Nicolai maps for quantum cosmology

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Abstract

We construct Nicolai maps for $N = 2$ supersymmetric extensions of minisuperspace models. It is shown that Nicolai maps exist for only a very restricted set of states. In the models considered these are the two states corresponding to the empty and the filled fermion sectors. The form of the Nicolai maps in these sectors is given explicitly, and it is shown that they have a natural stochastic interpretation. This result also suggests a probabilistic interpretation of the wave function.
Introduction

One of the most problematic features of quantum cosmology is that the Hamiltonian constraint is quadratic in the momenta. Two resulting difficulties are that of interpreting the wave function, and that of ascertaining the initial conditions for the Universe.

It has recently been discovered that the minisuperspace Wheeler-DeWitt equations for certain Bianchi models admit simple supersymmetric extensions, for which the Hamiltonian operator can be represented as the anticommutator of two first-order operators [1, 2, 3]. This means that the problematic second-order Hamiltonian constraint can be replaced by two first-order constraints. The possible consequences of this result have only begun to be explored. It is conceivable that both of the problems mentioned above could be disposed of using supersymmetry.

One of the remarkable features of supersymmetric theories is that they admit Nicolai maps, which means that they can be transformed into non-interacting bosonic theories [4]. When realistic boundary conditions are imposed, however, there are limits to the applicability of this result; only certain quantum states can actually be generated by Nicolai maps [5].

In supersymmetric quantum cosmology, the wave function of the Universe consists of a number of linearly independent components, corresponding to quantum states satisfying different boundary conditions. A question of considerable current interest is the choice of the boundary condition. The Hartle-Hawking no-boundary proposal is one possible prescription [6]. Here we investigate, in the context of $N = 2$ models, the consequences of requiring the existence of a Nicolai map. We find that this requirement allows just two states; one each in the empty and filled fermion sectors. Of these two states, at most one can be normalisable. It is interesting to note that the same two states are singled out by Lorentz invariance in some
mini-superspace models with extended $N = 4$ supersymmetry obtained from $N = 1$
supergravity by dimensional reduction\cite{7,8,9}.

Because Nicolai maps are naturally interpreted as stochastic processes, this ap-
proach suggests an interpretation of the wave function (in an appropriate represen-
tation) as a probability density whose integral is conserved in Euclidean time.

**Euclidean Quantum Mechanics**

It is well-known that the ground state of a quantum mechanical system can be
found by rotating to Euclidean time $\tau = it$ and performing a functional integral
over paths whose actions vanish in the far past. This is the motivation for the
Hartle-Hawking proposal for the ground state of quantum cosmology\cite{6}.

Since the initial conditions are set in the far past, it follows that the wave function
obtained by this procedure is invariant with respect to Euclidean time translation.
In other words, this procedure ensures that the wave function is annihilated by
the Hamiltonian operator for the Euclidean theory. But it is easily seen that the
Hamiltonian operator is exactly the same in the Euclidean formulation as in the
usual Lorentz formulation, so that the wave function is a zero-energy state of the
original “Lorentzian” theory.

Although the Hamiltonians for the two theories are represented by the same
differential operator, the momenta are not. In the usual Lorentzian theory, the
momentum operators are defined as $\hat{p}_\mu = -i\hbar \partial / \partial q^\mu$ so that

$$\hat{p}_\mu \exp \frac{i}{\hbar} S = \frac{\partial S}{\partial q^\mu} \exp \frac{i}{\hbar} S. \tag{1}$$

In the Euclidean theory, however, the momentum operators are defined as $\hat{\pi}_\mu =
-i\hat{p}_\mu = -\hbar \partial / \partial q^\mu$ so that

$$\hat{\pi}_\mu \exp -\frac{1}{\hbar} S_E = \frac{\partial S_E}{\partial q^\mu} \exp -\frac{1}{\hbar} S_E \tag{2}$$

where $S_E$ is the Euclideanised action.
The Euclidean momentum operators $\hat{\pi}_\mu$ are not hermitian, but in fact this is not necessary. In standard “Lorentzian” quantum mechanics, observables must be represented by hermitian operators so that they have real expectation values. In the Euclidean case, however, all wave functions are real and so any real operator will have a real expectation value. Consequently, observables in Euclidean quantum mechanics are represented by real (rather than hermitian) operators.

