Amplification of Quantum Meson Modes
in the Late Time of Chiral Phase Transition

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Abstract

It is shown that there exists a possibility of amplification of amplitudes of quantum pion modes with low momenta in the late time of chiral phase transition by using the Gaussian wave functional approximation in the O(4) linear sigma model. It is also shown that the amplification occurs in the mechanism of the resonance by forced oscillation as well as the parametric resonance induced by the small oscillation of the chiral condensate. These mechanisms are investigated in both the case of spatially homogeneous system and the spatially expanded system described by the Bjorken coordinate.
§1. Introduction

One of the recent interests in the context of the relativistic heavy ion collisions is to clarify the nature of matter at very high energy density. Especially, it is interesting to investigate the dynamics of the chiral phase transition in connection with the problem of the formation of a disoriented chiral condensate (DCC). Recently, Ikezi, Asakawa and one of the present authors (Y.T.) showed that there was a possibility of the formation of DCC by taking account of both the quantum fluctuations around the chiral condensate and the mode-mode coupling of quantum meson modes. In the formation of DCC, it is necessary that the low momentum components of the inhomogeneous chiral condensate were grown. Some authors pointed out that the parametric resonance was seen when the formation of DCC occurred.

Similarly, the amplitudes of quantum meson modes with low momenta may be grown in the chiral phase transition as well as the low momentum components of chiral condensate. In order to investigate the time evolution of the chiral condensate as an order parameter of chiral phase transition and of the fluctuation modes around it, it is suitable to adopt the time-dependent variational approach to dynamics of quantum fields in the O(4) linear sigma model in terms of a squeezed state or a Gaussian wave functional. This method is based on the time-dependent variational method with a Gaussian wave functional by using a functional Schrödinger picture.

In this paper, we will show that the quantum meson modes are amplified even in the late time of chiral phase transition in both the cases of uniform and spatially expanding system in the O(4) linear sigma model. Then, it will be pointed out that the possible mechanism of amplification is the resonance mechanism by the forced oscillation as well as the parametric resonance which was seen in the DCC formation. The unstable region in the parameter space is also presented concretely. This paper is organized as follows. In the next section, we recapitulate the time-dependent variational approach with a Gaussian wave functional to the O(4) linear sigma model. The time evolution of chiral condensate and the quantum meson modes are governed by the Klein-Gordon type equation of motion and the Liouville-von Neumann type equation of motion, respectively. The numerical results are also given in §3. In §4, the time evolution of quantum meson modes in the late time of chiral phase transition is investigated. In both the case of the uniform and the spatially expanding system, it is shown that there exists a possible solution of equation of motion for quantum meson modes, which reveals a parametric resonance and/or resonance by forced oscillation induced by the small oscillation of the chiral condensate. In §5, the unstable regions are depicted for quantum meson modes, in which the amplitudes of meson modes are amplified.
§2. Recapitulation of Gaussian wave functional approach to O(4) linear sigma model

In this section, we derive the equations of motion for the chiral condensate and quantum meson modes in the O(4) linear sigma model. We apply the time-dependent variational method based on the functional Schrödinger picture to the O(4) linear sigma model.

2.1. Time-dependent variational approach with a Gaussian wave functional

Following Refs. 16) and 17), we derive equations of motion for the chiral condensate and quantum fluctuations. We start with the following Hamiltonian:

\[ \hat{H} = \int d^3x \left\{ \frac{1}{2} \pi_a^2(x) + \frac{1}{2} (\nabla \varphi_a(x))^2 + \frac{m^2}{2} \varphi_a(x)^2 + \frac{\lambda}{24} (\varphi_a(x)^2)^2 - h \varphi_0(x) \right\}, \quad (2.1) \]

where \( a \) runs 0~3. The index 0 indicates the sigma field and 1~3 indicate the pion fields.

We adopt the following Gaussian wave functional as a trial wave functional in the framework of the functional Schrödinger picture:

\[ \Psi[\varphi(x)] = N \exp \left\{ i \langle \bar{\pi} | \varphi - \bar{\varphi} \rangle - \frac{1}{4} \left( G - i \Sigma \right) | \varphi - \bar{\varphi} \rangle \right\}, \quad (2.2) \]

where \( N \) is a normalization factor and \( G_{ab}(x, y, t), \Sigma_{ab}(x, y, t), \bar{\varphi}_a(x, t) \) and \( \bar{\pi}_a(x, t) \) define the real and imaginary parts of the kernel of the Gaussian and its average position and momentum. Here, we have used simple notations as

\[ \langle \bar{\pi} | \varphi \rangle = \int d^3x \sum_{a=0}^3 \bar{\pi}_a(x, t) \varphi_a(x), \]

\[ \langle \varphi | G | \varphi \rangle = \int d^3x d^3y \sum_{a,b=0}^3 \varphi_a(x) G_{ab}(x, y, t) \varphi_b(x), \quad (2.3) \]

