ASYMPTOTIC FREENESS OF UNITARY MATRICES IN TENSOR PRODUCT SPACES FOR INVARIANT STATES

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Abstract. In this paper, we pursue our study of asymptotic properties of families of random matrices that have a tensor structure. In [CGL17], the first- and second-named authors provided conditions under which tensor products of unitary random matrices are asymptotically free with respect to the normalized trace. Here, we extend this result by proving that asymptotic freeness of tensor products of Haar unitary matrices holds with respect to a significantly larger class of states. Our result relies on invariance under the symmetric group, and therefore on traffic probability.

As a byproduct, we explore two additional generalizations: (i) we state results of freeness in a context of general sequences of representations of the unitary group – the fundamental representation being a particular case that corresponds to the classical asymptotic freeness result for Haar unitary matrices, and (ii) we consider actions of the symmetric group and the free group simultaneously and obtain a result of asymptotic freeness in this context as well.

1. Introduction

1.1. Absorption Properties in Tensor Products. In this paper, our main aim is to study some of the mechanisms that give rise to asymptotic absorption properties of unitary random matrices. Roughly speaking, absorption phenomena refers to the observation that several interesting properties of free unitary operators remain unaffected by taking tensor products with other unitary operators.

A prototypical example of an absorption phenomenon is Fell’s absorption principle, which states that the left regular representation of a discrete group absorbs any unitary representation through tensor products (see, for instance, [Pis03, Proposition 8.1] for a precise statement). Combined with a classical computation due to Akemann and Ostrand [AO76], Fell’s absorption principle implies the following result, which has interesting applications in operator algebras (e.g., [Pis97]).

Proposition 1.1 (Norm Absorption). Let \((u_1, \ldots, u_L), L \geq 2\) be a Haar unitary system, i.e., free Haar unitary operators (see Definition 3). For
every unitary operators $v_1, \ldots, v_L$, one has

$$\left\| \sum_{\ell=1}^L u_\ell \otimes v_\ell \right\| = \left\| \sum_{\ell=1}^L u_\ell \right\| = 2 \sqrt{L - 1}.$$  

In recent years, the authors of the present paper have studied several problems in free probability in which asymptotic absorption phenomena arise at the level of random unitary matrices. For example, Collins and Male proved the following finite-dimensional version of Proposition 1.1:

**Proposition 1.2** ([CM14, Section 2.2.4]). For all $N \in \mathbb{N}$, let $U_1^{(N)}, \ldots, U_L^{(N)}$, $L \geq 2$ be independent $N \times N$ Haar unitary random matrices, and let $V_1, \ldots, V_L$ be unitary matrices of fixed dimension $M \in \mathbb{N}$. Almost surely, it holds that

$$\lim_{N \to \infty} \left\| \sum_{\ell=1}^L U_\ell^{(N)} \otimes V_\ell \right\| = \lim_{N \to \infty} \left\| \sum_{\ell=1}^L U_\ell^{(N)} \right\| = 2 \sqrt{L - 1}.$$  

Proposition 1.2 follows from the strong asymptotic freeness of independent Haar unitary matrices with respect to polynomials with scalar or matrix-valued coefficients, which is the central result in [CM14].

In a slightly different direction, Collins and Gaudreau Lamarre [CGL17] proved a general result which has the following proposition as a simple special case:

**Proposition 1.3.** For every $N \in \mathbb{N}$, let $U_1^{(N)}, \ldots, U_L^{(N)}$ be independent $N \times N$ Haar unitary random matrices, and let $V_1^{(M)}, \ldots, V_L^{(M)}$ be unitary matrices of arbitrary dimension $M = M(N)$, which may or may not depend on $N$. In the space $(M_N^N(\mathbb{C}) \otimes M_M^M(\mathbb{C}), \text{tr}_N \otimes \text{tr}_M)$ (where $\text{tr}_N = N^{-1} \text{Tr}$ denotes the normalized trace), the family

$$(U_1^{(N)} \otimes V_1^{(M)}, \ldots, U_L^{(N)} \otimes V_L^{(M)})$$  

converges almost surely and in expectation (Definition 5) as $N \to \infty$ to a Haar unitary system.

**Remark 1.** Clearly, the matrices $U_1^{(N)} \otimes 1$ have the same distribution (Definition 4) in the space $(M_N^N(\mathbb{C}) \otimes M_M^M(\mathbb{C}), \text{tr}_N \otimes \text{tr}_M)$ as the matrices $U_1^{(N)}$ in the space $(M_N^N(\mathbb{C}), \text{tr}_N)$.

The almost sure convergence of $(U_1^{(N)}, \ldots, U_L^{(N)})$ with respect to $\text{tr}_N$ to a Haar unitary system is a classical result in free probability [HP00, Voic91]. The fact that this is preserved after taking tensor products with arbitrary unitary matrices is a special case of the tensor freeness conditions introduced in [CGL17, Definition 1.4]. We refer to Section 2.3 for more details, including an elementary proof of Proposition 1.3.
Our main purpose in this paper is to study a generalization of the absorption property stated in Proposition 1.3 (see Theorem 1.4 below for a statement of our main result). The main departure of the present paper from Proposition 1.3 is that we consider asymptotic freeness of families of the form (I) with respect to states on \( M_N(\mathbb{C}) \otimes M_M(\mathbb{C}) \) other than the tensor product of traces \( tr_N \otimes tr_M \). Although this greater generality comes at a cost of making stricter assumptions on the matrices \( V_{\ell}^{(M)} \) that the \( U_{\ell}^{(N)} \) can absorb and replacing almost sure convergence with convergence in expectation, we show that an absorption property holds for a class of problems that go well beyond what can be explained by such simple criteria as the tensor freeness conditions of [CGL17].

1.2. **Representation Theory.** Representation theory has also played an important role in the study of asymptotic freeness for random matrices; see for example [Bia98, Col03]. The choice of \( V_{\ell}^{(M)} = U_{\ell}^{(N) \otimes K_1} \otimes U_{\ell}^{(N) \otimes K_2} \) in Equation (I) above (where \( \cdot \) denotes the entrywise complex conjugate) is a special case of the results that we treat, but it is of particular interest because it introduces additional symmetries arising from permutations of legs, and \( U_{\ell}^{(N)} \mapsto U_{\ell}^{(N) \otimes K_1} \otimes \overline{U_{\ell}^{(N) \otimes K_2}} \) is a group morphism. That is, we are working with the representation theory of the unitary group – irreducible representations can all be obtained by taking corners of the above, that can themselves be constructed with permutations (or, more generally, elements of the commutant for the action of the group). In turn, it becomes interesting and natural to study the asymptotic properties of random unitaries that arise from representation theory, as well as families combining such unitary and permutation operators. We are able to obtain asymptotic freeness in the first case, and asymptotic freeness with amalgamation in the latter case (see Theorem 1.5 below). We note that such questions are natural from the point of view of harmonic analysis over the free group; we refer to Section 5.1 for more details.

1.3. **Main Result and Corollaries.** In what follows, for every \( N \in \mathbb{N} \), we let \( \mathcal{U}_N \) denote the unitary group of dimension \( N \). We use \( \mathcal{X}_N \) to denote a subgroup of \( \mathcal{U}_N \), and we distinguish \( \mathcal{X}_N = \mathcal{O}_N \) and \( \mathcal{X}_N = \mathcal{S}_N \) in the cases of the orthogonal and permutation groups, respectively.

**Definition 1.** Let \( K \geq 1 \) be an integer.

- A family \( A_N = (A_1^{(N)}, \ldots, A_L^{(N)}) \) of random matrices in \( M_N(\mathbb{C})^{\otimes K} \) is said to be \( \mathcal{X}_N \)-invariant if

\[
A_N \overset{\text{law}}{=} \left( (U \otimes \cdots \otimes U) A_1^{(N)} (U^* \otimes \cdots \otimes U^*) \right)_{t=1,\ldots,L}
\]

for every \( U \in \mathcal{X}_N \).
A linear form $\phi_N : M_N(\mathbb{C})^{\otimes K} \to \mathbb{C}$ is said to be $X_N$-invariant if
$$\phi_N(A_1 \otimes \cdots \otimes A_K) = \phi_N(UA_1 U^* \otimes \cdots \otimes UA_K U^*)$$
for every $A_1, \ldots, A_K \in M_N(\mathbb{C})$ and $U \in X_N$.

Our main result regarding absorption in tensor products is the following.

**Theorem 1.4.** Let $K \geq 1$ be an integer. For every $N \in \mathbb{N}$, consider a family of unitary random matrices $W_N = (W_1, \ldots, W_L)$ in $M_N(\mathbb{C})^{\otimes K}$ of the form
$$W_\ell = U_\ell^{(N)} \otimes K_1 \otimes V_\ell^{(N)}, \quad \ell = 1, \ldots, L,$$
where

- $K = K_1 + K_2 + K_3$, with $K_1 \geq 1$, $K_2, K_3 \geq 0$ integers.
- $U_N = (U_1^{(N)}, \ldots, U_L^{(N)})$ is a family of $N \times N$ independent Haar unitary matrices ($U_\ell^{(N)^*}$ denotes the transpose of $U_\ell^{(N)}$).
- $V_N = (V_1^{(N)}, \ldots, V_L^{(N)})$ is a family of unitary random matrices in $M_N(\mathbb{C})^{\otimes K_3}$, independent of $U_N$.

Let $\psi_N : M_N^{\otimes K}(\mathbb{C}) \to \mathbb{C}$ be a state (see Definition 2). Assume that $\psi_N$ or $V_N$ is $S_N$-invariant. If $V_N$ satisfies the Mingo-Speicher bound (see Definition 12), then, in the space $(M_N(\mathbb{C})^{\otimes K}, \psi_N)$, the family $W_N$ converges in expectation as $N \to \infty$ to a Haar unitary system.

**Remark 2.** In Theorem 1.4, there is no loss of generality in assuming that $\psi_N$ and $V_N$ are both $S_N$-invariant. We refer to Section 2.4 for more details.

**Remark 3.** The Mingo-Speicher bound is a very powerful and fine property when analyzing the asymptotics of large random matrices by the method of moments. Its exact formulation is quite technical and requires several combinatorial definitions, which is why it is postponed until later in this article. Let us note however that the Mingo-Speicher bound holds when $V_\ell^{(N)}$ is a tensor product $V_\ell^{(N)} \otimes \cdots \otimes V_\ell^{(N)}$ of unitary matrices of dimension $N$; see Remark 20.

**Remark 4.** For $K_1 = 1$ and $K_2 = K_3 = 0$, Theorem 1.4 simply states that independent Haar unitary matrices are asymptotically $^\ast$-free with respect to any state, which has been proved for a large class of unitary invariant matrices in [CDM16].

Next, we state our results concerning representation theory.

**Theorem 1.5.** Let $(\lambda, \mu)$ be a signature, and let $\chi_{\lambda,\mu}$ be the character of the associated rational irreducible representation $(\rho_{\lambda,\mu}, V_{\lambda,\mu})$ of $U_N$, provided $N$ is large enough (see Section 5.2 for more details on this notation).
Let \( K \in \mathbb{N} \) and let \( \mathbf{U}_N \) be a family of i.i.d. \( N \times N \) Haar unitary random matrices. We denote
\[
(\mathbf{U}_N, \overline{\mathbf{U}_N}) := (U_1^{(N)}, \ldots, U_K^{(N)}, \overline{U_1^{(N)}}, \ldots, \overline{U_K^{(N)}}),
\]
where we recall that \( \overline{\cdot} \) denotes the entrywise complex conjugate. In the space \( \text{End}(V_\lambda \otimes V_\mu) \), the family \((\rho_\lambda \otimes \rho_\mu \mathbf{U}_N, \rho_\lambda \otimes \rho_\mu \overline{\mathbf{U}_N})\) converges in expectation as \( N \to \infty \) to a Haar unitary system.

Let \( \mathbf{U}_N \) be as above and \( d \in \mathbb{N} \) be an integer. The family
\[
\mathbf{U}_N \otimes d := (U_1^{(N) \otimes d}, \ldots, U_K^{(N) \otimes d})
\]
is asymptotically free with amalgamation over \( S_d \) in the tensor product representation \( \text{M}_N(\mathbb{C}) \otimes d \), as \( N \to \infty \).

Remark 5. The above theorem extends the result of [MP16] to the case of arbitrary sequences of irreducible representations (associated to a given signature) in the limit of large dimension.

1.4. Organization of Paper. The remainder of this paper is organized as follows. In Section 2, we recall basic notions and results in free probability that are used in this paper. Section 3 prepares the proof of the main result, while Section 4 supplies the actual proof. Sections 5 and 6 are devoted to applications of the main result, including the proof of Theorem 1.5.

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2. Background in Free Probability

In this section, we go over the basic definitions and results in free probability that are used in this paper. For a thorough introduction to the subject and its applications to random matrix theory, the reader is referred to [MS17, NS06, VDN92].
2.1. Non-commutative Probability and Haar Unitary Systems. Recall that a non-commutative probability space is defined as a pair \((A, \phi)\), where \(A\) is a unital algebra and \(\phi : A \to \mathbb{C}\) is a unital \((\phi(1) = 1)\) linear functional; elements of \(A\) are called non-commutative random variables.

**Definition 2.** A \(^*\)-probability space is a non-commutative probability space \((A, \phi)\), where \(A\) is a \(^*\)-algebra (i.e., a unital algebra endowed with an antilinear involution such that \((ab)^* = b^*a^*\) for any \(a, b \in A\)) and \(\phi\) is a state (i.e., \(\phi(aa^*) \geq 0, \forall a \in A\)). We say that \(\phi\) is tracial whenever \(\phi(ab) = \phi(ba)\) for any \(a, b \in A\).

A non-commutative random variable \(u\) in a \(^*\)-probability space \((A, \phi)\) is said to be unitary if \(u^*u = uu^* = 1\), and Haar unitary if it also satisfies \(\phi(u^n) = 0\) for all \(n \in \mathbb{Z}\setminus\{0\}\).

Recall that unital \(^*\)-subalgebras \(A_i\) (\(i \in I\)) of \(A\) are called \(^*\)-free if for every \(t \geq 1, i(1), \ldots, i(t) \in I\), and \(a_{i(1)} \in A_{i(1)}, \ldots, a_{i(t)} \in A_{i(t)}\), one has \(\phi(a_{i(1)} \cdots a_{i(t)}) = 0\) whenever \(i(1) \neq i(2) \neq \cdots \neq i(t)\) and \(\phi(a_{i(1)}) = \cdots = \phi(a_{i(t)}) = 0\). A family of non-commutative random variables \(x_i\) (\(i \in I\)) is said to be \(^*\)-free if the collection of unital \(^*\)-subalgebras generated by the \(x_i\) are \(^*\)-free.

**Definition 3.** A family \(u = (u_1, \ldots, u_L)\) of non-commutative random variables is called a Haar unitary system if the \(u_t\) are \(^*\)-free Haar unitary non-commutative random variables.

