The Geometry of Supersymmetric Quantum Mechanics

C.M. Hull

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, U.K.

ABSTRACT

One-dimensional sigma-models with $N$ supersymmetries are considered. For conventional supersymmetries there must be $N-1$ complex structures satisfying a Clifford algebra and the constraints on the target space geometry can be formulated in terms of these. In the cases in which the complex structures are simultaneously integrable, a conventional extended superspace formulation is given, with the geometry determined by a 2-form potential for $N = 2$, by a 1-form potential for $N = 3$ and a scalar potential for $N = 4$; for $N > 4$ it is given by a scalar potential satisfying differential constraints. This gives explicit constructions of models with $N = 3$ but not $N = 4$ supersymmetry and of $N = 4$ models in which the complex structures do not satisfy a quaternionic algebra. Generalisations with central terms in the superalgebra are also considered.
1. Sigma Models and Supersymmetry

The conditions for supersymmetry in one-dimensional sigma-models were given by Coles and Papadopoulos [1] and further studied in [2-6]. The analysis is similar to that of two-dimensional sigma-models, for which there is a rich relation between target space geometry and the amount of supersymmetry. The geometries of the two-dimensional supersymmetric models were first classified in [7] and have been studied extensively [7-21]. Remarkably, the conditions in one dimension are considerably weaker, giving a much wider range of geometries. The one-dimensional models have many applications. The moduli spaces for supersymmetric black holes are the target spaces of certain $d = 1$ supersymmetric sigma-models and the sigma-model describes the geodesic motion in the moduli space [2,5]. Supersymmetric quantum mechanics also arises in the light cone quantization of supersymmetric field theories [6]. The aim here is to study further the geometries of one-dimensional supersymmetric sigma-models and to give explicit constructions of certain classes of models.

The standard supersymmetry algebra in one dimension is

$$\{Q^I, Q^J\} = 2\delta^{IJ}H$$ (1.1)

where $\{Q^I; I = 1, \ldots, N\}$ are the supersymmetry charges and $H$ is the Hamiltonian. There are some generalisations, such as the twisted superalgebra

$$\{Q^I, Q^J\} = 2\eta^{IJ}H$$ (1.2)

for some metric $\eta^{IJ}$ or arbitrary signature [19,20], or the addition of extra terms

$$\{Q^I, Q^J\} = 2\eta^{IJ}H + Z^{IJ}$$ (1.3)

The extra generators $Z^{IJ}$ are central in some cases, and in others their commutators with $Q, H, Z$ lead to further generators and a larger algebra. The general algebra
can be written as

\[ \{ Q^I, Q^J \} = X^{IJ} \] (1.4)

and \( H \) and \( Z^{IJ} \) can then be defined by the trace and trace-free parts of \( X^{IJ} \) with respect to some metric \( \eta^{IJ} \). In some cases it is more natural to consider (1.4) rather than to split \( X \) into \( H \) and \( Z \). We will consider models in which each of these algebras arise.

The plan of the paper is as follows. In sections 2, 3 and 4, sigma-models with \( N = 1, N = 2 \) and \( N = 4 \) supersymmetries are discussed, reviewing the results of [1,2,5] and presenting extended superspace formulations which give simple derivations of some of the results of [5]. For conventional supersymmetry, there must be \( N - 1 \) complex structures satisfying certain conditions. The general \( N = 2 \) geometry is specified by a 2-form potential (considered further in section 6). Generalisations are discussed in which superalgebras such as (1.3) or (1.4) arise, extending the results of [3,4] to geometries with torsion, allowing the realisation of \( N = 2 \) supersymmetry on almost complex manifolds, or on manifolds with no almost complex structure (and so of arbitrary dimension and signature) which admit generalised Yano-Killing tensors. In section 5, new models with \( N = 3 \) supersymmetry but not \( N = 4 \) supersymmetry are found, which have three (almost) complex structures but only two of which lead to extra supersymmetries. In the case in which the three complex structures are simultaneously integrable, the geometry is given in terms of a 1-form potential, giving the local construction of all such models. In section 6, the conditions for \( N \) extended supersymmetry for general \( N \) are discussed, requiring the target space to be a manifold with a Clifford structures (a set of \( N - 1 \) complex structures satisfying a Clifford algebra) and to have a geometry which we refer to as Clifford Kähler with Torsion (CKT), in analogy with the nomenclature suggested in [12] of Kähler with Torsion (KT) for the geometry of (2,0) sigma-models [9] and Hyper-Kähler with Torsion (HKT) for the geometry of the (4,0) sigma-models first found in [7]. The case of \( N = 4 \) is considered in section 7. In the special case in which the three complex structures satisfy the algebra of
unit imaginary quaternions, the geometry was shown in [5] and in section 4 to be weak HKT. In the general case (in which the complex structures do not necessarily satisfy the quaternion algebra), the geometry for the case in which the complex structures are simultaneously integrable is shown to be given locally in terms of a scalar potential by an expression that reduces to the expression of section 4 and [5] for the quaternionic case. For \( N > 4 \), the geometry cannot be weak HKT, and the scalar potential must satisfy certain differential constraints; non-trivial examples of such geometries are known for \( N = 8 \) [2]. Under certain conditions similar to those derived in [5], these models can have \( OSp(N/1) \) superconformal symmetry.

2. \( \text{N}=1 \) one-dimensional supersymmetry

The simplest form of \( \text{N}=1 \) supersymmetric sigma model of [1] is defined on a \( D \)-dimensional manifold \( \mathcal{M} \) with metric \( g \) and a 3-form \( c \) (which is not closed in general) and has an \( \text{N}=1 \) superspace action

\[
I = -\frac{1}{2} \int dt d\theta \left( ig_{ij} DX^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} DX^i DX^j DX^k \right)
\]  

(2.1)

where

\[
D^2 = i \frac{d}{dt},
\]  

(2.2)

\( t \) is the worldline parameter, \( \theta \) is a real fermionic variable and \( X^i(t, \theta) \) is an unconstrained real superfield, which gives a map from superspace to \( \mathcal{M} \), with \( X^i \) real coordinates on \( \mathcal{M} \). The generalisation considered in [1] in which fermionic superfields are added will not be considered here.

Expanding the superfield \( X^i \) gives the component fields

\[
X^i = X^i| \quad \lambda^i = DX^i|
\]  

(2.3)

consisting of \( D \) scalar fields \( X^i \) and \( D \) real fermionic fields \( \lambda^i \). The component
The action is

$$I = \frac{1}{2} \int dt \left( g_{ij} \frac{d}{dt} X^i \frac{d}{dt} X^j + ig_{ij} \lambda^i \nabla^{(+)}_t \lambda^j - \frac{1}{3!} \partial_{[i} c_{jkl]} \lambda^i \lambda^j \lambda^k \lambda^l \right)$$

(2.4)

The covariant derivative $\nabla^{(+)}_t$ is the pull back of the target space covariant derivative with torsion $c$

$$\nabla^{(+)} = \nabla + \frac{1}{2} c$$

(2.5)

where

$$\Gamma^{(+)}_{jk} = \Gamma_{jk} + \frac{1}{2} c_{jk}$$

(2.6)

and $\Gamma$ is the Levi-Civita connection of the metric $g$. If $c$ is closed then this action can be obtained by dimensionally reducing the action [9] of (1,0) supersymmetric two-dimensional sigma models.

3. N=2 one-dimensional supersymmetry

3.1. Complex Geometry and Supersymmetry

There are two basic kinds of $N = 2$ models in 1 dimension. The $N = 2a$ models are constructed from unconstrained real $N = 2$ superfields while the $N = 2b$ models use complex chiral superfields. The dimensional reduction of (1,1) supersymmetric two dimensional sigma-models gives $N = 2a$ models while the reduction of (2,0) supersymmetric two dimensional sigma-models gives $N = 2b$ models. Both have been constructed in [1,2]. Here we shall consider only the $N=2b$ models and the special cases in which they have extra supersymmetry (such as the $N=4b$ and $N=8b$ models of [2]) and we shall begin by reviewing the results of [1,2].

