Constrained effective potential in hot QCD

by

C.P.Korthals Altes

Centre Physique Théorique au CNRS
Campus de Luminy, Route Léon Lachamps, B.P.907
F13288, Marseille, Cedex 2, France

ABSTRACT

Constrained effective potentials in hot gauge theory give the probability that a configuration p of the order parameter (Polyakov loop) occurs. They are important in the analysis of surface effects and bubble formation in the plasma. The vector potential appears non-linearly in the loop; in weak coupling the linear term gives rise to the traditional free energy graphs. But the non-linear terms generate insertions of the constrained modes into the free energy graphs, through renormalisations of the Polyakov loop. These insertions are gauge dependent and are necessary to cancel the gauge dependence of the free energy graphs. The latter is shown, through the BRST identities, to have again the form of constrained mode insertions. It also follows, that absolute minima of the potential are at the centergroup values of the loop. We evaluate the two-loop contributions for SU(N) gauge theories, with and without quarks, for the full domain of the N-1 variables.
1. Introduction

Study of high-temperature QCD\(^1\) has become urgent because of the building of the relativistic ion collider (RHIC). Apart from this, there is the vast area of early cosmology, where hot QCD has come into play. At these temperatures the Z(N) symmetry of quarkless QCD is broken spontaneously leaving us with N ordered phases. Adding quarks breaks this symmetry explicitly, but the deconfinement transition is still there.

Recently there has been interest\(^2\)\(^3\) in the surface tension that occurs in QCD with quarks only as an order-disorder interface, and, subsequently, in the surface tension between the different ordered phases in quarkless QCD. These ordered phases are characterized by the order parameter (the Polyakov loop) having a Z(N) center group value. Numerical work on this preceded our analytic approach\(^1\)\(^5\), and both are in reasonable agreement.

One key ingredient in these calculations is the profile \( p \) that the Polyakov loop \( p \) develops in between two Z(N) phases. The other key ingredient is the probability that a given profile occurs. This quantity is called a ”constrained” effective potential.

In this paper we will calculate the constrained effective potential \( U \), that is associated with the probability that a given constant mode \( \bar{p} \equiv \frac{1}{V} \int d \bar{x} \, p(\bar{x}) \) appears:

\[
\exp[-V \beta U(t)] \equiv \frac{\int DA \delta(t - \bar{p}) \exp \left(-\frac{1}{g^2} S(A)\right)}{\int DA \exp \left(-\frac{1}{g^2} S(A)\right)}
\]

\( U(t) \) as a function of \( t \) is computable in perturbation theory. It is accessible to Monte Carlo simulations and is gauge independent. We will give the perturbative rules for computing it; these rules include the ad-hoc prescription of ref 12.

In a previous paper\(^5\) we did compute the constrained effective action for a special one parameter set of the Polyakov loop, in the absence of quarks. This special set is called the ”q valley”. Its physical significance is due to the fact that it vehicles the tunneling effects giving rise to the surface tension between two ordered vacua\(^5\).

In this paper we will give the full result, including all values of the Polyakov loop outside the q valley, and including quarks. We show (in section 3) that the ad hoc insertion method in\(^12\) and\(^5\) follows from starting from the constrained path integral. Our results are useful for calculations of transition probabilities between various stable and unstable minima that appear in the quark case, e.g. for the case of three colours and six quarks\(^15\).

Apart from this we have been motivated by two reasons.
i) To lift some of the uncertainties in the literature on the pure gluon results. Those readers that are only interested in the end results can consult eqns 4.2, 4.3 (pure glue) 5.2 and 5.4 (fermions). Our pure glue results are in agreement with ref 5 eqn 5.1, 5.13 and 5.14.

ii) To gain some insight into the gauge-cancellations using BRST identities generalized to a non-trivial Polyakov loop background, and the accompanying stabilisation of the $Z(N)$ minima due to extra vertices induced by the background. There is some overlap with ref 13, but our methods are complementary. The gauge artifact cancellations are shown in 2.35 and 3.18.

As mentioned, there has been related work by various authors, with different motivations and different outcomes. With our method we compute in various gauge fixing choices and find the same result. This fortifies our confidence in our results. Besides, we find that the $Z(N)$ vacua stay stable to two loop order; if quarks are present the stable vacuum stays at where the Polyakov loop takes the value one, whereas metastable minima remain at roughly the location of the non-trivial center group values of the loop (see section 5 and 6).

In section 2 we explain in some detail the method of calculation with as main result eq. 2.35. The expectation value of the Polyakov loop to one loop order is computed in section 3, resulting in eq. 3.17 and 3.18. In section 4 the gluonic 2-loop results are given in 4.2 and 4.3. In section 5 the fermionic results are analysed. Section 6 contains conclusions and outlook. A few appendices elaborate on the results.
2. The method to compute the effective action $U$. 

This section is rather long so it is divided into six subsections. Section 2.a defines the effective action, section 2.b gives the saddle point method for the perturbative evaluation and section 2.c deals with the one loop results. In section 2.d the two loop results are explained; it is here that the intricacies of the non linearities of the Polyakov loop enter. How the final answer is gauge choice independent is detailed in sections 2.e and 2.f.

2.a. Effective potential $U$.

We will work in a finite box $V = L^3$ in a heat both of temperature $T \equiv \beta^{-1}$; in the box we have an $SU(N)$ Yang Mills field, and we work on the usual Euclidean formulation with periodic b.c.'s in the time direction. This means that the unitary NxN matrix $P(A_0(\vec{x}))$ defined by

$$P(A_0(\vec{x})) \equiv \mathcal{P}(\exp i \int_0^\beta d\tau A_0(\vec{x}\tau)) \equiv \lim_{n \to \infty} \prod_{k=1}^n \exp i \Delta \tau_k A_0(\vec{x}, \tau_k) \quad 2.1$$

- $\mathcal{P}$ being the path ordering, $\tau_{k+1} \equiv \tau_k + \Delta \tau$, $\Delta \tau \equiv \beta/n$ - is transformed by a periodic gauge transformation $\Omega$ like $\Omega P(A_0)\Omega^{-1}$, and has therefore gauge invariant eigenvalues. Another way of saying this that :

$$t_1(A_0(\vec{x})) \equiv \frac{1}{N} Tr P(A_0(\vec{x})), \ldots, t_{N-1}(A_0(\vec{x})) \equiv \frac{1}{N} Tr P^{N-1}(A_0(\vec{x})) \quad 2.1(a)$$

are gauge invariant. For $Tr P(A_0(\vec{x}))$, the "Polyakov loop" we have a straightforward interpretation once it is averaged in the Euclidean path integral formalism :

$$\langle t_1(A_0(\vec{x})) \rangle \equiv \frac{\int DA t_1(A_0(\vec{x})) \exp -\frac{1}{g^2} S(A)}{\int DA \exp -\frac{1}{g^2} S(A)} \quad 2.2$$

This average is the exponential of the free energy of an infinitely heavy quark (i.e. a fermion in the fundamental representation of $SU(N)$) at the point $\vec{x}$. $\langle t_1(\vec{x}) \rangle$ depends on the spatial boundary conditions. This simple relation between the average and the free energy $F_q$ of a single quark reads:

$$\langle t_1(A_0(\vec{x})) \rangle = \exp -\beta F_q \quad 2.3$$

From this relation \(^1\) follows the use of $< t_1(\vec{x}) >$ as an order parameter.
Averages of traces of higher powers of the loop $2.1 \ t_k(\vec{x})$ can be gotten by suitable linear combinations of free energies of "quarks" in higher representations of SU(N). Taking the SU(3) case as an example, one can relate

$$< TrP(A_0)^2 >= < TrP_6(A_0) > - < Tr\bar{P}(A_0) >$$

with $P_6$ the Polyakov loop in the sextet representation of SU(3). From now on we will be interested in spatial averages $\bar{t}_k \equiv \frac{1}{V} \int d\vec{x} \ t_k(A_0(\vec{x}))$, and we will define, following Fukuda et al$^{8,9}$, the constrained effective potential $U$ by:

$$\exp{\beta V} (U(t_1 \ldots t_{N-1}) + F) \equiv \int DA \prod_{k=1}^{N-1} \delta(t_k - \bar{t}_k) \exp{-\frac{1}{g^2}S(A)}$$

The left hand side is up to normalisation the probability that a given set of fixed numerical values $t_1 \ldots t_{N-1}$ for the corresponding spatial averages appears in our system$^{20}$. The normalisation is in terms of the free energy $F$, which one gets by doing the path integral 2.4 without the constraints. In ref 9 it was argued that this effective potential is numerically a very useful quantity, and easier accessible than say the usual free energy in the presence of sources $J_1 \ldots J_{N-1}$ for the $\bar{t}_k$:

$$\exp{\beta V F(J_1 \ldots J_{N-1})} \equiv \int DA \exp{-\frac{1}{g^2}S(A) + V \sum_{k=1}^{N-1} J_k \cdot \bar{t}_k}$$

This free energy is obviously related to 2.4 through the Laplace transform:

$$\exp{\beta V F(J_1 \ldots J_{N-1})} = \int dt_1 \ldots dt_{N-1} \exp{V J_1 t_1 + \ldots + V J_{N-1} t_{N-1}} \exp{-\beta V U(t_1 \ldots t_{N-1})}$$

$U$ and $F$ are clearly gauge invariant.

Consider the infinite volume limit in eqn 2.5. The ensuing saddle point equations give $U$ the same dependence on the $t_k$ as the effective potential $G = F - \frac{\partial F}{\partial J}$ has on the expectation values of the $t_k$. In fact one has

$$G = U + F(0)$$

in the infinite volume limit. The reader may ask at this point: why not discuss the effective potential instead of the constrained effective potential? A first reason for preferring the
latter is that it is given directly in terms of a path integral, eqn 2.4. Second, in a finite volume there are differences, which may become significant in perturbation theory, when there are infrared divergencies. Third, the study of surface effects necessitates the use of the constrained effective potential.

Some authors\textsuperscript{13} prefer to discuss the eigenvalues of the loop, rather than the traces. The Jacobian between the two is a van der Monde determinant, as explained in appendix D. We prefer traces, because they are easier to handle when taking higher order effects into account (see section 3).

The next subsection deals with the perturbative evaluation of the effective potential.

2.b. Perturbative evaluation.