2.2. Asymptotic Freeness of Random Matrices. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \(L^{\infty-} = L^{\infty-}(\Omega, \mathbb{C})\) denote the \(^*\)-algebra of random variables with finite moments of all orders. Given \(N \in \mathbb{N}\), let \(A \in M_N(L^{\infty-})\) be a random \(N \times N\) matrix with entries in \(L^{\infty-}\). If we are given a state \(\psi_N : M_N(\mathbb{C}) \to \mathbb{C}\), then there are two natural \(^*\)-probability spaces in which \(A\) can be studied: we can consider \(A\) an element of \((M_N(L^{\infty-}), \mathbb{E}[\psi_N])\), and for every \(\omega \in \Omega\), the realization \(A(\omega)\) of \(A\) is an element of \((M_N(\mathbb{C}), \psi_N)\).

Let \(X_i, X_i^*\) (\(i \in I\)) be a collection of non-commuting indeterminates. We call a non-commutative polynomial \(P \in \mathbb{C}[X_i, X_i^*]_{i \in I}\) a \(^*\)-polynomial (here, \(\mathbb{C}[X_i, X_i^*]_{i \in I}\) denotes the unital algebra freely generated by the collection of non-commuting indeterminates \(X_i\) and \(X_i^*\)). If \(P\) is a monomial, then it may also be called a \(^*\)-monomial.

**Definition 4.** Given a collection \(a = (a_i)_{i \in I}\) of non-commutative random variables in a \(^*\)-probability space \((A, \phi)\), the \(^*\)-distribution of \(a\) is defined as the linear functional \(\mu_a : \mathbb{C}[X_i, X_i^*]_{i \in I} \to \mathbb{C}\) determined by the relation
\[
\mu_a(P) = \phi(P(a)).
\]

**Definition 5.** For every \(N \in \mathbb{N}\), let \(A_N = (A_1^{(N)}, \ldots, A_L^{(N)})\) be a family of \(N \times N\) random matrices with entries in \(L^{\infty-}\). Let \(a = (a_1, \ldots, a_L)\)
be a family of non-commutative random variables in some \(\mathcal{A}_N\) as elements of the space \((M_N(C),\psi_N)\):

- \(\mathbf{A}_N \to \mathbf{a}\) almost surely if for almost every realization of \(\mathbf{A}_N\),
  \[
  \lim_{N \to \infty} \psi_N(P(\mathbf{A}_N)) = \phi(P(\mathbf{a}))
  \]
  for every \(*\)-polynomial \(P\);
- \(\mathbf{A}_N \to \mathbf{a}\) in expectation if for every \(*\)-polynomial \(P\),
  \[
  \lim_{N \to \infty} \mathbb{E}[\psi_N(P(\mathbf{A}_N))] = \phi(P(\mathbf{a})).
  \]

Note that here, ‘in expectation’ applies to the distribution, i.e. it tells that for any (self adjoint) polynomial, the expectation of its empirical eigenvalues distribution converges.

**Remark 6.** If the limiting family \(\mathbf{a} = (a_1, \ldots, a_L)\) in the above definition is \(*\)-free, then we say that \(\mathbf{A}_N\) is **asymptotically \(*\)-free** almost surely, in probability, or in expectation.

2.3. **Tensor Freeness.**

**Lemma 2.1** (Tensor Freeness). Let \(\mathbf{u} = (u_1, \ldots, u_L)\) be a Haar unitary system in \((\mathcal{A}, \phi)\), and let \(\mathbf{v} = (v_1, \ldots, v_L)\) be a family of unitary non-commutative random variables in \((\mathcal{B}, \psi)\). Then,

\[
\mathbf{w} = (u_1 \otimes v_1, \ldots, u_L \otimes v_L)
\]

is a Haar unitary system in \((\mathcal{A} \otimes \mathcal{B}, \phi \otimes \psi)\).

**Proof.** Clearly, the tensor products \(u_\ell \otimes v_\ell\) are unitary. Moreover, for any \(*\)-monomial \(M\), one has

\[
(\phi \otimes \psi)(M(\mathbf{w})) = \phi(M(\mathbf{u})) \times \psi(M(\mathbf{v})).
\]

If \(M\) is trivial (i.e., \(M(\mathbf{u}) = 1\) for any family \(\mathbf{u}\) of unitary operators), then \(\phi(M(\mathbf{u})) = \psi(M(\mathbf{v})) = 1\). Otherwise, the fact that \(\mathbf{u}\) is \(*\)-free implies that \((\phi \otimes \psi)(M(\mathbf{w})) = \phi(M(\mathbf{u})) = 0\). Thus, \(\mathbf{w}\) is a Haar unitary system. \(\Box\)

**Remark 7.** If we are given families of variables \((a_1, \ldots, a_L)\) and \((b_1, \ldots, b_L)\) in respective non-commutative probability spaces \((\mathcal{A}, \phi)\) and \((\mathcal{B}, \psi)\), and we assume that the \(a_\ell\) are \(*\)-free, then it is not necessarily the case that the tensor product collection

\[
(a_1 \otimes b_1, \ldots, a_L \otimes b_L)
\]

is \(*\)-free in \((\mathcal{A} \otimes \mathcal{B}, \phi \otimes \psi)\). Lemma 2.1 is a special case of a more general class of examples that satisfy the tensor freeness conditions [CGL17] Definition 1.4 and Proposition 1.5], which guarantees that the freeness present in one collection propagates to the tensor product collection.
We may now prove Proposition 1.3.

**Proof of Proposition 1.3.** For the sake of readability, let us denote

\[
U_N = (U_1^{(N)}, \ldots, U_L^{(N)}), \quad V_N = (V_1^{(M)}, \ldots, V_L^{(M)}),
\]

and

\[
W_N = (U_1^{(N)} \otimes V_1^{(M)}, \ldots, U_L^{(N)} \otimes V_L^{(M)}).
\]

By [HP00, Voic91], \(U_N\) converges to a Haar unitary system \(u = (u_1, \ldots, u_L)\) almost surely. Since unitary matrices are bounded in operator norm, every subsequence of \(N\) has a further subsequence along which \(V_M\) and \(W_N\) converge almost surely to some limiting families \(v = (v_1, \ldots, v_L)\) and \(w = (w_1, \ldots, w_L)\), respectively. Note that \(W_N\) is the sequence of tensor products of \(U_N\) and \(V_N\). Hence every limit \(w\) of the subsequences is of the form \(w_\ell = u_\ell \otimes v_\ell (1 \leq \ell \leq L)\), and satisfies the hypotheses of Lemma 2.1, so it is a Haar unitary system. Since there is a single possible limit for every subsequence, \(W_N\) converges almost surely to a Haar unitary system. Since the matrices of \(W_N\) are bounded in operator norm, the convergence also holds in expectation. □

### 2.4. Duality of Invariance

We now explain the claim made in Remark 2 that, in the context of Theorem 1.4, we can always assume that \(\psi_N\) and \(V_N\) are both \(S_N\)-invariant. Let \(B = (B_1, \ldots, B_L)\) be a collection of random matrices in \(M_N(C) \otimes K\) and \(\psi : M_N(C) \otimes K \to C\) be a linear form.

Suppose that \(B\) is \(X_N\)-invariant, and let \(U\) be a unitary matrix distributed according to the Haar measure on \(X_N\), independently of \(B\). Consider the collection

\[
\hat{B} := (U^{\otimes K}B_1U^{\ast \otimes K}, \ldots, U^{\otimes K}B_LU^{\ast \otimes K}).
\]  

By the invariance of \(B\), for every \(*\)-polynomial \(P\),

\[
E_U[\psi(P(\hat{B}))] = E_U[\psi(U^{\otimes K}P(B)U^{\ast \otimes K})]
\]

is equal in distribution to \(\psi(P(B))\), and since \(U\) is Haar distributed, the form defined as

\[
\hat{\psi}(A) := E_U[\psi(U^{\otimes K}AU^{\ast \otimes K})], \quad A \in M_N(C) \otimes K
\]

is \(X_N\)-invariant. Thus, if we are interested in the large \(N\) limits of expectations \(E[\psi(P(B))]\), then there is no loss of generality in assuming that \(\psi = \hat{\psi}\), and, in particular, that \(\psi\) is \(X_N\)-invariant.

Similarly, if \(\psi\) is \(X_N\)-invariant, then

\[
\psi(A) = \psi(U^{\otimes K}AU^{\ast \otimes K}), \quad A \in M_N(C) \otimes K
\]

for any \(U \in X_N\), and thus there is no loss of generality in replacing \(B\) by \(\hat{B}\), which is \(X_N\)-invariant if \(U\) is independent of \(B\) and Haar distributed.
2.5. Freeness with Amalgamation. The notion of freeness with amalgamation was introduced by Voiculescu as a generalization of freeness—see for example [VDN92, Section 3.8]—and appears naturally in several contexts of large random matrices. In particular, let us consider two matrices whose entries are non-commutative random variables in a space \((\mathcal{A}, \phi)\). If the entries of the respective matrices are free, then the two matrices themselves are free with amalgamation over scalar matrices [MS17, Section 9, Corollary 14]. Together with the so-called linearization trick, this result gives a powerful method to compute the spectral distribution of self-adjoint \(^*\)-polynomials in free variables [MS17, Section 10.3]. Moreover, freeness with amalgamation over the diagonal holds for independent permutation invariant matrices with variance profiles [Shl96, ACD+21] and appear in the second order distribution of certain Wigner and deterministic matrices [Mal21].

In a \(^*\)-algebra \(\mathcal{A}\), we pick a unital subalgebra \(B\) and we say that a unital linear functional \(E : A \to B\) is a conditional expectation of \(A\) onto \(B\) if it satisfies \(E(abc) = aE(b)c\) for all \(a, c \in B\), \(b \in A\). In other words, \(E\) can be seen as an orthogonal projection of \(A\) onto \(B\) with respect to an appropriate scalar product arising from a state preserved by \(E\). \(E\) is not always guaranteed to exist; however, in the case of von Neumann algebras, there are systematic existence theorems, and existence entails uniqueness. We mostly work in the context of finite dimensional algebras which are automatically von Neumann algebras, so the existence and uniqueness is granted in the cases of interest to us. For more details we refer to Theorem 4.2 of section IX-4 of [Tak03].

Next, we get to the definition of freeness with amalgamation. In the above context of \(B \subset A\) with \(1 \in B\) and a conditional expectation \(E\) from \(A\) onto \(B\), we consider an arbitrary index set \(I\) and take a family \((A_i)_{i \in I}\) of subalgebras satisfying \(B \subset A_i \subset A\). The family \((A_i)_{i \in I}\) is said to be free with amalgamation over \(B\) if and only if

\[
E(a_1 \ldots a_i) = 0
\]

whenever \(E(a_{i_1}) = 0\) and \(a_j \in A_{i_j}\), with \(i_1 \neq i_2, i_2 \neq i_3, \ldots\). For a systematic treatment, we refer to [MS17, Section 9.2]. One key example is as follows: if \(1 \in A_1, \ldots \subset A\) are free, then \(M_k(A_1), M_k(A_2), \ldots \in \mathcal{M}_k(\mathcal{A})\) are free with amalgamation over \(\mathcal{M}_k(\mathbb{C})\).

Next we get to the definition of conditional distribution.

**Definition 6.** Given a collection \(a = (a_i)_{i \in I}\) of non-commutative random variables in a \(^*\)-probability space \((\mathcal{A}, \phi)\) endowed with a conditional expectation \(E : A \to B\), the \(^*\)-conditional distribution of \(a\) is defined as the linear
functional $\mu_a : B\langle X_i, X_i^* \rangle_{i \in I} \to B$ determined by the relation
\[ \mu_a(P) = E(P(a)). \]

Finally, we can provide a definition of asymptotic freeness with amalgamation.

**Definition 7.** For every $N \in \mathbb{N}$, let $A_N = (A_1^{(N)}, \ldots, A_L^{(N)})$ be a family of non-commutative random variables in a $\ast$-probability space $(A^{(N)}, \phi^{(N)})$ endowed with a conditional expectation $E^{(N)} : A^{(N)} \to B$ – note here that we require $B$ to be the same for each $N$.

Let $a = (a_1, \ldots, a_L)$ be a family of non-commutative random variables in some $\ast$-probability space $(A, \phi)$ with a conditional expectation $E : A \to B$. Then, we say that $A_N \to a$ if
\[ \lim_{N \to \infty} E_N(P(A_N)) = E(P(a)) \]
for every $P \in B\langle X_i, X_i^* \rangle$, the set of all polynomials in $X_i$ and $X_i^*$ with coefficients from $B$. If, in addition to (4), the $\ast$-algebras $A_i := B\langle a_i, a_i^* \rangle$ are free with amalgamation over $B$ in $A$, then we say that $A_N$ is asymptotically free with amalgamation over $B$ in $A$.

3. Invariant states on tensor matrix spaces

3.1. **Proof Overview Part 1.** For any subgroup $\mathcal{X}_N$ of $\mathcal{U}_N$, the set of $\mathcal{X}_N$-invariant linear forms on $M_N(\mathbb{C})^{\otimes K}$ is a finite dimensional vector space. In particular, there exists a finite collection of $\mathcal{X}_N$-elementary linear forms $\text{Tr}_{N,1}, \text{Tr}_{N,2}, \ldots$ that are $\mathcal{X}_N$-invariant and such that for every other $\mathcal{X}_N$-invariant form $\psi_N$, one has
\[ \psi_N = \sum_i a_{N,i} \text{Tr}_{N,i} \]
for some scalars $a_{N,1}, a_{N,2}, \ldots$.

**Remark 8.** For the classical groups (such as $\mathcal{U}_N$, $\mathcal{O}_N$ and $\mathcal{S}_N$), the invariant linear forms are given by the Schur-Weyl duality. In the case of $\mathcal{S}_N$, we can compute the elementary forms and their associated constants $a_{N,i}$ explicitly by elementary means (see Proposition 3.1 and its proof).

The first step of the proof of Theorem 1.4 consists of identifying the $\mathcal{S}_N$-elementary linear forms. In Proposition 3.1 below, we prove that the latter are characterized by the set of partitions of $\{1, \ldots, 2K\}$ (which we denote $\mathcal{P}(2K)$), so that $\psi_N$ can be written as a sum of the form
\[ \psi_N = \sum_{\pi \in \mathcal{P}(2K)} a_{N,\pi} \text{Tr}_{N,T_\pi^\circ}. \]
A precise description of the $S_N$-elementary linear forms $\text{Tr}_{N,T_0}$ can be found in Definition 11.

Remark 9. A different description of these elementary functions can also be found in [Gab15].

