Revealing inherent quantum interference and entanglement of a Dirac Fermion

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The Dirac equation is critical to understanding the universe, and plays an important role in technological advancements. Compared to the stationary solution, the dynamical evolution under the Dirac Hamiltonian is less understood, exemplified by Zitterbewegung. Although originally predicted in relativistic quantum mechanics, Zitterbewegung can also appear in some classical systems, which leads to the important question of whether Zitterbewegung of Dirac Fermions is underlain by a more fundamental and universal interference behavior without classical analogs. We here reveal such an interference pattern in phase space, which underlies but goes beyond Zitterbewegung, and whose nonclassicality is manifested by the negativity of the phase-space quasiprobability distribution, and the associated pseudospin-momentum entanglement. We confirm this discovery by numerical simulation and an on-chip experiment, where a superconducting qubit and a quantized microwave field respectively emulate the internal and external degrees of freedom of a Dirac particle. The measured quasiprobability equalities well agree with the numerical simulation. Besides being of fundamental importance, the demonstrated nonclassical effects are useful in quantum technology.

To unify quantum mechanics and relativity, P. A. M. Dirac formulated a famous matter-wave equation for a spin-1/2 particle, known as the Dirac equation [1, 2]

$$i\hbar \frac{\partial \psi}{\partial t} = H_D \psi,$$

where $H_D = c\alpha \cdot \mathbf{p} + \beta mc^2$, $c$ denotes the light speed in the vacuum, $\alpha$ and $\beta$ are the $4 \times 4$ Dirac matrices, $m$ and $\mathbf{p}$ represent the rest mass and momentum of the particle, respectively, and $\psi$ is the wavefunction with 4 components. Unlike the Schrödinger equation, the Dirac equation is linear in both the time and space derivatives, satisfying the Lorentz-covariance, and includes the spin degree of freedom at the ab initio level by describing the wave function in terms of a spinor. These features have led to remarkable accomplishments, including predictions of the spin-1/2 feature of electrons and the existence of anti-particles indicated by the negative-energy component accompanying the positive one, and an accurate description of the spectrum of the hydrogen atom. These predictions are based on stationary solutions of the Dirac equation and show excellent agreements with experiments. Over the past century, this equation has been producing enduring profound influences on a wide variety of fields of modern science and technology, ranging from atomic physics to quantum electrodynamics [2], and from material engineering [3–6] to medical imaging [7].

Despite the fundamental importance of the Dirac equation, the physics underlying its dynamical solution has not been fully understood owing to the associated elusive phenomena, exemplified by Zitterbewegung (ZB) [8], the oscillatory motion of a Fermion, as a result of the interference between the positive and negative energy components. For a free electron, the predicted ZB has an amplitude on the order of the Compton wavelength, $\hbar/mc \sim 10^{-12}$ m, and thus cannot be unambiguously observed due to the restriction of the Heisenberg uncertainty principle. Although whether or not ZB really exists in relativistic quantum mechanics is still an open question [9–12], enduring efforts have been made to its simulations with different quantum systems, including circuit quantum electrodynamics [12], ion traps [13, 14], ultracold atoms [15–18], semiconductor quantum wells [19], graphene [20], and moiré excitons [21]. These investigations have shed new light on ZB, which itself, however, is not a unique character of Dirac Fermions as similar phenomena can also appear in some classical wave systems [22–27]. This leads us to consider the important question: Does ZB associated with Dirac particles have a deeper quantum origin that can manifest itself even without ZB? Answering this question is critical for understanding the dynamical behaviors of Dirac particles at a more fundamental level, but a deep exploration is still lacking.

We here present an investigation on this important issue, and unveil a universal quantum interference behavior in the position-momentum space. The nonclassicality of this behavior is manifested by the negativity of the phase-space quasiprobability distribution–Wigner function (WF), as well as by the quantum correlation between the spatial and internal degrees of freedom. These quantum signatures distinguish the ZB of the Dirac particle, obtained by integrating the WF over the momentum, from the trembling motion of classical wavepackets, and more importantly, can express themselves even in the absence of any negative component. We demonstrate this unique interference pattern with a circuit, where the spinorial characteristic of a Dirac particle is encoded in
the two lowest energy levels of a superconducting Xmon qubit, while the position and momentum are mapped to the quadratures of the photonic field. The measured WFs and entanglement entropy well agree with theoretical predictions. Furthermore, we simulate the Klein tunneling\cite{28, 29} in a linear potential field and observe mesoscopic superpositions of two separated wavepackets in phase space.

We focus on the simplest case, where the motion of a Dirac Fermion is confined to one dimension (1D). In this case, the Hamiltonian reduces to

$$H_D = c\sigma_y \hat{p} + mc^2\sigma_z. \tag{2}$$

We note that here the Pauli operators $\sigma_y$ and $\sigma_z$ endow the Dirac particle with a spinor characteristic, manifested by a two-component wavefunction, where the spatial position and momentum are correlated with the degree of freedom defined in an “internal space”, which will be referred to as pseudospin for simplicity. As the Hamiltonian commutes with the momentum operator, it is illustrative to uncover the physics in the momentum representation, where the momentum operator $\hat{p}$ can be taken as a parameter $p$. For a specific value of $p$, $H_D$ has two eigenvalues $\pm E_p = \pm \sqrt{p^2c^2 + m^2c^4}$, with the corresponding eigenstates $|\phi_\pm(p)\rangle = (\cos \phi_p, i \sin \phi_p)^T$ and $|\phi_\mp(p)\rangle = (i \sin \phi_p, \cos \phi_p)^T$, where $\tan(2\phi_p) = \frac{p}{mc}$. Suppose that the system is initially in the product state

$$|\psi(0)\rangle = \int dp \, \xi_p |p\rangle |X\rangle, \tag{3}$$

where $|\pm X\rangle = \frac{1}{\sqrt{2}} (1, \pm 1)^T$ and $\xi_p$ denotes the wave function in the momentum representation. Under the Dirac Hamiltonian, the system evolves as

$$|\psi(t)\rangle = \int dp \, |p\rangle \xi_p (\cos \varphi_t |X\rangle - i e^{-2i\phi_p} \sin \varphi_t |-X\rangle), \tag{4}$$

where $\varphi_t = E_p t/\hbar$. This directly yields the average position evolution,

$$\langle x(t)\rangle = \langle x(0)\rangle + \langle v(0)\rangle t$$

$$+ \hbar \int dp \, |\xi_p|^2 x_p (1 - \cos 2\varphi_t), \tag{5}$$

where $x_p = d\varphi_p/dp$. $\langle x(0)\rangle$ represents the average value of the initial position, and $\langle v(0)\rangle$ denotes the initial mean velocity. The ZB, manifested by the last term, is observable only in the intermediate regime where $mc$ is comparable with $p$.

