CHARACTERIZING THE GAUSSIAN COHERENCE BREAKING CHANNEL AND ITS PROPERTY WITH ASSISTANT ENTANGLEMENT INPUTS

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ABSTRACT. We give a characterization of arbitrary $n$-mode Gaussian coherence breaking channels (GCBCs) and show that the tensor product of a GCBC with an arbitrary Gaussian channel maps all input states into product states. The inclusion relations between the sets of GCBCs, Gaussian positive partial transpose channels (GPPTCs), entanglement breaking channels (GEBCs) and Gaussian classical-quantum channels (GCQCs) are displayed.

1. Introduction

Quantum coherence is one of important topics of the quantum theory and has been developed as an important quantum resource in recent years (Ref. [1]). Furthermore, it had been linked to quantum entanglement in many quantum phenomenon and plays an important role in fields of quantum biology and quantum thermodynamics ([2]–[8]).

In finite-dimensional systems, the theory of (quantifying) quantum coherence has been undertaken by many authors ([9]–[16]). Quantum operations relative to coherence have become important approaches and resources, such as incoherent operations. Recently, coherence breaking channels for the finite-dimensional case, as a subclass of inherent operations, are defined and characterized in [17]. As we know, continuous variable (CV) quantum systems are fundamental important from theoretical and experimental views. In particular, Gaussian states can be produced and managed experimentally. Several researchers focused on the theory of (quantifying) quantum coherence of Gaussian states ([14]–[20]). Furthermore, Gaussian incoherent channels also be characterized in [20]. The purpose of this paper is to characterize Gaussian coherence-breaking channels and discuss some relative topics.

The paper is organized as follows. In Section 2, we give a complete characterization of Gaussian coherence-breaking channels. Furthermore, we show the inclusion relation among the sets of Gaussian positive partial transpose channels (GPPTCs), entanglement breaking channels (GEBCs) and Gaussian classical-quantum channels (GCQCs).

Key words and phrases. Gaussian coherence breaking channels, assistant entanglement inputs.
In Section 3, we discuss the properties of the tensor product of a GCBC with an arbitrary Gaussian channel with entanglement inputs, and show that the tensor product of a GCBC and an arbitrary Gaussian channel maps all input states into product ones. Its connection to the channel capacity is also discussed.

2. Characterizing GCBCs

We first recall some necessary notions.

Let $H$ be the complex separable infinite-dimensional Hilbert space, $\mathcal{T}(H)$ the trace class operators on $H$. For the fixed reference basis $\{|i\rangle\}$, a state $\rho$ is incoherent if $\rho = \sum_i \lambda_i |i\rangle\langle i|$; otherwise, it is coherent. Denote by $\mathcal{I}_C$ the set of all incoherent states and $\mathcal{I}_G^C$ the set of all incoherent Gaussian states.

**Definition 2.1** A channel $\Phi : \mathcal{T}(H_A) \to \mathcal{T}(H_B)$ is called a Gaussian coherence-breaking channel (GCBC) if $\Phi(\rho)$ is always incoherent for any Gaussian input state $\rho \in \mathcal{T}(H_A)$.

The aim of this section is to give a characterization of GCBCs.

Recall that an $n$-mode Gaussian state $\rho$ is represented by

$$\rho = \frac{1}{(2\pi)^n} \int \! \! \int d^{2n}z \exp(-\frac{1}{4} z^T \nu z + id^T z) W(-z),$$

where $W(z)$ is the Weyl operator, $\nu$ is the covariance matrix (CM) of $\rho$, which is a $2n \times 2n$ real symmetric matrix, and $d$ is the displacement vector, which is a $2n$-dimensional real vector. So a Gaussian state is described by its covariance matrix and displacement and thus we write $\rho = \rho[\nu, d]$.

If $\Phi$ is an $n$-mode (Bosonic) Gaussian channel, then for any input $\rho = \rho[\nu, d]$, the output $\Phi(\rho)$ has the following covariance matrix and displacement:

$$K\nu K^T + M, \ Kd + \bar{d},$$

where $\bar{d}$ is a $2n$ dimensional real vector, $K, M$ are $2n \times 2n$ real matrices satisfying $M \geq \pm \frac{i}{2}(\Delta - K\Delta K^T)$, $K^T$ is the transpose of $K$ and

$$\Delta = \oplus^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So we write a Gaussian channel as $\Phi = \Phi(K, M, \bar{d})$.