The second step of the proof consists of bounding the decay rate of the constants $\alpha_{N,\pi}$ that appear in the above expansion for large $N$. In Proposition 3.2, we prove that there exist positive constants $\mathcal{L}(T_0^J)$ (see Definition 12) such that $\alpha_{N,\pi} = O(N^{-\mathcal{L}(T_0^J)/2})$ as $N \to \infty$.

The third and last step of our proof is to understand the growth rate of the $S_N$-elementary linear forms $\text{Tr}_{N,T_0}$ evaluated in the matrices $W_N$ defined in (2), especially as compared to $N^{\mathcal{L}(T_0^J)/2}$. This step is carried out in Section 4; see Section 4.1 for a detailed overview of this part of the argument.

The remainder of this section is devoted to the proof of the first two steps outlined above.

3.2. The $S_N$-Elementary Linear Forms.

3.2.1. Basis Elements. In order to describe the $S_N$-elementary linear forms, we first introduce several notions in graph theory. In what follows, given an integer $K \geq 1$, we use the notation $[K] = \{1, 2, \ldots, K\}$.

Definition 8. We say that a couple $(V, E)$ is a directed graph if $V$ is a set of vertices and $E$ is a multi-set of directed edges, i.e., ordered pairs of elements of $V$. More specifically, $(v, w) \in E \subset V^2$ means that there is a directed edge from $v$ to $w$, which we represent graphically as $v \to w$. We call $w$ the target of that edge, and $v$ the source. We allow $(V, E)$ to contain loops and multiple edges, and to be disconnected.

Let $K \geq 1$ be an integer. A linear graph of order $K$ consists of a triplet $T = (V, E, \gamma)$ that satisfies the following conditions.

- $(V, E)$ is a finite directed graph.
- $\gamma$ maps every element of $E$ to a unique number in $[K]$ (thus indicating that $e \in E$ is the $\gamma(e)$-th edge for every $e \in E$). We emphasize that multiple edges are associated with different numbers by $\gamma$, so that $\gamma$ is a bijection from the multi-set $E$ to $[K]$.

Remark 10. We note that the set $[K]$ can be replaced by any totally ordered set in the above definition.

Remark 11. We always consider linear graphs up to isomorphisms that preserve the order of the edges. That is, two linear graphs $T = (V, E, \gamma)$ and $T' = (V', E', \gamma')$ are considered equal if there is a directed graph isomorphism $\Phi : (V, E) \to (V', E')$ such that $\gamma(e) < \gamma'(\tilde{e})$ if and only if $\gamma'(\Phi(e)) < \gamma'(\Phi(\tilde{e}))$. 

Remark 12. A linear graph may be illustrated as follows
\[
T = \cdot \xrightarrow{3} \cdot \xleftarrow{2} \cdot \xleftarrow{1} \cdot \xrightarrow{\cdot \cdot \cdot} \cdot \xleftarrow{\cdot \cdot \cdot} \cdot
\]
In the above illustration, the dots represent the vertices, the arrows represent the directed edges (making this particular example a linear graph of order 3), and the value of \( \gamma \) at an edge is displayed above the edge in question.

Definition 9. We define the minimal linear graph of order \( K \), denoted \( T_0 = (V_0, E_0, \gamma_0) \), as the following linear graph. The vertices consist of the set \( V_0 = [2K] \), the \( K \) edges are given by \( e_k = (K + k, k) \) for \( 1 \leq k \leq K \), and we assign the order \( \gamma_0(e_k) = k \).

Remark 13. The minimal linear graph of order \( K \) is illustrated in Figure 1.

In the following definitions, we use \( \mathcal{P}(S) \) to denote the set of partitions of a set \( S \). In the special case where \( S = [K] \) for some integer \( K \in \mathbb{N} \), we simply denote \( \mathcal{P}(S) = \mathcal{P}(K) \).

Definition 10. Let \( T = (V, E, \gamma) \) be a linear graph and \( \pi \in \mathcal{P}(V) \) be a partition of its vertex set. We denote by \( T^\pi = (V^\pi, E^\pi, \gamma^\pi) \) the quotient graph of \( T \) for the partition \( \pi \), that is, the vertices \( V^\pi \) are the blocks of \( \pi \), every edge \( e = (v, w) \) of \( T \) induces the edge \( e^\pi = (C_v, C_w) \in E^\pi \), where \( C_v, C_w \in \pi \) are the blocks containing \( v \) and \( w \) respectively, and \( \gamma^\pi(e^\pi) = \gamma(e) \).

Remark 14. A quotient of \( T_0 \) is illustrated in Figure 2.

Remark 15. If a linear graph \( T \) of order \( K \) has no trivial component (i.e., single vertices with no edge), then it is a quotient of the minimal linear graph \( T_0 \), that is, \( T = T_0^\pi \) for some \( \pi \in \mathcal{P}(2K) \). In fact, since this partition is unique, the map \( \pi \mapsto T_0^\pi \) is a bijection between \( \mathcal{P}(2K) \) and the set of linear graphs of order \( K \) with no trivial component.

We may now finally define the \( S_N \)-elementary linear forms and state the first main result of this section.

Definition 11. Let \( N, K \in \mathbb{N} \). For every linear graph \( T \) of order \( K \), we introduce an associated linear form \( \text{Tr}_{N,T} : M_N(\mathbb{C})^{\otimes K} \rightarrow \mathbb{C} \) determined by the following relation: For every \( A_1, \ldots, A_K \in M_N(\mathbb{C}) \),
\[
\text{Tr}_{N,T}(A_1 \otimes \cdots \otimes A_K) = \sum_{\phi : V \rightarrow [N]} \prod_{e = (v,w) \in E} A_{\gamma(e)}(\phi(w), \phi(v)).
\]
Figure 2. The quotient $T_0^\pi$ for $K = 9$ and
\[ \pi = \{ \{1, 3, 13\}, \{2, 14, 15, 16\}, \{4, 11, 17\}, \{5, 10\}, \{6\}, \{7\}, \{8, 9, 18\}, \{12\} \}. \]
For instance, the first block $\{1, 3, 13\}$ means that the following vertices are equal: the targets of the 1st and 3rd edges and the source of the 4th edge (since $4 = 13 - 9 = 13 - K$).

(In the above, $\phi : V \to [N]$ denotes an arbitrary function from the set of vertices $V$ to $[N]$, so that (5) contains $N^{|V|}$ summands.) We call such $T_{N,T}$ linear graph (unormalized) $S_N$-elementary linear forms of order $K$.

Remark 16. In general $T_{N,T}$ is neither tracial nor a state. For a non-tracial counterexample, note that the linear graph $T = \cdot \xleftarrow{1} \cdot$ of order 1 is such that
\[ T_{N,T}(A) = \sum_{i,j=1}^N A(i,j), \quad A \in M_N(\mathbb{C}). \]
This is clearly not tracial for $N \geq 2$. For an example that fails to be a state, note that the linear graph $T = \cdot \xleftarrow{1} \cdot \xleftarrow{2}$ of order 2 is such that
\[ T_{N,T}(A_1 \otimes A_2) = \sum_{i,j,k=1}^N A_1(i,j)A_2(j,k) = \sum_{i,k=1}^N A_1A_2(i,k). \]
This linear form is not positive for $N \geq 2$.

Remark 17. Clearly, the $S_N$-elementary linear forms are invariant under order-preserving isomorphisms on the linear graphs. Moreover if a linear graph has a trivial component, then deleting that vertex changes the associated linear form by a multiplicative factor of $N$. Hence it is easy to see that, up to multiplicative constants, there is a finite number of $S_N$-elementary linear forms of order $K$.

Combining this observation with Remark 15 one expects that we need only consider $S_N$-elementary linear forms $T_{N,T}$ such that $T$ is a quotient of the minimal graph. The following proposition confirms that this is the case.

Proposition 3.1. The set of $S_N$-elementary linear forms of order $K$ generates the space of $S_N$-invariant linear forms on $M_N(\mathbb{C})^\otimes K$. In particular, for
every $S_N$-invariant form $\psi_N$, there exists constants $a_{N,\pi}$ (where $\pi \in \mathcal{P}(2K)$) such that

$$\psi_N = \sum_{\pi \in \mathcal{P}(2K)} a_{N,\pi} \text{Tr}_{N,T,T_0^\pi}.$$  

(6)

Proposition 3.1 is proved in Section 3.3

3.2.2. Control of the Coefficients. With the $S_N$-elementary linear forms identified in (6), the second main result of this section concerns the control of the coefficients $a_{N,\pi}$ for large $N$. In order to state this result, we introduce one more graph-theoretic notion.

Definition 12. Let $T = (V,E,\gamma)$ be a linear graph of order $K$.

(1) A cutting edge of a graph is an edge whose removal increases the number of connected components.

(2) A two-edge connected graph is a connected graph with no cutting edge.

(3) A two-edge connected component of a graph is a maximal connected sub-graph that is two-edge connected.

(4) The forest of two-edge connected components of a graph $T$ is the graph $\mathcal{F}(T)$ whose vertices are the two-edge connected components of $T$ and whose edges are the cutting edges of $T$, making links between the components that contain the source and the target of a cutting edge.

(5) A trivial component of $\mathcal{F}(T)$ is a component consisting of a single vertex.

We denote by $\mathcal{L}(T)$ the number of leaves in the forest of two-edge connected components $\mathcal{F}(T)$, with the convention that a trivial component has two leaves.

The following result, which is proved below in Section 3.4, contains our bound on the coefficients $a_{N,\pi}$ that appear in (6).

Proposition 3.2. For every $\pi \in \mathcal{P}(2K)$, as $N \to \infty$ it holds that

$$a_{N,\pi} = O(N^{-\mathcal{L}(T_0^\pi)/2}).$$  

(7)
Before proving Propositions 3.1 and 3.2, we take this opportunity to formulate the technical boundedness assumption on the matrices $V_N$ mentioned in the statement of Theorem 1.4, which is a direct consequence of the asymptotic (7):

**Definition 13** (Mingo-Speicher Bound). For each $N \geq 1$, let $A_N = (A_j^{(N)})_{j \in J}$ be a family of random matrices such that for every $j \in J$, there is an integer $K_j \geq 1$ such that $A_j^{(N)} \in M_N(\mathbb{C})^{\otimes K_j}$. We say that the sequence $A_N$, $N \geq 1$, satisfies the *Mingo-Speicher bound* if for every $n \geq 1, j_1, \ldots, j_n \in J$, and linear graph $T$ or order $K = K_{j_1} + \cdots + K_{j_n}$, there exists a constant $C > 0$ independent of $N$ such that

$$\mathbb{E} \left[ \text{Tr}_{N,T}(A_{j_1}^{(N)} \otimes \cdots \otimes A_{j_n}^{(N)}) \right] \leq C N^{C(T)/2}.$$  

**Remark 18.** The appellation *Mingo-Speicher bound* is inspired by a result of Mingo and Speicher that we state as Theorem 3.3 in Section 3.4 below.

### 3.3. Proof of Proposition 3.1

**3.3.1. Multi-Index Kernels.** For any integers $i, j = 1, \ldots, N$, we denote by $E_{i,j}$ the $(i,j)$-th elementary matrix of $M_N(\mathbb{C})$, that is,

$$E_{i,j}(m,n) = \delta_{i,m} \delta_{j,n}, \quad 1 \leq m, n \leq N,$$

where $\delta$ denotes the Kronecker delta function. A basis for $M_N(\mathbb{C})^{\otimes K}$ is given by the tensor products

$$E_{i,j} = E_{i_1,j_1} \otimes \cdots \otimes E_{i_K,j_K}, \quad i, j \in [N]^K,$$

and an element $B = \sum_{i,j \in [N]^K} B(i,j)E_{i,j} \in M_N(\mathbb{C})^{\otimes K}$ is generically denoted as $B = (B(i,j))_{i,j \in [N]^K}$.

Let $\psi_N : M_N(\mathbb{C})^{\otimes K} \to \mathbb{C}$ be an arbitrary linear form. We can write $\psi_N$ as a trace against a matrix, namely, for every $A \in M_N(\mathbb{C})^{\otimes K}$,

$$\psi_N(A) = \text{Tr} \left[ A B^t \right], \quad (8)$$

where $B = \sum_{i,j} \psi_N(E_{i,j})E_{i,j}$. If $\psi_N$ is $S_N$-invariant, then we can assume that the matrix $B$ in (8) is a $S_N$-invariant deterministic matrix. Indeed, if $V$ is a random matrix uniformly distributed on $S_N$, then by $S_N$-invariance

$$\psi_N(A) = \mathbb{E}_V \left[ \psi_N(V^{\otimes K} AV^*^{\otimes K}) \right] = \mathbb{E}_V \left[ \text{Tr} \left[ V^{\otimes K} AV^*^{\otimes K} B^t \right] \right] = \text{Tr} \left[ A \tilde{B}^t \right],$$

where $\tilde{B} = \mathbb{E}_V[V^*^{\otimes K} BV^{\otimes K}]$. Hence we may assume without loss of generality that $B = \tilde{B}$.

In the sequel, we denote pairs of multi-indices $(i,j) \in [N]^K \times [N]^K$ as elements of $[N]^{2K}$, that is,

$$(i,j) = (i_1, \ldots, i_K, i_{K+1}, \ldots, i_{2K}).$$
Given \((i, j) \in [N]^{2K}\), we use \(\ker(i, j) \in \mathcal{P}(2K)\) to denote the partition of \([2K]\) determined by the condition

\[
C \in \ker(i, j) \quad \text{if and only if} \quad i_k = i_\ell \text{ for every } k, \ell \in C.
\]

In words, the blocks of \(\ker(i, j)\) are the groups of indices for which the associated integers are equal.

**Example 1.** We have \(\ker(6, 1, 4, 1, 6, 2, 2, 2) = \ker(1, 2, 3, 2, 1, 4, 4, 4) = \{\{1, 5\}, \{2, 4\}, \{3\}, \{6, 7, 8\}\}

Our purpose for introducing these partitions is the following trivial fact: For any two pairs of multi-indices \((i, j), (i', j') \in [N]^{2K}\), there exists a permutation \(\sigma \in S_N\) such that \(\sigma(i, j) = (i', j')\) if and only if \(\ker(i, j) = \ker(i', j')\) (here, we denote \(\sigma(i, j) = (\sigma(i_1), \ldots, \sigma(i_{2K}))\)). Since we assume that the matrix \(B\) in (8) is permutation invariant, then it follows that \(B(i, j) = B(i', j')\) whenever \(\ker(i, j) = \ker(i', j')\). Consequently, if, for every \(\pi \in \mathcal{P}(2K)\), we denote by \(B_\pi\) the common value of \(B(i, j)\) for all \((i, j)\) such that \(\ker(i, j) = \pi\), and we define the matrix \(\xi_\pi \in M_N(\mathbb{C})^{\otimes K}\) as

\[
\xi_\pi(i, j) = \delta_{\ker(i,j)=\pi}, \quad i,j \in [N]^K,
\]

then we get the decomposition

\[
B = \sum_{\pi \in \mathcal{P}(2K)} B_\pi \xi_\pi.
\]

Thus for any \(A \in M_N(\mathbb{C})^{\otimes K}\), one has

\[
\psi_N(A) = \sum_{\pi \in \mathcal{P}(2K)} B_\pi \operatorname{Tr} [A \xi_\pi^\dagger].
\]

### 3.3.2. Injective Linear Forms and Möbius Inversion

With (10) established, it now remains to prove that each linear map \(A \mapsto \operatorname{Tr} [A \xi_\pi^\dagger]\) is a linear combination of the \(S_N\)-elementary linear forms. For this, we introduce the following modification of the \(\operatorname{Tr}_{N,T}\).

