Remnant Symmetry and the Confinement Phase in Coulomb Gauge

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We report on connections between the confining color Coulomb potential, center vortices, and the unbroken realization of remnant gauge symmetry in Coulomb gauge.

1. The Color Coulomb Potential

The color Coulomb potential is derived from the expectation value of the non-local part of the Coulomb-gauge Hamiltonian, in the presence of static external charges. This is a $1/R$ potential in electrodynamics, but the $R$-dependence may be different in non-abelian gauge theories. In this talk I will show that the confining behavior of the color Coulomb potential depends on the unbroken realization of a global gauge symmetry, which remains after imposing Coulomb gauge. I will also make some connections to the vortex confinement mechanism, and this theme will be further developed in Dan Zwanziger’s talk. The results outlined here are reported in much greater detail in ref. [1]; I will also touch on the studies in refs. [2] and [3].

Let $|\Psi_{qq}\rangle = \mathcal{T}(0)\phi(R)|\Psi_0\rangle$ denote a physical state in Coulomb gauge containing heavy quark-antiquark static charges, where $\Psi_0$ is the Yang-Mills vacuum wavefunctional. Then

$$\mathcal{E} = \langle \Psi_{qq}|H|\Psi_{qq}\rangle - \langle \Psi_0|H|\Psi_0\rangle = V_{coul}(R) + E_{se}$$ (1)

includes the $R$-dependent color Coulomb potential, plus self-energy contributions. It is natural to ask, first, whether $V_{coul}(R)$ is confining. If confining, is the potential asymptotically linear? If linear, does the Coulomb string tension $\sigma_{coul}$ equal the usual string tension $\sigma$ of the static quark potential? Finally, what about center vortices? What happens to the color Coulomb potential when vortices are removed? To address these questions, we define the correlator of two timelike Wilson (not Polyakov) lines in Coulomb gauge

$$G(R, T) = \langle \frac{1}{2} \text{Tr}[L(0, T)L(R, T)] \rangle$$

$$V(R, T) = -\frac{d}{dT} \log[G(R, T)]$$ (2)

where

$$L(\vec{x}, T) = \exp \left[ i \int_0^T dt A_0(\vec{x}, t) \right]$$ (3)

It is straightforward to show that

$$\mathcal{E} = V_{coul}(R) + E_{se} = \lim_{T \to 0} V(R, T)$$

$$\mathcal{E}_{min} = V(R) + E_{se} = \lim_{T \to \infty} V(R, T)$$ (4)

where $\mathcal{E}_{min}$ is the minimal energy state containing two static charges, and $V(R)$ is the static Coulomb potential. From this it follows, since $\mathcal{E} > \mathcal{E}_{min}$, that if $V(R)$ is confining, then so is $V_{coul}(R)$, with $V(R) \leq V_{coul}(R)$. This means that a confining Coulomb potential is a necessary (but not sufficient) condition for confinement, as first noted by Zwanziger [4]. With a lattice regularization

$$L(x, T) = U_0(x, a)U_0(x, 2a) \cdots U_0(x, T)$$

$$V(R, 0) = -\frac{1}{a} \log[G(R, 1)]$$ (5)
This allows us to obtain an estimate, exact in the continuum limit, of $V_{\text{coul}}(R)$, which we may compare to the static potential $V(R)$.

Figure 1, upper line of data points, shows our result for $V(R,0)$ at $\beta = 2.5$ [2]. This is clearly a linear potential; we find however that $\sigma_{\text{coul}} \approx 3\sigma$. The evidence, then, is that the color Coulomb potential \textit{overconfines}, a fact which is relevant to the gluon chain model, proposed by Greensite and Thorn [5]. As $T$ increases, one finds that the slope of $V(R,T)$ approaches $\sigma$ from above, as it should. The lower line of data points in Fig. 1 shows the result for $V(R,0)$ when center vortices are removed from lattice configurations by the method of de Forcrand and D’Elia [3]. It is clear that removing center vortices also removes the confining property of the color Coulomb potential. In Coulomb gauge, this property is attributed to the density of near-zero eigenvalues of the Faddeev-Popov operator; evidently vortex removal must alter this density drastically. This question will be addressed in Dan Zwanziger’s talk [7].

