The Dijkgraaf-Vafa prepotential
in the context of general Seiberg-Witten theory

H. Itoyama¹ and A. Morozov²

¹ Department of Mathematics and Physics, Osaka City University, Osaka, Japan
² Institute of Theoretical and Experimental Physics, Moscow

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Abstract

We consider the prepotential of Dijkgraaf and Vafa (DV) as one more (and in fact, singular) example of the Seiberg-Witten (SW) prepotentials and discuss its properties from this perspective. Most attention is devoted to the issue of complete system of moduli, which should include not only the sizes of the cuts (in matrix model interpretation), but also their positions, i.e. the number of moduli should be almost doubled, as compared to the DV consideration. We introduce the notion of regularized DV system (not necessarily related to matrix model) and discuss the WDVV equations. These definitely hold before regularization is lifted, but an adequate limiting procedure, preserving all ingredients of the SW theory, remains to be found.

1 Introduction

The recent papers [1] of R. Dijkgraaf and C. Vafa (DV) (see [2,3] for the prehistory and the follow-up in [4]-[7], its relation [8] to integrability [12]-[15] in general and to that of matrix models [16] in particular. The whole story is about the old claim [17] that "effective actions" (functions of coupling constants and background fields), obtained after functional integration over all fields, are the τ-functions [12,15] of classical (if fields are integrated out completely) or quantum (if some background fields are preserved and considered as subjects of further averaging, i.e. as operators) integrable hierarchies. The SW theory deals with specific situation, a boundary between classical and quantum, when the non-integrated fields are essentially the vacuum values, parametrizing non-trivial moduli spaces of vacua in SUSY theories. As usual, the classical moduli spaces are essentially quasiclassical objects (in particular, possess distinguished symplectic structures), and the corresponding effective actions (written in terms of prepotentials) should possess interpretation as "quasiclassical τ-functions". Empirically this is indeed the case (numerous examples are worked out since the claim was first put forward in [1], see [18]-[23]), but the general theory is still lacking, largely because of the lack of a relevant self-contained definition of a quasiclassical τ-function (in variance with the ordinary τ-function, which can be described in group-theory terms and/or as satisfying peculiar bilinear Hirota equations. See a discussion in [24]). All known definitions, including the hierarchy of Whitham equations [24,21,25], refer to additional data, like solution of original integrable equations and original τ-function, of which the prepotential is a "quasiclassical" approximation, or like the SW/Hitchin "fibrations" of complex manifolds over moduli spaces. Such definitions, though very important, especially for applications, are not fully satisfactory, because one and the same "quasiclassical" (in fact, the
low-energy, in most applications) limit arises for different original systems (moreover, original system can have different quasiclassical limits (phases)), while the SW/Hitchin pattern should finally arise as a solution to some problem, rather than serve as a definition of it.

So far, the two most promising approaches to an a priori definition of prepotentials (quasiclassical $\tau$-function) – as a peculiar class of special functions – are to consider them as solutions of some universal (like Hirota) equation or to represent them as series or integrals of some special type.

The so far best achievement on the first route was the formulation of (generalized) “spherical” WDVV equations \[27\] in \[28\] (see the followup in \[29\]-\[33\]). Unfortunately, there is more and more evidence that in their original (“spherical”) form, suggested in \[28\], these equations are true only for the hyperelliptic prepotentials, while their generalization, adequate to cover at least the elliptic level, remain unknown.\(^1\)

Taking the second route, one naturally arrives at considering matrix models, which: are, at large $N$, the natural approximations to ordinary QFT models, like Yang-Mills theories or models, based on Kac-Moody algebras (like conformal models of the WZNW family \[35\]); provide an integrable approximation (in the sense that their partition functions are $\tau$-functions at all $N$ \[36, 16\]); and since \[37\] are known to give rise to quasiclassical $\tau$-functions. The DV theory \[1\] is a new important step in this direction.

In ref.\[1\], in addition to important results for other branches of field theory, a new example of the SW theory is suggested, with

$$dS_{DV} = \sqrt{P_1^2(x) + f_{n-1}(x)} dx$$  \hspace{1cm} (1)

(motivated by the studies of SW theory in relation to Calabi-Yau manifolds \[2\] – the most natural arena for SW theory to act, (e.g.\[24\]), for which the prepotential can be found explicitly in the form of a double integral. This example appears to coincide with the spherical large-$N$ limit of partition function of Hermitean 1-matrix model. This fact at last provides a long-awaited bridge between matrix models and SW theory.

However, the bridge is still somewhat fragile, despite being heavily explored, exploited and strengthened by more and more examples. In our opinion, there are questions of principal importance of the following kind which have escaped these studies so far.

First, it is still unclear what happens to the correspondence for other examples of SW theory, where prepotentials are not (yet?) representable as double integrals. In fact, this subject is getting attention (see, e.g.\[1\]) and, hopefully, will be clarified in the near future.

Second, the SW prepotential acquires its full meaning only when its peculiar arguments, (which we refer to as flat moduli in this paper) are carefully specified. Unfortunately, no clear definition (neither conceptual nor technical) of all flat moduli is yet known in matrix models. (Below in this paper we shall see that there are actually two types of moduli in the DV theory, $S_i$ and $T_i$. Of these, $S_i$ has an interpretation as the number of eigenvalues, concentrated on the $i$-th cut, while the matrix-model definition of $T_i$ is still lacking.) In fact, the very theory of continuum limit of matrix models from the perspective of integrable systems is not well-developed. Since \[36, 37\] the subject has not attracted

\(^1\) Recently, the old example of Calogero system (associated with the $N = 2$ SUSY deformation of the $N = 4$ SUSY YM), where the breakdown of the spherical WDVV eqs. of \[28\] was first observed \[30\], got supplemented by another elliptic prepotential \[5\] (the Leigh-Strassler deformation \[34\] of the same $N = 4$ SUSY YM), which, according to the proof in \[31\], can not satisfy the spherical WDVV. (Ref.\[31\] lists all the solutions of WDVV equations of the form $f(a_i - a_j)$, and there is no elliptic solution, like $\zeta(a_i - a_j + b) + \zeta(a_i - a_j - b)$, announced to be a prepotential of LS deformation in ref.\[5\].)
much attention.

Third, no a priori characterization such as a new universal equation for the prepotential is yet provided. We shall see that the original DV prepotential (as a function of all moduli, $S_i$ and $T_i$) has some chances to satisfy just the usual spherical WDVV equations of [28] (with the usual problem expected for elliptic prepotentials, like the one in [3]). Moreover, even in this hyperelliptic case the raison d'etre for the WDVV equations from the matrix-model perspective remains obscure (from the SW theory perspective the reason is the existence of peculiar closed algebra of 1-forms [24]).

In the present paper we suggest to start the thorough study of DV prepotentials per se, irrespective of their exciting applications to Yang-Mills and string models, and make some first steps in this direction. Namely, after a brief discussion of generic SW theory and of the DV suggestion in s.2 and 3 respectively, we introduce in s.4 a regularized version of DV theory, – actually, the most direct generalization of original ansatz of ref.[8], – which does not have direct matrix-model representation, but instead deals with smooth hyperelliptic curves and holomorphic differentials on them. We define the full set of flat moduli, the prepotential (in implicit form, as usual in the SW theory) and residue formula [38] for its third derivatives. According to the general arguments of [29, 30] this (hyperellipticity, holomorphicity and residue formula) is enough to prove the validity of WDVV equations. In s.5 we briefly discuss the transition to unregularized DV system and the calculation of the CIV-DV prepotential. Much remains to be done, however, in order to come to appropriate understanding of the subject.

2 General Seiberg-Witten (SW) theory

Generically, the SW theory [8, 9] (in complex dimension one) includes the following ingredients:

Input data consist of a special family of spectral Riemann surfaces = complex curves (with or without singularities) and (a homotopical class of) a meromorphic 1-differential $dS$ on every surface with the property, that its moduli derivatives are less singular than $dS$ itself. In practice all the relevant examples can be considered as the limiting cases of the regularized ones, where the requirement is just that the moduli-derivatives of $dS$ are holomorphic 1-differentials (with no poles at all)[2]

Given such a SW family one can further make the following steps.

– Introduce specific flat coordinates on the moduli space (of the SW curves) by integrating $dS$ along the $A_p$ contours, $p$ runs from 1 to $g$, the genus of the curve (which is the same for the entire family),

$$a_p = \oint_{A_p} dS; \quad (2)$$

– Introduce specific (non-single-valued) function on the moduli space – the prepotential $F(a)$ by identifying its first $a_p$-derivatives with the integrals of $dS$ along the conjugated $B_p$ contours,

$$\frac{\partial F(a)}{\partial a_p} = b_p = \oint_{B_p} dS; \quad (3)$$

The self-consistency of such definition, i.e. the symmetricty of the matrix

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Note that $dS$ itself need not be holomorphic even in the regularized setting.
The SW fibration of complex curves over the SW moduli space possesses a distinguished symplectic structure
\[ \delta dS = \sum_p \delta a_p \wedge d\omega_p. \] (6)

It reflects the connection between SW theory and integrable systems [11], [18]-[23] (actually, \( dS \) is an eigenvalue of the Lax 1-form).

