Structure of passive states and its implication in charging quantum batteries

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Passive states are those which are energetically stable under the cyclic Hamiltonian, whereas thermal states are the subclass which are stable even in the asymptotic limit. In this article we study their geometrical state space structure and focus on their operational implementation. First we characterize the set of passive states which is convex, compact and forms a simplex polytope for non-degenerate Hamiltonian. This property enables us to design a complete set of witnesses to detect them. All the thermal states lie inside the set except the ones corresponding to $T = 0$ and $T = \infty$, which lie on the vertex. Structural instability of a passive state assures the existence of a natural number $n$, for which $\rho \otimes n \geq 1$ can charge a quantum battery (although $n$ copy cannot), in contrast to a thermal state. We also show that general ordering of the passive states on the basis of their charging capabilities is not possible. However, in some special cases, the majorization criterion gives sufficient order in the charging and discharging scenarios. On the other hand, the charging order among thermal states is very precise; the hotter one is better.

I. INTRODUCTION

A state $\rho$ is called passive if no work (ergotropy) can be extracted from it cyclically, or in other words, it is the lowest energetic state under any unitary evolution $U$, i.e., $Tr(\rho H) \leq Tr(U \rho U^\dagger H)$, where $H$ is the corresponding Hamiltonian of the system [1–3]. So a passive state necessarily follows the criterion that (i) $[\rho, H] = 0$ and (ii) $\epsilon_i > \epsilon_j$ implies $q_i \leq q_j \ \forall i, j$ where, $\epsilon$'s and $q$'s are the eigenvalues of the Hamiltonian and the system respectively. Although a single copy of the state cannot produce any work, but multiple copies can. However, if it retains passivity even in the asymptotic limit, then the state is called completely passive. These states are the only structurally stable states (satisfies stability condition which means that, equal energetic states have equal probability [2, 4]) and take the Gibbsian form associated with a virtual temperature $\beta \geq 0$, whereas in a single passive state, different $\beta_i \geq 0$ can be associated with the different energy levels. But it is not very intuitive why the passive and not the completely passive states are able to produce work from multiple copies under the reversible (unitary) operations.

Various physically motivated ideas have been provided in recent years to bring out distinction between them on the basis of their operational capabilities. For example, for every passive state, there exist some copies for which some $\beta_i$ of the composite system would be negative. This is not the case for a completely passive state, where a unique $\beta \geq 0$ exists, leading to the concept of temperature in the asymptotic limit [5]. In another study [6], a weaker cyclic process has been considered, based on which the passive state’s energy can be decreased further and the only states incapable of doing this are in the Gibbsian form. A large dimensional ancilla has been considered as a catalyst to show the energetic instability of the single passive states. An alternative definition of the thermal state is that it is the lowest energetic state under constant entropy or the highest entropic state under constant energy. On the contrary, there exists a unique passive state which is the highest energetic state under constant entropy or the lowest entropic state under constant energy called the maximum energetic passive state [7].

In this article, we have characterized the structure of the passive state space geometrically. Since these are diagonal in the energy basis, they can be considered as probability vector of $\mathbb{R}^d$. Passive states forms a convex polytope which is compact in nature and for the non-degenerate Hamiltonian it is simplex, where all the extreme points are generally passive, except two $(T = 0, \infty)$ which are thermal. If there exists degeneracy in the ground state, then $T = 0$ lies on the boundary instead of the vertex. All other thermal states lie inside the polytope and not on the boundary. Since the set is convex we are able to provide a witness operator for the diagonal states to detect an active state which can provide work under a unitary evolution.

We further provide an operational characterization of passive and completely passive states. We discuss how a Quantum Battery (QB) can be charged up via the passive states by energy conserving unitary. Here we have considered the most elementary QB i.e., a qubit. Quantum batteries were first introduced in [8], followed by many articles [9–14] regarding enhancement of charging power [15–17], work extraction [18, 19] and advantage in multiple usage of the battery incorporated with entanglement [20–22]. A battery can simply be charged up by only using the field energy where the unitaries are controlled by an arbitrary field parameter which acts cyclically for finite time. But under these circumstances, a passive battery cannot be charged and so one needs to consider some ancillary system. If we take arbitrary ancilla, then it can be trivially done by just an energy conserving swap operation. However, here our main aim is to consider passive states as ancilla since they are by themselves not useful at all. And in this charging scenario we have con-
sidered energy conserving unitary such that the supplied energy comes only from the passive ancilla states. It has been shown explicitly that if \( n \) (max) copies of a passive state fail to charge a \( QB \), then \((n+1)\) copies of it can. On the other hand, if a single thermal state cannot charge a \( QB \), then even infinite copies (thermal bath) cannot help. The reason behind this is the structural instability of passive states for the composite system.

It has been observed that the state having higher ground population is less useful in charging the \( QB \) state \(|0\>_B|0\>\). For an arbitrary battery, we derive a condition on the charger state. If we consider a thermal charger it should be at a higher temperature than the \( QB \). An ordering among the thermal chargers on the basis of their charging capabilities is possible; the hotter one is the better one. For the passive but not completely passive states, this kind of ordering is never possible for all \( QB \), but for some special kind of passive states, the majorization criteria sufficiently orders them. We also provide the activation criterion of a charger for the given battery such that the battery can be made useful in work extraction. Lastly, we focus on the discharging of a battery through the passive state using arbitrary unitary. Again, the majorization criterion provides a sufficient condition for discharging, which is just the opposite of the charging criterion.

II. FRAMEWORK

In this section we briefly introduce some basic thermodynamical concepts which will be used later.