and so on. In the functional Schrödinger picture, the operator \( \pi_a(x) \), which is a conjugate operator of the field operator \( \varphi_a(x) \), is regarded as \( -i \partial / \partial \varphi_a(x) \). The expectation value \( \langle \hat{O} \rangle \) for the field operator \( \hat{O} \) is easily calculated such as

\[ \langle \varphi_a(x) \rangle = \bar{\varphi}_a(x, t), \]

\[ \langle \pi_a(x) \rangle = \bar{\pi}_a(x, t), \]

\[ \langle \varphi_a(x) \varphi_b(y) \rangle = \bar{\varphi}_a(x, t) \bar{\varphi}_b(y, t) + G_{ab}(x, y, t), \]

\[ \langle \pi_a(x) \pi_b(y) \rangle = \bar{\pi}_a(x, t) \bar{\pi}_b(y, t) + \frac{1}{4} G_{ab}^{-1}(x, y, t) + 4 \Sigma \cdot G \cdot \Sigma_{ab}(x, y, t), \quad (2.4) \]
where we have used the following shorthanded notation:

\[ A \cdot B_{ab}(x, y, t) = \sum_{c=0}^{3} \int d^3z A_{ac}(x, z, t) B_{cb}(z, y, t). \]

It is understood that \( \bar{\varphi} \) represents the mean field which is identical with the chiral condensate in this model, and \( G \) represents quantum fluctuations around the mean field.

The time dependence of the Gaussian wave functional is determined by the time dependence of variational functions \( G, \Sigma, \bar{\varphi} \) and \( \bar{\pi} \), which is governed by the time-dependent variational principal:

\[ \delta \int dt \left\{ i \frac{\partial}{\partial t} - \hat{H} \right\} = 0. \] (2.5)

The variational equations give the canonical equations of motion for \( (\bar{\varphi}_a, \bar{\pi}_a) \) and \( (G_{ab}, \Sigma_{ab}) \) with the Hamiltonian \( \langle \hat{H} \rangle \). As a result, we obtain the following equations of motion:

\[ \dot{\bar{\varphi}} = \bar{\pi}, \]
\[ \dot{\bar{\pi}} = \left\{ (\nabla^2 - m^2 - \frac{\lambda}{6} \bar{\varphi}^2 - \frac{\lambda}{6} \text{tr}G(x, x)) - \frac{\lambda}{3}G(x, x) \right\} \bar{\varphi} + h\delta_{a0}, \] (2.6)
\[ \dot{G} = 2(\Sigma \cdot G + G \cdot \Sigma), \]
\[ \dot{\Sigma} = -\frac{1}{8}G^{-2} + 2\Sigma^2 + \frac{1}{2}(\nabla^2 + m^2 + \frac{\lambda}{12}(\bar{\varphi}^2 + \text{tr}G)) + \frac{\lambda}{6}((G + G) \bar{\varphi} + G). \] (2.7)

These are a set of basic equations of motion for the chiral condensate \( \bar{\varphi} \) and quantum fluctuations \( G \).

2.2. Reformulation of equations of motion for quantum fluctuations

In this subsection, we reformulate the equations of motion for quantum fluctuations in order to investigate clearly the time evolution of quantum meson modes. This reformulation has been firstly carried out in Ref.17. We introduce the reduced density matrix \( \mathcal{M}_{ab} \):

\[ \mathcal{M}_{ab}(x, y, t) + \frac{1}{2}\delta^3(x - y) = \begin{pmatrix} -i\langle \hat{\varphi}_a(x)\hat{\pi}_b(y) \rangle & \langle \hat{\varphi}_a(x)\hat{\varphi}_b(y) \rangle \\ \langle \hat{\pi}_a(x)\hat{\pi}_b(y) \rangle & i\langle \hat{\pi}_a(x)\hat{\varphi}_b(y) \rangle \end{pmatrix}, \] (2.8)

where \( \hat{\varphi}_a = \varphi_a - \bar{\varphi}_a \) and \( \hat{\pi}_a = \pi_a - \bar{\pi}_a \). Thus, the reduced density matrix can be expressed in terms of \( G \) and \( \Sigma \) as

\[ \mathcal{M} = \begin{pmatrix} -2iG \cdot \Sigma & G \\ \frac{1}{4}G^{-1} + 4\Sigma \cdot G \cdot \Sigma & 2i\Sigma \cdot G \end{pmatrix}. \] (2.9)

The time dependence of this reduced density matrix is governed by the Liouville-von Neumann type equation:

\[ i\dot{\mathcal{M}}_{ab}(x, y, t) = [\mathcal{H}, \mathcal{M}]_{ab}(x, y, t), \] (2.10)
where the generalized Hamiltonian $H_{ab}$ has particularly simple form as

$$H_{ab}(x, y, t) = \begin{pmatrix} 0 & \delta_{ab} \\ \Gamma_{ab}(x, t) & 0 \end{pmatrix} \delta^3(x - y)$$

(2.11)

with

$$\Gamma_{ab}(x, t) = -\nabla^2 \delta_{ab} + M^2_{ab}(x, t),$$

(2.12)

$$M^2_{ab}(x, t) = \left( m^2 + \frac{\lambda}{6} \vec{\varphi}(x, t) + \frac{\lambda}{6} \text{tr}G(x, x, t) \right) \delta_{ab} + \frac{\lambda}{3} \bar{\varphi}_a(x, t) \bar{\varphi}_b(x, t) + \frac{\lambda}{3} G_{ab}(x, x, t).$$

Equations of motion derived from (2.10) are equivalent to (2.7).