The third derivatives of the prepotential with respect to the flat moduli, which are the first moduli-derivatives of the period matrices, are given by residue formulas [38, 28], like
\[ F_{pqr} = \frac{\partial^3 F(a)}{\partial a_p \partial a_q \partial a_r} = \frac{\partial T_{pq}}{\partial a_r} \sum_{dS=0} \text{res} \frac{d\omega_p d\omega_q d\omega_r}{ddS}, \] (7)

where quadratic differential \( ddS \) is intimately related to the symplectic structure \( \delta dS \), and sum goes over residues at all the zeroes of \( dS \). If a meaning can be given to a closed associative algebra of holomorphic forms modulo \( ddS/dS \) and residue formula is rewritten in terms of a sum over its zeroes (this proves possible at least for all known examples of hyperelliptic SW families), eq.(6) can be used to derive the WDVV equations [27]-[33] for the prepotential \( F(a) \),
\[ \bar{F}_p \bar{F}_q^{-1} \bar{F}_r = \bar{F}_r \bar{F}_q^{-1} \bar{F}_p \] (8)

for all triples \( p, q, r \). Here \( \bar{F}_p \) denotes a (symmetric) matrix with the entries
\[ (\bar{F}_p)_{qr} = F_{pqr} = \frac{\partial^3 F(a)}{\partial a_p \partial a_q \partial a_r}. \] (9)

Since matrices \( \bar{F} \) are symmetric, eqs.(8) are equivalent to the symmetricity of the matrix \( \bar{F}_p \bar{F}_q^{-1} \bar{F}_r \). In fact, it is enough to check the equations for a given \( q \) (and all \( p, r \)), then they automatically hold for all values of \( q \). If the set of moduli of the SW system is sufficiently large for the WDVV eqs. to hold, we call the system complete.

The prepotential can be further extended to a larger moduli space. In the simplest case of such extension [20, 26] additional moduli are located at a single point \( x_0 \) on the spectral curve, where \( dS \) develops a singularity (generically, an essential one), and (2), (3) get supplemented by

\[ T_{pq} = \frac{\partial b_p}{\partial a_q} = T_{qp}, \] (4)

is guaranteed by the requirements, imposed on the moduli dependence of \( dS \). Actually, \( T_{pq} \) is the (always symmetric) period matrix of the curve, and derivatives of \( dS \) over flat moduli,
\[ d\omega_p = \frac{\partial dS}{\partial a_p}, \] (5)

are canonical 1-differentials for the set of \( A \)-contours, selected in (2), i.e. \( \oint A_p d\omega_q = \delta_{pq} \). Moreover, as already stated, the SW family can be always regularized in such a way that \( d\omega_p \) are holomorphic canonical 1-differentials.

A natural interpretation/generalization of \( dS \) is in terms of correlators of free fields on a Riemann surface (either
\[
\hat{T}_k = \text{res}_{x_0} (x - x_0)^k dS(x),
\]

\[
\frac{\partial \mathcal{F}(a)}{\partial T_k} = \frac{1}{k} \text{res}_{x_0} (x - x_0)^{-k} dS(x)
\]  

(10)

with all \( k \geq 0 \) (here some particular choice of coordinates in the vicinity of \( x_0 \) is implied, also logarithm is implicit in the formula for the \( T_0 \) derivative). If the singularity at \( x = x_0 \) is not an essential one, but just a finite-order pole, the number of additional moduli is finite. At least in this case the singularity can be considered as a result of degeneration of the non-singular moduli like (2) – this brings us back to the notion of regularized SW system\(^4\). Instead of taking simple singularities, one can consider larger sets of singular points \( x_0 \) and, further, make them form continuous sets, like cuts, closed contours and entire areas inside the contours. Certain development in this direction is due to the series of papers \(33\)-\(35\), in particular, the WDVV eqs. are shown to survive this type of generalizations. The DV theory \(1\) actually analyzes the case of the multicomponent contours (multicuts).

– On infinitely extended moduli space the prepotential still satisfies integrable equations of Whitham hierarchies \(2\), and finally can be considered as a peculiar quasiclassical limit of the logarithm of a \(\tau\)-function. With no surprise, this procedure is believed to be in parallel with the occurrence of the quasiclassical \(\tau\)-functions in the planar large-\(N\) limit of the matrix models (which, before taking the limit, are ordinary KP-like \(\tau\)-functions \(16\)). This parallel was recently pushed much closer to concrete statements in \(2\) and, even further, in \(3\). What is most interesting in this development – from the point of view of prepotential theory – is discovery of explicit expressions (in the form of a double limit of the logarithm of a \(\tau\) function \(10\)) of degree \(2\) – this brings us back to the notion of regularized SW system\(^4\). Instead of taking simple singularities, one can consider larger sets of singular points \( x_0 \) and, further, make them form continuous sets, like cuts, closed contours and entire areas inside the contours. Certain development in this direction is due to the series of papers \(33\)-\(35\), in particular, the WDVV eqs. are shown to survive this type of generalizations. The DV theory \(1\) actually analyzes the case of the multicomponent contours (multicuts).

An important example is description of fundamental matter in SW theory \(21\). When some \(2N_f\) branching points of the hyperelliptic spectral curve

\[
y^2 = \tilde{P}_{\tilde{N}_c}^2(x) - \Lambda^{2\tilde{N}_c}
\]
collide pairwise (this happens at the hypersurfaces of vanishing monopole masses in the original moduli space),

\[
\tilde{P}_{\tilde{N}_c}^2(x) - \Lambda^{2\tilde{N}_c} = Q_{N_f}(x) F_{2N_c}(x), \quad Q_{N_f}(x) = \prod_{\nu=1}^{N_f} (x - m_\nu),
\]

the newly emerging polynomial of degree \(2N_c = 2\tilde{N}_c - 2N_f\) can be represented as

\[
F_{2N_c}(x) = P_{N_c}(x) - \Lambda^{2(N_c - N_f)} Q_{N_f}(x)
\]

with some new polynomial \(P_{N_c}(x)\) of degree \(N_c\). The corresponding \(dS = \lambda \frac{dw}{w} = x d\log (P + y)\) \(22\) has simple poles at the points \(m_\nu\), i.e. describes a singular SW model. Original "pure gauge" SW model with \(\tilde{N}_c = N_c + N_f\) can be considered as a regularization of the singular SW model with \(N_c\) colors and \(N_f\) flavors, obtained by the blowing up of degenerated handles. This also explains, why the pole positions ("masses") \(m_\nu\) should necessarily be included into the set of SW moduli \(24\).
3 DV Theory

3.1 Generalities

The DV theory attracts a lot of attention because of its non-trivial message to physics of Yang-Mills theories: the two seemingly different procedures, (i) the non-perturbative analysis of the low-energy actions, performed (at least implicitly) with the help of sophisticated instanton calculus, and (ii) the rather naive analysis of the zero-mode sector (reduction to $d = 0$) in the large-$N$ limit, giving rise to the well-known planar matrix models \[42\], provide the same quantities. While being not very surprising conceptually (especially to those who believe in universality classes of effective actions and their integrable properties which reflect this universality and hidden symmetries), this is a very concrete statement, opening the field for detailed analysis and raising a lot of challenging questions.

Our goal is somewhat orthogonal to this line of thought about DV theory. We rather concentrate on its value for purely theoretical purposes, namely for the study of quasiclassical $\tau$-functions and their properties. The main achievement of the DV theory from the point of view of generic SW theory is that it seems to provide a set of examples – matrix models in the planar continuous limit – where all of the four interrelated ingredients of the SW theory (the set of curves, the form $dS$, the flat moduli $a_i = \int_{A_i} dS$ and the prepotential $F(a_i)$) have direct interpretation. In all previous examples of SW theory only some of these ingredients appeared in a natural way, while the others were built with the help of the SW theory itself: in SUSY YM theories at low energies the natural things are moduli and prepotential, while spectral curves are hidden structures (revealed in \[3\]); in brane patterns the natural things are instead the Lax forms (describing the shape of interacting branes, \[44,45\]) and their spectral curves; in Calabi-Yau studies again the curves are natural, as describing the geometry of Calabi-Yau 3-folds near the singularities, $dS$ then arises as restriction of the distinguished 3-form, but the prepotential should be introduced as some extra quantity, characterizing an associated SUSY sigma-model (in fact, one can consider this association, described sometime in terms of the "special geometry" \[46\] as another, not too different, formulation of SW theory); etc. The planar continuum limit of matrix models, however, seems to provide everything simultaneously: $dS$ is associated with the density $\rho(\lambda) d\lambda$ of the eigenvalues, the curves appear from algebraic equations of motion for $\rho(\lambda)$, the flat moduli are just the fractions of eigenvalues on every cut and the prepotential is nothing but the free energy $\log Z_{\text{planar}}$ of the model. The achievement of simultaneous description of all the ingredients bears an immediate fruit: in so obtained SW models there is a straightforward double-integral representation for the prepotential

$$F(a_i) = \sum_{i,j} \oint_{A_i} \oint_{A_j} dS(x) dS(x') G(x, x'), \quad a_i = \oint_{A_i} dS$$

with certain kernel (Green’s function) $G(x, x')$.

Unfortunately, there are still clouds over this idyllic picture. The most important one for the purposes of the present paper is somewhat unnatural restriction on the set of moduli. Obviously, the actual moduli of the planar-limit matrix model include more than just the numbers of eigenvalues on cuts; there are also cut’s positions. (The lengths of the cuts are expressible through the numbers of eigenvalues, but the positions of their centers remain independent variables). Without inclusion of all these moduli, there is no grounded hope of obtaining a complete SW system, in particular of obtaining any universal equations, like WDVV, for the prepotential. We consider our attempts to introduce back these ”lost” moduli in the remaining sections below. It looks like, after being introduced, they can indeed restore the WDVV equations. This is absolutely obvious for a ”regularized DV” system, to be introduced and analyzed in s.4 below, but instead the simple relation to matrix models can be lost.
after regularization. From this point of view it becomes clear that the original DV systems, coming from matrix models, are in fact examples of singular SW models, where many of moduli get localized at particular points (and then easily overlooked, as it happens in most analyses of the situation in the literature). Careful analysis of the singular limit is rather difficult, even with the shortcuts, making use of the prepotential theory \cite{20, 26}, see s.5 below.

Before proceeding to describe these results in the next sections 4 and 5, we briefly summarize the statements suggested in \cite{1} with some emphasis on the questions, which, in our opinion, deserve further investigation.

### 3.2 From matrix model to a SW model

The recipe is:

1) Take some matrix model,

\[ Z_N = \int e^{-Tr \mathcal{L}(\Phi)} d\Phi \]  

(There can be many matrix-valued $\Phi$-variables and multitrace operators can appear in the action).