To determine the conditions on the couplings of the action (2.1) required by $N=2$ supersymmetry, we follow [8] and express the second supersymmetry trans-
formation in terms of the N=1 superfield X as

\[ \delta X^i = \eta I^i_j DX^j, \quad (3.1) \]

where \( \eta \) is the parameter of the transformation. The \( N = 2 \) superalgebra will be satisfied if \( I \) satisfies

\[ I^2 = -1 \quad \text{(3.2)} \]

and

\[ \mathcal{N}^{k}_{ij}(I) = 0, \quad (3.3) \]

where \( \mathcal{N}(I) \) is the Nijenhuis tensor of \( I \), defined by

\[ \mathcal{N}^{k}_{ij}(I) \equiv I^l_i I^k_{lj} - I^l_j I^k_{il} \quad \text{(3.4)} \]

The condition (3.2) implies that \( I \) is an almost complex structure, requiring that the target space dimension be even, and (3.3) implies that the almost complex structure is integrable, and so is a complex structure.

It was shown in [1,2] that the action (2.1) is invariant under this transformation provided that

\[ g_{k\ell} I^k_i I^\ell_j = g_{ij} \]

\[ \nabla_\text{(+)} I^k_{ij} = 0 \]

\[ \partial_i (I^m_{j [c i |m|kl]}) - 2 I^m_{[i} |\partial_m c_{jkl]}| = 0 \quad \text{(3.5)} \]

An alternative derivation was given in [4]. The first condition is that the metric \( g \) is hermitian with respect to the complex structure \( I \). The last condition was written in [2] as

\[ i_I dc - \frac{2}{3} dt_I c = 0 \quad \text{(3.6)} \]

where \( i_I \) is the inner derivation with respect to the complex structure \( I \). This
acts on an $n$-form $\omega$ as
\[
\iota_I : \omega = \frac{1}{n!} \omega_{i_1 \ldots i_n} dX^{i_1} \wedge dX^{i_2} \ldots \wedge dX^{i_n} \to (\iota_I \omega),
\]
\[
(\iota_I \omega) = \frac{1}{(n-1)!} \omega_{j[i_2 \ldots i_{n-1}} I^{j]i_1]} dX^{i_1} \wedge dX^{i_2} \ldots \wedge dX^{i_n}
\]

If the metric is hermitian, $c$ is closed and the complex structure is covariantly constant with respect to the $\nabla^{(+)\text{connection}}$

\[
\nabla_i^{(+)} I^j_k = 0
\]

then these conditions are all satisfied and the model can be obtained by dimensional reduction of a $(2,0)$ supersymmetric sigma model in two dimensions. However, these conditions are much stronger than (3.5), so that there are many geometries that allow $d = 1, N = 2b$ models but not $(2,0) \ d = 2$ models.

The complex structure enables us to introduce complex coordinates $Z$, so that $X^i = (Z^\alpha, \bar{Z}^{\bar{\beta}}) \ (\alpha, \bar{\beta} = 1, \ldots, D/2)$ and the complex structure is constant

\[
I^i_j = \begin{pmatrix} i\delta^\alpha_{\beta} & 0 \\ 0 & -i\delta^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix}
\]

and the line element for the hermitian metric is $ds^2 = 2g_{\alpha\bar{\beta}} dZ^\alpha d\bar{Z}^{\bar{\beta}}$.

3.2. Generalised Symmetries

For (3.1) to be a symmetry, it is sufficient for (3.5) to be satisfied. If (3.5) is satisfied but (3.2),(3.3) are not, then the symmetry algebra will not be the usual supersymmetry algebra (1.1). For example, if $I^2 = +1$ and (3.3) holds, then $I$ is a real structure and the algebra is a twisted superalgebra (1.2) of the type studied in [19,20]. If $I$ is an almost complex structure satisfying (3.2) but not (3.3), then
the commutator of two transformations of the form (3.1) gives the new symmetry

$$\delta X^i = \rho N^i_{jk} DX^j DX^k$$ (3.10)

with bosonic parameter $\rho$. The superalgebra is then of the form (1.3) with $Z^{22}$ the charge generating this new symmetry. This is a central charge commuting with $Q, H$ and so the supersymmetry algebra closes without further generators, by an argument similar to that given for two-dimensional sigma models in [15-18].

In the general case in which (3.5) is satisfied but (3.2),(3.3) are not, the superalgebra is of the form (1.4) with $X^{11} = H, X^{12} = 0$, but $X^{22}$ generates the bosonic symmetry

$$\delta X^i = \rho (2iR^i_j \partial_t X^i + N^i_{jk} DX^j DX^k)$$ (3.11)

where

$$R^i_j = (I^2)^i_j$$ (3.12)

If the trace $(I^2)^i_i$ is non-zero, it can be set to one by scaling $I$, in which case the algebra takes the form (1.3) with $Z^{11} = H, Z^{12} = 0$, but $Z^{22}$ generating the symmetry (3.11) with $R^i_j = (I^2 - I)I_i^j$. Such symmetries have been considered in [3,4].

3.3. The Torsion-Free Case

If the torsion $c$ vanishes, then the second condition in (3.5) becomes the condition

$$\nabla_{(i} I^{k}_{j)} = 0$$ (3.13)

implying that $I_{ij}$ is a Yano Killing-tensor, as was pointed out in the context of one-dimensional supersymmetric sigma models in [3]. For an almost complex structure satisfying (3.13), the hermiticity condition in (3.5) together with (3.2) implies that
$\mathcal{M}, g, I$ is an almost Tachibana space [22]. The condition (3.13) implies that the Nijenhuis tensor can be written as

$$\mathcal{N}^{k}_{ij} = -I^k_i \nabla_i I^l_j$$

(3.14)

Then if in addition $I$ is a complex structure with $\mathcal{N}(I) = 0$, (3.14) implies that the complex structure must be covariantly constant, $\nabla I = 0$, and the space must be Kähler. (An almost Tachibana space with vanishing Nijenhuis tensor is also called a Tachibana space, but this is the equivalent to the Kähler condition.)

If $I$ is not an almost complex structure, but is a general Yano Killing-tensor, then the tensor $R^i_{\ j} = (I^2)^i_{\ j}$ is a Stackel-Killing tensor and the symmetry algebra is of the form (1.4) with $X^{22}$ generating the symmetry (3.11). This is the case analysed in [3], where a number of examples were considered. Models with $N = 2$ supersymmetry of this type can arise for odd dimensional target spaces as well as for even dimensional ones, and for Lorentzian signature target spaces, such as the Kerr-Newman black hole [3]. It would be interesting to investigate whether there are BPS states associated with the central charge $Z^{22}$.

### 3.4. N=2 Superspace Action

The $N = 2b$ one-dimensional supersymmetric model can be written in $N = 2$ superspace with coordinates $t, \theta^0, \theta^1$ and supercovariant derivatives $D_0, D_1$ satisfying

$$D_0^2 = i \frac{d}{dt}, \quad D_1^2 = i \frac{d}{dt}, \quad D_0 D_1 + D_1 D_0 = 0.$$  

(3.15)

It is useful to define $D = D_0 + iD_1$, $\theta = \theta^0 + i\theta^1$ so that

$$D^2 = 0, \quad \{D, \bar{D}\} = 2i \frac{d}{dt},$$  

(3.16)

and introduce chiral superfields $Z^\alpha$ and their complex conjugates $\bar{Z}^{\bar{\alpha}}$ satisfying

$$\bar{D} Z^\alpha = 0$$  

(3.17)
The most general superspace action is given by

\[ I = \frac{1}{4} \int dt \, d^2 \theta \left( i G_{\alpha \bar{\beta}} \mathcal{D} Z^{\alpha} \mathcal{D} \bar{Z}^{\bar{\beta}} + \frac{1}{2} B_{\alpha \bar{\beta}} \mathcal{D} Z^{\alpha} \mathcal{D} Z^{\bar{\beta}} + \frac{1}{2} \bar{B}_{\alpha \bar{\beta}} \mathcal{D} \bar{Z}^{\bar{\alpha}} \mathcal{D} \bar{Z}^{\bar{\beta}} \right), \tag{3.18} \]

for some \( G_{\alpha \bar{\beta}}, B_{\alpha \bar{\beta}}, \) with \( \bar{B}_{\alpha \bar{\beta}} = (B_{\alpha \bar{\beta}})^* \), plus a chiral superpotential term

\[ S = \int dt \, d\theta W(Z) + \int dt \, d\bar{\theta} \bar{W}(\bar{Z}) \tag{3.19} \]

for some holomorphic function \( W \). The action is invariant under

\[ B_{\alpha \bar{\beta}} \rightarrow B_{\alpha \bar{\beta}} + \partial_{[\alpha} \lambda_{\beta]} \tag{3.20} \]

and so only depends on the holomorphic field strength

\[ e_{\alpha \beta \gamma} = 3 \partial_{[\alpha} \partial_{\beta \gamma]} \tag{3.21} \]

