Shifted Appell sequences in Clifford analysis

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Abstract
This paper is a continuation of [D. Peña Peña, On a sequence of monogenic polynomials satisfying the Appell condition whose first term is a non-constant function, arXiv:1102.1833], in which we prove that for every monogenic polynomial \( P_k(x) \) of degree \( k \) in \( \mathbb{R}^{m+1} \) there exists a sequence of monogenic polynomials \( \{M_n(x)\}_{n \geq 0} \) satisfying the Appell condition such that \( M_0(x) = P_k(x) \).

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1 Preliminaries

The real Clifford algebra \( \mathbb{R}_{0,m} \) (see [6]) is the free algebra generated by the standard basis \( \{e_1, \ldots, e_m\} \) of the Euclidean space \( \mathbb{R}^m \), subject to the multiplication relations

\[
e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m,
\]

where \( \delta_{jk} \) denotes the Kronecker delta. The dimension of the real Clifford algebra \( \mathbb{R}_{0,m} \) is \( 2^m \), as is the case for the Grassmann algebra generated by \( \{e_1, \ldots, e_m\} \), but the difference is that now \( e_j^2 = -1 \) instead of \( e_j^2 = 0 \), creating a structure with similarities to the complex numbers.

A general element \( a \) of \( \mathbb{R}_{0,m} \) may be written as \( a = \sum_A a_A e_A \), \( a_A \in \mathbb{R} \), in terms of the basic elements \( e_A = e_{j_1} \ldots e_{j_k} \), defined for every subset \( A = \{j_1, \ldots, j_k\} \) of \( \{1, \ldots, m\} \) with \( j_1 < \cdots < j_k \). For the empty set, one puts
e_0 = 1, the latter being the identity element. Conjugation in \( \mathbb{R}_{0,m} \) is given by 
\[ a = \sum_A a_A \mathcal{e}_A, \]
with \( \mathcal{e}_A = e_{j_1} \ldots e_{j_k}, \)
\( e_j = -e_j, j = 1, \ldots, m. \)

One natural way to generalize the holomorphic functions to higher dimensions is by considering the null solutions of the fundamental first order differential operator \( \partial_x \) in \( \mathbb{R}^{m+1} \) given by
\[ \partial_x = \partial_{x_0} + \partial_{\mathbf{x}} = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j}, \]
called the generalized Cauchy-Riemann operator, and where \( \partial_{\mathbf{x}} \) is the Dirac operator in \( \mathbb{R}^m \). That is, an \( \mathbb{R}_{0,m} \)-valued function \( f \) defined and continuously differentiable in an open set \( \Omega \) of \( \mathbb{R}^{m+1} \), is said to be (left) monogenic in \( \Omega \) if and only if \( \partial_x f = 0 \) in \( \Omega \). In a similar way one also defines monogenicity with respect to the Dirac operator \( \partial_{\mathbf{x}} \) in \( \mathbb{R}^m \). Monogenic functions are a central object of study in Clifford analysis (see e.g. [4, 8, 13]).

An remarkable feature of the generalized Cauchy-Riemann operator \( \partial_x \) is that it gives a factorization of the Laplacian, i.e.
\[ \Delta_x = \sum_{j=0}^m \partial_{x_j}^2 = \partial_x \overline{\partial}_x = \overline{\partial}_x \partial_x, \]
and therefore every monogenic function is also harmonic. Observe that the operator \( \overline{\partial}_x = \partial_{x_0} - \partial_{\mathbf{x}} \) may be seen as the higher dimensional version of the well-known operator \( 2\partial_z = \partial_{x_0} - i\partial_{x_1} \). Furthermore, according to [12, 15], the hypercomplex derivative of a monogenic function \( f \) is defined as
\[ \frac{1}{2} \overline{\partial}_x f. \]
As a monogenic function \( f \) clearly satisfies \( \partial_{x_0} f = -\overline{\partial}_x f \), it easily follows that
\[ \frac{1}{2} \overline{\partial}_x f = \partial_{x_0} f = -\overline{\partial}_x f. \]

An important class of polynomial sequences is the class of Appell sequences which is defined as follows (see [1]). A polynomial sequence \( \{p_n(t)\}_{n \geq 0} \), i.e. the index of each polynomial equals its degree, is said to be an Appell sequence if it satisfies
\[ p'_n(t) = np_{n-1}(t), \quad n \geq 1. \]
Probably, the simplest example of an Appell sequence is the sequence \( \{t^n\}_{n \geq 0} \), other examples being the Bernoulli, the Euler and the Hermite polynomials.
Appell sequences have been recently introduced to the Clifford analysis setting (see e.g. [2, 3, 7, 9, 10, 11, 14, 16]). Namely, a sequence \( \{P_n(x)\}_{n \geq 0} \) of \( \mathbb{R}_{0,m} \)-valued polynomials forms an Appell sequence if the following conditions are satisfied:

(i) \( \{P_n(x)\}_{n \geq 0} \) is a polynomial sequence;

(ii) each \( P_n(x) \) is monogenic in \( \mathbb{R}^{m+1} \), i.e. \( \partial_x P_n(x) = 0 \) for all \( x \in \mathbb{R}^{m+1} \);

(iii) \( \frac{1}{2} \partial_x P_n(x) = nP_{n-1}(x) \), \( n \geq 1 \).

Note that the requirement of \( \{P_n(x)\}_{n \geq 0} \) being a polynomial sequence implies that the first term \( P_0(x) \) must be a constant. It is natural to ask whether one can construct sequences of monogenic polynomials satisfying the Appell condition (iii) but in which the first term is a monogenic polynomial in \( \mathbb{R}^{m+1} \) and not necessarily a constant. More precisely, we are interested in sequences \( \{M_n(x)\}_{n \geq 0} \) of \( \mathbb{R}_{0,m} \)-valued polynomials which are monogenic in \( \mathbb{R}^{m+1} \) fulfilling

\[
\frac{1}{2} \partial_x M_n(x) = nM_{n-1}(x), \quad n \geq 1, \tag{1}
\]

where \( M_0(x) \) is an arbitrary monogenic polynomial in \( \mathbb{R}^{m+1} \). These sequences will be called \textit{shifted Appell sequences of monogenic polynomials}.

This paper is a continuation of [13], where an example of these sequences was constructed for the case \( M_0(x) = P_k(x) \) being an arbitrary \( \mathbb{R}_{0,m} \)-valued homogeneous monogenic polynomial of degree \( k \) in \( \mathbb{R}^m \).

It is clear that the class of shifted Appell sequences of monogenic polynomials is a right \( \mathbb{R}_{0,m} \)-module under the usual addition of sequences and multiplication by Clifford numbers. Suppose now that \( P_\kappa(x) \) is an \( \mathbb{R}_{0,m} \)-valued polynomial of degree \( \kappa \) which moreover is monogenic in \( \mathbb{R}^{m+1} \). It is easy to check that \( P_\kappa(x) \) may be written as

\[
P_\kappa(x) = \sum_{k=0}^\kappa P_k(x),
\]

where \( P_k(x) \) denotes a homogeneous monogenic polynomial of degree \( k \) in \( \mathbb{R}^{m+1} \). Thus, on account of the previous remarks, we only need to prove:

\textbf{Theorem 1} Let \( P_k(x) \) be an \( \mathbb{R}_{0,m} \)-valued homogeneous polynomial of degree \( k \) which is monogenic in \( \mathbb{R}^{m+1} \). Then there exists a shifted Appell sequence of monogenic polynomials \( \{M_n(x)\}_{n \geq 0} \) such that \( M_0(x) = P_k(x) \).
2 Some fundamental results

Let $P(k)$ ($k \in \mathbb{N}_0$) be the set of all $\mathbb{R}_{0,m}$-valued homogeneous polynomials of degree $k$ in $\mathbb{R}^m$. This set contains the important subspace $M^+(k)$ consisting of all polynomials in $P(k)$ which are monogenic. That is, $P_k(x) \in M^+(k)$ if it is an $\mathbb{R}_{0,m}$-valued polynomial of degree $k$ and

$$P_k(tx) = t^k P_k(x), \quad \partial_x P_k(x) = 0, \quad x \in \mathbb{R}^m, \; t \in \mathbb{R}.$$

For a differentiable $\mathbb{R}$-valued function $\phi$ and a differentiable $\mathbb{R}_{0,m}$-valued function $g$, we have

$$\partial_x (\phi g) = \partial_x (\phi) g + \phi (\partial_x g). \quad (2)$$

Moreover, for a differentiable vector-valued function $f = \sum_{j=1}^m f_j e_j$, we also have

$$\partial_x (fg) = (\partial_x f) g - f (\partial_x g) - 2 \sum_{j=1}^m f_j (\partial_{x_j} g). \quad (3)$$

Let

$$\beta_k(n) = \begin{cases} n, & \text{if } n \text{ even} \\ 2k + m + n - 1, & \text{if } n \text{ odd} \end{cases}$$

for $n \geq 1$. Using the Leibniz rules (2)-(3) as well as Euler’s theorem for homogeneous functions, we can deduce the useful identity:

$$\partial_x (x^n P_k(x)) = -\beta_k(n)x^{n-1}P_k(x), \quad P_k(x) \in M^+(k), \quad n \geq 1. \quad (4)$$

Let us recall two basic results of Clifford analysis: the Cauchy-Kovalevskaya extension technique (see e.g. [4, 8]) and the Almansi-Fischer decomposition (see e.g. [8, 17]).

