ON THE LATTICE OF WEAKLY EXACT STRUCTURES

ROSE-LINE BAILLARGEON, THOMAS BRÜSTLE, MIKHAIL GORSKY,
AND SOUHEILA HASSOUN

Abstract. The study of exact structures on an additive category $\mathcal{A}$ is closely related to the study of closed additive sub-bifunctors of the maximal extension bifunctor $\text{Ext}^1$ on $\mathcal{A}$. We initiate in this article the study of “weakly exact structures”, which are the structures on $\mathcal{A}$ corresponding to all additive sub-bifunctors of $\text{Ext}^1$. We introduce weak counterparts of one-sided exact structures and show that a left and a right weakly exact structure generate a weakly exact structure. We further define weakly extriangulated structures on an additive category and characterize weakly exact structures among them.

We investigate when these structures on $\mathcal{A}$ form lattices. We prove that the lattice of substructures of a weakly extriangulated structure is isomorphic to the lattice of topologizing subcategories of a certain functor category. In the idempotent complete case, this characterises the lattice of all weakly exact structures. We study in detail the situation when $\mathcal{A}$ is additively finite, giving a module-theoretic characterization of closed sub-bifunctors of $\text{Ext}^1$ among all additive sub-bifunctors.

Contents

1. Introduction 2
2. Acknowledgements 5
3. Weakly exact and exact structures 6
3.1. Definitions 6
3.2. Example 7
3.3. Weakly exact structures 8
3.4. The left and right weakly exact structures 10
3.5. The maximal weakly exact structure 14
4. Sub-bifunctors and closed sub-bifunctors of $\text{Ext}^1$ 15
4.1. From weakly exact structures to bifunctors 15
4.2. From sub-bifunctors of $\text{Ext}^1$ to weakly exact structures 18
4.3. Example 20
4.4. Weakly exact structures as bimodules 21
5. Weakly extriangulated structures 22
6. Defects and topologizing subcategories 26
7. Lattice structures 29
7.1. Definitions 29
ON THE LATTICE OF WEAKLY EXACT STRUCTURES

1. INTRODUCTION

Exact structures go back to the work of Yoneda and early versions of exact structures in [Buch59, BuHo61] originated from studies of relative homological algebra in abelian categories, like studying resolutions with a different set of objects than the projectives. In these papers, a mix of structures has been considered, on one hand classes of morphisms satisfying certain properties ("h.f.class"), on the other hand certain ("closed") subfunctors of Ext$^1$. The authors considered also a weaker notion, an f.class, which omits the condition of being closed under composition of admissible monics and epics.

The "stand alone" concept of an exact structure as a class of short exact sequences in an additive category $\mathcal{A}$ satisfying certain axioms has been laid out by Quillen in [Qu73], however requiring that $\mathcal{A}$ be embedded into an abelian category. The independent version of these axioms was formulated by Keller in [Ke90], see also [GR92]. It allows to develop methods from homological algebra, and define derived categories, see [Ne90, Ke91]. Note that there exist different independant notions of "exact categories", like the "Barr-exact categories" or "effective regular categories", to not be confused with the one we consider in our work.

The comparison to subfunctors of Ext$^1$ has been re-considered in [AS] and then in [DRSS], with applications to exact structures originating from one-point extensions, a special case of exact structures associated with bimodule problems in [BrHi]. However, the lack of a unique maximum extension-functor for arbitrary additive categories was a limiting factor in these studies. If $\mathcal{A}$ has kernels and cokernels, the existence of the maximal exact structure was first proved by Sieg and Wegner [SW11]. Crivei [Cr11] extended the result to additive categories for which every split epimorphism has a kernel, but finally Rump [Ru11] showed that any additive category admits a unique maximal exact structure $\mathcal{E}_{\text{max}}$.

In [BHLR] a study of the family of all exact structures $\text{Ex}(\mathcal{A})$ on an additive category $\mathcal{A}$ was initiated. The existence of a unique maximum exact structure allows to turn $\text{Ex}(\mathcal{A})$ into a complete bounded lattice. On the side of bifunctors, this amounts to studying all closed sub-bifunctors of a unique maximum bifunctor $\mathcal{E}_{\text{max}}$ which corresponds to the exact structure $\mathcal{E}_{\text{max}}$. It is natural, on the bifunctor
side, to extend the study to all additive sub-bifunctors, which in turn raises the question to which structure of exact sequences they correspond. We study in this paper the corresponding systems of short exact sequences, which we call weakly exact structures, reminiscent of the f. classes studied in [Buch59]. We show that all the weakly exact structures on $\mathcal{A}$ form a lattice. When the underlying category $\mathcal{A}$ is additively finite, this lattice is a finite length modular lattice, a class of lattices studied recently by Haiden, Katzarkov, Kontsevich and Pandit in [HKKP] in connection with weight filtrations and the notion of semi-stability.

In order to find a general way to give proofs of various statements concerning exact and triangulated categories that would work for both of these classes of categories (or, rather, classes of structures on additive categories), Nakaoka and Palu [NP19] studied additive bifunctors $\mathcal{E} : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Ab}$ equipped with certain extra data called a realization. They found a set of axioms on triples consisting of an additive category, a bifunctor and a realization that unifies the axioms of exact and of triangulated categories. They called such structures extriangulated. Extensions in exact categories are realized by “admissible” kernel-cokernel pairs. In an extriangulated category this role is played by pairs of composable morphisms $f, g$ where $f$ is a weak kernel of $g$ and $g$ is a weak cokernel of $f$. Moreover, Nakaoka and Palu characterized all triples that define exact structures, in other words, closed additive sub-bifunctors of $\mathcal{E}_{\text{max}}$. Herschend, Liu and Nakaoka [HLN] introduced $n$-exangulated structures and proved that the choice of a 1-exangulated structure on an additive category is equivalent to the choice of an extriangulated structure. The set of axioms of 1-exangulated structures is slightly different from that of extriangulated categories. In Section 5, we consider 1-exangulated categories with one of the axioms removed. We prove that such weakly 1-exangulated, or weakly extriangulated structures naturally generalize weakly exact structures we defined earlier.

For a finite-dimensional algebra $\Lambda$, Buan [Bu01] studied closed sub-bifunctors of the bifunctor $\text{Ext}_1^\Lambda$ on the category $\text{mod} \Lambda$. He proved that they correspond to certain Serre subcategories of the category of finitely presented additive functors $(\text{mod} \Lambda)^{\text{op}} \to \text{Ab}$ (i.e. of finitely presented modules over $\text{mod} \Lambda$), defined as categories of contravariant defects in works of Auslander [A66, A78]. This result was later extended to exact structures on additive categories in [En18], see also [FG20].

We note that the definition of contravariant defects naturally extends to the setting of weakly exact structures. Ogawa [Og19] defined contravariant defects in the setting of extriangulated categories, and we further extend this notion to the framework of weakly extriangulated categories. By adapting arguments of Ogawa and Enomoto [En20], we prove that the category of defects of a weakly extriangulated structure on an additive category $\mathcal{A}$ is topologizing (in the sense of Rosenberg Ros) in the category $\text{coh}(\mathcal{A})$ of coherent right $\mathcal{A}$-modules. That means that it is closed under subquotients and finite coproducts. For coherent $\mathcal{A}$-modules, we
have a natural notion of subobjects and of quotients: these are defined object-wise (for objects in \( \mathcal{A} \)). This allows us to define topologizing subcategories of an arbitrary (not necessarily abelian) full subcategory \( \mathcal{C} \) of \( \text{coh}(\mathcal{A}) \) as full subcategories of \( \mathcal{C} \) which are topologizing in \( \text{coh}(\mathcal{A}) \).

Given a weakly extriangulated structure, all its substructures are uniquely characterized by their categories of defects, and each topologizing subcategory of a given category of defects defines a weakly extriangulated substructure. Weakly extriangulated substructures of a weakly exact structure are necessarily weakly exact. Thus, whenever we know that an additive category \( \mathcal{A} \) admits a unique maximal weakly exact structure, we can classify all weakly exact structures on \( \mathcal{A} \) in terms of topologizing structures in a certain abelian category.

Topologizing subcategories of an abelian category form a lattice. Topologizing subcategories of the (not necessarily abelian) category of defects of a weakly extriangulated structure on \( \mathcal{A} \) also form a lattice, which is an interval in the lattice of all topologizing subcategories of \( \text{coh}(\mathcal{A}) \). Note that Serre subcategories form a subposet, but not a sublattice of this lattice. Weakly extriangulated substructures of a weakly extriangulated structure also form a natural lattice, extending the lattice of weakly exact structures. We establish lattice isomorphisms between these several lattices, summarized in the following figure:

![Figure 1: Isomorphisms of lattices](image)

We introduce Section 3 the class \( \textbf{Wex}(\mathcal{A}) \) of all weakly exact structures on an additive category \( \mathcal{A} \). It turns out that, despite the fact that weakly exact structures are not closed under compositions, some of the properties of exact structures are
still valid, in particular, every weakly exact structure satisfies Quillen’s obscure axiom, see Proposition 3.8. Similar to exact structures, it is sometimes beneficial to dissect the set of axioms into two parts, leading to the notion of left/right weakly exact structure. We show that any pair of left and right weakly exact structure gives rise to a weakly exact structure, and that all such structures arise in that way.

The existence of a unique maximal weakly exact structure for any additive category is an open question. Since it is not known in general if there exist weakly exact structures larger than the maximal exact structure ($\mathcal{E}_{\text{max}}$), we study in this paper mainly the weakly exact structures that are included in $\mathcal{E}_{\text{max}}$, so we consider the interval $\text{Wex}(\mathcal{E}_{\text{max}}) := [\mathcal{E}_{\text{min}}, \mathcal{E}_{\text{max}}] \subseteq \text{Wex}(\mathcal{A})$. Given a weakly exact structure $\mathcal{W}$ on $\mathcal{A}$, constructing the the group $\mathcal{W}$ of $\mathcal{W}$–extensions yields a map $\Phi$ to category of bifunctors from $\mathcal{A}$ to abelian groups:

$$\Phi : \text{Wex}(\mathcal{A}) \rightarrow \text{BiFun}(\mathcal{A})$$

$$\mathcal{W} \mapsto \mathcal{W} = \text{Ext}^1_{\mathcal{W}}(-,-).$$

This function $\Phi$ induces lattice isomorphisms

$$\text{Wex}(\mathcal{E}_{\text{max}}) \leftrightarrow \text{BiFun}(\mathcal{E}_{\text{max}})$$

$$\cup \cup$$

$$\text{Ex}(\mathcal{A}) \leftrightarrow \text{CBiFun}(\mathcal{A})$$

where $\text{CBiFun}(\mathcal{A})$ denotes the subclass of closed sub-bifunctors of $\mathcal{E}_{\text{max}}$. Note that $\text{Ex}(\mathcal{A})$ is not a sublattice of $\text{Wex}(\mathcal{E}_{\text{max}})$, even if it is a subposet: the join operation we consider on both sets is different, as we illustrate by an example in Section 4.3.

When the underlying category $\mathcal{A}$ is additively finite and Krull-Schmidt, it is known that the lattice $\text{Ex}(\mathcal{A})$ is boolean, with each object $\mathbb{E}(S)$ determined by the choice of a set $S$ of Auslander-Reiten sequences. The larger lattice $\text{Wex}(\mathcal{A})$ however is not boolean, and it is interesting to characterise the members of $\text{Ex}(\mathcal{A})$ in module-theoretic terms, that is, describe the closed sub-bimodules of $\mathcal{E}_{\text{max}}$. We show that, when viewed as bimodules over the Auslander algebra of $\mathcal{A}$, elements in $\text{Ex}(\mathcal{A})$ can be characterized as follows: For every set $S$ of Auslander-Reiten sequences, the closed bimodule $\mathbb{E}(S)$ of $\mathcal{E}_{\text{max}}$ introduced above is the maximal submodule of $\mathcal{E}_{\text{max}}$ whose socle is $S$.

2. Acknowledgements

The authors would like to thank Shipeing Liu and Hiroyuki Nakaoka for helpful discussions contributing to this version of the work.

Most of this work was done while the first author was supported by an NSERC USRA grant. The second author was supported by Bishop’s University and NSERC of Canada, and the fourth author acknowledges support from the
"thésards étoiles" scholarship of ISM for outstanding PhD candidates. This work was completed during the third author’s participation at the Junior Trimester Program “New Trends in Representation Theory” at the Hausdorff Institute for Mathematics in Bonn. He is very grateful to the Institute for the perfect working conditions.

3. WEAKLY EXACT AND EXACT STRUCTURES

We introduce in this section the central topic of this paper, exact structures on additive categories. Then, we introduce weakly exact structures, which are a generalization of exact structures, and study some of their properties.

3.1. Definitions. We recall the definition of an exact structure on an additive category given by Quillen in [Qu73], using the terminology of [Ke90], see also [GR92]. We refer to [Bü10] for an exhaustive introduction to exact categories.

We fix an additive category $\mathcal{A}$ throughout this section. Many of the early versions of exact structures were formulated in the context of an abelian category $\mathcal{A}$, using all the short exact sequences on it. For a general additive category $\mathcal{A}$, the notion of short exact sequence is specified to be a kernel-cokernel pair $(i,d)$, that is, a pair of composable morphisms such that $i$ is kernel of $d$ and $d$ is cokernel of $i$. An exact structure on $\mathcal{A}$ is then given by a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{A}$ satisfying certain axioms which we recall below. We call admissible monic a morphism $i$ for which there exists a morphism $d$ such that $(i,d) \in \mathcal{E}$. An admissible epic is defined dually. Note that admissible monics and admissible epics are referred to as inflation and deflation in [GR92], respectively. We depict an admissible monic by $\xrightarrow{i}$ and an admissible epic by $\xrightarrow{d}$.

The pair $(i,d) \in \mathcal{E}$ is referred to as admissible short exact sequence, or short exact sequence in $\mathcal{E}$.

Definition 3.1. An exact structure $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs $(i,d)$ in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

(E0) For all objects $A$ in $\mathcal{A}$ the identity $1_A$ is an admissible monic;
(E0)$^{op}$ For all objects $A$ in $\mathcal{A}$ the identity $1_A$ is an admissible epic;
(E1) The class of admissible monics is closed under composition
(E1)$^{op}$ The class of admissible epics is closed under composition;
(E2) The push-out of an admissible monic $i : A \rightarrow B$ along an arbitrary morphism $t : A \rightarrow C$ exists and yields an admissible monic $s_C$:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow t & & \downarrow s_B \\
C & \xrightarrow{s_C} & S.
\end{array}
$$
The pull-back of an admissible epic $h$ along an arbitrary morphism $t$ exists and yields an admissible epic $p_B$

$$
\begin{array}{c}
P \\
P_A \\
A
\end{array}
\begin{array}{c}
p_B \\
t \\
h
\end{array}
\begin{array}{c}
P_B \\
C.
\end{array}
$$

An exact category is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. Note that $\mathcal{E}$ is an exact structure on $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure on $\mathcal{A}^{op}$.

We denote by $(\text{Ex}(\mathcal{A}), \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by containment $\mathcal{E} \subseteq \mathcal{E}'$. Note that $\text{Ex}(\mathcal{A})$ need not actually form a set, but by abuse of language, we still use the term poset when $\text{Ex}(\mathcal{A})$ is a class. The poset $(\text{Ex}(\mathcal{A}), \subseteq)$ always contains a unique minimal element, the split exact structure $\mathcal{E}_{min}$ which is formed by all split exact sequences, that is, sequences isomorphic to

$$
\begin{array}{ccc}
A & \xrightarrow{[1]} & A \oplus [0] \\
& & B
\end{array}
$$

(see [Bü10, Lemma 2.7]).

Moreover, every additive category admits a unique maximal exact structure $\mathcal{E}_{max}$, see [Ru11, Corollary 2]. When the category $\mathcal{A}$ is abelian, then $\mathcal{E}_{max}$ is formed by all short exact sequences in $\mathcal{A}$. The construction is more subtle for other classes of additive categories, we refer to [BHLR, Section 2.4] for a more detailed discussion.