Dimensional reduction of the superspace action [14] for the (2,0) model in two dimensions [9] gives an action of the form

\[ I = \frac{i}{2} \int dt \, d^2 \theta \left( k_\alpha \partial_t Z^{\alpha} - \bar{k}_{\bar{\alpha}} \partial_t \bar{Z}^{\bar{\alpha}} \right), \tag{3.22} \]

where \( k_\alpha \) is the potential introduced in [9], but this can be rewritten using (3.16),(3.17) as

\[ I = \frac{1}{4} \int dt \, d^2 \theta \left( k_{\alpha \bar{\beta}} + k_{\bar{\beta} \alpha} \right) \mathcal{D} Z^{\alpha} \mathcal{D} \bar{Z}^{\bar{\beta}}, \tag{3.23} \]

which is of the same form as the \( G_{\alpha \bar{\beta}} \) term in (3.18), and so terms of the form (3.22) are already included, and (3.18) is indeed the most general action.
The $N = 2$ superfields $Z^\alpha$ give the $N = 1$ superfields ($D \equiv D_0$)

$$Z^\alpha = Z^\alpha|_{\theta^1=0} \quad DZ^\alpha = D_0Z^\alpha = \frac{1}{2}DZ^\alpha|_{\theta^1=0} \quad (3.24)$$

Then the $\theta^1$ integral in (3.18) can be performed (see appendix for details) to give the $N = 1$ superspace action (2.1), with

$$g_{\alpha\beta} = 0, \quad g_{\alpha\bar{\beta}} = G_{\alpha\bar{\beta}} \quad (3.25)$$

and

$$c_{\alpha\beta\bar{\gamma}} = 2(g_{\alpha\bar{\gamma},\beta} - g_{\beta\bar{\gamma},\alpha}), \quad c_{\alpha\beta\gamma} = 12i\partial_{[\alpha}B_{\beta\gamma]} \quad (3.26)$$

together with the complex conjugate equations. Thus the geometry is completely specified in terms of (i) a hermitian metric $g_{\alpha\bar{\beta}}$ and (ii) a (3,0) form $e$ satisfying $\partial e = 0$, so that it can be expressed locally as $e = \partial B$ for some (2,0) form potential $B$. Then, in terms of the holomorphic exterior derivative $\partial$ with $d = \partial + \bar{\partial}$,

$$c = i(\partial - \bar{\partial})\omega + 4i(e - \bar{e}) \quad (3.27)$$

where $\omega(I)$ is the fundamental form constructed from $I$:

$$\omega(I) = \frac{1}{2}I_{ij}dX^i \wedge dX^j = ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta \quad (3.28)$$

If $e = 0$, then for a given hermitian metric, $c = i(\partial - \bar{\partial})\omega$ is the unique torsion 3-form such that the complex structure is covariantly constant,

$$\nabla_i^{(+)}P_{jk} = 0 \quad (3.29)$$

In the case in which $I$ is an almost complex structure, the unique torsion three-form for which the complex structure is covariantly constant, (3.29), is

$$c_{ijk} = 4N_{ijk} + I_{[mn,p]}I^m_i I^n_j I^k_p \quad (3.30)$$

Examples of $d = 1, N = 2$ supersymmetric models on almost complex manifolds with $c$ given by (3.30) arise from the dimensional reduction of the models of [15-18].
4. \textbf{N=4 one-dimensional supersymmetry}

4.1. \textbf{Conditions for \( N = 4 \) Supersymmetry}

One-dimensional N=4b supersymmetric sigma models arise from the dimensional reduction of the two-dimensional (4,0) supersymmetric sigma-model. The geometry of these \( D = 2 \) models was first found in [7], and the name hyper-Kähler with torsion (HKT) has been proposed for this geometry [12]. The geometry associated with the N=4b model in 1 dimension is not necessarily HKT, but satisfies weaker conditions [1,2]. The extended supersymmetry transformations can be written in terms of N=1 superfields as [8]

\[
\delta X^i = \eta^r I^i_r DX^j \tag{4.1}
\]

where \( \{ \eta^r; r = 1, 2, 3 \} \) are the supersymmetry parameters and \( \{ I_r; r = 1, 2, 3 \} \) are tensors on \( \mathcal{M} \). The conditions from the closure of the N=4 supersymmetry algebra are [1]

\[
I_r I_s + I_s I_r = -2\delta_{rs}\mathbf{1} \tag{4.2}
\]

\[
\mathcal{N}(I_r, I_s) = 0
\]

where \( \mathcal{N}(I_r, I_s) \) is the Nijenhuis tensor for the pair \( (I_r, I_s) \), so that the \( I_r \) are three complex structures that anti-commute with one another. The conditions for the invariance of the action are [1,2]

\[
g_{k\ell} I^k_I r I^{\ell}_r j = g_{ij} \tag{4.3}
\]

\[
\nabla^{(+)}_{(i} I^k_j) = 0
\]

\[
\iota_r dc - \frac{2}{3} dt_r c = 0,
\]

where \( \iota_r \) denotes inner derivation with respect to the complex structure \( I_r \) [2]. The metric is hermitian with respect to all complex structures.
A weak HKT manifold is a Riemannian manifold \( \{M, g, c\} \) equipped with a metric \( g \), a three-form \( c \) and three complex structures \( \{I_r; r = 1, 2, 3\} \) satisfying the algebra of imaginary unit quaternions

\[
I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t ,
\]

such that the metric is hermitian with respect to all complex structures and the complex structures are each covariantly constant with respect to the \( \nabla^{(+)} \) covariant derivative

\[
\nabla_k^{(+)} I_r \,^i_j = 0 .
\]

If in addition the three-form \( c \) is closed, then \( M \) has a strong HKT structure. The target space of two-dimensional (4,0)-supersymmetric sigma models has a strong HKT structure [7] while any weak HKT manifold solves all the conditions required by N=4b one-dimensional supersymmetry [2]. It was argued in [5] that the conditions for N=4b one-dimensional supersymmetry with (4.4) are equivalent to the ones for weak HKT geometry, but in [2] examples are given of models admitting N=4b supersymmetry (and in fact N=8b supersymmetry) but which are not weak HKT. We will now investigate further the general solution of the N=4b supersymmetry conditions, using superspace constructions, and aim to clarify the relation between weak HKT geometries and the \( N = 4 \) supersymmetry conditions.

4.2. N=2 Superspace Formulation

Suppose the complex structures satisfy the quaternion algebra (4.4). One of the extra supersymmetries, that parameterised by \( \eta_3 \), say, can be made manifest by using an \( N = 2 \) superspace formulation with chiral superfields, so that the action is of the form (3.18). The remaining two supersymmetry transformations can be written as

\[
\delta Z^\alpha = \eta J^\alpha_\beta \bar{Z}^{\bar{\beta}}
\]

(4.6)

together with the complex conjugate relation, where \( J = \frac{1}{2}(I_1 - iI_2) \) and \( \eta = \eta_1 + i\eta_2 \). This is consistent with the chirality constraint \( \mathcal{D}Z = 0 \) provided \( I_r \) are
complex structures and

\[ J^\alpha_{[\beta,\gamma]} = 0 \quad (4.7) \]

which is the condition that the Nijenhuis tensor \( \mathcal{N}(I_r, I_3) = 0 \) for \( r = 1, 2 \). This then implies that the remaining Nijenhuis tensors also vanish \([5,24]\). The condition (4.7) implies that locally there is some \( f^\alpha \) such that (4.6) can be written as

\[ \delta Z^\alpha = \eta \bar{D} f^\alpha \quad (4.8) \]

with

\[ J^\alpha_{\bar{\beta}} = \partial_{\bar{\beta}} f^\alpha \quad (4.9) \]

