Self-stabilizing Graph Exploration by a Single Agent

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Abstract

In this paper, we present two self-stabilizing algorithms that enable a single (mobile) agent to explore graphs. The agent visits all nodes starting from any configuration, \(i.e.,\) regardless of the initial state of the agent, the initial states of all nodes, and the initial location of the agent. We evaluate the algorithms using two metrics: cover time, which is the number of moves required to visit all nodes, and memory usage, which includes the storage needed for the state of the agent and the state of each node. The first algorithm is randomized. Given an integer \(c = \Omega(n)\), the cover time of this algorithm is optimal, \(i.e., O(m)\) in expectation, and the memory requirements for the agent and each node \(v\) are \(O(\log c)\) and \(O(\log(c + \delta_v))\) bits, respectively, where \(n\) and \(m\) are the numbers of nodes and edges, respectively, and \(\delta_v\) is the degree of \(v\). The second algorithm is deterministic. It requires an input integer \(k \geq \max(D, \delta_{\text{max}})\), where \(D\) and \(\delta_{\text{max}}\) are the diameter and the maximum degree of the graph, respectively. The cover time of this algorithm is \(O(m + nD)\), and it uses \(O(\log k)\) bits both for agent memory and each node.

1 Introduction

We focus on the exploration problem involving a single mobile entity, referred to as a mobile agent or simply an agent, within any undirected, simple, and connected graph \(G = (V, E)\). This agent, functioning as a finite state machine, migrates from node to node via edges at each time step. Upon visiting a node, the agent can access and modify the node’s local memory, known as a whiteboard. The graph is anonymous, \(i.e.,\) nodes lack unique identifiers. Our objective is to enable the agent to visit every node in the graph in as few steps as possible while minimizing the memory usage of both the agent and the whiteboards. This exploration problem, fundamental in the study of mobile computing entities, has been extensively studied \([13, 11, 17, 7, 8, 15]\). Exploration algorithms have frequently served as a foundation for solving other fundamental problems such as rendezvous, gathering, dispersion, and gossip sharing.

In this paper, we tackle the exploration problem under a more challenging setting: self-stabilizing exploration \([13, 8]\). We do not presuppose any specific initial global state (or configuration) of the network. This means that at the start of the exploration, (i) the agent’s location within \(G\) is arbitrary; (ii) the agent’s state is arbitrary, and (iii) the content of each whiteboard is arbitrary. The agent is required to visit all nodes in \(G\) from any potentially inconsistent configuration. Generally, an algorithm is considered self-stabilizing \([4]\) for problem \(P\) if it can solve \(P\) starting from any configuration. Self-stabilizing algorithms are capable of handling any type of transient faults, such as temporary memory corruption, making their design both practically and theoretically significant. We evaluate algorithms using two metrics: cover time, which is the number of moves required to visit all nodes, and memory usage, which includes the storage needed for the state of the agent and the state of each node.

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Generally speaking, several studies tackle a variety of problems involving mobile agents in the self-stabilizing setting [2, 8, 10]. In this setting, the number of agents in the graph is fixed. In our case (i.e., self-stabilizing exploration by a single agent), the number of agents is always exactly one; we do not consider configurations where no agent exists, or where two or more agents are present. Therefore, this setting may be particularly suitable for applications where physical robots operate in a field represented by an undirected graph, and the robots can leave information in some way at each intersection in the field.

One might think that this problem, self-stabilizing exploration by a single agent, fall outside the scope of distributed computing because only a single mobile agent is considered. However, we believe this is not the case, as the information accessible to the single agent is distributed throughout the entire graph. When minimizing agent memory, the agent must manage the necessary information distributed across the whiteboards throughout the graph. This situation often illustrates the trade-off between time complexity and agent memory, which is one of the essential aspects of distributed computing. Moreover, as mentioned earlier, exploration algorithms often serve as a fundamental building block for addressing other problems related to mobile agents. Therefore, our randomized and deterministic algorithms, introduced in this paper, could be used to solve various (more distributed) problems, such as rendezvous, gathering, gossiping, and leader election, in a self-stabilizing manner.

Throughout this paper, we denote the number of nodes, the number of edges, the diameter of a graph by \( n \), \( m \), and \( D \), respectively. We denote the degree of a node \( v \) by \( \delta_v \), and define \( \delta_{\min} = \min_{v \in V} \delta_v \) and \( \delta_{\max} = \max_{v \in V} \delta_v \).

### 1.1 Related Work

If we are allowed to use randomization, we can easily solve the self-stabilizing exploration with a well known strategy called the simple random walk. When the agent visits a node \( v \in V \), it simply chooses a node as the next destination uniformly at random among \( N(v) \), where \( N(v) \) is the set of all neighbors of \( v \) in \( G \). In other words, it moves to any node \( u \in N(v) \) with probability \( P_{v,u} = 1/\delta_v \). It is well known that the agent running this simple algorithm visits all nodes in \( G \) within \( O(\min(mn, mD \log n)) \) steps in expectation where \( n = |V| \), \( m = |E| \), and \( D \) is the diameter of \( G \). (See [4, 9].) Since the agent is oblivious (i.e., the agent does not bring any information at a migration between two nodes) and does not use whiteboards, the simple random walk is obviously a self-stabilizing exploration algorithm.

Ikeda, Kubo, and Yamashita [7] improved the cover time (i.e., the number of steps to visit all nodes) of the simple random walk by setting the transition probability as \( P'_{v,u} = \delta_u^{-1/2}/\sum_{w \in N(v)} \delta_w^{-1/2} \) for any \( u \in N(v) \). They proved that the cover time of this biased random walk is \( O(n^2 \log n) \) steps in expectation. However, we cannot use this result directly in our setting because the agent must know the degrees of all neighbors of the current node to compute the next destination. We can implement this random walk, for example, as follows: every time the agent visits node \( v \), it first obtains \( \delta_u \) for \( u \in N(v) \) by visiting all \( v \)'s neighbors in \( 2\delta_v \) steps, and then decides the next destination according to probability \( (P'_{v,u})_{u \in N(v)} \), which is now computable with \( (\delta_u)_{u \in N(v)} \). However, this implementation increases the cover time by a factor of at least \( \delta_{\min} \) and at most \( \delta_{\max} \). Whereas \( n^2\delta_{\max} \log n > mn \) always holds, \( n^2\delta_{\min} \log n < \min(mn, mD \log n) \) may also hold. Thus, we cannot determine which random walk has smaller cover time without detailed analysis. To bound the space complexity, we must know an upper bound \( \Delta \) on \( \delta_{\max} \) to implement this random walk. If the agent stores \( (\delta_u)_{u \in N(v)} \) on \( v \)'s whiteboard, it uses \( O(\log \Delta) \) bits in the agent-memory and \( O(\delta_v \log \Delta) \) bits in the whiteboard of each node \( v \). If the agent stores \( (\delta_u)_{u \in N(v)} \) only on the agent-memory, it uses \( O(\Delta \log \Delta) \) bits in the agent-memory.

The algorithm given by Priezhev, Dhar, Dhar, and Krishnamurthy [13], which is nowadays well known as the rotor-router, solves the self-stabilizing exploration deterministically. The agent is oblivious, but it uses only \( O(\log \delta_v) \) bits in the whiteboard of each node \( v \in V \). The edges \((\{v, u\})_{u \in N(v)}\) are assumed to be locally labeled by \( 0, 1, \ldots, \delta_v - 1 \) in a node \( v \). The whiteboard of each node \( v \) has one variable \( v.last \in \{0, 1, \ldots, \delta_v - 1\} \). Every time the agent visits a node \( v \), it increases \( v.last \) by one modulo \( \delta_v \) and moves to the next node via the edge labeled by the updated value of \( v.last \). This simple algorithm guarantees that starting from any configuration, the agent visits all nodes within \( O(mD) \)
steps \cite{17}. Masuzawa and Tixeuil \cite{8} also gave a deterministic self-stabilizing exploration algorithm. This algorithm itself is designed to solve the gossiping problem where two or more agents have to share their given information with each other. However, this algorithm has a mechanism to visit all the nodes starting from any configuration, which can be seen as a self-stabilizing exploration algorithm. The cover time and the space complexity for the whiteboards of this algorithm are asymptotically the same as those of the rotor-router, while it uses a constant space of the agent-memory, unlike oblivious algorithms such as the rotor-router.

In his seminal paper, Reingold \cite{14} proved that given positive integer \( N \), a Universal Exploration Sequence (UXS) with length \( \text{poly}(N) \) for (possibly non-simple) connected \( d \)-regular graphs with a size of at most \( N \) can be explicitly constructed in log-space and, hence, in polynomial time. Although we omit the definition of UXS here, from this result, we can immediately derive a self-stabilizing exploration algorithm for arbitrary graphs whose size is at most \( N \), whose cover time is polynomial in \( N \), with memory requirements of \( O(\log N) \) bits for the agent and zero for the whiteboards. One might think that Reingold’s UXS was designed for regular graphs, thus questioning its applicability to arbitrary graphs. However, this difference is not significant because we can virtually transform any arbitrary graph into a regular graph by adding self-loops (see \cite{15} for details). Later, Ta-shma and Zwick \cite{16} introduced the concept of a Strongly Universal Exploration Sequence (SUXS) and obtained results similar to those of Reingold, which allow us to improve the cover time of the above-mentioned self-stabilizing exploration algorithm from \( \text{poly}(N) \) to \( \text{poly}(n) \). Thus, the cover time no longer depends on a given upper bound \( N \) but only on the actual size \( n \).

