DENSITY OF POWER-FREE VALUES OF POLYNOMIALS

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Abstract. We establish asymptotic formulae for the number of $k$-free values of polynomials $F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ of degree $d \geq 2$ for any $n \geq 1$, including when the variables are prime, as long as $k \geq (3d+1)/4$. In addition we prove an asymptotics for $k = d - 1$ in the cases $d = 3, 4$. Thus we generalize a work of Browning, while we use a different sieving technique for the middle range of primes.

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1. Introduction

In this paper we consider power-free values of polynomials $F(x_1, \cdots, x_n)$ with integer coefficients and degree $d \geq 2$. Put

$$N_{F,k}(B) = \# \{ (x_1, \cdots, x_n) \in \mathbb{Z}^n : |x_i| \leq B \text{ for } i = 1, \cdots, n, F(x_1, \cdots, x_n) \text{ is } k\text{-free} \}.$$  

(1.1)

Our goal is to show that $N_{F,k}(B)$ satisfies an asymptotic formula provided that $F$ is not always divisible by $p^k$ for a fixed prime $p$, and $k$ is suitably large compared to $d$.

The case $k = d - 1$ is of particular interest. In the case of polynomials in a single variable, the first to establish the infinitude of $N_{F,d-1}(B)$ was Erdős [5]. His argument did not establish an asymptotic formula for $N_{F,d-1}(B)$; this had to wait until Hooley [10]. In the two variable case, the asymptotic formula for $N_{F,d-1}(B)$ is only established in the case when $F$ splits into linear factors over some extension of $\mathbb{Q}$; this was done by Hooley in [13].

For general $k$ and $n = 1$, various authors have worked on the problem of estimating $N_{F,k}(B)$. The record is a theorem of Browning [4], which asserts that the expected asymptotic formula for $N_{F,k}(B)$ holds when $k \geq (3d+1)/4$. The main point of our paper is to reduce the problem for general $n$ to the setting of Browning’s theorem.
In the case of multiple variables, most of the work has been done in the case of binary forms only. The asymptotic formula for \( N_{F,k}(B) \) for binary forms \( F \) was established for \( k \geq (d - 1)/2 \) by Greaves [7], \( k > (2\sqrt{2} - 1)d/4 \) by Filaseta [6], \( k > 7d/16 \) by Browning [4], and \( k > 7d/18 \) by Xiao [17]. For polynomials of more variables, Bhargava handled square-free values for discriminants of representations of some pre-homogeneous vector spaces in [1], and Bhargava, Shankar, Wang handled the case of discriminants of polynomials in [3]. Xiao handled the case of square-free values of decomposable forms in [18]. In fact, he obtained an asymptotic relation for \( N_{F,2}(B) \) when \( F \) is decomposable whenever \( d \leq 2n + 2 \).

For inhomogeneous polynomials of two variables the works of Hooley [12] and Browning [4] provided lower bounds for the number of \( k \)-free values and Hooley [13] managed to provide an asymptotic formula in certain cases. There are also several specific inhomogeneous polynomials of more variables whose power-free values were estimated asymptotically, e.g. [14], [15].

Our main result in this paper is the following theorem, which asserts that an asymptotic relation for \( N_{F,k}(B) \) holds whenever \( k \geq (3d + 1)/4 \), for any \( n \geq 1 \) and assuming only a natural condition on the divisibility of the polynomial \( F \). Observe that the lower bound for \( k \) is the same as in Browning’s theorem in [4].

**Theorem 1.1.** Let \( k \geq 2 \) be a positive integer and let \( F \) be a polynomial with integer coefficients and degree \( d \geq 2 \) in \( n \) variables, such that for all primes \( p \), there exists an integer \( n \)-tuple \((m_1, \ldots, m_n)\) such that \( p^k \nmid F(m_1, \ldots, m_n) \). Then there exists a positive number \( C_{F,k} \) such that the asymptotic relation

\[
N_{F,k}(B) \sim C_{F,k}B^n
\]

holds whenever \( k \geq (3d + 1)/4 \).

Here the constant term is given by the limit of an absolutely convergent infinite product

\[
C_{F,k} = \prod_p \left( 1 - \frac{\rho_F(p^k)}{p^{kn}} \right),
\]

which is positive under our assumptions. (The convergence is a well-known fact when \( n = 1 \) and follows from (2.5) for \( n \geq 2 \).) In particular, whenever \( d \geq 5 \), the quantity \( N_{F,d-1}(B) \) will satisfy the expected asymptotic formula.