3.2. Example. Consider the category $\mathcal{A} = \text{rep} \, Q$ of representations of the quiver $Q : 1 \rightarrow 2 \leftarrow 3$

Then the Hasse diagram of the poset of exact structures $\text{Ex}(\mathcal{A})$ has the shape of a cube (see [BHLR, Example 4.2] for detailed description of the different exact structures on $\mathcal{A}$):
Let us mention that by taking other forms of the quiver of type $A_3$ such as

$$Q : \begin{array}{ccc}
1 & \longleftarrow & 2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 2 \\
\end{array}$$

or

$$Q : \begin{array}{ccc}
1 & \longrightarrow & 2 \\
\downarrow & & \downarrow \\
3 & \longleftarrow & 3 \\
\end{array}$$

we get an isomorphic poset. In fact, $\text{Ex}(\mathcal{A})$ is a Boolean lattice in these cases, with $n$ Auslander-Reiten sequences in $\mathcal{A}$ giving rise to exactly $2^n$ exact structures and poset structure isomorphic to the power set of the set of Auslander-Reiten sequences in $\mathcal{A}$, see [En18].

3.3. Weakly exact structures.

**Definition 3.2.** Let $\mathcal{A}$ be an additive category. We define a weakly exact structure $\mathcal{W}$ on $\mathcal{A}$ as a class of kernel-cokernel pairs $(i, d)$ in $\mathcal{A}$ which is closed under isomorphisms and direct sums, and satisfies the axioms $(E0)$, $(E0)^{op},(E2)$ and $(E2)^{op}$ of Definition 3.1.

We denote by $(\text{Wex}(\mathcal{A}), \subseteq)$ the poset of all weakly exact structures on $\mathcal{A}$, ordered by containment.

**Lemma 3.3.** $\text{Ex}(\mathcal{A})$ is a subclass of $\text{Wex}(\mathcal{A})$.

*Proof.* Only the direct sum condition needs to be verified. But this is always satisfied for exact structures, by [Bü10, Proposition 2.9].

**Remark 3.4.** The proof of [Bü10, Proposition 2.9] makes heavy use of axioms $(E1)$ and $(E1)^{op}$, this makes us think that the property of being closed under direct sums does not follow from the remaining axioms for weakly exact structures.

We now state some of the properties for exact structures that also hold for weakly exact structures:

**Lemma 3.5.** Let $\mathcal{W}$ be a weakly exact structure and let $i$ and $i'$ be admissible monics of $\mathcal{W}$ forming the rows of a commutative square:

$$\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow f & & \downarrow f' \\
A' & \xrightarrow{i'} & B' \\
\end{array}$$

Then the following statements are equivalent:

(i) The square is a push-out.

(ii) $A \xrightarrow{i} B \oplus A' \xrightarrow{[f, f']^{\prime}} B'$ is a short exact sequence belonging to $\mathcal{W}$.

(iii) The square is both a push-out and a pull-back.
(iv) There exists a commutative commutative diagram with rows being conflations in $W$:

$$
\begin{array}{c}
A \xrightarrow{i} B \xrightarrow{p} C \\
\downarrow f \quad \downarrow f' \quad \downarrow 1_c \\
A' \xrightarrow{i'} B' \xrightarrow{p'} C
\end{array}
$$

**Proof.** One can easily verify that the proof of the statement for exact categories in [Bii10, Proposition 2.12] does not use axioms $(E1)$ or $(E1)^{op}$ when it is done in the order $(i) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. □

**Remark 3.6.** The dual of Lemma 3.5 is also true. For example, the dual of (i) implies (iv) would be: If $d$ and $d'$ are admissible epics of $W$ and $(g, d)$ is the push-out of $(d', g')$ then the following diagram exists, is commutative and has rows in $W$:

$$
\begin{array}{c}
A \xrightarrow{j} B \xrightarrow{d} C \\
\downarrow 1_A \quad \downarrow g \quad \downarrow g' \\
A' \xrightarrow{j'} B' \xrightarrow{d'} C'
\end{array}
$$

Commutative squares that are both a pushout and a pullback are called *bicartesian* squares.

**Lemma 3.7.** Let $W$ be a weakly exact structure on $A$. For any morphism of admissible short exact sequences

$$
\begin{array}{c}
A \longrightarrow B \longrightarrow C \\
\downarrow \quad \downarrow \quad \downarrow \\
A' \longrightarrow B' \longrightarrow C'
\end{array}
$$

in $W$, there exists a commutative diagram

$$
\begin{array}{c}
A \longrightarrow B \longrightarrow C \\
\downarrow \quad \downarrow \quad \downarrow \\
A' \longrightarrow E \longrightarrow C \\
\downarrow \quad \downarrow \quad \downarrow \\
A' \longrightarrow B' \longrightarrow C',
\end{array}
$$

where the middle row is also an admissible short exact sequence in $W$ and the top left and bottom right squares are bicartesian.
Proof. The same proof as in [Bü10, Lemma 3.1] applies here. □

Proposition 3.8. (Quillen’s obscure axiom for weakly exact structures)
Let \( \mathcal{W} \) be a weakly exact structure on an additive category \( \mathcal{A} \).

1. Consider morphisms \( A \xrightarrow{i} B \xrightarrow{j} C \) in \( \mathcal{A} \), where \( i \) has a cokernel and \( ji \) is an admissible monic of \( \mathcal{W} \). Then \( i \) is also an admissible monic of \( \mathcal{W} \).

2. Consider morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \) in \( \mathcal{A} \), where \( g \) has a kernel and \( gf \) is an admissible epic of \( \mathcal{W} \). Then \( g \) is also an admissible epic of \( \mathcal{W} \).

Proof. (1) The proof given in [Bü10, Proposition 2.16] also holds for weakly exact categories: Lemma 3.5 is the equivalent of [Bü10, Proposition 2.12]. One step in the proof of [Bü10, Proposition 2.16] is using axiom \((E1)\), but in fact, the composition of an admissible monic with an isomorphism gives an admissible monic because the class \( \mathcal{W} \) is closed under isomorphisms.

(2) The proof is done dually. □

Lemma 3.9. The split exact structure \( E_{\text{min}} \) forms the unique minimal element of the poset \( (\text{Wex}(\mathcal{A}), \subseteq) \).

Proof. The proof of [Bü10] Lemma 2.7 does not use axioms \((E1)\) and \((E1)^{op}\), so the statement of [BHLR] Prop 2.12 applies to weakly exact structures as well. □

3.4. The left and right weakly exact structures. In this subsection, we define left weakly exact structures and right weakly exact structures. We show that their combination gives a weakly exact structure and also that every weakly exact structure can be obtained in this way.

These definitions generalise the left and right exact structures introduced in [BC12, Definition 3.1] and studied in [HR20]. Rump is using left and right exact structures in [Ru11] to prove the existence of a unique maximal exact structure on any additive category. Unfortunately, his method does not apply to the case of weakly exact structures, and it is an open question if a unique maximal weakly exact structure exists for any additive category.

Definition 3.10. A right weakly exact structure on \( \mathcal{A} \) is a class of kernels \( I \) which is closed under isomorphisms and satisfies the following properties:

(Id) For all objects \( X \) in \( \mathcal{A} \) the identity \( 1_X \) and the zero monomorphism \( 0 \to X \) are in \( I \).
(P) The push-out of $f : X \to Y \in I$ along an arbitrary morphism $h : X \to X'$ exists and yields a morphism $f' \in I$:

\[
\begin{array}{c}
X \\
\downarrow h
\end{array}
\overset{f}{\longrightarrow}
\begin{array}{c}
Y \\
\downarrow h'
\end{array}
\]

(Q) Given $A \xrightarrow{a} B \xrightarrow{b} C$ with $ba \in I$ and $a$ has a cokernel, then $a$ is in $I$.

(S) $I$ is closed under direct sums of morphisms.

**Definition 3.11.** A left weakly exact structure on $\mathcal{A}$ is a class of cokernels $\mathcal{D}$ which is closed under isomorphisms and satisfies the following properties:

(Id$^{\text{op}}$) For all objects $X$ in $\mathcal{A}$ the identity $1_X$ and the zero epimorphism $X \to 0$ are in $\mathcal{D}$.

(P$^{\text{op}}$) The pullback of $f : C \to F \in \mathcal{D}$ along an arbitrary morphism $h : E \to F$ exists and yields a morphism $e \in \mathcal{D}$:

\[
\begin{array}{c}
B \\
\downarrow e
\end{array}
\overset{b}{\longrightarrow}
\begin{array}{c}
C \\
\downarrow f
\end{array}
\]

(Q$^{\text{op}}$) Given $A \xrightarrow{a} B \xrightarrow{b} C$ with $ba \in \mathcal{D}$ and $b$ has a kernel, then $b$ is in $\mathcal{D}$.

(S$^{\text{op}}$) $\mathcal{D}$ is closed under direct sums of morphisms.

**Remark 3.12.** Note that, contrary to exact structures (see [BHM10, Proposition 2.9]) the properties (S) and $(\text{S})^{\text{op}}$ above are not implied by the rest of the properties and we need to add them. These properties are necessary to ensure that we get a structure which is equivalent to an additive sub-bifunctor of Ext$^1$ as we show in Section 4. The reason behind this is that the Baer sum uses the direct sum of two short exact sequences in its construction.
Theorem 3.13. Let $\mathcal{A}$ be an additive category. A left weakly exact structure $\mathcal{D}$ on $\mathcal{A}$ can be combined with a right weakly exact structure $\mathcal{I}$ to form a weakly exact structure $\mathcal{W}$ given by the short exact sequences $\xymatrix{A \ar[r]^i & B \ar[r]^d & C}$ with $i \in I$ and $d \in \mathcal{D}$.

Proof. We adapt the proof of [Ru11, Theorem 1] to the case of weakly exact structures. Denote by $\mathcal{D}_I$ the class of morphisms in $\mathcal{D}$ that have a kernel in $I$, and let $I_D$ be the class of morphisms in $\mathcal{I}$ that have a cokernel in $\mathcal{D}$. Thus $\mathcal{W}$ is given by the short exact sequences $\xymatrix{A \ar[r]^i & B \ar[r]^d & C}$ with $i \in I_D$ and $d \in \mathcal{D}_I$.

(1) Suppose we have the following commutative diagram where the top row is in $\mathcal{W}$, the bottom row is a short exact sequence and $u, v, w$ are isomorphisms:

$$
\begin{align*}
E_1 : & \quad A \xymatrix{\ar[r]^i & B \ar[r]^d & C} \\
\downarrow^u & \quad \downarrow^v \quad \downarrow^w \\
E_2 : & \quad A' \xymatrix{\ar[r]^{i'} & B' \ar[r]^{d'} & C'}
\end{align*}
$$

Since $E_1 \in \mathcal{W}$ we have $i \in I$ and $d \in \mathcal{D}$. Since $I$ is closed under isomorphisms, we obtain $i' \in I$, and dually, $d' \in \mathcal{D}$. Therefore, $E_2 \in \mathcal{W}$ and $\mathcal{W}$ is closed under isomorphisms.

(2) Suppose that $E_1$ and $E_2$ are in $\mathcal{W}$. It is well-known that the direct sum $E_1 \oplus E_2 : A \oplus A' \xymatrix{\ar[r]^{i \oplus i'} & B \oplus B' \ar[r]^{d \oplus d'} & C \oplus C'}$ is a short exact sequence.

With $E_1, E_2 \in \mathcal{W}$ we have $i, i' \in I$ and $d, d' \in \mathcal{D}$. Axioms (S) and $(S)^{op}$ imply $i \oplus i' \in I$ and $d \oplus d' \in \mathcal{D}$, so $\mathcal{W}$ is closed under direct sums.

(3) For any object $X$ in $\mathcal{A}$, $\xymatrix{X \ar[r]^1 & X \ar[r]^0 & 0}$ and $\xymatrix{0 \ar[r]^0 & X \ar[r]^1 & X}$ are short exact sequences. By axiom (Id), $\xymatrix{X \ar[r]^1 & X \ar[r]^0 & 0}$ and $\xymatrix{0 \ar[r]^0 & X \ar[r]^1 & X}$ are exact sequences. By (Id) $\xymatrix{X \ar[r]^0 & 0 \ar[r] & \mathcal{D}}$ and $\xymatrix{X \ar[r]^1 & \mathcal{D}}$. Therefore, the two sequences are in $\mathcal{W}$ which means that $\mathcal{W}$ satisfies (E0) and $(E0)^{op}$.

(4) Let us show that $\mathcal{W}$ satisfies (E2). To this end, consider a short exact sequence $\xymatrix{A \ar[r]^i & B \ar[r]^d & C}$ in $\mathcal{W}$ and $f : A \xymatrix{\ar[r] & A'} \in \mathcal{A}$. By (P), the pushout of $i$ and $f$ exists and $i'$ belongs to $I$.

$$
\begin{align*}
\xymatrix{A \ar[r]^i \ar[d]_f & B \ar[r]^d \ar[d]_{\text{PO}} & C \ar[d]^g} \\
A' \ar[r]^{i'} & B'
\end{align*}
$$

Since $di = 0 = 0 \circ f$, the push-out property gives us that there exists a unique $d' : B' \xymatrix{\ar[r] & C}$ such that $d'i' = 0$ and $d'g = d$. Therefore, we have
the following commutative diagram.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f & & \downarrow g \\
A' & \rightarrow & B' \\
\end{array}
\quad \begin{array}{ccc}
& d & \rightarrow & C \\
& \downarrow 1_C & & \\
& & & \end{array}
\quad \begin{array}{ccc}
i & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
i' & \rightarrow & \rightarrow \\
\end{array}
\]

We know \(i' \in I\) and we want to show that \(i' \in I_D\). First, we show that \(d'\) is the cokernel of \(i'\): Let \(x : B' \rightarrow X\) be such that \(xi' = 0\), so \(xgi = xif = 0\). By the cokernel property of \(d\), there exists a unique \(h : C \rightarrow X\) such that \(hd = xg\), so \((xg)i = hdi = 0 = 0 \circ f\). By the pushout property, there exists a unique \(w : B' \rightarrow X\) such that \(wi' = 0\) and \(wg = xg\). Both \(x\) and \(hd'\) qualify for the defining properties of \(w\), so by uniqueness, we get \(x = hd'\).

Therefore, \(d' = \text{coker}(i')\).

Since \((i', d')\) is a kernel-cokernel pair, we get \(i' = \ker(d')\).

As \(d'g = d \in D\) and \(d'\) has a kernel, we infer from property \((Q)^{op}\) for \(D\) that \(d' \in D\), hence \(i' \in I_D\). Therefore, \(A' \rightarrow B' \rightarrow C' \in W\).

(5) Dually, \(W\) satisfies \((E2)^{op}\).

\[\square\]

**Proposition 3.14.** Every weakly exact structure \(W\) on \(A\) can be constructed from a right weakly exact structure and a left weakly exact structure as in Theorem 3.13.

More precisely, if \(I\) is the class of admissible monics of a weakly exact structure \(W\) and \(D\) is the class of admissible epics of \(W\), then \(I\) is a right weakly exact structure and \(D\) is a left weakly exact structure.

**Proof.** Let \(W\) be a weakly exact structure on an additive category \(A\). Let \(I\) be the class of admissible monics of \(W\) and \(D\) the class of admissible epics of \(W\).