**Definition 14.** Let \(T = (V, E, \gamma)\) be a linear graph of order \(K\). For every \(N \in \mathbb{N}\), we define the *injective linear form of order \(K\)*, denoted \(\operatorname{Tr}_{N,T}^\circ\) as

\[
\operatorname{Tr}_{N,T}^\circ(A_1 \otimes \cdots \otimes A_K) = \sum_{\phi: V \to [N]} \prod_{e \in \gamma(v)} A_{\gamma(e)}(\phi(w), \phi(v))
\]

for every \(A_1, \ldots, A_K \in M_N(\mathbb{C})\).

The relevance of injective linear forms comes from the following fact: If \(T = (V, E, \gamma)\) is such that \(T = T_0^\pi\) for some \(\pi \in \mathcal{P}(2K)\), then for every \(A_1, \ldots, A_K \in M_N(\mathbb{C})\), it holds that

\[
\operatorname{Tr}_{N,T}^\circ(A_1 \otimes \cdots \otimes A_K) = \operatorname{Tr} [(A_1 \otimes \cdots \otimes A_K) \xi_\pi^\dagger].
\]
(recall that \(\xi_\pi\) is defined in (9)). To see this, note that, on the one hand,

\[
\text{Tr} \left[ (A_1 \otimes \cdots \otimes A_K) \xi_{\pi}^{T} \right] = \sum_{\ker((i,j))=\pi} A_1(i_1, i_{K+1}) A_2(i_2, i_{K+2}) \cdots A_K(i_K, i_{2K}).
\]

On the other hand, if we enumerate the edges

\[
e_1 = (v_1, w_1), e_2 = (v_2, w_2), \ldots, e_K = (v_K, w_K)
\]

of a linear graph \(T = (V, E, \gamma)\) in such a way that \(\gamma(e_\ell) = \ell\) for every \(1 \leq \ell \leq K\), then for any injective map \(\phi : V \to [N]\), the multi-index

\[
(i, j) = (\phi(w_1), \ldots, \phi(w_K), \phi(v_1), \ldots, \phi(v_K))
\]

is such that \(\ker(i, j) = \pi\) if and only if \(T = T_{\pi}^\circ\).

We now conclude the proof of Proposition 3.1 by showing that injective linear forms can be written as linear combinations of \(S_N\)-elementary linear forms. Recall that the set \(P(2K)\) of partitions can be endowed with a natural partial order whereby \(\pi \leq \pi'\) if and only if every block of \(\pi\) is contained in a block of \(\pi'\). With this in mind, we note the following comparison between injective linear forms and \(S_N\)-elementary linear forms:

**Remark 19.** Note that (11) only differs from (5) in the requirement that the map \(\phi\) be injective. If \(T = T_{\pi}^\circ\) for some \(\pi \in P(2K)\) and \(\phi : V \to [N]\) is an arbitrary function (i.e., not necessarily injective), then the multi-index

\[
(i, j) = (\phi(w_1), \ldots, \phi(w_K), \phi(v_1), \ldots, \phi(v_K))
\]

satisfies \(\ker(i, j) \geq \pi\). In fact, for every \(\pi \in P(2K)\), one has

\[
\text{Tr}_{\pi, 0} = \sum_{\pi' \geq \pi} \text{Tr}_{\pi', 0}. \tag{12}
\]

Endowed with its natural order, the poset \(P(2K)\) forms a lattice \([\text{Sta12}, \text{Section 3.3}]\). In particular, by the Möbius inversion formula (dual form) \([\text{Sta12}, \text{Proposition 3.7.2}]\), (12) implies that

\[
\text{Tr}_{\pi, \pi} = \sum_{\pi' \geq \pi} \text{Mob}(\pi, \pi') \text{Tr}_{\pi', \pi}, \tag{13}
\]

where \(\text{Mob}\) denotes the Möbius function on \(P(2K)\) \([\text{Sta12}, \text{Section 3.7}]\). If we combine all that was shown in Section 3.3, then we see that

\[
\psi_N = \sum_{\pi' \in P(2K)} B_{\pi'} \text{Tr}_{\pi, \pi} = \sum_{\pi' \in P(2K)} B_{\pi'} \text{Mob}(\pi', \pi) \text{Tr}_{\pi', \pi} = \sum_{\pi \in P(2K)} a_{\pi, \pi} \text{Tr}_{\pi, \pi},
\]
where
\[
\alpha_{N,\pi} = \sum_{\pi' \leq \pi} B_{\pi'} \operatorname{Mob}(\pi', \pi),
\]
concluding the proof of Proposition 3.1.

3.4. **Proof of Proposition 3.2.** Since \(\psi_N\) is a state, we know that
\[
|\psi_N(A)| \leq \|A\|, \quad A \in \mathcal{M}_N(\mathbb{C})^\otimes K
\]
(c.f., [NS06 Proposition 3.8]). Moreover, we recall the following result of Mingo and Speicher.

**Theorem 3.3.** [MS12 Theorem 6] For any linear graph \(T\) of order \(K\),
\[
\sup_{A_1 \otimes \cdots \otimes A_K \in \mathcal{M}_K(\mathbb{C}) \otimes K, \|A_k\|=1, \forall k} \left| \operatorname{Tr}_{0} N,T_{\pi_0}(A) \right| = N^{\mathcal{L}(T)/2},
\]
with \(\mathcal{L}(T)\) as in Definition 12.

**Remark 20.** According to (16), any family of tensor products of unitary \(N \times N\) matrices satisfies the Mingo-Speicher bound.

Note that (15) implies that
\[
\sup_{A_1 \otimes \cdots \otimes A_K \in \mathcal{M}_K(\mathbb{C}) \otimes K, \|A_k\|=1, \forall k} \left| \psi_N(A) \right| = 1.
\]
Combining this fact with the suprema in (16) and the expansion in (6) suggests that the constants \(\alpha_{N,\pi}\) should be of order \(N^{-\mathcal{L}(T_0)/2}\). We can make this heuristic precise with the following three results, which we prove in Sections 3.4.1–3.4.3 below.

**Lemma 3.4.** For every \(\pi \in \mathcal{P}(2K)\), there exists a constant \(C^{|\pi|} > 0\) such that for every \(N \in \mathbb{N}\) and \(A \in \mathcal{M}_N(\mathbb{C})^\otimes K\),
\[
\left| \left( \sum_{\pi' \leq \pi} \alpha_{N,\pi'} \right) \operatorname{Tr}_{0} N,T_{\pi_0}(A) \right| \leq C^{|\pi|}\|A\|.
\]

**Lemma 3.5.** If \(\pi' \leq \pi\), then \(\mathcal{L}(T_{\pi}) \leq \mathcal{L}(T_{\pi'})\).

**Lemma 3.6.** For any \(\pi \in \mathcal{P}(2K)\), there are two constants \(0 < C_\pi < C'_\pi\) such that for every \(N \geq 2K\),
\[
C_\pi N^{\mathcal{L}(T_{\pi})/2} \leq \sup_{A_1 \otimes \cdots \otimes A_K \in \mathcal{M}_K(\mathbb{C}) \otimes K, \|A_k\|=1, \forall k} \left| \operatorname{Tr}_{0} N,T_{\pi_0}(A) \right| \leq C'_\pi N^{\mathcal{L}(T_{\pi})/2}.
\]
Indeed, if we denote $b_{N,\pi} = \sum_{\pi' \leq \pi} a_{N,\pi'}$, then Lemma 3.4 implies that $|b_{N,\pi} \Tr_{N,0}^0 (A)| \leq C(\pi)$ for any matrix $A$ with unit norm. If we combine this with Lemma 3.6 then we conclude that $b_{N,\pi} = O(N^{-L(T_0^\pi)/2})$. Given the relationship between the constants $b_{N,\pi}$ and $a_{N,\pi}$, it follows from the Möbius inversion formula [Sta12, Proposition 3.7.1] that

$$a_{N,\pi} = \sum_{\pi' \leq \pi} b_{N,\pi'} \Mob(\pi', \pi) = O(N^{-L(T_0^\pi)/2}),$$

where the last estimate follows from Lemma 3.5.

**Remark 21.** Using the same argument that we have just provided, if there exists some $\alpha > 0$ such that $C(\pi) N^{\alpha} \leq \sup_{A \in M_N(C) \otimes K} \|A\|_1, \forall k$, then we have that $a_{N,\pi} = O(N^{-\alpha})$. However, to the best of the authors’ knowledge, the order of the suprema (16) and (17) over all matrices $A \in M_N(C) \otimes K$ of norm one (instead of $A = A_1 \otimes \cdots \otimes A_K$) is unknown.

In order to complete the proof of Proposition 3.2, it now only remains to prove Lemmas 3.4–3.6.

3.4.1. **Proof of Lemma 3.4.** Let $\pi \in \mathcal{P}(2K)$ be fixed. Suppose that we construct random matrices $D^{(L)}$, $D^{(R)} \in M_N(C) \otimes K$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E}_D \left[ \|D^{(L)} D^{(R)}\|_1 \right] \leq C(\pi)$$

for some constant $C(\pi)$, and such that for every $\pi' \in \mathcal{P}(2K)$ and $A \in M_N(C) \otimes K$, one has

$$\mathbb{E}_D \left[ \Tr_{N,0}^0 (D^{(L)} A D^{(R)}) \right] = \delta_{\pi,\pi'} \Tr_{N,0}^0 (A),$$

where $\mathbb{E}_D$ denotes the expected value with respect to $D^{(L)}$ and $D^{(R)}$. Then, by (6) and (12), we see that

$$\mathbb{E}_D \left[ \psi_N (D^{(L)} A D^{(R)}) \right] = \left( \sum_{\pi' \leq \pi} a_{N,\pi'} \right) \Tr_{N,0}^0 (A),$$

and thus Lemma 3.4 is proved by (15).

We now construct $D^{(L)}$ and $D^{(R)}$. Suppose for now that we can write

$$D^{(L)} = D_1 \otimes \cdots \otimes D_K \quad \text{and} \quad D^{(R)} = D_{K+1} \otimes \cdots \otimes D_{2K},$$
where the $D_{\ell} \in \mathbb{M}_N(C)$ are diagonal. Then, for every linear graph $T = (V, E, \gamma)$ of order $K$ and matrix $A = A_1 \otimes \cdots \otimes A_K$, it holds that

$$
\mathbb{E}_D \left[ T_{N,T}^\phi \left( D^{(L)} A D^{(R)} \right) \right] 
= \sum_{\phi: V \to [N] \text{ injective}} \mathbb{E}_D \left[ \prod_{e = (v, w) \in E} D_{\gamma(e)}(\phi(w), \phi(w)) D_{K + \gamma(e)}(\phi(v), \phi(v)) \right] 
\times \prod_{e = (v, w) \in E} A_{\gamma(e)}(\phi(w), \phi(v)).
$$

We enumerate the edges of $T$ as $e_1 = (v_{K+1}, v_1), \ldots, e_K = (v_{2K}, v_K)$ in such a way that $\gamma(e_k) = k$ for each $1 \leq k \leq K$. For every injective map $\phi$, the partition $\text{ker} \ (\phi(v_1), \ldots, \phi(v_{2K}))$ of $[2K]$ does not depend on $\phi$ and is denoted $\pi$: two integer $\ell$ and $\ell'$ are in a same block of $\pi$ whenever $v_\ell = v_{\ell'}$. Then we can write

$$
\mathbb{E}_D \left[ \prod_{e = (v, w) \in E} D_{\gamma(e)}(\phi(w), \phi(w)) D_{K + \gamma(e)}(\phi(v), \phi(v)) \right] 
= \mathbb{E}_D \left[ \prod_{C \in \pi} \prod_{\ell \in C} D_{\ell}(\phi(v_\ell), \phi(v_\ell)) \right].
$$

and this quantity is independent of the choice of injective $\phi$. Our objective is to define the matrices $D_{\ell}$ in such a way that if $T = T_0^{\pi'}$, then (20) is equal to $\delta_{\pi, \pi'}$. We need two ingredients to make this construction.

Firstly, for every block $C \in \pi$, we define $\tilde{D}_C \in \mathbb{M}_N(C)$ as a diagonal matrix whose diagonal entries are i.i.d. random variables sampled according to the uniform measure on the complex roots of unity of order $|C|$. In particular, for every $i \in [N]$ and $n \in \mathbb{N}$,

$$
\mathbb{E} [\tilde{D}_C(i, i)^n] = \begin{cases} 
1 & \text{if } n \text{ is a multiple of } |C|, \\
0 & \text{otherwise.}
\end{cases}
$$

Furthermore, we assume that the matrices $(\tilde{D}_C)_{C \in \pi}$ are independent of each other.

Secondly, for every block $C \in \pi$, we define $\bar{D}_C \in \mathbb{M}_N(C)$ as a diagonal matrix whose diagonal entries are random variables satisfying the following conditions:

1. For every $i \in [N]$, it holds that

$$
\mathbb{E}[\bar{D}_C(i, i)^{|C|}] = 1 \quad \text{for every } C \in \pi,
$$

and if two blocks $C, C' \in \pi$ are distinct, then $\bar{D}_C(i, i)\bar{D}_{C'}(i, i) = 0$. 

$$
E_{\mathcal{D}} \left[ T_{N,T}^\phi \left( D^{(L)} A D^{(R)} \right) \right] = \sum_{\phi: V \to [N] \text{ injective}} \mathbb{E}_D \left[ \prod_{e = (v, w) \in E} D_{\gamma(e)}(\phi(w), \phi(w)) D_{K + \gamma(e)}(\phi(v), \phi(v)) \right] 
\times \prod_{e = (v, w) \in E} A_{\gamma(e)}(\phi(w), \phi(v)).
$$
(2) The collections \( (\bar{D}_C(i, i))_{C \in \pi} \) are independent of each other for different values of \( i \in [N] \).

(3) \( \sup_{N \in \mathbb{N}} \left( \sup_{C \in \pi, i \in [N]} \bar{D}_C(i, i) \right) < \infty \).

The existence of such variables is proved in Example 2 below. We also assume that the matrices \( (\bar{D}_C)_{C \in \pi} \) are independent of \( (\tilde{D}_C)_{C \in \pi} \).