Following Browning [4], we can also handle the case when we restrict the inputs to be primes. This extends an Erdős conjecture for \((d - 1)\)-free values of one-variable polynomials to multi-variable polynomials with \( d \geq 5 \). We thus obtain the following theorem:

**Theorem 1.2.** Let \( k \geq 2 \) be a positive integer and let \( F \) be a polynomial with integer coefficients and degree \( d \geq 2 \) in \( n \) variables, such that for all primes \( p \), there exists an integer \( n \)-tuple \((m_1, \ldots, m_n)\) such that \( p^k \nmid F(m_1, \ldots, m_n) \). Put

\[
N_{F,k}(B) = \#\{(p_1, \ldots, p_n) \in \mathbb{Z}^n : |p_i| \leq B \text{ for } 1 \leq i \leq n, F(p_1, \ldots, p_n) \text{ is } k\text{-free}\},
\]
where \( p_i \) is prime for \( 1 \leq i \leq n \). Then there exists a positive number \( C'_{F,k} \) such that the asymptotic relation
\[
N_{F,k}(B) \sim C'_{F,k} \frac{B^n}{(\log B)^n}
\]
holds whenever \( k \geq (3d + 1)/4 \).

Here we have
\[
C'_{F,k} = \prod_p \left(1 - \frac{\rho_F(p^k)}{\phi(p^k)n}\right),
\]
which is again positive under our assumptions.

The proof of Theorems 1.1 and 1.2 will largely rely on the affine determinant method of Heath-Brown [9], which is the same tool used by Browning in [4]. The main innovation in this paper is using sieve methods to handle medium sized primes which contribute to \( N_{F,k}(B) \) and \( N_{F,k}(B) \). Resolving the contribution of the middle range would also allow us to provide the expected asymptotic formula for \( N_{F,d-1} \) for the cases not covered in Theorem 1.1, namely \( d = 3, 4 \). As Hooley himself suggests in [12] this is possible using his early method from [11]. More precisely, one can estimate the last range of primes contributing to \( N_{F,d-1}(B) \) with the help of the Gallagher’s larger sieve.

**Theorem 1.3.** Let \( F \) be a polynomial with integer coefficients and degree \( d \geq 3 \) in \( n \) variables, such that for all primes \( p \), there exists an integer \( n \)-tuple \((m_1, \ldots, m_n)\) such that \( p^k \nmid F(m_1, \ldots, m_n) \). Then there exists a positive number \( C_{F,d-1} \) such that the following asymptotic relation holds
\[
N_{F,d-1}(B) \sim C_{F,d-1} B^n.
\]

Due to the error terms produced in the proof of Theorem 1.3, however, we could not resolve the prime input problem for the \((d - 1)\)-free values of a polynomial \( F(x_1, \ldots, x_n) \) of degree \( d = 3, 4 \).

2. **Preliminaries: the simple, Ekedahl’s and Gallagher’s larger sieves**

2.1. **Counting \( k \)-free values over integer inputs.** We shall find quantities \( N_1(B), N_2(B), N_3(B) \) such that
\[
(2.1) \quad N_1(B) - N_2(B) - N_3(X) \leq N_{F,k}(B) \leq N_1(B),
\]
and for which \( N_2(B), N_3(B) = o(B^n) \). This is similar to the simple sieve technique of Hooley.

Let \( \xi_1 = \frac{1}{nk} \log B \) and \( \xi_2 = B(\log B)^{1/2} \), and put
\[
(2.2) \quad N_1(B) = N_{1,k}(B) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}^n : |x_i| \leq B, p^k \nmid F(x_1, \ldots, x_n) \Rightarrow p > \xi_1\},
\]
\[
(2.3) \quad N_2(B) = N_{2,k}(B) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}^n : |x_i| \leq B, p^k \mid F(x_1, \ldots, x_n) \Rightarrow p > \xi_1, \exists \xi_1 < p \leq \xi_2 \text{ s.t. } p^2 \mid F(x_1, \ldots, x_n)\}
\]
and
\[ (2.4) \]
\[
N_3(B) = N_{3, k}(B) = \# \{ (x_1, \ldots, x_n) \in \mathbb{Z}^n : |x_i| \leq B, p^k | F(x_1, \ldots, x_n) \Rightarrow p > \xi_1, \\
p^2 \nmid F(x_1, \ldots, x_n) \text{ for } \xi_1 < p \leq \xi_2, \exists p > \xi_2 \text{ s.t. } p^k | F(x_1, \ldots, x_n) \}. 
\]
Then it is clear that (2.1) holds.

