THE MODIFIED COMPLEX BUSEMANN-PETTY PROBLEM ON SECTIONS OF CONVEX BODIES.

MARISA ZYMONOPOULOU

Abstract. The complex Busemann-Petty problem asks whether origin symmetric convex bodies in $\mathbb{C}^n$ with smaller central hyperplane sections necessarily have smaller volume. The answer is affirmative if $n \leq 3$ and negative if $n \geq 4$. Since the answer is negative in most dimensions, it is natural to ask what conditions on the $(n-1)$-dimensional volumes of the central sections of complex convex bodies with complex hyperplanes allow to compare the $n$-dimensional volumes. In this article we give necessary conditions on the section function in order to obtain an affirmative answer in all dimensions. The result is the complex analogue of [KYY].

1. Introduction

The Busemann-Petty problem was completely solved in the late 90’s as a result of a series of papers of many mathematicians ([LR], [Ba], [Gi], [Bo], [Lu], [Pa], [Ga], [Zh1], [K1], [K2], [Zh2], [GKS]; see [K5, p.3] for the history of the solution). The problem asks the following:

Suppose $K$ and $L$ are two origin symmetric convex bodies in $\mathbb{R}^n$ such that for every $\xi \in S^{n-1}$,

$$\text{Vol}_{n-1}(K \cap \xi^\perp) \leq \text{Vol}_{n-1}(L \cap \xi^\perp).$$

Does it follow that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L)?$$

The problem has an affirmative answer only if $n \leq 4$. Since the answer is negative in most dimensions, it is natural to ask what conditions on the $(n-1)$-dimensional volumes of central sections do allow to compare the $n$-dimensional volumes. Such conditions were found in [KYY]. The result is as follows.

For an origin symmetric convex body $K$ in $\mathbb{R}^n$ define the section function

$$S_K(\xi) = \text{Vol}_{n-1}(K \cap \xi^\perp), \ \xi \in S^{n-1}.$$ 

Suppose $K$ and $L$ are origin symmetric convex smooth bodies in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ with $\alpha \geq n - 4$. Then, the inequality

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi), \ \xi \in S^{n-1}$$

implies that $\text{Vol}_n(K) \leq \text{Vol}_n(L)$. If $\alpha < n - 4$ this is not necessarily true. Here, $\Delta$ is the Laplace operator on $\mathbb{R}^n$. 

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In this article we study the complex version of this problem. For \( \xi \in \mathbb{C}^n, \ |\xi| = 1 \) we denote by

\[
H_\xi = \{ z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^{n} z_k \xi_k = 0 \}
\]

the complex hyperplane perpendicular to \( \xi \).

Origin symmetric convex bodies in \( \mathbb{C}^n \) are the unit balls of norms on \( \mathbb{C}^n \). We denote by \( \| \cdot \|_K \) the norm corresponding to the body \( K = \{ z \in \mathbb{C}^n : \|z\|_K \leq 1 \} \).

We identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) using the mapping

\[
\xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2})
\]

and observe that under this mapping the complex hyperplane \( H_\xi \) turns into a \((2n - 2)\)-dimensional subspace of \( \mathbb{R}^{2n} \) orthogonal to the vectors

\[
\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \text{ and } \xi^\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}).
\]

Since norms on \( \mathbb{C}^n \) satisfy the equality

\[
\| \lambda z \| = |\lambda| \|z\|, \ \forall z \in \mathbb{C}^n, \ \forall \lambda \in \mathbb{C}^n,
\]

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies \( K \) in \( \mathbb{R}^{2n} \) that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each \( \theta \in [0, 2\pi] \) and each \( x = (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} \)

\[
\|x\|_K = \|R_\theta(x_{11}, x_{12}), \ldots, R_\theta(x_{n1}, x_{n2})\|_K, \tag{1}
\]

where \( R_\theta \) stands for the counterclockwise rotation of \( \mathbb{R}^2 \) by the angle \( \theta \) with respect to the origin. If a convex body satisfies (1) we will say that it is invariant with respect to all \( R_\theta \).

The complex Busemann-Petty problem ([KKZ]) can now be formulated as follows: Suppose \( K \) and \( L \) are origin symmetric invariant with respect to all \( R_\theta \) convex bodies in \( \mathbb{R}^{2n} \) such that

\[
\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi)
\]

for each \( \xi \) from the unit sphere \( S^{2n-1} \) of \( \mathbb{R}^{2n} \). Does it follow that

\[
\text{Vol}_{2n}(K) \leq \text{Vol}_{2n}(L) ?
\]

As it is proved in [KKZ], the answer is affirmative if \( n \leq 3 \) and negative if \( n \geq 4 \). In this article our aim is to extend the result from [KYY] to the complex case.

Let \( D \) be an origin symmetric convex body in \( \mathbb{C}^n \). For every \( \xi \in \mathbb{C}^n, \ |\xi| = 1 \), we define the section function

\[
S_{CD}(\xi) = \text{Vol}_{2n-2}(D \cap H_\xi), \ \forall \xi \in S^{2n-1}. \tag{2}
\]

Extending \( S_{CD} \) to the whole \( \mathbb{R}^{2n} \) as a homogeneous function of degree \(-2\) we prove the following:
Main Result. Suppose $K$ and $L$ are two origin symmetric invariant with respect to all $R\theta$ convex bodies in $\mathbb{R}^{2n}$. Suppose that $\alpha \in [2n - 6, 2n - 2)^n$, $n \geq 3$. If
\[ (-\Delta)^{\alpha/2} S_{CK}(\xi) \leq (-\Delta)^{\alpha/2} S_{CL}(\xi), \]
for every $\xi \in S^{2n-1}$. Then
\[ \text{Vol}_{2n}(K) \leq \text{Vol}_{2n}(L). \]
If $\alpha \in (2n - 7, 2n - 6)$ then one can construct two convex bodies $K$ and $L$ that satisfy (3), but $\text{Vol}_{2n}(K) > \text{Vol}_{2n}(L)$. This means that one needs to differentiate the section functions at least $2n - 6$ times and compare the derivatives in order to obtain the same inequality for the volume of the original bodies. Note that if $\alpha = 0$ the problem coincides with the original complex Busemann-Petty problem.