With these definitions in mind, for every \( \ell \in [2K] \), we define the diagonal matrix \( D_\ell = \bar{D}_C \tilde{D}_C \), where \( C_\ell \in \pi \) denotes the block that contains \( \ell \). On the one hand, since the entries of \( D_L \) and \( D_R \) are uniformly bounded in \( N \), it is clear that (18) holds true. On the other hand, (20) is now equal to

\[
\mathbb{E} \left[ \prod_{C \in \pi} \prod_{\ell \in C} \bar{D}_C(\phi(v_\ell), \phi(v_\ell)) \tilde{D}_C(\phi(v_\ell), \phi(v_\ell)) \right].
\]

(23)

If there exists distinct blocks \( C, C' \in \pi \) and \( \ell \in C, \ell' \in C' \) such that \( v_\ell = v = v_\ell' \), then the expectation in (23) contains the product

\[
\bar{D}_C(\phi(v), \phi(v)) \tilde{D}_{C'}(\phi(v), \phi(v)),
\]

and thus is equal to zero. Otherwise, if the fact that \( \ell \) and \( \ell' \) are in distinct blocks of \( \pi \) implies that \( v_\ell \neq v_\ell' \) (and thus \( \phi(v_\ell) \neq \phi(v_\ell') \) since \( \phi \) is injective), then by the independence assumptions on \( \bar{D}_C \) and \( \tilde{D}_C \) we can simplify (23) to

\[
\prod_{C \in \pi} \mathbb{E} \left[ \prod_{\ell \in C} \bar{D}_C(\phi(v_\ell), \phi(v_\ell)) \right] \mathbb{E} \left[ \prod_{\ell \in C} \tilde{D}_C(\phi(v_\ell), \phi(v_\ell)) \right].
\]

According to (21) and (22), this expression is one if \( v_\ell = v_\ell' \) whenever \( \ell \) and \( \ell' \) are in the same block of \( \pi \), and zero otherwise. In summary, (20) is equal to one if \( T = T_0^c \) and zero otherwise, concluding the proof.

**Example 2.** Let \( n \in \mathbb{N} \), and let \( X_1, \ldots, X_n \) be i.i.d. uniform random variables on \( \{0, 2\} \). Next, for every \( i \in [n] \), let

\[
Y_{i,j} = \begin{cases} 
X_i & \text{if } j = 0, \text{ and} \\
(2 - X_i) & \text{if } j = 1.
\end{cases}
\]

Then, for every binary sequence \( b = (b_1, \ldots, b_n) \in \{0, 1\}^n \), we let

\[
Z_b = \left( \prod_{i=1}^{n} Y_{i,b_i} \right)^{1/f(b)},
\]

where \( f : \{0, 1\}^n \to (0, \infty) \) is some function. By independence,

\[
\mathbb{E}[Z_b^{(b)}] = \prod_{i=1}^{n} \mathbb{E}[Y_{i,b_i}] = 1.
\]
for every \( b \). Moreover, if \( b, b' \in \{0, 1\}^n \) are distinct, which means that \( b_i \neq b'_i \) for some \( i \in [n] \), then the product \( Z_b Z_{b'} \) contains the factor

\[
Y_{i,b_i}^{1/ \ell(b)} Y_{i,b'_i}^{1/ \ell(b')} = X_i^{1/ \ell(b)} (2 - X_i)^{1/ \ell(b')} \quad \text{or} \quad (2 - X_i)^{1/ \ell(b')} X_i^{1/ \ell(b')},
\]

whence it is zero.

3.4.2. Proof of Lemma 3.5 Given that \( \pi' \leq \pi \), there exists a sequence of partitions \( \pi' = \pi_1 \leq \pi_2 \leq \cdots \leq \pi_n = \pi \) such that for each \( 1 \leq i \leq n - 1 \), \( \pi_{i+1} \) is obtained from \( \pi_i \) by joining two blocks of \( \pi_i \) into one. At the level of linear graphs, this corresponds to a sequence \( T_0^{\pi_1}, T_0^{\pi_2}, \ldots, T_0^{\pi_n} \) where each \( T_0^{\pi_{i+1}} \) is obtained from \( T_0^{\pi_i} \) by identifying two vertices in the latter.

On the one hand, if the two vertices that are joined together in \( T_0^{\pi_i} \) are in the same two-edge connected component, then \( \mathcal{L}(T_0^{\pi_{i+1}}) = \mathcal{L}(T_0^{\pi_i}) \) (i.e., the forest of two-edge connected components is unaffected by this operation).

On the other hand, if we identify two distinct two-edge connected components, then the forest \( \mathcal{F}(T_0^{\pi_{i+1}}) \) can be obtained from \( \mathcal{F}(T_0^{\pi_i}) \) by identifying the corresponding vertices. Since this process can only decrease the number of leaves, we conclude that \( \mathcal{L}(T_0^{\pi_{i+1}}) \geq \mathcal{L}(T_0^{\pi_i}) \).

Remark 22. By using the same argument presented here, it is easy to see that for general linear graphs \( T \) and \( T' \) (which may contain trivial components, unlike quotients of \( T_0 \), if \( T \) is a quotient of \( T' \) then \( \mathcal{L}(T) \leq \mathcal{L}(T') \).

3.4.3. Proof of Lemma 3.6 The upper bound is a direct consequence of Theorem 3.3 equation 13, and Lemma 3.5. To prove the lower bound, we present an adaptation of the example of optimality presented by Mingo and Speicher in [MS12] for their proof of Theorem 3.3 (see Example 7 and Section 5 therein).

Let \( \pi \in \mathcal{P}(2\mathcal{K}) \). We want to find matrices \( A_1, \ldots, A_K \in \mathbb{M}_N(\mathbb{C}) \) of unit norm such that \( \text{Tr}^0_{N,T_0^\pi}(A_1 \otimes \cdots \otimes A_K) \) is of order \( N^{\mathcal{L}(T_0^\pi)/2} \) for large \( N \).

Suppose that \( T_0^\pi \) satisfies the following:

- There are L_1 cutting edges adjacent to only one leaf in \( \mathcal{F}(T_0^\pi) \).
- There are L_2 cutting edges adjacent to two leaves in \( \mathcal{F}(T_0^\pi) \).
- There are L_3 isolated two-edge connected components (i.e., not connected to a cutting edge). We denote the vertex sets of these connected components as \( C_1, C_2, \ldots, C_{L_3} \subset \mathcal{V}_0^\pi \).

By Definition 12, it is easily seen that \( \mathcal{L}(T_0^\pi) = L_1 + 2(L_2 + L_3) \).

If we denote by \( e_k = (v_k, w_k) \) the k-th edge of \( T_0^\pi \) for every \( k \in [K] \), then up to permuting the order of the matrices \( A_k \) in the tensor product \( A_1 \otimes \cdots \otimes A_K \), or replacing some \( A_k \)'s by their transposes \( A_k^t \), we may assume that the following holds:

- The cutting edges adjacent to one leaf are \( e_1, \ldots, e_{L_1} \), and the cutting edges adjacent to two leaves are \( e_{L_1+1}, \ldots, e_{L_1+L_2} \).
For every $\ell \in [L_1]$, the target of $e_\ell$ (i.e., $w_\ell$) belongs to a leaf.

Let $\pi_0 = \ker(v_{1}, \ldots, v_{L_1})$ be the partition of $[L_1]$ (defined as above Example 2) such that $i \sim_\pi_0 j$ if and only if $v_i = v_j$. We enumerate the blocks of $\pi_0$ from 1 to $|\pi_0|$, and we use $\pi_0(\ell)$ to denote the number of the block containing $v_\ell$. For any $\ell = 1, \ldots, L_1$, let us define the matrix

$$A_\ell(i, j) = N^{-1/2} \delta_{j, \pi_0(\ell)}, \quad i, j = 1, \ldots, N.$$

Let $\mathbb{J}_K$ be the $2K \times 2K$ matrix whose entries are all $\frac{1}{2K}$, let $N = m_N 2K + r$, $m_N \in \mathbb{N}$, $0 \leq r < 2K$ be the Euclidean division of $N$ by $2K$, and let

$$B = (\mathbb{J}_K)^{\otimes m_N} \oplus 0_{r \times r},$$

where $0_{r \times r}$ denotes the $r \times r$ zero matrix (so long as $N \geq 2K$, this can be defined without problem).

Finally, we define the matrix $A = A_1 \otimes \cdots \otimes A_{L_1} \otimes B^{\otimes K-L_1}$. It is easy to see that $A_1, \ldots, A_{L_1}$ and $B$ all have unit norm. Moreover,

$$\text{Tr}^0_{N, \mathbb{J}_0}(A) = N^{-L_1/2} \left( \frac{1}{2K} \right)^{K-L_1} \sum_{\text{injective}_\leq \text{injective}} \prod_{\ell=1}^{L_1} \delta_{\phi(v_\ell), \pi_0(\ell)}$$

$$\times \prod_{k=L_1+1}^{K} \mathbb{1}_{\{ m2K+1 \leq \phi(w_k), \phi(v_k) \leq (m+1)2K \text{ for some } 0 \leq m \leq m_N \}},$$

where $\mathbb{1}$ denotes the indicator function. Thus, it suffices to prove that the number of injections $\phi : V_0^3 \rightarrow [N]$ such that

$$\prod_{\ell=1}^{L_1} \delta_{\phi(v_\ell), \pi_0(\ell)} \prod_{k=L_1+1}^{K} \mathbb{1}_{\{ m2K+1 \leq \phi(w_k), \phi(v_k) \leq (m+1)2K \text{ for some } 0 \leq m \leq m_N \}} = 1 \quad (24)$$

is at least of order $N^{L_1+L_2+L_3}$. In order to see this, we propose to define such injections $\phi$ by using the following procedure.

1. For every $1 \leq \ell \leq L_1$, let $\phi(v_\ell) = \pi_0(\ell)$.
2. Make an arbitrary choice of vertices $\tilde{v}_1 \in C_1, \tilde{v}_2 \in C_2, \ldots, \tilde{v}_{L_3} \in C_{L_3}$ in the isolated connected components of $T_0^3$.
3. Make an arbitrary choice for the values

$$\phi(w_1), \ldots, \phi(w_{L_1}), \phi(w_{L_1+1}), \ldots$$

$$\ldots, \phi(w_{L_1+L_2}), \phi(\tilde{v}_1), \ldots, \phi(\tilde{v}_{L_3}) \in [\pi_0 + 1, N - r],$$

except for the requirement that the values all be distinct.
(4) Let \( 1 \leq \ell \leq L_1 \), and let \( m \leq m_N \) be the integer such that \( m2K+1 \leq \phi(w_\ell) \leq (m + 1)2K \). For every vertex \( v \neq w_\ell \) in the leaf that the edge \( e_\ell \) is pointing to, choose \( m2K+1 \leq \phi(v) \leq (m + 1)2K \).

(5) Let \( L_1 + 1 \leq \ell \leq L_1 + L_2 \), and let \( m \leq m_N \) be the integer such that \( m2K+1 \leq \phi(w_\ell) \leq (m + 1)2K \). For every vertex \( v \neq w_\ell \) in one of the two leaves that \( e_\ell \) is connected to, choose \( m2K+1 \leq \phi(v) \leq (m + 1)2K \).

(6) Let \( 1 \leq \ell \leq L_3 \), and let \( m \leq m_N \) be the integer such that \( m2K+1 \leq \phi(\tilde{v}_\ell) \leq (m + 1)2K \). For every \( v \in C_\ell \setminus \{\tilde{v}_\ell\} \), choose \( m2K+1 \leq \phi(v) \leq (m + 1)2K \).

(7) Finally, for every vertex \( v \) for which \( \phi \) has not yet been defined, choose \( 1 \leq \phi(v) \leq 2K \).

Clearly, any injective \( \phi \) constructed according to those conditions satisfies (24). Since \( T_0^{\pi_0} \) is a quotient of the minimal graph \( T_0 \), the total number of vertices is at most \( 2K \). Thus, for any choice made in steps (1)–(3), there is always at least one way to select the values of \( \phi \) in such a way that steps (4)–(7) are also satisfied. Since there are

\[
\frac{(N - r - |\pi_0|)!}{(N - r - |\pi_0| - L_1 - L_2 - L_3)!} \sim N^{L_1+L_2+L_3}
\]

ways of selecting the values of \( \phi \) in step (3), the result is proved.

**Remark 23.** As before, the argument presented here can easily be generalized to an arbitrary linear graph \( T \) possibly containing trivial components, giving the statement

\[
\sup_{A=A_1 \otimes \ldots \otimes A_k \atop \text{s.t. } \|A_k\|=1, \forall k} \left| \text{Tr}_{N,T}^0(A) \right| \asymp N^{\ell(T)/2}
\]

for large \( N \).

4. **Proof of Theorem 1.4**

4.1. **Proof Overview Part 2.** As per Definitions \([\text{3}]\) and \([\text{5}]\) we aim to prove that for every nontrivial *-monomial \( M \), one has

\[
\mathbb{E} \left[ \psi_N(M(W_N)) \right] = o(1)
\]

as \( N \to \infty \), where we recall that \( W_N \) is the collection of matrices

\[
W^{(N)}_\ell = U^{(N)}_\ell \otimes K \otimes U^{(N)}_\ell \otimes V^{(N)}_\ell, \quad \ell \in [L].
\]

**Remark 24.** Recall that we call the *-monomial \( M \) trivial if \( M(u) = 1 \) for every family \( u \) of unitary operators, and nontrivial otherwise.
Remark 25. The family of random unitary matrices
\[(U^{(N)}_{\ell} \otimes U^{(N_1)}_{\ell} \otimes \ldots)_{\ell=1,\ldots,L}\]
is $\theta_N$-invariant [MS12, Lemma 15]. Therefore, since $U_N$ and $V_N$ are independent, if $V_N$ is $S_N$-invariant, then so is $W_N$. As per Remark 2 throughout our proof of (25), we assume without loss of generality that $\psi_N$ and $W_N$ are both $S_N$-invariant.

Thanks to Propositions 3.1 and 3.2, it suffices to show that for every linear graph $T$ of order $K$ that is a quotient of $T_\emptyset$, one has
\[N^{-\frac{\ell(T)}{2}} \mathbb{E} \left[ \text{Tr}_{N,T}(M(W_N)) \right] = o(1). \tag{25}\]

Our method of proof for this result, which we outline in the next few paragraphs, makes significant use of ideas from traffic probability (c.f., [ACD+21, CDM16, Mal20]).

The first step for the proof of (25) consists of a linearization procedure that exhibits a linear graph $T_M$ and a random matrix $A_M$ such that
\[N^{-\frac{\ell(T)}{2}} \mathbb{E} \left[ \text{Tr}_{N,T}(M(W_N)) \right] = N^{-\frac{\ell(T_M)}{2}} \mathbb{E} \left[ \text{Tr}_{N,T_M}(A_M) \right], \tag{26}\]
where $A_M$ is a tensor product of the matrices in $U_N$ and $V_N$. We note that this linearization procedure already appears in [Mal20, Definition 1.7]. However, since our proof depends on several specific details of the construction of $T_M$ and $A_M$, we provide a complete description of the linearization in Section 4.2.1 (see Definitions 15 and 16).