(1) First, we show that \(I\) and \(D\) are closed under isomorphisms. Suppose that the following diagram is commutative, \(i \in I\) and that \(u\) and \(v\) are isomorphisms.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow u & & \downarrow v \\
A' & \rightarrow & B' \\
\end{array}
\]

Since \(i\) is an admissible monic of \(W\), there exists \(d : B \rightarrow C \in D\) such that \((i, d) \in W\). We can show that \(i' = \ker(dv^{-1})\) and \(dv^{-1} = \text{cokernel}(i')\). So we have the following isomorphism of short exact sequences.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow u & & \downarrow v \\
A' & \rightarrow & B' \\
\end{array}
\quad \begin{array}{ccc}
& d & \rightarrow & C \\
& \downarrow 1_C & & \\
& & & \end{array}
\quad \begin{array}{ccc}
i & \rightarrow & \rightarrow \\
\downarrow & & \downarrow \\
i' & \rightarrow & \rightarrow \\
\end{array}
\]

\[\square\]
Since, \((i, d) \in W\) and \(W\) is closed under isomorphisms, then \((i', d') \in W\) so \(i' \in I\). Therefore, \(I\) is closed under isomorphisms. Dually, \(D\) is closed under isomorphisms.

(2) Since \(W\) satisfies \((E0)\) and \((E0)^{op}\), it is clear that \(I\) satisfies \((Id)\) and \(D\) satisfies \((Id)^{op}\).

(3) By Proposition 3.8, \(I\) satisfies \((Q)\) and \(D\) satisfies \((Q)^{op}\).

(4) Since \(W\) is closed under direct sums it is clear that \(I\) satisfies \((S)\) and \(D\) satisfies \((S)^{op}\).

\[ \square \]

**Remark 3.15.** In view of Proposition 3.14, we write a weakly exact structure \(W\) as \(W = (I, D)\) where \(I\) is the right weakly exact structure of kernels and \(D\) is the left weakly exact structure of cokernels in \(W\).

3.5. The maximal weakly exact structure. Rump’s construction of a maximal exact structure in [Ru11] does not generalise to the weakly exact structures. In fact, it is an open question if a unique maximal weakly exact structure exists for any additive category. However, under certain conditions on \(A\) we show that the maximal weakly exact structure coincides with the maximal exact structure; this is true under the same conditions as in Crivei’s characterisation of stable short exact sequences forming the maximal exact structure. We recall the necessary definitions first:

**Definition 3.16.** [RW77] A kernel \((A, f)\) is in an additive category \(A\) is called semi-stable if for every push-out square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{i} & & \downarrow{s_B} \\
C & \xrightarrow{s_C} & S \\
\end{array}
\]

the morphism \(s_C\) is also a kernel. We define dually a semi-stable cokernel. A short exact sequence \(A \rightarrow i \rightarrow B \rightarrow d \rightarrow C\) in \(A\) is said to be stable if \(i\) is a semi-stable kernel and \(d\) is a semi-stable cokernel. We denote by \(E_{sta}\) the class of all stable short exact sequences.

For any additive category \(A\), the class \(E_{sta}\) clearly satisfies conditions \((E2)\) and \((E2)^{op}\), but in general it need not satisfy \((E1)\) or the direct sum property \((S)\). Rump provides in [Ru15] an example showing that the class of semi-stable cokernels does not satisfy property \((S)\), thus the semi-stable cokernels do not form a left weakly exact structure. This does not imply that \(E_{sta}\) is not always closed for direct sum of conflations. However, it is known by [Cr11, Theorem 3.5] that \(E_{sta}\) forms an exact structure when \(A\) is weakly idempotent complete, that is, every section has a cokernel, or equivalently, every retraction has a kernel. Moreover, in this case,
the class of stable short exact sequences clearly forms the maximal weakly exact structure since no larger class will satisfy (E2) and (E2)$^{op}$. So we get that a unique maximum weakly exact structure exists and coincides with $\mathcal{E}_{\text{sta}}$ when $\mathcal{A}$ is weakly idempotent complete:

$$\mathcal{W}_{\text{max}} = \mathcal{E}_{\text{max}} = \mathcal{E}_{\text{sta}}$$

Crivei goes further in [Cr12, Theorem 3.4], characterising maximum exact structures using the idempotent (or Karoubian) completion $H: \mathcal{A} \to \hat{\mathcal{A}}$ of the category $\mathcal{A}$. Considering the maximal exact structure $\hat{\mathcal{E}}_{\text{max}}$ in $\hat{\mathcal{A}}$, he defines the notion of $\mathcal{A}$ being closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{\text{max}})$ (see [Cr12, Theorem 3.4] for details), and obtains the following result:

**Theorem 3.17.** [Cr12, Theorem 3.4] Let $\mathcal{A}$ be an additive category, and let $H: \mathcal{A} \to \hat{\mathcal{A}}$ be its idempotent completion. Then the class $\mathcal{E}_{\text{sta}}$ of stable short exact sequences of $\mathcal{A}$ defines an exact structure on $\mathcal{A}$ if and only if $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{\text{max}})$. In this case, $\mathcal{E}_{\text{sta}}$ is the maximal exact structure on $\mathcal{A}$.

Again, since the class of stable short exact sequences clearly forms the maximal class satisfying (E2) and (E2)$^{op}$, we get:

**Corollary 3.18.** Assume that $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{\text{max}})$. Then the class of stable short exact sequences forms the unique maximal weakly exact structure on $\mathcal{A}$:

$$\mathcal{W}_{\text{max}} = \mathcal{E}_{\text{max}} = \mathcal{E}_{\text{sta}}$$

We refer to [Cr12, Corollary 3.5] for an example of an additive category $\mathcal{A}$ which is not weakly idempotent complete, but satisfies that $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{\text{max}})$.

4. **Sub-bifunctors and closed sub-bifunctors of Ext$^1$**

We explore in this section the correspondence between exact structures and certain subfunctors of Ext$^1$$_\mathcal{A}$.

4.1. **From weakly exact structures to bifunctors.** Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$. The aim of this section is to associate with $\mathcal{W}$ an additive functor to the category of abelian groups

$$\mathbb{W} = \text{Ext}^1_{\mathcal{W}}(-, -): \mathcal{A}^{op} \times \mathcal{A} \to \text{Ab}.$$ 

In the following definition we review the classical construction for abelian categories, stated in [M65, chapter VII], and formulate it in our context.

**Definition 4.1.** Define for objects $A, C \in \mathcal{A}$ the set

$$\mathbb{W}(C, A) = \text{Ext}^1_{\mathcal{W}}(C, A) = \left\{ (i, d) \mid A \xrightarrow{i} B \xrightarrow{d} C \in \mathcal{W} \right\},$$

where $i$ is injective and $d$ is a surjective morphism.
where we denote by \((i,d)\) the usual equivalence class of the short exact sequence \((i,d)\). To define the action of the functor \(W\) on morphisms, let \(E = (i,d) \in W\) be a short exact sequence from \(A\) to \(C\), and \(a : A \to A'\) a morphism. Then we define the short exact sequence \(aE \in W\) to be obtained by taking the pushout along \(i\) and \(a\) (thus defining the image of \(E\) under the map \(W(C,a) : E : A \xrightarrow{i} B \xrightarrow{d} C\):

\[
E : \quad A \xrightarrow{i} B \xrightarrow{d} C
\]

\[
aE : \quad A' \xrightarrow{i'} PO \xrightarrow{d'} C
\]

Dually, for a morphism \(c : C' \to C\), the pullback \(Ec\) along \(d\) and \(c\) defines the image of \(E\) under the map \(W(c,A) : E : A \xrightarrow{i} B \xrightarrow{d} C\):

\[
Ec : \quad A \xrightarrow{i''} PB \xrightarrow{d''} C'
\]

\[
E : \quad A \xrightarrow{i} B \xrightarrow{d} C
\]

Moreover, we define on \(\mathcal{W}(C,A)\) an addition (Baer’s sum) by

\[
E_1 + E_2 = \nabla_A (E_1 \oplus E_2) \Delta_C
\]

where \(\nabla_A\) and \(\Delta_C\) are the codiagonal and diagonal maps, and \(E_1 \oplus E_2\) is the direct sum of \(E_1\) and \(E_2\) in \(\mathcal{W}(C \oplus C, A \oplus A)\).

Note that we use property (E2) (or (P)) from the definition of (right) exact structure to define the product \(aE\), dually we use \((E2)^\text{op}\) (or \((P)^\text{op}\)) to define the product \(Ec\). For the sum \(E_1 + E_2\), we employ \(E2\), \((E2)^\text{op}\), \((S)\) and \((S)^\text{op}\).

Given a left weakly exact structure \(D\) on \(A\) and objects \(A,C \in A\), we define

\[
\mathbb{D}_A(C) = \{ (i,d) \mid A \xrightarrow{i} B \xrightarrow{d} C \text{ is a short exact sequence with } d \in D \}\]

We also use the notation \(\text{Ext}^1_D(C,A) = \mathbb{D}_A(C)\). Dually, we define \(\text{Ext}^1_C(A) = \mathbb{I}_A(C)\) for a right weakly exact structure \(I\).

**Lemma 4.2.** Let \(D\) be a left weakly exact structure on \(A\). Then for each \(A \in A\), the construction in Definition 4.1 yields a functor \(\mathbb{D}_A = \text{Ext}^1_D(\_ , A) : A^{\text{op}} \to \text{Set}\). Dually, for every object \(C \in A\), a right weakly exact structure \(I\) defines a functor \(\mathbb{I}_C = \text{Ext}^1_C(\_ , \_ ) : A \to \text{Set}\).

**Proof.** Adapt [M65, Chapter VII], Lemma 1.3 (i) and (ii) to our context. \(\square\)

**Proposition 4.3.** Let \(W\) be weakly exact structure on \(A\). Then the construction in Definition 4.1 yields an additive bifunctor

\[
\mathbb{W} = \text{Ext}^1_W(\_ , \_ ) : A^{\text{op}} \times A \to \text{Ab}; (C, A) \mapsto \mathbb{W}(C,A).
\]
Proof. This is a classical result going back to Yoneda and Baer when $\mathcal{A}$ is the category of modules over a ring and $\mathcal{W} = \mathcal{E}_{\text{max}}$ is the class of all short exact sequences, see [Mac63, Chapter III]. When $\mathcal{A}$ is abelian and $\mathcal{W} = \mathcal{E}$ is an exact structure on $\mathcal{A}$, then $\mathcal{E}$ is what MacLane calls a proper class, and the result is given in Proposition 4.3 of [Mac63, Chapter XII]. For an exact structure $\mathcal{W} = \mathcal{E}$ on a general additive category $\mathcal{A}$, the result can be obtained using the embedding of [GR92, Prop. 9.1] and the same techniques as used in [DRSS, Section 1.2]. Finally, assume that $\mathcal{W}$ is a weakly exact structure on $\mathcal{A}$, and write $\mathcal{W} = (\mathcal{I}, \mathcal{D})$ with $\mathcal{I}$ a right weakly exact structure and $\mathcal{D}$ a left weakly exact structure as in Remark 3.15. The fact that $\mathcal{W}$ is a bifunctor then follows from Lemma 4.2 and [M65, Chapter VII, Lemma 1.3 (iii)]. The proof that $\mathcal{W}$ is an additive functor to the category of (big) abelian groups is shown as in [M65, Chapter VII, Theorem 1.5], noting that none of the proofs there is using condition (E1) or that $\mathcal{A}$ is abelian, all that is needed are the axioms of a weakly exact structure. □

Lemma 4.4. Let $\mathcal{V}$ and $\mathcal{W}$ be weakly exact structures on $\mathcal{A}$ with $\mathcal{V} \subseteq \mathcal{W}$. Then $\mathcal{V} = \text{Ext}^1_{\mathcal{V}}(-, -) : A^{\text{op}} \times A \to \text{Ab}$ is an additive sub-bifunctor of $\mathcal{W}$.

Proof. Since $\mathcal{V}$ is contained in $\mathcal{W}$, the set $\mathcal{V}(C, A)$ is contained in the abelian group $\mathcal{W}(C, A)$. We show that it is a subgroup: By Lemma 3.9, the set $\mathcal{V}(C, A)$ contains the zero element of $\mathcal{W}(C, A)$ which is given by the split exact sequence. Moreover, $\mathcal{V}(C, A)$ is closed under the addition in $\mathcal{W}(C, A)$. In fact, given short exact sequences $E_1$ and $E_2$ in $\mathcal{V}(C, A)$, we have that $E_1 \oplus E_2$ is in $\mathcal{V}(C \oplus C, A \oplus A)$ by definition of weakly exact structure. Axioms (E2) and (E2) imply then that $E_1 + E_2 = \nabla_A(E_1 \oplus E_2) \Delta_C$ lies in $\mathcal{V}(C, A)$. Finally, it is well-known that the additive inverse in $\mathcal{V}(C, A)$ of the element $E = A \overset{i}{\longrightarrow} B \overset{d}{\longrightarrow} C$ is given by the equivalence class of $-E = A \overset{-i}{\longrightarrow} B \overset{d}{\longrightarrow} C$. But $-E$ is the pushout of $E$ along $-1_A$, thus is contained in $\mathcal{V}$ when $E$ is. This shows that $\mathcal{V}(C, A)$ is a subgroup of $\mathcal{W}(C, A)$. Multiplication by morphisms is given by pullback and pushout, and since $\mathcal{V}$ is stable under these operations, we have that $\mathcal{V}$ is an additive sub-bifunctor of $\mathcal{W} : A^{\text{op}} \times A \to \text{Ab}; (C, A) \mapsto \mathcal{W}(C, A)$.

Remark 4.5. The construction in Definition 4.1 associates with Rump’s maximal exact structure $\mathcal{E}_{\text{max}}$ an additive bifunctor $\mathcal{E}_{\text{max}} = \text{Ext}^1_{\mathcal{E}_{\text{max}}}(-, -)$. In particular, when $\mathcal{A}$ is an abelian category, then $\mathcal{E}_{\text{max}}$ is the class of all short exact sequences, and we obtain Yoneda’s bifunctor

$$\mathcal{E}_{\text{max}} = \text{Ext}^1_{\mathcal{A}}(-, -).$$

Generalizing this notation, we write $\text{Ext}^1_{\mathcal{A}}(-, -) := \text{Ext}^1_{\mathcal{E}_{\text{max}}}(-, -)$ for any additive category $\mathcal{A}$.

Additive functors from a preadditive category $\mathcal{C}$ to the category of abelian groups are usually referred to as $\mathcal{C}$—modules, and they form an abelian category, see e.g. Theorem 4.2 in [Po73, chap. 3]. For $\mathcal{C} = A^{\text{op}} \times A$ and $\mathcal{W}$ a weakly exact structure
on $\mathcal{A}$, we can thus view $W = \text{Ext}^1_W(\cdot, \cdot)$ as an object in the abelian category $\text{BiFun}(\mathcal{A})$ of $\mathcal{A} - \mathcal{A}$-bimodules. We consider the partial order on $\text{BiFun}(\mathcal{A})$ given by

$$F \leq F' \iff F(C, A) \leq F'(C, A)$$

for all $A, C \in \mathcal{A}$, that is, $F(C, A)$ is a subgroup of $F'(C, A)$ for every pair of objects in $\mathcal{A}$. The construction in Definition 4.1 thus defines a map $\Phi$ from the weakly exact structures included in $\mathcal{E}_{\text{max}}$ on the additive category $\mathcal{A}$ to the $\mathcal{A} - \mathcal{A}$-bimodules:

$$\Phi : \text{Wex}(\mathcal{A}) \rightarrow \text{BiFun}(\mathcal{A})$$

$$W \mapsto W = \text{Ext}^1_W(\cdot, \cdot).$$

Lemma 4.4 shows that $\Phi$ is a morphism of posets. The elements in $\text{Ex}(\mathcal{A})$ are sent under the map $\Phi$ to subfunctors of $\text{Ext}^1_A(\cdot, \cdot) = \mathcal{E}_{\text{max}}$ that enjoy an additional property, namely they give rise to a long exact sequence of functors:

**Definition 4.6.** ([BuHo61, DRSS]) An additive sub-bifunctor $F$ of $\text{Ext}^1_A(\cdot, \cdot)$ is called **closed** if for any short exact sequence

$$0 \rightarrow \text{Hom}(X, A) \rightarrow \text{Hom}(X, B) \rightarrow \text{Hom}(X, C) \rightarrow F(X, A) \rightarrow F(X, B) \rightarrow F(X, C)$$

and

$$0 \rightarrow \text{Hom}(C, X) \rightarrow \text{Hom}(B, X) \rightarrow \text{Hom}(A, X) \rightarrow F(C, X) \rightarrow F(B, X) \rightarrow F(A, X)$$

are exact in the category of abelian groups. As noted in [BuHo61], the above sequences are always exact in all positions except $F(X, B)$, respectively $F(B, X)$, thus one could equivalently say $F$ is closed if the functors $F(X, \cdot)$ and $F(\cdot, X)$ are **middle-exact**, or using the terminology of [Rou, 4.1.1], (co-)**homological**.