2. The Remnant Symmetry Order Parameter

Non-abelian gauge theories can exist in a number of different phases; those that concern us here belong to one of three broad categories. First, there are massless phases, e.g. compact $QED_4$, and non-abelian lattice gauge theories at weak couplings, and dimension $D > 4$, where external charges are associated with long-range electric fields. Second, there are confined phases, e.g. SU(N) gauge theory with or without matter fields in the adjoint representation, in which color electric fields are squeezed into flux tubes. In these theories there exists a global $Z_N$ symmetry of the Lagrangian, which is unbroken at the quantum level. Finally, there are screened phases, where the color electric field falls off exponentially away from the source. This condition is found in theories with trivial center symmetry, such as SU(N) gauge theories with matter in the fundamental representation, or in $G_2$ pure gauge theory. The screened phase is also found when a non-trivial center symmetry is spontaneously broken, as in high-temperature gauge theory, and in theories with adjoint matter fields in the Higgs phase.

Large-scale field fluctuations are enhanced in the confined phase and suppressed in the screened phases, as can be verified by computing the appropriate observables. We are interested in studying the behavior of the Coulomb energy in these different phases, and for this purpose it is useful to introduce a new order parameter.

Minimal Coulomb gauge, which maximizes the average value of $\sum_{x,k} \text{Tr}[U_k(x)]$, does not fix the gauge completely. There is still freedom to carry out time-dependent transformations

$$U_k(x,t) \to g(t)U_k(x,t)g^\dagger(t)$$
$$U_0(x,t) \to g(t)U_0(x,t)g^\dagger(t + 1)$$

which we will call the \textit{remnant symmetry} of Coulomb gauge. Remnant symmetry is a global symmetry at fixed time; and it can therefore be spontaneously broken at a fixed time $t$ in an infinite spatial volume. If remnant symmetry is broken, this means that

$$\langle U_0(x,t) \rangle \neq 0$$
Figure 2. Plot of $Q$ vs. root inverse 3-volume, and extrapolation of $Q$ to infinite volume in $QED_4$, for $\beta = 0.7$ (confining phase) and $\beta = 1.3$ (massless phase).

and in consequence

$$\lim_{R \to \infty} G(R,1) > 0$$
$$\lim_{R \to \infty} V(R,0) = \text{finite const.}$$
$$\sigma_{\text{coul}} = 0$$

So Coulomb confinement or non-confinement can be understood as the symmetric or broken realization, respectively, of the remnant gauge symmetry (this is not a new idea, c.f. ref. [8]). Our order parameter for remnant symmetry breaking, denoted $Q$, is the modulus of the average timelike link variable, averaged over all sites at fixed time, i.e.

$$U_0^{av}(t) = \frac{1}{L^3} \sum \bar{x} U_0(\bar{x},t)$$
$$Q = \left\langle \sqrt{\frac{1}{2} \text{Tr}[U_0^{av}(t)U_0^{av\dagger}(t)]} \right\rangle$$

On general grounds, on an $L^3 \times L_t$ lattice,

$$Q = c + \frac{b}{L^{3/2}}$$

where $c = 0$ in the symmetric phase, and $c > 0$ in the broken phase. $Q > 0$ at infinite $L$ implies that $V_{\text{coul}}(R)$ is non-confining, and therefore $Q = 0$ is a necessary (but not sufficient) condition for confinement.

3. Q in Different Phases

It is interesting to apply this order parameter first in compact $QED_4$, where we know there is a transition from the confined to the massless phase around $\beta = 1.0$. Fig. 2 shows our result for $Q$ at couplings in the confining and massless phases. The values for $Q$ extrapolate to zero at infinite volume in the confined phase, and to a non-zero value in the massless phase, as expected.

