ZCPs) were introduced by Fan et al. [6] to overcome the limitation on the lengths of Golay complementary pairs (GCPs) [7], [8]. A ZCP refers to a pair of sequences of the same length $N$ having zero aperiodic auto-correlation sums for all time shifts $\tau$ satisfying $0 < |\tau| < Z$, where $Z$ is called zero-correlation zone (ZCZ) width. When $Z = N$, the resultant sequence pair reduces to a GCP. In the literature, there are direct constructions of GCPs and ZCPs with the aid of generalized Boolean functions (GBFs) [9], [10]. The idea of ZCPs introduced in [6] was generalized to ZCCSs by Feng et al. in [11]. A ZCCS refers to a set of $K$ codes, each of which consists of $M$ constituent sequences of identical length $L$, having ideal aperiodic auto- and cross-correlation properties inside the ZCZ width and satisfying the theoretical upper bound: $K \leq M \lfloor N/Z \rfloor$ [13]. When $Z = N$, the set is called a mutually orthogonal Golay complementary sets (MOGCSs) [4], which refers to collection of complementary codes (CCs) [14]–[16] with ideal cross-correlation properties. A set of CCCs is known as a MOGCSs with the equality $K = M$ [17]. The theoretical upper bound shows that an optimal ZCCS has larger set size as compared to CCCs provided $\lfloor \frac{N}{Z} \rfloor \geq 2$. Recently, several GBFs based constructions of optimal ZCCSs with power-of-two lengths have been reported in [18]–[21]. In the recent literature, two direct constructions of ZCCSs with NPT lengths can be found in [22] and [23], which produces sub-optimal ZCCS with $\lfloor \frac{N}{Z} \rfloor = 1$ and non-optimal ZCCSs for NPT lengths with $\lfloor \frac{N}{Z} \rfloor < 1$, respectively. To the best of our knowledge, the construction of optimal ZCCSs of NPT lengths with $\lfloor \frac{N}{Z} \rfloor \geq 2$, based on GBFs remains open. Other methods which are dependent on the existence of special sequences, known as indirect constructions [11], to construct ZCCSs can be found in [25]–[27]. The indirect constructions heavily rely on a series of sequence operations which may not be feasible for rapid hardware generation, especially, when the sequence lengths are large [5].

It is noted that the MAI in MC-CDMA system can be mitigated using zero-correlation properties of a ZCCS provided that all the received multiuser signals are roughly synchronous within the ZCZ width [19]. In addition to their applications in MC-CDMA [18], [19], [27], ZCCSs have also been employed as optimal training sequences in multiple-input multiple-output (MIMO) communications [28], [29]. The limitation on the set size of CCCs and the unavailability of optimal ZCCSs with NPT lengths using direct constructions in the existing literature are a major motivation of this work. Specifically, for the first time in the literature, we propose to use PBFs for direct construction of optimal ZCCS of lengths $p^{2m}$, where $p$ is a prime number and $m$ is a positive integer. A PBF [30] refers to an arbitrary mapping of the set of binary $m$-tuples to

| Table I |
|---|
| COMPARISON OF THE PROPOSED CONSTRUCTION WITH [5], [21]–[24]. |

| ZCCS | Method | Length $(N)$ | Constraints | Optimality |
|---|---|---|---|---|
| [5] | Direct | $2^m$ | $m \geq 2$ | Optimal |
| [21] | Direct | $2^m$ | $m \geq 2$ | Optimal |
| [22] | Direct | $2^m + 2^h$ | $m \geq 2, h > 0$ | Sub-optimal |
| [23] | Indirect | $L$, $L \geq 1$ | Optimal |

Theorem I

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real numbers. Being a natural generalization of GBFs, PBFs are also suitable for rapid hardware generation of sequences. A detailed comparison of the proposed construction with \([5], [21]–[24], [27]\) is given in TABLE I.