As in ref 5) we will Fourier transform the $\delta$-constraints. This will give us $N - 1$ variables $\lambda_k$, $k = 1 \ldots N - 1$, with:

$$\prod_{k=1}^{N-1} \delta(t_k - \bar{t}_k) = \int \prod_{k=1}^{N-1} d\lambda_k \ e^{i \lambda_k (t_k - \bar{t}_k)}$$

2.7

In the following we will drop the index $k$ for notational simplicity, but indicate the dependence of $\bar{t}$ on the potential $A_0$. Substituting 2.7 into 2.4 leaves us with a path integral in terms at the gauge invariant variables $\lambda$ and the gauge potentials $A$:

$$\exp{-\beta V_U(t)} = \int d\lambda \int DA \exp{i\lambda(t - \bar{t}(A_0)) - \frac{1}{g^2}S(A)}$$

2.8

To keep the bookkeeping of our degrees of freedom precise, we have to introduce boundary conditions, e.g. periodic boundary conditions.

We will search for a saddlepoint $\lambda = b$, $A = B$ and set therefore:

$$A_\mu = B_\mu + gQ_\mu$$

$$\lambda = b + gq$$

2.9

where $Q$ and $q$ are quantumvariables, expand the exponent in 2.8 and require that linear terms cancel. The integration over the quantum variables in 2.9 will leave us with a functional $Z(b, B)$ which stays invariant under a gauge transformation

$$A^\Omega = \Omega A \Omega^{-1} + \Omega \partial \Omega^{-1}$$
The latter can be rewritten in obvious way as

\[ B^\Omega = \Omega B\Omega^{-1} + \Omega \partial\Omega^{-1} \]  

2.10

together with

\[ Q^\Omega = \Omega Q\Omega^{-1} \]

It is then clear that indeed

\[ Z(b, B) = Z(b, B^\Omega) \]  

2.11

The path integral 2.8 still needs gauge fixing; in order to retain property 2.11, we will take the familiar background field gauge fixing:

\[ S_{g.f.} \equiv \frac{1}{2\xi} Tr(D_\mu(B)Q_\mu)^2 \]  

2.12

with the Faddeev-Popov term:

\[ \bar{\eta}D_\mu(B)D_\mu(B + Q)\omega \]  

2.13

Using 2.1 and 2.9 we find for the linear term in an obvious short hand notation:

\[ i(t - \bar{t}(B))q + ib\bar{t}^{(1)}(B)Q - S^{(1)}(B) \cdot Q = 0 \]  

2.14

This gives three saddle point equations, one for the \( q \)'s (2.15(a)), one for those \( Q \)'s, that couple to \( \bar{t}^{(1)}(B) \) and \( S^{(1)}(B) \) (2.15(b)) and finally those \( Q \)'s, that couple only to \( S^{(1)}(B) \):

\[ t - \bar{t}(B) = 0 \]  

2.15(a)

\[ ib\bar{t}^{(1)}(B) - S^{(1)}(B) = 0 \]  

2.15(b)

\[ S^{(1)}(B) = 0 \]  

2.15(c)

Since 2.11 determines \( B \) up to a gauge transformation we choose a solution \( B \) that is computationally most convenient:

\[ B_\mu = C\delta_{\mu 0} \]  

2.16

\( C \) is a space time independent diagonal matrix in the Lie algebra of \( SU(N) \); this satisfies 2.15 with \( b = 0 \). We emphasize this is just a choice, and will come back to it in section 2.e, where the gauge variance of our results is discussed.
So the eigenvalues \(C_i(i = 1 \ldots N)\) of \(C\) obey a constraint

\[
\sum_{i=1}^{N} C_i = 0
\]

and are related to the \(N - 1\) \(t_k\) \((k = 1, \ldots, N - 1)\) through \(t_k = \frac{1}{N} Tr \exp ikC\beta\).

The advantage of this choice for the saddle point lies in the easy diagonalisation of the quadratic part \(S^{(2)}\) of the action :

\[
S^{(2)} = i\frac{1}{V} \int d\vec{x}drTr\tilde{t}^{(1)}(C)Q_0(\vec{x}\tau) \\
+ \int d\vec{x}dr TrQ_\mu(-D^2(C) + (1 - \xi)D_\mu(C)D_\nu(C))Q_\nu + \\
2 \int d\vec{x}dr Tr\tilde{\eta}(-D^2(C))\omega
\]

\(D_\mu(C)\) is the covariant derivative : \(D_\mu(C) \equiv \partial_\mu + \delta_\mu_0[C,\]

Let us diagonalize the last two terms in 2.18, in terms of the colour basis \((\lambda_{ij})_{n,m} = \frac{1}{\sqrt{2}}\delta_{in},\delta_{jm}\) and \(\lambda^d \equiv \frac{1}{r_d} diag(1, \ldots, 1 - d, 0 \ldots, 0), Tr \lambda_d = 0\). Details on this Cartan basis are discussed in appendix D.

The Fourier transform of the \(Q's\) is defined as :

\[
Q_\mu(p) \equiv \frac{1}{V} \int d\vec{x} \int_{0}^{\beta} d\tau e^{-ip\vec{x} - ip^0\tau} Q_\mu(\vec{x}, \tau)
\]

with

\[
p^0 \equiv 2\pi Tn_0, \quad \vec{p} \equiv \frac{2\pi}{L}\vec{n} \quad |n_\mu| = 0, 1, 2, \ldots
\]

We write \(Q_\mu(p)\) in terms of colour components as :

\[
Q_\mu(p) = \sum_{i\neq j} Q_{ij}^\mu(p)\lambda_{ij} + \sum_{d=2}^{N} Q_{d}^\mu(p)\lambda^d
\]

and find for \(S^{(2)}\) in its diagonalized form, restoring the colour indices:

\[
S^{(2)} = i\sum_{l=1,d}^{N-1} q_l t_l^{(1)}(C)Q_0^d(0) + \frac{V}{2} T \sum_{i\neq j,n_\mu} Q_{ij}^\mu(p)((p_{ij}^2)^2\delta_{\mu\nu} - (1 - \xi)p^\mu p^\nu)Q_{ij}^\mu(-p) \\
+ \frac{V}{2} T \sum_{d=2}^{N} \sum_{n} Q_d^\mu(p)(p^2\delta_{\mu\nu} - (1 - \xi)p^\mu p^\nu)Q_d^\mu(-p) \\
+ VT \sum_{i\neq j,n_\mu} \bar{\eta}_{ij}(p)(p^2)^2\omega_{ij}(-p) \\
+ VT \sum_{d=2}^{N} \sum_{n_\mu} \bar{\eta}^d(p)p^2\omega^d(-p)
\]
In the first term the matrix \( \hat{r}_{1,d}^{(1)} = Tr \exp(\imath l C)\lambda_d \) results from expanding the l-th power of the Polyakov loop to first order in the quantum field \( Q_0^d(0) \). The symbol \( p_{\mu}^{ij} \) stands for \( p_0^{ij} = p_0 + C_i - C_j \), and \( p_k^{ij} = p_k \). This shift in the momentum is due to the covariant derivatives in 2.18.

Let us note that \( \tilde{n}^k(p = 0), \tilde{\omega}^k(p = 0) \) and \( \tilde{Q}^k_\mu(p = 0) \) are non-Gaussian variables. This is due to our periodic boundary conditions. We can use e.g. twisted b.c.s and overcome this problem to get the same thermodynamic limit as we will get here by ignoring the problem.

The \( N - 1 \) constraint variables \( q \) couple only to the \( N - 1 \) zero-momentum diagonal \( Q_0 \) variables, \( Q_0^d(0) \), in the first term of 2.21.

2.c. One loop result

The one loop result is obtained from the expression for \( S^{(2)} \) in 2.21, by substituting it in the exponent of 2.8.

We have first to integrate out the \( q \) variables. This will give us delta function constraints on the \( N-1 \) \( Q_0^d(0) \) variables, of the type \( \delta(Q_0^d(0)) \), and the one loop result is:

\[
V \beta U^{(1)}(t) = \frac{1}{2} Tr' \log(p_{ij}^2 \delta_{\mu\nu} - (1 - \zeta)p_{\mu}^{ij}p_{\nu}^{ij}) - Tr' \log p_{ij}^2
\]

2.22

The prime on the trace means we left out the eigenvalues corresponding to these \( N - 1 \) modes.

We are interested in the thermodynamic limit \( V \to \infty \). Then these \( N - 1 \) constraints, as argued in ref 5), are not important.

For completeness we mention the result\(^1\) :

\[
U^{(1)}(t) = \frac{2T^4\pi^2}{3} \sum_{i \neq j} C_{ij}^2 (1 - C_{ij})^2
\]

2.23

where \( C_{ij} \equiv \frac{C_i - C_j}{2\pi T} \), and have to be taken mod 1; the \( t \) are given in terms of the diagonal matrix \( C \) 2.16 :

\[
t_k \equiv \frac{1}{N} Tr e^{i k C_{i\beta}}
\]

2.24

Note that \( U^{(1)} \) is independent of the gauge choice, as it should. So far, all our labours did not lead to anything new. This changes in the next subsection.

2.d. Two loop results
For the two loop results we need to expand the action and the constraint to order $g^2$. To avoid clutter in the formulas we will stop indicating the colour degrees of freedom on $q, \bar{t}^{(2)}(C)$ etc. . . . This leads to:

$$S^{(3)} = i q \left( \frac{g^2}{2!} \bar{t}^{(2)}(C) \cdot \bar{Q}_0^2 + \frac{g^2}{3!} \bar{t}^{(3)}(C) \cdot \bar{Q}_0^3 \right) + \frac{g}{3!} S^{(3)}_{\text{inv}} \cdot Q^3 + \frac{g^2}{4!} S^{(4)}_{\text{inv}} \cdot Q^4 + 2(D_{\mu}(C) \bar{\eta})[Q_{\mu} \omega]$$  

2.25

The bars mean the average over the space-volume $V$.

$S^{(3)}$ is standard, except for the first term linear in $q$. Had the constraint been linear in the potential, the second and third derivative of $\bar{t}(C)$ would have been absent. Then we would have had - as in the one-loop case - as only effect from the $q$-integration the $\delta$-function constraint on the $N - 1$ $Q_0^d(\vec{p} = 0)$ variables; hence, in the thermodynamic limit, only the traditional two-loop contributions with propagators and vertices carrying momenta $p_{\mu}^{ij}$ instead of $p_{\mu}$, as depicted in the three graphs in fig 1.