From the structure of $M$, it can be checked that the reduced density matrix satisfies a relation $M^2 = 1/4$. The eigenvalues of the reduced density matrix are thus $\pm 1/2$. We introduce an eigenvector of $M_{ab}$, $(u_a, v_a)$, with eigenvalue $1/2$, i.e.,

$$\int d^3 y M_{ab}(x, y, t) \begin{pmatrix} u_b(y, t) \\ v_b(y, t) \end{pmatrix} = \pm \frac{1}{2} \begin{pmatrix} u_a(x, t) \\ v_a(x, t) \end{pmatrix}.$$  

(2.13)

Then, $(u_a^*, -v_a^*)$ is also an eigenvector with eigenvalue $-1/2$. The eigenvectors are conveniently normalized to

$$\sum_{n=0}^{3} \int d^3 x (u_n^{(a)}(x, t))^* v_n^{(a)}(x, t) + v_n^{(a)}(x, t)^* u_n^{(a)}(x, t)) = \pm \delta_{mn}$$

(2.14)

with a $\pm$ sign for eigenvalues $\pm 1/2$. For the particular normalization condition we have adopted the eigenvectors $u_a$ and $v_a$ provide the following spectral decomposition for the reduced density matrix $M_{ab}$:

$$M_{ab}(x, y, t) = \frac{1}{2} \sum_n \begin{pmatrix} (u_n^{(a)}(x, t))^* & v_n^{(a)}(y, t) \\ v_n^{(a)}(x, t) & u_n^{(a)}(y, t) \end{pmatrix}$$

$$+ \begin{pmatrix} (u_n^{(a)}(x, t))^* \\ -v_n^{(a)}(x, t) \end{pmatrix} \begin{pmatrix} -v_n^{(a)}(y, t) & u_n^{(a)}(y, t) \end{pmatrix},$$

(2.15)

where sum runs over eigenstates with eigenvalues $+1/2$ only. From Eqs.(2.9) and (2.15), the Gaussian kernel $G$ and $\Sigma$ can be expressed in terms of $u$ and $v$, for example,

$$G_{ab}(x, y, t) = \frac{1}{2} \sum_n (u_n^{(a)}(x, t) u_n^{(b)}(y, t)^* + u_n^{(a)}(x, t)^* u_n^{(b)}(y, t)).$$

(2.16)

Thus, $u_a$ can be regarded as mode functions for quantum meson modes.
The equations of motion for mode functions can also be written by using the generalized Hamiltonian matrix:

\[
\begin{align*}
    i\partial_t \begin{pmatrix} u_a(x,t) \\ v_a(x,t) \end{pmatrix} &= \begin{pmatrix} 0 & \delta_{ab} \\ \Gamma_{ab}(x,t) & 0 \end{pmatrix} \begin{pmatrix} u_b(x,t) \\ v_b(x,t) \end{pmatrix} .
\end{align*}
\]

(2.17)

The normalization condition (2.14) is preserved by these equations. Using the explicit form of the mean field operator \( \Gamma_{ab} \), we see that the mode function \( u_a \) satisfy the set of coupled Klein-Gordon-type equations

\[
\begin{align*}
    (\Box \delta_{ab} + M^2_{ab}(x,t)) u_b(x,t) &= 0 . \\
    (\Box + M^2_0(x,t) - \frac{\lambda}{3} \varphi_0(x,t)^2) \varphi_0(x,t) &= h , \\
    (\Box + M^2_a(x,t)) u_a(x,t) &= 0 .
\end{align*}
\]

(2.18)

The matrix elements \( M^2_{ab} \) are found to be diagonal in the isospin index, i.e., \( M^2_{ab} = M^2_a \delta_{ab} \). Let us assume that \( \varphi_0 \neq 0 \) and \( \varphi_1 = \varphi_2 = \varphi_3 = 0 \), that is, the chiral condensate points in the sigma direction. Then the Eqs.(2.6) and (2.18) read

\[
\begin{align*}
    \left\{ \Box + M^2_0(x,t) - \frac{\lambda}{3} \varphi_0(x,t)^2 \right\} \varphi_0(x,t) &= h , \\
    (\Box + M^2_a(x,t)) u_a(x,t) &= 0 .
\end{align*}
\]

(2.19)

(2.20)

It should be noted here that, if the condensate is uniform spatially with translation invariance, namely, the condensate depends on only time, it is possible to carry out the Fourier transformation for \( u_a(x,t) \) as follows:

\[
    u_a(x,t) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k u^k_a(t)e^{ik\cdot x} .
\]

(2.21)

Then, the above forms are substituted into Eq.(2.20) and \( \varphi_0 \) depends only on time, we found following equations:

\[
\begin{align*}
    \left\{ \partial_t^2 + M^2_0(x,t) - \frac{\lambda}{3} \varphi_0(t)^2 \right\} \varphi_0(t) &= h , \\
    \left\{ \partial_t^2 + k^2 + M^2_a(x,t) \right\} u^k_a(t) &= 0 .
\end{align*}
\]

(2.22)

(2.23)

These equations are investigated later.

