**Abstract.** We prove the existence of a 1-dimensional family of nondisplaceable Lagrangian tori in the smoothing of a 4-dimensional symplectic orbifold with $A_n$-type singularities. The proof uses the idea of toric degeneration to calculate the full potential functions of these tori and show that they carry nontrivial Floer cohomology groups.

**Contents**

1. Introduction 1
   Acknowledgements 4
2. Preliminaries 4
   2.1. Weak bounding cochains and potential functions 4
   2.2. One-point open Gromov-Witten invariant 8
3. Smoothing and resolution of $X_0$ 10
   3.1. Smoothing of $X_0$ 10
   3.2. Topology of $X_0(\epsilon)$ 11
   3.3. Minimal resolution of $X_0$ 12
4. $X_\epsilon$ as a hyperKähler manifold 13
5. Potential function in $X_0(\epsilon)$ 14
   5.1. Potential function in $X_\alpha$ 15
   5.2. Deformation invariance 15
6. General case of an $A_n$-type singularity 21
   6.1. Potential function in the minimal resolution 22
   6.2. Potential function in the smoothing 24
7. Examples in closed case 26
   7.1. Compactified moment polytope of an $A_n$-type singularity 27
   7.2. Lagrangian tori in the smoothing of a symplectic 4-orbifold 35
References 38

1. **Introduction**

A compact submanifold $L$ of a connected symplectic manifold $(M,\omega)$ is called nondisplaceable if it cannot be separated from itself by any Hamiltonian diffeomorphism. That is,

$$L \cap \phi(L) \neq \emptyset, \quad \forall \phi \in \text{Ham}(M,\omega).$$
When \( L \) is a Lagrangian submanifold, nondisplaceability of \( L \) is related with Lagrangian intersection Floer theory. The Floer cohomology groups \( HF(L) \) are generated by intersection points \( L \cap \phi(L) \) for a generic Hamiltonian perturbation \( \phi \). Under certain conditions these groups are invariant under the action of \( Ham(M, \omega) \). Hence nonvanishing of \( HF(L) \) implies nondisplaceability of \( L \). In a series of papers [6], [7] and [8] Fukaya-Oh-Ohta-Ono established a critical point theory of Floer cohomology groups of toric fibers and reduced the calculation of Floer cohomology to the calculation of potential functions with bulk deformation. Later in [9] they applied this theory to the Hirzebruch surface \( F_2 \) and then used toric degeneration methods to get a family of nondisplaceable Lagrangian tori in \( S^2 \times S^2 \), which are not toric fibers. In this paper we carry out the same idea in a slightly more general situation, involving symplectic orbifolds with \( A_n \)-type singularities. We formulate the main theorem as follows.

**Theorem 1.1.** Let \( X \) be a 4-dimensional symplectic orbifold with \( A_n \)-type singularities and let \( \hat{X} \) be its smoothing obtained by gluing \( A_n \)-Milnor fibers. Then there is a family of nondisplaceable Lagrangian tori in \( \hat{X} \) parameterized by open intervals.

Using this 1-dimensional family of mutually disjoint nondisplaceable Lagrangian tori we can produce linearly independent Calabi quasi-morphisms by the same method in Corollary 1.9 and Theorem 1.10 in [10]. That is, we have the following corollary of Theorem 1.1.

**Corollary 1.2.** Let \( X \) be a 4-dimensional symplectic orbifold with \( A_n \)-type singularities and let \( \hat{X} \) be its smoothing obtained by gluing \( A_n \)-Milnor fibers. Then \( \hat{X} \) carries a family of linearly independent Calabi quasi-morphisms parameterized by open intervals.

The proof of Theorem 1.1 starts with a local study of a certain family of Lagrangian tori in the smoothing of an \( A_n \)-type singularity. Consider the hypersurface

\[
X_{\epsilon, n} = \{ x^2 + y^2 + z^{n+1} = \epsilon \} \subseteq \mathbb{C}^3
\]

where \( \epsilon \geq 0 \) is a small complex number and when there is no confusion we omit the subscript \( n \).

For \( \epsilon \neq 0 \), \( X_{\epsilon, n} \) has an induced symplectic structure from \(( \mathbb{C}^3, \omega_{std}) \). All of them are smooth exact symplectic manifolds and they are exact symplectomorphic to each other.
For $\epsilon = 0$, $X_{0,n}$ has a singular point of $A_n$-type at the origin and it is biholomorphic to $\mathbb{C}^2/\mathbb{Z}_n$. Hence we have an induced Hamiltonian $T^2$-action on $X_{0,n}$ from the standard action on $\mathbb{C}^2$ such that $X_{0,n}$ is a symplectic toric orbifold and has the moment polytope shown in Figure 1 when $n = 3$. We obtain the smoothing $X_0(\epsilon)$ by cutting out a small neighborhood of the origin in $X_{0,n}$ and pasting back a proper neighborhood in $X_{\epsilon,n}$.

On the line $\{u_1 = u_2\}$, Floer-type invariants of corresponding Lagrangian tori are nonzero, see Section 4 in [1]. Therefore these tori are nondisplaceable in the orbifold $X_0$. Using toric degeneration we can show that in the smoothing of $X_0$ this family of Lagrangian tori are still nondisplaceable.

**Theorem 1.3.** In the smoothing $X_0(\epsilon)$, there is a one dimensional family of monotone nondisplaceable Lagrangian tori $L(u)$, where $u \in \mathbb{R}^+$.  

We first remark that $L(u)$ is not the type of a toric fiber since the smoothing $X_0(\epsilon)$ is no longer toric. But these Lagrangian tori are still parameterized by interior points in the moment polytope because we just doing a small deformation near the origin. We also remark that the smoothing $X_0(\epsilon)$ is symplectomorphic to the Milnor fiber $X_\epsilon$. And the existence of a family of monotone Lagrangian tori with nontrivial Floer cohomology in $X_\epsilon$ is known, see [10] by Lekili-Maydanskiy and Corollary 9.1 in [2] by Abouzaid-Auroux-Katzarkov. In this paper we provide an alternative proof of the noncompact case (Theorem 1.3) using toric degeneration. Moreover our method can be generalized to the global case of an arbitrary symplectic 4-orbifold with $A_n$-type singularities (Theorem 1.1).
The outline of proof of Theorem 1.3 is the following. For a fixed Lagrangian toric fiber $L(u)$ in $X_0$, the key ingredient of its potential function are moduli spaces of holomorphic disks with boundary on $L(u)$ with Maslov index 2. Hence our main task is to study those moduli spaces in different ambient spaces. First we resolve the singularity of $X_0$ in a toric way to get $X_\alpha$. Then we calculate the potential function of $L(u)$ in $X_\alpha$ combinatorially using the presentation of the moment polytope. Next we use deformation techniques to study how the corresponding moduli spaces vary when symplectic and complex structures change from $X_\alpha$ to $X_0(\epsilon)$ to get the potential function of $L(u)$ in $X_0(\epsilon)$. Then we can apply the critical point theory of potential functions to deduce that $L(u)$ carries nontrivial Floer cohomology groups and hence is non-displaceable.

Once we have the existence of non-displaceable Lagrangian tori in $X_{\epsilon,n}$ we can consider the global case in the smoothing of a 4-dimensional symplectic orbifold $X$ with $A_n$-type singularities. The goal is to show that after gluing $X_{\epsilon,n}$ to $X$, a subfamily of the above family of non-displaceable Lagrangian tori is still non-displaceable under the action of a larger group $\text{Ham}(X,\omega)$. This is done by using potential function with bulk deformation to deal with possible higher order terms.

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2. Preliminaries

Let $(M^{2n},\omega,\mathbb{T}^n,\mu)$ be a symplectic manifold with an effective Hamiltonian torus action and a moment map $\mu$. The image of $\mu$ is a convex polytope in $\mathbb{R}^n$. For an interior point $u$ in the moment polytope its preimage $\mu^{-1}(u)$ is an oriented Lagrangian torus corresponding to an orbit of the Hamiltonian $\mathbb{T}^n$ action. We refer to chapter B in [3] for more background and notations about symplectic toric manifolds.

The non-displaceability problem of toric fibers is well-studied by many people. In [7] and [8] Fukaya-Oh-Ohta-Ono related Lagrangian Floer cohomology groups to critical points of the potential function of a toric Lagrangian fiber. Later on Wilson and Woodward generalized similar methods to orbifolds and noncompact settings in [17] and [18]. In Abreu-Borman-McDuff’s work [4] they established the theory of probes to displace a toric fiber and many explicit examples were calculated by combining the method of probes and potential functions.

In this section we briefly set up background for the potential function theory and quote theorems from other literature.

2.1. Weak bounding cochains and potential functions. We first introduce the potential function of a closed relative spin Lagrangian $L \subset (M,\omega)$ without assuming $(M,\omega)$ is toric, mainly following section 2 in [9].

Definition 2.1. The universal Novikov ring $\Lambda_{0,\text{nov}}$ is defined to be

$$\Lambda_{0,\text{nov}} = \{ \sum_{i=1}^{\infty} a_i T^\lambda_i e^{\mu_i} | a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty, \mu_i \in \mathbb{Z} \}$$
where $T$ and $e$ are formal variables labeling the symplectic energy and Maslov index of a holomorphic disk. When we only consider holomorphic disks with the same Maslov index we insert $e = 1$ and get a subring

$$\Lambda_0 = \{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \}.$$ 

Also $\Lambda_+$ is defined to be

$$\Lambda_+ = \{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty, \}$$

which is a maximal ideal of $\Lambda_0$. We denote $\Lambda_0 - \Lambda_+$ as the subset consisting of all elements with a nonzero leading constant.

Let $(\Sigma; k + 1, l)$ be a bordered semistable curve of genus zero with $k + 1$ boundary marked points and $l$ interior marked points. We assume that the boundary marked points are labeled from 0 to $k$ in a counter-clockwise direction on $\partial \Sigma$. We now study the moduli space of stable maps

$$(\Sigma, \partial \Sigma) \to (M, L).$$

Let $\mathcal{M}_{k+1,l}(\beta, J)$ be the set of isomorphism classes of $J$-holomorphic stable maps for an almost complex structure $J$ and a homotopy class $\beta \in \pi_2(M, L)$. When $l = 0$ we write $\mathcal{M}_{k+1,0}(\beta, J) = \mathcal{M}_{k+1}(\beta, J)$. Under transversality assumptions we can associate an $A_{\infty}$-structure to $L$. Given $k$ smooth singular chains

$$[P_1, f_1], [P_2, f_2], \cdots, [P_k, f_k],$$

we consider the fiber product

$$\text{ev}_0 : \mathcal{M}_{k+1}(\beta, J)(\text{ev}_1, \text{ev}_2, \cdots, \text{ev}_k) \times (f_1, \cdots, f_k) (P_1 \times P_2 \times \cdots \times P_k) \to L$$

where $\text{ev}_i$ is the evaluation map of $i$-th boundary marked point. For each $\beta \in \pi_2(M, L)$ and $k \in \mathbb{Z}_{\geq 0}$, we define

$$m_{k, \beta} = [\mathcal{M}_{k+1}(\beta, J)(\text{ev}_1, \text{ev}_2, \cdots, \text{ev}_k) \times (f_1, \cdots, f_k) (P_1 \times P_2 \times \cdots \times P_k), \text{ev}_0]$$

and

$$m_k = \sum_{\beta \in \pi_2(M, L)} m_{k, \beta} \omega(\beta) e^{\mu(\beta)/2}.$$ 

Using Kuranishi structures on $\mathcal{M}_{k+1}(\beta, J)$ we know that $m_{k, \beta}$ are singular chains in $L$. Moreover we have $C[1]$ which is a completion of suitably chosen countably generated singular chain complex with $\Lambda_{0, \text{nov}}$ coefficients of $L$ such that

$$m_k : C[1] \otimes k \to C[1]$$

is a well-defined $A_{\infty}$-structure, see section 3 in [7]. By identifying the chain complex $C_*(L)$ and the cochain complex $C^{\dim L - *}(L)$ this $A_{\infty}$-structure can be regarded on the cochain level. Moreover by the following theorem we can transfer it to a canonical $A_{\infty}$-structure $\{m^{\text{can}}_k\}_{k=0,1,\cdots}$ on the classical cohomology level $H^*(L; \Lambda_{0, \text{nov}})$ with $\Lambda_{0, \text{nov}}$ coefficients, which is called the canonical model. See more details about the construction of canonical model in section 5.4 in [6].
Theorem 2.2. (Theorem A, [8]) For a closed relative spin Lagrangian $L \subset (M, \omega)$, there exists a canonical $A_\infty$-structure $m^{can} = \{m^k_{can}\}$ on the classical cohomology $H^* (L; \Lambda_{0,\text{nov}})$ which is homotopy equivalent to $(C[1], m = \{m_k\})$. Let $PD([L]) \in H^0 (L; \mathbb{C})$ be the Poincaré dual of the fundamental class of $L$ then it is the unit of the above canonical $A_\infty$-structure $m^{can}$.

Next we deform the $A_\infty$-structure by ambient cycles in the symplectic manifold. Using the moduli spaces $M_{k+1,l} (\beta, J)$ we can construct a series of operators

$$q_{\beta,l,k}^{can} : E_l H(M; \mathbb{C}) \otimes B_k H(L; \mathbb{C}) \to H(L; \mathbb{C}).$$

Here $H(L; \mathbb{C}), H(M; \mathbb{C})$ are singular homology groups, $B_k H(L; \mathbb{C}) = H(L; \mathbb{C})^\otimes k$ and $E_l H(M; \mathbb{C})$ is the subspace of $H(M; \mathbb{C})^\otimes l$ that is invariant under symmetric permutations. We sketch the construction as follows and refer to section 2 in [8] for details.

For singular chains $\bar{Q} = ((Q_1, g_1), \cdots, (Q_l, g_l))$ in $M$ and singular chains $\bar{P} = ((P_1, f_1), \cdots, (P_k, f_k))$ in $L$ we define the following fiber product

$$M_{k+1,l} (\beta, J; \bar{Q}, \bar{P}) = M_{k+1,l} (\beta, J; \bar{Q}, \bar{P})$$

$$= M_{k+1,l} (\beta, J; \bar{Q}, \bar{P}) = \frac{1}{!} \left( M_{k+1,l} (\beta, J; \bar{Q}, \bar{P}), \text{ev}_0 \right).$$

Those operators descend to $E_l C(M; \mathbb{C}) \otimes B_k C(L; \mathbb{C})$. Also we obtain a series of operators

$$q_{l,k} : E_l C(M; \Lambda_0) \otimes B_k C(L; \Lambda_0) \to C(L; \Lambda_0).$$

by setting $q_{l,k} = \sum_{\beta} q_{\beta,l,k} T^w(\beta)$. Taking a similar construction to the canonical model we get operators

$$q_{l,k}^{can} : E_l H(M; \Lambda_0) \otimes B_k H(L; \Lambda_0) \to H(L; \Lambda_0).$$

We will now use the operators $q = \{q_{l,k}^{can}\}$ to deform the $A_\infty$-structure $m^{can}$.