The reality of the Euclidean wave function can be seen most directly from its path-integral representation,

$$\Psi(q_1, \tau_1) = \int_{q(\tau_1) = q_1} d[q(\tau)] \exp -\frac{1}{\hbar} S_E[q(\tau)]$$

where the integral is taken over all histories for which $q(t') = q'$ and which satisfy some condition in the past which specifies the state. This suggests that $\Psi$ could be interpreted simply as a probability, rather than any kind of quantum amplitude. This interpretation appears even more natural when the model is supersymmetric, since then the probabilities automatically add up to one.

**Supersymmetry and Nicolai Maps**

From our point of view, Euclidean quantum mechanics is characterised by another important feature; if the theory is supersymmetric then generally only the Euclidean version will admit a Nicolai map.

Indeed, the original proof of Nicolai’s theorem was given in Euclidean spacetime. It is sometimes possible to rotate Nicolai maps from Euclidean to Lorentzian spacetime, but this generally involves a complex rotation of the bosonic variables $q^\mu$ as well. If these variables label points in a real configuration space, such as the components of the metric on a Riemannian manifold for example, it may be unreasonable to treat these as complex variables. In such cases we must therefore stay with the Euclidean formulation if we want to construct Nicolai maps.
Nicolai maps in supersymmetric Euclidean theories are naturally interpreted as stochastic differential equations which describe the evolution of the system in Euclidean time. From a physical point of view, this evolution is of little interest in itself; however the stationary state which is finally reached is of considerable interest, since it represents the ground state of both the Euclidean and the Lorentzian theory.

It has recently been found that a number of minisuperspace formulations of quantum cosmology admit simple supersymmetric extensions. Because the action for quantum cosmology is invariant under local reparametrisations of the time coordinate, any supersymmetry in the quantum theory should be local with respect to time.

A Nicolai map for a locally supersymmetric theory can be obtained by integrating out the remaining fermionic fields, after imposing some gauge condition for all local symmetries so that there is a contribution from only one member of each class of physically equivalent configurations. In particular, in the case of locally supersymmetric quantum mechanics there are fermionic gauge fields [11] and one can choose a gauge in which these vanish everywhere. (This is possible here because the gauge field and the parameter of infinitesimal fermionic supersymmetry transformations have the same number of components in this case. Note, however, that it would not be possible to eliminate the gauge field in higher dimensional theories such as superstrings or inhomogeneous supergravity.)

To avoid introducing ghosts, it is also necessary to fix the bosonic reparametrisation symmetry. This is done most simply be requiring that $\dot{\bar{\hat{N}}} = 0$ [12], after which we find that the theory has reverted to its original globally invariant form. The gauge fields have gone and the lapse function is constant; the only supersymmetry transformations which preserve these conditions are those parametrised by constants – i.e. rigid transformations. One is thus drawn to the conclusion that, in the case
of supersymmetric quantum mechanics, the local theory is just the global theory in disguise.

Nicolai maps can therefore be obtained directly from the rigid theory, without bothering to consider the locally invariant version. In particular, this will apply to supersymmetric descriptions of homogeneous cosmologies.

**N=2 Supersymmetric Non-Linear $\sigma$-Model**

The one-dimensional N=2 supersymmetric non-linear $\sigma$-model provides a supersymmetric description of a particle moving in a curved configuration space. Originally suggested by Witten as a toy model for understanding the quantum mechanics of general supersymmetric theories, it applies very neatly to the supersymmetric models of minisuperspace quantum cosmology which have recently attracted attention.

This model is described by a Lagrangian of the form

$$L \equiv \frac{1}{2}G_{\mu\nu} \frac{dq^\mu}{d\tau} \frac{dq^\nu}{d\tau} + \frac{1}{2}G^{\mu\nu} \Phi_{\mu} \Phi_{\nu} + \frac{1}{2}(\psi^*_\mu \frac{D\psi^\mu}{d\tau} - \frac{D\psi^*_\mu}{d\tau} \psi^\mu) + \Phi_{\mu\nu} \psi^{*\mu} \psi_{\nu} + \frac{1}{4}R_{\mu\nu\rho\sigma} \psi^{*\mu} \psi^{*\nu} \psi^\rho \psi^\sigma$$