2) Rewrite it in terms of eigenvalues:

\[ Z_N = \int e^{-S\{\lambda\}} \prod_s d\lambda_s \]  

(For non-eigenvalue model, the integral over unitary matrices can not be taken in any final form. Then $S\{\lambda\}$ is a formal series, derivable term by term with the help of unitary integrals, see \cite{47} for some relevant formulas).

3) Rewrite it further in terms of an integral over eigenvalue densities $\rho(\lambda)d\lambda = \sum_s \delta(\lambda - \lambda_s)d\lambda$.

\[ Z_N = \int e^{-S\{\rho(\lambda)\}} D\rho(\lambda) \]  

Note, that the density $\rho(\lambda)d\lambda$ is actually a 1-form. This step can be considered unnecessary for the rest of the DV construction, although it may be still important, at least conceptually.

4) Write down a classical equation of motion for $\rho(\lambda)$ (in principle, it can be extracted directly from $S\{\lambda\}$, introduced at step 2),

\[ \text{Diff}\{\rho(\lambda)\} = 0 \]  

Generically, this is a differential equation (a loop equation or the Ward identity, of primary importance in the theory of matrix models \cite{12, 43, 26, 14}).

5) In fact, there are two different forms of eq.\cite{15}, related by differentiation. DV prescription is to choose the option with teh derivative taken and ignore the constraints, imposed on integration constants (which would require that the level of Fermi surface is the same for all cuts). This step is crucial for the DV construction, and this is where it deviates from the usual studies of planar matrix models, allowing more solutions (more moduli); see \cite{4} for more details.

6) In a straightforward large-$N$ limit, equation \cite{15} becomes algebraic and its solution, $\rho_0(\lambda)d\lambda$, acquires interpretation in terms of a Riemann surface (complex spectral curve). The moduli of this solution, – positions of ramification points,– become parameters, e.g. the positions of the cuts. At
this step the information about multitrace terms in original action in (12) can be lost — as usual in the naive large-$N$ limit.

7) The classical action, evaluated at this solution,

$$\mathcal{F} \left( \text{moduli} \right) = S \{ \rho_0(\lambda) d\lambda \}$$

(16)

when considered as a function of moduli, should be a prepotential of some SW model.

8) The commonly adopted postulates are:

– The SW structure is defined by the 1-form $\rho_0(\lambda) d\lambda$:

$$dS_{DV} = \rho_0(\lambda) d\lambda$$

(17)

– Moduli are just the integrals of $\rho_0(\lambda) d\lambda$ along the cuts.

There are some immediate obvious questions to be asked about these statements.

What are the moduli? Why only "lengths" of the cuts, but not their "positions" are included in the DV considerations? In other words, what prevents one from considering all ramification points as moduli of the solutions to the algebraic limit of (15)?

Why does $\rho_0(\lambda) d\lambda$ provide a SW structure? Why are moduli derivatives holomorphic (in fact, not quite!) on the spectral curve? Why

$$\frac{\partial S \{ \rho_0(\lambda) d\lambda \}}{\partial (\text{moduli})} = B - \text{periods of } \rho_0(\lambda) d\lambda?$$

(18)

In fact, in the examples, the arising SW structure is singular in the sense that certain meromorphic differentials should be included, and some entries of the period matrix diverge. This requires some kind of regularization. However, the naive cut-off procedure, included into the DV recipe, while providing a fast short-cut to physical applications, can (and does) cause problems in theoretical investigation of the relevant structures and hidden symmetries. (This group of questions is partly addressed in ref.[4].) Is there any independent self-contained characterization of the prepotential $S \{ \rho_0(\lambda) d\lambda \}$ (like a WDVV-like equation)?

### 3.3 From SW model to a matrix model

Not too much seems to be currently known on the subject of this subsection. Leaving aside many obvious, but more detailed questions, at the beginning it is enough to ask:

Why should the prepotential be expressed by eq.(11) and what is the relevant kernel (Green function) $G(x, x')$?

In general SW theory, whenever a prepotential is homogeneous of degree 2,

$$\sum_I a_I \frac{\partial \mathcal{F}}{\partial a_I} = 2 \mathcal{F},$$

(19)

one always has [20]:

$$\mathcal{F}(a_I) = \frac{1}{2} \sum_I \oint_{A_I} dS \oint_{B_I} dS, \quad a_I = \oint_{A_I} dS$$

(20)
At the first glance this can seem to be the double-integral representation of interest. As a matter of principle, it is; however, there are two essential differences between eqs. (11) and (20), which allow to consider (11) as a significant step forward as compared to (20) – if it can be extended to sufficiently general circumstances.

First, in most examples, $F(a_I)$ is not really homogeneous of order 2, because of anomalies, responsible for logarithmic contributions to the prepotential. (In matrix models it is common to attribute these to the ”volume factors”; alternatively they are easily obtained from careful treatment of the double integrals along the same contours). To restore homogeneity of the prepotential, one needs to introduce a new modulus, $\Lambda$, but this makes the SW structure singular, thus more new moduli should be introduced and so on – often up to the full scaled Whitham theory. As a result, the set $\{I\}$ of moduli in eq.(20) is in fact infinitely large (includes, at least, all the integrals like (10)).

Second, eq.(20) contains integrals along the $A$ and $B$ contours, while only $A$-contours occur in eq.(11). Instead, however, a new object – the kernel $G(x, x')$ appears in (11). One can get a clue to understanding this last phenomenon by applying the obvious eq.(20) to the simplest example [26], when the infinite set of additional moduli is localized at a single point, as described by eqs.(10). Then

$$\sum_{k=0}^{\infty} \tilde{T}_k \frac{\partial F}{\partial T_k} = \sum_{k=0}^{\infty} \frac{1}{k} \oint_{x=x_0} \oint_{x'=x_0} \frac{(x-x_0)^k}{(x'-x_0)^k} dS(x)dS(x') =$$

$$-\oint_{x=x_0} \oint_{x'=x_0} \log(x-x')dS(x)dS(x')$$

(21)

(the term with $k = 0$ is actually logarithmic). This formula is exactly of the type (11), with the kernel $G(x, x') = \log(x-x')$. The next step towards understanding (11) should be to explain why the points $x$ and $x'$ in (21) can belong to different contours (cuts) when the singular points get distributed along continuous curves or areas and when (20) and (21) can be generalized to reproduce (11). See the discussion at the end of s.2 above.

4 Original SW anzatz and the Regularized Dijkgraaf-Vafa (DV) setting

4.1 The prepotential

Consider the differential $dS$ of the form

$$dS = \sqrt{\frac{P_{2n}(x)}{Q_{2n}(x)}} dx = \sqrt{\frac{2n}{x - M_L}} dx$$

(22)

Under the usual additional constraint

\footnote{In fact instead of $\log(x-x')$ one could easily get, say $\log \sinh(x-x')$ or anything else; the formulas (10) and thus (21) depend on the choice of local coordinates in the vicinity of $x_0$. Different types of matrix models (Hermitean, unitary, ...) with different Van-der-Monde determinants and thus different kernels $G(x, x')$ are in fact associated with different possibilities of global coordinatization of the simple bare spectral curves.}

\footnote{This condition is needed to eliminate the moduli-dependence of the pole at $x = \infty_{\pm}$ (note that $x = \infty$ describes two different points at two different sheets of the hyperelliptic Riemann surface). Instead of imposing (23) one can take the polynomial in the numerator to be $P_{2n-1}(x)$ of degree $2n - 1$, not $2n$. (Note, that exactly such $dS_{SU(2)} = \sqrt{\frac{2}{x(x-M_L)}} dx$, with $n = 1$ and one of the two $M_L$'s is equal to zero, was introduced in the very first example of SW theory in ref.)}
\[
\sum_{L=1}^{m} \beta_L = 0
\]  

(23)

This expression defines a regularized SW system of the hyperelliptic curves

\[
Y^2 = \mathcal{P}_{2n}(x)Q_{2n}(x) = \prod_{L=1}^{2n} (x - \beta_L) \prod_{L=1}^{2n} (x - M_L)
\]

(24)

of genus \(2n - 1\) with the \(2n - 1\) ramification points \(\beta_L\) with \(L = 1, \ldots, 2n - 1\) serving as moduli, and \(\beta_{2n}\) expressed through them by (23). The remaining \(2n\) ramification points \(M_L, L = 1, \ldots, 2n\) are kept fixed as external parameters of the family.

A convenient way to choose the \(A\)- and \(B\)-cycles and the associated flat moduli is\(^7\)

\[
S_i = \oint_{A_i} dS = 2 \int_{\beta_{2i-1}}^{\beta_{2i}} dS, \quad i = 1, \ldots, n,
\]

\[
T_i = \oint_{A_{n+i}} dS = 2 \int_{M_{2i-1}}^{M_{2i}} dS, \quad i = 1, \ldots, n-1
\]

(25)

and

\[
\frac{\partial F(S, T)}{\partial S_i} = \oint_{B_i} dS = 2 \int_{\beta_{2i}}^{M_{2n}} dS, \quad i = 1, \ldots, n,
\]

\[
\frac{\partial F(S, T)}{\partial T_i} = \oint_{B_{n+i}} dS = 2 \int_{M_{2i}}^{M_{2n}} dS, \quad i = 1, \ldots, n-1
\]

(26)

The \(2n - 1\) differentials

\[
dv_L = \frac{\partial dS}{\partial \beta_L} = -\frac{1}{2} \left( \frac{dS}{x - \beta_L} - \frac{dS}{x - \beta_{2n}} \right), \quad L = 1, \ldots, 2n-1
\]

(27)

are all holomorphic, their linear combinations with the coefficients from inverted matrix of \(A\)-periods provide canonical 1-differentials \(d\omega_L\), and the period matrix, made out of the \(B\)-periods of \(d\omega_L\), is symmetric:

\[
T_{IJ} = \oint_{B_J} d\omega_J = T_{JI}, \quad \oint_{A_I} d\omega_J = \delta_{IJ},
\]

\[
dv_K = \left( \oint_{A_I} dv_K \right) d\omega_I,
\]

\[
\oint_{A_I} dv_K = T_{IJ} \oint_{A_J} dv_K
\]

(28)

With such choice of the numerator the \(2n - 1\) moduli \(\beta_i\) are unconstrained and some formulas below are further simplified; e.g. the second term is absent at the r.h.s. of eq. (27). However, for the further discussion of the DV theory we need \(\mathcal{P}_{2n}(x)\) of degree \(2n\).