We first show that \( N_1(X) \) gives us a term with the expected order of magnitude. Put
\[
\rho_F(m) = \# \{ (m_1, \ldots, m_n) \in \mathbb{Z}^n : F(m_1, \ldots, m_n) \equiv 0 \pmod{m} \}. 
\]
By the same argument as in Lemma 2.1 of [18], we see that
\[ (2.5) \]
\[
\rho_F(p^k) = O_{d,n} \left( p^{n(k-1)} + p^{k(n-1)} \right). 
\]
Now put
\[
N(B; h) = \# \{ x \in \mathbb{Z} : |x| \leq B, h^k | F(x_1, \ldots, x_n) \}. 
\]
By standard properties of the Möbius function \( \mu \) we then see that
\[ (2.6) \]
\[
N_1(B) = \sum_{h \in \mathbb{N}} \mu(h) N(B; h) \\
= \sum_{h \in \mathbb{N}} \mu(h) \rho_F(h^k) \left\{ \frac{B^n}{h^{nk}} + O \left( \frac{X^{n-1}}{h^{k(n-1)}} + 1 \right) \right\}. 
\]
Since the summand vanishes when \( h \) is not square-free, we may assume that \( h \) is in fact square-free. Hence
\[
h \leq \prod_{p \leq \xi_1} p = \exp \left( \sum_{p \leq \xi_1} \log p \right) \leq e^{2\xi_1}. 
\]
By (2.5), we then see that
\[
N_1(B) = B^n \prod_{p \leq \xi_1} \left( 1 - \frac{\rho_F(p^k)}{p^{nk}} \right) + O \left( \sum_{h \leq e^{2\xi_1}} h^\epsilon (h^{k(n-1)} + h^{k(n-1)}) \left( B^{n-1} h^{-k(n-1)} + 1 \right) \right). 
\]
The big-\( O \) term can be evaluated to be
\[
O_{\epsilon} \left( B^\epsilon \left( B^{n-1+\frac{1}{nk}} + B^{n-1+\frac{k-n+1}{nk}} + B^{k(n-1)} + B^{n(k-1)} \right) \right), 
\]
and we see that this is \( O(B^{n-\delta}) \) for some \( \delta > 0 \). Thus we have
\[ (2.7) \]
\[
N_1(B) = C_{F,k} B^n + o(B^n) + O(B^{n-\delta}). 
\]
Next we give an estimate for \( N_2(B) \).

**Lemma 2.1.** Let \( N_2(B) \) be as in (2.3). Then
\[
N_2(B) = O_d \left( \frac{B^n}{\xi_1} + \frac{B^{n-1} \xi_2}{\log \xi_2} \right). 
\]
To prove this lemma, we will need the following result due to Ekedahl; the formulation below is due to Bhargava and Shankar [2].
Lemma 2.2 (Ekedahl). Let $\mathcal{B}$ be a compact region in $\mathbb{R}^n$ with positive measure, and let $Y$ be any closed subscheme of $A^n_{\mathbb{Q}}$ of co-dimension $k \geq 2$. Let $r$ and $M$ be positive real numbers. Then we have

$$\# \{v \in r\mathcal{B} \cap \mathbb{Z}^n : v \equiv (\text{mod } p) \in Y(\mathbb{F}_p) \text{ for some } p > M \} = O \left( \frac{r^n}{M^{k-1} \log M} + r^{n-k+1} \right).$$

Proof of Lemma 2.1. We fix the variables $x = (x_2, \cdots, x_n)$, and for each such choice, we put

$$F_x(x) = F(x, x_2, \cdots, x_n).$$

Define the function, for a polynomial $f$ in a single variable $x$, by

$$\rho_f(m) = \# \{ s \in \mathbb{Z}/m\mathbb{Z} : f(s) \equiv 0 \pmod{m} \}.$$ 

It is clear from the Chinese Remainder Theorem that $\rho_f$ is multiplicative. Observe that $\rho_f(p) \leq d$; the bound is independent of $p$. Moreover if $p \nmid \Delta(f)$, then for any positive integer $k$ we have $\rho_f(p^k) \leq d$. We now use Lemma 2.2 as follows. Let $G(x_1, \cdots, x_n) = \frac{\partial F}{\partial x_1}(x_1, \cdots, x_n)$. Define the variety $V_{F,G}$ to be

$$(2.8) \quad V_{F,G} = \{ (x_1, \cdots, x_n) \in \mathbb{C}^n : F(x_1, \cdots, x_n) = G(x_1, \cdots, x_n) = 0 \}.$$ 

Observe that $V_{F,G}$ is of co-dimension two and is defined over $\mathbb{Z}$. Put

$$N_p^*(B) = \# \{(x_1, \cdots, x_n) \in \mathbb{Z}^n : |x_i| \leq B, (x_1, \cdots, x_n) \pmod{p} \in V_{F,G}(\mathbb{F}_p)\}.$$ 

It follows from Lemma 2.2 that

$$\sum_{p > \xi_1} N_p^*(B) = O \left( \frac{B^n}{\xi_1 \log \xi_1} + B^{n-1} \right),$$ 

which is an acceptable error term.