**Proposition 4.7.** [DRSS, Prop 1.4] Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$. Then the bifunctor $\Phi(\mathcal{E})$ is closed.

### 4.2. From sub-bifunctors of $\text{Ext}^1_A$ to weakly exact structures.

We defined in the previous section a map

$$\Phi : \text{Wex}(\mathcal{A}) \rightarrow \text{BiFun}(\mathcal{A}).$$

The aim of this section is to construct a partial inverse function $\Psi$. Since we do not know if there is a maximal weakly exact structure for a general additive category, we need to restrict the construction to the interval of weakly exact structures included in $\mathcal{E}_{\text{max}}$, denoted

$$\text{Wex}(\mathcal{E}_{\text{max}}) := [\mathcal{E}_{\text{min}}, \mathcal{E}_{\text{max}}] \subseteq \text{Wex}(\mathcal{A}).$$
Likewise, we write $\text{BiFun}(E_{\text{max}})$ for the class of sub-objects of $E_{\text{max}}$ in $\text{BiFun}(A)$. Formulated in terms of posets, one can say

$$\text{BiFun}(E_{\text{max}}) := [E_{\text{min}}, E_{\text{max}}] \subseteq \text{BiFun}(A)$$

is the interval of all additive bifunctors between the minimum and the maximum exact structure on $A$. Note that for a weakly idempotent complete category $A$, or more generally under the conditions of Corollary 3.18, we have that $E_{\text{max}}$ is the maximal weakly exact structure on $A$, therefore $Wex(E_{\text{max}}) = Wex(A)$.

To define a map $\Psi$ on $\text{BiFun}(E_{\text{max}})$, we use the notion of $F$-exact pairs given in the following definition:

**Definition 4.8.** [BuHo61, DRSS] Let $F : A^{\text{op}} \times A \longrightarrow \text{Ab}$ be an additive sub-bifunctor of $\text{Ext}_A^1(-, -)$. Define a class $\mathcal{W}_F$ of short exact sequences by

$$\mathcal{W}_F := \{ A \xrightarrow{i} B \xrightarrow{d} C \text{ in } A \mid (i, d) \in F(C, A) \}.$$ 

The short exact sequences $(i, d)$ in $\mathcal{W}_F$ are called $F$-exact pairs.

**Proposition 4.9.** ([BuHo61, DRSS]) The construction in Definition 4.8 yields a map

$$\Psi : \text{BiFun}(E_{\text{max}}) \longrightarrow Wex(E_{\text{max}})$$

$$F \mapsto \mathcal{W}_F.$$ 

Moreover, the functions $\Phi$ and $\Psi$ induce mutually inverse poset isomorphisms

$$Wex(E_{\text{max}}) \bigcup \bigcup \text{BiFun}(E_{\text{max}})$$

$$\text{Ex}(A) \bigcup \bigcup \text{CBiFun}(A)$$

where $\text{CBiFun}(A)$ denotes the subclass of closed sub-bifunctors of $\text{Ext}_A^1$.

**Proof.** These results are mostly covered in [DRSS], some going back to [BuHo61]: Let $F : A^{\text{op}} \times A \longrightarrow \text{Ab}$ be an additive sub-bifunctor of $\text{Ext}_A^1(-, -)$. As discussed in Section 1.2 of [DRSS], the collection of $F$-exact pairs $\mathcal{W}_F$ is closed under isomorphisms and satisfies condition (E2) and $(E2)^{\text{op}}$ from Definition 3.1. Also conditions (E0) and $(E0)^{\text{op}}$ hold since the identity $1_A$ splits and thus represents the zero object which is clearly an $F$-exact pair. Moreover, Lemma 1.2 in [DRSS] ensures that $\mathcal{W}_F$ is closed under direct sums of $F$-exact pairs since $F$ is an additive functor. We have thus verified that $\mathcal{W}_F$ is a weakly exact structure, hence the map $\Psi$ is well-defined. It is also clear that $\Psi$ is a morphism of posets.

Proposition 1.4 in [DRSS] shows that the sub-bifunctor $F$ is closed precisely when the class $\mathcal{W}_F$ forms an exact structure, so the restriction of the map $\Psi$ to $\text{CBiFun}(A)$ maps exactly to $\text{Ex}(A)$.

Finally, it is easy to verify that the maps $\Phi$ and $\Psi$ are mutually inverse. \qed
Remark 4.10. If $\mathcal{A}$ is essentially small then $\text{BiFun}(\mathcal{E}_{\text{max}})$ forms a set: the choice of a sub-bifunctor of $\mathcal{E}_{\text{max}}$ is determined by the choice of a subgroup of $\mathcal{E}_{\text{max}}(C, A)$ for all objects $C, A$ in $\mathcal{A}$, which form a set up to isomorphism.

Since $\text{Wex}(\mathcal{E}_{\text{max}})$ is in bijection with $\text{BiFun}(\mathcal{E}_{\text{max}})$, we conclude that the class $\text{Wex}(\mathcal{E}_{\text{max}})$ of weakly exact substructures of $\mathcal{E}_{\text{max}}$ also forms a set in this case, and the same argument applies to the subclasses $\text{Ex}(\mathcal{A})$ of $\text{Wex}(\mathcal{E}_{\text{max}})$ and $\text{CBiFun}(\mathcal{A})$ of $\text{BiFun}(\mathcal{E}_{\text{max}})$.

4.3. Example. We reconsider here Example 3.2 in light of the bijection from the last proposition: Let $\mathcal{A} = \text{rep} \, Q$ be the category of representations of the quiver

$$Q : \quad 1 \longrightarrow 2 \longrightarrow 3$$

The Auslander-Reiten quiver of $\mathcal{A}$ is as follows:

$$\begin{array}{c}
P_1 \\
\downarrow \\
P_2 \quad \cdots \quad I_2 \\
\downarrow \\
P_3 \quad \cdots \quad S_2 \quad \cdots \quad S_1
\end{array}$$

There are (up to equivalence) exactly five non-split exact sequences with indecomposable end terms, where the first three are the Auslander-Reiten sequences:

- $(\alpha)$ \quad $0 \longrightarrow P_3 \overset{a}{\longrightarrow} P_2 \overset{c}{\longrightarrow} S_2 \longrightarrow 0$
- $(\beta)$ \quad $0 \longrightarrow S_2 \overset{e}{\longrightarrow} I_2 \overset{f}{\longrightarrow} S_1 \longrightarrow 0$
- $(\gamma)$ \quad $0 \longrightarrow P_2 \longrightarrow P_1 \oplus S_2 \longrightarrow I_2 \longrightarrow 0$
- $(\delta)$ \quad $0 \longrightarrow P_3 \longrightarrow P_1 \overset{d}{\longrightarrow} I_2 \longrightarrow 0$
- $(\epsilon)$ \quad $0 \longrightarrow P_2 \overset{b}{\longrightarrow} P_1 \longrightarrow S_1 \longrightarrow 0$

Up to isomorphism, an additive functor is uniquely determined by its values on indecomposable objects. To study additive sub-bifunctors of $\text{Ext}^1_{\mathcal{A}}$ it is therefore sufficient to examine the bimodule structure on the vector space generated by these five non-split exact sequences with indecomposable end-terms. It is depicted in the following diagram, which indicates the multiplication rules $\delta e = \alpha, a \delta = \gamma, e f = \gamma, c = \beta$ (see Definition 3.1) :

$$\begin{array}{cccccc}
c & e & f & a & \delta & c \\
\beta & \gamma & \alpha
\end{array}$$

From there it is easy to see that $\text{Ext}^1_{\mathcal{A}}$ admits 13 submodules (including the zero submodule and itself), and the submodule lattice is given in Figure 4.3, indicating each submodule by a set of generators. The eight closed submodules, corresponding to the eight exact structures on $\mathcal{A}$, are indicated in blue.
4.4. Weakly exact structures as bimodules. In Section 4.1, we explained how weakly exact structures give rise to bifunctors. In this subsection, we use these bifunctors to obtain bimodules over the Auslander algebra.

**Definition 4.11.** Let $\mathcal{A}$ be an additively finite, Hom-finite Krull-Schmidt category with indecomposables $X_1, \ldots, X_n$ and denote by $A = \text{End}(X)$ with $X = X_1 \oplus \cdots \oplus X_n$ its Auslander algebra. The Krull-Schmidt property implies that the additive category $A$ is weakly idempotent complete, thus as discussed in Section 3.5, we know that the maximum weakly exact structure coincides with the maximum weakly exact structure formed by the stable short exact sequences. The corresponding bifunctor $E_{\text{max}}$, evaluated at the object $X$ yields a bimodule 

$$B = E_{\text{max}}(X, X)$$

over the Auslander algebra $A$: We know that $B = E_{\text{max}}(X, X)$ is an abelian group, and elements of $A$ are morphisms $a : X \to X$, whose action on $B$ is described by the action of the bifunctor $E_{\text{max}}$.

More generally, let $W$ be a weakly exact structure on $\mathcal{A}$, and consider its associated bifunctor $W = \text{Ext}^1_W(-, -)$. We showed in Proposition 4.3 that the abelian group $B_W = W(X, X)$ forms a bimodule over the Auslander algebra $B$, and by Proposition 4.9, we obtain that $B_W$ is an $A - A$-subbimodule of $B$.

We denote by $\text{Bim}(B)$ the class of all sub-bimodules of $A_BA$; it forms a poset $(\text{Bim}(B), \subseteq)$ with inclusion as order relation.

**Example 4.12.** In the example studied in Section 4.3, the Auslander algebra $A$ is the algebra whose quiver is the Auslander-Reiten quiver with mesh relations, and the $A - A$-bimodule $B = E_{\text{max}}(X, X)$ is the Ext-bimodule on $A$, a five-dimensional bimodule with basis given by the elements $\alpha, \beta, \gamma, \delta, \epsilon$. The Figure 4.3 describes the bimodule lattice $(\text{Bim}(B), \subseteq)$ in this example.
5. Weakly extriangulated exact structures

In Section 4.2, we showed that additive sub-bifunctors of $E_{\text{max}}$ give rise to weakly exact sub-structures of the maximal exact structure $E_{\text{max}}$ on $\mathcal{A}$. Extriangulated structures [NP19] (or, equivalently, 1−exangulated structures [HLN]) are defined in terms of additive bifunctors $\mathcal{A}^{op} \times \mathcal{A} \to \text{Ab}$ equipped with an extra data and satisfying certain axioms. They generalize both exact and triangulated categories. In this Section, we define their weak version by removing one of the axioms and show that this covers weakly exact structures.

We first present the definition of 1−exangulated categories following [HLN].

**Definition 5.1.** Let $E: \mathcal{A}^{op} \times \mathcal{A} \to \text{Ab}$ be an additive bifunctor. Given a pair of objects $A,C \in \mathcal{A}$, we call an element $\delta \in E(C,A)$ an $E$−extension. When we want to emphasize $A$ and $C$, we also write $A_\delta C$.

Since $E$ is a bifunctor, each morphism $a \in \text{Hom}(A,A')$ induces the extension $a_*(\delta) := E(C,a)(\delta) \in E(C,A')$. Similarly, each morphism $c \in \text{Hom}(C',C)$ induces the extension $c^*(\delta) := E(c,A)(\delta) \in E(C',A)$.

Moreover, we have $E(c,a)(\delta) = c^*a_*(\delta) = a_*c^*(\delta)$.

By the Yoneda lemma, each extension $A_\delta C$ induces a pair of natural transformations $\delta^1: \text{Hom}(-,C) \to E(-,A)$ and $\delta^2: \text{Hom}(A,-) \to E(C,-)$.

Namely, for each $X \in \mathcal{A}$, we have

\[(\delta^1)_X: \text{Hom}(X,C) \to E(X,A), \quad c \mapsto c^*(\delta);\]
\[(\delta^2)_X: \text{Hom}(A,X) \to E(C,X), \quad a \mapsto a_*(\delta).\]

**Definition 5.2.** A morphism of extensions $A_\delta C \to B_\rho D$ is a pair of morphisms $(a,c) \in \text{Hom}(A,B) \times \text{Hom}(C,D)$ such that $a_*(\delta) = c^*(\rho)$.

**Definition 5.3.** A weak cokernel of a morphism $f: A \to B$ in $\mathcal{A}$ is a morphism $g: B \to C$ such that for all $X \in \mathcal{A}$, the induced sequence of abelian groups

$$\text{Hom}(C,X) \to \text{Hom}(B,X) \to \text{Hom}(A,X)$$

is exact, i.e. the sequence of functors

$$\text{Hom}(C,-) \to \text{Hom}(B,-) \to \text{Hom}(A,-)$$

is exact. Equivalently, $g$ is a weak cokernel of $f$ if $g \circ f = 0$ and for each morphism $h: B \to X$ such that $h \circ f = 0$, there exists a (not necessarily unique) morphism $l: C \to X$ such that $h = l \circ g$. Weak kernel is a weak cokernel in $\mathcal{A}^{op}$.

Note that weak (co)kernels satisfy the same factorization properties as usual (co)kernels, but without requiring uniqueness. Clearly, a weak (co)kernel $g$ of $f$ is a (co)kernel of $f$ if and only if $g$ is a monomorphism (resp. an epimorphism).
Definition 5.4. We call a pair of composable morphisms
\[ A \xrightarrow{f} B \xrightarrow{g} C \]
a weak kernel-cokernel pair if \( f \) is a weak kernel of \( g \) and \( g \) is a weak cokernel of \( f \).

By definition, in each weak kernel-cokernel pair as above the composition \( g \circ f \) is 0, so the pair can be understood as an element of the category \( C^{[0,2]}(A) \hookrightarrow C(A) \) of complexes over \( A \) concentrated in the degrees 0, 1 and 2.

Let \( C_w(A) \) be the full subcategory of \( C^{[0,2]}(A) \) with objects being weak kernel-cokernel pairs.

Consider morphisms of complexes in \( C_w(A) \)
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
1_A & & 1_B & & 1_C \\
A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C
\end{array}
\]
with leftmost and rightmost vertical morphisms being identities.

Lemma 5.5. For a diagram of the form (1), the following are equivalent:

- The morphism \( f^* = (1_A, b, 1_C) \) is an isomorphism in \( C_w(A) \);
- The morphism \( b \) is an isomorphism;
- The morphism \( f^* \) is a homotopy equivalence in \( C^{[0,2]}(A) \).

Here by homotopy equivalence in \( C^{[0,2]}(A) \) we mean that there exists a morphism \( g^* \) in \( C^{[0,2]}(A) \) and morphisms
\[
\phi_1 : B \to A, \quad \phi_2 : C \to B, \quad \psi_1 : B' \to A, \quad \psi_2 : C \to B'
\]
such that the pair \((\phi_1, \phi_2)\) yields a chain homotopy \( g^* \circ f^* \sim 1 \) and the pair \((\phi_1, \phi_2)\) yields a chain homotopy \( f^* \circ g^* \sim 1 \).

Proof. This is a reformulation of [HLN, Lemma 4.1], see also [HLN, Claim 2.8]. \(\square\)

Morphisms \( f^* = (1_A, b, 1_C) \) satisfying either of conditions in Lemma 5.5 define an equivalence relation on objects in \( C_w(A) \). We denote by \([A \xrightarrow{f} B \xrightarrow{g} C]\) the equivalence class of the complex \( A \xrightarrow{f} B \xrightarrow{g} C \) in \( C_w(A) \) under this equivalence.