**Theorem 2** Every $\mathbb{R}_{0,m}$-valued function $g(x)$ analytic in $\mathbb{R}^m$ has a unique monogenic extension $\text{CK}[g]$ to $\mathbb{R}^{m+1}$, which is given by

$$\text{CK}[g(x)](x) = \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \partial_x^j g(x).$$

**Remark:** Observe that a monogenic function $f(x)$ can be reconstructed by knowing its restriction to $\mathbb{R}^m$ using previous formula, i.e.

$$f(x) = \text{CK}[f(x)|_{x_0=0}](x).$$

It is also worth noting that

$$\frac{1}{2} \partial_x \text{CK}[g(x)](x) = -\partial_x \text{CK}[g(x)](x) = -\text{CK}[\partial_x g(x)](x). \quad (5)$$
Theorem 3 Let \( k \in \mathbb{N} \). Then
\[
P(k) = \bigoplus_{\nu=0}^{k} x^\nu M^+(k - \nu).
\]

Theorems 2 and 3 together with equality (4) will be essential for proving our main result.

3 Proof of the main result

We shall first introduce a collection \( \{ M^{k,\nu}_n(x) \}_{n \geq 0}, 0 \leq \nu \leq k \} \) of shifted Appell sequences of homogeneous monogenic polynomials whose first terms are
\[
M^{k,\nu}_0(x) = C K [ x^\nu P_{k-\nu}(x) ](x), \quad P_{k-\nu}(x) \in M^+(k - \nu), \quad 0 \leq \nu \leq k.
\]

From Theorem 2 we can deduce that \( M^{k,\nu}_n(x) \) is a homogeneous monogenic polynomials of degree \( k \) in \( \mathbb{R}^{m+1} \) and is of the form
\[
M^{k,\nu}_n(x) = \left( \sum_{j=0}^{\nu} \frac{\mu_{j}^{k,\nu}}{j!} x^j x^{\nu-j} \right) P_{k-\nu}(x), \quad n \geq 0.
\]

Lemma 1 Assume that \( P_{k-\nu}(x) \in M^+(k - \nu) \) where \( k, \nu \in \mathbb{N}_0, \nu \leq k \) and put
\[
\lambda^{k,\nu}_n = \frac{n!}{\prod_{s=1}^{\nu} \beta_{k-\nu}(\nu + s)},
\]
for \( n \geq 1 \) and \( \lambda^{k,\nu}_0 = 1 \) for \( n = 0 \). The sequence \( \{ M^{k,\nu}_n(x) \}_{n \geq 0} \) defined by
\[
M^{k,\nu}_n(x) = \lambda^{k,\nu}_n C K [ x^{\nu+n} P_{k-\nu}(x) ](x), \quad n \geq 0,
\]
is a shifted Appell sequence of homogeneous monogenic polynomials.

Proof. By Theorem 2 it follows that \( M^{k,\nu}_n(x) \) is a homogeneous monogenic polynomials of degree \( k + n \) in \( \mathbb{R}^{m+1} \) and is of the form
\[
M^{k,\nu}_n(x) = n! \left( \sum_{j=0}^{n-1} \frac{\lambda^{k,\nu}_{n-j}}{j!(n-j)!} x^j x^{\nu+n-j} + \sum_{j=n}^{\nu+n} \frac{\mu_{j-n}^{k,\nu}}{j!} x^j x^{\nu+n-j} \right) P_{k-\nu}(x), \quad n \geq 1.
\]
It only remains to show that $\{M_n^{k,\nu}(x)\}_{n \geq 0}$ satisfies the Appell condition (1). Indeed, using (5) and identity (4), we see at once that
\[
\frac{1}{2} \partial_x M_n^{k,\nu}(x) = -\lambda_n^{k,\nu} CK [\partial_x (x^{\nu+n} P_{k-\nu}(x))](x)
= \beta_{k-\nu}(\nu + n) \lambda_n^{k,\nu} CK [x^{\nu+n-1} P_{k-\nu}(x)](x)
= n \lambda_n^{k,\nu} CK [x^{\nu+n-1} P_{k-\nu}(x)](x) = n M_n^{k,\nu}(x).
\]
□

Remark: It should be noticed that $\{M_n^{k,0}(x)\}_{n \geq 0}$, which is the first sequence in the above collection, corresponds to the sequence constructed in [18].

