Nonexistence of self-similar singularities for the 3D incompressible Euler equations

Dongho Chae
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
e-mail: chae@skku.edu

Abstract
We prove that there exists no self-similar finite time blowing up solution to the 3D incompressible Euler equations. By similar method we also show nonexistence of self-similar blowing up solutions to the divergence-free transport equation in $\mathbb{R}^n$. This result has direct applications to the density dependent Euler equations, the Boussinesq system, and the quasi-geostrophic equations, for which we also show nonexistence of self-similar blowing up solutions.

1 Incompressible Euler equations
We are concerned here on the following Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3$.

\[
\begin{aligned}
(E) \quad \begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\
\text{div } v = 0, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\
v(x, 0) = v_0(x), & x \in \mathbb{R}^3
\end{cases}
\end{aligned}
\]

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x, t)$, $j = 1, 2, 3$, is the velocity of the flow, $p = p(x, t)$ is the scalar pressure, and $v_0$ is the given initial velocity, satisfying

*The work was supported partially by the KOSEF Grant no. R01-2005-000-10077-0.
\[ \text{div } v_0 = 0. \] There are well-known results on the local existence of classical solutions (see e.g. [21, 16, 7] and references therein). The problem of finite time blow-up of the local classical solution is one of the most challenging open problems in mathematical fluid mechanics. On this direction there is a celebrated result on the blow-up criterion by Beale, Kato and Majda (\cite{2}). By geometric type of consideration some of the possible scenarios to the possible singularity has been excluded (see \cite{8, 12, 14}). One of the main purposes of this paper is to exclude the possibility of self-similar type of singularities for the Euler system.

The system \((E)\) has scaling property that if \((v, p)\) is a solution of the system \((E)\), then for any \(\lambda > 0\) and \(\alpha \in \mathbb{R}\) the functions

\[ v_{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p_{\lambda, \alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \quad (1.1) \]

are also solutions of \((E)\) with the initial data \(v_{\lambda, \alpha}^0(x) = \lambda^\alpha v_0(\lambda x)\). In view of the scaling properties in (1.1), the self-similar blowing up solution \(v(x, t)\) of \((E)\) should be of the form,

\[ v(x, t) = \frac{1}{(T^* - t)^{\alpha+1}} V \left( \frac{x}{(T^* - t)^{\alpha+1}} \right) \quad (1.2) \]

for \(\alpha \neq -1\) and \(t\) sufficiently close to \(T^*\). Substituting (1.2) into \((E)\), we find that \(V\) should be a solution of the system

\[ (SE) \left\{ \begin{array}{l} \alpha \frac{V}{\alpha + 1} + \frac{1}{\alpha + 1} (x \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\
\text{div } V = 0 \end{array} \right. \]

for some scalar function \(P\), which could be regarded as the Euler version of the Leray equations introduced in \cite{18}. The question of existence of nontrivial solution to \((SE)\) is equivalent to the that of existence of nontrivial self-similar finite time blowing up solution to the Euler system of the form (1.2). Similar question for the 3D Navier-Stokes equations was raised by J. Leray in \cite{18}, and answered negatively by the authors of \cite{22}, the result of which was refined later in \cite{25}. Combining the energy conservation with a simple scaling argument, the author of this article showed that if there exists a nontrivial self-similar finite time blowing up solution, then its helicity should be zero (\cite{3}, see also \cite{23} for other related discussion). To the author’s knowledge, however, the possibility of self-similar blow-up of the form (1.2)
has never been excluded previously. In particular, due to lack of the laplacian term in the right hand side of the first equations of (SE), we cannot apply the argument of the maximum principle, which was crucial in the work [22] for the 3D Navier-Stokes equations. Using a completely different argument from those used in [3], or [22], we prove here that there cannot be self-similar blowing up solution to (E) of the form (1.2), if the vorticity decays sufficiently fast near infinity.

**Theorem 1.1** There exists no finite time blowing up self-similar solution to the 3D Euler equations of the form (1.2) for \( t \in (0, T^*) \) with \( \alpha \neq -1 \), if there exists \( p_1 > 0 \) such that the vorticity \( \Omega = \text{curl} \, V \in L^p(\mathbb{R}^3; \mathbb{R}^3) \) for all \( p \in (0, p_1) \).