From now on we are always using the canonical model and write $m = m^{can}$.

Definition 2.3. Let $b \in H^* (M; \Lambda_+) \text{ and } x_i \in H^* (L; \Lambda_0)$. We define

$$m^b_k (x_1, \cdots, x_k) = \sum_{l} q_{l,k} (b^\otimes x_1, \cdots, x_k).$$

It is proved in Lemma 3.8.39 in [8] that $m^b = \{m^b_k\}$ is also an $A_\infty$-structure. We call $m^b$ the $A_\infty$-structure with bulk deformation by $b$.

Remark 2.4. In section 11 in [8] the bulk-deformed $A_\infty$-structure $m^b$ has been generalized to the case where $b \in H^2 (M; \Lambda_0)$. The defining formula is similar but with a minor modification to deal with convergence problems. In particular when $b \in H^2 (M; \Lambda_0)$ we write

$$b = b_0 + b_+; \quad b_0 \in H^2 (M; \mathbb{C}), \quad b_+ \in H^2 (M; \Lambda_+).$$
and define
\[ q_{\beta,l,k} (b \otimes \ell, x_1, \ldots, x_k) = e^{b_0 \cap \beta} q_{\beta,l,k} (b \otimes \ell, x_1, \ldots, x_k) \]
where \( b_0 \cap \beta \) is the pairing between cochains and chains. We use this definition to define \( \{ m^b_k \} \) when \( b \in H^2(M; \Lambda_0) \). In this paper we always use \( b \in H^2(M; \Lambda_0) \).

An element \( b \in H^1(L; \Lambda_+ \otimes \Xi) \) is called a weak bounding cochain if it satisfies the \( A_\infty \)-Maurer-Cartan equation
\[
\sum_{k=0}^{\infty} m^b_k (b, \ldots, b) = 0 \mod PD([L]).
\]
We denote by \( \hat{\mathcal{M}}_{weak} (L, m^b) \) the set of weak bounding cochains \( b \) of \( m^b \) and \( \hat{\mathcal{M}}_{weak} (L) \) the set of pairs \( (b, b) \) such that \( b \) is a weak bounding cochain of \( m^b \). If \( \hat{\mathcal{M}}_{weak} (L) \neq \emptyset \) then we call \( L \) weakly unobstructed. In [7] the coefficients of \( b \) have been generalized from \( \Lambda_+ \) to \( \Lambda_0 \) using Cho’s idea of nonunitary flat line bundles. When the coefficient ring is extended, there is a gauge equivalence
\[ b \sim b' \in \hat{\mathcal{M}}_{weak} (L, m^b) \subset H^1(L; \Lambda_0) \]
due to the periodicity of integration on the leading part of \( b \). We directly use the extended definition and write \( \mathcal{M}_{weak} (L, m^b) \) as the set of equivalence classes of weak bounding cochains of \( m^b \), referring to section 2 in [9] for details.

**Definition 2.5.** For \( b \in \mathcal{M}_{weak} (L, m^b) \) we define the potential function \( \mathfrak{P}_b^L: \mathcal{M}_{weak} (L, m^b) \to \Lambda_+ \) by
\[
\sum_{k=0}^{\infty} m^b_k (b, \ldots, b) = \mathfrak{P}_b^L (b) \cdot PD([L]).
\]
We also define the bulk-deformed Floer coboundary operator \( \delta_b^L : H^* (L; \Lambda_0) \to H^* (L; \Lambda_0) \) by
\[
\delta_b^L (x) = \sum_{k,l \geq 0} m^b_{k+l+1} (b \otimes^k, x, b \otimes^l).
\]

**Lemma 2.6.** (Proposition 3.7.17, [6]) Let \( b \) be a weak bounding cochain of \( m^b \) then \( \delta_b^L \circ \delta_b^L = 0 \).

Therefore we can define the bulk-deformed Floer cohomology
\[
HF ((L; b, b), (L; b, b)) = \frac{Ker \delta_b^L}{Im \delta_b^L}.
\]

The weak bounding cochain and potential function are defined assuming the regularity of all \( \mathcal{M}_{k+l+1} (\beta, J) \). In this paper we always consider the case when \( (M, \omega) \) is a smooth symplectic 4-manifold and \( L \) is a Lagrangian torus. Therefore the transversality issue can be solved using Kuranishi structures. Moreover by dimension counting we have the following lemma.
Lemma 2.7. If $\beta$ is nontrivial with $\mu_L(\beta) \leq 0$ then $\mathcal{M}_{k+1}(\beta, J) = \emptyset$ for generic almost complex structure $J$.

Remark 2.8. The potential function $\mathfrak{PO}^L_b$ depends on a choice of $J$ but we omit it in the notation. If there is a family $\{J_t\}_{t \in [0,1]}$ connecting $J_0$ and $J_1$ without breaking moduli spaces of disks then this cobordism gives us $\mathfrak{PO}^{L_0,J_0}_b = \mathfrak{PO}^{L_1,J_1}_b$. For example when our Lagrangian torus is monotone with positive minimal Maslov number then we can show that the potential function does not depend on $J$.

Next we introduce the critical point theory for Lagrangian Floer cohomology with bulk deformation.

Theorem 2.9. (Theorem 2.3, [9]) Let $L$ be a Lagrangian torus in $(M, \omega)$. Suppose that

$$H^1(L; \Lambda_0) / H^1(L; 2\pi \sqrt{-1}\mathbb{Z}) \subset \tilde{\mathcal{M}}_{\text{weak}}(L, \mathfrak{m}^b)$$

and $b \in H^1(L; \Lambda_0)$ is a critical point of the potential function $\mathfrak{PO}^L_b$. Then we have

$$HF((L; b, b), (L; b, b)) \cong H(L).$$

In particular $L$ is nondisplaceable.

Note that if we assume $(M, \omega)$ is a 4-manifold and $L$ is a Lagrangian torus then the potential function has an explicit expression.

Theorem 2.10. (Theorem A.2, [9]) When $(M, \omega)$ is a 4-manifold and $L$ is a Lagrangian torus and $b = 0$ we have

$$\mathfrak{PO}^L(b) = \sum_{\mu_L(\beta) = 2} T^{\omega(\beta)} \exp(b \cap \partial \beta) \cdot \deg[ev_0 : \mathcal{M}_1(\beta) \rightarrow L]$$

$$= \sum_{\mu_L(\beta) = 2} c_\beta T^{\omega(\beta)} e^{x_1(e_1(\partial \beta)) + x_2(e_2(\partial \beta))}$$

(2.6)

where $e_1, e_2$ is a basis of $H^1(L(u); \mathbb{Z})$ and $b \cap \partial \beta$ is the pairing between $H^1(L(u); \mathbb{Z})$ and $H_1(L(u); \mathbb{Z})$. Also $b = x_1 e_1 + x_2 e_2$ for some $x_1, x_2 \in \Lambda_0$.

In toric case both $T^{\omega(\beta)}$ and $e^1(\partial \beta)$ can be calculated using combinatorial data from moment polytope but $c_\beta = \deg[ev_0 : \mathcal{M}_1(\beta) \rightarrow L]$ is not always easy to compute. Note that by Lemma 2.7 the Maslov index 2 disk is minimal for generic $J$ hence $\mathcal{M}_1(\beta)$ is a cycle and this mapping degree is well-defined. Next we introduce Chan-Lau’s work on the computation of this one-point open Gromov-Witten invariant.

2.2. One-point open Gromov-Witten invariant. For a symplectic toric manifold $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ and a Lagrangian fiber $L(u) = \mu^{-1}(u)$, we have the following Maslov index formula in [5].

Theorem 2.11. Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold with moment polytope $P$. Let $\partial P_i$ be the codimension one faces and $D_i = \mu^{-1}(\partial P_i)$ be the toric divisors. Let $L(u)$ be a Lagrangian fiber of the moment map. Then the Maslov index of any holomorphic disk with boundary on $L(u)$ is twice the sum of the intersection multiplicities of this disk with all toric divisors $D_i$. 

Let $D_i$ be a toric divisor. A class $\beta_i \in \pi_2(M, L(u))$ is called a meridian class around $D_i$ if

$$\beta_i \cap D_j = \delta_{ij}$$

where $\delta_{ij}$ is the Kronecker delta function. In toric case we have the following structure theorem proven in [5] and [7].

**Theorem 2.12.** (Theorem 11.1, [7]) Let $L = L(u)$ be a Lagrangian toric fiber and $J$ be the toric complex structure which we omit from the notation. Let $M^{reg}_1(\beta)$ be the subset of smooth maps in $M_1(\beta)$.

1. If $\mu(\beta) < 0$ or $\mu(\beta) = 0, \beta \neq 0$ then $M^{reg}_1(\beta)$ is empty.
2. If $\mu(\beta) = 2$ and $\beta$ is not a meridian class $\beta_i$ then $M^{reg}_1(\beta)$ is empty.
3. If $M_1(\beta)$ is not empty then there exist a finite collection of nonnegative integers $k_i$ and homology classes $E_i \in H_2(M; \mathbb{Z})$ such that

$$\beta = \sum i k_i \beta_i + \sum l E_l$$

where $E_l$ can be realized by a holomorphic sphere and $\beta_i$ is a meridian class with at least one positive $k_i$.

Note that here $M^{2n}$ is assumed to be of arbitrary dimension and $J$ is toric. In [7] the virtual fundamental chain $[M_1(\beta)]^{vir}$ has been constructed using a system of multisection perturbations. Such a perturbation only exists when given an upper bound on symplectic energy of those classes $\beta$. We also have an analogue to Lemma 2.7 from the previous subsection.

**Lemma 2.13.** For a nontrivial $\beta$ with $\mu(\beta) \leq 0$ the perturbed $M_1(\beta)$ is empty for toric complex structure $J$.

The expected dimension of $M_1(\beta)$ is $n + \mu(\beta) - 2$. In particular for $\mu(\beta) = 2$, the dimension of $M_1(\beta)$ is $n$. As a corollary of the above lemma $\mu(\beta) = 2$ is minimal and so the virtual fundamental chain $[M_1(\beta)]^{vir}$ is a cycle and hence the following mapping degree is well-defined

$$c_\beta = ev_0_* \left( [M_1(\beta)]^{vir} \right) \in H_n(L(u); \mathbb{Q}) \cong \mathbb{Q}.$$ We call $c_\beta$ the one-point open Gromov-Witten invariant with respect to the class $\beta$. In [7] it is shown that $c_\beta$ is independent of the perturbations used in defining the virtual fundamental class.

**Remark 2.14.** The perturbation method used to eliminate $M_1(\beta)$ when $\mu(\beta) = 0$ works in general toric cases. In our setting we will verify later that by topological constraints there is no nontrivial disk with Maslov index 0.

We say a symplectic manifold is nef or semi-Fano if its first Chern class is nonnegative on all holomorphic spheres. Then for a nef symplectic toric manifold Theorem 2.12.(3) tells us that a holomorphic disk with Maslov index 2 must be of the following type

$$\beta = \beta_i + \sum l k_l E_l, \quad k_l \in \mathbb{Z}$$

where $E_l$ is a chain of sphere classes with first Chern number zero and where $\beta_i$ is a meridian class.

Let $(M^4, \omega, T^2, \mu)$ be a compact nef toric surface and $L(u)$ be a Lagrangian fiber. Let $\beta \in \pi_2(M, L)$ be a class of the above type. Since the image of a holomorphic
disk is connected, either all $k_l = 0$ or $k_l \neq 0$. Therefore an above class $\beta$ gives us a sequence of integers $\{k_1, \cdots, k_m\}$.

**Definition 2.15.** We call a tuple $\{k_1, \cdots, k_m\}$ admissible with center $i$ if each $k_l$ is a nonnegative integer and

1. $k_l \leq k_{l+1} \leq k_l + 1$ when $l < i$;
2. $k_l \geq k_{l+1} \geq k_l - 1$ when $i \leq l$;
3. $k_l \leq 1$ and $k_m \leq 1$.

Then we have the following Theorem 1.2 from [4].

**Theorem 2.16.** Let $M$ be a compact nef toric Kähler surface and $L$ be a Lagrangian fiber. Let $\beta \in \pi_2(M, L)$ be a class such that $\beta = \beta_i + \sum_i k_i E_i$, where $E_i$ is a chain of $-2$ curves in $M$ which are toric prime divisors. Then the one-point open Gromov-Witten invariant $c_\beta$ is either 1 or 0 according to whether the tuple $\{k_1, \cdots, k_m\}$ is admissible or not.

In section 5 we use an analogue of this result in the case of an open symplectic manifold with convex boundary to calculate the full potential functions of some lagrangian tori.

## 3. Smoothing and resolution of $X_0$

First we use $X_{0,3}$ as an explicit example for the calculation of potential functions then in section 6 we generalize to the case $X_{0,n}$.

Recall that $X_0 = X_{0,3}$ is an affine variety in $\mathbb{C}^3$ with an $A_3$-type singularity at the origin, we can form the smoothing and resolution of it. Both of the smoothing and resolution of $X_0$ have the same link $S^3/\mathbb{Z}_4$, which is a hypersurface of contact-type. In this section we describe those constructions in detail.

### 3.1. Smoothing of $X_0$

The smoothing $X_0(\epsilon)$ is obtained by cutting a small neighborhood of the singular point and gluing back a similar neighborhood of $X_\epsilon$.

Let $(M, \omega)$ be a compact symplectic manifold with boundary. Then we say $\partial M$ is of contact-type if there exists a Liouville vector field $V$ defined near $\partial M$ such that it is outward pointing and $\omega = d\theta$, where $\theta = \omega(V, \cdot)$ and $\rho = \theta|_{\partial M}$ is a contact form.

We identify $X_0$ with $\mathbb{C}^2/\mathbb{Z}_4$. Let

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}_4 \cong X_0$$

be the quotient map. Let $B_\epsilon^3 \subseteq \mathbb{C}^3$ be the ball of radius $\epsilon$ centered at origin and $S_\epsilon^2$ be the boundary of $B_\epsilon^3$. Then $\pi(S_\epsilon^2)$ is a contact hypersurface in $X_0$, which is isomorphic to $S^3/\mathbb{Z}_4$ with standard contact structure. Let $Y_\epsilon = X_0 - \pi(B_\epsilon^3)$. We will now glue a Milnor fiber to $Y_\epsilon$ to get a smooth symplectic manifold.

The Milnor fiber $X_\epsilon$ is symplectomorphic to the plumbing of three copies of cotangent bundles of the 2-sphere. First let $S^2$ be the unit 2-sphere with round metric in $\mathbb{R}^3$, $T^*S^2$ be its cotangent bundle with standard symplectic form and $D_r(T^*S^2)$ be its disk bundle of radius $r > 0$. Following the $A_3$-diagram, we do the plumbing operation of three copies of $D_r(T^*S^2)$ with same radius $r$. Denote by

$$Z_r = D_r(T^*S^2) \bigcup D_r(T^*S^2) \bigcup D_r(T^*S^2)$$
as the plumbing. The boundary \( \partial Z_r \) is convex and diffeomorphic to \( S^3/\mathbb{Z}_4 \). After a deformation of the boundary we can make it to be \( S^3/\mathbb{Z}_4 \) with standard contact structure. That is, there is a manifold \( W \) which is diffeomorphic to \( S^3/\mathbb{Z}_4 \times [0, 1] \) with \( \partial W_0 = \partial Z_r \) and \( \partial W_1 = S^3/\mathbb{Z}_4 \) with standard contact structure. So we glue \( Z_r \) on \( \partial W_0 \) and glue \( \partial W_1 \) on \( \partial Y_r \) to get the smoothing, see Figure 2. This gluing does not change the toric symplectic structure outside a collar neighborhood of \( \partial Y_r \).