A few self-stabilizing algorithms were given for mobile agents to solve problems other than exploration. Blin, Potop-Butucaru, and Tixeuil \cite{2} studied the self-stabilizing naming and leader election problem. Masuzawa and Tixeuil \cite{8} gave a self-stabilizing gossiping algorithm. Ooshita, Datta, and Masuzawa \cite{10} gave self-stabilizing rendezvous algorithms.

If we assume a specific initial configuration, that is, if we do not require a self-stabilizing solution, the agent can easily visit all nodes within \( 2m \) steps with a simple depth-first traversal (DFT). Panaite and Pelc \cite{11} gave a faster algorithm, whose cover time is \( m + 3n \) steps. They assume that the nodes are labeled by the unique identifiers. Their algorithm uses \( O(m \log n) \) bits in the agent-memory, while it does not use whiteboards. Sudo, Baba, Nakamura, Ooshita, Kakugawa, and Masuzawa \cite{15} gave another implementation of this algorithm: they removed the assumption of the unique identifiers and reduced the space complexity on the agent-memory from \( O(m \log n) \) bits to \( O(n) \) bits by using \( O(n) \) bits in each whiteboard. It is worthwhile to mention that these algorithms \cite{11,15} guarantee the termination of the agent after exploration is completed, whereas designing a self-stabilizing exploration algorithm with termination is impossible. Self-stabilization and termination contradict each other by definition: if an agent-state that yields termination exists, the agent never completes exploration when starting exploration with this state. If such state does not exist, the agent never terminates the exploration.

In the classical or standard distributed computing model (excluding mobile agents), the self-stabilizing token circulation problem, particularly self-stabilizing depth-first token circulation (DFTC), has been extensively studied \cite{6,8,12}. Introduced by Huang and Chen \cite{5}, this problem was addressed with a self-stabilizing DFTC algorithm using \( O(\log n) \) bits per process, which was later reduced to \( O(\log \delta_{\max}) \) bits by Datta, Johnen, Petit, and Villain \cite{3}. Petit \cite{12} developed a time-optimal (i.e., \( O(n) \)-time) self-stabilizing DFTC algorithm that also requires \( O(\log n) \) bits per process. However, these algorithms are not directly applicable to self-stabilizing exploration by a single agent because the network models are fundamentally different. In the standard model, \( n \) computational processes can communicate with each other via communication links in parallel, whereas in our model, only a single agent computes and updates the states of nodes in the network, potentially requiring more time to solve problems. On the other hand, one of the main challenges for self-stabilizing token circulation in the standard model is maintaining exactly one token starting from any configuration where there may be no tokens or where two or more tokens may exist. As mentioned above, this challenge does not apply to our model, where there is always a single agent in any configuration. Yet, many techniques from standard distributed computing might be adaptable for mobile agent algorithms. For example, our self-stabilizing exploration algorithms employ the technique of repeatedly recoloring graph nodes to resolve variable inconsistencies,
are summarized in Tables 1 and 2.

The cover times and the space complexities of the proposed algorithms and the existing algorithms can easily observe that the cover time is lower bounded by \(\Omega(m)\). One algorithm whose cover time is close to this lower bound with as small complexity of agent-memory and whiteboards as possible. Thus, we do not have trade-off between the cover time and the space complexity. However, the knowledge of an upper bound on \(\max(D, \Delta)\) is no longer guaranteed. However, the knowledge of an upper bound on \(\max(D, \Delta)\) is not a strong assumption because the space complexity increases only logarithmically in \(k\): we can assign any large \(\text{poly}(n)\) value for \(k\) to satisfy \(k \geq \max(D, \Delta)\) while keeping the space complexity of the agent-memory.

### 1.2 Our Contribution

In this paper, we investigate how short a cover time we can achieve in a self-stabilizing setting. One can easily observe that the cover time is lower bounded by \(\Omega(m)\): any deterministic algorithm requires \(\Omega(m)\) steps and any randomized algorithm requires \(\Omega(m)\) steps in expectation before the agent visits all nodes. (For completeness, we will prove this lower bound in Appendix A.) Our goal is to give an algorithm whose cover time is close to this lower bound with as small complexity of agent-memory and whiteboards as possible.

We give two self-stabilizing exploration algorithms \(\mathcal{R}_c\) and \(\mathcal{D}_k\), where \(c\) and \(k\) are the design parameters. The cover times and the space complexities of the proposed algorithms and the existing algorithms are summarized in Tables 1 and 2.

Algorithm \(\mathcal{R}_c\) is a randomized algorithm that achieves a cover time of \(O(m \cdot \min(D, \frac{n}{c} + 1, D + \log n))\) steps in expectation, utilizes \(O(\log c)\) bits of agent memory, and requires \(O(\log \delta_c + \log c)\) bits for the whiteboard of each node. Thus, we have trade-off between the cover time and the space complexity. The larger \(c\) we use, the smaller cover time we obtain. In particular, the expected cover time is \(O(m \log n)\) steps if we set \(c = \Omega(D/\log n)\), and it becomes optimal (i.e., \(O(m)\) steps) if we set \(c = \Omega(n)\). This means that we require the knowledge of \(\Omega(n)\) value to make \(\mathcal{R}_c\) time-optimal. Fortunately, this assumption can be ignored from a practical point of view: even if \(c\) is extremely larger than \(n\), the overhead will be just an additive factor of \(\log c\) in the space complexity. Thus, any large \(c \in \text{poly}(n) \cap \Omega(n)\) is enough to obtain the optimal cover time and the space complexity of \(O(\log n)\) bits both in the agent memory and whiteboards. Moreover, irrespective of parameter \(c \geq 2\), the cover time is \(O(mD)\) steps with probability 1.

Algorithm \(\mathcal{D}_k\) is a deterministic algorithm. The cover time of \(\mathcal{D}_k\) is \(O(m + nD)\) steps, which does not depend on parameter \(k\), while the agent uses \(O(\log k)\) bits both for the agent-memory and the whiteboard of each node. Thus, we do not have trade-off between the cover time and the space complexity. However, unlike \(\mathcal{R}_c\), we require an upper bound on the diameter and the maximum degree of the graph, that is, \(\mathcal{D}_k\) requires \(k \geq \max(D, \Delta)\) to solve a self-stabilizing exploration. If \(k < \min(D, \Delta)\), the correctness of \(\mathcal{D}_k\) is no longer guaranteed. However, the knowledge of an upper bound on \(\max(D, \Delta)\) is not a strong assumption because the space complexity increases only logarithmically in \(k\): we can assign any large \(\text{poly}(n)\) value for \(k\) to satisfy \(k \geq \max(D, \Delta)\) while keeping the space complexity of the agent-memory.

### Table 1: Randomized self-stabilizing graph exploration algorithms for a single agent.

| Algorithm          | Expected Cover Time | Agent Memory | Memory on node v |
|--------------------|---------------------|--------------|-----------------|
| Simple Random Walk | \(O(\min(mn, mD \log n))\) | 0            | 0               |
| Biased Random Walk | \(O(n^2\delta_{\max} \log n)\) | \(O(\log \Delta)\) | \(O(\delta_c \log \Delta)\) |
| \(\mathcal{R}_c\) (require \(\Delta \geq \delta_{\max}\)) | \(O(m \cdot \min(D, \frac{n}{c} + 1, D + \log n))\) | \(O(\log c)\) | \(O(\log(\delta_c + c))\) |

### Table 2: Deterministic self-stabilizing graph exploration algorithms for a single agent.

| Algorithm          | Cover Time | Agent Memory | Memory on node v |
|--------------------|------------|--------------|-----------------|
| Rotor-router \([13]\) | \(O(mD)\) | 0            | \(O(\log \delta_c)\) |
| UXs \([14]\) (require \(N \geq n\)) | polynomial in \(N\) | \(O(\log N)\) | 0               |
| SUXS \([15]\) (require \(N \geq n\)) | polynomial in \(n\) | \(O(\log N)\) | 0               |
| 2-color DFT \([8]\) | \(O(mD)\) | \(O(1)\) | \(O(\log \delta_c)\) |
| \(\mathcal{D}_k\) (require \(k \geq \max(D, \delta_{\max})\)) | \(O(m + nD)\) | \(O(\log k)\) | \(O(\log k)\) |

a common approach in the design of self-stabilizing algorithms (see Dolev, Israeli, and Moran \([5]\)).
and \( v \)'s whiteboard bounded by \( O(\log n) \) bits. For example, consider the case that we set \( k = 2^{500} \).

Then, \( D_k \) can fail only if \( D \geq 2^{500} \), which is too large to consider in practice. This extremely large value for \( k \) results in the increase of the memory usage only by 500 bits in the agent and whiteboards.

It remains open if there is a deterministic self-stabilizing exploration algorithm with optimal cover time, i.e., \( O(m) \) steps.

2 Preliminaries

Let \( G = (V, E, p) \) be a simple, undirected, and connected graph where \( V \) is the set of nodes and \( E \) is the set of edges. The edges are locally labeled at each node: we have a family of functions \( p = (p_e)_e \in V \) such that each \( p_e : \{v, u\} \mid u \in N(v) \} \rightarrow \{0, 1, \ldots , \delta_v - 1\} \) uniquely assigns a port number to every edge incident to node \( v \). Two port numbers \( p_u(e) \) and \( p_v(e) \) are independent of each other for edge \( e = \{u, v\} \in E \). The node \( u \) neighboring to node \( v \) such that \( p_u(\{v, u\}) = q \) is called the \( q \)-th neighbor of \( v \) and is denoted by \( N(v, q) \). For any two nodes \( u, v \), we define the distance between \( u \) and \( v \) as the length of a shortest path between them and denote it by \( d(u, v) \). We also define the set of the \( i \)-hop neighbors of \( v \) as \( N_i(v) = \{u \in V \mid d(u, v) \leq i\} \). Note that \( N_0(v) = \{v\} \) and \( N_1(v) = N(v) \cup \{v\} \).

