Research Article

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Critical Concave Convex Ambrosetti–Prodi Type Problems for Fractional $p$-Laplacian

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Abstract: In this paper, we consider a class of critical concave convex Ambrosetti–Prodi type problems involving the fractional $p$-Laplacian operator. By applying the linking theorem and the mountain pass theorem as well, the interaction of the nonlinearities with the first eigenvalue of the fractional $p$-Laplacian will be used to prove existence of multiple solutions.

Keywords: Variational Methods, Fractional Equations, Nonlinear Elliptic Equations, Quasilinear Equations, Critical Exponents

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $s \in (0, 1)$ and $N > sp$. In this paper, we investigate the existence of multiple solutions for the nonlocal problem

$$\begin{cases} (-\Delta)_p^s u = -\lambda |u|^{q-2} u + a |u|^{p-2} u + b |u^+|^{p^*_s-2} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $(-\Delta)_p^s u(x)$ is the fractional $p$-Laplacian operator defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x, \varepsilon)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N.$$ 

In (1.1), $\lambda > 0$ is a parameter, $a, b > 0$ are real constants, $1 < q < p$, $p^*_s = \frac{pN}{N-sp}$ is the critical Sobolev exponent for $(-\Delta)_p^s$, and $u^+ := \max\{0, u\}$ denotes the positive part of $u$, while the negative part of $u$ will be denoted $u^- = \min\{0, u\}$. Consequently, $u = u^+ + u^-$. The term $(u^+)$ appears for the first time in the classical paper by Ruf and Srikanth [34]. But our problem is also related to two classical local problems, namely, the Ambrosetti–Prodi and the Brezis–Nirenberg problems. On these subjects, see the excellent books [22] and [39], respectively.

In 1972, Ambrosetti and Prodi [4] considered the Dirichlet boundary value problem

$$-\Delta u = g(u) + f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$

(AP)
where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(-\Delta\) denotes the Laplacian operator, \(f : \mathbb{R}^N \to \mathbb{R}\) is a \(C^1\) function, \(g : \mathbb{R} \to \mathbb{R}\) is \(C^2\), convex and satisfies
\[
0 < g_- = \lim_{t \to -\infty} g'(t) < \lambda_1 < g_+ = \lim_{t \to +\infty} g'(t) < \lambda_2,
\]
with \(\lambda_1\) and \(\lambda_2\) denoting the first and second eigenvalues of \((-\Delta, H^1_0(\Omega))\). They proved the existence of a \(C^1\) manifold \(M\) in \(C^{0,\alpha}(\overline{\Omega})\), which splits the space into two open sets \(O_0\) and \(O_2\) with the following properties:
(i) if \(f \in O_0\), problem (AP) has no solution;
(ii) if \(f \in M\), problem (AP) has exactly one solution;
(iii) if \(f \in O_2\), problem (AP) has exactly two solutions.
Many authors have extended this result in different ways, and we would like to apologize if we omit some important contributions, but we cite, e.g., the papers \([2, 6, 8, 9, 14, 18, 19, 27, 34, 37]\) and references therein. All these results show the role of the interaction between \(g\) and the eigenvalues of \((-\Delta, H^1_0(\Omega))\).

On the other hand, in 1983, Brezis and Nirenberg \([11]\) studied the Dirichlet boundary problem
\[
-\Delta u = au + |u|^{2^* - 2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]
where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(a > 0\) and \(2^* = \frac{2N}{N-2} (N \geq 3)\) is the Sobolev critical exponent. As before, denoting by \(\lambda_j (j = 1, 2, \ldots)\) the eigenvalues of \(-\Delta\), the authors proved that there exists \(a_0 > 0\) such that
(i) if \(a_0 < a < \lambda_1\), \(N = 3\) and \(\Omega = B_1(0)\), problem (BN) has at least one positive solution;
(ii) if \(a < \lambda_1\) and \(N \geq 4\), problem (BN) has at least one positive solution.
Among many works extending or complementing the above result for both local and nonlocal operators, we mention, e.g., \([3, 4, 7, 10, 14, 16, 28, 30, 32, 36]\). But we would like to highlight Capozzi, Fortunato and Palmieri \([15]\), where the authors proved that problem (BN) has at least one nontrivial solution for all \(a > 0\) if \(N \geq 5\) and for all \(a \neq \lambda_j\) if \(N = 4\).

In the interesting work \([21]\), de Paiva and Presoto established a multiplicity result for the Dirichlet boundary problem
\[
-\Delta u = -\lambda |u|^{q-2}u + au + (u^r)^{p-1} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]
where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(a, \lambda > 0\) and \(1 < q < 2 < p \leq 2^*\). (See also \([20]\) for the subcritical case and \([24, 31]\) for related problems.) The main goal of the present paper is to prove the result obtained in \([21]\) for the fractional \(p\)-Laplacian operator extending the results obtained in \([13, 29]\).

### 2 Notations and Preliminary Stuff

For any measurable function \(u : \mathbb{R}^N \to \mathbb{R}\) the Gagliardo seminorm is defined by
\[
[u]_{s,p} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
We consider the fractional Sobolev space
\[
W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \}
\]
edu with the norm \(\|u\|_{W^{s,p}} = ([u]_{L^p}^p + [u]_{s,p}^p)^{\frac{1}{p}}\). Since solutions should be equal to zero outside of \(\Omega\), it is natural to consider the closed linear subspace given by
\[
X^{s}_p = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}
\]
equivalently renormed by setting \(\| \cdot \|_{X^s_p} = ([ \cdot ]_{s,p})^{\frac{1}{p}}\). The embedding \(X^s_p \hookrightarrow L^r(\Omega)\) is continuous for \(r \in [1, p^{*}_s]\) and compact for \(r \in [1, p^{*}_s)\).

We define, for all \(u, v \in X^s_p\), the operator \(A : X^s_p \to (X^s_p)^*\) by
\[
A(u) \cdot v = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, dx \, dy.
\]
Definition 2.1. We say that \( u \in X^s_p \) is a weak solution to (1.1) if
\[
A(u) \cdot v = -\lambda \int_{\Omega} |u|^{q-2} uv \, dx + a \int_{\Omega} |u|^{p-2} uv \, dx + b \int_{\Omega} (u^+)^{p'_s-1} v \, dx \quad \text{for all } v \in X^s_p.
\]

Since the action functional \( I_{\lambda,s} : X^s_p \to \mathbb{R} \) is given by
\[
I_{\lambda,s}(u) = \frac{1}{p} \|u\|^p_{X^s_p} + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \frac{a}{p} \int_{\Omega} |u|^p \, dx - \frac{b}{p'_s} \int_{\Omega} (u^+)^{p'_s} \, dx,
\]
we have
\[
I'_{\lambda,s}(u) \cdot v = A(u) \cdot v + \lambda \int_{\Omega} |u|^{q-2} uv \, dx - a \int_{\Omega} |u|^{p-2} uv \, dx - b \int_{\Omega} (u^+)^{p'_s-1} v \, dx,
\]
thus implying that critical points of \( I_{\lambda,s} \) are weak solutions of (1.1).

In the local problem studied by de Paiva and Presoto [21], the driving operator used is the standard Laplacian on the Sobolev space \( H^1_0(\Omega) \). In our work, in order to obtain two solutions of opposite constant sign for problem (1.1) as in [21], we apply the mountain pass theorem to the positive and negative parts of the functional \( I_{\lambda,s} \). In view of the essential differences in the functional setting \( (H^1_0(\Omega) \) versus \( X^s_p \) and \( \Delta \) versus \( (-\Delta)^s \)), it was necessary to obtain a result (see Theorem 2) relating the local minimizers in different spaces \( (C_0^0(\Omega) \) versus \( X^s_p \)) to obtain the geometrical conditions of the mountain pass, whose proof was inspired by the works [12] and [35] for semilinear and quasilinear problems, respectively. (The space \( C_0^0(\Omega) \) will be defined in Section 3).

A third solution to problem (1.1) was obtained via the linking theorem (see [33]) adapting arguments found in Miyagaki, Motreanu and Pereira [29], de Paiva and Presoto [21], but mainly in de Figueiredo and Yang [19]. The presence of the fractional \( p \)-Laplacian makes the application of approximate eigenfunctions impossible, which were used in [21]. Furthermore, since an explicit formula for minimizers of the best constant of the Sobolev immersion \( X^s_p \to L^{p'_s}(\Omega) \) is not available, other difficulties arise. Partial solutions were obtained by Chen, Mosconi and Squassina [17], Mosconi, Perera, Squassina and Yang [30] and also by Brasco, Mosconi and Squassina [10].

In order to obtain the geometric conditions of the linking theorem, we define
\[
\lambda^* = \inf \{ \|u\|^p_{X^s_p} : u \in W, \|u\|^p_{L^p(\Omega)} = 1 \}, \quad \text{where} \quad W = \{ u \in X^s_p : A(\varphi_1) \cdot u = 0 \},
\]
with \( \varphi_1 \) the first eigenfunction of \( (-\Delta)^s \), positive and \( L^p \)-normalized associated with the first eigenvalue \( \lambda_1 \).

Following ideas of Alves, Carrião and Miyagaki [1] and Anane and Tsouli [5] (see also [15]), it is not difficult to obtain the next result; see [13] for details.

Proposition 2.2. \( \lambda_1 < \lambda^* \).

We are now in a position to establish the main result of this paper.

Theorem 1. Suppose \( \lambda_1 < a < \lambda^* \), \( b > 0 \), \( 1 < q < p \), and assume that one of the following conditions holds:

(i) \( N > sp^2 \) and \( 1 < p \leq \frac{2N}{N+2} \),

(ii) \( N > sp((p-1)^2 + p) \) and \( p > \frac{2N}{N+2} \).

Then problem (1.1) has at least three nontrivial solutions if \( \lambda > 0 \) is small enough.

Remark 2.3. In Theorem 1, we consider only two of the six possibilities below,

(a) \( 1 < p \leq \frac{2N}{N+2} \), and

(1) \( N > sp^2 \),

(2) \( N = sp^2 \),

(3) \( sp < N < sp^2 \),

(b) \( [p > \frac{2N}{N+2}, N > sp((p-1)^2 + p)] \) and

(1) \( N > sp^2 \),

(2) \( N = sp^2 \),

(3) \( sp < N < sp^2 \),

since it is not difficult to verify that the situations (a) (2), (a) (3), (b) (2) and (b) (3) are incompatible.
3 \textbf{C}^0 \text{ versus } W^{s,p} \text{ Minimization for Polynomial Growth}

The main result of this section is a local minimization equivalent for functionals defined in the fractional Sobolev space \( X^s_p \) with polynomial growth nonlinearity, following ideas developed by Barrios, Colorado, de Pablo and Sánchez [7], Giacomoni, Prashanth and Sreenadh [23] and Iannizzotto, Mosconi and Squassina [25]. The result we prove is more general than those found in [7] and [25] since we allow \( p > 1 \).