However, the linear terms in $q$ in eqn 2.25 do contribute through a zero-momentum insertion, using the identity:

$$\int dQ \int dq e^{-iq\bar{t}^{(1)}(1)Q} i q = \int \frac{dQ}{\bar{t}^{(1)}} \frac{\partial}{\partial Q} \left( \int dq e^{-iq\bar{t}^{(1)}(1)Q} \right) = \int \frac{dQ}{\bar{t}^{(1)}^2} \delta(Q) \frac{\partial}{\partial Q}$$  

2.26

We have suppressed the $C$ dependence in $\bar{t}^{(1)}$ to simplify notation, and $Q \equiv Q_0(0)$. This identity is of course nothing but the expansion of the $\delta$ function in the defining equation 2.4.* It gives us the zero-momentum insertions through the derivative.

The identity 2.26 carries no volume factors, since the $\bar{t}$ has none right from the outset, see eqn 2.4 and above. Moreover, when it acts on a given term with a product of $Q$’s, only terms linear in $Q_0^d(0)$ will survive the $\delta$-function in 2.26. If there are $n$ powers of $q$, then 2.26 changes to one where $n$ derivatives of $Q$ act to the right.

After these remarks we go through the usual procedure of expanding $S^{(3)}$ out of the exponential in 2.8 and doing the path integral, by contracting $Q$-fields. Apart from the terms, that give rise to the diagrams in fig 1, we have from the derivative in eqn 2.26 acting

* The last step in this identity, the integration by parts, is strictly only permitted if the boundary conditions are such that the $Q$’s are Gaussian variables.
on $S^{(3)}$ and $(S^{(3)})^2$:

$$\exp -\beta V U = \int D'QD\bar{\eta}D\omega \exp \left[ -Q \cdot S^{(2)} \cdot Q - \bar{\eta} \cdot D^2(C) \cdot \omega \right]$$

$$g^2 \left[ \frac{1}{2!} \frac{1}{\| \bar{\ell}^{(1)}(C) \|} \left\{ \frac{1}{2!} \bar{\ell}^{(3)}(C) \sum_p Q_0(p)Q_0(-p) ight. 
- 2\bar{\ell}^{(2)}(C) \sum_p Q_0(p)Q_0(-p) 
+ \left. \left( \frac{1}{2!} S^{(3)}_{\mu\nu\lambda}(C) \sum_p Q_\mu(p)Q_\nu(-p) + \sum_p D_0(C) \bar{\eta}(p)\omega(-p) \right) \right\} 
+ \frac{1}{2!} \frac{1}{\| \bar{\ell}^{(1)}(C) \|^3} \left( \bar{\ell}^{(2)}(C) \right)^2 \sum_p Q_0(p)Q_0(-p) \right]$$

$\| \bar{\ell}^{(1)}(C) \|$ is the determinant of the matrix $\bar{\ell}^{(1)}(C)$ in the first term of eqn 2.21. This determinant is the van der Monde determinant formed from the matrix $C$ in eqn 2.16, as shown at the end of appendix D.

Without knowledge of the explicit colour structure of eqn 2.27 the reader can easily check the following. All but one of the contractions in 2.27 are one loop contractions and hence of order $0(1)$. The only one that is $O(V)$ comes from the two-loop contraction we get from the second term in the straight brackets; we have presented this term pictorially in fig 2 and denote it by $U^{(2)}_p$. The reader can easily verify this, using B.2 for the propagator.

This contraction consists of two factors; one is given by the renormalisation of the Polyakov loop:

$$g^2 \bar{\ell}^{(2)}(C) \sum_p \langle Q_0(p)Q_0(-p) \rangle$$

2.28(a)

The other is given by the zero-momentum insertion and equals:

$$\frac{\partial}{\partial C} U^{(1)}(C)$$

2.28(b)

with $U^{(1)}(C)$ given in 2.23. Hence $U^{(2)}_p$ is the product of 2.28(a) and (b).

Thus to two loop order $U^{(2)}$ consists of two parts:

$$U^{(2)} = U^{(2)}_f + U^{(2)}_p$$

2.29
$U^{(2)}_f$ is given by the graphs in fig 1 with the topology of the graphs contributing to the free energy but with the energies shifted through the background $C$, as described underneath 2.21. $U^{(2)}_P$ is given by the 1-loop result 2.23 with the argument $C$ shifted by the renormalisation of the loop 2.28(a) (see fig.2).

$U^{(2)}_f$ is known to depend on the gauge choice $\xi$. $U^{(2)}_P$ depends on the gauge choice only through 2.28(a), since 2.28(b) is independent of $\xi$ (see 2.23). Therefore there must be a relation between the gauge artifacts in both terms, in order to cancel and give a $\xi$-independent $U^{(2)}$.

This relation is provided in the next subsection.

2.e. BRST and gauge variation.

In this subsection we will establish a relation between the BRST prediction for the gauge - variance of $U^{(2)}_f$ and the zero-momentum insertion 2.28 (fig 2(a)). In fact they are the same as a very simple argument will show.

In fig 1 we have the three diagrams that contribute $U^{(2)}_f$ to $U^{(2)}$. In the $\xi$ gauge there are contributions linear, quadratic and cubic in $(1 - \xi)$ from the propagators. The latter two are absent, as shown in appendix A. The terms linear in $(1 - \xi)$ do not cancel. Rather, a simple observation shows they come in with a factor 3 from graph 1a, 2 from 1b and 1 from 1c. Combining this with the combinatorial factors in front we see in fig 3(a) that $U^{(2)}_f$ has a term linear in $(1 - \xi)$ of the form:

$$-\frac{(1 - \xi)}{2} T \sum_{po} \int \frac{d^{n-1}p}{(2\pi)^{n-1}} \left( \sum_{i,j} \frac{p_{ij}^{ij} \Pi^{(ij)}_{\mu\nu} p_{ij}^{ij}}{(p^{ij})^4} + \sum_{d=2}^{N} \frac{p_{\mu}^{(d)} \Pi^{(d)}_{\mu\nu} p_{\nu}}{p^d} \right)$$

where $\Pi_{\mu\nu}$ is the one-loop self energy, and the momenta $p^{ij}$ have colour shifted Matsubara frequencies defined underneath 2.21.

The first sum is over the off-diagonal propagators, the second over the diagonal ones (in colour space).

In the zero temperature case we obviously would obtain zero for the coefficient, because the BRST identities$^{10}$ tell us that the self energies are transverse.

However, at non-zero temperature the BRST identities are not constrained by Lorentz (Euclidean) invariance. Although the BRST identities are local and therefore still valid at finite temperature their consequences are different. In our case, with a background field, they take the form$^{11}$:

$$0 = \frac{\delta \Gamma}{\delta Q} \cdot \frac{\delta \Gamma}{\delta J} + \text{ghost contribution} + \text{eventual matter contributions}$$

30
Γ is the one-particle irreducible generating functional in an external field C; at Q = 0 it is related to U_f by Γ = βVU_f, since C is constant in space time. J is the external source coupling to the gauge variation of Q, the quantum field, through the term:

\[ J \cdot D(Q + C) \omega \]  

The dot means summation over all degrees of freedom.

To obtain information on the self energy Π in eqn 2.30 one derives the BRST identity with respect to the ghost-field ω and the quantum field Q; setting all fields and sources to zero we get:

\[ 0 = \frac{\delta^2 \Gamma^{(1)}}{\delta Q \delta Q} \cdot \frac{\delta^2 \Gamma^{(0)}}{\delta J \delta \omega} + \frac{\delta \Gamma^{(1)}}{\delta Q} \cdot \frac{\delta^3 \Gamma^{(0)}}{\delta J \delta \omega \delta Q} \]  

This is the result to one-loop order. The first term in eqn 2.32 contains a double derivative with respect to the quantum fields. This double derivative equals the self-energy Π. All other terms are identically zero to this order. At zero temperature Lorentz invariance renders the last term in eqn 2.32 identically zero, and the one-loop self energy is transverse. However at finite temperature \( \frac{\delta \Gamma^{(1)}}{\delta Q_0(z)} \) is not vanishing, and - by inspection - it equals the zero momentum insertion \( \beta V \frac{\delta U^{(1)}}{\delta C} \) (see fig 3(c)); precisely through this zero-momentum insertion the renormalisation effect of the Polyakov loop did couple to the effective action, eqn 2.28.

Note that only the temporal component survives in the second term of 2.32.

So we have that both the gauge artifacts of the graphs in fig 1 and the full renormalisation of the Polyakov loop couple to one and the same thing: the zero-momentum insertion of \( Q_0 \) into the 1-loop result.

In order to make contact with the next section on the renormalisation of the Polyakov-loop we rewrite 2.32 in momentum space, with momenta and indices of the Cartan basis explicit:

\[ 0 = ip^{ij}_{\mu} \Pi^{ij}_{\mu \nu} - i \frac{\delta U^{(1)}}{\delta Q_0^d(0)} f^{ij,ji,d}_{\delta \nu,0} 1 \leq i \neq j \leq N \]  

and

\[ 0 = ip^d_{\mu} \Pi^d_{\mu \nu} \quad d = 2, \ldots, N \]  

The colour-shifted momenta \( p^{ij} \) are defined below 2.21. \( Q_0^d(0) \) is the Euclidean space-time average of \( Q_0^d(x) \) To obtain eqn 2.33(b) we used the fact that \( \frac{\delta \Gamma^{(1)}}{\delta Q_0^d} \) vanishes identically because of colour conservation. So only the off diagonal self-energies are non-transverse.
The terms linear in the gauge parameter \((1 - \xi)\) in \(U_f^{(2)}\) are proportional to the zero-momentum insertion in 2.33(a):

\[
\frac{\delta U^{(1)}(C)}{\delta Q^d_0(0)} = \sum_{k,l} \hat{B}_3(C_{kl}) f^{kl,lk,d}.
\]

When combining 2.33(a) and 2.34 we use the identity\(^*\):

\[
\sum_{d=2}^N f^{kl,lk,d} f^{ij,ji,d} = \frac{1}{2}(\delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk}).
\]

So finally substituting 2.33(a) into 2.30 we obtain for the coefficient of \((1 - \xi)\) in \(U_f^{(2)}\):

\[
U_f^{(2)} = U_f^{(2)}(\xi = 1) - 2(1 - \xi)g^2N \sum_{i<j} \hat{B}_1(C_{ij}) \hat{B}_3(C_{ij}).
\]

Both in 2.34 and 2.35 we used eq. C.5 (with d=4 and k=1, 2 respectively) from Appendix C.