2. The Fourier analytic approach

Throughout this paper we use the Fourier transform of distributions. The Schwartz class of the rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$ is denoted by $S(\mathbb{R}^n)$, and the space of distributions over $S(\mathbb{R}^n)$ by $S'(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi$. A distribution is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle$ for every test function $\phi$ and every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n-p$. A distribution $f$ is called positive definite if, for every test function $\phi$, $\langle f, \phi \ast \overline{\phi}(-x) \rangle \geq 0$. By Schwartz’s generalization of Bochner’s theorem, this is equivalent to $\hat{f}$ being a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi$.

We denote by $\Delta$ the Laplace operator on $\mathbb{R}^n$, and by $|\cdot|_2$ the Euclidean norm in the proper space. Then the fractional powers of the Laplacian are defined by
\[ \left( (-\Delta)^{\alpha/2} f \right)^\wedge = \frac{1}{(2\pi)^n} |x|^\alpha_2 \hat{f}(x), \]
where the Fourier transform is considered in the sense of distributions.

A compact set $K \subset \mathbb{R}^n$ is called a star body, if every straight line that passes through the origin crosses the boundary of the set at exactly two points and the boundary of $K$ is continuous in the sense that the Minkowski functional of $K$, defined by
\[ \|x\|_K = \min \{ \alpha \geq 0 : x \in \alpha K \} \]
is a continuous function on $\mathbb{R}^n$. Using polar coordinates it is possible to obtain the following polar formula of the volume of the body:
\[
\text{Vol}_n(K) = \int_{\mathbb{R}^n} \chi(||x||_K)dx = \frac{1}{n} \int_{S^{n-1}} ||\theta||_K^n d\theta.
\]

A star body \(K\) in \(\mathbb{R}^n\) is called \(k\)-smooth (infinitely smooth) if the restriction of \(\|x\|_K\) to the sphere \(S^{n-1}\) belongs to the class of \(C^k(S^{n-1})\) \((C^\infty(S^{n-1}))\). It is well-known that one can approximate any convex body in \(\mathbb{R}^n\) in the radial metric, \(d(K, L) = \sup\{|\rho_K(\xi) - \rho_L(\xi)|, \xi \in S^{n-1}\}\), by a sequence of infinitely smooth convex bodies. The proof is based on a simple convolution argument (see for example [Sch, Theorem 3.3.1]). It is also easy to see that any convex body in \(\mathbb{R}^{2n}\) invariant with respect to all \(R_\theta\) rotations can be approximated in the radial metric by a sequence of infinitely smooth convex bodies invariant with respect to all \(R_\theta\). This follows from the same convolution argument, because invariance with respect to \(R_\theta\) is preserved under convolutions. This approximation argument allows us to consider only infinitely smooth origin symmetric convex bodies for the solution to the problem.

If \(D\) is an infinitely smooth origin symmetric star body in \(\mathbb{R}^n\) and \(0 < k < n\), then the Fourier transform of the distribution \(\|x\|_D^{-p}\) is a homogeneous function of degree \(-n + k\) on \(\mathbb{R}^n\), whose restriction to the sphere is infinitely smooth (see [K5, Lemma 3.16]).

We use a spherical version of Parseval's identity, established in [K3] (see also [K5, Lemma 3.22]).

**Proposition 1.** Let \(K\) and \(L\) be two infinitely smooth origin symmetric convex bodies in \(\mathbb{R}^n\) and \(0 < p < n\). Then

\[
\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\|x\|_L^{-n+p})^\wedge(\xi) d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p} dx.
\]

Let \(H\) be an \((2n - 2)\)-dimensional subspace of \(\mathbb{R}^{2n}\) and \(p \leq 2n - 2\). We fix an orthonormal basis \(\{e_1, e_2\}\), in the orthogonal subspace \(H^\perp\). For any convex body \(D\) in \(\mathbb{R}^{2n}\) we define the function \(A_{D,H,p}\) as a function on \(\mathbb{R}^2\) such that

\[
A_{D,H,p}(u) = \int_{D \cap H_u} |x|_2^{-p} dx, \ u \in \mathbb{R}^2,
\]

where \(H_u = \{x \in \mathbb{R}^{2n} : (x, e_1) = u_1, (x, e_2) = u_2\}\).

If the body \(D\) is infinitely smooth and \(0 \leq p < 2n - m - 2\), then \(A_{D,H,p}\) is \(m\)-times continuously differentiable near the origin. This can be seen from an argument similar to [K5, Lemma 2.5].
In addition, if we consider the action of the distribution \(|u|_2^{-q-2}/\Gamma(-q/2)\) on \(A_{D,H,p}\), we may apply a standard regularization argument (see [GS, p. 71-74]) and define the function
\[
q \mapsto \left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle.
\] (6)

For \(q \in \mathbb{C}\) with \(\text{Re } q \leq 2n - p - 3\), the function is an analytic function of \(q\). If \(q < 0\)
\[
\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^2} |u|_2^{-q-2} A_{D,H,p}(u) du.\] (7)

If \(q = 2d\), \(d \in \mathbb{N} \cup \{0\}\), then
\[
\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle = (-1)^d \pi^{d} \Delta^d A_{f,D,H}(0),\] (8)

where \(\Delta = \sum_{i=1}^{2} \partial^2/\partial u_i^2\) is the 2-dimensional Laplace operator (see [GS, p. 71-74]). Note that the function (4) is equal, up to a constant, to the fractional power of \(\Delta^{q/2} A_{D,H,p}\) (see [KKZ, p. 6-7] or [K4, p. 6-7] for complete definition).