A. Active states

Let \( \mathcal{H} \) be the Hilbert space of the system and \( B(\mathcal{H}) \) be the set of bounded operators on \( \mathcal{H} \).

Definition 1 [3]: A state \( \sigma \in \mathcal{H} \) is said to be active under the Hamiltonian \( H \) if \( \max_{U \in B(\mathcal{H})} [Tr(\sigma H) - Tr(U \sigma U^\dagger H)] > 0 \).

For a \( d \) dimensional state \( \sigma = \sum_{k=1}^{d} a_k |k\><k| \), with the given Hamiltonian \( H = \sum_{i=1}^{d} \epsilon_i |\epsilon_i\><\epsilon_i| \), the maximum amount of extractable work will be \( W = \sum_{i,k} \epsilon_k a_k (|\langle k|\epsilon_i\rangle|^2 - \delta_{ik}) \), where \( a_k \geq a_{(k+1)} \), \( \forall k \) and \( \epsilon_i \leq \epsilon_{(i+1)} \), \( \forall i \). So an active state is one from which we can always extract work.

B. Passive and Thermal states

Definition 2 [2]: A state \( \rho \in \mathcal{H} \) commuting with the Hamiltonian \( H \) is said to be passive under \( H \) if \( \max_{U \in B(\mathcal{H})} [Tr(\rho H) - Tr(U \rho U^\dagger H)] = 0 \).

Generalizing this, an \( n \)-copy passive state is one which remains passive under \( n \) copies of the state but becomes active under \((n+1)\) copies, for all possible \( U \in B(\mathcal{H}^\otimes n+1) \). A passive state should be diagonal in energy basis. Moreover the population in energy levels should be assigned in non-increasing order with the increase of energies. So the state \( \rho = \sum_{i=1}^{d} p_i |i\><i| \) with the Hamiltonian \( H = \sum_{i=1}^{d} \epsilon_i |i\><i| \) is said to be passive iff \( \forall i, \epsilon_{i(i+1)} \geq \epsilon_i \) and \( p_{i+1} \leq p_i \). It is evident from the above expression that maximum work \( W \) that can be extracted from the passive state is zero.

Definition 3 [2]: A state \( \rho_{cp} \in \mathcal{H} \) commuting with the Hamiltonian \( H \) is said to be completely passive under \( H \) iff \( \max_{U \in B(\mathcal{H}^\otimes n)} [Tr(\rho_{cp} H^\otimes n) - Tr(U \rho_{cp} U^\dagger H^\otimes n)] = 0, \forall n \in \mathbb{Z}_+ \).

In other words, the state \( \rho_{cp} = \sum_{i=1}^{d} \lambda_i |i\><i| \) is completely passive under the Hamiltonian \( H = \sum_{i=1}^{d} \epsilon_i |i\><i| \), iff for any two sets of non-negative integers \( N = [n_i] \) and \( M = [m_i] \),

\[
\sum_{i=1}^{d} n_i = \sum_{i=1}^{d} m_i
\]

and,

\[
\sum_{i=1}^{d} n_i \epsilon_i \leq \sum_{i=1}^{d} m_i \epsilon_i
\]

implies,

\[
\prod_{i} \lambda_i^{n_i} \geq \prod_{i} \lambda_i^{m_i}.
\]

So this state is passive under any number of copies.

Definition 4 [2]: A state \( \tau \in \mathcal{H} \) is said to be thermal under the Hamiltonian \( H \) iff \( S(\tau) > S(\tau^*), \forall \tau^* \in \mathcal{H} \), such that, \( Tr(\tau^* H) = Tr(\tau H) \).

Alternatively, a state \( \tau \in \mathcal{H} \) is said to be thermal under the Hamiltonian \( H \) iff \( Tr(\tau H) < Tr(\tau^* H), \forall \tau^* \in \mathcal{H} \), such that, \( S(\tau^*) = S(\tau) \).

So a thermal state \( \tau \) corresponding to a given state \( \rho \) is an equal energetic, maximum entropic state or an equal entropic, minimum energetic state.

For a passive state \( \rho = \sum_{i=1}^{d} p_i |i\><i| \), one can rewrite \( p_{i+1} = p_i \exp[-\beta_i (\epsilon_{i+1} - \epsilon_i)] \). When all \( \beta_i \)'s are same, one can associate a temperature \( \beta = \frac{1}{k_B T} \) and the state is said to be thermal, i.e., it follows Gibbs distribution. This is a completely passive state for all non-degenerate Hamiltonians.
III. STRUCTURE OF PASSIVE STATES

It is known that passive states are those which are energetically stable under a unitary operation. Here we would like to study in detail their structure. Below we will show that they form a convex and compact set. The complementary states, namely active states, are thermodynamically important for work extraction and can be detected by activity witness operators.

A. Convexity of passive states

A set $S$ is said to be convex iff $\forall x, y \in S, px + (1-p)y \in S$, where $0 \leq p \leq 1$.

Lemma 1: The set of all passive states for a fixed Hamiltonian forms a convex set.

Proof: Let $\rho = \sum_{k=1}^{d} r_k |k\rangle \langle k|$ and $\sigma = \sum_{k=1}^{d} q_k |k\rangle \langle k|$ be two different passive states in $\mathcal{H}$ under the Hamiltonian $H = \sum_{k=1}^{d} \epsilon_k |k\rangle \langle k|.$

From passivity of $\rho$ and $\sigma$ we can write, $r_{k+1} \leq r_k$ and $q_{k+1} \leq q_k$, $\forall k \in 1, 2, \cdots, d - 1$ with $\epsilon_{k+1} \geq \epsilon_k$.

