Sampled-Data Implementation of Derivative-Dependent Control Using Artificial Delays

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Abstract—We study a sampled-data implementation of linear controllers that depend on the output and its derivatives. First, we consider an LTI system of relative degree \( r \geq 2 \) that can be stabilized using \( r-1 \) output derivatives. Then, we consider PID control of a second-order system. In both cases, the Euler approximation is used for the derivatives giving rise to a delayed sampled-data controller. Given a derivative-dependent controller that stabilizes the system, we show how to choose the parameters of the delayed sampled-data controller that preserves the stability under fast enough sampling. The maximum sampling period is obtained from LMIs that are derived using Taylor’s expansion of the delayed terms with the remainders compensated by appropriate Lyapunov–Krasovskii functionals. Finally, we introduce the event-triggering mechanism that may reduce the amount of sampled control signals used for stabilization.

Index Terms—Sampled-data control, time-delay, linear system, event-triggered control, LMIs.

I. INTRODUCTION

Control laws that depend on output derivatives are used to stabilize systems with relative degrees greater than one. To estimate the derivatives, which can hardly be measured directly, one can use the Euler approximation \( y(t) \approx (y(t) - y(t-\tau))/\tau \). This replaces the derivative-dependent control with the delay-dependent one [2]–[5]. It has been shown in [6] that such approximation preserves the stability if \( \tau > 0 \) is small enough. Similarly, the output derivative in PID controller can be replaced by its Euler approximation. The resulting controller was studied in [7] and [8] using the frequency domain analysis.

In this paper, we study sampled-data implementation of the delay-dependent controllers. For double-integrators, this has been done in [9] using complete Lyapunov–Krasovskii functionals with a Wirtinger-based term and in [10] via impulsive system representation and looped-functionals. Both methods lead to complicated linear matrix inequalities (LMIs) containing many decision variables. In this paper, we obtain simpler LMIs for more general systems and prove their feasibility for small enough sampling periods.

A simple Lyapunov-based method for delay-induced stabilization was proposed in [11] and [12]. The key idea is to use Taylor’s expansion of the delayed terms with the remainders in the integral form that are compensated by appropriate terms in the Lyapunov–Krasovskii functional. This leads to simple LMIs feasible for small delays if the derivative-dependent controller stabilizes the system.

In this paper, we study sampled-data implementation of two types of derivative-dependent controllers. In Section II, we consider an LTI system of relative degree \( r \geq 2 \) that can be stabilized using \( r-1 \) output derivatives. In Section III, we consider PID control of a second-order system. In both cases, the Euler approximation is used for the derivatives giving rise to a delayed sampled-data controller. Assuming that the derivative-dependent controller exponentially stabilizes the system with a decay rate \( \alpha' > 0 \), we show how to choose the parameters of its sampled-data implementation that exponentially stabilizes the system with any decay rate \( \alpha < \alpha' \) if the sampling period is small enough. The maximum sampling period is obtained from LMIs that are derived using the ideas of [11] and [12]. Finally, we introduce the event-triggering mechanism that may reduce the amount of sampled control signals used for stabilization [13]–[17]. In the preliminary paper [1], we studied delayed sampled-data control for systems with relative degree two.

Notations: \( \mathbb{N} = \mathbb{N} \cup \{0\} \), \( I_r = \{1, \ldots, r\} \in \mathbb{R}^r \), \( I_{r/s} \in \mathbb{R}^{r/s} \) is the identity matrix, \( \otimes \) stands for the Kronecker product, \( |x| = \max\{n \in \mathbb{N} | n \leq x \} \) for \( x \in \mathbb{R} \), \( \text{col} \{a_1, \ldots, a_n\} \) denotes the column vector composed from the vectors \( a_1, \ldots, a_n \). For \( p \in \mathbb{R} \), \( f(h) = O(h^p) \) if there exist positive \( M \) and \( h_0 \) such that \( |f(h)| \leq Mh^p \) for \( h \in (0, h_0) \).

A. Auxiliary Lemmas

Lemma 1 (Exponential Wirtinger inequality [18]): Let \( f \in \mathcal{H}_1(a, b) \) be such that \( f(a) = 0 \) or \( f(b) = 0 \). Then

\[
\int_a^b e^{2\alpha t} f(t) W f(t) \, dt \\
\leq e^{2\alpha b} \left( \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\alpha t} f^2(t) \, dt \right)
\]

for any \( \alpha \geq 0 \) and \( b \geq 0 \).