Now put

$$N_p^+(B) = \# \{(x_1, \cdots, x_n) \in \mathbb{Z}^n : |x_i| \leq B, p^2 | F(x_1, \cdots, x_n), p \nmid G(x_1, \cdots, x_n)\}.$$ 

It then follows that

$$N_2(B) \leq \sum_{\xi_1 < p \leq \xi_2} N_p^+(B) + \sum_{\xi_1 < p \leq \xi_2} N_p^*(B),$$

and since we have already estimated the second sum, it suffices to estimate the former. For fixed $(x_2, \cdots, x_n)$, the solutions to $f(x) = F(x, x_2, \cdots, x_n) \equiv 0 \pmod{p^2}$ contributing to $N_p^+(X)$ must satisfy $p \nmid \Delta(f)$; in particular, the number of solutions is at most $d$. We then have that

$$\sum_{\xi_1 < p \leq \xi_2} N_p^+(B) = O_d \left( B^{n-1} \sum_{\xi_1 < p \leq \xi_2} \left( \frac{B}{p^2} + 1 \right) \right) = O_d \left( \frac{B^n}{\xi_1} + \frac{B^{n-1}\xi_2}{\log \xi_2} \right).$$ 

$\square$
2.2. **Proof of Theorem 1.3.** With the same values of $\xi_1$ and $\xi_2 = B(\log B)^{1/2}$ we already have $N_{1,d-1}(B) = C_{F,d-1}B^n + o(B^n) + O(B^{n-d})$, $N_{2,d-1}(B) = O(B^n/\sqrt{\log B})$. Here we deal with $N_{3,d-1}(B)$. We first observe that

$$N_{3,d-1}(B) \leq \sum_{|x| \leq B} \sum_{p \mid x, \rho^d m = n} 1.$$  

Let $F_m(x) = F(x, m_2, \ldots, m_n)$ with $m = (m_2, \ldots, m_n)$, also put

$$P(B, f) = \sum_{n \leq B} \sum_{\rho^d m = n} 1$$

for the polynomial $f(x)$ with integer coefficients. Then we notice that

$$N_{3,d-1}(B) \ll \sum_{|m_2|, \ldots, |m_n| \leq B} \sup_{m} P(B, F_m) \ll B^{n-1} \sup_{|m_i| \leq B} P(B, F_m),$$

so we reduce our task to providing an upper bound for the sum $P(B, F_m)$.

Now we notice that $P(B, f)$ is exactly the sum $P_2(B)$ for an irreducible polynomial $f(x)$ of degree $d$ estimated by Hooley in chapter 4.3 of [11]. However, it is easy to see that Hooley’s argument, an application of Galagher’s larger sieve using a Hasse-Weil bound and an average estimate of the function $\rho_f(m)$, is also valid if $\deg F_m \leq d$ and $F_m$ is reducible. Moreover the estimate is uniform on the coefficients of $f$ and depends only on its degree. Thus by Hooley’s bound we have $P(B, F_m) \ll_d B^{1/2}B^{1/2}$ where $\xi_3 \ll B^d \xi_2^{-(d-1)}$. Therefore $P(B, F_m) \ll_d B(\log B)^{-(d-1)/4}$ and we conclude that $N_{3,d-1}(B) = o(B^n)$. Theorem 1.3 then follows from (2.1).

We will show that the prime input problem cannot be solved with the same method regardless of the choice of $\xi_2$. Indeed, from the definition of the quantity $N_3(B)$ in (2.4) we see that the identity $F(x_1, \ldots, x_n) = \mu \rho^{d-1} \rho^d$ for $|x_1| \leq B$ and $p > \xi_2$ yields $B^d \gg F_2^{d-1}$, i.e.

$$\xi_2 \ll B^{d/(d-1)}.$$  

(If we choose $\xi_2 \gg B^{d/(d-1)}$, then $N_{3,d-1}(B) = O(1)$ and we only need to take care of the size of $N_{2,d-1}(B)$. Then it is necessary to have $\xi_2/\log \xi_2 \ll B$, but this is impossible for so huge $\xi_2$.)