The terms in the variation of the action (3.18) involving \( \eta \) are

\[ \delta I = \frac{1}{4} \int dt d^2 \theta \eta \left( - \frac{i}{2} \hat{\nabla}_\alpha J_{\bar{\beta}\gamma} \bar{D} Z^\alpha \bar{D} \bar{Z}^\bar{\beta} \bar{D} \bar{Z}^\bar{\gamma} 
+ 2 J_{(\alpha\bar{\beta})} \partial_t \bar{Z}^\alpha \bar{D} \bar{Z}^\bar{\beta} 
+ \frac{1}{2} e_{\alpha\beta\gamma} J^\gamma_{\bar{\gamma}} \bar{D} Z^\alpha \bar{D} Z^\beta \bar{D} \bar{Z}^\bar{\gamma} 
- \frac{1}{6} e_{\bar{\alpha}\bar{\beta}\bar{\gamma},\alpha} f^\alpha \bar{D} \bar{Z}^\alpha \bar{D} \bar{Z}^\bar{\beta} \bar{D} \bar{Z}^\bar{\gamma} \right), \quad (4.10) \]

where \( \hat{\nabla} \) is the covariant derivative preserving \( I_3 \), \( \hat{\nabla} I_3 = 0 \), so that the torsion 3-form for the connection \( \hat{\nabla} \) is given by \( i(\partial - \bar{\partial})\omega_3 \), where \( \omega_3 \) is the two-form constructed from \( I_3 \). The terms involving \( \bar{\eta} \) are obtained by complex conjugation. The terms involving \( \eta \) must vanish separately from those involving \( \bar{\eta} \), which requires

\[ J_{(\alpha\bar{\beta})} = 0 \]
\[ e = 0 \]
\[ \hat{\nabla} i_{I^j_k} = 0 \quad (4.11) \]

so that \( \nabla^{(+)} = \hat{\nabla} \) and the space is weak HKT. The same result was obtained studying the conditions (4.3) in complex coordinates in \([5]\). Thus the only \( N = 4b \) models for which the complex structures satisfy the quaternion algebra are those with weak HKT target spaces.
4.3. N=4 Superspace

The one-dimensional N=4b supersymmetry multiplet can be written in N=4 superspace with coordinates \( \{ t, \theta^0, \theta^r; r = 1, 2, 3 \} \) and the constraints

\[
D_r X^i = I_{r j}^i D_0 X^j
\]  
(4.12)

The supersymmetry derivatives satisfy the algebra

\[
D_0^2 = i \frac{d}{dt}, \quad D_0 D_r + D_r D_0 = 0 \tag{4.13}
\]

\[
D_s D_r + D_r D_s = 2i \delta_{rs} \frac{d}{dt}.
\]

The action

\[
I = -\frac{1}{2} \int dt d\theta^0 \left( ig_{ij} D_0 X^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} D_0 X^i D_0 X^j D_0 X^k \right) \tag{4.14}
\]

was given in [2].

A more useful action can be given for the case in which the three complex structures are simultaneously integrable, that is, there is a local coordinate choice for which all three complex structures have constant components. The construction is very similar to that of [8]. Here we will discuss the special case in which the complex structures satisfy the quaternion algebra (4.4), and will defer the general case until section 7.3. It is convenient to use two complex fermionic superspace coordinates \( \theta^a, a = 1, 2 \), instead of four real ones, with the supercovariant derivatives \( D_a \) and their complex conjugates \( \bar{D}^a \) satisfying

\[
\{ D_a, D_b \} = 0, \quad \{ D_a, \bar{D}^b \} = 2i \partial_0 \delta_a^b \tag{4.15}
\]

Choosing a coordinate system in which the complex structures satisfying (4.4) take
the convenient form

\[ I_1 = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \otimes 1, \quad I_2 = \left( \begin{array}{cc} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{array} \right) \otimes 1, \quad I_3 = \left( \begin{array}{cc} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{array} \right) \otimes 1 \] (4.16)

where 1 is the \( n \times n \) identity matrix, the constraints can be written as follows. The complex dimension must be even, and the fields \( Z^\alpha (\alpha = 1, \ldots, 2n) \) split into two sets \( z^A, w^A \) where \( A = 1, \ldots, n \) and satisfy the constraints

\[ \bar{D}^a z^A = 0, \quad \bar{D}^a w^A = 0 \]
\[ D_2 w^A = -i\bar{D}^1 z^A, \quad D_2 z^A = i\bar{D}^1 w^A \] (4.17)

These are a truncation of the \( N = 4 \) twisted chiral constraints of [8] (given by restricting to negative chirality).

The general \( N=4 \) superspace action for this twisted chiral \( N=4b \) multiplet is then

\[ I = \frac{1}{4} \int dt d^4 \theta L(z, \bar{z}, w, \bar{w}) \] (4.18)

for an arbitrary function \( L \). Integrating over \( \theta^2, \bar{\theta}_2 \) gives an \( N=2 \) superspace action \( D_2 \bar{D}_2 L \) which can be rewritten using (4.17) to be of the form (3.18) with

\[ e_{ijk} = 0 \] (4.19)

and metric \( g_{\alpha \bar{\beta}} = G_{\alpha \bar{\beta}} \) given by

\[ g_{z^A z^B} = \partial_{w^A} \partial_{\bar{w}^B} L + \partial_{z^A} \partial_{\bar{z}^B} L \]
\[ g_{w^A \bar{w}^B} = \partial_{z^A} \partial_{\bar{z}^B} L + \partial_{w^A} \partial_{\bar{w}^B} L \]
\[ g_{z^A \bar{w}^B} = -\partial_{z^B} \partial_{\bar{w}^A} L + \partial_{z^A} \partial_{\bar{w}^B} L \]
\[ g_{w^A \bar{z}^B} = \partial_{w^A} \partial_{\bar{z}^B} L - \partial_{z^A} \partial_{\bar{w}^B} L \] (4.20)

giving a simple superspace derivation of the result of [5]. We have recovered the result of [5] that the geometry of any weak HKT space with simultaneously integrable
complex structures can be given locally in terms of a potential $L$ by (4.20). However, many HKT spaces do not have simultaneously integrable complex structures (e.g. any non-trivial hyper-Kähler space) and in those cases the geometry is not given by an unconstrained potential in general (e.g. for hyper-Kähler spaces, the geometry is given by a Kähler potential satisfying highly non-trivial constraints).

It is perhaps worth noting that the conditions (given by equation (38) of [8]) for a function $L(z, \bar{z}, w, \bar{w})$ to determine a two-dimensional off-shell (4,4) supersymmetric sigma-model constructed from twisted chiral multiplets are precisely the conditions that the metric $g_{\alpha\bar{\beta}}$ defined by (4.20) vanish.

Introducing the notation $Z^{Au} = \{z^A, w^A\}$ where $u = 1, 2$ so that $Z^{A1} = z^A$ and $Z^{A2} = w^A$, and the complex conjugate $\bar{Z}^{A}\star$, the constraints (4.17) can be written as

$$\bar{D}^a Z^{Au} = 0, \quad D_a Z^{Au} = -i \epsilon^{ab} \epsilon^{uv} \bar{D}^b \bar{Z}^{A}_v$$ (4.21)

This is the form of the constraint used in [6], where the action (4.18) was also considered. The metric (4.20) can be rewritten as

$$g^{Au B} = \frac{\partial^2 L}{\partial Z^w \partial Z^x} \left( \delta^C_A \delta^D_B \delta^v_u \delta^v_x - \delta^D_A \delta^C_B (\delta^v_u \delta^w_x - \delta^w_u \delta^v_x) \right)$$ (4.22)

* Complex conjugation raises or lowers the indices $a, b$ and $u, v$ but not the real indices $A, B$. 
5. **N=3 one-dimensional supersymmetry**

5.1. **Complex Structures and Supersymmetry**

We now return to the relation between the argument of [5] that N=4 supersymmetry (with (4.4)) implies weak HKT geometry and the explicit examples of models in [2] with \( N \geq 4 \) supersymmetry whose geometry is not weak HKT, but is in fact a geometry with 7 complex structures that was termed OKT in [2] (see section 7). The resolution lies in an important difference between the supersymmetric sigma model geometries in 1 dimension [1]. In both cases, there are \( m \) complex structures satisfying a Clifford algebra

\[
I_r I_s + I_s I_r = -2 \delta_{rs}
\]  

(5.1)

where \( r = 1, ..., m \), giving the possibility of \( N = m + 1 \) supersymmetries in \( D = 1 \) and \( (m + 1, 0) \) supersymmetry in \( D = 2 \). If the complex structures are each covariantly constant with respect to some connection \( \hat{\nabla} \) (possibly with torsion)

\[
\hat{\nabla} I_r = 0
\]