Since
\[
H^1 \left( S^3/\mathbb{Z}_4; \mathbb{Q} \right) = H^2 \left( S^3/\mathbb{Z}_4; \mathbb{Q} \right) = 0
\]
the glued symplectic form does not depend on parameters \( \epsilon \) and \( r \) up to a symplectomorphism by Moser’s theorem. Hence we get a family of smooth symplectic manifolds \( X_0(\epsilon) \) with parameter \( \epsilon \). All of them are exact symplectic manifolds and symplectomorphic to \( X_r \). However the complex structures on \( X_0(\epsilon) \) depend on \( \epsilon \). In particular, outside a neighborhood of the gluing region \( X_0(\epsilon) \) is equal to \( X_0 \) and hence it is toric. We can still denote by \( L(u) \) as the preimage of \( u \) under the moment map where \( u \) is not close to the origin. In this case \( L(u) \) is a Lagrangian torus because the symplectic structure is unchanged outside the gluing region.

3.2. Topology of \( X_0(\epsilon) \). In this subsection we specify some homotopy groups which will be used later. Note that since \( X_0(\epsilon) \) is diffeomorphic to \( X_r \), it suffices to study the topology of one of these two. The vanishing cycle of \( X_r \) is a wedge of three spheres, intersecting with respect to the \( A_3 \)-diagram and each has self intersection number \(-2\). We denote the corresponding homology classes by \( E_1, E_2, E_3 \). In \( X_0(\epsilon) \) they correspond to three zero sections in the plumbing operation. In singularity theory it is proved that \( X_r \) is homotopy equivalent to its vanishing cycle. Hence we have
\[
\pi_1 \left( X_0(\epsilon) \right) = \pi_1 \left( S^2 \vee S^2 \vee S^2 \right) = 0
\]
and
\[
\pi_2 \left( X_0(\epsilon) \right) = \pi_2 \left( S^2 \vee S^2 \vee S^2 \right) = \mathbb{Z}^3.
\]

Next we fix a Lagrangian torus \( L(u) \) outside the gluing neighborhood of the origin and consider the relative homotopy group \( \pi_2 \left( X_0(\epsilon), L(u) \right) \). We have a long exact sequence
\[
0 = \pi_2(L(u)) \to \pi_2 \left( X_0(\epsilon) \right) \to \pi_2 \left( X_0(\epsilon), L(u) \right) \to \pi_1(L(u)) \to \pi_1 \left( X_0(\epsilon) \right) = 0
\]
Hence
\[
\pi_2 \left( X_0(\epsilon), L(u) \right) \cong \pi_2 \left( X_0(\epsilon) \right) \oplus \pi_1(L(u)) \cong \mathbb{Z}^5.
\]
Let \( \beta_1, \beta_2, \beta_3 \) be the classes of topological disks with boundary on \( L(u) \) such that \( \beta_i \cap E_j = \delta_{ij} \). We care about when an element \( \beta \in H_2 \left( X_0(\epsilon), L(u) \right) \) can be represented as
\[
\beta = m_1 \beta_1 + m_2 \beta_2 + m_3 \beta_3 + k_1 E_1 + k_2 E_2 + k_3 E_3; \quad m_i, k_i \in \mathbb{Z}.
\]
When calculating the potential function, we need to discuss which homotopy class \( \beta \) with \( \mu_L(\beta) = 2 \) can be represented holomorphically, where \( \mu_L(\beta) \) is the Maslov index of a disk with Lagrangian boundary condition.

3.3. Minimal resolution of \( X_0 \). Along with the smoothing \( X_0(\epsilon) \) we also have the minimal resolution \( X_\alpha \) of \( X_0 \). One good thing is that \( X_\alpha \) is diffeomorphic to \( X_\epsilon \). So we can view them as the same smooth manifolds with different symplectic and complex structures. Recall that \( X_0 \) has a symplectic toric orbifold structure induced from the standard toric structure on \( \mathbb{C}^2 \) with moment map

\[
\mu([z_1, z_2]) = \left(|z_1|^2, \frac{3}{4}|z_1|^2 + \frac{1}{4}|z_2|^2 \right)
\]

and moment polytope shown in Figure 1. The toric minimal resolution \( X_\alpha \) can be obtained by a Hirzebruch-Jung resolution, which has the moment polytope shown in Figure 4. For more details we refer section 4.3 in [1].

Geometrically it is obtained by consecutively blowing up the singular locus of \( X_0 \) three times. There are three exceptional spheres intersecting with respect to the \( A_3 \)-diagram. Also \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is the energy parameter where three exceptional spheres have symplectic energy \( (\alpha_1, \alpha_2, \alpha_3) \) respectively. Here we divide the constant \( 2\pi \) in the energy parameter for notational simplicity.

Let \( P \) be the moment polytope of \( X_\alpha \) under Hirzebruch-Jung resolution. It is an intersection of 5 convex domains \( P = \cap_{i=0}^4 P_i \), where \( P_i = \{ l_i \geq 0 \} \). The defining affine functions \( l_i \) are

1. \( l_0 = u_1 \),
2. \( l_1 = u_2 - \alpha_1 \),
3. \( l_2 = 2u_2 - u_1 - \alpha_2 \),
4. \( l_3 = 3u_2 - 2u_1 - \alpha_3 \),
5. \( l_4 = 4u_2 - 3u_1 \),

where \( \alpha_i \) are small real parameters.

![Figure 4. Moment polytope for \( X_\alpha \).](image-url)
We denote the preimage of $\partial P_i$ by $D_i$ and call it a toric divisor. Then $D_1, D_2, D_3$ are three exceptional spheres with area $\alpha_1, \alpha_2, \alpha_3$ and they lie in the same homology classes with vanishing cycles $E_1, E_2, E_3$ respectively. By a direct calculation we have that $\langle c_1(X), D_i \rangle = 0, \quad i = 1, 2, 3$.

The divisors $D_0$ and $D_4$ are not compact. If we identify $X_0$ with $\mathbb{C}^2/\mathbb{Z}_4$ then they are the quotients of two complex lines $\{z_1 = 0\}$ and $\{z_2 = 0\}$.

To calculate the potential function of $L(u)$ in $X_\alpha$ we care about those elements $\beta \in \pi_2(X_\alpha, L(u))$ with Maslov index 2. In this case we have five toric divisors $D_i, i = 0, \cdots, 4$ and five meridian classes $\beta_i$ where $\beta_i \cap D_j = \delta_{ij}$. By Theorem 2.11 we have $\mu_L(\beta_i) = 2, i = 0, \cdots, 4$. Note that $X_\alpha$ and $X_\epsilon$ are diffeomorphic. By our previous calculation we have

$$\pi_2(X_\alpha, L(u)) \cong \mathbb{Z}^5$$

hence the meridian classes $\beta_i$ are generators of $\pi_2(X_\alpha, L(u))$ with Maslov index 2. Therefore the Lagrangian torus $L(u)$ has minimal Maslov number 2 in $X_\alpha$ by Lemma 2.13. A general class $\beta \in \pi_2(X_\alpha, L(u))$ with Maslov index 2 can be represented as either

$$\beta = \beta_i + k_1D_1 + k_2D_2 + k_3D_3; \quad m_i, k_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

or

$$\beta = \beta_i, \quad i = 0, 4.$$ 

One key difference between $X_\alpha$ and $X_\epsilon$ is that each component of the vanishing cycle of $X_\epsilon$ is a Lagrangian sphere and in $X_\alpha$ those spheres are holomorphic symplectic submanifolds, which we will discuss more later on from the perspective of a hyperKähler manifold.

4. $X_\epsilon$ as a HyperKähler Manifold

Note that $X_\epsilon$ is a Kähler surface with an induced Kähler structure from $\mathbb{C}^3$. Moreover it carries an $S^2$-family of Kähler forms making it into a hyperKähler manifold. In [11] and [12], Kronheimer showed that the minimal resolution of $\mathbb{C}^2/G$ is diffeomorphic to an ALE space, where $G \subseteq SU(2)$ is a finite subgroup. In this section we recall basic definitions and properties of hyperKähler manifolds and ALE spaces. Besides the original literature [11] and [12], we also suggest section 7 in [14] as a good reference on hyperKähler manifolds in the symplectic point of view.

**Definition 4.1.** Let $(M, g)$ be a Riemannian manifold. It is called a hyperKähler manifold if there are three orthogonal covariant constant almost complex structures $I, J, K$ satisfying the quaternion relation $IJK = -1$.

Therefore a hyperKähler manifold is Kähler with respect to each of the complex structures $I, J, K$, with corresponding Kähler forms

$$\omega_I = g(I \cdot, \cdot), \quad \omega_J = g(J \cdot, \cdot), \quad \omega_K = g(K \cdot, \cdot).$$

Indeed, there is an $S^2$-family of Kähler forms for any hyperKähler manifold. Given a vector $u = (u_I, u_J, u_K) \in S^2 \subseteq \mathbb{R}^3$ we have a complex structure $I_u = u_I I + u_J J + u_K K$ and a Kähler form

$$\omega_u = u_I \omega_I + u_J \omega_J + u_K \omega_K.$$
Definition 4.2. An ALE space (asymptotically locally Euclidean) is a hyperKähler 4-manifold with precisely one end at infinity that is isometric to \(\mathbb{C}^2/G\) for a finite subgroup \(G \subseteq SU(2)\). The metric on \(\mathbb{C}^2/G\) differs from the Euclidean metric by order \(O(r^{-4})\) terms and has an appropriate decay in the derivatives.

In our setting, the complex surface \(X_\alpha\) is the minimal resolution of \(X_0 = \mathbb{C}^2/\mathbb{Z}_4\) hence the underlying smooth manifold carries a hyperKähler structure by Theorem 4.3 below. The following existence and uniqueness theorems for hyperKähler structures are Theorem 1.1 and Theorem 1.3 in [11].

Theorem 4.3. Let \(X\) be the underlying smooth manifold of some minimal resolution of \(\mathbb{C}^2/G\). Let three cohomology classes \(\kappa_1, \kappa_2, \kappa_3 \in H^2(X; \mathbb{R})\) be given which satisfy the nondegeneracy condition
\[
\forall \Sigma \in H_2(X; \mathbb{Z}) \quad \text{with} \quad \Sigma \cdot \Sigma = -2,
\]
\[
\exists i \in \{1, 2, 3\} \quad \text{with} \quad \kappa_i(\Sigma) \neq 0.
\]
Then there exists on \(X\) an ALE hyperKähler structure for which the cohomology classes of the Kähler forms \([\omega_i]\) = \(\kappa_i\).

Theorem 4.4. If \(X_1\) and \(X_2\) are two ALE hyperKähler 4-manifolds, and there is a diffeomorphism \(X_1 \to X_2\) under which the cohomology classes of the Kähler forms agree, then \(X_1\) and \(X_2\) are isometric.

We start with \((X_\alpha, \omega_\alpha, J^{\text{toric}})\) which is the toric minimal resolution of the \(A_3\)-type singularity. Then \(E_1, E_2, E_3\) are the only classes which have self-intersection \(-2\). Set \(\kappa_1 = [\omega_\alpha], \kappa_2 = \kappa_3 = [0]\). Since \([\omega_\alpha](E_i) = \alpha_i \neq 0\), these three cohomology classes satisfy the nondegeneracy condition and we have an ALE hyperKähler structure on \(X_\alpha\), which we denote by \(\omega_1, \omega_2, \omega_K\). Then by the uniqueness theorem we know that \(\omega_\alpha = \omega_I\) as symplectic forms up to a symplectomorphism. (Another way to see \(\omega_\alpha = \omega_I\) is to connect them in the Kähler cone and apply Moser’s theorem.) Hence we have the following proposition.

Proposition 4.5. The Milnor fiber \(X_\epsilon\) is a hyperKähler 4-manifold. It admits an \(S^2\)-family of Kähler forms
\[
\omega_\alpha = u_I\omega_I + u_J\omega_J + u_K\omega_K,
\]
where \(\omega_I\) is the induced symplectic form from \(\mathbb{C}^3\). And when equipped with \(\omega_I\) it is symplectomorphic to \(X_\alpha\). In particular the cohomology class of \(\omega_I\) can be chosen such that the three exceptional spheres have symplectic energy \((\alpha_1, \alpha_2, \alpha_3)\) respectively.

In the following section we will use this relation between \(X_\alpha\) and \(X_\epsilon\) to produce a cobordism argument to compute two potential functions.

5. Potential function in \(X_0(\epsilon)\)

Fix a Lagrangian torus \(L(u)\) outside the gluing neighborhood of \(X_0(\epsilon)\), we will now calculate the potential function of \(L(u)\) with respect to some compatible almost complex structures \(J\).
5.1. **Potential function in** $X_\alpha$. First we calculate the potential function when the ambient space is $X_\alpha$, using combinatorial data from its moment polytope.

**Theorem 5.1.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian fiber of $X_\alpha$. Then the full potential function $\mathfrak{PO}_{\alpha}^{L(u)}(b) = \mathfrak{PO}_{\alpha}(u_1, u_2; y_1, y_2)$ is

\[
(5.1) \quad \mathfrak{PO}_{\alpha}(y_1, y_2) = \frac{1}{4} T^{-4} y_1^{-1} T^{-3} y_2^{-2} + \sum_{\mu \leq 2} c_\beta T^{\omega(\beta)} e^{x_1(e_1(\beta)) + x_2(e_2(\beta))}
\]

\[
(5.2) \quad = \sum_{\mu \leq 2} c_\beta T^{\omega(\beta)} y_1^{x_1(e_1(\beta))} y_2^{x_2(e_2(\beta))}
\]

where we set $y_1 = e^{x_1}, y_2 = e^{x_2}$ for $b = x_1 c_1 + x_2 c_2 \in H^1(L(u); \Lambda_0)$.

**Proof.** In this calculation we always use the toric complex structure $J$ on $X_\alpha$ and we omit it in the notation. Let $\mathcal{M}_1(X_\alpha, L(u); \beta, k_1, k_2, k_3)$ be the moduli space of holomorphic disks with boundary on $L(u)$ representing the class $\beta = \beta_i + k_1 D_1 + k_2 D_2 + k_3 D_3$. Here $\beta_i$ are meridian classes with respect to $D_i$. Let $c_\beta$ be corresponding one-point open Gromov-Witten invariant. Then in this setting the potential function has the following formula.

