ROUTH REDUCTION OF PALATINI GRAVITY IN VACUUM

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ABSTRACT. An interpretation of Einstein-Hilbert gravity equations as Lagrangian reduction of Palatini gravity is made. The main techniques involved in this task are those developed in a previous work [6] for Routh reduction in classical field theory. As a byproduct of this approach, a novel set of conditions for the existence of a vielbein for a given metric is found.

1. INTRODUCTION

The relationship between Einstein-Hilbert and Palatini formulation of gravity has been studied in several places. Nevertheless, the main theoretical tool used in the discussion of the connection between these formulations of gravity appears to be some flavor of Hamiltonian reduction. For instance, [12] and [28] use ADM formalism [1] in order to establish the connection; it has been explored also in [10, 21], where the correspondence is set by using a Hamiltonian structure on the set of fields at the boundary.

From this viewpoint, it becomes interesting to find a reduction scheme relating the Lagrangian formulation of Palatini and Einstein-Hilbert gravity directly, without the detour through Hamiltonian formalism. So far there exists two ways to implement reduction at the Lagrangian level, namely Lagrange-Poincaré reduction [9, 8, 14] and Routh reduction [25, 11, 24, 15, 4, 6]. Moreover, there are physical considerations that can be said in support of this kind of reduction: They deal not only with the reduction problem, but also with the reconstruction problem, and it is argued in [27] that reconstruction can be relevant from the physical point of view.

The problem with these approaches to Lagrangian reduction is that they work in the setting of classical variational problems, that is, variational problems where the velocity space is a jet space of the field bundle, and where the restrictions imposed on the fields are prescribed by the contact structure of the jet space. In the present work we want to study the reduction of a variational problem of a more general nature, namely, by using the formulation of Palatini gravity given in [3], where it was interpreted as an example of the so called Griffiths variational problems [19, 20]. In this approach, the field bundle is the bundle of frames (whose sections are the vielbein) but the jet space is replaced by a submanifold of the jet space of the frame bundle, namely, by the submanifold corresponding to the torsion zero constraint; also, the contact structure is changed by a set of differential constraints implementing the metricity conditions. Therefore, it is necessary to find a formulation of a Lagrangian reduction scheme taking into consideration these characteristics; because of its versatility, we will choose to work with the Routh reductions as formulated in [6].
These considerations set the purposes of the following article: In one hand, to carry out a proof of concept for the generalization of Routh reduction to variational problems more general than those corresponding to first order field theory, generalizing the techniques employed in [6]; on the other hand, to apply Routh reduction of field theory in the context of a meaningful example, namely, a formulation of gravity with basis.

In order to be more specific about the nature of this generalization, let us briefly describe how Routh reduction in field theory is performed:

- First, a unified formulation along the lines of [17] is constructed, and its ability for representing the extremals of the original variational problem is proved. This procedure must be done both for the original variational problem and the reduced one. The set of differential forms encoding the restrictions to be imposed on the fields are used at this stage.
- The momentum map is defined in the unified setting, and its momentum level sets are determined. It should be proved that the equations of motion can be naturally restricted to these sets.
- A connection on the bundle obtained by quotienting out the symmetry must be provided. Because of the characteristics of the contact structure on a jet space, this connection allows us to split the fields of the unified formalism. The splitting induced by the chosen connection allows us to define the Routhian and the force term for the reduced system.
- It is necessary to set a common ground for the comparison of the extremals of the unreduced and reduced variational problems. This is done by considering an affine subbundle of the bundle of forms on a fibred product of bundles; the factors in this product are the bundles of the unreduced and the reduced system.
- The equivalence between the extremals of the original variational problem and the reduced variational problem is checked by a map involving the translation along the force term (in the space of forms associated to the unified formalism). In the reconstruction of the extremals of the unreduced system from the reduced dynamics, it is necessary to impose some integrability conditions.

The two first items can also be done for Griffiths variational problems; when trying to reproduce the third item in this generalized context, we have to face the problem that the splitting induced by the chosen connection strongly depends on features of the contact structure. Nevertheless, the metricity constraints can be formulated using forms belonging to the contact structure, and so the hopes of reproducing the third item in this context increase. A solution to this problem is provided by Lemma 8.1. No difficulties must arise from the last two items, as they are based on geometrical operations of general nature; the main results of the article are Theorem 10.3 and Theorem 10.10, which describe reduction and reconstruction respectively.

The paper is organized as follows. In Section 2, we will review some geometrical tools necessary for the construction of the variational problem for Palatini gravity we will use in this article; the actual construction of this variational problem, as well as the associated unified problem, is done in Section 3. The symmetry considerations necessary to carry out the reduction are discussed in Section 4. Section 5 is technical, and contains some calculations used in the reduction and reconstruction theorems. In Section 6, the results achieved in the previous section are employed in the search of identifications between geometrical structures present in both the reduced and unreduced spaces: A remarkable fact in this vein is that the metricity constraints correspond after projection onto the quotient, with the contact structure of a jet bundle. Construction of the first order formalism for Einstein-Hilbert
gravity (and its correspondence with the usual second order formalism) is delayed until Section 7; also, an unified formalism for this variational problem is discussed in this section. The choice of a connection induces a splitting in the contact structure on the jet space of the frame bundle; in Section 8 the effects of this splitting in the variational formulation of Palatini gravity are analyzed. The Routhian is constructed in Section 9. It is shown that the Routhian for Palatini gravity is the (first order) Einstein-Hilbert Lagrangian. Finally, in Section 10 the reduction theorem and the reconstruction theorem are proved. The main result of this section is the notion of flat condition for a metric, which is a helpful hypothesis in the proof of the reconstruction theorem.

Notations. We are adopting here the notational conventions from [29] when dealing with bundles and its associated jet spaces. Also, if $Q$ is a manifold, $\Lambda^p(Q) = \wedge^p(T^*Q)$ denotes the $p$-th exterior power of the cotangent bundle of $Q$. Moreover, for $k \leq l$ the set of $k$-horizontal $l$-forms on the bundle $\pi: P \to N$ is

$$\wedge^k_l(Q) := \{ \alpha \in \wedge^l(Q) : v_1, \ldots, v_k \alpha = 0 \text{ for any } v_1, \ldots, v_k \pi\text{-vertical vectors} \}.$$ 

For the same bundle, the set of vectors tangent to $P$ in the kernel of $T\pi$ will be represented with the symbol $V_\pi \subset TP$. In this regard, the set of vector fields which are vertical for a bundle map $\pi: P \to N$ will be indicated by $\mathfrak{X}^{\pi\ast}(P)$. The space of differential $p$-forms, sections of $\Lambda^p(Q) \to Q$, will be denoted by $\Omega^p(Q)$. We also write $\Lambda^\ast(Q) = \bigoplus_{j=1}^{\dim Q} \Lambda^j(Q)$. If $f: P \to Q$ is a smooth map and $\alpha_x$ is a $p$-covector on $Q$, we will sometimes use the notation $\alpha_f \circ T_x f$ to denote its pullback $f^\ast \alpha_x$. If $P_1 \to Q$ and $P_2 \to Q$ are fiber bundles over the same base $Q$ we will write $P_1 \times_Q P_2$ for their fibred product, or simply $P_1 \times P_2$ if there is no risk of confusion. Unless explicitly stated, the canonical projections onto its factor will be indicated by

$$\text{pr}_i: P_1 \times P_2 \to P_i, \quad i = 1, 2.$$ 

Given a manifold $N$ and a Lie group $G$ acting on $N$, the symbol $[n]_G$ for $n \in N$ will indicate the $G$-orbit in $N$ containing $n$; the canonical projection onto its quotient will be denoted by

$$p_N^G: N \to N/G.$$ 

Also, if $\mathfrak{g}$ is the Lie algebra for the group $G$, the symbol $\xi_N$ will represent the infinitesimal generator for the $G$-action associated to $\xi \in \mathfrak{g}$. Finally, Einstein summation convention will be used everywhere.

2. Geometrical tools for Palatini gravity

We will give a brief account of the construction carried out in [5]. The basic bundle is the frame bundle $\tau: LM \to M$ on the spacetime manifold $M$ ($\dim M = m$); because it is a principal bundle with structure group $GL(m)$, we can lift this action to the jet bundle $J^1\tau$, 

so that we obtain a commutative diagram

\[
\begin{array}{ccc}
J^1 \tau & \xrightarrow{\tau_{10}} & J^1 \tau \\
\downarrow{\tau_1} & \downarrow{\tau_1} & \downarrow{\tau_1} \\
LM & \xrightarrow{\tau_{10}} & C(LM) \\
\downarrow{\tau} & \downarrow{\tau} & \downarrow{\tau} \\
M & \xrightarrow{\tau_1} & M
\end{array}
\]

where \( C(LM) := J^1 \tau / GL(m) \) is the so called connection bundle of \( LM \), whose sections can be naturally identified with the principal connections of the bundle \( \tau \) (for details, see [7] and references therein). It is interesting to note that there exists an affine isomorphism

\[
F : J^1 \tau \rightarrow LM \times_M C(LM) : J^1 s \mapsto \left( s(x), [j^1 s]_{GL(m)} \right)
\]

and under this correspondence, the \( GL(m) \)-action is isolated to the first factor in the product, namely

\[
F \left( J^1 s \cdot g \right) = \left( s(x) \cdot g, [j^1 s]_{GL(m)} \right).
\]

It means that a section of the bundle \( \tau_1 \) is equivalent to a connection on \( LM \) plus a moving frame \( (X_1, \cdots, X_m) \) on \( M \); although this moving frame has no direct physical interpretation, we can associate a metric to it, namely, in contravariant terms,

\[
g := \eta^{ij} X_i \otimes X_j
\]

for some nondegenerate symmetric matrix \( \eta \) (see Equation (2.1) below). It is the same to declare that the metric \( g \) is the unique metric on \( M \) making the moving frame \( (X_1, \cdots, X_m) \) (pseudo)orthonormal, with the signature given by \( \eta \).

The tautological form \( \tilde{\theta} \in \Omega^1(LM, \mathbb{R}^m) \) can be pulled back along \( \tau_{10} \) to a 1-form \( \theta := \tau_{10}^* \tilde{\theta} \) on \( J^1 \tau \); moreover, the Cartan form \( \tilde{\omega} \in \Omega^1(J^1 \tau, \mathfrak{gl}(m)) \), given by the formula

\[
\tilde{\omega}|_{j^1 s} := T_{j^1 s} \tau_{10} - T_{x^1} \circ T_{j^1 s} \tau_1,
\]

gives rise to a \( \mathfrak{gl}(m) \)-valued 1-form \( \omega \) on \( J^1 \tau \), by using the identification

\[
V \tau \simeq LM \times \mathfrak{gl}(m).
\]

By means of the canonical basis \( \{ e_i \} \) on \( \mathbb{R}^m \) and \( \{ E^i_j \} \) on \( \mathfrak{gl}(m) \), where

\[
(E^i_j)^q_p := \delta^q_j \delta^i_p,
\]

we can define the collection of 1-forms \( \{ \theta^i, \omega^i \} \) on \( J^1 \tau \) such that

\[
\theta = \theta^i e_i, \quad \omega = \omega^i E^i_j.
\]

We also have the formula

\[
\tilde{\omega} = \omega^i (E^i_j)_{j^1 \tau},
\]
where $A_{J^1\tau} \in \mathfrak{X}^{V_{GL(m)}}(J^1\tau)$ is the infinitesimal generator associated to $A \in \mathfrak{gl}(m)$ for the lifted action. It can be proved that $\omega$ is a connection form for a principal connection on the bundle

$$p^*_{J^1\tau} : J^1\tau \to C(LM).$$

Let us define

$$\theta_0 := \theta^1 \wedge \cdots \wedge \theta^m;$$

because every $u \in LM$ is a collection $u = (X_1, \cdots, X_m)$ of vectors on $\tau(u) \in M$, and $\theta^i$ is a $\tau_1$-horizontal 1-form on $J^1\tau$, we can define the set of forms

$$\theta_{i_1 \cdots i_k} \mid_{j_1^s} := X_{i_1} \cdots X_{i_k} \cdot \theta_0 \mid_{j_1^s},$$

for $1 \leq i_1, \cdots, i_k \leq m$, where $j_1^s \in J^1\tau$ is any element such that $u = \tau_{10}(j_1^s)$.

Additionally, let us fix a matrix

$$\begin{bmatrix}
-1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix} \in GL(m)$$

and let $\eta_{ij}$ its $(i,j)$-entry; we will represent with the symbol $\eta^{ij}$ the $(i,j)$-entry of its inverse. With these ingredients we can construct the *Palatini Lagrangian*

$$\mathcal{L}_{PG} := \eta^{kp} \theta_{ik} \wedge \Omega^k_p,$$

where $\Omega := \Omega^i_j E^j_i$ is the curvature of the canonical connection $\omega$. This $m$-form will determine the dynamics of the vacuum gravity in this formulation.