The form of the multi-mode incoherent Gaussian operations were described implicitly in [20] without any proof. However, to obtain a characterization of GCBCs, we need formulate the multi-mode incoherent Gaussian operations exactly. We do this in the following lemma.
Note that, for any $2 \times 2$ real orthogonal matrix $O$, we have $O \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} O^T =$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ according to $O$ is symplectic or not.

**Lemma 2.1** An $n$-mode Gaussian channel $\Phi = \Phi(K, M, \bar{d})$ satisfies $\Phi(\mathcal{I}_G^C) \subseteq \mathcal{I}_G^C$ if and only if $\bar{d} = 0$ and there exists a permutation $\pi$ on $\{1, 2, \ldots, n\}$ such that

$$K = (P_\pi \otimes I_2)(\oplus_{i=1}^n t_i O_i),$$

(2.2)

$$M = \oplus_{i=1}^n \lambda_{\pi(i)} I_2,$$  

(2.3)

where $P_\pi$ is the $n \times n$ permutation matrix corresponding to $\pi$, $O_i$s are some $2 \times 2$ orthogonal matrices and $\lambda_i \geq \frac{1}{2}|t_i^2 - 1|$ whenever $O_i$ is symplectic or $\lambda_i \geq \frac{1}{2}|t_i^2 + 1|$ otherwise.

If $K$ and $M$ have respectively the form in Eq.(2.2) and Eq.(2.3), let $\Phi_{i \rightarrow \pi(i)} = \Phi(K_{\pi(i)}, \lambda_{\pi(i)} I_2, 0)$ be the Gaussian channel from the $i$th mode into the $\pi(i)$th mode determined by $K_{\pi(i)}$, $\lambda_{\pi(i)} I_2$ and $\bar{d} = 0$. Then $\Phi(K, M, \bar{d}) = \otimes_{i=1}^n \Phi_{i \rightarrow \pi(i)}$ and every $\Phi_{i \rightarrow \pi(i)}$ is an incoherent Gaussian channel between $i$th and $\pi(i)$th one-mode systems by [20, Theorem 2]. Then the following corollary is obvious.

**Corollary 2.2** An $n$-mode Gaussian channel $\Phi$ satisfies $\Phi(\mathcal{I}_G^C) \subseteq \mathcal{I}_G^C$ if and only if there exists a permutation $\pi$ on $\{1, 2, \ldots, n\}$ and one-mode incoherent Gaussian channel $\Phi_{i \rightarrow \pi(i)}$ from $i$th mode into $\pi(i)$th mode for each $i$ such that

$$\Phi = \otimes_{i=1}^n \Phi_{i \rightarrow \pi(i)}.$$  

**Proof of Lemma 2.1.** Noting that a single-mode Gaussian state is incoherent (thermal) if and only if it has a diagonal covariance matrix, zero displacement and squeezing; and every $n$-mode Gaussian incoherent state is a tensor product of single-mode Gaussian incoherent states ([19], [20]). Then, for any $n$-mode Gaussian channel $\Phi = \Phi(K, M, \bar{d})$, if $\Phi(\mathcal{I}_G^C) \subseteq \mathcal{I}_G^C$, it is obvious that $\bar{d} = 0$.

Write $K = (A_{ij})_{n \times n}$ and $M = (M_{ij})_{n \times n}$ with $A_{ij}$, $M_{ij}$ are $2 \times 2$ real matrices. Note that an $n$-mode incoherent Gaussian state $\rho$ has the covariance matrix

$$V_\rho = \text{diag}(r_1 I_2, r_2 I_2, \ldots, r_n I_2).$$

Since $\Phi$ maps arbitrary incoherent states to incoherent ones, we have that, for each set $\{r_i\}_{i=1}^n$ corresponding to an input incoherent Gaussian state, there exist a set $\{s_i\}_{i=1}^n$
corresponding to the output state such that

\[ KV_0 K^T + M = (A_{ij})_{n \times n} (\bigoplus_{i=1}^n r_i I_2) (A_{ij})_{n \times n}^T + M \]

\[ = \begin{pmatrix}
\sum_{j=1}^n r_j A_{1j} A_{1j}^T & \sum_{j=1}^n r_j A_{1j} A_{2j}^T & \cdots & \sum_{j=1}^n r_j A_{1j} A_{nj}^T \\
\sum_{j=1}^n r_j A_{2j} A_{1j}^T & \sum_{j=1}^n r_j A_{2j} A_{2j}^T & \cdots & \sum_{j=1}^n r_j A_{2j} A_{nj}^T \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^n r_j r_j A_{nj} A_{1j}^T & \sum_{j=1}^n r_j A_{nj} A_{2j}^T & \cdots & \sum_{j=1}^n r_j A_{nj} A_{nj}^T
\end{pmatrix} + M \]

\[ = \bigoplus_{i=1}^n s_i I_2 \]

