The geometry of quantum states is an old and a rich
topic \[^1\]. In this paper, I explore the space of unitary
orbits, which I define following a recent paper by Boya
and Dixit \[^2\] as the following set
\[ \varrho \equiv \{ U \rho U^\dagger \} \ \forall \ U, \] (1)
where \( \rho \) is a density matrix and \( U \) is an unitary oper-
ator in the same space, and \( \varrho \) represents the set of all
states that are connected by all possible unitary oper-
ations; hence I call it an unitary orbit.

An unitary transformation is a transformation of ba-
sis, which leaves the spectrum unchanged. Any two den-
sity matrices that share a spectrum, belong to a single
unitary orbit; conversely if \( \rho_1 \in \varrho \) and \( \rho_2 \in \varrho \), then
\( \text{Spec}(\rho_1) = \text{Spec}(\rho_2) \). Therefore an unitary orbit is
uniquely identified by the spectrum of the states it con-
tains. Consequently, all states on an unitary orbit will
also have the same scalar measures such as the von Neu-
mann entropy and purity. On the other hand, the analy-
sis here has more depth than a study of scalar measures
alone because two different unitary orbits can share a sin-
gle value for entropy or purity \[^2\]. It may be tempting
to simply study the space of the eigenvalues. In fact that
is all I am doing, but I am taking advantage of the nice
properties of unitary transformations to connect various
density states.

Let me now briefly discuss the familiar example of the
space of one qubit represented by the points inside
of the Bloch sphere, shown on the left in Fig 1. The outer
most shell of the Bloch sphere represents the set pure
states, the center point represents the fully mixed state,
and everything else is represented by the space inside of
the sphere. The states on each of the spherical shells of
the Bloch sphere are connected to each other via unitary
transformations. To construct the geometry of unitary
orbits, I contract each shell of the Bloch sphere into a
single point. That is, all of the states that are unitarily
connected are now represented by an unitary orbit. The
geometry for the unitary orbit, for a qubit, simplifies to
line, shown on the right in Fig 1. The two end points
of the line are the fully mixed state labeled as \( O \) and the
set of all pure states labeled as \( P \).

One can carry out such a procedure any dimensional
systems. For a qutrit, a three-level system, the geometry
of states is given by an eight dimensional manifold. The
geometry of unitary orbit, however, is given by a triangle
\[^2\]. In general the manifold that represents the set of
state for a \( d \)-dimensional system grows as \( d^2 - 1 \) \[^2\],
while the manifold of unitary orbit grows linearly \( d - 1 \).
Furthermore the manifold for unitary orbits is a simplex
with \( d \) vertices. One can think of each unitary orbit as a
\( d + 1 \) dimensional manifold contracted to a point.

In this paper I want to analyze the space of unitary
orbits for composite systems with three different cases
in mind. First, I want look at the unitary orbits of a
composite system that contain product states, followed
by classically correlated states, and finally entanglement
within unitary orbits. Before I proceed, I should remark
that generally in quantum information theory one is in-
terested in local operations. The unitary orbits of com-
posite system that I consider here are due to the actions
of global unitary transformations.

\[ \text{Theorem.} \] The subspace of unitary orbits contain-
ing product states a \( \sum_i d_i - n \) dimensional surface in
a \( \prod_i d_i - 1 \) dimensional manifold, where \( d_i \) is the dimen-
sions of \( i \)th subsystem.

\[ \text{Proof.} \] Suppose that an unitary orbit, \( \varrho_p \), contains
a \( n \)-partite product state written as
\( \rho_p = \bigotimes_i \rho_i^{(i)} \equiv \]
$\rho^{(1)} \otimes \rho^{(2)} \otimes \rho^{(3)} \otimes \cdots \otimes \rho^{(n)}$. Diagonalize each of the subparts above by local unitary transformations, $u^{(i)}$, to yield $\rho_{pd} = \bigotimes_{i} u^{(i)} \rho^{(i)} u^{(i)\dagger} = \rho^{(i)}_d$. The spectrum of $\rho_{pd}$ is simply the outer product of the spectrums of each of the subsystem. Local unitary transformations are a subset of global unitary transformations, therefore $\rho_p$ and $\rho_{pd}$ belong to the same unitary orbit and share the same spectrum. In fact, any unitary orbit that contains a product states has a spectrum that is factorable in terms of the spectrum of the subparts. Each of the subparts above has $d_i - 1$ independent parameters (eigenvalues); the total number of independent parameters for each of the subpart, $\sum_i (d_i - 1) = \sum_i d_i - n$. While the number of independent parameters for a generic unitary orbit is given by $\prod_i d_i - 1$. \( \square \)

Let me illustrate this with a simple example of a two-qubit composite system. But first, a word of caution is necessary. If an unitary orbit contains a product state that does not mean that all states on the that orbit are of product form. A simple example is a maximally entangled state: it is a pure state belonging to the unitary orbit of pure states, yet it is not of product form. But, of course, the unitary orbit containing the maximally entangled state will also contain pure product states.