Lemma 2 (Jensen’s inequality [19]): Let \( \rho : [a, b] \to [0, \infty) \) and \( f : [a, b] \to \mathbb{R}^n \) be such that the integration concerned is well defined. Then, for any \( 0 < Q \in \mathbb{R}^{n \times n} \)

\[
\int_a^b \rho(s) f(s) \, ds \leq \int_a^b \rho(s) f^T(s) Q f(s) \, ds.
\]
II. DERIVATIVE-DEPENDENT CONTROL USING DISCRETE-TIME MEASUREMENTS

Consider the LTI system
\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^n, y \in \mathbb{R}^l \] (1)
with relative degree \( r \geq 2 \), i.e.,
\[ CA^r B = 0, \quad i = 0, 1, \ldots, r - 2, \quad CA^r B \neq 0. \] (2)

Relative degree is how many times the output \( y(t) \) needs to be differentiated before the input \( u(t) \) appears explicitly. In particular, (2) implies
\[ y^{(i)} = CA^i x, \quad i = 0, 1, \ldots, r - 1. \] (3)

To prove (3), note that it is trivial for \( i = 0 \) and, if it has been proved for \( i < r - 1 \), it holds for \( i + 1 \)
\[ y^{(i+1)} = (y^{(i)})^{(1)} \equiv (CA^i x)^{\prime} = CA^i [Ax + Bu] \equiv CA^{i+1} x. \]

For LTI systems with relative degree \( r \), it is common to look for a stabilizing controller of the form
\[ u(t) = \bar{K}_0 y(t) + \bar{K}_1 y(t) + \cdots + \bar{K}_{r-1} y^{(r-1)}(t) \] (4)
with \( \bar{K}_i \in \mathbb{R}^{m \times l} \) for \( i = 0, \ldots, r - 1 \).

Remark 1: The control law (4) essentially reduces the system’s relative degree from \( r \) to \( r - 1 \). Indeed, the transfer matrix of (1) has the form
\[ W(s) = \frac{\beta_0 s^{n-r} + \cdots + \beta_n}{s^{n} + \alpha_1 s^{n-1} + \cdots + \alpha_n} \]
with \( \beta_r = CA^r B \neq 0 \). Taking \( u(t) = \bar{K}_i \dot{u}_0(t) + \bar{K}_1 \dot{u}_0(t) + \cdots + \bar{K}_{r-1} u_0^{r-1}(t) \), one has
\[ \bar{y}(s) = \frac{(\beta_0 s^{n-r} + \cdots + \beta_n)(K_{r-1} s^{r-1} + \cdots + K_0)}{s^{n} + \alpha_1 s^{n-1} + \cdots + \alpha_n} \bar{u}_0(s) \]
where \( \bar{y} \) and \( \dot{u}_0 \) are the Laplace transforms of \( y \) and \( u_0 \). If \( \beta_0 K_{r-1} \neq 0 \), the latter system has relative degree one. If it can be stabilized by \( u_0 = \bar{K} \dot{y} \), then (1) can be stabilized by (4) with \( \bar{K}_0 = \bar{K} K \).

The controller (4) depends on the output derivatives, which are hard to measure directly. Instead, the derivatives can be approximated by the finite differences
\[ \dot{y}(t) \approx \frac{y(t) - y(t - \tau_i)}{\tau_i}, \] \[ \ddot{y}(t) \approx \frac{ \frac{y(t) - y(t - \tau_i)}{\tau_i} - \frac{y(t - \tau_i) - y(t - \tau_2)}{\tau_2} }{\tau_2 - \tau_i}, \]
This leads to the delay-dependent control
\[ u(t) = \bar{K}_0 y(t) + \bar{K}_1 y(t - \tau_i) + \cdots + \bar{K}_{r-1} y(t - \tau_{r-1}) \] (5)
where the gains \( \bar{K}_0, \ldots, \bar{K}_{r-1} \) depend on the delays \( 0 < \tau_1 < \cdots < \tau_{r-1} \). If (1) can be stabilized by the derivative-dependent control (4), then it can be stabilized by the delayed control (5) with small enough delays [6]. In this paper, we study the sampled-data implementation of (5)
\[ u(t) = \bar{K}_0 y(t_k) + \sum_{i=1}^{r-1} \bar{K}_i y(t_k - q_i h) + \bar{K}_{r-1} y(t_k - \tau_{r-1}) \] (6)
with \( h > 0 \) is a sampling period, \( t_k = k h \) are sampling instants, \( 0 < q_1 < \cdots < q_{r-1} \), \( q_i \in \mathbb{N} \) are discrete-time delays, and \( y(t) = 0 \) if \( t < 0 \).

In the next section, we prove that if (1) can be stabilized by the derivative-dependent controller (4), then it can be stabilized by the delayed sampled-data controller (6) with a small enough sampling period \( h \). Moreover, we show how to choose appropriate sampling period \( h \), controller gains \( \bar{K}_0, \ldots, \bar{K}_{r-1} \), and discrete-time delays \( q_1, \ldots, q_{r-1} \).

