A SHORT REMARK ON THE SURJECTIVITY OF THE COMBINATORIAL LAPLACIAN ON INFINITE GRAPHS

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ABSTRACT. Applying a well-known theorem due to Eidelheit, we give a short proof of the surjectivity of the combinatorial Laplacian on connected locally finite undirected simplicial graph $G$ with countably infinite vertex set $V$ established in [1]. In fact, we show that every linear operator on $K^V$ which has finite hopping range and satisfies the pointwise maximum principle is surjective.

1. Introduction

In [1] Ceccherini-Silberstein, Coornaert, and Dodziuk showed that on a connected locally finite simplicial undirected graph $G$ with a countably infinite vertex set $V$ the combinatorial Laplacian on the real valued functions on $V$, $\Delta_G : \mathbb{R}^V \to \mathbb{R}^V$ defined by

$$\forall v \in V : \Delta_G f(v) = f(v) - \frac{1}{\text{deg}(v)} \sum_{v \sim w} f(w),$$

is surjective. Two vertices $v, w \in V$ of $G$ are adjacent, $v \sim w$, if $(v, w)$ is an edge of $G$. Recall that a graph is locally finite if for every vertex $v$ of $G$ the number $\text{deg}(v)$ of adjacent vertices is finite and that $G$ is connected if for any pair of different vertices $v, w$ there is a finite number of edges $(v_0, v_1), \ldots, (v_{n-1}, v_n)$ of $G$ with $v \in \{v_0, v_n\}$ and $w \in \{v_0, v_n\}$. Finally, $G$ is simplicial if it does not have any loops, i.e. $(v, v)$ is not an edge of $G$ for any vertex $v$, and $G$ is undirected if $(w, v)$ is an edge of $G$ whenever $(v, w)$ is an edge of $G$.

In [1] the surjectivity of the Laplacian was proved by a Mittag-Leffler argument and an application of the (pointwise) maximum principle for $\Delta_G$. There is a vast amount of literature dealing with a systematic study of the Mittag-Leffler procedure and its applications to a wide range of surjectivity problems, see e.g. [4] and the references therein. As noted in [1], the prove of surjectivity of $\Delta_G$ extends to more general operators, e.g. to operators of the form $\Delta_G + \lambda$, where $\lambda : V \to [0, \infty)$.

The aim of this note is to give a very short proof of the following generalisation of the above result.

Theorem 1. Let $G$ be a locally finite connected graph (which may be directed or undirected) with countably infinite vertex set $V$. Every linear operator $A : \mathbb{R}^V \to \mathbb{K}^V$ which has finite hopping range and satisfies the pointwise maximum principle is surjective.

Note that $\Delta_G$ has finite hopping range and satisfies the pointwise maximum principle. (For a precise definition of finite hopping range and the pointwise maximum principle see section 2.) The proof of the above theorem relies on a well-known result due to Eidelheit.

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2. Proof of Theorem 1

We equip $\mathbb{K}^\mathbb{N}$ with its usual Fréchet space topology, i.e. the locally convex topology defined by the increasing fundamental system of seminorms $(p_k)_{k \in \mathbb{N}}$ given by
\[
\forall f = (f_j)_{j \in \mathbb{N}} \in \mathbb{K}^\mathbb{N} : p_k(f) = \sum_{j=1}^{k} |f_j|.
\]
As usual, we denote this Fréchet space by $\omega$. The dual space $\omega'$ of $\omega$ is given by the space of finitely supported sequences $\varphi$. For $j \in \mathbb{N}$ we set $\pi_j : \omega \to \mathbb{K}, (f_m)_{m \in \mathbb{N}} \mapsto f_j$ so that $\varphi = \text{span}\{\pi_j; j \in \mathbb{N}\}$.

