More than four decades ago Sherrington and Kirkpatrick [1] (SK) introduced the spin glass Hamiltonian
\[ H = -\frac{1}{2} \sum_{i\neq j} J_{ij} S_i S_j \]
where the spins \( S_i \) and \( S_j \) of \( N \) Ising spins \( (S_i = \pm 1) \). The bonds \( J_{ij} \) are independent random variables with zero mean and standard deviations \( N^{-1/2} \), which fixes the spin glass temperature to \( T_0 = 1 \).

The TAP approach [2][3] gives the Gibbs potential \( G(T, m_i) = U - TS \) in terms of the local magnetizations \( m_i \) and the temperature \( T \) with the energy
\[ U = N \left( w - \frac{1}{2T} (1 - q_2)^2 \right) \]
and the entropy
\[ S = \sum_i s_0(m_i) - \frac{N}{4T^2} (1 - q_2)^2 \]
where \( s_0(m) = -\frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2} \) and where
\[ q_k = N^{-1} \sum_i m_i^k \quad (k = 2, 4). \]

The TAP equations, which are given by \( G_i \equiv \partial G / \partial m_i = 0 \), determine the \( N \) magnetizations \( m_i \).

As shown by the present author [3][4], the \( m_i \) have to satisfy the two convergence criteria
\[ c_1 \equiv T^2 - 1 + 2q_2 - q_4 > 0, \quad c_2 \equiv T^2 - 2q_2 + 2q_4 > 0. \]

Criterion \( c_1 \) is generally accepted and is related to the de Almeida Thouless condition [5] for the SK solution. Criterion \( c_2 \) is controversial [6][7].

These criteria are of some importance for the present work. They result from an application of a theorem of Pastur [8], which requires the in-dependency of the variables \( m_i \) from the bonds \( J_{ij} \). For a Gibbs potential, it is indeed these magnetizations \( m_i \) which are the free and independent variables. Note, that for every thermodynamic stability analysis one has to study the influence of all possible \( m_i \) values including the \( J_{ij} \) independent values. This requirement is also essential for the integration procedure used in this work. Thus the application of the theorem of Pastur is justified (Compare[10][11]) and \( c_2 > 0 \) is a necessary (but probably not sufficient [11]) convergence condition for the expansion [3].

Further support for validity of both criteria result from the fact, that they are necessary to prove the positiveness of the entropy \( S(T, m_i) \). Simple examples leading to a negative entropy, if \( c_2 < 0 \), can easily be constructed (see [12]). Consequences of the criteria [4] to the \( T \)-dependence of \( q_2 \) and \( q_4 \) has been presented in [13].

The present approach is related to the studies [7][14][15] of the complexity
\[ \Sigma(T, \Omega) = N^{-1} \log \mathcal{N}(T, \Omega) \]
which describes the extensive number \( \mathcal{N} \) of solutions of TAP equations \( G_i \)
\[ \mathcal{N}(T, \Omega) = \int_{-1}^{1} \prod_i \left( \int \mathcal{C}(\Omega) |\delta G_i/\partial m_i| \right) \left( \delta w - \frac{1}{2N} \sum_i m_i^2 \right) \left( \delta q_4 - \frac{1}{N} \sum_i m_i^4 \right) \]
Constrains are considered in the term \( \mathcal{C}(\Omega) \) which is chosen in this work as
\[ \mathcal{C} = \delta(q_2 - \frac{1}{N} \sum_i m_i^2) \delta(q_4 - \frac{1}{N} \sum_i m_i^4) \]
and the set of constrained variables is \( \Omega = \{q_2, q_4, w\} \). The inclusion of \( q_4 \) is new, but is essential to take into account the restrictions due to the criteria [4]. Note that \( q_4 \) is a sum of single particle terms and the modifications due to such terms are simple. The use of \( w \) instead of the Gibbs potential is technical and simplifies the calculation.