First by Cho-Oh’s classification result in [5], the Maslov index of a holomorphic disk with boundary on $L(u)$ is twice the sum of intersection numbers with toric divisors. Note that since $X_\alpha$ is nef, by Theorem 2.12 we know that a class $\beta \in \pi_2 (X_\alpha, L(u))$ with $\mu(\beta) = 2$ and $\mathcal{M}_1 (\beta) \neq \emptyset$ it is of the following type

\[
\beta = \beta_i, \quad i = 0, 4;
\]

or

\[
\beta = \beta_i + k_1 D_1 + k_2 D_2 + k_3 D_3; \quad i = 1, 2, 3; \quad k_i \in \mathbb{Z}.
\]

Next we calculate $c_\beta, \omega(\beta), e_1(\beta), e_2(\beta)$ for above $\beta$.

By Theorem 4.5 and Theorem 4.6 in [7],

\[
\omega(\beta_i) = l_i(u); \quad i = 0, 1, 2, 3, 4;
\]

\[
\omega(D_i) = \alpha_i; \quad i = 1, 2, 3;
\]

\[
e_j(\beta_i) = \partial l_i / \partial u_j; \quad i = 0, 1, 2, 3, 4; \quad j = 1, 2.
\]

Here $l_i$ are defining affine functions of the moment polytope in Figure 4.

For $c_\beta$, the one-point open Gromov-Witten invariant, we discuss in two cases. If $\beta = \beta_i$ then $c_\beta = 1$ for all $i$, which has been calculated in Theorem 4.5 and Theorem 4.6 in [7]. If $\beta = \beta_i + k_1 D_1 + k_2 D_2 + k_3 D_3; i = 1, 2, 3$, we apply Theorem 2.16 of Chan and Lau on admissible tuples.

By direct calculation the only admissible tuples $\{k_1, k_2, k_3\}$ are listed as follows.

1. when $i = 1$, $\{0, 0, 0\}; \{1, 0, 0\}; \{1, 1, 0\}; \{1, 1, 1\}$.
2. when $i = 2$, $\{0, 0, 0\}; \{0, 1, 0\}; \{1, 1, 0\}; \{1, 1, 1\}; \{0, 1, 1\}; \{1, 1, 1\}$.
3. when $i = 3$, $\{0, 0, 0\}; \{0, 0, 1\}; \{0, 1, 1\}; \{1, 1, 1\}$. 

The original result of Chan and Lau assume that the toric manifold is compact. We modify it to our setting which is an open symplectic toric manifold with convex boundary.

**Lemma 5.2.** The correspondence between admissible tuples and one-point open Gromov-Witten invariant works in $X_\alpha$.

*Proof.* The proof of Theorem 1.2 in [4] relates the one-point open Gromov-Witten invariant to the local Gromov-Witten invariant. Let $K$ be a large compact set in $X_\alpha$ containing all exceptional divisors. Then $X_\alpha - K$ has the standard symplectic and complex structures as in the end of $\mathbb{C}^2/\mathbb{Z}_4$. In particular $X_\alpha$ has a convex boundary at infinity. When we fix a Lagrangian torus $L(u)$ all holomorphic disks with boundary on $L(u)$ are contained in some large neighborhood of $L(u)$ by maximal principle. Outside this large neighborhood there is no other toric divisors. Therefore the correspondence between admissible tuples and one-point open Gromov-Witten invariant works in this compact large neighborhood.

Another way to see this is that first we can compactify the moment polytope of $X_\alpha$. If the compactification is singular then we take a toric desingularization. The desingularization may fail to be nef. But if the new toric divisors produced from desingularization do not intersect our chain of $-2$ spheres then the correspondence between admissible tuples and one-point open Gromov-Witten still works. In our setting this can be done using the natural compactification of the moment polytope of $X_\alpha$. We refer to the proof of Theorem 1.2 in [4] for a similar argument. \qed

Combining all the information above we obtain the full potential function for a Lagrangian fiber $L(u)$, which finishes the proof. \qed

**Remark 5.3.** In the case of a Fano symplectic manifold, the potential function has only finitely many terms. But in the case of a nef symplectic manifold, the potential function can have infinitely many terms. In the above setting of a nef toric surface, it has finitely many terms because of the relation between nonvanishing of $c_3$ and admissible tuples in [4]. Note that there are always finitely many admissible tuples of a given length.

Once we have the full potential function in the ambient space $X_\alpha$, we want to use it to calculate the full potential function in $X_0(\epsilon)$. Note that $X_0(\epsilon)$ is obtained by cutting and pasting in a small neighborhood of the origin. For a fixed fiber $L(u)$ we can choose the neighborhood so small that $L(u)$ is not affected. Hence we can still use the coordinates system $(u_1, u_2)$ to represent a Lagrangian torus in $X_0(\epsilon)$. Similarly the coordinates $(x_1, x_2)$ can be used after we have identified $H^1(L(u); \Lambda_0)$ in the two different ambient spaces. Therefore under this identification the potential function of $L(u)$ in $X_0(\epsilon)$ can be regarded as

$$\mathcal{P}_D^{L(u)}(b) = \mathcal{P}_D(u_1, u_2; x_1, x_2) : (u_1, u_2) \times \Lambda_0^2 \to \Lambda_+.$$ 

And it is calculated explicitly as follows.

**Theorem 5.4.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian torus in $X_0(\epsilon)$ using the above identification. The full potential function $\mathcal{P}_D^{L(u)}(b)$ of $X_0(\epsilon)$ is

$$(5.3) \quad \mathcal{P}_D(u_1, u_2; x_1, x_2) = \mathcal{P}_D(y_1, y_2) = y_1^3 T^{u_1} + y_1^{-3} y_2 T^{-u_1+2u_2} + 4y_2 T^{u_2} + 6y_1^{-1} 2^{T^{-u_1+2u_2}} + 4y_1^{-2} y_2^3 T^{-2u_1+3u_2},$$
where we set \( y_1 = e^{x_1}, y_2 = e^{x_2} \) for \( b = x_1e_1 + x_2e_2 \in H^1 (L(u); \mathbb{A}_0) \).

**Remark 5.5.** By comparing those two potential functions we note that (5.3) is obtained from (5.1) by formally setting \( \alpha = 0 \). An intuitive explanation is that the symplectic structure of \( X_0(\epsilon) \) is exact hence the exceptional spheres have energy zero. One can think \( X_0(\epsilon) \) has a “ghost” moment polytope as shown in Figure 1 even though it is not toric. Also there are three additional “ghost” facets corresponding exact spheres \( E_1, E_2, E_3 \).

We consider the classes
\[
\beta = \beta_i, \quad i = 0, 4;
\]
and
\[
\beta = \beta_i + k_1E_1 + k_2E_2 + k_3E_3; \quad i = 1, 2, 3
\]
in \( \pi_2 (X_0(\epsilon), L(u)) \). To simplify the notation, we denote \( c_{i, k} \) as the one-point open Gromov-Witten invariant of the class \( \beta = \beta_i + k_1E_1 + k_2E_2 + k_3E_3 \), where \( k = (k_1, k_2, k_3) \) is a tuple of three integers. In the following section we give a rigorous proof of the invariance of those mapping degrees, using deformation techniques in hyperKähler geometry.

### 5.2. Deformation invariance.

The smoothing \( X_0(\epsilon) \) is made by gluing a small neighborhood of a Milnor fiber \( X_\epsilon \) to \( X_0 - (B_\epsilon \cap X_0) \). We fix the symplectic and complex structure outside this neighborhood and do the deformation inside, see Figure 2. In this neighborhood we want to deform both the symplectic and complex structures from \( X_\alpha \) to \( X_\epsilon \) using the fact that the underlying manifold is hyperKähler.

Recall that \( \omega_\epsilon \) is the induced symplectic form on \( X_\epsilon \) and \( \omega_\ell \) is the symplectic form on \( X_\alpha \). Then the deformation from \( X_\alpha \) to \( X_0(\epsilon) \) can be obtained by a family of gluing operations. We keep the symplectic and complex structures outside some neighborhood and in the gluing region we deform from \( X_\alpha \) to \( X_\epsilon \), see Figure 2. First pick a smooth path \( \omega_t \) of symplectic forms connecting \( \omega_\ell \) and \( \omega_\epsilon \). For example we can take
\[
\omega_t = (1 - t)\omega_\ell + t\omega_\epsilon
\]
and normalize the coefficients to make that this path lies on the \( S^2 \)-family of symplectic forms we introduced before. Then with respect to this family of symplectic forms we can always assume that the top gluing region has convex boundary. After rearranging the boundary we can glue this family to the standard bottom region, see Figure 2. Next we define the corresponding moduli spaces \( \mathcal{M}_1 (\omega_t; \beta_i, k_1, k_2, k_3) \) with respect to some compatible almost complex structures, which we will specify in the proof of following proposition.

**Proposition 5.6.** The mapping degree of
\[
ev_0 : \mathcal{M}_1 (\omega_t; L(u); \beta_i, k_1, k_2, k_3) \to L(u)
\]
is independent of \( t \).

**Proof.** The idea is to use a cobordism argument. It suffices to show that for any time \( t \) there a breaking of disks or spheres will not appear. Note that when we deform the symplectic structure we have to deform the almost complex structure because there may not exist a single \( J \) such that \( J \) is compatible with \( \omega_t \) for all \( t \).

Let \( J_t \) be a smooth family of almost complex structures such that \( J_t \) is \( \omega_t \)-tame. The existence of such a family \( J_t \) can be obtained as follows. Let \( J_\epsilon (\omega_t) \) be the
space of $\omega_t$-tame almost complex structures, then for $|t' - t''| \leq t$ small enough we have that
\[
\bigcap_{t' \leq t \leq t''} \mathcal{J}_r(\omega_t) \neq \emptyset, \quad \forall t' < t''
\]
Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a division of $[0, 1]$ such that
\[
\bigcap_{t_k \leq t \leq t_{k+1}} \mathcal{J}_r(\omega_t) \neq \emptyset.
\]
Such a division exists because $\mathcal{J}_r(\omega_t)$ is open in the space of almost complex structures and our path is compact. We start with $\omega_0 = \omega_j$ and $J_0$ is the toric complex structure on $X_n$. For a fixed division as above we pick $J_{t_k} \in \cap_{0 \leq t < t_k} \mathcal{J}_r(\omega_t)$ and connect $J_0$ with $J_{t_k}$ by a path $J_s$ in $\mathcal{J}_r(\omega_0)$. When we deform the symplectic structure from $\omega_0$ to $\omega_1$, the almost complex structure $J_{t_k}$ is unchanged. Hence we get a cobordism from $M_1(\omega_0; J_0; \beta; k_1, k_2, k_3)$ to $M_1(\omega_0; J_{t_k}; \beta, k_1, k_2, k_3)$. Next we get a cobordism from $M_1(\omega_0; J_{t_k}; \beta, k_1, k_2, k_3)$ to $M_1(\omega_{t_k}; J_{t_k}; \beta, k_1, k_2, k_3)$ since $J_{t_k}$ is $\omega_{t_k}$-tame for all $0 \leq t \leq t_k$. That is, we first fix symplectic structure and deform the almost complex structure then fix a common compatible almost complex structure and deform the symplectic structure, see Figure 5. We extend this procedure step by step to get a cobordism from $M_1(\omega_1; J_0; \beta, k_1, k_2, k_3)$ to $M_1(\omega_1; J_1; \beta, k_1, k_2, k_3)$ where $J_0$ is the toric complex structure and $J_1$ is the complex structure on $X_\tau$.

The initial manifold is $(X_\alpha, \omega_1, J_0)$. In this case by Lemma 2.13 there is no holomorphic disk of nonpositive Maslov index hence all our moduli spaces carry a fundamental cycle and the mapping degrees are well-defined. By dimension formulas
\[
dim M(\beta, J) = 2 + \mu(\beta) - 3 = \mu(\beta) - 1, \quad \beta \in \pi_2(X_\alpha, L(u))
\]
and
\[
dim M(E, J) = 4 + 2c_1(E) - 6 = -2, \quad E \in H_2(X_\alpha)
\]
we can choose our one-parameter family to be generic to avoid disk bubbles of negative Maslov index and sphere bubbles.

Next we prove that during the deformation there is no disk bubble of Maslov index zero. Let
\[
\beta = \beta_t + k_1E_1 + k_2E_2 + k_3E_3.
\]
Then we have that
\[
\omega(\beta) = \omega(\beta_t) + k_1\omega(E_1) + k_2\omega(E_2) + k_3\omega(E_3)
\]
\[
\leq l_\alpha(u) + \alpha_t + k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3
\]
(5.4)
because in our deformation the symplectic energy of class $E_i$ decrease from $\alpha_i$ to zero. Now suppose that there is a disk bubble of nonpositive Maslov index. That is, our holomorphic disk of class $\beta$ splits into two disks of classes $\beta'$ and $\beta''$ respectively where
\[
\beta = \beta' + \beta'', \quad \mu(\beta'') \leq 0.
\]
The image of those singular disks contains several disk and sphere components. There exist one disk component which intersect $E_i$ once since the total class $\beta$ intersects $E_i$ once. This component consumes most of the symplectic energy, which is roughly $l_\alpha(u) + \alpha_t$. Other components have very small symplectic energy, which is less than $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$. Moreover for the disk bubble we have that
\[
\omega(\beta'') \leq k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3.
\]
Figure 5. Deformation of symplectic and complex structures.

Since we only have finitely many admissible tuples \( \{k_1, k_2, k_3\} \) as a priori we can make the right-hand side of the above inequality uniformly small. As a consequence the image of the disk bubble in the class \( \beta'' \) can not escape the undeformed region. But in the undeformed region there is no holomorphic disk of nonpositive Maslov index. Therefore when we deform the complex and symplectic structures there is no disk bubble. So firstly we know that for each \( t \) the mapping degree is well-defined and secondly those mapping degrees do not depend on \( t \).

A direct consequence is the following.

**Theorem 5.7.** For each \( \beta, k_1, k_2, k_3 \) the mapping degree of

\[
e_{v_0} : M_1 ((X_0 (\epsilon), L (u)); J; k_1, k_2, k_3) \rightarrow L (u)
\]

is equal to \( c_{i,k} \), where \( c_{i,k} \) is the mapping degree of

\[
e_{v_0} : M_1 ((X_\alpha, L (u)); J_0; \beta, k_1, k_2, k_3) \rightarrow L (u).
\]

**Remark 5.8.** The hyperKähler structure that connects the family of resolutions \( X_\alpha \) and the family of smoothings \( X_\epsilon \) can be regarded as a simultaneous resolution of \( X_\epsilon \). In the complex 2-dimensional case we can connect the exact symplectic form and the symplectic form of resolution without passing through the orbifold \( X_0 \). But in the complex 3-dimensional case this does not work because in the \( A_2 \)-type singularity case, for example, we need to replace a Lagrangian 3-sphere with a symplectic 2-sphere. Further techniques are needed, like symplectic surgery to study conifold transitions, see [15] and [16].