An algorithm is defined as a 3-tuple \( P = (\phi, M, W) \), where \( M \) is the set of states for the agent, \( W = (W_k)_{k \in \mathbb{N}} \) is the family such that \( W_k \) is the set of states for each node with degree \( k \), and \( \phi \) is a function that determines how the agent updates its state (i.e., agent memory) and the state of the current node (i.e., whiteboard). At each time step, the agent is located at exactly one node \( v \in V \), and moves through an edge incident to \( v \). Every node \( v \in V \) has a whiteboard \( w(v) \in W_{k_v} \), which the agent can access freely when it visits \( v \). The function \( \phi \) is invoked every time the agent visits a node or when the exploration begins. Suppose that the agent with state \( s \in M \) has moved to node \( v \) with state \( w(v) = x \in W_{k_v} \) from \( u \in N(v) \). Let \( p_u = p_v(\{u, v\}) \). The function \( \phi \) takes 4-tuple \( (\delta_v, p_u, s, x) \) as the input and returns 3-tuple \( (p_{out}, s', x') \) as the output. Then, the agent updates its state to \( s' \) and \( w(v) \) to \( x' \), after which it moves via port \( p_{out} \), that is, it moves to \( v' \) such that \( p_{out} = p_v(\{v, v'\}) \). At the beginning of an execution, we let \( p_{in} \) be an arbitrary integer in \( \{0, 1, \ldots , \delta_v - 1\} \) where \( v \) is the node that the agent exists on. Note that if algorithm \( P \) is randomized one, function \( \phi \) returns the probabilistic distributions for \( (p_{out}, s', x') \).

Given a graph \( G = (V, E) \), a configuration (or a global state) consists of the location of the agent, the state of the agent (including \( p_{in} \)), and the states of all the nodes in \( V \). Algorithm \( P \) is a self-stabilizing exploration algorithm for a class \( \mathcal{G} \) of graphs if for any graph \( G = (V, E, p) \in \mathcal{G} \), the agent running \( P \) on \( G \) eventually visits all the nodes in \( V \) at least once starting from any configuration. Note that, by the above definition, any self-stabilizing exploration algorithm ensures that the single agent visits every node infinitely often.

We measure the efficiency of algorithm \( P \) by three metrics: the cover time, the agent memory space, and the whiteboard memory space. All the above metrics are evaluated in the worst-case manner with respect to graph \( G \) and an initial configuration. The cover time is defined as the number of moves that the agent makes before it visits all nodes. If algorithm \( P \) is a randomized one, the cover time is evaluated in expectation. The memory spaces of the agent and the whiteboard on node \( v \) are just defined as \( \log_2 |M| \) and \( \log_2 |W_{k_v}| \), respectively.

This paper presents two algorithm descriptions. For simplicity, instead of giving a formal 3-tuple \( (\phi, M, W) \), we specify the set of agent variables, the set of whiteboard variables, and the pseudocode of instructions that the agent performs. In addition to the agent variables, the agent must convey the program counter of the pseudocode and the call stack (or the function stack) when it migrates between nodes to execute an algorithm consistently. Thus, all the agent variables, the program counter, and the call stack constitute the state of the agent. Conforming to the above formal model, the state of the agent changes only when it migrates between nodes. Therefore, the program counter of each state is restricted to one of the line numbers corresponding to the instructions of migration. For example, in Algorithm 1 explained in Section 3, the domain of the program counter in the agent states are \( \{13, 17, 19\} \). We can ignore the space for the program counter and the call stack to evaluate the space complexity because they require only \( O(1) \) bits. (Our pseudocodes do not use a recursive function.) Whiteboard variables
are stored in the whiteboard \(w(v)\) of each node \(v\). All the whiteboard variables in \(w(v)\) constitute the state of \(w(v)\). For clarification, we denote an agent variable \(x\) by \(\text{self}.x\) and a whiteboard variable \(y\) in \(w(v)\) by \(v.y\).

Throughout this paper, we call the node the agent currently exists on the current node and denote it by \(v_{\text{cur}}\). We call the port number of the current node via which the agent migrates to \(v_{\text{cur}}\) in-port and denote it by \(p_{\text{in}}\). As mentioned above, the function \(\phi\) can use \(p_{\text{in}}\) to compute 3-tuple \((p_{\text{out}}, s', x')\). In other words, the agent can always access \(p_{\text{in}}\) to update its variables and the whiteboard variables of the current node and compute the destination of the next migration.

### 3 Randomized Exploration

In this section, we give a randomized self-stabilizing exploration algorithm \(\mathcal{R}_c\). The cover time and space complexity of \(\mathcal{R}_c\) depend on the design parameter \(c \geq 2\). Specifically, the cover time is \(O(m)\) if \(c = \Omega(n)\), while it is \(O(m \log n)\) if \(c = \Omega(D/\log n)\). The cover time is \(O(mD)\) with probability 1, regardless of how small \(c\) may be. Algorithm \(\mathcal{R}_c\) requires \(O(\log c)\) bits of agent memory and \(O(\log c + \log \delta_e)\) bits for each node \(v \in V\). If \(c = 2\), the algorithm becomes deterministic and is nearly identical to the exploration algorithm given by [8].

The idea of algorithm \(\mathcal{R}_c\) is simple. The agent tries to traverse all the edges in every \(2m\) moves according to the Depth-First-Traversal (DFT). However, we have an issue for executing DFT. When the agent visits a node for the first time, it has to mark the node as an already-visited node. However, we must deal with an arbitrary initial configuration in a self-stabilizing setting, thus some of the nodes may be marked even at the beginning of an execution. To circumvent this issue, we use colors and let the agent make DFTs repeatedly. The agent and all nodes maintain their color in a variable \(\text{color} \in \{1, 2, \ldots, c\}\). Every time the agent begins a new DFT, it chooses a new color \(\rho\) uniformly at random from all the colors except for the current color of the agent. In the new DFT, the agent changes the colors of all the visited nodes to \(\rho\). Thus, the agent can distinguish the visited nodes and the unvisited nodes with their colors: it interprets that the nodes colored \(\rho\) are already visited and the nodes having other colors have not been visited before in the current DFT. Of course, the agent may make incorrect decision because one or more nodes may be colored \(\rho\) when the agent chooses \(\rho\) for its color at the beginning of a new DFT. Thus, the agent may not perform a complete DFT. However, the agent can still make a progress for exploration to a certain extent in this DFT: Let \(r\) be the node at which the agent is located when it begins the DFT, let \(S\) be the set of the nodes colored \(\rho\) at that time, and let \(R\) be the set of all nodes in the connected component including \(r\) of the induced subgraph \(G[V \setminus S]\); then it visits all nodes in \(R\) and their neighbors. Hence, as we will see later, the agent visits all nodes in \(V\) by repeating DFTs a small number of times, which depends on how large \(c\) is.

Figure 1 shows an example of one DFT where \(r = v_1\) and \(S = \{v_6, v_9\}\). The agent tries to make a DFT on the graph starting from node \(v_1\). Unfortunately, whenever the agent visits node \(v_6\) or \(v_9\), which are colored \(\rho\), it immediately backtracks to the previous node. This is because it mistakenly thinks that it has already visited these nodes even when it visits them for the first time in the DFT. Thus, it never visits \(v_8\) or \(v_{11}\) in this DFT, which is unreachable from node \(v_1\) in the induced subgraph \(G[V \setminus S]\). However, it visits all the other nodes. Note that the index of the nodes are depicted in the figure only for simplicity of explanation, and we do not assume the existence of any unique identifiers of the nodes.
Algorithm 1: $R_c$