Finally, let us describe a decomposition on $\mathfrak{gl}(m)$ induced by $\eta$. In fact, this matrix yields to a compact real form $u$ in $\mathfrak{gl}(m, \mathbb{C})$, given by

$$u = \{ \xi \in \mathfrak{gl}(m, \mathbb{C}) : \xi^T \eta + \eta \xi = 0 \}$$

and thus we have a Cartan decomposition

$$\mathfrak{gl}(m, \mathbb{C}) = u \oplus \mathfrak{s}.$$

Given the inclusion

$$\mathfrak{gl}(m) \subset \mathfrak{gl}(m, \mathbb{C}),$$

we obtain the decomposition

$$\mathfrak{gl}(m) = \mathfrak{t} \oplus \mathfrak{p}.$$
The canonical forms defined in the previous section allow us to set the torsion form

\[ T := \left( d\theta^i + \omega^i_k \wedge \theta^k \right) \otimes e_j \in \Omega^2 \left( J^1 \tau, \mathbb{R}^m \right). \]

Now, every connection \( \Gamma : M \to C(LM) \) gives rise to a section \( \sigma_\Gamma : LM \to J^1 \tau \) of the bundle \( \tau_{10} : J^1 \tau \to LM \), as the equivariance of the following diagram shows.

\[
\begin{array}{ccc}
LM & \xrightarrow{\tau_0} & J^1 \tau \\
\downarrow{\tau_0} & & \downarrow{\sigma_\Gamma} \\
C(LM) & \xrightarrow{p_{GL(m)}} & M
\end{array}
\]

The interesting fact is that the pullback form \( \sigma_\Gamma^* T \) coincides with the torsion of the connection \( \Gamma \). Additionally, it can be proved that \( T \) is a 1-horizontal form on \( \tau_{10} : J^1 \tau \to LM \), so that there exists a maximal (with respect to the inclusion) submanifold \( \mathcal{T}_0 : \mathcal{T}_0 \hookrightarrow J^1 \tau \) such that

1. \( \mathcal{T}_0 \) is transversal to the fibers of \( \tau_{10} : J^1 \tau \to M \) (namely, \( T_{j_1s}(\mathcal{T}_0) \oplus V_{j_1s} \tau_1 = T_{j_1s}(J^1 \tau) \)), and
2. it annihilates the torsion, i.e.

\[ \iota_{\mathcal{T}_0} T = 0. \]

The transformation properties of the form \( T \) allow us to conclude that \( \mathcal{T}_0 \) is \( GL(m) \)-invariant. The connections associated to sections of \( J^1 \tau \) taking values in \( \mathcal{T}_0 \) are torsionless, so that the zero torsion restriction can be achieved through the requirement that these sections would take values in this submanifold. Accordingly, we can use the affine isomorphism \( F : J^1 \tau \to LM \times C(LM) \), to define the bundle of torsionless connections as the bundle \( \mathcal{T}' : C_0(LM) \to M \) obtained by restricting \( F \) to \( \mathcal{T}_0 \)

\[ C_0(LM) := \text{pr}_2 \left( F \left( \mathcal{T}_0 \right) \right). \]

Moreover, the following lemma can be proved using standard facts about principal bundles [22].

**Lemma 2.1.** The submanifold \( \mathcal{T}_0 \subset J^1 \tau \) is a principal subbundle of the \( GL(m) \)-bundle \( p_{GL(m)} : J^1 \tau \to C(LM) \), associated to the isomorphism \( \text{id} : GL(m) \to GL(m) \).

These considerations give rise to the commutative diagram

\[
\begin{array}{ccc}
LM & \xrightarrow{\tau} & C_0(LM) \\
\downarrow{\tau_0} & & \downarrow{\iota_{\mathcal{T}_0}} \\
\mathcal{T}_0 & \xrightarrow{p_{GL(m)}} & GL(m)
\end{array}
\]
Remark 2.2 (Local description for $\mathcal{S}_0$). Let $(x^\mu, e^\nu_k)$ be a set of adapted coordinates for $LM$ induced on $\tau^{-1}(U)$ by a set of coordinates $(x^\mu)$ on $U \subset M$; as usual, it induces coordinates $(x^\mu, e^\nu_k, e^\nu_{k\sigma})$ on $\tau^{-1}_1(U)$. On this open set we have

$$T = e^\nu_k e^\sigma_{k\nu} dx^\mu \wedge dx^\nu \otimes e_i$$

(where $(e^\mu_k)$ is the inverse matrix of $(e^\nu_k)$), so that the set $\mathcal{S}_0 \cap \tau^{-1}_1(U)$ is described by the constraints

$$e^\nu_k e^\sigma_{k\nu} = e^\nu_k e^\sigma_{k\mu}.$$ 

On the other hand, the metricity condition has differential nature: As we mentioned before, matrix $\eta$ determines a factorization of $gl(m)$ in a subalgebra $\frak{k}$ (the subalgebra of $\eta$-Lorentz transformations) and an invariant subspace $p$. The explicit formulas for this decomposition are given by the projectors

$$A_k := \frac{1}{2} \left( A - \eta A^T \eta \right), \quad A_p := \frac{1}{2} \left( A + \eta A^T \eta \right)$$

for every $A \in gl(m)$. The metricity condition is imposed on a section $\Sigma : M \to J^1\tau$ by requiring that

$$\Sigma^* \omega_p = 0,$$

where $\omega_p$ is the $p$-component of the canonical connection $\omega$ with respect to this decomposition. Taking into account the affine isomorphism $F : J^1\tau \to LM \times_M C(LM)$, this constraint means that the parallel transport of the connection $\eta_2 \circ F \circ \Sigma : M \to C(LM)$ leaves invariant the metric associated to the vielbein $pr_1 \circ F \circ \Sigma : M \to LM$ (see Equation (6.5) below).

2.2. Contact-like structure and admissible sections. The scheme we will use for Routh reduction relies on the notion of unified formulation of a variational problem; it is a necessary step in order to avoid issues regarding the non regularity of the Lagrangian to be reduced [15]. In this vein, we should mention that our approach to the unified formalism is strongly based on the groundbreaking work of Gotay [17]. Although the aim in the previously cited article is to extend the definition of the Poincaré-Cartan form to a generalized family of variational problems (the so called Griffiths variational problems, see [19, 20] and references), it can be readily seen that it can serve as a generalization of the unified formalism (as defined in [2, 13, 26] and references therein) to these kind of variational problems, namely, when restricted to the particular case of the classical variational problem (the variational problem underlying the first order classical field theory, see [18]), this construction reduces to the construction associated to the usual formulation of the unified formalism.

There are two crucial differences between a classical variational problem and a more general Griffiths variational problem:

- First of all, a classical variational problem (of first order) is formulated in a first order jet bundle, whereas a Griffiths variational problem can use in principle any bundle.
- More important (at least from the viewpoint of the present work) is the fact that the sections are integral for some characteristic set of forms. In the classical case, these set of forms are the contact forms, allowing us to restrict the set of forms to be varied to the holonomic sections of the jet bundle; the contact structure is replaced by another set of forms in the more general case.
Remark 2.3 (On the description of Griffiths variational problems). As it is said in the previous paragraph, a Griffiths variational problem consists into three kind of data: A bundle $p : W \to M$, whose sections will be the fields of the theory, a Lagrangian form $\lambda \in \Omega^m(W)$ setting the dynamics, and a set of forms $\mathcal{S} \subset \Omega^* (W)$ (more precisely, an exterior differential system) describing the set of differential restrictions on the fields. Accordingly, we will often specify a variational problem of this kind with the symbol

$$(p : W \to M, \lambda, \mathcal{S}).$$

The variational problem underlying such triple consist into finding the extremals of the action

$$S[s] := \int_M s^* \lambda$$

where the sections $s : M \to W$ of the bundle $p$ must be integral for the set of forms in $\mathcal{S}$, namely,

$$s^* \alpha = 0$$

for every $\alpha \in \mathcal{S}$.

The variational problem we will consider here for the Palatini gravity is not a classical one; it will differ from a variational problem of this kind in both of the aspects mentioned above:

- The relevant bundle is not the first order jet $J^1 \tau$; instead, it is the subset $\mathcal{S}_0$ consisting into the jets associated to torsionless connections. Due to this fact, we will consider the pullback of the canonical forms and the restriction of maps from $J^1 \tau$ to $\mathcal{S}_0$; unless explicitly stated, the new forms and maps will be indicated with the same symbols. An exception to this rule will be the restriction of the bundle maps $\tau_{10}$ and $\tau_1$, which will be indicated as $\tau'_{10} : \mathcal{S}_0 \to LM$ and $\tau'_1 : \mathcal{S}_0 \to M$.

- The forms we will use for the restriction of the sections of $\tau'_1 : \mathcal{S}_0 \to M$ are not the whole set of contact forms $\{\omega^i_j\}$, but a geometrically relevant subset, namely, the components of the metricity forms $\omega_0$.

In order to establish the unified version of the equations of motion for Palatini gravity, it will be necessary to define the metricity subbundle $I_{PG}$ on $\mathcal{S}_0$,

$$I_{PG}^m := \{\eta^{ik} \beta_{kp} \wedge \omega^p_i : \beta_{ij} \in \wedge^{m-1}_1 (T^* \mathcal{S}_0), \beta_{ij} - \beta_{ji} = 0\} \subset \wedge^m_2 (T^* \mathcal{S}_0),$$

where $\wedge^{m-1}_1 (T^* \mathcal{S}_0)$ indicates the set of $\tau'_1$-horizontal $(m-1)$-covectors on $\mathcal{S}_0$. With the metricity subbundle in mind, we can define the affine subbundle

$$W_{PG} := \mathcal{L}_{PG} + I_{PG}^m \subset \wedge^m_2 (T^* \mathcal{S}_0),$$

which comes with the projection

$$\tau_{PG} : W_{PG} \to \mathcal{S}_0 : \alpha \in \wedge^m_2 \left( T^*_{j} \mathcal{S}_0 \right) \mapsto j^1_\alpha s.$$  

Because this is a subbundle in the set of $m$-forms on $\mathcal{S}_0$, it has a canonical $m$-form $\lambda_{PG}$ on it given by

$$\lambda_{PG|\alpha} (w_1, \cdots, w_m) := \alpha (T_\alpha \tau_{PG}(w_1), \cdots, T_\alpha \tau_{PG}(w_m)), \quad w_1, \cdots, w_m \in T_\alpha (W_{PG}).$$
3. THE VARIATIONAL PROBLEM FOR PALATINI GRAVITY

The variational problem we will work with in the present article is the following.

**Definition 3.1** (Griffiths variational problem for Palatini gravity). The variational problem for Palatini gravity is given by the action

\[ S[\sigma] := \int_{U} \sigma^* \mathcal{L}_{PG}, \]

where \( \sigma : U \subset M \rightarrow \mathcal{T}_0 \) is any section of \( \tau'_1 \) such that \( \sigma^* \omega_p = 0 \). According to Remark 2.3 it is described by the triple \( (\tau'_1 : \mathcal{T}_0 \rightarrow M, \mathcal{L}_{PG}, \langle \omega_p \rangle) \), where \( \langle \cdot \rangle \) indicates the exterior differential system generated by the set of forms enclosed in the brackets.

The relevance of the unified formalism in dealing with variational problems is guaranteed by the following result [5].

**Proposition 3.2.** A section \( s : U \subset M \rightarrow \mathcal{T}_0 \) is critical for the variational problem established in Definition 3.1 if and only if there exists a section \( \Gamma : U \subset M \rightarrow W_{PG} \) such that

1) \( \Gamma \) covers \( s \), i.e. \( \tau_{PG} \circ \Gamma = s \), and
2) \( \Gamma^* \left( X \cdot d\lambda_{PG} \right) = 0 \), for all \( X \in \mathcal{X}^{V(\tau'_1 \circ \tau_{PG})}(W_{PG}) \).

\( \Gamma \) is called a solution of the Palatini gravity equations of motion.

**Remark 3.3.** Although the proof in [5] refers to sections of \( \tau : J^1 \tau \rightarrow M \), it can be also readily adapted to cover this case; in this regard, see Appendix [B].

The situation described by Proposition 3.2 is summarized in the following diagram:

\[
\begin{array}{ccc}
W_{PG} & \xrightarrow{\tau_{PG}} & \mathcal{T}_0 \\
\downarrow{\tau'_1 \circ \tau_{PG}} & & \downarrow{\tau'_1} \\
M & \xrightarrow{\Gamma} & \mathcal{T}_0
\end{array}
\]

We will see below (Section 7) that the same can be done for (first order) Einstein-Hilbert variational problem; the reduction and reconstruction theorems (see Section 10) will be proved using these lifted systems.

4. SYMMETRY AND MOMENTUM

We now discuss the presence of natural symmetries and their momentum maps for the unified formulation of Palatini gravity.

As we said above (see Lemma 2.1), there exists a \( GL(m) \)-action on \( \mathcal{T}_0 \); nevertheless, the Lagrangian \( \mathcal{L}_{PG} \) is preserved by the action of the subgroup \( K \subset GL(m) \) composed of the linear transformations keeping invariant the matrix \( \eta \),

\[
K := \left\{ A = \left( A^i_j \right) : \eta_{ij}A^i_kA^k_j = \eta_{kl} \right\}.
\]

We can lift the \( GL(m) \)-action to \( \wedge^m (\mathcal{T}_0) \); it results that the subbundle \( I_{PG}^m \) is also preserved by the action of \( K \), and so

\[ K \cdot W_{PG} \subset W_{PG}. \]

It is our aim to find a momentum map for this action, in the sense of the following definition.
Definition 4.1. A momentum map for the action of $K$ on $W_{PG}$ is a map

$$J: W_{PG} \to \Lambda^{m-1}(T^*W_{PG}) \otimes \mathfrak{k}^*$$

over the identity in $W_{PG}$ such that

$$\xi_{W_{PG}} \lrcorner d\lambda_{PG} = -dJ_\xi,$$

where $J_\xi$ is the $(m-1)$-form on $W_{PG}$ whose value at $\alpha \in W_{PG}$ is $J_\xi(\alpha) = \langle J(\alpha), \xi \rangle$.

A momentum map is $Ad^*$-equivariant if it satisfies

$$\langle J(g\alpha), Ad_{g^{-1}}\xi \rangle = g\langle J(\alpha), \xi \rangle.$$

Thus, we obtain Noether’s theorem in this setting:

Proposition 4.2. The momentum map $J$ is conserved along solutions of the Palatini gravity equations of motion.

Proof. Recall that $\Gamma: U \subset M \to W_{PG}$ is a solution for the Palatini gravity equations of motion if and only if

$$\Gamma^*(Z \lrcorner d\lambda_{PG}) = 0$$

for any $\tau_1^t \circ \tau_{PG}$-vertical vector field $Z$. Then for each $\xi \in \mathfrak{k}$ we have

$$d(\Gamma^*J_\xi) = \Gamma^*(dJ_\xi) = \Gamma^*(-\xi_{W_{PG}} \lrcorner d\lambda_{PG}) = 0,$$

and therefore the momentum is conserved along solutions. □

Accordingly, we think of a “momentum” $\hat{\mu}$ as an element $\hat{\mu} \in \Omega^{m-1}(W_{PG}, \mathfrak{gl}(m)^*)$, i.e. as a $\mathfrak{gl}(m)^*$-valued $(m-1)$-form on $W_{PG}$; a conserved value $\hat{\mu}$ of the momentum map is a closed one, i.e. $d\hat{\mu} = 0$.