Definition 5.6. (cf. [HLN, Definition 2.22]) Let \( \mathfrak{s} \) be a correspondence which associates an equivalence class
\[
\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]
\]
in \( C(A) \) to each extension \( \delta = A \delta C \). Such \( \mathfrak{s} \) is called a realization of \( \mathcal{E} \) if it satisfies the following condition for any \( \mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C] \) and any \( \mathfrak{s}(\rho) = [A' \xrightarrow{f'} B' \xrightarrow{g'} C'] \):

For any morphism of extensions \((a, c) : \delta \to \rho\), there exists a morphism \(b : B \to B'\) such that \(f^\bullet = (a, b, c)\) is a morphism in \(C^{[0, 2]}(A)\):

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{1_A} & \circ & \downarrow{b} & \circ & \downarrow{1_C} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C
\end{array}
\]

Such \(f^\bullet\) is called a lift of \((a, c)\).

We say that \([A \xrightarrow{f} B \xrightarrow{g} C]\) realizes \(\delta\) whenever we have \(s(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]\).

Each weak kernel-cokernel pair \(A \xrightarrow{f} B \xrightarrow{g} C\) realizing an extension \(\delta\) induces a pair of sequences of functors

(2) \(\text{Hom}(C, -) \to \text{Hom}(B, -) \to \text{Hom}(A, -) \to \mathbb{E}(C, -)\);  
(3) \(\text{Hom}(-, A) \to \text{Hom}(-, B) \to \text{Hom}(-, C) \to \mathbb{E}(-, A)\).

**Definition 5.7.** (cf. [HLN, Definition 2.22])

A realization \(s\) is called exact if the following two conditions are satisfied:

- (R1) For each extension \(\delta\), for each \(A \xrightarrow{f} B \xrightarrow{g} C\) realizing \(\delta\), both sequences (2) are exact (i.e. exact when applied to each object in \(A\));
- (R2) For each object \(A \in A\), we have \(s(A0) = [A \xrightarrow{1_A} A \to 0], \ s(0_A) = [0 \to A \xrightarrow{1_A} A]\).

**Remark 5.8.** Note that since we require realizations to be given by weak kernel-cokernel pairs, sequences (2) are automatically exact at \(\text{Hom}(B, -)\), resp. at \(\text{Hom}(-, B)\). In other words, condition (R1) concerns only exactness at \(\text{Hom}(A, -)\), resp. at \(\text{Hom}(-, A)\).

**Remark 5.9.** By [HLN, Proposition 2.16], condition (R1) does not depend on the choice of a representative in the equivalence class \(s(\delta)\).

**Definition 5.10.** ([HLN, Definition 2.23], [NP19, Definition 2.15, Definition 2.19]) Let \(s\) be an exact realization of \(\mathbb{E}\). Pairs \(\delta, s(\delta)\) are called (distinguished) \(\mathbb{E}\)-triangles. If a complex

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

is a representative in \(s(\delta)\) for some \(\delta\), it is called a conflation. In this case, the morphism \(f\) is called an inflation and the morphism \(g\) is called a deflation.

**Lemma 5.11.** ([HLN, Proposition 3.2]) The class of conflations and the class of \(\mathbb{E}\)-triangles are both closed under direct sums and direct summands.
Since we consider weak kernel-cokernel pairs as complexes, we can consider mapping cones and cocones of morphisms between them. We use the minor modification of the usual definition that was considered in [HLN] and applies only for certain morphisms.

**Definition 5.12.** [HLN Definition 2.27] Let $f^* = (1_A, b, c)$ be a morphism in $C^{[0,1]}(A)$:

$$
\begin{array}{c}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\xrightarrow{1_A} & b & \downarrow & c & \\
A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'.
\end{array}
$$

Its mapping cone $M_f^*$ is defined to be the complex

$$
\begin{bmatrix}
B \\
C \oplus B' \xrightarrow{c g'}
\end{bmatrix} \rightarrow C'.
$$

In other words, this is the usual mapping cone of the morphism of complexes

$$
\begin{array}{c}
B & \xrightarrow{g} & C' \\
\xrightarrow{b} & c & \\
B' & \xrightarrow{g'} & C'.
\end{array}
$$

The mapping cocones of morphisms of the form $(a, b, 1_C)$ are defined dually.

**Definition 5.13.** ([HLN Definition 2.32 for $n = 1$]) A 1-exangulated category is a triplet $(A, E, s)$ of an additive category $A$, additive bifunctor $E : A^{op} \times A \rightarrow Ab$, and its exact realization $s$, satisfying the following conditions.

(EA1) The composition of two inflations is an inflation. Dually, the composition of two deflations is a deflation.

(EA2) For each $\rho \in E(C', A)$ and $c \in \text{Hom}(C, C')$, for each pair of realizations $A \xrightarrow{f} B \xrightarrow{g} C$ of $c^* \rho$ and $A \xrightarrow{f'} B' \xrightarrow{g'} C'$ of $\rho$, the morphism $(1_A, c) : c^* \rho \rightarrow \rho$ admits a good lift $f^* = (1_A, b, c)$, in the sense that $M_f^*$ realizes $f_* \rho$.

(EA2)$^{op}$ Dual of (EA2).

**Proposition 5.14.** ([HLN Proposition 4.3]) A triplet $(A, E, s)$ is a 1-exangulated category if and only if it is an extriangulated category as defined by Nakaoka and Palu [NP19].

This result motivates the following definition.
Definition 5.15. A weakly extriangulated (= weakly $1$-exangulated) structure on an additive category $\mathcal{A}$ is a pair $(\mathcal{W}, s)$ of an additive bifunctor $\mathcal{W} : \mathcal{A} \times \mathcal{A} \to \text{Ab}$ and its exact realization $s$) satisfying axioms (EA2) and (EA2)$^\text{op}$.

Lemma 5.16. A weakly exact structure $\mathcal{W}$ on $\mathcal{A}$ defines a weakly extriangulated structure $(\mathcal{A}, \mathcal{W}, s)$.

Proof. Using Lemma 3.7, all the arguments from [NP19, Example 2.13], except for those concerning (ET4) and (ET4)$^\text{op}$, apply here word for word. That means that a weakly exact structure defines a pair of a bifunctor and its exact realization. Axioms (EA2) and (EA2)$^\text{op}$ follow directly from axioms (E2) and (E2) combined with Lemma 3.5 and its dual. □

We can also characterize weakly exact structures among weakly extriangulated ones.

Lemma 5.17. (cf. [NP19, Corollary 3.18]) Let $(\mathcal{A}, \mathcal{W}, s)$ be a weakly extriangulated category, in which each inflation is monomorphic, and each deflation is epimorphic. If we let $\mathcal{W}$ be the class of conflations given by the $\mathcal{W}$-triangles, then $(\mathcal{A}, \mathcal{W})$ is a weakly exact category.

Proof. If an inflation in a conflation is monomorphic, it is not just a weak kernel of the deflation, but the actual kernel. Similarly, if a deflation is epimorphic, it is the cokernel of an inflation. Therefore, if each inflation is monomorphic, and each deflation is epimorphic, all conflations are kernel-cokernel pairs. From the exactness of the realization, it follows that the class of conflations is closed under direct sums and axioms (E0) and (E0)$^\text{op}$ are satisfied. Axioms (EA2) and (EA2)$^\text{op}$ imply the axioms (E2) and (E2)$^\text{op}$ by Lemma 3.5 and its dual. □

6. Defects and topologizing subcategories

In this section, we extend the notion of contravariant defects to the setting of weakly extriangulated categories. These categories were used in [Bu01, En18, En20, FG20] to classify exact structures on an additive category and, more generally, extriangulated substructures of an extriangulated structure. We show that their results extend to our framework. First, let us recall some necessary notions.

Definition 6.1. Let $\mathcal{A}$ be an essentially small additive category. Contravariant additive functors $\mathcal{A}^{\text{op}} \to \text{Ab}$ to the category of abelian groups are called right $\mathcal{A}$-modules. They form an abelian category Mod $\mathcal{A}$. Dually, left $\mathcal{A}$-modules are covariant additive functors to abelian groups, they form an abelian category that can be seen as $\text{Mod} \mathcal{A}^{\text{op}}$.

These categories have enough projectives. Those are precisely the direct summands of direct sums of representable functors $\text{Hom}(-, A) \in \text{Mod} \mathcal{A}$, resp. $\text{Hom}(A, -) \in \text{Mod} \mathcal{A}^{\text{op}}$. 
We will work with certain full subcategories of categories of \( A \)-modules. First, we need to recall several classical definitions:

**Definition 6.2.** An \( A \)-module \( M \) is called *finitely generated* if admits an epi-morphism \( \text{Hom}(-,X) \twoheadrightarrow M \) from a representable functor. It is moreover *finitely presented* if it is a cokernel of a morphism of representable functors. A module is called *coherent* if it is finitely presented and each of its finitely generated submodule is also finitely presented. Note that every finitely generated submodule of a coherent module is automatically coherent.

By definition, we have a chain of embeddings of full additive categories

\[
\text{coh}(A) \hookrightarrow \text{fp}(A) \hookrightarrow \text{fg}(A) \hookrightarrow \text{Mod}A,
\]

where the first three categories are the categories of coherent, finitely presented and finitely generated right \( A \)-modules, respectively.

The category of finitely presented modules \( \text{fp}(A) \) is known to be abelian if and only the category \( A \) has weak kernels. The category of coherent modules behaves better, as the following standard fact shows:

**Proposition 6.3.** ([He97, Proposition 1.5], see also [Fi16, Appendix B]) The category \( \text{coh}(A) \) is abelian and the canonical embedding \( \text{coh}(A) \hookrightarrow \text{Mod}A \) is exact. In particular, \( \text{coh}(A) \) is closed under kernels, cokernels and extensions in \( \text{Mod}(A) \).

Two more important full subcategories of categories of modules over abelian categories has been studied thoroughly since 1950s and 1960s: the category of effaceable functors, studied already by Grothendieck [Gr57], and the category of defects introduced by Auslander [A66, A78, ARS]. These notions have been generalized to the setting of exact categories (see e.g. [Ke90, Fi16, En18]) and, by Ogawa [Og19] and Enomoto [En20], to that of extriangulated categories. Ogawa’s definition uses only part of the axioms of extriangulated categories, and so we can formulate it in our broader context.

Let \( (A, W, s) \) be a weakly extriangulated category.

**Definition 6.4.** We say that a module \( F \in \text{Mod}A \) is *weakly effaceable with respect to \((W, s)\)* if the following condition is satisfied:

For any \( Z \in A \) and any \( z \in F(Z) \), there exists a deflation \( g : Y \rightarrow Z \) such that \( F(g)(z) = 0 \).

**Definition 6.5.** Given a conflation \( X \xrightarrow{f} Y \xrightarrow{g} Z \), we define its *contravariant defect* to be the cokernel of \( \text{Hom}(-, g) : \text{Hom}(-, Y) \rightarrow \text{Hom}(-, Z) \) in \( \text{Mod}A \).

\(^1\)Full subcategories of abelian categories, which are closed under kernels, cokernels and extensions, are sometimes also called *wide* subcategories.
We denote by $\text{Eff} \mathcal{W}$ the category of weakly effaceable functors and by $\text{def} \mathcal{W}$ the full subcategory of right $\mathcal{A}$–modules isomorphic to defects of conflations. If $(\mathcal{A}, \mathcal{W}, \mathcal{s})$ corresponded to a weakly exact structure $\mathcal{W}$ on $\mathcal{A}$, we also write $\text{Eff} \mathcal{W} := \text{Eff} \mathcal{W}$ and $\text{def} \mathcal{W} := \text{Eff} \mathcal{W}$.

For abelian categories endowed with maximal exact structures, the following two statements are standard, see e.g. [Gr57], resp. [ARS].

**Lemma 6.6.** The category $\text{Eff} \mathcal{W}$ is closed under subquotients and finite direct sums in $\text{Mod} \mathcal{A}$.

**Proof.** Let

$$0 \to F \xrightarrow{\mu} G \xrightarrow{\nu} H \to 0$$

be a short exact sequence in $\text{Mod} \mathcal{A}$. Assume that $G$ is weakly effaceable with respect to $(\mathcal{W}, \mathcal{s})$. Let $Z$ be an object of $\mathcal{A}$. Choose an element $z \in F(Z)$ and a deflation $f : P \to Z$ such that

$$0 = G(f) \circ \mu(Z)(z) = \mu(P) \circ F(f)(z).$$

Since $\mu$ is monic, $F(f)(z) = 0$. Thus, $F$ is weakly effaceable with respect to $(\mathcal{W}, \mathcal{s})$. So $\text{Eff} \mathcal{W}$ is closed under subobjects. The rest is proved by similar straightforward diagram chasing.

**Lemma 6.7.** The category $\text{def} \mathcal{W}$ is closed under kernels and cokernels in $\text{Mod} \mathcal{A}$.

**Proof.** The same argument as in [Og19, Lemma 2.6] applies here. A morphism of defects of two conflations gives rise to a morphism $(a, c)$ of these conflations. Then the kernel is given by the defect of the mapping cone of any good lift of the morphism $(1, c)$ and the cokernel is given by the defect of the mapping cocone of any good lift of the morphism $(a, 1)$.

The following notion was introduced by Rosenberg [Ros] in his works on non-commutative algebraic geometry and reconstruction of schemes.

**Definition 6.8.** A full subcategory of an abelian category is called topologizing if it is closed under subquotients and finite direct sums.

**Proposition 6.9.** Let $(\mathcal{A}, \mathcal{W}, \mathcal{s})$ be a weakly extriangulated category. We have

$$\text{def} \mathcal{W} = \text{Eff} \mathcal{W} \cap \text{coh}(\mathcal{A})$$

and this category is topologizing.

**Proof.** The same argument as in the proof of [En20, Proposition 2.9] applies here. The only difference is that in our generality $\text{Eff} \mathcal{W}$ is not closed under extensions in $\text{Mod} \mathcal{A}$, but only under finite direct sums.

For $\mathcal{A}$–modules, we have natural notions of subobjects, quotients and extensions: these are defined object-wise (for objects in $\mathcal{A}$).
**Definition 6.10.** We say that a subcategory of an arbitrary (not necessarily abelian) full subcategory $C$ of $\text{coh}(\mathcal{A})$ is **topologizing** if it is closed under sub-quotients (considered object-wise) and finite direct sums. Equivalently, it is topologizing if it is a full subcategory of $C$ which is topologizing in $\text{coh}(\mathcal{A})$.

Similarly, we say that a subcategory of $C$ is **Serre** if it is topologizing and closed under extensions; equivalently, if it is a full subcategory of $C$ and a Serre subcategory in $\text{coh}(\mathcal{A})$.

Note that this definition ensures that a Serre subcategory of $C$ is abelian.

**Corollary 6.11.** Let $(\mathcal{A}, \mathcal{W}, s)$ be a weakly extriangulated category and let $(\mathcal{A}, \mathcal{W}', s|_{\mathcal{W}})$ be a weakly extriangulated substructure on $\mathcal{A}$ (that is, $\mathcal{W}'$ is an additive sub-bifunctor of $\mathcal{W}$). Then the category $\text{def} \mathcal{W}'$ is a topologizing subcategory of $\text{def} \mathcal{W}$.

**Corollary 6.12.** Let $\mathcal{W}'$ be a weakly exact substructure of a weakly exact structure $\mathcal{W}$. Then the category $\text{def} \mathcal{W}'$ is a topologizing subcategory of $\text{def} \mathcal{W}$.

### 7. Lattice structures

We study in this section lattice structures on the different posets introduced in the previous parts of this article.

