Localizability with Range-Difference Measurements

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Abstract—The physical position is crucial in location-aware services or protocols based on geographic information, where localization is performed given a set of sensor measurements for acquiring the position of an object with respect to a certain coordinate system. In this paper, we revisit the long-standing localization methods for locating a radiating source from range-difference measurements, or equivalently, time-difference-of-arrival measurements from the perspective of least squares (LS). In particular, we focus on the spherical LS error model, where the error function is defined as the difference between the squared true distance from a signal receiver (sensor) to the source and its squared measured value, and the resulting spherical LS estimation problem. This problem has been known to be challenging due to the non-convex nature of the hyperbolic measurement model. First of all, we prove that the existence of least-squares solutions is universal and that solutions are bounded under some assumption on the geometry of the sensor placement. Then a necessary and sufficient condition is presented for the solution characterization based on the method of Lagrange multipliers. Next, we derive a characterization for the uniqueness of the solutions incorporating a second-order optimality condition. The solution structures for some special cases are also established, contributing to insights on the effects of the Lagrangian multipliers on global solutions. These findings establish a comprehensive understanding of the localizability with range-difference measurements, which are also illustrated with numerical examples.

Keywords: localization; least squares estimation; quadratic function minimization; time difference of arrival (TDoA)

I. INTRODUCTION

Location-based services and protocols [3] can be seen in a large number of applications, ranging from wireless communication [4], [8], [56], internet-of-things [9], transportation [10] to advertising or social networks [11]. It is worthwhile to note that, amid the proliferation of localization-based applications in mobile networks, we have seen constantly surging interests in localization. This trend has been boosting a batch of researches on distributed localization [12], [13], [14], [54], indoor localization [15], [16], [17], simultaneous localization and mapping [18], [19], to name a few. Localization problems focus on acquiring the position of an object in a certain coordinate system based on measurement data from a set of sensors. The methods for localization vary, among which localization using field geometry about objects’ relative placement (also known as range-based localization methods) [20], [21] are practically prevalent since the geometric information of an object can be easily measured by sensors. The localization methods utilizing time difference of arrival (TDoA) [24], [25], [26], [27], [1], [2], [5], [6], [7] are important examples of the lateration approach for passive localization. It performs with high accuracy since it does not rely on time synchronization between the source and receivers.

In this paper, we consider the problem of locating a radiating source from TDoA measurements. In an ideal uniform medium, radiation travels through the medium at a known velocity, and therefore the measurement of range differences from the source is accessible from the TDoA measurement. Source localization from range-difference measurements has been extensively studied by means of least squares methods. In particular, finding the least squares solution with respect to the spherical error criterion [47] can be formulated as an optimization problem with a quadratic objective function and some constraints. The solution structure presented for the solution characterization based on the method of Lagrange multipliers has been an active research topic in the area of localization. A direct method is discarding the quadratic constraints [48], [32], which gives rise to an unconstrained least squares (ULS) problem. An extension of of the ULS method is to explore the constraint to reduce the error of the ULS solution for generating the final estimate [46], [53]. The spherical-interpolation method is a commonly used approach proposed that solves the CLS problem approximately [42], [43], [52], by which optimization is done by alternating restricted minimization over two disjoint subsets of the variables. The subspace minimization method [43], [52] is another way to solve the CLS problem approximately. In this method, orthogonal projection is used to eliminate the challenging quadratic constraint. As a revisit to the methods reviewed above, it is pointed by [42] that the spherical-interpolation method and the subspace minimization method are identical to the unconstrained LS method in the sense that these
methods generate identical solutions. The reference \cite{45} presents a closed-form localization technique, termed the spherical-intersection method, which has similar formulation to the spherical-interpolation one. What is worth noticing is that the spherical-intersection method gives consideration to quadratic constraint but fails in obtaining a global minimizer too as it by its nature is alternating optimization. Some solvers are iterative methods, such as \cite{40,41}. At each step a location estimate is improved by solving a local LS estimation problem using the Taylor-series method. However, proper initialization is needed to avoid false local minima \cite{55}. A different category of solvers for the CLS problem is based on the Lagrange multiplier technique. In \cite{54,55}, numerical global solution searching algorithms are developed in virtue of necessary conditions for the global optimality of the CLS problem, as a consequence of the the analysis on generalized trust region problems offered in \cite{33}. The solution finding amounts to finding roots of a 4th order polynomial in the two-dimensional or a 6th order in the three-dimensional localizations. Such extensions are possible because the CLS problem is closed to a generalized trust region problem in form. To attack the global solution, the methods \cite{54,55} need an exhaustive search for all the suspected, which is numerically less efficient.

There are a few key questions remaining to be answered for this range-difference based localizations: (i) How are the solutions to the CLS problem related to the localization measurements? (ii) When does the CLS problem for range-difference based localizations have a global/unique solution? (iii) How can the global/uniq

ue solution be characterized? In this paper, we establish a series of results attempting to address these questions for range-difference based localization problems utilizing the persistence of excitation on the localization measurements as well as an extended and structured Karush-Kuhn-Tucker (KKT) analysis. The main contributions of our paper are threefold as follows.

(i). We show that in the absence of measurement noises, the persistence of excitation on the localization measurements describes whether or not the coordinate of a radiating source can be exactly recovered from TDoA measurements, which depends on the rank of the Jacobi matrix of the model, and that for a practical CLS problem where measurement noise is taken into account the CLS solutions always exist and under some assumption on the Jacobi matrix of the measurement model the solution set is bounded. In principle, the condition suggests that a larger-sized sensor array inclines to lead to a bounded CLS solution.

(ii). We develop a necessary and sufficient condition for a global CLS solution in terms of a group of KKT conditions. Moreover, we also establish a characterization for the uniqueness of an optimal solution. The uniqueness result is attached to a stricter condition on the curvature of a Lagrangian function.

(iii). The theoretical results are developed in this paper inspire us to consider some special cases, and uncover insightful findings on the position of the Lagrangian multiplier affecting how the global minimizers can be solved in these cases. They pave a way for us to compute a global minimizer in a relative easy manner. In this part, we also draw remarks on the multiplicity nature of CLS solutions.

Simulation examples are carried out to illustrate our theory. The derivation of our results are partly inspired by \cite{33}. In particular, the reference \cite{33} gives characterization of the global minimizer of the generalized trust region problem and its uniqueness in terms of the Lagrange multiplier. However, the additional positivity constraint, see \cite{55}, leads to challenging difficulty in analysis. The global optimal solution has been investigated in \cite{34}, and a sufficient condition and a necessary one are offered separately. The two conditions do not coincide in general and contain conservativeness, which can be seen from a concrete example in that work. In contrast, our results close the gap. In addition, we develop a condition for the solution uniqueness. To the best of our knowledge, this is the first time that a characterization of the uniqueness is given in the literature.

The remainder of the paper is organized as follows. In Section II, we introduce range-difference localization and how it is formulated into a CLS localization problem, as well as the problems of interest in the paper. In Section III, we develop characterizations of solutions to the CLS localization. Specifically, the development mainly includes a feasibility condition for the problem, a characterization of a global solution and a uniqueness characterization. We also give comparison between our results and the main related ones in literature. In Section IV, we establish a few findings on the structural properties of global solutions in some special cases. We also give some numerical examples to illustrate our theory. Finally, some concluding remarks are drawn in Section VI. Most of the proofs of the main results can be found in Appendices.

Notations. We use $\sigma_{\min}(X)$ and $\sigma_{\max}(X)$ to denote the eigenvalues of a square real matrix $X$, which have the smallest and largest magnitude, respectively. For $x,y \in \mathbb{R}$, $x \vee y$ and $x \wedge y$ stand for the maximum and minimum of $x$ and $y$, respectively. For a real-valued function $h(x): \mathbb{R}^n \to \mathbb{R}$, we use $\nabla h$ to denote the gradient of $h$. For a set $A$, we use $\overline{A}$ to denote the closure of $A$. Let $A$ be a metric space, which is a set equipped with a metric $d$. Suppose that $\emptyset \neq B \subset A$ and $x \in A$. The distance of $x$ from $A$ is defined as $d(x; A) = \inf\{d(x,y) : y \in A\}$.

II. Problem Statement

A. Range-difference Based Localization

We consider a radiating source and an array consisting of $m+1$ sensors that collect signals emitting from the source. We denote the coordinate, with respect to an Euclidean coordinate system, of the source by $x \in \mathbb{R}^n$ and denote the coordinate of sensor $i$ by $a_i \in \mathbb{R}^n$. In particular we assume
a0 = 0 for sensor 0, i.e., sensor 0 is set to be at the origin of the coordinate system. We use di to denote the range-difference measurement from sensor i, for i = 1, ..., m, to sensor 0. We adopt the additive measurement error model, in which the measurements of the range differences are modeled as di, ai and x can be given as

\[ d_i = \|a_i - x\| - \|x\| + r_i, \]  

where \( \| \cdot \| \) denotes the \( \ell_2 \) norm of a vector and \( r_i \) captures the measurement noise contained in \( a_i \), which is also called “equation error” [55]. Therefore, a natural localization problem arises, on identifying the unknown source \( x \) from the measurement data \( \{d_i\} \) and the sensor localizations \( \{a_i\} \). We postulate that the additive measurement errors have mean zero and are independent of the range difference observation and the source location, as routinely assumed in the literature, such as [47].