**Remark 1.1** For example, if \( \Omega \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3) \) and there exist constants \( R, K \) and \( \delta > 0 \) such that \( |\Omega(x)| \leq Ke^{-\delta|x|} \) for \( |x| > R \), then we have \( \Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3) \) for all \( p \in (0, 1) \). Indeed, for all \( p \in (0, 1) \), we have
\[
\int_{\mathbb{R}^3} |\Omega(x)|^p \, dx = \int_{|x| \leq R} |\Omega(x)|^p \, dx + \int_{|x| > R} |\Omega(x)|^p \, dx \\
\leq |B_R|^{1-p} \left( \int_{|x| \leq R} |\Omega(x)| \, dx \right)^p + K^p \int_{\mathbb{R}^3} e^{-p\delta|x|} \, dx < \infty,
\]
where \( |B_R| \) is the volume of the ball \( B_R \) of radius \( R \).

**Remark 1.2.** We note that there is no integrability condition imposed on the velocity \( V \) itself in the above theorem. In particular, \( V \) does not need to decay at infinity. For example, if \( \text{curl} \, V = \Omega \) has compact support in \( \mathbb{R}^3 \) with \( \text{div} \, V = 0 \), we have by the Biot-Savart law,
\[
V(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \Omega(y)}{|x - y|^3} \, dy + \nabla h(x)
\]
for a harmonic function \( h(x) \) in \( \mathbb{R}^3 \). Choosing \( h(x) = 2x_1^2 - x_2^2 - x_3^2 \), for example, we have \( V(x) \), which grows to infinite in the positive \( x_1 \)-direction.

The proof of Theorem 1.1 will follow as a corollary of the following more general theorem.

**Theorem 1.2** Suppose there exists \( T > 0 \) such that we have a representation of the vorticity of the solution, \( v \in C([0, T); C^1(\mathbb{R}^3; \mathbb{R}^3)) \), to the 3D Euler...
equations by

\[ \omega(x, t) = \Psi(t)\Omega(\Phi(t)x) \quad \forall t \in [0, T) \]  

(1.3)

where \(\Psi(\cdot) \in C([0, T]; (0, \infty))\), \(\Phi(\cdot) \in C([0, T]; \mathbb{R}^{3 \times 3})\) with \(\det(\Phi(t)) \neq 0\) on \([0, T)\); \(\Omega = \text{curl} V\) for some \(V\), and there exists \(p_1 > 0\) such that \(\Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3)\) for all \(p \in (0, p_1)\). Then, necessarily either \(\det(\Phi(t)) \equiv \det(\Phi(0))\) on \([0, T)\), or \(\Omega = 0\).

**Proof.** By consistency with the initial condition, \(\omega_0(x) = \Psi(0)\Omega(\Phi(0)x)\), and hence \(\Omega(x) = \Psi(0)^{-1}\omega_0([\Phi(0)]^{-1}x)\) for all \(x \in \mathbb{R}^3\). We can rewrite the representation (1.3) in the form,

\[ \omega(x, t) = G(t)\omega_0(F(t)x) \quad \forall t \in [0, T), \]  

(1.4)

where \(G(t) = \Psi(t)/\Psi(0)\), \(F(t) = [\Phi(0)]^{-1}\Phi(t)\). In order to prove the theorem it suffices to show that either \(\det(F(t)) = 1\) for all \(t \in [0, T)\), or \(\omega_0 = 0\), since \(\det(F(t)) = \det(\Phi(t))/\det(\Phi(0))\). Let \(a \mapsto X(a, t)\) be the particle trajectory mapping, defined by the ordinary differential equations,

\[ \frac{\partial X(a, t)}{\partial t} = v(X(a, t), t) \quad ; \quad X(a, 0) = a. \]

We set \(A(x, t) := X^{-1}(x, t)\), which is called the back to label map(6), satisfying

\[ A(X(a, t), t) = a, \quad X(A(x, t), t) = x. \]  

(1.5)

Taking curl of the first equation of (E), we obtain the vorticity evolution equation,

\[ \frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v. \]