We start showing a regularization result that will be useful in the proof of Theorem 2. Its proof is similar to that of [13, Lemma 3.1].

\textbf{Proposition 3.1.} Suppose \(|g(t)| \leq C(1 + |t|^{q-1})\) for some \( 1 \leq q \leq p^*_s \) and \( C > 0 \). Let \((v_\epsilon)_{\epsilon(0,1)} \subseteq X^s_p \) be a bounded family of solutions in \( X^s_p \) to the problem

\[
\begin{cases}
(-\Delta)^s u = \left(\frac{1}{1 - \xi_\epsilon}\right) g(u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

with \( \xi_\epsilon \leq 0 \). Then \( \sup_{\epsilon(0,1)} \| v_\epsilon \|_{L^\infty(\Omega)} < \infty \).

**Proof.** For \( 0 < k \in \mathbb{N} \), we define

\[ T_k(s) = \begin{cases}
s + k & \text{if } s \leq -k, \\
0 & \text{if } -k < s < k, \\
s - k & \text{if } s \geq k,
\end{cases} \]

\[ \Omega_k = \{ x \in \Omega : |v_\epsilon(x)| > k \}. \]

Observe that \( T_k(v_\epsilon) \in X^s_p \) and \( \| T_k(v_\epsilon) \|_{X^s_p} \leq C \| v_\epsilon \|_{X^s_p} < \infty \) for a constant \( C > 0 \).

Taking \( T_k(v_\epsilon) \) as a test-function, we obtain

\[
A(v_\epsilon) \cdot T_k(v_\epsilon) = \int_{\Omega} \left(\frac{1}{1 - \xi_\epsilon}\right) g(v_\epsilon) T_k(v_\epsilon) \, dx \leq \int_{\Omega} C \| T_k(v_\epsilon) \| \, dx + C \int_{\Omega} \| v_\epsilon \|_{X^s_p}^{p-1} |T_k(v_\epsilon)| \, dx.
\]

Now consider \( 1 < \theta_1 < \theta_2 < p, p < \frac{\theta_2}{\theta_1} + 1 \) and \( (p^*_s - 1) \theta_1 < p^*_s \) such that \( \theta_1^{-1} + \theta_2^{-1} + \theta_3^{-1} = 1 \). Therefore, by applying Hölder’s inequality, we obtain

\[
A(v_\epsilon) \cdot T_k(v_\epsilon) \leq C \left( \int_{\Omega} \| T_k(v_\epsilon) \|^{\theta_2} \, dx \right)^{\frac{1}{\theta_2}} \| \Omega_k \|^{\frac{1}{\theta_3}}.
\]

Denote

\[ T(x, y) = \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2} (v_\epsilon(x) - v_\epsilon(y)) (T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y))}{|x - y|^{N+sp}}. \]

Noting that the inequality

\[ |s - t|^{p-2} (s - t)(T_k(s) - T_k(t)) \geq |T_k(s) - T_k(t)|^p \]

holds since both \( T_k(s) \) and \( s - T_k(s) \) are non-decreasing functions, we obtain

\[ T(x, y) \geq \frac{|T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y)|^p}{|x - y|^{N+sp}}. \]

Therefore, we have the estimate

\[
A(v_\epsilon) \cdot T_k(v_\epsilon) \geq \int_{\mathbb{R}^N} \frac{|T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \| T_k(v_\epsilon) \|_{X^s_p}^p.
\]

It follows from the continuous immersion \( X^s_p \hookrightarrow L^{\theta_3}(\Omega) \) that (for a constant \( C_1 > 0 \))

\[
C_1 \left( \int_{\Omega} \| T_k(v_\epsilon) \|^{\theta_2} \, dx \right)^{\frac{1}{\theta_2}} \leq A(v_\epsilon) \cdot T_k(v_\epsilon).
\]
So (3.2) and (3.3) guarantee the existence of $C > 0$ such that
\[
\int_{\Omega} |T_k(v_\varepsilon)|^{\beta_2} \, dx \leq C |\Omega_k|^{\frac{\beta_2}{\beta_1}} = C |\Omega_k|^{\frac{\beta}{\beta_1}},
\]
where $\beta = \frac{\theta}{m - \theta (p - 1)} > 1$, the last inequality being a consequence of $p < \frac{\theta}{\theta_1} + 1$.

Since, for all $s \in \mathbb{R}$, we have $|T_\varepsilon(s)| = (|s| - k)(1 - \chi_{[-k, k]})$, we conclude that, if $0 < k < h \in \mathbb{N}$, then $\Omega_k \subset \Omega_h$. Therefore,
\[
\int_{\Omega} |T_k(v_\varepsilon)|^{\beta_2} \, dx = \int_{\Omega_k} (|v_\varepsilon| - k)^{\beta_2} \, dx \geq \int_{\Omega_h} (|v_\varepsilon| - k)^{\beta_2} \, dx \geq (h - k)^{\beta_2} |\Omega_h|.
\]

Defining, for $0 < k \in \mathbb{N}$, $\phi(k) = |\Omega_k|$, it follows
\[
\phi(h) \leq C(h - k)^{-\beta_1} \phi(k)^{\frac{1}{\beta_1}}, \quad 0 < k < h \in \mathbb{N}.
\]

Considering the sequence $(k_n)$ defined by $k_0 = 0$ and $k_n = k_{n-1} + \frac{d}{\sqrt{n}}$, where $d = 2^{\beta} C^{\frac{1}{\beta_1}} |\Omega|^{\frac{1}{\beta_1 - 1}}$, we have $0 \leq \phi(k_n) \leq \phi(0)^{\left(d |\Omega|^{-\frac{1}{\beta_1}}\right)}$ for all $n \in \mathbb{N}$. Thus, $\lim_{n \to \infty} \phi(k_n) = 0$. Since $\phi(k_n) \geq \phi(d)$ implies $\phi(d) = 0$, we have $|v_\varepsilon(x)| \leq d$ a.e. in $\Omega$ for all $\varepsilon \in (0, 1)$. We are done. \qed

We recall the definitions of the spaces $C_0^0(\overline{\Omega})$ and $C_0^0(\overline{\Omega})$. Let $\delta : \overline{\Omega} \to \mathbb{R}^+$ be given by $\delta(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$. Then, for $0 < \alpha < 1$, we have
\[
C_0^0(\overline{\Omega}) = \left\{ u \in C_0^0(\Omega) : \frac{u}{\delta^\alpha} \text{ has a continuous extension to } \overline{\Omega} \right\},
\]
\[
C_0^0, \alpha(\overline{\Omega}) = \left\{ u \in C_0^0(\overline{\Omega}) : \frac{u}{\delta^\alpha} \text{ has a } \alpha\text{-Hölder extension to } \overline{\Omega} \right\}
\]
with the respective norms
\[
\|u\|_{0, \alpha, \delta} = \left\| \frac{u}{\delta^\alpha} \right\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u\|_{\alpha, \delta} = \|u\|_{0, \delta} + \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.
\]

Let us consider the Dirichlet problem
\[
\begin{cases}
(-\Delta)^\frac{s}{2} u = f(u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{3.4}
\]
where $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded, smooth domain, $s \in (0, 1)$, $p > 1$ and $f \in L^\infty(\Omega)$.

The next two results can be found in Iannizzotto, Mosconi and Squassina [26, Theorems 1.1 and 4.4], respectively. They will play a major role in the proof of Theorem 2.

**Proposition 3.2.** There exist $a \in (0, s)$ and $C_\Omega > 0$ depending only on $N, p, s$, with $C_\Omega$ also depending on $\Omega$, such that, for all weak solution $u \in X^0_p$ of (3.4), $u \in C^a(\overline{\Omega})$ and
\[
\|u\|_{C^a(\overline{\Omega})} \leq C_\Omega \|u\|_{L^\infty(\Omega)}^{\frac{1}{2} \delta^a}.
\]

**Proposition 3.3.** Let $u \in X^0_p$ satisfy $|(-\Delta)^{\frac{s}{2}} u| \leq K$ weakly in $\Omega$ for some $K > 0$. Then $|u| \leq (C_\Omega K)^{\frac{1}{2} \delta^a}$ a.e. in $\Omega$ for some $C_\Omega = C(N, p, s, \Omega)$.

The proof of the next result is similar to that of [13, Theorem 1]. We emphasize that, only in this section, $\delta$ represents the function defined by $\delta(x) = \text{dist}(x, \partial \Omega)$.

**Theorem 2.** Suppose that $g \in C(\Omega)$ satisfies
\[
|g(t)| \leq C(1 + |t|^{q - 1}) \quad \text{for some } 1 \leq q \leq p_1^* \text{ and } C > 0,
\]
and consider the functional $\Phi : X^0_p \to \mathbb{R}$ defined by
\[
\Phi(u) = \frac{1}{p} \|u\|_{X^0_p}^p - \int_{\Omega} G(u) \, dx,
\]
where $G(t) = \int_0^t g(s) \, ds$. 