(4)

where $G_{\mu\nu}$ is the metric of the configuration space in which the particle moves, and $\Phi$ is a potential on this space. The covariant derivatives of the fermion fields are defined as

$$\frac{D\psi^\mu}{d\tau} = \frac{d\psi^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} \psi^\nu \frac{dq^\rho}{d\tau} \quad \frac{D\psi^*_\mu}{d\tau} = \frac{d\psi^*_\mu}{d\tau} - \Gamma_{\mu\rho\nu} \psi^\nu \frac{dq^\rho}{d\tau}$$

(5)

where $\Gamma^\mu_{\nu\rho}$ is the usual Christoffel connection on the configuration space. The Riemann curvature tensor is defined as

$$R_{\mu\nu\rho\sigma} \equiv G_{\mu\lambda}(\Gamma^\lambda_{\nu\sigma,\rho} - \Gamma^{\lambda}_{\nu\rho,\sigma} + \Gamma^{\lambda}_{\tau\rho} \Gamma_{\nu\sigma} - \Gamma^{\lambda}_{\tau\sigma} \Gamma_{\nu\rho})$$

(6)

This model can be quantised using either the canonical formalism or path integrals. We will briefly discuss the former. In the canonically quantised theory, observables are represented by operators which act on a multi-component wave
function. The different components of the wave-function correspond to solutions with different fermion number.

The wave function can be represented as a linear combination of differential forms on the configuration space \([14]\). In this representation, the operators corresponding to the variables \(\psi^\mu\) and \(\psi^*_\mu\) act on a differential form \(\omega\) according to the rule

\[
\hat{\psi}^\mu \omega = dq^\mu \wedge \omega, \quad \hat{\psi}^*_\mu \omega = -\hbar i_\mu \omega
\]

where \(i_\mu \omega\) denotes the interior product of \(\omega\) with the tangent vector \(\frac{\partial}{\partial q^\mu}\).

The supersymmetry generators can be obtained from the action by the Noether procedure. With the form of the Lagrangian given above, the supersymmetry generators are represented by the operators

\[
\hat{Q}_- = -\hbar e^{\Phi/\bar{h}} d e^{-\Phi/\bar{h}}, \quad \hat{Q}_+ = \hbar e^{-\Phi/\bar{h}} \delta e^{\Phi/\bar{h}}
\]

where \(d\) is the usual exterior derivative, and \(\delta\) is its adjoint with respect to the standard inner product for differential forms, \(\langle \omega | \eta \rangle = \int \omega \wedge \eta\).

It will be useful to consider how these operators are affected by a canonical transformation of the classical theory. In particular, if the Lagrangian \(L\) is augmented or diminished by the \(\tau\)-derivative of \(\Phi + \frac{1}{2} \psi^*_\mu \psi^\mu\),

\[
L \mapsto L_\pm = L \pm \frac{d}{d\tau}(\Phi + \frac{1}{2} \psi^*_\mu \psi^\mu)
\]

then we obtain new representations of the quantum theory. These particular choices of the boundary action are needed below (eq(19)) in order to ensure the existence of Nicolai maps \([4]\).

The wave functions \(\omega_\pm\) in the new representations are related to the wave function \(\omega\) of the original representation by the transformation

\[
\omega \mapsto \omega_\pm = e^{\mp \Phi/\bar{h} \tau} \omega.
\]
For the sake of notational simplicity, we have omitted the operator $\frac{1}{2} \hat{\psi}_\mu \hat{\psi}^\mu / \hbar$ from the exponent. It is easily seen that this omission affects only the relative normalisations of the different components of the wave function.

The supersymmetry generators transform in a corresponding manner. Going to the “+” representation, one has

$$\hat{Q} \mapsto \hat{Q}_+ = e^{-\Phi / \hbar} \hat{Q} e^{\Phi / \hbar} = -\hbar d, \quad (11)$$

$$\hat{Q}^* \mapsto \hat{Q}^*_+ = e^{-\Phi / \hbar} \hat{Q}^* e^{\Phi / \hbar} = \hbar e^{-2\Phi / \hbar} \delta e^{2\Phi / \hbar}, \quad (12)$$

while going to the “−” representation gives

$$\hat{Q} \mapsto \hat{Q}_- = e^{+\Phi / \hbar} \hat{Q} e^{-\Phi / \hbar} = -\hbar e^{+2\Phi / \hbar} d e^{-2\Phi / \hbar}, \quad (13)$$

$$\hat{Q}^* \mapsto \hat{Q}^*_− = e^{+\Phi / \hbar} \hat{Q}^* e^{-\Phi / \hbar} = \hbar \delta. \quad (14)$$