A. Stability Conditions

Introduce the errors due to sampling
\[ \delta_0(t) = y(t_k) - y(t), \quad t \in [t_k, t_{k+1}), k \in \mathbb{N}_0 \]
\[ \delta_i(t) = y(t_k - q_i h) - y(t - q_i h), \]
where \( i = 1, \ldots, r - 1 \). Following [12], we employ Taylor’s expansion with the remainder in the integral form
\[ y(t - q_i h) = \sum_{j=0}^{r-1} \frac{y^{(j)}(t)}{j!} (-q_i h)^j + \kappa_i(t) \]
where
\[ \kappa_i(t) = \frac{(-1)^r}{(r - 1)!} \int_{t - q_i h}^{t} (s - t + q_i h)^{r-1} y^{(r)}(s) \, ds. \]
Combining these representations with (3), we rewrite (6) as follows:
\[ u = [\bar{K}_0, \bar{K}] \bar{M} \bar{C} x + \bar{K}_0 \delta_0 + K \delta + K \kappa \]
where \( \delta = \text{col} \{\delta_1, \ldots, \delta_{r-1}\}, \kappa = \text{col} \{\kappa_1, \ldots, \kappa_{r-1}\}, \) we obtain
\[ M = \begin{bmatrix} I_l & 0 & \cdots & 0 \\ I_l & -q_1 I_l & \cdots & (-q_1 h)^{-1} I_l \\ I_l & -q_2 I_l & \cdots & (-q_2 h)^{-1} I_l \\ \vdots & \vdots & \ddots & \vdots \\ I_l & -q_{r-1} I_l & \cdots & (-q_{r-1} h)^{-1} I_l \end{bmatrix} \]
\[ K = \begin{bmatrix} K_1 & K_2 & \ldots & K_{r-1} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \]
The closed-loop system (1), (6) takes the form
\[ \dot{x} = Dx + BK_0 \delta_0 + BK \delta + BK \kappa \]
\[ D = A + B[\bar{K}_0, \bar{K}] \bar{M} \bar{C} \]
Using (3), the closed-loop system (1), (4) can be written as follows:
\[ \dot{x} = \bar{D} x, \quad \bar{D} = A + B[\bar{K}_0, \bar{K}] \bar{C}. \]
Choosing
\[ [K_0, K_1, \ldots, K_{r-1}] = [\bar{K}_0, \ldots, \bar{K}_{r-1}] M^{-1} \]
we obtain \( D = \bar{D} \). (The Vandermonde-type matrix \( M \) is invertible, since the delays \( q_1 h \) are different. If (1), (4) is stable, \( \bar{D} \) must be Hurwitz and (9) will be stable for zero \( \delta_0, \delta, \kappa \). The following theorem provides LMIs guaranteeing that \( \delta_0, \delta, \kappa \) do not destroy the stability of (9).

**Theorem 1**: Consider the LTI system (1) subject to (2).

i) For given sampling period \( h > 0 \), discrete-time delays \( 0 < q_1 < \cdots < q_{r-1} \), controller gains \( K_0, \ldots, K_{r-1} \in \mathbb{R}^{m \times l} \), and decay rate \( \alpha > 0 \), let there exist positive-definite matrices \( P \in \mathbb{R}^{n \times n}, W_0, W_1, R_i \in \mathbb{R}^{m \times m} (i = 1, \ldots, r - 1) \) such that \( \Phi \leq 0 \), where

\(^3\)MATLAB codes for solving the LMIs are available at https://github.com/ AntonSelivanov/TAC18
Theorem 1(ii) explicitly defines the controller parameters $K_1, u$ for small $h > 0$. PB $R_1 \in \mathbb{R}^{u, 0}$, $Cx_0 \in \mathbb{P}$, $K_1$ from (6) was transmitted and $\hat{u}_k = \hat{u}_{k-1}$ otherwise. The signal $u(t_k)$ is transmitted if its relative change since the last transmission is large enough, namely if

$$
(u(t_k) - \hat{u}_{k-1})^T \Omega (u(t_k) - \hat{u}_{k-1}) > \sigma \alpha^T (t_k) \Omega u(t_k)
$$

where $\hat{u}_k = u(t_k)$ and

$$
\hat{u}_k = \begin{cases} u(t_k), & (13) \text{ is true} \\ \hat{u}_{k-1}, & (13) \text{ is false}. \end{cases}
$$

Theorem 2: Consider the system (12) subject to (2). For given sampling period $h > 0$, discrete-time delays $0 < q_1 < \cdots < q_{r-1}$, controller gains $K_0, K_{r-1} \in \mathbb{R}^{r \times r}$, event-triggering threshold $\sigma \in [0, 1)$, and decay rate $\alpha > 0$, let there exist positive-definite matrices $P \in \mathbb{R}^{n \times n}, \Omega, W_0, W_r, R_l \in \mathbb{R}^{m \times m}$ (i = 1, ..., r-1) such that $\Phi \preceq 0$, where

$$
\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_r \\ \Phi_1^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_r^T & 0 & \cdots & 0 \\ \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}
$$

$$
\begin{bmatrix} 
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
$$

with $M, K, C$ are defined in (8) and $\Phi, H$ are given in Theorem 1. Then, the event-triggered controller (6), (13), (14) exponentially stabilizes the system (12) with the decay rate $\alpha$.

Proof is given in Appendix B.

Remark 3: The event-triggering mechanism (13), (14) is constructed with respect to the control signal. This allows to reduce the workload of a controller-to-actuator network. To compensate the event-triggering error, we add (29) to $\dot{V}$, which leads to the second additional block-columns and block-rows in the LMI (confer $\Phi$ of Theorem 1 and $\Phi_e$ of Theorem 2). One can study the event-triggering mechanism with respect to the measurements by replacing $y(t_k)$, $y(t_k - q_i h)$ with $\tilde{y}_k = y(t_k) + e_k, \tilde{y}_k - \tilde{y}_k - q_i h = y(t_k - q_i h) + e_k - e_k - q_i h$ in (6). This may reduce the workload of a sensor-to-controller network but would require to add expressions similar to (29) to $\dot{V}$ for each error $e_k, e_k - q_i, \cdots, e_k - q_{r-1}$. This would lead to more complicated LMIs with two additional block-columns and block-rows for each error. We study the event-triggering mechanism with respect to the control for simplicity.