II. PRELIMINARY

In this section, we present some basic definitions and lemmas to be used in the proposed construction. Let \(y_1 = (y_{1,0}, y_{1,1}, \cdots, y_{1,N-1})\) and \(y_2 = (y_{2,0}, y_{2,1}, \cdots, y_{2,N-1})\) denote a pair of sequences with complex components. For an integer \(\tau\), define \([5]\)

\[
\theta(y_1, y_2)(\tau) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} y_{1,i} y_{2,i}^* e^{2j\pi i \tau / N}, \quad 0 \leq \tau < N,
\]

\[
\theta(y_1, y_2)(\tau) = 0, \quad -N < \tau < 0,
\]

otherwise.

The functions \(\theta(y_1, y_2)\) and \(\theta(y_1, y_1)\) are called the aperiodic cross-correlation function (ACCF) between \(y_1\) and \(y_2\), and the aperiodic auto-correlation function (AACF) of \(y_1\), respectively. Let \(S = \{S_0, S_1, \cdots, S_{K-1}\}\) be a set of \(K\) codes or ordered sets defined as

\[
S_\mu = (s_{\mu 0}^0, s_{\mu 1}^0, \cdots, s_{\mu M-1}^0),
\]

where \(s_{\mu i}^0\) \((0 \leq \nu \leq M-1, 0 \leq \mu \leq K-1)\) is the \(\nu\)-th element which we assume is a complex-valued sequence of length \(N\) in \(S_\mu\). For \(S_{\mu_1}, S_{\mu_2} \in S\) \((0 \leq \mu_1, \mu_2 \leq K-1)\), the ACCF between \(S_{\mu_1}\) and \(S_{\mu_2}\) is defined as

\[
\theta(S_{\mu_1}, S_{\mu_2})(\tau) = \sum_{\nu=0}^{M-1} \theta(s_{\nu 1}^\mu, s_{\nu 2}^\mu)(\tau).
\]

Definition 1 ([5]): Code set \(S\) is called a ZCCS if

\[
\theta(S_{\mu_1}, S_{\mu_2})(\tau) = \begin{cases} MN, & \tau = 0, \mu_1 = \mu_2, \\ 0, & 0 < |\tau| < Z, \mu_1 = \mu_2, \\ 0, & |\tau| > Z, \mu_1 \neq \mu_2, \end{cases}
\]

where \(Z\) is called ZCC width. We denote a ZCCS with the parameters \(K, N, M,\) and \(Z\) by the notation \((K, N, Z)\)-ZCCS\(_N^M\). For \(K = M\) and \(Z = N\), a \((K, Z)\)-ZCCS\(_N^M\) is called a set of CCCs and we denote it by \((K, K, N)\)-CCC.

We call a \((K, Z)\)-ZCCS\(_N^M\) optimal if it achieves the equality in the upper-bound, given by \(K \leq \frac{M}{2}\) \([13]\).

A. Generalized Boolean Functions (GBFs)

Let \(x_0, x_1, \cdots, x_m-1\) denote \(m\) variables which take values from \(\mathbb{Z}_2\). A monomial of degree \(i\) \((0 \leq i \leq m)\) is defined as the product of any \(i\) distinct variables among \(x_0, x_1, \cdots, x_m-1\). Let us assume that \(A_i\) denotes the set of all monomials of degree \(i\), where

\[
A_i = \left\{ x_0^{r_0} x_1^{r_1} \cdots x_{m-1}^{r_{m-1}} : r_0 + r_1 + \cdots + r_{m-1} = i, (r_0, r_1, \cdots, r_{m-1}) \in \mathbb{Z}_2^m \right\}.
\]

A function \(f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q\) is said to be a GBF if it can uniquely be expressed as a linear combination of the monomials in \(A_m\), where the coefficient of each monomial is drawn from \(\mathbb{Z}_q\), where \(\mathbb{Z}_q\) denotes the set of integers modulo \(q\). The highest degree monomial with non-zero coefficient present in the expression of \(f\) determine the order of \(f\). As an example, \(2x_0x_1 + x_1 + 1\) is a second order GBF of two variables \(x_0\) and \(x_1\) over \(\mathbb{Z}_3\). We denote the graph of a second-order GBF \(f\) by \(\bar{G}(f)\) \([13]\). It contains \(m\) vertices which are denoted by the \(m\) variables of \(f\). The edges in the \(\bar{G}(f)\) are determined by the second-degree monomials present in the expression of \(f\) with non-zero coefficients. There is an edge of weight \(w\) between the vertices \(x_\alpha\) and \(x_\beta\) of \(\bar{G}(f)\) if the expression of \(f\) contains the term \(wx_\alpha x_\beta\). Let \(\psi(f)\) denotes the complex-valued sequence corresponding to a GBF \(f\) and it is defined as \([14]\),