Whichever representation is used, the Hamiltonian operator is related to the supersymmetry generators by the anticommutation relation

$$\hat{Q}_± \hat{Q}^*_± + \hat{Q}^*_± \hat{Q}_± = -2 \hat{H}_± \quad (15)$$

and the Euclidean-time evolution of the wave function is governed by the Schrödinger equation

$$-\hbar \frac{\partial \omega_±}{\partial \tau} = \hat{H}_± \omega_±. \quad (16)$$

**Initial Conditions for Nicolai Maps.**

In the path-integral quantisation of the non-linear $\sigma$-model, every Euclidean history of the variables $q, \psi, \psi^*$ is assigned a definite weight. A purely bosonic theory can be obtained by integrating out the fermionic variables $\psi, \psi^*$. This integration yields a factor $J$ which, in general, depends on the configuration of the bosonic variables $q^\mu(\tau)$.

As shown by Nicolai, the supersymmetry of the underlying model ensures that the factor $J$ may be interpreted as the Jacobian of a transformation from a set
of non-interacting variables $\xi^a(\tau)$ to the variables $q^\mu(\tau)$ appearing in the bosonic functional integral; this transformation is known as a Nicolai map. In the case of Euclidean supersymmetric quantum mechanics, the Nicolai map takes the form of a first-order stochastic differential equation, in which the variables $\xi^a$ can be interpreted as uncorrelated Gaussian noise [15].

To ensure that the Nicolai map is one-to-one, it is necessary to impose some kind of boundary conditions on the bosonic variables $q^\mu$; for a stochastic interpretation, the most natural choice is to specify the values of these variables at some initial time $\tau_0$.

However, Nicolai's theorem will not work unless the boundary conditions are invariant under either one or other of the supersymmetry generators. To ensure the existence of a Nicolai map, it is therefore necessary to supplement the initial conditions on $q^\mu$ with Dirichlet initial conditions on either the variables $\psi^\mu$ or else their conjugates $\psi^*_\mu$ at $\tau_0$ [5]. Note that different choices of initial conditions will result in different Nicolai maps.

It is useful to interpret these initial conditions in terms of the operators of the canonical formalism. Setting the initial values of $q^\mu$ means that, at time $\tau_0$, the wave function $\omega$ is in an eigenstate of the operator $\hat{q}^\mu(\tau_0)$. Hence $\omega(\tau_0)$ vanishes except at the specified value of $q^\mu$.

We have a choice of fermionic initial conditions. Imposing Dirichlet conditions on $\psi^\mu$ means the wave function $\omega$ must satisfy

$$0 = \hat{\psi}^\mu \omega = dq^\mu \wedge \omega \quad \text{at} \quad \tau_0 \quad \mu = 1, \ldots, n$$

and consequently $\omega$ must be purely of degree $n$ at time $\tau_0$, where $n$ is the dimensionality of the configuration space. It turns out that the Hamiltonian operator commutes with the fermion number operator $\hat{\psi}^\mu \hat{\psi}^*_\mu$ which yields the degree of the form, and so $\omega$ remains an $n$-form for all $\tau$. 
The alternative choice is to impose Dirichlet initial conditions on the variables $\psi^*_\mu$ at $\tau_0$. When the wave function is represented by differential forms, these conditions are written

$$0 = \widehat{\psi}^*_\mu \omega = -\hbar i_\mu \omega \quad \text{at} \quad \tau_0 \quad \mu = 1, \ldots n. \quad (18)$$

In this case, therefore, $\omega$ must be a pure 0-form at time $\tau_0$. Again, this property is preserved for all time $\tau$ since the Hamiltonian respects fermion number.

The initial conditions necessary for the existence of a Nicolai map therefore require that all but two of the components of the wave function vanish identically. In other words, these two components alone can be generated from Nicolai maps.

**Construction of Nicolai Maps for Quantum Cosmology.**

Nicolai maps have been constructed for the supersymmetric non-linear $\sigma$-models described above, and quantum cosmological models can be treated as a special case of these. As remarked earlier, it suffices to consider globally supersymmetric descriptions.