\(^7\) Actually, it is the one which is well adjusted for taking the limit of small \(\rho_i\) when \(\beta_{2i-1} = \gamma_i - \rho_i\) and \(\beta_{2i} = \gamma_i + \rho_i\). See discussion in s.5 below.
4.2 Residue formula

Since moduli $\beta_L$ are just the branching points, the derivation of residue formula in the present case is especially simple. As usual in the case of hyperelliptic systems, it is based on the expression for the hyperelliptic period matrix derivative over the position of a branching point [48]:

$$\frac{\partial T_{IJ}}{\partial \beta_L} = \hat{\omega}_I(\beta_L)\hat{\omega}_J(\beta_L) - \hat{\omega}_I(\beta_{2n})\hat{\omega}_J(\beta_{2n}), \quad L = 1, \ldots, 2n - 1. \quad (29)$$

Here $\hat{\omega}_I(\beta_L)$ denote the value of canonical 1-differential at the branching point $\beta_L$, i.e. in the vicinity of this point, where $x = \beta_L + \xi^L$, the differential becomes $d\omega(x) = \hat{\omega}_I(\beta_L)d\xi^L + O(\xi^L)d\xi_L$. The second term at the r.h.s. of eq. (29), arises because the constraint (23) requires $\beta_{2n}$ to vary whenever any other $\beta_L$ is changed.

Eq. (29) can be used to find the third derivative of the prepotential with respect to the flat moduli $\{a_L\} = \{S_i, T_i\}$. The constraint (23) makes intermediate formulas look heavier than they actually are, but it does not affect explicitly the final expression (35):

$$F_{IJK} = \frac{\partial^3 F}{\partial a_I \partial a_J \partial a_K} = \frac{\partial T_{IJ}}{\partial a_K} = \sum_{L=1}^{2n-1} \frac{\partial \beta_L}{\partial a_K} \frac{\partial T_{IJ}}{\partial \beta_L} =$$

$$= \sum_{L=1}^{2n-1} \frac{\partial \beta_L}{\partial a_K} (\hat{\omega}_I(\beta_L)\hat{\omega}_J(\beta_L) - \hat{\omega}_I(\beta_{2n})\hat{\omega}_J(\beta_{2n})) \quad (30)$$

Actually, expression in brackets at the r.h.s. is equal to

\footnote{For the sake of completeness, we use the chance to correct an impurity in the derivation of residue formula for the pure $N = 2$ SUSY gauge theory in 4d, which is present in the (incorrect) formula (64) of Appendix C in ref. [28]. Simultaneously we generalize that proof to include the fundamental matter hypermultiplets. The relevant family of hyperelliptic curves is given by}

$$y^2 = P_{N_c}(x) - Q_{N_f}(x) = \prod_{\gamma=1}^{2N_c} (x - \gamma)$$

Differentiating both sides of this equation, first, by $h_l$ (the coefficients of $P_{N_c}(x)$) and, second, by $x$, and putting further $x = \lambda_\alpha$, we get respectively:

$$2P(\lambda_\alpha)\lambda_\alpha^{-1} - \prod_{\gamma \neq \alpha} (\lambda_\alpha - \lambda_\gamma) \frac{\partial \lambda_\alpha}{\partial h_l}$$

and

$$2P(\lambda_\alpha)P'(\lambda_\alpha) - Q'(\lambda_\alpha) = \prod_{\gamma \neq \alpha} (\lambda_\alpha - \lambda_\gamma)$$

Since at $x = \lambda_\alpha$ we have $g(\lambda_\alpha) = 0$ and $P^2(\lambda_\alpha) = Q'(\lambda_\alpha)$, one can deduce that

$$\frac{\partial \lambda_\alpha}{\partial h_l} = \frac{-\lambda_\alpha^{-1}}{2P(\lambda_\alpha) - P'(\lambda_\alpha)}$$

Starting from eq. (65) the derivation of ref. [28] is already correct, and one arrives at residue formula of the form

$$\frac{\partial^3 F}{\partial a_I \partial a_J \partial a_K} = \sum_{\lambda_\alpha} \frac{\hat{\omega}_I(\lambda_\alpha)\hat{\omega}_J(\lambda_\alpha)\hat{\omega}_K(\lambda_\alpha)}{(P'(\lambda_\alpha) - \frac{1}{2} \frac{Q'(\lambda_\alpha)}{P(\lambda_\alpha) + y(\lambda_\alpha)})/y(\lambda_\alpha)} = \sum_{dx=0}^{\text{res}} \frac{d\omega(d\omega/d\omega_y) d\omega_y}{dxd\log(P + y)}$$

In this case the quadratic differential $ddS = dxd\log(P + y) \equiv d\lambda d\log w$ is associated with the SW symplectic structure $d\lambda \land d\log w$. 

11
\[-2 \sum_{dS=0} \text{res} \frac{dw_I(x)dw_J(x)dv_L(x)}{dSd\log P_{2n}} \]  \hspace{1cm} (31)

where sum is over all zeroes of \( dS \) (i.e. over all the points \( \beta_L, L = 1, \ldots 2n \)). Indeed, the ratio

\[-2 \frac{dv_L(x)}{dS(x)} = \frac{1}{x - \beta_L} - \frac{1}{x - \beta_{2n}} \]  \hspace{1cm} (32)

has double poles at \( \beta_L \) and \( \beta_{2n} \) and no other singularities, while

\[ d\log P_{2n} = \sum_{K=1}^{2n} \frac{dx}{x - \beta_K} \]  \hspace{1cm} (33)

has simple poles at all the branching points \( \beta_L \) (where \( dx \) has simple zeroes). Then the entire ratio in (31) possesses just two simple poles at \( \beta_L \) and \( \beta_{2n} \), and only these two contribute to the sum with coefficients \( \hat{\omega}_I(\beta_L)\hat{\omega}_J(\beta_L) \) and \( -\hat{\omega}_I(\beta_{2n})\hat{\omega}_J(\beta_{2n}) \) respectively.

It remains to substitute

\[ \frac{\partial a_K}{\partial \beta_L} = \oint_{A_K} \frac{dS}{\partial \beta_L} = \oint_{A_K} dv_L \]  \hspace{1cm} (34)

and use the relation (28) between \( dv_L(x) \) and \( d\omega_K(x) \), in order to obtain finally the residue formula for the system (22) in the form:

\[ F_{IJK} = \frac{\partial^3 F}{\partial a_I \partial a_J \partial a_K} = \sum_{dS=0} \text{res} \frac{dw_I dw_J dw_K}{dSd\log P_{2n}} = -\sum_{dP_{2n}=0} \text{res} \frac{dw_I dw_J dw_K}{dSd\log P_{2n}} \]  \hspace{1cm} (35)

In particular, we see that in the present case the quadratic differential \( ddS = dSd\log P_{2n} \), and it is associated in the usual way with canonical SW symplectic structure

\[ \sum_I d\omega_I(x) \wedge \delta a_I = \sum_I dv_I(x) \wedge \delta \beta_I = dS(x) \wedge \delta \log P_{2n}(x) \]  \hspace{1cm} (36)

Eq.(35) can be further used for straightforward derivation of the WDVV equations on the lines of (28)-(30).

4.3 WDVV equations (derivation)

Residue formula (35) expresses the prepotential’s third derivatives over flat moduli, \( F_{IJK} \), as an action of certain linear operator, \( \mathcal{L} \), on the product \( dw_Id\omega_J d\omega_K \) of three canonical holomorphic differentials. Imagine now, that an ”algebra” of such differentials can be defined, i.e. that

\[ dw_I dw_J \overset{\mathcal{L}}{=} C_{IJ}^L d\omega_L \]  \hspace{1cm} (37)

with some coefficients \( C_{IJ}^L \). Then, using the linearity of operation \( \mathcal{L} \),

\[ F_{IJK} = \mathcal{L}(dw_Id\omega_Jd\omega_K) \overset{\mathcal{L}}{=} C_{IJ}^L \mathcal{L}(d\omega_Kd\omega_L) = C_{IJ}^L \eta_{KL} \]  \hspace{1cm} (38)

or

\[ F_{IJK} = \mathcal{L}(dw_Id\omega_Jd\omega_K) = \sum_{dP_{2n}=0} \text{res} \mathcal{L}(d\omega_Kd\omega_L) = \sum_{dP_{2n}=0} \text{res} C_{IJ}^L \eta_{KL} \]  \hspace{1cm} (39)

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In particular, we see that in the present case the quadratic differential \( ddS = dSd\log P_{2n} \), and it is associated in the usual way with canonical SW symplectic structure

\[ \sum_I d\omega_I(x) \wedge \delta a_I = \sum_I dv_I(x) \wedge \delta \beta_I = dS(x) \wedge \delta \log P_{2n}(x) \]  \hspace{1cm} (36)

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or

\[ F_{IJK} = \mathcal{L}(dw_Id\omega_Jd\omega_K) = \sum_{dP_{2n}=0} \text{res} \mathcal{L}(d\omega_Kd\omega_L) = \sum_{dP_{2n}=0} \text{res} C_{IJ}^L \eta_{KL} \]  \hspace{1cm} (39)
\[
\left( \tilde{C}_I \right)_j^L = C_I^L = \tilde{F}_I \eta^{-1}
\]  
(39)

If, further, the product in (37) was associative, i.e. the matrices \( \tilde{C}_I \) commute,

\[
\tilde{C}_I C_K = C_K \tilde{C}_I
\]  
(40)

then

\[
\tilde{F}_I \eta^{-1} \tilde{F}_K = \tilde{F}_K \eta^{-1} \tilde{F}_I
\]  
(41)

This is basically the way to derive the WDVV equations. It, however, remains to correct two important details.