The TDOA techniques, as means of passive localization, has a large number of applications in different positioning systems [56, 57, 58, 59]. For instance, they have been widely used for sound source localization, where the goal is to estimate the coordinates of a sound source using acoustic signals received by an array of microphones. The microphones are mounted at fixed and known positions \( a_i \), with one of them selected as the reference node. The sound source locates at an unknown position \( x \). The TDOA measurement \( d_i \) is actually time delay that the sound takes to get received by the \( i \)th and the reference nodes (microphones), multiplied by the propagation speed of sound in appropriate medium, see Figure 1. Then a corresponding hyperbolic surface can be derived from a TDOA measurement, for which all points on the surface will have the same distance difference from the said pair of nodes.

Figure 1. Illustration of range-difference measurements in an array consisting of three sensors (i.e., \( m = 2 \)). The array is a line one. The solid dot “*” represents the radiating source and the hollow dots “o” represent the sensors.

B. Persistence of Excitation of the Spherical LS Error Model

The spherical LS model built on the distance error from the hypothesized source location to every sensor, a transformed model of (1), has a simpler expression on the localization \( x \) than (1). Consider the distance from the source to the \( i \)th sensor. The measured value from the TDOA measurement model (1) can be computed as

\[ d_{ix} = d_i + \|x\| \]

and the true value is \( \|a_i - x\| \). The spherical LS error function is defined as the difference of the squares of \( d_{ix} \) and \( \|a_i - x\| \):

\[ e_i = \frac{1}{2}d_{ix}^2 - \frac{1}{2}\|a_i - x\|^2. \]

By (1), we obtain the spherical LS model

\[ e_i = d_i \|x\| + a_i^\top x - b_i, \text{ for } i = 1, \ldots, m, \]  

where \( b_i = \frac{1}{2}(\|a_i\|^2 - d_i^2) \). By (1) and (2), we have \( e_i = \|a_i - x\| r_i + \frac{d_i^2}{2} \).

Suppose that the measurement errors were absent (i.e., \( r_i \equiv 0 \)). Then model (1) defines a branch of a hyperbola, of which \( a_0 \) and \( a_0 \) are the two foci and \( d_i \) is the focal distance. For a practical TDoA problem, the true position of the source must exist, which means that the \( m \) equations have at least one intersection. Meanwhile, the spherical LS error \( e_i \) of the spherical LS model (2) turns to be zero. Here one basic concept persistence of excitation of a given set of locations \( \{a_i : i = 0, \ldots, m\} \) at a source location \( x \in \mathbb{R}^n \) lies in, whether or not the exact \( x \) can be uniquely recovered from \( \{a_i : i = 0, \ldots, m\} \) in some sense. When the answer is affirmative, we call the measurement locations \( \{a_i : i = 0, \ldots, m\} \) are persistently exciting (PE) at \( x \). The concept of persistence of excitation is highly related to localizability of network localization systems and rigidity of graphs as they are all about whether the location(s) of node(s) can be uniquely determined using edge information.

Since the global persistence of excitation is difficult to check for nonlinear models (1) and (2), here we instead introduce a little weaker condition (the local persistence of excitation conception) of input signals in system identification [58] and derive a verifiable condition for the measurement locations \( \{a_i : i = 0, \ldots, m\} \) to a source location.

Definition 1: [58] Definition 3] Consider the spherical LS model (2) in the absence of noises, i.e., \( r_i \equiv 0 \). The set of measurement locations \( A := \{a_i : i = 0, \ldots, m\} \) is said to be locally PE with respect to c at a source location \( x \in \mathbb{R}^n \setminus A \) if there holds that \( d_i\|\hat{x}\| + a_i^\top \hat{x} - b_i = 0, i = 1, \ldots, m \) for any \( \hat{x} \) in a sufficiently small neighborhood of \( x \), then \( \hat{x} = x \). This local PE is a prerequisite for developing numerical algorithms to estimate \( x \), otherwise intrinsic ambiguity persists: there are at least two different possible source locations that correspond to measurements \( d_i \) even in the absence of noises. Actually, the local PE issue aims to find the condition on the measurement locations under which the solution to the spherical LS model (2) is unique in a small neighborhood of the true position of the source.

For models (1) and (2), in the absence of measurement noises, the measurement location set \( \{a_i, i = 0, \ldots, m\} \) is locally PE if their Jacobian matrices at the true location are
of column full rank \([53]\). The Jacobian matrix \(J_0(x)\) of \([4]\) is
\[
J_0(x) = - \begin{bmatrix}
\frac{a_1 - x}{||a_1 - x||} & \cdots & \frac{a_n - x}{||a_n - x||}
\end{bmatrix},
\]
while the Jacobian matrix \(J(x)\) of \([2]\) is:
\[
J(x) = \begin{bmatrix}
\frac{w_1}{||x||} & \cdots & \frac{w_m}{||x||}
\end{bmatrix}
\begin{bmatrix}
||a_1 - x|| x^T + (a_1 - x)^T ||x|| \\
\vdots \\
||a_n - x|| x^T + (a_n - x)^T ||x||
\end{bmatrix}
= 1 ||x|| \begin{bmatrix}
||a_1 - x|| x^T + (a_1 - x)^T ||x|| \\
\vdots \\
||a_n - x|| x^T + (a_n - x)^T ||x||
\end{bmatrix}
= - \begin{bmatrix}
||a_1 - x|| \\
\vdots \\
||a_n - x||
\end{bmatrix} J_0(x)
\]
where \(w_j = ||a_j - x|| - ||x||\) for \(j = 1, \ldots, m\) are applied in the second equality. This means that the model \([1]\) is equivalent to the model \([2]\) in the absence of measurement noises when \(x \neq a_i\) for all \(i = 1, \ldots, m\). The measurement locations \(\{a_i, i = 1, \ldots, m\}\) are locally PE if they satisfy special geometric relations, which is given in the following proposition.

**Proposition 1:** Suppose that \(r_i \equiv 0\). The measurement location set \(\{a_i : i = 0, \ldots, m\}\) is locally PE with respect to \([2]\) at a generic source location \(x \in \mathbb{R}^n\) if

1) for \(n = 2\), all of measurement locations \(\{a_i : i = 0, \ldots, m\}\) are not collinear;

2) for \(n = 3\), \(\{a_i : i = 0, \ldots, m\}\) are not coplanar.

**Proof.** By the inverse function theorem \([59]\), \(J(x)\) being nonsingular implies that there exists an open neighborhood \(U\) around \(x\) there is a diffeomorphism mapping \(U\) to \(e(U)\), which further implies local PE. If \(\{a_i : i = 1, \ldots, m\}\) is not locally PE with respect to \([2]\) \(x \in \mathbb{R}^n\), we have that \(J(x)\) (because \(x \in \mathbb{R}^n\) / \(A\) so does \(J_0(x)\)) does not have full column rank, that is, for any \(i_1, \ldots, i_n \in \{1, \ldots, m\}\), there exist real numbers \(\beta_1, \ldots, \beta_n\) that are not all zero, such that
\[
\sum_{i=1}^{n} \beta_i \left( \frac{x}{||x||} + \frac{a_i - x}{||a_i - x||} \right) = 0.
\]
This is further equivalent to the condition that the rank of
\[
\begin{bmatrix}
\frac{x}{||x||} + \frac{a_1 - x}{||a_1 - x||} & \cdots & \frac{x}{||x||} + \frac{a_n - x}{||a_n - x||}
\end{bmatrix}
\]
is strictly less than \(n\), or its determinant is zero. Let \(V\) be the set of \(x\) resulting in vanishing determinants of all so defined matrices associated with any \(n\) location selections from \(\{a_i : i = 1, \ldots, m\}\). Since all of measurement locations \(\{a_i : i = 0, \ldots, m\}\) are not collinear for \(n = 2\) or not coplanar for \(n = 3\), there exists at least one collection, termed \(i_1', \ldots, i_n'\), implying that vectors \(a_{i_1'}, \ldots, a_{i_n'}\) are linearly independent.

Therefore the set of \(x\), leading to that the determinant of
\[
\begin{bmatrix}
\frac{x}{||x||} + \frac{a_1 - x}{||a_1 - x||} & \cdots & \frac{x}{||x||} + \frac{a_n - x}{||a_n - x||}
\end{bmatrix}
\]
vanishes, is proper, which further ensures \(V\) to be proper. Therefore the location set \(\{a_i : i = 0, \ldots, m\}\) is locally PE with respect to \([2]\) at a generic \(x \in \mathbb{R}^n\), which completes the proof.

\[\blacksquare\]

(C) Practical CLS Solution from the Spherical LS Error Model

A fundamental approach to estimate \(x\) is to minimize the summed spherical least squares errors \(e_i\)'s, i.e.,
\[
(\text{LS}) : \min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} (d_i||x|| + a_i^T x - b_i)^2, \quad (5)
\]
The LS problem is nonconvex. Letting \(y = [||x||, x^T]^T\), we transform the LS problem equivalently into the following constrained least squares (CLS) problem
\[
\min_{y \in \mathbb{R}^{n+1}} f(y) \quad (6a)
\]
(CLS): subject to \(g(y) = 0\),
\[
[6b] \quad \parallel y \parallel_1 \geq 0.
\]
In this problem, \(f(y) := ||Ay - b||^2\), where
\[
A = \begin{bmatrix}
d_1 & a_1^T \\
\vdots & \vdots \\
d_m & a_m^T
\end{bmatrix}
\]
\[
b = [b_1, \ldots, b_m]^T, \quad g(y) := y^T D y
\]
with
\[
D = \text{diag}(1, -I_n),
\]
where \(I_n\) denotes an identity matrix of size \(n\) and \(\text{diag}(\cdot, \cdot, \cdot)\) denotes a diagonal matrix with the arguments in brackets on the main diagonal, and the notation \(||x||, i \leq p\), is used to denote the \(i\)th element of \(x\) for any given \(x \in \mathbb{R}^p\). The interpretation of each equation in the CLS problem is as follows: the least square criterion is given by \([6b]\) and \([6c]\) describe the geometric constraints \(||x||^2 = [x_1^2 + \cdots + x_n^2]\) and \(||x|| \geq 0\). An optimal value \(f^*\) of the CLS problem is defined as
\[
f^*_{\text{CLS}} = \inf \{ f(y) : [y]_1 \geq 0, g(y) = 0, y \in \mathbb{R}^{n+1} \}. \quad (9)
\]

D. Problems of Interest

To find the least squares solution with respect to the spherical error criterion \([47]\), it resorts to solving the CLS problem \([6]\), which was firstly studied in this reorganization idea in \([42], [43], [44]\) and was investigated widely in later work, such as \([45], [46], [47], [48], [32], [49], [34]\).