It follows that for each pair \((k, l)\),

\[ \sum_{j=1}^n r_j A_{kj} A_{kj}^T + M_{kk} = s_k I_2; \]

\[ \sum_{j=1}^n r_j A_{kj} A_{lj}^T + M_{kl} = 0 \quad (k \neq l). \]

Furthermore, by the arbitrariness of \((r_1, \ldots, r_n)\), we must have \(M_{kl} = 0 \quad (k \neq l)\),

\[ A_{kj} A_{lj}^T = 0 \quad (k \neq l) \quad (2.4) \]

and

\[ M_{kk} = m_k I_2, \quad A_{kj} = t_{kj} O_{kj} \quad (2.5) \]

for some real numbers \(m_k, t_{kj}\) and \(2 \times 2\) orthogonal matrix \(O_{kj}, j = 1, 2, \ldots, n\). Thus by \(s_k > 0\) and Eqs.(2.4)-(2.5), for each \(k\), there exists unique \(j_k\) such that \(t_{kj_k} \neq 0\); that is, for each \(k\), all \(A_{kj} = 0\) except \(A_{kj_k} = t_{kj_k} O_{kj_k}\). Obviously, \(j_k \neq j_l\) whenever \(l \neq k\).

Therefore there exists a permutation \(\pi\) of \((1, 2, \ldots, n)\) so that \(j_k = \pi(k)\).

Write \(t_{\pi(k)} = t_{k\pi(k)}, \quad \lambda_{\pi(k)} = m_k, \quad O_{k\pi(k)} = O_{\pi(k)}\) and let \(P_\pi\) be the \(n \times n\) permutation matrix corresponding to \(\pi\). We can describe the structures of \(K = (A_{ij})_{n \times n}\) and \(M\) as

\[ K = (P_{\pi} \otimes I_2)(\bigoplus_{i=1}^n t_i O_i) \quad (2.6) \]

and

\[ M = \bigoplus_{i=1}^n \lambda_{\pi(i)} I_2. \quad (2.7) \]

As \(M \geq \pm \frac{i}{2} (\Delta - K \Delta K^T)\), one gets \(\lambda_i I_2 \geq \pm \frac{1}{2} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) - t_i^2 O_i \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) O_i^T\),

and this holds if and only if \(\lambda_i \geq \frac{1}{2} |t_i^2 - 1|\) when \(O_i \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) O_i^T = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)\)

and \(\lambda_i \geq \frac{1}{2} (t_i^2 + 1)\) when \(O_i \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) O_i^T = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)\).

Conversely, assume that \(\Phi = \Phi(K, M, \tilde{d})\) is a Gaussian channel with \(K = (P_{\pi} \otimes I_2)(\bigoplus_{i=1}^n t_i O_i)\), \(M = \bigoplus_{i=1}^n \lambda_{\pi(i)} I_2\) and \(\tilde{d} = 0\), where \(\pi\) is a permutation of \((1, 2, \ldots, n)\), \(O_i\)s are some \(2 \times 2\) orthogonal matrices and \(\lambda_i \geq \frac{1}{2} |t_i^2 - 1|\) with taking \(-\) or \(+\) depending
on $\mathcal{O}_i$ is symplectic or not, then $\Phi = \otimes_{i=1}^{n} \Phi_i \rightarrow \pi(i) (t_{\pi(i)} \mathcal{O}_{\pi(i)}, \lambda_{\pi(i)} I_2, 0)$ which clearly sends every $n$-mode Gaussian incoherent state to an $n$-mode Gaussian incoherent, completing the proof.