Theorem 5.4 directly follows Theorem 5.7. That is, as stated in Theorem 5.4, we have

\[
\mathcal{P} \mathcal{O}^{L(u)} (y_1, y_2) = y_1 T^{u_1} + y_1^{-3} y_2 T^{-3u_1+4u_2} + 4y_2 T^{u_2} + 6y_1^{-1} y_2^{-2} T^{-u_1+2u_2} + 4y_1^{-2} y_2^{-3} T^{-2u_1+3u_2}.
\]

When \( u_1 = u_2 \), we have

\[
\mathcal{P} \mathcal{O}^{L(u)} (y_1, y_2) = (y_1 + y_1^{-3} y_2^4 + 4y_2 + 6y_1^{-1} y_2^2 + 4y_1^{-2} y_2^3) T^{u_1}.
\]
Set
\[ 0 = \frac{\partial \mathcal{P}(u)}{\partial y_1}(y_1, y_2) = (1 - 3y_1^{-4}y_2^4 - 6y_1^{-2}y_2^2 - 8y_1^{-3}y_2^3) T^{u_1}, \]
and
\[ 0 = \frac{\partial \mathcal{P}(u)}{\partial y_2}(y_1, y_2) = (4y_1^{-3}y_2^3 + 4 + 12y_1^{-1}y_2 + 12y_1^{-2}y_2^2) T^{u_1}. \]
That is,
\[
\begin{align*}
3y_1^{-4}y_2^4 + 6y_1^{-2}y_2^2 + 8y_1^{-3}y_2^3 & = 1 \\
y_1^{-3}y_2^3 + 3y_1^{-1}y_2 + 3y_1^{-2}y_2^2 & = -1
\end{align*}
\]
Set \( a = y_1^{-1}y_2 \), we have
\[
\begin{align*}
3a^4 + 6a^2 + 8a^3 & = 1 \\
a^3 + 3a + 3a^2 & = -1
\end{align*}
\]
One solution is \( a = -1 \). Hence by Theorem 2.9 we will prove that when \( u_1 = u_2 \) the Lagrangian torus \( L(u) \) is nondisplaceable. The only thing we need to verify is that \( H^1(L(u); \Lambda_0)/H^1(L(u); 2\pi \sqrt{-1} \mathbb{Z}) \subset \hat{\mathcal{M}}_{\text{weak}}(L(u)) \).

**Lemma 5.9.** Let \((M, \omega)\) be a symplectic 4-manifold and \( L \subset M \) be a Lagrangian torus. Then we have
\[
H^1(L; \Lambda_0)/H^1(L; 2\pi \sqrt{-1} \mathbb{Z}) \cong \hat{\mathcal{M}}_{\text{weak}}(L)
\]
with respect to a generic almost complex structure \( J \).

**Proof.** We want to show that for each \( b \in H^1(L; \Lambda_0) \)
\[
\sum_{k=0}^{\infty} m_k(b, \cdots, b) = \sum_{k, \beta} m_{k, \beta}(b, \cdots, b) = 0 \ mod \ PD([L]).
\]
Note that the \( A_\infty \)-structure is constructed using the moduli spaces \( \mathcal{M}_{k+1}(\beta, J) \). First we consider the naive moduli space \( \mathcal{M}(\beta, J) \) without marked points. The dimension is given by
\[
dim_k \mathcal{M}(\beta, J) = 2 + \mu_L(\beta) - 3 = \mu_L(\beta) - 1.
\]
Therefore by perturbing \( J \) there is no holomorphic disk with \( \mu_L(\beta) \leq 0 \). When \( \mu_L(\beta) \geq 4 \) by a degree calculation we have \( m_{k, \beta} = 0 \) since our Lagrangian is a 2-dimensional torus. So we only consider the contribution from holomorphic disks with \( \mu_L(\beta) = 2 \).

In this case it is calculated that
\[
m_{k, \beta}(b, \cdots, b) = \frac{c_\beta}{k!} (\partial \beta \cap b)^k : PD([L])
\]
here \( m_{k, \beta} \) is the canonical model and we refer to Appendix 1 in [9] for a detailed calculation. So the Maurer-Cartan equation becomes
\[
\sum_{k=0}^{\infty} m_k(b, \cdots, b) = \sum_{\mu_L(\beta) = 2} c_\beta e^{\partial \beta / \beta} T^{\omega(\beta)} : PD([L]),
\]
which means \( b \) is always a weak bounding cochain. \( \square \)
Remark 5.10. The above calculation works when the bulk deformation $b = 0$ and $m = m^{\text{can}}$ is the canonical model. When the bulk deformation $b \in H^2(M; \Lambda_0)$ does not intersect our Lagrangian the same statement is also true by same calculation. That is,
\[ H^1 (L; \Lambda_0) / H^1 (L; 2\pi \sqrt{-1} \mathbb{Z}) \cong \tilde{\mathcal{M}}_{\text{weak}} (L, m^b). \]
We refer to Proposition 3.1 in [10] for the calculation with bulk deformation. In this paper we only use the bulk deformation $b \in H^2(M; \Lambda_0)$.

In Proposition 5.6 we showed that there is no disk bubble of nonpositive Maslov index for all $\{J_t\}_{t \in [0,1]}$. Therefore by applying Lemma 5.9 to $(X_0(\epsilon), J_1)$ we have that
\[ H^1 (L; \Lambda_0) / H^1 (L; 2\pi \sqrt{-1} \mathbb{Z}) \cong \tilde{\mathcal{M}}_{\text{weak}} (L, m^b). \]
for any $b \in H^2(X_0(\epsilon); \Lambda_0)$. In conclusion we find a continuous family of non-displaceable Lagrangian tori $L(u)$ parameterized by $\mathbb{R}^+$, which finishes the proof of Theorem 5.4. Moreover, these tori are monotone Lagrangians hence there is no holomorphic disk with nonpositive Maslov index. By Remark 2.8 the potential function of $L(u)$ does not depend on $J$.

**Proposition 5.11.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian torus in $X_0(\epsilon)$ with $u_1 = u_2$. Then $L(u)$ is monotone with proportion $2/u_1$.

**Proof.** By an earlier calculation in section 3.2 we have $\pi_2 (X_0(\epsilon), L(u)) \cong \mathbb{Z}^5$ and it is generated by meridian classes $\beta_i$. For $\beta_i$ we have
\[ \mu_{L(u)} (\beta_i) = 2, \quad \omega (\beta_i) = \lim_{\alpha \to 0} l_i (u). \]
When $u_1 = u_2$ we have
\[ \lim_{\alpha \to 0} l_i (u) = u_1 = u_2, \quad i = 0, \cdots, 4 \]
hence $L(u)$ is a monotone Lagrangian torus. \qed

6. **General case of an $A_n$-type singularity**

In section 5 we explicitly worked out the potential function of certain tori in the smoothing of a $A_3$-type singularity. In this section we generalize the situation to a hypersurface with $A_n$-type singularity.

Consider the family of hypersurfaces
\[ X_{\epsilon, n} = \{ x^2 + y^2 + z^{n+1} = \epsilon \} \subseteq \mathbb{C}^3. \]
When $\epsilon = 0$ the surface $X_0 = X_{0, n}$ has an $A_n$-type singularity at the origin and we can identify it with $\mathbb{C}^2 / \mathbb{Z}_n$. Here $\mathbb{Z}_n \cong \{ \eta \in \mathbb{C} \mid \eta^n = 1 \}$ acts on $\mathbb{C}^2$ by
\[ \eta \cdot (z_1, z_2) = (\eta^{-1} z_1, \eta z_2). \]
Hence $X_0$ has a symplectic toric orbifold structure induced from the standard toric structure on $\mathbb{C}^2$ with moment map
\[ \mu ([z_1, z_2]) = \left( |z_1|^2, \frac{n}{n+1} |z_1|^2 + \frac{1}{n+1} |z_2|^2 \right) \]
and moment polytope shown in Figure 6.

Next we want to show that in the smoothing of $X_0$ there is a line of non-displaceable Lagrangian tori parameterized by $\{u_1 = u_2\}$. The strategy of proof is almost the same with the case of an $A_3$-type singularity but combinatorially harder.
Figure 6. Moment polytope for $X_0 = \{x^2 + y^2 + z^{n+1} = 0\}$.

6.1. **Potential function in the minimal resolution.** By consecutively blowing up the singular locus $n$ times we get the minimal resolution $X_\alpha$ of $X_0$. It has $n$ exceptional spheres intersecting with respect to the $A_n$-diagram. Here $\alpha = (\alpha_1, \cdots, \alpha_n)$ are the energy parameters of exceptional spheres.

We choose the resolution in a toric way such that $X_\alpha$ has moment polytope

$$P = \bigcap_{k=1}^{n+1} P_k = \bigcap_{k=1}^{n+1} \{l_k \geq 0\}$$

defined by $n+2$ affine functions

1. $l_0 = u_1$;
2. $l_k = ku_2 - (k-1)u_1 - \alpha_k, \quad k = 1, \cdots, n$;
3. $l_{n+1} = (n+1)u_2 - nu_1$.

Let $D_k = \mu^{-1}(\partial P_k)$ be the toric divisors then $D_1, \cdots, D_{n-1}$ are exceptional spheres with first Chern number zero. Hence $X_\alpha$ is a nef surface. Let $(u_1, u_2) \in P$ and $L(u)$ be the Lagrangian torus fiber. We can calculate the potential function of $L(u)$ in $X_\alpha$ as we did in section 5.

**Theorem 6.1.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian fiber of $X_\alpha$. Then the full potential function $\Psi^L(u)(b) = \Psi_\alpha(u_1, u_2; y_1, y_2)$ is

$$(6.1) \quad \Psi_\alpha(y_1, y_2) = y_1 T^{u_1} + y_1^{-n} y_2^{n+1} T^{-nu_1 + (n+1)u_2} + \sum_{k=1}^{n} y_1^{1-k} y_2^{k + 1/(1-k)} T^{nu_1 + ku_2 - \alpha_k} (1 + H_k(T^{\alpha_1}, T^{\alpha_2}, \cdots, T^{\alpha_n})).$$

Here we set $y_1 = e^{x_1}, y_2 = e^{x_2}$ for $b = x_1 e_1 + x_2 e_2 \in H^1(L(u); \Lambda_0)$ and

$$H_k(T^{\alpha_1}, T^{\alpha_2}, \cdots, T^{\alpha_n})$$

are polynomials in terms of $T^{\alpha_1}, T^{\alpha_2}, \cdots, T^{\alpha_n}$ corresponding to admissible tuples with center $k$. 
Proof. We calculate the potential function by counting Maslov index 2 holomorphic disks. The minimal resolution $X_{\alpha}$ is diffeomorphic to the Milnor fiber $X_{\epsilon}$ hence both of them are homotopy equivalent to a wedge of $n$ spheres. As a consequence we have

$$\pi_2(X_{\alpha}, L(u)) \cong \pi_2(X_{\alpha}) \oplus \pi_1(L(u)) \cong \mathbb{Z}^{n+2}.$$ 

Let $\beta_k$ be meridian classes, that is, $\beta_k \cap D_j = \delta_{kj}$. Then by Cho-Oh’s Maslov index formula we have that $\mu_{L(u)}(\beta_k) = 2$, $k = 0, \ldots, n+1$. Hence the meridian classes are generators of $\pi_2(X_{\alpha}, L(u))$. By Theorem 2.12, a general class $\beta$ of Maslov index 2 with $\mathcal{M}_1(\beta) \neq \emptyset$ is of the following type

$$\beta = \beta_k, \quad k = 0, n+1;$$

or

$$\beta = \beta_k + s_1D_1 + s_2D_2 + \cdots + s_nD_n, \quad k = 1, \ldots, n; \quad s_k \in \mathbb{Z}_{\leq 0}.$$ 

The potential function has the formula

$$\mathcal{P}O^L(b) = \sum_{\mu_{L(\beta)} = 2} c_{\beta} T^{\omega(\beta)} y_1^{e_1(\beta_1)} y_2^{e_2(\beta_2)}$$

where

$$\omega(\beta_i) = \lambda_i(u); \quad i = 0, \ldots, n+1;$$

$$\omega(D_i) = \alpha_i; \quad i = 1, \ldots, n;$$

$$e_j(\beta_i) = \partial \lambda_i / \partial u_j; \quad i = 0, \ldots, n+1; \quad j = 1, 2.$$ 

and $c_{\beta}$ is the one-point open Gromov-Witten invariant. The first two terms in (6.1) come from the classes $\beta_0$ and $\beta_{n+1}$. Other terms come from classes

$$\beta = \beta_k + s_1D_1 + s_2D_2 + \cdots + s_nD_n.$$ 

Then we apply Theorem 2.16 on admissible tuples. The number $c_{\beta}$ is either one or zero according to whether the tuple $\{s_1, \ldots, s_n\}$ is admissible or not. When all $s_k = 0$ we have the term $1$ in $1 + H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$. Otherwise we have a term

$$T^{s_1\alpha_1 + \cdots + s_n\alpha_n}$$

in $1 + H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$. For a fixed integer $n$ we always have a finite collection of admissible tuples hence $H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$ is a polynomial. This finishes our calculation in the minimal resolution setting. \hfill $\square$

Remark 6.2. Given positive integers $n$ and $k$ one can easily list all the admissible tuples with center $k$ just by definition. But writing down the full expression of the polynomials $H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$ would be too long. In the following subsection we deform the symplectic structure to be exact hence in the limit $H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$ becomes a positive integer as $\alpha = (\alpha_1, \ldots, \alpha_n)$ tends to zero. We will show that these integers have good combinatorial properties. This ensures the existence of critical points of the potential functions.
6.2. Potential function in the smoothing. The smoothing $X_0(\varepsilon)$ is obtained by cutting a neighborhood of singular point in $X_0$ and pasting back a similar neighborhood of Milnor fiber $X_\varepsilon$. It is diffeomorphic to $X_\varepsilon$ and $X_\alpha$ but with different symplectic and complex structures. By previous calculations we have

$$\pi_2(X_\alpha, L(u)) \cong \mathbb{Z}^{n+2}$$

and the generators are meridian classes. Note that we do small deformation inside a neighborhood of the vanishing cycles and there are finitely many admissible tuples when $n$ is fixed. This enables us to use the same deformation technique in Proposition 5.6 to calculate the potential function in $X_0(\varepsilon)$.

**Theorem 6.3.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian fiber of $X_0(\varepsilon)$ under the coordinate identification between $X_0(\varepsilon)$ and $X_\alpha$. Then the full potential function $PO\,(b)$ is

$$PO\,(u_1, u_2; y_1, y_2) = y_1 T^{u_1} + y_2^{n+1} T^{-n u_1 + (n+1) u_2} + \sum_{k=1}^n y_1^{-k} y_2^k T^{(1-k) u_1 + k u_2} (1 + H_k).$$

(6.2)

Here we set $y_1 = e^{x_1}, y_2 = e^{x_2}$ for $b = x_1 e_1 + x_2 e_2 \in H^1(L(u); \Lambda_0)$ and $H_k = \lim_{\alpha \to 0} H_k(T^{\alpha_1}, T^{\alpha_2}, \ldots, T^{\alpha_n})$ are limits of the polynomials as the energy parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ tend to zero.