The \(\xi = 1\) contribution in 2.35 is trivial to obtain and will be discussed in section 4.

Let us note that the presence of fermions does not change the form of the BRST identity 2.32. So when fermions are present gauge artifacts can be treated the same way. The fermionic contribution shows up additively in the \(\hat{B}_3\) factor in 2.35, as is clear from fig. 3 (c).

2.f Comparison of gauge variation of \(U_f^{(2)}\) and \(U_P^{(2)}\)

In this section we have found that both the renormalisation of the Polyakov loop, and the gauge artifacts in the free energy part of the effective action are coupled to the same object: the temporal and colour diagonal zero-momentum insertions into the one loop result 2.23 (see 2.28 (b) and the combination of 2.30 and 2.33(a)). The gauge variations of \(U_P^{(2)}\) and \(U_f^{(2)}\) have the same structure. For the former we combine 2.28(a) and (b), for the latter 2.30(a) and 2.33(a). In section 3, eq. 3.18, we will see that evaluation of the renormalisation of the loop (2.28(a)) actually matches the gauge artifacts in 2.35, but with opposite sign.

Therefore we will have complete cancellation of the gauge artifacts in the sum of the two, in two loop order. One expects this to continue for any order.

\(^*\) As follows by inserting the definition D.2 for the structure constants and using the trace properties of the \(\lambda^{ij}\).
3. Renormalisation of the Polyakov-loop

The renormalisation of the loop is not the renormalisation of the numerical value
\[ < \tilde{t}_1 > \equiv \frac{1}{V} \int d\vec{x} < t_1(\vec{x}) > . \]

Rather, what renormalizes is the relation between \( < \tilde{t}_1 > \) and the particular choice of saddlepoint \( C \) that we made in eqn 2.16 in the phase of the loop. This renormalisation can - and will - contain a reflection of our gauge choice 2.12.

As a general expectation one would say that the phase of the loop renormalizes without any extra ultraviolet infinities, beyond the usual ones that renormalise couplings and fermion masses. This should be so, because all these effects are thermal in nature.

In fact it turns out, that the expectation values of the gauge invariant traces \( \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{N-1} \) indeed stay finite through one loop. From them one can reconstruct the finite renormalizations of the eigenvalues of the loop easily. As a caveat let us look how proceeding through the renormalization of the gauge variant unitary matrix \( P(A_0) \) leads to a correction with unwanted properties.

So take the special unitary diagonal matrix \( P(A_0) = e^{iC+gQ_0} \). It becomes after adding the one loop correction in fig 2.a a diagonal matrix which is not special, not unitary and not finite! Nevertheless the expectation value of its trace \( \tilde{t}_1 \) is finite as it should. The same holds true for the quantum average of the \( k^{th} \) power of the Polyakov loop and taking its trace \( \tilde{t}_k \) (see appendix B).

We have for the average of the loop matrix to order \( g^2 \):
\[
\frac{1}{V} \int d\vec{x} < P(A_0) > = e^{iC\beta} - g^2 \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 e^{iC\tau_2} \left< Q_0(\tau_2) e^{iC(\tau_1-\tau_2)} Q_0(\tau_1) e^{iC(\beta-\tau_1)} \right> + O(g^4)
\]

where we used the definition 2.1 for \( P(A_0) \), 2.9 and 2.16. There is no \( \vec{x} \) dependence after contraction of the \( Q_0 \) fields in 3.1, so we drop any reference to it for notational convenience.

The order \( g^2 \) correction receives a contribution from all diagonal quantum fields \( Q_0^d \) equal to:
\[
- g^2 \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 e^{iC\beta} \left< Q_0^d(\tau_2) Q_0^d(\tau_1) \right>
\]

because the diagonal \( Q_0^d \) do commute with the diagonal \( e^{iC\tau} \). Since
\( \Delta_{00}(\tau_2 - \tau_1) \equiv \left\langle Q^d_0(\tau_2)Q^d_0(\tau_1) \right\rangle \)

\[
= T \sum_{p_0} \int \frac{d\vec{p}}{(2\pi)^3} \left[ \frac{1}{p_0^2 + \vec{p}^2} - (1 - \xi) \frac{p_0^2}{(p_0^2 + \vec{p}^2)^2} \right] e^{ip_0(\tau_2 - \tau_1)} \tag{3.3}
\]

we find with dimensional regularisation for the sum of all diagonal contributions to 3.2:

\[
-g^2 \frac{1}{2} \frac{N - 1}{N} \left\{ \frac{\beta^2}{2} T \int \frac{d^{n-1}\vec{p}}{(2\pi)^{n-1}} \left[ \frac{1}{\vec{p}^2} + 2T \sum_{p_0 \neq 0} \frac{1}{ip_0} \Delta_{00}(p_0, \vec{p}) \right] \right\} \tag{3.4}
\]

The first term is zero; the second as well, since \( \Delta_{00} \) is even in \( p_0 \).

So we are left with the off-diagonal contributions \( Q^{ij}_0(\tau_2)\lambda^{ij}(i \neq j) \). Let us denote the product of two \( \lambda \)'s by:

\[
\lambda^{ij}\lambda^{ji} \equiv D^{ii} \quad i \neq j \text{ fixed} \tag{3.5}
\]

and

\[
\lambda^{ji}\lambda^{ij} \equiv D^{jj} \quad i \neq j \text{ fixed} \tag{3.6}
\]

\( D^{ii} \) is the diagonal matrix, with all entries zero except the \( i^{th} \) diagonal element, which equals 1/2.

We use the identity

\[
e^{iC\tau}\lambda^{ij}e^{-iC\tau} = e^{i(C_i - C_j)\tau}\lambda^{ij} \tag{3.7}
\]

and the propagator for the \( Q^{ij}_0 \) excitation:

\[
\Delta^{ij}_{00} \equiv \left\langle Q^{ij}_0(\tau_2)Q^{ij}_0(\tau_1) \right\rangle = T \sum_{p_0} \int \frac{d\vec{p}}{(2\pi)^3} \left\{ \frac{1}{(p_0^{ij})^2 + \vec{p}^2} - (1 - \xi) \frac{(p_0^{ij})^2}{((p_0^{ij})^2 + \vec{p}^2)^2} \right\} e^{ip_0(\tau_2 - \tau_1)} \tag{3.8}
\]

Notice the occurrence of \( p_0^{ij} \equiv p_0 + C_i - C_j \). The time dependence of the propagator is of course periodic.

With the help of 3.5 to 3.8 we can rewrite the \( O(g^2) \) contribution in the form (see appendix B for details):

\[
g^2 \sum_{1 \leq i \leq j \leq N} \left\{ \left( e^{-i(C_i - C_j)\beta} - 1 \right) D^{ii} + \left( e^{i(C_i - C_j)\beta} - 1 \right) D^{jj} \right\} \Delta^{ij}_{(2)} - \frac{\beta}{i} (D^{ii} - D^{jj}) \Delta^{ij}_{(1)} \right\} e^{iC\beta} \tag{3.9}
\]
with

\[ \Delta_{ij}^{(r)} \equiv T \sum_{p_0} \int \frac{d^{n-1} \vec{p}}{(2\pi)^{n-1}} \frac{1}{(p_0^{ij})_r} \Delta_{00}(p_0^{ij}, \vec{p}) \]  

3.10

The terms proportional \( D^{ii}, D^{jj} \) come from \( i < j \), \( i > j \) respectively.

The factors \( \frac{1}{(p_0^{ij})_r} \) in 3.10 stem from the integrations over the \( \tau \)-variables: they are there because of the non-locality of the loop. Observe in 3.9 that the term with \( r=1 \) has a similar momentum structure as in 2.30, i.e. the loop on the left in fig 3.c proportional to \( \hat{B}_1 \) as in 2.35. We will see below, that the part with \( r=2 \) is projected out, after taking the trace of eq. (3.9).

Obviously the renormalisation of the loop in 3.9 gives again a diagonal matrix and so eqn 3.1 can be rewritten as

\[ \frac{1}{V} \int d\vec{x} < P(A_0) > = \exp i(C + g^2 \delta C)\beta + 0(g^4) \]  

3.11

with \( \delta C \) given by the matrix

\[ \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \left[ T \left\{ \left( e^{-i(C_i - C_j)\beta} - 1 \right) D^{ii} + \left( e^{i(C_i - C_j)\beta} - 1 \right) D^{jj} \right\} \Delta_{ij}^{(2)} \right. \]

\[ \left. + (D^{ii} - D^{jj})\Delta_{ij}^{(1)} \right] \]  

3.12

We note that the trace of \( \delta C \) is not zero! It is:

\[ Tr\delta C = \frac{T}{2i} \sum_{1 \leq i \neq j \leq N} (\cos(C_i - C_j)\beta - 1)\Delta_{ij}^{(2)} \]  

3.13

Note that \( \Delta_{ij}^{(2)} \) is logarithmically divergent whereas \( \Delta_{ij}^{(1)} \) is finite (see eqns C.5, C.15 and C.17). So our caveat at the beginning of this section has come true: as a matrix the expectation value of the Polyakov loop has become infinite, non-unitary and not special.

One might argue, that the Polyakov loop as a matrix is not gauge invariant; in particular under a periodic and constant (in \( \vec{x} \)) gauge transformations 3.11 is similarity transformed, and so is the exponent \( C + g^2 \delta C \). But the trace 3.13 stays invariant, so the infinity cannot be absorbed by the gauge transformation!

On the contrary, when we first calculate the expectation value of the traces of the \( N - 1 \) powers of the loop, \( \bar{t}_1, \bar{t}_2 \ldots \bar{t}_{N-1} \), there is no infinity, and the traces are real. To see this, let us start with \( SU(2) \) (\( N = 2 \)). Then computing the trace of 3.9 gives us

\[ \langle \bar{t}_1 \rangle = \cos \beta(C_1 - C_2) - g^2 \frac{1}{2} \beta \Delta_{12}^{(1)} \sin(C_1 - C_2)\beta \]  

3.14(a)
with
\[ \Delta^{ij}_{(1)} = \frac{1}{4\pi}(2 + (1 - \xi)) \left( C_{ij} - \frac{1}{2} \right) \]  
\[ \text{3.14(b)} \]

The trace projects out the infinite terms! The result 3.14 coincides with that of Belyaev\(^{12}\) and that of ref 5.