If 0 is a local minimum of $\Phi$ in $C^2_0(\overline{\Omega})$, that is, there exists $r_1 > 0$ such that
\[ \Phi(0) \leq \Phi(z) \quad \text{for all } z \in X^s_p \cap C^2_0(\overline{\Omega}), \quad \|z\|_{0,\delta} \leq r_1, \]
then 0 is a local minimum of $\Phi$ in $X^s_p$, that is, there exists $r_2 > 0$ such that
\[ \Phi(0) \leq \Phi(z) \quad \text{for all } z \in X^s_p, \quad \|z\|_{X^s_p} \leq r_2. \]

Proof. Let us consider initially the subcritical case $q < p^*_s$. By contradiction, denoting
\[ \hat{B}_\epsilon = \{ z \in X^s_p : \|z\|_{X^s_p} \leq \epsilon \}, \]
let us suppose that, for any $\epsilon > 0$, there exists $u_\epsilon \in \hat{B}_\epsilon$ such that
\[ \Phi(u_\epsilon) < \Phi(0). \quad (3.6) \]
Since $\Phi : \hat{B}_\epsilon \to \mathbb{R}$ is weakly lower semicontinuous, there exists $v_\epsilon \in \hat{B}_\epsilon$ such that $\inf_{u \in \hat{B}_\epsilon} \Phi(u) = \Phi(v_\epsilon)$. It follows from (3.6) that
\[ \Phi(v_\epsilon) = \inf_{u \in \hat{B}_\epsilon} \Phi(u) \leq \Phi(u_\epsilon) < \Phi(0). \]
We will show that $v_\epsilon \to 0$ in $C^2_0(\overline{\Omega})$ as $\epsilon \to 0$ since this implies, for $r_1 > 0$, the existence of $z \in C^2_0(\overline{\Omega})$ such that $\|z\|_{0,\delta} < r_1$ and $\Phi(z) < \Phi(0)$, contradicting our hypothesis. Since $v_\epsilon$ is a critical point of $\Phi$ in $X^s_p$, by Lagrange multipliers, it is not difficult to verify that
\[ \Phi'(v_\epsilon) = \xi_\epsilon A(v_\epsilon) \quad (3.7) \]
implies $\xi_\epsilon \leq 0$. Thus, it follows from (3.7) that $v_\epsilon$ satisfies
\[ \begin{cases} (-\Delta^s_{p}) v_\epsilon + \left( \frac{1}{1 - \xi_\epsilon} \right) g^\epsilon(v_\epsilon) = 0 & \text{in } \Omega, \\ v_\epsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (Pv) \]
If $\|v_\epsilon\|_{X^s_p} \leq \epsilon < 1$, Proposition 3.1 shows the existence of a constant $C_1 > 0$, not depending on $\epsilon$, such that
\[ \|v_\epsilon\|_{L^\infty(\Omega)} \leq C_1. \quad (3.8) \]
Since $\xi_\epsilon \leq 0$, (3.5) and (3.8) imply that $\|g^\epsilon(v_\epsilon)\|_{L^\infty((0,1))} \leq C_2$ for some constant $C_2 > 0$. Proposition 3.2 yields $\|v_\epsilon\|_{C^0(\overline{\Omega})} \leq C_3$ for $0 < \beta \leq s$ and a constant $C_3$ not depending on $\epsilon$. Now, it follows from the Arzelà–Ascoli theorem the existence of a sequence $(v_\eta)$ such that $v_\epsilon \to 0$ uniformly as $\epsilon \to 0$. Passing to a subsequence, we can suppose that $v_\epsilon \to 0$ a.e. in $\Omega$ and, therefore, $v_\epsilon \to 0$ uniformly in $\overline{\Omega}$. But now it follows from Proposition 3.3 that
\[ \|v_\epsilon\|_{L^\infty(\Omega)} \leq C \sup_{x \in (0,1)} |g^\epsilon(v_\epsilon(x))| \]
for a constant $C > 0$. The proof of the subcritical case is complete.

We now consider the critical case $q = p^*_s$. As before, we argument by contradiction. For this, we define $g_k, G_k : \mathbb{R} \to \mathbb{R}$ by
\[ g_k(s) = g(t_k(s)) \quad \text{and} \quad G_k(t) = \int_0^t g_k(s) \, ds, \]
with $t_k$ given by
\[ t_k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ s & \text{if } -k < s < k, \\ k & \text{if } s \geq k. \end{cases} \]
Considering $\Phi_k \in C^1(X^s_p, \mathbb{R})$ given by
\[ \Phi_k(u) = \frac{\|u\|^p_{X^s_p}}{p} - \int_{\Omega} G_k(t) \, dt, \]
it follows \( \Phi_k(u) \rightarrow \Phi(u) \) as \( k \rightarrow \infty \). Thus, for any \( \varepsilon \in (0, 1) \), there exists \( k_\varepsilon \geq 1 \) such that \( \Phi_k(u_\varepsilon) < \Phi(0) \), and the subcritical growth of \( g_k \) guarantees the existence of \( u_\varepsilon \in B_\varepsilon \) such that

\[
\Phi_k(u_\varepsilon) = \inf_{u \in B_\varepsilon} \Phi_k(u) \leq \Phi_k(u_\varepsilon) < \Phi(0).
\]

As in the subcritical case, we find \( \xi_\varepsilon \leq 0 \) such that \( u_\varepsilon \) is a weak solution to problem \((P_v)\) with \( u_\varepsilon \) instead of \( v_\varepsilon \).

From the definition of \( g_k \) and since \( \|u_\varepsilon\|_{X^p} \leq \varepsilon < 1 \), by applying Proposition 3.1, we obtain \( \|g_k(u_\varepsilon)\|_{L^{\infty}(\Omega)} \leq C_2 \) for a constant \( C_2 > 0 \). It follows from Proposition 3.2 that \( \|u_\varepsilon\|_{C^{0,N/2p}(\bar{\Omega})} \leq C_3 \) for \( 0 < \beta \leq s \), the constant \( C_3 \) not depending on \( \varepsilon \). By applying the Arzelà–Ascoli theorem, the conclusion is now obtained as in the subcritical case.

\[
\text{Remark 3.4. If 0 is a strict local minimum in } C^1_0(\bar{\Omega}), \text{ it follows that 0 is also a strict local minimum in } X^p_0.
\]

### 4 Positive and Negative Solutions

Most of the results in this section are standard. Therefore, our presentation will be only schematic.

We denote

\[
S_{p,s} = \inf \left\{ \frac{\|u\|^p_{X^p_s}}{\left( \int_{\Omega} |u|^{p^*} \, dx \right)^{\frac{p}{p^*}} ; \ u \in X^p_s, \ u \neq 0 \right\}
\]

the best constant of the immersion \( X^p_s \rightarrow L^{p^*}(\Omega) \); see [30].

If we take care of the operator \( A \), the proof of the next two results is similar to that exposed in [21].

**Lemma 4.1.** If \( a > \lambda_1, b > 0, 1 < q < p \) and \( \lambda > 0 \), then any \((PS)\)-sequence of \( I_{\lambda,s} \) is bounded in \( X^p_s \).

**Lemma 4.2.** If \( a > \lambda_1, b > 0, 1 < q < p \) and \( \lambda > 0 \), then \( I_{\lambda,s} \) satisfies the \((PS)\)-condition at any level \( C \) such that

\[
C < \frac{S_{p,s}}{N} b^{\frac{m}{p^*}} S_{p,s}^\frac{N}{p^*}.
\]

We now consider the positive part of the functional \( I_{\lambda,s} \). That is, \( I^+_s : X^p_s \rightarrow \mathbb{R} \) given by

\[
I^+_s(u) = \frac{1}{p} \|u\|^p_{X^p_s} + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - a \int_{\Omega} |u|^p \, dx - b \int_{\Omega} |u|^{p^*} \, dx.
\]

Of course, \( I^+_s \in C^1(X^p_s, \mathbb{R}) \), and it holds, for all \( u, h \in X^p_s \),

\[
(I^+_s)'(u) \cdot h = A(u) \cdot h + \lambda \int_{\Omega} |u|^{q-1} h \, dx - a \int_{\Omega} |u|^{p-1} h \, dx - b \int_{\Omega} |u|^{p^*-1} h \, dx.
\]

Furthermore, critical points of \( I^+_s \) are weak solutions to the problem

\[
\begin{cases}
(-\Delta)^s_p u = -\lambda |u|^{q-1} + a |u|^{p-1} + b (u^+)^{p^*-1} & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( a, b, \lambda > 0, 1 < q < p \) and \( u^+ = \max\{u, 0\} \).

We now recall the following elementary inequality, which has a straightforward proof.

**Lemma 4.3.** For all \( u : \mathbb{R}^N \rightarrow \mathbb{R}, p > 1 \) and \( x, y \in \mathbb{R}^N \), it holds

\[
\begin{align*}
&(i) \ |u^+(x) - u^+(y)|^p \leq |u(x) - u(y)|^{p-2}(u(x) - u(y))(u^+(x) - u^+(y)), \\
&(ii) \ |u^+(x) - u^-(y)|^p \leq |u(x) - u(y)|^{p-2}(u(x) - u(y))(u^+(x) - u^-(y)).
\end{align*}
\]

If \( u \) is a critical point of \( I^+_s \), then \( (I^+_s)'(u) \cdot h = 0 \) for all \( h \in X^p_s \). Taking \( h = u^- \), it follows from Lemma 4.3 that

\[
0 = (I^+_s)'(u) \cdot u^- = A(u) \cdot u^- \geq \int_{\mathbb{R}^N} \frac{|u^+(x) - u^-(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \|u^-\|^p_{X^p_s}
\]

and therefore \( u^- = 0 \). Thus, a critical point \( u \) of \( I^+_s \) satisfies \( u = u^+ \geq 0 \).
As in the proof of Lemma 4.2, we have that $I_{\lambda,s}^*$ satisfy the (PS)-condition at any level
\[ C < \frac{S}{N} b^{\frac{p-N}{p}} S_{p,s} \] (4.2)
for any $\lambda > 0$. The definition of the functional $J$ was removed.

**Lemma 4.4.** If $a, b > 0$ and $1 < q < p$, then the trivial solution $u = 0$ is a strict local minimizer of $I_{\lambda,s}$ for any $\lambda > 0$.

**Proof.** According to Remark 3.4, it is enough to show that $u = 0$ is a strict local minimum of $I_{\lambda,s}^*$ in $C_b^0(\overline{\Omega})$. Take $u \in C_b^0(\overline{\Omega}) \setminus \{0\}$, and consider the functional
\[ I_{\lambda,s}^*(u) = \frac{1}{p} \|u\|_{X_p}^p + \frac{\lambda}{q} \frac{\|u\|_{C_{p,s}}}{q} \int_\Omega |u|^q \, dx - \frac{a}{p} \int_\Omega |u|^p \, dx - \frac{b}{p_s} \int_\Omega |u|^{p_s} \, dx. \]
So, for positive constants $C_1$ and $C_2$, we have
\[ I_{\lambda,s}^*(u) \geq \frac{1}{p} \|u\|_{X_p}^p + \left( \frac{\lambda}{q} - \frac{aC_1}{p} \|u\|_{C_{p,s}}^{p-q} - \frac{bC_2}{p_s} \|u\|_{C_{p,s}}^{p_s-q} \right) \int_\Omega |u|^q \, dx \]
This implies for all $u \neq 0$ with $\|u\|_{C_{p,s}}$ sufficiently small,
\[ I_{\lambda,s}^*(u) > 0 = I_{\lambda,s}^*(0). \]

**Lemma 4.5.** If $\lambda_1 < a$, $1 < q < p$ and $b > 0$, then, for any fixed $\Lambda > 0$, there exists $t_0 = t_0(\Lambda) > 0$ such that $I_{\lambda,s}^*(t_\varphi_1) < 0$ for all $t \geq t_0$ and $\lambda < \Lambda$.