Notation: $next_R(v) = (p_n + 1) \mod \delta_v$

```
while True do
  Choose self.color uniformly at random from \{1, 2, \ldots, c\} \{self.color\}
  $v_{cur}.color \leftarrow$ self.color
  GoForward(0)
  while $(v_{cur}.parent = \bot \land p_n = \delta_{v_{cur}} - 1)$ do
    if $v_{cur}.color \neq$ self.color then
      $v_{cur}.color \leftarrow$ self.color
      $v_{cur}.parent \leftarrow p_n$
      GoForward($next_R(v_{cur})$)
    else
      if $v_{cur}.parent = next_R(v_{cur})$ then
        $v_{cur}.parent \leftarrow \bot$
        Migrate to $N(v_{cur}, next_R(v_{cur}))$
      // type-II backtracking
      else
        GoForward($next_R(v_{cur})$)
  
function GoForward($q$):
  Migrate to node $N(v_{cur}, q)$ // forward move
  if $v_{cur}.color =$ self.color then
    Migrate to node $N(v_{cur}, p_n)$ // type-I backtracking

3.1 Implementation

The pseudocode of $R_c$ is shown in Algorithm 1. The agent maintains one variable self.color $\in \{1, 2, \ldots, c\}$, while each node $v$ maintains two variables $v.parent \in \{\bot, 0, 1, \ldots, \delta_v - 1\}$ and $v.color \in \{1, 2, \ldots, c\}$. As mentioned before, the agent uses variables self.color and $v.color$ to distinguish already-visited nodes and unvisited nodes in the current DFT. It also uses whiteboard variable $v.parent$ to go back to the node at which it began the current DFT. Specifically, whenever the agent finds a node having different color from self.color, it changes $v_{cur}.color$ to self.color while it simultaneously substitutes $p_n$ for $v_{cur}.parent$ (Lines 7,8). Thus, the agent performs a DFT with making a spanning tree on graph $G$. We say that node $u$ is the parent of node $v$ when $u = N(v, v.parent)$.

In an execution of this algorithm, there are three kinds of agent-migration between nodes:

**Forward Moves:** The agent executes this kind of migration when it tries to find an unvisited node in the current DFT, that is, a node not colored self.color. The agent makes a forward move only in Line 17 in function GoForward(), which is invoked in Lines 4, 9, and 15.

**Backtracking (Type I):** The agent executes this kind of migration when the agent makes a forward move, but the destination node already has color self.color. By this migration, the agent backtracks to the previous node at which it made a forward move (Line 19).

**Backtracking (Type II):** The agent executes this kind of migration when it thinks that it has already visited all nodes in $N(v_{cur})$. Specifically, it backtracks to the parent of $v_{cur}$ when $v_{cur}.parent = (p_n + 1) \mod \delta_v$ (Line 13).

Every time the agent begins a new DFT, it chooses a new color uniformly at random from \{1, 2, \ldots, c\} \{self.color\} (Line 2). At this time, the starting node of this DFT, say $r$, satisfies $r.parent = \bot$ because the while-loop at Lines 5-15 ends if and only if $v_{cur}.parent = \bot$ holds. Thereafter, it performs a DFT
Lemma 1. If the agent running $R_c$ makes a forward move from node $v \in V$ to node $u \in N(v)$, thereafter no type-II backtracking to $v$ occurs before the agent makes type-II backtracking from $u$ to $v$.

Proof. By the definition of $R_c$, the set $\{v, \text{parent}\}_{v \in V}$ always contains a path from $v_{\text{cur}}$ to $v$ during the period from the move to the type-II backtracking from $u$ to $v$. Therefore, the agent never makes type-II backtracking from any $w \in N(v) \setminus \{u\}$ to $v$ during the period. 

Lemma 2. Starting from any configuration, the agent running $R_c$ changes its color (i.e., begins a new DFT) within $8m + n = O(m)$ rounds with probability 1.

Proof. Let $C$ be an any configuration and let $\rho$ be the color of the agent (i.e., $\rho = \text{self.color}$) in $C$. Let $M_F(v)$, $M_{B1}(v)$, and $M_{B2}(v)$ be the numbers of forward moves, type-I backtracking, and type-II backtracking that the agent makes at $v \in V$, respectively, until it changes $\text{self.color}$ from $\rho$ in the execution starting from $C$. Whenever the agent makes type-II backtracking at $v$, it changes $v$.parent to $\perp$. Thus, it never makes type-II backtracking twice at $v$, that is, $M_{B2}(v) \leq 1$. Next, we bound $M_F(v)$. By Lemma 1, once the agent makes a forward move from $v$ to $N(v,i)$, the following migrations involving $v$ must be type-II backtracking from $N(v,i)$ to $v$, the forward move from $v$ to $N(v,i+1)$, type-II backtracking from $N(v,i+1)$ to $v$, the forward move from $v$ to $N(v,i + 2)$, and so on. Therefore, the agent makes type-II backtracking at $v$ or observes $v$.parent $= \perp \land p_m = \delta_v$ (and changes its color) before it makes forward moves $\delta_v$ times. Since the agent makes type-II backtracking at $v$ at most once, we also have $M_F(v) \leq 2\delta_v$. Type-I backtracking occurs only after the agent makes a forward move. Therefore, we have

$$\sum_{v \in V} M_{B1}(v) \leq \sum_{v \in V} M_F(v).$$

To conclude, we have

$$\sum_{v \in V} (M_F(v) + M_{B1}(v) + M_{B2}(v)) \leq \sum_{v \in V} 2M_F(v) + \sum_{v \in V} M_{B2}(v) = 8m + n.$$

Lemma 3. Suppose now that the agent running $R_c$ changes its color to $\rho$ and begins a new DFT at node $r$. Let $S \subseteq V$ be the set of the nodes colored $\rho$ at that time and let $R$ be the set of the nodes in the connected component including $r$ in the induced subgraph $G[V \setminus S]$. Then, the agent visits all nodes in $R$ and their neighbors in this DFT with probability 1. Moreover, this DFT finishes (i.e., the agent changes its color) at node $r$.

Proof. By the definition of $R_c$, the set $\{v, \text{parent}\}_{v \in V}$ always contains a path from $v_{\text{cur}}$ to $r$. Therefore, this DFT terminates only at $r$. Thus it suffices to show that the agent visits all nodes in $R \cup \bigcup_{v \in R} N(v)$ in this DFT. Let $u$ be any node in $R \cup \bigcup_{v \in R} N(v)$. By definition of $R$, there exists a path $v_0, v_1, v_2, \ldots, v_l$ in $G$ where $r = v_0$, $u = v_l$ and $v_i \notin S$ for all $i = 1, 2, \ldots, l - 1$. As mentioned in the proof of Lemma 2, once the agent makes a forward move from $v$ to $N(v,i)$, the following migrations involving $v$ must be type-II backtracking from $N(v,i)$ to $v$, the forward move from $v$ to $N(v,i+1)$, type-II backtracking from $N(v,i+1)$ to $v$, the forward move from $v$ to $N(v,i+2)$, and so on. Moreover,
the agent makes the first forward move at $r$ to $N(r, 0)$ at Line 4 and makes the first forward move at each $v_i$ for $i = 1, \ldots, l - 1$ to $N(v_i, (v_i, \text{parent} + 1) \mod \delta_i)$ at Line 9 because we have $v_i \notin R$ for $i = 0, 1, \ldots, l - 1$. This means that for each $i = 0, 1, \ldots, l - 1$, the agent makes a forward move exactly once from $v_i$ to each $u \in N(v_i)$ except for $v_i$’s parent in the current DFT. In particular, the agent makes a forward move from $v_{i-1}$ to $v_i = u$, thus the agent visits $u$ in this DFT.

In the DFT mentioned in Lemma 3, the agent changes the colors of all nodes in $R' = R \cup \bigcup_{v \in R} N(v)$ to $p$. In the next DFT, the agent must choose a color different from $p$, ensuring that the agent will visit all nodes in $R'$ and their neighbors (plus possibly many more nodes). Consequently, the number of the nodes visited in the $i$-th DFT monotonically increases with respect to $i$. Therefore, we obtain the following corollary.

**Corollary 1.** The agent running $R_c$ visits all nodes before it changes its color $D + 1$ times.

**Theorem 1.** Algorithm $R_c$ is a randomized self-stabilizing exploration algorithm for all simple, undirected, and connected graphs. Irrespective of $c$, the cover time is $O(m \cdot D/c)$ steps with probability 1. The expected cover time is $O(m \cdot \min(D/c, 2 + \log n))$ steps. The agent memory space is $O(\log c)$ and the memory space of each node $v$ is $O(\log c + \log \delta_i)$.

**Proof.** The first and the second claims of this theorem immediately follow from Lemma 2 and Corollary 1. The forth claim is trivial (See the list of variables in Algorithm 1). In the rest of this proof, we prove the third claim of this lemma: the agent visits all nodes within $O(m \cdot \min(D/c, 2 + \log n))$ steps in expectation. Let $X$ denote how many times the agent changes its color before it visits all nodes in $V$. By Lemma 2 and $\Pr[X \leq D + 1] = 1$, it suffices to show $E[X] = O(n/c + 1)$ and $E[X] = O(D/c + \log n)$. We exclude the case $c = 2$ without loss of generality because these equalities immediately follow from $X \leq D + 1$ when $c = 2$.

First, we prove $E[X] = O(n/c + 1)$. For any $i \geq 1$, we define the $i$-th DFT as the DFT that the agent executes just after it changes its color $i$ times and define $V_i$ as the set of nodes that the agent visits in the $i$-th DFT. Fix $i \geq 1$ and let $V' \subseteq V$ be an arbitrary superset of $V_i$ such that $|V' \setminus V_i| = [c/2]$ holds and the subgraph induced by $V'$ is connected. We define $V' = V$ if $|V_i| + [c/2] > n$. Let $i$ be an integer such that the agent chooses the color for the $(i + 1)$-st DFT is different from any color of $|V' \setminus V_i|$, the agent visits all nodes of $V'$ in the $(i + 1)$-st DFT, thus we have $|V_{i+1}| \geq \max(|V_i| + [c/2], n)$. Such an event occurs with probability $p \geq 1 - (|V' \setminus V_i|)/(c - 1) \geq 1 - [c/2]/(c - 1) = \Theta(1)$. (We have assumed $c \geq 3$ above.) Hence, $|V_i|$ will increase at least by $[c/2]$ or reach $n$ at $1/p$ DFTs in expectation until $V_i = V$ holds, resulting in $E[X] \leq [n/c]/p = O(n/c + 1)$.