The second step consists of isolating the contributions of the families $U_N$ and $V_N$ to the expression on the right-hand side of (26) (see Lemma 4.1 and (33)). Our main tool for this is a splitting lemma that appears in [Mal20]. As it turns out, the contribution of $V_N$ can be controlled thanks to the Mingo-Speicher bound assumption (Definition 13), and the contribution of $U_N$ can be reduced to the asymptotic analysis of the injective trace of tensor products of i.i.d. Haar unitary random matrices (see (36)).

The third and final step in the proof of (25) consists of showing that, due to the special structure of the graph $T_M$ (which depends on the $*$-monomial $M$), the contributions of $U_N$ to (26) must vanish in the large $N$ limit. This part of our argument makes crucial use of a precise asymptotic for the injective trace of Haar unitary matrices from [CDM16] (see Proposition 4.2).

We now proceed to the proof of (25).

4.2. Proof of (25).

4.2.1. Linearization. Let $T = (V, E, \gamma)$ be a linear graph of order $K$, and let $M \in \mathbb{C}\langle X_\ell, X_\ell^* \rangle_{\ell \in [L]}$ be a nontrivial $*$-monomial, which we write as
\[M(X) = X_{\delta(1)}^{\epsilon(1)} \cdots X_{\delta(p)}^{\epsilon(p)} \tag{27}\]
for some $p \in \mathbb{N}, \delta(1), \ldots, \delta(p) \in [L]$, and $\epsilon(1), \ldots, \epsilon(p) \in \{1, \ast\}$. 

**Definition 15** (Linearized Graph). As usual, let us enumerate $\Gamma$’s edges as 

$$e_1 = (v_1, w_1), e_2 = (v_2, w_2), \ldots, e_k = (v_k, w_k),$$

with the convention that $\gamma(e_k) = k$. We define $\Gamma_M = (V_M, E_M, \gamma_M)$ from $\Gamma$ by replacing each edge

$$e_k = \cdot \leftarrow v_k \cdot , \quad k \in [K]$$

by the following sequence of $p$ edges (with $p - 1$ new vertices):

$$p_k = \begin{cases} \cdot \leftarrow \cdot \leftarrow v_{k-1} \cdot \leftarrow v_k \cdot & \text{if } k \leq K_1 \text{ or } k > K_1 + K_2, \\ \cdot \leftarrow \cdot \leftarrow v_{K_1} \leftarrow v_{K_1+1} \cdot & \text{if } k \in [K_1 + 1, K_1 + K_2]. \end{cases} \tag{28}$$

Thus the vertex set $V_M$ consists of the vertices of $\Gamma$, with an additional $p - 1$ new vertices for each $e_k$. The edges are denoted $e_{k,i}$, where $k \in [K]$ and $i \in [p]$, so that $\gamma_M(e_{k,i}) = (k, i)$, as illustrated in (28) and (29). We take the alphabetical order on the set of pairs $(\ell, i)$, i.e. $(\ell, i) < (\ell', i')$ if and only if either $\ell < \ell'$, or $\ell = \ell'$ and $i < i'$. 

**Definition 16** (Linearized Matrix). Let us denote $\tilde{K}_1 = (K_1 + K_2)p$ and $\tilde{K}_2 = K_3p$. Define the matrices $B_1 \in \mathbb{M}_N(\mathbb{C})^{\otimes \tilde{K}_1}$ and $B_2 \in \mathbb{M}_N(\mathbb{C})^{\otimes \tilde{K}_2}$ as

$$B_1 = \left( \begin{pmatrix} U_{\delta(1)}^{(N)} \end{pmatrix}^{\epsilon(1)} \otimes \cdots \otimes \begin{pmatrix} U_{\delta(p)}^{(N)} \end{pmatrix}^{\epsilon(p)} \right)^{\otimes (K_1 + K_2)}$$

and

$$B_2 = \left( \begin{pmatrix} V_{\delta(1)}^{(N)} \end{pmatrix}^{\epsilon(1)} \otimes \cdots \otimes \begin{pmatrix} V_{\delta(p)}^{(N)} \end{pmatrix}^{\epsilon(p)} \right).$$

We define the matrix $A_M = B_1 \otimes B_2$. 

We note that, by definition of $\text{Tr}_{N, \Gamma}$ (i.e., [5]), the edges $e_1, \ldots, e_{K_1}$ in $\Gamma$ are associated with the matrices $U_{\delta(1)}^{(N)}$, the edges $e_{K_1 + 1}, \ldots, e_{K_1 + K_2}$ are associated with the matrices $U_{\delta(p)}^{(N)}$, and $e_{K_1 + K_2 + 1}, \ldots, e_K$ are associated with the $V_{\delta(p)}^{(N)}$. Thus, by comparing the definition of the $*$-monomial $M$ with the matrices $B_1$ and $B_2$ in the above definition, it is clear that (26) holds. We refer to the passage following [Mal20, Definition 1.7] for more details. 

**Remark 26.** It can be noted that $\Gamma$ and $\Gamma_M$ have the same forest of two-edge connected components, up to replacing every cutting edge by a sequence of $p$ consecutive cutting edges. In particular, $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma_M)$. 


4.2.2. Reduction via Injective Traces and Splitting Lemma. Given two linear graphs $\tilde{T}$ and $T'$, we denote $T' \geq \tilde{T}$ if $T'$ is a quotient of $\tilde{T}$. According to (12),
\[
\mathbb{E} \left[ \text{Tr}_{N,T}^N (A_M) \right] = \sum_{T' \geq T_M} \mathbb{E} \left[ \text{Tr}_{N,T'}^0 (A_M) \right].
\] (30)
From this point on until the end of the proof of (25), we fix a linear graph $T' = (E', V', \gamma')$ such that $T' \geq T_M$.

By Remarks 22 and 26, we see that $N - \frac{L(T)}{2} = N - \frac{L(T_M)}{2} \leq N - \frac{L(T')}{2}$. Consequently, by (30), in order to prove (25) it suffices to establish that $N - \frac{L(T')}{2} \mathbb{E} \left[ \text{Tr}_{N,T'}^0 (A_M) \right] = o(1).$ (31)
In order to do so, we must understand the contributions of the families $U_N$ and $V_N$ to $\text{Tr}_{N,T'}^0 (A_M)$.

**Definition 17.** For $j = 1, 2$, let us denote by $T_j$ the linear graph obtained from $T'$ by:
- considering only the edges numbered $(k,i)$ for $k = 1, \ldots, \tilde{K}_1$ and $i = 1, \ldots, p$ for $T_1$,
- considering only the edges numbered $(k,i)$ for $k = \tilde{K}_1+1, \ldots, \tilde{K}_1 + \tilde{K}_2$ and $i = 1, \ldots, p$ for $T_2$,
and deleting all other edges. Hence $T_1$ is of order $\tilde{K}_1$, whereas $T_2$ is of order $\tilde{K}_2$. Note that $V'$, the vertex set of $T'$, is also the vertex set of $T_1$ and $T_2$.

The following result, which is a direct application of [Mal20, Lemma 2.21], splits the term $\mathbb{E} \left[ \text{Tr}_{N,T'}^0 (A_M) \right]$ into two injective traces involving the matrices in $U_N$ and $V_N$ separately.

**Lemma 4.1.** With the notation of Definitions 16 and 17, we have that
\[
\mathbb{E} \left[ \text{Tr}_{N,T'}^0 (A_M) \right] = \frac{(N - |V'|)!}{N!} \times \mathbb{E} \left[ \text{Tr}_{N,T_1}^0 (B_1) \right] \times \mathbb{E} \left[ \text{Tr}_{N,T_2}^0 (B_2) \right].
\] (32)
**Proof.** As per Remark 25, we may assume that the matrices in $V_N$ are $S_N$-invariant. Suppose first that we can write
\[
V_l^{(N)} = V_{l,1}^{(N)} \otimes \cdots \otimes V_{l,K_3}^{(N)}, \quad \ell = 1, \ldots, L
\]
for some $S_N$-invariant $N \times N$ matrices $V_{l}^{(N)}$. Then, we can write $B_1$ and $B_2$ as tensor products of $N \times N$ matrices, where the matrices in $B_1$ are independent of those in $B_2$. In this case the result follows directly from [Mal20, Lemma 2.21] (therein, $T_N^0[T' \cdot \cdot \cdot]$ is used to denote $\frac{1}{N} \text{Tr}_{N,T'}^0$, and the vertex sets $V_1$ and $V_2$ can be taken to be both equal to $V'$, since we allow connected components consisting of a single vertex in $T_1$ and $T_2$).
Since $\text{Tr}^0_{N,T'}$ and the expression of $\text{Mal20}$, Equation (2.14) are linear, we conclude that the result holds for general $B_1 \otimes B_2$ by representing the latter as a sum of tensor products of $S_N$-invariant $N \times N$ matrices.

The proof of (25) is therefore reduced to showing that

$$N^{-\mathcal{L}(T')/2} \mathbb{E}[\text{Tr}^0_{N,T'}(A_M)] = N^{\eta(T')} (1 + o(1)) \times \prod_{i=1,2} N^{-\mathcal{L}(T_i)/2} \mathbb{E}[\text{Tr}^0_{N,T_i}(B_i)] = o(1)$$

(33)

as $N \to \infty$, where

$$\eta(T') = \frac{1}{2} (\mathcal{L}(T_1) + \mathcal{L}(T_2) - \mathcal{L}(T') - 2|V'|).$$

(34)

Note that the quotient relation between linear graphs induces a partial order that makes the set of linear graphs of a fixed order a lattice. Thus, although $T_2$ may contain trivial components, the same argument used in (12) yields

$$\mathbb{E}[\text{Tr}^0_{N,T_2}(B_2)] = \sum_{\tilde{T} \geq T_2} \mathbb{E}[\text{Tr}^0_{N,\tilde{T}}(B_2)].$$

(35)

This then implies by Möbius inversion $\text{Sta12}$, Proposition 3.7.2 that

$$\text{Tr}^0_{N,T_2}(B_2) = \sum_{\tilde{T} \geq T_2} \text{Mob}(T_2, \tilde{T}) \text{Tr}^0_{N,\tilde{T}}(B_2).$$

Since $T_2 \leq \tilde{T}$ implies that $N^{-\mathcal{L}(T_2)/2} \leq N^{-\mathcal{L}(\tilde{T})/2}$ (Remark 22), the assumption that $V_N$ satisfies the Mingo-Speicher bound (Definition 13) implies that the term $N^{-\mathcal{L}(T_2)/2} \mathbb{E}[\text{Tr}^0_{N,T_2}(B_2)]$ is bounded. Thus, (33) follows if we show that $\eta(T') \leq 0$, and that

$$N^{-\mathcal{L}(T_1)/2} \mathbb{E}[\text{Tr}^0_{N,T_1}(B_1)] = o(1).$$

(36)

This is the subject of Sections 4.2.3 and 4.2.4 respectively.

4.2.3. Analysis of $\eta(T')$. We begin with some definitions.

**Definition 18.** Let $\mathcal{C}$ be the set of connected components of the graphs $T_1$ and $T_2$, called *colored components*. Let $\mathcal{G} = (V, \mathcal{E})$ be the undirected graph, called the *graph of colored components*, defined as follows:

1. The vertices of $V$ are the connected components in $\mathcal{C}$.
2. Let $C_1, C_2 \in \mathcal{C}$ be connected components of $T_1$ and $T_2$, respectively. For every vertex $v \in V'$ of $T'$ that is in both $C_1$ and $C_2$, we associate an undirected edge in $\mathcal{E}$ connecting $C_1$ and $C_2$.

**Definition 19.** Let $\tilde{T} = (\tilde{V}, \tilde{E})$ be a graph. For every $v \in \tilde{V}$, we let $\deg_{\tilde{T}}(v)$ denote the number of edges in $\tilde{E}$ that are adjacent to $v$. 
Given that there is a one-to-one correspondence between the vertices of \( T' \) and the edges of \( \mathcal{G} \), it is easy to see that

\[
2|V'| = \sum_{C \in \mathcal{C}} \deg_{\mathcal{G}}(C).
\]

Hence, since \( \mathcal{L} \) is additive with respect to connected components, we can reformulate \( \eta(T') \) as

\[
\eta(T') = \frac{1}{2} \left( \sum_{C \in \mathcal{C}} \left( \mathcal{L}(C) - \deg_{\mathcal{G}}(C) \right) - \mathcal{L}(T') \right).
\]  

In order to analyze this quantity, we propose a modification of the graph \( \mathcal{G} \).

Let \( C_0 \in \mathcal{C} \) be a connected component with no cutting edge, and which is a leaf in the graph \( \mathcal{G} \). Since \( C_0 \) has no cutting edge, then \( \mathcal{L}(C_0) = 2 \).

Since \( C_0 \) is a leaf in \( \mathcal{G} \), the single edge in \( \mathcal{E} \) adjacent to it adds a contribution of 2 to the quantity \( \sum_{C \in \mathcal{C}} \deg_{\mathcal{G}}(C) \). In particular, if we remove \( C_0 \) and its adjacent edge from \( \mathcal{G} \), then the quantity \( \sum_{C \in \mathcal{C}} \left( \mathcal{L}(C) - \deg_{\mathcal{G}}(C) \right) \) remains unchanged.

Let \( \mathcal{G}_0 := \mathcal{G} \), and for every \( n \geq 1 \), let \( \mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n) \) be the graph obtained from \( \mathcal{G}_{n-1} \) by removing all connected components with no cutting edges that are leaves in \( \mathcal{G}_{n-1} \), as well as their adjacent edges. Clearly, there exists some \( m \geq 1 \) such that \( \mathcal{G}_m = \mathcal{G}_{m+1} = \mathcal{G}_{m+2} = \cdots \), namely, the first \( m \) such that \( \mathcal{G}_m \) has no leaf which is a connected component with no cutting edge. We refer to \( \mathcal{G}_m \) in the sequel as the pruning of \( \mathcal{G} \). By arguing as in the previous paragraph, we see that

\[
\eta(T') = \frac{1}{2} \left( \sum_{C \in \mathcal{V}_m} \left( \mathcal{L}(C) - \deg_{\mathcal{G}_m}(C) \right) - \mathcal{L}(T') \right). \tag{37}
\]

For every \( C \in \mathcal{V}_m \), let \( \mathcal{L}'(C) \) denote the number of leaves in \( \mathcal{F}(C) \) that do not contain a vertex that is attached to another connected component \( C' \in \mathcal{V}_m \setminus \{C\} \), and let \( d(C) := \mathcal{L}(C) - \mathcal{L}'(C) \) be the remaining leaves. We claim that for every \( C \in \mathcal{V}_m \),

\[
d(C) \leq \deg_{\mathcal{G}_m}(C) \quad \text{and} \quad \sum_{C \in \mathcal{V}_m} \mathcal{L}'(C) \leq \mathcal{L}(T'). \tag{38}
\]

Indeed, the first inequality holds since there are no connected components without cutting edges in \( \mathcal{V}_m \) (and thus \( \mathcal{L}(C) \) is actually equal to the number of leaves in \( \mathcal{F}(C) \)), and the second inequality is valid because every leaf counted by \( \mathcal{L}'(C) \) must already appear in \( \mathcal{F}(T') \). By combining (37) with (38), we finally conclude that \( \eta(T') \leq 0 \), as desired.
4.2.4. Limiting Injective Forms of Haar Unitary Matrices. To conclude the proof of (25), it now only remains to establish (36). For this, we must understand the asymptotic behaviour of the term $E\left[\text{Tr}_{N,T_1}(B_1)\right]$ for large $N$. In order to state the result we need, we recall a few more notions from graph theory.