**Proof.** For a fixed $\Lambda > 0$, our hypotheses guarantee that we can choose $t_0 = t_0(\Lambda) > 0$ such that, if $t \geq t_0$ and $\lambda < \Lambda$, it follows $I_{\lambda,s}^*(t_\varphi_1) < 0$.

Now we prove that (1.1) has at least one positive solution.

**Proposition 4.6.** Suppose that $\lambda > 0$, $1 < q < p$, $\lambda_1 < a$ and $b > 0$. There exists $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$, then problem (1.1) has at least one positive solution.

**Proof.** We observe that a non-negative weak solution of (1.1) is a critical point of the functional $I_{\lambda,s}^*$. We now apply the mountain pass theorem. The geometric conditions of this theorem are consequences of Lemmas 4.4 and 4.5. We now prove the existence of $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$, then $I_{\lambda,s}^*$ satisfies the (PS)-condition at level
\[ C_+^* = \inf_{g \in \Gamma^*} \max_{u \in \mathcal{U}(g)} I_{\lambda,s}^*(u), \]
where $\Gamma^* = \{ g \in C((0,1), X_p) : g(0) = 0, \ g(1) = t_0 \varphi_1 \}$, with $t_0$ obtained in Lemma 4.5.

In order to do that, we observe that, for all $0 \leq t \leq 1$, our hypotheses imply that
\[ \inf_{g \in \Gamma^*} \max_{u \in \mathcal{U}(g)} I_{\lambda,s}^*(u) \leq \max_{u \in \mathcal{U}(g_0)} I_{\lambda,s}^*(u) = \max_{t \in (0,1]} I_{\lambda,s}^*(g_0(t)) \leq \frac{\lambda_0 C_1}{q} \| \varphi_1 \|^q_{L^q(\Omega)}, \]
from which the existence of $\lambda_0 > 0$ follows such that
\[ 0 \leq C_+^* < \frac{S}{N} b^{\frac{p-N}{p}} S_{p,s} \lambda_0 \] for all $0 < \lambda < \lambda_0 < \Lambda$.

with $\Lambda$ as in Lemma 4.5. The proof is complete as a consequence of (4.2).

In order to show the existence of a negative solution for $I_{\lambda,s}$, we consider $I_{\lambda,s}^- : X_{p,s}^\infty \to \mathbb{R}$ given by
\[ I_{\lambda,s}^-(u) = \frac{1}{p} \|u\|_{X_p}^p + \frac{\lambda}{q} \frac{\|u\|_{C_{p,s}}}{q} \int_\Omega |u|^q \, dx - \frac{a}{p} \int_\Omega |u|^p \, dx, \]
where $u^- = \text{min}(u, 0)$. 


Of course, $I_{A,s} \in C^1(X_{p}^s, \mathbb{R})$ and critical points of $I_{A,s}$ are weak solutions to the problem

\[
\begin{cases}
(-\Delta)^s_p u = -\lambda |u|^{q-1} + a |u|^{p-1} & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]  

(4.3)

As before, a critical point $u$ of $I_{A,s}$ satisfies $u = u^{-} \leq 0$.

Observe that $I_{A,s}$ satisfies the (PS)-condition at all levels for any $\lambda > 0$ since the nonlinearity in (4.3) does not have fractional critical power.

**Lemma 4.7.** If $0 < a$ and $1 < q < p$, then $u = 0$ is a strict local minimizer of $I_{A,s}$ for all $\lambda > 0$.

**Proof.** According to Remark 3.4, it is enough to show that $u = 0$ is a strict local minimum of $I_{A,s}$ in $C_0^0(\Omega)$. It is not difficult to verify that, for all $u \in C_0^0(\Omega) \setminus \{0\}$ and a fixed $\lambda > 0$, we have

\[
I_{A,s}(u) \geq \frac{1}{p} \|u\|_{X_p^s}^p + \left(\frac{\lambda}{q} - \frac{a}{p} \|u\|_{B_0}^{p-q}\right) \int_{\Omega} |u|^q \ dx.
\]

Taking $R = \left(\frac{\lambda}{p-a} \right)^{\frac{1}{q-1}}$, it follows $\frac{a \|u\|_{X_p^s}^{p-q}}{p} < \frac{\lambda}{q}$ and for all $u \neq 0$, $I_{A,s}(u) > 0 = I_{A,s}(0)$ if $\|u\|_{0,\delta} < R$. \qed

The proof of the next result is analogous to that of Lemma 4.5.

**Lemma 4.8.** If $\lambda_1 < a$ and $1 < q < p$, then, for any fixed $\Lambda > 0$, there exists $t_0' = t_0'(<\Lambda > 0$ such that

\[
I_{A,s}(-t\varphi_1) < 0 \quad \text{for all } t \geq t_0'
\]

and $\Lambda < \Lambda$.

The proof of the next result is similar to that of Proposition 4.6. In the proof, the inequality

\[
0 \leq C_{\alpha} = \frac{\lambda(t_0')^q}{q} \|\varphi_1\|_{L^q(\Omega)}^q
\]

for all $\lambda > 0$

plays an essential role, with $C_{\alpha}$ defined analogously to $C_{\alpha}^s$.

**Proposition 4.9.** Suppose that $\lambda > 0$, $1 < q < p$, $\lambda_1 < a < \lambda^*$ and $b > 0$. There exists $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$, then problem (1.1) has at least one negative solution.

## 5 A Third Solution via Linking Theorem

In contrast to the previous section, the proof of the existence of a third solution to (1.1) is much more intricate and also technical. We obtain a third solution by applying the linking theorem and a series of previously obtained results that will be useful in our proof.

We suppose $0 < s < 1$, $N > sp$, $\Lambda > 0$, $\lambda_1 < a < \lambda^*$ and $b > 0$. The proof of our first result is simple.

**Proposition 5.1.** If $\text{span}\{\varphi_1\}$ denotes the space generated by the first (positive, $L^p$-normalized) eigenfunction of $(-\Delta)^s_p$, then $X_p^s = W \oplus \text{span}\{\varphi_1\}$.

We recall that $W = \{u \in X_p^s : A(\varphi_1) \cdot u = 0\}$ and $\lambda^* = \inf\{\|u\|_{X_p^s}^p : u \in W, \|u\|_{L^p(\Omega)} = 1\}$.

The next result will be used to prove that the geometric conditions of the linking theorem are satisfied.

**Proposition 5.2.** Suppose that $a < \lambda^*$. Then there exist $a > 0$ and $\rho > 0$ such that $I_{A,s}(u) \geq \alpha$ for any $u \in W$ with $\|u\|_{X_p^s} = \rho$.

**Proof.** Since $a > 0$, the immersion $X_p^s \hookrightarrow L^r(\Omega)$ for $r \in [p, p_s^*)$ combined with the definition of $\lambda^*$ implies (after some calculations) that

\[
I_{A,s}(u) \geq \frac{1}{p} \left(1 - \frac{a}{\lambda^*}\right) \|u\|_{X_p^s}^p - \frac{bC_{\alpha}^s}{p_s} \|u\|_{X_p^s}^{p_s} \geq \|u\|_{X_p^s}^p (A - B \|u\|_{X_p^s}^{p_s-p}),
\]

where $A = \frac{1}{p_s} (1 - \frac{a}{\lambda^*}) > 0$ and $B = \frac{bC_{\alpha}^s}{p_s} > 0$. If $0 < \rho < \left(\frac{\rho}{2}\right)^{\frac{1}{p'}}$, then $I_{A,s}(u) \geq \rho^p (A - B \rho^{p_s-p}) = \alpha > 0$, and we conclude that $I_{A,s}(u) \geq \alpha$ for all $u \in W$ satisfying $\|u\|_{X_p^s} = \rho$. \qed
In order to apply the linking theorem with respect to the decomposition given in Proposition 5.1, we need to prove the existence of a vector $e \in W$ satisfying the hypotheses of that result. We recall that $S_{p,s}$ was defined in (4.1).

We now state the following result, which can be found in [30, Proposition 2.1]. See also [10].

**Proposition 5.3.** Let $1 < p < \infty$, $s \in (0, 1)$, $N > sp$.

(i) There exists a minimizer for $S_{p,s}$.

(ii) For every minimizer $U$, there exist $x_0 \in \mathbb{R}^N$ and a constant sign monotone function $u_\varepsilon : [0, \infty) \to \mathbb{R}$ such that $U(x) = u_\varepsilon (|x| - x_0)$.

(iii) For every non-negative minimizer $U \in X^s_p$ and $v \in X^s_p$, we have

$$
\int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^{p-2}(U(x) - U(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy = S_{p,s} \int_{\mathbb{R}^N} U^{p_{s}^{-1}} v \, dx.
$$

Applying Proposition 5.3, we fix a radially symmetric non-negative decreasing minimizer $U = U(r)$ of $S_{p,s}$. Multiplying $U$ by a positive constant, we may assume that

$$
(-\Delta)^s_p U = U^{p_{s}^{-1}}.
$$

It follows from (4.1) that

$$
\|U\|_{X^s_p}^p = \|U\|_{L^p(\mathbb{R}^N)}^p = (S_{p,s})^{-\frac{N}{p}}.
$$

For any $\varepsilon > 0$, the function

$$
U_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{N}{p}}} U \left( \frac{|x|}{\varepsilon} \right)
$$

is also a minimizer of $S_{p,s}$ satisfying (5.1) and (5.2), so after a rescaling, we may assume that $U(0) = 1$. Henceforth, $U$ will denote such a function and $\{U_\varepsilon\}_{\varepsilon > 0}$ the associated family of minimizers given by (5.3).

Since an explicit formula for a minimizer of $S_{p,s}$ is unknown, we make use of some asymptotic estimates obtained by Brasco, Mosconi and Squassina [10]; see also [30, Lemma 2.2].