The construction of a momentum map for the action on $W_{PG}$ is standard [18]:

Lemma 4.3. The map $J: W_{PG} \to \Lambda^{m-1}(T^*W_{PG}) \otimes \mathfrak{k}^*$ defined by

$$\langle J(\alpha), \xi \rangle = \xi_{W_{PG}}(\alpha) \lrcorner \lambda_{PG}|\alpha,$$

for each $\xi \in \mathfrak{k}$, is an $Ad^*$-equivariant momentum map for the $\mathfrak{gl}(m)$-action on $W_{PG}$.

Now, because

$$T \tau_{PG} \circ \xi_{W_{PG}} = \xi_{\tau_{PG}} \circ \tau_{PG},$$

then for every $\alpha \in W_{PG}$ it results that

$$\langle J(\alpha), \xi \rangle = \xi_{W_{PG}}(\alpha) \lrcorner \lambda_{PG}|\alpha \xi_{\tau_{PG}} \circ \tau_{PG},$$

$$= \xi_{\tau_{PG}} \circ \tau_{PG},$$

$$= \xi_{\tau_{PG}} \circ \tau_{PG},$$

$$= i_0 \left[ \xi_{\tau_{PG}} \circ \tau_{PG} \left( \eta^{ij} \omega_{jk} \wedge \Omega_{ij}^k + \eta^{ij} \beta_{pq} \wedge \omega_{ij}^q \right) \right],$$

$$= 0$$

for all $\xi \in \mathfrak{k}$. It means that the unique allowed momentum level set for this symmetry is $J = 0$; accordingly, the isotropy group of this level set is $K$, and

$$J^{-1}(0) = W_{PG}.$$
matrix \( \eta \) (see Section 2). The connection \( \omega_K \) on the bundle \( p_K^{LM} : LM \to \Sigma \) is induced by this decomposition, namely
\[
\omega_K := \pi_K \circ \omega_0,
\]
where \( \pi_K : \mathfrak{gl}(m) \to \mathfrak{k} \) is the canonical projector onto the \( \mathfrak{k} \)-factor in the Cartan decomposition and \( \omega_0 \) is a connection form on the principal bundle \( \tau : LM \to M \). The \( K \)-invariance of the factor \( p \), \( \text{Ad}_A p \subset p \ \forall A \in K \)
ensures us that it has the expected properties of a connection.

Finally, let us identify a candidate for the reduced bundle. In order to proceed, consider the adjoint bundle \( \tilde{\tau}_K : \tilde{\tau} \to \Sigma \); then, the following result holds.

**Proposition 4.4.** The map
\[
\Upsilon_{\omega} : J^1 \tau \longrightarrow (p_K^{LM})^* \left( J^1 \tau \Sigma \times _\Sigma \text{Lin} \left( \tau^\Sigma_T M, \tilde{\mathfrak{e}} \right) \right),
\]
\[
\Upsilon_{\omega} \left( j^1_x (s), j^1_x (s)_K, [s(x), \omega_K \circ T_s]_K \right) \to \left( j^1_x (s), j^1_x (s)_K, [s(x), \omega_K \circ T_s]_K \right).
\]
is a bundle isomorphism.

The inverse of \( \Upsilon_{\omega} \) is given by
\[
(4.1) \quad \Upsilon_{\omega}^{-1} \left( e, j^1_x [e, \tilde{\xi}]_K \right) = \left[ v \in T_x M \longmapsto (T_x \Xi)(v_x) \right] + \left( \tilde{\xi}(v_x) \right)_L (e),
\]
where \( (\cdot)^H, e \in LM \), is the horizontal lift associated to \( \omega_K \).

The map \( \Upsilon_{\omega} \) enjoys a useful property: under this identification, the action of \( K \) on \( J^1 \tau \) is simply
\[
g \cdot (e, j^1_x [e, \tilde{\xi}]_K) = (R \cdot e, j^1_x [e, \tilde{\xi}]_K).
\]
This is a direct consequence of the equivariance of the principal connection \( \omega_K \). As a result, we get the following corollary.

**Corollary 4.5.** There is an identification
\[
J^1 \tau / K \simeq J^1 \tau \Sigma \times _\Sigma \text{Lin} \left( \tau^\Sigma_T M, \tilde{\mathfrak{e}} \right).
\]

**Remark 4.6.** The choice of a connection on the bundle \( p_K^{LM} \) is allowing us to establish a relationship between the quotient space \( J^1 \tau / K \) and the jet bundle of the metric bundle \( J^1 \tau \Sigma \), the latter being the relevant bundle in the Einstein-Hilbert approach to relativity. It will be studied in detail in Section 6.

Motivated by these considerations, we are in position to define what is the Lagrangian quotient bundle for Palatini gravity.

**Definition 4.7 (Quotient bundle for Palatini gravity).** The bundle \( J^1 \tau \Sigma \times _\Sigma \text{Lin} \left( \tau^\Sigma_T M, \tilde{\mathfrak{e}} \right) \) is the quotient bundle for Palatini gravity.

In the next Sections we will explore a further simplification for this bundle, as well as a reduction for the Lagrangian responsible of the dynamics on these bundles.
5. Local Coordinates Expressions

Here we will obtain some identities allowing us to write down the isomorphism $\Upsilon^{-1}_\omega$ in local terms. In order to proceed, we fix a coordinate chart on $M$, inducing coordinates $(x^\mu, e^\mu_k)$ on $LM$. As usual, we will indicate with $(x^\mu, e^\mu_k, e^\mu_{k\sigma})$ the coordinates induced on $J^1\pi$. It can be proved that there exists a set of coordinates $(x^\mu, g^{\mu\nu}, \Gamma^\sigma_{\mu\nu})$ on $J^1\pi / K = \Sigma \times C(LM)$ and adapted to this decomposition, namely

$$p^L_M (x^\mu, e^\mu_k) = \left(x^\mu, \eta^{ij} e^j_k e^i_l\right).$$

In terms of these coordinates, we have

$$p^L_M (x^\mu, e^\mu_k, e^\mu_{k\sigma}) = \left(x^\mu, \eta^{ij} e^j_k e^i_l, -e^k_\mu e^\sigma_{k\sigma}\right).$$

It means in particular that

$$TP^L_M \left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\mu}$$

and

$$(5.1) \quad TP^L_M \left(\frac{\partial}{\partial e^\mu_k}\right) = \left(\eta^{pq} e^p_k \delta^\sigma_\mu + \eta^{kp} e^\sigma_\mu \delta^p_\sigma\right) \frac{\partial}{\partial g^{\sigma\rho}}.$$ 

On the other hand, a principal connection on $LM$ can be written as

$$\omega_0 = -e^j_\mu \left(d e^\mu_k - f^\mu_{k\alpha} dx^\alpha\right) E^k_j,$$

where $(f^\mu_{k\sigma})$ is a collection of local functions on $M$; its Christoffel symbols will be

$$\Gamma^\sigma_{\rho\mu} = -e^j_\mu f^\sigma_{j\rho}.$$ 

Given our definition of the connection $\omega_K$ on the $K$-bundle $p^L_M : LM \to \Sigma$, its components become

$$[(\omega_0)_e]_{k\mu} = -\eta_{lp} \left(\eta^{pq} e^q_j \eta^{lp} e^p_\mu \left(d e^\mu_k - f^\mu_{k\alpha} dx^\alpha\right)\right).$$

Now we will find the horizontal lift defined by $\omega_K$ for vector fields on $\Sigma$.

**Proposition 5.1.** The horizontal lift of vector fields on $\Sigma$ associated to the connection $\omega_K$ is locally given by

$$\left(\frac{\partial}{\partial x^\alpha}\right)_H = \frac{\partial}{\partial x^\alpha} + \frac{1}{2} g^{\beta\rho} e^\beta_\rho \left(g^{\alpha\sigma} \Gamma^\beta_{\alpha\mu} - g^{\alpha\beta} \Gamma^\sigma_{\alpha\mu}\right) \frac{\partial}{\partial e^\mu_k},$$

$$\left(\frac{\partial}{\partial g^{\mu\nu}}\right)_H = \frac{1}{4} g^{\beta\rho} e^\beta_\rho \left(\delta^{\sigma\beta}_\mu \delta^{\beta\rho}_k + \delta^{\sigma\rho}_\mu \delta^{\beta\rho}_k\right) \frac{\partial}{\partial e^\mu_k}.$$ 

**Proof.** See Appendix C. \hfill \Box

This proposition has the following consequence, that will be important to work with the reduction of the Palatini variational problem.

**Corollary 5.2.** Let $(x^\mu, g^{\mu\nu}, \delta^\alpha_\sigma)$ be the induced coordinates on $J^1\pi$. Then

$$(5.2) \quad \left(\frac{\partial}{\partial x^\sigma} + g^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}}\right)_H = \frac{\partial}{\partial x^\sigma} + \frac{1}{2} g^{\beta\rho} e^\beta_\rho \left[\Gamma^\beta_{\alpha\sigma} - g^{\alpha\beta} \Gamma^\beta_{\alpha\sigma}\right] \frac{\partial}{\partial e^\mu_k}.$$
Proof. According to Proposition 5.1, we have that
\[
\left(\frac{\partial}{\partial x} + g_{\rho}^\sigma \frac{\partial}{\partial g^{\mu \nu}}\right)^H = \\
= \frac{\partial}{\partial x} + \frac{1}{2} g^{\beta \rho} e_k^\rho \left( g^{\alpha \beta} T_{\alpha \sigma} - g^{\alpha \beta} T^{\alpha}_{\sigma} \right) \frac{\partial}{\partial e_k^\sigma} + \frac{1}{4} g^{\mu \nu} g^{\rho \delta} e_k^\rho \left( \delta^{\alpha}_{\mu} \delta^\beta_{\nu} + \delta^{\alpha}_{\nu} \delta^\beta_{\mu} \right) \frac{\partial}{\partial e_k^\sigma}
\]

as required. \(\square\)

Let us now introduce coordinates on the vector bundle \(\tilde{\mathfrak{e}}\). In order to do this, let us suppose that \((\phi = (x^\mu), U)\) is a coordinate chart on \(M\); then it is also a trivializing domain for the principal bundle \(\mathcal{L}M\), where

\[ t_U : \tau^{-1}(U) \to U \times GL(m) : u = (X_1, \cdots, X_m) \mapsto (x^\mu (\tau(u)), e_k^{\mu}(u)) \]

if and only if

\[ X_k = e_k^{\mu}(u) \frac{\partial}{\partial x^\mu}. \]

Therefore we can define the coordinate chart \((\phi_k, \tau^{-1}_k(U))\). In order to proceed, we use the correspondence between the space of sections of the adjoint bundle \(\Gamma_{\mathcal{L}E}\) and the set of \(p^M_K\)-vertical \(K\)-invariant vector fields on \(LM\). Therefore, taking the base \(\{E^\rho_\sigma\} \) on \(gl(m)\) such that

\[ (E^\rho_\sigma)^\beta_\alpha = \delta^\beta_\sigma \delta^\rho_\alpha, \]

we can define the set of \(GL(m)\)-invariant \(\tau\)-vertical vector fields \(\tilde{E}^\rho_\sigma\) whose flow \(\Phi^\rho_\sigma_{\tau} : \tau^{-1}(U) \to \tau^{-1}(U)\) is given by

\[ \Phi^\rho_\sigma_{\tau} (u) := t^{-1}_U \left( \tau(u), \exp \left( t E^\rho_\sigma \right) e_k^{\rho}(u) \right); \]

it means that, locally, these vector fields are such that

\[ (5.3) \hspace{1cm} T_{\phi_k} \left( \tilde{E}^\rho_\sigma (u) \right) = e_k^{\rho} \frac{\partial}{\partial e_i^\sigma}. \]

In the following we will adopt the usual convention according to which the map \(T t_U\) is not explicitly written, namely, where

\[ \frac{\partial}{\partial e_i^\sigma} \quad \text{and} \quad \frac{\partial}{\partial e_i^\sigma} \]

are identified. We can write down any \(p^M_K\)-vertical \(K\)-invariant vector field \(Z\) on \(LM\) as

\[ Z = A^\rho_\sigma E^\rho_\sigma; \]

then, using Equation (5.1), we obtain the following result.

**Lemma 5.3.** The vector field on \(\tau^{-1}(U)\) given by

\[ Z = A^\rho_\sigma E^\rho_\sigma \]

is \(p^M_K\)-vertical if and only if

\[ g^{\sigma \alpha} A^\rho_\alpha + g^{\rho \alpha} A^\sigma_\alpha = 0. \]
Proof. In fact, we have that

\[ 0 = T_u p_K ^{LM} (A^\sigma _{\alpha } \tilde{E}^\sigma _{\rho }) (u) \]

\[ = A^\sigma _{\alpha } T_u p_K ^{LM} (\tilde{E}^\sigma _{\rho }) (u) \]

\[ = A^\sigma _{\alpha } \varepsilon ^\sigma _{i} (u) T_u p_K ^{LM} \left( \frac{\partial}{\partial \varepsilon ^\sigma _{i}} \right) \]

\[ = A^\sigma _{\alpha } \varepsilon ^\sigma _{i} (u) \eta ^{\alpha } \left[ \varepsilon ^\sigma _{p} (u) \delta ^\rho _{p} + \varepsilon ^\sigma _{q} (u) \delta ^\rho _{q} \right] \frac{\partial}{\partial g^{\alpha \sigma \rho}}. \]

and the identity follows. \( \square \)

Therefore, we will have that

\[ \phi _{t} ([u, B]_{K}) := (x^\mu (u), \eta ^{ik} e^\mu _{k} (u) e^\nu _{i} (u), \alpha A^\sigma _{\alpha } ([u, B]_{K})) \]

if and only if \( g^{\sigma \alpha } A^\rho _{\alpha } + g^{\rho \alpha } A^\sigma _{\alpha } = 0 \) and

\[ [u, B]_{K} = A^\sigma _{\alpha } \tilde{E}^\sigma _{\rho } (u). \]

In order to relate the coordinates \( A^\rho _{\sigma } \) with the element \([u, B]_{K}\), we need to look closely to the identification between \( \Gamma ^{K} \) and the set of \( \mu ^{K} \)-vertical \( K \)-invariant vector fields on \( LM \).