**Proof.** The proof is parallel to the proof in Theorem 5.4. We connect two symplectic structures on $X_\alpha$ and $X_0(\varepsilon)$ by a path. Then we produce a cobordism between corresponding moduli spaces of holomorphic disks. Using the fact that there are finitely many admissible tuples we can give an energy-estimate argument to exclude disk bubbles of Maslov index 0. Then this cobordism gives us the invariance of mapping degrees. Therefore the potential function in $X_0(\varepsilon)$ is obtained as a limit of the potential function in $X_\alpha$. □

The coefficient $1 + H_k$ is the number of admissible tuples of length $n$ with center $k$ hence it depends on both $n$ and $k$. Actually they are the binomial coefficients

$$C(n, k) = \frac{n!}{(n-k)!k!}.$$

**Proposition 6.4.** The number $1 + H_{n,k}$ of admissible tuples of length $n$ with center $k$ is $C(n+1, k)$.

**Proof.** The first few numbers can be listed easily and we use them to apply induction. For example in $A_3$-case we have

$$1 + H_{3,k} : 1, 4, 6, 4, 1.$$  

And in $A_4$-case we have

$$1 + H_{4,k} : 1, 5, 10, 10, 5, 1.$$  

Note that the binomial coefficients are determined by the recursive formula

$$C(n+1, k) = C(n, k-1) + C(n, k)$$
hence we divide the set of admissible tuples into subsets to count the numbers. Let
\[ \{ \{ s_1, s_2, \cdots, s_n \} \}_k \]
be the set of admissible tuples of length \( n \) with center \( k \). It is a disjoint union of
four subsets with different constraints
\[
\{ \{ s_1 = 0, s_2, \cdots, s_n = 0 \} \}_k;
\]
\[
\{ \{ s_1 = 0, s_2, \cdots, s_n = 1 \} \}_k;
\]
\[
\{ \{ s_1 = 1, s_2, \cdots, s_n = 0 \} \}_k;
\]
\[
\{ \{ s_1 = 1, s_2, \cdots, s_n = 1 \} \}_k.
\]
We have the following bijections between sets
\[
\{ \{ s_1 = 0, s_2, \cdots, s_n = 0 \} \}_k \cong \{ \{ s_1, s_2, \cdots, s_{n-2} \} \}_{k-1};
\]
\[
\{ \{ s_1 = 0, s_2, \cdots, s_n \} \}_k \cong \{ \{ s_1, s_2, \cdots, s_{n-1} \} \}_{k-1};
\]
\[
\{ \{ s_1 = 1, s_2, \cdots, s_n = 1 \} \}_k \cong \{ \{ s_1 = 0, s_2, \cdots, s_n = 0 \} \}_k.
\]
The first bijection is given by forgetting \( s_1 \) and \( s_n \). The second bijection is given
by forgetting \( s_1 \). The third bijection is given by subtracting 1 in each slot.
Now suppose that the number \( 1 + H_{n,k} = C(n + 1, k) \) for \( n < N \). Then by the
bijections above we know that
\[
| \{ \{ s_1 = 0, s_2, \cdots, s_N = 0 \} \}_k | = C(N - 1, k - 1);
\]
\[
| \{ \{ s_1 = 0, s_2, \cdots, s_N \} \}_k | = C(N, k - 1);
\]
\[
| \{ \{ s_1 = 1, s_2, \cdots, s_N = 1 \} \}_k | = C(N - 1, k - 1).
\]
Combining the first two equalities we have that
\[
| \{ \{ s_1 = 0, s_2, \cdots, s_N = 1 \} \}_k | = C(N, k - 1) - C(N - 1, k - 1) = C(N - 1, k - 2).
\]
Moreover we have the following bijection by symmetry of admissible tuples
\[
\{ \{ s_1 = 1, s_2, \cdots, s_N = 0 \} \}_k \cong \{ \{ s_1 = 0, s_2, \cdots, s_N = 1 \} \}_{n-k+1}.
\]
Therefore
\[
| \{ \{ s_1 = 1, s_2, \cdots, s_N = 0 \} \}_k | = C(N - 1, N - k - 1) = C(N - 1, k).
\]
Summing all these four equalities we have that
\[
| \{ \{ s_1, s_2, \cdots, s_N \} \}_k |
\]
\[
\quad = C(N - 1, k - 1) + C(N - 1, k - 2) + C(N - 1, k) + C(N - 1, k - 1)
\]
\[
\quad = C(N + 1, k).
\]
Since this counting formula is correct for small \( n \) we complete the induction
procedure and finish the calculation of the number of admissible tuples of length \( n \)
with center \( k \).

As a direct consequence, in the smoothing of an \( A_n \)-type singularity there are
exactly \( 2^{n+1} \) homotopy classes
\[
\beta = \beta_k + s_1 E_1 + s_2 E_2 + \cdots + s_n E_n
\]
which can be represented holomorphically and pass through a generic point on \( L(u) \)
onece. This was already known in [13].
Using these integers we have the explicit expression of our potential function

$$P^L(u) (b) = P(u_1, u_2; y_1, y_2)$$

\begin{equation}
(6.4)
= \sum_{k=0}^{n+1} C(n+1, k) y_1^{1-k} y_2^k T^{1-k} u_1^1 + ku_2^1.
\end{equation}

Next we look for its critical points when $u_1 = u_2$. By setting partial derivatives to be zero we have

$$0 = \frac{\partial P^L(u)}{\partial y_1} (y_1, y_2) = \sum_{k=0}^{n+1} C(n+1, k) (1-k) y_1^{-k} y_2^k T^u_1$$

and

$$0 = \frac{\partial P^L(u)}{\partial y_2} (y_1, y_2) = \sum_{k=0}^{n+1} C(n+1, k) k y_1^{-k} y_2^{-k} T^u_1.$$

We claim that $y_1^{-1} y_2 = -1$ is a solution to these two equations. For the second equation it follows from the fact that

$$\left( \frac{d}{dx} (1 + x)^{n+1} \right) |_{x=-1} = 0.$$

And the first one follows from the fact that

$$(1 + x)^{n+1} |_{x=-1} = 0$$

combined with the second one. Therefore when $u_1 = u_2$ the potential functions always have critical points and the corresponding Lagrangian tori are nondisplaceable. Note that the meridian classes $\beta_k$ generate $\pi_2 (X_0(\epsilon), L(u))$ and

$$\mu_{L(u)} (\beta_k) = 2, \quad \omega (\beta_k) = u_1 = u_2, \quad \forall k = 0, \ldots, n + 1$$

therefore when $u_1 = u_2$ these tori are also monotone.

Remark 6.5. The above calculation in $X_0 (\epsilon)$ can be regarded as a local study of nondisplaceable Lagrangian tori in the smoothing of an $A_n$-type singularity. First we can generalize it to a closed toric symplectic orbifold with a special presentation of its moment polytope, like the natural compactification of the moment polytope and the weighted projective planes $\mathbb{P} (1, n, n+1)$. Further more, by not assuming the toric condition, for a 4-dimensional symplectic orbifold with $A_n$-type singularities we can perform the smoothing operation and then find a family of nondisplaceable Lagrangian tori, which will be carried out in later sections. But in those cases the nondisplaceable Lagrangian tori are parameterized by a finite length interval, not $\mathbb{R}^+$. Also there may only be a unique monotone Lagrangian torus due to the existence of new disk classes of higher energy.

7. Examples in closed case

In this section we consider symplectic 4-orbifolds with $A_n$-type singularities and study Lagrangian tori near the smoothing of the singularities.
7.1. Compactified moment polytope of an $A_n$-type singularity. We start with the case of a natural compactification of the moment polytope of an $A_n$-type singularity. This compact polytope is obtained by adding one extra facet $P_\kappa = \{ l_\kappa = -u_2 + \kappa \geq 0 \}$ where $\kappa$ is a positive number.

First we perform the explicit calculation for the compactification of $X_{0,3}$. We denote the corresponding symplectic orbifold by $\overline{X}_{0,3}$. It has two orbifold singularities, one at the origin and the other at the top right corner. We only smooth the origin and denote the resulting orbifold by $\overline{X}_{0,3}(\epsilon)$.

The appearance of the new toric divisor $D_\kappa = \mu^{-1}(\partial P_\kappa)$ gives a new meridian disk class $\beta_\kappa$ hence there is a new term in the potential function. To deal with this new term we will modify the potential function using bulk deformations. In the following we calculate the potential function with two different bulk deformations, one has the property that it is easy to find explicit critical points and the other is easy to be used in the case of a general orbifold.

**Theorem 7.1.** Let $L(u) = L(u_1, u_2)$ be a Lagrangian torus in $\overline{X}_{0,3}(\epsilon)$ parameterized by an interior point in the moment polytope of $\overline{X}_{0,3}$. Consider the bulk deformation by

$$b = vPD(D_0) + wPD(D_1) \in H^2(\overline{X}_{0,3}(\epsilon); \Lambda_0).$$

Then the full potential function $\mathfrak{PD}_b^{L(u)}(b)$ of $L(u)$ is

$$\mathfrak{PD}_{b}^{L(u)}(b) = \mathfrak{PD}(y_1, y_2) = e^v y_1 T^{u_1} + (e^w + e^{v-w} + 2e^v) y_2 T^{u_2} + (1 + 2e^w + 2e^{v-w} + e^v) y_1^{-1} y_2^{-1} T^{-u_1 - 2u_2} + (2 + e^w + e^{v-w}) y_1^{-2} y_2^{-3} T^{-2u_1 + 3u_2} + y_1^{-3} y_2^{-4} T^{-3u_1 + 4u_2} + y_2^{-1} T^{-u_2 + \kappa},$$

where we set $y_1 = e^{x_1}, y_2 = e^{x_2}$ for $b = x_1 e_1 + x_2 e_2 \in H^1(L(u); \Lambda_0)$.

Note that $D_0 = \mu^{-1}(\partial P_0) = \mu^{-1}\{ u_1 = 0, 0 \leq u_2 \leq \kappa \}$ is a smooth divisor and $D_1$ is one of the exceptional spheres with $D_0 \cap D_1 = 1$. Hence for $(v, w) \in \Lambda_0^2$ we have

$$b = vPD(D_0) + wPD(D_1) \in H^2(\overline{X}_{0,3}(\epsilon); \Lambda_0)$$

which can be used to define the bulk-deformed $A_\infty$-structure and corresponding potential function.
Before giving the proof we first calculate the critical points of the above potential function when \( u_1 = u_2 \) and \(-u_1 - u_2 + \kappa > 0\). Set

\[
0 = T^{-u_1} \frac{\partial \mathcal{P}^{L(u)}_b}{\partial y_1} (y_1, y_2) = e^v + (1 + e^v + 2e^w + 2e^{v-w}) \left(-y_1^{-2}y_2^2\right) + (2 + e^w + e^{v-w}) \left(-2y_1^{-3}y_2^3\right) - 3y_1^{-4}y_2^4
\]

and

\[
0 = T^{-u_1} \frac{\partial \mathcal{P}^{L(u)}_b}{\partial y_2} (y_1, y_2) = (2e^v + e^{v-w} + e^w) + (1 + e^v + 2e^w + 2e^{v-w}) \left(2y_1^{-1}y_2\right) + (2 + e^w + e^{v-w}) \left(3y_1^{-2}y_2^2\right) + 4y_1^{-3}y_2^3 - y_2^{-2T-u_1-u_2+\kappa}.
\]

By multiplying \( e^w \) on both sides we get two equations of four variables in terms of \( y_1, y_2, e^v, e^w \). The idea to solve these equations is to assign values to \( y_1 \) and \( y_2 \) such that the equations above become polynomials in terms of \( e^v \) and \( e^w \). Then we solve these polynomial equations to assure that the previous choices of \( y_1 \) and \( y_2 \) are critical points.

For example when \( y_1^{-1}y_2 = 1 \) we have

\[
\begin{align*}
4e^v - 4 - T^{-u_1-u_2+\kappa} &= 0 \\
4e^{2w} + 8e^w + 4 + T^{-u_1-u_2+\kappa} &= 0
\end{align*}
\]

Explicit solutions of \( e^v \) and \( e^w \) are

\[
\begin{align*}
e^v &= 1 + \frac{1}{4}T^{-u_1-u_2+\kappa} \in \Lambda_0 - \Lambda_+ \\
e^w &= -1 \pm \frac{1}{2} \sqrt{-T^{-u_1-u_2+\kappa}} \in \Lambda_0 - \Lambda_+
\end{align*}
\]

Therefore \( y_1^{-1}y_2 = 1 \) are critical points of the potential function with bulk deformation by \( b = vPD(D_0) + wPD(D_1) \), where \( v \) and \( w \) are solved above. We remark that to solve \( v, w \in \Lambda_0 \) we need that \( e^v, e^w \in \Lambda_0 - \Lambda_+ \) to use the logarithm function.

Next we calculate the potential function with another bulk deformation which can be used in general cases.

**Theorem 7.2.** Let \( L(u) = L(u_1, u_2) \) be a Lagrangian torus in \( \overline{X_{0,3}}(\epsilon) \) parameterized by an interior point in the moment polytope of \( \overline{X_{0,3}} \). Consider the bulk deformation by

\[
b = vPD(D_1) + wPD(D_3) \in H^2(\overline{X_{0,3}}(\epsilon) ; \Lambda_0) .
\]

Then the full potential function \( \mathcal{P}^{L(u)}_b(b) \) of \( L(u) \) is

\[
\mathcal{P}^{L(u)}_b(b) = y_1T^{u_1} + (e^v + e^{-v} + e^w + e^{-w}) y_2T^{u_2}
\]

\[
+ (2 + e^v+w + e^{-v+ w} + e^{-v-w} + e^w) y_1^{-1}y_2^{-2T^{-u_1+2u_2}}
\]

\[
+ (e^v + e^{-w} + e^w + e^{-v}) y_1^{-2}y_2^{-3T^{-2u_1+3u_2}} + y_1^{-3}y_2^{-4T^{-3u_1+4u_2}} + y_2^{-1}T^{-u_2+\kappa},
\]

where we set \( y_1 = e^{x_1}, y_2 = e^{x_2} \) for \( b = x_1e_1 + x_2e_2 \in H^1(L(u) ; \Lambda_0) \).
Proof. In the calculation of the potential function without bulk deformation we calculated the one-point open Gromov-Witten invariants of all disk classes with Maslov index 2, which are mapping degrees of some evaluation maps. When we have bulk deformation we count mapping degrees with bulk effect. Here we explain the calculation through examples and refer to section 7 in [9] and Theorem 24.6 in [10] for more detail.

The bulk effect comes from intersection numbers of a disk class and the bulk class. For example, the term $e^{v+w}y_1^{-1}y_2^{3}T^{-u_1+2u_2}$ comes from the contribution of the class $\beta = \beta_2 + D_2$. We have following intersection numbers

$$\beta \cap D_1 = 1, \quad \beta \cap D_3 = 1.$$ 

Therefore the bulk effect of $ePD(D_1)$ is

$$\sum_{l=0}^{\infty} \frac{1}{l!} (\beta \cap vD_1)^l PD(L(u)) = e^{v}PD(L(u))$$

and the bulk effect of $wPD(D_3)$ is

$$\sum_{l=0}^{\infty} \frac{1}{l!} (\beta \cap wD_3)^l PD(L(u)) = e^{w}PD(L(u)).$$

So the total bulk effect of $b$ on the class $\beta = \beta_2 + D_2$ is $e^{v+w}PD(L(u))$. Note that in the definition of the potential function we modulo $PD(L(u))$ hence we get the coefficient $e^{v+w}$ in the term $e^{v+w}y_1^{-1}y_2^{3}T^{-u_1+2u_2}$.