**7.1. Definitions.** We recall the following well known notions:

**Definition 7.1.** A poset $P$ is called a **join-semilattice** if for every pair $(p, q)$ of elements of $P$ there exists a supremum $p \lor q$ (also called join). It is called a **meet-semilattice** if for every pair $(p, q)$ of elements of $P$ there exists an infimum $p \land q$ (also called meet). Finally, $P$ is a **lattice** if it is both a join-semilattice and a meet-semilattice. Equivalently, a lattice is a set $P$ equipped with two binary operations $\lor$ and $\land : P \times P \to P$ satisfying the following axioms:

1. $\lor$ is associative and commutative,
2. $\land$ is associative and commutative,
3. $\land$ and $\lor$ satisfy the following property:
   \[
   m \lor (m \land n) = m = m \land (m \lor n) \quad \text{for all } m, n \in P.
   \]

A lattice is called **complete** if all its subsets have both a join and a meet, similar for semilattices. A **bounded lattice** is a lattice that has a greatest element (also called maximum) and a least element (also called minimum).

**Remark 7.2.** As a consequence of the axioms above we have the following property for lattices:

\[
\lor m = m \quad \text{and} \quad \land m = m \quad \text{for all } m \in P.
\]
Definition 7.3. A lattice \((P, \leq, \wedge, \vee)\) is modular if the following property is satisfied for all \(r, s, t \in P\) with \(r \leq s\):
\[
s \wedge (r \vee t) = r \vee (s \wedge t).
\]

Definition 7.4. Let \(P\) and \(Q\) be two lattices, then a function \(f : P \to Q\) is a morphism of lattices if for all \(m, n \in P\) one has:
\[
f(m \vee n) = f(m) \vee f(n) \quad \text{and} \quad f(m \wedge n) = f(m) \wedge f(n).
\]
An isomorphism of lattices is a bijective morphism of lattices (in which case its inverse is also an isomorphism).

Definition 7.5. Let \((P, \leq)\) be a partially ordered set with a unique minimal element \(0\). An atom is an element \(a \in P\) with \(a > 0\) and such that \(0 \leq x \leq a\) implies \(x = 0\) or \(x = a\). In other words, atoms are the elements that are directly above the minimal element.

7.2. Lattices of right and left weakly exact structures. In this subsection we study a lattice structure on the class of all right (or left) weakly exact structures. These results generalise the one obtained in [HR20, Proposition 8.4] on the complete lattice structure of the class of (strong) one-sided exact structures.

Definition 7.6. We denote by \(LW(A)\) (respectively \(RW(A)\)) the class of all left (right) weakly exact structures on \(A\).

Lemma 7.7. Let \(\{L_i\}_{i \in \omega}\) (respectively \(\{R_i\}_{i \in \omega}\)) be a family of left (right) weakly exact structures on \(A\). Then the intersection \(\bigcap_{i \in \omega} L_i\) (\(\bigcap_{i \in \omega} R_i\)) is also a left (right) weakly exact structure.

Proof. Same as Lemma 5.2 of [BHLR]. \(\square\)

Proposition 7.8. Let \(A\) be an additive category. Then \(LW(A)\) and \(RW(A)\) are complete meet-semi lattices.

Proof. Let \(L\) and \(L'\) be two left weakly exact structures on \(A\). The partial order on \(LW(A)\) is given by containment. We define the meet given by \(L \wedge L' = L \cap L'\). These operations define the structure of complete meet-semilattice on \(LW(A)\) by Lemma 7.7. \(\square\)

Remark 7.9. If there exists a unique maximal left weakly exact structure \(L_{\text{max}}\) on \(A\), then \(LW(A)\) is a complete lattice (similarly for \(RW(A)\)). In this case, the join can be defined by the usual construction
\[
L \vee_L L' = \bigcap\{L'' \in LW(A) \mid L \subseteq L'', L' \subseteq L''\}.
\]

The intersection in the definition of the join is well defined since the set includes \(L_{\text{max}}\) by assumption. These operations define a lattice structure on \(LW(A)\). Since the lattice has a minimal element \(L_{\text{min}}\), formed by all retractions, and a maximal element \(L_{\text{max}}\), it is a bounded lattice. Likewise, any interval in the poset \(LW(A)\) forms a complete bounded lattice.
Remark 7.10. The constructions in Section 3.4 can be reformulated in terms of the lattices studied in this section as follows: As stated in Remark 3.15, there is a splicing function

\[ s : \text{Wex}(\mathcal{A}) \longrightarrow \text{LW}(\mathcal{A}) \times \text{RW}(\mathcal{A}), \quad \mathcal{W} \mapsto (\mathcal{L}_\mathcal{W}, \mathcal{R}_\mathcal{W}) \]

where \( \mathcal{L}_\mathcal{W} := \{ d \mid (i, d) \in \mathcal{W} \} \) is the class of all \( \mathcal{W} \)-cokernels or \( \mathcal{W} \)-admissible deflations and \( \mathcal{R}_\mathcal{W} := \{ i \mid (i, d) \in \mathcal{W} \} \) is the class of all \( \mathcal{W} \)-kernels or \( \mathcal{W} \)-admissible inflations.

Moreover, Theorem 3.13 shows that there is a gluing function:

\[ g : \text{LW}(\mathcal{A}) \times \text{RW}(\mathcal{A}) \longrightarrow \text{Wex}(\mathcal{A}), \quad (\mathcal{L}, \mathcal{R}) \mapsto \mathcal{W}(\mathcal{L}, \mathcal{R}) \]

where \( \mathcal{W}(\mathcal{L}, \mathcal{R}) \) is formed by all short exact sequences \((i, d)\) in \( \mathcal{A} \) with \( i \in \mathcal{R}, \ d \in \mathcal{L} \).

The maps \( s \) and \( g \) are not bijective, but it seems interesting to study their properties.

7.3. Lattice of weakly exact structures.

7.3.1. Lattice of exact structures revisited. We know by [BHLR, Theorem 5.3] that the class of exact structures on an additive category \( \text{Ex}(\mathcal{A}) \) forms a lattice. In order to study the properties of this lattice, we show that it is isomorphic to the lattice of closed additive sub-bifunctors of \( \text{Ext}^1_{\mathcal{A}}(-,-) \) defined in Section 4.

Theorem 7.11. [BHLR 5.1, 5.3, 5.4] Let \( \mathcal{A} \) be an additive category. The poset \( \text{Ex}(\mathcal{A}) \) of exact structures \( \mathcal{E} \) on \( \mathcal{A} \) forms a bounded complete lattice

\[ (\text{Ex}(\mathcal{A}), \subseteq, \wedge, \vee_E) \]

under the following operations:

1. The partial order is given by containment \( \mathcal{E}' \subseteq \mathcal{E} \)
2. The meet \( \wedge \) is defined by \( \mathcal{E} \wedge \mathcal{E}' = \mathcal{E} \cap \mathcal{E}' \)
3. The join \( \vee_E \) is defined by \( \mathcal{E} \vee_E \mathcal{E}' = \bigcap \{ \mathcal{F} \in \text{Ex}(\mathcal{A}) \mid \mathcal{E} \subseteq \mathcal{F}, \mathcal{E}' \subseteq \mathcal{F} \} \).

7.3.2. Lattice structure on the class of all weakly exact structures of a given additive category.

Lemma 7.12. Let \( \{ \mathcal{W}_i \}_{i \in \omega} \) be a family of weakly exact structures on \( \mathcal{A} \). Then the intersection \( \cap_{i \in \omega} \mathcal{W}_i \) is also a weakly exact structure.

Proof. Same as Lemma 5.2 of [BHLR]. \( \square \)

Theorem 7.13. Let \( \mathcal{A} \) be an additive category and \( \mathcal{E}_{\text{max}} \) the maximal exact structure on \( \mathcal{A} \). Then the weakly exact structures that are included in \( \mathcal{E}_{\text{max}} \) form a complete bounded lattice:

\[ (\text{Wex}(\mathcal{E}_{\text{max}}), \subseteq, \wedge, \vee_W) \]
Proof. It follows from Lemma 7.12 that \( \text{Wex}(\mathcal{A}) \) forms a meet semi-lattice: \((\text{Wex}(\mathcal{A}), \subseteq, \wedge)\) with order relation given by inclusion and meet operation given by inclusion. Moreover, the weakly exact structures that are included in \( \mathcal{E}_{\text{max}} \) form a complete bounded lattice \((\text{Wex}(\mathcal{E}_{\text{max}}), \subseteq, \wedge, \vee)\) where the join \( \vee \) is defined by
\[
\mathcal{W} \vee \mathcal{W}' = \bigcap \{ \mathcal{V} \in \text{Wex}(\mathcal{A}) \mid \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}' \subseteq \mathcal{V} \}
\]
This join is well-defined for \( \text{Wex}(\mathcal{E}_{\text{max}}) \) since the set includes \( \mathcal{E}_{\text{max}} \) by assumption. Since the lattice \( \text{Wex}(\mathcal{E}_{\text{max}}) \) has a minimal element \( \mathcal{E}_{\text{min}} \) and a maximal element \( \mathcal{E}_{\text{max}} \), it is a bounded lattice. \( \square \)

Remark 7.14. While the partial order and the meet coincide for \( \text{Ex}(\mathcal{A}) \) and \( \text{Wex}(\mathcal{A}) \), the join \( \vee_{\mathcal{F}} \) is different from the join for weakly exact structures since we intersect over a smaller set, making the join larger when both are viewed in the poset \( \text{Wex}(\mathcal{E}_{\text{max}}) \):
\[
\mathcal{E} \vee_{\mathcal{F}} \mathcal{E}' \leq \mathcal{E} \vee_{\mathcal{F}} \mathcal{E}'
\]
for all \( \mathcal{E}, \mathcal{E}' \in \text{Ex}(\mathcal{A}) \). In fact, in the example from Section 4.3 if we consider the exact structures \( \mathcal{E} = \langle \alpha \rangle \) and \( \mathcal{E}' = \langle \gamma \rangle \), then \( \mathcal{E} \vee_{\mathcal{F}} \mathcal{E}' = \langle \alpha, \gamma \rangle \) which is strictly smaller than \( \mathcal{E} \vee_{\mathcal{F}} \mathcal{E}' = \langle \alpha, \gamma, \delta \rangle \). This shows that \( \text{Ex}(\mathcal{A}) \) is a meet-subsemilattice of \( \text{Wex}(\mathcal{E}_{\text{max}}) \), but it is not a sublattice in general.

We now describe the join of two weakly exact structures in a more constructive way, motivated by the sum of bifunctors:

Definition 7.15. Let \( \mathcal{W}_1, \mathcal{W}_2 \in \text{Wex}(\mathcal{E}_{\text{max}}) \) be two weakly exact structures contained in \( \mathcal{E}_{\text{max}} \). Then, \( \mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 \) is defined as \( \mathcal{W} := \bigcup_{A,C \in \mathcal{A}} \mathcal{W}(C, A) \) where
\[
\mathcal{W}(C, A) := \{ \eta_1 + \eta_2 \mid \eta_1 \in \mathcal{W}_1(C, A), \eta_2 \in \mathcal{W}_2(C, A) \}
\]
with \( \mathcal{W}_k(C, A) := \{ \eta : A \xrightarrow{i} B \xrightarrow{a} C \mid \eta \in \mathcal{W}_k \} \) for \( k = 1, 2 \). Here, for \( \eta_1 \in \mathcal{W}_1(C, A) \) and \( \eta_2 \in \mathcal{W}_2(C, A) \), the sum \( \eta_1 + \eta_2 := \nabla_A(\eta_1 \oplus \eta_2)\Delta_C \) is the Baer sum for short exact sequences. Since \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are included in \( \mathcal{E}_{\text{max}} \) and the Baer sum in well defined in \( \mathcal{E}_{\text{max}} \), we have \( \mathcal{W} \subseteq \mathcal{E}_{\text{max}} \).

Proposition 7.16. Let \( \mathcal{W}_1, \mathcal{W}_2 \) be two weakly exact structures contained in \( \mathcal{E}_{\text{max}} \). Then

(a) \( \mathcal{W}_1 + \mathcal{W}_2 \) is weakly exact
(b) \( \mathcal{W}_1 + \mathcal{W}_2 \) is the join \( \mathcal{W}_1 \vee \mathcal{W}_2 \) in the lattice \( \text{Wex}(\mathcal{E}_{\text{max}}) \).

Proof. For part (a), \( \mathcal{W} \) has to satisfy properties (E0), (E0)$^{op}$, (E2), (E2)$^{op}$ and needs to be closed under direct sums.

To show (E0), let \( X \) be any object of \( \mathcal{A} \). Since \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are weakly exact structures, by (E0) for \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) the short exact sequence \( \mathcal{E} : X \xrightarrow{1} X \xrightarrow{0} 0 \) is in \( \mathcal{W}_1 \) and in \( \mathcal{W}_2 \). Since \( \mathcal{E} + \mathcal{E} = \mathcal{E} \) with the first \( \mathcal{E} \) in \( \mathcal{W}_1 \) and the second in \( \mathcal{W}_2 \), we obtain that \( \mathcal{E} \) is in \( \mathcal{W} \) by definition. The proof for (E0)$^{op}$ is dual.
For \((E2)\), suppose we have \(\eta : A \xrightarrow{i} B \xrightarrow{d} C \in \mathcal{W}\) and \(a : A \rightarrow A' \in \mathcal{A}\).

We show that the push-out \(a\eta\) of \(\eta\) by \(a\) exists and is in \(\mathcal{W}\). Since \(\eta \in \mathcal{W}(C,A)\), there exist \(\eta_1 \in \mathcal{W}_1(C,A)\) and \(\eta_2 \in \mathcal{W}_2(C,A)\) such that \(\eta = \eta_1 + \eta_2\). Using Lemma 1.4(iii)) we have \(a\eta_1 = a(\eta_1 + \eta_2) = a\eta_1 + a\eta_2\). Since \(\mathcal{W}_1\) and \(\mathcal{W}_2\) satisfy (E2) we have \(a\eta_1 \in \mathcal{W}_1(C,A')\) and \(a\eta_2 \in \mathcal{W}_2(C,A')\). Therefore, \(a\eta = a\eta_1 + a\eta_2 \in \mathcal{W}(C,A') \subseteq \mathcal{W}\). The proof for \((E2)^{op}\) is dual.

For the direct sums, let \(\alpha\) and \(\beta\) be in \(\mathcal{W}\). We want to show that \(\alpha + \beta \in \mathcal{W}\).

Suppose that \(\alpha : A \xrightarrow{i} B \xrightarrow{d} C \in \mathcal{W}(C,A)\) and \(\beta : D \xrightarrow{j} E \xrightarrow{e} F \in \mathcal{W}(F,D)\). Then there exist \(\alpha_1 \in \mathcal{W}_1(C,A)\), \(\alpha_2 \in \mathcal{W}_2(C,A)\), \(\beta_1 \in \mathcal{W}_1(F,D)\) and \(\beta_2 \in \mathcal{W}_2(F,D)\) such that \(\alpha = \alpha_1 + \alpha_2\) and \(\beta = \beta_1 + \beta_2\), hence
\[
\alpha + \beta = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) = (\nabla_A(\alpha_1 + \alpha_2)\Delta_C) + (\nabla_D(\beta_1 + \beta_2)\Delta_F).
\]

Since \(\mathcal{W}_1\) and \(\mathcal{W}_2\) are closed under direct sums, we get \(\alpha_1 + \beta_1 \in \mathcal{W}_1(C \oplus F, A \oplus D)\) and \(\alpha_2 + \beta_2 \in \mathcal{W}_2(C \oplus F, A \oplus D)\), so \((\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) \in \mathcal{W}(C \oplus F, A \oplus D)\). We have \((\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) = \nabla_{A \oplus D}(\alpha_1 + \beta_1 \oplus (\alpha_2 + \beta_2))\Delta_{C \oplus F}\) and \((\nabla_D(\beta_1 + \beta_2)\Delta_F)\) is the diagram for \((\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\) \(\mathcal{W}(C \oplus F, A \oplus D)\). This means that \(\alpha + \beta = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) \in \mathcal{W}(C \oplus F, A \oplus D) \subseteq \mathcal{W}\). Therefore \(\mathcal{W}\) is closed under direct sums and it is a weakly exact structure.