If the body \(D\) is origin symmetric the function \(A_{D,H,p}\) is even and for \(0 < q < 2\) we have (see also [K5, p. 39])
\[
\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{0}^{2\pi} \left( \int_{0}^{\infty} \frac{A_{D,H,p}(t\theta) - A_{D,H,p}(0)}{t^{1+q}} dt \right) d\theta.\] (9)

The following proposition is a generalization of [K4], (see also [KKZ, Proposition 4]) with \(k = 2\). We prove it using a well-known formula (see for example [GS, p. 76]): for any \(v \in \mathbb{R}^2\) and \(q < -1\),
\[
(v_1^2 + v_2^2)^{-q-2} = \frac{\Gamma(-q/2)}{2\Gamma((-q - 1)/2)\pi^{1/2}} \int_{0}^{2\pi} |(v, u)|^{-q-2} du.\] (10)

**Proposition 2.** Let \(D\) be an infinitely smooth origin symmetric convex body in \(\mathbb{R}^{2n}\). If \(-2 < q < 2n - 2\), \(0 \leq p \leq 2n - q - 3\). Then for every \((2n - 2)\)-dimensional subspace \(H\) of \(\mathbb{R}^{2n}\)
\[
\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle = \frac{2^{-q-2}}{\pi \Gamma(2n^2 + 2)(2n - q - p - 2)} \int_{S^{2n-1} \cap H^\perp} (\|x\|_D^{-2n+q+p+2} |x|_2^{-p})^\wedge (\theta) d\theta.\] (11)
Also, for every \( d \in \mathbb{N} \cup \{ 0 \}, \ d < n - 1 \)

\[
\Delta^d A_{D,H,p}(0) = \frac{(-1)^d}{8\pi^d(n-d-1)} \int_{S^{2n-1}\cap H^\perp} (\|x\|_D^{2n+2d+p+2}|x|_2^{-p})^\wedge(\eta) d\eta.
\]

(12)

**Proof.** First we assume that \( q \in (-2, -1) \). Then

\[
\left\langle \frac{u_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^2} |u_2^{-q-2} A_{D,H,p}(u)| du
\]

Using the expression (5) for the function \( A_{D,H,p} \), writing the integral in polar coordinates and then using (10), we see that the right-hand side of the latter equation is equal to

\[
\frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^n} ((x,e_1)^2 + (x,e_2)^2)^{-q/2} |x|_2^{-p} \chi(\|x\|_D) dx
\]

\[
= \frac{1}{\Gamma(-q/2)(n-q-p-2)} \int_{S^{n-1}} ((\theta,e_1)^2 + (\theta,e_2)^2)^{-q/2} \|\theta\|_D^{-n+q+2} d\theta
\]

\[
= \frac{1}{2\Gamma(-q/2)\pi^{n/2}(n-q-p-2)} \times
\int_{S^{n-1}} \|\theta\|_D^{-n+q+2} \left( \int_0^{2\pi} |(u_1e_1 + u_1e_2, \theta)|^{-q-2} du \right) d\theta
\]

\[
= \frac{1}{2\Gamma(-q/2)\pi^{n/2}(n-q-p-2)} \times
\int_0^{2\pi} \left( \int_{S^{n-1}} |\|\theta\|_D^{-n+q+2} (u_1e_1 + u_2e_2, \theta)|^{-q-2} d\theta \right) du.
\]

(13)

Let us show that the function under the integral over \([0, 2\pi]\) is the Fourier transform of \(\|x\|_D^{-n+q+2} |x|_2^{-p} \) at the point \(u_1 e_1 + u_2 e_2\). For any even test function \(\phi \in \mathcal{S}^{(\mathbb{R}^n)}\), using the well-known connection between the Fourier and Radon transforms (see [K5, p.27]) and the expression for the Fourier transform of the distribution \( |z|_2^{q-1} \) (see [K5, p.38]), we get

\[
\left\langle \|x\|_D^{-n+q+2} |x|_2^{-p}, \phi \right\rangle = \int_{\mathbb{R}^n} \|x\|_D^{-n+q+2} |x|_2^{-p} \hat{\phi}(x) dx
\]

\[
= \int_{S^{n-1}} \|\theta\|_D^{-n+q+2} \left( \int_0^{\infty} r^{q+1} \hat{\phi}(r\theta) dr \right) d\theta
\]

\[
= \frac{1}{2} \int_{S^{n-1}} \|\theta\|_D^{-n+q+2} \left( \int_0^{\infty} r^{q+1} \hat{\phi}(r\theta) dr \right) d\theta
\]

\[
= \frac{2^{q+2}\sqrt{\pi}}{2\Gamma((q+2)/2)} \int_{S^{n-1}} \|\theta\|_D^{-n+q+2} \left( \int_0^{\infty} r^{q+1} \hat{\phi}(r\theta) dr \right) d\theta
\]

\[
= \frac{2^{q+1}\sqrt{\pi}\Gamma((q+2)/2)}{2\Gamma((-q-1)/2)} \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} |(\theta, y)|^{-q-2} \|\theta\|_D^{-n+q+2} d\theta \right) \phi(y) dy.
\]
Since $\phi$ is an arbitrary test function, this proves that, for every $y \in \mathbb{R}^n \setminus \{0\}$,
\[
\left(\|x\|_D^{-n+q+p+2}|x|_2^{-p}\right)^\wedge(y)
\]
\[
= 2^{q+2}\pi \Gamma((q+2)/2) \frac{\Gamma((-q-1)/2)}{\Gamma((q+2)/2)} \int_{S^{n-1}} |(\theta, y)|^{-q-2}\|\theta\|_D^{-n+q+p+2} \, d\theta.
\]
Together with (13), the latter equality shows that
\[
\left\langle \frac{|u|^{q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle
\]
\[
= \frac{2^{-q-2}\pi^{-1}}{\Gamma((q+2)/2)(n-q-p-2)} \int_{S^{n-1} \cap H^\perp} \left(\|x\|_D^{-n+q+p+2}|x|_2^{-p}\right)^\wedge(\theta) \, d\theta,
\]
because in our notation $S^{n-1} \cap H^\perp = [0,2\pi]$.