Therefore, $pr_{k+1} + (1-p)q_{k+1} \leq pr_k + (1-p)q_k$, $\forall k \in 1, 2, \cdots, d - 1$ where $0 \leq p \leq 1$.

Hence, the passive states form a convex set with a fixed Hamiltonian.

B. Thermal states (except $T = 0, \infty$) do not lie on the boundary of the convex set

In the Appendix, we show that for a general $d$ dimensional system governed by a $d$ dimensional Hamiltonian, $d$ number of extreme points would exist. Among them, $e_1$ and $e_d$ are the only two thermal states of temperature $T = 0(\beta = \infty)$ and $T = \infty(\beta = 0)$ respectively.

Lemma 2: There does not exist any thermal state except $T = 0$ and $T = \infty$ which lie on the boundary of the convex set $S$.

Proof: In the $d$ dimensional passive state, extreme points are represented as $\{e_j = (\frac{1}{j}, \frac{1}{j}, \cdots, \frac{1}{j}, 0)\}$.

Let a general thermal state of inverse temperature $\beta$ lie on the $(d - 1)$ dimensional boundary, which can be constructed by the convex combination of the $(d - 1)$ number of extreme points i.e.,

$$
\tau_\beta(t_1, \cdots t_d) = \sum_{j=1, j \neq i}^{d} p_j e_j \text{ such that } \sum_{j=1, j \neq i}^{d} p_j = 1,
$$

where $i^{th}(t_i)$ and $(i+1)^{th}(t_{i+1})$ element would be equal to $\sum_{j>i}^{d} \frac{p_j}{j}$.

So

$$
e^{-\beta e_i} = \frac{e^{-\beta e_i}}{z} = \frac{e^{-\beta e_{i+1}}}{z}.
$$

Since the Hamiltonian is non-degenerate, $\epsilon_i \neq \epsilon_{i+1}$ and the only solutions are $\beta = 0, \infty$. So all other thermal states do not lie on the boundary of the convex set of passive states.

C. Compactness of the set of all passive states

A set is said to be compact iff it is closed and bounded. To show that the set is closed, its closure should coincide with the set itself.

Lemma 3: The set of all passive states for a given Hamiltonian are closed.

Proof: In the convex set of $d$-dimensional passive states, there will be exactly $d$ number of extreme points of the polygon. If $e_i \forall i \in [1, d]$ are the extreme points for the set of passive states, then the boundary points will be formed by any $(d - 1)$ number of extreme points. These points are passive in nature, and so their convex combination which lie on the boundary will also be passive. Hence, the set is closed in nature.

Lemma 4: The set of all passive states for a given Hamiltonian will be bounded.

Proof: For the above $d$-dimensional convex and closed set $S$ the extreme points are, $e_1 = (1, 0, 0\ldots 0)^T, e_2 = (\frac{1}{2}, \frac{1}{2}, 0\ldots 0)^T, \ldots, e_d = (\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d})^T$.

We define $M = \max(x \in S) |x - e_1| = \sqrt{\sum_{k=1}^{d} |p_k^e - p_k^c|^2}$.

One can show that the maximum value $M = \sqrt{\frac{d-1}{d}}$ is attained for $x = e_d$. Evidently, $M(x, y) = \sqrt{\sum_{k=1}^{d} |p_k^x - p_k^y|^2}, \forall x, y \in S$, will be bounded by $M$. Hence, the set is bounded.

The geometrical structure of passive states in more detail has been given in the Appendix.

D. Activity witness

The state outside this set $S$ is called active, useful for work extraction under unitary. Since the state space of the passive state is convex and compact we always can give some witnesses to detect them.

Theorem 1: For any active state $\rho(\notin S)$, diagonal in energy Eigen basis, (where, $S$ is the set of all passive states for a given Hamiltonian), $\exists$ a Hermitian operator
proof: From Lemma 1, 3 and 4, we can conclude that the set of all passive states, for a given Hamiltonian, is convex and compact in nature, which as a consequence of Hahn-Banach theorem, assures the existence of a Hermitian operator $W$, namely, a witness operator for activity.

The passive states in any arbitrary dimension $d$, for a given Hamiltonian, will form a polytope $P_d \subset \mathbb{R}^d$, which will lie on the $(d - 1)$ dimensional hyperplane in $\mathbb{R}^d$ due to the probability constraint. The facets of this polytope will behave as witness operators for the active states diagonal in the energy basis. In general, for the set of $d$-dimensional passive states, there will be $(d + 1)$ number of witness operators which are $d \times d$ matrices denoted as, $[W_0, W_i(i+1)]$, $\forall i \in 1, 2, ..., d$. Among these, $W_0$ will be a trivial one, with $[W_0]_{d,d} = 1$, and 0 otherwise.

Now, a general $W_i(i+1)$ will be the witness operator with $[W_i(i+1)]_{i,i} = 1$, $[W_i(i+1)]_{(i+1),i} = -1$ and 0 otherwise $\forall i \in 1, 2, ..., n$.

IV. CHARGING OF A QUANTUM BATTERY THROUGH PASSIVE STATE

In general, charging could be done through an arbitrary unitary, where the corresponding field supplies the energy. However, instead we have studied how the energetically passive states could boost up the quantum batteries in the finite dimensional case. To exploit the passive states, we have considered a joint unitary which is energy conserving.