2.3. Geometry for the spatially expansion

In this subsection, we derive the equations of motion that is taken account of the geometry of one-dimensional spatial expansion in \( z \)-direction.\(^{20} \) For this purpose, the convenient variables, namely, the proper time \( \tau \) and usual rapidity variable \( \eta \) are defined by

\[
    \tau = \sqrt{t^2 - z^2} , \quad \eta = \frac{1}{2} \ln \frac{t + z}{t - z} .
\]

(2.24)
for one dimensional scaling case.\textsuperscript{21}) D’Alembertian in Eqs.\,(2.19) and (2.20) is rewritten by using these variables:

\[ \square = \frac{1}{\tau} \frac{\partial^2}{\partial \tau} - \frac{1}{\tau^2} \frac{\partial^2}{\partial \eta} - \frac{\partial^2}{\partial \perp^2}, \]  

where \( \partial^2_{\perp} = \partial^2_{x} + \partial^2_{y}. \)

If the condensate does not depend on \((x_\perp, \eta)\), namely, the condensate depends on only proper time \(\tau\), the quantum fluctuation modes \(u_a\) can be expressed as follows:\textsuperscript{7)}

\[ u_a(x_\perp, \eta, \tau) = \frac{1}{\sqrt{\tau}} \int_k u_a^{[k]}(\tau) e^{ik_\eta \eta} e^{ik_\perp \cdot x_\perp}, \quad (2.26) \]

where we have defined \( \int_k = \int d^2(k_\perp)dk_\eta/(2\pi)^{3/2}. \) Then, the equations of motion (2.19) and (2.20) are recast into the following equations:

\[ \left( \frac{\partial^2}{\partial \tau^2} + \frac{1}{\tau} \frac{\partial}{\partial \tau} + M_0^2(\tau) - \frac{\lambda}{3} \phi_0^2(\tau) \right) \phi_0(\tau) = h, \quad (2.27) \]

\[ \left[ \frac{\partial^2}{\partial \tau^2} + \frac{k_\eta^2 + 1/4}{\tau^2} + k_\perp^2 + M_a^2(x_\perp, \eta, \tau) \right] u_a^{[k]}(\tau) = 0, \quad (2.28) \]

where momentum \(k_\eta\) appears in the combination \((k_\eta^2 + 1/4)/\tau^2\).

\section{Numerical calculation}

Let us demonstrate qualitatively the time evolution of the mean field, which is in the sigma-direction only, and of quantum meson mode functions in the case of an uniform system in Eqs.\,\,(2.22) and (2.23) without spatially expansion. We assume \(\bar{\varphi}_i = 0\) with \(i = 1 \sim 3.\)

In the numerical calculation, we adopt the box normalization with the spatial length being \(L\) in each direction. We then impose the periodic boundary conditions for the fluctuation modes, namely the allowed values of momenta are \(k_x = (2\pi/L)n_x\) and so on, where \(n_x\) is integer. The fluctuation modes labeled by \((n_x, n_y, n_z)\) are included each direction up to \(n^2 = n_x^2 + n_y^2 + n_z^2 = 8^2.\) This corresponds to the momentum cutoff \(\Lambda \sim 1\text{GeV}(900\text{ MeV})\) since we have adopted the collisional region as \(L^3 = (10\text{ fm})^3.\) Further, we assume that fluctuation modes of the pion fields are identical one another, which are denoted as \(u_1^k = u_2^k = u_3^k \equiv u_\pi^k.\)

In Fig.1, the time evolution of chiral condensate is depicted. In order to avoid the complexity of problems with respect to the initial conditions in the relativistic heavy ion collisions, we only demonstrate the time evolution qualitatively with \(\bar{\varphi}_0(t = 0) = 40\text{ MeV}\) and \(\dot{\bar{\varphi}}_0 = 0.\) It is shown that the chiral condensate approaches to the vacuum value gradually.
Fig. 1. The time evolution of the chiral condensate is depicted. The horizontal and vertical axes represent time and the value of chiral condensation, respectively.

Fig. 2. The time evolution of the fluctuation mode with $n = 0$ in the $\sigma$-direction is depicted.

Fig. 3. The time evolution of the fluctuation mode with $n = 0$ in the $\pi$-direction is depicted.

Fig. 4. The time evolution of the fluctuation mode with $n = 1$ in the $\pi$-direction is depicted.

Fig. 5. The time evolution of the fluctuation mode with $n = 2$ in the $\pi$-direction is depicted.