It uses the correspondence

\[ V \tau \simeq LM \times gl(m) \]

given by

\[ (u, B) \mapsto \left. \frac{d}{dt} \right|_{t=0} (u \cdot \exp (-tB)). \]

In coordinates it reads

\[ (u = (X_1, \ldots, X_m), B = (B^i_1)) \mapsto -B^j_1 e^\rho _i \frac{\partial}{\partial e^\rho _j}, \]

and using Equation (5.3) it becomes

\[ (u = (X_1, \ldots, X_m), B = (B^i_1)) \mapsto -B^j_1 e^\rho _i \varepsilon ^\sigma _{j} \tilde{E}^\sigma _{\rho }. \]

Therefore, it results that

\[ \tilde{A}^\sigma _{\rho } (u, B) = -e^\rho _j B^j_1 e^\sigma _j \]

is a \( GL(m) \)-invariant function on \( LM \times gl(m) \) when \( GL(m) \) acts on \( gl(m) \) by the adjoint action; therefore, it gives us the set of functions \( A^\sigma _{\rho } \) on \( \tau ^{-1}_t(U) \subset \mathfrak{t} \) that completes the coordinates \( \phi _{t} \).

Lemma 5.4. The map \( \phi _{t} : \tau ^{-1}_t(U) \to U \times \mathbb{R}^{2m^2} \) given by

\[ \phi _{t} ([u, B]_{K}) = (x^\mu (u), \eta ^{ik} e^\mu _{k} (u) e^\nu _{i} (u), -e^i_1 (u) B^j_1 e^\sigma _j (u)) \]

defines a set of coordinates on \( \tau ^{-1}_t(U) \).

Proof. According to the previous discussion, it is only necessary to prove that for any \( B \in \mathfrak{t} \), i.e. such that

\[ \eta ^{ik} B^j_1 + \eta ^{jk} B^i_1 = 0, \]

the corresponding element on \( T_u LM \),

\[ Z = -e^i_1 (u) B^j_1 e^\sigma _j (u) \tilde{E}^\sigma _{\rho } \]
verifies the constraint
\[ T_{a\rho}^{\text{LM}}(Z) = 0. \]
But it follows that
\[ g^\rho \epsilon_{a} B_{j}^\rho e_j^\sigma - g^\sigma \epsilon_{a} B_{j}^\rho e_j^\rho = -g^\rho \epsilon_{a} B_{j}^\rho e_j^\rho + g^\sigma \epsilon_{a} B_{j}^\rho e_j^\rho = -g^\rho \epsilon_{a} B_{j}^\rho e_j^\rho e_k^\rho = -\delta_{ik} B_{j}^k \epsilon_{a} B_{j}^\rho e_j^\rho = 0, \]
as required. \[ \Box \]

6. Metricity and Contact Structures on the Quotient Space

In this section we will use the local expressions obtained in the Section 5 in order to study the relationship between the quotient bundle
\[ J^1 \tau / K \cong \Sigma \times C(\text{LM}) \]
and the bundle \( J^1 \tau \times \Sigma \text{Lin} \left( \tau^\Sigma \text{T}M, \tilde{\text{F}} \right) \). So far, we have the diagram

\[
\begin{array}{ccc}
J^1 \tau & \xrightarrow{\Upsilon_{\omega}} & (p_K^{\text{LM}})^* \left( J^1 \tau \times \Sigma \text{Lin} \left( \tau^\Sigma \text{T}M, \tilde{\text{F}} \right) \right) \\
\downarrow p_{K}^{\text{LM}} & & \downarrow p_{t23} \\
J^1 \tau / K & \xrightarrow{T_{\omega}} & J^1 \tau \times \Sigma \text{Lin} \left( \tau^\Sigma \text{T}M, \tilde{\text{F}} \right) \\
\Sigma \times C(\text{LM}) & \xrightarrow{G_{\omega}} & \end{array}
\]

(6.1)

defining the diffeomorphism \( G_{\omega} \): here \( T_{\omega} \) is the map induced by \( \Upsilon_{\omega} \). In short, we will prove that the introduction of a connection on the bundle \( p_K^{\text{LM}} : \text{LM} \rightarrow \Sigma \) allows us to split a principal connection on \( \tau : \text{LM} \rightarrow M \) into horizontal and vertical degrees of freedom. Moreover, this splitting will be powerful enough to relate the metricity forms \( \omega_{\rho} \) and the contact structure on the quotient bundle \( \tau_{\Sigma} : \Sigma \rightarrow M \).

First, let us stress that Lemma 5.4 allows us to set coordinates on the bundle
\[ \overline{p} : \text{Lin} \left( \tau^\Sigma \text{T}M, \tilde{\text{F}} \right) \rightarrow \Sigma. \]
In fact, any element \((g_x, \alpha) \in \text{Lin} \left( \tau^\Sigma \text{T}M, \tilde{\text{F}} \right)\) admits coordinates \((x^\mu, \tilde{g}^{\mu\nu}, A_{\alpha\rho}^\mu)\) if and only if \((x^\mu, \tilde{g}^{\mu\nu})\) are the corresponding coordinates for \(g_x \in \Sigma\) and
\[ \alpha \left( \frac{\partial}{\partial x^\sigma} \right) = A_{\alpha\rho}^\mu \tilde{e}_\mu (e_x), \]
where \(e_x \in \text{LM}\) is any element in \((p_K^{\text{LM}})^{-1} (g_x)\).
It is important to see the isomorphism \( \Upsilon_{\omega} \) restricted to \( \mathcal{T}_0 \). In order to properly set this result, let us construct the pullback bundles

\[
(p_{LM}^*)^* \left( J^1 \tau_\Sigma \times \Sigma \operatorname{Lin} \left( \tau_\Sigma^\tau T M, \tilde{\mathbf{t}} \right) \right) \longrightarrow J^1 \tau_\Sigma \times \Sigma \operatorname{Lin} \left( \tau_\Sigma^\tau T M, \tilde{\mathbf{t}} \right)
\]

\[
LM \quad \xrightarrow[p_{LM}]{\quad \xrightarrow{p_{LM}^*} \quad} \quad \Sigma
\]

and

\[
(p_{LM}^*)^* \left( J^1 \tau_\Sigma \right) \quad \xrightarrow[p_{2}]{\quad \xrightarrow{p_{1}} \quad} \quad J^1 \tau_\Sigma
\]

\[
LM \quad \xrightarrow[p_{LM}^*]{\quad \xrightarrow{p_{LM}^*} \quad} \quad \Sigma
\]

The zero torsion submanifold \( \mathcal{T}_0 \) has some nice properties regarding the decomposition induced by the connection \( \omega_K \).

**Proposition 6.1.** The canonical projection

\[
\text{pr}_\Sigma : (p_{LM}^*)^* \left( J^1 \tau_\Sigma \times \Sigma \operatorname{Lin} \left( \tau_\Sigma^\tau T M, \tilde{\mathbf{t}} \right) \right) \longrightarrow (p_{LM}^*)^* \left( J^1 \tau_\Sigma \right)
\]

\[
\left( e, j^1, x^\alpha, e_\beta \right) \quad \xrightarrow{\quad \xrightarrow{\quad} \quad} \quad \left( e, j^1, \xi_\beta \right)
\]

restricted to the submanifold

\[
\mathcal{T}_0' := \Upsilon_{\omega} (\mathcal{T}_0) \subset \left( p_{LM}^* \right)^* \left( J^1 \tau_\Sigma \times \Sigma \operatorname{Lin} \left( \tau_\Sigma^\tau T M, \tilde{\mathbf{t}} \right) \right)
\]

is a diffeomorphism between \( \mathcal{T}_0' \) and \( (p_{LM}^*)^* \left( J^1 \tau_\Sigma \right) \).

**Proof.** The proof of this proposition will be local. Using Equation (5.2) and the coordinates introduced above, we have that

\[
\frac{\partial}{\partial x^\alpha} + e^\mu_{\alpha \sigma} \frac{\partial}{\partial e^\mu_k} =
\]

\[
= \left( \frac{\partial}{\partial x^\sigma} + g^\mu_{\alpha \nu} \frac{\partial}{\partial g^\mu \nu} \right)^H + A_{\rho \sigma} \tilde{E}_\mu (e_\lambda)
\]

\[
= \frac{\partial}{\partial x^\sigma} + \frac{1}{2} g_{\beta \rho} e^\rho_k \left[ g^\mu_{\sigma \beta} + \left( g_{\alpha \mu} \Gamma^\beta_{\alpha \sigma} - g_{\alpha \beta} \Gamma^\mu_{\alpha \sigma} \right) \right] \frac{\partial}{\partial e^\mu_k} - A_{\rho \sigma} e^\rho_k \frac{\partial}{\partial e^\mu_k},
\]

namely

\[
e^\mu_{\alpha \sigma} = \frac{1}{2} g_{\beta \rho} e^\rho_k \left[ g^\mu_{\sigma \beta} + \left( g_{\alpha \mu} \Gamma^\beta_{\alpha \sigma} - g_{\alpha \beta} \Gamma^\mu_{\alpha \sigma} \right) \right] - A_{\rho \sigma} e^\rho_k.
\]

Then it follows that, for the \( K \)-invariant functions \( \Gamma^\mu_{\nu \sigma} \),

\[
\Gamma^\mu_{\rho \sigma} = - e^\rho_k e^\mu_{k \sigma}
\]

\[
= - \frac{1}{2} g_{\beta \rho} \left[ g^\mu_{\sigma \beta} + \left( g_{\alpha \mu} \Gamma^\beta_{\alpha \sigma} - g_{\alpha \beta} \Gamma^\mu_{\alpha \sigma} \right) \right] + A_{\rho \sigma} e^\mu_k.
\]

(6.2)
It means that the set $\mathcal{T}_0'$ is locally given by the equation

$$\frac{1}{2}g_{\beta\sigma} \left[ g_\mu^\beta + \left( g_{\alpha\mu} \Gamma_\alpha^\beta - g_{\alpha\beta} \Gamma_\alpha^\mu \right) \right] - \frac{1}{2}g_{\beta\rho} \left[ g_\sigma^\beta + \left( g_{\alpha\sigma} \Gamma_\alpha^\beta - g_{\alpha\beta} \Gamma_\alpha^\sigma \right) \right] + A_{\rho\sigma} - A_{\beta\rho} = 0.$$ 

Let us define the set of quantities $A_{\mu\nu\sigma} := g_{\mu\rho} A_{\nu\sigma}^\rho$; then using this equation and the fact that $A_{\mu\nu\sigma} + A_{\nu\mu\sigma} = g_{\mu\rho} A_{\nu\sigma}^\rho + g_{\nu\rho} A_{\mu\sigma}^\rho = 0$, we can conclude, from Proposition A.1, that the elements $A_{\nu\sigma}$ are uniquely determined by the fact that they belong to $\mathcal{T}_0'$. In other words, the set $(pr_2)^{-1} (e, j^1, x, s) \cap \mathcal{T}_0'$ consists into a single element.

Restricting $G_\omega$ to $\mathcal{T}_0$ (see Diagram (6.1)), we obtain the following result, that permits us to reconstruct Levi-Civita connection from a section of the reduced bundle.

**Proposition 6.2.** Let

$$\sigma : M \to J^1_{\Sigma} \times_{\Sigma} \text{Lin} \left( \tau^* \Sigma T M, \tilde{\mathfrak{t}} \right)$$

be a section of the composite map

$$J^1_{\Sigma} \times_{\Sigma} \text{Lin} \left( \tau^* \Sigma T M, \tilde{\mathfrak{t}} \right) \to \Sigma \xrightarrow{\tau} M$$

such that $pr_1 \circ \sigma : M \to J^1_{\Sigma}$ is a holonomic section and

$$\text{Im} \sigma \subset pr_{23} (\mathcal{T}_0').$$

Then

$$\Gamma_\sigma := pr_2 \circ G_\omega \circ \sigma : M \to C_0 (\text{LM})$$

is the Levi-Civita connection associated to the metric $g_\sigma := pr_1 \circ G_\omega \circ \sigma$.

**Proof.** Locally, the map $G_\omega$ is given by Equation (6.2). Therefore, from the proof of the previous Proposition and using Proposition A.1, we will have that the elements $g_{\mu\rho} \Gamma_{\nu\sigma}^\rho$ are uniquely determined by the set of equations

$$\Gamma_{\mu\rho\sigma} - \Gamma_{\rho\mu\sigma} = 0$$

and

$$\Gamma_{\mu\rho\sigma} + \Gamma_{\rho\mu\sigma} = -g_{\mu\rho} g_{\sigma\beta} g_{\alpha\beta}.$$ 

It means that

$$\Gamma_{\mu\rho\sigma} = -\frac{1}{2} \left( g_{\mu\rho} g_{\sigma\beta} g_{\alpha\beta} + g_{\rho\alpha} g_{\sigma\beta} g_{\mu\beta} \right).$$

Now, using the definition $g_{\mu\nu,\sigma} := -g_{\mu\rho} g_{\nu,\sigma}^\rho$, we obtain

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g_{\mu\alpha} (g_{\alpha\nu,\sigma} + g_{\rho,\sigma\alpha} - g_{\sigma\alpha,\nu}).$$
Because \( p_1 \circ \sigma \) is holonomic, we have that
\[
g'^{\mu \nu}_\sigma = \frac{\partial g^{\mu \nu}}{\partial \sigma},
\]
as required.