The problem is nonconvex as the Hessian matrix of a quadratic term of one of the constraints is not positive semi-definite. To the best of our knowledge, most of the existing methods solve an approximate of \([6]\) with some
information loss, e.g., one classical way is to discard the two quadratic constraints (6b) and (6c), and investigate the resulting ULS problem [39, 32, 50, 34, 35]. In 35, one sufficient condition and one necessary condition are provided respectively for the CLS solution, but there is no explicit and complete characterization for global CLS solutions.

In this paper, we are concerned about solvability of the CLS problem, the solution characterization and CLS solution uniqueness. We are also interested in exploring the solution structures for some special cases. Answers to these questions will constitute our understanding of CLS localizability and will further facilitate the development of numerical algorithms for the exact CLS solution.

III. CHARACTERIZATION OF CLS LOCALIZABILITY

In this part, we present conditions on the localizability of range-difference based measurements using CLS methods.

A. CLS Solution Existence

First we discuss the existence and boundedness of CLS solutions. In reality we hope that the solution set is bounded so that when some solving algorithm is implemented it would not output arbitrarily large solution. As will be shown in the following, the boundedness of CLS solutions has connections with the Jacobian of the TDoA measurement model (2).

Lemma 1: The CLS localization problem has a solution. Let Assumption 1 hold. A vector \( y \in \mathbb{R}^n \) such that

\[
(A^\top A + \lambda D)v = 0,
\]

for \( v \) satisfies \( v^\top Dv \leq 0 \)

(10b)

(iii). It holds that \( y = 0 \) and there does not exist \( y' \in \mathbb{R}^{n+1} \) and \( \lambda \in \mathbb{R} \) such that \( y' \neq 0 \), \( [y']_1 > 0 \), \( g(y') = 0 \) and (10b) hold.

We revisit the ULS problem, an approximate of the CLS problem, and note that the rank of \( A^\top A \) dictates the number of solutions: when \( A^\top A \) is nonsingular there is a unique solution of the ULS approximation. However, the situation is not the same for the CLS localization itself. The nonsingularity of \( A^\top A \) guarantees existence of a global solution of the CLS localization, but not necessarily uniqueness. Our further result uncovers that uniqueness of CLS solution follows from a stricter condition, see Theorem 2. We use the following example to illustrate this fact.

Example 1: Consider a line-shaped (see Figure 1) sensor array consisting of four sensors (i.e., \( m = 3 \)) monitoring a radiating source in a 2-dimension space, i.e., \( n = 2 \). The coordinates of the sensors are \( a_0 = 0, a_1 = \left[ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right]^\top \), \( a_2 = \left[ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right]^\top \), \( a_3 = \left[ 0, -\frac{2}{\sqrt{6}} \right]^\top \). We let the range-differences measurements be as follows:

\[
d_1 = \frac{1}{\sqrt{3}}, d_2 = \frac{1}{\sqrt{3}}, d_3 = \frac{1}{\sqrt{3}}.
\]

Then we compute

\[
A = \left[ \begin{array}{cc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0
\end{array} \right], \quad b = \left[ \begin{array}{c}
\frac{1}{\sqrt{6}} \\
0
\end{array} \right].
\]

It can be verified that in this example Assumption 1 is satisfied with \( A^\top A = I \). Figure 2 specifies the contour lines to display the value of \( f(y) \) under the constraints (6b) and (6c) on a plane spanned by \([y]_2 \) and \([y]_3 \). As we see in the figure, all points on a ring (the ring is marked with a red arrow) achieve the minimum of \( f \). This indicates that solutions of the CLS problem are not unique.

![Figure 2](image-url)
Since $A^\top A \geq 0$, the inequality $v^\top (A^\top A + \lambda D)v \geq 0$ holds for all $\lambda \leq 0$ satisfying $v^\top Dv \leq 0$. Thus, the set $I$ is not empty. The next result gives more refined descriptions of the set $I$.

Lemma 2: Let Assumption 1 hold. The set $I$ can be written in the form of $I := (-\infty, \lambda_u]$ with a constant $\lambda_u \geq 0$. Moreover, $\lambda_u$ can be computed by solving a linear matrix inequality (LMI) problem as follows:

\[
\begin{align}
\text{maximize} & \quad \lambda \\
\text{subject to} & \quad A^\top A + \lambda D \succeq 0.
\end{align}
\]

Proof. We shall show that $I$ is convex and therefore it is an interval. Since $I$ is not empty, let $\lambda_1 \in I$. For any $\lambda \in (-\infty, \lambda_1)$, we obtain

\[v^\top (A^\top A + \lambda D)v > v^\top (A^\top A + \lambda_1 D)v \geq 0, \quad \forall v^\top Dv < 0.
\]

Hence, $\lambda \in I$. Finally, we conclude $\lambda_u \geq 0$ due to $0 \in I$.

The above lemma, as a preliminary result of the value range of the multipliers $\lambda$’s that satisfy (10b), depicts an interval it may locate in. It is succeeded by a concise edition of the CLS solution characterization as follows:

Corollary 1: Let Assumption 1 hold. A vector $y \in \mathbb{R}^{n+1}$ is a global optimal solution of the CLS Problem if and only if it falls into one of the following two cases:

(i). The vector $y$ satisfies $g(y) = 0$ with $[y]_1 > 0$ and there exists a $\lambda \in I$ such that $(A^\top A + \lambda D)y = A^\top b$.

(ii). It holds that $y = 0$ and there does not exist $y' \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$ such that $y' \neq 0$, $[y']_1 > 0$, $g(y') = 0$ and (10a) hold.

Proof. The proof follows from Theorem 1 and the definition of the set $I$.

B. CLS Localization Uniqueness

In this part, on top of the existence of the CLS solutions, we will present a sufficient and necessary condition for the solutions’ uniqueness. Then we further present a finding that relates the Lagrange multiplier to the multiplicity of the CLS solutions.

The solution characterization in Theorem 1 involves a KKT condition and a second-order condition on the Hessian matrix $A^\top A + \lambda^* D$ of a Lagrangian function. The next result is on uniqueness of the global optimal solutions of problem (6). We will find that uniqueness follows a stricter second-order condition. The proof is reported in Appendix B.

Theorem 2: Let Assumption 1 hold. A vector $y \in \mathbb{R}^{n+1}$ is the unique global optimal solution of the CLS Problem if and only if it falls into one of the following two cases:

(i). The vector $y$ satisfies $g(y) = 0$ with $[y]_1 > 0$ and there exists a $\lambda \in \mathbb{R}$ such that

\[
(\lambda D)^y = A^\top b,
\]

\[
v^\top (A^\top A + \lambda D)v > 0, \quad \forall v \text{ satisfies } v^\top Dv < 0 \text{ and } v^\top Dy \not= 0.
\]

The conditions in Lemma 1 are sufficient for the existence of a bounded global solution of the CLS problem, but not necessary. The conditions in Theorems 1 and 2 are both exact characterizations of some certain properties of the global solutions.

The following example demonstrates the gap between Lemma 1 and Theorems 1 and 2.

Example 2: Let Assumption 1 hold. The CLS problem has bounded global solutions. Observe that $A^\top A + D = \text{diag}(2, 3, 3, 3)$ and $A^\top b = [4\sqrt{2}, 0, 0]^\top$. We then find that $\lambda^* = 1$ and any $y^*$, which satisfies $[y^*]_1 = \sqrt{\frac{3}{2}}$ and $\|y^*\| = \sqrt{\frac{3}{2}}$, meet the conditions (10). By Theorem 1 the problem has global solutions, illustrated by Figure 2. On the other hand, for any nonzero vector $v \in \text{span}\left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right)$, we have $v^\top (A^\top A + \lambda^* D)v = 0$.

The condition (13b) is not met, therefore, by Theorem 2 the problem’s solution is not unique. Figure 2 is evidence of this claim.

We next consider another example of an array consisting of four sensors. Assume that the coordinates of them are $a_0 = a_1 = 0$, $a_2 = [4, 0]^\top$, $a_3 = [0, 4]^\top$, and that the range-difference measurements are $d_1 = 4$, $d_2 = d_3 = 0$.

Then we obtain that

\[A = 4I, \quad b = [-8, 8, 8]^\top.
\]

By Theorem 1 we have $y^* = \frac{2-\sqrt{2}}{2}[\sqrt{2}, 1, 1]^\top$ and $\lambda^* = -16(3 + 2\sqrt{2})$ satisfy (10a) and (10b), and therefore $\frac{2-\sqrt{2}}{2}[\sqrt{2}, 1, 1]^\top$ is a minimizer of the CLS problem. Numerical search suggests that $\frac{2-\sqrt{2}}{2}[\sqrt{2}, 1, 1]^\top$ is the unique minimizer, see Figure 3. Since $\lambda^* = -16(3 + 2\sqrt{2}) < 0$, for any $v^\top Dv < 0$, we have $v^\top (A^\top A + \lambda^* D)v > 0$, which theoretically supports our numerical result.