Remark. By the proof of lemma 2.1, it is ease to get a characterization of the a Gaussian channel $\Phi = \Phi(K, M, \bar{d})$ sends every product Gaussian state into a product Gaussian state if and only if there exists a a permutation $\pi$ on $\{1, 2, \ldots, n\}$ such that $K = (P_{\pi} \otimes I_2)(\oplus_{i=1}^{n} K_i)$, $M = \oplus_{i=1}^{n} M_{\pi(i)}$, where $P_{\pi}$ is the $n \times n$ permutation matrix corresponding to $\pi$, and in turn, if and only if $\Phi = \otimes_{i=1}^{n} \Phi_i \rightarrow \pi(i)$, where, for each $i$, $\Phi_i \rightarrow \pi(i)$ is a Gaussian channel from the $i$th mode to the $\pi(i)$th mode. In fact, if $K, M$ have the mentioned form, then, for arbitrary $n$-mode product Gaussian state $\rho = \otimes_{i=1}^{n} \rho_i$ with its covariance matrix $\oplus_{i=1}^{n} \nu_i$, the covariance matrix of $\Phi(\rho)$ is
\begin{equation}
K((\oplus_{i=1}^{n} \nu_i)K^T + M = \oplus_{i=1}^{n} (K_{\pi(i)} \nu_i K_{\pi(i)}^T + M_{\pi(i)}), \tag{2.8}
\end{equation}
which implies that $\Phi(\rho)$ is a product state.

Next we give a characterization of $n$-mode GCBCs which reveals that a Gaussian channel is coherence-breaking if and only if it collapses to a Gaussian incoherent state.

**Theorem 2.3** An $n$-mode Gaussian channel $\Phi(K, M, \bar{d})$ is coherence-breaking if and only if $K = 0$, $\bar{d} = 0$ and there exist scalars $\lambda_i \geq \frac{1}{2}$ such that
\begin{equation}
M = \text{diag}(\lambda_1 I_2, \lambda_2 I_2, \ldots, \lambda_n I_2). \tag{2.9}
\end{equation}

**Proof.** The “if” part is obvious, let us check the “only if” part. If an $n$-mode Gaussian channel $\Phi = \Phi(K, M, \bar{d})$ is coherence-breaking, then $\Phi$ is incoherent. By Lemma 2.1 and Corollary 2.2, $\Phi = \otimes_{i=1}^{n} \Phi_i \rightarrow \pi(i)$, where $\Phi_i \rightarrow \pi(i) = \Phi_i \rightarrow \pi(i) (K_{\pi(i)}, M_{\pi(i)}, 0)$ with
\begin{align*}
K_i &= t_i \mathcal{O}_i, \quad M_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix} \tag{2.9}
\end{align*}
for some $2 \times 2$ real orthogonal matrices $\mathcal{O}_i$ and real numbers $t_i$ and $\lambda_i \geq \frac{1}{2} |t_i^2 \pm 1|$. So, to complete the proof, it is enough to check that each $t_i = 0$.

It is clear that each $\Phi_i \rightarrow \pi(i)$ is Gaussian coherence breaking. Assume on the contrary $t_i \neq 0$ for some $i$. As we know, a $2 \times 2$ real orthogonal matrix has one of the following forms:
\begin{align*}
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \tag{2.10}
\end{align*}
For a Gaussian state $\rho$ in $\pi^{-1}(i)$th mode, suppose that its CM is
\begin{equation}
\nu_{\rho} = \begin{pmatrix} a & c \\ c & b \end{pmatrix},
\end{equation}
where \( a \geq 0, b \geq 0, \) and \( ab \geq c^2 + \frac{1}{4}. \)

If \( \mathcal{O}_i \) has the first form in (2.10), then \( t_i \mathcal{O}_i \nu \mathcal{O}_i^T + M_i \) has to be diagonal for any \( \rho. \) It follows from a short computation that

\[
\begin{align*}
t_i[-a \cos \theta \sin \theta - c \sin^2 \theta + c \cos^2 \theta + b \cos \theta \sin \theta] &= 0 \tag{2.11}
\end{align*}
\]

holds for all real numbers \( a, b, c \) with \( a \geq 0, b \geq 0 \) and \( ab \geq c^2 + \frac{1}{4}. \) As \( t_i \neq 0 \) in Eq (2.11), taking \( c \neq 0 \) and \( a = b \) leads to \( \cos^2 \theta = \sin^2 \theta, \) that is, \( \cos \theta = \pm \sin \theta. \) Thus \( (b - a) \cos \theta \sin \theta \) is always zero in Eq (2.11). Taking \( a \neq b, \) we get either \( \cos \theta = 0 \) or \( \sin \theta = 0. \) It follows from \( \cos \theta = \pm \sin \theta \) that \( \cos \theta = \sin \theta = 0, \) a contradiction.

Similarly, if \( \mathcal{O}_i \) has the second form, one can also get a contradiction.