Next, we prove $E[X] = O(D/c + \log n)$. By Lemma 3, the agent always begins DFTs at the same node. We denote this node by $r$. For any node $v$, consider a shortest path $p = v_0, v_1, \ldots, v_l$ from $r$ to $v$, i.e., $v_0 = r$ and $v_l = v$. Let $Y$ be the maximum integer such that $v_Y$ has already been visited. Note that $Y$ is monotonically non-decreasing. For each iteration of DFT, we say that the iteration succeeds if by this iteration, $Y$ increases by at least $[c - 1]/2$ or reaches to $l$. The agent visits $v = v_l$ before it performs a $\chi = D/([c - 1]/2)$ successful iterations of DFT. Since the agent chooses a new color uniformly at random among $c - 1$ colors when it begins a new DFT, each iteration of DFT succeeds with a probability of at least $1/2$. Suppose the agent repeats a DFT $2\chi + [16 \log n]$ times, and let $Z$ be the number of successful iterations among those DFTs. By Chernoff bound, $Z \geq E[Z]/2 \geq \chi$ holds with a probability of at least $1 - \exp(-E[Z]/8) \geq 1 - \exp(-2 \log n) = 1 - n^{-2}$. Thus, the agent visits any node $v$ in $O(D/c + \log n)$ DFTs with probability $1 - n^{-2}$. Thus, $X = O(D/c + \log n)$ with high probability, yielding $E[X] = O(D/c + \log n)$.

**4 Deterministic Exploration**

The randomized algorithm $R_c$, which is presented in the previous section, achieves self-stabilizing exploration in a small number of steps in expectation. The key idea is to repeat DFTs with changing the color of the agent randomly. Unfortunately, we cannot make use of the same idea if we are not allowed
Specifically, the agent attempts to construct a Breadth First Search (BFS) tree rooted at some node $v$. Throughout this section, we denote by $dist(u,v)$ the distance between two nodes $u$ and $v$, i.e., the length of a shortest path from $u$ to $v$.

In this section, we give a deterministic self-stabilizing exploration algorithm $D_k$ with design parameter $k$. We must satisfy $k \geq \max(D, \delta_{\max})$, otherwise this algorithm no longer guarantees the correctness. Thus we require the knowledge of an upper bound of the diameter $\max(D, \delta_{\max})$ of graph $G$. Unlike $R_c$, the cover time of this algorithm does not depend on parameter $k$: the agent visits all nodes within $O(m+nD)$ steps starting from any configuration. Algorithm $D_k$ requires $O(\log k)$ bits both in the agent memory and the whiteboard of each node $v \in V$. Throughout this section, we denote by $dist(u,v)$ the distance between two nodes $u$ and $v$, i.e., the length of a shortest path from $u$ to $v$.

Whereas $R_c$ is based on depth first traversals, $D_k$ is based on Breadth First Traversals (BFTs). Specifically, the agent attempts to construct a Breadth First Search (BFS) tree rooted at some node $r \in V$. Constructing a BFS tree by a single agent in a self-stabilizing setting poses several challenges, especially when bounding the time complexity by $O(m+nD)$ steps. In the rest of this section, we first introduce some terminologies and then explain the challenges posed by self-stabilization and how we address them.

We place the detailed implementation of $D_k$ in the appendix. Although we present all key ideas of $D_k$ and the proofs necessary to verify the correctness of our main theorem in the main body, the detailed implementation found in Section[13] especially the pseudocode of $D_k$, may be helpful for understanding $D_k$ in more detail.
4.1 Consistent Tree

In $D_k$, the agent attempts to maintain a BFS tree that it can periodically traverse. However, due to memory constraints, managing such a tree within the graph is non-trivial. Specifically, at each node $v$, the agent cannot store information about the set of $v$’s children in the tree because only $O(\log k)$ bits are available on $v$’s whiteboard. In this subsection, we will explain how to embed a tree in the graph while adhering to the memory constraints.

Each node $v \in V$ maintains whiteboard variables $v\text{.parent} \in \{0, 1, 2, \ldots, \delta_v - 1\}$, $v\text{.firstNameChild}, v\text{.nextChild} \in \{-1, 0, 1, \ldots, k - 1\}$, and $v\text{.level} \in \{0, 1, \ldots, k\}$. A node $v$ is valid if $v\text{.firstNameChild} = -1$ or there is a sequence of nodes $c_1, c_2, \ldots, c_\ell \in N(v)$ with $\ell \geq 1$ that satisfies the following conditions:
- $c_i\text{.parent} = v$ and $c_i\text{.level} = v\text{.level} + 1$ for all $i \in [1, \ell]$,
- $p_v(\{v, c_i\}) < p_v(\{v, c_{i+1}\})$ and $c_i\text{.nextChild} = p_v(\{v, c_{i+1}\})$ for all $i \in [1, \ell - 1]$,
- $N(v, v\text{.firstNameChild}) = c_1$, and
- $c_\ell\text{.nextChild} = -1$.

We call those nodes $c_1, c_2, \ldots, c_\ell \in N(v)$ the children of $v$. An example of a valid node and its children is shown in Figure 3.

A tree $T = (V_T, E_T)$ rooted at $v \in V$ is consistent if $r\text{.level} = 0$, all nodes in $V_T$ are valid, and for any $u, v \in V_T$, $u$ is a child of $v$ if and only if $(u, v) \in E_T$ holds. By definition, we observe the following proposition.

**Observation 1.** For any configuration $C$ and any node $v \in V$, there is at most one consistent tree that contains $v$ in $C$.

We say that a node $v \in V$ is consistent if there is a consistent tree $T = (V_T, E_T)$ such that $v \in V_T$. Observation 1 implies that the consistent tree that contains a consistent node $v$ is uniquely determined.

We denote such a tree by $T(v)$. Notice $T(v)$ may not be rooted at $v$.

In the remainder of this paper, for any tree $T = (V_T, E_T)$, we use $v \in T$ and $e \in T$ to denote $v \in V_T$ and $e \in E_T$, respectively. We define $|T|$ as the number of nodes in the tree, i.e., $|T| = |V_T|$.

Note that a consistent tree $T = (V_T, E_T)$ is not necessarily a subgraph of a BFS tree. For example, there may be some node $v \in V_T$ for which $v\text{.level} > \text{dist}(r, v)$. Additionally, there could be a node $u \notin V_T$ such that $\text{dist}(r, u) < \text{dist}(r, w)$ for some $w \in V_T$.

4.2 Tree Expansion

Our deterministic algorithm $D_k$ expands $T(v_{\text{cur}})$ by incorporating new nodes one by one. The first challenge is determining how to expand $T(v_{\text{cur}})$. Consider the scenario where the agent is located at a consistent node $v \in V$, and explores an edge $e \in \{v, u\} \notin T(v)$. If $u \notin T(v)$, the agent should incorporate the destination node $u$ into $T(v)$ by setting $u\text{.parent} \leftarrow p_v(e)$ and $u\text{.level} = v\text{.level} + 1$, initializing $u\text{.firstNameChild} = -1$, and updating $u\text{.nextChild}$ and $u\text{.firstNameChild}$ appropriately. However, the agent must refrain from making these changes if $u$ is already part of $T(v)$. A significant challenge arises here because when the agent visits node $u$, another node $w \in T(v)$ may be in the same state as $u$, complicating the determination of whether $u$ belongs to $T(v)$.

We resolve this challenge by maintaining two agent variables $v\text{.stage} \in \{0, 1, 2, 3, 4, 5\}$, and one whiteboard variable $v\text{.color} \in \{B, W, R\}$ for each $v \in V$. Here, $B$, $W$, and $R$ stand for black, white, and red, respectively. When $v\text{.stage} = i$, the agent is said to be in phase $i$. Suppose the agent begins phase 0 at node $r \in V$. In this phase, the agent visits all neighbors of $r$, incorporating them such that $T(r)$ becomes the star centered at $r$, containing all nodes in $N_1(r)$. For any $i \geq 1$, let $V(r, i)$ be the set of nodes with level $i$ in $T(r)$, and $V_{\text{target}}(r, i)$ be the set of nodes that are neighbors to a node in $V(r, i)$ and are not included in $T(r)$. The objective of phase $i$ is to incorporate all nodes in $V_{\text{target}}(r, i)$ into $T(r)$.

Each phase consists of five sub-phases, or stages, managed by the variable $v\text{.stage} \in \{1, 2, 3, 4, 5\}$, described as follows. (Figure 4 illustrates a typical example.)