Remark 4: Taking $\Omega = \omega I$ with large $\omega > 0$, one can show that $\Phi_e \leq 0$ and $\Phi_e \leq 0$ are equivalent for $\sigma = 0$. This happens since the event-triggered control (6), (13), (14) with $\sigma = 0$ degenerates into periodic sampled-data control (6). Therefore, an appropriate $\sigma$ can be found by increasing its value from zero while preserving the feasibility of the LMIs from Theorem 2.

C. Example

Consider the triple integrator $\ddot{y} = u$, which can be presented in the form (1) with

$$
A = \begin{bmatrix} 0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
0 \\
1 \\
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\
\end{bmatrix}. \quad (15)
$$

These parameters satisfy (2) with $r = 3$. The derivative-dependent control (4) with

$$
\begin{bmatrix} \hat{K}_0 = -2 \times 10^{-4} \\
\hat{K}_1 = -0.06 \\
\hat{K}_2 = -0.342 \\
\end{bmatrix}
$$

where $\hat{K}_k = \hat{u}_k = u(t_k)$ if $u(t_k)$ from (6) was transmitted and $\hat{u}_k = \hat{u}_{k-1}$ otherwise. The signal $u(t_k)$ is transmitted if its relative change since the last transmission is large enough, namely if

$$
\frac{u(t_k) - \hat{u}_{k-1}}{\max(1, \hat{u}_{k-1})} > \sigma \alpha^T (t_k) \Omega u(t_k)
$$

where $\hat{u}_k = u(t_k)$ and

$$
\hat{u}_k = \begin{cases} u(t_k), & (13) \text{ is true} \\ \hat{u}_{k-1}, & (13) \text{ is false}. \end{cases}
$$

MATLAB codes for solving the LMIs are available at https://github.com/AntonSellovich/7AC16

2Note that $q_i = \lfloor h \hat{q}^{-1} \rfloor \in \mathbb{N}$ for small $h > 0$.
stabilizes the system (1), (15). The LMIs of Theorem 1 are feasible for
\[ h = 0.044,\quad q_1 = 30,\quad q_2 = 60,\quad \alpha = 10^{-3} \]
\[ K_0 \approx -0.265,\quad K_1 \approx 0.483,\quad K_2 \approx -0.219 \]
where \( K_i \) are calculated using (11). Therefore, the delayed-sampled
controller (6) also stabilizes the system (1), (15).

Consider now the system (12), (15). The LMIs of Theorem 2 are feasible for \( h = 0.042,\quad \sigma = 2 \times 10^{-3} \) with the same control gains \( K_0, K_1, K_2 \), delays \( q_1, q_2 \), and decay rate \( \alpha \). Thus, the event-triggered control (6), (13), (14) stabilizes the system (12), (15). Performing numerical simulations for ten randomly chosen initial conditions \( \|x(0)\|_\infty \leq 1 \), we find that the event-triggered control (6), (13), (14) requires to transmit on average 455.6 control signals during 100 s.

The amount of transmissions for the sampled-data control (6) is given by \( \frac{100}{1 + \frac{1}{\sigma}} = 2273 \). Thus, the event-triggering mechanism reduces the workload of the controller-to-actuator network by almost 80% preserving the decay rate \( \alpha \). Note that \( \sigma > 0 \) leads to a smaller sampling period \( h \). Therefore, the event-triggering mechanism requires to transmit more measurements through sensor-to-controller network. However, the total workload of both networks is reduced by over 37%.

### III. EVENT-TRIGGERED PID CONTROL

Consider the scalar system
\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b \dot{u}(t) \]
and the PID controller
\[ u(t) = \tilde{k}_p y(t) + \tilde{k}_i \int_0^t y(s) ds + \tilde{k}_d \dot{y}(t). \]

Here, we study sampled-data implementation of the PID controller (17) that is obtained using the approximations
\[ \int_0^t y(s) ds \approx \int_0^{t_k} y(s) ds \approx h \sum_{j=0}^{k-1} y(t_j), \quad t \in [t_k, t_{k+1}) \]
\[ \ddot{y}(t) \approx \ddot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-q})}{q h} \]
where \( h > 0 \) is a sampling period, \( t_k = kh \), \( k \in \mathbb{N}_0 \) are sampling instants, \( q \in \mathbb{N} \) is a discrete-time delay, and \( y(t_{k-q}) = 0 \) for \( k < q \).