As we have noted, ZB itself does not manifest quantum effects, but is closely related to quantum entanglement between the internal and spatial degrees of freedom, produced by their coupling. Under the time evolution, the populations of two states $|\pm X\rangle$ become increasingly balanced, and the entropy tends to 1. Due to this entanglement, the spatial quantum interference appears when the WF is correlated with the projection of the pseudospin along some basis, e.g., $\{|\pm B\rangle\}$. The WFs associated to $|\pm B\rangle$ are respectively

$$W_\pm(x, p) = \frac{1}{\pi \hbar} \int dv \, \phi_\pm^*(p + v) \phi_\pm(p - v) e^{-2ivx/\hbar}, \tag{6}$$

where $\phi_\pm(p) = \xi_p \langle \pm B | (\cos \varphi_t |X\rangle - i e^{-2i\phi_p} \sin \varphi_t |-X\rangle)$. During the evolution, the wavepacket is continually deformed under the competition between the momentum-dependent and static energy terms in the Dirac Hamiltonian, which leads to a nonlinear dependence of the energy on the momentum. This nonlinear process evolves an initial Gaussian wavepacket to a non-Gaussian one, manifesting pseudospin-dependent quantum interference signatures. The ZB phenomenon appears as the integral of the weighted mixture of the two WFs over the momentum, which reflects the classical probability distribution, but does not manifest the underlying quantum nature. It should be noted that the presence of ZB is challenged by the claim that the positive and negative components could not be assigned to a single particle\cite{9}, however, recent experimental evidence indicates nature does not prohibit the existence of a quantum superposition of a particle with its antiparticle\cite{30}. We further note that even when the Fermion keeps in the positive branch, there still exists phase-space quantum interference, though ZB disappears.

The simulation is performed with a superconducting qubit of angular frequency $\omega_0$ that encodes the internal state of the simulated spinor, whose position and momentum are mapped onto the two quadratures of the microwave field stored in a bus resonator, defined as $\hat{x} = \frac{1}{\sqrt{2}} (a^\dagger + a)$ and $\hat{p} = \frac{1}{\sqrt{2}} (a^\dagger - a)$, where $a^\dagger$ and $a$ denote the creation and annihilation operators for the photonic field of angular frequency $\omega_r$. If we take $\hbar = 1$, $\hat{x}$ and $\hat{p}$ satisfy the same commutation relation as the position and momentum operators. The qubit is subjected to two longitudinal parametric modulations with amplitudes $\varepsilon_j$ and angular frequencies $\nu_j$ ($j = 1, 2$), and a transverse continuous microwave driving with an amplitude $\Omega$ (Fig. 1(a)). With the choice $\omega_r = \omega_0 + 2\nu_1$, the resonator is coupled to the qubit at the second upper sideband of the first modulation with the effective strength $\eta = \lambda J_2(\mu)/2$\cite{31}, where $J_n(\mu)$ is the $n$th Bessel function of the first kind, with $\mu = \varepsilon_1/\nu_1$. When $\Omega \gg 2\eta$, the transverse drive effectively transforms the rotating-wave interaction into an equal combination of rotating- and counter-rotating-wave interactions, simulating the coupling between the internal and external degrees of freedom of the spinor. Then the resulting effective Hamiltonian reduces to the form of Eq. (2), with the correspondences $\sigma_y = i|g\rangle \langle e| - i|e\rangle \langle g|$, $\sigma_z = |e\rangle \langle e| - |g\rangle \langle g|$, $c^* = \sqrt{2\eta}$ and $m^* c^* s^2 = \sigma_z/4$, where $c^*$ and $m^*$ denote the effective light speed and mass of the Dirac particle in the simulation, respectively (see Supplementary Material).
Before the experiment, both the resonator and the spinor qubit are initialized to their ground states. The experiment starts with the application of a pulse to the resonator, translating its state along the $p$-axis in phase space by an amount of $p_0 = 2$, and thus transforming the initial vacuum state to the coherent state $|\sqrt{n}\rangle^0$. A $\pi/2$ rotation is performed on the test qubit, transforming it from $|g\rangle$ to $|X\rangle$ at the operating frequency $\omega_0/(2\pi) = 5.26$ GHz. The initial qubit-resonator state is pictorially shown in Fig. 1(b). After this initial state preparation, a parametric modulation with frequencies $\nu_1/(2\pi) = 160$ MHz and $\nu_2/(2\pi) = 33.4$ MHz is applied to the qubit. These modulations, together with the transverse microwave driving at the frequency $\omega_0$, couple the qubit to the bus resonator with a fixed frequency of $\omega_r/(2\pi) = 5.584$ GHz, effectively realizing the Dirac Hamiltonian in the rotating frame. The ratio between the effective momentum and mass of the Dirac particle is variable by adjusting the modulation amplitudes $\varepsilon_1$ and/or $\varepsilon_2$. Detailed system parameters are shown in the Supplementary Material.

We investigate the interference behaviors with the choice $\omega = \sqrt{2}\eta p_0$. After a preset evolution time, both the parametric modulations and microwave driving are switched off. This is followed by Wigner tomography, realized by performing a phase-space displacement, $D(\gamma) = e^{\gamma a^\dagger - \gamma^* a}$, to the resonator and then tuning an ancilla qubit on resonance with the resonator. The photon number population of the displaced resonator field, $P_n(\gamma)$, inferred from the measured Rabi oscillation signals, directly yields the WF, $W(x, p) = \frac{1}{2} \sum_n (-1)^n P_n(\gamma)$.

During the evolution the initial Gaussian wavepacket is split into two parts, propagating towards opposite directions. The WF associated with each qubit state displays a region of negativity that is a purely quantum-mechanical effect. This result can be interpreted as follows. Under the Dirac Hamiltonian, each component accumulates a phase that nonlinearly depends on the “momentum” and “mass” as $\sqrt{p^2 + (m^* c^*)^2} c^* t$. Such a process corresponds to a non-Gaussian operation, turning a
Gaussian state to a non-Gaussian state. The two rods sprouting from the bulk of the distorted wavepacket interfere with each other, resulting in a negative quasiprobability distribution in the region between them. The lower panels show the probability distribution $P(x)$ with respect to the quadrature $x$, obtained by integrating $\hat{W}(x,p)$ over $p$.

Another important feature associated with the simulated particle is the production of quantum entanglement between its internal and external degrees of freedom. To quantitatively characterize the behavior, we present the von Neumann entropy of the test qubit, $S = -\text{tr}(\rho_q \log_2 \rho_q)$, measured for different evolution time $t$ in Fig. 2(d), where $\rho_q$ denotes the reduced density operator of this qubit. The measured results agree with the numerical simulation (blue line), where the small fluctuations are due to the fast Rabi oscillations. ZB is manifested in the time-evolving mean position, which is related to $P(x)$ by $\langle x \rangle = \int \text{d}x \ x P(x)$. This evolution, inferred from the measured WF, is presented in Fig. 2(e), which coincides with the simulation (green line), confirming that ZB has a deeper root that is of pure quantum characteristic.