(5.2)

then the complex structures must commute with the holonomy group of this connection, and if the holonomy is irreducible this implies that the complex structures must form a division algebra, so that the only possibilities are \( m = 0, m = 1 \) and \( m = 3 \) (the octonion algebra cannot be represented by a set of matrices). Thus in \( D = 2 \), the only \( (N, 0) \) supersymmetries of standard type that can arise for rigid supersymmetric sigma-models are those for \( N = 1, 2, 4 \) [7]. (Other possibilities such as \((3,0)\) can arise for the more general sigma-models of the type considered in [7].) In particular, given two covariantly constant complex structures \( I_1, I_2 \), their product \( I_3 = I_1 I_2 \) must also be a covariantly constant complex structure and \((3,0)\) supersymmetry implies \((4,0)\). However, in \( D = 1 \) the complex structures are not
necessarily covariantly constant, but satisfy the weaker condition

\[ \nabla_{(i}^{(+)} I^k_{j)} = 0 \]  \hspace{1cm} (5.3)

Consider the case of \( N = 3 \) models with two complex structures \( I_1, I_2 \) satisfying (4.2),(4.3). The product \( I_3 = I_1 I_2 \) is an almost complex structure but in general it will not satisfy the supersymmetry conditions (4.3), so that \( N = 3 \) does not necessarily imply \( N = 4 \) supersymmetry [1]. However, if \( I_3 = I_1 I_2 \) is a complex structure satisfying these constraints, then the \( I_r \) satisfy the quaternion algebra (4.4) and the geometry must be weak HKT, as we saw in the last section. In particular, for the \( N=8 \) supersymmetric models of [2], the 7 complex structures satisfy a Clifford algebra but not a division algebra, so that the product of any two complex structures is not a complex structure corresponding to a supersymmetry, and so the OKT geometries need not be weak HKT, and none of the OKT examples in [2] are. More generally, a target space which is not weak HKT can have \( N \geq 3 \) supersymmetry provided that for any two complex structures \( I_1, I_2 \) that correspond to supersymmetries, the product \( I_1 I_2 \) does not lead to a supersymmetry.

In this section we will investigate \( N = 3b \) models in \( D = 1 \) further. These are of a different type to the \( N = 3 \) models constructed in [1], which are based on a real \( N = 2 \) supermultiplet, and require the target space structure group to be reducible. We require two complex structures \( I_1, I_2 \) satisfying (4.2),(4.3). The supersymmetry corresponding to \( I_1 \) can be made manifest by using \( N = 2 \) superspace with chiral superfields and action (3.18). The extra supersymmetry transformation corresponding to \( J = I_2 \) can be written as

\[ \delta Z^\alpha = \eta J^\alpha_\beta \bar{D} Z^{\bar{\beta}} \]  \hspace{1cm} (5.4)

which is of the same form as (4.6), but now with the important difference that \( \eta \) is the real parameter corresponding to the third supersymmetry, whereas \( \eta \) was the complex parameter \( \eta = \eta_1 + i\eta_2 \) in (4.6). Moreover, \( J = I_2 \) here, whereas in (4.6)
we had \( J = \frac{1}{2}(I_1 - iI_2) \). Again

\[
J^\alpha_{[\bar{\beta}, \bar{\gamma}]} = 0
\]  

(5.5)

so that locally there is some \( f^\alpha \) such that (5.4) can be written as

\[
\delta Z^\alpha = \eta \bar{D} f^\alpha
\]  

(5.6)

with

\[
J^\alpha_{\bar{\beta}} = \partial_{\bar{\beta}} f^\alpha
\]  

(5.7)

The conditions for invariance of the action (3.18) are

\[
J_{(\alpha\beta)} = 0, \quad \bar{\nabla}_{\bar{\alpha}} J_{\alpha\beta} = e_{\alpha\beta\gamma} J^\gamma_{\bar{\alpha}}
\]  

(5.8)

\[
\bar{e}_{\bar{\alpha}\bar{\beta}\bar{\gamma}, \alpha} = 0
\]

These are found from requiring that the sum of (4.10) and its complex conjugate vanish for real \( \eta \). Thus there will be \( N = 3 \) supersymmetry for any hermitian manifold with \( (3,0) \) form \( e = \partial B \) and an extra complex structure \( J \) provided the conditions (5.8) are satisfied. We will now construct a large class of \( N = 3 \) models satisfying these constraints and which are not \( N = 4 \) supersymmetric.

5.2. \( N=3 \) Superspace Construction

In the special case in which the two complex structures \( I_1, I_2 \) are simultaneously integrable, we can use a superspace formulation similar to that used in section 4.3. The supercovariant derivatives can be taken to be \( \mathcal{D}, \bar{\mathcal{D}}, \tilde{\mathcal{D}} \) with \( \tilde{\mathcal{D}} \) real, satisfying

\[
\{\mathcal{D}, \mathcal{D}\} = 0, \quad \{\mathcal{D}, \bar{\mathcal{D}}\} = 2i\partial_t, \\
\{\tilde{\mathcal{D}}, \mathcal{D}\} = 0, \quad \tilde{\mathcal{D}}^2 = i\partial_t
\]  

(5.9)

As in section 4.3, we take the complex dimension of the target space to be be even, and the fields \( Z^\alpha (\alpha = 1, ..., 2n) \) split into two sets \( z^A, w^A \) where \( A = 1, ..., n \). If
the complex structures are taken to be of the form

\[
I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \mathbf{1}, \quad I_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \otimes \mathbf{1}
\]

(5.10)

where \( \mathbf{1} \) is the \( n \times n \) identity matrix, then the constraints are

\[
\bar{D}z^A = 0, \quad \bar{D}w^A = 0
\]

\[
\tilde{D}w^A = -\frac{1}{\sqrt{2}} \bar{D}z^A, \quad \tilde{D}z^A = \frac{1}{\sqrt{2}} \bar{D}w^A
\]

(5.11)

which can be rewritten in terms of \( Z^{Au} = \{z^A, w^A\} \) where \( u = 1, 2 \), \( Z^{A1} = z^A \) and \( Z^{A2} = w^A \), as

\[
\bar{D}^a Z^{Au} = 0, \quad \tilde{D} Z^{Au} = \frac{1}{\sqrt{2}} \epsilon^{uv} \bar{D} \bar{Z}^A_v
\]

(5.12)

The general action can be written in terms of an unconstrained 1-form potential \( k_i(X) = (k_{Au}, \bar{k}_A^u) \) as

\[
I = -\frac{1}{2} \int dt d^3\theta \left( k_{Au} \bar{D} Z^{Au} + \bar{k}_A^w \tilde{D} \bar{Z}_u^A \right)
\]

(5.13)

Note that a term of the form \( h_{Au} \tilde{D} Z^{Au} \) could be rewritten using the constraint (5.12) to be proportional to \( \bar{D} \bar{Z} \) instead of \( \tilde{D} Z \), and so can be absorbed into the \( \bar{k} \) term in (5.13).

Integrating over the third \( \theta \) gives an \( N = 2 \) superspace action of the form (3.18) with

\[
g_{Au B^v} = \frac{1}{\sqrt{2}} \left( \epsilon^{uv} \left[ \frac{\partial k_B v}{\partial Z^A u} - \frac{\partial k_A u}{\partial Z_B v} \right] + \epsilon_{wu} \left[ \frac{\partial \bar{k}_B v}{\partial Z^A w} - \frac{\partial \bar{k}_A w}{\partial Z_B v} \right] \right)
\]

(5.14)

and

\[
B_{Au B^v} = \frac{1}{\sqrt{2}} \left[ \epsilon_{wu} \frac{\partial k_B v}{\partial Z^A w} - \epsilon_{wu} \frac{\partial k_A u}{\partial Z_B w} \right]
\]

(5.15)

This gives the general construction of \( N = 3 \) models with two simultaneously integrable complex structures in terms of a single potential \( k \), and in the general
case with $B \neq 0$ the complex structures will not be covariantly constant but the weaker conditions (5.8) will be satisfied. If $B = 0$, then the complex structures are covariantly constant and their product will be a third covariantly constant complex structure and the space will be weak HKT, with N=4 supersymmetry.