For \( SU(N) \) we find for the \(< \hat{t}_k >\) (see appendix B):
\[ < \hat{t}_k > = \frac{1}{N} Tr \langle (P(A_0))^k \rangle = \frac{1}{N} Tr e^{i k C} \]
\[ - \frac{g^2}{N} \beta k \sum_{i,j} Tr(D^{jj} - D^{ii})e^{i k C} \Delta^{(ij)}_{(1)} \]
\[ k = 1, \ldots, N - 1 \]  
\[ \text{3.15} \]

All infinite contributions do cancel in the trace. These \( N - 1 \) results in eqn 3.15 can be summarized by adding to \( C \) a diagonal traceless matrix \( \delta C \) (not to be confused with the matrix \( \delta C \) in 3.11 and 3.12), and to demand that the \( k \)-th power of this matrix gives us the trace computed in 3.15:
\[ < \hat{t}_k > = \frac{1}{N} Tr \exp ik(C + \delta C) \beta + C(g^4) \]  
\[ \text{3.16} \]

Identifying 3.16 with 3.15 gives for a diagonal element:
\[ \frac{\delta C_i}{2\pi T} = \frac{g^2}{(4\pi)^2}(2 + 1 - \xi) \sum_j B_1(C_{ij}) \]  
\[ \text{3.17} \]

where \( B_1(x) \equiv x - \frac{1}{2} \epsilon(x) \). (See appendix C).

From the anti-symmetry of \( \Delta^{ij}_{(1)} \) in its argument \( C_{ij} \), it follows that the loop does not renormalize when \( C = 0 \), as expected. This is useful for knowledge of absolute minima (see section 6). It also shows that \( \sum_{i=1}^{N} \delta C_i = 0 \), i.e. the matrix \( \delta C \) is indeed traceless (and real and finite).

To find the full result for the Polyakov loop inserted \( U_{P}^{(2)} \) we combine 2.28(a) and (b) with 3.17 to get:
\[ U_{P}^{(2)} = 2(2 + (1 - \xi))g^2 N \sum_{i<j} \hat{B}_1(C_{ij}) \hat{B}_3(C_{ij}) \]  
\[ \text{3.18} \]

Eqns 3.17 and 3.18 are the main results of this section*. They generalize the results in

---

* To obtain 3.18 more in line with the derivation of 2.35, take the insertion from 2.28 as \( \sum_k < \hat{t}_k > < \frac{\partial}{\partial \epsilon^{(1)}_{(1)k}(A_0)} S_{\text{int}} > \). By inserting into 3.15 a complete set of diagonal generators \( \lambda_d \) one retrieves the matrix \( \hat{t}_k^{(1)} \) defined below 2.21. It factors out and therefore cancels the same matrix in the denominator of the average. Use of 2.34 and the subsequent identity for the f-symbols yields 3.18.
ref. 5 beyond the $q$-valley. They will be used when evaluating the full constrained effective action in the next two sections.
4. The gluonic constrained effective potential

In this section we will give the one and two loop results. Everything is expressed in terms of the Bernouilli polynomials $B_k(x)$, that we have compiled in appendix C.

The result for an $SU(N)$ gauge theory becomes to two loop order:

$$U = U^{(1)} + U^{(2)}$$

with

$$U^{(1)} = \frac{4\pi^2}{3} T^4 \sum_{i<j} \tilde{B}_4(C_{ij})$$

and from 2.35 and 3.18:

$$U^{(2)} = U^{(2)}_f(\xi = 1) + U^{(2)}_P(\xi = 1),$$

with the righthand side given in eq. (4.7) and (4.8). As anticipated all gauge artifacts have dropped out. The variable $C_{ij}$ equals

$$C_{ij} \equiv \frac{C_i - C_j}{2\pi T}$$

as in eqn 2.23, and $\tilde{B}_4$, $\tilde{B}_2$ are defined in eqns C.13 and C.14.

A few remarks. $U^{(1)}$ and $U^{(2)}$ are both of order $N^2$ for a generic value of $C_{ij}$. However, for the specific choice in ref 5: $C_1 = C_2 = \ldots = C_{N-1} = \frac{q}{N}$ and $C_N = -\frac{(N-1)}{N}q$ we have only $N-1$ terms:

$$U^{(1)} = \frac{4\pi^2}{3} T^4 (N-1) \tilde{B}_4(q)$$

$$U^{(2)} = \frac{4\pi^2}{3} T^4 \cdot -5 \frac{g^2 N}{(4\pi)^2} (N-1) \tilde{B}_4(q)$$

So in this one parameter space the potentiel is $O(N)$ smaller, and we will denote it by the "$q$-valley".

The reader can see this in the $SU(3)$ example in fig 4.

The $q$-valley formulae 4.5 and 4.6 were derived in ref 5 (eqn 5.14 in ref 5). Remarkable is the simplicity of 4.6: it just renormalizes multiplicatively the one loop result 4.5. This is lost outside of the $q$-valley.

The actual derivation of $U_f^{(2)}$ in 4.3 is astonishingly simple in $\xi = 1$ gauge. First we compute the contribution from the 3 graphs in fig 1. Using momentum conservation like
in A.3, all integrands are of the form $\frac{1}{l_i l_j}$, where $l_i$, $l_j$ are two different momenta from the set of momenta $l_1$, $l_2$ and $l_3$ flowing through the lines of fig 1a and b. The answer is:

$$U_f = g^2 \sum_{b,c,a} |f^{bca}|^2 (\hat{B}_2(C_c)\hat{B}_2(C_a) - \hat{B}_2^2(0))$$  \hspace{1cm} (4.7)

The colour indices $b, c, a$ run through the basis $(ij)$ and $d$. $C_c = C_i - C_j$ if $c = (i, j)$, $C_c = 0$ if $c = d$ ($d$ labelling a diagonal generator).

The contribution from the Polyakov loop insertion ($\xi = 1$) is from 3.18:

$$U_P^{(2)} = 4g^2 N \left[ \sum_{i<j} \hat{B}_3(C_{ij})\hat{B}_1(C_{ij}) \right]$$ \hspace{1cm} (4.8)

where the $\hat{B}_k$ are momentum integrals defined in appendix C and up to some multiplicative factors identical to the Bernoulli polynomials used before. Now the total $U_f + U_P$ reduces in the q-valley to 4.6, using the formulae in appendix C, and properties of the structure constants in appendix D.

We would like to draw the readers attention to a remarkable, but unwanted property of $U_f$ in 4.7. Let us write out its content in the q-valley:

$$U_f^{(2)} = \frac{g^2 N}{4} \frac{(N - 1)}{3} T^4 [3q^2(1 - q)^2 - 2q(1 - q)]$$ \hspace{1cm} (4.9)

We see that the linear term causes an absolute minimum, where the value of $U_f^{(2)}$ is negative. But remember the discussion in section 2.d: the free energy graphs alone follow from the $\delta$ function constraint linear in the potential. As 4.7 is normalised by the the free energy graphs at C=0 we would expect the constrained path integral to give us a non-negative result! The fact, that it is not, just illustrates once more how gauge dependent effective potentials can give deceptive information. As another example\(^5\) take this absolute minimum $q_m$ of $U$ to two loop order. It will be of order $O(g^2)$. But by charge conjugation we get $-q_m$ and it is not hard to see, that there we have another absolute minimum: charge conjugation seems spontaneously broken!

Both unwanted properties do disappear when we add 4.8, to get 4.6. More generally, also outside the q-valley our result 4.3 is non-negative and has only $Z(N)$ minima.
5. Fermions

In this section we add $n_f$ quarks (taken to be massless) in the fundamental representation of $SU(N)$. As fermions they are anti-periodic in time $\tau$. The action reads

$$S = \int d\vec{x} d\tau \bar{q}(\partial + A)q$$

for every flavour. The Z(N) invariance is now broken through the boundary conditions: a gauge transformation which is periodic modulo a Z(N) phase will change the anti-periodicity.

We go through the same arguments as in section 2.b, so introduce background and fluctuation variables for the gauge fields. The next step is to compute the contribution $U^{(1)}_q$ of the quarks to the constrained effective potential; this has been done\(^1\) and gives:

$$U^{(1)}_q(t) = -\frac{4}{3} \pi^2 T^4 n_f \left( \sum_i \hat{B}_4 \left( \frac{C_i}{2\pi T} + \frac{1}{2} \right) - \frac{N}{16} \right)$$

for $n_f$ quark flavours. The minus sign in front of 5.2 is due to the fermionic determinant, the Bernoulli function appears just as in the gluon case 4.2. Its argument is shifted over $\frac{1}{2}$ because of the anti-periodicity. The result is normalised to zero for $C_i = 0$. In what follows we will use the abbreviation:

$$C^f_i = \frac{C_i}{2\pi T} + \frac{1}{2}$$

To two loop order we will find-like we did for the pure gluon case in section 2.d - that the renormalisation of the Polyakov loop couples to the zero-momentum insertion of $U^{(1)}_q$. Likewise the arguments in section 2.e are valid for the linear gauge variation of the fermion case (see figs1, 2, and 3).

The two loop result for $n_f$ quarks becomes:

$$U^{(2)}_q = -n_f 4 \pi^2 T^4 \frac{g^2}{(4\pi)^2} \left[ \sum_{i \neq j} \left\{ B_2(C_{ij})(B_2(C^f_i) + B_2(C^f_j)) - B_2(C^f_i)B_2(C^f_j) \right\} \right. + \left. \frac{N-1}{N} \sum_i (2B_2(0)B_2(C^f_i) - B_2^2(C^f_i)) \right.$$  

$$- \frac{8}{3} \sum_i B_3(C^f_i) \sum_j B_1(C_{ij}) + (N^2 - 1) \frac{5}{144} \right]$$

\(^1\)
The terms in the braces come from the configuration in fig 1(d) where the gluon is off-diagonal. The next term stems from the diagonal gluons (see eqn D.7). The one but last term in 5.4 comes from the zero momentum insertion into 5.2 using eqn 3.17, and restores the minimum at $C_i = 0$. Thus, also in the fermion case charge conjugation and non-negativity are restored! But the $Z(N)$ minima are no longer degenerate, because the symmetry is explicitly broken.

In the $q$-valley both gluon contribution and fermion contribution are of order $N$, when $q \simeq N/2$. This explains why the fermion contribution is comparable to the gluon contribution (fig 6).