Although ZB appears only when the system is in a superposition of positive and negative components, phase-space quantum interference is actually a universal inherent characteristic of Dirac Fermions, which can manifest itself even without negative components. This point can be illustrated with the representative example, where the momentum has a Gaussian distribution, centered at $p_0$ with the spread $\delta p$. When restricted to the positive branch, the system state can be written as

$$|\psi(t)\rangle = \int \text{d}p \ e^{i\theta_p(t)} \xi_p \ |p\rangle \ |\phi_+(p)\rangle, \quad (7)$$

where $\xi_p = (\delta p \sqrt{2\pi})^{-1/2} e^{-(p-p_0)^2/(2\delta p)^2}$. This implies that the internal freedom degree is necessarily entangled with the momentum except for a plane wave with $\delta p \to 0$. As it is experimentally difficult to prepare such an entangled state, we reveal the associated quantum feature by numerical simulation. The entanglement entropy as a function of $\delta p$ is shown in Fig. 3(a), which is independent of the evolution time. We here have set $\hbar = c = 1$ and $m = p_0 = 1$. Unlike the entropy, the phase-space interference pattern is time-dependent. To clearly reveal such an interference behavior, we assume that $\delta p = 1$, and $\theta_p(0) = 0$. The unconditional WFs, for different evolution times under the ideal Dirac Hamiltonian, are shown in Fig. 3(b). Unexpectedly, the WF displays a time-evolving quantum interference pattern, even without correlating the result with the internal state. We note that the phase-space quantum interference effects were previously predicted for some special states with both positive and negative components [12], but the presence of such effects without ZB has not been revealed. The WFs correlated with $|g\rangle$ and $|e\rangle$, together with the evolutions of mean position and entanglement entropy, are displayed in the Supplementary Material.

Pushing one step further, we simulate Klein tunneling in a linear potential field [27, 28]. It was first noted by Klein that a relativistic electron may exhibit a counterintuitive behavior when confronted with a semi-infinite step potential with $V = 0$ and $V_0$ for $x < 0$ and $x \geq 0$, respectively. This occurs in the regime $V_0 > E + mc^2$, for which the electron can propagate through the barrier without damping, where it is transformed into a positron, where $E$ denotes the initial kinetic energy. In our setup, it is not easy to engineer the step-shaped potential. However, a linear potential can be added to the Dirac Hamiltonian in situ by applying a continuous microwave to the resonator, given by $V = \sqrt{2} \epsilon x$, where $\epsilon$ is the amplitude of the drive, and set to be $2\pi \times 0.39$ MHz in our experiment. For simplicity, the simulation is performed for the choice $\epsilon_2 = 0$. Figures 4(a) and 4(b) showcase the WFs of the resonator correlated with the states $|g\rangle$ and $|e\rangle$ of the spinor qubit, respectively, and Fig. 4(c) presents the result irrespective of the qubit’s state, all measured after an evolving time $t = 288$ ns. As expected, the linear potential drags the phase-space evolution down along the $p$-axis by an amount $\sqrt{2} \epsilon t$, but does not affect the motion along the $x$-axis. The resulting cat-like state was previously predicted to exist as a solution of a relativistic spin-1/2 charged particle in an external magnetic field [33], but has not been characterized in previous simulations [34, 35]. To illustrate this phenomenon more clearly, we display the time evolutions of the measured $\langle x \rangle$ and $\langle p \rangle$ in Fig. 4(d) and Fig. 4(e), respectively, where the dots and diamonds respectively denote the results for wavepackets along the positive and negative directions.
of the $x$-axis. The measured results imply that the two wavepackets have the same momentum at each moment, but move along the opposite directions. This can be explained as follows. For $x > 0$, the momentum of the Dirac particle is given by $p = (E - V)/m$ ($c = 1$), with the group velocity $v_g = dE/dp = p/(E - V)$ [29]. When the massless particle moves from left to right with $E < V$, $p$ is assigned with its negative solution, so that $v_g$ is positive.

In conclusion, we have performed an investigation on the dynamical evolution of the Dirac Fermion, showing that the competition between its dynamic and static energies leads to a time-evolving phase-space quasi-probability distribution, which underlies its spatial motion. The quantum interference signatures appear in the phase space spanned by position and momentum, but disappear when the momentum is traced out. We demonstrate this discovery with numerical simulations and with a circuit experiment, where a superconducting qubit represents the internal degree of freedom, which is coupled to the microwave field in a resonator that encodes the spatial degree of freedom. The measured negativities of the WFs and entanglement entropy distinguish the ZB of the Dirac particle from that exhibited by the classical systems. In addition to fundamental interest, the demonstrated nonclassicality can serve as a resource for quantum-enhanced sensing [36].

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[1] P. A. M. Dirac, The quantum theory of the electron, Proceedings of the Royal Society of London. Series A. Containing Papers of a Mathematical and Physical Character 117, 610 (1928).
[2] B. Thaller, The Dirac Equation (Springer Berlin Heidelberg, Berlin, Heidelberg, 1992).
[3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81, 109 (2009).
[4] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Two-dimensional gas of massless Dirac fermions in graphene, Nature 438, 197 (2005).
[5] C.-H. Park, L. Yang, Y.-W. Son, M. L. Cohen, and S. G. Louie, Anisotropic behaviours of massless Dirac fermions in graphene under periodic potentials, Nat. Phys. 4, 213 (2008).
[6] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice, Nature 483, 302 (2012).
[7] T. Beyer, D. W. Townsend, T. Brun, P. E. Kinahan, M. Charron, R. Roddy, J. Jerin, J. Young, L. Byars, and R. Nutt, A combined PET/CT scanner for clinical oncology, J. Nucl. Med. 41, 1369 (2000).
[8] E. Schrödinger, Über die kräftefreie bewegung in der relativistischen quantenmechanik, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. 24, 418 (1930).
[9] L. L. Foldy and S. A. Wouthuysen, On the Dirac theory
of spin 1/2 particles and its non-relativistic limit, Phys. Rev. 78, 29 (1950).

[10] P. Krekora, Q. Su, and R. Grobe, Relativistic electron localization and the lack of Zitterbewegung, Phys. Rev. Lett. 93, 043004 (2004).

[11] Z.-Y. Wang and C.-D. Xiong, Zitterbewegung by quantum field-theory considerations, Phys. Rev. A 77, 044502 (2008).

[12] J. S. Pedernales, R. Di Candia, D. Ballester, and E. Solano, Quantum simulations of relativistic quantum physics in circuit QED, New J. Phys. 15, 055008 (2013).

[13] L. Lamata, J. León, T. Schäitz, and E. Solano, Dirac equation and quantum relativistic effects in a single trapped ion, Phys. Rev. Lett. 98, 253005 (2007).

[14] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, Quantum simulation of the Dirac equation, Nature 463, 68 (2010).

[15] J. Y. Vaishnav and C. W. Clark, Observing Zitterbewegung with ultracold atoms, Phys. Rev. Lett. 100, 153002 (2008).

[16] L. J. LeBlanc, M. C. Beeler, K. Jiménez-García, A. R. Perry, S. Sugawa, R. A. Williams, and I. B. Spielman, Direct observation of Zitterbewegung in a Bose–Einstein condensate, New J. Phys. 15, 073011 (2013).