Before stage 1 follows: Here, we refer to the movement from a parent to a child as a forward move, and all nodes in $V_u$ from a child to a parent as backtracking.

During Stages 1 and 2, all nodes in $V_u$ are red, and all nodes in $T$ which nodes join $V_u$'s parent. Suppose the agent backtracks from a node $u$, it moves to $u$'s parent, say $v$, and $u$ is the $i$-th child of $v$. Before backtracking, the agent tentatively memorizes $u$.\text{\texttt{nextChild}} in its own variable $\text{\texttt{self.nextChild}}$. Thus, after backtracking, it can go to the $i + 1$-st child of $v$ if one exists. If $u$ is the last child of $v$, $\text{\texttt{self.nextChild}}$ must be $\bot$ at that time. Then, the agent returns to $v$'s parent, or terminates the circulation if $v = r$. This process clearly circulates $T(r)$, moving through every edge in the tree exactly twice, requiring $2|T(r)| - 2$ steps.

In Stages 2, 4, and 5, each time the agent visits a node $v$ at level $\text{\texttt{self.phase}}$, it must visit all of $v$'s neighbors. At this point, the agent moves to the neighbors in descending order of the port numbers; specifically, it visits $u_{3i-1}, u_{3i-2}, \ldots, u_0$ in this order, where $u_i = N(v, i)$ for $i \in [0, δ_v - 1]$. This ordering helps the agent efficiently incorporate red neighbors of $v$ into $T(r)$ during Stage 5. Initially, the agent sets both $v.\text{\texttt{firstChild}}$ and $\text{\texttt{self.nextChild}}$ to $\bot$. Thereafter, each time the agent visits a red neighbor $u_i$, it first updates $u_i.\text{\texttt{nextChild}} \leftarrow \text{\texttt{self.nextChild}}$ in addition to $u_i.\text{\texttt{level}} = v.\text{\texttt{level}} + 1$, $u_i.\text{\texttt{parent}} = p_n$, and $u_i.\text{\texttt{firstChild}} \leftarrow \bot$. After returning to $v$, it updates $\text{\texttt{self.nextChild}} \leftarrow i$ and $v.\text{\texttt{firstChild}} \leftarrow i$.

Figure 4: How to expand tree $T(r)$ in phase 3. A number in a circle represents the level of the corresponding node. An arrow from node $u$ to $v$ indicates that $u$ is a child of $v$. The orange region represents which nodes join $T(r)$.

- **Stage 1**: The agent circulates within $T(r)$ (i.e., visits all nodes in $T(r)$ in $2|T(r)| - 2$ steps), coloring all nodes white.
- **Stage 2**: The agent recirculates within $T(r)$. During the circulation, each time it visits a node $v$ at level $i$, it visits each neighbor $u \in N(v)$ and colors $u$ red if $u$ is black.
- **Stage 3**: The agent recirculates within $T(r)$, this time coloring all nodes in $T(r)$ black.
- **Stage 4**: The agent recirculates within $T(r)$. Similar to Stage 2, as it visits each node $v$ at level $i$, it visits all of $v$'s neighbors, this time changing the color of white nodes to red.
- **Stage 5**: In the final circulation of $T(r)$, similar to Stage 2, the agent visits all nodes that neighbor a node at level $i$ in $T(r)$, incorporating any red nodes it encounters into $T(r)$. Thereafter, it updates $\text{\texttt{self.phase}} \leftarrow (\text{\texttt{self.phase}} + 1) \mod k$.
Once this process is completed, all red nodes in \( N(v) \) are incorporated into \( T(r) \), ensuring that \( T(r) \) remains consistent.

**Lemma 4.** If the agent begins phase \( i \) at a consistent node \( r \) with level 0, \( V_{\text{target}}(r, i) \) is incorporated into \( T(r) \) before the end of phase \( i \). Phase \( i \) ends in \( O(n + \sum_{v \in V(r, i)} \delta_v) \) steps.

**Proof.** The correctness of the first part of the lemma, concerning the incorporation of \( V_{\text{target}}(r, i) \) into \( T(r) \) in phase \( i \), follows from the above discussion. We derive the second part from the fact that: (i) each of the five circulations on \( T(r) \) requires only \( 2|T(r)| - 2 = O(n) \) steps, and (ii) in Stages 2, 4, and 5, for each node \( v \in T(r, i) \), the agent moves from \( v \) to each \( u \in N(v) \) and vice versa, which requires exactly \( 3 \cdot 2 \cdot \sum_{v \in V(r, i)} \delta_v \) steps in total. \( \square \)

**Corollary 2.** Once the agent begins phase 0, it visits all nodes within \( O(m + nD) \) steps.

**Proof.** Let \( r \) be the node at which the agent begins phase 0. As defined, phase 0 completes within \( O(\delta_r) \) steps. During the subsequent phases 1, 2, ..., \( D - 1 \), by Lemma 4, all nodes are incorporated into \( T(r) \) within \( O(m + nD) \) steps in total. \( \square \)

### 4.3 Towards Fast Reset

Since we seek a self-stabilizing solution, we must address arbitrary initial configurations where the current node may not be consistent, or \( T(v_{\text{cur}}) \) significantly deviates from a BFS tree structure. How can we ensure that the agent visits all nodes within \( O(m + nD) \) steps starting from such a configuration? According to Corollary 2, this is feasible if the agent can reset \texttt{self.phase} to zero and initiate phase 0 within every \( \Theta(m + nD) \) steps with a sufficiently large hidden constant. However, the agent does not know the values of \( n, m, D \); it only knows an upper bound \( k \geq \max(D, \delta_{\text{max}}) \), which may significantly exceed \( D \), i.e., \( k = \omega(D) \). Therefore, we require a sophisticated mechanism to reset \texttt{self.phase} to zero within \( O(m + nD) \) steps from any configuration, without harming tree expansion so that the validity of Corollary 2 is preserved.

It is straightforward to handle configurations where the current node, \( v_{\text{cur}} \), is not consistent. As mentioned earlier, the agent attempts to circulate the consistent tree \( T(v_{\text{cur}}) \) in every stage of each phase, assuming that \( v_{\text{cur}} \) is consistent. If \( v_{\text{cur}} \) is not consistent, the agent will encounter some inconsistency within \( O(m) \) steps during the circulation process, or \( v_{\text{cur}} \) becomes consistent. For instance, an inconsistency is detected when the agent moves from a node \( v \) to \( u \), one of \( v \)'s children, but either \( u\.\text{parent} \neq p_u\{(u, v)\} \) or \( u\.\text{level} \neq v\.\text{level} + 1 \) holds. The agent begins phase 0 whenever it detects any such \textit{local} inconsistency. (See Section 3 for detailed rules on detecting inconsistencies). Hence, in the rest of this section, we assume that \( v_{\text{cur}} \) is consistent without loss of generality.

The main challenge is handling configurations where \( v_{\text{cur}} \) is consistent, yet \( T(v_{\text{cur}}) \) significantly deviates from a BFS tree structure. To illustrate the difficulty of this scenario, consider the configuration shown in Figure 5, where the agent begins phase 3 at a node \( r = v_0 \) and \( T(r) = \{(v_3, v_2), (v_2, v_1), (v_1, v_0)\} \).

In this configuration, edges \( \{v_0, v_1\}, \{v_0, v_2\}, \ldots, \{v_0, v_{n-2}\} \) are never utilized in subsequent tree circulations, as \( v_2 \) is the only child of \( v_0 \). Consequently, exactly one node is added to \( T(r) \) in each phase, requiring \( \Omega(n^2) \) steps to visit all nodes and reset \texttt{self.phase} to zero, despite \( D = 2 \) and \( m = O(n) \).

If we ignore time complexity, this challenge can be easily addressed. For example, consider what happens if we require \( T(v_{\text{cur}}) \) to have some kind of BFS tree structure in addition to being consistent.
Specifically, let $r$ be the root of $T(v_{\text{cur}})$ and suppose we require: (i) for any $v \in T(r)$, $v.\text{level} = \text{dist}(r, v)$, and (ii) all nodes $v \in N_H$ are contained in $T(v_{\text{cur}})$, where $H$ is the height of $T(r)$, i.e., $H = \max \{u.\text{level} | u \in T(r)\}$. Then, the agent can verify these conditions in each circulation of $T(v_{\text{cur}})$ by visiting all neighbors of each node in $T(v_{\text{cur}})$ and reset $\text{self.phase}$ to zero if it finds any violations of these conditions. However, this may require $\Omega(m)$ steps in each circulation, resulting in $\Omega(mD)$ steps in the entire execution of the algorithm.

We address the second challenge (i.e., deviation from the BFS structure) by adding the following three rules:

- **Rule 1**: In Stage 5 of each phase, the agent checks whether it has incorporated at least one node into $T(v_{\text{cur}})$. If no node has been incorporated, indicating that $T(v_{\text{cur}})$ can no longer be expanded, the agent resets $\text{self.phase}$ to zero.

- **Rule 2**: In Stage 1 of each phase $i$, the agent resets $\text{self.phase}$ to zero if it finds a node $v \in V(r, i)$ such that $v.\text{firstChild} \neq \bot$.

- **Rule 3**: As explained above, in Stage 5 of each phase $i$, the agent visits all neighbors of each node $v \in V(r, i)$. The agent resets $\text{self.phase}$ to zero if it finds a black neighbor $u \in N(v)$ with $u.\text{level} < v.\text{level} = 1 = i - 1$.