Substituting these approximations into (17), we obtain the sampled-data controller
\[ u(t) = k_p y(t_k) + k_i \sum_{j=0}^{k-1} y(t_j) + k_d y(t_{k-q}) \]
\[ t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0 \]
with \( y(t_{k-q}) = 0 \) for \( k < q \) and
\[ k_p = \tilde{k}_p + \frac{k_i}{q h}, \quad k_i = \tilde{k}_i, \quad k_d = -\frac{\tilde{k}_d}{q h}. \]

Similarly to Section II-B, we introduce the event-triggering mechanism to reduce the amount of transmitted control signals. Namely we consider the system
\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b \dot{u}(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0 \]
where \( \dot{u}_k \) is the event-triggered control: \( \dot{u}_k = u(t_k) \)
\[ \dot{u}_k = \begin{cases} u(t_k), & \text{if (22) is true} \\ \dot{u}_{k-1}, & \text{if (22) is false} \end{cases} \]
with \( u(t) \) from (18) and the event-triggering condition
\[ (u(t_k) - \dot{u}_{k-1})^2 > \sigma u^2(t_k). \]

Here, \( \sigma \in [0, 1) \) is the event-triggering threshold.

**Remark 5:** We consider the event-triggering mechanism with respect to the control signal, since the event-triggering with respect to the measurements \( \dot{u}_k = u(t_k) + e_k \) leads to an accumulating error in the integral term
\[ \int_0^{t_k} y(s) ds \approx h \sum_{j=0}^{k-1} y_j = h \sum_{j=0}^{k-1} y(t_j) + h \sum_{j=0}^{k-1} e_j. \]

### A. Stability Conditions

To study the stability of (20) under the event-triggered PID control (18), (21), (22), we rewrite the closed-loop system in the state space. Let \( x_1 = y, \quad x_2 = \dot{y}, \) and
\[ x_3(t) = (t - t_k) y(t_k) + h \sum_{j=0}^{k-1} y(t_j), \quad t \in [t_k, t_{k+1}). \]

Introduce the errors due to sampling
\[ v(t) = x(t_k) - x(t), \quad \delta(t) = y(t_k) - y(t - q h). \]

Using Taylor’s expansion for \( y(t - q h) \) with the remainder in the integral form, we have
\[ y(t_{k-q}) = y(t - q h) + \delta(t) = y(t) - \dot{y}(t) q h + \kappa(t) + \delta(t) \]
where
\[ \kappa(t) = \int_{t_k}^{t_k+q h} (s - t + q h) \dot{y}(s) ds. \]

Using these representations in (18), we obtain
\[ u(t_k) = k_p x_1(t_k) + k_i x_2(t_k) + k_d y(t_{k-q}) \]
\[ = [k_p + k_d, -q h k_d, k_i] x + [k_p, 0, k_d] v + k_d (\kappa + \delta). \]

Introduce the event-triggering error \( e_k = \dot{u}_k - u(t_k) \). Then, the system (20) under the event-triggered PID control (21), (22), (23) can be presented as follows:
\[ \dot{x} = Ax + Av + B k_d (\kappa + \delta) + B e_k \]
\[ y = C x \]
for \( t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \) where
\[ A = \begin{bmatrix} 0 & 0 & 1 \\ -a_2 + b k_p + k_d & -a_1 - q h b k_d & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]
\[ B = \begin{bmatrix} 0 \\ b k_p + b k_i \\ 0 \end{bmatrix}, \]
\[ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \]

Note that the “integral” term in (18) requires to introduce the error due to sampling \( v \) that appears in (24) but was absent in (9). The analysis of \( v \) is the key difference between Theorem 2 and the next result.

**Theorem 3:** Consider the system (20).

i) For given sampling period \( h > 0, \) discrete-time delay \( q > 0, \) controller gains \( k_p, k_i, k_d, \) event-triggering threshold \( \sigma \in [0, 1), \) and decay rate \( \alpha > 0, \) let there exist positive-definite matrices \( P, S \in \mathbb{R}^{3 \times 3} \) and nonnegative scalars \( W, R, \) such that\(^*\)

\[ \Psi \leq 0, \]

\(^*\)MATLAB codes for solving the LMIs are available at https://github.com/antonSelivanov/7AC18.
where $\Psi = \{\Psi_{ij}\}$ is the symmetric matrix composed from

$$
\Psi_{11} = PA + A^T P + 2\alpha P + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} W k_3^2 h^2 e^{2\alpha h} \\
\Psi_{12} = PA \sqrt{h}, \quad \Psi_{13} = \Psi_{14} = \Psi_{15} = PB \\
\Psi_{16} = \begin{bmatrix} k_p + k_d \\ -q h k_d \end{bmatrix} \omega \sigma, \quad \Psi_{26} = \begin{bmatrix} k_p \\ 0 \end{bmatrix} \omega \sigma \sqrt{h} \\
\Psi_{17} = A^T G, \quad \Psi_{22} = -\frac{\pi^2}{4} S h, \quad \Psi_{27} = A^T G \sqrt{h} \\
\Psi_{36} = \Psi_{46} = \omega \sigma, \quad \Psi_{37} = \Psi_{47} = B^T G \\
\Psi_{33} = -W \frac{\pi^2}{4} e^{-2\alpha q h}, \quad \Psi_{44} = -R \frac{4}{(qh)^2} e^{-2\alpha q h} \\
\Psi_{55} = -\omega, \quad \Psi_{66} = -\omega \sigma, \quad \Psi_{77} = -G \\
G = h^2 e^{2\alpha h} S + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} R k_3^2 (qh)^2
$$

with $A$, $A_1$, $B$, $C$ given in (25). Then, the event-triggered PID controller (18), (21), (22) exponentially stabilizes the system (20) with the decay rate $\alpha$.

ii) Let there exist $k_p, k_i, k_d$ such that the PID controller (17) exponentially stabilizes the system (16) with a decay rate $\alpha'$. Then, the event-triggered PID controller (18), (21), (22) with $k_p, k_i, k_d$ given by (11) and $q = [h^{1/2}]$ exponentially stabilizes the system (20) with any given decay rate $\alpha < \alpha'$ if the sampling period $h > 0$ and the event-triggering threshold $\sigma \in [0, 1]$ are small enough.