The following corollary is on the relations between the multiplier $\lambda$ in (10) or (13) and the multiplicity of the CLS solutions. It unveils that whether a CLS problem has a single unique solution is dictated by the position of $\lambda$ in $I$. It directly follows from Corollary 1, Theorem 2 and Lemma 2.

Corollary 2: Let Assumption 1 hold. The CLS problem has multiple global solutions only if there exists a vector
y ∈ ℝⁿ⁺¹ such that \( g(y) = 0 \) with \([y]_1 > 0\) and \((A^TA + λ_uD)y = A^Tb\). On the other hand, a vector \( y \in ℝⁿ⁺¹ \) is a single unique global solution if it falls into one of the following two cases:

(i). The vector \( y \) satisfies \( g(y) = 0 \) with \([y]_1 > 0\) and there exists a \( λ < λ_u \) such that \((A^TA + λD)y = A^Tb\).

(ii). \( y = 0 \) and there exists no \( y' \neq 0 \), where \([y']_1 > 0\), and \( λ' \in \mathcal{I} \) such that \((A^TA + λ'D)y = A^Tb\).

Remark 1: The reference [33] gives characterization of the global minimizer of the generalized trust region problem and its uniqueness in terms of the Lagrange multiplier. In our paper, the existence of global solutions to the CLS localization and solution uniqueness is derived partly based on [33]. On one hand, the problems respectively investigated in the two papers have similar forms. The additional positivity constraint (6c) require more elaborate analysis to derive conditions for CLS solution and its uniqueness. On the other hand, the conditions in our paper are less strict than those of [33] for the generalized trust region problem since adding (6c) into the constraints defines a narrowed feasible set.

C. Literature Comparison and Discussions

In this part, we present comparison between our aforementioned results and the related ones in literature.

First note that a characterization of the global solutions to the CLS problem was presented in [34], giving a sufficient condition and a necessary condition respectively. There is a high degree of similarity between the CLS problem and generalized trust region problems (GTRS) [33], a class of problems minimizing a quadratic function subject to a quadratic equality constraint that has been reviewed by us in above. A characterization of the global solution was obtained in [33]. In the following remarks, we will compare the results of this paper with those of [34] and [33], respectively.

In [34], a sufficient condition, i.e., [34] Lemma 3.1, and a necessary condition, i.e., [34] Theorem 3.1, for solutions of the CLS problem were presented. The gaps in between them two are discussed as follows: (i) the sufficient condition requires \( A^TA + λD \) to have no negative eigenvalues while the necessary condition is relaxed, allowing \( A^TA + λD \) to have at most one negative eigenvalue; (ii) the possibility of \( y_{CLS} = 0 \) is excluded from the sufficient condition while not excluded in the necessary one. These gaps are not trivial in the sense that there may exist cases where a global minimizer \( y_{CLS} \) fails to satisfy the sufficient condition as well as cases where the necessary condition does not lead to a solution. In Example 2 we demonstrate two cases where the gaps exist.

Example 3 (Example 2 Cont’d): In this example, we continue to study the cases in Example 2. First, we consider the same scenarios to the first case of Example 2. We have shown that \( λ^* = 1 \) and any \( y^* \) that satisfies \([y']_1 = \sqrt{\frac{3}{2}}\) and \( \|y^*\| = \sqrt{\frac{3}{2}}\) meet the conditions (10). Then, we have \( A^TA + λ^*D = diag(2, 0, 0) \). The positive semidefiniteness of \( A^TA + λ^*D \) indicates the conservativeness of the necessary condition of [34] Lemma 3.1.

We next consider the second scenario of Example 2. It has been computed in Example 2 that \( y = \frac{2}{\sqrt{3}}[\sqrt{2}, 1, 1]^T \) is an optimal solution in this case (see Figure 3) and \( λ = -16(3 + 2\sqrt{2}) \) satisfies (10a) and (10b). However, simple computation reveals that, having a negative eigenvalue, \( A^TA + λD \) does not hold. It suggests conservativeness of the sufficient condition of [34] Lemma 3.1.

The CLS problem has a close relation with the GTRS problem as they have a similar form. The form of the GTRS problem is as follows [33]:

\[
\text{(GTRS): } \begin{align*}
& \text{minimize } f(y) \\
& y \in ℝⁿ⁺¹ \\
& \text{subject to } g(y) = 0.
\end{align*}
\]

Compared to the CLS problem, the GTRS does not contain the constraint \([y]_1 ≥ 0\). By [33] Theorems 3.2], a vector \( y_{GTRS}^* \) is a global minimizer if and only if \( g(y_{GTRS}) = 0 \) and there exists a \( λ_{GTRS}^* \in ℝ \) such that (10a) and \( A^TA + λ_{GTRS}^*D \) hold. Moreover by [33] Theorems 3.3], in case of \( λ_{GTRS}^* = 1 \), \( y_{GTRS}^* \) is the unique solution.

A comparative study to the characterization of solutions to the GTRS and the CLS problems reveals some close observations. First, Theorems 1 and 2 of our paper and Theorems 3.2 and 4.1 of [33] shared similar KKT conditions because the CLS and the GTRS problems have similar forms and it is allowed to have analytic characterization of the solutions using the same Lagrange function. On the other hand, these problems have different second-order conditions because for the CLS problem the constraint (6c) gets rid of the lower napple of the double cone \( \{y : y^TDy = 0\} \) and this allows negative definiteness of the Hessian matrix of (22) in some subspace.

1 A double cone is a quadratic surface. Each single cone placing apex to apex is called a napple.
IV. STRUCTURED CLS LOCALIZATIONS

The characterization of CLS global minimizers established in Theorem 1 paves the way for us to locate the minimizers by resorting to seeking for feasible λ's. A viable way for doing so is to verify the conditions (i)–(ii) of Theorem 1 in order: first search multipliers λ's satisfying (10b), substituting each λ into (10a), then solve a solution of (10a) and check the feasibility of (10b) and (6c). Besides, we can do better in some special cases, where a global minimizer of the CLS problem can be computed easily. This is done on the basis of the theory developed in the previous section. In what follows, we will proceed to present how it is realized.

To guarantee that the CLS problem exists solutions, we assume that Assumption 1 holds in the rest of this section, and will elaborate how λCLS is positioned in different situations.

A. Technical Preparations

Define the following set

\[ \mathcal{I}_1 = \{ \lambda \in \mathbb{R} : A^T A + \lambda D > 0 \}. \]  

(15)

Obviously \( \mathcal{I}_1 \) is a non-empty bounded interval. This interval can be formally represented by \( [\lambda_l, \lambda_u] \), where \( \lambda_l \in \mathbb{R} \) and is the solution of the following LMI problem:

minimize \( \lambda \in \mathbb{R} \)
subject to \( A^T A + \lambda D \geq 0. \)

By [33] Theorem 5.3], the closure of \( \mathcal{I}_1 \), denoted by \( \overline{\mathcal{I}_1} := [\lambda_l, \lambda_u] \), has the following property:

\[ \overline{\mathcal{I}_1} = \{ \lambda \in \mathbb{R} : A^T A + \lambda D \geq 0 \}, \]

and \( A^T A + \lambda D \geq 0 \) is singular for \( \lambda = \lambda_l \) and \( \lambda_u \). In the presence of the specific form of \( D \) given in (3), we further reach claims that \( \lambda_l < 0 \) and that, as an eigenvalue of \( A^T A + \lambda_l D \), zero has a geometric multiplicity \( M_0 = 1 \). This claim can be proved in the following way. If \( M_0 \) were greater than 1, we consider two distinct subspaces \( \mathcal{U} \) and \( \mathcal{V} \), defined as \( \mathcal{U} := \{ u \in \mathbb{R}^{n+1} : (A^T A + \lambda_l D) u = 0 \} \) and \( \mathcal{V} := \{ v \in \mathbb{R}^{n+1} : (D + I) v = 0 \} \). By calculation, the dimensions of \( \mathcal{U} \) and \( \mathcal{V} \) satisfy \( \dim(\mathcal{U}) = M_0 \) and \( \dim(\mathcal{V}) = n \). By the Grassmann’s formula [51] 2.18 Corollary],

\[ \dim(\mathcal{U}) + \dim(\mathcal{V}) = \dim(\mathcal{U} \cup \mathcal{V}) + \dim(\mathcal{U} \cap \mathcal{V}). \]  

(16)

Since \( \dim(\mathcal{U} \cup \mathcal{V}) \leq n+1 \), we have that \( \dim(\mathcal{U} \cap \mathcal{V}) \geq M_0 - 1 \geq 1 \). However, for any \( v \in \mathcal{V} \setminus \{0\} \), \( v^T (A^T A + \lambda_l D) v = v^T A^T A v - \lambda_l ||v||^2 > 0 \). Hence \( v \notin \mathcal{U} \) and \( \mathcal{U} \cap \mathcal{V} = \{0\} \), which reaches a contradiction.