On the other hand, if we want to solve the prime input problem for $N_{F,d-1}(B)$, we would like that Hooley’s upper bound satisfies $N_{3,d-1}(B) < B^{n-1}B^{d+1/2} \xi_2^{-(d-1)/2} < B^n/(\log B)^n$, i.e. $\xi_2 \gg (\log B)^{2n/(d-1)}$. At the same time, to manage the error term from Lemma 2.1, we would need $\xi_2/\log \xi_2 \ll B/(\log B)^n$. Plugging in the latter estimate for $\xi_2$ we obtain $\log \xi_2 \gg (\log B)^{n+2n/(d-1)}$ which contradicts (2.9), since $n \geq 1$. Thus, in order to adapt Hooley’s methods to the prime input case, further ideas are required.

2.3. **Counting $k$-free values over prime inputs.** We shall seek an analogue to (2.1) for the case of prime inputs. Put $N_1(B), N_2(B), N_3(B)$ for the analogues to (2.2), (2.3), and (2.4) in the case of prime inputs. We begin with an estimate
for $\mathcal{N}_1(B)$. Since we are interested in prime inputs, we should modify the function $\rho_F(m)$. In particular, the only way for $m$ with all prime coordinates to have two coordinates to share a common factor is if they are equal, and thus $m$ lies on one of the hyperplanes in $\mathbb{R}^n$ defined by $x_i = x_j$ for some $i < j$. These points are negligible in the box $[-B, B]^n \cap \mathbb{Z}^n$, so we may assume that $\gcd(m_i, m_j) = 1$ for all $i < j$. We shall thus put

\begin{equation}
(2.10) \quad \rho_F^*(m) = \#\{m \in \mathbb{Z}^n : |m_i| \leq m - 1, \text{ for } 1 \leq i \leq n, \gcd(m_i, m_j) = 1 \text{ for all } i < j, F(m) \equiv 0 \pmod{m}\}.
\end{equation}

It is immediate that $\rho_F^*(p^k) = O_{d,n} \left( p^{k(n-1)} \right)$. Moreover, the number of possible elements in $(\mathbb{Z}/p^k\mathbb{Z})^n$ such that no coordinate is divisible by $p$ is $\phi(p^k)^n = (p^k - p^{k-1})^n$. Similar to (2.6), we find that

\begin{align*}
\mathcal{N}_1(B) &= \sum_{p \in \mathbb{N} \atop p | h \Rightarrow p > \xi_1'} \mu(h) \rho_F^*(p^k) \left( \frac{B^n}{\phi(p^k)^n (\log B)^n} + O_c \left( \frac{B^n}{\phi(p^k)^n (\log B)^{n-1} \exp(c \sqrt{\log B})} \right) \right),
\end{align*}

by the Siegel-Walfisz theorem. We thus find that

\begin{align*}
\mathcal{N}_1(B) &= \frac{B^n}{(\log B)^n} \prod_{p \leq \xi_1'} \left( 1 - \frac{\rho_F^*(p^k)}{\phi(p^k)^n} \right) + O_c \left( \frac{B^n (\xi_1')^{nk}}{(\log B)^{n-1} \exp(c \sqrt{\log B})} \right).
\end{align*}

Since $\exp(c \sqrt{\log B})$ is eventually larger than any power of $\log B$, we may take $\xi_1' = (\log B)^{\frac{n^{r+2}}{k}}$ say. We then see that $\mathcal{N}_1(B)$ approximates our main term, bearing in mind that in a similar way we have $C'_{F,k} > 0$. Now the proof of the analogue of Lemma 2.1 for $\mathcal{N}_3(B)$ carries through as before, with $\xi_1$ replaced with $\xi_1'$ and $\xi_2$ replaced with $B \exp(-c \sqrt{\log B})$. The remaining quantity $\mathcal{N}_3(B)$ will be estimated by the upper bound of $\mathcal{N}_3(B)$, which is obtained at the end of the paper in subsection 3.1.

In the next section, we shall derive a version of the global determinant method which applies to affine varieties, refining Heath-Brown’s Theorem 15 in [9]. It is well-known that the global determinant method produces estimates which are uniform in the coefficients of the polynomials involved and produces power-saving error terms. Thus, the proof of Theorems 1.1 and 1.2 in the case $d > 4$ can be carried out simultaneously.

3. Global affine determinant method

In [8], Heath-Brown introduced a novel technique which is widely applicable to the subject of counting rational points on algebraic varieties, which is now known as the $p$-adic determinant method. It is a generalization of the original determinant method of Bombieri and Pila, which is based on real analytic arguments. This was later refined by Salberger in [16]. Salberger’s named his refinement the global determinant method, in reference to the fact that it uses multiple primes simultaneously.

In [9], Heath-Brown gave a different refinement of his $p$-adic determinant method in [8]. Originally the $p$-adic determinant method could only treat projective hypersurfaces, but in [9] Heath-Brown showed that an analogous version exists for affine varieties. Browning combined Salberger’s global determinant method with the affine method for surfaces in [4], in application to power-free values of polynomials in one
or two variables.