**Definition 20.** Let $\tilde{T}$ be a linear graph. For every edge $e = (v, w)$ in $\tilde{T}$, we denote $e^t = (w, v)$.

1. A *path* of $\tilde{T}$ (also called a *walk*) is a sequence of edges $e_i$ of $\tilde{T}$, $i = 1, \ldots, n$, and an order of passage for each step $t_i \in \{1, t\}$ such that the target of $e_{t_i}^i$ is the source of $e_{t_{i+1}}^i$.
2. A *cycle* of $\tilde{T}$ (also called *closed walk*) is a path such that the target of $e_{t_n}^n$ is the source of $e_{t_1}^1$ (with the same notation as above).
3. A *circuit* of $\tilde{T}$ is a cycle where no edge is visited twice.
4. A *simple cycle* of $\tilde{T}$ is a cycle where no vertex is visited twice, except for the first (and last) vertex.
5. We say that $\tilde{T}$ is a *forest of cacti* whenever each edge belongs to exactly one simple cycle.
6. A forest of cacti is said to be *well oriented* when the edges of a same cycle follow the same orientation.

**Remark 27.** It is worth noting here that the notion of cactus presented in the above definition differs from an arguably more common definition, which is to assume that every edge belongs to *at most* one simple cycle.

**Definition 21.** Let $\tilde{T}$ be a well oriented forest of cacti of order $K$, and let $\tilde{\delta} : [K] \to [L]$ and $\tilde{\epsilon} : [K] \to \{1, \ast\}$ be two labellings of $\tilde{T}$’s edges.

1. If $\tilde{\delta}(k_1) = \cdots = \tilde{\delta}(k_n)$ for every simple cycle $e_{k_1}, \ldots, e_{k_n}$, then we say that $(\tilde{T}, \tilde{\delta}, \tilde{\epsilon})$ is *well colored*.
2. If every simple cycle of $\tilde{T}$ is of even size and the values of $\tilde{\epsilon}$ alternate along indices of each cycle (i.e., we can enumerate the edges of
every simple cycle \( e_{k_1}, \ldots, e_{k_{2n}} \) in such a way that \( \varepsilon(k_i) = 1 \) if \( i \) is even and \( \varepsilon(k_i) = \ast \) if \( i \) is odd), then we say that \((\tilde{T}, \tilde{\delta}, \tilde{\varepsilon})\) is alternated.

If \((\tilde{T}, \tilde{\delta}, \tilde{\varepsilon})\) is both well colored and alternated, we say that it is valid.

The following proposition, which is a special case of a more general result in [CDM16], is based on the Weingarten calculus [Col03, CS06]. It can also be derived from the limiting traffic distribution of a single Haar unitary matrix and the rule of traffic independence [Mal20].

**Proposition 4.2** ([Mal20] Proposition 3.7). Let \( \tilde{T} \) be a linear graph of order \( K \), and let \( \tilde{\delta} : [K] \to [L] \) and \( \tilde{\varepsilon} : [K] \to \{1, \ast\} \) be labellings of \( \tilde{T} \)'s edges. If we denote by \( c(\tilde{T}) \) the number of connected components of \( \tilde{T} \), then the limit

\[
\tau_U(\tilde{T}, \tilde{\delta}, \tilde{\varepsilon}) = \lim_{N \to \infty} N^{-c(\tilde{T})} \mathbb{E}\left[ \text{Tr}^{\tilde{T}}_{N,1} \left( U^{(N)}_{\tilde{\delta}(1)} \otimes \cdots \otimes U^{(N)}_{\tilde{\delta}(K)} \right) \right]
\]

exists and is finite. More precisely, we have that

\[
\tau_U(\tilde{T}, \tilde{\delta}, \tilde{\varepsilon}) = \mathbb{1}_{\{(\tilde{T}, \tilde{\delta}, \tilde{\varepsilon}) \text{ is valid}\}} \times \prod_{\text{\emph{\text{e}} \text{ simple cycle of} \ \tilde{T}}} (-1)^{k_{\varepsilon}-1} \frac{(2k_\varepsilon - 2)!}{(k_\varepsilon - 1)!k_\varepsilon!},
\]

where the above product is taken over all simple cycles of \( \tilde{T} \), and \( 2k_\varepsilon \) denotes the length of a particular cycle \( \varepsilon \).

We recall that the edges of \( T_1 \) are enumerated by pairs of the form

\( (k, i), \quad k \in [K_1 + K_2], i \in [p] \)

endowed with the alphabetical order. Moreover, we recall that the \( \ast \)-monomial \( M \) is written as

\[
M(X) = X^{\varepsilon(1)}_{\delta(1)} \cdots X^{\varepsilon(p)}_{\delta(p)}
\]

for some \( \delta : [p] \to [L] \) and \( \varepsilon : [p] \to \{1, \ast\} \) (see (27)). We note that \( \delta \) and \( \varepsilon \) naturally induce a labelling of \( T_1 \)'s edges (which we also denote as \( \delta \) and \( \varepsilon \) for simplicity) as follows: For every \( k \in [K_1 + K_2] \) and \( i \in [p] \), we let

\[
\delta(k, i) = \delta(i) \quad \text{and} \quad \varepsilon(k, i) = \varepsilon(i).
\]

Thus, it follows from Proposition 4.2 that

\[
\lim_{N \to \infty} N^{-c(T_1)} \mathbb{E}\left[ \text{Tr}^{\tilde{T}}_{N,1} (B_1) \right] = \mathbb{1}_{\{(T_1, \delta, \varepsilon) \text{ is valid}\}} \times \prod_{\text{\emph{\text{e}} \text{ simple cycle of} \ \tilde{T}_1}} (-1)^{k_{\varepsilon}-1} \frac{(2k_\varepsilon - 2)!}{(k_\varepsilon - 1)!k_\varepsilon!}.
\]

We remark that the renormalization of \( N^{-c(T_1)} \) in (40) is different from \( N^{-\Omega(T_1)/2} \), which is what we use in (36). However, it is clear from Definition 12 that \( \Omega(T_1) \geq 2c(T_1) \), and that there are cases where \( \Omega(T_1) = 2c(T_1) \) (for
instance when $T_1$ has no cutting edge). Therefore, the asymptotic (36) is proved if we show that $(T_1, \delta, \epsilon)$ is not a valid well oriented forest of cacti. In order to prove this, we compare some basic properties of valid well oriented forests of cacti with the structure that the nontrivial $\ast$-monomial $M$ imposes on $T_1$.

The first property of well oriented forests of cacti that is of interest to us is a type of nested simple cycle structure. This can be described effectively using noncrossing partitions.

**Definition 22.** A partition $\sigma \in P(n)$ ($n \in \mathbb{N}$) is said to be noncrossing if no two blocks cross each other, that is, there exist no two blocks $C \neq \tilde{C}$ in $\sigma$ and $i, j \in C$, $i, j \in \tilde{C}$ such that $i < \tilde{i} < j < \tilde{j}$. A block $C = \{i_1 < \cdots < i_m\}$ in a noncrossing partition is said to be inner if there exist another block $\tilde{C} = (\tilde{i}_1 < \cdots < \tilde{i}_n)$ such that $\tilde{i}_1 < i_1 < \tilde{i}_n$, which also implies that $\tilde{i}_1 < i_m < \tilde{i}_n$.

**Lemma 4.3.** Every connected component of a well oriented forest of cacti $\tilde{T}$ has a circuit that follows the orientation of $\tilde{T}$’s edges. In particular, if $c = (e_1, e_2, \ldots, e_n)$ is such a circuit, then the partition $\sigma_c$ defined by $i \sim_{\sigma_c} j \iff e_i$ and $e_j$ are in the same simple cycle

is a noncrossing partition with blocks of even size.

**Proof.** The existence of the circuit follows from the fact that each vertex in a forest of cacti must be contained in a unique simple cycle. Next, suppose by contradiction that there exists $i < i' < j < j'$ such that $i \sim_{\sigma_c} j$ and $i' \sim_{\sigma_c} j'$. Then, $e_i$ is part of a simple cycle that is entirely contained in the sequence $e_i, e_{i+1}, \ldots, e_{i'}, \ldots, e_j$, and which does not contain $e_{j'}$. However, since $i' \sim_{\sigma_c} j'$, this means that $e_{i'}$ must be part of two distinct simple cycles (one of which contains $e_{j'}$), which is a contradiction. \(\square\)

Let $\tilde{T}_0$ be the disjoint union of the graphs represented by the paths $p_k$ defined in (28)-(29) for $k \in [K_1 + K_2]$ and the vertices coming from the paths $p_k$ for $k > K_1 + K_2$ (i.e., we include all vertices that appear in the paths $p_k$ for $k > K_1 + K_2$, but we exclude the edges). With notations as in (28) and (29), for $k \leq K_1$ we call $v_k$ and $w_k$ respectively the source and the target of $p_k$, whereas for $k \in [K_1 + 1, K_1 + K_2]$ we interchange the role of the vertices, calling $v_k$ the target and $w_k$ the source of $p_k$. We recall (Definitions 15 and 17) that $T_1$ is a quotient of $\tilde{T}_0$.

With this in mind, we have the following consequence of Lemma 4.3.

**Lemma 4.4.** If $T_1$ is valid, then it is an edge-disjoint union of circuits that are compositions of the paths $(p_k)_{k \leq K_1 + K_2}$ (here, we say that two paths $p$ and $p'$ can be composed if the target of the last step of $p$ is the source of the first step of $p'$).
Proof. If \( T_1 \) is a forest of cacti, then there is no vertex of odd degree. In particular, every vertex of degree one in \( \tilde{T}_0 \) (which is either a target or source of a path \( p_k \)) is identified with at least one other such vertex in the quotient \( T_1 \). We may then first form a quotient of \( \tilde{T}_0 \) by composing the paths whose targets and sources have been identified as in \( T_1 \), and then obtain \( T_1 \) by adding more identifications.

Next, in order to better understand the structure that \( M \) imposes on \( T_1 \), we introduce the concept of a word induced by a path.

**Definition 23.** Let \( \tilde{T} \) be a graph of order \( K \) with two labellings \( \tilde{\delta} : [K] \to [L] \) and \( \tilde{\epsilon} : [K] \to \{1, *\} \). Let \( p = (e_{k_1}, \ldots, e_{k_n}) \) be a path of \( \tilde{T} \) which follows the edge orientations in \( \tilde{T} \). We denote by \( M_p \) the \( * \)-monomial

\[
M_p(X) = X_{\tilde{\delta}(k_1)}^{\tilde{\epsilon}(k_1)} \cdots X_{\tilde{\delta}(k_n)}^{\tilde{\epsilon}(k_n)}
\]

Lemma 4.3 has the following consequence.

**Lemma 4.5.** Let \((\tilde{T}, \tilde{\delta}, \tilde{\epsilon})\) be valid. If \( \epsilon \) is a circuit as in Lemma 4.4 of \( \tilde{T} \) which agrees with the edge orientations in \( \tilde{T} \), then \( M_\epsilon \) is trivial.

**Proof.** Suppose first that \( \epsilon \) is a simple cycle. Since \((\tilde{T}, \tilde{\delta}, \tilde{\epsilon})\) is valid, it is well colored and alternated, and thus we can write

\[
M_\epsilon(X) = X_t^* X_t^* \cdots X_t^* \text{ or } X_t^* X_t^* \cdots X_t^* X_t
\]

for some integer \( m \in \mathbb{N} \) and index \( \ell \in [L] \). Clearly, this \( * \)-word reduces to 1 when evaluated in unitary operators.

More generally, let us denote \( \epsilon = (e_1, \ldots, e_n) \), and let the noncrossing partition \( \sigma_\epsilon \in \mathcal{P}(n) \) be defined as in Lemma 4.3. Suppose that \( \tilde{\epsilon} \) denotes the (smaller) circuit obtained from \( \epsilon \) by removing the edges contained in any inner block of \( \sigma_\epsilon \). By repeating the argument used in the case where \( \epsilon \) was a simple cycle, it is easy to see that \( M_\epsilon(u) = M_{\tilde{\epsilon}}(u) \) for any family of unitary operators \( u \), since every inner block of \( \sigma_\epsilon \) corresponds to an uninterrupted simple cycle within \( \epsilon \). By removing each inner block from \( \sigma_\epsilon \) one by one, we are eventually left with a simple cycle, concluding the proof.

We now have all the necessary ingredients to conclude the proof of (25). Recalling from (27) that \( M = X_{s(1)}^{\epsilon(1)} \cdots X_{s(p)}^{\epsilon(p)} \), we define the mirrored word

\[
M_{\text{mrr}} = X_{s(p)}^{\epsilon(p)} \cdots X_{s(1)}^{\epsilon(1)}
\]

Suppose by contradiction that \((T_1, \delta, \epsilon)\) is valid. For each \( k \in [K_1 + K_2] \), it holds that \( M_{p_k} = M \) for \( k \leq K_1 \) and \( M_{p_k} = M_{\text{mrr}} \) for \( k \in [K_1 + 1, K_1 + K_2] \).

By Lemma 4.4, \( T_1 \) is quotient of a well oriented forest of cacti \( T'_1 \) such that if \( \epsilon \) is a circuit of a connected component of \( T'_1 \) which agrees with the edge orientations in \( T'_1 \), then it must be the case that \( M_\epsilon \) is a non commutative
product $\omega_\epsilon$ of powers of $M$ and $M_{\text{mirr}}$, such as $M^{\theta_1} M_{\text{mirr}}^{\theta_2} \cdots M_{\text{mirr}}^{\theta_{p-1}} M_{\text{mirr}}^{\theta_p}$ for $\theta_i \geq 1$.