The first step in the construction of a Nicolai map is to obtain a purely bosonic theory by integration of the fermions subject to the appropriate initial conditions. It is important here to use a form of the action which is invariant under the same supersymmetry transformations as the initial conditions; if the initial conditions (17) are imposed on $\psi^\mu$, then the $L_+$ form of the Lagrangian must be used, while if the conditions (18) are imposed then $L_-$ must be used. Note that the Lagrangian must have one of the forms specified in (9) if there is to exist a Nicolai map [4].

Integration of the fermions then yields the following expression for the weight of a path $q^\mu(\tau)$ in configuration space;

$$P[q(\tau)] = \int d[\psi^*, \psi] \exp -\frac{1}{\hbar} \left\{ \int_{\tau_0}^\infty (L_{\pm}[q, \psi^*, \psi]) \right\}$$

$$= J[q] \exp -\frac{1}{\hbar} \left\{ \int_{\tau_0}^\infty \frac{1}{2} \left( \frac{dq^\mu}{d\tau} \pm G^{\mu\nu} \Phi_\nu \right)^2 d\tau \right\} \quad (20)$$
where the functional \( J[q] \) depends on the Ricci curvature scalar of the configuration space according to the rule \[16\]

\[
J[q] = \exp \pm \frac{1}{2\hbar} \int_{t_0}^{\infty} (G^{\mu\nu} \Phi_{,\mu\nu} - \frac{1}{4} R) d\tau. \tag{21}
\]

In fact the functional \( J[q] \) is the Jacobian for the transformation \( \xi^a(\tau) \mapsto q^\mu(\tau) \) defined by the differential equation

\[
\frac{\Delta q^\mu}{d\tau} = \mp \Gamma^\mu_{\nu}(q) \Phi_{,\nu} + e^\mu_a(q) \cdot \xi^a \tag{22}
\]

where \( e^i_a(q) \) is a vielbein field on the configuration space with the property that

\[
G_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} = \text{constant}. \tag{23}
\]

The differentials \( \Delta q^\mu = dq^\mu + g^{\mu\rho} \Gamma^\rho_{\nu\rho} d\tau \) and \( e^\mu_a(q) \cdot \xi^a d\tau \) are defined so that they transform covariantly in the Itô calculus \[15\]. It follows by a change of variable that the weight for a given history \( \xi^a(\tau) \) of the new variable is

\[
\mathcal{P}[\xi(\tau)] = \exp -\frac{1}{2\hbar} \left\{ \int_{t_0}^{\infty} \eta_{ab} \xi^a \xi^b d\tau \right\} \tag{24}
\]

so that \( \xi^a(\tau) \), for positive definite \( \eta_{ab} \), can be interpreted as an uncorrelated Gaussian noise process.

At this point we must acknowledge a conceptual difficulty arising from this result; namely, that the minisuperspace metric \( G_{\mu\nu} \) and consequently the “noise” metric \( \eta_{ab} \) both have a Lorentzian signature due to the fact that the scale factor for the Universe is always a time-like variable. Consequently, expression \[24\] for the weight of the path \( \xi(\tau) \) is unbounded and can only be integrated by first rotating \( \xi^0 \) to the imaginary axis. In eq.(24) this rotation turns \( \xi^0 \) into noise with a positive intensity (like the other components) and changes \( G^{\mu\nu} \), which can be represented as

\[
G^{\mu\nu} = e^\mu_a e^\nu_b (\xi^a \xi^b) d\tau,
\]

to

\[
\tilde{G}^{\mu\nu} = e^\mu_a e^\nu_b \delta^{ab}.
\]
In fact this approach provides a definite prescription for dealing with the problem of unbounded action, which arises in different guises in all attempts to quantise Einstein gravity.

The stochastic differential equation (22) is the Nicolai map for the model, subject to the corresponding set of initial conditions (17) or (18). After rotating to imaginary \(\xi^0\), this is a Langevin equation for the particle whose motion in minisuperspace represents the Euclidean time evolution of the Universe. The associated Fokker-Planck equation takes the form

\[
\frac{\partial P}{\partial \tau} = \frac{1}{2} [\tilde{G}^{\mu\nu}(\bar{\hbar} P_\mu \pm 2\Phi_\mu P)]_\mu
\]  

(25)

where \(P\) is positive. Moreover, the integral of \(P(q, \tau)\) over the minisuperspace is conserved with respect to Euclidean time. These features justify referring to \(P\) as the probability distribution function for the Universe.