We now give a complete statement of the global determinant method in the affine setting, recovering the generality given by Salberger in [16] and Xiao in [17] (in the setting of weighted projective spaces).

For positive numbers $B_1, \cdots, B_n$ all exceeding one, put

$$S(F; B_1, \cdots, B_n) = \{ x \in \mathbb{Z}^n : |x_i| \leq B_i, F(x) = 0 \}.$$  

Put $X$ for the hypersurface in $\mathbb{A}^n$ defined by $F$. For a prime $p$ and a point $P \in X(\mathbb{F}_p)$, put

$$S_p(F; B_1, \cdots, B_n; P) = \{ x \in S(F; B_1, \cdots, B_n) : x \equiv P \pmod{p} \}.$$  

For a vector of non-negative integers $e = (e_1, \cdots, e_n)$, put $x^e = x_1^{e_1} \cdots x_n^{e_n}$. Let $E$ be a finite set of vectors in $\mathbb{Z}^n$ with non-negative entries. Put

$$s_p = \#S_p(F; B; P)$$  

and $E = \#E$. Consider the $s_p \times E$ matrix $M$ whose $ij$-th entry is the $i$-th monomial in $E$ evaluated at the $j$-th element of $S_p(F; x; P)$.

We pick the same exponent set $E$ as Heath-Brown in [9]. In particular, write

$$F(x_1, \cdots, x_n) = \sum_{r} a_r x^r,$$  

and let $\mathcal{P}(F)$ be the Newton polyhedron of $F$. Put

$$T = \max_{r \in \mathcal{P}(F)} B^r,$$  

where $B = (B_1, \cdots, B_n)$ is a vector in $\mathbb{R}^n$ with $B_i \geq 1$ for $1 \leq i \leq n$. Then there is at least one element $m \in \mathcal{P}(F)$ such that $T = B^m$. We pick a parameter $\lambda > 0$ and put $\tau = \lambda \log T$. We then define our exponent set $E$ to be:

$$E = \left\{ e \in \mathbb{Z}^n : e_i \geq 0 \text{ for } 1 \leq i \leq n, \sum_{i=1}^{n} e_i \log B_i \leq \tau, e_i < m_i \text{ for at least one } i \right\}$$  

We shall abuse notation and also refer to $E$ as a set of monomials. It was shown by Heath-Brown in [9] that all non-trivial linear combinations of monomials whose exponent vectors lie in $E$ leads to a polynomial which is not divisible by $F$.

Put

$$W = \exp \left( \frac{\prod_{1 \leq i \leq n} \log B_i}{\log T} \right)^{1/(n-1)}.$$  

We can now state the main result of this section, which is the global determinant method analogue of Heath-Brown’s Theorem 15 in [9]:

**Theorem 3.1.** Let $B = (B_1, \cdots, B_n) \in \mathbb{R}^n$ be a vector of positive numbers of size at least 1. Let $X$ be a hypersurface defined over $\mathbb{Z}$ in $\mathbb{A}^n$ which is irreducible over $\mathbb{Q}$ and defined by an irreducible polynomial $F$. Let $\mathfrak{U}$ be a finite set of primes and put

$$\mathcal{Q} = \prod_{p \in \mathfrak{U}} p.$$
For each prime \( p \in \mathcal{U} \), let \( P_p \) be a non-singular point in \( X(\mathbb{F}_p) \) and put \( \mathcal{U} = \{ P_p : p \in \mathcal{U} \} \).

Let \( \mathcal{E} \) be as given in (3.1). Then

(a) Let \( \varepsilon > 0 \). If \( WT^\varepsilon < Q \leq WT^{2\varepsilon} \),
then there is a hypersurface \( Y(U) \) containing \( S(F; B; U) \), not containing \( X \) and defined by a primitive polynomial \( G \), whose degree is \( O_{d,n,\varepsilon}(1) \) and whose height is at most \( O_{d,n,\varepsilon}(\log T) \).

(b) If \( X \) is geometrically integral, then there exists a hypersurface \( Y(U) \) containing \( S(F; B; U) \), not containing \( X \) and defined by a primitive polynomial \( G \) whose degree satisfies

\[
\deg G = O_{d,n}(1 + \frac{Q}{W} \log T Q).
\]

Theorem 3.1 can be proved using the same arguments as in [16] or [17]. The key insight is that the proofs in [16] and [17] producing large divisors of the discriminants essentially reduces to the affine case, so they remain valid if one starts with the affine case.