\[
\psi(f) = (\omega_{q1}^0, \omega_{q1}^1, \cdots, \omega_{q2^m-1}) = (\omega_{q1}^0, \omega_{q1}^1, \cdots, \omega_{q2^m-1}),
\]

where \(\omega_q\) denotes exp \((2\pi \sqrt{-1}/q)\).

Lemma 1: (Construction of CCC [13])

Let \(f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q\) be a second-order GBF. Let us assume that \(\bar{G}(f)\) contains the vertices \(x_{i_0}, x_{i_1}, \cdots, x_{i_{k-1}}\) such that after performing a deletion operation on those vertices, the resulting graph reduces to a path. Let the edges in the path have identical weight of \(\frac{1}{2}\) and \(t = (t_0, t_1, \cdots, t_{k-1})\) be the binary representation of the integer \(t\). Define the CC, \(C_t\) to be

\[
f + \frac{q}{2}\left( (d + t) \cdot x + dx_{i_0} \right) : d \in \{0, 1\}^k, d \in \{0, 1\}^i\}
\]

and \(\bar{C}_t\) to be

\[
f + \frac{q}{2}\left( (d + t) \cdot x + dx_{i_0} \right) : d \in \{0, 1\}^k, d \in \{0, 1\}^i\}
\]

where \((\cdot, \cdot)\) denotes the dot product between two real-valued vector \((\cdot)\) and \((\cdot, \gamma)\) is the label of either end vertex in the path, \(x = (x_{i_0}, x_{i_1}, \cdots, x_{i_{k-1}})\), \(\bar{x} = (1-x_{i_0}, 1-x_{i_1}, \cdots, 1-x_{i_{k-1}})\), and \(d = (d_0, d_1, \cdots, d_{k-1})\). Then \(\psi(C_t), \psi(\bar{C}_t) : 0 \leq t < 2^k\) forms \((2^{k+1}, 2^{k+1}, 2^m)\)-CCC, where \(\psi(\cdot)\) denotes the complex conjugate of \(\psi(\cdot)\).

B. Pseudo-Boolean Functions (PBFs)

A function \(F : \{0, 1\}^m \rightarrow \mathbb{R}\) is said to be a PBF if it can be uniquely expressed as a linear combination of monomials in \(\mathbb{A}_m\) with the coefficients drawn from \(\mathbb{R}\), where \(\mathbb{R}\) denotes the set of real numbers. Therefore, PBFs are a natural generalization of GBFs [30]. As an example, \(\frac{1}{2}x_0 x_1 + x_0 + 1\) is a second-order PBF of two variables \(x_0\) and \(x_1\) but not a GBF. Let \(f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q\) be a GBF of the variables \(x_0, x_1, \cdots, x_{m-1}\). Let us assume that \(p\) denotes a prime number and define the following PBFs with the help of the GBF \(f\):

\[
F^\lambda = f + \frac{\lambda q}{p} \left( x_m + 2x_{m+1} + \cdots + 2^{s-1}x_{m+s-1} \right),
\]

\[
G^\lambda = \tilde{f} \left( x_m + 2x_{m+1} + \cdots + 2^{s-1}x_{m+s-1} \right),
\]
where $s \in \mathbb{Z}^+$ which denotes the set of all positive integers, $2 \leq p < 2^{s+1}$, and $\lambda = 0, 1, \ldots, p-1$. From [3], it is clear that $F^\lambda$ and $G^\lambda$ are PBFs of $m+s$ variables $x_0, x_1, \ldots, x_{m+s-1}$. From [3], it can be observed that the PBFs $F^\lambda$ and $G^\lambda$ reduce to $\mathbb{Z}_q$-valued GBFs if $p$ divides $q$.

III. PROPOSED CONSTRUCTION OF Z-COMPLEMENTARY CODE SET

In this section, we shall present our proposed construction of ZCCS using PBFs. To this end, we first present a lemma which will be used in our proposed construction.