To show part (b), recall that the join \(\mathcal{W}_1 \lor_W \mathcal{W}_2\) is the smallest (by inclusion) weakly exact structure on \(\mathcal{A}\) containing both \(\mathcal{W}_1\) and \(\mathcal{W}_2\). We have that \(\mathcal{W}_1 \subseteq \mathcal{W}_1 + \mathcal{W}_2\) since \(\eta_1 = \eta_1 + 0 \in \mathcal{W}_1 + \mathcal{W}_2\) for any \(\eta_1 \in \mathcal{W}_1\). Likewise for \(\mathcal{W}_2\), so \(\mathcal{W}_1 + \mathcal{W}_2\) contains both \(\mathcal{W}_1\) and \(\mathcal{W}_2\), hence by definition of the join, \(\mathcal{W}_1 \lor_W \mathcal{W}_2 \subseteq \mathcal{W}_1 + \mathcal{W}_2\).

To show the converse inclusion, let \(\mathcal{W}\) be any weakly exact structure containing both \(\mathcal{W}_1\) and \(\mathcal{W}_2\). Since \(\mathcal{W}\) satisfies the direct sum property (S), we have \(\eta_1 \oplus \eta_2 \in \mathcal{W}\) for all \(\eta_1 \in \mathcal{W}_1, \eta_2 \in \mathcal{W}_2\). By definition of Baer sum and property (E2) and \((E2)^{op}\) for \(\mathcal{W}\) we have \(\eta_1 + \eta_2 \in \mathcal{W}\). This shows \(\mathcal{W}_1 + \mathcal{W}_2 \subseteq \mathcal{W}\) for all \(\mathcal{W}\) containing both \(\mathcal{W}_1\) and \(\mathcal{W}_2\), so this also holds for the smallest one (their intersection) : \(\mathcal{W}_1 + \mathcal{W}_2 \subseteq \mathcal{W}_1 \lor_W \mathcal{W}_2\).

\[\square\]

**Proposition 7.17.** Let \(\alpha\) be an Auslander-Reiten sequence in \(\mathcal{A}\), and denote by \(\mathcal{E}_\alpha = \{X \oplus Y \mid X \in \mathcal{E}_{\min}, Y \in \text{add}(\alpha)\}\) the (weakly) exact structure generated by \(\alpha\). Then \(\mathcal{E}_\alpha\) is an atom of both lattices \((\text{Ex}(\mathcal{A}), \subseteq, \lor, \land, \lor_E)\) and \((\text{Wex}(\mathcal{A}), \subseteq, \land, \lor_W)\).

**Proof.** This property amounts to showing that the Auslander-Reiten sequence lies in the socle of the bifunctor \(\text{Ext}^1_{\mathcal{A}}(-,-)\), that is, multiplication with morphisms does not generate any new non-split sequences. This is a well-known property of almost split sequences. \(\square\)
7.4. Lattice of additive sub-bifunctors of $\text{Ext}^1_{\mathcal{A}}$. In Section 4 we discussed additive sub-bifunctors of $\text{Ext}^1_{\mathcal{A}} := \mathbb{E}_{\max} = \text{Ext}^1_{\mathcal{E}_{\max}}$ and closed additive sub-bifunctors, and we denote these classes respectively by $\text{BiFun}(\mathbb{E}_{\max})$ and $\text{CBiFun}(\mathcal{A})$. In this section, we construct lattice structures of both classes.

**Theorem 7.18.** The additive sub-bifunctors of $\mathbb{E}_{\max}$ form a lattice $(\text{BiFun}(\mathbb{E}_{\max}), \leq, \wedge, \vee)$.

*Proof.* For $F, F' \in \text{BiFun}(\mathbb{E}_{\max})$, we write $F \leq F'$ if $F$ is a sub-bifunctor of $F'$. The meet of $F$ and $F'$ is given by the sub-bifunctor $F \wedge F'$ of $\mathbb{E}_{\max}$ satisfying $(F \wedge F')(C, A) = F(C, A) \cap F'(C, A)$ for all $A, C \in \mathcal{A}$.

The join is given by the sub-bifunctor $F \vee_{bf} F'$ of $\mathbb{E}_{\max}$ satisfying $(F \vee_{bf} F')(C, A) = F(C, A) + F'(C, A)$ for all $A, C \in \mathcal{A}$, where the sum is the sum of abelian subgroups of $\mathbb{E}_{\max}(C, A)$. Since $\text{BiFun}(\mathbb{E}_{\max})$ has a maximal element $\mathbb{E}_{\max}$, one can show similarly to the proof of Proposition 7.16 that the join can also be expressed by

$$F \vee_{bf} F' = \wedge \{ G \in \text{BiFun}(\mathbb{E}_{\max}) \mid F \leq G, F' \leq G \}.$$ 

□

7.4.1. Lattice of closed additive sub-bifunctors. As discussed in Proposition 4.9 for any additive category $\mathcal{A}$ there is a bijection between exact structures and closed additive sub-bifunctors of $\mathbb{E}_{\max}$. We already know that the exact structures form a lattice [BHLLR, Theorem 5.3]. In this section we define a lattice structure on the class $\text{CBiFun}(\mathcal{A})$ of closed additive sub-bifunctors of $\mathbb{E}_{\max}$.

**Lemma 7.19.** [DRSS, corollary 1.5] Consider a family $\{ F_i \}_{i \in I}$ of closed sub-bifunctors of $\mathbb{E}_{\max}$. Then the intersection $\bigcap_{i \in I} F_i$ is a closed sub-bifunctor of $\mathbb{E}_{\max}$-bifunctor, given by $\{ \bigcap_{i \in I} F_i \} (C, A) = \bigcap_{i \in I} F_i(C, A)$ on objects.

**Remark 7.20.** If $F$ and $F'$ are closed bifunctors in $\text{CBiFun}(\mathcal{A})$ then their sum $F + F'$ is the sub-bifunctor of $\mathbb{E}_{\max}$ given by $\{ F + F' \} (C, A) = F(C, A) + F'(C, A)$ on objects. Note that the sum of closed sub-bifunctors is not always closed.

**Theorem 7.21.** For an additive category $\mathcal{A}$, the closed additive sub-bifunctors of $\mathbb{E}_{\max}$ form a complete bounded lattice $(\text{CBiFun}(\mathcal{A}), \leq, \wedge, \vee_{\text{obj}})$.

*Proof.* The lattice structure is given as follows: the meet is defined by

$$F \wedge F' = F \cap F'$$

while the join is defined by

$$F \vee_{\text{obj}} F'' = \bigcap \{ F'' \in \text{CBiFun}(\mathcal{A}) \mid F \leq F'', F' \leq F'' \},$$
which is well defined since the intersection is always a non-empty, containing $E_{\text{max}}$. Lemma 7.19 ensures that $\text{CBiFun}(\mathcal{A})$ forms a closed meet-semilattice, and the definition of join turns it into a closed lattice, which is bounded by $E_{\text{min}}$ below and $E_{\text{max}}$ above. □

Remark 7.22. The closed sub-bifunctors $(\text{CBiFun}(\mathcal{A}), \leq)$ form a subposet of $(\text{BiFun}(E_{\text{max}}), \leq)$. However, $(\text{CBiFun}(\mathcal{A}), \leq, \wedge, \vee_{\text{cbf}})$ is not a sublattice of $(\text{BiFun}(E_{\text{max}}), \leq, \wedge, \vee_{\text{bf}})$ because their joins are different. In fact, for $F, F' \in \text{CBiFun}(\mathcal{A})$, the join $F \vee_{\text{bf}} F' = F + F'$ is not necessarily closed. As discussed in Remark 7.14, the join of $<\alpha>$ with $<\gamma>$ in $\text{BiFun}(E_{\text{max}})$ is $<\alpha, \gamma>$ which is not closed. The join of $<\alpha>$ with $<\gamma>$, in $\text{CBiFun}(\mathcal{A})$ is $<\alpha, \gamma, \delta>$. In general, for $F, F' \in \text{CBiFun}(\mathcal{A})$ we have that $F \vee_{\text{bf}} F' \leq F \vee_{\text{cbf}} F'$.

7.5. Lattice of bimodules over the Auslander algebra. We return now to the study of the bimodule $B$ over the Auslander algebra $A$ defined in Section 4.4. As is the case for any module over a ring, recall that the set $\text{Bim}(B)$ of sub-bimodules of $B$ forms a complete bounded modular lattice $(\text{Bim}(B), \leq, \wedge_{\text{Bim}}, \vee_{\text{Bim}})$, where the meet is given by intersection and the join is given by the sum $N + N'$ of sub-bimodules.

Definition 7.23. An element $N \in \text{Bim}(B)$ is said to be a closed bimodule if there exists a closed sub-bifunctor $F$ of $\text{Ext}^1_{E_{\text{max}}}$ such that $Ev_X(F) = N$ where $Ev_X : \text{CBiFun}(\mathcal{A}) \rightarrow \text{Bim}(B)$

$$F \mapsto F(X, X)$$

is the evaluation at the object $X \in \mathcal{A}$.

Lemma 7.24. The intersection of two closed sub-bimodules of $B$ is again closed.

Proof. Let $N$ and $P$ be two closed sub-bimodules of $B$ such that $\Phi(F) = N$ and $\Phi(G) = P$. We consider the sub-bifunctor $H$ of $\text{Ext}^1_{E_{\text{max}}}$ given by the meet of $F \land G = H$. By Lemma 7.19 $H$ is closed. Since

$$N \cap P = F(X, X) \cap G(X, X) = H(X, X),$$

the intersection is a closed sub-bimodule of $B$. □

Theorem 7.25. The subset $\text{Cbim}(B)$ of closed sub-bimodules of $B$ forms a complete bounded lattice

$$(\text{Cbim}(B), \leq, \cap, \vee_{\text{Cbim}}).$$

Proof. First this class is a poset ordered by inclusion. Second it is a meet-semilattice using the associative, commutative intersection of modules. Third, it is a join-semilattice using the following operation

$$\vee_{\text{Cbim}} : \text{Cbim}(B) \times \text{Cbim}(B) \rightarrow \text{Cbim}(B)$$
An operation $\wedge$ on the lattice of weakly exact structures is defined by $N \wedge P = \bigcap \{ R \in \text{Cbim}(B) | N \subset R, P \subset R \}$, which is associative commutative and satisfies the following property:

$$P \vee (P \wedge N) = N = N \wedge (N \vee P) \quad \text{for all } N, P \in \text{Cbim}(B).$$

The intersection in this definition of the join is well defined since the set includes $B$ by assumption. These operations define a lattice structure on $\text{Cbim}(B)$. Since the lattice has a minimal element 0 and a maximal element $B$, it is a bounded lattice. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a family of weakly exact structures in $\text{Cbim}(B)$. Their meet is given by $\bigcap_{\lambda \in \Lambda} N_\lambda$ and the join is given by

$$\bigcap \{ N'' \in \text{Cbim}(B) | N_\lambda \subseteq N'', \forall \lambda \in \Lambda \}.$$

Therefore, the lattice is complete.

In the setting of this subsection, the bimodule $B = \text{E}_{\text{max}}(X,X)$ is finite-dimensional, thus $B$ and all of its submodules have a non-zero socle. We know from Proposition 7.17 that the Auslander-Reiten sequences lie in the socle of the bimodule $B$, and since all non-projective objects admit an Auslander-Reiten sequence in $\mathcal{A}$ ending there, one can derive that the socle is precisely formed by all Auslander-Reiten sequences in $\mathcal{A}$. Based on Auslander’s concept of defects, Enomoto shows in [En18] that the lattice $\text{Cbim}(B)$ is an atomic lattice, in fact it is a boolean lattice determined by its atoms, the Auslander-Reiten sequences in $\mathcal{A}$ (see also [FG20, Theorem 2.26]).

Reformulated in module-theoretic terms, that means that the closed sub-bimodules of $B = \text{E}_{\text{max}}(X,X)$ are uniquely determined by their socle, and for every choice of elements in the socle, there is a unique closed sub-bimodule of $B$ having precisely these elements as its socle. If the socle is formed by a set $S$ of Auslander-Reiten sequences, we can thus denote by $\mathcal{E}(S)$ the subbimodule of $B$ determined by $S$. For all elements $\sigma \in S$, denote by $\mathcal{E}_\sigma$ the bimodule corresponding to the exact structure $\mathcal{E}_\sigma$ introduced in Proposition 7.17. Since the lattice $\text{Cbim}(B)$ is atomic, we conclude that

$$\mathcal{E}(S) = \bigvee_{\sigma \in S} \mathcal{E}_\sigma.$$ 

There may be several submodules of $B$ with the same socle $S$, but only one of them is closed. As explained in the proof of [FG20, Theorem 2.26], this closed submodule with socle $S$ corresponds to a Serre subcategory $\mathcal{S}$ generated by the simple objects contained in the set $S$. All other submodules of $B$ with socle $S$ correspond to certain subcategories of $\mathcal{S}$, but only the closed one is given by the abelian length category formed by all extensions of its simple objects. In other words, $\mathcal{E}(S)$ is maximal, so we derive the following result:

**Proposition 7.26.** For every set $S$ of Auslander-Reiten sequences, the closed bimodule $\mathcal{E}(S)$ of $B$ introduced above is the maximal submodule of $B$ whose socle is $S$. 
This fact is illustrated nicely in the example in Section 4.3. It is also shown independently for Nakayama algebras in [BHT, Theorem 6.9].

7.6. **Lattice of topologizing subcategories.** Topologizing subcategories of an abelian category $\mathcal{C}$ form a complete lattice. The order is given by the canonical inclusion of categories and the meet is given by the usual intersection. This is a complete semi-lattice and, therefore, it has a canonical join operation upgrading it to a complete lattice. It is straightforward to check from the definitions that the join is given by the closure of the union by finite direct sums:

$$\bigvee: \text{Top}(\mathcal{C}) \times \text{Top}(\mathcal{C}) \rightarrow \text{Top}(\mathcal{C})$$

$$(T, T') \mapsto \oplus\{T \cup T'\}.$$  

Since this lattice has a canonical minimal element, it is moreover bounded.

By definition, each Serre subcategory of an abelian category is topologizing. Thus, Serre subcategories form a subposet of the lattice of topologizing subcategories. By similar arguments this subposet admits a lattice structure, with the join given by the closure of the union by finite extensions. Since the closure of the union by finite direct sums is, in general, not extension-closed, the join of Serre subcategories in the lattice of topologizing subcategories is different from their join in the lattice of Serre subcategories. In other words, the lattice of Serre subcategories is a subposet, but not a sublattice of the lattice of all topologizing subcategories.

Given a topologizing subcategory $\mathcal{C}$ of the category $\text{coh}(\mathcal{A})$, its topologizing subcategories in the sense of definition 6.10 form a lattice, which is an interval in the lattice of all topologizing subcategories in $\text{coh}(\mathcal{A})$. Serre subcategories of $\mathcal{C}$ form a lattice, which is an interval in the lattice of all Serre subcategories in $\text{coh}(\mathcal{A})$. It is a subposet, but not a sublattice of the lattice of topologizing subcategories of $\mathcal{C}$.

We formulate this observation explicitly in the case of the categories of defects of weakly extriangulated structures:

**Proposition 7.27.** Let $\mathcal{A}$ be an essentially small category and $(\mathcal{W}, s)$ a weakly extriangulated structure on it, then the topologizing subcategories of $\text{def } \mathcal{W}$ form a bounded complete lattice

$$(\text{Top}(\mathcal{W}), \subseteq, \bigcap, \bigvee).$$

Serre subcategories of $\text{def } \mathcal{W}$ also form a lattice, which is a subposet, but not a sublattice of $(\text{Top}(\mathcal{W}))$.

7.7. **Lattices of extriangulated and weakly extriangulated substructures.** Let $\mathcal{A}$ be an essentially small additive category. We consider the class of all weakly extriangulated structures on $\mathcal{A}$.