Let us define
\[
\mathcal{B}''_0 := G_\omega^{-1} (\Sigma \times C_0 (LM)) = \overline{\varTheta}_\omega \left( p_{K1}^{JI} (\mathcal{B}_0) \right);
\]
then, we need to draw our attention to the diagram in Figure 1. As a consequence of

**Figure 1.** Maps involved in the Routh reduction of Palatini gravity

Formula (6.4), we obtain the following corollary; in short, it says that in the reduced bundle
\[
J^1 \Sigma \times \Sigma \text{Lin} \left( \tau^\Sigma_{LM} \bar{\mathbf{r}} \right),
\]
the degrees of freedom associated to the factor \( \text{Lin} \left( \tau^\Sigma_{LM} \bar{\mathbf{r}} \right) \) are superfluous.

**Corollary 6.3.** The map
\[
pr_1 \mid _{\mathcal{B}''_0} : \mathcal{B}''_0 \to J^1 \Sigma
\]
is a bundle isomorphism over the identity on \( \Sigma \).

**Proof.** Locally, composite map \( \overline{\varTheta}_\omega \circ p_{K1}^{JI} \) is given by
\[
\overline{\varTheta}_\omega \circ p_{K1}^{JI} \left( \left[ x^\mu, e_i^\nu, e_\rho^\sigma \right]_K \right) = (x^\mu, \eta^{ij} e_i^\mu e_j^\nu, g^{\mu \nu}_\sigma),
\]
where coordinates \( g^{\mu \nu}_\sigma \) are calculated using Equation (6.3).
As we mentioned above, the splitting induced by the connection form $\omega_k$ allows us to relate the metricity forms with a contact structure on the quotient bundle.

**Proposition 6.4.** The metricity forms are $(pr_2 \circ pr_\Sigma \circ Y_\omega)$-horizontal (also $(pr_1 \circ \overline{\omega}_A \circ \rho^H K)$-horizontal). In fact,

$$T \rho^H K \circ \omega_B = (pr_2 \circ pr_\Sigma \circ Y_\omega)^* \overline{\omega} = (pr_1 \circ \overline{\omega}_A \circ \rho^H K)^* \overline{\omega}$$

where $\overline{\omega}$ is the contact form on $J^1 \Sigma$.

**Proof.** In local coordinates, we have that

$$(pr_2 \circ pr_\Sigma \circ Y_\omega)(x^\mu, e^\nu_k, e^\nu_{kA}) = (x^\mu, g^{\mu \nu}, g^{\mu \sigma}_{\rho})$$

where $g^{\mu \nu}$ is calculated using Equation (6.2). On the other hand, the metricity forms have the following local expression \[3\]

$$(6.5) \quad \eta^i \omega^j - \eta^j \omega^i = e^i \epsilon^j e^l \left[ dg^{\mu \nu} + \left( g^{\mu \sigma}_{\rho} \Gamma^\rho_{\sigma \nu} + g^{\nu \sigma}_{\rho} \Gamma^\rho_{\sigma \mu} \right) dx^\rho \right].$$

Using Equation (6.2), it follows that

$$\eta^i \omega^j - \eta^j \omega^i = e^i \epsilon^j e^l \left( dg^{\mu \nu} - g^{\mu \nu} dx^\sigma \right),$$

namely, the metricity condition is horizontal with respect to the projection

$$pr_2 \circ pr_\Sigma \circ Y_\omega : J^1 \tau \to J^1 \Sigma,$$

and the form in the base manifold is nothing but the generator of the contact structure. \[23\]

### 7. First order variational problem for Einstein-Hilbert gravity

We have enough background to define a variational problem on $J^1 \Sigma$ for Einstein-Hilbert gravity. The *Einstein-Hilbert Lagrangian form* will be defined as the unique 2-horizontal $m$-form $\mathcal{L}^{(1)}_{EH}$ on $J^1 \Sigma$ such that

$$\left( pr_1 \circ \overline{\omega}_A \circ \rho^H K \right)^* \mathcal{L}^{(1)}_{EH} = \iota_B^* \mathcal{L}_{PG}.$$

Recall also that in local terms, Palatini Lagrangian \[22\] can be written as

$$(7.1) \quad \mathcal{L}_{PG} = \epsilon_{\mu_1 \cdots \mu_{m-2} \gamma} \sqrt{\det g} \left[ e^k \eta^i \omega^j \wedge \cdots \wedge d \bar{x}^{\mu_{m-2}} \wedge \left( d \bar{\Gamma}^\gamma_{\rho \delta} \wedge d x^\rho + \Gamma^\rho_{\sigma \rho} \Gamma^\gamma_{\rho \beta} d x^\beta \wedge d x^\delta \right) \right].$$

We are pursuing here to establish the equivalence between the classical variational problem associated to the Lagrangian density $\mathcal{L}^{(1)}_{EH} : J^2 \tau \to \wedge^m \mathcal{M}$ (see \[16\]) and the variational problem $\left( J^1 \Sigma, \mathcal{L}^{(1)}_{EH}, \mathcal{L}_{EH} \right)$. As we have said above, the main difference between these variational problems is related to the nature of the Lagrangian form: In the latter, this form is not a horizontal form on $J^1 \Sigma$, meanwhile in the former case the Lagrangian form on $J^2 \Sigma$ is specified through a Lagrangian density, that gives rise to a horizontal form on this jet bundle. The following lemma tells us how these Lagrangians are related.

**Lemma 7.1.** It results that

$$\mathcal{L}_{EH} = h \mathcal{L}^{(1)}_{EH}$$

where $h : \Omega^m (J^1 \Sigma) \to \Omega^m (J^2 \Sigma)$ is the horizontalization operator \[23\].
Proof. Recall that the horizontalization operator is defined by the map
\[ h_{j^1s} := T_{x^j} s \circ T_{j^1s} (\tau_\Sigma)_2, \]
where \((\tau_\Sigma)_2 : J^2 \tau_\Sigma \rightarrow M\) is the canonical projection of the 2-jet bundle of the metric bundle onto \(M\). In terms of the coordinates \((x^\mu, g^{\mu\nu}, g^{\mu\nu}_a, g^{\mu\nu}_{\alpha\beta})\) on \(J^2 \tau_\Sigma\), we have that
\[ hdg^{\mu\nu} = g^{\mu\nu}_a dx^a, \quad hdg^{\mu\nu}_a = g^{\mu\nu}_{\alpha\beta} dx^\beta. \]
The result follows from a (rather lengthy) calculation, using expression (7.1) and the formula for the Christoffel symbols (6.4). □

The occurrence of the horizontalization operator in this lemma is crucial for our purposes, as the following proposition shows.

Theorem 7.2. Let \(\pi : E \rightarrow M\) be a bundle on a (compact) manifold \(M\) of dimension \(m\). For any \(\alpha \in \Omega^m (J^1 \pi)\) and any section \(s \in \Gamma \pi\), we have that
\[ \int_M \left( j^k s \right)^* \alpha = \int_M \left( j^{k+1} s \right)^* h\alpha. \]
Proof. It follows from the formula
\[ T_x j^k s = T_x j^k s \circ T_{j^1s} \pi_{k+1} \circ T_x j^{k+1} s, \]
that holds for every \(x \in M\) and \(s \in \Gamma \pi\). □

It is immediate to prove the desired equivalence.

Corollary 7.3. The classical variational problem specified by the Lagrangian density \(L_{EH}\) on \(J^2 \tau_\Sigma\) and the variational problem \((J^1 \tau_\Sigma, L_{EH}(1)\), \(\mathcal{C}_{\Sigma} \)) have the same set of extremals.

Proof. From Lemma 7.1 and using Theorem 7.2, we see that \(g : M \rightarrow \Sigma\) is an extremal for the action integral
\[ g \mapsto \int_M \left( j^2 g \right)^* L_{EH} \]
if and only if it is an extremal for the action integral
\[ g \mapsto \int_M \left( j^1 g \right)^* \lambda_{EH}, \]
as required. □

As usual [17], the equations of motion of this variational problem can be lifted to a space of forms on \(J^1 \tau_\Sigma\). Let us define the affine subbundle
\[ W_{EH} := L_{EH}(1) + I^E_{\text{con,2}} \subset \wedge^m (J^1 \tau_\Sigma). \]
Here, for every \(j^1s \in J^1 \tau_\Sigma\),
\[ I^E_{\text{con,2}} |_{j^1s} = L \left\{ \alpha_{\sigma(x)} \circ (T_{j^1s} (\tau_\Sigma)_1)_{10} - T_{s} \circ T_{j^1s} (\tau_\Sigma)_1 \right\} \wedge \beta : \]
\[ \alpha_{\sigma(x)} \in T^*_{\sigma(x)} \Sigma, \beta \in \left( \Lambda^{m-1} (J^1 \tau_\Sigma) \right)_{j^1s}, \]
is the corresponding fiber for the contact subbundle on \(J^1 \tau_\Sigma\). The canonical map will be denoted by
\[ \tau_{EH} : W_{EH} \rightarrow J^1 \tau_\Sigma. \]
We will indicate with $\lambda_{EH}$ the pullback of the canonical $m$-form on $\wedge^m (J^1 \gamma_2)$ to $W_{EH}$. Then we have a result analogous to Proposition 3.2 in the context of (first order) Einstein-Hilbert formulation.

**Proposition 7.4.** A section $s : U \subset M \rightarrow J^1 \gamma_2$ is a critical holonomic section for the variational problem \((J^1 \gamma_2, \mathcal{L}^{(1)}_{EH}, \mathcal{F}_{con})\) if and only if there exists a section $\Gamma : U \subset M \rightarrow W_{EH}$ such that

1) $\Gamma$ covers $s$, i.e. $\gamma_{EH} \circ \Gamma = s$, and
2) $\Gamma^* (X \cdot d\lambda_{EH}) = 0$, for all $X \in \mathfrak{X}(\gamma_{EH}) (W_{EH})$.

**Remark 7.5.** This proposition provides us with a unified formalism for Einstein-Hilbert gravity, based on the first order formulation. For the corresponding formalism associated to the second order formulation, see [16].

8. CONTACT BUNDLE DECOMPOSITION FOR PALATINI GRAVITY

Now we will recall some general facts regarding the decomposition induced for the connection $\omega_K$ on the bundle of forms $W_{PG}$ defined in Equation (2.3). The contact structure on $J^1 \tau$ gives rise to the contact subbundle on $\mathcal{F}_0$ given by

$$I^m_{\text{con},2} = \mathcal{L} \left\{ \alpha_{s(x)} \circ (T_{j_1} \gamma_1 T_{j_2} \gamma_2 - T_{j_2} \gamma_1 T_{j_1} \gamma_2) \wedge \beta : \right. \\
\left. \alpha_{s(x)} \in T^*_s (LM), \beta \in \Lambda_{1}^{m-1} (\mathcal{F}_0) \right\},$$

(8.1)

where $\mathcal{L}$ indicates linear closure. There is a splitting of $I^m_{\text{con},2}$ induced by the choice of a connection on the principal bundle $p_{LM}^K : LM \rightarrow \Sigma$. Its construction is as follows. We denote by $\omega_K \in \Omega^1 (LM, \mathfrak{t})$ the chosen connection and consider the following splitting of the cotangent bundle:

$$T^* (LM) = (p_{LM}^K)^* (T^* \Sigma) \oplus (LM \times \mathfrak{t}^*).$$

The identification is obtained as follows:

$$(p_{LM}^K)^* (T^* \Sigma) \oplus (LM \times \mathfrak{t}^*) \rightarrow T^* (LM),
\quad (e, \alpha_{[\omega]}, \sigma) \mapsto \alpha_{[\omega]} = \alpha_{[\omega]} \circ T_{\omega} p_{LM}^K + (\sigma, \omega_K (\cdot)).$$

Accordingly, we have an splitting of contact bundle (8.1)

$$I^m_{\text{con},2} = I^m_{\text{con},2} \oplus I^m_{\mathfrak{t}^*,2},$$

(8.2)

with

$$\widetilde{I}^m_{\text{con},2} = \mathcal{L} \left\{ \tilde{\alpha}_{[\omega]}(x) \circ T_{s(x)} p_{LM}^K \circ (T_{j_1} \gamma_1 T_{j_2} \gamma_2 - T_{j_2} \gamma_1 T_{j_1} \gamma_2) \wedge \beta : \right. \\
\left. \tilde{\alpha}_{[\omega]}(x) \in T^*_s (LM), \beta \in \Lambda_{1}^{m-1} (\mathcal{F}_0) \right\},$$

$$I^m_{\mathfrak{t}^*,2} = \left\{ \langle \sigma \wedge \omega_K, (T_{j_1} \gamma_1 T_{j_2} \gamma_2 - T_{j_2} \gamma_1 T_{j_1} \gamma_2) \rangle : \sigma \in \Lambda_{1}^{m-1} (\mathcal{F}_0 \otimes \mathfrak{t}^*) \right\}.$$

The symbol $\langle \cdot \wedge \cdot \rangle$ denotes the natural contraction, defined as follows: For elements of the form $\alpha_1 \otimes \nu, \alpha_2 \otimes \eta$ with $\nu, \eta \in \mathfrak{t}$ and $\alpha_1, \alpha_2$ forms, we have $\langle \alpha_1 \otimes \nu \wedge \alpha_2 \otimes \eta \rangle = \langle \nu, \eta \rangle \alpha_1 \wedge \alpha_2$. For a general element in the linear closure, extend linearly.

We can split our metricity subbundle $I^m_{PG}$ using the inclusion

$$I^m_{PG} \subset I^m_{\text{con},2},$$
namely

$$I^m_{PG} = \bigcup (I^m_{PG} \cap I_{\text{con,2}}^m) \oplus (I^m_{PG} \cap I_{\text{con,2}}^m).$$

But we have the following fact.