This, taking dot product with \(\omega\), leads to

\[ \frac{\partial |\omega|}{\partial t} + (v \cdot \nabla)|\omega| = \alpha|\omega|, \]  

(1.6)

where \(\alpha(x, t)\) is defined as

\[ \alpha(x, t) = \begin{cases} \sum_{i,j=1}^{3} S_{ij}(x, t)\xi_i(x, t)\xi_j(x, t) & \text{if } \omega(x, t) \neq 0 \\ 0 & \text{if } \omega(x, t) = 0 \end{cases} \]
with
\[ S_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right), \quad \text{and} \quad \xi(x, t) = \frac{\omega(x, t)}{|\omega(x, t)|}. \]

We note that (1.6) was previously derived in [8]. In terms of the particle trajectory mapping we can rewrite (1.6) as
\[ \frac{\partial}{\partial t} |\omega(X(a, t), t)| = \alpha(X(a, t), t)|\omega(X(a, t), t)|. \quad (1.7) \]

Integrating (1.7) along the particle trajectories \( \{X(a, t)\} \), we have
\[ |\omega(X(a, t), t)| = |\omega_0(a)| \exp \left[ \int_0^t \alpha(X(a, s), s)ds \right]. \quad (1.8) \]

Taking into account the simple estimates
\[ -\|\nabla v(\cdot, t)\|_{L^\infty} \leq \alpha(x, t) \leq \|\nabla v(\cdot, t)\|_{L^\infty} \quad \forall x \in \mathbb{R}^3, \]
we obtain from (1.8) that
\[ |\omega_0(a)| \exp \left[ -\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq |\omega(X(a, t), t)| \]
\[ \leq |\omega_0(a)| \exp \left[ \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right], \]
which, using the back to label map, can be rewritten as
\[ |\omega_0(A(x, t))| \exp \left[ -\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq |\omega(x, t)| \]
\[ \leq |\omega_0(A(x, t))| \exp \left[ \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right]. \quad (1.9) \]

Combining this with the self-similar representation formula in (1.4), we have
\[ |\omega_0(A(x, t))| \exp \left[ -\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq G(t)|\omega_0(F(t)x)| \]
\[ \leq |\omega_0(A(x, t))| \exp \left[ \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right]. \quad (1.10) \]

5
Given \( p \in (0, p_1) \), computing \( L^p(\mathbb{R}^3) \) norm of the each side of (1.10), we derive

\[
\|\omega_0\|_{L^p} \exp \left[ -\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq G(t)[\det(F(t))]^{-\frac{4}{p}}\|\omega_0\|_{L^p} \\
\leq \|\omega_0\|_{L^p} \exp \left[ \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right],
\]

(1.11)

where we used the fact \( \det(\nabla A(x, t)) \equiv 1 \). Now, suppose \( \Omega \neq 0 \), which is equivalent to assuming that \( \omega_0 \neq 0 \), then we divide (1.11) by \( \|\omega_0\|_{L^p} \) to obtain

\[
\exp \left[ -\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq G(t)[\det(F(t))]^{-\frac{1}{p}} \\
\leq \exp \left[ \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right].
\]

(1.12)

If there exists \( t_1 \in (0, T) \) such that \( \det(F(t_1)) \neq 1 \), then either \( \det(F(t_1)) > 1 \) or \( \det(F(t_1)) < 1 \). In either case, setting \( t = t_1 \) and passing \( p \searrow 0 \) in (1.12), we deduce that

\[
\int_0^{t_1} \|\nabla v(\cdot, s)\|_{L^\infty} ds = \infty.
\]

This contradicts with the assumption that the flow is smooth on \( (0, T) \), i.e \( v \in C([0, T); C^1(\mathbb{R}^3; \mathbb{R}^3)) \). \( \square \)

**Proof of Theorem 1.1** We apply Theorem 1.2 with

\[
\Phi(t) = (T_s - t)^{-\frac{1}{\alpha+1}}I, \quad \text{and} \quad \Psi(t) = (T_s - t)^{-1},
\]

where \( I \) is the unit matrix in \( \mathbb{R}^{3 \times 3} \). If \( \alpha \neq -1 \) and \( t \neq 0 \), then

\[
\det(\Phi(t)) = (T_s - t)^{-\frac{3}{\alpha+1}} \neq T_s^{-\frac{3}{\alpha+1}} = \det(\Phi(0)).
\]