Letting \( \mathcal{I}_2 := (\infty, \lambda_l) \), we have that \( \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \) and that, for any \( \lambda \in \mathcal{I}_2 \), the matrix \( A^T A + \lambda D \) is nonsingular, or to be more specific, it has exactly one negative eigenvalue and \( n \) positive eigenvalues. We call \( \mathcal{I}_1 \) the positive-definite interval (PDI) for an optimal Lagrangian multiplier and call \( \mathcal{I}_2 \) the indefinite interval (IDI), and call \( \lambda_l \) and \( \lambda_u \) the left singular point (LSP) and right singular point (RSP), respectively. We use \( y(\lambda) \) to denote a solution to (10b), where the argument \( \lambda \) in \( y(\lambda) \) reminds that it is a solution under a given \( \lambda \in \mathcal{I}_1 \cup \mathcal{I}_2 \), and denote

\[ h(\lambda) := g(y(\lambda)). \]  

(17)

We will show that the function \( h(\lambda) \) exhibits some nice properties and develop analysis in some special cases on the ground of them. In what follows, we will present these cases one by one. Let us first recall notions that have been defined and will be used frequently in the sequel.

**Notations.** The functions \( f, g \) and \( h \) are denoted in (6) and (17), respectively. The parameter \( b \) is defined in (2).

The sets \( \mathcal{I} \) and \( \mathcal{I}_1 \) are defined in (11) and (15). We have knowledge of \( \mathcal{I}_1 \) that it is an open and bounded interval. The set \( \overline{\mathcal{I}_1} \) is the closure of \( \mathcal{I}_1 \), and \( \lambda_l \) and \( \lambda_u \) are the left and right end points of \( \overline{\mathcal{I}_1} \), respectively. The set \( \mathcal{I}_2 = \mathcal{I} \setminus \overline{\mathcal{I}_1} \). The variables \( y_{GTRS} \in \mathbb{R}^{n+1} \) and \( \lambda_{GTRS} \in \mathbb{R}^n \) relate to the GTRS problem (14). They satisfy \( g(y_{GTRS}) = 0 \) and (10b) and \( A^T A + \lambda_{GTRS} D \geq 0 \) in this situation \( y_{GTRS} \) is a global minimizer of the GTRS problem. Similarly \( \lambda_{CLS} \in \mathbb{R}^{n+1} \) and \( \lambda_{CLS} \in \mathbb{R}^n \) are two variables that meet the conditions in Theorem 1.

**Table I**

| Notation | Description |
|----------|-------------|
| \( \mathcal{I} \) | interval of the Lagrangian multiplier defined in (11) |
| \( \mathcal{I}_1 \) | positive-definite interval (PDI) defined in (15) |
| \( \mathcal{I}_2 \) | indefinite interval (IDI), \( \mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1 \) |
| \( \lambda_l, \lambda_u \) | left singular point (LSP)/right singular point (RSP) of \( \mathcal{I}_1 \) |
| \( y_{GTRS}, \lambda_{GTRS} \) | global minimizer of the GTRS problem (14) |
| \( y_{CLS}, \lambda_{CLS} \) | variables that satisfy the conditions in Theorem 1 |

B. The Case of Collinear Source Position \( (b = 0) \)

We consider the case of \( b = 0 \). This case is special in the sense that it allows us to exactly obtain the solution immediately. When \( b = 0 \), one can, in a straightforward manner, derive the following result (The proof is skipped because of its simplicity):

**Lemma 3:** If \( b = 0 \), we have that \( h(\lambda) = 0 \) for \( \lambda \in \mathcal{I}_1 \cup \mathcal{I}_2 \).

It can be readily validated that \( y = 0 \) and any \( \lambda \in \mathcal{I}_1 \cup \mathcal{I}_2 \) satisfies condition (i) of Theorem 2. Hence, in this case we conclude that the origin is the unique solution to the CLS problem. Other facts that we would like to point out are that the uniqueness of \( \lambda_{CLS} \) is not needed for the uniqueness of \( y_{CLS} \) and that when \( b = 0 \) the origin is also the unique minimizer of \( f(y) \).

The position of the CLS solution for \( b = 0 \) can also be inferred geometrically. By (2) we have \( ||a_i|| = d_i \) for all sensor i. The geometry in Figure 1 alludes that the best estimate of the radiating source’s position is collinear with \( a_0 \) and any \( a_i \). To make sure that Assumption 1 holds, \( a_0 \)
and $a_1, \ldots, a_m$ cannot be in a line. As a consequence, the origin must be the unique best position estimate.

Next we turn to consider the case of $b \neq 0$, which needs to be treated by more refined analysis.

C. Searching CLS solution over PDI

In case of $b \neq 0$, $h(\lambda)$ is strictly monotonic on $I_1$. The property was originally derived in [33, Theorem 5.2] and is rephrased as follows:

Lemma 4: [33, Theorem 5.2] Assume that $b \neq 0$. The function $h(\lambda)$ is strictly decreasing on $I_1$.

We revisit the GTRS problem [14], Theorem 3.2 of [33] reads that a vector $y^*_{GTRS} \in \mathbb{R}^{n+1}$ is a global minimizer of the GTRS problem if and only if $g(y^*_{GTRS}) = 0$ and there exists an $\lambda^*_{GTRS} \in I_1$ such that (10a) holds. It can be further concluded that, if in addition $[y^*_{GTRS}] > 0$ holds, $y^*_{GTRS}$ is also a global minimizer of the CLS problem.

Based on the above conclusions, we can first search $\lambda^*_{GTRS}$ over $I_1$, if any, and then compute $y^*_{GTRS}$. Lemma 4 reveals that $h(\lambda)$ is monotonically decreasing on $I_1$. If $A^T A + \lambda^*_{GTRS} D > 0$, a bisection algorithm can be applied for locating $\lambda^*_{GTRS}$ [34] by evaluating the sign of $h(\lambda)$. Then $y^*_{GTRS} = g(\lambda^*_{GTRS}) = (A^T A + \lambda^*_{GTRS} D)^{-1} A^T b$ can be computed naturally. By examining whether $[y^*_{GTRS}]$ is positive or not, one can determine whether $y^*_{GTRS}$ is a global minimizer to the CLS problem. To be precise, $y^*_{GTRS}$ is a global solution to (9) in the presence of $[y^*_{GTRS}] > 0$. For the case of $A^T A + \lambda^*_{GTRS} D$ being positive semidefinite and singular, one needs other analysis for locating $\lambda^*_{GTRS}$, which will be treated in the following subsection.

D. Searching CLS solution on Singular Points

If $A^T A + \lambda^*_{GTRS} D$ is positive semidefinite and singular, the sign of $h(\lambda)$ does not change on $I_1$. In this situation, by [33, Theorem 5.4], the following claims can be established:

(i). The value of $\lambda^*_{GTRS}$ is pushed towards an endpoint, either $\lambda_1$ or $\lambda_n$, of $I_1$. More precisely, $\lambda^*_{GTRS} = \lambda_1$ in case of $h(\lambda) > 0$ on $I_1$ and $\lambda^*_{GTRS} = \lambda_1$ otherwise.

(ii). The limit $\lim_{\lambda \rightarrow \lambda^*_{GTRS}} h(\lambda)$ exists and $\lim_{\lambda \rightarrow \lambda^*_{GTRS}} g(\lambda) = y_1$ for some $y_1 \in \mathbb{R}^{n+1}$.

It is pointed out by the reference [33] that $y_1$ is not necessarily a solution to (15). In general, it requires a subtle treatment to compute a $y^*_{GTRS}$. Besides what has been concluded in [33], in the presence of the specific form of the matrix $D$, see (8), we can get access to better properties related to $h(\lambda)$ and $y^*_{GTRS}$, which make it possible to determine the sign of $[y^*_{GTRS}]$ even prior to solving the exact value. By knowing the sign of $[y^*_{GTRS}]$, we can determine whether $y^*_{GTRS}$ is also a CLS solution. The whole computation procedure will be detailed for the rest of this subsection.

First we focus on how the sign of $h(\lambda)$ affects that of $[y^*_{GTRS}]$. The analysis is divided into two parts on the ground of either $h(\lambda) > 0$ or $h(\lambda) < 0$.

1) The case $h(\lambda) < 0$ for $\lambda \in I_1$: In this case, $\lambda^*_{GTRS} = \lambda_1$. We choose $z^{-}$ with $\|z^{-}\| = 1$ such that $(A^T A + \lambda_1 D)z^{-} = 0$.

Since 0 is an eigenvalue of $A^T A + \lambda_1 D$ with geometric multiplicity 1, $z^{-}$ is unique up to sign. Then $y^*_{GTRS} = y_1 + \alpha z^{-}$ for some $\alpha \in \mathbb{R}$ (Recall that $\lim_{\lambda \rightarrow \lambda^*_{GTRS}} g(\lambda) = y_{1}$). Here $\alpha$ is chosen as either of the roots of the following quadratic equation (see [33, Section 5]):

$$g(z^{-}) \alpha^2 + 2y_{1}^T Dz^{-} \alpha + g(y_{1}) = 0.$$  \hspace{1cm} (18)

The roots have the following property. The proof can be found in Appendix C.

Lemma 5: The quadratic equation (18) has two distinct roots.

To distinguish the two roots of (18), without loss of generality, we denote them by $\alpha_1^-$ and $\alpha_2^-$, where $\alpha_1^- > \alpha_2^-$, and have the following claim.

Lemma 6: One of $[y_1 + \alpha_1^- z^{-}]$ and $[y_1 + \alpha_2^- z^{-}]$ is negative and the other is positive, i.e., $[y_1 + \alpha_1^- z^{-}] [y_1 + \alpha_2^- z^{-}] < 0$.

The proof is given in Appendix C.

To summarize, when $h(\lambda) < 0$ for $\lambda \in I_1$, we have $\lambda^*_{CLS} = \lambda_1$ and can solve the CLS problem by resorting to Lemma 6. We obtain $y_1$ through $\lim_{\lambda \rightarrow \lambda_1} g(\lambda) = y_1$ and choose a unit eigenvector $z^{-}$ of $A^T A + \lambda_1 D$ with respect to eigenvalue 0. Then we compute $\alpha_1^-$ and $\alpha_2^-$ by solving (18). Lemma 6 guarantees that either $y_1 + \alpha_1^- z^{-}$ or $y_1 + \alpha_2^- z^{-}$ is a global minimizer to (6).