Fig. 6. The time evolution of the fluctuation mode with $n = 1$ in the $\sigma$-direction is depicted.

with oscillation. In Fig. 2, the time evolution of the amplitude of quantum sigma meson mode with $n = 0$, that is $u_{0}^{k=0}$, is shown. This mode function only oscillates. On the other hand, as is seen in Fig. 3, the amplitude of the quantum pion mode with $n = 0$, that is $u_{1}^{k=0}$, is amplified. The amplitude of pion mode with $n = 1$ is also amplified weakly as is seen in Fig. 4. However, as is seen, for example, in Fig. 5, the amplification is not realized in the modes with $n \geq 2$. Also, the sigma meson modes with $n \geq 1$ as well as $n = 0$ are not amplified.
as is seen in Fig. 6. Thus, we conclude that, in this parameterization, the amplitudes of the lowest and the first excited pion modes are amplified, but other quantum meson modes are not grown.

Next, let us demonstrate qualitatively the time evolution of mean field and of quantum meson mode functions in Eqs. (2.27) and (2.28) with spatially expansion. We assume the same conditions with respect to the chiral condensate as those in the case of the homogeneous condensate, namely $\bar{\varphi}_0 \neq 0$ and $\bar{\varphi}_i = 0$ for $i = 1 \sim 3$, and the same initial condition for $\tau$ instead of $t$. We then impose the periodic boundary conditions for the fluctuation modes, $k_x = (2\pi/L)n_x$, $k_y = (2\pi/L)n_y$ and $k_\eta = (2\pi/\alpha)n_\eta$, where $n_x, n_y, n_\eta$ are integer, $L$ is spatial length in $x$-$y$ direction and $\alpha$ is dimensionless parameter. The fluctuation modes labeled by $(n_x, n_y, n_\eta)$ are counted up to $(2\pi/L)^2(n_x^2 + n_y^2) + (2\pi/\alpha)^2(n_\eta^2 + \frac{1}{4})^2 \leq 1$GeV, where we take $L = 10$ fm and $\alpha = 4$. In Fig. 7, the time evolution of chiral condensate with the expanding geometry is depicted. It is seen that the behavior of damped oscillation is realized quicker than the case without spatially expansion. In Figs. 8 and 9, the time evolution of the lowest ($n = 0$) and the first excited ($n = 1$) pion modes is depicted. They show that the amplitudes

Fig. 7. The time evolution of the chiral condensate with one dimensional expansion is depicted.

Fig. 8. The time evolution of the fluctuation mode with $n = 0$ in the $\pi$-direction is depicted.

Fig. 9. The time evolution of the fluctuation mode with $n = 1$ in the $\pi$-direction is depicted.

Fig. 10. The time evolution of the fluctuation mode with $n = 0$ in the $\sigma$-direction is depicted.
of these modes are amplified weakly. However, the sigma modes are not amplified even the lowest sigma mode \((n = 0)\) as is seen in Fig. 10.

It should be here noted that the amplification occurs in the late time of chiral phase transition both the case of no spatially expansion and of the one-dimensional spatial expansion. Namely, even when the chiral condensate oscillates around its vacuum value with small amplitude, there are amplified solutions of pion modes. The mechanism of amplification is clarified in the next section.

§4. Late time of chiral phase transition

In this section, we investigate the time evolution of the chiral condensate and quantum meson modes based on Eqs.(2.22) and (2.23) or (2.27) and (2.28) in the late time of chiral phase transition. Especially, we concern with the amplification of quantum meson modes, which was shown in the previous section. It is shown that possible mechanism may be an amplification by the forced oscillation as well as the parametric resonance.

4.1. Linear approximation around static configurations without expansion

Hereafter, we assume that fluctuation modes of the pion fields are identical one another as was assumed in the previous subsection. Also, the condensate depends on time only. When the explicit chiral symmetry breaking term, \(h\), is small, the static solutions in Eq.(2.22) and (2.23) are given as

\[
\bar{\varphi}_0 = \varphi_0 \equiv \sqrt{\frac{3}{\lambda}} M_0^2 - \frac{h}{2M_0^2},
\]

\[
u^k_a(t) = \psi^k_a(t) = \frac{1}{\sqrt{2E^k_a}} e^{-iE^k_a t},
\]

(4.1)

where \(E^k_a = \sqrt{k^2 + M_0^2}\). The Fourier mode \(1/\sqrt{2E^k_a}\) has been determined by the normalization condition (2.14).