For simplicity we have taken a completely uncharged battery state $|0\rangle_B$. Although by an arbitrary unitary the battery states can be charge maximally to $|1\rangle_B$, but here the assistance of passive states could impose some restriction from practical point of view due to energy conserving unitary. Throughout the process we will take the Hamiltonian of the battery as $H_B = |1\rangle_B \langle 1\mid$ and the Hamiltonian of the passive state (Charger) as $H_C = \sum_{i=0}^{d-1} \hat{v}_i |i\rangle \langle i|$. Let the initial state of the $QB$ be represented by the probability vector $\rho_B \equiv (1, 0)^T$ and the $d$ dimensional passive state be given by the probability vector $\rho_C \equiv (q_0, q_1 \ldots q_{d-1})^T$. The combined initial state can be written as (it is a conventional matrix representation and not an actual tensor product)

$$\rho_B \otimes \rho_C = \begin{pmatrix} q_0 & 0 \\ q_1 & 0 \\ q_2 & 0 \\ \vdots & \vdots \\ q_{d-1} & 0 \end{pmatrix},$$

where the sum of the columns determine the battery state while sum of the rows give the charger state. Off diagonal elements having the same energy can be interchanged under the energy conserving unitary which is the only allowed unitary operation in this scenario. So the final state of the battery is given by

$$\rho_B = (q_0, 1 - q_0)^T.$$ 

Now let us consider another charger having state $\rho_C' \equiv (q_0', q_1', \ldots, q_{d-1}')^T$. The majorization criterion [23] gives a sufficient condition of a better charger for the given battery state $|0\rangle_B$ i.e., if $\rho_C \prec \rho_C'$, then $q_0 \leq q_0'$ which implies that the unprimed charger is able to charge more than the primed one. So one can say that a more disordered state is more useful in this scenario.

If we consider an arbitrary passive $QB$ then the following theorem gives the charging condition of a passive charger.

**Theorem 2:** If the passive $QB$ state is $\rho_B = (p_0, p_1)^T$, then a passive charger $\rho_C = (q_0, q_1 \ldots q_{d-1})^T$ is able to charge the battery if and only if $\frac{p_0}{p_1} > \min_i \{ \frac{q_i}{q_{i+1}} \}$, $\forall i \in [0, d-2]$.

**Proof:** The joint state of the battery and the charger is
\[ \rho_B \otimes \rho_C \equiv \begin{pmatrix} p_{0q_0} & p_{1q_0} \\ p_{0q_1} & p_{1q_1} \\ p_{0q_2} & p_{1q_2} \\ \vdots & \vdots \\ p_{0q_{d-1}} & p_{1q_{d-1}} \end{pmatrix}. \] (2)

Since the pair \( p_{0q_k} \) and \( p_{1q_k-1} \), \( \forall k \in [1, d - 1] \) are the coefficients of equal energetic states, they can be interchanged by the energy conserving unitary. If any one of the pairs follow \( p_{0q_k} > p_{1q_k-1} \), then charging is possible. This leads to the necessary condition for charging: \( \frac{p_{0q}}{p_{1q}} \geq \min \{ \frac{q}{q_i} \}, \forall i \in [0, d - 2] \).

The probability of a \( d \) dimensional passive state can be written as \( p_k = e^{\beta_{k+1} p_{k+1}} \), \( \forall k \in [0, d - 2] \). For a general passive state, the set \( \{ \beta_i \}_{i=1}^{d-1} \) can take any positive value without maintaining any particular order. Below we provide ordering between the particular type of passive states on the basis of charging.

**Corollary 1:** An arbitrary passive QB is characterized by inverse temperature \( \beta_h \) and the charging states (\( \rho_C \) and \( \rho'_C \)) have been taken such that \( \beta_h > \max \{ \beta_i \} \) and \( \beta_h > \max \{ \beta'_i \} \). So if \( \rho'_C < \rho_C \) then \( \rho'_C \) is a better charger than \( \rho_C \).

**Proof:** Since these states are able to charge, they must satisfy\( \frac{p_{0q}}{p_{1q}} \geq \min \{ \frac{q}{q_i} \}, \forall i \in [0, d - 2] \). If a given charging state satisfies \( \frac{p_{0q}}{p_{1q}} \geq \max \{ \frac{q}{q_i} \} \), then all the equal energetic pairs in matrix (2) would swap their positions and the resultant battery state would be given by

\[ \tilde{\rho}_B(q) = (p_0 - \delta(q), p_1 + \delta(q)), \] (3)

where \( \delta(q) = p_0 \sum_{i=1}^{d-1} q_i - p_1 \sum_{i=0}^{d-2} q_i \). If \( \rho'_C < \rho_C \) then \( \delta(q') \geq \delta(q) \) which implies \( Tr(\tilde{\rho}_B(q') H_B) \geq Tr(\tilde{\rho}_B(q) H_B) \).

For the completely passive or thermal states, all \( \beta \) are equal and hence charging of a battery is possible only when the battery state is colder than the charger i.e., \( \beta_b > \beta \).

**Corollary 2:** A hotter thermal state is a better charger than a colder one.

**Proof:** Let us consider a \( d \) dimensional \( \beta \)-thermal charger which can be written as

\[ \tau_C = \begin{pmatrix} q_0 \\ q_0 e^{-\beta} \\ q_0 e^{-2\beta} \\ \vdots \\ q_0 e^{-(d-1)\beta} \end{pmatrix}, \] (4)

Probability constraint gives \( q_0 = \frac{1}{1 + x + x^2 + \cdots + x^{d-1}} \) and \( q_{d-1} = \frac{1}{1 + y + y^2 + \cdots + y^{d-1}} \) where \( y = \frac{1}{x} = e^\beta \). If \( T < T' \) which implies \( \beta > \beta' \leftrightarrow y > y' \leftrightarrow x < x' \) gives \( q_0 > q'_0 \) as well as \( q_{d-1} < q'_{d-1} \).