Proof is given in Appendix C.

Remark 6: The event-triggered control (18), (21), (22) with $\sigma = 0$ degenerates into sampled-data control (18). Therefore, Theorem 3 with $\sigma = 0$ gives the stability conditions for the system (16) under the sampled-data PID control (18).

Remark 7: Appropriate values of $h$ and $\sigma$ can be found in a manner similar to Remarks 2 and 4.

B. Example

Following [8], we consider (16) with $\alpha_1 = 8.4$, $\alpha_2 = 0$, $b = 35.71$. The system is not asymptotically stable if $u = 0$. The PID controller (17) with $k_p = -10$, $k_i = -40$, $k_d = -0.65$ exponentially stabilizes it with the decay rate $\alpha' \approx 10.4$.

Theorem 3 with $\sigma = 0$ (see Remark 6) guarantees that the sampled-data PID controller (18) can achieve any decay rate $\alpha < \alpha'$ if the sampling period $h > 0$ is small enough. Since $\alpha'$ is on the verge of stability, $\alpha$ close to $\alpha'$ requires to use small $h$. Thus, for $\alpha = 10.3$, the LMIs of Theorem 3 are feasible with $h \approx 10^{-7}$, $q = 4272$, and $k_p$, $k_i$, $k_d$ given by (19). To avoid small sampling periods, we take $\alpha = 5$.

For $\sigma = 0$, $\alpha = 5$, and each $q = 1, 2, 3, \ldots$, we find the maximum sampling period $h > 0$ such that the LMIs of Theorem 3 are feasible. The largest $h$ corresponds to

$$
\alpha = 5, \quad \sigma = 0, \quad q = 7, \quad h = 4.7 \times 10^{-3} \\
k_p \approx 29.76, \quad k_i \approx -40, \quad k_d \approx 19.76
$$

where $k_p$, $k_i$, $k_d$ are calculated using (19). Remark 6 implies that the sampled-data PID controller (18) stabilizes (16).

Theorem 3 remains feasible for

$$
\alpha = 5, \quad \sigma = 9 \times 10^{-3}, \quad q = 7, \quad h = 4 \times 10^{-3} \\
k_p \approx -33.21, \quad k_i \approx -40, \quad k_d \approx 23.21
$$

where $k_p$, $k_i$, $k_d$ are calculated using (19). Thus, the event-triggered PID control (18), (21), (22) exponentially stabilizes (20). Performing numerical simulations in a manner described in Section II-C, we find that the event-triggered PID control requires to transmit on average 628.4 control signals during 10 s. The sampled-data controller (18) requires $\left\lceil \frac{10}{\pi} \right\rceil + 1 = 2128$ transmissions. Thus, the event-triggering mechanism reduces the workload of the controller-to-actuator network by more than 70%. The total workload of both networks is reduced by more than 26%.

APPENDIX A

PROOF OF THEOREM 1

(i) Consider the functional

$$
V = V_0 + V_{50} + V_{5} + V_x
$$

with

$$
V_0 = x^T P x \\
V_{50} = h^2 e^{2\alpha h} \int_{t_k}^{t} e^{-2\alpha (t-s)} \frac{\dot{y}(s)}{K_0} W_0 K_0 \frac{y(s)}{K_0} ds \\
= \frac{-\pi^2}{4} \int_{t_k}^{t} e^{-2\alpha (t-s)} \left(y(s) - y(t_k)\right)^T K_0^T W_0 \left(y(s) - y(t_k)\right) ds \\
= \frac{-\pi^2}{4} \int_{t_k}^{t} e^{-2\alpha (t-s)} \left(y(s) - y(t_k - q h)\right)^T K_0^T W_0 \left(y(s) - y(t_k - q h)\right) ds \\
V_5 = \sum_{i=1}^{r-1} \int_{t_k - q h}^{t} e^{-2\alpha (t-s)} (s - t + q h)^r \left(y_r(s)\right)^T K_r^T R_r K_r y_r(s) ds
$$