In summary, \( t_i = 0 \) for all \( i \) and hence \( K = 0. \) We complete the proof. \( \square \)

Now let us turn to considering the relationship between some known classes of Gaussian channels. A picture about inclusion relations between GCBCs, GPPTCs, GEBCs, GCQCs and GQCCs will be shown. Without loss of generality, in the remained part of the section, we assume that all Gaussian channels involved have zero displacement vectors.

We begin by recalling the definitions of several kinds of Gaussian channel.

Let \( H_A, H_B, H_E \) be separable infinite-dimensional Hilbert spaces. A Gaussian channel \( \Phi : \mathcal{T}(H_A) \to \mathcal{T}(H_B) \) is called a Gaussian entanglement-breaking channel (GEBC) if \( \Phi \otimes I(\rho_{AE}) \) is always separable for any Gaussian input state \( \rho_{AE} \in \mathcal{T}(H_A \otimes H_E); \Phi \) is called a Gaussian positive partial transpose channel (GPPTC) if \( \Phi \otimes I(\rho_{AE}) \) has the positive partial transpose for any Gaussian input state \( \rho_{AE} \) (21); \( \Phi \) is called a Gaussian classical-quantum channel (GCQC) if \( \Phi(\rho) = \int_X \langle x|\rho|x \rangle \rho_x dx, \) where \( dx \) is the Lebesgue measure and \( \{|x>: x \in X\} \) is the Dirac’s system satisfying \( \langle x|y \rangle = \delta(x-y), \) which is a direct analogy of CQ maps in the finite-dimensional systems(Ref. \[23, 24\]).

We denote by \( \Omega_{GPPT}^G, \Omega_{EB}^G, \Omega_{CQ}^G \) and \( \Omega_{CB}^G \) the sets of GPPTCs, GEBCs, GCQCs and GCBCs respectively.

To uncover the inclusion relation between the above kinds of Gaussian channels, let us start from a general Gaussian channels. The matrix \( M \) for an \( n \)-mode Gaussian channel \( \Phi(K, M) \) satisfies

\[
M \geq \pm \frac{i}{2}(\Delta - K\Delta K^T),
\]

where \( \Delta = \bigoplus_{i=1}^n \Delta_i \) with \( \Delta_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \) Notice also that a Gaussian channel \( \Phi(K, M) \) is entanglement-breaking (GEBC) if and only if there exist matrices \( M_1, M_2 \) such that (Theorem 12.35 in Ref. \[23\])

\[
M = M_1 + M_2, \quad M_1 \geq \pm \frac{i}{2} \Delta, \quad M_2 \geq \pm \frac{i}{2} K\Delta K^T.
\]
Moreover, $\Phi$ is a GPPTC if and only if $M \geq \frac{i}{2}(\Delta \pm K\Delta K^T)$ (Ref. [24]). Then, it follows that

$$\Omega^G_{EB} \subseteq \Omega^G_{PPT}. \tag{2.12}$$

Holevo ([23, 24]) showed that $\Phi(K, M, \bar{d})$ is a GCQC if and only if $K\Delta K^T = 0$. A direct observation and Theorem 2.3 reveal that

$$\Omega^G_{CB} \subseteq \Omega^G_{CQ}.\tag{2.13}$$

Next let us check that $\Omega^G_{CQ} \subseteq \Omega^G_{EB}$. If $M$ satisfies that $M \geq \pm \frac{i}{2}K\Delta K^T$, taking $M_1 = M - (\pm \frac{i}{2}K\Delta K^T)$ and $M_2 = \pm \frac{i}{2}K\Delta K^T$. It follows that $M_1 \geq 0$. So $\Omega^G_{CQ} \subseteq \Omega^G_{EB}$.

In summary, we have

**Proposition 2.4.** The following inclusion relations are true.

$$\Omega^G_{CB} \subseteq \Omega^G_{CQ} \subseteq \Omega^G_{EB} \subseteq \Omega^G_{PPT}.\tag{2.12}$$

3. Property of GCBCs with assistant entanglement inputs

Next we are interested in the Gaussian coherence breaking channels with assistant entanglement inputs. Such property is helpful to understand relative problems on channel capacities ([23]-[37]). Without loss of generality, we assume that the Gaussian channels in this section are those with zero displacement. We first present an observation as follows.

**Theorem 3.1** For any Gaussian channel $\Psi$ on a system $H_E$ and any $n$-mode Gaussian coherence breaking channel $\Phi$ on a system $H_A$, $\Phi \otimes \Psi(\rho^{AE})$ is always a product state for any input Gaussian state $\rho^{AE}$ on the composite system $H_A \otimes H_E$.