First, (37) can not be true for 1-forms; instead one should better write

\[
d\omega_I(x)d\omega_J(x) = \frac{1}{2} C_{IJ}^L d\omega_L(x)d\eta(x)
\]  
(42)

with quadratic differentials at both sides of the equation. This requires \( d\eta(x) \) to be also a holomorphic 1-differential,

\[
d\eta(x) = \eta_M d\omega_M(x)
\]  
(43)

and

\[
\eta_{KL} = \mathcal{L}(d\omega_K d\omega_L d\eta) = \eta_M \mathcal{F}_{KLM}, \quad \tilde{\eta} = \eta_M \tilde{\mathcal{F}}_M
\]  
(44)

If \( \eta_M = \delta_{MJ} \) we get from (14) the WDVV equations in the form of (8).

Second, corrected eq.(42) is still too strong a requirement to be realized in a SW model. Fortunately, we can add at the r.h.s. any quadratic differential, annihilated by the linear operation \( \mathcal{L} \). From an explicit form of the residue formula (35) it is clear that

\[
\mathcal{L}(\ldots dP_{2n}) = 0
\]  
(45)

This means that instead of (42) one can demand:

\[
d\omega_I(x)d\omega_J(x) = C_{IJ}^L d\omega_L(x)d\eta(x) + D_{IJ}^L d\omega_L(x) \frac{dP_{2n}(x)}{Y(x)},
\]  
(46)

while \( d\omega_I(x) \) are nothing but polynomials of degree \( 2n - 2 \) in \( x \) divided by \( Y(x) \). Therefore at the l.h.s. of eq.(46) we can encounter any polynomial of degree \( 2(2n - 2) = 4n - 4 \) (on top of \( Y^2(x) \) in denominator), with \( 4n - 3 \) different coefficients, while at the r.h.s. we have exactly \( (2n - 1) + (2n - 2) \) adjustment parameters \( C_{IJ}^L \) and \( D_{IJ}^L \) to match them. (Note that \( D_{IJ}^L d\omega_L(x)Y(x) \) needs to be a polynomial of degree \( (4n - 4) - (2n - 1) = 2n - 3 \). Thus there are only \( 2n - 2 \) adjustment parameters \( D_{IJ}^L \).) This completes the derivation of the WDVV equations in the hyperelliptic regularized SW model (22).

\[ ^{9} \text{Note, that if the } T \text{-moduli were not taken into account, and instead one would keep only the differentials which remain holomorphic after the transition to the singular DV limit (see the next section below), we would actually stay with polynomial of degree } n - 1 \text{ instead of } 2n - 2, \text{ as representatives of } d\omega_i \text{'s, while residue formula would still contain entire } dP_{2n}/dx \text{ of degree } 2n - 1. \text{ Then all the coefficients } D_{IJ}^L \text{ would need to vanish, and just } n \text{ adjustment parameters} \]
5 Towards the theory of the CIV-DV prepotential

5.1 Lifting regularization

Original DV setting appears from (22) in the specific singular limit, when all $M_L \to \infty$.

In this DV limit the genus $2n - 1$ curve is degenerated into a pair of genus $n - 1$ curves,

$$y^2(x) = \prod_{L=1}^{2n} (x - \beta_L) \quad (47)$$

and

$$\tilde{y}^2(x) = \prod_{L=1}^{2n} (x - M_L), \quad (48)$$

connected by two long tubes (turning just into two punctures at $x = \infty_{\pm}$ on every curve in extreme limit). The first curve (47) is where the (unregularized) DV theory lives

$$dS_{DV} = \lim_{all \: M_L \to \infty} dS \cdot \sqrt{\prod_{L} (-M_L)} = y(x) dx, \quad (49)$$

while the second curve (48) can be used to describe the regularization procedure in a way, consistent with the SW theory (in particular, preserving its completeness)\[10\].

The crucial problem with the limiting procedure is to describe it in such a way, that the set of $2n - 1$ moduli $\beta_L$ of the curve (47) gets naturally split into two parts: the $n$ moduli associated with the 1-differentials which remain holomorphic on (47) (in fact, DV recipe requires to include also a meromorphic differential with simple poles at $\infty_{\pm}$ and the remaining $n - 1$ moduli – the remnants of our $T_i$ which are not considered in the conventional DV approach. Such splitting is not automatically guaranteed by our separation (25) of $S$ and $T$-moduli in regularized theory, despite that the limit of $S$’s is definitely correct.\[11\]

The next question is whether any information about the $T$-moduli survives the limit, (which should be because these moduli before the limit certainly survive as parameters in $C_{ij}$ would not be sufficient to make the “algebra” closed. Thus there would be no reason for the WDVV equations to hold, and the singular DV system, if treated naively, would be incomplete. It is still an open question, whether its completeness can be restored with the help of adequate introduction of $T$-moduli, which can nicely survive the limiting procedure without destroying generic properties of SW system.

Note, that the points $x = \infty$ (the two points on the two sheets of the hyperelliptic curve), where $M_L$ are prescribed to condense in the DV limit, are in no way distinguished on the curve (22); one can shift them to any other position by a rational transformation of $x$ and $Y(x)$. However, after such a transformation, the singular nature of $dS_{DV}$ becomes transparent. Further introduction of an extra cut-off parameter $\Lambda$ (to make the planar matrix model formally finite), does not save the whole situation: this prescription (its real meaning is to smitutate the splitting between different $M_L$’s), if formally applied, breaks a lot of the SW structure. The minimal price, payed by conventional DV prescription, is to allow canonical differentials with simple poles at $x = \infty$, and to allow some $\Lambda$-dependent terms (and sometime even those containing $\sqrt{S}$ without any $\Lambda$-dependence.) in the prepotential, which are not taken in subsequent applications.

To give just one example of non-trivial alternative, define $S$’s and $T$’s in (23) with the help of another basis of canonical cycles: take instead of $A_i$ (from $\beta_{2i-1}$ to $\beta_{2i}$), $\tilde{A}_i$ (from $M_{2i-1}$ to $M_{2i}$), $B_i$ (from $\beta_{2i}$ to $M_{2n}$), $\tilde{B}_i$ (from $M_{2i-1}$ to $M_{2i}$) the symplectically conjugate set, consisting of $A_i, \tilde{A}_i - A_i, B_i + \tilde{B}_i, \tilde{B}_i$. Then the definition of the $S$ moduli remains unchanged, but those of $T$’s, and notably, of $\partial F/\partial S_i$ (now defined with new $T$’s kept fixed under differentiation) become different. The definitions can still coincide in the limit, in particular, the difference in the formulas for $\partial F/\partial S_i$ can seem not too big: these are given by integrals from $\beta_{2i}$ to one and the same point $M_{2n}$ if one uses the basis (22), and from $\beta_{2i}$ to the $i$-dependent point $M_{2i}$ if the alternative basis is used. In the limit when all $M_i \to \infty$ these prescriptions do not seem to differ too much, but since the limit is highly singular, they actually can lead to very different formulas.
the DV setting), and whether the formulas for prepotential’s $T$-derivatives and residue formula for its third derivatives can also be preserved.

Full solution of this problem remains for the future research (which can actually reveal different universality classes with different prepotentials, corresponding to different degenerations of the "hidden-sector" curve \( [18] \) in the limit). The rest of this section contains comments on variety of notation, useful for attacking this kind of problem, and on its simplest thinkable solution. Potential problem with such solution (and instead confirming the option of different universality classes) will be illustrated in a separate publication \([19]\).

5.2 Various parametrizations of the DV family

On the "hidden-sector" curve \([18]\), \( dS \) can be represented as

\[
dS = \frac{dx}{\tilde{y}(x)} \sum_{k=0}^{\infty} u_{k-n} x^{n-k}
\]

The sum here comes from the large-$x$ expansion of \( \sqrt{P_{2n}(x)} \):

\[
\sqrt{P_{2n}(x)} = \sum_{k=0}^{\infty} u_{k-n} x^{n-k} = P_n(x) + \sum_{k=1}^{\infty} u_k x^{-k}
\]

The new polynomial

\[
P_n(x) = \prod_{i=1}^{n} (x - \alpha_i),
\]

which naturally emerges here, is of degree \( n \). Its roots \( \alpha_i \) are functions of \( \beta_L \), subjected to the usual \( U(1) \) constraint

\[
\sum_{i=1}^{n} \alpha_i = 0,
\]

and the remaining \( (2n - 1) - (n - 1) = n \) moduli can be encoded – in this new parametrization – either by the first \( n \) of the variables \( u_k \) (i.e. by \( u_1, u_2, \ldots, u_n \)) or by the \( n \) coefficients of still another polynomial \( f_{n-1}(x) \), of degree \( n - 1 \), accounting for the difference between \( P_{2n}(x) \) and \( P_n(x) \):

\[
y^2(x) = P_{2n}(x) = P_n^2(x) + f_{n-1}(x) = P_n^2(x) + u_1 Q_{n-1}(x)
\]

Here

\[
Q_{n-1}(x) = x^{n-1} + O(x^{n-2}) = \prod_{\nu=1}^{n-1} (x - m_{\nu})
\]

\footnote{It is this polynomial that is identified within the DV theory \([18]\) with the derivative of the polynomial superpotential \( W_{n+1}(x) \) of the \( N = 1 \) SUSY theory in 4d, which, in turn, provides an action of \( d = 0 \) matrix model, describing the average over constant fields:

\[
P_n(x) = \frac{1}{g_{n+1}} W_{n+1}(x) = \prod_{i=1}^{n} (x - \alpha_i), \quad W_{n+1}(x) = g_{n+1} \left( x^{n+1} + O(x^{n-1}) \right)
\]}
has unit coefficient in front of the \(x^{n-1}\) term (while the coefficient in the corresponding term in \(f_{n-1}(x)\) is \(u_1\)). Also, its root decomposition reminds us of another example of SW theory [21], the one gauge theories with massive hypermultiplets in the fundamental representation of the gauge group (in this particular case \(N_c = n\) and \(N_f = N_c - 1\)).