Lemma 2: ([31]) Let $\lambda$ and $\lambda'$ be two non-negative integers, where $0 \leq \lambda \neq \lambda' < p$, $p$ is a prime number as defined in Section-II. Then

$$
\sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda')\alpha} = 0.
$$

For $0 \leq t < 2^k$ and $0 \leq \lambda < p$, we define the following sets of PBFs:

$$
U_t^\lambda = \left\{ F^\lambda + \frac{q}{2} \left( (d + t) \cdot x + d \gamma, d \in \{0, 1\} \right) \right\},
$$

and

$$
V_t^\lambda = \left\{ G^\lambda + \frac{q}{2} \left( (d + t) \cdot \bar{x} + \bar{d} \bar{\gamma}, d \in \{0, 1\} \right) \right\}.
$$

Let us assume that

$$
F^{d.t.d} = \frac{1}{2}(d + t) \cdot x + d \gamma, \quad G^{d.t.d} = \frac{1}{2}(d + t) \cdot \bar{x} + \bar{d} \bar{\gamma}.
$$

As per our assumption, for any choice of $d, t \in \{0, 1\}^k$, and $d \in \{0, 1\}$, the functions $F^{d.t.d}$ and $G^{d.t.d}$ are $\mathbb{Z}_q$-valued GBFs of $m$ variables. For any choice of $d, t \in \{0, 1\}^k, d \in \{0, 1\}$, and $\lambda \in \{0, 1, \ldots, p-1\}$, the functions $F^{d.t.d, \lambda}$ and $G^{d.t.d, \lambda}$ are PBFs of $m$ variables. We define $\psi(F^{d.t.d, \lambda})$, the complex-value sequence corresponding to $F^{d.t.d, \lambda}$,

$$
\psi(F^{d.t.d, \lambda}) = (\omega_q^{F^{d.t.d, \lambda}(0)}, \omega_q^{F^{d.t.d, \lambda}(1)}, \ldots, \omega_q^{F^{d.t.d, \lambda}(2^m-1)}),
$$

where

$$
F^{d.t.d, \lambda} = F^{d.t.d}(r_0, r_1, \ldots, r_{m+s-1}),
$$

and

$$\omega_q^{F^{d.t.d, \lambda}(r_0, r_1, \ldots, r_{m+s-1})} = \frac{2^m}{2} \sum_{\alpha=0}^{p-1} \omega_p^{\lambda r_0 + \lambda' r_1 + \cdots + \lambda' r_{m+s-1}}.
$$

From [12], it can be observed that $\omega_q^{F^{d.t.d, \lambda}}$ is a root of the polynomial: $z^{d-1}$, where $d = \text{lcm}(p, q)$, denotes a positive integer given by the least common multiple (lcm) of $p$ and $q$. Therefore, the components of $\psi(F^{d.t.d, \lambda})$ are given by the roots of the polynomial: $z^{d-1}$. From (11) and (12), we have

$$
\psi(F^{d.t.d, \lambda}) = \left( \omega_q^{F^{d.t.d, \lambda}(0)}, \omega_q^{F^{d.t.d, \lambda}(1)}, \ldots, \omega_q^{F^{d.t.d, \lambda}(2^m-1)} \right).
$$

Similarly, we can also obtain $\psi_{2m-s-p^m}(G^{d.t.d, \lambda})$ as

$$
\psi_{2m-s-p^m}(G^{d.t.d, \lambda}) = (\omega_q^{G^{d.t.d, \lambda}(0)}, \omega_q^{G^{d.t.d, \lambda}(1)}, \ldots, \omega_q^{G^{d.t.d, \lambda}(2^m-1)}).
$$

Let us also define $\psi_{2m-s-p^2m}(F^{d.t.d, \lambda})$ which is defined to be obtained from $\psi(F^{d.t.d, \lambda})$ by removing its last $2^{m+s} - p^m$ components.

$$
\psi_{2m+s-p^2m}(F^{d.t.d, \lambda}) = (\omega_q^{F^{d.t.d, \lambda}(0)}, \omega_q^{F^{d.t.d, \lambda}(1)}, \ldots, \omega_q^{F^{d.t.d, \lambda}(2^m-1)}).
$$