**Proof.** Suppose $\Psi = \Psi(X_\Psi, Y_\Psi)$ and $\rho_{AE}$ has the CM $\begin{pmatrix} A & C \\ C^T & E \end{pmatrix}$. Then the output $\Phi \otimes \Psi(\rho^{AE})$ has the CM $\nu_{\text{out}}$ of the form

$$\nu_{\text{out}} = \begin{pmatrix} 0 & 0 \\ 0 & X_\Psi \end{pmatrix} \begin{pmatrix} A & C \\ C^T & E \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & X_\Psi^T \end{pmatrix} + \left( \oplus_{i=1}^n \lambda_i I_2 \, 0 \right),$$

where $\lambda_i$ is the eigenvalue of thermal state in the $i$th mode of the system $A$. It follows that

$$\nu_{\text{out}} = \left( \oplus_{i=1}^n \omega_i I_2 \right) \oplus \left( X_\Psi E X_\Psi^T + Y_\Psi \right).$$

Thus, $\Phi \otimes \Psi(\rho^{AE}) = \Phi(\rho_A) \otimes \Psi(\rho_E)$ is a product state. \qed

The observation says that the set of GCBCs is a proper subset of the set of GEBCs. In order to link to applications in the theory of channel capacities, it is needed to discuss the property of the $k$-tensor-product for GCBCs and arbitrary channels with entangled inputs.
Suppose that the Gaussian channel \( \Psi \) has the form \( \Psi(X, Y) \) with the zero displacement vector and \( \Phi \otimes \Psi \) acts on bipartite system \( H_A \otimes H_B \). It follows that the Gaussian channel \( (\Phi \otimes \Psi)^\otimes k \) can be described based on covariance matrices of input states as follows

\[
\nu_{\text{in}} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & X_\Psi \end{pmatrix} \otimes k \nu_{\text{in}} \begin{pmatrix} 0 & 0 \\ 0 & X_\Psi^T \end{pmatrix} \otimes k + \begin{pmatrix} \oplus_{i=1}^n \omega_i I_2 \\ 0 \\ 0 \end{pmatrix} \otimes k ,
\]

where \( \text{diag} \nu_{\text{in}} = \text{diag}(\nu_{AB}, \nu_{AB}, \ldots, \nu_{AB}) \), \( \nu_{AB} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \) is the covariance matrix for input states of \( \Phi \otimes \Psi \), \( \omega_i \) is the eigenvalue of thermal state in the \( i \)th mode of the system \( A \). It follows that the CM \( \nu_{\text{out}} \) of the output state

\[
\nu_{\text{out}} = \begin{pmatrix} \oplus_{i=1}^n \omega_i I_2 \\ 0 \\ X_\Psi B X_\Psi^T + Y_\Psi \end{pmatrix} \otimes k .
\]

Without loss of generality, write \( \nu_{\text{mod}} = (m_{ij}) \) the modulation covariance matrix for the Gaussian channel \( (\Phi \otimes \Psi)^\otimes k \) (for example, see \cite{31, 34}). Similar to the above discussion, we have that the modulated output state has the covariance matrix \( \bar{\nu}_{\text{out}} \) of the form

\[
\bar{\nu}_{\text{out}} = \nu(\Phi \otimes \Psi)^\otimes k(\rho[\nu_{\text{in}} + \nu_{\text{mod}}]) = \begin{pmatrix} \oplus_{i=1}^n \omega_i I_2 \\ 0 \\ X_\Psi (B + \nu'_{\text{mod}}) X_\Psi^T + Y_\Psi \end{pmatrix} \otimes k ,
\]

where \( \nu'_{\text{mod}} \) is the submatrix of \( \nu_{\text{mod}} \) corresponding to \( B \).

Recall that the capacity of quantum channels is a core topic in quantum information theory, and the additivity question for channel capacity arises from the following one: whether or not the entangled inputs can improve the classical capacity of quantum channels \( [22]-[37] \). The classical \( \chi \)-capacity of a channel \( \Phi \) is defined as

\[
C_\chi(\Phi) = \sup_{\{p_i, \rho_i\}} \left[ S\left( \sum_i p_i (\Phi(\rho_i)) \right) - \sum_i p_i S(\Phi(\rho_i)) \right] ,
\]

where \( S \) denotes the quantum entropy. The classical capacity \( C(\Phi) \) is defined by

\[
C(\Phi) = \lim_{k \to \infty} \frac{1}{k} C_\chi(\Phi^\otimes k) .
\]