Eq. 25 can also be derived directly from the canonically quantised theory described earlier. Recall that initial conditions on \(\psi^\mu\) imply that the wave function is a \(n\)-form for all \(\tau\), and consequently \(\hat{Q}_+ \omega_+\) vanishes in the “+” representation. It follows from (16) that

\[
\frac{\partial \omega_+}{\partial \tau} = \frac{1}{2\hbar} \hat{Q}_+ \hat{Q}_+^* \omega_+ = -\frac{1}{2} \delta d(\delta \omega_+ + 2i A \omega_+)
\]  

(26)

where \(i A \omega_+\) stands for the interior product of the \(n\)-form \(\omega_+\) with the tangent vector \(G^{\mu\nu} \Phi_\nu \frac{\partial}{\partial q^\mu}\). The Hodge star of \(\omega_+\), denoted \(* \omega_+\), is then a pure scalar and satisfies

\[
\frac{\partial (* \omega_+)}{\partial \tau} = -\frac{1}{2} \delta [\hbar d(* \omega_+) + 2d \Phi \wedge (* \omega_+)]
\]  

(27)

Alternatively if one chooses the initial condition on \(\psi_\mu^*\) then, using the “−” representation, we know that \(\hat{Q}_- \omega_-\) vanishes for all \(\tau\) and therefore

\[
\frac{\partial \omega_-}{\partial \tau} = \frac{1}{2\hbar} \hat{Q}_-^* \hat{Q}_- \omega_- = -\frac{1}{2} \delta [\hbar d \omega_- - 2d \Phi \wedge \omega_-]
\]  

(28)

Identifying \(P\) with either \(\omega_-\) or \(* \omega_-\) then reproduces eq. (25), provided that \(G^{\mu\nu}\) and \(\delta\) are analytically continued according to the prescription defined earlier.
It is important to note that the wave function only satisfies a conservation equation such as (27) or (28) if we use the $\omega_+$ or $\omega_-$ representation; there is no conservation equation in the canonically-related $\omega$ representation.

We recall that the original motive for considering the Euclidean version of the theory was to enable us to find the ground state for quantum cosmology, which is given by the $\tau \rightarrow \infty$ limit. As $\tilde{G}^{\mu\nu}$ is positive definite, $P(q, \tau)$ approaches a stationary distribution of the form

$$P_0(q) = A \exp \pm 2\Phi(q)/\hbar.$$  

provided that this distribution is normalisable with the invariant measure

$$\left(\text{Det}(\tilde{G}_{\mu\nu})\right)^{1/2} dq^n = |\text{Det}(G_{\mu\nu})|^{1/2} dq^n.$$

Note that the distribution $P_0(q)$ is a static solution of eq.(25) even if we use the original minisuperspace metric $G^{\mu\nu}$ instead of its positive-definite analytic continuation $\tilde{G}^{\mu\nu}$.

Let us close by giving two examples of static solutions by considering the case of Bianchi type IX. It turns out that there are in this case two different supersymmetric Lagrangians (4) compatible with the same Lagrangian in the classical limit [17]. The two superpotentials are

$$\Phi = \frac{1}{6} \left( e^{\beta_1} + e^{\beta_2} + e^{\beta_3} \right) - 2\hbar(\beta_1 + \beta_2 + \beta_3)$$  

and

$$\tilde{\Phi} = \frac{1}{6} \left( e^{\beta_1} + e^{\beta_2} + e^{\beta_3} - 2e^{\beta_1+\beta_2} - 2e^{\beta_2+\beta_3} - 2e^{\beta_3+\beta_1} \right) - 2\hbar(\beta_1 + \beta_2 + \beta_3)$$

where the variables $\beta^i$ parametrise the metric by

$$ds^2 = -N(t)^2 + \frac{1}{6\pi} \sum_{i=1}^{3} e^{2\beta^i} \omega^i \omega^i$$

with the lapse function $N(t)$ and the basis 1-forms $\omega^i$ satisfying

$$d\omega^1 = \omega^2 \wedge \omega^2$$  

and cyclic permutations.
The terms proportional to $\hbar$ in eqs (30),(31) are quantum corrections whose given explicit form is obtained only if the larger $N = 4$ supersymmetry suggested by dimensional reduction of $N = 1$ supergravity is imposed [8]. In the present context of $N = 2$ supersymmetry the form of these terms could be left arbitrary, reflecting an operator-ordering ambiguity. It is interesting, however, that with $\Phi$ as given, the probability distribution $\exp -2\Phi/\hbar$ is normalisable. This state can be interpreted as a wormhole state.