Next we have computed $Q$ in an SU(2) gauge theory with a scalar field, of fixed modulus $|\phi| = 1$, in the adjoint representation

$$S = \beta \sum_{plaq} \frac{1}{2} \text{Tr}[UUU^\dagger U^\dagger]$$
$$+ \frac{\gamma}{4} \sum_{x,\mu} \phi^a(x)\phi^b(x+\hat{\mu}) \frac{1}{2} \text{Tr}[\sigma^a U_{\mu}(x)\sigma^b U_{\mu}^\dagger(x)]$$

In this case it is also well known that there are two distinct phases in the $\beta - \gamma$ coupling plane, namely, a confining phase and a Higgs phase, and in this case again we find that $Q = 0$ in the confined phase, and $Q > 0$ in the Higgs phase, upon extrapolation to infinite volume.

At finite temperature, in pure SU(2) gauge theory, we encounter something unanticipated. We have calculated $V_{\text{coul}}(R)$ and $Q$ in the high-temperature deconfined phase, expecting to see
Instead, the opposite result was obtained, as seen in Figs. 3 and 4. This result is, however, a little less surprising when we remember that the non-local, Coulomb part of the Hamiltonian depends only on links in the three-space directions, and not on timelike links. The links in the space directions still form a confining ensemble, since spacelike Wilson loops have an area law even in the deconfined phase. From this point of view, it is rather natural that the instantaneous Coulomb potential is confining, even if the static quark potential is not.

As a check of this reasoning, we remove center vortices from the high-temperature ensemble of lattices; this is known to remove the area law for spacelike Wilson loops, so the spacelike links at fixed time are no longer a confining ensemble. On computing the color Coulomb potential, we indeed find that $\sigma_{\text{coul}} = 0$ asymptotically.

Finally, we have computed $Q$ in SU(2) gauge theory with a radially-frozen Higgs field in the fundamental representation. For the SU(2) gauge group, the action can be written

\begin{equation}
S = \beta \sum_{\text{plaq}} \frac{1}{2} \text{Tr} [UUU^\dagger] + \gamma \sum_{x,\mu} \frac{1}{2} \text{Tr} [\phi^\dagger(x)U_\mu(x)\phi(x + \hat{\mu})] (12)
\end{equation}

where $\phi$ is SU(2)-group valued, and $\phi\phi^\dagger = I$. In this theory the Wilson loop never acquires an area law, at any $\gamma > 0$. There is a theorem by Fradkin and Shenker which says that it is always possible to follow a path from the small-$\gamma$ "confinement-like" region of the $\beta-\gamma$ phase diagram, into the large-$\gamma$ Higgs region, which avoids any non-analyticity in local gauge-invariant observables, such as the free energy. Thus, according to the textbook definition, the Higgs and confinement-like regions do not constitute distinct phases of the lattice theory.

The $Q$ operator tells a different story. We find, in fact, two distinct regions: a region of unbroken remnant symmetry, where $Q = 0$, and a region of broken symmetry, with $Q > 0$, as shown in Fig. 5. The boundary between these regions is formed by the solid line, which is a line of first-order phase transitions, continuing into the dashed line, which is not associated with any non-analyticity in the free energy. There is a discontinuity in $Q$ across the solid line. $Q$ is zero at the dashed line, and then rises to non-zero values as $\gamma$ increases, as illustrated in Fig. 6. This behavior is strongly reminiscent of magnetization in a second order phase transition.

The non-analytic behavior of the $Q$ order parameter, across the remnant-symmetry breaking transition, is not at odds with the Fradkin-Shenker theorem. When expressed in gauge-invariant form, $Q$ is a highly non-local observable, whose behavior is not covered by this theorem. Nevertheless, we would like to better understand the nature of a symmetry-breaking transition which is not accompanied by any singularity in thermodynamic quantities. It was suggested by Langfeld that this is an example of a Kertész line.

A Kertész line is a line of percolation transitions; the original example comes from the Ising model. In the Ising model, in the absence of an external magnetic field, there is a phase transition from a $Z_2$ symmetric phase to an ordered phase. This transition can be expressed, in different variables, as a transition from a percolating phase at low temperature, to a non-percolating phase.
phase at high temperature. In the presence of a magnetic field, the partition function and thermodynamic observables become analytic in temperature; there is no thermodynamic phase transition. Nevertheless, the percolation transition persists, and traces out a Kertész line in the temperature-magnetic field plane, completely separating the phase diagram into two regions. The interesting question, in the gauge-Higgs theory, is what sort of objects are actually percolating. Based in part on results reported by Bertle and Faber [12], Langfeld [13] conjectured that the unbroken remnant symmetry region is a region of percolating center vortices, which cease percolating in the broken symmetry region.