Under interaction, the battery state moves from \( \rho_B = (p_0, p_1) \) to \( \tilde{\rho}_B = (p_0 - \delta(q), p_1 + \delta(q)), \) where \( \delta(q) = (p_0 - p_1) + (p_1 q_{d-1} - p_0 q_0) \). From the above it is clear that the thermal charger having higher temperature, boosts the battery’s energy more i.e., \( Tr(\tilde{\rho}_B(q) H_B) < Tr(\tilde{\rho}_B(q') H_B) \).

**Ordering of the passive states:** Here in this section, the ordering between the passive states on the basis of their charging capabilities is investigated. We have seen that if the battery state is \( |0\>_B |0\>_C \), the charger having lower ground state population (i.e., \( q_0 \)) can charge up more. But this parameter alone does not specify complete order. Moreover, if we are restricted only to the thermal states, then the hotter one is the better charger than the colder one for all passive QB. But in general there does not exist any function \( f : \mathbb{R}^d \to \mathbb{R} \) on the charging states which is able to order them on the basis of charging capability for all the battery states simultaneously. Here we provide an example of chargers and batteries for which individual charger is better for the individual battery. Let \( \rho_C = (0.5, 0.4, 0.1)^T \) and \( \rho'_C = (0.5, 0.3, 0.2)^T \). If the battery state is \( \rho_B = (0.6, 0.4, 0.1)^T \) then the excited state probability is increased by \( \delta(q) = 0.04 \) and \( \delta(q') = 0.24 \) respectively. However, if the battery state is \( \rho_B = (0.8, 0.2, 0.0)^T \), then the excited state probability is increased by \( \delta(q) = 0.22 \) and \( \delta(q') = 0.24 \) respectively. From this we can conclude that there does not exist any function which can characterize the passive states on the basis of charging capability for all passive QB states simultaneously.

Now we address the question that if a single copy of a charger is unable to charge a QB, whether multiple copies can? Such possibilities arise since adding \( n \) copies creates more scope to swap between the equal energetic states by using joint unitary \( U_{B_{C_1 \cdots C_n}} \neq U_B \otimes U_{C_1} \otimes \cdots \otimes U_{C_n} \).

**Corollary 3:** If a thermal state cannot charge a QB, then it’s multiple copies also cannot.

**Proof:** If the charging is not possible by a thermal state, it means the probability of the battery state satisfies \( \frac{p_0}{p_1} \leq \frac{q}{q_i} = e^\beta \), where \( (p_0, p_1) \) is the spectrum of the battery state \( \rho_B \) and \( \beta \) is the virtual temperature of the corresponding thermal charger \( \tau_C \). So we will prove that if the single copy of a thermal state cannot charge, it’s multiple copies also cannot, i.e.,

\[ \begin{align*}
Tr(\rho_B H_B) &= \max U \{ Tr[\tau_C \{ U (\rho_B \otimes \tau_C^0) U^\dagger \} H_B] \} \\
&= \max U \{ Tr[\tau_C \{ U (\rho_B \otimes \tau_C^0) U^\dagger \} H_B] \}, \end{align*} \]

where, \( U \) is energy conserving unitary i.e., \( [U, H_{BC}] = 0 \).

For a thermal state the probability of energy \( \epsilon \), is given by \( T_\epsilon = t_0 e^{-\beta (\epsilon - \epsilon_0)} \), where \( t_0 \) and \( \epsilon_0 \) is the ground state
probability and the corresponding energy respectively. One of the basic features of the thermal state is that the occupying probability for equal energetic Eigen states is equal. Since \( \tau_C \) is a thermal state, it’s \( n \) copy also remains thermal at same temperature where probability of the ground state can be defined as \( t_0 = q_0^n \). Now the probability ratio for the \( r \) and \( r+1 \) energy levels is given by

\[
\frac{t_r}{t_{r+1}} = \frac{t_0 e^{-\beta t_r}}{t_0 e^{-\beta t_{r+1}}} = e^\beta,
\]

which satisfies the no charging condition. Thus charging is not possible for the given battery \( \rho_B \), and even multiple usage of the thermal state cannot enhance the battery energy under joint unitary.

**Theorem 3:** If \( Tr\{Tr_C\{U(\rho_B \otimes \rho_C)U^\dagger\}H_B\} = Tr(\rho_B H_B) \) then \( \exists n \in \mathbb{Z}_+ \) s.t. \( Tr\{Tr_C\{U(\rho_B \otimes \rho_C^{\otimes n})U^\dagger\}H_B\} = Tr(\rho_B H_B) \) but \( Tr\{Tr_C\{U(\rho_B \otimes \rho_C^{\otimes (n+1)})U^\dagger\}H_B\} > Tr(\rho_B H_B) \) where, \( [U, H_{BC}] = 0 \).