For $SU(3)$ we plotted in fig 4 and 5 the full potential 4.2, 4.3 and 5.2, 5.4 with $n_f = 0, 2$ respectively. The ”$q$-valley” is seen on the border of the admitted values of the loop. It is plotted for $n_f = 6$ in fig 6.
6. Conclusion and outlook

In this paper we established the perturbative expansion of the constrained effective action, to two-loop order in the large volume limit. Our result is consistent with the following prescription, to all orders:

i) Compute the sum \( U_f(C) \) of all 1-PI free energy diagrams, with the energies shifted by \( C \), and obtain
\[
\tilde{U}(C) \equiv U_f(C) - U_f(0)
\] 6.1

ii) Compute the renormalisation of \( C \) to all orders to obtain
\[
C = C(C_r)
\] 6.2

iii)
\[
\tilde{U}(C(C_r)) \equiv U(C_r)
\] 6.3

is the final answer.

Of course the result 6.3 is intuitively expected because of the relation 2.6 between \( U \) and the effective potential \( G \).

As a function of \( C_r \), \( U \) will not depend on the gauge choice. \( U \) is therefore perfectly well defined in contrast to statements made by some authors. Of course, it may be that in three and higher loop order infrared divergencies will show up. They fall into two categories. The first one concerns divergencies due to the diagonal gluon propagators. The off diagonal propagators are protected by a value of \( C_{ij} \) of order 1. The second category corresponds to small values of \( C_{ij} \) of order \( g \). The first category can be compared to the infrared divergencies in an Abelian \( U(1)^{N-1} \) theory and is less severe. Some inroads into the latter have been made in ref 6). The second one is truly non Abelian and more severe.

The other important issue was that the effective potential is non-negative and has \( Z(N) \) minima as absolute minima. A discussion of the latter can be found in ref. 13). Here we want to point out that the two are related. To this end, suppose
\[
U(C_r) \geq 0
\] 6.4
to all orders, because, at least formally, \( \exp(-\beta V U(C_r)) \) is a probability. We would like to know whether \( U(0) \) is an absolute minimum, i.e. whether \( U(0) = 0 \). According to 6.1, 6.2 and 6.3:

\[
U(C_r = 0) = 0 \tag{6.5}
\]

is equivalent to:

\[
F(C(C_r = 0)) = F(0) \tag{6.6}
\]

The simplest way to have 6.6 fulfilled is

\[
C(C_r = 0) = 0 \tag{6.7}
\]

This is actually the case to two loop order (eqn 3.17).

The authors of ref (13) have found a very ingenuous gauge choice, which they call Static Background Gauge (SBG):

\[
S_{g.f.} = \frac{1}{2\xi} \left( \frac{1}{\xi'} D_0(C) Q_0 + D_i Q_i \right)^2 \tag{6.8}
\]

In this gauge \( \xi \) is fixed, \( \xi' \to 0 \). It looks very probable that in this gauge the Polyakov loop does not renormalize to any order of perturbation theory. If so, then indeed 6.5 is true to any order in perturbation theory.

With our gauge choice we found in section 3, that only traces of powers of loops are finite. The infinities in 3.12 are not ultra violet in nature; they stem from the factors \( \frac{1}{(\rho_0')^2} \) that originate in the \( \tau \)-integrations in 3.1, that is, in the non-locality of the loop. It has consequences for the spontaneous breaking of charge conjugation as discussed in ref.5).

Let us look first at the SU(2) case. Though the potential \( A_0 \) becomes \( -A_0^T \) under charge conjugation (so in particular the colour diagonal matrix \( C \) will just flip sign), this will not affect the quantum average 3.14(a), since it is even in \( C \). So the average of the trace of the loop is charge conjugation even, and no charge conjugation breaking effects can be measured with it. In general, for any \( N \), charge conjugation leaves \( < P(A_0) > \) invariant as long as it is real.

In between different vacua however the order parameter takes on complex values, (except for SU(2) of course), but this lies outside the scope of this paper. We will come back to the problem of computing the constrained effective action when the argument is a profile, as a function of one spatial variable\(^{16}\).
In fig 4 and 5 the reader will find the effective one loop potential for $SU(3)$, with and without quarks. How the metastable state\textsuperscript{15) in fig 6, near $C = \frac{2\pi}{3}$, is sensitive to two loop contributions is shown in the same figure.

The effects of lattice artifacts will be evaluated in ref 17). The Montecarlo evaluation could be done eventually with the multicanonical ensemble\textsuperscript{18) 19) this method avoids the difficulty of the peak between two vacua (see fig 6), which bars the MC-access to this domain of $q$-values. Of course the massless fermions remain a practical problem.

Some time ago\textsuperscript{15), it was pointed out that the total one loop effective action, i.e. the sum of the free energy at $C=0$, $F(0)$, and $U^{(1)}(C)$ lead for high enough fermion number $n_f$ and appropriate values of $C$ in the $q$-valley to absurd thermodynamical properties, in particular near the metastable minimum in fig 6. If we are only interested in local minima the presence of fermions is needed.

But in a Montecarlo simulation one can apply an external field coupled to the Polyakov loop.

In this latter context it is amusing to observe that the same happens for a system with gluons alone. For example, the maximum in fig 3 corresponding to the center of the admitted values for $SU(3)$, has negative entropy: explicitely, the free energy for $N^2 - 1$ gluons\textsuperscript{1) is $-\frac{1}{45}\pi^2T^4(N^2-1)$. Adding to this the value of the one loop result for the effective potential 2.23 in the maximum gives for $SU(2)\pi^2\frac{T^4}{60}$ and for $SU(3)\frac{8\pi^2}{405}T^4$, producing a negative entropy. This negative entropy may be a very serious default of the perturbative approach. Since at this order the free energy is just counting the degrees of freedom, the extremist would say the negative entropy indicates that we have overlooked hitherto unknown degrees of freedom in QCD. However, already the first derivative of the effective potential with respect to the order parameter shows peculiar behaviour: instead of the intuitively expected flat part- corresponding to the spontaneous breaking of the symmetry-it has smooth behaviour. This smooth behaviour is the opposite of what happens in Mean Field approximation, namely overcooling, and it fits with the fact that the second derivative of the effective potential is indeed convex (again, in contrast to Mean Field). Note that the effective potential is only convex in terms of the trace of the loop, not in terms of the background field $C$! Clearly some important physics remains to be understood.

ACKNOWLEDGEMENTS

I’m indebted to Tanmoy Bhattacharya, Andreas Gocksch, Bernd Grossmann, Frithjof Karsch, Keijo Kajantie, Robert Pisarski and Jay Watson for useful correspondence and
discussions, J. Ignatius for discussions and providing me with some figures.
Appendix A. **Gauge terms proportional to** \((1 - \xi)^2\) **AND** \((1 - \xi)^3\)

Gauge terms proportional to \((1 - \xi)\) are treated in the main text, section 2.e.

Terms proportional to \((1 - \xi)^2\) are involving contractions of the 3-vertices with two momenta. This contraction simplifies considerably the vertex. Consider diagram (a) in fig 1. Its contribution to the \((1 - \xi)^2\) term equals:

\[
\frac{g^2}{12} \int_{1,2} \left\{ \frac{1}{l_1l_2l_3} \left( l_1^2 (l_2 \cdot l_3)^2 - 2(l_1 \cdot l_2)(l_2 \cdot l_3)(l_3 \cdot l_1) + l_2^2 (l_1 \cdot l_3)^2 \right) + (1 \rightarrow 2, \ 2 \rightarrow 3, \ 3 \rightarrow 1) + (1 \rightarrow 3, \ 2 \rightarrow 1, \ 3 \rightarrow 2) \right\}
\]

Because of the symmetry of the integration summation \(\int_{1,2}\) over \(l_1\) and \(l_2\) and colour, the two terms in A.1, not explicitly written, give the same result as the first one we will now further analyse.

We use momentum conservation:

\[
l_1 + l_2 + l_3 = 0
\]

A.2

to write

\[
2(l_i \cdot l_j) = l_k^2 - l_i^2 - l_j^2
\]

A.3

for any triple \((i, j, k) = (1, 2, 3)\).

The first term \(I\) in A.1 gives then:

\[
I \equiv \frac{g^2}{48} \int_{1,2} \left( \frac{l_1^2}{l_1l_2l_3} + \frac{l_2^2}{l_1l_2l_3} + \frac{l_3^2}{l_1l_2l_3} - 2 \frac{l_1^2 l_2 l_3}{l_1l_2l_3} - 2 \frac{l_2^2 l_1 l_3}{l_1l_2l_3} + 2 \frac{l_3^2 l_1 l_2}{l_1l_2l_3} \right)
\]

A.4

The last four terms add up to:

\[
\frac{g^2}{48} \hat{B}^2 \cdot \hat{B}^2
\]

A.5

using

\[
\int_{1,2} \frac{1}{l_2} = 0
\]

A.6

The first two terms give an equal contribution upon integration; the second is seen to be, using A.2:

\[
\frac{g^2}{48} \int_{1,2} \left( \frac{l_1^2 + 2l_1 \cdot l_2 + l_2^2}{l_1l_2l_3} \right) = \frac{g^2}{48} \int_{1,2} \left( \frac{1}{l_2} + \frac{2l_1 \cdot l_2}{l_1l_2l_3} + \frac{1}{l_1l_2} \right)
\]

A.7

The second term equals:

\[
\frac{g^2}{48} \int_{1,2} \frac{2l_{10}l_{20}}{l_1l_2}
\]

A.8

28
because the correlation \( \vec{l}_1 \cdot \vec{l}_2 \) is odd in \( \vec{l}_1 \) (and \( \vec{l}_2 \)). Using A.4 to A.8 we get:

\[
I = \frac{g^2}{48} (4\hat{B}_1 \cdot \hat{B}_3 + 3\hat{B}_2 \cdot \hat{B}_2) \tag{A.9}
\]

The dots mean summation over colour degrees of freedom:

\[
\hat{B}_i \cdot \hat{B}_j = \sum_{a,b,c} |f_{a,b,c}|^2 B_i(C_a) B_j(C_b)
\]

The third term in A.1 is through exchange \((1 \to 2)\) identical to the first one in A.4 we just computed.