**Lemma 5.4.** There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that, for all $r \geq 1$, $c_1 \leq U(r) \leq c_2 \frac{r^{N/p}}{r^{N/p}}$ and $\frac{U(\theta r)}{U(r)} \leq \frac{1}{2}$.

In order to apply the linking theorem with respect to the decomposition given by Proposition 5.1, we consider the family of functions $\{U_\varepsilon\}_{\varepsilon > 0}$ as defined in (5.3). Without loss of generality, we suppose that $0 \in \Omega$ (boldface removed). From now on, let us consider $\theta$ as given in Lemma 5.4. For $\varepsilon, \delta > 0$, we define, as in Chen, Mosconi and Squassina [17],

$$
m_{\varepsilon, \delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta \delta)}
$$

and also

$$
g_{\varepsilon, \delta}(t) := \begin{cases} 
0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta \delta), \\
\frac{m_{\varepsilon, \delta}(t - U_\varepsilon(\theta \delta))}{m_{\varepsilon, \delta}(t - U_\varepsilon(\delta))} & \text{if } U_\varepsilon(\theta \delta) \leq t \leq U_\varepsilon(\delta), \\
t - U_\varepsilon(\delta)(m_{\varepsilon, \delta}^{-1} - 1) & \text{if } t \geq U_\varepsilon(\delta), 
\end{cases}
$$

Since

$$
G_{\varepsilon, \delta}(t) := \int_{0}^{t} (g_{\varepsilon, \delta}(\tau))^{\frac{1}{p}} \, d\tau = \begin{cases} 
0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta \delta), \\
m_{\varepsilon, \delta}(t - U_\varepsilon(\theta \delta)) & \text{if } U_\varepsilon(\theta \delta) \leq t \leq U_\varepsilon(\delta), \\
t & \text{if } t \geq U_\varepsilon(\delta), 
\end{cases}
$$

it is not difficult to verify that the functions $g_{\varepsilon, \delta}$ and $G_{\varepsilon, \delta}$ are non-decreasing and absolutely continuous. We now define the non-increasing, absolutely continuous and radially symmetric function $u_{\varepsilon, \delta}(r) = G_{\varepsilon, \delta}(U_\varepsilon(r))$ which satisfies

$$
u_{\varepsilon, \delta}(r) = \begin{cases} 
U_\varepsilon(r) & \text{if } r \leq \delta, \\
0 & \text{if } r \geq \theta \delta.
\end{cases}
$$
According to Ambrosio and Isernia [4, p. 17], for any \( \delta \leq r \leq \theta \delta \), it holds

\[
0 \leq m_{e, \delta}(U_e(r) - U_e(\theta \delta)) = U_e(\delta) \left[ \frac{U_e(r) - U_e(\theta \delta)}{U_e(\delta) - U_e(\theta \delta)} \right] \leq U_e(\delta).
\]

Therefore, from the definition of \( G_{e, \delta} \) and (5.4), it follows that

\[
u_{e, \delta}(r) = \begin{cases} U_e(r) & \text{if } r < \theta \delta, \\ 0 & \text{if } r \geq \theta \delta. \end{cases}
\]

We denote by \( P_1^{e} \), \( P_2^{e} \) the projections of \( X^{e}_\delta \) in \( \text{span} \{ \varphi_1 \} \) and \( W \), respectively, and define \( e_{e, \delta} = P_1^{e}u_{e, \delta} \in W \) and claim that \( e_{e, \delta} \) is a continuous function. As shown in [10], we know that \( U \in L^{\infty}(\mathbb{R}^N) \cap C^{0}(\mathbb{R}^N) \). Since \( e_{e, \delta} = u_{e, \delta} - P_1^{e}u_{e, \delta} \), our claim is proved.

We now want to show that we can take \( e = e_{e, \delta} \) in the linking theorem. So we need to show that \( e_{e, \delta}(0) > 0 \) for \( \varepsilon > 0 \) sufficiently small. In order to do that, we obtain the inequalities of the next result using arguments similar to those used in [16], with the exception of (5.7). Estimate (5.7) follows from [4, Lemma 2.4]. Therefore, we state the following result.

**Lemma 5.5.**

\[
\|P_1^{e}u_{e, \delta}\|_{L^{\infty}(\Omega)} \leq \begin{cases} \| \varphi_1 \|_{L^{\infty}(\Omega)}^p C_1 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } p = \frac{2N}{N+3}, \\ \| \varphi_1 \|_{L^{\infty}(\Omega)}^p C_1 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } 1 < p < \frac{2N}{N+3}, \end{cases}
\]

\[
\|e_{e, \delta}\|_{L^{1}(\Omega)} \leq \begin{cases} C_0 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } p = \frac{2N}{N+3}, \\ C_0 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } 1 < p < \frac{2N}{N+3}, \end{cases}
\]

\[
\|e_{e, \delta}\|_{L^{p}(\Omega)}^p \leq \begin{cases} K_1 e^{p} & \text{if } p = \frac{2N}{N+3} \text{ and } N > sp^2, \\ K_1 e^{p} & \text{if } 1 < p < \frac{2N}{N+3} \text{ and } N > sp^2, \end{cases}
\]

\[
\|e_{e, \delta}\|_{L^{p-1}(\Omega)}^{p-1} \leq \begin{cases} K_2 e \frac{\log(\frac{\varepsilon}{\delta})}{p}^{p-1} & \text{if } p = \frac{2N}{N+3}, \\ K_2 e \frac{\log(\frac{\varepsilon}{\delta})}{p}^{p-1} & \text{if } 1 < p < \frac{2N}{N+3}, \end{cases}
\]

\[
\left| \int_{\Omega} (|e_{e, \delta}|^p - |u_{e, \delta}|^p) \, dx \right| \leq \begin{cases} K_3 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } p = \frac{2N}{N+3}, \\ K_3 e^{N} + K_4 e^{N(p^{1}-1)} & \text{if } 1 < p < \frac{2N}{N+3}, \end{cases}
\]

We now fix \( K > 0 \).

**Lemma 5.6.** There exist \( \varepsilon(K) > 0 \) and \( \sigma > 0 \) such that \( B_{\varepsilon}(0) \subset \{ x : e_{e, \delta}(x) > K \} = \Omega_{e, K} \) for all \( 0 < \varepsilon \leq \varepsilon(K) \).

As a consequence,

\[
\left| \int_{\Omega_{e, K}} |e_{e, \delta}|^p \, dx - \int_{\Omega} |u_{e, \delta}|^p \, dx \right| \leq \begin{cases} K_4 e^{N} \frac{\log(\frac{\varepsilon}{\delta})}{p} & \text{if } p = \frac{2N}{N+3}, \\ K_4 e^{N} + K_4 e^{N(p^{1}-1)} & \text{if } 1 < p < \frac{2N}{N+3}, \end{cases}
\]

**Proof.** It follows from the definition of \( u_{e, \delta} \), Lemma 5.4 and (5.5) that

\[
e_{e, \delta}(0) \geq \frac{1}{\varepsilon^{p-1}} - \|P_1^{e}u_{e, \delta}\|_{L^{\infty}(\Omega)} \to +\infty
\]

as \( \varepsilon \to 0 \); the proof of the claim is complete because \( e_{e, \delta} \) is continuous. The proof of the estimates is obtained by applying [19, Lemma 2.4].
It follows from Lemma 5.6 that there exists $\varepsilon_0 > 0$ such that $e_{\varepsilon, \delta} \neq 0$, $0 < \varepsilon \leq \varepsilon_0$. Thus, in the linking theorem, we can take $e = e_{\varepsilon, \delta}$. If $\varepsilon \in (0, \varepsilon_0]$, take $R_1, R_2 > 0$, and define

$$Q_{e,R_1,R_2} = \{ u \in X^s_p : u = u_1 + re_{\varepsilon, \delta}, u_1 \in \text{span}(\varphi_1) \cap \overline{B}_{R_1}(0), \ 0 \leq r \leq R_2 \}. \tag{5.10}$$

Let $\partial Q_{e,R_1,R_2}$ be the boundary of $Q_{e,R_1,R_2}$ in the finite-dimensional space $\text{span}(\varphi_1) \oplus \text{span}(e_{\varepsilon, \delta})$. We denote $O(\varepsilon^\omega)$ for $\omega \geq 0$ if $|O(\varepsilon^\omega)| \leq C \varepsilon^\omega$ for some $C > 0$ not depending on $\varepsilon > 0$. We remark that $O(\varepsilon^\omega)$ is not always positive.

The next elementary result can be found in Mosconi, Perera, Squassina and Yang [30, p. 17].

**Lemma 5.7.** Given $k > 1$ and $p - 1 < \tau < p$, there exists a constant $C = C(k, q) > 0$ such that

$$|a + b|^p \leq k|a|^p + |b|^p + C|a|^{p^*}\tau|b|^\tau \quad \text{for all } a, b \in \mathbb{R}. \tag{5.11}$$

As an immediate consequence of Lemma 5.7, we have the following result.

**Lemma 5.8.** There exist constants $C_1, C_2 > 0$ such that

$$|a + b|^p \leq C_1|a|^p + C_2|b|^p \quad \text{for all } a, b \in \mathbb{R}. \tag{5.12}$$

The proof of the next result is obtained by applying Lemma 5.7 with $a = \|re_{\varepsilon, \delta}\|_{L^p(\Omega)}$ and $b = \|u_1 + re_{\varepsilon, \delta}\|_{L^p(\Omega)}$ and considering the cases $0 < \tau < 1$ and $\tau > 1$. In the latter case, we then apply Lemma 5.8 and once again Lemma 5.7 with $a = \|re_{\varepsilon, \delta}\|$ and $b = \|u_1\|_{X^s_p}$.