Following the previous work [7][14][15] and adopting the notation of [15] the calculation of the complexity leads to
\[ \Sigma(T) = \Sigma_0(\Omega) + \log \int dm \ e^{\mathcal{C}(\Omega, m)} \]
where
\[
\Sigma_0(\Omega) = -\lambda q_2 - \mu q_4 - \Delta(1 - q_2) - \frac{\Delta^2}{2\beta^2} - \frac{1}{2} \log(2\pi \beta^2 q_2) + \frac{\beta^2 u^2 q_2^2}{4}
\]
\[
\mathcal{L}(\Omega, m) = \lambda m^2 + \mu m^4 - \frac{[\tanh^{-1}(m) - \Delta m]^2}{2\beta^2 q_2} - \log(1 - m^2)
\]

(9)  (10)

The variables \(\lambda, \Delta, \mu, u\) enter via the Fourier representations of the \(\delta\)-functions in the calculation. In the limit \(N \to \infty\) steepest decent methods are applied. The stationary of \(\Sigma(T)\) with respect to \(\lambda, \Delta, \mu, u\) leads to
\[
q_2 = \langle \langle m^2 \rangle \rangle, \quad q_4 = \langle \langle m^4 \rangle \rangle
\]
\[
\beta w = -\frac{\beta^2}{2} u q_2^2 - q_2 \Delta - \beta^2 q_2 (1 - q_2)
\]
\[
\Delta = -\frac{\beta^2}{2} (1 - q_2) + \frac{1}{2 q_2} \langle \langle m \tanh^{-1}(m) \rangle \rangle - \frac{\beta^2}{2} u q_2
\]

(11)  (12)  (13)

with
\[
\langle \langle F(m) \rangle \rangle = \frac{1}{\int dm \mathcal{E}(\Omega, m)} \int dm F(m) e^{\mathcal{E}(\Omega, m)}
\]

(14)

The set of Eqs. (8-13) correspond to the Eqs. (56-61) of [15] with the replacements \(f \to w, \phi_0 \to 0, B \to 0\) and the additional terms proportional to \(\mu\) resulting from the inclusion of \(q_4\). The apparent differences of Eq. (5) and Eq. (56) of [15] result from a simplification using Eq. (12).

Setting \(B = 0\) corresponds to an exclusion of a none physical solution. For more details of the calculation and for the performed approximations it is referred to the previous work [7, 14, 15].

Note, however, that long as \(q_2\) and \(q_4\) satisfy the criteria [4], the value of the determinant \(\det(\partial G_i/\partial m_j)\) in Eq. (6) is always positive. All previous work disregards the modulus with not completely satisfying arguments.

As first application the pioneer work [14] on the total complexity \(\Sigma_{tot}(T)\) is reanalyzed, which describes the total number of TAP states. Setting \(u = 0\) in Eqs. (8-13) the resulting equations determine the complexity \(\Sigma(T; q_2, q_4)\) for fixed values of \(q_2\) and \(q_4\). These equations are numerical investigated for all possible values of \(q_2, q_4\) and for all temperatures \(T < 1\).

As example \(\Sigma(T = .4; q_2, q_4)\) is plotted in the \(q_2 - q_4\) plane in Fig. 1. The region of allowed \(q_2 - q_4\) values is restricted by \(q_2^2 \leq q_4 \leq q_2\) and by the criteria [4]. The cyan and the red boundaries represent the lines \(c_1 = 0\) and \(c_2 = 0\), respectively. The physical relevant region \(c_2 > 0\) is above the red boundary. The region below the red line with \(c_2 < 0\) has no physical significance.

The absolute maximum of \(\Sigma(T; q_2, q_4)\) in the \(q_2 - q_4\) plane, representing \(\Sigma_{tot}(T)\), can generally be located in

FIG. 1: Contour-plot of the complexity \(\Sigma(T; q_2, q_4)\) at \(T = .4\): The cyan and the red boundaries represent the lines \(c_1 = 0\) and \(c_2 = 0\). The region above the red line is the relevant area with \(c_2 > 0\). The maximum value represents the total complexity \(\Sigma_{tot}(T = .4)\) and its position is indicated by the green dot. The black dot indicates the position of \(g_0\).

FIG. 2: Contour-plot of the complexity \(\Sigma(T; q_2, q_4)\) far above, near and far below the critical temperature \(T_1 = .367\): Red Dots mark boundary maxima and green dots denote maxima in the interior (compatible with both criteria [4]). Black dots indicate the position of \(g_0\).
the interior or on the boundary of the relevant region. For temperatures \( T \geq T_1 \) the maximum is within the region and for \( T \leq T_1 \) the maximum is located on the boundary \( c_2 = 0 \). The numerical value of the critical temperature \( T_1 \) is given by

\[
T_1 = 0.367. \tag{15}
\]

The coordinates of the maxima are determined by \( \mu = 0 \) and by \( \partial \Sigma / \partial q_2 = 0 \) or by \( c_2 = 0 \). At \( T_1 \) the internal maximum coincides with the boundary maximum.