We will now need the following lemma which is a consequence of Theorem 3.1, and was stated without proof in [4]:

**Lemma 3.2.** Let \( X \) be a geometrically integral affine surface of degree \( d \) in \( \mathbb{A}^3 \) defined by an integral polynomial \( F \), and let \( B = (B_1, B_2, B_3) \) be a vector of positive numbers exceeding one. Then for any \( \varepsilon > 0 \) there exists a collection of integral polynomials \( \Gamma \), defined over \( \mathbb{Z} \), such that

(a) \( \# \Gamma = O_{d,\varepsilon}(W^{1+\varepsilon}) \),
(b) Each polynomial \( G \in \Gamma \) is co-prime with \( F \),
(c) The number of points in \( X(\mathbb{Z}; B) \) not lying on \( \{ G = 0 \}, G \in \Gamma \) is at most \( O_{d,\varepsilon}(W^{2+\varepsilon}) \), and
(d) Each polynomial \( G \in \Gamma \) has degree \( O_{d,\varepsilon}(1) \).

**Proof.** The proof given here follows the proof of Lemma 2.8 in [16] and Theorem 1.1 in [17]. By Theorem 3.1, there exists an affine surface \( Y_0 \) defined over \( \mathbb{Z} \), which contains \( X(\mathbb{Z}; B) \) but does not contain \( X \) as a component, which is defined by a polynomial \( G_0 \) of degree \( O_d(W^{1+\varepsilon}) \).

Let \( p_1 < \cdots < p_{t+1} \) be the sequence of increasing consecutive primes satisfying \( p_1 > \log(B_1B_2B_3) \), and

\[ p_1 \cdots p_t \leq WT^\varepsilon < p_1 \cdots p_{t+1}. \]

Write \( Q_t = \prod_{i=1}^{t+1} p_i \).

Then following the arguments in [17], we see that \( Q_t = O(W^{1+\varepsilon} \log W) \).

We now begin constructing the set of polynomials \( \Gamma \). By Theorem 3.1, for each
point $P_{p_1} \in X(\mathbb{F}_{p_1})$ there exists a surface $Y(P_{p_1})$ containing $X(\mathbb{Z}; B; P_{p_1})$ but does no contain $X$ as a component of degree
\[ O_d \left( (1 + p_1^{-1}W) \log W \right). \]

$\Gamma(P_{p_1})$ is defined by a polynomial $G_{P_{p_1}}$ with integer coefficients. Put $\Gamma(P_{p_1})$ for the set of irreducible factors $g$ of $G_{P_{p_1}}$ for which the intersection of $\{g = 0\} \cap X$ contains a curve which lies in the intersection $Y_0 \cap X$. Now put $\Gamma^{(1)} = \bigcup_{P_{p_1} \in X(\mathbb{F}_{p_1})} \Gamma(P_{p_1})$. Note that $\# \Gamma^{(1)}$ is bounded by the number of components of $Y_0 \cap X$, which is bounded by $O_d(W^{1+\varepsilon})$ by Bezout’s theorem. Moreover, since for each $g \in \Gamma^{(1)}$ is a divisor of $G_{P_{p_1}}$ for some $P_{p_1} \in X(\mathbb{F}_{p_1})$, its degree is at most $O_d \left( (1 + p_1^{-1}W) \log W \right)$.

Likewise, for any point $P_1 \in X(\mathbb{F}_{p_1})$ and $P_2 \in X(\mathbb{F}_{p_2})$, there exists a polynomial $G_{P_1, P_2}$ defining a surface $Y(P_1, P_2)$ of degree $O_{d, \varepsilon} \left( (1 + (p_1p_2)^{-1}W) \log W \right)$ which contains $X(\mathbb{Z}; B; P_1, P_2)$. Now put $\Gamma^{(2)}$ for the collection of polynomials $g$ dividing $G_{P_1, P_2}$ for some $P_1 \in X(\mathbb{F}_{p_1}), P_2 \in X(\mathbb{F}_{p_2})$ and such that $\{g = 0\} \cap X$ contains a curve which is also contained in $Y_0 \cap X$. Again, we see that
\[ \deg g = O_{d, \varepsilon} \left( (1 + (p_1p_2)^{-1}W) \log W \right) \]

and
\[ \# \Gamma^{(2)} = O_{d, \varepsilon} \left( W^{1+\varepsilon} \right). \]

We continue this process until we reach $t + 1$, and set $\Gamma = \Gamma^{(t+1)}$. We then see that
\[ \# \Gamma = O_{d, \varepsilon} \left( W^{1+\varepsilon} \right) \]

and for each $g \in \Gamma$ we have
\[ \deg g = O_{d, \varepsilon}(1) \]

by part a) of Theorem 3.1.