Therefore, Lemma 4.5 implies that $\omega_\epsilon$ is trivial. Let $u = (u_1, \ldots, u_L)$ be a Haar unitary system. Since $M(u) \neq 1$ (as $M$ is a nontrivial $^*$-monomial), and by Nielsen-Schreier theorem, the group generated by $M(u)$ and $M_{\text{mirr}}(u)$ is either $\mathbb{Z}$ or $F_2$. The group cannot be $F_2$, because if that were the case, then $\omega_\epsilon$ would not be trivial. Thus, the group generated by $M(u)$ and $M_{\text{mirr}}(u)$ must be $\mathbb{Z}$. Consequently, without loss of generality, we can assume there exists a $k \in \mathbb{Z}$, such that $M = (M_{\text{mirr}}^k) = (M^k)_{\text{mirr}}$. But the mirror operation is an involution, so $M_{\text{mirr}} = M_k$ and then $M = M^{2k}$. Since $M$ is nontrivial this is absurd.

We therefore finally conclude that $(\tau_1, \delta, \epsilon)$ cannot be valid, whence (33) holds, as desired.

5. PROOF OF THEOREM 1.5

5.1. Asymptotic representation theory. This manuscript deals with asymptotic freeness for tensors. In the case of unitary operators, it is possible to turn this question into a problem of harmonic analysis over the free group. This is what we would like to discuss in this section.

Voiculescu established asymptotic freeness for sequences of group in the large dimension limit in [Voi91, Voi98]. However, long before his asymptotic freeness results in the nineties, he had already studied the limit of unitary groups, from the slightly different point of view of representation theory [SV75]. His result here generalized results of Thoma [Tho64], and it was discovered in [VK82] that finite dimension group approximation was a natural way to prove the results.

A way to reformulate some questions of this paper is: consider sequences of unital functions of positive type (sometimes called positive definite) $\phi_N$ on the unitary group $U_N$. Under which conditions will $\phi_N$ be asymptotically free almost surely for i.i.d. Haar unitary variables in $U_N$?

We restrict our question slightly further – without however missing any example provided by the results contained in this manuscript – by requiring in addition the limit to be Haar distributions, i.e. all non-trivial moments tend to zero. An important observation is that if $\phi_N$ satisfy this condition, then the same will hold true for any polynomial in these, as soon as it does not have any constant component. If one wants to ensure that this polynomial operation remains a state, it is enough to require that the coefficients of each product be non negative, and that they add up to 1. Indeed, in terms of representation theory, taking a product corresponds to a tensor product, and taking a barycenter (with rational coefficients) corresponds to taking
direct sums of representations. Put differently, this paper can be interpreted as saying that many states that are not tracial yield also asymptotic freeness.

5.2. **Asymptotic freeness for any representation.** We begin by proving the asymptotic freeness of \((U_N, U^*_N)\) with respect to arbitrary irreducible rational representations. We first recall a result of Mingo and Popa.

**Theorem 5.1** ([MP16]). The family \((U_N, U^*_N) := (U_1^{(N)}, \ldots, U_K^{(N)}, U_1^{(N)*}, \ldots, U_K^{(N)*})\), converges to a Haar unitary system almost surely and in expectation as \(N \to \infty\) in the space \((\mathbb{M}_N(\mathbb{C}), \text{tr}_N)\).

**Remark 28.** [MP16] states in more generality the result for unitarily invariant random matrices in the sense of expectation, together with the second order asymptotic freeness (see Corollary 20 and Proposition 38 therein). This implies that the variance of the \(*\)-distribution is of order \(N^{-2}\), and thus almost sure convergence.

It is known that irreducible representations of \(\mathfrak{U}(N)\) are in a one to one correspondence with the signatures associated with their characters, i.e., sequences \(\lambda_1 \geq \ldots \geq \lambda_N\) of integers. If \(\lambda_N \geq 0\), then the associated representation is polynomial, otherwise it is rational.

For the purpose of asymptotics, it is convenient to characterize the irreducible representation by a pair of Young tableaux (known as the signatures, e.g., [Z73]). That is, given a sequence \(\lambda_1 \geq \ldots \geq \lambda_N\) of integers, if we let \(l \in [N]\) be the largest index such that \(\lambda_l \geq 0\), then the data

\[(\lambda, \mu) := (\lambda_1, \ldots, \lambda_l), (\mu_1, \ldots, \mu_{N-l})\]

with \(\lambda_1 \geq \ldots \geq \lambda_l \geq 0\) and

\[\mu_1 = -\lambda_N \geq \mu_2 = -\lambda_{N-1} \geq \ldots \geq \mu_{N-l} = -\lambda_{l+1} > 0\]

characterizes the rational irreducible representation. Calling \(l(\lambda)\) (resp. \(l(\mu)\)) the length, i.e., the number of non-zero elements of the sequence of integers \((\lambda_1, \ldots, \lambda_l)\) (resp. \((\mu_1, \ldots, \mu_{N-l})\)), we have that \(l(\lambda) + l(\mu) \leq N\). In other words, in order to pass from the highest weights in the Cartan-Weyl theory to the representation with signatures \((\lambda, \mu)\), one has to pad “zero” highest weights in the middle of the sequence in a unique way to ensure completion into a non-increasing sequence of \(N\) integers, as follows:

\[\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_{N-l} = \lambda_1, \ldots, \lambda_{l(l(\lambda))}, \underbrace{0, \ldots, 0}_{N-l(l(\lambda))-l(\mu)}, \mu_1, \ldots, \mu_{l(\mu)}\]

Conversely, a pair of tableaux \((\lambda, \mu)\) characterizes a rational irreducible representation of the \(N\)-dimensional unitary group as soon as \(l(\lambda) + l(\mu) \leq N\); indeed, for fixed choices of \((\lambda, \mu)\), we are interested in the behaviour of the
sequence of irreducible representations of $\mathcal{U}(N)$ associated to $(\lambda, \mu)$ when $N \to \infty$.

**Proof of Theorem 1.5 Part 1.** Given a non-trivial $^*$-monomial $M$, it follows from Theorem 5.1 that $\text{tr}_{N}(M(U_N, \overline{U}_N)) \to 0$ in expectation as $N \to \infty$. In addition, according to [EI16, Lemma 3.5],

$$\chi_{\lambda, \mu}(U) = \left(\text{tr}_{N}(U)\right)^{1(\lambda)}\left(\text{tr}_{N}(\overline{U})\right)^{1(\mu)}\left(1 + \mathcal{O}(N^{-1})\right), \quad U \in \mathcal{U}_N$$

where the error term $\mathcal{O}(N^{-1})$ is uniform in $U \in \mathcal{U}_N$. This concludes the proof of asymptotic freeness with respect to $\chi_{\lambda, \mu}$. □

5.3. **Asymptotic freeness with amalgamation.** We now conclude this section by proving the statement in Theorem 1.5 regarding asymptotic freeness of $\mathcal{U}^\otimes N$ with amalgamation over $S_d$. Let $K \in \mathbb{N}$. Given $2K$ unitary matrices $U_1, \ldots, U_{2K} \in M_N(\mathbb{C})^\otimes d$, we consider the representation of the group $\mathbb{F}_{2K} \times S_d$, where each generator $u_i$ of the free group $\mathbb{F}_{2K}$ is sent to $U^\otimes d_i$, and $\sigma \in S_d$ acts by permutation of legs of the tensor (we use $\rho^{(N)}(\cdot)$ to denote the function that maps each permutation $\sigma$ to the associated matrix $\rho^{(N)}(\sigma)$ that permutes the legs of the tensor).

**Theorem 5.2.** The map $w_1 \to U_1, \ldots, w_K \to U_K, w_{K+1} \to U^{1}_{1}, \ldots, w_{2K} \to U^{1}_{2K}$ extends to a random representation of the free group on $2K$ generators $\mathbb{F}_{2K}$ in $(\mathbb{C}^N)^{\otimes d}$. Likewise, the map $\sigma \to \rho^{(N)}(\sigma)$ yields a representation of $S_d$ in $(\mathbb{C}^N)^{\otimes d}$ and these two representations commute, therefore we have a random representation of the group $\mathbb{F}_{2K} \times S_d$. This random representation converges pointwise to the character associated to the left regular representation of $\mathbb{F}_{2K} \times S_d$ as $N \to \infty$ (i.e. 1 for the neutral element, and zero for all others).

**Proof.** It is enough to prove that for a non trivial word $w \in \mathbb{F}_{2K} \times S_d$, $\text{tr}_{N}^{\otimes d}\rho^{(N)}(w) \to 0$. If the $\sigma$ component is not the identity, the character is bounded above by a product of normalized traces of unitaries times $N^{-|\sigma|}$ therefore it goes to zero. If $\sigma$ component is the identity, then the value of the character is $\text{tr}_{N}(\cdot)^d$, where $\cdot$ is obtained from the random representation of $\mathbb{F}_{2K}$. In this case, asymptotic freeness is known, and this quantity converges either to zero or one, depending on whether the word is trivial or not. The power $d$ of this quantity converges to the same limit and this concludes the proof. □

**Remark 29.** Note that we may as well say that this is a matrix model for the group $\mathbb{F}_{2K} \times S_d$, or microstates (we refer to to the book [MS17] for a comprehensive introduction to microstates; see also [AGZ10]).

From Theorem 5.2 we then obtain the desired result as corollary:
Proof of Theorem 1.5 Part 2. The asymptotic freeness of $U_N \otimes S_d$ with amalgamation over $S_d$ follows directly from Theorem 5.2, modulo the following two facts:

1) $tr_N \otimes d$ restricted to $S_d$ converges to the regular trace, as a consequence of Theorem 5.2 (see also [Col03]).

2) The free product of $\mathbb{Z} \times S_d$, $2K$-times, amalgamated over $S_d$ under the canonical identification, is isomorphic to $\mathbb{F}_{2K} \times S_d$.

(As a remark, we point out that this result yields another proof of the asymptotic freeness with respect to arbitrary characters, which we have proved in Section 5.2.)

□

6. DISCUSSION

6.1. Strong Asymptotic Freeness. Given that absorption properties regarding asymptotic $*$-freeness of tensor products hold with rather general assumptions, it is natural to wonder if a similar phenomenon occurs with strong asymptotic freeness. Unfortunately, the following counterexample shows that strong asymptotic $*$-freeness is not as easily preserved by tensor products.

Example 3. Let $U_1^{(N)}, \ldots, U_L^{(N)}$ be independent $N$ by $N$ Haar unitary random matrices, where $L \geq 2$. We claim that $U_1^{(N)} \otimes U_1^{(N)}, \ldots, U_L^{(N)} \otimes U_L^{(N)}$ are not strongly asymptotically $*$-free. To see this, let $u_1, \ldots, u_L$ be the limits in $*$-distribution of the $U_i^{(N)}$, and let $v_1, \ldots, v_L$ be the limits of the $\overline{U_i^{(N)}}$. If strong asymptotic $*$-freeness holds, then

$$\left\|U_1^{(N)} \otimes \overline{U_1^{(N)}} + \cdots + U_L^{(N)} \otimes \overline{U_L^{(N)}}\right\| \to \left\|u_1 \otimes v_1 + \cdots + u_L \otimes v_L\right\|.$$

According to Fell’s absorption principle (in particular, Proposition 1.1), the fact that the $u_i$ are $*$-free Haar unitary variables implies that

$$\left\|u_1 \otimes v_1 + \cdots + u_L \otimes v_L\right\| = \left\|u_1 + \cdots + u_L\right\| = 2\sqrt{L-1}.$$

For each $N$, let $e_1, \ldots, e_N$ denote the canonical basis of $\mathbb{C}^N$, and let us define $\xi_N = (e_1 \otimes e_1 + \cdots + e_N \otimes e_N)$. It is easy to see that for any unitary matrix $U$, $(U \otimes U)\xi_N = \xi_N$. Therefore,

$$\left\|U_1^{(N)} \otimes \overline{U_1^{(N)}} + \cdots + U_L^{(N)} \otimes \overline{U_L^{(N)}}\right\| \geq \sqrt{L},$$

which is a contradiction as soon as $L > 2$. For the case $L = 2$ we cannot derive a contradiction immediately, however we can get one along the same lines by exhibiting three or more free elements in the free group generated by two elements, and reason along the same lines as the argument above.
6.2. Renormalizations. In the framework of traffic spaces \cite{Mal20}, one considers families of random matrices $A_N = (A_j)_{j \in J}$ such that

$$\tau_{A_N}(j, T) := N^{-c(T)/2} E \left[ \text{Tr}_{N,T}(A_{j_1} \otimes \cdots \otimes A_{j_n}) \right]$$

is of order 1 for large $N$, where $c(T)$ is the number of connected components of $T$; see for instance Proposition 4.2. This is very different from the renormalization which naturally arises in this article, namely by defining

$$\zeta_{A_N}(j, T) := N^{-\mathcal{E}(T)/2} E \left[ \text{Tr}_{N,T}(A_{j_1} \otimes \cdots \otimes A_{j_n}) \right],$$

where we recall that $\mathcal{E}(T)$ is the number of leaves of the tree of two-edge connected components of $T$ (Definition 12). It is interesting to note that our main result leads to an analogue of the asymptotic traffic independence in this regime:

**Proposition 6.1.** Let $A_N$ and $B_N$ be two independent $S_N$-invariant families of random elements of tensor matrix spaces, as in Definition 13, and let $\zeta_{A_N}$ and $\zeta_{B_N}$ be defined as in (42). If $A_N$ and $B_N$ satisfy the Mingo-Speicher bound, then so do the joint family $A_N \cup B_N$. If $\zeta_{A_N}$ and $\zeta_{B_N}$ converges pointwise as $N \to \infty$, then so does $\zeta_{A_N \cup B_N}$.

**Remark 30.** The limit $\zeta_{A_N \cup B_N}$ depends only on $\zeta_{A_N}$ and $\zeta_{B_N}$, and it differs from the so-called traffic free product of the individual distributions.

**Proof of Proposition 6.1.** We denote by $\zeta_{A_N}^0$ the function defined as $\zeta_{A_N}$ with $\text{tr}_{N,T}^2$ instead of $\text{tr}_{N,T}$. Then $\zeta_{A_N}$ is bounded if and only if $\zeta_{A_N}^0$ is bounded since they are related by Möbius formulas and by Lemma 3.5. Let $B_1 = A_{j_1} \otimes \cdots \otimes A_{j_n}$ and $B_2 = B_{j_1'} \otimes \cdots \otimes B_{j_{m'}}$, for some indices $j_k$’s and $j'_k$’s. From (30), changing only the definitions of $B_1$ and $B_2$ the computation remains valid, and we get: for any linear graph $T$,

$$\zeta_{A_N \cup B_N}^0(j \cup j', T) = 1(\eta(T) = 0) \zeta_{A_N}^0(j, T_1) \zeta_{B_N}^0(j', T_2) + o(1),$$

where $T_1$ and $T_2$ are the subgraphs of $T$ consisting of edges associated with $A_N$ and $B_N$ respectively, and we recall that $\eta(T)$ is defined as in (34). □

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