We have proved (14) under the assumption that $q \in (-2,-1)$. However, both sides of (14) are analytic functions of $q \in \mathbb{C}$ in the domain where $-2 < \text{Re} q < 2n - 2$. This implies that the equality (14) holds for every $q$ from this domain (see [K5, p.61] for the details of a similar argument).

Putting $q = 2m$, $m \in \mathbb{N} \cup \{0\}$, $m < n - 1$ in (14) and applying (8) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we get the second formula. □

Brunn’s theorem (see for example [K5, Theorem 2.3]) states that for an origin symmetric convex body and a fixed direction, the central hyperplane section has the maximal volume among all the hyperplane sections perpendicular to the given direction. As a consequence we have the following generalization proved in [KKZ, Lemma 1] for $p = 0$.

**Proposition 3.** Suppose $D$ is a 2-smooth origin symmetric convex body in $\mathbb{R}^{2n}$, then the function $A_{D,H,p}$ is twice differentiable at the origin and
\[
\Delta A_{D,H,p}(0) \leq 0.
\]
Moreover, for any $q \in (0,2)$,
\[
\left\langle \frac{|u|^{q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle \geq 0.
\]

**Proof.** Differentiability follows from the same argument as in [K5, Lemma 2.4].

The body $D$ is origin symmetric and convex, so, to prove the first inequality we need to observe that the function $u \mapsto A_{D,H,p}(u)$, $u \in \mathbb{R}^2$, attains its maximum at the origin:

If $p = 0$ then it follows immediately from Brunn’s theorem (see [K5, Theorem 2.3] and [KKZ, Lemma 1].)
Let $p > 0$. Since $|x|_2^{-p} = p \int_0^\infty \chi(z|x|_2)z^{q-1}dz$, we have that for any $u \in \mathbb{R}^2$
\[
A_{D,H,p}(u) = \int_{D \cap H_u} |x|_2^{-p}dx = p \int_{D \cap H_u} \int_0^\infty \chi(z|x|_2)z^{q-1}dzdx
\]
\[
= p \int_0^\infty z^{q-1} \int_{D \cap H_u} \chi(z|x|_2)dxdz
\]
\[
= p \int_0^\infty z^{q-1} \int_{B(1/z) \cap H_u} \chi(\|x\|_D)dxdz,
\]
where $B(1/z)$ is the unit ball of radius $1/z$. Applying Brunn’s theorem to the body $B(1/z) \cap D$, we have that the latter integral is
\[
\leq p \int_0^\infty z^{q-1} \int_H \chi(\|x\|_{B(1/z) \cap D})dxdz = A_{D,H,p}(0).
\]
If $q \in (0, 2)$ then $\Gamma(-q/2) < 0$. Hence, for the second inequality we use \cite{10} to get that
\[
\frac{\left| |u|_2^{-q/2} \right|}{\Gamma(-\frac{q}{2})} \cdot A_{D,H,p}(u)
\]
\[
= \frac{1}{\Gamma(-\frac{q}{2})} \int_0^{2\pi} \left( \int_0^\infty \frac{A_{D,H,p}(t\theta) - A_{D,H,p}(0)}{t^{1+q}}dt \right) d\theta \geq 0,
\]
since $A_{D,H,p}(u) \leq A_{D,H,p}(0)$, for every $u \in \mathbb{R}^2$.

3. DISTRIBUTIONS OF THE FORM $|x|_2^{-\beta} \|x\|_D^{-\gamma}$

As in the modified real Busemann-Petty problem the solution is closely related to distributions of the form $|x|_2^{-\beta} \|x\|_D^{-\gamma}$.

First, we need a simple observation. The following lemma is crucial for the solution of the problem.

**Lemma 1.** For every infinitely smooth origin symmetric invariant with respect to all $R_\theta$ convex body $D$ in $\mathbb{R}^{2n}$ and every $\xi \in S^{2n-1}$, the Fourier transform of the distribution $|x|_2^{-\beta} \|x\|_D^{-\gamma}$, $0 < \beta, \gamma < 2n$ is a constant function on $S^{2n-1} \cap H_\xi$.

**Proof.** The proof (see \cite[Theorem 1]{KKZ}, when $\beta = 0$) is based on the following observation:

The body $D$ is invariant with respect to all $R_\theta$. So, because of the connection between the Fourier transform and linear transformations, the Fourier transform of $|x|_2^{-\beta} \|x\|_D^{-\gamma}$ is also invariant with respect to all $R_\theta$. This implies that it is a constant function on $S^{2n-1} \cap H_\xi$ because this circle can be represented as the set of all the rotations $R_\theta$, $\theta \in [0, 2\pi]$, of the vector $\xi \in S^{2n-1}$.

As a consequence of the above we have that
\[
\int_{S^{2n-1} \cap H_\xi} \left( |x|_2^{-\beta} \|x\|_D^{-\gamma} \right)^\wedge (\theta) d\theta = 2\pi \left( |x|_2^{-\beta} \|x\|_D^{-\gamma} \right)^\wedge (\xi).
\]
Lemma 2. Let $D$ be an origin symmetric invariant with respect to all $R_\theta$ convex body in $\mathbb{R}^{2n}$, $n \geq 3$. If $q \in (-2, 2]$ and $0 \leq p < 2n - q - 3$ then $|x|_2^{-p}||x||^{-2n+p+q+2}_D$ is a positive definite distribution.

Proof.

If $p = 0$ then by [KKZ, Theorem 3], $(||x||^{-2n+q+2}_D)^\wedge \geq 0$, since $2n - q - 2 \in [2n - 4, 2n)$.