The smallest composite system is the two-qubit system with $d = 4$. The space of the unitary orbits for the two-qubit case is confined to a tetrahedron the three-dimensional space $\mathbb{C}^4$. I am interested to identify the region of the tetrahedron that is occupied by unitary orbits that contain product states. To solve this problem I need to match the eigenvalues of a generic two-qubit state to the eigenvalues of a qubit state in outer product with itself

$$S\text{pec}(\rho^{AB}) = S\text{pec}(\rho^A) \times S\text{pec}(\rho^B)$$

$$\begin{align*}
1 + a + b + ab & \quad \begin{cases} 1 + \sqrt{6}a + \sqrt{2}b + z \\ 1 - \sqrt{6}a + \sqrt{2}b + z \\ 1 - 2\sqrt{2}a + z \\ 1 - 3z 
\end{cases} \\
1 + a - b - ab & \quad \begin{cases} 1 + \sqrt{6}a - \sqrt{2}b + z \\ 1 - \sqrt{6}a - \sqrt{2}b + z \\ 1 - 2\sqrt{2}a + z \\ 1 - 3z 
\end{cases} \\
1 - a + b - ab & \quad \begin{cases} 1 + \sqrt{6}a + \sqrt{2}b + z \\ 1 - \sqrt{6}a + \sqrt{2}b + z \\ 1 - 2\sqrt{2}a + z \\ 1 - 3z 
\end{cases} \\
1 - a - b + ab & \quad \begin{cases} 1 + \sqrt{6}a - \sqrt{2}b + z \\ 1 - \sqrt{6}a - \sqrt{2}b + z \\ 1 - 2\sqrt{2}a + z \\ 1 - 3z 
\end{cases}
\end{align*}$$

Above the spectrum of a two-qubit states is given in terms of three parameters as $\mathbb{C}^4$. The constraint of Eq. \( \ref{eq:spec} \) leads to equation $z^2 + z + \frac{1}{\sqrt{2}}(z - 1) (\sqrt{3}x + y) + 3xy - y^2 = 0$ with two solutions for $z$. We only consider the positive solution, plotted in Fig. \( \ref{fig:tetrahedron} \). The solution of positive $z$ shows that the set of unitary orbits containing products states are confined to a two-dimensional plane in a three-dimensional manifold.

Note that the surface of unitary orbits containing product states runs along two of the six possible sides. The surface then divide the total space into two regions. This is a necessary condition for the plane containing unitary orbits with product states. I shall prove this now.

I would like make a few remarks about unitary orbits with product states embedded within the space all unitary orbits. Once again, the dimensions for the unitary orbit manifold is given by the product of dimensions of each subspace, while the dimensions of the manifold that represent the unitary orbits that contain product states is given by the sum of the number of independent parameters of each subspace. The surface of unitary orbits containing product states for $d = 6$ (a qubit-qutrit system) is a 3-dimensional surface in a 5-dimensional volume. For $d = 8$ (three qubit system) case with, the surface of unitary orbits containing product states is also a 3-dimensional surface, but in a 7-dimensional volume. Roughly speaking, the set of correlated states in the three-qubit system is much higher than the correlations in the qubit-qutrit system for product states as the base measure. The space of unitary orbits grows much faster than the subspace that contains product states; this should be cautionary statement for an analysis that depends on only product states; as it may lack generality.

Next I want to look at the case of classically correlated states (not separable states), as defined by Henderson and Vedral, and independently by Ollivier and Zurek. \( \rho_{cl} = \sum_{i=1}^{d_A} a_i |i\rangle \langle i| \otimes \rho^B. \) Where $a_i$ are classical weights satisfying $\sum_i a_i = 1$ and $d_A$ is dimensions of subsystems $A$.

**Theorem.** All bipartite-unitary orbits contain classically correlated states.

**Proof.** The classically correlated state is block diagonal, but it would fruitful to fully diagonalize it yielding the eigenvalues of any the unitary orbit containing it. Consider a set of unitary operators in the space of $B$ that diagonalize the ith state, $\rho^B_i \rightarrow u^B_i \rho^B_i u^B_i\dagger = \sum_j b^B_{ij} |j\rangle \langle j|$, where $b^B_{ij}$ is the jth eigenvalues of $\rho^B_i$. Let me now construct the following control-unitary operator $U_{cd} = \sum_i |i\rangle \langle i| \otimes u_i$. Applying $U$ to $\rho_{cl}$ yields a diagonal
transformation there exists an operator \( W \) and the basis \( \lambda_i \) where \( a \) are eigenvalues \( \lambda_i \) and \( b_j^{(i)} \) are the eigenvalues of state \( \rho_{cl, diag} \), and therefore the this is the form of the eigenvalues a classically correlated state and of the unitary orbit to which contains a classically correlated state.