[17] C. Qu, C. Hamner, M. Gong, C. Zhang, and P. Engels, Observation of Zitterbewegung in a spin-orbit-coupled Bose-Einstein condensate, Phys. Rev. A 88, 021604(R) (2013).

[18] M. Hasan, C. S. Madasu, K. D. Rathod, C. C. Kwong, C. Miniatura, F. Chevy, and D. Wilkowski, Wave packet dynamics in synthetic non-abelian gauge fields, Phys. Rev. Lett. 129, 130402 (2022).

[19] J. Schliemann, D. Loss, and R. M. Westervelt, Zitterbewegung of electronic wave packets in III-V zinc-blende semiconductor quantum wells, Phys. Rev. Lett. 94, 206801 (2005).

[20] T. M. Rusin and W. Zawadzki, Theory of electron Zitterbewegung in graphene probed by femtosecond laser pulses, Phys. Rev. B 80, 045416 (2009).

[21] I. R. Lavor, D. R. da Costa, L. Covaci, M. V. Milošević, F. M. Peeters, and A. Chaves, Zitterbewegung of moiré excitons in twisted MoS2/WSe2 heterobilayers, Phys. Rev. Lett. 127, 106801 (2021).

[22] X. Zhang, Observing Zitterbewegung for photons near the Dirac point of a two-dimensional photonic crystal, Phys. Rev. Lett. 100, 113903 (2008).

[23] J. Otterbach, R. G. Unanyan, and M. Fleischhauer, Confining stationary light: Dirac dynamics and Klein tunneling, Phys. Rev. Lett. 102, 063602 (2009).

[24] S. Longhi, Photonic analog of Zitterbewegung in binary waveguide arrays, Optics Letters 35, 235 (2010).

[25] Y. Chen, R.-Y. Zhang, Z. Xiong, Z. H. Hang, J. Li, J. Q. Shen, and C. T. Chan, Non-abelian gauge field optics, Nat. Commun. 10, 3125 (2019).

[26] F. Dreisow, M. Heinrich, R. Keil, A. Tünnemann, S. Nolte, S. Longhi, and A. Szameit, Classical simulation of relativistic Zitterbewegung in photonic lattices, Phys. Rev. Lett. 105, 143902 (2010).

[27] T. L. Silva, E. R. F. Taillebois, R. M. Gomes, S. P. Walsborn, and A. T. Avelar, Optical simulation of the free Dirac equation, Phys. Rev. A 99, 023332 (2019).

[28] O. Klein, Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac, Zeit. Phys. 53, 157 (1929).

[29] N. Dombey, Seventy years of the Klein paradox, Phys. Rep. 315, 41 (1999).

[30] R. Aaij et al. (LHCb Collaboration), Observation of the mass difference between neutral charm-meson eigenstates, Phys. Rev. Lett. 127, 111801 (2021).

[31] R.-H. Zheng, W. Ning, Y.-H. Chen, J.-H. Lü, L.-T. Shen, K. Xu, Y.-R. Zhang, D. Xu, H. Li, Y. Xia, F. Wu, Z.-B. Yang, A. Miranowicz, N. Lambert, D. Zheng, H. Fan, F. Nori, and S.-B. Zheng, Emergent Schrödinger cat states during superradiant phase transitions, arXiv:2207.0551 (2022).

[32] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Synthesizing arbitrary quantum states in a superconducting resonator, Nature 459, 546 (2009).

[33] A. Bermudez, M. A. Martin-Delgado, and E. Solano, Mesoscopic superposition states in relativistic Landau levels, Phys. Rev. Lett. 99, 123602 (2007).

[34] R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. García-Ripoll, E. Solano, R. Blatt, and C. F. Roos, Quantum simulation of the Klein paradox with trapped ions, Phys. Rev. Lett. 106, 060503 (2011).

[35] T. Salger, C. Grossert, S. Kling, and M. Weitz, Klein tunneling of a quasirelativistic Bose-Einstein condensate in an optical lattice, Phys. Rev. Lett. 107, 240401 (2011).

[36] H. Kwon, K. C. Tan, T. Volkoff, and I. B. Spielman, Direct observation of Zitterbewegung in a spin-orbit-coupled Bose–Einstein condensate, Phys. Rev. A 88, 021604(R) (2013).

[37] H. Kwon, K. C. Tan, T. Volkoff, and H. Jeong, Nonclassicality as a quantifiable resource for quantum metrology, Phys. Rev. Lett. 122, 040503 (2019).
Supplementary Material for
“Revealing inherent quantum interference and entanglement of a Dirac Fermion”

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S1. SYNTHESIS OF THE DIRAC HAMILTONIAN

A. Theoretical model

To realize the spinor-momentum coupling, the excitation energy of the test qubit is periodically modulated as (ℏ = 1 hereafter)

\[ \omega_q(t) = \omega_0 + \varepsilon_1 \cos(\nu_1 t) + \varepsilon_2 \cos(\nu_2 t), \]  

(S1)

where \( \omega_0 \) is the mean excitation energy, and \( \varepsilon_j \) and \( \nu_j \) (\( j = 1, 2 \)) are the corresponding modulation amplitude and angular frequency, respectively. In addition to interaction with the resonator, the qubit is driven by a continuous microwave with frequency \( \omega_0 \). The system Hamiltonian is given by

\[ H = H_0 + H_I, \]  

(S2)