In what follows, we first observe that these three rules do not invalidate Corollary 2 and then demonstrate that these rules ensure a high frequency of resetting $\text{self.phase}$.

**Observation 2.** Corollary 2 still holds even with the addition of the above three rules.

*Proof.* Suppose the agent begins phase 0 at a node $r$. The first rule is applicable only after the agent has incorporated all nodes into $T(r)$ and thus does not affect the corollary. The second rule is never triggered because the tree expansion proceeds hop by hop after initiating phase 0. The third rule is also never applicable in the execution subsequent to resetting $\text{self.phase}$ to zero, because at the beginning of each phase $i$, $T(v_{\text{cur}})$ forms a shortest path (i.e., BFS) tree over $G[N_r(r)]$. This configuration ensures that the difference in levels between any two nodes $u, v \in V'$, where $V'$ is the set of nodes in $T(v_{\text{cur}})$ at the start of phase $i$, satisfies $|u.\text{level} - v.\text{level}| \leq 1$. Moreover, all black nodes that the agent visits in Stage 5 must be within $V'$: all neighbors of the nodes in $V(r, i)$ that are not part of $V'$ are colored red in either Stage 2 or 4. Thus, the condition of the third rule is never satisfied. $\square$

**Lemma 5.** Starting from any configuration, the agent begins phase 0 in $O(m + nD)$ steps.

*Proof.* As mentioned above, the agent begins phase 0 within $O(m)$ steps by finding a local inconsistency if the current node is not consistent. Thus, we assume the current node is consistent. Since each phase requires only $O(m)$ steps, starting from any configuration, the agent initiates some phase $i$ at some node, say $r$, within $O(m)$ steps. By the second rule, we can assume without loss of generality that at that time all nodes $v \in V(r, i)$ do not have children in $T(r)$, i.e., $v.\text{firstChild} = \bot$; otherwise, the agent begins phase 0 in the next $O(m)$ steps.

In the rest of this proof, we show that the agent resets $\text{self.phase}$ to zero before it enters phase $i + 2D$, thereby supporting the lemma by Lemma 1. Assume for contradiction that it enters phase $i + 2D$ and focus on the configuration $C$ at that time. Let $V_{\text{just}} = \bigcup_{j \leq i} V(r, j)$, any node in $V(r, i)$, and $v$ any node in $V(r, i + D + 1)$. Note that the first rule guarantees $V(r, i + D + 1) \neq \emptyset$. After initiating phase $i$, the agent expands $T(r)$ hop by hop, avoiding the region of $V_{\text{just}}$. This implies that the distance between $u$ and $v$ in the induced subgraph $G[V \setminus V_{\text{just}}]$ is at least $D + 1$; otherwise, $v.\text{level} \leq i + D$ holds, contradicting $v \in V(r, i + D + 1)$. Since $\text{dist}(u, v) \leq D$, a shortest path $v_0, v_1, \ldots, v_{\ell}$ between $v$ and $u$ must contain a node in $V_{\text{just}}$, where $v_0 = v$, $v_{\ell} = u$, and $\ell \leq D$ holds. Let $s$ be the smallest index such that $v_s \in V_{\text{just}}$, i.e., $v_0, v_1, \ldots, v_{s-1} \notin V_{\text{just}}$. Since $v = v_0$ and $v_{s-1}$ belong to the same connected component in $G[V \setminus V_{\text{just}}]$, $v_{s-1}$ is contained in $T(r)$ and $v_{s-1} \in V(r, i')$ holds for some $i' \in [i + 1, i + D + s]$ in configuration $C$. Note that we have $i' \neq i$ here because otherwise $v.\text{level} \leq i + s - 1 < i + D + 1$ holds, a contradiction with the fact $v \in V(r, i + D + 1)$. Therefore, in Stage 5 of phase $i'$ satisfies $\ell' \leq i + D + s < i + D + \ell \leq i + 2D$, where $\ell'$ is the level of $v_{s-1}$ in configuration $C$. $\square$
the agent finds a black node with a level at most \( i - 1 \), i.e., it visits \( v_s \in V_{<i} \). Since \( i' \geq i + 1 \), the agent detects \( v_s, \text{level} < i' - 1 \) and triggers the third rule, resetting \( \text{self.phase} \) to zero, a contradiction. \( \square \)

The following theorem is derived from Observation 2 and Lemma 5.

**Theorem 2.** Algorithm \( \mathcal{D}_k \) solves the self-stabilizing exploration problem for all simple, undirected, and connected graphs with a diameter and maximum degree of at most \( k \). The cover time is \( O(m + nD) \) steps, regardless of the value of \( k \). The memory requirement is \( O(\log k) \) for both the agent and each node.
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A Lower Bound

By definition, the cover time of any exploration algorithm is trivially $\Omega(n)$. In this section, we give a better lower bound: the cover time of any (possibly randomized) algorithm is $\Omega(m)$. Specifically, we prove the following theorem.

**Theorem 3.** Let $\mathcal{P}$ be any exploration algorithm. For any positive integers $n, m$ such that $n \geq 3$ and $n - 1 \leq m \leq n(n + 1)/2$, there exists a simple, undirected, and connected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ such that the agent running $\mathcal{P}$ on $G$ starting from any node in $V$ requires at least $(m - 1)/4$ steps to visit all nodes in $V$ in expectation.

**Proof.** For simplicity, we assume that $\mathcal{P}$ is a randomized algorithm. However, the following proof also holds without any modification even if $\mathcal{P}$ is a deterministic one.

Let $G' = (V', E')$ be any simple, undirected, and connected graph such that $|V'| = n - 1$ and $|E'| = m - 1$. There must be such a graph because $n - 1 \geq 2$ and $m - 1 \geq (n - 1) - 1$. Suppose that the agent runs $\mathcal{P}$ on $G'$ starting from any node $r \in V'$. Let $X_{u,v}$ be the indicator random variable such that $X_{u,v} = 1$ if the agent traverses edge $\{u, v\}$ (from $u$ to $v$ or vice versa) in the first $(m - 1)/2$ steps, and $X_{u,v} = 0$ otherwise. Clearly, the agent traverses at most $(m - 1)/2$ edges in the first $(m - 1)/2$ steps. Therefore, $\sum_{(u,v) \in E'} X_{u,v} \leq (m - 1)/2$ always hold, thus $\sum_{(u,v) \in E} E[X_{u,v}] = E[\sum_{(u,v) \in E'} X_{u,v}] \leq (m - 1)/2$. This yields that there exists an edge $\{u, v\} \in E'$ such that $\Pr[X_{u,v} = 1] = E[X_{u,v}] \leq 1/2$.

Let $\{u, v\} \in E'$ such that $\Pr[X_{u,v} = 1] \leq 1/2$ and define $G = (V, E)$ as the graph that we obtain by modifying $G'$ as follows: remove edge $\{u, v\}$, introduce a new node $w \notin V'$, and add two edges $\{w, u\}$ and $\{w, v\}$. Formally, $V = V' \cup \{w\}$ and $E = E' \cup \{(w, u), (w, v)\}\setminus\{(u, v)\}$. By definition, graph $G$ is simple, undirected, and connected. Then, the agent running $\mathcal{P}$ on $G$ starting from node $r$ requires at least $(m - 1)/4$ steps in expectation to visit all nodes in $V$ because the agent does not visit node $w$ in the first $(m - 1)/2$ steps with probability at least $1/2$.

B Detail Implementation of Deterministic Algorithm

The pseudocode of $D_k$ is presented in Algorithms 1, 2, and 3. Algorithm 1 gives the list of variables, the main routine, and a subroutine Initialize(). While Algorithms 2 and 3 give other three subroutines.

In addition to the variables we explained above (i.e., self.phase, self.stage, self.firstChild, v.parent, v.firstChild, v.nextChild, v.level, v.color), we have three agent variables self.level $\in \{0, 1, \ldots, k\}$, self.error $\in \{\text{False}, \text{True}\}$, self.expanded $\in \{\text{False, True}\}$, self.mode $\in \{-1, 1\}$ and one whiteboard variable v.port $\in \{0, 1, \ldots, \delta\}$ for each $v \in V$. We use self.level to maintain the distance between the current node $v_{cur}$ and the root of $T(v_{cur})$ in $T(v_{cur})$. We use self.error and self.expanded to reset self.phase to zero, as we will see later. Two variables self.mode and v.port are just temporary variables: self.mode is used to remember the last movement is a forward move or not, and v.port is used to visit all neighborhoods of the current node in functions Initialize() and Expand().

The main routine is simply structured. In phase 0, the agent executes function Initialize() and sets self.stage to 1 (Lines 2-3). In Initialize(), the agent simply initializes self.level and v.level for all $v \in N_1(v_{cur})$ and makes the parent-child relationship between $v_{cur}$ and each of its neighbors (Lines 13-21). Thus, at the end of phase 0, $T(v_{cur})$ is the tree rooted at $v_{cur}$ such that every $u \in N(v_{cur})$ is a child of $v_{cur}$ and no node outside $N_1(v_{cur})$ is included in $T(v_{cur})$. In phase $i \geq 1$, it executes function Circulate() and increments self.stage by one; if self.stage is already 5 before incrementation, self.stage is reset to 1 (Lines 4-6). We will explain Circulate() later. Every time self.stage is reset to 1, the agent begins the next phase, that is, increments self.phase by one modulo $k$ (Line 10). To implement the first rule (Rule 1 in Section 1.3), the agent always remembers in self.expanded whether it added a new node to $T(v_{cur})$ in the current phase. If self.expanded is still False at the end of a phase except for phase 0, the agent resets self.phase to 0 and begins phase 0 (Line 8). In addition, the agent always substitutes True for self.error when it finds any inconsistency in Circulate(). This variable self.error is also used for implementing the second and third rules (Rules 2 and 3 in Section 1.3). Once self.error = True holds,