Then by checking other intersection numbers of $b$ and all disk classes with nonzero one-point open Gromov-Witten invariant we get the full potential function with bulk deformation $b$. Theorem 7.1 can be proved in the same way. \qed

Next we calculate the critical points of the above potential function when $u_1 = u_2$ and $-u_1 - u_2 + \kappa > 0$. Set

$$0 = T^{-u_1} \frac{\partial PD_{\beta}}{\partial y_1} (y_1, y_2) = 1 + (2 + e^{v+w} + e^{-v+w} + e^{-v-w} + e^{v-w}) (-y_1^{-2} y_2^{2})$$

$$+ (e^{w} + e^{-w} + e^{v} + e^{-v}) (-2y_1^{-3} y_2^{3}) - 3y_1^{-4} y_2^{4}$$

and

$$0 = T^{-u_1} \frac{\partial PD_{\beta}}{\partial y_2} (y_1, y_2) = (e^{v} + e^{-v} + e^{w} + e^{-w})$$

$$+ (2 + e^{v+w} + e^{-v+w} + e^{-v-w} + e^{v-w}) (2y_1^{-1} y_2)$$

$$+ (e^{w} + e^{-w} + e^{v} + e^{-v}) (3y_1^{-2} y_2^{2})$$

$$+ 4y_1^{-3} y_2^{3} - y_2^{-2} T^{-u_1-u_2+\kappa}.$$ 

We want to find solutions $(y_1, y_2; v, w)$. The idea is to assign suitable values to $(y_1, y_2)$ such that we get two polynomials in terms of $(e^{v}, e^{w})$ which have common zeroes. For example, denote by

$$\begin{align*}
A &= e^{v+w} + e^{-v+w} + e^{-v-w} + e^{v-w} \\
B &= e^{w} + e^{-w} + e^{v} + e^{-v}
\end{align*}$$
when \( y_1 = 1, y_2 = 2 \) we have
\[
\begin{cases}
4A + 16B = -55 \\
4A + 13B = -40 + \frac{1}{4} T^{-u_1-u_2+\kappa}
\end{cases}
\]
from (7.5) and (7.6). Solving \( A \) and \( B \) we get
\[
\begin{cases}
A = e^{v+w} + e^{-v+w} + e^{-v-w} + e^{v-w} = \frac{25}{4} + \frac{1}{3} T^{-u_1-u_2+\kappa} \in \Lambda_0 - \Lambda_+ \\
B = e^{w} + e^{-w} + e^{v} + e^{-v} = -5 - \frac{1}{12} T^{-u_1-u_2+\kappa} \in \Lambda_0 - \Lambda_+
\end{cases}
\]
By multiplying \( e^{v+w} \) on both sides we get
\[
\begin{cases}
e^{2v+w} + e^{2w} + 1 + e^{2v} = \left( \frac{25}{4} + \frac{1}{3} T^{-u_1-u_2+\kappa} \right) e^{v+w} \\
e^{2v+w} + e^{w} + e^{v+2w} + e^{v} = \left( -5 - \frac{1}{12} T^{-u_1-u_2+\kappa} \right) e^{v+w}
\end{cases}
\]
This is a system of cubic polynomials with two variables \( e^{v} \) and \( e^{w} \) and coefficients in \( \Lambda_0 \). By checking the resultant we know that there exist some solutions. More explicitly we can solve them as follows.

We set \( X = e^{v} \) and \( Y = e^{w} \) for notational simplicity. Then the above equations become
\[
\begin{cases}
X^2(Y + 1) - AXY + Y^2 + 1 = 0 \\
X^2Y + X(Y^2 - BY + 1) + Y = 0
\end{cases}
\]
First we solve for \( X \) by using the root formula for a quadratic polynomial, which is a combination of algebraic expressions of \( Y \).
\[
X = \frac{AY \pm \sqrt{A^2Y^2 - 4(Y + 1)(Y^2 + 1)}}{2(Y + 1)}
\]
Then we plug the solution with plus sign into the second equation and get
\[
[AY + \sqrt{A^2Y^2 - 4(Y + 1)(Y^2 + 1)}]Y + 2[AY + \sqrt{A^2Y^2 - 4(Y + 1)(Y^2 + 1)}](Y^2 + 1 - BY)(Y + 1) + 4Y(Y + 1)^2 = 0
\]
(7.7)

By rearranging terms and squaring both sides of this equation we get a new polynomial of \( Y \) of higher degree, which fits in the following algebraic lemma. Note that both of the highest order term and the constant term are in \( \Lambda_0 - \Lambda_+ \) and nonzero. We remark that here we use \( A, B \) for notational simplicity and they are not related to \( A, B \) in Proposition 7.4.

**Lemma 7.3.** Consider the polynomial equation
\[
X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 = 0
\]
where the coefficients \( a_{n-1}, \cdots, a_0 \in \Lambda_0 \) and \( 0 \neq a_0 \in \Lambda_0 - \Lambda_+ \). Then it has at least one solution in \( \Lambda_0 - \Lambda_+ \).
Proof. Let $\Lambda$ be the fraction field of $\Lambda_0$. It is algebraically closed when the ground field of our Novikov rings are algebraically closed. In this paper we always assume that the ground field is $\mathbb{C}$. Therefore the above polynomial equation has at least one solution $X = f(T)/g(T)$ in $\Lambda$, where $f(T), g(T) \in \Lambda_0$. We divide common powers of the $T$ variable and assume that at least one of $f(T)$ and $g(T)$ is in $\Lambda_0 - \Lambda_+$. By plugging $X = f(T)/g(T)$ in the equation we have

$$f(T)^n + \sum_{k=1}^{n-1} a_k g(T)^{n-k} f(T)^k + a_0 g(T)^n = 0.$$  

If both $f(T), g(T) \in \Lambda_0 - \Lambda_+$ then $X = f(T)/g(T) \in \Lambda_0 - \Lambda_+$ since elements in $\Lambda_0 - \Lambda_+$ are units. If $f(T) \in \Lambda_0 - \Lambda_+$ but $g(T)$ not then we have a contradiction since $f(T)^n \in \Lambda_0 - \Lambda_+$ but the other two terms are in $\Lambda_+$. If $g(T) \in \Lambda_0 - \Lambda_+$ but $f(T)$ not then we have a contradiction since $a_0 g(T)^n \in \Lambda_0 - \Lambda_+$ but the other two terms are in $\Lambda_+$. Therefore both $f(T), g(T) \in \Lambda_0 - \Lambda_+$ and their quotient is in $\Lambda_0 - \Lambda_+$. 

By Lemma 7.3 we can solve $Y \in \Lambda_0 - \Lambda_+$. Then we plug $Y$ back into

$$X^2 Y + X(Y^2 - BY + 1) + Y = 0$$

to solve $X$. Since $Y \neq 0$ and $Y \in \Lambda_0 - \Lambda_+$ we can apply Lemma 7.3 again to get $X \in \Lambda_0 - \Lambda_+$ which completes our solution.

Therefore $(y_1, y_2) = (1, 2)$ is a critical point of the potential function with above bulk deformation. It follows that when $u_1 = u_2$ and $-u_1 - u_2 + \kappa > 0$ the deformed Floer cohomology $HF(L(u); b, b)$ with above bulk deformation is nonzero and $L(u)$ is nondisplaceable.

This calculation can be generalized to the natural compactification of the moment polytope of an $A_n$-type singularity. Note that we already have the explicit formula without bulk deformation in the local case in (6.4). Hence for a Lagrangian torus $L(u)$ in $\overline{X_{0,n}}(\epsilon)$ the potential function can be obtained by adding one term of higher order. That is,

$$\Psi^{L(u)}(b) = \Psi^{(u_1, u_2; y_1, y_2)}$$

(7.8)

$$= \sum_{k=0}^{n+1} C(n+1, k) y_1^{1-k} y_2 k T^{(1-k) u_1 + k u_2} + y_2^{-1} T^{-u_2 + \kappa}.$$  

We consider the bulk deformation by

$$b = vPD(D_1) + wPD(D_n) \in H^2(\overline{X_{0,n}}(\epsilon); \Lambda_0).$$

Then the potential function with bulk deformation is

$$\Psi^{(u_1, u_2; x_1, x_2; v, w)} = \Psi^{(y_1, y_2)}$$

(7.9)

$$= y_1^{1-u_1} + \sum_{k=1}^{n} P'_k(e^v, e^w) y_1^{1-k} y_2 k T^{(1-k) u_1 + k u_2} + y_1^{-n} y_2^{n+1} T^{-u_1 + (n+1) u_2} + y_2^{-1} T^{-u_2 + \kappa}$$

where $P'_k(e^v, e^w)$ are Laurent polynomials. In particular the degrees of $e^v$ and $e^w$ are at least negative one. Calculation of this potential function is the same with $\overline{X_{0,1}}(\epsilon)$ but the bulk effect of $P'_k(e^v, e^w)$ is more complicated. To find the critical points we give an algebraic argument without knowing $P'_k(e^v, e^w)$ explicitly.
When \( u_1 = u_2 \) and \(-u_1 - u_2 + \kappa > 0\) we consider the partial derivatives
\[
0 = T^{u_1} \frac{\partial \mathcal{O}_{L(u)}^{(u)}}{\partial y_1} (y_1, y_2) = 1 + \sum_{k=1}^{n} P_k(e^v, e^w)(1-k)y_1^{-k}y_2^{-1} - ny_1^{-n-1}y_2^{n+1}
\]
and
\[
0 = T^{u_1} \frac{\partial \mathcal{O}_{L(u)}^{b}}{\partial y_2} (y_1, y_2) = \sum_{k=1}^{n} P_k(e^v, e^w)ky_1^{1-k}y_2^{-k-1} + (n+1)y_1^{-n}y_2^{-2T^{u_1} - u_2 + \kappa}.
\]

\[\text{Proposition 7.4.} \] There exist \((e^v, e^w) \in (\Lambda_0 - \Lambda_+)^2\) such that the above two equations have solutions \((y_1, y_2) \in (\Lambda_0 - \Lambda_+)^2\).

\[\text{Proof.} \] First we discuss some intersection number formulas to analyze \(P_k(e^v, e^w)\).

For a disk class \(\beta = \beta_k\), \(k \neq 1, n\) we have
\[
D_1 \cap \beta = \begin{cases} 
1, & s_1 = 0 \\
-1, & s_1 = 1, \ s_2 = 0 \\
0, & s_1 = 1, \ s_2 = 1 
\end{cases}
\]
and
\[
D_n \cap \beta = \begin{cases} 
0, & s_{n-1} = 0 \\
1, & s_{n-1} = 1, \ s_n = 0 \\
-1, & s_{n-1} = 1, \ s_n = 1 
\end{cases}
\]
And symmetrically when \(k = n\) we have
\[
D_1 \cap \beta = \begin{cases} 
0, & s_2 = 0 \\
1, & s_2 = 1, \ s_1 = 0 \\
-1, & s_2 = 1, \ s_1 = 1 
\end{cases}
\]
and
\[
D_n \cap \beta = \begin{cases} 
1, & s_n = 0 \\
-1, & s_n = 1, \ s_{n-1} = 0 \\
0, & s_n = 1, \ s_{n-1} = 1 
\end{cases}
\]
Lastly when \(k \neq 1, n\) we have
\[
D_1 \cap \beta = \begin{cases} 
0, & s_2 = 0 \\
1, & s_2 = 1, \ s_1 = 0 \\
-1, & s_2 = 1, \ s_1 = 1 \\
0, & s_2 = 2, \ s_1 = 1 
\end{cases}
\]
and

\[ D_n \cap \beta = \begin{cases} 
0, & s_{n-1} = 0 \\
1, & s_{n-1} = 1, \ s_n = 0 \\
-1, & s_{n-1} = 1, \ s_n = 1 \\
0, & s_{n-1} = 2, \ s_n = 1 
\end{cases} \]

Therefore \( e^v, e^w \) only has degree one and negative one terms in \( P_k(e^v, e^w) \). By multiplying \( e^{v+w} \) to (7.10) and (7.11) we always get a system of cubic equations. Then we use the same method of solving (7.5) and (7.6) to solve these equations. By regarding \( y_1 \) and \( y_2 \) as parameters we first solve for \( e^v \) in terms of \( e^w \) then plug in the other equation. After rearranging we get a polynomial in \( e^w \) with coefficients in \( \mathbb{C}[y_1, y_2, y_1^{-1}, y_2^{-1}] \). Next we explain this method explicitly and show that by choosing generic \( y_1, y_2 \) we get a new polynomial of \( e^w \) that fits in Lemma 7.3.

From above calculation of intersection numbers we know that in \( P_k(e^v, e^w) \) there are 9 possible terms

\[ e^v, e^w, e^{-v}, e^{-w}, e^{v+w}, e^{v-w}, e^{-v+w}, e^{-v-w}, constants \]

and they all have nonnegative integer coefficients. We multiply \(-e^{v+w} \) to (7.10) and multiply \( e^{v+w} \) to (7.11) to get two polynomial equations

\[ 0 = -e^{v+w} + e^{v+w} \sum_{k=1}^{n} P_k(e^v, e^w)(k-1)y_1^{-k}y_2^k + ny_1^{-n}y_2^n e^{v+w} \]  \hfill (7.12)

and

\[ 0 = e^{v+w} \sum_{k=1}^{n} P_k(e^v, e^w)k y_1^{-k}y_2^{-1} + (n+1)y_1^{-n}y_2^n e^{v+w} - y_2^{-2}T^{-u_1-u_2+6}e^{v+w} \]  \hfill (7.13)

We set \( e^v = X \) and \( e^w = Y \) and get

\[ \begin{align*}
X^2(AY^2 + BY + C) + X(DY^2 + EY + F) + GY^2 + HY + I &= 0 \\
X^2(A'Y^2 + B'Y + C') + X(D'Y^2 + E'Y + F') + G'Y^2 + H'Y + I' &= 0
\end{align*} \]  \hfill (7.14)

where \( A, B, C, D, E, F, G, H, I, A', B', C, D, F', G', H', I' \) are nonconstant elements in \( \mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}] \) and only \( E' \) has the \( T \) parameter.