**Lemma 8.1.** For every $j^1s \in \mathcal{F}$

$$I^m_{PG} \subset I_{\text{con,2}}^m.$$

**Proof.** Let us work in the coordinates considered above; therefore, we have Equation (5.1) for the projector $T^p \mathcal{L}^M$ and also

$$T_{j^1s} \tau_{10} - T_s \circ T_{j^1s} \tau_1 = \frac{\partial}{\partial \epsilon^e_k} \otimes (de^\mu_k - e^\mu_{ek} dx^\alpha).$$

Then

$$T_{s(x)} T^p \mathcal{L}^M \circ \left( T_{j^1s} \tau_{10} - T_s \circ T_{j^1s} \tau_1 \right) =$$

$$= T_{s(x)} T^p \mathcal{L}^M \left( \frac{\partial}{\partial \epsilon^e_k} \right) \otimes (de^\mu_k - e^\mu_{ek} dx^\alpha)$$

$$= \frac{\partial}{\partial \epsilon^e_k} \otimes \left[ \eta_{kq} (e^\rho_{kq} de^\sigma_k + e^\sigma_k de^\rho_k) - \eta_{kq} (e^\sigma_k e^\rho_k + e^\rho_k e^\sigma_k) dx^\alpha \right]$$

$$= \frac{\partial}{\partial \epsilon^e_k} \otimes \left[ dg^\rho^\sigma - \left( e^\rho_{e\beta} e^\sigma_k + e^\sigma_k e^\rho_{e\beta} \right) dx^\alpha \right]$$

$$= \frac{\partial}{\partial \epsilon^e_k} \otimes \left[ dg^\rho^\sigma + \left( e^\rho_{e\beta} \Gamma^\sigma_{\beta e\alpha} + e^\sigma_k e^\rho_{e\beta} \right) dx^\alpha \right]$$

that will be rise to a set of generators of the bundle $I^m_{PG}$ (see Equation (6.5)).

This result is compatible with the fact that the whole subbundle $W_{PG}$ is in the zero level set for the momentum map. We will return on that below.

### 9. First Order Einstein-Hilbert Lagrangian as Routhian

Because $J \equiv 0$ the Routhian density will coincide with the Lagrangian $L_{PG}$. This density will induce a density on the reduced bundle, which we will define next.

First, we write $p : \text{Lin} \left( \tau_{e}^1 TM, \tilde{\epsilon} \right) \rightarrow \Sigma$ for the obvious projection. In principle, the bundle $\text{Lin} \left( \tau_{e}^1 TM, \tilde{\epsilon} \right)$ would be the field bundle for the reduced system; nevertheless, we will show next (see Lemma [9.2] below) that the Routhian, namely, the Lagrangian form for this reduced system, will be horizontal for the projection onto the jet space of the base bundle $\Sigma$.

In particular, one can consider the map:

$$q : J^1(\tau_{e} \circ \mathcal{P}) \rightarrow J^1 \tau_{e} \times \text{Lin} \left( \tau_{e}^1 TM, \tilde{\epsilon} \right),$$

$$j^1 \sigma \rightarrow (j^1 (\mathcal{P} \circ \sigma), \sigma(x))$$

projecting onto the quotient bundle for Palatini gravity. So, we can formulate the reduced system as a first order field theory by taking the bundle $\tau_{e} \circ \mathcal{P} : \text{Lin} \left( \tau_{e}^1 TM, \tilde{\epsilon} \right) \rightarrow M$ as the basic field bundle. Nevertheless, there are some identifications that will permit us to simplify further this basic bundle.
In order to proceed, let us use the connection $\omega_K$ to define the maps fitting in the following diagram:

\[
\begin{array}{c}
\mathcal{T}_0 \xrightarrow{\omega} \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right) \\
\mathcal{T}_0 / K \xrightarrow{g_\omega := \Omega_0 |_{\mathcal{T}_0 / K}} J^1(\tau_{\Sigma} \circ \tilde{\tau})
\end{array}
\]

The definitions are as follows:

- $f_\omega : \mathcal{T}_0 \rightarrow \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right)$,
- $j^1_s : \rightarrow [s(x), \omega_K \circ T_s]_K$,
- $g_\omega : \mathcal{T}_0 / K \rightarrow J^1(\tau_{\Sigma} \times \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right))$,
- $[j^1_s]_K : \rightarrow (j^1_s (p^M_{\Sigma} \circ s), [s(x), \omega_K \circ T_s]_K)$.

The map $g_\omega$ is the identification from Corollary 4.5. Since the Lagrangian density $\mathcal{L}_{PG}$ is invariant under $K$, it defines a reduced density on $\mathcal{T}_0 / K$ which can be seen as a density on $J^1(\tau_{\Sigma} \times \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right))$. We will denote it by $\mathcal{L}^{\prime}_{PG}$:

\[
\left(\mathcal{L}^{\prime}_{PG}ight)^{\omega} \mathcal{L}_{PG} = \mathcal{L}_{PG}, \quad \mathcal{L}_{PG} \in \Omega_m(\mathcal{S} \times \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right)).
\]

**Definition 9.1.** The $m$-form $\mathcal{L}_{PG} \in \Omega_m(\mathcal{S} \times \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right))$ is the Routhian for the variational problem $(\mathcal{T}_0, \mathcal{L}_{PG})$.

Then, we are ready to prove a characteristic property for the Routhian associated to the reduction of Palatini gravity.

**Lemma 9.2.** The Routhian $\mathcal{L}_{PG}$ is $pr_1$-horizontal, where $pr_1 : J^1(\tau_{\Sigma} \times \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right)) \rightarrow J^1(\tau_{\Sigma})$ is the projection onto the first factor of the fibred product.

**Proof.** It follows from Equations (6.2) and (6.4) that

\[
pr_1^{\prime}\mathcal{L}^{(1)}_{EH} = \mathcal{L}_{PG},
\]

as required. □

In short, Routhian $\mathcal{L}_{PG}$ does not depend on the fiber coordinates $A_{\mu \rho}^\sigma$ of the bundle $\tau : \text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right) \rightarrow \Sigma$; it is just the pullback along $pr_1$ of the first order Lagrangian for Einstein-Hilbert gravity.

In the usual Routh reduction, the reduced Routhian is a $m$-form on $J^1(\tau_{\Sigma} \circ \tilde{\tau})$; in this case, Lemma 9.2 allows us to consider the form $\mathcal{L}^{(1)}_{EH}$ on $J^1(\tau_{\Sigma})$ as the Routhian. Therefore, we can forget about the degrees of freedom associated to the factor $\text{Lin}\left(\tau^\alpha_{\Sigma} TM, \tilde{\tau}\right)$ in the quotient bundle, and take as the quotient bundle for Palatini gravity the jet bundle $J^1(\tau_{\Sigma})$; this is the way in which we will proceed from this point.
10. EINSTEIN-HILBERT GRAVITY AS ROUTh REDUCTION OF PALATINI GRAVITY

We will devote the present section to establish the two main results of the article, namely, Theorem 10.3 regarding reduction of Palatini gravity and Theorem 10.5 dealing with reconstruction of metrics verifying Einstein equations of gravity. The strategy, as we mention in the introduction, is to compare the equations of motion (lifted to the corresponding spaces of forms \(W_{EH}\) and \(W_{PG}\)) in a bundle containing every relevant degree of freedom; this role is played below by the pullback bundle \(F^*_{\omega}(W_{EH})\). So, let us define

\[
F_{\omega} := \text{pr}_1 \circ g_\omega \circ p^{j_1s}_K : \mathcal{T}_0 \rightarrow J^1\tau_2,
\]

namely

\[
F_{\omega}(j_1s) = j_1^1(p^{LM}_K \circ s)
\]

for every \(j_1s \in \mathcal{T}_0\). Then we have the diagram

\[
\begin{array}{ccc}
W_{PG} & \xrightarrow{\wedge \nolimits^m_T(T^*\mathcal{T}_0)} & F_{\omega}(W_{EH}) \\
\pi_{PG} & & \text{pr}^\omega \\
\mathcal{T}_0 & \xrightarrow{\tau^\omega} & \mathcal{T}_0 \\
& \xrightarrow{\text{pr}^\omega} & \mathcal{T}_0 \\
& \xrightarrow{\text{pr}^\omega} & \mathcal{T}_0 \\
& \xrightarrow{\tau_{EH}} & W_{EH}
\end{array}
\]

where

\[
\tilde{F}_{\omega} : F_{\omega}(W_{EH}) \rightarrow \wedge \nolimits^m_T(T^*\mathcal{T}_0) : (j_1s, \rho) \mapsto \rho \circ T_{j_1s}F_{\omega}
\]

and

\[
\text{pr}^\omega : F_{\omega}(W_{EH}) \rightarrow \mathcal{T}_0, \quad \text{pr}^\omega : F_{\omega}(W_{EH}) \rightarrow W_{EH}
\]

are the canonical projections of the pullback bundle.

**Lemma 10.1.** The bundle map \(\tilde{F}_{\omega}\) is an affine bundle isomorphism on \(\mathcal{T}_0\) between \(W_{PG}\) and \(F^*_{\omega}(W_{EH})\).

**Proof.** It is consequence of Equation 9.1 and Proposition 6.4. □

We will use Diagram (10.1) as a mean to compare the equations of motion of Palatini gravity and Einstein-Hilbert gravity; the idea is to use Propositions 3.2 and 7.4 in order to represent these equations in terms of the spaces of forms \(W_{PG}\) and \(W_{EH}\) respectively, and to pull them back to the common space \(F^*_{\omega}(W_{EH})\). Crucial to this strategy is the following result.

**Proposition 10.2.** It is true that

\[
\tilde{F}^*_{\omega} \lambda_{PG} = (pr^\omega)^* \lambda_{EH}.
\]

**Proof.** Let \((j_1s, \rho) \in F^*_{\omega}(W_{EH})\) be an arbitrary element in this pullback bundle; then using Diagram (10.1) we will have that

\[
\lambda_{PG}|_{\rho \circ T_{j_1s}F_{\omega} \circ T_{j_1s}F_{\omega}} = \left( \rho \circ T_{j_1s}F_{\omega} \circ T_{j_1s}F_{\omega} \right) \lambda_{PG} = \lambda_{EH}|_{\rho \circ T_{j_1s}F_{\omega} \circ T_{j_1s}F_{\omega}}.
\]
where it was used that $\pi_{EH} : W_{EH} \to J^1\tau_{\Sigma}$ is the restriction of the canonical projection
$$\tau^\pi_{J^1\tau_{\Sigma}} : \wedge^2 \left( T^* J^1\tau_{\Sigma} \right) \longrightarrow J^1\tau_{\Sigma}$$
to $W_{EH}$. This identity proves the Proposition. □

10.1. **Routh reduction of Palatini gravity.** We are now ready to prove the first result on Routh reduction of Palatini gravity.

**Theorem 10.3.** Let $\hat{Z} : U \subset M \to \mathcal{T}_0$ be a section that obeys the Palatini gravity equations of motion. Then the section
$$F_\omega \circ \hat{Z} : U \to J^1\tau_{\Sigma}$$
is holonomic and obeys the Einstein-Hilbert gravity equations of motion.

**Proof.** The idea of the proof is encoded in the following diagram

(10.2)

Using Proposition 3.2, we construct $\Gamma : U \to W_{PG}$ out of $\hat{Z}$; the Palatini gravity equations of motion will become
$$\Gamma^* \left( Z \land d\lambda_{PG} \right) = 0$$
for any $Z \in \mathfrak{X}^V(\tau_{\Sigma} \circ \pi_{PG}) (W_{PG})$. Using Lemma 10.1 we can define
$$\Gamma' := \left( F_\omega \right)^{-1} \circ \Gamma : U \longrightarrow F_\omega^* (W_{EH});$$
then the Palatini equations of motion translate into
$$\left( \Gamma' \right)^* \left( Z' \land d\tilde{F}_\omega \lambda_{PG} \right) = 0$$
for any $Z' \in \mathfrak{X}^V(\tau_{\Sigma} \circ \pi_{PG}) (F_\omega^* (W_{EH}))$. Then using Proposition 10.2 and the fact that $\text{pr}_{\varphi_2}^\varphi : F_\omega^* (W_{EH}) \to W_{EH}$ is a submersion, we can conclude that the section
$$\tilde{\Gamma} := \text{pr}_{\varphi_2}^\varphi \circ \Gamma' : U \to W_{EH}$$
obeys the equations of motion
$$\tilde{\Gamma}^* \left( \tilde{Z} \land d\lambda_{EH} \right) = 0,$$
where $\tilde{Z} \in \mathfrak{X}^V((\tau_{\Sigma} \circ \pi_{PG}) (W_{EH})$ is an arbitrary vertical vector field on $W_{EH}$. Also, using Diagram (10.2), we have that
$$\pi_{EH} \circ \tilde{\Gamma} = \pi_{EH} \circ \text{pr}_{\varphi_2}^\varphi \circ \Gamma' = F_\omega \circ \text{pr}_{\varphi_1}^\varphi \circ \Gamma' = F_\omega \circ \hat{Z}.$$ The theorem then follows from Proposition 7.4. □
10.2. ...and reconstruction. It is time now to give a (somewhat partial) converse to Theorem 10.3. That is, given a section \( \zeta : U \subset M \to \Sigma \) such that \( j^1 \zeta : U \to J^1 \Sigma \) is extremal for the Einstein-Hilbert variational problem, find a section

\[
\hat{Z} : U \to \mathscr{T}_0
\]

such that \( F_\omega \circ \hat{Z} = j^1 \zeta \) and \( \hat{Z} \) is an extremal for the Palatini variational problem. From Figure 1 it is clear that we need to lift the section \( j^1 \zeta \) through the quotient map \( \mathcal{T}_0 \to \mathcal{T}_0/K \), which has the structure of a principal bundle on \( \mathcal{T}_0/K \). It is clear that any principal bundle can be trivialized by a convenient restriction of the base space. As discussed in [6], it is not the way in which this kind of reconstruction problems are solved. Rather, the problem of lifting sections along the projection in a principal bundle is reduced to the problem of deciding if certain connection is flat; moreover, it is expected that this connection is related to the connection used to define the Routhian. We will present in this section a theory of reconstruction along these lines. With this goal in mind, we will recall here some of the details developed in [6]; for proofs we refer to the original article. We begin with a pair of diagrams (10.3):

Then we have the following result.