**Lemma 5.9.** Suppose that $u_1 \in \text{span}(\varphi_1)$ and $\tau \in (p - 1, p)$. Then there exists a constant $C_* > 0$ such that

$$\frac{1}{p} \|u_1 + re_{\varepsilon, \delta}\|_{X^p_s}^p \geq \frac{a}{p} \|u_1 + re_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \tag{5.13}$$

$$\leq \frac{1}{p} \|u_1\|_{X^p_s}^p - \frac{a}{p} \|u_1\|_{L^p(\Omega)}^p + C_* \|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p + C_* \|e_{\varepsilon, \delta}\|_{L^p(\Omega)} \|u_1\|_{X^p_s}^\tau. \tag{5.14}$$

The next result follows immediately by considering $f : [0, \infty) \to \mathbb{R}$ given by $f(t) = \frac{Bt^p}{p} - \frac{Cp^*}{p} t^{p^*}$. \tag{5.15}

**Lemma 5.10.** For constants $B > 0$ and $C > 0$, we have

$$\max_{t \geq 0} \left( \frac{Bt^p}{p} - \frac{Cp^*}{p^*} t^{p^*} \right) = \frac{B}{N} \left( \frac{B}{Cp^*} \right)^{\frac{1}{p^*}}. \tag{5.16}$$

To estimate $I_{A,s}$ on $\partial Q_{e,R_1,R_2}$, we apply Lemmas 5.7 to 5.10 and the two next results. The first is a special case of the one proved by Chen, Mosconi and Squassina [17, Lemma 2.11], while the proof of the second can be found in [30, Lemma 2.7].

**Lemma 5.11.** For any $\beta > 0$ and $0 < 2\varepsilon \leq \delta \leq \theta^{-1} \text{dist}(0, \partial \Omega)$, we have

$$\|u_{\varepsilon, \delta}\|_{L^p(\Omega)} \leq \begin{cases} C \beta \varepsilon^{N \cdot \frac{p - p'}{p}} |\log(\frac{\delta}{\varepsilon})| & \text{if } \beta = \frac{p^*}{p}, \\ C \beta \varepsilon^{N \cdot \frac{p - p'}{p}} \delta^{N \cdot \frac{p - p'}{p}} & \text{if } \beta < \frac{p^*}{p}, \\ C \beta \delta^{N \cdot \frac{p - p'}{p}} & \text{if } \beta > \frac{p^*}{p}, \end{cases} \tag{5.17}$$

where $p' = \frac{p}{p - 1}$.

Observe that, taking $\beta = 1$, we obtain

$$\|u_{\varepsilon, \delta}\|_{L^p(\Omega)} \leq \begin{cases} C_1 \varepsilon^{N \cdot \frac{p}{p - 1}} |\log(\frac{\delta}{\varepsilon})| & \text{if } p = \frac{2N}{N + 2}, \\ C_1 \delta^{N \cdot \frac{p}{p - 1}} & \text{if } 1 < p < \frac{2N}{N + 2}, \\ C_1 \varepsilon^{N \cdot \frac{p - p'}{p - 1}} \delta^{N \cdot \frac{p - p'}{p - 1}} & \text{if } p > \frac{2N}{N + 2}. \tag{5.18} \end{cases}$$

Since $(p^*_s - 1) \frac{p}{p - 1} > p^*_s$, it also follows from Lemma 5.11 by taking $\beta = p^*_s - 1$ that

$$\|u_{\varepsilon, \delta}\|_{L^{p^*_s - 1}(\Omega)}^{p^*_s - 1} \leq C \varepsilon^{N \cdot \frac{p}{p - 1}}. \tag{5.19}$$
Lemma 5.12. There exists a constant $C = C(N, p, s) > 0$ such that, for any $0 < \varepsilon \leq \frac{\delta}{2}$,

$$
\|u_{\varepsilon, \delta}\|_{W_0^{1,p}(\Omega)}^p \leq S_{p, s}^N + \frac{C}{\delta^N} \left( \frac{\varepsilon}{\delta} \right)^{\frac{N}{p-1}}, \tag{5.13}
$$

$$
\|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \geq \begin{cases}
\frac{1}{2} \varepsilon^p \log \left( \frac{\delta}{\varepsilon} \right), & N = sp^2, \\
\frac{1}{2} \varepsilon^p, & N > sp^2,
\end{cases}
$$

$$
\|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \geq S_{p, s}^N - \frac{C}{\delta^N} \left( \frac{\varepsilon}{\delta} \right)^{\frac{N}{p-1}}.
$$

In our development, we need an auxiliary result that was obtained in [19, Lemma 2.5] when $s = 1$ and $p = 2$. It is easily adapted for $s \in (0, 1)$ and $p > 1$.

Lemma 5.13. Let $u, v \in L^p(\Omega)$ with $p \leq r < p^*_s$. If $\omega \subset \Omega$ and $u + v > 0$ on $\omega$, then

$$
\left| \int_{\omega} (u + v)^r \, dx - \int_{\omega} |u|^r \, dx - \int_{\omega} |v|^r \, dx \right| \leq C \int_{\omega} (|u|^{r-1} + |v|^{r-1}) \, dx,
$$

where $C$ depends only on $r$.

Adapting ideas from Miyagaki, Motreanu and Pereira [29] and de Figueiredo and Yang [19], we obtain the desired estimate $I_{\lambda, s}$ on $\partial Q_{\varepsilon, R_1, R_2}$.

Proposition 5.14. Consider $Q_{\varepsilon, R_1, R_2}$ defined in (5.10). There exist $R_1 > 0$ and $R_2 > 0$ large enough such that

$$
I_{\lambda, s}(u) \leq \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx \quad \text{for all } u \in \partial Q_{\varepsilon, R_1, R_2},
$$

for all $\varepsilon > 0$ small enough and all $\lambda > 0$.

Proof. We write $\partial Q_{\varepsilon, R_1, R_2} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with

$$
\begin{align*}
\Gamma_1 &= B_{R_1} \cap \text{span}\{\varphi_1\}, \\
\Gamma_2 &= \{ u \in X_{p}^s : u = u_1 + re_{\varepsilon, \delta}, u_1 \in \text{span}\{\varphi_1\}, \|u_1\|_{X_p^s} = R_1, 0 \leq r \leq R_2 \}, \\
\Gamma_3 &= \{ u \in X_{p}^s : u = u_1 + R_2 e_{\varepsilon, \delta}, u_1 \in \text{span}\{\varphi_1\}, \|u_1\|_{X_p^s} \leq R_1 \}.
\end{align*}
$$

We consider $I_{\lambda, s}$ in the three parts of the boundary $Q_{\delta, R_1, R_2}$. If $u \in \Gamma_1$, then $u = t\varphi_1$, and we obtain

$$
I_{\lambda, s}(u) \leq \frac{|t|^p}{p} (\lambda_1 - a) \int_{\Omega} |\varphi_1|^p \, dx + \frac{\lambda}{q} \int_{\Omega} |t\varphi_1|^q \, dx \leq \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx \quad \text{for all } u \in \Gamma_1,
$$

as desired. If $u \in \Gamma_2$, then $u = u_1 + re_{\varepsilon, \delta} \in \text{span}\{\varphi_1\} \cap \text{span}\{e_{\varepsilon, \delta}\}$, with $\|u_1\|_{X_p^s} = R_1$. Since $P_{\delta}^s$ is bounded in $X_p^s$, there exist $C_1 > 0$ such that $\|e_{\varepsilon, \delta}\|_{X_p^s} = \|P_{\delta}^s u_{\varepsilon, \delta}\|_{X_p^s} \leq C_1 \|u_{\varepsilon, \delta}\|_{X_p^s}$. It follows then from (5.13) the existence of $C > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0 = \min\{\frac{\delta}{2}, \varepsilon_0\}$, we have

$$
\|e_{\varepsilon, \delta}\|_{X_p^s} \leq C \|u_{\varepsilon, \delta}\|_{X_p^s} \leq C_1 \|u_{\varepsilon, \delta}\|_{X_p^s} + C_1 \frac{C}{\delta^N} \left( \frac{\varepsilon}{\delta} \right)^{\frac{N}{p-1}}.
$$

We conclude that $\eta := \sup_{0 < \varepsilon \leq \varepsilon_0} \|e_{\varepsilon, \delta}\|_{X_p^s}$ is finite. In order to satisfy the condition $R_2 \|e_{\varepsilon, \delta}\| > \rho$ in the linking theorem for $0 < \varepsilon \leq \varepsilon_0$ small enough, with $\delta < \delta_0 \frac{\min(0, \rho)}{2}$ as in Lemma 5.11 and $\rho > 0$ as in Proposition 5.2, we must have $R_2 \eta \geq R_2 \|e_{\varepsilon, \delta}\|_{X_p^s} > \rho$, showing that $\frac{p}{q}$ is a lower bound for $R_2$. Thus, we define $r_0 = \max\{\frac{p}{q}, 1\}$ and consider two cases.

(a) $0 \leq r \leq r_0$. Since all norms are equivalent on finite-dimensional spaces,

$$
I_{\lambda, s}(u) \leq \frac{1}{p} \left( 1 - \frac{a}{\lambda_1} \right) R_1^p + C \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx.
$$

Since $a > \lambda_1$, the result is obtained in this case for all $R_1 > 0$ large enough to be fixed later.
(b) \( r > r_0 \). We suppose \( R_1 \geq 1 \) and denote

\[
K(R_1) := \frac{1}{r_0} \sup \{\|u_1\|_{L^\infty(\Omega)} : u_1 \in \text{span}(\varphi_1), \|u_1\|_{X^p} = R_1\} \in [c_0 R_1, c_1 R_1],
\]

with positive constants \( c_0, c_1 \). We introduce the open set \( \Omega_{r, \delta} = \{x \in \Omega : e_{r, \delta}(x) > K(R_1)\} \). It follows from (5.9) that \( 0 \in \Omega_{r, \delta} \) if \( e \in (0, \varepsilon_0) \). If \( x \in \Omega_{r, \delta} \) and \( r > r_0 \), we have \( e_{r, \delta}(x) > \frac{|u_1(x)|}{r} \geq -\frac{u_1(x)}{r} \) a.e. in \( \Omega \). That is,

\[
e_{r, \delta}(x) + \frac{u_1(x)}{r} > 0 \tag{5.15}
\]

for all \( x \in \Omega_{r, \delta}, r > r_0 \) and \( u_1 \in \text{span}(\varphi_1) \) with \( \|u_1\|_{X^p} = R_1 \). We now take \( R_2 = 2 R_1 \). By Lemma 5.9 and using estimate (5.14), we can write

\[
I_{\lambda, s}(u) \leq \frac{1}{p} \left( 1 - \frac{a}{\lambda_1} \right) R_1^p + C_s r^p \left( C_1^p S_{p, s}^{\frac{N}{p}} + C_0 \right) + 2^{p-r} C_s R_1^p \|e_{r, \delta}\|_{L^p(\Omega)}^{p-r} - \frac{b}{p^s} \int_{\Omega} \left( \frac{u_1}{r} + e_{r, \delta} \right)^{p^s} + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx.
\]

Now, applying Lemma 5.13 for \( u = \frac{u_1}{r}, v = e_{r, \delta} \) and \( \omega = \Omega_{r, \delta} \), we obtain

\[
I_{\lambda, s}(u) \leq \frac{1}{p} \left( 1 - \frac{a}{\lambda_1} \right) R_1^p + C_s r^p \left( C_1^p S_{p, s}^{\frac{N}{p}} + C_0 \right) + 2^{p-r} C_s R_1^p \|e_{r, \delta}\|_{L^p(\Omega)}^{p-r} - \frac{b}{p^s} \int_{\Omega} \left[ |e_{r, \delta}|^{p^s} \Omega_{r, \delta} \right] + C_2 \left( \|e_{r, \delta}\|_{L^p(\Omega)}^{p^s} + \|e_{r, \delta}\|_{L^p(\Omega)}^{p^s} - R_1^p \right) + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx
\]

for a positive constant \( C_2 > 0 \).