By root formula we solve for \( X \) from the first equation in term of \( Y \) then plug in the second equation. We choose one of the root

\[ X = \frac{-(DY^2 + EY + F) - \Delta}{2(AY^2 + BY + C)} \]

and get

\[ \begin{align*}
[(DY^2 + EY + F) + \Delta]^2(A'Y^2 + B'Y + C') + 2[-(DY^2 + EY + F) - \Delta](D'Y^2 + E'Y + F')(AY^2 + BY + C) + 4(G'Y^2 + H'Y + I')(AY^2 + BY + C)^2 &= 0
\end{align*} \]  \hfill (7.15)

where

\[ \Delta = \sqrt{(DY^2 + EY + F)^2 - 4(AY^2 + BY + C)(GY^2 + HY + I)}. \]

After rearranging (7.15) we want to get a polynomial in \( Y \) which fits in Lemma 7.3. That is, we need to check that the coefficients of the highest order term and
constant term are nonzero in $\Lambda_0 - \Lambda_+$. First we check the coefficient of the highest order. By rearranging (7.15) we have that

$$[(DY^2 + EY + F)^2 + \Delta^2](A'Y^2 + B'Y + C')$$

$$-2(DY^2 + EY + F)(DY^2 + E'Y + F')(AY^2 + BY + C)$$

$$+4(G'Y^2 + H'Y + I')(AY^2 + BY + C)^2 =$$

$$2\Delta[(DY^2 + E'Y + F')(AY^2 + BY + C) - (DY^2 + EY + F)(A'Y^2 + B'Y + C')].$$

Then we square both side of (7.16) to get a polynomial in $Y$. By direct calculation we know that the coefficient of the highest order term is

$$16[A^2(DA' - D'A)(DG' - D'G) + A^2(G'A - A'G)^2]$$

which depends on $A, A', D, D', G, G'$. Note that $A, A'$ are the coefficients of $e^{2v+2w}$ in (7.12) and (7.13), $D, D'$ are the coefficients of $e^{v+2w}$ and $G, G'$ the coefficients of $e^{2w}$. Next we calculate them out explicitly. We assume that $n \geq 4$ in the following.

By above intersection number formulas we have that when $k = 1$

$$P_1(e^v, e^w) = e^v + e^{-v} + e^w + e^{-w} + n - 3$$

and symmetrically when $k = n$

$$P_n(e^v, e^w) = e^v + e^{-v} + e^w + e^{-w} + n - 3.$$

So $P_1(e^v, e^w)$ and $P_n(e^v, e^w)$ only contribute to $D, D'$. When $2 \leq k \leq n - 1$ the coefficient of $e^{v+w}$ in $P_k(e^v, e^w)$ is the number of admissible tuples of length $n$ with center $k$ satisfying

$$s_1 = 0, s_2 = 1, s_{n-1} = 1, s_n = 0$$

by above intersection number formulas. This number is $C(n - 3, k - 2)$ by Proposition 6.4. Similarly the coefficient of $e^{-v+w}$ in $P_k(e^v, e^w)$ is the number of admissible tuples of length $n$ with center $k$ satisfying

$$s_1 = 1, s_2 = 1, s_{n-1} = 1, s_n = 0.$$ 

This number is $C(n - 3, k - 2)$. The third case is the coefficient of $e^w$ in $P_k(e^v, e^w)$, which is the number of admissible tuples of length $n$ with center $k$ satisfying

$$s_2 = 0, s_{n-1} = 1, s_n = 0$$

or

$$s_1 = 1, s_2 = 2, s_{n-1} = 1, s_n = 0.$$ 

When $k = 1, n$ this number is 1 as we have the explicit expressions for $P_1(e^v, e^w)$ and $P_n(e^v, e^w)$. When $k = 2, n - 1$ this number is $n - 3$. When $3 \leq k \leq n - 2$ this number is $p_{n,k} := C(n-3, k-3) + C(n-2, k-1) - C(n-3, k-2)$ where $C(n-3, k-3)$ is the number satisfying the first condition and $C(n-2, k-1) - C(n-3, k-2)$ is the number satisfying the second condition.
Therefore we have that

\[
A = \sum_{k=2}^{n-1} C(n - 3, k - 2)(k - 1)y_1^{-k}y_2^k, \\
A' = \sum_{k=2}^{n-1} C(n - 3, k - 2)ky_1^{-k}y_2^{k-1}, \\
D = (n - 3)[y_1^{-2}y_2^2 + (n - 2)y_1^{-n+1}y_2^{n-1}] \\
+ \sum_{k=3}^{n-2} p_{n,k}(k - 1)y_1^{-k}y_2^k + (n - 1)y_1^{-n}y_2^n, \\
D' = 1 + (n - 3)[2y_1^{-1}y_2 + (n - 1)y_1^{-2}y_2^{n-2}] \\
+ \sum_{k=3}^{n-2} p_{n,k}ky_1^{-k}y_2^{k-1} + n y_1^{-n}y_2^{n-1}, \tag{7.17}
\]

\[
G = \sum_{k=2}^{n-1} C(n - 3, k - 2)(k - 1)y_1^{-k}y_2^k, \\
G' = \sum_{k=2}^{n-1} C(n - 3, k - 2)ky_1^{-k}y_2^{k-1}. 
\]

In particular \( A = G \) and \( A' = G' \). This shows that the coefficient of the highest order term is a nontrivial element in \( \mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}] \). We denote this nontrivial element by \( f_1(y_1, y_2, y_1^{-1}, y_2^{-1}) \).

By a similar calculation we can show that the constant term in (7.16) is also a nontrivial element in \( \mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}] \), which is written as \( g_1(y_1, y_2, y_1^{-1}, y_2^{-1}) \). Hence we can choose \( y_1, y_2 \) to be positive complex numbers such that those coefficients are nonzero in \( \mathbb{C} \), which makes that (7.16) fits in Lemma 7.3 to solve \( Y \in \Lambda_0 - \Lambda_+ \). Next we redo the above process but first solve for \( Y \) by root formula involving \( X \). By the same calculation we get two nontrivial elements \( f_2(y_1, y_2, y_1^{-1}, y_2^{-1}) \) and \( g_2(y_1, y_2, y_1^{-1}, y_2^{-1}) \) in \( \mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}] \). Since \( f_1, g_1, f_2, g_2 \) are nontrivial the union of their zeroes can not be the whole \( \mathbb{C}^* \times \mathbb{C}^* \). Then we choose suitable \( y_1, y_2 \) such that all of \( f_1, g_1, f_2, g_2 \) are nonzero complex numbers. By Lemma 7.3 we know that all \( X \)-coordinates and \( Y \)-coordinates of solutions of (7.14) are in \( \Lambda_0 - \Lambda_+ \). Hence we get solutions \((e^u, e^w) = (X, Y)\) in \((\Lambda_0 - \Lambda_+)^2\) with respect to previous choices of \( y_1, y_2 \).

\[\square\]

Once we have desired solutions \( y_1, y_2, e^u, e^w \) for the critical points equations it follows that there is a 1-dimensional family of nondisplaceable Lagrangian tori in \( X_{0,n}(\epsilon) \).

This proposition is rather technical. It involves the calculation of a combination of admissible tuples and bulk deformation. We will use it in the following subsection to deal with the case of a symplectic 4-orbifold.

**7.2. Lagrangian tori in the smoothing of a symplectic 4-orbifold.** Finally we consider the case of a symplectic orbifold. Let \( X \) be a 4-dimensional symplectic orbifold. We say it has an \( A_n \)-type singularity at \( x \in X \) if there is a neighborhood \( U \) of \( x \) such that \( U \) is equal to a neighborhood of 0 in \( \mathbb{C}^2/\mathbb{Z}_n \) and where \( x \) corresponds
to the origin. Here \( \mathbb{Z}_n \cong \{ \eta \in \mathbb{C} \mid \eta^n = 1 \} \) acts on \( \mathbb{C}^2 \) by
\[
\eta \cdot (z_1, z_2) = (\eta^{-1} z_1, \eta z_2).
\]
By the Darboux-Weinstein theorem, an equivariant version of Darboux theorem, we can make the neighborhood small to assume the symplectic form on \( U \) is induced from the standard one on \( \mathbb{C}^2 \). Hence we can perform the smoothing by cutting and pasting a Milnor fiber as we did in the local case. Then we get a smooth symplectic manifold \( \hat{X} \).

Without loss of generality we assume that \( X \) has only one singular point. In the following we want to show that the family of nondisplaceable Lagrangian tori in the Milnor fiber is still nondisplaceable in \( \hat{X} \), see Figure 2. Let \( L(u) \) be a Lagrangian torus in the above family, its potential function in \( \hat{X} \) looks like
\[
\Psi_b^{L(u)} = \Psi_{b,0}^{L(u)} + G(y_1, y_2, u, T)
\]
where \( \Psi_{b,0}^{L(u)} \) is the potential function of \( L(u) \) with bulk
\[
b = vPD(D_1) + wPD(D_n) \in H^2(\overline{X_{0,n}}(\epsilon) ; \Lambda_0) .
\]
in the Milnor fiber and \( G(y_1, y_2, u, T) \) are higher order terms of \( T \). The appearance of \( G(y_1, y_2, u, T) \) is due to new disk classes in \( \pi_2(\hat{X}, L(u)) \) of Maslov index 2, possibly infinitely many. Since we glued in an arbitrarily small Milnor fiber the new disk classes give contributions of higher energy terms. Moreover the new disk classes are outside the glued region hence do not intersect \( D_1 \) and \( D_n \). Therefore there is no \( e^v, e^w \) terms in \( G(u, T) \). This is important to assure that we can use the information of \( \Psi_{b,0}^{L(u)} \) to show that the full potential function \( \Psi_b^{L(u)} \) has some critical points without knowing \( G(y_1, y_2, u, T) \) explicitly.

**Theorem 7.5.** Let \( X \) be a 4-dimensional symplectic orbifold with an \( A_n \)-type singularity and let \( \hat{X} \) be its smoothing obtained by gluing an \( A_n \)-Milnor fiber. Then there is a family of nondisplaceable Lagrangian tori in \( \hat{X} \) parameterized by an open interval.

**Proof.** We glue \( X_{0,n}(\epsilon) \) to \( X \) to smooth the \( A_n \)-type singularity. In previous sections we showed that there is a family of nondisplaceable Lagrangian tori \( L(u) \) in \( X_{0,n}(\epsilon) \) parameterized by an open interval. We choose a generic almost complex structure \( J \) on \( \hat{X} \) such that all holomorphic disks with boundary on \( L(u) \) have positive Maslov index. Then with respect to \( J \) the potential function of \( L(u) \) in \( \hat{X} \) is
\[
\Psi_b^{L(u)} = \Psi_{b,0}^{L(u)} + G(y_1, y_2, u, T)
\]
where the bulk deformation is
\[
b = vPD(D_1) + wPD(D_n) \in H^2(\overline{X_{0,n}}(\epsilon) ; \Lambda_0) \subset H^2(\hat{X} ; \Lambda_0)
\]
and \( \Psi_{b,0}^{L(u)} \) is calculated in (7.9) since it is invariant of almost complex structures. Note that \( D_1 \) and \( D_n \) are classes in \( X_{0,n}(\epsilon) \), which correspond to the first one and last one in the chain of Lagrangian spheres in the \( A_n \)-diagram. The new disk classes do not intersect \( D_1 \) and \( D_n \).

Then the equations for critical points are
\[
0 = \frac{\partial}{\partial y_1} \Psi_{b,0}^{L(u)} + \frac{\partial}{\partial y_1} G(y_1, y_2, u, T)
\]
and

\begin{equation}
0 = \frac{\partial}{\partial y_2} \Phi \mathcal{O}_{b,0}^{L(u)} + \frac{\partial}{\partial y_2} G(y_1, y_2, u, T).
\end{equation}

Note that we assume that new disk classes have more energy than those disk classes in \(X_{0,n}(c)\) when we fix \(u = (u_1, u_2)\) in this family \(L(u)\). And in \(\Phi \mathcal{O}_{b,0}^{L(u)}\) all terms have the same energy parameter \(T^{|w|}\). So we divide \(T^{|w|}\) on both sides of above two equations such that there is no \(T\) parameter in \(\Phi \mathcal{O}_{b,0}^{L(u)}\) and in \(\frac{\partial}{\partial y_i} G(y_1, y_2, u, T), i = 1, 2\) the parameter \(T\) has positive power.

The idea to solve those equations is the same with the case of \(X_{0,n}\), choosing appropriate \(y_1, y_2\) such that two polynomial equations of \(e^v, e^w\) have common solutions. First we set \(X = e^v, Y = e^w\) and multiply \(-e^{v+w}\) on both sides of (7.19) and multiply \(e^{v+w}\) on both sides of (7.20) to get

\[
\begin{cases}
X^2(AY^2 + BY + C) + X(DY^2 + EY + F) + GY^2 + HY + I = 0 \\
X^2(A'Y^2 + B'Y + C') + X(D'Y^2 + E'Y + F') + G'Y^2 + H'Y + I' = 0
\end{cases}
\]

where \(A, B, C, D, F, G, H, I, A', B', C, D, F', G', H', I'\) are nonconstant elements in \(\mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}]\). In particular they are exactly the same terms as in Proposition 7.9 since there is no terms of \(e^v\) and \(e^w\) in \(G(y_1, y_2, u, T)\). The only changed terms are \(E\) and \(E'\), which involve the information from \(G(y_1, y_2, u, T)\).

We use the same method in Proposition 7.4 to solve those two new equations. That is, we first solve for \(X\) in terms of \(Y\) and plug in the second equation. After rearranging we get a polynomial of \(Y\). To use Lemma 7.3 we need to check that \(a_{n-1}, \cdots, a_1 \in \Lambda_0\) and \(a_0 \in \Lambda_0 - \Lambda_+\). By the calculation in Proposition 7.4 we know that \(E\) and \(E'\) do not appear in the coefficients of the highest order term or the constant term. Moreover \(E\) and \(E'\) are always in \(\Lambda_0\) if \(y_1, y_2\) are in \(\Lambda_0\). Therefore the coefficients of the highest order term and the constant term are exactly the same nontrivial elements in \(\mathbb{Z}[y_1, y_2, y_1^{-1}, y_2^{-1}]\) as in Proposition 7.4. We can choose positive complex numbers such that the new polynomial of \(Y\) satisfies the conditions in Lemma 7.3. Hence the equations for critical points have desired solutions \(y_1, y_2, e^v, e^w\).

Note that in the beginning we choose a generic almost complex structure \(J\) on \(\hat{X}\) such that all holomorphic disks with boundary on \(L(u)\) have positive Maslov indices. Therefore by Lemma 5.9 the condition

\[
H^1(L; \Lambda_0) / H^1(L; 2\pi \sqrt{-1} \mathbb{Z}) \cong \tilde{\mathcal{M}}_{\text{weak}}(L, m^b).
\]

is satisfied. In conclusion the corresponding Lagrangian tori are nondisplaceable in \(\hat{X}\), which finishes our proof of Theorem 7.5.

\begin{remark}
The family of nondisplaceable Lagrangian tori \(L(u)\) in the local case is parameterized by an open interval. When we glue this local neighborhood to an orbifold \(X\) the family of Lagrangian tori \(L(u)\) which are still nondisplaceable in \(X\) is parameterized by a shorter open interval. This is because we assume that disk classes outside this neighborhood have more energy. Like in the case of \(X_{0,n}\), the family of nondisplaceable Lagrangian tori \(L(u)\) in the Milnor fiber is parameterized by \(\mathbb{R}^+\) but in \(X_{0,n}\) they are parameterized by an open interval, which depends on the parameter \(\kappa\) of the compactification.
\end{remark}
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