If $E$ is any Fréchet space and $A : E \to \omega$ linear and continuous we set $A_j := \pi_j \circ A, j \in \mathbb{N}$, so that $A_j \in E'$. A straightforward calculation gives that the transpose $A^t : \varphi \to E'$ is given by $A^t(y) = \sum_{j=1}^{\infty} y_j A_j$, where only finitely many $y_j$'s do not vanish. Thus, the linear independence of $(A_j)_{j \in \mathbb{N}}$ is equivalent to the injectivity of $A^t$. By the Hahn-Banach Theorem, the later is equivalent to $A$ having dense range. We recall the following theorem due to Eidelheit (see e.g. [2] or [3, Theorem 26.27]).

Eidelheit's Theorem Let $E$ be a Fréchet space, $(p_k)_{k \in \mathbb{N}}$ be an increasing fundamental system of seminorms on $E$ and let $A : E \to \omega$ be linear and continuous. Then, $A$ is surjective if, and only if, for $(A_j)_{j \in \mathbb{N}}$ defined as above the following conditions are satisfied.

i) $(A_j)_{j \in \mathbb{N}}$ is linearly independent.

ii) For every $k \in \mathbb{N}$

\[
\text{dim}(\{\phi \in E'; \exists c > 0 : |\phi| \leq c p_k\} \cap \text{span}\{A_j; j \in \mathbb{N}\}) < \infty.
\]

For $E = \omega$ with the increasing fundamental system of seminorms $(p_k)_{k \in \mathbb{N}}$ given by (1), it is immediate that for $y \in \varphi$ there is $c > 0$ with $|\langle y, f \rangle| \leq c p_k(f)$ for all $f \in \omega$ precisely when $y \in \varphi_k := \{x = (x_j)_{j \in \mathbb{N}} \in \omega; x_j = 0 \text{ for all } j > k\}$. Since $\varphi_k$ is obviously finite dimensional, by Eidelheit's Theorem, a linear continuous operator $A : \omega \to \omega$ is surjective if and only if $(A_j)_{j \in \mathbb{N}}$ is linearly independent.

Now, let $G$ be a locally finite connected graph with countably infinite vertex set $V$. For each $v \in V$ and $n \in \mathbb{N}$ we define $U_n(v)$ to be the union of $\{v\}$ with the set of all endpoints of paths in $G$ starting in $v$ and of length not exceeding $n$ together with the set of starting points of paths in $G$ ending in $v$ and of length not exceeding $n$.

Definition 1. Let $G$ be a graph with vertex set $V$ and let $A : \mathbb{K}^V \to \mathbb{K}^V$ be linear.

i) $A$ has finite hopping range if for every $v \in V$ there is $n \in \mathbb{N}$ such that the following implication holds

\[
\forall f, g \in \mathbb{K}^V : f|_{U_n(v)} = g|_{U_n(v)} \Rightarrow A(f)(v) = A(g)(v).
\]

ii) $A$ satisfies the pointwise maximum principle if for every $v \in V$ there is $n \in \mathbb{N}$ such that for each $f \in \mathbb{K}^V$ with $A(f)(v) = 0$ the implication

\[
|f(v)| = \max_{U_n(v)} |f(w)| \Rightarrow \forall w \in U_n(v) : |f(w)| = |f(v)|
\]

holds.

Enumeration of the vertices $V = \{v_k; k \in \mathbb{N}\}$ of $G$ clearly gives an isomorphism of $\mathbb{K}^V$ onto $\mathbb{K}^\mathbb{N}$. In order to keep notation simple, for a linear mapping $A : \mathbb{K}^V \to \mathbb{K}^V$ we denote by $A$ also the linear operator on $\mathbb{K}^\mathbb{N}$ induced by the canonical isomorphism between $\mathbb{K}^V$ and $\mathbb{K}^\mathbb{N}$. Since the linear operator $A : \omega \to \omega$ is continuous if and only if $A_j \in \varphi$ for all $j \in \mathbb{N}$, using that $G$ is connected and locally finite, a
straight forward calculation shows that \( A : \omega \to \omega \) is continuous if and only if the inducing \( A : \mathbb{K}^V \to \mathbb{K}^V \) has finite hopping range.