We consider the case where \( p > \frac{2N}{N-2} \) and \( N > sp((p - 1)^2 + p) > sp^2 \) since the case \( 1 < p \leq \frac{2N}{N-2} \) and \( N > sp^2 \) is analogous. Estimates (5.6), (5.7), (5.8) and Lemma 5.6 combined with Lemma 5.12 imply that

\[
I_{\lambda, s}(u) \leq \frac{1}{p} \left( 1 - \frac{a}{\lambda_1} \right) + 2^{p-r} C_s e^{s(p-r)} + C_s r^p \left( C_1^p S_{p, s}^{\frac{N}{p}} + C_0 \right) - \frac{b}{p^s} \int_{\Omega} \left[ |e_{r, \delta}|^{p^s} \Omega_{r, \delta} \right] + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx,
\]

where \( R_1 = e^{-y} \) with \( y \) chosen so that \( 0 < y < \min \left\{ \frac{N-sp}{p(p-1)(p+1)}, \frac{N-sp}{p-2} \right\} \). Taking

\[
A = \left( 1 - \frac{a}{\lambda_1} \right) + 2^{p-r} C_s e^{s(p-r)}, \quad B = C_s r^p \left( C_1^p S_{p, s}^{\frac{N}{p}} + C_0 \right),
\]

\[
C = bS_{p, s}^{\frac{N}{p}} + O(e^{\frac{N-sp}{p-r}}) - bC_2 e^{\frac{N-sp}{p-r}} - bC_2 e^{\frac{N-sp}{p-r}} - y
\]

it is easy to see that there exists \( \varepsilon_1 > 0 \) small enough such that, for all \( 0 < \varepsilon < \varepsilon_1 < \varepsilon_0 \), we have \( A < 0 \) and \( C > 0 \), so

\[
I_{\lambda, s}(u) \leq \frac{A}{p} R_1^p + \frac{B}{p} - \frac{C}{p^s} R_1^{p^s} + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx.
\]

By applying Lemma 5.10 to the function \( h(r) = \frac{Br^s}{p} - \frac{C R_1^s}{p^s} \), we obtain

\[
I_{\lambda, s}(u) \leq \frac{A}{p} R_1^p + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx + \frac{1}{N} \left( \frac{C_s r^p \left( C_1^p S_{p, s}^{\frac{N}{p}} + C_0 \right) - bC_2 e^{\frac{N-sp}{p-r}} - y}{bS_{p, s}^{\frac{N}{p}} + O(e^{\frac{N-sp}{p-r}})} \right) \left( \frac{R_1^s}{R_2^s} \right)^{\frac{N}{p}}.
\]

Therefore, since \( A < 0 \) and there exists \( \varepsilon_1 > 0 \) small enough such that, if \( 0 < \varepsilon < \varepsilon_1 < \varepsilon_1 \), then \( R_1 > 0 \) is large enough, we have the result for \( u \in \Gamma_2 \). Finally, if \( u \in \Gamma_3 \), then \( u = u_1 + R_2 e_{r, \delta} \in \text{span}(\varphi_1) \oplus \text{span}(e_{r, \delta}) \), with \( \|u_1\|_{X^p} \leq R_1 \). We recall that \( R_2 = 2 R_1 \). By (5.14), since \( \|e_{r, \delta}\|_{X^p} \) is bounded, we have

\[
I_{\lambda, s}(u) \leq \left[ C_s (C_1^p S_{p, s}^{\frac{N}{p}} + C_0) + \frac{C_2}{2} \right] R_2^2 - \frac{b}{p^s} \left( \frac{u_1}{R_2} + e_{r, \delta} \right)^{p^s} + \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx. \tag{5.16}
\]


Since the norms are equivalent on finite-dimensional spaces, there exists a constant $C_3 > 0$ such that, if $\|u_1\|_{X_2^p} \leq R_1$, then $\|u_1\|_{L^\infty(\Omega)} \leq C_3\|u_1\|_{X_2^p} \leq C_3 R_1$. Furthermore,

$$\Omega^*_e := \{ x \in \Omega : e_{e,\delta}(x) > \frac{C_3}{2} + 1 \} \uplus \{ x \in \Omega : e_{e,\delta}(x) > C_3 + 1 \} := D_\varepsilon, \quad (5.17)$$

with $|D_\varepsilon| > 0$. So, for all $x \in \Omega^*_e$, it follows from (5.15), (5.17) and $R_2 = 2R_1$ that

$$\frac{u_1(x)}{R_2} + e_{e,\delta}(x) > \frac{u_1(x)}{R_2} + C_3 R_1 \frac{1}{R_2} + 1 \geq \frac{u_1(x)}{R_2} + \frac{\|u_1\|_{L^\infty(\Omega)}}{R_2} + 1 \geq 1. \quad (5.18)$$

Substituting (5.17) and (5.18) into (5.16), we obtain

$$I_{\lambda,s}(u) \leq \left[ C_s(c_{\ast}^p \frac{p}{p-s} + C_0) + C_3 \frac{C_2}{2^r} \right] R_2^p - \frac{b}{p^s} R_2^p \frac{1}{s} |D_\varepsilon| + \frac{\lambda}{q} \int_\Omega |u|^q \, dx \quad \text{for all } u \in \Gamma_3.$$ 

So, by choosing $\varepsilon > 0$ small enough, we obtain $R_2$ (and also $R_1$) large enough, and our proof is complete.

**Remark 5.15.** It is noteworthy to stress that $R_1$, $R_2$ and $\varepsilon$ in Proposition 5.14 do not depend on $\lambda$.

We also recall the following elementary inequality (see [38, p. 122]).

**Lemma 5.16.** For $p > 1$, there exists a positive constant $K_p$, depending on $p$, such that, for all $a, b \in \mathbb{R}$,

$$||b|^p - |a|^p - |b - a|^p| \leq K_p (|b - a|^{p-1}|a| + |a|^{p-1}|b - a|).$$

By adapting the proof of [29, Lemma 6.1], we obtain the following result. Some details are to be found there.

**Lemma 5.17.** Suppose that $N > sp$. For $\varepsilon > 0$ small enough, the following estimate is true.

$$I_{\lambda,s}(u) \leq \frac{\varepsilon}{N} \left( \frac{1}{b} \frac{X - \varepsilon}{p} \left( \frac{\|e_{e,\delta}\|_{X_2^p}^p - a \|e_{e,\delta}\|_{L^p(\Omega)}^p}{\|e_{e,\delta}\|_{L^p(\Omega)}^p} \right) \frac{b}{p} + \lambda \int_\Omega |u|^q \, dx \right) + \frac{\varepsilon}{N} \left( K_0 e^s(p-1) \log(\frac{x}{\varepsilon}) \right) R_2^p$$

$$\quad \quad + \left\{ \begin{array}{ll} K_0 e^s(p-1) \log(\frac{x}{\varepsilon}) R_2^p & \text{if } p = \frac{2N}{N + s} \text{ and } N > sp^2, \\
K_0 e^s(p-1) & \text{if } 1 < p < \frac{2N}{N + s} \text{ and } N > sp^2, \\
K_0 e^s(p-1) + K_0 e^s & \text{if } p > \frac{2N}{N + s} \text{ and } N > sp^2, \end{array} \right.$$ 

for all $u \in Q_{R_1, R_2}$, where the constant $K_0 > 0$ does not depend on $\varepsilon$.

**Proof.** By applying Lemma 5.16 for $a = u_1(x)$ and $b = u_1(x) + re_{e,\delta}(x)$, the equivalents of norms in finite-dimensional spaces guarantees that

$$\frac{1}{p} \|u_1 + re_{e,\delta}\|_{X_2^p}^p - a \|u_1 + re_{e,\delta}\|_{L^p(\Omega)}^p \leq \frac{1}{p} \left( \|u_1\|_{X_2^p}^p - a \|u_1\|_{L^p(\Omega)}^p + \frac{p}{p} \left( \|e_{e,\delta}\|_{X_2^p}^p - a \|e_{e,\delta}\|_{L^p(\Omega)}^p \right) \right)$$

$$\quad \quad + 2C_1 e^{s(p-1)} \log\left( \frac{x}{\varepsilon} \right) R_2^p \\|u_1\|_{X_2^p} R_2^p + CR_2 \|u_1\|_{X_2^p} \|e_{e,\delta}\|_{L^p(\Omega)}.$$ 

Thus, by applying Lemma 5.9 and estimate (5.7), since $a > \lambda_1$, we obtain

$$I_{\lambda,s}(u) \leq \frac{p}{p} \left( \|e_{e,\delta}\|_{X_2^p}^p - a \|e_{e,\delta}\|_{L^p(\Omega)}^p \right) - \frac{b}{p^s} \int_\Omega (u^+) \, dx + \lambda \int_\Omega |u|^q \, dx$$

$$\quad \quad + \left\{ \begin{array}{ll} 2C_1 e^{s(p-1)} \log\left( \frac{x}{\varepsilon} \right) R_2^p & \text{if } p = \frac{2N}{N + s} \text{ and } N > sp^2, \\
2C_1 e^{s(p-1)} & \text{if } 1 < p < \frac{2N}{N + s} \text{ and } N > sp^2, \\
C_1 e^{s(p-1)} + C_1 e^s & \text{if } p > \frac{2N}{N + s} \text{ and } N > sp^2. \end{array} \right.$$ 