Hence, we conclude that \( \Omega = 0 \) by Theorem 1.2. In this case, there is no finite time blow-up for \( v(x, t) \), since the vorticity is zero. \( \square \)
2 Divergence-free transport equation

The previous argument in the proof of Theorem 1.1 can also be applied to the following transport equations by a divergence-free vector field in \(\mathbb{R}^n\), \(n \geq 2\).

\[
(TE) \begin{cases}
\frac{\partial \theta}{\partial t} + (v \cdot \nabla) \theta = 0, \\
\text{div} \, v = 0, \\
\theta(x, 0) = \theta_0(x),
\end{cases}
\]

where \(v = (v_1, \ldots, v_n) = v(x, t)\), and \(\theta = \theta(x, t)\). In view of the invariance of the transport equation under the scaling transform,

\[
v(x, t) \mapsto v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1}t),
\]

\[
\theta(x, t) \mapsto \theta^{\lambda, \alpha, \beta}(x, t) = \lambda^\beta \theta(\lambda x, \lambda^{\alpha+1}t)
\]

for all \(\alpha, \beta \in \mathbb{R}\) and \(\lambda > 0\), the self-similar blowing up solution is of the form,

\[
v(x, t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha + 1}}} V \left( \frac{x}{(T_* - t)^{\frac{1}{\alpha + 1}}} \right), \tag{2.1}
\]

\[
\theta(x, t) = \frac{1}{(T_* - t)^\beta} \Theta \left( \frac{x}{(T_* - t)^{\frac{1}{\alpha + 1}}} \right) \tag{2.2}
\]

for \(\alpha \neq -1\) and \(t\) sufficiently close to \(T_*\). Substituting (2.1) and (2.2) into the above transport equation, we obtain

\[
(ST) \begin{cases}
\beta \Theta + \frac{1}{\alpha + 1} (x \cdot \nabla) \Theta + (V \cdot \nabla) \Theta = 0, \\
\text{div} \, V = 0.
\end{cases}
\]

The question of existence of suitable nontrivial solution to (ST) is equivalent to the that of existence of nontrivial self-similar finite time blowing up solution to the transport equation. We will establish the following theorem.

**Theorem 2.1** Suppose there exist \(\alpha \neq -1, \beta \in \mathbb{R}\) and solution \((V, \Theta)\) to the system (ST) with \(\Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)\) for some \(p_1, p_2\) such that \(0 < p_1 < p_2 \leq \infty\). Then, \(\Theta = 0\).
This theorem is a corollary of the following one.

**Theorem 2.2** Suppose there exists \( T > 0 \) such that there exists a representation of the solution \( \theta(x,t) \) to the system (TE) by

\[
\theta(x,t) = \Psi(t)\Theta(\Phi(t)x) \quad \forall t \in [0,T)
\]

where \( \psi(\cdot) \in C([0,T);(0,\infty)) \), \( \Phi(\cdot) \in C([0,T);\mathbb{R}^{n\times n}) \) with \( \det(\Phi(t)) \neq 0 \) on \( [0,T) \); there exists \( p_1 < p_2 \) with \( p_1, p_2 \in (0,\infty) \) such that \( \Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n) \). Then, necessarily either \( \det(\Phi(t)) \equiv \det(\Phi(0)) \) and \( \psi(t) \equiv \psi(0) \)

\( \square \)