Let us investigate the time-dependent solutions around the above static configurations. The condensate \(\bar{\varphi}_0\) and quantum meson modes \(u^k_a\) can be expanded around the static solutions of Eq.(4.1):

\[
\bar{\varphi}_0(t) = \varphi_0 + \delta \varphi(t),
\]

\[
u^k_a(t) = \psi^k_a(t) + \delta \psi^k_a(t).
\]

(4.2)

Here, we consider the late time of the chiral phase transition. Then, \(|\delta \varphi(t)|\) is small compared with the vacuum value \(\varphi_0\). Further, we assume that \(|\delta \psi^k_a| \ll |\psi^k_a|\). Also, in general, the
fluctuation $G$ given in (2.4) is small compared with $\varphi^2_0$. Since $G$ can be written in the form in (2.16), we conclude that the following relation should be satisfied:

$$\bar{\varphi}_0^2 \gg \frac{1}{(2\pi)^3} \int d^3k |u_a^k|^2,$$

$$|\bar{\varphi}_0 \delta \varphi(t)| \gg \frac{1}{(2\pi)^3} \int d^3k(u_a^k(t) \delta u_a^k(t)^* + u_a^k(t)^* \delta u_a^k(t)).$$

(4.3)

Under the above approximation, the equation of motion for $\delta \varphi(t)$ and $\delta u_a^k(t)$ are obtained from Eqs.(2.22) and (2.23). For the condensate $\delta \varphi_0(t)$,

$$(\partial_t + M_0^2)\delta \varphi(t) = 0$$

(4.4)

is obtained. A solution of (4.4) is written as

$$\delta \varphi(t) = -\delta \sigma \cos(M_0 t + \phi),$$

(4.5)

where $\delta \sigma$ and $\phi$ are constants. We rewrite hereafter $a = 0$ and $1 \sim 3$ into $\sigma$ and $\pi$. Let us introduce the following dimensionless variables:

$$\delta \tilde{u}_a^k(t') = \sqrt{2M_0} \delta u_a^k(t), \quad \gamma t' = M_\sigma t,$$

$$\omega_\sigma^2 = \frac{\gamma^2(k^2 + M_\sigma^2)}{M_\sigma^2}, \quad \omega_\pi^2 = \frac{\gamma^2(k^2 + M_\pi^2)}{M_\pi^2},$$

$$h_\sigma = \frac{\lambda \varphi_0^2}{k^2 + M_\sigma^2} \cdot \frac{\delta \sigma}{\varphi_0}, \quad h_\pi = \frac{1}{3} \cdot \frac{\lambda \varphi_0^2}{k^2 + M_\pi^2} \cdot \frac{\delta \sigma}{\varphi_0},$$

$$F_\sigma = \frac{\gamma^2 \lambda \varphi_0^2}{M_\sigma^2} \left( \frac{M_\sigma^2}{k^2 + M_\sigma^2} \right)^{1/4} \cdot \frac{\delta \sigma}{\varphi_0},$$

$$F_\pi = \frac{\gamma^2 \lambda \varphi_0^2}{3M_\pi^2} \left( \frac{M_\pi^2}{k^2 + M_\pi^2} \right)^{1/4} \cdot \frac{\delta \sigma}{\varphi_0},$$

(4.6)

where we put $\gamma = 2$. If we set up $\phi = 0$ without the loss of generality, then the equations of motion for the meson modes derived from (2.23), (4.1) and (4.2) can simply be expressed as

$$\left[ \frac{d^2}{dt'^2} + \omega_\alpha^2 [1 - h_\alpha \cos(\gamma t')] \right] \delta \tilde{u}_\alpha^k(t') = F_\alpha \cos(\gamma t') e^{-i\omega_\alpha t'},$$

(4.7)

where $\alpha = \sigma$ or $\pi$. If $F_\sigma$ ($F_\pi$) is negligible, then Eq.(4.7) is reduced to Mathieu’s equation. In this case, the existence of the unstable solution for $\delta \tilde{u}_\alpha^k(t')$ may be expected. This phenomena is well known as parametric resonance in classical mechanics. On the other hand, the forced oscillation may be realized by the effect of $F_\sigma$ ($F_\pi$) even if the model parameters do not offer the unstable regions for the parametric resonance.
4.2. **Linear approximation around static solutions with expansion**

In this subsection, we investigate the case with the expanding geometry. We derive the equations of motion with expansion using the same formalism in §4.1 in the late time of chiral phase transition. The solution of (2-28) is given as

\[ u_a^{[k]}(\tau) = u_a^{[k]}(\tau) \equiv u_a^{[k]} e^{-iE_a^{[k]} \tau}, \tag{4.8} \]

where \( E_a^{[k]} = \sqrt{(k^2 + 1/4)/\tau^2 + k_\perp^2 + M_a^2} \). We expand the proper-time dependent variables \( \bar{\varphi}_0(\tau) \) and \( u_a^{[k]}(\tau) \) in (2-26) around \( \varphi_0 \) and the above solution (4.8), respectively:

\[ \bar{\varphi}_0(\tau) = \varphi_0 + \delta \varphi_0(\tau), \]

\[ u_a^{[k]}(\tau) = u_a^{[k]}(\tau) + \delta u_a^{[k]}(\tau). \tag{4.10} \]