We control the term $-\frac{b}{p^s} \int_\Omega (u^+) \, dx$ by applying estimates (5.11) and (5.12) and conclude the existence of a constant $C_6 > 0$ such that

$$-\frac{b}{p^s} \int_\Omega (u^+) \, dx \leq \frac{b p^s}{p} \left( \int_\Omega (e_{e,\delta})^p \, dx \right) + \left\{ \begin{array}{ll} C_6 e^{\frac{N}{p-1}} \log\left( \frac{x}{\varepsilon} \right) R_2^p & \text{if } p = \frac{2N}{N + s}, \\
C_6 e^{\frac{N}{p-1}} + C_6 e^{\frac{N}{p-1}} R_2^p & \text{if } 1 < p < \frac{2N}{N + s}, \\
C_6 e^{\frac{N}{p-1}} & \text{if } p > \frac{2N}{N + s}. \end{array} \right.$$
Thus, there exists a constant $K_0 > 0$ such that

$$I_{\lambda, \delta}(u) \leq \frac{p}{p} \left( \|e_{\varepsilon, \delta}\|_{X_p}^p - a\|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \right) - \frac{br_{\lambda}^p}{p_p} \int_{\Omega} |u_{\varepsilon, \delta}|^p dx + \frac{\lambda}{q} \int_{\Omega} |u|^q dx$$

$$+ \begin{cases} K_0 \varepsilon^{p(1-\log(\varepsilon)^p)} & \text{if } p = \frac{2N}{N+3} \text{ and } N > sp^2, \\ K_0 \varepsilon^{p(1-\log(\varepsilon)^p)} & \text{if } 1 < p < \frac{2N}{N+3} \text{ and } N > sp^2, \\ K_0 \varepsilon^{p(1-\log(\varepsilon)^p)} + K_0 \varepsilon^8 & \text{if } p > \frac{2N}{N+3} \text{ and } N > sp^2. \end{cases} \tag{5.19}$$

By applying Lemma 5.10, the function $f : [0, +\infty) \to \mathbb{R}$ given by

$$f(t) = \frac{t^p}{p} \left( \|e_{\varepsilon, \delta}\|_{X_p}^p - a\|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \right) - \frac{br_{\lambda}^p}{p_p} \|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p$$

admits its maximum at a point $t_M$ so that

$$f(t_M) = \frac{1}{N} \left( \frac{1}{1} \right)^{\frac{N}{p} - \frac{N}{p}} \left( \frac{\|e_{\varepsilon, \delta}\|_{X_p}^p - a\|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p}{\|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p} \right)^{\frac{N}{p}}.$$

Our result now follows from (5.19).

Also, the next result adapts a similar result obtained in [29, Lemma 6.2].

**Lemma 5.18.** Suppose that $N > sp^2$ and $1 < p < \frac{2N}{N+3}$. Then we have

$$c_s := \inf_{h \in P} \sup_{u \in Q_{\varepsilon, R_1, R_2}} I_{\lambda, \delta}(h(u)) < \frac{S}{N} \frac{1}{b^\frac{N}{p}} \frac{1}{S_{p, a}}$$

for all $\varepsilon > 0$ and $\lambda > 0$ small enough, where $\Gamma = \{ h \in C(\overline{Q}_{\varepsilon, R_1, R_2}, X^p_p) : h = \text{id} \in \partial Q_{\varepsilon, R_1, R_2} \}$. The same result is also valid if $N > sp((p-1)^2 + p)$ and $p > \frac{2N}{N+3}$.

**Proof.** Since $h = \text{id} \in Q_{\varepsilon, R_1, R_2}$, assures that is enough to prove

$$I_{\lambda, \delta}(u) < \frac{S}{N} \frac{1}{b^\frac{N}{p}} \frac{1}{S_{p, a}} \text{ for all } u \in Q_{\varepsilon, R_1, R_2}. \tag{5.20}$$

By applying Lemma 5.8 for $a = u_{\varepsilon, \delta} - P_s^u u_{\varepsilon, \delta} = e_{\varepsilon, \delta}$ and $b = P_s^u u_{\varepsilon, \delta}$, for some constant $c_0 > 0$, we have

$$\|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \geq \frac{1}{c_0} \|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p - \|P_s^u u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p.$$

Since $N > sp^2$, it follows from Lemma 5.12 that $\|u_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \geq c_2 \varepsilon^{sp}$ for a constant $c_2 > 0$. Taking into account (5.5), we obtain

$$\|e_{\varepsilon, \delta}\|_{L^p(\Omega)}^p \geq \frac{c_2}{c_0} \varepsilon^{sp} - \begin{cases} c_1 \varepsilon^{N(p-1)+sp} & \text{if } N > sp^2, \\ c_1 \varepsilon^{N(1-\log(\varepsilon)^p)} & \text{if } 1 < p < \frac{2N}{N+3}, \\ c_1 \varepsilon^{N(p-1)+sp} & \text{if } p > \frac{2N}{N+3}. \end{cases}$$

But Lemma 5.12 yields

$$\|u_{\varepsilon, \delta}\|_{L^{p^*}(\Omega)}^{p^*} \geq \frac{S}{S_{p, a}} + O(\varepsilon^p).$$

Now, mimicking [16, p. 286], we obtain

$$\|e_{\varepsilon, \delta}\|_{X_p}^p - \|u_{\varepsilon, \delta}\|_{X_p}^p \leq c_3 \|e_{\varepsilon, \delta}\|_{X_p}^{p-1} \|P_s^u u_{\varepsilon, \delta}\|_{L^p(\Omega)} + c_3 \|P_s^u u_{\varepsilon, \delta}\|_{L^p(\Omega)}.$$

A new application of Lemma 5.12 and estimate (5.5) give

$$\|e_{\varepsilon, \delta}\|_{X_p}^p \leq \|u_{\varepsilon, \delta}\|_{X_p}^p + \begin{cases} c_4 \varepsilon^p & \text{if } N = \frac{2N}{N+3}, \\ c_4 \varepsilon^{p_2} & \text{if } 1 < p < \frac{2N}{N+3}, \\ c_4 \varepsilon^{p_2} & \text{if } p > \frac{2N}{N+3}. \end{cases}$$
Case 1: Suppose \( N > sp^2 \) and \( p = \frac{2N}{N+2} \). By applying the mean value theorem to \( f(t) = (1 + t)\frac{N-sp}{N} \), we conclude that

\[
|1 - (1 + S_{p,s}^N O(\frac{N}{N+1}))^{\frac{N-sp}{N}}| = O(\frac{N}{N+1}).
\]

Then adding and subtracting \((1 + S_{p,s}^N O(\frac{N}{N+1}))^{\frac{N-sp}{N}} S_{p,s}^N\), we have

\[
\frac{\|e,\delta\|_{X_p^0}^p - a\|e,\delta\|_{L_p^2(\Omega)}^p}{\|u,\delta\|_{L_p^2(\Omega)}^p} \leq S_{p,s} + \frac{\varepsilon sp[O(\frac{N}{N+1}) + O(\frac{N-sp}{N}) + (c_s + ac_s)\varepsilon(\frac{N-sp}{N})\log(\frac{\delta}{c})]}{(S_{p,s}^N + O(\frac{N}{N+1}))^{\frac{N-sp}{N}}},
\]

and

\[
\|e,\delta\|_{X_p^0}^p - a\|e,\delta\|_{L_p^2(\Omega)}^p < S_{p,s}
\]

follows if \( \varepsilon > 0 \) is small enough. Therefore,

\[
\frac{s}{N} \left( \frac{1}{b} \right)^{\frac{N-sp}{N}} \left( \frac{\|e,\delta\|_{X_p^0}^p - a\|e,\delta\|_{L_p^2(\Omega)}^p}{\|u,\delta\|_{L_p^2(\Omega)}^p} \right) < S_{p,s}^N N \left( \frac{1}{b} \right)^{\frac{N-sp}{N}} S_{p,s}^N.
\]

The boundedness of \( Q_{e,R_1,R_2} \) and Lemma 5.17 guarantee that (5.20) is true for \( \varepsilon > 0 \) and \( \lambda > 0 \) small enough.

Case 2: \( N > sp^2 \) and \( 1 < p < \frac{2N}{N+2} \) or \( N > sp((p-1) + p) \) and \( p > \frac{2N}{N+2} \). The proofs are analogous to that of case 1. The conclusion follows once inequality (5.21) is obtained.

\[ \square \]

### 6 Proof of Theorem 1

**Proof.** The positive and negative solutions were obtained in Section 4. Let us denote \( u_1 \) the positive solution and \( u_2 \) the negative solution of problem (1.1) for every \( \lambda > 0 \) sufficiently small. Obviously, these two solutions are distinct. In order to find a third nontrivial solution \( u_3 \) for problem (1.1) whose existence depends on the parameter \( \lambda \) small enough, we apply the linking theorem to the functional \( I_{\lambda,s} : X_p^s \to \mathbb{R} \). Indeed, the geometric conditions of the linking theorem follow from Proposition 5.2 and Lemma 5.14. According to Lemmas 4.2 and 5.18, \( I_{\lambda,s} \) satisfies the (PS)\(_{C_s}\) condition, with

\[
c_s = \inf_{h \in C} \sup_{u \in Q_{e,R_1,R_2}} I_1(h(u)) < \frac{s}{N} b^{\frac{N-sp}{N}} S_{p,s}^N
\]

and \( \Gamma = \{ h \in C(Q_{e,R_1,R_2}, X_p^s) ; h = \text{id} \text{ em } \partial Q_{e,R_1,R_2} \text{, if } \varepsilon > 0, \lambda > 0 \text{ are small enough and } R_1 > 0, R_2 > 0 \text{ are large enough. Therefore, the solution } u_3 \text{ is obtained by applying the linking theorem. Since}

\[
I_{\lambda,s}(u_1) = I_{\lambda,s}^*(u_1) := C_1^* \leq \frac{\lambda I_1^q}{q} \int_\Omega \varphi_1^q \, dx < a \leq c_s = I_{\lambda,s}(u_3),
\]

we conclude that \( u_1 \neq u_3 \). By the same reasoning, we conclude that \( u_2 \neq u_3 \). We are done.

\[ \square \]

Observe that it follows from (6.1) and the analogous equation for \( I_{\lambda,s}^* \) that, in the case \( 0 < \lambda < qa/ \int_\Omega \varphi_1^q \, dx \), then solutions \( u_1, u_2 \) and \( u_3 \) are distinct for \( \alpha > 0 \) given in Proposition 5.2.

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