We can now prove the main result of this paper:

Proof of Theorem 7. Suppose that $P_k(x)$ is a $\mathbb{R}_{0,m}$-valued homogeneous polynomial of degree $k$ which is monogenic in $\mathbb{R}^{m+1}$. From Theorem 2 we have that
\[
P_k(x) = CK \left.P_k(x)\right|_{x_0 = 0} (x).
\]
It is clear that $P_k(x)|_{x_0 = 0} \in P(k)$. Consequently, by Theorem 3 there exists unique $P_{k-\nu}(\xi) \in M^+(k-\nu)$ such that
\[
P_k(x)|_{x_0 = 0} = \sum_{\nu = 0}^{k} x^{\nu} P_{k-\nu}(x).
\]
From the above it follows that
\[
P_k(x) = \sum_{\nu = 0}^{k} CK [x^{\nu} P_{k-\nu}(x)](x).
\]
Define
\[
\{M_n(x)\}_{n \geq 0} = \left\{ \sum_{\nu = 0}^{k} M_n^{k,\nu}(x) \right\}_{n \geq 0}.
\]
Lemma 4 now shows that $\{M_n(x)\}_{n \geq 0}$ is a shifted Appell sequence of monogenic polynomials with first term $M_0(x) = P_k(x)$. □

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References

[1] P. Appell, Sur une classe de polynômes. Ann. Sci. École Norm. Sup. (2) 9 (1880), 119–144.

[2] S. Bock and K. Gürlebeck, On a generalized Appell system and monogenic power series. Math. Methods Appl. Sci. 33 (2010), no. 4, 394–411.

[3] S. Bock, K. Gürlebeck, R. Lávička and V. Souček, The Gelfand-Tsetlin bases for spherical monogenics in dimension 3, arXiv:1010.1615 [math.CV], 2010, submitted.

[4] F. Brackx, R. Delanghe and F. Sommen, Clifford analysis. Research Notes in Mathematics, 76, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[5] I. Caç ao and H. R. Malonek, On complete sets of hypercomplex Appell polynomials. International Conference on Numerical Analysis and Applied Mathematics, AIP-Proceedings, 2008, 647–650.

[6] W. K. Clifford, Applications of Grassmann’s Extensive Algebra. Amer. J. Math. 1 (1878), no. 4, 350–358.

[7] H. De Bie and F. Sommen, Hermite and Gegenbauer polynomials in superspace using Clifford analysis. J. Phys. A 40 (2007), no. 34, 10441–10456.

[8] R. Delanghe, F. Sommen and V. Souček, Clifford algebra and spinor-valued functions. Mathematics and its Applications, 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[9] M. I. Falcão, J. F. Cruz and H. R. Malonek, Remarks on the generation of monogenic functions. 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, K. Gürlebeck and C. Körne (eds.), Weimar, Germany, 12–14 July 2006.

[10] M. I. Falcão and H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. International Conference on Numerical Analysis and Applied Mathematics, AIP-Proceedings, 2007, 738–741.

[11] N. Gürlebeck, On Appell sets and the Fueter-Sce mapping. Adv. Appl. Clifford Algebr. 19 (2009), no. 1, 51–61.
[12] K. Gürlebeck and H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. Complex Variables Theory Appl. 39 (1999), no. 3, 199–228.

[13] K. Gürlebeck and W. Sprössig, Quaternionic and Clifford calculus for physicists and engineers. Wiley and Sons Publications, Chichester, 1997.

[14] R. Lávička, Canonical bases for $\mathfrak{sl}(2, \mathbb{C})$-modules of spherical monogenics in dimension 3. Arch. Math.(Brno) 46 (2010) (5), 339-349.

[15] H. R. Malonek, Selected topics in hypercomplex function theory. Clifford algebras and potential theory, 111–150, Univ. Joensuu Dept. Math. Rep. Ser., 7, Univ. Joensuu, Joensuu, 2004.

[16] H. R. Malonek and M. I. Falcão, Special monogenic polynomials - Properties and applications. International Conference on Numerical Analysis and Applied Mathematics, AIP-Proceedings, 2007, 764–767.

[17] H. R. Malonek and G. Ren, Almansi-type theorems in Clifford analysis. Math. Methods Appl. Sci. 25 (2002), no. 16-18, 1541–1552.

[18] D. Peña Peña, On a sequence of monogenic polynomials satisfying the Appell condition whose first term is a non-constant function, arXiv:1102.1833 [math.CV], 2011, submitted.