The middle term in A.1 equals:

\[
-\frac{g^2}{48} \int_{1,2} (l^6_3 + l^6_2 + l^6_1 - l^4_3 l^2_1 - l^4_3 l^2_2 - l^4_1 l^2_2 - l^4_1 l^2_3 - l^4_2 l^2_1 - l^4_2 l^2_3 + 2l^2_3 l^2_2 l^2_1) \frac{1}{l^4_1 l^4_2 l^4_3} \tag{A.10}
\]

The first term in A.10 gives

\[
-\frac{g^2}{48} \int_{1,2} l^4_3 \frac{1}{l^4_1 l^4_2} \tag{A.11}
\]

whereas the other terms all cancel, using symmetry in the integration variables, and the result is for A.1:

\[
\frac{g^2}{8} \left( 4\hat{B}_1 \cdot \hat{B}_3 + 3\hat{B}_2 \cdot \hat{B}_2 - \frac{1}{2} \int_{1,2} \frac{l^4_3}{l^4_1 l^4_2} \right) \tag{A.12}
\]

The \((1 - \xi)^2\) contribution from diagram \((b)\) in fig 1 can be worked out with the same tricks and gives the same result as in A.12, but with opposite sign. That is: the \((1 - \xi)^2\) term from fig 1 is zero.

The \((1 - \xi)^3\) term is zero, because the 3-vertex vanishes when contracted with all three incoming momenta.
Appendix B. Quantum average of Polyakov loops

In this appendix the trace of the quantum average of the $n$–th power of the Polyakov-loop is worked out.

Let us introduce some notation. The matrix of order $g^2$ in eqn 3.1 is called $L_2$ :

$$L_2 \equiv -g^2 \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 e^{iC\tau_2} \frac{1}{V} \int d\vec{x} Q_0(\tau_2 \vec{x}) e^{iC(\tau_1 - \tau_2)} Q_0(\tau_1 \vec{x}) e^{iC(\beta - \tau_1)}$$ \hspace{1cm} (B.1)

Remember $C$ is a diagonal $N \times N$ matrix with $i$\textsuperscript{th} entry $C_i$.

Now we have for the propagator in a finite volume from 2.21 :

$$\langle Q_0(p_0\vec{p})Q_0(p'_0\vec{p}')\rangle = \frac{1}{T} \delta_{p_0,-p'_0} \delta_{\vec{p},-\vec{p}'} \frac{1}{V} \left( \frac{1}{p_0^2 + \vec{p}^2} - (1 - \xi) \frac{p_0^2}{(p_0^2 + \vec{p}^2)^4} \right)$$ \hspace{1cm} (B.2)

So :

$$\frac{1}{V} \int d\vec{x} \langle Q_0(\tau_1 \vec{x})Q_0(\tau_2 \vec{x}) \rangle = T \sum_{p_0} \int \frac{d\vec{p}}{(2\pi)^3} \left( \frac{1}{p_0^2 + \vec{p}^2} - (1 - \xi) \frac{p_0^2}{(p_0^2 + \vec{p}^2)^4} \right) e^{ip_0(\tau_1 - \tau_2)}$$ \hspace{1cm} (B.3)

in the infinite volume limit.

In calculating $Tr < L >$ we push $e^{iC\tau_2}$ through $Q_0^{ij}(\tau_2)\lambda^{ij}$ to the right, which causes an extra phase $e^{i(C_1 - C_j)\tau_2}$ to $Q_0^{ij}(\tau_2)$. This phase adds to the Fourier coefficient $e^{ip_0\tau_2}$ of $Q_0^{ij}(\tau_2)$; the same happens to the Fourier coefficient of $Q_0^{ij}(\tau_1)$, it becomes $e^{i(p'_0 - (C_1 - C_j))\tau_2}$.

Having done this, integration over $\tau_2$ and $\tau_1$ as in B.1, gives eqn 3.9 in the text.

When computing the quantum average of $Tr P^n(A)$, $(n \geq 2)$ to order $g^2$, we also need the first order matrix :

$$L_1 \equiv ig \int_0^\beta d\tau e^{iC\tau} Q_0(\tau) e^{-iC(\beta - \tau)}$$ \hspace{1cm} (B.4)

To order $g^2$ we get (using $P(A_0) = e^{iC\beta} + L_1 + L_2$) :

$$\langle Tr P^n(A) \rangle = n Tr \langle L_2 \rangle e^{i(n-1)C\beta}$$
$$+ (n - 1) Tr \langle L_1 L_1 \rangle e^{i(n-2)C\beta}$$
$$+ (n - 2) Tr \langle L_1 e^{iC\beta} L_1 \rangle e^{i(n-3)C\beta}$$
$$+ \ldots + Tr \langle L_1 e^{-iC\beta} L_1 \rangle + 0(g^4)$$ \hspace{1cm} (B.5)
We now push in each term the matrices $e^{iC_{ij}}$ to the right, to have the two potentials next to one another. This leads to extra phases as below eqn B.1, and B.5 becomes, for a fixed pairing $\langle Q^{ij}_{0}(\tau_2)Q^{ji}_{0}(\tau_1) \rangle \equiv \Delta^{ij}_{00}$:

$$
\langle TrP(A)^n \rangle = -g^2 \left[ \int_0^\beta d\tau_1 \int_0^\tau_2 d\tau_2 \left\{ nTrD^{ij}_{00}e^{ip^{ij}(\tau_2-\tau_1)}e^{inC_\beta} \right. \\
+ nTrD^{jj}_{00}e^{ip^{jj}(\tau_2-\tau_1)}e^{inC_\beta} \right] \\
+ \int_0^\beta d\tau_2 \int_0^\beta d\tau_1 \left\{ (n-1)Tr \left( D^{ij}_{00}e^{ip^{ij}(\tau_2-\tau_1)}e^{inC_\beta} \\
+ D^{ij}_{00}e^{ip^{jj}(\tau_2-\tau_1)}e^{inC_\beta} \right) \\
+ (n-2)Tr \left( D^{ij}_{00}e^{ip^{ij}(\tau_2-\tau_1)}e^{inC_\beta} \\
+ D^{ij}_{00}e^{ip^{jj}(\tau_2-\tau_1)}e^{inC_\beta} \right) \\
+ \ldots \\
+ \left. Tr \left( D^{ij}_{00}e^{ip^{ij}(\tau_2-\tau_1)}e^{inC_\beta} \\
+ D^{ij}_{00}e^{ip^{jj}(\tau_2-\tau_1)}e^{inC_\beta} \right) \right\} \right]
$$

B.6

Do the $\tau$-integrals and obtain, using the notation in 3.10:

$$
\frac{1}{N} Tr \langle P^n(A_0) \rangle = -\frac{g^2}{N} \left[ \frac{\beta}{\pi} nTr \left( D^{ii} - D^{jj} \right) \Delta^{ij}_{1}e^{inC_\beta} \\
+ n \left( TrD^{ij}_{(2)}e^{inC_\beta} \left( 1 - e^{-iC_{ij}} \right) \\
+ TrD^{jj}_{(2)}e^{inC_\beta} \left( 1 - e^{iC_{ij}} \right) \right) \\
+ \Delta^{ij}_{(2)}|1 - e^{iC_{ij}}|^2 \left\{ (n-1)Tr \left( D^{ii}e^{inC_\beta}e^{-iC_{ij}} + D^{jj}e^{inC_\beta}e^{iC_{ij}} \right) \\
+ (n-2)Tr \left( D^{ii}e^{inC_\beta}e^{-i2C_{ij}} + D^{jj}e^{inC_\beta}e^{i2C_{ij}} \right) \\
+ \ldots \\
+ Tr \left( D^{ii}e^{inC_\beta}e^{-i(n-1)C_{ij}} + D^{jj}e^{inC_\beta}e^{i(n-1)C_{ij}} \right) \right\} \right] \right]
$$

B.7

The terms in the braces in B.7 do add up pairwise: the first and the last to $n e^{i(\frac{n}{2}-1)C_{ij}} e^{i(\frac{n}{2}+C_i+C_j)\beta}$, the second and the one but last to $n e^{-i(\frac{n}{2}-1)C_{ij}} e^{i(\frac{n}{2}+C_i+C_j)\beta}$, etc... Thus gathering all terms proportional to $\Delta^{ij}_{(2)}$, we obtain as coefficient:

$$
n e^{i\frac{n}{2}(C_i+C_j)\beta} \left\{ e^{i\frac{n}{2}C_{ij}} - e^{i(\frac{n}{2}-1)C_{ij}} + \left( 2 - e^{iC_{ij}} - e^{-iC_{ij}} \right) \left( e^{i(\frac{n}{2}-1)C_{ij}} + e^{i(\frac{n}{2}-2)C_{ij}} + e^{i(\frac{n}{2}-3)C_{ij}} + \ldots \right) + c\cdot c. \right\} \right.
$$

B.8
What is in the braces in B.8 adds up to zero. All what remains is the first term in B.7, which is nothing but eqn 3.15 in the main text.

Amusingly, we can find another form of 3.17:

$$\frac{\delta C_i - \delta C_j}{2\pi T} = \frac{g^2 N}{(4\pi)^2}(2 + 1 - \xi) B_1(C_{ij})$$

3.18

This form is only true in special directions in the Lie algebra, given by the generalized hypercharges:

$$Y_k = \frac{1}{N} diag(k, k, ...k, k-N, k-N, ...., k-N).$$

The entry k is integer, running from 1 to N-1. It is repeated N-k times, whereas k-N is repeated k times. In these directions the sign function has the wanted property

$$\sum_l (\epsilon(C_{il}) - \epsilon(C_{jl})) = N\epsilon(C_{ij}).$$
Appendix C Bernoulli polynomials

The Bernoulli polynomials come about naturally in thermal field theory. We take from ref (14)

\[ B_{2k}(x) \equiv \sum_{r=1}^{\infty} \frac{1}{r^{2k}} (-)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \cos r 2\pi x \]  

and

\[ 2kB_{2k-1}(x) \equiv B'_{2k}(x). \]  

Explicitely one has on the interval \(-1 \leq x \leq 1\):

\[ B_4(x) = x^2(1 - |x|)^2 - \frac{1}{30} \]  
\[ B_3(x) = x^3 - \frac{3}{2} x^2 \epsilon(x) + \frac{1}{2} x \]  
\[ B_2(x) = x^2 - |x| + \frac{1}{6} \]  
\[ B_1(x) = x - \frac{1}{2} \epsilon(x). \]  

\( \epsilon(x) \) is the sign function \( x/|x| \).