Example 4: Consider a sensor array consisting of five sensors (i.e., $m = 4$) monitoring a radiating source in a 3-dimension space, i.e., $n = 3$. The coordinates of the sensors are $a_0 = 0$, $a_1 = [-1,0,0]^T$, $a_2 = [1,-1,0]^T$, $a_3 = [1,1,1]^T$ and $a_4 = [0,1,0]^T$. We set the range-differences measurements to be:

$$d_1 = -1, d_2 = 1, d_3 = 0, d_4 = -1.$$  \hspace{1cm}

Then we compute

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}.$$  \hspace{1cm}

It can be verified that $A^T A$ is nonsingular so the problem has solutions by Lemma 6. We obtain $I_1 = (-1,0.2679)$ using cvx in Matlab. The Matlab simulation reads that $h(\lambda) < 0$ and it is strictly decreasing for $\lambda \in I_1$. By the result in this part, we have $\lambda^*_{GTRS} = \lambda_1 = -1$ and $z^{-} = \frac{1}{\sqrt{2}} [2, -1, 0]^T$. Figure 4 plots the values of $y(\lambda)$ for $\lambda \in (-1,1]$ and reads that $y(\lambda)$ converges to $y_1 \approx [-0.25, 0.5, 0.5]^T$ when $\lambda$ approaches $-1$. By solving (18), we obtain $\alpha_1^- = 2\sqrt{21} - \sqrt{6}$.
and $\alpha_2 = -2\sqrt{2} - \sqrt{6}$. Then we find that $y_\ast + \alpha_1^+ z^- = [2\sqrt{14} - 2, -\sqrt{14} + 4, \sqrt{14} - 4]^\top$ and $y_\ast + \alpha_2^- z^+ = [-2\sqrt{14} - 2, \sqrt{14} + 4, -\sqrt{14} - 4]^\top$, which verifies Lemma 4. Finally, we conclude that $y_\ast + \alpha_1^+ z^-$ is a solution.

2) The case of $h(\lambda) > 0$ for $\lambda \in \mathcal{I}_1$: In this case, $\lambda_{\text{GTRS}} = \lambda_u$. Similarly, we choose a vector $z^+$ with $\|z^+\| = 1$ such that $(A^\top A + \lambda u D)z^+ = 0$. Then $y_{\text{GTRS}} = y_\ast + \alpha z^+$, where $\alpha$ is a solution of the quadratic equation:

$$g(z^+)\alpha^2 + 2y_\ast^\top Dz^+ \alpha + g(y_\ast) = 0.$$ (19)

Without loss of generality, we denote the roots (which is not necessarily distinct) by $\alpha_1^+$ and $\alpha_2^+$, where $\alpha_1^+ \geq \alpha_2^+$. The following result holds.

**Lemma 7:** The values of $[y_\ast + \alpha_1^+ z^+]_1$ and $[y_\ast + \alpha_2^+ z^+]_1$ are simultaneously negative or positive, i.e., $[y_\ast + \alpha_1^+ z^+]_1$ and $[y_\ast + \alpha_2^+ z^+]_1 > 0$. Moreover, it is true that $[y(\lambda)]_1$, $[y_\ast + \alpha_1^+ z^+]_1$, and $[y_\ast + \alpha_2^+ z^+]_1$ have the same sign for any $\lambda \in \mathcal{I}_1$.

The proof is given in Appendix C.

To summarize, when $h(\lambda) > 0$ for $\lambda \in \mathcal{I}_1$, $\lambda_{\text{GTRS}} = \lambda_u$. We compute $y_\ast$ through $\lim_{\lambda \to \lambda_u} y(\lambda) = y_\ast$ and choose a unit eigenvector $z^+$ of $A^\top A + \lambda u D$ with respect to the eigenvalue 0. Then $y_{\text{GTRS}} = y_\ast + \alpha_2^+ z^+$ or $y_\ast + \alpha_1^+ z^+$. To decide whether $y_{\text{GTRS}}$ is a global minimizer of the CLS problem, we need to identify the sign of $[y_{\text{GTRS}}]_1$. Lemma 7 allows us to do this by simply examining the sign of $[y(\lambda)]_1$ for $\lambda \in \mathcal{I}_1$. When $0 \in \mathcal{I}_1$, a simple means is to check the sign of $[y(0)]_1 = [(A^\top A)^{-1} A b]^\top$.

**Example 5 (Example 7 Cont’d):** We continue to consider Example 1 In this example, $\mathcal{I}_1 = (-1, 1)$. Our Matlab simulation reads that $h(\lambda) > 0$ is strictly decreasing for $\lambda \in \mathcal{I}_1$, see Figure 5. By the result in this part, we have $\lambda_{\text{GTRS}} = \lambda_u = 1$, which is consistent with the result in Example 4. Moreover, Figure 5 plots the values of $y(\lambda)$ for $\lambda \in [0, 1)$ and reads that $y(\lambda)$ converges to $y_\ast \approx [0.145, 0, 0]^\top$ when $\lambda$ approaches 1, and that $[y_\ast]_1$ and $[y(\lambda)]_1$’s are all positive numbers.

Figure 5. In Example 5, the Matlab simulation reads that $h(\lambda) > 0$ is strictly decreasing for $\lambda \in [0, 1)$ and $y(\lambda)$ converges to $y_\ast \approx [0.145, 0, 0]^\top$. In addition, it can be observed that $[y_\ast]_1$ and $[y(\lambda)]_1$’s are all positive numbers.

V. NUMERICAL SIMULATION

We search optimal solutions of the CLS problems by utilizing the results we develop in the paper. The methods compared with ours are as follows: (a) ULS: unconstrained least squares solution [50]; (b) SDP-I: inner-product semidefinite relaxation algorithm [52]; (c) CFS: the classical TDOA closed-form solution by Chan and Ho [46]; (d) SUM-MPR: modified polar representation solved by the successive unconstrained minimization algorithm [55].

**Example 6.** Let the reference sensor sit at the origin and the other four sensors locate in

\[
\begin{align*}
    a_1 &= [-1 1]^\top, & a_2 &= [-1 4]^\top, \\
    a_3 &= [-4 6]^\top, & a_4 &= [-6 7]^\top.
\end{align*}
\]

The coordinates of the source is set as $[-5 2]^\top$. The range-difference measurements $d_i, i = 1, \ldots, 4$ are corrupted by i.i.d. Gaussian noises $r_i \sim \mathcal{N}(0, \sigma^2)$. For each given $\sigma$, we run $N = 1000$ times and calculate the average squared error $\frac{1}{N} \sum_{i=1}^N \|\hat{x}^+(\omega_i) - x^+\|^2$ where $x^+$ is the true coordinates of the source and $\hat{x}^+(\omega_i)$ is our coordinate estimate in the $i$-th experiment. In the subsequence, we treat the average squared
error as an approximate of the mean squared error (MSE), i.e.,
\[
E \left[ \| \hat{x} - x^* \|^2 \right] \approx \frac{1}{N} \sum_{i=1}^{N} \| \hat{x}(\omega_i) - x^* \|^2.
\]
(20)

The performance comparison among our algorithm and the other methods is depicted in Fig. [6] where we take X-axis as $10 \log(1/\sigma^2)$ and Y-axis as the root mean squared error (RMSE). We see from the figure that when the intensity of the noises is low, our algorithm performs comparable with the CFS and SUM-MPR algorithms, which are superior to the ULS and SDP-I methods. With the increase of noise intensity, our algorithm has the lowest RMSE.

Example 7. In this example, we leave the other settings unchanged but move four sensors far away from the origin by subtracting $[100 \ 100]^\top$ from the coordinates in Example 1. The results are plotted in Fig. [7]. In this setup, our estimator outperforms the other estimators by several orders of magnitude. This may be because the distances from the other four sensors to the origin are much larger than those between one another, making the matrix $A$ ill-conditioned. As such, the ULS solution $(A^\top A)^{-1}A^\top b$ will be far away from the feasible set $\{ y \mid y^\top D_y = 0 \}$ and has a poor performance. For the SDP-I and CFS algorithms, noticing that their performances in both examples are nearly the same as that of the ULS solution.

VI. CONCLUSIONS

In this work, we were concerned about the problem of radiating source localization from range-difference measurements. We placed the attention to a spherical least squares approach that squares the range-difference measurements. By utilizing this model, one can formulate a CLS range-difference based localization problem, which is a nonconvex optimization problem. Our first result suggested that the resulting localization problem have bounded global solutions under some rank condition. A necessary and sufficient condition for a global CLS solution was derived by means of the Lagrange multiplier technique. The uniqueness of a global solution can be equivalently checked using a stricter second-order condition. Consequently, by examining these two conditions, we have complete knowledge of the multiplicity nature of CLS solutions. Finally, we studied the structural properties of global CLS solutions for some special cases. Our results contribute to finding out the locations of the global solutions in a convenient manners. Numerical algorithms for computing the CLS solutions by means of our research findings worth future research. In addition, analysis on asymptotic properties of the statistical estimator generated from the CLS solution will be an interesting problem.

APPENDIX A. PROOF OF LEMMA 1

For any vector $y$ such that $y^\top A^\top A y = 0$, where $A$ is given in (7), it holds that $f(y)$ is constant. In addition, since $A^\top A \succeq 0$, for any unbounded sequence $(y_k)_{k \in \mathbb{N}}$, \(\liminf_{k \to \infty} f(y_k)\) is unbounded or constant, therefore the CLS problem has at least a global solution in $\mathbb{R}^{n+1}$.