If \( \Phi \) is a Gaussian channel, its Gaussian classical \( \chi \)-capacity is defined as

\[
C^G_\chi(\Phi) = \sup_{\rho, \tilde{\rho}} S(\Phi(\tilde{\rho})) - \int \mu(\omega) S(\Phi(\rho(\omega))) d\omega ,
\]

where the input \( \rho(\omega) \) is \( n \)-mode Gaussian state, \( \tilde{\rho} = \int \mu(dz)\rho(z) \) is the so-called averaged signal state, \( \mu \) is a probability measure. Note that the right side of Eq (3.4) may be infinite. It is obvious that the Gaussian classical capacity is a lower bound of the classical capacity. The additivity of channel capacity can be described as

\[
C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi) ,
\]
where $\Phi$ and $\Psi$ are channels and $C$ denotes the channel capacity (35). The additivity of classical capacity for GEBCs had been discussed in (26, 27). Here, we are interested in giving a shorter proof of the additivity of classical capacity of the tensor product of a GCBC with a Gaussian channel. For the Gaussian channel $\Psi$ of the form $\Psi(X_\Psi, Y_\Psi)$ and the GCBC $\Phi$, it follows from (3.1) that

$$C_\chi((\Phi \otimes \Psi)^\otimes k) = S(\rho[\bar{\nu}_{\text{out}}]) - S(\rho[\nu_{\text{out}}]) = kS(\rho[\oplus_{i=1}^n \omega_i I_2] \otimes \rho[X_\Psi(B + \nu_{\text{mod}})X_\Psi^T + Y_\Psi]) - kS(\rho[X_\Psi B X_\Psi^T + Y_\Psi])$$

$$= kS(\Phi(\bar{\rho}_A)) + kS(\Psi(\bar{\rho}_E)) - kS(\Phi(\rho_A)) - kS(\Psi(\rho_E)) = kC_\chi(\Phi) + kC_\chi(\Psi).$$

It follows that the Gaussian classical capacity and Gaussian classical $\chi$-capacity are both additive for the tensor product of a GCBC and arbitrary a Gaussian channel.

4. Conclusion

(Need rewrite) We have give a complete characterization of arbitrary $n$-mode Gaussian coherence breaking channels (GCBCs), which constitute a proper subclass of Gaussian entanglement breaking channels. Indeed, Gaussian coherence breaking channels have more rigorous properties, comparing with Gaussian entanglement breaking ones. A obvious witness is our observation that the tensor product of a GCBC and arbitrary a Gaussian channel maps all input states into product states. We also establish the inclusion relations among GCBCs and other common kinds of Gaussian channels. Furthermore, we discuss the property of the $k$-tensor-product of GCBCs and arbitrary channels with entangled inputs, and show a simple application in the research on the theory of the channel capacity.