**Proof:** Let us consider a \((d+1)\) dimensional passive state \( \rho_C \equiv (q_0, q_1, \ldots, q_d)^T \), which is unable to charge up the passive battery \( \rho_B \equiv (p_0, p_1)^T \) i.e.,

\[
\frac{p_0}{p_1} < \min_i \{ \frac{q_i}{q_{i+1}} \}; \quad \forall i \in [0, d-1].
\]

If we consider the \((r+1)\) copy of the charging state the probability ratio of \( kr \) and \( kr+1 \) energy levels is given by

\[
\frac{q_0}{q_1} \times \left( \frac{q_{k-1} q_{k+1}}{q_k^2} \right)^r\]

where \( k \in [1, d-1] \). Population ratio of the next consecutive energy levels is given by

\[
\frac{q_1}{q_2} \times \left( \frac{q_k^2}{q_{k-1} q_{k+1}} \right)^r\]

Since it is a passive state, if \( q_k^2 > q_{k-1} q_{k+1} \) then for some finite value of \( r \) Eq. [6] would satisfy the charging condition i.e., \( \frac{p_0}{p_1} > \frac{q_0}{q_1} \times \left( \frac{q_{k-1} q_{k+1}}{q_k^2} \right)^r \). But if \( q_k^2 < q_{k-1} q_{k+1} \) then Eq. [7] would satisfy the charging condition for some other value of \( r \) i.e., \( \frac{p_0}{p_1} > \frac{q_1}{q_2} \times \left( \frac{q_{k-1} q_{k+1}}{q_k^2} \right)^r \).

Therefore any passive battery can be charge up by the usage of multiple copies of the passive charger, whereas if the single copy of the completely passive state cannot charge then it’s multiple usage also can not.

A passive battery would be called active only when population of the excited state becomes more than the ground state. Then one can extract work from it only through a unitary operation. Even if a passive state has charging capability (by Theorem 2) there is no guarantee that it would make the battery active.

**Theorem 4:** To activate a qubit \( QB \), the condition for a 3d charger is given by

\[
\frac{p_0}{p_1} < \max\left\{ \frac{1 - 2q_0}{1 - 2q_1}, \frac{1 - 2q_0 - 2q_1}{1 - 2q_2} \right\}. \quad (8)
\]

**Proof:** The composite system of battery and charger is

\[
\rho_B \otimes \rho_C \equiv \begin{pmatrix} p_0 & p_1 q_0 & p_1 q_1 & p_1 q_2 \\ p_0 q_0 & p_1 & p_1 q_1 & p_1 q_2 \\ p_0 q_1 & p_1 q_1 & p_1 & p_1 q_2 \\ p_0 q_2 & p_1 q_2 & p_1 q_2 & p_1 \end{pmatrix}.
\]

The charging conditions are (i) \( \frac{p_0}{p_1} > \frac{q_0}{q_1} \) or (ii) \( \frac{p_0}{p_1} > \frac{q_1}{q_2} \). If condition (i) is satisfied, then after the action of energy conserving unitary the probability of the excited state would be

\[
\tilde{p}_1 = p_1 + (p_0 q_1 - p_1 q_0).
\]

Condition of active state gives

\[
p_1 + (p_0 q_1 - p_1 q_0) > \frac{1}{2} \Rightarrow \frac{p_0}{p_1} < \frac{1 - 2q_0}{1 - 2q_1}. \quad (10)
\]

In the same way it can be shown that satisfying condition (ii) gives

\[
\tilde{p}_1 = p_1 + (p_0 q_2 - p_1 q_1),
\]

and activation implies

\[
\frac{p_0}{p_1} < \frac{1 - 2q_1}{1 - 2q_2}. \quad (11)
\]

Simultaneous satisfaction of condition (i) and (ii) makes the state active if

\[
\frac{p_0}{p_1} < \frac{1 - 2q_0 - 2q_1}{1 - 2q_1 - 2q_2}. \quad (12)
\]

From the above equations, a charger would be called an activator for the given passive \( QB \) iff

\[
\frac{p_0}{p_1} < \max\left\{ \frac{1 - 2q_0}{1 - 2q_1}, \frac{1 - 2q_0 - 2q_1}{1 - 2q_2}, \frac{1 - 2q_0}{1 - 2q_1}, \frac{1 - 2q_0 - 2q_1}{1 - 2q_2} \right\}.
\]

This can be easily generalized for a charger of any dimension.

**V. DISCHARGING OF A QUANTUM BATTERY THROUGH THE PASSIVE STATE**

An arbitrary passive \( QB \) cannot discharge or be erased via a field unitary only. Consideration of ancilla is necessary to diminish its energy by redistributing energy and entropy further. Here we focus on the discharging
The best ancilla is $|0\rangle_D |0\rangle$ through which any $QB$ can be discharged completely by applying a swap unitary. But how much discharging is possible in the presence of other passive states? We show below that the ordering among all passive states in the discharging scenario is exactly inverse to that in the special charging case (Theorem 2 and Corollary 1).

**Theorem 4:** A $(d+1)$ dimensional passive discharger $\rho_D$ would discharge a $QB$ if and only if it satisfies $\rho_D \prec \rho_D$. Further if $\rho_D \prec \rho_D$ then $\rho_D$ would be a better discharger than $\rho_D$.

**Proof:** Let an arbitrary passive $QB$ be denoted by $\rho_B \equiv (p_0, p_1)^T$ and a $(d+1)$ dimensional passive discharger by $\rho_D \equiv (q_0 q_1 \ldots q_d)^T$. The composite system is given by

$$\rho_B \otimes \rho_D \equiv \begin{pmatrix} p_0 q_0 & p_1 q_0 \\ p_0 q_1 & p_1 q_1 \\ \vdots & \vdots \\ p_0 q_k & p_1 q_k \\ p_0 q_{k+1} & p_1 q_{k+1} \\ \vdots & \vdots \\ p_0 q_{d-k} & p_1 q_{d-k} \\ p_0 q_d & p_1 q_d \end{pmatrix}. \quad (13)$$