To show the rest, consider nonzero vectors in a form of $y = [\|x\|, x^\top]^\top$ for some $x$ and let the true position of the source be $x^*$. When $y^\top A^\top A y > 0$ for all nonzero $y$, for any unbounded such sequence $(y_k)_{k \in \mathbb{N}}$, \(\liminf_{k \to \infty} f(y_k)\) is unbounded, which shows the boundedness of the solution set. To evaluate whether $y^\top A^\top A y > 0$ is zero or not for all nonzero $y$, we only need to consider unit vectors, i.e., vectors with $\|x\| = 1$. As such, in the rest, we let $\|x\| = 1$ in $y$ without loss of generality. Let
\[
\Psi := \begin{bmatrix} \|a_1 - x^*\| - \|x^*\| \\ \vdots \\ \|a_m - x^*\| - \|x^*\| \end{bmatrix}, \quad \Sigma := \begin{bmatrix} a_1^\top x \\ \vdots \\ a_m^\top x \end{bmatrix},
\]
and $r := [r_1, \ldots, r_m]^\top$. Notice that $A y = \Psi + \Sigma x + r$. Therefore, if $r \in \mathcal{O} := \{-\Sigma x : \|x\| = 1\}$, there exists some nonzero $y$ satisfying $y^\top A^\top A y = 0$. The Lebesgue measure of $\Omega$ is zero, the solution set is bounded almost
surely for any set of \{d_i, i = 1, \ldots, m\}.

APPENDIX B. PROOFS OF THEOREMS 1 AND 2

B.1. Proof of Theorem 7

We define the Lagrangian function
\[ L(y, \lambda) = f(y) + \lambda g(y). \]  
(21)

Notice that \( \nabla_y L(y, \lambda) \) is well defined. Next we divide the proof into two steps.

Necessity. Suppose \( y^* \) is a global optimal solution of the optimization problem (6). We consider two cases.

When \( y^* \neq 0 \), the conditions \( g(y^*) = 0 \) and \( [y^*]_1 > 0 \) are readily met, and linear independence condition qualification is also satisfied. The KKT optimality condition ensures that there exists a multiplier \( \lambda^* \) such that \( \nabla_y L(y^*, \lambda^*) = 0 \), the condition (10a) follows. It further leads to the following result: for any \( y \) with \([y]_1 \geq 0\),
\[ L(y, \lambda^*) = L(y^*, \lambda^*) + (y - y^*)^T (A^TA + \lambda^* D)(y - y^*). \]  
(22)

Since \( L(y, \lambda^*) = f(y) \) whenever \( g(y) = 0 \), we have that
\[ v^T (A^TA + \lambda^* D)v \geq 0 \]  
holds for \( v \in \mathcal{T} := \{v : g(v^*) + v = 0, [v]_1 \geq -[y^*]_1\} \). We will use this fact to show (10b). To this end, denote \( \mathcal{S} := \{v : v^T Dv \leq 0\} \) and partition \( \mathcal{S} \) into a few disjoint subsets, where
\[ \mathcal{S}_1 = \{v : v^T D y^* > 0, v^T Dv < 0\}, \]  
\[ \mathcal{S}_2 = \{v : v^T D y^* < 0, v^T Dv < 0\}, \]  
\[ \mathcal{S}_3 = \{v : v^T D y^* = 0, v^T Dv = 0\} \]  
(24) (25) (26)
and \( \mathcal{S}_4 = \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3) \).

For any \( v \in \mathcal{S}_1 \), there always exists a constant \( \psi > 0 \) such that \( g(y^* + \psi v) = g(y^*) + 2\psi v^T D y^* + \psi^2 v^T Dv = g(y^*) \). Next we will show \( [\psi v]_1 \geq -[y^*]_1 \). Suppose that
\[ [\psi v]_1 + [v]_1 \leq 0 \]  
are true. Then \(([y^*]_1 + [\psi v]_1)[y^*]_1 \leq 0\), by which we further have
\[ v^T D y^* = \frac{1}{\psi} (y^* + \psi v)^T D y^* = \frac{1}{\psi} ([y^*]_1 + [\psi v]_1)[y^*]_1 = \frac{1}{\psi} [y^* + \psi v]|_{2n+1}^T [y^*]|_{2n+1} \leq \frac{1}{\psi} ([y^*]_1 + [\psi v]_1)[y^*]_1 + \frac{1}{\psi} \| [y^* + \psi v]|_{2n+1} \| \| [y^*]|_{2n+1} \| \]  
(27)

The above inequality contradicts \( v \in \mathcal{S}_1 \), Therefore, it is clear that \( \psi v \in \mathcal{T} \). Then by (23), we conclude that
\[ v^T (A^TA + \lambda^* D)v \geq 0 \]  
for any \( v \in \mathcal{S}_1 \).

For any \( v \in \mathcal{S}_2 \), we obtain that \(-v \in \mathcal{S}_1 \). Therefore, \( v^T (A^TA + \lambda^* D)v \geq 0 \) holds for any \( v \in \mathcal{S}_2 \). Similarly, for any \( v \in \mathcal{S}_3 \), we have that \( g(y^* + \psi v) = 0 \) for any \( \phi \in \mathbb{R} \) and \([\psi v]_1 + [y^*]_1 \geq 0 \) when \( \phi \) is sufficiently small. It implies that \( v^T (A^TA + \lambda^* D)v \geq 0 \) also holds for \( v \in \mathcal{S}_4 \). So far we have shown (10b) for the case of \( y^* \neq 0 \).

When \( y^* = 0 \), we will prove the result by contradiction. Suppose that there exists at least a \( \tilde{y} \neq 0 \) such that \( g(\tilde{y}) = 0 \), \([\tilde{y}]_1 > 0 \), and there corresponds a \( \lambda^* \) satisfying (10). Then by (22) we have
\[ f(0) = L(0, \lambda^*) = L(\tilde{y}, \lambda^*) + \tilde{y}^T (A^TA + \lambda^* D)\tilde{y} = f(\tilde{y}) + \tilde{y}^T (A^TA + \lambda^* D)\tilde{y}. \]  
Since \( \tilde{y}^T D\tilde{y} = 0 \) and by Assumption 1
\[ \tilde{y}^T (A^TA + \lambda D)\tilde{y} > 0. \]
It thus follows that \( f(0) > f(\tilde{y}) \), which contradicts the hypothesis.

 Sufficiency. Suppose that \( y^* \in \mathbb{R}^{n+1} \) and \( \lambda^* \in \mathbb{R} \) is a vector and a multiplier, respectively, which satisfy (i). Then we have \( \nabla_y L(y^*, \lambda^*) = 0 \), which further yields the result of (22). Let \( y \neq y^* \) be any given vector satisfying \( g(y) = 0 \) and \([y]_1 \geq 0\). Then \( y \) and \( y^* \) have the following geometric relation
\[ \nabla g(y^*)(y - y^*) = 2y^T D(y - y^*) = 2y^T D y = 2[y]_1 y_1 - 2([y]|_{2n}) y|_{2n} \geq 2[y]_1 y_1 - 2\|[y]|_{2n} ||[y]|_{2n} || = 0. \]  
This also implies
\[ (y - y^*)^T \nabla^2 g(y^*)(y - y^*) = -2y^T D y \leq 0. \]  
(28)
In other words, by (23), we can find a vector \( v = y - y^* \) satisfying \( \nabla^2 g(y^*) v \leq 0 \), which together with (22) implies that
\[ L(y, \lambda^*) = L(y^*, \lambda^*) + v^T (A^TA + \lambda^* D)v \geq L(y^*, \lambda^*), \]  
where the inequality follows from (10). Observing that \( g(y) = g(y^*) = 0 \), then the above inequality yields that
\[ f(y^*) \leq f(y), \]  
which shows that \( y^* \) is a global optimal solution of problem (6).

Next we will show that (ii) is a sufficient condition for the optimality of \( y \). Suppose that \( y^* = 0 \) is not an optimal solution to problem (6). By Lemma 1 we can find at least a vector \( \tilde{y} \neq 0 \) that is a global minimizer of (5). By the argument used for showing the necessity, we conclude that the conditions in (10) hold for \( \tilde{y} \), which contradicts (ii).
B.2. Proof of Theorem 2

The proof resembles that of Theorem 1. Here we only provide a sketch.

**Sufficiency.** We first suppose that \( y^* \in \mathbb{R}^{n+1} \) and \( \lambda^* \in \mathbb{R} \) is a vector and a multiplier, respectively, that satisfy (i). Then we have \( \nabla_y L(y^*, \lambda^*) = 0 \), further yielding that, for any \( y \) with \( \|y\|_1 \geq 0 \), the equality (22) holds. For any given vector \( y \neq y^* \) satisfying \( g(y) = 0 \) and \( \|y\|_1 \geq 0 \), by (28), we have \((y-y^*)^T \nabla g(y^*)(y-y^*) < 0\). When \((y-y^*)^T \nabla g(y^*)(y-y^*) < 0\), we have

\[
(y-y^*)^T Dy^* = y^T Dy^* > [y^*]_1 - \|[y^*]_2n ||[y]_2n || = 0.
\]

By (22) and (13b),

\[
f(y) = L(y, \lambda^*) = L(y^*, \lambda^*) + (y-y^*)^T (A^T A + \lambda^* D)(y-y^*) > L(y^*, \lambda^*) = f(y^*);
\]

and when \((y-y^*)^T \nabla g(y^*)(y-y^*) = 0\), due to Assumption 1,

\[
f(y) = L(y, \lambda^*) = L(y^*, \lambda^*) + (y-y^*)^T (A^T A + \lambda^* D)(y-y^*) = L(y^*, \lambda^*) + (y-y^*)^T A^T A(y-y^*) > L(y^*, \lambda^*) = f(y^*).
\]

The above altogether imply that \( y^* \) is the unique global optimal solution of (5). For (ii), notice that \( y = 0 \) is a minimizer in this situation and that (10) is a necessary condition for a \( y \neq 0 \) being an solution to problem (5), both by Theorem 1. They altogether show that the origin is the unique solution.