Similarly, we can also obtain $\psi_{2m+s-p^2m}(G^{d.t.d, \lambda})$ as

$$
\psi_{2m+s-p^2m}(G^{d.t.d, \lambda}) = (\omega_q^{G^{d.t.d, \lambda}(0)}, \omega_q^{G^{d.t.d, \lambda}(1)}, \ldots, \omega_q^{G^{d.t.d, \lambda}(2^m-1)}).
$$

Theorem 1: Let $f: \mathbb{Z}_m^2 \rightarrow \mathbb{Z}_m^2$ be a GBF as defined in Lemma 7. Then the set of codes

$$
\left\{ \psi_{2m-s-p^2m}(U_t^\lambda), \psi_{2m+s-p^2m}(V_t^\lambda) : 0 \leq t < 2^k, 0 \leq \lambda < p \right\},
$$

forms $(p2^k+1, 2m)$-ZCCS$^{2p^m}$.

Proof: In [13], [14], and [15], each of the parentheses below a complex-valued sequence contains $2m$ components of the complex-valued sequence. It can be observed that the $2m$ components in the $i$-th parentheses of $\psi_{2m-s-p^2m}(F^{d.t.d, \lambda})$ and $\psi_{2m+s-p^2m}(G^{d.t.d, \lambda})$ represent the complex-valued sequences $\omega_q^{(i-1)} \psi(F^{d.t.d})$ and $\omega_q^{(i-1)} \psi(G^{d.t.d})$, respectively, where $i = 1, 2, \ldots, p$. Using [9], [14], Lemma 7 and Lemma 2 the ACCF between $\psi_{2m-s-p^2m}(U_t^\lambda)$ and $\psi_{2m+s-p^2m}(U_p^\lambda)$ for $\tau = 0$ can be derived as follows:

$$
\theta(\psi_{2m-s-p^2m}(U_t^\lambda), \psi_{2m+s-p^2m}(U_p^\lambda))(0) = \sum_{d,d'} \theta(\psi_{2m-s-p^2m}(F^{d.t.d, \lambda}), \psi_{2m-s-p^2m}(F^{d.t'.d', \lambda'}))(0)
$$

$$
= \sum_{d,d'} \theta(\psi(F^{d.t.d}), \psi(F^{d.t'.d'}))(0) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda') \alpha}
$$

$$
= \sum_{d,d'} \theta(\psi(C_1), \psi(C_1'))(0) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda') \alpha}
$$

$$
= \begin{cases}
\begin{aligned}
p^{2m+k+1}, & t \neq t', \lambda = \lambda', \\
0, & t = t', \lambda \neq \lambda', \\
0, & t \neq t', \lambda \neq \lambda', \\
0, & t = t', \lambda \neq \lambda'.
\end{aligned}
\end{cases}
$$

(17)
Again, Using (9), (14), and Lemma 7, the ACCF between $\psi_{2m+p-2m}(U_t^k)$ and $\psi_{2m+p-2m}(V_t^k)$ for $0 < |\tau| < 2^m$ can be derived as

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k)) = \theta(\psi(C_t), \psi(C_t'))(\tau) \sum_{\alpha=0}^{p-1} \omega_p^{\lambda-\lambda'} \alpha \quad \text{(18)}
$$

From Lemma 7 we have

$$
\theta(\psi(C_t), \psi(C_t'))(\tau) = 0, \quad 0 < |\tau| < 2^m.
$$

From (18) and (19), we have

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k)) = 0, \quad 0 < |\tau| < 2^m.
$$

From (17) and (20), we have

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k)) = \begin{cases}
 p^{2m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\
 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m.
\end{cases}
$$

Similarly, it can be shown that

$$
\theta(\psi_{2m+p-2m}(V_t^k), \psi_{2m+p-2m}(V_t^k)) = \begin{cases}
 p^{2m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\
 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m.
\end{cases}
$$

From Lemma 7, (9), (10), (14), and (15), the ACCF between $\psi_{2m+p-2m}(U_t^k)$ and $\psi_{2m+p-2m}(V_t^k)$ for $\tau = 0$ can be derived as