This parallel implies still another parametrization of the same DV curve [54]:

\[
\begin{align*}
    w + \frac{Q_{n-1}(x)}{w} &= 2P_n(x), \\
    w - \frac{Q_{n-1}(x)}{w} &= 2y(x)
\end{align*}
\] (56)

The difference between the two versions of SW theory is that for the \(N = 2\) SUSY gauge theory in 4d the SW differential \(dS_4 \cong \log w dx\) (for its 5d counterpart [22], instead, \(dS_5 \cong \log \log x\), while

\[
dS_{DV} = y(x)dx \cong w dx
\] (57)

Discussion of the obviously emerging chain of differentials \(dS_{DV}, dS_4, dS_5\) remains beyond the scope of the present paper.

Yet another representation of the same polynomial \(f_{n-1}(x) = u_1Q_{n-1}(x)\), convenient for the study of perturbative (logarithmic) contribution to the prepotential, is

\[
f_{n-1}(x) = P_n(x) \sum_{i=1}^{n} \frac{\tilde{S}_i}{x - \alpha_i} = \sum_{i=1}^{n} \tilde{S}_i \prod_{j \neq i}(x - \alpha_j)
\] (58)

In perturbative limit \(\tilde{S}_i = S_i + O(S^2)\) and

\[
dS_{DV}^{\text{pert}} = P_n(x)dx + \sum_{i=1}^{n} \frac{S_i}{x - \alpha_i}dx.
\] (59)

Significance of parameters \(\alpha_i\) is in that these are the ones which are kept constant in the definition of conventional (unregularized) CIV-DV [3, 1] prepotential, \(\mathcal{F}(S_i|\alpha_i)\),

\[
S_i = \oint_{A_i} dS_{DV},
\]

\[
\left.\frac{\partial \mathcal{F}}{\partial S_i}\right|_{\text{constant } \alpha's} = \int_{\gamma_i + \rho_i}^{\gamma_i - \rho_i} dS_{DV},
\] (60)

\(i = 1, \ldots, n\). The main problem of taking the DV limit from the regularized case is to identify the \(T\)-moduli in such a way (ways?) that condition of \(T\)'s being fixed smoothly transforms in the limit into condition of \(\alpha\)'s being fixed).

Along with the points \(\alpha_i\) it is useful to introduce another set of points, namely, the positions \(\gamma_i\) of the cut centers, so that

\[
y^2 = \prod_{L=1}^{2n} (x - \beta_L) = \prod_{i=1}^{n}((x - \gamma_i)^2 - \rho_i^2)
\] (61)

i.e. ramification points \(\{\beta_L\} = \{\gamma_i \pm \rho_i\}\), and
\[ \sum_{i=1}^{n} \gamma_i = 0. \] (62)

An advantage of this parametrization is that for the small cut lengths \( \rho_i \) the flat moduli \( S_i \) are also small, and expansion of the prepotential in powers of \( S_i \) is related to its expansion in powers of \( \rho_i \) (actually, \( \rho_i^2 \)). This later one is rather straightforward to obtain, and can serve as the first step towards the calculation of \( F(S|\alpha) \).

### 5.3 Calculating the prepotential

Evaluation of the (unregularized) CIV-DV prepotential is a tedious procedure, involving the following steps.

1. First of all, one needs to evaluate the integral

\[ \int dS_{DV}(x) = \int \sqrt{P_{2n}(x)} dx = \int \prod_{j=1}^{n} \sqrt{(x - \gamma_j)^2 - \rho_j^2} dx \] (63)

as a function of \( \gamma \)'s and \( \rho \)'s. The answer can be found in the form of a power series in \( \rho_j^2 \) with sophisticated \( \gamma \)-dependent coefficients, calling for a representation theory interpretation. Definite integrals between \( \gamma_i - \rho_i \) and \( \gamma_i + \rho_i \) provide \( S_i/2g_{n+1} \), while integrals between \( \gamma_i + \rho_i \) and \( \Lambda \) (where \( \Lambda \) is somewhat like the common gathering point for all \( \lambda_i \)'s on their way to infinity) provide \( \frac{1}{2g_{n+1}} \frac{\partial F}{\partial S_i} \) at constant \( \alpha \)'s.

In order to integrate these formulas and obtain \( F(S|\alpha) \), one now needs to switch from \( \gamma \)'s to \( \alpha \)'s.

2. The expression of \( \gamma \)'s by \( \alpha \)'s and \( \rho \)'s arises from comparison of the coefficients in front of \( x^{2n-2}, x^{2n-3}, \ldots, x^n \) at both sides of the equation

\[ \prod_{i}^{n} ((x - \gamma_j)^2 - \rho_j^2) = \left[ \prod_{i}^{n} (x - \alpha_j) \right]^2 + f_{n-1}(x) \] (64)

It is easy to see that \( \sigma_i \equiv \gamma_i - \alpha_i = O(\rho_i^2) = O(S) \). After solving this system of \( n - 1 \) equations (in the form of infinite series in \( \rho \)'s, sometime summable), expressions for \( \gamma \)'s can be substituted into the formulas for \( S_i \) and \( \partial F/\partial S_i \), derived at the previous step.

3. Now expressions for \( S_i(\rho^2|\alpha) \) should be inverted to provide \( \rho_i^2(S|\alpha) \), which are further substituted into the formula for \( \partial F/\partial S_i \) to provide these derivatives in the form of expansion in powers of \( S \)'s with coefficients, made out of \( \alpha \)'s (which presumably still have a straightforward representation-theory interpretation).

The resulting expressions are already easy to integrate and we obtain \( F(S|\alpha) \) in a form of expansion in powers of \( S \)'s with coefficients made out of \( \alpha \)'s:

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13 In fact, it appears to be that step 1 and step 2 have closer relation to representation theory than the prepotential itself (relation can be spoiled at step 3). It can happen that the connection of the prepotential theory to the finite-\( N \) matrix models and ordinary \( \tau \)-functions can be found more easily at the level of \( S(\gamma, \rho) \) or \( S(\alpha, \rho) \) (not a big surprise, since the prepotential itself, being a quasiclassical \( \tau \)-function, does not need to have a transparent group-theoretical structure). A similar (related?) phenomenon can be observed in ...
\[2\pi i F(S|\alpha) = 4\pi ig_{n+1} \left( W_{n+1}(\Lambda) \sum_i S_i - \sum_i W_{n+1}(\alpha_i)S_i \right) - (\sum_i S_i)^2 \log \Lambda + \]
\[+ \frac{1}{2} \sum_{i=1}^{n} S_i^2 \left( \log \frac{S_i}{4} - \frac{3}{2} \right) - \frac{1}{2} \sum_{i<j} n \left( S_i^2 - 4S_iS_j + S_j^2 \right) \log \alpha_{ij} + \sum_{k=1}^{\infty} \frac{1}{(i\pi g_{n+1})^k} F_{k+2}(S|\alpha), \] (65)

where \( F_{k+2}(S|\alpha) \) are polynomials of degree \( k + 2 \) in \( S \)'s with \( \alpha \)-dependent coefficients, and \( W_{n+1}'(x) = P_n(x) \).

5.4 T-moduli

We suggest to supplement these 3 steps by the final one aimed at replacing the \( \alpha \) dependence by that on the flat moduli \( T_i, i = 1, \ldots, n - 1 \). An immediate problem, as we already discussed, is to find an adequate definition of these in the singular DV limit.

A possible guess could be to take just
\[ \tilde{T}_k = \text{res}_{s} x^{-k} dS_{DV} = \text{res}_{s} x^{-k} P_n(x) dx = \]
\[= e_{n-k-1}(\alpha) \equiv (-)^{n-k-1} \sum_{i_1 < \ldots < i_{n-k-1}} \alpha_{i_1} \ldots \alpha_{i_{n-k-1}}, \] (66)

\( k = 1, \ldots, n - 1 \) (obviously, \( T_0 = \sum_{i=1}^{n} S_i \) would be the case) and
\[ \frac{\partial F}{\partial T_k} = \frac{1}{k} \text{res}_{S} x^{+k} dS_{DV}. \] (67)

A problem with such an ansatz is that individual differences \( \alpha_{ij} = \alpha_i - \alpha_j \), appearing in (65), are difficult to express through symmetric polynomials of \( \alpha \)'s such as \( e_{n-k-1}(\alpha) \). A somewhat more sophisticated guess based on the previous work with Whitham hierarchies [26] can be to replace \( x^{\pm k} \) in these formulas by \( w^{\pm k/n} \) with \( w \) defined in (66).

Whatever the right formulas are, they should obey consistency conditions ( \( \partial^2 F/\partial S_i \partial T_j \) and \( \partial^2 F/\partial T_i \partial T_j \) being symmetric) and the proper form of the residue formula for the third derivatives should be found. Afterwards one can proceed to the WDVV equations, which can hypothetically survive after the DV limit is taken.

6 Conclusion

To conclude, we outlined a program for the study of DV prepotential from the perspective of SW theory. Its main non-trivial ingredients include the need to study the regularized DV theory [22]; introduce additional \( T \)-moduli and relate them to the ones arising from simple [20, 26] and sophisticated [33] Whitham hierarchies; to check the validity of the WDVV equations (and, probably, discover their adequate generalization to elliptic situation); and to find a representation-theory interpretation of periods (and may be even the prepotential). If fulfilled, this program can provide non-trivial conceptual clues for developments of theory of effective-action as well as helpful machinery to some more concrete questions such as relations to finite-N matrix models [16], KP/Toda \( \tau \)-functions [12], and instanton calculus of [39, 41].
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References

[1] R. Dijkgraaf and C. Vafa, *Nucl. Phys. B644* (2002) 3-20, hep-th/0206257, *Nucl. Phys. B644* (2002) 21-39, hep-th/0207106, hep-th/0208043.