**Lemma 7.28.** Let $\{\mathcal{W}_i\}_{i \in \omega}$ be a family of weakly extriangulated structures on $\mathcal{A}$. Then the intersection $\cap_{i \in \omega} \mathcal{W}_i$ is also a weakly extriangulated structure.
ON THE LATTICE OF WEAKLY EXACT STRUCTURES 38

Proof. Similar to Lemma 5.2 of [BHLR]. □

Theorem 7.29. Let \((\mathcal{A}, W, s)\) be a weakly extriangulated category. Then all its weakly extriangulated substructures form a bounded complete lattice:

\[ (WET(\mathcal{A}), \leq, \bigwedge, \bigvee) \]

Proof. We consider the set \(WET(\mathcal{A})\) of all the additive sub-bifunctors of \(W\) on the essentially small category \(\mathcal{A}\). They are ordered by

\[ W \leq W' \iff W(C, A) \subseteq_{\text{Ab}} W'(C, A) \text{ for all } A, C \in \mathcal{A} \]

that is, \(W(C, A)\) is a subgroup of \(W'(C, A)\) for every pair of objects in \(\mathcal{A}\). It follows from 7.28 that \((WET(\mathcal{A}), \leq, \bigwedge)\) is a meet semi-lattice with the meet 

\[ (W \wedge W')(C, A) = W(C, A) \cap W'(C, A), \forall A, C \in \mathcal{A}, \text{ by using the intersection of abelian groups.} \]

It also forms a join semi-lattice where the join is defined by

\[ W \lor W' = \bigwedge \{ V \in WET(\mathcal{A}) \mid W \subseteq V, W' \subseteq V \} \]

This join is well-defined for \(WET(\mathcal{A})\) since the set includes \(W\) by assumption, and so \(WET(\mathcal{A})\) is a complete meet semi-lattice: \(W\) is its unique maximal element. These operations satisfy the axioms of 7.1 and form then a structure of a complete lattice. Moreover the lattice structure defined above on \(WET(\mathcal{A})\) has a minimal element given by the split weakly extriangulated structure \(W_{\text{min}}\), so it is a bounded lattice. □

Corollary 7.30. Let \((\mathcal{A}, E, s)\) be an extriangulated category. Then all the additive sub-bifunctors of \(E\) form a bounded complete lattice.

7.8. Isomorphisms of lattices.

7.8.1. The three large isomorphic lattices.

Theorem 7.31. Let \(\mathcal{A}\) be an additive category. The map \(\Phi : W \mapsto \text{Ext}^1_W(-, -)\) induces a lattice isomorphism

\[ (\text{Wex}(E_{\text{max}}), \subseteq, \cap, \lor_W) \cong (\text{BiFun}(E_{\text{max}}), \leq, \land, \lor_{\text{bf}}). \]

Proof. We have already shown in Proposition 4.9 that \(\Phi\) is an isomorphism of posets. We need to verify that it preserves the meet and the join. Let \(W\) and \(W'\) be two weakly exact structures, then \(W \wedge W'\) is also an exact structure. Let \(A\) and \(C\) be two objects in \(\mathcal{A}\).

\[
\begin{align*}
\text{Ext}^1_{W \wedge W'}(C, A) &= \{ (i, d) \mid A \xrightarrow{i} B \xrightarrow{d} C \in W \wedge W' \} \\
&= \{ (i, d) \mid (i, d) \in W \} \cap \{ (i, d) \mid (i, d) \in W' \} \\
&= \text{Ext}^1_W(C, A) \cap \text{Ext}^1_W(C, A)
\end{align*}
\]
Therefore the two sub-bifunctors $\text{Ext}^1_{\mathcal{W} \Delta \mathcal{W}'}(-, -)$ and $\text{Ext}^1_{\mathcal{W}}(-, -) \wedge \text{Ext}^1_{\mathcal{W}'}(-, -)$ coincide, which shows that $\Phi$ is a morphism of meet-semilattices. Moreover, the join is defined in both lattices in the same way using intersections (meet), hence $\Phi$ is a morphism of lattices.

**Theorem 7.32.** Consider the setting of an additively finite category $\mathcal{A}$ as in Section 4.4 and the bimodule $B$ over the Auslander algebra $A$ defined there. Then the evaluation map yields an isomorphism of lattices

$$\text{Ev}_X : \text{BiFun}(\mathbb{E}_{\max}) \rightarrow \text{Bim}(B)$$

$$F \mapsto F(X, X)$$

**Proof.** (1) We first show that the map is well defined:

As $F$ is an additive sub-bifunctor of $\mathbb{E}_{\max}$, we get that $F(X, X)$ is a sub-bimodule of the $A - A$-bimodule $B = \mathbb{E}_{\max}(X)$, which shows $F(X, X) \in \text{Bim}(B)$.

(2) Injectivity: Consider two bifunctors $F, G \in \text{BiFun}(\mathbb{E}_{\max})$ such that their images under $\text{Ev}_X$ are equal as $A - A$-bimodules: $F(X, X) = G(X, X)$. Decomposing $X$ into indecomposables $X \cong X_1 \oplus \cdots \oplus X_n$, we consider the idempotent elements $e_i$ in $A$ given by projection $pr_i : X \rightarrow X_i$ onto $X_i$ and followed by inclusion $in_i : X_i \rightarrow X$ of $X_i$. Being equal as bimodules implies that also their images $F(X_i, X_i)$ under the maps $F(e_i, e_j)$ are equal, thus $F(X_i, X_j) = G(X_i, X_j)$ for all $i, j = 1, \ldots, n$. Since the functors $F, G$ are additive, and every element in $\mathcal{A}$ decomposes uniquely as a direct sum of the $X_i$’s, this shows that $F = G$ as subfunctors of $\mathbb{E}_{\max}$.

(3) Surjectivity: Let $N \in \text{Bim}(B)$ be a sub-bimodule of $B$. As explained in the injectivity part, this yields subgroups $e_j N e_i$ of $\mathbb{E}(X_j, X_i)$ for all $i, j = 1, \ldots, n$. Setting $F(X_j, X_i) := e_j N e_i$ allows then to define an additive sub-bifunctor $F \in \text{BiFun}(\mathbb{E}_{\max})$ with $\text{Ev}_X(F) = N$.

(4) Morphism of posets: If $F$ is a sub-bifunctor of $F'$ then $F(X, X)$ is a sub-bimodule of $F'(X, X)$.

(5) Morphism of lattices: The meet of $F, F'$ in $\text{BiFun}(\mathbb{E}_{\max})$ is given by intersection $(F \wedge_{bf} F')(C, A) = F(C, A) \cap_{bf} F'(C, A)$ for any two objects $A, C$ of $\mathcal{A}$. Applying to $A = C = X$ yields the meet $(F \wedge_{bf} F')(X, X)$ in the lattice $\text{Bim}(B)$.

We conclude that $\text{Ev}_X$ induces an isomorphism of lattices. □

**Corollary 7.33.** If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the three lattice structures we defined on $\text{Wex}(\mathcal{A}), \text{BiFun}(\mathcal{A})$ and $\text{Bim}(B)$ are isomorphic.

**Proof.** Combine 7.31 and 7.32 □
7.8.2. The three small isomorphic lattices.

**Theorem 7.34.** Let $\mathcal{A}$ be an additive category. The map $\Phi : \mathcal{E} \mapsto \text{Ext}^1_{\mathcal{A}}(-,-)$ induces a lattice isomorphism between $(\mathcal{E}(\mathcal{A}), \subseteq, \cap, \lor)$ and $\text{CBiFun}(\mathcal{A}), \leq, \land, \lor)$.

**Proof.** Same as for Theorem 7.31. $\blacksquare$

**Theorem 7.35.** If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the two lattices $(\text{CBiFun}(\mathcal{A}), \leq, \land, \lor)$ and $(\text{Cbim}(\mathcal{B}), \subseteq, \cap, \lor)$ are isomorphic.

**Proof.** As already verified in Theorem 7.32, the evaluation map $\text{Ev}_X$ preserves the order and the meet-semi-lattice structure. But the join for closed sub-bimodules is given by intersections on both sides, therefore $\text{Ev}_X$ also preserves the join-semi-lattice structure. $\blacksquare$

**Corollary 7.36.** If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the three lattice structures defined above on $\mathcal{E}(\mathcal{A}), \text{CBiFun}(\mathcal{A})$ and $\text{Cbim}(\mathcal{B})$ are isomorphic.

**Proof.** By 7.34 and 7.35. $\blacksquare$

7.8.3. General isomorphism of lattices.

**Proposition 7.37.** Let $(\mathcal{A}, \mathcal{W}, s)$ be a weakly extriangulated category. Then there is a lattice isomorphism between the lattice of additive sub-bifunctors of $\mathcal{W}$ and the lattice of topologizing subcategories of $\text{def} \mathcal{W}$.

**Proof.** The proof of [En20, Theorem B], with Step 3 removed, applies word for word. $\blacksquare$

**Corollary 7.38.** Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$. Then there is a lattice isomorphism between the interval $[\mathcal{W}^{\text{add}}, \mathcal{W}]$ in the lattice of weakly exact structures on $\mathcal{A}$ and the lattice of topologizing subcategories of $\text{def} \mathcal{W}$.

**Corollary 7.39.** When the category $\mathcal{A}$ admits a unique maximal weakly exact structure $\mathcal{W}^{\text{max}}$, the lattice of weakly exact structures on $\mathcal{A}$ is isomorphic to the lattice of topologizing subcategories of $\text{def} \mathcal{W}^{\text{max}}$.

In particular we get the following summarising result:

**Corollary 7.40.** Let $\mathcal{A}$ be an idempotent complete essentially small additive category, then the following four lattices are isomorphic:

$\mathcal{Wex}(\mathcal{A}) \cong \text{BiFun}(\mathcal{A}) \cong \text{Bim}(\mathcal{B}) \cong \text{def} \mathcal{E}^{\text{max}}$.

**Proof.** It follows from 3.18, 7.33 and 7.39. $\blacksquare$

Note that when $\mathcal{A}$ is idempotent complete, we can use arguments from [En18, En19, FG20] instead. In particular, this approach would give another proof of the existence of $\mathcal{W}^{\text{max}}$ in this generality.
References

[A66] M. Auslander, Coherent functors, 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) 189–231 Springer, New York.

[A78] M. Auslander, Functors and morphisms determined by objects, in Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), 1–244. Lecture Notes in Pure Appl. Math., 37, Dekker, New York, 1978.

[ARS] M. Auslander, I. Reiten, S. O. Smalø, Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1997.

[AS] M. Auslander, O. Solberg, Relative homology and representation theory I, Relative homology and homologically finite subcategories, Comm. Alg. 21 (9) (1993), 2995–3031.

[BC12] S.Bazzoni, S.Crivei, One-sided exact categories, Journal of Pure and Applied Algebra, 2012.

[BHLR] Th.Brüstle, S.Hassoun, D.Langford, S.Roy, Reduction of exact structures, Journal of Pure and Applied Algebra, no. 06212, 2019.

[BHT] Th.Brüstle, S.Hassoun, A.Tattar Intersections, sums, and the Jordan-Hölder property for exact categories, arXiv:2006.03505

[BrHi] Th.Brüstle, L.Hille, Matrices over upper triangular bimodules and Δ−filtered modules over quasi-hereditary algebras, Colloq. Math. 83 (2000), no. 2, 295–303.

[Bu01] A.Buan, Closed sub-bifunctors of the Extension functor, J.Algebra 244, 407–428 (2001).

[Buch59] David A. Buchsbaum, A note on homology in categories, Ann. of Math. (2) 69 (1959), 66–74.

[Bü10] T.Bühler, Exact categories. Expo. Math. 28 (2010), no. 1, 1–69.

[BuHo61] M.C.R.Butler and G.Horrocks, Classes of Extensions and Resolutions, Philosophical Transactions of the Royal Society of London, Series A. Mathematical and Physical Sciences, no. 1039, vol. 254, 1961.

[Cr11] S.Crivei, Maximal exact structures on additive categories revisited, Math. Nachr. 285 (2012), no. 4, 440–446.

[Cr12] S.Crivei, When stable short exact sequences define an exact structure on an additive category, arXiv: 1209.3423.

[Da02] B.A.Davey, Introduction to lattices and order, Cambridge University Press, 2002.

[DRSS] P.Dräxler, I.Reiten, S.O.Smalo, O.Solberg, Exact Categories and Vector Space Categories, Transactions of the American Mathematical Society, vol.351, no.2, 1999.

[Gr57] A. Grothendieck, Sur quelques points d’algèbre homologique, Tôhoku Math. J. (2) 9 (1957), 119–221.

[En18] H.Enomoto,Classifications of exact structures and Cohen-Macaulay-finite algebras, Advances in Mathematics, 335 (2018), 838-877.

[En19] H. Enomoto, Relations for Grothendieck groups and representation-finiteness, J. Algebra 539 (2019), 152–176.

[En20] H. Enomoto, Classifying substructures of extriangulated categories via Serre subcategories, preprint, arXiv: 2005.13381, 2020.

[FG20] X.Fang, M.Gorsky, Exact structures and degeneration of Hall algebras, arXiv: 2005.12130, 2020.

[Fi16] L. Fiorot, N-Quasi-Abelian Categories vs N-Tilting Torsion Pairs, preprint, arXiv:1602.08253.

[GR92] P.Gabriel and A.V.Roıter, Representations of Finite-dimensional Algebras, in: Algebra, VIII, Encyclopedia Mathematical Sciences, vol. 73, Springer, Berlin, 1992 (with a chapter by B. Keller), pp. 1–177.
[HKKP] F. Haiden, L. Katzarkov, M. Kontsevich, P. Pandit, *Semistability, modular lattices, and iterated logarithms*, arXiv:1706.01073

[HR20] R. Henrard, A.-C. van Roosmalen, *On the obscure axiom for one-sided exact categories*, to appear.

[HLN] H. Herschend, L. Liu, H. Nakaoka, *n−exangulated categories*, arXiv:1709.06689, 2017.

[He97] I. Herzog, *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. London Math. Soc. 74 (1997), 503–558.

[INP] O. Iyama, H. Nakaoka, Y. Palu, *Auslander–Reiten theory in extriangulated categories*, arXiv:1805.03776

[Ke90] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. 67 (1990), 379–417.

[Ke91] B. Keller, *Derived categories and universal problems*, Comm. Algebra 19 (1991), no. 3, 699–747.

[LN] Y. Liu, H. Nakaoka, *Hearts of twin cotorsion pairs on extriangulated categories*, J. Algebra 528 (2019) 96–149.

[Mac63] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963.

[M65] B. Mitchell, *Theory of Categories*, Pure and Applied Mathematics, A Series of Monographs and Textbooks, vol. 17, no. 65-22761, 1965.

[Ne90] A. Neeman, *The derived category of an exact category*, J. Algebra 135, no. 2, 388–394, 1990.

[NP19] H. Nakaoka and Y. Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, Volume LX-2, 2019.

[Og19] Y. Ogawa, *Auslander’s defects over extriangulated categories: an application for the General Heart Construction*, preprint, arXiv:1911.00259, 2019.

[Po73] N. Popescu, *Abelian categories with applications to rings and modules*, London Mathematical Society Monographs, No. 3. Academic Press, London–New York, 1973. xii+467 pp.

[Qu73] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[RW77] F. Richman and E. A. Walker, *Ext in pre-Abelian categories*, Pacific J. Math. 71(1977), 521–535.

[Ros] A. L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, MIA 330, Kluwer Academic Publishers Group, Dordrecht, 1995. xii+315 pp. ISBN: 0-7923-3575-9

[Rou] R. Rouquier, Dimensions of triangulated categories, J. K-Theory 1 (2008), 193–256.

[Ru11] W. Rump, *On the maximal exact structure of an additive category*, Fund. math. 214 (2011), no. 1, 77–87.

[Ru15] W. Rump, *Stable short exact sequences and the maximal exact structure of an additive category*, Fund. Math. 228 (2015).

[SW11] D. Sieg, S.-A. Wegner, *Maximal exact structures on additive categories*, Math. Nachrichten 284 (2011), no. 16, 2093–2100.