6. One-dimensional supersymmetry for general N

6.1. Clifford Structures and Supersymmetry

In this section we will consider sigma models with $N$ supersymmetries for any $N$, so that the $N = 1$ model (2.1) is invariant under an additional $N - 1$ supersymmetry transformations

$$\delta X^i = \eta^a I_a i^j DX^j$$

(6.1)

where $\{\eta^a; a = 1, \ldots, N - 1\}$ are the supersymmetry parameters. The conditions required by the closure of the supersymmetry algebra are

$$I_a I_b + I_b I_a = -2\delta_{ab}$$

(6.2)

and

$$N(I_a, I_b) = 0$$

(6.3)

and the conditions required by the invariance of the action are

$$g_{k \ell} I_a^k i I_a^\ell j = g_{ij}$$

$$\nabla^{(+)} I_a^{k \ell} j = 0$$

(6.4)

$$\iota_a dc - \frac{2}{3} d\iota_a c = 0 ,$$

where $\iota_a$ denotes inner derivation with respect to $I_a$. We shall call a set of $m$ complex structures satisfying (6.3) a Clifford Structure, and call a Riemannian
manifold \(\{\mathcal{M}, g, c\}\) equipped with metric \(g\), antisymmetric tensor \(c\), and complex structures \(\{I_a\}\) that obey the compatibility conditions (6.3) and (6.4) a *Clifford Kähler with Torsion* manifold, or CKT for short. The name *Octonionic Kähler with Torsion* (OKT) was suggested in [2] for the special case in which \(m = 7\).

The more general case in which the first condition in (6.2) is satisfied but (6.3) is not, so that the \(I_r\) are almost complex structures, will lead to an enlarged supersymmetry algebra of the form (1.3). We will refer to geometries satisfying (6.2),(6.4) but not (6.3) as *Almost Clifford Kähler with Torsion* manifold, or ACKT for short. This constitutes a generalisation of the almost Tachibana spaces that arose in section 3.2.

More generally, if neither (6.2) nor (6.3) is satisfied but (6.4) holds, then the superalgebra is of the form (1.4) with \(X^{00} = H, X^{0r} = 0\), but \(X^{rs}\) generates the bosonic symmetry

\[
\delta X^i = \rho \left( 2i(R_{rs})^i_j \partial_t X^i + \mathcal{N}(I_r, I_s)^i_{jk} DX^j DX^k \right)
\]

(6.5)

where

\[
R_{rs} = \frac{1}{2} \{I_r, I_s\}
\]

(6.6)

We will restrict ourselves to the CKT case in what follows.

Which values of \(N\) can arise? In [2], models with \(N = 0, 1, 2, 4, 8\) were constructed, and in section 5 models with \(N = 3\) were found. Supersymmetry transformations of the form (6.1) satisfying (6.2) can be found for any \(N\). For example, consider the case in which all complex structures are constant matrices in some coordinate system. Then the complex structures satisfy a Clifford algebra and can be realised as gamma matrices, which must be real. If the target space dimension is \(D = 2^{d/2}\), then the condition (6.3) is satisfied by \(d + 1\) complex gamma matrices \((\gamma_a)^i_j\) satisfying the Clifford algebra corresponding to \(O(d+1)\). If \(m\) is the number of these that can be chosen to be simultaneously real, then these can be used to construct a realisation of \(N = m + 1\) extended supersymmetry. For example, for
$D = 2$ there are 3 complex gamma matrices satisfying (6.2), which can be taken to be $\gamma_a = i\sigma_a$ for $a = 1, 2, 3$, but of these only $i\sigma_2$ is real, so that $m = 1$ and only $N = 2$ supersymmetry is possible. For $D = 2$, $m = 1$, for $D = 4$, $m = 3$, for $D = 8$, $m = 7$ and it is clear that $m$ can be made arbitrarily large by taking $D$ large enough. There are manifolds admitting Clifford structures for arbitrarily large values of $m$.

Supersymmetry with any $N$ can be realised on flat space with $c = 0$ and $g_{ij} = \delta_{ij}$. For given $D$, there are $m$ real gamma matrices, and the transformation corresponding to a given one will preserve the free action provided (6.4) are satisfied with $c = 0$ and $g_{ij} = \delta_{ij}$, which will be the case if the corresponding gamma matrix is anti-symmetric. For general geometries and general Clifford structure, the conditions become more and more restrictive the higher the value of $N$. There are non-trivial examples for $N = 8$ [2], but it seems likely that for high enough $N$ the geometry will be required to be trivial. The $N = 16$ models considered in [2] that arise for black hole moduli spaces have flat target spaces.

Given $m = N - 1$ almost complex structures satisfying (6.2), the products

$$I_{rs} \equiv I_r I_s = \frac{1}{2}[I_r, I_s] = -I_{sr} \quad (6.7)$$

are also almost complex structures, and further tensors can be formed by taking anti-symmetrised products:

$$I_{rs\ldots t} \equiv I_r I_s \ldots I_t = I_{[rs\ldots t]} \quad (6.8)$$

The tensors $I_{r_1\ldots r_n}$ (with $n \leq m$) are almost complex structures ($I^2 = -\mathbf{1}$) for $n = 4k + 1, 4k + 2$ and are almost real structures or almost product structures ($I^2 = \mathbf{1}$) for $n = 4k, 4k + 3$, where $k = 1, 2, 3, \ldots$. These generate the enveloping algebra of the Clifford algebra. Note that the set of all almost complex complex structures constructed in this way do not satisfy a Clifford algebra in general; for
Consider the case of $m = 3$. There are three almost complex structures $I, J, K$ say, which anti-commute with each other and which each squares to $-I$. If $IJ = K$, then they satisfy the quaternion algebra (4.4). If not, then we can define the products

$$\tilde{K} = IJ, \quad \tilde{I} = JK, \quad \tilde{J} = KI$$

(6.10)

each of which is an almost complex structure. Moreover,

$$[I, \tilde{I}] = 0, \quad [J, \tilde{J}] = 0, \quad [K, \tilde{K}] = 0$$

(6.11)

and there are several subalgebras which are isomorphic to the quaternion algebra (4.4), such as those generated by $(I, J, \tilde{K})$ or $(\tilde{I}, \tilde{J}, \tilde{K})$. The $\tilde{I}$ will play a role in the following section.

In the case in which there are $n$ complex structures which are simultaneously integrable, i.e. that can be simultaneously taken to be constant matrices in a suitable coordinate system, then for $n = 1$, we have seen that the geometry is determined by a 2-form potential, for $n = 2$ it is determined by a 1-form potential, and that for the special case of $n = 3$ in which the complex structures satisfy a quaternion algebra, by a 0-form potential. In the next section, we will generalise this and show that for the general case of $n = 3$ the geometry is again determined by a 0-form potential, and for higher $n \geq 3$ it is given by a 0-form potential satisfying certain differential constraints.

In [5], the conditions for sigma-models with $N$ supersymmetries to have superconformal supersymmetry were found for $N = 1$ and $N = 2$, and for $N = 4$ sigma-models with complex structures satifying the $SU(2)$ algebra (4.4). These generalise to give the corresponding conditions for any $N$, for an $N$-supersymmetric sigma-model to be invariant under the superconformal group $OSp(N/1)$. The analysis is similar to that in [5] and will be given elsewhere.
An extended superspace form of the sigma-model for general $N$ can be given, following [2]. Let $X$ be a map from the superspace with coordinates $\{t; \theta^0, \theta^r, r = 1, \ldots, N - 1\}$ into a CKT manifold $M$. Then we impose the constraints

$$D_r X^i = I_r^i D_0 X^j \quad (6.12)$$

where $\{D_0, D_r; r = 1, \ldots, N - 1\}$ are the supersymmetry derivatives satisfying

$$D_0^2 = i \frac{d}{dt}$$

$$D_0 D_r + D_r D_0 = 0 \quad (6.13)$$

$$D_s D_r + D_r D_s = 2i \delta_{rs} \frac{d}{dt}.$$

These constraints are consistent provided (6.2),(6.3) are satisfied. An action for this multiplet is

$$I = -\frac{1}{2} \int dt d\theta^0 \left( i g_{ij} D_0 X^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} D_0 X^i D_0 X^j D_0 X^k \right). \quad (6.14)$$

and will be independent of $\theta^a$ and hence fully supersymmetric provided (6.4) are satisfied.