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References

[1] A. Streltsov, G. Adesso and M. B. Plenio, Colloquium: Quantum Coherence as a Resource, arXiv:1609.02439v1 [quant-ph] 8 Sep 2016
[2] J. S. Ivan, K. K. Sabapathy, and R. Simon, Nonclassicality breaking is the same as entanglement breaking for bosonic Gaussian channels, Phys. Rev. A 88(2013), 032302
[3] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring Quantum Coherence with Entanglement, Phys. Rev. Lett. 115(2015), 020403
[4] M. B. Plenio and S. F. Huelga, Dephasing-assisted transport: quantum networks and biomolecules, New J. Phys. 10(2008), 113019
[5] F. Levi and F. Mintert, A quantitative theory of coherent delocalization, New J. Phys. 16(2014), 033007
[6] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, Proc. Natl. Acad. Sci. U.S.A. 112(2015), 3275
[7] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Limitations on the Evolution of Quantum Coherences: Towards Fully Quantum Second Laws of Thermodynamics, Phys. Rev. Lett. 115(2015), 210403
[8] A. Misra, U. Singh, S. Bhattacharya, and A. K. Pati, Energy cost of creating quantum coherence, Phys. Rev. A 93(2016), 052335
[9] R. Glauber, Coherent and Incoherent States of the Radiation Field, Phys. Rev. 131(1963), 2766
[10] E. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. 10(1963), 277
[11] A. Monras, A. Checĩnska, and A. Ekert, Witnessing quantum coherence in the presence of noise, New J. Phys. 16(2014), 063041
[12] J. Áberg, Catalytic Coherence, Phys. Rev. Lett. 113(2014), 150402
[13] D. Girolami, Observable Measure of Quantum Coherence in Finite Dimensional Systems, Phys. Rev. Lett. 113(2014), 170401
[14] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. 113(2014), 140401
[15] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A 53(1996), 2046
[16] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Fidelity and Trace-Norm Distances for Quantifying Coherence, Phys. Rev. A 91(2015), 042120
[17] K.F. Bu, Swati, U. Singh, and J. D. Wu, Coherence-breaking channels and coherence sudden death, Phys. Rev. A 94(2016), 052335
[18] Y.-R. Zhang, L.-H. Shao, Y.-M. Li, H. Fan, Quantifying coherence in infinite-dimensional systems, Phys. Rev. A 93 (2016), 012334
[19] D. Buono, G. Nocerino, G. Petrillo, G. Torre, G. Zonzo, F. Illuminati, Quantum coherence of Gaussian states, arXiv:1609.00913 [quant-ph]
[20] J.W. Xu, Quantifying coherence of Gaussian states, Phys. Rev. A 93(2016), 032111
[21] M. Horodecki, P. W. Shor, and M. B. Ruskai, Entanglement breaking channels, Rev. Math. Phys. 15(2003), 629
[22] P.W. Shor, J. Math. Phys. 43, 4334 (2002); C. King, J. Math. Phys. 43(2002), 4641; C. King, IEEE Trans. Inf. Theory 49(2003), 221; K. Matsumoto and F. Yura, J. Phys. A 37(2004), L167
[23] A. S. Holevo, Gaussian classical-quantum channels: gain of entanglement-assistance, Problems of information transmission, v. 50(2014), 1-15
[24] A. S. Holevo, Information capacity of quantum observable, Problems of Information Transmission March, Volume 48(2012), Issue 1, pp 1-10
[25] A. S. Holevo Quantum systems, channels, information: a mathematical introduction. De Gruyter studies in mathematical physics. Walter de Gruyter GmbH Co. KG, Berlin. 16(2012), P. 11-349
[26] A. S. Holevo, Shirokov M. E. On Shor’s channel extension and constrained channels. Comm. Math. Phys. 249(2004), P. 417-430
[27] M. E. Shirokov, The Holevo capacity of infinite dimensional channels and the additivity problem. Comm. Math. Phys. 262(2006), P. 131-159
[28] A. S. Holevo, Entanglement breaking channels in infinite dimensions. Probl. Inf. Transmiss. 44 (3)(2008), P. 3-18
[29] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, H. P. Yuen, Classical capacity of the lossy bosonic channel: the exact solution. Phys. Rev. Lett. 92 (2)(2009), P. 0279021
[30] C. Lupo, O.V. Pilyavets, S. Mancini, Capacities of lossy bosonic channel with correlated noise. New J. Phys. 11(2009), P. 0630231-06302318
[31] J. Schäfer, E. Karpov, R. García Patrón , O.V. Pilyavets, N. J. Cerf, Equivalence relations for the classical capacity of singlemode Gaussian quantum channels. Phys. Rev. Lett. 111(2013), P. 0305031-0305035
[32] J. Eisert, M. M. Wolf, Gaussian quantum channels. In: Quantum information with continuous variables of atoms and light, edited by N. J. Cerf, G. Leuchs, E. S. Polzik, Imperial College Press, London. 1(2007), P. 23-42
[33] T. Hiroshima, Additivity and multiplicativity properties of some Gaussian channels for Gaussian Inputs. Phys. Rev. A 73(2006), P. 0123301-0123309
[34] J. Schäfer, E. Karpov, O. V. Pilyavets, N. J. Cerf, Classical capacity of phase-sensitive Gaussian quantum channels, arXiv:1609.04119 [quant-ph]
[35] C. H. Bennett, C. A. Fuchs, J. A. Smolin, Entanglement-enhanced classical communication on a noisy quantum channel, in: Quantum Communication, Computing and Measurement, Proc. QCM96, ed. by O. Hirota, A. S. Holevo and C. M. Caves, New York: Plenum 1997, pp. 79-88
[36] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Classical Capacity of the Lossy Bosonic Channel: The Exact Solution, Phys. Rev. Lett. 92(2004), 027902
[37] O. V. Pilyavets, C. Lupo, and S. Mancini. Methods for Estimating Capacities and Rates of Gaussian Quantum Channels. IEEE Transactions on Information Theory, 58(9)(2012):6126-6164

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