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where
\[ H_0 = \omega_r a^\dagger a + [\omega_0 + \varepsilon_1 \cos(\nu_1 t)] |e\rangle \langle e|, \]
\[ H_I = \varepsilon_2 \cos(\nu_2 t) |e\rangle \langle e| + (-\lambda a^\dagger |g\rangle \langle e| + \Omega e^{i\theta} e^{i\omega_0 t} |g\rangle \langle e|) + \text{H.c.}, \]
\[ (S3) \]
\( \lambda \) is the qubit-resonator coupling strength, and \( \Omega (\theta) \) denotes the amplitude (phase) of the classical driving field. We here assume \( \omega_r = \omega_0 + 2\nu_1 \). With this setting, the resonator interacts with the qubit at the second sideband associated with the first modulation, while the microwave drive works at the carrier. Performing the transformation
\[ U_0 = \exp \left( i \int_0^t H_0 \, dt \right), \]
\[ (S4) \]
we obtain the system Hamiltonian in the interaction picture
\[ H'_I = \varepsilon_2 \cos(\nu_2 t) |e\rangle \langle e| + \sum_{o=-\infty}^{\infty} J_o(\mu) \left\{ -\lambda \exp[-i(o-2)\nu_1 t] a^\dagger + \Omega e^{i\theta} \exp(-i\nu_1 t) \right\} |g\rangle \langle e| + \text{H.c.}, \]
\[ (S5) \]
where \( \mu = \varepsilon_1/\nu_1 \). To clarify the underlying physics clearly, we use the Jacobi-anger expansion
\[ \exp [i\mu \sin(\nu_1 t)] = \sum_{o=-\infty}^{\infty} J_o(\mu) \exp (i\nu_1 t), \]
\[ (S6) \]
with \( J_o(\mu) \) being the \( o \)th Bessel function of the first kind, we obtain
\[ H'_I = \varepsilon_2 \cos(\nu_2 t) |e\rangle \langle e| + \sum_{o=-\infty}^{\infty} J_o(\mu) \left\{ -\lambda \exp[-i(o-2)\nu_1 t] a^\dagger + \Omega e^{i\theta} \exp(-i\nu_1 t) \right\} |g\rangle \langle e| + \text{H.c.}, \]
\[ (S7) \]
For \( \lambda, \Omega J_0(\mu) \ll \nu_1 \), the qubit interacts with the resonator at the upper second-order sideband modulation with \( o = 2 \), and is driven at the carrier with \( o = 0 \). With the fast oscillating terms being discarded, \( H'_I \) reduces to
\[ H'_I = K \sigma_\theta + \frac{1}{2} \varepsilon_2 \cos(\nu_2 t) \sigma_z - \eta e^{-i\theta} a^\dagger (\sigma_\theta - i\sigma_{\theta+\pi/2}) + \text{H.c.}, \]
\[ (S8) \]
where \( K = \Omega J_0(\mu) \), \( \eta = \lambda J_2(\mu)/2 \), \( \sigma_\theta = e^{i\theta} |g\rangle \langle e| + e^{-i\theta} |e\rangle \langle g| \), and \( \sigma_z = |e\rangle \langle e| - |g\rangle \langle g| \). With the assumption \( \nu_2 = 2K \gg \eta, \varepsilon_2/2 \) and under the further transformation \( \exp(iK \sigma_\theta t) \), the system Hamiltonian can be well approximated by
\[ H''_I = -\eta e^{-i\theta} a^\dagger \sigma_\theta + \text{H.c.} + \omega \sigma_z, \]
\[ (S9) \]
where \( \omega = \varepsilon_2/4 \). When \( \theta = \pi/2 \), this effective Hamiltonian has the same form as the 1+1 Dirac equation of a spin-1/2 particle, with the correspondence \( c^* = \sqrt{2\eta} \) and \( m^* c^2 = \omega \). Herein \( c^* \) and \( m^* \) denote the effective light speed and mass of the Dirac particle in the simulation, respectively.

### B. System parameters

The whole electronics and wiring of the used superconducting 5-qubit sample [S1, S2] are shown in Fig. S1. Each frequency-tunable Xmon qubit has an individual flux line for its dynamic tuning and a microwave drive for controllably flipping its states. All the qubits are capacitively coupled to a bus resonator with coupling strength \( \lambda_j \approx 2\pi \times 20 \text{ MHz} \) (\( j = 1, 2 \)) and every qubit has a readout resonator for reading out its states. The bus resonator \( R \) is a superconducting coplanar waveguide resonator with fixed frequency \( \omega_r/(2\pi) = 5.5835 \text{ GHz} \), which is measured when all used qubits stay in ground states at their respective idle frequencies (See below, unused qubits are always at their sweet points \( \approx 6 \text{ GHz} \)). More detailed parameters of the sample are listed in Tab. S1.

In this experiment, we use two qubits (\( Q_1, Q_2 \)) and the bus resonator (\( R \)). One qubit (\( Q_1 \)) is used to realize the effective Dirac model, and the other qubit (\( Q_2 \)) is used to measure the photon number distribution of the bus resonator by resonant population exchange (see Sec. S4A for details). For initialization, we wait for 300 \( \mu \text{s} \) to make sure all qubits return to their ground states, and then bias them to their idle frequencies, where single qubit gates and qubit states measurements are performed.

We use two independent XY signal channels controlled by Digital-to-Analog converters (DACs), to generate two microwave sequences. These two low-frequency microwave sequences are mixed with a continuous microwave by the In-phase and Quadrature (IQ) mixer to generate two tone pulses. This continuous microwave is a carrier microwave emitted by the microwave source (MS) and has a frequency of 5.21 GHz. One-tone pulse is used to drive the test
TABLE S1. Qubit and resonator characteristics. The test qubit, auxiliary qubit, and bus resonator are denoted by \( Q_1, Q_2, \) and \( R \), respectively. The idle frequencies of \( Q_j \) (\( j = 1, 2 \)) are marked by \( \omega_{10}/(2\pi) \), where the pulses for initial state preparation and tomographic pulses for qubit measurement are applied. The energy relaxation time, the Ramsey Gaussian dephasing time, and the spin echo Gaussian dephasing time are \( T_1, T^*_2 \) and \( T_{SE}^2 \), respectively, all measured at the idle frequency of each qubit. The coupling strength \( \lambda_j \) between the qubits \( Q_j \) and bus resonator \( R \) is measured by their population exchange rate at resonance. The symbol \( \alpha \) is the anharmonicity of the qubits, \( \omega_{ro} \) is the bare frequency of qubit’s readout resonator, and \( F_g (F_e) \) is the probability of reading out \( Q_j \) in \( |g\rangle (|e\rangle) \) when it is prepared in \( |g\rangle (|e\rangle) \).

During the procedure of qubit’s initial state preparation and measurement, some errors are caused by decoherence.
To calibrate these errors, we use a calibration matrix, defined as

\[ F_j = \begin{pmatrix} F_{g,j} & 1 - F_{e,j} \\ 1 - F_{g,j} & F_{e,j} \end{pmatrix}, \]

(S10)

where \( F_{g,j} \) and \( F_{e,j} \) are the readout fidelities of qubits \( Q_j \) (see Tab. S1), to reconstruct the readout results. Defining \( F = F_1 \otimes F_2 \) as a two-qubit calibration matrix. We rewrite the measurement results of two qubits into a column vector \( P_m \), and the calibrated measurement result is then \( P = F^{-1} \cdot P_m \).

### S2. Numerical Simulations of Phase-Space Quantum Interference

#### A. Interference between Positive and Negative Components

To further confirm the experimental results indeed reflect the quantum dynamics of Dirac Fermions, we perform numerical simulations based on the ideal Dirac Hamiltonian. The resonator WFs correlated with the qubit’s \( |g⟩ \) and \( |e⟩ \) states after 330 ns are respectively shown in Figs. S2(a) and S2(b), while the result irrespective of the qubit state is displayed in Fig. S2(c). In the numerical simulations, the Hamiltonian parameters and the system initial state are the same as those for the experimental simulations displayed in Fig. 2 of the main text. The numerical results with the full Hamiltonian are displayed in Fig. S3. To compensate for the difference due to the Stark shifts produced by the discarded off-resonant couplings, we perform proper phase-space rotations, which, however, do not change the quantum effects in any way. The numerical simulations of the evolutions of the mean position \( \langle x \rangle \) and entropy are displayed in Fig. S4. All these numerical results agree well with the experimental simulations shown in Fig. 2 of the main text.