The points which lie on the complement $Z$ of the union of the integral points on the surfaces defined by polynomials in $\Gamma$ in $X(\mathbb{Z}; B)$ can be estimated as follows. If $x \in Z$, then there exists $0 \leq j \leq t + 1$ such that the irreducible component of $Y(P_1, \ldots, P_j)$ containing $x$ and the irreducible component of $Y(P_1, \ldots, P_{j+1})$ containing $x$ differ. Then $x$ lies on a 0-dimensional variety defined by $F, g_1, g_2$ where $g_1, g_2$ are distinct polynomials such that $g_1 | G(P_1, \ldots, P_j)$ and $g_2 | G(P_1, \ldots, P_{j+1})$. Thus, the number of such $x$ is bounded by
\[ O_d \left( (1 + (p_1 \cdots p_{j+1})^{-1}W) \log W \right) \]

by Bezout’s theorem. Since $t = O_d \left( \frac{\log T}{\log \log T} \right)$, it follows that
\[ \# Z = O_{d, \varepsilon} \left( W^{2+\varepsilon} \right) \]

as desired. \qed

The following lemma, given as Lemma 2 in [4], then follows:

**Lemma 3.3.** Let $f(x)$ be a polynomial of degree $d$ with integer coefficients, and let $B_1, B_2, B_3$ be positive real numbers. Put $M(f; B)$ for the number of integer solutions to the equation
\[ f(x) = vz^k \]
satisfying 
\[ B_1/2 \leq x < B_1, B_2/2 \leq y < B_2, B_3/2 \leq z < B_3. \]

Then
\[ M(f; B) = O_{d, \varepsilon, k} \left( (B_1 B_2 B_3)^\varepsilon W \left( W + B_1 B_3^{-k} + B_3^{1/d} \right) \right). \]

We remark that in [4] Browning had insisted that \( f \) be irreducible, but this is not a necessary assumption. Indeed, the surface defined by \( f(x) = vz^k \) is geometrically integral whenever \( f \) is not identically zero, therefore the machinery established by Salberger in [16] will be applicable.

We may now finalize the proofs of Theorem 1.1 and Theorem 1.2 when \( d \geq 5 \).

3.1. **Estimate of \( N_3(B) \).** Following the strategy of Browning in [4], we may fix \( n-1 \) variables, say \( x_2, \ldots, x_n \), and reduce the problem to the single variable case. For a positive number \( H \), put
\[ R(F; B, H) = \# \{ x \in \mathbb{Z}^n : |x_i| \leq B, y \ll B^{d/k}, H/2 \leq z < H, F(x) = yz^k \}. \]

We then see, by dyadic summation, that
\[ N_3(B) \ll (\log B)^{\varepsilon} \sup_{\xi < H \ll B^{d/k}} R(F; B, H). \]

Put \( F_m(x) = F(x, m_2, \ldots, m_n) \), where \( m = (m_2, \ldots, m_n) \). Then
\[ (3.3) \quad R(F; B, H) \leq \sum_{|m_2|, \ldots, |m_n| \leq B} R(F_m; B, H). \]

Put \( H = B^n \). We then see from (3.2) that
\[ W = \exp \left( \sqrt{\frac{(\log B)\eta (\log B)(d \log B - \eta k \log B)}{d \log B}} \right) = B^{\sqrt{\eta(1-k\eta/d)}}. \]

We now apply Lemma 3.3 to see that
\[ R(F_m; B, H) \ll (\log B)^{2} \left( BH \frac{B^d}{H^k} \right)^{\varepsilon} B^{\sqrt{\eta(1-k\eta/d)}} \left( B^{\sqrt{\eta(1-k\eta/d)}} + B^{1-k\eta} + B^{\eta/d} \right). \]

It thus follows that
\[ N_3(B) = O_{d, n, \varepsilon} \left( B^{n-1+2\sqrt{\eta(1-k\eta/d)}+\varepsilon} \right). \]

We note that for any \( v > 0 \), the quadratic function
\[ y = x(1-vx) \]

is concave down and decreasing on the positive real line. Therefore, the maximum value for the term \( \sqrt{\eta(1-k\eta/d)} \) is achieved when \( \eta \) is minimum. From Lemma 2.1 we see that we can take \( \eta \geq 1 \), so the maximum occurs when \( \eta = 1 \). Thus the maximum value is
\[ \sqrt{1-k/d}. \]

We want to have \( N_3(B) = O(B^{n-\varepsilon}) \), so that we prove both Theorem 1.1 and Theorem 1.2, we thus require
\[ 2\sqrt{1-k/d} < 1, \]
which is equivalent to \( k > 3d/4 \). This finishes the proofs.
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