**Lemma 10.4.** There exists a section \( s : M \to P \) covering the section \( \zeta : M \to P/G \) if and only if \( \zeta^* P \) is a trivial bundle.

Using that \( \zeta^* P \) is a principal bundle, being trivial can be characterized in terms of a flat connection [22]:

**Theorem 10.5.** Let \( \pi : P \to M \) be a \( G \)-principal bundle with \( M \) simply connected. Then \( P \) is trivial if and only if there exists a flat connection on \( P \).

If \( M \) is not simply connected, then one can ask for a flat connection with trivial holonomy and obtain a similar result. For the sake of simplicity, we will assume that \( M \) is simply connected to apply Theorem 10.5 when needed. For later use, we also observe that the section constructed in the proof of Theorem 10.5 has horizontal image w.r.t. the given connection.

We now wish to apply the previous discussion to the case of the bundle \( p^\mathcal{T}_0 : p^J_1 \big|_{\mathcal{T}_0} : \mathcal{T}_0 \to \mathcal{T}_0/K \). We have the situation depicted in Diagram 10.4 (left): \( Z : M \to \mathcal{T}_0/K \) is a given section and \( \zeta : M \to \Sigma \) is the induced section. The basic question we want to address
is the following: does there exist a section \( \hat{Z} : M \to \mathcal{T}_0 \) such that \( \rho_K^{-1} \circ \hat{Z} = Z \)?

(10.4)

Now, using the fact that \( \mathcal{T}_0 / K \simeq \Sigma \times C_0 (LM) \), \( \mathcal{T}_0 \simeq LM \times C_0 (LM) \), we have that \( Z = (\xi, \Gamma) \) is composed by the metric times the Levi-Civita connection \( \Gamma \); therefore, we will have that

\[
\hat{Z} = \left( \hat{\xi}, \Gamma \right)
\]

where \( \hat{\xi} : M \to LM \) is some lift of the section \( \xi : M \to \Sigma \). Then, we can then construct the pullback bundle \( \xi^* (LM) \) (Diagram (10.4), right) and particularize Lemma 10.4 to conclude the following:

**Lemma 10.6.** Assume that \( M \) is simply connected. If \( \xi^* (LM) \) admits a flat connection then there exists a section \( \hat{Z} : M \to \mathcal{T}_0 \) such that

\[
(\rho_K^{LM} \circ \tau_{10}) \circ \hat{Z} = \xi
\]

and \( \hat{Z}^* \omega_\xi = 0 \). Conversely, every such section gives rise to a flat connection on \( \xi^* (TM) \).

**Proof.** Because \( \xi^* (LM) \) is a \( K \)-principal bundle, Theorem 10.5 and Lemma 10.4 allow us to find a section \( \hat{\xi} : M \to LM \) iff there exists a flat connection on it. Thus if \( \omega^\xi \) is flat, we can construct a lift \( \hat{\xi} : M \to LM \) for \( \xi \) and so

\[
\hat{Z} = \left( \hat{\xi}, \Gamma \right)
\]

is the desired lift to \( \mathcal{T}_0 \), where \( \Gamma : M \to C (LM) \) is the Levi-Civita connection for \( \xi \).

Conversely, let us suppose that we have a lift

\[
\hat{\xi} := \tau_{10} \circ \hat{Z} : M \to LM
\]

for the metric \( \xi : M \to \Sigma \). Recall that, for every \( (x, u) \in \xi^* (LM) \),

\[
T_{(x,u)} \xi^* (LM) = \left\{ (v_x, V_u) : T_x \xi (v_x) = T_u \rho_K^{LM} (V_u) \right\} \subset T_x M \times T_u (LM).
\]

Then we construct the following \( K \)-invariant distribution \( H \) on \( \xi^* (LM) \): If \( k \in K \) fullfills the condition \( u = \hat{\xi} (x) \cdot k \), then

\[
H_{(x,u)} := \left\{ (v_x, T_x \hat{\xi} (v_x) R_k \left( T_x \hat{\xi} (v_x) \right)) : v_x \in T_x M \right\}.
\]

It can be shown that it defines a flat connection on \( \xi^* (LM) \).  \( \square \)
So, in order to find a lift for the section \( \zeta \), it is sufficient to construct a flat connection on the \( K \)-principal bundle \( \zeta^+ (LM) \).

To this end, we will define
\[
\omega^\natural := \pi_k \circ (pr_2)^* \omega_0 \in \Omega^1 (\zeta^+ (LM), \mathfrak{k})
\]
where \( \omega_0 \in \Omega^1 (LM, \mathfrak{gl}(m)) \) is a principal connection on \( LM \) and \( \pi_k : \mathfrak{gl}(m) \to \mathfrak{k} \) is the canonical projection onto \( \mathfrak{k} \). Lemma 10.6 allows us to establish the following definition, inspired in the analogous concept from regular Routh reduction.

**Definition 10.7** (Flat condition for Palatini gravity). We will say that a metric \( \zeta : M \to \Sigma \) satisfies the flat condition respect the principal connection \( \omega_0 \in \Omega^1 (LM, \mathfrak{gl}(m)) \) if and only if the associated connection \( \omega^\natural \) is flat.

**Remark 10.8.** This condition yields to a relationship between the metric \( \zeta : M \to \Sigma \) and the principal connection \( \omega_0 \); the physical relevance of this relationship remains unclear for the author. Mathematically, it means that, even in the case that the bundle \( p^M_K : LM \to \Sigma \) is nontrivial, it could be the case when restricted to the image of the section \( \zeta \).

Also, it is necessary to establish the following result regarding the map \( F_\omega \).

**Lemma 10.9.** The following diagram commutes
\[
\begin{array}{ccc}
\mathcal{J}_0 & \xrightarrow{F_\omega} & J^1 \tau_\Sigma \\
\downarrow p^0_K & & \downarrow (\tau_\Sigma)_{10} \\
\mathcal{J}_0/K & \xrightarrow{\tau_{10}} & \Sigma
\end{array}
\]

*Proof.* In fact, for \( j^1 s \in \mathcal{J}_0 \) we have
\[
(\tau_\Sigma)_{10} (F_\omega (j^1 s)) = (\tau_\Sigma)_{10} (j^1_s (p^M_K \circ s)) = [s(x)]_K,
\]
and also
\[
\tau_{10} (p^0_K (j^1_s)) = \tau_{10} ([j^1_s]_K) = [s(x)]_K.
\]
and the lemma follows.

With this in mind, we are ready to formulate the reconstruction side of this version of Routh reduction for Palatini gravity.

**Theorem 10.10** (Reconstruction in Palatini gravity). Let \( \zeta : M \to \Sigma \) be a metric satisfying the flat condition and the Einstein-Hilbert equations of motion. Then there exists a section
\[
\hat{Z} : M \to \mathcal{J}_0
\]
that is extremal of the Griffiths variational problem for Palatini gravity.

*Proof.* The holonomic lift
\[
j^1 \zeta : M \to J^1 \tau_\Sigma
\]
is extremal for the variational problem \( (J^1 \tau_\Sigma, Z_{EH}^{(1)} c_\text{con}) \); then, by Proposition 7.4, there exists a section
\[
\hat{\Gamma} : M \to W_{EH}
\]
such that \( \tau_{EH} \circ \bar{\Gamma} = j^1 \zeta \) and

\[
(10.6) \quad \bar{\Gamma}^\ast (X \cdot d\lambda_{EH}) = 0
\]

for all \( Z \in \mathfrak{X}^V((\tau_\Sigma) \circ \tau_{EH}) (W_{EH}) \).

On the other hand, by Lemma 10.6 we have a lift

\[
\tilde{Z} : M \to \mathcal{R}_0
\]

such that

\[
\tau_{10} \circ p_{\mathcal{R}_0} \circ \tilde{Z} = \zeta;
\]

by Diagram (10.5) we have that

\[
(10.7) \quad \zeta = \tau_{10} \circ p_{\mathcal{R}_0} \circ \tilde{Z} = (\tau_\Sigma)_{10} \circ F_\omega \circ \tilde{Z}.
\]

We will define the map

\[
\Gamma' := (\tilde{Z}, \bar{\Gamma}) : M \to \mathcal{R}_0 \times W_{EH}
\]

and show that it is a section of \( \text{pr}_1^\ast : F_\omega^\ast (W_{EH}) \to \mathcal{R}_0 \); namely, we have to show that

\[
F_\omega \circ \tilde{Z} = \pi_{EH} \circ \bar{\Gamma}.
\]

It is important to this end to note that the conclusion of Proposition 6.4 can be translated to this context into

\[
T_{P_L}^M \circ \omega_{\bar{\pi}} = F_\omega^\ast \omega;
\]

moreover, by Lemma 10.6 we know that \( \tilde{Z}^\ast \omega_{\bar{\pi}} = 0 \), so

\[
(\tilde{F}_\omega \circ \tilde{Z})^\ast \omega = \tilde{Z}^\ast (\tilde{F}_\omega^\ast \omega) = T_{P_L}^M \circ (\tilde{Z}^\ast \omega_{\bar{\pi}}) = 0.
\]

Then the section

\[
F_\omega \circ \tilde{Z} : M \to J^1 \tau_\Sigma
\]

is holonomic; finally, from Equation (10.7) we must conclude that

\[
F_\omega \circ \tilde{Z} = j^1 \zeta.
\]

But \( j^1 \zeta = \pi_{EH} \circ \bar{\Gamma} \) by construction of the section \( \bar{\Gamma} \); then

\[
F_\omega \circ \tilde{Z} = \pi_{EH} \circ \bar{\Gamma}
\]

and \( \Gamma' \) is a section of \( F_\omega^\ast (W_{EH}) \), as required.

Now define the section

\[
\Gamma := \tilde{F}_\omega (\tilde{Z}, \bar{\Gamma}) : M \to W_{PG};
\]

Therefore, for any \( Z \in \mathfrak{X}^V((\tau_1 \circ \tau_{PG}) (W_{PG}) \) that is \( \text{pr}_2 \circ F_\omega^{-1} \)-projectable, we have that

\[
\Gamma^\ast (Z \cdot d\lambda_{PG}) = (\Gamma')^\ast \left( (TF_\omega^{-1} \circ Z) \cdot dF_{\omega}^\ast \lambda_{PG} \right)
\]

\[
= (\Gamma')^\ast \left( (TF_\omega^{-1} \circ Z) \cdot d(\text{pr}_2^\ast \lambda_{EH}) \right)
\]

\[
= (\text{pr}_2 \circ \Gamma')^\ast \left( (T\text{pr}_2 \circ TF_\omega^{-1} \circ Z) \cdot d\lambda_{EH} \right)
\]

\[
= 0
\]

because \( \bar{\Gamma} \) obeys Equation (10.6). \( \square \)
11. Conclusions and outlook

We were able to adapt the Routh reduction scheme developed in [6] to the case of affine gravity with vielbeins. It suggests that this formalism could be fit to deal with Griffiths variational problems more general than the classical, at least with cases when the differential restrictions are a subset of those imposed by the contact structure. Extensions of this scheme to gravity interacting with matter fields will be studied elsewhere.

Appendix A. An important algebraic result

First we want to state the following algebraic proposition.

Proposition A.1. Let \( \{ c_{ijk} \} \) be a set of real numbers such that
\[
\begin{align*}
\pm c_{ijk} &= b_{ijk} \\
\pm c_{ijk} &= a_{ijk}
\end{align*}
\]
for some given set of real numbers \( \{ a_{ijk} \} \) and \( \{ b_{ijk} \} \) such that \( b_{ijk} \mp b_{jik} = 0 \) and \( a_{ijk} \mp a_{ikj} = 0 \). Then
\[
c_{ijk} = \frac{1}{2} \left( a_{ijk} + a_{jki} - a_{kij} + b_{ijk} + b_{kj} - b_{jki} \right)
\]
is the unique solution for this linear system.