The term $V_x$, introduced in [12], compensates Taylor’s remainders $\kappa_i$, whereas $V_0$ and $V_5$ introduced in [9], compensate the sampling errors $\delta_t$ and $\delta$. The Wirtinger inequality (see Lemma 1) implies $V_{50} \geq 0$ and $V_5 \geq 0$. Using (9) and (3), we obtain

$$
\dot{V}_0 + 2 \alpha V_0 = 2 x^T P \left[D x + B K_0 \delta_0 + B K \delta + B K \kappa + 2 \alpha x^T P x \right] \\
\dot{V}_{50} + 2 \alpha V_{50} = h^2 e^{2\alpha h} x^T (K_0 C A)^T W_0 (K_0 C A) x \\
- \sum_{i=1}^{r-1} \delta_i^T K_0^T W_0 K_0 \delta_i \\
\dot{V}_5 + 2 \alpha V_5 = h^2 e^{2\alpha h} \sum_{i=1}^{r-1} x^T (K_i C A)^T W_i (K_i C A) x \\
- \frac{-\pi^2}{4} \int_{t_k}^{t} e^{-2\alpha (t-s)} \frac{\dot{y}(s)}{K_0} W_0 \left(y(s) - y(t_k - q h)\right) ds
$$
Using \( y(r) = CA^{-1} \dot{x} \) [which follows from (3)] and Jensen’s inequality (see Lemma 2) with \( \rho(s) = (s - t + q, h)^{-1} \), we have

\[
\dot{V}_k + 2aV_k = \sum_{i=1}^{r-1} (q_i h_i y_i(t)) T^T K_i R_i K_i y_i(t) - \sum_{i=1}^{r-1} r \int_{t-q_i h_i}^t e^{-2a(t-s)} (s - t + q_i h_i)^{-1} \times (y_i(t)) T^T K_i R_i K_i y_i(s) \, ds \\
\leq \sum_{i=1}^{r-1} (q_i h_i) \dot{x}^T (A^{-1})^T C^T K_i R_i K_i C A^{-1} \dot{x} - \sum_{i=1}^{r-1} (r_i h_i) e^{-2a}(t-s) K_i T^T K_i R_i K_i.
\]

Summing up, we obtain

\[
\dot{V} + 2aV \leq \varphi^T \Phi \varphi + \dot{x}^T (A^{-1})^T H(A^{-1}) \dot{x} \quad (27)
\]

where

\[
\varphi = \{x, K_0 \delta_0, \ldots, K_{r-1} \delta_{r-1}, K_1 \kappa_1, \ldots, K_{r-1} \kappa_{r-1}\} \quad (28)
\]

and \( \Phi \) is obtained from \( \Phi \) by removing the last block-column and block-row. Substituting (9) for \( \dot{x} \) and applying the Schur complement, we find that \( \Phi \leq 0 \) guarantees \( \dot{V} \leq -2aV \). Since \( V(t_k) \leq V(t_0) \), the latter implies exponential stability of the system (9) and, therefore, (1), (6).

ii) Since \( q_i = O(h^{\frac{1}{1}}) \), [12, Lemma 2.1] guarantees \( M^{-1} = O(h^{\frac{1}{1}}) \), which implies \( K_i = O(h^{\frac{1}{1}}) \) for \( i = 0, \ldots, r - 1 \). Since \( D = D \) [with \( D \) defined in (10)] and (1), (4) is exponentially stable with the decay rate \( \alpha \), there exists \( P > 0 \) such that \( PD + DT^T P + 2aP < \alpha \) for any \( \alpha > \alpha'. \) Choose \( W_0 = O(h^{-\frac{1}{2}}) \), \( W_i = O(h^{-\frac{1}{2}}) \), and \( R_i = O(h^{-\frac{1}{2}}) \) for \( i = 1, \ldots, r - 1 \). Applying the Schur complement to \( \Phi \leq 0 \), we obtain

\[
PD + DT^T P + 2aP + O(h^{\frac{1}{2}}) < 0
\]

which holds for small \( h > 0 \). Thus, (i) guarantees (ii).

**Appendix B**

**Proof of Theorem 2**

Denote \( e_k = \dot{u}_k - u(t_k), k \in \mathbb{N}_0 \). The event-triggering mechanism (13), (14) guarantees

\[
0 \leq \sigma u^T (t_k) \Omega u(t_k) - e_k^T \Omega e_k \quad (29)
\]

Substituting \( \dot{u}_k = u(t_k) + e_k \) into (12) and using (7), we obtain [cf. (9)]

\[
\dot{x} = DX + BK_0 \delta_0 + BK_1 \delta + BK_k + B e_k, \quad t \in [t_k, t_{k+1}]
\]

with \( D \) given in (9). Consider \( V \) from (26). Calculations similar to those from the proof of Theorem 1 lead to [cf. (27)]

\[
\dot{V} + 2aV \leq \dot{V} + 2aV + \sigma u^T (t_k) \Omega u(t_k) - e_k^T \Omega e_k \\
\leq \varphi^T \tilde{\Phi} \varphi + \dot{x}^T (A^{-1})^T H(C A^{-1}) \dot{x} + \sigma \dot{\alpha}^T (t_k) \Omega u(t_k)
\]

where \( \varphi = \{ \varphi, e_k \} \) with \( \varphi \) from (28) and \( \tilde{\Phi} \) is obtained from \( \Phi \) by removing the blocks \( \Phi_r \) with \( i \in [4, 0) \) or \( j \in [4, 6) \). Substituting (30) for \( \dot{x} \) and (7) for \( u(t_k) \) and applying the Schur complement, we find that \( \Phi_r \leq 0 \) guarantees \( \dot{V} \leq -2aV \). Since \( V(t_k) \leq V(t_k) \), the latter implies exponential stability of the system (30) and, therefore, (12) under the controller (6), (13), (14).