Let $p > 0$. If $q \in (-2, 0)$ then by equation (7) and Proposition 2 (formula (11)) we have that
\[
\frac{2^{-q/2}}{\pi \Gamma\left(\frac{q+2}{2}\right)(2n - q - p - 2)} \int_{S^{2n-1} \cap H^\perp} (||x||^{-2n+q+p+2}_D|x|_2^{-p})^\wedge (\theta)d\theta
\]
\[
= \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^2} |u|_2^{-q/2} A_{D,H,p}(u)du \geq 0.
\]

By Lemma 1, the Fourier transform of the distribution $|x|_2^{-p}||x||^{-2n+p+q+2}_D$ is a constant function on $S^{2n-1} \cap H^\perp_\xi$. So,
\[
(|x|_2^{-p}||x||^{-2n+p+q+2}_D)^\wedge \geq 0,
\]
since $\Gamma\left(\frac{q+2}{2}\right) > 0$, $\Gamma(-\frac{q}{2}) > 0$ and $q < 2n - p - 2$.

Now, if $q = 0$, (12) and (15) give that
\[
A_{D,H,p}(0) = \frac{1}{4\pi(n - 1)} \left(|x|_2^{-p}||x||^{-2n+p+q+2}_D\right)^\wedge (\xi) \geq 0.
\]

For the case where $q \in (0, 2)$ we use Proposition 2 and the Remark to get that
\[
\frac{|u|_2^{-q/2}}{\Gamma\left(-\frac{q}{2}\right)} A_{D,H,p}(u)
\]
\[
= \frac{2^{-q-1}}{\Gamma\left(\frac{q+2}{2}\right)(2n - q - p - 2)} (||x||^{-2n+q+p+2}_D|x|_2^{-p})^\wedge (\xi).
\]

Then, by the generalization of Brunn’s theorem, Proposition 3, the desired follows.

Lastly, if $q = 2$, (12) and (15) imply that
\[
\Delta A_{D,H,p}(0) = \frac{-1}{4\pi(n - 2)} \left(||x||^{-2n+p+4}_D|x|_2^{-p}\right)^\wedge (\xi).
\]

Combining this with Brunn’s generalization, since the Laplacian of the function $A_{D,H,p}$ at 0 is non-positive, we have that
\[
\left(|x|_2^{-p}||x||^{-2n+p+4}_D\right)^\wedge (\xi) \geq 0.
\]
Before we prove the main result of this article we need the following:

**Lemma 3.** Let $D$ be an infinitely smooth origin symmetric invariant with respect to all $R_\theta$ convex body in $\mathbb{R}^{2n}$ and $\alpha \in \mathbb{R}$. Then

$$(-\Delta)^{\alpha/2}SC_D(\xi) = \frac{1}{4\pi(n-1)}(|x|^2\|x\|^{-2n+2})^\wedge(\xi) \quad (16)$$

**Proof.** Let $\xi \in S^{2n-1}$. As proved in [KKZ, Theorem 1], using the same idea as in Lemma 1 (with $r = 0$)

$$\text{Vol}_{2n-2}(D \cap H_\xi) = \frac{1}{4\pi(n-1)}(\|x\|^{-2n+2})^\wedge(\xi). \quad (17)$$

By the definition of the section function of $D$, and equation (17) we obtain the following formula:

$$SC_D(\xi) = \frac{1}{4\pi(n-1)}(\|x\|^{-2n+2})^\wedge(\xi). \quad (18)$$

We extend $SC_D$ to the whole $\mathbb{R}^{2n}$ as a homogeneous function of degree $-2$ and apply the definition of the fractional powers of the Laplacian. Then, since $\|x\|^{-2n+2}$ is an even distribution, equation (16) immediately follows. □

**4. The solution of the problem.**

We consider the affirmative and negative part of the main result separately. The proof follows by the next two theorems.

**Theorem 1.** *(AFFIRMATIVE PART)* Let $K$ and $L$ be two infinitely smooth origin symmetric invariant with respect to all $R_\theta$ convex bodies in $\mathbb{R}^{2n}$. Suppose that $\alpha \in [2n - 6, 2n - 2)$, $n \geq 3$. Then for every $\xi \in S^{2n-1}$

$$(-\Delta)^{\alpha/2}SC_K(\xi) \leq (-\Delta)^{\alpha/2}SC_L(\xi) \quad (19)$$

implies that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L).$$

**Proof.** The bodies $K$ and $L$ are infinitely smooth and invariant with respect to all $R_\theta$ convex bodies. So by equation (16) the condition in (19) can be written as

$$\left(|x|^2\|x\|^{2n+2}\right)^\wedge(\xi) \leq \left(|x|^2\|x\|^{2n+2}\right)^\wedge(\xi). \quad (20)$$

We apply Lemma 2 with $p = \alpha$ and $q = 2n - \alpha - 4$ so that the distribution $|x|^2\|x\|^{2n+2}$ is positive definite. By Bochner’s theorem this implies that its Fourier transform is a non-negative function on $\mathbb{R}^{2n} \setminus \{0\}$. By [K5, Lemma 3.16], it is also continuous, since $K$ is infinitely smooth. Multiply both sides
in (20) by \(|\alpha|\parallel x \parallel^{-2} K\) and integrate over the unit sphere \(S^{2n-1}\). Then we can apply Parseval’s spherical version, Proposition 1, to get that
\[
\int_{S^{2n-1}} \| x \parallel^{-2} K dx \leq \int_{S^{2n-1}} \| x \parallel^{-2} L dx + 2.
\] (21)

Then, by a simple application of Hölder’s inequality on formula (21) and the polar formula of the bodies (see Section 2.) we obtain the affirmative answer to the problem, since
\[
2n \text{Vol}_{2n}(K) \leq \left(2n \text{Vol}_{2n}(K)\right)^{1/n} \left(2n \text{Vol}_{2n}(L)\right)^{(n-1)/n}.
\]

□

To prove the negative part we need the following lemma.