[2] G. Veneziano and S. Yankielowicz, *Phys. Lett. B113* (1982) 231; V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Nucl. Phys. B229* (1983) 407; *Phys. Lett. 217B* (1989) 103; K. Konishi, *Phys. Lett. B135* (1984) 439; N. Seiberg, *Phys. Lett. B206* (1988) 75; M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Comm. Math. Phys.* 165 (1994) 311, hep-th/9309140; K. A. Intriligator, R. G. Leigh and N. Seiberg, *Phys. Rev. D50* (1994) 1092, hep-th/9403198; K. A. Intriligator and N. Seiberg, *Nucl. Phys. Proc. Suppl.* 45 BC (1996) 1, hep-th/9509066.

[3] F. Cachazo, K. A. Intriligator and C. Vafa, *Nucl. Phys. B603* (2001) 3, hep-th/0103067; F. Cachazo and C. Vafa, hep-th/0206017.

[4] L. Chekhov and A. Mironov, hep-th/0209085.

[5] N. Dorey, T. J. Hollowood, S. Prem Kumar and A. Sinkovics, hep-th/0209089, hep-th/0209093; N. Dorey, T. J. Hollowood and S. Prem Kumar, hep-th/0210239.

[6] R. Dijkgraaf, S. Gukov, V. Kazakov and C. Vafa, hep-th/0210238; F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, hep-th/0211170.

[7] M. Aganagic and C. Vafa, hep-th/0209138; F. Ferrari, hep-th/0210135, hep-th/0211069; H. Fuji and Y. Ookouchi, hep-th/0210148; D. Berenstein, hep-th/0210183; A. Gorsky, hep-th/0210281; R. Argurio, V. L. Campos, G. Ferretti and R. Heise, hep-th/0210291; J. McGreevy, hep-th/0211009; R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, hep-th/0211017; H. Suzuki, hep-th/0211052; I. Bena and R. Roiban, hep-th/0211073; Y. Demasure and R. A. Janik, hep-th/0211082; M. Aganagic, A. Klemm, M. Marino and C. Vafa, hep-th/0211098; R. Gopakumar, hep-th/0211100; S. Naculich, H. Schnitzer and N. Wyllard, hep-th/0211123; R. Dijkgraaf, A. Neitzke and C. Vafa, hep-th/0211194; Y. Tachikawa, hep-th/0211189.
B. Feng, hep-th/0211202.
A. Klemm, M. Marino and S. Theisen, hep-th/0211216.

[8] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19 (Erratum-ibid. B430 (1994) 485), hep-th/9407087.

[9] N. Seiberg and E. Witten, hep-th/9607163.

[10] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048;
P. Argyres and A. Farragi, Phys. Rev. Lett. 74 (1995) 3931, hep-th/9411057;
A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 283, hep-th/9505074;
N. Seiberg, Phys. Lett. B388 (1996) 753-760.

[11] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466, hep-th/9505033.

[12] M. Jimbo, T. Miwa and M. Sato, Holonomic Quantum Fields, I-V: Publ. RIMS, Kyoto Univ., 14 (1978) 223; 15 (1979) 201, 577, 871, 1531.;
E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation Groups for Soliton Equations, in Nonlinear Integrable Systems–Classical Theory and Quantum Theory, edited by M. Jimbo and T. Miwa, World Scientific, Singapore, 1983.

[13] B. Dubrovin, I. Krichever and S. Novikov, in Sovremennye problemy matematiki (VINITI), Dynamical Systems, 4 (1985) 179;
N. Hitchin, Duke Math. J. 54 (1987) 91;
E. Markman, Comp. Math. 93 (1994) 255;
A. Gorsky and N. Nekrasov, Nucl. Phys. B436 (1995) 582-608, hep-th/9401017; hep-th/9401021;
A. Levin and M. Olshanetsky, Comm. Math. Phys. 188 (1997) 449-466, alg-geom/9605005;
R. Donagi, alg-geom/9705010;
J. C. Hurtubise and E. Markman, Comm. Math. Phys. 223 (2001) 533-552, math.AG/9912161.

[14] S. N. Ruijsenaars, Comm. Math. Phys. 115 (1988) 127-165;
O. Babelon and D. Bernard, Phys. Lett. B317 (1993) 363-368;
V. Fock, in Geometry and Integrable Models, World Scientific, (eds. P. Pyatov and S. Solodukhin), 1995, p.20;
V. Fock and A. Rosly, Am. Math. Soc. Transl. 191 (1999) 67-86, math/9802054;
V. Fock, A. Gorsky, N. Nekrasov and V. Roubtsov, JHEP 0007 (2000) 028, hep-th/9906235;
A. Gorsky and V. Rubtsov, hep-th/0103004;
A. Mironov and A. Morozov, Phys. Lett. B490 (2000) 173-179, hep-th/0005280, ibidem, B475 (2000) 71-76, hep-th/9912088; hep-th/0001168; Phys. Lett. B524 (2002) 217-226, hep-th/0107114.

[15] A. Morozov and L. Vinet, Int. J. Mod. Phys. A13 (1998) 1651-1708, hep-th/9409093;
A. Mironov, A. Morozov and L. Vinet, Theor. Math. Phys. 100 (1995) 890-899, hep-th/9312213;
S. Kharchev, A. Mironov and A. Morozov; q-alg/9501013;
A. Mironov, Theor. Math. Phys. 114 (1998) 127, q-alg/9711006.

[16] A. Morozov, Rus. Phys. Uspekhi 37 (1994) 1-55, hep-th/9303139; hep-th/9502091.

[17] A. Morozov, Rus. Phys. Uspekhi 35 (1992) 671;
A. Gerasimov, D. Lebedev and A. Morozov, Int. J. Mod. Phys. A6 (1991) 977-988;
A. Mironov, Int. J. Mod. Phys. A9 (1994) 4355-3306, hep-th/9312212.
[18] E. Martinec and N. Warner, *Nucl. Phys.* **B459** (1996) 97-112, hep-th/9509161; hep-th/9511052.
T. Nakatsu and K. Takasaki, *Mod. Phys. Lett.* **A11** (1996) 157-168, hep-th/9509162.
R. Donagi and E. Witten, *Nucl. Phys.* **B460** (1996) 299, hep-th/9510101.
T. Eguchi and S. Yang, *Mod. Phys. Lett.* **A11** (1996) 131-138, hep-th/9510183.
E. Martinec, *Phys. Lett.* **B367** (1996) 91, hep-th/9510204.
A. Gorsky and A. Marshakov, *Phys. Lett.* **B375** (1996) 127, hep-th/9510224.
C. Gomez, R. Hernandes and E. Lopez, *Phys. Lett.* **B386** (1996) 115, hep-th/9512017.
*Nucl. Phys. B501* (1997) 109, hep-th/9608104.
H. Itoyama and A. Morozov, hep-th/9601168.
C. Ahn and S. Nam, *Phys. Lett.* **B387** (1996) 304, hep-th/9603028.
G. Bonelli and M. Matone, hep-th/9605090.
E. D’Hoker, I. Krichever and D. Phong, *Nucl. Phys.* **B489** (1997) 179, hep-th/9609041.

[19] H. Itoyama and A. Morozov, *Nucl. Phys.* **B477** (1996) 855, hep-th/9511120.

[20] H. Itoyama and A. Morozov, *Nucl. Phys.* **B491** (1997) 529, hep-th/9512161.

[21] N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484, hep-th/9408099.
P. C. Argyres and A. D. Shapere, *Nucl. Phys.* **B461** (1996) 437, hep-th/9509175.
A. Gorsky, A. Marshakov, A. Mironov, A. Morozov, *Phys. Lett.* **B380** (1996) 75-80, hep-th/9603140.

[22] N. Nekrasov, *Nucl. Phys.* **B531** (1998) 323-344, hep-th/9609219.
A. Marshakov and A. Mironov, *Nucl. Phys.* **B518** (1998) 59-91, hep-th/9711156.
H. W. Braden, A. Marshakov, A. Mironov, A. Morozov, *Nucl. Phys.* **B558** (1999) 371-390, hep-th/9902203.

[23] A. Klemm, hep-th/9705131.
I. Krichever and D. Phong, hep-th/9708170.
E. D’Hoker and D. Phong, hep-th/9709053; hep-th/9804120;
S. Ketov, hep-th/9710083.
A. Cappelli, P. Valtancoli and L. Vergnano, *Nucl. Phys.* **B524** (1998) 469-501, hep-th/9710248.
C. Gomez and R. Hernandez, hep-th/9711102.
H. Kanno and Y. Ohta, hep-th/9801036.
R. Carroll, hep-th/9712110.
J. Edelstein, M. Marino and J. Mas, *Nucl. Phys.* **B541** (1999) 671-697, hep-th/9805172.
J. Edelstein, M. Gomez-Reino, M. Marino and J. Mas, *Nucl. Phys.* **B574** (2000) 587-619, hep-th/9911113.
A. Gorsky, S. Gukov and A. Mironov, *Nucl. Phys.* **B518** (1998) 689-713, hep-th/9710239.
A. Morozov, hep-th/9810031; hep-th/9903087.
H. W. Braden, A. Marshakov, A. Mironov, A. Morozov, hep-th/9906240.
H. W. Braden and A. Marshakov, *Nucl. Phys.* **B595** (2001) 417-466, hep-th/0009068.
A. Marshakov, *Seiberg-Witten Theory and Integrable Systems*, World Scientific, Singapore, 1999;
A. Gorsky and A. Mironov, hep-th/0011197.