Consider an arbitrary state \( \rho = \sum_{ij} \lambda_{ij} |v_{ij}\rangle \langle v_{ij}| \), with eigenvalues \( \lambda_{ij} \) and (generally not separable) basis \( |v_{ij}\rangle \), where \( i \) and \( j \) run up to the dimensions of spaces \( A \) and \( B \) respectively. By the definition of an unitary transformation there exists an operator \( W \) that connects the basis \( \{|ij\rangle\} \) to the basis \( \{|v_{ij}\rangle\} \), as \( W |v_{ij}\rangle = |ij\rangle \).

Next, rewrite the eigenvalues \( \lambda_{ij} \) as \( \lambda_{ij} = a_i b_j^{(i)} \), where \( a_i \equiv \sum_j \lambda_{ij} \) and \( b_j^{(i)} \equiv \lambda_{ij} / a_i \). By definition both \( a_i \) and \( b_j^{(i)} \) are positive numbers less than (or equal to) 1 satisfying the conditions \( \sum_j b_j^{(i)} = \sum_j \lambda_{ij} / a_i = \sum_j \lambda_{ij} / a_j = 1 \) and \( \sum_i a_i = \sum_i \lambda_{ij} = 1 \). The two conditions are the trace condition for \( \rho_{cl,diag} \) and its subsystems. And finally we can connect a generic state \( \rho \) to a classically correlated state \( \rho_{cl} \) by the aid unitary transformations \( U_{cd} \) and \( W \).

\[
U_{cd}^\dagger W \rho W U_{cd} = U_{cd}^\dagger \left( \sum_{ij} a_i b_j^{(i)} |ij\rangle \langle ij| \right) U_{cd} = U_{cd}^\dagger \rho_{cl, diag} U_{cd} = \rho_{cl}. \tag{4}
\]

This completes the proof that any unitary orbit containing a generic state \( \rho \) also contains a classically correlated state \( \rho_{cl} \). \( \Box \)

There is no real difference between the space \( AB \) and space \( BA \) apart from labeling; the result above can be extended for classical correlated states in the space of \( B \) by swapping the definitions for \( a_i \) and \( b_j^{(i)} \).

Before I conclude this paper I would like to discuss unitary orbits that contain states with entanglement briefly. Above I showed that each unitary orbit contains classically correlated states, which are unentangled. To perform this analysis on equal footing I consider the maximum entanglement an orbit can posses. Verstraete, Audenaert, and De Moor [2] have calculated the states with maximally entanglement for two qubits. They further showed that these state maximize Negativity [8], concurrence [9, 10], and relative entropy [11].

In Fig. 3 I have plotted equi-negative surfaces within the tetrahedron. Note in the region near the fully mixed state there is no entanglement however almost all unitary orbits do not contain product states (see Fig. 2). This is a simple and method to see the visualize how entanglement is distributed in the state space. There are more applications of the tools developed above for studying entanglement. However I will refrain myself from discussing them presently. These topics will be discussed in greater detail in an upcoming paper [12].

The concept of unitary orbit is rather simple, yet it has allowed much insight into how multipartite systems are correlated. I showed that the set of unitary orbits containing product states is a very small subset of set of all unitary orbits. Which in return means that the set product states is a very small subset of set of all states. Any realistic model of a multipartite quantum system should not assume that the parts are simplify separable. That assumption leads to gross simplification which could undermine a great deal of physical effects. This was one the major goal of this paper. There has been a great deal of discussion regarding initially product state assumption in open quantum dynamics [13, 14, 15, 16]. I have not definitively shown that when initially product state assumption is retained one is restricted to a small subspace of the total state space.

The other immediate application of the calculations above is seen in the derivation of the non-Markovian master equation due Rodríguez-Rosario and Sudarshan [17]. In their derivation they start with a generic state of the system and the environment, which is assumed to be of product form at some point in its (unitary) history. This assumption cannot hold for a generic state. However, now that we know that all unitary orbits contain classically correlated states. The derivation is fully generalized by simply making the substitution of classically correlated states in place of product states in Rodríguez-Rosario and Sudarshan’s derivation.

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I have made one additional contraction by ordering the eigenvalues from large to small (top to bottom in Eq. 2). In that sense, I have contracted twenty-four tetrahedrons (permutations of four) into one tetrahedron [2]. The negative solution therefore belongs to a space that has been contracted by permutation symmetry.

If an $a_i = 0$ for $i = k$, that means that all $\lambda_{kj} = 0$ for all $j$. The only requirement we have so far set is that $\lambda_{ij} = a_i b_j^{(i)}$. When the l.h.s is zero, $a_i$ on the r.h.s is also zero, and $b_j^{(i)}$ can be anything. Another way to look at this is by considering $a_i$ as classical weights. When the classical weight is zero then the corresponding term for the subsystem $B$ can be anything or neglected altogether.