**Necessity.** First consider \( y^* \neq 0 \). Since \( y^* \) is the unique optimal solution, similar to (22), we have that \( v^T (A^T A + \lambda^* D)v \geq 0 \) holds for \( v \in \mathbb{R} \setminus \{0\} \). Since \( \{v : v^T Dv < 0, v^T Dy^* \neq 0\} = S_1 \cup S_2 \), where \( S_1 \) and \( S_2 \) are given in (22) and (25), following the proof of Theorem 1, we eventually obtain (i). For \( y^* = 0 \), the proof directly follows from that of Theorem 1.

The above derivations altogether conclude the claim. ■

APPENDIX C. PROOFS OF LEMMAS IN SECTION IV

This appendix contains proofs of some lemmas in Section IV.

C.1. Proof of Lemma 5

When \( h(\lambda) < 0 \) for \( \lambda \in \mathcal{I}_1 \), it has been shown in (23) that \( g(z^-) > 0 \). Since \( g(y(\lambda)) = h(\lambda) \), by continuity we have

\[
g(y_*) \leq 0.
\]

We consider the following two cases, respectively.

Suppose that \( g(y_*) < 0 \). It is straightforward to see that (18) has two different roots: one is positive and the other is negative.

Suppose that \( g(y_*) = 0 \). First we shall show that \( y^* Dz^- \neq 0 \). Computation suggests that

\[
|y^* Dz^-| = |[y^*]_1 [z^-]_1 - [y^*]_2 [z^-]_2+n_1| \\
\geq |[y^*]_1 [z^-]_1| - |[y^*]_2+n_1 [z^-]_2+n_1| \\
\geq |[y^*]_1 [z^-]_1| - |[y^*]_2+n_1| |[z^-]_2+n_1|
\]

Observe that \( |[y^*]_1| = |[y^*]_2+n_1| \) due to \( g(y_*) = 0 \) and \( [z^-]_1 > |[z^-]_2+n_1| \) due to \( g(z^-) > 0 \). Combining these facts, we obtain that \( y^* Dz^- > 0 \), implying \( y^* Dz^- \neq 0 \).

Since \( g(y_*) = 0 \) and \( g(z^-) > 0 \), we finally conclude that (18) has a zero root and a nonzero root.

We conclude the result by completing the analysis for the above two cases. ■

C.2. Proof of Lemma 6

The following technical lemma is functional to the proof.

**Lemma 8:** Let \( y, \bar{y} \in \mathbb{R}^{n+1} \) be two vectors satisfying \( g(y) = g(\bar{y}) = 0 \). Then the following statement are true:

(i). If \( [y]_1 [\bar{y}]_1 \geq 0 \), it holds that \( g(\psi y + (1 - \psi) \bar{y}) \geq 0 \) for any \( \psi \in [0, 1] \).

(ii). If \( [y]_1 [\bar{y}]_1 \leq 0 \), it holds that \( g(\psi y + (1 - \psi) \bar{y}) \leq 0 \) for any \( \psi \in [0, 1] \).

**Proof.** First we prove (i). We begin with an observation that

\[
g(\psi y + (1 - \psi) \bar{y}) = \psi^2 g(y) + (1 - \psi)^2 g(\bar{y}) + 2\psi(1 - \psi) y^T D \bar{y}
\]

\[
= 2\psi(1 - \psi) |[y]_1 | [\bar{y}]_1 - 2\psi(1 - \psi) |[y]_2+n_1 | [\bar{y}]_2+n_1
\]

\[
\geq 2\psi(1 - \psi) [y]_1 [\bar{y}]_1 - 2\psi(1 - \psi) |[y]_2+n_1 ||[\bar{y}]_2+n_1|
\]

where the last inequality follows from the Cauchy-Schwarz inequality (22) and the last equality holds due to the form of \( D \) given in (8) and \( g(y) = g(\bar{y}) = 0 \).

To show (ii), similarity, we have

\[
g(\psi y + (1 - \psi) \bar{y}) = 2\psi(1 - \psi) |[y]_1 | [\bar{y}]_1 - 2\psi(1 - \psi) |[y]_2+n_1 | [\bar{y}]_2+n_1
\]

\[
\leq 2\psi(1 - \psi) [y]_1 [\bar{y}]_1 + 2\psi(1 - \psi) |[y]_2+n_1 ||[\bar{y}]_2+n_1|
\]

which completes the proof. ■

**Proof of Lemma 6.** It is clear that neither \( y_1 + \alpha_1 z^- \) nor \( y_1 + \alpha_2 z^- \) is 0, since otherwise (10a) is violated given that \( b \neq 0 \). Notice that \( y_* \) can be expressed as a linear...
combination of \( y_\ast + \alpha_1^\top z^- \) and \( y_\ast + \alpha_2^\top z^- \) as follows:

\[
y_\ast = \frac{\alpha_1^\top}{\alpha_1 - \alpha_2} (y_\ast + \alpha_2^\top z^-) + \frac{-\alpha_2^\top}{\alpha_1 - \alpha_2} (y_\ast + \alpha_1^\top z^-). \tag{31}
\]

In (31), \( \alpha_1^\top \) and \( \alpha_2^\top \) cannot happen to be both positive or both negative. Therefore, \( \frac{\alpha_1^\top}{\alpha_1 - \alpha_2}, \frac{-\alpha_2^\top}{\alpha_1 - \alpha_2} \in [0, 1] \) and the above combination is a convex one.

In virtue of (29), we will proceed by considering the case in which the equality sign of (29) holds and the case in which the strict inequality sign of (29) holds, respectively. Suppose that \( g(y_\ast) < 0 \). Then, the form of (18) suggest that \( \alpha_1 > 0 \) and \( \alpha_2 < 0 \). If \( [y_\ast + \alpha_1^\top z^-]_1, [y_\ast + \alpha_2^\top z^-]_1 \geq 0 \), then by Lemma 3, \( g(y_\ast) \geq 0 \), which contradicts the hypothesis. Suppose that \( g(y_\ast) = 0 \). By (30), we consider two cases, i.e., \( y_\ast^\top Dz^- > 0 \) and \( y_\ast^\top Dz^- < 0 \), respectively. First, assume that \( y_\ast^\top Dz^- > 0 \). Then, \( \alpha_1^\top \) and \( \alpha_2^\top \) were larger than 0. It suggests from computation that

\[
y_\ast^\top Dz^- = \frac{\alpha_2^\top y_\ast^\top Dz^-}{\alpha_2 - \alpha_1} = \frac{1}{\alpha_2} \left[ [y_\ast]_1 [y_\ast + \alpha_2^\top z^-]_1_1 
- [y_\ast^\top 2:n+1 + \alpha_2^\top z^-]_2:n+1 + 1 \right] 
- \frac{1}{\alpha_2} \left[ [y_\ast]_1 [y_\ast + \alpha_1^\top z^-]_1_1 
- [y_\ast^\top 2:n+1 + \alpha_1^\top z^-]_2:n+1 + 1 \right] = 0,
\]

where the last equality follows since \( g(y_\ast) = g(y_\ast + \alpha_2^\top z^-) = 0 \). It contradicts the hypothesis. On the other hand, when we assume that \( y_\ast^\top Dz^- < 0 \), we reach a similar contradiction, which concludes the result.

\[\text{C.3. Proof of Lemma 7}\]

When \( h(\lambda) > 0 \) for \( \lambda \in I_1 \), it has been shown in [33] that \( g(z^-) < 0 \). In addition, by continuity, we have \( g(y_\ast) \geq 0 \).

The first claim can be proved by contradiction. The procedure is similar to the proof of Lemma 6 and is omitted.

Next we prove the second claim. As \( \alpha_1^\top \) and \( \alpha_2^\top \) are the roots of (19). Therefore, \( \alpha_1^\top \) and \( \alpha_2^\top \) are not both positive or both negative since \( g(y_\ast)/g(z^-) \leq 0 \). Then \( y_\ast \) is a convex combination of \( y_\ast + \alpha_1^\top z^- \) and \( y_\ast + \alpha_2^\top z^- \). It further implies that \( [y_\ast]_1, [y_\ast + \alpha_1^\top z^-]_1 \) and \( [y_\ast + \alpha_2^\top z^-]_1 \) have the same sign. In addition, \( g(\lambda) \) is a rational function for \( \lambda \in I_1 \) [33] and \( z^- \) is a connected set. We then obtain the image of \( I_1 \) under the function \( y \), denoted by \( [y]_1 \), is connected. If there were a \( \lambda \in I_1 \) such that \([y]_1, [y_\ast]_1 \) have distinct signs, we could find another \( \lambda \in I_1 \) satisfying \([y]_1 = 0 \). Notice that \( g(y(\lambda^\top)) = -\|g(y(\lambda^\top))_2:n+1\|^2 \leq 0 \), which contradicts \( g(y(\lambda^\top)) := h(\lambda^\top) > 0 \) for all \( \lambda \in I_1 \) and completes the proof.

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