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k)) = \begin{cases}
 p^{2m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\
 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m.
\end{cases}
$$

From Lemma 7 we have

$$
\theta(\psi(C_t), \psi(C_t'))(0) = 0.
$$

From (23) and (24), we have

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k))(0) = 0.
$$

From Lemma 7, (9), (10), (14), (15), and (24), the ACCF between $\psi_{2m+p-2m}(U_t^k)$ and $\psi_{2m+p-2m}(V_t^k)$ for $0 < |\tau| < 2^m$ can be derived as

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k))(\tau) = \begin{cases}
 p^{2m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\
 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\
 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m.
\end{cases}
$$

From (25) and (26), we have

$$
\theta(\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k))(\tau) = 0, \quad 0 < |\tau| < 2^m.
$$

The obtained results in (20), (22), and (27) show that the following set of codes

$$
\{\psi_{2m+p-2m}(U_t^k), \psi_{2m+p-2m}(V_t^k) : 0 \leq t < 2^k, 0 \leq \lambda < p\}
$$

forms $p^{2k+1}$-ZCCSs. The proposed construction is optimal as it satisfies the equality $K = M = \frac{N}{2^k}$.

Remark 1: For $p = 2, \delta = lcm(p, g) = q$, and the PBFs $F^\lambda$ and $G^\lambda$ become GFs of $m+s$ variables over $\mathbb{Z}_q$. For the same value of $p$, from Theorem 1, we obtain $(2^{k+2}, 2^m)$-ZCCSs which is optimal and the components of each codeword from a code in $(2^{k+2}, 2^m)$-ZCCSs are drawn from the roots of the polynomial: $z^q - 1$. Therefore, the proposed construction also generates ZCCSs of length in the form of power-of-two over the ring $\mathbb{Z}_q$.

Let us illustrate the Theorem 1 with the following example:

**Example 1:** Let us assume that $q = 2, p = 3, m = 3, k = 1$ and $s = 2$. Let us take the GBF $f : \{0, 1\}^3 \rightarrow \mathbb{Z}_2$ as follows:

$$
f(x_1, x_2, x_3) = G(f(x_1, x_0) = 0) \text{ and } G(f(x_1, x_0) = 1) \text{ give a path with } x_2 \text{ as one of the end vertices.}
$$

From (8), we have

$$
F^\lambda = x_1x_2 + \frac{2\lambda}{3}(x_3 + x_4), \quad G^\lambda = x_1x_2 + \frac{2\lambda}{3}(x_3 + x_4),
$$

where $\lambda = 0, 1, 2$. From (9) and (10), we have

$$
U_t^\lambda = \{F^\lambda + d_0x_0 + d_0x_0 + d_0x_2 : d_0, d_1 \in \{0, 1\} \}
$$

and $V_t^\lambda = \{G^\lambda + d_0x_0 + d_0x_0 + d_0x_2 : d_0, d_1 \in \{0, 1\} \},
$$

where $(t_0)$ is the binary vector representation of $t$. Therefore, $\{\psi_8(U_t^0), \psi_8(V_t^0) : 0 \leq t \leq 1, 0 \leq \lambda \leq 2\}$ forms $(12, 8)$-ZCCSs which also optimal. The components of each code word from a code in $(12, 8)$-ZCCSs are drawn from the roots of the polynomial: $z^8 - 1$, where $\delta = lcm(p, q) = lcm(2, 3) = 6$.

Remark 2: From (14) and (15), we see that $\psi_{2m+p-2m}(U_t^k)$ and $\psi_{2m+p-2m}(V_t^k)$ can also be expressed as the concatenation of $\psi_{2m+p-2m}(C_t)$ and $\psi_{2m+p-2m}(C_t')$, respectively, where $i = 1, 2, \ldots, p$. Therefore, the proposed PBF generators establish a link between the proposed direct construction and the indirect constructions of ZCCSs which are obtained by performing concatenation operation on the CCCs from [4].

IV. CONCLUSIONS

In this paper, we have developed a direct construction of optimal ZCCSs with NPT lengths. Unlike the current state-of-the-art which can only generate sub-optimal ZCCSs with NPT lengths, the novelty of this work stems from the use of PBFs.

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