**Appendix C**

**Proof of Theorem 3**

i) Consider the functional

\[
V = V_0 + V_c + V_\delta + V_e
\]

with

\[
V_0 = x^TPx \quad V_c = h^2 e^{2a} \int_{t_k}^t e^{-a(t-s)} \dot{x}^T (s) S \dot{x}(s) \, ds \\
- \frac{\pi^2}{4} \int_{t_k}^t e^{-a(t-s)} v^T (s) S v(s) \, ds \\
V_\delta = W_k^2 h^2 e^{2a} \int_{t_k}^t e^{-a(t-s)} \{y(s) - y(t_k - qh)\}^2 \, ds \\
V_e = R_k^2 \int_{t_k}^t e^{-a(t-s)} (s - t + qh)^2 \{y(s)\}^2 \, ds.
\]

The Wirtinger inequality (see Lemma 1) implies \( V_c \geq 0 \) and \( V_\delta \geq 0 \).

Using the representation (24), we obtain

\[
\dot{V}_0 + 2aV_0 = 2x^T P[Ax(t) + Av(t) + B \delta(k + \delta) + Be_k] \\
+ 2a x^T P x
\]

\[
\dot{V}_c = h^2 e^{2a} \int_{t_k}^t e^{-a(t-s)} \dot{x}^T (s) S \dot{x}(s) \, ds \\
- \frac{\pi^2}{4} \int_{t_k}^t e^{-a(t-s)} v^T (s) S v(s) \, ds \\
\dot{V}_\delta = W_k^2 h^2 e^{2a} (x_2(t))^2 - W_k^2 \frac{\pi^2}{4} e^{-a(t-s)} \dot{y}(s) \, ds \\
\dot{V}_e = R_k^2 \int_{t_k}^t e^{-a(t-s)} (s - t + qh)^2 \{y(s)\}^2 \, ds
\]

Using Jensen’s inequality (see Lemma 2) with \( \rho(s) = (s - t + qh) \), we obtain

\[
\dot{V}_c + 2aV_c = R_k^2 (qh)^2 \{y(s)\}^2 \\
- R_k^2 \int_{t_k}^t e^{-a(t-s)} (s - t + qh) \{y(s)\}^2 \, ds \\
\leq R_k^2 (qh)^2 \{x_2(t)^2\} - R_k^2 \frac{\pi^2}{4} (qh)^2 e^{-a(t-s)} \dot{y}(s) \, ds
\]

For \( \omega \geq 0 \), the event-triggering rule (21), (22) guarantees

\[
0 \leq \omega \sigma u^2 (t_k) - \omega e_k^2
\]

Thus, we have

\[
\dot{V} + 2aV \leq \dot{V} + 2aV + \omega \sigma u^2 (t_k) - \omega e_k^2 \\
\leq \psi^T \tilde{\Psi} \psi + \dot{x}^T (t) G \dot{x}(t) + \omega \sigma u^2 (t_k)
\]

where \( \psi = \{x, v/\sqrt{h}, \delta, \delta, \kappa, e_k\} \) and \( \tilde{\Psi} \) is obtained from \( \Psi \) by removing the last two block-columns and block-rows. Substituting (24) for \( \dot{x} \) and (23) for \( u(t_k) \) and applying the Schur complement, we find that \( \Phi \leq 0 \) guarantees \( \dot{V} \leq -2aV \). Since \( V(t_k) \leq V(t_k) \), the latter implies exponential stability of the system (24) and, therefore, (18), (20)–(22).

(ii) The closed-loop system (16), (17) is equivalent to \( \dot{x} = \tilde{A} x \) with

\[
\tilde{A} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-a_2 + b \delta & -a_1 + b \delta & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad x = \begin{bmatrix}
y \\
y \\
y(s) ds
\end{bmatrix}
\]
Since \( q = O\left( \frac{1}{\sqrt{h}} \right) \), relations (19) imply \( k_p = O\left( \frac{1}{\sqrt{h}} \right), k_d = O\left( \frac{1}{\sqrt{h}} \right) \). Since (16), (17) is exponentially stable with the decay rate \( \alpha' \) and (19) implies \( A = \bar{A} \), there exists \( P > 0 \) such that \( PA + A^T P + 2\alpha P < 0 \) for any \( \alpha < \alpha' \). Choose \( S = O\left( \frac{1}{\sqrt{h}} \right), W = O\left( \frac{1}{\sqrt{h}} \right), R = O\left( \frac{1}{\sqrt{h}} \right), \) and \( \omega = O\left( \frac{1}{\sqrt{h}} \right) \). Applying the Schur complement to \( \Psi \leq 0 \), we obtain
\[
PA + A^T P + 2\alpha P + O\left( \frac{1}{\sqrt{h}} \right) + \sigma F < 0
\]
with some \( F \) independent of \( \sigma \). The latter holds for small \( h > 0 \) and \( \sigma \geq 0 \). Thus, (i) guarantees (ii).

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