**Lemma 4.** Let \(\alpha \in (2n-7, 2n-6)\). There exists an infinitely smooth origin symmetric convex body \(L\) with positive curvature, so that \(|\alpha|\parallel x \parallel^{-2} L\) is not a positive definite distribution.

We postpone the proof of Lemma 4 until the end of this section to show that the existence of such a body gives a negative answer to the problem.

**Theorem 2.** (NEGATIVE PART) Suppose there exists an infinitely smooth, origin symmetric convex body \(L\) for which \(|\alpha|\parallel x \parallel^{-2} L\) is not a positive definite distribution. Then one can construct an origin symmetric convex body \(K\) in \(\mathbb{R}^{2n}, n \geq 3\), so that together with \(L\) they satisfy (17), for every \(\xi \in S^{2n-1}\) but

\[
\text{Vol}_{2n}(K) > \text{Vol}_{2n}(L).
\]

**Proof.** The body \(L\) is infinitely smooth, so by [K5, Lemma 3.16] the Fourier transform of the distribution \(|\alpha|\parallel x \parallel^{-2} L\) is a continuous function on the unit sphere \(S^{2n-1}\). Moreover there exists an open subset \(\Omega\) of \(S^{2n-1}\) in which \((|\alpha|\parallel x \parallel^{-2} L)^{\wedge} < 0\). Since \(L\) is invariant with respect to all \(R_\theta\) we may assume that \(\Omega\) is also invariant we respect to rotations \(R_\theta\).

We use a standard perturbation procedure for convex bodies, see for example [K5, p.96] (similar argument was used in [KKZ, Lemma 5]). Consider a non-negative infinitely differentiable even function \(g\) supported on \(\Omega\) that is also invariant with respect to rotations \(R_\theta\). We extend it to a homogeneous function of degree \(-\alpha - 2\) on \(\mathbb{R}^{2n}\). By [K5, Lemma 3.16] its Fourier transform is an even homogeneous function of degree \(-2n + \alpha + 2\) on \(\mathbb{R}^{2n}\), whose restriction to the sphere is infinitely smooth: \((g(x/|x|_2)|x|_2^{-\alpha-2})^{\wedge}(y) = h(y/|y|_2)|y|_2^{2n+\alpha+2}\), where \(h \in C^\infty(S^{2n-1})\).

We define a body \(K\) so that
\[
\| x \parallel^{-2n+2} K = \| x \parallel^{-2n+2} L + \varepsilon |x|_2^{-2n+2} h\left(\frac{x}{|x|_2}\right),
\]
for small enough $\varepsilon > 0$ so that the body $K$ is strictly convex. Note that $K$ is also invariant with respect to all $R_\theta$. We multiply both sides by $\frac{1}{4\pi(n-1)}|x|^2$ and apply Fourier transform. Then

$$(-\Delta)^{\alpha/2} S_{CK}(\xi) = (-\Delta)^{\alpha/2} S_{CL}(\xi) + \frac{\varepsilon(2\pi)^{2n}}{4\pi(n-1)}|x|_2^{-\alpha-2}g\left(\frac{x}{|x|_2}\right)$$

$$\leq (-\Delta)^{\alpha/2} S_{CL}(\xi),$$

since $g$ is non-negative.

On the other hand, we multiply both sides of (22) by $\left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge$ and integrate over the sphere,

$$\int_{S^{2n-1}} \left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge(\theta)(-\Delta)^{\alpha/2} S_{CK}(\theta)d\theta$$

$$= \int_{S^{2n-1}} \left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge(\theta)(-\Delta)^{\alpha/2} S_{CL}(\theta)d\theta$$

$$+ \varepsilon \frac{(2\pi)^{2n}}{4\pi(n-1)} \int_{S^{2n-1}} \left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge(\theta)g(\theta)d\theta$$

$$> \int_{S^{2n-1}} \left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge(\theta)(-\Delta)^{\alpha/2} S_{CL}(\theta)d\theta,$$

since $\left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge < 0$ on the support of $g$. Using equation (16) and the spherical version of Parseval’s identity, the latter becomes

$$\int_{S^{2n-1}} \|x\|_L^{-2}\|x\|_K^{-2n+2} > \int_{S^{2n-1}} \|x\|_L^{-2n} dx.$$  

As in Theorem [1] we apply Hölder’s inequality and the polar representation of the volume to obtain the desired inequality for the volumes of the bodies.  

\[\Box\]

**Proof of Lemma [4].** The construction of the body follows similar steps as in [KYY]. We put $q = 2n - \alpha - 4$, so $q \in (2, 3)$. From the definition of the fractional derivatives, Proposition [2] and the Remark we see that for $\xi \in S^{2n-1}$ we need to construct a convex body $D$ so that

$$\int_0^{2\pi} \int_0^\infty t^{-q-1} \left(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\right)dt d\theta < 0$$

since $\Gamma(-\frac{q}{2}) > 0$ for $q \in (2, 3)$.