They are related to integrals of the type:

\[ \hat{B}_{d-2k} \equiv T \sum_{n_0} \int \frac{d^{d-1}\bar{P}}{(2\pi)^{d-1}} \frac{1}{((2\pi T n_0 + 2\pi T x)^2 + P^2)^k} \]  

\[ \hat{B}_{d-2k+1} \equiv T \sum_{n_0} \int \frac{d^{d-1}\bar{P}}{(2\pi)^{d-1}} \frac{(2\pi T n_0 + 2\pi T x)}{((2\pi T n_0 + 2\pi T x)^2 + P^2)^k} \]  

\[ \hat{B}_d \equiv T \sum_{n_0} \int \frac{d^{d-1}\bar{P}}{(2\pi)^{d-1}} \log((2\pi T n_0 + 2\pi T x)^2 + P^2) - \log((2\pi T n_0)^2 + P^2)) \]  

d is the number of dimensions of space time. For \( d = 4 \) we have

\[ \hat{B}_4(x) = \frac{2\pi^2}{3} T^4 \left( B_4(x) + \frac{1}{30} \right) \]  
\[ \hat{B}_3(x) = \frac{2\pi}{3} T^3 B_3(x) \]  
\[ \hat{B}_2(x) = \frac{T^2}{2} B_2(x) \]
\[ \hat{B}_1(x) = -\frac{T}{4\pi} B_1(x) \quad \text{C.10} \]

Note that \( \hat{B}_4 \) and \( \hat{B}_2 \) are even in \( x \), \( \hat{B}_3 \) and \( \hat{B}_1 \) odd. Hence \( \hat{B}_2 \) and \( \hat{B}_1 \) are not analytic in \( x = 0 \).

In deriving 4.3 from 4.7 and 4.8 we have used the identities:

\[ B_1(x)B_3(x) = \tilde{B}_4(x) + \frac{1}{4}\tilde{B}_2(x) \quad \text{C.11} \]

and

\[ (B_2(x))^2 = \tilde{B}_4(x) + \frac{1}{3}\tilde{B}_2(x) + B_2^2(0) \quad \text{C.12} \]

and

\[ \tilde{B}_2(x) \equiv B_2(x) - \frac{1}{6} \quad \text{C.13} \]

Since it occurs frequently we define

\[ \tilde{B}_4 \equiv B_4 + \frac{1}{30} \quad \text{C.14} \]

For the evaluation of \( \Delta_{ij}^{(1)} \) in eqn 3.10 it is useful to do the \( d-1 \) momentum integrations first, to find a \( \zeta \)-function of the type\(^\text{14})\):

\[ \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \quad \text{C.15} \]

with \( z = 4 - d \). Using\(^\text{14})\)

\[ \zeta(-z, q) = -\frac{B'_{z+2}(q)}{(z+1)(z+2)} = -\frac{B_{z+1}(q)}{z+1} \quad \text{C.16} \]

valid for \( z \) non-negative, one finds

\[ T \sum_{n_0} \int \frac{d\vec{p}}{(2\pi)^{d-1}} \frac{1}{p_0^{ij}(p^{ij})^2} = \frac{1}{2\pi}B_1(C_{ij}) \quad \text{C.17} \]

as in 3.14(b).

The gauge term in \( \Delta_{ij}^{(1)} \) follows immediately from C.5 and C.10.

\( \Delta_{ij}^{(2)} \) will have terms like in C.15, but now with \( z=5-d \). This gives us a pole\(^\text{14}) \) at \( d=4 \).
Appendix D Some group theory relations and two relations for the gluon potential

We introduce two unitarily related bases on the Lie algebra of $SU(N)$.

1) The Gell Mann basis $\lambda_a$, which is hermitean, traceless and normalised

$$Tr\lambda_a\lambda_b = \frac{1}{2}\delta_{ab} \quad D.1$$

2) The Cartan basis $\lambda^{ij}$, $\lambda^d\quad (i \neq j), \quad d = 2, 3, \ldots, N$, with :

$$(\lambda^{ij})_{kl} \equiv \frac{1}{\sqrt{2}}\delta_{ik}\delta_{jl}, \quad \lambda_d = \frac{1}{r_d}\text{diag}(1, 1, \ldots, 1 - d, 0, 0, 0), \quad r_d \equiv \sqrt{2d(d - 1)}.$$  

In both cases we define the structure constants $f$ as :

$$if_{abc} \equiv 2TrM_a[M_b,M_c] \quad D.2$$

In basis 2) we have

$$|f_{ij,jk,ki}|^2 = \frac{1}{2} \quad D.3$$

and

$$\sum_{d=2}^{N} |f_{ij,ji,d}|^2 = 1 \quad D.4$$

All other structure constants vanish in basis 2).

It is useful to note that there are $2N(N - 1)(N - 2)$ $f$’s like in D.3, and that there are $3N(N - 1)$ sums like in D.4.

So, in basis 1) we have

$$\sum_{a,b,c} |f_{a,b,c}|^2 = N(N^2 - 1) \quad D.5$$

and in basis 2) we find again :

$$\sum_{a,b,c} |f_{a,b,c}|^2 = \frac{1}{2} \cdot 2N(N - 1)(N - 2) + 3N(N - 1) = N(N^2 - 1) \quad D.6$$

as it should, because of the unitary relationship between the two sets.

For the fermionic two-loop contribution 5.4 we need another identity for the coefficients $r_k$ defined under D.1 :

$$\frac{(d - 1)^2}{r_d^2} + \sum_{k=d+1}^{N} \frac{1}{r_k^2} = \frac{1}{2} \frac{N - 1}{N} \quad D.7$$
D.7 is true for all \( d = 2, \ldots, N-1 \). It is useful when computing the exchange of all diagonal gluons in fig 1(d).

Let us finally relate the determinant \( ||t_{k,d}^{(1)}(C)|| \) (appearing in the first term of eqn 2.21) to the van der Monde determinant

\[
\prod_{1 \leq i \leq j \leq N} (\exp \imath C_i - \exp \imath C_j) \quad D.8
\]

This equality is true up to a factor independent of \( C \). This determinant was defined as the Jacobian between the variables \( t_1(C), t_2(C), \ldots t_{N-1}(C) \), eqn 2.1(a), and the variables \( C_d \equiv \text{Tr} \lambda_d \), \( d = 2, \ldots, N \). From this it follows immediately, that a generic matrix element reads:

\[
t_{k,d}^{(1)}(C) = \frac{k}{N r_d} (\exp \imath k C_1 + \exp \imath k C_2 + \ldots - (d-1) \exp \imath k C_d) \quad D.9
\]

From this expression it is easy to see that the determinant of the \((N-1)\times(N-1)\) matrix \( t_{k,d}^{(1)}(C) \) can be written as the determinant of the \( N \times N \) van der Monde matrix, by adding and subtracting columns, and D.9 follows.
REFERENCES

1) D. Gross, R. Pisarski, L. Yaffe, Rev. Mod. Phys. 53 (1981) 43
N. Weiss, Phys. Rev. D. 24 (1981) 75 ; D 25 (1982) 2667
2) S. Huang, Y. Potuin, C. Rebbi, and S. Sanielevici, Phys. Rev. D 42 (1990) 2864
3) K. Kajantie, L. Kärkainen, K. Rummukainen, Nucl. Phys. B 357 (1991) 693
4) T. Bhattacharya, A. Gocksch, C.P. Korthals Altes, R.D. Pisarski, Phys. Rev. Lett. 66 (1991) 998
5) T. Bhattacharya, A. Gocksch, R.D. Pisarski, Nucl. Phys. B 383, (1992) 497
6) K. Enqvist, K. Kajantie, Z. Phys. C 47 (1990) 291
7) R. Anishetty, J. Phys. G. 10 (1984) 439
8) R. Fukuda, E. Kyriakopoulos, Nucl. Phys. B 85 (1975) 354
9) L. O'Reifearteigh, A. Wipf, H. Yoneyama, Nucl. Phys. B 271 (1986) 653
10) C. Becchi, R. Rouet, R. Stora, Ann. Phys. (N.Y.) 98 (1976) 287
11) C.P. Korthals Altes, in Progress in Gauge Theory, Cargèse 1983, eds ’t Hooft et al, Plenum 1984.
12) V.M. Belyaev, Phys. Lett. B 254 (1991) 153
13) A. Gocksch, R.D. Pisarski, BNL-GP-1/93
14) I.M. Ryshik, I.S. Gradsteijn, Table of Integrals Series and Products, 9.622, Academic Press (1965)
15) V. Dixit, M. Ogilvie, Phys. Lett. B 269, 353 (1991)
16) To be published.
17) T. Bhattacharya, C.P. Korthals Altes, in preparation.
18) B. Berg, T. Neuhaus, Phys. Rev. Lett. 68 (1992) 9
19) B. Grossmann, M.L. Laursen, in ”Dynamics of First Order Phase Transitions”, eds H. J. Herrmann, W. Janke, F. Karsch, World Scientific, 1992
20) C.P. Korthals Altes, in Hot Summer Daze, Proceedings of the BNL Summer Study on QCD at non-zero temperature and density, Eds A.Gocksch and R.Pisarski, World Scientific, 1992
Fig 1: Graphs obtained from 2.12 with the topology of the free energy (a,b,c). The two loop fermion graph is shown in 1.d. The symmetry factor is explicitly written. Continuous (dashed) lines are gluons (ghosts). Constant background gauge is used, hence all energies are shifted through a constant amount. Their contribution $U_f^{(2)}$ is given in 4.7 (for $\xi = 1$) and 2.35 (if $\xi \neq 1$) for the gluons, and in 5.4 for the fermions.
Fig 2: Zero momentum insertion accompanied by renormalisation of the Polyakov-loop, $U_p^{(2)}$ in eqn.2.28. The dot on the loops are the zero-momentum insertions 2.28(b).
Fig 3: Relation between gauge variation of the free energy graphs (a), the gauge variation of the gluon self energy (b), and the zero-momentum insertion into the (cross-hatched) one loop free energy (c) through the BRST identity 2.32. The double bars represent the contraction of the momentum of the line into the vertex. The loop in the left part of fig.3(c) equals the gauge variation of the renormalisation of the Polyakov loop in fig 2.
Fig 4: The potential 4.2 and 4.3 for $N = 3$ and $n_f = 0$, as function of real and imaginary part of the loop. $V_{eff}$ stands for $\frac{3}{8\pi^2} U$. 
Fig 5: As in Fig 4, with $n_f = 2$, using 5.2 and 5.4.
Fig 6: The q-valley profile for the case $N=3$ and $n_f = 6$. Solid curve is the one loop result 4.2 and 5.2., the dotted curve the one and two loop result at $\alpha_s=0.1$. The dashed curve is the pure gluon result.