**Proof.** Similarly to the proof of Theorem 1.2 the representation \([2.3] \) reduces to the form,

\[
\theta(x,t) = G(t)\theta_0(F(t)x),
\]

where \( G(t) = \psi(t)/\psi(0) \), \( F(t) = \Phi(t)[\Phi(0)]^{-1} \). By standard \( L^p \)-interpolation and the relation between \( \theta_0 \) and \( \Theta \) by \( \theta_0(x) = \psi(0)\Theta(\Phi(0)x) \), we have that \( \Theta \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n) \) implies \( \theta_0 \in L^p(\mathbb{R}^n) \) for all \( p \in [p_1, p_2] \). As in the proof of Theorem 1.2 we denote by \( \{X(a,t)\} \) and \( \{A(x,t)\} \) the particle trajectory map and the back to label map respectively, each one of which is defined by \( v(x,t) \). As the solution of the first equation of (TE) we have \( \theta(X(a,t),t) = \theta_0(a) \), which can be rewritten as \( \theta(x,t) = \theta_0(A(x,t)) \) in terms of the back to label map. This, combined with \([2.4] \), provides us with the relation

\[
\theta_0(A(x,t)) = G(t)\theta_0(F(t)x). \quad (2.5)
\]

Using the fact, \( \det(\nabla A(x,t)) = 1 \), we compute \( L^p(\mathbb{R}^n) \) norm of \([2.5] \) to have

\[
\|\theta_0\|_{L^p} = |G(t)||\det(F(t))|^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n} |\theta(F(t)x)|^p|\det(F(t))|dx \right)^{\frac{1}{p}}
\]

\[
= |G(t)||\det(F(t))|^{-\frac{1}{p}}\|\theta_0\|_{L^p} \quad (2.6)
\]

for all \( t \in [0,T) \) and \( p \in [p_1, p_2] \). Suppose \( \theta_0 \neq 0 \), which is equivalent to \( \Theta \neq 0 \), then we divide \([2.6] \) by \( \|\theta_0\|_{L^p} \) to obtain \( |G(t)|^p = \det(F(t)) \) for all \( t \in [0,T) \) and \( p \in [p_1, p_2] \), which is possible only if \( G(t) = \det(F(t)) = 1 \) for all \( t \in [0,T) \). Hence, \( \psi(t) \equiv \psi(0) \), and \( \det(\Phi(t)) \equiv \det(\Phi(0)) \). \( \square \)

**Proof of Theorem 2.1** We apply Theorem 2.2 with

\[
\Phi(t) = (T_* - t)^{-\gamma}, \quad \psi(t) = (T_* - t)^{-\beta},
\]

8
where $I$ is the unit matrix in $\mathbb{R}^{n \times n}$. Then,

$$
\det(\Phi(t)) = (T_* - t)^{-n/(\alpha + 1)} \neq \det(\Phi(0)) = T_*^{-n/(\alpha + 1)} \quad \text{if} \quad \alpha \neq -1, t \neq 0.
$$

Hence, by Theorem 2.2 we have $\Theta = 0$. \qed

Below we present some examples of fluid mechanics, where we can apply similar argument to the above to prove nonexistence of nontrivial self-similar blowing up solutions.

A. The density-dependent Euler equations

The density-dependent Euler equations in $\mathbb{R}^n$, $n \geq 2$, are the following system.

$$
\begin{align*}
\frac{\partial \rho v}{\partial t} + \text{div} \left( \rho v \otimes v \right) &= -\nabla p, \\
\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho &= 0, \\
\text{div} \ v &= 0, \\
v(x, 0) &= v_0(x), \quad \rho(x, 0) = \rho_0(x),
\end{align*}
$$

where $v = (v_1, \cdots, v_n) = v(x, t)$ is the velocity, $\rho = \rho(x, t) \geq 0$ is the scalar density of the fluid, and $p = p(x, t)$ is the pressure. We refer to section 4.5 in [19] for more detailed introduction of this system. Here we just note that this system reduces to the homogeneous Euler system of the previous section when $\rho \equiv 1$. The question of finite time blow-up for the system is wide open even in the case of $n = 2$, although we have local in time existence result of the classical solution and its finite time blow-up criterion (see e.g. [1, 4]). The system $(E_1)$ has scaling property that if $(v, \rho, p)$ is a solution of the system $(E_1)$, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$
v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad \rho^{\lambda, \alpha, \beta}(x, t) = \lambda^\beta \rho(\lambda x, \lambda^{\alpha+1} t), \quad \text{(2.7)}
$$

$$
p^{\lambda, \alpha, \beta}(x, t) = \lambda^{2\alpha+\beta} p(\lambda x, \lambda^{\alpha+1} t) \quad \text{(2.8)}
$$

