How to Accurately Extract the Running Coupling of QCD from Lattice Potential Data

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By (a) using an expression for the lattice potential of QCD in terms of a continuum running coupling and (b) globally parameterizing this coupling to interpolate between 2- (or higher-) loop QCD in the UV and the flux tube prediction in the IR, we can perfectly fit lattice data for the potential down to one lattice spacing and at the same time extract the running coupling to high precision. This allows us to quantitatively check the accuracy of 2-loop evolution, compare with the Lepage-Mackenzie estimate of the coupling extracted from the plaquette, and determine the scale $r_0$ ten times more accurately than previously possible. For pure SU(3) we find that the coupling scales on the percent level for $\beta \geq 6$.

1. Introduction

The string picture predicts that the static potential of QCD behaves for large distances like

$$V(r) = \sigma r - \frac{e_{IR}}{r} + \text{const} + \ldots$$  \hspace{1cm} (1)

(ignoring string breaking in the presence of dynamical fermions), where $\sigma$ is the string tension and the “IR charge” $e_{IR} = \pi/12$ for the simplest string. In the UV $V(r)$ is Coulomb like, with a running charge that is known to 2 loops in terms of the $\Lambda$-parameter. The only known general method of obtaining quantitative information on $V(r)$ is through Monte Carlo (MC) simulations, which nowadays reach relative accuracies of almost $10^{-4}, 10^{-3}$ at short, respectively, long distances. In contrast, the methods used so far extract $\sigma$ from these data with an error of several percent, $\Lambda$ with a much larger error. To obtain $\sigma$ one basically fits to an Ansatz of the form (1). For $\Lambda$ one does something different, namely, one defines a running coupling in the force scheme via

$$r^2 V'(r) \equiv C_F \alpha_F(r) \hspace{1cm} (\text{with } C_F = (N^2 - 1)/2N \text{ for } SU(N)),$$

and then fits numerical derivatives of the potential data to a 2-loop formula for $\alpha_F(r)$. 2-loop evolution is just becoming good at the shortest distances presently available, where however lattice artifacts, that one does not know how to take into account properly, prevent one from seeing it. With these methods one would therefore hardly benefit from more precise data or smaller lattice spacings. Here we present a method that allows for a unified fit of the whole $r$ range by incorporating lattice artifacts and the running of the coupling in a fundamental way. Details and a complete set of references can be found in [1].

2. The Method

The general idea is to express the lattice potential in terms of a continuum running coupling; more precisely there are three ingredients:

- Use the V-scheme, where the running coupling is defined via the continuum static potential

$$\hat{V}(q) = r^{-4} \pi C_F \alpha_V(q) \hspace{1cm} q^2$$  \hspace{1cm} (2)

- Parameterize $\alpha_V(q)$ to take into account 2-loop QCD in the UV in terms of $\Lambda$, the string prediction (1) in the IR in terms of $\sigma$, and to have another parameter for the crossover from UV to IR.

- Express the lattice potential as

$$V_{\alpha^2}(r) = V_0 - 4 \pi C_F \int_{-\pi/a}^{\pi/a} dq \frac{e^{-iqr}}{q^2} \alpha_V(q)$$  \hspace{1cm} (3)

where $a\hat{q}_i = 2 \sin(aq_i/2)$ for the usual (unimproved) gluon action.

The three parameters in $\alpha_V(q)$ and the constant $V_0$ are then fitted by matching (3) to the MC data. Though very plausible, (3) must be considered an Ansatz. The spectacular success of
our fits shows that it incorporates the lattice effects at an astonishing accuracy. But before presenting these fits we describe the

3. Parameterization of $\alpha_V(q)$

The following Ansatz for $\alpha_V(q)$ satisfies the 2-loop $\beta$-function equation and has no Landau pole (as long as $c_0 \geq 1, c_1 \geq 1$ and $c > 0$):

$$\frac{1}{\beta_0 \alpha_V(q)} = \ln \left[ 1 + \frac{q^2}{A^2} \ln^b(c_0 + \frac{q^2}{A^2} \lambda(q)) \right], \quad (4)$$

where $\lambda(q) = \ln^b(c_1 + \frac{q^2}{A^2})$, and $b = \frac{\beta_1}{\beta_0^2}$ is a ratio of the first two coefficients of the $\beta$-function. Of the three dimensionless parameters $c_0, c_1$ and $c$, the first two are fixed in terms of other parameters by matching $[\mathbb{3}]$ and $[\mathbb{4}]$ to $[\mathbb{5}]$,

$$\ln^b c_0 = \frac{C_F A^2}{2 \beta_0 \sigma}, \quad \ln^b c_1 = \left( 1 - \frac{2 \beta_0}{C_F} \right) \frac{c_0 \ln^{b+1} c_0}{2b} \quad (5)$$

while $c$ is the crossover parameter to be fitted.

One can show that by iterating the log’s in $[\mathbb{5}]$ in a suitable way, it is possible, in principle, to incorporate QCD to any number of loops, in terms of the (unknown) higher coefficients of the $\beta$-function and other, non-perturbative parameters.