In addition to Fig. 1, which gives an overview, some details are presented in Fig. 2 for \( T = 0.6 \), for \( T = 0.37 \) and for \( T = 0.075 \). The internal maxima and the boundary maxima are marked by a green and red dots, respectively. (On the scale of Fig. 1 these two minima are not separated.)

\( \Sigma_{tot}(T) \) has two branches \( \Sigma^i \) and \( \Sigma^b \) resulting from the two different maxima. Fig. 3 shows the \( T \) dependence of these branches. Both have continuations from \( T_1 \) to the irrelevant temperatures regions. The difference of their extremal values is of the order of \( 10^{-4} \) according to Fig. 1.

The quantity characteristic for the transition is \( c_2 \), the \( c_2 \) for the internal maxima, which tends to zero for \( T \to T_1 \) from above. Via the inclusion of \( q_1 \) this relevant variable enters in the theory. Previous work does not include this variable and it is natural, that the present results have not been found earlier.

In this context it should be pointed out that the results for \( \Sigma_{tot}(T) \) are identical to the results of Bray and Moore [14] for \( T \geq T_1 \) and similar for \( T \leq T_1 \). However, the interpretations and conclusions differ considerably.

The presented results are new and have the important consequence that nearly all TAP solutions are marginal stable for temperatures near and below \( T_1 \)[16]. This conclusion is based on the exponential increase of the total number of TAP solutions and the fact, that the marginal stability implies a vanishing eigenvalue of the Hessian and a divergence of a mode of the susceptibility matrix [3]. Thus the system is critical for all these temperatures. Recalling that critical slowing down is a general feature of critical systems, the marginal stability gives an explanation for the slow dynamics found generally for spin glasses.

The calculation includes the non physical regime with \( c_1 > 0 \) but with \( c_2 < 0 \). Note that the numerical results are nearly independent of the sign of \( c_2 \). The interpretation of the results, however, is more transparent and natural, if one accepts that \( c_2 > 0 \) is a necessary condition.

Let us present some additonal conclusions resulting from the present approach. Averages [14] performed with the extremal values of the parameters represent an average over all TAP states. According to Eqs. [1] and [2] these 'white' averages for the Gibbs potential \( g_{av} \) and the energy \( u_{av} \) per spin are given by

\[
g_{av} = \hat{w} - \frac{\beta}{4} (1 - \hat{q}_2)^2 - T \langle \langle s_0(m) \rangle \rangle \tag{16}
\]

and by

\[
u_{av} = \hat{w} - \frac{\beta}{2} (1 - \hat{q}_2)^2
\]

where \( \hat{q}_2 \) and \( \hat{w} \) and \( \langle \langle \cdots \rangle \rangle \) denote the extremal values of \( q_2 \) and \( w \) and the white averages, respectively.

The temperature dependence for \( T \leq 1 \) of \( g_{av} \) and of \( u_{av} \) is plotted in Fig. 4 and in Fig. 5, respectively. The strange increase of \( u_{av} \) with decreasing \( T \) results from the fast increasing number of TAP solutions with high energies. These white averages have therefore no physical significance or any relevance for low temperatures.

An interesting alternative is the averaging, which leads to the lowest value \( g_0 \) of the Gibbs potential consistent with both, the existence of TAP solutions and the validity of the criteria [4].
To attack this problem the complete set of Eqs. (8-13) and $g = w - \beta/4(1-q_2)^2 - T\langle s_0(m) \rangle$ is needed. Keeping $q_2, q_4$ and $u$ constant the parameters $\lambda, \Delta, \mu$ are determined numerically with Eqs. (11) and (12). Repeating this procedure for all possible values of $q_2, q_4$ and $u$ the dependence of the complexity $\Sigma$ and of $w$ on these quantities is obtained according to Eq. (9) and Eq. (12), respectively. The setting $\Sigma = 0$ ensures at least the presence of one TAP state in the thermodynamic limit $N \to \infty$. Finally the minimum $g_0$ of the Gibbs potential is determined in the region of allowed values of $q_2$ and $q_4$. The resulting $g_0$ and the corresponding energy $u_0$ are plotted in Fig. 5 and in Fig. 6, respectively. Both quantities exhibits the expected temperature dependence and are similar to the results of the replica approach [17]. This is remarkable as both approaches use different averages. A second interesting point is the fact that the location of $g_0$ is again on the boundary $c_2 = 0$. (Compare Fig. 1 and Fig. 2)

This fact is again an indication for the relevance of criterion $c_2 > 0$.

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Moreover, it is generally impossible to conclude anything on the convergence of a series from a partial re-summation, as done in [6].

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