**Proposition 2.** Let \( G \) be a locally finite connected graph with countably infinite vertex set \( V \) and let \( A : \mathbb{K}^V \to \mathbb{K}^V \) be linear. If \( A \) has finite hopping range and satisfies the pointwise maximum principle, then \( (A_j)_{j \in \mathbb{N}} \) is a linearly independent sequence of continuous linear forms on \( \omega \).

**Proof.** As pointed out above, the claim is equivalent to the injectivity of \( A' \). For \( k \in \mathbb{N} \) we define

\[
M_k : \omega \to \omega, (f_j)_{j \in \mathbb{N}} \mapsto (f_1, \ldots, f_k, 0, \ldots).
\]

Then \( M_k \) is a continuous linear operator on \( \omega \) with \( M_k^t = M_k|_{\mathbb{R}^t} \).

Fix \( y \in \varphi \) with \( A'(y) = 0 \). Then there is \( k \in \mathbb{N} \) with \( y \in \varphi_k \) and

\[
\forall f \in \varphi_k : 0 = \langle y, A(f) \rangle = \langle M_k^t(y), (A_{|\varphi_k}(f)) \rangle = \langle y, (M_k \circ A_{|\varphi_k})(f) \rangle
\]

where \( \langle \cdot, \cdot \rangle_k \) denotes the duality bracket between \( \varphi_k \) and \( \varphi_k \) and \( (M_k \circ A_{|\varphi_k})^t \) the transpose of

\[
M_k \circ A_{|\varphi_k} : \varphi_k \to \varphi_k
\]

with respect to this duality. From the above we conclude that \((M_k \circ A_{|\varphi_k})^t(y) = 0\) so that \( y = 0 \) if we can show that \((M_k \circ A_{|\varphi_k})^t\) is injective. Since \( \varphi_k \) is finite dimensional, the later holds precisely when \( M_k \circ A_{|\varphi_k} \) is injective, which will be proved as in [1] by the pointwise maximum principle for \( A \).

For \( j \in \mathbb{N} \) we define \( N(j) = \{l \in \mathbb{N}; v_l \in U_n(v_j)\} \), where \( n \) is chosen as in Definition II ii) for \( v = v_j \). Let \( f \in \varphi_k \) satisfy \((M_k \circ A)(f) = 0\) and let \( 1 \leq k_0 \leq k \) be such that \( |f_{k_0}| = \max\{|f_j| : j \in \mathbb{N}\} \). By \( 0 = (M_k \circ A)(f)(k_0) = A(f)(k_0) \) and the pointwise maximum principle we conclude \( |f_j| = M \) for all \( j \in N(k_0) \).

If there is \( j \in N(k_0) \) with \( j > k \) then \( M = 0 \), since \( f \) vanishes. If \( N(k_0) \subseteq \{1, \ldots, k\} \) it follows again from the maximum principle that \( |f_j| = M \) for all \( j \in \bigcup_{l \in N(k_0)} N(l) \). Using that \( G \) is connected, a finite numbers of iterations of this process finally yields that \(|f_j| = M\) for some \( j \in \{k + 1, \ldots\} \), where \( f \) vanishes so that \( M = 0 \), i.e. \( f = 0 \).

**Proof of Theorem 1.** Theorem 1 follows now immediately from the considerations following Eidelheit’s Theorem and the above proposition.

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**References**

[1] Ceccherini-Silberstein, Tullio and Coornaert, Michel and Dodziuk, Józef, *The surjectivity of the combinatorial Laplacian on infinite graphs*, Enseign. Math. (2), vol. 58(1-2):125–130, 2012.

[2] Eidelheit, M., *Zur Theorie der Systeme linearer Gleichungen*, Studia Math., 6:139–148, 1936.

[3] Meise, Reinhold and Vogt, Dietmar, *Introduction to functional analysis*, volume 2 of Oxford Graduate Texts in Mathematics, The Clarendon Press Oxford University Press, New York 1997.

[4] Wengenroth, Jochen, *Derived functors in functional analysis*, Lecture Notes in Mathematics, volume 1810, Springer-Verlag, Berlin, 2003.

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