We define the function

$$f(|u|) = (1 - |u|^2 - N|u|^2)_{\frac{1}{n-\alpha}}, \quad u \in \mathbb{R}^2$$

and consider the body $D$ in $\mathbb{R}^{2n}$ as

$$D = \{(x_1, x_2, \ldots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} : |\bar{x}|_2 = |(x_{n1}, x_{n2})|_2 \in [-\alpha N, \alpha N],$$
\[
\left( \sum_{j=1}^{n-1} x_{ij}^2 \right)^{1/2} \leq f(|x|) \}
\]

where \(a_N\) is the first positive root of the equation \(f(t) = 0\). From its definition, the body \(D\) is strictly convex with an infinitely smooth boundary. We choose \(\xi \in S^{2n-1}\) in the direction of \(\bar{x}\). For \(u \in \mathbb{R}^2\) with \(|u|_2 \in [0, a_N]\), we write equation (5) in polar coordinates and get that

\[
A_{D,H,\xi}(u) = \int_{S_u^{2n-3}} \int_0^{f(|u|_2)} \left( r^2 + |u|_2^2 \right)^{-\frac{n}{2}} r^{2n-3} dr d\theta
\]

where \(|S_u^{2n-3}|\) is the volume of the \((2n-3)\)-dimensional unit sphere. Note that if \(|u|_2 > a_N\) then \(A_{D,H,\xi}(u) = 0\) Moreover, if \(u = t\theta, \ t \in [0, \infty)\) and \(\theta \in S^1\), the parallel section function \(A_{D,H,\xi}(t\theta)\) is independent of \(\theta\) since

\[
A_{D,H,\xi}(t\theta) = |S_t^{2n-3}| \int_0^{f(t)} \left( r^2 + t^2 \right)^{-\frac{n}{2}} r^{2n-3} dr.
\]

Hence, we need to prove that the above construction of the body \(D\) gives that

\[
\int_0^\infty t^{-q-1} \left( A_{D,H,\xi}(t\theta) - A_{D,H,\xi}(0) - \Delta A_{D,H,\xi}(0) \right) dt < 0.
\] (24)

Note that the condition \(|u|_2 \in [0, a_N]\) is now equivalent to \(t \in [0, a_N]\). In order to prove the above first compute

\[
A_{D,H,\xi}(0) = \frac{|S_u^{2n-3}|}{2n - \alpha - 2}
\]

and

\[
\Delta A_{D,H,\xi}(0) = -|S_u^{2n-3}| \left[ \frac{1}{2n - \alpha - 2} + \frac{\alpha}{2n - \alpha - 4} \right].
\]

Let \(\beta_N\) be the positive root of the equation \(1 - t^2 - N t^4 = t^{q+1}\). We split the integral in (24) in three parts: \([0, \beta_N]\), \([\beta_N, a_N]\) and \([a_N, \infty)\) and work separately. It is not difficult to see that for large \(N, a_N, \beta_N \approx N^{-\frac{4}{3}}\). Also, for every \(t \in [0, \alpha_N]\), \(f(t) > 0\) and \(f(t) \geq t\) if and only if \(t \in [0, \beta_N]\).

For the first part, the interval \([0, \beta_N]\), since \(f(t) \geq t\), the 2-dimensional parallel section function \(A_{D,H,\xi}\) can be easily estimated if we split it into two integrals. For the second we use the inequality \((1+x)^{-\gamma} \leq 1 - \gamma x + \frac{\gamma(\gamma+1)}{2} x^2\), for \(\gamma > 0\) and \(0 < x < 1\). Then

\[
\int_0^t (r^2 + t^2)^{-\frac{n}{2}} r^{2n-3} dr \leq \int_0^t r^{\alpha+2n-3} dr = \frac{t}{2n - \alpha - 2}
\]
We now use the definition of the function \( f \) where

\[
\int_t^f (r^2 + t^2)^{-\frac{\alpha}{2}} r^{2n-3} dr \leq \int_t^f \left[ 1 - \frac{\alpha t^2}{2 r^2} + \frac{\alpha(t + 2)}{4 r^4} \right] r^{2n-\alpha-3} dr
\]

\[
= \frac{r^{2n-\alpha-2}}{2n-\alpha-2} - \frac{\alpha}{2} t^2 \frac{r^{2n-\alpha-4}}{2n-\alpha-4} + \frac{\alpha(t + 2)}{4} t^4 \frac{r^{2n-\alpha-6}}{2n-\alpha-6} \bigg|_t^f
\]

\[
= \frac{f^{2n-\alpha-2}(t)}{2n-\alpha-2} - \frac{\alpha}{2} t^2 f^{2n-\alpha-4}(t) + \frac{\alpha(t + 2)}{4} t^4 f^{2n-\alpha-6}(t) - C t^{2n-\alpha-2},
\]

where \( C = \frac{1}{2n-\alpha-2} - \frac{p}{2(2n-\alpha-4)} + \frac{\alpha(t + 2)}{4(2n-\alpha-6)} > 0 \), since \( n \geq 3 \) and \( \alpha \in (2n - 7, 2n - 6) \).

We now use the definition of the function \( f \) and the inequality \((1 - x)^\gamma \geq 1 - \gamma x(1 - x)^{\gamma - 1}\), for \( 0 < \gamma < 1 \) and \( 0 < x < 1 \). We then write

\[
\int_t^f \left[ 1 - \frac{\alpha t^2}{2 r^2} + \frac{\alpha(t + 2)}{4 r^4} \right] r^{2n-\alpha-3} dr \leq \frac{1 - t^2 - N t^4}{2n-\alpha-2} - \frac{\alpha t^2}{2} \frac{1 - t^2 - N t^4}{2n-\alpha-4} + \frac{\alpha(t + 2)}{4} t^4 \frac{1 - t^2 - N t^4}{2n-\alpha-6} - C t^{2n-\alpha-2}
\]

\[
\leq \frac{1 - t^2 - N t^4}{2n-\alpha-2} - \frac{\alpha t^2}{2(2n-\alpha-4)} + \frac{\alpha(t + 2)}{4(2n-\alpha-6)} t^4 (1 - t^2 - N t^4) - C t^{2n-\alpha-2}
\]

Hence, we have that

\[
\int_0^{\beta_N} t^{-\gamma - 1} \left( A_{D, H_\xi, \alpha}(t \theta) - A_{D, H_\xi, \alpha}(0) - \Delta A_{D, H_\xi, \alpha}(0) \left( \frac{t^2}{2} \right) \right) dt
\]

\[
= \int_0^{\beta_N} t^{-\gamma - 1} \left( C t^{2n-\alpha-2} - D t^4 + E \frac{t^4 (t^2 + N t^4)}{(1 - t^2 - N t^4)^{2n-\alpha-2}} \right) dt,
\]

where \( E = \frac{\alpha}{2(2n-\alpha-2)} > 0 \), \( F = \frac{\alpha(t + 2)}{4(2n-\alpha-6)} > 0 \) and \( D = \frac{N}{2n-\alpha-2} - \frac{\alpha(t + 2)}{4(2n-\alpha-6)} > 0 \), for \( N \) large enough.