Corresponding to (4.3), the following approximations are adopted:

\[ \bar{\varphi}_0^2 \gg \frac{1}{\tau} \cdot \frac{1}{(2\pi)^3} \int d^2k_\perp \int dk_\eta |u_a^{[k]}|^2, \]

\[ \bar{\varphi}_0 \delta \varphi_0(\tau) \gg \frac{1}{\tau} \cdot \frac{1}{(2\pi)^2} \int d^2k_\perp \int dk_\eta (u_a^{[k]}(\tau) \delta u_a^{[k]}(\tau) + u_a^{[k]}(\tau) \delta u_a^{[k]}(\tau)). \tag{4.11} \]

Under the above approximation, the equations of motion for \( \delta \varphi_0(\tau) \) and \( \delta u_a^{[k]}(\tau) \) are obtained from Eqs.(2.27) and (2.28) as

\[ \left\{ \left( \partial_\tau^2 + \frac{1}{\tau^2} \partial_\tau + M_\sigma^2(\tau) \right) \right\} \delta \varphi_0(\tau) = 0, \tag{4.12} \]

\[ \left( \partial_\tau^2 + \frac{k^2 + 1/4}{\tau^2} + k_\perp^2 + M_\sigma^2 + \lambda \varphi_0 \cdot \delta \varphi_0(\tau) \right) \delta u_a^{[k]}(\tau) = -\lambda \varphi_0(\tau) \cdot \delta \varphi_0(\tau) u_a^{[k]} e^{-iE_a^{[k]} \tau}, \]

\[ \left( \partial_\tau^2 + \frac{k^2 + 1/4}{\tau^2} + k_\perp^2 + M_\pi^2 + \frac{\lambda}{3} \varphi_0 \cdot \delta \varphi_0(\tau) \right) \delta \varphi_\pi(\tau) = -\frac{\lambda}{3} \varphi_0(\tau) \cdot \delta \varphi_0(\tau) u_\pi^{[k]} e^{-iE_a^{[k]} \tau}. \tag{4.13} \]

Here, we have omitted \( \partial E_a^{[k]}/\partial \tau \) because we will replace \( \tau \) into \( \tau_c \) in the second term in (2.25) in order to stress the effect of friction due to the spatial expansion, where \( \tau_c \) may be taken as the proper time when the hadronization occurs. Thus, by putting \( \tau \) into \( \tau_c \) of the second term in the left-hand side in (4.12), the amplitude of chiral condensate around the vacuum value reveals a behavior of a damped oscillation. As is well known, this equation has three-type solutions. For \( (1/\tau_c)^2 < 4M_\sigma^2 \), we obtain

\[ \delta \varphi_0(\tau) = -\delta \varphi e^{-\tau/2\tau_c} \cos(\omega \tau + \phi), \tag{4.14} \]
where $\delta \sigma$ and $\phi$ are determined by the initial conditions and

$$
\omega = \sqrt{M^2 - \frac{1}{4\tau_c^2}} .
$$

(4-15)

Here, we adopt $\phi = 0$ for simplicity. We define the following dimensionless variables:

$$
\delta \tilde{u}_{\alpha}^{[k]}(t') = \sqrt{2\omega} \delta u_{\alpha}^{[k]}(\tau) , \quad \omega \tau \equiv \gamma t' , \quad \mu = \frac{\gamma}{\sqrt{4M^2\tau_c^2 - 1}} ,
$$

$$
\omega_{\alpha}^2 = \frac{\gamma^2}{\omega^2} \left( \frac{(k_0^2 + 1/4)/\tau_c^2 + k_1^2 + M_{\alpha}^2}{\omega^2} \right) ,
$$

$$
h_{\sigma} = \frac{\lambda \varphi_0^2}{(k_0^2 + 1/4)/\tau_c^2 + k_1^2 + M_{\sigma}^2} \cdot \frac{\delta \sigma}{\varphi_0} , \quad F_{\sigma} = \frac{\gamma^2 \lambda \varphi_0^2}{\omega^2} \cdot \sqrt{2\omega}u_{\sigma}^{[k]} \cdot \frac{\delta \sigma}{\varphi_0} ,
$$

$$
h_{\pi} = \frac{1}{3} \cdot \frac{\lambda \varphi_0^2}{(k_0^2 + 1/4)/\tau_c^2 + k_1^2 + M_{\pi}^2} \cdot \frac{\delta \sigma}{\varphi_0} , \quad F_{\pi} = \frac{\gamma^2 \lambda \varphi_0^2}{3\omega^2} \cdot \sqrt{2\omega}u_{\pi}^{[k]} \cdot \frac{\delta \sigma}{\varphi_0} ,
$$

(4-16)

where $\alpha = \sigma$ or $\pi$. By introducing the above dimensionless variables, $\delta \varphi_0$ and the equation of motion for quantum meson modes in (4-13) are recast into

$$
\delta \varphi_0(t') = -\delta \sigma e^{-\mu t'} \cos(\gamma t') ,
$$

(4-17)

$$
\left\{ \frac{d^2}{dt'^2} + \omega_{\alpha}^2 \left[ 1 - h_{\alpha} \cos(\gamma t') e^{-\mu t'} \right] \right\} \delta \tilde{u}_{\alpha}^{[k]}(t') = F_{\alpha} \cos(\gamma t') e^{-\mu t'} e^{-i\omega \mu t'} .
$$

(4-18)

The unstable regions in which the amplified solutions of (4-7) and (4-18) exist will be investigated in the next section numerically.