are also solutions of $(E_1)$ with the initial data

$$
v^{\lambda, \alpha}_0(x) = \lambda^\alpha v_0(\lambda x), \quad \rho^{\lambda, \alpha, \beta}_0(x) = \lambda^\beta \rho_0(\lambda x).
$$
In view of the scaling properties in (2.7), we should check if there exists nontrivial solution \((v(x, t), \rho(x, t))\) of \((E_1)\) of the form,

\[
v(x, t) = \frac{1}{(T_\ast - t)^{\frac{\alpha}{\alpha + 1}}} V \left( \frac{x}{(T_\ast - t)^{\frac{1}{\alpha + 1}}} \right), \tag{2.9}
\]

\[
\rho(x, t) = \frac{1}{(T_\ast - t)^{\beta}} R \left( \frac{x}{(T_\ast - t)^{\frac{1}{\alpha + 1}}} \right) \tag{2.10}
\]

for \(\alpha \neq -1\) and \(t\) sufficiently close to \(T_\ast\). The solution \((v, \rho)\) of the form (2.9)-(2.10) is called the self-similar blowing up solution of the system \((E_1)\).

The following theorem establish the nonexistence of nontrivial self-similar blowing up solution of the system \((E_1)\), which is immediate from Theorem 2.2.

**Theorem 2.3** Suppose there exist \(\alpha \neq -1\) and solution \((v, \rho)\) to the system \((E_1)\) of the form (2.9)-(2.10), for which there exists \(p_1, p_2\) with \(0 < p_1 < p_2 \leq \infty\) such that \(R \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)\). Then, \(R = 0\).

**B. The 2D Boussinesq system**

The Boussinesq system for the inviscid fluid flows in \(\mathbb{R}^2\) is given by

\[
\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\nabla p + \theta e_1, \\
\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta &= 0, \\
\text{div } v &= 0, \\
v(x, 0) &= v_0(x), \quad \theta(x, 0) = \theta_0(x)
\end{align*}
\]

where \(v = (v_1, v_2) = v(x, t)\) is the velocity, \(e_1 = (1, 0)\), and \(p = p(x, t)\) is the pressure, while \(\theta = \theta(x, t)\) is the temperature function. The local in time existence of solution and the blow-up criterion of the Beale-Kato-Majda type has been well known(see e.g. [15, 34]). The question of finite time blow-up is open until now. Here, we exclude the possibility of self-similar finite time blow-up for the system. The system \((B)\) has scaling property that if \((v, \theta, p)\) is a solution of the system \((B)\), then for any \(\lambda > 0\) and \(\alpha \in \mathbb{R}\) the functions

\[
v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1}t), \quad \theta^{\lambda, \alpha}(x, t) = \lambda^{2\alpha+1} \theta(\lambda x, \lambda^{\alpha+1}t), \tag{2.11}
\]

\[
\begin{align*}
\frac{\partial v^{\lambda, \alpha}}{\partial t} + (v^{\lambda, \alpha} \cdot \nabla)v^{\lambda, \alpha} &= -\nabla^{\lambda, \alpha} p^{\lambda, \alpha} + \theta^{\lambda, \alpha} e_1, \\
\frac{\partial \theta^{\lambda, \alpha}}{\partial t} + (v^{\lambda, \alpha} \cdot \nabla)\theta^{\lambda, \alpha} &= 0, \\
\text{div } v^{\lambda, \alpha} &= 0, \\
v^{\lambda, \alpha}(x, 0) &= v_0^{\lambda, \alpha}(x), \quad \theta^{\lambda, \alpha}(x, 0) = \theta_0^{\lambda, \alpha}(x),
\end{align*}
\]

where \(v^{\lambda, \alpha} = (v_1^{\lambda, \alpha}, v_2^{\lambda, \alpha}) = v^{\lambda, \alpha}(x, t)\) is a solution of the system \((B)\), then for any \(\lambda > 0\) and \(\alpha \in \mathbb{R}\) the functions
are also solutions of (B) with the initial data

er
\[ v_0^{\lambda,\alpha}(x) = \lambda^\alpha v_0(\lambda x), \quad \theta_0^{\lambda,\alpha}(x) = \lambda^{2\alpha+1} \theta_0(\lambda x). \]