6.2. $N=2$ Superspace Formulation

The models with $N \geq 2$ supersymmetry can be written in $N = 2$ superspace with action (3.18). It will prove useful to rewrite the action using the chiral constraint $DZ = 2DZ$ as

$$I = \frac{1}{4} \int dt d^2 \theta \Omega_{ij} DX^i DX^j \quad (6.15)$$

where the antisymmetric tensor $\Omega_{ij}$ has components

$$\Omega_{\alpha\beta} = B_{\alpha\beta}, \quad \Omega_{\alpha\bar{\beta}} = \bar{B}_{\alpha\bar{\beta}}, \quad \Omega_{\alpha\bar{\beta}} = -\Omega_{\bar{\beta}\alpha} = i g_{\alpha\bar{\beta}} \quad (6.16)$$

so that the two-form $\Omega = \frac{1}{2} \Omega_{ij} dX^i \wedge dX^j$ is given by

$$\Omega = \omega + B + \bar{B} \quad (6.17)$$

where $B = \frac{1}{2} B_{\alpha\beta} dZ^\alpha \wedge dZ^\beta$. Thus the geometry is specified by the choice of
an arbitrary real 2-form \( \Omega \) potential, so that the model is defined by a complex manifold with a 2-form, i.e. by the triple \((\mathcal{M}, I, \Omega)\). The metric is defined by the (1,1) part of \( \Omega \) and \( B \) by the (2,0) part, and \( \Omega \) is defined up to the transformations (3.20)

\[
\Omega \rightarrow \Omega + \partial \lambda + \bar{\partial} \bar{\lambda}
\]  

(6.18)

where \( \lambda \) is an arbitrary (1,0) form.

The action (6.15) can be expanded into \( N = 1 \) superspace to give

\[
I = \frac{1}{4} \int dt d\theta \left( 3\Omega_{ijk} DX^i DX^j DX^k - 2i\Omega_{ij} DX^i \partial_t X^j \right)
\]

(6.19)

where \( \Omega_{ijk} \equiv \Omega_{[ij,k]} \),

\[
\hat{D} \equiv \frac{1}{2i}(\mathcal{D} - \bar{\mathcal{D}})
\]

(6.20)

and the chiral constraint implies

\[
\hat{D} X^i = I^i_j DX^j
\]

(6.21)

Using the notation of [23] that for any tensor \( T_{ij\ldots kl} \),

\[
T_{ij\ldots kl} = T_{ij\ldots km} I^m_l
\]

(6.22)

the action (6.19) can be rewritten as

\[
I = \frac{1}{4} \int dt d\theta \left( 3\Omega_{ijk} DX^i DX^j DX^k - 2i\Omega_{ij} DX^i \partial_t X^j \right)
\]

(6.23)

The fact that, for any tensor \( \sigma_{ij} \),

\[
\int dt d\theta \sigma_{ij} DX^i \partial_t X^j = \int dt d\theta \left( \sigma_{(ij)} DX^i \partial_t X^j - \frac{i}{2} \sigma_{[ij,k]} DX^i DX^j DX^k \right)
\]

(6.24)
can be used with $\sigma_{ij} = \Omega_{ij}$ to rewrite the action (6.23) as

$$I = \frac{1}{4} \int dt \, d\theta \left( [3\Omega_{ijk} - \Omega_{ij,k}] DX^i DX^j DX^k - 2i\Omega_{ij} DX^i \partial_t X^j \right)$$  \hspace{1cm} (6.25)

so that, comparing with (2.1), we have

$$g_{ij} = \Omega_{ij} = \Omega_{k(i} I^k_{j)}$$
$$c_{ijk} = -3[3\Omega_{ijk} - \Omega_{ij,k}] = -3[I^l_{[i} \Omega_{jk]l} + 3\partial_{[k}(I^l_{i} \Omega_{j]l}])$$ \hspace{1cm} (6.26)

In form notation, $c$ is

$$c = -\iota_I d\Omega + \frac{1}{2} d\iota_I \Omega$$ \hspace{1cm} (6.27)

Finally, note that for $N > 2$ supersymmetry with $N - 1$ complex structures $I_r$, one can choose any one of them and work in the corresponding $N = 2$ superspace, giving a 2-form $\Omega^r$ for that complex structure, and in this way one can construct $N - 1$ 2-forms $\Omega^r_{ij}$.

7. Integrable Complex Structures and Extended Superspace

7.1. Complex Structures and Clifford Algebras

In this section we will examine the case in which there are $m$ simultaneously integrable complex structures $I_a$, each of which satisfies the conditions (6.2),(6.3),(6.4) for $N = m + 1$ supersymmetry. In such a case, there is a coordinate choice in which the $I_a$ are all real constant matrices satisfying the Clifford algebra (6.2), and the superspace constraints become of the conventional kind. The superspace action then leads to a simplification of the geometry (for example, for $N = 4$ supersymmetry, the metric and torsion are given in terms of a scalar potential $L$).
Real $D \times D$ matrices satisfying (6.2) can be constructed from the basic real $2 \times 2$ matrices $\sigma_1, \sigma_3, \epsilon = i\sigma_2$. For $D = 2$, the only real matrix satisfying (6.2) is $\epsilon$, for $D = 4$ a set of three matrices $I_r$ ($r = 1, 2, 3$) satisfying (6.2) is given by

$$\epsilon \otimes \mathbb{1}, \quad \sigma_1 \otimes \epsilon, \quad \sigma_3 \otimes \epsilon$$

and these in fact satisfy the quaternion algebra (4.4). An alternative set $\tilde{I}_r$ is given by

$$\mathbb{1} \otimes \epsilon, \quad \epsilon \otimes \sigma_1, \quad \epsilon \otimes \sigma_3$$

and these two sets commute: $[I_r, \tilde{I}_s] = 0$. For $D = 8$, a set of 7 matrices satisfying (6.2) can be constructed from the $I_r, \tilde{I}_r$:

$$\epsilon \otimes \mathbb{1}, \quad \sigma_1 \otimes I_r, \quad \sigma_3 \otimes \tilde{I}_s$$

More generally, if for some $D$ there are two commuting sets of Clifford structures $I_r, \tilde{I}_r$, $r = 1, \ldots, m$ for some $m$, a Clifford structure for dimension $2D$ (i.e. with $2D \times 2D$ matrices) is given by the $2m + 1$ matrices (7.3).

7.2. N=3 SUPERSYMMETRY

In this case there are simultaneously integrable complex structures $I, J$ and the superspace constraints are

$$D_1 X^i = I^i_j DX^j, \quad D_2 X^i = J^i_j DX^j$$

with $D = D_0$. The product $\tilde{K} \equiv IJ$ is also a complex structure (as $IJ = -JI$), and $I, J, \tilde{K}$ satisfy the algebra of unit imaginary quaternions and are simultaneously integrable. In general $\tilde{K}$ will not satisfy the conditions (6.4) so that the action will be invariant under the supersymmetry transformations corresponding to $I$ and $J$ but not $\tilde{K}$.
The general form of the $N = 3$ action (with simultaneously integrable complex structures) is given in terms of an arbitrary 1-form potential $k$ by (5.13), which can be rewritten as

$$I = \frac{1}{4} \int dt \, d\theta^0 d\theta^1 d\theta^2 k_i DX^i$$  \hspace{1cm} (7.5)

using the constraints (7.4). Performing the $\theta^2$ integral gives the $N = 2$ superspace action

$$I = \frac{1}{2} \int dt \, d\theta^0 d\theta^1 k_{[i,k]} J^k DX^i DX^j$$  \hspace{1cm} (7.6)

so comparing with (6.15) gives

$$\Omega_{ij} = J^{k} [i, k] j \delta_{k}$$  \hspace{1cm} (7.7)

where $k_{ij} = 2k_{[i,j]}$, or

$$\Omega = 2t \, J \, dk$$  \hspace{1cm} (7.8)

The metric and torsion are then given by (6.26), so that

$$g_{ij} = k_{k(i} \tilde{\Omega}^{k}_{j)} + I^{k} (i, J^l_{j}) k_{kl}$$  \hspace{1cm} (7.9)

The condition for the complex structure $\tilde{\Omega}$ to give a fourth supersymmetry is that

$$dt \tilde{\Omega}^i k = 0$$  \hspace{1cm} (7.10)

so that locally there is a scalar $L$ such that

$$k_i = \tilde{\Omega}^j_{i} \partial_j L$$  \hspace{1cm} (7.11)

as will be seen in the next section.
7.3. N=4 Supersymmetry

In this case there are simultaneously integrable complex structures $I, J, K$ and the superspace constraints are

$$D_1 X^i = I^i_k DX^k, \quad D_2 X^i = J^i_j DX^j, \quad D_3 X^i = K^i_j DX^j \quad (7.12)$$