If $p_0 q_d < p_1 q_d$, then there exists some positive integer $k$ such that

$$p_1 q_{k+1} \leq p_0 q_{d-k} < p_1 q_k$$

holds, where $k \in [0, \frac{d}{2}]$ when $d$ is even, or $k \in [0, \frac{d+1}{2}]$ when $d$ is odd. So the composite system evolves to

$$U(\rho_B \otimes \rho_D)U^\dagger \equiv \begin{pmatrix} p_0 q_0 & p_0 q_{d-k} \\ p_0 q_1 & p_0 q_{d-k+1} \\ \vdots & \vdots \\ p_0 q_k & p_0 q_d \\ \vdots & \vdots \\ p_1 q_0 & p_1 q_{d-k} \\ p_1 q_1 & p_1 q_{d-k} \\ \vdots & \vdots \\ p_1 q_k & p_1 q_d \end{pmatrix}. \quad (14)$$

and the final state is given by $\tilde{\rho}_D(q) \equiv (\tilde{\rho}_0, \tilde{\rho}_1)$, where

$$\tilde{\rho}_0(q) = p_0 \sum_{i=0}^{d-k-1} q_i + p_1 \sum_{i=0}^{k} q_i$$

$$= p_0 (1 - \sum_{i=d-k}^{d} q_i) + p_1 \sum_{i=0}^{k} q_i$$

$$= p_0 + p_1 \sum_{i=0}^{d} q_i - p_0 \sum_{i=d-k}^{d} q_i.$$

If $\rho_D < \rho_D$ then $Tr\{\tilde{\rho}_D(q) H_B\} \geq Tr\{\tilde{\rho}_D(q) H_B\}$ which means that the more ordered state is a better discharger.

**VI. DISCUSSION**

In this paper we have shown that the state space of the passive states forms a convex-compact set. For the non degenerate Hamiltonian it is simplex in nature where all the extreme points are passive except at $T = 0$ and $T = \infty$. Furthermore, we have shown that no other thermal state lies on the boundary of the set. Any state outside this set is called active. For the diagonal states, we have given finite number of witness operators to detect them. We have also discussed how passive states can be useful to charge up the quantum batteries, and provided a criterion for it. Under some additional constraints on the charger states, the majorization criterion is able to order them sufficiently on the basis of their charging capabilities, and the maximally mixed state turns out to be the universal charger for the $QB$. In the case of a thermal charger, the hotter one is always able to charge more than the colder one, with the battery having temperature lower than both. However, there does not exist any such order among the passive states for all batteries simultaneously and there cannot exist any function defined on the passive charger which can order them on the basis of charging capability. We have provided an explicit example to support this. Furthermore, we have shown that if a single copy of a thermal state cannot charge a $QB$, then a complete bath of the same also cannot. But due to the structural instability of the passive states, any $(n+1)$ copy of the state is able to charge although $n$ copies cannot. Moreover, we have studied the reverse process, namely, discharging of quantum batteries under an arbitrary global unitary. It turns out that the majorization criterion sufficiently provides order on the discharging capabilities of the passive states.

In this article we have considered only qubit battery but one can generalize it for arbitrary dimension. It would also be interesting to investigate explicitly how multiple copies of the charger would effect the charging of a quantum battery. How many minimum number of copies of a passive state are required to charge a battery, and whether this problem is decidable or not could be a future direction.

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VII. APPENDIX

A. Structure of Passive states

In this section we give a graphical representation of the set of passive states for arbitrary dimension under degenerate as well as non-degenerate Hamiltonian.

1. 2D passive states

a. Non-Degenerate Hamiltonian: A diagonal state \( \rho \) can be defined by \( \rho = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \), \( p_1 + p_2 = 1 \), under the non-degenerate Hamiltonian \( H = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \). Without loss of generality, from now on we will write them by probability vectors. According to the definition of passive states if \( 0 \leq \epsilon_1 \leq \epsilon_2 \) then \( p_1 \geq p_2 \). All diagonal states with \( p_1 \geq \frac{1}{2} \) are passive. The two extreme points, \( \epsilon_1 = (1, 0)^T \) and \( \epsilon_2 = \left( \frac{1}{2}, \frac{1}{2} \right)^T \) are thermal states of \( T = 0 \) and \( T = \infty \) temperature respectively, while all other passive states are completely passive or thermal of different temperatures.
b. **Degenerate Hamiltonian**: Due to the equality of energy Eigen values, all diagonal states are passive having zero ergotropy. Extreme points are $e_1 = (1 \ 0)^T$ and $e_2 = (0 \ 1)^T$. Both extreme points are passive and $\tau_\beta = (\frac{1}{2} \frac{1}{2})^T$ is the only thermal state lying in the middle of the line joining these two extreme points.

2. **3D passive states**

**a. Non-Degenerate Hamiltonian**: In 3D Hilbert space a diagonal state is represented by probability vector $\rho = (p_1 \ p_2 \ p_3)^T$; $p_1 + p_2 + p_3 = 1$ under the non-degenerate Hamiltonian $H = (\epsilon_1 \ \epsilon_2 \ \epsilon_3)^T$. The state is said to be passive if $p_1 \geq p_2 \geq p_3$ holds for energy eigen values $0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3$. The region of these passive states will lie in a 2D plane, where the extreme points are $e_1 = (1 \ 0 \ 0)^T$, $e_2 = (\frac{1}{2} \frac{1}{2} \ 0)^T$ and $e_3 = (\frac{1}{3} \frac{1}{3} \frac{1}{3})^T$ as depicted in Fig 3 (VII A 2 a) by the red region. Among these three extreme points $e_1$ and $e_3$ are thermal states of $T = 0$ and $T = \infty$ temperature respectively, where $e_2$ is not thermal but passive.

b. **Degenerate Hamiltonian**:

**Case-I(Complete degeneracy)**: Hamiltonian : $(\epsilon \ \epsilon \ \epsilon)^T$ ; where all the diagonal states (meshed region in all the figures) are passive and extreme points $e_1 = (1 \ 0 \ 0)^T$, $e_2 = (0 \ 1 \ 0)^T$ and $e_3 = (0 \ 0 \ 1)^T$ are also passive. For complete degeneracy, maximally mixed state is the state whose free energy is lowest and becomes the only thermal state among the whole state space.