In view of the scaling properties in (2.11), the self-similar blowing-up solution \((v(x,t), \theta(x,t))\) of (B) should of the form,

\[ v(x,t) = \frac{1}{(T_* - t)^{\alpha+1}} V \left( \frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right), \quad (2.13) \]

\[ \theta(x,t) = \frac{1}{(T_* - t)^{2\alpha+1}} \Theta \left( \frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}} \right), \quad (2.14) \]

where \(\alpha \neq -1\). We have the following nonexistence result of such type of solution.

**Theorem 2.4** There exists no nontrivial solution \((v, \theta)\) of the system (B) of the form (2.13)-(2.14), if there exists \(p_1, p_2 \in (0, \infty), p_1 < p_2\), such that \(\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)\), and \(V \in H^m(\mathbb{R}^2), m > 2\).

**Proof.** Similarly to the proof of Theorem 2.1, we first conclude \(\Theta = 0\), and hence \(\theta(\cdot, t) \equiv 0\) on \([0, T_*)\). Then, the system (B) reduces to the 2D incompressible Euler equations, for which we have global in time regular solution for \(v_0 \in H^m(\mathbb{R}^2), m > 2\) (see e.g. [17]). Hence, we should have \(v(\cdot, t) \equiv 0\) on \([0, T_*)\). \(\square\)

**Note added to the proof.** Similar proof to the above leads to the nonexistence of self-similar blowing up solution to the axisymmetric 3D Euler equations with swirl of the form (1.2), if \(\Theta = rV^\theta\) satisfies the condition of Theorem 2.4, and \(\text{curl } V \in H^m(\mathbb{R}^3), m > 5/2\), where \(r = \sqrt{x_1^2 + x_2^2}\), and \(V^\theta\) is the angular component of \(V\). Indeed, applying Theorem 2.2 to the \(\theta\)-component of the Euler equations, \(\frac{\partial}{\partial t}(rv^\theta) = 0\), we show that \(v^\theta = 0\) as in the above proof, and then we use the global regularity result for the 3D axisymmetric Euler equations without swirl ([20] [24]) to conclude that \((v^r, v^3)\) is also zero.
C. The 2D quasi-geostrophic equation

The following 2D quasi-geostrophic equation (QG) models the dynamics of the mixture of cold and hot air, and the fronts between them.

\[
\begin{aligned}
\frac{\partial \theta}{\partial t} + (v \cdot \nabla) \theta &= 0, \\
v &= -\nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta = \nabla^\perp \left( \int_{\mathbb{R}^2} \frac{\theta(y,t)}{|x-y|} \, dy \right), \\
\theta(x,0) &= \theta_0(x),
\end{aligned}
\]

where \( \nabla^\perp = (-\partial_2, \partial_1) \). Besides its physical significance, mainly due to its similar structure to the 3D Euler equations, there have been many recent studies on this system (see e.g. [9, 10, 11] and references therein). Although the question of finite time singularities is still open, some type of scenarios of singularities have been excluded ([10, 11, 13]). Here we exclude the scenario of self-similar singularity. The system (QG) has the scaling property that if \( \theta \) is a solution of the system, then for any \( \lambda > 0 \) and \( \alpha \in \mathbb{R} \) the functions

\[
\theta^{\lambda,\alpha}(x,t) = \lambda^\alpha \theta(\lambda x, \lambda^{\alpha+1}t)
\]

are also solutions of (QG) with the initial data \( \theta^{\lambda,\alpha}_0(x) = \lambda^\alpha \theta_0(\lambda x) \). Hence, the self-similar blowing up solution should be of the form,

\[
\theta(x,t) = \frac{1}{(T_* - t)^{\alpha+1}} \Theta \left( \frac{x}{(T_* - t)^{\alpha+1}} \right)
\]

for \( t \) sufficiently close \( T_* \) and \( \alpha \neq -1 \). Applying the same argument as in the proof of Theorem 2.1, we have the following theorem.