This conjecture that the gauge-Higgs theory is divided into percolating and non-percolating phases has been verified in ref. [3], which introduces a percolation observable $s_w$. To define this observable, let $f_p$ denote the fraction of all P-plaquettes on the lattice, which lie in the P-vortex containing the P-plaquette $p$. Then $s_w$ denotes the value of $f_p$ averaged over all P-plaquettes. This is, in a sense, the fraction of the total P-vortex area contained in the "average" P-vortex.

If all P-plaquettes on the lattice lie a single percolating vortex, then $s_w = 1$. If vortices do not percolate at all, then $s_w = 0$ on an infinite lattice. We have percolation, in infinite volumes, when $s_w > 0$.

Ref. [3] did not actually use a radially frozen Higgs field; the matter action of the gauge-Higgs theory was instead taken to be

$$S_{\text{matter}} = \sum_x \left[ \Phi^\dagger(x) \Phi(x) + \lambda (\Phi^\dagger(x) \Phi(x) - 1)^2 \right] - \kappa \sum_{\mu,x} (\Phi^\dagger(x) U_\mu(x) \Phi(x + \hat{\mu}) + c.c.)$$

(13)

where $\Phi$ is a two-component massive scalar field in the fundamental representation of SU(2). The result for $s_w$ vs. $\kappa$ at $\beta = 0.25$, $l = 1.0$ is shown in Fig. 4 along with a number of other observables such as vortex density and $O_{GH} = \langle \Phi^\dagger(x) U_\mu(x) \Phi(x + \hat{\mu}) \rangle$. In local gauge-invariant observables, such as $O_{GH}$, there is no obvious transition as $\kappa$ increases. But in the center vortex percolation variable $s_w$, a sharp transition is seen.

Our findings in the gauge-Higgs theory bear on the question: In what sense does real QCD, or any gauge theory with matter in the funda-
Figure 7. Vortex percolation order parameter $s_w$ (open circles) vs. gauge-Higgs coupling $\kappa$, in the gauge-Higgs theory of eq. (13). The other couplings ($\beta = 0.25$, $\lambda = 1.0$) are fixed. Also shown is the vortex density, and the one-link observable $O_{GH}$. The "sm" data points refer to a type of smoothing, as explained in ref. [3].

mental representation, confine color? As already noted, in such theories there are no transitions in the free energy which isolate the Higgs from the confinement-like regions of the phase diagram; it seems possible to continuously interpolate from one region to another.

In contrast, we find that these phases are indeed physically different, and separated by a sharp symmetry-breaking transition. The confinement-like phase is distinguished from the Higgs phase by its unbroken realization of remnant symmetry, by a confining color Coulomb potential, and by percolating center vortices.

4. Conclusions

We have found that the color Coulomb potential in pure SU(2) gauge theory is linearly rising, with a slope which is roughly three times larger than the usual asymptotic string tension. This overconfinement is essential to the gluon-chain scenario of QCD string formation.

The confinement property of the color Coulomb potential is a consequence of the unbroken realization of remnant gauge symmetry in Coulomb gauge. Center symmetry breaking, which takes $\sigma \to 0$, does not necessarily imply remnant symmetry breaking. In at least two cases, namely, the high-temperature deconfined phase and the confinement-like phase of gauge-fundamental Higgs theory, we find that $\sigma = 0$ coexists with $\sigma_{coul} > 0$. In the latter theory, the transition to the Higgs phase is a remnant-symmetry breaking, center vortex deperecolation transition.

In every case, center vortex removal also sends $\sigma_{coul} \to 0$, suggesting that there may be a deep relationship between the center vortex and Gribov horizon confinement scenarios. This relationship will be explored more fully in Dan Zwanziger’s talk [7] at this meeting, and in ref. [14].

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