**Case-II(Ground state degeneracy)**: $H=(\epsilon \ \epsilon \ \epsilon_3)^T$ ; If $0 \leq \epsilon \leq \epsilon_3$ then the elements of passive state would be $p_1 \geq p_3$ and $p_2 \geq p_3$. This region has been marked in green in Fig 4 (VII A 2 b).

Extreme points : $e_1 = (1 \ 0 \ 0)^T$, $e_2 = (0 \ 1 \ 0)^T$, $e_3 = (\frac{1}{2} \frac{1}{2} \ 0)^T$.

Owing to the degeneracy in ground state, all the states on the line joining $e_1$ and $e_2$ are passive, out of which only one is thermal i.e., $\tau_\infty = (\frac{1}{2} \frac{1}{2} \ 0)^T$ of $\beta = \infty$ temperature. The last extreme point $e_3$ is the thermal state of $T = \infty$ virtual temperature.

**Case-III(Excited state degeneracy)**: $H = (\epsilon_1 \ \epsilon \ \epsilon)^T$ ; If $0 \leq \epsilon_1 \leq \epsilon$ then from passivity we have $p_1 \geq p_2$ and $p_1 \geq p_3$. In this case the passive state region lies in a plane depicted in yellow in Fig 5 (VII A 2 b).

Extreme points: $e_1 = (1 \ 0 \ 0)^T$, $e_2 = (\frac{1}{2} \frac{1}{2} \ 0)^T$, $e_3 = (\frac{1}{2} \ 0 \ \frac{1}{2})^T$, $e_4 = (\frac{1}{3} \frac{1}{3} \frac{1}{3})^T$. 

Figure 3. 3D (Non-Degenerate Hamiltonian): The mesh region denotes the total diagonal state space where as the red region denotes the space of passive states.
Figure 4. 3D (Degenerate Hamiltonian): The green area shows the passive state region for 3D system where ground state is degenerate.

Figure 5. 3D (Degenerate Hamiltonian): The yellow region depicts the passive state space for 3D system where excited state has degeneracy.

$e_1$ and $e_4$ are thermal states of $T = 0$ and $T = \infty$ temperature respectively, whereas the other two extreme points are passive in nature.

It may be noted that the non-degenerate passive state space is always a subset of any degenerate passive state space, shown in Fig 6 (VII A 2 b). It is true for any arbitrary dimension.
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Figure 6. Passive states region for Non-degenerate case depicted by black is always subset of the passive states region for all and any degenerate cases. Black+cyan is the region of ground state degeneracy whereas black+cyan+yellow is the region of excited state degeneracy.

3. 4D passive states

**Non-Degenerate Hamiltonian:** Given Hamiltonian $H = (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4)^T$ under the condition $0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4$. In 4D Hilbert Space all diagonal states can be represented as $\rho = (p_1 p_2 p_3 p_4)^T$; $p_1 + p_2 + p_3 + p_4 = 1$. A passive state would follow $p_1 \geq p_2 \geq p_3 \geq p_4$ and the corresponding region would lie in 3D volume as shown in Fig 7 (VII A 3).

Extreme points: $e_1 = (1 0 0 0)^T$, $e_2 = (\frac{1}{3} \frac{1}{2} 0 0)^T$, $e_3 = (\frac{1}{3} \frac{1}{3} \frac{1}{3} 0)^T$, $e_4 = (\frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3})^T$.

$e_1$ and $e_4$ are the thermal states of $T = 0$ and $T = \infty$ temperature respectively whereas $e_1$ and $e_3$ are passive.

Seven Different types of degeneracy may occur in the 4D Hamiltonian. In all cases, it is possible to plot the passive state region and the extreme points very easily.

4. d-D passive states

**Non-Degenerate Hamiltonian:** A general non-degenerate Hamiltonian can be defined by, $H = (\epsilon_1 \epsilon_2 \ldots \epsilon_d)^T$ and a general diagonal state $\rho = (p_1 p_2 \ldots p_d)^T$ with the condition $p_1 + p_2 + \ldots + p_d = 1$. By the definition of passive state $p_1 \geq p_2 \geq \ldots \geq p_d$ if $\epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_d$.

The diagonal and passive state space would occur in $(d-1)$ dimensional space due to the normalization constraint. Extreme points would always follow the same pattern just as before. There would be $d$ number of extreme points, among which two are thermal ($T = 0$ and $T = \infty$) and other $(d-2)$ are passive states.

Extreme points: $e_1(T = 0) = (1 \ 0 \ldots \ 0)^T$, $e_2 = (\frac{1}{2} \frac{1}{2} 0 \ldots 0)^T$, $e_3 = (\frac{1}{3} \frac{1}{3} \frac{1}{3} \ldots 0)^T$, $e_d(T = \infty) = (\frac{1}{d} \frac{1}{d} \frac{1}{d} \ldots \frac{1}{d})^T$.
Figure 7. 4D (Non-Degenerate Hamiltonian): Passive state region for 4D system in non-degenerate Hamiltonian.