![FIG. S2. Numerical simulations of phase-space quantum interference, governed by the effective Hamiltonian Eq. (S9).](image)

(a), (b) WFs correlated with the basis states \( |g⟩ \) and \( |e⟩ \) of the test qubit; (c) WFs irrespective of the test qubit’s state, all obtained after an evolution time of 330 ns. Lower panels: Probability distributions \( P(x) \) with respect to the quadrature \( x \), obtained by integrating the corresponding WFs over \( p \).

#### B. Interference between Positive Components with Different Momentums

As stated in the main text, quantum interference is actually a universal inherent characteristic of Dirac Fermions, which can manifest itself even without negative components. When the initial state is restricted to the positive branch, the system state can be written as

\[ |ψ_+(t)⟩ = \int dp \, e^{-iE_p t} \xi_p |p⟩ |φ_+(p)⟩. \]

(S11)

We assume that \( |ξ_p|^2 \) satisfies Gaussian distribution, i.e., \( |ξ_p|^2 = 1/(σp√2π)e^{-(p-p_0)^2/(2σp^2)} \), with \( p_0 = 1, σp = 1 \). To demonstrate the phase-space quantum interference, we project the system state along the basis \( \{|B_±⟩\} = \{|e⟩,|g⟩\} \) and deduce the WF signal, given by

\[ \mathcal{W}_±(x,p,t) = \frac{1}{π} \int dv \, ϕ_±^*(p + v)ϕ_±(p - v)e^{-2ivx}, \]

(S12)
FIG. S3. Numerical simulations of phase-space quantum interference, governed by the full Hamiltonian Eq. (S2). (a), (b) WFs correlated with the basis states $|g\rangle$ and $|e\rangle$ of the test qubit; (c) WFs irrespective of the test qubit’s state, all deduced after an evolution time of 330 ns. Lower panels: Probability distributions $P(x)$ with respect to the quadrature $x$, obtained by integrating the corresponding WFs over $p$.

FIG. S4. Numerical simulation of Zitterbewegung. (a) Mean position $\langle x \rangle$ versus time. (b) Entropy versus time. The solid and dashed lines represent the results of full and effective dynamics (governed by full and effective Hamiltonians), respectively.

where $\varphi_{\pm}(p) = \xi^*_p e^{-iE_p t}\langle B_{\pm} | \phi_+ (p) \rangle$. Numerical simulations of such WFs are shown in Fig. S5, where we can see phase-space quantum interference patterns for different evolution times. The simulations are performed based on the ideal Dirac Hamiltonian with $m = 1$, and by setting $\hbar = c = 1$. An unexpected result is that the WFs, obtained irrespective of the internal state, also displays a time-evolving quantum interference pattern. This result is in distinct contrast with the case that the system is initially in the superposition of positive and negative components, for which the WFs exhibit a quantum interference pattern that appears only when correlating to the internal state. Such an unconditional feature further confirms the universality of the phase-space quantum interference behavior.

Based on Eq. (S11), the entropy is calculated by

$$S' = -\log_2 \left( \int dp \, |\xi_p|^2 |\phi_+ (p)\rangle \langle \phi_+ (p) | \right) ,$$  \hspace{1cm} (S13)$$

The numerical result is presented in Fig. S6(a), which confirms that the internal and spatial freedom degrees remain highly correlated with a time-independent entanglement entropy. Additionally, the corresponding mean position $\langle x \rangle$ versus time is shown in Fig. S6(b), which exhibits a linear increase without ZB. These results unambiguously demonstrate that the Dirac particle evolves with a time-dependent quantum interference pattern, which is hidden in the entangled internal-spatial state, and expresses itself in the position-momentum quasiprobability distribution, correlated to the internal state.
FIG. S5. **Time-evolving WFs associated with the positive branch.** The upper and middle rows respectively denote the WFs correlated with the internal states $|g\rangle$ and $|e\rangle$, and the lower row shows the results irrespective of the internal state. The momentum is supposed to have a Gaussian distribution, centered at $p_0 = 1$ with the spread $\delta p = 1$. The simulated Dirac particle has a mass of $m = 1$. The results for $t = 0, 2, \text{ and } 4$ are shown in (a), (b), and (c), respectively. For simplicity, $\hbar$ and $c$ in the Dirac Hamiltonian are both set to be 1.

FIG. S6. **Entanglement entropy evolution and spatial motion for the positive branch.** The mean position $\langle x \rangle$ is calculated by integrating $xW(x, p, t)$ over phase space.

### S3. OPTIMIZATION OF DRIVING PULSES

#### A. Optimized Qubit Driving

To achieve a periodic modulation of the qubit frequency, we first measure the resonant frequency $\omega_{10}$ versus the Z-line pulse amplitude (ZPA) for the test qubit, the results are shown in Fig. S7 [S3]. Here we set the detuning between the idle frequency of the $Q_1$ and the resonator frequency being $\Delta/(2\pi) \approx 320$ MHz, and the frequency of the first modulation pulse $\nu_1/(2\pi) = 160$ MHz. In such a case, the working frequency of the qubit is the idle frequency point, avoiding a large rising edge at the beginning of the pulse. Considering the frequency of the second frequency modulation pulse $\nu_2/(2\pi) = 33.4$ MHz, this parameter setting will effectively reduce the high-energy level excitation caused by the modulation pulse. Subsequently, we modulate the qubit in the frequency with amplitude $\varepsilon_1 = 2\pi \times 130$ MHz, centering around its idle frequency according to the $\omega_q$ versus ZPA curves obtained previously.

Considering that the nonlinear curve of frequency versus ZPA and the limited DACs bandwidth lead to imperfect
waveforms of the periodically modulated excitation energy, we modify the frequency-modulated pulse as

$$\omega_q(t) = \omega_0 + \delta + \varepsilon_1 \cos(\nu_1 t + \phi_1) + \varepsilon_2 \cos(\nu_2 t + \phi_2),$$  \hspace{1cm} (S14)

making the experiment dynamics closer to the Dirac dynamics. We traverse these two phases ($\phi_1, \phi_2$) between 0 and $2\pi$, set the initial state of the system to be $|\psi\rangle = \sqrt{\frac{1}{2}}(|e\rangle + |g\rangle) \otimes |0\rangle$ in both the ZB and Klein tunneling simulations, and finally obtain the one which is closest to the ideal situation. Here we use the 2-norm as an index to measure the agreement between the experimental population data of qubit state $|e\rangle$ and the theoretical values. Then, we optimize the $\delta$ and other parameters in the same way. In this case, the optimized population is shown in Fig. S8, and the experimental data are in good agreement with the ideal ones, which proves the correctness of our method.

For the transverse field driving the qubit, we want the microwave to arrive at the qubit with an initial phase $\pi/2$. Therefore, we use a similar approach to optimize this phase. In addition, in the application of modulated pulses, the oscillation center frequency of the qubit may deviate slightly from the working frequency $\omega_0$, so we mildly tune the microwave frequency (about 1 $\sim$ 2 MHz) of the transverse field to make the results more predictable.