Proof. From first equation we see that
\[
\pm c_{jik} = c_{ijk} - b_{ijk}.
\]
The trick now is to form the following combination
\[
2c_{ijk} - b_{ijk} - b_{kij} + b_{jki}
\]
where in the permutation of indices was used the remaining condition. \( \square \)

Appendix B. Proof of Proposition 3.2

In order to do this proof, it will be necessary to bring some facts from [5]. First, we have the bundle isomorphism on \( \mathscr{T}_0 \)
\[
W_{PG} \cong E_2
\]
where \( p'_2 : E_2 \to \mathscr{T}_0 \) is the vector bundle
\[
E_2 := \Lambda^{m-1}_1(\mathscr{T}_0) \otimes S^*(m),
\]
with \( S^*(m) := (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \) the set of symmetric forms on \( \mathbb{R}^m \), and
\[
\Lambda^{m-1}_1(\mathscr{T}_0) := \{ \gamma \in \Lambda^{m-1}(\mathscr{T}_0) : \gamma \text{ is horizontal respect to the projection } \tau'_1 : \mathscr{T}_0 \to M \}.
\]
The bundle \( E_2 \) is a bundle of forms with values in a vector space; therefore, it has a canonical \((m-1)\)-form
\[
\Theta := \Theta_{ij} e^i \circ e^j.
\]
Using the structure equations for the canonical connection on \( J^1 \tau \) (pulled back to \( \mathscr{T}_0 \)), we have that the differential of the Lagrangian form \( \lambda_{PG} \) is given by
\[
(B.1) \quad d\lambda_{PG} |_p = \left[ 2\eta^{kp}(\omega_p)|^i_k \wedge \theta_{il} - (\omega_p)|^i_k \wedge \eta^{kp} \theta_{il} + \eta^{ip} \Theta_{il} |^j_\beta \right] \wedge \Omega'_{p} + \eta^{ik} \left[ d\Theta_{ij} |^j_\beta + \eta^{ij} \eta_{il} \Theta_{ij} |^j_\beta \wedge (\omega_k)|^i_q \wedge (\omega_p)|^j_k \right].
\]
The equations of motion
\begin{equation}
\Gamma^\nu (X_\lambda \lambda_{\nu \mu}^\alpha) = 0, \quad X \in \mathcal{X}^{\nu} (t_0^{\nu \mu}) \left( W_{\nu \mu} \right)
\end{equation}
are obtained by choosing a convenient set of vertical vector fields; because of the identification given above, it is sufficient to give a set of vertical vector fields on \( \mathcal{H}_0 \) and on \( E_2 \). It results that a global basis of vertical vector fields on \( \mathcal{H}_0 \) is
\( B' := \left\{ A_J \tau, M_{rs} \left( \Theta^r, \left( E^s_j \right)_\nu \right)_L^M \right\} : \theta \in \mathfrak{gl}(m), M_{pq} - M_{qp} = 0 \); in fact, the equation defining \( \mathcal{H}_0 \)
is invariant by the \( GL(m) \)-action, and also
\begin{equation}
\left( \Theta^{t}, \left( E^{v}_{j} \right)^{L}_{M} \right) \cdot \left( \theta^{l}_{\nu} \gamma_{\nu}^{\mu} - \theta^{l}_{\nu} \gamma_{\nu}^{\mu} \right) = \epsilon_{j}^{v} \left( \gamma_{\nu}^{\mu} - \gamma_{\nu}^{\mu} \right).
\end{equation}
Given that \( E_2 \) is a vector bundle on \( \mathcal{H}_0 \), any section \( \beta : \mathcal{H}_0 \rightarrow E_2 \) gives rise to a vertical vector field; the equations of motion associated to these kind of vector fields are the metricity conditions
\( \omega_{\nu} = 0 \).
Therefore, fixing an Ehresmann connection on the bundle \( p'_2 : E_2 \rightarrow \mathcal{H}_0 \), we can produce the set of vertical vector fields on \( E_2 \)
\begin{equation}
\left( A_{\mathcal{H}_0} \right)_t^p = M_{rs} \left( \Theta^r, \left( E^s_j \right)_L^M \right)_t^p ;
\end{equation}
the equations of motion associated to \( M \) are
\begin{equation}
\eta^{l}_{\nu} \gamma_{\nu}^{\mu} \left( M_{pq} \theta_{pq} \right) = 0.
\end{equation}
The unique solution of these equations is \( \theta_{pq} = 0 \). In fact, by writing
\begin{equation}
\Gamma^{\nu} \theta_{pq} = \eta^{l}_{\nu} N_{l_{pq}} \theta_{t}
\end{equation}
and taking into account the symmetry properties of \( M_{pq} \), we have that the set of quantities \( N_{pq} \) must satisfy
\( N_{pqr} - N_{qpr} = 0, \quad N_{pqr} + N_{opr} = 0; \)
by Proposition A.1, it results that \( N_{pqr} = 0 \), as desired. The rest of the equations of motion can be calculated in the same fashion that in the \( J^1 \tau \) case; therefore, the equations (B.2) are equivalent to the equations for the extremals of the Palatini variational problem.

APPENDIX C. PROOF OF PROPOSITION 5.1

First, let us write down
\begin{equation}
\left( \frac{\partial}{\partial x^\mu} \right)_t^p = M^v_{\mu} \frac{\partial}{\partial x^v} + N^v_{\mu k} \frac{\partial}{\partial e^v_k}.
\end{equation}
Then from
\begin{equation}
T p^L_{j_k} \left( \left( \frac{\partial}{\partial x^\mu} \right)_t^p \right)_t^p = \frac{\partial}{\partial x^\mu},
\end{equation}
it results
\( M^v_{\mu} = \delta^v_{\mu} \).
the condition
\[ \omega_k \left( \frac{\partial}{\partial x^\mu} \right)^H = 0 \]
implies
\[ N^\nu_{\mu k} \left( \eta^{\mu k} e_\nu - \eta^{\nu k} e_\mu \right) + \eta^{\nu q} e_\sigma f_{q\mu} - \eta^{\mu q} e_\sigma f_{q\mu} = 0. \]

In order to understand this equation for the unknowns \( N^\nu_{\mu k} \), let us change of variables through the formula
\[ N^\nu_{\mu k} = g_{\alpha \rho} e^\rho_k N^\nu_{\mu}; \]
in terms of these new variables, and the Christoffel symbols of the connection \( \omega_0 \)
\[ \Gamma^\sigma_{\alpha \mu} = -e^\sigma_{\alpha k} f_{k\mu}, \]

the previous equations can be expressed as
\[ 0 = g_{\alpha \rho} e^\rho_k N^\nu_{\mu} \left( \eta^{\mu k} e_\nu - \eta^{\nu k} e_\mu \right) - \eta^{\nu q} e_\sigma f_{q\mu} - \eta^{\mu q} e_\sigma f_{q\mu} \Gamma^\sigma_{\alpha \mu} + \eta^{\nu q} e_\sigma f_{q\mu} \Gamma^\sigma_{\alpha \mu} \]
\[ = N^\nu_{\mu} \left( g_{\alpha \rho} e^\rho_k \eta^{\mu k} e_\nu - g_{\alpha \rho} e^\rho_k \eta^{\nu k} e_\mu \right) + \left( \eta^{\nu q} e_\sigma f_{q\mu} - \eta^{\mu q} e_\sigma f_{q\mu} \right) \Gamma^\sigma_{\alpha \mu} \]
\[ = N^\nu_{\mu} \left( g_{\alpha \rho} e^\rho_k \eta^{\mu k} e_\nu - g_{\alpha \rho} e^\rho_k \eta^{\nu k} e_\mu \right) \Gamma^\sigma_{\alpha \mu} + \left( \eta^{\nu q} e_\sigma f_{q\mu} - \eta^{\mu q} e_\sigma f_{q\mu} \right) \Gamma^\sigma_{\alpha \mu} \]
\[ = N^\sigma_{\nu} \left( e^\mu_{\rho} e_\sigma f_{\alpha \mu} - e^\mu_{\rho \sigma} e_\alpha f_{\mu \sigma} \right) + \left( e^\mu_{\rho} e_\sigma f_{\alpha \mu} - e^\mu_{\rho \sigma} e_\alpha f_{\mu \sigma} \right) \Gamma^\sigma_{\alpha \mu} \]
\[ = \left( e^\sigma_{\alpha \mu} \right) \left( N^\sigma_{\nu} + g_{\alpha \rho} \Gamma^\sigma_{\alpha \mu} \right). \]
The operator in the left is essentially an antisymmetrizer, because of the formula
\[ e^\mu_{\rho} e_\sigma f_{\alpha \mu} - e^\mu_{\rho \sigma} e_\alpha f_{\mu \sigma} = \delta^\mu_{\beta} \delta^\nu_{\sigma} - \delta^\nu_{\beta} \delta^\mu_{\sigma}; \]
therefore
\( (C.2) \)
\[ N^\sigma_{\nu} \Gamma^\sigma_{\alpha \mu} = S^\alpha_{\beta \mu}, \]

where
\[ S^\alpha_{\beta \mu} - S^\beta_{\alpha \mu} = 0. \]

Finally, from the condition \( (C.1) \) we obtain
\[ N^\sigma_{\nu} \left( \eta^{\kappa \mu} e^\beta_{\alpha \mu} \delta_{\sigma} + \eta^{\kappa \sigma} e^\alpha_{\kappa \sigma} \delta_{\alpha} \right) = 0 \]
or, in terms of the variables \( N^\nu_{\mu} \)
\[ 0 = g_{\nu \beta} e^\beta_{\mu} N^\nu_{\mu} \left( \eta^{\kappa \mu} e^\beta_{\alpha \mu} \delta_{\sigma} + \eta^{\kappa \sigma} e^\alpha_{\kappa \sigma} \delta_{\alpha} \right) \]
\[ = N^\nu_{\mu} \left( g_{\nu \beta} e^\beta_{\mu} \eta^{\kappa \mu} e^\beta_{\alpha \mu} \delta_{\sigma} + g_{\nu \beta} e^\beta_{\mu} \eta^{\kappa \sigma} e^\alpha_{\kappa \sigma} \delta_{\alpha} \right) \]
\[ = N^\nu_{\mu} \left( g_{\nu \beta} e^\beta_{\mu} \delta_{\sigma} + g_{\nu \beta} \delta_{\alpha} \delta_{\sigma} \right) \]
\[ = N^\nu_{\mu} \left( \delta_{\sigma} \delta_{\alpha} + \delta_{\nu} \delta_{\sigma} \right) \].

From Equation \( (C.2) \) it results that
\[ g_{\alpha \rho} \Gamma^\sigma_{\alpha \mu} + g_{\alpha \rho} \Gamma^\sigma_{\alpha \mu} - 2S^\alpha_{\beta \mu} = 0, \]
or equivalently
\[ N^\alpha_{\beta \mu} = \frac{1}{2} \left( g_{\alpha \rho} \Gamma^\sigma_{\alpha \mu} - g_{\alpha \rho} \Gamma^\sigma_{\alpha \mu} \right). \]
Therefore, we have

\[
\left( \frac{\partial}{\partial x^\mu} \right)^H = \frac{\partial}{\partial x^\mu} + \frac{1}{2} g^{\beta \rho} \frac{\partial}{\partial e_\beta^k} (g^{\alpha \sigma} \Gamma^k_{\alpha \mu} - g^{\alpha \beta} \Gamma^k_{\alpha \mu}) \frac{\partial}{\partial e_\rho^k}.
\]

Additionally, we need to construct the horizontal lifts

\[
\left( \frac{\partial}{\partial g^{\mu \nu}} \right)^H = P^\sigma_{\mu \nu} \frac{\partial}{\partial x^\sigma} + Q^\sigma_{\mu \nu k} \frac{\partial}{\partial e^k_\beta}
\]

with \( P^\sigma_{\mu \nu} - P^\sigma_{\nu \mu} = 0 \), \( Q^\sigma_{\mu \nu k} - Q^\sigma_{\nu \mu k} = 0 \). The equation

\[
(C.3) \quad T_{PK}^{LM} \left( \frac{\partial}{\partial g^{\mu \nu}} \right)^H = \frac{\partial}{\partial g^{\mu \nu}}
\]

and the identity (5.1) imply

\[
P^\sigma_{\mu \nu} \frac{\partial}{\partial x^\sigma} + Q^\sigma_{\mu \nu k} \left( \eta^{k \rho} e_q^\rho \delta^\alpha_{\sigma} + \eta^{k \sigma} e^\alpha_{q \delta} \right) \frac{\partial}{\partial g^{\alpha \rho}} = \frac{\partial}{\partial g^{\mu \nu}},
\]

namely

\[
P^\sigma_{\mu \nu} = 0
\]

and (given the symmetry properties of \( g^{\mu \nu} \))

\[
\frac{1}{2} \left( \delta^\rho_{\mu} \delta^\mu_{\nu} + \delta^\rho_{\nu} \delta^\mu_{\nu} \right) = Q^\sigma_{\mu \nu k} \left( \eta^{k \rho} e_q^\rho \delta^\alpha_{\sigma} + \eta^{k \sigma} e^\alpha_{q \delta} \right) = \eta^{k \rho} e_q^\rho Q^\alpha_{\mu \nu k} + \eta^{k \sigma} e^\alpha_{q \delta} Q^\rho_{\mu \nu k}.
\]

The horizontality condition

\[
\omega_K \left( \frac{\partial}{\partial g^{\mu \nu}} \right)^H = 0
\]

will be equivalent to

\[
(C.5) \quad \left( \eta^{p \rho} e^\rho_{\sigma} - \eta^{k \rho} e^\rho_{\delta} \right) Q^\sigma_{\mu \nu k} = 0.
\]

These conditions can be understood by introducing the variables

\[
Q^{\rho \alpha}_{\mu \nu} := \eta^{k \rho} e_q^\rho \delta^\alpha_{\sigma} Q^\alpha_{\mu \nu k};
\]

then, Equation (C.4) becomes

\[
Q^{\rho \alpha}_{\mu \nu} + Q^{\alpha \rho}_{\mu \nu} = \frac{1}{2} \left( \delta^\rho_{\mu} \delta^\mu_{\nu} + \delta^\rho_{\nu} \delta^\mu_{\nu} \right)
\]

and Equation (C.5) is equivalent to

\[
Q^{\rho \alpha}_{\mu \nu} - Q^{\alpha \rho}_{\mu \nu} = 0.
\]

Therefore

\[
Q^{\rho \alpha}_{\mu \nu} + Q^{\alpha \rho}_{\mu \nu} = \frac{1}{2} \left( Q^{\rho \alpha}_{\mu \nu} + Q^{\alpha \rho}_{\mu \nu} \right) + \frac{1}{2} \left( Q^{\rho \alpha}_{\mu \nu} - Q^{\alpha \rho}_{\mu \nu} \right) = \frac{1}{4} \left( \delta^\rho_{\mu} \delta^\mu_{\nu} + \delta^\rho_{\nu} \delta^\mu_{\nu} \right)
\]
and so

\[
\left( \frac{\partial}{\partial g^\mu \nu} \right)^H = Q^\mu_{\nu k} \frac{\partial}{\partial e^k}
\]

\[
= \frac{1}{4} \eta_{\mu \rho} \left( \delta^\rho_\delta^\alpha_\mu + \delta^\alpha_\delta^\rho_\mu \right) \frac{\partial}{\partial e^k}
\]

\[
= \frac{1}{4} \delta^\alpha_\beta \delta^\rho_\kappa \left( \delta^\kappa_\mu \delta^\alpha_\nu + \delta^\alpha_\nu \delta^\mu_\rho \right) \frac{\partial}{\partial e^\alpha}.
\]

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