4. Results

The fitting and error analysis is somewhat involved $[\mathbb{4}]$. We here only mention this: Since $c$ is quite strongly correlated with $\Lambda$, it is better to use $\alpha_V(q^*)$ instead of $\Lambda$ as independent fit parameter. Here $q^*$ is some UV scale; we chose $q^* = 3.4018/a$ for easy comparison with the Lepage-Mackenzie estimate $[\mathbb{2}]$ of $\alpha_V(q^*)$ extracted from the plaquette $W_{11}$. We performed fits for various theories, with gauge group SU(3) and SU(2), without and with fermions (in the latter cases there was no sign of string breaking). In all cases we could obtain $\chi^2/N_{\text{DF}} \approx 1$ including all points down to $r/a = 1$, which is impossible to achieve with Coulomb + Linear (C+L) type fits. In fig. 1 we show one of our fits. For our detailed results we again refer to $[\mathbb{4}]$; in table 1 we compare just two quantities with previous estimates (we should mention that the $\beta = 6.8$ data used are preliminary $[\mathbb{4}]$). One is $\alpha_V(q^*)$, the other the scale $r_0$ $[\mathbb{4}]$ defined by $C_F \alpha_F(r_0) = 1.65$. [The rhs of this equation is chosen so that $r_0 \approx 0.5 \text{ fm}$.]

We can use the exact relation between $\alpha_V(q)$ and $\alpha_F(r)$ to calculate $r_0$. We thereby retain the conceptual advantages of using a scale like $r_0$ without the errors from derivatives of lattice data.

In fig. 2 we show our running coupling and compare it with estimates of $\alpha_V(q)$ at various scales extracted by the Lepage-Mackenzie method from various Wilson loops (the rightmost circle corresponding to $W_{11}$) and Creutz ratios. We also show the 2-loop approximation to our $\alpha_V(q)$.

5. Discussion and Conclusion

The good agreement between our more precise and the simpler Lepage-Mackenzie estimate of $\alpha_V(q^*)$ increases the confidence in both methods.

Our fits show at present hardly any sign of systematic errors. The only way to check for such errors, then, is to come up with other parameterizations of $\alpha_V(q)$ that give similarly good or better fits. We have done so by taking 3-loop effects into account. It seems that slight systematic errors exist only at the “edges” of $\alpha_V(q)$, i.e. for the parameters $\alpha_V(q^*)$ and $\sigma$. Those of the former are quoted as the second error in table 1. In the intermediate region $\alpha_V(q)$, and therefore $r_0$, does not seem to have significant systematic errors.
Table 1
Our and previous determinations of $\alpha_V(q^*)$ and $r_0$

| Group   | $n_f$ | $\beta$ | $\alpha_V(q^*)$ from our fit | $W_{11}$ | $r_0/a$ from our fit | C+L fit |
|---------|-------|---------|-----------------------------|----------|---------------------|---------|
| SU(3)   | 0     | 6.0     | 0.1466(19)(8)               | 0.1519(35)| 5.30(1)             | 5.44(26)|
| SU(3)   | 0     | 6.4     | 0.1293(7)(7)               | 0.1302(22)| 9.54(3)             | 9.90(54)|
| SU(3)   | 0     | 6.8     | 0.1140(2)(4)               | 0.1153(15)| 14.56(5)            |         |
| SU(2)   | 0     | 2.85    | 0.1687(8)(9)               | 0.1712(50)| 19.0(3)             | 20.6(14)*|
| SU(3)   | 2W    | 5.3     | 0.1902(19)(12)             | 0.1991(79)| 3.62(1)             | 3.7(2)  |
| SU(3)   | 2S    | 5.6     | 0.1685(14)(9)              | 0.1788(57)| 5.20(1)             | 5.2(2)  |

The $\alpha_V(q^*)$ from $W_{11}$ is quoted with error $\alpha_V(q^*)^{3/2}$. W, S indicates Wilson, staggered fermions.

* From simulations using the Schrödinger functional coupling $\alpha_{SF}(q)$, see ref. [5].

Figure 2. Fit result for $\alpha_V(q)$ for $\beta = 6$ pure SU(3) (the solid lines delineating the error), its 2-loop approximation (dotted), and results from the method of ref. [2] (circles). $a^{-1} = 2.0$ GeV.

In contrast to $\sqrt{\sigma}/\Lambda$, the quantity $r_0\sqrt{\sigma}$ scales very well; it equals about 1.14 for the $n_f = 2$ theories of table 1, 1.17 for pure SU(2), and 1.17–1.19 for pure SU(3). For the latter $r_0\sqrt{\sigma}$ therefore scales at the 1% level for $\beta \geq 6.0$. At the edges of $\alpha_V(q)$ the scaling violations are slightly larger, but we expect more precise data and analysis to further decrease them.

As one might suspect from fig. 2 and table 1, we are beginning to gain control of $\alpha_V(q)$ at the fraction of percent level on all momentum scales (except perhaps in the far IR). For intermediate momenta below about 5 GeV this is basically already the case; for larger momenta 2-loop evolution is becoming good, so there is hope to soon achieve this accuracy also in the UV.

The bad scaling properties of quantities involving $\Lambda$ are not surprising: We found that the error of $\Lambda$ is quite large — without leading to a large error in $\alpha_V(q)$ itself — because $\Lambda$ is strongly correlated with higher order and non-perturbative contributions at an intermediate range, appearing here in the form of the fit parameter $c$.

For the future we note that our method can also be used to fit results from improved gluon actions (with the obvious modifications in eq. (3)), and can be extended to incorporate string breaking.

Finally, it should be clear that our $\alpha_V(q)$ is useful for potential models of heavy quarks and anywhere one wants to extend perturbation theory without running into the Landau pole.

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