![FIG. S7. Frequence of the test qubit versus Z line bias. The desired modulation of $\omega_q(t)$ can be realized by mapping it to the modulation of Z bias.](image)

B. Optimized Bus Resonator Driving

The resonator is driven by using a flattop envelope with a resonance frequency, described as $\Omega_r(a + a^\dagger)$, where $\Omega_r$ is the Rabi frequency of the pulse. During the preparation of the initial state in the ZB simulation experiment, the presence of a dispersion coupling (form of $\sigma_z a^\dagger a$) between the test qubit and the bus resonator is $\lambda_2^2/\Delta \approx 2\pi \times 1.25$ MHz. So we reduce the pulse time and increase the Rabi frequency of the pulse to avoid the initial state rotation caused by the dispersion interaction. Lengths of pulse for initial state preparation and tomography pulse are 24 ns and 60 ns, respectively.

For pulses of a specific length, we apply them with different amplitudes to the resonator and measure the photon number to fit the slope between the Rabi frequency ($\Omega_r$) and the pulse amplitude. Up to now, together with the initial phase of the microwave pulse, we can implement arbitrary displacement operators $D(\gamma)$ for Wigner measurements.

The Klein tunneling experiment requires a continuous pulse on the resonator during the dynamics. There is a fixed resonator frequency shift induced by the non-resonant coupling, thus we adjust the microwave frequency to about 0.75 MHz to get better results.

The qubit-state-dependent resonator frequency shift in the dynamical process is still unavoidable; therefore, additional correction of the results is necessary so as to make the results more intuitive (see Sec. S5).
FIG. S8. **Validity check of the Rabi model.** For the present Rabi model, the effective frequency of the qubit and the effective coupling strength are $\omega/(2\pi) = 2.2$ MHz and $\eta = 2\pi \times 0.78$ MHz, respectively. We fix the initial state as $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |0\rangle$.
After optimizing parameters $\varepsilon_1$, $\varepsilon_2$, $\phi_1$, $\phi_2$, and $\delta$ appropriately, in the experiment, we measure the population of the test qubit state $|e\rangle$, marked as $P_e(t)$, at different times, and compare it with the numerical results.

C. Full Pulse Sequence

The pulse sequence is shown in Fig. S9, including three steps: 1. Initial state preparation; 2. Dirac dynamics; 3. Quantum state measurement. For clarity of reading, the real-time scales are not used in the figure.

FIG. S9. **Sketch of the Pulse Sequences.** The test qubit $Q_1$ is driven by a transverse microwave field with the Rabi frequency $\Omega/(2\pi) \approx 20.03$ MHz through the XY-line (red), and two Sine longitudinal modulations apply to the Z-line (purple) with the modulation frequencies $\nu_{1(2)}/(2\pi) = 160$ (33.4) MHz and the amplitudes $\varepsilon_{1(2)} = 2\pi \times 130$ (8.8) MHz in the ZB simulation. While in the Klein tunneling simulation, we set $\varepsilon_2 = 0$ to simulate massless particles. The orange dashed box represents the pulse sequence of $R^{xy}$ used in the ZB simulation, which is replaced by the series of yellow dashed boxes in the Klein tunneling simulation, and outside the dashed box there are the same pulse sequences for these two simulations. The evolution times $t$ of the two experiments are 330 ns and 288 ns, respectively. The auxiliary qubit $Q_2$ is used to measure the resonator photon number. After the displacement operation pulse, we bias $Q_2$ to the resonator frequency for a given time $\tau$ and then deduce the photon number distribution of the resonator according to the result of the Rabi oscillation. Finally, a multiplexing tone is applied to the readin line to simultaneously measure two qubits’ states.
S4. CHARACTERIZATION OF THE QUBIT-RESONATOR STATE

A. Photon-Number Distribution

All the WF values in the main text are deduced from the photon number distribution. We use the same method as in Ref. \[S4\]. After the Dirac dynamics and displacement operation, we bias the auxiliary qubit $Q_2$ to the frequency of the bus resonator for a given time $\tau$. Then, we bias it to the idle frequency and measure. The excited state population $P^a_\tau(\tau)$ of $Q_2$ is defined as

$$P^a_\tau(\tau) = \frac{1}{2} \left[ 1 - P^a_\tau(0) \sum_{n=0}^{n_{\max}} P_n e^{-\kappa_n \tau} \cos(2\sqrt{n}\lambda_2 \tau) \right], \quad (S15)$$

where $P_n$ denotes the photon-number distribution probability, $n_{\max}$ is the cutoff of the photon number, and $\kappa_n = n^l/T_{1,p} \ (l = 1)$ [S5–S9] is the empirical decay rate of the $n$-photon state, $\lambda_2$ is the coupling strength between auxiliary qubit $Q_2$ and resonator. It is worth noting that due to the finite detuning $\Delta = 2\pi \times 470$ MHz between the auxiliary qubit $Q_2$ and bus resonator $R$, as the number of photons increases, ancilla qubit $Q_2$ inevitably interacts with the resonator during the Dirac dynamics, the ground state population $P^a_\tau(0)$ may not be 1 at $\tau = 0$. Referring to [S5], the potentially slight excitation can be ignored and thus $P^a_\tau(0)$ is introduced to make the fit more accurate. In the experiment, $P^a_\tau(0) \leq 0.05$, $\tau$ is taken every 2 ns from 0 to 200 ns. We use the least square method to fit the experimental data according to Eq. (S15) to obtain the actual photon number distribution and further obtain the value of the WF. However, with the further increase of the photon number, the entanglement between the auxiliary qubit $Q_2$ and the resonator can not be ignored, and the photon number and its distribution in the resonator can not be obtained accurately. It is also worth noting that although ancilla qubits can be biased further down to increase the detuning in the Dirac dynamics, the performance of the qubits deteriorates and the frequency changes more widely, affecting the measurement results. Ergo, the photon number capability of the present system is about 20, resulting in the boundary of phase-space in the main text.

B. Wigner Matrix Elements

The WF is given by

$$W(x, p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n P_n(\gamma), \quad (S16)$$

$$P_n(\gamma) = \langle n | D(-\gamma) \rho D(\gamma) | n \rangle, \quad (S17)$$

where $x = \sqrt{2} \text{Re}(\gamma)$, $p = \sqrt{2} \text{Im}(\gamma)$, $D(\gamma) = \exp(\gamma a^* - \gamma^* a)$. The photon number distribution $P_n$ is inferred by the Rabi oscillation signal [S9]. And then we calculate the Wigner matrix according to Eq. (S16). We adjust the Wigner tomography pulse based on $\gamma$ to realize the measurement of WF values at different positions in phase-space. The WF conditional on test qubit states $|e\rangle$ and $|g\rangle$ are given by

$$W_{e(g)}(x, p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n P_{n}^{e(g)}(\gamma), \quad (S18)$$

$$P_{n}^{e(g)}(\gamma) = \frac{1}{P_k} \langle n | \otimes \langle e(g) | D(-\gamma) \rho D(\gamma) | e(g) \rangle \otimes | n \rangle, \quad (S19)$$

with $P_k$ being the population of the test qubit $Q_1$ in $|k\rangle$ state at time $t$.