**Theorem 2.5** There exists no nontrivial solution \( \theta \) to the system (QG) of the form (2.16), if there exists \( p_1, p_2 \in (0, \infty), \ p_1 < p_2, \) such that \( \Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2) \).

**References**

[1] Beirão da Veiga and A. Valli, *Existence of \( C^\infty \) solutions of the Euler equations for nonhomogeneous fluids*, Comm. P.D.E., 5, (1980), pp.95-107.
[2] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys., **94**, (1984), pp. 61-66.

[3] D. Chae, *Remarks on the blow-up of the Euler Equations and the Related Equations*, Comm. Math. Phys., **245**, no. 3, (2003), pp. 539-550.

[4] D. Chae and J. Lee, *Local existence and blow-up criterion of the inhomogeneous Euler equations*, J. Math. Fluid Mech., **5**, (2003), 144-165.

[5] D. Chae and H-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations*, Proc. Roy. Soc. Edinburgh Sect. A **127**, no. 5, (1997), pp.935-946.

[6] P. Constantin, *An Eulerian-Lagrangian approach for incompressible fluids: local theory*. J. Amer. Math. Soc. **14**, no. 2, (2001), pp. 263-278.

[7] P. Constantin, *A few results and open problems regarding incompressible fluids*, Notices Amer. Math. Soc. **42**, No. 6, (1995), pp. 658-663.

[8] P. Constantin, C. Fefferman and A. Majda, *Geometric constraints on potential singularity formulation in the 3-D Euler equations*, Comm. P.D.E, **21**, (3-4), (1996), pp. 559-571.

[9] P. Constantin, A. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar*, Nonlinearity, **7** (1994), 1495-1533.

[10] D. Córdoba, *Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation*, Ann. of Math. **148**, (1998), pp. 1135-1152.

[11] D. Córdoba, C. Fefferman, *Growth of solutions for QG and 2D Euler equations*, J. Amer. Math. Soc. **15**, (2002), no. 3, pp. 665-670.

[12] D. Córdoba, C. Fefferman, *On the collapse of tubes carried by 3D incompressible flows*, Comm. Math. Phys., **222**, no. 2, (2001), pp. 293-298.

[13] D. Córdoba, C. Fefferman and R. De La LLave, *On squirt singularities in hydrodynamics*, SIAM J. Math. Anal., **36**, no. 1, (2004), pp. 204-213.

[14] J. Deng, T. Y. Hou and X. Yu, *Geometric and Nonblowup of 3D Incompressible Euler Flow*, Comm. P.D.E, **30**, (2005), pp. 225-243.
[15] W. E. and C. Shu, Small scale structures in Boussinesq convection, Phys. Fluids, 6, (1994), pp. 48-54.

[16] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, J. Func. Anal. 9, (1972), pp. 296-305.

[17] T. Kato and G. Ponce, On nonstationary flows of viscous and ideal fluids in $L^p_2(\mathbb{R}^2)$, Duke Math. J., 55, (1987), pp. 487-499.

[18] J. Leray, Essai sur le mouvement d’un fluide visqueux emplissant l’espace, Acta Math. 63 (1934), pp. 193-248.

[19] P. L. Lions, Mathematical Topics in Fluid Mechanics: Vol. 1 Incompressible Models, Clarendon Press, Oxford, (1996).

[20] A. Majda, Vorticity and the mathematical theory of incompressible fluid flow, Comm. Pure Appl. Math., 39, (1986), pp. 187-220.

[21] A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge Univ. Press. (2002).

[22] J. Nečas, M. Ružička and V. Šverák, On Leray’s self-similar solutions of the Navier-Stokes equations, Acta Math. 176, (1996), pp. 283-294.

[23] Y. Pomeau and D. Sciamarella, An unfinished tale of nonlinear PDEs: Do solutions of 3D incompressible Euler equations blow-up in finite time?, Physica D, 205, (2005), pp. 215-221.

[24] X. Saint Raymond, Remarks on axisymmetric solutions of the incompressible Euler system, Comm. P. D. E. 19, (1994), pp.321-334.

[25] T-P. Tsai, On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates, Arch. Rat. Mech. Anal., 143, no. 1, (1998), pp. 29-51.