[24] R. Hirota, *Bilinear Forms of Soliton Equations*, in Nonlinear Integrable Systems—Classical Theory and Quantum Theory, edited by M. Jimbo and T. Miwa, World Scientific, Singapore, 1983;
A. Gerasimov, S. Khoroshkin, D. Lebedev, A. Mironov and A. Morozov, *Int. J. Mod. Phys.* **A10** (1995) 2589-2614, hep-th/9405011.

21
[25] I. Krichever, *Comm. Pure Appl. Math.* 47 (1992) 437, hep-th/9205110; *Comm. Math. Phys.* 143 (1992) 415;
B. Dubrovin, *Nucl. Phys.* B379 (1992) 627; *Comm. Math. Phys.* 145 (1992) 195.

[26] A. Gorsky, A. Marshakov, A. Mironov, A. Morozov, *Nucl. Phys.* B527 (1998) 690-716, hep-th/9802007.

[27] E. Witten, *Nucl. Phys.* B340 (1990) 281; *Surveys Diff. Geom.* 1 (1991) 243;
R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* B352 (1991) 59;
B. Dubrovin, *Lecture Notes in Math.* 1620, Springer, Berlin, 1996, 120-348, hep-th/9407018;
E. Getzler, alg-geom/9612004.

[28] A. Marshakov, A. Mironov and A. Morozov, *Phys. Lett.* B389 (1996) 43-52, hep-th/9607108.

[29] A. Marshakov, A. Mironov and A. Morozov, *Mod. Phys. Lett.* A12 (1997) 773-788, hep-th/9701014.

[30] A. Marshakov, A. Mironov and A. Morozov, *Int. J. Mod. Phys.* A15 (2000) 1157-1206, hep-th/9701123.

[31] H. W. Braden, A. Marshakov, A. Mironov, A. Morozov, *Phys. Lett.* B448 (1999) 195-202, hep-th/9812078.

[32] E. Getzler, math.AG/9801003; math.AG/9812026;
A. Morozov, *Phys. Lett.* B427 (1998) 93-96, hep-th/9711194;
G. Bertoldi and M. Matone, *Phys. Lett.* B425 (1998) 104-106, hep-th/9712039; *Phys. Rev.* D57 (1998) 6483-6485, hep-th/9712109;
A. Mironov and A. Morozov, *Phys. Lett.* B424 (1998) 48-52, hep-th/9712177;
K. Ito and S.-K. Yang, *Phys. Lett.* B433 (1998) 56-62, hep-th/9803126;
J. M. Isidro, *Nucl. Phys.* B539 (1999) 379-402, hep-th/9805051;
J. Kalayci and Y. Nutku, *J. Phys.* A31 (1998) 723, hep-th/9810076;
R. Martini and P. K. H. Gragnet, *J. Nonlinear Math. Phys.* 6 (1999), No.1, 1-4, hep-th/9901160;
A. Veselov, *Phys. Lett.* A261 (1999) 297-302, hep-th/9902142, hep-th/0105020;
S. Natanzon, *J. Geom. Phys.* 39 (2001) 323-336, hep-th/9904103;
Y. Ohta, *J. Math. Phys.* 41 (2000) 6042-6047, hep-th/9905126;
L. Hoevenaars, P. Kersten and R. Martini, *Phys. Lett.* B503 (2001) 189, hep-th/0012133;
A. Dzhamay, hep-th/0003034;
L. Hoevenaars and R. Martini, *Lett. Math. Phys.* 57 (2001) 175-183, hep-th/0102190;
B. de Wit and A. Marshakov, *Theor. Math. Phys.* 129 (2001) 1504-1510, hep-th/0105289;
H. Aratyn and J. van de Leur, hep-th/0111243;
A. Marshakov, *Theor. Math. Phys.* 132 (2002) 895-933, hep-th/0201267;
H. W. Braden and A. Marshakov, *Phys. Lett.* B541 (2002) 376-383, hep-th/0205308;
L. Hoevenaars, hep-th/0202007.

[33] M. Mineev-Weinstein and A. Zabrodin, *J. Nonlin. Math. Phys.* 8 (2001) 212-218;
M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, *Phys. Rev. Lett.* 84 (2000) 5106-5109, nlin.SI/0001007;
I. K. Kostov, I. Krichever, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, hep-th/0005259;
A. Boyarsky, A. Marshakov, O. Ruchayskiy, P. Wiegmann and A. Zabrodin, *Phys. Lett.* B515 (2001) 483-492, hep-th/0105260;
A. Marshakov, P. Wiegmann and A. Zabrodin, *Comm. Math. Phys.* 227 (2002) 131-153.
[34] R. Leigh and M. J. Strassler, *Nucl. Phys.* B496 (1997) 132-148, hep-th/9611020.

[35] E. Witten, *Nucl. Phys.* B223 (1983) 422-432; *ibidem*, 433-444; *Comm. Math. Phys.* 92 (1984) 455-472; V. Knizhnik and A. Zamolodchikov, *Nucl. Phys.* B247 (1984) 83-103; A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, *Int. J. Mod. Phys.* A5 2495-2589, 1990.

[36] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Nucl. Phys.* B357 (1991) 565; Yu. Makeenko, A. Marshakov, A. Mironov and A. Morozov, *Nucl. Phys.* B356 (1991) 574-628; M. Bowick, A. Morozov and D. Shevitz, *Nucl. Phys.* B354 (1991) 496-530; A. Marshakov, A. Mironov and A. Morozov, *Phys. Lett.* B265 (1991) 91-107; *Mod. Phys. Lett.* A7 (1992) 1345-1360, hep-th/9201010; *Phys. Lett.* B274 (1992) 280, hep-th/9201011. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, *Phys. Lett.* B275 (1992) 311-314, hep-th/9111037; *Nucl. Phys.* B380 (1992) 181-240, hep-th/9201013. S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Nucl. Phys.* B397 (1993) 339, hep-th/9203043; *Int. J. Mod. Phys.* A10 (1995) 2015, hep-th/9312210. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, *Nucl. Phys.* B404 (1993) 717, hep-th/9208044.

[37] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Mod. Phys. Lett.* A8 (1993) 1047-1062, hep-th/9208046.

[38] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, 1978.

[39] A. Losev, G. Moore, N. Nekrason and S. Shatashvili, *Nucl. Phys. Proc. Suppl.* 46 (1996) 130-145, hep-th/9509151. A. Losev, N. Nekrason and S. Shatashvili, *Nucl. Phys.* B534 (1998) 549-611, hep-th/9711108. M. Marino and G. Moore, *Comm. Math. Phys.* 199 (1998) 25, hep-th/9802183. A. Losev, N. Nekrason and S. Shatashvili, hep-th/9909204; *Class. Quant. Grav.* 17 (2000) 1181-1187, hep-th/9911099. J. Edelstein, M. Gomez-Reino and M. Marino, *Advances in Theoretical and Mathematical Physics* 4 (2000) 503-543, hep-th/0006113; *Journal of High Energy Physics* 0101 (2001) 004, 1-16, hep-th/0011227.

[40] N. Nekrason, hep-th/0206161.

[41] R. Flume and R. Poghossian, hep-th/0208176.

[42] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, *Comm. Math. Phys.* 59 (1978) 35-31; J. Ambjorn, L. Chekhov, S. Kristjansen and Yu. Makeenko, *Nucl. Phys.* B404 (1993) 127 (Erratum-ibid. B449 (1995) 681), hep-th/9302014. G. Akemann, *Nucl. Phys.* B482 (1996) 403, hep-th/9606004. G. Bonnet, F. David and B. Eynard, *J. Phys.* A33 (2000) 6739, cond-mat/0003324.

[43] A. Mironov and A. Morozov, *Phys. Lett.* 252B (1990) 47; J. Ambjorn, J. Jurkiewicz and Yu. Makeenko, *Phys. Lett.* 251B (1990) 517. H. Itouya and Y. Matsuo, *Phys. Lett.* 255B (1991) 202. F. David, *Mod. Phys. Lett.* A5 (1990) 1019.
[44] E.Witten, *Nucl.Phys.* **B460** (1996) 335, [hep-th/9510135];
D.-E.Diaconescu, *Nucl.Phys.* **B503** (1997) 220-238, [hep-th/9608163];
A.Hanany and E.Witten, *Nucl.Phys.* **B492** (1997) 152-190, [hep-th/9611230];
A.Gorsky, *Phys.Lett.* **B410** (1997) 22, [hep-th/9612238];
E.Witten, *Nucl.Phys.* **B500** (1997) 3-42, [hep-th/9703160]; *J.Geom.Phys.* **22** (1997) 103-133, [hep-th/9610234].

[45] A.Marshakov, M-Martellini and A.Morozov, *Phys.Lett.* **B418** (1998) 294-302, [hep-th/9706050].

[46] E.Cremmer, C.Kounnas, A.van Proeyen, J.P.Derendinger, S.Ferrara, B.de With and L.Girardello,
*Nucl.Phys.* **B250** (1985) 385;
B.de Wit, P.G.Lauwers and A.Van Proeyen, *Nucl.Phys.* **B255** (1985) 569;
A.Strominger, *Comm.Math.Phys.* **133** (1990) 163;
B.de Wit, [hep-th/9601044];
W.Lerche, P.Mayr and N.Warner, [hep-th/0208039].

[47] A.Morozov, *Mod.Phys.Lett.* **A7** (1992) 3503-3508, [hep-th/9209074];
S.Shatashvili, *Comm.Math.Phys.* **154** (1993) 421-432, [hep-th/9209083];
A.Mironov, A.Morozov and G.Semenoff, *Int.J.Mod.Phys.* **A11** (1996) 5031-5080.

[48] J.Fay, *Theta Functions on Riemann Surfaces*, Lect.Notes Math., **352**, Springer, N.Y., 1973;
V.Knizhnik, *Soviet Physics Uspekhi* **32** (1989) 945-971;
D.Lebedev and A.Morozov, *Nucl.Phys.* **B302** (1988) 163.

[49] H.Itoyama and A.Morozov, [hep-th/0211259].