In this way, we can get all the information about the resonator state. Then, we use the CVX toolbox based on MATLAB [S10] to reconstruct the corresponding density matrix $\rho_k$ to calculate the average position $\langle x \rangle$ and average momentum $\langle p \rangle$ of the resonator.

As mentioned in the main text, a larger number of photons requires Wigner tomography within a larger phase-space region. Due to the hardware-limited pulse amplitude, we can increase the size of $|\gamma|$ by increasing the length of the tomography pulse, at the cost of increasing the dispersive interaction time. Considering the finite detuning between the auxiliary qubit and the resonator, our ZB simulation ends at 330 ns. With the improvement of the hardware, such as larger pulse amplitude, or better qubits performance, we could simulate the ZB behavior within a longer time scale.
S5. CORRECTION OF PHASE-SPACE ROTATIONS DUE TO STARK SHIFTS

After reconstructing the density matrix $\rho_k$ from the Wigner tomography $W_k$, we add a rotation operator on the corresponding density matrix numerically. The rotation operator is defined as

$$U_k = \exp \left[ -i(\theta_{k,0} + \theta_k t_f) a\dagger a \right],$$  

(S20)

$$\rho'_k = U_k \rho_k U_k^\dagger, \quad (k = e, g)$$  

(S21)

where $\theta_{k,0}$ is used to counteract the resonator states rotation due to the initial state preparation when the test qubit in $|k\rangle$ state, $\theta_k$ cancels out the rotation caused by the difference between the experimental frame and the effective Hamiltonian frame in Eq. (S9), $t_f$ is the end time of each simulation, and $\rho'_k$ is the resonator density matrix after rotation when the test qubit in $|k\rangle$ state. In the Klein tunneling simulation experiment, because the initial state of the resonator is a vacuum state, we set $\theta_{k,0} = 0$. Here we choose $(\theta_{e,0}, \theta_{g,0}, \theta_e, \theta_g, t_f) = (0.260, 0.045, 1.510, 1.217, 330 \text{ ns})$ in the ZB simulation and $(\theta_{e,0}, \theta_{g,0}, \theta_e, \theta_g, t_f) = (0, 0, 1.402, 1.162, 280 \text{ ns})$ in the Klein tunneling simulation. Note $\theta_e$ and $\theta_g$ are slightly different because the dispersive interaction during Wigner tomography (time scale 60 ns, causing phase difference $2\lambda^2/\Delta \times 60 \text{ ns} \sim 0.3$). While both $\theta_e$ and $\theta_g$ are around 1.3, offsetting the resonator phase induced by the frame difference, and therefore demonstrating that the rotation of our data is reasonable. The unconditional resonator WF is obtained by adding two rotated WFs according to the corresponding qubits state population:

$$W'(t) = P^e_k W'_e(t) + P^g_k W'_g(t),$$  

(S22)

where $W'_k(t)$ is calculated from $\rho'_k$. The complete Wigner data are shown in Fig. S10 for ZB simulation and Fig. S11 for Klein tunneling simulation. For the selection of the time point, we try to make the population of states $|e\rangle$ and $|g\rangle$ of the test qubit close to 0.5, making the normalized data of the auxiliary qubit accurate and convenient to fit the photon number distribution of the resonator.

![Fig. S10. Wigner functions for Zitterbewegung.](image-url)

FIG. S10. Wigner functions for Zitterbewegung. The WFs correlated with the basis states $|g\rangle$ and $|e\rangle$ of the test qubit displayed in (a) and (b), respectively, and (c) show the result irrespective of the test qubit’s state. These five columns show the WFs at different times: 0, 90, 178, 240, and 330 ns.

S6. DISCRIMINATION BETWEEN POSITIVE AND NEGATIVE COMPONENTS IN KLEIN TUNNELING

In the Klein tunneling simulation, we need to show the measured position average $\langle x \rangle$ and momentum average $\langle p \rangle$ for the positive and negative wavepackets. Here, we simply use $x = 0$ as a boundary to distinguish positive and
negative wavepackets. The measured $\langle x \rangle$ and $\langle p \rangle$ for positive wavepackets can be calculated by

\[
\langle x \rangle = \frac{\int_{t_1}^{t_2} \int_{0}^{+\infty} x W(x,p) \, dx \, dp}{\int_{t_1}^{t_2} \int_{0}^{+\infty} W(x,p) \, dx \, dp},
\]

(S23)

\[
\langle p \rangle = \frac{\int_{t_1}^{t_2} \int_{0}^{+\infty} p W(x,p) \, dx \, dp}{\int_{t_1}^{t_2} \int_{0}^{+\infty} W(x,p) \, dx \, dp},
\]

(S24)

where $W(x,p)$ is shown in Fig. S11(c). At $t = 0$ and 76 ns, the positive and negative wavepackets cannot be fully distinguished, so the data for these two mean positions $\langle x \rangle$ are abandoned in Fig. 3(d) of the main text. The negative wavepacket can be obtained in the same way,

\[
\langle x \rangle = \frac{\int_{t_1}^{t_2} \int_{-\infty}^{0} x W(x,p) \, dx \, dp}{\int_{t_1}^{t_2} \int_{-\infty}^{0} W(x,p) \, dx \, dp},
\]

(S25)

\[
\langle p \rangle = \frac{\int_{t_1}^{t_2} \int_{-\infty}^{0} p W(x,p) \, dx \, dp}{\int_{t_1}^{t_2} \int_{-\infty}^{0} W(x,p) \, dx \, dp}.
\]

(S26)

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[S1] C. Song et al., “Continuous-variable geometric phase and its manipulation for quantum computation in a superconducting circuit,” Nat. Commun. 8, 1061 (2017).
[S2] W. Ning et al., “Deterministic entanglement swapping in a superconducting circuit,” Phys. Rev. Lett. 123, 060502 (2019).
[S3] W. Liu et al., “Synthesizing three-body interaction of spin chirality with superconducting qubits,” Appl. Phys. Lett. 116, 114401 (2020).
[S4] R.-H. Zheng et al., “Emergent Schrödinger cat states during superradiant phase transitions,” arXiv:2207.05512 (2022).
[S5] M. Hofheinz et al., “Synthesizing arbitrary quantum states in a superconducting resonator,” Nature 459, 546–549 (2009).
[S6] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, D. J. Wineland, “Generation of nonclassical motional states of a trapped atom,” Phys. Rev. Lett. 76, 1796–1799 (1996).
[S7] D. Leibfried, R. Blatt, C. Monroe, D. Wineland, “Quantum dynamics of single trapped ions,” Rev. Mod. Phys. 75, 281–324 (2003).
[S8] D. Lv et al., “Quantum simulation of the quantum Rabi model in a trapped ion,” Phys. Rev. X 8, 021027 (2018).
[S9] M.-L. Cai et al., “Observation of a quantum phase transition in the quantum Rabi model with a single trapped ion,” Nat. Commun. 12, 1126 (2021).

[S10] M. Grant, S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.0 beta. http://cvxr.com/cvx, September (2013).