§5. Unstable regions for quantum meson modes

In this section, we numerically show the unstable regions in which the absolute values of the amplitudes $\delta \tilde{u}$ derived from solutions of the equations of motion for quantum meson

Fig. 11. The unstable regions are depicted in the usual Mathieu equation.
modes in (4.7) without spatial expansion and in (4.18) with spatially expansion, respectively, are amplified. This phenomena are seen in the lowest and the first excited pion modes as was demonstrated in the previous numerical calculations.

Hereafter, we omit the subscript $\alpha$. In Fig.11, the unstable regions, which are represented in the dark area on the figure, for the usual Mathieu equation are depicted, which corresponds to the case $F_{\alpha} = 0$ in (4.7). The horizontal axis represents $\omega^2 h/2$ and the vertical axis represents $\omega^2$, respectively. On the other hand, for $F_{\alpha} \neq 0$, the unstable regions are added to those given in the usual Mathieu equation because of the effect of the right-hand side in (4.7) which reveals the effect of the forced oscillation or the beat. Here and hereafter, we obtain the unstable regions, in which the amplification of the quantum fluctuation mode functions occurs, as follows: We performed the time-integration in a certain time interval of the magnitude of the quantum meson mode function, which means a kind of the time average. Then, we compare a time-integration in a certain time step with the proceeding one. If the value of time-integration is larger than the last one for all time steps under consideration, then we decided that this mode is unstable. In Figs.12 and 13, the cases $F_{\alpha} = 1.0$ and 3.0 are shown, where the time-integration is taken in 10 fm/c in order to judge the unstable or stable solutions up to 100 fm/c. From these results, it is obviously found that the unstable regions become wider due to the effects of the forced oscillation and/or the beat induced by the oscillation of the chiral condensate.

For the case in Eq.(4.18) where the friction term appears in the equation of motion because the system has the spatially expanding geometry, the unstable regions are depicted with $F_{\alpha} = 0$, 1 and 3 in Figs.14, 15 and 16, respectively. The parameter $\mu$ is determined by adopting $M_{\sigma} = 500$ MeV and $\tau_c = 10$ fm/c. The time interval for the time-integration to determine the unstable regions in our approach is taken as rather short time, 4 fm/c, because the unstable regions are disappeared by strong damping effect for long time past. We calculate the time evolution up to 40 fm/c. In Fig.14 with $F_{\alpha} = 0$, the result for the

![Fig. 12. The unstable regions are depicted for the solutions in Eq.(4.7) with $F_{\alpha} = 1.0$.](image1)

![Fig. 13. The unstable regions are depicted for the solutions in Eq.(4.7) with $F_{\alpha} = 3.0$.](image2)
Fig. 14. The unstable regions are depicted in the Mathieu equation with friction term in (4.18), in which we set $F_\alpha = 0$.

Fig. 15. The unstable regions are depicted for the solutions in Eq.(4.18) with $F_\alpha = 1.0$.

Fig. 16. The unstable regions are depicted for the solutions in Eq.(4.18) with $F_\alpha = 3.0$.

unstable regions is well known because this case corresponds to the case of the Mathieu equation with a friction term. In Fig. 15 and 16, we introduce the effect of $F_\alpha$ in the realistic situation in Eq.(4.18) which gives the forced oscillation or the beat. However, these effects only give a small modification. The reason is that the friction works strongly in this model. As a result, the amplification of the quantum meson modes with spatially expanding geometry is not so strong, although the amplification appears. This results mean that the effect of forced oscillation or the beat ceases to work for spatially expanding geometry.

§6. Summary

We have demonstrated that the amplitudes of quantum pion modes are amplified even in the late time of chiral phase transition by the mechanism of the forced oscillation as well as the parametric resonance in the framework of the $O(4)$ linear sigma model. The basic equations of motion have been derived in the time-dependent variational approach with the Gaussian wavefunctional, in which the equations of motion of the chiral condensate and quantum meson mode functions have been formulated in the form of the Klein-Gordon type
equation and the Liouville-von Neumann type equation of motion, respectively, in both the cases of the uniform and the one-dimensional spatially expanding systems. These equations of motion have been solved numerically, and it has been shown that the amplified solutions for the quantum pion modes with low momenta exist really.

We have investigated the mechanism for this amplification phenomena, and have pointed out that there is a possibility of the resonance by forced oscillation or the beat as well as the parametric resonance induced by the oscillation of the chiral condensate around its vacuum value, while the effect of the forced oscillation ceases to work due to the strong friction for the case that has the spatially expanding geometry. Also, by numerical calculation, we have concretely given the parameter regions in which the unstable solutions for the quantum meson mode functions exist. Of course, it is necessary to know the initial conditions in order to judge whether the amplification of quantum meson mode occurs or not, namely, the parameters are in the unstable region or not, in the realistic relativistic heavy ion collisions.

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