Now, in order to obtain an upper bound for (25) we need to estimate four different integrals. The first one simply gives \( \frac{C}{T} \beta_N^2 \simeq C_1 N^{-\frac{3}{4}} \), and the
second $\frac{D}{q}\beta_N^{-q+1} \simeq D_1 N^{\frac{q-4}{4}}$, for large $N$. For the third one, we make a change of variables, $u = N^{\frac{4}{q}} t$ and get

$$E \int_0^{\beta_N} \frac{t^{-q+1}(t^2 + Nt^4)}{(1 - t^2 - Nt^4)^{\frac{4}{2n-\alpha}}} dt = EN^\frac{q-1}{4} \int_0^{\beta_N N^\frac{4}{4}} \frac{u^{-q+3}(u^{-\frac{1}{2}} + u^2)}{(1 - u^2 N^{-\frac{3}{2}} - u^4)^{\frac{4}{2n-\alpha}}} du \leq E_1 N^\frac{q-1}{4},$$

since $\beta_N N^\frac{4}{4} \to 1$ as $N \to \infty$ and the integral $\int_0^1 \frac{u^{-q+5}}{(1-u^4)^{\frac{4}{2n-\alpha}}} du$ converges.

We apply the same change of variables, $u = N^{\frac{4}{q}} t$, for the last integral and find that it is comparable to $N^\frac{q-1}{4}$.

$$F \int_0^{\beta_N} \frac{t^{-q+3}(t^2 + Nt^4)}{(1 - t^2 - Nt^4)^{\frac{4}{2n-\alpha}}} dt = FN^\frac{q-1}{4} \int_0^{\beta_N N^\frac{4}{4}} \frac{u^{-q+3}(u^2 N^{-\frac{3}{2}} + u^4)}{(1 - u^2 N^{-\frac{3}{2}} - u^4)^{\frac{4}{2n-\alpha}}} du. \tag{26}$$

The integrand in the latter is a positive increasing function of $u$ and $\beta_N N^\frac{4}{4} \to 1$ as $N \to \infty$. So, we can roughly bound the integral from below by a positive constant and have that equation (26) is greater than $F_1 N^\frac{q-1}{4}$, where $F_1 > 0$.

In the second interval, we use the fact that $A_{D,H,\alpha}(t\theta) \leq A_{D,H,\alpha}(0)$ since central sections have maximum volume. Then, since $t << 1$, we have that

$$\int_{\beta_N}^{\alpha_N} t^{-q-1} \left(A_{D,H,\alpha}(t\theta) - A_{D,H,\alpha}(0) - \Delta A_{D,H,\alpha}(0) \frac{t^2}{2}\right) dt \leq \int_{\beta_N}^{\alpha_N} t^{-q-1} \left(\frac{1}{2n - \alpha - 2} + \frac{\alpha}{2(2n - \alpha - 4)}\right) t^2 dt < A \int_{\beta_N}^{\alpha_N} t^{-q-1} dt.$$

Recall that $\alpha_N$ and $\beta_N$ are the positive solutions of the equations $f(t) = 0$ and $1 - t^2 - Nt^4 = t^{q+1}$ respectively, and that for large $N$, $\alpha_N \simeq N^{-\frac{4}{q}}$.

Then, it is not difficult to see that

$$A \int_{\beta_N}^{\alpha_N} t^{-q-1} dt \leq \frac{A}{(\alpha_N + \beta_N)(1 + N(\alpha_N^2 + \beta_N^2))} \simeq AN^{-\frac{1}{4}},$$

see [KYY, p.204] for details.

Lastly, for the interval $[\alpha_N, \infty)$, we use the fact that $A_{D,H,\epsilon,\alpha}(t\theta) = 0$. Then, we have that

$$\int_{\alpha_N}^{\infty} t^{-q-1} \left(-A_{D,H,\epsilon,\alpha}(0) - \Delta A_{D,H,\epsilon,\alpha}(0) \frac{t^2}{2}\right) dt = \int_{\alpha_N}^{\infty} \left[-\frac{t^{-q-1}}{2n - \alpha - 2} + \left(\frac{2}{2n - \alpha - 2} + \frac{\alpha}{2n - \alpha - 4}\right) \frac{t^{-q+1}}{2}\right] dt$$

$$= -A_1 \alpha_N^{-q} + A_2 \alpha^{-q+2} \simeq -A_1 N^\frac{q}{4} + A_2 N^\frac{q-2}{4},$$
where $A_1, A_2 > 0$.

Combining all the above estimations, for $N$ large enough, we obtain the following upper bound for the integral in (24),

$$\int_0^\infty t^{-q-1} \left( A_{D,H,\xi,\alpha}(t\theta) - A_{D,H,\xi,\alpha}(0) - \Delta A_{D,H,\xi,\alpha}(0) \frac{t^2}{2} \right) dt d\theta$$

$$< C_1 N^{-\frac{q}{4}} + D_1 N^{\frac{q-4}{4}} + E_1 N^{\frac{q+2}{4}} - F_1 N^{q-1} + AN^{-\frac{q}{4}} - E_1 N^{\frac{q}{4}} + A_2 N^{\frac{q-2}{4}},$$

which clearly shows that it is negative since all the constants are positive and $q \in (2, 3)$.

\[\square\]

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Marisa Zymonopoulou, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

*E-mail address: marisa@@math.missouri.edu*