The case in which $IJ = K$ and $I, J, K$ satisfy the algebra of unit imaginary quaternions has been analysed in section 4.3. Here we will not assume this, so that in general there are three additional complex structures defined by the products

$$\tilde{K} = IJ, \quad \tilde{I} = JK, \quad \tilde{J} = KI \quad (7.13)$$

and there is a real structure ($R^2 = \mathbb{1}$)

$$R = IJK \quad (7.14)$$

If the complex structures satisfy (4.4), the dimension is $D = 4n$ for some $n$ and the complex structures can be taken to be $I_r \times \mathbb{1}_n$ where $I_r$ are the $4 \times 4$ real matrices (7.1) and $\mathbb{1}_n$ is the $n \times n$ identity matrix. If the complex structures do not satisfy (4.4), the dimension is $D = 8n$ for some $n$ and the complex structures can be taken to be

$$\epsilon \otimes \mathbb{1} \otimes \mathbb{1}, \quad \sigma_1 \otimes \epsilon \otimes \mathbb{1}, \quad \sigma_3 \otimes \mathbb{1} \otimes \epsilon \quad (7.15)$$

In the coordinate system in which the complex structures are all constant, the general form of the $N = 4$ superspace action is

$$I = \frac{1}{4} \int dt \, d\theta^0 \, d\theta^1 \, d\theta^2 \, d\theta^3 \, L(X) \quad (7.16)$$

for some potential $L$. Performing the $\theta^3$ integral gives the $N = 3$ superspace action

$$I = \frac{1}{4} \int dt \, d\theta^0 \, d\theta^1 \, d\theta^2 \, L_{ij} K^i j DX^i \quad (7.17)$$
which is of the same form as (7.5), with the 1-form potential given by

\[ k_i = L_{ij} K^j_i \]  

(7.18)

or

\[ k = \iota_K dL \]  

(7.19)

so that the geometry 2-form is given by

\[ \Omega = 2\iota_J d(\iota_K dL) \]  

(7.20)

Then the metric and torsion are given by (6.26). Defining

\[ P^{ij}_{kl} = \tilde{I}^i_k I^j_l + \tilde{J}^i_k J^j_l + \tilde{K}^i_k K^j_l \]  

(7.21)

this gives

\[ g_{jk} = \frac{1}{2} L_{ml} P^{ml}_{(kj)} - \frac{1}{2} R^i_l \partial_k \partial_l L \]  

(7.22)

and

\[ I^i_{[j} B^j_{k]} = \frac{1}{2} L_{ml} P^{ml}_{[kj]} - \frac{1}{2} R^i_l \partial_k \partial_l L \]  

(7.23)

In the special case of a quaternionic structure with \( I = \tilde{I}, J = \tilde{J}, K = \tilde{K} \) and \( R = -1 \), then the space is weak HKT, \( B = 0 \) and the expression for the metric is given by

\[ g_{kl} = \frac{1}{2} \left( L_{,kl} + [I^i_k I^j_l + J^i_k J^j_l + K^i_k K^j_l] L_{,ij} \right) \]  

(7.24)

as in [5], and agreeing with the results of section 4.3.
7.4. N>4 Supersymmetry

For any model with $N \geq 4$ supersymmetry and simultaneously integrable complex structures, the action can be written in the $N = 4$ superspace corresponding to any three complex structures in terms of a scalar potential (with different potentials for different sets of three). If any three of the complex structures satisfy the quaternion algebra (4.4), then the geometry must be weak HKT and there can be no more than 4 supersymmetries unless the holonomy of the connection $\Gamma^{(+)}$ is trivial. For $N > 4$, the action can be written in full extended superspace in a same way similar to that used in [8], giving an explicit expression for the potential $L$ as a multiple contour integral. Alternatively, the conditions for the $N = 4$ action to have further supersymmetries leads to differential constraints on $L$ (similar to those in [8], the general solution of which is given by the multiple contour integral expression. Details will be given elsewhere.

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APPENDIX

For any $N = 2$ superspace action of the form

$$I = \int dt \, d\theta^1 \, d\theta^2 \, L$$

(A.1)

the $\theta^2$ integral gives

$$I = \int dt \, d\theta \, \frac{1}{2i} (\mathcal{D} - \bar{\mathcal{D}}) L$$

(A.2)

(Recall that $\mathcal{D} = D_0 + iD_1$ and $D = D_1$.) For

$$L = iG_{\alpha\beta} \mathcal{D} Z^\alpha \bar{\mathcal{D}} \bar{Z}^\beta$$

(A.3)
this gives

\[ \frac{1}{2i}(D - \bar{D})L = -2iG_{\alpha\beta}(DZ^\alpha \partial_t \bar{Z}^\beta + \partial_t Z^\alpha D\bar{Z}^\beta) \]

\[ - 4G_{\alpha\beta,\gamma} DZ^\alpha DZ^\gamma D\bar{Z}^\beta - 4G_{\alpha\beta,\gamma} D\bar{Z}^\alpha D\bar{Z}^\gamma DZ^\beta \]  

where the chiral constraint \( DZ = 2DZ \) has been used. For

\[ L = \frac{1}{2} B_{\alpha\beta} DZ^\alpha DZ^\beta \]  

it is useful to write

\[ (D - \bar{D})L = 2DL - 2DL \]  

(A.6)

to obtain

\[ \frac{1}{2i}(D - \bar{D})L = -4iB_{[\alpha,\beta,\gamma]} DZ^\alpha DZ^\beta DZ^\gamma + iDL \]  

(A.7)

and the term \( iDL \) is a surface term in the superspace action, which can be discarded.

REFERENCES

1. R. Coles & G. Papadopoulos, Class. Quantum Grav. 7 (1990) 427.

2. G. W. Gibbons, G. Papadopoulos and K.S. Stelle, Nucl. Phys. B 508, 623 (1997) hep-th/9706207.

3. G.W. Gibbons, R.H. Rietdijk & J.W. van Holten, Nucl. Phys. B404 (1993) 42.

4. F. de Jonghe, K. Peeters & K. Sfetsos, Class. Quantum Grav. 14 (1997) 35.

5. J. Michelson and A. Strominger, hep-th/9907191.

6. S. Hellerman and J. Polchinski, hep-th/9908202.

7. C.M. Hull, Lectures on Nonlinear Sigma Models and Strings, Lectures given in Super Field Theories, eds. H.C.Lee et al, (Plenum, 1987).
8. S.J. Gates, C.M. Hull & M. Roček, Nucl. Phys. B248 (1984) 157.
9. C.M. Hull and E. Witten, Phys. Lett. 160B (1985) 398.
10. P.S. Howe & G. Papadopoulos, Commun. Math. Phys. 151 (1993) 467.
11. P.S. Howe & G. Papadopoulos, Nucl. Phys. B289 (1987) 264; Class. Quantum Grav. 5 (1988) 1647.
12. P.S. Howe & G. Papadopoulos, Phys. Lett. B379 (1996) 80.
13. P.S. Howe & G. Papadopoulos, Nucl. Phys. B381 (1992) 360.
14. M. Dine and N. Seiberg, Phys. Lett. B180 (1986) 364.
15. B. deWit and P. van Nieuwenhuizen, Nucl. Phys. B312 (1989) 58.
16. G.W. Delius, M.Roček, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B324 (1989) 523.
17. P.S. Howe and G. Papadopoulos, Commun. Math. Phys. 151 (1993) 467.
18. G. Papadopoulos, Nucl. Phys. B448 (1995) 199, hep-th/9503063.
19. C. M. Hull, Actions for (2,1) Sigma Models and Strings, Nucl. Phys. B509 (1998) 252, hep-th/9702067.
20. Mohab Abou Zeid and C. M. Hull, The Geometry of sigma models with twisted supersymmetry, hep-th/9907046.
21. E. Kiritsis, C. Counnas & D. Lüst Int. Journ. Mod. Phys. A9 (1994) 1361.
22. K. Yano, Differential Geometry on Complex and Almost Complex Spaces, Pergamon, Oxford, 1965.
23. C.N. Pope, M.F. Sohnius and K.S. Stelle, Nucl. Phys. B283, 192 (1987).
24. K. Yano and M. Ako, Hokkaido Math J. 1 (1972) 63.