The distribution $\exp -2\Phi/\hbar$, on the other hand, is not normalisable. However this problem is not as serious as it first seems. Defining a scale factor

$$a = \exp \left[ (\beta^1 + \beta^2 + \beta^3)/3 \right]$$

we see that, with $\beta^1 - \beta^2$ and $\beta^1 + \beta^2 - 2\beta^3$ fixed, the normalisation integral converges as $a \to 0$ when $\Phi$ has the form given in (29). Although the integrand diverges as $a \to \infty$, this is not a serious concern as the role played by matter fields (which we have neglected) becomes important for large $a$ and eqs (29),(31) can no longer be expected to provide a good approximation to the true wave function.

Interestingly, the probability distributions given above are precisely the same as those obtained recently by Bene and Graham [17] using a different and more conventional approach, in which the wave function is interpreted as the square root of the probability density.

**Discussion and Conclusion**

We have shown that the existence of an N=2 supersymmetric extension of the minisuperspace Wheeler-DeWitt equations suggests a stochastic interpretation of the resulting solutions. It appears likely that a stochastic interpretation of a similar kind will also apply to more general cosmological models with various types of extended supersymmetries (including full supergravity) and work to establish this is in progress. In such cases the existence of Nicolai maps will be assured only if
the initial conditions we impose are invariant under some non-trivial subalgebra of the full supersymmetry algebra \[\mathfrak{g}\]. Assuming that this subalgebra includes the generators of Lorentz transformations, it follows that boundary conditions for Nicolai maps should be Lorentz invariant. In particular, this means that Dirichlet initial conditions must be imposed on all the components of either the left-handed or the right-handed spinors (though not on both, or the path integral would be overdetermined). If chirality is preserved by the equations of motion, these same components will vanish throughout the subsequent evolution. We can therefore surmise that, in such cases, Nicolai maps will exist only for states in which spinors of a certain chirality vanish identically.

For example, let us consider the case of pure $N = 1$ supergravity without a cosmological constant. (Our analysis will also apply to minisuperspace models with $N = 4$ supersymmetry obtained from this theory by dimensional reduction). In this case, the chirality of the spinor fields is preserved by their equations of motion. Consequently, if we impose Dirichlet initial conditions on the left-handed spinors $\overline{\psi}^{A'}$ in order to obtain a state admitting a Nicolai map, then the $\overline{\psi}^{A'}$ will vanish at all subsequent times. The wave-function will then depend only on the right-handed spinors $\psi^A$ and the bosonic variables. However, the momenta conjugate to the $\psi^A$ are linear combinations of the $\overline{\psi}^{A'}$, which vanish identically. It follows that the wave function must in fact be independent of the $\overline{\psi}^{A'}$, and therefore is simply a function of the bosonic variables.

We therefore conclude that any boundary conditions which permit Nicolai maps in this model will give rise to states in one of the bosonic sectors. (There are two bosonic sectors, depending on whether we choose $\psi^A$ or $\overline{\psi}^{A'}$ as the canonical coordinates. These correspond to the empty and filled fermion states discussed earlier.) Interestingly, these are precisely the sectors which are found to admit physical solutions in the Bianchi I case considered by D’Eath, Hawking and Obrégon \[\text{I}\] and the
Bianchi IX case \([3, 8]\) as well as the more general Bianchi class A models studied by Asano, Tanimoto and Yoshino \([18]\).

Note that the above argument doesn’t work if we include matter or a cosmological constant, since then the chirality of spinors is not preserved by their equations of motions. In these cases, therefore, we would *not* expect states in the bosonic sectors to admit Nicolai maps. But in the models of this kind which have been studied so far \([3, 17]\), it turns out that there are in fact no physical states in these sectors!

In view of these connections, it is tempting to speculate that the physical states in supersymmetric quantum cosmological models are precisely those which admit Nicolai maps. This would lend considerable support to the stochastic interpretation of the wave function outlined above. However, much further work is needed to clarify this question.

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