the agent immediately halts Circulate (Line 25) and resets self.phase to 0 and begins phase 0 (Line 8).

When the agent invokes Circulate() at a node $r$ in phase $i$, it circulates all nodes in $T(r)$ unless it finds any inconsistency and setting self.error ← True. This function Circulate() simply implements the movement strategy specified in Section 4.2: the agent always moves to $N(v, next_D(v))$ unless $next_D(v) = \bot$ (Lines 32), where $next_D(v)$ is defined in Algorithm 3. Only exception is when the agent visits a node $v$ with $v.level = self.phase$. Then, it simply goes back to the parent node (after invoking Expand() in Stages 2, 4, and 5) (Line 28–30). The agent makes a move inside the tree by invoking MoveInTree(), in which self.level is adequately updated (Line 35). In Stage 1 (resp. 3), it always changes the color of the visited node to white (resp. black). In Stages 2, 4, and 5, whenever the agent visits a node with level = self.phase, it tries to add the neighbors of the current node according to the strategy mentioned in Section 4.2 by invoking Expand() (Line 29).

In MoveInTree(), every kind of local inconsistency regarding the parent-child relationship is detected. Specifically, the agent detects an inconsistency at the migration from $u$ to $v$ if (i) self.level $\neq v.level$, (ii) self.nextChild $\neq \bot$, but self.nextChild $\leq p_m$, (iii) the agent has made a forward move from $u$ to $v$, but $p_m \neq v.parent$, (iv) the agent has made backtracking from $u$ to $v$, but $p_m = v.parent$. If an inconsistency is detected, the agent sets True for self.error (Line 42). Moreover, this function implements the second rule (Rule 2 in Section 4.3): it sets self.error ← True if the agent visits a node with the level self.phase that has one or more children in Stage 1.

Function Expand() (Lines 43-61) is a simple implementation of the strategy explained in Sections 4.2 and the third rule (Rule 3 in Section 4.3).
Algorithm 2: Main Routine of $D_k$ and function $\text{Initialize()}$

Variables in Agent Memory:
- $\text{self.phase, self.level} \in \{0, 1, \ldots, k - 1\}$,
- $\text{self.stage} \in \{1, 2, 3, 4, 5\}$,
- $\text{self.nextChild} \in \{\bot, 0, 1, \ldots, k - 1\}$,
- $\text{self.error, self.expanded} \in \{\text{False, True}\}$,
- $\text{self.mode} \in \{-1, 1\}$

Variables in $v$’s Whiteboard:
- $\text{v.parent} \in \{0, 1, \ldots, \delta_v - 1\}$,
- $\text{v.firstChild, v.nextChild} \in \{\bot, 0, 1, \ldots, k - 1\}$,
- $\text{v.color} \in \{W, B, R\}$,
- $\text{v.level} \in \{0, 1, \ldots, k\}$,
- $\text{v.port} \in \{-1, 0, \ldots, \delta_v\}$

Note: Whenever the agent decides to move via a port not in $[0, \delta_{v_{\text{cur}}} - 1]$, it immediately detects the error and jumps to Line 8.

1. while True do
2. 1. if $\text{self.phase} = 0$ then
3. 2. Initialize(); $\text{self.stage} \leftarrow 1$
4. 3. if $\text{self.phase} \geq 1$ then
5. 4. Circulate()
6. 5. $\text{self.stage} \leftarrow (\text{self.stage} \mod 5) + 1$
7. 6. if $\left(\text{self.error} = \text{True} \lor (\text{self.phase} \geq 1 \land \text{self.stage} = 1 \land \text{self.expanded} = \text{False})\right)$ then
8. 7. $\text{self.phase} \leftarrow 0$; $\text{self.error} \leftarrow \text{False}$
9. 8. else if $\text{self.stage} = 1$ then
10. 9. $\text{self.phase} \leftarrow (\text{self.phase} + 1) \mod k$
11. 10. $\text{self.expanded} \leftarrow \text{False}$
12. function $\text{Initialize()}$: 12
13. 12. $\text{self.level} \leftarrow 0$; $v_{\text{cur}.level} \leftarrow 0$; $v_{\text{cur}.port} \leftarrow 0$
14. 13. $v_{\text{cur}.firstChild} \leftarrow 0$
15. 14. while $v_{\text{cur}.port} < \delta_{v_{\text{cur}}}$ do
16. 15. $\text{self.nextChild} \leftarrow \begin{cases} v_{\text{cur}.port} + 1 & \text{if } v_{\text{cur}.port} < \delta_{v_{\text{cur}}} - 1 \\ \bot & \text{otherwise} \end{cases}$
17. 16. Migrate to $N(v_{\text{cur}}, v_{\text{cur}.port})$
18. 17. $v_{\text{cur}.parent} \leftarrow p_{\text{in}}$; $v_{\text{cur}.level} \leftarrow 1$
19. 18. $v_{\text{cur}.firstChild} \leftarrow \bot$; $v_{\text{cur}.nextChild} \leftarrow \text{self.nextChild}$
20. 19. Migrate to $N(v_{\text{cur}}, p_{\text{in}})$
21. 20. $v_{\text{cur}.port} \leftarrow \min(v_{\text{cur}.port} + 1, \delta_{v_{\text{cur}}})$
Algorithm 3: Functions Circulate() and MoveInTree() in $D_k$

Notation:

\[ \text{next}_D(v) = \begin{cases} \text{self}.\text{nextChild} & \text{if } \text{self}.\text{nextChild} \neq \bot \lor v.\text{level} = 0 \\ v.\text{firstChild} & \text{else if } v.\text{parent} = p_{in} \land v.\text{firstChild} \neq \bot \\ v.\text{parent} & \text{otherwise} \end{cases} \]

\[ \text{colorUpdate}(a) = \begin{cases} W & \text{self}.\text{stage} = 1 \\ B & \text{self}.\text{stage} = 3 \\ a & \text{otherwise} \end{cases} \]

22 function Circulate():
23 \hspace{1em} v_{\text{cur}}.\text{color} \leftarrow \text{colorUpdate}(v_{\text{cur}}.\text{color})
24 \hspace{1em} \text{MoveInTree}(v_{\text{cur}}.\text{firstChild})
25 \hspace{1em} \textbf{while } \text{next}_D(v_{\text{cur}}) \neq \bot \land \text{self}.\text{error} = \text{False} \textbf{ do}
26 \hspace{2em} v_{\text{cur}}.\text{color} \leftarrow \text{colorUpdate}(v_{\text{cur}}.\text{color})
27 \hspace{2em} \textbf{if } \text{self}.\text{level} = \text{self}.\text{phase} \textbf{ then}
28 \hspace{3em} \textbf{if } \text{self}.\text{stage} \in \{2, 4, 5\} \textbf{ then}
29 \hspace{4em} \text{Expand}()
30 \hspace{4em} \text{MoveInTree}(v_{\text{cur}}.\text{parent})
31 \hspace{2em} \textbf{else}
32 \hspace{3em} \text{MoveInTree(} \text{next}_D(v_{\text{cur}}) \text{)}

33 function MoveInTree(q):
34 \hspace{1em} \text{self}.\text{mode} \leftarrow \begin{cases} -1 & \text{v.level} > 0 \land q = v_{\text{cur}}.\text{parent} \\ 1 & \text{otherwise} \end{cases}
35 \hspace{1em} \text{self}.\text{level} \leftarrow \min(k, \max(0, \text{self}.\text{level} + \text{self}.\text{mode}))
36 \hspace{1em} \textbf{if } \text{self}.\text{mode} = 1 \textbf{ then}
37 \hspace{2em} \text{self}.\text{nextChild} \leftarrow \bot
38 \hspace{1em} \textbf{else}
39 \hspace{2em} \text{self}.\text{nextChild} \leftarrow v_{\text{cur}}.\text{nextChild}
40 \hspace{1em} \text{Migrate to } N(v_{\text{cur}}, q)
41 \hspace{1em} \textbf{if } \left( \text{self}.\text{level} \neq v_{\text{cur}}.\text{level} \lor (\text{self}.\text{nextChild} \neq \bot \land \text{self}.\text{nextChild} \leq p_{in}) \right)
42 \hspace{2em} \lor (\text{self}.\text{mode} = 1 \land p_{in} \neq v_{\text{cur}}.\text{parent}) \lor (\text{self}.\text{mode} = -1 \land p_{in} = v_{\text{cur}}.\text{parent})
43 \hspace{2em} \lor (\text{self}.\text{stage} = 1 \land \text{self}.\text{phase} = v_{\text{cur}}.\text{level} \land v_{\text{cur}}.\text{firstChild} \neq \bot) \textbf{ then}
44 \hspace{2em} \text{self}.\text{error} \leftarrow \text{True}
Algorithm 4: Function Expand() in $D_k$

43 function Expand():
44 $v_{cur}.port \leftarrow \delta_{v_{cur}} - 1$
45 $v_{cur}.firstName \leftarrow \perp$; self.nextChild $\leftarrow \perp$
46 while $v_{cur}.port \geq 0$ do
47 \hspace{1em} if $v_{cur}.port \neq v_{cur}.parent$ then
48 \hspace{2em} Migrate to $N(v_{cur}, v_{cur}.port)$
49 \hspace{2em} if (self.stage = 2 \land v_{cur}.color = B) \lor (self.stage = 4 \land v_{cur}.color = W) then
50 \hspace{3em} $v_{cur}.color \leftarrow R$; Migrate to $N(v_{cur}, p_{in})$
51 \hspace{2em} else if self.stage = 5 \land v_{cur}.color = R then
52 \hspace{3em} $v_{cur}.level \leftarrow self.level + 1$; $v_{cur}.parent \leftarrow p_{in}$; $v_{cur}.firstName \leftarrow \perp$
53 \hspace{3em} $v_{cur}.nextChild \leftarrow self.nextChild$
54 \hspace{3em} Migrate to $N(v_{cur}, p_{in})$
55 \hspace{3em} self.nextChild $\leftarrow p_{in}$
56 \hspace{3em} $v_{cur}.firstName \leftarrow p_{in}$
57 \hspace{3em} self.expanded $\leftarrow True$
58 \hspace{2em} else if self.stage = 5 \land v_{cur}.color = B \land v_{cur}.level < self.phase - 1 then
59 \hspace{3em} self.error $\leftarrow True$
60 \hspace{3em} Migrate to $N(v_{cur}, p_{in})$
61 \hspace{2em} end if
62 $v_{cur}.port \leftarrow \max(-1, v_{cur}.port - 1)$